The Similarity Invariants of non-lightlike Frenet curves in the Minkowski 3-space

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Abstract

In this paper, we firstly introduce the group of similarity transformations in the Minkowski-3 space. We describe differential- geometric invariants of a non-lightlike Frenet curve with respect to the group of similarity transformations of the Minkowski 3-space. We show extension of fundamental theorem for non-lightlike Frenet curves under the group of similarity of the Minkowski 3-space.

Keywords : Minkowski space, Similarity invariants, non-lightlike curves.
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1 Introduction

Any Euclidean motion of the Euclidean 3-space $E^3$ is an affine transformation that preserves the distances and the angles. Any similarity of $E^3$ is an affine transformation that preserves not lengths, but the angles and ratios between lengths. Therefore, the group of similarities is the smallest extension of the Euclidean motion group. On the other hand, it is well-known that a Frenet curve is established modulo an Euclidean motion of $E^3$ by curvature and torsion.

Differential geometric invariants of Frenet curves up to the group of similarities can be used for an analysis of their local shape. It was studied these differential geometric invariants by [2] and [3] in the Euclidean 3-space $E^3$.

This paper mentions to differential geometric invariants forming a non-lightlike Frenet curve with respect to similarity of the Minkowski 3-space. First, we give the

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primary informations about similarity transformation and Lorentzian space. Then, by describing the similarity transformations (which are called $p$-similarity) in the Minkowski 3-space, we prove that these transformations preserve causal characters and the angles. We examine two invariants of a non-lightlike Frenet curve which are called $p$-shape curvature and $p$-shape torsion up to the group of $p$-similarity like curvature and torsion which are invariant with respect to Euclidean motion in the standard Euclidean space $\mathbb{E}^3$. We also show the relationship between the focal curvatures of non-lightlike Frenet curves and these invariants in the Minkowski 3-space. We give the uniqueness theorem which states that two non-lightlike Frenet curves having same the $p$-shape curvature and same the $p$-shape torsion are equivalent modulo a $p$-similarity. Furthermore, we obtain the existence theorem that is a process for constructing a non-lightlike Frenet curve by its $p$-shape curvature and $p$-shape torsion under some initial condition. Lastly, we give examples about construction of a non-lightlike Frenet curve with a given $p$-shape.

2 Preliminaries

We consider the standard 3-dimensional Euclidean affine space $\mathbb{E}^3$ with an associated vector space $\mathbb{R}^3$. A map $\varphi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ is called an affine transformation iff $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation where $\varphi$ can be stated by

$$\varphi (x y) = \varphi (x) \varphi (y)$$

for every $x, y \in \mathbb{E}^3$ (see [1]).

A similarity of Euclidean affine space $\mathbb{E}^3$, of ratio $|\mu|$, is a decomposition of a homothety and an orthogonal map. Any similarity $F : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ is determined by

$$F(x) = \mu A x + b,$$

where $\mu$ is a real constant, $A$ is a fixed orthogonal $3 \times 3$ matrix with $\det(A) = 1$ and $b = (b_1, b_2, b_3)^T \in \mathbb{R}^3$ is a translation vector. If $\mu$ is a positive (negative) real number, the transformation $F$ is called orientation-preserving (reversing) similarity. Every similarity is an affine transformation of $\mathbb{E}^3$ that preserves angles (see [1]).

Now, let us give some basic notions of the Lorentzian geometry. Let $x = (x_1, x_2, x_3)^T$, $y = (y_1, y_2, y_3)^T$ and $z = (z_1, z_2, z_3)^T$ be three arbitrary vectors in the Minkowski space $\mathbb{E}_1^3$. The Lorentzian inner product of $x$ and $y$ can be stated as

$$x \cdot y = x^T I^* y$$

where $I^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then, the norm of the vector $x$ is represented by $\|x\| = \sqrt{x \cdot x}$. The Lorentzian vector product $x \times y$ of $x$ and $y$ is defined as follows:

$$x \times y = \begin{bmatrix} -1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$
The hyperbolic and Lorentzian unit spheres are
\[H^2_0 = \{ \mathbf{x} \in E^3_1 : \mathbf{x} \cdot \mathbf{x} = -1 \}\] and \[S^2_1 = \{ \mathbf{x} \in E^3_1 : \mathbf{x} \cdot \mathbf{x} = 1 \}\] respectively. There are two components \(H^2_0\) passing through \((1, 0, 0)\) and \((-1, 0, 0)\) a future pointing hyperbolic unit sphere a past pointing hyperbolic unit sphere, and they are denoted by \(H^2_{0^+}\) and \(H^2_{0^-}\), respectively.

**Theorem 1** Let \(\mathbf{x}\) and \(\mathbf{y}\) be vectors in the Minkowski 3-space.

(i) If \(\mathbf{x}\) and \(\mathbf{y}\) are future-pointing (or past-pointing) timelike vectors, then \(\mathbf{x} \times \mathbf{y}\) is a spacelike vector, \(\mathbf{x} \cdot \mathbf{y} = -\|\mathbf{x}\| \|\mathbf{y}\| \cosh \theta\) and \(\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sinh \theta\) where \(\theta\) is the hyperbolic angle between \(\mathbf{x}\) and \(\mathbf{y}\).

(ii) If \(\mathbf{x}\) and \(\mathbf{y}\) are spacelike vectors satisfying the inequality \(|\mathbf{x} \cdot \mathbf{y}| < \|\mathbf{x}\| \|\mathbf{y}\|\), then \(\mathbf{x} \times \mathbf{y}\) is timelike, \(\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta\) and \(\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sin \theta\) where \(\theta\) is the angle between \(\mathbf{x}\) and \(\mathbf{y}\).

(iii) If \(\mathbf{x}\) and \(\mathbf{y}\) are spacelike vectors satisfying the inequality \(|\mathbf{x} \cdot \mathbf{y}| > \|\mathbf{x}\| \|\mathbf{y}\|\), then \(\mathbf{x} \times \mathbf{y}\) is spacelike, \(\mathbf{x} \cdot \mathbf{y} = -\|\mathbf{x}\| \|\mathbf{y}\| \cosh \theta\) and \(\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \sinh \theta\) where \(\theta\) is the hyperbolic angle between \(\mathbf{x}\) and \(\mathbf{y}\).

(iv) If \(\mathbf{x}\) and \(\mathbf{y}\) are spacelike vectors satisfying the equality \(|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\| \|\mathbf{y}\|\), then \(\mathbf{x} \times \mathbf{y}\) is lightlike.

It can be seen [5] and [8] for further Lorentzian notion.

### 3 Similarities in the Minkowski 3-Space

Now, we define similarity transformation in \(E^3_1\). A pseudo-similarity (in short \(p\)-similarity) of Minkowski 3-space \(E^3_1\) is a decomposition of a homothety and an pseudo-orthogonal map. Any \(p\)-similarity \(f : E^3_1 \rightarrow E^3_1\) is determined by

\[f(\mathbf{x}) = \mu \mathbf{A}\mathbf{x} + \mathbf{b},\] (2)

where \(\mu\) is a real constant, \(\mathbf{A}\) is a fixed pseudo-orthogonal \(3 \times 3\) matrix with \(\det(\mathbf{A}) = 1\) and \(\mathbf{b} = (b_1, b_2, b_3)^T \in \mathbb{R}^3_1\) is a translation vector. We can find using (1) that the transformation \(\vec{f}\) is equal to \(\vec{f}(\mathbf{x}) = \mu \mathbf{A}\mathbf{x}\). Therefore, we have \(|\vec{f}(\mathbf{x})| = |\mu| \|\mathbf{x}\|\), where \(|\mu|\) is called \(p\)-similarity ratio of the transformation \(f\). \(p\)-similarity transformations are a group under the composition of maps and we denote by \(\text{Sim}(E^3_1)\).

**Theorem 2** The \(p\)-similarity transformations preserve the causal characters and angles.

**Proof.** Let \(f\) be a \(p\)-similarity. Then, since we can write the equation

\[\vec{f}(\mathbf{x}) \cdot \vec{f}(\mathbf{x}) = \mu^2 (\mathbf{A}\mathbf{x} \cdot \mathbf{A}\mathbf{x}) = \mu^2 (\mathbf{x} \cdot \mathbf{x}),\] (3)
\( f \) preserves causal character in \( \mathbb{E}^3_1 \).

Let \( x \) and \( y \) be future-pointing (or past-pointing) timelike vectors and \( \theta, \gamma \) be the angle between \( x, y \) and \( \vec{f}(x), \vec{f}(y) \) respectively. Since \( \vec{f}(x) \) and \( \vec{f}(y) \) have same causal characters with \( x \) and \( y \), we can find the following equation from Theorem 1:

\[
\vec{f}(x) \cdot \vec{f}(y) = -\|\vec{f}(x)\| \|\vec{f}(y)\| \cosh \gamma
\]

Therefore, it can be said from Theorem 1 that we have

\[
\mu^2 (x \cdot y) = -\mu^2 \|x\| \|y\| \cosh \gamma
- \|x\| \|y\| \cosh \theta = -\|x\| \|y\| \cosh \gamma
\]

\[
\cosh \theta = \cosh \gamma.
\]

From here, we have \( \theta = \gamma \). If \( x \) and \( y \) are spacelike vectors satisfying the inequality \( |x \cdot y| < \|x\| \|y\| \), then

\[
\|\vec{f}(x)\| \|\vec{f}(y)\| = \mu^2 \|x\| \|y\| > \mu^2 |x \cdot y| = \|\vec{f}(x) \cdot \vec{f}(y)\|.
\]

Therefore, it can be said from Theorem 1 that we have \( \theta = \gamma \) similar to (4).

It can also be found that \( \theta \) is equal to \( \gamma \) in case of condition (iii) in the Theorem 1. As a consequence, Every \( p \)-similarity transformation preserves the angle between any two vectors. ■

### 4 Geometric invariants of non-lightlike Frenet curves up to group of \( p \)-similarities

Let \( \alpha : t \in I \rightarrow \alpha(t) \in \mathbb{E}^3_1 \) be a non-lightlike curve of class \( C^3 \) and \( \kappa_\alpha \) and \( \tau_\alpha \) show curvature and torsion of \( \alpha \), respectively. We denote image of \( \alpha \) under \( f \in \text{Sim}(\mathbb{E}^3_1) \) by \( \beta \), i.e. \( \beta = f \circ \alpha \). Then \( \beta \) can be stated as

\[
\beta(t) = \mu A\alpha(t) + b, \quad t \in I.
\]

The arc length functions of \( \alpha \) and \( \beta \) starting at \( t_0 \in I \) are

\[
s(t) = \int_{t_0}^{t} \left| \frac{d\alpha(u)}{du} \right| du, \quad s^*(t) = \int_{t_0}^{t} \left| \frac{d\beta(u)}{du} \right| du = |\mu| s(t).
\]

The Frenet formulas of \( \alpha \) in the Minkowski 3-space is

\[
\frac{d}{ds} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ \varepsilon_{\vec{e}_1} \kappa_\alpha & 0 & \tau_\alpha \\ 0 & \varepsilon_{\vec{e}_1} \tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}
\]

where \( \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \) is Frenet frame of \( \alpha \) and \( \varepsilon_{\vec{x}} = \vec{x} \cdot \vec{x} \) (see [6] and [9]). In this section, we denote by primes the differentiation with respect to \( s \). The curvature \( \kappa_\alpha \) and torsion \( \tau_\alpha \) of non-lightlike curve \( \alpha \) is given by

\[
\kappa_\alpha(s) = \|\alpha' \times \alpha''\|, \quad \tau_\alpha(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}.
\]

4
From (5), (6) and (8), we can calculate the curvature \( \kappa_\beta (|\mu| s) \) and torsion \( \tau_\beta (|\mu| s) \) as follow

\[
\kappa_\beta = \| \beta' \times \beta'' \| = \left\| \frac{\mu}{|\mu|} A \alpha' \times \frac{1}{|\mu|} A \alpha'' \right\|
= \frac{1}{|\mu|} \kappa_\alpha (s)
\] (9)

and similarly

\[
\tau_\beta = \frac{1}{\mu} \tau_\alpha (s).
\] (10)

Since we have \( ds^* = |\mu| ds \) from (6), we get \( \kappa_\alpha ds = \kappa_\beta ds^* \) and \( |\tau_\alpha| ds = |\tau_\beta| ds^* \).

Let \( \sigma_\alpha \) and \( \sigma_\beta \) be spherical arc-length parameters of \( \alpha \) and \( \beta \), respectively. Then, we can find that

\[
d\sigma_\alpha = \kappa_\alpha ds = \kappa_\beta ds^* = d\sigma_\beta.
\] (11)

Thus, spherical arc-length element \( d\sigma_\alpha \) is invariant under the group of the \( p \)-similarities of \( \mathbb{E}^3_1 \). The derivative formulas of \( \alpha \) with respect to \( \sigma_\alpha \) are given by

\[
\frac{d\alpha}{d\sigma_\alpha} = \frac{1}{\kappa_\alpha} e_1,
\quad \frac{d^2\alpha}{d\sigma_\alpha^2} = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \frac{d\alpha}{d\sigma_\alpha} + \frac{1}{\kappa_\alpha} e_2
\] (12)

and

\[
\frac{d}{d\sigma_\alpha} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & e_{e_3} & 0 \\ 0 & e_{e_1} e_{\kappa_\alpha} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}
\] (13)

via (7) and (11). Similarly, for the non-lightlike curve \( \beta \) we also have

\[
\frac{d^2\beta}{d\sigma_\beta^2} = -\frac{d\kappa_\beta}{\kappa_\beta d\sigma_\beta} \frac{d\beta}{d\sigma_\beta} + \frac{1}{\kappa_\beta} e_2^*
\] (14)

where \( \{ e_1^*, e_2^*, e_3^* \} \) is a Frenet frame field along the non-lightlike curve \( \beta \). From (9), (10) and (11) we can write

\[
-\frac{d\kappa_\beta}{\kappa_\beta d\sigma_\beta} = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \quad \text{and} \quad \frac{\tau_\beta}{\kappa_\beta} = \frac{|\mu|}{\mu} \frac{\tau_\alpha}{\kappa_\alpha}.
\]

If we take \( \mu > 0 \), i.e. the \( p \)-similarity is an orientation-preserving transformation, we get \( \frac{\tau_\beta}{\kappa_\beta} = \frac{\tau_\alpha}{\kappa_\alpha} \). Thus, we obtain the following Lemma from above calculations.

**Lemma 3** The functions \( \bar{\kappa}_\alpha = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \) and \( \bar{\tau}_\alpha = \frac{\tau_\alpha}{\kappa_\alpha} \) are invariants under the group of the orientation-preserving \( p \)-similarities of the Minkowski 3-space.

Using (12) and (13) the invariants \( \bar{\kappa}_\alpha \) and \( \bar{\tau}_\alpha \) can take the form

\[
\bar{\kappa}_\alpha (\sigma_\alpha) = \frac{d^2\alpha}{d\sigma_\alpha^2} \cdot \frac{d\alpha}{d\sigma_\alpha} = \frac{d\alpha}{d\sigma_\alpha} \cdot \frac{d\alpha}{d\sigma_\alpha}.
\] (15)
\[ \tilde{\kappa}_\alpha (\sigma_\alpha) = \det \left( \frac{d\alpha}{d\sigma_\alpha}, \frac{d^2\alpha}{d\sigma_\alpha^2}, \frac{d^3\alpha}{d\sigma_\alpha^3} \right) \left\| \frac{d\alpha}{d\sigma_\alpha} \right\|^3. \]  

**Definition 4** Let \( \alpha : I \rightarrow \mathbb{E}^3_1 \) be non-lightlike Frenet curve of the class \( C^3 \) parameterized by a spherical arc length parameter \( \sigma_\alpha \). Let \( \kappa_\alpha (\sigma_\alpha) \) and \( \tau_\alpha (\sigma_\alpha) \) be respectively the curvature and torsion of \( \alpha \). The functions
\[ \tilde{\kappa}_\alpha = -\frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} \quad \text{and} \quad \tilde{\tau}_\alpha = \frac{\tau_\alpha}{\kappa_\alpha} \]  
are p-shape curvature and p-shape torsion of \( \alpha \). The ordered pair \( (\tilde{\kappa}_\alpha, \tilde{\tau}_\alpha) \) is called a (local) p-shape of the non-lightlike curve \( \alpha \) in the Minkowski 3-space.

### 4.1 The relation between focal curvatures and p-shape of \( \alpha \)

Let \( \alpha : I \rightarrow \mathbb{E}^3_1 \) be a unit speed non-lightlike Frenet curve with the Frenet frame \( e_1, e_2, e_3 \) and let \( s \) be an arc length parameter of \( \alpha \). The curve \( \gamma : I \rightarrow \mathbb{E}^3_1 \) consisting of the centers of the osculating sphere of the curve \( \alpha \) is called the focal curve of \( \alpha \). The focal curve can be represented by
\[ \gamma (s) = \alpha (s) + f_1 (s) e_2 + f_2 (s) e_3 \]
where \( f_1 \) and \( f_2 \) are smooth functions called focal curvature of \( \alpha \). Then, we have the following theorem from [9].

**Theorem 5** Let \( \alpha \) be a non-lightlike Frenet curve in \( \mathbb{E}^3_1 \), the radius and center of the osculating sphere of \( \alpha \) at \( \alpha (s) \) are
\[ r = \sqrt{\left( \varepsilon_{e_2} \right)^2 + \left( \varepsilon_{e_3} \right)^2 \left( \kappa' \right)^2 \left( \kappa^2 \tau \right)} \quad \text{and} \quad \gamma (s) = \alpha (s) + \frac{\varepsilon_{e_1} \varepsilon_{e_2}}{\kappa} e_2 + \frac{\varepsilon_{e_1} \varepsilon_{e_3}}{\tau} \left( \frac{1}{\kappa} \right)' e_3 \]
where \( e_2 \) and \( e_3 \) are normal and binormal vector fields of the curve at \( \alpha (s) \).

Using Theorem 5 we state that the focal curvatures \( f_1 \) and \( f_2 \) of the non-lightlike curve \( \alpha \) are equal to
\[ \frac{\varepsilon_{e_1} \varepsilon_{e_2}}{\kappa_\alpha} \quad \text{and} \quad \frac{1}{\tau_\alpha} \left( \frac{\varepsilon_{e_1} \varepsilon_{e_3}}{\kappa_\alpha} \right)' \]  
respectively. Now, we can show the relation between the focal curvatures and p-shape curvature and torsion.

**Proposition 6** Let \( \alpha : I \rightarrow \mathbb{E}^3_1 \) be a unit speed non-lightlike Frenet curve with the non-zero curvature \( \kappa \) and torsion \( \tau \). Then,
\[ \tilde{\kappa}_\alpha = \varepsilon_{e_1} \varepsilon_{e_2} f'_1 \quad \text{and} \quad \tilde{\tau}_\alpha = \varepsilon_{e_1} \varepsilon_{e_3} \frac{f'_1 f_1}{f_2}. \]
Proof. From (17) and (18) we can write
\[ \tilde{\kappa}_\alpha = - \frac{d\kappa_\alpha}{\kappa_\alpha d\sigma_\alpha} = - \frac{1}{\kappa_\alpha^2} \frac{d\kappa_\alpha}{ds} = \left( \frac{1}{\kappa_\alpha} \right)' = \varepsilon_{e_1} \varepsilon_{e_2} f_1' \]
and
\[ \tilde{\tau}_\alpha = \frac{\tau_\alpha}{\kappa_\alpha} = \varepsilon_{e_1} \varepsilon_{e_2} f_1' \varepsilon_{e_1} \varepsilon_{e_2} f_2 \left( \varepsilon_{e_1} \varepsilon_{e_2} f_1 \right)' = \varepsilon_{e_1} \varepsilon_{e_3} f_1' f_3. \]

\[ \text{Proof.} \]

5 Uniqueness Theorem

Two non-lightlike Frenet curves which have the same torsion and the same positive curvature are always equivalent up to Lorentzian motion. This notion can be extended with respect to \( \text{Sim}(\mathbb{E}^3) \) for the non-lightlike Frenet curves which have the same p-shape torsion and p-shape curvature, in the Minkowski 3-space \( \mathbb{E}^3 \).

**Theorem 7** (Uniqueness Theorem) Let \( \alpha, \alpha^* : I \to \mathbb{E}^3 \) be two non-lightlike Frenet curves of class \( C^3 \) parameterized by the same spherical arc length parameter \( \sigma \) and have the same causal character, where \( I \subset \mathbb{R} \) is an open interval. Suppose that \( \alpha \) and \( \alpha^* \) have the same p-shape curvature \( \kappa = \tilde{\kappa}^* \) and the same p-shape torsion \( \tau = \tilde{\tau}^* \) for any \( \sigma \in I \). Then there exists a p-similarity \( f \) of \( \mathbb{E}^3 \) such that \( \alpha^* = f \circ \alpha \).

**Proof.** Let \( \kappa, \kappa^* \) and \( \tau, \tau^* \) be the curvature and the torsion of the \( \alpha, \alpha^* \). Since \( \alpha \) and \( \alpha^* \) have the same shape curvature \( \kappa = \tilde{\kappa}^* \), we have
\[ \frac{d\kappa}{\kappa} = \frac{d\kappa^*}{\kappa^*} \quad \text{or} \quad \log \kappa = \log \kappa^* + \log \mu \]
where \( \mu \in \mathbb{R}^+ \). Then, we find \( \kappa = \mu \kappa^* \) for any \( \sigma \in I \). Using \( \tilde{\tau} = \tilde{\tau}^* \) we get \( \tau = \mu \tau^* \) for any \( \sigma \in I \). Let \( e_i, e_i^*, i = 1, 2, 3, \) be a Frenet frame on \( \alpha, \alpha^* \) and we choose any point \( \sigma_0 \in I \). There exists Lorentzian motion \( \varphi \) of \( \mathbb{E}^3 \) such that
\[ \varphi \left( \alpha \left( \sigma_0 \right) \right) = \alpha^* \left( \sigma_0 \right) \quad \text{and} \quad \varphi \left( e_i \left( \sigma_0 \right) \right) = -\varepsilon_{e_i} e_i^* \left( \sigma_0 \right) \quad \text{for} \quad i = 1, 2, 3. \]

Let’s consider the function \( \Psi : I \to \mathbb{R} \) defined by
\[ \Psi \left( \sigma \right) = \| \varphi \left( e_1 \left( \sigma \right) \right) + \varepsilon_{e_1} e_1^* \left( \sigma \right) \|^2 + \| \varphi \left( e_2 \left( \sigma \right) \right) + \varepsilon_{e_2} e_2^* \left( \sigma \right) \|^2 + \| \varphi \left( e_3 \left( \sigma \right) \right) + \varepsilon_{e_3} e_3^* \left( \sigma \right) \|^2. \]

Then
\[ \frac{d\Psi}{d\sigma} = 2 \left( \frac{d}{d\sigma} \varphi \left( e_1 \left( \sigma \right) \right) + \varepsilon_{e_1} \frac{d}{d\sigma} e_1^* \left( \sigma \right) \right) \cdot \left( \varphi \left( e_1 \left( \sigma \right) \right) + \varepsilon_{e_1} e_1^* \left( \sigma \right) \right) + \\
+ 2 \left( \frac{d}{d\sigma} \varphi \left( e_2 \left( \sigma \right) \right) + \varepsilon_{e_2} \frac{d}{d\sigma} e_2^* \left( \sigma \right) \right) \cdot \left( \varphi \left( e_2 \left( \sigma \right) \right) + \varepsilon_{e_2} e_2^* \left( \sigma \right) \right) + \\
+ 2 \left( \frac{d}{d\sigma} \varphi \left( e_3 \left( \sigma \right) \right) + \varepsilon_{e_3} \frac{d}{d\sigma} e_3^* \left( \sigma \right) \right) \cdot \left( \varphi \left( e_3 \left( \sigma \right) \right) + \varepsilon_{e_3} e_3^* \left( \sigma \right) \right). \]
Using $\|\varphi (e_i)\|^2 = \|e_i\|^2 = \|e_i^*\|^2 = 1$ we can write
\[
\frac{d\Psi}{d\sigma} = 2\varepsilon_{e_1} \left[ \left( \varphi \left( \frac{d}{d\sigma} e_1 \right) \right) \cdot e_i^* + \varphi (e_1) \cdot \left( \frac{d}{d\sigma} e_i^* \right) \right] +
2\varepsilon_{e_2} \left[ \left( \varphi \left( \frac{d}{d\sigma} e_2 \right) \right) \cdot e_i^* + \varphi (e_2) \cdot \left( \frac{d}{d\sigma} e_i^* \right) \right] +
2\varepsilon_{e_3} \left[ \left( \varphi \left( \frac{d}{d\sigma} e_3 \right) \right) \cdot e_i^* + \varphi (e_3) \cdot \left( \frac{d}{d\sigma} e_i^* \right) \right].
\]

From (13), we get
\[
\frac{d\Psi}{d\sigma} = (2\varepsilon_{e_1} + 2\varepsilon_{e_2}\varepsilon_{e_3}^*) \left[ \varphi (e_2) \cdot e_i^* \right] + (2\varepsilon_{e_1} + 2\varepsilon_{e_2}\varepsilon_{e_3}) \left[ \varphi (e_1) \cdot e_i^* \right]
\]
\[
(2\varepsilon_{e_2} \tilde{\tau} + 2\varepsilon_{e_3}\varepsilon_{e_1}\tilde{\tau}^*) \left[ \varphi (e_3) \cdot e_i^* \right] + (2\varepsilon_{e_2} \tilde{\tau}^* + 2\varepsilon_{e_3}\varepsilon_{e_1} \tilde{\tau}) \left[ \varphi (e_2) \cdot e_i^* \right].
\]

Since $\alpha$ and $\alpha^*$ have the same causal characters and $\tilde{\tau} = \tilde{\tau}^*$, we can write
\[
2\varepsilon_{e_1} + 2\varepsilon_{e_2}\varepsilon_{e_3} = 0, \quad 2\varepsilon_{e_1} + 2\varepsilon_{e_2}\varepsilon_{e_3} = 0
2\varepsilon_{e_2} \tilde{\tau} + 2\varepsilon_{e_3}\varepsilon_{e_1} \tilde{\tau}^* = 0, \quad 2\varepsilon_{e_2} \tilde{\tau}^* + 2\varepsilon_{e_3}\varepsilon_{e_1} \tilde{\tau} = 0.
\]

Therefore, we find $\frac{d\Psi}{d\sigma} = 0$ for any $\sigma \in I$. On the other hand, we know $\Psi (\sigma_0) = 0$ and thus we have $\Psi (\sigma) = 0$ for any $\sigma \in I$. As a result, we can say that
\[
\varphi (e_i (\sigma)) = -\varepsilon_{e_i} e_i^* (\sigma), \quad \forall \sigma \in I, \quad i = 1, 2, 3. \tag{19}
\]

The map $g = \mu \varphi : E_1^3 \to E_1^3$ is a $\mu$-similarity of $E_1^3$. We examine the other function $\Phi : I \to \mathbb{R}$ such that

\[
\Phi (\sigma) = \left\| \frac{d}{d\sigma} g (\alpha (\sigma)) + \varepsilon_{e_1} \frac{d}{d\sigma} \alpha^* (\sigma) \right\|^2 \quad \text{for} \quad \forall \sigma \in I.
\]

Taking derivative this function with respect to $\sigma$ we get
\[
\frac{d\Phi}{d\sigma} = 2g \left( \frac{d^2 \alpha}{d\sigma^2} \right) \cdot g ( \frac{d\alpha}{d\sigma} ) + 2\varepsilon_{e_1} \left[ g \left( \frac{d^2 \alpha}{d\sigma^2} \right) \cdot \frac{d\alpha^*}{d\sigma} \right] +
2 \frac{d^2 \alpha^*}{d\sigma^2} \cdot g \left( \frac{d\alpha}{d\sigma} \right) + 2\varepsilon_{e_1} \left[ \frac{d^2 \alpha^*}{d\sigma^2} \cdot \frac{d\alpha^*}{d\sigma} \right].
\]

Since the function $\varphi$ is linear map and we have (12) and (19), we can write
\[
\frac{d\Phi}{d\sigma} = 2\varepsilon_{e_1} \varepsilon_{e_1}^* \mu^2 \frac{\kappa}{\kappa^*} - 2\varepsilon_{e_1} \varepsilon_{e_1}^* \mu^2 \frac{\kappa}{\kappa^*} - 2\varepsilon_{e_1} \varepsilon_{e_1}^* \mu^2 \frac{\kappa}{\kappa^*} + 2\varepsilon_{e_1} \varepsilon_{e_1}^* \frac{\kappa}{(\kappa^*)^2}.
\]

Using $\mu = \frac{\kappa}{\kappa^*}$ and $\varepsilon_{e_1}^2 = 1$, we have $\frac{d\Phi}{d\sigma} = 0$. Also, we can find
\[
\frac{d}{d\sigma} g (\alpha (\sigma_0)) = g \left( \frac{1}{\kappa} e_1 (\sigma_0) \right) = -\varepsilon_{e_1} \frac{1}{\kappa^*} e_i^* (\sigma_0)
\]

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and we know
\[ \frac{d}{d\sigma} \alpha^*(\sigma_0) = \frac{1}{\kappa^*} \mathbf{e}_1^*(\sigma_0). \]

Then, we conclude that \( \Phi(\sigma_0) = 0 \). Hence, \( \Phi(\sigma) = 0 \) for \( \forall \sigma \in I \). This means that
\[ \frac{d}{d\sigma} g(\alpha(\sigma)) = -\varepsilon \mathbf{e}_1 \frac{d}{d\sigma} \alpha^*(\sigma) \]
or equivalently \( \alpha^*(\sigma) = -\varepsilon \mathbf{e}_1 g(\alpha(\sigma)) + \mathbf{b} \) where \( \mathbf{b} \) is a constant vector. Then, the image of the non-lightlike curve \( \alpha \) under the p-similarity transformation \( f = \vartheta \circ (-\varepsilon \mathbf{e}_1 g) \), where \( \vartheta : E^3_1 \rightarrow E^3_1 \) is a translation function determined by \( \mathbf{b} \), is the non-lightlike curve \( \alpha^* \).

\textbf{Remark 8} If the curves \( \alpha, \alpha^* \) are taken as the timelike curve in the Theorem 7, p-similarity transformation \( f \) which provide the relation \( \alpha^* = f \circ \alpha \) is orientation-preserving transformation. However, when the curves \( \alpha, \alpha^* \) are the spacelike curve, p-similarity transformation \( f \) is orientation-reversing transformation. Thus, two timelike Frenet curves which have the same p-shape curvature and the same p-shape torsion are equivalent according to orientation-preserving p-similarity and two spacelike Frenet curves which have the same p-shape curvature and the same p-shape torsion are equivalent according to orientation-reversing p-similarity.

Is it possible to say that two spacelike Frenet curves are equivalent under orientation-preserving p-similarity? We can see the answer with the following theorem.

\textbf{Theorem 9} Let \( \alpha, \alpha^* : I \rightarrow E^3_1 \) be two spacelike Frenet curves of class \( C^3 \) parameterized by the same spherical arc length parameter \( \sigma \), where \( I \subset \mathbb{R} \) is an open interval. Suppose that \( \alpha \) and \( \alpha^* \) have the same p-shape curvature \( \tilde{\kappa} = \tilde{\kappa}^* \) and \( \tilde{\tau} = -\tilde{\tau}^* \) for the p-shape torsions \( \tilde{\tau}, \tilde{\tau}^* \). Then there exists an orientation-preserving p-similarity \( f \) of \( E^3_1 \) such that \( \alpha^* = f \circ \alpha \).

\textbf{Proof.} The proof is similar to the proof of the Theorem 7. Let \( \mathbf{e}_i, \mathbf{e}_i^* \), \( i = 1, 2, 3 \), be a Frenet frame field on \( \alpha, \alpha^* \) and we choose any point \( \sigma_0 \in I \). If \( \mathbf{e}_2 \) and \( \mathbf{e}_2^* \) are timelike vectors, There exists Lorentzian motion \( \varphi \) of \( E^3_1 \) such that
\[ \varphi(\alpha(\sigma_0)) = \alpha^*(\sigma_0), \quad \varphi(\mathbf{e}_1(\sigma_0)) = \mathbf{e}_1^*(\sigma_0) \quad \text{and} \quad \varphi(\mathbf{e}_i(\sigma_0)) = -\mathbf{e}_i^*(\sigma_0) \quad \text{for} \quad i = 2, 3. \]

Let’s consider the function \( \Psi : I \rightarrow \mathbb{R} \) defined by
\[ \Psi(\sigma) = \|\varphi(\mathbf{e}_1(\sigma)) - \mathbf{e}_1^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_2(\sigma)) + \mathbf{e}_2^*(\sigma)\|^2 + \|\varphi(\mathbf{e}_3(\sigma)) + \mathbf{e}_3^*(\sigma)\|^2. \]

Then
\[ \frac{d\Psi}{d\sigma} = 2(\tilde{\tau} + \tilde{\tau}^*) (\varphi(\mathbf{e}_3) \cdot \mathbf{e}_2 + \varphi(\mathbf{e}_2) \cdot \mathbf{e}_3) \]
\[ = 0. \]
Due to $\Psi (\sigma_0) = 0$, we can write
\[ \varphi (e_1(\sigma)) = e_1^*(\sigma) \quad \text{and} \quad \varphi (e_i(\sigma)) = -e_i^*(\sigma) \quad \text{for} \quad i = 2, 3 \quad \forall \sigma \in I. \]

The map $g = \mu \varphi : E_1^2 \to E_1^2$ is a p-similarity of $E_1^3$. We examine the other function $\Phi : I \to \mathbb{R}$ such that
\[ \Phi (\sigma) = \left\| \frac{d}{d\sigma} g (\alpha (\sigma)) - \frac{d}{d\sigma} \alpha^* (\sigma) \right\|^2 \quad \text{for} \quad \forall \sigma \in I. \]

Since we have $\frac{d\Phi}{d\sigma} = 0$ and $\Phi (\sigma_0) = 0$, we get $\Phi (\sigma) = 0$ for any $\sigma \in I$. Namely, we can write
\[ \frac{d}{d\sigma} g (\alpha (\sigma)) = \frac{d}{d\sigma} \alpha^* (\sigma) \text{ or equivalently } \alpha^* (\sigma) = g (\alpha (\sigma)) + b \text{ where } b \text{ is a constant vector.} \]

In the same way, if we take $e_3$ and $e_3^*$ as timelike vectors, we can find orientation-preserving p-similarity $f$ which provides $\alpha^* = f \circ \alpha$ such that the functions $\Psi$ and $\Phi$ are respectively defined by
\[ \Psi (\sigma) = \left\| \varphi (e_1(\sigma)) - e_1^*(\sigma) \right\|^2 + \left\| \varphi (e_2(\sigma)) - e_2^*(\sigma) \right\|^2 + \left\| \varphi (e_3(\sigma)) - e_3^*(\sigma) \right\|^2 \]
\[ \Phi (\sigma) = \left\| \frac{d}{d\sigma} g (\alpha (\sigma)) - \frac{d}{d\sigma} \alpha^* (\sigma) \right\|^2 \quad \text{for} \quad \forall \sigma \in I. \]

6 Construction of the non-lightlike Frenet curves by curves on the Lorentzian and hyperbolic unit sphere

Let $c : I \to S^2_1$ be non-lightlike spherical curve with $\sigma$ arc length parameter of $c$. The orthonormal frame \{ $c (\sigma), t (\sigma), q (\sigma)$ \} along $c$ is called Sabban frame of $c$ if \[ t (\sigma) = \frac{dc}{d\sigma} \] is unit tangent vector of $c$ and \[ q (\sigma) = \varepsilon_q c (\sigma) \times t (\sigma). \] Then we state spherical Frenet-Serret formulas of the non-lightlike curve $c$.

If the curve $c$ is timelike curve, i.e. $t (\sigma)$ is timelike vector, we have the following spherical Frenet-Serret formulas of $c$:
\[
\frac{d}{d\sigma} \begin{bmatrix} c \\ t \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & k_g \\ 0 & k_g & 0 \end{bmatrix} \begin{bmatrix} c \\ t \\ q \end{bmatrix} \quad \text{(20)}
\]

If $q (\sigma)$ is timelike vector, we have the following spherical Frenet-Serret formulas of $c$:
\[
\frac{d}{d\sigma} \begin{bmatrix} c \\ t \\ q \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & k_g \\ 0 & k_g & 0 \end{bmatrix} \begin{bmatrix} c \\ t \\ q \end{bmatrix} \quad \text{(21)}
\]
If $c : I \rightarrow H^2_0$ is spacelike spherical curve with $\sigma$ arc length parameter of $c$, then spherical Frenet-Serret formulas of $c$ are

$$
\frac{d}{d\sigma} \begin{bmatrix} c \\ t \\ q \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & k_g \\ 0 & -k_g & 0 \end{bmatrix} \begin{bmatrix} c \\ t \\ q \end{bmatrix}
$$

(22)

since $c(\sigma)$ is timelike vector. $k_g(\sigma) = \varepsilon_q \det \begin{pmatrix} c(\sigma), t(\sigma), \frac{dt}{d\sigma}(\sigma) \end{pmatrix}$ is the geodesic curvature of $c$ in the three different spherical Frenet-Serret formulas.

Let $k : I \rightarrow \mathbb{R}$ be a function of class $C^1$. We can describe a non-lightlike curve $\alpha : I \rightarrow \mathbb{E}^3_1$ given by

$$
\alpha(\sigma) = b \int e^{\int k(\sigma) d\sigma} c(\sigma) d\sigma + a,
$$

(23)

where $a$ is a constant vector and $b$ is a real constant. The fact that $\sigma$ is arc spherical length parameter of $\alpha$ can be easily seen because we have $\frac{\frac{d\alpha}{d\sigma}}{\left\| \frac{d\alpha}{d\sigma} \right\|} = c(\sigma)$. Then, we can state a description of all Frenet curves in Minkowski 3-space.

**Proposition 10** The non-lightlike curve $\alpha$ defined by (23) is a Frenet curve with shape curvature $\tilde{\kappa} = k(\sigma)$ and shape torsion $\tilde{\tau} = \varepsilon_q k_g(\sigma)$ in the Minkowski 3-space. Furthermore, all non-lightlike Frenet curves can be obtained in this way.

**Proof.** First, from (23) we can write

$$
\frac{d\alpha}{d\sigma} = be^{\int k(\sigma) d\sigma} c(\sigma), \quad \frac{d^2\alpha}{d\sigma^2} = be^{\int k(\sigma) d\sigma} \left[ k(\sigma) c(\sigma) + \frac{dc}{d\sigma} \right],
$$

$$
\frac{d^3\alpha}{d\sigma^3} = be^{\int k(\sigma) d\sigma} \left[ \left\{ k^2(\sigma) + \frac{dk}{d\sigma} \right\} c(\sigma) + 2k(\sigma) \frac{dc}{d\sigma} + \frac{d^2c}{d\sigma^2} \right].
$$

Then, since we have

$$
\frac{d\alpha}{d\sigma} \times \frac{d^2\alpha}{d\sigma^2} = b^2 e^{2\int k(\sigma) d\sigma} \left( c(\sigma) \times \frac{dc}{d\sigma} \right) \neq 0,
$$

this means that $\alpha$ is non-lightlike Frenet curve. Using (15) and (16) we find that

$$
\tilde{\kappa} = k(\sigma) \quad \text{and} \quad \tilde{\tau} = \det \begin{pmatrix} c, \frac{dc}{d\sigma}, \frac{dt}{d\sigma} \end{pmatrix} = \varepsilon_q k_g(\sigma).
$$

Conversely, suppose that $\alpha : I \rightarrow \mathbb{E}^3_1$ is a non-lightlike regular curve parameterized by a spherical arc length parameter $\sigma$. Denote by $\kappa(\sigma)$ and $\tau(\sigma)$ the curvature and the torsion of $c$, respectively. Let $c$ be the spherical indicator of $\alpha$ such that $c : I \rightarrow \mathbb{E}^3_1$ is given by

$$
c(\sigma) = e_1(\sigma) = \frac{\frac{d\alpha}{d\sigma}}{\left\| \frac{d\alpha}{d\sigma} \right\|} = \kappa(\sigma) \frac{d\alpha}{d\sigma}.
$$

(24)
We can say that \( \sigma \) is an arc length parameter of \( c \) and \( k_\sigma = \varepsilon_\varphi \det \left( c (\sigma), t (\sigma), \frac{dt (\sigma)}{d\sigma} \right) = \varepsilon_\varphi \tilde{\tau} \) is the geodesic curvature of \( c \). If we take \( k (\sigma) = \tilde{\kappa} (\sigma) \), then

\[
\int e^{\frac{1}{\tilde{\kappa} (\sigma)}} c (\sigma) \, d\sigma = \int e^{\frac{1}{\tilde{\kappa} (\sigma)}} c (\sigma) \, d\sigma = e^{b_0} \int \frac{1}{\tilde{\kappa} (\sigma)} d\sigma = e^{b_0} \int \frac{d\alpha}{d\sigma} = e^{b_0} \alpha (\sigma) + a_0
\]

where \( b_0 \) is a real constant and \( a_0 \) is a constant vector. Hence, we can write

\[
\alpha (\sigma) = b \int e^{\frac{1}{\tilde{\kappa} (\sigma)}} c (\sigma) \, d\sigma + a.
\]

\[\blacksquare\]

**Theorem 11 (Existence Theorem)** Let \( z_i : I \to \mathbb{R}, i = 1, 2, \) be two functions of class \( C^1 \) and \( e^0_1, e^0_2, e^0_3 \) be an right-handed orthonormal triad of vectors at a point \( x_0 \) in the Minkowski 3-space \( \mathbb{E}^3_1 \). Up to a \( p \)-similarity with center \( x_0 \) there exists a unique non-lightlike Frenet curve \( \alpha : I \to \mathbb{E}^3_1 \) such that \( \alpha \) satisfies the following conditions:

(i) There is a \( \sigma_0 \in I \) such that \( \alpha (\sigma_0) = x_0 \) and the Frenet frame of \( \alpha \) at \( x_0 \) is \( \{e^0_1, e^0_2, e^0_3\} \).

(ii) \( \tilde{\kappa} (\sigma) = z_1 (\sigma) \) and \( \tilde{\tau} (\sigma) = \varepsilon_\varphi z_2 (\sigma) \) for any \( \sigma \in I \).

**Proof.** We consider following system of differential equations with respect to the vectorial functions \( t (\sigma), c (\sigma) \) and \( q (\sigma) \)

\[
\frac{dX}{d\sigma} (\sigma) = M (\sigma) X (\sigma)
\]

where if \( t (\sigma), c (\sigma) \) or \( q (\sigma) \) is timelike vector, the matrix \( M \) is respectively equal to

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & z_2 \\
0 & z_2 & 0
\end{bmatrix},
\begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & z_2 \\
0 & -z_2 & 0
\end{bmatrix}, \text{ or } \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & z_2 \\
0 & z_2 & 0
\end{bmatrix},
\]

and \( X (\sigma) = [c (\sigma) \ t (\sigma) \ q (\sigma)] \). The system \((25)\) has an unique solution \( X (\sigma) \) which satisfies initial conditions \( X (\sigma_0) = [e^0_1 \ e^0_2 \ e^0_3] \) for \( \sigma_0 \in I \). If \( I \) is the unit matrix and \( X^t \) is the transposed matrix of \( X (\sigma) \), then

\[
\frac{d}{d\sigma} (\Gamma X^t \Gamma X) = \Gamma \frac{d}{d\sigma} X^t X + \Gamma X^t \frac{d}{d\sigma} X
\]

\[
= \Gamma X^t M^t \Gamma X + \Gamma X^t \Gamma M X
\]

\[
= \Gamma X^t (M^t + \Gamma M) X.
\]
Since we can find $M^t I^t + I^t M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where $I^t$ is respectively equal to

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

when $c(\sigma)$, $t(\sigma)$ or $q(\sigma)$ is timelike vector. Also, we have $I^t X^t (\sigma_0) I^t X (\sigma_0) = I$ since $\{e_0^1, e_2^0, e_3^0\}$ is the orthonormal frame. As a result, we find $I^t X^t (\sigma) I^t X (\sigma) = I$ for any $\sigma \in I$. This say us that the vectorial fields $t(\sigma)$, $c(\sigma)$ and $q(\sigma)$ form a right-handed orthonormal frame field.

Let $\alpha : I \to \mathbb{E}_3^1$ be the regular non-lightlike curve given by

$\alpha (\sigma) = b \int_{\sigma_0}^{\sigma} e^{\int z_1(\sigma) d\sigma} c(\sigma) d\sigma + x_0, \quad \sigma \in I, \ b > 0.$

Using proposition (10) we get that the Frenet frame field of $\alpha$ is

$\{e_1(\sigma) = c(\sigma), e_2(\sigma) = t(\sigma), \ e_3(\sigma) = q(\sigma)\}$

and Frenet frame of $\alpha$ at $x_0 = \alpha (\sigma_0)$ is

$\{e_1^0 (\sigma_0) = c(\sigma_0), e_2^0 (\sigma_0) = t(\sigma_0), \ e_3^0 (\sigma_0) = q(\sigma_0)\}.$

Besides, the functions $z_1$ and $\varepsilon_{x_3} z_2$ is respectively the p-shape curvature and p-shape torsion of $\alpha$. ■

From Theorems 7 and 11 we get the following theorem which is an analogue of the fundamental theorem of Frenet curves.

**Theorem 12** Let $z_i : I \to \mathbb{R}$, $i = 1, 2$, be two functions of class $C^1$. Up to p-similarity there exists an unique non-lightlike Frenet curve with p-shape curvature $z_1$ and p-shape torsion $z_2$.

### 6.1 Forming a non-lightlike Frenet curve from its p-shape

Let $\alpha : I \to \mathbb{E}_3^1$ be non-lightlike Frenet curve with spherical arc length parameter $\sigma$ such that the ordered pair $(\tilde{\kappa}_\alpha, \tilde{\tau}_\alpha)$ is p-shape of the $\alpha$ defined by (17). From Theorem 12 we have that $\alpha$ is uniquely determined by its p-shape with respect to p-similarity in the Minkowski 3-space. First we define fixed right-handed orthonormal triad of non-lightlike vectors $e_1^0, e_2^0, e_3^0$. When $t(\sigma)$, $c(\sigma)$ or $q(\sigma)$ is timelike vector, we take respectively differential equations

$$\frac{dc}{d\sigma} = t(\sigma), \quad \frac{dt}{d\sigma} = c(\sigma) + \varepsilon q_{\tilde{\tau}_\alpha} q(\sigma), \quad \frac{dq}{d\sigma} = \varepsilon q_{\tilde{\tau}_\alpha} t(\sigma)$$ (26)
Then, the system

\[
\frac{dc}{d\sigma} = -t(\sigma), \quad \frac{dt}{d\sigma} = -c(\sigma) + \varepsilon_q \tilde{t}_a q(\sigma), \quad \frac{dq}{d\sigma} = -\varepsilon_q \tilde{t}_a t(\sigma) \quad (27)
\]

\[
\frac{dc}{d\sigma} = -t(\sigma), \quad \frac{dt}{d\sigma} = c(\sigma) + \varepsilon_q \tilde{t}_a q(\sigma), \quad \frac{dq}{d\sigma} = \varepsilon_q \tilde{t}_a t(\sigma). \quad (28)
\]

The unique solution of one of these differential equations with initial conditions \(e_0^0, e_0^0, e_3^0\), determine a spherical non-lightlike curve \(c = c(\sigma)\) such that \(c(\sigma_0) = e_0^0\) for some \(\sigma_0 \in I\). Let \(\rho(\sigma) = \int_{\sigma_0}^{\sigma} \tilde{t}_a(\sigma) d\sigma\) for fixed \(\sigma_1 \in I\). Using the equation (23) and proposition [10] we can find the non-lightlike curve

\[
\alpha(\sigma) = \alpha_0 + \int_{\sigma_0}^{\sigma} e^{\rho(\sigma)} c(\sigma) d\sigma \quad (29)
\]

passes through a point \(\alpha_0 = \alpha(\sigma_0)\). Now, we show few examples of the non-lightlike Frenet curves constructed by above procedure.

**Example 13** Let \(p\)-shape \((\tilde{t}_a, \tilde{r}_a)\) of the \(\alpha : I \to \mathbb{E}_3^1\) be \(0, a\), where \(a \neq 0\) is real constant. We can find \(\rho(\sigma) = 0\) for any \(\sigma \in I\). We take the unit vector \(t(\sigma)\) as timelike vector. Choose initial conditions

\[
e_1^0 = \left(0, -\frac{1}{\sqrt{1 + a^2}}, \frac{a}{\sqrt{1 + a^2}}\right), \quad e_2^0 = (1, 0, 0), \quad e_3^0 = \left(0, \frac{a}{\sqrt{1 + a^2}}, \frac{1}{\sqrt{1 + a^2}}\right). \quad (30)
\]

Then, the system (26) describes a spherical timelike curve \(c : I \to S_1^2\) defined by

\[
c(\sigma) = \left(\frac{1}{\sqrt{1 + a^2}} \sinh \left(\sqrt{1 + a^2} \sigma\right), -\frac{1}{\sqrt{1 + a^2}} \cosh \left(\sqrt{1 + a^2} \sigma\right), \frac{a}{\sqrt{1 + a^2}}\right) \quad (31)
\]

with \(c(0) = e_1^0\), in the Minkowski 3-space. Solving the equation (29) we obtain the spacelike Frenet curve parameterized by

\[
\alpha(\sigma) = \left(\frac{1}{1 + a^2} \cosh \left(\sqrt{1 + a^2} \sigma\right), -\frac{1}{1 + a^2} \sinh \left(\sqrt{1 + a^2} \sigma\right), \frac{a}{\sqrt{1 + a^2}}\right), \quad \sigma \in I.
\]

If the unit vector \(c(\sigma)\) is timelike vector, Choose another initial conditions

\[
e_1^0 = \left(-\sqrt{\frac{a^2 + 2}{a^2 - 1}}, 0, \frac{1}{\sqrt{a^2 - 1}}\right), \quad e_2^0 = (0, 1, 0), \quad e_3^0 = \left(\frac{1}{\sqrt{a^2 - 1}}, 0, \sqrt{\frac{a^2 + 2}{a^2 - 1}}\right)
\]

where \(a^2 > 1\). Then, the system (27) describes a spherical spacelike curve \(c : I \to H_0^2\) defined by

\[
c(\sigma) = \left(-\sqrt{\frac{a^2 + 2}{a^2 - 1}}, \frac{1}{\sqrt{a^2 - 1}} \sin \left(\sqrt{a^2 - 1} \sigma\right), \frac{1}{\sqrt{a^2 - 1}} \cos \left(\sqrt{a^2 - 1} \sigma\right)\right)
\]
with \( c(0) = e^0_1 \), in the Minkowski 3-space. Solving the equation (29) we obtain the timelike Frenet curve given by

\[
\alpha(\sigma) = \left( -\frac{\sqrt{a^2+2}}{\sqrt{a^2-1}}\sigma, \frac{1}{a^2-1}\cos\left(\sqrt{a^2-1}\sigma\right), -\frac{1}{a^2-1}\sin\left(\sqrt{a^2-1}\sigma\right) \right).
\]

Let the unit vector \( q(\sigma) \) be timelike vector. Choose another initial conditions

\[
e^0_1 = \left( -\frac{1}{\sqrt{a^2-1}}, 0, \frac{\sqrt{a^2+2}}{\sqrt{a^2-1}} \right), \quad e^0_2 = (0, 1, 0), \quad e^0_3 = \left( \frac{\sqrt{a^2+2}}{\sqrt{a^2-1}}, 0, \frac{1}{\sqrt{a^2-1}} \right)
\]

where \( a^2 > 1 \). Then, the system (28) describes a spherical spacelike curve \( c : I \to S^2_1 \) defined by

\[
c(\sigma) = \left( -\frac{1}{\sqrt{a^2-1}}\cosh\left(\sqrt{a^2-1}\sigma\right), \frac{1}{\sqrt{a^2-1}}\sinh\left(\sqrt{a^2-1}\sigma\right), \frac{\sqrt{a^2+2}}{\sqrt{a^2-1}} \right)
\]

with \( c(0) = e^0_1 \), in the Minkowski 3-space. Solving the equation (29) we obtain the spacelike Frenet curve given by

\[
\alpha(\sigma) = \left( -\frac{1}{a^2-1}\sinh\left(\sqrt{a^2-1}\sigma\right), \frac{1}{a^2-1}\cosh\left(\sqrt{a^2-1}\sigma\right), \frac{\sqrt{a^2+2}}{\sqrt{a^2-1}} \right).
\]

**Example 14** Let \( \alpha : I \to \mathbb{E}_3^1 \) be a non-lightlike Frenet curve with p-shape \((\kappa_\alpha, \tau_\alpha) = (b, a)\) where \( a \neq 0 \) and \( b \neq 0 \) are real constants. Choosing initial conditions (30) as in the Example 13, we get the same spherical timelike curve (31) which is a circle with a radius \( 1/\sqrt{1 + a^2} \). Also, we have \( \rho(\sigma) = \int_0^\sigma b d\sigma = b\sigma \) for \( \sigma \in I \). Solving the equation (29) we obtain

\[
\alpha(\sigma) = \left( \frac{(b - m)e^{(b+m)\sigma} - (b + m)e^{(b-m)\sigma}}{2m(b^2 - m^2)}, \frac{(b + m)e^{(b-m)\sigma} + (b - m)e^{(b+m)\sigma}}{2m(m^2 - b^2)}, \frac{ae^{b\sigma}}{bm} \right)
\]

where \( m = \sqrt{1 + a^2} \).

**Example 15** Let \( \alpha : I \to \mathbb{E}_3^1 \) be a non-lightlike Frenet curve with p-shape \((\kappa_\alpha, \tau_\alpha) = (1/\sigma, a)\) where \( a \neq 0 \) is real constants. Because of \( \rho(\sigma) = \ln\sigma \), the parametric equation of the non-lightlike curve \( \alpha \) is given by

\[
\alpha(\sigma) = \left( \frac{t \cosh t - \sinh t}{(1 + a^2)^{3/2}}, \frac{\cosh t - \sinh t}{(1 + a^2)^{3/2}}, \frac{at^2}{2(1 + a^2)^{3/2}} \right)
\]

where \( t = \sqrt{1 + a^2}\sigma \). As in the Example 13 we take the same spherical timelike curve \( c = c(\sigma) \) parameterized by (31).
References

[1] M. Berger: *Geometry I*. Springer, New York 1998.

[2] R. Encheva and G. Georgiev, *Shapes of space curves*, J. Geom. Graph. 7 (2003), 145-155.

[3] R. P. Encheva and G. H. Georgiev, *Similar Frenet curves*, Results in Mathematics, vol. 55, no. 3-4, pp.359–372, 2009.

[4] G. H. Georgiev, *Rational Generalized Offsets of Rational Surfaces*, Mathematical Problems in Engineering, Volume 2012, 15 pages.

[5] W. Greub, *Linear Algebra*, 3rd ed., Springer Verlag, Heidelberg, 1967.

[6] Jun-Ichi Inoguchi, *Biharmonic curves in Minkowski 3-space*, International Journal of Mathematics and Mathematical Sciences 21 (2003): 1365-1368.

[7] S. Izumiya and N. Takeuchi, *Generic properties of helices and Bertrand curves*, J. Geom. 74 (2002), 97-109.

[8] B. O’Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press Inc., London, 1983.

[9] M. Ozdemir, A. A. Ergin, *Spacelike Darboux Curves in Minkowski 3-Space*, Differ. Geom. Dyn. Syst. 9, 131-137, 2007.