Strong-coupling-expansion analysis of the false-vacuum decay rate of the lattice $\phi^4$ model in $1+1$ dimensions

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Abstract.
Strong-coupling expansion is performed for the lattice $\phi^4$ model in $1+1$ dimensions. Because the strong-coupling limit itself is not solvable, we employed numerical calculations so as to set up unperturbed eigensystems. Restricting the number of Hilbert-space bases, we performed linked-cluster expansion up to eleventh order. We carried out alternative simulation by means of the density-matrix renormalization group. Thereby, we confirmed that our series-expansion data with a convergence-acceleration trick are in good agreement with the simulation result. Through the analytic continuation to the domain of negative biquadratic interaction, we obtain the false-vacuum decay rate. Contrary to common belief that tunneling phenomenon lies out of perturbative treatments, our series expansion reproduces the instanton-theory behaviour for high potential barrier. For shallow barrier, on the contrary, our result tells that the relaxation is no more described by instanton, but the decay rate acquires notable enhancement.

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1. Introduction

Suppose that a system is placed at a certain metastable state surrounded by local potential minimum, the system would be unstable to decay to a global minimum assisted by either quantum or thermal fluctuations. Such processes are called false-vacuum decay and metastability relaxation, and they are considered to be non-perturbative in nature. Hence, in order to calculate the decay rate (life time), ingenious treatments have been invented so far [1, 2, 3, 4]. Those treatments rely on semi-classical approximation. That is, the treatments take into account quadratic fluctuations around the field configuration which extremizes the Euclidean action. Such field configurations are called instanton, bounce and (critical) droplet. Therefore, those treatments, just like the WKB approximation in wave mechanics, are not justified for strong fluctuations (namely, short life time). In addition, it is quite cumbersome to improve the approximation systematically.

In order to compensate the above drawback, first-principle calculation scheme free from any biased errors would be desirable. As for discrete variable model (kinetic Ising model), actually, remarkable tour de force scheme was invented by Günther et al. [5]. They introduced the so-called constrained-transfer-matrix method, which meets nonequilibrium situation. Then, they carried out extensive numerical calculations of the transfer matrix. In consequence, they extracted imaginary part of the free energy, which is to be identified as the decay rate. To the best of our knowledge, it is the first ab initio approach to the decay rate in the presence of many-body correlations. Their result supports the aforementioned analytic theory based on the droplet picture. (Besides this, Monte-Carlo simulation has been utilized to evolve the relaxation processes [6, 7], where the number of Monte-Carlo steps is interpreted as time progression. Though the interpretation is, in a strict sense, not fully justified, the simulation result is fairly in accordance with the droplet picture actually.)

On the contrary, as for continuous-variable model such as the $\phi^4$ model, the above approach does not apply, and so far, no attempt at ab initio calculation has been reported. For quantum-mechanics level ($0 + 1$ dimension), however, a number of substantial progresses are made [11]: Suzuki and Yasuta obtained a compact expression for the decay (tunnelling) rate based on the weak-coupling expansion and succeeding Borel resummation [12, 13]. They succeeded in calculating the tunnelling rate beyond instanton calculus. Alternatively, from the weak coupling expansion, Karrlein and Kleinert obtained, remarkably enough, strong-coupling series by means of the so-called variational perturbation [14]. Both approaches pursue first-principle calculation scheme beyond instanton calculus. As a consequence, these theories clarified how the instanton description fails for low potential barrier; true decay rate is suppressed owing to inter-instanton interaction. At present, extension to many-body case appears to be unsuccessful [13].

The aim of this paper is to investigate the false-vacuum decay rate for many-body
system through series expansion. We studied the lattice $\phi^4$ model in $1 + 1$ dimensions,

$$\mathcal{H} = \sum_i \left( \frac{1}{2} \pi_i^2 + \frac{1}{2} (\phi_i - \phi_{i+1})^2 + \frac{1}{2} \phi_i^2 + g \phi_i^4 \right).$$

(1)

with the canonical commutation relations $[\phi_i, \pi_j] = i \delta_{ij}$, $[\phi_i, \phi_j] = 0$ and $[\pi_i, \pi_j] = 0$. Note that for $g < 0$, the potential is not bounded below, and renders the state $\phi \approx 0$ unstable (false vacuum). The decay rate due to the quantum fluctuations is our concern. We will show that in contrast to $0 + 1$ dimension mentioned above, the decay rate is enhanced owing to inter-instanton interaction.

The present paper is organized as follows. In the next section, we calculate the decay rate by means of strong-coupling expansion. We explain methodological details, and check the validity by means of an alternative simulation. In the last section, we summarize the present study.

2. Results and discussions

In this section, we will calculate the false-vacuum decay rate of the model (1) through strong-coupling expansion. To begin with, we will formulate the basis of the expansion.

2.1. Strong-coupling expansion

Making use of the rescalings $\phi \rightarrow g^{-1/6} \phi$ and $\pi \rightarrow g^{1/6} \pi$, we arrive at the expression,

$$\mathcal{H} = g^{1/3} h,$$

where,

$$h = \sum_i \left( \frac{1}{2} \pi_i^2 + \phi_i^4 + \frac{1}{g^{2/3}} \left( \frac{1}{2} (\phi_i - \phi_{i+1})^2 + \frac{1}{2} \phi_i^2 \right) \right).$$

(3)

According to the formula, the quadratic potential terms are regarded as perturbations, and so the ground-state energy is expanded in terms of the strong-coupling parameter $\lambda = 1/g^{2/3}$,

$$E_g = g^{1/3} e_g,$$

where,

$$e_g = \sum_{n=0} a_n \lambda^n.$$ 

(4)

Note that the unperturbed Hamiltonian $h|_{\lambda=0}$ is biquadratic. Hence, it is not quite straightforward to perform perturbation with respect to this limit. Here, however, we will manage the perturbation expansion with the aid of computer calculations.
2.2. Linked-cluster expansion

The unperturbed Hamiltonian is a collection of independent anharmonic oscillators, and the perturbation introduces coupling among them. In such case, the linked-cluster expansion is useful to generate perturbation series. The linked-cluster expansion is a method of, so to speak, computer-aided diagrammatic expansion [8, 9, 10]. To perform cluster expansion, we should set up unperturbed eigensystems nevertheless. For that purpose, we must diagonalize the Hamiltonian of each local anharmonic oscillator $h_i = \pi_i^2/2 + \phi_i^4$. We carried out the diagonalization in the following way: (a) An oscillator spans infinite-dimensional Hilbert space. In order to perform computer simulation, we need to restrict the number of bases. For that purpose, we prepare low-lying $M = 400$ states of harmonic oscillator with quadratic potential $\Omega^2 \phi^2/2$; namely, $\{|n\rangle_\Omega\}$ ($n = 0, \cdots, M - 1$). Note that the diagonalization of $h_i$ is now manageable, because the Hilbert space is spanned by finite number of bases just prepared. Here, $\Omega$ is a freely tunable variational parameter, and we had adjusted it so as to minimize $\Omega \langle 0 | h_i | 0 \rangle_\Omega$; namely, we chose $\Omega = 6^{1/3}$. (This idea is a reminiscence of Feynman and Kleinert [15], who calculated the thermodynamics of anharmonic oscillator by replacing biquadratic potential with an optimal quadratic one.) (b) With respect to the Hilbert-space frame $\{|n\rangle_\Omega\}$ ($n = 0 \sim M - 1$), we represented the anharmonic-oscillator Hamiltonian $h_i$, and diagonalized it to obtain the energy levels and the eigenvectors. (c) Provided that those eigenvectors are at hand, we carried out secondary Hilbert-space truncation: We extracted low-lying $m$ eigenvectors among $M$, and discarded the others. Henceforth, those $m$ vectors are to be used to span the (intra-oscillator) Hilbert space. (Such Hilbert space restriction scheme originates in Wilson, who diagonalized huge cluster of conduction electrons [16].)

To summarize, we truncated the intra-oscillator bases through two steps. First, we had utilized the eigenvectors of a harmonic oscillator to span the Hilbert-space frame. Those are not very efficient, and so, we prepared rather huge number of $M = 400$ bases in practice. In that sense, the second truncation is significant, where we had remained only low-lying $m$ bases after solving the eigensystems of the intra-site Hamiltonian $h_i$. These bases turned out to be very efficient (see below), and only $m = 10 \sim 25$ bases are necessary so as to achieve reliable calculations in the succeeding linked-cluster expansion. (Note that to perform the linked-cluster expansion, we need to store, in computer memory, huge Hilbert-space vector for clusters consisting many oscillators.)

Before going into cluster expansion, we will check the reliability of the Hilbert-space restrictions. We treat a single anharmonic oscillator (namely, we ignore the inter-oscillator coupling) with respect to the restricted Hilbert space mentioned above. We used the ordinary Rayleigh-Schrödinger perturbation theory, because the system is of one-body problem. The strong-coupling perturbation coefficients are reported in the literature [17]. We observed following encouraging features: First, the choice of $M = 400$ is sufficient. Namely, it reaches the limit of numerical round-off error (we used extended precision of 16-byte real number), and further increase of $M$ just alters final few digits.
Secondly, we found that rather small \( m \) yields precise data. For example, \( m = 15 \), which would seem exceedingly small, reproduces the perturbation coefficients reported in [7] with high precision of order \( \sim 10^{-17} \) (that is not relative but absolute error). Moreover, the precision is maintained even for high-order perturbation coefficients. For example, the choice of \( M = 400 \) and \( m = 25 \), for which the simulation takes ten minutes or so, is sufficient to reproduce the full result of [7].

Encouraged by these findings, we performed the linked-cluster expansion for the lattice \( \phi^4 \) model (3). We obtained the perturbation series up to eleventh order. The strong-coupling series is given by,

\[
e_g = 0.66798625915577710827096201688 \\
+ 0.4310063501425947306095738275\lambda \\
- 0.1014880952111863294125944502\lambda^2 \\
+ 0.04803845646443637442034775341\lambda^3 \\
- 0.029018513979643624653232757064\lambda^4 \\
+ 0.01977791330895673863274529570\lambda^5 \\
- 0.014454753622894705466341917665\lambda^6 \\
+ 0.01106139124598227911409431586\lambda^7 \\
- 0.0087493465269972\lambda^8 \\
+ 0.007096745791805\lambda^9 \\
- 0.005871428\lambda^{10} + 0.00493622\lambda^{11}
\]

(6)

with uncertainties only in the final digits.

2.3. Resummation and its verification with DMRG

In the above, we obtained strong-coupling series expansion for \( e_g \) (6). We plotted the result in figure [1]. We had truncated the series at various orders, which are indicated for respective curves. We see that the curves start to deviate at \( \lambda \approx 1 \), and higher-order data exhibit even worse convergence. Hence, it is suggested that the series (6) has finite convergence radius \( |\lambda| \sim 1 \). In order to go beyond the convergence bound and extract meaningful physics, we have to process our data with some resummation trick.

We found that Aitken’s \( \delta^2 \)-process [18],

\[
S'_n = S_n - \frac{(S_n - S_m)^2}{S_n - 2S_m + S_l},
\]

(7)

is very useful to accelerate the convergence of our series. Here, \( S_l, S_m \) and \( S_n \) are three successive partial sums truncated at respective orders. We plotted the resumed results in figure [2]. The symbol such as 5-6-7 indicates that the data are accelerated with the partial sums of \( S_5, S_6 \) and \( S_7 \). We see that the data exhibit pronounced convergence improvement. Our data may be valid up to \( \lambda \approx 2 \).

In order to check the convergence bound more definitely, we performed an alternative first-principle simulation with the density-matrix renormalization group.
Our algorithm is standard. As for a comprehensive overview of this algorithm, interested readers may consult with a proceeding [19]. Full account of technical details specific to the 1 + 1-dimensional scaler field theory will be found in our paper [20]. (In this paper, we studied field $\phi$ confined within the rigid-wall potential $V(\phi)$. In order to match the present case, one has to replace $V(\phi)$ with $\phi^4$.) The numerical error was checked thoroughly in [20], and it was found to reach $10^{-7}$. We monitored the performance in the present case as well, and found that the precision is maintained. The error would be far less than the symbol size shown in the plot.

The first-principle data are shown in figure 2 as well. We see that our resummed data are valid up to $\lambda \approx 2$ fairly definitely.

Finally, we mention a singularity occurring at $\lambda \approx -2(< 0)$; see figure 2. It is noteworthy that for $\lambda < 0$, the potential becomes double-well form. Therefore, at a certain critical $\lambda$, there would be an Ising-type phase transition. The singularity found in our data may indicate the onset of the transition. Determination of the critical point for the lattice $\phi^4$ model is attracting considerable attention recently in the context of quantum ferroelectric transition [21]. We will pursue this issue elsewhere, and in the present paper, we will not go into details any further.

### 2.4. Analytic continuation to $g < 0$: false-vacuum decay rate

In the above, we attained good convergence of the series expansion (6) with the aid of the convergence-acceleration trick (7). Armed by this achievement, in this subsection, we access the domain of $g < 0$ through the analytic continuation $g \to -g$. For $g < 0$, potential is not bounded below, and exhibits a local potential minimum in the vicinity of $\phi = 0$ (metastability). Because the series expansion (6) is an irrational function in terms of $g$, the analytic continuation renders imaginary part in the ground-state energy. Thereby, from it, we can read off the false-vacuum decay rate. In practice, the analytic continuation is done through the path $g \exp(i\theta)$ of $\theta = 0 \to \pi$. That is, the term $g^{1/3}$ gives rise to the contribution $g/2 + i\sqrt{3}g/2$ after $g \to -g$.

So far, as to calculate the decay rate, the instanton technique has been used. The technique is justified for sufficiently large potential barrier (small $g$). In the following we will show that our series-expansion approach covers the instanton theory.

In figure 4, we plotted the false-vacuum decay rate multiplied by $g$, namely, $g\text{Im}E_g(-g)$, against $1/g$. The factor $g$ should kill the prefactor of a dominant exponential contribution: That is, the instanton theory predicts that the decay rate should obey the formula,

$$\text{Im}E_g(-g) \propto \frac{1}{g} \exp(-S/g),$$

where $S$ denotes the Euclidean action of one instanton. We solved the instanton solution numerically, and obtained the estimate,

$$S = 1.1891027(5).$$
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To be concrete, we will sketch the calculation method. First of all, one must reformulate the Hamiltonian formalism into the Lagrangian formalism in the Euclidean space-time. Thereby, we considered the system with 28 sites and imaginary time $\beta = 28$. The imaginary time is discretized into 8000 intermediate time slices. (Note that now, the field is defined in the discretized space-time.) Because the instanton solution (field configuration) should minimize the Euclidean action, the problem reduces to the minimization of multi-dimensional function. That computation is readily achieved by the utilities supplied in simulation guide books such as [18]. The amount of error is estimated with changing the system sizes and discretization intervals.

We would like to draw reader's attention to the fact that the formula (8) has an essential singularity at $g = 0$. That is why we had selected the strategy of approaching from $g \to \infty$ rather than from $g = 0$.

Let us turn to the discussion of our result of figure 3. As is mentioned above, the instanton result (8) is validated for large $1/g$. As a matter of fact, for $1/g > 1$, our data approaches the instanton prediction; we had drawn the slope of (8). In this respect, the convergence-acceleration trick (7) is crucial in our study, because it enables us to attain good convergence up to $1/g \sim 3$, which appears to reach the instanton region.

For $1/g < 1$, on the other hand, our data indicate rapid enhancement of the decay rate; namely, the curve starts to deviate from the instanton prediction. It is to be stressed that our treatment is justified for strong-coupling limit ($1/g \ll 1$). Therefore, it is found that the inter-instanton correlation gives rise to enhancement of relaxation. This feature is to be contrasted with that of $0+1$ dimension, where the inter-instanton correlation results in suppression of decay rate. Enhancement in $1+1$ dimensions was speculated in the former study [13], where the authors utilized the weak-coupling expansion and the Borel technique. Although their series does not show any indication of convergence, their result actually captures a signal of relaxation enhancement.

According to Kleinert, in the regime $g \gg 1$, the decay process is governed by ‘sliding’ rather than instanton [22]. Nevertheless, we stress that the present series-expansion approach covers both instanton ($1/g > 1$) and sliding ($1/g < 1$) regimes in a unified way. Moreover, Our series is readily improved systematically just by performing cluster expansion further.

In the above, we found that at $1/g \approx 1$, there exists a crossover boundary separating two distinctive regimes. Our result supports the previous proposal of [13]. The authors calculated the effective potential, and found that for $g > 1.17$, the potential barrier is smeared out by quantum fluctuations. Their criterion would be sensible for separating instanton and sliding phases.

In figure 3, we see that the data of 9-10-11 and 8-9-10 show poor convergence. That may possibly be due to the fact that our 10th and 11th perturbation coefficients have rather few significant figures available.

Finally, we recollect past findings for the $\phi^4$ theory in continuous space time. Brézin and Parisi completed the instanton calculation, and obtained the formula $\text{Im}E_g(-g) = (0.0815435/g) \exp(-1.4626121/g)$ [23]. We notice that the instanton action
is similar to that of our lattice model. Perhaps, the decay process would be identical between the lattice model and the continuous-field theory.

### 3. Summary and discussions

So far, several *ab initio* approaches have been proposed in order to calculate the decay rate beyond semiclassical approximation. In particular, the $\phi^4$ model in $0 + 1$ dimension has come under through investigation [11, 12, 13, 14], while the extension to many-body case remains unsuccessful. In the present paper, by means of the strong-coupling expansion, we studied the $(1 + 1)$-dimensional lattice $\phi^4$ model [1]. We demonstrated that the linked-cluster expansion method works very efficiently, provided that Hilbert-space restriction is processed properly. In addition, we found that the convergence-acceleration trick [7] is significant. In fact, the convergence-accelerated sum reproduces the first-principle data for considerably wide range $\lambda < 2$.

Based on the above achievements, we surveyed the domain of metastability through the analytic continuation $g \to -g$. We are concerned in the false-vacuum decay rate $\text{Im} E_g(-g)$; see figure [3]. Our result indicates that there are two regimes. For $g < 1$, our result obeys the prediction by the instanton theory. It is to be stressed that the convergence acceleration [7] is significant to reach the instanton regime. For shallow potential barrier $g > 1$, the relaxation is no more described by instanton, but the relaxation rate acquires notable enhancement. According to Kleinert, for $g > 1$, the relaxation is driven by sliding rather than instanton. Nevertheless, we stress that our series-expansion approach does cover both regions with a unified framework, and it is readily improved systematically just by continuing the perturbation further. It would be promising that the present method is applied to other wide class of metastable systems.

As is mentioned above, we had performed the density-matrix-renormalization-group simulation as well. From the simulation data, we are able to extract perturbation coefficients by polynomial fitting. This technique is applicable to those models that even possess complicated interactions and spatial inhomogeneity. Tunnelling phenomena assisted by an impurity is of current interest [24]. However, ambiguities in estimating fitting errors are not fully resolved at present. It is left for future study.

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Figure 1. Strong-coupling series $e_g(\lambda)$ is plotted. The series is truncated at various orders. Sudden deviation may indicate the convergence bound. Convergence-accelerated data are presented in figure 2.

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Figure 2. The same as figure 1, but the data are convergence-accelerated by the formula (7). The symbol such as \(l-m-n\) indicates that the data are processed with use of three partial sums \(S_l\), \(S_m\) and \(S_n\). We also presented a first-principle simulation result by means of the density-matrix renormalization group. We confirm that our series achieves good convergence over the range \(\lambda < 2\).
Figure 3. False-vacuum decay rate (multiplied by $g$) $g \text{Im} E_g(-g)$ is plotted. The symbol $l-m-n$ indicates that the data are convergence-accelerated with use of three partial sums $S_l$, $S_m$ and $S_n$. We plotted a slope $\exp(-S/g)$ which is predicted by the instanton theory; see text. Note that the instanton treatment is justified for large $1/g$. As a matter of fact, our series expansion obeys the prediction for $1/g > 1$. For $1/g < 1$, on the contrary, our result exhibits notable enhancement. Hence, it is suggested that the inter-instanton interaction rather enhances the relaxation.