Estimate and asymptotic of the solution for the $p$-Laplacian parabolic equation double non-linear type with damping

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Abstract. In this article, using the solution to the Hamilton-Jacobi equation, we allegedly investigate the estimate and asymptotic of solutions for a parabolic equation with double nonlinearity with damping with a variable coefficient. An estimate for the weak solution and the asymptotic of regular, unbounded and finite solutions of the stationary equation are obtained. The condition for spatial localization of the solution to the Cauchy problem is found.

1. Introduction

When studying nonlinear processes, it is interesting to analyze the influence of various factors on the speed of propagation of the temperature wave front.

In [1], it was shown how the presence of absorption or heat release in a medium affects the speed of propagation of the temperature wave front. In this case, the volumetric heat absorption reduces the front propagation speed and in some cases leads to the stopping of the temperature wave. It is obvious that the movement of the medium, leading to convective heat transfer, should also affect the spatial localization of the temperature wave propagation velocity. Therefore, it is interesting to consider the influence of other factors leading to the emergence of new phenomena. Of particular interest is the study of nonlinear processes with damping, which has been studied in a large number of works (see [1, 2] and the references therein).

In this work in the domain $Q = \{(t, x): t > 0, x \in R\}$ the following Cauchy problem is investigated

$$L(u) = -\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)^{p-1} \frac{\partial u}{\partial x} - g(x) \left| \frac{\partial u}{\partial x} \right|^p = 0, \quad (1)$$

$$u|_{t=0} = u_0(x) \geq 0, \quad x \in R, \quad (2)$$

for a parabolic equation with damping [2].

Equation (1) describes the processes of nonlinear heat conduction, diffusion, filtration, biological population and various other processes [1-10]. A special case of the equation ($m = 1, \ p = 2$ without damping) was proposed by Lei Benson [16] to describe the processes of oil and gas. The case $p = 2, \ p_1 = 0, \ g(x) = 1$. Is known as equalized porous medium and the case. $q_1 = 0, \ g(x) = 1$. Hamilton-Jacobi equation with absorption [1-5]. A distinctive feature of the Cauchy problem under study is the degeneration of equation (1), which is why it is in a domain where $u = 0$ or $\frac{\partial u}{\partial x} = 0$ may not have a
solution in the classical sense. Therefore, it is of interest to study a weak solution from a physically meaningful class with the property

\[
\left| \frac{\partial^2 u^k}{\partial x} \right|^{p-2} \frac{\partial u^m}{\partial x} \in C(Q), \; u \geq 0,
\]

and satisfying problem (1), (2) in the sense of distribution [1-6].

Various qualitative properties of the solution to problem (1) and nonlinear phenomena for particular values of numerical parameters and for \( g(x) = 1 \) have been intensively studied by many authors [1-33]. In particular, in [2], it was considered the case \( k = m > 1, \; p > 2, \; p > p_1, \; q_1 + p_1m > m(p-1) > 1, \; g(x) = 1 \). Applying the standard iterative method, the author gave a sufficient condition for the existence of singular self-similar solutions to equation (1) and classification of these singular self-similar solutions.

Questions of the existence of a global solution, estimates, asymptotic of the solution for \( t \rightarrow \infty \) and nonlinear phenomena arising at different values of numerical parameters have been intensively studied by many authors (see [1-10] and the literature cited there). A number of works are devoted to the study of the qualitative properties of the solution of the problem under consideration for particular values of the numerical parameters in the case when the initial value \( u_0(x) \) is smooth. There are a number of papers devoted to the global solvability and in solvability of the Cauchy problem (1), (2) in various function spaces. As an example, we cite, in particular, the works of Huashui Zhan [2], Z. Wu, J. Zhao, J. Yun, Y.M. Qi [18] and works [1, 28,29], where absorption has a power-law gradient nonlinearity. In [2] the problem (1), (2) was considered in the case \( k = m, \; p > 2, \; m > 1 \) and \( p > p_1, \; q_1 + p_1m > m(p-1) > 1, \; g(x) = 1 \). In this article, using the standard iterative Picard method, a sufficient condition for the existence of singular self-similar solutions is given. In addition, a classification of these singular self-similar solutions is given. H. Zhan [8] applying the standard iterative method in the case \( k = m \) got a sufficient condition for the existence of singular self-similar solutions to equation (1). The author also gives a classification of singular self-similar solutions. The qualitative properties of the solution to problem (1) and nonlinear phenomena for various particular values of numerical parameters are intensively studied by many authors [10-18]. In [19], the critical curves of an equation with double nonlinearity in a no divergence form with a nonlinear boundary flow are investigated. Namely in it the critical curve of global existence and the critical curve of Fujita were obtained. The asymptotic of the solution is established for the critical value of the parameter \( q_1 = p - 1 + p/N \) for the problem (1), (2) in the case \( k = 1, \; m = 1 \) in [18]. In [13], the solution for the system of the reaction-diffusion equation with double nonlinearity in the presence of a source was investigated. The self-similar approach is used to process the qualitative properties of a nonlinear reaction-diffusion system. It is shown that there are some parameter values for which the effect of the finite velocity of the perturbation and spatial localization of the solution can take place. In [20], the asymptotic behavior of solutions of the Cauchy problem for nonlinearity, which describes heat diffusion with nonlinear heat absorption at a critical value of the parameter, was investigated.

Samarskiy A A and Sobol I M [15], proposed numerical schemes and a method for numerical solution based on the sweep method of self-similar analysis for porous medium equation \( (k = 1, \; p = 2, \; m > 1) \). In [9], this method extended to for numerical solution of wave type structures in nonlinear diffusion medium with damping.

In this paper, we study the spatial localization of the solution, the asymptotic representation of regular and unbounded and finite solutions of the stationary equation (1), depending on the value of the numerical parameters \( k, p, m, p_1, q_1 \) of the medium. A method for estimating the solution based on the solution of the corresponding Hamilton-Jacobi equation is the first order asymptotic properties of solutions are obtained depending on the values of numerical parameters characterizing a nonlinear medium by solving a first-order equation, which solves the problem of choosing initial approximations for the numerical solution of the problem. It is shown that some of these properties of solutions to equation (1) can be established by solving the following Hamilton-Jacobi equation.
The solution of which is relatively simpler than the solution of the original second-order equation. It is easy to verify that equation (3) has a particular solution

$$\tilde{u}_z = c \pm \frac{k^{(2-p)m/2}(p-2m)}{k+1} k(p-2)+m-(mp_1+q_1) \int_0^y \frac{1}{\gamma_1(p-2)+m-(mp_1+q_1)} dy^{\gamma_1},$$

(4)

here \(c\) is a constant of integration

$$\gamma_1 = \frac{p-p_1}{k(p-2)+m-(mp_1+q_1)}.$$

If \(k(p-2)+m=(mp_1+q_1)\), then

$$\tilde{u}(x) = \exp \left( -\frac{jm^{\gamma_1}}{(k+2)k^{p-2}(k(p-2)+m)} \frac{1}{p-p_1} \int_x^y \frac{1}{\gamma_1(p-2)+m-(mp_1+q_1)} dy \right).$$

(5)

In this paper, using the solution of the Hamilton-Jacobi equation, an estimate of the weak solution is allegedly obtained, the spatial localization property of the Cauchy problem for problem (1), (2), the asymptotic of regular, unbounded and finite solutions of the stationary equation (1) are investigated. It is shown that in the critical case when \(k(p-2)+m-(mp_1+q_1)=0\) the nature of the solution changes and has the form of an exponential function. It is shown that functions \(u_+(x), \tilde{u}(x)\) are upper solutions of the Cauchy problem (1), (2). Functions \(u_+(x), \tilde{u}(x)\) were used as an initial approximation when solving the problem by the iterative method.

2. Evaluation of the solution and spatial localization of the solution

We restrict ourselves to the following theorem, which is proved by the method of comparison of solutions. It is shown that the qualitative property of the solution to problem (1), (2) depends on the convergence or divergence of the integral \(\int_0^1 (\theta(y))^{-\rho} dy\).

**Theorem 1.** Let function \(\theta(x)\) satisfy the conditions \(\theta(x) \leq g(x), \theta'(x) \geq 0, x \in (0, \infty)\) and

$$\int_0^\infty (\theta(y))^{-\rho} dy = \infty.$$

Let

$$u_+(x) = \left[ a - b \int_0^1 (\theta(y))^{-\rho} dy \right]^{\gamma_1}, a > 0,$$

where the notation \((c)_+ = \max(0, c)\) is used and

$$\gamma_1 = \frac{p-p_1}{k(p-2)+m-(mp_1+q_1)},$$

(6)

Then the solution to problem (1), (2) satisfies the estimate \(u(x) \leq u_+(x)\) in \(Q = \{(t, x) : t > 0, x \in R\}\), where function \(u_+(x)\) is defined above and the solution has the property of spatial localization [1].

Proof. The proof of the theorem is based on the principle of comparison of solutions. As a comparison, consider the function
\[ u_*(x) = (a - b \phi)_1^p, \quad \phi = \int_0^1 \frac{1}{\theta^{p_{-1}}} (y) dy. \]

To use the principle of comparison, it is necessary to show that for the function \( u_*(x) \) being compared, condition \( L(u_*(t, x)) \leq 0 \) is satisfied in \( D = \{ (t, x) : t > 0, \int_0^t \theta^{p_{-1}} (y) dy < a/b \} \). Let us calculate \( L(u_*(t, x)) \). We have after simple calculations:

\[
\frac{d}{dx} \left( \frac{d_{u^*}^m}{dx} \right) = -b \gamma_1 m(\gamma_1 k)^{p-2} (a - b \phi)^{\gamma_1 - p} \phi'^{p-1},
\]

\[
\frac{d}{dx} \left( \frac{d_{u^*}^m}{dx} \right) = -b \gamma_1 m(b \gamma_1 k)^{p-2} (a - b \phi)^{\gamma_1 - p} \times \left[ -b(\gamma_1 z - (p - 1)\phi' + (a - b \phi) d dx (\phi'^{p-1}) \right],
\]

\[
\frac{d_{u^*}^m}{dx} = (bm \gamma_1)^p (a - b \phi)^{\gamma_1 (mp_1 + q_1) - p_1} \phi'^{p_1}.
\]

\( z = k(p - 2) + m \). Substituting the obtained expressions into equation (3) for \( L(u_*(t, x)) \) we have

\[ L(u_*(t, x)) = -b \gamma_1 m(b \gamma_1 k)^{p-2} (a - b \phi)^{\gamma_1 - p} \left[ -b(\gamma_1 z - (p - 1)\phi' + (a - b \phi) d dx (\phi'^{p-1}) \right] -
\]

\[
- g(x)(bm \gamma_1)^p (a - b \phi)^{\gamma_1 (mp_1 + q_1) - p_1} \phi'^{p_1}.
\]

Let

\[ \gamma_1 z - p = \gamma_1 (mp_1 + q_1) - p_1, \text{i.e.} \gamma_1 = \frac{p - p_1}{k(p - 2) + m - (mp_1 + q_1)}. \]

Then

\[ L(u_*(t, x)) = -b \gamma_1 m(b \gamma_1 k)^{p-2} (a - b \phi)^{\gamma_1 - p} \left[ -b(\gamma_1 z - (p - 1)\phi' - (a - b \phi) d dx (\phi'^{p-1})) \right] \leq 0,
\]

because

\[ \phi' = \theta^{p_{-1}}(x), \quad \phi(x) = \int_0^x \frac{1}{\theta(y)} dy. \]

By the condition of Theorem 1, if \( b^{p_{-1}} = (m \gamma_1)^p / \gamma_1^{p_{-1}} k^{p_{-2}} m(\gamma_1 z - (p - 1)) \) in \( D \) then from (11) we have:

\[ L(u_*(t, x)) = -b \gamma_1 m(b \gamma_1 k)^{p-2} (a - b \phi)(p_1 - p) \frac{d}{dx} (\phi'^{p-1}) +
\]

\[ + [b \gamma_1^{p_{-1}} k^{p_{-2}} m(\gamma_1 z - (p - 1) \theta(x) - g(x))] (a - b \phi)^{\gamma_1 - p} \leq 0,
\]

here \( D = \{ (t, x) : t > 0, \int_0^t \theta^{p_{-1}} (y) dy < a/b \} \). By the condition of Theorem 1 \( \phi(x) \leq g(x) : \theta(x) \geq 0, \frac{d}{dx} (\phi'^{p-1}) \geq 0 \). Then, according to the principle of comparison of solutions, we have
Simple calculations show that
\[ \gamma_1 = \frac{p - p_1}{k(p-2) + m - (mp_1 + q_1)} \quad \text{in} \quad Q = \{(t,x) : t > 0, x \in \mathbb{R} \}.
\]

From the last inequality we have \( u(t,x) \equiv 0 \) when \( \int_0^1 \Theta^{p-p_1}(y)dy \geq a/b \). This means that for the solution of problem (1), (2) the solution is spatially localized. Theorem proved.

For example, consider the case \( g(x) = x^\sigma \). Note that to as in equation (1) in the case \( p_1 = 0, m = 1, p = 2 \) is called Emden-Fowler [31], which arose from astrophysics, and in the case \( p_1 = 0, m = 1, p = 2, \sigma = 1/2 \) the Thomas-Fermi equation from atomic physics (see, for example, [31]).

From the last inequality we have \( u(t,x) \equiv 0 \) by
\[
x \geq \left[ \frac{(\sigma + p - p_1)}{p - p_1} \right] \left( \frac{a}{b} \right)^{\frac{p-p_1}{p-p_1}}, \quad p - p_1 > 0, \quad \sigma + p - p_1 > 0.
\]

This means that for the solution of the problem (1), (2) there is a spatial localization of the solution. The last inequality makes it possible to analyze the localization area of the solution depending on the value of the numerical parameters of the nonlinear medium and the effect of damping.

3. Asymptotic of the solution to a stationary equation
The stationary equation (1) mentioned above is called the generalized Emden-Fowler equation.

**Theorem 2.** Let \( \frac{\theta(x)}{g(x)} \rightarrow 1 \), at \( x \rightarrow \infty, \quad \gamma_1 > 0 \). Then the finite solution of the stationary equation (3) for \( \phi(x) \rightarrow \frac{a}{b} \) has the following asymptotic representation
\[
u(x) = c(a - b\phi(x))^{\gamma_1}(1 + o(1)),
\]
where \( c = \left[ \frac{k(p^-p \gamma_1 + m\gamma_1)}{k(p-2) + m - (mp_1 + q_1)} \right] \left( \frac{1}{k(p-2) + m - (mp_1 + q_1)} \right) \), number \( \gamma_1 \) is defined above.

Proof. The proof of the theorem is carried out by comparing solutions. The function to compare is taken as
\[
u(x) = (a - b\phi(x))^{\gamma_1},
\]
which is the solution of the Hamilton-Jacobi equation (3). The solution of the stationary equation (1) is found in the form \( u(t,x) = \nu(x)w(x), \quad \nu(x) = (a - b\phi(x))^{\gamma_1}, \quad \tau = -\ln(a - b\phi(x)), \quad \phi^{p-p_1} = \theta(x) \). The meaning of this transformation is that \( \tau \rightarrow \infty \) by \( \phi \rightarrow \frac{a}{b} \) and the problem is reduced to the asymptotic stability of the solution of equation (1) for \( \tau \rightarrow \infty \). Further simple calculations give the following
\[
\left| \frac{du}{dx} \right|^{p-2} \left( \frac{du}{dx} \right) = b^{p-1}(\phi')^{p-1}(a - b\phi)^{\gamma_1(k(p-2) + m - (p-1)Lw},
\]
where \( Lw = \left| w^k - k\gamma_1 w^k \right|^{p-2} \left( w^m - m\gamma_1 w^m \right) \). Simple calculations show that
\[
\frac{d}{dx} \left( \frac{du^k}{dx} \right)^{\gamma_1} \frac{du^m}{dx} = b^{p-1} \frac{d}{dx} (\phi^r)^{p-1} (a-b\phi)^{\left[ k(p-2) + m \right]} (p-1) L_w + \\
+ b^r \phi^r (a-b\phi)^{\left[ k(p-2) + m \right] (p)} \left( \frac{d}{d\tau} L_w - [(k(p-2) + m)\gamma_1 - (p-1)] L_w \right), \tag{15}
\]

\[
\left( \frac{du^m}{dx} \right)^{\gamma_m} = b^{\gamma_m} (a-b\phi)^{\gamma_m} (p+\gamma_m) \left| w^m \gamma_w^m \right|^p \cdot \tag{16}
\]

Substituting the obtained expressions in (3), we have

\[
b^{p-1} \frac{d}{dx} (\phi^r)^{p-1} (a-b\phi)^{\left[ k(p-2) + m \right] (p-1)} L_w + \\
b^r (a-b\phi)^{\left[ k(p-2) + m \right] (p)} \left( \frac{d}{d\tau} L_w - [(k(p-2) + m)\gamma_1 - (p-1)] L_w \right) = \\
g(x) b^{\gamma_m} (a-b\phi)^{\gamma_m} \left| w^m \gamma_w^m \right|^p \cdot \tag{17}
\]

Given that, \( \phi^r = \theta(x) \) and \( (k(p-2) + m)\gamma_1 - p = (\gamma_m p - 1) p_1 + \gamma_1 q_i \), the last equation is rewritten as

\[
b^{p-1} (\phi^r)^{p-1} (a-b\phi) L_w + \\
b^r \left( \frac{d}{d\tau} L_w - [(k(p-2) + m)\gamma_1 - (p-1)] L_w \right) = b^{\gamma_m} (a-b\phi)^{\gamma_m} \left| w^m \gamma_w^m \right|^p \cdot \tag{18}
\]

\[
a-b\phi \to 0, \quad \frac{d}{dx} (\phi^r)^p (a-b\phi) \to 0. \text{ From (17) we get}
\]

\[
\frac{d}{d\tau} L_w - [(k(p-2) + m)\gamma_1 - (p-1)] L_w = b^{\gamma_m} (a-b\phi)^{\gamma_m} \left| w^m \gamma_w^m \right|^p \cdot \tag{19}
\]

Analysis of the solution of equation (18) shows that all solutions of equation (18), tending to a constant must satisfy the algebraic equation

\[
[(k(p-2) + m)\gamma_1 - (p-1)] k\gamma_w c^m b^{p-2} = (a-b\phi)^{\gamma_m} c^m \cdot \tag{20}
\]

i.e. \( w = c \)

Because \( \frac{d}{d\tau} L_w = 0 \) by \( w = c \). Therefore, for sufficiently large ones, the following asymptotic representation of the finite solutions of equation (1) takes place

\[
u(x) = \left[ \frac{b^{p-2} \gamma_1 b^{p-1} k^{2-p} m^{p-1}}{(k(p-2) + m)\gamma_1 - (p-1)} \right]^{1 \left[ k(p-2) + m \right] \gamma_m} (a-b\phi)^{\gamma_1} \cdot \tag{21}
\]

Theorem 2 proved.

Let \( \int_{\gamma_1}^{\infty} \frac{1}{\theta^p} (y) dy < \infty \), then \( \int_{\gamma_1}^{\infty} \frac{1}{\theta^{p-1}} (y) dy \to 0 \), by \( x \to \infty \), \( \gamma_1 > 0 \).

Consider the function

\[
\tilde{u}(x) = \left[ \int_{\gamma_1}^{\infty} \frac{1}{\theta^{p-1}} (y) dy \right] \gamma_1, \quad \gamma_1 = \frac{p-p_1}{k(p-2) + m - (p_1 + q_i)}. \tag{22}
\]
In this case, it is fair.

3.1. Asymptotic of regular solutions

Case 1. \( \int_{-\infty}^{\infty} \theta^{p-\eta} (y) dy = \infty, \ x_0 \geq 0. \)

Theorem 3. Let \( \gamma_1 < 0, \ (m+k(p-2))-(p-1) < 0, \ \theta^{p-\eta_1} \theta' \left( \int_{0}^{\infty} \theta^{p-\eta} (y) dy \right) \rightarrow 0 \) with \( x \rightarrow \infty, \)

\( \int_{0}^{\infty} \theta^{p-\eta} (x) dx = \infty. \) Then, as \( x \rightarrow \infty, \) the regular solutions of the stationary equation (1) have the following asymptotic representation:

\[
\lim_{x \to \infty} u(t, x) = \frac{\gamma_1^{p-\eta_1+1} k^{2-p} m^{\eta_1-1}}{\gamma_1^{p-\eta_1} (m+k(p-2))-(p-1)} \left[ \int_{0}^{\infty} \theta^{p-\eta} (y) dy \right]^{\gamma_1} \left( 1 + o(1) \right). \tag{22}
\]

3.2. Asymptotic of unbounded solutions

Theorem 4. Let \( \gamma_1 < 0, \ (m+k(p-2))-(p-1) < 0, \ \int_{0}^{\infty} \theta^{p-\eta} (y) dy < \infty, \)

\( \theta^{p-\eta_1} \theta' \left( \int_{0}^{\infty} \theta^{p-\eta} (y) dy \right) \rightarrow 0 \) with \( x \rightarrow \infty. \) Then, as \( x \rightarrow \infty \) the unbounded solutions of the stationary solution to equation (1) have the following asymptotic representation:

\[
\lim_{x \to \infty} u(t, x) = \frac{\gamma_1^{p-\eta_1+1} k^{2-p} m^{\eta_1-1}}{\gamma_1^{p-\eta_1} (m+k(p-2))-(p-1)} \left[ \int_{0}^{\infty} \theta^{p-\eta} (y) dy \right]^{\gamma_1} \left( 1 + o(1) \right). \tag{23}
\]

Proof. The solution of equation (1) will be found in the form

\[
u(x) = \frac{1}{\theta^{p-\eta}} \int_{0}^{\infty} \theta^{p-\eta} (y) dy, \quad \tau = -\ln \left[ \int_{0}^{\infty} \theta^{p-\eta} (y) dy \right]. \tag{24}
\]

It is clear that \( \tau(x) \to \infty \) by \( x \to \infty \) by virtue of the convergence of the integral \( \int_{0}^{\infty} \theta^{p-\eta} (y) dy. \) Put (24) in (3) and after the following simple calculations we have

\[
\frac{d u^{k-\eta-1}}{dx} = \frac{1}{\theta^{p-\eta}} L w, \quad \text{where} \quad L w = (w_k^m - m \gamma_1 w_k^m) \left[ \gamma_1^{m+k(p-2)} - \gamma_1 k w_k^p \right]^{-p-2} \tag{25}
\]

\[
\frac{d}{dx} \left( \frac{d u^{k-\eta-1}}{dx} \right) = \frac{1}{\theta^{p-\eta}} \left[ \gamma_1^{m+k(p-2)} \right]^{p-2} - \gamma_1 \left[ \gamma_1^{m+k(p-2)} \right]^{p-1} \theta^{p-\eta} L w \tag{26}
\]
\[ + \frac{p-1}{p-p_1} \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \int_0^{p_1-1} \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w, \]

\[ + \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \frac{du^m}{dx} \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} w^m \]

\[ \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} w^m \]

\[ \frac{d}{d\tau} L_1 w + \left( -\phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \frac{d}{d\tau} L_1 w = \left[ g(x) \theta(x) \right] w^m - m \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ = \frac{d}{d\tau} L_1 w = \left[ g(x) \theta(x) \right] w^m - m \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \text{Since when } x \to \infty, \int_0^\infty \frac{1}{\theta^{p-p_1}} (y) \, dy \to \infty \text{ therefore } \tau \to \infty. \text{ Therefore, according to the condition of the theorem, equation (31) takes } \tau \to \infty \text{ the form} \]

\[ \frac{d}{d\tau} L_1 w - \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \left( \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \text{where } L_1 w = w^m - m \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \text{Obviously, the solution of equation (32) tends to a constant } w = c, \text{ since } \frac{d}{d\tau} L_1 c = 0, \text{ then } c \text{ is the solution of an algebraic equation} \]

\[ \left[ \phi(m+k(p-2)) \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \text{from here we find} \]

\[ \frac{\gamma_1}{\gamma_1(p-1)(m+k(p-2))-(p-1)} \left[ \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \text{Therefore, the stationary solution of equation (3) has an asymptotic representation} \]

\[ u(x) = \left[ \int_0^\infty \frac{1}{\theta^{p-p_1}} \, dy \right) \theta^{p-p_1} \frac{d}{d\tau} L_1 w \right) \]

\[ \text{Theorem 4 proved.} \]

\[ \text{Case 2. } g(x) = x^\sigma, \sigma \in R. \]

\[ \text{Corollary 1. Let} \]
\[ \sigma + p - p_1 > 0, \quad p - p_1 > 0, \quad \gamma_1 > 0, \quad b = \left[ \frac{k^{2-p} \gamma_1^{p-1} m^{p-1}}{\gamma_1^2 (k (p-2) + m) - (p-1)} \right]^{1-p/p_1}. \]

Then, for the solution of problem (1), (2) with the property of spatial localization, estimate
\[ u(t,x) \leq u_0(t,x) \text{ in } Q = \{ (t,x) : t > 0, x \in \mathbb{R} \} \]
where \( u_0(x) = (a - b \varphi(x))^\gamma \) holds, and for the front, estimate
\[ x \geq \left[ \frac{\sigma + p - p_1}{p - p_1} (a/b) \right]^{1-p/p_1}, \quad a > 0, \quad b = \left[ \frac{k^{2-p} \gamma_1^{p-1} m^{p-1}}{\gamma_1^2 (k (p-2) + m) - (p-1)} \right]^{1-p/p_1}. \]

In particular, for a porous medium \( (p = 2, k = 1) \) for the front, we have estimate
\[ x \geq \left[ \frac{\sigma + 2 - p_1}{2 - p_1} (a/b) \right]^{1-p/p_1}, \quad a > 0, \quad \gamma_1 = \frac{2 - p_1}{m - (mp_1 + q_1)}, \quad b = \left[ \frac{\gamma_1^{p-1} m^{p-1}}{\gamma_1^2 (k (p-2) + m) - (p-1)} \right]^{1-p/p_1}. \]

Accordingly, we can write out the asymptotic of the solution for cases
\[ \varphi(x) = \frac{p - p_1}{\sigma + p - p_1} x \to \infty, \]
which corresponds to case \( p - p_1 > 0, \quad \sigma + p - p_1 > 0, \quad \gamma_1 < 0. \)

**Corollary 2.** The asymptotic of regular solutions for \( x \to \infty \) has the form \( u(x) \to c [\varphi(x)]^\gamma, \)
\[ c = \left[ \frac{\gamma_1^{p-1} k^{2-p} m^{p-1}}{\gamma_1^2 (k (p-2)) - (p-1)} \right]^{1-p/p_1}, \quad \gamma_1 = \frac{2 - p_1}{m - (mp_1 + q_1)}, \quad b = \left[ \frac{\gamma_1^{p-1} m^{p-1}}{\gamma_1^2 (k (p-2) + m) - (p-1)} \right]^{1-p/p_1}. \]

4. **Evaluation of solutions in the critical case**

Let's call \( k (p-2) + m - (mp_1 + q_1) = 0 \) this a critical case. In this case, the nature and evaluation of the decision changes. We show that what is the function
\[ u_2(x) = \exp \left\{ - \left[ \frac{\gamma m^{p-1}}{k^{p-2} (k (p-2) + m)} \right]^{1-p/p_1} \int_{x_0}^{x} [\theta(y)]^{1-p/p_1} dy \right\}, \]
is the upper solution of equation (1) at the critical value of the parameter.

**Theorem 5.** Let in equation (1) \( k (p-2) + m - (mp_1 - q_1) = 0, \quad \theta(x) > 0, \quad u_0(x) \leq Au_2(x), \quad A > 0, \quad x \in \mathbb{R}. \) Then for the solution of the problem (1), (2) the estimate \( u(t,x) \leq u_2(x) \) in \( Q \) is valid.

Proof. The proof of the theorem is based on the principle of comparing solutions. As a comparison, consider the function
\[ u_2(x) = A \exp \left\{ - \left[ \frac{\gamma m^{p-1}}{k^{p-2} (k (p-2) + m)} \right]^{1-p/p_1} \int_{x_0}^{x} [\theta(y)]^{1-p/p_1} dy \right\}. \]

To use the principle of comparing solutions, we show that in \( D = \{ t,x : t > 0, 0 < x < \infty \} \). The condition is met \( L(u_2(x)) \leq 0 \). In fact, direct calculations show that by virtue of the condition of theorem 5, that
\[ L(u_2(x)) = -a \theta^\prime(x) \hat{\theta}(x)^{p-1} \exp \left( -a \int_{x_0}^{x} \hat{\theta}(y)^{p-1} \, dy \right) < 0, \quad a = \left[ \frac{m \gamma^{p-1}}{k^{p-2} (k (p-2)+m)} \right]^{1/p-1} \]

in \( D = \{(t,x) : t > 0, 0 < x < \infty \} \). By virtue of the principle of comparing solutions, theorem 5 is proved.

5. Conclusion
The Cauchy problem for the \( p \)-Laplacian equation with a variable damping coefficient depending on the spatial variable arising in various applications studied. A method is proposed and substantiated for obtaining an estimate of the solution and the front based on the method of standard equations and the Hamilton-Jacobi equation corresponding to a nonlinear second-order equation. Based on the effects of spatial localization, the finite velocity of propagation of the disturbance, the asymptotes of regular, finite, unbounded solutions are obtained. These results allow us to solve the problem of the initial approximation in the numerical solution of the problem by the iterative method. The emergence of a critical case in which the behavior of the solution changes is shown.

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