Revisiting Clifford algebras and spinors II: Weyl spinors in $\mathbb{C}\ell_{3,0}$ and $\mathbb{C}\ell_{0,3}$ and the Dirac equation

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Abstract

This paper is the second one of a series of three and it is the continuation of [1]. We review some properties of the algebraic spinors in $\mathbb{C}\ell_{3,0}$ and $\mathbb{C}\ell_{0,3}$ and how Weyl, Pauli and Dirac spinors are constructed in $\mathbb{C}\ell_{3,0}$ (and $\mathbb{C}\ell_{0,3}$, in the case of Weyl spinors). A plane wave solution for the Dirac equation is obtained, and the Dirac equation is written in terms of Weyl spinors, and alternatively, in terms of Pauli spinors. Finally the covariant and contravariant undotted spinors in $\mathbb{C}\ell_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$ are constructed. We prove that there exists an application that maps $\mathbb{C}\ell_{3,0}^+$, viewed as a right $\mathbb{H}$-module, onto $\mathbb{C}\ell_{0,3}^+$, but now viewed as a left $\mathbb{H}$-module.

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Introduction

The theory of spinors was developed practically in an independent way by mathematicians and physicists. On the one hand, E. Cartan in 1913 wrote a treaty about spinors [2], after he has originally discovered them as entities that carry representations of the rotation groups associated to finite-dimensional vector spaces. He was investigating linear representations of simple groups. On the other hand, spinors were introduced in physics in order to describe the wave function of quantum systems with spin. W. Pauli, in 1926, described the electron wave function with spin by a 2-component spinor in his non-relativistic theory [3]. After, in 1928, P. A. M. Dirac used a 4-component spinor to investigate the relativistic quantum mechanics formalism [4]. With the increasing use of spinors in physical theories, L. Infeld and B. L van der Waerden [5] wrote a treaty, but their formalism is not so simple for an undergraduate student in mathematics, or physics, to learn. In physical theories, spinors are fundamental entities describing matter, constituted by leptons and quarks, since they are spin 1/2 fermions [6] naturally described by Dirac spinors. From the algebraic viewpoint, spinors are elements of a lateral minimal ideal of a Clifford algebra. This was introduced by C. Chevalley [7].

The main aim of this paper is to formulate the paravector model of spacetime, using algebraic Weyl spinors, and to describe Weyl spinors in the Clifford algebras $\mathcal{C}^{\ell}_{3,0}$ and $\mathcal{C}^{\ell}_{0,3}$. Consequently Dirac spinors are naturally introduced, together with (algebraic) Pauli spinors. This paper is organized as follows: In Sec. 1, we present some brief mathematical preliminaries concerning Clifford algebras. In Sec. 2 contravariant and covariant, dotted and undotted Weyl spinors are introduced in $\mathcal{C}^{\ell}_{3,0}$, together with the spinorial transformations associated to each one of them. In this way, Dirac spinors are naturally presented as elements of the Clifford algebra $\mathcal{C}^{\ell}_{3,0}$ over $\mathbb{R}^3$. In Sec. 3 the Dirac-Hestenes equation (DHE) in $\mathcal{C}^{\ell}_{3,0}$ is introduced. DHE is written as two coupled Weyl equations, using Weyl spinors, and alternatively, using Pauli spinors. The null tetrad in spacetime is introduced, using Weyl spinors to construct the paravector model of spacetime. In Sec. 4 we construct contravariant and covariant (Weyl) dotted spinors in $\mathcal{C}^{\ell}_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$ and define an application that maps $\mathcal{C}^{\ell+}_{0,3}$, viewed as a right module (over the quaternion ring $\mathbb{H}$) onto $\mathcal{C}^{\ell+}_{0,3}$, but now viewed as a left $\mathbb{H}$-module.

1 Preliminaries

Let $V$ be a finite $n$-dimensional real vector space. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) := \bigoplus_{k=0}^{n} \Lambda^k(V)$ of multivectors over $V$ ($\Lambda^k(V)$ denotes the space of the antisymmetric $k$-tensors). The reversion of $\psi \in \Lambda(V)$, denoted by $\tilde{\psi}$, is an algebra anti-automorphism given by $\tilde{\psi} = (-1)^{[k/2]} \psi$ ([k] denotes the integer part of $k$) while the main automorphism or graded involution of $\psi$, denoted by $\hat{\psi}$, is an algebra automorphism given by $\hat{\psi} = (-1)^k \psi$. The conjugation, denoted by $\bar{\psi}$, is defined to be the reversion followed by the main automorphism. If $V$ is endowed with a non-degenerate, symmetric, bilinear map $g : V \times V \rightarrow \mathbb{R}$, it is possible to extend $g$ to $\Lambda(V)$. Given $\psi = u_1 \wedge \cdots \wedge u_k$ and

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1Dirac spinors are defined as the sum of two elements, called Weyl spinors, that respectively carry two non-equivalent representations of the group SL(2,C).

2Pauli spinors are elements of the representation space of the group SU(2).

3Or parity operator.
We now define the Weyl spinors \[8\]:

Let \( \psi \) be an element of \( V \) with its \( p \)-vector part given by \( \langle \psi \rangle_p := \psi_p \). The Clifford product between \( v \in V \) and \( \psi \in \Lambda(V) \) is given by \( v\psi = v \wedge \psi + v \cdot \psi \). The Grassmann algebra \( \Lambda(V), g \) endowed with this product is denoted by \( \mathcal{C}(V, g) \) or \( \mathcal{C}_{p,q} \), the Clifford algebra associated to \( V \approx \mathbb{R}^{p,q} \), \( p + q = n \).

2 Weyl spinors in \( \mathcal{C}_{3,0} \)

In this section Weyl spinors and spinorial metrics are constructed.

2.1 Weyl spinors and spinor metrics

Let \( \{e_1, e_2, e_3\} \) be an orthonormal basis of \( \mathbb{R}^3 \). The Clifford algebra \( \mathcal{C}_{3,0} \) is generated by \( \{1, e_1, e_2, e_3\} \), such that

\[
g(e_i, e_j) = 2\delta_{ij} = (e_i e_j + e_j e_i), \quad i, j = 1, 2, 3. \tag{1}\]

An arbitrary element of \( \mathcal{C}_{3,0} \) can be written as

\[
\psi = s + a^i e_i + a^2 e_2 + a^3 e_3 + b^{12} e_{12} + b^{13} e_{13} + b^{23} e_{23} + p e_{123}, \quad s, a^i, b^{ij}, p \in \mathbb{R}. \tag{2}\]

Let \( f_\pm = \frac{1}{2}(1 \pm e_3) \) be primitive idempotents of \( \mathcal{C}_{3,0} \), clearly satisfying the relations \( f_+ f_- = f_- f_+ = 0 \) and \( f_\pm^2 = f_\pm \). The matrix representation

\[
f_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 f_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 f_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}\]

is used heretofore. Each one of the four elements above generates a (left or right) minimal ideal.

The isomorphism

\[
\mathcal{C}_{3,0} f_+ \simeq \mathcal{C}^+_{3,0} f_+ \tag{4}\]

is not difficult to see, since

\[
\mathcal{C}^+_{3,0} f_+ \ni \phi f_+ = \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix} f_+ \\
= \begin{pmatrix} w_1 & 0 \\ w_2 & 0 \end{pmatrix} \simeq \begin{pmatrix} w_1 & w_3 \\ w_2 & w_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}_{3,0} f_+. \tag{5}\]

Then there is redundancy if \( \psi f_+ \), with \( \psi \) given by eq. (2), is written. From the isomorphism in (4) we need only to use \( \psi f_+ \), with \( \psi = s + b^{12} e_{12} + b^{13} e_{13} + b^{23} e_{23} \in \mathcal{C}^+_{3,0} \).

We now define the Weyl spinors [5]:

- **Contravariant undotted spinor (CUS):**

  \[
  \mathcal{K} = \psi f_+ \tag{6}\]

Such spinor is written as

\[
\mathcal{K} = \psi f_+ = (s + b^{12} e_{12} + b^{13} e_{13} + b^{23} e_{23}) f_+ \\
= (s + b^{12} e_{123})(f_+) + (b^{13} + b^{23} e_{123})(e_1 f_+) \\
= k^1(f_+) + k^2(e_1 f_+). \tag{7}\]
where
\[ k^1 = s + b^{12}e_{123} \quad \text{and} \]
\[ k^2 = b^{13} + b^{23}e_{123}. \] (8)

The CUS are chosen to be written in this form because their components commute with the basis \( \{ f_+, e_1 f_+ \} \) of the algebraic spinors. Therefore all spinor components are written as elements of the center of \( C\ell_{3,0} \), that is well-known to be isomorphic to \( \Lambda^0(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3) \).

From the spinor \( K \), other three types of spinors in \( C\ell_{3,0} \) are constructed:

- **Covariant undotted spinor** (CVUS):
  \[ K^* := e_1 K \] (9)

By this definition, the expression
\[ K^* = e_1 (k_1 f_+ + k_2 e_1 f_+) = e_1 (f_- k^1 + f_- (-e_1) k^2) = (-k^2) f_+ + (k^1) (f_+ e_1) \] (10)

can immediately be written. Since \( K^* \in f_+ C\ell_{3,0} \), we write
\[ K^* = k_1 (f_+) + k_2 (f_+ e_1), \] (11)
from where the relation
\[ \begin{align*}
  k_1 &= -k^2 \\
  k_2 &= k^1
\end{align*} \] (12)
follows. Note that these relations are the ones obtained in the classical approach [5, 9].

Now, given \( K^* \in f_+ C\ell_{3,0} \) and \( \eta = \eta^1 f_+ + \eta^2 e_1 f_+ \in C\ell_{3,0} f_+ \), the spinorial metric associated to the idempotent \( f_+ \) is defined:

\[ G_{f_+} : f_+ C\ell_{3,0} \times C\ell_{3,0} f_+ \to f_+ C\ell_{3,0} f_+ \cong \mathbb{C} f_+ \]
\[ (K, \eta) \mapsto K^* \eta = (-k^2 f_+ + k^1 f_+ e_1) (\eta^1 f_+ + \eta^2 e_1 f_+) \] (13)

which results in the expression
\[ G_{f_+} (K, \eta) = K^* \eta = (-k^2 \eta^1 + k^1 \eta^2) (f_+) \] (14)
This definition coincides with the classical one [5, 9], where the scalar product has mixed and antisymmetric components. The idempotent \( f_+ \) is the unit of the algebra \( f_+ C\ell_{3,0} f_+ \cong \mathbb{C} \).

From the spinor \( K \) we also define the

- **Contravariant dotted spinor** (CDS):
  \[ \overline{K} := e_1 \overline{K} \] (15)
This definition results in the expression

\[
\mathbf{K} = e_1(k_1 f_+ + k_2 e_1 f_+) = e_1(f_+ \mathbf{k} + f_1 e_1 \mathbf{k}') = \mathbf{k}^1(e_1 f_+) + \mathbf{k}^2 f_- = \mathbf{k}^3(f_+ - e_1) + \mathbf{k}^2(f_-). \tag{16}
\]

But \( \mathbf{K} \in f_-\mathbb{C}\ell_{3,0} \), and it is possible to write

\[
\mathbf{K} = \mathbf{k}'_1(f_- - e_1) + \mathbf{k}'_2(f_-). \tag{17}
\]

Then the relation

\[
\mathbf{k}'_1 = \mathbf{k}^1, \quad \mathbf{k}'_2 = \mathbf{k}^2 \tag{18}
\]

is obtained. Besides \(^4\), \( \mathbf{k}\Lambda = (a + b e_{123}) = (a + b e_{321}) = a - b e_{123} \), which suggests the notation \(^5\)

\[
\mathbf{k}\Lambda = \mathbf{k}\Lambda. \tag{19}
\]

Finally the

- **Covariant dotted spinor** (CVDS) is constructed:

\[
\mathbf{K}^\ast := (e_1\mathbf{K}) \tag{20}
\]

from where it can be shown that

\[
\mathbf{K}^\ast = \frac{(e_1\mathbf{K})}{-(e_1)} = \frac{(e_1\mathbf{K})}{(f_- - e_1) + (e_1 f_+)} = \frac{(e_1 f_+ \mathbf{k} + f_1 e_1 \mathbf{k}')}{(\mathbf{k}^1 f_- - \mathbf{k}^2 f_+ e_1)} = \frac{(-\mathbf{k}'_1)(e_1 f_-) + (\mathbf{k}'_2)(f_-)}{\mathbf{k}^3(f_+ - e_1) + \mathbf{k}^2(f_-)}. \tag{21}
\]

As \( \mathbf{K}^\ast \in \mathbb{C}\ell_{3,0} f_- \), we can write

\[
\mathbf{K}^\ast = (\mathbf{K}'_1)(e_1 f_-) + (\mathbf{K}'_2)(f_-). \tag{22}
\]

Therefore the following relation is obtained:

\[
\begin{align*}
\mathbf{K}'_1 & = \frac{-\mathbf{k}'_2}{\mathbf{k}^3}, \\
\mathbf{K}'_2 & = \frac{\mathbf{k}'_1}{\mathbf{k}^3}.
\end{align*} \tag{23}
\]

\(^4\) \( A = 1, 2 \).

\(^5\) Denoting \( \Lambda^0(\mathbb{R}^3) \) the subspace of the scalars and \( \Lambda^3(\mathbb{R}^3) \) the space of the pseudoscalars, the isomorphism \( \Lambda^0(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3) \simeq \mathbb{C} \) is evident, since \( (e_1^2 = -1) \). The notation \( k^\Lambda = \mathbf{k}\Lambda \) is also immediate, since if \( e_{123} \) is denoted by \( \mathbf{I} \simeq i \in \mathbb{C} \), the reversion in \( \Lambda^0(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3) \) is equivalent to the \( \mathbb{C} \)-conjugation.
These relations are the ones obtained in van der Waerden paper [5], and unceasingly exposed in Penrose seminal works [9, 10, 11, 12, 13].

Now, given $K \in f_-C_\ell_{3,0}$ and $\eta^* = \eta^{i'}e_1f_+ + \eta^{2'}f_- \in C_\ell_{3,0}f_-$, the spinorial metric associated to the idempotent $f_-$ is obtained:

$$G_{f_-} : f_-C_\ell_{3,0} \times C_\ell_{3,0}f_- \rightarrow f_-C_\ell_{3,0}f_- \simeq \mathbb{C}f_-$$

$$(K, \eta^*) \mapsto \overline{K}\eta^* = (\eta^{i'}k^{i'}e_1 + \eta^{2'}k^{2'}f_-)(-\eta^{i''}\epsilon_1 f_+ + \eta^{i'}f_-), \quad (24)$$

which results in

$$G_{f_-} = \overline{K}\eta^* = (\eta^{i'}k^{i'} - \eta^{2'}k^{2'})f_- \quad (25)$$

The expressions for the four algebraic Weyl spinors, as elements of a lateral ideal of $C_\ell_{3,0}$, are listed below:

- **Contravariant undotted spinor** (CUS):
  $$K = k^1(f_+) + k^2(e_1f_+) \in C_\ell_{3,0}f_+ \quad (26)$$

- **Covariant undotted spinor** (CVUS):
  $$K^* = e_1K = k_1(f_+) + k_2(f_+e_1) \in f_+C_\ell_{3,0} \quad (27)$$

- **Contravariant dotted spinor** (CDS):
  $$\overline{K} = e_1\overline{k} = \overline{k}^{i'}(f_-e_1) + \overline{k}^{2'}(f_-) \in f_-C_\ell_{3,0} \quad (28)$$

- **Covariant dotted spinor** (CVDS):
  $$\overline{K}^* = -(\overline{K})e_1 = (e_1\overline{K}) = \overline{k}^{i'}(e_1f_-) + \overline{k}^{2'}(f_-) \in C_\ell_{3,0}f_- \quad (29)$$

The following diagram illustrates how we can pass from one ideal to the others, obtaining in this way all the four Weyl spinors, using (anti-)automorphisms of $C_\ell_{3,0}$:

![Diagram](image-url)

We then have the correspondence between the formalism of this section and the notation exhibited in [9]:

$$\mathbb{A} \longleftrightarrow C_\ell_{3,0}f_+, \quad \mathbb{A}_+ \longleftrightarrow f_+C_\ell_{3,0}, \quad \mathbb{A}' \longleftrightarrow f_-C_\ell_{3,0}, \quad \mathbb{A}'_+ \longleftrightarrow C_\ell_{3,0}f_- \quad (30)$$
2.2 Spinorial tranformations

An arbitrary element \( R \in \mathcal{C} \ell_{3,0} \) can be written as
\[
R = s + v^i e_i + b^{ij} e_{ij} + p e_{123} = \alpha + \beta e_{12} + \gamma e_{13} + \delta e_{23},
\]
where \( \alpha = s + p e_{123}, \quad \beta = b^{12} - v^3 e_{123}, \quad \gamma = b^{13} + v^2 e_{123}, \) and \( \delta = b^{23} - v^1 e_{123} \).

Under the action of \( R \), a CUS \( K \) transforms as
\[
RK = R(\psi f_+) = k^1(Rf_+) + k^2(Re_1 f_+),
\]
where we denote:
\[
Rf_+ = (\alpha + \beta e_{123})f_+ + (\gamma + \delta e_{123})(e_1 f_+),
Re_1 f_+ = (\alpha - \beta e_{123})f_+ + (-\gamma + \delta e_{123})(e_1 f_+).
\]

A matrix representation \( \rho : \mathcal{C} \ell_{3,0} \rightarrow \mathcal{M}(2, \mathbb{C}) \) of \( R \) is given by:
\[
\rho(R) = \begin{pmatrix} \alpha + \beta i & -\gamma + \delta i \\ -\gamma + \delta i & \alpha - \beta i \end{pmatrix}.
\]
It follows that
\[
det \rho(R) = \alpha^2 + \beta^2 + \gamma^2 + \delta^2.
\]

Under the automorphism and antiautomorphisms of \( \mathcal{C} \ell_{3,0} \), respectively the graded involution, the reversion and the conjugation, the multivector \( R \in \mathcal{C} \ell_{3,0} \) transforms as:
\[
\hat{R} = \sigma + \beta e_{12} + \gamma e_{13} + \delta e_{23},
\]
\[
\tilde{R} = \sigma - \beta e_{12} - \gamma e_{13} - \delta e_{23},
\]
\[
\bar{R} = \alpha - \beta e_{12} - \gamma e_{13} - \delta e_{23}.
\]

Then we have the relation
\[
RR = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = det \rho(R)
\]

Given \( R \in \mathfrak{spin}_+(1,3) \), i.e., \( RR = 1 \), we see that \( det \rho(R) = 1 \) and \( R \in SL(2, \mathbb{C}) \). Then the isomorphism
\[
\mathfrak{spin}_+(1,3) \simeq SL(2, \mathbb{C})
\]
is explicitly exhibited.

From the condition \( RR = 1 \) it follows that \( R = R^{-1} \), and the following transformation rules are verified in this formalism:
\[
\mathcal{K} \quad \mapsto \quad RK, \\
\mathcal{K}^* \quad \mapsto \quad e_1(\overline{RK}) = e_1K\overline{R} = K^*R^{-1}, \\
\overline{K} \quad \mapsto \quad e_1(\overline{RK}) = e_1\overline{K}\overline{R} = \overline{K}(\overline{R}) = \overline{K}\overline{R}^{-1}, \\
\overline{K}^* \quad \mapsto \quad (\overline{RK})^* = \overline{K}\overline{R}^*.
\]

which permits to represent
\[
\mathcal{K} \quad \mapsto \quad RK \\
\mathcal{K}^* \quad \mapsto \quad K^*R^{-1} \\
\overline{K} \quad \mapsto \quad \overline{K}(\overline{R})^{-1} \\
\overline{K}^* \quad \mapsto \quad \overline{K}\overline{R}^*.
\]
In this way it is proved that the transformations of CUS, CVUS, CDS and CVDS under $\text{Spin}(1,3) \simeq \text{SL}(2,\mathbb{C})$ are the same as pointed in the classical formalism \[5,9\]. Indeed, we see that
\[
\rho(\hat{R}) = [\rho(\hat{R})^\dagger]^{-1},
\]
from where expressions \[39\] follow. Therefore we have the correspondence:

\[
\mathcal{K} \leftrightarrow \begin{pmatrix} k_1 & 0 \\ k_2 & 0 \end{pmatrix}, \quad \overline{\mathcal{K}} \leftrightarrow \begin{pmatrix} 0 & \bar{k}'_1 \\ \bar{k}'_2 & \bar{k}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \bar{k}_2 & \bar{k}_1 \end{pmatrix},
\]
\[
\mathcal{K}^* \leftrightarrow \begin{pmatrix} -k_2 & k_1 \\ 0 & 0 \end{pmatrix}, \quad \overline{\mathcal{K}}^* \leftrightarrow \begin{pmatrix} 0 & \bar{k}'_1 \\ \bar{k}'_2 & \bar{k}_2 \end{pmatrix}.
\]

In order to write the four Weyl spinors (and subsequently the Dirac spinor), it is enough to consider the ideal $\mathcal{C} \ell_{3,0} f_+$ and, using the right and left Clifford product by $e_1$, the reversion and the graded involution, the other ideals $f_+, \mathcal{C} \ell_{3,0}, \mathcal{C} \ell_{3,0} f_-, f_- \mathcal{C} \ell_{3,0}$ are immediately obtained. The other three types of Weyl spinors are respectively elements of these ideals.

## 3 Dirac theory in the paravector model of $\mathcal{C} \ell_{3,0}$

In this section we introduce, according to Hestenes \[14,15,16,17\] and Lounesto \[18,19\], the algebraic description of the Dirac spinor and present the Dirac-Hestenes equation (DHE). We first reproduce some important results that are in, e.g. \[18,19\].

The Dirac equation for a quantum relativistic particle of mass $m$, described by $\psi$, in a background with electromagnetic potential $A$, is written as\(^6\) \[20\]

\[
\gamma^\mu(i\partial_\mu - eA_\mu)\psi = (i\partial - eA)\psi = m\psi, \quad \psi \in \mathbb{C}^4
\]

An element $\Psi \in \mathcal{C} \ell_{1,3}^+ \simeq \mathcal{M}(4,\mathbb{C})$ is written as

\[
\Psi = c + c^{01}\gamma_{01} + c^{02}\gamma_{02} + c^{03}\gamma_{03} + c^{12}\gamma_{12} + c^{13}\gamma_{13} + c^{23}\gamma_{23} + c^{0123}\gamma_{0123}.
\]

From the standard representation of $\gamma_\mu$ \[20,21\] we have the correspondence:

\[
\rho(\Psi) = \begin{pmatrix}
    c - ic^{12} & c^{13} - ic^{23} & -c^{03} + i0^{123} & -c^{01} + ic^{02} \\
    -c^{13} + ic^{23} & c + ic^{12} & -c^{01} - ic^{02} & c^{03} + i0^{123} \\
    -c^{03} + ic^{0123} & c^{01} + ic^{02} & c - ic^{12} & c^{13} - ic^{23} \\
    -c^{01} - ic^{02} & c^{03} + i0^{123} & c^{13} - ic^{23} & c + ic^{12}
\end{pmatrix}
\equiv \begin{pmatrix}
    \phi_1 & -\phi^*_2 & \phi_3 & \phi^*_4 \\
    \phi_2 & \phi^*_1 & \phi_4 & -\phi^*_3 \\
    \phi_3 & \phi^*_4 & \phi_1 & -\phi^*_2 \\
    \phi_4 & -\phi^*_3 & \phi_2 & \phi^*_1
\end{pmatrix} \in \mathcal{M}(4,\mathbb{C})
\]

A Dirac spinor $\psi$ can be expressed as an element of the left ideal $(\mathbb{C} \otimes \mathcal{C} \ell_{1,3})f$, where $f = \frac{1}{4}(1 + \gamma_0)(1 + i\gamma_1)$. Since we have the isomorphism

\[
(\mathbb{C} \otimes \mathcal{C} \ell_{1,3})f \simeq \mathcal{C} \ell_{3,0} \simeq \mathcal{C} \ell_{3,0} f_+ \oplus \mathcal{C} \ell_{3,0} f_-,
\]

it is easy to see that the definition of a Dirac spinor as the sum of a CUS (an element of $\mathcal{C} \ell_{3,0} f_+$) and a CVDS (an element of $\mathcal{C} \ell_{3,0} f_-$) immediately follows.

\(^6\)It will be used natural units, such that $\hbar = 1$ and $c = 1$. 
A Dirac spinor can also be written as

$$\psi = \Phi \frac{1}{2}(1 + i\gamma_{12}) \in (\mathbb{C} \otimes \mathcal{C}_{1,3})f,$$  \hspace{1cm} (46)

where $\Phi = \Phi \frac{1}{2}(1 + \gamma_0) \in \mathcal{C}_{1,3}(1 + \gamma_0)$ is two times the real part of $\psi$. Therefore the following expression is obtained [18, 19]:

$$(\mathbb{C} \otimes \mathcal{C}_{1,3})f \ni \psi \simeq \mathbb{C} \otimes \begin{pmatrix}
\phi_1 & 0 & 0 & 0 \\
\phi_2 & 0 & 0 & 0 \\
\phi_3 & 0 & 0 & 0 \\
\phi_4 & 0 & 0 & 0 \\
\end{pmatrix} \simeq \mathbb{C} \otimes \begin{pmatrix}
\phi_1 & 0 \\
\phi_2 & 0 \\
\phi_3 & 0 \\
\phi_4 & 0 \\
\end{pmatrix} = \left( \begin{array}{c}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 \\
\end{array} \right) \in \mathbb{C}^4, \hspace{1cm} (47)$$

Besides, $4 \text{Re}(i\Phi) = \Psi \gamma_2 \gamma_1$, and the spinor $\Phi \in \mathcal{C}_{1,3}(1 + \gamma_0)$ is decomposed in even and odd parts:

$$\Phi = \Phi_0 + \Phi_1 = (\Phi_0 + \Phi_1) \frac{1}{2}(1 + \gamma_0) = \frac{1}{2}(\Phi_0 + \Phi_1 \gamma_0) + \frac{1}{2}(\Phi_1 + \Phi_0 \gamma_0), \hspace{1cm} (48)$$

It follows that $\Phi_0 = \Phi_1 \gamma_0$ and $\Phi_1 = \Phi_0 \gamma_0$. Taking the real part in $\mathbb{C} \otimes \mathcal{C}_{1,3}$ of eq. (42) one obtains

$$\exists \Phi \gamma_2 \gamma_1 - eA\Phi = m\Phi, \hspace{1cm} (49)$$

which can be decomposed again in even and odd parts, respectively:

$$\exists \Phi_0 \gamma_2 \gamma_1 - eA\Phi_0 = m\Phi_1,$$

$$\exists \Phi_1 \gamma_2 \gamma_1 - eA\Phi_1 = m\Phi_0. \hspace{1cm} (50)$$

The **Dirac-Hestenes equation** is written, denoting $\Psi = \Phi_1 = \Phi_0 \gamma_0$, as:

$$\exists \Psi \gamma_2 \gamma_1 - eA\Psi = m\Psi \gamma_0, \hspace{1cm} \Psi \in \mathcal{C}^+_{1,3} \hspace{1cm} (51)$$

All elements of the equation above are now multivectors of $\mathcal{C}^+_{1,3}$.

In order to simplify the notation we write

$$\Psi = c + c^1 \gamma_{01} + c^2 \gamma_{02} + c^3 \gamma_{03} + b^1 \gamma_{23} + b^2 \gamma_{31} + b^3 \gamma_{12} + b^0 \gamma_{0123}, \hspace{1cm} (c, b, c^\mu, b^\mu \in \mathbb{R})$$

$$= c^0 + c^k \gamma_{0k} - b^k \mathbb{J} e_k - b^0 \mathbb{J}, \hspace{1cm} (52)$$

where $\mathbb{J} = e_{123} = \gamma_{0123}$. In this way, given the electromagnetic potential $A = A^\mu \gamma_\mu$, it is valid the expression $\gamma_0 A = A^0 + A^k \gamma_{0k} = A^0 - A^k \mathbb{J} e_k$. Denoting $A = A^k \mathbb{J} e_k$, it follows that

$$A = A^0 - A. \hspace{1cm} (53)$$

Left multiplying eq. (51) by $\gamma_0$,

$$\gamma_0 \exists \Psi \gamma_2 \gamma_1 - e\gamma_0 A\Psi = m\gamma_0 \Psi \gamma_0, \hspace{1cm} (54)$$

and using the above notation we obtain

$$(\partial^0 - \partial^k \mathbb{J} e_k) \Psi \mathbb{J} e_3 - e(A^0 - A) \Psi = m\gamma_0 \Psi \gamma_0, \hspace{1cm} \Psi \in \mathcal{C}_{3,0} \simeq \mathcal{C}^+_{1,3}, \hspace{1cm} (55)$$

But $\gamma_0 \Psi \gamma_0$ is the **parity operator**, which will be denoted by $\Psi^P = \gamma_0 \Psi \gamma_0 = \hat{\Psi}$. Therefore,

$$\hat{\Psi} = c^0 - c^k \gamma_{0k} + b^k \mathbb{J} e_k - b^0 \mathbb{J}, \hspace{1cm} (56)$$
Denoting \( \partial^0 = \partial_t \) and \( \phi = A^0 \), eq. (55) is rewritten as

\[
(\partial_t - \partial^k e_k) \Psi e_3 - e(\phi - A) \Psi = m \hat{\Psi}, \quad \Psi \in \mathcal{C} \ell_{3,0},
\]

and then

\[
\partial_t \Psi + \nabla \Psi = \left[ e(A - \phi) \Psi - m \hat{\Psi} \right] e_3, \quad \Psi \in \mathcal{C} \ell_{3,0}.
\]

Eq. (58) is the **Dirac equation** written in \( \mathcal{C} \ell_{3,0} \). It does not contradict the impossibility of using \( 2 \times 2 \) matrices in order to describe the relativistic electron, because in the non-relativistic theory the wave function is represented by a complex 2-dimensional vector, with four real parameters, and the matrices act only by left multiplication. In the formalism presented in this section, \( \psi \) is represented by a \( 2 \times 2 \) complex matrix, with eight real parameters, where the matrices act by left and/or right multiplication.

3.1 The plane wave solution of the Dirac equation in \( \mathcal{C} \ell_{3,0} \)

The Dirac equation (eq. (58)) is written, in the absence of external fields, as:

\[
\partial_t \Psi + \nabla \Psi = -m \hat{\Psi} e_3, \quad \Psi \in \mathcal{C} \ell_{3,0}.
\]

Since eq. (60) is invariant under Lorentz transformations, we first solve the equation in a rest referential and, after this, a boost \( L \) is applied. In the Pauli algebra \( \mathcal{C} \ell_{3,0} \), the action of the momentum operator \( p \) on the wave function \( \Psi \in \mathcal{C} \ell_{3,0} \) is given by

\[
p \Psi = \nabla \Psi e_3.
\]

The eigenvector of \( p \) is \( p \in \mathbb{R}^3 \) and in the rest referential, \( p = 0 \). Then eq. (60) is written as

\[
\partial_t \Psi = -m \hat{\Psi} e_3
\]

Consider the solution of eq. (62) to be of the type

\[
\Psi = \Psi_0 \exp(-\mathfrak{J} e_3 \omega).
\]

Substituting this solution is eq. (62) we obtain

\[
\Psi_0 \omega = m \hat{\Psi}_0.
\]

It follows that, in the case of even multivectors, the relation \( \Psi_0 = \hat{\Psi}_0 \) is satisfied, and in these conditions \( \omega = m \). For odd multivectors, \( \Psi_0 = -\hat{\Psi}_0 \) and consequently \( \omega = -m \).

We first investigate elements of \( \Lambda^0(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3) \hookrightarrow \mathcal{C} \ell_{3,0} \). For \( \Psi_0 \in \Lambda^0(\mathbb{R}^3) \), the solution is given (up to scalars) by:

\[
\Psi = \exp(-\mathfrak{J} e_3 m t).
\]

On the other hand, for \( \Psi_0 \in \Lambda^2(\mathbb{R}^3) \), there are three possibilities: \( \Psi_0 = e_1 e_2, \Psi_0 = e_1 e_3 \) and \( \Psi_0 = e_2 e_3 \) (the general case is obtained by linearity).
1. The expression $e_1 e_2 = 3 e_3 = \exp(3 e_3 \pi/2)$ shows that the choice $e_1 e_2$ only adds the phase factor to the spinor (wave function).

2. The choice $\Psi_0 = e_2 e_3$ is obtained from $-(e_1 e_2)(e_1 e_3) = e_2 e_3$, and it only adds a phase factor to the third choice:

3. $\Psi_0 = e_1 e_3$.

Then the second solution of eq. (62) is given by

$$\Psi = e_1 e_3 \exp(-3 e_3 m t).$$

The two solutions given by eqs. (65) and (66) are positive frequency solutions \[4, 20\].

In order to obtain the other two solutions of the Dirac equation in $\mathcal{C} \ell_{3,0}$, we must investigate elements of $\Lambda^1(\mathbb{R}^3) \oplus \Lambda^3(\mathbb{R}^3) \mapsto \mathcal{C} \ell_{3,0}$. For $\Psi_0 \in \Lambda^3(\mathbb{R}^3)$, we have the solution:

$$\Psi = \mathcal{J} \exp(3 e_3 m t).$$

For the fourth solution, let $\Psi_0 \in \Lambda^1(\mathbb{R}^3)$. Then the solution is obtained if eq. (67) is left multiplied by $e_1 e_2$, $e_2 e_3$ or $e_1 e_3$, since the general case is obtained by linearity. From the same reason cited in the last paragraph, after eq. (65), all the choices other than $e_1 e_3$ are redundant, in the sense that they only add a phase factor in $\Psi$. With the choice $e_1 e_3$, we have $e_1 e_3 \mathcal{J} = e_2$. Then the fourth solution of eq. (62) is given by

$$\Psi = e_2 \exp(3 e_3 m t).$$

A general solution of eq. (62) is given by the linear span of the four solutions obtained, listed below:

$$\Psi^{(+)} = \exp(-3 e_3 m t)
\Psi^{(+)} = e_1 e_3 \exp(-3 e_3 m t)
\Psi^{(-)} = \mathcal{J} \exp(3 e_3 m t)
\Psi^{(-)} = e_2 \exp(3 e_3 m t)$$

It is worth to note that the general solution of $\Psi$ is given, up the phase factor $\Psi \mapsto \Psi \exp(\alpha 3 e_3)$. The general solution is given by the linear combination (with $\mathbb{C}$-coefficients) of the solutions given by eqs. (69). The $\mathbb{C}$-coefficients are of the form

$$(c + d 3 e_3), \quad c, d \in \mathbb{R}.$$

They right multiply the functions. The left linear combination is forbidden, since the operator $\nabla$ does not commute with this possibility. For a particle with momentum $p$ we obtain the solution if the boost $L = L(p)$ is applied \[24\]:

$$\Psi^{(+)} = L(p) \exp[-3 e_3 (E t - p \cdot x)]
\Psi^{(+)} = L(p) e_1 e_3 \exp[-3 e_3 (E t - p \cdot x)]
\Psi^{(-)} = L(p) \mathcal{J} \exp[3 e_3 (E t - p \cdot x)]
\Psi^{(-)} = L(p) e_2 \exp[3 e_3 (E t - p \cdot x)]$$

3.2 Dirac spinors

Penrose denotes a Dirac spinor as an element of $G^A \oplus G_A$. Since the Dirac spinor has four $\mathbb{C}$-components, it suggests that the Dirac spinor can be described as a multivector of $\mathcal{C} \ell_{3,0}$. In the present formalism, the Dirac spinor $\psi$ is an element of $\mathcal{C} \ell_{3,0} f_+ \oplus \mathcal{C} \ell_{3,0} f_- \simeq \mathcal{C} \ell_{3,0}$. Indeed,

$$\mathcal{C} \ell_{3,0} \ni \psi = \psi(f_+ + f_-) = \psi f_+ + \psi e_1 f_+ e_1 = K + L e_1,$$
where
\[
K = \psi f_+ \quad \text{and} \quad L = \psi (e_1 f_+).
\] (73)

### 3.3 Decomposition of the Dirac equation in terms of Weyl spinors

The Dirac equation in \(C\ell_{3,0}\), eq. (58), is led to two Weyl equations [21]. Indeed, consider eq. (58) without external fields:
\[
(\partial_t + \nabla)\psi e_3 = m\dot{\psi}, \quad \psi \in C\ell_{3,0}.
\] (74)

Besides, consider the decomposition \(\psi = \psi f_+ + \psi f_-\), where \(f_\pm = \frac{1}{2}(1 \pm e_3)\). We write
\[
\xi := \psi f_+ \in C\ell_{3,0}f_+
\]
\[
\hat{\eta} := \psi f_- \in C\ell_{3,0}f_-
\] (75)
since \(\hat{f}_+ = f_-\). The correspondence with the notation used in the last subsection is given by
\[
\xi = K \quad \text{and} \quad \hat{\eta} = Le_1.
\] (76)

It follows that
\[
(\partial_t + \nabla)(\xi + \hat{\eta})e_3 = m(\dot{\xi} + \dot{\eta}),
\] (77)
and then
\[
(\partial_t + \nabla)\xi \mathcal{J} - (\partial_t + \nabla)\hat{\eta} \mathcal{J} = m\dot{\xi} + m\dot{\eta},
\] (78)
where the relations \(\xi e_3 = \xi\) and \(\hat{\eta} e_3 = -\hat{\eta}\) follows from the fact that \(\xi \in C\ell_{3,0}f_+\) and \(\hat{\eta} \in C\ell_{3,0}f_-\).

Separating the terms in \(C\ell_{3,0}f_+\) and in \(C\ell_{3,0}f_-\) two equations are obtained:
\[
(\partial_t + \nabla)\xi \mathcal{J} = m\eta,
\] (79)
\[
-(\partial_t + \nabla)\hat{\eta} \mathcal{J} = m\dot{\xi}.
\] (80)

Taking the graded involution of the last equation, the Dirac equation in terms of the Weyl spinors gives the following coupled system:

\[
\begin{bmatrix}
(\partial_t + \nabla)\xi \mathcal{J} = m\eta \\
(\partial_t - \nabla)\eta \mathcal{J} = m\dot{\xi}
\end{bmatrix}
\] (81)

This result is presented, e.g., in [20, 21].

### 3.4 Decomposition of the Dirac equation in terms of Pauli spinors

A multivector \(\psi \in C\ell_{3,0}\) is written as
\[
C\ell_{3,0} \ni \psi = a + a^e e_i + a^ij e_{ij} + pe_{123} = a + a^{12}e_{12} + a^{23}e_{23} + a^{13}e_{13} + (a^3 + a^1e_{13} + a^2e_{23} + pe_{12})e_3 = \phi + \chi e_3, \quad \phi, \chi \in C\ell^+_{3,0}.
\] (82)

If we substitute in eq. (82) it follows that
\[
(\partial_t + \nabla)(\phi + \chi e_3)\mathcal{J}e_3 = m(\phi - \chi e_3),
\] (83)
This equation is separated in even and odd parts:

\[
\begin{align*}
\partial_t \phi & \mathcal{e}_1 + \nabla \chi \mathcal{J} = m \phi \\
\partial_t \chi & \mathcal{e}_3 + \nabla \phi \mathcal{J} = -m \chi
\end{align*}
\]  

(84)
a system of two coupled equations.

3.5 Paravectors of Minkowski spacetime obtained from Weyl spinors in \(C\ell_{3,0}\)

An arbitrary paravector \(\nu \in \mathbb{R} \oplus \mathbb{R}^3\) is written as

\[
\nu := K \mathcal{e}_1 \mathcal{K} = K \mathcal{e}_1 \tilde{\mathcal{K}} = K \mathcal{\tilde{K}}.
\]  

(85)

From eqs. (6) and (15), we obtain:

\[
K \mathcal{\tilde{K}} = \begin{pmatrix}
k^1 \mathcal{K}' & k^1 \mathcal{K}' \\
k^2 \mathcal{K}' & k^2 \mathcal{K}'
\end{pmatrix}.
\]  

(86)

Given

\[
\mathcal{K} = k^1 f_+ + k^2 e_1 f_+ \in C\ell_{3,0} f_+
\]  

(87)
and

\[
\tilde{\mathcal{K}} = \mathcal{k}' (f_- e_1) + \mathcal{k}' (f_-) \in C\ell_{3,0} f_-
\]  

(88)
we have the expression

\[
K \mathcal{\tilde{K}} = K \mathcal{e}_1 \mathcal{K} = k^1 \mathcal{k}' f_+ + k^1 \mathcal{k}' f_+ e_1 + k^2 \mathcal{k}' f_+ e_1 + k^2 \mathcal{k}' f_- e_1 + k^2 \mathcal{k}' f_- e_1 + k^2 \mathcal{k}' f_-
\]  

(89)

Using the representation in eq. (86), the correspondence

\[
\begin{array}{c c c c}
f_+ & \leftrightarrow & o^A o^{A'}, & f_+ e_1 & \leftrightarrow & o^A i^{A'}, & f_- e_1 & \leftrightarrow & i^A o^{A'}, & f_- & \leftrightarrow & i^A i^{A'}
\end{array}
\]  

(90)
is obtained. The idempotents of \(C\ell_{3,0}\), constructed from the vectors \(e_1\) and \(e_3\), can be identified with the null tetrad given in [9]. In this way, the null tetrad is constructed from spinors in Minkowski spacetime using the Clifford algebra \(C\ell_{3,0}\).

A paravector \(a \in \mathbb{R} \oplus \mathbb{R}^3 \in C\ell_{3,0}\) can be written as

\[
a = 2K \mathcal{e}_1 \mathcal{K} = 2K \mathcal{\tilde{K}}
\]  

(91)

Indeed, an operatorial spinor \(\psi \in C\ell_{3,0}^+\) is given by,

\[
\begin{align*}
a & = 2K \mathcal{\tilde{K}} = 2\hat{\psi} f_+ f_+ \mathcal{\tilde{\psi}} = 2\hat{\psi} f_+ \mathcal{\tilde{\psi}} = \psi(1 + \mathcal{e}_3) \mathcal{\tilde{\psi}} \\
& = \psi \psi + \psi \mathcal{e}_3 \mathcal{\tilde{\psi}} \\
& = a^0 + a^i e_i, \quad (i = 1, 2, 3).
\end{align*}
\]  

(92)

The paravector \(a\) points to the future:

\[
\begin{align*}
\psi \mathcal{\tilde{\psi}} &= a^0 \\
&= (a + b e_{12} + c e_{13} + d e_{23})(a - b e_{12} - c e_{13} - d e_{23}) \\
&= a^2 + b^2 + c^2 + d^2 \\
&> 0.
\end{align*}
\]  

(93)
Besides, from the relation $(a^i)^2 = (\psi\bar{\psi})^2 = (a^0)^2$, the paravector $a$ is null. Indeed,

$$a^2 := (a^0)^2 - (a^i)^2 = 0. \quad (94)$$

The last expression in eq. (92) follows from the property that it always possible to write $x \in \mathbb{R}^3$ as:

$$x = x^i e_i = \psi e_3 \bar{\psi}, \quad (95)$$

which is the composition of a rotation with a dilation $|a|$. Eq. (95), multiplied by $\hbar/2$, defines the spin density.

From eq. (89), two paravectors $a$ and $b$ are written as:

$$a = k^1 \mathcal{K}^i f_+ + k^1 \mathcal{K}^j f_+ e_1 + k^2 \mathcal{K}^i f_- e_1 + k^2 \mathcal{K}^j f_-$$

$$b = 0 + a^i e_i,$$

and their respective conjugation are given by

$$\hat{a} = k^1 \mathcal{K}^i f_- - k^1 \mathcal{K}^j f_+ e_1 - k^2 \mathcal{K}^i f_+ e_1 + k^2 \mathcal{K}^j f_+ f_-$$

$$0 - a^i e_i,$$

$$\hat{b} = r^1 \mathcal{R}^i f_- - r^1 \mathcal{R}^j e_1 f_- + r^2 \mathcal{R}^i f_+ e_1 + r^2 \mathcal{R}^j f_+ f_-$$

$$0 - b^i e_i. \quad (96)$$

The Clifford relation, for paravectors, is naturally obtained:

$$a \hat{b} + \hat{b} a = (k^1 \mathcal{K}^i r^2 \mathcal{R}^j + k^1 \mathcal{K}^j r^2 \mathcal{R}^i + k^2 \mathcal{K}^i r^3 \mathcal{R}^j + k^2 \mathcal{K}^j r^3 \mathcal{R}^i)$$

$$= 2(a^0 b^0 - a^i b_i)$$

$$= 2g(a, b). \quad (98)$$

### 4 The Clifford algebra $\mathcal{C}l_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$

Consider $\mathbb{R}^{0,3}$ and an orthonormal frame field $\{e_1, e_2, e_3\}$. The Clifford algebra $\mathcal{C}l_{0,3}$ is generated by $\{1, e_1, e_2, e_3\}$, that satisfies

$$g(e_i, e_j) = -\delta_{ij} = \frac{1}{2} (e_i e_j + e_j e_i), \quad (i, j = 1, 2, 3). \quad (99)$$

In particular $e^2 = -1$.

We first take the redundant dimensions out of the formalism, proving the

**Proposition**: $\mathcal{C}l_{0,3}f_+ \simeq \mathcal{C}l_{0,3}^+ f_+$, where $f_+ = \frac{1}{2} (1 + 3)$ and $f := e_1 e_2 e_3$. $\blacktriangleleft$

**Proof**: The left minimal ideal $\mathcal{C}l_{0,3}f_+$ is isomorphic to $\mathbb{H}$, as an algebra. Besides, an arbitrary element of $\mathcal{C}l_{0,3}$ is written as

$$A = a^0 + a^k e_k + b^1 e_1 + b^2 e_2 + b^3 e_3 + b^0 e_1 e_2 e_3 = a^0 + a^k e_k - b^k e_k e_1 e_2 e_3 + b^0 e_1 e_2 e_3. \quad (100)$$

It is easily seen that

$$Af_+ = [(a^0 + b^0) + (a^k - b^k)e_k]f_+$$

$$= [(a^0 + b^0) + (a^k - b^k)e_k e_1 e_2 e_3]f_+ \quad (101)$$
Therefore, given \( A f_+ \in \mathcal{C}_{0,3} f_+ \), and writing \( A' = (a^0 + b^0) + (a^k - b^k) \epsilon_k \epsilon_{123} \), we see that \( A' \in \mathcal{C}_{0,3}^+ \) and that \( A f_+ = A' f_+ \). This shows that \( \mathcal{C}_{0,3} f_+ \hookrightarrow \mathcal{C}_{0,3}^+ f_+ \). The another inclusion follows immediately, since \( \mathcal{C}_{0,3}^+ \) is the even subalgebra of \( \mathcal{C}_{0,3} \).

Now consider an even element \( Q \in \mathcal{C}_{0,3}^+ \) given by

\[
Q = a + b \epsilon_{12} + c \epsilon_{13} + d \epsilon_{23}
\]

\[
= (a + b \epsilon_{12}) + c(\epsilon_{13} - d \epsilon_{12})
\]

\[
= k^1 + \epsilon_{13} k^2.
\]  

(102)

Another possibility to describe Weyl spinors is to consider the algebra \( \mathcal{C}_{0,3} \cong \mathbb{H} \oplus \mathbb{H} \). A spinor \( K = Q f_+ \in \mathcal{C}_{0,3} f_+ \) is expressed as a

- **Contravariant undotted spinor** (CUS):
  \[
  K = (k^1 f_+ + \epsilon_{13} k^2 f_+)
  \]
  (103)

We also define the

- **Contravariant dotted spinor** (CDS):
  \[
  \bar{K} = \left( f_+ \bar{k}^1 - f_+ \bar{k}^2 \epsilon_{13} \right)
  \]
  (104)

Left multiplying the conjugate of \( K \) by \( \epsilon_{13} \), we obtain

\[
\epsilon_{13} \bar{K} = \epsilon_{13} (f_+ \bar{k}^1 + f_+ \bar{k}^2 \epsilon_{13})
\]

\[
= f_+ k^1 \epsilon_{13} + f_+ k^2.
\]  

(105)

When the last expression is multiplied by another spinor \( \eta \in \mathcal{C}_{0,3} f_+ \), giving

\[
\epsilon_{13} \bar{K} \eta = k^1 \eta^1 \epsilon_{13} f_+ - k^1 \eta^2 f_+ + k^2 \eta^1 f_+ + k^2 \eta^2 \epsilon_{13} f_+,
\]  

(106)

the spinor metric is obtained in \( \mathcal{C}_{0,3} \):

\[
G(K, \eta) := 2 \langle (\epsilon_{13} \bar{K}) \eta \rangle_0 = (k^2 \eta^1 - k^1 \eta^2) \bar{f}_+
\]

(107)

Now consider the application \( \sigma : \mathcal{C}_{0,3}^+ \to \mathcal{C}_{0,3}^+ \) given by

\[
\sigma(Q) = \epsilon_{32} \bar{Q} \epsilon_{23},
\]

(108)

where \( Q = \tilde{Q} \). The map \( \sigma \) takes \( \mathcal{C}_{0,3}^+ \), viewed as a left-module, onto \( \mathcal{C}_{0,3}^+ \), but now viewed as a right-module. Indeed, from eq. (102) we have:

\[
\sigma(k^1 + \epsilon_{13} k^2) = \sigma(a + b \epsilon_{12} + c \epsilon_{13} + d \epsilon_{23})
\]

\[
= \epsilon_{32} (a + b \epsilon_{12} + c \epsilon_{13} + d \epsilon_{23}) \epsilon_{23}
\]

\[
= (a + b \epsilon_{12}) + (c - d \epsilon_{12}) \epsilon_{13}
\]

\[
= k^1 + k^2 \epsilon_{13}.
\]  

(109)
For a spinor $\mathcal{K} = Qf_+ \in \mathcal{C}^{+}_{0,3}f_+$, it follows that

\[
\sigma(\mathcal{K}) = \sigma(Qf_+) = e_{32}(Qf_+)e_{23} = e_{32}(f_+Qe_{23}) = f_+e_{32}Qe_{23} = f_+\sigma(Q) = f_+(k^1 + k^2 e_{13}).
\] (110)

In this way,

\[
\sigma(\mathcal{K})e_{13} = f_+(k^1 e_{13} - k^2) = f_+(-k^2 + k^1 e_{13}) = \mathcal{K}^*.
\] (111)

The spinor metric is alternatively defined as:

\[
G(\psi, \phi) := \langle \sigma(\psi)e_{13}\bar{\phi}\rangle_{0\oplus 3} = \frac{1}{2}\langle \sigma(\psi)e_{13}\bar{\phi} + e_{21}\sigma(\psi)e_{13}\bar{\phi}e_{12} \rangle.
\] (112)

The algebra $\mathcal{C}^{+}_{0,3}$ is not so natural as $\mathcal{C}^{+}_{3,0}$ to describe a lorentzian spacetime. Indeed it is suitable to investigate an euclidian space $\mathbb{R}^4$, since given $u \in \mathbb{R}^4$ we have,

\[
uu = u_0^2 + \bar{u}^2, \quad u_0 \in \mathbb{R}, \quad \bar{u} \in \mathbb{R}^3.
\] (113)

Besides, $\mathcal{C}_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$ is a semi-simple algebra, and the ring $\mathbb{H}$ is not commutative. It is then necessary to treat the right and left product by $\mathbb{H}$. We proved that there exists an application $\sigma : \mathcal{C}^{+}_{0,3} \rightarrow \mathcal{C}^{+}_{0,3}$ that maps a left $\mathbb{H}$-module onto a right $\mathbb{H}$-module.

5 Concluding Remarks

We introduced the covariant and contravariant, dotted and undotted Weyl spinors in $\mathcal{C}^{+}_{3,0}$ and the two last ones in $\mathcal{C}^{+}_{0,3}$, where we constructed an application that maps $\mathcal{C}^{+}_{0,3}$, viewed as a left $\mathbb{H}$-module, onto $\mathcal{C}^{+}_{0,3}$, but now viewed as a right $\mathbb{H}$-module. The correspondence between the idempotents that generate the four lateral minimal ideals in $\mathcal{C}^{+}_{3,0}$ is obtained, if the antioperators in $\mathcal{C}^{+}_{3,0}$ act on the four types of Weyl spinors, respectively elements of the minimal lateral ideals in $\mathcal{C}^{+}_{3,0}$. The null tetrad is obtained in the paravector model of Minkowski spacetime. The Dirac equation in $\mathcal{C}^{+}_{3,0}$ is also presented and discussed. The plane wave solutions of such equation are constructed and the Dirac theory is formulated using the $\mathcal{C}^{+}_{3,0}$ structure, where operators, vectors and tensors are unified described by multivectors and multiforms in the Clifford formalism. The decomposition of the Dirac equation into a system of coupled equations, written in terms of Weyl, and alternatively, Pauli spinors, is also presented.

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