2-blocks in strongly biconnected directed graphs

Raed Jaberi

Abstract
A directed graph \( G = (V, E) \) is called strongly biconnected if \( G \) is strongly connected and the underlying graph of \( G \) is biconnected. A strongly biconnected component of a strongly connected graph \( G = (V, E) \) is a maximal vertex subset \( L \subseteq V \) such that the induced subgraph on \( L \) is strongly biconnected. Let \( G = (V, E) \) be a strongly biconnected directed graph. A 2-edge-biconnected block in \( G \) is a maximal vertex subset \( U \subseteq V \) such that for any two distinct vertices \( v, w \in U \) and for each edge \( b \in E \), the vertices \( v, w \) are in the same strongly biconnected components of \( G \setminus \{b\} \). A 2-strongly biconnected block in \( G \) is a maximal vertex subset \( U \subseteq V \) of size at least 2 such that for every pair of distinct vertices \( v, w \in U \) and for every vertex \( z \in V \setminus \{v, w\} \), the vertices \( v \) and \( w \) are in the same strongly biconnected component of \( G \setminus \{v, w\} \). In this paper we study 2-edge-biconnected blocks and 2-strong biconnected blocks.

Keywords: Directed graphs, Graph algorithms, Strongly biconnected directed graphs, 2-blocks

1. Introduction

Let \( G = (V, E) \) be a directed graph. A 2-edge block of \( G \) is a maximal vertex subset \( L_e \subseteq V \) with \( |L_e| > 1 \) such that for each pair of distinct vertices \( x, y \in L_e \), \( G \) contains two edge-disjoint paths from \( x \) to \( y \) and two edge-disjoint paths from \( y \) to \( x \). A 2-strong block of \( G \) is a maximal vertex subset \( B_s \subseteq V \) with \( |B_s| > 1 \) such that for each pair of distinct vertices \( x, y \in B_s \) and for every vertex \( w \in V \setminus \{x, y\} \), \( x \) and \( y \) belong to the same strongly connected component of \( G \setminus \{w\} \). \( G \) is called strongly biconnected if \( G \) is strongly connected and the underlying graph of \( G \) is biconnected. This class of directed graphs was introduced by Wu and Grumbach [11]. A strongly biconnected component of \( G \) is a maximal vertex subset \( C \subseteq V \) such that the induced subgraph on \( C \) is strongly biconnected [11]. Let \( G = (V, E) \) be a strongly biconnected directed graph. An edge \( e \in E \) is a b-bridge if the subgraph \( G \setminus \{e\} = (V, E \setminus \{e\}) \) is not strongly biconnected. A vertex
$w \in V$ is a b-articulation point if $G \setminus \{w\}$ is not strongly biconnected, where $G \setminus \{w\}$ is the subgraph obtained from $G$ by deleting $w$. $G$ is 2-edge-strongly-biconnected (respectively, 2-vertex-strongly biconnected) if $|V| > 2$ and $G$ has no b-bridges (respectively, b-articulation points). A 2-edge-biconnected block in $G$ is a maximal vertex subset $U \subseteq V$ such that for any two distinct vertices $v, w \in U$ and for each edge $b \in E$, the vertices $v, w$ are in the same strongly biconnected components of $G \setminus \{b\}$. The 2-edge blocks of $G$ are disjoint [5]. Notice that 2-edge-biconnected blocks are not necessarily disjoint, as shown in Figure 1.

![Figure 1: A strongly biconnected directed graph $G$. The vertex subset $\{9, 10, 11, 12, 13, 14, 15\}$ is a 2-edge block of $G$. Notice that the vertices 15, 12 are not in the same 2-edge-biconnected block because 15 and 12 are not in the same strongly biconnected component of $G \setminus \{(2, 15)\}$. Moreover, $G$ has two 2-edge-biconnected blocks $U_1 = \{9, 10, 11, 12, 13, 14\}$ and $U_2 = \{4, 15\}$. $U_1$ and $U_2$ share vertex 4.](image)

A 2-strong-biconnected block in $G$ is a maximal vertex subset $U \subseteq V$ of size at least 2 such that for every pair of distinct vertices $v, w \in U$ and for every vertex $z \in V \setminus \{v, w\}$, the vertices $v$ and $w$ are in the same strongly biconnected component of $G \setminus \{z\}$.

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Blocks, articulation points, and bridges of an undirected graph can be calculated in $O(n + m)$ time [2, 10, 8]. In [4], Georgiadis presented a linear time algorithm to test whether a directed graph is 2-vertex-connected. Strong articulation points and strong bridges of a directed graph can be computed in $O(n + m)$ time [7, 1]. Jaberi [5] presented algorithms for computing 2-strong blocks, and 2-edge blocks of a directed graph. Georgiadis et al. [2, 3] gave linear time algorithms for determining 2-edge blocks and 2-strong blocks. Wu and Grumbach [11] introduced the concept of strongly biconnected directed graphs and the concept of strongly biconnected components. Jaberi [6] studied b-bridges in strongly biconnected directed graphs. In this paper we study 2-edge-biconnected blocks and 2-strong biconnected blocks.

2. 2-edge-biconnected blocks

In this section we study 2-edge-biconnected blocks and present an algorithm for computing them. Let $G = (V, E)$ be a strongly biconnected directed graph. For every pair of distinct vertices $x, y \in V$, we write $x \leftrightarrow y$ if for any edge $b \in E$, the vertices $x, y$ belong to the same strongly biconnected component of $G \setminus \{b\}$. A 2-edge-biconnected block in $G$ is a maximal subset of vertices $U \subseteq V$ with $|U| > 1$ such that for any two vertices $x, y \in U$, we have $x \leftrightarrow y$. A 2-edge-strongly-biconnected component in $G$ is a maximal vertex subset $C_{2eb} \subseteq V$ such that the induced subgraph on $C_{2eb}$ is 2-edge-strongly biconnected. Note that the strongly biconnected directed graph in Figure 1 contains one 2-edge-strongly biconnected component $\{9, 10, 11, 12, 13, 14\}$, which is a subset of the 2-edge-biconnected block $\{9, 10, 11, 12, 13, 14, 4\}$.

**Lemma 2.1.** Let $G = (V, E)$ be a strongly biconnected directed graph and let $C_{2eb}$ be a 2-edge-strongly biconnected component of $G$. Then $C_{2eb}$ is a subset of a 2-edge-biconnected block of $G$.

**Proof.** Let $x$ and $y$ be distinct vertices in $C_{2eb}$ and let $e \in E$. Let $G[C_{2eb}]$ be the induced subgraph on $C_{2eb}$. By definition, the subgraph obtained from $G[C_{2eb}]$ by deleting $e$ is still strongly biconnected. Therefore, we have $x \leftrightarrow y$.

2-edge blocks are disjoint [5]. Note that 2-edge biconnected blocks are not necessarily disjoint. But any two of them share at most one vertex, as illustrated in Figure 1.

**Lemma 2.2.** Let $U_1, U_2$ be distinct 2-edge-biconnected blocks of a strongly biconnected directed graph $G = (V, E)$. Then $|U_1 \cap U_2| \leq 1$. 

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Proof. Assume for the purpose of contradiction that $|U_1 \cap U_2| > 1$. Let $x \in U_1 \setminus (U_1 \cap U_2)$ and let $y \in U_2 \setminus (U_1 \cap U_2)$. Let $v, w \in U_1 \cap U_2$ with $v \neq w$ and let $b \in E$. Notice that $x, v$ belong to the same strongly connected component of $G \setminus \{b\}$ since $x 
leftrightarrow v$. Moreover, $v, y$ belong to the same strongly connected component of $G \setminus \{b\}$. Consequently, $x, y$ are in the same strongly connected component of $G \setminus \{b\}$. Then, the vertices $x, y$ do not lie in the same strongly biconnected component of $G \setminus \{b\}$. Suppose that $C_x, C_y$ are two strongly biconnected components of $G \setminus \{b\}$ such that $x \in C_x$ and $y \in C_y$. There are two cases to consider.

1. $v \in C_x \cap C_y$. In this case $w \notin C_x \cap C_y$. Suppose without loss of generality that $v \notin C_x$. Then $w, y$ are not in the same strongly biconnected components of $G \setminus \{b\}$. But this contradicts that $v 
leftrightarrow y$

2. $v \notin C_x \cap C_y$. Suppose without loss of generality that $v \in C_x$. Then $v, y$ do not lie in the same strongly biconnected component of $G \setminus \{b\}$. But this contradicts that $v 
leftrightarrow y$

□

Using similar arguments as in Lemma 2.2, we can prove the following.

Lemma 2.3. Let $G = (V, E)$ be a strongly biconnected directed graph and let $\{w_0, w_1, \ldots, w_t\} \subseteq V$ such that $w_0 
leftrightarrow w_t$ and $w_{i-1} \nleftrightarrow w_i$ for $i \in \{1, 2, \ldots, t\}$. Then $\{w_0, w_1, \ldots, w_t\}$ is a subset of a 2-edge biconnected block of $G$.

Lemma 2.4. Let $G = (V, E)$ be a strongly biconnected directed graph and let $v, w$ be two distinct vertices in $G$. Let $b$ be an edge in $G$ such that $b$ is not a $b$-bridge. Then, the vertices $v, w$ are in the same strongly biconnected component of $G \setminus \{b\}$

Proof. Immediate from definition. □

Algorithm 2.5 shows an algorithm for computing all the 2-edge biconnected blocks of a strongly biconnected directed graph.

The correctness of this algorithm follows from Lemma 2.2, Lemma 2.3, and Lemma 2.4.

Theorem 2.6. Algorithm 2.3 runs in $O(n^3)$ time.

Proof. The b-bridges of $G$ can be computed in $O(nm)$ time [6]. Strongly biconnected components can be calculated in linear time [11]. Lines 7–11 take $O(b.n^2)$, where $b$ is the number of b-bridges in $G$. The time required for building $G^{eb}$ in lines 12–16 is $O(n^2)$. Moreover, the blocks of an undirected graph can be found in linear time using Tarjan’s algorithm. [6, 8]. □
Algorithm 2.5.
Input: A strongly biconnected directed graph $G = (V, E)$.
Output: The 2-edge biconnected blocks of $G$.
1. Compute the b-bridges of $G$
2. If $G$ has no b-bridges then
3. Output $V$.
4. else
5. Let $L$ be an $n \times n$ matrix.
6. Initialize $L$ with 1s.
7. for each b-bridge $b$ of $G$ do
8. calculate the strongly biconnected components of $G \setminus \{b\}$
9. for each pair $(x, y) \in V \times V$ do
10. if $x, y$ in different strongly biconnected components of $G \setminus \{b\}$ then
11. $L[x, y] \leftarrow 0$.
12. $E^{eb} \leftarrow \emptyset$.
13. for every pair $(x, y) \in V \times V$ do
14. if $L[x, y] = 1$ and $L[y, x] = 1$ then
15. $E^{eb} \leftarrow E^{eb} \cup \{(x, y)\}$
16. $G^{eb} \leftarrow (V, E^{eb})$
17. Compute all the blocks of size $\geq 2$ in $G^{eb}$ and output them.

3. 2-strong-biconnected blocks

In this section we illustrate some properties of 2-strong-biconnected blocks. The strongly biconnected directed graph in Figure 2 has two 2-strong biconnected blocks $L_1 = \{1, 2, 3, 4\}$ and $L_2 = \{3, 4, 5, 6\}$. Note that $L_1$ and $L_2$ share two vertices. The intersection of any two distinct 2-strong biconnected blocks contains at most 2 vertices. Note also that the subgraph induced by the 2-strong biconnected block $L_1$ has no edges.

Let $G = (V, E)$ be a strongly biconnected directed graph. A 2-vertex-strongly biconnected component $C_{2sb}$ is a maximal vertex subset $C_{2sb} \subseteq V$ such that the induced subgraph on $C_{2sb}$ is 2-vertex-strongly biconnected. Each 2-vertex-strongly biconnected component $C_{2sb}$ of $G$ is a subset of a 2-strong-biconnected-block of $G$. Furthermore, each 2-vertex-strongly biconnected component $C_{2sb}$ of $G$ is 2-vertex connected. Therefore, the subgraph induced by $C_{2sb}$ contains at least $2|C_{2sb}|$ edges. In contrast to, the subgraphs induced by the 2-strong-biconnected blocks do not necessarily contain edges.
Figure 2: A strongly biconnected directed graph $G = (V, E)$. The vertices 2 and 6 are in the same 2-strong block of $G$ but they do not belong to the same 2-strong-biconnected-block of $G$ since they are not in the same strongly biconnected component of $G \setminus \{4\}$. Moreover, $G$ has two 2-strong biconnected blocks $L_1 = \{1, 2, 3, 4\}$ and $L_2 = \{3, 4, 5, 6\}$. Note that $|L_1 \cap L_2| = 2$
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