Carving model-free inference

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Abstract: Many scientific studies are modeled as hierarchical procedures where the starting point of data-analysis is based on pilot samples that are employed to determine parameters of interest. With the availability of more data, the scientist is tasked with conducting a meta-analysis based on the augmented data-sets, that combines explorations from the pilot stage with a confirmatory study in the second stage. Casting these two-staged procedures into a conditional framework, inference is based on a carved likelihood. Such a likelihood is obtained in Fithian et al. (2014) by conditioning the law of the augmented data upon the selection carried out on the first stage data. Though the concept of carving is more general, the theory for valid inference in this previous work is strongly tied to parametric models for the data, an example being the ubiquitous Gaussian model.

Our focus in the current paper is to take a step towards model-free inference while integrating explorations with fresh samples in a data-efficient manner. Towards this goal, we provide results that validate carved inference in an asymptotic regime for a broad class of parameters.

1. Introduction

Many scientific analyses, if not all, proceed in a hierarchical fashion starting with exploration of meaningful structure in data, possibly through multiple experiments. From a statistical inferential perspective, the findings of these model-free explorations in the preliminary stages may guide the statistician to a set of parameters corresponding to a model or certain statistical functionals of interest. For example, consider large-scale problems in sequence models where inference is pursued for promising genes based on whether or not they are differentially expressed between two groups in experiments of an exploratory nature; see Storey and Tibshirani (2003). In regression settings, inference may be pursued only for a subset of genetic variants that potentially explain the variability in the phenotype. See Consortium et al. (2017) where a multivariate regression model is used report the variants with significant associations with gene expressions, the response. Such experiments may be carried out in multiple stages, which we broadly divide into two, an exploratory study preceding an investigation of effect sizes of these discoveries. The confirmatory analysis may be carried out upon availability of more data or arise out of interest to integrate these discoveries into a second related data set, which can be described with a similar generative mechanism.

It is well acknowledged that naive interval estimates that use an augmented data-set from both stages, but ignore selection in the first stage analysis do not have the classical coverage guarantees. On the other hand, inference based
only on the confirmatory data set, dubbed as “data-splitting” does not utilize the left-over information in the pilot data from selection. Formalized in Fithian et al. (2014), splitting leads to inadmissible decision rules in most of these scenarios. A solution that overcomes selection-bias and typically dominates data-splitting is aligned along the conditional approach; this is coined as data-carving in Fithian et al. (2014). Carved inference in the hierarchical paradigm uses a conditional law: the law of an appropriately chosen statistic (based on the augmented data-set) conditioned upon the selection rule applied to data from only the first stage. Equivalently, we may consider a scenario that involves reserving some samples for a confirmatory analysis, with selection conducted on only a fraction of the entire data-set. In fact, a motivation to “not use” all the data in selection arises from the need to reserve some information for follow-up inference, failing which the interval estimates might be infinitely long. This is emphasized from a practical standpoint in the selective inferential literature in Tian et al. (2016); Panigrahi et al. (2018).

Methods implementing a conditional approach to counter the effect of winner’s curse are introduced in Zöllner and Pritchard (2007); Zhong and Prentice (2008); Reid et al. (2017) for sequence models and in Lee et al. (2016); Tibshirani et al. (2016) for regression settings. The focus of these papers is inference under the modeling assumption that data is generated from a Gaussian distribution. When no adaptive inference is involved, it is often easy to infer based on the limiting distribution of a statistic even though the data-generative model deviates from normality. However, evident from Tian and Taylor (2018), a Central Limit Theorem (CLT) for such statistics is not trivial under a conditional law when the generative distribution belongs to a (model-free) non-parametric family. To allow a transfer of the CLT to the conditional regime of inference, Tian and Taylor (2018) propose adding heavy-tailed noise to data in the selection stage and call the noise “randomization”.

In hierarchical experiments where it is only optimal to combine the data involved in the initial investigations with new samples, conditional inference is equivalent to a randomization scheme. However, the behavior of such a randomization is asymptotically equivalent to that of a Gaussian random variable and thereby, quite different from the heavy-tailed perturbation schemes in previous works. Before describing the main results of the paper, we discuss the conditional law under carving, introduce the implicit randomization in carving and pose the question on asymptotic validity of conditional inference through a canonical example.

1.1. Carving: an implicit randomization

Below, we describe a canonical example of estimating only those effects in a many-means problem that meet “statistical significance” criteria post a scan across Z-statistics in pilot samples. Let \( \tilde{\zeta}_n^{(j)} = \sum_{i=1}^{n} \tilde{\zeta}_{i,n}^{(j)} \); \( j = 1, 2, \ldots, d \) be the means based on \( d \) triangular arrays of observations such that the vector

\[
\zeta_{i,n} = c(\tilde{\zeta}_{1,n}^{(1)}, \tilde{\zeta}_{i,n}^{(2)}, \ldots, \tilde{\zeta}_{i,n}^{(d)})^T \sim i.i.d. \mathbb{P}_n; \ i = 1, 2, \ldots, n
\]

where distribution \( \mathbb{P}_n \) belongs to a non-parametric family.
\begin{align}
\{ P_n : \mathbb{E}_{P_n}(Z - \mathbb{E}_{P_n}[Z])(Z - \mathbb{E}_{P_n}[Z])^T = \Sigma, \mathbb{E}_{P_n}[\|Z\|^2] < \infty \} \quad \text{(1.1)}
\end{align}

Define the mean functional for the \( d \) arrays as \( \beta_n^{(j)} = \mathbb{E}_{P_n}[\zeta_n^{(j)}] \) for \( j = 1, 2, \ldots, d \).

In the above setting, consistent with the two-staged experimental set-up, we observe \( n_1 \) samples in the initial stage. With the availability of an additional \( n_2 \) data samples, we pursue inference about the mean functionals \( \beta_n^{(j)} \) for which the test statistic \( \sqrt{n_1}Z_n^{(j)} \) (denoted by \( Z_n^{(j)} \)) based on the initial \( n_1 \) samples exceeds a threshold. That is,
\[
\{ \sqrt{n_1}Z_n^{(j)} > \lambda^{(j)} \equiv Z_n^{(j)} > \lambda^{(j)} \};
\]
\( \lambda^{(j)} \) being an appropriately chosen threshold. Equivalently, we infer about \( \beta_n^{(j)} \) in the second stage when data is indicative of a positive effect for the \( j \)-th variable from the pilot analysis.

Let the total number of data points be \( n = n_1 + n_2 \) and the ratio of additional samples in the second stage to the first stage ones be \( \rho^2 = n_2/n_1 \). The selection in the pilot samples in terms of \( Z_n^{(j)} \equiv \sqrt{n}\zeta_n^{(j)} \), the test statistic based on the augmented data, can be expressed as: infer about \( \beta_n^{(j)} \) whenever
\[
Z_n^{(j)} + \left( \sqrt{n}\zeta_n^{(j)} - \sqrt{n}\zeta_n^{(j)} \right) > \sqrt{1 + \rho^2} \cdot \lambda^{(j)},
\]
thereby noting that carving is equivalent to adding a perturbation term to \( Z_n^{(j)} \). The perturbation given by
\begin{align}
W_n^{(j)} = \sqrt{n}\zeta_n^{(j)} - \sqrt{n}\zeta_n^{(j)} , \quad j = 1, 2, \ldots, d \quad \text{(1.3)}
\end{align}
can be interpreted as an implicit randomization, that is inferred from using only a fraction of the data in the first-stage scanning. Borrowing the term “randomization” from Tian and Taylor (2018), we point out that a randomization variable \( W_n^{(j)} \), drawn from a heavy-tailed distribution and independent of data, is added to the test-statistic in this prior work.

The behavior of the randomization inherited from the hierarchical design of experiment is very different from heavy-tailed randomization schemes in Markovic and Taylor (2016); Tian and Taylor (2018). Lemma 1 describes the asymptotic distribution of the implicitly acting randomization \( W_n^{(j)} ; j = 1, \ldots, d \).

**Lemma 1.** When a \( \mathbb{R}^d \)-valued triangular array \( \{ \zeta_n ; i = 1, 2, \ldots, n \} \) is generated from a non-parametric model in (1.1), then the implicit randomization \( W_n = (W_n^{(1)}, \ldots, W_n^{(d)}) \) with \( W_n^{(j)} \) defined in (1.3) is asymptotically distributed as \( \mathcal{N}(0, \rho^2\Sigma) \), where \( \Sigma \) is the asymptotic covariance of \( Z_n = (Z_n^{(1)}, \ldots, Z_n^{(d)}) \). Further, the covariance of \( W_n \) with the statistic \( Z_n \) equals 0 for each \( n \in \mathbb{N} \).

When \( P_n \equiv \mathcal{N}(\beta_n, \Sigma) \), it is easy to note that \( W_n \sim \mathcal{N}(0, \rho^2\Sigma) \) and independent of \( Z_n \) for each \( n \in \mathbb{N} \).

\subsection{1.2. A carved likelihood}

Under the screening example described in Section 1.1, we introduce few notations in order to discuss the carved likelihood. Denote the set of selected indices as \( E \) and the cardinality of the selected set as \( |E| \). Let \( (Z_n^{(E)}, Z_n^{(E^c)}) \) represent
the vectors of test-statistics \((\sqrt{n}c_n^{(E)}, \sqrt{n}c_n^{(-E)})\) in \(\mathbb{R}^{|E|}\) and \(\mathbb{R}^{d-|E|}\), corresponding to the selected and dropped indices respectively. Similar notations hold for the randomization and threshold vectors.

Next, define the random variables:

\[
T_n^{(E)} = Z_n^{(E)} + W_n^{(E)} - \sqrt{1 + \rho^2} \cdot \lambda^{(E)}; \quad R_n^{(-E)} = Z_n^{(-E)} + W_n^{(-E)}.
\]

Denote the centered and scaled version of random variables \(Z_n = (\bar{Z}_n^{(E)}, \bar{Z}_n^{(-E)})\) as \(\bar{Z}_n = (\bar{Z}_n^{(E)}, \bar{Z}_n^{(-E)})^T = \Sigma^{-1/2}(Z_n - \sqrt{n}\beta_n)\).

Finally, observe that \(\{\bar{E} = E, R_n^{(-E)} = R_n\} = \{T_n^{(E)} > 0, R_n^{(-E)} = R_n\}\) based on the linear map:

\[
\begin{pmatrix}
W_n^{(E)} \\
W_n^{(-E)}
\end{pmatrix} = -\left(\Sigma^{1/2}Z_n + \sqrt{n}\beta_n\right) + Q_E T_n^{(E)} + r_E(R_n^{(-E)})
\]

with \(Q_E = [I \ 0]^T, \quad r_E(R_n^{(-E)}) = (\sqrt{1 + \rho^2} \cdot \lambda^{(E)}, R_n^{(-E)})^T\).

Inference, modeled along this two-stage analysis, is based upon a carved likelihood stated in Lemma 2. The next result, Lemma 3 derives this likelihood under a Gaussian generative model.

**Lemma 2.** Let \(f(\cdot)\) denote the density of \(Z_n\) and \(g_{z_n}(\cdot)\) denote the density of randomization \(W_n|Z_n = z_n\). Conditioned on selection of \(E\) coupled with some additional information in the form of statistics \(R_n^{(-E)} = R_n\), the carved likelihood for the centered statistic \(Z_n\) at an observed realization \(z_n, f(z_n)\) equals

\[
\frac{f(z_n) \times \int g_{z_n}(-(\Sigma^{1/2}z_n + \sqrt{n}\beta_n) + Q_E t_n + r_E(R_n))1_{t_n > 0} dt_n}{\int f(z_n) \times \int g_{z_n}(-(\Sigma^{1/2}z_n + \sqrt{n}\beta_n) + Q_E t_n + r_E(R_n))1_{t_n > 0} dt_n dz_n^T}.
\]

**Lemma 3.** Under a Gaussian model when the statistic \(Z_n\) is distributed as a Gaussian variable with mean 0 and covariance \(I\), the carved likelihood \(f(\cdot)\) in Lemma 2 at \(z_n\) is proportional to

\[
\exp(-z_n^T z_n/2) \times \prod_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n}e_n^{(j)} \right) / \mathbb{E} \Phi \left[ Q_j z_n + \sqrt{n}e_n^{(j)} \right]
\]

where \(Q_j\) is the \(j\)-th row of the matrix \(Q = -(Q_E^T \Sigma^{-1} Q_E / \rho^2)^{-1/2} Q_E^T \Sigma^{-1/2} / \rho^2\) and \(e_j\) is the vector in \(\mathbb{R}^{|E|}\) with 1 in the \(j\)-th coordinate and 0’s elsewhere

\[
\sqrt{n}e_n^{(j)}(\beta_n) = e_j^T \left( Q_E^T \Sigma^{-1} Q_E / \rho^2 \right)^{-1/2} Q_E^T \Sigma^{-1} \left(-\sqrt{n}\beta_n + r_E(R_n)\right) / \rho^2.
\]

### 1.3. Asymptotic validity of carved inference

Under Gaussian generative models in the canonical example, we are equipped to construct an exact pivot for marginal inference about \(\beta_n^{(j)}\). Aligned along the ideas introduced in Lee et al. (2016); Tian and Taylor (2018), the construction of this pivot is based upon an application of the probability integral transform of a conditional law.

Denoted as \(P_j(Z_n; \sqrt{n}\beta_n)\), a pivot to infer about \(\beta_n^{(j)}\) when \(P_n \equiv N(\beta_n, \Sigma)\) is derived in Proposition 1 (see A) and equals

\[
P_j(Z_n; \sqrt{n}\beta_n) = P_j(P_E^{(1)}(\Sigma^{1/2}Z_n + \sqrt{n}\beta_n); \sqrt{n}\beta_n),
\]

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with $P_j$ and the matrix $P_E^{(j)}$ defined in Appendix A and $P_j(\mathcal{Z}_n; \sqrt{n}\beta_n) \sim \text{Unif}(0, 1)$. Deferring details to the Appendix, note that $P_j(\cdot; \sqrt{n}\beta_n)$ is a function of the statistic $\mathcal{Z}_n$ and parameters $\beta_n$ and exact inference based upon this pivot is tied to a Gaussian model for the data.

If this were the classical settings, with no selection operating on the data, the usual CLT:

$$
\left| E_{\mathcal{F}}[H \circ P_j(\mathcal{Z}_n; \sqrt{n}\beta_n)] - E_{\mathcal{F}}[H \circ P_j(\mathcal{Z}_n; \sqrt{n}\beta_n)] \right| = O(n^{-1/2}),
$$

for any bounded, continuous $H$ would enable model-free inference for a sufficiently smooth pivot. Does weak convergence of pivots $P_j(\cdot; \sqrt{n}\beta_n)$ for $j \in E$ transfer from the unconditional law to the conditional law, when data is carved for inference? We address precisely this question in our work, formalized as 1.

Denote $E_{\mathcal{F}}[\cdot]$ and $E_{\Phi}[\cdot]$ as the expectations computed with respect to the carved densities $\tilde{\ell}(\cdot)$ of $\mathcal{Z}_n$ in Lemma 2 and 3, under a distribution $P_n$ and a Gaussian model respectively.

**Question 1.** Let $P_j(\mathcal{Z}_n; \sqrt{n}\beta_n)$ be a pivotal quantity to infer about $\beta_n^{(j)}$, with a distribution $F \equiv \text{Unif}(0, 1)$ under the conditional likelihood $\tilde{\ell}(\cdot)$ in Lemma 3, when $\mathcal{Z}_n$ is distributed as a Gaussian random variable. Then, is it still asymptotically distributed as $F$ when $\mathcal{Z}_n$ is based on a triangular array from a non-parametric family of distributions in (1.1)? Under what conditions is the difference between the expectations

$$
\left| E_{\mathcal{F}}[H \circ P_j(\mathcal{Z}_n; \sqrt{n}\beta_n)] - E_{\Phi}[H \circ P_j(\mathcal{Z}_n; \sqrt{n}\beta_n)] \right| = o(1)?
$$

### 1.4. Towards model-free inference: contributions

Our results provide answers for Question 1 under a randomization in (1.3) that arises out of a hierarchical design of experiments. More generally, we give significantly improved finite-sample bounds that provide a rate of weak convergence under additive Gaussian perturbations for a broad class of parameters.

We recognize that objects fundamental to addressing asymptotic validity are relative differences between the selection probabilities under a Gaussian and a non-parametric generative model. For a real-valued parameter sequence $\{\beta_n : n \in \mathbb{N}\}$, the relative difference equals

$$
\frac{E_{\Phi}\left[h(\mathcal{Z}_n)\Pi_{j=1}^{[E]} \tilde{\Phi}\left(Q_j, \mathcal{Z}_n + \sqrt{n} \alpha_n^{(j)}(\beta_n)\right)\right] - E_{\Phi}\left[h(\mathcal{Z}_n)\Pi_{j=1}^{[E]} \tilde{\Phi}\left(Q_j, \mathcal{Z}_n + \sqrt{n} \alpha_n^{(j)}(\beta_n)\right)\right] - E_{\Phi}\left[\Pi_{j=1}^{[E]} \tilde{\Phi}\left(Q_j, \mathcal{Z}_n + \sqrt{n} \alpha_n^{(j)}(\beta_n)\right)\right]}{E_{\Phi}\left[\Pi_{j=1}^{[E]} \tilde{\Phi}\left(Q_j, \mathcal{Z}_n + \sqrt{n} \alpha_n^{(j)}(\beta_n)\right)\right]},
$$

where $h$ is a bounded function with some smoothness properties and the probability $\Pi_{j=1}^{[E]} \tilde{\Phi}\left(Q_j, \mathcal{Z}_n + \sqrt{n} \alpha_n^{(j)}(\beta_n)\right)$ is defined in Lemma 3. The rate of weak convergence of pivots under the conditional likelihood $\tilde{\ell}(\mathcal{Z}_n)$ crucially depends on the rate at the above difference decays to 0. The problem, therefore, gets only harder for parameter sequences $\{\beta_n : n \in \mathbb{N}\}$ under which the probability of selection in the denominator decays to 0. An important contribution of our results is that we are able to prove weak convergence of pivots in a challenging regime of rare selections with exponentially decaying probabilities. Sections 2,
3 and 4 provide conditions under which these conditional inferential procedures are valid in an asymptotic sense in sequence and multivariate models.

Finally, the bounds in the paper validating a transfer of CLT from the unconditional to the conditional law are by no means limited to prototypical screening example in Section 1.1. Our results enable model-free conditional inference post many variable selection mechanisms in both marginal and regression settings where parameters may or may not be related to each other. Example include top-D, step-up screenings, LASSO, elastic net, forward stepwise etc.. The concluding Section 5 illustrates the wide applicability of our results.

2. Selection probabilities under Gaussian randomization

We take a step towards answering Question 1 by considering an equivalent Gaussian randomization scheme based on the asymptotic behavior of the randomization in (1.3). Making explicit the relative difference terms we analyze to prove weak convergence of pivots, we derive a rate of decay of the selection probability under additive Gaussian perturbations– the denominator of these relative differences.

2.1. An equivalent Gaussian randomization scheme

For the rest of the paper (with a slight abuse of notation), it suffices to consider the statistic

\[ Z_n = \sum_{i=1}^{n} Z_{i,n} = \sum_{i=1}^{n} \zeta_{i,n} / \sqrt{n} \tag{2.1} \]

where \( Z_n \) is the sum of i.i.d. random variables \( \{ \zeta_{i,n} = c(\zeta_{i,n}^{(1)}, \ldots, \zeta_{i,n}^{(d)})^T \in \mathbb{R}^d; i = 1, 2, \ldots, n \} \) scaled by \( 1/\sqrt{n} \), with each \( \zeta_{i,n}, i \in \{1, 2, \ldots, n\} \) drawn from a distribution in the family

\[ \{ P_n : \mathbb{E}_{P_n}(Z) = 0, \mathbb{E}_{P_n}(Z - \mathbb{E}_{P_n}[Z])(Z - \mathbb{E}_{P_n}[Z])^T = I, \mathbb{E}_{P_n} \| Z \|^3 < \infty \} \tag{2.2} \]

with mean 0 and identity covariance.

We first consider a perturbation scheme in which we add independent Gaussian noise to the \( Z \)-statistic. That is, we consider the same selection rule as our canonical example:

\[ \{ W_n^{(j)} > \sqrt{1 + \rho^2} \cdot \lambda^{(j)} - Z_n^{(j)}, j = 1, 2, \ldots, d \} \]

except that the randomization variable \( W_n \) is now independent of the statistic \( Z_n \) and is exactly distributed as a Gaussian:

\[ W_n \sim \mathcal{N}(0, \rho^2 \Sigma) \text{ and independent of } Z_n \text{ for each } n \in \mathbb{N}; \rho^2 \equiv n_2/n_1. \tag{2.3} \]

Observe that the randomization in (2.3) is asymptotically equivalent to the implicitly acting randomization in (1.3), as verified by Lemma 1.

We denote the conditional likelihood of \( Z_n \) under the Gaussian randomization in (2.3) as \( \ell^{(N)}(\cdot) \); we use superscript \( (N) \) to distinguish the conditional likelihood under a normally distributed perturbation \( W_n \) from that induced by the implicit randomization in the carved likelihood in (1.3). The conditional likelihood \( \ell^{(N)}(z_n) \) of \( Z_n \) at a realized value \( z_n \) under randomization (2.3) equals

\[ \ell(z_n) \times \prod_{j=1}^{d} \Phi \left( Q_j, z_n + \sqrt{n} \alpha_n^{(j)} \right) / \mathbb{E}_{\hat{P}_n} \left[ \prod_{j=1}^{d} \Phi \left( Q_j, Z_n + \sqrt{n} \alpha_n^{(j)} \right) \right] \tag{2.4} \]
where $\ell(\cdot)$ is the pre-selective (unconditional) law of $Z_n$; $Q$, $\alpha_n$ are as derived in Lemma 3. This is obtained by plugging in a Gaussian law for the randomization in Lemma 2 and using the fact that it is independent of $Z_n$. Under a Gaussian generative model for the triangular arrays, the likelihood in (2.4) matches with the carved likelihood for $Z_n$ in Lemma 3.

Knowledge of the rate at which the tail of the carved randomization converges to a Gaussian tail provides answers to Question 1 if we can first answer Question 2 on a transfer of CLT from $\ell(\cdot) \rightarrow \tilde{\ell}^{(N)}(\cdot)$ Let $\tilde{E}_{\nu}[\cdot]$ compute the expectation with respect to (2.4) when the triangular array of observations $\{Z_i; i=1,2,\cdots,n\}$ is independently and identically distributed as $P_n$ from a model-free family in (2.2). Similarly, let $\tilde{E}_\nu[\cdot]$ denote the expectation under the conditional likelihood $\tilde{\ell}^{(N)}(\cdot)$ when $Z_n \sim N(0,1)$, which coincides with the carved likelihood in Lemma 3.

**Question 2.** What is the rate of weak convergence of a pivot $P_i(\cdot; \sqrt{n}\beta_n)$ for $\beta_n^{(i)}$ in Proposition 1 (Appendix A) whenever $j \in E$? Equivalently, what is a rate of decay for the difference:

$$\tilde{E}_{\nu}[H \circ P_j(Z_n; \sqrt{n}\beta_n)] - \tilde{E}_{\nu}[H \circ P_j(Z_n; \sqrt{n}\beta_n)]: j \in E$$

$\tilde{E}$ is the expectation computed with respect to the conditional law $\tilde{\ell}^{(N)}$.

In the following two sections, we focus on providing an answer for Question 2, thereby enabling model-free inference under additive Gaussian randomizations. To show a transfer of CLT from $\ell(\cdot) \rightarrow \tilde{\ell}^{(N)}(\cdot)$, we formalize in Theorem 1 the dependence of weak convergence of the pivot on two relative difference terms.

**Theorem 1.** Let $H$ be a bounded function that satisfies $\|H\|_\infty \leq K$, $Z_n$ be the statistic in (2.1) and $\tilde{E}$ be the expectation computed with respect to the likelihood under the law $\tilde{\ell}^{(N)}(\cdot)$, induced by randomization scheme in (2.3). If

$$\left| \tilde{E}_{\nu}[g_i(Z_n)\Pi_{j=1}^n \Phi \left( Q_j Z_n + \sqrt{n}\alpha_n^{(j)} \right)] - \tilde{E}_{\nu}[g_i(Z_n)\Pi_{j=1}^n \Phi \left( Q_j Z_n + \sqrt{n}\alpha_n^{(j)} \right)] \right|$$

$$\leq R_n^{(i)}, i=1,2$$

where $g_1(Z_n) = 1, g_2(Z_n) = H \circ P_j(Z_n; \sqrt{n}\beta_n)$, then we have

$$\left| \tilde{E}_{\nu}[H \circ P_j(Z_n; \sqrt{n}\beta_n)] - \tilde{E}_{\nu}[H \circ P_j(Z_n; \sqrt{n}\beta_n)] \right| \leq (K \cdot R_n^{(1)} + R_n^{(2)}).$$

Whenever $R_n^{(k)} = o(1), k=1,2$, we have a CLT under the conditional law $\tilde{\ell}^{(N)}(\cdot)$ facilitating valid asymptotic inference for $\beta_n^{(j)}$ via the pivot $P_j(Z_n; \sqrt{n}\beta_n)$.

**2.2. Decay rate of selection probability**

Before proceeding further, we consider two sets of real-valued parameters in $\mathbb{R}^d$ that govern the generative models.
(LA) **Local Alternatives:** $\mathcal{N} = \{ \{\beta_n, n \in \mathbb{N} \} : \sqrt{n}\alpha_n(\beta_n) = O(1) \}$. Under this set of parameters, the selection probability $\mathbb{P}_\Phi \left[ \prod_{j=1}^{\lfloor E \rfloor} \Phi \left( Q_j, Z_n + \sqrt{n}\alpha_n(j) \right) \right]$ is bounded away from 0. This set of parameter sequences is previously considered in Tian and Taylor (2018), but with heavier tailed randomization schemes.

(RA) **Rare Alternatives:** $\mathcal{R} = \{ \{\beta_n, n \in \mathbb{N} \} : (I + QQ^T)^{-1/2} \sqrt{n}\alpha_n(\beta_n) = a_n; a_n > 0 \}$

for all $n \to \infty$, $a_n = o(n^{1/2})$. This set contains sequences which lead to selection events of vanishing probability. Specifically for the screening example in Section 1, note that $QQ^T = I/\rho^2$ and thus, the set $\mathcal{R}$ consists of sequences $\beta_n$ such that $\sqrt{n}\alpha_n(\beta_n)$ is parameterized as $a_n\tilde{\alpha}$ with $a_n$, $\tilde{\alpha}$ described above.

**Remark 1.** Classically, the parameters in (RA) are termed as local alternatives. In this paper, we abuse this naming convention and call them “rare alternatives” since they induce rare selection events with probabilities decaying to 0.

Note that we exclude the uninteresting parameters where $(I + QQ^T)^{-1/2} \sqrt{n}\alpha_n(\beta_n)$ converge to $-\infty$, since the selection event has a probability converging to 1 with accumulating data. Unconditional inference (without adjusting for any selection) is asymptotically valid in this regime.

We will note that a fairly direct application of the Stein’s approximation establishes a transfer of CLT from $\ell \to \ell^{(N)}$ for a local alternative sequence. The parameters for which weak convergence necessitates a more intricate analysis belong to $\mathcal{R}$ in (RA). In this rare regime of inference, the selection probability and hence, the denominator of the relative differences in Theorem 1 decays to 0. A crucial step towards establishing a rate for the difference of expectations in Question 2 under rare alternatives is therefore an analysis of the selection probability under the randomization in (2.3).

Theorem 2 establishes a rate of decay for the Gaussian selection probability $\mathbb{P}_\Phi \left[ \prod_{j=1}^{\lfloor E \rfloor} \Phi \left( Q_j, Z_n + \sqrt{n}\alpha_n(j) \right) \right]$, the denominator of the relative differences. Before that, we state a key result in Lemma 4, providing some necessary rates to derive Theorem 2. We define $L : \mathbb{R}^d \to \mathbb{R}$ and $U : \mathbb{R}^d \to \mathbb{R}$

\[
L(Z_n; \sqrt{n}\alpha_n) = \frac{2^{|E|}\exp(-(Q Z_n + \sqrt{n}\alpha_n)^T(Q Z_n + \sqrt{n}\alpha_n)/2)}{(2\pi)^{|E|/2}\prod_{j=1}^{\lfloor E \rfloor} \left\{ \sqrt{4 + (Q_j Z_n + \sqrt{n}\alpha_n(j))^2 + (Q_j Z_n + \sqrt{n}\alpha_n(j))} \right\}},
\]

\[
U(Z_n; \sqrt{n}\alpha_n) = \frac{2^{|E|}\exp(-(Q Z_n + \sqrt{n}\alpha_n)^T(Q Z_n + \sqrt{n}\alpha_n)/2)}{(2\pi)^{|E|/2}\prod_{j=1}^{\lfloor E \rfloor} \left\{ \sqrt{2 + (Q_j Z_n + \sqrt{n}\alpha_n(j))^2 + (Q_j Z_n + \sqrt{n}\alpha_n(j))} \right\}}.
\]

Define $\mathcal{L}(QQ^T, Q^TQ; \tilde{\alpha}) = \left\{ \prod_{j=1}^{\lfloor E \rfloor} \frac{1}{(I + QQ^T)^{-1}(I + QQ^T)^{-1}} \right\}^{-1}$. Note in the rest of the paper, for vectors $a, b \in \mathbb{R}^d$: inequality $a > b \implies a_i > b_i$ for $1 \leq i \leq d$; similarly $a < b \implies a_i < b_i$ for $1 \leq i \leq d$.

**Lemma 4.** For a sequence of parameters in (RA) such that $\sqrt{n}\alpha_n(j)/\sqrt{n}\alpha_n(j') \to \tilde{\alpha}_{jj'}$ for all $j, j' \in \{1, 2, \cdots, \lfloor E \rfloor \}$ as $n \to \infty$ and for $q \in (0, 1)$ such that $q^2 \geq \lambda_{max}$ where $\lambda_{max}$ denotes the largest eigen value of $QQ^T(I + QQ^T)^{-1}$
\[
\lim \frac{\left| \Pi_{j=1}^{[E]} \sqrt{n} \alpha_n^{(j)} \right| \cdot \exp(\sqrt{n} \alpha_n^{T}(I + QQ^{T})^{-1}\sqrt{n} \alpha_n/2)}{\mathbb{E}_{\Phi} \left[ F_1(Z_n; \sqrt{n} \alpha_n)1_{\{|QZ_n|<\sqrt{n} \alpha_n\}} \right]} = \mathcal{L}(QQ^{T}, Q^{T}Q; \bar{\alpha})
\]

where \( F_1(Z_n; \sqrt{n} \alpha_n) = L(Z_n; \sqrt{n} \alpha_n); F_2(Z_n; \sqrt{n} \alpha_n) = U(Z_n; \sqrt{n} \alpha_n). \)

**Theorem 2.** Under the rare alternatives in (RA) and \( \mathcal{L}(QQ^{T}, Q^{T}Q; \bar{\alpha}) \) defined in Lemma 4, we have

\[
\mathbb{E}_{\Phi} \left[ \Pi_{j=1}^{[E]} \bar{\Phi} \left( Q_j Z_n + \sqrt{n} \alpha_n^{(j)} \right) \right] = \exp(-\sqrt{n} \alpha_n^{T}(I + QQ^{T})^{-1}\sqrt{n} \alpha_n/2) \\
\times \left( \Pi_{j=1}^{[E]} \sqrt{n} \alpha_n^{(j)} \right)^{-1} \cdot \mathcal{L}(QQ^{T}, Q^{T}Q) + o(1).
\]

### 3. Weak convergence in sequence models

Our first analysis of weak convergence is in the sequence model settings, considered previously in Reid et al. (2017) for non-hierarchical investigations. This corresponds to a scenario where the parameters are assumed not to have any relationship with one another.

Assume that the \( d \) test statistics \( \{Z_n^{(j)}, 1 \leq j \leq d\} \) in the screening example are independent. Under independence, the density of randomization \( W_n|Z_n = z_n \) in Lemma 2 decouples into a coordinate-wise density \( g_{z_n}(\omega_n) = \Pi_j g_{z_n}^{(j)}(\omega_n^{(j)}). \)

Further, the density of \( Z_n \) also decouples into \( \Pi_{j=1}^{d} \ell(z_n^{(j)}) \). Assuming (without loss of generality) the variance of the test statistics to be unity (that is, \( \Sigma = I \)), the carved likelihood of \( Z_n^{(j)} \) - the \( j \)-th coordinate of \( Z_n \) at \( z_n^{(j)} \) now takes the form:

\[
\hat{\ell}(z_n^{(j)}) = \ell(z_n^{(j)}), \quad \int_{\ell(z_n^{(j)'})} g_{z_n^{(j)}}^{(j)}(-z_n^{(j)} + \sqrt{n} \beta_n^{(j)} + \ell_n^{(j)} + \sqrt{1 + \rho^2} \cdot \lambda_n^{(j)}1_{\ell_n^{(j)} > 0} dt_n^{(j)} \\
\times \ell(z_n^{(j)}) \int_{\ell(z_n^{(j)'})} g_{z_n^{(j)}}^{(j)}(-z_n^{(j)} + \sqrt{n} \beta_n^{(j)} + \ell_n^{(j)} + \sqrt{1 + \rho^2} \cdot \lambda_n^{(j)}1_{\ell_n^{(j)} > 0} dt_n^{(j)} dz_n^{(j)'})
\]

whenever \( j \in E; \ell(z_n^{(j)}) \) is the pre-selective (unconditional) law of the centered, scaled Z-statistic, that in limit is distributed as a standard normal. Carved inference for \( \beta_n^{(j)} \) is thus separable, not depending on \( \beta_n^{(j)}, j' \neq j \) and hence, in agreement with a sequence model.

In the same setting, under a normal model when \( Z_n \sim N(0, I) \), observe that \( W_n \) in (1.3) is independent of data and is exactly distributed as \( N(0, \rho^2 I) \). Plugging in normal densities for \( \ell(\cdot), g_{z_n}(\cdot) \) in (3.1), the carved likelihood of \( Z_n^{(j)} \) under normality and independence equals

\[
\hat{\ell}(z_n^{(j)}) = \phi(z_n^{(j)}), \quad \mathbb{E}_{\Phi} \left[ \Phi \left( \sqrt{1 + \rho^2} \lambda_n^{(j)}/\rho - (z_n^{(j)} + \sqrt{n} \beta_n^{(j)}/\rho) \right) \right].
\]

With inference reducing to a univariate law in (3.1) and (3.2), we use a univariate Stein’s approximation (see Chatterjee (2014)) to investigate weak convergence in the sequence model. On the other hand, validating asymptotic
inference for $\{\beta_n^{(j)}, j \in E\}$ under a likelihood in Lemma 2, not assuming independence, requires a more intricate analysis involving a multivariate version of the Stein’s equation. We treat the sequence and multivariate models separately in the current section and Section 4 respectively due to a fundamental difference in the behavior of solutions to the Stein’s equation for a multivariate approximation versus the univariate counterpart, see Raič (2004).

### 3.1 Selection probability under Gaussian perturbation

Back to the discussion on asymptotic validity in the sequence model, a pivot to infer about $\beta_n^{(j)}$ when $Z_n^{(j)} \sim N(0, 1)$ equals

$$
P_j(Z_n^{(j)}, \sqrt{n}\beta_n^{(j)}) = \int_{-\infty}^{\infty} \exp(-z^2/2) \cdot \bar{\Phi} \left( -\sqrt{1 + \rho^2} \cdot \lambda^{(j)}/\rho - (z + \sqrt{n}\beta_n^{(j)})/\rho \right) / \sqrt{2\pi} \, dz \cdot \bar{E}_\Phi \left[ \bar{\Phi} \left( -\sqrt{1 + \rho^2} \cdot \lambda^{(j)}/\rho - (z + \sqrt{n}\beta_n^{(j)})/\rho \right) \right],
$$

(3.3)
a function of $Z_n^{(j)}$ and $\sqrt{n}\beta_n^{(j)}$. See Proposition 2 in Appendix A for a derivation of the above pivot, obtained via the probability integral transform of the conditional law in (3.2). Working towards an answer for Question 2, the conditional likelihood in (3.1) for $Z_n^{(j)}$ under a Gaussian perturbation in (3.3) equals

$$
\tilde{\ell}(N)(z_n^{(j)}) = \ell(z_n^{(j)}) \times \bar{\Phi} \left( -z_n^{(j)} + \sqrt{n}\beta_n^{(j)}/\rho \right) / \bar{E}_\Phi \left[ \bar{\Phi} \left( -z_n^{(j)} + \sqrt{n}\beta_n^{(j)}/\rho \right) \right]
$$

setting the threshold $\lambda^{(j)} = 0$.

To present results in the sequence model, whenever $j \in E$, we let

$$
Z_n^{(j)} = \sum_{i=1}^n Z_{i,n}^{(j)} = \sum_{i=1}^n \zeta_{i,n}^{(j)}/\sqrt{n},
$$

(3.5)

where $\{\zeta_{i,n}^{(j)}, 1 \leq i \leq n\}$ is an i.i.d. array from a non-parametric family of distributions with zero-mean and some conditions on existence of moments.

Corollary 1. Under the conditions of Theorem 1, let the functions $g_1(Z_n^{(j)}) = 1$ and $g_2(Z_n^{(j)}) = \mathcal{H} \circ P_j(Z_n^{(j)}, \sqrt{n}\beta_n^{(j)}); P_j(\cdot; \sqrt{n}\beta_n^{(j)})$ is defined in (3.3). Whenever

$$
\left| \bar{E}_\Phi \left[ g_1(Z_n^{(j)}) \bar{\Phi} \left( -(Z_n^{(j)} + \sqrt{n}\beta_n^{(j)})/\rho \right) \right] - \bar{E}_\Phi \left[ g_2(Z_n^{(j)}) \bar{\Phi} \left( -(Z_n^{(j)} + \sqrt{n}\beta_n^{(j)})/\rho \right) \right] \right| \leq R_n^{(1)}
$$

for $l = 1, 2$ and for $Z_n^{(j)}$ in (3.5), the following holds:

$$
\left| \bar{E}_\Phi \left[ H \circ P_j(Z_n^{(j)}, \sqrt{n}\beta_n^{(j)}) \right] - \bar{E}_\Phi \left[ \bar{H} \circ P_j(Z_n^{(j)}, \sqrt{n}\beta_n^{(j)}) \right] \right| \leq \left( K \cdot R_n^{(1)} + R_n^{(2)} \right).
$$

For the sequence model, define the set of local alternatives as:

$$
\mathcal{N} = \{ \{\beta_n^{(j)}, n \in \mathbb{N} \} : \sqrt{n}\beta_n^{(j)} = O(1) \}.
$$

The rare alternatives, on the other hand, are parameterized by...
\[ \mathcal{R} = \{ \{ \beta_n^{(j)}, n \in \mathbb{N} \} : \sqrt{n} \beta_n^{(j)} = b_n \beta, \beta < 0, b_n \to \infty \text{ as } n \to \infty, b_n = o(n^{1/2}) \}. \quad (3.7) \]

The definitions of local and rare alternatives in (3.6) and (3.7) are consistent with those for the multivariate model in (LA) and (RA), when \( \Sigma = I \).

Clearly, the denominators in the relative differences in Corollary 1 decay to 0 under \( \mathcal{R} \) and are bounded away from 0 under \( \mathcal{N} \). Corollary 2 provides a rate of decay for the denominator in the relative differences in Corollary 1 under the rare regime. This can be seen as a univariate version of Theorem 2 with \( E = \{ j \}, |E| = 1, Q \) as scalar \(-1/\rho \) and \( |\sqrt{n} \alpha_n^{(j)}| = |\sqrt{n} \beta_n^{(j)}|/\rho \).

**Corollary 2.** Under parameters in \( \mathcal{R} \) with a parameterization in (3.7), we have
\[
\mathbb{E}_\Phi \left[ \Phi(- (\mathcal{Z}_n^{(j)} + \sqrt{n} \beta_n^{(j)})/\rho) \right] = |\sqrt{n} \beta_n^{(j)}|^{-1} \exp(-n \beta_n^{(j)}^2/2(1 + \rho^2)) \left( \frac{\sqrt{1 + \rho^2}}{\sqrt{2\pi}} + o(1) \right).
\]

### 3.2. Steining selection probabilities: preliminaries

We adopt a univariate Stein’s approximation to establish a rate of decay for the numerators of the relative differences in Corollary 1. Together with the rate of decay of the denominator in Corollary 2, we obtain a rate of weak convergence in the sequence model. Below, we provide some necessary background on the Stein’s approximation. For a bounded function \( F(\cdot) : \mathbb{R} \to \mathbb{R} \) with some smoothness properties, the Stein’s identity
\[
\mathbb{E} \left[ F(\mathcal{Z}_n^{(j)}) \right] - \mathbb{E} \left[ F(\mathcal{Z}_n^{(j)}) \right] = \mathbb{E} \left[ f_F(\mathcal{Z}_n^{(j)}) \right] - \mathbb{E} \left[ f_F(\mathcal{Z}_n^{(j)}) \right]
\]
evaluates the differences in expectations under a non-parametric distribution \( \mathbb{P}_n \) and the standard Gaussian distribution where \( f_F(\cdot) \) is the Stein’s function:
\[
f_F(z) = \exp(z^2/2) \int_{-\infty}^{z} \left( F(t) - \mathbb{E} \left[ F(\mathcal{Z}_n^{(j)}) \right] \right) \exp(-t^2/2) dt; \quad Z \sim \mathcal{N}(0, 1). \quad (3.8)
\]

We denote a “leave-one” out statistic based on \( \mathcal{Z}_n^{(j)} \) in (3.5) as \( \mathcal{Z}_n^{(i,j)} = \mathcal{Z}_n^{(j)} - \mathcal{Z}_n^{(i)} \) such that \( \mathcal{Z}_n^{(i,j)}, \mathcal{Z}_n^{(i)} \) are independent random variables. Define \( Q_i : \mathbb{R} \to \mathbb{R} \) as
\[
Q_i(t) = \mathbb{E} \left[ \mathcal{Z}_n^{(i)} - 1_{\{0 \leq t \leq \mathcal{Z}_n^{(i)} \}} \right]. \quad (3.9)
\]

**Lemma 5.** Under the assumption that the triangular array \( \{ \zeta_{i,n}, i \in \{ 1, 2, \ldots, n \} \} \) are i.i.d. from a distribution \( \mathbb{P}_n \in \{ P_n : \mathbb{E} P_n (Z - \mathbb{E} P_n [Z])^2 = 1, \mathbb{E} P_n [Z^3] < \infty \} \),
\[
\int_{-\infty}^{\infty} \mathbb{E} \left[ |t| + |\mathcal{Z}_n^{(i)}| Q_i(t) dt \right] = n^{-3/2} \left( \mathbb{E} [\zeta_{1,n}^3] / 2 + \mathbb{E} [\zeta_{1,n}^2] \mathbb{E} [\zeta_{1,n}^2] \right).
\]

When \( \mathbb{P}_n \) belongs to the family \( \{ P_n : \mathbb{E} P_n (Z - \mathbb{E} P_n [Z])^2 = 1, \mathbb{E} P_n [Z^3] < \infty \} \),
\[
\int_{-\infty}^{\infty} \mathbb{E} \left[ (|t| + |\mathcal{Z}_n^{(i)}|) \left( 1 + (|t| + |\mathcal{Z}_n^{(i)}|)^2 \right) \right] Q_i(t) dt = n^{-3/2} \left( \mathbb{E} P_n [\zeta_{1,n}^3] \right)
\]
\[+ 2 \cdot \mathbb{E} \left[ \mathbb{E} P_n [\zeta_{1,n}^3] \mathbb{E} [\zeta_{1,n}^2] + o(1) \right].
\]
Next, observing
\[
\mathbb{E}_n [f_Z^{(j)}(Z_n^{(j)})] = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mathbb{E}_n [f_Z(Z_n^{(j)} + t)]Q_i(t)dt, \quad \text{and}
\mathbb{E}_n [f_Z'(Z_n^{(j)})] = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \mathbb{E}_n [f_Z'(Z_n^{(j)})]Q_i(t)dt,
\]
the right-hand side of the Stein’s identity can be bounded above as
\[
\left| \mathbb{E}_n \left[ f_Z(Z_n^{(j)}) - Z_n^{(j)} f_Z(Z_n^{(j)}) \right] \right| \leq \sum_{i=1}^{n} \int_{-\infty}^{\infty} \left| \mathbb{E}_n [f_Z'(Z_n^{(j)} + t)] - \mathbb{E} [f_Z'(Z_n^{(j)})] \right| Q_i(t)dt
\leq n \cdot \int_{-\infty}^{\infty} \sup_{\alpha \in [0,1]} \left| \mathbb{E}_n \left[ (t - Z_{t,n}^{(j)}) f_Z' \left( \alpha t + (1 - \alpha) Z_{i,n}^{(j)} + Z_{i,n}^{(j)} \right) \right] \right| Q_i(t)dt. \quad (3.10)
\]

For certain functions \( F_k, k = 1,2 \), we state some results about the smoothness of the corresponding Stein’s functions. These will be crucial in establishing a CLT in the next section.

**Lemma 6.** If \( F : \mathbb{R} \to \mathbb{R} \) is defined as
\[
F(z) = g(z; \sqrt{n}\beta_n^{(j)}) \cdot \Phi \left( -z + \sqrt{n}\beta_n^{(j)} \right) / \rho
\]
where the function \( g(\cdot; \sqrt{n}\beta_n^{(j)}) \) is bounded and has a uniformly bounded first derivative whenever \( \sqrt{n}\beta_n^{(j)} = O(1) \), then, the second derivative of the Stein’s function for \( F \) is uniformly bounded under local alternatives \( N \) in (3.6).

Define the functions \( L, U : \mathbb{R} \to \mathbb{R}^+ \):
\[
L(z; \sqrt{n}\beta_n^{(j)}) = 2\phi((z + \sqrt{n}\beta_n^{(j)}) / \rho) / \left\{ \sqrt{1 + (z + \sqrt{n}\beta_n^{(j)})^2} / \rho^2 - (z + \sqrt{n}\beta_n^{(j)}) / \rho \right\},
U(z; \sqrt{n}\beta_n^{(j)}) = 2\phi((z + \sqrt{n}\beta_n^{(j)}) / \rho) / \left\{ \sqrt{2 + (z + \sqrt{n}\beta_n^{(j)})^2} / \rho^2 - (z + \sqrt{n}\beta_n^{(j)}) / \rho \right\}
\]
the univariate versions of the functions in (2.5) with \( |E| = 1 \), \( Q_j z_n = -z_n^{(j)} / \rho \) and \( \sqrt{n}\alpha_n^{(j)} = -\sqrt{n}\beta_n^{(j)} / \rho \).

**Lemma 7.** Under rare alternatives \( R \) in (3.7) and for any \( c \in (0,1) \), let \( F_k : \mathbb{R} \to \mathbb{R}^+ \); \( k = 1,2 \) represent
\[
F_1(z; \sqrt{n}\beta_n^{(j)}) = g(z; \sqrt{n}\beta_n^{(j)}) \cdot L(z; \sqrt{n}\beta_n^{(j)}) 1_{\{z \in [c\sqrt{n}\beta_n^{(j)}, -c\sqrt{n}\beta_n^{(j)}]\}};
F_2(z; \sqrt{n}\beta_n^{(j)}) = g(z; \sqrt{n}\beta_n^{(j)}) \cdot U(z; \sqrt{n}\beta_n^{(j)}) 1_{\{z \in [c\sqrt{n}\beta_n^{(j)}, -c\sqrt{n}\beta_n^{(j)}]\}};
\]
where \( \|g\|_{\infty} \leq K_1, \|g'\|_{\infty} \leq K_2\sqrt{n}\beta_n^{(j)} \). Then, for sufficiently large \( n \), the second derivative of the Stein’s functions \( f_{F_k}, k = 1,2 \) on the interval \( [c\sqrt{n}\beta_n^{(j)}, -c\sqrt{n}\beta_n^{(j)}] \) is bounded as:
\[
|f_{F_k}''(z)| \leq (1 + z^2) |f_{F_k}(z)| - cK_1\sqrt{n}\beta_n^{(j)} \mathbb{E}_n \left[ f_k(Z; \sqrt{n}\beta_n^{(j)}) / g(Z; \sqrt{n}\beta_n^{(j)}) \right] - \sqrt{n}\beta_n^{(j)} \left( K_2 + K_1 / \rho^2 + c(1 + 1 / \rho^2) \right) K_1 / 2 \rho \sqrt{n}\beta_n^{(j)}
\]
\( k = 1,2 \).
3.3. A transfer of CLT in sequence models

In this section, we prove a CLT in the sequence model for a pivot in (3.3). First, we make an observation on the smoothness property of this pivot in Lemma 8.

Lemma 8. The pivot \( P_j(Z_n^{(j)}, \beta_n^{(j)}) \) under a sequence model in (3.3) has a uniformly bounded first derivative.

We consider a family of functions \( H \in \{ h : \mathbb{R} \rightarrow \mathbb{R} : \| h \|_\infty \leq K_1, \| h' \|_\infty \leq K_2 \} \) in Theorems 3 and 6. Note, we denote constants as \( C(\cdot); C_k(\cdot); C^{(j)}(\cdot), k, l = 1, 2, \ldots \) and these constants depend on the parameters mentioned within braces. Our first result is a direct application of the Stein’s bound to prove weak convergence of the pivot in (3.3) under a local alternative.

Theorem 3. For a sequence of parameters \( \{ \beta_n : n \in \mathbb{N} \} \) in (3.6), the bounds for the relative differences \( R_n^{(k)}, k = 1, 2 \) in Corollary 1 equal \( C_k(p; K_2)/\sqrt{n} \) whenever \( P_n \in \{ P_n : \mathbb{E}_{P_n}(Z - \mathbb{E}_{P_n}[Z])^2 = 1, \mathbb{E}_{P_n}[|Z|] < \infty \} \). Further,

\[
\left| \mathbb{E}_{P_n}[H \circ P_j(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})] - \mathbb{E}_{\tilde{P}}[H \circ P_j(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})]\right| \leq (K_1 + C_1 + C_2)/\sqrt{n}
\]

Next, we take up the more challenging rare alternatives. Under an additive Gaussian randomization, Theorem 6 demonstrates a transfer of CLT to the conditional paradigm under rare parameters \( R \) in (3.7), with no further restriction on the rate of decay of \( b_n \). We remark that this is a significant improvement over previously known results in Markovic and Taylor (2016) which were restricted to the local alternatives for Gaussian perturbations.

For weak convergence in the rare regime, we need some additional assumptions beyond the existence of third moments for our data generative family. We assume that the statistic \( Z_n \) in (3.5) is the sum of i.i.d. centered random variables \( \{ \zeta_{i,n}, i \in \{ 1, 2, \ldots, n \} \} \) from a distribution in the family

\[
\{ P_n : \mathbb{E}_{P_n}(Z - \mathbb{E}_{P_n}[Z])^2 = 1, \mathbb{E}_{P_n}[\exp(\gamma|Z| + \eta Z^2)] < \infty \text{ for some } \gamma, \eta > 0 \}.
\]

That is, we restrict the model-free family in Theorem 3 to distributions with finite exponential moments in a neighborhood of 0. Lemma 9 and Theorem 4 analyze the dominant term in the Stein’s bound in (3.10) under parameters in \( R \) and thus, lead to the weak convergence rates in Theorem 5 and 6.

Lemma 9. The Stein’s functions for non-negative valued, bounded functions \( F_1(\cdot; \sqrt{n}\beta_n^{(j)}) \) and \( F_2(\cdot; \sqrt{n}\beta_n^{(j)}) \) defined in Lemma 7 satisfy the following decaying condition under true underlying parameter sequence \( \{ \beta_n^{(j)} : n \in \mathbb{N} \} \) in (3.7)

\[
f_{F_k}(z) \leq C(p, K_1, c) \exp(-n\beta_n^{(j)2}/2(1 + \rho^2))/\sqrt{n}\beta_n^{(j)}|; \ k = 1, 2.
\]

Theorem 4. The sequence \( \{ B_n : n \in \mathbb{N} \} \) defined as

\[
B_n = n^{3/2}|\beta_n^{(j)}| \cdot \int_{-\infty}^{\infty} \sup_{\alpha \in [0, 1]} \mathbb{E}_{P_n} \left[ \left( |t| + |Z_n^{(j)}| L(1 + \alpha t + (1 - \alpha)Z_{1,n}^{(j)} + Z_{2}^{(j)} + \sqrt{n}\beta_n^{(j)}) \right)
\right.
\]
\[
\left. 1_{\{ c\sqrt{n}\beta_n^{(j)} < \alpha t + (1 - \alpha)Z_{1,n}^{(j)} + Z_{2}^{(j)} < -c\sqrt{n}\beta_n^{(j)} \}} \right] Q_1(t) dt.
\]
satisfies 
\[ B_n \leq C(c, \rho) \cdot \beta_n^{(j)} \exp(-n\beta_n^{(j)2}/2(1 + \rho^2))/\sqrt{n}\beta_n^{(j)} \]
under parameters \( \{ \beta_n : \sqrt{n}\beta_n^{(j)} = b_n\beta, \beta < 0, b_n \to \infty \text{ as } n \to \infty, b_n = o(n^{1/2}) \} \) in (3.7) and under a data-generative distribution in (3.12).

**Theorem 5.** Let \( F_k : \mathbb{R} \to \mathbb{R}; k = 1, 2 \) be the functions defined in Lemma 7. Under parameters in Theorem 4 

\[ \left| \mathbb{E}_{\pi_n} \left[ F_k(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) \right] - \mathbb{E}_\Phi \left[ F_k(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) \right] \right| \leq \frac{\exp \left( -n\beta_n^{(j)2}/2(1 + \rho^2) \right)}{\sqrt{n}\beta_n^{(j)}} R_{n,k} \]

where the rate sequence \( R_{n,k} = C_k^{(1)}(c, K_1, K_2, \rho)/\sqrt{n} + C_k^{(2)}(c, K_1, K_2, \rho)\beta_n^{(j)} \).

Now, we state the CLT theorem that derives the rates of convergence of \( R_n^{(1)}, R_n^{(2)} \) in Corollary 1, when the selection probability vanishes at an exponentially fast rate. The bound in Theorem 6 proves weak convergence of the pivot in (3.3) under parameters in (3.7) or equivalently, when \( \sqrt{n}\beta_n^{(j)} \to -\infty \) as \( n \to \infty \) and \( \beta_n^{(j)} = o(1) \).

**Theorem 6.** Under the rare alternatives in (3.7), the bounds \( R_n^{(k)} \) in Corollary 1 equal 
\[ C_k^{(1)}(c, K_1, K_2, \rho)/\sqrt{n} + C_k^{(2)}(c, K_1, K_2, \rho) + C_k^{(3)}(c, K_1, K_2, \rho)/\sqrt{n}\beta_n^{(j)} \]
\( k = 1, 2 \), under a distribution in (3.12). Further, 
\[ \left| \tilde{\mathbb{E}}_{\pi_n} [H \circ \mathcal{P}_j(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})] - \mathbb{E}_\Phi [H \circ \mathcal{P}_j(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})] \right| \leq C_1/\sqrt{n} + C_2/\sqrt{n}\beta_n^{(j)} + C_3/\sqrt{n}\beta_n^{(j)} \]
where \( C_j = K_1 \cdot C_1^{(j)} + C_2^{(j)}, j = 1, 2, 3 \).

4. **A multivariate Stein’s approximation**

Next, we provide an answer for Question 2 under a multivariate model described in Section 1.2. The asymptotic results presented in the section extend to selective inference post regularized regression, an important extension from sequence models to the setting of linear models made for Gaussian data in Lee and Taylor (2014); Lee et al. (2016); Tibshirani et al. (2016).

Unlike the sequence model setting considered in Section 3, the likelihood ratio and pivots are now functions of the statistic \( Z_n \in \mathbb{R}^d \); see Lemma 2 and the pivot in Proposition 1. To provide rates for the bounds on the relative differences in Theorem 1, we turn to a multivariate version of the Stein’s approximation, based upon the construction of an exchangeable pair of random variables in Chatterjee and Meckes (2007).

4.1. **A Stein’s bound**

Denoting \( F \) as a bounded and thrice-differentiable function, let \( f_F(\cdot) \) denote the corresponding Stein’s function, a solution to the differential equation
\[ F(z) - \mathbb{E}_\Phi[F(Z_n)] = \text{Tr}(\nabla^2 f_F(z)) - z^T \nabla f_F(z); \text{ Tr}(\cdot) \text{ denotes Trace}. \]
Based upon $Z_n$ in (2.1), denote a leave-one statistic as $Z_n^{(i)} = Z_n - Z_{i,n}$. Construct
\[ Z_n' = Z_n - Z_{I,n} + Z_{I,n}' = Z_n - \zeta_{I,n}/\sqrt{n} + \zeta_{I,n}'/\sqrt{n}, \]
\[ \zeta_{I,n}' \] an independent copy of $\zeta_{I,n}$ and $I$ is a uniform variable over the indices \( \{1, 2, \cdots, n\} \), independent of $Z_n$. Note that with this construction, $Z_n'$ is distributed as $Z_n$ and $(Z_n', Z_n)$ forms an exchangeable pair. Further, the pair satisfies the below conditional moment conditions for $\lambda = 1/n$:
\[ E_{P_n}|Z_n' - Z_n|Z_n| = -\lambda Z_n = -Z_n/n \quad (4.1) \]
\[ E_{P_n}[(Z_n' - Z_n)(Z_n' - Z_n)^T|Z_n]/2\lambda = I + E_{P_n}[A|Z_n]. \quad (4.2) \]
where random matrix $A$ equals $A = n \cdot E_{P_n}[(Z_n' - Z_n)(Z_n' - Z_n)^T|Z_n]/2 - I$.

Let $\zeta_{I,n}'$ denote the $i$-th coordinate of the vector $\zeta_{I,n} \in \mathbb{R}^d$. A Stein’s bound in Chatterjee and Meckes (2007) (Equation 11) based upon the exchangeable pair $(Z_n', Z_n)$ is
\[ |E_{P_n}[F(Z_n)] - E_{\Phi}[F(Z_n)]| \leq \left| E_{P_n}[\text{Tr}(A\nabla^2 f_F(Z_n))] \right| + n \cdot |E_{P_n}[R_n(Z_n, Z_n')]| \quad (4.3) \]
where the remainder term $R_n(Z_n', Z_n)$ equals
\[ \frac{1}{6n^{3/2}} \sum_{i,j,l} \sup_{\alpha \in [0,1]} (\zeta_{I,n}' - \zeta_{I,n}) \cdot \zeta_{I,n}' \cdot (\zeta_{I,n}' - \zeta_{I,n}')(\zeta_{I,n}' - \zeta_{I,n}')(\nabla^3 f_F(\alpha Z_n + (1 - \alpha) \cdot Z_n'))_{i,j,l}. \]

Simplifying the two terms in (4.3), Lemma 10 states a finite sample bound based on a third-order partial derivative of the Stein’s function $f_F(\cdot)$ where $(\nabla^3 f_F(\cdot))_{i,j,l}$ denotes the $(i,j,l)$-th entry of the third-order tensor.

**Lemma 10.** For a bounded function $F : \mathbb{R}^d \to \mathbb{R}$ that is thrice differentiable, a multivariate Stein’s bound for $|E_{P_n}[F(Z_n)] - E_{\Phi}[F(Z_n)]|$ equals
\[ \sum_{k \in \{1,2\}} \frac{A_k}{\sqrt{n}} \sum_{i,j,l} E_{P_n} \left[ |\zeta_{1,n}'| \cdot |\zeta_{1,n}'| \cdot |\zeta_{1,n}'| \cdot \sup_{\alpha \in [0,1]} |\nabla^3 f_F(Z_n^{(1)} + \Delta^{(k)}(\zeta_{1,n}/\sqrt{n}; \zeta_{1,n}'/\sqrt{n})|_{i,j,l}| \right] \\
+ \sum_{k \in \{1,2\}} \frac{B_k}{\sqrt{n}} \sum_{i,j,l} E_{P_n} \left[ |\zeta_{1,n}'| \cdot \sup_{\alpha \in [0,1]} |\nabla^3 f_F(Z_n^{(1)} + \Delta^{(k)}(\zeta_{1,n}/\sqrt{n}; \zeta_{1,n}'/\sqrt{n})|_{i,j,l}| \right] \]
where $\Delta^{(1)} = \alpha \zeta_{1,n}/\sqrt{n} + (1 - \alpha)\zeta_{1,n}'/\sqrt{n}$ and $\Delta^{(2)} = \alpha \zeta_{1,n}/\sqrt{n}; A_k, B_k = O(1), k = 1, 2$.

### 4.2. Transfer of CLT in multivariate model

The pivot considered in this section is $P_j(Z_n; \sqrt{n} \beta_n)$, defined in Proposition 1. We describe the smoothness properties of this pivot based upon the conditional law in Lemma 11.

**Lemma 11.** The $k$-th derivative of the pivot $P_j(Z_n; \sqrt{n} \beta_n)$ in Proposition 1 satisfies
\[ \|\partial^k P_j(\cdot; \sqrt{n} \beta_n)\| \leq \|M Z_n\|^k + \|N \sqrt{n} \beta_n\|^k; j \in E \]
for some matrices $M, N$. 

First, let us turn to the local alternatives \( \mathcal{N} \) in \((LA)\). The selection probability under an additive Gaussian perturbation, \( \mathbb{E}_g \left[ \prod_{j=1}^L \Phi \left( Q_j, \mathcal{Z}_{n} + \sqrt{n} \alpha_n^{(j)} \right) \right] \), the denominator in the relative differences in Theorem 1 is bounded away from 0 under the local alternatives. Therefore, deriving a rate of decay for the numerators in the relative differences

\[
\left| \mathbb{E}_g \left[ g_l(\mathcal{Z}_{n}) \prod_{j=1}^L \Phi \left( Q_j, \mathcal{Z}_{n} + \sqrt{n} \alpha_n^{(j)} \right) \right] - \mathbb{E}_{\mathcal{P}_n} \left[ g_l(\mathcal{Z}_{n}) \prod_{j=1}^L \Phi \left( Q_j, \mathcal{Z}_{n} + \sqrt{n} \alpha_n^{(j)} \right) \right] \right|, \quad (4.4)
\]

establishes a weak convergence result; \( g_l(\cdot) \) is defined in Theorem 1.

Theorem 7 states a weak convergence result for the pivot under local alternatives and a data generative family with finite moments up to the fifth order.

**Theorem 7.** Let \( \mathcal{Z}_{n} \) be the statistic in (2.1) based on a triangular array from

\[
\{ P_n : \mathbb{E}_P(\mathcal{Z} - \mathbb{E}_P(\mathcal{Z}))(\mathcal{Z} - \mathbb{E}_P(\mathcal{Z}))^T = I, \mathbb{E}_P(\| \mathcal{Z} \|^2) < \infty \}.
\]

For \( \mathcal{H} \in \{ h : \mathbb{R} \to \mathbb{R} : \| h \|_\infty \leq K_1, \| h' \|_\infty \leq K_2, \| h'' \|_\infty \leq K_3 \} \), the bounds \( R_n \) in Theorem 1 equal \( C_1(\mathbb{K}_1, \mathbb{K}_2, \rho)/\sqrt{n}, l = 1, 2 \) and

\[
| \mathbb{E}_P. [\mathcal{H} \circ \mathcal{P}_j(\mathcal{Z}_{n}; \sqrt{n} \beta_n)] - \mathbb{E}_P. [\mathcal{H} \circ \mathcal{P}_j(\mathcal{Z}_{n}; \sqrt{n} \beta_n)] | \leq (K_1 \cdot C_1 + C_2)/\sqrt{n},
\]

under a parameter sequence in \((LA)\).

In the rare regime, the selection probability decays at an exponentially fast rate, derived in Theorem 2. Combined with the vanishing rate of the denominator of the relative differences in Theorem 1, the asymptotic validity of conditional inference for the more challenging parameters in \((RA)\) hinges on a rate at which

\[
\left( \prod_{j=1}^L | \sqrt{n} \alpha_n^{(j)} | \right) \cdot \exp(\sqrt{n} \alpha_n^{(j)} (1 + Q Q^T)^{-1} \sqrt{n} \alpha_n^{(j)}/2) \]

\[
\left| \mathbb{E}_g \left[ g_l(\mathcal{Z}_{n}) \prod_{j=1}^L \Phi \left( Q_j, \mathcal{Z}_{n} + \sqrt{n} \alpha_n^{(j)} \right) \right] - \mathbb{E}_{\mathcal{P}_n} \left[ g_l(\mathcal{Z}_{n}) \prod_{j=1}^L \Phi \left( Q_j, \mathcal{Z}_{n} + \sqrt{n} \alpha_n^{(j)} \right) \right] \right|
\]

decays to 0. Analogous to the assumptions we make in the sequence setting, we impose a stronger assumption on the family of distributions in Theorem 7 in order to prove asymptotic validity of pivotal inference:

\[
\{ P_n : \mathbb{E}_P(\mathcal{Z} - \mathbb{E}_P(\mathcal{Z}))(\mathcal{Z} - \mathbb{E}_P(\mathcal{Z}))^T = I, \mathbb{E}_P(\exp(\gamma^T|| \mathcal{Z} || + \eta^T || Z ||^2)) < \infty \text{ for some } \gamma, \eta > 0 \}. \quad (4.5)
\]

Next, adopting a similar line of proof for Theorems 5 and 6, it suffices to analyze the Stein’s bound in Lemma 10 for functions \( F_k^{(i)} : \mathbb{R}^d \to \mathbb{R}^+, k = 1, 2; t = 1, 2 \) such that

\[
F_1^{(i)}(z; \sqrt{n} \alpha_n) = g_1(z) \cdot L(z; \sqrt{n} \alpha_n) \cdot 1_{\{ q_{Z_n} < q_{\sqrt{n} \alpha_n} \}},
\]

\[
F_2^{(i)}(z; \sqrt{n} \alpha_n) = g_2(z) \cdot U(z; \sqrt{n} \alpha_n) \cdot 1_{\{ q_{Z_n} < q_{\sqrt{n} \alpha_n} \}},
\]

for \( q, L(\cdot), U(\cdot) \) defined in Lemma 4 and \( g_1, g_2 : \mathbb{R}^d \to \mathbb{R}^+ \) satisfying \( g_1(z) \equiv 1 \) and \( g_2(z) = \mathcal{H} \circ \mathcal{P}_j(z; \sqrt{n} \beta_n) \).

Theorem 8 establishes a bound for the dominant term in analyzing the bounds in Lemma 10 for the functions \( F_k^{(i)} : \mathbb{R}^d \to \mathbb{R}^+, k = 1, 2; t = 1, 2 \). Before stating the theorem, we introduce few notations. We let \( \Delta(\zeta_{1,n}, \sqrt{n}, \zeta_{1,n}/\sqrt{n}) \) represent both the terms \( \Delta^{(1)}, \Delta^{(2)} \) in Lemma 10 and define...
Let $\mu_n(W_n) = -(I + (1-t)QQ^T)^{-1}QW_n + \sqrt{n}a_n$. Let $\mu_n(t)$ denote the $i$-th coordinate of $\mu_n$. Finally, define a sequence $\{B_n(i,j,l) : n \in \mathbb{N}\}$ such that

$$B_n(i,j,l) \leq n^{3/2} \cdot C(\rho, \|\alpha_n\|^3) \cdot \exp(-\sqrt{n}a_n(I + QQ^T)^{-1/2} \sqrt{n}a_n/2).$$

**Theorem 9.** Let $Z_n$ be based on a triangular array from the family of distributions in (4.5). Under parameters in (RA), $R_n^{(j)}$ in Theorem 1 equals $n \cdot C_1(K_2, K_3, Q, \rho)\|\alpha_n\|^3, l = 1, 2$. Further,

$$|\mathbb{E}_{\alpha_n} [H \circ \mathcal{P}_j(Z_n; \sqrt{n} \beta_n)] - \mathbb{E}_{\alpha_n} [H \circ \mathcal{P}_j(Z_n; \sqrt{n} \beta_n)]]| \leq n\|\alpha_n\|^3 \cdot (K_1 \cdot C_1 + C_2)$$

whenever $H \in \{h : R \rightarrow \mathbb{R} : \|h\|_\infty \leq K_1, \|h\|_\infty \leq K_2, \|h\|_\infty \leq K_3\}$.

For alternatives in (RA) inducing rare selections, Theorem 9 proves a weak convergence of the conditional pivot whenever $a_n = o(n^{1/6})$ or equivalently when

$$\|\alpha_n\|^3 = o(n^{-1}).$$

### 5. A carved CLT

After analyzing an asymptotically equivalent Gaussian randomization scheme in a sequence and a multivariate setting, we now answer Question 1 for data-carved investigations. Coupled with the rate of transfer of CLT from $\ell \rightarrow \ell^{(N)}$, weak convergence in carved inference is determined by conditions under which the tail of the implicit randomization in (1.3) converges to a Gaussian tail.

First, we revisit the sequence model settings in Section 3. Theorem 10 establishes validity of carved inference in (3.12) for parameters in (3.6) and a subset of rare alternatives in (3.7) such that $b_n = o(n^{1/6})$.

**Theorem 10.** Let $H$ belong to a class of functions considered in Theorem 6 and let $\mathcal{P}_j(\cdot; \sqrt{n} \beta_n^{(j)})$ be a pivot in (3.3). When $Z_n^{(j)}$ is based on an i.i.d. triangular array from (3.12)

$$|\mathbb{E}_{\alpha_n} [H \circ \mathcal{P}_j(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)})] - \mathbb{E}_{\alpha_n} [H \circ \mathcal{P}_j(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)})]| = o(1)$$

under the parameters $\beta_n^{(j)}$ such that $\sqrt{n} \beta_n^{(j)} = O(1)$ or $\sqrt{n} \beta_n^{(j)} = b_n\beta, \beta < 0, b_n \rightarrow \infty$ as $n \rightarrow \infty$ and $b_n = o(n^{1/6})$.

We provide examples of some large-scale inference problems in the sequence model where the application of CLT under the conditional law allows a relaxation of the limiting distributional assumption of a Gaussian model.

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Example 5.1. Selective reporting of difference in means: In many problems including gene selection in microarray data analysis described in Storey and Tibshirani (2003); Fan and Ren (2006), two-sample test-statistics are commonly used to select from d genes a subset that shows differential expression across two or more biological conditions. Modeled into the two-stage experimental set-up, we pursue inference with an additional \( n_2 \) samples for only those genes that showed statistically significant effects based on the \( n_1 \) initial samples in the screening tests. Casting this framework into the many-means problem, we have for each gene, two groups of observations

\[
X^{(j)}_{1,n}, X^{(j)}_{2,n}, \ldots, X^{(j)}_{n,n} \overset{i.i.d.}{\sim} \mathcal{P}^{(1)}_n, \quad Y^{(j)}_{1,n}, Y^{(j)}_{2,n}, \ldots, Y^{(j)}_{n,n} \overset{i.i.d.}{\sim} \mathcal{P}^{(2)}_n, \quad j \in \{1, 2, \ldots, d\},
\]

where \( \mathcal{P}^{(1)}_n, \mathcal{P}^{(2)}_n \) belong to a non-parametric family of distributions \( \mathcal{P} \) with variance \( \sigma_j \). We selectively report inference for the difference in the mean functionals

\[
\beta^{(j)}_n(\mathcal{P}^{(1)}_n, \mathcal{P}^{(2)}_n) = \mathbb{E}_{\mathcal{P}^{(1)}_n}[X^{(j)}_n] - \mathbb{E}_{\mathcal{P}^{(2)}_n}[Y^{(j)}_n] = \mu^{(j)}_{1,n} - \mu^{(j)}_{2,n}
\]

whenever the two-sample \( Z \)-test exceeds a threshold \( \lambda^{(j)} \); that is,

\[
\bar{Z}^{(j)}_n = \sqrt{n_1}(\bar{X}^{(j)}_{n_1} - \bar{Y}^{(j)}_{n_1})/\sqrt{2\sigma^{(j)}} > \lambda^{(j)} \iff \bar{Z}^{(j)}_n + W^{(j)}_n > \lambda^{(j)}
\]

for implicitly introduced randomizations

\[
W^{(j)}_n = \sqrt{n} \left( (\bar{X}^{(j)}_{n_1} - \bar{Y}^{(j)}_{n_1})/\sqrt{2\sigma^{(j)}} - (\bar{X}^{(j)}_{n_1} - \bar{Y}^{(j)}_{n_1})/\sqrt{2\sigma^{(j)}} \right).
\]

Consider a parameterization \( \{ (\mu^{(j)}_{1,n}, \mu^{(j)}_{2,n}) : \sqrt{n}(\mu^{(j)}_{1,n} - \mu^{(j)}_{2,n}) = b_n \bar{\mu} \} \). Assuming independence across the \( d \) groups of observations, we provide inference for \( \beta^{(j)}_n, j \in E \) under the likelihood in (3.1). Whenever \( b_n = o(n^{1/6}) \), carved inference for the difference of means \( \{ \beta^{(j)}_n, j \in E \} \) based on the pivot in (3.3) is asymptotically valid for a non-parametric distribution \( \mathcal{P}^{(b)}_n \) in (3.12), \( k = 1, 2 \).

In the two-stage setting, the randomization arises naturally from a hierarchical design of experiment with a split of the augmented samples involved in selection. An alternate scheme of randomizing is the asymptotically equivalent Gaussian randomization in (2.3), with \( W^{(j)}_n \) independent of \( Z^{(j)}_n, j \in \{1, 2, \ldots, d\} \). Under this additive Gaussian perturbation, asymptotic inference extends to a larger class of rare alternatives. Theorem 6 in this case proves weak convergence of the pivot whenever \( b_n = o(n^{1/2}) \) for any distribution in the family (3.12).

Example 5.2. Carved inference for D-Largest effects: Applications in Zeggini et al. (2007); Zhong and Prentice (2008) amongst many others acknowledge that confidence intervals are likely to be reported for the largest effects in a genome-wide scan. For example, in the identification of risk loci in the genome, selecting the largest effects is equivalent to an event where the log-adds ratios corresponding to a subset of risk loci exceed a threshold. Motivated by these examples, we look at a naive thresholding rule that selectively reports the D-largest effects.

We follow the same notations and set-up as the canonical example in the introductory section. Cast into the carved settings, we consider inference for the selected mean functionals in an augmented sample of size \( n_1 + n_2 \) based on a screening that uses \( n_1 \) pilot samples.
Denote the reported set of $D$-largest effects by $E$. Let the $(D+1)$-th ordered (in decreasing order) test-statistic based on $n_1$ samples be denoted by $Z^{(D+1)}_n$ and let the signs of screened statistics, $\{\text{sign}(W^{(j)}_n + Z^{(j)}_n) = s^{(j)}, j \in E\}$ be denoted by the vector $s^{(E)}$. Observe that the selection event
\[
\{\hat{E} = E, s^{(E)} = s_E, R_n^{(-E)} = R_n, Z^{(D+1)}_n = Z^{(D+1)}\} \equiv \{s_E T_n^{(E)} > 0, s^{(E)} = s_E, R_n^{(-E)} = R_n, Z^{(D+1)}_n = Z^{(D+1)}\}
\]
based on a linear map:
\[
\begin{pmatrix}
W^{(E)}_n \\
W^{(-E)}_n
\end{pmatrix}
= -\left(\Sigma^{1/2} z_n + \sqrt{n} \beta_n\right) + Q_E T_n^{(E)} + r_E (Z^{(D+1)}_n, R_n^{(-E)})
\]
\[
Q_E = \begin{pmatrix} \mathbf{I} & 0 \end{pmatrix}^T, \quad r_E (Z^{(D+1)}_n, R_n^{(-E)}) = (\sqrt{1 + \rho^2} |Z^{(D+1)}_n| \cdot s^{(E)}, R_n^{(-E)})^T;
\]
and $R_n^{(-E)} = W^{(-E)}_n + Z^{(-E)}_n$. To provide marginal inference for $\beta^{(j)}_n$ in the sequence model, the carved likelihood $\ell(z^{(j)}_n)$ is proportional to
\[
\ell(z^{(j)}_n) \cdot g_{z^{(j)}_n} (-z^{(j)}_n + \sqrt{n} \rho^{(j)}_n) + \ell^{(j)} + \sqrt{1 + \rho^2} \cdot s^{(j)}_n |Z^{(D+1)}_n| \cdot s^{(j)}_n |Z^{(D+1)}_n| 1_{s^{(j)}_n |z^{(j)}_n| > 0} \, dt_n.
\]
Under a Gaussian perturbation in (2.3), the above likelihood, denoted as $\bar{\ell}^{(N)}$, equals
\[
\ell(z^{(j)}_n) \cdot \Phi(z^{(j)}_n) / \rho + \sqrt{1 + \rho^2} \cdot s^{(j)}_n |Z^{(D+1)}_n| / \rho 1_{s^{(j)}_n |z^{(j)}_n| > 0}.
\]

The transfer of CLT results from $\ell \rightarrow \bar{\ell}^{(N)}$ and $\ell \rightarrow \bar{\ell}$ allow us to provide model-free inference in an asymptotic sense. Note that we are not limited to valid inference for the local alternatives in (3.6). The rate of weak convergence, provided in Theorems 6 and 10 extends to weak convergence in the rare regime. The validity of inference in this asymptotic paradigm is controlled by the rate of decay of parameters $\beta_n$, defined in (3.7).

Continuing with the example above, we now look at inference for the largest $D$-effects selected via a step-up procedure with $D$ chosen in a data-adaptive fashion. As a specific example, consider the BH-$q$ procedure that successfully controls for false discoveries in the multiple testing framework; see Benjamini et al. (2009). Denote the ordered $Z$-statistics based on the $n_1$ initial samples as
\[
Z^{(1)}_n \geq Z^{(2)}_n \cdots \geq Z^{(d)}_n
\]
and a set of corresponding thresholds for each of these statistics, again in decreasing order
\[
\tau^{(1)} \geq \tau^{(2)} \cdots \geq \tau^{(d)}.
\]

Conditional on the selected indices $E$ coupled with additional information: the values of the statistics for the selected indices $s^{(E)}$, the (adaptive) rejection threshold $D$ and the values of the statistics corresponding to the rejected set $R_n^{(-E)}$:
\[
\{\hat{E} = E, s^{(E)} = s_E, D = D_0, R_n^{(-E)} = R_n\},
\]
the likelihood $\bar{\ell}(z)$ is proportional to
\[
\ell(z_n) \cdot g_{z_n} (-\Sigma^{1/2} z_n + \sqrt{n} \beta_n) + Q_E t_n + r_E (D_0, R_n) 1_{s_E: t_n > 0} \, dt_n.
\]
where $Q_E = [I \ 0]^T$, $r_E(D_0, R_n) = (\sqrt{1 + \rho^2 \cdot \tau(D_0)}, R_n(-E))^T$. In the sequence model, the carved likelihood decouples and hence, marginal inference about $\sqrt{n}\beta_n^{(j)}$ is based upon

$$
\ell(z_n^{(j)}) \cdot \int g_{x_n^{(j)}}(-z_n^{(j)} + \sqrt{n}\beta_n^{(j)}) + \ell_n^{(j)} + \sqrt{1 + \rho^2 \cdot s_n^{(j)} \tau(D_0)} \cdot 1_{s_n^{(j)} t_n^{(j)} > 0} dt_n^{(j)}.
$$

Specifically, for the Benjamini-Hochberg step-up selection, the cut-off threshold $\tau(D_0)$ equals $\Phi^{-1}(1 - D_0 \alpha/2d)$ when we consider two-sided p-values in the screening. Applying the weak convergence results in the paper allows us to provide limiting inference based on the pivot

$$
\mathcal{P}(Z_n^{(j)}, \beta_n^{(j)}) = \int_{\mathbb{R}}^{\infty} \exp(-z^2/2) \cdot \Phi \left( \sqrt{1 + \rho^2 \cdot s_n^{(j)} \tau(D_0)} / \rho - (z + \sqrt{n}\beta_n^{(j)}) / \rho \right) dz
$$

When we do not assume independence amongst the d-groups of observations, a multivariate setting describes the examples discussed above. Theorem 11 proves validity of carved inference based upon a pivot in Proposition 1 under a likelihood in Lemma 2.

**Theorem 11.** Let $\mathcal{H}$ belong to a class if functions considered in Theorem 9. Let $Z_n$ be the statistic in (2.1) based upon an i.i.d. triangular array from any distribution in the non-parametric family in (4.5) and randomized by $W_n$ defined in (1.3). Then,

$$
|E_{\mathcal{H}}[\mathcal{H} \circ \mathcal{P}_j(Z_n; \sqrt{n}\beta_n)] - E_{\mathcal{H}}[\mathcal{H} \circ \mathcal{P}_j(Z_n; \sqrt{n}\beta_n)]| = o(1).
$$

under parameters $\{\beta_n : n \in \mathbb{N}\}$ in (LA) and in (RA) such that $a_n = o(n^{1/6})$.

We conclude this section with an example of inference post a carved version of elastic net in the regression setting, where an application of Theorem 11 allows valid effect size calibration in an asymptotic sense.

**Example 5.3.** Carved inference for regression: A variable selection problem in the regression settings can be cast as:

$$
\min_{\beta \in \mathbb{R}^p} \ell_p(y_{1n}, X_{1n}; \beta) + P_\lambda(\beta)
$$

where $\hat{\beta}_n = (y_i, X_i) \in \mathbb{R}^{e+1}, i = 1, 2, \ldots, n$ are i.i.d. from $\mathbb{P}$, an unknown distribution in a non-parametric family with finite covariance matrix. In the above optimization objective, $\ell_p(y_{1n}, X_{1n}; \beta)$ is a loss function based upon the n1 pilot samples, $\rho^2 = n_2/n_1$ as set in the canonical example and $P_\lambda(\beta)$ is a regularizing penalty with tuning parameter $\lambda$. As a concrete example, consider the elastic net that solves (5.2) with

$$
\ell_p(y_{1n}, X_{1n}; \beta) = \frac{(1 + \rho^2)}{2\sqrt{n}} \|y_{1n} - X_{1n}\beta\|^2_2; P_{\lambda, \eta}(\beta) = \lambda \|\beta\|_1 + \frac{\eta}{2} \|\beta\|^2_2.
$$

Noting that $\ell_0(y_{1n}, X_{1n}; \beta)$ represents the loss when selection is based on all of the data, the optimization in elastic net can be represented in terms of the augmented data as:
minimize $\ell_0(y_n, X_n; \beta) + (\ell_\rho(y_{n_1}, X_{n_1}; \beta) - \ell_0(y_n, X_n; \beta)) + P_{\lambda, \eta}(\beta)$.

Interpreting $(\ell_0(y_n, X_n; \beta) - \ell_\rho(y_{n_1}, X_{n_1}; \beta))$ as a perturbation to the variable selection algorithm, a randomization term is identified in Markovic and Taylor (2016) as

$$W_n = \left. \frac{\partial}{\partial \beta} (\ell_0(y_n, X_n; \beta) - \ell_\rho(y_{n_1}, X_{n_1}; \beta)) \right|_{\beta = \beta_n},$$

where $\tilde{\beta}^{\lambda, \eta}$ is the solution to the carved optimization (5.2). Defining

$$\beta_E = \arg \min_{\alpha_E} \mathbb{E}_{\tilde{F}_n}[(Y - x_E^T \alpha_E)^2] = \mathbb{E}_{\tilde{F}_n}[x_E x_E^T]^{-1} \mathbb{E}_{\tilde{F}_n}[x_E Y],$$

the best linear parameters in selected model $E$ as a relevant target of interest post selection, consider marginal inference for $\beta^{(j)}_E$, the $j$-th coordinate of $\beta_E$.

Define a centered and scaled version of the statistic $Z_n = (\tilde{\beta}_E, N_E)^T \equiv \sqrt{n} T_n$ in Proposition 2 of Panigrahi et al. (2016) where

$$\tilde{\beta}_E = \sqrt{n} (X_E^T X_E)^{-1} X_E^T y, N_E = X_E^T (y - X_E \tilde{\beta}_E) / \sqrt{n}.$$  

Denoted as $Z_n = \Sigma^{-1}(Z_n - \sqrt{n} \beta_n)$ where $\mathbb{E}_{\tilde{F}_n}[Z_n] = \sqrt{n} \beta_n$, it follows from this prior work that $Z_n$ satisfies a CLT and can be cast as a statistic in (2.1).

Let the signs of active coefficients $\tilde{\beta}^{\lambda, \eta}_E$ be $s_E$ and let the inactive sub-gradient variables $\partial ||b||_1$ at the solution $\tilde{\beta}^{\lambda, \eta}_E$ be denoted by $R_{n}^{(-E)}$. Analogous to the prototypical example in the introduction, we define variables $T_n^{(E)}$ such that

$$W_n = \left[ \begin{array}{cc} U & 0 \\ V & I \end{array} \right] (\Sigma_1/2 Z_n + \sqrt{n} \beta_n) + \left( \begin{array}{c} U + \eta I \\ V \end{array} \right) T_n^{(E)} + \left( \begin{array}{c} \lambda s_E \parallel \parallel \\ \tilde{R}_n^{(-E)} \parallel \parallel \end{array} \right).$$

The above map is derived from the K.K.T. map associated with the query, with $U, V$ defined as the probability limits of the matrix expectations $\mathbb{E}_{\tilde{F}}[X_E^T X_E]/n$ and $\mathbb{E}_{\tilde{F}}[X_E^T X_E]/n$.

Observe that the selection constraints associated with observing an active set of variable $E$ with the corresponding signs and the inactive coordinates of the subgradient: $\{\tilde{E} = E, s_E = s_E, R_{n}^{(-E)} = R_n\}$ are equivalent to constraints $\{\text{sign}(T_n^{(E)}) = s_E, R_{n}^{(-E)} = R_n\}$. Thus, a carved likelihood for the statistic $Z_n$ conditional on the output $\tilde{E} = E, s_E = s_E, R_{n}^{(-E)} = R_n$ from solving (5.2) is proportional to

$$\frac{\ell(\tilde{z}_n) \times \int g_{\tilde{z}_n}(P_E(1/2 \tilde{z}_n + \sqrt{n} \tilde{\beta}_n) + Q_E t_n + r_E(R_n)) \mathbb{1}_{\text{sign}(t_n) = s_E} dt_n}{\int \ell(z'_n) \times \int g_{z'_n}(P_E(1/2 z'_n + \sqrt{n} \beta_n) + Q_E t_n + r_E(R_n)) \mathbb{1}_{\text{sign}(t_n) = s_E} dt_n dz'_n},$$

$$P_E = - \left[ \begin{array}{cc} U & 0 \\ V & I \end{array} \right], Q_E = \left( \begin{array}{c} U + \eta I \\ V \end{array} \right), r_E = \left( \begin{array}{c} \lambda s_E \\ \tilde{R}_n^{(-E)} \parallel \parallel \end{array} \right).$$

Validity of a pivot, constructed when $Z_n$ is distributed exactly as a Gaussian random variable, now holds for broad classes of model-free families. More importantly, asymptotic inference extends beyond local alternatives to certain rare selections, proved in Theorems 9 and 11.
6. Concluding perspectives and future directions

In a modern scientific paradigm, the first stage of many investigations is of an exploratory nature, identifying some parameters of interest. With the arrival of new data-samples or availability of related data-sets, consolidating these explorations from the pilot stage through interval estimates is a natural goal. Modeling such investigations into a hierarchical set-up, selection rules applied to samples in the preliminary stage lead to a naturally induced perturbation scheme, asymptotically distributed as a Gaussian random variable.

In the current paper, we provide rates of weak convergence of pivots that validate asymptotic conditional inference in an iterative approach to experimentation and validation. We are thus able to provide a basis for model-free inference in hierarchical investigations, cast into a conditional framework. An important contribution of our “transfer of CLT” theorems to the conditional paradigm is an extension of asymptotic inference to a class of parameters that induce to rare selections. In a practical scenario, this class of results enables valid non-parametric inference in a post-selective setting for effects that might not be easy to detect.

While we proved asymptotic validity of exact pivots constructed through probability integral transforms of a conditional law under normal distributions, our technique of proof is not limited to this pivot. In fact, it is of interest to investigate other classes of pivots, like an approximate pivotal quantity in Panigrahi et al. (2017); the main take away being that the performance of pivots under parametric distributions now transfers to model-free families.

Finally, we conclude by noting that we assumed a known covariance structure in the generative family of distributions. The question if asymptotic inference is still valid under self-normalized statistics is an important direction to pursue. For example, in the simple sequence model, this translates to substituting the sample standard deviation as an estimate for the true \( \sigma \). Prior work in Tian and Taylor (2018) advocates a transfer of consistency of such sample estimates to a post-selective setting under local alternatives, defined in the paper. Together with the Slutsky’s theorem, a transfer of CLT is then justified with plug-in estimates for \( \Sigma \). However, the transfer of consistency might no longer be obvious under vanishing selection probabilities. Recognizing these questions of potential interest, we hope to address some of these in the future with the current work as a first attempt to resolve validity of conditional inference under Gaussian perturbations in the rare regime.

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8. Appendix

A. Proofs in introductory section

Proof. Lemma 1: This is easy to see by the usual application of CLT and that $E_{Z_n}[W_n] = 0, E_{Z_n}[W_n^TW_n] = \rho^2 \Sigma$. Finally, the covariance term between $Z_n$ and $W_n$ is 0 for each $n \in N$. In particular, this indicates asymptotic independence between the two. □

Proof. Lemma 2: Before selection, the joint likelihood of $(Z_n, W_n)$ at $(z_n, \omega_n)$ is proportional to $\ell(z_n) \cdot g_{\omega_n}(\omega_n)$, $g$ being the density of randomization conditional on data-statistics $Z_n = z_n$. Note that the event $\{E = E, R_n^{(-E)} = R_n\}$ is equivalent to $\{T_n^{(E)} > 0, R_n^{(-E)} = R_n\}$ where the inequality is in a coordinate-wise fashion. Now, applying the linear map in (1.4), we have through a change of variables formula that the density of $(Z_n, T_n^{(E)})$ at $(z_n, t_n)$ conditional on $\{E = E, R_n^{(-E)} = R_n\}$ is proportional to

$$\ell(z_n) \cdot g_{\omega_n}(-\Sigma^{1/2}z_n + \sqrt{\rho} \beta_n) + Q_E t_n + r_E(R_n)) \cdot 1_{t_n > 0}.$$ Integrating out $T_n^{(E)}$, we have the carved likelihood of $Z_n$ at realized value $z_n$. □

Proof. Lemma 3: Under a Gaussian model for the test-statistics $Z_n$, the conditional law of $W_n|Z_n = z_n$ equals the unconditional law for the randomization, a $N(0, \rho^2 \Sigma)$-distribution; this is due to independence following from $\text{Cov}(W_n, Z_n) = 0$. Hence, we have the carved likelihood in Lemma 2 at $z_n$ proportional to

$$\exp(-z_n^Tz_n/2) \cdot \int_{t_n > 0} \exp(-(-\Sigma^{1/2}z_n + \sqrt{\rho} \beta_n) + Q_E t_n + r_E(R_n))^T \Sigma^{-1}$$

$$ \cdot (-\Sigma^{1/2}z_n + \sqrt{\rho} \beta_n) + Q_E t_n + r_E(R_n))/2\rho^2 dt_n.$$ Now, the integral over the variables $t_n$ equals $P_{\Phi}[\hat{T}_n^{(E)} > 0]$ where $\hat{T}_n^{(E)} \sim N(\hat{\mu}, \hat{\Sigma})$ with $\hat{\Sigma} = Q_E^T \Sigma^{-1} Q_E/\rho^2$ and $\hat{\Sigma}^{-1} \hat{\mu} = Q_E^T \Sigma^{-1}(\Sigma^{1/2}z_n + \sqrt{\rho} \beta_n - r_E(R_n))/\rho^2$. This leads to the carved law in Lemma 3 with

$$ P_{\Phi}[\hat{T}_n^{(E)} > 0] = P_{\Phi}[\hat{\Sigma}^{-1/2}(\hat{T}_n^{(E)} - \hat{\mu}) > -\hat{\Sigma}^{-1/2}\hat{\mu}] = \prod_{j=1}^P \Phi \left( Q_j z_n + \sqrt{\rho} \alpha_n^{(j)} \right) .$$
Exact pivot under a Gaussian model: For \( j \in E \), to obtain a pivot for \( \beta_n^{(j)} \), denote \( Z_n^{(-j)} \) as the vector of the \( Z \)-statistics except the \( j \)-th one. Define
\[
N_n^{(j)} = Z_n^{(-j)} - \Sigma_{-j,j} Z_n^{(j)} / \sigma_j^2; \quad \text{Cov}(Z_n^{(-j)}, Z_n^{(j)}) = \Sigma_{-j,j}, \quad \text{Var}(Z_n^{(j)}) = \sigma_j^2.
\]

Finally, let \( P_E^{(j)} = \left[ \begin{array}{cc} 1 & 0 \\ \Sigma_{-j,j} / \sigma_j^2 & I \end{array} \right] \) so that \( P_E^{(j)} (Z_n^{(j)}, N_n^{(j)})^T = (Z_n^{(j)}, Z_n^{(-j)})^T \). Propositions 1 and 2 derive an exact pivot under a Gaussian model for the data in the multivariate and sequence settings respectively.

**Proposition 1.** Under the model \( Z_n \sim N(\sqrt{n}\beta_n, \Sigma) \), a pivot for \( \beta_n^{(j)} \) whenever \( j \in E \) is \( P_j(\Sigma^{1/2} Z_n + \sqrt{n}\beta_n; \sqrt{n}\beta_n) = P_j((Z_n^{(j)}, N_n^{(j)})^T; \sqrt{n}\beta_n) \) that equals
\[
\int_{\mathbb{R}^j} \exp(- (z - \sqrt{n}\beta_n^{(j)})^2 / 2\sigma_j^2) \cdot \int_{t_n > 0} I_{z, N_n^{(j)}}(t_n) dt_n dz
\]
where the function \( I_{z, N_n^{(j)}}(t_n) : \mathbb{R}^j \to \mathbb{R} \) is defined as
\[
\exp(-(\cdot P_E^{(j)}(z, N_n^{(j)})^T + Q_E t_n + r_E(R_n))^T \Sigma^{-1} (\cdot P_E^{(j)}(z, N_n^{(j)})^T + Q_E t_n + r_E(R_n))/2\rho^2).
\]

**Proof. Proposition 1:** We include a proof for \( j = 1 \) assuming that \( 1 \in E \).

With \( P_E^{(1)} = \left[ \begin{array}{cc} \Sigma_{-1,1} / \sigma_1^2 & 0 \\ \Sigma_{-1,1} & I \end{array} \right] \), note that the linear map in (1.4) can be written as
\[
\left( W_n^{(E)} W_n^{(-E)} \right)^T = -P_E^{(1)} (Z_n^{(1)} N_n^{(1)})^T + Q_E T_n^{(E)} + r_E(R_n^{(-E)}).\]
Denote the mean and covariance of \( N_n^{(1)} \) as \( E[N_n^{(1)}] \) and \( \Sigma_{N_n^{(1)}} \) respectively. The joint likelihood of \( (Z_n^{(1)}, N_n^{(1)}, T_n^{(E)}) \) at \( (z, N, T_n) \) conditional on \( \{ E = E, R_n^{(-E)} = R_n \} \), using a change of variables from \( W_n \to (Z_n^{(1)}, N_n^{(1)}, T_n^{(E)}) \) is proportional to
\[
\exp(-(z - \sqrt{n}\beta_n^{(1)})^2 / 2\sigma_1^2) \exp(-(N - E[N_n^{(1)}])^T \Sigma_{N_n^{(1)}} (N - E[N_n^{(1)})/2)
\]
\[
\exp(-(\cdot P_E^{(1)}(z, N)^T + Q_E t_n + r_E(R_n))^T \Sigma^{-1} (\cdot P_E^{(1)}(z, N)^T + Q_E t_n + r_E(R_n))/2\rho^2) I_{t_n > 0}.
\]

Conditioning further on \( N_n^{(1)} \) and marginalizing over \( T_n^{(E)} \), the carved density for \( Z_n^{(1)} \) at \( z \) conditional on \( \{ E = E, R_n^{(-E)} = R_n, N_n^{(1)} = N_n \} \) is therefore proportional to
\[
\exp(-(z - \sqrt{n}\beta_n^{(1)})^2 / 2\sigma_1^2) \cdot \int_{t_n > 0} \exp(-(\cdot P_E^{(1)}(z, N)^T + Q_E t_n + r_E(R_n))^T \Sigma^{-1} (\cdot P_E^{(1)}(z, N)^T + Q_E t_n + r_E(R_n))/2\rho^2) dt_n.
\]

Applying a probability integral transform of the carved law of \( Z_n^{(1)} \) with the above density, a pivot that is distributed as Unif(0, 1) is given by:
\[
P_j((Z_n^{(1)}, N_n^{(1)})^T; \sqrt{n}\beta_n) = \frac{\int_{\mathbb{R}^j} \exp(-(z - \sqrt{n}\beta_n^{(1)})^2 / 2\sigma_1^2) \cdot \int_{t_n > 0} I_{z, N_n^{(1)}}(t_n) dt_n dz}{\int_{\mathbb{R}^j} \exp(-(z' - \beta_n^{(1)})^2 / 2\sigma_1^2) \cdot \int_{t_n > 0} I_{z', N_n^{(1)}}(t_n) dt_n dz'}.
\]

We conclude the proof by noting that
\[
P_j(P_E^{(1)}(\Sigma^{1/2} Z_n + \sqrt{n}\beta_n; \sqrt{n}\beta_n) = P_j((Z_n^{(j)}, N_n^{(j)})^T; \sqrt{n}\beta_n).
\]
This follows from the linear transformation: \( Z_n = \Sigma^{-1/2}(P_E^{(1)}(Z_n^{(1)}, N_n^{(1)}) - \sqrt{n}\beta_n). \)
Proposition 2. Under a sequence model when \((Z_n^{(1)}, \ldots, Z_n^{(d)}) \sim N(\sqrt{n} \beta_n, I)\), a pivot to infer about \(\beta_n^{(j)}\) under the likelihood in (3.2) equals

\[
P_j(Z_n^{(j)}, \beta_n^{(j)}) = \frac{\int_{z_n^{(j)}}^{\infty} \exp(-z^2/2) \cdot \Phi \left( \frac{1 + \rho^2 \cdot \lambda^{(j)}}{\rho} - (z + \sqrt{n} \beta_n^{(j)})/\rho \right) / \sqrt{2\pi} dz}{\Phi \left( \frac{1 + \rho^2 \cdot \lambda^{(j)}}{\rho} - (z + \sqrt{n} \beta_n^{(j)})/\rho \right)}.
\]

Proof. Proposition 2: Let \(j = 1\) without loss of generality. The proof can be seen through the observation that under \(\Sigma = I\), the expression

\[
\exp(-(-P^{(1)}(z, N)^T + Q) \Sigma^{-1} (-P^{(1)}(z, N)^T + Q) + r E_n + r (R_n)/2 \rho^2))
\]

in Proposition 1 decouples into \(\exp(-(-z_n^{(1)} + t_n^{(1)} + \sqrt{1 + \rho^2 \lambda^{(j)}/2} \cdot P(z_n^{(1)}, t_n^{(1)}); f(z^{(1)}, t^{(1)}); f(z^{(-1)}, t^{(-1)}))\) does not depend on \(z_n^{(1)}, t_n^{(1)}\). Thus, it follows that a pivot for \(\beta_n^{(1)}\) equals

\[
\int_{z_n^{(1)}}^{\infty} \exp(-z_n^{(1)} - \sqrt{n} \beta_n^{(1)})^2/2 \cdot \int_{t_n^{(1)} > 0} \exp(-(-z_n^{(1)} + t_n^{(1)} + \sqrt{1 + \rho^2 \lambda^{(j)}/2} \cdot P(z_n^{(1)}, t_n^{(1)}); f(z^{(1)}, t^{(1)}); f(z^{(-1)}, t^{(-1)}))\)
\]

This proves the Lemma by the variable substitution with the centered variable: \(Z^{(1)} = Z^{(1)} - \sqrt{n} \beta_n^{(1)}\) and the observation that \(\Phi \left( \frac{1 + \rho^2 \cdot \lambda^{(j)}}{\rho} - (z + \sqrt{n} \beta_n^{(j)})/\rho \right)\) equals

\[
\int_{t_n^{(1)} > 0} (2\pi)^{-1/2} \rho^{-1} \exp(-(-z_n^{(1)} + t_n^{(1)} + \sqrt{1 + \rho^2 \lambda^{(j)}}/2 \rho^2) dt_n^{(1)}).
\]

\[\square\]

B. Proofs in Section 2

Proof. Theorem 1: To see a proof of the above, denote the likelihood ratio at \(z_n\) between the conditional law in (2.4) and its unconditional counterpart by LR, with the subscript denoting the generative model for \(Z_n\). This equals

\[
LR_{P_n}(z_n) = \prod_{j=1}^{[E]} \Phi \left( Q_j, z_n + \sqrt{n} \alpha_n^{(j)} \right) / \Phi \left( Q_j, z_n + \sqrt{n} \alpha_n^{(j)} \right)
\]

under a generative model in (2.2). Under a Gaussian model for \(Z_n\), when \(\{z_{\ell, n}, 1 \leq \ell \leq n\}\) is an i.i.d. array distributed as \(N(0, I)\), this likelihood ratio equals

\[
LR_{\Phi}(z_n) = \prod_{j=1}^{[E]} \Phi \left( Q_j, z_n + \sqrt{n} \alpha_n^{(j)} \right) / \Phi \left( Q_j, z_n + \sqrt{n} \alpha_n^{(j)} \right).
\]

Noting \(\tilde{\mathbb{E}}_{\Phi}[f(Z_n)] = \mathbb{E}_{P_n}[f(Z_n) LR_{P_n}(Z_n)]; \tilde{\mathbb{E}}_{\Phi}[f(Z_n)] = \mathbb{E}_{\Phi}[f(Z_n) LR_{\Phi}(Z_n)]\), we see that the difference in expectations

\[
|\tilde{\mathbb{E}}_{\Phi}(\mathcal{H} \circ P_j(Z_n; \sqrt{n} \beta_n)) - \tilde{\mathbb{E}}_{\Phi}(\mathcal{H} \circ \mathcal{P}_j(Z_n; \sqrt{n} \beta_n))|
\]

equals \(\mathbb{E}_{P_n}[\mathcal{H} \circ P_j(Z_n; \sqrt{n} \beta_n) LR_{P_n}(Z_n)] - \mathbb{E}_{\Phi}[\mathcal{H} \circ P_j(Z_n; \sqrt{n} \beta_n) LR_{\Phi}(Z_n)]\).

Decomposing the above difference into the following two terms:

\[
(T_1): \mathbb{E}_{P_n}[\mathcal{H} \circ P_j(Z_n; \sqrt{n} \beta_n) LR_{P_n}(Z_n)] - \mathbb{E}_{\Phi}[\mathcal{H} \circ P_j(Z_n; \sqrt{n} \beta_n) LR_{\Phi}(Z_n)]
\]

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\[ (T2) : \left| \mathbb{E}_\pi [H \circ \mathcal{P}_j (Z_n; \sqrt{n} \beta_n)] LR_\Phi (Z_n) \right| - \mathbb{E}_\Phi [H \circ \mathcal{P}_j (Z_n; \sqrt{n} \beta_n)] LR_\Phi (Z_n) \]

the difference is bounded above by \((T1) + (T2)\). Using the fact that \( \mathcal{H}(\cdot) \) is uniformly bounded by \( K \), we bound \((T1)\) as:

\[
\int |H \circ \mathcal{P}_j (z; \sqrt{n} \beta_n)| \prod_{j=1}^{[E]} \Phi (Q_j Z_n + \sqrt{n} \alpha_n^{(j)}) - \mathbb{E}_\Phi \left[ \prod_{j=1}^{[E]} \Phi (Q_j Z_n + \sqrt{n} \alpha_n^{(j)}) \right] \]

\[
\leq K \cdot \left[ \mathbb{E}_\Phi \left[ \prod_{j=1}^{[E]} \Phi (Q_j Z_n + \sqrt{n} \alpha_n^{(j)}) \right] - \mathbb{E}_\Phi \left[ \prod_{j=1}^{[E]} \Phi (Q_j Z_n + \sqrt{n} \alpha_n^{(j)}) \right] \right]
\]


\[
\mathcal{E}_\Phi \left[ H \circ \mathcal{P}_j (\zeta_n; \sqrt{n} \alpha_n) \right] LR_\Phi (\zeta_n) - \mathbb{E}_\Phi \left( H \circ \mathcal{P}_j (\zeta_n; \sqrt{n} \alpha_n) \right) LR_\Phi (\zeta_n)
\]

Plugging \( LR_\Phi (\cdot) \) in \((T2)\), the second relative difference equals:

\[
\mathbb{E}_\Phi \left[ H \circ \mathcal{P}_j (\zeta_n; \sqrt{n} \alpha_n) \right] LR_\Phi (\zeta_n) - \mathbb{E}_\Phi \left( H \circ \mathcal{P}_j (\zeta_n; \sqrt{n} \alpha_n) \right) LR_\Phi (\zeta_n)
\]

Clearly, with the above decomposition of terms, Theorem 1 follows.

**Proof.** **Lemma 4:** To prove the result, observe that

\[
\mathbb{E}_\Phi \left[ L(\zeta_n; \sqrt{n} \alpha_n) \right]_{\{Q \zeta_n < \sqrt{n} \alpha_n \}}
\]

equals

\[
2^{[E]} \int \frac{\exp(-Qz + \sqrt{n} \alpha_n)(z + \sqrt{n} \alpha_n)/2}{(2\pi)^{[E]} \prod_{j=1}^{[E]} \sqrt{4 + (Q_j z + \sqrt{n} \alpha_n^{(j)})^2 + (Q_j z + \sqrt{n} \alpha_n^{(j)})}} \exp(-z^T z/2)
\]

\[
\times 1_{\{Q_0 \alpha < z < \sqrt{n} \alpha \}} dz
\]

where \( I_n \) is a sequence of integrals defined as

\[
\int \frac{2^{[E]} \left( \prod_{j=1}^{[E]} \sqrt{n} \alpha_n^{(j)} \right) \exp(-z - \tilde{\mu} \sqrt{n} \alpha_n)^T (I + Q^T Q) (z - \tilde{\mu} \sqrt{n} \alpha_n)/2}{(2\pi)^{[E]} \prod_{j=1}^{[E]} \sqrt{4 + (Q_j z + \sqrt{n} \alpha_n^{(j)})^2 + (Q_j z + \sqrt{n} \alpha_n^{(j)})}} \times 1_{\{Q_0 \alpha < z < \sqrt{n} \alpha \}} dz
\]

with \( \tilde{\mu} \sqrt{n} \alpha_n = -(I + Q^T Q)^{-1} Q^T \sqrt{n} \alpha_n. \) The proof for the L-1 convergence of \( L(\zeta_n; \sqrt{n} \alpha_n) \) at the specified exponential rate is now complete by showing that the integral \( I_n \) converges to \( \mathcal{L}(QQ^T, Q^T Q; \tilde{\alpha}) \) as \( n \to \infty. \)

Defining a change of variables map \( z \to t \) such that

\[
t = z + (I + Q^T Q)^{-1} Q^T \sqrt{n} \alpha_n
\]

we see that the set \( \{z : -q \sqrt{n} \alpha_n < z < q \sqrt{n} \alpha_n \} \) can be now written in terms of \( t \) as \( \{t : l_n < Q^t < u_n \} \) where the boundaries satisfy

\[
u_n = q \sqrt{n} \alpha_n + Q(I + Q^T Q)^{-1} Q^T \sqrt{n} \alpha_n > 0;
\]

\[
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\]
\( l_n = -q\sqrt{n}\alpha_n + Q(I + Q^T Q)^{-1}Q^T\sqrt{n}\alpha_n < -(q^2 I - Q(I + Q^T Q)^{-1}Q^T)\sqrt{n}\alpha_n < 0. \)

Based on observing \( Q + (I + QQ^T)^{-1}\sqrt{n}\alpha_n = Q + \sqrt{n}\alpha_n \) and denoting \( Q_j, t + (I + QQ^T)^{-1}_j\sqrt{n}\alpha_n = v_j(\alpha_n) \) we use the change of variables in (8.1) to write

\[
I_n = \int_{(2\pi)^E} \left\{ \prod_{j=1}^{E} \frac{2|\sqrt{n}\alpha_n^{|j|}}{\sqrt{4 + (v_j(\alpha_n))^2 + v_j(\alpha_n)}} \cdot 1_{\{l_n < q^2 I - Q < u_n\}} \right\} dt.
\]

Observing that the function \( g^{(j)}: \mathbb{R} \to \mathbb{R}; g^{(j)}(x) = 1/\{\sqrt{4 + x} \} \) is monotonically decreasing on \([l_{j,n}, u_{j,n}]\), under parameter sequences in \( \mathcal{R} \), the sequence

\[
\left\{ \frac{2|\sqrt{n}\alpha_n^{|j|}}{\sqrt{4 + (1 + q)^2n\alpha_n^{|j|^2} + (1 + q)|\sqrt{n}\alpha_n^{|j|}}} : \frac{2|\sqrt{n}\alpha_n^{|j|}}{\sqrt{4 + (1 - q)^2n\alpha_n^{|j|^2} + (1 - q)|\sqrt{n}\alpha_n^{|j|}}} \right\}
\]

is bounded below and above by the sequences

\[
\left\{ \frac{2|\sqrt{n}\alpha_n^{|j|}}{\sqrt{4 + (1 + q)^2n\alpha_n^{|j|^2} + (1 + q)|\sqrt{n}\alpha_n^{|j|}}} : \frac{2|\sqrt{n}\alpha_n^{|j|}}{\sqrt{4 + (1 - q)^2n\alpha_n^{|j|^2} + (1 - q)|\sqrt{n}\alpha_n^{|j|}}} \right\}
\]

respectively. From the fact that the bounding sequences are convergent and uniformly bounded below and above by constants \( \delta^{(j)}_L, \delta^{(j)}_U > 0 \), we conclude

\[
\left( \min_j \frac{|\delta^{(j)}_L|}{(2\pi)^E} \cdot \frac{1}{\int\exp(-t^T(I + Q^T Q)t/2)1_{\{l_n < q^2 I - Q < u_n\}} dt} \right) \leq I_n \leq \left( \max_j \frac{|\delta^{(j)}_U|}{(2\pi)^E} \cdot \frac{1}{\int\exp(-t^T(I + Q^T Q)t/2)1_{\{l_n < q^2 I - Q < u_n\}} dt} \right)
\]

Finally, applying Pratt’s convergence Lemma, we conclude that

\[
I_n \to \left( \left( \prod_{j=1}^{E} \frac{1}{\sqrt{\det(I + Q^T Q)}} \right)^{-1} \cdot (2\pi)^E \cdot \frac{1}{\sqrt{\det(I + Q^T Q)}} \right)^{-1} \text{ as } n \to \infty.
\]

A computation along similar lines for function \( U(Z_n; \sqrt{n}\alpha_n) \) concludes the proof. \( \square \)

**Proof. Theorem 2:** To prove this, bound the Gaussian tail:

\[
\Pi_{j=1}^{E} \Phi \left( Q_j, Z_n + \sqrt{n}\alpha_n^{|j|} \right) \geq L(Z_n; \sqrt{n}\alpha_n)1_{\{|qZ_n| < q\sqrt{n}\alpha_n\}}
\]

\[
\Pi_{j=1}^{E} \Phi \left( Q_j, Z_n + \sqrt{n}\alpha_n^{|j|} \right) \leq U(Z_n; \sqrt{n}\alpha_n)1_{\{|qZ_n| < q\sqrt{n}\alpha_n\} + 1_{\{|qZ_n| > q\sqrt{n}\alpha_n\}}}
\]

from below and above using \( L, U : \mathbb{R}^d \to \mathbb{R} \), defined in Lemma 4. Next, we use a Chernoff bound on the tail probability to sandwich the selection probability

\[
\mathbb{E}_\Phi \left[ \Pi_{j=1}^{E} \Phi \left( Q_j, Z_n + \sqrt{n}\alpha_n^{|j|} \right) \right]
\]

between

\[
\mathbb{E}_\Phi \left[ L(Z_n; \sqrt{n}\alpha_n)1_{\{|qZ_n| < -q\sqrt{n}\alpha_n\}} \right], \mathbb{E}_\Phi \left[ U(Z_n; \sqrt{n}\alpha_n)1_{\{|qZ_n| < -q\sqrt{n}\alpha_n\}} \right] + 2 \cdot \exp(-q^2 \sqrt{n}\alpha_n^{|j|}(QQ^T)^{-1}\sqrt{n}\alpha_n^{|j|}/2).
\]

In particular, with the defined choice of \( q \in (0, 1) \) that satisfies \( q^2 \geq \lambda_{\text{max}} \), \( \lambda_{\text{max}} \) being the largest eigen value of \( QQ^T(I + QQ^T)^{-1} \), we have

\[
\left( \prod_{j=1}^{E} \sqrt{n}\alpha_n^{|j|} \right) \exp(\sqrt{n}\alpha_n^{|j|}Q(I + QQ^T)^{-1}\sqrt{n}\alpha_n^{|j|}/2) \cdot \exp(-q^2 \sqrt{n}\alpha_n^{|j|}(QQ^T)^{-1}\sqrt{n}\alpha_n^{|j|}/2) = o(1)
\]

Thus, follows the conclusion of Theorem 2 from Lemma 4 and the Chernoff bound. \( \square \)
C. Proofs under sequence model in Section 3

Proof. Lemma 6: Under the condition that the function \( g(\cdot) \) is bounded and has a uniformly bounded first derivative whenever \( \sqrt{n^2\beta_{ni}} = O(1) \), the function \( F \) is bounded and has a uniformly bounded first derivative. The second derivative of the Stein’s function \( f_F(\cdot) \) for bounded function \( F : \mathbb{R} \to \mathbb{R} \), defined in (3.8) satisfies

\[
\frac{d^2}{dz^2} f_F(z) = (1 + z^2) f_F(z) + z(F(z) - \mathbb{E}_\Phi[F(Z)]) + F'(z), \quad Z \sim \mathcal{N}(0, 1); k = 1, 2.
\]

An alternative representation of the Stein’s function \( f_F(z) \) is given by

\[
-2\pi \exp(z^2/2) \Phi(z) \cdot f_z^\infty F(t) (1 - \Phi(t)) dt - \sqrt{2\pi} \exp(z^2/2) (1 - \Phi(z)) \cdot F'(t) \Phi(t) dt
\]

and finally, observe that \( F(z) - \mathbb{E}_\Phi[F(Z)] = f_z^\infty F(t) \Phi(t) dt - f_z^\infty F'(t) \Phi(t) dt \).

With this, we compute a bound on the second derivative of \( f_f(\cdot) \) as follows:

\[
|f_f(z)| \leq |(1 + z^2) f_F(z) + z(F(z) - \mathbb{E}_\Phi[F(Z)]) + F'(z)|
\]

\[
\leq \left| (z - \sqrt{2\pi} (1 + z^2) \exp(z^2/2) (1 - \Phi(z))) \int_z^\infty F'(t) \Phi(t) dt \right| + \left| F'(z) \right| \leq C \cdot \| F' \|_{\infty}
\]

The conclusion now follows from using the bounds \( \| F' \|_{\infty} \leq \bar{B}_k, k = 1, 2 \). \( \square \)

Proof. Lemma 7: This proof follows by computing the first derivative of \( F_1 \) on \( [c\sqrt{n^2\beta_{ni}}, -c\sqrt{n^2\beta_{ni}}] \), which equals

\[
L(z; \sqrt{n^2\beta_{ni}}) \cdot \left( g'(z; \sqrt{n^2\beta_{ni}}) - (z + \sqrt{n^2\beta_{ni}}) g(z; \sqrt{n^2\beta_{ni}})/\rho^2 + \frac{g(z; \sqrt{n^2\beta_{ni}})}{\rho \cdot \sqrt{4 + (z + \sqrt{n^2\beta_{ni}})^2/\rho^2}} \right).
\]

A similar expression follows for the derivative of \( F_2 \) replacing \( L(z; \sqrt{n^2\beta_{ni}}) \) with \( U(z; \sqrt{n^2\beta_{ni}}) \). Thus, using an expression for the second derivative of \( f_F \)

\[
\frac{d^2}{dz^2} f_F(z) = (1 + z^2) f_F(z) + z(F(z) - \mathbb{E}_\Phi[F(Z)]) + F'(z), \quad Z \sim \mathcal{N}(0, 1),
\]

the bounds on the function \( g \) and the rate of growth of the first derivative of \( g \), we have the result in Lemma 7. \( \square \)

Proof. Lemma 8: The pivot in Proposition 2 (with \( \lambda = 0 \))

\[
\mathcal{P}(Z^{(1)}, \rho^{(1)}) = \int_{Z^{(1)}} \exp(-z^2/2) \cdot \Phi \left( \sqrt{1 + \rho^2 \cdot \lambda^{(1)}/\rho} - (z + \sqrt{n^2\beta_{ni}})/\rho \right) \cdot \sqrt{2\pi} \ dz
\]

equals

\[
\int_{Z^{(1)}} \int \exp(-z^2/2) \exp(-(t - (z + \sqrt{n^2\beta_{ni}})^2/2\rho^2)) 1_{t > 0} dt dtdz.
\]

We can see that the pivot can be expressed as

\[
\frac{\mathbb{E}[Z_1 > Z^{(1)}_{1}(T = t) | T = t]}{\mathbb{P}[T > 0]} \cdot \mathbb{E}[|Z_1 > Z^{(1)}_{1}| T = t] 1_{T > 0}
\]

where \( (Z_1, T) \) are random variables with a density proportional to

\[
\exp(-z^2/2) \exp(-(t - (z + \sqrt{n^2\beta_{ni}})^2/2\rho^2)).
\]

Simplifying the pivot, we have

\[
(\mathbb{P}[T > 0])^{-1} \cdot \mathbb{E}[|Z_1 > Z^{(1)}_{1}| T = t] 1_{T > 0}
\]
are bounded above by $\kappa$ as $F$ for a positive valued

constant, say $C$, whenever $\sqrt{n}\delta_n = O(1)$. We are thus left to bound the

numerators of the relative differences. Letting the functions $F_k(\cdot) : \mathbb{R} \to \mathbb{R}$, $k = 1, 2$

represent

$$F_k(z) = g_k(z; \sqrt{n}\delta_n) \cdot \Phi\left(-\frac{z + \sqrt{n}\delta_n^j}{\rho}\right)$$

where $g_1(z) = 1$ and $g_2(z) = \mathcal{H} \circ \mathcal{P}(z; \sqrt{n}\delta_n^j)$, we note from the result in Proposition

8 that $F_k$ satisfies the conditions in Lemma 6. Thus, for constants $B_k(\rho, K_k)$,

$k = 1, 2$ depending on $\rho, K_2$, we have $\|f_{F_k}^n\|_\infty \leq B_k(\rho, K_2)$; $k = 1, 2$. Now, the

Stein’s bound in (3.10) and the result in Lemma 5 yield the bound on $\Delta_n$ as

$$n \cdot \sup_{-\infty < a < 0} \mathbb{E}_{\mathbb{P}} \left[ (t - Z_{t,n}^j) f_{F_k}'' \left( at + (1 - \alpha)Z_{t,n}^j + Z_{t,n}^{(i,j)} \right) \right] Q_i(t) dt \leq f_{F_k}(z) \leq \begin{cases} \exp(z^2/2) & \text{if } z \leq 0 \\ \exp(z^2/2) \Phi(F_k(Z)) & \text{if } z \geq 0. \end{cases}$$

for a positive valued $F_k$. Next, note that

$$\int_{-\infty}^z F_k(t) \exp(-t^2/2) dt \leq D_k K_1 \exp\left(-n\beta_n^{(j)} / (2(1 + \rho^2)) \right) \cdot \Phi\left( \frac{\sqrt{1 + \rho^2} z}{\rho} - \frac{\sqrt{n}\beta_n^j}{\rho} \right)$$

$$D_1 = 1/\left\{ \sqrt{1 + (1-c)^2 n\beta_n^j} / (1 - c) \sqrt{n}\beta_n^j, 1 \right\}$$

Further, observe that:

$$\mathbb{E}_\Phi[F_k(Z)] \leq K_1 \exp\left(-n\beta_n^{(j)} / (2(1 + \rho^2)) \right) \frac{\sqrt{1 + \rho^2}}{\sqrt{2\pi}} + o(1) \leq K_1 \cdot \delta \exp\left(-n\beta_n^{(j)} / (2(1 + \rho^2)) \right)$$

for some constant $\delta$. With these observations, we conclude–
\[
\begin{align*}
f_{F_k}(z) & \leq D_k \cdot K_1 \exp(-n\beta_n^{(j)} z^2/2(1 + \rho^2)) \exp(z^2/2) \cdot \Phi \left( \sqrt{1 + \rho^2} \cdot z/\rho - \sqrt{n\beta_n^{(j)}} \right) 1_{z < 0} + K_1 \cdot \delta \exp(-n\beta_n^{(j)} z^2/2(1 + \rho^2)) \exp(z^2/2) \cdot \Phi(z)/|\sqrt{n\beta_n^{(j)}}| \cdot 1_{z > 0}
\end{align*}
\]

and thus, follows the result that the Stein’s functions for \(F_k\), \(k = 1, 2\) satisfy

\[
f_{F_k}(z) \leq C(\rho, K_1, c) \cdot \exp(-n\beta_n^{(j)} z^2(1 + \rho^2))/|\sqrt{n\beta_n^{(j)}}|; \ k = 1, 2
\]

where \(C(\rho, K_1, c)\) is a constant dependent on \(\rho, K_1, c\).

**Proof.** **Theorem 4:** Observing that the function \(\alpha(x) = 1/\sqrt{4 + x^2} = x\) is monotonically increasing on the interval \([c\sqrt{n\beta_n^{(j)}}, +c\sqrt{n\beta_n^{(j)}}]\) and that the sequence \(\left\{-\sqrt{n\beta_n^{(j)}}/\left(\sqrt{4 + (1-c)^2n\beta_n^{(j)}^2}/\rho^2 - (1-c)\sqrt{n\beta_n^{(j)}}/\rho\right); n \in \mathbb{N}\right\}\) is uniformly bounded by \(\delta\), we have:

\[
\begin{align*}
\mathcal{B}_n &= n^{1/2}|\beta_n^{(j)}| \cdot \int_{a \in (0, 1)} \mathbb{E}_{\lambda_n} \left[ (|t| + |Z_{i,n}^{(j)}|) L(a(t + (1-\alpha)Z_{i,n}^{(j)} + Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}})) 
\right. \\
& \qquad \left. \cdot 1_{\{c\sqrt{n\beta_n^{(j)}} < \alpha(t + (1-\alpha)Z_{i,n}^{(j)} + Z_{n}^{(j)} + c\sqrt{n\beta_n^{(j)}})\}} \right] Q_1(t) dt \\
& \leq n^{1/2}|\beta_n^{(j)}| \cdot \int \mathbb{E}_{\lambda_n} \left[ (|t| + |Z_{i,n}^{(j)}|) \exp((|t| + |Z_{i,n}^{(j)}|)(|Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}}|)/\rho^2 \\
- (Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}})^2/2\rho^2) \times \left(\sqrt{4 + (1-c)^2n\beta_n^{(j)}^2}/\rho^2 - (1-c)\sqrt{n\beta_n^{(j)}}/\rho\right)^{-1} \right] Q_1(t) dt \\
& \leq \delta n \cdot \int \mathbb{E}_{\lambda_n} \left[ (|t| + |Z_{i,n}^{(j)}|) \exp((|t| + |Z_{i,n}^{(j)}|)(|Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}}|)/\rho^2 - (Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}})^2/2\rho^2) \right] Q_1(t) dt.
\end{align*}
\]

Using tower property of expectation,

\[
\mathbb{E}_{\lambda_n} \left[ (|t| + |Z_{i,n}^{(j)}|) \cdot \exp((|t| + |Z_{i,n}^{(j)}|)(|Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}}|)/\rho^2 - (Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}})^2/2\rho^2) \right] \text{ equals}
\]

\[
\mathbb{E}_{\lambda_n} \left[ (|t| + |Z_{i,n}^{(j)}|) \cdot \mathbb{E} \left[ \exp((|t| + |Z_{i,n}^{(j)}|)(|Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}}|)/\rho^2 - (Z_{n}^{(j)} + \sqrt{n\beta_n^{(j)}})^2/2\rho^2) \right] \right] Z_{i,n}^{(j)}.
\]

To compute the inner expectation, we use the following facts

- the independence of \((Z_{n}^{(i,j)}, Z_{i,n}^{(j)})\) so that the conditional expectation (conditioning on \(Z_{i,n}\)) equals the unconditional expectation.

- a moderate deviation bound on the expectation of \(\mathbb{E}_{\lambda_n}[\exp(b_n^2 \cdot \Psi(Z_{n}^{(i,j)}/b_n))]\) with

\[
Psi(z) = b_n^{-1}(|t| + |Z_{i,n}^{(j)}|(|z + |\beta|)/\rho^2 - (z + \beta)^2/2\rho^2)
\]

under the parameterization \(\sqrt{n\beta_n} = b_n\beta\). Note that, this bound is valid under the assumption of existence of exponential moments of \(\zeta_{i,n}\) in a neighborhood of 0, which would imply that \(\mathbb{E}_{\lambda_n}[\exp(\alpha|Z_{n}^{(i,j)})|] < \infty\) for any \(\alpha > 0\). This bound yields

\[
\mathbb{E}_{\lambda_n}[\exp(b_n^2 \cdot \Psi(Z_{n}^{(i,j)}/b_n))] \leq \exp(-b_n^2 \inf \{z^2/2 - \Psi(z)\}).
\]
Using the above two facts, the inner expectation equals
\[
\mathbb{E}_{\nu_n} \left[ \exp \left( \left| t \right| + \left| Z_{i,n}^{(1)} \right| (\left| Z_{i,n}^{(2)} \right| + \sqrt{n} \beta_{i,n}^{(1)} \right) / \rho^2 - (Z_{i,n}^{(1)} + \sqrt{n} \beta_{i,n}^{(1)})^2 / 2 \rho^2 \right) \right] \\
\leq \exp(-n \beta_{i,n}^{(1)} / 2(1 + \rho^2)) \exp\left( -\frac{\sqrt{n} \beta_{i,n}^{(1)}}{\rho^2(1 + \rho^2)} \left( \left| t \right| + \left| Z_{i,n}^{(1)} \right| \right) \right) + \frac{\left| Z_{i,n}^{(1)} \right|^2}{2 \rho^2(1 + \rho^2)} .
\]

Based on our assumptions on the generative family in (3.12), \( \mathbb{E}_{\nu_n} \left[ \exp(C_1 |Z_{i,n}^{(1)}| + C_2 |Z_{i,n}^{(2)}|) \right] = O(1) \) and \( \mathbb{E}_{\nu_n} \left[ |Z_{i,n}^{(1)}| \exp(C_1 |Z_{i,n}^{(1)}|) \right] = O(n^{-1/2}) \) for sufficiently large \( n \). Thus, \( B_n \) is bounded above by
\[
2 \delta \cdot n \exp(-n \beta_{i,n}^{(1)} / 2(1 + \rho^2)) \int \mathbb{E}_{\nu_n} \left[ \left| t \right| + |Z_{i,n}^{(1)}| \exp\left( -\frac{\sqrt{n} \beta_{i,n}^{(1)}}{\rho^2(1 + \rho^2)} \left( \left| t \right| + \left| Z_{i,n}^{(1)} \right| \right) \right) \right] dt .
\]
\[
\leq 2 \delta \cdot n \exp(-n \beta_{i,n}^{(1)} / 2(1 + \rho^2)) \cdot \mathbb{E}_{\nu_n} \left[ Z_{i,n}^{(1)} \exp\left( -\frac{\sqrt{n} \beta_{i,n}^{(1)}}{\rho^2(1 + \rho^2)} |Z_{i,n}^{(1)}| \right) \right] \\
\times \int \exp(-\sqrt{n} \beta_{i,n}^{(1)} |t| / \rho^2(1 + \rho^2) + t^2 / \rho^2(1 + \rho^2)) dt
\]
\[
+ 2 \delta \cdot n \cdot \exp(-n \beta_{i,n}^{(1)} / 2(1 + \rho^2)) \cdot \mathbb{E}_{\nu_n} \left[ \exp\left( -\frac{\sqrt{n} \beta_{i,n}^{(1)}}{\rho^2(1 + \rho^2)} |Z_{i,n}^{(1)}| \right) \right] \\
\times \int |t| \exp(-\sqrt{n} \beta_{i,n}^{(1)} |t| / \rho^2(1 + \rho^2) + t^2 / \rho^2(1 + \rho^2)) dt + O(n^{-1/2})(TA) + (TB)
\]

We are left to compute

**(TA):** \( \int \exp(-\sqrt{n} \beta_{i,n}^{(1)} |t| / \rho^2(1 + \rho^2) + t^2 / \rho^2(1 + \rho^2)) dt \)

and

**(TB):** \( \int |t| \exp(-\sqrt{n} \beta_{i,n}^{(1)} |t| / \rho^2(1 + \rho^2) + t^2 / \rho^2(1 + \rho^2)) dt \).

A final step proceeds with the observation that the functions in the integrands in **(TA)**, **(TB)** are symmetric functions about 0 and an increasing function on the positive axis. Thus, we conclude by noting that **(TA)** equals
\[
\int \exp\left( -\sqrt{n} \beta_{i,n}^{(1)} |t| / \rho^2(1 + \rho^2) + t^2 / \rho^2(1 + \rho^2) \right) dt = O(1).
\]

\[
\int \mathbb{E} \left[ |Z_{i,n}^{(2)}| \exp\left( |Z_{i,n}^{(2)}| / \rho^2(1 + \rho^2) - \sqrt{n} \beta_{i,n}^{(1)} |Z_{i,n}^{(1)}| / \rho^2(1 + \rho^2) \right) \right] = O(n^{-1}).
\]
Proceeding similarly, we have (TB) equals
\[
\int |t| \exp \left( -\sqrt{n} \beta_n^{(j)} / \rho^2 (1 + \rho^2) + t^2 / \rho^2 (1 + \rho^2) \right) Q_t(t) dt \\
= \int_0^\infty \int_0^\infty |z| \exp \left( -\sqrt{n} \beta_n^{(j)} / \rho^2 (1 + \rho^2) + t^2 / \rho^2 (1 + \rho^2) \right) dt d\mathbb{P}_n(z) \\
- \int_0^\infty \int_0^\infty |z| \exp \left( -\sqrt{n} \beta_n^{(j)} / \rho^2 (1 + \rho^2) + t^2 / \rho^2 (1 + \rho^2) \right) dt d\mathbb{P}_n(z) \\
\leq \int_0^\infty \int_0^\infty |z|^3 \exp \left( z^2 / \rho^2 (1 + \rho^2) - \sqrt{n} \beta_n^{(j)} / \rho^2 (1 + \rho^2) \right) f_z(z) d\mathbb{P}_n(z) \\
\leq \mathbb{E} \left[ |Z_{i,n}^{(j)}|^3 \right] \exp \left( Z_{i,n}^{(j)} / \rho^2 (1 + \rho^2) - \sqrt{n} \beta_n^{(j)} / \rho^2 (1 + \rho^2) \right) = O(n^{-3/2}).
\]

With this, we conclude that \( B_n = O(\beta_n^{(j)} \exp(-n \beta_n^{(j)} / 2(1 + \rho^2) / \sqrt{n} \beta_n^{(j)}) \).

**Proof. Theorem 5:** From the Stein’s bound in (3.10), we derive an upper bound on the difference of expectations as
\[
\Delta_n = n \cdot \int_0^\infty \sup_{\alpha \in [0,1]} \mathbb{E}_{\mathbb{P}_n} \left[ (t - Z_{i,n}^{(j)}) f_{F_1} (\alpha t + (1 - \alpha) Z_{i,n}^{(j)} + Z_{i,n}^{(i,j)}) \right] Q_i(t) dt \\
\]
where \( f_{F_1} (\cdot) \) is the Stein’s function in (3.8) for bounded function
\[
F_1(z; \sqrt{n} \beta_n^{(j)}) = g(z; \sqrt{n} \beta_n^{(j)}) \cdot L(z; \sqrt{n} \beta_n^{(j)})
\]
where \( E[F_1(Z)] < \infty \) for \( Z \sim \mathcal{N}(0, 1) \). Without loss of generality, we assume that \( F_1 \) is non-negative valued. This is possible by noting that we can always write \( F_1(\cdot) = F_1^+(\cdot) - F_1^-(\cdot) \), where \( F_1^+(z) = \max(F_1(z), 0) \) and \( F_1^-(z) = -\min(F_1(z), 0) \) and bound the differences
\[
\mathbb{E}_{\mathbb{P}_n} \left[ F_1^+(Z_{i,n}^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] - \mathbb{E}_{\Phi} \left[ F_1^+(Z_{i,n}^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] \quad \text{and}
\]
\[
\mathbb{E}_{\mathbb{P}_n} \left[ F_1^-(Z_{i,n}^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] - \mathbb{E}_{\Phi} \left[ F_1^-(Z_{i,n}^{(j)}; \sqrt{n} \beta_n^{(j)}) \right].
\]

Using a bound on the second derivative of \( f_{F_1} \) on \([c \sqrt{n} \beta_n^{(j)}; -c \sqrt{n} \beta_n^{(j)}] \) in Lemma 7 follows an upper bound on
\[
\Delta_n \leq B_{1,n} + C_1(K_1, c) \cdot B_{2,n} + C_2(\rho, K_1, K_2, c) \cdot B_{3,n} + C_3(\rho, K_1, c) \cdot B_{4,n} / \sqrt{n} \beta_n^{(j)}.
\]
The constans in the bound are:
\[
C_1(K_1, c) = cK_1, \quad C_2(\rho, K_1, K_2, c) = K_2 + K_1 / \rho^2 + c(1 - 1 / \rho^2) K_1, \quad C_3 = K_1 / 2 \rho.
\]
The non-trivial bounds are given by
\[
B_{1,n} = n \cdot \int_0^\infty \sup_{\alpha \in [0,1]} \mathbb{E}_{\mathbb{P}_n} \left[ \mathbb{E}_{\mathbb{P}_n} \left[ |t| + |Z_{i,n}^{(j)}| \right] \left( 1 + (\alpha t + (1 - \alpha) Z_{i,n}^{(j)} + Z_{i,n}^{(i,j)})^2 \right) \right] Q_i(t) dt \\
B_{2,n} = n^{3/2} |\beta_n^{(j)}| \cdot \mathbb{E}_{\Phi} \left[ L(Z; \sqrt{n} \beta_n^{(j)}) \mathbb{1} \{ c \sqrt{n} \beta_n^{(j)} < Z < -c \sqrt{n} \beta_n^{(j)} \} \right] \int_0^\infty \mathbb{E}_{\mathbb{P}_n} \left[ |t| + |Z_{i,n}^{(j)}| \right] Q_i(t) dt \\
B_{3,n} = n^{3/2} |\beta_n^{(j)}| \cdot \int_0^\infty \sup_{\alpha \in [0,1]} \mathbb{E}_{\mathbb{P}_n} \left[ \mathbb{E}_{\mathbb{P}_n} \left[ |t| + |Z_{i,n}^{(j)}| L(1 + \alpha t + (1 - \alpha) Z_{i,n}^{(j)} + Z_{i,n}^{(i,j)}) \right. \sqrt{n} \beta_n^{(j)} ) \right. \mathbb{1} \{ c \sqrt{n} \beta_n^{(j)} < \alpha t + (1 - \alpha) Z_{i,n}^{(j)} + Z_{i,n}^{(i,j)} < -c \sqrt{n} \beta_n^{(j)} \} \right] Q_i(t) dt.
\]
Now, based on Lemma 5 and Lemma 9, we obtain a rate of decay for $B_{1,n}$:

$$B_{1,n} \leq C(\rho, K_1, \epsilon) \cdot \frac{n \exp(-n\beta_n^{(j)}/2(1+\rho^2))}{\sqrt{n}|\beta_n^{(j)}|} \int_{-\infty}^{\infty} \mathbb{E}_{\gamma_n} \left( |t| + |Z_{n,n}^{(j)}| \right)
\left( 1 + |t| + |Z_{n,n}^{(j)}| + Z_{n,n}^{(j)} \right)^2 \right) Q_n(t) dt
= \exp(-n\beta_n^{(j)} / 2(1+\rho^2)) / |n\beta_n^{(j)}| \cdot O(1).$$

For the second bounding term $B_{2,n}$, using Lemma 5 and the fact that

$$\mathbb{E}_{\Phi} \left[ L(Z; \sqrt{n}\beta_n^{(j)}) \right] \leq \beta(\rho, c) \cdot \exp(-n\beta_n^{(j)} / 2(1+\rho^2)) / \sqrt{n}|\beta_n^{(j)}|,$$

it follows that $B_{2,n} \leq \beta_n^{(j)} \cdot \exp(-n\beta_n^{(j)} / 2(1+\rho^2)) / \sqrt{n}|\beta_n^{(j)}| \cdot O(1)$. Finally, the rate of decay of the term $B_{3,n} = B_n$ is derived in Theorem 4 as

$$\beta_n^{(j)} \exp(-n\beta_n^{(j)} / 2(1+\rho^2)) / \sqrt{n}|\beta_n^{(j)}|,$$

from which we conclude that

$$\mathbb{E}_{\gamma_n} \left[ F_1(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) \right] - \mathbb{E}_{\gamma_n} \left[ F_1(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) \right] \leq \frac{\exp(-n\beta_n^{(j)} / 2(1+\rho^2))}{\sqrt{n}|\beta_n^{(j)}|} R_{n,1}$$

where $R_{n,1} \leq C(1)/\sqrt{n} + C(2)|\beta_n^{(j)}|$. An exactly similar proof follows for the function $F_2$ which proves the Theorem.

**Lemma 12.** Under the parameters in $\mathcal{R}$ and for a $c \in [0, 1]$, we have

$$n\beta_n^{(j)} \exp(n\beta_n^{(j)} / 2(1+\rho^2)) \cdot \mathbb{E}_{\Phi}[U(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) 1_{\{c \sqrt{n}\beta_n^{(j)} \leq z_n^{(j)} < -c \sqrt{n}\beta_n^{(j)}\}}]
- \mathbb{E}_{\Phi}[L(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) 1_{\{c \sqrt{n}\beta_n^{(j)} \leq z_n^{(j)} < -c \sqrt{n}\beta_n^{(j)}\}}] = O(1).$$

**Proof. Lemma 12:** To prove the Lemma, we compute the integral

$$\mathbb{E}_{\Phi} \left[ \left( U(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) - L(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)}) \right) 1_{\{c \sqrt{n}\beta_n^{(j)} \leq z_n^{(j)} < -c \sqrt{n}\beta_n^{(j)}\}} \right]
= \frac{1}{\pi} \int \exp(-z + \sqrt{n}\beta_n^{(j)} / 2 / \rho^2) \cdot \left( \sqrt{4 + (z + \sqrt{n}\beta_n^{(j)})^2 / \rho^2} - (z + \sqrt{n}\beta_n^{(j)}) / \rho \right)^{-1}
\times \left( \sqrt{4 + (z - \sqrt{n}\beta_n^{(j)})^2 / \rho^2} - (z - \sqrt{n}\beta_n^{(j)}) / \rho \right)^{-1}
\times \frac{1}{\sqrt{2 + (z + \sqrt{n}\beta_n^{(j)})^2 / \rho^2 + \sqrt{4 + (z + \sqrt{n}\beta_n^{(j)})^2 / \rho^2}}}
\exp(-z^2 / 2) 1_{\{c \sqrt{n}\beta_n^{(j)} \leq z < -c \sqrt{n}\beta_n^{(j)}\}} dz
\leq \frac{2 \exp(-n\beta_n^{(j)} / 2(1+\rho^2))}{(2 + 2)n\beta_n^{(j)} / 2} \int \frac{\exp(-(1+\rho^2)(z + \sqrt{n}\beta_n^{(j)})^2 / 2 / \rho^2)}{2\pi}
\times \left( \sqrt{2 + (z + \sqrt{n}\beta_n^{(j)})^2 / \rho^2} - (z + \sqrt{n}\beta_n^{(j)}) / \rho \right)^{-1}
\times \left( \sqrt{2 + (z - \sqrt{n}\beta_n^{(j)})^2 / \rho^2} - (z - \sqrt{n}\beta_n^{(j)}) / \rho \right)^{-1}
\exp(-z^2 / 2) 1_{\{c \sqrt{n}\beta_n^{(j)} \leq z < -c \sqrt{n}\beta_n^{(j)}\}} dz
= C \cdot \frac{\exp(-n\beta_n^{(j)} / 2(1+\rho^2))}{n\beta_n^{(j)} / 2} \cdot D_n,\]
where $D_n$ equals
\[
\int \frac{n\beta_n^{(j)} \cdot z}{\sqrt{2\pi}} \exp\left(- (1 + \rho^2)(z + \sqrt{n}\beta_n^{(j)}/(1 + \rho^2))^2 / 2\rho^2\right)\left(\sqrt{4 + (z + \sqrt{n}\beta_n^{(j)})^2} - z - (z + \sqrt{n}\beta_n^{(j)})/\rho\right)^{-1} \times \left(\sqrt{2 + (z + \sqrt{n}\beta_n^{(j)})^2} - z - (z + \sqrt{n}\beta_n^{(j)})/\rho\right)^{-1} 1\{c\sqrt{n}\beta_n^{(j)} < z < -c\sqrt{n}\beta_n^{(j)}\} \, dz.
\]

Noting again that the sequence
\[
n\beta_n^{(j)} \frac{1}{\sqrt{2 + (z + \sqrt{n}\beta_n^{(j)})^2} - z - (z + \sqrt{n}\beta_n^{(j)})/\rho}\left(\sqrt{4 + (z + \sqrt{n}\beta_n^{(j)})^2} - z - (z + \sqrt{n}\beta_n^{(j)})/\rho\right)^{-1} \frac{n\beta_n^{(j)} \cdot z}{\sqrt{2\pi}} \exp\left(- (1 + \rho^2)(z + \sqrt{n}\beta_n^{(j)}/(1 + \rho^2))^2 / 2\rho^2\right)
\]
can be bounded by a convergent sequence using the increasing nature of functions $\alpha(x) = 1/(\sqrt{4 + x^2} - x), \beta(x) = 1/(\sqrt{2 + x^2} - x)$ on $[c\sqrt{n}\beta_n^{(j)}, -c\sqrt{n}\beta_n^{(j)}]$, it follows that $D_n = o(1)$. Thus, follows the conclusion of the Lemma. \hfill \Box

**Proof. Theorem 6:** Let $F_l(t) : \mathbb{R} \to \mathbb{R}, \ l = 1, 2$ represent a function
\[
F_l(z; \sqrt{n}\beta_n) = g_l(z; \sqrt{n}\beta_n^{(j)}) \cdot \Phi\left(-(z + \sqrt{n}\beta_n^{(j)})/\rho\right), \ l = 1, 2
\]
where $g_1(z; \sqrt{n}\beta_n^{(j)}) = 1$ and $g_2(z; \sqrt{n}\beta_n^{(j)}) = H \circ P(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})$. Note, we can assume (without any loss of generality) that $F_l, \ l = 1, 2$ is non-negative valued with a similar reasoning as provided in the proof of Theorem 5. Under parameter sequences in $\mathcal{R}_n$, observing from Theorem 2 that
\[
\sqrt{n}\beta_n^{(j)} |\exp(\nu(\beta_n^{(j)} \cdot z)/(1 + \rho^2))| \cdot \Phi\left(-(W_n + \sqrt{n}\beta_n^{(j)})/\rho\right) = \frac{\sqrt{1 + \rho^2}}{\sqrt{2\pi}} + o(1)
\]
it suffices to find a bound for $l = 1, 2$ on
\[
\sqrt{n}\beta_n^{(j)} |\exp(\nu(\beta_n^{(j)} \cdot z)/(1 + \rho^2))| \cdot \Phi\left(-(W_n + \sqrt{n}\beta_n^{(j)})/\rho\right) = \frac{\sqrt{1 + \rho^2}}{\sqrt{2\pi}} + o(1)
\]
Using a moderate deviation tail-bound on the probability $\mathbb{P}_{\mathbb{P}_n}[|Z_n^{(j)}| > c\sqrt{n}|\beta_n^{(j)}|]$ and a Chernoff-bound for the Gaussian probability
\[
\mathbb{P}_{\mathbb{P}_n}[|Z_n^{(j)}| > c\sqrt{n}|\beta_n^{(j)}|, \mathbb{P}_n[|Z_n^{(j)}| > c\sqrt{n}|\beta_n^{(j)}|] \leq \exp(-c^2 n|\beta_n^{(j)}|^2/2).
\]
In particular, for choice of $c \in [0, 1]$ satisfying $c^2 > 1/(1 + \rho^2)$, we have
\[
\sqrt{n}|\beta_n^{(j)}| \exp\left(n|\beta_n^{(j)}|^2/(1 + \rho^2)\right) \mathbb{P}_{\mathbb{P}_n}[|Z_n^{(j)}| > c\sqrt{n}|\beta_n^{(j)}|] = o(1),
\]
\[
\sqrt{n}|\beta_n^{(j)}| \exp\left(n|\beta_n^{(j)}|^2/(1 + \rho^2)\right) \mathbb{P}_{\mathbb{P}_n}[|Z_n^{(j)}| > c\sqrt{n}|\beta_n^{(j)}|] = o(1),
\]
and hence, we are left to obtain a bound on
\[
\sqrt{n}|\beta_n^{(j)}| \exp\left(n|\beta_n^{(j)}|^2/(1 + \rho^2)\right) \mathbb{P}_{\mathbb{P}_n}\left[F_1(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})\right] \left[F_1(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})\right] 1\{c\sqrt{n}|\beta_n^{(j)}| < Z_n < -c\sqrt{n}|\beta_n^{(j)}|\}
\]
\[
- \mathbb{E}_{\Phi}\left[F_1(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})\right] 1\{c\sqrt{n}|\beta_n^{(j)}| < Z_n < -c\sqrt{n}|\beta_n^{(j)}|\}.
\]
Sandwiching $F_1(z; \sqrt{n}\beta_n^{(j)})$ between $g(z; \sqrt{n}\beta_n^{(j)}) \cdot U(z; \sqrt{n}\beta_n^{(j)})$ and $g(z; \sqrt{n}\beta_n^{(j)}) \cdot L(z; \sqrt{n}\beta_n^{(j)})$ on the set $[c\sqrt{n}\beta_n^{(j)}, -c\sqrt{n}\beta_n^{(j)}]$, we can bound the above by maximum of the differences, denoted as (D1) and (D2) which equal respectively
\[
\sqrt{n}|\beta_n^{(j)}| \exp\left(n|\beta_n^{(j)}|^2/(1 + \rho^2)\right) \mathbb{E}_{\mathbb{P}_n}\left[F_2(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})\right] - \mathbb{E}_{\Phi}\left[F_1(Z_n^{(j)}; \sqrt{n}\beta_n^{(j)})\right]
\]
and
\[ \sqrt{n} |\beta_n^{(j)}| \exp \left( n |\beta_n^{(j)}|^2 / 2(1 + \rho^2) \right) \left| \mathbb{E}_n \left[ F_1(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] - \mathbb{E}_\Phi \left[ F_2(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] \right|; \]

\[ F_1, F_2 \text{ are as defined in Lemma 7 and Theorem 5 with } g(z; \sqrt{n} \beta_n^{(j)}) \equiv g_t(z; \sqrt{n} \beta_n^{(j)}). \]

Bounding (D1) through the following decomposition as below:

\[ \sqrt{n} |\beta_n^{(j)}| \exp(n |\beta_n^{(j)}|^2 / 2(1 + \rho^2)) \left| \mathbb{E}_n \left[ F_1(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] - \mathbb{E}_\Phi \left[ F_2(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] \right| \]

\[ \sqrt{n} |\beta_n^{(j)}| \exp(n |\beta_n^{(j)}|^2 / 2(1 + \rho^2)) \left| \mathbb{E}_\Phi \left[ F_2(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] - \mathbb{E}_\Phi \left[ F_1(Z_n^{(j)}; \sqrt{n} \beta_n^{(j)}) \right] \right|, \]

the Stein’s bound in Theorem 5 and the result in Lemma 12 prove that

\[ \max((D1), (D2)) \leq C_{i}^{(1)} / \sqrt{n} + C_{i}^{(2)} |\beta_n^{(j)}| + C_{i}^{(3)} / \sqrt{n} |\beta_n^{(j)}|; l = 1, 2, \]

which in turn yield bounds for the relative differences \( R_n^{(1)}, R_n^{(2)} \) and provide a rate for weak convergence whenever \( \beta_n^{(j)} = o(1) \).

\[ \square \]

**D. Proofs under a multivariate model in Section 4 and 5**

**Proof. Lemma 10:** With indices \( i, j, l \) running over \( \{1, \cdots, d\} \), the remainder \( n \cdot \mathbb{E}_n [\mathcal{R}_n] \) simplifies as:

\[ \frac{1}{6n^{3/2}} \sum_{i,j,l} \sum_{k=1}^{n} \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}')(\zeta_{j,n} - \zeta_{j,n}')(\zeta_{k,n} - \zeta_{k,n}') \right] \times \sup_{\alpha \in [0,1]} (\nabla^3 f_P(\mathcal{Z}_n - (1 - \alpha) \cdot (\zeta_{i,n} - \zeta_{i,n}'/\sqrt{n}))_{i,j,l} \right] = \frac{1}{2n} \cdot \sum_{i,j,l} \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}')(\zeta_{j,n} - \zeta_{j,n}')(\zeta_{k,n} - \zeta_{k,n}') \times \sup_{\alpha \in [0,1]} (\nabla^3 f_P(\mathcal{Z}_n - (1 - \alpha) \cdot (\zeta_{i,n} - \zeta_{i,n}'/\sqrt{n}))_{i,j,l} \right] \leq \frac{1}{2n} \cdot \sum_{i,j,l} \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}')(\zeta_{j,n} - \zeta_{j,n}')(\zeta_{k,n} - \zeta_{k,n}') \times \sup_{\alpha \in [0,1]} (\nabla^3 f_P(\mathcal{Z}_n - (1 - \alpha) \cdot (\zeta_{i,n} - \zeta_{i,n}'/\sqrt{n}))_{i,j,l} \right] \right| (8.2) \]

Observe that \( \mathbb{E}_n [\text{Tr}(A \nabla^2 f_P(\mathcal{Z}_n))] = \sum_{i,j} \mathbb{E}_n [A_{i,j}(\nabla^2 f_P(\mathcal{Z}_n))_{j,i}] \) simplifies as:

\[ \frac{1}{2n} \cdot \sum_{i,j,k=1}^{n} \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}')(\zeta_{j,n} - \zeta_{j,n}')(\zeta_{k,n} - \zeta_{k,n}') |\mathcal{Z}_n| \right] - 2 \cdot (\delta_{i,j} - \delta_{j,i}) \left( \nabla^2 f_P(\mathcal{Z}_n) \right)_{j,i} \right] = \frac{1}{2n} \cdot \sum_{i,j,k} \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}') \left( \nabla^2 f_P(\mathcal{Z}_n) \right)_{j,i} \right] - \delta_{i,j} \left( \nabla^2 f_P(\mathcal{Z}_n) \right)_{j,i} \right] \]

\[ = \frac{1}{2} \cdot \sum_{i,j} \left\{ \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}' - \delta_{i,j}) \left( \nabla^2 f_P(\mathcal{Z}_n) \right)_{j,i} \right] \right\} \]

\[ \leq \frac{1}{2n} \cdot \sum_{i,j} \left\{ \mathbb{E}_n \left[ (\zeta_{i,n} - \zeta_{i,n}' - \delta_{i,j}) \left( \nabla^2 f_P(\mathcal{Z}_n) \right)_{j,i} \right] \right\} \right| (8.3) \]

Based on (8.2) and (8.3), a bound is given by

\[ \sum_{k \in \{1,2\}} A_k n \sum_{i,j,l} \mathbb{E}_n \left[ (\zeta_{i,n}' - \zeta_{i,n}' - \delta_{i,j}) \left( \nabla^2 f_P(\mathcal{Z}_n) + \Delta^k \right) (\zeta_{i,n} + \zeta_{i,n}'/\sqrt{n})_{j,i} \right] \]
\[ + \sum_{k \in \{1, 2\}} \frac{B_k}{\sqrt{n}} \sum_{i,j,l} E_{\hat{\Phi}} \left[ \left| \frac{\partial}{\partial n} + \Delta^{(k)}(\xi_{1,n}/\sqrt{n}, \xi_{1,n}/\sqrt{n}) \right| \right], \]

as stated in the Lemma in order to bound \[ \| \hat{\Phi}(F(Z_n)) - \hat{\Phi}(F(Z_n)) \|. \]

**Proof. Lemma 11:** Fixing \( j = 1 \), we can re-write the pivot in Proposition 1 as

\[ \int_{-\infty}^{\infty} \exp(-z - \sqrt{\beta_n(1)^2}z^2) \cdot f_{\frac{1}{2}}(t) z_n(t) dt \]

where the random variables \((Z_1, T)\) have a joint density at \((z, t)\) proportional to:

\[ \exp(-z - \sqrt{\beta_n(1)^2}z^2) \cdot \exp(-F(z, N_n(1))T + Q_0T + r_0T)/2!, \]

That is, \((Z_1, T) \sim N(\mu|N_n(1), \Sigma); \) where \( \mu \) is an affine function of \( N_n(1) \). Let the conditional distribution of \( Z_1|T = t \) be a \( N(a^T t + b^T N_n(1) + c, \sigma^2) \) for vectors \( a, b, c \) and variance \( \sigma^2 \) determined from the implied joint-normal distribution. Similarly, let the marginal normal density of \( T \) at \( t \) be denoted by \( f_T(t; N_n(1)) \), a function of \( N_n(1) \). Substituting with the centered and scaled version of the statistic \((Z_n(1), N_n(1))\) through the transformation \( Z_n = \Sigma^{-1/2}(F(z, N_n(1))T - \sqrt{\beta_n}) \), the pivot in Proposition 1 has a representation of the form

\[ P_1(Z_n; \sqrt{\beta_n}) = \int f_T(t; U Z_n + V \sqrt{\beta_n}) 1_{1 > 0} dt \]

where \( f_T(t; U Z_n + V \sqrt{\beta_n}) \equiv N(U Z_n + V \sqrt{\beta_n}, \Sigma_2) \), the marginal density of \( T \) as a function of \((Z_n, \beta_n)\) and \( \rho T Z_n + q T + s T \sqrt{\beta_n} + r, U Z_n + V \sqrt{\beta_n} \) are linear maps in \((Z_n, \sqrt{\beta_n})\).

The \( k \)-th derivative of the pivot with respect to \( Z_n \) is seen to be linear combinations of the mean of a truncated normal distribution,

\[ \mathbb{E}[Y|Y > -\Sigma_2]^{-1/2} (U Z_n + V \sqrt{\beta_n}) \],

which in turn, grows linearly in the threshold vector. Thus, for some matrices \( M, N \), we can finally bound \[ \| \phi^k P_1(\cdot; \sqrt{\beta_n}) \| \leq \| M Z_n \| + \| N \sqrt{\beta_n} \|. \]

**Proof. Theorem 7:** In the bound derived in Lemma 10, the \((i, j, l)\) entry of \( \nabla^3 f_F(w) \) for a bounded and thrice differentiable function \( F \) equals

\[ \frac{1}{2} \int_0^1 2^{1/2} E_{\hat{\Phi}} \left[ \partial_{n}^3 F(\sqrt{t} w + \sqrt{1-t} Z; \sqrt{\alpha_n})_{i,j,l} \right] dt; Z \sim N(0, I) \]

\[ = \int_0^1 2^{1/2} \sqrt{\frac{t}{1-t}} E_{\hat{\Phi}} \left[ \partial_{n}^3 F(\sqrt{t} w + \sqrt{1-t} Z; \sqrt{\alpha_n})_{i,j,l} \right] dt. \]  

(8.4)

Set \( F_l(Z_n; \sqrt{\alpha_n}) = g_l(Z_n; \hat{\Phi}(Q_f Z_n + \sqrt{\alpha_n}) \) for \( l = 1, 2 \) in order to bound (4.4). Using (8.4) and the rate of growth of the pivot in Lemma 11, we can further bound the bounds in Lemma 10:

\[ \sum_{k \in \{1, 2\}} \frac{A_k}{\sqrt{n}} \sum_{i,j,l} E_{\hat{\Phi}} \left[ \partial_{n}^3 F(\sqrt{t} w + \sqrt{1-t} Z; \sqrt{\alpha_n})_{i,j,l} \right] dt. \]
takes a similar form. Clearly, constants dependent on $K_L$ for some matrices we compute the integral with respect to $\zeta_{i,n}$. Letting $\Delta$ up to the fifth order, the above bound equals $\zeta_i \in \{1, 2\}; \zeta_i, l = 1, 2$ are constants dependent on $K_2, K_3, \rho$. Thus, follows a transfer of CLT under local alternatives with rate equal to $n^{-1/2}$.

**Proof. Theorem 8:** Applying Fubini’s theorem to the expectation in $B_n(i, j, l)$, we compute the integral with respect to $Z_n^{(1)}$ as our starting point. Thus, we have

\[
\begin{align*}
E_{\mathbb{P}_n}[|Z_1| \mathbb{E}_{|Z_1|} \mathbb{E}_{\eta_1}] & \leq \exp(-\langle QW_n(t) + \sqrt{\alpha_n} \rangle^T (I + (1 - t)QQ^T)^{-1} (QW_n(t) + \sqrt{\alpha_n})/2) dt \\
& = \int_0^1 \sup_{\alpha \in [0, 1]} \left| \zeta_i \int |Z_1| \mathbb{E}_{|Z_1|} \mathbb{E}_{\eta_1} \exp(-\langle QW_n(t) + \sqrt{\alpha_n} \rangle^T V^{-1/2}(QW_n(t) + \sqrt{\alpha_n})/2 dt \right| \\
& \leq \exp(-\langle Q\sqrt{\Delta} + \sqrt{\alpha_n} \rangle^T V^{-1/2}(I + tV^{-1/2}QQ^TV^{-1/2})^{-1} V^{-1/2}(Q\sqrt{\Delta} + \sqrt{\alpha_n})) \times O(\Pi_{k=1, 2, 3}(\|L_t^{(k)} \sqrt{\alpha_n}\| + \|M_t^{(k)} \Delta\|));
\end{align*}
\]

for some matrices $L_t^{(k)}, M_t^{(k)}; k = 1, 2, 3$. Finally, using the matrix identity:

\[
V^{-1/2}(I + tV^{-1/2}QQ^TV^{-1/2})^{-1} V^{-1/2} = (I + QQ^T)^{-1},
\]

we have the bound on $B_n$ as

\[
\begin{align*}
& \int_0^1 \sup_{\alpha \in [0, 1]} \left| \zeta_i \int |Z_1| \mathbb{E}_{|Z_1|} \mathbb{E}_{\eta_1} \prod_{k=1, 2, 3} (\|L_t^{(k)} \sqrt{\alpha_n}\| + \|M_t^{(k)} \Delta(\zeta_i, \sqrt{\alpha_n}, \zeta_i, \sqrt{\alpha_n})\|) \right| dt \\
& \times \exp(-\langle Q\sqrt{\Delta} + \sqrt{\alpha_n} \rangle^T (I + QQ^T)^{-1} (Q\sqrt{\Delta} + \sqrt{\alpha_n})/2) dt \mathbb{P}_n(\zeta_i, \zeta_i, \sqrt{\alpha_n}) dt \\
& \leq \int_0^1 \sup_{\alpha \in [0, 1]} \left| \zeta_i \int |Z_1| \mathbb{E}_{|Z_1|} \mathbb{E}_{\eta_1} \prod_{k=1, 2, 3} (\|L_t^{(k)} \sqrt{\alpha_n}\| + \|M_t^{(k)} \Delta(\zeta_i, \sqrt{\alpha_n}, \zeta_i, \sqrt{\alpha_n})\|) \right| dt \\
& \times \exp(-\langle Q\sqrt{\Delta}(\zeta_i, \sqrt{\alpha_n}, \zeta_i, \sqrt{\alpha_n}) + \sqrt{\alpha_n} \rangle^T (I + QQ^T)^{-1} (Q\sqrt{\Delta}(\zeta_i, \sqrt{\alpha_n}, \zeta_i, \sqrt{\alpha_n}) + \sqrt{\alpha_n})/2) \mathbb{P}_n(\zeta_i, \zeta_i, \sqrt{\alpha_n}) dt.
\end{align*}
\]
\[
\begin{align*}
= \exp(-\sqrt{n\alpha^T(I+QQ^T)^{-1}}\sqrt{n\alpha_n/2}) \int_0^1 \frac{\sqrt{t}}{2\sqrt{(1-t)}} \, dt \times \mathbb{E}_{\phi_n} \left[ |\zeta_{1,n}^l||\zeta_{1,n}^l||\zeta_{1,n}^r| \right] \\
\sup_{t\in[0,1],a\in[0,1]} \prod_{k=1,2,3} (||\zeta_{k}^l\sqrt{n\alpha}|| + ||M_k^l\Delta(\zeta_{1,n}/\sqrt{n},\zeta_{1,n}/\sqrt{n})||)
\exp(-\sqrt{n\alpha^T}\sqrt{Q}\sqrt{\Delta(\zeta_{1,n}/\sqrt{n},\zeta_{1,n}/\sqrt{n})}).
\end{align*}
\]

Observing that \( \Delta \) is a linear combination of \( \zeta_{1,n}/\sqrt{n} \) and \( \zeta_{1,n}/\sqrt{n} \), the assumption of existence of exponential moments in a neighborhood of 0 for the generative triangular array allows us to conclude that

\[B_n(i,j,l) \leq C(r, Q)n^{3/2}\|\alpha_n\|^3 \exp(-\sqrt{n\alpha^T(I+QQ^T)^{-1}}\sqrt{n\alpha_n/2}).\]

Proof. Theorem 9: We will focus on the function \( F_{1(t)} \); \( t = 1,2 \) with an exact similar calculation following for \( F_{2(t)} \). The first step involves computing the third order derivatives of the Stein’s function, the \( (i,j,l) \) entry of \( \nabla^3 f_{1(t)}(w) \) which equals

\[
\frac{1}{2} \int_0^1 t^{1/2} \mathbb{E}_{\mathbb{P}} \left[ \nabla^3 F_{1(t)}(\sqrt{t}w + \sqrt{1-t}Z; \sqrt{n\alpha_n}) \right]_{i,j,l} \left[ \{Q(\sqrt{t}w + \sqrt{1-t}Z) < q\sqrt{n\alpha_n}\} \right] dt
\]

\[
= \int_0^1 \frac{\sqrt{t}}{2\sqrt{(1-t)}} \mathbb{E}_{\mathbb{P}} \left[ \nabla^3 F_{1(t)}(\sqrt{t}w + \sqrt{1-t}Z; \sqrt{n\alpha_n}) \right]_{i,j,l} \left[ \{Q(\sqrt{t}w + \sqrt{1-t}Z) < q\sqrt{n\alpha_n}\} \right] dt.
\]

Denoting \( R(W_n, Z) = W_n + \sqrt{(1-t)Z} \) where \( W_n = \sqrt{t}Z_n^{(1)} + \sqrt{t}\Delta \) and

\[
\Psi(\sqrt{n\alpha_n}) = \sqrt{t} \left\{ \prod_{j=l}^{q} \sqrt{4 + (Q_j R(W_n, Z) + \sqrt{n\alpha_n})^2 + (Q_j R(W_n, Z) + \sqrt{n\alpha_n})^2} \right\},
\]

we note that \( \nabla^2 F_{1(t)}(R(W_n, Z); \sqrt{n\alpha_n})_{i,j,l} \) can be expressed as a linear combination of terms of the form

\[
\left\{ (P_1 \cdot R(W_n, Z) + P_2 \cdot \sqrt{n\alpha_n})(P_3 \cdot R(W_n, Z) + P_4 \cdot \sqrt{n\alpha_n})^T \right\}_{i,j,l} \cdot \Psi(\sqrt{n\alpha_n})
\]

\[
\exp(-Q \cdot R(W_n, Z) + \sqrt{n\alpha_n})^T(Q \cdot R(W_n, Z) + \sqrt{n\alpha_n})/2).
\]

We can now write the \( (i,j,l) \)-th entry of the third derivative of the Stein’s function:

\[
\int_0^1 \frac{\sqrt{t}}{2\sqrt{(1-t)}} \mathbb{E}_{\mathbb{P}} \left[ Z^t \left\{ (P_1 \cdot R(W_n, Z) + P_2 \cdot \sqrt{n\alpha_n})(P_3 \cdot R(W_n, Z) + P_4 \cdot \sqrt{n\alpha_n})^T \right\}_{j,l} \cdot \Psi(\sqrt{n\alpha_n})
\times \exp(-Q \cdot R(W_n, Z) + \sqrt{n\alpha_n})^T(Q \cdot R(W_n, Z) + \sqrt{n\alpha_n})/2 \right]_{i,j,l} \cdot \left\{ Q \cdot R(W_n, Z) < q\sqrt{n\alpha_n} \right\}
\]

The strict decreasing nature of the function \( 1/(\sqrt{4+w^2+w}) \) on the set \( \{Qw < q\sqrt{n\alpha_n}\} \) allows us to bound from below and above the sequence

\[
\frac{|\sqrt{n\alpha_n}^{(l)}|}{\sqrt{4 + (Q_j R(W_n, Z) + \sqrt{n\alpha_n})^2 + (Q_j R(W_n, Z) + \sqrt{n\alpha_n})^2}} \text{ by } \frac{|\sqrt{n\alpha_n}^{(l)}|}{\sqrt{4 + (1-q)\alpha_n^{(l)} + (1-q)\alpha_n^{(l)}}}
\]

\[
\frac{|\sqrt{n\alpha_n}^{(l)}|}{\sqrt{4 + (1-q)\alpha_n^{(l)} + (1-q)\alpha_n^{(l)}}} \text{ in a point-wise sense. Observing that both the bounding sequences are convergent and hence, uniformly bounded, the above integral is bounded by}
\]

\[\text{imn-gener. ver. 2014/10/16 file: main-2.tex date: January 8, 2019} \]
\[ K \left( \prod_{j=1}^{e} |\tilde{\alpha}_{n}^{(j)}| \right)^{-1} \times \int_{0}^{1} \frac{\sqrt{t}}{2(1-t)^{1/2} \| \bar{\mu}_{n}(t) \|} \mathbb{E}_{\bar{\nu}} \left[ \mathbb{Z}^{j} \left\{ (P_{1} R(W_{n}, \bar{Z}) + P_{2} \sqrt{\tilde{\alpha}_{n}}) \right\} \right] dt \]

\[ (P_{3} R(W_{n}, \bar{Z}) + P_{4} \sqrt{\tilde{\alpha}_{n}})^{T} \] \times \exp(-Q \cdot R(W_{n}, \bar{Z}) + \sqrt{\tilde{\alpha}_{n}})^{T} (Q \cdot R(W_{n}, \bar{Z}) + \sqrt{\tilde{\alpha}_{n}})/2 \right] dt \]

\[ K \] being a constant. This further equals

\[ K \left( \prod_{j=1}^{e} |\tilde{\alpha}_{n}^{(j)}| \right)^{-1} \exp(-(QW_{n} + \sqrt{\tilde{\alpha}_{n}})^{T}(I + (1 - t)QQ^{T})^{-1}(QW_{n} + \sqrt{\tilde{\alpha}_{n}})/2) \]

\[ \int_{0}^{1} \frac{\sqrt{t}}{2(1-t)^{1/2}} \times \mathbb{E}_{\bar{\nu}} \left[ (Z_{j}) \left\{ (P_{1} R(W_{n}, \bar{Z}) + P_{2} \sqrt{\tilde{\alpha}_{n}}) \right\} (P_{3} R(W_{n}, \bar{Z}) + P_{4} \sqrt{\tilde{\alpha}_{n}})^{T} \right] dt \]

where \( \tilde{Z} \sim N(\mu_{n}, \Sigma(t)) \); \( \bar{\mu}_{n}(W_{n}) = -(I + (1 - t)QQ^{T})^{-1}Q^{T}\delta(Z_{n}, \sqrt{\tilde{\alpha}_{n}}). \)

\[ \tilde{\Sigma}(t) = (I + (1 - t)QQ^{T})^{-1} \delta(Z_{n}, \sqrt{\tilde{\alpha}_{n}}) = Q\sqrt{\tilde{\alpha}_{n}}^{(1)} + \Delta(\tilde{\alpha}_{n}/\sqrt{n}) + \sqrt{\tilde{\alpha}_{n}}. \]

Finally, using the moments of a multivariate Gaussian random variable, it suffices to analyze linear combinations of:

\[ \left( \prod_{j=1}^{e} |\tilde{\alpha}_{n}^{(j)}| \right)^{-1} \sum_{K=1,2} \sum_{i,j,l} C^{(i,j,l)}(K_{2}, K_{3}, Q, \rho) B_{n}(i, j, l)/\sqrt{n} \]

in computing \( |\nabla^{3}F_{1}(Z_{n})^{(1)} + \Delta(\tilde{\alpha}_{n}/\sqrt{n})|\tilde{\alpha}_{n}(t), i, j, l|\); the coefficients in the linear combination depend on \( (K_{2}, K_{3}, Q, \rho) \). Plugging in the dominant term in the above calculation into the Stein’s bound from Lemma 10, we are left to bound

\[ \left( \prod_{j=1}^{e} |\tilde{\alpha}_{n}^{(j)}| \right)^{-1} \sum_{K=1,2} A_{k} \cdot \sum_{i,j,l} C^{(i,j,l)}(K_{2}, K_{3}, Q, \rho) B_{n}(i, j, l)/\sqrt{n} \]

for constants \( C^{(i,j,l)}(K_{2}, K_{3}, Q, \rho) \). Applying Theorem 8, we conclude that

\[ |\mathbb{E}_{\bar{\nu}} \left[ g_{i}(Z_{n}) \prod_{j=1}^{e} |\tilde{\alpha}_{n}^{(j)}| \Phi(Q_{j}, Z_{n} + \sqrt{\tilde{\alpha}_{n}}) \right] - \mathbb{E}_{\bar{\nu}} \left[ g_{i}(Z_{n}) \prod_{j=1}^{e} \Phi(Q_{j}, Z_{n} + \sqrt{\tilde{\alpha}_{n}}) \right] | \]

\[ = O \left( \left( \prod_{j=1}^{e} |\tilde{\alpha}_{n}^{(j)}| \right)^{-1} n\|\tilde{\alpha}_{n}\|^{3} \exp(-\sqrt{\tilde{\alpha}_{n}})(I + QQ^{T})^{-1}\sqrt{\tilde{\alpha}_{n}(n)}/2 \right). \]

Combined with the rate of decay of the denominator in the relative differences in Theorem 2, the conclusion of the current Theorem follows. \( \square \)

**Proof. Theorems 10 and 11:** We prove both the theorems below. First, we derive dependence of the difference in expectations, now depending on a third relative difference that captures the rate at which the tail of the randomization in (1.3) converges to the Gaussian counterpart. Denoting the carved likelihood ratio under generative model (2.2) between the carved law in Lemma 2 and its unconditional counterpart by \( LR_{n}^{z} \)

\[ LR_{n}^{z}(z_{n}) = \frac{\int g_{z_{n}}(\Sigma^{1/2} z_{n} + \sqrt{\tilde{\alpha}_{n}}) + Q_{E} t_{n} + r_{E}(R_{n})1_{t_{n} > 0} dt_{n}}{\int \ell'(z'_{n}) \times \int g_{z_{n}}(\Sigma^{1/2} z_{n} + \sqrt{\tilde{\alpha}_{n}}) + Q_{E} t_{n} + r_{E}(R_{n})1_{t_{n} > 0} dt_{n} dz'_{n}} \]
and noting \( \widetilde{E}_P[f(Z_n)] = E_P[f(Z_n)LR_{P,n}^*(Z_n)] \), the difference in expectations

\[
\left| \widetilde{E}_P[H \circ P(Z_n; \sqrt{n} \beta_n)] - \widetilde{E}_P[H \circ P(Z_n; \sqrt{n} \beta_n)] \right|
\]

equals \( E_P[H \circ P(Z_n; \sqrt{n} \beta_n)LR_{P,n}^*(Z_n)] - E_P[H \circ P(Z_n; \sqrt{n} \beta_n)LR_{P,n}(Z_n)] \).

The above term can be bounded from above by (T1) + (T2) + (T3). Terms (T1), (T2) are derived in the proof of Theorem 1. The term (T3) is given by

\[
\left| E_P[H \circ P(Z_n; \sqrt{n} \beta_n)LR_{P,n}^*(Z_n)] - E_P[H \circ P(Z_n; \sqrt{n} \beta_n)LR_{P,n}(Z_n)] \right|
\]

where \( LR_{P,n}(z_n) = \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) / E_P \left[ \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) \right] \). Using the uniform bound on \( H(\cdot) \), term (T3) is bounded above by

\[
2K_1 \cdot E_P \left[ \left| \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) - E_P \left[ \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) \right] \right| \right]
\]

where \( E_P \left[ \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) \right] \) is bounded above by \( 2K_1 \cdot R_{n,3}^0 \). Terms \( P_{P,n}[T_{n,E}^0 > 0] = \int g_{Z_n}(-z_n; \beta_n) + Q_{E} t_n + r_{E}(R_n)) \) have a joint-density proportional to \( \ell(z_n) \cdot g_{Z_n}(-z_n; \beta_n) + Q_{E} t_n + r_{E}(R_n)) \).

Thus, using the bounds \( R_{n,1}^1, R_{n,2}^2 \) for (T1), (T2) respectively and the above bound \( R_{n,3}^0 \) on (T3), we observe that

\[
\left| E_P[H \circ P(Z_n; \sqrt{n} \beta_n)] - E_P[H \circ P(Z_n; \sqrt{n} \beta_n)] \right| \leq (K_1 \cdot R_{n,1}^1 + R_{n,2}^2 + 2K_1 \cdot R_{n,3}^0).
\]

Theorems 3 and 6 give already bounds on \( R_{n,1}^1, R_{n,2}^2 \) in the sequence setting. Similarly, Theorems 7 and 9 provide bounds for the relative differences \( R_{n,1}^1, R_{n,2}^2 \) in the multivariate model.

The proof for Theorems 10 and 11 is thus complete by proving that \( R_{n,3}^0 = o(1) \). Note that conditional on \( Z_n \), the variable \( T_{n,E}^0 \) satisfies a CLT; this follows from the fact that we know \( Z_n, T_{n,E}^0 \) jointly satisfy a CLT. Under a parameterization \( \sqrt{n} \alpha_n = a_n \), a representation result for the probability of an open and convex set on a moderate deviation scale in Einmahl et al. (2004) shows that

\[
P_{P,n}[T_{n,E}^0 > 0] = \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) \cdot \exp(O(n^{-1/2} a_n^3))(1 + o(1)),
\]

\[
P_{P,n}[T_{n,E}^0 > 0] = \Pi_{j=1}^{|E|} \Phi \left( Q_j z_n + \sqrt{n} \alpha_n \right) \cdot \exp(O(n^{-1/2} a_n^3))O(1)
\]

for sufficiently large \( n \). Thus, whenever \( a_n = o(n^{1/6}) \) or equivalently, \( \sqrt{n} \alpha_n = o(n^{2/3}) \), it follows that \( R_{n,3}^0 = o(1) \). The proof for the sequence setting follows similarly.

In this case, we are left to bound the term

\[
R_{n,3}^0 = 2K_1 \cdot \frac{E_P \left[ \Phi (-Z_n + \sqrt{n} \beta_n) / \rho - P_{P,n}[W_n > -(Z_n + \sqrt{n} \beta)] Z_n] \right]}{E_P \left[ P_{P,n}[W_n > -(Z_n + \sqrt{n} \beta)] Z_n] \right]}
\]

which is \( o(1) \) whenever \( b_n = o(n^{1/6}) \) using the moderate deviation approximation for \( d = 1 \).