Chiral zero modes in non local domain walls

C. D. Fosco and G. Torroba

Centro Atómico Bariloche and Instituto Balseiro
Comisión Nacional de Energía Atómica
8400 Bariloche, Argentina.

March 27, 2022

Abstract

We study a generalization of the Callan-Harvey mechanism to the case of a non local mass. Using a 2+1 model as a concrete example, we show that both the existence and properties of localized zero modes can also be consistently studied when the mass is non local. After dealing with some general properties of the resulting integral equations, we show how non local masses naturally arise when radiative corrections are included. We do that for a 2 + 1 dimensional example, and also evaluate the zero mode of the resulting non local Dirac operator.
1 Introduction

The Callan-Harvey mechanism [1] explains the existence and properties of the fermionic zero modes that appear whenever the mass of a Dirac field in $2k + 1$ dimensions ($k = 1, 2, \cdots$) has a domain-wall like defect. The zero modes due to this phenomenon are localized, concentrated around the domain wall, and chiral from the point of view of the domain-wall world-volume (a $2k$-dimensional theory).

This mechanism has found many interesting applications, both to phenomenological issues [2] and theoretical elaborations [3, 4]. Dynamical domain walls have been considered in [5] and the very interesting case of a dynamical supersymmetric model with this kind of configuration has been discussed in [6]. Remarkably, lattice versions of the zero modes, the so-called ‘domain wall fermions’ [7] have also been precursors of the overlap Dirac operator [8], a sensible way to put chiral fermions on a spacetime lattice.

In spite of the fact that the Callan-Harvey mechanism has been extended in many directions, the existence of chiral zero modes has been so far only studied for the case of local mass terms, namely, those where the mass is a local function of the spacetime coordinates.

In this article, we consider a different generalization of the problem, namely, the case of a non local mass in $2 + 1$ dimensions. This is a non trivial modification of the usual assumptions of the Callan-Harvey mechanism, which does, however, arise naturally in some applications: we show that explicitly for a case where the fields are dynamical, and radiative corrections are included.

The structure of the article is as follows: in section 2 we set up the general framework, defining the class of systems we shall consider, in terms of an associated non-homogeneous integral equation for the zero mode. In section 3 a concrete realization of the kind of non local mass discussed earlier is exhibited, evaluating its zero mode solutions. Section 4 contains our conclusions.

2 Callan-Harvey mechanism for a non local mass

In the standard Callan-Harvey mechanism [1] one considers a fermion field in $2+1$ dimensions coupled to a domain wall like defect, with the Euclidean
action

\[ S(\bar{\psi}, \psi) = \int d^3x \bar{\psi}[\not{\partial} + m(x)]\psi. \] (1)

We use Euclidean coordinates \( x = (x_0, x_1, x_2) \) \( (x_0 \text{ is the Euclidean time}) \) and \( \not{\partial} = \gamma_\mu \partial_\mu \), where the \( \gamma \)-matrices are chosen according to the convention:

\[ \gamma_0 = \sigma_3 \quad \gamma_1 = \sigma_1 \quad \gamma_2 = \sigma_2. \] (2)

The local mass \( m(x) \) contains a topological defect; in the simplest case of a rectilinear static defect \([3]\), they have the characteristic shape:

\[ m(x) \sim \Lambda \sigma(x_2), \] (3)

where \( \sigma(x_2) \equiv \text{sign}(x_2) \). Therefore the domain wall, which is the interface between two regions with different signs for \( m(x) \), is the \( x_1 \) axis.

From the Dirac operator \( \mathcal{D} = \not{\partial} + m(x_2) \), (4)

we can construct the hermitian operator \( \mathcal{H} = \mathcal{D}^\dagger \mathcal{D} \). The form of \( \mathcal{H} \) suggests the introduction of the adjoint operators

\[ a = \partial_2 + m(x_2), \quad a^\dagger = -\partial_2 + m(x_2) \] (5)

in terms of which

\[ \mathcal{H} = (a^\dagger a - \not{\partial}^2)P_L + (aa^\dagger - \not{\partial}^2)P_R \] (6)

and

\[ \mathcal{D} = (a + \not{\partial})P_L + (a^\dagger + \not{\partial})P_R \] (7)

where \( P_L = \frac{1}{2}(1 + \gamma_2) \), \( P_R = \frac{1}{2}(1 - \gamma_2) \). Expanding \( \psi(x) \) in the complete set of eigenfunctions of \( a^\dagger a \) and \( aa^\dagger \), there appears \([3]\) a massless left fermion, localized over the domain wall. Its \( x_2 \) dependence is dictated by the fact that it is a zero mode of the \( a \) operator, and it dominates the low energy dynamics of the system.

We want to generalize this phenomenon to include a non local mass (in the following section we present a model that motivates this generalization). Namely, rather than \([4]\), the Dirac operator shall be

\[ \tilde{\mathcal{D}}(x, y) = \not{\partial} \delta(x - y) + M(x, y). \] (8)
Little can be said about the existence of a fermionic zero mode before we make some hypotheses to restrict the form of $M(x, y)$. We assume that the system has translation invariance in the coordinates $\hat{x} \equiv (x_0, x_1)$ and that $M(x, y)$ consists of a local domain wall like part, plus a non local term with a strength controlled by a parameter $\lambda$:

$$M(x, y) = m(x_2) \delta(x - y) - \lambda \int \frac{d^2 k}{(2\pi)^2} e^{ik \cdot (\hat{x} - \hat{y})} \gamma_k(x_2, y_2). \quad (9)$$

We are looking for a zero mode $\Psi(x)$, so that:

$$\langle x| \hat{D}|\Psi \rangle = \int d^3 y \ \hat{D}(x, y)\Psi(y) = 0. \quad (10)$$

Taking advantage of the translation invariance in $\hat{x}$, we use ‘separation of variables’ to look for solutions of the form:

$$\Psi(x) = \chi(\hat{x}) \psi(x_2), \quad (11)$$

where $\chi(\hat{x})$ is a massless spinor, which is left-handed from the point of view of the two-dimensional world defined by $\hat{x}$, i.e.,

$$\hat{\partial} \chi(\hat{x}) = 0 \quad P_R \chi(\hat{x}) = 0. \quad (12)$$

There is an essential difference regarding the space of solutions to the equations above in Euclidean and Minkowski spacetimes. Indeed, in Euclidean spacetime, it leads to analytic functions of $x_0 + ix_1$, while in the Minkowski case one has ‘left-mover’ solutions. Keeping this distinction in mind, we continue working with the Euclidean version.

Substituting (11) into (10) and comparing with (7), we arrive to a non local version of the kernel for the annihilation operator

$$a(x_2, y_2) = [\partial_2 + m(x_2)] \delta(x_2 - y_2) - \lambda \gamma_k(x_2, y_2). \quad (13)$$

The zero mode $\psi(x_2)$, must then satisfy the equation

$$\langle x|a|\psi \rangle = [\partial_2 + m(x_2)] \psi(x_2) - \lambda \int_{-\infty}^{+\infty} dy_2 \gamma_k(x_2, y_2)\psi(y_2) = 0. \quad (14)$$

Following the method of variation of parameters, we use the ansatz

$$\psi(x_2) = \psi_0(x_2) \varphi(x_2) \quad (15)$$
where \( \psi_0 \) is the zero mode for the local part in (14), which satisfies
\[
[\partial_2 + m(x_2)] \psi_0(x_2) = 0 \quad \psi_0(x_2) = N \exp[-\int_0^{x_2} ds \, m(s)] ,
\]
and \( N \) is a normalization constant.

The function \( \varphi(x_2) \), which modulates \( \psi_0(x_2) \) must then satisfy the equation
\[
\partial_2 \varphi(x_2) - \lambda \int_{-\infty}^{+\infty} dy_2 \left[ \psi_0(x_2)^{-1} \gamma_k(x_2, y_2) \psi_0(y_2) \right] \varphi(y_2) = 0 .
\]

By integrating over \( x_2 \), the previous equation can be written in a more convenient form as an integral equation
\[
\varphi(x_2) - \lambda \int_{-\infty}^{+\infty} dy_2 \tilde{\gamma}_k(x_2, y_2) \varphi(y_2) = \varphi(0) ,
\]
where the kernel \( \tilde{\gamma}_k \) is
\[
\tilde{\gamma}_k(x_2, y_2) = \int_0^{x_2} dz_2 \psi_0(z_2)^{-1} \gamma_k(z_2, y_2) \psi_0(y_2) .
\]

This is a homogeneous integral equation, which can be conveniently rewritten as an equivalent non-homogeneous set of equations. Indeed, introducing linear operators, (18) can be rewritten as follows:
\[
(I - \lambda T) \varphi = c
\]
where
\[
(I \varphi)(x_2) \equiv \varphi(x_2) , \quad c \equiv \varphi(0) ,
\]
\[
(T \varphi)(x_2) \equiv \int dy_2 \tilde{\gamma}_k(x_2, y_2) \varphi(y_2) .
\]

We see that the problem can be discussed in terms of (20), which is an inhomogeneous system of the Fredholm type. The condition \( c \equiv \varphi(0) \) has to be verified, of course, after solving (20) for arbitrary \( c \).

The reason for this procedure is that, had we used the original homogeneous system, we should have had to introduce non compact operators, and the theory for this kind of operator is substantially poorer than for the compact case.
2.1 Properties of the solutions

We have shown that the physical restrictions imposed on the non local mass lead us to integral equations of the Fredholm type. The existence and properties of solutions of these kinds of equations are very well known [9]. In particular, the theorem of the Fredholm alternative provides information that is useful to our purposes.

The Fredholm alternative [9] states that if $A = I - \lambda T$, where $T$ is a compact operator on a Hilbert space $H$, then the following alternative holds:

- either $A\varphi_0 = 0$ has only the trivial solution, in which case $A\varphi = c$ has a unique solution $\forall c \in H$
- or $A\varphi_0 = 0$ has $q$ linearly independent solutions $\varphi_i \in H$. Then $A^\dagger \tilde{\varphi}_0 = 0$ also has $q$ linearly independent solutions $\tilde{\varphi}_i \in H$. In this case $A\varphi = c$ is solvable iff $(c, \tilde{\varphi}_i) = 0 \ \forall i = 1, \ldots, q$.

In the second alternative, the general non-homogeneous solution is

$$\varphi = \varphi_p + \sum_{i=1}^{q} a_i \varphi_i$$  \hspace{1cm} (23)

where $\varphi_p$ is a particular solution and $a_i$ are arbitrary constants.

Note that, when the first alternative holds true, it implies in particular that the solution will be unique when $c$ is replaced by $\varphi(0)$. Solutions corresponding to $c \neq \varphi(0)$ are not solutions of the system: $(I - \lambda T)\varphi = c$, $c \equiv \varphi(0)$, equivalent to the original homogeneous equation (18), and may, therefore, be discarded.

Any true solution of the system will also verify a subsidiary equation, obtained by setting $x_2 = 0$ in (18):

$$\int_{-\infty}^{+\infty} dy_2 \tilde{\gamma}_k(0, y_2)\varphi(y_2) = 0$$  \hspace{1cm} (24)

for any $\lambda \neq 0$. This equation shall be true whenever the equations $(I - \lambda T)\varphi = c$, $c \equiv \varphi(0)$ are both true (since it is derived from them), and any solution will automatically verify it.

In our case, we want to study the effect of the non local term, the strength of which is controlled by the value of $\lambda$. Close enough to the local mass case, $\lambda$ can be made arbitrarily small, so $\lambda^{-1}$ is not an eigenvalue of $T$ and
consequently $A\varphi = 0$ has only the trivial solution. Therefore, if $T$ is a Fredholm operator, \((18)\) will have a unique solution. The functional space $H$ to which $\varphi(x)$ belongs is restricted by the condition
\[
\int dx_2 \left( \psi_0(x_2) \right)^2 \left( \varphi(x_2) \right)^2 < \infty , \tag{25}
\]
because $\psi(x)$ itself has to be normalizable. This becomes a Hilbert space, and it contains the zero mode of the local operator ($\varphi = \text{const}$), when the scalar product used in $H$ is the one defined by \((25)\), namely,
\[
(f,g) \equiv \int dx_2 \left( \psi_0(x_2) \right)^2 [f(x_2)]^* g(x_2) . \tag{26}
\]

Of course, things may be different if $H$ were equipped with the standard $L^2$ scalar product, since then a constant function would not have a finite norm, although it would verify \((26)\). We assume that the norm defined by \((26)\) is used in what follows.

When all the Fredholm’s hypotheses are satisfied for a general kernel $\tilde{\gamma}_k(x_2, y_2)$, it is possible to find a perturbative solution. Indeed, writing
\[
\varphi(x_2) = \varphi(0) + \lambda \int_{-\infty}^\infty dy_2 \tilde{\gamma}_k(x_2, y_2) \varphi(y_2) , \tag{27}
\]
and successively replacing $\varphi(y_2)$ by $\varphi(0) + \lambda \int_{-\infty}^\infty dy_2 \tilde{\gamma}_k(x_2, y_2) \varphi(y_2)$ on the right hand side, we obtain the Neumann series \[9\]
\[
\varphi(0) + \lambda K_1(x_2) \varphi(0) + \cdots + \lambda^n K_n(x_2) \varphi(0) + \ldots \tag{28}
\]
where
\[
K_j(x_2) = \int_{-\infty}^\infty dy_2 \gamma^{(j)}(x_2, y_2) \tag{29}
\]
and
\[
\gamma^{(1)}(x_2, y_2) = \tilde{\gamma}_k(x_2, y_2) , \quad \gamma^{(2)}(x_2, y_2) = \int_{-\infty}^\infty dz_2 \tilde{\gamma}_k(x_2, z_2) \tilde{\gamma}_k(z_2, y_2)
\]
\[
\ldots \quad \gamma^{(n+1)}(x_2, y_2) = \int_{-\infty}^\infty dz_2 \tilde{\gamma}_k(x_2, z_2) \gamma^{(n)}(z_2, y_2) . \tag{30}
\]
It can be shown \[9\] that the Neumann series converges uniformly to the solution $\varphi(x)$ when
\[
|\lambda| < \frac{1}{\mu} , \tag{31}
\]
\[ \mu^2 = \int_{-\infty}^{\infty} dx_2 dy_2 [\tilde{\gamma}_k(x_2, y_2)]^2. \]  

So choosing \( \lambda \) to satisfy (31), we are allowed to represent \( \varphi \) by the expansion

\[ \varphi(x_2) = \varphi(0) + \lambda K_1(x_2) \varphi(0) + \cdots + \lambda^n K_n(x_2) \varphi(0) + \ldots \]  

In subsection 3.3 we shall use another method to find a solution, which is useful when the kernel is separable \[9\]. Since this condition may be satisfied by an operator which is not of the Fredholm type, it allows us to explore rather different situations.

### 2.2 Example

After the previous general analysis, we now consider the effects of quantum corrections for a specific choice of both the local and non local parts of \( M(x, y) \) in \[9\]. A natural generalization of the purely local mass case is to have \( m(x_2) = \Lambda \sigma(x_2) \) and a \( \gamma_k(x_2, y_2) \) which is ‘strongly diagonal’ and symmetric in \( (x_2, y_2) \), i.e.:

\[ \gamma_k(x_2, y_2) = \frac{1}{2} \left[ \sigma(x_2) + \sigma(y_2) \right] \delta_N(x_2 - y_2), \]  

where \( \delta_N \) is an approximation of Dirac’s delta: \( \delta_N(x_2 - y_2) \to \delta(x_2 - y_2) \) when \( N \to \infty \). Note that \( \gamma_k(x_2, y_2) \to \text{sign}(x_2) \delta(x_2 - y_2) \) when \( N \to \infty \), so that the non local term reduces to a local domain wall mass.

We then adopt ‘natural’ units such that \( \Lambda = 1 \), and study the particular case \( |x_2| \leq L = \Lambda^{-1} \). Regarding \( \delta_N \), we use a truncation of the one-dimensional completeness relation:

\[ \delta_N(x_2, y_2) = \sum_{n=0}^{N} \varphi_n(x_2) \varphi_n^\dagger(y_2), \]  

where \( \{ \varphi_n \} \) is a complete set of functions. We chose \( \varphi_n(x) \) to be the harmonic oscillator’s eigenfunctions.

The normalized zero mode corresponding to the local part is

\[ \psi_0(x_2) = N_0 \exp \left[ - \int_0^{x_2} dt \, m(t) \right] = \frac{1}{\sqrt{1 - e^{-|x_2|}}} \]  

8
and the corrected zero mode \( \psi(x_2) = \psi_0(x_2) \varphi(x_2) \) is then determined by \( \varphi(x_2) \). The integral equation for \( \varphi(x_2) \) becomes

\[
\varphi(x_2) = \varphi(0) + \lambda \int_{-1}^{1} dy_2 \tilde{\gamma}(x_2, y_2) \varphi(y_2)
\]  

(37)

where

\[
\tilde{\gamma}(x_2, y_2) = \frac{1}{2} \int_{0}^{x_2} dz_2 e^{\frac{|z_2|}{N}} \sum_{n=0}^{N} \varphi_n(z_2) \varphi_n^\dagger(y_2) e^{-|y_2|}.
\]  

(38)

From equations (31) and (32), we see that the Neumann series converges, in this case, for \(|\lambda| < 0.0990\). So we assume \( \lambda = 0.01 \), and we are ready to calculate pertubatively. Since its expression in terms of analytic functions is not very illuminating, we present, in Figure 1, the numerical results of the first two iterations, taking \( \varphi(0) = 1 \) and \( N = 3 \).

In Figure 2 we show the zero mode (36) and the one with \( \lambda^2 \)-order corrections. We see that the corrected zero mode continues to be localized over the domain wall, although it is no longer a symmetric function of \( x_2 \).
Besides, this perturbative method introduces only smooth corrections in the zero mode, as expected.

3 A physical model for the non local mass

A natural way of generating a non local mass of the form (9) is to couple the Dirac field to a defect that is allowed to fluctuate, with a dynamics given by a scalar field action. In this context, quantum corrections to the classical configurations generate a non local term in the Dirac operator, already at the one loop order. We shall now derive such a term, and study how the classical fermionic zero mode is affected by its presence.

3.1 Description of the system

We consider the Euclidean action

\[ S(\phi, \bar{\psi}, \psi) = S_B(\phi) + S_F(\bar{\psi}, \psi, \phi) \]  

(39)

where

\[ S_B(\phi) = \int d^3x \left[ \frac{1}{2} (\partial \cdot \phi)^2 + V(\phi) \right] , \]  

(40)
\[ S_F(\bar{\psi}, \psi, \phi) = \int d^3x \bar{\psi} [\not\partial + g\phi(x)] \psi \]  

and \( V(\phi) \) is the quartic potential:

\[ V(\phi) = \frac{g^2}{2} \left( \phi^2 - \frac{\kappa^2}{g^2} \right)^2. \]  

The reason for choosing this scalar potential is that if we only consider the scalar field, there exists a domain wall like solution (known as the kink configuration) of the form [10]

\[ \phi(x_2) = \frac{\kappa}{g} \tanh(\kappa x_2). \]  

Now, for the system (39), the Euler-Lagrange equations of motion are

\[ [\not\partial + g\phi(x)] \psi = 0 \]

\[ -\partial^2 \phi(x) + V'(\phi(x)) + g\bar{\psi}(x)\psi(x) = 0. \]  

It is immediate to verify that the configuration given by the domain wall (43) and the chiral zero mode

\[ \psi(x) = P_L \chi(\hat{x}) \exp \left( - \int_{0}^{x_2} dt g \phi(t) \right), \]

with \( \chi \) as in (12), is a self-consistent solution to the coupled system of equations (44).

Quantum corrections appear in the vacuum expectation values (VEVs) \( \langle \phi \rangle \equiv \langle \Omega | \phi | \Omega \rangle \), \( \langle \psi \rangle \equiv \langle \Omega | \psi | \Omega \rangle \), where \( | \Omega \rangle \) is the interacting vacuum. The most direct way of calculating these expectation values is within the context of the effective action \( \Gamma(\varphi, \bar{\chi}, \chi) \) [11]. Here \( \varphi, \bar{\chi} \) and \( \chi \) denote the so called ‘classical fields’ corresponding to \( \phi, \bar{\psi} \) and \( \psi \), respectively.

### 3.2 One loop calculations

It is useful to consider the loopwise expansion [11]

\[ \Gamma(\varphi, \bar{\chi}, \chi) = S(\varphi, \bar{\chi}, \chi) + \Gamma_1(\varphi, \bar{\chi}, \chi) + \ldots \]  

11
where $S$ is the classical action and $\Gamma_1$ contains one-loop diagrams that introduce $h$-order corrections in the classical tree graphs. After a direct calculation we find

$$\Gamma_1 = -\text{Tr} \ln \mathcal{D} + \frac{1}{2} \text{Tr} \ln \left[ \frac{\delta^2 S_B(\varphi)}{\delta \phi(x_1) \delta \phi(x_2)} - 2 g^2 \bar{\chi}(x_1) \mathcal{D}^{-1}(x_1, x_2) \chi(x_2) \right]$$

(47)

where $\mathcal{D} = \hat{\partial} + g \varphi$. For simplicity, we will approximate the field $\varphi(x_2)$ by taking

$$\varphi(x_2) = \frac{\kappa}{g} \tanh(\kappa x_2).$$

As a result, the logarithms of operators in (47) can be obtained by Fourier-transforming in $\hat{x}$ and expanding in adequate complete sets of functions in $x_2$. As before, these are eigenvectors of $a^\dagger a$ and $aa^\dagger$, with $a = \partial_2 + g \varphi(x_2)$. We shall note them as $\psi_n(x_2)$ and $\tilde{\psi}_n(x_2)$:

$$a^\dagger a \psi_n(x_2) = \lambda_n^2 \psi_n(x_2), \quad aa^\dagger \tilde{\psi}_n(x_2) = \lambda_n^2 \tilde{\psi}_n(x_2).$$

(48)

Only $a$ has a normalizable zero mode

$$a \psi_0(x_2) = 0, \quad \psi_0(x_2) = \sqrt{\frac{\kappa}{2}} \sech(\kappa x_2).$$

(49)

The rest of the $a^\dagger a$ and $aa^\dagger$ spectra coincide, and consist of a continuum separated by a finite gap $\kappa^2 > 0$ from the zero mode. We also need to introduce the operators

$$b = \partial_2 + 2 \kappa \tanh(\kappa x_2), \quad b^\dagger = -\partial_2 + 2 \kappa \tanh(\kappa x_2),$$

(50)

with eigenfunctions $\xi_n(x_2), \tilde{\xi}_n(x_2)$:

$$b^\dagger b \xi_n(x_2) = \mu_n^2 \xi_n(x_2), \quad bb^\dagger \tilde{\xi}_n(x_2) = \mu_n^2 \tilde{\xi}_n(x_2).$$

(51)

In terms of these operators,

$$\Gamma_1 = -\frac{1}{2} \text{Tr} \ln \left[ -\hat{\partial}^2 + a^\dagger a \right] - \frac{1}{2} \text{Tr} \ln \left[ -\hat{\partial}^2 + a a^\dagger \right] + \frac{1}{2} \text{Tr} \ln \left[ -\hat{\partial}^2 + b^\dagger b \right]$$

$$+ \frac{1}{2} \text{Tr} \ln \left[ \delta(x_1 - x_2) - 2 g^2 \int_y \Delta_\varphi(x_1, y) \bar{\chi}(y) \mathcal{D}^{-1}(y, x_2) \chi(x_2) \right].$$

(52)
The quantum Dirac operator is then
\[ \tilde{D}(x,y) = -\frac{\delta^2 \Gamma}{\delta \bar{\chi}(x) \delta \chi(y)} \big|_{\chi = \bar{\chi} = 0}. \] (53)

Considering only one-loop corrections given by (52), we have
\[ \tilde{D}(x,y) = D(x,y) + \left( \mu^2 \right) \Delta \varphi(y,x) D^{-1}(x,y). \] (54)

In Figure 3 we show the Feynman diagram that represents the first order quantum corrections to the fermion self-energy (second term in (54)).

Using the expansions
\[ \langle x|D^{-1}|y \rangle = \int \frac{d^2 \hat{p}}{(2\pi)^2} e^{i \hat{p} \cdot (\hat{x} - \hat{y})} \left\{ \langle x_2|\psi_0\rangle \langle \psi_0|y_2 \rangle \frac{-i \hat{p}}{\hat{p}^2 + \lambda_n^2} \mathcal{P}_R + \sum_{n=1}^{\infty} \left[ \langle x_2|\psi_n\rangle \langle \psi_n|y_2 \rangle \frac{\lambda_n}{\hat{p}^2 + \lambda_n^2} \mathcal{P}_R + \langle x_2|\bar{\psi}_n\rangle \langle \bar{\psi}_n|y_2 \rangle \frac{\lambda_n}{\hat{p}^2 + \lambda_n^2} \mathcal{P}_L \right] \right\} \] (55)

and
\[ \langle x|\Delta \varphi|y \rangle = -\int \frac{d^2 \hat{k}}{(2\pi)^2} e^{i \hat{k} \cdot (\hat{x} - \hat{y})} \sum_{n=0}^{\infty} \langle x_2|\xi_n\rangle \langle \xi_n|y_2 \rangle \frac{1}{\hat{k}^2 + \mu_n^2}, \] (56)

we obtain
\[ \tilde{D}(x,y) = D(x,y) + \int \frac{d^2 \hat{k}}{(2\pi)^2} e^{i \hat{k} \cdot (\hat{x} - \hat{y})} \Upsilon_k(x_2, y_2) \] (57)
with
\[
\gamma_k(x_2, y_2) = -g^2 \left\{ \sum_{n=0}^{\infty} \xi_n(x_2) \psi_0(x_2) \xi_n^\dagger(y_2) \psi_0^\dagger(y_2) J(\hat{k}; \mu_n, 0) \mathcal{P}_R + \sum_{n=0,m=1}^{\infty} [\xi_n(x_2) \psi_m(x_2) \xi_n^\dagger(y_2) \psi_m^\dagger(y_2) J(\hat{k}; \mu_n, \lambda_m) \mathcal{P}_L] \right. \\
+ \sum_{n=0,m=1}^{\infty} \xi_n(x_2) \tilde{\psi}_m(x_2) \xi_n^\dagger(y_2) \tilde{\psi}_m^\dagger(y_2) J(\hat{k}; \mu_n, \lambda_m) \mathcal{P}_R \\
+ \sum_{n=0,m=1}^{\infty} \xi_n(x_2) \tilde{\psi}_m(x_2) \xi_n^\dagger(y_2) \tilde{\psi}_m^\dagger(y_2) I(\hat{k}; \mu_n, \lambda_m) \mathcal{P}_L \left. \right\},
\]
(58)

and the loop integrals
\[
I(\hat{k}; M_1, M_2) = \int \frac{d^2 \hat{p}}{(2\pi)^2} \frac{M_2}{[(\hat{k} - \hat{p})^2 + M_1^2](\hat{p}^2 + M_2^2)} 
\]
(59)
\[
J(\hat{k}; M_1, M_2) = \int \frac{d^2 \hat{p}}{(2\pi)^2} \frac{-i \hat{p}}{[(\hat{k} - \hat{p})^2 + M_1^2](\hat{p}^2 + M_2^2)}. 
\]
(60)

We used a discrete notation for the sum over eigenvalues, but of course an integral over the continuous part of the spectrum is implicitly assumed.

At this stage we recognize, as was anticipated, that the Dirac operator \[\mathcal{D} \] is of the form given by \[\mathcal{L} \] and \[\mathcal{L} \].

### 3.3 Quantum corrections to the fermionic zero mode

In the previous subsection we saw how the quantum fluctuations generate non-localities in the Dirac operator \[\mathcal{D} \]; now we want to find its zero mode \[\Psi(x): \langle x|\mathcal{D}|\Psi \rangle = 0 \], and compare it with \[\Psi \]. Repeating the steps that led to \[\Psi \], we obtain \[\lambda = g^2 \] and
\[
\gamma_k(x_2, y_2) = \sum_{n=0,m=1}^{\infty} \xi_n(x_2) \psi_m(x_2) \xi_n^\dagger(y_2) \tilde{\psi}_m^\dagger(y_2) I(\hat{k}; \mu_n, \lambda_m),
\]
(61)
because in $\Upsilon_k$ only the term proportional to $P_k$ and $I(\hat{k}; \mu_n, \lambda_m)$ gives a non-zero contribution to $|10\rangle$. In effect, $P_R\chi = 0$ and the term proportional to $J(\hat{k}; \mu_n, \lambda_m) \propto \hat{k}$ also yields a vanishing contribution when applied to $\chi$, after integration over $x_2$.

On the other hand, for $n = 0$, $I(\hat{k}; \mu_0, \lambda_m) = I(\hat{k}; 0, \lambda_m)$ is IR divergent. This divergence can be associated with the massless bosonic zero mode

$$\xi_0(x_2) = \sqrt{\frac{3\kappa}{4}} \text{sech}^2(\kappa x_2)$$

(62)

living in a defect of infinite extension in the $x_2$-direction; it contributes to the propagator $\Delta_\phi$ in the Feynman graph in Figure 3. Using an IR cutoff $\epsilon \to 0$, the divergent part of the regularized loop integral becomes

$$I_\epsilon(\hat{k}; 0, \lambda_m) = -\frac{1}{2\pi} \frac{\lambda_m}{k^2 + \lambda_m^2} \ln \epsilon.$$  

(63)

In other words, in the limit of infinite size, the dominant contribution to $\gamma_k$ comes from the zero mode, and we can write

$$\gamma_k(x_2, y_2) \approx \left( \sum_{m=1}^{\infty} I_\epsilon(\hat{k}; 0, \lambda_m) \psi_m(x_2) \tilde{\psi}_m^\dagger(y_2) \right) \xi_0(x_2) \xi_0^\dagger(y_2).$$

(64)

Furthermore, the $m > 1$ modes give exponentially decreasing corrections in $\lambda_m$. This can be seen by applying the quantum Dirac operator $\tilde{D}$ to the zero mode $\psi_0$ (obtained in Eq. (15)) of the classical operator $D$:

$$\langle x_2 | \tilde{D} | \psi_0 \rangle = 0 + \int_{-\infty}^{+\infty} dy_2 \gamma_k(x_2, y_2) \langle y_2 | \psi_0 \rangle.$$  

(65)

From (64), the integral over $y_2$ in (65) is proportional to

$$\int_{-\infty}^{+\infty} dy_2 \tilde{\psi}_m(y_2) \psi_m(y_2) \xi_0^\dagger(y_2) = \frac{3\sqrt{\pi}}{8} \left( \frac{\lambda_m}{\kappa} \right)^2 \text{sech}[\frac{\pi}{2} \sqrt{(\frac{\lambda_m}{\kappa})^2 - 1}].$$

(66)

Consequently, we can approximate (64) by considering only the $m = 1$ term:

$$\gamma_k(x_2, y_2) \approx I_\epsilon(\hat{k}; 0, \kappa) \xi_0(x_2) \xi_0^\dagger(y_2) \psi_1(x_2) \tilde{\psi}_1^\dagger(y_2).$$

(67)

Later in this section, we will indicate how to construct the corrected zero mode when other terms in (64) are taken into account. The presence of $\psi_1$
in $\gamma_k(x_2, y_2)$ tells us that quantum fluctuations will mix the first two modes corresponding to the local mass case. Substituting (67) into (19), we see that

$$\tilde{\gamma}_k(x_2, y_2) = \frac{3\kappa}{8\pi} I_\epsilon(\hat{k}; 0, \kappa) \left(1 - \text{sech}(\kappa x_2)\right) \text{sech}^3(\kappa y_2). \quad (68)$$

This is a separable kernel, because it is of the form $\tilde{\gamma}_k(x_2, y_2) = \alpha(x_2) \beta(y_2)$. The Fredholm integral equation with separable kernel may be solved, without using the Fredholm alternative \[9\], by the following method. Defining $C \equiv \frac{3\kappa}{8\pi} g^2 I_\epsilon(\hat{k}; 0, \kappa)$ we obtain, from (18)

$$\varphi(x_2) + C \left(\text{sech}(\kappa x_2) - 1\right) \int_{-\infty}^{+\infty} dy_2 \text{sech}^3(\kappa y_2) \varphi(y_2) = \varphi(0). \quad (69)$$

Multiplying (69) by $\text{sech}^3(\kappa x_2)$, integrating over $x_2$, and introducing the constant

$$X \equiv \int_{-\infty}^{+\infty} dy_2 \text{sech}^3(\kappa y_2) \varphi(y_2) \quad (70)$$

the result is

$$X \left(1 - C \frac{\pi}{2} - \frac{4}{3}\right) = \frac{\pi}{2\kappa} \varphi(0). \quad (71)$$

As $C \gg 1$,

$$-C X \approx \frac{\pi}{\pi - 8/3} \varphi(0). \quad (72)$$

On the other hand, from (69),

$$\varphi(x_2) + \left(1 - \text{sech}(\kappa x_2)\right) (-C X) = \varphi(0). \quad (73)$$

Therefore we arrive to a result which is independent of $C$ in the large size limit:

$$\varphi(x_2) \approx \varphi(0) \left[1 + \frac{\pi}{\pi - 8/3} \left(\text{sech}(\kappa x_2) - 1\right)\right]. \quad (74)$$

The normalized zero mode including quantum corrections is, according to (13)

$$\psi(x_2) = N \left[1 + \frac{\pi}{\pi - 8/3} \left(\text{sech}(\kappa x_2) - 1\right)\right] \text{sech}(\kappa x_2), \quad (75)$$

where $N$ is a numerical normalization constant.
Figure 4: Classical (dashed line) and quantum-corrected (full line) zero mode profiles.

It should be clear from the previous derivation that the infrared singularity that arises in the loop integral \( I(\hat{k}; 0, \kappa) \) does not propagate to the physical results; namely, those obtained for large values of the infrared cutoff.

In Figure 4 we plot \( \psi_0(x_2) \) and \( \psi(x_2) \) (given by (75)); there, it becomes evident the quantum mixture of the eigenmodes \( \psi_0 \) and \( \psi_1 \). In addition, the dispersions corresponding to the distributions \( |\psi_0(x_2)|^2 \) and \( |\psi(x_2)|^2 \) are \( \sigma_{\text{loc}} \approx 0.907/\kappa \) and \( \sigma_{\text{nonloc}} \approx 1.850/\kappa \), respectively. Since the local fermion mass term is \( \kappa \tanh(\kappa x_2) \), \( \kappa \) is the parameter that controls the localization of the domain wall and consequently of the fermion zero modes, as seen from the previous dispersion formulas.

Besides, from Figure 4 it is clear that the quantum correction given by (75) are nonperturbative; in fact, this result cannot be obtained by resuming the Neumann series (33). Quantum effects mix the zero localized mode \( \psi_0 \) with the nonlocalized mode \( \psi_1 \) giving, as a result, a nontrivial localized function with two nodes and broader than the local zero mode one. Of course, one would expect the inclusion of more modes should result in an extra distortion of the zero mode.

The generalization of this calculation to include a finite number of terms from (64), associated to the contribution of the \( \psi_{m>1} \) modes, should proceed as follows: applying the same procedure as before to an expression of the
form
\[
\varphi(x) + \sum_m a_m(x) \int_{-\infty}^{+\infty} dy_2 e_m(y_2) \varphi(y_2) = \varphi(0),
\] (76)
leads to the linear system
\[
X_n + \sum_m a_{nm} X_m = c_n,
\] (77)
where
\[
X_n = \int_{-\infty}^{+\infty} dy_2 e_n(y_2) \varphi(y_2),
\]
a_{nm} = \int_{-\infty}^{+\infty} dy_2 e_n(y_2) a_m(y_2),
\]
\[c_n = \varphi(0) \int_{-\infty}^{+\infty} dy_2 e_n(y_2).\] (78)
In terms of the solutions \{X_n\}, \varphi would become
\[
\varphi(x) = \varphi(0) - \sum_m X_m a_m(x_2).
\] (79)

We conclude this section with a comment on the relation between \text{dim ker}(a) and \text{dim ker}(a^\dagger). In the local case, only \text{a} has a normalizable zero mode; consequently the index relation
\[
\text{dim ker}(a) - \text{dim ker}(a^\dagger) = 1
\] (80)
is satisfied. This is characteristic of a harmonic oscillator. We saw in (75) that the quantum-corrected destruction operator \text{a} also has a normalizable zero mode; besides, it’s easy to see that the corrected operator \text{a}^\dagger does not have such a mode. As a result, quantum fluctuations to one-loop order preserve the relation [80]. It should be noted that in the compactified case, both \text{a} and \text{a}^\dagger have zero modes, so
\[
\text{dim ker }a - \text{dim ker }a^\dagger = 0.
\] (81)
The consequences of these index relations are studied, for example, in [12].
4 Conclusions

We have analyzed a generalization of the Callan-Harvey mechanism to the case of a non local mass in 2+1 dimensions, showing that for a certain set of assumptions about the non locality, there continues to exist a fermionic zero mode.

We have first considered a quite general non local term, deriving a linear integral equation for a function which modulates the zero mode of the local case, and accounts for the effect of the non local domain wall. Considering a defect of finite size, the Fredholm alternative theorem applies and there is a unique, localized chiral zero mode. Moreover, perturbation theory may be applied to calculate it: for the concrete example of a strongly diagonal mass, we have calculated the first few terms in a perturbative expansion, showing that they lead to negligible modifications with respect to the local mass case.

Finally, we have shown how radiative corrections for a system with a Dirac field coupled to a scalar field do generate a non local mass in the Dirac operator. In this case, we have been able to derive non perturbative quantum corrections to the fermionic zero mode, which should be relevant in the infinite size limit. That solution shows the phenomenon of a ‘splitting’ of the zero mode for the local case into a corrected zero mode in three stripes, as seen in Figure[4]. Of course, this is a reflection of the fact that the quantum corrected zero mode is a minimum of the one loop effective action, rather than of the classical action.

Acknowledgments

G. T. is supported by CNEA, Argentina. C. D. F. is supported by CONICET (Argentina), and by a Fundación Antorchas grant.

References

[1] C. G. Callan and J. A. Harvey, Nucl. Phys. B250, 427 (1985).

[2] T. L. Ho, J. Fulco, J. R. Schrieffer and F. Wilczek, Phys. Rev. Lett. 52, 1524 (1984);
D. Boyanovsky, E. Dagotto, E. Fradkin, Nucl. Phys. B285 [FS19], 340 (1987).
M. Stone, A. Garg and P. Muzikar, Phys. Rev. Lett. 55, 2328 (1985).
A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys.
Rev. B50, 7526 (1994).

[3] C. D. Fosco and A. Lopez, Nucl. Phys. B 538, 685 (1999).

[4] C. D. Fosco, A. Lopez and F. A. Schaposnik, Nucl. Phys. B 582, 716
(2000).

[5] E. Fradkin, C. D. Fosco and A. Lopez, Phys. Lett. B451: 31-37 (1999).

[6] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, New J. Phys. 4, 31
(2002).
A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, Nucl. Phys. B 648,
174 (2003).

[7] D. B. Kaplan, Phys. Lett. B288 342 (1992).
For a review of Kaplan’s formulation, see: K. Jansen,
Phys. Rept. 273: 1-54 (1996).

[8] R. Narayanan and H. Neuberger, Nucl. Phys. B443, 305 (1995). See also:
R. Narayanan, H. Neuberger and P. Vranas, Nucl. Phys. Proc. Suppl.
47: 596-598 (1996); R. Narayanan and H. Neuberger, Nucl. Phys. Proc.
Suppl. 47: 591-595 (1996).

[9] See, for example: L. Debnath and P. Mikusiński, Introduction to Hilbert
Spaces, with Applications, second edition, Academic Press, San Diego,
1999, chapter 5.

[10] R. Rajaraman, Solitons and Instantons. An Introduction to Solitons and
Instantons in Quantum Field Theory, Elsevier Science Publishers, Am-
sterdam, 1982, chapter 2.

[11] We follow the notation and conventions of: J. Zinn-Justin, Quantum
Field Theory and Critical Phenomena, second edition, Clarendon Press,
Oxford, 1993, chapter 6.

[12] K. Fujikawa, Phys. Rev. A 52, 3299 (1995).