Isolated points of the Zariski space

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Abstract. Let $D$ be an integral domain and $L$ be a field containing $D$. We study the isolated points of the Zariski space $\text{Zar}(L|D)$, with respect to the constructible topology. In particular, we completely characterize when $L$ (as a point) is isolated and, under the hypothesis that $L$ is the quotient field of $D$, when a valuation domain of dimension 1 is isolated; as a consequence, we find all isolated points of $\text{Zar}(D)$ when $D$ is a Noetherian domain and, under the hypothesis that $D$ and $D'$ are Noetherian, local and countable, we characterize when $\text{Zar}(D)$ and $\text{Zar}(D')$ are homeomorphic. We also show that if $V$ is a valuation domain and $L$ is transcendental over $V$ then the set of extensions of $V$ to $L$ has no isolated points.

1. Introduction

Let $D$ be an integral domain with quotient field $K$, and let $L$ be a field containing $K$. The Zariski space of $L$ over $D$, denoted by $\text{Zar}(L|D)$, is the set of all valuation rings containing $D$ and having quotient field $L$. O. Zariski introduced this set (under the name abstract Riemann surface) and endowed it with a natural topology (later called the Zariski topology) during its study of resolution of singularities; in particular, he used the compactness of the Zariski space to reduce the problem of gluing infinitely many projective models to the gluing of only...
finitely many of them [30, 31]. Later on, it was showed that Zar(L|D) enjoys even deeper topological properties: in particular, it is a spectral space, meaning that there is always a ring $R$ such that Spec$(R)$ (endowed with the Zariski topology) is homeomorphic to Zar(L|D), and an example of such an $R$ can be found using the Kronecker function ring construction [5, 6, 8]. Beyond being a very natural example of a spectral space “occurring in nature”, the Zariski topology can also be used, for example, to study representation of integral domains as intersection of overrings [19, 20, 21], or in real and rigid algebraic geometry [15, 24].

As a spectral space, two other topologies can be constructed on Zar(L|D) starting from the Zariski topology: the inverse and the constructible (or patch) topology. Both of them give rise to spectral spaces (in particular, they are compact); furthermore, the constructible topology gains the property of being Hausdorff, and plays an important role in the topological characterization of spectral spaces (see for example Hochster’s article [14]). The constructible topology can also be studied through ultrafilters [7], and this point of view allows to give many examples of spectral spaces, for example by finding them inside other spectral spaces (see [21, Example 2.2(1)] for some very general constructions, [27] for examples in the overring case, and [10, 9] for examples in the setting of semistar operations).

In this paper, we want to study the points of Zar(L|D) that are isolated, with respect to the constructible topology. Our starting point is a new interpretation of a result about the compactness of spaces in the form Zar(K|D) \ {V} [26, Theorem 3.6], where $K$ is the quotient field of $D$: in particular, we show that if $V$ is isolated in Zar(L|D), where $L$ is a field containing $V$, then $V$ is the integral closure of $D[x_1, \ldots, x_n]_M$ for some $x_1, \ldots, x_n \in L$, where $M$ is a maximal ideal of $D[x_1, \ldots, x_n]$ (Theorem 3.4). Through this result, we characterize when $L$ is an isolated point of Zar(L|D)cons (i.e., Zar(L|D) endowed with the constructible topology; Proposition 4.1) and, under the hypothesis that $L = K$ is the quotient field of $D$, when the one-dimensional valuation overrings are isolated (Theorem 5.2).

In Section 6, we study the isolated points of the constructible topology when $D$ is a Noetherian domain and $L = K$ is its quotient field. Theorem 6.3 gives a complete characterization: $V \in$ Zar(K|D) = Zar(D) is isolated if and only if the center $P$ of $V$ on $D$ has height at most 1 and $P$ is contained in only finitely many prime ideals; in particular, this cannot happen if $D$ is local and of dimension at least 3. In the countable case, we also give a complete characterization of when Zar(D)cons \cong Zar(D')cons under the hypothesis that $D$ and $D'$ are Noetherian and local (Theorem 6.11).

The last two sections of the paper explore the case of extension of valuations. Section 7 studies the case where $D$ itself is a field: in particular, we show that if the transcendence degree of $L$ over $D$ is at least 2 then Zar(L|D)cons has no isolated points, improving [3, Theorem 4.45]. In Section 8, we show that if $V$ is a valuation domain that is not a field and $K(X)$ is the field of rational functions
in one indeterminate, then the set of extensions of $V$ to $K(X)$ has no isolated points (Theorem 8.2), and as a consequence we further extend [3, Theorem 4.45] to $\text{Zar}(L|D)^{\text{cons}}$ when $D$ is an arbitrary integral domain (Theorem 7.3 and Corollary 8.6).

2. Notation and preliminaries

Throughout the paper, all rings will be commutative, unitary and will have no zero-divisors (that is, they are integral domains). We usually denote by $D$ such a domain and by $K$ its quotient field; we use $\bar{D}$ to denote the integral closure of $D$ in $K$.

2.1. Spectral spaces. A topological space $X$ is spectral if it is homeomorphic to the prime spectrum of a ring, endowed with the Zariski topology; spectral spaces can also be characterized in a purely topological way (see [14] and [4]). Among their properties, spectral spaces are always compact and have a basis of open and compact sets. If $\Delta \subseteq X$, we denote by $\text{Cl}(\Delta)$ the closure of $\Delta$. The topology of $X$ induces an order such that $x \leq y$ if and only if $y \in \text{Cl}(x)$. If $Y \subseteq X$, the closure under generalization of $Y$ if the set $Y^{\text{gen}} := \{x \in X \mid x \leq y \text{ for some } y \in Y\}$, where $\leq$ is the order induced by the topology, and $Y$ is closed by generalizations if $Y = Y^{\text{gen}}$.

If $X$ is a spectral space, the inverse topology on $X$ is the coarsest topology such that the open and compact subsets of $X$ are closed. We denote by $X^{\text{inv}}$ the space $X$, endowed with the inverse topology. A subset $Y \subseteq X$ is closed in the inverse topology if and only if it is compact in the starting topology and closed by generalizations; in particular, if $Y$ is compact in the starting topology then its closure in the inverse topology is $Y^{\text{gen}}$.

If $X$ is a spectral space, the constructible topology (or patch topology) on $X$ is the coarsest topology such that the open and compact subsets of $X$ are closed. We denote by $X^{\text{cons}}$ the space $X$, endowed with the constructible topology; if $Y \subseteq X$, we denote by $Y^{\text{cons}}$ the subset $Y$ considered with respect to the constructible topology, and by $\text{Cl}^{\text{cons}}(Y)$ the closure of $Y$ in $X^{\text{cons}}$. If $Y = \text{Cl}^{\text{cons}}(Y)$, then $Y$ is compact in the starting topology; conversely, if $Y$ is closed in the starting topology or in the inverse topology, then it is closed also in the constructible topology.

Both $X^{\text{inv}}$ and $X^{\text{cons}}$ are spectral spaces, and in particular compact spaces; moreover, $X^{\text{cons}}$ is Hausdorff and zero-dimensional.

A map $f : X \longrightarrow Y$ of spectral spaces is a spectral map if $f^{-1}(\Omega)$ is open and compact for every open and compact subset $\Omega$ of $Y$; in particular, a spectral map is continuous. If $f$ is both spectral and closed, then it is also proper, and in particular $f^{-1}(\Omega)$ is compact for every compact subset $\Omega$ of $Y [4, 5.3.7(i)]$. If $f : X \longrightarrow Y$ is a spectral map, then it is spectral also when $X$ and $Y$ are both endowed with the inverse topology, and when they are both endowed with the constructible topology [4, Theorem 1.3.21]. In the latter case, $f$ is also closed, since it is a continuous map between Hausdorff compact spaces.
2.2. Isolated points. If \( X \) is a topological space, a point \( p \in X \) is isolated in \( X \) if \( \{p\} \) is an open set. If \( X \) has no isolated points, then \( X \) is said to be perfect. The set of points that are not isolated in \( X \) is a closed set, called the derived set of \( X \).

If \( \Omega \subseteq X \) and \( p \in \Omega \) is isolated in \( X \), then \( p \) is also isolated in \( \Omega \); if \( \Omega \) is open, then \( p \) is isolated in \( X \) if and only if \( p \) is isolated in \( \Omega \).

2.3. Valuation domains. A valuation domain is an integral domain \( V \) such that, for every \( x \neq 0 \) in the quotient field of \( V \), at least one of \( x \) and \( x^{-1} \) is in \( V \). Any valuation domain is local; we denote the maximal ideal of \( V \) by \( \mathfrak{m}_V \). If \( L \) is a field containing the quotient field \( K \) of \( V \), an extension of \( V \) to \( L \) is a valuation domain \( W \) having quotient field \( L \) such that \( W \cap K = V \). We denote the set of extension of \( V \) to \( L \) by \( E(L|V) \); this set is always nonempty (see e.g. [12, Theorem 20.1]).

If \( D \) is an integral domain and \( L \) is a field containing \( D \), the Zariski space (or Zariski-Riemann space) of \( L \) over \( D \), denoted by \( \text{Zar}(L|D) \), is the set of all valuation domains containing \( D \) and having quotient field \( L \). The Zariski space \( \text{Zar}(L|D) \) is always nonempty. When \( L \) is the quotient field of \( D \), we denote \( \text{Zar}(L|D) \) simply by \( \text{Zar}(D) \), and we call its elements the valuation overrings of \( D \).\footnote{An overring of \( D \) is, more generally, a ring contained between \( D \) and its quotient field.} If \( D' \) is the integral closure of \( D \) in \( L \), then \( \text{Zar}(L|D) = \text{Zar}(L|D') \); in particular, \( \text{Zar}(D) = \text{Zar}(D) \). A valuation ring in \( \text{Zar}(L|D) \) is minimal if it is minimal with respect to containment.

The Zariski-Riemann space \( \text{Zar}(L|D) \) can be endowed with a natural topology, called the Zariski topology, which is the topology generated by the basic open sets

\[
\mathcal{B}(x_1, \ldots, x_n) := \{V \in \text{Zar}(L|D) \mid x_1, \ldots, x_n \in V\},
\]

as \( x_1, \ldots, x_n \) range among the elements of \( L \); we use the notation \( \mathcal{B}^L(x_1, \ldots, x_n) \) if we need to underline the field \( L \). Under this topology, \( \text{Zar}(L|D) \) is a spectral space whose order is the opposite of the containment order \([6, 5]\); in particular, the minimal valuation rings in \( \text{Zar}(L|D) \) are maximal with respect to the order induced by the Zariski topology. As a spectral space, we can define the inverse and the constructible topology on \( \text{Zar}(L|D) \); a set \( \Delta \subseteq \text{Zar}(L|D) \) is closed with respect to the inverse topology if and only if it is compact with respect to the Zariski topology and \( \Delta = \{W \in \text{Zar}(L|D) \mid W \supseteq V \text{ for some } V \in \Delta\} \) [8, Remark 2.2 and Proposition 2.6].

Since \( \mathcal{B}(z_1, \ldots, z_n) = \mathcal{B}(z_1) \cap \cdots \cap \mathcal{B}(z_n) \) for every \( z_1, \ldots, z_n \in L \), a basis of the constructible topology of \( \text{Zar}(L|D) \) is the family of the sets in the form \( \mathcal{B}(x_1, \ldots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c \), as \( x_1, \ldots, x_n, y_1, \ldots, y_m \) range in \( L \). In particular, \( V \) is isolated in \( \text{Zar}(L|D)^{\text{cons}} \) if and only if

\[
\{V\} = \mathcal{B}(x_1, \ldots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c = \text{Zar}(L|D[x_1, \ldots, x_n]) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c
\]

for some \( x_1, \ldots, x_n, y_1, \ldots, y_m \in L \).
If \( L' \subseteq L \) is a field extension and \( D \subseteq L' \), we have a restriction map
\[
\rho : \text{Zar}(L|D) \rightarrow \text{Zar}(L'|D),
\]
\[
V \mapsto V \cap L'.
\]
The map \( \rho \) is surjective due to Chevalley’s extension theorem (see e.g. [1, Theorem 5.21] or [12, Theorem 19.5]), and is a spectral map since \( \rho^{-1}(B^L(x)) = B^L(x) \). Therefore, it is spectral and closed with respect to the constructible topology (on both sets). In particular, if \( V \in \text{Zar}(L'|D) \), then \( \mathcal{E}(L|V) = \rho^{-1}(V) \); hence, \( \mathcal{E}(L|V) \) is always closed in \( \text{Zar}(L|D) \text{cons} \), and in particular it is compact both in the Zariski and the constructible topology.

Since, by definition, the spectrum \( \text{Spec}(D) \) is a spectral space (when endowed with the Zariski topology), we can define the inverse and the constructible topology also on \( \text{Spec}(D) \). For every ideal \( I \) of \( D \), set \( \mathcal{V}(I) := \{ P \in \text{Spec}(D) \mid I \subseteq P \} \) and \( \mathcal{D}(I) := \text{Spec}(D) \setminus \mathcal{V}(I) \); then, a basis of \( \text{Spec}(D) \text{cons} \) is given by the intersections \( \mathcal{D}(aD) \cap \mathcal{V}(I) \), as \( a \) ranges in \( D \) and \( I \) among the finitely generated ideals of \( D \) [4, Theorem 12.1.10(iv)].

For every field \( L \), we can define a map
\[
\gamma : \text{Zar}(L|D) \rightarrow \text{Spec}(D),
\]
\[
V \mapsto m_V \cap D,
\]
which is called the center map. When \( \text{Zar}(L|D) \) and \( \text{Spec}(D) \) are endowed with the Zariski topology, \( \gamma \) is spectral (in particular, continuous; see [32, Chapter VI, §17, Lemma 1] or [5, Theorem 4.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [12, Theorem 19.6]) and closed [5, Theorem 2.5], so in particular it is proper. Therefore, \( \gamma \) is a spectral map also when \( \text{Zar}(L|D) \) and \( \text{Spec}(D) \) are endowed with their respective constructible topologies.

3. General results

We begin by establishing some general criteria to determine which valuation domains are isolated in \( \text{Zar}(D) \).

Let \( D \) be an integral domain: a prime ideal is called essential if \( D_P \) is a valuation domain, and \( D_P \) is said to be an essential valuation overring of \( D \). We shall need the following weaker notion: we say that a prime ideal \( P \) of \( D \) is almost essential if there is a unique valuation overring of \( D \) having center \( P \); equivalently, \( P \) is almost essential if and only if the integral closure of \( D_P \) is a valuation domain \( V \). When this happens, we say that \( V \) is an almost essential valuation overring of \( D \).

In the context of almost essential primes and valuation overrings, isolated valuation rings correspond to isolated prime ideals.

**Proposition 3.1.** Let \( D \) be an integral domain, and let \( P \) be an almost essential prime ideal of \( D \); let \( V \) be the valuation overring with center \( P \). Then, \( V \) is isolated in \( \text{Zar}(D) \text{cons} \) if and only if \( P \) is isolated in \( \text{Spec}(D) \text{cons} \).
Proof. Let \( \gamma : \text{Zar}(D) \rightarrow \text{Spec}(D) \) be the center map. If \( P \) is isolated in \( \text{Spec}(D)^{\text{cons}} \), then \( \{P\} \) is open and thus, as \( \gamma \) is continuous, \( \{V\} = \gamma^{-1}(\{P\}) \) is open in \( \text{Zar}(D)^{\text{cons}} \), i.e., \( V \) is isolated. Conversely, if \( V \) is isolated then \( \text{Zar}(D) \setminus \{V\} \) is closed, with respect to the constructible topology, and thus \( \gamma(\text{Zar}(D) \setminus \{V\}) = \text{Spec}(D) \setminus \{P\} \) is closed in \( \text{Spec}(D)^{\text{cons}} \). Hence, \( \{P\} \) is open and \( P \) is isolated in \( \text{Spec}(D)^{\text{cons}} \), as claimed.

Corollary 3.2. Let \( D \) be a Prüfer domain, and let \( V \) be a valuation overring of \( D \) with center \( P \). Then, \( V \) is isolated in \( \text{Zar}(D)^{\text{cons}} \) if and only if \( P \) is isolated in \( \text{Spec}(D)^{\text{cons}} \). In particular, \( \text{Zar}(D)^{\text{cons}} \) is perfect if and only if \( \text{Spec}(D)^{\text{cons}} \) is perfect.

Proof. Since \( D \) is a Prüfer domain, every valuation overring is essential. The claim follows from Proposition 3.1.

In general, almost essential valuation overrings are rare; for example, if \( D \) is Noetherian, no prime ideal of height 2 or more can be almost essential. For this reason, we need more general results; the first step is connecting isolated valuation rings with compactness.

Proposition 3.3. Let \( X \) be a spectral space, and let \( x \) be a maximal element with respect to the order induced by the topology. Then, the following are equivalent:

(i) \( x \) is isolated in \( X^{\text{cons}} \);
(ii) \( X \setminus \{x\} \) is compact, with respect to the starting topology;
(iii) \( X \setminus \{x\} \) is closed, with respect to the inverse topology.

Proof. Let \( Y := X \setminus \{x\} \).

The equivalence of (ii) and (iii) follows from the fact that \( Y \) is closed by generalizations.

If (i) holds, then \( \{x\} \) is an open set in the constructible topology, and thus \( Y \) is closed; since \( X^{\text{cons}} \) is compact, it follows that \( Y \) is compact in the constructible topology and thus also in the Zariski topology (which is coarser). Thus, (ii) holds.

Conversely, if (iii) holds, then \( Y \) is closed also in the constructible topology; hence, \( \{x\} \) is open and \( x \) is isolated. Thus, (i) holds.

In particular, the previous proposition applies when \( X = \text{Zar}(L|D) \) and \( V \) is a minimal element of \( \text{Zar}(L|D) \), with respect to containment. In this case, the fact that \( \text{Zar}(L|D) \setminus \{V\} \) is compact has very strong consequences.

Theorem 3.4. Let \( D \) be an integral domain and let \( V \in \text{Zar}(L|D) \). Then, the following are equivalent.

(i) \( V \) is isolated in \( \text{Zar}(L|D)^{\text{cons}} \);
(ii) there are \( x_1, \ldots, x_n \in L \) and a maximal ideal \( M \) of \( D[x_1, \ldots, x_n] \) such that \( V \) is the integral closure of \( D[x_1, \ldots, x_n]_M \) and \( M \) is isolated in \( \text{Spec}(D[x_1, \ldots, x_n])^{\text{cons}} \),
(iii) there are $x_1, ..., x_n \in L$ and a prime ideal $P$ of $D[x_1, ..., x_n]$ such that $V$ is the integral closure of $D[x_1, ..., x_n]_P$ and $P$ is isolated in $\text{Spec}(D[x_1, ..., x_n])^{\text{cons}}$.

**Proof.** Let $X$ be an indeterminate over $D$, and let $R := D + XL[[X]]$. By the reasoning in the proof of [28, Proposition 3.3] (or by Lemma 4.2 below) the Zariski space $\text{Zar}(L|D)^{\text{cons}}$ is homeomorphic to $(\text{Zar}(R) \setminus \{L((X))\})^{\text{cons}}$, which is open in $\text{Zar}(R)^{\text{cons}}$; in particular, a $W \in \text{Zar}(L|D)$ is isolated with respect to the constructive topology if and only if $W + XL[[X]]$ is isolated in $\text{Zar}(R)^{\text{cons}}$. Therefore, without loss of generality we can suppose that $L$ is the quotient field of $D$.

(i) $\implies$ (iii) Since $V$ is isolated, there are $x_1, ..., x_k, y_1, ..., y_m \in L$ such that $\{V\} = \text{Zar}(D[x_1, ..., x_k]) \cap \mathcal{B}(y_1)^c \cap ... \cap \mathcal{B}(y_m)^c$. In particular, $V$ is a minimal valuation overring of $D[x_1, ..., x_k]$. By Proposition 3.3, $\text{Zar}(D[x_1, ..., x_k]) \setminus \{V\}$ is compact, with respect to the Zariski topology; therefore, by [26, Theorem 3.6], there are $x_{k+1}, ..., x_n \in L$ such that $V$ is the integral closure of

$$D[x_1, ..., x_k][x_{k+1}, ..., x_n]_M = D[x_1, ..., x_n]_M$$

for some maximal ideal $M$ of $D[x_1, ..., x_n]$. Hence, $M$ is almost essential in $D[x_1, ..., x_n]$, and by Proposition 3.1, $M$ is isolated in $\text{Spec}(D[x_1, ..., x_n])^{\text{cons}}$. Thus (ii) holds.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (i) The set $\text{Zar}(D[x_1, ..., x_n]) = \mathcal{B}(x_1, ..., x_n)$ is open in the constructive topology, and thus $V$ is isolated in $\text{Zar}(D)^{\text{cons}}$ if and only if it is isolated in $\text{Zar}(D[x_1, ..., x_n])^{\text{cons}}$. By hypothesis, $P$ is almost essential for $D[x_1, ..., x_n]$, and thus by Proposition 3.1 the integral closure $V$ of $D[x_1, ..., x_n]_P$ is isolated, as claimed.

**4. Dimension 0**

In this section, we study when the field $L$ is isolated in $\text{Zar}(L|D)^{\text{cons}}$. If $L$ is the quotient field of $D$, then $L$ is an essential valuation overring of $D$, and thus one can reason through Proposition 3.1; however, it is possible to use a more general approach.

A domain $D$ with quotient field $K$ is said to be a **Goldman domain** (or a $G$-domain) if $K$ is a finitely generated $D$-algebra, or equivalently if $K = D[u]$ for some $u \in K$.

**Proposition 4.1.** Let $D$ be an integral domain with quotient field $K$, and let $L$ be a field extension of $K$. Then, $L$ is isolated in $\text{Zar}(L|D)^{\text{cons}}$ if and only if $D$ is a Goldman domain and $K \subseteq L$ is an algebraic extension.

**Proof.** Suppose first that the two conditions hold. Then, $K = D[u]$ for some $u \in K$; since $K \subseteq L$ is algebraic, it follows that $\mathcal{B}(u) = \text{Zar}(L|K) = \{L\}$. Hence, $L$ is isolated in $\text{Zar}(L|D)^{\text{cons}}$.

Conversely, suppose that $L$ is isolated. By Theorem 3.4, there are $x_1, ..., x_n \in L$ such that $L$ is the integral closure of $D[x_1, ..., x_n]_M$ for some maximal ideal
\(M\); since \(M\) must have height 0, \(F := D[x_1, \ldots, x_n]\) must be a field such that \(F \subseteq L\) is algebraic.

Suppose that \(F\) is transcendental over \(K\): then, we can take a transcendence basis \(y_1, \ldots, y_k\) of \(F\) over \(K\). By construction, \(F\) is algebraic over the quotient field of \(D[y_1, \ldots, y_k]\); since \(F\) is a field, it is a Goldman domain, and thus by [16, Theorem 22] so should be \(D[y_1, \ldots, y_k]\), against [16, Theorem 21]. Thus, \(F\) is algebraic over \(K\). Applying again [16, Theorem 22] to the extension \(D \subseteq F\), we see that \(D\) is a Goldman domain; furthermore, \(L\) is algebraic over \(F\) and thus over \(K\). The claim is proved. \(\square\)

The previous result can be used to give some necessary conditions for \(V\) to be isolated. We premise a lemma.

**Lemma 4.2.** Let \(D\) be an integral domain, \(L\) be a field containing \(D\), and let \(W \in \text{Zar}(L|D)\). Let \(\pi : W \to W/m_W\) be the quotient map. Then, the map

\[
\overline{\pi} : \{Z \in \text{Zar}(L|D) \mid Z \subseteq W\} \to \text{Zar}(W/m_W|D/(m_W \cap D)),
\]

\(Z \mapsto \pi(Z)\)

is a homeomorphism, when both sets are endowed with either the Zariski or the constructible topology.

**Proof.** Let \(Z \in \text{Zar}(L|D)\): then, \(\ker \pi = m_W \subseteq Z\) since \(Z\) and \(W\) are valuation domains with the same quotient field and \(Z \subseteq W\). Hence, \(\pi(Z) = Z/m_W\) is a valuation ring containing \(D/(m_W \cap D)\); moreover, since \(W\) is a localization of \(Z, W/m_W\) is a localization of \(Z/m_W\) and thus \(W/m_W\) is the quotient field of \(\pi(Z)\). Hence, \(\overline{\pi}\) is well-defined.

Moreover, if \(Z' \in \text{Zar}(W/m_W|D/(m_W \cap D))\), then \(\overline{\pi}^{-1}(Z') = \pi^{-1}(Z')\) is the pullback of \(Z'\) along the quotient map \(W \to W/m_W\). Thus, \(Z\) is a valuation domain by [11, Proposition 1.1.8(1)], and its quotient field is \(L\) by [11, Lemma 1.1.4(10)]. Hence \(\overline{\pi}\) is surjective. Furthermore, if \(Z \in \text{Zar}(L|D)\) and \(Z \subseteq W\), then \(\ker \pi \subseteq Z\) and thus \(\pi^{-1}(\pi(Z)) = Z\); hence, \(\overline{\pi}\) is bijective.

Let now \(x \in W/m_W\). Then, \(Z \in \overline{\pi}^{-1}(\mathcal{B}(x))\) if and only if \(x \in \pi(Z)\). Since \(\ker \pi \subseteq Z\), this happens if and only if \(Z\) contains all of \(\pi^{-1}(x)\); thus, for every \(y \in \pi^{-1}(x)\), we have \(\overline{\pi}^{-1}(\mathcal{B}(x)) = \mathcal{B}(y)\), and likewise \(\overline{\pi}(\mathcal{B}(y)) = \mathcal{B}(\pi(x))\) for every \(x \in L\). Hence, \(\overline{\pi}\) is continuous and open when both \(\{Z \in \text{Zar}(L|D) \mid Z \subseteq W\}\) and \(\text{Zar}(W/m_W|D/(m_W \cap D))\) are endowed with the Zariski topology, and thus it is a homeomorphism. It follows that it is also a homeomorphism when both sets are endowed with the constructible topology, as claimed. \(\square\)

**Proposition 4.3.** Let \(V \in \text{Zar}(D)\) be a valuation domain with center \(P\) on \(D\). If \(V\) is isolated in \(\text{Zar}(D)_{\text{cons}}\), then the field extension \(D_P/PD_P \subseteq V/m_V\) is algebraic.

**Proof.** Consider \(\Delta := \{W \in \text{Zar}(D) \mid W \subseteq V\}\). Since \(m_V \cap D = P\), by Lemma 4.2, the quotient map \(V \to V/m_V\) induces a homeomorphism between \(\Delta_{\text{cons}}\) and \(\text{Zar}(V/m_V|D/P)_{\text{cons}}\), and thus \(V/m_V\) is isolated in \(\text{Zar}(V/m_V|D/P)_{\text{cons}}\). Let \(F\) be the quotient field of \(D/P\): then, \(F = (D/P)_P = D_P/PD_P\). By Proposition 4.1, \(F \subseteq V/m_V\) must be algebraic, as claimed. \(\square\)
Corollary 4.4. Let $D$ be an integral domain, let $\gamma : \text{Zar}(D) \to \text{Spec}(D)$ be the center map and let $V \in \text{Zar}(D)$. If $V$ is isolated in $\text{Zar}(D)^{\text{cons}}$, then $V$ is minimal in $\gamma^{-1}(\gamma(V))$.

Proof. Let $P := \gamma(V)$. If $V$ is not minimal, then $V/\mathfrak{m}_V$ is not minimal in $\text{Zar}(V/\mathfrak{m}_V|D_P/P D_P)$; hence, the extension $D_P/P D_P \subseteq V/\mathfrak{m}_V$ cannot be algebraic, against Proposition 4.3.

5. Dimension 1

We now analyze the case where the valuation ring $V$ has (Krull) dimension 1; however, the methods we use only work when $V$ is a valuation overring of $D$, i.e., only for the space $\text{Zar}(D) = \text{Zar}(K|D)$, where $K$ is the quotient field of $D$. Unlike in the proof of Theorem 3.4, we cannot use [28, Proposition 3.3] to extend these results to arbitrary Zariski spaces $\text{Zar}(L|D)$, because that construction changes the dimension of the valuation domains involved.

The idea of this section is to study the maximal ideals of the finitely generated algebras $D[x_1, \ldots, x_n]$.

Proposition 5.1. Let $(D, \mathfrak{m})$ be an integrally closed local domain, and let $T \neq D$ be a finitely generated $D$-algebra contained in the quotient field $K$ of $D$. If $\mathfrak{m}T \neq T$, then no maximal ideal of $T$ above $\mathfrak{m}$ has height 1.

Proof. Let $T := D[x_1, \ldots, x_n]$, $\mathfrak{m}$; we proceed by induction on $n$.

Suppose $n = 1$, and let $x := x_1$; then, $x \notin D$. If $x^{-1} \notin D$, then $x \in \mathfrak{m}$, and thus $\mathfrak{m}T = T$, a contradiction. Hence, $x, x^{-1} \notin D$. By [25, Theorem 6], the ideal $\mathfrak{p} := \mathfrak{m}T$ is prime but not maximal; since every maximal ideal of $T$ above $\mathfrak{m}$ must contain $\mathfrak{p}$, it follows that no such maximal ideal can have height 1.

Suppose that the claim holds up to $n - 1$; let $A := D[x_1, \ldots, x_{n-1}]$, so that $T = A[x_n]$; without loss of generality, $A \neq D$ and $x_n \notin A$. Let $M$ be a maximal ideal of $T$ above $\mathfrak{m}$. If $x_n$ is integral over $A$, then $T$ is integral over $A$, and thus the height of $M$ is equal to the height of $M \cap A$, which is not equal to 1 by induction.

Suppose that $x_n$ is not integral over $A$. Let $A'$ be the integral closure of $A$; then, $T \subseteq A'[x_n]$ is an integral extension, and since $x_n$ is not integral over $A$ it follows that $A' \subseteq A'[x_n]$. Take a maximal ideal $M'$ of $A'[x_n]$ above $M$. Let $N := M' \cap A'$; then, $N$ is a nonzero prime ideal of $A'$, and thus $A'' := (A')_N$ is a local integrally closed domain with maximal ideal $(A')_N \neq (0)$. Then, the ring $A''[x_n]$ is the quotient ring of $A'[x_n]$ with respect to the multiplicatively closed set $A'[x_n] \setminus N$, the set $M'' := M'A''[x_n]$ is a maximal ideal, and $N(A')_N \subseteq M''$. Applying the case $n = 1$ to $A''$ and $A''[x_n]$, it follows that the height of $M''$ is not 1; since the height of $M''$ is the same of the height of $M'$ and of $M$, it follows that the height of $M$ is not 1, as claimed.

Theorem 5.2. Let $D$ be an integral domain, and let $V \in \text{Zar}(D)$ be a valuation overring of dimension 1. Then, $V$ is isolated in $\text{Zar}(D)^{\text{cons}}$ if and only if $V$ is a localization of $D$ and its center on $D$ is isolated in $\text{Spec}(D)^{\text{cons}}$. 

Proof. Since Zar(D) = Zar(\overline{D}), we can suppose without loss of generality that D is integrally closed.

If the two conditions hold, then V is isolated by Proposition 3.1.

Suppose that V is isolated in Zar(D)_{cons}. Let P be the center of V on D, and suppose that V \neq D_p. Since V is also isolated in Zar(D_p)_{cons}, by Theorem 3.4 there are x_1, ..., x_n \in K \setminus D_p such that V is the integral closure of D_p[x_1, ..., x_n]_M, where M is a maximal ideal of D_p[x_1, ..., x_n]. However, m_V \cap D_p[x_1, ..., x_n] = M, and thus M \cap D_p = PD_p, so that PD_p \cdot D_p[x_1, ..., x_n] \neq D_p[x_1, ..., x_n]; by Proposition 5.1, M cannot have height 1. However, the dimension of the integral closure of D_p[x_1, ..., x_n]_M is exactly the height of M; hence, this contradicts the fact that V has dimension 1. Thus, V = D_p. The fact that P is isolated in Zar(D)_{cons} now follows from Proposition 3.1.

Corollary 5.3. Let D be an integral domain, and let V \in Zar(D) be a minimal valuation overring of D. If \text{dim}(V) = 1 and V is isolated in Zar(D)_{cons}, then the center of V on D has height 1.

Proof. The claim is a direct consequence of Theorem 5.2.

Theorem 5.2 does not work when V has dimension 2 or more, as the next example shows.

Example 5.4. Let F be a field, take two independent indeterminates X and Y, and consider D := F + XF(Y)[[X]], i.e., D is the ring of all power series with coefficients in F(Y) such that the 0-degree coefficient belongs to F. Then, D is a one-dimensional local integrally closed domain (its maximal ideal is XF(Y)[[X]]), and its valuation overrings are its quotient field, F(Y)[[X]] and the rings in the form W + XF(Y)[[X]], where W belongs to Zar(F(Y)[F])\{F(Y)\}, i.e., W is either F[Y]_{(f)} for some irreducible polynomial f \in F[Y] or W = F[Y^{-1}]_{(f^{-1})}.

Each of these W + XF(Y)[[X]] is isolated in Zar(D)_{cons}, since each W is isolated in Zar(F(Y)[F]) (this follows, for example, by applying Theorem 6.3 below to F[Y] or to F[Y^{-1}]). However, since every W + XF(Y)[[X]] has dimension 2, it can’t be a localization of D = \overline{D}.

6. The Noetherian case

In this section, we want to characterize the isolated points of Zar(D)_{cons} when D is a Noetherian domain. If D is integrally closed, this is a straightforward consequence of Theorem 5.2; to extend it to the non-integrally closed case, we need a few lemmas. (Note that the integral closure of a Noetherian domain is not necessarily Noetherian; see e.g. [18, Example 5, page 209].)

Lemma 6.1. Let D be an integral domain. Let P be a prime ideal of D and let \Delta \subseteq \text{Spec}(D). If \text{P} = \bigcap\{Q \mid Q \in \Delta\}, then \text{P} \in \text{Cl}^{\text{cons}}(\Delta).

Proof. Let \Omega = \text{D(aD)} \cap \forall(J) be a basic subset of \text{Spec}(D)_{cons} containing P, where a \in D and J is a finitely generated ideal. We claim that \Omega \cap \Delta \neq \emptyset.
Indeed, $\Delta \subseteq \mathcal{V}(J)$ since $J \subseteq P$ and $P \subseteq Q$ for every $Q \in \Delta$. Moreover, since $a \notin P$, there must be a $\overline{Q} \in \Delta$ such that $a \notin \overline{Q}$; thus, $\overline{Q} \in \mathcal{D}(aD) \cap \mathcal{V}(J) \cap \Delta = \Omega \cap \Delta$. In particular, $\Omega \cap \Delta \neq \emptyset$ and $P \in \text{Cl}^\text{cons}(\Delta)$.

**Lemma 6.2.** Let $A \subseteq B$ be an integral extension, and let $P \in \text{Spec}(A)$, $Q \in \text{Spec}(B)$ be such that $Q \cap A = P$. If $\bigcap\{P' \in \text{Spec}(A) \mid P' \supseteq P\} = P$, then $\bigcap\{Q' \in \text{Spec}(B) \mid Q' \supseteq Q\} = Q$.

**Proof.** Let $I := \bigcap\{Q' \in \text{Spec}(B) \mid Q' \supseteq Q\}$, and suppose $I \neq Q$; then, $Q \subseteq I$ and $\mathcal{V}(I) = \mathcal{V}(Q) \setminus \{Q\}$. Consider the canonical map of spectra $\phi : \text{Spec}(B) \longrightarrow \text{Spec}(A)$: then, $\phi$ is closed (with respect to the Zariski topology) [2, Chapter V, §2, Remark (2)], and thus $\phi(\mathcal{V}(I))$ is closed in $\text{Spec}(A)$.

By the lying over and the going up theorems, every $P' \supseteq P$ belongs to $\phi(\mathcal{V}(I))$, while $P \notin \phi(\mathcal{V}(I))$; hence, $\phi(\mathcal{V}(I)) = \mathcal{V}(P) \setminus \{P\}$. However, the condition $\bigcap\{P' \in \text{Spec}(A) \mid P' \supseteq P\} = P$ shows that $\mathcal{V}(P) \setminus \{P\}$ is not closed (its closure is $\mathcal{V}(P)$), a contradiction. Hence, $I = Q$, as claimed.

**Theorem 6.3.** Let $D$ be a Noetherian domain, and let $V \in \text{Zar}(D)$; let $P$ be the center of $V$ on $D$. Then, $V$ is isolated in $\text{Zar}(D)^\text{cons}$ if and only if $\text{h}(P) \leq 1$ and $\mathcal{V}(P)$ is finite.

**Proof.** Suppose first that $V$ is isolated in $\text{Zar}(D)^\text{cons}$.

If $\text{dim}(V) > 1$, then $V$ is not Noetherian. By Theorem 3.4, $V$ is the integral closure of $D[x_1, \ldots, x_n]$, for some $x_1, \ldots, x_n \in V$ and some maximal ideal $M$. However, $D[x_1, \ldots, x_n]$ is Noetherian, and thus so is $D[x_1, \ldots, x_n]$: hence, its integral closure is a Krull domain, which can't be a non-Noetherian valuation domain, a contradiction.

If $\text{dim}(V) = 0$, then $V = K$. By Proposition 4.1, $D$ must be a Goldman domain; by [16, Theorem 146], $\mathcal{V}(P)$ is finite.

If $\text{dim}(V) = 1$, then by Theorem 5.2 $V$ is the localization of $\overline{D}$ at a prime ideal of $Q$ of height 1; hence, $V$ is an essential prime ideal of $\overline{D}$ and thus $Q$ is isolated in $\text{Spec}(\overline{D})^\text{cons}$ by Proposition 3.1.

Let $P := Q \cap D$. If $\mathcal{V}(P)$ is infinite, then $P$ is the intersection of all the prime ideals properly containing it (since $D/P$ is not a Goldman domain); by Lemma 6.2, the same property holds for $Q$, and thus by Lemma 6.1, $Q$ is not isolated in $\text{Spec}(D)^\text{cons}$. This is a contradiction, and thus $\mathcal{V}(P)$ must be finite.

Conversely, suppose the two conditions hold and let $\mathcal{V}(P) = \{P, Q_1, \ldots, Q_n\}$. For each $i$, let $y_i \in Q_i \setminus P$ and let $x_i := 1/y_i$; then, $A := D[x_1, \ldots, x_n]$ is a Noetherian domain such that $PA$ is a maximal ideal of $A$ of height $\leq 1$; moreover, since $\mathfrak{m}_V \cap D = P$, each $x_i$ belongs to $V$, and thus $V \subseteq \text{Zar}(A)$ and $\mathfrak{m}_V \cap A = PA$.

The subspace $\text{Zar}(A) = \mathcal{B}(x_1, \ldots, x_n)$ is an open set of $\text{Zar}(D)^\text{cons}$; therefore, all isolated points of $\text{Zar}(A)^\text{cons}$ are also isolated in $\text{Zar}(D)^\text{cons}$.

If $P$ has height 0, then $A = K = V$ and thus $V$ is isolated. Suppose that $\text{h}(P) = 1$.

Since $A$ is Noetherian, $\{PA\} = \mathcal{V}(PA)$ is an open subset of $\text{Spec}(A)^\text{cons}$; hence, $\gamma_A^{-1}(PA)$ is an open subset of $\text{Zar}(A)^\text{cons}$, where $\gamma_A : \text{Zar}(A) \longrightarrow \text{Spec}(A)$.
Proposition 6.5. Let \( \mathfrak{m} \) be a Noetherian local domain of dimension at least 3. Then, \( \text{Zar}(\mathfrak{m}) \) is perfect.

Proof. Suppose \( V \) is isolated in \( \text{Zar}(\mathfrak{m}) \). By Theorem 6.3, its center \( P \) must have height 1 and \( V(P) \) must be finite. However, since \( P \) has height 1 and the maximal ideal \( \mathfrak{m} \) of \( D \) has height at least 3, there is at least one prime ideal between \( P \) and \( \mathfrak{m} \), and since \( D \) is Noetherian there must be infinitely many of them [16, Theorem 144], a contradiction. Hence, no \( V \) can be isolated, and \( \text{Zar}(\mathfrak{m}) \) is perfect. \( \square \)

We now want to show that, when \( D \) is countable, there are few possible topological structures for \( \text{Zar}(\mathfrak{m}) \). The one-dimensional case is very easy.

Proposition 6.5. Let \( (D, \mathfrak{m}) \) and \( (D', \mathfrak{m}') \) be two Noetherian local domains of dimension 1. The following are equivalent:

(i) \( |\text{Max}(D)| = |\text{Max}(D')| \);
(ii) \( \text{Zar}(D) \approx \text{Zar}(D') \);
(iii) \( \text{Zar}(D)^{\text{cons}} \approx \text{Zar}(D')^{\text{cons}} \).

Proof. Since \( D \) is Noetherian and one-dimensional, \( \text{Max}(D) \) is a principal ideal domain with finitely many maximal ideals; hence, \( \text{Zar}(D) = \text{Zar}(\text{Max}(D)) \approx \text{Spec}(\text{Max}(D)) \), and the homeomorphism holds both in the Zariski and in the constructible topology.

Hence, if \( |\text{Max}(D)| = |\text{Max}(D')| \) then \( \text{Spec}(D) \approx \text{Spec}(D') \) and thus \( \text{Zar}(D) \) and \( \text{Zar}(D') \) are homeomorphic in both the Zariski and the constructible topology. Conversely, if \( \text{Zar}(D) \approx \text{Zar}(D') \) (in any of the two topologies) then in particular they have the same cardinality, which is equal to \( |\text{Max}(D)| + 1 = |\text{Max}(D')| + 1 \); thus, \( |\text{Max}(D)| = |\text{Max}(D')| \). The claim is proved. \( \square \)

For larger dimension, we need to join the previous theorems with the topological characterization of the Cantor set. We isolate a lemma.

Lemma 6.6. Let \( D \) be a countable domain. Then, \( \text{Zar}(D)^{\text{cons}} \) is metrizable.

Proof. The space \( \text{Zar}(D)^{\text{cons}} \) is compact and Hausdorff, hence normal [29, Theorem 17.10] and, in particular, regular. Furthermore, the family of sets \( B(t) \) and \( B(t)^{c} \) (as \( t \) ranges in the quotient field of \( D \)) form a subbasis of \( \text{Zar}(D)^{\text{cons}} \), and thus \( \text{Zar}(D)^{\text{cons}} \) is second countable. By Urysohn’s metrization theorem (see e.g. [29, Theorem 23.1]), \( \text{Zar}(D)^{\text{cons}} \) is metrizable. \( \square \)
Lemma 6.9. Let $m$ be a local Noetherian domain of dimension at least 3. Then, $\text{Zar}(D)^\text{cons} \simeq \text{Zar}(D')^\text{cons}$.

Proof. Both $\text{Zar}(D)^\text{cons}$ and $\text{Zar}(D')^\text{cons}$ are Boolean spaces, hence totally disconnected and compact; they are also perfect by Corollary 6.4 and metrizable by Lemma 6.6.

By [29, Theorem 30.3], any two spaces with these properties are homeomorphic; hence, $\text{Zar}(D)^\text{cons} \simeq \text{Zar}(D')^\text{cons}$. □

To study the case of dimension 2, we need two further lemmas.

Lemma 6.8. Let $(D, m)$ be a local Noetherian domain with $\dim(D) > 1$. If $D$ is countable, then $D$ has exactly countably many prime ideals of height 1.

Proof. By [16, Theorem 144], there are infinitely many prime ideals between $(0)$ and $m$, and thus $D$ has infinitely many prime ideals of height 1.

Moreover, every prime ideal is generated by a finite set, and thus the number of prime ideals of height 1 is at most equal to the number of finite subsets of $D$. Since $D$ is countable, so is the set of its finite subsets; the claim is proved. □

Lemma 6.9. Let $(D, m)$ be a local Noetherian domain of dimension 2 with quotient field $K$, and let $X$ be the set of isolated points of $\text{Zar}(D)^\text{cons}$. Then:

(a) a valuation overring of $D$ belongs to $X$ if and only if its center has height 1;
(b) $X$ is nonempty and compact, with respect to the Zariski topology;
(c) if $D$ is countable, then $X$ is countable;
(d) $\text{Cl}^\text{cons}(X) = X \cup \{K\}$;
(e) the only isolated point of $(\text{Zar}(D) \setminus X)^\text{cons}$ is $K$;
(f) $\text{Zar}(D) \setminus (X \cup \{K\})$ is closed and perfect, with respect to the constructible topology.

Proof. (a) Let $V \in \text{Zar}(D)$. If $V$ is isolated, then its center has height at most 1 by Theorem 6.3, but the height can’t be 0 since $V((0))$ is infinite. Conversely, if $P := m \cap D$ has height 1, then $V(P) = \{P, m\}$ is finite, and thus $V \in X$, by Theorem 6.3.

(b) Let $X_1$ be the set of all height 1 prime ideals of $D$: by the previous point, $X = \gamma^{-1}(X_1)$. Since $\gamma$ is surjective, and $X_1$ is nonempty, also $X$ is nonempty. Furthermore, since $D$ is a Noetherian ring, $\text{Spec}(D)$ is a Noetherian space with respect to the Zariski topology (i.e., all its subsets are compact; see [4, Theorem 12.4.3] or [1, Chapter 6, Exercises 5–8]). Since $\gamma$ is a spectral closed map, it is proper, and thus the counterimage of any compact subset of $\text{Spec}(D)$ is compact; therefore, $X = \gamma^{-1}(X_1)$ is compact with respect to the Zariski topology, as claimed.

(c) By Lemma 6.8, $X_1$ is countable; furthermore, $\gamma^{-1}(P)$ is finite for every $P \in X_1$, since it is in bijective correspondence with the set of maximal ideals of the integral closure of $D_P$. Since $X = \gamma^{-1}(X_1)$, it follows that $X$ is countable.

(d) Since $X$ is compact, the set $X^\text{gen} = \{W \in \text{Zar}(D) \mid W \supseteq V \text{ for some } V \in X\}$ is closed in the inverse topology, and thus in the constructible topology; since
every element of $X$ is a one-dimensional valuation ring, furthermore, $X^{\text{gen}} = X \cup \{K\}$. Hence, $C^{\text{cons}}(X) \subseteq X \cup \{K\}$.

If they are not equal, then $C^{\text{cons}}(X) = X$. However, $X$ is infinite (since $X_1$ is infinite, by Lemma 6.8) and discrete (by definition, all its points are isolated) and thus it is not compact with respect to the constructible topology; this is a contradiction, since a closed set of a compact set is compact. Thus, $C^{\text{cons}}(X) = X \cup \{K\}$, as claimed.

(e) The set $\text{Zar}(D) \setminus (X \cup \{K\})$ is open, with respect to the constructible topology (by part (d)), and its elements are not isolated in $\text{Zar}(D)^{\text{cons}}$, therefore, none of its elements can be isolated in $(\text{Zar}(D) \setminus X)^{\text{cons}}$. On the other hand, let $x \in m, x \neq 0$: then, $D[x^{-1}]$ is a Noetherian domain of dimension 1, and its maximal ideals are extensions of prime ideals of $D$ of height 1. Therefore, if $V \in \text{Zar}(D[x^{-1}])$ has dimension 1 then the center of $V$ on $D$ has height 1, and thus it is an isolated point of $\text{Zar}(D)$, i.e., $\mathcal{B}(x^{-1}) = \text{Zar}(D[x^{-1}]) \subseteq X \cup \{K\}$, and $\mathcal{B}(x^{-1}) \cap (\text{Zar}(D) \setminus X) = \{K\}$. Since $\mathcal{B}(x^{-1})$ is open in $\text{Zar}(D)^{\text{cons}}$, it follows that $K$ is isolated in $(\text{Zar}(D) \setminus X)^{\text{cons}}$.

(f) is a direct consequence of (e). \hfill \Box

Note that the set $X$ of the previous proposition is not compact with respect to the constructible topology, as it is discrete and infinite.

**Proposition 6.10.** Let $(D, m)$ and $(D', m')$ be two countable Noetherian local domains of dimension 2. Then, $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$.

**Proof.** Denote by $K, K'$ the quotient fields of $D$ and $D'$, respectively.

Let $X$ be the set of isolated points of $\text{Zar}(D)^{\text{cons}}$, and let $C := \text{Zar}(D) \setminus (X \cup \{K\})$: then, $C$ is closed in $\text{Zar}(D)^{\text{cons}}$. Define in the same way $X'$ and $C'$ inside $\text{Zar}(D')$; then, $C'$ is closed.

As in the proof of Proposition 6.7, by Lemma 6.9(f) $C^{\text{cons}}$ and $(C')^{\text{cons}}$ are totally disconnected, perfect, compact and metrizable (with respect to the constructible topology), and thus they are homeomorphic. Let $\phi_C : C^{\text{cons}} \longrightarrow (C')^{\text{cons}}$ be a homeomorphism.

The set $X$ is discrete and countable, and the unique nonisolated point of $X \cup \{K\}$ is $K$; since the same holds for $X'$ and $K'$, any bijection $X \longrightarrow X'$ extends to a homeomorphism $\phi_X : (X \cup \{K\})^{\text{cons}} \longrightarrow (X' \cup \{K'\})^{\text{cons}}$ by setting $\phi_X(K) = K'$. Define

$$
\phi : \text{Zar}(D)^{\text{cons}} \longrightarrow \text{Zar}(D')^{\text{cons}},
\quad V \longmapsto \begin{cases}
\phi_C(V) & \text{if } V \in C, \\
\phi_X(V) & \text{if } V \in X \cup \{K\}.
\end{cases}
$$

By construction, $\phi$ is bijective, and $\phi$ is a homeomorphism when restricted to $C$ and to $X \cup \{K\}$. Since these two sets are closed, by [29, Theorem 7.6] $\phi$ is a homeomorphism. In particular, $\text{Zar}(D)^{\text{cons}} \simeq \text{Zar}(D')^{\text{cons}}$. \hfill \Box

We summarize the previous results in the following theorem.
Theorem 6.11. Let \((D, m)\) and \((D', m')\) be two countable Noetherian local domains. Then, \(\text{Zar}(D)_{\text{cons}} \simeq \text{Zar}(D')_{\text{cons}}\) if and only if one of the following conditions hold:

(a) \(\dim(D) = \dim(D') = 1\) and \(|\text{Max}(D)| = |\text{Max}(D')|\);
(b) \(\dim(D) = \dim(D') = 2\);
(c) \(\dim(D) \geq 3\) and \(\dim(D') \geq 3\).

Proof. If \(D\) and \(D'\) satisfy one of the conditions, then \(\text{Zar}(D)_{\text{cons}} \simeq \text{Zar}(D')_{\text{cons}}\) by, respectively, Proposition 6.5, Proposition 6.10 and Proposition 6.7.

Suppose now that \(\text{Zar}(D)_{\text{cons}} \simeq \text{Zar}(D')_{\text{cons}}\).

If \(\dim(D) = 1\), then \(\text{Zar}(D)\) is finite, and thus so must be \(\text{Zar}(D')\); hence, \(\dim(D') = 1\), and \(|\text{Max}(D)| = |\text{Max}(D')|\) by Proposition 6.5.

Suppose \(\dim(D), \dim(D') \geq 2\). By Corollary 6.4 and Lemma 6.8, \(\text{Zar}(D)_{\text{cons}}\) has isolated points if and only if \(\dim(D) = 2\); therefore, \(\dim(D) = 2\) if and only if \(\dim(D') = 2\), and \(\dim(D) \geq 3\) if and only if \(\dim(D') \geq 3\). The claim is proved. \(\square\)

7. When \(D\) is a field

In this section we analyze the isolated points Zariski space \(\text{Zar}(L|D)_{\text{cons}}\) when \(D = K\) is a field. Note that, if \(L\) is algebraic over \(K\), then \(\text{Zar}(L|K)\) is just a point (\(L\) itself); thus, the only interesting case is when \(\text{trdeg}(L/K) \geq 1\).

We start by connecting the isolated points of \(\text{Zar}(L|D)_{\text{cons}}\) and of \(\text{Zar}(L'|D)_{\text{cons}}\), where \(L' \subseteq L\) is an algebraic extension.

Proposition 7.1. Let \(V\) be a valuation domain, and \(L' \subseteq L\) be an algebraic extension such that \(V \subseteq L'\). Let \(\rho : \text{Zar}(L|V) \rightarrow \text{Zar}(L'|V)\) be the restriction map, and let \(\mathcal{X} \subseteq \text{Zar}(L|V)\) be a subset such that \(\rho^{-1}(\rho(\mathcal{X})) = \mathcal{X}\). Then, the following hold.

(a) If \(W\) is isolated in \(\mathcal{X}^\text{cons}\), then \(\rho(W)\) is isolated in \(\rho(\mathcal{X})^\text{cons}\).
(b) If \(\rho(\mathcal{X})\) is perfect and \(|\rho(\mathcal{X})| > 1\), then \(\mathcal{X}\) is perfect.

In particular, the previous statements apply to \(\mathcal{X} = \text{Zar}(L|V)\) and \(\mathcal{X} = \mathcal{E}(L|V)\).

Proof. (a) Let \(W\) be an isolated point of \(\mathcal{X}^\text{cons}\), and let \(W' := W \cap L' = \rho(W)\).

Suppose first that \(L\) is finite and normal over \(L'\). Let \(G\) be the group of \(L'\)-automorphisms of \(L\); then, every \(\sigma \in G\) is continuous when seen as a map from \(\text{Zar}(L'|V)^\text{cons}\) to itself. Moreover, \(\rho(\sigma(Z)) = \rho(Z)\) for every \(Z \in \text{Zar}(L|V)\), and thus \(\sigma\) restricts to a self-homeomorphism of \(\mathcal{X}\).

Since \(G\) acts transitively on \(\rho^{-1}(W')\) (see e.g. [12, Corollary 20.2]) and \(W \in \rho^{-1}(W')\) is isolated, all points of \(\rho^{-1}(W')\) are isolated in \(\mathcal{X}\); hence, \(\rho^{-1}(W')\) is open in \(\mathcal{X}^\text{cons}\). Since \(\rho : \text{Zar}(L|V) \rightarrow \text{Zar}(L'|V)\) is a closed map (with respect to the constructible topology), it is also closed when seen as a map \(\mathcal{X} \rightarrow \rho(\mathcal{X})\); therefore, \(\rho(\mathcal{X} \setminus \rho^{-1}(W')) = \rho(\mathcal{X}) \setminus \{W'\}\) is closed in \(\rho(\mathcal{X})\), with respect to the constructible topology, and thus \(W'\) is an isolated point of \(\rho(\mathcal{X})^\text{cons}\), as claimed.

Suppose now that \(L\) is finite over \(L'\), and let \(F\) be the normal closure of \(L'\). Let \(\rho_0 : \text{Zar}(F|V) \rightarrow \text{Zar}(L|V)\) be the restriction map. Since \(W\) is isolated
in $X$, the set $\rho^{-1}_0(W)$ is open in $\rho^{-1}_0(X)$; moreover, $\rho^{-1}_0(W)$ is finite since $[F : L] < \infty$. Therefore, $\rho^{-1}_0(W)$ is a discrete subspace of $\rho^{-1}_0(X)$, and in particular each $Z \in \rho^{-1}_0(W)$ is isolated. Applying the previous part of the proof to the extension $L' \subset F$ and to any such $Z$, we obtain that $Z \cap L' = W \cap L' = \rho(W)$ is isolated, as claimed.

Suppose now that $L' \subsetneq L$ is arbitrary. Since $W$ is isolated in $X$, there are $x_1, \ldots, x_n, y_1, \ldots, y_m \in L$ such that $\{W\} = \mathcal{B}(x_1, \ldots, x_n) \cap \mathcal{B}(y_1) \cap \cdots \cap \mathcal{B}(y_m) \cap X$. Let $F := L'(x_1, \ldots, x_n, y_1, \ldots, y_m)$; then, $W \cap F$ is isolated in $\{Z \cap F \mid Z \in X\}$. Since $[F : L'] < \infty$, we can apply the previous part of the proof, obtaining that $W \cap F \cap L' = W \cap L' = \rho(W)$ is isolated in $\rho(X)$, as claimed.

(b) Suppose that $X$ is not perfect: then, there is a $W \in X$ that is isolated. By the previous part of the proof, it would follow that $W \cap L'$ is isolated in $\rho(X)$.

Since $\rho(X)$ has more than one point, this is impossible, and so $X$ is perfect.

The “in particular” statement follows from the fact that $\text{Zar}(L|V)$ and $\mathcal{E}(L|V)$ satisfy the hypothesis on $X$.

**Corollary 7.2.** Let $V$ be a valuation domain and $L' \subsetneq L$ be an algebraic extension; suppose that $V \subsetneq L'$ and that $L'$ is transcendental over the quotient field of $V$. If $\text{Zar}(L'|V)^{\text{cons}}$ (respectively, $\mathcal{E}(L'|V)^{\text{cons}}$) is perfect, then $\text{Zar}(L|V)^{\text{cons}}$ (resp., $\mathcal{E}(L|V)^{\text{cons}}$) is perfect.

**Proof.** It is enough to apply Proposition 7.1(b) to $X = \text{Zar}(L|V)$ or $X = \mathcal{E}(L|V)$, using the hypothesis that $L'$ is transcendental over the quotient field of $V$ to guarantee that $|\text{Zar}(L'|V)| > 1$ and $|\mathcal{E}(L'|V)| > 1$.

The following result completely settles the problem of finding the isolated points when $\text{trdeg}(L/K) \geq 2$, generalizing [3, Theorem 4.45] and solving the authors’ Conjecture A (in an even more general formulation). Note that the first case in the proof is exactly [3, Theorem 4.45], but we give a new proof of it using Theorem 6.3.

**Theorem 7.3.** Let $K \subsetneq L$ be a field extension with $\text{trdeg}(L/K) \geq 2$. Then, $\text{Zar}(L|K)^{\text{cons}}$ is perfect.

**Proof.** Suppose first that $L = K(x_1, \ldots, x_n)$ is a finitely generated purely transcendental extension of $K$, with transcendence basis $x_1, \ldots, x_n$. Suppose there exists an isolated point $W$ of $\text{Zar}(L|K)^{\text{cons}}$. By Proposition 4.1, $W \neq L$.

For each $i$, at least one of $x_i$ and $x_i^{-1}$ belongs to $W$; let it be $t_i$. Then, $W \in \text{Zar}(K[t_1, \ldots, t_n])$, and so $W$ is isolated in $\text{Zar}(K[t_1, \ldots, t_n])^{\text{cons}}$. Let $P$ be the center of $W$ on $K[t_1, \ldots, t_n]$: since $K[t_1, \ldots, t_n]$ is Noetherian, by Theorem 6.3 $P$ has height 1 and $\mathcal{V}(P)$ is finite.

Since $K[t_1, \ldots, t_n]$ is isomorphic to a polynomial ring, every maximal ideal of $K[t_1, \ldots, t_n]$ has height $n > 1$ [16, Section 3.2, Exercise 3], and thus $P$ is not maximal. However, $K[t_1, \ldots, t_n]$ is an Hilbert ring, and thus every non-maximal prime ideal is the intersection of the maximal ideals containing it [16, Theorem...
147]; in particular, this happens for \( P \), and thus \( V(P) \) must be infinite. This is a contradiction, and so \( \text{Zar}(L|K)^{\text{cons}} \) is perfect.

Suppose now that \( L \) has finite transcendence degree over \( K \), let \( x_1, \ldots, x_n \) be a transcendence basis of \( L \) and let \( L' := K(x_1, \ldots, x_n) \). By the previous part of the proof, \( \text{Zar}(L'|K)^{\text{cons}} \) is perfect; since \( L' \subseteq L \) is algebraic, by Corollary 7.2 also \( \text{Zar}(L|K)^{\text{cons}} \) is perfect.

Take now any extension \( L \) of \( K \), and suppose that \( W \) is an isolated point of \( \text{Zar}(L|K)^{\text{cons}} \). Then, there are \( x_1, \ldots, x_n, y_1, \ldots, y_m \in L \) such that \( \{ W \} = \mathcal{B}(x_1, \ldots, x_n) \cap \mathcal{B}(y_1) \cap \cdots \cap \mathcal{B}(y_m) \). Take two elements \( a, b \in L \) that are algebraically independent over \( K \), and let \( L' := K(a, b, x_1, \ldots, x_n, y_1, \ldots, y_m) \): then, \( 2 \leq \text{trdeg}(L'/K) < \infty \). Set \( V := W \cap L' \): then, \( \{ V \} = \mathcal{B}(x_1, \ldots, x_n) \cap \mathcal{B}(y_1) \cap \cdots \cap \mathcal{B}(y_m) \), and thus \( V \) is isolated in \( \text{Zar}(L'|V)^{\text{cons}} \). However, by the previous part of the proof, \( \text{Zar}(L'|V)^{\text{cons}} \) is perfect, a contradiction. Hence, \( \text{Zar}(L|K)^{\text{cons}} \) is perfect.

When the transcendence degree of \( L \) over \( K \) is 1, the picture is very different, because it may even happen that all elements of \( \text{Zar}(L|K)^{\text{cons}} \) (except \( L \) itself) are isolated. Compare the next results with [26, Corollary 5.5(a)] and [28, Proposition 4.2].

**Proposition 7.4.** Let \( K \) be a field. Then all points of \( \text{Zar}(K(X)|K) \), except \( K(X) \), are isolated with respect to the constructible topology.

**Proof.** The points of \( \text{Zar}(K(X)|K) \) are \( K(X), K[X^{-1}]_{(X^{-1})} \), and the rings \( K[X]_{(f(X))} \), where \( f(X) \) is an irreducible polynomial of \( K[X] \). The first one is not isolated by Proposition 4.1; on the other hand, \( \{ K[X]_{(f(X))} \} = \mathcal{B}(f(X)^{-1}) \) and \( \{ K[X^{-1}]_{(X^{-1})} \} = \mathcal{B}(X)^{-1} \), and thus these domains are isolated, as claimed.

**Lemma 7.5.** Let \( D \) be an integral domain with quotient field \( K \), and let \( L' \subseteq L \) be two extensions of \( K \). Let \( V \in \text{Zar}(L'|D) \). If \( V \) is isolated in \( \text{Zar}(L'|D)^{\text{cons}} \) and \( \mathcal{E}(L'|V) \) is finite, then every \( W \in \mathcal{E}(L'|V) \) is isolated in \( \text{Zar}(L|D)^{\text{cons}} \).

**Proof.** Let \( \rho : \text{Zar}(L|D) \longrightarrow \text{Zar}(L'|D) \) be the restriction map. Then, \( \mathcal{E}(L'|V) = \rho^{-1}(V) \) is open in \( \text{Zar}(L'|D)^{\text{cons}} \) since \( V \) is isolated. Moreover, \( \mathcal{E}(L'|V) \) is finite by hypothesis, and, since the constructible topology is Hausdorff, all its points are isolated in \( \text{Zar}(L|D)^{\text{cons}} \).

**Proposition 7.6.** Let \( K \) be a field and let \( L \) be an extension of \( K \) with \( \text{trdeg}(L/K) = 1 \). Let \( V \in \text{Zar}(L|K) \), \( V \neq L \). Then the following are equivalent:

(i) \( V \) is isolated in \( \text{Zar}(L|K)^{\text{cons}} \).

(ii) there exists a finitely generated extension \( L' \) of \( K \) such that \( L' \subseteq L \) and \( \mathcal{E}(L|V \cap L') = \{ V \} \);

(iii) there exists a finitely generated extension \( L' \) of \( K \) such that \( L' \subseteq L \) and \( \mathcal{E}(L|V \cap L') \) is finite.

**Proof.** (i) \( \longrightarrow \) (ii) Since \( V \) is isolated, we have
\[
\{ V \} = \Omega := \mathcal{B}(x_1, \ldots, x_n) \cap \mathcal{B}(y_1) \cap \cdots \cap \mathcal{B}(y_m)
\]
for some \( x_1, \ldots, x_n, y_1, \ldots, y_m \in L \). Let \( L' = K(x_1, \ldots, x_n, y_1, \ldots, y_m) \). Then every extension of \( V \cap L' \) to \( L \) belongs to \( \Omega \), and thus it is equal to \( V \). Hence, \( L' \) is the required field.

(ii) \( \implies \) (iii) is obvious.

(iii) \( \implies \) (i) Since \( \mathcal{E}(L|V \cap L') \) is finite, \( L' \subseteq L \) must be algebraic and so \( K \subseteq L' \) is transcendental; take any \( X \in L' \) that is transcendental over \( K \). Since \( K \subseteq L' \) is finitely generated, \( K(X) \subseteq L' \) must be a finite extension.

Since \( V \neq L \), we have \( V \cap K(X) \neq K(X) \); by Proposition 7.4, \( V \cap K(X) \) is isolated in \( \text{Zar}(K(X)|K) \). Moreover, since \( K(X) \subseteq L' \) is a finite extension, \( \mathcal{E}(L'|V \cap K(X)) \) is finite; by Lemma 7.5, all points of \( \mathcal{E}(L'|V \cap K(X)) \) are isolated in \( \text{Zar}(L'|K)^{\text{cons}} \), and in particular this happens for \( V \cap L' \). We can now apply Lemma 7.5 to \( V \cap L' \) and \( L \), obtaining that all elements of \( \mathcal{E}(L|V \cap L') \) (in particular, \( V \)) are isolated in \( \text{Zar}(L|K)^{\text{cons}} \). \( \square \)

Proposition 7.7. Let \( K \) be a field and let \( L \) be an extension of \( K \) with \( \text{trdeg}(L/K) = 1 \). Let \( \mathcal{X} := \text{Zar}(L|K) \setminus \{L\} \). Then, the following are equivalent:

(i) all points of \( \mathcal{X} \) are isolated in \( \text{Zar}(L|K)^{\text{cons}} \);

(ii) for every \( X \in L \), transcendental over \( K \), the set \( \mathcal{E}(L|V) \) is finite for every \( V \in \text{Zar}(K(X)|K) \);

(iii) there is an \( X \in L \), transcendental over \( K \), such that the set \( \mathcal{E}(L|V) \) is finite for every \( V \in \text{Zar}(K(X)|K) \).

Proof. (i) \( \implies \) (ii) Take any \( X \in L \) that is transcendental over \( K \), and let \( V \in \text{Zar}(K(X)|K) \). The space \( \mathcal{E}(L|V) \) is closed in \( \text{Zar}(L|V)^{\text{cons}} \), and thus it is compact. Since all its points are isolated, it is also discrete; hence, \( \mathcal{E}(L|V) \) is finite.

(ii) \( \implies \) (iii) is obvious.

(iii) \( \implies \) (i) Apply Proposition 7.6, (iii) \( \implies \) (i) with \( L' = K(X) \) to each \( V \in \mathcal{X} \). \( \square \)

Corollary 7.8. Let \( K \) be a field and let \( L \) be a finitely generated extension of \( K \) such that \( \text{trdeg}(L/K) = 1 \). Then, all points of \( \text{Zar}(L|K) \setminus \{L\} \) are isolated in \( \text{Zar}(L|K)^{\text{cons}} \).

Proof. It is enough to apply Proposition 7.7. \( \square \)

Remark 7.9. Let \( K \subseteq L \) be a transcendental extension of degree 1, and let \( V \in \text{Zar}(L|K) \). Let \( X \in L \) be transcendental over \( K \). By Proposition 7.6, if \( \mathcal{E}(L|V \cap K(X)) \) is finite, then \( V \) is isolated in \( \text{Zar}(L|K)^{\text{cons}} \); however, unlike in Proposition 7.7, the converse does not hold, i.e., \( \mathcal{E}(L|V \cap K(X)) \) may be infinite even if \( V \) is isolated.

For example, let \( W = K[X](x) \) (or more generally, we can take any \( W \in \text{Zar}(K(X)|K) \), \( W \neq K(X) \)). Since \( W \) is a discrete valuation ring, using [17] (see also [13, Section 3]), it is possible to construct a chain \( K(X) \subset F_0 \subset F_1 \subset \cdots \) of extensions of \( K(X) \) such that:

- the extensions \( K(X) \subset F_0 \) and \( F_i \subset F_{i+1} \) are finite, for each \( i > 0 \);
- \( W \) has two extensions to \( F_0 \), say \( W_1 \) and \( W_2 \);
- \( W_1 \) has only one extension to \( F_1 \), for each \( i > 0 \);
8. Extensions of valuations

In this section, we extend the results of the previous section from the case where \(D = K\) is a field to the case where \(D = V\) is a valuation domain. In particular, we want to study the set \(\mathcal{E}(L|V)\) of extensions of \(V\) to \(L\).

The most important case is when \(L = K(X)\) is the field of rational functions.

If \(V\) is a valuation domain with quotient field \(K\) and \(s \in K\), we set

\[ V_s := \{ \phi \in K(X) \mid \phi(s) \in V \}. \]

Then, \(V_s\) is an extension of \(V\) to \(K(X)\), and it is possible to analyze quite thoroughly its algebraic properties (see for example [22, Proposition 2.2] for a description when \(V\) has dimension 1).

The following lemma is a partial generalization of [22, Theorem 3.2], of which we follow the proof.

**Lemma 8.1.** Let \(V\) be a valuation domain with quotient field \(K\), and let \(U\) be an extension of \(V\) to the algebraic closure \(\overline{K}\). Let \(s, t \in \overline{K}\). Then, \(U_s \cap K(X) = U_t \cap K(X)\) if and only if \(s\) and \(t\) are conjugated over \(K\).

**Proof.** If \(s\) and \(t\) are conjugated, there is a \(K\)-automorphism \(\sigma\) of \(\overline{K}\) sending \(s\) to \(t\).

Setting \(\overline{\sigma(\sum a_iX^i)} := \sum \sigma(a_i)X^i\), we can extend \(\sigma\) to a \(K(X)\)-automorphism \(\overline{\sigma}\) of \(\overline{K(X)}\) such that \(\overline{\sigma}(\phi(t)) = \sigma(\phi(s))\) for every \(\phi \in \overline{K(X)}\); in particular, if \(\phi \in K(X)\) then \(\overline{\sigma}(\phi) = \phi\) and thus \(\phi(s) \in V\) if and only if \(\phi(t) \in V\), i.e., \(\phi \in U_s \cap K(X)\) if and only if \(\phi \in U_t \cap K(X)\). Therefore, \(U_s \cap K(X) = U_t \cap K(X)\).

Conversely, suppose that \(s\) and \(t\) are not conjugated, and let \(p(X)\) be the minimal polynomial of \(s\) over \(K\) then, \(p(t) \neq 0\), and thus there is a \(c \in K, c \neq 0\), such that \(v(c) > u(p(t))\) (where \(v\) and \(u\) are, respectively, the valuations with respect to \(V\) and \(U\) and \(u|_K = v\)). Then, \(q(X) := \frac{p(X)}{c} \in K(X)\) belongs to \(U_s\) (since \(q(s) = 0 \in V\) but not to \(U_t\) (since \(u(q(t)) = u(p(t)) - v(c) < 0\)). Hence, \(U_s \cap K(X) \neq U_t \cap K(X)\), as claimed. \(\square\)

**Theorem 8.2.** If \(V\) is a valuation domain that is not a field, then \(\mathcal{E}(K(X)|V)^{\text{cons}}\) is perfect.

**Proof.** Suppose first that \(K\) is algebraically closed. By [23, Theorem 7.2], for all extensions \(W\) of \(V\) to \(K(X)\) there is a sequence \(E = \{s_v\}_{v \in \Lambda}\) (where \(\Lambda\) is a
well-ordered set without maximum) such that
\[ W = V_E = \{ \phi \in K(X) \mid \phi(s_v) \in V \text{ for all large } v \} \]
and \( W \neq V_{s_v} \) for every \( v \). In particular, the elements \( \phi(s_v) \) are either eventually in \( V \) or eventually out of \( V \) \( \text{(by [23, Proposition 3.2]; see also the proof of Theorem 3.4 therein). Take } \psi \in K(X) \text{; then, if } W \not\subseteq B(\psi) \text{ it must be } \psi(s_v) \in V \text{ eventually, and thus } B(\psi) \text{ contains } V_{s_v} \text{ for all large } v \text{; on the other hand, if } W \not\subseteq B(\psi)^c \text{ then } \psi(s_v) \not\in V \text{ eventually, and thus } B(\psi)^c \text{ contains } V_{s_v} \text{ for all large } v. \]
Now let
\[ \Omega := B(\psi_1, \ldots, \psi_n) \cap B(\phi_1)^c \cap \cdots \cap B(\phi_m)^c \cap E(K(X)|V) \]
be a basic open set of \( E(K(X)|V)^\text{cons} \) containing \( W \). For every \( i \), there is an index \( N_i \) such that \( \psi_i(s_v) \in V \) for all \( v \geq N_i \); likewise, for every \( j \) there is a \( M_j \) such that \( \phi_j(s_v) \not\in V \) for all \( v \geq M_j \). Therefore, for every \( \nu \geq \max\{N_1, \ldots, N_n, M_1, \ldots, M_m\} \), we have \( V_{s_v} \subseteq \Omega \). Hence \( W \) belongs to the closure of \( \{V_{s_v}\}_{s \in \Lambda} \subseteq E(K(X)|V) \), with respect to the constructible topology. It follows that \( W \) is not isolated in \( E(K(X)|V)^\text{cons} \) and, since \( W \) was arbitrary, \( E(K(X)|V)^\text{cons} \) is perfect.

Suppose now that \( K \) is any field. Let \( W \subseteq E(K(X)|V) \), and suppose that \( W \) is isolated in \( E(K(X)|V)^\text{cons} \). Let \( \rho : E(\overline{K}(X)|V) \rightarrow E(K(X)|V) \) be the restriction map. Since \( W \) is isolated and \( \rho \) is continuous, \( \rho^{-1}(W) \) is open. Let \( W' \subseteq \rho^{-1}(W) \) and let \( U := W' \cap \overline{K} \); then, \( U \) is an extension of \( V \) to \( \overline{K} \).

By the previous part of the proof, for every open neighborhood \( \Omega \) of \( W' \) there is a \( s \in \overline{K} \) such that \( U_s \neq W' \) and \( U_s \subseteq \Omega \); since \( \rho^{-1}(W) \) is open, it follows that for every such \( \Omega \) there is a \( U_s \subseteq \rho^{-1}(W) \) with these properties. Therefore, the set \( \Delta := \{ U_s \subseteq \rho^{-1}(W) \mid s \in \overline{K} \} \) is dense in \( \rho^{-1}(W) \). Since \( U_s \cap K(X) = U_t \cap K(X) = W \) for every \( U_s, U_t \in \Delta \), by Lemma 8.1 \( \Delta \) is finite; since \( E(K(X)|V)^\text{cons} \) is Hausdorff, it follows that \( \Delta = \rho^{-1}(W) \), and in particular \( \rho^{-1}(W) \) is finite. Hence, all its points are isolated. However, this contradicts the fact that \( E(\overline{K}(X)|V)^\text{cons} \) is perfect; thus, also \( E(K(X)|V)^\text{cons} \) must be perfect. \( \square \)

The theorem above allows to determine the isolated points of \( \text{Zar}(K(X)|D)^\text{cons} \) for every integral domain \( D \).

**Proposition 8.3.** Let \( D \) be an integral domain that is not a field, and let \( J \) be the intersection of the nonzero prime ideals of \( D \).

(i) If \( J = (0) \), then \( \text{Zar}(K(X)|D)^\text{cons} \) is perfect.

(ii) If \( J \neq (0) \), then the only isolated points of \( \text{Zar}(K(X)|D)^\text{cons} \) are \( K[X]_{f(X)} \) (where \( f(X) \) is an irreducible polynomial of \( K[X] \)) and \( K[X]_{(X^{-1})} \).

**Proof.** Let \( W \subseteq \text{Zar}(K(X)|D) \). If \( W \cap K \neq K \), then \( E(K(X)|W \cap K) \) is perfect (when endowed with the constructible topology) by Theorem 8.2. Since \( W \) belongs to this set, it is not isolated in \( \text{Zar}(K(X)|D)^\text{cons} \).
Suppose that $W \cap K = K$. If $W = K(X)$, then $W$ is not isolated by Proposition 4.1, since $K(X)$ is not algebraic over $K$. Thus let $W \neq K(X)$.

Suppose that $J = (0)$, and suppose that $W$ is isolated in $\text{Zar}(K(X)|D)^{\text{cons}}$. Since $K \subseteq W$, we have $m_W \cap D = (0)$; by Lemma 4.2, the quotient map of $W$ onto its residue field induces a homeomorphism between the spaces $\Delta := \{ Z \in \text{Zar}(K(X)|D) \mid Z \subseteq W \}$ and $\text{Zar}(W/m_W|D)$, where $W$ is sent to $W/m_W$. Since $W$ is isolated in $\text{Zar}(K(X)|D)^{\text{cons}}$, it is also isolated in $\Delta^{\text{cons}}$, and thus $W/m_W$ must be an isolated point of $\text{Zar}(W/m_W|D)^{\text{cons}}$. By Proposition 4.1, $D$ must be a Goldman domain, against the hypothesis $J = (0)$. Therefore, $W$ is not isolated and $\text{Zar}(K(X)|D)^{\text{cons}}$ is perfect.

Suppose now that $J \neq (0)$, and let $j \in J$, $j \neq 0$. Then, $D[j^{-1}] = K$, and thus $B(j^{-1}) = \mathcal{E}(K(X)|K) = \text{Zar}(K(X)|K)$ is a clopen subset of $\text{Zar}(K(X)|D)^{\text{cons}}$; in particular, $W \in \mathcal{E}(K(X)|K)$ is isolated in $\text{Zar}(K(X)|D)^{\text{cons}}$ if and only if it is isolated in $\text{Zar}(K(X)|K)^{\text{cons}}$. The claim now follows from Proposition 7.4. □

To conclude the paper, we extend Theorem 7.3 to valuation domains.

**Theorem 8.4.** Let $V$ be a valuation domain with quotient field $K$, and let $L$ be a field extension of $K$ such that $\text{trdeg}(L/K) \geq 2$. Then, $\mathcal{E}(L|V)^{\text{cons}}$ and $\text{Zar}(L|V)^{\text{cons}}$ are perfect.

**Proof.** We first show that $\mathcal{E}(L|V)^{\text{cons}}$ is perfect: suppose that is not, and let $W$ be an isolated point.

Suppose that $L = K(x, z_2, \ldots)$ is purely transcendental over $K$, where $x, z_2, \ldots$, is a transcendence basis. Take an $m \in m_V \subseteq m_W$: then, at least one of $mx$ and $x^{-1}$ belongs to $m_W$. Let $z_1$ be that element. Then, $z_1, z_2, \ldots$ is also a transcendence basis of $L$.

Let $L’ := K(z_1, z_2, \ldots)$ be the extension of $K$ obtained adjoining all the elements of this basis except $z_2$. Then, $z_1^{-1} \in L’ \setminus W$, and thus $W \cap L’ \neq L’$; since, by construction, $L = L’(X)$, by Theorem 8.2 $\mathcal{E}(L|W \cap L’)^{\text{cons}}$ is perfect. Since $\mathcal{E}(L|W \cap L’) \subseteq \mathcal{E}(L|W)$, all the elements of $\mathcal{E}(L|W \cap L’)$ (in particular, $W$) are not isolated in $\mathcal{E}(L|V)^{\text{cons}}$. This is a contradiction, and thus $\mathcal{E}(L|V)^{\text{cons}}$ is perfect.

Suppose now that $L$ is arbitrary: then, we can find a purely transcendental extension $L’$ of $K$ such that $L’ \subseteq L$ is algebraic. By the previous part of the proof, $\mathcal{E}(L’|V)^{\text{cons}}$ is perfect; by Corollary 7.2, also $\mathcal{E}(L|V)^{\text{cons}}$ is perfect. Therefore, $\mathcal{E}(L|V)^{\text{cons}}$ is always perfect.

Finally, Zar($L|V$) is the union of $\mathcal{E}(L|V_0)$, as $V_0$ ranges among the valuation overlings of $V$; since each of these is perfect with respect to the constructible topology (by the previous part of the proof), then also Zar($L|V$) is perfect, as claimed. □

**Corollary 8.5.** Let $V$ be a valuation domain with quotient field $K$, suppose $V \neq K$, and let $L$ be a transcendental field extension of $K$. Then, $\mathcal{E}(L|V)^{\text{cons}}$ is perfect.

**Proof.** If $\text{trdeg}(L/K) \geq 2$ the claim follows from Theorem 8.4. If $\text{trdeg}(L/K) = 1$, let $X \in L$ be transcendental over $K$. By Theorem 8.2, $\mathcal{E}(K(X)|V)^{\text{cons}}$ is perfect; by Corollary 7.2, also $\mathcal{E}(L|V)^{\text{cons}}$ is perfect. □
**Corollary 8.6.** Let $D$ be an integral domain, and let $L$ be a transcendental extension of the quotient field $K$ of $D$. If $\text{trdeg}(L/K) \geq 2$, then $\text{Zar}(L|D)^{\text{cons}}$ is perfect.

**Proof.** Any $W \in \text{Zar}(L|D)$ belongs to $E(L|V)$ for some $V \in \text{Zar}(D)$. By Theorem 8.4, all $E(L|V)^{\text{cons}}$ are perfect and thus no $W$ is isolated. Hence, $\text{Zar}(L|D)^{\text{cons}}$ is perfect. □

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