Deformations of the Whitham systems in the almost linear case.

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Abstract

We consider deformations of the Whitham systems in the case when the initial system is close to linear one. It appears that the almost linear case requires a special procedure of the deformation of the Whitham system to make all the constructions stable in the linear limit. We suggest here a special deformation scheme which gives a stable deformation of the Whitham system for an almost linear initial system.

1 General deformation schemes.

We will consider deformations of the Whitham systems for non-linear differential equations which are close to linear ones for small values of some "non-linearity parameters" \( \lambda \).

As is well known the Whitham method ([1, 2, 3]) is connected with the slow modulations of periodic or quasiperiodic \( m \)-phase solutions of nonlinear systems

\[
F^i(\varphi, \varphi_t, \varphi_x, \ldots) = 0, \quad i = 1, \ldots, n, \quad \varphi = (\varphi^1, \ldots, \varphi^n)
\]

which are represented usually in the form

\[
\varphi^i(x,t) = \Phi^i(k(U)x + \omega(U)t + \theta_0, U)
\]

In these notations the functions \( k(U) \) and \( \omega(U) \) play the role of the "wave numbers" and "frequencies" of \( m \)-phase solutions and \( \theta_0 \) are the initial phase shifts. The parameters of the solutions \( U = (U^1, \ldots, U^N) \) can be chosen in arbitrary way, however, we assume that they do not change under arbitrary shifts of the initial phases \( \theta_0 \) of solutions.

The functions \( \Phi^i(\theta) \) satisfy the system

\[
F^i(\Phi, \omega^\alpha \Phi_{\theta^\alpha}, k^\beta \Phi_{\theta^\beta}, \ldots) \equiv 0, \quad i = 1, \ldots, n
\]

and we choose for every \( U \) some function \( \Phi(\theta, U) \) as having "zero initial phase shifts". A full set of \( m \)-phase solutions of (1.1) can then be represented in the form (1.2). For \( m \)-phase
solutions of (1.1) we have then \( k(U) = (k^1(U), \ldots, k^m(U)) \), \( \omega(U) = (\omega^1(U), \ldots, \omega^m(U)) \), \( \theta_0 = (\theta^1, \ldots, \theta^m) \), where \( U = (U^1, \ldots, U^N) \) are parameters of the solution. We require also that all the functions \( \Phi^i(\theta, U) \) are \( 2\pi \)-periodic with respect to every \( \theta^\alpha, \alpha = 1, \ldots, m \). Let us denote by \( \Lambda \) the family of the functions \( \Phi(\theta, U) \) which depend on the parameters \( U \) in a smooth way and satisfy system (1.3) for all \( U \). We will assume also that \( \Lambda \) is the maximal family having these properties.

As it is well known the famous area of the theory of integrable systems connected with \( m \)-phase solutions was started by the fundamental work of S.P. Novikov (\( \Pi \)) where the algebro-geometric approach to the theory of quasi-periodic solutions was invented. An introduction of the finite-gap potentials as the stationary points of higher \( KdV \)-equations and the investigation of their algebro-geometric properties has become a corner stone of the algebro-geometric approach in the theory of solitons. As is well known by now the theory of algebro-geometric solutions has become much larger area then the pure theory of integrable systems and there are many other wide areas of mathematical physics where the varieties of Novikov potentials play a basic role. The algebro-geometric methods have become very efficient also in the theory of slow modulations of quasi-periodic solutions and play in fact the fundamental role in their consideration.

We are going to consider here dispersion corrections to the Whitham systems and pay the special attention to the "almost linear" case.

In Whitham approach the parameters \( U \) become slow functions of \( x \) and \( t \): \( U = U(X, T) \), where \( X = \epsilon x, T = \epsilon t \) (\( \epsilon \to 0 \)).

The functions \( U(X, T) \) should satisfy in this case some system of differential equations (Whitham system) which makes possible the construction of the corresponding asymptotic solution. More precisely (see \( \Pi \)), we try to find the asymptotic solutions

\[
\varphi^i(\theta, X, T) = \sum_{k \geq 0} \Psi^i(k, \frac{S(X, T)}{\epsilon} + \theta, X, T) \epsilon^k
\]

(1.4)

(where all \( \Psi(k) \) are \( 2\pi \)-periodic in \( \theta \)) which satisfy the system (1.1), i.e.

\[
F^i(\varphi, \epsilon \varphi_T, \epsilon \varphi_X, \ldots) = 0, \quad i = 1, \ldots, n
\]

The function \( S(X, T) = (S^1(X, T), \ldots, S^m(X, T)) \) is called a "modulated phase" of solution (1.3).

It is easy to see that the function \( \Psi(0)(\theta, X, T) \) should belong to the family of \( m \)-phase solutions of (1.1) at every \( X \) and \( T \). We have then

\[
\Psi(0)(\theta, X, T) = \Phi(\theta + \theta_0(X, T), U(X, T))
\]

(1.5)

and

\[
S^\alpha_X(X, T) = \omega^\alpha(U), \quad S^\alpha_X(X, T) = k^\alpha(U)
\]

as follows from the substitution of (1.4) into system (1.3).

The functions \( \Psi(k)(\theta, X, T) \) are defined from the linear systems
\( \hat{L}^i_{j[U,\theta_0]}(X, T) \Psi^j_{(k)}(\theta, X, T) = f^j_{(k)}(\theta, X, T) \) (1.6)

where \( \hat{L}^i_{j[U,\theta_0]}(X, T) \) is a linear operator given by the linearization of system (1.3) on solution (1.5). The resolvability conditions of systems (1.6) can be written as the orthogonality conditions of the functions \( f_{(k)}(\theta, X, T) \) to all the ”left eigen vectors” (the eigen vectors of adjoint operator) \( \kappa^{(q)}_{[U(X, T)]}(\theta + \theta_0(X, T)) \) of the operator \( \hat{L}^i_{j[U,\theta_0]}(X, T) \) corresponding to zero eigen-values. The resolvability conditions of (1.6) for \( k = 1 \)

\[ L^i_{j[U,\theta_0]}(X, T) \Psi^j_{(1)}(\theta, X, T) = f^j_{(1)}(\theta, X, T) \] (1.7)

together with the relations \( k_n^2 = \omega_n^2 \) give the Whitham system for \( m \)-phase solutions of (1.1) which plays the central role in the slow modulations approach.

Let us say that the resolvability conditions of (1.6) can in fact be rather complicated in a general multi-phase case. Indeed, we need to investigate the eigen-spaces of the operators \( \hat{L}_{[U,\theta_0]} \) and \( \hat{L}^\dagger_{[U,\theta_0]} \) on the space of \( 2\pi \)-periodic functions which can be rather non-trivial in the multi-phase situation. Thus even the dimensions of kernels of \( \hat{L}_{[U,\theta_0]} \) and \( \hat{L}^\dagger_{[U,\theta_0]} \) can depend in non-smooth way on the values of \( U \) so we can have a rather complicated picture on the \( U \)-space ([15, 16, 32, 33]).

These difficulties do not usually appear in the one-phase situation (\( m = 1 \)) where the behavior of eigen-values of \( \hat{L}_{[U,\theta_0]} \) and \( \hat{L}^\dagger_{[U,\theta_0]} \) is usually rather regular. It is natural to introduce some regularity conditions on the space of one-phase solutions in this situation which will play an important role in the construction of asymptotic series (1.4). We will assume here that the parameters \( k \) and \( \omega \) can be considered (locally) as the independent parameters on the family \( \Lambda \) and the total family of solutions of (1.3) depends on \( N = 2 + s \) \((s \geq 0)\) parameters \( U^\nu \) and the initial phase \( \theta_0 \).

Easy to see then that the functions \( \Phi_\theta(\theta + \theta_0(X, T), U(X, T)) \) and \( \nabla_\xi \Phi(\theta + \theta_0(X, T), U(X, T)) \) where \( \xi \) is any vector in the space of parameters \( U^\nu \) tangential to the surface \( k = const, \omega = const \) belong to the kernel of the operator \( \hat{L}^i_{(X,T)j} \).

Let us represent the space of parameters \( U \) in the form \( U = (k, \omega, n) \) where \( k \) is the wave number, \( \omega \) is the frequency of one-phase solution, and \( n = (n^1, \ldots, n^s) \) are some additional parameters (if they exist).

**Definition 1.1.**

We call the family \( \Lambda \) a full regular family of one-phase solutions of (1.1) if

1) The functions \( \Phi_\theta(\theta, k, \omega, n), \Phi_\nu((\theta, k, \omega, n) \) are linearly independent and give the full basis in the kernel of the operator \( \hat{L}^i_{[\theta_0,k,\omega,n]} \); 

2) The operator \( \hat{L}^i_{[\theta_0,k,\omega,n]} \) has exactly \( s + 1 \) linearly independent ”left eigen vectors” 

\( \kappa^{(q)}_{[U]}(\theta + \theta_0) = \kappa^{(q)}_{[k,\omega,n]}(\theta + \theta_0) \)

depending on the parameters \( U \) in a smooth way and corresponding to zero eigen-values.
The Whitham system is defined in this regular situation by the orthogonality conditions of the discrepancy $f(1)(\theta, X, T)$ to the "left eigen-vectors" $\kappa^{(q)}_{(k, \omega, n)}(\theta + \theta_0)$, $q = 1, \ldots, s + 1$.

As is well known, in many examples the Whitham system gives restrictions on the parameters $U^\nu(X, T)$ of zero approximation $\Psi(0)(\theta, X, T)$ and leaves free the parameter $\theta_0(X, T)$. This fact was formulated in [47] as a general Lemma for the case of the full regular family of $(m$-phase) solutions $\Lambda$. In fact, it is also true in a more general $(m$-phase) situation even without the requirements of "regularity" of the family $\Lambda$. Let us prove here the corresponding Lemma.

**Lemma 1.1.**

The orthogonality conditions of all the "left eigen vectors" of $\hat{L}_j^i(U, \theta_0)$ corresponding to zero eigen values to the discrepancy $f(1)(\theta, X, T)$ give restrictions on the functions $U^\nu(X, T)$ only and do not involve the functions $\theta_0^0(X, T), \ldots, \theta_0^m(X, T)$.

Proof.

Let us represent the main term of expansion (1.4) in the form

$$\phi^i_{(0)}(\theta, X, T) = \Phi^i(\theta + S(X, T) + \epsilon \theta_0(X, T), U(X, T))$$

Easy to see then that the part of $f(1)$ containing the functions $\theta_0^\alpha(X, T)$ has the form

$$\tilde{f}^i(1)(\theta, X, T) = \frac{\partial F^i}{\partial \omega^\alpha} \theta_0^\alpha_T - \frac{\partial F^i}{\partial k^\alpha} \theta_0^\alpha_X$$

where the notations $\partial F^i/\partial \omega^\alpha$ and $\partial F^i/\partial k^\alpha$ mean that we don’t consider the dependence of the functions $\Phi^i$ on $k$ and $\omega$ in (1.3) and differentiate $F^i$ only with respect to the "explicit" $k^\alpha$ and $\omega^\alpha$ in (1.3) and then put $\varphi = \Phi(\theta, k, \omega, n)$.

However, the full derivatives

$$\frac{d}{d\omega^\alpha} F^i(\Phi, \omega^\alpha \Phi_{\theta^\alpha}, \ldots), \frac{d}{dk^\alpha} F^i(\Phi, \omega^\alpha \Phi_{\theta^\alpha}, \ldots)$$

(including the differentiation of $\Phi$ with respect to $\omega$ and $k$) are identically zero on the family $\Lambda$ according to (1.3). So we can write that

$$\tilde{f}^i(1) = \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \frac{\delta F^i}{\delta \Phi^j(\theta')} \Phi^j_{\omega^\alpha}(\theta', X, T) \theta_0^\alpha_T + \frac{\delta F^i}{\delta \Phi^j(\theta')} \Phi^j_{k^\alpha}(\theta', X, T) \theta_0^\alpha_X \right) \frac{d^m \theta'}{(2\pi)^m}$$

i.e.

$$\tilde{f}^i(1) = \hat{L}_j^i(U, \theta_0) \Phi^j_{\omega^\alpha}(\theta, X, T) \theta_0^\alpha_T + \hat{L}_j^i(U, \theta_0) \Phi^j_{k^\alpha}(\theta, X, T) \theta_0^\alpha_X$$

We have then that $\tilde{f}(1)(\theta, X, T)$ always belongs to the image of $\hat{L}_j^i(U, \theta_0)$ without any restriction on the functions $\theta_0^\alpha(X, T)$.
Lemma 1.1 is proved.

It is easy to see also from the proof of the Lemma that the functions $\theta_0^\alpha(X,T)$ generate the additions $\Phi_{\omega^\alpha}\theta_0^\alpha T$ and $\Phi_{k^\alpha}\theta_0^\alpha X$ to the function $\Psi_{(1)}(\theta, X, T)$ which in the main order is equivalent to the effective "renormalization"

$$\omega^\alpha \rightarrow \omega^\alpha + \epsilon \theta_0^\alpha T, \quad k^\alpha \rightarrow k^\alpha + \epsilon \theta_0^\alpha X$$

of the parameters $(\omega, k)$ of zero approximation $\Phi(\theta + \theta_0, U)$.

According to the deformation procedure used in [47] the parameters $\theta_0(X, T)$ become in fact unnecessary after a "renormalization" of the phase $S(X, T)$ when the unnecessary "renormalization freedom" disappears.

The Whitham system is a so-called system of Hydrodynamic Type, which can be written in the form

$$A_\nu^\mu(U) U_\mu^\nu = B_\mu^\nu(U) U_\lambda^\mu$$

with some matrices $A(U)$ and $B(U)$. In generic case the system (1.8) can be resolved w.r.t. the time derivatives of $U$ and written in the evolution form

$$U_\nu^\mu = V_\mu^\nu(U) U_\lambda^\mu, \quad \nu = 1, \ldots, N$$

(1.9)

(where $V = A^{-1}B$).

Lagrangian properties of the Whitham system were investigated by Whitham [3] who suggested also a method of "averaging" of a Lagrangian function to get a Lagrangian function for the Whitham system.

Another important procedure is the procedure of "averaging" of local Hamiltonian structures suggested by B.A. Dubrovin and S.P. Novikov [18, 29, 30]. The Dubrovin-Novikov procedure gives a field-theoretical Hamiltonian structure of Hydrodynamic Type for system (1.9) with a Hamiltonian function having the hydrodynamic form $H = \int h(U)dX$. The Dubrovin-Novikov bracket for system (1.9) has the form

$$\{U_\nu^\alpha(X), U_\mu^\beta(Y)\} = g_\nu^\mu(U) \delta'(X - Y) + b_\lambda^\nu(U) U_\lambda^\mu \delta(X - Y)$$

(1.10)

which is called also a local Poisson bracket of Hydrodynamic Type.

The Hamiltonian properties of systems (1.9) are strongly correlated with their integrability properties. Thus it was proved by S.P. Tsarev [31] that all the diagonalizable systems (1.9) having the Dubrovin-Novikov Hamiltonian structure can in fact be integrated (S.P. Novikov conjecture). Actually the same is true also for the diagonalizable systems (1.9) having more general weakly-nonlocal Mokhov-Ferapontov or Ferapontov Hamiltonian structures. Let us say also here that the Dubrovin-Novikov procedure of averaging of local Poisson brackets can be generalized also to the weakly-nonlocal case.

The construction of asymptotic series (1.4) for the case of a full regular family of (one-phase) solutions of (1.1) can be represented in a regular way. Namely, provided that the Whitham system is satisfied we find the first correction $\Psi_{(1)}(\theta, X, T)$ at every $X$ and $T$ from system (1.7). The function $\Psi_{(1)}(\theta, X, T)$ is defined modulo the linear combination
\[ c^{(1)}(X, T) \Phi_\theta(\theta, X, T) + \sum_{l=1}^{s} d_l^{(1)}(X, T) \Phi_n^l(\theta, X, T) \]  

(1.11)

of the eigen-vectors of \( \hat{L}_{j[U, \theta_0]} \) corresponding to zero eigen-values. The coefficients \( c^{(k)}(X, T) \) and \( d_l^{(k)}(X, T) \) arising at every step \( k \) and the initial phase \( \theta_0(X, T) \) can be used to provide resolvability of systems (1.6) in the higher orders so we can hope to find recurrently all the corrections \( \Psi_{(k)}(\theta, X, T) \).

The structure of the recurrent procedure can be constructed in regular way. Let us note first of all that it was pointed out by J.C. Luke that the values \( c^{(1)}(X, T) \) are actually not involved in the resolvability conditions of (1.6) for \( k = 2 \) and the order \( k = 2 \) gives restrictions on the initial phase \( \theta_0(X, T) \) instead ([4], see also [15, 26, 27, 28]).

Let us say that this statement can be generalized in fact for all the orders \( k \geq 1 \). Let us prove here the corresponding Lemma which will not actually require the ”full regular family” \( \Lambda \) of solutions of (1.1) and can be formulated in fact in the most general (\( m \)-phase) situation. We will assume just that all the corrections \( \Psi^{(1)}, \ldots, \Psi^{(k)} \) are found in asymptotic solution (1.4) in general \( m \)-phase case and we can write the general solution \( \Psi^{(k)} \) of (1.6) in the form

\[ \Psi_{(k)}(\theta, X, T) = \Psi'_{(k)}(\theta, X, T) + \sum_{\alpha=1}^{m} c^{(k)}_{\alpha}(X, T) \Phi^{(\alpha)}(\theta, X, T) + \sum_{l'} d_{l'}^{(k)}(X, T) Q_{l'}(\theta, X, T) \]

where \( \Psi'_{(k)}(\theta, X, T) \) is normalized in some way. Here the functions \( \Phi^{(\alpha)}(\theta, X, T) \) always belong to the kernel of the operator \( \hat{L}_{j[U, \theta_0]} \) and \( Q_{l'}(\theta, X, T) \) denote all the other (linearly independent) vectors from \( \text{Ker} \hat{L}_{j[U, \theta_0]} \). In our notations we put here also

\[ \Phi(\theta, X, T) \equiv \Phi(\theta + \theta_0(X, T), U(X, T)) = \Psi_{(0)}(\theta, X, T) \]

Lemma 1.2.

The functions \( c^{(k)}_{\alpha}(X, T) \) do not appear in resolvability conditions of system (1.6) in the order \( k + 1 \).

Proof.

Assume first that \( k \geq 2 \). Let us look at the terms in \( f_{k+1}(\theta, X, T) \) which contain the functions \( c^{(k)}_{\alpha}(X, T) \). Let us divide these terms in three groups: \( f'_{k+1}, f''_{k+1}, f'''_{k+1} \) in the following way:

1) There are terms corresponding to the ”correction” of the value \( F^i(\varphi, k^\alpha \varphi_{\theta^\alpha}, \omega^{\beta} \varphi_{\theta^\beta}, \ldots) \) as a result of corrections of \( \Psi_{(0)} \) in the \( k \)-th order. To describe these terms it is convenient to use again the fact that the correction \( e^k c^{(k)}_{\alpha}(X, T) \Phi_{\theta^{\alpha}}(\theta, X, T) \) to \( \Psi_{(0)}(\theta, X, T) \) is equivalent (modulo the terms of order \( e^{2k} \)) to the correction \( e^k c^{(k)}_{\alpha}(X, T) \) to the phase \( S^\alpha(X, T) \) of \( \Psi_{(0)}(\theta, X, T) \).
So the corresponding corrections of $k$-th and $k+1$-th orders to $F^i$ will have the form:

\[ 
\int_0^{2\pi} \ldots \int_0^{2\pi} \frac{\delta F^i(\theta)}{\delta \varphi^j(\theta')} |_{\varphi=\Phi(\theta,X,T)} \epsilon^k \sum_{\alpha=1}^m c_{\alpha T}^{(k)}(X,T) \Phi^{\alpha_0}(\theta', X, T) \frac{d^{m+1}\theta'}{(2\pi)^m} 
\]

This correction has order $\epsilon^k$ but it does not appear in $f_{(k)}$ since it describes in fact the freedom of the choice of $\Psi_{(k)}(\theta, X, T)$ on the $k$-th step. It is equal to zero since all $\Phi^{\alpha_0}$ belong to the kernel of operator $\hat{L}_{[U,\theta_0]}$.

b) In the same way as in Lemma 1.1 the corrections of order $(k+1)$ to $F^i$ can be represented in the form

\[ 
\epsilon^{k+1} \sum_{\alpha=1}^m \frac{\partial F^i}{\partial \omega^\alpha}(X,T) c_{\alpha T}^{(k)}(X,T) + \epsilon^{k+1} \sum_{\alpha=1}^m \frac{\partial F^i}{\partial k^\alpha} c_{\alpha X}^{(k)}(X,T) 
\]

where the notations $\partial^i$ mean again

\[ 
\frac{\partial F^i(\varphi, \omega^\beta \varphi_\beta, \ldots)}{\partial \omega^\alpha} |_{\varphi=\Phi(\theta,X,T)}, \quad \frac{\partial F^i(\varphi, \omega^\beta \varphi_\beta, \ldots)}{\partial k^\alpha} |_{\varphi=\Phi(\theta,X,T)} 
\]

We denote a correction of this type by $-\epsilon^{k+1} f_{(k+1)}^i(\theta, X, T)$ according to our notations. Again we can state that

\[ 
f_{(k+1)}^i = \sum_{\alpha=1}^m c_{\alpha T}^{(k)}(X,T) \hat{L}_{j[U,\theta_0]}^i \Phi^{\alpha_0} + \sum_{\alpha=1}^m c_{\alpha X}^{(k)} (X,T) \hat{L}_{j[U,\theta_0]}^i \Phi^{\alpha_0} 
\]

in the same way as in Lemma 1.1 and we get then that $f_{(k+1)}^i$ always belongs to the image of operator $\hat{L}_{[U,\theta_0]}$.

Let us consider now the other two groups of terms in $f_{(k+1)}$ containing $c_{\alpha}^{(k)}(X,T)$.

II) The second group represent the correction of $f_{(1)}[\Psi_{(0)}](\theta, X, T)$ as a result of the corrections of $\Psi_{(0)}$ in the $k$-th order. The interesting part of this correction of order $(k+1)$ has again the form

\[ 
\epsilon^{k+1} \int_0^{2\pi} \ldots \int_0^{2\pi} \sum_{\alpha=1}^m \frac{\delta f_{(1)}^i(\theta, X, T)}{\delta \Psi_{(0)}(\theta', X, T)} c_{\alpha}^{(k)}(X,T) \Phi^{\alpha_0}(\theta', X, T) \frac{d^{m+1}\theta'}{(2\pi)^m} 
\]

and is equal in fact to $\epsilon^{k+1} \sum_{\alpha=1}^m c_{\alpha}^{(k)}(X,T) f_{(1)\theta_0}^i(\theta, X, T)$. So we have

\[ 
f_{(k+1)}^{II}(\theta, X, T) = \sum_{\alpha=1}^m c_{\alpha}^{(k)}(X,T) f_{(1)\theta_0}^i(\theta, X, T) 
\]

This correction has the form of a shift of the phase of $f_{(1)}$ and belongs to $f_{(k+1)}$.

III) The third group is generated by the terms in $f_{(k+1)}$ which contain the functions $\Psi_{(1)}(\theta, X, T)$ and $\Psi_{(k)}(\theta, X, T)$. Easy to see that the interesting part of these terms can be written in the form
\[
\begin{align*}
f^{IIIi}_{(k+1)}(\theta, X, T) &= -\int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \frac{\delta^2 F^i(\theta)}{\delta \varphi^l(\theta') \delta \varphi^j(\theta'')} |_{\varphi = \Phi(\theta, X, T)} \times \\times \Psi^j_{(1)}(\theta'', X, T) \sum_{\alpha=1}^{m} c^{(k)}_{\alpha}(X, T) \Phi^{l}_{\theta^{\alpha}}(\theta', X, T) \frac{d^m \theta'}{(2\pi)^m} \frac{d^m \theta''}{(2\pi)^m} \\
\text{i.e.}
\begin{align*}
f^{IIIi}_{(k+1)}(\theta, X, T) &= -\int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \sum_{\alpha=1}^{m} c^{(k)}_{\alpha}(X, T) \Phi^{l}_{\theta^{\alpha}}(\theta', X, T) \times \\times \frac{\delta L^j_{(\theta, \theta'')}}{\delta \varphi^l(\theta')} \Psi^j_{(1)}(\theta'', X, T) \frac{d^m \theta'}{(2\pi)^m} \frac{d^m \theta''}{(2\pi)^m} \\
\text{where the distribution } L^j_{(\theta, \theta'')} \text{ gives an "integral representation" of the operator } \hat{L}^j_{[U, \theta_0]}.
\end{align*}
\end{align*}
\]

In a translationally invariant (in \( \theta \)) case it’s not difficult to prove then the relation

\[
\begin{align*}
f^{IIIi}_{(k+1)}(\theta, X, T) &= -\int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \sum_{\alpha=1}^{m} c^{(k)}_{\alpha}(X, T) \times \frac{\delta L^j_{(\theta, \theta'')}}{\delta \varphi^l(\theta')} \Psi^j_{(1)}(\theta'', X, T) \frac{d^m \theta'}{(2\pi)^m} \frac{d^m \theta''}{(2\pi)^m} + \\
&+ \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \sum_{\alpha=1}^{m} c^{(k)}_{\alpha}(X, T) L^j_{(\theta, \theta'')} \Psi^j_{(1)}(\theta'', X, T) \frac{d^m \theta''}{(2\pi)^m}
\end{align*}
\]

So we get that

\[
\begin{align*}
f^{IIIi}_{(k+1)}(\theta, X, T) + f^{IIIi}_{(k+1)}(\theta, X, T) &= \hat{L}^j_{i} \sum_{\alpha=1}^{m} c^{(k)}_{\alpha}(X, T) \Psi^j_{(1)}(\theta^{\alpha}, X, T)
\end{align*}
\]

which belongs to the image of \( \hat{L}^j_{i} \). Thus we get the statement of the Lemma for \( k \geq 2 \).

Let us consider now the case \( k = 1 \). Let us represent for simplicity the solution \( \Psi^j_{(1)}(\theta, X, T) \) in the form

\[
\Psi^j_{(1)}(\theta, X, T) = \Psi^j_{(1)}(\theta, X, T) + \sum_{\alpha=1}^{m} c^{(1)}_{\alpha}(X, T) \Phi^{l}_{\theta^{\alpha}}(\theta, X, T)
\]

where the freedom connected with vectors \( Q^{(1)}_{\alpha}(\theta, X, T) \) is included in \( \Psi^j_{(1)}(\theta, X, T) \).

It is not difficult to see then that in order \( \epsilon^2 \) we will have all the terms described above for \( k \geq 2 \) and one extra term:

\[
\begin{align*}
f^{IVi}_{(2)}(\theta, X, T) &= -\int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \frac{\delta^2 F^i(\theta)}{\delta \varphi^l(\theta') \delta \varphi^j(\theta'')} |_{\varphi = \Phi(\theta, X, T)} \times \\times \sum_{\alpha, \beta=1}^{m} c^{(1)}_{\alpha}(X, T) c^{(1)}_{\beta}(X, T) \Phi^{j}_{\theta^{\alpha}}(\theta', X, T) \Phi^{l}_{\theta^{\beta}}(\theta'', X, T) \frac{d^m \theta'}{(2\pi)^m} \frac{d^m \theta''}{(2\pi)^m}
\end{align*}
\]
containing the functions $c^{(1)}(X, T)$.

Consider the expansion of the values $F^i(\Phi, \omega^a \Phi_g, \ldots)$ under a small shift of phase $\theta \rightarrow \theta + \delta \theta$ where $\delta \theta = c^{(1)} \delta z, \delta z \rightarrow 0$. First of all we know that all the orders of this expansion should be equal to zero on $\Lambda$ since the values $F^i(\Phi, \omega^a \Phi_g, \ldots)$ remain zero on $\Lambda$ after this shift.

The second order of this expansion consists of two parts. The first part is equal to $-f_{(2)}^{IVi} (\delta z)^2$ while the second one is equal to

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\delta F^i(\theta)}{\delta \varphi^j(\theta')} \delta \varphi^j(\theta') d\theta \cdots d\theta \sum_{\alpha, \beta = 1}^m c^{(1)}(X, T) c^{(1)}(X, T) (\delta z)^2 \Phi_{\theta=\theta^j}(\theta', X, T) \frac{d^m \theta'}{(2\pi)^m}$$

We get then

$$f_{(2)}^{IVi} = \hat{L}_{(U, R)}^{i} \sum_{\alpha, \beta = 1}^m c^{(1)}(X, T) c^{(1)}(X, T) \Phi_{\theta=\theta^j}(\theta, X, T)$$

i.e. $f_{(2)}^{IV}$ belongs to $Im \hat{L}_{(U, R)}$. We obtain thus the statement of the Lemma for all $k \geq 1$. Lemma 1.2 is proved.

For the case of a full regular family of (one-phase) solutions of system (1.1) we can see then a regular scheme of construction of asymptotic solution (1.4) if the Whitham system (1.8) is satisfied. Namely, we find the correction $\Psi^{(1)}(\theta, X, T)$ from the system (1.7) modulo linear combination (1.11) and then try to find the correction $\Psi^{(2)}(\theta, X, T)$. The resolvability conditions for system (1.6) for $k = 2$ then give us restrictions on the functions $\theta_0(X, T), d_i^{(1)}(X, T), l = 1, \ldots, s$. Provided that the corresponding conditions are satisfied we find the solutions $\Psi^{(2)}(\theta, X, T)$ modulo a linear combination of the same type. Now at every step $k > 2$ we will have the restrictions on the functions $c^{(k-2)}(X, T), d_i^{(k-1)}(X, T)$ and obtain the solution $\Psi^{(k)}(\theta, X, T)$ modulo the linear combination

$$c^{(k)}(X, T) \Phi_{\theta}(\theta, X, T) + \sum_{l=1}^{s} d_i^{(k)}(X, T) \Phi_{\omega^l}(\theta, X, T) \quad (1.12)$$

So we get a regular way of the recurrent construction of functions $\Psi^{(k)}(\theta, X, T)$.

The procedure described above looks quite natural. Indeed, the functions $\theta_0(X, T), c^{(1)}(X, T), c^{(2)}(X, T), \ldots$ represent in fact corrections to the modulated phase $S(X, T)$ while the functions $d_i^{(1)}(X, T), d_i^{(2)}(X, T), \ldots$ represent corrections to the parameters $n = (n^1, \ldots, n^s)$ of the zero approximation $\Psi_{(0)}(\theta, X, T)$. The Whitham system (1.8) or (1.3) gives restrictions on the functions $\omega(X, T) = S_T, k(X, T) = S_X$ and $n_l(X, T)$ as a resolvability condition of system (1.7) for the first correction $\Psi^{(1)}(\theta, X, T)$. It is natural then that the resolvability conditions of system (1.6) for $k = 2$ give restrictions on the corrections $\theta_0(X, T), c^{(1)}(X, T), c^{(2)}(X, T), \ldots$ and $d_i^{(1)}(X, T), d_i^{(2)}(X, T), \ldots$ to $S(X, T)$ and $n_l(X, T)$ in a successive order.
The Whitham solution (1.4) can be rewritten also in the form:

\[
\phi^i(\theta, X, T, \epsilon) = \Phi^i \left( \frac{S(X, T, \epsilon)}{\epsilon} + \theta, S_X(X, T, \epsilon), S_T(X, T, \epsilon), n(X, T, \epsilon) \right) + \\
+ \sum_{k \geq 1} \tilde{\Psi}^i_{(k)} \left( \frac{S(X, T, \epsilon)}{\epsilon} + \theta, X, T \right) \epsilon^k
\]

(1.13)

where we allow the regular \( \epsilon \)-dependence

\[
S(X, T, \epsilon) = \sum_{k \geq 0} S(k)(X, T) \epsilon^k, \quad n'(X, T, \epsilon) = \sum_{k \geq 0} n_l(k)(X, T) \epsilon^k
\]

of the phase \( S \) and the parameters \((k, \omega, n)\) of the zero approximation \( \Psi(0)(\theta, X, T) \) such that

\[
k(X, T, \epsilon) = S_X(X, T, \epsilon), \quad \omega(X, T, \epsilon) = S_T(X, T, \epsilon)
\]

In this approach the functions \( \tilde{\Psi}_{(k)}(\theta, X, T) \) can be normalized in some way while the functions \( S(k), n_l(k) \) can be used to provide the resolvability conditions of systems (1.6). For instance, the normalization of \( \tilde{\Psi}_{(k)}(\theta, X, T) \) used in [47] required that the main term of (1.13) gives ”the best approximation” to solution (1.4) such that the rest of series (1.13) is orthogonal to the functions

\[
\Phi_{\theta} \left( \frac{S(X, T, \epsilon)}{\epsilon} + \theta, S_X(X, T, \epsilon), S_T(X, T, \epsilon), n(X, T, \epsilon) \right)
\]

and

\[
\Phi_{n'} \left( \frac{S(X, T, \epsilon)}{\epsilon} + \theta, S_X(X, T, \epsilon), S_T(X, T, \epsilon), n(X, T, \epsilon) \right)
\]

at every \( \epsilon \).

The functions \( \tilde{\Psi}_{(k)}(\theta, X, T) \) satisfy linear systems analogous to (1.6), i.e.

\[
\tilde{L}_j[H(S_{(0)}, n_{(0)})] \tilde{\Psi}_{(k)}^j(\theta, X, T) = \tilde{f}_{(k)}^j(\theta, X, T)
\]

(1.14)

The functions \( \tilde{f}_{(k)} \) are slightly different from \( f_{(k)} \) since a ”part of \( \epsilon \)-dependence” is included now in the main term of (1.13).

Let us formulate here the Lemma ([47]) about the systems on the functions \( S(k)(X, T), n_l(k)(X, T), (k \geq 1) \) arising in this approach.

**Lemma 1.3.**

*In the case of a full regular family of (one-phase) solutions of (1.1) the functions \( S(k)(X, T), n_l(k)(X, T) \) satisfy the linearized Whitham system on the functions \( S_{(0)}(X, T), n_{(0)}(X, T) \) with an additional right-hand part depending on the functions \( S_{(0)}, \ldots, S_{(k-1)}, n_{(0)}, \ldots, n_{(k-1)} \).*
In fact, it is not difficult to show that the functions $c^{(k-1)}(X, T)$, $d^{(k)}_t(X, T)$, $(k \geq 1)$ satisfy rather similar systems in this case.

Let us say that we can give a definition of a full regular family of $m$-phase solutions of (1.1) also for the case $m > 1$ if we require the properties of Definition 1.1 for generic $k = (k^1, \ldots, k^m)$ and $\omega = (\omega^1, \ldots, \omega^m)$. Let us give here the corresponding definition.

**Definition 1.1’.**

We call the family $\Lambda$ a full regular family of $m$-phase solutions of (1.1) if

1) The functions $\Phi^{\theta}(\theta, k, \omega, n)$, $\Phi^{nl}(\theta, k, \omega, n)$ are linearly independent and give for generic $k$ and $\omega$ the full basis in the kernel of the operator $\hat{L}^{i}_{ij}[\theta_0, k, \omega, n]$;

2) The operator $\hat{L}^{i}_{ij}[\theta_0, k, \omega, n]$ has for generic $k$ and $\omega$ exactly $m + s$ linearly independent ”left eigen vectors”

$$\kappa^{(q)}_{(U)}(\theta + \theta_0) = \kappa^{(q)}_{(k, \omega, n)}(\theta + \theta_0)$$

depending on the parameters $U$ in a smooth way and corresponding to zero eigen-values.

All the constructions described above can be used also for the full regular family of $m$-phase solutions if we require that the orthogonality conditions

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \kappa^{(q)}_{[U]}(\theta, X, T) f^{i}_{(k)}(\theta, X, T) \frac{d^m \theta}{(2\pi)^m} = 0 \quad (1.15)$$

give the necessary and sufficient conditions of resolvability of systems (1.6). This is a serious requirement and it can be shown ([15, 16, 32, 33]) that it is not satisfied in the general case. However, there exist examples where this requirement is satisfied in $m$-phase situation so the same procedure of construction of asymptotic series (1.4) (or (1.13) can be used in this case.

Let us assume in our further considerations that we have a full regular family of one-phase or $m$-phase solutions of (1.1) and the compatibility conditions of system (1.6) are defined by orthogonality conditions (1.15). Let us say again, that this situation is given by rather specific examples in a multi-phase case.

As far as we know dispersive corrections to the Whitham systems were first considered by M.Y. Ablowitz and D.J. Benney ([5], also [6]-[7]) where the first consideration of a multi-phase Whitham method was also made. As was pointed out in [5] the higher corrections in Whitham method satisfy more complicated equations including ”dispersive terms” and the Whitham system (1.8) should in fact contain also the higher derivatives (”dispersion”) being considered in the next orders of $\epsilon$.

In [47, 48] a general procedure of deformation of the Whitham systems based on a renormalization of parameters was suggested. The deformations of the Whitham systems appeared in [47, 48] have the so-called Dubrovin-Zhang form and were considered in connection with B.A. Dubrovin problem of deformations of Frobenius manifolds. Let us say here some words about this deformation scheme.
The higher corrections to Topological Quantum Field theories require the deformations (38, 40, 42) of the Hydrodynamic Type hierarchies (1.9) having the form

\[ U_\nu = V_\mu(U) U_\mu_X + \sum_{k\geq 2} v^{(k)}(U, U_X, \ldots, U_{kX}) \epsilon^{k-1} \]  

where all \( v^{(k)} \) are smooth functions polynomial in the derivatives \( U_X, \ldots, U_{kX} \) and having degree \( k \) according to the following gradation rule:

1) All the functions \( f(U) \) have degree 0;
2) The derivatives \( U_\mu_X \) have degree \( k \);
3) The degree of the product of two functions having certain degrees is equal to the sum of their degrees.

Deformation (1.16) of system (1.9) implies also the deformation of the corresponding (bi-)Hamiltonian structures (1.10)

\[ \{U_\nu(X), U_\mu(Y)\} = \{U_\nu(X), U_\mu(Y)\}_0 + \]  

\[ + \sum_{k\geq 2} \sum_{s=0}^{k} B^{(k,s)}(U, U_X, \ldots, U_{(k-s)X}) \delta^{(s)}(X - Y) \]  

where all \( B^{(k,s)} \) are polynomial w.r.t. derivatives \( U_X, \ldots, U_{(k-s)X} \) and have degree \((k-s)\).

We call deformations of form (1.16)–(1.17) deformations of Dubrovin-Zhang type. Let us say that form (1.16)–(1.17) is not the only possible form of deformation of the Whitham system. For instance, a Lorentz-invariant scheme for nonlinear Klein-Gordon equation was considered in [49].

However, as we will see, the deformation procedure used in [47, 48, 49] is not very good for "almost linear systems" where a non-linearity is rather small. This instability is connected with the general instability of the Whitham approximation in the higher orders for the case of the small amplitude of oscillations which was pointed out by A.C. Newell in [20]. In the next chapter we will suggest a deformation scheme of the Whitham system for such "almost linear" systems which should describe the slow modulations of periodic (or quasiperiodic) solutions in this situation. We will use here the deformations of Dubrovin-Zhang form (1.16) although other types of gradation rules are also possible as well.

In general a deformation of the Whitham system can be (in new notations) described in the following way:

We look for a solution of (1.1) having the form

\[ \varphi(\theta, X, T) = \Phi(S(X, T) + \theta, S_X, S_T, n) + \sum_{k\geq 1} \Psi(k)(S(X, T) + \theta, X, T) \]  

where the functions \( \Psi(k) \) are now local functionals of \( (k, \omega, n) \) and there derivatives having gradation degree \( k \). We omit now the parameter \( \epsilon \) although we put first \( S = S(X, T, \epsilon), n = n(X, T, \epsilon) \). Now the higher derivatives of \( (k, \omega, n) \) play the role of small parameters.
in the expansion according to chosen gradation rule. The Dubrovin-Zhang gradation rule
implies the following simple definitions:

1) The functions \( k^\alpha(X, T) = S^\alpha_X(X, T), \omega^\alpha(X, T) = S^\alpha_T(X, T), \) and \( n'(X, T) \) have
degree 0;
2) Every differentiation with respect to \( X \) adds 1 to the degree of a function;
3) The degree of the product of two functions having certain degrees is equal to the
sum of their degrees.

The functions \( \Psi_{(k)}(\theta, X, T) \) are defined from the linear systems
\[
\hat{L}^i_{[S_X, S_T, n_j]} \Psi^j_{(k)}(\theta, X, T) = f^i_{(k)}(\theta, X, T) \tag{1.19}
\]
where \( f^i_{(k)}(\theta, X, T) \) is the discrepancy having gradation \( k \) according to the rules introduced
above.

The functions \( \Psi_{(k)} \) are uniquely normalized by the conditions
\[
\int_0^{2\pi} \ldots \int_0^{2\pi} \sum_{i=1}^n \Phi_{\alpha}^i(\theta, S_X, S_T, n) \Psi^i_{(k)}(\theta, X, T) \frac{d^m\theta}{(2\pi)^m} = 0 \tag{1.20}
\]
\[
\int_0^{2\pi} \ldots \int_0^{2\pi} \sum_{i=1}^n \Phi_{\omega}^i(\theta, S_X, S_T, n) \Psi^i_{(k)}(\theta, X, T) \frac{d^m\theta}{(2\pi)^m} = 0 \tag{1.21}
\]
k \geq 1, \ (\alpha = 1, \ldots, m, \ l = 1, \ldots, s), and are local functionals of \( (k, \omega, n) \) and there
derivatives having gradation degree \( k \).

The ”renormalized” modulated phase \( S(X, T) \) and parameters \( n(X, T) \) satisfy now
the deformed Whitham system
\[
S^\alpha_{TT} = \sum_{k \geq 1} \sigma^\alpha_{(k)}(k, \omega, n, k_X, \omega_X, n_X, \ldots) \tag{1.22}
\]
\[
n^l_T = \sum_{k \geq 1} \eta^l_{(k)}(k, \omega, n, k_X, \omega_X, n_X, \ldots) \tag{1.23}
\]
where \( \sigma^\alpha_{(k)}, \eta^l_{(k)} \) are general polynomials in derivatives \( k_X, \omega_X, n_X, k_{XX}, \omega_{XX}, n_{XX}, \ldots \)
(with coefficients depending on \( (k, \omega, n) \)) having degree \( k \).

The functions \( \sigma_{(1)}, \eta_{(1)} \) coincide with the right-hand part of the Whitham system
(1.1) and the functions \( \sigma_{(k)}, \eta_{(k)} \) are defined by orthogonality conditions arising on the
\( k \)-th order of (1.19). Let us remind again that we work here with a full regular family
of \( (m \text{-phase}) \) solutions of (1.1). Besides that we imply that the orthogonality conditions
of \( f_{(k)} \) to all ”regular” left eigen-vectors \( k^{(q)} \) introduced in Definition 1.1’ are equivalent
to the resolvability conditions of the system (1.19). The deformation of the Whitham
system can be rewritten also in parameters \( (k, \omega, n) \) in obvious way
\[
k^\alpha_T = \omega^\alpha_X
\]
\[ \omega_T^n = \sum_{k \geq 1} \sigma^n_{(k)}(k, \omega, n, k_X, \omega_X, n_X, \ldots) \quad (1.24) \]
\[ n_T' = \sum_{k \geq 1} \eta(k)(k, \omega, n, k_X, \omega_X, n_X, \ldots) \]

2 Deformation procedure for almost linear systems.

Let us say however that the regular procedure of deformation formulated above does not behave well in the case of "almost linear" systems (1.1). This means that the procedure of deformation and the corresponding asymptotic series (1.18) do not have a good limit for systems (1.1) when

\[ F^i(\lambda, \varphi, \varphi_t, \varphi_x, \ldots) = F^i_0(\varphi, \varphi_t, \varphi_x, \ldots) + F^i_1(\lambda, \varphi, \varphi_t, \varphi_x, \ldots) \]
where \( F^i_0 \) are linear in \( (\varphi, \varphi_t, \varphi_x, \ldots) \) and \( F^i_1(\lambda, \varphi, \varphi_t, \varphi_x, \ldots) \to 0 \) when \( \lambda \to 0 \).

A reason for such a behavior is that the operator \( \hat{L}_j^i(\lambda) \) can now be represented in the form

\[ \hat{L}_j^i(\lambda) = \hat{L}_{0j}^i + \hat{L}_{1j}^i(\lambda) \]

where the operator \( \hat{L}_{0j}^i \) has in fact a larger number of ("left" and "right") eigenvectors corresponding to zero eigen-values than the operator \( \hat{L}_{1j}^i \) (see [20], Chp. 2).

Indeed, let us consider for instance the system

\[ \varphi_{tt} - \varphi_{xx} + \varphi + \lambda \varphi^3 = 0 \quad , \quad \lambda \to 0 \quad (2.1) \]

It is well known that system (2.1) has a two-parametric family of one-phase solutions

\[ \varphi = \Phi_\lambda(kx + \omega t + \theta_0, \mu) \quad , \quad (\mu = \omega^2 - k^2) \]
depending on the parameters \( k \) and \( \omega \).

Since the amplitude of the solution depends on \( \mu \) in a singular way near the point \( \lambda = 0 \) it is more convenient to use the parameters \( k \) and \( A = \Phi_{\text{max}} - \Phi_{\text{min}} \) in this situation. So we will write

\[ \varphi = \Phi_\lambda(kx + \omega(k, A, \lambda)t + \theta_0, A) \]
in the new notations. If we choose the initial phase such that the function \( \Phi_\lambda(\theta, A) \) has a local maximum at \( \theta = 0 \) then we will have the additional condition \( \Phi_\lambda(\theta, A) = \Phi_\lambda(-\theta, A) \).

We assume also that \( \Phi_\lambda(\theta, A) \equiv \Phi_\lambda(\theta + 2\pi, A) \) as usually and we have in the limit \( \lambda \to 0 \)

\[ \lim_{\lambda \to 0} \Phi_\lambda(\theta, A) = A \cos \theta \]
The dependence $\mu(A)$ disappears in the limit $\lambda \to 0$ and we have $\mu = 1$ for $\lambda = 0$.

The operator

$$\hat{L}(\lambda) = (\omega^2 - k^2) \frac{d^2}{d\theta^2} + 1 + 3\lambda \Phi_\lambda^2(\theta)$$

has only one (both ”left” and ”right”) eigen-vector $\kappa_\lambda(\theta, A) = \Phi_{\lambda, \theta}(\theta, A)$ corresponding to zero eigen-value. We have

$$\lim_{\lambda \to 0} \Phi_{\lambda, \theta}(\theta, A) = -A \sin \theta$$

which is an eigen-vector of the operator $\hat{L}_0$. However, the function $\cos \theta$ is also an eigen-vector (both ”left” and ”right”) of the operator $\hat{L}_0 = d^2/d\theta^2 + 1$ corresponding to zero eigen-value. As a result, the operator $\hat{L}(\lambda)$ has an eigen-vector (both ”left” and ”right”) $\zeta_\lambda(\theta, A)$ corresponding to a ”small” eigen-value $\nu(\lambda, A)$, such that $\nu(\lambda, A) \to 0$ for $\lambda \to 0$.

We obtain then that even if the resolvability conditions of systems (1.19) are satisfied the solutions $\Psi_k(\theta, X, T)$ can be singular at $\lambda \to 0$ if the right-hand parts $f_k(\theta, X, T)$ are not orthogonal to the eigen-vector $\zeta_\lambda(\theta, A)$. We can see then that it’s natural to require orthogonality of all $f_k$ to both the eigen-vectors $\kappa_\lambda(\theta, A)$ and $\zeta_\lambda(\theta, A)$ to make the procedure regular in the limit $\lambda \to 0$.

In this situation we can not require anymore the normalization conditions (1.20) and just require that all the corrections $\Psi_k$ in expansion (1.18) are regular functions at $\lambda \to 0$. Using these requirements we can obtain now both the deformed Whitham system for system (1.1) and normalization conditions for coefficients $c_k(X, T)$ arising in the definition of the function $\Psi_k(\theta, X, T)$.

Let us consider now the corresponding procedure in a more general formulation. We will omit for simplicity the additional parameters $n$ and consider a one-phase situation.

Let us assume that we have an ”almost linear” system (1.1) and a full regular family of (one-phase) periodic solutions

$$\varphi(x, t) = \Phi_i^0(kx + \omega(k, A, \lambda)t + \theta_0, A)$$

Here we choose the parameters $(k, A)$ instead of $(k, \omega)$ in the limit $\lambda \to 0$ where $A$ is some parameter playing the role of amplitude. We will also assume that the dependence $\omega(k, A, \lambda)$ becomes the dispersion relation $\omega(0)(k)$ for some branch of the spectrum of linear system

$$F_0^i(\varphi, \varphi_t, \varphi_x, \ldots) = 0 \quad (2.2)$$

and the function $\Phi_\lambda(\theta, k, A)$ becomes the corresponding solution $A \Phi_0(\theta, k)$ of the system

$$F_0^i(\Phi, \omega(0)(k) \Phi_\theta, k \Phi_\phi, \ldots) = 0 \quad (2.3)$$

It is convenient to assume also that all the systems (1.1), (2.2), (2.3) are written in a real form and both the dispersion relation $\omega(0)(k)$ and the functions $\Phi_0^i(\theta, k)$ are real.
functions. We also have that $\Phi_0(\theta + 2\pi, k) \equiv \Phi_0(\theta, k)$ and the function $\Phi_{0,\theta}(\theta, k)$ is proportional to $\Phi_0(\theta, k)$ in this case.

The function $\Phi_{\lambda,\theta}(\theta, k, A)$ is a ”right” eigen-vector of the operator $\hat{L}(\lambda)$ corresponding to zero eigen-value and we require that there is also exactly one ”left” eigen vector $\kappa_{[\lambda, k, A]}(\theta)$ of $\hat{L}(\lambda)$ corresponding to zero eigen-value for all $\lambda$. The limit of $\Phi_{\lambda,\theta}(\theta, k, A)$ for $\lambda \to 0$ is equal to $A \Phi_{0,\theta}(\theta, k)$ and we denote $\kappa_{0[k]}(\theta)$ the limit of the (normalized) vector $\kappa_{[\lambda, k, A]}(\theta)$ for $\lambda \to 0$. Besides that, we assume that there exist ”left” and ”right” real eigenvectors $\zeta_{[\lambda, k, A]}(\theta)$ and $\xi_{[\lambda, k, A]}(\theta)$ of the operator $\hat{L}(\lambda)$ corresponding to a ”small” eigen-value $\nu(\lambda, k, A)$ which give the additional ”left” and ”right” real eigen-vectors $\zeta_{0[k]}(\theta)$ and $\xi_{0[k]}(\theta)$ of operator $\hat{L}_0$ corresponding to zero eigen-value. As we said already the existence of the vectors $\xi_{[\lambda, k, A]}$ and $\zeta_{[\lambda, k, A]}$ is connected with the existence of additional real eigenvectors $\Phi_{0,\theta}(\theta, k)$ (or $\Phi_0(\theta, k)$) and $\kappa_{0[k]}(\theta)$ lying in the kernel of the operators $\hat{L}_0$ and $\hat{L}_0^*$ respectively.

We assume that the branch (0) of the spectrum of $\hat{L}_0$ is non-degenerate and

$$\frac{\partial \hat{L}_0(\omega, k)}{\partial \omega} |_{\omega=\omega(0)(k)} \Phi_0(\theta, k) \neq 0$$

(the notation $\partial'/\partial'\omega$ means here that we don’t keep $\omega$ and $k$ connected by the dispersion relation and consider them as free parameters after the substitution $\partial/\partial t \to \omega \partial/\partial \theta$, $\partial/\partial x \to k \partial/\partial \theta$).

We have the identity

$$\int_0^{2\pi} \kappa_{0[k]}(\theta) \frac{\partial \hat{L}_0^i(\omega, k)}{\partial \omega} |_{\omega=\omega(0)(k)} \Phi_0^i(\theta, k) \frac{d\theta}{2\pi} = 0 \quad (2.4)$$

Indeed, expression (2.4) is a limit of the expression

$$\int_0^{2\pi} \kappa_{[\lambda, k, A]}(\theta) \frac{\partial F^i(\lambda, \varphi, \omega, \varphi_\theta, \ldots, k \varphi_\theta, \ldots)}{\partial \omega} |_{\varphi=\Phi_\lambda(\theta, k, A(\omega, k))} \frac{d\theta}{2\pi}$$

(in parameters $(\omega, k)$) at $\lambda \to 0$. This expression according to (1.3) is equal to

$$- \int_0^{2\pi} \kappa_{[\lambda, k, A]}(\theta) \frac{\partial \hat{L}_0^i(\lambda, \omega, k)}{\partial \omega} \Phi_\lambda^i(\theta, k, A(\omega, k)) \frac{d\theta}{2\pi}$$

which is identically zero.

However, for the vector $\zeta_{0[k]}$ we don’t have an identity like (2.4) in generic situation, so we will imply

$$\int_0^{2\pi} \zeta_{0[k]}(\theta) \frac{\partial \hat{L}_0^i(\omega, k)}{\partial \omega} |_{\omega=\omega(0)(k)} \Phi_0^i(\theta, k) \frac{d\theta}{2\pi} \neq 0 \quad (2.5)$$

and

$$\int_0^{2\pi} \zeta_{[\lambda, k, A]}(\theta) \frac{\partial F^i(\lambda, \varphi, \omega, \varphi_\theta, \ldots, k \varphi_\theta, \ldots)}{\partial \omega} |_{\varphi=\Phi_\lambda(\theta, k, A)} \frac{d\theta}{2\pi} = O(1) \quad , \lambda \to 0 \quad (2.6)$$
We can suggest now a recurrent procedure of construction of asymptotic solutions \((1.18)\) and the deformation of the Whitham system in the almost linear case. As we said already we are going to require the orthogonality of the discrepancies \(f_{(k)}(\theta, X, T)\) to both the "left" eigen-vectors \(\kappa_{\lambda[k,A]}(\theta + \theta_0)\) and \(\zeta_{\lambda[k,A]}(\theta + \theta_0)\) at each step. We don’t put normalization conditions \((1.20)\) now and use the freedom in the coefficients \(c^k(X, T)\) and \(\theta_0(X, T)\) to provide orthogonality of \(f_{(k)}\) and \(\zeta_{\lambda[k,A]}\). So we try to find a deformation of the Whitham system in the form

\[
k_T = (\omega(k, A))_X, \quad A_T = \sum_{k \geq 1} a_k(k, A, k_X, A_X, \ldots)
\]

where all \(a_k\) are polynomial in \((k_X, A_X, k_{XX}, A_{XX}, \ldots)\) and have degree \(k\). Every function \(a_k(k, A, k_X, A_X, \ldots)\) is found as previously from the orthogonality conditions of \(f_{(k)}\) and \(\kappa_{\lambda}(\theta, X, T)\) in the \(k\)-th order. The system

\[
k_T = (\omega(k, A))_X, \quad A_T = a_1(k, A, k_X, A_X)
\]

coinsides with the Whitham system for the system \((1.1)\).

The functions \(\theta_0(X, T)\) and \(c^{(1)}(X, T), c^{(2)}(X, T), \ldots\) are defined from the orthogonality conditions of \(f_{(1)}\) and \(f_{(2)}, f_{(2)}, \ldots\) to the "left" eigen-vector \(\zeta_{\lambda}(\theta, X, T)\) respectively. It’s not difficult to obtain the form of the systems arising on the function \(\theta_0(X, T)\) and \(c^{(k)}(X, T)\). Indeed, as we saw in the proof of Lemma 1.1 the part of \(f_{(1)}\) containing the function \(\theta_0(X, T)\) has the form

\[
\tilde{f}_{(1)}(\theta, X, T) = - \frac{\partial F^i}{\partial \omega} \theta_{0T} - \frac{\partial F^i}{\partial k} \theta_{0X}
\]

So we obtain that the equation on \(\theta_0(X, T)\) is the first order linear differential equation which can be written in the form

\[
Q_\lambda(k, A) \theta_{0T} + P_\lambda(k, A) \theta_{0X} = R_\lambda(k, A, k_X, A_X)
\]

where

\[
Q_\lambda(k, A) = \int_0^{2\pi} \zeta_{\lambda[k,A]}(\theta) \frac{\partial F^i}{\partial \omega}(\theta, k, A) \frac{d\theta}{2\pi}, \quad P_\lambda(k, A) = \int_0^{2\pi} \zeta_{\lambda[k,A]}(\theta) \frac{\partial F^i}{\partial k}(\theta, k, A) \frac{d\theta}{2\pi}
\]

It can be also seen from the proof of the Lemma 1.2 that all the systems on the functions \(c^{(k)}(X, T)\) can be written in a similar form, i.e.

\[
Q_\lambda(k, A) c^{(k)}_{0T} + P_\lambda(k, A) c^{(k)}_{0X} + H_\lambda(k, A, k_X, A_X) c^{(k)} = R^{(k)}_{\lambda}[k, A, c^{(1)}, \ldots, c^{(k-1)}], \quad k \geq 2
\]

\[
Q_\lambda(k, A) c^{(1)}_{0T} + P_\lambda(k, A) c^{(1)}_{0X} + H_\lambda(k, A, k_X, A_X) c^{(1)} + W_\lambda(k, A) c^{(1)} c^{(1)} = R^{(1)}_{\lambda}[k, A]
\]

where \(Q_\lambda(k, A)\) is a non-vanishing function at \(\lambda \to 0\) according to \((2.3)\).
For deformed Whitham system (2.7) and a corresponding solution of the Cauchy problem it is natural to consider also the Cauchy problem for systems (2.8)-(2.9) and define the functions \(\theta_0(X, T)\), \(c^{(k)}(X, T)\). So, finally we get a recurrent procedure of deformation of the Whitham system and the construction of asymptotic solutions (1.18) which is regular at \(\lambda \to 0\). Let us note that the deformed Whitham system here is different in general case from that described in the first chapter because of the different normalization of the corrections \(\Psi^{(k)}(\theta, X, T)\).

It is not difficult to understand why normalization conditions (1.20) are not very good in the almost linear situation. Indeed, in the case close to linear one the dependence \(A(k, \omega, \lambda)\) becomes a singular function for \(\lambda \to 0\) since \(k\) and \(\omega\) are not independent in a linear case. So, the rigid fixation of phase \(S(X, T)\) by conditions (1.20) leads to quite an ”unstable” behavior of the main term in (1.18) depending on \(S_T\) and \(S_X\) as on parameters. As a result, all the higher corrections in (1.18) become singular functions at \(\lambda \to 0\). Relations (2.8)-(2.9) describe in this case the corrections to \(S(X, T)\) which should not be included in the parameters of the main approximation to keep all the terms stable in the limit \(\lambda \to 0\).

To avoid this difficulty arising in the Whitham approach it was suggested by A.C. Newell not to keep a rigid dispersion relation between \(\omega\), \(k\), and \(A\) and correct the Whitham equations in the almost linear case \([20]\).

Indeed, we can see now that the most natural way is in fact not to put a rigid connection between the phase \(S(X, T)\) and the parameters \(\omega, k\) of the main approximation

\[
\Psi^{(0)}(\theta, X, T) = \Phi \left( \theta + \theta_0(X, T), k(X, T), A(k(X, T), \omega(X, T)) \right)
\]

if we don’t want to introduce the additional parameters \(\theta_0(X, T), c^{(k)}(X, T)\).

Indeed we can try to write the asymptotic solution of (1.1) in the form

\[
\varphi = \Phi \left( S(X, T) + \theta, k, A \right) + \sum_{k \geq 1} \Psi^{(k)}(S(X, T) + \theta, X, T) \tag{2.10}
\]

where we don’t require the exact relations \(S_T = \omega(k, A)\), \(S_X = k\) anymore. Instead, we allow now corrections to the phase \(S(X, T)\) and put these relations just in the main order of our asymptotic expansion. This approach gives us the possibility to put again the normalization conditions (1.20) and include the corrections generated by the functions \(\theta_0(X, T), c^{(k)}(X, T)\) to the phase \(S(X, T)\).

The type of the corrections to the phase \(S(X, T)\) depends on a deformation type we choose. For a deformation of type (1.16) it is convenient to put

\[
S_X = k, \quad S_T = \omega(k, A) + \sum_{k \geq 1} \omega^{(k)}(k, A, k_X, A_X, \ldots) \tag{2.11}
\]

where all \(\omega^{(k)}\) are polynomial in \((k_X, A_X, k_{XX}, A_{XX}, \ldots)\) and have degree \(k\). As previously, we assume here that any function of \(k\) and \(A\) has degree 0 and every differentiation with respect to \(X\) adds 1 to the degree of a function.
Now we put again normalization conditions (1.20) and look for both the deformation of dispersion relation (2.11) and the Whitham system which has now the form

\[ k_T = \frac{d}{dX} \left( \omega(k, A) + \sum_{k \geq 1} \omega(k) \right), \quad A_T = \sum_{k \geq 1} a(k) \]

System (2.12) is a closed system on the functions \( k(X, T), A(X, T) \) and it's natural to consider this system as a full deformation of the Whitham system in the almost linear case.

Now the procedure of construction of asymptotic series (2.10) looks similar to the previous one and we look for the functions \( \Psi(k) \) as for local functionals of \( (k, A, kX, A_X, \ldots) \) polynomial in derivatives of \( (k, A) \) and having degree \( k \). The functions \( \Psi(k) \) satisfy systems analogous to (1.19) and are uniquely determined by normalization conditions (1.20). A difference in this approach is that we require now the orthogonality of all the functions \( f(k) \) to both the "left" eigen-vectors \( \kappa_\lambda(\theta, X, T) \) and \( \zeta_\lambda(\theta, X, T) \) of the operator \( \hat{L}_i^j(\lambda) \) corresponding to zero and "small" eigen-values. The functions \( \omega(k) \) and \( a(k) \) are uniquely determined by the orthogonality conditions in the \( k \)-th order which gives the recurrent procedure of construction of the deformation of Whitham system (2.12) and asymptotic series (2.10).

Let us call system (2.12) the deformation of the Whitham system in an almost linear case.

In the limit \( \lambda \to 0 \) system (2.12) gives a system describing slow modulations of solutions of a purely linear system (1.1). All the functions \( f(k) \) coincide in this case with their first Fourier harmonics and in fact only two linearly independent functions of \( \theta \) \( (\Phi_0(\theta), \kappa_0(\theta), \zeta_0(\theta)) \) arise at every \( X \) and \( T \) after the substitution of \( \Phi_0(\theta) \) in initial system (1.1). The orthogonality of all \( f(k) \) to both the functions \( \kappa_0(\theta), \zeta_0(\theta) \) gives in this case the relations \( f(k) \equiv 0 \) and we have \( \Psi(k) \equiv 0 \) in this situation.

All the sums in (2.11) and (2.12) contain just the finite number of terms in the linear case and the exact solution of (1.1) is given by the relation

\[ \varphi(\theta, X, T) = A(X, T) \Phi(S(X, T) + \theta, k(X, T)) \]

where the functions \( S(X, T), A(X, T), k(X, T) \) satisfy the systems (2.11)-(2.12).

For the \( \lambda \)-expansion of systems (2.11)-(2.12) the Fourier expansion of the functions \( \Phi_\lambda(\theta, k, A), \kappa_\lambda[k,A](\theta), \zeta_\lambda[k,A](\theta) \) can be used. Indeed, the higher harmonics of functions \( \Phi_\lambda, \kappa_\lambda, \zeta_\lambda \) decrease usually as some power of \( \lambda \) so we need just a finite number of Fourier harmonics at a given order of \( \lambda \). This approach can be connected then with the well known method of including of a non-linearity to the slowly modulated solutions of linear systems (see for instance [3]). Thus, the method of derivation of modulation equations for the parameters of the first Fourier harmonic of the solution ([3], Chp. 15-16) can be considered as a calculation of the first \( \lambda \)-correction in the system (2.11)-(2.12) in this case.

Finally, let us say just some words about multi-phase situation. As we told above, a multi-phase situation can be much more complicated for the construction of asymptotic
totic solutions (1.18) or (2.10). So all our remarks should be addressed to some special cases when the orthogonality of the discrepancies \( f_{(k)}(\theta, X, T) \) to the “regular” left eigen-vectors \( \kappa^{(q)}_{\lambda[k,A]}(\theta) \) of the operator \( \hat{L}_j(\lambda) \) is sufficient for the determination of the corrections \( \Psi_{(k)}(\theta, X, T) \).

We will assume according to our definition that we have \( m \) linearly independent ”regular” left eigen-vectors \( \kappa^{(q)}_{\lambda[k,A]}(\theta) \) of the operator \( \hat{L}(\lambda) \) corresponding to zero eigen-values which is equal to the number of functions \( \Phi_{\lambda, \theta=0}(\theta, k, A) \) giving the ”regular” right eigen-vectors of \( \hat{L}(\lambda) \) corresponding to zero eigen-values. We assume also that we have \( m \) parameters \( A = (A^1, \ldots, A^m) \) playing the role of amplitudes (say the amplitudes of the main Fourier harmonics \( \{\cos(\theta^\alpha + \theta^\alpha_0)\} \) in the expansion of \( \Phi_{\lambda}(\theta) \)) and we can express the function \( \omega^\alpha \) in the form \( \omega^\alpha = \omega^\alpha(k, A, \lambda) \) for rather small \( \lambda \). It is natural to accept then that in the limit \( \lambda \to 0 \) we have

\[
\Phi_{\lambda}(\theta, k, A) \to \sum_{\alpha=1}^{m} A^{\alpha} g^\alpha \cos(\theta^\alpha + \theta^\alpha_0)
\]

where \( g^\alpha = (g^\alpha_1, \ldots, g^\alpha_n)^t \) are some constant vectors and the functions \( \omega^\alpha(k, A, \lambda) \) become \( m \) independent dispersion relations \( \omega^\alpha = \omega^\alpha(k^\alpha) \) also independent of \( A \).

According to our general approach we will assume that we have \( m \) additional ”regular” left real eigen-vectors \( \zeta^{(q)}_{\lambda[k,A]}(\theta) \), \( (q = 1, \ldots, m) \) of \( \hat{L}(\lambda) \) corresponding to ”small” eigen-values \( \nu^{(q)}(\lambda, k, A) \) which give \( m \) additional left eigen-vectors \( \kappa^{(q)}_{0[k]}(\theta) \) of the operator \( \hat{L}_0 \) corresponding to zero eigen-values. The number of vectors \( \zeta^{(q)}_{0[k]} \) corresponds in this case to \( m \) additional right eigen-vectors \( g^\alpha \cos(\theta^\alpha + \theta^\alpha_0) \) of the operator \( \hat{L}_0 \) corresponding to zero eigen-values.

As in the one-phase situation we will require now orthogonality of the functions \( f_{(k)}(\theta, X, T) \) to both the sets of the left eigen-vectors \( \{\kappa^{(q)}_{\lambda[k,A]}(\theta)\} \) and \( \{\zeta^{(q)}_{\lambda[k,A]}(\theta)\} \) and try to find a deformation of the Whitham system and the dispersion relation in the form

\[
S_X^\alpha = k^\alpha \qquad S_T^\alpha = \omega^\alpha(k, A) + \sum_{k \geq 1} \omega^\alpha_{(k)}(k, A, k_X, A_X, \ldots)
\]

\[
k_T^\alpha = \frac{d}{dX} \left( \omega^\alpha(k, A) + \sum_{k \geq 1} \omega^\alpha_{(k)}(k, A, k_X, A_X, \ldots) \right) \quad A_T^\alpha = \sum_{k \geq 1} a^\alpha_{(k)}(k, A, k_X, A_X, \ldots)
\]

where all the functions \( \omega^\alpha_{(k)}, a^\alpha_{(k)} \) satisfy the same requirements as in the one-phase situation.

Using our assumptions we can try then to find the functions \( \Psi_{(k)} \) as local functionals of \( (k, A) \) and their derivatives and repeat all the steps of one-phase situation. All the functions \( a^\alpha_{(k)}, \omega^\alpha_{(k)} \) will be uniquely determined in this case. However, we should remind again that this assumption is rather serious in the \( m \)-phase situation and in fact is not valid in general case.
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