We consider the influence of mode - mode coupling in the inflaton field on the spectrum of primordial fluctuations. To this end, we formulate a phenomenological model where the inflaton fluctuations are treated as a fluid undergoing turbulent motion. Under suitable assumptions, it is possible to estimate the size and scale of fluctuations in velocity, which upon reheating induce corresponding fluctuations in the radiation energy density. For De Sitter inflation the resulting spectrum is scale invariant on all scales of interest. The amplitude of the resulting spectrum is compatible with known observational limits. This suggests that the hypothesis of a extremely weakly coupled inflation could be relaxed without affecting the predictions of the model. Principal PACS No 98.80.Cq; additional PACS nos: 04.62.+v, 11.10.Wx, 47.27

I. INTRODUCTION

The object of this paper is to present models of fully nonlinear fluctuations during the inflationary era conducing to a scale invariant primordial density contrast spectrum with an amplitude consistent with COBE observations.

Inflationary models were originally introduced as a solution for the so-called puzzles of standard hot Big Bang cosmology [1], namely the horizon, flatness and photon to baryon ratio problem [2] [3]. A period of exponential expansion of the Universe was postulated, whereby the ratio of the Universe increased by a factor $e^{60}$ or so, followed by a period of reheating, during which the temperature of radiation was raised up to a final value of $10^{9}$ GeV or more (we assume units with $\hbar = c = k = 1$) [4]. From then on, cosmic evolution followed the lines of standard cosmology [5].

Although many implementations of inflation have been proposed, most attention has been devoted to the simplest scenario, which was Linde’s chaotic inflation with a single inflaton field [6]. In this model, inflation is powered by a scalar field $\phi$, the inflaton, slowly rolling down an effective potential $V(\phi)$. We shall simplify the model further by assuming the Universe is described by a spatially flat, Friedmann - Robertson - Walker geometry. Inflation begins when the effective potential becomes the dominant form of energy density, and ends with the decay of the inflaton onto radiation. During inflation the potential energy $V(\phi)$ acts as an effective cosmological constant. It can be said that no satisfactory contending explanation of the cosmic puzzles is available [7].

Soon after the original proposal, it was realized that inflation could perform a subtler task: to provide a framework for explaining the origin of primordial density fluctuations [8]. Quantum fluctuations of the inflaton field distort the reheating surface, inducing a primordial density contrast (see [1], we shall review this argument in greater detail below)

$$\frac{\delta \rho}{\rho} \sim \frac{H}{\phi} \delta \phi$$

$H$ being the inflationary Hubble parameter. A similar argument shows that gravitational waves are also being created, with an amplitude $h \sim H/m_p$. The validity of Eq. (1) has been corroborated by several different methods [8] (for a dissenting viewpoint, see [9]).

In order to obtain a concrete prediction from Eq. (1) we must estimate the quantum fluctuations $\delta \phi$. The usual approach treats these fluctuations as a free field (for example, in the seminal paper by Starobinsky [10]) in its (De Sitter invariant) vacuum state or very close to it [11]. Then a simple calculation within quantum field theory in curved spaces allows us to evaluate the $\delta \phi$ in Eq. (1) as $\delta \phi \sim H$ [12]. All quantities in the right hand side of Eq. (1) are evaluated as the relevant mode leaves the horizon. There are well defined ways to relate the spectrum of scalar and tensor primordial fluctuations to a given potential, which are generally described as the potential reconstruction program [13].

As our understanding of the formalism and the complexity of observational data progresses [15], it becomes clear that simple potentials such as single powers of $\phi$ are not rich enough to account for observations. Thus the potential reconstruction program generally assumes more complex functional forms, depending upon several parameters [14]. In these more general potentials their higher derivatives may not be negligible, leading to nonlinear interactions between fluctuations. In such a case, the usual way of estimating the primordial density contrast would be invalid. The relevance of non linear fluctuations was already taken up in Ref. [16]. Also several authors [17, 20] have considered models where fluctuation self interactions were treated perturbatively.
The main goal of this paper is to develop ways of estimating the primordial density contrast in chaotic inflation models without presupposing that couplings among fluctuations are negligible. Of course, working out the structure of quantum fluctuations on even simple states of fully nonlinear quantum field theory is a daunting task \cite{21}. However, a simpler alternative is available, namely, the application of hydrodynamics to describe the macroscopic behavior of quantum fluctuations. This is possible because these fluctuations, as far as it is relevant to our present concerns, may be described by a \( c \)-number energy momentum tensor subject both to the usual conservation laws and the Second Law of thermodynamics \cite{13}. There is therefore an equivalent fluid description, consisting of a classical fluid whose energy momentum tensor and equation of state reproduce the observed ones for the quantum fluctuations. Solving the dynamics of this equivalent fluid yields answers to all relevant questions concerning the behavior of the actual quantum fluctuations. In this paper we shall not press the issue of whether the equivalent fluid is anything more than a convenient computational device.

Along with the assumption of free inflaton fluctuations, we must question whether these fluctuations are in their vacuum state. This assumption usually rests on the so-called cosmic no hair theorem, which states that any reasonable initial quantum state for the field will quickly relax to a De Sitter invariant state, or its closest available analogue \cite{22}. The application of this theorem in models with a hundred or more e-foldings of inflation is justified, but these models must be rejected on the grounds that they also predict a value of the density parameter \( \Omega \) unacceptably close to one (see Appendix and Ref. \cite{23}; it is also possible to develop inflationary models in open universes, see Ref. \cite{24}). When the duration of inflation is close to minimal, it becomes likely that some large but observable scales will leave the horizon before the cosmic no hair theorem is able to operate. It is important therefore to set a realistic initial condition such as a Planckian distribution of particles with temperature \( T \). The relevance of thermal effects in inflation was already pointed out (in a different context) in ref. \cite{25}.

An immediate consequence of energy momentum conservation and the Second Law is that when velocities are low, the phenomenological fluid may be described within the Eckart theory of dissipative fluids \cite{26,27} (for an analysis of the limitations of Eckart’s theory see \cite{28}). It follows that it obeys a continuity equation and a curved space time Navier-Stokes equation. The model is then defined by giving the equation of state and the viscosity of the equivalent fluid. The advantage we gain is that these are features that can be computed locally. As far as the relevant scales are much below the curvature radius, it is possible to use for them their standard flat space time values. At high temperatures we obtain the equation of state for radiation, \( p = (1/3)\rho \), and a dynamic viscosity \( \eta \sim T^3 \). Since the speed of sound, being close to light speed, is much higher than the characteristic speed of the fluid, the flow may be considered incompressible. There will be fluctuations in velocity, nevertheless, and these are the ones responsible for density fluctuations, as it is well known \cite{1}.

As we shall show below, conditions in the early stages of inflation are such that, for generic initial conditions, the flow of the equivalent fluid is highly turbulent, meaning that the corresponding Reynolds number is well over a thousand. The possible role of turbulence in the formation of cosmic structures has been studied in some detail even before inflationary models were introduced (see \cite{29,30} for a textbook account). These early attempts were abandoned because there were no natural mechanisms for the production of primordial turbulence, and the density contrast predicted was generally too high. After inflation was proposed, the matter was taken up again by Goldman and Canuto \cite{31}, who studied longitudinal turbulence excited by density fluctuations in the radiation and matter dominated eras. Our work should not be understood as a continuation of this line of research, but rather as a variation on conventional models of primordial fluctuation generation (cfr. \cite{1,9,11}) whereby quantum fluctuations of the inflaton field leave their imprint on the primordial density field and are subsequently washed away. In other words, the underlying physics in our model is the same as in these more familiar approaches to primordial fluctuation generation: we do not question the ultimate quantum origin of the fluctuations (another difference with the cosmic turbulence theory from the fifties), but simply borrow insights from hydrodynamics to describe the macroscopic behavior of these fluctuations, rather than rely on possibly oversimplified linearized microscopic models. The conditions of validity of our procedure are the assumptions that the energy momentum tensor of fluctuations is a \( c \)-number quantity (which ought to be true at any scale below Planck’s) and the Second Law (which, unlike the Third, has not been challenged yet to our knowledge).

A second goal of this paper is to demonstrate the application of the hydrodynamic approach by discussing some simple solutions to the non linear Navier-Stokes equation, and the resulting spectra of primordial fluctuations. Since the general solution to the Navier-Stokes equation is certainly not available, this requires the appeal to physical insights to simplify the problem.

The basic mechanism of non linear hydrodynamic evolution is the energy transport between eddies of different size through mode-mode coupling, and the viscous dissipation of small scale eddies. The fundamental issue regarding model building is to obtain a closed form energy balance equation, which allows us to follow in time the shape of the energy spectrum. This usually involves some closure hypothesis to reduce the infinite hierarchy of dynamical equations for velocity correlation functions to a manageable set \cite{32}. Lacking anything better, we shall fall back on the time-honored hypothesis that the effect of smaller eddies at any given scale may be simulated by a scale dependent effective viscosity \cite{33,34}; for concreteness, we shall follow Heisenberg’s 1948 formulation of this idea \cite{36}.
The Heisenberg theory admits some very simple solutions with the property of self-similarity. These have been worked out by Chandrasekhar [37] and generalized to Friedmann - Robertson - Walker (FRW) backgrounds by Tomita et al [38]. These solutions agree with the Kolmogorov 1941 theory in the inertial range [35], failing to reproduce observations for very small eddies. Fortunately, we shall only require the solution in the opposite limit of very large eddies, where it is trustworthy. With minor adjustments, Tomita’s analysis of turbulence decay in FRW space times also provides a solution to the evolution of our equivalent fluid. Self-similarity is an appealing feature to us, as we do expect that a generic flow will be eventually brought to some sort of steady state by the inflationary expansion.

The task at hand is then to study the evolution of a typical eddy as it is blown up by the universal expansion, exchanging and dissipating energy while inside the horizon, and freezing when outside, until it reaches the reheating hypersurface and delivers its energy to radiation. By assuming that the turbulent velocity fluctuations in the eddy produce fluctuations in the energy density of radiation in the usual way, we shall be able to relate the primordial density contrast to the features of the original self similar turbulence. The resulting spectrum may be matched against the known data on the cosmic microwave background [13], providing a crucial test of the inner consistency and viability of the approach. Our conclusion shall be that, insofar as the horizon remains constant during inflation, the spectrum of primordial density fluctuations produced by self similar flows is strictly scale invariant ($n = 1$, see [40]) at large scales. Quantitative agreement with observations may be obtained without any special fine tuning.

A few comments on the possibility of deriving present day spectra from early Universe hydrodynamics are in order; after all, one of the main criticisms against the conventional cosmic turbulence theory was that, even if turbulence were efficiently generated in the Early Universe, it would decay and become uninteresting well before recombination. This criticism does not apply to the present work, since we shall show that the results of customary inflationary models and self similar flows concerning the primordial density field immediately after reheating are identical. Further evolution of the primordial density contrast after reheating is well described by the theory of linear density fluctuations in an expanding Universe, a subject properly covered by many textbooks (cfr. [1,9,26,41]); the result, for both the conventional inflationary models and the present ones, is that these fluctuations in the primordial density contrast survive to recombination time.

The key point in our analysis is the evolution of the flow during inflation, as the different modes leave the horizon and "freeze", and we discuss this issue at some length. The amount of turbulence at reheating is ample enough to seed fluctuations at the level suggested by COBE data, provided the initial temperature is large enough. The lower bound in the temperature, however, is not so large that would invalidate the vacuum dominance condition which is a presupposition of Inflation. Of course, whether Inflation is likely to happen or not is a difficult question (cfr. [42]) lying beyond the scope of this paper.

To conclude, the main objective of this paper is to show that it is possible to develop sensible models of inflation where inflaton fluctuations evolve nonlinearly and are very far from their vacuum states. The connection of the physics of primordial fluctuations to hydrodynamics opens up a wealth of new interesting phenomena, such as intermittency in the primordial spectrum [24,43] and Burgers turbulence [44], with a strong potential impact on our understanding of the evolution of cosmic structures. Moreover, it is appealing to be able to account for a macroscopic phenomenon, such as fluctuation generation on super horizon scales, mostly on macroscopic terms (for an independent attempt in this direction, see [45]). Most importantly, by not assuming that higher derivatives of the potential are a priori negligible, we avoid a possible conflict within the potential reconstruction program [14].

The rest of the paper is organized as follows. In next section we provide a brief summary of hydrodynamics in flat and expanding universes, in order to set up the language for the rest of the paper. In section III we proceed to discuss the equivalent fluid description of inflaton fluctuations, and how to extract the primordial density contrast therefrom. As a simple application of the method, we consider briefly the case of free fluctuations, showing that the model leads back to the conventional results. In section IV we present Chandrasekhar’s self-similar solutions and their generalization to expanding universes, deriving the corresponding scale invariant primordial contrast; we also discuss how relevant these solutions are as actual descriptions of the inflationary period. We state our main conclusions in the final section.

II. HYDRODYNAMIC FLOWS

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1 We wish to point out that the applicability of Kolmogorov’s spectrum to large scale turbulence should not be taken for granted [39].
A. Flows in flat space time

The equations governing the dynamics of a fluid in local thermodynamic equilibrium are the continuity and Navier-Stokes ones, which, in the case of flat space time, read:

$$\frac{\partial \rho}{\partial t} + (\mathbf{U} \cdot \nabla) \rho = 0$$  \hspace{1cm} (2)

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{U}$$  \hspace{1cm} (3)

where we have assumed incompressibility, valid when typical velocities are much smaller than the sound velocity; $\nu = \eta/\rho$ is the kinematic shear viscosity. The transition from laminar to turbulent motion can be universally described by the dimensionless "Reynolds" number:

$$R = \frac{UL}{\nu}$$

where $U$ is a typical velocity and $L$ a typical length scale. This number represents the order of magnitude of the ratio of the inertial to the viscous term. Low Reynolds numbers correspond to laminar motion, while high ones suggest turbulent behavior.

In general, the velocity profile displays variations in space and time. This implies that the flow must be described probabilistically. Thus, each quantity involved in (2-3) is divided in its mean value and a fluctuation from it; for example, we write $\mathbf{U} = \bar{\mathbf{U}} + \mathbf{u}$, where $\mathbf{u}$ stands for the fluctuating part of the velocity. In the case where motion is isotropic, the mean value $\bar{\mathbf{U}}$ for the velocity must be zero, since otherwise there would be a preferred direction.

To analyze the system’s behavior, we define the two-point one-time correlation function for the velocity:

$$R_{ij}(\mathbf{x}, \mathbf{x}', t) = \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}', t) \rangle$$  \hspace{1cm} (4)

In the case of homogeneous and isotropic motion, this correlation must be only a function of the time $t$ and the distance between $\mathbf{x}$ and $\mathbf{x}'$, i.e. $R_{ij}(\mathbf{x}, \mathbf{x}', t) = R_{ij}(r, t)$, where $r = |\mathbf{x} - \mathbf{x}'|$. Observe that $R_{ii}(0, t)$ (summation over repeated indices must be understood) is twice the average energy density of the flow at time $t$. From (3) we obtain the equation that this correlation must obey, namely:

$$\frac{\partial}{\partial t} R_{ij}(r, t) = T_{ij}(r, t) + P_{ij}(r, t) + 2\nu \nabla^2 R_{ij}(r, t)$$  \hspace{1cm} (5)

where

$$P_{ij}(r, t) = \frac{1}{\rho} \left( \frac{\partial}{\partial r_i} \langle p(\mathbf{x}, t) u_j(\mathbf{x}', t) \rangle - \frac{\partial}{\partial r_j} \langle p(\mathbf{x}', t) u_i(\mathbf{x}, t) \rangle \right)$$  \hspace{1cm} (6)

and

$$T_{ij}(r, t) = \frac{\partial}{\partial r_k} \langle u_i(\mathbf{x}, t) u_k(\mathbf{x}, t) u_j(\mathbf{x}', t) - u_i(\mathbf{x}, t) u_k(\mathbf{x}', t) u_j(\mathbf{x}', t) \rangle$$  \hspace{1cm} (7)

The tensor $T_{ij}$ comes form the inertia term in Navier-Stokes equation and, as it involves a product of third order in the velocity, reflects the fact that there is not a close set of equation for the correlations of successive orders but there is a hierarchy of equations instead. The problem of closing that hierarchy is known as the "moment closure problem". Let us call $\Phi_{ij}(k, t)$ the Fourier transform of $R_{ij}(r, t)$. Then the energy density becomes

$$\frac{1}{2} R_{ii}(0, t) = \int E(k, t) \, dk,$$

where

$$E(k, t) = \frac{1}{2} \int \Phi_{ii}(k, t) \, k^2 \, d\Omega(k)$$  \hspace{1cm} (8)

is the energy density stored in eddies of size $k^{-1}$. Defining $\Gamma_{ij}$ as the Fourier transform of $T_{ij}$, we obtain from (3) the equation of balance of the energy spectrum:
\[- \frac{\partial}{\partial t} E(k,t) = T(k,t) + 2\nu k^2 E(k,t) \tag{9}\]

where

\[ T(k,t) = -\frac{1}{2} \int \Gamma_{ik}(k,t) k^2 d\Omega(k) \tag{10}\]

The inertia term \( T(k,t) \) is the one that contains the mode-mode interaction, and its effect is to drain energy from the more energetic modes -typically the bigger ones- to the ones where there is major viscous dissipation -the smaller ones-.

**B. Flows in expanding universes**

For a curved space-time, in particular that described by a Friedmann - Robertson - Walker (FRW) background metric with zero spatial curvature (\( ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \)), the generalization of the above arguments has been considered by many authors [44][45]. We follow Tomita et al.’s analysis [58], in which they obtain the solution for the energy spectrum in the case of homogeneous, isotropic and incompressible turbulence.

In a generic space time, we describe fluid flow from the energy density \( \rho \), pressure \( p \) and four velocity \( U^i \). The symmetries of the FRW solution suggest using instead the comoving three velocity \( \nu \) for the energy spectrum in the case of homogeneous, isotropic and incompressible turbulence.

For a non relativistic incompressible fluid, with shear viscosity \( \eta \), the Navier-Stokes equation reads: 

\[ \frac{\partial \rho}{\partial t} + 3\frac{\dot{a}}{a} (\rho + p) = 0 \tag{11} \]

\[ \frac{\partial \mathbf{u}}{\partial t} + \left[ (\mathbf{u} \cdot \nabla) + \frac{\partial \ln ((\rho + p)a^3)}{\partial t} \right] \mathbf{u} = -\frac{\nabla p}{a^2 (\rho + p)} + \frac{1}{a^2} \nu \nabla^2 \mathbf{u} \tag{12} \]

where we have assumed that \( p + \rho \) depends only on time. For the physical three velocity \( \mathbf{v} \), the corresponding Navier-Stokes equation reads:

\[ \frac{\partial \mathbf{v}}{\partial t} + \left[ \frac{1}{a} (\mathbf{v} \cdot \nabla) + \frac{\partial \ln ((\rho + p)a^4)}{\partial t} \right] \mathbf{v} = -\frac{\nabla p}{a (\rho + p)} + \frac{1}{a^2} \nu \nabla^2 \mathbf{v} \tag{13} \]

In obtaining (11)-(13) we have neglected possible perturbations to the FRW metric. The corresponding equations considering fluctuations in the metric (\( g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu} \)) have been obtained by Weinberg [26]. The continuity equation is not corrected by gravitational perturbations, while in the Navier-Stokes equation the metric fluctuations appear explicitly only within the shear viscosity term. It can be demonstrated that these terms involving metric fluctuations are negligible for scales that are inside the horizon [53]. For scales bigger than the Hubble radius, since dissipation through viscosity is not effective anyway, we may still use the unperturbed Navier - Stokes equation.

The operation of Fourier transforming in the case of a Robertson-Walker cosmology is done in terms of comoving wave-numbers. In doing so, the following equation for the energy spectrum is obtained:

\[- \frac{\partial}{\partial t} E(k,t) = T(k,t) + 2 \left\{ \frac{\nu k^2}{a^2} + \frac{\partial \ln ((\rho + p)a^4)}{\partial t} \right\} E(k,t) \tag{14} \]

where the relationship between \( E(k,t) \) and \( \Phi_{ij}(k,t) \) as well as between \( T(k,t) \) and \( \Gamma_{ij}(k,t) \) is the same as that for a flat space time, if we define \( R_{ij} \) and \( T_{ij} \) from correlations of physical quantities, as follows:

\[ R_{ij}(r,t) = a^2 \langle u_i(x,t)u_j(x + r,t) \rangle, \tag{15} \]

\[ T_{ij}(r,t) = a^2 \frac{\partial}{\partial r_k} \langle (u_i(x,t)u_k(x,t)u_j(x + r,t)) - \langle u_i(x,t)u_k(x + r,t)u_j(x + r,t) \rangle \rangle \tag{16} \]
III. EQUIVALENT FLUID FOR INFLATON FLUCTUATIONS

After establishing the basic necessary notions for the description of hydrodynamic flows, our goal is to associate an equivalent fluid description to inflaton fluctuations, and to derive the spectrum of primordial density fluctuations at reheating therefrom. We shall discuss in the following sections some non trivial instances of this method.

A. The inflaton as a fluid

To describe the inflaton field from the point of view of an equivalent fluid, we need to obtain the energy density, pressure and velocity of this fluid as functionals of the state of the field. To this end, our starting point will be that in the rest frame of the fluid (quantities in this frame being labelled by a curl), the field ought to be spatially constant

\[ \nabla \phi = 0 \]  

To obtain the fluid four-velocity, we make a boost to the comoving frame. Then, the boost’s characteristic velocity will be the one we are seeking for. By the condition (17) we obtain:

\[ u_i = -\frac{\partial_i \phi}{\phi} \]  

which is generalized to the covariant form

\[ u_\mu = -\frac{\partial_\mu \phi}{\sqrt{-\partial_\rho \phi \partial^\rho \phi}} \]  

The energy density in the rest frame must be:

\[ \tilde{\rho} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + V(\phi) \]

Using the Lorentz transformations with the four-velocity [19] we obtain the general form for the energy density:

\[ \rho = -\frac{1}{2} \partial_\rho \phi \partial^\rho \phi + V(\phi) \]  

Finally, we obtain the pressure imposing an equality between the energy-momentum tensor for a perfect fluid (see for example [26]) and that for a minimally coupled scalar field (see [13]). The resulting pressure is:

\[ p = -\frac{1}{2} \partial_\rho \phi \partial^\rho \phi - V(\phi) \]  

Since the possibility of deriving a Navier - Stokes equation for the equivalent fluid rests on the conservation of $T_{\mu \nu}$, in principle only the whole inflaton field can be thus represented. However, under the approximation that the homogeneous part of the inflaton essentially contributes an effective cosmological constant, the background energy momentum tensor $T_{0\mu} = \Lambda g_{0\mu}$ is independently conserved (even if $\Lambda$ were not constant, conservation fails only on scales too large to be cosmologically relevant), and we can associate an equivalent fluid to the inhomogeneous quantum fluctuations $\delta \phi$ alone. For this fluid, we find the physical velocity ($v^i = au^i = a^{-1}u_i$)

\[ v^i_k = k_{\text{phys}} \left( \frac{\delta \phi_k}{\delta \phi_0} \right) \]  

where $\phi_0$ is the homogeneous background. We interpret this equation to mean that stochastic averages of the fluid velocity are to be identified with (symmetric) quantum expectation values of the operator in the right hand side [51,52]. As far as the equation of state is concerned, the free energy for a massive scalar field in the high temperature limit ($T \gg m$) [53] gives us the relationship between the pressure and the energy density, which turns out to be that for radiation, $p = \frac{1}{3} \rho$. This means that the energy density for this fluid redshifts proportional to $a^{-4}$. As this result has been obtained for a flat space-time, it is valid for scales smaller than the curvature radius. When scales are bigger than the Hubble radius, which in turn takes place when the high temperature limit is no longer valid, the field’s equation of motion $\square \phi + V'(\phi) = 0$ turns out to be
\[ \frac{d^2 \phi_0}{dt^2} + 3H \frac{d \phi_0}{dt} + V'(\phi_0) = 0 \]

for the background field \( \phi_0 \) (where we have neglected spatial derivatives) and

\[ \frac{d^2 (\delta \phi)}{dt^2} + 3H \frac{d (\delta \phi)}{dt} + V''(\phi_0)\delta \phi = 0 \]  

(23)

for the fluctuation \( \delta \phi \) (where we have neglected spatial derivatives as well as non linear terms). The time derivative of the equation for the background field gives us an equation of motion for \( \dot{\phi}_0 \):

\[ \frac{d^2 \dot{\phi}_0}{dt^2} + 3H \frac{d \dot{\phi}_0}{dt} + V''(\phi_0)\dot{\phi}_0 = 0 \]  

(24)

where we have used the constancy of \( H \) during inflation. Comparing (23) and (24) we see that, as \( \dot{\phi}_0 \) and \( \delta \phi \) obey the same equation of motion, they must be related by: \( \delta \phi = \dot{\phi}_0 f(\tau) \), which implies that the ratio \( \frac{\delta \phi}{\dot{\phi}_0} \) must be independent of time (cfr. [54]). For our fluid description, this means that when scales are much bigger than the Hubble radius, the velocity \( u_i \) (equation (18)) must remain constant, which in turn means that the physical three velocity \( v_i = au_i \) must redshift proportional to \( a^{-1} \). As these scales are frozen out because they are outside the horizon, they cannot interact among them or be dissipated by viscosity. Thus, the Navier-Stokes equation (13) reduces to

\[ \frac{\partial v}{\partial t} + \frac{\partial \ln \left( \rho + p \right)}{\partial t} a^4 v = 0 \]  

(25)

We obtain \( v \propto a^{-1} \) when \( (p + \rho) \propto a^{-3} \), corresponding to the equation of state of matter: \( p = 0 \). Thus, when the scales are well outside the horizon, our fluid behaves as pressureless dust, which agrees with the well known prediction based on Virial’s theorem for the equation of state of a field undergoing oscillations [53], which occurs at the final period of inflation. We must point out that the hypothesis of incompressibility is no longer valid for an equation of state of this type. Nevertheless, for scales bigger than the Hubble radius, which cannot decay through non linear interaction or dissipation by viscosity, equation (25) is still valid, regardless of the ratio of typical velocities to the speed of sound.

**B. Transport coefficients**

The framework to obtain transport coefficients for our fluid is linear response theory. In the limit of slowly variations in space and time of the magnitudes involved in the equation of conservation for the energy-momentum tensor, the system’s response while it is slightly displaced from equilibrium can be alternatively described by Navier-Stokes and continuity equations as well as by equilibrium expectation values of correlation functions. Matching these two descriptions, one obtains the Kubo formula for the shear viscosity [56]:

\[ \eta = \frac{1}{6} \lim_{w,k \to 0} \left[ \frac{1}{w} \int dt \int d^3r \ e^{i(k \cdot r - wt)} \left\langle \left[ \pi_{ij}(r,t), \pi_{ij}(0,0) \right] \right\rangle_{eq} \right] \]  

(26)

where \( \pi_{ij} \) are the traceless spatial-spatial components of the energy-momentum tensor:

\[ \pi_{ij} = T_{ij} - \frac{1}{3} \delta_{ij} T^k_k \]

For a minimally coupled scalar field the commutator involved in (26) turns out to be:

\[ \left[ \pi_{ij}(x), \pi_{ij}(0) \right] = \left[ \partial_i \phi(x) \partial_j \phi(x), \partial_i \phi(0) \partial_j \phi(0) \right] - \frac{1}{3} \left[ \partial_i \phi(x) \partial^k \phi(x), \partial_j \phi(0) \partial^k \phi(0) \right] \]  

(27)

These commutators can be evaluated from retarded and advanced Green functions for a massive scalar field coupled to other fields as well as to itself, which must satisfy:

\[ \left( \Box + \Gamma \frac{\partial}{\partial t} + m^2 \right) G_{R,A}(x,x') = \delta^4(x-x') \]
The first term involved in (27) turns then out to be the commutator of the product of two fields as:

\[ G(x, x') = -\frac{2i\Gamma}{(2\pi)^3} \int \frac{d^3p}{|p_o - i\Gamma|^2 - p^2 - m^2} \frac{e^{-i(p_0(t-t')-p(x-x'))}p_0}{|p_0 + i\Gamma|^2 - p^2 - m^2} \]

The mean value for the anticommutator of the field \( G_1(x, x') = \langle \{\phi(x), \phi(x')\} \rangle \) is obtained through the Kubo-Martin-Schwinger’s theorem:

\[ G_1(x, x') = -\frac{2i\Gamma}{(2\pi)^3} \int \frac{d^3p}{|p_o - i\Gamma|^2 - p^2 - m^2} \frac{e^{-i(p_0(t-t')-p(x-x'))}p_0 \operatorname{coth} (\beta p_o/2)}{|(p_o + i\Gamma)^2 - p^2 - m^2|} \]

FROM THESE, USING WICK’S THEOREM AND THE C-NUMBER CHARACTER OF THE MEAN VALUE FOR THE COMMUTATOR, WE OBTAIN THE COMMUTATOR OF THE PRODUCT OF TWO FIELDS AS:

\[ \langle \{\phi(x), \phi(x')\} \rangle = \langle \{\phi(x), \phi(x')\} \rangle \langle \{\phi(x), \phi(x')\} \rangle \]

The first term involved in (27) turns then out to be

\[
\int dt \ d^3r \ e^{i(k \cdot x - wt)} \langle \{\partial_t \phi(x), \partial_t \phi(x), \partial^2 \phi(0) \partial^2 \phi(0) \} \rangle =
\]

\[
-\frac{4\Gamma^2}{(2\pi)^2} \iint \frac{d^4p}{|p_o - i\Gamma|^2 - p^2 - m^2} \frac{\{\partial_i \phi(x), \partial_j \phi(x), \partial^i \phi(0) \partial^j \phi(0) \}}{|(w - p_o) p_0 \operatorname{coth} (\beta p_o/2)/|w - p_o|^2| - p^2 - m^2|^2|} \]

AND A SIMILAR FORMULA FOR THE SECOND TERM IN (27). THIS GIVES FOR THE SHEAR VISCOSITY IN THE HIGH TEMPERATURE LIMIT \( T \gg m, \Gamma \):

\[ \eta = \text{const} \ T \ \Gamma^2 \]

(28)

The thermal width \( \Gamma \) comes from the imaginary part of the self-energy. From dimensional analysis, it must be proportional to the temperature since the only relevant scale in the high temperature limit is the temperature itself (the constant of proportionality must be much less than unity for the consistency of the high temperature limit). Assuming that the only present interaction is the one coming from a \( \sigma \phi^4 \) term, \( \Gamma \) must include two \( \sigma \) insertions, which means that \( \Gamma \) must be proportional to \( \sigma^2 \) (we will estimate it as \( \Gamma \sim \sigma^2 T \)). The shear dynamic viscosity becomes

\[ \eta \sim \sigma^4 T^3 \]

(29)

A similar analysis allows us to evaluate the bulk viscosity, which turns out to be zero for a fluid with an equation of state of the type \( p = \frac{1}{2} \rho \) [59], in agreement with our previous assumptions.

C. Conversion of hydrodynamic fluctuations into primordial density contrast

Having described the quantum field as a fluid, we will analyze the resulting spectrum of density inhomogeneities. To do so, we assume that at some time \( t_1 \) during the beginning of the inflationary phase, when all the scales relevant to cosmology were inside the horizon, the scalar field fluctuations were undergoing hydrodynamic fluctuations. Once the scales leave the Hubble radius, their energy cannot be dissipated by viscosity or by nonlinear coupling. Thus, equation (14) means that they evolve according to:

\[ E(k, t > t_{out}) = E(k, t = t_{out}) \left( \frac{((p + \rho)^2)_{out}}{((p + \rho)(t))^2} \right)^2 \]

(30)

WHERE THE SUBSCRIPT "out" REFERS TO THE TIME WHEN EACH SCALE LEAVES THE HORIZON.

The definition of \( E(k) \) (equation 8) can be written in terms of the Fourier transform of the velocity; since the flow is statistically isotropic and homogeneous:
\[ \langle v'(k)v'(k') \rangle = \left( \frac{E(k)}{4\pi k^2} \right) \delta^3(k + k') \]  

(31)

Combining (30) and (31) we can obtain the r.m.s. value for the scalar field velocity at the time of reheating, which will be the r.m.s. value of the perturbation in the radiation’s streaming velocity. This perturbation will in turn produce fluctuations in the energy density of radiation, which will evolve in the usual way. The theory of relativistic very large wavelength fluctuations predicts \( \delta \sim a^2 \), where \( \delta = \delta\rho/\rho \) is the density contrast, and thus \( \dot{\delta} \sim H\delta \), while the continuity equation yields \( \dot{\delta} \sim v/l \) on a scale of physical size \( l \). Consistency of these two pictures leads to the relationship between the velocity at the time of reheating and the fluctuation in the energy density as

\[ \frac{\delta \rho}{\rho} \bigg|_{reh} = \frac{v_{reh}}{H_{reh}} \]  

(32)

where \( H_{reh} \) is the Hubble parameter at reheating. Following this fluctuations up to the time they reenter the Hubble radius, assuming that their size is such that they are always unstable (they must be always bigger than the Jeans’s length), they grow following the law (see for example [1,26]):

\[ \delta \equiv \frac{\delta\rho}{\rho} \bigg|_{ent} = \frac{\delta\rho}{\rho} \bigg|_{reh} \left( \frac{a_{eq}}{a_{reh}} \right)^2 \left( \frac{a_{ent}}{a_{eq}} \right) = H_{reh}^2 \left( \frac{a_{ent}}{a_{eq}} \right)^2 \frac{\delta \rho}{\rho} \bigg|_{reh} \]  

(33)

where the subscript "ent" means the time each scale reenters the Hubble radius (the second equation holds even if the entering time occurs before matter-radiation equality). Combining equations ((32)-(33)), we obtain the density contrast predicted by this theory at the time the modes reenter the Hubble radius:

\[ \langle \delta_k \delta_{k'} \rangle_{ent} = \frac{H_{reh}^2 a_{reh}^2 E(k, t = t_{reh})}{4\pi k^4} \delta^3(k + k') \]  

(34)

This is the main result of this paper, as it relates the density contrast to a hydrodynamic variable. We shall see a nontrivial application of this formula in next section, but before, it is convenient that we pause to show explicitly how the familiar results relating to free field fluctuations are recovered in this language. Probably the most important feature of a theory where inflaton fluctuations are free is that each mode evolves independently of the other ones. Immediately after leaving the horizon they freeze, a situation that can be described phenomenologically assigning to the mode the effective equation of state of dust. This implies that

\[ E(k, t = t_{reh}) = E(k, t = t_{out}) \left( \frac{a(t = t_{out})}{a(t = t_{reh})} \right)^2 \]  

(35)

\[ \langle \delta_k \delta_{k'} \rangle_{ent} = \frac{E(k, t = t_{out}(k))}{4\pi k^4} \delta^3(k + k') \]  

(36)

Let us compare this expression to the usual one in terms of quantum fluctuations. First we use Eq. (31), neglecting any variation of \( H \) or of the velocities during reheating, to get

\[ \langle \delta_k \delta_{k'} \rangle_{ent} = \left. \langle v'(k)v'(k') \rangle \right|_{t = t_{out}(k)} \]  

(37)

We now relate the physical velocity to field fluctuations according to Eq. (22); at \( t = t_{out}(k) \), \( k_{phys} = H \), and this reduces to

\[ \langle \delta_k \delta_{k'} \rangle_{ent} = \left( \frac{H}{\phi} \right)^2 \left( \delta\phi_k \delta\phi_{k'} \right)_{t = t_{out}(k)} \]  

(38)

which is the conventional result [34].

This shows the agreement between the fluid description and the conventional approach in this case, although of course it is only in the nonlinear case where we expect the hydrodynamic formalism to bring definite advantages.
IV. SELF SIMILAR FLOWS AND NONLINEAR FLUCTUATIONS

In the previous sections we set up the general formalism whereby we can associate to the evolution of quantum fluctuations during inflation an equivalent fluid description, and derive the corresponding primordial density contrast from hydrodynamic variables. Of course, to put the formalism to actual use, we must be able to solve the Navier-Stokes equations, which is in itself almost as daunting as solving the fundamental quantum field theory. However, there is in the hydrodynamic case a century of lore to draw upon \[38\], and some well tested approximations leading to relatively simple solutions. In this section, we shall demonstrate the equivalent fluid method by investigating the spectra resulting from one of these solutions, namely self similar flows. Towards the end of the section, we shall discuss the relevance of these solutions to actual cosmology.

A. Self similar flows in flat and expanding universes

As we have seen in the previous section (Eq.(34)), the key element in deriving the primordial density contrast is the energy spectrum \(E(k)\) (Eq.(3)), which is the solution of the balance equation (Eq.(3)). In it, the right hand side contains the viscous dissipation as well as the inertial force \(T(k, t)\). The overall effect of this term is to transfer energy from a given scale to smaller ones through mode - mode coupling; thus it is natural to model the action of the inertia term as a source of viscous dissipation, where the effective turbulent viscosity for a given mode depends on the motion of all smaller eddies \[34\]. By providing closure, that is, writing this effective viscosity in terms of the spectrum itself, a closed evolution equation for \(E(k)\) is obtained. Concretely, Heisenberg \[36\] proposed the ansatz

\[
\int_0^k T(k', t) \, dk' = 2\nu(k, t) \int_0^k E(k', t) \, k'^2 \, dk' \tag{39}
\]

where

\[
\nu(k, t) = A \int_k^\infty \frac{E(k', t)}{k'^3} \, dk' \tag{40}
\]

and \(A\) is a dimensionless constant. With this hypothesis (known as the Heisenberg hypothesis) as the solution to the closure problem, Chandrasekhar \[37\] has obtained the energy spectrum for decaying turbulence, assuming that there is a stage in the decay where the bigger eddies have sufficient amount of energy to maintain an equilibrium distribution, thus requiring that the solution for the spectrum should be self-similar. With this consideration into account he obtained an energy spectrum:

\[
E(k, t) = \frac{1}{A^2 k_0^3 t_0^7} \sqrt{\frac{t_0}{t}} F \left( \frac{k \sqrt{t}}{k_0 \sqrt{t_0}} \right) \tag{41}
\]

where \(k_0\) and \(t_0\) are initial conditions (namely, the wave number corresponding to the bigger eddy and its typical time of evolution). The function \(F\) obeys the equation

\[
\frac{1}{4} \int_0^x \left[ F(x') - x' \frac{dF(x')}{dx'} \right] dx' = \left\{ \nu k_0^2 t_0 + \int_x^\infty \frac{\sqrt{F(x')}}{x'}^{3/2} \, dx' \right\} \int_0^x F(x') x'^2 \, dx' \tag{42}
\]

which predicts a Kolmogorov type behavior for an inviscid fluid \((R \to \infty, R = \frac{1}{\nu k_0 t_0})\) in the ultraviolet limit:

\[
F(x) \to \text{const} \, x^{-5/3} \, (\nu = 0 \, , \, x \to \infty) \tag{43}
\]

While for nonzero viscosity:

\[
F(x) \to \text{const} \, x^{-7} \, (\nu \neq 0 \, , \, x \to \infty) \tag{44}
\]

In the infrared limit, \(F\) has the universal behavior \(F(x) = 4x \, (x \ll 1)\), and thus we find a linear energy spectrum for \(k \sqrt{t} \ll k_0 \sqrt{t_0}\). Chandrasekhar’s self similar solutions are easily generalized to flows in expanding Universes. The dependence on time and wave-number for the self similar energy spectrum is \[38\]

\[
E(k, t) = v_1^2 \left( \frac{(p + \rho) a^4}{(p + \rho) a_0^4} \right)^2 \lambda^2 F(\lambda k) \tag{43}
\]
where the subscript $i$ refers to "initial" and $\lambda$ and $v_t$ are respectively the Taylor’s microscale and an average turbulent velocity, defined as:

\[
\lambda_i^2(t) = 5 \frac{\int E(k, t) \, dk}{\int E(k, t) \, k^2 \, dk} \quad \frac{1}{2} v_i^2(t) = \int E(k, t) \, dk
\] (44)

Their time evolution must follow the law:

\[
\lambda_i^2(t) = \lambda_i^2 + 10 \int_{t_i}^t \frac{\eta}{(p+\rho)a^2} \, dt \quad v_t = v_{ti} \left( \frac{(p+\rho)_i a_i^4}{(p+\rho)a^4} \right) \frac{\lambda_i}{\lambda(t)}
\] (45)

The viscosity for our fluid, at least at high temperature, is given by Eq. (29). The equation which determines the function $F(\lambda k)$ in (43) turns out to be the same as in flat space time, Eq. (42), which means that assuming Heisenberg’s hypothesis the spectrum is linear in $k$ for scales much bigger than the Taylor’s microscale $\lambda_i^2 k^2$ for $\lambda k \ll 1$

\[
E(k, t) = 4 v_{ti}^2 \left( \frac{(p+\rho)_i a_i^4}{(p+\rho)a^4} \right)^2 \lambda_i^2 k \quad \text{for} \quad \lambda k \ll 1
\] (46)

B. Nonlinear inflationary models

We now want to place a self similar solution in the context of a inflationary scenario where, instead of regarding the inflaton fluctuations as free, we shall substitute them by an equivalent fluid, whose evolution we will assume to be self similar. We discuss at the end whether this last assumption is a reasonable one.

As before, we will assume a duration of inflation close to the minimum value $(N_{\text{min}} \simeq 60$, where $N$ stands for the number of e-folds), which can be justified by the expected quadrupole anisotropy as well as by the ratio of the present to the critical density (see Appendix). By this assumption, a scale whose present size equals the horizon leaves the Hubble radius soon after the beginning of inflation.

Unless in the free field case, here we cannot deal with each mode independently, but we must treat the whole flow subject to a phenomenological equation of state. Let us assume the self similar flow sets in at a time $t_1$ when the temperature $T >> H$ (we discuss whether this is a suitable assumption below); and that the present horizon scale leaves the horizon at or around time $t_1$. Then it is valid to use the high temperature limit for length scales close to the present horizon while they leave the Hubble radius during the inflationary phase. The fluid’s equation of state in this limit is of the $p = \frac{1}{3} \rho$ type, which means that the product $(p+\rho)a^4$ remains constant throughout the universal expansion. The factor $\left( (p+\rho)a^4 \right)_{\text{out}}$ involved in (41) is then independent of the particular scale being considered within this group. Thus, by (30) and (46) we can obtain the energy spectrum for these scales while they are outside the horizon:

\[
\frac{E(k, t > t_{\text{out}}(k))}{E(k_0, t > t_{\text{out}}(k_0))} = \frac{E(k, t = t_{\text{out}}(k))}{E(k_0, t = t_{\text{out}}(k_0))}
\] (47)

where (cfr. (44))

\[
E(k, t = t_{\text{out}}) = 4 v_{ti}^2(t_1) \lambda^2(t_1) k
\] (48)

$\lambda(t_1)$ being the comoving Taylor’s microscale at the time the self similar flow sets in. As at the initial time $t_1$ the only relevant scale is the temperature, we expect the initial Taylor’s microscale to be the inverse of the temperature.

\[^2\text{We wish to point out an ambiguity concerning the meaning of Heisenberg’s hypothesis in the case of curved spaces. For flat space time, the proportionality between the integral up to a certain wave number $k$ of the inertia and the viscous forces is given by (29) and (40). In the case of a FRW space time, the autosimilar solution required by Tomita et al. (13) needs a time dependent dimensionless constant $A$ proportional to $na^2$ for the consistency of the solution. This product does remain constant only if the dynamic shear viscosity evolves in time proportional to $a^{-2}$. Thus unless this is the case, the solution we described looks like a natural curved space generalization of the Heisenberg-Chandrasekhar solution, but does not admit the same physical interpretation.}\]
at that time, i.e. \( \lambda_{\text{phys}}(t_1) \sim \frac{1}{\eta(t_1)} \). By (42) we can obtain the comoving Taylor’s microscale at later times, such that most of the scales are still inside the horizon:

\[
\lambda^2(t) = \lambda^2(t_1) + 10 \int_{t_1}^{t} \frac{\eta}{(p + \rho) a^2} dt \simeq \lambda^2(t_1) + 10 \frac{\eta(t_1)}{(p + \rho)(t_1)} \frac{1}{Ha^2(t_1)} \tag{49}
\]

where we have used the fact that for a viscosity dependence upon time given by (29) the main contribution to the integral is given by its lower limit, provided there is a difference between the times \( t \) and \( t_1 \) of more than an e-fold. Since

\[
\frac{\eta(t_1)}{\lambda^2(t_1)H(p + \rho)(t_1)} \sim \sigma^4 T(t_1) \frac{1}{H}
\]

this means that the comoving Taylor’s microscale freezes in its initial value, unless coupling is very strong. Let us take as reference scale \( k_0 \) in Eq. (34) the inverse Taylor’s microscale (which is the last scale to leave the horizon in the high temperature regime). Then, from Eq. (30)

\[
\frac{E(k_0, t_{\text{reh}})}{E(k_0, t = t_{\text{out}}(k_0))} \sim \left( \frac{a(T_{\text{phys}} = H)}{a(t_{\text{reh}})} \right)^2 \tag{50}
\]

From Eqs. (17) and (55),

\[
E(k, t_{\text{reh}}) \sim \left( \frac{a(T_{\text{phys}} = H)}{a(t_{\text{reh}})} \right)^3 \frac{4\eta^2(t_1)}{a(t_{\text{reh}})} \lambda^2(t_1)k \sim \left( \frac{2\eta(t_1)}{a(t_{\text{reh}})H} \right)^2 k \tag{51}
\]

and so from Eq. (34)

\[
\langle \delta_k \delta_k' \rangle_{\text{ent}} = \frac{v_t^2(t_1)}{\pi} \frac{1}{k^3} \delta^3(k + k') \tag{52}
\]

That is, a scale invariant Harrison - Zel’dovich spectrum [61] with amplitude \( v_t \).

On the other hand, let us recall that the fluid velocity is related to the underlying field description by \( v \sim k_{\text{phys}} \delta \phi/\dot{\phi} \). Now the typical wave number is \( k_{\text{phys}} \sim T \), and at high temperature also \( \delta \phi \sim T \). From the background evolution, we have \( \dot{\phi} \sim V'(\phi)/3H \sim m_{\phi}^2 H/3H \). The number of e-foldings \( N \sim H \delta \phi / \dot{\phi} \sim \phi^2 / m_{\phi}^2 \), so \( \phi \sim \sqrt{N} m_{\phi} , \phi \sim m_{\phi} H \), and

\[
v_t(t_1) \sim \frac{T^2(t_1)}{m_{\phi} H} \tag{53}
\]

Since the ratio is necessarily less than one, the hypothesis of a non relativistic incompressible flow is seen to be consistent. The constraint of \( k \ll \lambda^{-1} \) reduces to a minimum scale above which we obtain scale invariance. Jeans’s length imposes a lower limit bigger than this (we are assuming that the scales are always unstable while they are outside the horizon, which is valid if they are bigger than the Jeans’s length).

Finally, combining the estimates for the initial Taylor’s microscale and the turbulent velocity, we obtain a Reynolds number:

\[
R = \frac{4}{3} \frac{\lambda_{\text{phys}}(t_1) v_t(t_1) \rho(t_1)}{\eta(t_1)} \sim \frac{v_t(t_1)}{\sigma^4} \tag{54}
\]

suggesting highly turbulent motion, specially for small couplings (the self similar solutions are not dependent on high Reynolds numbers anyway). The estimate of the viscosity must be taken with a grain of salt, nevertheless, since we have ignored non perturbative effects [61].

Observations suggest that density fluctuations have a scale invariant spectrum with an amplitude \( \delta \rho / \rho \sim 10^{-5} \) over a range of scales going from maybe as low as 100 Mpc up to the present horizon scale at \( t_{\text{now}} = 3000 \) Mpc \( \sim 10^{31} GeV^{-1} \) [15 62]. Recall Eq. (63) from the Appendix, relating the size of the Universe at the time \( t_{\text{exit}} \) when the actual horizon’s scale left the inflationary horizon, to the Hubble parameter during inflation. At the time \( t_{\text{exit}} \), the physical Taylor microscale is at most \( \lambda = \left( \frac{a(t_{\text{exit}})}{a(t_1)} \right) T^{-1}(t_1) = 10^{-2} H^{-1} \), since a larger value would narrow too much the regime where the spectrum is scale invariant. So

\[
\frac{T(t_1)}{\sqrt{m_{\phi} H}} = \left( \frac{a(t_{\text{exit}})}{a(t_1)} \right) 10^2 \left( \frac{H}{10^{15} GeV} \right)^{1/2} = 10^{-26} \left( \frac{a(t_f)}{a(t_1)} \right) \]

12
We obtain agreement with observations (cfr.\(52\) and \(33\)) provided
\[
\left(\frac{a(t_f)}{a(t_i)}\right) \sim 10^{23} = e^{53}
\]

Since the total run of inflation is some sixty e-foldings (no more than 70 even in the extreme case \(H = m_p\)) this is not hard to obtain, provided self similarity sets in a few e-foldings after the beginning of inflation.

We have thus set an explicit model where interacting inflaton fluctuations lead to a density contrast in agreement with observations. We could rest our case at this point, but before that, we would like to discuss briefly how likely it is that this solution was actually realized in the Early Universe.

C. Self similar flows and our Universe

As we have seen above, a self similar turbulent flow pattern in the equivalent fluid could explain the scale invariant spectrum of primordial density fluctuations observed at scales above 100 Mpc, provided the self similar regime sets early enough. Our goal in this final section is to discuss whether this is a likely assumption regarding our own Universe.

In general, it is known that turbulent flows tend to relax towards self similarity, but it is difficult to estimate how long does it take to get there. In a rough approximation, there are two issues involved, first, on what time scale the Heisenberg closure condition becomes valid, and then, how long does it take the closed equation for the spectrum to yield self similarity.

If posed in these terms, we may observe the analogy with the problem of equilibration in the theory of dilute gases. In the latter, the fundamental description of the dynamics is the BBGKY hierarchy \(\[33\]. However, after a time of the order of a collision time the hierarchy can be closed, turning into the Boltzmann equation, which in turn leads to equilibrium in times of the order of the mean free time.

In the turbulence problem, the time scale of a typical eddy is
\[
\tau = \frac{\lambda}{v_t} = \frac{m_p H}{T^3} \sim \frac{1}{H} \left(\frac{H}{T v_t}\right)
\]

We may take this as an estimate of the mean free time. The collision time can be estimated as the characteristic time for the smallest eddies in the flow (we visualize a collision as the exchange of a small eddy between two larger ones). According to Landau - Lifshitz \(\[64\]), this is
\[
\tau_{\text{min}} = \tau R^{-3/4} \ll \tau
\]

It is simple to find parameters leading to \(\tau\) of the order of an e-folding, and therefore \(\tau_{\text{min}} \ll H^{-1}\); for example, take \(H = 10^6 \text{GeV}, T(t_1) = 10^{10} \text{GeV}, v_t(t_1) = 10^{-5}\), with \(T_{\text{reh}} = 10^{12} \text{GeV}\). These solutions may not be available in models with such simple potentials as \(\sigma \dot{\phi}^4\), but they are feasible in multiparameter models such as those invoked in the potential reconstruction program \(\[14\]. While this does not constitute a proof, it makes it plausible that a self similar solution may be realized in the early stages of inflation.

Once the motion becomes self-similar, it may last for a rather long time. From the equation of motion Eq. \((14)\) we expect the energy in the flow to be dissipated in a time scale of the order of \(\tau_{\text{rel}} = \nu^{-1} \lambda_{\text{phys}}^2\). With \(\lambda \sim T^{-1}\) and \(\nu \sim \eta/ (\rho + p) \sim \sigma^4 T^{-1}\), we obtain
\[
\tau_{\text{rel}} \sim \frac{1}{\sigma^4 T} \sim R \tau
\]

This time will be generally larger than an e-folding, and might be even larger than the whole duration of inflation if \(R\) is large enough.

We conclude that self similarity will be easily achieved for scales around the Taylor microscale or smaller, and will propagate to larger scales after several e-foldings. At larger scales there could be deviations from scale invariance, associated with the transient behavior of the flow. We may conjecture that as the fluid cools down typical velocities will decrease, and thereby the primordial density contrast too. Thus the model will naturally yield higher power at larger scales, reproducing the behavior of multiple inflation models \(\[33\).

It bears mention that the same situation occurs in the usual treatments of the free field case. From Eq. \((38)\), we may conclude that the spectrum of primordial density fluctuations has an amplitude \(\delta_k \sim (H^2/ \dot{\phi}) \sqrt{1 + n_k}\), where \(n_k\) is the occupation number for that mode in the initial state of the inflaton field. Vacuum dominance requires \(n_k \ll 1\) for \(k \geq \sqrt{m_p H}\), but places no real restriction on larger scales which, in minimal models of inflation, are still observable. Thus we only obtain the conventional result (corresponding to \(n_k = 0\) for all modes) for specially chosen initial conditions.
V. FINAL REMARKS

In present conventional approaches, density inhomogeneities arise from primordial fluctuations in the inflaton field, ultimately of quantum origin. Fluctuations are treated as a free field, thus forcing upon us the assumption that higher derivatives of the inflaton potential are negligible. In this paper we sought a direct estimate of the primordial density contrast generated in a nonlinear inflationary model. Instead of assuming fluctuations to behave as a free field, we consider them to be coupled, so that they can be described phenomenologically as a fluid. We showed that there are flow patterns for this fluid that reproduce observations at very large scales.

While our model takes advantage of many insights from earlier studies of the role of turbulence in structure formation (see [31,38] etc.), it explores a whole new aspect of the problem in the sense that it places turbulence at the origin of the primordial fluctuations, rather than being excited from them. It is therefore free from the criticisms that are usually raised against the turbulence theory of structure formation [29,30]. To the contrary, our model successfully reproduces the results of the conventional approach on very large scales, that is, a scale invariant spectrum with a density contrast of about $10^{-5}$. While it is not completely free of fine tuning, the fine tuning involved is the same as in the usual scenario.

In our view, the main result of our work is not that a self similar solution should be our final description of fluctuations during inflation, but rather that it is possible to make sense of the physics of fluctuations even in rather general potentials. The self similar solutions we have explored in some detail should be seen as an ideal case which will more or less approximate actual flow patterns; indeed, the same could be said of the De Sitter invariant vacuum as a description of the actual state of the field in free theories.

The connection of hydrodynamics to fluctuation generation has some interest of its own, as it provides an alternative to brute force quantum field theoretic calculations, and also yields physical insight on the macroscopic behavior of quantum fields in the Early universe. The equivalent fluid method may be used to advantage also in other regimes, such as the pre-inflationary Universe and the reheating era, where the strong back reaction phase has so far been untractable [4,66]. Moreover, it opens up a wealth of new phenomena, such as intermittence [43,34] and shocks [44], which are not apparent in the customary treatments. We will continue our research in this field, which promises a most rewarding dialogue between cosmology, astrophysics, and nonlinear physics at large.

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VII. APPENDIX

The minimal amount of inflation necessary to solve the homogeneity problem is obtained by the condition that a scale of the size of the horizon at present ($\sim 3000 Mpc$) should have been inside the Hubble radius at the beginning of inflation. The Hubble radius during inflation is approximately constant. A scale whose physical size at present is $\lambda(t_0)$ was, at the end of reheating $\lambda(t_{reh}) = \lambda(t_0) a(t_{reh}) / a(t_0) = \lambda(t_0) T(t_0) \approx \lambda(t_0) 2.35 \cdot 10^{-25} \left( \frac{10^{12} GeV}{T_{reh}} \right)$

Write $T_{reh} = \Gamma \sqrt{m_p H}$, where $\Gamma \leq 1$ and $H$ is the Hubble parameter during inflation. Then towards the end of inflation we have $\lambda(t_f) = \lambda(t_0) 2.35 \cdot 10^{-25} \left( \frac{10^5 GeV}{H} \right)^{1/2}$

This scale left the horizon at a time $t_{exit}$ with $\lambda(t_{exit}) = (a(t_{exit})/a(t_f)) \lambda(t_f) = H^{-1}$. So

$$1 = \left( \frac{a(t_{exit})}{a(t_f)} \right) \lambda(t_0) 2.35 \cdot 10^{-20} GeV \left( \frac{H}{10^{5} GeV} \right)^{1/2}$$
In particular, for the present horizon scale we get
\[
1 = \left( \frac{a(t_{\text{exit}})}{a(t_f)} \right) 10^{21} \left( \frac{H}{10^5 \text{GeV}} \right)^{1/2} \tag{57}
\]

Therefore, defining \( N_{\text{min}} = \ln \left[ a(t_f)/a(t_{\text{exit}}) \right] \), which would make \( t_{\text{exit}} = t_i \) for the scale of the horizon at present
\[
N_{\text{min}} = \ln \left[ 10^{21} \left( \frac{H}{10^5 \text{GeV}} \right)^{1/2} \right] = 64.4 + \ln \sqrt{\frac{H}{m_p}} \tag{58}
\]

On the other hand, inflation could not have lasted much more than this because otherwise the present density should be so fine tuned to the critical one that would contradict observations as well as most speculations on dark matter’s density. This can be seen from one of the Friedmann’s equations:
\[
\Omega(t) - 1 = \frac{k}{(a(t)H(t))^2}
\]

where \( \Omega \) is the ratio of the density to the critical one and \( k \) is the spatial curvature of the FRW metric. This means that the ratio (assuming instantaneous reheating):
\[
\frac{\Omega(t_0) - 1}{\Omega(t_i) - 1} = \frac{(a(t_i)H(t_i))^2}{(a(t_0)H(t_0))^2} \approx \exp \left[ 69.06 + \ln \sqrt{\frac{H}{m_p}} - N \right] \tag{59}
\]

So, if we set \( \Omega(t_i) \) to lie in the interval \((0, 1)\), we conclude that if \( \Omega(t_0) \) is not fine tuned to 1, \( N \) should not have been much more than 60.

There is another argument supporting that inflation should not have spanned much more than its minimum duration based on the comparison between the expected quadrupole anisotropy and the detected one (see [23]).

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