Color Dynamics On Phase Space

A. Bonasera*
Laboratorio Nazionale del Sud, Istituto Nazionale di Fisica Nucleare, Via S. Sofia 44, 95123 Catania, Italy &
Cyclotron Institute, Texas A&M University, College Station, TX 77843-3366,USA.

Abstract

We describe the properties of quark matter at zero temperature and finite baryon densities within microscopic Vlasov/molecular dynamics approaches. We use an inter-quark Richardson’s potential consistent with the indications of Lattice QCD calculations. The color degrees of freedom are explicitly taken into account. We explicitly demonstrate that the Vlasov approach alone is insufficient in the hadronization region. In order to overcome this problem we prepare the initial condition for many events using molecular dynamics with frictional cooling and a Thomas-Fermi approximation to the Fermi motion. These events are averaged and propagated in time using the Vlasov approach. We find some evidence for a second order phase transition from nuclear to quark matter at high baryon densities. An order parameter suitable to describe the phase transition is discussed. At low densities the quark condensate into approximately color white clusters (nucleon).

PACS : 12.39.Pn 24.85.1p

*Bonasera@lns.infn.it
One of the open problems in theoretical nuclear and particle physics is how to obtain the well known nuclear properties starting from the quark degrees of freedom [1]. This also includes the possibility of understanding the basic free nucleon-nucleon interaction from quark and gluons dynamics. Some kind of solution to this problem is becoming more and more needed with the new experiments done or planned using ultra-relativistic heavy ions at CERN and soon at RHIC. The search for a quark-gluon plasma (QGP) in such collisions is in fact one of the new and most exciting directions in physics at the border between nuclear and particle physics [3]. Quantum ChromoDynamics (QCD) because of its difficulties (numerical and conceptual), has been applied so far to some limited cases such as quark matter at zero baryon ($\rho_B$) density and high temperature ($T$) [1,2]. Furthermore in relativistic heavy ion collisions (RHIC) dynamics plays surely an important role and accordingly the theory should be dynamical.

Recently [3], we have proposed a dynamical approach based on the Vlasov equation [4,5] to reproduce hadron masses and the properties of nuclear matter at finite $\rho_B$. Some works in the same spirit are discussed in [6,7]. Our approach needs as inputs the interaction potential among quarks, which was borrowed from phenomenology i.e. the Richardson’s potential [8], and the quark masses which were fitted to reproduce known meson masses such as the $\pi$, the $\phi$, the $\eta_c$ etc. When the particles are embedded in a dense medium such as in nuclear matter (NM) the potential becomes screened in a similar fashion as ions and electrons in condensed matter do, i.e. Debye screening (DS) [3,5].

It is the purpose of this paper to refine that approach in one important aspect which is the treatment of the color degrees of freedom. In the previous works [3] color degrees of freedom were implicitly taken into account through the use of a Debye radius that effectively screens the $qq$ interaction potential. In the present paper we give to the quarks explicitly a color (using the Gell-Mann matrices) and follow their dynamics in phase space solving the Vlasov equation (VE). Thus screening will be dynamically obtained. In general the self screening obtained in Vlasov dynamics is inadequate, which is the reason why it was not adopted in the earlier attempts [3]. In fact we will show explicitly that the Vlasov approach alone gives a good description of the system at large densities only, i.e. in the QGP region. In order to overcome this problem we adopted the following strategy. We first prepare the initial conditions using molecular dynamics (MD) with frictional cooling for many events. The events are averaged and care is taken of antisymmetrization. These are the initial conditions for the Vlasov evolution. Since the VE fulfills the Liouville theorem, the initial antisymmetrization and eventual clustering obtained in the cooling process are maintained.

We outline our approach on purely classical grounds, however the same results can be obtained within the Wigner transform formalism [4,5] of the quantum BBGKY-hierarchy in the limit $\hbar \to 0$.

The exact (classical) one-body distribution function $f_1(r,p,t)$ satisfies the equation (BBGKY hierarchy) [3]:

$$\partial_t f_1 + \frac{\vec{p}}{E} \cdot \nabla_r f_1 = \int d(2) \nabla_r V(r,r_2) \nabla_p f_2(r,r_2,p,p_2,t)$$  \hspace{1cm} (1)

$E = \sqrt{p^2 + m_i^2}$ is the energy and $m_i = 10 MeV$ is the (u,d) quark mass. Here we assume the potential to be dependent on the relative coordinates only. A generalization to include
a momentum dependent part is straightforward. \( f_2 \) is the two-body distribution function, which in the classical limit reads:

\[
f_2(\mathbf{r}, \mathbf{r}_2, \mathbf{p}, \mathbf{p}_2, t) = \sum_{\alpha \neq \beta} Q \delta(\mathbf{r} - \mathbf{r}_\alpha) \delta(\mathbf{p} - \mathbf{p}_\alpha) \times \delta(\mathbf{r}_2 - \mathbf{r}_\beta) \delta(\mathbf{p}_2 - \mathbf{p}_\beta)
\]

where \( Q = q + \bar{q} \) is the total number of quarks and anti quarks (\( \bar{q} = 0 \) in this work). Inserting this equation into Eq.(1) gives:

\[
\partial_t f_1 + \frac{\mathbf{p} \cdot E}{E} \cdot \nabla_\mathbf{r} f_1 - \nabla_\mathbf{r} U \cdot \nabla_\mathbf{p} f_1 = 0 \tag{3}
\]

Where \( U = \sum_j V(\mathbf{r}, \mathbf{r}_j) \) is the exact potential. Let us now define \( f_1 \) and \( U \) as sums of an ensemble averaged quantity plus the deviation from this average:

\[
f_1 = \bar{f}_1 + \delta f_1; \quad U = \bar{U} + \delta U \tag{4}
\]

Substituting these equations in Eq.(3) and ensemble averaging gives:

\[
\partial_t \bar{f}_1 + \frac{\mathbf{p} \cdot E}{E} \cdot \nabla_\mathbf{r} \bar{f}_1 - \nabla_\mathbf{r} \bar{U} \cdot \nabla_\mathbf{p} \bar{f}_1 = < \nabla_\mathbf{r} \delta U \nabla_\mathbf{p} \delta f_1 > \tag{5}
\]

where one recognizes in the lhs the Vlasov term and in the rhs the Balescu-Lennard collision term \([5,9]\). The mean-field is given by:

\[
\bar{U}(\mathbf{r}) = \frac{1}{N_{ev}} \sum_{\mathbf{r}_j} V(\mathbf{r}, \mathbf{r}_j) \tag{6}
\]

For the purpose of this work we neglect the collision term in Eq.(5) and note that such term will be essential when dealing with RHIC. In agreement to LQCD calculations \([1,8]\) the interacting potential \( V(\mathbf{r}) \) for quarks is (\( \hbar = 1 \)):

\[
V(r_{ij}) = 3\sum_{a=1}^{8} \frac{\lambda^a_i \lambda^a_j}{2} \left[ \frac{8\pi}{33 - 2n_f} \Lambda (\Lambda r_{ij} - f(\Lambda r_{ij})) + \frac{8\pi}{9} \Lambda \sigma_{q\bar{q}} \right] \delta(\mathbf{r}_{ij}) \tag{7}
\]

and \([8]\)

\[
f(t) = 1 - 4 \int \frac{dq}{q} \frac{e^{-qt}}{[\ln(q^2 - 1)]^2 + \pi^2} \tag{8}
\]

We fix the number of flavors \( n_f = 2 \) and the parameter \( \Lambda = 0.25 GeV \). In Eq.(7) we have added to the Richardson’s potential the chromomagnetic term (ct), very important to reproduce the masses of the hadrons in vacuum. Since in this work we will be dealing with finite nuclei, the ct can be neglected, we only notice that with the parameters choice discussed here, the hadron masses can be reproduced by suitably tuning the ct term \([3]\).

The \( \lambda^a \) are the Gell-Mann matrices. From lattice calculations we expect that there is no color transport for distances of the order of \( 0.2 - 0.3 fm \), which are distances much shorter than the ones we will be dealing with in this paper. Thus we will use the \( \lambda_3,8 \) only commuting diagonal Gell-Mann matrices (Abelian approximation) \([4]\).

Numerically the VE equation(5) is solved by writing the one body distribution function as:
\[ f_i(r, p, t) = \frac{1}{n_{tp}} \sum_{i}^{N} \delta(r - r_i(t))\delta(p - p_i(t)) \]  

(9)

\[ N = Qn_{tp} \]  

is the number of such terms. Actually, \( N \) is much larger than the total quark number \( Q \), so that we can say that each quark is represented by \( n_{tp} \) terms called test particles (tp). Notice how well this fits in the previous discussion if we put \( n_{tp} = N_{ev} \). Inserting Eq.(9) in the Vlasov equation gives the Hamilton equations of motion (eom) for the tp [4]. The total number of tp (or corresponding \( N/Q \) events) used in this work ranges from 5000 to 50000, and \( Q = 150-300 \).

Initially, we distribute randomly the tp in a sphere of radius \( R = r_0B^{1/3} \) (the radius of the nucleus) in coordinate space and \( p_f \) in momentum space. \( r_0B = \left( \frac{3}{4\pi\rho_B} \right)^{1/3}, A = Q/3 \) and \( \rho_B \) is the baryon density. \( p_f \) is the Fermi momentum estimated in a simple Fermi gas model by imposing that a cell in phase space of size \( h = 2\pi \) can accommodate at most two identical quarks of different spins, flavors and colors. A simple estimate gives the following relation between the quark density \( n_q \), and the Fermi momentum:

\[ n_q = \frac{g_{q}}{6\pi^2}p_f^3 \]  

(10)

The degeneracy number \( g_q = n_f \times n_c \times n_s \), where \( n_c \) is the number of colors and \( n_s \) is the number of spins [4]. For quarks and anti quarks 3 different colors are used red, yellow and blue (r,y,b) [1].

In figure(1), we plot the total energy per nucleon (top) and energy density (in units of the Fermi gas energy density [2]) vs. baryon density. The full triangles give the results obtained by randomly distributing the tp as described above. We notice that a minimum at about \( \rho_c = 2.08fm^{-3} \) is found with \( E_t/A = 2GeV/A \). Such a minimum is at much higher density and energy than expected for the ground state (gs) of a nucleus (\( \rho_0 = 0.16fm^{-3} \) and \( E_t/A = 0.938 - 0.016GeV/A \)). An important property of the system that we have described above is the following. If we rotate the quarks in color space, regardless of their position in r-space, the total energy will remain the same. This is indeed a "pure" Vlasov solution and demonstrates explicitly the in-capability of the Vlasov approach to give clustering of quarks into nucleons. However this result is already instructive since it gives us an hint on where the quark and nuclear matter are located, i.e. above and below \( \rho_c \) respectively. This result is qualitatively in agreement with the Hartree-Fock (HF) calculations of refs. [10,11] (compare to fig.1 in [10]). For a discussion on why the \( E_t/A \) increases at low densities in the HF/VE approaches we also refer to [10].

Of course distributing randomly the quarks in a sphere in r and p-space is not the most economical way to find the real gs of the system. In MD one searches for a minimum energy by introducing a friction term. The friction acts in such a way to lead the particles to a configuration for which the potential energy is a minimum. We cannot use this technique for our system since we are dealing with fermions and the friction term will destroy the initial antisymmetrization. In order to overcome this difficulty we adopt the following strategy. First we prepare \( N_{ev} \) events as above, and we evolve them numerically solving the eom but with friction included. Because of the large number of particles interacting with attractive and repulsive forces, the quarks will slowly evolve to new positions where the potential is lower while keeping the initial root mean square radius approximately constant. When the averaged potential (over events) reaches a given value \( V_{min} \), we look for the two closest
particles \((j, k)\) to a quark \(i\) in the same event. For these three quarks we know what the local density and the number of colors are. For instance if in a certain region we find two red and one blue quark, we use \(n_c = 2\) in Eq.(10) and calculate the local density from the knowledge of the distances of the 3 quarks. In this way the Fermi momentum is defined locally similar to the procedure used in nuclear or atomic physics (Thomas-Fermi approximation) [12]. We repeat this for all quarks \(i\) in all events and calculate the total energy for this state. We let the system evolve again with friction included to a lower potential energy \(V_{\text{min}} = V_{\text{min}} - \delta V\), where \(\delta V\) is a constant. We calculate the local density and local color numbers again and apply the Thomas-Fermi approximation to obtain the new total energy. We stop the procedure when the total energy is a minimum. The initial conditions so obtained are then propagated in time using the VE in order to maintain the initial antisymmetrization. We will show below, see fig.3, that the initial clustering obtained in MD is preserved as well during the Vlasov evolution.

The open triangles in fig.(1) are the result of the minimization. Notice especially at low densities the large decrease of the total energy of the system as compared to the Vlasov result. Now the calculated total energy at the nuclear gs is very close to the experimental value indicated by the full circle. However, we find slightly lower energies at lower densities, i.e. the gs of the nucleus is shifted in our calculations at about \(1/10\) of the experimental one. This should not be surprising in view of the simple potential that we have used. Also we have not tried to look for a best set of fitting parameters in these exploratory studies. At low \(\rho_B\), the global invariance for rotations in color space is lost, i.e. if we exchange the colors of two distant quarks, the total energy of the system will change. At high densities (larger than \(2 fm^{-3}\)) the Vlasov and MD solutions are the same. This can be also seen in the bottom part of the figure where energy densities are given. We would like to stress again the qualitative agreement to ref. [10] where a stochastic method had been used to calculate the g.s. energy of the system. This is quite evident if one compares our fig.1 to fig.1 of [10].

The energy density displayed in fig.1 (bottom) is a smooth function apart some fluctuations around \(2 fm^{-3}\) density. From this result we can exclude a first order phase transition but a second order phase transition might be possible.

In order to check if a second order phase transition occurs we define an order parameter. As we discussed above the color degrees of freedom play an important role for our system. When the quarks of different colors are in suitable positions in r-space the potential energy largely decreases. It is also very important that the system is locally color white because in this way \(n_c = 3\) and the kinetic part will also decrease. Thus for extremely small densities the quarks should condensate in clusters of 3 and zero net color. When the density changes this picture gradually modifies and at very high densities it does not matter where the quarks are located and which color they have. Looking at Eq.(7) we see that this fact is a consequence of the \(\frac{3}{2}\sum_{a=3,8} \lambda_i^a \lambda_j^a\) term (equal to \(-1\) for identical color quarks and \(1/2\) otherwise) . Thus we define an order parameter \(M_c\) as:

\[
M_c = \frac{1}{N} \sum_{i=1}^{N} \sum_{a=3,8} (\lambda_j^a \lambda_k^a + \lambda_i^a \lambda_j^a + \lambda_i^a \lambda_k^a) = M_{cr} + \frac{1}{N} \sum_{i=1}^{N} \sum_{a=3,8} (\lambda_j^a \lambda_k^a + \lambda_i^a \lambda_k^a) \tag{11}
\]

Where \(j, k\) are the two quarks closest in r-space to quark \(i\) as before. In Eq.(11) we have also defined a ”reduced” order parameter \(M_{cr}\) which tells us the color of the closest particle \(j\) to quark \(i\). From the properties of the \((3,8)\) Gell-Mann matrices it is easy to
derive the following results for $M_c$. If the 3 closest quarks have the same color, $M_c = -3$. We stress that this case is practically impossible to be obtained because the corresponding potential energy would be very large and repulsive. If the 3 closest quark states have two colors $M_c = 0$, this case is also recovered if the colors are randomly distributed such as in the ”pure” Vlasov solution. The case of 3 different color quarks gives $M_c = 3/2$. The last is the ideal case of well isolated white nucleons. If this last case is recovered in the calculations at small densities then the system is locally invariant for rotation in color space. i.e. if we rotate the color states inside the nucleon, the total energy of the system will remain constant.

Using similar arguments it is simple to show that if the closest particle to quark $i$ has always a different color $M_{cr} = 1/2$, if the two closest quarks have the same color $M_{cr} = -1$. If the closest quark color is randomly chosen $M_{cr} = 0$.

In figure (2) top, we plot the order parameter (opportunely normalized) vs. density (divided by a critical density—see below) for the MD case. The displayed $M_c$ are always positive i.e. it never happens that 3 equal quark color states are on average in the same region in r-space. In the top part of the figure we have distinguished two cases. The first one indicated by the full triangles corresponds to calculations where the average potential energy is larger than zero i.e. the linear term in Eq.(7) is dominant, small $\rho_B$. The full squares correspond to the case where the Coulomb term is dominant, large $\rho_B$ and negative mean field. The first case can be rather well fitted by the relation:

$$M_c \propto \left| 1 - \frac{\rho_B}{\rho_c} \right|^{\beta}$$

Where $\rho_c = 2.08 \, fm^{-3}$ is the critical density and the two curves correspond to the critical exponent values $\beta = 1/3$ (full line) and $1/2$ (dashed line). The latter is the expected value of the critical exponent $\beta$ in a mean field approach [13]. We notice that in ref. [1] a phase transition of first order was found. However the potential and the kinetic term (non-relativistic) used there are quite different from ours. In LQCD calculations for Fermions at zero temperature the order of the phase transition depends on the quark masses. For small quark masses such as ours, the transition found is second order [2][4].

In order to better understand the behavior of $M_c$ we have repeated the calculations by turning off the Coulomb term (open triangles) or the linear part (open squares). At low densities the linear term is larger while the Coulomb term is dominant at high densities. The two terms are equally important around the critical density. However, there is an important difference between the two cases. In fact at low densities the kinetic part is rather small compared to the potential one, while it is rather large at high densities, Eq.(10). Furthermore, at high densities the strong coupling constant entering the Coulomb potential, Eq.(8), vanishes logarithmically [5]. Thus it is clear that the order parameter increases again at high densities because of the friction used to lower the potential energy. But as soon as we ”turn on” the kinetic term we expect the bonds to be quickly broken. This is shown in figure (3) where the time evolution of $M_c$ is given. At high density we have a large value of $M_c$ (full circles) at time $t=0fm/c$. But such a value quickly decreases to a minimum value of about 0.3. The $M_c$ obtained at low densities is rather large and constant (open circles), while the one obtained at the critical density slightly increases (open squares). This result also shows that the VE can keep the initial clustering obtained from the MD initial conditions for a time sufficient to perform calculations for heavy ion collisions at relativistic energies. It also implies that the $M_c$ values obtained at high $\rho_B$ are an artifact of the calculation and...
we expect $M_c$ to saturate at about 0.3-0.4 at high densities. This is exactly the behavior expected for a order parameter, i.e. a power law dependence for densities below the critical one and constant otherwise. It is also instructive to notice that the two closest quarks have always a different color for low densities as it is suggested by $M_c$ in the bottom part of figure(2). The $M_c$ starts to be smaller than 1/2 for densities larger than $\rho_0$ and approaches zero very slowly with increasing density.

From figures (2) and (3) some important consequences can be derived.

i) The order parameter is never equal to 3/2 i.e. isolated white nucleons. As it is shown in figure 2, $M_c$ is still increasing with decreasing densities. Thus in the limit of very small densities we should get color white objects. However such a limit is hard to reach because of numerical fluctuations due to the confining potential. The maximum calculated $M_c$ value is about 0.9, a value in between 2 and 3 color states. Of course it is not always the same cluster (nucleon) to have $n_c = 2$ or 3, but rather the number of colors in a cluster changes dynamically between 2 and 3. In order to understand this behavior imagine to have two clusters with one color exchanged, $(r_1, y_1, y_2)$ and $(r_2, b_1, b_2)$, located at very large distances. The contribution to the potential of cluster 1 due to cluster 2 is zero for the red quark (-1+1/2+1/2) and 3/2 for each of the yellow quarks. Thus the two cluster will be attracted towards each other at low densities. On the other hand there is a repulsion between the two equal color quarks in the two clusters. These quarks will be pushed away from their original cluster and eventually the white color will be established. In other words the ”color migration” binds the clusters.

ii) The system is not locally ($M_c = 3/2$) nor globally ($M_c = 0$) color invariant. This is true at high densities as well, but there the potential energy is negligible as compared to the kinetic one and color invariance is (approximately) restored.

In conclusion in this work we have discussed microscopic Vlasov/MD approaches to finite nuclei starting from quark degrees of freedom with colors. In order to obtain the correct initial conditions we have introduced a method based on MD with frictional cooling plus a Thomas-Fermi approximation for the Fermi motion. We have shown that the method is able to describe at least qualitatively the well known features of nuclei near the ground state. At high densities a second order phase transition from nuclear to quark matter is predicted. Such a transition is due to the restoration of global color invariance at high densities and we have defined an order parameter accordingly. The approach can be refined in order to obtain a better description of the ground state of the nucleus. This can be used to simulate heavy ion collisions at ultra-relativistic energies after the introduction of a suitable collision term. Our approach can be very useful for the understanding of the quark gluon plasma formation and its signatures.

**ACKNOWLEDGMENTS**

We thank prof. J.B. Natowitz and the colleagues at the Cyclotron Institute-Texas A&M University for warm hospitality and financial support.
REFERENCES

[1] B.Povh, K.Rith, C. Scholz, F.Zetsche, *Particles and Nuclei: an introduction to the physical concepts*, Springer, Berlin, 1995.

[2] C.Y. Wong, *Introduction to High-Energy Heavy ion Collisions*, World Scientific Co., Singapore, 1994; L.P. Csernai *Introduction to Relativistic Heavy ion Collisions*, John Wiley and Sons, New York, 1994.

[3] A. Bonasera, "Quark Dynamics on Phase Space", nucl-th/9905023, "Vlasov Dynamics of Dense Quark Matter", nucl-th/9908036, Phys.Rev. C60(1999)65212.

[4] A. Bonasera, F. Gulminelli and J. Molitoris, Phys. Rep. 243, (1994)1; G. Bertsch, S. Dasgupta, Phys. Rep. 160(1988)189, and references therein.

[5] E.M. Lifshitz and L.P. Pitaevskii, *Physical Kinetics*, Pergamon Press 1991.

[6] T.Vetter, T.Biro and U.Mosel, Nucl.Phys. A581(1995)598; S.Loh, T.Biro, U.Mosel and M.Thoma, Phys.Lett. B387(1996)685; S.Loh, C.Greiner, U.Mosel and M.Thoma, Nucl.Phys. A619(1997)321; S.Loh, C.Greiner and U.Mosel, Phys.Lett. B404(1997)238; C.T. Traxler, U. Mosel and T.S. Biro Phys. Rev. C59(1999)1620.

[7] M.Hofmann et al. e-print nucl-th/9908030.

[8] J.L. Richardson, Phys.Lett. 82B(1979)272.

[9] M.Belkacem, V.Latora and A. Bonasera, Phys.Lett. 326B (1994)21.

[10] C.J.Horowitz, E.J. Moniz and J.W. Negele, Phys. Rev. D31(1985)1689.

[11] G.Roepke, D.Blaschke and H.Schulz, Phys. Rev. D34(1986)3499.

[12] P.Ring, P.Schuck, *The Nuclear Many-Body Problem*, Springer-Verlag, New York, 1980.

[13] K.Huang, *Statistical Mechanics*, John Wiley & Sons, New York,1987.

[14] F. R. Brown, F. P. Butler, H. Chen, N. H. Christ, Z. Dong, W. Schaffer, L. I. Unger, and A. Vaccarino, Phys.Rev.Lett. 65(1990) 2491.
FIG. 1. Energy per nucleon (top) and energy density of the quarks (bottom) vs. baryon density. The full triangles refer to Vlasov and the open ones to the molecular dynamics initializations (see text). The full circles give the values of the nuclear matter ground state.
FIG. 2. Order parameter (top) and “reduced” order parameter (bottom) vs. $\frac{\rho_B}{\rho_c}$. Critical density $\rho_c = 2.08 \text{fm}^{-3}$. The top part refers to the full calculation when the average potential is larger than zero (full triangles) or less than zero (full squares). The lines are given by $|1 - \frac{\rho_B}{\rho_c}|^\beta$ with $\beta = 1/3$ (full line) or $\beta = 1/2$ (dashed line). Also calculations are reported where the linear part of the potential is turned off (open squares), or the Coulomb part is off (open triangles).
FIG. 3. Order parameter vs. time calculated in molecular dynamics plus Vlasov approach at three initial densities: $0.033 fm^{-3}$ (open circles), $29.8 fm^{-3}$ (full circles), and $2.08 fm^{-3}$ (open squares).