ESTIMATES OF THE BERGMAN DISTANCE ON
DINI-SMOOTH BOUNDED PLANAR DOMAINS

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Abstract. Precise estimates for the Bergman distances of Dini-smooth bounded planar domains are given. These estimates imply that on such domains the Bergman distance almost coincides with the Carathéodory and Kobayashi distances.

1. Results

In [7, Proposition 8], the first named author found optimal estimates for Carathéodory and Kobayashi distances, $c_D$ and $k_D$, on Dini-smooth bounded planar domains $D$. In this paper we shall prove similar estimates for the Bergman distance $b_D$. For convenience of the reader, the definitions of these three distances, as well as of Dini-smoothness, are given in the next section.

Proposition 1. Let $D$ be a Dini-smooth bounded planar domain. Then there exists a constant $c > 1$ such that

$$\sqrt{2} \log \left(1 + \frac{|z - w|}{c \sqrt{d_D(z)d_D(w)}}\right) \leq b_D(z, w) \leq \sqrt{2} \log \left(1 + \frac{c |z - w|}{\sqrt{d_D(z)d_D(w)}}\right), \quad z, w \in D.$$ 

By [7, Proposition 8], the same result holds for $\sqrt{2} c_D$ and $\sqrt{2} k_D$ instead of $b_D$. So, we have the following

Corollary 2. If $D$ is a Dini-smooth bounded planar domain, then the differences $b_D - \sqrt{2} c_D$ and $b_D - \sqrt{2} k_D$ are bounded.

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Note that Proposition 1 is equivalent to

**Proposition 1’.** Let $D$ be a Dini-smooth bounded planar domain. There exists a constant $c > 1$ such that:

- if $|z - w|^2 > d_D(z)d_D(w)$, then
  \[ \log \frac{|z - w|^2}{d_D(z)d_D(w)} - c < \sqrt{2}b_D(z, w) < \log \frac{|z - w|^2}{d_D(z)d_D(w)} + c; \]
- if $|z - w|^2 \leq d_D(z)d_D(w)$, then
  \[ \frac{|z - w|}{c\sqrt{d_D(z)d_D(w)}} \leq b_D(z, w) \leq \frac{c|z - w|}{\sqrt{d_D(z)d_D(w)}}. \]

**Remark.** (a) The Dini-smoothness is essential as an example of a $C^1$-smooth bounded simply connected planar domain shows (see [6, Example 2]).

(b) One of the missing properties of $b_D$ in comparison with $c_D$ and $l_D$ is monotonicity under inclusion of (planar) domains. However, the invariants $M_D$ and $K_D$ share this property which allows us to modify the approach from [7].

(c) Results in $\mathbb{C}^n$ in the spirit of Proposition 1 and Corollary 3 can be found in [1] and [3], respectively, where the strictly pseudoconvex domains are treated. Note also that the Levi pseudoconvex corank one domains are considered in [3]. As can be expected, our estimates are more precise than these in [1] and [3], when the two points, $z$ and $w$, are close to each other.

(d) It follows by the second statement of Proposition 1’ that

\[ \frac{1}{cd_D(u)} \leq \liminf_{z, w \to u \atop z \neq w} \frac{b_D(z, w)}{|z - w|}, \quad u \in D. \]

This inequality agrees with the fact that (cf. [5] Lemma 4.3.3 (e)])

\[ \limsup_{z, w \to u \atop z \neq w} \frac{b_D(z, w)}{|z - w|} \leq \beta_D(u; 1) \]

(cf. [5] Lemma 4.3.3 (e)]) and the equality (see [4] Remark, p. 11)

\[ \lim_{u \to \partial D} \beta_D(u; 1)d_D(u) = \frac{\sqrt{2}}{2}. \]

Recall now another comparison result between $c_D$ and $k_D$ (see [7], Proposition 9): if $D$ is a finitely connected bounded planar domain
without isolated boundary points, then

\[ \lim_{w \to \partial D, z \neq w} \frac{c_D(z, w)}{k_D(z, w)} = 1 \text{ uniformly in } z \in D. \]

Similar results for \( c_D, k_D, l_D \) and \( b_D \) in the strictly pseudoconvex case can be found in [9, Theorem 1] and [8, Proposition 4].

The next proposition shows that (11) remains true if we replace \( c_D \) or \( k_D \) by \( b_D / \sqrt{2} \).

**Proposition 3.** If \( D \) is a finitely connected bounded planar domain without isolated boundary points, then

\[ \lim_{w \to \partial D, z \neq w} \frac{b_D(z, w)}{c_D(z, w)} = \lim_{w \to \partial D, z \neq w} \frac{b_D(z, w)}{k_D(z, w)} = \sqrt{2} \text{ uniformly in } z \in D. \]

**Remark.** The isolated boundary points condition is essential. Indeed, if \( p \) is an isolated boundary point of a planar domain \( D \neq \mathbb{C} \setminus \{p\} \), then \( c_D = c_D \cup \{p\} \) and \( b_D = b_D \cup \{p\} \), but \( k_D(z, w) \to \infty \) as \( w \to p \) and \( z \in D \) is fixed.

2. Definitions

1. A boundary point \( p \) of a planar domain \( D \) is said to be Dini-smooth if \( \partial D \) near \( p \) is given by a Dini-smooth curve \( \gamma : [0, 1] \to \mathbb{C} \) with \( \gamma' \neq 0 \) (i.e., \( \int_0^1 \frac{\omega(t)}{t} dt < \infty \), where \( \omega \) is the modulus of continuity of \( \gamma' \)). A planar domain is called Dini-smooth if all its boundary points are Dini-smooth.

2. Let \( D \) be a domain in \( \mathbb{C}^n \).

The Bergman distance \( b_D \) of \( D \) is the integrated form of the Bergman metric \( \beta_D \), i.e.,

\[ b_D(z, w) = \inf_{\gamma} \int_0^1 \beta_D(\gamma(t); \gamma'(t)) dt, \quad z, w \in D, \]

where the infimum is taken over all smooth curves \( \gamma : [0, 1] \to D \) with \( \gamma(0) = z \) and \( \gamma(1) = w \).

Recall that

\[ \beta_D(z; X) = \frac{M_D(z; X)}{K_D(z)}, \quad z \in D, \ X \in \mathbb{C}^n, \]

where

\[ M_D(z; X) = \sup \{ |f'(z)X| : f \in L^2_h(D), \|f\|_{L^2(D)} \leq 1, \ f(z) = 0 \} \]

\( 1 \)Any \( C^1 \)-smooth bounded planar domain is such a domain.
and
\[ K_D(z) = \sup\{|f(z)| : f \in L^2_h(D), \|f\|_{L^2(D)} \leq 1\} \]
is the square root of the Bergman kernel on the diagonal (we assume that \( K_D > 0 \); for example, this holds if \( D \) is bounded).

The Carathéodory distance \( c_D \) and the Lempert function \( l_D \) of \( D \) are defined as follows:
\[ c_D(z, w) = \sup\{\tanh^{-1}|f(w)| : f \in \mathcal{O}(D, \mathbb{D}), \text{ with } f(z) = 0\}, \]
\[ l_D(z, w) = \inf\{\tanh^{-1}|\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \varphi(\alpha) = w\}, \]
where \( \mathbb{D} \) is the unit disc.

The Kobayashi distance \( k_D \) is the largest pseudodistance not exceeding \( l_D \). It is well-known that \( k_D \) is the integrated form of Kobayashi metric \( \kappa_D \) defined by
\[ \kappa_D(z; X) = \inf\{|\alpha| : \exists \varphi \in \mathcal{O}(D, D) \text{ with } \varphi(0) = z, \alpha \varphi'(0) = X\}. \]

Note that \( c_D \leq k_D \leq l_D \) and \( c_D \leq b_D \). On the other hand, \( k_D = l_D \) for any planar domain \( D \) (cf. [5, Remark 3.3.8(e)]).

We refer to [5] for other basic properties of the above invariants.

3. Proofs

To prove Proposition 1, we shall need the following

**Lemma 4.**
\[ \log \left(1 + \frac{|z - w|}{\sqrt{(1 - |z|^2)(1 - |w|^2)}}\right) \leq \frac{b_D(z, w)}{\sqrt{2}} \]
\[ \leq k_D(z, w) \leq \log \left(1 + \frac{2|z - w|}{\sqrt{(1 - |z|^2)(1 - |w|^2)}}\right). \]

**Proof.** We have that
\[ \sqrt{2}b_D(z, w) = 2k_D(z, w) = \log \frac{1 + \frac{|z - w|}{1 - \bar{z}w}}{1 - \frac{|z - w|}{1 - \bar{z}w}} = \]
\[ \log \left(1 + \frac{2|z - w|}{|1 - \bar{z}w| - |z - w|}\right) = \log \left(1 + 2|z - w| \frac{|1 - \bar{z}w| + |z - w|}{(1 - |z|^2)(1 - |w|^2)}\right). \]
It remains to use that
\[ |1 - \bar{z}w|^2 = (1 - |z|^2)(1 - |w|^2) + |z - w|^2 \]
and hence
\[ \sqrt{(1 - |z|^2)(1 - |w|^2)} \leq |1 - \bar{z}w| \leq \sqrt{(1 - |z|^2)(1 - |w|^2) + |z - w|^2} \]
Remark. The constants 1 and 2 in front of $|z-w|$ in the lower and upper estimates in Lemma 4 are sharp. To see this, let 
\[ \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)} \rightarrow 0 \text{ and } \infty, \text{ respectively.} \]

Proof of Proposition 1. Let $D \supset (z_n) \rightarrow p$ and $D \supset (w_n) \rightarrow q$. It is enough to find a constant $c > 1$ such that the respective estimates for $b_D(z_n, w_n)$ hold for any $n$.

Note that, by [7] Proposition 5 and Corollary 6, for any neighborhood $U$ of $p$ there exist a neighborhood $V \subset U$ and a constant $c_1 > 0$ such that

\[ (2) \ |\sqrt{2}b_D(z, w) + \log d_D(z) + \log d_D(w)| < c_1, \quad z \in D \cap V, w \in D \setminus U. \]

This inequality provides the wanted constant if $D \ni p \neq q \in D$, or $p \in \partial D$, or $p \in D$, or $p \in D$, or $p \in \partial D \ni q \neq q \in \partial D$.

For a planar domain $\Omega$, set $\beta_\Omega(z) = \beta_\Omega(z; 1)$, $M_\Omega(z) = M_\Omega(z; 1)$ and $\kappa_\Omega(z) = \kappa_\Omega(z; 1)$.

If $p = q \in D$, then the continuity of $\beta_D$ implies that

\[ \frac{b_D(z_n, w_n)}{|z_n - w_n|} \rightarrow \beta_D(p) > 0 \]

and we may easily find the desired constant.

It remains to consider the most difficult case $p = q \in \partial D$. Some of our arguments will be closed to that in the proof of [7, Proposition 5].

This proof allows us to assume that $p = 1$ and

\[ \{ z \in \mathbb{D} : |z-1| < r \} =: E_r \subset D \subset \mathbb{D} \]

for some $r > 0$ (after an appropriate conformal map). Then

\[ (3) \ \sqrt{2}\frac{\kappa_D^2(z)}{\kappa_{E_r}(z)} = \frac{M_D(z)}{K_{E_r}(z)} \leq \beta_D(z) \leq \frac{M_{E_r}(z)}{K_D(z)} = \sqrt{2}\frac{\kappa_{E_r}^2(z)}{\kappa_D(z)}, \quad z \in E_r \]

(the both equalities hold because $E_r$ is a simply connected domain).

Fix an $r_1 \in (0, r)$. The localization of the Kobayashi metrics from [2] Theorem 2.1 and Lemma 2.2] implies that

\[ (4) \ \kappa_D(z) > (1 - c_2 d_D(z)) \kappa_{E_r}(z), \quad z \in E_{r_1}, \]

for some constant $c_2 > 0$. Choose an $r_2 \in (0, r_1]$ with $3c_2 r_2 \leq 1$. Then

\[ \sqrt{2}(1 - c_2 d_D(z)) \kappa_D(z) < \beta_D(z) < \sqrt{2}(1 + \frac{3}{2} c_2 d_D(z)) \kappa_D(z), \quad z \in E_{r_2} \]

Since $\kappa_D(z) = \frac{\beta_D(z)}{\sqrt{2}} = \frac{1}{1 - |z|^2}$, it follows for $c_3 = \frac{3}{2} c_2$ that

\[ (5) \ \frac{\beta_D(z)}{3} < \beta_D(z) - 2c_3 < \beta_D(z) < \beta_D(z) + 3c_3, \quad z \in E_{r_2}. \]
We may assume that $z_n, w_n \in E_{r_3}$, where $r_3 \in (0, r_2)$ is such that if $\alpha_n$ is the shorter arc with endpoints $z_n$ and $w_n$ of the circle through $z_n$ and $w_n$ which is orthogonal to the unit circle, then $\alpha_n \subset E_{r_2}$. Hence
\[
b_D(z_n, w_n) < \int_{\alpha_n} \left( \frac{\sqrt{2}}{1 - |z|^2} + 3c_3 \right) dl
\]
\[
= b_D(z_n, w_n) + 3c_3l(\alpha_n) < b_D(z_n, w_n) + 6c_3|z_n - w_n|
\]
for any $n$.

Now, using Lemma 4 and the inequalities
\[
1 - |z|^2 \geq 1 - |z| = d_D(z) \geq d_D(z), \quad z \in D,
\]
it is easy to find a constant $c > 1$ such that the upper estimate for $b_D(z_n, w_n)$ in Proposition 1 holds for any $n$.

It is left to manage with the lower estimate. Shrinking $r_3$ (if necessary), we may assume that
\[
1 - |z|^2 < 2d_D(z) = 2d_D(z), \quad z \in E_{r_3}.
\]

Consider the set $A$ of all $n$ for which there exists a smooth curve $\gamma_n : [0, 1] \to D$ such that $\gamma_n(0) = z_n$, $\gamma_n(1) = w_n$, $\gamma_n(0, 1) \not\subset D_{r_2}$ and
\[
b_D(z_n, w_n) + |z_n - w_n| > \int_0^1 \beta_D(\gamma_n(t); \gamma'_n(t)) dt.
\]
For any $n \in A$ we may find a number $t_n \in (0, 1)$ such that $|u_n - 1| = r_2$, where $u_n = \gamma(t_n)$. By (2), there exists a constant $c_4 > 0$, which does not depend on $n \in A$, such that
\[
b_D(z_n, w_n) + |z_n - w_n| > b_D(z_n, u_n) + b_D(u_n, w_n)
\]
\[
> - \frac{\log d_D(z_n)}{\sqrt{2}} - \frac{\log d_D(w_n)}{\sqrt{2}} - c_4.
\]
This inequality easily provides a constant $c > 1$ for which the lower estimate for $b_D(z_n, w_n)$ in Proposition 1 holds for any $n \in A$.

Let now $n \not\in A$. Then, using (3) and the formula
\[
\kappa_{B_2}^2(w; Y) = \frac{||Y||^2}{1 - ||w||^2} + \frac{||\langle w, Y \rangle||^2}{(1 - ||w||^2)^2},
\]
we get that
\[
b_D(z_n, w_n) + |z_n - w_n| \geq \sqrt{2}\kappa_{B_2}(z_n, w_n),
\]
where $\kappa_{B_2}$ is the pseudodistance arising from the Finsler pseudometric $\hat{\kappa}_{B_2}(w; Y) = (\kappa_{B_2}(w; Y) - 2c_2||Y||)^+$. Applying [1, Theorem 1.1] to $\kappa_{B_2}$
and $\hat{\kappa}_{B_2}$, we may find a constant $c_5 > 0$ such that $0 < k_{B_2} - \hat{\kappa}_{B_2} < c_5$.

It follows from here and $\beta_D = \sqrt{2k_{B_2}}|_{D \times \{0\}}$ that

$$b_D(z_n, w_n) + |z_n - w_n| > b_D(z_n, w_n) - \sqrt{2c_5}$$

which, together with Lemma 4 and (6), easily implies the lower estimate in Proposition 1 if $|z_n - w_n|^2 > d_D(z_n)d_D(w_n)$.

To prove the lower estimate in Proposition 1 when $n \not\in A$ and $|z_n - w_n|^2 \leq d_D(z_n)d_D(w_n)$, it suffices to observe that (5) leads to $3b_D(z_n, w_n) \geq b_D(z_n, w_n)$ and then to apply Lemma 4 and (6).

So, Proposition 1 is completely proved.

Proof of Proposition 3. By the K"{o}be uniformization theorem, we may assume that $\partial D$ consists of disjoint circles. Using Proposition 1, Corollary 2, (1) and compactness, it is enough to prove that

$$\lim_{z, w \rightarrow p} \frac{b_D(z, w)}{k_D(z, w)} = \sqrt{2}$$

for any point $p \in \partial D$.

Applying an inversion, we may assume that the outer boundary of $D$ is the unit circle and $p = 1$. Then (3) and (4) imply

$$\lim_{z \rightarrow 1} \frac{\beta_{E_r}(z)}{\beta_D(z)} = 1 = \lim_{z \rightarrow 1} \frac{\kappa_{E_r}(z)}{\kappa_D(z)}.$$  

The first equality shows that $\liminf_{z, w \rightarrow 1, z \neq w} \frac{b_{E_r}(z, w)}{b_D(z, w)} \geq 1$.

To get that

$$\limsup_{z, w \rightarrow 1, z \neq w} \frac{b_{E_r}(z, w)}{b_D(z, w)} \leq 1,$$

we shall follow the proof of [9, Proposition 3]. Fix an $\varepsilon > 0$ and choose an $r_1 \in (0, r)$ such that

$$\beta_{E_r}(z) < (1 + \varepsilon)\beta_D(z), \quad z \in E_{r_1}.$$  

Combining the argument in the case $n \not\in A$ from the previous proof and the estimates from Proposition 1, we may find an $r_2 \in (0, r_1)$ such that if $z, w \in E_{r_2}$ and $\gamma : [0, 1] \rightarrow D$ is a smooth curve for which $\gamma(0) = 1$, $\gamma(1) = w$ and

$$\int_0^1 \beta_D(\gamma(t); \gamma'(t))dt \leq (1 + \varepsilon)b_D(z, w),$$
then $\gamma([0,1]) \subset E_{r_1}$. It follows that
\[
b_{E_r}(z, w) \leq \int_0^1 \beta_{E_r}(\gamma(t); \gamma'(t)) dt
\]
\[
\leq (1 + \varepsilon) \int_0^1 \beta_D(\gamma(t); \gamma'(t)) dt \leq (1 + \varepsilon)^2 b_D(z, w), \quad z, w \in E_{r_2}.
\]
To obtain (7), it remains to let $\varepsilon \to 0$.

So, $\lim_{z, w \to 1 \atop z \neq w} b_{E_r}(z, w) = 1$.

On the other hand, [7, Proposition 8] gives the estimates from Proposition 1 for $2k_D$ instead of $\sqrt{2}b_D$. Then we get as above that
\[
\lim_{z, w \to 1 \atop z \neq w} \frac{\kappa_{E_r}(z, w)}{\kappa_D(z, w)} = 1.
\]
Now, the equality $b_{E_r} = \sqrt{2}k_{E_r}$ completes the proof.

REFERENCES

[1] Z. M. Balogh, M. Bonk, Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, Comment. Math. Helv. 75 (2000), 504–533.
[2] F. Forstneric, J.-P. Rosay Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings, Math. Ann. 279 (1987), 239–252.
[3] G. Herbort, Estimation of the Carathéodory distance on pseudoconvex domains of finite type, whose boundary has a Levi form of corank at most one, Ann. Polon. Math. 109 (2013), 209–260.
[4] M. Jarnicki, N. Nikolov, Behavior of the Carathéodory metric near strictly convex boundary points, Univ. Iag. Acta Math. XL (2002), 7–12.
[5] M. Jarnicki, P. Pflug, Invariant distances and metrics in complex analysis, de Gruyter Exp. Math. 9, de Gruyter, Berlin.
[6] N. Nikolov, P. Pflug, P. J. Thomas, Upper bound for the Lempert function of smooth domains, Math. Z. 266 (2010), 425–430.
[7] N. Nikolov, Estimates of invariant distances on "convex" domains, Ann. Mat. Pura Appl., DOI 10.1007/s10231-013-0345-7.
[8] N. Nikolov, Comparison of invariant functions on strongly pseudoconvex domains, J. Math. Anal. Appl., DOI: 10.1016/j.jmaa.2014.07.007.
[9] S. Venturini, Comparison between the Kobayashi and the Carathéodory distances on strongly pseudoconvex bounded domains in $\mathbb{C}^n$, Proc. Am. Math. Soc. 107 (1989), 725–730.
