STABILITY OF LOGARITHMIC DIFFERENTIAL ONE-FORMS.

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Abstract. This article deals with the irreducible components of the space of codimension one foliations in a projective space defined by logarithmic forms of a certain degree. We study the geometry of the natural parametrization of the logarithmic components and we give a new proof of the stability of logarithmic foliations, obtaining also that these irreducible components are reduced.

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1. Introduction.

We consider differential one-forms of logarithmic type $\omega = F \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i}$ where, for $i = 1, \ldots, m$, $F_i$ is a homogeneous polynomial of a fixed degree $d_i$ in variables $x_0, \ldots, x_n$, with complex coefficients, $F = \prod_j F_j$, and $\lambda_i$ are complex numbers such that $\sum_i d_i \lambda_i = 0$. Such an $\omega$ defines a global section of $\Omega^1_{\mathbb{P}^n}(d)$ for $d = \sum_i d_i$. Also, $\omega$ satisfies the Frobenius integrability condition $\omega \wedge d\omega = 0$.

Fixing $d = (m; d_1, \ldots, d_m)$ denote $L_n(d) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$ the collection of all such logarithmic one-forms and $\mathcal{L}_n(d) \subset \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) = \mathbb{P}^N$ the corresponding closed projective variety. It is easy to see that $\mathcal{L}_n(d)$ is an irreducible algebraic variety. Also, $\mathcal{L}_n(d)$ is contained in the subvariety $\mathcal{F}_n(d) \subset \mathbb{P}^N$ of integrable one-forms of degree $d$.

Here the motivating problem is to describe the irreducible components of $\mathcal{F}_n(d)$.

It was proved by Omegar Calvo in [2] that, for any $d$, the variety of logarithmic forms $\mathcal{L}_n(d)$ is an irreducible component of the moduli space $\mathcal{F}_n(d)$ of codimension one algebraic foliations of degree $d$ in $\mathbb{P}^n(\mathbb{C})$. In other words, the logarithmic one-forms enjoy a stability condition among integrable forms. Actually, the results of [2] hold for more general ambient varieties than projective spaces.

In this article we will provide another proof of O. Calvo’s theorem, in case the ambient space is a complex projective space. Our strategy will be to calculate the tangent space $T(\omega)$ of $\mathcal{F}_n(d)$ at a general point $\omega \in \mathcal{L}_n(d)$. The main results are stated in Theorems 24 and 25.

This method is completely algebraic and provides further information, especially the fact that $\mathcal{F}_n(d)$ results generically reduced along the irreducible component $\mathcal{L}_n(d)$.

The logarithmic components are the closure of the image of a multilinear map $\rho$, defined in Section 4, from a product of projective spaces into a projective space. We describe the base locus of $\rho$ in Section 5, and study its generic injectivity in Section 6. Our proof requires a detailed analysis of the derivative of $\rho$, started in Section 7. Another important ingredient is the resolution of the ideal of various strata of the singular scheme of a logarithmic form; this is carried out in Section 8. The end of the proof is achieved in Section 9, where we distinguish two cases, depending on whether or not $d$ is balanced.

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2. Notation.

We shall use the following notations:

\( \mathbb{C}^{n+1} \) = complex affine space of dimension \( n + 1 \).

\( \mathbb{P}^n \) = complex projective space of dimension \( n \).

\( S_n = \mathbb{C}[x_0, \ldots, x_n] \) = graded ring of polynomials with complex coefficients in \( n + 1 \) variables.

When \( n \) is understood we denote \( S_n = S \).

\( S_n(d) \) = homogeneous elements of degree \( d \) in \( S_n \).

When \( n \) is understood we denote \( S_n(d) = S(d) \).

Recall that one has \( S_n(d) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \).

\( \Omega^q_X = \) sheaf of algebraic differential \( q \)-forms on an algebraic variety \( X \).

\( \Omega^q(X) = \) the set of rational \( q \)-forms on \( X \) (with \( X \) an irreducible variety).

It is a vector space over the field \( \mathbb{C}(X) \) of rational functions of \( X \).

\( \Omega^q_n = H^0(\mathbb{C}^{n+1}, \Omega^q_{\mathbb{C}^{n+1}}) \).

A typical element of \( \Omega^q_n \) is \( \omega = \sum_{i=0}^n a_i \, dx_i \) with \( a_i \in S_n \).

More generally, a typical element of \( \Omega^q_n \) may be written in the usual way as \( \sum_{|J|=q} a_J \, dx_J \) with \( a_J \in S_n \).

where \( J = \{j_1, \ldots, j_q\} \) with \( j_1 < \cdots < j_q \).

When \( n \) is understood we denote \( \Omega^q_n = \Omega^q \).

\( \Omega^q_n(d) = \{\sum_{|J|=q} a_J \, dx_J, \ a_J \in S_n(d - q)\} \).

In particular, \( dx_i \) is homogeneous of degree one.

The exterior derivative is an operator of degree zero, i.e. it preserves degree.

\( H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) \) = projective one-forms of degree \( d \).

It follows from the Euler exact sequence that \( \omega = \sum_i a_i \, dx_i \in \Omega^1_{\mathbb{P}^n}(d) \) is projective if and only if it contracts to zero with the Euler or radial vector field \( R = \sum_{i=0}^n x_i \, \frac{\partial}{\partial x_i} \), that is, if \( \sum_i a_i x_i = 0 \).

\( \mathbb{P}^n(d) = \mathbb{P}(H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))) \).

\( F_n(d) = \{\omega \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))/\omega \wedge d\omega = 0\} \) = the set of integrable projective one-forms in \( \mathbb{P}^n \) of degree \( d \), and

\( F_n(d) \subset \mathbb{P}^n(d) \) the projectivization of \( F_n(d) \).

\( \mathbb{P}^n(d) = \mathbb{P}\Lambda(d) \times \prod_{i=1}^n \mathbb{P}S_i(d_i) \).
3. Logarithmic one-forms.

1. Definition. Fix natural numbers $n, d$ and $m$. Let 
$$d = (m; d_1, \ldots, d_m)$$
be a partition of $d$ into $m$ parts, that is, for $i = 1, \ldots, m$ each $d_i$ is a natural number and $\sum_{i=1}^{m} d_i = d$. Let us normalize so that $d_i \geq d_{i+1}$ for all $i < m$. We denote
$$P(m, d)$$
the set of all such partitions of $d$ into $m$ parts.

2. Definition. Fix $d = (m; d_1, \ldots, d_m) \in P(m, d)$. A differential one-form $\omega \in \Omega^1_n$ is logarithmic of type $d$ if
$$\omega = (\prod_{j=1}^{m} F_j) \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \sum_{i=1}^{m} \lambda_i \left( \prod_{j \neq i} F_j \right) dF_i$$
where $F_i \in S_n(d_i)$ is a non-zero homogeneous polynomial of degree $d_i$ and the $\lambda_i$ are complex numbers.

3. Definition. It will be convenient to use the following notation. For $d$ and $F_i \in S_n(d_i)$ as above,
$$F = (F_1, \ldots, F_m), \quad F = \prod_{j=1}^{m} F_j,$$
$$\hat{F}_i = \prod_{j \neq i} F_j = F/F_i, \quad \hat{F}_{ij} = \prod_{k \neq i, k \neq j} F_k = F/F_i F_j, \quad (i \neq j),$$
or, more generally, for a subset $A \subset \{1, \ldots, m\}$ we write
$$\hat{F}_A = \prod_{j \notin A} F_j$$
Hence a logarithmic one-form may be written
$$\omega = F \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i. \quad \text{(3.1)}$$
We denote $\hat{d}_i = \sum_{j \neq i} d_j$ the degree of $\hat{F}_i$ and, more generally, $\hat{d}_A = \sum_{j \notin A} d_j$ the degree of $\hat{F}_A$.

4. Proposition. For $\omega$ a logarithmic one-form as above,
\begin{enumerate}
  \item $\omega$ is homogeneous of degree $d = \sum_{i=1}^{m} d_i$.
  \item $\omega$ is integrable.
  \item $\langle R, \omega \rangle = (\sum_{i=1}^{m} d_i \lambda_i) F$. In particular, $\omega$ is projective if and only if
    $$\sum_{i=1}^{m} d_i \lambda_i = 0.$$
\end{enumerate}
Proof. a) Since the exterior derivative is of degree zero, each term in the sum \( \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i \) is homogeneous of degree \( d \), hence the claim.

b) For each polynomial \( G \), the rational one-form \( dG/G \) is closed. It follows that \( \omega/F = \sum_{i=1}^{m} \lambda_i \hat{F}_i dF_i / F_i \) is closed, hence integrable. A short calculation shows that the product of a rational function with an integrable rational one-form is an integrable rational one-form. Therefore, \( \omega = F \omega/F \) is integrable.

c) Euler’s formula implies that \( \langle R, dG \rangle = eG \) for \( G \in S_n(e) \). By linearity of contraction we have \( \langle R, \omega \rangle = \langle R, \sum_{i} \lambda_i \hat{F}_i dF_i \rangle = \sum_{i} \lambda_i \hat{F}_i F_i = (\sum_{i} d_i \lambda_i) F \).

\[ \square \]

5. Proposition. Suppose \( \omega \) is logarithmic as in 3.1. Then,

a) \( d\omega = (dF/F) \wedge \omega = \sum_{1 \leq i,j \leq m} \lambda_j \hat{F}_{ij} dF_i \wedge dF_j = \sum_{1 \leq i < j \leq m} (\lambda_j - \lambda_i) \hat{F}_{ij} dF_i \wedge dF_j \).

b) \( F \) is an integrating factor of \( \omega \): \( d(\omega/F) = 0 \), or, equivalently, \( F d\omega - dF \wedge \omega = 0 \).

c) Each hypersurface \( F_i = 0 \) is an algebraic leaf of \( \omega \), that is, \( dF_i/F_i \wedge \omega \) is a regular 2-form (i.e. without poles). Hence \( dF_i \wedge \omega = 0 \) on the hypersurface \( F_i = 0 \).

Proof. These follow by straightforward calculations, left to the reader. \( \square \)

4. The logarithmic components and their parametrization.

As before, we fix natural numbers \( n, d \) and \( m \) and a partition \( \mathbf{d} = (m; d_1, \ldots, d_m) \) of \( d \).

For a complex vector space \( V \) we denote \( \mathbb{P}V = V - \{0\}/\mathbb{C}^* \) the corresponding projective space of one-dimensional subspaces of \( V \). Let \( \pi : V - \{0\} \to \mathbb{P}V \) be the canonical projection. If \( X \subset V \) we call \( \mathbb{P}X = \pi(X - \{0\}) \subset \mathbb{P}V \) the projectivization of \( X \).

As in Section 2, we denote

\[ \mathbb{P}^n(d) = \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) \]

the projective space of sections of \( \Omega^1_{\mathbb{P}^n}(d) \). This is the ambient projective space that contains the set of integrable forms \( \mathcal{F}_n(d) \) and the logarithmic components that we will investigate.

6. Definition. Let \( L_n(\mathbf{d}) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) \) denote the set of all logarithmic projective one-forms of type \( \mathbf{d} \) in \( \mathbb{P}^n \), and \( \mathbb{P}L_n(\mathbf{d}) \subset \mathbb{P}^n(d) \) its projectivization. We denote

\[ \mathcal{L}_n(\mathbf{d}) \subset \mathbb{P}^n(d) \]

the Zariski closure of \( \mathbb{P}L_n(\mathbf{d}) \).

If \( \omega \) is a non-zero logarithmic form, the corresponding projective point \( \pi(\omega) \) will be denoted simply by \( \omega \) when the danger of confusion is small.

Let

\[ \Lambda(\mathbf{d}) = \{ (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m / \sum_{i=1}^{m} d_i \lambda_i = 0 \} \]
which is a hyperplane in $\mathbb{C}^m$.

7. **Definition.** Consider the map

$$\mu : V_n(d) := \Lambda(d) \times \prod_{i=1}^{m} S_n(d_i) \to H^0(\mathbb{P}^n, \Omega^{1}_{\mathbb{P}^n}(d))$$

such that

$$\mu((\lambda_1, \ldots, \lambda_m), (F_1, \ldots, F_m)) = \sum_{i=1}^{m} \lambda_i \hat{F}_i \ dF_i$$

and

$$\rho : \mathbb{P}^n(d) := \mathbb{P}\Lambda(d) \times \prod_{i=1}^{m} \mathbb{P}S_n(d_i) \to \mathbb{P}^n(d) = \mathbb{P}H^0(\mathbb{P}^n, \Omega^{1}_{\mathbb{P}^n}(d))$$

such that

$$\rho(\pi(\lambda_1, \ldots, \lambda_m), (\pi(F_1), \ldots, \pi(F_m))) = \pi(\sum_{i=1}^{m} \lambda_i \hat{F}_i \ dF_i).$$

8. **Remark.**
   a) $\mu$ is a multi-linear map. By Proposition 4, the image of $\mu$ is $L_n(d)$.
   b) The induced map $\rho$ from a product of projective spaces into a projective space is only a rational map. Later we will determine the base locus $B(\mu) = \{(\mu(\lambda), F) = 0\}$ of $\mu$. Anyway, it is clear that the image of $\rho$ is $\mathbb{P}L_n(d)$. Hence $L_n(d)$ is the closure of the image of $\rho$. Therefore, $L_n(d)$ is a projective irreducible variety.

5. **Base locus.**

Let $B(\mu) = \mu^{-1}(0)$. Then $B(\mu) \subset V_n(d)$ is an affine algebraic set, and we intend to describe its irreducible components.

Let us remark that the multilinearity of $\mu$ implies that $B(\mu)$ is stable under the natural action of $(\mathbb{C}^\times)^{m+1}$ on $V_n(d)$.

From the multilinearity of $\mu$ it follows that $Z = \{(\lambda, F) \in V_n(d) / \lambda = 0 \text{ or } F_i = 0 \text{ for some } i\}$ is contained in $B(\mu)$. We denote $B = B(\mu) - Z$ and

$$B(\rho) = \pi(B) \subset \mathbb{P}^n(d)$$

the base locus of $\rho$.

An example of a point in the base locus is the following. Suppose $d_1 = \cdots = d_m$. It is then clear that if $F_1 = \cdots = F_m$ then $(\lambda, F) \in B(\mu)$. More generally, each string of equal $d_i$'s gives elements of $B(\mu)$: if $d_i = d_j$ for all $i, j \in A$, where $A \subset \{1, \ldots, m\}$, then taking $F_i = F_j$ for all $i, j \in A$, $\sum_{i \in A} d_i \lambda_i = 0$, $\lambda_j = 0$ for $j \notin A$, we obtain that $(\lambda, F) \in B(\mu)$.

These examples generalize as follows: suppose our $d_i$'s may be written as

$$d_i = \sum_{j=1}^{m'} e_{ij}d'_j, \quad i = 1, \ldots, m, \quad (5.1)$$
where $m' \in \mathbb{N}$, $d'_j \geq 1$ and $e_{ij} \geq 0$ are integers. Let $\lambda \in \Lambda_n(d)$ such that $\sum_{i=1}^{m} e_{ij} \lambda_i = 0$ for $j = 1, \ldots, m'$, and take $F$ such that

$$F_i = \prod_{j=1}^{m'} G^e_{ij}$$

for some $G_j \in S_n(d'_j)$, $j = 1, \ldots, m'$. Then,

$$\sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m'} dG_j / G_j = \sum_{j=1}^{m'} \left( \sum_{i=1}^{m} \lambda_i e_{ij} \right) dG_j / G_j = 0$$

and we obtain elements in the base locus.

We will see now that this construction accounts for all the irreducible components of the base locus.

9. Definition. We denote $F(d)$ the collection of all decompositions of $d$ as in 5.1, that is, let

$$F(d) = \{ (m', e, d') / m' \in \mathbb{N}, e \in \mathbb{N}^{m \times m'}, d' \in (\mathbb{N} - \{0\})^{m'}, d = e \cdot d', e \text{ without zero columns} \}$$

In 5.1, for each $i$ there exists $j$ such that $e_{ij} > 0$; that is, all rows of $e$ are non-zero. This follows from $d_i > 0$. If the $j$-th column of $e$ is zero then in the decomposition 5.1 the terms $e_{ij}d'_j$ are zero and do not contribute, so this zero column may be disregarded.

Let us remark that $F(d)$ is finite: we have, $d = \sum d_i = \sum_{i,j} e_{ij}d'_j \geq \sum_j d'_j \geq m'$, hence $m'$ is bounded. Also, 5.1 implies $e_{ij} \leq d_i/d'_j \leq d_i$, so all $e_{ij}$ are also bounded.

For $\varphi = (m', e, d') \in F(d)$ denote the (Segre-Veronese) map

$$\nu_\varphi : \prod_{j=1}^{m'} S_n(d'_j) \to \prod_{i=1}^{m} S_n(d_i)$$

$$\nu_\varphi(G_1, \ldots, G_{m'}) = (F_1, \ldots, F_m)$$

such that $F_i = \prod_{j=1}^{m'} G^e_{ij}$. Also, let

$$\Lambda(e) = \{ \lambda \in \Lambda(d) / \lambda \cdot e = 0 \}$$

which is a linear subspace of $\mathbb{C}^m$ of dimension $m - \text{rank}(e)$.

Notice that $\lambda \cdot e = 0$ implies $\lambda \cdot d = 0$. For $\varphi \in F(d)$ let

$$B_\varphi = \Lambda(e) \times \text{im } \nu_\varphi \subset V_n(d)$$

By the calculation 5.3 we know that $B_\varphi \subset B(\mu)$ for all $\varphi \in F(d)$.

Each $B_\varphi$ is clearly irreducible. Next we will see, first, that $B(\mu) = \bigcup_{\varphi \in F(d)} B_\varphi$. And, second, we will determine when there are inclusions among the $B_\varphi$’s, thus characterizing the irreducible components of the base locus.

Let us first recall from [14], Lemme 3.3.1, page 102, the following
10. **Proposition.** Let \( F_i \in S_n(d_i), \ i = 1, \ldots, m, \) be irreducible distinct (modulo multiplicative constants) homogeneous polynomials. If \( \lambda_i \in \mathbb{C} \) are such that
\[
\sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = 0
\]
then \( \lambda_i = 0 \) for all \( i \). That is, the rational one-forms \( dF_i/F_1, \ldots, dF_m/F_m \) are linearly independent over \( \mathbb{C} \).

11. **Corollary.** Let \( (\lambda, F) \in V_n(d) \) with the \( F_i \) distinct and irreducible, and \( \lambda \neq 0 \). Then \( (\lambda, F) \notin B(\mu) \).

12. **Proposition.** With the notations above, we have \( B(\mu) = Z \cup \bigcup_{\varphi \in F(d)} B_\varphi \).

**Proof.** Let \( (\lambda, F) \in B = B(\mu) - Z \). Write each \( F_i \) as a product of distinct irreducible homogeneous polynomials:
\[
F_i = \prod_{j=1}^{m'} G_j^{e_{ij}}
\]
We allow some \( e_{ij} = 0 \). Denote \( d'_j \) the degree of \( G_j \). Taking degree we obtain \( d = e d' \).

Repeating the calculation of 5.3 we have
\[
0 = \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m'} e_{ij} \frac{dG_j}{G_j} = \sum_{j=1}^{m'} \sum_{i=1}^{m} \lambda_i e_{ij} \frac{dG_j}{G_j}
\]
(5.4)

Since the \( G_j \) are irreducible, Proposition 10 implies that \( \sum_{i=1}^{m} \lambda_i e_{ij} = 0 \) for all \( j = 1, \ldots, m' \). Therefore, \( (\lambda, F) \in B_\varphi \) with \( \varphi = (m', e, d') \in F(d) \), as claimed. \( \square \)

Regarding possible inclusions among the \( B_\varphi \)'s, we make the following

13. **Definition.** For \( \varphi_1 = (m_1, e_1, d_1), \ \varphi_2 = (m_2, e_2, d_2) \in F(d) \) we write \( \varphi_2 \preceq \varphi_1 \) if \( \text{rank}(e_1) = \text{rank}(e_2) \) and there exists \( e_3 \in \mathbb{N}_{m_1 \times m_2} \) such that \( e_2 = e_1 \cdot e_3 \).

Then we have

14. **Proposition.** For \( \varphi_1, \varphi_2 \in F(d), \ B_{\varphi_2} \subset B_{\varphi_1} \) if and only if \( \varphi_2 \preceq \varphi_1 \).

**Proof.** Suppose \( B_{\varphi_2} \subset B_{\varphi_1} \). Choose an element \( (\lambda, F) \in B_{\varphi_2} \), that is, \( \lambda \cdot e_2 = 0 \) and \( F_i = \prod_{k=1}^{m_2} H_k^{e_{2ik}} \) for all \( i \), for some \( H_k \). We may take this element so that the \( H_k \)'s are irreducible. By our hypothesis, \( (\lambda, F) \in B_{\varphi_1} \) and we also have \( F_i = \prod_{j=1}^{m_1} G_j^{e_{1ij}} \) for all \( i \), for some \( G_j \). By unique factorization and the irreducibility of the \( H_k \), \( G_j = \prod_{k=1}^{m_2} H_k^{e_{jk}} \) for some \( e_{3jk} \in \mathbb{N} \). A simple calculation now gives \( e_2 = e_1 \cdot e_3 \).

Also, the equality \( e_2 = e_1 \cdot e_3 \) just obtained easily implies \( \Lambda(e_1) \subset \Lambda(e_2) \). Since we are assuming \( B_{\varphi_2} \subset B_{\varphi_1} \), we also have \( \Lambda(e_2) \subset \Lambda(e_1) \). Hence \( \Lambda(e_1) = \Lambda(e_2) \), and therefore \( \text{rank}(e_1) = \text{rank}(e_2) \).

Conversely, suppose \( \varphi_2 \preceq \varphi_1 \). Then \( e_2 = e_1 \cdot e_3 \) and \( \text{rank}(e_1) = \text{rank}(e_2) \) imply, as before, that \( \Lambda(e_1) = \Lambda(e_2) \). Also, the condition \( e_2 = e_1 \cdot e_3 \) easily implies that \( \text{im} \nu_{\varphi_2} \subset \text{im} \nu_{\varphi_1} \). Hence \( B_{\varphi_2} \subset B_{\varphi_1} \). \( \square \)
15. **Corollary.** The irreducible components of \( B(\rho) \) are the \( \pi(B_\varphi) \) for \( \varphi \) a maximal element of the finite ordered set \((F(d), \leq)\).

6. **Generic injectivity.**

Suppose \((\lambda, F), (\lambda', F') \in V_n(d)\) are such that \( \mu(\lambda, F) = \mu(\lambda', F') \neq 0 \), that is,

\[
F \sum_{i=1}^{m} \lambda_i \frac{dF_i}{F_i} = \omega = F' \sum_{i=1}^{m} \lambda'_i \frac{dF'_i}{F'_i}.
\]

Next we discuss conditions that imply that \((\lambda, F) = (\lambda', F')\).

Let’s observe that if the partition \( d \) contains repeated \( d_i \)’s then the generic injectivity may hold only up to order. More precisely, suppose \( A \subset \{1, \ldots, m\} \) is such that \( d_i = d_j \) for all \( i,j \in A \). For each permutation \( \sigma \in S_m \) such that \( \sigma(j) = j \) for \( j \notin A \), clearly we have \( \mu(\lambda, F) = \mu(\sigma.\lambda, \sigma.F) \) for all \((\lambda, F) \in V_n(d)\). For \( e \in \mathbb{N} \) let \( A_e = \{i/d_i = e\} \). Then the non-empty \( A_e \) form a partition of \( \{1, \ldots, m\} \). Let \( S(e) = \{\sigma \in S_m/\sigma(j) = j, \forall j \notin A_e\} \) and \( S(d) = \prod S(e) \). Then the subgroup \( S(d) \subset S_m \) acts on \( V_n(d) \) and \( \mu \) is constant on its orbits. By injectivity up to order we will of course mean injectivity of the induced map with domain \( V_n(d)/S(d) \).

16. **Proposition.** The rational map

\[
\rho : \mathbb{P}^n(d) \dashrightarrow \mathcal{L}_n(d) \subset \mathbb{P}^n(d)
\]

as in Definition 7, is generically injective (up to order).

**Proof.** We will prove the existence of a non-empty Zariski open \( U \subset X \) such that \( \rho|U \) is injective morphism (up to order). It is easy to see, using that \( \rho \) is a dominant map of irreducible varieties, that the existence of such a \( U \) implies that there exists a non-empty Zariski open \( V \subset \mathcal{L}_n(d) \) such that \( \rho : \rho^{-1}(V) \to V \) is injective (up to order).

Consider the Zariski open \( S(d) \)-stable \( U \subset V_n(d) \) of points \((\lambda, F)\) such that the \( F_i \) are irreducible and all distinct. Hence, for \((\lambda, F), (\lambda', F') \in U \) distinct (up to order),

\[
F = \prod F_i \neq F' = \prod F'_i.
\]

Suppose \( \mu(\lambda, F) = \omega = \mu(\lambda', F') \neq 0 \). Then \( \omega \) has two integrating factors \( F \) and \( F' \), and therefore has a rational first integral \( f = F/F' \). It follows that \( \omega \) has infinitely many algebraic leaves (the fibers of \( f \)).

On the other hand, if \((\lambda_1 : \cdots : \lambda_m) \in \mathbb{P}^{m-1}(\mathbb{C}) - \mathbb{P}^{m-1}(\mathbb{Q})\), Proposition (3.7.8) from [14] implies that \( \omega \) has only finitely many algebraic leaves.

Let \( U_0 = \{\langle \lambda, F \rangle \in U/\lambda \in \mathbb{P}^{m-1}(\mathbb{C}) - \mathbb{P}^{m-1}(\mathbb{Q})\} \).

Consider the restriction \( \rho : U \to \mathcal{L}_n(d) \) and \( \tilde{\rho} : U/S(d) \to \mathcal{L}_n(d) \) the induced map.

We obtain that if \( \omega = \mu(\lambda, F) \) with \((\lambda, F) \in U_0 \) then \( \tilde{\rho}^{-1}(\omega) = \{\langle \lambda, F \rangle\} \).

This implies, first, that since \( \rho \) has a fiber of dimension zero, \( \dim(U) = \dim(\mathcal{L}_n(d)) \) and the general fiber of \( \rho \) is finite. Also, since the (open analytic) set \( U_0 \) is Zariski dense in \( U \) (because \( \mathbb{C} - \mathbb{Q} \) is dense in \( \mathbb{C} \)), \( U_0 \) is not contained in the branch divisor of \( \tilde{\rho} \) and hence \( \tilde{\rho} \) has degree one, and therefore is birational, as claimed.
7. Derivative of the Parametrization.

With the notation of Definition 7, let
\[(\lambda, F) = ((\lambda_1, \ldots, \lambda_m), (F_1, \ldots, F_m)) \in V_n(d)\]
be a point in the vector space \(V_n(d)\) domain of \(\mu\).

Let \((\lambda', F') = ((\lambda'_1, \ldots, \lambda'_m), (F'_1, \ldots, F'_m)) \in V_n(d)\) represent a tangent vector
\[(\lambda, F) + \epsilon(\lambda', F'), \quad \epsilon^2 = 0,\]
to \(V_n(d)\) at \((\lambda, F)\).

From the multilinearity of \(\mu\) we easily obtain the following formula for its derivative:
\[
d\mu(\lambda, F)(\lambda', F') = \sum_i \lambda'_i \hat{F}_i \, dF_i + \sum_{i \neq k} \lambda_i \, F'_k \hat{F}_{ik} \, dF_i + \sum_i \lambda_i \, \hat{F}_i \, dF'_i \] \quad (7.1)

17. **Remark.** By Proposition 4 b), the image of \(\mu\) is contained in the variety of integrable projective forms \(F_n(d) \subset H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))\). Hence for each \((\lambda, F) \in V_n(d)\) we have an inclusion of vector spaces
\[
\text{im} \, d\mu(\lambda, F) \subset T_{F_n(d)}(\omega) = \{ \alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) / \omega \wedge d\alpha + \alpha \wedge d\omega = 0 \} \] \quad (7.2)
where \(\omega = \mu(\lambda, F)\) and \(T_{F_n(d)}(\omega)\) denotes de tangent space of \(F_n(d)\) at the point \(\omega\).

Our main task in Section 9 will be to show that this inclusion is actually an equality, for a sufficiently general \((\lambda, F) \in V_n(d)\).

18. **Definition.** It is convenient now to introduce the following notation:
- \(\omega = \mu(\lambda, F) = \sum_{i=1}^m \lambda_i \hat{F}_i \) (a logarithmic one-form),
- \(\eta = \omega / F = \sum_{i=1}^m \lambda_i \, dF_i / F_i\) (the corresponding rational logarithmic one-form),
- \(\alpha = d\mu(\lambda, F)(\lambda', F') = \sum_i \lambda'_i \hat{F}_i \, dF_i + \sum_{i \neq k} \lambda_i \, F'_k \hat{F}_{ik} \, dF_i + \sum_i \lambda_i \, \hat{F}_i \, dF'_i\),
- \(\beta = \alpha / F = \sum_i \lambda'_i \, dF_i / F_i + \sum_{i \neq k} \lambda_i \, F'_k / F_k \, dF_i / F_i + \sum_i \lambda_i \, dF'_i / F_i\).

19. **Proposition.** With the notations above, we have
\[
\beta = \eta' + (G/F)\eta + d(H/F) \]
where
\[
\eta' = \sum_{i=1}^m \lambda'_i \, dF_i / F_i, \quad G = \sum_{i=1}^m \hat{F}_i \, F'_i / F_i \in S_n(d), \text{ and} \\
H = \sum_{i=1}^m \lambda_i \, \hat{F}_i \, F'_i / F_i \in S_n(d). \]

**Proof.** We add and subtract to \(\beta\) the sum \(\sum_i \lambda_i \, F'_i / F_i^2 \, dF_i\). A straightforward calculation gives the proposed expression. □
8. Singular ideals of logarithmic one-forms and their resolution.

For $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^{1}(d))$ denote $S(\omega) \subset \mathbb{P}^n$ the scheme of zeros of $\omega$ and $\mathcal{I} = \mathcal{I}_\omega \subset \mathcal{O}_{\mathbb{P}^n}$ the corresponding ideal sheaf. Considering $\omega$ as a morphism $\mathcal{O}_{\mathbb{P}^n} \to \Omega_{\mathbb{P}^n}^{1}(d)$, $\mathcal{I}$ is defined as the image of the dual morphism $T_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n}$. Also, if $\omega = \sum_{i=0}^{n} a_i dx_i$ then $\mathcal{I}$ corresponds to the homogeneous ideal generated by $a_0, \ldots, a_n \in S_n(d-1)$.

We keep the notation of Definitions 2 and 3.

Let $(\lambda, F) \in V_n(d)$ and $\omega = F \sum_{i=1}^{m} \lambda_i dF_i/F_i = \sum_{i=1}^{m} \lambda_i \tilde{F}_i dF_i$ the corresponding logarithmic one-form.

We denote $X_i = \{ x \in \mathbb{P}^n / F_i(x) = 0 \}$ the hypersurface defined by $F_i$.

For $i \neq j$,

$$X_{ij} = X_i \cap X_j = \{ x \in \mathbb{P}^n / F_i(x) = F_j(x) = 0 \}$$

and, more generally, for a subset $A \subset \{1, \ldots, m\}$,

$$X_A = \bigcap_{i \in A} X_i$$

For $1 \leq r \leq m$ we write

$$X^{(r)} = \bigcup_{|A|=r} X_A$$

and we shall use especially the following particular cases

$$X^{(1)} = \bigcup_{i=1}^{m} X_i, \quad X^{(2)} = \bigcup_{i<j} X_{ij}, \quad X^{(3)} = \bigcup_{i<j<k} X_{ijk}.$$  

20. Remark. For our purposes we will be able to assume that the $F_i \in S_n(d_i)$ are general. We shall assume, more precisely, that each $F_i$ is smooth irreducible and that $X^{(1)}$ is a normal crossings divisor. Hence, each $X_A$ is a smooth complete intersection of codimension $|A|$, and thus the strata $X^{(r)}$ are of codimension $r$, singular only along $X^{(r+1)}$.

It is shown in [8] and [3] that for $\omega$ logarithmic as above, with all $\lambda_i \neq 0$,

$$S(\omega) = X^{(2)} \cup P$$

with $P \subset \mathbb{P}^n - X^{(1)}$ closed, and $P$ is a finite set if $\omega$ is general. Let’s revisit the argument, under the assumptions of Remark 20. First, since clearly $\tilde{F}_i$ vanishes on $X^{(2)}$ for all $i$, we have $X^{(2)} \subset S(\omega)$. Since $\omega = \lambda_i \tilde{F}_i dF_i$ on $X_i$, we see that $(X^{(1)} - X^{(2)}) \cap S(\omega) = \emptyset$. As for the zeros of $\omega$ in the complement of $X^{(1)}$, they are the same as the zeros of $\eta = \omega/F = \sum_{i=1}^{m} \lambda_i dF_i/F_i$, which is a section of the locally free sheaf $E = \Omega_{\mathbb{P}^n}^{1}(\log X^{(1)})$ of rank $n$ (see [9], [12], [15], [11]). Considering the $F_i$ (hence the divisor $X^{(1)}$) as fixed, the space of global sections of $E$ has dimension $m - 1$, and these sections correspond...
bijectively with the residues \((\lambda_1, \ldots, \lambda_m)\), satisfying \(\sum_i d_i \lambda_i = 0\), as it follows from taking cohomology in the exact sequence \([9]\) or \([11]\), p. 170:
\[
0 \to \Omega^1_{\mathbb{P}^n} \to E \to \oplus_{i=1}^m \mathcal{O}_{X_i} \to 0.
\]

For general \((\lambda_1, \ldots, \lambda_m)\) as above, the corresponding section \(\eta\) of \(E\) has a finite set \(P\) of simple zeros. Further, the cardinality of \(P\) (see \([8]\)) is the degree of the top Chern class \(c_n(E)\), computable from the exact sequence above.

Coming back to the study of the resolution of the ideal \(I_\omega\), let us denote
\[
\mathcal{J}^{(r)} = I(X^{(r)}) \subset \mathcal{O}_{\mathbb{P}^n}
\]
the ideal sheaf of regular functions vanishing on \(X^{(r)}\), and
\[
J^{(r)} = \bigoplus_{k \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{J}^{(r)}(k)) \subset S_n
\]
the corresponding saturated homogeneous ideal.

Our arguments to prove stability of logarithmic forms will rely on the following results regarding the ideals \(J^{(2)}\).

21. **Proposition.** Under the hypothesis of Remark 20,

a) \(J^{(2)}\) is generated by \(\{\hat{F}_i, \ 1 \leq i \leq m\}\).

b) The relations among the generators of a) are generated by
\[
F_j \hat{F}_j - F_i \hat{F}_i, \quad 1 \leq i < j \leq m,
\]
and also by the subset
\[
R_j = F_j \hat{F}_j - F_1 \hat{F}_1, \quad 2 \leq j \leq m.
\]

c) We have a resolution of \(J^{(2)}\)
\[
0 \to \mathcal{O}(-d)^{m-1} \xrightarrow{\delta_0} \bigoplus_{1 \leq i \leq m} \mathcal{O}(-\hat{d}_i) \xrightarrow{\delta_1} J^{(2)} \to 0
\]
where, denoting \(\{e_i\}\) the respective canonical basis,
\[
\delta_0(e_j) = F_j e_j - F_1 e_1 \quad \text{for} \quad 2 \leq j \leq m,
\]
\[
\delta_1(e_i) = \hat{F}_i \quad \text{for} \quad 1 \leq i \leq m.
\]

**Proof.** a) We are assuming that the \(F_i\) are generic. This implies in particular that each ideal \(< F_i, F_j >\) is prime. Then, \(J^{(2)} = \bigcap_{1 \leq i < j \leq m} < F_i, F_j >\). Let us denote \(J = < \hat{F}_1, \ldots, \hat{F}_m >\). It is clear that \(J \subset J^{(2)}\). We shall prove that \(J^{(2)} \subset J\) by induction on \(m\). The case \(m = 2\) is trivial. The inductive hypothesis, applied to \(F_1, \ldots, F_{m-1}\), may be written as \(\bigcap_{1 \leq i < j \leq m-1} < F_i, F_j > \subset < \hat{F}_1, \ldots, \hat{F}_{m-1} >\). Take an element \(G \in \bigcap_{1 \leq i < j \leq m} < F_i, F_j > = \bigcap_{1 \leq i < j \leq m} < F_i, F_j > \cap \bigcap_{1 \leq i < m} < F_i, F_m >\). Using the inductive hypothesis, we may write \(G = \sum a_i \hat{F}_i\), and we also have \(G \in < F_i, F_m >\) for \(i < m\). Since \(\hat{F}_{jm} \in < F_i, F_m >\) for \(j \neq i\), it follows that \(a_i \hat{F}_{im} \in < F_i, F_m >\) for \(i < m\).
Since \( \langle F_i, F_m \rangle \) is prime, we have \( a_i = b_i F_i + c_i F_m \). Then, \( G = \sum_{i<m} (b_i F_i + c_i F_m) \hat{F}_{im} = \sum_{i<m} (b_i \hat{F}_m + c_i \hat{F}_i) \in J \), as wanted.

b) and c) Using the relations \( R_j \) of b) we write down the complex in c). The proof will be complete if we show that this complex is exact. The surjectivity of \( \delta_1 \) follows from a). Looking at the matrix of \( \delta_0 \) it is easy to see that the determinant of the minor obtained by removing row \( j \) is precisely \( \hat{F}_j \), for \( j = 1, \ldots, m \). Then this complex is the one associated to the maximal minors of a matrix of size \( m \times m - 1 \). Since in our case, by a), the ideal of minors vanishes in codimension two, the complex is exact (see [1] (5), [10] (20.4)). \( \square \)

22. Remark. Let \( X \) be an algebraic variety, \( J \subset \mathcal{O}_X \) a sheaf of ideals, and \( E \) a locally free sheaf on \( X \). Let \( Y \subset X \) denote the subvariety corresponding to \( J \). Taking global sections on the exact sequence \( 0 \to E \otimes J \to E \to E \otimes \mathcal{O}_Y = E|_Y \to 0 \) we obtain an identification of \( H^0(X, E \otimes J) \) with the global sections of \( E \) vanishing on \( Y \), that is, with the kernel of the restriction map \( H^0(X, E) \to H^0(Y, E|_Y) \).

23. Proposition. Let \( \alpha \in \Omega^1_n(d) \) be a 1-form of degree \( d \) in \( \mathbb{C}^{n+1} \). Denote \( \bar{X}^{(2)} \subset \mathbb{C}^{n+1} \) the cone over \( X^{(2)} \).

a) \( \alpha \) vanishes on \( \bar{X}^{(2)} \) if and only if it may be written as

\[
\alpha = \sum_{i=1}^m \hat{F}_i \alpha_i
\]

for some \( \alpha_i \in \Omega^1_n(d_i) \).

b) \( \alpha \) is projective (see Section 2) and vanishes on \( X^{(2)} \) if and only if it may be written as

\[
\alpha = \sum_{i=1}^m \lambda_i' \hat{F}_i dF_i + \sum_{i=1}^m \hat{F}_i \gamma_i
\]

where \( \lambda_i' \in \mathbb{C} \), \( \sum_{i=1}^m d_i \lambda_i' = 0 \) and \( \gamma_i \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d_i)) \) are projective 1-forms of respective degrees \( d_i \).

Proof. a) By Remark 22, we need to determine \( H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d) \otimes J^{(2)}) \). The stated result then follows from Proposition 21 c), by tensoring with \( \Omega^1_{\mathbb{P}^n}(d) \) and taking global sections.
b) Suppose \( \alpha \) is also projective, that is, \( \langle R, \alpha \rangle = 0 \), where \( R \) is the radial vector field. From a) we have

\[
\sum_{i=1}^m \hat{F}_i < R, \alpha_i >= 0.
\]

This is a relation among the \( \hat{F}_i \) with coefficients \( < R, \alpha_i > \) homogeneous of degrees \( d_i \). By Proposition 21 c), by tensoring with \( \mathcal{O}_{\mathbb{P}^n}(d) \) and taking global sections, this relation
is a linear combination of the relations $R_i$ of Proposition 21 b), that is,

$$< R, \alpha_1 >, \ldots, < R, \alpha_m >) = \sum_{2 \leq i \leq m} a_i R_i.$$

This means that

$$< R, \alpha_1 > = (\sum_j a_j) F_1, \quad < R, \alpha_i > = -a_i F_i, \quad i = 2, \ldots, m.$$

Hence $a_i$ has degree zero, i. e. $a_i \in \mathbb{C}$, for all $i$. Define $\lambda_i' = a_i/d_i$ for $i = 2, \ldots, m$, $\lambda_1' = -(\sum_j a_j)/d_1$ and $\gamma_i = \alpha_i - \lambda_i'd_F_i$. It follows that $< R, \gamma_i > = 0$ and hence $\alpha$ may be written as stated.

9. **Surjectivity of the derivative and main Theorem.**

As in Remark 17 we denote the derivative of $\mu$ at the point $\mu(\lambda, F)$

$$d\mu(\lambda, F) : V_n(d) \to T(\omega)$$

where $\omega = \mu(\lambda, F)$ and

$$T(\omega) = T_{F_n(d)}(\omega) = \{ \alpha \in H^n(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d)) / \omega \wedge d\alpha + \alpha \wedge d\omega = 0 \}$$

denotes the Zariski tangent space of $F_n(d)$ at the point $\omega$.

Our main objective is to prove the following:

24. **Theorem.** Let $n, d, m$ and $d \in P(m, d)$ be as in Definition 1. Suppose $n \geq 3$. Then the derivative $d\mu(\lambda, F) : V_n(d) \to T(\omega)$ is surjective for $(\lambda, F) \in V_n(d)$ general.

**Proof.** The proof will be obtained through various steps, including several Propositions of independent interest.

25. **Theorem.** If $n \geq 3$, the set of logarithmic forms $L_n(d) \subset F_n(d)$, as in Definition 6, is an irreducible component of $F_n(d)$. Furthermore, the scheme $F_n(d)$ is reduced generically along $L_n(d)$.

**Proof.** Follows from Theorem 24 by the same arguments as in [6] or [7].

Let us now start with several steps towards the proof of Theorem 24.
26. **Remark.** A typical element $\alpha$ in the image of $d\mu(\lambda, F)$ as in 7.1

$$\alpha = \sum_i \lambda'_i \tilde{F}_i \, dF_i + \sum_{i \neq j} \lambda_i \tilde{F}_{ij} \, dF_i + \sum_i \lambda_i \tilde{F}_i \, dF'_i$$

may be written

$$\alpha = \sum_i \tilde{F}_i (\lambda'_i \, dF_i + \lambda_i \, dF'_i) + \sum_{i \neq j} \tilde{F}_{ij} (\lambda_i \tilde{F}'_j \, dF_i + \lambda_j \tilde{F}'_i \, dF_j)$$

or

$$\alpha = \sum_i \tilde{F}_i (\lambda'_i \, dF_i + \lambda_i \, dF'_i) + \sum_{i < j} \tilde{F}_{ij} (\lambda_i \tilde{F}'_j \, dF_i + \lambda_j \tilde{F}'_i \, dF_j)$$

Let us observe that the first sum is zero on $X^{(2)}$ (hence on $X^{(3)}$) and the second sum is zero on $X^{(3)}$. The idea of our proofs, leading to Theorem 24, will be based on this observation.

Our strategy to characterize the elements $\alpha \in T(\omega)$ will be this: first we shall determine $\alpha|_{X^{(3)}}$, next we shall determine $\alpha|_{X^{(2)}}$, and finally we show that $\alpha$ may be written as in 7.1 for some $\lambda'$ and $F'$, and therefore $\alpha$ belongs to the image of $d\mu(\lambda, F)$.

In order to carry out this plan, let us start with some Propositions, some of them of independent interest.

27. **Proposition.** For $\omega \in F_n(d)$ and $\alpha \in H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d))$, the following conditions are equivalent:

a) $\omega \wedge d\alpha + \alpha \wedge d\omega = 0$, that is, $\alpha \in T(\omega)$.

b) $d\omega \wedge d\alpha = 0$.

Further, for $\omega$ logarithmic, $\eta = \omega/F$ and $\beta = \alpha/F$,

c) $\eta \wedge d\beta = 0$.

d) $d(\eta \wedge \beta) = 0$.

Proof. From a) one obtains b) by applying exterior derivative. Conversely, from b) one obtains a) by contracting with the radial vector field. The equivalence with c) follows from Proposition 5 by a straightforward calculation. The equivalence of c) and d) follows from the fact that $\eta$ is closed. \qed

28. **Proposition.** Let $\omega = \mu(\lambda, F)$ be a logarithmic form and $\alpha \in T(\omega)$. Assume that $X^{(1)}$ is normal crossings, with smooth irreducible components $X_i$, as in Remark 20. Then $\alpha|_{X^{(3)}} = 0$, that is, $\alpha(x) = 0$ for all $x \in X^{(3)}$.

Proof. Let us denote, for $1 \leq i < j \leq m$,

$$U_{ij} := X_{ij} - X^{(3)} = \{x \in \mathbb{P}^n/F_i(x) = F_j(x) = 0, \ F_k(x) \neq 0 \text{ for } k \notin \{i, j\}\}$$

and, similarly, for $1 \leq i < j < k \leq m$,

$$U_{ijk} := X_{ijk} - X^{(4)}$$
Since the set of zeros of \( \alpha \) is closed, it is enough to see that \( \alpha \) is zero on \( X^{(3)} - X^{(4)} \), which is the disjoint union of the \( U_{ijk} \). Notice that \( dF_i, dF_j, dF_k \) are linearly independent on \( U_{ijk} \) because of the normal-crossings hypothesis. Since clearly \( \omega|_{X^{(2)}} = 0 \), the relation \( \omega \wedge d\alpha + \alpha \wedge d\omega = 0 \) reduces to \( \alpha(x) \wedge d\omega(x) = 0 \) for each \( x \in X^{(2)} \). We may assume that \( \lambda_i \neq \lambda_j \) for \( i \neq j \) without losing generality. Then it follows from Proposition 5 a) that
\[
\alpha \wedge dF_i \wedge dF_j = 0 \quad (9.3)
\]
on \( U_{ij} \), and hence on its closure \( X_{ij} \). This means that
\[
\alpha(x) \in \mathbb{C}.dF_i(x) + \mathbb{C}.dF_j(x) \subset \Omega^1_{\mathbb{P}^n}(x) \quad (9.4)
\]
for \( x \in X_{ij} \). Therefore, for \( x \in U_{ijk} \) we have
\[
\alpha(x) \in (\mathbb{C}.dF_i(x) + \mathbb{C}.dF_j(x)) \cap (\mathbb{C}.dF_i(x) + \mathbb{C}.dF_k(x)) \cap (\mathbb{C}.dF_j(x) + \mathbb{C}.dF_k(x)).
\]
Due to the normal crossings hypothesis this last intersection of two-dimensional subspaces is zero, hence \( \alpha(x) = 0 \) for \( x \in U_{ijk} \), as wanted. \( \square \)

29. Proposition. With the notation and hypothesis of Proposition 28, for each ordered pair \((i, j)\) with \( 1 \leq i, j \leq m \) and \( i \neq j \), there exists \( \alpha_{ij} \in S_m(d_j) \) such that
\[
\alpha = \tilde{F}_{ij} (A_{ij} \, dF_i + A_{ji} \, dF_j) \text{ on } X_{ij}.
\]

Proof. This will follow easily combining that \( X_{ij} \) is a smooth complete intersection of codimension two in a projective space, and the fact that \( \alpha|_{X^{(3)}} = 0 \) that we just proved.

Suppose \( J = \langle A, B \rangle \) is the ideal generated by general homogeneous polynomials \( A \) and \( B \) of respective degrees \( a \) and \( b \). Let \( Y \subset \mathbb{P}^n \) be the set of zeroes of \( J \). We have an exact sequence \((13), II.8) \)
\[
0 \to J/J^2 = \mathcal{O}_Y(-a) \oplus \mathcal{O}_Y(-b) \xrightarrow{\delta} \Omega^1_{\mathbb{P}^n}|_Y \to \Omega^1_Y \to 0
\]
Tensoring with \( \mathcal{O}_Y(d) \) and taking global sections we obtain that an element \( \alpha|_Y \in H^0(Y, \Omega^1_{\mathbb{P}^n}(d)|_Y) \) which belongs to the image of \( H^0(\delta) \), may be written as \( A'dA + B'dB \) for \( A' \in H^0(Y, \mathcal{O}_Y(d-a)) \) and \( B' \in H^0(Y, \mathcal{O}_Y(d-b)) \). By \( [13] \), Ex. III (5.5), \( A' \) and \( B' \) are represented by homogeneous polynomials of respective degrees \( d-a \) and \( d-b \).

For each \((i, j)\), \( \alpha|_{X_{ij}} \) belongs to the image of the corresponding \( H^0(\delta) \), by 9.4. Hence, we know that \( \alpha = A_{ij}' \, dF_i + A_{ji}' \, dF_j \) on \( X_{ij} \), for homogeneous polynomials \( A_{ij}' \) of degree \( d - d_i \). Now, \( \alpha|_{X^{(3)}} = 0 \) by Proposition 28, and in particular \( \alpha = 0 \) on \( X_{ijk} \) for all \( k \). Since \( dF_i \) and \( dF_j \) are linearly independent at all points of \( X_{ijk} \) by the normal crossings hypothesis, it follows that \( A_{ij}' \) and \( A_{ji}' \) are divisible by \( \tilde{F}_{ij} \) and we obtain the claim. \( \square \)

30. Corollary. With the notation of Proposition 29, define
\[
\alpha' = \sum_{i<j} \tilde{F}_{ij} (A_{ij} \, dF_i + A_{ji} \, dF_j) \in \Omega^1_n(d)
\]
Then \( \alpha'|_{X^{(2)}} = \alpha|_{X^{(2)}}. \)
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(But notice that \( \alpha' \) may not satisfy 7.2; see the Proof of Corollary 35).

**Proof.** Follows from Proposition 29 since \( \hat{F}_{ij} \) vanishes on \( X_{hk} \) if \( \{h, k\} \neq \{i, j\} \). \( \square \)

31. **Corollary.** We keep the notation of Proposition 29. Then any \( \alpha \in T(\omega) \) may be written as

\[
\alpha = \sum_{i<j} \hat{F}_{ij} (A_{ij} \ dF_i + A_{ji} \ dF_j) + \sum_i \hat{F}_i \ \alpha_i
\]

for some \( \alpha_i \in \Omega^1_n(d_i) \).

**Proof.** For \( \alpha \in T(\omega) \), take \( \alpha' \) as in Corollary 30. Then \( \alpha - \alpha' \in \Omega^1_n(d) \) vanishes on \( \tilde{X}^{(2)} \) and hence, by Proposition 23 a), may be written as \( \sum_{i=1}^m \hat{F}_i \alpha_i \) for some \( \alpha_i \in \Omega^1_n(d_i) \). \( \square \)

We would like to obtain further information on the \( A_{ij} \)'s and the \( \alpha_i \)'s. For this, we will use again that \( \alpha \) satisfies \( \omega \wedge d\alpha + \alpha \wedge d\omega = 0 \) as in 7.2.

32. **Proposition.** Suppose \( n \geq 3 \). With notation as in Corollary 31, for each \( j = 1, \ldots, m \) there exists \( F'_j \in S_n(d_j) \) such that

\[
A_{ij} = \lambda_i \ F'_j \text{ on } X_{ij}
\]

for all \( (i, j) \) with \( 1 \leq i, j \leq m \) and \( i \neq j \).

**Proof.** The calculation is nicer working with the equivalent condition \( d\beta \wedge \eta = 0 \), where \( \beta = \alpha/F \) and \( \eta = \omega/F \); see Proposition 27 c). We have:

\[
\beta = \sum_{i \neq j} \frac{A_{ij}}{F_j} \frac{dF_i}{F_i} + \sum_i \frac{\alpha_i}{F_i}
\]

\[
d\beta = \sum_{i \neq j} \frac{d(A_{ij})}{F_j} \wedge \frac{dF_i}{F_i} + \sum_i \frac{d(\alpha_i)}{F_i}
\]

\[
d\beta \wedge \eta = \sum_{i \neq j, k} \lambda_k \ d\left( \frac{A_{ij}}{F_j} \right) \wedge \frac{dF_i}{F_i} \wedge \frac{dF_k}{F_k} + \sum_{i, k} \lambda_k \ d\left( \frac{\alpha_i}{F_i} \right) \wedge \frac{dF_k}{F_k} =
\]

\[
\sum_{i \neq j, k} \lambda_k \ d\left( \frac{A_{ij}}{F_j} \right) \wedge \frac{dF_i}{F_i} \wedge \frac{dF_k}{F_k} + \sum_{i \neq j} \lambda_j \ d\left( \frac{A_{ij}}{F_j} \right) \wedge \frac{dF_i}{F_i} \wedge \frac{dF_j}{F_j} +
\]

\[
\sum_{i \neq k} \lambda_k \ d\left( \frac{\alpha_i}{F_i} \right) \wedge \frac{dF_k}{F_k} + \sum_k \lambda_k \ d\left( \frac{\alpha_k}{F_k} \right) \wedge \frac{dF_i}{F_i} = 0
\]

Let’s replace

\[
d\left( \frac{A_{ij}}{F_j} \right) = \frac{dA_{ij}}{F_j} - \frac{A_{ij} \ dF_j}{F_j}, \ d\left( \frac{\alpha_i}{F_i} \right) = \frac{d\alpha_i}{F_i} - \frac{dF_i}{F_i} \wedge \frac{\alpha_i}{F_i}
\]
and multiply by $F^2$. After some straightforward calculation we obtain:

$$
F \sum_{i \neq j \neq k} \lambda_k \hat{F}_{ijk} dA_{ij} \wedge dF_i \wedge dF_k + \sum_{i \neq k} \lambda_k \hat{F}_{ik} dA_{ik} \wedge dF_i \wedge dF_k + \sum_{i \neq j \neq k} \lambda_k \hat{F}_{ijk} A_{ij} dF_i \wedge dF_j \wedge dF_k + F \sum_{j \neq k} \lambda_k \hat{F}_{jk} d\alpha_j \wedge dF_k + \sum_k \lambda_k \hat{F}_k^2 d\alpha_k \wedge dF_k + \sum_{j \neq k} \lambda_k \hat{F}_{jk} \alpha_j \wedge dF_j \wedge dF_k = 0
$$

Now we choose $r$ such that $1 \leq r \leq m$ and restrict to $X_r$, that is, we reduce modulo $F_r$. We get:

$$
\hat{F}_r \left( \sum_{i \neq r} \lambda_r \hat{F}_{ir} dA_{ir} \wedge dF_i \wedge dF_r + \sum_{i \neq k \neq r} \lambda_k \hat{F}_{irk} A_{ir} dF_i \wedge dF_r \wedge dF_k + \lambda_r \hat{F}_r d\alpha_r \wedge dF_r + \sum_{k \neq r} \lambda_k \hat{F}_{rk} \alpha_r \wedge dF_r \wedge dF_k \right) = 0 \quad (9.5)
$$

Since $\hat{F}_r$ is not zero on the irreducible variety $X_r$, we may cancel this factor out.

Next, choose $s$ such that $1 \leq s \leq m$, $s \neq r$, and further restrict to $X_r \cap X_s = X_{rs}$ to obtain:

$$
\lambda_r \hat{F}_{sr} dA_{sr} \wedge dF_s \wedge dF_r + \sum_{k \neq r \neq s} \lambda_k \hat{F}_{srk} A_{sr} dF_s \wedge dF_r \wedge dF_k + \sum_{i \neq r \neq s} \lambda_s \hat{F}_{irs} A_{sr} dF_i \wedge dF_r \wedge dF_s + \lambda_s \hat{F}_{rs} \alpha_r \wedge dF_r \wedge dF_s = 0 \quad (9.6)
$$

And, once more, choose $t$ such that $1 \leq t \leq m$, $t \neq s \neq r$. Restricting to $X_r \cap X_s \cap X_t = X_{rst}$ we get:

$$
\hat{F}_{rst}(\lambda_t A_{st} - \lambda_s A_{tr}) dF_r \wedge dF_s \wedge dF_t = 0
$$

By the genericity of the $F_i$’s, $X_{rst}$ is irreducible, and we may cancel out the factor $\hat{F}_{rst} \neq 0$. By the normal crossing hypothesis we may also cancel out $dF_r \wedge dF_s \wedge dF_t \neq 0$.

Therefore,

$$A_{sr}/\lambda_s = A_{tr}/\lambda_t \quad \text{on } X_{rst} \quad (9.7)$$

for all distinct $1 \leq r, s, t \leq m$.

Let us fix $r$, $1 \leq r \leq m$. We consider the natural restriction maps

$$S_n(d_r) = H^0(\mathbb{P}^n, \mathcal{O}(d_r)) \rightarrow H^0(X_r, \mathcal{O}(d_r)) \rightarrow H^0(X_{rs}, \mathcal{O}(d_r)) \rightarrow H^0(X_{rst}, \mathcal{O}(d_r)).$$

For $s = 1, \ldots, m$, $s \neq r$, the polynomials $A_{sr}/\lambda_s \in S_n(d_r)$ (all of the same degree $d_r$) define, by restriction to the hypersurfaces $X_{rs} \subset X_r$, sections $A_{sr}/\lambda_s \in H^0(X_{rs}, \mathcal{O}(d_r))$. By 9.7 these sections coincide on the pairwise intersections $X_{rs} \cap X_{rt} = X_{rst}$. Hence
this collection defines a section of $\mathcal{O}(d_r)$ on the (reducible) variety $D_r = \cup_{s \neq r} X_{rs} \subset X_r$. By Lemma 33 below, with $X = X_r$ and $D = D_r$, there exists $F'_r \in S_n(d_r)$, such that $A_{sr}/\lambda_s = F'_r$ on $X_{rs}$, for each $s \neq r$, as claimed.

33. **Lemma.** Let $n \geq 3$, and let $X \subset \mathbb{P}^n$ be a smooth irreducible hypersurface of degree $e$. For $m \geq 1$ and $i = 1, \ldots, m$ let $D_i \subset X$ be smooth irreducible distinct hypersurfaces. We consider the (reducible) hypersurface $D = \cup_{1 \leq i \leq m} D_i \subset X$. Then the natural restriction map

$$H^0(X, \mathcal{O}(e)) \rightarrow H^0(D, \mathcal{O}(e))$$

is surjective.

**Proof.** In the exact sequence $0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0$ we tensor by $\mathcal{O}_X(e)$ and take cohomology. Since $\mathcal{O}_X(-D)(e) = \mathcal{O}_X(-d)(e) = \mathcal{O}_X(e - d)$ for some $d$, and $H^1(X, \mathcal{O}_X(e - d)) = 0$ (see e. g. [13], Exercise III, (5.5)), we obtain the claim. 

34. **Corollary.** Let $n \geq 3$. Any $\alpha \in T(\omega)$ may be written as

$$\alpha = \sum_{i \neq j} \lambda_i \tilde{F}_{ij} F'_j \, dF_i + \sum_i \tilde{F}_i \, \alpha_i.$$ 

for some $F'_i \in S_n(d_i)$ and $\alpha_i \in \Omega^1_n(d_i)$.

**Proof.** Follows from Corollary 31 and Proposition 32.

35. **Corollary.** Let $n \geq 3$. Any $\alpha \in T(\omega)$ may be written as

$$\alpha = \bar{\alpha} + \sum_i \tilde{F}_i \, \gamma_i,$$

where $\bar{\alpha}$ belongs to the image of $d\mu(\lambda, \mathbf{F})$, $\gamma_i \in \Omega^1_n(d_i)$ and $\sum_i \tilde{F}_i \, \gamma_i \in T(\omega)$.

**Proof.** Using Corollary 34, then adding and substracting $\sum_i \lambda_i \tilde{F}_i \, dF'_i$, we have:

$$\alpha = \sum_{i \neq j} \lambda_i \tilde{F}_{ij} F'_j \, dF_i + \sum_i \tilde{F}_i \, \alpha_i$$

$$= \sum_{i \neq j} \lambda_i \tilde{F}_{ij} F'_j \, dF_i + \sum_i \lambda_i \tilde{F}_i \, dF'_i + \sum_i \tilde{F}_i \, (\alpha_i - \lambda_i \, dF'_i)$$

$$= d\mu(\lambda, \mathbf{F})(0, \mathbf{F}') + \sum_i \tilde{F}_i \, \gamma_i$$

taking $\gamma_i = \alpha_i - \lambda_i \, dF'_i$. Since $\alpha, \bar{\alpha} \in T(\omega)$, we have $\alpha - \bar{\alpha} = \sum_i \tilde{F}_i \gamma_i \in T(\omega)$, as claimed.
36. Remark. Corollary 35 implies that to prove Theorem 24 we are reduced to showing that any \( \alpha \in T(\omega) \) of the form \( \alpha = \sum_i \hat{F}_i \gamma_i \), with \( \gamma_i \in \Omega^1_n(d_i) \), belongs to the image of \( d\mu(\lambda, F) \).

To this end, let us first prove the following

37. Proposition. Let \( \alpha \in T(\omega) \) be of the form

\[
\alpha = \sum_j (\hat{F}_j)^e \gamma_j
\]

(9.8)

with \( e \in \mathbb{N}, e \geq 1 \), and \( \gamma_j \in \Omega^1_n(d - ed_j) \). Then, for \( 1 \leq i, j \leq m \), \( i \neq j \), there exist \( \lambda_j \in \mathbb{C}, D_{ij} \in S_n(d_j - ed_j) \) and \( \epsilon_j \in \Omega^1_n(d_j - ed_j) \), such that

\[
\gamma_j = \lambda_j \ dF_j + \sum_{i \neq j} \hat{F}_{ij} D_{ij} \ dF_i + \hat{F}_j \ \epsilon_j
\]

for \( j = 1, \ldots, m \). In case \( e \geq 2 \), all \( \lambda_j = 0 \).

Proof. Let us use once more that \( \alpha \) satisfies \( 7.2 \ \omega \wedge da + \alpha \wedge d\omega = 0 \). We may apply to our present \( \alpha \) the calculation in the Proof of Proposition 32, with \( A_{ij} = 0 \) and \( \alpha_j = (\hat{F}_j)^{e-1} \gamma_j \), for all \( i, j \). Then it follows from equation 9.6 that

\[
\gamma_j \wedge dF_i \wedge dF_j = 0 \quad \text{on } X_{ij}, \quad \text{for all } i \neq j,
\]

since \( \lambda_j \neq 0 \), and \( \hat{F}_{ij} \neq 0 \) on \( X_{ij} \). Then,

\[
\gamma_j = B_{ij} dF_i + C_{ij} dF_j \quad \text{on } X_{ij}
\]

for some \( B_{ij} \in S_n(d - ed_j - d_i) \) and \( C_{ij} \in S_n((1 - e)d_j) \). Notice that \( C_{ij} \in S_n(0) = \mathbb{C} \) if \( e = 1 \), and \( C_{ij} = 0 \) if \( e \geq 2 \), since \( (1 - e)d_j < 0 \).

Now we fix \( j \) and vary \( i \neq j \). On \( X_{ij} \cap X_{kj} = X_{ijk} \) we have \( B_{ij} dF_i + C_{ij} dF_j = B_{kj} dF_k + C_{kj} dF_j \). From the normal crossings hypothesis we obtain, for all \( i \neq k \):

a) \( B_{ij} = B_{kj} = 0 \) on \( X_{ijk} \), and

b) \( C_{ij} = C_{kj} \)

From b), \( C_{ij} \) does not depend on \( i \) and we may denote \( C_{ij} = \lambda_j' \). As noticed above, \( C_{ij} = \lambda_j' = 0 \) in case \( e \geq 2 \).

On the other hand, a) implies that \( B_{ij} = \hat{F}_{ij} D_{ij} \) on \( X_{ij} \) for some \( D_{ij} \in S_n(d_j - ed_j) \). Therefore,

\[
\gamma_j = \lambda_j' dF_j + \hat{F}_{ij} D_{ij} dF_i \quad \text{on } X_{ij}
\]

for all \( j \) and all \( i \neq j \). Let \( \gamma_j' = \gamma_j - (\lambda_j' dF_j + \sum_{i \neq j} \hat{F}_{ij} D_{ij} dF_i) \in \Omega^1_n(d - ed_j) \). Then \( \gamma_j' \) is zero on \( D_j = \bigcup_{i \neq j} X_{ij} \subset X_j \), hence there exists \( \epsilon_j \in \Omega^1_n(d_j - ed_j) \) such that \( \gamma_j' = \hat{F}_j \ \epsilon_j \) on \( X_j \). Denoting \( J_j \cong O(-d_j) \) the ideal sheaf of \( X_j \), we have \( H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(d_j)(J_j)) \cong H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 0 \). Therefore the equality \( \gamma_j' = \hat{F}_j \ \epsilon_j \) holds in \( \mathbb{P}^n \), and this implies our claim.

\( \square \)
38. Corollary. If \( \alpha \in T(\omega) \) is divisible by \((\hat{F}_1)^c\), that is, \( \alpha = (\hat{F}_1)^c \gamma_1 \) for some \( \gamma_1 \in \Omega^1_n(d - e\hat{d}_1) \), then there exist \( \lambda'_1 \in \mathbb{C}, D_i \in S_n(d_1 - e\hat{d}_1) \), for \( i > 1 \), and \( \epsilon_1 \in \Omega^1_n(d_1 - e\hat{d}_1) \), such that

\[
\alpha = (\hat{F}_1)^c(\lambda'_1 \, dF_1 + \sum_{i > 1} \hat{F}_{i1} \, D_i \, dF_i + \hat{F}_1 \, \epsilon_1).
\]

In case \( e \geq 2 \), \( \lambda'_1 = 0 \).

Proof. It follows immediately from Proposition 37 applied to the case \( \gamma_j = 0 \) for \( j > 1 \). \( \square \)

9.1. End of the proof: balanced case.

39. Definition. Let \( \mathbf{d} = (m; d_1, \ldots, d_m) \in P(m, d) \). We say that \( \mathbf{d} \) is balanced if \( d_i < \sum_{j \neq i} d_j = \hat{d}_i \) for all \( i = 1, \ldots, m \). Equivalently, if \( 2d_i < d \) for all \( i \).

Notice that if \( \mathbf{d} \) is not balanced then there exists a unique \( i \) such that \( 2d_i \geq d \). Since we normalized \( \mathbf{d} \) so that \( d_1 \geq d_2 \geq \cdots \geq d_m \) (see Definition 1), it follows that \( \mathbf{d} \) is balanced if and only if \( 2d_1 < d \).

40. Theorem. Suppose \( \mathbf{d} \in P(m, d) \) is balanced. Let \( (\lambda, \mathbf{F}) \in V_n(\mathbf{d}) \) be general and \( \omega = \mu(\lambda, \mathbf{F}) \). Then, for any \( \alpha \in T(\omega) \) such that \( \alpha = \sum \hat{F}_i \, \gamma_i \), with \( \gamma_i \in \Omega^1_n(d_i) \), there exists \( \lambda' = (\lambda'_1, \ldots, \lambda'_m) \in \mathbb{C}^m \), with \( \sum_{i=1}^m d_i \lambda'_i = 0 \), such that

\[
\alpha = \sum_{i=1}^m \lambda'_i \, \hat{F}_i \, dF_i.
\]

In particular,

\[
\alpha = d\mu(\lambda, \mathbf{F})(\lambda', 0)
\]

belongs to the image of \( d\mu(\lambda, \mathbf{F}) \).

Proof. We apply Proposition 37 with \( e = 1 \). Since \( \mathbf{d} \) is balanced, \( d_j - \hat{d}_j < 0 \) for all \( j \) and then \( D_{ij} = 0 \) and \( \epsilon_j = 0 \) for all \( i, j \). Hence \( \gamma_j = \lambda'_j \, dF_j \) for all \( j \), as claimed. \( \square \)

It follows from Remark 36 that the proof of Theorem 24 is now complete, if \( \mathbf{d} \) is balanced.

9.2. End of the proof: general case. When \( \mathbf{d} \) is not balanced, Theorem 40 is not true; we may have an \( \alpha \in T(\omega) \) such that \( \alpha|_{X^{(2)}} = 0 \) but \( \alpha \) is not logarithmic as in Theorem 40. For example, take \( F'_1 = G_1 \, \hat{F}_1 \) where \( G_1 \) is any homogeneous polynomial of degree \( d_1 - \hat{d}_1 > 0 \), and \( F'_j = 0 \) for \( j > 1 \). Then \( \alpha = d\mu(\lambda, \mathbf{F})(0, F') \) satisfies this condition, as it easily follows from 7.1. Notice that this \( \alpha \) is divisible by \( \hat{F}_1 \).

In Theorem 42 we will see that any \( \alpha \in T(\omega) \) such that \( \alpha|_{X^{(2)}} = 0 \) may be written in a special form that still implies it belongs to the image of \( d\mu(\lambda, \mathbf{F}) \).
41. **Definition.** Let \( d \in P(m, d) \). We define
\[
    r(d) = \max \{ e \in \mathbb{N} / d_1 \geq e \hat{d}_1 \} = [d_1/\hat{d}_1]
\]
the integer part of \( d_1/\hat{d}_1 \).

Notice that \( d \) is balanced when \( r(d) = 0 \).

42. **Theorem.** Fix \( d \in P(m, d) \). Let \((\lambda, F) \in V_n(d)\) be general and \( \omega = \mu(\lambda, F) \). Then, any \( \alpha \in T(\omega) \) such that \( \alpha = \sum_i \hat{F}_i \gamma_i \), with \( \gamma_i \in \Omega^1_n(d_i) \), may be written as
\[
    \alpha = d\mu(\lambda, F)(\lambda', F')
\]
where \( \lambda' \in C^m \) is such that \( \sum_{i=1}^m d_i \lambda'_i = 0 \), \( F'_j = 0 \) for \( j > 1 \), and
\[
    F'_1 = \sum_{e=1}^{r(d)} G_e \hat{F}_1^e
\]
where \( G_e \) are homogeneous polynomials of respective degrees \( d_1 - e\hat{d}_1 \), for \( e = 1, \ldots, r(d) \).

**Proof.** By Proposition 37 with \( e = 1 \),
\[
    \alpha = \sum_j \lambda_j^I \hat{F}_j dF_j + \sum_{ij} \hat{F}_{ij} \hat{F}_j D_{ij} dF_i + \sum_j \hat{F}_j \hat{F}_j \epsilon_j. \tag{9.9}
\]
In the current unbalanced case, \( d_1 - \hat{d}_1 \geq 0 \) and \( d_i - \hat{d}_i < 0 \) for \( i > 1 \), as in Definition 9.2. Hence \( D_{ij} = 0 \) and \( \epsilon_j = 0 \) for \( j > 1 \). Also, since \( \sum_j \lambda_j^I \hat{F}_j dF_j = d\mu(\lambda, F)(\lambda', 0) \), it is enough to consider
\[
    \alpha = \alpha^{(1)} = \sum_{i>1} \hat{F}_{i1} \hat{F}_i D_{i1} dF_i + \hat{F}_1 \hat{F}_1 \epsilon_1 = \hat{F}_1 (\sum_{i>1} \hat{F}_{i1} D_{i1} dF_i + \hat{F}_1 \epsilon_1) \tag{9.10}
\]
which is divisible by \( \hat{F}_1 \) (the last term is actually divisible by \( \hat{F}_1^2 \)).

What we shall do is to express \( \alpha^{(1)} \) as the sum of an element of the image of \( d\mu(\lambda, F) \) (of the claimed shape) plus an \( \alpha^{(2)} \in T(\omega) \) divisible by \( \hat{F}_1^2 \). Next we repeat the argument and express \( \alpha^{(2)} \) as the sum of another element of the image of \( d\mu(\lambda, F) \) plus an \( \alpha^{(3)} \in T(\omega) \) divisible by \( \hat{F}_1^3 \). After at most \( r(d) \) iterations this process ends, since \( \alpha^{(r(d)+1)} = 0 \) by degree reason, and hence we obtain the claimed expression for the original \( \alpha \).

The essential step is to pass from \( \alpha^{(e)} \) to \( \alpha^{(e+1)} \), for \( 1 \leq e \leq r(d) \). To carry out this step, let us assume that \( \alpha \) is divisible by \( \hat{F}_1^e \), that is,
\[
    \alpha = \alpha^{(e)} = \hat{F}_1^e \left( \sum_{i>1} \hat{F}_{i1} D_{i1} dF_i + \hat{F}_1 \epsilon_1 \right). \tag{9.11}
\]
as in Corollary 38.

Now we apply to \( \alpha \) the calculation in the Proof of Proposition 32 with
\[
    A_{ij} = \hat{F}_1^e D_{ij}, \quad \alpha_j = \hat{F}_1^e \epsilon_j,
\]
that is:
\[
    A_{i1} = \hat{F}_1^e D_{i1} \quad \text{for } i > 1, \quad \alpha_1 = \hat{F}_1^e \epsilon_1,
\]
\[ A_{ij} = 0, \quad \alpha_j = 0 \quad \text{for } j > 1. \]

From equation 9.5 with \( r = 1 \) we get

\[
\hat{F}_1 \left( \sum_{i \neq 1} \lambda_i \hat{F}_{i1} d(\hat{F}_1^e D_{i1}) \wedge dF_i \wedge dF_1 + \sum_{i \neq k \neq 1} \lambda_k \hat{F}_{ik} \hat{F}_1^e D_{i1} dF_i \wedge dF_1 \wedge dF_k + \right)
\begin{align*}
\lambda_1 \hat{F}_1 d(\hat{F}_1^e \epsilon_1) \wedge dF_1 + & \sum_{k \neq 1} \lambda_k \hat{F}_{1k} \hat{F}_1^e \epsilon_1 \wedge dF_1 \wedge dF_k \right) = 0 \quad (9.12)
\end{align*}
\]

We have \( d(\hat{F}_1^e D_{i1}) = e \hat{F}_1^e D_{i1} d\hat{F}_1 + \hat{F}_1^e dD_{i1} \). Also, \( d\hat{F}_1 \wedge dF_i = (\sum_{j \neq 1} \hat{F}_{ij} dF_j) \wedge dF_i \), so that \( \hat{F}_{11} d\hat{F}_1 \wedge dF_i = \sum_{j \neq 1, j \neq i} \hat{F}_{ij} \hat{F}_{11} dF_j \wedge dF_i = \hat{F}_1 \sum_{j \neq 1, j \neq i} \hat{F}_{ij} dF_j \wedge dF_i \). Replacing these into 9.12, we obtain, on \( X_1 \):

\[
\hat{F}_1^{e+1} \left( \sum_{i \neq j \neq 1} e \lambda_i \hat{F}_{ij} D_{i1} dF_j \wedge dF_i \wedge dF_1 + \sum_{i \neq 1} \lambda_i \hat{F}_{i1} dD_{i1} \wedge dF_i \wedge dF_1 + \right)
\begin{align*}
\sum_{i \neq j \neq 1} \lambda_j \hat{F}_{ij} D_{i1} \wedge dF_i \wedge dF_j + & e \lambda_1 \hat{F}_1 \wedge \epsilon_1 \wedge dF_1 + \lambda_1 \hat{F}_1 d\epsilon_1 \wedge dF_1 + \right)
\begin{align*}
\sum_{i \neq 1} \lambda_i \hat{F}_{1i} \epsilon_1 \wedge dF_i \wedge dF_1 \right) = 0 \quad (9.13)
\end{align*}
\]

Now we cancel the factor \( \hat{F}_1^{e+1} \) on \( X_1 \) and then restrict to \( X_{1st} \) for \( 1, s, t \) distinct. After straightforward calculation we obtain, on \( X_{1st} \):

\[
(e \lambda_1 + \lambda_s) D_{s1} = (e \lambda_1 + \lambda_t) D_{s1}
\]

Then the collection \( \{ D_{s1}/(e \lambda_1 + \lambda_s) \in S_n(d_1 - e\bar{d}_1) \}_{s \neq 1} \) defines a section of \( \mathcal{O}(d_1 - e\bar{d}_1) \) on \( \cup_{s \neq 1} X_{1s} \subset X_1 \). Hence, there exists \( G_e \in S_n(d_1 - e\bar{d}_1) \) such that

\[
D_{s1} = (e \lambda_1 + \lambda_s) G_e
\]

on \( X_{1s} \) for all \( s \neq 1 \). Then, with the notation of 9.11,

\[
\sum_{i > 1} \hat{F}_{i1} D_{i1} dF_i + \hat{F}_1 \epsilon_1 - \sum_{i > 1} \hat{F}_{i1} (e \lambda_1 + \lambda_i) G_e \ w dF_i = 0
\]

on \( \cup_{s \neq 1} X_{1s} \subset X_1 \), and hence is divisible by \( \hat{F}_1 \). We obtain

\[
\alpha = \hat{F}_1^e \sum_{i > 1} \hat{F}_{i1} (e \lambda_1 + \lambda_i) G_e \ w dF_i + \hat{F}_1^{e+1} \bar{\epsilon}_1 \quad (9.14)
\]

for some \( \bar{\epsilon}_1 \in \Omega_1^1(d_1 - e\bar{d}_1) \).

Denote \( \mathbf{F}' = (\hat{F}_1^e G_e, 0, \ldots, 0) \). Combining 9.14 with

\[
d\mu(\lambda, \mathbf{F})(0, \mathbf{F}') = \sum_{i > 1} \lambda_i \hat{F}_1^e G_e \hat{F}_{i1} dF_i + \lambda_1 \hat{F}_1 d(\hat{F}_1^e G_e)
\]

(see 7.1), one immediately obtains

\[
\alpha = d\mu(\lambda, \mathbf{F})(0, \mathbf{F}') + \alpha^{(e+1)}
\]
with $\alpha^{(e+1)} = \hat{F}_1^{e+1} (\bar{\epsilon}_1 - \lambda_1 dG_e)$. Now, $\alpha^{(e+1)} \in T(\omega)$ because $\alpha$ and $d\mu(\lambda, F)(0, F')$ belong to $T(\omega)$. Since $\alpha^{(e+1)}$ is divisible by $\hat{F}_1^{e+1}$, by Corollary 38, it may be written as in 9.11 with exponent $e + 1$. Hence we may apply again the previous procedure to $\alpha^{(e+1)}$. This proves the essential iterative step and implies our statement.

It follows from Remark 36 that the proof of Theorem 24 is now complete, for any $d$. 

\[\square\]
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