The Parameter Estimation of Double Exponential Pulse Based on a Least Infinity Norm Fitting Method

Yuewu Shi*, Wei Wang, Zhizhen Zhu, Xin Nie

State Key Laboratory of Intense Pulsed Radiation Simulation and Effect (Northwest Institute of Nuclear Technology) Xi’an, China

*Corresponding author: shiyuwei@nint.ac.cn

Abstract. This paper presents an estimation method of double exponential pulse (DEP) between the physical parameters rise time ($t_r$), full width at half maximum amplitude ($t_{FWHM}$) and the mathematical parameters $\alpha$, $\beta$. A newly fitting method based on the least infinity norm criterion is proposed to deal with the estimation problem of DEP. The calculation process and equation of parameters of this method is proposed based on an m-th-order polynomial fitting model. This estimation method is compared with the least square method by the same data and fitting function. The results show that the maximum estimation error of parameters of double exponential pulse obtained by the least infinity norm method is 1.5 %.

Keywords: Fitting method, Minimum infinity norm, relative error, parameter estimation, double exponential pulse.

1. Introduction

Double exponential function is widely used to describe High altitude electromagnetic pulse (HEMP) [1], ultrawide-band pulse (UWB) [2] and lightning electromagnetic pulse (LEMP) [3] mathematically. Though the Double exponential pulse (DEP) is just a simplification of measured impulsive quantities, the two groups of parameters rise time ($t_r$), full width at half maximum amplitude ($t_{FWHM}$) and the mathematical parameters $\alpha$, $\beta$ of double exponential function are often need to be transformed into each other. In particular, the estimated results of $\alpha$, $\beta$ is often used to reconstruct the waveform of electromagnetic pulse (EMP) with different $t_r$ and $t_{FWHM}$, and as the input of the numerical calculation, coupling simulation and effect evaluation.

An approximate closed-form formula of parameters $t_r$ and $t_{FWHM}$ with the help of the Least-squares method and Nelder-Mead algorithm is applied in [4]. The estimation error of $t_r$ and $t_{FWHM}$ is 7% and 3% respectively. And the formula for the reverse transformation is supplemented in [5]. The approximation error is less than 3.8%. However, the estimation is somewhat rough. For the purpose of reduce the fitting error, a novel piecewise relationship is established by statistical means [6]. This method utilizes 4 assistants to realize the transform, which divide the translation formula into 4 parts and the overall estimation error is less than 2.0%. Similarly, another relationship is introduced in the discussion of the dependences of pulse shape properties [7]. The overall estimation error is less than 1.7%. Since the piecewise relationships obtained by statistical method have reasonable estimation errors, it’s not very straight forward. Furthermore, a simplified translation system of non-piecewise
equations with 2 groups of correction terms is proposed [8]. The overall estimation error is less than 2.9%. This method is neither the simplest nor the accurate.

Least square method or least square method combined with another algorithm is used in all these approaches mentioned above for the fitting except one [7] did not explain specifically which fitting method was chosen. However, the maximum relative error is used to judge the performance of estimation results in these relationships without exception. The results obtained by the least square method are not the optimal solution under this judgment criterion. Therefore, a fitting method based on the least infinity norm of error or relative errors need to be developed to deal with this problem.

The least infinity norm criterion of errors has been applied to polynomial function approximations to simplify the complex function [9]. However, the least infinity norm criterion of relative error is not taken into consideration, and there is no fitting method for discrete data with similar criteria. Thus, under the judgment norm of maximum error or relative error in the parameter estimation of DEP, a new fitting method need to develop that can control the maximum error or the maximum relative error of all the points to the least and find the optimal solution.

This paper is focused on the parameter estimation of DEP, to achieve a group of estimation equations that are uniform and accurate. In section II, the parameter calculation process of fitting function is proposed based on the polynomial fitting model. In section III, this fitting method is used to deal with the parameter estimation of double exponential pulse. In Section IV, we present some conclusions.

2. Fitting method based on the least infinity norm

For a group of calibration input data $x$ and output data $y$, as in (1) an (2),

$$x = (x_1, x_2, \cdots, x_n)^T,$$

$$y = (y_1, y_2, \cdots, y_n)^T.$$  \hspace{1cm} (1) \hspace{1cm} (2)

The least infinity norm of relative error criterion can be expressed to find the least minimum infinity norm of errors $\delta$ and its corresponding parameters:

$$\delta = \left| \frac{y - Y}{Y} \right|_\infty,$$  \hspace{1cm} (3)

Where $Y$ is the fitting results.

It is difficult to obtain the analytical solution of this fitting criterion directly. Though it is similar to the approximations problem of continuous function, for which the “Remes” algorithm [10] can be used for the least infinity norm of error, a calculation process of the parameters is established with a polynomial fitting function under the least infinity norm of relative error.

Assume that the fitting model is an m-th-order polynomial as follows:

$$Y = f(x, k_1, k_2, \cdots, k_m) = \sum_{i=0}^{m+1} k_i x^{i-1}$$  \hspace{1cm} (4)

Consider the case of $m+1$ points first. When the number of fitting points and parameters are the same, we can substitute the fitting data into the function, and the data points can be fitted entirely.

Consider the number of points as $m+2$. If there is a curve $C_1$, and the errors of the fitting results are the same in curve $C_1$ for all of the data points, and the points are distributed at both sides of the curve alternately, then the maximum relative error of $C_1$ is the least. To confirm it, another curve $C_2$ can be assumed to exist such that the maximum relative error is the least. Obviously, the maximum relative
error of C2 must be smaller than C1. It means the distances of all the m+2 points to C2 are closer than
C1, because the error at every point is the same and all of them are the maximum. Therefore, if the
fitting function is continuous, C1 and C2 must have at least m+1 intersections as shown in Fig. 1.
Regarding the value of m+1 intersections as constant and substituting them into C1 and C2, it is not
difficult to find that C1 and C2 are the same.
Based on the consequence above, if the number of points is m+2, the parameters can be calculated
using the following equations.

\[
\begin{align*}
\frac{f(x_1, k_1, k_2, \ldots, k_m) - y_1}{y_1} &= \frac{f(x_2, k_1, k_2, \ldots, k_m) - y_2}{y_2} \\
\frac{f(x_3, k_1, k_2, \ldots, k_m) - y_3}{y_3} &= \frac{f(x_4, k_1, k_2, \ldots, k_m) - y_4}{y_4} \\
\vdots \\
\frac{f(x_m, k_1, k_2, \ldots, k_m) - y_m}{y_m} &= \frac{f(x_{m+1}, k_1, k_2, \ldots, k_m) - y_{m+1}}{y_{m+1}}
\end{align*}
\]  

(5)

These equations can be simply expressed as a matrix equation:

\[
A \mathbf{k} = \mathbf{b}
\]  

(6)

Where \( \mathbf{k} \) is the parameters vector; \( \mathbf{b} \) is an \( m+1 \) dimensional column vector.

\[
b_i = 2y_i y_{i+1}
\]  

(7)

\( A \) is an \( m+1 \) dimensional coefficient matrix shown in (8).

\[
A_{ij} = y_{i+1} x_i^{-1} + y_i x_{i+1}^{-1}
\]  

(8)

Therefore, \( \mathbf{k} \) can be calculated by equation (9).

\[
\mathbf{k} = A^{-1} \mathbf{b}
\]  

(9)

Consider the case of \( n \) (\( n>m+2 \)) points; if a curve C is the optimal solution of m+2 points and the
maximum relative error is larger than the relative errors of the rest of the points, then C is the optimal
solution of m points. Because in the m+2 points, the maximum relative fitting error of C is the least
among the maximum relative fitting errors of all the curves, and the maximum relative fitting errors of
C from the m+2 points are equal to the maximum relative fitting errors of C from all the points, we can
find the curve as long as we find m+2 points among all the n points that satisfy the constraints
mentioned above.

The calculation process of the parameters can be divided into 3 steps:

(1) Choose m+2 points among n fitting data points randomly.

(2) Calculate the parameters of the fitting model by (9).

(3) Verify the function produced in step ii. If the maximum relative error of the m points is equal to
that of all the points, then it is the function that satisfies the fitting criterion. If not, change to another
group of m+2 points and repeat the steps until the optimal function is found.

3. Parameter estimation of double exponential pulse
The double exponential pulse is described by (10).
\[ E(t) = E_0 k(e^{-\alpha t} - e^{-\beta t}) \cdot u(t), \]  

(10)

Where \( E_0 \) is the amplitude of the pulse, \( \alpha \) and \( \beta \) are parameters to characterize the pulse shape; \( u(t) \) is a step unit function. The creation parameter \( k \) can be calculated by

\[ k = e^{-\frac{\ln \alpha - \ln \beta}{\alpha - \beta}} - e^{-\frac{\ln \alpha - \ln \beta}{\alpha - \beta}}. \]  

(11)

The definition of \( t_r \) and \( t_{FWHM} \) are shown in Fig.1.

Figure 1. Pulse shape and the definition of \( t_r \) and \( t_{FWHM} \)

Parameters \( \alpha t_r \) and \( \alpha t_{FWHM} \) are constant at the same ratio of \( \beta/\alpha \) or \( t_{FWHM}/t_r \) [6]. The high precision values of \( \alpha t_r \), \( \alpha t_{FWHM} \), \( \beta t_r \) corresponding to \( \beta/\alpha \) or \( t_{FWHM}/t_r \) with accuracy of 15 effective digits are calculated with iterative algorithms, as shown in Fig.3, where the \( \beta/\alpha \) range from 1.1 to 2, 2 to 10, 10 to 100 and 100 to 1000 respectively. And the \( \beta/\alpha \) take a value at every 0.01, 0.1, 1, 10 for the dense data with \( \beta/\alpha \) ranging from 1.1 to 2, 2 to 10, 10 to 100 and 100 to 1000 respectively.

Based on the statistical data, we can obtain the conversion equation of parameters by polynomial fit. As the maximum relative error criterion is used to judge the estimation results, the least relative infinity norm fitting method proposed above is suitable for the fitting. The coefficients of polynomial \( k_i \) take 4 significant digits in the estimation with an exception which are indicated in the following.

In this method, the calculated quantity increases sharply with the increase of the fitted data and the order of the polynomial function. Thus, relatively sparse data are selected for fitting and relatively dense data are used for the verifying of fitting results. The \( \beta/\alpha \) take a value at every 0.1, 1, 10, 100 for the sparse data with \( \beta/\alpha \) ranging from 1.1 to 2, 2 to 10, 10 to 100 and 100 to 1000 respectively. And the \( \beta/\alpha \) take a value at every 0.01, 0.1, 1, 10 for the dense data with \( \beta/\alpha \) ranging from 1.1 to 2, 2 to 10, 10 to 100 and 100 to 1000 respectively.
A sixth order polynomial function is used as the fitting model, as shown in equation (12). The fitting results are shown in Table 1. Where $k_i$ is the multinomial coefficient in equation (12), $\delta$ is the maximum fitting relative error as shown in equation (3). The parameters of “Knows” and “Parameter” rows denote the given parameters and parameters to be estimated respectively, and the parameters of “$x$” and “$Y$” rows in Table 1 denote $x$ and $Y$ in the fitting equation. The same is true for Table 2 and Table 3.

$$Y = f(x, k_1, k_2, \ldots, k_7) = \sum_{i=1}^{7} k_i x^{i-1}. \quad (12)$$

Based on the fitting results in Table 1, the two groups of parameters $t_r$, $t_{FWHM}$ and $\alpha$, $\beta$ can be transformed each other. However, the maximum estimation error of the four parameters is 4.9%, the precision need to be improved.

**Table 1.** Results of parameters $\alpha t_r$, $\alpha t_{FWHM}$ with known $\alpha$, $\beta$ and $\alpha t_{FWHM}$, $\beta t_r$ with known $t_r$, $t_{FWHM}$

| Knows Parameter | $\alpha & \beta$ | $t_r$ | $t_{FWHM}$ | $\alpha$ | $\beta$ |
|-----------------|-------------------|------|------------|-----------|---------|
| $x$             | $\alpha/\beta$    | $\alpha/\beta$ | $t_r/t_{FWHM}$ | $t_r/t_{FWHM}$ |         |
| $Y$             | $\alpha t_r$      | $t_{FWHM}$ | $\alpha t_{FWHM}$ | $\beta t_r$ |         |
| $k_1$           | 1.981E-4           | 6.991E-1 | 7.467E-1   | 2.112E+0   |         |
| $k_2$           | 2.007E+0           | 4.281E+0 | -9.615E+0  | 2.677E+0   |         |
Except the fitting method, the fitting model is also one of the factors that affect the estimation. Though the polynomial fitting function is given in the proposed least infinity norm method, the parameter $x$ and $Y$ can be replaced by different expressions that contain parameters to be estimated. A forth order polynomial function is used as the fitting model. The fitting results are shown in Table 2.

$$Y = f(x, k_1, k_2, \ldots, k_5) = \sum_{i=1}^{5} k_i x^{i-1}. \tag{13}$$

Based on Table 2, the $t_r$, $t_{FWHM}$ and $\alpha$ can be calculated by calculate the parameter $\alpha$ and $\alpha \beta$ firstly. Though $\alpha \beta$ has high estimation accuracy, the accuracy of $\beta$ is related to the estimation accuracy of $\alpha$. As a result, either the maximum estimation error of $\alpha$ or the maximum estimation error of $\beta$ is larger than 2.5%.

Further, we find that the fitting results get improved by replacing the parameter $1/\alpha^2 t_{FWHM}^2$ in “Y” row with $1/\alpha^4 t_{FWHM}^4$. As is shown in Table 3, the maximum estimation error of the parameters is 1.5%.Fig.3 (a) and Fig.3 (b) compare the estimation results of Table 3 and the exact values of parameters $a_{FWHM}$ and $\beta t_r$ with different $\beta / \alpha$ and $t_{FWHM} / t_r$. Fig.4 (a) and Fig.4 (b) are the fitting errors of parameters $t_{FWHM}$, $t_r$ and $\alpha$, $\beta$ respectively.

### Table 2. Results of $1/\beta^2 t_r^2$, $\alpha^2 t_{FWHM}^2$ with known $\alpha$, $\beta$ and $1/\alpha^2 t_{FWHM}^2$, $\alpha \beta t_r^2$ with known $t_r$, $t_{FWHM}$

| Parameter | $t_r$ | $t_{FWHM}$ | $\alpha$ | $\beta$ |
|-----------|-------|------------|----------|----------|
| $x$       | $\alpha / \beta$ | $\alpha / \beta$ | $t_r / t_{FWHM}$ | $t_r / t_{FWHM}$ |
| $Y$       | $1/\beta^2 t_r^2$ | $\alpha^2 t_{FWHM}^2$ | $1/\alpha^2 t_{FWHM}^2$ | $\alpha \beta t_r^2$ |
| $k_1$     | 2.175E-1 | 4.910E-1 | 1.956E+0 | 6.222E-5 |
| $k_2$     | 4.146E+0 | 5.639E+0 | -3.123E-1 | 1.495E+0 |
| $k_3$     | -8.177E+0 | -4.586E+0 | -1.497E+2 | -2.087E+0 |
| $k_4$     | 1.510E+1 | 1.067E+1 | 1.001E+3 | 8.148E+0 |
| $k_5$     | -8.555E+0 | -6.543E+0 | -2.114E+3 | -4.649E+0 |
| $\delta \%$ | 2.8 | 1.3 | 4.8 | 0.2 |
| $\delta P \%$ | 1.4 | 0.7 | 2.5 | 2.6 |

### Table 3. Results of $1/\beta^2 t_r^2$, $\alpha^2 t_{FWHM}^2$ with known $\alpha$, $\beta$ and $1/\alpha^4 t_{FWHM}^4$, $\alpha \beta t_r^2$ with known $t_r$, $t_{FWHM}$

| Parameter | $t_r$ | $t_{FWHM}$ | $\alpha^*$ | $\beta$ |
|-----------|-------|------------|------------|----------|
| $x$       | $\alpha / \beta$ | $\alpha / \beta$ | $t_r / t_{FWHM}$ | $t_r / t_{FWHM}$ |
| $Y$       | $1/\beta^2 t_r^2$ | $\alpha^2 t_{FWHM}^2$ | $1/\alpha^4 t_{FWHM}^4$ | $\alpha \beta t_r^2$ |
| $k_1$     | 2.175E-1 | 4.910E-1 | 4.061E+0 | 6.222E-5 |
| $k_2$     | 4.146E+0 | 5.639E+0 | -2.696E+1 | 1.495E+0 |
| $k_3$     | -8.177E+0 | -4.586E+0 | -2.890E+1 | -2.087E+0 |
| $k_4$     | 1.510E+1 | 1.067E+1 | 5.989E+2 | 8.148E+0 |
| $k_5$     | -8.555E+0 | -6.543E+0 | -1.273E+3 | -4.649E+0 |
| $\delta \%$ | 2.8 | 1.3 | 5.7 | 0.2 |
| $\delta P \%$ | 1.4 | 0.7 | 1.4 | 1.5 |
The coefficients of polynomial in the estimation of $\alpha$ takes 5 significant digits, as the 5th significant digit has a great influence on the accuracy of the estimation.

Figure 3. The estimation results of Table 3 and the exact values of parameters $t_{FWHM}$ and $t_r$ with different $\beta/\alpha$ and $t_{FWHM}/t_r$: (a) The results with different $\beta/\alpha$; (b) The results with different $t_{FWHM}/t_r$.

Figure 4. The fitting errors of parameters $t_{FWHM}$, $t_r$ and $\alpha$, $\beta$ of the results in Table 3: (a) Fitting errors of $t_{FWHM}$, $t_r$; (b) Fitting errors of $\alpha$, $\beta$.

According to the estimated results above, the parameter estimation can be described by (14) and (15). For example, the parameters of high-altitude electromagnetic pulse (HEMP) recommended in IEC 6100-2-9 are 2.5ns/23ns and 4×10$^7$/6×10$^8$, the estimated parameters obtained by (14) and (15) are 2.45ns/2.3ns and 4.01×10$^7$/5.89×10$^8$.

i) Knows $\alpha$ and $\beta$: 

-estimation errors of $t_r$ and $t_{FWHM}$ (%)
-estimation errors of $t_{FWHM}$

-estimation errors of $\alpha$ and $\beta$ (%)
\[ t_r = \frac{1}{\beta} \left[ 0.2175 + 4.146 \frac{\alpha}{\beta} - 8.177 \left( \frac{\alpha}{\beta} \right)^2 + 15.10 \left( \frac{\alpha}{\beta} \right)^3 - 8.553 \left( \frac{\alpha}{\beta} \right)^4 \right]^{1/2} \]

\[ t_{\text{FWHM}} = \frac{1}{\alpha} \left[ 0.4910 + 5.639 \frac{\alpha}{\beta} - 0.4586 \left( \frac{\alpha}{\beta} \right)^2 + 10.67 \left( \frac{\alpha}{\beta} \right)^3 - 6.543 \left( \frac{\alpha}{\beta} \right)^4 \right]^{1/2} \] (14)

ii) Knows \( t_r \) and \( t_{\text{FWHM}} \):

\[
\begin{align*}
\alpha &= \frac{1}{t_{\text{FWHM}}} \left[ 4.0616 - 26.963 t_r - 4.190 \left( \frac{t_r}{t_{\text{FWHM}}} \right)^2 + 59.899 \left( \frac{t_r}{t_{\text{FWHM}}} \right)^3 - 1273.2 \left( \frac{t_r}{t_{\text{FWHM}}} \right)^4 \right]^{1/4} \\
\beta &= \frac{1}{\alpha t_r} \left[ 6.222 \times 10^{-5} + 1.495 t_r - 2.087 \left( \frac{t_r}{t_{\text{FWHM}}} \right)^2 + 8.418 \left( \frac{t_r}{t_{\text{FWHM}}} \right)^3 - 4.649 \left( \frac{t_r}{t_{\text{FWHM}}} \right)^4 \right]^{1/4} 
\end{align*}
\] (15)

4. Conclusions

In this paper, a new parameter estimation method of double exponential pulse fitting method is proposed. To deal with this problem, a fitting method based on the least infinity norm criterion is given. The results show that the overall relative estimation error is less than 1.5%. As the limit of the fitting model, we believe that the estimation error can be smaller when the calculation process of the fitting method with arbitrary model established.

| Numble | Scheme 1 | Scheme 2 | Scheme 3 | Scheme 4 | Scheme 5 |
|--------|----------|----------|----------|----------|----------|
| 1      | 456      | 456      | 123      | 123      | 123      |
| 2      | 789      | 213      | 644      | 644      | 644      |
| 3      | 213      | 654      | 649      | 649      | 649      |

All manuscripts must be in English, also the table and figure texts, otherwise we cannot publish your paper. Please keep a second copy of your manuscript in your office. When receiving the paper, we assume that the corresponding authors grant us the copyright to use the paper for the book or journal in question. Should authors use tables or figures from other Publications, they must ask the corresponding publishers to grant them the right to publish this material in their paper.

References

[1] J. van der Geer, J.A.J. Hanraads, R.A. Lupton, The art of writing a scientific article, J. Sci. Commun. 163 (2000) 51-59.
[2] W. Strunk Jr., E.B. White, The Elements of Style, third ed., Macmillan, New York, 1979.
[3] G.R. Mettam, How to prepare an electronic version of your article, in: B.S. Jones, R.Z. Smith (Eds.), Introduction to the Electronic Age, E-Publishing Inc., New York, 1999, pp. 281-304.
[4] C. D. Smith and E. F. Jones, “Load-cycling in cubic press,” in Shock Compression of Condensed Matter-2001, AIP Conference Proceedings 620, edited by M. D. Furnish et al. American Institute of Physics, Melville, NY, 2002, pp. 651–654.
[5] P.G. Clem, M. Rodriguez, J.A. Voigt and C.S. Ashley, U.S. Patent 6,231,666. (2001)
[6] Information on http://www.weld.labs.gov.cn
[7] Ianoz, "Review of HEMP activities in Europe (1970–1995)", IEEE Trans. Electromagn. Compat., vol. 55, no. 3, pp. 412-421, 2013.
[8] İ. Güvenç and H. Arslan, "A review on multiple access interference cancellation and avoidance for IR-UWB", Signal Processing, Vol. 87, no. 4, pp. 623-653, Apr. 2007.

[9] Z. Jiang; B.Zhou and Y. Liu, "A multiresolution time-domain method for LEMP calculation and comparison with FDTD", IEEE Trans. Electromagn. Compat., Vol. 56, no. 2, pp. 419-426, Apr. 2014.

[10] M. Camp and H. Garbe, "Parameter estimation of double exponential pulse (EMP UWB) with least squares and Nelder Mead algorithm", IEEE Trans. Electromagn. Compat., vol. 46, no. 4, pp. 675-678, Nov. 2004.

[11] M. Magdowski and R. Vick, "Estimation of the mathematical parameters of Double-Exponential pulses using the Nelder–Mead algorithm", IEEE Trans. Electromagn. Compat., Vol. 52, no. 4, pp.1060-1062, Nov. 2010.

[12] C. Mao and H. Zhou, "Novel parameter estimation of double exponential pulse (EMP, UWB) by statistical means", IEEE Trans. Electromagn. Compat., vol. 50, no. 1, pp.97–100, Feb. 2008.

[13] G. Wu, "Shape properties of pulses described by double exponential function and its modified forms", IEEE Trans. Electromagn. Compat., Vol. 56, no. 4, pp. 923-931, Aug. 2014.

[14] Y. Shi et al, "A simplified method for parameter estimating of double exponential pulse", 2016 Asia Pacific International Symposium on Electromagnetic Compatibility, vol. 01, pp. 537-539, 2016.

[15] S. Gong et al, "Two-dimensional extrapolation technique combined adaptive frequency-sampling method with the best uniform rational approximation", Journal of Electromagnetic Waves and Applications, Vol. 27, no. 18, pp.2395-2406, Oct. 2013.

[16] T. YAMAMOTO and T. TAKEDA, "A complement proposal for optimization of subgroup parameters", Journal of Nuclear Science and Technology, Vol. 43, no. 7, pp. 765-773, Nov. 2006.