Self-Localized Solutions of the Kundu-Eckhaus Equation in Nonlinear Waveguides

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In this paper we numerically analyze the 1D self-localized solutions of the Kundu-Eckhaus equation (KEE) in nonlinear waveguides using the spectral renormalization method (SRM) and compare our findings with those solutions of the nonlinear Schrödinger equation (NLSE). We show that single, dual and N-soliton solutions exist for the case with zero optical potentials, i.e. V = 0. We also show that these soliton solutions do not exist, at least for a range of parameters, for the photorefractive lattices with optical potentials in the form of V = L cos^2(x) for cubic nonlinearity. However, self-stable solutions of the KEE with saturable nonlinearity do exist for some range of parameters. We compare our findings for the KEE with those of the NLSE and discuss our results.

I. INTRODUCTION

The study of nonlinear wave equations appear everywhere in applied mathematics and theoretical physics including engineering and bio-sciences. These equations provide good examples of dynamical systems which possess diverse phenomena. The list of this diverse phenomena include but are not limited to solitary waves, rogue waves, the formation of singularities, dispersive turbulence and the propagation of chaos, just to name a few. Nonlinear waves occur in physical and natural systems and the studies of nonlinear optics and fiber optics, water and atmospheric waves, and turbulence in hydrodynamics and plasmas represent their important applications.

Some well-known nonlinear partial differential equations are the Korteweg-de Vries equation, nonlinear Klein-Gordon equation, nonlinear Schrödinger equation (NLSE) etc. In this paper we study the self-localized solutions of the Kundu-Eckhaus equation (KEE), which is an NLSE like equation. Widely accepted form of the KEE has two additional terms compared to the cubic NLSE. These terms are the quintic nonlinear term which accounts for the higher order nonlinearity and the Raman-effect term which accounts for the self-frequency shift of the waves.

KEE equation can adequately model the propagation of ultrashort pulses in nonlinear and quantum optics, which can possibly be used to describe the optical properties of the femtosecond lasers and can be used in femtochemistry studies. In mechanics, KEE is capable of examining the stability of Stokes waves in weakly nonlinear dispersive media. In plasma physics it can be used to model ion-acoustic waves. Some extensions of the NLSE, similar to the form of the KEE, where quintic nonlinearity is not included but third order dispersion and gain and loss terms are included are also used as models in the soliton-similariton laser studies [1,2]. Some analytical solutions of the KEE, including self-localized solutions of sech type, are analytically obtained by utilizing different techniques such as Darboux and Backlund transformations, the first integral method and exp-function method [1,3]. While single and dual self-localized solitons are derived, to our best knowledge it remains an open question if KEE admits N-soliton solutions.

With these motivations, in this paper we study the self-localized solutions of the KEE numerically. For this purpose we implement spectral renormalization scheme to investigate the self-localized solutions of the KEE. We compare our finding with their NLSE counterparts and discuss our results.

II. METHODOLOGY

A. Spectral Renormalization Method for the Kundu-Eckhaus Equation

Self-localized solutions of many nonlinear systems can be found by different computational techniques. These include but are not limited to shooting, self-consistency and relaxation [4,5]. One of the most popular methods is the Petviashvili’s method. In Petviashvili’s method, the governing nonlinear equation is transformed into Fourier space as in the case of general Fourier spectral schemes [6,7,23], and a convergence factor is determined according to the degree of the nonlinear term [6,27]. This method was first introduced by Petviashvili and applied to the Kadomtsev-Petviashvili equation [27]. Later, it has been applied to many other systems for modeling many different phenomena such as dark and gray solitons and lattice vortices, just to name a few [7,28]. Petviashvili’s method works well for nonlinearities with fixed homogeneity only therefore this method is extended to spectral renormalization method (SRM), which is can be used to find the localized solutions in waveguides with other types of nonlinearities [6,29]. Later another extension which is known as compressive spectral renormalization method (CSRM) is proposed [8], in order to obtain stable self-localized solutions in nonlinear waveguides with missing spectral data.

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The SRM essentially transforms the governing equation into wavenumber space by means of Fourier transform and couples it to a nonlinear integral equation. This nonlinear integral equation is basically an energy conservation principle used in the iterations in the wavenumber space \([7]\). Due to this coupling, the initial conditions converge to the self-localized solutions of the nonlinear system modeled \([2]\). SRM is efficient, is easy to implement and it can be applied to many different dynamic nonlinear models with different higher-order nonlinearities \([7]\).

In this section we apply the SRM to the KEE to obtain its self-localized solutions in waveguides. We start with the KEE in the form of

\[
\hat{\xi}_z + \xi_{xx} + 2 |\xi|^2 \xi + \beta^2 |\xi|^4 \xi - 2 \beta i \left( |\xi|^2 \right)_x \xi - V(x) \xi = 0
\] (1)

where \(z\) is the propagation direction of optical pulse, \(x\) is the transverse coordinate, \(i\) denotes the imaginary number and \(\xi\) is complex amplitude of the optical field. Eq. (1) can be rewritten as

\[
i\xi_z + \xi_{xx} - V(x) \xi + N(|\xi|^2) \xi = 0
\] (2)

where \(N(|\xi|^2) = 2 |\xi|^2 + \beta^2 |\xi|^4 - 2 \beta i \left( |\xi|^2 \right)_x\). Using the ansatz, \(\xi(x, z) = \eta(x, \mu) \exp(i \mu z)\), where \(\mu\) shows the soliton eigenvalue, the KEE becomes

\[-\mu \eta + \eta_{xx} - V(x) \eta + N(|\eta|^2) \eta = 0\] (3)

Furthermore taking the 1D Fourier transform of \(\eta\) one can obtain

\[
\hat{\eta}(k) = F[\eta(x)] = \int_{-\infty}^{+\infty} \eta(x) \exp(i k x) dx
\] (4)

For a zero optical potential, \(V = 0\), the Fourier transform of Eq. (3) in 1D yields to

\[
\hat{\eta}(k) = \frac{F \left[ N(|\eta|^2 \eta) \right]}{\mu + |k|^2}
\] (5)

The formula given in Eq. (5) may be applied iteratively to find the self-localized solutions of the model equation. This procedure was proposed by Petviashvili in \([27]\) for the first time. However the iterations of Eq. (5) may grow unboundedly or it may tend to zero as discussed in \([7]\). This problem can be addressed by introducing a new variable in the form \(\eta(x) = \alpha \xi(x)\). The 1D Fourier transform of this new variable reads \(\hat{\eta}(k) = \alpha \hat{\xi}(k)\). Using these substitutions, Eq. (5) becomes

\[
\hat{\xi}(k) = \frac{F \left[ N(|\alpha|^2 |\xi|^2) \xi \right]}{\mu + |k|^2} = R_\alpha \hat{\xi}(k)
\] (6)

and corresponding iteration scheme can be written as

\[
\hat{\xi}_{j+1}(k) = \frac{F \left[ N(|\alpha_j|^2 |\xi_j|^2) \xi_j \right]}{\mu + |k|^2}
\] (7)

An algebraic condition on the parameter \(\alpha\) can be obtained using the energy conservation principle for the normalization part of the SRM. By multiplying both sides of Eq. (6) with the \(\hat{\xi}^*(k)\), which is the complex conjugate of \(\xi(k)\), and integrating to evaluate the total energy, one can obtain the algebraic condition as

\[
\int_{-\infty}^{+\infty} \left| \hat{\xi}(k) \right|^2 dk = \int_{-\infty}^{+\infty} \hat{\xi}^*(k) R_\alpha \hat{\xi}(k) dk
\] (8)

which is the normalization constraint. This constraint guarantees that the scheme converges to a stable self-localized solitons. The procedure of obtaining self-localized solutions of a nonlinear system, which is applied to KEE in this paper, summarized above is known as the SRM \([7]\). Using a single or multi-Gaussians as initial conditions, Eq. (5) and the normalization constraint given in Eq. (8) is applied iteratively to find the profile for each iteration count. Iterations are continued until the parameter \(\alpha\) convergences.

Returning to a more general setting, the nonzero potentials \((V \neq 0)\) are widely used as models for various optical media i.e. nondefected or defected photonic crystals. To avoid singularity of the scheme, once can add and subtract a \(p \eta\) term with \(p > 0\) from Eq. (3) \([7]\). Then the 1D Fourier transform of Eq. (3) becomes

\[
\tilde{\eta}(k) = \frac{(p + |\mu|) \tilde{\eta}}{p + |k|^2} - \frac{F[V \eta] - F \left[ N(|\eta|^2 \eta) \right]}{p + |k|^2}
\] (9)

which is the iteration scheme for the KEE with a nonzero optical potential. In this paper we are specifically interested in photorefractive solitons of the KEE which are of practical use. Therefore, considering the 1D versions of the photorefractive solitons of the NLSE like equation first reported in \([30]\), we set the optical potential as \(V = I_o \cos^2(x)\) and the nonlinear term as \(N(|\eta|^2) = -1/(1 + 2 |\eta|^2)\) for the cubic nonlinearity and \(N(|\eta|^2) = -1/(1 + 2 |\eta|^2 + \beta^2 |\eta|^4 - 2 \beta i \left( |\eta|^2 \right)_x\) for the saturable nonlinearity. As before, we can define a new parameter \(\eta(x) = \alpha \xi(x)\) and find its Fourier transform as \(\hat{\eta}(k) = \alpha \hat{\xi}(k)\). Using these substitutions iteration formula for saturable nonlinearity becomes

\[
\hat{\xi}_{j+1}(k) = \frac{(p + |\mu|) \hat{\xi}_j - F[V \xi_j]}{p + |k|^2} + \frac{1}{p + |k|^2} \left( F \left[ 1 + 2 |\alpha_j|^2 |\xi_j|^2 + \beta^2 |\alpha_j|^4 |\xi_j|^4 - 2 \beta i F^{-1}[ik F[|\alpha_j|^2 |\xi_j|^2]] \right] \right)
\] (10)

The algebraic condition SRM for nonzero potentials can be attained by multiplying both sides of Eq. (10) with
\(\hat{\xi}^*(k)\) and integrating to evaluate the total energy, which leads to the normalization constraint as

\[
\int_{-\infty}^{+\infty} |\hat{\xi}(k)|^2 dk = \int_{-\infty}^{+\infty} |\hat{\xi}^*(k)R_0(\xi(k))|dk
\]  

(11)

As in the case of zero potentials, an initial condition in the form of a single or multi-Gaussians converges to self-localized states of the model equation when Eq. (10) and Eq. (11) are applied iteratively. Iterations can be performed until the parameter \(\alpha\) converges with a specified upper error bound. The reader is referred to [7] for a more comprehensive discussion and application of SRM to NLSE like equation and to second-harmonic generation analysis. We present our results for the KEE in the next section.

III. RESULTS AND DISCUSSION

A. Self-Localized Soliton Solutions of the KEE for Zero Optical Potential

First we concentrate on the KEE with no optical potential, which can be obtained by setting \(V = 0\). The nonlinear term for this case is taken as \(N(\xi^4) = 2|\xi|^2 + \beta^2|\xi|^4 - 2\beta i \left(\frac{|\xi|^2}{\pi}\right)\) in Eq. (2). The parameters of the computations are selected as \(p = 10, \mu = 1, I_0 = 0.1\).

In Figure 1, we compare the single humped self-localized soliton solution of the KEE with its NLSE counterpart. For this numerical solution we use \(N = 2048\) spectral components and define the convergence as the normalized change of \(\alpha\) to be less than \(10^{-15}\). The initial condition for this simulation is simply a Gaussian in the form of \(\exp(-(x-x_0)^2)\) where \(x_0\) is taken as 0. SRM converges to the exact single sech type soliton solution presented in [7] within few iteration steps. The self-localized soliton solution of the KEE is more slender, that is its peak value is achieved for a narrower profile which can be measured using -3 dB mainlobe width, compared to its NLSE counterpart. This is expected since the quintic nonlinear term in the KEE leads to higher order nonlinearity of the solutions.

In Figure 2, the dual humped self-localized soliton solution of the KEE with its NLSE counterpart are compared. Again, for this numerical solution we use \(N = 2048\) spectral components. The convergence is defined as the normalized change of \(\alpha\) to be less than \(10^{-15}\) as before. The initial condition for this simulation is simply two Gaussians in the form of \(\exp(-(x-x_0)^2) + \exp(-(x-x_1)^2)\) where \(-x_0 = x_1 = 10\). SRM converges to the dual sech type soliton solution within few iteration steps. The self-localized soliton solution of the KEE are more slender, that is its peak value is achieved for a narrower profile which can be measured using -3 dB mainlobe width, compared to its NLSE counterpart for which analytical expressions are known. Again this is an expected result due to higher order nonlinearity of the KEE compared to the NLSE.

In Figure 3, the self-localized soliton solution of the KEE with 4 humps with its NLSE counterpart are compared. Same number of spectral components are used as before. The convergence is defined as the normalized change of \(\alpha\) to be less than \(10^{-7}\) for this simulation. The initial condition for this simulation is simply four Gaussians in the form of \(\exp(-(x-x_0)^2) + \exp(-(x-x_1)^2) + \exp(-(x-x_2)^2) + \exp(-(x-x_3)^2)\) where \(x_0 = 0, x_1 = 10, x_2 = -10, x_3 = -30\). SRM converges to the sech type soliton solution with 4 humps within few iteration steps. The self-localized soliton solu-
tion of the KEE for this case is more slender as before. If we add more Gaussians (N-Gaussians) in the initial condition we observe that SRM converges to self-localized N-soliton solutions. Our findings support that KEE with no potential admits analytical N-soliton solutions.

B. Self-Localized Soliton Solutions of the KEE for Nonzero Optical Potential

Secondly, we concentrate on the KEE with an optical potential in the form of $V = I_o \cos^2(x)$. The nonlinear term for this case is again cubic-quintic-Raman nonlinearity, $N(|\xi|^2) = 2|\xi|^2 + \beta^2 |\xi|^4 - 2\beta i \left(|\xi|^2\right)_x$ which is used in practice [30]. The iteration formula becomes Eq. (10). Calculations are performed for $p = 10, \mu = 1, I_0 = 0 − 10$. Within the range of $I_0 \approx 4 − 5.5$ stable self-localized solitons of the KEE are observed, while within the same range there is no such solitons of the NLSE.

Turning our attention to the KEE-like equation given in Eq. (2) for a nonzero optical potential of $V = I_o \cos^2(x)$ and the saturable nonlinear term given as $N(|\xi|^2) = -1/(1 + 2|\xi|^2 + \beta^2 |\xi|^4 - 2\beta i \left(|\xi|^2\right)_x)$, which is used in practice [30], the iteration formula becomes Eq. (10). Calculations are performed for $p = 10, \mu = 1, I_0 = 0 − 10$. Within the range of $I_0 \approx 4 − 5.5$ stable self-localized solitons of the KEE are observed, while within the same range there is no such solitons of the NLSE.

A self-localized solution of the KEE with saturable nonlinearity for $I_o = 4$ is depicted in Figure 4 with the convergence criteria of $\alpha$ to be less than $10^{-15}$. This soliton is similar to the solitons of the NLSE with saturable nonlinearity observed in practice and given in [30]. Stability and propagation characteristics of these solitons which rely on power calculations can be the subject of future research.

IV. CONCLUSION AND FUTURE WORK

In this paper we have numerically analyzed the 1D self-localized solutions of the KEE in nonlinear waveguides using the SRM and have compared our findings with those solutions of the nonlinear Schrödinger equation. We showed that single, dual and N-soliton solutions of the KEE do exist for the case with zero optical potential, i.e. $V = 0$. We have also showed that single, dual and N-soliton solutions of the KEE do not exist, for at least a range of parameters, for $V = I_o \cos^2(x)$ type potentials for cubic nonlinearity. However self-stable solutions of the KEE with saturable nonlinearity do exist for some range of parameters. Our results can be used in the derivation of the analytical N-soliton solutions, as well as studying the propagation and stability characteristics of the pulses in nonlinear optics and physics.
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