Incidence of the boundary shape in the tunnelling exponent of electrons into fractional quantum Hall edges

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Experiments on tunneling into fractional quantum Hall droplets systematically found tunneling exponents smaller than those predicted by the ordinary chiral Luttinger liquid theory. In this note, we consider the effects of a smooth boundary, and propose a modification of the theory that predicts a reduced exponent, in qualitative agreement with experiments. Such modification consists on a self-interaction cubic term, containing higher derivatives of the chiral boson field, that provides a non-universal one loop correction to the propagator.

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The study of tunnelling of electrons into both incompressible and compressible quantum Hall states has been the subject of intense research from both the theoretical and experimental sides (see [1] and references therein). It is found that the tunnelling conductivity is non-Ohmic, $I \propto V^\alpha$, with $\alpha$ being a function of the filling fraction $\nu$. This behavior can be understood within the chiral Luttinger liquid description of the edge physics advanced by Wen in [2]. Although tunnelling experiments in fractional quantum Hall effect (FQHE) systems have shown certain degree of agreement with the theoretical predictions obtained from the chiral Luttinger liquid picture [2], there remain certain discrepancies which have been addressed by different authors [1, 3, 4, 5, 6, 7, 8]. In particular, a reduction of the order of 10\% from the theoretical prediction for the tunnelling exponent has been observed experimentally, as well as the absence of the theoretically predicted plateau structure of this exponent [3, 6, 8]. These issues have been the subject of intense debate [1].

Here we present an alternative derivation of the boundary effective action by including the effects of a smooth boundary, and propose a modification of the theory that predicts a reduced exponent, in qualitative agreement with experiments. Such modification consists on a self-interaction cubic term, containing higher derivatives of the chiral boson field, that provides a non-universal one loop correction to the propagator.

The action of a two dimensional system of charged particles (charge $-e$) moving in a constant magnetic field $A_a = -\frac{B}{2} \epsilon_{abc} x^b$ and an electric potential $V(x^a)$ is

$$S = \sum_p \int dt \left( \frac{m}{2} \dot{x}_p^a \dot{x}_p^a + \frac{eB}{2} \epsilon_{abc} x_p^a x_p^b - eV(x^a_p) \right)$$

where $x^a_p$ denote the particle positions, $a = 1, 2$ being the space directions and $p$ being a discrete particle index.

The electric potential will be chosen to be approximately constant inside some bounded region $\mathcal{M}$, vanishing smoothly outside the boundary $\partial \mathcal{M}$ through a region $\mathcal{E}$ of width $W$. This region, in which the electric field $E_\nu = \partial_\nu V$ is non-vanishing, defines the edge of the sample. The density $\rho_0$ will be chosen, consistently with the assumed form of the sample, as approximately constant.
inside $\mathcal{M}$ and decreasing within the edge $\mathcal{E}$ to be zero outside.

To see the effect of $V(x^a)$ at the edge, we approximate it in the neighborhood of each particle by a linear function $V(x^a) = -E_a x^a$, assuming that the length in which changes in the electric field are measurable is much larger than the cyclotron radius, i.e. $r \ll E/|\nabla E|$. Then we have an effectively constant electric field around the particle and we see that it can be eliminated through the boost $\tilde{x}^a = x^a + v^a t$, where $v_a = \epsilon_{ab} E_b / B$. Then the classical motion is given by circular orbits with cyclotron radius $eB/m$ and radii related to the initial velocities by $r = |\tilde{x}_{\text{in}}| m / eB$, the centers of which are moving with velocity $-v^a$ transverse to the electric field, i.e. parallel to the edge.

As a conclusion, for a strong magnetic field the action can be written as

$$S = \sum_p \frac{eB}{2} \int dt \epsilon_{ab} \tilde{x}^a_{\mu} \tilde{x}^b_{\nu}$$

where the new variables $\tilde{x}^a_{\mu}$ are given by the above expressions in terms of the position of the particle and the electric field in its neighborhood. Note that the velocity $v^a_{\mu}$ is parallel to $\partial \mathcal{M}$, being zero for the particles outside $\mathcal{E}$.

If the mean distance between the particles $a = \sqrt{\rho_0}$ is small compared to all the external scales involved into the problem (i.e. the lengths at which variations on the electric field or on the density are measurable), we can take the continuum limit. Then, we can replace the variables $\tilde{x}^a_{\mu}$ by fields $\tilde{x}^a(y^i, t)$ where the continuous label $y^i$ is fixed by the condition that $\tilde{x}^a(y^i, t = 0) = \tilde{y}^a$. Then the sum on $p$ can be replaced by an integral

$$S = \int dt d^2 y \rho_0 \frac{eB}{2} \epsilon_{ab} \partial_i \tilde{x}^a \tilde{x}^b$$

In the continuum theory, the aforementioned condition can be characterized by $E/\partial_i E \gg a$ and $\rho_0/\partial \rho_0 \gg a$, where $\partial$ represents the derivative in some spatial direction. Since these derivatives can differ significantly when taken in different directions, so does the accuracy of the limit.

To be concrete: at the edge, if the curvature of the boundary is small, then we have $E/\partial_i E \gg a$ (and the same for $\rho_0$), and the limit works in the longitudinal direction. On the other hand, for a sharp enough edge we may have $E/\partial_i E \sim a$, and we most somehow pay attention to the discrete structure on the transverse direction. We will accomplish this bellow by re-discretizing the transverse integration.

Here it is important to stress that $\tilde{y}$ is the initial position of the fluid element in its particular “rest frame” in which the electric potential term vanishes. Note that the operation of taking the continuum limit introduces the function $\rho_0$, which was part of the initial conditions, into the action of the system in the form of a measure in space.

The resulting action is invariant under the group of measure preserving diffeomorphisms in the $y^i$ plane, i.e. changes of coordinates such that $y'_i = y_i + \epsilon_{ij} \partial_j \mathcal{N} / \rho_0$. The corresponding conserved generator is

$$Q = \frac{eB}{2} \int d^2 y \epsilon_{ab} \epsilon^{ij} \partial_j \tilde{x}^a \tilde{x}^b$$

since it is conserved for any function $\mathcal{N}$, its integrand is conserved, i.e.

$$\epsilon_{ab} \epsilon^{ij} \partial_j \tilde{x}^a \tilde{x}^b = \epsilon_{ab} \epsilon^{ij} \partial_j \tilde{x}^b \tilde{x}^a \big|_{t=0} = 2$$

this relation is a constraint in the values of the fields $\tilde{x}^a$ that has to be solved to obtain the independent dynamical degrees of freedom of the system.

The next step is to solve the constraint (5) and rewrite the action in terms of the dynamical degrees of freedom. Here and in what follows, we will use as parameters the filling fraction $\nu = \rho_0 / eB$ and the effective non-commutativity parameter $\theta = 1/(2\pi \rho_0)$.

If we parameterize the $\tilde{x}^a$ field with the help of a new field $\phi$ according to

$$\tilde{x}^a = y^a + \theta \epsilon^{ab} \partial_b \phi + \frac{\theta^2}{2} \epsilon^{abc} \partial_c \phi \partial_b \phi + \mathcal{O}(\theta^3)$$

then (5) is solved up to order $\theta^3$ in the interior of $\mathcal{M}$. We see in (6) that $\phi$ is defined up to the addition of an arbitrary function of time $\phi \rightarrow \phi + g(t)$, and that its derivatives account for the deviation of the particles from the classical center of their motion.

At the edge, we take (6) as a parametrization of $x^a$, which is a solution of the constraint up to subleading contributions containing derivatives of $\theta$.

Replacing this solution in the action (3) we obtain

$$S = \int dt d^2 y \rho_0 \nu \frac{eB}{2} \epsilon_{ab} \partial_i \phi \partial_i \phi$$

We reintroduce the effect of the electric field by boosting back $\tilde{y}^i = \tilde{y}^i - v^i t$, with the velocity defined above. We end up with the Lagrangian description of a fluid, where each fluid element is labelled with its initial position and it is followed along its motion through the plane. Here the velocity is a function of $y^i$, almost constant within $\mathcal{E}$ and vanishing outside the sample.

In the special case in which $\nu, \theta$ are constants the action is a total derivative. The usual procedure is to assume this and then impose a boundary to the region in which the fluid moves. We choose a different way here, allowing the parameters to change in space and introducing
through them the information about the geometry, without adding any boundary. A further partial integration gives

\[ S = -\frac{1}{8\pi^2} \int dt \, dy \, \dot{\phi}^2 (\partial_j (v^{-1}) \phi \partial_i (\partial_t - v^a \partial_a) \phi + + \frac{1}{3} \partial_j (\theta v^{-1}) \epsilon^{bc} \partial_b (\partial_t - v^a \partial_a) \phi \partial_i \phi \partial_c \phi) \]  
\number{8}

Here we see that, wherever the sample is homogeneous, the derivatives in front of each term vanish, leaving us without any dynamics. On the other hand, the dynamical degrees of freedom localize at places where there is a change in the properties of the material (noncommutativity and/or filling fraction). This is precisely what happens at the edge of the sample, and it is at the core of our argument.

To stress this point, let us suppose that the parameters change as a step function, then their derivatives will provide delta functions leading us to a one dimensional boundary theory. But we see that our treatment is more general in the sense that it includes the possibility of smooth edges.

The action \number{8} is written completely in terms of the dynamical degrees of freedom \( \phi \). Note that this is not a boundary theory because it is defined in the full two dimensional space. Nevertheless, the degrees of freedom are bounded to the region \( \mathcal{S} \) in which there is a change in the parameters \( \nu \) and \( \theta \) (i.e. in the density), and propagate chirally along their level surfaces.

To proceed further, we define a strip in this setup by the condition that the parameters are independent of one of the cartesian coordinates \( y' = (x, y) \), say \( x \). Then the action becomes

\[ S = -\frac{1}{8\pi^2} \int dt \, dx \, dy \left( \partial_y (v^{-1}) \phi \partial_x (\partial_t - v \partial_x) \phi + + \frac{1}{2} \partial_y (\theta v^{-1}) \partial_y \phi (\partial_t - v \partial_x) (\partial_x \phi)^2 \right) \]  
\number{9}

To recover the discrete structure in the direction \( y \) transverse to the edge, we proceed as follows:

- Cut the space in the \( y \) direction in slices at \( y_n \) of width \( \Delta y \sim a \).
- Replace the integral by a sum over the values of the integrand evaluated at each slice, and define \( \phi_n(x) \equiv \phi(x, y_n) \).
- Replace all the \( y \) derivatives by its finite difference approximation \( \partial_y \phi(x, y_n) = (\phi(x, y_n) - \phi(x, y_{n-1})) / \Delta y \).

With this method we obtain an effective theory for the fields \( \phi_n(x) \) living on each slice \( n \), with action

\[ S = -\frac{1}{8\pi^2} \int dt \, dx \, \sum_{n=0}^{N} \left( \kappa_n L_F (\phi_n) + + \kappa'_n (L_S (\phi_n) + L_T (\phi_n, \phi_{n-1})) \right) \]  
\number{10}

where we have defined the free, self interaction and interaction Lagrangians as

\[ L_F (\phi_n) = \phi_n \partial_x (\partial_t - v_n \partial_x) \phi_n \]
\[ L_S (\phi_n) = \phi_n (\partial_t - v_n \partial_x) (\partial_x \phi_n)^2 \]
\[ L_T (\phi_n, \phi_{n-1}) = -\phi_{n-1} (\partial_t - v_{n-1} \partial_x) (\partial_x \phi_{n-1})^2 \]  
\number{11}

and the constants \( \kappa_n \) and \( \kappa'_n \) are given by

\[ \kappa_n = \frac{1}{v'_n} - \frac{1}{v_{n-1}}, \quad \kappa'_n = \frac{1}{2\Delta y} \left( \frac{\theta_n}{v_n} - \frac{\theta_{n-1}}{v_{n-1}} \right) \]  
\number{12}

It is important to note that the field \( \phi_n \) will enter into the action only when the constants \( \kappa_n, \kappa'_n \) are non-vanishing, i.e. if there is a change in the properties of the sample between the slices \( n \) and \( n + 1 \).

The gauge invariance is now \( \phi_n \rightarrow \phi_n + g(t) \) adding the same \( g(t) \) to all the \( \phi_n \).

Note that when the sample has a sharp edge, i.e. if the density is changing in a region whose width is smaller than the slicing length \( W < \Delta y \sim a \), then the whole procedure is not applicable and the boundary theory corresponds to the usual chiral boson theory.

Let us suppose that the edge is wider than the slicing length, so that we have to keep a finite number of terms of the sum in \number{10}, \( n = 1, \cdots, N \).

We have then a single non-dynamical field \( \psi \equiv \phi_{-1} \) in the interaction term \( L_T (\phi_0, \phi_{-1}) \), all other fields appearing in the \( L_T (\phi_n, \phi_{n-1}) \) terms being dynamical. The integration of this multiplier field will enforce a constraint on the field \( \phi_0 \) to which it is coupled. When solved, \( \phi_0 = f(x + v_0 t) \), and replaced in the action, it implies that the interaction term with \( \phi_1 \) takes the form

\[ -\frac{1}{8\pi^2} \kappa'_1 \int dt \, dx \, f(x + v_0 t) (\partial_t - v_1 \partial_x) (\partial_x \phi_1)^2 \]  
\number{13}

which is different from zero provided that the two velocities are not equal. Since we have assumed that the electric field in the edge zone is approximately constant in regions of size \( a \sim \Delta y \), hence \( v_1 \approx v_2 \) and we can discard this term within the present approximation.

We have then obtained an action very similar to the original one but without any Lagrange multiplier

\[ S = -\frac{1}{8\pi^2} \int dt \, dx \, \sum_{n=1}^{N} \left( \kappa_n L_F (\phi_n) + \kappa'_n (L_S (\phi_n) + + (\delta n - 1) L_T (\phi_n, \phi_{n-1})) \right) \]  
\number{14}
where the lower bound of the sum has changed and the coefficient \((\delta_{n1} - 1)\) ensures that there is no interaction term for the field \(\phi_1\).

Let us consider the case of an almost sharp boundary, in which the density can be well approximated by choosing \(N = 1\). In that case the action is

\[
S = -\frac{1}{8\pi^2} \int dt dx \left( k \phi \partial_x (\partial_t - v \partial_x) \phi + \kappa' \phi (\partial_t - v \partial_x)(\partial_x \phi)^2 \right) \tag{15}
\]

where we have dropped the subindexes since we have a single field. This is a chiral boson theory with a cubic higher derivative self interaction, the latter arising essentially from the smoothness of the edge.

To see the effect of this last term in the physical properties, we compute the propagator since it is directly related to the tunnelling exponent \([2]\).

In Fourier space we write for the Feynmann propagator

\[
G(p, \omega_p) = \frac{2\pi}{\kappa p (\omega_p - vp) + i\epsilon} \tag{16}
\]

while the vertex proportional to

\[-k'\delta(p + q + r)(\omega_p - vp) q r \tag{17}\]

Then, the one loop corrected propagator reads

\[
G'(p, \omega_p) = \frac{2\pi}{\kappa p (\omega_p - vp) - \delta G(p, \omega_p)} \tag{18}
\]

We consider those Feynmann diagrams whose contribution to the corrected propagator is

\[
\delta G(p, \omega_p) = \pi \lambda^2 \frac{\kappa^2}{\kappa'^2} (\omega_p - vp) p \tag{19}
\]

where \(\lambda\) is a momentum cutoff. Replaced in (16) this gives

\[
G(p, \omega_p) = \frac{1}{(\kappa + \delta \kappa)p (\omega_p - vp)} \tag{20}
\]

where the non-universal correction to the level is given by

\[
\delta \kappa = -\pi \frac{\kappa^2}{\kappa'^2} \Lambda^2 \tag{21}
\]

A non vanishing correction to the velocity arises from the remaining diagrams but it produces no physical consequences.

Since the tunnelling exponent is directly related to the level by \(\alpha \approx \kappa\), the appearance of a correction for the latter implies, to first order, a correction to the tunnelling exponent. This is our main result.

To estimate the magnitude of this correction, we need to relate the momentum cutoff \(\Lambda\) to the minimal space distance measurable \(a\), which naturally leads to the identification \(\Lambda = \pi/a\). Using (22) we then get for this choice of the cutoff

\[
\delta \kappa = -\frac{\pi}{(4\alpha^2 \rho_0)^2} = -\frac{\pi}{16} \approx -0.196 \tag{22}
\]

where we have further identified \(\rho_0 = 1/a^2\). The predicted dependence on the density of this non-universal correction could in principle be tested experimentally.

Putting all together this leads to a linear dependence of the tunnelling exponent \(\alpha\) as a function of \(1/\nu\) which is very close to the experimental fit presented in \([3]\) (see for example Fig. 3 of this reference). It can be observed that the departure of our result from the experimental data becomes greater for increasing \(1/\nu\), which could be attributed to the need to work with \(N > 1\) in \([12]\). This could in turn be related to the need to consider a wider boundary region \(W > a\). The analysis of the higher \(N\) case will be presented separately \([10]\).

To summarize, we have constructed a boundary effective field theory for a fractional quantum Hall effect droplet by including the granularity of the fluid (the interparticle distance \(a\) and the effects of a smooth boundary, starting from the microscopic model. By further assuming a space dependent density to define the boundary region, we have computed the correction to one loop order of the tunnelling exponent which is good agreement with experimental results for the case of an almost sharp boundary.

Wider boundary regions can be easily treated within the present approach, by simply taking higher values of \(N\) in (14).

Other approaches to include a boundary in a Chern-Simons description of a quantum Hall droplet have been proposed \([11]\). In this context, it would be interesting to study the connection between the two approaches.

It should be mentioned that the experimental data in \([3]\) was reanalyzed in \([6]\) in view of the results of \([8]\), and certain degree of agreement with the plateau structure was obtained. Here we propose an alternative description which provides a reasonable agreement with the raw experimental data as presented in \([3]\). It would be interesting to study the higher order theory that we have constructed in the present Letter along the lines of the approach presented in ref. \([8]\). This could provide a closer agreement between theoretical and experimental results.

It would be also interesting to analyze the consequences of the higher order corrections induced by the smoothness of the boundary that we have obtained here in the transition between plateaux states.

The approach proposed in this note has been formalized carefully starting from a microscopic description and developed further in \([12]\).
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[13] This procedure may have some connection with the so-called boundary reconstruction effect, which has been studied from both the experimental and analytical point of views in [7].