Estimating the number of superimposed sinusoids

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Abstract—Estimation of the number of superimposed sinusoids in the presence of noise is an important model order selection (MOS) problem in statistical signal processing. In this paper, we propose a new approach to the design of MOS algorithms for estimating the number of superimposed sinusoids. Our proposed approach is partially based on the minimum error probability criterion. Also, we pay a lot of attention to the performance and consistency analysis of the MOS algorithms. In this study, an error probability is used as a universal performance measure of the MOS algorithms. We propose a theoretical framework that makes it possible to provide consistency analysis and to obtain closed-form expressions for the approximated error probabilities of a wide range of MOS algorithms. As an example, we applied this framework to the consistency and performance analysis of several MOS algorithms for estimating the number of superimposed sinusoids. Using the obtained results, we provide a parametric optimization of the presented MOS algorithms. Finally, we examine a quasilikelihood approach to the design and performance analysis of the MOS algorithms. The proposed theoretical framework is used to find the scope of the quasilikelihood approach.

Index Terms—model order selection, sinusoids in noise, superimposed sinusoids, minimum error probability criterion, abridged error probability, number of signals estimating, maximum likelihood, penalty term, quasilikelihood approach.

I. INTRODUCTION

In this paper, we study the problem of estimating the number of superimposed sinusoids. This important kind of model order selection (MOS) problem [1]–[3] arises in signal processing and its applications such as radar and sonar theory [4], [5], communication theory [6], astronomy [7], spectral analysis [2], and many others.

Modern MOS algorithms for estimating the number of superimposed sinusoids are based on various optimality criteria. For example, the Akaike information criterion (AIC) [8], the Bayesian information criterion (BIC) [9], [10], the Minimum Description Length (MDL) criterion [11], the maximum a posteriori criterion (MAP) [12], the Exponentially embedded families (EEF) criterion [13]–[15], the Penalizing adaptively likelihood (PAL) criterion [16], Residual Ratio Thresholding (RRT) [17], etc. Good surveys on the MOS criteria can be found, for instance, in [1], [18], and [3]. Also, there are some lesser-known approaches, see, for example, the nonparametric MOS procedure in [19] and the tensor approach in [20], [21]. In some rare cases, the maximum likelihood method can be directly used to estimate the model order, for example, [22]–[25]. The MOS criteria define the structure of the correspond-

The problem of estimating the number of superimposed sinusoids is a parametric MOS problem [3]. In a typical parametric MOS problem, the true model is the most appropriate model [3]. Hence, in this paper, we formulate the problem of estimating the number of superimposed sinusoids as a problem of finding the true number of superimposed sinusoids. In this case, the probability of the event that the estimation of the model order is not equal to the true model order (error probability) can be used as a performance measure of the MOS algorithms. One can equivalently use the probability of the complementary event known as the probability of correct detection (or correct estimation or correct model order estimation, etc). The error probability and other equivalent performance measures are very important for the problem of estimating the number of superimposed sinusoids [12]–[14], [18], [20], [21], [26]–[28].

If the goal is to find the true model order (true number of superimposed sinusoids), then it is also important to minimize the error probability. However, most existing MOS criteria do not minimize the error probability. In this paper, we propose a new approach to designing MOS algorithms. This approach is partially based on the minimum error probability criterion. Hence, MOS algorithms based on the proposed approach have better performance (in terms of the error probability) compared to algorithms based on the previously known criteria.

A significant part of this work is devoted to the performance analysis and consistency analysis of the MOS algorithms. We use the conception of the error probability to provide a universal performance measure for MOS algorithms. Next, we develop theoretical and numerical frameworks that make it possible to calculate this performance measure for a wide range of MOS algorithms. Also, the provided results allow to obtain consistency conditions for a wide range of MOS algorithms.

The problem of estimating the number of superimposed sinusoids involves the use of maximum likelihood (ML) frequency estimates in the MOS algorithms. This fact significantly increase the computational complexity of the MOS algorithms. Thus, we propose the so-called quasilikelihood (QL) approach to the design of the MOS algorithms. On the one hand, the proposed QL design approach can solve the mentioned computational complexity problem, but on the other hand, this approach has some limitations. We use the developed theoretical and numerical frameworks to find these limitations and justify the use of QL design approach. Moreover, the proposed theoretical framework makes it possible to determine whether a MOS algorithm is robust to errors in the used frequency values.

The paper is organized as follows. Section II presents the
formulation of the problem. Section III covers the modern penalty terms considered in this paper and presents a new approach to the design of MOS algorithms. A theoretical framework for the consistency and performance analysis is described in Section IV In Section V we collect the results connected with the QL approach. We present the main problem of one type of MOS algorithms based on the proposed design approach in Section VI, also, we present some possible solutions to this problem. Moreover, in Section VI, we bring together all the results previously obtained in this paper and formulate ways of future developing the proposed approach to the design of MOS algorithms. Section VII presents numerical results supplementing the theoretical ones.

II. Problem Formulation

We start from the general formulation of the problem of estimating the number of superimposed signals. Let the observed data \( x(t) \) be a signal composed of an unknown number of signals \( \nu \) at unknown parameters, and corrupted by the additive white Gaussian noise (AWGN). Thus, one can represent \( x(t) \) as

\[
x(t) = s_{\nu_0}(t, \Theta_{\nu_0}) + \sigma_0 n(t),
\]

where \( t \in \{t_1, \ldots, t_N\} \), \( N_s \) denotes the number of samples; \( \nu \in \{1, \ldots, \nu_s\} \) denotes the number of signals; \( \Theta_{\nu} = (\theta_1, \ldots, \theta_{\nu_s}) \) is a vector of the unknown parameters of the \( \nu \)-th signal, \( \mu_s \) is the number of the unknown parameters of the \( \nu \)-th signal, \( \Theta_{\nu} = (\theta_1, \ldots, \theta_{\nu_s}) \); \( n(t) \) is the AWGN, i.e., \( \{n(t)\}_{t=1}^{N_s} \) are independent identically distributed Gaussian random variables with zero means and unit variances; \( \sigma \) denotes a noise level. We denote the true values of the unknown parameters by subscript \( 0 \).

The problem is to estimate the unknown number of signals using the observed data \( x(t) \). This problem can also be interpreted as a MOS problem.

An algorithm of estimating the number of signals (hereafter also called a MOS algorithm) is essentially a function from the space of observed signals \( x(t) \) into the set \( \{1, \ldots, \nu_s\} \). Note that each MOS algorithm can be represented as

\[
\hat{\nu} = \arg \min_{\nu} R(\nu, x(t)),
\]

where \( R = R(\nu, x(t)) \) is a decision function that completely defines the corresponding MOS algorithm, \( \hat{\nu} \) is the estimate of the number of signals. In this paper, we use the decision function \( R \) and formalism [2] to define all the MOS algorithms. For many modern algorithms, the function \( R \) is the penalized log-likelihood function, thus we also use the following common representation of the decision function [13]:

\[
R(\nu, x(t)) = -L_\nu(\Theta_{\nu}, x(t)) + \Psi_{\nu},
\]

where \( L_\nu = L_\nu(\Theta_{\nu}, x(t)) \) is the logarithm of the likelihood function, \( \Theta_{\nu} \) denotes the maximum likelihood (ML) estimate of the block vector \( \Theta_{\nu} \), and \( \Psi_{\nu} \) is the so-called penalty term.

Let us consider the problem of estimating the number of modulated sinusoids embedded in AWGN. This problem includes the problem of estimating the number of superimposed sinusoids in AWGN as a special case. Model [1] of the observed data for the problem of estimating the number of modulated sinusoids can be represented as follows:

\[
x(t) = \sum_{i=1}^{\nu_0} a_{0i} f_i(t) \cos(\omega_{0i} t - \varphi_{0i} + \Psi_i(t)) + \sigma_0 n(t).
\]

where \( a_i \in \mathbb{R}^1 \) is the amplitude, \( \omega_i \in \Omega_i = (\omega_{1i}, \omega_{2i}) \) is the frequency, and \( \varphi_i \in [0, 2\pi] \) is the phase shift of the \( i \)-th signal, \( f_i(t) \), \( \Psi_i(t) \) denote the amplitude and phase envelope respectively. We assume that amplitudes and phases of the signals in [1] are unknown, frequencies of these signals are known or unknown or partially (see the definition below) unknown, and the noise level is known or unknown. The amplitude and phase envelopes we assume to be known.

In this paper, we use the traditional definition for the SNR \( z_i \) of the \( i \)-th signal in [1]

\[
z_i = a_{0i}^2/2\sigma_0^2, \quad z_i, dB = 10 \log_{10} \left( a_{0i}^2/2\sigma_0^2 \right).
\]

Under the above assumptions we can write the log-likelihood function for the observed data [1] as follows

\[
L_\nu = \frac{N_s}{2} \ln \left( 2\pi \sigma^2 \right) - \sum_{i=1}^{N_s} \left( x(t) - s_{\nu}(t, \Theta_{\nu}) \right)^2 / 2\sigma^2.
\]

Firstly, discuss the case of the known noise level. Some terms in [6] do not affect the algorithms considered in this paper, thus we can omit these terms and rewrite [6] without losing the generality as

\[
L_\nu = \frac{1}{\sigma^2} \sum_{i=1}^{N_s} x(t) s_{\nu}(t, \Theta_{\nu}) - \frac{1}{2\sigma^2} \sum_{i=1}^{N_s} s_{\nu}^2(t, \Theta_{\nu}),
\]

Next, suppose that the noise level is unknown. In this case, we should substitute the ML estimate of the noise level instead of \( \sigma^2 \) in [6]

\[
L_\nu(\hat{\sigma}_0^2) = - \frac{N_s}{2} \ln \left( \sum_{i=1}^{N_s} \left( x(t) - s_{\nu}(t, \hat{\Theta}_{\nu}) \right)^2 / N_s / 2\pi \right) - \frac{N_s}{2}.
\]

Finally, let us use model [4], substitute maximum likelihood estimates of the unknown amplitudes and phases to [7] and [8], and rewrite the result using the block matrix notation

\[
\hat{L}_\nu = \frac{1}{\hat{\sigma}_0^2} X_{2\nu}^T C_{2\nu}^{-1} X_{2\nu},
\]

\[
\hat{L}_\nu(\hat{\sigma}_0^2) = \frac{N_s}{2} \ln \left( \frac{\sum_{i=1}^{N_s} (x(t) - \hat{X}_{2\nu}^T C_{2\nu}^{-1} \hat{X}_{2\nu})^2}{N_s / 2\pi} \right) - \frac{N_s}{2},
\]

where the superscript \( T \) denotes the transpose; \( X_n = (X_i)_{i=1}^n \),

\[
X_i = \begin{cases} X_{ai}, & \text{if } i \text{ is even}, \\ X_{ai}, & \text{if } i \text{ is odd}; \end{cases}
\]

\[
C_{ij} = \begin{cases} C_{aij}, & \text{if } i \text{ is even and } j \text{ is even}, \\ C_{aij}, & \text{if } i \text{ is odd and } j \text{ is odd}, \\ C_{aij}, & \text{if } i \text{ is even and } j \text{ is odd}, \\ C_{aij}, & \text{if } i \text{ is odd and } j \text{ is even}; \end{cases}
\]

\[
\begin{align*}
X_{si} & = \sum_{t=1}^{N_x} x(t) f_i(t) \left( \cos(\omega_{i}t + \Psi_{i}(t)) \right), \\
\end{align*}
\]

In the case of unknown or partially unknown frequencies, we need to replace the frequencies in (9) by their estimates. In this paper, we study two different (in terms of frequency estimation) approaches to the design of MOS algorithms.

For the first approach (ML approach), we should substitute ML estimates of frequencies \( \hat{\omega}_i \) into (9) instead of the unknown frequencies \( \omega_i \): \( L_{\nu} = L_{\nu}(\hat{\omega}_i, x(t)) \), where \( \hat{\omega}_i = (\hat{\omega}_i)_{i=1}^{\nu} \). The ML approach is universal and well-known.

For the second approach (the quasi-likelihood (QL) approach, see [29], [30]) we should substitute arbitrary values \( \omega_i^* \in \Omega_i \) of frequencies into (9) instead of the unknown frequencies \( \omega_i \): \( L_{\nu} = L_{\nu}(\omega_i^*, x(t)) \). This simple estimate of frequencies we call a blind (BL) estimate. Moreover, if one uses BL estimates instead of the ML ones in some MOS algorithms we call such MOS algorithms QL algorithms (see, for example, [24], [25]).

The BL estimates and the QL approach can be useful for an estimation problem in which there are some a priori known domains \( \Omega_i \) containing unknown frequencies \( \omega_i \) and these domains are small enough. We call such an estimation problem a problem of estimating the number of modulated sinusoids with partially unknown frequencies.

III. ALGORITHMS DESIGN

Definition. A non-adaptive MOS algorithm is the MOS algorithm whose decision function has the following structure:

\[
R = -L_{\nu} + \gamma \kappa \nu,
\]

here \( \kappa = \phi(\nu)/\nu, \phi(\nu) \) denotes the number of independently adjusted parameters within the model (see, [8]), and \( \gamma \) is a constant or a function of \( N_s, \kappa \) or the other parameters. The generalized information criterion includes several different criteria (AIC, BIC, MDL, criterion in [28], etc.).

Non-adaptive algorithms have some flaws. In particular, non-adaptive algorithms are high-SNR inconsistent (see [18] or Theorem 2 in this paper). Adaptive MOS algorithms can solve the inconsistency flaw and thus they can be more efficient in many scenarios. Penalty functions of the adaptive algorithms do not have any fixed structure. In [18] and other sources one can find a wide range of adaptive MOS algorithms based on the various approaches. Here we provide algorithms based on the EEF [13], g-MDL [31], NMDL [32], and RRT [17] criteria as examples (note that, the RRT-based algorithm is used only in the case of the unknown noise level).

As mentioned in Section II, the error probability (or equivalent measures) is a very important performance measure of MOS algorithms for estimating the number of superimposed sinusoids. However, almost all modern MOS algorithms (both adaptive and non-adaptive) are not based on the minimum error probability criterion (or some related criteria). In practical applications, the minimum error probability criterion is also very important for most signal processing systems because the errors of the used MOS algorithm usually affect the performance of the system as a whole.

Here we present a new approach to the design of the MOS algorithms partially based on the minimum error probability criterion. Let us provide the following step-by-step description of this approach. Firstly, one should design a parametric family of MOS algorithms, i.e., it is needed to design a structure of the MOS algorithm that is defined up to some vector of parameters (the tuning parameters). This structure should generate consistent MOS algorithms at least on some subset (the consistent subset) of the tuning parameters. Note that each MOS algorithm from the designed parametric family corresponds to a single value of the tuning parameters vector. Secondly, it is needed to obtain the error probability of each consistent MOS algorithm from the designed family, i.e., we need to obtain the error probability of the corresponding MOS algorithm for each value of the tuning parameters vector from the consistent subset. Finally, one should find the optimum value of the tuning parameters vector in the sense of minimum error probability. The criteria based on the described approach are called the parametric minimum error probability (PMEP) criteria.

In this paper, we focus on two examples of applications of the PMEP approach and two corresponding penalty terms that were first proposed and used for the particular problems in [22], [23], [33], [34]. The first penalty term is called an invariant random penalty term, see [33, Eq. (30)]. Using representation (2) the MOS algorithm with the invariant random penalty term (PMEP-IR algorithm) can be defined by the following decision function

\[
R_{IR} = -L_{\nu} + \kappa_{IR} \nu \max_{i \in \{1,...,N\}} \left( L_i - L_{i-1} \right) ,
\]

where \( L_0 \) is the logarithm of the likelihood function in the case of \( \nu = 0 \) (i.e., under the assumption that there is no signal in the observed data), \( \kappa_{IR} > 0 \) is a tuning parameter.

The second penalty term is called an inverse penalty term, see [33, Eq.31] . The MOS algorithm with the inverse penalty term (PMEP-I algorithm) can be defined as

\[
R_{I} = -\left( L_{\nu} - L_0 \right)^{\kappa_{I}} / \nu ,
\]

where \( \kappa_{I} > 0 \) is a tuning parameter.

Following the above procedure, we should first find the subset of the tuning parameters \( \kappa_{IR} \) and \( \kappa_{I} \) such that algorithms (13) and (14) are consistent. Next, it is needed to find in the consistent set the optimal values of these tuning parameters for the problems under consideration.

It will be shown in Section VI that algorithms (13), (14) have some significant limitations. However, it is also important to note that these algorithms are only special cases of the
The PEAP approach, and we consider them here only as examples of the implementation of the PEAP approach.

IV. Consistency and Performance Analysis

A. Properties of the likelihood functions

First of all, let us provide a general framework that is needed for the further consistency and performance analysis of the MOS algorithms. Note that, the results below are obtained using the results given in Appendix (including a new representation for the Gram-Schmidt orthogonalization, see Theorem 4 in Appendix A-A).

Represent $\tilde{L}_\nu$ as

$$\tilde{L}_\nu = \frac{\nu}{2} \sum_{i=1}^{\nu} \left( \hat{L}_i - \hat{L}_{i-1} \right) = \frac{\nu}{2} \sum_{i=1}^{\nu} (\tilde{l}_i + \tilde{l}_i^*) = \frac{\nu}{2} \sum_{i=1}^{\nu} \tilde{l}_i^2, \quad (15)$$

where $\hat{L}_0 = 0$, $\tilde{l}_1^2 = \frac{1}{\sigma_0^2} X_i^T C_i^{-1} X_i$, $l_1^2 = l_1^2 + l_1^*$, and

$$l_1^2 = \frac{1}{\sigma_0^2} \left( X_i^T C_i^{1/2} X_i - X_i^T C_1^{1/2} X_i - 1 \right),
\text{and}

$$l_1^2 = \frac{1}{\sigma_0^2} \left( X_i^T C_i^{1/2} X_i - X_i^T C_{i-1}^{1/2} X_{i-1} - 1 \right), \quad (16)$$

Expressions for random variables $\{l_1^2 \}_{i=1}^N$ and $\{l_1^2 \}_{i=1}^N$ (up to sign) in (10) can be obtained in the same way as in [33] Eq. 37

$$l_{ci} = \frac{\gamma_{ci}}{\sigma_0^2} \left( X_{ci} - X_{ci}^T C_{ci}^{-1} R_{ci} \right) / \sqrt{\Theta_{ci}},$$

$$l_{si} = \frac{\gamma_{si}}{\sigma_0^2} \left( X_{si} - X_{si}^T C_{si}^{-1} R_{si} \right) / \sqrt{\Theta_{si}}, \quad (17)$$

where for all $i$:

$$\Theta_{ci} = C_{ci} - R_{ci} C_{ci}^{-1} R_{ci}, \quad \Theta_{si} = (C_{si}, \ldots, C_{si})^T, \quad \gamma_i = \pm 1.$$

From (17) and the definition of the random variables $\{X_i\}$ one can conclude that random variables $\{D_{ci}, D_{si}\}$ are Gaussian as the linear combinations of the Gaussian random variables. Note that random variables $\{D_{ci} \}_{i=1}^N$, $\{D_{si} \}_{i=1}^N$ can be considered as random fields $\{D_{ci}(\omega_i)\}_{i=1}^N, \{D_{si}(\omega_i)\}_{i=1}^N$, where $\omega_i = (\omega_j)_{j=1}^1$, Theorem 5 (see Appendix A-B) and the above results lead to the following consequences.

Theorem 1 The random fields from the following set

$$\left\{ \{l_1^2(\omega_i)\}_{i=1}^N \cup \{l_1^2(\omega_i)\}_{i=1}^N \right\}$$

are independent and unit-variance Gaussian random fields.

Also, if one suppose that $\omega_i = \omega_0, \forall i$, then the following result can be obtained using Theorem 6 (see Appendix A-B)

$$E(l_{ci}(\omega_0)) = E(l_{si}(\omega_0)) = 0, \quad \forall i > \nu_0. \quad (18)$$

Next, rewrite $\tilde{L}_\nu$ and $\hat{L}_\nu$ by analogy with (15):

$$\tilde{L}_\nu = \frac{\nu}{2} \sum_{i=1}^{\nu} (l_{ci}^2 + l_{si}^2), \quad \hat{L}_\nu = \frac{\nu}{2} \sum_{i=1}^{\nu} (l_{ci}^2 + l_{si}^2), \quad (19)$$

where random variables $l_{ci}^2$ and $l_{si}^2$ in (19) are obtained from $l_{ci}$ and $l_{si}$ in (15) by the replacement of $\omega_i$ by $\omega_i^*$; $l_{ci}$ and $l_{si}$ in (19) are obtained from $l_{ci}$ and $l_{si}$ in (15) by the replacement of $\omega_i$ by $\hat{\omega}_i$.

Taking into account Theorem 1 we can conclude that

$$\left\{ \{l_{ci}^2 \}_{i=1}^N \cup \{l_{si}^2 \}_{i=1}^N \right\}$$

are independent unit-variance Gaussian random variables for all $\{\omega^*_i\}$. Thus, cumulative distribution function (CDF) $F^*_i(x)$ and probability density function (PDF) $W^*_i(x)$ of $l_{ci}^2 + l_{si}^2$ can be represented as

$$F^*_i(x) = F_{NSQ}(x, d_{ci}^2 + d_{si}^2), W^*_i(x) = W_{NSQ}(x, d_{ci}^2 + d_{si}^2), \quad (20)$$

where $F_{NSQ}(x, \lambda)$ denotes CDF of the noncentral chi-square random variable with two degrees of freedom and noncentrality parameter $\lambda$; $W_{NSQ}(x, \lambda)$ denotes PDF of this random variable and $d_{ci}^2 = E(l_{ci}^2), d_{si}^2 = E(l_{si}^2)$, respectively. In particular, statistical properties of random variables $\{l_{ci}^2 \}_{i=1}^N \cup \{l_{si}^2 \}_{i=1}^N$ in the case of known frequencies can be obtained from (20) if we put $\omega^*_i = \omega_0 i$ for all $i$.

Now study statistical properties of random variables $\{l_{ci}^2 \}_{i=1}^N \cup \{l_{si}^2 \}_{i=1}^N$ for the ML method. In this case, we assume that the signals in (4) are orthogonal, i.e.,

$$\sum_{i=1}^{N_0} s_i(t, \theta_i) s_j(t, \theta_j) = \begin{cases} E_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \quad (21)$$

in terms of (11). Also, we assume that the energies of signals $E_i$ do not depend on $\omega_i$ and $\varphi_i$.

Taking into account condition (21), Theorem 1 and results from (34) one can conclude that random variables $\{l_{ci}^2 \}_{i=1}^N \cup \{l_{si}^2 \}_{i=1}^N$ are independent. Next, using a general result from (34) Eq. (25)) for our particular case we can obtain an approximate expression for CDF of $l_{ci}^2$

$$\hat{F}_{ci}(x) \approx \Phi \left( \frac{x - \bar{z}_{ci}}{\bar{z}_{ci}} \right) \exp \left[ -\frac{-\bar{z}_{ci}}{\bar{z}_{ci}} \exp \left( -\frac{x}{\bar{z}_{ci}} \right) \right], \quad \text{if } i \leq \nu_0,
\exp \left[ -\frac{-\bar{z}_{ci}}{\bar{z}_{ci}} \exp \left( -\frac{x}{\bar{z}_{ci}} \right) \right], \quad \text{if } i > \nu_0, \quad (22)$$

for $x > 0$, and $\hat{F}_{ci}(x) = 0$ for $x < 0$. Here, $z_{ci} = z_i \cos(\varphi_0)$. The accuracy of formula (22) is increasing with increasing the SNR or $\xi_{ci}$. The CDF $\hat{F}_{ci}(x)$ for the random variable $\hat{L}_{ci}^2$ can be obtained from (22) by changing $\xi_{ci}$ to $\xi_{ci}$ and $z_{ci}$ to $z_{ci} = z_i \sin(\varphi_0)$.

Finally, the CDF and PDF for the random variable $l_{ci}^2 + l_{si}^2$ can be written as

$$\hat{F}_{i}(x) = \int_{-\infty}^{\infty} \hat{F}_{ci}(x - y) d\hat{F}_{si}(y), \quad \hat{W}_{i}(x) = \frac{d\hat{F}_{i}(x)}{dx}. \quad (23)$$

For brevity, in this paper we shall assume that $V_i = \bar{l}_{ci}^2 + \bar{l}_{si}^2$, $\hat{F}_i(x) = F^*_i(x)$, $W_i(x) = W^*_i(x)$ if we study the QL design approach or that $V_i = \bar{l}_{ci}^2 + \bar{l}_{si}^2$, $\hat{F}_i(x) = \hat{F}_i(x)$, $W_i(x) = \hat{W}_i(x)$ if we study the ML design approach or that $V_i = l_{ci}^2 + l_{si}^2$ if we study the general case.

B. Consistency

Now find the ranges of the values of tuning parameters $\kappa_{ir}$ and $\kappa_i$, which provide the consistency of MOS algorithms [13] and [14], respectively. Let us represent SNRs (5) of all signals
as $z_i = z_{\nu_0 i}$. Next, we can write a definition of the SNR-consistency. In terms of (2), this definition can be written as

$$\Pr \left( \bigcap_{i \neq \nu_0} \left\{ R(\nu_0, x(t)) - R(\nu, x(t)) < 0 \right\} \right) \xrightarrow{z_0 \to \infty} 1. \quad (24)$$

Note that the concept of SNR-consistency (24) generalizes the same concept in [18].

Suppose that the noise level is known. In the case of known frequencies and in the case of the QL design approach to the estimation of unknown frequencies, using representation (15) we can rewrite (24) as $\Pr (\mathcal{C}) \xrightarrow{z_0 \to \infty} 1$, where $\mathcal{C}$ denotes the following intersection of events

$$\bigcap_k \left\{ \sum_{i=\nu_0+1}^{\nu_0+k} V_i < \kappa\nu_k \max_{i \in \{1, \ldots, N\}} V_i < \sum_{i=\nu_0-k+1}^{\nu_0} V_i \right\} \quad (25)$$

for algorithm (13), or the next intersection of events

$$\bigcap_k \left\{ \frac{\sum_{i=\nu_0+1}^{\nu_0+k} V_i}{\nu_0+k} < \frac{\sum_{i=1}^{\nu_0-k+1} V_i}{\nu_0-k+1} \right\} \quad (26)$$

for algorithm (14), respectively. In (25) and (26) the number $1 \leq k \leq N - \nu_0$ for the left-hand inequalities, and $1 \leq k \leq \nu_0 - 1$ for the right-hand inequalities, respectively.

Taking into account (17) one can conclude that

$$d_i^2 = \left( d_{ni}^2 + \frac{2}{\sigma_i^2} \right) z_{\nu_0}^2, \quad (27)$$

here $d_i^2 = E(V_i)$, and $d_{ni}^2$ does not depend on $z_0$ for all $i$.

Let us formulate without proof the next simple lemma.

Lemma 1 Suppose $\{\mathcal{A}_i\}_{i=1}^N$ is a finite set of events. If all probabilities $\{\Pr (\mathcal{A}_i)\}_{i=1}^N$ tend to unity, then the probability $\Pr \left( \bigcap_{i=1}^N \mathcal{A}_i \right)$ also tends to unity.

In the case of known frequencies, using (18), (25), (26), and Lemma 1 one can find that condition (24) is satisfied for algorithms (13) and (14) if and only if the following inequalities

$$\kappa I_R k \max_{i \in \{1, \ldots, N\}} d_{ni}^2 < \sum_{i=\nu_0-k+1}^{\nu_0} d_{ni}^2 + \sum_{i=\nu_0}^{\nu_0+k+1} d_{ni}^2 \left( \frac{1}{\nu_0-k+1} \right), \quad (28)$$

hold for all $1 \leq k \leq \nu_0 - 1$.

Inequalities (28) can be replaced by the following ones

$$\kappa I_R < \rho, \kappa I > \ln N / \ln (\rho / N + 1), \quad (29)$$

where $d_{\min}^2 = \min d_{ni}^2$, $d_{\max}^2 = \max d_{ni}^2$, and $\rho = d_{\min}^2 / d_{\max}^2$ (please, see more about the parameter $\rho$ in Section VI). Inequalities (29) reject some sufficient values of the tuning parameters, but they are quite simpler than inequalities (28). One can conclude that, if the tuning parameters $\kappa I_R$ and $\kappa I$ satisfy conditions (28) or (29), then the corresponding algorithms are SNR-consistent.

In the case of the QL design approach to the estimation of frequencies, one can find that for $i > \nu_0$ expectations $d_i^2$ can be non-zero (see Section VII), thus in this case, we get consistency conditions as follows

$$\sum_{i=\nu_0+k}^{\nu_0} d_{ni}^2 < \kappa I_R \sum_{i=\nu_0-k+1}^{\nu_0} d_{ni}^2 + \sum_{i=\nu_0}^{\nu_0+k+1} d_{ni}^2 \left( \frac{1}{\nu_0-k+1} \right), \quad (30)$$

for all $1 \leq k \leq N - \nu_0$ for the left-hand inequalities, and for all $1 \leq k \leq \nu_0 - 1$ for the right-hand inequalities, respectively.

Let us study the case of the known noise level. Comparing (7) and (10) one can write

$$\tilde{\mathcal{L}}_i \left( \sigma^2 \right) - \tilde{\mathcal{L}}_{i-1} \left( \sigma^2 \right) \xrightarrow{z_0 \to \infty} d_{\sigma ni}^2, \quad (31)$$

where terms $d_{\sigma ni}^2$ can be obtained from $d_{0 ji}^2 = E(\omega_{ij}^2 + \omega_{ij}^2)$ in accordance with formula (27). We can use formulas (9), (10), and (31), representation (15), and also the approach to the consistency analysis presented in this subsection (see (24)-(27)) to obtain the consistency conditions for algorithms (13) and (14) for the case of known frequencies and unknown noise level:

$$\kappa I_R < 1 / (N - \nu_0), \kappa I > 0. \quad (32)$$

Note that consistency conditions (32) do not contain the parameter $\rho$ or any other similar parameters. Next, if the value of $N$ grows, then the range of turning parameters where the PMP-ER algorithm is consistent decreases. Consistency conditions for the QL design approach can be obtained by replacing in (30) terms $d_{\min}^2$ by terms $d_{\max}^2$ from (31).

Finally, let us formulate the following conjecture: in the case of the ML design approach to the estimation of unknown frequencies one can use the same consistency analysis as in the case of the known frequencies. In particular, consistency conditions (28), (29), and (32) are valid in the case of the ML approach. This conjecture is based on the consistency of the ML estimates of frequencies.

Consistency analysis of MOS algorithms proposed in this subsection can be extended to the case when the number of samples tends to infinity. This extension can be done directly if the decision function of the MOS algorithm depends on the observed data and on the number of samples only through the likelihood function (9) or (10). In particular, for algorithms (13) and (14), one can directly obtain consistency conditions similar to (28), (29), (30), (32) and (30) with (31) using consistency analysis presented in this subsection.

It is important to note that the presented consistency analysis can be extended to a wide range of MOS algorithms. In particular, for SNR-consistency, this extension can be done directly if the decision function of the MOS algorithm depends on the observed data only through the likelihood function.
(see [9] or [10]) and/or the energy \( \sum_{t=1}^{N} x^2(t) \) of the observed data. For example, decision functions of EEF, g-MDL, NMDL algorithms (see [13] TABLE II), and decision function \( (44) \) satisfy this condition.

C. Abridged Error Probability

First of all, introduce the following definition. The error probability of the MOS algorithm \( p_e \) is the probability that the estimate obtained by this algorithm does not match with the true number of signals, i.e., \( p_e = \Pr(\hat{\nu} \neq \nu_0) \). Using representation \( (2) \) we can rewrite the error probability \( p_e \) as

\[
P_e = 1 - \Pr \left( \bigcap_{i=1, i \neq \nu_0}^N \{ R(\nu_0) < R(\nu_0) \} \right) \tag{33}
\]

Performance analysis by the error probability \( (33) \) is quite difficult. Some approximations of the error probability were introduced in recent works (for example, \([18, \text{TABLE II}]\)). However, these approximations are primarily made for non-adaptive MOS algorithms. Moreover, these approximations do not take into account the general design specifics of the MOS algorithms. Namely, in practice applications, the different values of the errors of the MOS algorithms have different effects on the performance of the system as a whole. For example, in the case of \( |\hat{\nu} - \nu_0| = 1 \) the degradation of the system performance as a whole is usually quite less than in the case of \( |\hat{\nu} - \nu_0| > 1 \). Hence, one may choose the next additional condition for the most efficient MOS algorithms

\[
\frac{p_e(\hat{\nu} - \nu_0 > 1)}{p_e(\hat{\nu} - \nu_0 = 1)} \rightarrow 0, \tag{34}
\]

as the signal-to-noise ratio (SNR) or the number of samples tends to infinity.

Taking into account \( (34) \) we use an abridged error probability \( (23) \) as a universal approximation to the error probability \( (33) \)

\[
p_a \triangleq 1 - \Pr \left( R(\nu_0) < R(\nu_0+1), R(\nu_0) < R(\nu_0-1) \right). \tag{35}
\]

Let us list some common properties of the abridged error probability.

P.1. In the case of \( 1 < \nu_0 < N \), the abridged error probability \( (35) \) is a lower bound of the error probability \( (33) \).

P.2. In the case of \( \nu_0 = 2, N = 3 \), the error probability \( (33) \) is reduced to the abridged error probability \( (35) \).

P.3. Assume that the absolute values of \( R(\nu_0-1) \) and \( R(\nu_0+1) \) grow with the SNR or the number of samples faster than all terms from the set \( \{ R(i) \}_{i=1}^{\nu_0-2} \cup \{ R(i) \}_{i=\nu_0+2}^N \), i.e., if the SNR or the number of samples tends to infinity, then

\[
\frac{|R(i)|}{|R(\nu_0-1)|} \rightarrow 0 \quad \text{and} \quad \frac{|R(i)|}{|R(\nu_0+1)|} \rightarrow 0 \tag{36}
\]

for all \( i \in \{1, \ldots, \nu_0-2, \nu_0+2, \ldots, N\} \). Under assumption \( (36) \), abridged error probability \( (35) \) tends to the error probability \( (33) \) as the SNR or the number of samples tends to infinity.

Note that, if condition \( (36) \) is not fulfilled, then condition \( (34) \) is also not fulfilled, i.e., the truth of condition \( (36) \) is a necessary property for an efficient algorithm. In practice, even if condition \( (36) \) is not satisfied (for example, for all GIC algorithms condition \( (36) \) is not satisfied), the abridged error probability can be an accurate enough approximation to the error probability (see, for example, Fig. [7] and [9]). In particular, if instead of zero in conditions \( (36) \) there is a sufficiently small constant, then the approximation with increasing SNR will also be sufficiently accurate.

D. Theoretical calculation of the abridged error probability

Let us calculate the abridged error probability (determined \( (35) \) for algorithms \([12, 13, 14]\) in the case of known noise level. The results obtained in this section generalize the ones in \([33]\) see eq. 46, 51, 53] that were obtained for the particular case of the known frequencies.

1. Algorithms based on the GIC. Using \( (35) \) and results from Section IV-A we obtain the abridged error probability of algorithm \([12]\)

\[
p_{agiC} = 1 - \Pr(V_{\nu_0+1} < 2\nu_0, V_{\nu_0} > 2\nu_0) \tag{37}
\]

\[
= 1 - \Pr(V_{\nu_0+1} < 2\nu_0) Pr(V_{\nu_0} > 2\nu_0) = 1 - F_{\nu_0+1}(2\nu_0) + F_{\nu_0+1}(2\nu_0) F_{\nu_0}(2\nu_0). \tag{38}
\]

Note that, if the SNR or \( \nu_0 \) tends to infinity, then using \( (15) \) and \( (17) \) we get \( \Pr(V_{\nu_0} > 2\nu_0) \rightarrow 1 \). Combining this result with \( (37) \) one obtain

\[
p_{agiC} \rightarrow 1 - \Pr(V_{\nu_0+1} < 2\nu_0) = 1 - F_{\nu_0+1}(2\nu_0). \tag{38}
\]

If we do not consider the QL design approach, then the right side of \( (38) \) does not depend on SNR.

Result \( (37) \) can be used to design new MOS algorithms. Let us provide some examples. Firstly, we fix some arbitrary values of SNRs \( z = (z_i) \) and calculate the corresponding values of \( (d_{\nu_0}^z) \) using \( (11) \) and \( (17) \). Next, we find an optimal value of the parameter \( T \) (from \( (12) \)) as

\[
T_{opt}(z) = \arg \min_T p_{agiC}(T, z). \tag{39}
\]

This approach guarantees that the MOS algorithm \([12]\) has the smallest abridged error probability for given SNRs values \( z \). Moreover, if we have some estimation \( \hat{z} \) of the SNRs, then we can use \( \hat{z} \) instead of \( z \) in \( (39) \).

2. PMEP-IR algorithm. Firstly, rewrite the abridged error probability \( (35) \) for algorithm \([13]\) as

\[
p_{aIR} = 1 - \Pr(V_{\nu_0+1} < \kappa_{IR}: \max V_i < \kappa_{IR}: \max V_i). \tag{35}
\]

Using this formula and the result \([33, \text{Eq. } 50] \) we can obtain the following equation for the abridged error probability of algorithm \([13]\)

\[
p_{aIR} = 1 - \int_0^{\infty} W_{\nu_0}(x) F_{\max}(\frac{x}{\kappa_{IR}}) F_{\nu_0+1}(x) dx
\]

\[
+ \int_0^{\infty} W_{\nu_0+1}(x) F_{\max}(\frac{x}{\kappa_{IR}}) F_{\nu_0}(\frac{x}{\kappa_{IR}}) F_{\nu_0}(x) dx, \tag{40}
\]
where $F_{\text{max}}$ denotes the CDF of $\max_{i \in \{1, \ldots, N\}} V_i$, and can be calculated as
\[
F_{\text{max}}(x) = \prod_{i=1}^{N} F_i(x).
\]

3. PMEP-I algorithm. The abridged error probability for algorithm (14) can be rewritten as
\[
p_{a,i} = 1 - \Pr \left( \frac{V_{0,i} + 1}{B} > V_{0,i} - \sum_{i=1}^{v_0-1} V_i \leq V_{0,i}/A \right) = 1 - \int_{0}^{\infty} W_{0,i}(y) \int_{0}^{\infty} W_{0,i}(x) \times [F_{\Sigma}(x/A) - F_{\Sigma}(y/B - x)] \, dx \, dy,
\]
where $A = \sqrt{\frac{\nu_{0,i} - 1}{\nu_{0,i} + 1}} - 1$, $B = \sqrt{\frac{\nu_{0,i} + 1}{\nu_{0,i} - 1}} - 1$, $F_{\Sigma}$ is the CDF of $\sum_{i=1}^{\nu_{0,i}-1} V_i$.

Formulas (20), (23) with abridged error probability formulas (37), (40), (41) allow us to provide a performance analysis of the MOS algorithms based on (12), (13) and (14).

It is important to note that the presented theoretical performance analysis can be extended to a wide range of MOS algorithms. In particular, one can use presented performance analysis to study the performance of algorithm (14).

V. QUASILIKELIHOOD APPROACH

A. Frequency errors and MOS algorithms robustness

Let us start with the case of known frequencies. In this case, using result (37), statistical properties of random variables $\{l_{ci} \}_{i=1}^{N}$, $\{l_{si} \}_{i=1}^{N}$ (see Section VI-A), and property P.1. of the abridged error probability one can find the next theorem.

**Theorem 2** If $\nu_0 < N$, then the error probability of any non-adaptive MOS algorithm tends to some non-zero constant with the SNR or the number of samples tends to infinity.

Suppose that the BL estimates of frequencies $\omega_i^*$ are used in some non-adaptive MOS algorithm, then the following theorem can be stated.

**Theorem 3** If $\omega_i^* \neq \omega_{0,i}$ and $\nu_0 < N$, then the error probability of the non-adaptive MOS algorithm tends to unity with the SNR tends to infinity.

**Proof:** We give here a sketch of the proof. If the condition $\omega_i = \omega_{0,i}$ is not fulfilled, then equalities (18) are also broken. Moreover, in this case, the expectations from (18) depend on the true values of the amplitudes. Indeed, suppose that $\omega_i = \omega_{0,i}^*$, $E(l_{ci}^*)$ and $E(l_{si}^*)$ can be rewritten (up to sign) as
\[
E \left( \frac{X_i^{*} - X_i^{*} T C_{i-1}^{*} r_i^{*}}{\sigma_0 \sqrt{E_i^{*}}} \right) = Q_{0,0} \left( \frac{r_i^{*} - C_{i-1}^{*} T C_{i-2}^{*} r_i^{*}}{\sqrt{E_i^{*}}} \right),
\]
where $X_i^* = \{X_i^{*} \}_{j=1}^{N}$, $r_i^* = \{r_i^{*} \}_{j=1}^{N}$, $C_i^{*} = \{C_{i,a,b}^{*} \}_{a,b=1}^{N}$, $C_i^{*} = \{C_{i,a,b}^{*} \}_{a,b=1}^{N}$, $C_i^{*} = \{C_{i,a,b}^{*} \}_{a,b=1}^{N}$.

Finally, for $i > \nu_0$ equation (42) can be rewritten as
\[
E(l_{ci}^*) = Q_{0,0} k_{ci}, E(l_{si}^*) = Q_{0,0} k_{si},
\]
where $i > \nu_0$, $k_{ci}$ and $k_{si}$ do not depend on the true values of the amplitudes, noise level, and SNR.

Summing up, on the one hand, result (43) show that for $i > \nu_0$ the expected values of the random variables $l_{ci}^*$ and $l_{si}^*$ grow with SNR, but, on the other hand, the penalty terms of the non-adaptive algorithms still do not depend on the SNR. These facts imply Theorem 3.

Suppose we design the MOS algorithm in the case of a priori known frequencies. If the error probability of this algorithm saves its behavior (for SNR tends to infinity) in the case of some errors in using frequencies values (i.e., the error probability, for example, still monotonically tends to zero or some non-zero constant), then we call such MOS algorithm the BL-robust algorithm.

Theorem 3 shows that non-adaptive MOS algorithms are not BL-robust. Note that, the numerical analysis in Section VII shows that algorithms EEF, gMDL, NMDL, and RRT are also not BL-robust (see Fig. 3 for the known noise level and Fig. 4 for the unknown noise level).

B. Quasilikelihood performance analysis

Note that the performance analysis based on the formula (23) considers all statistical information about the ML estimates of the frequencies, and vice versa the analysis based on the formula (20) does not take into account this information. On the other hand, comparing results (20), (23) we can conclude that the theoretical performance analysis in the case of the ML estimation of the frequencies is quite more difficult than in the case of the BL estimation. Moreover, in practice, one may use some fast or simplified modifications of the maximum likelihood method for estimating the frequencies, thus the actual probability distribution of the estimates of frequencies can be very different from the distribution of the exact ML estimates. These reasons motivate us to propose a new approach to the performance analysis of the MOS algorithms, and this approach is called QL performance analysis.

Substitute $F_{\text{max}}(x)$ from (20) to the formulas for the abridged error probabilities (37), (40), (41). Note that these formulas are still valid for any values of $\omega_i^*$ due to Theorem 1.

Next, we can represent these probabilities as functions of the frequencies errors $p_{a,i} = \text{c}(\Delta \omega_i)$, $p_{a,i} = \text{c}(\Delta \omega_i)$, where $\Delta \omega_i = \omega_i - \omega_{0,i}$.

Finally, we provide a performance measure that uses these functions only: $p_{aq} = \sum_{i=1}^{N} p_{a,i} (\Delta \omega_i) \sigma(\Delta \omega_i)$, where $\{\Delta \omega_i \}_{i=1}^{N}$ is a set of the frequencies errors and $\sigma(\Delta \omega)$ is a loss function (for instance, $\sigma(\Delta \omega) = \Delta \omega^2$). The measure $p_{aq}$ shows the impact of the different errors in the used frequencies values over the MOS algorithm performance. The loss
function allows specifying the errors contributions. Moreover, if the probabilities of frequencies errors \( p_{i \Delta \omega} \) are known, then one can also use them and calculate weighted average abridged error probability as \( p_{\text{waa}} = \sum_{i=1}^{N} p_{i \Delta \omega} \cdot p_i(\Delta \omega_i) \mathcal{L}(\Delta \omega_i) \).

C. Practical applications

Let us provide some practical examples of the application of the QL approach. Consider situations when the frequencies of sinusoids in (4) are a priori known, but true values \( \omega_{0i} \) of these frequencies can deviate from the known values due to some causes. Describe some common causes for such deviations. The first cause is the Doppler effect. If the source (or sources) of the sinusoids in (4) has some radial velocity relative to the receiver, then the frequencies \( \omega_{0i} \) of the observed sinusoids in (4) will be shifted from their known values [37], [38]. The second cause is the hardware imperfection, for example, drift and/or offsets of frequencies of local oscillators [39].

In practice, the first way is to treat this case as a MOS problem with unknown frequencies, thus it is necessary to estimate the unknown frequencies and use these estimates in (2) or (10). This way involves computational difficulties. The second way is to use a priori known frequency values even though the true frequencies can deviate from them. Usually, the upper bound of possible deviations of the true frequency values from a priori known frequency values can be found. For example, it is usually possible to set some upper bound on the possible speed of the signal source in a given practical scenario. Using this upper bound of possible deviations and QL performance analysis, one can choose between the first and the second way (see, for example, Fig. 7). Note that if the second way is chosen, then one should use BL-robust MOS algorithms to avoid the problems described in Section V-A.

VI. Future work

Formulas [29] indirectly point to one of the main problems of algorithms [13], (14) and other similar algorithms: their performance degrades with increasing the value of \( u = \max(a_{0i}/\min a_{0i}) \). Let us call this ratio amplitudes unevenness. Using (17) one can conclude that the increasing \( u \) implies decreasing \( p \). We have done a deeper numerical analysis of algorithms [13], (14) and it also approved the problem (for example, see Fig. 8). On the other hand, for example, GIC algorithms are free from such problems. Let us propose a qualitative explanation of this phenomenon. First of all, we represent the decision function (see (2)) as

\[
R(\nu, x(t)) = \sum_{i=1}^{N} \left[ R(i, x(t)) - R(i-1, x(t)) \right]
\]

\[
= \sum_{i=1}^{N} \left[ \left( \tilde{L}_{i-1} - \tilde{L}_{i} \right) + (\Psi_i - \Psi_{i-1}) \right] = \sum_{i=1}^{N} -\frac{V_i}{2} + p_i,
\]

here we use representations (3) and (15). Let us characterize the contribution of individual terms \( V_i \) and \( p_i \) to the final value of decision function \( R \) by the values of their expectations: \( d_{V_i}^2 = E(V_i) \) and \( \delta_i = E(p_i) \).

Next, we consider two types of penalty functions: local and global penalty functions. We assume that each term \( p_i \) of the local penalty function can depend on the observed data \( x(t) \) only through the term \( V_i \). On the other hand, each term \( p_i \) of the global penalty function can, for example, statistically depend on all terms in \( \{V_j\}_{j=1}^{\nu=\nu_0} \). It implies that all values in the set \( \{d_{V_i}^2\}_{i=1}^{\nu} \) or \( \{d_{V_{\nu,j}}^2\}_{j=1}^{\nu=\nu_0} \) can make some contribution to \( \delta_i \). This leads to both the advantages of the global penalty over the local one and the disadvantages. On the one hand, the global penalty can improve the performance of the MOS algorithm since the global penalty can enhance the growth of the decision function for \( \nu > \nu_0 \) (with SNR increasing). For the local penalty, if \( \nu > \nu_0 \), then \( d_{V_i}^2 = 0 \) due to Theorem 6. It is important to note that one can extend Theorem 2 to the case of the local penalty term. On the other hand, the global penalty can make the MOS algorithm sensitive to the unevenness of the signal amplitudes, because the global penalty can suppress the contribution of the weak signals to the decision function if \( u \gg 1 \) (i.e., \( \rho \ll 1 \)). For the local penalty, the terms corresponding to the strong signals do not affect the terms corresponding to the weak signals.

A. Complex penalty

These observations indicate the advisability of using the local and global penalty functions together in a single complex penalty function, moreover, the contribution of the global part of the penalty function should be significant only for sufficiently large SNR (compared to the value of \( 1/\rho \)).

Based on the results given in this paper, we introduce a weak optimality criterion (WOC) for MOS algorithms (instead of the minimum error probability criterion). According to WOC, the MOS algorithm must satisfy the following conditions.

WOC1: Consistency (see (24) for definition) at least for a certain range of tuning parameters.

WOC2: BL-robustness (see Section V).

WOC3: Assumption (35).

WOC4: Tuning parameters should be chosen to minimize the abridged error probability.

WOC5: Low sensitivity to the amplitudes unevenness.

If some penalty term contains both local and global components, then we call this term a complex penalty term. Let us demonstrate an example of a PMEP algorithm based on the complex penalty term (PMEPc algorithm)

\[
R_c = -L_\nu + \kappa_{1c}L_\nu + \kappa_{2c}p \left( \max_{i \in \{1, \ldots, N\}} (L_i - L_{i-1}) \right)^{\kappa_{3c}}, \quad (44)
\]

where \( \kappa_{1c} \), \( \kappa_{2c} \), and \( \kappa_{3c} \) are tuning parameters. Note that, parameters \( \kappa_{1c} \) and \( \kappa_{2c} \) serve to minimize error probability, and parameter \( \kappa_{3c} \) serves to provide a balance between the local \( \kappa_{1c}L_\nu \) and global \( \kappa_{2c}p \) components of the complex penalty term for different values of SNR.

Next, let us describe the steps that need to be taken to provide optimality of algorithm (44) in the sense of WOC. Firstly, we need to find ranges of tuning parameters such that PMEPc MOS algorithm (44) is consistent (see WOC1). One can directly use the methods presented in Section IV-B to find these ranges. Secondly, we need to find the values of the tuning parameters for which algorithm (44) satisfies BL-robustness (WOC2), WOC3, and low sensitivity to the
amplitudes unevenness (WOC5) conditions taking into account the results of the first step. Essentially, it is necessary to choose (according to SNR) such a balance between the local and global penalties using tuning parameters so that algorithm (44) satisfies all these three conditions. Finally, we find values of tuning parameters that minimize the abridged error probability of algorithm (44) and belong to the ranges obtained as a result of the first two steps. One can use methods described in Section IV-D to find an analytical expression for the abridged error probability of algorithm (44). This expression can help to solve problems formulated both in the second and final steps.

B. Adaptation of tuning parameters

Note that the described problem of the high sensitivity of PMEP MOS algorithms (13) and (14) to amplitudes unevenness can be solved in a different way for a wide range of practical applications. Suppose that $z_1, z_2, \ldots$ are true values of SNRs and they can tend to infinity independently (not only as specified in Section IV-B). Let the parameter $u$ be known or can be estimated or can be bounded in some way. In this case, one can fix the high sensitivity of PMEP MOS algorithms (13) and (14) to amplitudes unevenness using the true value of $u$ (or its estimate, or its lower bound). For example, in the case of PMEP-IR algorithm (13), we should change the tuning parameter $\kappa_{IR}$ to the following $\hat{u}\kappa_{IR}$, where $\hat{u}$ is a true value or an estimate or a lower bound of the parameter $u$.

VII. SIMULATIONS

In this section, we numerically study the problem of estimating the number of superimposed sinusoids as a special case of problem (4), thus we assume that $f_i(t) = 1$ and $\Psi_i(t) = 0$ for all $i$. We use settings and notations similar to those of [26]–[28], [40]. We suppose that $\alpha_{0i} = 2\pi(0.2 + (i - 1)/N)$, $\varphi_{01} = 0$, $\varphi_{02} = -\pi/8$, $\varphi_{03} = -\pi/6$, for $i > 3$ $\varphi_{0i} = -\pi/(i + 3)$, and $\forall \alpha_{0i} = \alpha$. For all simulations, it is supposed (without loss of generality) that $\sigma_0 = 1$, thus we tune the SNRs of algorithms (45) by the amplitudes only. We provide simulations for six cases: a priori known frequencies, unknown frequencies (each iteration requires ML estimation of frequencies), partially unknown frequencies (in this case we use the BL estimates of frequencies only), and for a priori known and unknown noise level. In the case of unknown frequencies, one has to numerically estimate these frequencies, however, it is a complex computational problem. We use an approach that is similar to that presented in [40] to resolve these difficulties.

Besides the above settings, we need to find optimal in the sense of minimum error probability values of the tuning parameters for the numerical studies of the PMEP algorithms. Firstly, we find

$$\kappa_{IR}^{opt/abr} = \arg \min_{\kappa_{IR}} p_{aIR}(\kappa_{IR}), \quad \kappa_{I}^{opt/abr} = \arg \min_{\kappa_{I}} p_{aI}(\kappa_{I}),$$

(45)

where abridged error probabilities $p_{aIR}(\kappa_{IR})$ and $p_{aI}(\kappa_{I})$ are defined by formulas (40) and (41), respectively, in the case of $N = 5$, $v_0 = 3$, and SNR = −4 dB. Moreover, we find optimal values of tuning parameters $\kappa_{IR}$ and $\kappa_{I}$ in limited ranges obtained in [29] and [32] to ensure that resulting MOS algorithms with tuning parameters (45) are consistent. Secondly, we use numerical methods to calculate error probabilities $p_{eIR}(\kappa_{IR})$ and $p_{eI}(\kappa_{I})$ of algorithms (13) and (14) in the case of unknown frequencies, $N = 5$, $v_0 = 3$, and SNR = −4 dB. Next, we find $\kappa_{IR}^{opt} = \arg \min_{\kappa_{IR}} p_{eIR}(\kappa_{IR})$, $\kappa_{I}^{opt} = \arg \min_{\kappa_{I}} p_{eI}(\kappa_{I})$, Finally, we obtain the following result

$$\kappa_{IR}^{opt} \approx \kappa_{IR}^{opt/abr} = 0.25, \quad \kappa_{I}^{opt} \approx \kappa_{I}^{opt/abr} = 3.$$
Fig. 2. Error probabilities $p_e$ of the different MOS algorithms versus SNR. A priori known frequencies and unknown noise level case.

Fig. 3. Abridged error probabilities $p_a$ and error probabilities $p_e$ of the different MOS algorithms versus SNR. In this figure, all presented MOS algorithms are designed within the QL approach. Here, we suppose that $\Delta \omega = 0.0025$. A priori known noise level case.

$p_e$ calculated numerically for PMEP-IR, PMEP-I, GIC, EEF, gMDL, NMDL algorithms in the case of a priori known frequencies and unknown noise level, all as functions of SNR. Note that, in the case of the unknown noise level we use the form of EEF algorithm from [14].

Now let us study the QL approach to the design of MOS algorithms (the case of partially unknown frequencies). Figures 3 and 4 show the same dependencies with the same settings as in Fig. 1 and Fig. 2, respectively. However, in this case, we use the BL estimates $\omega^*_i = \omega_{0i} + \Delta \omega$ (with the error in all BL estimates $\Delta \omega = 0.0025 \approx 5\% (\omega_{0i+1} - \omega_{0i})$) instead of $\omega_{0i}$ to design all the studied MOS algorithms. Fig. 3 confirms (see the dash-dotted line and triangles) Theorem 3, i.e., it confirms that the GIC algorithm is not robust to errors in used frequencies values. Moreover, this figure shows that the EEF, gMDL, and NMDL algorithms are also not robust to the errors in used frequencies values (see asterisks and pentagrams). Figure 4 confirms these conclusions in the case of the unknown noise level. Also, in this particular case, Fig. 3 shows that starting with $\text{SNR} = -4$ dB the error probability can also be well approximated by the abridged error probability.

Figures 5 and 6 show our results in the case of a priori unknown frequencies (the ML approach) and a priori known and unknown noise level, respectively. These figures show numerically calculated values of abridged error probabilities $p_a$ and error probabilities $p_e$ of the MOS algorithms versus SNR. The values of probabilities $p_a$ and $p_e$ are calculated based on the identical realizations of the observed data, thus the statistical variations of these values are the same. Figures 5 and 6 confirm the abridged error probability as a good approximation of the error probability starting with $\text{SNR} = -6$ dB.

Note that, from our theoretical and numerical studies, partially shown in Figures 3 - 6, it follows that the MOS algorithms that demonstrate poor performance in the case
of BL estimate (see Fig. 3 and 4) also demonstrate poor performance in the case of ML estimate of frequencies (see Fig. 3 and 9) and vice versa. This fact confirms the QL performance analysis proposed in Section V-B.

Overall figures 1 - 6 show a significant advantage of the studied PMEP algorithms (PMEP-IR, PMEP-I) in the case of SNR > 0 (design with a priori known frequencies and the QL design approach) and in the case of SNR > 4 (the ML design approach). This advantage can be crucial in many practical applications.

Next, compare the QL and ML approaches to the design of MOS algorithms in the case of the known noise level. Figure 7 shows the abridged error probabilities of GIC, PMEP-IR, and PMEP-I algorithms and error probabilities of EEF and NMDL algorithms in the case of the QL approach versus errors in the used BL frequencies estimates \( \Delta \omega_i \). In Fig. 7 we suppose that for all \( i \): \( \Delta \omega_i = \Delta \omega \). Also, in Fig. 7 we plot the error probabilities of the same MOS algorithms, but in the case of the ML design approach (horizontal bold lines). Note that in Fig. 7 the lines of the same type refer to the same algorithms. For all the cases we set SNR = 0dB. Solid and dashed horizontal lines almost coincide.

Comparing the plotted curves in Fig. 7 in pairs (a pair of curves of the same type corresponds to the same MOS algorithm), we can conclude that for each MOS algorithm, there is an interval of errors \( \Delta \omega_i \) such that the MOS algorithm designed using the QL approach has better performance than one designed using the ML approach. We call these intervals the BL intervals. If all possible errors in frequencies estimates in a particular practical scenario are less than the corresponding BL intervals, then one can use the BL estimates instead of the ML estimates to design the MOS algorithm without loss of its performance. Here we list the BL intervals obtained from Fig. 7: the BL interval is 0.006 for the PMEP-IR algorithm; 0.006 for the PMEP-I algorithm; 0.005 for the GIC algorithm; 0.007 for the EEF, gMDL, NMDL algorithms. Comparing these values of BL intervals and the value of the error in the BL estimates \( \Delta \omega = 0.0025 \) which is chosen for numerical simulations presented in Fig. 5 one concludes that the robustness problem described in Theorem 5 has a practical meaning. Next, consider these results according to practical applications from Section V-C. Suppose that, deviations of frequencies from their a priori known values occur due to the motion of the source (or sources) of observed signal \( 4 \). Using the Doppler shift formula (see for example [37]) one can conclude that if all speeds of sources satisfy the following condition \( v_{source} < 0.0039c \) (here \( c \) – is the propagation speed of waves in the medium), then one can use the QL approach instead of the ML approach without any losses in performance. Let us clarify this condition for the case of electromagnetic waves in a vacuum: \( v_{source} < 10^5 \text{ m/s} \). Thus, in this case, one can use BL estimates instead of ML estimates for all practical applications.

Study the impact of amplitudes unevenness on the performance of algorithms PMEP-I and PMEP-IR. We introduce amplitudes unevenness to our model with a multiplier \( m_{un} \geq 1 \) as follows: \( a_{un} = m_{un}a_i \), while \( \forall v_i > 1: a_{un} = a, \) i.e., \( u = m_{un} \). Fig. 8 shows abridged error probabilities \( p_a \) calculated for PMEP-IR (40) algorithm (performance of PMEP-I show a similar behavior) for different values of \( m_{un} \in \{1, 1.25, 1.5, 1.75, 2\} \) and for GIC algorithm (37) for comparison. Fig. 8 confirms the high sensitivity of algorithms containing only the global penalty function to amplitudes unevenness (see a group of thin solid lines). On the other hand, the bold dashed line shows the performance of the PMEP-IR algorithm for \( m_{un} = 2 \) and in the case of using the tuning parameter adaptation, as suggested in Section VI-B. Comparing the solid and dashed lines, we conclude that the use of an adapted tuning coefficient \( \tilde{w}_{IR} \) can remove the sensitivity of the PMEP-IR algorithm to amplitude unevenness. Our additional studies show that the amplitudes unevenness problem is still partially valid in the case of the unknown noise level. One can explain this using an analysis similar to the one given in Section VII. However, there are some differences between the case of known noise level and the case of the unknown noise level. For example, if in the situation shown in Fig. 8 the noise level is unknown, then the PMEP-IR
frequencies. Thus, the quality of the approximation is almost
starting with SNR

\[ \text{SNR}_{\text{approximated using the abridged error probability starting with}} \]

this case. Fig. 9 shows that the error probability can be well
\[ \Delta \]
partially unknown frequencies, we suppose that
\[ N \]
amplitudes unevenness and without any modification. The MOS algorithm will retain its performance for any value of
amplitudes unevenness and without any modification.

Finally, let us study the quality of approximation of the error
probability by the abridged error probability for big values of
\( \mathcal{N} \) and \( \nu_0 \). We study the cases of a priori known and
partially unknown frequencies. We set \( \mathcal{N} = 50, \nu_0 = 25 \)
for numerical studies depicted in Fig. 8. Figure 8 shows abridged error probabilities \( p_a \) theoretically calculated for the
PMEP-IR (see (40)), PMEP-I (41), GIC (37) algorithms and
error probabilities \( p_e \) calculated numerically for PMEP-IR,
PMEP-I, and GIC algorithms, all as functions of SNR. For the
case of partially unknown frequencies, we suppose that
\( \Delta \omega = 0.0025 \) and we use red color to present results for
this case. Fig. 8 shows that the error probability can be well
approximated using the abridged error probability starting with
SNR = -6 dB in the case of a priori known frequencies and
starting with SNR = -4 dB in the case of partially unknown
frequencies. Thus, the quality of the approximation is almost
independent of values of \( \mathcal{N} \) and \( \nu_0 \).

VIII. CONCLUSIONS

In this paper, we proposed new approaches to the design,
consistency analysis, and performance analysis of MOS algo-
rithms and applied these approaches to the classical prob-
lems of estimating the number of modulated sinusoids with
unknown amplitudes, phases, known, unknown, and partially
unknown frequencies in the presence of noise with known and
unknown levels.

APPENDIX A
PRELIMINARY RESULTS

A. A new non-iterative formula for the Gram-Schmidt process

Let \( \mathbb{V} \) be a vector space over the field \( \mathbb{C} \) with an inner
product \( \langle \cdot, \cdot \rangle \). Suppose \( A_1, \ldots, A_N \) are linearly independent
vectors in \( \mathbb{V} \). Let \( n \) and \( N \) be natural numbers, and 1 \( \leq \)
\( n \leq N \). Suppose for any \( n, G_n = (G_{ij})_{i,j=1}^n \in \mathbb{C}^{n \times n} \) is a Gram
matrix with dimension \( n \times n \) whose element \( G_{ij} = \langle A_i, A_j \rangle \) is the
inner product of \( A_i \) and \( A_j \). \( g_n = (G_{1,n}, \ldots, G_{n-1,n})^T \in \mathbb{C}^{n-1 \times 1} \).

Now let us define vectors \( B_1 \in \mathbb{V}, \ldots, B_N \in \mathbb{V} \) as

\[ B_n = \left( A_n - \sum_{i=1}^{n-1} A_i \left( G_{n-1}^{-1} g_n \right)_i \right) / \sqrt{S_n}, \quad (46) \]

if \( 2 \leq n \leq N \), \( B_1 = A_1 / \sqrt{G_{1,1}} \).

Here \( S_n = G_{n,n} - g_n^T G_{n-1}^{-1} g_n \) is the Schur complement
\[ \text{[41]} \]
the superscript \( H \) denotes the Hermitian transpose,
\( (G_{n-1}^{-1} g_n)_i \) denotes the \( i \)-th component of the vector \( G_{n-1}^{-1} g_n \).

Let us prove that formulas (46) define the Gram-Schmidt
ortonormalization process.

Theorem 4 Suppose that \( \tilde{B}_1, \ldots, \tilde{B}_N \) are orthonormal vec-
tors produced by applying the Gram-Schmidt process to the
vectors \( A_1, \ldots, A_N \), and that the vectors \( B_1, \ldots, B_N \) are
defined by equations (46); then \( \tilde{B}_n = B_n \) for any \( 1 \leq n \leq N \).

Proof: Please see Appendix B-A.

B. Preliminary probabilistic results

Let \( X_1, \ldots, X_N \) be linearly independent real valued random
variables with finite variances. Suppose for any \( 1 \leq n \leq N \),
\( X_n = (X_1, \ldots, X_n)^T \) is a random vector, \( C_n = (C_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n} \)
is a covariance matrix of the vector \( X_n \) (the element
\( C_{ij} = \text{cov}(X_i, X_j) \) is the covariance between \( X_i \) and \( X_j \)),
\( \mu_n = (\mu_1, \ldots, \mu_n)^T \) is the mean vector of \( X_n \).

Despite the fact that random variables with arbitrary expec-
tations do not form a vector space, one can find that applying
the Gram-Schmidt process to random variables \( X_1, \ldots, X_N \)
produces uncorrelated random variables with unit variances.
Thus Theorem 4 implies the following result.

Theorem 5 If random variables \( \{Y_n\}_{n=1}^N \) are defined as

\[ Y_n = \gamma_n \left( X_n - X_{n-1}^T C_{n-1}^{-1} c_n \right) / \sqrt{S_n}, \gamma_1 = \gamma_1 / \sqrt{C_{1,1}}, \quad (47) \]
where $\gamma = \pm 1$, then the random variables $\{Y_n\}_{n=1}^N$ can be produced by applying the Gram-Schmidt process to the random variables $\{X_j\}_{j=1}^N$, and for any $1 \leq n, m \leq N$, the following equalities hold
\begin{align*}
\text{cov}(Y_n, Y_m) &= 0, \text{ if } n \neq m; \\
\text{cov}(Y_n, Y'_n) &= 1, \text{ if } n = m. 
\end{align*}
(48)

(Partially similar theorem can be found in [35] or [33] Statement 1.)

Next, suppose that random variables $\{X_j\}_{j=1}^n$ have the following structure
\begin{equation}
X_j = \sum_{i=1}^{N_x} x(t) s_j(t, \theta_j).
\end{equation}
Here $x(t) = \sum_{i=1}^{n_0} a_{0i} s_i(t, \theta_{0i}) + \xi(t)$, $s_i(t, \theta_i)$ is a signal with a vector of parameters $\theta_i, a_{0i} \in \mathbb{R}$, and $\xi(t)$ is the AWGN.

**Theorem 6** If $n > n_0$ and $\theta_i = \theta_0$ for all $i$ then
\begin{equation}
E(\mathcal{Y}_0) = E(\mathcal{X}_0 - \mathcal{X}_0^T \mathcal{C}_0^{-1} \mathcal{C}_0) = 0,
\end{equation}
where $E$ denotes an expectation, $\mathcal{X}_0 = \sum_{i=1}^{N_x} x(t) s_j(t, \theta_j), \mathcal{C}_0 = \text{cov}(\mathcal{X}_0, \mathcal{X}_0)$, $\mathcal{C}_0 = (\mathcal{C}_{0ij})_{i,j=1}^n, \mathcal{X}_0 = (X_0)_j$.

(This result is a generalization of [35] Eq. 40)

**Proof:** Please see Appendix B.

**APPENDIX B**

**Proofs**

**A. Proof of the Theorem 2**

The case $n = 1$ is obvious. Let us consider the case $n > 1$. Using the non-iterative formula for the Gram-Schmidt process from [41], we get
\begin{equation}
\hat{B}_n = \frac{1}{\sqrt{|G_n|}} \begin{bmatrix}
G_{1,1} & \cdots & G_{1,n-1} & A_1 \\
\vdots & \ddots & \vdots & \vdots \\
G_{n-1,1} & \cdots & G_{n-1,n-1} & A_{n-1} \\
G_{n,1} & \cdots & G_{n,n-1} & A_n 
\end{bmatrix},
\end{equation}
(49)
where $|\cdot|$ denotes the determinant of a matrix. The determinant in the second factor at the right-hand side of (49) has to be formally expanded along the last column.

Next, using block matrices, we can represent (49) as
\begin{equation}
\hat{B}_n = \sqrt{\frac{|G_{n-1}|}{|G_n|}} \begin{bmatrix}
\mathcal{G}_{n-1} \\
|G_n|
\end{bmatrix}
\begin{bmatrix}
A_n + \frac{1}{|G_{n-1}|} \sum_{i=1}^{n-1} (-1)^{i+n} A_i \mathcal{G}_{n-1}^{(i)} \mathcal{G}_{n-1}^{(i)} 
\end{bmatrix},
\end{equation}
(50)
where $\mathcal{G}_{n-1}^{(i)}$ is the matrix obtained by deleting the $i$-th row from $G_{n-1}$.

Let us expand the determinant of each block matrix $\mathcal{G}_{n-1}^{(i)}$ in (50) along the row $g_n^H$
\begin{equation}
\hat{B}_n = \sqrt{\frac{|G_{n-1}|}{|G_n|}} \begin{bmatrix}
\mathcal{G}_{n-1} \\
|G_n|
\end{bmatrix}
\begin{bmatrix}
A_n + \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} (-1)^{i+k} A_i M_{i,k}^{[n-1]} G_{k,n} 
\end{bmatrix},
\end{equation}
(51)
where $M_{i,k}^{[n-1]}$ is the $i, k$-th minor of $G_{n-1}$, i.e., $M_{i,k}^{[n-1]}$ is the determinant of the submatrix that is obtained by deleting the $i$-th row and the $k$-th column from $G_{n-1}$.

Now let us rewrite $B_n$ [46] using $M_{i,k}^{[n-1]}$:
\begin{equation}
B_n = \left( A_n + \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} (-1)^{i+k} A_i M_{i,k}^{[n-1]} G_{k,n} \right) / \sqrt{S_n}.
\end{equation}
(52)
Comparing (51) and (52), we get
\begin{equation}
\hat{B}_n = \sqrt{|G_n| / |G_{n-1}|} = B_n / \sqrt{S_n}.
\end{equation}
(53)

Now to complete the proof of Theorem 3 we need to prove the following equality:
\begin{equation}
\left\langle \hat{B}_n, \hat{B}_n \right\rangle = \left\langle B_n, B_n \right\rangle.
\end{equation}
(54)
It is well known that $\left\langle \hat{B}_n, \hat{B}_n \right\rangle = 1$ (see [41]). Equality $\left\langle B_n, B_n \right\rangle = 1$ can be proved by direct calculations, thus we assume that equality (54) holds.

Comparing (53) and (54), we get proof of Theorem 3.

**B. Proof of the Theorem 6**

Firstly, the following result can be obtained by direct calculations: $\text{cov}(X_0, X'_0) = \sum_{i=1}^{N_x} s_i(t, \theta_i) s_j(t, \theta_j)$.

If $n > n_0$ and $\theta_i = \theta_0$ for all $i$, then one can find that
\begin{align}
E(X_0) &= \left( \sum_{i=1}^{n_0} \sum_{t=1}^{n} a_{0i} s_i(t, \theta_0) s_j(t, \theta_0) \right) \\
&= \left( \sum_{i=1}^{n_0} a_{0i} C_{0i,j} \right)^n = (a_{00}^{-1} C_{00}^{-1} c_0^{-1})^T,
\end{align}
(55)
where $(a_{00})_i = \begin{cases} a_{0i}, & \text{if } i \leq n_0, \\ 0, & \text{if } i > n_0. \end{cases}$

Equation (55) allows to prove Theorem 6 as follows
\begin{align}
E(\mathcal{Y}_0) &= E(X_0) - E(X_0^{-1})^T C_0^{-1} c_0 \\
&= a_{00}^{-1} c_0 - a_{00}^{-1} C_{00}^{-1} c_0 = 0.
\end{align}

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