N-BLACK HOLE STATIONARY AND AXIALLY SYMMETRIC SOLUTIONS OF THE EINSTEIN/MAXWELL EQUATIONS

GILBERT WEINSTEIN

Abstract. The Einstein/Maxwell equations reduce in the stationary and axially symmetric case to a harmonic map with prescribed singularities $\varphi: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_\mathbb{C}$, where $\Sigma$ is a subset of the axis of symmetry, and $\mathbb{H}^2_\mathbb{C}$ is the complex hyperbolic plane. Motivated by this problem, we prove the existence and uniqueness of harmonic maps with prescribed singularities $\varphi: \mathbb{R}^n \setminus \Sigma \to \mathbb{H}$, where $\Sigma$ is a submanifold of $\mathbb{R}^n$ of codimension $\geq 2$, and $\mathbb{H}$ is a classical Riemannian globally symmetric space of noncompact type and rank one. This result, when applied to the black hole problem, yields solutions which can be interpreted as equilibrium configurations of multiple co-axially rotating charged black holes held apart by singular struts.

1. Introduction

Let $(M, g)$ be a four-dimensional Lorentzian manifold, and let $F$ be a two-form on $M$. Consider the Einstein/Maxwell field equations:

\begin{align*}
\text{Ric}_g - \frac{1}{2} R_g g &= 2 T_F \\
F &= dA \\
d^\ast F &= 0,
\end{align*}

where $\text{Ric}_g$ is the Ricci curvature tensor of the metric $g$, $R_g$ its scalar curvature, and $T_F$ is the energy-momentum-stress tensor of the electromagnetic field $F$:

\begin{equation}
T_F(X, Y) = \frac{1}{2} \left( i_X F \cdot i_Y F + i_X \ast F \cdot i_Y \ast F \right)
= i_X F \cdot i_Y F - \frac{1}{2} |F|^2 g(X, Y),
\end{equation}

Here, $i_X$ denotes inner multiplication by the vector $X$, $\ast$ is the Hodge star operator, $\sigma \cdot \tau$ denotes the inner product, and $|\sigma|^2$ the norm of $k$-forms.

This research was supported in part by NSF Grant DMS-9404523.

The author would like to express his thanks for the support and hospitality of the Erwin Schrödinger Institute where part of this work was completed.
These are given in local coordinates by:
\begin{align}
*\sigma_{\mu_1\ldots\mu_4-k} &= \frac{1}{k!}\varepsilon^{\mu_1\ldots\mu_k\nu_1\ldots\nu_k}\sigma_{\nu_1\ldots\nu_k\mu_1\ldots\mu_4-k}, \\
\sigma \cdot \tau &= -\varepsilon^{\mu_1\ldots\mu_k}(\sigma \wedge *\tau) = \frac{1}{k!}\sigma_{\mu_1\ldots\mu_k}\tau_{\mu_1\ldots\mu_k}, \\
|\sigma|^2 &= \sigma \cdot \sigma = \frac{1}{k!}\sigma^{\mu_1\ldots\mu_k}\sigma_{\mu_1\ldots\mu_k},
\end{align}
where, according to the Einstein summation convention, repeated indices are summed from 0 to 3, and \(\varepsilon\) is the volume form of \(g\). Note that \(\text{tr} T_F = 0\), hence taking the trace of Equations (1.1) yields \(R_g = 0\). Thus, Equation (1.8) can be rewritten as
\[ \text{Ric}_g = 2T_F. \]

We are using rationalized units in which \(4\pi G = 1\), where \(G\) is the gravitational constant. We will seek asymptotically flat, stationary and axially symmetric solutions of Equations (1.1)–(1.3).

These equations reduce, using an idea originally due to Ernst [7], to an axially symmetric harmonic map \(\varphi: \mathbb{R}^3 \setminus \Sigma \rightarrow \mathbb{H}_C^2\), where \(\Sigma = A \setminus \bigcup_{j=1}^N I_j\) consists of the axis of symmetry \(A \subset \mathbb{R}^3\) with \(N\) closed intervals \(I_j\) removed, and \(\mathbb{H}_C^2 = SU(1,2)/S(U(1) \times U(2))\) is the complex hyperbolic plane, see [12, 3]. Each interval \(I_j\) corresponds in \((M, g)\) to a connected component of the event horizon. Thus \(N\) will denote, throughout this paper, the number of black holes. The space \(\mathbb{H}_C^2\) has negative sectional curvature, and finite energy harmonic maps into spaces of negative sectional curvature are well understood, see [17, 18]. Nevertheless, this problem is analytically interesting due to the fact that the boundary conditions for \(\varphi\) on \(\Sigma\) and at \(r \to \infty\) are singular, and hence \(\varphi\) has infinite energy. For the case \(N = 1\), there is a family of solutions known in closed form, the Kerr-Newman black holes, see [2]. We note that the problem has \(4N - 1\) parameters: \(N\) masses \(m_j\), \(N - 1\) distances \(d_j\), \(N\) angular momenta \(L_j\), and \(N\) electric charges \(q_j\).

Special cases of the Ernst reduction have been used to prove non-existence results for the Einstein equations. In early work along this line, Weyl investigated the vacuum static case and showed that the equations reduce to a single linear equation. The case \(N = 1\) was known to have the Schwarzschild solution. Superimposing two such solutions, Weyl obtained new solutions which could be interpreted as equilibrium configurations of a pair of black holes. However, with Bach [1] in 1921, they showed that an obstruction arose as a conical singularity along the axis separating the two black holes. Having interpreted this singularity as the gravitational force, they computed its value and verified that the result was asymptotic to the Newtonian gravitational force in the appropriate limit. The reduction was also used by Robinson [10], following work of Carter [2], to prove that within the \(N = 1\) vacuum case, the Kerr solutions were unique. Mazur [12] and Bunting [3] independently, generalized this uniqueness result to the charged \(N = 1\) case.
In [20, 21], we used the Ernst reduction to construct vacuum solutions which could be viewed as nonlinear generalizations of the Weyl solutions. Clearly, it is of interest to find out whether the obstruction found by Weyl in the vacuum nonrotating case persists also for all possible choice of the parameters, i.e., whether the force between rotating black holes is always positive. It has been conjectured for some time that this is so, see [15, Problem 14], but the evidence is limited. Indeed, it is unlikely that the angle deficiencies can in general be computed exactly. We mention here a number of partial results, e.g., [10], where the small angular momentum case is treated, [22], where non-existence is proved in the extreme regime, and also [9], where Y. Li and G. Tian proved non-existence in the case when the solution admits an involutive symmetry.

In the present paper, we begin the generalization of this work to the Einstein/Maxwell Equations. We prove the existence of a unique \((4N - 1)\)-parameter family of solutions to the reduced problem. Our main tool is the study of harmonic maps with prescribed singularities into classical globally symmetric spaces of noncompact type and rank one [23], i.e., the real, complex, and quaternionic hyperbolic spaces, see [13]. However, the results of [23] do not apply directly to the problem considered here. In [23], we restricted our attention to bounded domains \(\Omega \subset \mathbb{R}^n\), and to singular sets \(\Sigma\) compactly contained in \(\Omega\), while here \(\Omega = \mathbb{R}^n\) and \(\Sigma\) is unbounded. The necessary generalization is achieved in two steps. First, we consider the problem on a large ball \(B \subset \mathbb{R}^n\), allowing, however, the singular set \(\Sigma\) to meet \(\partial B\). Then, we let \(B\) exhaust \(\mathbb{R}^n\). In this second step, we need the boundary data at infinity in \(\mathbb{R}^n\) to be given by a harmonic map which admits those singularities which lie along the unbounded components of \(\Sigma\). In the application to black holes, this map is obtained from the Kerr-Newman solutions.

The main motivation for this generalization is an attempt to come to a better understanding of the force between rotating black holes, i.e., the magnitude of the conical singularities on the bounded components of the axis. An important question in this respect is the dependence of this force on the parameters. It should be pointed out that the force can vanish in the case of the Einstein/Maxwell Equations. Indeed, even in the static case, equilibrium can be achieved with masses and charges being equal [11, 14]. However, these configurations are extreme, i.e., all the event horizons are degenerate, see Section 2.3. Consequently, one could still pose the question of whether the force is positive for all nondegenerate Einstein/Maxwell black holes [15]. In a future paper, we will study the regularity properties of these maps. This is a necessary step in order to complete the application of the results presented here to black holes.

The plan of the paper is as follows. In Section 2 we review the Ernst Reduction as it applies to the Einstein/Maxwell Equations. We carry out the reduction in two steps. First in Section 2.1 we reduce the equations in the presence of one Killing field. Then, the second reduction, with two
2. The Generalized Ernst Reduction for the Einstein/Maxwell Equations

In this section, we describe the generalized Ernst reduction for the Einstein/Maxwell equations in the stationary and axially symmetric case. The main result in this section is Theorem 1, based on which we state the Reduced Problem for the stationary and axially symmetric Einstein/Maxwell Equations in terms of harmonic maps with prescribed singularities. The computations are carried out mostly in the exterior algebra formalism which is particularly well suited to the Maxwell Equations.

2.1. Axial Symmetry. We first describe the connection between the axially symmetric Einstein/Maxwell Equations and harmonic maps into $\mathbb{H}^2_C$. We note that the arguments given here could be applied to any spacelike Killing field.

**Definition 1.** Let $(M, g)$ be an oriented 4-dimensional Lorentzian manifold, and let $F$ be a two-form on $M$. We say that $(M, g, F)$ is $SO(2)$-symmetric if $SO(2)$ acts effectively on $(M, g)$ as a group of isometries leaving $F$ invariant. The axis in an $SO(2)$-symmetric spacetime, is the set of points fixed by the action of $SO(2)$. We say that $(M, g, F)$ is axially symmetric if it is $SO(2)$ symmetric, and it has a nonempty axis.

Let $(M, g, F)$ be a simply connected $SO(2)$-symmetric solution of the Einstein/Maxwell Equations. Let $\xi$ be the Killing field generator, then $\mathcal{L}_\xi F = 0$. Note that if $(M, g)$ is causal, i.e., admits no closed causal curves, then $\xi$ either is spacelike, or vanishes. Define the one-forms $\alpha = i_\xi F$, and $\beta = i_\xi*F$. Then, we find $d\alpha = -i_\xi dF + \mathcal{L}_\xi F = 0$. Thus, there is a function $\chi$ such that $d\chi = \alpha$. Similarly $d\beta = 0$, hence there is a function $\psi$ such that $d\psi = \beta$. Note that $\chi$ and $\psi$ are determined only up to constants. Now, define the one-form $\gamma = \chi d\psi - \psi d\chi$, and observe that $d\gamma = 2\alpha \wedge \beta$.

It is easy to see from (1.5) that for any $k$-form $\sigma$ and one-form $\theta$, we have:

$$i_\theta * \sigma = * (\sigma \wedge \theta),$$

where we have used the metric $g$ to identify the tangent space $T_pM$ and its dual $T^*_pM$. Using $[\mathcal{L}_\xi, *] = 0$, and $** = (-1)^{k+1}$ on $k$-forms, it follows that

$$\delta (\sigma \wedge \xi) = (-1)^{k+1} \mathcal{L}_\xi \sigma + \delta \sigma \wedge \xi,$$
where $\delta = *d*$ is the divergence operator on forms. Thus, if we define the twist of $\xi$ by $\omega = *(d\xi \wedge \xi)$, we find

$$*d\omega = \delta (d\xi \wedge \xi) = -\mathcal{L}_\xi d\xi + \delta d\xi \wedge \xi = \delta d\xi \wedge \xi.$$ 

Now for any Killing field $\xi$, we have $\delta \xi = 0$, and

$$(2.1) \quad \delta d\xi = -2 \text{tr} \nabla^2 \xi = 2i\xi \text{Ric}_g.$$ 

Hence in view of Equation (1.2), we obtain $*d\omega = 4i\xi T_F \wedge \xi$. Furthermore, in view of (1.3), $i\xi T_F = -i\alpha F - (1/2)|F|^2 \xi$. Thus, we have $*d\omega = 4\xi \wedge i\alpha F$.

On the other hand, $\beta = *(F \wedge \xi)$, hence

$$*(\alpha \wedge \beta) = -i\alpha * \beta = \xi \wedge i\alpha F.$$ 

It follows that

$$(2.2) \quad d\omega = 2d\gamma,$$ 

i.e., the one-form $\omega - 2\gamma$ is closed. We conclude that there is a function $v$, also determined up to a constant, such that $2dv = \omega - 2\gamma$.

Let $M' = \{ x \in M; |\xi|^2 > 0 \}$, and let $u = -\log |\xi|$ on $M'$. Then, we have $i_\xi d\xi = -d\xi + L_\xi \xi = 2e^{-2u} du$. Thus, since $du = -\Delta u$, we have $\delta(e^{-2u} du) = 2e^{-2u} |du|^2 - 2e^{-2u} \Delta u$. On the other hand, $i_\xi d\xi = -*(d\xi \wedge \xi)$, hence using Equations (2.1), (1.3), and (1.4), we obtain:

$$\delta(i_\xi d\xi) = -i_\xi \delta d\xi + |d\xi|^2 = -2(|\alpha|^2 + |\beta|^2) + |d\xi|^2.$$ 

Furthermore, since $\omega = i_\xi * d\xi$, we find

$$|\omega|^2 = *(d\xi \wedge i_\xi d\xi \wedge \xi) + *(d\xi \wedge d\xi \wedge i_\xi \xi) = 4e^{-4u} |du|^2 - e^{-2u} |d\xi|^2$$

i.e., $|d\xi|^2 = 4e^{-2u} |du|^2 - e^{-2u} |\omega|^2$. We conclude that

$$(2.3) \quad \Delta u - \frac{1}{2} e^{4u} |\omega|^2 - e^{2u} (|\alpha|^2 + |\beta|^2) = 0.$$ 

Now, $\delta \omega = *(d\xi \wedge d\xi)$, and

$$2 du \cdot \omega = e^{2u} * (d\xi \wedge i_\xi d\xi \wedge \xi) + e^{2u} * (d\xi \wedge d\xi \wedge i_\xi \xi)$$

$$= -2 du \cdot \omega + *(d\xi \wedge d\xi),$$

i.e., $4 du \cdot \omega = *(d\xi \wedge d\xi)$. Thus, we find that $\delta(e^{4u} \omega) = -4e^{4u} du \cdot \omega + e^{4u} \delta \omega = 0$, which we write as

$$(2.4) \quad \text{div}(e^{4u} \omega) = 0,$$ 

where we have put $\text{div} \sigma = -\delta \sigma$ for any one-form $\sigma$. In addition, since $\alpha = -*(F \wedge \xi)$, we find $\delta \alpha = -d*(F \wedge \xi) = d\xi \cdot F$, and

$$2 du \cdot \alpha = -e^{2u} * (d\xi \wedge i_\xi * F \wedge \xi) - e^{2u} * (d\xi \wedge F \wedge i_\xi \xi)$$

$$= e^{2u} \beta \cdot \omega + d\xi \cdot F.$$ 

It follows that $\delta(e^{2u} \alpha) = -2e^{2u} du \cdot \alpha + e^{2u} \delta \alpha = -e^{4u} \beta \cdot \omega$, i.e.,

$$(2.5) \quad \text{div}(e^{2u} \alpha) - e^{4u} \beta \cdot \omega = 0.$$
Similarly, $\delta(e^{2u} \beta) = e^{4u} \alpha \cdot \omega$, i.e.,

$$\text{div}(e^{2u} \beta) + e^{4u} \alpha \cdot \omega = 0. \tag{2.6}$$

Substituting the definitions of $v$, $\chi$ and $\psi$ into Equations (2.3)–(2.6), it follows that $\varphi = (u, v, \chi, \psi)$ satisfies the following system of equations:

$$\Delta u - 2e^{4u} |\nabla v + \chi \nabla \psi - \psi \nabla \chi|^2 = 0 \tag{2.7}$$
$$\text{div}(e^{4u} (\nabla v + \chi \nabla \psi - \psi \nabla \chi)) = 0 \tag{2.8}$$
$$\text{div}(e^{2u} \nabla \chi) - 2e^{4u} \nabla \psi \cdot (\nabla v + \chi \nabla \psi - \psi \nabla \chi) = 0 \tag{2.9}$$
$$\text{div}(e^{2u} \nabla \psi) + 2e^{4u} \nabla \chi \cdot (\nabla v + \chi \nabla \psi - \psi \nabla \chi) = 0. \tag{2.10}$$

It is now clear that for every subset $\Omega \subset M'$, $\varphi = (u, v, \chi, \psi)$ is a critical point of the functional:

$$E_{\Omega}(\varphi) = \int_{\Omega} \left\{ |\nabla u|^2 + e^{4u} |\nabla v + \chi \nabla \psi - \psi \nabla \chi|^2 + e^{2u} (|\nabla \chi|^2 + |\nabla \psi|^2) \right\} \, d\mu_g,$$

where $d\mu_g$ is the volume element of the metric $g$. Thus, if we choose an ‘upper half-space model’ for $\mathbb{H}^2_C$, i.e., $\mathbb{R}^4$ with the metric given by the line element:

$$ds^2 = du^2 + e^{4u} (dv + \chi \, d\psi - \psi \, d\chi)^2 + e^{2u} (d\chi^2 + d\psi^2), \tag{2.11}$$

then the map $\varphi: (M', g) \to \mathbb{H}^2_C$ is a harmonic map, see Section 3 and [23].

2.2. Stationary and Axially Symmetric Solutions. We now turn to the case where $(M, g, F)$ admits an additional symmetry.

**Definition 2.** Let $(M, g)$ be an oriented and time-oriented 4-dimensional Lorentzian manifold and let $F$ be a two-form on $M$. Let $G$ be a group acting on $M$. The orbit of a point $p \in M$ is degenerate if the isotropy subgroup at $p$ is nontrivial. The axis is the set of points whose orbits are degenerate. We say that $(M, g, F)$ is stationary and $SO(2)$-symmetric if the group $G = \mathbb{R} \times SO(2)$ acts effectively on $(M, g)$ as a group of isometries leaving $F$ invariant and such that the orbits of points not on the axis are timelike two-surfaces. We say that $(M, g, F)$ is stationary and axially symmetric if it is stationary and $SO(2)$-symmetric and has a nonempty axis.

Assume that $(M, g, F)$ is stationary and axially symmetric, and let $\xi$ be the Killing field generator of the $SO(2)$-symmetry normalized so that its orbits are closed circles of length $2\pi |\xi|$. Let $\tau$ be a linearly independent generator. Then, we have $[\xi, \tau] = 0$, and as before, if $(M, g)$ is causal, then $\xi$ is either spacelike or vanishes, i.e., $\xi$ is spacelike outside the axis. We will now prove that if $(M, g, F)$ is a stationary and axially symmetric solution of the Einstein/Maxwell Equations, then:

(i) $F$ and $*F$ vanish on the orbits, i.e., $F(\xi, \tau) = *F(\xi, \tau) = 0$;
(ii) the distribution of planes orthogonal to the orbits of $G$ is integrable.
To see (i), note that \([\mathcal{L}_{\tau}, i_\epsilon] = i_{[\tau, \epsilon]} = 0\), hence we have
\[
di_t \alpha = -i_\tau d\alpha + \mathcal{L}_\tau i_\epsilon F = 0.
\]
Since \(i_\tau \alpha = 0\) on the axis it follows that \(F(\xi, \tau) = i_\tau \alpha = 0\) everywhere. Similarly, \(\ast F(\xi, \tau) = i_\tau \beta = 0\). To show (ii), it suffices by Frobenius Theorem to show that:
\[
(2.12) \quad \ast (\xi \wedge \tau \wedge d\xi) = 0, \quad \ast (\xi \wedge \tau \wedge d\tau) = 0.
\]
However, in view of (2.2) and (i), we have
\[
d \ast (\xi \wedge \tau \wedge d\xi) = di_\tau \omega = -i_\tau d\omega - \mathcal{L}_\tau \omega = 4i_\tau (\alpha \wedge \beta) = 0,
\]
where \(\omega\) is the twist of \(\xi\). Thus, \(\ast (\xi \wedge \tau \wedge d\xi)\) is constant, but since it vanishes on the axis, it is identically zero. Similarly, \(\ast (\xi \wedge \tau \wedge d\tau) = 0\).

Let \(Q\) be an integral surface of the distribution of planes orthogonal to the orbits, and let \(h\) be the metric induced by \(g\) on \(Q\). Then, the quotient space \(M/G\), with its quotient metric can be identified with \((Q, h)\). We choose the orientation on \(Q\) so that \(\ast (\xi \wedge \tau)\) is positively oriented. The map \(\varphi: M' \to \mathbb{R}^2_G\) is invariant under \(G\), hence reduces to a map, which we also denote by \(\varphi\), on the quotient \(Q' = Q \cap M'\). Indeed, let \(\zeta\) be an arbitrary Killing field generator in the Lie algebra of \(G\). We have \(\zeta e^{-2u} = 2g([\zeta, \xi], \xi) = 0\), and hence \(\zeta u = 0\). Also, \(\zeta = i_\zeta \alpha = F(\zeta, \zeta) = 0\), and similarly \(\zeta \psi = 0\). Finally, \(\gamma(\zeta) = \zeta \beta(\zeta) - \psi \alpha(\zeta) = 0\), hence \(2\zeta v = i_\zeta (\omega - 2\gamma) = \ast (d\xi \wedge \xi \wedge \zeta) - 2\gamma(\zeta) = 0\).

Define \(\sigma = \xi \wedge \tau\), then on \(M'\) we have \(|\sigma|^2 = |\xi|^2 |\tau|^2 - (\xi \cdot \tau)^2 < 0\). Let \(\rho^2 = -|\sigma|^2\), then \(\rho\) is invariant under \(G\), and thus reduces to a function on \(Q'\). It follows that for every subset \(\Omega \subset Q',\) the map \(\varphi: (Q', h) \to \mathbb{R}^2_G\) is a critical point of the functional:
\[
E'_\Omega(\varphi) = \int \Omega \left[ |\nabla u_h|^2 + \epsilon^4 u |\nabla v + \chi \nabla \psi - \psi \nabla \chi|_h^2 + \epsilon^2 u (|\nabla \chi|_h^2 + |\nabla \psi|_h^2) \right] \rho d\mu_h,
\]
where \(|\_|_h\) is the norm with respect to the metric \(h\), and \(d\mu_h\) is the volume element of the metric \(h\). Indeed, if \(\Omega \subset Q'\) is such a subset, and \(f\) is a function on \(M'\) invariant under \(G\), then \(2\pi \int_G f d\mu_h = \int_G f d\mu_g\), where \(G \cdot \Omega\) is the orbit of \(\Omega\) under \(G\).

Next, we show that \(\Delta_h \rho = 0\). Since \(\Delta_g \rho = \rho^{-1} \text{div}_h(\rho \nabla \rho)\), it suffices to prove that \(\Delta_g \rho = \rho^{-1} |d\rho|^2\). To see this, note first that \(\rho^2 = -\sigma(\xi, \tau) = i_\xi i_\tau \sigma\). Thus, we obtain:
\[
(2.13) \quad 2 \rho \delta d\rho = d i_\xi i_\tau \delta = i_\xi i_\tau d\sigma = -\ast (\ast d\sigma \wedge \sigma).
\]
Therefore, we see that
\[
2 \rho \Delta_g \rho + 2 |d\rho|^2 = -\delta(2 \rho d\rho) = \ast d(\ast d\sigma \wedge \sigma) = \sigma \delta d\sigma - |d\sigma|^2,
\]
or equivalently
\[
(2.14) \quad \Delta_g \rho = \frac{1}{2 \rho} (\sigma \delta d\sigma - |d\sigma|^2 - 2 |d\rho|^2).
\]
On the other hand, since $d\sigma = d\xi \land \tau - \xi \land d\tau$, we have, in view of Equation (2.12), that $i_\xi * d\sigma = *(d\sigma \land \xi) = 0$, and similarly $i_\tau * d\sigma = 0$. We deduce, using (2.13), that

\[ 4\rho^2 |d\rho|^2 = *(i_\tau d\sigma \land *d\sigma \land i_\xi \sigma) = - * (d\sigma \land *d\sigma \land i_\tau i_\xi \sigma) = -\rho^2 |d\sigma|^2, \]

i.e.,

(2.15) \quad |d\sigma|^2 = -4 |d\rho|^2

Furthermore, we claim that $\sigma \cdot \delta d\sigma = 0$. Indeed, in view of Equation (2.1):

$$
\delta d\sigma = *d(i_\tau * d\xi - i_\xi * d\tau) = \delta d\xi \land \tau + \xi \land \delta d\tau = 2(i_\xi \text{Ric}_g \land \tau + \xi \land i_\tau \text{Ric}_g),
$$

and consequently,

$$
\sigma \cdot \delta d\sigma = i_\tau i_\xi \delta d\sigma = 2 |\tau|^2 \text{Ric}_g(\xi, \xi) - 4 (\xi \cdot \tau) \text{Ric}_g(\xi, \tau) + 2 |\xi|^2 \text{Ric}_g(\tau, \tau).
$$

Introducing $\tilde{\alpha} = i_\tau F$, and $\tilde{\beta} = i_\tau * F$, we can use Equations (1.8) and (1.4) to write

\[ \text{Ric}_g(\xi, \xi) = |\alpha|^2 + |\beta|^2, \]
\[ \text{Ric}_g(\xi, \tau) = \alpha \cdot \tilde{\alpha} + \beta \cdot \tilde{\beta}, \]
\[ \text{Ric}_g(\tau, \tau) = |\tilde{\alpha}|^2 + |\tilde{\beta}|^2, \]

from which it follows that:

(2.16) \quad \sigma \cdot \delta d\sigma = 2 |\tau|^2 (|\alpha|^2 + |\beta|^2) - 4 (\xi \cdot \tau)(\alpha \cdot \tilde{\alpha} + \beta \cdot \tilde{\beta}) + 2 |\xi|^2 (|\alpha|^2 + |\beta|^2).

In addition, we have:

\[ |\xi|^2 F = \xi \land \alpha + i_\xi * \beta \]
\[ |\xi|^2 * F = \xi \land \beta - i_\xi * \alpha, \]

and hence,

(2.17) \quad |\xi|^2 \tilde{\alpha} - (\xi \cdot \tau) \alpha = i_\tau i_\xi * \beta

(2.18) \quad |\xi|^2 \tilde{\beta} - (\xi \cdot \tau) \beta = -i_\tau i_\xi * \alpha.

A computation similar to the one leading to (2.15) yields:

\[ |i_\tau i_\xi * \alpha|^2 = \rho^2 |\alpha|^2, \quad |i_\tau i_\xi * \beta|^2 = \rho^2 |\beta|^2. \]

Thus, taking the norm squared of both sides of (2.17) and (2.18), and adding the results, we obtain:

\[ |\xi|^4 (|\tilde{\alpha}|^2 + |\tilde{\beta}|^2) - 2 |\xi|^2 (\xi \cdot \tau)(\alpha \cdot \tilde{\alpha} + \beta \cdot \tilde{\beta}) + |\xi|^2 |\tau|^2 (|\alpha|^2 + |\beta|^2) = 0, \]

which, in view of (2.16), implies $\sigma \cdot \delta d\sigma = 0$ as claimed. Substituting this result back into (2.14), and taking into account (2.15), we obtain $\Delta_g \rho = \rho^{-1} |d\rho|^2$ as required.

Therefore, if $d\rho \neq 0$, the function $\rho$ can be used locally as a harmonic coordinate for the metric $h$ on $Q'$. Let $z$ be a conjugate harmonic coordinate, i.e., a function on $Q'$ such that $|dz|^2 = |d\rho|^2$, $dz \cdot d\rho = 0$, and $d\rho \land dz$ is
positively oriented. Then, in the \((\rho, z)\)-coordinate system, the metric \(h\) takes the form:

\[
h_{ab}dx^a dx^b = e^{2\lambda}(d\rho^2 + dz^2),
\]

where \(\lambda = -\log |d\rho|\). Let \(\tilde{h} = e^{-2\lambda}h\), then \(\tilde{h}\) is a flat metric on \(Q',\) and since the functional \(E'^2_\Omega\) is conformally invariant, we have that \(\varphi\) is a critical point of:

\[
\tilde{E}_\Omega(\varphi) = \int_{\Omega} \left[ |\nabla u|^2_h + e^{4u} |\nabla v + \chi \nabla \psi - \psi \nabla \chi|^2_h + e^{2u}(|\nabla \chi|^2_h + |\nabla \psi|^2_h) \right] \rho d\mu_{\tilde{h}}.
\]

This gives a semilinear elliptic system of partial differential equations for the four unknowns \((u, v, \chi, \psi)\).

We have obtained that the metric \(g\) must be locally of the form given by the line element:

\[
ds^2 = -\rho^2 e^{2u} dt^2 + e^{-2u} (d\phi - w dt)^2 + e^{2\lambda}(d\rho^2 + dz^2),
\]

where \(w = -e^{2u}(\xi \cdot \tau)\). All the metric coefficients can be determined from the map \(\varphi\). Indeed, \(u\) is obtained directly from \(\varphi\), and we will now obtain equations for the gradient of \(w\) and \(\lambda\). These quadratures are to be integrated after the harmonic map system has been solved.

Define \(\eta = \tau + w\xi\), and observe that \(\eta \cdot \xi = 0\), while \(|\eta|^2 = \eta \cdot \tau = -e^{2u}\rho^2\). Furthermore, we have \(dw = 2w du + e^{2u}i_\tau d\xi = e^{2u}i_\eta d\xi\). Since \(d\xi = 2\xi \wedge du - e^{2u}*(\xi \wedge \omega)\), and \(\xi \wedge \eta = \sigma\), we find \(i_\eta d\xi = e^{2u}*(\omega \wedge \sigma)\), and hence

\[
i_\tau i_\xi d\omega = -\rho^2 e^{4u} \omega
\]

The operator \(\rho^{-1}i_\tau i_\xi \ast\) restricted to forms tangential to \(Q\) is the Hodge star operator of the metric \(h\). Since in two dimensions, the star operator restricted to one-forms is conformally invariant, \(\rho^{-1}i_\tau i_\xi \ast\) is also the Hodge star operator \(\ast\) of the flat metric \(\tilde{h}\) restricted to one-forms. Now, on one-forms in two dimensions \(\ast \ast = -1\), hence we can rewrite Equation \((2.20)\) as:

\[
dw = e^{4u} \rho \ast \omega.
\]

To derive the equations for \(d\lambda\), we must now use of the components of the Einstein/Maxwell Equations in the \(\rho z\)-plane:

\[
\text{Ric}_g(\partial_\rho, \partial_\rho) - \text{Ric}_g(\partial_z, \partial_z) = 2\left(i_{\partial_\rho} F \cdot i_{\partial_\rho} F - i_{\partial_z} F \cdot i_{\partial_z} F\right)
\]

\[
\text{Ric}_g(\partial_\rho, \partial_z) = 2i_{\partial_\rho} F \cdot i_{\partial_z} F.
\]

A straightforward coordinate computation leads to:

\[
\lambda_\rho - u_\rho = \rho \left[u_\rho^2 - u_z^2 + \frac{1}{4} e^{4u}(\omega_\rho^2 - \omega_z^2) + e^{2u}(\chi_\rho^2 - \chi_z^2 + \psi_\rho^2 - \psi_z^2)\right]
\]

\[
\lambda_z - u_z = 2\rho \left[u_\rho u_z + \frac{1}{4} e^{4u} \omega_\rho \omega_z + e^{2u}(\chi_\rho \chi_z + \psi_\rho \psi_z)\right].
\]
Definition 3. We collect here a few definitions related to the notion of asymptotic flatness. A data set \((\tilde{M}, \tilde{g}, \tilde{k}, \tilde{E}, \tilde{B})\) for the Einstein/Maxwell Equations consists of a 3-manifold \(\tilde{M}\), a Riemannian metric \(\tilde{g}\) on \(\tilde{M}\), a symmetric twice covariant tensor \(\tilde{k}\) on \(\tilde{M}\), and two vectors \(\tilde{E}\) and \(\tilde{B}\) on \(\tilde{M}\), satisfying the Constraint Equations:

\[
\bar{R} - \bar{k}_{ij} \bar{k}^{ij} + (\text{tr } \bar{k})^2 = |\bar{E}|^2 + |\bar{B}|^2,
\]

\[
\nabla^i \bar{k}_{ij} - \nabla_j \text{tr } \bar{k} = 2 \varepsilon_{ijk} \bar{E}^j \bar{B}^k,
\]

\[
\nabla^i \bar{E}_i = 0,
\]

\[
\nabla^i \bar{B}_i = 0,
\]

where \(\varepsilon\) and \(\nabla\) denote respectively the volume form and the covariant derivative of \(\tilde{g}\). In addition, we require that \(\int_S \tilde{g}(\tilde{B}, n) = 0\) for every closed 2-surface \(S\) non-homologous to zero in \(\tilde{M}\), where \(n\) is the unit normal to \(S\).

A 3-manifold \(\tilde{M}\) is topologically Euclidean at infinity, if there is a compact set \(K \subset \tilde{M}\), such that each connected component of \(\tilde{M} \setminus K\), henceforth called an end, is homeomorphic to the complement of a ball in \(\mathbb{R}^3\). The data set \((\tilde{M}, \tilde{g}, \tilde{k}, \tilde{E}, \tilde{B})\) is strongly asymptotically flat if \(\tilde{M}\) is topologically Euclidean at infinity, and if there is on each end a coordinate system \((x^1, x^2, x^3)\), in which \(\tilde{g}, \tilde{k}, \tilde{E}, \tilde{B}\), take the asymptotic forms:

\[
\tilde{g}_{ij} = \left(1 + \frac{2m}{r}\right) \delta_{ij} + o_2(r^{-3/2}), \quad \tilde{k}_{ij} = o_1(r^{-5/2}),
\]

\[
\tilde{E}_i = \frac{q x^i}{r^3} + o_1(r^{-5/2}), \quad \tilde{B}_i = o_1(r^{-5/2}),
\]

where \(m\) and \(q\) are constants. Here, a function \(f\) is \(o_1(r^{-j})\) if \(\partial^j f = o(r^{-l-j})\) for \(j = 0, \ldots, k\), and \(r = \sqrt{\sum (x^i)^2}\). A spacetime is globally hyperbolic if it admits a Cauchy hypersurface, i.e., a hypersurface which intersects every inextendible causal curve exactly once. Given a data set \((\tilde{M}, \tilde{g}, \tilde{k}, \tilde{E}, \tilde{B})\) which is strongly asymptotically flat, the Cauchy Problem for the Einstein/Maxwell Equations, consists of finding a globally hyperbolic spacetime \((M, g)\), a two-form \(F\), and an embedding \(\iota: \tilde{M} \rightarrow M\) such that \((M, g, F)\) is a solution of the Einstein/Maxwell Equations, \(\tilde{g}\) and \(\tilde{k}\) are respectively the first and second fundamental forms of \(\iota\), and such that \(\tilde{E} = \iota_\tau F\), and \(\tilde{B} = \iota_\tau \ast F\), where \(\tau\) is the future pointing unit normal of \(\tilde{M}\) in \(M\). The triple \((M, g, F)\) is called the Cauchy development of the data \((\tilde{M}, \tilde{g}, \tilde{k}, \tilde{E}, \tilde{B})\). A triple \((M, g, F)\) will be called asymptotically flat if it is the Cauchy development of strongly asymptotically flat data. A domain of outer communications in an asymptotically flat spacetime \((M, g)\) is a maximal connected open set \(O \subset M\) such that from each point \(p \in O\) there are both future and past directed timelike curves to asymptotically flat regions of \((M, g)\). An event horizon is the boundary of a domain of outer communications.

If \((\tilde{M}, g)\) is asymptotically flat and globally hyperbolic, so is any domain of outer communications \(M\) in \((\tilde{M}, g)\), since \(M\) is then the intersection of future and past sets. We will restrict our attention to such a domain. We may then assert that \((M, g)\) is causal. We assume that \(M\) is simply connected. If in addition, \((M, g, F)\) is a solution of the Einstein/Maxwell
Equations which is stationary and axially symmetric, then there is a unique Killing field generator $\tau$, which is future directed, and such that $|\tau|^2 \to -1$ at spacelike infinity. Defining now $\xi$ and $\rho$ as before, it can be shown, using $\Delta_h \rho = 0$, that $\rho$ has no critical points, and hence can be used as a harmonic coordinate on all of $Q'$. Furthermore, the event horizon $H \subset \tilde{M}$ can be characterized by the conditions: $|\xi|^2 > 0$ and $\rho = 0$.

If $|d\sigma|^2$ vanishes identically on some component $H_0$ of $H$, then we say that $H_0$ is degenerate. If a component $H_0$ of $H$ is nondegenerate, then by (2.15), we have $|d\rho|^2 = |dz|^2 \neq 0$ on $H_0$. We will see in Section 2.3 that each component $H_0$ of $H$ is a Killing horizon in the sense that the null generators of $H_0$ are tangent to a Killing vector $\eta_0$ which is non-zero on $H_0$, see [2, 3]. Furthermore, $e^{2u}|d\rho|^2$ tends on $H_0$ to a constant $\kappa_0$, the surface gravity of $H_0$, defined by the equation $d(|\eta_0|^2) = -2\kappa_0\eta_0$ which holds on $H_0$. Thus, $H_0$ is degenerate if and only if $\kappa_0 = 0$ and the definition of a degenerate horizon adopted here coincides with the standard one, see [2]. If a component $H_0$ of $H$ is nondegenerate, then $H_0 \cap Q$ is an interval on the boundary of $Q'$; otherwise, it is a point. Indeed, the intersection of $H_0$ with a spacelike hypersurface must have the topology of a 2-sphere, see [3]. We will assume that the event horizon $H$ consists of $N$ nondegenerate components, which we denote by $H_j$, $1 \leq j \leq N$.

Let $A$ be the $z$-axis in $\mathbb{R}^3$, and consider $Q' \times SO(2) = \mathbb{R}^3 \setminus A$ with the flat metric $\rho^2 d\phi \otimes d\phi + \tilde{h}$. Let $\Sigma = A \setminus \bigcup_{j=1}^N I_j$ where $I_j$ are the open intervals on the boundary of $Q'$ corresponding to the $N$ components $H_j$ of the event horizon $H$. Since $|\xi|^2 > 0$ on $I_j$, the map $\varphi$ can be extended continuously across $I_j$. Let $x \in I_j$, then since $I_j$ is of codimension 2, one can show, using a cut-off as in Lemma 5, that $\varphi$ is weakly harmonic and has finite energy in a sufficiently small ball centered at $x$. It then follows from regularity theory for harmonic maps [17, 18] that $\varphi$ is smooth in that ball. Thus, $\varphi: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_C$ is an axially symmetric harmonic map. We have proved:

**Theorem 1.** Let $(M, g, F)$ be a solution of the Einstein/Maxwell Equations which is stationary and axially symmetric. Assume that $M$ is simply connected, that $(M, g)$ is the domain of outer communications of an asymptotically flat globally hyperbolic spacetime, and that every component of the event horizon in $(M, g)$ is nondegenerate. Define $\varphi = (u, v, \chi, \psi)$, and $\Sigma$ as above. Then $\varphi: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_C$ is an axially symmetric harmonic map. In addition, the metric $g$ is of the form (2.19), and the metric coefficients $w$ and $\lambda$ satisfy Equations (2.21) and (2.22).

2.3. Physical Parameters. Let $S$ be an oriented spacelike two-sphere in $M$. We define the mass $m(S)$, angular momentum $L(S)$, and charge $q(S)$,
contained in $S$ by:

\begin{align*}
(2.23) \quad m(S) &= -\frac{1}{8\pi} \int_S \ast d\tau, \\
(2.24) \quad L(S) &= \frac{1}{16\pi} \int_S \ast d\xi, \\
(2.25) \quad q(S) &= -\frac{1}{4\pi} \int_S \ast F.
\end{align*}

The mass and the angular momentum are standard Komar integrals associated with the Killing fields $\xi$ and $\tau$, see [2], and the definition of the charge is classical. We will always perform the integration over a submanifold of revolution, so that the integral can be computed on the quotient $Q$. Let $1 \leq k \leq 3$, let $\theta$ be a $k$-form and let $\Lambda$ be a $k$-dimensional submanifold in $M$ which is the orbit of a $(k-1)$-dimensional submanifold $\Gamma \subset Q$ under the subgroup $SO(2) \subset G$, i.e., $\Lambda = SO(2) \cdot \Gamma$, then we find:

\begin{equation}
(2.26) \quad \int_{\Lambda} \theta = -2\pi \int_{\Gamma} i_{\xi} \theta,
\end{equation}

It is well-known that Stokes’ Theorem can be used in (2.23)–(2.25) to relate the conserved quantities of two spheres $S$ and $S'$ which are homologous in $M$.

Indeed, suppose that $S$ and $S'$ satisfy $\partial \Lambda = S - S'$ for some 3-dimensional submanifold $\Lambda$, then:

\begin{equation}
(2.27) \quad q(S) - q(S') = -\frac{1}{4\pi} \int_{\Lambda} d \ast F = 0,
\end{equation}

so that the charge $q(S)$ actually depends only on the homology class of $S$.

In our situation, the second homology group is generated by spacelike cross-sections $S_j$ of the $N$ components $H_j$ of the event horizon $H$. Hence $q(S)$ is determined by the homology class of $S$ and by the $N$ parameters $q_j = q(S_j)$, $j = 1, \ldots, N$, which we may call the electric charges of the $N$ black holes. By (2.26), we find:

\begin{equation}
q_j = \frac{1}{2} \int_{I_j} d\psi.
\end{equation}

In view of (2.27), $\psi$ is constant along each component of $\Sigma$. Let $\Sigma_j$, $j = 1, \ldots, N + 1$, denote the $N + 1$ connected components of $\Sigma$ in order of increasing $z$. We may assume that $q_0 = \psi|_{\Sigma_1} = 0$, and $\psi|_{\Sigma_j} = q_{j-1}$ for $j = 2, \ldots, N + 1$. According to Equation (1.2), we also have:

\begin{equation}
\frac{1}{2} \int_{I_j} d\chi = \frac{1}{4\pi} \int_{S_j} F = \frac{1}{4\pi} \int_{\partial S_j} A = 0,
\end{equation}

which simply expresses the absence of magnetic charge. As a consequence, $\chi$ is constant throughout $\Sigma$, and we may as well assume $\chi|_{\Sigma} = 0$. 


In analogy, we define the angular momenta:

\[ L_j = L(S_j) = \frac{1}{16\pi} \int_{S_j} \star d\xi = \frac{1}{8} \int_{I_j} \omega = \frac{1}{4} \int_{I_j} (dv + \gamma). \]

However, since the electromagnetic field carries angular momentum \( L(S) \) is not determined by these parameters. Indeed, if \( \partial \Lambda = S - S' \) and if \( \Lambda = SO(2) \cdot \Omega \), then:

\[ L(S) - L(S') = \frac{1}{16\pi} \int_{\Lambda} d* d\xi = \frac{1}{2} \int_{\Omega} \alpha \wedge \beta. \]

Nevertheless, since \( \gamma \) vanishes on \( \Sigma \), it is easy to see from (2.28), that \( v \) is constant on each component of \( \Sigma \). We introduce the \( N \) parameters \( \lambda_j = \int_{I_j} dv \), and note that \( L_j = (\lambda_j + l_j)/4 \) where \( l_j = \int_{I_j} \gamma \). We may assume that \( \lambda_0 = v|_{\Sigma_1} = 0 \), and \( v|_{\Sigma_j} = \lambda_{j-1} \) for \( j = 2, \ldots, N + 1 \).

Finally, define the \( N \) masses:

\[ m_j = m(S_j) = \frac{1}{4} \int_{I_j} i_\xi \star d\tau. \]

Equation (2.21) implies that \( w \) is constant on each \( I_j \). The \( N \) constants \( w_j = w|_{I_j} \) may be interpreted as the angular velocities of the \( N \) black holes. Now, in view of (2.13), we have:

\[ dz = \star d\rho = \frac{1}{2} \star d\sigma, \]

and therefore,

\[ i_\xi \star d\tau = -2 dz + * (d\xi \wedge \tau). \]

Since \( d\xi = 2 \xi \wedge du - e^{2u} \star (\xi \wedge \omega) \), we have

\[ * (d\xi \wedge \tau) = -2 \rho \star du - w\omega. \]

Combining Equations (2.29) and (2.30), we obtain:

\[ i_\xi \star d\tau = -2 dz - w\omega - 2 \rho \star du. \]

The last term vanishes on \( I_j \), and thus we obtain:

\[ m_j = \frac{1}{4} \int_{I_j} (2 dz + w\omega) = \mu_j + 2 w_j L_j, \]

where \( 2 \mu_j = \int_{I_j} dz \) is the length of \( I_j \) in the metric \( \tilde{h} \). Again, \( m(S) \) is not determined by these, since the electromagnetic field carries energy. Let \( \partial \Lambda = S - S' \) where \( \Lambda = SO(2) \cdot \Omega \), then we find:

\[ m(S) - m(S') = \frac{1}{4} \int_{\Omega} d(w\omega + 2 \rho \star du) \]

\[ = \frac{1}{2} \int_{\Omega} \left\{ e^{2u}(|\alpha|^2 + |\beta|^2) \rho d\rho dz + 2 w\alpha \wedge \beta \right\}. \]

For future reference, we also introduce the \( N - 1 \) parameters \( r_j = \int_{\Sigma_j} dz \), \( j = 2, \ldots, N \).
We conclude this section by verifying as mentioned earlier that $H_j$ is degenerate if and only if its surface gravity vanishes. If we let $\eta_j = \tau + w_j^2$, then $\eta_j$ is a Killing field tangent to $H_j$, which is null but non-zero on $H_j$. Thus, $H_j$ is a Killing horizon. Let $\kappa_j$ be the surface gravity of $H_j$, then

$$\frac{|d(|\eta_j|^2)|^2}{4|\eta_j|^2} \to \kappa_j^2,$$

on $H_j$, see [2, p. 150]. However, $|\eta_j|^2 = -\rho^2 e^{2u} + e^{-2u}(w - w_j)^2$, hence in view of (2.21),

$$\left|d(|\eta_j|^2)\right|^2 = 4\rho^2 e^{4u} |d\rho|^2 + O(\rho^4).$$

Therefore we conclude that $e^{2u} |d\rho|^2 \to \kappa_j$ on $H_j$ as claimed in Section 2.1 see also [2, p. 192].

2.4. **Boundary Conditions.** In order to complete the reduction, and formulate a reduced problem for the stationary and axially symmetric Einstein/Maxwell Equations, we need to derive boundary conditions for $\varphi$ on $\Sigma$ and as $r \to \infty$ in $\mathbb{R}^3$, where $r = \sqrt{\rho^2 + z^2}$. Since $u$ behaves like $\log \rho$ near $\Sigma$ and as $r \to \infty$, these boundary conditions will be singular. They may be viewed as prescribed singularities for the map $\varphi$ on $\Sigma$ and at infinity. In particular, $E_{\mathbb{R}^3}(\varphi)$ the total energy of $\varphi$ will be infinite.

Let $u_0$ be the potential of a uniform charge distribution of $\Sigma$, normalized so that $u_0 + \log \rho \to 0$ as $r \to \infty$ in $\mathbb{R}^3$. The map $(u_0,0,0,0)$ is the solution of the corresponding problem when $\lambda_j = q_j = 0$, i.e., the Weyl solution. Define $N + 1$ singular harmonic maps $\varphi_j = (u_0,\lambda_{j-1},0,q_j) : \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_\kappa$, $j = 1,\ldots,N + 1$. Note that each maps into a geodesic $\gamma_j(t) = (t,\lambda_j,0,q_j)$ of $\mathbb{H}^2_\kappa$. Furthermore, the distance $\text{dist}(\varphi(x),\varphi_j(x))$ remains bounded in a neighborhood of $\Sigma_j$. Indeed, if $x \in \Sigma_j$ is not an end point of $\Sigma_j$, then in a ball $B$ small enough about $x$, the function $u + \log \rho$ is bounded, and the functions $v, \chi_j$ and $\psi$ are even functions of $\rho$. It is straightforward now using (2.11) to check that $\text{dist}(\varphi,\varphi_j)$ is bounded on $B$. A similar argument can be used at the end points. Finally, in order to get a uniform estimate on $\Sigma$, one uses the asymptotic flatness of $(M,g,F)$, see the boundary conditions in [2]. This motivates the following definition 2.11.

**Definition 4.** Let $\mathbb{H}$ be a real, complex, or quaternionic hyperbolic space, let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, let $\Sigma \subset \Omega$ be a closed smooth submanifold of codimension at least 2, possibly with boundary, let $\Sigma' \subset \Sigma$, and let $\varphi, \varphi' : \Omega \setminus \Sigma \to \mathbb{H}$ be harmonic maps. We say that $\varphi$ and $\varphi'$ are asymptotic near $\Sigma'$ if there is a neighborhood $\Omega'$ of $\Sigma'$ such that $\text{dist}(\varphi,\varphi') \in L^\infty(\Omega' \setminus \Sigma')$.

Thus, in this terminology, $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$.

Now let $\Sigma = \Sigma_1 \cup \Sigma_{N+1}$ be the union of the unbounded components of $\Sigma$. $\mathbb{R}^3 \setminus \Sigma$ is the domain of a harmonic map $\tilde{\varphi}$ corresponding to a Kerr-Neu
solution with one black hole, such that \( \tilde{\varphi} \) is asymptotic to \( \varphi_1 \) near \( \Sigma_1 \), and asymptotic to \( \varphi_{N+1} \) near \( \Sigma_{N+1} \). The map \( \tilde{\varphi} \) can be written down explicitly. Introduce the global parameters \( q, \mu, m, \) and \( a \) defined by:

\[
q = \sum_{j=1}^{N} q_j, \quad 2\mu = 2 \sum_{j=1}^{N} \mu_j + \sum_{j=1}^{N-1} r_j, \quad ma = \frac{1}{4} \sum_{j=1}^{N} \lambda_j, \quad m^2 - a^2 - q^2 = \mu^2.
\]

Here \( q \) is the total charge, \( 2\mu \) is the length, in the Euclidean metric of \( \mathbb{R}^3 \), of the interval of the \( z \)-axis from the south pole of \( I_1 \) to the north pole of \( I_{N+1} \), \( m \) and \( a \) are the mass and angular momentum per mass of the corresponding Kerr-Newman spacetime. Define the Schwarzschild-type elliptical coordinates \((s, \theta)\) on \( \mathbb{R}^3 \) defined by:

\[
\rho^2 = (s - m + \mu)(s - m - \mu) \sin^2 \theta, \quad z = (s - m) \cos \theta,
\]

where we have assumed without loss of generality that the point \( z = 0 \) on \( A \) is the center of the interval \( I = A \setminus \Sigma \). Then, \( \tilde{\varphi} = (\tilde{u}, \tilde{v}, \tilde{\chi}, \tilde{\psi}) \) is given by:

\[
(2.31) \quad \tilde{u} = \log \sin \theta + \frac{1}{2} \log \left[ s^2 + a^2 + a^2(2ms - q^2)(s^2 + a^2 \cos^2 \theta)^{-1} \sin^2 \theta \right]
\]

\[
\tilde{v} = ma \cos \theta (3 - \cos^2 \theta) - \frac{a^2(q^2s - ma^2 \sin^2 \theta) \cos \theta \sin^2 \theta}{s^2 + a^2 \cos^2 \theta} + 2ma
\]

\[
\tilde{\chi} = -qas (s^2 + a^2 \cos^2 \theta)^{-1} \sin^2 \theta
\]

\[
\tilde{\psi} = q(s^2 + a^2 \cos^2 \theta)^{-1}(s^2 + a^2) \cos \theta + q.
\]

Furthermore, the asymptotic flatness of \((M, g, F)\) implies that \( \text{dist}(\varphi, \tilde{\varphi}) \to 0 \) as \( r \to \infty \) in \( \mathbb{R}^3 \), see [2].

**Definition 5.** Assume that \( \Sigma \subset \mathbb{R}^n \), and let \( \varphi, \varphi': \mathbb{R}^n \setminus \Sigma \to \mathbb{H} \) be harmonic maps. We say that \( \varphi \) and \( \tilde{\varphi} \) are asymptotic at infinity, if \( \text{dist}(\varphi, \tilde{\varphi}) \to 0 \) as \( x \to \infty \) in \( \Omega \setminus \Sigma \).

We can now formulate the reduced problem for the stationary and axially symmetric Einstein/Maxwell Equations. Let \( 4N - 1 \) parameters

\[
(2.32) \quad (\mu_1, \ldots, \mu_N, r_1, \ldots, r_{N-1}, \lambda_1, \ldots, \lambda_N, q_1, \ldots, q_N)
\]

be given, with \( \mu_j, r_j > 0 \). Construct the set \( \Sigma = \bigcup_{j=1}^{N} I_j \), where \( I_j \) are \( N \) intervals on the \( z \)-axis, of lengths \( 2\mu_j \), distance \( r_j \) apart, and let \( \Sigma = \Sigma_1 \cup \Sigma_{N+1} \). Define the harmonic maps \( \varphi_j = (u_0, \lambda_j, 0, q_j) \), and the map \( \tilde{\varphi}: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H} \) according to Equations (2.31). We associate with the given parameters (2.32), the data \((\varphi_1, \ldots, \varphi_{N+1}, \tilde{\varphi})\) consisting of these \( N+2 \) singular harmonic maps. The \( 4N - 1 \) physical parameters, \( m_j, d_j, L_j \), and \( q_j \), are determined only a posteriori. Note that the parameters (2.32) are nonetheless geometric invariants.

**Reduced Problem.** Given a set of \( 4N - 1 \) parameters as in (2.32), prove the existence of a unique axially symmetric harmonic map \( \varphi: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_C \).
such that $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for each $j = 1, \ldots, N + 1$, and such that $\varphi$ is asymptotic to $\tilde{\varphi}$ at infinity.

This problem will be solved in Section 3; see the Main Theorem and its Corollary. Clearly, of at least equal importance is the converse: given a solution of the Reduced Problem, is there a corresponding solution of the Einstein/Maxwell Equations which is stationary and axially symmetric, and which is a simply connected domain of outer communications of a globally hyperbolic and asymptotically flat spacetime with $N$ nondegenerate components to the event horizon? This question will be examined briefly in Section 4.

3. Harmonic Maps with Prescribed Singularities

In this section, we solve the Reduced Problem for the stationary and axially symmetric Einstein/Maxwell Equations. We state a somewhat more general result, whose proof, however, requires no additional effort.

Let $\Omega \subset \mathbb{R}^n$, and let $\Sigma$ be a smooth closed submanifold of $\mathbb{R}^n$ of codimension at least 2. Let $\mathbb{H}$ be one of the classical globally symmetric space of noncompact type and rank one, i.e., either real, complex, or quaternionic hyperbolic space of real dimension $m$. For our purpose, we take a harmonic map $\varphi = (\varphi^1, \ldots, \varphi^m): \Omega \setminus \Sigma \to \mathbb{H}$ to mean a map $\varphi \in C^\infty(\Omega \setminus \Sigma; \mathbb{H})$, which is for each $\Omega' \Subset \Subset \Omega \setminus \Sigma$ a critical point of the energy functional:

$$E_{\Omega'} = \int_{\Omega'} |d\varphi|^2,$$

where $|d\varphi|^2 = \sum_{k=1}^n \langle \partial_k \varphi, \partial_k \varphi \rangle$ is the energy density, and $\langle \cdot, \cdot \rangle$ denotes the metric of $\mathbb{H}$. This is equivalent to requiring $\varphi$ to satisfy in $\Omega \setminus \Sigma$ the following elliptic system of nonlinear partial differential equations:

$$\Delta \varphi^a + \sum_{k=1}^n \Gamma^a_{bc}(\varphi) \partial_k \varphi^b \partial_k \varphi^c = 0,$$

where $\Gamma^a_{bc}$ are the Christoffel symbols of $\mathbb{H}$.

**Definition 6.** Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let $\Sigma$ be a smooth closed submanifold of $\Omega$ of codimension at least 2. Let $\gamma$ be a geodesic in $\mathbb{H}$. We say that a harmonic map $\varphi: \Omega \setminus \Sigma \to \mathbb{H}$ is a $\Sigma$-singular map into $\gamma$ if

(i) $\varphi(\Omega \setminus \Sigma) \subset \gamma(\mathbb{R})$

(ii) $\varphi(x) \to \gamma(+\infty)$ as $x \to \Sigma$

(iii) There is a constant $\delta > 0$ such that $|d\varphi(x)|^2 \geq \delta \text{dist}(x, \Sigma)^{-2}$ in some neighborhood of $\Sigma$.

We will assume that we are given $N + 1$ disjoint smooth closed connected submanifolds $\Sigma_j \subset \mathbb{R}^n$ of codimension at least 2, possibly with boundary. Let $J = \{1 \leq j \leq N + 1: \Sigma_j \text{ bounded}\}$, and let $\Sigma = \bigcup_{j \notin J} \Sigma_j$ be the union of the unbounded components of $\Sigma$. For $p \in \mathbb{H}$, and $\gamma$ a geodesic in $\mathbb{H}$, let $\text{dist}(p, \gamma)$ denote the distance from $p$ to $\gamma(\mathbb{R})$, i.e., $\inf_t \text{dist}(p, \gamma(t))$. 


Definition 7. Singular Dirichlet data for the harmonic map problem with prescribed singularities on \( \mathbb{R}^n \setminus \Sigma \) into \( \mathbb{H} \) consists of \( N + 2 \) harmonic maps \((\varphi_1, \ldots, \varphi_{N+1}, \tilde{\varphi})\) satisfying:

(i) \( \varphi_j: \mathbb{R}^n \setminus \Sigma \to \mathbb{H} \) is \( \Sigma \)-singular into a geodesic \( \gamma_j \) of \( \mathbb{H} \).

(ii) \(|\text{dist}(p_0, \varphi_j) - \text{dist}(p_0, \varphi_{j'})| \leq C\), for all \( 1 \leq j, j' \leq N + 1 \), where \( p_0 \) is some fixed point of \( \mathbb{H} \).

(iii) \( \tilde{\varphi}: \mathbb{R}^n \setminus \tilde{\Sigma} \to \mathbb{H} \) is asymptotic to \( \varphi_j \) near \( \Sigma_j \) for \( j \notin J \).

(iv) \( \text{dist}(\tilde{\varphi}(x), \gamma_j) \to 0 \) as \( x \to \Sigma_j \), for \( j \notin J \) outside a compact subset of \( \mathbb{R}^n \).

The following result is our main existence theorem.

Main Theorem. Let \((\varphi_1, \ldots, \varphi_{N+1}, \tilde{\varphi})\) be singular Dirichlet data on \( \mathbb{R}^n \setminus \Sigma \). Then there exists a unique harmonic map \( \varphi: \mathbb{R}^n \setminus \Sigma \to \mathbb{H} \), such that for each \( j = 1, \ldots, N + 1 \), \( \varphi \) is asymptotic to \( \varphi_j \) near \( \Sigma_j \), and such that \( \varphi \) is asymptotic to \( \tilde{\varphi} \) at infinity.

In other words, given a harmonic map \( \tilde{\varphi} \) with the correct asymptotic behavior at infinity, we can modify it to a harmonic map \( \varphi \) with prescribed singular behavior near the compact components of \( \Sigma \). It is easily seen that the data \((\varphi_1, \ldots, \varphi_{N+1}, \tilde{\varphi})\) defined in Section 2.4 constitute singular Dirichlet data on \( \mathbb{R}^3 \setminus \Sigma \). Thus, as an immediate corollary to the Main Theorem, we obtain:

Corollary. The Reduced Problem for the stationary and axially symmetric Einstein/Maxwell Equations has a unique solution.

We note that the axial symmetry of the solution \( \varphi \) follows immediately from the uniqueness statement in the Main Theorem.

In [21], the case \( \mathbb{H} = \mathbb{H}^2 \) corresponding to the vacuum equations was studied, and in [23] we proved a version of this theorem for maps \( \varphi: \Omega \setminus \Sigma \to \mathbb{H} \) where \( \Omega \) is bounded and \( \Sigma \) is compactly contained in \( \Omega \). The proof of our Main Theorem, which combines the ideas in both these papers, is divided into three steps. In step one we consider the problem on a ball \( B \subset \mathbb{R}^n \), but allow \( \Sigma \) to extend to \( \partial B \). Besides some minor technical points there are no new difficulties in this step. In step two, we use the known map \( \tilde{\varphi} \) to obtain pointwise a priori bounds on \( d(\varphi, \varphi_j) \) and \( d(\varphi, \tilde{\varphi}) \) which are independent of the size of \( B \). This allows us to complete the proof in step three.

3.1. Preliminaries. The primary tools for the proof of the Main Theorem are the Busemann functions on Cartan-Hadamard manifolds, see [6, 8]. We note that \( \mathbb{H} \) is such a manifold with pinched sectional curvatures \(-4 \leq \kappa \leq -1\). Let \( \gamma \) be a geodesic in \( \mathbb{H} \), then \( f_\gamma(p) \), the Busemann function associated with \( \gamma \) evaluated at \( p \in \mathbb{H} \), is the renormalized distance between \( p \) and the ideal point \( \gamma(+\infty) \in \partial \mathbb{H} \). It is defined by:

\[
f_\gamma(p) = \lim_{t \to \infty} \left( \text{dist}(p, \gamma(t)) - t \right).
\]
It is not difficult to see that the limit is obtained uniformly for \( p \) in compact subsets of \( \mathbb{H} \). It is well-known that \( f_\gamma \) is analytic, convex, and has gradient of constant length one. The level sets, which we denote by \( S_\gamma(t) = \{ p \in \mathbb{H}; f_\gamma(p) = t \} \), are called horospheres. They are diffeomorphic to \( \mathbb{R}^{m-1} \).

We will also use the horoballs defined by \( B_\gamma(t) = \{ p \in \mathbb{H}; f_\gamma(p) \leq t \} \), and the geodesic balls \( B_R(p) = \{ q \in \mathbb{H}; \text{dist}(p,q) \leq R \} \). Two geodesics \( \gamma \) and \( \gamma' \) are said to be asymptotic if \( \text{dist}(\gamma(t),\gamma'(t)) \) is bounded for \( t \geq 0 \).

This is clearly an equivalence relation. The boundary of \( \mathbb{H} \) is defined to be the set of equivalence classes of geodesics in \( \mathbb{H} \). We will denote the equivalence class of \( \gamma \) in \( \mathbb{H} \) by \( \gamma(+\infty) \). Let \( \gamma \) be a geodesic in \( \mathbb{H} \). We denote the reverse geodesic \( t \mapsto \gamma(-t) \) by \( -\gamma \), and we also write \( \gamma(-\infty) = -\gamma(+\infty) \). We will introduce an analytic coordinate system in \( \mathbb{H} \) adapted to \( \gamma \). Let \( u = f_\gamma \). Let \( \Phi_t : S_{-\gamma}(0) \to \mathbb{R}^{m-1} \) be an analytic coordinate system on \( S_{-\gamma}(0) \) centered at \( \gamma(0) \). The integral curves of \( \nabla f_\gamma \) are geodesics parameterized by arclength. We ‘drag’ the coordinate system \( \Phi_0 \) along these integral curves. More precisely, let \( \Phi_t \) be the analytic flow generated by this vector field, then \( \Phi_{-t} \) maps \( S_{-\gamma}(t) \) to \( S_{-\gamma}(0) \), hence \( v_t = \Phi_0 \circ \Phi_{-t} \) is an analytic coordinate system on \( S_{-\gamma}(t) \). Define \( v : \mathbb{H} \to \mathbb{R}^{m-1} \) by \( v|_{S_{-\gamma}(t)} = v_t \), and let \( \Phi = (u,v) : \mathbb{H} \to \mathbb{R}^m \), then \( \Phi \) is an analytic coordinate system on \( \mathbb{H} \).

In these coordinates, the metric of \( \mathbb{H} \) reads:

\[ ds^2 = du^2 + Q_p(dv), \]

where for each \( p \in \mathbb{H} \), \( Q_p \) is a positive quadratic form on \( \mathbb{R}^{m-1} \). We note that the convexity of \( f_{-\gamma} \) implies that for each non-zero \( X \in \mathbb{R}^{m-1} \), and each fixed \( v \in \mathbb{R}^{m-1} \), \( Q_{\Phi^{-1}(u,v)}(X) \) is a positive increasing function of \( u \).

The following lemmas were proved in [23]. They are quoted here without proofs. Except for Lemma 5, they depend only on the bound \(-4 \leq \kappa \leq -1\) for the sectional curvatures of \( \mathbb{H} \).

**Lemma 1.** For any \( (u,v) \in \mathbb{R}^m \) and any \( X \in \mathbb{R}^{m-1} \), there holds:

\[ 2Q_{\Phi^{-1}(u,v)}(X) \leq \nabla_u (Q_{\Phi^{-1}(u,v)}(X)) \leq 4Q_{\Phi^{-1}(u,v)}(X). \]

**Lemma 2.** Let \( \gamma \) be a geodesic in \( \mathbb{H} \). Then for any \( t_0 \in \mathbb{R} \), and any \( T \geq 0 \), we have

\[ B_\gamma(-t_0 + T) \cap B_{-\gamma}(t_0 + T) \subset B_R(\gamma(t_0)), \]

where \( R = T + \log 2 \).

**Lemma 3.** Let \( \gamma \) and \( \gamma' \) be geodesics in \( \mathbb{H} \), such that \( \gamma(-\infty) = \gamma'(-\infty) \) and \( \gamma(+\infty) \neq \gamma'(+\infty) \). Then for some \( d \in \mathbb{R} \), we have:

\[ \lim_{t \to \infty} (f_{-\gamma} - f_\gamma) \circ \gamma'(t) = d \]

\[ \lim_{t \to -\infty} (f_{-\gamma} + f_\gamma) \circ \gamma'(t) = 0. \]
Lemma 4. Let $\gamma$ be a geodesic in $\mathbb{H}$. Then, for any $t_0 \in \mathbb{R}$, and any $T \geq (1/2) \log 2$, we have
\[
\langle \nabla f_\gamma(p), \nabla f_{-\gamma}(p) \rangle > 0, \quad \forall p \in S_\gamma(-t_0 + T) \setminus B_{\gamma}(t_0 + T)
\]
where $\langle \cdot, \cdot \rangle$ denotes the metric on $\mathbb{H}$.

Lemma 5. Let $\gamma$ be a geodesic in $\mathbb{H}$. Then there is an analytic coordinate system $\Phi = (u, v): \mathbb{H} \to \mathbb{R}^m$ with $u = f_{-\gamma}$ such that in these coordinates, the metric of $\mathbb{H}$ is given by:
\[
ds^2 = du^2 + Q_p(dv),
\]
and satisfies the following conditions:

(i) Let $R > 0$, $t_0 \in \mathbb{R}$, and let $\gamma'$ be a geodesic in $\mathbb{H}$ with $\gamma'(-\infty) = \gamma(-\infty)$. Then, there exists a constant $c \geq 1$ such that for all $t \geq t_0$, and all $p \in B_R(\gamma'(t))$, there holds:
\[
\frac{1}{c} Q_{\gamma'(t)}(X) \leq Q_p(X) \leq c Q_{\gamma'(t)}(X), \quad \forall X \in \mathbb{R}^{m-1}.
\]

(ii) For all $t, t' \in \mathbb{R}$, the set $S_{-\gamma}(t) \cap B_{\gamma}(t')$ is star-shaped in these coordinates with respect to its ‘center’, the point where $\gamma'$ intersects $S_{-\gamma}$.

The following lemma is a slight generalization of Lemma 8 in [23].

Lemma 6. Let $B = B_R \subset \mathbb{R}^n$ be the ball of radius $R$ centered at the origin, and let $\Sigma$ be a smooth closed submanifold of $\mathbb{R}^n$ of codimension at least 2. Suppose that $u \in C^\infty(\overline{B} \setminus \Sigma)$ satisfies $\Delta u \geq 0$, and $0 \leq u \leq 1$ in $B \setminus \Sigma$. Then, for any $R' < R$, there holds
\[
\int_{B_{R'}} |\nabla u|^2 \leq C,
\]
where $C$ is a constant that depends only on $R, R'$, and $n$. Furthermore, we have:
\[
\sup_{B \setminus \Sigma} u \leq \sup_{\partial B \setminus \Sigma} u.
\]

Proof. If $\chi \in C_0^{0,1}(B \setminus \Sigma)$ with $0 \leq \chi \leq 1$, then integrating by parts the inequality $\chi^2 u \Delta u \geq 0$, and using $0 \leq u \leq 1$, we obtain:
\[
\int_B \chi^2 |\nabla u|^2 \leq 4 \int_B |\nabla \chi|^2.
\]
Define the cut-off function $\chi_\varepsilon$ by:
\[
\chi_\varepsilon = \begin{cases} 
2 - \log r / \log \varepsilon & \text{if } \varepsilon^2 \leq r \leq \varepsilon \\
0 & \text{if } r \leq \varepsilon^2 \\
1 & \text{if } r \geq \varepsilon,
\end{cases}
\]
where \( r(x) = \text{dist}(x, \Sigma) \), and let \( \chi' \in C_0^\infty(B) \) be another cut-off satisfying \( 0 \leq \chi' \leq 1 \) and \( \chi' = 1 \) on \( B_R' \). Now, take \( \chi = \chi \varepsilon \chi' \) in (3.4), and then let \( \varepsilon \to 0 \) to get:

\[
\int_B (\chi')^2 |\nabla u|^2 \leq 4 \int_B |\nabla \chi'|^2 ,
\]

see \([23\text{, Lemma 8}]\). The estimate (3.2) follows immediately. Now, the same argument shows that \( \Delta u \geq 0 \) weakly throughout \( B \), hence, by the maximum principle, we obtain (3.3) over \( B_{R'} \) instead of \( B \). Finally, let \( R' \to R \) to get (3.3).

3.2. Step One. In this step, we restrict our attention to a ball \( B \subset \mathbb{R}^n \)
large enough so that (iv) in Definition 7 holds near \( \partial B \). We prove the existence of a solution \( \varphi : B \setminus \Sigma \to \mathbb{H} \) in a manner similar to the proof of [23, Theorem 1]. We will assume all the hypotheses of the Main Theorem. Some of the elements in the proof of Proposition 1 will be used also in the next steps.

**Proposition 1.** Let \( B \subset \mathbb{R}^n \) be a large enough ball. Then there is a unique harmonic map \( \varphi : B \setminus \Sigma \to \mathbb{H} \) such that \( \varphi \) is asymptotic to \( \varphi_j \) near \( \Sigma_j \) for each \( j = 1, \ldots, N + 1 \), and \( \varphi = \varphi_0 \) on \( \partial B \setminus \Sigma \).

**Proof.** The proof follows closely that of Proposition 2 in [23]. We begin with the uniqueness. Let \( \varphi \) and \( \varphi' \) be two such maps, then clearly \( \text{dist}(\varphi, \varphi') \in L^\infty(B \setminus \Sigma) \), and \( \varphi = \varphi' \) on \( \partial B \setminus \Sigma \). Consider the function \( u = \text{dist}(\varphi, \varphi')^2 \).

We have that \( \Delta u \geq 0 \) on \( B \setminus \Sigma \), see [10], \( u \) is bounded on \( B \setminus \Sigma \), and \( u|_{\partial B \setminus \Sigma} = 0 \). Thus, by Lemma 6 it follows that \( u = 0 \) and hence \( \varphi = \varphi' \) on \( B \setminus \Sigma \).

For the existence proof, we set-up a variational approach. Pick a smooth open cover of \( \mathbb{R}^n \) which separates the \( \Sigma_j \)'s, i.e., a collection \( \{\Omega_j\}_{j=1}^{N+1} \) of smooth open sets such that \( \bigcup_j \Omega_j = \mathbb{R}^n \), \( \Sigma_j \subset \Omega_j' \), and \( \Sigma_j \cap \Omega_j'' = \emptyset \) for \( j \neq j' \). It can easily be arranged that if \( j \in J \) then \( \Omega_j \subset B \). Let \( \Omega_j = \Omega_j' \cap B \). Let \( \{\chi_j\}_{j=1}^{N+1} \), be a partition of unity subordinate to the cover \( \{\Omega_j\}_{j=1}^{N+1} \) of \( B \) such that \( \chi_j = 1 \) near \( \Sigma_j \).

We may assume, without loss of generality, that all the geodesics \( \gamma_j \) have the same initial point on \( \partial \mathbb{H} \). Indeed, if this is not the case, we can write \( \varphi_j = \gamma_j \circ U_j \), where \( U_j \) is a harmonic function on \( B \setminus \Sigma \). Since \( \mathbb{H} \) has strictly negative curvature, there are geodesics \( \gamma_j' \) with a common initial point \( p_\infty = \gamma_j'(-\infty) \in \partial \mathbb{H} \) such that \( \gamma_j \) and \( \gamma_j' \) are asymptotic. Let \( \varphi_j' = \gamma_j' \circ U_j \), then it is easy to see that \( \varphi_j \) and \( \varphi_j' \) are asymptotic near \( \Sigma_j \), so that we may replace \( \varphi_j \) by \( \varphi_j' \). We may also assume that the \( U_j \)'s are all equal. Indeed, they differ at most by constants, since, by (ii) of Definition 7 \( U_j - U_{j'} \) is a bounded harmonic function on \( \mathbb{R}^n \). So we may replace \( \varphi_j' \) by \( \varphi_{j'} = \gamma_j' \circ u_j \), where \( U_j = u_0 + c_j \) for some \( N + 1 \) constants \( c_j \). Again \( \varphi_{j'} \) is asymptotic to \( \varphi_j \) near \( \Sigma_j \). Note that the other conditions in Definition 7 are not affected by these changes, and that the singular Dirichlet data \( (\varphi''_1, \ldots, \varphi''_{N+1}, \varphi_0) \) is
equivalent to \((\varphi_1, \ldots, \varphi_{N+1}, \tilde{\varphi})\). We will assume these changes have already been carried out.

Let \(\Phi = (u, v)\) be the coordinate system on \(\mathbb{H}\) with \(u = f_{-\gamma_1}\) given in Lemma 3. We will identify any map \(\varphi: B \setminus \Sigma \to \mathbb{H}\) with its parameterization \(\Phi \circ \varphi = (u, v)\) whenever no confusion can arise. We have \(\varphi_j = (u_0, w_j)\) where \(w_j \in \mathbb{R}^{m-1}\) are constants. We write \(u_0 = \sum_{j=1}^{N+1} u_j\) where \(u_j\) is a harmonic function on \(\mathbb{R}^n\) which is singular only on \(\Sigma_j\). Without loss of generality, we may assume that for \(j \in J\), \(u_j(x) = 0\) on \(\partial B\), and in particular \(u_j > 0\) in \(B\).

For any domain \(\Omega\), define the norms:

\[
\|u\|_{\Omega} = \left( \int_{\Omega} \left\{ u^2 + |\nabla u|^2 \right\} \right)^{1/2},
\]

\[
\|v\|_{\varphi_j;\Omega} = \left( \int_{\Omega} \left\{ |v|^2 + Q_{\varphi_j}(\nabla v) \right\} \right)^{1/2},
\]

where \(Q_{\varphi_j}(\nabla v) = \sum_{k=1}^n Q_{\varphi_j(x)}(\nabla_k v(x))\) for \(x \in \Omega \setminus \Sigma_j\). We will use the Hilbert spaces:

\[
H_1(\Omega) = \{ u: \Omega \to \mathbb{R}; \|u\|_\Omega < \infty \}
\]

\[
H_1^\varphi(\Omega; \mathbb{R}^{m-1}) = \{ v: \Omega \to \mathbb{R}^{m-1}; \|v\|_{\varphi_j;\Omega} < \infty \},
\]

and the subspaces \(H_{1,0}(\Omega)\) and \(H_{1,0}^\varphi(\Omega; \mathbb{R}^{m-1})\), defined to be the closure in \(H_1(\Omega)\) of \(C^\infty_0(\Omega)\), and the closure in \(H_1^\varphi(\Omega; \mathbb{R}^{m-1})\) of \(C^\infty_0(\Omega \setminus \Sigma_j; \mathbb{R}^{m-1})\), respectively.

We write \(\tilde{\varphi} = (\tilde{u}, \tilde{v})\), and introduce the function \(\tilde{v}_0\) obtained by truncating \(\tilde{v}\) near the bounded components of \(\Sigma\) and setting it to its prescribed values \(w_j \in \mathbb{R}^{m-1}\):

\[
\tilde{v}_0 = \left( \prod_{j \in J} (1 - \chi_j) \right) \tilde{v} + \sum_{j \in J} \chi_j w_j.
\]

Note that \(\tilde{v}_0 = w_j\) near \(\Sigma_j\) for \(j \in J\), hence clearly \(\tilde{v}_0 - w_j \in H_1^\varphi(\Omega_j; \mathbb{R}^{m-1})\).

We also decompose \(u_0 = U_0 + \tilde{U}_0\) into \(U_0 = \sum_{j \in J} u_j\), the contribution from the bounded components of \(\Sigma\), and \(\tilde{U}_0 = \sum_{j \notin J} u_j\), the contribution from the unbounded components. Clearly, \(\Delta U_0 = \Delta \tilde{U}_0 = 0\) on \(B \setminus \Sigma\) and \(U_0 = 0\) on \(\partial B\). Since \(\tilde{\varphi}\) is a harmonic map, \(\tilde{u}\) is subharmonic on \(\mathbb{R}^n \setminus \tilde{\Sigma}\), hence \(\Delta(\tilde{u} - \tilde{U}_0) \geq 0\) on \(\mathbb{R}^n \setminus \tilde{\Sigma}\). Furthermore, \(|\tilde{u} - \tilde{U}_0|\) is bounded on the ball \(B_{R+1}\). Indeed, on the one hand, for \(j \notin J\), we have \(|\tilde{u} - \tilde{U}_0| \leq |\tilde{u} - u_0| + U_0 \leq \text{dist}(\tilde{\varphi}, \varphi_j) + C_j\) on \(\Omega_j' \cap B_{R+1}\) for some constant \(C_j\). On the other hand, for \(j \in J\), we have that both \(\tilde{u}\) and \(\tilde{U}_0\) are bounded on \(\Omega_j\). Hence it follows from Lemma 4 that \(\tilde{u} - \tilde{U}_0 \in H_1(B)\). In addition, \(\tilde{u}\) satisfies the equation \(\Delta \tilde{u} = (\partial/\partial u)(Q_{\tilde{\varphi}}(\nabla \tilde{v}))\), and Lemma 4 implies that:

\[
Q_{\tilde{\varphi}}(\nabla \tilde{v}) \leq \frac{1}{2} \Delta(\tilde{u} - \tilde{U}_0),
\]
in $B_{R+1} \setminus \tilde{\Sigma}$. Multiplying by a cut-off $\chi$ as in the proof of Lemma 5 and integrating over $B$, we conclude, after taking the appropriate limit, that

\begin{equation}
(3.7) \quad \int_B Q\varphi(\nabla \tilde{v}) < \infty.
\end{equation}

Define $\mathcal{H}$ to be the space of maps $\varphi = (u, v): B \setminus \Sigma \to \mathbb{H}$ satisfying:

\begin{align*}
\begin{cases}
u - \tilde{u} - U_0 \in H_{1,0}(B); \\
\tilde{v} \in \bigcap_{j=1}^{N+1} H_{1,0}^j(B; \mathbb{R}^{m-1}); \\
\text{dist}(\varphi, \varphi_j) \in L^\infty(\Omega_j \setminus \Sigma_j), \quad \forall j = 1, \ldots, N + 1.
\end{cases}
\end{align*}

For maps $\varphi \in \mathcal{H}$ define:

\begin{equation}
F(\varphi) = \int_B \left\{ |\nabla (u - u_0)|^2 + Q\varphi(\nabla v) \right\}.
\end{equation}

For $R > 0$, define $\mathcal{H}_R$ to be the space of maps $\varphi \in \mathcal{H}$ such that $\text{dist}(\varphi, \varphi_j) \leq R$ for a.e. $x \in \Omega_j \setminus \Sigma_j$ for $j = 1, \ldots, N + 1$. We will need the following lemma which is a direct consequence of Lemma 5.

**Lemma 7.** Let $R > 0$, then there is a $c > 1$ such that for all $\varphi \in \mathcal{H}_R$, there holds for each $j = 0, \ldots, N$ and for all $x \in \mathbb{R}^{m-1}$:

\begin{equation}
\frac{1}{c} Q \varphi_j(x)(X) \leq Q \varphi(x)(X) \leq c Q \varphi_j(x)(X), \quad \text{for a.e. } x \in \Omega_j \setminus \Sigma_j.
\end{equation}

We already know that $\int_{\Omega_j} Q \varphi_j(\nabla \tilde{v}_0) < \infty$ for $j \in J$, and it follows from Lemma 4 and (3.7) that the same holds also for $j \notin J$. Indeed, since $\tilde{\varphi}$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for $j \notin J$, there is a constant $c \geq 1$ such that

\begin{equation}
\int_{\Omega_j} Q \varphi_j(\nabla \tilde{v}) \leq c \int_{\Omega_j} Q \varphi(\nabla \tilde{v}) < \infty.
\end{equation}

Since $\chi_j$ is smooth and $\nabla \chi_j = 0$ near $\Sigma$, we obtain that $\int_{\Omega_j} Q \varphi_j(\nabla \tilde{v}_0) < \infty$ as claimed. Now, let $\varphi = (u, v) \in \mathcal{H}$, then there is a constant $c \geq 1$ such that:

\begin{equation}
(3.8) \quad \int_B Q \varphi(\nabla v) \leq c \sum_{j=0}^{N} \int_{\Omega_j} Q \varphi_j(\nabla v) < \infty.
\end{equation}

Since also $u - u_0 = (u - \tilde{u} - U_0) + (\tilde{u} - \tilde{U}_0) \in H_1(B)$, we conclude that $F$ is finite on $\mathcal{H}$.

It is straightforward to check that a minimizer $\varphi \in \mathcal{H}$ of $F$ is a harmonic map on $B \setminus \Sigma$, see [23]. Hence, by the regularity theory for harmonic maps [17] [18], we have $\varphi \in C^\infty(\overline{B} \setminus \Sigma; \mathbb{H})$. Furthermore, $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for $j = 1, \ldots, N + 1$ by construction, and $\varphi = \tilde{\varphi}$ on $\partial B \setminus \Sigma$. Thus, to prove Proposition 1, it suffices to show that $F$ admits a minimizer in $\mathcal{H}$.

It can be shown, exactly as in [23], Proposition 1, that $F$ admits a minimizing sequence
We turn to the proof of \( \varphi^{(ii)} \). The bound \( \text{dist}(\tilde{\varphi}_k, \varphi_j) \leq R \) in \( \Omega_j \), it is shown that, perhaps along a further subsequence, \( \tilde{\varphi}_k - \tilde{\varphi}_0 \) also converges weakly in each \( H^1_0(B, \mathbb{R}^{m-1}) \), and pointwise a.e. in \( B \). Let \( \varphi = (u, v) \in \mathcal{H}_R \) be the weak limit. Then clearly we have

\[
(3.9) \quad \int_B |\nabla (u - u_0)|^2 \leq \liminf \int_B |\nabla (\tilde{\varphi}_k' - u_0)|^2.
\]

Furthermore, \( Q_\varphi \) is equivalent to \( Q_\varphi_j \) on \( \Omega_j \), hence we have

\[
(3.10) \quad \int_B Q_\varphi(\nabla v) = \lim \int_B Q_\varphi(\nabla v, \nabla \tilde{\varphi}_k') 
\[
\leq \liminf \left[ \left( \int_B \chi_{k'} Q_\varphi(\nabla v) \right)^{1/2} \left( \int_B Q_{\tilde{\varphi}_k'}(\nabla \tilde{\varphi}_k') \right)^{-1/2} \right],
\]

where

\[
\chi_k = \begin{cases} Q_\varphi(\nabla \tilde{\varphi}_k)/Q_{\tilde{\varphi}_k}(\nabla \tilde{\varphi}_k) & \text{if } \tilde{\varphi}_k \neq 0 \\ 1 & \text{otherwise.} \end{cases}
\]

However \( \chi_{k'} \to 1 \) pointwise a.e. in \( B \), and \( \chi_{k'} \) is bounded. Thus, we conclude that

\[
\lim \int_B \chi_{k'} Q_\varphi(\nabla v) = \int_B Q_\varphi(\nabla v),
\]

and therefore \((3.10)\) implies:

\[
\int_B Q_\varphi(\nabla v) \leq \left( \int_B Q_\varphi(\nabla v) \right)^{1/2} \liminf \left( \int_B Q_{\tilde{\varphi}_k'}(\nabla \tilde{\varphi}_k') \right)^{1/2}.
\]

Combining this with \((3.9)\) we obtain

\[
F(\varphi) \leq \liminf F(\varphi_{k'}) = \inf_{\mathcal{H}_R} F,
\]

and it follows that \( \varphi \) is a minimizer in \( \mathcal{H}_R \).

Thus, it remains to prove that for some \( R > 0 \) large enough, \( \inf_{\mathcal{H}_R} F = \inf_{\mathcal{H}_R} F \). This is the content of the next lemma.

**Lemma 8.** There is a constant \( R > 0 \) such that for every \( \varepsilon > 0 \), and every \( \varphi \in \mathcal{H} \), there is \( \varphi' \in \mathcal{H}_R \) such that \( F(\varphi') \leq F(\varphi) + \varepsilon \).

**Proof of Lemma** Let \( \mathcal{H}^* \) be the space of maps \( \varphi = (u, v) \in \mathcal{H} \) such that \( v = w_j \) in a neighborhood of \( \Sigma_j \) for \( j \in J \), \( \varphi = \tilde{\varphi} \) outside some compact set \( K \subset B \), and \( v = w_j \) in a neighborhood of \( \Sigma_j \cap K \) for \( j \not\in J \). Lemma \( \text{[23 Lemma 11]} \) will follow immediately from: (i) for each \( \varphi \in \mathcal{H} \) and each \( \varepsilon > 0 \) there is \( \varphi' \in \mathcal{H}^* \) such that \( F(\varphi') \leq F(\varphi) + \varepsilon \); and (ii) for each \( \varphi \in \mathcal{H}^* \) there is \( \varphi' \in \mathcal{H}_R \) such that \( F(\varphi') \leq F(\varphi) \). The proof of (i), a standard approximation argument, is practically unchanged from \( \text{[23 Lemma 11]} \). We turn to the proof of (ii). The bound \( \text{dist}(\varphi, \varphi_j) \leq R \) will be achieved consecutively on each \( \Omega_j \) beginning with all \( j \not\in J \). The case \( j \in J \) is simpler. Assume without loss of
generality that $1 \notin J$. Introduce a 'dual' coordinate system \( \Phi = (\bar{u}, \bar{v}) \) with \( \bar{u} = f_{\gamma_1} \) as given by Lemma \ref{lemma}. Write
\[
dS^2 = d\bar{u}^2 + \bar{Q}_\rho(d\bar{v}).
\]
for the metric of \( H \) in these coordinates. Set \( \bar{u}_0 = -u_1 + \sum_{j=2}^{N+1} u_j \). Let \( \varphi \in \mathcal{H}^* \) and write \( \Phi \circ \varphi = (\bar{u}, \bar{v}) \). Then, since \( |\nabla u|^2 + Q_\varphi(\nabla v)| = |\nabla \bar{u}|^2 + \bar{Q}_\varphi(\nabla \bar{v}) \), we find that in \( B \setminus \Sigma \):
\[
|\nabla (u - u_0)|^2 + Q_\varphi(\nabla v) = |\nabla (\bar{u} - \bar{u}_0)|^2 + \bar{Q}_\varphi(\nabla \bar{v})
- 2 \nabla (u + \bar{u}) \cdot \nabla u_1 - 2 \sum_{j=2}^{N+1} \nabla (u - \bar{u}) \cdot \nabla u_j + 4 \sum_{j=2}^{N+1} \nabla u_j \cdot \nabla u_1.
\]
The idea is now to integrate this identity over \( B \), use the right hand side to truncate \( \bar{u} - \bar{u}_0 \), and the left hand side to truncate \( u - u_0 \). The convexity of the Busemann functions \( u \) and \( \bar{u} \) on \( H \), expressed in the monotonicity of \( Q \) and \( \bar{Q} \), ensures that these truncations do not increase \( F \). This can be viewed as a weak form of a variational maximum principle.

Some care, however, needs to be taken because of the singularities on \( \Sigma \). Integrating first over \( B \setminus \Sigma^\varepsilon \), where \( \Sigma^\varepsilon = \{ x \in B \colon \text{dist}(x, \Sigma) \leq \varepsilon \} \), we obtain:
\[
\int_{B \setminus \Sigma^\varepsilon} \| \nabla (u - u_0) \|^2 + Q_\varphi(\nabla v) = \int_{B \setminus \Sigma^\varepsilon} \| \nabla (\bar{u} - \bar{u}_0) \|^2 + \bar{Q}_\varphi(\nabla \bar{v}) + I_\varepsilon(\varphi),
\]
where:
\[
(3.11) \quad I_\varepsilon(\varphi) = \int_{B \setminus \Sigma^\varepsilon} \left\{ -2 \text{div}((u + \bar{u}) \nabla u_1) - 2 \sum_{j=2}^{N+1} \text{div}((u - \bar{u}) \nabla u_j) + 4 \sum_{j=2}^{N+1} \nabla u_j \cdot \nabla u_1 \right\}.
\]
A tedious but straightforward verification shows that \( I_\varepsilon(\varphi) \to C \) as \( \varepsilon \to 0 \), where \( C \) is a constant independent of \( \varphi \in \mathcal{H}^* \), yielding an integral identity:
\[
(3.12) \quad F(\varphi) = \int_B \left\{ |\nabla (\bar{u} - \bar{u}_0) |^2 + \bar{Q}_\varphi(\nabla \bar{v}) \right\} + C.
\]
Note that if \( \varepsilon > 0 \) is small enough, we can decompose \( \partial(B \setminus \Sigma^\varepsilon) \) into a disjoint union \( \bigcup_{j=1}^{N+1} \partial(\Sigma_j^\varepsilon) \cup (\partial B \setminus \Sigma^\varepsilon) \). Then, for example, the first term in (3.11) can be integrated by parts:
\[
(3.13) \quad -2 \int_{B \setminus \Sigma^\varepsilon} \text{div}((u + \bar{u}) \nabla u_1) = -2 \left( \int_{\partial B \setminus \Sigma^\varepsilon} + \sum_{j=2}^{N+1} \int_{\partial \Sigma_j^\varepsilon} \right) (u + \bar{u}) \frac{\partial u_1}{\partial n}.
\]
For the first integral on the right hand side of (3.13), we have, as $\varepsilon \to 0$:

\[(3.14) \quad \int_{\partial B \setminus \Sigma^\varepsilon} (u + \bar{u}) \frac{\partial u_1}{\partial n} \to \int_{\partial B} (u + \bar{u}) \frac{\partial u_1}{\partial n},\]

which clearly is independent of $\varphi \in \mathcal{H}^*$. The integral on the right hand side of (3.14) is finite since $u + \bar{u}$ is bounded in $\Omega_1 \setminus \Sigma_1$. Next, since $\varphi \in \mathcal{H}^*$, there is a compact set $K \subset B$ such that $\varphi = \tilde{\varphi}$ outside $K$ and $\tilde{v} = w_1$ in a neighborhood of $\Sigma_1 \cap K$. We find:

\[\int_{\partial \Sigma_1^j \setminus K} (u + \bar{u}) \frac{\partial u_1}{\partial n} = \int_{\partial \Sigma_1^j \setminus K} (u + \bar{u}) \frac{\partial u_1}{\partial n},\]

since for $x \in \partial \Sigma_1^j \setminus K$ and $\varepsilon > 0$ small enough, $\varphi(x)$ lies along the geodesic $\gamma_1$ where $u + \bar{u} = 0$. However for $x \in \partial \Sigma_1^j \setminus K$, we have $\varphi(x) = \tilde{\varphi}(x)$ which implies $(u(x) + \bar{u}(x)) \leq \text{dist}(\tilde{\varphi}(x), \gamma_1) \to 0$ as $\varepsilon \to 0$, by (iv) of Definition 7. Thus, we obtain

\[\int_{\partial \Sigma_1^j \setminus K} (u + \bar{u}) \frac{\partial u_1}{\partial n} \to 0.\]

Finally, in view of Lemma 3 we have that, for $2 \leq j \leq N + 1$, $\lim_{\varepsilon \to 0} (u + \bar{u})|_{\partial \Sigma_j} = d_j$, for some constants $d_j$. Hence the last sum in (3.14) also tends to zero since $u_1$ is regular in $\Omega_j$ for $j = 2, \ldots, N + 1$. The other terms in (3.11) are handled similarly. See [23, Proof of Lemma 13] for more details.

Now, truncate $\bar{u} - \bar{u}_0$ above at

\[\bar{T}_1 = \sup_{\partial B} (\bar{u} - \bar{u}_0) + 1,\]

i.e., define a map $\varphi' = (u', v') \in \mathcal{H}$ by $\Phi \circ \varphi' = (\bar{u}', \bar{v})$ where

\[\bar{u}' - \bar{u}_0 = \min\{\bar{u} - \bar{u}_0, \bar{T}_1\}.\]

Clearly the map $\varphi' \in \mathcal{H}$ and satisfies:

\[(3.15) \quad \varphi'(x) \in B_{\gamma_1}(\bar{u}_0(x) + \bar{T}_1), \quad \forall x \in B \setminus \Sigma,\]

and also, in view of (3.12), $F(\varphi') \leq F(\varphi)$. Let $c_1 = \sup_{\Omega_j} \sum_{j=2}^{N+1} u_j$, then we obtain from (3.16):

\[(3.16) \quad \varphi'(x) \in B_{\gamma_1}(-u_1(x) + c_1 + \bar{T}_1), \quad \forall x \in \Omega_1 \setminus \Sigma_1.\]

Next truncate $u' - u_0$ above at

\[T_1 = \max\left\{\sup_{\partial B} (u - u_0) + 1, \bar{T}_1, (1/2) \log 2\right\},\]

to get a map $\varphi'' = (u'', v'') \in \mathcal{H}$ satisfying:

\[\varphi''(x) \in B_{-\gamma_1}(u_0(x) + T_1), \quad \forall x \in B \setminus \Sigma\]

and $F(\varphi'') \leq F(\varphi')$. As before it follows that:

\[(3.17) \quad \varphi''(x) \in B_{-\gamma_1}(u_1(x) + c_1 + T_1), \quad \forall x \in \Omega_1 \setminus \Sigma_1.\]
From Lemma 3 we obtain, as in [23, Lemma 11], that the bound \(3.16\) still holds for \(\varphi''\):

\[
(3.18) \quad \varphi''(x) \in B_{\gamma_j}(-u_1(x) + c_1 + T_1), \quad \forall x \in \Omega_1 \setminus \Sigma_1.
\]

Hence, combining (3.17) and (3.18) with Lemma 2 we conclude:

\[
(3.19) \quad \varphi''(x) \in B_{R_1}(\varphi_1(x)), \quad \forall x \in \Omega_1 \setminus \Sigma_1,
\]

where \(R_1 = c_1 + T_1 + \log 2\).

Consider now the map \(\varphi''|_{\cup_{j=2}^{N+1} \Omega_j}\), and notice that from (3.19) we can obtain a pointwise estimate for \(\varphi''\) on \(\partial(\bigcup_{j=2}^{N+1} \Omega_j)\). Thus we can proceed by induction to obtain a map \(\varphi''' \in \mathcal{H}\) satisfying \(F(\varphi''') \leq F(\varphi)\), and for each \(j = 1, \ldots, N + 1:\)

\[
\varphi''(x) \in B_{R_j}(\varphi_j(x)), \quad \forall x \in \Omega_j \setminus \Sigma_j,
\]

for some constants \(R_j\) depending only on the data \((\varphi_1, \ldots, \varphi_{N+1}, \tilde{\varphi})\), Setting \(R = \max_j R_j\), we have obtained \(\varphi''' \in \mathcal{H}_R\) with \(F(\varphi''') \leq F(\varphi)\). This completes the proof of Lemma 3 and of Proposition 1 \(\Box\)

Remark. It is important to note that the a priori bounds \(\text{dist}(\varphi, \varphi_j) \leq R\) in \(\Omega_j \setminus \Sigma_j\) given by Lemma 3 depend on the radius of \(B\), hence cannot be used to obtain a solution on \(\mathbb{R}^n \setminus \Sigma\). This is remedied in the next step where we obtain a priori bounds independent of the size of \(B\).

3.3. Step Two. In this step, we furnish the main ingredient in the proof of the Main Theorem. We establish uniform pointwise a priori bounds for \(\text{dist}(\varphi, \tilde{\varphi}) < \infty\), and for \(\text{dist}(\varphi, \varphi_j) < \Sigma_j\) near \(\varphi: B \setminus \Sigma \to \mathbb{H}\) is the solution given by Proposition 1. As in Proposition 1, \(B \subset \mathbb{R}^n\) is a sufficiently large ball, but we now use a slightly different open cover of \(\mathbb{R}^n\). For \(j \notin J\), \(\tilde{\varphi}\) is asymptotic to \(\varphi_j\) near \(\Sigma_j\), hence we can choose \(\Omega_j'\) to be a neighborhood of \(\Sigma_j\) such that \(\text{dist}(\tilde{\varphi}, \varphi_j)\) is bounded on \(\Omega_j' \setminus \Sigma_j\). However, to get an open cover of \(\mathbb{R}^n\), we add another open set \(\Omega'\) which we may choose so that \(\Sigma_j \cap \Omega' = \emptyset\) for all \(1 \leq j \leq N + 1\). We set \(\Omega_j = \Omega_j' \cap B\), and \(\Omega = \Omega' \cap B\). We also change the normalization of the harmonic functions \(u_j\) for \(j \in J\), so that \(u_j(x) \to 0\) as \(r = |x| \to \infty\) in \(\mathbb{R}^n\).

Proposition 2. There is a constant \(R > 0\) independent of \(B\) such that if \(\varphi\) is the solution given by Proposition 1 then

\[
(3.20) \quad \text{dist}(\varphi, \tilde{\varphi}) \leq R, \quad \text{in} \quad \Omega,
\]

\[
(3.21) \quad \text{dist}(\varphi, \varphi_j) \leq R, \quad \text{in} \quad \Omega_j \setminus \Sigma_j, \text{ for } j = 1, \ldots, N + 1.
\]

Proof. Define on \(B \setminus \Sigma\) the function:

\[
\nu = \sqrt{1 + \text{dist}(\varphi, \tilde{\varphi})^2} - U_0,
\]

where recall that \(U_0 = \sum_{j \in J} u_j\), defined on page 24, is the contribution to \(u_0\) from the bounded components of \(\Sigma\). Note that since the function \(\text{dist}(\cdot, \cdot)^2\) is convex on \(\mathbb{H} \times \mathbb{H}\), we have \(\Delta \nu \geq 0\) on \(B \setminus \Sigma\), see [19]. We claim that \(\nu\) is
bounded, and therefore Lemma 6 applies. To see this, fix first \( j \not\in J \), and note that \( \varphi \) is asymptotic to \( \varphi_j \), which is asymptotic to \( \tilde{\varphi} \) near \( \Sigma_j \). Hence \( \text{dist}(\varphi, \tilde{\varphi}) \) is bounded in \( \Omega_j \). Also, \( U_0 \) is bounded in \( \Omega_j \), and hence \( \nu \) is bounded there. Similarly, \( \nu \) is bounded on \( \tilde{\Omega} \). Now, let \( j \in J \), and observe that \( \text{dist}(\gamma_j(0), \tilde{\varphi}) \) is bounded on \( \Omega_j \). Thus, since \( \text{dist}(\varphi_j, \gamma_j(0)) = u_j \), we see that

\[
\nu \leq 1 + \text{dist}(\varphi, \varphi_j) + \text{dist}(\varphi_j, \gamma_j(0)) + \text{dist}(\gamma_j(0), \tilde{\varphi}) - U_0
\]

\[
= 1 + \text{dist}(\varphi, \varphi_j) + \text{dist}(\gamma_j(0), \tilde{\varphi}) - \sum_{j' \not\in J} u_{j'}
\]

Since \( \varphi \) is asymptotic to \( \varphi_j \) near \( \Sigma_j \), and \( u_{j'} \) is bounded in \( \Omega_j \setminus \Sigma_j \) for \( j' \neq j \), we obtain \( \nu \leq C_j \) in \( \Omega_j \setminus \Sigma_j \). Hence, we conclude \( \nu \leq C \) in \( \bar{B} \setminus \Sigma \). Similarly, we obtain

\[
\nu \geq -\text{dist}(\gamma_j(0), \tilde{\varphi}) - \text{dist}(\varphi, \varphi_j) - \sum_{j' \not\in J} u_{j'} \geq -C.
\]

Applying Lemma 6 and using the fact that \( \varphi = \tilde{\varphi} \) and \( U_0 = 0 \) on \( \partial B \setminus \Sigma \), we deduce that:

\[
(3.22) \quad \nu \leq \sup_{\partial B \setminus \Sigma} \nu \leq 1
\]

It follows that

\[
(3.23) \quad \text{dist}(\varphi, \tilde{\varphi}) \leq \sqrt{(1 + U_0)^2 - 1}.
\]

Now there is a \( T \) such that \( U_0 \leq T \) on \( \bigcup_{j \not\in J} \Omega_j' \cup \tilde{\Omega}' \). Thus, we obtain immediately \( \text{dist}(\varphi, \tilde{\varphi}) \leq \bar{R} \) on \( \tilde{\Omega} \) with \( \bar{R} = 1 + T \). Furthermore, for \( j \not\in J \) there are constants \( T_j > 0 \) such that \( \text{dist}(\tilde{\varphi}, \varphi_j) \leq T_j \) on \( \Omega_j' \setminus \Sigma_j \). Thus, from (3.23), we obtain the pointwise a priori estimate:

\[
\text{dist}(\varphi, \varphi_j) \leq R_j,
\]

in \( \Omega_j \setminus \Sigma_j \) for \( j \not\in J \), where \( R_j = 1 + T + T_j \) is clearly independent of the radius of \( B \). Using these estimates, we can now bound \( \text{dist}(\varphi, \varphi_j) \) on \( \partial \Omega_j \) for \( j \in J \). Therefore, for \( j \in J \), the same argument, using the bounded subharmonic functions \( v_j = \sqrt{1 + \text{dist}(\varphi, \varphi_j)^2} \) on \( \Omega_j \setminus \Sigma_j \), yields \( \text{dist}(\varphi, \varphi_j) \leq R_j \), for some \( R_j \) independent of the radius of \( B \). Setting \( R = \max_j R_j \), Proposition 2 follows.

3.4. **Proof of the Main Theorem.** The proof of the uniqueness statement is almost the same as in Proposition 1. If \( \varphi \) and \( \varphi' \) are two solutions, then they are asymptotic near each \( \Sigma_j \), and asymptotic at infinity, hence \( \text{dist}(\varphi, \varphi')^2 \) is a bounded subharmonic function on \( \mathbb{R}^n \setminus \Sigma \). From Lemma 6 it follows that for any ball \( B \subset \mathbb{R}^n \) of radius \( R \) centered at the origin:

\[
\sup_{B \setminus \Sigma} \text{dist}(\varphi, \varphi') \leq \sup_{\partial B \setminus \Sigma} \text{dist}(\varphi, \varphi'),
\]

Since the right hand side tends to zero as the radius \( R \) of \( B \) tends to \( \infty \), it follows that \( \text{dist}(\varphi, \varphi') = 0 \), and hence \( \varphi = \varphi' \).
To prove the existence of a solution, we choose a sequence of radii \( R_k \to \infty \), let \( B_k \) be the ball of radius \( R_k \) centered at the origin, and denote by \( \tilde{\varphi}_k = (\tilde{u}_k, \tilde{v}_k) \) the solutions given by Proposition 1 on \( B_k \). We will use the uniform pointwise a priori bounds given in Proposition 2 to prove the convergence of \( \tilde{\varphi}_k \) to a solution on \( \mathbb{R}^n \setminus \Sigma \).

We use the open cover \( \{ \Omega'_j, \Omega''_j \} \) introduced in section 3.3. For any ball \( B \), define the space \( \mathcal{H}'(B) \) as the space of maps \( \varphi: B \setminus \Sigma \to \mathbb{H} \) satisfying

\[
\begin{align*}
\{ & u - u_0 \in H^1_1(B); \\
& v - \tilde{v}_0 \in \bigcap_{j=1}^{N+1} H^1_{\psi_j}(B; \mathbb{R}^{m-1}); \\
& \text{dist}(\varphi, \varphi_j) \in L^\infty(\Omega'_j \setminus \Sigma_j), \quad \forall j = 1, \ldots, N + 1,
\end{align*}
\]

and for \( R > 0 \) denote by \( \mathcal{H}'_R(B) \) the space of maps \( \varphi \in \mathcal{H}'(B) \) satisfying:

\[
\begin{align*}
\text{dist}(\varphi, \varphi) & \leq R, \quad \text{in } \overline{\Omega}, \\
\text{dist}(\varphi, \varphi_j) & \leq R, \quad \text{in } \Omega_j \setminus \Sigma_j, \quad \forall j = 1, \ldots, N + 1.
\end{align*}
\]

Now, fix \( B = B_{k_0} \), let \( B' = B_{k_0+1} \), and consider the sequence of maps \( \tilde{\varphi}_k |_{B'} \) for \( k \geq k_0 \). By Proposition 2 there is \( R > 0 \) such that \( \tilde{\varphi}_k \in \mathcal{H}'_R(B') \). It follows that on \( \bigcup_{j=1}^{N+1} \Omega_j \setminus \Sigma_j \), we have \( |\tilde{u}_k - u_0| \leq R \). Similarly, on \( \Omega \) we have \( |\tilde{u}_k - u_0| \leq |\tilde{u}_k - \tilde{u}| + |\tilde{u} - u_0| \leq R' \) where \( R' = R + \sup_{\Omega} |\tilde{u} - u_0| \).

Thus, we conclude that \( |\tilde{u}_k - u_0| \leq C \) on \( B \setminus \Sigma \), where \( C = \max\{R, R'\} \). Furthermore, we have \( \Delta(\tilde{u}_k - u_0) \geq 0 \) on \( B \setminus \Sigma \). Now the argument in the proof of Lemma 6 shows that there is a constant \( C' \) independent of \( k \) such that

\[
\int_{B'} |\nabla(\tilde{u}_k - u_0)|^2 \leq C',
\]

and similarly, using \( Q_{\tilde{\varphi}_k}(\nabla \tilde{v}_k) \leq (1/2)\Delta(\tilde{u}_k - u_0) \), we deduce, as in the argument leading to \( 3.7 \), that

\[
\int_{B'} Q_{\tilde{\varphi}_k}(\nabla \tilde{v}_k) \leq C''.
\]

Thus, we have a uniform bound \( F(\tilde{\varphi}_k) \leq C'' \). It is straightforward now to show that there is a subsequence, without loss of generality \( \tilde{\varphi}_k \), which converges weakly in \( \mathcal{H}'(B') \) and pointwise a.e. in \( B' \), see 2.4 Proof of Proposition 1.

Repeating this argument for each \( k_0 \), and using a diagonal sequence, it is clear that we can choose a subsequence, without loss of generality \( \tilde{\varphi}_k \) again, which converges pointwise a.e. in \( \mathbb{R}^n \setminus \Sigma \), and which for each ball \( B \subset \mathbb{R}^n \), converges weakly in \( \mathcal{H}'(B) \) to a map \( \varphi \in \mathcal{H}'_R(B) \). For each open set \( O \subset \mathbb{R}^n \setminus \Sigma \), \( \tilde{\varphi}_k |_O \) is a family of smooth harmonic maps with uniformly bounded energy into \( \mathbb{H} \), which maps into a fixed compact subset of \( \mathbb{H} \). Hence, using standard harmonic map theory, one can obtain uniform priori bounds in \( C^{2,\alpha}(O) \) for \( \tilde{\varphi}_k \), and hence we deduce that a subsequence converges uniformly in \( O \) together with two of its derivatives. We conclude that \( \varphi \) is a harmonic map. It remains to see that \( \varphi \) is asymptotic to \( \tilde{\varphi} \) at infinity. However, from
the estimate (3.23) on $\nu$ which holds for each $\tilde{\varphi}_k$, we deduce that the same holds of $\varphi$:

$$\text{dist}(\varphi, \tilde{\varphi}) \leq \sqrt{U_0(2 + U_0)}$$

Since $U_0(x) \to 0$ as $x \to \infty$, this shows that $\text{dist}(\varphi, \tilde{\varphi}) \to 0$ as $x \to \infty$ in $B \setminus \Sigma$. This completes the proof of the Main Theorem.

4. $N$-Black Hole Solutions of the Einstein/Maxwell Equations

In this section, starting from a solution $\varphi: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_C$ of the Reduced Problem, we construct a solution $(M, g, F)$ of the Einstein/Maxwell Equations.

4.1. The Spacetime Metric $g$. Let $\varphi: \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_C$, be a solution of the Reduced Problem. Then $\varphi = (u, v, \chi, \psi)$ satisfies on $\mathbb{R}^3 \setminus \Sigma$ the system (2.7)–(2.10) of partial differential equations, where however now, the Laplacian, the divergence, the norms, and the inner-product are all with respect to the Euclidean metric of $\mathbb{R}^3$.

Introduce cylindrical coordinates $(\rho, \phi, z)$ in $\mathbb{R}^3$, and denote by $\xi$ and $\zeta$ the vector fields $\partial/\partial \phi$ and $\partial/\partial z$ respectively. Let $*$ be the Hodge star operator of the Euclidean metric in $\mathbb{R}^3$ and $\star = \rho^{-1} i_\xi *$ the Hodge star operator of the flat metric $ds^2 = d\rho^2 + dz^2$ in the $\rho z$-plane. Setting $\omega = 2 (dv + \chi \, d\psi - \psi \, d\chi)$ it follows from (2.8) that the two-form $e^4 u * \omega$ is closed on $\mathbb{R}^3 \setminus \Sigma$, and hence, since $\omega$ is invariant under $\xi$, the one-form $e^{4u} i_\xi * \omega$ is closed on $\mathbb{R}^3 \setminus \Sigma$. Since $\mathbb{R}^3 \setminus \Sigma$ is simply connected, there is a function $w$, defined up to an additive constant such that

$$dw = e^{4u} i_\xi * \omega = \rho e^{4u} * \omega,$$

see (2.21).

Now let $T_\varphi$ be the stress tensor of the map $\varphi$:

$$T_\varphi(X, Y) = \langle d\varphi(X), d\varphi(Y) \rangle - \frac{1}{2} |d\varphi|^2 X \cdot Y,$$

for $X, Y \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{H}^2_C$. Clearly $T_\varphi$ is symmetric, and since $\varphi$ is a harmonic map $T_\varphi$ is also divergence free. Thus, since $\zeta$ is a Killing field in $\mathbb{R}^3$, $i_\zeta T_\varphi$ is a divergence free vector field on $\mathbb{R}^3 \setminus \Sigma$. As before, it follows that $i_\xi * i_\xi T_\varphi$ is a closed one-form on $\mathbb{R}^3 \setminus \Sigma$, hence there is a function $\lambda$, defined up to an additive constant such that

$$d(\lambda - u) = -2 i_\xi * i_\xi T_\varphi.$$
and compare with Equations (2.22). It is clear that $w$ and $\lambda$ defined in this way are axially symmetric. Furthermore, we deduce from (4.1) that $e^{4u} \omega = d\phi \wedge dw$. Thus, Equation (2.10) implies that the two-form $e^{2u} * d\psi + w \, d\chi \wedge d\phi$ is closed. It follows that the one-form $e^{2u} i_\xi * d\psi - w \, d\chi$ is closed, and hence there is a function $\tilde{\chi}$ such that

$$d\tilde{\chi} = e^{2u} i_\xi * d\psi - w \, d\chi.$$

Let $A = \{z\}$ be the $z$-axis and let $g$ be the metric on $M' = \mathbb{R} \times (\mathbb{R}^3 \setminus A)$ given by the line element:

$$ds^2 = -\rho^2 e^{2u} \, dt^2 + e^{-2u} (d\phi - w \, dt)^2 + e^{2\lambda} (d\rho^2 + dz^2).$$

Define the one-form $A$ on $M'$ by:

$$A = - (\chi \, d\phi + \tilde{\chi} \, dt),$$

and let

$$F = dA = dt \wedge d\tilde{\chi} + d\phi \wedge d\chi.$$

We have that $i_\xi F = d\chi$, and $i_\tau F = d\tilde{\chi}$, where $\tau = \partial / \partial t$. Clearly $\xi$ and $\tau$ are commuting Killing fields of $(M', g)$ which generate an abelian two-parameter group $G = \mathbb{R} \times SO(2)$ of isometries leaving $F$ invariant with timelike two-dimensional orbits, and hence $(M', g, F)$ is stationary and axially symmetric. Furthermore, it is not difficult to check that $\omega$ is the twist of $\xi$, and that $i_\xi * F = d\psi$ where here $*$ denotes the Hodge star operator of the metric $g$. It is then quite straightforward to verify that $(M', g, F)$ satisfies the Einstein/Maxwell equations. Finally, since the $4N - 1$ parameters $\mu_j, r_j, \lambda_j$, and $q_j$ are geometric invariants, these solutions form a $(4N - 1)$-parameter family of stationary and axially symmetric solutions of the Einstein/Maxwell Equations.

4.2. Conclusion. We have shown the existence a $(4N - 1)$-parameter family of solutions of the Einstein/Maxwell Equations which are stationary and axially symmetric, and which should be interpreted as $N$ co-axially rotating charged black holes in equilibrium possibly held apart by singular struts. In order to complete this interpretation, it is necessary to show that the metric $g$ and the field $F$ can be extended smoothly across the axis of symmetry $\Sigma$. This requires first the smoothness of the metric and field components across $\Sigma$, which would follow from the smoothness of $e^u, v, \chi$ and $\psi$ on $\Sigma \setminus \partial \Sigma$, where $\partial \Sigma$ denotes the set of $2N$ endpoints of $\Sigma$; less regularity is expected on $\partial \Sigma$, see [21]. For the target $\mathbb{H} = \mathbb{H}^2_\mathbb{R}$ corresponding to the Einstein/Vacuum Equations this was established in [20]. Regularity of harmonic maps with prescribed singularities into $\mathbb{H}^2_\mathbb{R}$ was studied in further generality in [10].

Even after smoothness is established, there is still the possibility of a conical singularity on the bounded components of $\Sigma$. As in the vacuum case, this is to be interpreted as a singular strut holding the black holes apart, and the angle deficiency can be related to the force between these. Finally, to prove that $g$ is asymptotically flat, the decay estimate obtained at infinity must be sharpened, see [21].
These questions will be addressed in a future paper.

REFERENCES

[1] R. Bach and H. Weyl, Neue Lösungen der Einsteinschen Gravitationsgleichungen, Mathematische Zeitschrift 13 (1921), 132–145.
[2] B. Carter, Black Hole Equilibrium States, in Black Holes, edited by C. DeWitt and B. S. DeWitt, Gordon and Breach Science Publishers, New York, 1973.
[3] B. Carter, Bunting Identity and Mazur Identity for Nonlinear Systems Including the Black Hole Equilibrium System, Comm. Math. Phys. 99 (1985), 563–591.
[4] P. T. Chruściel, "No Hair" Theorems — Folklore, Conjectures, Results, to appear in Proc. of AMS/CMS Conf. on Diff. Geom. and Math. Phys., Vancouver, August 1993, Eds. J. Beem and K. Duggal, Contemporary Mathematics, AMS, 1994.
[5] P. T. Chruściel and R. M. Wald, On the Topology of Black Holes, ESI preprint, Vienna, October 1994.
[6] P. Eberlein and B. O’Neill, Visibility Manifolds, Pacific J. Math. 46 (1973), No. 1, 45–109.
[7] F. J. Ernst, New Formulation of the Gravitational Field Problem, Phys. Rev. Letters 167 (1968), 1175–1178.
[8] E. Heintze and H. C. Im Hop, Geometry of Horospheres, J. Diff. Geom. 12 (1977), 481–491.
[9] Y. Li and G. Tian, Nonexistence of Axially Symmetric, Stationary Solution of Einstein Vacuum Equation with Disconnected Symmetric Event Horizon, Manuscripta Math. 73 (1991), 83–89.
[10] , Regularity of Harmonic Maps with Prescribed Asymptotic Behavior and Applications, Comm. Math. Phys. 149 (1992), No. 1, 1–30.
[11] S. D. Majumdar, A Class of Exact Solutions of Einstein’s Field Equations, Phy. Rev. 72 (1947), 390–398.
[12] P. O. Mazur, Proof of Uniqueness of the Kerr-Newman Black Holes, J. Phys. A 15 (1982), 3173–3180.
[13] G. D. Mostow, Strong Rigidity of Locally Symmetric Spaces, Annals of Mathematics Studies, No. 78, Princeton University Press, Princeton, 1973.
[14] A. Papapetrou, A Static Solution of the Equations of the Gravitational Field for an Arbitrary Charge-Distribution, Proc. Roy. Irish Acad. A51 (1947), 191–204.
[15] R. Penrose, Some Unsolved Problems in Classical General Relativity, in Seminar on Differential Geometry, edited by S. T. Yau, Annals of Mathematics Studies no. 102, Princeton University Press, Princeton 1982.
[16] D. C. Robinson, Uniqueness of the Kerr Black Hole, Phys. Rev. Letters 34 (1975), 905–906.
[17] R. Schoen and K. Uhlenbeck, A Regularity Theory for Harmonic Maps, J. Diff. Geom. 17 (1982), 307–335.
[18] , Boundary Regularity and the Dirichlet Problem for Harmonic Maps, J. Diff. Geom. 18 (1983), 253–268.
[19] R. Schoen and S. T. Yau, Compact Group Actions and the Topology of Manifolds with Non-Positive Curvature, Top. 18 (1979), 361–380.
[20] G. Weinstein, On Rotating Black Holes in Equilibrium in General Relativity, Comm. Pure Appl. Math. 43 (1990), 903–948.
[21] The Stationary Axisymmetric Two-Body Problem in General Relativity, Comm. Pure Appl. Math. 45 (1992), 1183–1203.
[22] , On the Force between Rotating Co-Axial Black Holes, Trans. Amer. Math. Soc. 342 (1994), No. 2, 899–906.
[23] ______, On the Dirichlet Problem for Harmonic Maps with Prescribed Singularities, Duke Math. J. 77 (1995), No. 1, 135–165.

Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama 35205

E-mail address: weinstein@math.uab.edu