GALOIS ORBITS OF TORSION POINTS NEAR ATORAL SETS

V. DIMITROV AND P. HABEGGER

ABSTRACT. We prove that the Galois equidistribution of torsion points of the algebraic torus $G_m^d$ extends to the singular test functions of the form $\log |P|$, where $P$ is a Laurent polynomial having algebraic coefficients that vanishes on the unit real $d$-torus in a set whose Zariski closure in $G_m^d$ has codimension at least 2. Our result includes a power saving quantitative estimate of the decay rate of the equidistribution. It refines an ergodic theorem of Lind, Schmidt, and Verbitskiy, of which it also supplies a purely Diophantine proof. As an application, we confirm Ih’s integrality finiteness conjecture on torsion points for a class of atoral divisors of $G_m^d$.

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Date: March 25, 2022.
2010 Mathematics Subject Classification. 11J83, 11R06, 14G40, 37A45, 37P30.
1. Introduction

1.1. Main results. Let \( d \geq 1 \) be an integer and let \( G_m^d \) denote the \( d \)-dimensional algebraic torus with base field \( \mathbb{C} \). We will identify \( G_m^d \) with \( (\mathbb{C}\setminus\{0\})^d \), the group of its \( \mathbb{C} \)-points.

Let \( \zeta \in G_m^d \) be a torsion point, i.e., a point of a finite order. We define

\[
\delta(\zeta) = \inf\{|a| : a \in \mathbb{Z}^d \setminus \{0\} \text{ with } \zeta^a = 1\}
\]

where, here and throughout the article, \( | \cdot | \) denotes the maximum-norm; we refer to Section 2 for the notation \( \zeta^a \).

It is well-known that the Galois orbit \( \{\zeta' : \sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})\} \) becomes equidistributed in \( G_m^d \) with respect to the Haar measure as \( \delta(\zeta) \to \infty \). More precisely, if \( f : G_m^d \to \mathbb{R} \) is a continuous function with compact support, then

\[
\frac{1}{|\mathbb{Q}(\zeta) : \mathbb{Q}|} \sum_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} f(\zeta^\sigma) \to \int_{[0,1]^d} f(e(x)) dx
\]

as \( \delta(\zeta) \to \infty \) where

\[
e(x) = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_d})
\]

for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).

Our aim is to investigate the equidistribution result for test functions \( f = \log |P| \) where \( P \) is a Laurent polynomial in \( d \) unknowns and with algebraic coefficients. Such \( P \) may vanish on \((S^1)^d\), where \( S^1 = \{z \in \mathbb{C} : |z| = 1\} \) is the unit circle, and so \( f \) is not defined everywhere. But for \( \delta(\zeta) \) large in terms of \( P \), Laurent’s Theorem [26] also known as the Manin–Mumford Conjecture for \( G_m^d \), implies that \( P \) does not vanish at any conjugate of \( \zeta \). See also [40] for another proof by Sarnak and Adams. Moreover, the integral of \( f \) over \((S^1)^d\) exists as the singularity is merely logarithmic. It is known as the Mahler measure

\[
m(P) = \int_{[0,1]^d} \log |P(e(x))| dx,
\]

see for instance Section 3.4 in [41] for the convergence of this integral for arbitrary \( P \in \mathbb{C}[X_{1}^{\pm1}, \ldots, X_{d}^{\pm1}] \setminus \{0\} \).

A torsion coset of \( G_m^d \) is the translate of a connected algebraic subgroup of \( G_m^d \) by a point of finite order. We call a torsion coset proper if it does not equal \( G_m^d \).

We call \( P \in \mathbb{C}[X_{1}^{\pm1}, \ldots, X_{d}^{\pm1}] \setminus \{0\} \) essentially atoral if the Zariski closure of

\[
\{(z_1, \ldots, z_d) \in (S^1)^d : P(z_1, \ldots, z_d) = 0\}
\]

in \( G_m^d \) is a finite union of irreducible algebraic sets of codimension at least 2 and proper torsion cosets.

For example, if \( d = 1 \) then \( P \) is essentially atoral if and only if it does not vanish at any point of infinite multiplicative order in \( S^1 \).

Lind–Schmidt–Vertitskiy define the notion of an atoral Laurent polynomial \( P \in \mathbb{Z}[X_{1}^{\pm1}, \ldots, X_{d}^{\pm1}] \setminus \{0\} \) in Definition 2.1 [32]. An atoral Laurent polynomial is essentially atoral in our sense. Moreover, if \( P \) is irreducible then it is atoral if and only if
the intersection of its zero locus with \( (S^1)^d \) has dimension at most \( d - 2 \) as a semi-algebraic set, cf. by Proposition 2.2 [32]. A related, but not quite equivalent, definition of atoral Laurent polynomials with complex coefficients was introduced earlier by Agler–McCarthy–Stankus [1].

Let \( \overline{Q} \) denote the algebraic closure of \( Q \) in \( C \). We are ready to state our first result.

**Theorem 1.1.** For each essentially atoral \( P \in \overline{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\} \) there exists \( \kappa > 0 \) with the following property. Suppose \( \zeta \in G_m^d \) has finite order with \( \delta(\zeta) \) sufficiently large. Then \( P(\zeta^a) \neq 0 \) for all \( a \in \operatorname{Gal}(Q(\zeta)/Q) \) and

\[
\frac{1}{|Q(\zeta) : Q|} \sum_{\sigma \in \operatorname{Gal}(Q(\zeta)/Q)} \log |P(\zeta^a)| = m(P) + O(\delta(\zeta)^{-\kappa})
\]

as \( \delta(\zeta) \to \infty \), where the implicit constant depends only on \( d \) and \( P \).

Theorem 8.8 below is a more precise version of this result. In particular, we allow \( \sigma \) to range over subgroups of \( \operatorname{Gal}(Q(\zeta)/Q) \) whose index and conductor grow sufficiently slow, the conductor is defined in Section 3. Moreover, \( \kappa \) depends only on \( d \) and the number of non-zero terms appearing in \( P \). Our method of proof allows one to determine an explicit value for \( \kappa \).

Torsion points in \( G_m^d \) are characterized as the algebraic points of height zero; see Section 2 for the definition of the height \( h : G_m^d(\mathbb{Q}) \to [0, \infty) \). Bilu [4] proved that Galois orbits of algebraic points \( a \in G_m^d \) of small height satisfy an analogous equidistribution statement as (1.2), asymptotically as \( h(a) \to 0 \) and \( \delta(a) \to \infty \); the definition (1.1) extends naturally to non-torsion points and may take infinity as a value. It is natural to ask whether Theorem 1.1 admits a suitable generalization to points of small height. Autissier’s example [2] rules out the verbatim generalization already for \( G_m \). He constructed a sequence \((a_n)_{n \in \mathbb{N}}\) of pairwise distinct algebraic numbers whose height tends to 0 but such that \( \frac{1}{|Q(a_n) : Q|} \sum_{\sigma} \log |\sigma(a_n)| - 2 | \) tends to 0 for \( n \to \infty \). But the integral of the corresponding test function against the unit circle is \( \log 2 \). An interesting problem still arises if the test function has at worst a logarithmic singularity of real codimension at least 2 on \((S^1)^d\). Suppose that \(|f(z)| = O(|\log(|P(z)|^2 + |Q(z)|^2)|)\) on an open neighborhood of \((S^1)^d\) in \( G_m^d \), where \( P \) and \( Q \) are non-constant and co-prime Laurent polynomials with algebraic coefficients, and that \( f \) vanishes on the complement of a compact set in \( G_m^d \). One may then ask about comparing the average of \( f \) over the Galois orbit of \( a \in G_m^d(\mathbb{Q}) \) with the average of \( f \) over \((S^1)^d\): is their difference bounded by \( \ll \kappa h(a) + \delta(a)^{-1}\), for some \( \kappa > 0 \) depending only on \( P \) and \( Q \)? We also mention Chambert-Loir and Thuillier’s Théorème 1.2 [10] which is a general equidistribution result for points of small height, allowing \( \log |P| \) as a test function if the zero locus of \( P \) in \( G_m^d \) is a finite union of torsion cosets. In this paper we allow \( \log |P| \) as a test function if \( P \) is essentially atoral but we average over points of finite order.

Our Theorem 1.1 recovers a variant of the result of Lind–Schmidt–Vérbisitky [32]. In their work, the sum is not over the Galois orbit of a single point of finite order but rather over a finite subgroup \( G \) of \( G_m^d \). For this purpose we define

\[
\delta(G) = \inf \{ |a| : a \in \mathbb{Z}^d \setminus \{0\} \text{ such that } \xi^a = 1 \text{ for all } \xi \in G \}.
\]
Each finite subgroup of $G_d^m$ is a disjoint union of Galois orbits. This observation allows us to recover the Theorem of Lind, Schmidt, and Verbitskiy with an estimate on the decay rate.

**Theorem 1.2.** Let $P \in \mathbb{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ be essentially atoral. There exists $\kappa > 0$ such that for any finite subgroup $G \subset G_d^m$ we have

$$
1 \sum_{\substack{\xi \in G \\ P(\xi) \neq 0}} \log |P(\xi)| = m(P) + O(\delta(G)^{-\kappa})
$$

where the implicit constant depends only on $d$ and $P$.

To relate (1.5) to the expression in Lind, Schmidt, and Verbitskiy’s Theorem 1.3 [32] we refer to Lemma 2.1 [31] as well as the comments on page 1063 and 1064 [32]. Note that $G$ is $\Omega$ and $\#G$ is $|\mathbb{Z}^d/\Gamma|$ in the notation of [32].

Lind, Schmidt, and Verbitskiy’s approach is based on an in-depth study [43, 31, 32] of an associated dynamical system: the algebraic $\mathbb{Z}^d$-action on a closed, shift-invariant subgroup of $(S^1)^{\mathbb{Z}^d}$ whose dual is $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] / (P)$. The atoral condition, in the sense of [32], turns out to be equivalent to the existence of a non-trivial summable homoclinic point.

Theorem 1.2 may be read as a strong quantitative estimate on the growth of periodic points for such dynamical systems. The refinement to Galois orbits, Theorem 1.1, does not seem to be directly possible by the homoclinic method, nor does it seem to follow formally from the case (1.5) of finite subgroups, which is where the dynamical method applies.

Our method of proof draws its origins in work of Duke [16]. It differs from the method of Lind, Schmidt, and Verbitskiy. However, it is striking that the notion of atoral appears crucially in both approaches.

The first-named author [11] was able to prove Theorem 1.2 for a general Laurent polynomial when $G$ equals the group of $N$-torsion elements in $G_d^m$.

Let us return to Galois orbits. We believe that the hypothesis on $P$ being essentially atoral is also unnecessary in Theorem 1.1 on Galois orbits. The next conjecture sums up our expectations. It is related to Schmidt’s Conjecture [42, Remark 21.16(2)].

**Conjecture 1.3.** For each $P \in \overline{\mathbb{Q}}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ there exists $\kappa > 0$ with the following property. Suppose $\xi \in G_d^m$ has finite order with $\delta(\xi)$ sufficiently large. Then $P(\xi^\sigma) \neq 0$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}(\xi) / \mathbb{Q})$ and

$$
\frac{1}{[\overline{\mathbb{Q}}(\xi) : \mathbb{Q}]} \sum_{\sigma \in \text{Gal}(\overline{\mathbb{Q}}(\xi) / \mathbb{Q})} \log |P(\xi^\sigma)| = m(P) + O(\delta(\xi)^{-\kappa})
$$

as $\delta(\xi) \to \infty$, where the implicit constant depends only on $d$ and $P$.

For $d = 1$ this conjecture follows from work of M. Baker, Ih, and Rumely [3], see their statement around (6). They use a version of Baker’s deep estimates on linear forms in logarithms. Already the case $d = 2$ and $P(X_1, X_2) = X_1 + X_1^{-1} + X_2 + X_2^{-1} - 3$ is open.
1.2. **Ih’s conjecture on integral torsion points.** As another application of our results we derive a special case of Ih’s Conjecture [3] in the multiplicative setting. Let \( P \in \mathbb{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \). A special case of Ih’s Conjecture predicts that the set of torsion points \( \zeta \in \mathbb{G}_m^d \) such that \( P(\zeta) \) is an algebraic unit is not Zariski dense in \( \mathbb{G}_m^d \), unless the zero set of \( P \) in \( \mathbb{G}_m^d \) is itself a finite union of proper torsion cosets. M. Baker, Ih, and Rumely [3] cover the case \( d = 1 \) for univariate polynomials. Their approach runs through a similar limiting statement as our Theorem 1.1 for univariate polynomials.

Here we solve a case of Ih’s Conjecture for essentially atoral polynomials with integral coefficients.

**Corollary 1.4.** Let \( K \subset \mathbb{C} \) be a number field with ring of integers \( \mathbb{Z}_K \) and let \( P \in \mathbb{Z}_K[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \backslash \{0\} \). Suppose that the zero set of \( P \) in \( \mathbb{G}_m^d \) is not a finite union of torsion cosets. Suppose in addition that \( \tau(P) \) is essentially atoral for all field embeddings \( \tau : K \to \mathbb{C} \). Then there exists \( B \geq 1 \) such that if \( \zeta \in \mathbb{G}_m^d \) has finite order and \( P(\zeta) \) is an algebraic unit, then \( \delta(\zeta) \leq B \).

Ih’s Conjecture expects the existence of \( B \) without assuming that each \( \tau(P) \) is essentially atoral. Observe that the result of M. Baker, Ih, and Rumely is not a direct consequence of this corollary, as we do not allow univariate polynomials that vanish at a point of infinite multiplicative order on the unit circle. Our approach does not depend on the theory of linear forms in logarithms.

A special class of atoral polynomials, to which our results apply *a fortiori*, are the irreducible integer Laurent polynomials \( P \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \backslash \{0\} \) that are not fixed up-to a monomial factor and up-to a sign by the involution sending each \( X_i \) to \( 1/X_i \). We call these \( P \) asymmetric. They are atoral in the sense of Lind–Schmidt–Verbitskiy, see the proof of Proposition 2.2 [32]. Hence an asymmetric Laurent polynomial is essentially atoral. The converse is false as the Laurent polynomial

\[
X_1 + X_1^{-1} + X_2 + X_2^{-1} - 4.
\]

is essentially atoral; indeed, its zero locus on \((S^1)^2\) consists of the single point \((1, 1)\).

If \( K = \mathbb{Q} \), Corollary 1.4 in the case of an asymmetric, and thus necessarily irreducible Laurent polynomial \( P \), can be deduced as follows from the Manin–Mumford Conjecture for \( \mathbb{G}_m^d \). Indeed, if \( \gamma \) is a unit in the ring of algebraic integers of a cyclotomic field, then \( \eta = \sqrt[d]{\gamma} \) is an algebraic integer whose Galois conjugates lie on \( S^1 \). So \( \eta \) is a root of unity by Kronecker’s Theorem, see Theorem 1.5.9 [5]. We consider the zero \( (\eta, \zeta) \) of \( P(X_1^{-1}, \ldots, X_d^{-1}) - X_0 P(X_1, \ldots, X_d) \), which is irreducible and defines an algebraic subset of \( \mathbb{G}_m^d \) none of whose geometric irreducible components is a torsion coset. A similar argument applies if \( K \) is a totally real number field.

1.3. **Overview of the proof.** We close the introduction by describing the method of proof of Theorem 1.1 which builds upon work of the second-named author [20] and is related to the approach of Duke [16]. The basic idea is to reduce the multivariate statement in Theorem 1.1 to the univariate case. Whereas we worked with torsion points of prime order in [20], the main technical difficulty in this paper is that we allow torsion points of arbitrary order.

Any torsion point \( \zeta \in \mathbb{G}_m^d \) of order \( N \) takes on the form \((\zeta^{a_1}, \ldots, \zeta^{a_d})\) where \( \zeta = e(1/N) \) is a root of unity of order \( N \) and \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \). The precise manner
how the non-unique \( a \) is chosen is delicate and will be discussed below. The notation \( \zeta = \zeta^a \) will be quite useful. A non-boldface \( \zeta \) denotes a root of unity and boldface \( \zeta \) suggests a torsion point of \( G^d_m \).

If \( \tilde{P} \) is as in Theorem 1.1 but for simplicity with coefficients in \( K = Q \), we define the univariate polynomial

\[
Q(X) = P(X^a) = P(X^{a_1}, \ldots, X^{a_d}) \in Q[X^{\pm 1}].
\]

Multiplying \( Q \) by a power of \( X \) turns out to be harmless, so one can assume that \( Q \) is a polynomial. The values \( |P(\zeta^\sigma)| \) equal the values of \( |Q(\zeta^\sigma)| \) as \( \sigma \) ranges over \( \text{Gal}(Q(\zeta)/Q) \).

The univariate case and root separation (Section 4). Let us suppose for the moment that \( \zeta = \zeta^a \) is a root of unity. It is classical that the Galois conjugates of \( \zeta \) are equidistributed around the unit circle; we recall of these facts in Section 3. So (1.2) holds for \( f(z) = \log |Q(z)| \) provided \( Q \) has no zero on the unit circle. In Proposition 4.5 we make convergence quantitative for such \( Q \). Roughly speaking, for all \( \epsilon > 0 \) we have

\[
\frac{1}{[Q(\zeta) : Q]} \sum_{\sigma} \log |Q(\zeta^\sigma)| = m(Q) + O_{P, \epsilon} \left( \frac{|a|^{1+\epsilon}}{N^{1-\epsilon}} \right)
\]

where \( \sigma \) runs over \( \text{Gal}(Q(\zeta)/Q) \). Actually, the hypothesis on \( Q \) is slightly weaker as we allow it to vanish at roots of unity, if all \( Q(\zeta^\sigma) \neq 0 \). This hypothesis is ultimately a reflection of the hypothesis that the multivariate \( P \) is essentially atoral in Theorem 1.1. Indeed, in the univariate case, being essentially atoral boils down to not vanishing at any point of infinite multiplicative order in \( S^1 \). The hypothesis on \( Q \) is crucial for our method to work. The main difficulty we encounter in the average (1.7) are exceptionally small values of \( Q \) at some \( \zeta^\sigma \). The burden is to show, in a uniform sense, that no complex root \( z \) of \( Q \) can be too close to \( \zeta^\sigma \) in a suitable sense.

If \( z \) is itself a root of unity, doing this is straightforward as \( |z - 1| \gg 1/\text{ord}(z) \).

The difficulty lies in the case when \( z \) has infinite multiplicative order. Here it is tempting to apply a version of Baker’s Theorem on linear forms in logarithms, as did M. Baker, Ih, and Rumely [3]. However, and as already discussed by Duke in Section 3 [16] this seems unhelpful for the problem at hand. Indeed, estimates on linear forms in two logarithms such as [27] lead to a factor \( [Q(z) : Q]^2 = O(|a|^2) \) in a bound for any member of the sum in (1.7). This is not good enough for our application as \( |a|^2/[Q(\zeta) : Q] \) may spoil the average in (1.7).

Our solution is to use the banal inequality \( |z - \zeta| \geq |z| - 1 \) which lies at the heart of the method here and in [20]. As \( z \) is no root of unity, and as \( Q \) does not vanish at points of infinite multiplicative order on \( S^1 \), we have \( |z| \neq 1 \) and so the banal inequality provides a non-trivial lower bound. We now explain how it leads to a useful estimate on \( |z - \zeta| \) via lower bounding \( |z| - 1 \).

If \( z \) is close to the unit circle, then \( |z| - 1 \) is approximately \( |z - 1/\zeta| \). In [20] a result of Mahler [33] on the separation of roots of an integer polynomial led to a suitable lower bound for \( |z - 1/\zeta| \). In that paper, the second-named author used his counting result on approximations to a set definable in an \( o \)-minimal structure. This allowed to make Mahler’s estimate uniform over the various zeros \( z \) of \( Q \).
The main tool of the present paper is a uniform generalization of Mahler’s inequality for the separation of several pairs of roots of \( Q \). Such a generalization was obtained by Mignotte [34]. In Section 4, we give a variant of Mignotte’s theorem that is tailored to our application and is self-contained. We thus bypass the o-minimal theory used in [20]. We still require Bombieri, Masser, and Zannier’s Theorem [6] to be mentioned below. Moreover, our Theorem 1.1 is effective in nature.

A possible approach towards Conjecture 1.3 lies in extending (1.7) to \( Q \) that are allowed to vanish at any point of \( S^1 \). As observed, we lack a suitable lower bound for \(|z - \zeta|\) if \( z \) is an algebraic number of infinite multiplicative order on the unit circle. As suggested in the similar setting of Lemma 4.2 [20], it turns out that the \( z \) of interest have small height \( h(z) \). We therefore propose the following conjecture.

**Conjecture 1.5.** For all \( B \geq 1 \) and \( \epsilon > 0 \) there exists a constant \( c = c(B, \epsilon) > 0 \) with the following property. Let \( z \in \mathbb{C} \) be an algebraic number with \(|z| = 1\) and \( h(z) \leq B/D \) where \( D = |Q(z) : Q| \). If \( \zeta \in \mathbb{C} \setminus \{z\} \) is a root of unity of order \( N \), then \( \log |\zeta - z| \geq -cD^{1+\epsilon}N^\epsilon \).

The crux of this conjecture is its best-possible dependency on the degree \( D \). In comparison, the state-of-the-art results in the theory of linear forms in two logarithms of algebraic numbers in the \( D \)-aspect, such as Laurent, Mignotte, and Nesterenko’s Théorème 3 [27], have only a quadratic dependency on \( D \).

**Equidistribution of torsion points (Section 5).** We return to the case \( \zeta = \zeta^a \) of a general torsion point in \( \mathbb{G}_{m, d}^\ast \) of order \( N \). The exponent vector \( a \) used to define \( Q \) as in (1.6) depends on \( \zeta \). For this reason it is important that the error term in Proposition 4.5 is explicit in terms of \( Q \). Moreover, it is important to choose \( a \) with \(|a|\) as small as possible. For fixed \( \zeta \), the exponent \( a \) is well-defined up-to addition of an element in \( NZ^d \). So clearly we may assume \(|a| \leq N \), although this is not good enough in view of (1.7). Fortunately, there is a second degree of freedom, namely we can replace \( \zeta \) by any Galois conjugate of itself.

This leads us to classical questions of equidistribution of the Galois orbit of \( \zeta \); we compile the necessary statements in Section 3. Using the Erdős–Turán Theorem and the theory of Gauß sums, Lemma 3.7 produces \( a \) with \(|a| = O(N\delta(\zeta)^{-1/(3d)})\) such that \( \zeta^a \) is a Galois conjugate of \( \zeta \).

Let us return to the error term in (1.7). One factor \( N \) cancels out and the error term becomes \( N^{2\epsilon}\delta(\zeta)^{-(1+\epsilon)/(3d)} \). The innocuous \( \epsilon \) in (1.7) is ultimately responsible for the factor \( N^{2\epsilon} \). Although \( \delta(\zeta) \leq N \), there is no non-trivial bound in the reverse direction and \( N^{2\epsilon}\delta(\zeta)^{-(1+\epsilon)/(3d)} \) could explode.

**Factoring \( \zeta \) (Section 5).** The solution to this problem is described in Section 5. In Proposition 5.1, we factor \( \zeta \) into a product \( \eta\xi \) where \( \xi \) has finite order \( M \) such that \( \xi = e(a/M) \) where \(|a| = O(M^{1-\kappa}) \). Moreover, the order of \( \eta \) is bounded from above by a small power of \( N \). The power saving obtained in the exponent of \( N \) is small even when compared to the saving obtained for \(|a|\). The methods employed come from the geometry of numbers and slopes of lattices in \( \mathbb{R}^d \).

We will replace \( \zeta \) by \( \xi \) and the univariate polynomial \( Q(X) = P(X^a) \) by \( P(\eta^aX^a) \). This last transformation does not change the height or the monomial structure of \( Q \). But it can change the field generated by its coefficients as the order of \( \eta \) and hence its
field of definition vary as $\zeta$ varies. For this reason, we must keep track of the base field of $Q$ throughout the whole argument.

**Putting everything together (Sections 6, 7, 8).** In Sections 6 and 8 we put all ingredients together to prove the final result. Here we apply a result of Bombieri, Masser, and Zannier [6] on the intersections of a subvariety in $G^d_{m_1}$ of codimension at least 2 with all 1-dimensional algebraic subgroups of $G^d_{m_1}$. Roughly speaking, this result shows that if $P$ is essentially atoral, then for “most” choices of $a$ the univariate polynomial $Q$ as in (1.6) does not vanish at any point of infinite multiplicative order on $S^1$. Recall that this property of $Q$ was crucial to deduce (1.7). Bombieri, Masser, and Zannier’s result is related to the study of unlikely intersections, for an overview we refer to Zannier’s book [46]. Another tool that makes an appearance is Lawton’s Theorem [29].

The intermediate Section 7 contains a weak version of a result of Hlawka [22] on the numerical integration of a continuous, multivariate function. The results obtained there are useful in connection with the function attaching the Mahler measure to a non-zero polynomial.

**Appendices.** In Appendix A we give a quantitative version of Lawton’s Theorem [29] regarding the convergence of a sequence of Mahler measures. Unfortunately, we are not able to use the very closely related theorem in [20] as we require additional uniformity. The arguments in this appendix follow closely Lawton’s strategy. Finally, in the second Appendix we show how to deduce Theorem 1.2, the Theorem of Lind–Schmidt–Verbitskiy, from our Theorem 1.1.

The results mentioned above, in particular the theorem of Bombieri, Masser, and Zannier, also play an important role in Le’s approach [30]. The question on how small a sum of roots of unity can be was raised by Myerson [37] in connection with a combinatorial question [35, 36] which was later studied by Duke [16]. Dubickas [15] has more recent work in this direction for sums of 2 and 3 roots of unity of prime order.

**Acknowledgments.** The authors thank Pierre Le Boudec for references regarding Gauß sums, Peter Sarnak for the reference to Le’s [30], and Shouwu Zhang for pointing out Chambert-Loir and Thuillier’s work [10]. We also thank the referee for carefully reading this text and for providing many valuable comments that led to improvements of the text and some simplifications. The authors thank François Brunault, Antonin Guilloux, Mahya Mehrabdollahi, and Riccardo Pengo for pointing out a mistake in an earlier attempt to prove Lemma A.3(i) and for providing the reference to Dobrowolski’s work [12]. Vesselin Dimitrov gratefully acknowledges support from the European Research Council via ERC grant GeTeMo 617129. Philipp Habegger has received funding from the Swiss National Science Foundation project n° 200020_184623.
2. Notation and preliminaries

Apart from the notation already introduced we use $\mathbb{N}$ to denote the natural numbers $\{1, 2, 3, \ldots\}$. If $x = (x_1, \ldots, x_m)$ with all $x_i$ elements in an abelian group $G$ and if $A = (a_{ij})_{i,j} \in \text{Mat}_{m,n}(\mathbb{Z})$ we write $x^A = (x_1^{a_{11}} \cdots x_m^{a_{1m}}, \ldots, x_1^{a_{m1}} \cdots x_m^{a_{mn}}) \in G^n$. So if $B \in \text{Mat}_{n,p}(\mathbb{Z})$, then $(x^A)^B = x^{AB}$. For a commutative ring $R$ with 1 we let $R^\times$ denote its group of units. Euler’s function $\varphi$ maps $N \in \mathbb{N}$ to the cardinality of $(\mathbb{Z}/N\mathbb{Z})^\times$. The group of all roots of unity in $\mathbb{C}^\times$ is $\mu_\infty$. We often identify $G_m^d$ with the set of its complex points $(\mathbb{C}^\times)^d$ and let 1 denote the unit element $(1, \ldots, 1) \in G_m^d$. If $\zeta \in G_m^d$ is a torsion point, we write $\text{ord}(\zeta)$ for its order. We write $\langle \cdot, \cdot \rangle$ for the Euclidean inner product on $\mathbb{R}^d$, $| \cdot |_2$ for the Euclidean norm on $\mathbb{R}^d$, and $| \cdot |$ for the maximum-norm on $\mathbb{R}^d$ and $\text{Mat}_{m,n}(\mathbb{R})$. We define $\log^+ x = \log \max\{1, x\}$ for all $x \geq 0$.

The constants implicit in Vinogradov’s notation $\ll_{x,y,z,\ldots} \gg_{x,y,z,\ldots}$, and in $O_{x,y,z,\ldots}(\cdots)$ depend only on the values $x, y, z, \ldots$ appearing in the subscript.

Let $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$, then $|P|$ denotes the maximum-norm of the coefficient vector of $P$ and we set $|0| = 0$. Recall that $m(P)$ is the Mahler measure of $P$. It follows from Corollaries 4 and 6 in Chapter 3.4 [41] that $\exp(m(P))$ is at most the Hermitian norm of the coefficient vector of $P$. Suppose $P$ has at most $k \geq 1$ non-zero terms, we find

$$m(P) \leq \log |P| + \frac{1}{2} \log k.$$  

The following result [13, Corollary 2] of Dobrowolski and Smyth provides a reverse inequality of the same quality.

**Theorem 2.1 (Dobrowolski–Smyth).** Suppose $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ has at most $k \geq 2$ non-zero terms with $k$ an integer. Then $m(P) \geq \log |P| - (k - 2) \log 2$.

Therefore,

$$(2.2) \quad |m(P) - \log |P|| \ll k$$

with absolute implied constant. Observe that if $P$ is a polynomial, then $m(P) \geq \log |P| - \log(2) \sum_{i=1}^d \deg_{X_i} P$ by the classical Lemma 1.6.10 [5]. So (2.2) is stronger when the number of terms in $P$ is known to be bounded, which is often the case in our work.

Let $x$ be an element of a number field $K$. The absolute logarithmic Weil height, or just height, of $x$ is

$$h(x) = \frac{1}{[K : \mathbb{Q}]} \sum_v |K_v : \mathbb{Q}_v| \log \max\{1, |x|_v\};$$

here $v$ runs over all places of $K$ normalized such that $|2|_v = 2$ for an infinite place $v$ and $|p|_v = 1/p$ if $v$ lies above the rational prime $p$, the completion of $K$ with respect to $v$ is $K_v$ and the completion of $\mathbb{Q}$ with respect to the restriction of $v$ is $\mathbb{Q}_v$. Let $P$ be a non-zero Laurent polynomial with coefficients $x_0, \ldots, x_n \in K$. The absolute logarithmic Weil height, or just height, of $P$ is

$$h(P) = \frac{1}{[K : \mathbb{Q}]} \sum_v |K_v : \mathbb{Q}_v| \log \max\{|x_0|_v, \ldots, |x_n|_v\}.$$
See Chapter 1 [3] for more details on heights. For example, \( h(x) \) and \( h(P) \) are well-defined for \( x \in \overline{Q} \) and \( P \in \overline{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \), i.e., the values do not depend on the number field \( K \) containing \( x \) and the coefficients of \( P \), respectively. Moreover \( h(P) = h(\lambda P) \) for all \( \lambda \in \overline{Q}^\times \).

3. Quantitative Galois equidistribution for torsion points

We need a strong enough quantitative version of the Galois equidistribution of torsion points \( \zeta \) of \( G_m^d \), with a power saving discrepancy in \( \delta(\zeta) \) defined in \([1,1]\).

Different approaches are possible and we opt to use the Erdős–Turán–Koksma bound. This reduces the problem to the estimation of certain exponential sums, which happen to be Gauß sums that can be explicitly evaluated.

Let \( N \in \mathbb{N} \). For a divisor \( f \in \mathbb{N} \) of \( N \) we work with the canonical surjective, homomorphism \( (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/f\mathbb{Z})^\times \) induced by reducing modulo \( f \).

The conductor \( f_G \) of a subgroup \( G \) of \( (\mathbb{Z}/N\mathbb{Z})^\times \) is the least positive integer \( f \mid N \) such that \( G \) contains \( \ker((\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/f\mathbb{Z})^\times) \). Observe that \( (\mathbb{Z}/N\mathbb{Z})^\times : G \leq \varphi(f_G) \).

Certainly, \( f_G \) is well-defined as \( \ker((\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times) \) is the trivial subgroup. Moreover, \( f_G(\mathbb{Z}/N\mathbb{Z})^\times = 1 \). But one should take care as the conductor of \( G = \{1\} \) is \( N/2 \) for \( N \equiv 2 \) (mod 4).

The group \( (\mathbb{Z}/N\mathbb{Z})^\times \) is naturally isomorphic to the Galois group of \( \mathbb{Q}(\zeta) / \mathbb{Q} \), where \( \zeta \) is a root of unity of order \( N \). Let \( L \subset \mathbb{Q}(\zeta) \) be the fixed field of \( G \). Then \( L \) lies in the fixed field of \( \ker((\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/f\mathbb{Z})^\times) \) which equals \( \mathbb{Q}(\zeta_{f_G}) \) where \( \zeta_{f_G} \) is a root of unity of order \( f_G \).

Let \( \ell \geq 1 \) be an integer and \( \zeta_{f} \) of order \( f \). We claim \( L \subset \mathbb{Q}(\zeta_{f}) \) if and only if \( f_G \mid f \). Indeed, if the inclusion holds, then \( L \subset \mathbb{Q}(\zeta_{f}) \cap \mathbb{Q}(\zeta_{f_G}) \). It is well-known that the intersection is generated by a root of unity of order \( \gcd(f, f_G) \). By minimality of \( f_G \) we find \( f_G \mid f \). The converse direction follows as \( \mathbb{Q}(\zeta_{f_G}) \subset \mathbb{Q}(\zeta_{f}) \) if \( f_G \mid f \).

So \( f_G \) is the greatest common divisor of all \( f \) for which \( L \subset \mathbb{Q}(\zeta_{f}) \). Equivalently \( f_G \) is the greatest common divisor of all \( f \mid N \), for which \( \ker((\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/f\mathbb{Z})^\times) \subset G \).

By class field theory, \( f_G \) is the finite part of the conductor of the abelian extension \( L/\mathbb{Q} \).

The next lemma collects some classical facts on Gauß sums. We write \( f_{\chi} = f_{\ker \chi} \) for a character \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \). We recall that \( e(\cdot) \) was defined in \([1,3]\).

Lemma 3.1. Let \( N \in \mathbb{N} \) and say \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \) is a character. For \( k \in \mathbb{Z} \) we define \( \tau = \sum_{\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(\sigma)e(k\sigma/N) \), then the following hold true.

(i) If \( \gcd(k, N) = 1 \) then \( |\tau| \leq f_{\chi}^{1/2} \).

(ii) For unrestricted \( k \) we set \( N' = N / \gcd(k, N) \). Then

\[ |\tau| \leq \frac{\varphi(N)}{\varphi(N')}f_{\chi}^{1/2}. \]

Proof. If \( k = 1 \), part (i) follows directly from Lemma 3.1, Section 3.4 [23]. The more general case \( \gcd(k, N) = 1 \) follows as \( \sum_{\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(\sigma)e(k\sigma/N) = \sum_{\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(k'\sigma)e(\sigma/N) \) where \( kk' \equiv 1 \) (mod \( N \)) and since \( \chi \) is completely multiplicative.
To prove (ii) set $N' = N / \gcd(k, N)$ and $k' = k / \gcd(k, N)$. Then $\tau$ is
\[
\sum_{\sigma \in ((\mathbb{Z}/N\mathbb{Z})^\times)^{\times}} \chi(\sigma)e \left( \frac{k'}{N'}\sigma \right) = \sum_{\sigma' \in ((\mathbb{Z}/N'\mathbb{Z})^\times)^{\times}} \left( \sum_{\sigma \equiv \sigma' (\mod N')} \chi(\sigma) \right)e \left( \frac{k'}{N'}\sigma \right).
\]
The inner sum on the right runs over a coset of the kernel of $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N'\mathbb{Z})^\times$. Since $\chi$ is a character, the inner sum equals 0 if the said kernel does not lie in the kernel of $\chi$. In this case, $\tau = 0$ and we are done.

Otherwise, $\ker((\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N'\mathbb{Z})^\times) \subset \ker \chi$, and then $f_{\chi} | N'$. We find moreover that $\chi$ factors through a character $\chi': (\mathbb{Z}/N'\mathbb{Z})^\times \to \mathbb{C}^\times$ and $f_{\chi'} | f_{\chi}$. As the kernel of $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N'\mathbb{Z})^\times$ has order $\varphi(N)/\varphi(N')$ we have
\[
\tau = \frac{\varphi(N)}{\varphi(N')} \sum_{\sigma' \in ((\mathbb{Z}/N'\mathbb{Z})^\times)^{\times}} \chi'(\sigma')e \left( \frac{k'}{N'}\sigma \right).
\]
Part (ii) now follows from (i) since $\gcd(k', N') = 1$. \qed

**Lemma 3.2.** Let $N \in \mathbb{N}$, let $G$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$, and let $k \in \mathbb{Z}$. We define $N' = N / \gcd(k, N)$, then
\[
\frac{1}{|G|} \sum_{\sigma \in G} e(k\sigma / N) \leq \frac{[((\mathbb{Z}/N\mathbb{Z})^\times)^{\times}] : G}{\varphi(N')} f_G^{1/2}.
\]
Proof. Let $\chi_1', \ldots, \chi_m' : (\mathbb{Z}/N\mathbb{Z})^\times / G \to \mathbb{C}^\times$ be all characters and $m = [((\mathbb{Z}/N\mathbb{Z})^\times)^{\times}] : G]$. Then $\sum_{i=1}^m \chi_i'(\sigma) = 0$ for all $\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times / G$ except for the neutral element, where this sum equals $m$. Write $\chi_i$ for $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times / G$ composed with $\chi_i'$. Then $\sum_{i=1}^m \chi_i(\sigma) = 0$ if and only if $\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times / G$, otherwise this sum is $m$. Therefore,

\[
\sum_{\sigma \in G} e(k\sigma / N) = \frac{1}{m} \sum_{i=1}^m \sum_{\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi_i(\sigma)e(k\sigma / N)
\]
and lemma 3.1(ii) implies
\[
\left| \sum_{\sigma \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi_i(\sigma)e(k\sigma / N) \right| \leq \frac{\varphi(N)}{\varphi(N')} f_{\chi_i}^{1/2}.
\]
Note that $G \subset \ker \chi_i$ because $\chi_i$ factors through $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times / G$. So $f_{\chi_i} \leq f_G$, by the minimality of $f_{\chi_i}$. The current lemma now follows from (3.1). \qed

Let $d, n \in \mathbb{N}$ and $x_1, \ldots, x_n \in [0,1]^d$. The discrepancy of $(x_1, \ldots, x_n)$ is

\[
D(x_1, \ldots, x_n) = \sup_B \left| \frac{\# \{i : x_i \in B \}}{n} - \text{vol}(B) \right|
\]
where $B$ ranges over all products $\prod_{i=1}^d [\alpha_i, \beta_i)$ with $0 \leq \alpha_i < \beta_i \leq 1$. Note that the discrepancy lies in $[0,1]$. In some references such as [21], the discrepancy is not normalized by dividing by $n$ and can be greater than 1.
In the next proposition we bound from above the discrepancy of the Galois orbit of a point of finite order in $G^d_m$ using the Gauß sum estimates above. Below, $d_0(N)$ denotes the number of divisors of a natural number $N$.

**Proposition 3.3.** Let $\zeta \in G^d_m$ have order $N$ and let $G$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$ such that 
\[ \{\zeta^i : \sigma \in G\} = \{e(x_i) : 1 \leq i \leq \#G\} \text{ with all } x_i \text{ in } [0,1)^d. \]

(i) We have
\[
D(x_1, \ldots, x_{\#G}) \ll_d \left( (\mathbb{Z}/N\mathbb{Z})^\times : G \right) f_G^{1/2} \frac{(\log 2\delta(\zeta))^{d-1} \log \log 3\delta(\zeta)}{\delta(\zeta)^{1/2}}.
\]

(ii) If $d = 1$, then
\[
D(x_1, \ldots, x_{\#G}) \ll \left( (\mathbb{Z}/N\mathbb{Z})^\times : G \right) f_G^{1/2} \frac{\log(2N)d_0(N)}{\varphi(N)}.
\]

**Proof.** We abbreviate $n = \#G$. We fix $a \in \mathbb{Z}^d$ with $\zeta = e(a/N)$. Then $N$ and the entries of $a$ are coprime. Let $H \geq 4$ be an integer. We use the Erdős–Turán–Koksma inequality, Theorem 5.21 [21], to bound the discrepancy $D = D(x_1, \ldots, x_n)$ as follows

\[
D \ll_d \frac{1}{H} + \sum_{b \in \mathbb{Z}^d \setminus \{0\} : |b| \leq H} \frac{1}{r(b)} \sum_{\sigma \in G} e\left( \frac{\langle a, b \rangle}{N} \right).
\]

here $r(b_1, \ldots, b_d) = \max\{1, |b_1|\} \cdots \max\{1, |b_d|\}$.

By Lemma 3.2, the expression inside the modulus is at most $C / \varphi(N / \gcd(\langle a, b \rangle, N))$ with $C = \left( (\mathbb{Z}/N\mathbb{Z})^\times : G \right) f_G^{1/2}$. We have $\varphi(M) \gg M/\log \log(3 + M)$ for all integers $M \geq 1$ with an absolute and effective implicit constant, see for example Theorem 15 [39]. Therefore,

\[
D \ll_d \frac{1}{H} + C \sum_{b \in \mathbb{Z}^d \setminus \{0\} : |b| \leq H} \frac{1}{r(b)} \frac{\gcd(\langle a, b \rangle, N)}{N} \log \log(3 + N / \gcd(\langle a, b \rangle, N)).
\]

If $b \in \mathbb{Z}^d \setminus \{0\}$ with $|b| \leq H$, then
\[
\left\langle a, \frac{N}{\gcd(\langle a, b \rangle, N)} b \right\rangle = N \langle a, b \rangle / \gcd(\langle a, b \rangle, N) \in \mathbb{Z}
\]

which implies $\zeta^{bN/\gcd(\langle a, b \rangle, N)} = 1$. So $N / \gcd(\langle a, b \rangle, N) \geq \delta / |b| > 0$ where $\delta = \delta(\zeta)$. As $t \mapsto (\log \log(3 + t))/t$ is decreasing on $t > 0$ we find

\[
D \ll_d \frac{1}{H} + C \frac{1}{\delta} \sum_{b \in \mathbb{Z}^d \setminus \{0\} : |b| \leq H} \frac{|b|}{r(b)} \log(3 + \delta).
\]

The sum of $|b|/r(b)$ over all $b \in \mathbb{Z}^d$ with $1 \leq |b| \leq H$ is $\ll_d H(\log H)^{d-1}$, so we find

\[
D \ll_d \frac{1}{H} + C \frac{\log \log(3\delta)}{\delta} H(\log H)^{d-1}.
\]

Part (i) follows by fixing $H$ to be the least integer with $H \geq \delta^{3/2}$ and $H \geq 4$. 


In part (ii) we have \( d = 1 \). We may assume \( N \geq 4 \) as the discrepancy is at most 1. Here \( a \) is coprime to \( N \) and so \( \gcd(ab,N) = \gcd(b,N) \). In \( \text{Lemma 3.2} \) we take \( H = N \) and use again \( \text{Lemma 3.2} \) with \( C \) as before to find
\[
D \ll \frac{1}{N_0} + \sum_{b=1}^{N_0} \frac{C}{b\varphi(N/\gcd(b,N))} \leq \frac{1}{N_0} + \sum_{g|N} \frac{C}{g\varphi(N/g)} \sum_{\epsilon=1}^{N/g} 1.
\]
In the sum over \( g \) we have \( g\varphi(N/g) \geq \varphi(N) \) and the harmonic sum is \( \ll \log N \). So \( D \ll \frac{1}{N_0} + C (\log N) d_0(N)/\varphi(N) \), which implies (ii).

\[\square\]

A variant of the case \( d = 1 \) already appears in \( \text{Lemma 1.3} \) \( [3] \), it is attributed to Pomerance. The discrepancy bound in (i) depends on \( \delta(\zeta) \). But \( \delta(\zeta) \) is always bounded above by \( N \). So estimates involving \( N \) are stronger than estimates involving \( \delta(\zeta) \). However, there can be no upper bound for the discrepancy in terms of the order \( N \).

For \( d = 1 \) we have \( \delta(\zeta) = N \). If \( [(\mathbb{Z}/N\mathbb{Z})^\times : G] \) and \( f_G \) are fixed, the decay of the discrepancy is \( 1/N \) up to terms of subpolynomial growth. This fact will be important.

The total variation of a real valued function \( F \) whose domain contains the interval \([a, b]\) with \( a \leq b \) is
\[
\text{Var}_a^b(F) = \sup_{a \leq x_0 \leq \cdots \leq x_m \leq b} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|.
\]
For \( a = 0 \) and \( b = 1 \) we abbreviate \( \text{Var}(F) = \text{Var}_a^b(F) \).

The next lemma requires Koksma’s inequality.

**Lemma 3.4.** Let \( F : [0,1] \to \mathbb{R} \) be a function with \( \text{Var}(F) < \infty \). If \( N \geq 1 \) is an integer and \( G \) is a subgroup of \( (\mathbb{Z}/N\mathbb{Z})^\times \) such that \( \{ \zeta^\sigma : \sigma \in G \} = \{ e(x_i) : 1 \leq i \leq \#G \} \) with all \( x_i \) in \([0,1]\), then
\[
\left| \frac{1}{\#G} \sum_{i=1}^{\#G} F(x_i) - \int_0^1 F(x)dx \right| \ll [(\mathbb{Z}/N\mathbb{Z})^\times : G] f_G^{1/2} \log(2N) d_0(N)/\varphi(N) \text{Var}(F).
\]

**Proof.** The claim follows from \( \text{Theorems 1.3 and 5.1} \) in Chapter 2 \( \text{[24]} \) together with \( \text{Proposition 3.3(ii)} \).

**3.1. A univariate average.**

**Lemma 3.5.** Let \( \alpha \in \mathbb{C} \) and \( r > 0 \). For \( x \in [0,1] \) we define
\[
F_{a,r}(x) = \log \max (r, |e(x) - \alpha|).
\]
Then \( F_{a,r} : [0,1] \to \mathbb{R} \) satisfies \( \text{Var}(F_{a,r}) \leq 3 \log(1 + 2/r) \).

**Proof.** We abbreviate \( F = F_{a,r} \). By elementary geometry we can find \( m \leq 3 \) and \( 0 = x_0 < x_1 < \cdots < x_m = 1 \) such that \( F \) is monotone on all \([x_{i-1}, x_i]\). Then \( \text{Var}_{x_{i-1}}^{x_i}(F) = |F(x_i) - F(x_{i-1})| \) and \( \text{Var}(F) = \sum_{i=1}^{m} \text{Var}_{x_{i-1}}^{x_i}(F) \). We have \( \log \max \{ r, |\alpha| - 1 \} \leq F(x) \leq \log \max \{ r, |\alpha| + 1 \} \) for all \( x \in [0,1] \). Hence \( \text{Var}_{x_{i-1}}^{x_i}(F) \leq \log \max \{ r, |\alpha| - 1 \} \) which we see is at most \( \log(1 + 2/r) \) by considering the cases \( |\alpha| \geq 1 + r \) and \( |\alpha| < 1 + r \). Thus \( \text{Var}(F) \leq 3 \log(1 + 2/r) \).

\[\square\]
The value $r$ serves as a truncation parameter. We now apply Koksma’s inequality to $F_{a,r}$.

**Lemma 3.6.** Let $\zeta \in \mu_\infty$ have order $N$ and let $G$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Let $\alpha \in \mathbb{C}$ and $r \in (0, 1]$, then

$$
\frac{1}{|G|} \sum_{\sigma \in G} \log |\zeta^\sigma - \alpha| = \log^+ |\alpha| + O \left( \left( (\mathbb{Z}/N\mathbb{Z})^\times : G \right) f_G^{1/2} \log(2N) d_0(N) \frac{\varphi(N)}{\varphi(N)} + r \right) \left( \log \frac{r}{2} \right).
$$

**Proof.** We let $I$ denote the left-hand side of (3.4). Then $I = I_1 + I_2$ with

$$
I_1 = \frac{1}{|G|} \sum_{i=1}^{|G|} F_{a,r}(x_i) \quad \text{and} \quad I_2 = \frac{1}{|G|} \sum_{|e(x_i) - \alpha| \leq r} - \log r
$$

with the $x_i \in [0, 1)$ as in Lemma 3.5 and $F_{a,r}$ as in Lemma 3.5. The integrals below are understood to be over subsets of $[0, 1]$. Applying Lemma 3.4 to $F_{a,r}$ yields

$$
I_1 = \int_0^1 F_{a,r}(x) dx + O \left( \left( (\mathbb{Z}/N\mathbb{Z})^\times : G \right) f_G^{1/2} \log(2N) d_0(N) \frac{\varphi(N)}{\varphi(N)} \right) \left( \log \frac{r}{2} \right).
$$

The set of $x \in [0, 1]$ with $|e(x) - \alpha| \leq r$ is of the form $\emptyset, [a, b]$, or $[0, a] \cup [b, 1]$. So its characteristic function has total variation at most 2. Lemma 3.4 applied to this characteristic function yields

$$
I_2 = - \int_{|e(x) - \alpha| \leq r} \log r dx + O \left( \left( (\mathbb{Z}/N\mathbb{Z})^\times : G \right) f_G^{1/2} \log(2N) d_0(N) \frac{\varphi(N)}{\varphi(N)} \right).
$$

The sum of the integrals in (3.5) and (3.6) equals

$$
\int_0^1 \log |e(x) - \alpha| dx - \int_{|e(x) - \alpha| \leq r} \log |e(x) - \alpha| dx.
$$

Jensen’s formula, Proposition 1.6.5 [5], implies that the first integral equals $\log^+ |\alpha|$. To complete the proof it suffices to show that the second integral is $O(r \log r/2)$.

The integral is non-positive as $r \leq 1$ and we may assume that it is non-zero. First assume, $|\alpha| \leq 1/2$, then $r \geq 1/2$. In this case $|e(x) - \alpha| \geq 1/2$ and the integral is $O(r \log r/2)$. Second, say $|\alpha| > 1/2$. Lemma 11.6.1 [3] implies $|e(x) - \alpha| \geq |\alpha|^{1/2} |e(x) - e(y)| \geq 2^{-1/2} |e(x - y) - 1|$ where $a = |\alpha| e(y)$ and $|x - y| \leq 1/2$. There is an absolute and effectively computable constant $C > 0$ with $|e(x - y) - 1| \geq C|x - y|$ and thus $|e(x) - \alpha| \geq 2^{-1/2} C|x - y|$. In the integral we have $r \geq |e(x) - \alpha|$ and so the desired bound follows from elementary analysis. \qed

### 3.2. A Galois conjugate near 1

We will also need an estimate on the minimal distance of a Galois conjugate of a torsion point to the unit element.

**Lemma 3.7.** Let $\zeta \in G_m^d$ have order $N$ and let $G$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. There exist $\sigma \in G$ and $a \in \mathbb{Z}^d$ with $\zeta = e(\alpha / N), |\alpha| < N$, and

$$
\frac{|\alpha|}{N} \ll \frac{\left((\mathbb{Z}/N\mathbb{Z})^\times : G\right)^{1/2} f_G^{1/(2d)}}{\delta(\zeta)^{1/(3d)}}.
$$
Proof. Let \( \zeta = e(b/N) \) with \( b \in \mathbb{Z}^d \), the entries of \( b \) and \( N \) have no common prime divisor. Suppose \( x_1, \ldots, x_n \) are as in Proposition 3.3 coming from the \( \zeta^\sigma \) as \( \sigma \) ranges over \( G \) where \( n = \#G \). There exists \( c(d) > 0 \) depending only on \( d \) with \( D(x_1, \ldots, x_n) \leq c(d)[(\mathbb{Z}/N\mathbb{Z})^\times : G]^f 1/2 \delta(\zeta)^{-1/3} \). We set \( \kappa = 2c(d)^{1/d}[(\mathbb{Z}/N\mathbb{Z})^\times : G]^{1/d} f_G^{1/(2d)} \delta(\zeta)^{-1/(3d)} \).

There is nothing to show if \( \kappa \geq 1 \). Otherwise, by the definition of the discrepancy the hypercube \([0, \kappa)^d \) contains some \( x_i = a/N \). Hence \( a \) satisfies, \( |a| < N \), (3.7), and \( e(a/N) = \zeta^\sigma^{-1} \) for some \( \sigma \in G \). \( \square \)

4. **Theorem of Mahler–Mignotte**

In this section, we firstly establish the separation of pairs of roots of an integer polynomial. Theorem 4.1 below was shown by Mahler [33] for the case \( k = 1 \) of a single pair of roots. Mignotte [34] generalized Mahler’s inequality to products over several disjoint pairs of roots (see his Theorem 1). We reproduce here a lightened version of Mignotte’s theorem that is suitable for our needs. The proof is an adaptation of Mahler’s original argument about a single pair, guided by the principle that Liouville’s Inequality bounds an algebraic number at an arbitrary set of places in terms of the height. Let us also mention G"uting’s proof [19] of a less precise earlier result involving the length of a polynomial instead of the Mahler measure.

Let \( Q \in \mathbb{C}[X] \) be a non-zero univariate polynomial. By Jensen’s formula its Mahler measure equals

\[
(4.1) \quad m(Q) = \log |a_0| + \sum_{i=1}^D \log^+ |z_i|
\]

if \( Q = a_0(X - z_1) \cdots (X - z_D) \) and where the \( z_i \) are complex. If \( Q \) is non-constant, we let \( \text{disc}(Q) \) denote its discriminant as a degree \( \deg Q \) polynomial.

**Theorem 4.1.** Let \( Q \in \mathbb{C}[X] \setminus \mathbb{C} \) be of degree \( D \) and with no repeated roots. If \( z_1, \ldots, z_k, z_1', \ldots, z_k' \) are pairwise distinct complex roots of \( Q \), then

\[
(4.2) \quad \sum_{j=1}^k - \log |z_j - z_j'| \leq \frac{D + 2k}{2} \log D - \frac{k}{2} \log 3 + (D - 1)m(Q) - \frac{1}{2} \log |\text{disc}(Q)|
\]

with strict inequality for \( k \geq 1 \).

**Proof.** We modify Mahler’s argument as follows.

Both sides of (4.2) are invariant under multiplication \( Q \) by a non-zero scalar. So we may assume that \( Q \) is monic. After possibly swapping \( z_j \) with \( z_j' \) we may assume \( |z_j| \geq |z_j'| \) for all \( j \).

We augment \( z_1, \ldots, z_k \) to all complex roots \( z_1, \ldots, z_D \) of \( Q \). Then we consider the Vandermonde determinant

\[
V = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_D \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{D-1} & z_2^{D-1} & \ldots & z_D^{D-1} \end{pmatrix},
\]
which is non-zero as \( z_1, \ldots, z_D \) are pairwise distinct. For \( j \in \{1, \ldots, k\} \), let \( i_j > k \) be the index with \( z'_j = z_{i_j} \). For these \( j \), we subtract the \( i_j \)-th column from the \( j \)-th column and factoring each difference \( z_j - z_{i_j} \) out of the determinant with the identities 
\[
z_j^m - z_{i_j}^m = (z_j - z_{i_j})(z_j^{m-1} + z_j^{m-2}z_{i_j} + \cdots + z_{i_j}^{m-1}), \quad 1 \leq m \leq D - 1.
\]
We obtain an expression
\[
V = W \prod_{j=1}^{k} (z_j - z_{i_j}) = W \prod_{j=1}^{k} (z_j - z'_j),
\]
where \( W \neq 0 \) is the determinant of the matrix having
\[
\begin{pmatrix}
0 \\
1 \\
z_j + z'_j \\
\vdots \\
z_j^{D-2} + z_j^{D-3}z'_j + \cdots + z_j^{D-2}
\end{pmatrix}
\]
for its \( j \)-th column, \( j \in \{1, \ldots, k\} \), and the same entries as in the Vandermonde matrix in the remaining columns. By Hadamard’s inequality, \(|W|\) is bounded from above by the product of the Hermitian norms of all these columns. The \( j \)-th column, for some \( j \in \{1, \ldots, k\} \), has Hermitian norm
\[
\sqrt{\sum_{m=0}^{D-2} |z_j^m + z_j^{m-1}z'_j + \cdots + z_j^m|^2} \leq \sqrt{\sum_{m=0}^{D-2} (m+1)^2 \max\{1, |z_j|, |z'_j|\}^{D-2}}
\]
\[
< \sqrt{D^3/3} \cdot \max\{1, |z_j|\}^{D-1}
\]
where we used \(|z'_j| \leq |z_j|\). The Hermitian norm of the \( j \)-th column with \( j \in \{k+1, \ldots, D\} \) is at most \( \sqrt{D} \max\{1, |z_j|\}^{D-1} \).

Applying Hadamard’s inequality, using these two bounds, and taking the logarithm yields
\[
\log |W| \leq \frac{k}{2} \log \left( \frac{D^3}{3} \right) + \frac{D-k}{2} \log D + (D-1) \sum_{j=1}^{D} \log^+ |z_j|
\]
\[
= \frac{D+2k}{2} \log D - \frac{k}{2} \log 3 + (D-1)m(Q)
\]
as \( Q \) is monic; the inequality is strict for \( k \geq 1 \). If \( k = 0 \) no column operations are necessary and \( V = W \).

The squarefree polynomial \( Q \) has discriminant \( \text{disc}(Q) = V^2 \). Consequently \(|V| = |\text{disc}(Q)|^{1/2} \), and in view of (4.3) we have
\[
\sum_{j=1}^{k} - \log |z_j - z'_j| = \log |W| - \log |V|
\]
\[
\leq \frac{D+2k}{2} \log D - \frac{k}{2} \log 3 + (D-1)m(Q) - \frac{1}{2} \log |\text{disc}(Q)|
\]
with a strict inequality for \( k \geq 1 \). This concludes the proof.

While Theorem 4.1 suffices for our needs here, we remark that it is possible to relax the hypothesis to having \( z_1, \ldots, z_k \) pairwise distinct and \( \{z_1, \ldots, z_k\} \cap \{z'_1, \ldots, z'_k\} = \emptyset \), at the cost of a slightly worse upper bound (4.2).

The following corollary holds for integral polynomials that are not necessarily squarefree.

**Corollary 4.2.** Let \( Q \in \mathbb{Z}[X] \setminus \mathbb{Z} \) be of degree \( D \). If \( z_1, \ldots, z_k, z'_1, \ldots, z'_k \) are pairwise distinct complex roots of \( Q \), then

\[
(4.4) \quad \sum_{j=1}^{k} - \log |z_j - z'_j| \leq \frac{D + 2k}{2} \log D - \frac{k}{2} \log 3 + (D - 1)m(Q)
\]

with strict inequality for \( k \geq 1 \).

**Proof.** Again we may assume \( k \geq 1 \). We begin by splitting off the squarefree part of \( Q \). More precisely, we factor \( Q = \tilde{Q}R \) where \( \tilde{Q}, R \in \mathbb{Z}[X] \) and \( \tilde{Q} \) is squarefree and vanishes at all complex roots of \( Q \). The discriminant \( \text{disc}(\tilde{Q}) \) is a non-zero integer, and so \( |\text{disc}(\tilde{Q})| \geq 1 \). Moreover, \( m(\tilde{Q}) \geq 0 \). Theorem 4.1 applied to \( \tilde{Q} \) and \( 1 \leq \deg \tilde{Q} \leq D \) imply that the sum on the left of (4.4) is at most \( \frac{1}{2}(D + 2k) \log D - \frac{k}{2} \log 3 + (D - 1)m(\tilde{Q}) \). The corollary follows from \( m(\tilde{Q}) = m(Q) - m(R) \leq m(Q) \). \( \square \)

4.1. A repulsion property of the unit circle. A key point in [20] is that while Mahler’s theorem does not give a strong enough bound for the distance of a complex root of \( Q \in \mathbb{Z}[X] \setminus \{0\} \) to an \( N \)-th root of unity (the product \( (X^N - 1)Q(X) \) has an exceedingly large degree), it can be used to bound the distance from the unit circle to the locus of roots of \( P \) lying off the unit circle. With Corollary 4.2, this repulsion property of the unit circle can be strengthened as follows.

**Lemma 4.3.** Let \( Q \in \mathbb{Z}[X] \setminus \mathbb{Z} \) and \( Q = a_0(X - z_1) \cdots (X - z_D) \) where \( z_1, \ldots, z_D \in \mathbb{C} \). Then

\[
(4.5) \quad \sum_{j=1}^{D} \frac{1}{|z_j| - 1} \leq D \log \left( \frac{3 + \sqrt{5}}{2} \right) + 2D \log(2D) + 4Dm(Q)
\]

\[
\leq 4D(\log(2D) + m(Q)).
\]

Before we come to the proof let us remark that \( |z| - 1 \) is the distance \( \text{dist}(z, S^1) \) of \( z \in \mathbb{C} \) to the unit circle \( S^1 \). Thus inequality (4.5) can be restated as providing

\[
\frac{1}{D} \sum_{j=1}^{D} \frac{1}{\text{dist}(z_j, S^1)} \leq \log \left( \frac{3 + \sqrt{5}}{2} \right) + 2\log(2D) + 4m(Q).
\]

Our result suggests that the unit circle repels roots of \( Q \) that lie off the unit circle. Related estimates are implicit in work of Dubickas [14], cf. his Theorem 2.
Proof. The second bound in (4.5) is elementary, so it suffices to prove the first one.

Say \( Q = a_0 Q_1 \cdots Q_n \) where each \( Q_i \in \mathbb{Z}[X] \) is irreducible of positive degree with \( a_0 \in \mathbb{Z} \). Observe that \( m(Q_i) \leq m(Q) \) and \( \sum_i \deg Q_i = \deg Q \). So it suffices to prove (4.5) for \( Q \) irreducible in \( \mathbb{Z}[X] \). We may also assume \( Q(0) \neq 0 \).

We will apply Corollary \([4.2]\) to the polynomial \( \tilde{Q} \in \mathbb{Z}[X] \) constructed from \( Q \) in the following manner. If \( Q(1/X)^{2D} \neq \pm Q \) we take \( \tilde{Q} = Q(X)Q(1/X)^{D} \) and \( \tilde{Q} = Q \) otherwise. So \( \tilde{D} = \deg \tilde{Q} = \delta D \) and \( m(\tilde{Q}) = \delta m(Q) \) with \( \delta \geq 2 \) in the first case and \( \delta = 1 \) in the second case. For any root \( z \) of \( \tilde{Q} \) we also have \( \tilde{Q}(1/\overline{z}) = 0 \).

The following basic observation for a complex number \( z \) will prove useful. We have \(|z - 1/\overline{z}| \leq 1 \) if and only if \( \phi^{-1} \leq |z| \leq \phi \) with \( \phi = (1 + \sqrt{5})/2 \) the golden ratio.

Let \( w_1, \ldots, w_k \) be the roots of \( \tilde{Q} \) without repetition such that \( \phi^{-1} \leq |w_j| < 1 \). Then \( w_j' = 1/\overline{w_j} \) is a root of \( \tilde{Q} \) for each \( j \in \{1, \ldots, k\} \) with \(|w_j'| > 1 \). Corollary \([4.2]\) yields

\[
(4.6) \quad \sum_{j=1}^{k} \log^+ \left| w_j - 1/\overline{w_j} \right| \leq \delta D \log(\delta D) + \delta^2 D m(Q)
\]

because \( k \leq \tilde{D}/2 = \delta D/2 \) and \( m(Q) \geq 0 \).

Suppose \( z_j \) is a root of \( Q \) with \(|z_j| \neq 1 \) and \( \phi^{-1} \leq |z_j| \leq \phi \). Then \( z_j \in \{ w_j, 1/\overline{w_j} \} \) for some unique \( l \). The mapping \( j \mapsto l \) is at worst \( 2 \)-to-1 and injective if \( \delta = 2 \) as \( Q \) is irreducible.\(^1\) This leads to the factor \( 2/\delta \) in

\[
(4.7) \quad \sum_{|z_j| \neq 1 \atop 1/\phi \leq |z_j| \leq \phi} \log^+ \frac{1}{|z_j - 1/\overline{z_j}|} \leq \frac{2}{\delta} \sum_{l=1}^{k} \log^+ \frac{1}{|w_l - 1/\overline{w_l}|}
\]

For a complex number \( z \) with \(|z| \geq \phi^{-1} \) we have \(|z - 1/\overline{z}| = |z + 1/|z||z| - 1| \leq (1 + \phi)||z| - 1| \). This allows us to get

\[
\sum_{|z_j| \neq 1 \atop 1/\phi \leq |z_j| \leq \phi} \log^+ \frac{1}{||z_j| - 1|} \leq s \log(1 + \phi) + \sum_{|z_j| \neq 1 \atop 1/\phi \leq |z_j| \leq \phi} \log^+ \frac{1}{|z_j - 1/\overline{z_j}|}
\]

where \( s \) is the number of terms in the first sum. There are at most \( D - s \) other roots of \( Q \) and if \(|z_j| < \phi^{-1} \) or \(|z_j| > \phi \) we get \( \log^+ 1/||z| - 1| \leq \log(1 + \phi) \). Together with (4.6) and (4.7) we find

\[
\sum_{|z_j| \neq 1} \log^+ \frac{1}{||z_j| - 1|} \leq D \log(1 + \phi) + 2D \log(\delta D) + 2\delta D m(Q).
\]

We have established (4.5) for \( Q \) as \( \delta \leq 2 \). \( \square \)

Next we generalize our bound to a polynomial with coefficients in a number field. Recall that \( h(Q) \) is the absolute logarithmic projective Weil height of a non-zero polynomial \( Q \) with algebraic coefficients.

\(^1\)Indeed, if \( z_j, z_k \in \{ w_l, 1/\overline{w_l} \} \) with \( z_j \neq z_k \), then \( z_j = 1/\overline{z_k} \). So \( \tilde{Q} = Q \) and hence \( \delta = 1 \) in this case.
Corollary 4.4. Let $F \subset \mathbb{C}$ be a number field and let $Q \in F[X] \setminus F$ and $Q = a_0(X - z_1) \cdots (X - z_D)$ where $z_1, \ldots, z_D \in \mathbb{C}$. Then

$$\sum_{j=1}^{D} \log^+ \frac{1}{|z_j| - 1} \leq 10D[F : Q]^2(\log(2D) + h(Q)).$$

Proof. Let $\tilde{Q}$ be the product of the $Q$-Galois conjugates of $Q$. Then $\tilde{Q}$ has rational coefficients and degree $\tilde{D} \leq D[F : Q]$. Let $\lambda \in \mathbb{N}$ such that $\lambda \tilde{Q}$ is integral with content $1$. For the projective height we find $h(\tilde{Q}) = \log |\lambda \tilde{Q}|$. Together with Lemma 1.6.7 we get $m(\lambda \tilde{Q}) \leq \frac{1}{2} \log(1 + \tilde{D}) + h(\tilde{Q})$. As all $Q$-Galois conjugates of $Q$ have the same projective height we use elementary estimates at local places to find

$$h(\tilde{Q}) \leq [F : Q] \log(1 + \tilde{D}) + [F : Q]h(Q).$$

By Lemma 4.3 applied to $\lambda \tilde{Q}$, the sum $\sum_{j=1}^{D} \log^+ 1/|z_j| - 1$ is at most

$$4\tilde{D} \left( \log(2\tilde{D}) + \frac{1}{2} \log(1 + \tilde{D}) + [F : Q] \log(1 + \tilde{D}) + [F : Q]h(Q) \right).$$

We use $1 + \tilde{D} \leq 2\tilde{D} \leq 2[D[F : Q] \leq (2D)^{[F : Q]}$ to complete the proof. \hfill $\square$

4.2. Averages over roots of unity. In this subsection we apply the repulsion property of the unit circle, Corollary 4.4, to estimate the norm of cyclotomic integers of the form $Q(\zeta)$, where $\zeta$ is a varying root of unity and $Q$ is a moderately controlled univariate polynomial with algebraic coefficients and without zeros in $S^1 \setminus \mu_\infty$. This gives a fairly uniform solution of the one dimensional essentially atoral case and forms the basis for the higher dimensional case to be taken up in the next sections.

Proposition 4.5. Let $F \subset \mathbb{C}$ be a number field and let $Q \in F[X] \setminus \{0\}$ be of degree at most $D \geq 1$ with no roots in $S^1 \setminus \mu_\infty$. Let $\zeta \in \mu_\infty$ be of order $N$ and $G$ a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $Q(\zeta^\sigma) \neq 0$ for all $\sigma \in G$. Then

$$\frac{1}{\#G} \sum_{\sigma \in G} \log |Q(\zeta^\sigma)| = m(Q) + O \left( [F : Q]^2[(\mathbb{Z}/N\mathbb{Z})^\times : G]^{1/2}D(\log(2D) + h(Q)) \frac{(\log 2N)^3d_0(N)}{N} \right).$$

Proof. We may assume that $Q$ is non-constant and $D = \deg Q$. Let $Q = a_0(X - z_1) \cdots (X - z_D)$. The idea is that each given root $z_j$ may get within distance of $1/N^2$ to at most a single conjugate of $\zeta$.

We call $z_j$ exceptional if $|\zeta^{\sigma_j} - z_j| \leq 1/N^2$ for some $\sigma_j \in G$. As $|\zeta - \zeta'| \geq 4/N$ for distinct roots of unity $\zeta, \zeta'$ of order $N$ we see that $\sigma_j$ is uniquely determined by $z_j$. Note that $\zeta^{\sigma_j} \neq z_j$ because $Q(z_j) = 0 \neq Q(\zeta^{\sigma_j})$. 

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We apply Lemma 3.6 with $a = z_j$ and $r = 1/N^2$. Thus
\begin{equation}
\frac{1}{\#G} \sum_{\sigma \in G} \log |\zeta^\sigma - z_j| = \log+ |z_j| + \frac{1}{\#G} \log |\zeta^{\sigma_j} - z_j| + O \left( \left[ (\mathbb{Z}/N\mathbb{Z})^\times : G \right] f_G^{1/2} \frac{(\log 2N)^2 d_0(N)}{\varphi(N)} + \frac{\log 2N}{N^2} \right),
\end{equation}
if $z_j$ is exceptional, otherwise the same bound without the term $(\#G)^{-1} \log |\zeta^{\sigma_j} - z_j|$ holds true. As $1/N^2 \leq 1/\varphi(N)$ we merge $(\log 2N)/N^2$ into the first term of the error term. Summing (4.9) over all $j \in \{1, \ldots, D\}$ and adding $\log |a_0|$ gives
\begin{equation}
\frac{1}{\#G} \sum_{\sigma \in G} \log |Q(\zeta^\sigma)| = m(Q) - \frac{1}{\#G} \sum_{j=1}^{D'} \log \frac{1}{|\zeta^{\sigma_j} - z_j|} + O \left( \left[ (\mathbb{Z}/N\mathbb{Z})^\times : G \right] f_G^{1/2} D \frac{(\log 2N)^2 d_0(N)}{\varphi(N)} \right),
\end{equation}
the dash signifies that we only sum over those $j$ for which $z_j$ is exceptional.

To bound the dashed sum we require Corollary 4.4. If $z_j$ is exceptional, then $|\zeta^{\sigma_j} - z_j| \leq 1$. Therefore, the dashed sum is non-negative.

First, we consider the subsum over all exceptional $z_j \notin \mu_\infty$. Then $|z_j| \neq 1$ and $|\zeta^{\sigma_j} - z_j| \geq ||z_j| - 1|$ by the reverse triangle inequality. By Corollary 4.4 we find
\begin{equation}
0 \leq \sum_{j=1}^{D'} \log \frac{1}{|\zeta^{\sigma_j} - z_j|} = \sum_{j=1}^{D'} \log+ \frac{1}{||z_j| - 1|} = O \left( [F : Q]^2 D (\log (2D) + h(Q)) \right).
\end{equation}

Second, we consider the subsum over all exception $z_j \in \mu_\infty$, which is harmless. Recall that $\zeta^{\sigma_j} \neq z_j$. Since the order of $z_j$ is $\ll |Q(z_j) : Q|^2 \leq (D[F : Q])^2$ and the order of $\zeta^{\sigma_j}$ is $N$ we find $|\zeta^{\sigma_j} - z_j| \gg N^{-1} (D[F : Q])^{-2}$. On the other hand, $|\zeta^{\sigma_j} - z_j| \leq N^{-2}$ and hence $N \ll (D[F : Q])^2$. We obtain the crude estimate $|\zeta^{\sigma_j} - z_j| \gg (D[F : Q])^{-4} \gg (2D)^{-4[F : Q]}$ and finally bound the at most $D$ terms below separately to get
\begin{equation}
0 \leq \sum_{j=1}^{D'} \log \frac{1}{|\zeta^{\sigma_j} - z_j|} = O \left( [F : Q] D \log (2D) \right).
\end{equation}

We divide the sum of (4.11) and (4.12) by $\#G$ to find
\begin{equation*}
0 \leq \frac{1}{\#G} \sum_{j=1}^{D'} \log \frac{1}{|\zeta^{\sigma_j} - z_j|} = O \left( [F : Q]^2 D (\log (2D) + h(Q)) \frac{|(\mathbb{Z}/N\mathbb{Z})^\times : G|}{\varphi(N)} \right).
\end{equation*}
The proposition follows from (4.10) and $\varphi(N) \gg N/\log \log (3N)$, a consequence of Theorem 15 [39].

Proposition 4.5 and ultimately Theorem 4.1 may be viewed as our input from transcendence theory. If this or a comparable bound held without the restrictive condition that $Q$ has no roots in $S^1 \setminus \mu_\infty$ then it could be used to attack Conjecture 1.3. We were
unable to prove or disprove that a suitable version of Theorem 4.5 extends to general polynomials. Progress on Conjecture 1.5 could indicate a path towards this goal.

5. GEOMETRY OF NUMBERS

Let $d \geq 1$ and suppose $\zeta \in \mathbb{C}^d_m$ has order $N$. It would be useful if $\zeta$ had a Galois conjugate close to the unit element 1. If the distance were at most a small power of $N^{-1}$, this conjugate could be used to help reduce the multivariate Theorem 1.1 to the univariate Proposition 4.5, cf. [20].

Unfortunately, such a conjugate need not exist. Take for example $\zeta = e(1/p, 1/p^n)$ where $p$ is a prime and $n \in \mathbb{N}$, here $N = p^n$. Any conjugate of $\zeta$ has distance $\gg 1/p$ to 1 regardless of the value of $n$. The problem is that $\zeta$ is up-to a point of order $p$

We overcome this difficulty by constructing a factorization $\zeta = \eta \zeta$ into torsion points $\eta$ and $\zeta$ that satisfy the following properties for prescribed $\epsilon > 0$. First, the order of $\eta$

is small relative to $N$, more precisely it is $O_d,e(N^\epsilon)$. Second, some Galois conjugate of $\zeta$ is at distance at most $O_d,e(N^{-\kappa(\epsilon)})$ to 1. Here $\kappa(\epsilon)$ is expected to be small for small $\epsilon$, but we will see that $\kappa(\epsilon)/\epsilon$ is large. This is of central importance for our application.

We use the geometry of numbers to construct this factorization. An important tool

is the slope of a lattice.

A lattice $\Lambda$ in $\mathbb{R}^d$ is a finitely generated and discrete subgroup of $\mathbb{R}^d$. The rank of $\Lambda$

is denoted by $\text{rk}(\Lambda)$ and its determinant by $\det(\Lambda)$. We consider the set

$$A = \{(r, \log \det(\Omega)) : r \in \mathbb{Z} \text{ and } \Omega \text{ is a subgroup of } \Lambda \text{ with } \text{rk}(\Omega) = r\}$$

and use the convention $\det(\{0\}) = 1$. In contrast to the convention in Arakelov theory, we have no sign in front of $\log \det(\Lambda)$. Observe that the second coordinate is bounded from below on $A$. In Proposition 1, Stuhler [45] proved that for each $j \in \{0, \ldots, \text{rk}(\Lambda)\}$ there exists a lattice $\Lambda_j \subset \Lambda$, possibly non-unique, with $\log \det(\Lambda_j)$ minimal. The lower boundary of the convex hull of $A$ is the graph of a piece-wise linear, continuous, convex function $P : [0, \text{rk}(\Lambda)] \to \mathbb{R}$. As $\Lambda_0 = \{0\}$ and $\Lambda_{\text{rk}(\Lambda)} = \Lambda$

we find $P(0) = 0$ and $P(\text{rk}(\Lambda)) = \log \det(\Lambda)$.

For each $j \in \{1, \ldots, \text{rk}(\Lambda)\}$, the slope of $P$ on $[j-1, j]$ is

$$\mu_j(\Lambda) = P(j) - P(j-1).$$

By convexity we have

$$\mu_1(\Lambda) \leq \mu_2(\Lambda) \leq \cdots \leq \mu_{\text{rk}(\Lambda)}(\Lambda).$$

Moreover, $\mu_1(\Lambda) + \cdots + \mu_j(\Lambda) = P(j) - P(0) = P(j)$ for all $j$ as $P(0) = \log \det(\Lambda_0) = 0$.

Assume $\Lambda \neq \{0\}$ and let $\nu \in (0, 1/2]$ be a parameter. Suppose that

$$\mu_j(\Lambda) < \nu^{\text{rk}(\Lambda)-j+1} \log \det(\Lambda)$$

for all $j \in \{1, \ldots, \text{rk}(\Lambda)\}$. Taking the sum yields

$$\log \det(\Lambda) < (\nu + \nu^2 + \cdots + \nu^{\text{rk}(\Lambda)}) \log \det(\Lambda).$$

As $\nu \in (0, 1/2]$ we must have $\det(\Lambda) < 1$. 

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Let us now assume \( \det(\Lambda) \geq 1 \), then there exists a unique \( j_0 \in \{0, \ldots, \operatorname{rk}(\Lambda) - 1\} \) such that

\[
(5.1) \quad \mu_k(\Lambda) < \nu^{\operatorname{rk}(\Lambda) - k + 1} \log \det(\Lambda) \text{ for all } 1 \leq k \leq j_0 \quad \text{and} \quad \mu_{j_0+1}(\Lambda) \geq \nu^{\operatorname{rk}(\Lambda) - j_0} \log \det(\Lambda).
\]

We write \( \Lambda(\nu) \) for the rank \( j_0 \) lattice \( \Lambda_{j_0} \), indicating its dependency on \( \nu \). It satisfies \( \operatorname{rk}(\Lambda/\Lambda(\nu)) \geq 1 \).

Note that \( \mu_{j_0}(\Lambda(\nu)) < \mu_{j_0+1}(\Lambda(\nu)) \) if \( j_0 \geq 1 \). Therefore, \( \Lambda(\nu) \) appears in the Harder–Narasimhan filtration of \( \Lambda \) as considered by Stuhler \([45]\) and Grayson \([18]\), if we include \( \{0\} \) as a member of the filtration. In particular, \( \Lambda(\nu) \) is the unique lattice in \( \Lambda \) of rank \( \operatorname{rk}(\Lambda(\nu)) \) and minimal determinant.

Here are two simple properties.

First, for the Euclidean norm \( |\cdot|_2 \) we claim

\[
(5.2) \quad \log |v|_2 \geq \nu^{\operatorname{rk}(\Lambda/\Lambda(\nu))} \log \det(\Lambda) \quad \text{for all } v \in \Lambda \setminus \Lambda(\nu).
\]

Indeed, the lattice \( \Lambda' \) generated by \( \Lambda(\nu) \) and \( v \) contains \( \Lambda(\nu) \) strictly. We must have \( \operatorname{rk}(\Lambda') > \operatorname{rk}(\Lambda) \), as \( \det(\Lambda') \) would otherwise be strictly less than \( \det(\Lambda) \). (This shows in particular that \( \Lambda/\Lambda(\nu) \) is torsion free; a well-known property of the Harder–Narasimhan filtration.) So \( \operatorname{rk}(\Lambda') = \operatorname{rk}(\Lambda) + 1 \) and by convexity of \( P \) we find \( \log \det(\Lambda') \geq \log \det(\Lambda(\nu)) + \mu_{j_0+1}(\Lambda) \). On the other hand, \( \det(\Lambda') \leq \det(\Lambda'(\nu) \cap v\mathbb{Z}) \leq \det(\Lambda(\nu)) \det(v\mathbb{Z}) \) is well-known, for a proof see Proposition 2 \([45]\). We conclude \( \log \det(v\mathbb{Z}) \geq \mu_{j_0+1}(\Lambda) \). Now \( \det(v\mathbb{Z}) = |v|_2 \), so \((5.2)\) follows from \((5.1)\).

Second, \((5.1)\) implies

\[
(5.3) \quad \log \det(\Lambda(\nu)) \leq \mu_1(\Lambda) + \cdots + \mu_{j_0}(\Lambda) \leq 2\nu^{\operatorname{rk}(\Lambda/\Lambda(\nu))} \log \det(\Lambda).
\]

We now make things more concrete. Let \( \zeta \in \mathbb{G}_m^d \) have order \( N \) and set

\[
\Lambda_\zeta = \{ u \in \mathbb{Z}^d : \zeta^u = 1 \}.
\]

We consider the homomorphism \( \mathbb{Z}^d \to \mathbb{G}_m \) defined by \( u \mapsto \zeta^u \) and see that \( \mathbb{Z}^d/\Lambda_\zeta \) is isomorphic to the finite subgroup of \( \mathbb{G}_m \) generated by the coordinates of \( \zeta \). So \( \mathbb{Z}^d/\Lambda_\zeta \) is cyclic of order \( N \). In particular, \( \Lambda_\zeta \) is a lattice in \( \mathbb{R}^d \) of rank \( d \) with \( \det(\Lambda_\zeta) = |\mathbb{Z}^d : \Lambda_\zeta| = N \geq 1 \). The saturation

\[
(5.4) \quad \tilde{\Lambda}_\zeta(\nu) = \{ u \in \mathbb{Z}^d : \text{there is } n \in \mathbb{Z} \setminus \{0\} \text{ such that } nu \in \Lambda_\zeta(\nu) \}
\]

of \( \Lambda_\zeta \) in \( \mathbb{Z}^d \) will also be useful for us. It is a lattice of the same rank as \( \Lambda_\zeta \).

For any lattice \( \Lambda \subset \mathbb{R}^d \) of positive rank we set

\[
(5.5) \quad \lambda_1(\Lambda) = \min \{|u| : u \in \Lambda \setminus \{0\}\}
\]

where as usual \( |\cdot| \) denotes the maximum-norm. It is convenient to define \( \lambda_1(\{0\}) = \infty \).

**Proposition 5.1.** Let \( \nu \in (0, 1/4] \) and let \( \zeta \in \mathbb{G}_m^d \) be of order \( N \). There exists \( V \in \text{GL}_d(\mathbb{Z}) \) and a decomposition \( \zeta = \eta \xi \) with \( \eta \) and \( \xi \in \mathbb{G}_m^d \) of finite order \( E \) and \( M \), respectively, such that the following holds. We abbreviate \( r = \operatorname{rk}(\Lambda_\zeta/\Lambda_\zeta(\nu)) \in \{1, \ldots, d\} \).

(i) We have \( E / N, M / N, \) and \( E \leq N^{2d+1+r} \). In particular, \( Q(\eta, \zeta) = Q(\zeta) \).
(ii) We have $|V| \ll_d N^{2u^{1+r}}$ with $\xi^V = (1, \ldots, 1, \xi')$ and $\xi' \in G_m$.
(iii) If $G$ is a subgroup of $(\mathbb{Z}/M\mathbb{Z})^\times$ there exist $a \in \mathbb{Z}'$ and $\sigma \in G$ such that $\xi' = e(a\sigma/M)$.

\begin{equation}
|a| < M, \quad \text{and} \quad \frac{|a|}{M} \ll_d \frac{|(\mathbb{Z}/M\mathbb{Z})^\times : G| f_{1/2}^1}{N^{u't/(6d)}}.
\end{equation}

(iv) With the definition (vii), we have $\delta(\xi) \geq d^{-1/2} \min \{\lambda_1(\tilde{\Lambda}(v)), N^{u't/2}\}$. Moreover, if $r = d$, or equivalently $\Lambda(v) = \{0\}$, then $V$ is the identity matrix.

**Proof.** We abbreviate $\Lambda = \Lambda_\xi$ as well as $\Lambda(v) = \Lambda_\xi(v)$ and $\tilde{\Lambda}(v) = \tilde{\Lambda}_\xi(v)$. Note that $\det(\Lambda) = N$. We can find a collection of $d - r = \text{rk}(\tilde{\Lambda}(v))$ linearly independent vectors in $\tilde{\Lambda}(v)$ whose norms are at most $\ll_d \det(\tilde{\Lambda}(v))$ by applying Minkowski’s Second Theorem, see Theorem V in Chapter VIII [9], and using $\lambda_1(\Lambda) \geq 1$. By appending suitable standard basis vectors of $\mathbb{Z}_d$ we find $d$ linearly independent vectors in $\mathbb{Z}^d$. By Corollary 2, Chapter I.2 [9] applied to $\mathbb{Z}^d$ and these vectors we get a basis of $\mathbb{Z}^d$ whose entries have norm $\ll_d \det(\tilde{\Lambda}(v))$. By the said corollary, the original linearly independent vectors can be expressed via an triangular matrix in terms of the new basis vectors. So the first $\text{rk}(\tilde{\Lambda}(v))$ entries of this basis are a basis of the saturated group $\tilde{\Lambda}(v)$. Thus there exists $V \in \text{GL}_d(\mathbb{Z})$ whose first $\text{rk}(\tilde{\Lambda}(v))$ columns constitute a basis of $\tilde{\Lambda}(v)$ and

\begin{equation}
|V| \ll_d \det(\tilde{\Lambda}(v)).
\end{equation}

As $\det(\tilde{\Lambda}(v)) \leq \det(\Lambda(v))$, the bound for $|V|$ in (ii) follows from (5.3). We write $\xi^V = (\eta', \xi')$ where $\eta' \in G_{d-r}'$ and $\xi' \in G_m'$ both have finite order dividing $N$. We take $\eta$ and $\xi$ from the assertion to equal $(\eta', 1, \ldots, 1)^{V-1}$ and $(1, \ldots, 1, \xi')^{V-1}$, respectively.

Observe that $[(\tilde{\Lambda}_\xi : \Lambda(v))\tilde{\Lambda}(v) \subset \Lambda(v) \subset \Lambda$. So the first $\text{rk}(\tilde{\Lambda}(v))$ entries of $\xi[\tilde{\Lambda}(v) : \Lambda(v)]V$ are $\eta'[\tilde{\Lambda}(v) : \Lambda(v)]$. This implies that $E = \text{ord}(\eta)$ from the assertion satisfies $E | [\tilde{\Lambda}(v) : \Lambda(v)]$ and thus $E \leq \det(\Lambda(v)) \leq N^{2u^{1+r}}$ by (5.3).

To verify (iii) let us fix $v \in \mathbb{Z}^r \setminus \{0\}$ such that $\xi^{Vv} = 1$ and $|v| = \delta(\xi')$. Then $\xi^{V'v} = 1$ where $V' \in \text{Mat}_{d'}(\mathbb{Z})$ consists of the final $r$ columns of $V$. Raising to the $E$-th power to kill $\eta$ yields $\xi^{E V'v} = 1$. Therefore, $E V'v \in \Lambda$. Note that $E V'v \notin \Lambda(v)$, indeed otherwise $V'v$ would lie in the saturation $\tilde{\Lambda}(v)$. This is impossible as no non-trivial linear combination of columns of $V'$ lies in $\tilde{\Lambda}(v)$ which is generated by the first $\text{rk}(\tilde{\Lambda}(v))$ columns of $V$. Thus (5.2) implies $|E V'v|_2 \geq N^{e'}$. By (5.7)

$$|EV'v| \ll_d |E||v| \ll_d |E||v| \ll_d [\tilde{\Lambda}(v) : \Lambda(v)]|\det(\tilde{\Lambda}(v))||v| = \det(\Lambda(v))|v|$$

we conclude $N^{e'} \ll_d \det(\Lambda(v))|v|$. The determinant bound in (5.3) gives

$$\delta(\xi') = |v| \gg_d N^{e' - 2u^{1+r}} \gg_d N^{e'/2}$$

and the last inequality used $v \leq 1/4$.
To complete the proof of (iii) let $G$ be a subgroup of $(\mathbb{Z}/M\mathbb{Z})^\times$ where $M = \text{ord}(\zeta) = \text{ord}(\zeta')$. By Lemma 3.7 applied to $\zeta'$ there are $a \in \mathbb{Z}'$ and $\sigma \in G$ with $\zeta' = e(a\sigma/M), |a| < M$, and

$$\frac{|a|}{M} \leq \frac{[(\mathbb{Z}/M\mathbb{Z})^\times : G]^{1/3} f_G^{1/3}}{\delta(\zeta')^{1/3}} \leq \frac{[(\mathbb{Z}/M\mathbb{Z})^\times : G]^{1/2}}{N^{1/2}(8d)}.$$ 

It remains to check (iv). Say $v \in \mathbb{Z}^d \setminus \{0\}$ with $\zeta^v = 1$ and $|v| = \delta(\zeta)$. Then $\zeta^v = \eta \zeta^v = \eta^v$. Thus $E \nu \in \Lambda$ and there are two cases to consider. If $v \in \tilde{\Lambda}(v)$, then $|v|_2 \geq \lambda_1(\tilde{\Lambda}(v))$ by definition. Otherwise, $v \notin \tilde{\Lambda}(v)$ in which case $E \nu \notin \Lambda(v)$ by saturation. Here we can use (5.2) and the bound for $E$ from (i) to conclude $|v|_2 \geq E^{-1} N^{1/2} \geq N^{1/2}$. So $|v| \geq |v|_2 / \sqrt{d} \geq N^{1/2} / \sqrt{d}$, as claimed in (iv).

The situation simplifies in the following two cases. If $r = d$, then $\zeta = \zeta' = \eta = 1 = M, E = 1$, and $V$ is the identity matrix. If $N$ is a prime, then $E = 1$ as $E | N$ and $E \leq N^{2v^{1+r}} < N$ by part (i) above. Thus again $\zeta = \zeta'$ and $\eta = 1$.

6. A preliminary result

Let $d \geq 1$ be an integer.

**Definition 6.1.** We use the convention $\inf \emptyset = \infty$. For $u \in \mathbb{Z}^d$ we define

$$(6.1) \quad \rho(u) = \inf \{|v| : v \in \mathbb{Z}^d \setminus \{0\} \text{ and } \langle u, v \rangle = 0\}.$$ 

For a Laurent polynomial $P \in \overline{\mathbb{Q}}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ we define

$$B(P) = \inf \{ B \in \mathbb{N} : \text{if } \eta \in (\mu_\infty)^d, z \in S^1 \setminus \mu_\infty \text{ is algebraic, and } u \in \mathbb{Z}^d \text{ with } P(\eta z^u) = 0 \text{ then } \rho(u) \leq B \}.$$ 

Let us spell this out for $d = 1$. Then $\rho(u) = 1$ for $u = 0$ and $\rho(u) = \infty$ otherwise. If $P$ vanishes at a point $S^1$ of infinite order, then $B(P) = \infty$. Conversely, if $P$ does not vanish at any point of $S^1 \setminus \mu_\infty$ then we have $B(P) = 1$. In particular, if $d = 1$ and $P$ is essentially atoral, then $B(P) = 1$.

Let $\zeta \in G_m^d$ have order $N$ and say $v \in (0, 1/2]$. Below we make use of the canonically determined lattice $\Lambda_\zeta(v)$ attached to $(\zeta, v)$ as in Section 5. Recall that $\lambda_1(\tilde{\Lambda}_\zeta(v))$ is the least positive Euclidean norm of a vector in the saturation of $\Lambda_\zeta(v)$ in $\mathbb{Z}^d$. For technical reasons we work with

$$(6.2) \quad \tilde{\lambda}(\zeta; v) = \min \left\{ \lambda_1(\tilde{\Lambda}_\zeta(v)), N^{d/2} \right\}.$$ 

For example, if $\Lambda_\zeta(v)$ is $\{0\}$, then the minimum equals $N^{d/2}$.

An important goal is to generalize Proposition 4.5 to multivariate polynomials. Proposition 6.2 below is a step in this direction.

**Proposition 6.2.** Let $K \subset \mathbb{C}$ be a number field, $0 < v \leq 1/(128d^2)$, and suppose $P \in K[X_1, \ldots, X_d] \setminus \{0\}$ has at most $k$ non-zero terms for an integer $k \geq 2$ and satisfies $B(P) < \infty$. Let $\zeta \in G_m^d$ have order $N$ and suppose $G$ is a subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$ with $P(\zeta^v) \neq 0$ for all $\sigma \in G$. Then the following properties hold true with $r = d - \text{rk}(\Lambda_\zeta(v)) \geq 1$. 

---

**Note:** The above text contains mathematical expressions and definitions related to algebraic number theory and lattice theory, focusing on the study of Galois orbits and atoral sets. The text is structured to present a series of propositions and definitions, each building upon the previous ones, culminating in a significant goal of generalizing a previous result to multivariate polynomials. The notation and concepts used are consistent with the field of algebraic number theory, including the use of conventions for denoting norms, orders, and lattice norms, which are crucial for understanding the implications of the propositions presented.
Proof. We may assume that $P$ is non-constant. Part (i) follows with ample margin from Proposition 4.3 with $Q = P$ and $F = K$. Indeed, we use require the standard estimate $d_0(N) \ll \epsilon N^e$ which holds for all $\epsilon > 0$. We refrain from stating better bounds in (i) for the purpose of better comparability with the bounds in part (ii).

We split the proof of part (ii) up into 5 steps.

**Step 1: Reduction to the univariate case.** We write $L$ for the fixed field of $G$ in $Q(\zeta)$. Note that $G$ is the Galois group of $\text{Gal}(Q(\zeta)/L) = \text{Gal}(L(\zeta)/\mathbb{Z})$.

By Proposition 5.1 applied to $\zeta$ we obtain $V \in \text{GL}_d(\mathbb{Z})$ and a decomposition $\zeta = \eta \zeta$. Let $E = \text{ord}(\eta)$ and $M = \text{ord}(\zeta)$. By (i) of Proposition 5.1 we find

$$E \leq N^{2^{1+r}}$$

and thus $M \geq N/E \geq N^{1-2^{1+r}}$.

The group used in Proposition 5.1(iii) is obtained as follows; we denote it with $H$ to avoid a clash of notation with $G$ from above. Let $H$ be the subgroup of $(\mathbb{Z}/M\mathbb{Z})^\times$ corresponding to $\text{Gal}(Q(\zeta)/\mathbb{Z}) \cap L(\eta))$. By Galois theory, see for example Theorem VI.1.12 [25], the restriction homomorphism $\text{Gal}(L(\zeta)/L) \to \text{Gal}(Q(\zeta)/\mathbb{Z}) \cap L(\eta))$ is an isomorphism. Using this isomorphism we will identify $H$ with $\text{Gal}(L(\zeta)/\mathbb{Z}) \cap L(\eta))$.

For future reference we estimate the conductor of $H \subset (\mathbb{Z}/M\mathbb{Z})^\times$. The fixed field of $H$ in $Q(\zeta)$ is $Q(\zeta) \cap L(\eta)$. By the characterization of $f_G$, the field $L$ is contained in $Q(e(1/f_G))$. So $Q(\zeta) \cap L(\eta) \subset Q(e(1/M)) \cap Q(e(1/f_G), e(1/E))$ since $\zeta$ has order $M$ and $\eta$ has order $E$. This final intersection is generated by a root of unity of order $\text{gcd}(M, \text{lcm}(f_G, E))$. We conclude

$$f_H \leq \text{lcm}(f_G, E) \leq f_G E \leq f_G N^{2^{1+r}}$$

having used (6.4).

We use basic Galois theory to compute

$$\frac{1}{\#G} \sum_{\sigma \in G} \log |P(\zeta^\sigma)| = \frac{1}{[L(\eta) : L]} \sum_{\tau \in \text{Gal}(L(\eta)/L)} \sum_{\sigma \in \text{Gal}(L(\zeta)/L) \cap L(\eta) : \tau \sigma \tau^{-1} \sigma \in \text{Gal}(L(\eta)/L)} \frac{1}{\#H} \log |P(\eta^\tau \zeta^\sigma)|.$$ 

Observe that the inner sum is over a coset of $\tau H$ of $H$ inside $(\mathbb{Z}/M\mathbb{Z})^\times$; here $\tau \in (\mathbb{Z}/M\mathbb{Z})^\times$ restricts to the restriction $\tau|_{L(\eta) \cap L(\zeta)}$. Below, $\tau$ is as in the outer sum. The
inner sum equals

\[(6.6) \quad S_\tau = \frac{1}{\#H} \sum_{\sigma \in H} \log |P(\eta^\tau \xi^{w})| = \frac{1}{\#H} \sum_{\sigma \in H} \log |P(\eta^\tau \xi^{w})|,\]

that is

\[(6.7) \quad \frac{1}{\#G} \sum_{\sigma \in G} \log |P(\xi^{w})| = \frac{1}{[L(\eta) : L]} \sum_{\tau \in \text{Gal}(L(\eta)/L)} S_\tau.\]

By Proposition 5.1(iii) applied to \(H\) we get \(a \in \mathbb{Z}^r\) satisfying (5.6) and \(\sigma_0 \in H\) with \(\xi^V = (1, \ldots, 1, e(a\sigma_0/M))\). We extend \(a\) to the left by \(d - r\) zeros and obtain a row vector \((0, a) \in \mathbb{Z}^d\). We set \(u = (0, a)V^{-1} \in \mathbb{Z}^d\) and use Proposition 5.1(ii) to get \(\xi = e(u\sigma_0/M)\). Let us set

\[(6.8) \quad Q = P(\eta^\tau X^u)X^l\]

in the unknown \(X\); it depends on \(\tau\) and the exponent \(l\) is chosen to make sure that \(Q\) is a polynomial. So \(0 \neq |P(\eta^\tau \xi^{w})| = |Q(e(\tau\sigma_0/M))|\) and in particular \(Q \neq 0\). We may assume that \(Q(0) \neq 0\). The coefficients of \(Q\) lie in \(F = K(\eta)\) and \(Q\) has at most \(k\) non-zero terms as \(P\) has at most this many non-zero terms. All this allows us to rewrite (6.6) using a univariate polynomial, \(\sigma_0\) above is absorbed by the sum

\[(6.9) \quad S_\tau = \frac{1}{\#H} \sum_{\sigma \in H} \log |Q(e(\tau \sigma_0/M))|.\]

**Step 2: Non-vanishing of \(Q\) on \(S^1 \setminus \mu_\infty\).** Suppose \(w \in \mathbb{Z}^d \setminus \{0\}\) satisfies \(\langle u, w \rangle = 0\) and \(|w| = \rho(u)\). Recall that \(\xi = e(u\sigma_0/M)\), so \(\xi^w = 1\). Thus \(|w| \geq \delta(\xi)\) and Proposition 5.1(iv) together with (6.2) yield

\[(6.10) \quad \rho(u) = |w| \geq d^{-1/2}\lambda(\xi;\nu).\]

Let \(z \in S^1 \setminus \mu_\infty\) be algebraic. If \(Q(z) = 0\) then \(P(\eta^\tau z^w) = 0\) by (6.8). By Definition 6.1 we have \(\rho(u) \leq B(\tilde{P})\). This and (6.10) contradict the lower bound \(\lambda(\xi;\nu) > d^{1/2}B(\tilde{P})\) in the hypothesis. Hence \(Q(z) \neq 0\).

Thus \(Q\), having algebraic coefficients, does not vanish at any point of \(S^1 \setminus \mu_\infty\). As \(\rho(u) > 1\) we also have \(u \neq 0\).

**Step 3: Bounding quantities in preparation of Proposition 4.5.** This step is mainly bookkeeping. We aim to apply Proposition 4.5 to \(Q\), the root of unity \(e(\tau/M)\), and the subgroup \(H \subset (\mathbb{Z}/MZ)^\times\) to determine the asymptotic behaviour of \(S_\tau\). To proceed we bound the various quantities below separately:

\[(6.11) \quad [(\mathbb{Z}/MZ)^\times : H] \leq [(\mathbb{Z}/NZ)^\times : G]N^{2v^{1+r}},\]

\[f_H \leq f_G N^{2v^{1+r}},\]

\[\deg(Q) \ll_d \deg(P) \min\{[(\mathbb{Z}/NZ)^\times : G]f_G^{1/2}N^{1-v^{1-r}/(10d)}, N^2\},\]

\[h(Q) = h(P),\]

\[K(\eta) : Q = [F : Q] \leq [K : Q]N^{2v^{1+r}},\]

Note that \(\#H = [Q(\xi) : Q(\xi) \cap L(\eta)] = [Q(\xi) : Q]/[Q(\xi) \cap L(\eta) : Q] \geq [Q(\xi) : Q]/[L(\eta) : Q]\) and since \(|Q(\xi) : Q| = \#(\mathbb{Z}/MZ)^\times\) we find \([(\mathbb{Z}/MZ)^\times : H] \leq [L(\eta) : Q] \leq [L : Q]E.\) The first bound follows from (6.4) and as \([L : Q] = [(\mathbb{Z}/NZ)^\times : G].\)
We already proved the bound for $f_H$ in (6.5).

Next comes $\deg(Q)$. Observe that

$$\deg(Q) \ll_d |a||V^{-1}| \deg(P) \ll_d |a||V|^{d-1} \deg(P)$$

$$\ll_d [(Z/MZ)^\times : H]f_H^{1/2} \deg(P)N^{1+2(d-1)\nu^1+r-v'/(6d)}$$

$$\ll_d [(Z/NZ)^\times : G]f_G^{1/2} \deg(P)N^{1+2\nu^1+r+1}N^{d-1}v^{-r-v'}/(6r)$$

having used the bounds in Proposition 5.1 $M \leq N$, and the first two bounds in (6.11). As $v \leq 1/(128d^2)$ the exponent of $N$ is at most $1 + (2d+1)\nu^1+r-v'/(6d) \leq 1 - v'/10d$ and thus we obtain

$$\deg(Q) \ll_d [(Z/NZ)^\times : G]f_G^{1/2} \deg(P)N^{1-v'/(10d)}$$

which is part of the third inequality in (6.11). The bound $\deg(Q) \ll_d \deg(P)N^2$ is proved similarly, but requires only the trivial estimate $|a| < M \leq N$ from (5.6) and $|V^{-1}| \ll_d N^{2\nu^1+r}$.

We claim that the coefficients of $P(\eta^mX^n)$ are equal to the coefficients of $P$ up-to multiplication by a root of unity. In view of the definition of the height (2.4) this will imply the fourth claim in (6.11). Indeed, it suffices to rule out that two distinct monomials in $P$ lead to the same power of $X$ after the substitution. Hence it suffices to verify $\rho(u) > \deg P$. But this follows from (6.11) and as $\lambda(\zeta; v) > d^{1/2} \deg P$ by hypothesis.

The degree of the number field $F$ containing the coefficients of $Q$ satisfies

$$[F : Q] = [K(\eta) : Q] \leq [K : Q][Q(\eta) : Q] \leq [K : Q]E \leq [K : Q]N^{2\nu^1+r}$$

where we used (6.4). This implies the fifth claim in (6.11).

**Step 4: Applying Proposition 4.5 in the univariate case.** Our aim is to determine the asymptotics of (6.9). We use the bounds from the last step to control the error term in (4.8) arise in Proposition 4.5 applied to $Q, e(\tau/M)$, and $H$. By (6.11) the error is

$$\ll [F : Q]^2[(Z/MZ)^\times : H]f_H^{1/2} \deg(Q)(\log(2\deg Q) + h(Q))\log(2M)^3d_0(M)M$$

$$\ll_d [K : Q]^2[(Z/NZ)^\times : G]f_G^{1/2} \deg(P)(\log(2N^2\deg P) + h(P))N^{9\nu^1+r+1-v'/(10d)}\log(2M)^3d_0(M)N$$

where we use $\deg Q \ll N^2 \deg P$ to bound $\log(2\deg Q)$ from above and the lower bound for $M$ in (6.4).

The exponent of $N$ is $9\nu^1+r-v'/(10d) \leq -v'/(19d)$ as $v \leq 1/(128d^2) \leq 1/(256d)$. As $M \mid N$ we find $d_0(M) \leq d_0(N)$. It is well-known that $d_0(N) \ll \epsilon N^\epsilon$ for all $\epsilon$. We also anticipate log($2N^2$) coming from log($2\deg P$) to find

$$\log(2N^2)N^{9\nu^1+r-v'/(10d)}\log(2M)^3d_0(M) \ll_d \epsilon N^{-v'/(20d)}.$$
Applying Proposition 4.5 and recalling \( m(Q) = m(P(\eta^TX^u)) \) we find
\[
(6.12) \quad S_\tau = m(P(\eta^TX^u)) + O_{d,u} \left( \frac{[K : Q][\langle Z/NZ \rangle^\times : G]^2f_G\deg(P)^2(1 + h(P))}{N^{u'/20d}} \right).
\]

Step 5: Applying a quantitative version of Lawton’s Theorem. To determine the asymptotics of the Mahler measure we apply our quantitative variant of Lawton’s Theorem, Theorem A.1 to \( P(\eta^T(X_1, \ldots, X_d)) \neq 0 \). This polynomial has the same degree and number of terms as \( P \). The exponent vector satisfies \( \rho(u) \geq d^{-1/2}\tilde{\lambda}(\zeta;v) \) by (6.10). Our hypothesis implies \( \rho(u) \geq \deg P \), as required by Theorem A.1. We find
\[
(6.13) \quad m(P(\eta^TX^u)) = m(P(\eta^T(X_1, \ldots, X_d))) + O_{d,k} \left( \frac{\deg(P)^{16d^2}}{\lambda(\zeta;v)^{1/(16k-1)}} \right).
\]

The Mahler measure of \( P \) and \( P(\eta^T(X_1, \ldots, X_d)) \) are equal as translating by \( \eta^T \in (S^1)^d \) does not affect the value of the integral.

By combining (6.12) and (6.13) we conclude
\[
S_\tau = m(P) + O_{d,k,u} \left( \frac{[K : Q][\langle Z/NZ \rangle^\times : G]^2f_G\deg(P)^2(1 + h(P))}{N^{u'/20d}} \right) + \frac{\deg(P)^{16d^2}}{\lambda(\zeta;v)^{1/(16k-1)}}.
\]

The proposition follows from (6.7). \( \square \)

We now explain why the situation simplifies when the order \( N \) of \( \zeta \) is a prime number. In this case, after the proof of Proposition 5.1 we observed that \( \eta = 1 \) and \( \zeta = \xi \). In the proof above, inequality (6.10) can be replaced by \( \rho(u) \geq \delta(\xi) \). So the hypothesis on \( \xi \) in (ii) of the proposition can be replaced by \( \delta(\xi) > \max\{B(P), \deg P\} \); see also the argument near (6.13). This is certainly satisfied for \( \delta(\xi) \to \infty \). Moreover, \( \tilde{\lambda}(\xi;v) \) can be replaced by \( \delta(\xi) \) in (6.3). From this point it is not difficult to deduce Theorem 1.1 when \( N \) is a prime.

The remaining argument is required to treat general \( N \). We need to keep track of extra information such as \( [K : Q] \), \( [\langle Z/NZ \rangle^\times : G] \), \( f_G \), and the dependency on \( P \) to anticipate a monomial change of coordinates.

### 7. EQUI-DISTRIBUTION

Proposition 6.2 closes in on Theorem 1.1. Indeed, suppose that for some choice of \( \nu \) the value \( \tilde{\lambda}(\zeta;\nu) \) grows polynomially in \( \delta(\xi) \). Then the error term of (6.3) tends to 0 as \( \delta(\xi) \to \infty \) and we are done.

However, consider the following example, already found in the beginning of Section 5. Suppose \( n \geq 2 \) and \( \zeta_p \) and \( \zeta_p^n \) are roots of unity of order \( p \) and \( p^n \), respectively. Say \( \zeta = (\zeta_p, \zeta_p^n) \), it has order \( p^n \). The lattice \( \Lambda_\zeta \) contains \( (p,0) \) and this vector has minimal positive norm in \( \Lambda_\zeta \). For \( n \) large enough in terms of \( \nu \) we have \( \Lambda(\nu) = (p,0)Z \) and \( \tilde{\Lambda}(\nu) = (1,0)Z \). Thus \( \lambda_1(\Lambda_\zeta(\nu)) = 1 \) and this yields \( \tilde{\lambda}(\zeta;\nu) = 1 \).

This example suggests a monomial change of coordinates which we will do in the next section. In the current section we lay the groundwork for this change of coordinates.
7.1. Numerical integration. We require a higher dimensional replacement of the Koksma bound, Theorem 5.4 \[21\]. The classical analog is called the Koksma–Hlawka Inequality and applies to functions of bounded variation in the sense of Hardy and Krause. Let \( \theta : U \to \mathbb{R} \) be a function whose domain \( U \) is a non-empty subset of \( \mathbb{R}^d \). In this subsection we use the more rudimentary modulus of continuity of \( \theta \) defined by

\[
\omega(\theta; t) = \sup_{x,y \in U, |x-y| \leq t} |\theta(x) - \theta(y)|
\]

for all \( t \geq 0 \); as usual \(| \cdot |\) denotes the maximum-norm on \( \mathbb{R}^d \). We define \( \omega(\theta; t) = 0 \) if \( U = \emptyset \). We will use it to estimate a mean in terms of the corresponding integral in Proposition 7.1. Hlawka \[22\] has a related and more precise result. For the reader’s convenience we give a self-contained treatment that suffices for our purposes.

**Proposition 7.1.** Let \( \theta : [0, 1]^d \to \mathbb{R} \) be a continuous function and let \( x_1, \ldots, x_n \in [0, 1]^d \) with discrepancy \( D = D(x_1, \ldots, x_n) \). Then

\[
\frac{1}{n} \left| \sum_{j=1}^n \theta(x_j) - \int_{[0,1]^d} \theta(x) dx \right| \leq (1 + 2^{d+1}) \omega(\theta, D^{1/(d+1)}).
\]

**Proof.** Both sides of (7.2) are invariant under adding a constant function to \( \theta \). So we may assume \( \theta(0) = 0 \).

Let \( T \geq 1 \) be an integral parameter to be determined below. We write \([0, 1]^d\) as a disjoint union of \( T^d \) half-open hypercubes \( Q_j \) with side length \( 1/T \). Let \( \overline{Q}_j \) denote the closure of \( Q_j \) in \([0, 1]^d\). The Mean Value Theorem tells us that for each \( j \) there exists \( y_j \in \overline{Q}_j \) such that \( \int_{Q_j} \theta(x) dx = \text{vol}(Q_j) \theta(y_j) = T^{-d} \theta(y_j) \).

For each \( j \) we write \( n_j = \# \{ i \in \{1, \ldots, n\} : x_i \in Q_j \} \). So

\[
\frac{1}{n} \left| \sum_{j=1}^n \theta(x_j) - \sum_{j=1}^n n_j \theta(j) \right| \leq \frac{1}{n} \sum_{j=1}^n \sum_{x_j \in Q_j} |\theta(x_j) - \theta(y_j)| \leq \frac{1}{n} \sum_{j=1}^n \omega(\theta; 1/T) n_j = \omega(\theta; 1/T).
\]

On the other hand, \( \frac{1}{n} \sum_{j} n_j \theta(y_j) \) equals

\[
\sum_{j} \frac{n_j}{n} T^d \int_{Q_j} \theta(x) dx = \sum_{j} \int_{Q_j} \theta(x) dx = \int_{[0,1]^d} \theta(x) dx + T^d \sum_{j=1}^n \delta_j \int_{Q_j} \theta(x) dx
\]

where \( \delta_j = n_j/n - T^{-d} \). The definition of discrepancy implies \( |\delta_j| \leq D \). Hence

\[
\frac{1}{n} \left| \sum_{j} n_j \theta(y_j) - \int_{[0,1]^d} \theta(x) dx \right| \leq T^d D \int_{[0,1]^d} |\theta(x)| dx \leq T^{d+1} D \omega(\theta; 1/T)
\]

where we used \( |\theta(x)| \leq T \omega(\theta; 1/T) \) for all \( x \in [0, 1]^d \); recall that \( \theta(0) = 0 \).

We apply the triangle inequality to (7.3) and (7.4) and conclude that the left-hand side of (7.2) is at most \( (1 + T^{d+1} D) \omega(\theta; 1/T) \). To complete the proof observe that \( 0 < D \leq 1 \) and fix \( T = \lceil D^{-1/(d+1)} \rceil \) which satisfies \( D^{-1/(d+1)} \leq T \leq D^{-1/(d+1)} + 1 \). \( \square \)
7.2. Averaging the Mahler measure. This subsection is purely in the complex setting. Let $P \in \mathbb{C}[X_1, \ldots, X_d] \setminus \{0\}$ have at most $k \geq 2$ non-zero terms, where $k$ is an integer.

Let $l \in \{1, \ldots, d - 1\}$. For $x \in \mathbb{R}^l$ we define $P_{e(x)} = P(e(x), Y_1, \ldots, Y_{d-l}) \in \mathbb{C}[Y_1, \ldots, Y_{d-l}]$. Next we construct an auxiliary Laurent polynomial $\hat{P}$ in $l$ variables whose value at $e(x)$ is comparable to $|P_{e(x)}|$. For $i \in \mathbb{Z}^{d-l}$ we denote $p_i \in \mathbb{C}[X_1, \ldots, X_l]$ the coefficients of $P$ taken as a Laurent polynomial in $X_{l+1}, \ldots, X_d$ and define

$$
\hat{P} = \sum_i p_i(X_1, \ldots, X_l)\overline{p_i}(X_{l+1}^{-1}, \ldots, X_1^{-1}) \in \mathbb{C}[X_l^{\pm 1}, \ldots, X_1^{\pm 1}]
$$

where the bar denotes complex conjugation.

**Lemma 7.2.** In the notation above the following properties hold true:

\begin{enumerate}[(i)]
  \item The Laurent polynomial $\hat{P}$ has at most $k^2$ non-zero terms.
  \item The product $(X_1 \cdots X_l)^{\text{deg} P} \hat{P}$ is a polynomial of degree at most $(l + 1) \deg P$.
\end{enumerate}

**Proof.** If each $p_i$ consists of $k_i$ non-zero terms, then $\hat{P}$ consists of at most $\sum_i k_i^2$ terms. Since $\sum_i k_i \leq k$ we find that $\hat{P}$ has at most $k^2$ non-zero terms. This implies part (i).

Part (ii) follows from (7.5). \qed

Observe that $\hat{P}(e(x)) = \sum_i |p_i(e(x))|^2 \geq 0$. As $|P_{e(x)}|$ is the maximum of $|p_i(e(x))|$ as $i$ varies, we find

$$
\frac{1}{k^{1/2}} \hat{P}(e(x))^{1/2} \leq |P_{e(x)}| \leq \hat{P}(e(x))^{1/2}.
$$

So $P_{e(x)} = 0$ if and only if $\hat{P}(e(x)) = 0$.

The main result of this subsection is

**Proposition 7.3.** Assume $P \in \mathbb{C}[X_1, \ldots, X_d] \setminus \mathbb{C}$ has at most $k$ non-zero terms for an integer $k \geq 2$. Let $l \in \{1, \ldots, d - 1\}$ and let $\hat{P}$ be as above. Suppose $x_1, \ldots, x_n \in [0, 1]^l$ with discrepancy $\mathcal{D} = \mathcal{D}(x_1, \ldots, x_n)$. If $P_{e(x_i)} \neq 0$ for all $i \in \{1, \ldots, n\}$, then

$$
\frac{1}{n} \sum_{i=1}^{n} m(P_{e(x_i)}) = m(P) + O_{d,k} \left( \deg(P)\mathcal{D}^{1/(16(d+1)k^2)} + m(\hat{P}) - \frac{1}{n} \sum_{i=1}^{n} \log \hat{P}(e(x_i)) \right).
$$

By a theorem of Boyd [8], the Mahler measure is a continuous function in the coefficients of a non-zero polynomial of fixed degree (below in Lemma A.5 we prove that it is even Hölder continuous). Therefore, if the $P_{e(x_i)}$ in the proposition above are uniformly bounded away from 0, then the average on the left in (7.7) converges to the integral $\int_{[0,1]^l} m(P_{e(x)}) dx$ as the discrepancy tends to 0. But even when $|P| = 1$ it is conceivable that $|P_{e(x_i)}|$ is small for some $x_i$, then $P_{e(x_i)}$ is near the Mahler measure’s logarithmic singularity. This happens if and only if $\hat{P}(e(x_i))$ is small by (7.6). The proposition states that we can handle the mean for arbitrary $x_i$ if we can control the logarithmic mean of $\hat{P}$ over the $e(x_i)$.

The proof follows a series of lemmas. We first note a useful property of the modulus of continuity as defined in (2.1). Let $\theta : [0,1]^d \to \mathbb{R} \cup \{-\infty\}$ be a function and
We apply (7.8) to Lemma 7.5. Let us verify (7.10) to conclude the proof.

Indeed, say \( x, y \in [0, 1]^d \) with \( |x - y| \leq t \). To bound \( |\theta_c(x) - \theta_c(y)| \) from above by the right-hand side of (7.8) we may assume \( \theta_c(x) = \theta_c(y) \). By continuity of \( \theta_c \) there is for all small enough \( \epsilon > 0 \) a \( z \in [0, 1]^d \) on the line segment connecting \( x \) and \( y \) with 
\[
c + \epsilon = \theta_c(z) = \theta(z).
\]
Then \( |\theta_c(x) - \theta_c(y)| = \theta(x) - c = |\theta(x) - \theta(z)| + |\theta(z) - c| \leq \omega(\theta|_{0,1}^{\alpha_0}; t) + \epsilon \). Our claim (7.8) follows as \( \epsilon \) can be made arbitrarily small.

Let \( P \) and \( k \) be as in Proposition 7.3 and assume in addition that \( |P| = 1 \).

**Lemma 7.4.** Let \( x_1, \ldots, x_n \in [0, 1]^d \) have discrepancy \( D = D(x_1, \ldots, x_n) \). If \( r \in (0, 1] \), then
\[
\frac{1}{n} \# \{ i \in \{ 1, \ldots, n \} : |P(e(x_i))| \leq r \} \leq \|_{d,k} r^{1/(2k)} + \deg(P) D^{1/(d+1)}/r.
\]

**Proof.** For \( x \in [0, 1]^d \) we set
\[
\chi(x) = \max\{0, 2 - |P(e(x))|/r\}
\]
and this defines a continuous function on \([0, 1]^d\) with values in \([0, 2]\).

We note that \( \chi(x) \geq 1 \) if \( |P(e(x))| \leq r \). As \( \chi \) is non-negative the average \( \frac{1}{n} \sum_{i=1}^n \chi(x_i) \) is at least the proportion of the \( i \) amongst \( \{1, \ldots, n\} \) such that \( |P(e(x_i))| \leq r \). On the other hand, Lemma A.3(i) implies
\[
(7.10) \quad \int_{[0,1]^d} \chi(x) dx \leq 2\text{vol}(\{ x \in [0, 1]^d : |P(e(x))| < 2r \}) \leq \|_{d,k} r^{1/(2(k-1))} \leq \|_{d,k} r^{1/(2k)}.
\]

We will apply Proposition 7.1 to bound the proportion on the left in (7.9). Say \( t > 0 \), let us verify
\[
(7.11) \quad \omega(\chi; t) \leq \|_{d,k} \deg(P) t/r.
\]
We apply (7.8) to \( \theta(x) = 2 - |P(e(x))|/r \) and \( c = 0 \). Say \( x, y \in \theta^{-1}((0, \infty)) \) with \( |x - y| \leq t \), so in particular \( |P(e(x))| < 2r \) and \( |P(e(y))| < 2r \). Then \( |\theta(x) - \theta(y)| = |P(e(x)) - P(e(y))|/r \leq \|_{d,k} \deg(P) t/r \), where we used \( |x - y| \leq t \) and \( |P| = 1 \). We obtain (7.11).

Let us set \( t = D^{1/(d+1)} \). We apply numerical integration, Proposition 7.1 and use (7.10) to conclude the proof. \(\square\)

In the next lemma we truncate the singularity of \( x \mapsto \log |P(e(x))| \) using a parameter \( r \) and bound the modulus of continuity of the resulting function.

**Lemma 7.5.** Let \( r \in (0, 1], \) for \( x \in [0, 1]^d \) we define \( \psi(x) = \max\{ \log r, \log |P(e(x))| \} \) as above (7.8). Then \( \psi : [0, 1]^d \to \mathbb{R} \) is continuous and for all \( t > 0 \) we have
\[
\omega(\psi; t) \leq \|_{d,k} \frac{\deg(P) t}{r}.
\]

**Proof.** Clearly, \( \psi \) is continuous on \([0, 1]^d\). We apply (7.8) to \( \theta(x) = \log |P(e(x))| \) and \( c = \log r \). Say \( x, y \in [0, 1]^d \) with \( |P(e(x))| \geq |P(e(y))| \geq r \) and \( |x - y| \leq t \). Then as in
the proof of Lemma 7.4 we find \(|P(e(y))/P(e(x))| - 1| \ll_{d,k} \deg(P)t/|P(e(x))| \ll_{d,k} \deg(P)t/r. Applying the logarithm and using \(0 \leq \log s \leq s - 1\) for all \(s \geq 1\) yields

\[ |\log |P(e(x))| - \log |P(e(y))|| \ll_{d,k} \frac{\deg(P)t}{r}, \]

as desired.}

\[ \square \]

**Lemma 7.6.** We keep the notation of Lemma 7.5. Then

\[ |m(P) - \int_{[0,1]^d} \psi(x)dx| \ll_{d,k} r^{1/(4k)}. \]

**Proof.** The absolute value in question is

\[ E = \left| \int_{\Sigma} \log |P(e(x))|dx - \text{vol}(\Sigma) \log r \right| \]

where \(\Sigma = S(P,r) = \{x \in [0,1]^d : |P(e(x))| < r\}\) in the notation of (A.2). Hence

\[ \text{vol}(\Sigma) \ll_{d,k} r^{1/(2k-1)} \] by Lemma A.3(i). So

\[ E \ll_{d,k} \int_{\Sigma} |\log |P(e(x))||dx + r^{1/(2k)} \]

as \(r \leq 1\). To bound the final integral we use Lemma A.4 which implies \(E \ll_{d,k} r^{1/(4k-1)} + r^{1/(2k)} \ll_{d,k} r^{1/(4k)}\).}

\[ \square \]

**Lemma 7.7.** Let \(x_1, \ldots, x_n \in [0,1]^d\) with \(P(e(x_i)) \neq 0\) for all \(i\) and discrepancy \(D = D(x_1, \ldots, x_n)\). We set

\[ \epsilon = \left| m(P) - \frac{1}{n} \sum_{i=1}^{n} \log |P(e(x_i))| \right|. \]

If \(r \in (0,1]\), then

\[ \frac{1}{n} \sum_{i=1}^{n} \left| \log |P(e(x_i))| \right| \ll_{d,k} \deg(P)D^{1/(d+1)} r^{-2} + r^{1/(4k)} + \epsilon. \]

**Proof.** By the triangle inequality and with \(\psi\) as in Lemma 7.5 we have

\[ \left| \frac{1}{n} \sum_{i=1}^{n} \psi(x_i) - \log |P(e(x_i))| \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} \psi(x_i) - \int_{[0,1]^d} \psi(x)dx \right| + \left| \int_{[0,1]^d} \psi(x)dx - m(P) \right| + \epsilon. \]

We use Proposition 7.1 and Lemma 7.5 with \(t = D^{1/(d+1)}\) to bound the first term on the right by \(\ll_{d,k} \deg(P)D^{1/(d+1)} / r\). The second term is \(\ll_{d,k} r^{1/(4k)}\) by Lemma 7.6.

The term on the left equals \(\frac{1}{n} \sum_{i=1}^{n} \left| \log r - \log |P(e(x_i))| \right|\). Observe that \(- \log |P(e(x_i))| = |\log |P(e(x_i))|| \) in this sum as \(r \leq 1\). We rearrange and find

\[ \frac{1}{n} \sum_{i=1}^{n} \left| \log |P(e(x_i))| \right| \ll_{d,k} \deg(P)D^{1/(d+1)} r^{-1} + r^{1/(4k)} + \frac{|\log r|}{n} \left( \sum_{i=1}^{n} 1 \right) + \epsilon. \]
By Lemma 7.4 the term corresponding to the sum over $i$ on the right is $\ll_{d,k} r^{1/(2k)} |\log r| + \deg(P) D^{1/(d+1)} r^{-1} |\log r|$. Combining our bounds and absorbing $|\log r|$ in an appropriate power of $r^{-1}$ we find
\[
\frac{1}{n} \sum_{|P(e(x_i))|<r} |\log |P(e(x_i))|| \ll_{d,k} \deg(P) D^{1/(d+1)} r^{-2} + r^{1/(4k)} + \epsilon,\]
as desired. □

After this warming-up we prove variants of Lemmas 7.5 and 7.6 where $\log |\cdot|$ is replaced by the Mahler measure. We also truncate at the parameter $r$.

**Lemma 7.8.** Let $r \in (0,1]$, for $x \in [0,1]^d$ we define $\mu(x) = \max\{\log r, m(P_{e(x)})\}$ as above (7.8) where we interpret the Mahler measure of 0 as $–\infty$. Then $\mu : [0,1]^d \to \mathbb{R}$ is continuous and for all $t > 0$ we have
\[
\omega(\mu; t) \ll_{d,k} \left(\frac{\deg(P) t}{r}\right)^{1/(8k)} (1 + |\log r|).
\]

**Proof.** By Boyd’s Theorem [8] the Mahler measure is continuous on the space of non-zero polynomials of bounded degree. Thus $\mu$ is continuous on $[0,1]^d$. Observe that $\omega(\mu; t) \ll_k 1 + |\log r|$ as $m(P_{e(x)}) \ll_k 1$ by (2.1). So we may assume that $\deg(P) t/r$ is sufficiently small in terms of $d$ and $k$.

We again use (7.8), this time with $\theta(x) = m(P_{e(x)})$ and $c = \log r$. Let $x, y \in [0,1]^d$ with $m(P_{e(x)}) \geq \log r$ and $m(P_{e(y)}) \geq \log r$ and $|x – y| \leq t$. Then $|P_{e(x)}| \gg_k r$ and $|P_{e(y)}| \gg_k r$ by (2.1). As in the proof of Lemma 7.4 we find $|P_{e(x)} – P_{e(y)}| \ll_{d,k} \deg(P) t$. Since $\deg(P) t/r$ is smaller than some prescribed constant depending only on $d$ and $k$ we have $|P_{e(x)} – P_{e(y)}| / \min\{|P_{e(x)}|, |P_{e(y)}|\} \leq 1/2$. Lemma A.5 implies
\[
|m(P_{e(x)}) – m(P_{e(y)})| \ll_{d,k} \left(\frac{|P_{e(x)} – P_{e(y)}|}{\min\{|P_{e(x)}|, |P_{e(y)}|\}}\right)^{1/(8(k-1))} \ll_{d,k} \left(\frac{\deg(P) t}{r}\right)^{1/(8(k-1))},
\]
as desired. □

By Lemma 7.2(i) $\tilde{P}$ has at most $k^2$ non-zero terms, so $\sup_{x \in [0,1]^d} |\tilde{P}(e(x))| \leq k^2 |\tilde{P}|$. We let $\tilde{P}$ denote the polynomial from part (ii) of the said lemma divided by $|P|$, so $|\tilde{P}| = 1$. There exists $i$ with $|p_i| = |P| = 1$. The definition of the Mahler measure implies
\[
m(p_i) \leq \sup_{x \in [0,1]^d} \log |p_i(e(x))| \leq \frac{1}{2} \sup_{x \in [0,1]^d} \log |\tilde{P}(e(x))| \leq \frac{1}{2} (2 \log k + \log |\tilde{P}|).
\]

Using $|p_i| = 1$ and the Theorem of Dobrowolski–Smyth, Theorem 2.1 we conclude $m(p_i) \geq –(k – 2) \log 2$. Thus $|\tilde{P}| \gg_k 1$. Bounding $|\tilde{P}|$ from above is more straightforward. Indeed, $|\tilde{P}| \ll_k 1$ by (7.5) and since $|P| = 1$. Therefore,
\[
(7.12) \quad 1 \ll_k |\tilde{P}| \ll_k 1.
\]
Lemma 7.9. We keep the notation of Lemma 7.8. Then

\[ |m(P) - \int_{[0,1]} \mu(x) dx| \ll_{d,k} r^{1/(2k^2)}. \]

Proof. We recall that \( |\hat{P}| \) equals \( \hat{P} \) up-to a monomial factor. By (7.6), (7.12), and Theorem 2.4 there exists \( c > 0 \) depending only on \( k \) such that \( |\hat{P}(e(x))| \geq cr^2 \) implies \( m(P_e(x)) \geq \log r \). By Fubini’s Theorem we have \( \int_{[0,1]} m(P_e(x)) dx = m(P) \), so the absolute value in question is

\[ \mathcal{E} = \left| \int_{\Sigma} m(P_e(x)) dx - \text{vol}(\Sigma) \log r \right| \]

where \( \Sigma = S(\hat{P}, cr^2) \); indeed \( m(P_e(x)) = \mu(x) \) for all \( x \in [0,1] \setminus \Sigma \).

Note that \( \text{vol}(\Sigma) \ll_{d,k} r^{1/(k^2-1)} \) by Lemma A.3(i) applied to \( \hat{P} \). So

\[ (7.13) \quad \mathcal{E} \ll_{d,k} r^{1/(k^2-1)} |\log r| + \int_{\Sigma} m(P_e(x)) dx \ll_{d,k} r^{1/k^2} + \int_{\Sigma} m(P_e(x)) dx. \]

To bound the integral in (7.13) from above we will replace \( m(P_e(x)) \) by \( \log |P_e(x)| \).

Say \( x \in \Sigma \) and \( P_e(x) \neq 0 \), then \( |m(P_e(x)) - \log |P_e(x)|| \ll_k 1 \) by (2.2) and thus \( |m(P_e(x))| \ll_k 1 + |\log |P_e(x)|| \). The function \( x \mapsto |\log |P_e(x)|| \) is integrable over \([0,1] \) in the sense of Lebesgue and so is \( x \mapsto |m(P_e(x))| \); both take the value \(+\infty\) on a measure zero subset of \([0,1]\). We find

\[ \mathcal{E} \ll_{d,k} r^{1/k^2} + \int_{\Sigma} \left( 1 + |\log |P_e(x)|| \right) dx. \]

From (7.6) and (7.12) we deduce \( |\log |P_e(x)|| \ll_k |\log |\hat{P}(e(x))|| + 1 \) if \( \hat{P}(e(x)) \neq 0 \). So \( \mathcal{E} \ll_{d,k} r^{1/k^2} + \int_{\Sigma} \left( 1 + |\log |\hat{P}(e(x))|| \right) dx \). By Lemma A.4 applied to \( \hat{P} \) and the volume estimate for \( \Sigma \), the integral on the right is \( \ll_{d,k} r^{1/(2(k^2-1))} \ll_{d,k} r^{1/(2k^2)} \), as desired.

Proof of Proposition 7.9 If we scale \( P \) by a factor \( \lambda \), then \( \hat{P} \) is scaled by \( |\lambda|^2 \). So the proposition is invariant under non-zero scaling and we may assume \( |P| = 1 \). Later on we will choose the parameter \( r \) in terms of \( \text{deg}(P) \) and \( D \). In the meantime we assume that \( r \in (0, 1/2) \).

Observe that \( \int_{[0,1]} m(P_e(x)) dx = m(P) \). We want to bound \( \mathcal{E} = |m(P) - n^{-1} \sum_{i=1}^{n} m(P_e(x_i))| \) from above.

We replace the Mahler measure with \( \mu(\cdot) \) coming from Lemma 7.8. Indeed, the triangle inequality implies

\[ \mathcal{E} \leq \left| m(P) - \int_{[0,1]} \mu(x) dx \right| + \left| \int_{[0,1]} \mu(x) dx - \frac{1}{n} \sum_{i=1}^{n} \mu(x_i) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \mu(x_i) - m(P_e(x_i)) \right|. \]

The first term on the right is \( \ll_{d,k} r^{1/(2k^2)} \) by Lemma 7.9 applied to \( P \). By Proposition 7.1 applied to \( \mu \) and \( t = D^{1/(d+1)} \) and Lemma 7.8 the second term is \( \ll_{d,k} r^{1/(2k^2)} \).
We get here \( \epsilon \). By (2.2) we may replace and a monomial leaves \( E \ll \cdot \) applied to \( \cdot \) on \( k \). If \( \deg(e(x)) = \deg(P(x)) \cdot \) and that \( E \ll \cdot \) in a multiple \( \cdot \) is absorbed in the first term in \( \cdot \). We return to the total error term \( E \). We choose \( r = \frac{1}{2} D^{1/(8d+1)} \), then the proposition follows as \( D \leq 1 \).

8. ENDGAME

In this section we prove a stronger version of Theorem [1] from the introduction.
8.1. Preliminaries. Suppose $P \in \overline{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$. For $V \in \text{GL}_d(\mathbb{Z})$ we set $Q \in \overline{Q}[X_1, \ldots, X_d]$ to be $P(X^{V^{-1}})$ multiplied by a suitable monomial in $X_1, \ldots, X_d$ such that $Q$ is coprime to $X_1 \cdots X_d$. Let $l \in \{0, \ldots, d-1\}$. For $z = (z_1, \ldots, z_l) \in \mathbb{C}^l$ we set

$$P_{V,z} = Q(z_1, \ldots, z_l, X_1, \ldots, X_{d-l}) \quad (8.1)$$

this is a polynomial in $d-l$ variables. It is sometimes useful to allow $l = 0$ in which case $P_{V,z} = Q$.

The following lemma requires a result of Bombieri, Masser, and Zannier [6] and relies crucially on the hypothesis that $P$ is essentially atoral.

Lemma 8.1. Suppose $P \in \overline{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ is essentially atoral. There exists $c \geq 1$ depending only on $P$ and $d$ such that for all $\zeta \in \mathbb{C}_m^d$ with $\delta(\zeta) \geq c$, for all $V \in \text{GL}_d(\mathbb{Z})$, and all $l \in \{0, \ldots, d-1\}$, we have $P_{V,\eta} \neq 0$ and $\mathcal{B}(P_{V,\eta}) \leq c|V^{-1}|$ where $\zeta^V = \{\eta\} \times G_{m-l}^d$.

Proof. The Zariski closure $W$ in $G_m^d$ of all algebraic zeros of $P$ in $(S^1)^d$ is defined over $\overline{Q}$.

By hypothesis, $P$ is essentially atoral. So each irreducible component of the Zariski closure of all complex roots of $P$ on $(S^1)^d$ is of codimension at least 2 in $G_m^d$ or a proper torsion coset of $G_m^d$. Therefore, each irreducible component of $W$ is also of this type.

Let $\zeta \in G_m^d$ be of finite order with $\delta(\zeta) \geq c$, where $c$ is to be determined, and $\zeta^V = \{\eta\} \times G_{m-1}^d$ with $V$ and $l$ as in the hypothesis.

Let $\eta' \in G_{m-1}^d$ be of finite order, $z \in S^1 \setminus \mu_\infty$ be algebraic, and $u \in \mathbb{Z}^{d-l}$ with $P_{V,u}(\eta'z^u) = 0$. We must find $v'' \in \mathbb{Z}^{d-l} \setminus \{0\}$ with $|v''| \leq c|V^{-1}|$ such that $\langle u, v'' \rangle = 0$. The existence of such a $v''$ establishes in particular $P_{V,\eta} \neq 0$.

Now $P(x) = 0$ for the algebraic point $x = (\eta, \eta'z^u)^V \in (S^1)^d$. So $x$ is contained in an irreducible component $W'$ of $W$ and in a 1-dimensional algebraic subgroup of $G_m^d$.

If $\dim W' \leq d-2$, we apply Bombieri, Masser, and Zannier’s Theorem 1.5 [6] to $X = W'$. We get a proper torsion coset of $G_m^d$ containing $x$ and coming from a finite set depending only on $W'$. We find $v \in \mathbb{Z}^d \setminus \{0\}$ with $|v| \ll_{d,P} 1$ and $x^v = 1$.

If $W'$ is a proper torsion coset of $G_m^d$ where there exists $v \in \mathbb{Z}^d \setminus \{0\}$, depending only on $W'$ and thus only on $P$, such that $y^v = 1$ holds for all $y \in W'$. Again we find $|v| \ll_{d,P} 1$ and $x^v = 1$.

In either case we have

$$1 = x^v = (\eta, \eta'z^u)^{V^{-1}v} = \eta^v(\eta'z^u)^{v''} \text{ where } V^{-1}v = \left(\begin{array}{c} v' \\ v'' \end{array}\right) \in \mathbb{Z}^l \times \mathbb{Z}^{d-l}. \quad (8.2)$$

In particular, $\langle u, v'' \rangle = 0$ as $z$ has infinite order.

If $v'' \neq 0$, then we are done. Indeed, $|v''| \leq |V^{-1}v| \leq d|V^{-1}||v|$ and $|v|$ is bounded from above solely in terms of $P$ and $d$.

Let us assume $v'' = 0$ and derive a contradiction. Note $l \geq 1$ as $v$ cannot be 0. Then $v' \neq 0$ and by equality (8.2) we find $\eta' = 1$. Recall that $\eta$ consists of the first $l$ coordinates of $\zeta^V$. Thus $\zeta^v = 1$ and hence $\delta(\zeta) \leq |v|$ where $|v| \ll_{d,P} 1$. But $\delta(\zeta) \geq c$, a contradiction for large enough $c$. □
Lemma 8.3. Let $P \in \mathbb{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ and $\xi \in \mathbb{G}_m^d$ be of finite order. The pair $(P, \xi)$ is called $c$-admissible if for all $V \in \textrm{GL}_d(\mathbb{Z})$ and all $l \in \{0, \ldots, d-1\}$ we have $P_{V, l} \neq 0$ and $\mathcal{B}(P_{V, l}) \leq c|V^{-1}|$ where $\xi^V \in \{\eta\} \times \mathbb{G}_m^{-1}$.

The case $l = 0$ yields in particular $\mathcal{B}(P) \leq c$ if there exists $\xi$ such that $(P, \xi)$ is $c$-admissible.

Let $P$ be an essentially atoral Laurent polynomial with algebraic coefficient. By Lemma 8.1 there exists $c \geq 1$ such that $(P, \xi)$ is $c$-admissible for all $\xi \in \mathbb{G}_m^d$ of finite order with $\delta(\xi) \geq c$.

In the definition of admissibility, it will be useful to keep track of $\xi$ when passing in down in an induction step. The next lemma makes this precise.

Lemma 8.3. Let $P \in \mathbb{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ and let $\xi \in \mathbb{G}_m^d$ be of finite order such that $(P, \xi)$ is $c$-admissible with $c \geq 1$. Say $l \in \{0, \ldots, d-1\}, V \in \textrm{GL}_d(\mathbb{Z})$, and $\xi^V = (\eta, \xi)$ with $\eta \in \mathbb{G}_m^l$ and $\xi \in \mathbb{G}_m^{-1}$. Then $(P_{V, l}, \xi)$ is $(cd|V^{-1}|)$-admissible.

Proof. Throughout the proof we use that $|\cdot|$ is the maximum-norm on matrices.

We abbreviate $R = P((\eta, X_1, \ldots, X_{d-1})^V)$ which equals $P_{V, l}$ up-to a monomial factor. It suffices to show that $(R, \xi)$ is $c$-admissible.

To this end say $k \in \{0, \ldots, d - l - 1\}, W \in \textrm{GL}_{d-l}(\mathbb{Z})$, and $\xi^W = \{\eta'\} \times \mathbb{G}_m^{-l-k}$ with $\eta' \in \mathbb{G}_m$. We must bound $\mathcal{B}(R_{W, \eta'})$. So say $z \in S^1 \setminus \mu_\infty, u \in \mathbb{Z}^{d-l-k}$, and $\eta'' \in \mathbb{G}_m^{-l-k}$ is of finite order with $R_{W, \eta'}(\eta''z^u) = 0$. Then $R((\eta', \eta''z^u)^W) = 0$ and hence $P((\eta', \eta''z^u)^W) = 0$. We abbreviate $W' = \begin{pmatrix} E_l & 0 \\ 0 & W \end{pmatrix}$ with $E_l$ the $l \times l$ unit matrix. So $P((\eta', \eta''z^u)^W) = 0$ which means $P_{VW', (\eta, \eta')}((\eta''z^u)^W) = 0$.

Observe that $\xi^VW = (\eta, \xi)^W = (\eta, \xi^W) = (\eta, \eta', *)$. By hypothesis $(P, \xi)$ is $c$-admissible. Therefore, $\mathcal{B}(P_{VW', (\eta, \eta')}) \leq c|VW^{-1}| = c|W^{-1}V^{-1}| \leq cd|V^{-1}|W^{-1}| = cd|V^{-1}|W^{-1}|$. In other words, there exists $v \in \mathbb{Z}^{d-l-k} \setminus \{0\}$ with $|v| \leq cd|V^{-1}|W^{-1}$ and $\langle u, v \rangle = 0$. Thus $\mathcal{B}(R_{W, \eta'}) \leq cd|V^{-1}|W^{-1}|$, as desired. In particular, $R_{W, \eta'} \neq 0$.

Lemma 8.4. Let $P \in \mathbb{Q}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \setminus \{0\}$ and let $\xi \in \mathbb{G}_m^d$ be of finite order such that $(P, \xi)$ is $c$-admissible with $c \geq 1$. Say $l \in \{1, \ldots, d-1\}$ and let $\tilde{P} \in \mathbb{Q}[X_1^{\pm 1}, \ldots, X_l^{\pm 1}]$ be as in (7.5) and $\xi \in \{\eta\} \times \mathbb{G}_m^{-1}$. Then $(\tilde{P}, \eta)$ is $c$-admissible.

Proof. Suppose $V \in \textrm{GL}_l(\mathbb{Z})$ such that $\eta^V = (\eta', *)$ where $\eta \in \mathbb{G}_m^{l'}$ where $l' \in \{0, \ldots, l-1\}$. Following the definition of admissibility and recalling (8.1) we are in the following situation. There is $\eta'' \in \mathbb{G}_m^{-l'}, z \in S^1 \setminus \mu_\infty$ algebraic, and $u' \in \mathbb{Z}^{-l'}$ such that $\tilde{P}((\eta', \eta''z^{u'})^{V^{-1}}) = 0$.

It follows from the definition of $\tilde{P}$ that $P((\eta', \eta''z^{u'})^{V^{-1}}, X_{l+1}, \ldots, X_d) = 0$ as a polynomial in $X_{l+1}, \ldots, X_d$. We extend $\tilde{V} = \begin{pmatrix} V & 0 \\ 0 & E_{d-l} \end{pmatrix}$ where $E_{d-l}$ is the $(d - l) \times (d - l)$ unit matrix. Then $P((\eta', \eta''z^{u'}, z^{u''})^{\tilde{V}^{-1}}) = 0$ for all $u'' \in \mathbb{Z}^{d-l}$. 
By hypothesis, \((P, \zeta)\) is \(c\)-admissible and \(\tilde{\zeta}^V = (\eta^V, *) = (\eta', *, *)\). Now \(P_{V, \eta'}(\eta''z^u, z^{u''}) = 0\), so by definition there exist \(v'\in \mathbb{Z}^{d-l}, v'' \in \mathbb{Z}^{d-l}\), not both zero, such that \(\langle u', v' \rangle + \langle u'', v'' \rangle = 0\) and \(\| (v', v'') \| \leq c|V^{-1}| = c|V^{-1}|\) for the maximum-norm.

As we are free to vary \(u''\) we see that \(\{u'\} \times \mathbb{Q}^{d-l}\) is contained in a finite union of proper vector subspaces of \(\mathbb{Q}^d\), each defined as the kernel of \(\langle v, (v', v'') \rangle\) with \(v', v''\) as above. So \(\{u'\} \times \mathbb{Q}^{d-l} \subset V\) for one of these vector spaces \(V\) defined by some \((v', v'')\). We must have \(v'' = 0\) and hence \(\langle u', v' \rangle = 0\). Then \(v' \neq 0\) and as \(|v'| \leq c|V^{-1}|\) we conclude that \(\tilde{P}\) is \(c\)-admissible.

Here are some basic estimates involving \(P_{V, \eta'}\).

**Lemma 8.5.** Let \(P \in \overline{\mathbb{Q}}[X_1, \ldots, X_d] \{0\}, l, \) and \(V\) be as near the beginning of this subsection. Say \(\eta \in \mathbb{C}_m\) has finite order and \(P_{V, \eta} \neq 0\). The following hold true.

(i) We have \(\deg P_{V, \eta} \ll_d |V|^{d-1} \deg P\).

(ii) We have \(h(P_{V, \eta}) \leq \log(k) + h(P)\) where \(k \geq 2\) is an upper bound for the number of non-zero terms of \(P\).

**Proof.** Both parts follow are elementary consequences of the degree and the height of a polynomial. For (i) we require \(|V^{-1}| \ll_d |V|^{d-1}\).

For (ii) we note that \(Q\) from the beginning of this subsection has the same coefficients and thus the same height as \(P\). We decompose \(h(P_{V, \eta})\) in local heights as into (2.4). The triangle inequality at the archimedean places leads to \(\log k\).\[\]

We continue with further basic estimates involving \(\tilde{P}\) as in (7.5).

**Lemma 8.6.** Let \(K \subset \mathbb{C}\) be a number field and suppose \(P \in K[X_1, \ldots, X_d] \{0\}\) has at most \(k \geq 2\) terms, where \(k\) is an integer. Say \(l \in \{1, \ldots, d-1\}\) and suppose \(\tilde{P} \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_l^{\pm 1}]\) then the following properties hold true:

(i) We have \(\tilde{P} \in K'[X_1^{\pm 1}, \ldots, X_l^{\pm 1}]\) where \(K'\) is a number field such that \(K \subset K' \subset \mathbb{C}\) and \([K' : Q] \leq [K : Q]^2\).

(ii) We have \(h(\tilde{P}) \ll_k 1 + h(P)\).

**Proof.** To see (i) we note that the coefficients of \(\tilde{P}\) are contained in the subfield \(K'\) of \(\mathbb{C}\) generated by all elements of \(K\) and their complex conjugates.

Finally, for (ii) we remark that each \(p_i\) as in (7.5) has at most \(k\) terms and that there are at most \(k\) non-zero \(p_i\). Using the local decomposition of the height together with the ultrametric and archimedean triangle inequality yields the claim.\[\]

8.2. **Completion of proof.** The next lemma will setup a monomial change of coordinates. We recall that \(\Lambda_{\xi}(v)\) was defined in (5.4) and \(\lambda_1(\Lambda_{\xi}(v))\) in (5.5).

**Lemma 8.7.** Suppose \(\xi \in \mathbb{C}_m^d\) has order \(N\) and let \(\delta \geq 1, \epsilon \in (0, 1/2], v_1, \ldots, v_{d-1} \in (0, 1/2]\) with \(v_1 + \cdots + v_{d-1} \leq 1/2\). Then there exist \(l \in \{0, \ldots, d-1\}\) and \(V \in \text{GL}_d(\mathbb{Z})\) such that the following hold.

(i) We have \(|V| \ll_d \delta^{2\epsilon d - 1}\) and \(V\) is the unit matrix if \(l = 0\).
(ii) We have \( \zeta^V = (\eta, \xi) \) where \( \eta \in C^l_m, \xi \in \mathbb{C}^{d-l}, \text{ord}(\eta) \leq N^{v_1 + \cdots + v_l}, \) and either \( l = d - 1 \) and \( \xi \in \mu_\infty \) has order at least \( N^{1/2} \) or \( l \leq d - 2 \) and \( \lambda_1(\Lambda_{\xi_l}(v_l + 1)) > \delta^{d-l-1}. \)

**Proof.** Set \( \xi_1 = \zeta \) and let \( V_0 \) be the unit matrix in \( GL_d(\mathbb{Z}) \). For all \( l \in \{1, \ldots, d - 1\} \) with \( \lambda_1(\Lambda_{\xi_l}(v_l)) \leq \delta^{d-l} \) we will construct inductively \( V_l \in GL_d(\mathbb{Z}), \xi_{l+1} \in \mathbb{C}^{d-l} \) of order at most \( N \), and \( \eta_l \in \mathbb{C}_m \) of order at most \( N^{v_l} \) such that \( \zeta^V_l = (\eta_1, \ldots, \eta_l, \xi_{l+1}) \) and

\[
|V_l| \ll_d \delta^{d-l + \cdots + d-l} \tag{8.3}
\]

Suppose \( \lambda_1(\Lambda_{\xi_l}(v_l)) \leq \delta^{d-l} \), there exists \( v \in \Lambda_{\xi_l}(v_l) \setminus \{0\} \) such that \( |v| \leq \delta^{d-l} \) and \( v \) is primitive. Note that \( [\Lambda_{\xi_l}(v_l) : \Lambda_{\xi_l}(v_l)] \in \Lambda_{\xi_l}, \) so \( \text{ord}(\xi_l^v) \leq [\Lambda_{\xi_l}(v_l) : \Lambda_{\xi_l}(v_l)] \leq \text{det}(\Lambda_{\xi_l}(v_l)) \leq N^{2v_l^2} \leq N^{v_l} \) by (5.3) and since \( \text{det}(\Lambda_{\xi_l}) \leq N \). As in the proof of Proposition 5.1 we can realize \( v \) as the first column of a matrix \( V_l \in GL_{d+1-l}(\mathbb{Z}) \) with \( |V_l| \ll_d |v| \ll_d \delta^{d-l} \). Let \( E_{l-1} \) denote the \((l-1) \times (l-1)\) unit matrix and set

\[
V_l = V_{l-1} \begin{pmatrix} E_{l-1} & 0 \\ 0 & V_l' \end{pmatrix} \in GL_d(\mathbb{Z}).
\]

By step \( l - 1 \) we have \( \zeta^{V_{l-1}} = (\eta_1, \ldots, \eta_{l-1}, \xi_l) \). We define \( \xi_{l+1} \) via \( \zeta^{V_l} = (\eta_1, \ldots, \eta_l, \xi_{l+1}), \) note \( \eta_l = \xi_l^\ast \). Finally, \( |V_l| \ll_d |V_{l-1}| |V_l'| \ll_d \delta^{d-1 + \cdots + d-1}, \) which completes our construction.

Let us consider the largest \( l \in \{0, \ldots, d - 1\} \) for which \( \lambda_1(\Lambda_{\xi_l}(v_l)) \leq \delta^{d-l} \) and define \( V = V_l. \) Then (i) holds by (8.3) as \( \epsilon \leq 1/2. \) To verify (ii) observe that \( \zeta^V = (\eta_1, \ldots, \eta_l, \xi) \) with \( \xi = \xi_{l+1} \in \mathbb{C}^{d-l} \) has order \( N \) and \( (\eta_1, \ldots, \eta_l) \) has order at most \( N^{v_1 + \cdots + v_l} \leq N^{1/2}. \) If \( l = d - 1, \) then \( \xi \) is a root of unity of order at least \( N^{1/2}. \) Otherwise \( l \leq d - 2 \) and \( \lambda_1(\Lambda_{\xi_{l+1}}(v_{l+1})) > \delta^{d-l-1}, \) because the construction cannot continue.

We are ready to prove a theorem that will quickly imply our Theorem 1.1 and its refinements.

**Theorem 8.8.** Let \( c \geq 1, \) let \( K \subset \mathbb{C} \) be a number field, and suppose \( P \in K[X_1, \ldots, X_d]\setminus\{0\} \) has at most \( k \) terms for an integer \( k \geq 2. \) There are constants \( C = C(d, k) \geq 1 \) and \( \kappa = \kappa(d, k) > 0 \) depending only on \( d \) and \( k \) with the following property. Let \( \xi \in \mathbb{C}^d_m \) have order \( N \) and suppose \( G \) is a subgroup of \( (\mathbb{Z}/N\mathbb{Z})^\times \) with \( P(\xi) \neq 0 \) for all \( \sigma \in G. \) If \( (P, \xi^\sigma) \) is \( c \)-admissible for all \( \sigma \in G \) and if

\[
\delta(\xi) \geq C \max\{c, \deg P\}^C
\]

then

\[
\frac{1}{|G|} \sum_{\sigma \in G} \log |P(\xi^\sigma)| = m(P) + O_{d,k} \left( \frac{|K : Q|^2d^2((\mathbb{Z}/N\mathbb{Z})^\times : G)^2f_{G} \max\{c, \deg P\}^{16d^2} (1 + h(P))}{\delta(\xi)^k} \right).
\]

**Proof.** The case \( d = 1 \) follows from Proposition 6.2(i) as \( \delta(\xi) = N \) in this case and as \( B(P) < \infty. \) So we may assume \( d \geq 2. \) We may also assume that \( P \) is non-constant.

We work with the parameters \( v_1, \ldots, v_{d-1} \in (0, 1/(128d^2)], \epsilon \in (0, 1/2] \) in this proof. They are assumed to be small in terms of \( d \) and \( k \) but independent of \( P \) and \( \xi. \)
We may assume that \( \epsilon \) is small in terms of the \( v_l, c, \ldots, c \leq v_l^d \) for all \( l \). We determine them during the argument.

We apply Lemma \( 8.7 \) to \( \xi, \delta = \delta(\xi), \epsilon \), and the \( v_l \). Say \( l, V, \eta \), and \( \zeta \) are given by this lemma, in particular \( \xi^V = (\eta, \zeta) \) and \( |V| \ll_d \delta(\xi)^{2d-1} \). We have

\[
\text{ord}(\eta) \leq N^{v_1 + \cdots + v_l}.
\]

The case \( l = 0 \) is straightforward. Here \( V \) is the unit matrix, \( \xi = \zeta \), and \( \lambda(\xi, v_1) > \delta(\xi)^{2d-1} \) as we are in case \( d-l = d \geq 2 \) of Lemma \( 8.7 \). As \( (P, \zeta) \) is \( c \)-admissible we have \( B(P) \leq c \). We will apply Proposition \( 6.2 \) to \( P \) and \( v = v_1 \), so we must verify \( \lambda(\xi, v_1) > d^{1/2} \max\{c, \deg P\} \).

This inequality is satisfied if \( \delta(\xi) \) is as in \((8.4)\) with \( C \) large in terms of \( \epsilon, v_1, d, \) and \( k \).

**Step 1: A monomial change of coordinates.** From now on we assume \( l \geq 1 \), i.e., \( l \in \{1, \ldots, d-1\} \). We have \( \xi^V = (\eta, \zeta) \in G_{m}^{l} \times G_{m}^{d-l} \). This time we apply Proposition \( 6.2 \) to \( P_{V, \eta} \in K(\eta)[X_1, \ldots, X_{d-1}], \xi \), and \( v_{l+1} \). Note that \( P_{V, \eta} \neq 0 \) as \( (P, \xi) \) is \( c \)-admissible; this polynomial has at most \( k \) non-zero terms. By Lemma \( 8.5 \) the pair \( (P_{V, \eta}, \xi) \) is \((cd|V-1|)\)-admissible. Observe \( |V^{-1}| \ll_d |V|^{d-1} \ll_d \delta(\xi)^{2d-1(d-1)} \). So the said pair is \( c_1 c_2 \delta(\xi)^{2d-1d} \)-admissible; here and below \( c_1, c_2, \ldots \) denote positive constants that depend only on \( d \). In particular, \( B(P_{V, \eta}) \leq c_1 c_2 \delta(\xi)^{2d-1d} \) and \( B(P_{V, \eta}) < \infty \).

If \( d-l = 1 \) we will apply Proposition \( 6.2 \) and there is nothing further to check. But for \( d-l \geq 2 \) we must verify the hypothesis in the second part of this proposition. This step is similar as in the case \( l = 0 \). Indeed, by Lemma \( 8.7 \) we have

\[
(8.5) \quad \lambda(\xi, v_{l+1}) \geq \delta(\xi)^{\min\{e^{d-l-1}, v_{l+1}^d/2\}} = \delta(\xi)^{e^{d-l-1}}
\]
as \( e \leq v_{l+1}^d/2 \). Observe that

\[
\deg P_{V, \eta} \ll_d |V|^{d-1} \deg P \ll_d \delta(\xi)^{2d-1d} \deg P,
\]
by Lemma \((8.5)\). To apply Proposition \( 6.2 \) we must verify

\[
\lambda(\xi, v_{l+1}) > c_2 \delta(\xi)^{2d-1d} \max\{c, \deg P\}.
\]
We may assume \( e^{d-l-1} - 2c_{d-1}d \geq e^{d-l-1}/2 \). By \((8.4)\) and \((8.5)\) the desired inequality is satisfied when \( C \) is large in terms of \( e, d, \) and \( k \). We may thus apply Proposition \( 6.2 \).

Observe that

\[
h(P_{V, \eta}) \ll_k 1 + h(P)
\]

\[
[K(\eta) : Q] \leq \text{ord}(\eta)[K : Q] \leq N^{v_1 + \cdots + v_l}[K : Q]
\]

\[
\text{ord}(\zeta) \geq N/\text{ord}(\eta) \geq N^{1-(v_1 + \cdots + v_l)} \geq N^{1/2}
\]
by Lemma \((8.5)\) and Lemma \( 8.7 \). Recall that \( \xi^V = (\eta, \zeta) \). So \( \xi^u = 1 \) for some \( u \in \mathbb{Z}^{d-l} \) with \( |u| = \delta(\xi) \), hence \( \xi^V(0, u) = 1 \). We conclude \( \delta(\xi) \leq |V(0, u)| \leq d|V|^{d(\xi)} \).

We find

\[
\delta(\xi) \gg_d \delta(\xi)^{1-2d-l} \gg_d \delta(\xi)^{1/2}
\]
as we may assume \( e^{d-l} \leq 1/4 \).
We must choose a group $G$ in Proposition 6.2 we will denote it by $H$ here. Let $L$ denote the fixed field of $G$ in $Q(\xi)$. We may naturally identify $\text{Gal}(L(\xi)/L)$ with a subgroup of $(Z/MZ)^\times = \text{Gal}(Q(\xi)/Q)$ with $M = \text{ord}(\xi)$. We take $H$ the subgroup of $(Z/MZ)^\times$ thus identified with $\text{Gal}(L(\xi)/L(\eta) \cap L(\eta)) \cong \text{Gal}(L(\xi)/L(\eta))$. The fixed field of $H$ in $Q(\xi)$ is contained in $L(\eta)$, so

$$
[(Z/MZ)^\times : H] \leq [L(\eta) : Q] \leq \text{ord}(\eta)[L : Q] = \text{ord}(\eta)[(Z/NZ)^\times : G] \\
\leq N^{\nu_1 + \cdots + \nu_l}[(Z/NZ)^\times : G]
$$

having used the bound for the order of $\eta$ from Lemma 8.7. Moreover, the conductor of $H$ satisfies

$$
f_H \leq \text{lcm}(f_G, \text{ord}(\eta)) \leq f_G \text{ord}(\eta) \leq f_G N^{\nu_1 + \cdots + \nu_l}.
$$

To cover the case $l = d - 1$ we set $\nu_d = 1/(128d^2)$. By applying Proposition 6.2 to $P_{V, \eta, \xi}$, and $\nu_{l+1}$ and using the various estimates above, in particular (8.5), we find

$$
\frac{1}{#H} \sum_{\sigma \in H} \log |P_{V, \eta}(\xi^\sigma)| = m(P_{V, \eta}) \\
+ O_{d,k} \left( \frac{|K : Q|[(Z/NZ)^\times : G]f_G \deg(P)^2(1 + h(P))N^{(2+2+1)(\nu_1 + \cdots + \nu_l)}\delta(\xi)4^{d-1}d}{N^{\nu_{l+1}/(40d)}} \right) \\
+ O_{d,k} \left( \frac{\deg(P)^{16d^2}}{\delta(\xi)^{d-1}1/(16k)-32d^{-1}d^2} \right)
$$

here we used $r \leq d$ and $M \geq N^{1/2}$; the third line can be omitted if $l = d - 1$ as then we apply Proposition 6.2(i).

At this point we reap the benefit of having split the error term in Proposition 6.2 into a part depending on $N$ and a part depending on $\delta(\xi)$. Indeed, the order of $\eta$, which we bound in terms of $N$, does not affect the term involving $\delta(\xi)$. Recall that $\delta(\xi) \leq N$, but there can be no meaningful lower bound for $\delta(\xi)$ in terms of $N$. Introducing a dependency on $N$ in the part containing $\delta(\xi)$ would spoil the result.

We use the crude bound $\delta(\xi) \leq N$ and we assume the parameters satisfy

$$
5(\nu_1 + \cdots + \nu_l) + 4\varepsilon^{d-1}d \leq \frac{\nu_{l+1}}{80d}
$$

and

$$
32\varepsilon^{d-1}d^3 \leq \frac{\varepsilon^{d-1}d}{32k}.
$$

We now combine both contributions to the error term and get

(8.7)

$$
\frac{1}{#H} \sum_{\sigma \in H} \log |P_{V, \eta}(\xi^\sigma)| = m(P_{V, \eta}) + O_{d,k} \left( \frac{|K : Q|[(Z/NZ)^\times : G]f_G \deg(P)^{16d^2}(1 + h(P))}{\delta(\xi)^{d}} \right)
$$

if

$$
k \leq \min \left\{ \frac{\nu_{l+1}}{80}, \frac{\varepsilon^{d-1}d}{32k} \right\}.
$$

Later we may shrink $\kappa$. 
Step 2: Induction on $d$. Recall that $\zeta^V = (\eta, \xi)$ and $|P(\xi^\sigma)| = |P_{V, \eta^\tau}(\xi^\sigma)|$ for all $\sigma \in G$. We still assume $l \geq 1$ and we find, as in \ref{step:1}, that

\begin{equation}
\frac{1}{\#G} \sum_{\sigma \in G} \log |P(\xi^\sigma)| = \frac{1}{[L(\eta) : L]} \sum_{\tau \in \text{Gal}(L(\eta)/L)} \frac{1}{\#H'} \sum_{\sigma \in H'} \log |P_{V, \eta^\tau}(\xi^\sigma)|
\end{equation}

with $\tilde{\tau}$ a lift of $\tau$ to $\text{Gal}(L(\xi)/L)$.

The estimates from Step 1 hold equally for all conjugates $\eta^\sigma, \xi^\sigma$. Indeed, for example $\Lambda_{\xi}(v_{l+1})$ and $\delta(\xi)$ are Galois invariant and $(P, \xi^\sigma)$ is $c$-admissible. So we may apply \ref{step:1} to $P_{V, \eta^\tau}$ when summing over $\sigma \in H$.

We set $Q$ to equal $P(X^{V-1})$ times a monomial such that $Q$ is a polynomial coprime to $X_1 \cdots X_d$. So $\deg Q \ll_d |V-1| \deg P$ and recall that $|V-1| \ll_d \delta(\xi)^{2e^{d-1}d}$. We apply the construction \ref{construction} to $Q$ and $l$ and obtain $\bar{Q}$. Recall Lemma \ref{lem:admissible} and write $\bar{Q}$ for $\bar{Q}$ times the monomial from part (ii) of this lemma. Then $\bar{Q}$ has at most $k^2$ non-zero terms and using also Lemma \ref{lem:admissible} we find

\begin{equation}
\bar{Q} \in K'[X_1^{\pm 1}, \ldots, X_d^{\pm 1}] \text{ where } |K' : Q| \leq |K : Q|^2,
\end{equation}

\begin{equation}
\deg \bar{Q} \ll_d \deg Q \ll_d |V-1| \deg P \ll_d \delta(\xi)^{2e^{d-1}d} \deg P, \text{ and } h(\bar{Q}) \ll_d 1 + h(Q) \ll_d 1 + h(P).
\end{equation}

By Lemma \ref{lem:admissible} with $l = 0$, the pair $(Q, (\eta^\sigma, \xi^\sigma))$ is $c_3 c \delta(\xi)^{2e^{d-1}d}$-admissible for all $\sigma \in G$. Now $(\bar{Q}, \eta^\tau)$ is also $c_3 c \delta(\xi)^{2e^{d-1}d}$-admissible by Lemma \ref{lem:admissible} for all $\sigma$.

We want to apply this theorem to $\bar{Q}$ and $\eta \in G_{\eta} \subset G$ by induction, recall $l \leq d - 1$. For this we must verify

$$\delta(\eta) \geq c_4 C(l, k^2) \delta(\xi)^{2e^{d-1}dC(l, k^2)} \max\{c, \deg P\}^{C(l, k^2)}$$

having used the bound for $\deg \bar{Q}$ in \ref{deg:Q}. By symmetry we have, as above and in \ref{lem:admissible}, the bound $\delta(\eta) \gg_d \delta(\xi)/|V| \gg_d \delta(\xi)^{1/2}$. So it suffices to check

\begin{equation}
\delta(\xi)^{-4e^{d-1}dC(l, k^2)} \geq c_5 C(l, k^2)^2 \max\{c, \deg P\}^{2C(l, k^2)}.
\end{equation}

We may assume that $1 - 4e^{d-1}dC(l, k^2) \geq 1/2$ as we may choose $\epsilon$ small in terms of $d$ and $C(l, k^2)$. So \ref{delta:xi} follows from \ref{lem:admissible} if $C = C(d, k)$ is large enough in terms of $d$ and $k$.

By induction and \ref{deg:Q} we have

$$\frac{1}{\#H'} \sum_{\tau \in H'} \log |\bar{Q}(\eta^\tau)| = m(\bar{Q})$$

$$+ O_{d,k} \left( \frac{|K : Q|^2 |(\mathbb{Z}/E\mathbb{Z})^\times : H'|^2 f_{H'} \deg(P)^{16d^2 \delta(\xi)^{32e^{d-1}d^3} (1 + h(P))}}{\delta(\eta)^{3C(l, k^2)}} \right)$$

here $E = \text{ord}(\eta)$ and $H' \subset (\mathbb{Z}/E\mathbb{Z})^\times$ is the subgroup identified with $\text{Gal}(Q(\eta)/Q(\eta) \cap L) \cong \text{Gal}(L(\eta)/L)$. Note that $|(\mathbb{Z}/E\mathbb{Z})^\times : H'| = \left|Q(\eta) \cap L : Q\right| \leq \left|L : Q\right| =
By Lemma 8.1 there is an asymptotic estimate
\[ \tilde{Q} = \kappa \]
for \( \tilde{Q} \) as mentioned in (8.11). The theorem follows on combining this asymptotic estimate with (8.7) and (8.8), when \( \kappa \) is sufficiently large in terms of \( d \) and \( P \).

Recall that \( P \) equals \( P(X^{V-1}) \) up to a monomial factor. We will soon apply Proposition 7.3 to \( Q \). Consider \( (x_1, \ldots, x_{\#H'}) \), with each \( x_i \in [0,1]^d \), a tuple of discrepancy \( D \) as in (8.2), where the \( e(x_i) \) are the \( \eta_i \). Proposition 7.3 together with (8.11) imply
\[
\frac{1}{\#H'} \sum_{\tau \in H'} m(P_{V, \eta_i}) = m(Q) + O_{d,k} \left( \frac{|K : Q|^{2d} ([Z/NZ] \times G)^2 f_G \deg(P)^{16d^2} (1 + h(P))}{\delta(\zeta)^{1/(k^2)/4}} \right).
\]

By Proposition 7.3, parts (i) and (ii), we find
\[
D \ll_d ([Z/\{1\}]^{\times} : H') f_H^{1/2} \delta(\eta)^{-1/3} \ll_d ([Z/NZ]^{\times} : G) f_G^{1/2} \delta(\zeta)^{-1/6}.
\]

From above we find \( \deg Q \ll_d |V^{1/2}| \) deg \( P \ll_d \delta(\zeta)^{2e^{-d}d} \) deg \( P \). The Mahler measure is invariant under a monomial change of coordinates by Corollary 8, Chapter 3.4 [41], thus \( m(P) = m(Q) \). As \( \#H' = [L(\eta) : L] \) we get
\[
\frac{1}{[L(\eta) : L]} \sum_{\tau \in H'} m(P_{V, \eta_i}) = m(P) + O_{d,k} \left( \frac{|K : Q|^{2d} ([Z/NZ] \times G)^2 f_G \deg(P)^{16d^2} (1 + h(P))}{\delta(\zeta)^{\min\{1/(192(d+1)^2) - 2e^{-d}d\}, 1/(k^2)/4}} \right).
\]

We shrink \( \epsilon \) a final time to achieve \( 1/(192(d+1)^2) - 2e^{-d}d > 1/(200(d+1)^2) \). The theorem follows on combining this asymptotic estimate with (8.7) and (8.8), when \( \kappa = \kappa(d,k) \) is small in terms of \( \kappa(l,k^2), d, \) and \( k \).

To prove Theorem 1.1 we can multiply \( P \) by any monomial, so we may assume that it is a polynomial. Thus the theorem is a direct consequence of the following more precise corollary.

Corollary 8.9. Let \( K \subset \mathbb{C} \) be a number field and suppose \( P \in K[X_1, \ldots, X_d] \setminus \{0\} \) is essentially atoral and has at most \( k \) non-zero terms for an integer \( k \geq 2 \). There exists an \( \kappa = \kappa(d,k) > 0 \) with the following property. Suppose \( \zeta \in G_m^d \) has order \( N \) and suppose \( G \) is a subgroup of \( (Z/NZ)^{\times} \) and \( \delta(\zeta) \) is large in terms of \( d, P, [K : Q], f_G, \) and \( ([Z/NZ]^{\times} : G) \). Then \( P(\zeta^\sigma) \neq 0 \) for all \( \sigma \in G \) and
\[
\frac{1}{\#G} \sum_{\sigma \in G} \log |P(\zeta^\sigma)| = m(P) + O_{d,k} \left( \frac{|K : Q|^{2d} ([Z/NZ]^{\times} : G)^2 f_G \deg(P)^{16d^2} (1 + h(P))}{\delta(\zeta)^{x}} \right).
\]

Proof. By Lemma 8.1 there is \( c \geq 1 \), depending only on \( P \), such that \( (P, \zeta) \) is \( c \)-admissible for all \( \zeta \in G_m^d \) of finite order with \( \delta(\zeta) \geq c \).

Suppose \( \zeta \in G_m^d \) has finite order and \( P(\zeta) = 0 \). By the Manin–Mumford Conjecture, \( \delta(\zeta) \) is bounded in terms of \( d \) and \( P \) only. Hence for \( \delta(\zeta) \) sufficiently large in
terms of these quantities we have \( P(\zeta) \neq 0 \) and the same also holds with \( \zeta \) replaced by a Galois conjugate. Our corollary now follows from Theorem 8.8. \( \square \)

Proof of Corollary 8.9 We may assume that \( K/Q \) is Galois and that \( P \) is a polynomial. The product \( P' \) for \( \tau(P) \) as \( \tau \) ranges over Gal(K/Q) has rational coefficients. The coefficients are even integers as the coefficients of \( P \) lie in \( \mathbb{Z}_K \).

The Mahler measure of any non-zero, integral polynomial is non-negative. By a theorem attributed to Boyd [7], Lawton [28], Smyth [44], the fact that the zero set of \( \tilde{P} \) of Lemma A.4(i) [20].

Recall the definition of \( \zeta \). Suppose \( \zeta \in G_m^d \) has order \( N \). Take for \( G \) the subgroup of \( (\mathbb{Z}/N\mathbb{Z})^\times \) associated to \( \text{Gal}(Q(\zeta)/K \cap Q(\tilde{\zeta})) \). Then \( [\mathbb{Z}/N\mathbb{Z}^\times : G] \leq [K : Q] \). As \( \zeta \) varies, there are only finitely many possibilities for the number field \( K \cap Q(\tilde{\zeta}) \), being a subfield of the field \( K \). So \( f_C \) is bounded from above solely in terms of \( K \). For any \( \tau \in \text{Gal}(K/Q) \) choose an extension \( \bar{\tau} \in \text{Gal}(K(\tilde{\zeta})/Q) \). We apply Corollary 8.9 to the polynomial \( \tau(P) \) which is essentially atoral by hypothesis. If \( \delta(\tilde{\zeta}) = \delta(\zeta) \) is large enough in terms of the fixed data, then

\[
\frac{1}{#G} \sum_{\sigma \in G} \log |\tau(P)(\tilde{\zeta}^\sigma)| = m(\tau(P)) + o(1)
\]

as \( \delta(\tilde{\zeta}) \to \infty \), here and below the implied constant is independent of \( \zeta \).

The average of the left-hand side over \( \tau \in \text{Gal}(K/Q) \) equals the left-hand side in

\[
\frac{1}{[K(\zeta) : Q]} \sum_{\sigma : K(\tilde{\zeta}) \to C} \log |\sigma(P(\zeta))| = \frac{1}{[K : Q]} \sum_{\tau \in \text{Gal}(K/Q)} m(\tau(P)) + o(1).
\]

As the Mahler measure is additive, the average on the right-hand side is \( m(P')/[K : Q] > 0 \). But the left-hand side vanishes if \( P(\tilde{\zeta}) \) is an algebraic unit. In this case, we see that \( \delta(\tilde{\zeta}) \) is bounded from above. \( \square \)

APPENDIX A. A theorem of Lawton re-revisited

The following theorem makes explicit a result of Lawton [29]. It is a more precise version of the second-named author’s result [20] which is unfortunately insufficient for our purposes. We closely follow the proof presented in [20] which itself is based on Lawton’s approach [29]. We also show how to correct an inaccuracy in the proof of Lemma A.4(i) [20].

Recall the definition of \( \rho(\cdot) \) in (6.1) where \( d \geq 1 \) is an integer.

**Theorem A.1.** Suppose \( P \in K[X_1, \ldots, X_d] \setminus \{0\} \) has at most \( k \) non-zero terms for an integer \( k \geq 2 \). For \( a = (a_1, \ldots, a_d) \in \mathbb{Z}^d \setminus \{0\} \) with \( \rho(a) > \deg P \) we have

\[
m(P(X^{a_1}, \ldots, X^{a_d})) = m(P) + O_{dk} \left( \frac{\deg(P)^{16d^2}}{\rho(a)^{1/(16(k-1))}} \right)
\]

where the implicit constant depends only on \( d \) and \( k \).

In the univariate case \( d = 1 \) we have \( \rho(a) = \infty \) for all \( a \in \mathbb{Z} \setminus \{0\} \) by definition. Then we should interpret (A.1) as stating \( m(P(X^a)) = m(P) \). This identity is an easy consequence of (4.1). So throughout this subsection we assume \( d \geq 2 \).
We did not strive to obtain the best-possible exponent in $\rho(a)^{1/(16(k-1))}$ that our method can produce.

We must assume $\rho(a) > \deg P$ to avoid interaction of coefficients in $P(X^a_1, \ldots, X^a_d)$. Indeed, take for example $P = X_1(X_2 - 1 + \epsilon)$ with $\epsilon \in (0, 1)$ small and $a = (1, 0)$. Then $P(X, 1) = X\epsilon$ whose Mahler measure is $\log \epsilon$. On the other hand $m(P) = m(X_2 - 1 + \epsilon) = \log \max\{1, |1 - \epsilon|\} = 0$ by Jensen’s formula. The difference
$$m(P(X, 1)) - m(P) = \log \epsilon$$
is unbounded as $\epsilon \to 0$. This does not contradict our theorem as $\rho(a) = 1$.

The Lebesgue measure on $\mathbb{R}^d$ is $\nu(\cdot)$. For $P \in C[X^\pm_1, \ldots, X^\pm_d]$ and $r > 0$ we define
\begin{equation}
S(P, r) = \{x \in [0, 1)^d : |P(e(x))| < r\}
\end{equation}
where $e$ is as in (1.3).

Dobrowolski extended Lawton’s Theorem 1 [29] to polynomials that are not necessarily monic.

**Theorem A.2** (Dobrowolski, Theorem 1.1 [12]). Suppose $P \in C[X] \setminus \{0\}$ has at most $k$ non-zero terms for an integer $k \geq 2$. Then $\nu(S(P, r)) \ll_k \min\{1/r, |P|^{1/(k-1)}\}$ for all $r > 0$.

Dobrowolski requires that $P$ as at least 2 non-zero terms. But it is convenient to allow $P$ to have a single term, as above. It is also convenient to apply the estimate in the case $P = 0$, we then interpret the minimum to be 1.

Until the end of this appendix and if not stated otherwise we assume that $P \in C[X_1, \ldots, X_d] \setminus \mathbb{C}$ has at most $k$ non-zero terms for an integer $k \geq 2$ and $|P| = 1$.

**Lemma A.3.**
\begin{enumerate}
  
  \item If $r > 0$ then $\nu(S(P, r)) \ll_{d,k} r^{1/(2(k-1))}$.
  
  \item We have $\int_{[0,1)^d} \left| \log |P(e(x))| \right|^2 \, dx \ll_{d,k} 1$.
\end{enumerate}

**Proof.** To ease notation we drop $d, k$ in the subscript $\ll_{d,k}$.

Because of the trivial bound $\nu(S(P, r)) \leq 1$ we may assume $r \leq 1$.

The case $d = 1$ follows from Theorem A.2. So let us now assume $d \geq 2$. We consider $P$ as a polynomial in the unknown $X_d$ and coefficients among $C[X_1, \ldots, X_{d-1}]$. We pick a coefficient $P_i$ with maximal norm, i.e., $P$ has a term $P_i X_d^s$ such that $P_i \in C[X_1, \ldots, X_{d-1}]$ and $|P_i| = |P| = 1$.

For $x' \in \mathbb{R}^{d-1}$ we let $P_{e(x')}$ denote $P(e(x'), X) \in C[X]$. Recall that
$$S(P, r) = \{(x', t) \in [0, 1)^{d-1} \times [0, 1) : |P_{e(x')}(e(t))| < r\}.$$ We splice the hypercube and apply Fubini’s Theorem to find
$$\nu(S(P, r)) = \int_{[0,1)^{d-1}} \nu(S(P_{e(x')}, r)) \, dx'.$$

The measure zero set of $x' \in [0, 1)^{d-1}$ with $P_{e(x')}$ is harmless. By Theorem A.2 we find
$$\nu(S(P, r)) \ll \int_{[0,1)^{d-1}} \min \left\{ 1, \frac{r}{|P_{e(x')}|} \right\}^{1/(k-1)} \, dx'.$$
The coefficient of $X^i$ in $P_e(x')$ is $P_i(e(x'))$. So $|P_e(x')| \geq |P_i(e(x'))|$ and

$$\text{(A.3)} \quad \text{vol}(S(P, r)) \ll \int_{[0,1]^{d-1}} \min \left\{1, \frac{r}{|P_i(e(x'))|}\right\}^{1/(k-1)} d'x' = I_1 + r^{1/(k-1)} I_2$$

where

$$I_1 = \int_{|P_i(e(x'))| < r} d'x' \quad \text{and} \quad I_2 = \int_{|P_i(e(x'))| \geq r} \frac{d'x'}{|P_i(e(x'))|^{1/(k-1)}};$$

both integrals are over subsets of $[0,1]^{d-1}$. We will bound $I_1$ and $I_2$ from above.

We have $I_1 = \text{vol}(S(P_i, r))$. This lemma applied by induction to $P_i$, a polynomial in $d-1$ variables with at most $k$ non-zero terms and $|P_i| = 1$, yields

$$\text{(A.4)} \quad I_1 \ll r^{1/(2(k-1))}.$$

To bound $I_2$ we consider real numbers $r = r_0 < r_1 < \cdots < r_{n+1} = k + 1$, with $r_{n+1} \leq r_n + \delta$ where $\delta > 0$ is a small parameter. We split the domain of integration up into measurable parts

$$\Sigma_n = \left\{x' \in [0,1)^{d-1} : r_n \leq |P_i(e(x'))| < r_{n+1}\right\} \quad \text{for} \quad n \in \{0, \ldots, N\}.$$

Observe that $|P_i(e(x'))| \leq k < r_{n+1}$ for all $x'$. Thus

$$\text{(A.5)} \quad I_2 = \sum_{n=0}^{N} \int_{\Sigma_n} \frac{d'x'}{|P_i(e(x'))|^{1/(k-1)}} \leq \sum_{n=0}^{N} \frac{\text{vol}(\Sigma_n)}{r_n^{1/(k-1)}} = \sum_{n=0}^{N} a_n b_n$$

where $a_n = r_n^{-1/(k-1)}$ and $b_n = \text{vol}(\Sigma_n)$.

As the $\Sigma_n$ are pairwise disjoint, the partial sums satisfy

$$B_n = \sum_{l=0}^{n} b_l = \text{vol}(\bigcup_{l=0}^{n} \Sigma_l) \leq \text{vol}(\{x' \in [0,1)^{d-1} : |P_i(e(x'))| < r_{n+1}\}) = \text{vol}(S(P_i, r_{n+1})).$$

Hence we have the trivial bound $B_n \leq 1$. As in the bound for $I_1$ we apply this lemma by induction to $P_i$ and find

$$\text{(A.6)} \quad B_n \leq \text{vol}(S(P_i, r_{n+1})) \ll r_{n+1}^{1/(2(k-1))}.$$

Summation by parts implies

$$I_2 \leq \sum_{n=0}^{N} a_n b_n = a_N B_N - \sum_{n=0}^{N-1} B_n (a_{n+1} - a_n) \leq (k+1)^{1/(k-1)} + \sum_{n=0}^{N-1} B_n (a_n - a_{n+1});$$

we used the trivial bounds $a_N = r_N^{1/(k-1)} \leq (k+1)^{1/(k-1)}$ and $B_N \leq 1$. By (A.6) and the definition of $a_n$ we find

$$I_2 \ll 1 + \sum_{n=0}^{N-1} r_n^{1/(2(k-1))} (r_n^{-1/(k-1)} - r_{n+1}^{-1/(k-1)}).$$

We use the mean value theorem to bound

$$r_n^{-1/(k-1)} - r_{n+1}^{-1/(k-1)} \ll r_n^{-1/(k-1)} - r_{n+1}^{-1/(k-1)} \ll r_n^{-1/(k-1)} - r_{n+1}^{-1/(k-1)};$$

for the second bound we assume, as we may, that $\delta \leq r$ and so $r_{n+1} \leq r_n + \delta \leq 2r_n$.

Thus $I_2 \ll 1 + \int_{r}^{k+1} t^{-1/(2(k-1))} dt \ll r^{-1/(2(k-1))}$.
This bound together with (A.4) implies \( l_1 + r^{1/(k-1)} I_2 \lesssim r^{1/(2(k-1))} \). Therefore, \( \text{vol}(S(P, r)) \lesssim r^{1/(2(k-1))} \) by (A.3), completing the induction step and the proof of (i).

We define \( p_n(x) = \min\{n, \log |P(e(x))|/|x| \geq 0 \} \) where \( n \geq 0 \) is an integer. We must find an upper bound for the non-decreasing sequence \( I_n = \int_{[0,1]^d} p_n(x)\,dx \). Observe that \( |P(e(x))| \leq k|P| = k \), so if \( n \geq (\log k)^2 \), then \( |P(e(x))| \leq e\sqrt{n} \). We fix \( m \) to be the least integer with \( m \geq 1 + (\log k)^2 \), so \( m \geq 2 \). Say \( n \geq m \). Then \( p_n \) equals \( n \) on \( S(P, e^{-\sqrt{n}}) \) and it equals \( p_{n+1} \) outside this set. Thus

\[
I_{n+1} - I_n = \int_{S(P, e^{-\sqrt{n}})} (p_{n+1}(x) - p_n(x))\,dx \leq \text{vol}(S(P, e^{-\sqrt{n}})) \lesssim e^{-\lambda \sqrt{n}}
\]

from part (i), here \( \lambda = 1/(2(k-1)) \). A telescoping sum trick shows

\[
I_n - I_m \lesssim \sum_{l \geq m} e^{-\lambda \sqrt{l}} \lesssim \int_{m-1}^{\infty} e^{-\lambda \sqrt{l}}\,dl \lesssim 1.
\]

The initial term satisfies \( I_m \leq m \ll 1 \) as \( m \) depends only on \( k \), this completes the proof. \( \square \)

A more careful analysis should lead to \( \text{vol}(S(P, r)) \lesssim_{d,k} (1 + |\log r|)^{d-1} r^{1/(k-1)} \) for all \( r > 0 \) in part (i) of Lemma A.3. But this improvement has little effect on the main results of the current work.

Brunault, Guilloux, Mehrabdollahi, and Pengo pointed out that the second-named author’s argument for Lemma A.4(i) [20] leads (for \( k \geq 2 \)) to an estimate \( O(y f(n)/(2(k-1))) \) where \( f(n) \) depends on the number of variables \( n \), as opposed to the claimed bound \( O(y^{1/(2(k-1))}) \). However, the claimed bound holds true by Lemma A.3(i). Alternatively and in the proof of Lemma A.3(i) one can replace Dobrowolski’s Theorem 1.1 [12] by Lawton’s Theorem 1 [29] which is sufficient for the applications in [20].

**Lemma A.4.** If \( r > 0 \) then

\[
\int_{S(P, r)} \left| \log |P(e(x))| \right| \,dx \lesssim_{d,k} r^{1/(4(k-1))}.
\]

**Proof.** As \( |P(e(x))| \leq |P|k \leq k \) for all \( x \in [0,1]^d \) we may assume \( r \leq 1 \).

With \( \Sigma = S(P, r) \) we find

\[
0 \leq -\int_{\Sigma} \log |P(e(x))|\,dx = -\sum_{n=0}^{\infty} \int_{2^n+1}^{2^{n+1}} \log |P(e(x))|\,dx \\
\leq \sum_{n=0}^{\infty} \log \left( \frac{2^{n+1}}{r} \right) \text{vol}(S(P, r/2^n)).
\]

Let \( \lambda = 1/(2(k-1)) \leq 1/2 \). We use Lemma A.3(i) to bound \( \text{vol}(S(P, r/2^n)) \lesssim_{d,k} (r/2^n)^{\lambda} \). Note that \( \log(2t) \lesssim t^{\lambda/2} \) on \( t \in [1, \infty) \). We take \( t = 2^n/r \geq 1 \) and conclude

\[
-\int_{\Sigma} \log |P(e(x))|\,dx \lesssim_{d,k} \sum_{n=0}^{\infty} \left( \frac{r}{2^n} \right)^{\lambda/2} \lesssim_{d,k} r^{\lambda/2}.
\]

\( \square \)
Boyd [8] proved that the Mahler measure is continuous on the non-zero polynomials of fixed degree. Here we show that the Mahler measure is Hölder continuous away from 0. For the next lemma we momentarily drop our usual assumptions on $P$.

**Lemma A.5.** Suppose $P, Q \in \mathbb{C}[X_1, \ldots, X_d]\{0\}$ such that $P$ and $Q$ both have at most $k$ non-zero terms for an integer $k \geq 2$. If $\delta = |P - Q|/|Q| \leq 1/2$, then

$$m(P) \leq m(Q) + C(d, k)\delta^{1/(8(k-1))}$$

where $C(d, k) > 0$ is effective and depends only on $d$ and $k$.

**Proof.** It suffices to prove the lemma when $|Q| = 1$; indeed, just replace $P$ and $Q$ by $P/|Q|$ and $Q/|Q|$, respectively, to reduce to this case.

Suppose for the moment that $x \in \mathbb{R}^d$ with $P(e(x))Q(e(x)) \neq 0$. Then $|P(e(x)) - Q(e(x))| \leq 2k|P - Q|$ and so

$$(A.7) \quad \log \left| \frac{P(e(x))}{Q(e(x))} \right| \leq \left| \frac{P(e(x))}{Q(e(x))} \right| - 1 \leq 2k \frac{\delta}{|Q(e(x))|}.$$ 

where the first inequality used $\log t \leq t - 1$ for all $t > 0$. The difference of Mahler measures $m(P) - m(Q)$ can thus be written as

$$\int_{[0,1]^d} (\log |P(e(x))| - \log |Q(e(x))|) dx + \int_{\Sigma} (\log |P(e(x))| - \log |Q(e(x))|) dx$$

with $\Sigma = S(Q, \delta^{1/2})$.

The first integral is at most $2k\delta^{1/2}$ by (A.7). We proceed by bounding the second integral $I$ from above. First, we note that $|P(e(x))| \leq k|P| \leq 3k/2$ as $|P - Q| \leq \delta \leq 1/2$ and thus $|P| \leq 3/2$. So

$$I \leq \log (3k/2) \text{vol}(\Sigma) - \int_{\Sigma} \log |Q(e(x))| dx \leq \log (3k/2) \text{vol}(\Sigma) + c_0\delta^{1/(8(k-1))}$$

where we applied Lemma [A.4] to $Q$ and $\delta^{1/2}$, the case $Q$ constant being trivial; here $c = c(d, k) > 0$. Finally, Lemma [A.3(i)] yields $\text{vol}(\Sigma) = \text{vol}(S(Q, \delta^{1/2})) \ll_d \delta^{1/(4(k-1))}$ and the lemma follows as $\delta \leq 1$. $\square$

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $b \in \mathbb{N}_0$ let $C^b(\mathbb{R}^d)$ denote the set of real valued functions on $\mathbb{R}^d$ whose derivatives exist and are continuous up-to and including order $b$. For a multiindex $i = (i_1, \ldots, i_d) \in \mathbb{N}_0^d$ we set $\ell(i) = i_1 + \cdots + i_d$. If $g \in C^b(\mathbb{R}^d)$ and $\ell(i) \leq b$, we set $\partial^i g = (\partial/\partial x_1)^{i_1} \cdots (\partial/\partial x_d)^{i_d} g \in C^0(\mathbb{R}^d)$ and

$$|g|_{C^b} = \max_{i \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |\partial^i g(x)| \in \mathbb{R} \cup \{\infty\}.$$ 

We recall the construction of $f_r$ in [20] depending on the parameter $r \in (0, 1/2]$. This function lies in $C^b(\mathbb{R}^d)$ and equals $\log |P(e(\cdot))|$ away from the singularity, i.e., the locus where $P(e(\cdot))$ vanishes.

We fix the anti-derivative $\phi$ of $x^b(1 - x)^b$ on $[0, 1]$ with $\phi(0) = 0$ and multiply it with a positive number to ensure $\phi(1) = 1$. Then we extend it by 0 on $x < 0$ and by 1 for $x > 1$ to obtain a non-decreasing step function $\phi \in C^b(\mathbb{R})$ with support $[0, 1]$. 
Finally, we rescale and define \( \phi_r(x) = \phi((2/r)^2 x - 1)/3 \). So \( \phi_r \) is a non-decreasing function which vanishes on \((-\infty,(r/2)^2]\), equals 1 on \([r^2,\infty)\), and satisfies
\[
\left| \frac{d^i \phi_r}{dx^i} \right|_{C^0} \ll_b r^{-2i} \quad \text{for all} \quad 0 \leq i \leq b, \quad \text{hence} \quad |\phi_r|_{C^b} \ll_b r^{-2b}.
\]
The function \( \phi_r \) takes values in \([0,1]\). Moreover, we define
\[
\psi_r(x) = \begin{cases} 
\frac{1}{r} \phi_r(x) \log x & : x > 0, \\
0 & : x \leq 0.
\end{cases}
\]
which vanishes on \((-\infty, (r/2)^2]\), coincides with \( \frac{1}{r} \log x \) on \([r^2,\infty)\), and satisfies
\[
|\psi_r|_{C^b} \ll_b r^{-2b} |\log r|.
\]
We consider \( g : x \mapsto |P(e(x))|^2 \), then
\[
|g|_{C^b} \ll_{k,b} (\deg P)^b.
\]
Next we compose \( f_r = \psi_r \circ g \in C^b(\mathbb{R}^d) \), so for \( x \in \mathbb{R}^d \) we have
\[
f_r(x) = \begin{cases} 0 & : \text{if } |P(e(x))| \leq r/2, \\
\log |P(e(x))| & : \text{if } |P(e(x))| \geq r.
\end{cases}
\]
By Lemma A.5 \[20\], which follows from the chain rule, together with \( \text{(A.8)} \) and \( \text{(A.9)} \) we find
\[
|f_r|_{C^b} \ll_{k,b} r^{-2b} |\log r| (\deg P)^b.
\]
For the following lemmas we suppose \( b \geq d + 1 \). As above we have \( r \in (0,1/2] \).

**Lemma A.6.** Suppose \( a \in \mathbb{Z}^d \setminus \{0\} \), then
\[
\int_0^1 f_r(as) ds = \int_{[0,1)^d} f_r(x) dx + O_{d,k,b} \left( \frac{|\log r| (\deg P)^b}{r^{2b} \rho(a)^{b-d}} \right).
\]

We follow and adapt the proof of Lemma A.6 \[20\].

**Proof.** For \( m \in \mathbb{Z}^d \) let \( \hat{f}_r(m) \) denote the Fourier coefficient of \( f_r \). By Theorem 3.2.9(a) \[17\] with derivative up-to order \( b \) and using \( |\partial_\ell f_r| \leq |\partial_\ell f_r|_{C^0} \leq |f_r|_{C^b} \) where \( \ell(i) = b \) we conclude \( \left| \hat{f}_r(m) \right| \ll_{d,b} |f_r|_{C^b} |m|^{-b} \) if \( m \neq 0 \). So \( \left| \hat{f}_r(m) \right| \ll_{d,b} r^{-2b} |\log r| (\deg P)^b |m|^{-b} \) for all \( m \in \mathbb{Z}^d \setminus \{0\} \) by \( \text{(A.10)} \). Then
\[
\sum_{|m| \geq \rho(a)} \left| \hat{f}_r(m) \right| \ll_{d,k,b} \frac{1}{|m|^b} \sum_{|m| \geq \rho(a)} \frac{|\log r| (\deg P)^b}{r^{2b} \rho(a)^{b-d}} \ll_{k,b} \frac{1}{|m|^b} \sum_{|m| \geq \rho(a)} \frac{|\log r| (\deg P)^b}{r^{2b} \rho(a)^{b-d}}
\]
as \( b \geq d + 1 \). In particular, the Fourier coefficients of \( f_r \) are absolutely summable and the Fourier series converges absolutely and uniformly to \( f_r \), see Proposition 3.1.14 \[17\]. Hence
\[
\int_0^1 f_r(as) ds = \sum_{m \in \mathbb{Z}^d} \int_{0}^{1} \hat{f}_r(m) e^{2\pi i a.m/s} ds = \int_{[0,1)^d} f_r(x) dx + \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \hat{f}_r(m).
\]
The lemma follows from (A.11) as only those $m$ with $|m| \geq \rho(a)$ contribute to the final sum. \hfill \Box

Lemma A.7. Suppose $a \in \mathbb{Z}^d \setminus \{0\}$ such that $\rho(a) > \deg P$. For all $s \in [0,1)$, up-to finitely many exceptions, we have $|P(e(as))| \neq 0$ and

$$\int_0^1 \log |P(e(as))| ds = \int_0^1 f_r(as) ds + O_k \left( r^{1/(k-1)} |\log r| \right).$$

We follow and adapt the proof of Lemma A.7 [20].

Proof. Say $a = (a_1, \ldots, a_d)$ with $\rho(a) > \deg P$. Then the coefficients of the univariate Laurent polynomial $Q = P(X^{a_1}, \ldots, X^{a_d})$ are precisely the coefficients of $P$. Hence $|Q| = |P| = 1$ and $Q$ has at most $k$ non-zero terms. The first claim follows as $P(e(as)) = Q(e(s))$ for all $s \in \mathbb{R}$ and since $Q \neq 0$. For the second claim we note that the difference of the two integrals equals

$$\int_{S(Q,r)} (\log |Q(e(s))| - f_r(as)) ds$$

with $S(Q,r)$ as in (A.2). Note that $\int_{S(Q,r)} \log |Q(e(s))| ds \leq 0$ as $r \leq 1$. Recall Theorem A.2 which yields $\text{vol}(S(Q,r)) \ll_k r^{1/(k-1)}$. As in the proof of Lemma 4 [29], cf. also Theorem 7, Appendix G [41], we find

$$\int_{S(Q,r)} \log |Q(e(s))| ds \geq -C r^{1/(k-1)} |\log r|,$$

where $C > 0$ depends only on $k$. Finally, by the definition of $f_r$ we find $\log(r/2) \leq f_r(as) \leq 0$ if $|Q(e(s))| < r$. Thus $\int_{S(Q,r)} f_r(as) ds$ is also $O_k(r^{1/(k-1)} |\log r|)$. \hfill \Box

Lemma A.8. We have

$$\left| \int_{[0,1]^d} (f_r(x) - \log |P(e(x))|) dx \right| \ll_{d,k} r^{1/(4(k-1))}.$$

We follow and adapt the proof of Lemma A.8 [20].

Proof. We have

$$\left| \int_{[0,1]^d} (f_r(x) - \log |P(e(x))|) dx \right| = \left| \int_{[0,1]^d} (\phi_r(|P(e(x))|^2) - 1) \log |P(e(x))| dx \right|$$

$$\leq \int_{[0,1]^d} |\phi_r(|P(e(x))|^2) - 1| |\log |P(e(x))||^2 dx$$

$$\leq \left( \int_{[0,1]^d} |\phi_r(|P(e(x))|^2) - 1|^2 dx \right)^{1/2} \left( \int_{[0,1]^d} |\log |P(e(x))||^2 dx \right)^{1/2}$$

by definition and where we used the Cauchy-Schwarz inequality in the last step. The second integral on the final line is $\ll_{d,k} 1$ by Lemma A.3(ii). The first integral is

$$\int_{S(P,r)} |\phi_r(|P(e(x))|^2) - 1|^2 dx \leq \text{vol}(S(P,r)) \ll_{d,k} r^{1/(2(k-1))}$$

by Lemma A.3(i) and $|P| = 1$. We take the square root to complete the proof. \hfill \Box
Proof of Theorem [A.2]. As stated below Theorem [A.1] we may assume \( d \geq 2 \). As we have seen in the proof of Lemma [A.7] the condition \( \rho(a) > \deg P \) guarantees \( P(X^{a_1}, \ldots, X^{a_d}) \neq 0 \). We may also assume that \( P \) is non-constant. Moreover, replacing \( P \) by \( P/|P| \) leaves \( m(P(X^{a_1}, \ldots, X^{a_d})) - m(P) \) invariant. So it suffices to prove the theorem if \( |P| = 1 \).

We fix the parameters \( b = 4d \geq d + 1 \) and \( r = \rho(a)^{-1/4}/2 \leq 1/2 \).

We write \(|m(P(X^{a_1}, \ldots, X^{a_d})) - m(P)| \) as \( \int_0^1 \log |P(e(as))| ds - \int_{[0,1]^d} \log |P(e(x))| dx \) and find that it is at most

\[
\left| \int_0^1 f_r(as) ds - \int_{[0,1]^d} f_r(x) dx \right| + \left| \int_0^1 (\log |P(e(as))| - f_r(as)) ds \right| + \left| \int_{[0,1]^d} (f_r(x) - \log |P(e(x))|) dx \right|.
\]

Then by Lemmas [A.6, A.7] and [A.8] this sum is

\[
\ll_{d,k} \frac{\log r \cdot (\deg P)^2}{r^{2b}} + \frac{\log \rho(a)}{\rho(a)^{1/4(k-1)}} + \frac{1}{\rho(a)^{1/16(k-1)}}.
\]

By our choice of \( r \) and \( \rho(a) \geq 2 \), the sum is

\[
\ll_{d,k} \frac{\log \rho(a)}{\rho(a)^{b-d-\frac{b}{2}}} (\deg P)^{b^2} + \frac{\log \rho(a)}{\rho(a)^{1/4(k-1)}} + \frac{1}{\rho(a)^{1/16(k-1)}}.
\]

Finally, as \( b = 4d \) the sum is

\[
\ll_{d,k} (\deg P)^{16d^2} \frac{\log \rho(a)}{\rho(a)^{d}} + \frac{\log \rho(a)}{\rho(a)^{1/4(k-1)}} + \frac{1}{\rho(a)^{1/16(k-1)}}.
\]

\[\square\]

APPENDIX B. RECOVERING THE THEOREM OF LIND, SCHMIDT, AND VERBITSKY

In this appendix we recover from our work a variant of Lind, Schmidt, and Verbitsky’s Theorem 1.1 [32]. This variant is stated in the introduction as Theorem 1.2. For a finite subgroup \( G \subset \mathbb{G}_m^d \). Recall that we defined \( \delta(G) \) in (1.4).

Lemma B.1. Let \( G \) be a finite subgroup of \( \mathbb{G}_m^d \). If \( a \in \mathbb{Z}^d \setminus \{0\} \), then

\[
\frac{1}{\# G} \# \{ \zeta \in G : \zeta^a = 1 \} \leq \frac{|a|}{\delta(G)}.
\]

Proof. We will detect \( \zeta^a = 1 \) using the character \( \chi(\zeta) = \zeta^a \) of \( G \). The image \( \chi(G) \) is a cyclic subgroup of \( \mathbb{C}^\times \) of order \( E \), say. For \( \zeta \in G \), the sum \( \sum_{k=0}^{E-1} \chi(\zeta)^k = 0 \) equals \( E \) if \( \zeta^a = 1 \) and vanishes otherwise. The number of solutions \( \zeta \in G \) of \( \zeta^a = 1 \) is thus

\[
\sum_{\zeta \in G} \frac{1}{E} \sum_{k=0}^{E-1} \chi(\zeta)^k = \frac{1}{E} \sum_{k=0}^{E-1} \sum_{\zeta \in G} \chi(\zeta)^k = \frac{1}{E} \sum_{k=0}^{E-1} \# G \sum_{\zeta \in \chi(G)} \zeta^k = \# G.
\]

We conclude the proof as \( \zeta^a E = \chi(\zeta)^E = 1 \) for all \( \zeta \in G \) and hence \( E \geq \delta(G)/|a| \). \( \square \)

Lemma B.2. Let \( G \) be a finite subgroup of \( \mathbb{G}_m^d \).
(i) If $T \geq 1$, then
\[
\frac{1}{\#G} \# \{ \zeta \in G : \delta(\zeta) \leq T \} \leq \frac{3^d T^{d+1}}{\delta(G)}.
\]
(ii) If $\kappa > 0$, then
\[
\frac{1}{\#G} \sum_{\zeta \in G} \delta(\zeta)^{-\kappa} \leq \frac{4^d}{\delta(G)^{\kappa/(d+1+\kappa)}}.
\]

Proof. Any $\zeta \in G$ with $\delta(\zeta) \leq T$ satisfies $\zeta^a = 1$ for some $a \in \mathbb{Z}^d \setminus \{0\}$ and $|a| \leq T$. The number of such $a$ is at most $(2T + 1)^d \leq 3^d T^d$ and each $a$ leads to at most $|a| \# G/\delta(G) \leq T \# G/\delta(G)$ different $\zeta$ by Lemma B.1. This implies (i).

For the second assertion we split up the elements in $G$ into those with $\delta(\zeta) \leq T$ and those with $\delta(\zeta) > T$; here $T \geq 1$ is a parameter to be chosen. For the lower range, we use the trivial lower bound $\delta(\zeta) \geq 1$ and part (i) to obtain
\[
\frac{1}{\#G} \sum_{\zeta \in G, \delta(\zeta) \leq T} \delta(\zeta)^{-\kappa} \leq \frac{3^d T^{d+1}}{\delta(G)}.
\]

For the higher range, we have
\[
\frac{1}{\#G} \sum_{\zeta \in G, \delta(\zeta) > T} \delta(\zeta)^{-\kappa} \leq \frac{1}{T^\kappa}.
\]

The lemma follows by taking the sum of these two bounds with $T = \delta(G)^{1/(d+1+\kappa)}$. \hfill \Box

Proof of Theorem 1.2. Without loss of generality we can assume that $P$ is a polynomial.

Any finite subgroup of $G_m^d$ is defined over $\mathbb{Q}$, i.e., it is map to itself under the action of the absolute Galois group of $\mathbb{Q}$, see Corollary 3.2.15 [5]. We decompose $G$ into a disjoint union $G_1 \cup \cdots \cup G_m$ of Galois orbits. It is useful to fix a representative $\xi_i \in G_i$ for each $i \in \{1, \ldots, m\}$ and define $N_i = \text{ord}(\xi_i)$. For these $i$ all elements in $G_i$ have the same order and the Galois action is the natural action of $(\mathbb{Z}/N_i \mathbb{Z})^\times$ on $G_i$. Moreover, $\# G_i = \varphi(N_i)$. Note that $\delta$ is constant on each $G_i$ as $\delta(\zeta^\sigma) = \delta(\zeta)$ for all field automorphisms $\sigma$.

Let $T \geq 1$ be a parameter depending on $\delta(G)$ and large in terms of $P, d$ which we will fix in due time. We split our average (1.5) up into those $\zeta$ with $\delta(\zeta) \leq T$ and those with $\delta(\zeta) > T$.

First, we will show that the sum
\[
(B.1) \quad \frac{1}{\#G} \sum_{\zeta \in G, \delta(\zeta) \leq T, P(\zeta) \neq 0} \log |P(\zeta)| = \frac{1}{\#G} \sum_{i=1}^m \sum_{\sigma \in (\mathbb{Z}/N_i \mathbb{Z})^\times} \log |P(\xi_i^\sigma)|
\]
is negligible. Say $P(\xi_i) \neq 0$. Then $P(\xi_i)$ lies in a number field of degree $\varphi(N_i)$ over $\mathbb{Q}$. So
\[
\sum_{\sigma \in (\mathbb{Z}/N_i \mathbb{Z})^\times} \log |P(\xi_i^\sigma)| \leq \sum_{\sigma \in (\mathbb{Z}/N_i \mathbb{Z})^\times} |\log |P(\xi_i^\sigma)|| \leq 2\varphi(N_i) h(P(\xi_i)) \ll_p \varphi(N_i)
\]
where we used the height (2.3) and its basic properties. So the absolute value of (B.1) is at most
\[ \ll_{P} \frac{1}{\#G} \sum_{i=1}^{m} \varphi(N_i) \ll_{P} \frac{1}{\#G} \sum_{\xi \in G, \delta(\xi) \leq T} 1 \ll_{d,P} T^{d+1} \delta(G). \]

by Lemma [B.2(i)].

The remaining sum is
\[ \frac{1}{\#G} \sum_{i=1}^{m} \sum_{\delta(\xi_i) > T} \log |P(\xi_i^\sigma)|; \]

note that \( P(\xi_i^\sigma) \neq 0 \) for \( T \) large enough by Theorem [1.1]. We use this theorem to rewrite the inner sum as
\[ \frac{1}{\#G} \sum_{i=1}^{m} \varphi(N_i) \left( m(P) + O_{d,P}(\delta(\xi_i)^{-\kappa}) \right) \]

\[ = \frac{1}{\#G} \left( \sum_{\xi \in G, \delta(\xi) > T} 1 \right) m(P) + O_{d,P} \left( \frac{1}{\#G} \sum_{\xi \in G, \delta(\xi) > T} \delta(\xi)^{-\kappa} \right) \]

\[ = \left( 1 - \frac{1}{\#G} \sum_{\xi \in G, \delta(\xi) \leq T} 1 \right) m(P) + O_{d,P} \left( \delta(G)^{-\frac{\kappa}{d+1}} \right) \]

where we used Lemma [B.2(ii)]. The remaining average in the last line is \( O_{d}(T^{d+1}/\delta(G)) \) by Lemma [B.2(i)].

We combine this estimate with the first bound (B.2) to conclude that the average (1.5) equals
\[ m(P) + O_{d,P}(T^{d+1}\delta(G)^{-1} + \delta(G)^{-\frac{\kappa}{d+1}}) \]

The theorem follows with the choice \( T = c\delta(G)^{1/(2(d+1))} \) where \( c \geq 1 \) is sufficiently large in terms of \( d \) and \( P \). The exponent \( \kappa \) in (1.5) is \( \min\{1/2, \kappa/(d+1+\kappa)\} \) in the notation here. \(\square\)

We leave to the interested reader the task of generalizing the previous theorem to polynomials defined over an arbitrary number field.

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March 25, 2022

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, 40 ST. GEORGE STREET, TORONTO ON, M25 2E5, CANADA

Email address: vesselin.dimitrov@gmail.com

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BASEL, SPIEGELGASSE 1, 4051 BASEL, SWITZERLAND

Email address: philipp.habegger@unibas.ch