Abstract. The complex Lie superalgebras $\mathfrak{g}$ of type $D(2, 1; a)$ are usually defined for “non-singular” values of the parameter $a$, for which they are simple. In this paper we introduce five suitable integral forms of $\mathfrak{g}$, that are well-defined at those singular values too, giving rise to “singular specializations” that are no longer simple. This extends (in five different ways) the classically known $D(2, 1; a)$ family. Basing on this construction, we perform the parallel one for complex Lie supergroups and describe their singular specializations (or “degenerations”) at singular values of the parameter. This is done via a general construction based on suitably chosen super Harish-Chandra pairs, which suits the Lie group theoretical framework; nevertheless, it might also be realized by means of a straightforward extension of the method introduced in [R. Fiorese, F. Gavarini, Chevalley Supergroups, Mem. Amer. Math. Soc. 215 (2012), no. 1014, 1–77] and [F. Gavarini, Chevalley Supergroups of Type $D(2, 1; a)$, Proc. Edin. Math. Soc. 57, (2014), 465–491] to construct “Chevalley supergroups”, which is fit for the context of algebraic supergeometry.

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1. Introduction

In the classification of simple, finite dimensional Lie superalgebras over $\mathbb{C}$ a special one-parameter family occurs, whose elements $g_a$ depend on a parameter $a \in \mathbb{C} \setminus \{0, -1\}$ and are said to be of type $D(2, 1; a)$. Roughly speaking, these are “generically non-isomorphic”, namely there is a group of isomorphisms $\Gamma (\cong \mathfrak{S}_3)$ freely acting on the family $\{g_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$. This notation has two origins: (i) this Lie superalgebra is just $\mathfrak{osp}(4, 2)$ when $a \in \{1, -2, -\frac{1}{2}\}$, and (ii) this becomes a family of Lie algebras over a field of characteristic $2$, as was shown in [KV]. A drawback of this notation is that, à priori, one does not see from it the built-in $\mathfrak{S}_3$-symmetry; in this respect, the notation introduced by I. Kaplansky [Kap] instead, that is $\Gamma(A, B, C)$, seems to be more reasonable; however, Kaplansky’s notation also has a defect, that is one cannot guess out of it any particular property besides $\mathfrak{S}_3$-symmetry. In this paper we adopt Kac’ notation $D(2, 1; a)$ since it definitely seems, nowadays, the most commonly used and known in literature. Notice also that the Cartan matrix in [KV] is essentially the same as the one we use in [3.1.1].

On top of each of the (simple) Lie superalgebras $g_a$ one can construct a corresponding Lie supergroup, say $G_a$; this can be done via the equivalence between super Harish-Chandra pairs and Lie supergroups (like, e.g., in [Ga3]), or also — in an algebro-geometric setting — via the construction of “Chevalley supergroups” (as in [FG] and [Ga1]). Any such $G_a$ has $g_a$ as its tangent Lie superalgebra, and overall they form a family $\{G_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ bearing again a $\mathfrak{S}_3$-action that integrates the $\mathfrak{S}_3$-action on the family $\{g_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$.

The starting point of the present paper is the following remark: the definition of $g_a$, if suitable (re)formulated, still makes sense for the “singular values” $a = 0$ and $a = -1$ alike. Indeed, one can describe $g_a$ at “non-singular” values of the parameter $a$ choosing a suitable basis — hence a corresponding integral form — and then use that same basis to define $g_a$ at singular values as well. The aim of this article is to show this dependency on the choice of integral form of the Lie superalgebra $g_a$ and its corresponding supergroup. In fact, we present five (out of many) possible ways to perform such a step, i.e. five choices of bases (hence of integral forms) that lead to different outcomes. The remarkable fact then is that in each case the new Lie superalgebras $g_a$ we find at “exceptional values” of $a$ are non simple; in this way we extend the old family $\{g_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ of simple Lie superalgebras to five larger families, indexed by $a \in \mathbb{C}$, whose elements coincide for non-singular values of $a$ but do not for the singular ones.

Indeed, our construction is more precise, as instead of working with Lie superalgebras $g_a$ indexed by a single parameter $a \in \mathbb{C} \setminus \{0, -1\}$ — later extended to $a \in \mathbb{C}$ — we rather deal with a two-dimensional multiparameter $\sigma \in V := \{(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3 \mid \sum \sigma_i = 0\}$. For each $\sigma \in V$ we define a Lie superalgebra $g(\sigma)$ via Kac’ standard presentation (cf. [K]) in terms of a matrix $A$ depending on $\sigma$; so we still use Kac’ language, but sticking closer to Kaplansky-Scheunert’s point of view, as in [Kap] and [Sc]. Thus we have a full family of Lie superalgebras $\{g(\sigma)\}_{\sigma \in V}$, forming a bundle over $V$; naturally endowed with an action
by $G := \mathbb{C}^* \times \mathfrak{g}_3$ via Lie superalgebra isomorphisms. For each $\sigma$ in the “general locus” $V \setminus (\bigcup_{i=1}^3 \{\sigma_i = 0\})$ we have $g(\sigma) \cong g_a$ for some $a \in \mathbb{C} \setminus \{0, -1\}$ — roughly, given by the line in $V$ through $\sigma$ and $0 := (0, 0, 0)$ — so the original family $\{g_a\}_{a \in \mathbb{C} \setminus \{0, -1\}}$ of simple Lie superalgebras of type $D(2, 1; a)$ is taken into account; in addition, the $g_a$’s are well-defined also at singular values $\sigma \in \bigcup_{i=1}^3 \{\sigma_i = 0\}$, but there they are non-simple instead. On the other hand, at non-singular values we can change basis of $g(\sigma)$: thus, with four other different choices of basis the corresponding $\mathbb{C}$–spans yield new Lie superalgebras for all $\sigma \in V$. These form four more bundles (depending on the chosen basis) of Lie superalgebras over $V$, which all coincide with $\{g(\sigma)\}_{\sigma \in V}$ on the general locus of $V$ but not on the singular one, where their fibers are again non-simple and non-isomorphic to those of $\{g(\sigma)\}_{\sigma \in V}$.

We would better point out, here, the key point of the whole story. The problem with “critical” values of the parameter $a$ in Kac’ family of Lie superalgebras of type $D(2, 1; a)$ is that, in terms of Kac’ construction, the very definition of the Lie superalgebra is problematic for these critical values. Therefore, to overcome this obstruction we have to resort to a different description of these Lie superalgebras: to this end, we select a specific $\mathbb{C}$–basis, and define the Lie superalgebra as its $\mathbb{C}$–span, which makes sense for critical values too. The critical step then is the choice of such a basis: it is irrelevant at non-singular values, but it makes a difference at singular ones. As we can choose different bases, we end up with families of Lie superalgebras that coincide at non-singular values — when we still have to do with Kac’ original objects — but definitely differ from each other at singular ones.

As a second step, we perform the same operation at the level of Lie supergroups. Namely, for each $\sigma \in V$ we “complete” the Lie superalgebra $g(\sigma)$ to form a super Harish-Chandra pair, and then take the corresponding (complex holomorphic) Lie supergroup: this yields a family $\{G(\sigma)\}_{\sigma \in V}$ of Lie supergroups, with $G(\sigma)$ isomorphic to $G_a$ for a suitable $a \in \mathbb{C} \setminus \{0, -1\}$ for non-singular values of $\sigma$, while $G_a$ is not simple for singular values instead. Moreover, the group $G := \mathbb{C}^* \times \mathfrak{g}_3$ freely acts on this family via Lie supergroup isomorphisms.

In other words, we complete the “classical” family (with $G$–action) provided by the simple Lie supergroups $G_a$’s (isomorphic to suitable $G_a$’s) by suitably adding new, non-simple Lie supergroups at singular values of $\sigma$. Moreover, the same construction applies to the other four, above mentioned families of Lie superalgebras indexed by $V$ that complete the family $\{g(\sigma)\}_{\sigma \in V \setminus (\bigcup_{i=1}^3 \{\sigma_i = 0\})}$. In short, we have then five bundles of Lie supergroups over $V$, each endowed with a $G$–action, that complete in different ways the family of simple Lie supergroups springing out of Kac’ original construction.

Finally, we remark that all this analysis might be reformulated in the formal language of deformation theory of supermanifold — e.g., as treated in [Va] — thus leading to describe the moduli space of structures of the family of the supergroups $G(\sigma)$, etc. However, this goes beyond the scope of the present article; we leave it to further, separate investigations.

This article is organized as follows. In Section 2, we briefly recall the basic algebraic background necessary for this article, in particular, certain language about supermathematics. In Section 3, we introduce our Lie superalgebras $g(\sigma)$ of type $D(2, 1; a)$. Several integral forms of the Lie superalgebra $g(\sigma)$ are introduced in Section 4. In particular, as an application, the structure of their singular degenerations is studied in detail (Theorems 4.1.1, 4.2.1, 4.3.1, 4.4.1 and 4.4.2). Section 5 is the highlight of this article, where we introduce and analyze the Lie supergroups whose Lie superalgebras are studied in Section 4. The structure of the corresponding Lie supergroups is also analyzed (Theorems 5.1.1, 5.2.1, 5.3.1, 5.4.1 and 5.5.1).
As the main objects treated in this article have many special features, most of the above descriptions are given in a down-to-earth manner, so that even the readers who are not familiar with the subject could follow easily our exposition.

2. Preliminaries

In this section, we recall the notions and language of Lie superalgebras and Lie supergroups. Our purpose is to fix the terminology, but everything indeed is standard matter.

2.1. Basic superobjects. All throughout the paper, we work over the field \( \mathbb{C} \) of complex numbers (nevertheless, immediate generalizations are possible), unless otherwise stated. By \( \mathbb{C} \)-supermodule, or \( \mathbb{C} \)-super vector space, any \( \mathbb{C} \)-module \( V \) endowed with a \( \mathbb{Z}_2 \)-grading \( V = V_0 \oplus V_1 \), where \( \mathbb{Z}_2 = \{0, 1\} \) is the group with two elements. Then \( V_0 \) and its elements are called even, while \( V_1 \) and its elements odd. By \( |x| \) or \( p(x) \) \((\in \mathbb{Z}_2)\) we denote the parity of any homogeneous element, defined by the condition \( x \in V_{|x|} \).

We call \( \mathbb{C} \)-superalgebra any associative, unital \( \mathbb{C} \)-algebra \( A \) which is \( \mathbb{Z}_2 \)-graded: so \( A \) has a \( \mathbb{Z}_2 \)-splitting \( A = A_0 \oplus A_1 \), and \( A_a A_b \subseteq A_{a+b} \). Any such \( A \) is said to be commutative if \( xy = (-1)^{|x||y|}yx \) for all homogeneous \( x, y \in A \); so, in particular, \( z^2 = 0 \) for all \( z \in A_1 \). All \( \mathbb{C} \)-superalgebras form a category, whose morphisms are those of unital \( \mathbb{C} \)-algebras preserving the \( \mathbb{Z}_2 \)-grading; inside it, commutative \( \mathbb{C} \)-superalgebras form a subcategory, that we denote by \( \text{salg} \). We denote by \( \text{alg} \) the category of (associative, unital) commutative \( \mathbb{C} \)-algebras, and by \( \text{mod} \) that of \( \mathbb{C} \)-modules. Note also that there is an obvious functor \( (\cdot)_0 : \text{salg} \to \text{alg} \) given on objects by \( A \mapsto A_0 \).

We call Weil superalgebra any finite-dimensional commutative \( \mathbb{C} \)-superalgebra \( A \) such that \( A = \mathbb{C} \oplus \mathfrak{N}(A) \) where \( \mathbb{C} \) is even and \( \mathfrak{N}(A) = \mathfrak{N}(A)_0 \oplus \mathfrak{N}(A)_1 \) is a \( \mathbb{Z}_2 \)-graded nilpotent ideal (the nilradical of \( A \)). Every Weil superalgebra \( A \) is endowed with the canonical epimorphisms \( p_A : A \twoheadrightarrow \mathbb{C} \) and \( u_A : \mathbb{C} \hookrightarrow A \), such that \( p_A \circ u_A = id_\mathbb{C} \). Weil superalgebras over \( \mathbb{C} \) form a full subcategory of \( \text{salg}_\mathbb{C} \), denoted by \( \text{Wsalg}_\mathbb{C} \) or \( \text{Wsalg}_\mathbb{C} \). Finally, let \( \text{Walg}_\mathbb{C} := \text{Wsalg}_\mathbb{C} \cap \text{alg}_\mathbb{C} \) — also denoted by \( \text{Walg} \) — be the category of Weil algebras (over \( \mathbb{C} \)), i.e., the full subcategory of all totally even objects in \( \text{Wsalg}_\mathbb{C} \) — namely, those whose odd part is trivial. Then the functor \( (\cdot)_0 : \text{salg} \to \text{alg} \) obviously restricts to a similar functor \( (\cdot)_0 : \text{Wsalg} \to \text{Walg} \) given again by \( A \mapsto A_0 \).

2.2. Lie superalgebras. By definition, a Lie superalgebra is a \( \mathbb{C} \)-supermodule \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) with a \( \text{(Lie super)bracket} \ [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \ (x, y) \mapsto [x, y] \), which is \( \mathbb{C} \)-bilinear, preserving the \( \mathbb{Z}_2 \)-grading and satisfies the following (for all homogeneous \( x, y, z \in \mathfrak{g} \)):

\[
\begin{align*}
(a) & \quad [x, y] + (-1)^{|x||y|}[y, x] = 0 \quad \text{(anti-symmetry)}; \\
(b) & \quad (-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[y, [x, z]] + (-1)^{|z||y|}[z, [x, y]] = 0 \quad \text{(Jacobi identity)}.
\end{align*}
\]

In this situation, we write \( Y^{(2)} := 2^{-1}[Y, Y] \ (\in \mathfrak{g}_0) \) for all \( Y \in \mathfrak{g}_1 \).

All Lie \( \mathbb{C} \)-superalgebras form a category, denoted by \( \text{slie}_\mathbb{C} \) or just \( \text{slie} \), whose morphisms are \( \mathbb{C} \)-linear, preserving the \( \mathbb{Z}_2 \)-grading and the bracket. Note that if \( \mathfrak{g} \) is a Lie \( \mathbb{C} \)-superalgebra, then its even part \( \mathfrak{g}_0 \) is automatically a Lie \( \mathbb{C} \)-algebra.

Lie superalgebras can also be described in functorial language. Indeed, let \( \text{Lie}_\mathbb{C} \) be the category of Lie \( \mathbb{C} \)-algebras. Then every Lie \( \mathbb{C} \)-superalgebra \( \mathfrak{g} \in \text{slie}_\mathbb{C} \) defines a functor
2.3. Lie supergroups. We shall now recall, in steps, the notion of complex holomorphic “Lie supergroups”, as a special kind of “supermanifold”.

2.3.1. Supermanifolds. By superspace we mean a pair \( S = (|S|, \mathcal{O}_S) \) of a topological space \(|S|\) and a sheaf of commutative superalgebras \( \mathcal{O}_S \) on it such that the stalk \( \mathcal{O}_{S,x} \) of \( \mathcal{O}_S \) at each point \( x \in |S| \) is a local superalgebra. A morphism \( \phi : S \to T \) between superspaces \( S \) and \( T \) is a pair \((|\phi|, \phi^*)\) where \(|\phi| : |S| \to |T|\) is a continuous map of topological spaces and the induced morphism \( \phi^* : \mathcal{O}_T \to |\phi|_*\mathcal{O}_S \) of sheaves on \( |T| \) is such that \( \phi^*_x(\mathfrak{m}_{|\phi|(x)}) \subseteq \mathfrak{m}_x \), where \( \mathfrak{m}_{|\phi|(x)} \) and \( \mathfrak{m}_x \) denote the maximal ideals in the stalks \( \mathcal{O}_{S,x} \) and \( \mathcal{O}_{T,x} \) respectively.

As basic model, the superspace \( \mathbb{C}^{p|q} \) is defined to be the topological space \( \mathbb{C}^p \) endowed with the following sheaf of commutative superalgebras: \( \mathcal{O}_{\mathbb{C}^{p|q}}(U) := \mathcal{H}_{\mathbb{C}^p}(U) \otimes_{\mathbb{C}} \Lambda_{\mathbb{C}}(\xi_1, \ldots, \xi_q) \) for any open set \( U \subseteq \mathbb{C}^p \), where \( \mathcal{H}_{\mathbb{C}^p} \) is the sheaf of holomorphic functions on \( \mathbb{C}^p \) and \( \Lambda_{\mathbb{C}}(\xi_1, \ldots, \xi_q) \) is the complex Grassmann algebra on \( q \) variables \( \xi_1, \ldots, \xi_q \) of odd parity.

A (complex holomorphic) supermanifold of (super)dimension \( p|q \) is a superspace \( M = (|M|, \mathcal{O}_M) \) such that \(|M|\) is Hausdorff and second-countable and \( M \) is locally isomorphic to \( \mathbb{C}^{p|q} \), i.e., for each \( x \in |M| \) there is an open set \( V_x \subseteq |M| \) with \( x \in V_x \) and \( U \subseteq \mathbb{C}^p \) such that \( \mathcal{O}_M|_{V_x} \cong \mathcal{O}_{\mathbb{C}^{p|q}}|_U \) (in particular, it is locally isomorphic to \( \mathbb{C}^{p|q} \)). A morphism between holomorphic supermanifolds is just a morphism (between them) as superspaces.

We denote the category of (complex holomorphic) supermanifolds by \( \text{(hsmfd)} \).

Let now \( M \) be a holomorphic supermanifold and \( U \) an open subset in \(|M|\). Let \( \mathcal{I}_M(U) \) be the (nilpotent) ideal of \( \mathcal{O}_M(U) \) generated by the odd part of the latter: then \( \mathcal{O}_M/\mathcal{I}_M \) defines a sheaf of purely even superalgebras over \(|M|\), locally isomorphic to \( \mathcal{H}_{\mathbb{C}^p} \). Then \( M_{rd} := (|M|, \mathcal{O}_M/\mathcal{I}_M) \) is a classical holomorphic manifold, called the underlying holomorphic (sub)manifold of \( M \); the standard projection \( s \mapsto \tilde{s} := s + \mathcal{I}_M(U) \) (for all \( s \in \mathcal{O}_M(U) \)) at the sheaf level yields an embedding \( M_{rd} \hookrightarrow M \), so \( M_{rd} \) can be seen as an embedded sub(super)manifold of \( M \). The whole construction is clearly functorial in \( M \).

Finally, each “classical” manifold can be seen as a “supermanifold”, just regarding its structure sheaf as one of superalgebras that are actually totally even, i.e. with trivial odd part. Conversely, any supermanifold enjoying the latter property is actually a “classical” manifold, nothing more. In other words, classical manifolds identify with those supermanifolds \( M \) that actually coincide with their underlying (sub)manifolds \( M_{rd} \).

2.3.2. Lie supergroups and the functorial approach. A group object in the category \( \text{(hsmfd)} \) is called (complex holomorphic) Lie supergroup. These objects, together with the obvious morphisms, form a subcategory among supermanifolds, denoted \( \text{(Lsgrp)}_C \).

Lie supergroups — as well as supermanifolds — can also be conveniently studied via a functorial approach that we now briefly recall (cf. [BCF] or [Ga3] for details).

Let \( M \) be a supermanifold. For every \( x \in |M| \) and every \( A \in \text{(Wsalg)} \) we set \( M_{A,x} := \text{Hom}_{\text{(salg)}}(\mathcal{O}_{M,x}, A) \) and \( M_A = \bigsqcup_{x \in |M|} M_{A,x} \); then we define \( W_M : \text{(Wsalg)} \to \text{(set)} \) to be the “Weil-Berezin” functor given by \( A \mapsto M_A \) and \( \rho \mapsto \rho^{(M)} \) with \( \rho^{(M)} : M_A \to M_B \),
$x_A \mapsto \rho \circ x_A$. Overall, this provides a functor $B : (\text{hsmfd}) \to ([\text{Wsalg}], \text{(set)})$ given on objects by $M \mapsto W_M$; we can now refine still more.

Given a finite dimensional commutative algebra $A_0$ over $\mathbb{C}$, a (complex holomorphic) $A_0$–manifold is any manifold that is locally modelled on some open subset of some finite dimensional $A_0$–module, so that the differential of every change of charts is an $A_0$–module isomorphism. An $A_0$–morphism between two $A_0$–manifolds is any morphism whose differential is everywhere $A_0$–linear. Gathering all $A_0$–manifolds (for all possible $A$), and suitably defining morphisms among them, one defines the category $(A_0 – \text{hmfd})$ of all “$A_0$–manifolds”.

The first key point now is that each functor $W_M$ actually is valued into $(A_0 – \text{hmfd})$. Furthermore, let $[[([\text{Wsalg}], (A_0 – \text{hmfd}))]]$ be the subcategory of $[[([\text{Wsalg}], (A_0 – \text{hmfd}))]]$ with the same objects but whose morphisms are all natural transformations $\phi : G \to H$ such that for every $A \in (\text{Wsalg})$ the induced $\phi_A : G(A) \to H(A)$ is $A_0$–smooth. Then the second key point is that if $\phi : M \to N$ is a morphism of supermanifolds, then $\phi_A$ is a morphism in $[[([\text{Wsalg}], (A_0 – \text{hmfd}))]]$. The final outcome is that we have a functor $S : (\text{hsmfd}) \to [[([\text{Wsalg}], (A_0 – \text{hmfd}))]]$, given on objects by $M \mapsto W_M$; the key result is that “this embedding is full and faithful”, so that for any two supermanifolds $M$ and $N$ one has $M \cong N$ if and only if $S(M) \cong S(N)$, i.e. $W_M \cong W_N$.

Still relevant to us, is that the embedding $S$ preserves products, hence also group objects. Therefore, a supermanifold $M$ is a Lie supergroup if and only if $S(M) := W_M$ takes values in the subcategory (among $A_0$–manifolds) of group objects — thus each $W_M(A)$ is a group.

Finally, in the functorial approach the “classical” manifolds (i.e., totally even supermanifolds) can be recovered as follows: in the previous construction one simply has to replace the words “Weil superalgebras” with “Weil algebras” everywhere. It then follows, in particular, that the Weil-Berezin functor of points $W_M$ of any holomorphic, manifold $M$ is actually a functor from $\text{Walg}$ to $(A_0 – \text{hmfd})$; one can still see it as (the Weil-Berezin functor of points of) a supermanifold — that is totally even, though — by composing it with the natural functor $(\rho) : (\text{Wsalg}) \to (\text{Walg})$. On the other hand, given any supermanifold $M$, say holomorphic, the Weil-Berezin functor of points of its underlying submanifold $M_{rd}$ is given by $W_{M_{rd}}(A) = W_M(A)$ for each $A \in (\text{Walg})$, or in short $W_{M_{rd}} = W_M|_{(\text{Walg})}$.

Finally, it is worth stressing that the functorial point of view on supermanifolds was originally developed — by Leites, Berezin, Deligne, Molotkov, Voronov and many others — in a slightly different way. Namely, they considered functors defined, rather than on Weil superalgebras, on Grassmann (super)algebras. Actually, the two approaches are equivalent: see [BCF] for a detailed, critical analysis of the matter.

There are some advantages in restricting the focus onto Grassmann algebras. For instance, they are the sheaf of the superdomains of dimensione 0|q — i.e., “super-points”. Therefore, if $M$ is a supermanifold considered as a super-ringed space, its description via a functor defined on Grassmann algebras (only) can be really seen as the true restriction of the functor of points of $M$, considered as a super-ringed space. Moreover, using Grassmann algebras is consistent with the development of differential super-calculus “à la De Witt”.

On the other hand, the use of Weil superalgebras has the advantage that one can use it to perform differential calculus on Weil-Berezin’s functors, much in the spirit of Weil’s approach to differential calculus in algebraic geometry — something one cannot achieve working with Grassmann algebras only: e.g., the tangent bundle to a supermanifold, or “super-vectors” (rather than super-points) and “super-jets”, or point-supported distributions, or Weil’s Transitivity Theorem, etc. Note also that some peculiar properties for Grassmann algebras are still available for Weil superalgebras: e.g., the existence of “body” and “soul”, key tools in all
the theory (for instance, for any Lie supergroup $G$ this implies the existence of a semidirect product splitting of the group $G(A)$ of $A$-points of $G$). See [BCF] for further details. In addition, Weil-Berezin functors based on Weil superalgebras (rather than Grassmann algebras only) have been also extended to a broader class of superspaces (including supermanifolds), cf. [AHW]. So the approach via Weil superalgebras seems, in a sense, more powerful.

2.4. Super Harish-Chandra pairs and Lie supergroups. A different way to deal with Lie supergroups (or algebraic supergroups) is via the notion of “super Harish-Chandra pair”, that gathers together the infinitesimal counterpart — that of Lie superalgebra — and the classical (i.e. “non-super”) counterpart — that of Lie group — of the notion of Lie supergroup. We recall it shortly, referring to [Ga3] (and [Ga2]) for further details.

2.4.1. Super Harish-Chandra pairs. We call super Harish-Chandra pair — or just “sHCp” in short — any pair $(G, g)$ such that $G$ is a (complex holomorphic) Lie group, $g$ a complex Lie superalgebra such that $g_0 = Lie(G)$, and there is a (holomorphic) $G$-action on $g$ by Lie superalgebra automorphisms, denoted by $Ad : G \longrightarrow Aut(g)$, such that its restriction to $g_0$ is the adjoint action of $G$ on $Lie(G) = g_0$ and the differential of this action is the restriction to $Lie(G) \times g = g_0 \times g$ of the adjoint action of $g$ on itself. Then a morphism $(\Omega, \omega) : (G', g') \longrightarrow (G'', g'')$ between sHCp’s is given by a morphism of Lie groups $\Omega : G' \longrightarrow G''$ and a morphism of Lie superalgebras $\omega : g' \longrightarrow g''$ such that $\omega|_{g_0} = d\Omega$ and $\omega \circ Ad_g = Ad_{\Omega(g)} \circ \omega$ for all $g \in G$.

We denote the category of all super Harish-Chandra pairs by $(sHCp)$.

2.4.2. From Lie supergroups to sHCp’s. For any $A \in (Wsalg)$, let $A[\varepsilon] := A[x]/(x^2)$, with $\varepsilon := x \mod (x^2)$ being even. Then $A[\varepsilon] = A \oplus A \varepsilon \in (Wsalg)$, and there exists a natural morphism $p_A : A[\varepsilon] \longrightarrow A$ given by $(a + a' \varepsilon)^{p_A} = a$. For a Lie supergroup $G$, thought of as a functor $G : (Wsalg) \longrightarrow (groups)$ — i.e. identifying $G \cong W_G$ — let $G(p_A) : G(A[\varepsilon]) \longrightarrow G(A)$ be the morphism associated with $p_A : A[\varepsilon] \longrightarrow A$. Then there exists a unique functor $Lie(G) : (Wsalg) \longrightarrow (groups)$ given on objects by $Lie(G)(A) := Ker(G(p_A)_A)$. The key fact now is that $Lie(G)$ is actually valued in the category (Lie) of Lie algebras, i.e. it is a functor $Lie(G) : (Wsalg) \longrightarrow (Lie)$. Furthermore, there exists a Lie superalgebra $g$ — identified with the tangent superspace to $G$ at the unit point — such that $Lie(G) = L_g$ (cf. [222]). Moreover, for $A \in (Wsalg)$ one has $Lie(G)(A) = Lie(G(A))$, the latter being the tangent Lie algebra of the Lie group $G(A)$.

Finally, the construction $G \mapsto Lie(G)$ for Lie supergroups is actually natural, i.e. provides a functor $Lie : (Lsgrp)_C \longrightarrow (sLie)$ from Lie supergroups to Lie superalgebras.

On the other hand, each Lie supergroup $G$ is a group object in the category of (holomorphic) supermanifolds; therefore, its underlying submanifold $G_{rd}$ is in turn a group object among (holomorphic) manifolds, i.e. it is a Lie group. More precisely, the naturality of the construction $G \mapsto G_{rd}$ provides a functor from Lie supergroups to (complex) Lie groups.

On top of this analysis, if $G$ is any Lie supergroup then $(G_{rd}, Lie(G))$ is a super Harish-Chandra pair; more precisely, we have a functor $\Phi : (Lsgrp)_C \longrightarrow (sHCp)$ given on objects by $G \mapsto (G_{rd}, Lie(G))$ and on morphisms by $\phi \mapsto (\phi_{rd}, Lie(\phi))$.

2.4.3. From sHCp’s to Lie supergroups. The functor $\Phi : (Lsgrp)_C \longrightarrow (sHCp)$ has a quasi-inverse $\Psi : (sHCp) \longrightarrow (Lsgrp)_C$ that we can describe explicitly (see [Ga3], [Ga2]).
Indeed, let \( \mathcal{P} := (G, g) \) be a super Harish-Chandra pair, and let \( B := \{ Y_i \}_{i \in I} \) be a \( \mathbb{C} \)-basis of \( g_1 \). For any \( A \in (\text{Wsalg}) \), we define \( G_p(A) \) as being the group with generators the elements of the set \( \Gamma^p_A := G(A) \bigcup \{(1 + \eta_i Y_i)\}_{(i,\eta_i) \in I \times \mathcal{A}_1} \) and relations

\[
1_g = 1 \quad , \quad g' \cdot g'' = g' \circ g'' \\
(1 + \eta_i Y_i) \cdot g = g \cdot (1 + c_{ji} \eta_j Y_{ji}) \cdot \cdots \cdot (1 + c_{jk} \eta_j Y_{jk})
\]

with \( \text{Ad}(g^{-1})(Y_i) = c_{ji} Y_{ji} + \cdots + c_{jk} Y_{jk} \)

\[
(1 + \eta_i Y_i) \cdot (1 + \eta_j Y_j) = \left(1 + \eta_i \eta_j Y_{ij}\right) \cdot (1 + \eta_j Y_j) \cdot (1 + \eta_i Y_i)
\]

for \( g, g', g'' \in G(A) \), \( \eta_i, \eta_j, \eta''_i, \eta_j \in \mathcal{A}_1 \), \( i, j \in I \). This defines the functor \( G_p \) on objects, and one then defines it on morphisms as follows: for any \( \varphi : A' \rightarrow A'' \) in \( (\text{Wsalg}) \) we let \( G_p(\varphi) : G_p(A') \rightarrow G_p(A'') \) be the group morphism uniquely defined on generators by \( G_p(\varphi)(g') := G(\varphi)(g') \), \( G_p(\varphi)(1 + \eta_i Y_i) := (1 + \varphi(\eta_i) Y_i) \).

One proves (see [Ga3], [Ga2]) that every such \( G_p \) is in fact a Lie supergroup — thought of as a special functor, i.e. identified with its associated Weil-Berezin functor. In addition, the construction \( \mathcal{P} \mapsto G_p \) is natural in \( \mathcal{P} \), i.e. it yields a functor \( \Psi : (\text{sHCp}) \rightarrow (\text{Lsgrp})_\mathbb{C} \); moreover, the latter is a quasi-inverse to \( \Phi : (\text{Lsgrp})_\mathbb{C} \rightarrow (\text{sHCp}) \).

3. Lie superalgebras of type \( D(2,1;\sigma) \)

In this section, we introduce the complex Lie superalgebras that in Kac’ classification (cf. [K]) are labeled as of type \( D(2,1;a) \); we do follow Kac’ approach, but starting with a \( \mathfrak{G}_3 \)-symmetric Dynkin diagram, which makes evident the internal \( \mathfrak{G}_3 \)-symmetry of the family of all these Lie superalgebras — in fact, we recover Kaplansky-Scheunert’s presentation of them (see [Kap] and [SE]). We remark that this approach is essentially the same as starting with some Cartan matrix, where the existence of its internal \( \mathfrak{G}_3 \)-symmetry is less evident.

Then, choosing special \( \mathbb{Z} \)-integral forms of these objects, we find the “degenerations” (i.e., singular specializations) of these integral forms at critical points of the parameter space.

3.1. Definition via Dynkin diagram.

The Lie superalgebras we are interested in depend on a parameter, which can be conveniently given by a triple \( \sigma := (\sigma_1, \sigma_2, \sigma_3) \in (\mathbb{C}^*)^3 \cap \{ \sigma_1 + \sigma_2 + \sigma_3 = 0 \} \). This enters in the very definition of each Lie superalgebra \( g = g_\sigma \), which is given by a presentation as in [K].

3.1.1. Dynkin diagram I. For any given \( \sigma := (\sigma_1, \sigma_2, \sigma_3) \in (\mathbb{C}^*)^3 \cap \{ \sigma_1 + \sigma_2 + \sigma_3 = 0 \} \) we consider the Dynkin diagram
To this diagram, one associates the so-called Cartan matrix given by

$$A_\sigma = (a_{i,j})_{i,j=1,2,3} = \begin{pmatrix} 0 & -\sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & -\sigma_1 \\ -\sigma_2 & -\sigma_1 & 0 \end{pmatrix}$$

This Cartan matrix, up to some minor detail, seems to be first appeared in [KV]. It was shown that there is a simple Lie algebra (not superalgebra!) defined over a field $k$ of characteristic 2 associated with this Cartan matrix. Notice that it was parametrized by $(\sigma_1, \sigma_2, \sigma_3) = -(a + 1, a, 1)$ with $a \in k \setminus \overline{F}_2$.

Let $\{ \mathfrak{h}, \Pi^\vee = \{ H_\beta \}_{i=1,2,3}; \Pi = \{ \beta \}_{i=1,2,3} \}$ be the realization of $A_\sigma$, that is

1. $\mathfrak{h}$ is a $\mathbb{C}$-vector space,
2. $\Pi^\vee$ is the set of simple coroots, a basis of $\mathfrak{h}^*$,
3. $\Pi$ is the set of simple roots, a basis of $\mathfrak{h}^*$,
4. $\beta_j(H_\beta) = a_{i,j}$ for all $1 \leq i, j \leq 3$.

The Lie superalgebra $\mathfrak{g} = \mathfrak{g}_\sigma$ is, by definition, the simple Lie superalgebra generated by $\{ H_\beta, X_{\pm\beta} \}_{i,j=1,2,3}$ satisfying, at least the relations (for $1 \leq i, j \leq 3$)

$$[H_\beta_i, H_\beta_j] = 0 \quad [H_\beta_i, X_{\pm\beta_j}] = \mp \beta_j(H_\beta_i) X_{\pm\beta_j}$$

$$[X_{\beta_i}, X_{-\beta_j}] = \delta_{i,j} H_\beta_i \quad [X_{\pm\beta_i}, X_{\pm\beta_i}] = 0$$

with parity $|H_\beta_i| = \bar{0}$ and $|X_{\pm\beta_i}| = \bar{1}$ for all $i$. We remark that the set $\Delta^+$ of positive roots has the following description:

$$\Delta^+ = \{ \beta_1, \beta_2, \beta_3, \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_3 + \beta_1, \beta_1 + \beta_2 + \beta_3 \}$$

The dual $\mathfrak{h}^*$ of the Cartan subalgebra has the following description: let $\{ \varepsilon_i \}_{i=1,2,3} \subset \mathfrak{h}^*$ be an orthogonal basis normalized by the conditions $(\varepsilon_i, \varepsilon_i) = -\frac{1}{2} \sigma_i$ ($i = 1, 2, 3$). One can verify that $(\beta_i, \beta_j) = -\sigma_k$ with $\{ i, j, k \} = \{ 1, 2, 3 \}$, where the simple roots are

$$\beta_1 = -\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad \beta_2 = \varepsilon_1 - \varepsilon_2 + \varepsilon_3 \quad \beta_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon_3$$

**Remark 3.1.1.** Let $\Delta^+_0$ and $\Delta^+_1$ be the set of even (resp. odd) positive roots. One has

$$\Delta^+_0 = \{ \beta_1 + \beta_2, \beta_2 + \beta_3, \beta_3 + \beta_1 \} = \{ 2\varepsilon_i \mid 1 \leq i \leq 3 \} \quad \Delta^+_1 = \{ \beta_1, \beta_2, \beta_3, \theta \}$$

where $\theta = \beta_1 + \beta_2 + \beta_3 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ is the highest root.

We set now

$$X_{2\varepsilon_1} := [X_{\beta_2}, X_{\beta_3}] \quad X_{2\varepsilon_2} := [X_{\beta_3}, X_{\beta_1}] \quad X_{2\varepsilon_3} := [X_{\beta_1}, X_{\beta_2}]$$

$$X_{-2\varepsilon_1} := -[X_{-\beta_2}, X_{-\beta_3}] \quad X_{-2\varepsilon_2} := -[X_{-\beta_3}, X_{-\beta_1}] \quad X_{-2\varepsilon_3} := -[X_{-\beta_1}, X_{-\beta_2}]$$

$$H_{2\varepsilon_1} := -(H_{\beta_2} + H_{\beta_3}) \quad H_{2\varepsilon_2} := -(H_{\beta_3} + H_{\beta_1}) \quad H_{2\varepsilon_3} := -(H_{\beta_1} + H_{\beta_2})$$
It can be checked that, for \( i, j \in \{1, 2, 3\} \), one has
\[
[X_{2i}, X_{-2j}] = \sigma_i \delta_{i,j} X_{2e_i}, \quad [H_{2e_i}, X_{\pm 2e_j}] = \pm 2 \sigma_i \delta_{i,j} X_{\pm 2e_j}
\] (3.1)
which implies that each \( a_i := \mathbb{C}X_{2e_i} \oplus \mathbb{C}X_{-2e_i} \) (for \( 1 \leq i \leq 3 \)) is a Lie sub-
(super)algebra, with \( [a_j, a_k] = 0 \) for \( j \neq k \), and \( a_i \) is isomorphic to \( \mathfrak{sl}_2 \) since \( \sigma_i \neq 0 \). In
particular, the even part \( g_0 \) of the Lie superalgebra \( g \) can be described as \( g_0 = \bigoplus_{i=1}^{3} a_i \).

Now, for \( 1 \leq i \leq 3 \), if we set \( X_\theta = [X_{2i}, X_{\beta_i}] \in g_\theta \) and \( X_{-\theta} = [X_{-2i}, X_{-\beta_i}] \in g_{-\theta} \),
the following identities hold:
\[
\sum_{i=1}^{3} X_{\pm \theta}^i = 0, \quad [X_\theta, X_{-\theta}] = -\sigma_j \sigma_k \sum_{i=1}^{3} H_{\beta_i}.
\] (3.2)
These formulas imply that there exists \( X_\theta \in g_\theta \) and \( X_{-\theta} \in g_{-\theta} \) such that
\[
X_\theta^i = \sigma_i X_\theta, \quad X_{-\theta}^i = \sigma_i X_{-\theta}
\] (3.3)
for any \( 1 \leq i \leq 3 \). Hence, setting \( H_\theta = -(H_{\beta_1} + H_{\beta_2} + H_{\beta_3}) \), one has also \( [X_\theta, X_{-\theta}] = H_\theta \).
Moreover, it also follows that \( H_{2e_1} + H_{2e_2} + H_{2e_3} = 2 H_\theta \).

The odd part \( g_1 \) of the Lie superalgebra \( g := g_{\sigma} \) is the \( C \)-span of \( \{ X_{\pm \theta} \}_{i=1,2,3} \cup \{ X_{\pm \theta} \} \).
Now, \( g_1 \) as \( g_0 := sl_2^{(1)} \)-module is isomorphic to \( \boxtimes_1 \boxtimes_2 \boxtimes_3 \), where \( \boxtimes_i := \mathbb{C}|+\rangle \oplus \mathbb{C}|\rangle \)
is the tautological 2–dimensional module over the \( i \)–th copy \( sl_2^{(i)} \) of \( sl_2 \). An explicit isomorphism
is described as follows:
\[
\begin{align*}
|+\rangle \otimes |+\rangle \otimes |+\rangle & \quad \mapsto \quad X_\theta, \\
|+\rangle \otimes |-\rangle \otimes |+\rangle & \quad \mapsto \quad X_{\beta_2}, \\
|\rangle \otimes |+\rangle \otimes |-\rangle & \quad \mapsto \quad X_{-\beta_1}, \\
|\rangle \otimes |-\rangle \otimes |+\rangle & \quad \mapsto \quad X_{-\beta_3},
\end{align*}
\]

\textbf{Remark 3.1.2.} By our normalization, the non-trivial actions of each \( a_i \) on \( \square \) are given by
\[
H_{2e_i} |\pm\rangle = \pm \sigma_{\pm} |\pm\rangle, \quad X_{\pm 2e_i} |\mp\rangle = \sigma_{\pm} |\mp\rangle.
\]

Note that this realization was known to I. Kaplansky [Kap] and was denoted by \( \Gamma(A, B, C) \) for suitable \( A, B, C \); it has been explained in an accessible form in [Sze].

\textbf{3.1.2. The Lie bracket} \( g_1 \times g_1 \rightarrow g_0 \). The \( g_0 \)-module structure we described in the previous
subsection inspired us to think of describing the Lie superalgebra \( g_{\sigma} \) completely in terms
of \( sl_2 \) (such a construction was known to M. Scheunert [Sze], as we explain below). To
be precise, the only structure we are left to describe is the restriction \( [\ , \ ] : g_1 \times g_1 \rightarrow g_0 \)
in terms of “\( sl_2 \)-language”. Clearly, it is enough to record only the non-zero values of this
bracket among basis elements; these are the following:
\[
\begin{align*}
\langle [+\rangle \otimes [+\rangle \otimes [+\rangle, [+\rangle \otimes [-\rangle \otimes [-\rangle \rangle & = -X_{2e_1}, \\
\langle [+\rangle \otimes [+\rangle \otimes [+\rangle, [-\rangle \otimes [+\rangle \otimes [-\rangle \rangle & = -X_{2e_2}, \\
\langle [+\rangle \otimes [+\rangle \otimes [+\rangle, [-\rangle \otimes [-\rangle \otimes [+\rangle \rangle & = -X_{2e_3}, \\
\langle [+\rangle \otimes [-\rangle \otimes [+\rangle, [-\rangle \otimes [-\rangle \otimes [-\rangle \rangle & = H_{\beta_1}, \\
\langle [+\rangle \otimes [-\rangle \otimes [+\rangle, [-\rangle \otimes [+\rangle \otimes [-\rangle \rangle & = H_{\beta_2}, \\
\langle [+\rangle \otimes [-\rangle \otimes [+\rangle, [-\rangle \otimes [-\rangle \otimes [+\rangle \rangle & = H_{\beta_3}, \\
\langle [-\rangle \otimes [+\rangle \otimes [-\rangle, [-\rangle \otimes [-\rangle \otimes [-\rangle \rangle & = -X_{-2e_1}, \\
\langle [-\rangle \otimes [+\rangle \otimes [+\rangle, [-\rangle \otimes [-\rangle \otimes [-\rangle \rangle & = -X_{-2e_2}, \\
\langle [+\rangle \otimes [-\rangle \otimes [+\rangle, [-\rangle \otimes [+\rangle \otimes [-\rangle \rangle & = -X_{-2e_3}.
\end{align*}
\]
Let us interpret these formulas purely in terms of $\mathfrak{sl}_2$–theory.

Let $\psi: \square^2 \cong S^2 \square \oplus \Lambda^2 \square \rightarrow \wedge^2 \square \cong \mathbb{C}$ be the projection defined by

$$|\pm\rangle \otimes |\pm\rangle \mapsto 0, \quad |\pm\rangle \otimes |\mp\rangle \mapsto \pm \frac{1}{2} \sigma f.$$

For $\sigma \in \mathbb{C}^*$, we define the linear map $p: \square^3 \cong S^2 \square \oplus \Lambda^2 \square \rightarrow S^2 \square \cong \mathfrak{sl}_2$ by

$$p(u, v, w) := \sigma(\psi(w, v).u - \psi(w, u).v) \quad \forall u, v, w \in \square.$$

One can write down this map explicitly as follows:

$$|+\rangle \otimes |+\rangle \mapsto \sigma e, \quad |\pm\rangle \otimes |\mp\rangle \mapsto - \frac{1}{2} \sigma h, \quad |-\rangle \otimes |-\rangle \mapsto - \sigma f$$

where $\{e, h, f\}$ is the standard $\mathfrak{sl}_2$–triple, i.e. $[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f$.

Now, for each $i \in \{1, 2, 3\}$, denote by $p_i: \square^3 \rightarrow \mathfrak{sl}_2^{(i)} \cong \mathfrak{a}_i$ the above map with scalar factor $\sigma$ given by $-2\sigma_i \in \mathbb{C}^*$. To be precise, the map $p_i: \square^3 \rightarrow \mathfrak{a}_i$ is defined by

$$|+\rangle \otimes |+\rangle \mapsto -2 X_{+2i}, \quad |\pm\rangle \otimes |\mp\rangle \mapsto H_{2i}, \quad |-\rangle \otimes |-\rangle \mapsto 2 X_{-2i}.$$

It can be verified that the Lie superbracket $[ , ]$ on $\mathfrak{g}_1 \times \mathfrak{g}_1$ can be expressed as

$$[\otimes_{i=1}^3 u_i, \otimes_{i=1}^3 v_i] = \sum_{\tau \in \mathfrak{S}_3} \psi(u_{\tau(1)}, v_{\tau(1)}) \psi(u_{\tau(2)}, v_{\tau(2)}) p_{\tau(3)}(u_{\tau(3)}, v_{\tau(3)}).$$  \hspace{1cm} (3.4)

**Remark 3.1.3.** All of the above realization of $\mathfrak{g}_\sigma$ in terms of $\mathfrak{sl}_2$–theory actually does work for any $\sigma \in \mathbb{C}^3 \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$. Therefore, here and henceforth we extend our Lie superalgebra $\mathfrak{g}_\sigma$ to any $\sigma \in \mathbb{C}^3 \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$.

In fact, this outcome can be achieved via a different approach, that is described in detail in [Se2], Ch. I, Example 5. Indeed, the construction there starts from scratch with the (classical) Lie algebra $\mathfrak{g}_0 := \mathfrak{sl}_2^{(3)}$ and its standard action on $U := \mathbb{H}_{i=1}^3 \square_i$; then one constructs a suitable $\mathfrak{g}_0$–valued bilinear bracket $P$ on $U$, that depends on $\sigma \in \mathbb{C}^3$; finally, one gives degree 0 to $\mathfrak{g}_0$ and 1 to $\mathfrak{g}_1 := U$, and provides $\mathfrak{g} := \mathfrak{g}_0 \oplus U$ with the bilinear bracket $[ , ]_\mathfrak{g}$ uniquely given by the Lie bracket of $\mathfrak{g}_0$, the $\mathfrak{g}_0$–action on $\mathfrak{g}_1$ and the bracket $P$ on $\mathfrak{g}_1$. In the end, one proves that this bilinear bracket $[ , ]_\mathfrak{g}$ makes $\mathfrak{g}$ into a Lie superalgebra if and only if the condition $\sigma_1 + \sigma_2 + \sigma_3 = 0$ is fulfilled.

The following statement is proved in [loc. cit.] again:

**Proposition 3.1.4.** Let $\sigma, \sigma' \in \mathbb{C}^3 \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$. The Lie superalgebras $\mathfrak{g}_\sigma$ and $\mathfrak{g}_{\sigma'}$ are isomorphic iff there exists $\tau \in \mathfrak{S}_3$ such that $\sigma'$ and $\tau \sigma$ are proportional. Moreover, the Lie superalgebra $\mathfrak{g}_\sigma$ is simple iff $\sigma \in (\mathbb{C}^*)^3$.

By this reason, the case $\sigma \in (\mathbb{C}^*)^3$ will be said to be “general” or “generic”. In short, the isomorphism classes of our $\mathfrak{g}_\sigma$’s are in bijection with the orbits of the $\mathfrak{S}_3$–action in the space $\mathbb{P}(\sum_{i=1}^3 \sigma_i = 0) \cup \{0\} \cong \mathbb{P}_\mathbb{C}^1 \cup \{\ast\}$ — a complex projective line plus an extra point.

### 3.1.3. Dynkin diagram II

Here, for the reader’s convenience, we relate our Dynkin diagram with a more familiar one. This can be achieved by applying the odd reflection with respect to the root $\beta_2$, due to V. Serganova (see [Sc2]).

To begin with, set $\alpha_1 = \beta_2 + \beta_3, \quad \alpha_2 = - \beta_2, \quad \alpha_3 = \beta_1 + \beta_2$. Then $\Pi' := \{\alpha_i\}_{i=1,2,3}$ is a set of simple roots of $\mathfrak{g}$, which is not Weyl-group conjugate to $\Pi$; the corresponding set of
coroots \((\Pi')^\vee = \{ h_i \}_{i=1,2,3}\) should be taken as \(h_1 = H_{2e_1}, h_2 = H_{2e_2}, h_3 = H_{2e_3}\). With such a choice, the associated Cartan matrix \(A'_\sigma := (\alpha_j(h_i))_{i,j=1,2,3; i \neq j}\) is given by
\[
A'_\sigma = \begin{pmatrix}
2 \sigma_1 & -\sigma_1 & 0 \\
-\sigma_1 & 0 & -\sigma_3 \\
0 & -\sigma_3 & 2 \sigma_3
\end{pmatrix} = \sigma_1 \begin{pmatrix}
2 & -1 & 0 \\
-1 & 0 & -\frac{\sigma_3}{\sigma_1} \\
0 & -\frac{\sigma_3}{\sigma_1} & 2 \frac{\sigma_3}{\sigma_1}
\end{pmatrix}
\]
where the second equality is available only if \(\sigma_1 \neq 0\). Thus, for \(\sigma_1 \neq 0\), our original \(g = g_\sigma\) can be also defined via the following Dynkin diagram with \(a := \frac{\sigma_1}{\sigma_3}\)

\[\begin{array}
\alpha_1 & \alpha_2 & \alpha_3 \\
\end{array}\]

\[\begin{array}
1 & \times & a \\
\end{array}\]

a well-known Dynkin diagram of type \(D(2,1;a)\) — the unique one with just one odd vertex.

With respect to \(\Pi'\), the set of positive roots is given by
\[\Delta'^+ = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2 \alpha_2 + \alpha_3 \}\]
while the coroots can be expressed as
\[
\begin{align*}
h_1 &= H_{\beta_2+\beta_3} = -(H_{\beta_2} + H_{\beta_3}) , \\
h_2 &= H_{-\beta_2} = H_{\beta_2} , \\
h_3 &= H_{\beta_2+\beta_1} = -(\Pi'_{\beta_2} + \Pi'_{\beta_1}) \\
\end{align*}
\]
\[
\begin{align*}
h_{\alpha_1+\alpha_2} &= -H_{\beta_3} = h_1 + h_2 , \\
h_{\alpha_1+\alpha_2+\alpha_3} &= H_{\beta_1+\beta_2+\beta_3} = -H_{\beta_1} - H_{\beta_2} - H_{\beta_3} = h_1 + h_2 + h_3 \\
h_{\alpha_1+2\alpha_2+\alpha_3} &= H_{\beta_1+\beta_3} = -H_{\beta_1} - H_{\beta_3} = h_1 + 2h_2 + h_3
\end{align*}
\]

### 3.2. Further bases of \(g_\sigma\).

In [3.1.1] we have introduced a basis \(\{H_\beta\}_{\beta \in \Pi} \cup \{X_\beta\}_{\pm \beta \in \Delta^+}\) of \(g_\sigma\). In this subsection, we provide other bases of \(g_\sigma\) for generic \(\sigma\); for singular values of \(\sigma\) instead, these new “bases” — more precisely, some slightly larger spanning sets — provide new singular degenerations. In order to do this, we record hereafter some formulas for the Lie brackets on elements of these spanning sets.

### 3.2.1. A second basis.

Let \(\sigma \in \mathbb{C}^3 \cap \{ \sigma_1 + \sigma_2 + \sigma_3 = 0 \}\) be generic, i.e. \(\sigma \in (\mathbb{C}^*)^3\). The Lie superalgebra \(g_\sigma\) is, by definition, the complex simple Lie superalgebra associated to the Cartan matrix \(A_\sigma\) like in [3.1.1]. Thus, letting \((\mathfrak{h}, \Pi') = \{H'_{\beta_i}\}_{i=1,2,3}, \Pi = \{\beta_i\}_{i=1,2,3}\) be a realization of \(A_\sigma\) (as before), our Lie superalgebra \(g = g_\sigma\) is generated by \(\mathfrak{h}\) and \(\{X'_{\pm \beta_i}\}_{\beta_i \in \Pi}\) satisfying, at least, the relations (for \(i, j \in \{1,2,3\}\))
\[
\begin{align*}
[H'_{\beta_1}, H'_{\beta_2}] &= 0 , \\
[H'_{\beta_1}, X'_{\pm \beta_1}] &= \pm \beta_j (H'_{\beta_1}) X'_{\pm \beta_j} , \\
[X'_{\beta_1}, X'_{-\beta_1}] &= \delta_{i,j} H'_{\beta_i} , \\
[X'_{\pm \beta_1}, X'_{\pm \beta_1}] &= 0
\end{align*}
\]
with parity \(|H'_{\beta_i}| = 0\) and \(|X_{\pm \beta_i}| = 1\) for all \(i = 1,2,3\). So far we have just changed symbols for the generators: comparing with [3.1.1] we just have \(H'_{\beta_1} = H_{\beta_1}\) and \(X'_{\pm \beta_1} = X_{\pm \beta_1}\). Now instead we introduce new basis elements \(X'_{2e_1}, \in g_{\mp 2e_1}\) and \(H'_{2e_1} \in \mathfrak{h}\) via the relations
\[
\begin{align*}
\sigma_1 X'_{2e_1} &= [X'_{\beta_2}, X'_{\beta_3}] , \\
\sigma_2 X'_{2e_2} &= [X'_{\beta_3}, X'_{\beta_1}] , \\
\sigma_3 X'_{2e_3} &= [X'_{\beta_1}, X'_{\beta_2}] \\
\sigma_1 X'_{-2e_1} &= [-X'_{\beta_2}, X'_{\beta_3}] , \\
\sigma_2 X'_{-2e_2} &= [-X'_{\beta_3}, X'_{\beta_1}] , \\
\sigma_3 X'_{-2e_3} &= [-X'_{\beta_1}, X'_{\beta_2}] \\
\sigma_1 H'_{2e_1} &= -(H'_{\beta_1} + H'_{\beta_2}) , \\
\sigma_2 H'_{2e_2} &= -(H'_{\beta_3} + H'_{\beta_1}) , \\
\sigma_3 H'_{2e_3} &= -(H'_{\beta_3} + H'_{\beta_2})
\end{align*}
\]
It can be checked that, for \(i,j \in \{1,2,3\}\), one has
\[
[X'_{2e_1}, X'_{-2e_j}] = \delta_{i,j} H'_{2e_i} , \\
[H'_{2e_1}, X'_{\pm 2e_j}] = \pm 2 \delta_{i,j} X'_{\pm 2e_j}
\] (3.5)
which implies that each \( a'_i := \mathbb{C}X'_{2i} + \mathbb{C}H'_{2i} + \mathbb{C}X'_{-2i} \) (for \( 1 \leq i \leq 3 \)) is a Lie super-algebra, with \([a'_i, a'_k] = 0\) for \( j \neq k\), isomorphic to \( \mathfrak{g}_2\). In particular, the even part \( \mathfrak{g}_0\) of the Lie superalgebra \( \mathfrak{g}\) can be described as \( \mathfrak{g}_0 = \bigoplus_{i=1}^3 a'_i\).

By the analysis in (3.1.1) it turns out that \([X'_{2\epsilon_i}, \epsilon_{\beta'}]\) \( \in \mathfrak{g}_0\) and \([X'_{-2\epsilon_i}, \epsilon_{\beta'}]\) \( \in \mathfrak{g}_{-\theta}\) are independent of \( i\); whence we set \( X'_\theta := [X'_{2\epsilon_i}, \epsilon_{\beta'}]\) and \( X'_{-\theta} := [X'_{-2\epsilon_i}, \epsilon_{\beta'}]\). It follows that \([X'_\theta, X'_{-\theta}] = H'_\theta\), where \( H'_\theta := -(H'_{\beta_1} + H'_{\beta_2} + H'_{\beta_3})\); furthermore, we also record that \( \sigma_1 H'_{2\epsilon_1} + \sigma_2 H'_{2\epsilon_2} + \sigma_3 H'_{2\epsilon_3} = 2 H'_\theta\).

**Remark 3.2.1.** For \( \alpha \in \Delta_0^+\), the elements \( X'_{\pm \alpha} \) and \( H'_\alpha\) are related to the elements defined in (3.1.1) by \( X'_{\pm \alpha} = \sigma_i^{-1} X_{\epsilon_i \pm \alpha}, \) \( H'_\alpha = \sigma_i^{-1} H_{\epsilon_i \alpha}, \) if \( \alpha = 2 \epsilon_i \) for some \( 1 \leq i \leq 3\), than that is \( \alpha \in \Delta_0^+\), and \( X'_{\pm \alpha} = X_{\pm \alpha}, \) \( H'_\alpha = H_\alpha, \) if \( \alpha \in \Delta_0^+\) (cf. Remark 3.1.1).

By this remark, it follows that an isomorphism between the odd part \( \mathfrak{g}_1\) and \( \boxtimes_1 \boxtimes_2 \boxtimes_3\), viewed as \( \mathfrak{g}_0\)-module, is given completely by the same formula as in (3.1.1); one just has to literally replace each \( X_{\pm \alpha}\) (in (3.1.1) with \( X'_{\pm \alpha}\).

**Remark 3.2.2.** By our normalization, the non-trivial actions of \( a'_i\) on \( \boxtimes_i\) are given by

\[ H'_{2\epsilon_i}, |\pm\rangle = \pm |\pm\rangle, \quad X'_{\pm 2\epsilon_i}, |\mp\rangle = |\pm\rangle. \]

3.2.2. The Lie bracket \( \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0\). Hereafter we record the non-trivial commutation relations in the new basis \( \{X'_{\pm \alpha}\}_{\alpha \in \Delta_0^+} \cup \{H'_{\beta_i}\}_{i=1,2,3}\) which might be useful for later purpose:

\[
\begin{align*}
[X'_{\pm 2\epsilon_1}, |\pm\rangle \otimes |\pm\rangle\rangle &= -\sigma_1 X'_{2\epsilon_1}, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = \sigma_1 X'_{2\epsilon_1} \\
[X'_{\pm 2\epsilon_2}, |\pm\rangle \otimes |\pm\rangle\rangle &= -\sigma_2 X'_{2\epsilon_2}, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = \sigma_2 X'_{2\epsilon_2} \\
[X'_{\pm 2\epsilon_3}, |\pm\rangle \otimes |\pm\rangle\rangle &= -\sigma_3 X'_{2\epsilon_3}, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = \sigma_3 X'_{2\epsilon_3} \\
[X'_{\pm \epsilon_1}, |\pm\rangle \otimes |\pm\rangle\rangle &= H'_\beta_2, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = H'_\beta_2 \\
[X'_{\pm \epsilon_2}, |\pm\rangle \otimes |\pm\rangle\rangle &= H'_\beta_3, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = H'_\beta_3 \\
[X'_{\pm \epsilon_3}, |\pm\rangle \otimes |\pm\rangle\rangle &= -\sigma_1 X'_{-2\epsilon_1}, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = \sigma_1 X'_{-2\epsilon_1} \\
[X'_{\pm \epsilon_2}, |\pm\rangle \otimes |\pm\rangle\rangle &= -\sigma_2 X'_{-2\epsilon_2}, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = \sigma_2 X'_{-2\epsilon_2} \\
[X'_{\pm \epsilon_3}, |\pm\rangle \otimes |\pm\rangle\rangle &= -\sigma_3 X'_{-2\epsilon_3}, \quad [(+\rangle \otimes |\mp\rangle \otimes |\mp\rangle\rangle = \sigma_3 X'_{-2\epsilon_3} \\
\end{align*}
\]

3.2.3. Coroots in another root basis. As in (3.1.3) we fix now a different basis of simple roots, namely \( \Pi' := \{\alpha_i\}_{i=1,2,3}\) with \( \alpha_1 := \beta_2 + \beta_3, \) \( \alpha_2 := -\beta_2, \) \( \alpha_3 := \beta_1 + \beta_2\), whose corresponding set of coroots is \( \{\Pi'\}^\vee = \{h'_i\}_{i=1,2,3}\) with \( h'_i = H'_{2\epsilon_1}, \) \( h'_2 = H'_{2\epsilon_2}, \) \( h'_3 = H'_{2\epsilon_3}\).

With this choice, the Cartan matrix \( A''_\sigma := (\alpha_j(h'_i))_{i,j=1,2,3}\) is given explicitly by

\[
A''_\sigma = \begin{pmatrix}
2 & -1 & 0 \\
-\sigma_1 & 0 & -\sigma_3 \\
0 & -\sigma_3 & 2\sigma_1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \sigma_1 & 0 \\
0 & 0 & \sigma_3
\end{pmatrix}
\begin{pmatrix}
2 & -1 & 0 \\
-\sigma_1 & 0 & -\sigma_3 \\
0 & -\sigma_3 & 2\sigma_1
\end{pmatrix}
\]

where the second equality makes sense only if \( \sigma_1 \sigma_3 \neq 0\). Thus, for \( \sigma_1 \sigma_3 \neq 0\), our original \( \mathfrak{g} = \mathfrak{g}_\sigma\) can be also defined via the same Dynkin diagram as in (3.1.3).

The set of positive roots with respect to \( \Pi'\) is

\[
\Delta^{\Pi'} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + 2\alpha_2 + \alpha_3\}\]
and the corresponding coroots, in terms of the new generators $H'_{\alpha_i}$ (for all $i$), are given by

\[
\begin{align*}
&h_1' = H'_{2\varepsilon_1}, \quad h_2' = H'_{-\beta_2} = H'_{\beta_2}, \quad h_3' = H'_{2\varepsilon_3}, \\
h'_{\alpha_1+\alpha_2} = -H'_{\beta_3} = \sigma_1 h'_1 + h'_2, \quad h'_{\alpha_1+\alpha_2+\alpha_3} = -H'_{\beta_1+\beta_2+\beta_3} = \sigma_1 h'_1 + h'_2 + \sigma_3 h'_3, \\
&\sigma_2 h'_{\alpha_1+2\alpha_2+3\alpha_3} = \sigma_2 H'_{\beta_1+3\beta_2} = -H'_{\beta_1} - H'_{\beta_3} = \sigma_1 h'_1 + 2h'_2 + 3\sigma_3 h'_3.
\end{align*}
\]

Notice also that, as a consequence, we have $\sigma_2 H'_{2\varepsilon_2} = \sigma_1 h'_1 + 2h'_2 + 3\sigma_3 h'_3$.

3.2.4. A third basis. Let again $\sigma \in \mathbb{C}^3 \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$ be generic, i.e. $\sigma \in (\mathbb{C}^*)^3$. As a third basis for $\mathfrak{g}_\sigma$ we choose now a suitable mixture of the two ones considered in §3.1.1 and §3.2.1 above. Namely, let us consider

\[
\{H'_{2\varepsilon_i}\}_{i=1,2,3} \cup \{X_\alpha\}_{\alpha \in \Delta}
\]

By the previous analysis, this is yet another $\mathbb{C}$-basis of $\mathfrak{g}_\sigma$. In addition, the previous results also provide explicit formulas for the Lie brackets among elements of this new basis; we shall write them down explicitly — and use them — later on.

4. Integral forms & degenerations for Lie superalgebras of type $D(2,1;\sigma)$

Let $\mathfrak{l}$ be any Lie (super)algebra over a field $\mathbb{K}$, and $R$ any subring of $\mathbb{K}$. By integral form of $\mathfrak{l}$ over $R$, or (integral) $R$-form of $\mathfrak{l}$, we mean by definition any Lie $R$-sub(super)algebra $\mathfrak{t}_R$ of $\mathfrak{l}$ whose scalar extension to $\mathbb{K}$ is $\mathfrak{l}$ itself; in other words $\mathbb{K} \otimes_R \mathfrak{t}_R \cong \mathfrak{l}$ as Lie (super)algebras over $\mathbb{K}$. In this subsection we introduce five particular integral forms of $\mathfrak{l} = \mathfrak{g}_\sigma$, and study some remarkable specializations of them. Let $\Delta := \Delta^+ \cup (-\Delta^+)$ be the root system of $\mathfrak{g}_\sigma$.

As a matter of notation, hereafter for any $\sigma := (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3$ we denote by $\mathbb{Z}[\sigma]$ the (unital) subring of $\mathbb{C}$ generated by $\{\sigma_1, \sigma_2, \sigma_3\}$.

The reader may observe that the choice of a $\mathbb{Z}[\sigma]$-form becomes very important when one considers a singular degeneration: one cannot speak instead of the singular degeneration, in that any degeneration actually depends not only on the specific specialization value taken by $\sigma$ but also on the previously chosen $\mathbb{Z}[\sigma]$-form. Some specific features of this phenomenon are presented in Theorems 4.1.1, 4.2.1, 4.3.1, etc.

4.1. First family: the Lie superalgebras $\mathfrak{g}(\sigma)$.

4.1.1. The integral $\mathbb{Z}[\sigma]$-form $\mathfrak{g}_{\mathbb{Z}[\sigma]}$. Let us consider the system of $\mathbb{C}$-linear generators $B_\mathbb{Z} := \{H_{2\varepsilon_1}\}_{i=1,2,3} \cup \{H_\theta\} \cup \{X_\alpha\}_{\alpha \in \Delta}$ of $\mathfrak{g} := \mathfrak{g}_\sigma$, for any $\sigma := (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3$ such that $\sigma_1 + \sigma_2 + \sigma_3 = 0$ (cf. §3.1). The $\mathbb{Z}[\sigma]$-submodule

\[
\mathfrak{g}(\sigma) := \sum_{i=1}^3 \mathbb{Z}[\sigma] H_{2\varepsilon_i} + \mathbb{Z}[\sigma] H_\theta + \sum_{\alpha \in \Delta} \mathbb{Z}[\sigma] X_\alpha = \sum_{b \in B_\mathbb{Z}} \mathbb{Z}[\sigma] b
\]

of $\mathfrak{g}$ is clearly — thanks to the identity $H_{2\varepsilon_1} + H_{2\varepsilon_2} + H_{2\varepsilon_3} = 2 H_\theta$ — a free $\mathbb{Z}[\sigma]$-module, with basis $B_\mathbb{Z} \setminus \{H_{2\varepsilon_i}\}$ for any $1 \leq i \leq 3$. The explicit formulas for the Lie bracket given in §3.1 show that $\mathfrak{g}(\sigma)$ is a $\mathbb{Z}[\sigma]$-subsuperalgebra of $\mathfrak{g}$ hence also an integral $\mathbb{Z}[\sigma]$-form of the latter. Thus (4.1) defines a Lie superalgebra over $\mathbb{Z}[\sigma]$ for any possible point $\sigma \in V := \{\sigma \in \mathbb{C}^3 \mid \sum_{i=1}^3 \sigma_i = 0\}$; hence we can think of all these $\mathfrak{g}(\sigma)$’s as a family of Lie
superalgebras indexed over the complex plane $V$. Moreover, taking $g(σ)_C := C \otimes_{\mathbb{Z}[σ]} g(σ)$ for all $σ \in V$ we find a more regular situation, in a sense that now these (extended) Lie superalgebras all share $C$ as their common ground ring. In particular, if $σ_i \neq 0$ for all $i \in \{1, 2, 3\}$ we have $g(σ)_C \cong g_σ$ as given in [3.1.1].

In order to formalize the description of the family $\{g(σ)_C\}_{σ \in V}$, we proceed as follows. Let $\mathbb{Z}[x] := \mathbb{Z}[V] \cong \mathbb{Z}[x_1, x_2, x_3]/(x_1 + x_2 + x_3)$ be the ring of global sections of the $\mathbb{Z}$-scheme associated with $V$. In the construction of $g(σ)$, formally replace $x$ to $σ$ (hence the $x_i$'s to the $σ_i$'s): this does make sense, and provides a meaningful definition of a Lie superalgebra over $\mathbb{C}[x] := \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[x]$, denoted by $g(x)$, and then also $g(σ)_C := \mathbb{C}[x] \otimes_{\mathbb{Z}[x]} g(x)$ by scalar extension. Now definitions imply that, for any $σ \in V$, we have a Lie $\mathbb{C}[σ]$-superalgebra isomorphism

$$g(σ) \cong \mathbb{Z}[σ] \otimes_{\mathbb{Z}[x]} g(x)$$

— through the ring isomorphism $\mathbb{Z}[σ] \cong \mathbb{Z}[x]/(x_i − σ_i)_{i=1,2,3}$ — and similarly

$$g(σ)_C \cong \mathbb{C} \otimes_{\mathbb{C}[x]} g(x)_C$$

as Lie $\mathbb{C}$-superalgebras, through the ring isomorphism $\mathbb{C} \cong \mathbb{C}[x]/(x_i − σ_i)_{i=1,2,3}$.

In geometrical language, all this can be formulated as follows. The Lie superalgebra $g(x)_C$ — being a free, finite rank $\mathbb{C}[x]$-module — defines a coherent sheaf $L_{g(σ)}$ of Lie superalgebras over $\text{Spec}(\mathbb{C}[x])$. Moreover, there exists a unique fibre bundle over $\text{Spec}(\mathbb{C}[x])$, say $\mathbb{L}_{g(σ)}$, whose sheaf of sections is exactly $L_{g(σ)}$. This fibre bundle can be thought of as a (total) deformation space over the base space $\text{Spec}(\mathbb{C}[x])$, in which every fibre can be seen as a “deformation” of any other one, and also any single fibre can be seen as a degeneration of the original Lie superalgebra $g(x)_C$. Moreover, the fibres of $\mathbb{L}_{g(σ)}$ on $\text{Spec}(\mathbb{C}[x]) = V \cup \{\ast\}$ are, by definition, given by $(\mathbb{L}_{g(σ)})_σ = \mathbb{C} \otimes_{\mathbb{C}[x]} g(x)_C \cong g(σ)_C$ for any closed point $σ \in V \subseteq \text{Spec}(\mathbb{C}[x])$, while for the generic point $\ast \in \text{Spec}(\mathbb{C}[x])$ we have $(\mathbb{L}_{g(σ)})_\ast = \mathbb{C}(x) \otimes_{\mathbb{C}[x]} g(x)_C \quad (= : g(σ)_C)$. Finally, it follows from our construction that these sheaf and fibre bundle do admit an action of $\mathbb{C}^* \times \mathfrak{S}_3$, that on the base space $\text{Spec}(\mathbb{C}[x]) = V \cup \{\ast\}$ simply fixes $\{\ast\}$ and is the standard $(\mathbb{C}^* \times \mathfrak{S}_3)$-action on $V$.

By construction and Proposition 3.1.2 when $σ_1, σ_2, σ_3 \in \mathbb{C}^*$, we have that the fibre $(\mathbb{L}_{g(σ)})_σ \cong g(σ)_C \cong g_σ$ is simple as a Lie superalgebra; instead, at each closed point of the “singular locus” $\bigcup_{i=1}^3 \{σ_i = 0\}$ the fibre is non-simple. This follows from direct inspection, for which we need to look at the complete multiplication table of $g(σ)_C$.

4.1.2. Non-trivial bracket relations for $g(σ)_C$. Resuming from formulas in [3.1] for the Lie brackets among the elements of the $\mathbb{C}$-spanning set $B_\delta := \{H_{2ε_1}, H_{2ε_2}, H_{2ε_3}, H_0\} \cup \{X_α\}_{α \in Δ}$ of $g(σ)_C$, we find the following table:

\[
\begin{align*}
[H_{2ε_1}, H_{2ε_2}] &= 0 , & [H_{2ε_1}, X_{±2ε_2}] &= ±2σ_iδ_{i,j} X_{±2ε_j}, \\
[X_{2ε_1}, X_{2ε_2}] &= 0 , & [X_{−2ε_1}, X_{−2ε_2}] &= 0 , & [X_{2ε_1}, X_{−2ε_2}] &= σ_iδ_{i,j} H_{2ε_i}
\end{align*}
\]
\[ [H_{2e_i}, X_{\pm \beta}] = \pm(-1)^{\delta_{ij}/2} \sigma_i X_{\pm \beta_j}, \quad [H_{2e_i}, X_{\pm \theta}] = \pm \sigma_i X_{\pm \theta} \]

\[ [H_{\theta}, X_{\pm 2e_i}] = \pm \sigma_i X_{2e_i}, \quad [H_{\theta}, X_{\pm \beta}] = \mp \sigma_i X_{\pm \beta_i}, \quad [H_{\theta}, X_{\pm \theta}] = 0 \]

\[ [X_{2e_i}, X_{\beta_j}] = \delta_{ij} \sigma_i X_{\theta}, \quad [X_{2e_i}, X_{-\beta_j}] = (1 - \delta_{ij}) \sigma_i X_{\theta_k}, \quad [X_{2e_i}, X_{-\theta}] = 0 \]

\[ [X_{\pm \beta_j}, X_{\beta_j}] = (1 - \delta_{ij}) X_{2e_k}, \quad [X_{-\beta_j}, X_{-\beta_j}] = -(1 - \delta_{ij}) X_{-2e_k} \]

\[ [X_{\beta_i}, X_{\theta}] = 0, \quad [X_{\beta_i}, X_{-\theta}] = X_{-2e_i}, \quad [X_{-\beta_i}, X_{\theta}] = X_{2e_i}, \quad [X_{-\beta_i}, X_{-\theta}] = 0 \]

for all \( i, j \in \{1, 2, 3\} \), with \( k \in \{1, 2, 3\} \setminus \{i, j\} \).

In particular, these explicit formulas lead to the following

**Theorem 4.1.1.** Let \( \sigma \in V \) as above, and set \( a_i := \mathbb{C}X_{2e_i} \oplus \mathbb{C}H_{2e_i} \oplus \mathbb{C}X_{-2e_i} \) as defined after Remark 3.1.1 for all \( i = 1, 2, 3 \).

1. If \( \sigma_i = 0 \) and \( \sigma_j \neq 0 \neq \sigma_k \) for \( \{i, j, k\} = \{1, 2, 3\} \), then \( a_i \subseteq g(\sigma) \) (a Lie ideal), \( a_i \cong \mathbb{C}^{\oplus 3} \) and \( g(\sigma) \mathbb{C} \) is the universal central extension of \( \mathfrak{psl}(2|2) \) by \( a_i \) (cf. Theorem 4.7 in [IK]); in other words, there exists a short exact sequence of Lie superalgebras

\[
0 \rightarrow \mathbb{C}^{\oplus 3} \cong a_i \rightarrow g(\sigma) \mathbb{C} \rightarrow \mathfrak{psl}(2|2) \rightarrow 0
\]

A parallel result also holds true when working with \( g(\sigma) \) over the ground ring \( \mathbb{Z}[\sigma] \).

2. If \( \sigma_h = 0 \) for all \( h \in \{1, 2, 3\} \), i.e. \( \sigma = 0 \), then \( (g(0))_C = \mathbb{C}^{\oplus 9} \) is the center of \( g(0) \mathbb{C} \), and the quotient \( g(0) \mathbb{C} / (g(0))_C \) is \( \mathbb{C}^{\oplus 8} \) is Abelian; in particular, \( g(0) \mathbb{C} \) is a central extension of \( \mathbb{C}^{\oplus 8} \) by \( \mathbb{C}^{\oplus 9} \), i.e. there exists a short exact sequence of Lie superalgebras

\[
0 \rightarrow \mathbb{C}^{\oplus 9} \cong (g(0))_C \rightarrow g(0) \mathbb{C} \rightarrow \mathbb{C}^{\oplus 8} \rightarrow 0
\]

A parallel result also holds true when working with \( g(0) \) over the ground ring \( \mathbb{Z}[0] = \mathbb{Z} \).

*Proof.* The claim follows at once by direct inspection of the formulas in §4.1.2 above.

\[ \square \]

4.2. Second family: the Lie superalgebra \( g'(\sigma) \).

4.2.1. The integral \( \mathbb{Z}[\sigma] \)-form \( g'_{\mathbb{Z}[\sigma]} \). Look now at \( B'_\theta := \{ H'_{2e_1}, H'_{2e_2}, H'_{2e_3}, H'_\theta \} \cup \{ X'_\alpha \}_{\alpha \in \Delta} \), that is a second \( \mathbb{C} \)-spanning set of \( g := g(\sigma) \). For any \( \sigma \in V \) as before, the \( \mathbb{Z}[\sigma] \)-submodule

\[ g'(\sigma) := \sum_{i=1}^3 \mathbb{Z}[\sigma] H'_{2e_i} + \mathbb{Z}[\sigma] H'_\theta + \sum_{\alpha \in \Delta} \mathbb{Z}[\sigma] X'_\alpha = \sum_{b' \in B'_\theta} \mathbb{Z}[\sigma] b' \]  \hspace{1cm} (4.2)

is not a free \( \mathbb{Z}[\sigma] \)-module in this case — contrary to what happened with \( g(\sigma) \). The formulas in §3.2 prove that \( g'(\sigma) \) is also an integral \( \mathbb{Z}[\sigma] \)-form of \( g \).
Notice that the above mentioned formulas do make sense for any possible \( \sigma \) (such that \( \sum_{i=1}^{3} \sigma_i = 0 \)), i.e. without assuming \( \sigma \neq 0 \). Therefore (3.2) defines a Lie superalgebra over \( \mathbb{Z}[\sigma] \) for any possible \( \sigma \in V := \{ \sigma \in \mathbb{C}^3 \mid \sum_{i=1}^{3} \sigma_i = 0 \} \); thus all these \( g'(\sigma) \)'s form a family indexed over \( V \). Moreover, taking \( g'(\sigma)_c := \mathbb{C} \otimes_{\mathbb{Z}[\sigma]} g'(\sigma) \) we find a more regular situation, as now the latter Lie superalgebras all share \( \mathbb{C} \) as ground ring. In particular, we find \( g'(\sigma)_c \cong g_{\sigma} \) (as given in (3.1.1) for all possible \( \sigma \in V \). 

The family \( \{ g'(\sigma)_c \}_{\sigma \in V} \) can be described in a formal way, similar to what we did in §4.1.1 keeping the same notation, in particular \( Z[x] := Z[V] \cong Z[x_1, x_2, x_3] / (x_1 + x_2 + x_3) \). In the construction of \( g'(\sigma) \), replace \( x \) with \( \sigma \); this yields a definition of a Lie superalgebra over \( Z[x] \), denoted by \( g'(x) \), and also \( g'(\sigma)_c := \mathbb{C}[\sigma] \otimes_{Z[x]} g'(x) \) by scalar extension. Then definitions imply that, for any \( \sigma \in V \), we have a Lie \( Z[\sigma] \)-superalgebra isomorphism

\[
g'(\sigma) \cong Z[\sigma] \otimes g'(x)
\]

— through the ring isomorphism \( Z[\sigma] \cong Z[x] / (x_i - \sigma_i)_{i=1,2,3} \) — and similarly

\[
g'(\sigma)_c \cong \mathbb{C} \otimes g'(x)_c
\]
as Lie \( \mathbb{C} \)-superalgebras, through the ring isomorphism \( \mathbb{C} \cong \mathbb{C}[x] / (x_i - \sigma_i)_{i=1,2,3} \).

One can argue similarly as in §4.1.1 to have a geometric picture of the above description: this amounts to literally replacing \( g(\sigma) \) with \( g'(\sigma) \), hence we leave it to the reader.

4.2.2. Non-trivial bracket relations for \( g'(\sigma)_c \). From the formulas in §3.2 for the Lie brackets among the elements of the \( \mathbb{C} \)-spanning set \( B_\theta := \{ H'_{2e_1}, H'_{2e_2}, H'_{2e_3}, H'_\theta \} \cup \{ X'_\alpha \}_{\alpha \in \Delta} \) of \( g'(\sigma)_c \) we find the following table:

\[
\begin{align*}
[H'_{2e_1}, H'_{2e_2}] &= 0 \quad , \quad [H'_{2e_1}, X'_{\pm 2e_j}] = \pm 2 \delta_{i,j} X'_{\pm 2e_j} \\
[X'_{2e_1}, X'_{2e_2}] &= 0 \quad , \quad [X'_{-2e_1}, X'_{-2e_2}] = 0 \quad , \quad [X'_{2e_1}, X'_{-2e_2}] = \delta_{i,j} H'_{2e_i} \\
[H'_{2e_i}, X'_{\pm \beta_j}] &= \pm (-1)^{\delta_{i,j}} X'_{\pm \beta_j} \quad , \quad [H'_{2e_i}, X'_{\pm \theta}] = \pm X'_{\pm \theta} \\
[H'_{\theta}, X'_{\pm 2e_i}] &= \pm \sigma_i X'_{2e_i} \quad , \quad [H'_{\theta}, X'_{\pm \beta_i}] = \mp \sigma_i X'_{\pm \beta_i} \quad , \quad [H'_{\theta}, X'_{\pm \theta}] = 0 \\
[X'_{2e_i}, X'_{\beta_j}] &= \delta_{i,j} X'_{\theta} \quad , \quad [X'_{2e_i}, X'_{-\beta_j}] = (1 - \delta_{i,j}) X'_{\beta_k} \\
[X'_{-2e_i}, X'_{\beta_j}] &= (1 - \delta_{i,j}) X'_{-\beta_k} \quad , \quad [X'_{-2e_i}, X'_{-\beta_j}] = \delta_{i,j} X'_{-\theta} \\
[X'_{2e_i}, X'_{\theta}] &= 0 \quad , \quad [X'_{2e_i}, X'_{-\theta}] = X'_{-\beta_i} \quad , \quad [X'_{-2e_i}, X'_{\theta}] = X'_{\beta_i} \quad , \quad [X'_{-2e_i}, X'_{-\theta}] = 0 \\
[X'_{\beta_i}, X'_{\beta_j}] &= (1 - \delta_{i,j}) \sigma_i X'_{2e_k} \quad , \quad [X'_{\beta_i}, X'_{-\beta_j}] = -(1 - \delta_{i,j}) \sigma_i X'_{-2e_k} \\
[X'_{\beta_i}, X'_{\theta}] &= \delta_{i,j} (\sigma_i H'_{2e_i} - H'_{\theta}) \\
[X'_{\beta_i}, X'_{-\theta}] &= \sigma_i X'_{-2e_i} \quad , \quad [X'_{-\beta_i}, X'_{\theta}] = -\sigma_i X'_{2e_i} \quad , \quad [X'_{-\beta_i}, X'_{-\theta}] = 0 \\
[X'_{\theta}, X'_{\theta}] &= 0 \quad , \quad [X'_{\theta}, X'_{-\theta}] = H'_{\theta} \quad , \quad [X'_{-\theta}, X'_{-\theta}] = 0
\end{align*}
\]

for all \( i, j \in \{1, 2, 3\} \), with \( k \in \{1, 2, 3\} \setminus \{i,j\} \).

As a consequence, these explicit formulas yield the following
Theorem 4.2.1. Given $\sigma \in V$, consider $a'_i := \mathbb{C} X'_{2i_1} \oplus \mathbb{C} H'_{2i_1} \oplus \mathbb{C} X'_{-2i_1}$ as above, for all $i \in \{1, 2, 3\}$.

1. If $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then setting
   $$b'_i := \left( \sum_{\alpha \neq \pm 2i_1} \mathbb{C} X'_\alpha \right) \oplus \left( \sum_{j \neq i} \mathbb{C} H'_{2j} \right)$$
we have $b'_i \triangleq g'(\sigma)_c$ (a Lie ideal), $a'_i \triangleq g'(\sigma)_c$ (a Lie subsuperalgebra), and there exist isomorphisms $b'_i \cong \mathfrak{psl}(2|2)$, $a'_i \cong \mathfrak{sl}_2$ and $g'(\sigma)_c \cong \mathfrak{sl}_2 \ltimes \mathfrak{psl}(2|2)$ — a semidirect product of Lie superalgebras. In other words, there is a split short exact sequence
   $$0 \longrightarrow \mathfrak{psl}(2|2) \cong b'_i \longrightarrow g'(\sigma)_c \longrightarrow a'_i \cong \mathfrak{sl}_2 \longrightarrow 0$$
   A parallel result also holds true when working with $g'(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$.

2. If $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, i.e. $\sigma = 0$, then $(g'(0)_c)_0 \cong \mathfrak{sl}_2^{\mathbb{Z}}$ as Lie (super)algebras, the Lie (super)bracket is trivial on $(g'(0)_c)_1$ and $(g'(0)_c)_1 \cong \mathfrak{sl}_2^{\mathbb{Z}}$ as modules over $(g'(0)_c)_0 \cong \mathfrak{sl}_2^{\mathbb{Z}}$, and finally $g'(0)_c \cong (g'(0)_c)_0 \ltimes (g'(0)_c)_1 \cong \mathfrak{sl}_2^{\mathbb{Z}} \ltimes \mathfrak{sl}_2^{\mathbb{Z}}$ — a semidirect product of Lie superalgebras. In other words, there is a split short exact sequence
   $$0 \longrightarrow \mathfrak{sl}_2^{\mathbb{Z}} \cong (g'(0)_c)_1 \longrightarrow g'(0)_c \longrightarrow (g'(0)_c)_0 \cong \mathfrak{sl}_2^{\mathbb{Z}} \longrightarrow 0$$
   A parallel result holds true when working with $g'(0)$ over the ground ring $\mathbb{Z}[0]$.

Proof. The claim follows at once by direct inspection of the formulas in 4.3.2 above. □

4.3. Third family: the Lie superalgebras $g''(\sigma)$.

4.3.1. The integral $\mathbb{Z}[\sigma]$-form $g''_{\mathbb{Z}[\sigma]}$. Pick now $B_{g''} := \{H'_{2\epsilon_1}, H'_{2\epsilon_2}, H'_{2\epsilon_3}, H'_\theta\} \cup \{X_\alpha\}_{\alpha \in \Delta}$, that is a third $\mathbb{C}$-spanning set of $g := g_\sigma$ for all generic $\sigma$ (cf. 3.2.4). Now for any $\sigma \in V := \mathbb{C}^3 \setminus \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$ we consider the $\mathbb{Z}[\sigma]$-submodule of $g_\sigma$,
   $$g''(\sigma) := \sum_{i=1}^3 \mathbb{Z}[\sigma] H'_{2i} + \mathbb{Z}[\sigma] H'_\theta + \sum_{\alpha \in \Delta} \mathbb{Z}[\sigma] X_\alpha$$
(4.3)
that, like in 4.3.1, is not a free $\mathbb{Z}[\sigma]$-module once more. The key point is that the explicit formulas for the Lie bracket among the spanning elements of $g''(\sigma)$ — given in 4.3.2 below — show that $g''(\sigma)$ is also an integral $\mathbb{Z}[\sigma]$-form of $g_\sigma$. Inside the latter, we define
   $$b''_i := \left( \mathbb{Z}[\sigma] H'_\theta + \sum_{i=1}^3 \mathbb{Z}[\sigma] H'_{2i} \right) \oplus \bigoplus_{\alpha \neq \pm 2i_1} \mathbb{Z}[\sigma] X_\alpha$$
(4.4)
for all $i \in \{1, 2, 3\}$, that actually is a Lie subsuperalgebra.

Once more, the above mentioned formulas do make sense for any possible $\sigma$ (such that $\sum_{i=1}^3 \sigma_i = 0$), i.e. including singular values; therefore (4.3) defines a Lie superalgebra over $\mathbb{Z}[\sigma]$ for any possible $\sigma \in V := \{\sigma \in \mathbb{C}^3 | \sum_{i=1}^3 \sigma_i = 0\}$, and all these $g''(\sigma)$'s form a family indexed over $V$. Moreover, taking $g''(\sigma)_c := \mathbb{C} \otimes \mathbb{Z}[\sigma] g''(\sigma)$ we find a family of Lie superalgebras that all share $\mathbb{C}$ as ground ring.

Keeping notation as before, the family $\{g''(\sigma)_c\}_{\sigma \in V}$ can be described in a formal way, taking its “version over $\mathbb{Z}[x]”$, denoted by $g''(x)$ — just replacing the complex parameters $\sigma_1, \sigma_2, \sigma_3) := x$ adding to zero —
and its complex-based counterpart \( g''(x)_c := \mathbb{C}[x] \otimes_{\mathbb{Z}[x]} g''(x) \). Then the very construction implies that, for any \( \sigma \in V \), one has a Lie \( \mathbb{Z} [\sigma] \)-superalgebra isomorphism

\[
g''(\sigma) \cong \mathbb{Z} [\sigma] \otimes \mathbb{Z}[x] g''(x)
\]

— through \( \mathbb{Z}[\sigma] \cong \mathbb{Z}[x] / (x_i - \sigma_i)_{i=1,2,3} \) — and similarly

\[
g''(\sigma)_c \cong \mathbb{C} \otimes \mathbb{Z}[x] g''(x)_c
\]
as Lie \( \mathbb{C} \)-superalgebras, through \( \mathbb{C} \cong \mathbb{C}[x] / (x_i - \sigma_i)_{i=1,2,3} \). Finally, the reader can easily provide a geometric description of the family of the \( g''(\sigma)_c \)'s, just like in [3.1.1].

4.3.2. Non-trivial bracket relations for \( g''(\sigma)_c \). From the formulas given in the previous sections, we find for the Lie brackets among the elements of the set

\[
B_{g''} = \{ H'_{2e_1}, H'_{2e_2}, H'_{2e_3}, H'_{\theta} \} \cup \{ X_\alpha \}_{\alpha \in \Delta}
\]

— which spans \( g''(\sigma)_c \) over \( \mathbb{C} \) — the following table:

\[
\begin{align*}
[H'_{2e_1}, H'_{2e_1}] &= 0, \quad [H'_{2e_1}, X_{\pm 2e_1}] = \pm 2 \delta_{e_1j} X_{\pm 2e_j}, \\
[X_{2e_1}, X_{2e_2}] &= 0, \quad [X_{2e_1}, X_{-2e_2}] = 0, \quad [X_{2e_2}, X_{-2e_1}] = \delta_{i,j} \sigma_i^2 H'_{2e_i} \\
[H'_{2e_1}, X_{\pm j}] &= \pm (-1)^{i,j} \delta_{i,j} X_{\pm j}, \quad [H'_{2e_1}, X_{\pm \theta}] = \pm X_{\pm \theta} \\
[H'_{\theta}, X_{\pm 2e_1}] &= \pm \sigma_i X_{2e_i}, \quad [H'_{\theta}, X_{\pm \theta}] = \mp \sigma_i X_{\pm i}, \quad [H'_{\theta}, X_{\pm \theta}] = 0 \\
[X_{2e_1}, X_{\beta}] &= \delta_{i,j} \sigma_i X_{\theta}, \quad [X_{2e_1}, X_{-\beta}] = (1 - \delta_{i,j}) \sigma_i X_{\beta} \\
[X_{-2e_1}, X_{\beta}] &= (1 - \delta_{i,j}) \sigma_i X_{-\beta}, \quad [X_{-2e_1}, X_{-\beta}] = \delta_{i,j} \sigma_i X_{\theta} \\
[X_{2e_1}, X_{\theta}] &= 0, \quad [X_{2e_1}, X_{-\theta}] = \sigma_i X_{-\beta}, \quad [X_{-2e_1}, X_{\theta}] = \sigma_i X_{\beta}, \quad [X_{-2e_1}, X_{-\theta}] = 0 \\
[X_{\beta}, X_{\beta}] &= (1 - \delta_{i,j}) X_{2e_i}, \quad [X_{-\beta}, X_{-\beta}] = -(1 - \delta_{i,j}) X_{-2e_i} \\
[X_{\beta}, X_{\theta}] &= \delta_{i,j} (\sigma_i H'_{2e_i} - H'_{\theta}) \\
[X_{\beta}, X_{-\theta}] &= \delta_{i,j} (\sigma_i H'_{-2e_i} - H'_{\theta}) \\
[X_{\theta}, X_{\theta}] &= 0, \quad [X_{\theta}, X_{-\theta}] = H'_{\theta}, \quad [X_{-\theta}, X_{-\theta}] = 0
\end{align*}
\]

for all \( i, j \in \{1,2,3\} \), with \( k \in \{1,2,3\} \setminus \{i,j\} \).

It follows by construction that for general values of \( \sigma \) one has \( g''(\sigma)_c \cong g_{\sigma} \) — indeed, switching from either side amounts to making a change of basis, nothing more; in particular, \( g''(\sigma)_c \) is simple for all general \( \sigma \). Instead, at singular values of \( \sigma \) one has non-simple degenerations, that are explicitly described by the following result:

**Theorem 4.3.1.** Let \( \sigma \in V \), and let \( (b''_i)_c := \mathbb{C} \otimes_{\mathbb{Z}[\sigma]} b''_i \) for all \( i \), with \( b''_i \) as in (4.1).

1. If \( \sigma_i = 0 \) and \( \sigma_j \neq 0 \neq \sigma_k \) for \( \{i,j,k\} = \{1,2,3\} \), then there exists a split short exact sequence

\[
0 \longrightarrow ( C X_{2e_i} \oplus C X_{-2e_i} ) \longrightarrow g''(\sigma)_c \longrightarrow (b''_i)_c \longrightarrow 0
\]

so that \( g''(\sigma)_c \cong (b''_i)_c \ltimes ( C X_{2e_i} \oplus C X_{-2e_i} ) \), and a second short exact sequence

\[
0 \longrightarrow \mathfrak{psl}(2|2) \longrightarrow (b''_i)_c \longrightarrow \mathbb{C} \mathbb{S} \longrightarrow 0
\]
A parallel result also holds true when working with $\mathfrak{g}''(\sigma)$ over the ground ring $\mathbb{Z}[\sigma]$. 

(2) If $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, i.e. $\sigma = 0$, then there exists a first short exact sequence

$$0 \to \bigoplus_{i=1}^{3} (\mathbb{C} X_{+2i} \oplus \mathbb{C} X_{-2i}) \to \mathfrak{g}''(0) \to b''_C \to 0$$

where

$$b''_C := \sum_{i=1}^{3} \mathbb{C} \mathcal{H}_{2i} \oplus \bigoplus_{\alpha \in \Delta_1} \mathbb{C} \mathcal{X}_\alpha$$

and a second short exact sequence

$$0 \to \bigoplus_{\alpha \in \Delta_1} \mathbb{C} \mathcal{X}_\alpha \to b''_C \to \sum_{i=1}^{3} \mathbb{C} \mathcal{H}_{2i} \to 0$$

where $\bigoplus_{\alpha \in \Delta_1} \mathbb{C} \mathcal{X}_\alpha$ is an Abelian subalgebra of $b''_C$. A parallel result also holds true when working with $\mathfrak{g}''(0)$ over the ground ring $\mathbb{Z}[0] = \mathbb{Z}$.

Proof. The claim follows at once by direct inspection of the formulas in (4.3.2) above. $\square$

4.4. Degenerations from contractions: the $\widehat{\mathfrak{g}}(\sigma)$’s and the $\widehat{\mathfrak{g}}'(\sigma)$’s. We finish our study of remarkable integral forms of $\mathfrak{g}_0$ by introducing some other more, that all are obtained through a general construction; when specializing these forms, one obtains again degenerations of the kind that is often referred to as “contraction”.

We start with a very general construction. Let $R$ be a (commutative, unital) ring, and let $\mathcal{A}$ be an “algebra” (not necessarily associative, nor unitary), in some category of $R$–bimodules, for some “product” – we assume in addition that

$$\mathcal{A} = \mathcal{F} \oplus \mathcal{C} \quad \text{with} \quad \mathcal{F} \cdot \mathcal{F} \subseteq \mathcal{F}, \quad \mathcal{F} \cdot \mathcal{C} \subseteq \mathcal{C}, \quad \mathcal{C} \cdot \mathcal{F} \subseteq \mathcal{C}, \quad \mathcal{C} \cdot \mathcal{C} \subseteq \mathcal{F} \quad (4.5)$$

Choose now $\tau$ be a non-unit in $R$, and correspondingly consider in $\mathcal{A}$ the $R$–submodules

$$\mathcal{F}_\tau := \mathcal{F}, \quad \mathcal{C}_\tau := \tau \mathcal{C}, \quad \mathcal{A}_\tau := \mathcal{F} + \tau \mathcal{C} \quad (4.6)$$

Fix also a (strict) ideal $I \subseteq R$; then set $R_I := R/I$ for the corresponding quotient ring, and use notation $\mathcal{A}_{\tau,I} := \mathcal{A}_\tau/I\mathcal{A}_\tau \cong (R/I) \otimes_R \mathcal{A}_\tau$, $\mathcal{F}_{\tau,I} := \mathcal{F}_\tau/I\mathcal{F}_\tau \cong (R/I) \otimes_R \mathcal{F}$ and $\mathcal{C}_{\tau,I} := \mathcal{C}_\tau/I\mathcal{C}_\tau \cong (R/I) \otimes_R \mathcal{C}_\tau \cong (R/I) \otimes_R (\tau \mathcal{C})$. By construction we have $\mathcal{A}_{\tau,I} = \mathcal{F}_{\tau,I} \oplus \mathcal{C}_{\tau,I}$ as an $R_I$–module; moreover,

$$\mathcal{F}_{\tau,I} \cdot \mathcal{F}_{\tau,I} \subseteq \mathcal{F}_{\tau,I}, \quad \mathcal{F}_{\tau,I} \cdot \mathcal{C}_{\tau,I} \subseteq \mathcal{C}_{\tau,I}, \quad \mathcal{C}_{\tau,I} \cdot \mathcal{F}_{\tau,I} \subseteq \mathcal{C}_{\tau,I}, \quad \mathcal{C}_{\tau,I} \cdot \mathcal{C}_{\tau,I} \subseteq \mathcal{F}_{\tau,I}$$

where the last identity comes from $\mathcal{C}_{\tau,I} \cdot \mathcal{C}_{\tau,I} = \tau^2 (\mathcal{C} \cdot \mathcal{C}) \subseteq \tau^2 \mathcal{F} = \tau^2 \mathcal{F}_{\tau}$ and we write $\tau := (\tau \bmod I) \in R/I$. In particular, if $\tau \in I$ then $\mathcal{C}_{\tau,I} \cdot \mathcal{C}_{\tau,I} = \{0\}$ and we get

$$\mathcal{A}_{\tau,I} = \mathcal{F}_{\tau,I} \ltimes \mathcal{C}_{\tau,I} \quad (4.7)$$

where $\mathcal{C}_{\tau,I}$ bears the $\mathcal{F}_{\tau,I}$–bimodule structure induced from $\mathcal{A}$ and is given a trivial product, so that it sits inside $\mathcal{A}_{\tau,I}$ as a two-sided ideal, $\mathcal{F}_{\tau,I}$ being a semidirect product splitting.

In short, for $\tau \in I$ this process leads us from the initial object $\mathcal{A}$, that splits into $\mathcal{A} = \mathcal{F} \oplus \mathcal{C}$ as $R$–module, to the final object $\mathcal{A}_{\tau,I} = \mathcal{F}_{\tau,I} \ltimes \mathcal{C}_{\tau,I}$, now split as a semidirect product. Following [DR], §2 — and references therein — we shall refer to this process as “contraction”, and also refer to $\mathcal{A}_{\tau,I}$ as to a “contraction of $\mathcal{A}$”.

We apply now the above contraction procedure to a couple of integral forms of $\mathfrak{g}_0$. 


First consider the case \( A := \mathfrak{g}(\mathfrak{x}) \), \( F := \mathfrak{g}(\mathfrak{x})_0 \) and \( C := \mathfrak{g}(\mathfrak{x})_1 \); here the ground ring is \( R := \mathbb{Z}[\mathfrak{x}] \), and we choose in it \( \tau := x_1 x_2 x_3 \); the ideal \( I \) generated by \( x_1 - \sigma_1 \), \( x_2 - \sigma_2 \) and \( x_3 - \sigma_3 \). In this case, the “blown-up” Lie superalgebras in (4.6) reads \( \mathfrak{g}(\mathfrak{x})_\tau = \mathfrak{g}(\mathfrak{x})_0 \oplus (\tau \mathfrak{g}(\mathfrak{x})_1) \), that we write also with the simpler notation \( \hat{\mathfrak{g}}(\mathfrak{x}) := \mathfrak{g}(\mathfrak{x})_\tau \). Now, this provides yet another coherent sheaf of Lie superalgebras over \( V \), whose fibre \( \hat{\mathfrak{g}}(\sigma) \) at each non-singular (closed) point \( \sigma \) is again a \( \mathbb{Z}[\sigma] \)-integral form of our initial complex Lie superalgebra \( \mathfrak{g}_\sigma \). At singular points instead, the fibres of this sheaf — i.e., the singular specializations of \( \mathfrak{g}(\mathfrak{x})_\tau \) — are described by a slight variation of Theorem 4.1.1 taking into account that the Lie bracket on the odd part will be trivial, because we realize them as contractions of \( \hat{\mathfrak{g}}(\mathfrak{x}) := \mathfrak{g}(\mathfrak{x})_\tau \). Similarly occurs if we work over \( \mathbb{C} \), i.e. we consider \( A := \mathfrak{g}(\mathfrak{x})_c \), \( F := (\mathfrak{g}(\mathfrak{x})_c)_0 \) and \( C := (\mathfrak{g}(\mathfrak{x})_c)_1 \) with ground ring \( R := \mathbb{C}[\mathfrak{x}] \).

Hereafter we give the exact statement on singular specializations, focusing on the complex case. As a matter of notation, we set \( \hat{\alpha}_i := a_i \) (\( := C X_{2i} \oplus CH_{2i} \oplus C X_{-2i} \)), cf. Remark 3.1.1 for all \( i = 1, 2, 3 \).

**Theorem 4.4.1.** Let \( \sigma \in V := \mathbb{C}^{x^3} \cap \{ \sigma_1 + \sigma_2 + \sigma_3 = 0 \} \) as before. Then

1. If \( \sigma_i = 0 \) and \( \sigma_j \neq 0 \neq \sigma_k \) for \( \{ i,j,k \} = \{ 1,2,3 \} \), then \( \hat{\alpha}_i \leq \hat{\mathfrak{g}}(\sigma)_c \) is a central Lie ideal in \( \hat{\mathfrak{g}}(\sigma)_c \), with \( \hat{\alpha}_i \cong \mathfrak{g}(\mathfrak{c})^{*3} \), while \( \hat{\alpha}_i \cong \hat{\mathfrak{g}}(\sigma)_c \cong sl_2 \) for \( \{ j,k \} = \{ 1,2,3 \} \setminus \{ i \} \); moreover, we have a semidirect product splitting

\[
\hat{\mathfrak{g}}(\sigma)_c \cong (\hat{\mathfrak{g}}(\sigma)_c)_0 \ltimes (\hat{\mathfrak{g}}(\sigma)_c)_1
\]

with \( (\hat{\mathfrak{g}}(\sigma)_c)_0 \cong \mathbb{C}^{*3} \oplus sl_2 \oplus sl_2 \) while \( (\hat{\mathfrak{g}}(\sigma)_c)_1 \) is endowed with trivial Lie bracket and \( (\hat{\mathfrak{g}}(\sigma)_c)_1 \cong (\mathfrak{m} \oplus \mathfrak{n}) \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \) — where \( \mathfrak{m} \) is the trivial representation — as a module over \( (\hat{\mathfrak{g}}(\sigma)_c)_0 \cong \mathbb{C}^{*3} \oplus sl_2 \oplus sl_2 \).

A parallel result also holds true when working with \( \mathfrak{g}(\sigma) \) over the ground ring \( \mathbb{Z}[\sigma] \).

2. If \( \sigma_i = 0 \) for all \( h \in \{ 1,2,3 \} \), i.e. \( \sigma = 0 \), then \( \mathfrak{g}(0)_c \) is Abelian.

A parallel result holds true when working with \( \hat{\mathfrak{g}}(0)_c \) over the ground ring \( \mathbb{Z}[0] = \mathbb{Z} \).

**Proof.** The claim follows directly from Theorem 4.1.1 once we take also into account the fact that the \( \hat{\mathfrak{g}}(\sigma)_c \)'s are specializations of \( \hat{\mathfrak{g}}(\mathfrak{x})_c \), and for singular values of \( \sigma \) any such specialization is indeed a contraction of \( \hat{\mathfrak{g}}(\sigma)_c \), of the form \( \hat{\mathfrak{g}}(\sigma)_c = (\mathfrak{g}(\mathfrak{x})_c)_{\tau,I} \) for the element \( \tau := x_1 x_2 x_3 \) and the ideal \( I := \{ (x_i - \sigma_i)_{i=1,2,3} \} \). Otherwise, we can deduce the statement directly from the explicit formulas for (linear) generators of \( \mathfrak{g}(\mathfrak{x})_c \): indeed, the latter are easily obtained as slight modifications — taking into account that odd generators must be “rescaled” by the coefficient \( \tau := x_1 x_2 x_3 \) — of the similar formulas in §4.1.2. \( \square \)

As a second instance, we consider the case \( A := \mathfrak{g}(\mathfrak{x}) \), \( F := \mathfrak{g}(\mathfrak{x})_0 \) and \( C := \mathfrak{g}(\mathfrak{x})_1 \); the ground ring is again \( R := \mathbb{Z}[\mathfrak{x}] \), and again we choose in it \( \tau := x_1 x_2 x_3 \) and the ideal \( I \) generated by \( x_1 - \sigma_1 \), \( x_2 - \sigma_2 \) and \( x_3 - \sigma_3 \). In this second case, we have again a “blown-up” Lie superalgebra as in (4.6), that now reads \( \mathfrak{g}(\mathfrak{x})_\tau = \mathfrak{g}(\mathfrak{x})_0 \oplus (\tau \mathfrak{g}(\mathfrak{x})_1) \), for which we use the simpler notation \( \hat{\mathfrak{g}}(\mathfrak{x}) := \mathfrak{g}(\mathfrak{x})_\tau \). This gives one more coherent sheaf of Lie superalgebras over \( V \), whose fibre \( \hat{\mathfrak{g}}(\sigma) \) at each non-singular \( \sigma \in V \) is a new \( \mathbb{Z}[\sigma] \)-integral form of the complex Lie superalgebra \( \mathfrak{g}_\sigma \) we started with. Instead, the singular specializations of \( \mathfrak{g}(\mathfrak{x})_\tau \) are described by a variant of Theorem 4.2.1 taking into due account that the odd part now will have trivial Lie bracket, in that those fibres are now realized as suitable “contractions” of \( \hat{\mathfrak{g}}(\mathfrak{x}) := \mathfrak{g}(\mathfrak{x})_\tau \). The same holds if we work on the complex field, i.e. we deal with
\[ \mathcal{A} := g'(x)_c, \quad \mathcal{F} := (g'(x)_c)_0 \quad \text{and} \quad \mathcal{C} := (g'(x)_c)_1 \quad \text{with ground ring} \quad R := \mathbb{C}[x]. \] The exact statement on singular specializations (which is the same at any singular point in \( V \), this time), focused on the complex case, is given below. As a matter of notation, we set now \( \hat{a}' := a'_i := \mathbb{C}X_{2i} \oplus \mathbb{C}H_{2i} \oplus \mathbb{C}X_{2i} \), cf. \((3.2.1)\) for all \( i = 1, 2, 3 \).

**Theorem 4.4.2.** Let \( \sigma \in V =: \mathbb{C}^{x_3} \cap \{ \sigma_1 + \sigma_2 + \sigma_3 = 0 \} \) as before. Assume that \( \sigma \) is singular, i.e. \( \sigma_i = 0 \) for some \( i \in \{1, 2, 3\} \). Then \( (\hat{g}'(\sigma)_c)_{\sigma} \cong \mathfrak{s}_2^{\mathfrak{B}} \) as Lie superalgebras, the Lie bracket is trivial on \( (\hat{g}'(\sigma)_c)_1 \) and \( (\hat{g}'(\sigma)_c)_1 \cong \mathfrak{B} \), as modules over \( (\hat{g}'(\sigma)_c)_0 \cong \mathfrak{s}_2^{\mathfrak{B}} \); finally, we have semidirect product splittings

\[
\hat{g}'(\sigma)_c \cong (\hat{g}'(\sigma)_c)_0 \triangleright (\hat{g}'(\sigma)_c)_1 \cong \mathfrak{s}_2^{\mathfrak{B}} \triangleright (\mathfrak{B}).
\]

In other words, there exists a split short exact sequence

\[
0 \longrightarrow \square \longrightarrow (\hat{g}'(\sigma)_c)_1 \longrightarrow \hat{g}'(\sigma)_c \longrightarrow (\hat{g}'(\sigma)_c)_0 \cong \mathfrak{s}_2^{\mathfrak{B}} \longrightarrow 0.
\]

A parallel result also holds true when working with \( \hat{g}'(\sigma) \) over the ground ring \( \mathbb{Z}[\sigma] \).

**Proof.** Here again, the claim follows directly from Theorem 4.1.1.2 together with the fact that each \( \hat{g}'(\sigma)_c \) is a specialization of \( \hat{g}'(x)_c \), and for singular values of \( \sigma \) any such specialization is indeed a contraction of \( \hat{g}'(\sigma)_c \), namely of the form \( \hat{g}'(\sigma)_c = (g(x)_c)_{\tau_1} \), for the element \( \tau := x_1 x_2 x_3 \) and the ideal \( I := \{ (x_i - \sigma_i)_{i=1,2,3} \} \). As alternative method, one can deduce the statement via a direct analysis of the explicit formulas for (linear) generators of \( g(x)_c \), which are easily obtained as slight modification — taking into account the “rescaling” of odd generators by the coefficient \( \tau := x_1 x_2 x_3 \) — of the formulas in \((1.2.2)\). \( \Box \)

**Remark 4.4.3.** We considered five families of Lie superalgebras, denoted by \( \{ g(\sigma)_c \}_{\sigma \in V} \), \( \{ \hat{g}'(\sigma)_c \}_{\sigma \in V} \), \( \{ \hat{g}'(\sigma)_c \}_{\sigma \in V} \), \( \{ \hat{g}'(\sigma)_c \}_{\sigma \in V} \), \( \{ \hat{g}'(\sigma)_c \}_{\sigma \in V} \), all being indexed by the points of \( V \). Now, our analysis shows that these five families share most of their elements, namely all those indexed by “general points” \( \sigma \in V \setminus \left( \bigcup_{i=1,2,3} \{ \sigma_i = 0 \} \right) \). On the other hand, the families are drastically different at all points in the “singular locus” \( S := V \cap \bigcup_{i=1,2,3} \{ \sigma_i = 0 \} \). In other words, the five sheaves \( L_{g_c[x]} \), \( L_{\hat{g}'_c[x]} \), \( L_{\hat{g}'_c[x]} \), \( L_{\hat{g}'_c[x]} \) and \( L_{\hat{g}'_c[x]} \) of Lie superalgebras over \( \text{Spec} \ (\mathbb{C}[x]) \cong V \cup \{ \star \} \) ( \( \cong \mathbb{A}^2 \cup \{ \star \} \) ) share the same stalks on all “general” points (i.e., those outside \( S \)) and have different stalks instead on “singular” points (i.e., those in \( S \)). Likewise, the five fibre bundles \( L_{g_c[x]} \), \( L_{\hat{g}'_c[x]} \), \( L_{\hat{g}'_c[x]} \), \( L_{\hat{g}'_c[x]} \) and \( L_{\hat{g}'_c[x]} \) over \( \text{Spec} \ (\mathbb{C}[x]) \) share the same fibres on all general points and have different fibres on singular points.

The outcome is, loosely speaking, that our construction provides five different “completions” of the (more or less known) family \( \{ g_\sigma \}_{\sigma \in V \setminus S} \) of simple Lie superalgebras, by adding — in five different ways — some new non-simple extra elements on top of each point of the “singular locus” \( S \).

Finally, recall that the original complex Lie algebras \( g_\sigma \) of type \( D(2,1; \sigma) \) were described by Scheunert (see \([Sc\), Ch. 1, §1, Example 5]) for any \( \sigma \in V \), i.e. also for singular values of \( \sigma \). On the general locus \( V \setminus S \), Scheunert’s \( g_\sigma \) coincides with the Lie superalgebra (for the same \( \sigma \)) of any one of our five families above. On the singular locus instead — i.e., for any \( \sigma \in S \) — a straightforward comparison one shows that \( g_\sigma \) coincides with \( \hat{g}'(\sigma)_c \), the
Lie superalgebra (over $\sigma$) of our second family. In this sense, our $g'(\sigma)$ is a $\mathbb{Z}[\sigma]$–form of $g_{\sigma}$ for any $\sigma \in V$, whilst the other four families provide us different $\mathbb{Z}[\sigma]$–forms of $g_{\sigma}$ only on the general locus $V \setminus S$, that is a dense open subset in $V$.

5. Lie supergroups of type $D(2,1;\sigma)$: presentations and degenerations

In this section, we introduce (complex) Lie supergroups of type $D(2,1;\sigma)$, basing on the five families of Lie superalgebras introduced in [4] and following the approach of [2.4.2].

For simplicity, we formulate everything over $\mathbb{C}$, but the reader may see some subtleties to discuss about the Chevalley groups over a $\mathbb{Z}[\sigma]$–algebra. The latter had been discussed in [FG] and [Gal] for some basis, i.e. for one particular choice of $\mathbb{Z}[\sigma]$–integral form (though with slightly different formalism); in the present case everything works similarly, up to paying attention to the $\sigma$–dependence of the commutation relations of the $\mathbb{Z}[\sigma]$–form one chooses (cf. [4]). The details are left to the reader.

5.1. First family: the Lie supergroups $G_{\sigma}$.

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{C}^3 \cap \{ \sum_i \sigma_i = 0 \} =: V$, let $g = g(\sigma)_C$ be the complex Lie superalgebra associated with $\sigma$ as in [4.1] and $g_0$ its even part. We recall that $g$ is spanned over $\mathbb{C}$ by $\{ H_{2\sigma_1}, H_{2\sigma_2}, H_{2\sigma_3}, H_0 \} \cup \{ X_{\alpha} \}_\Delta$. Like in [4.1], we set $a_i := \mathbb{C} X_{2\sigma_i} \oplus \mathbb{C} H_{2\sigma_i} \oplus \mathbb{C} X_{-2\sigma_i}$ for each $i$, and all these being Lie subalgebras of $g(\sigma)_C$, with $(g(\sigma)_C)_0 = \oplus_{i=1}^3 a_i$. When $\sigma_i \neq 0$, the Lie algebra $a_i$ is isomorphic to $sl_2$: an explicit isomorphism is realized by mapping $X_{2\sigma_i} \mapsto \sigma_i e$, $H_{2\sigma_i} \mapsto \sigma_i h$ and $X_{-2\sigma_i} \mapsto \sigma_i f$, where $\{ e, h, f \}$ is the standard basis $sl_2$. When $\sigma_i = 0$ instead, $a_i \cong \mathbb{C}^{\geq 3}$ becomes the 3-dimensional Abelian Lie algebra.

Let us now set $A_i := SL_2$ if $\sigma_i \neq 0$ and $A_i := \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}$ if $\sigma_i = 0$, and define $G := \times_{i=1}^3 A_i$ — a complex Lie group such that $Lie(G) = (g(\sigma)_C)_0$. One sees that the adjoint action of $(g(\sigma)_C)_0$ onto $g(\sigma)_C$ integrates to a Lie group action of $G$ onto $g(\sigma)_C$ again, so that the pair $P_{\sigma} := (G, g(\sigma)_C)$ — endowed with that action — is a super Harish-Chandra pair (cf. [2.4.1]); note that its dependence on $\sigma$ lies within all its constituents: the structure of $G$, the Lie superalgebra $g(\sigma)_C$, and the action of the former onto the latter.

Finally, we let

$$G_{\sigma} := G_{P_{\sigma}}$$

be the complex Lie supergroup associated with the super Harish-Chandra pair $P_{\sigma}$ trough the category equivalence given in [2.4.3].

5.1.1. A presentation of $G_{\sigma}$. We shall now provide an explicit presentation by generators and relations for the supergroups $G_{\sigma}$, i.e. for the abstract groups $G_{\sigma}(A)$, $A \in (Wsalg)_C$.

To begin with, inside each subgroup $A_i$ we consider the elements $x_{2\sigma_i}(c) := \exp(c X_{2\sigma_i})$, $h_{2\sigma_i}(c) := \exp(c H_{2\sigma_i})$, $x_{-2\sigma_i}(c) := \exp(c X_{-2\sigma_i})$ for every $c \in \mathbb{C}$; then $\Gamma_i := \{ x_{2\sigma_i}(c), h_{2\sigma_i}(c), x_{-2\sigma_i}(c) \}_{c \in \mathbb{C}}$ is a generating set for $A_i$.

We define also elements $h_\theta(c) := \exp(c H_\theta)$ for all $c \in \mathbb{C}$: then the commutation relations $[H_{2\sigma_1}, H_{2\sigma_2}] = 0$ and $H_{2\sigma_1} + H_{2\sigma_2} + H_{2\sigma_3} = 2 H_\theta$ inside $g(\sigma)_C$, together imply the group relations $h_{2\sigma_1}(c) h_{2\sigma_2}(c) h_{2\sigma_3}(c) = h_\theta(c)^2$ for all $c \in \mathbb{C}$.
The complex Lie group $G_+$ is clearly generated by
\[ \Gamma_0 := \left\{ x_{2\epsilon}(c), h_{2\epsilon}(c), h_\theta(c), x_{-2\epsilon}(c) \right\}_{c \in \mathbb{C}} \]
(the $h_\theta(c)$’s might be dropped, but we prefer to add them too as generators).

In addition, when we consider $G$ as a (totally even) supergroup and we look at it as a group-valued functor $G : (\mathsf{Wsalg})_\mathbb{C} \rightarrow (\mathsf{grps})$, the abstract group $G(A)$ of its $A$–points — for $A \in (\mathsf{Wsalg})_\mathbb{C}$ — is generated by the set
\[ \Gamma_0(A) := \left\{ x_{2\epsilon}(a), h_{2\epsilon}(a), h_\theta(a), x_{-2\epsilon}(a) \right\}_{a \in A_0} \quad (5.1) \]

Note that here the generators do make sense — as operators in $\text{GL}(A \otimes \mathfrak{g}((\sigma)_\mathbb{C}))$, but also formally — since $A = \mathbb{C} \oplus \mathfrak{h}(A)$ (cf. \[2.3.1\]), so each $a \in A$ reads as $a = c + n_a$ for some $c \in \mathbb{C}$ and a nilpotent $n_a \in \mathfrak{h}(A)$, hence $\exp(a X_{2\epsilon}) = \exp(c X_{2\epsilon}) \exp(n_a X_{2\epsilon})$, etc., are all well-defined.

Following the recipe in \[2.4.3\] in order to generate the group $G_\sigma(A) := G_{\tau_\sigma}(A)$, beside the subgroup $G(A)$ we need also all the elements of the form $(1 + \eta Y_i)$ with $(i, \eta_i) \in I \times A_\mathbb{C}$ — cf. \[2.4.3\] — where now the $\mathbb{C}$–basis $\mathcal{Y}_i \in \mathfrak{y}_\mathbb{C}$ is $\{ Y_i \}_{i \in I} = \{ X_{\pm \theta}, X_{\pm \beta_i} \}_{i = 1, 2, 3}$. Therefore, we introduce notation $x_{\pm \theta}(\eta) := (1 + \eta X_{\pm \theta})$, $x_{\pm \beta_i}(\eta) := (1 + \eta X_{\pm \beta_i})$ for all $\eta \in A_\mathbb{C}$, $i \in \{1, 2, 3\}$, and we consider the set $\Gamma_1(A) := \{ x_{\pm \theta}(\eta), x_{\pm \beta_i}(\eta) \} \eta \in A_\mathbb{C} \}$.

Now, taking into account that $G(A)$ is generated by $\Gamma_0(A)$, we can modify the set of relations given in \[2.4.3\] by letting $g \in G(A)$ range inside the set $\Gamma_0(A)$: then we can find the following full set of relations (where hereafter we freely use notation $e^Z := \exp(Z)$):

\[ 1_c = 1 \quad , \quad g' \cdot g'' = g' \circ g'' \quad (\forall g', g'' \in G(A)) \]
\[ h_{2\epsilon}(a) x_{\pm \beta_j}(\eta) h_{2\epsilon}(a)^{-1} = x_{\pm \beta_j}(e^{\pm (1 - \delta_{ij}) \sigma_i a \eta}) \]
\[ h_\theta(a) x_{\pm \beta_j}(\eta) h_\theta(a)^{-1} = x_{\pm \beta_j}(e^{\pm \sigma_i a \eta}) \]
\[ h_\theta(a) x_{\pm \beta_j}(\eta) h_\theta(a)^{-1} = x_{\pm \beta_j}(e^{\pm \sigma_i a \eta}) \]
\[ x_{2\epsilon}(a) x_{\pm \beta_j}(\eta) x_{2\epsilon}(a)^{-1} = x_{\pm \beta_j}(\eta) x_{\theta}(\delta_{ij} \sigma_i a \eta) \]
\[ x_{2\epsilon}(a) x_{-\beta_j}(\eta) x_{2\epsilon}(a)^{-1} = x_{-\beta_j}(\eta) x_{-\beta_k}(1 - \delta_{ij}) \sigma_i a \eta \]
\[ x_{-2\epsilon}(a) x_{\beta_j}(\eta) x_{-2\epsilon}(a)^{-1} = x_{\beta_j}(\eta) x_{-\beta_k}(1 - \delta_{ij}) \sigma_i a \eta \]
\[ x_{-2\epsilon}(a) x_{-\beta_j}(\eta) x_{-2\epsilon}(a)^{-1} = x_{-\beta_j}(\eta) x_{-\theta}(\delta_{ij} \sigma_i a \eta) \]
\[ x_{2\epsilon}(a) x_{\theta}(\eta) x_{2\epsilon}(a)^{-1} = x_{\theta}(\eta) \]
\[ x_{2\epsilon}(a) x_{-\theta}(\eta) x_{2\epsilon}(a)^{-1} = x_{-\theta}(\eta) x_{-\beta_j}(\sigma_i a \eta) \]
\[ x_{-2\epsilon}(a) x_{\theta}(\eta) x_{-2\epsilon}(a)^{-1} = x_{\theta}(\eta) x_{\beta_j}(\sigma_i a \eta) \]
\[ x_{-2\epsilon}(a) x_{-\theta}(\eta) x_{-2\epsilon}(a)^{-1} = x_{-\theta}(\eta) x_{\beta_j}(\sigma_i a \eta) \]
\[ x_{\beta_j}(\eta_i) x_{\beta_j}(\eta_j) = x_{\beta_j}((1 - \delta_{ij}) \eta_j^i \eta_i \eta_j^i - \beta_j(\eta_i) x_{\beta_j}(\eta_j) \]
\[ x_{-\beta_j}(\eta_i) x_{-\beta_j}(\eta_j) = x_{-2\epsilon}((1 - \delta_{ij}) \eta_j^i \eta_i \eta_j^i) - \beta_j(\eta_i) x_{-\beta_j}(\eta_j) \]
\[ x_{\beta_j}(\eta_i) x_{-\beta_j}(\eta_j) = x_{-2\epsilon}((1 - \delta_{ij}) \eta_j^i \eta_i \eta_j^i) - \beta_j(\eta_i) x_{\beta_j}(\eta_j) \]
\[ x_{-\beta_j}(\eta_i) x_{\beta_j}(\eta_j) = x_{2\epsilon}((1 - \delta_{ij}) \eta_j^i \eta_i \eta_j^i) - \beta_j(\eta_i) x_{\beta_j}(\eta_j) \]
\[ x_{-\beta_j}(\eta_i) x_{-\beta_j}(\eta_j) = x_{2\epsilon}((1 - \delta_{ij}) \eta_j^i \eta_i \eta_j^i) - \beta_j(\eta_i) x_{\beta_j}(\eta_j) \]
\[ x_{-\beta_j}(\eta_i) x_{\theta}(\eta) = x_{\theta}(\eta) x_{-\beta_j}(\eta_i) \]
\[ x_{-\beta_j}(\eta_i) x_{-\theta}(\eta) = x_{-\theta}(\eta) x_{\beta_j}(\eta_i) \]
5.1.2. Singular specializations of the supergroup(s) $G_\sigma$. From the very construction of the supergroups $G_\sigma$, we get that

$G_\sigma$ is simple (as a Lie supergroup)

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$ such that $\sigma_i \neq 0$ for all $i \in \{1, 2, 3\}$.

This follows from the presentation of $G_\sigma$ in §5.1.1 above, or it can be seen as a direct consequence of the relation $\text{Lie}(G_\sigma) = g(\sigma) = g_\sigma$ and of Proposition 3.1.4.

On the other hand, the situation is different at “singular values” of the parameter $\sigma$, as the following shows:

**Theorem 5.1.1.** Let $\sigma \in V$ as usual.

1. If $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then $A_i \leq G_\sigma$, $A_i \cong \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}$ and $G_\sigma$ is a central extension of $\text{PSL}(2|2)$ by $A_i$; in other words, there exists a short exact sequence of Lie supergroups

$$1 \longrightarrow \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \cong A_i \longrightarrow G_\sigma \longrightarrow \text{PSL}(2|2) \longrightarrow 1$$

2. If $\sigma_h = 0$ for all $h \in \{1, 2, 3\}$, i.e. $\sigma = 0$, then $(G_\sigma)_{rd} \cong (\mathbb{C} \times \mathbb{C}^* \times \mathbb{C})^3$ is the center of $G_\sigma$, and the quotient $G_\sigma/(G_\sigma)_{rd} \cong \mathbb{C}^{\mathbb{Z}^8}$ is Abelian; in particular, $G_\sigma$ is a central extension of $\mathbb{C}^{\mathbb{Z}^8}$ by $(\mathbb{C} \times \mathbb{C}^* \times \mathbb{C})^3$, i.e. there exists a short exact sequence of Lie supergroups

$$1 \longrightarrow (\mathbb{C} \times \mathbb{C}^* \times \mathbb{C})^3 \cong (G_\sigma)_{rd} \longrightarrow G_\sigma \longrightarrow \mathbb{C}^{\mathbb{Z}^8} \longrightarrow 1$$

**Proof.** The claim follows directly from the presentation of $G_\sigma$ given in §5.1.1 above, or also from the relation $\text{Lie}(G_\sigma) = g(\sigma)$ along with Theorem 4.1.1. \qed

5.2. Second family: the Lie supergroups $G'_\sigma$.

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, let $\mathfrak{g}' := g'(\sigma)$ be the complex Lie superalgebra associated with $\sigma$ as in §4.2.1 and let $\mathfrak{g}'_0$ be its even part. Fix the $\mathbb{C}$-basis $\{X_{2i}, H_{2i}, X'_{2i}\}_{i=1,2,3}$ of $\mathfrak{g}_0$ as in §3.2 and set $a'_i := \mathbb{C} X_{2i} \oplus \mathbb{C} H_{2i} \oplus \mathbb{C} X'_{2i}$ for each $i$: each one of these is a Lie subalgebra of $\mathfrak{g}'_0$, with $\mathfrak{g}'_0 = a'_1 \oplus a'_2 \oplus a'_3$. Moreover, each Lie algebra $a'_i$ is isomorphic to $\mathfrak{sl}_2$, an explicit isomorphism being given by $X_{2i} \mapsto e$, $H_{2i} \mapsto h$ and $X'_{2i} \mapsto f$, where $\{e, h, f\}$ is the standard basis $\mathfrak{sl}_2$. It follows that $\mathfrak{g}'_0$ is isomorphic to $\mathfrak{sl}_2^{\mathbb{Z}^3}$.

For each $i \in \{1, 2, 3\}$, let $A'_i$ be a copy of $\text{SL}_2$, and set $G' := A'_1 \times A'_2 \times A'_3$. By the previous analysis, $\text{Lie}(G')$ is isomorphic to $\mathfrak{g}'_0$ and the $\text{Lie}(G')$–action lifts to a holomorphic $G'$–action on $\mathfrak{g}'$ again: in fact, one easily sees that this action is faithful too. With this action, $\mathcal{P}' := (G', \mathfrak{g}')$ is a super Harish-Chandra pair (cf. §2.4.1), which overall depends on $\mathcal{P}_\sigma$ (although $G'$ alone does not). Finally, we define

$$G'_\sigma := G'_\sigma.$$
to be the complex Lie supergroup associated with the super Harish-Chandra pair $\mathcal{P}_\sigma'$ via the equivalence of categories given in [2.4.3].

5.2.1. A presentation of $G'_\sigma$. The supergroups $G'_\sigma$ can be described in concrete terms via an explicit presentation by generators and relations of all the abstract groups $G'_\sigma(A)$, with $A$ ranging in $(\text{Wsalg})_C$. To this end, we first consider the Lie group $G' = A'_1 \times A'_2 \times A'_3$ with $A'_i \cong \text{SL}_2$. Letting $\exp : \mathfrak{g}'_0 \cong \text{Lie}(G') \longrightarrow G'$ be the exponential map, we consider $x'_{2c_i}(c) := \exp(cX'_{2c_i})$, $h'_{2c_i}(c) := \exp(cH'_{2c_i})$, $x'_{-2c_i}(c) := \exp(cX'_{-2c_i})$ and $h'_0(c) := \exp(cH'_0)$ for all $c \in \mathbb{C}$. Note that the commutation relations $[H'_{2c_i}, H'_{2c_j}] = 0$ and $\sigma_1H'_{2c_1} + \sigma_2H'_{2c_2} + \sigma_3H'_{2c_3} = 2H'_0$ inside $\mathfrak{g}'(\sigma)_C$ together imply inside $G'_+ \subset G'$ the group relations 

$$h'_{2c_1}(\sigma_1c)h'_{2c_2}(\sigma_2c)h'_{2c_3}(\sigma_3c) = h'_0(c)^2$$

for all $c \in \mathbb{C}$.

The complex Lie group $G'$ is clearly generated by the set

$$G'_0 := \left\{ x'_{2c_i}(c), h'_{2c_i}(c), h'_0(c), x'_{-2c_i}(c) \mid c \in \mathbb{C} \right\}$$

(where the $h'_0(c)$'s might be discarded, but we prefer to keep them). Then, looking at $G'$ as a (totally even) supergroup thought of as a group-valued functor $G' : (\text{Wsalg})_C \longrightarrow (\text{grps})_C$, each abstract group $G'(A)$ of its $A$-points — for $A \in (\text{Wsalg})_C$ — is generated by the set

$$G'_0(A) := \left\{ x'_{2c_i}(a), h'_{2c_i}(a), h'_0(a), x'_{-2c_i}(a) \mid a \in A'_0 \right\}$$

(5.2)

Following [2.4.3], we need as generators of $G'_\sigma(A) := G'_{\sigma'}(A)$ all the elements of $G'(A)$ and all those of the form $x'_{\pm}(\eta) := (1 + \eta X'_{\pm})$ or $x'_{\pm}(\eta) := (1 + \eta X'_{\pm})$ with $\eta \in A_i$ and $i \in \{1,2,3\}$ — since now we fix $\{Y^j_i\}_{i \in I} = \{X'_{\pm}, X'_{\pm,j} \}_{i=1,2,3}$ as our $\mathbb{C}$-basis of $\mathfrak{g}'_0$; we denote the set of all the latter by $G'_1(A) := \{ x'_{\pm}(\eta), x'_{\pm,j}(\eta) \mid \eta \in A_i \}$.

Implementing the recipe in [2.4.3] and recalling that $G'(A)$ is generated by $G'_0(A)$, we can now slightly modify the relations presented in [2.4.3] and consider instead the following, alternative full set of relations among the generators of $G'_\sigma(A)$:

$$1_G = 1$$

$$\begin{align*}
g' \cdot g'' & = g' \cdot g'' \quad (\forall \ g', g'' \in G'(A)) \\
h'_{2c_i}(a)x'_{\pm,j}(\eta)h'_{2c_i}(a)^{-1} & = x'_{\pm,j}(e^{(\pm i)\eta} \cdot a) \\
h'_{2c_i}(a)x'_{\pm,j}(\eta)h'_{2c_i}(a)^{-1} & = x'_{\pm,j}(e^{\pm i} \cdot a) \\
h'_0(a)x'_{\pm,j}(\eta)h'_0(a)^{-1} & = x'_{\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x'_{\pm,j}(\eta)x'_{2c_i}(a)^{-1} & = x'_{\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x'_{-\pm,j}(\eta)x'_{2c_i}(a)^{-1} & = x'_{-\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x'_{\pm,j}(\eta)x'_{-2c_i}(a)^{-1} & = x'_{\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x'_{-\pm,j}(\eta)x'_{-2c_i}(a)^{-1} & = x'_{-\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x''_{\pm,j}(\eta)x'_{2c_i}(a)^{-1} & = x''_{\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x''_{-\pm,j}(\eta)x'_{2c_i}(a)^{-1} & = x''_{-\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x''_{\pm,j}(\eta)x'_{-2c_i}(a)^{-1} & = x''_{\pm,j}(e^{\pm i} \cdot a) \\
x'_{2c_i}(a)x''_{-\pm,j}(\eta)x'_{-2c_i}(a)^{-1} & = x''_{-\pm,j}(e^{\pm i} \cdot a)
\end{align*}$$

$$x'_{\pm,j}(\eta_1 x'_{\pm,j}(\eta_2)) = x'_{2c_k}((1-\delta_{i,j}) \cdot \sigma_i \eta'_i \eta_j \cdot x'_{\pm,j}(\eta_2))x'_{\pm,j}(\eta_1)$$
Indeed, this follows from the presentation of $5.2.2$. Let $\{i,j,k\} = \{1,2,3\}$. 

5.2.2. Singular specializations of the supergroup(s) $G'_\sigma$. By construction, for the supergroups $G'_\sigma$ we have that $G'_\sigma$ is simple (as a Lie supergroup) for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{V}$ such that $\sigma_i \neq 0$ for all $i \in \{1,2,3\}$. Indeed, this follows from the presentation of $G'_\sigma$ in $5.2.1$ above, but also as a fallout of the relation $\text{Lie}(G'_\sigma) = \mathfrak{g}'(\sigma)_\mathbb{C} = \mathfrak{g}_\sigma$ along with Proposition $3.1.1$.

The situation is different at “singular values” of the parameter $\sigma$; the precise result is

**Theorem 5.2.1.** Let $\sigma \in \mathbb{V} =: \mathbb{C}^{3} \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$ as above.

(1) If $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i,j,k\} = \{1,2,3\}$, then letting $B'_i$ be the Lie subsupergroup of $G'_\sigma$ defined on every $A \in (\text{Wsalg})_\mathbb{C}$ by

$$B'_i(A) := \left\{ \left\{ h'_i(a), x'_{a}(b), x'_{\pm}\eta(1) \right\} \right\} \bigg\{ a \in A_{\sigma}, b \in A_{\overline{\sigma}}, \eta \in A_{i} \bigg\},$$

we have $B'_i \subseteq G'_\sigma$ (a normal Lie subsupergroup), $A'_i \leq G'_\sigma$ (a Lie subsupergroup), and there exist isomorphisms $B'_i \cong \mathbb{PGL}(2|\mathbb{C})$, $A'_i \cong \mathbb{SL}(2|\mathbb{C})$ — a semidirect product of Lie supergroups. In short, there exists a split short exact sequence

$$1 \longrightarrow \mathbb{PGL}(2|\mathbb{C}) \cong B'_i \longrightarrow G'_\sigma \longrightarrow A'_i \cong \mathbb{SL}(2|\mathbb{C}) \longrightarrow 1.$$

(2) If $\sigma_h = 0$ for all $h \in \{1,2,3\}$, i.e. $\sigma = 0$, then letting $(G'_\sigma)_1$ be the Lie subsupergroup of $G'_\sigma$ defined on every $A \in (\text{Wsalg})_\mathbb{C}$ by

$$(G'_\sigma)_1(A) := \left\{ \left\{ x'_\alpha(\eta) \right\} \right\} \bigg\{ \alpha \in \Delta_1 \bigg\},$$

then $G'_\sigma_1 \cong \mathbb{SL}^{2} \times \mathbb{C}$ and $(G'_\sigma)_1 \cong \mathbb{C}^{1,0} \times \mathbb{C}^{0,1} \cong \mathbb{C}^{1,0} \times \mathbb{C}^{0,1}$ as Lie (super)groups, $(G'_\sigma)_1 \cong \mathbb{C}^{1,0} \times \mathbb{C}^{0,1}$ as modules over $(G'_\sigma)_0 \cong \mathbb{SL}(2|\mathbb{C})$ — where $\square_i := \mathbb{C}^+ \oplus \mathbb{C}^-$ is the tautological 2–dimensional module over the $i$–th copy $\mathbb{SL}_2(i)$ of $\mathbb{SL}_2$ (for $i = 1,2,3$) — and $G'_\sigma \cong (G'_\sigma)_1 \times (G'_\sigma)_1 \cong \mathbb{SL}_2 \times \mathbb{SL}_2 \times \mathbb{SL}_2$. In other words, there is a split short exact sequence

$$1 \longrightarrow \mathbb{C}^{1,0} \longrightarrow G'_\sigma \longrightarrow (G'_\sigma)_0 \cong BL \longrightarrow 1.$$

**Proof.** Like for Theorem $5.1.1$, the present claim can be obtained from the presentation of $G'_\sigma$ in $5.2.1$ or otherwise from the relation $\text{Lie}(G'_\sigma) = \mathfrak{g}'(\sigma)_\mathbb{C}$ along with Theorem $4.2.1$.
5.3. Third family: the Lie supergroups $G''_\sigma$.

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, let $g'' := g''(\sigma)_c$ be the complex Lie superalgebra associated with $\sigma$ as in [1.3.1] and let $g''_0$ be its even part. Fix the elements $X_\alpha, H'_2, H'_h$ (with $\alpha \in \Delta$, $i \in \{1,2,3\}$) of $g$ as in [3] and set $a''_i := \mathbb{C} X_{2e_i} \oplus \mathbb{C} H'_2 \oplus \mathbb{C} X_{-2e_i}$ for each $i$: the latter are Lie subalgebras of $g''_0$ such that $g''_0 = a''_1 \oplus a''_2 \oplus a''_3$. Moreover, every $a''_i$ is isomorphic to $sl_2$ when $\sigma_i \neq 0$ — an explicit isomorphism being given by $X_{2e_i} \mapsto \sigma_i e$, $H'_{2e_i} \mapsto h$ and $X_{-2e_i} \mapsto \sigma_i f$, where $\{e, h, f\}$ is the standard basis $sl_2$ — while for $\sigma_i = 0$ it is isomorphic to the Lie subalgebra of $b_+ \oplus b_-$, with $b_+ := \mathbb{C} e + \mathbb{C} h$ and $b_- := \mathbb{C} h + \mathbb{C} f$ being the standard Borel subalgebras inside $sl_2$, with $\mathbb{C}$-basis $\{(e, 0), (h, h), (0, f)\}$.

Let $B_\pm$ be the Borel subgroup of $SL_2$ of all upper, resp. lower, triangular matrices, and let $S$ be the subgroup of $B_+ \times B_-$ whose elements are all the pairs of matrices $(X_+, X_-)$ such that the diagonal parts of $X_+$ and of $X_-$ are the same. For each $i \in \{1,2,3\}$, let $A''_i$ (depending on $\sigma_i$) respectively be a copy of $SL_2$ if $\sigma_i \neq 0$ and a copy of $S$ otherwise; then set $G'' := A''_1 \times A''_2 \times A''_3$. The adjoint action of $g''_0 \cong \text{Lie}(G'')$ on $g''$ lifts to a holomorphic $G''$–action on $g''$, which is faithful again; then the pair $\mathcal{P}_\sigma'' := (G'', g'')$ with this action is a super Harish-Chandra pair, in the sense of [2.4.1].

At last, we can define $G''_\sigma := G''_{\sigma''}$ as being the complex Lie supergroup associated with the super Harish-Chandra pair $\mathcal{P}_\sigma''$ through the equivalence of categories given in [2.4.3].

5.3.1. A presentation of $G''_\sigma$. In order to describe the supergroups $G''_\sigma$, we aim now for an explicit presentation by generators and relations of the abstract groups $G''_\sigma(A)$, for all $A \in (\text{Wsalg})_C$. To start with, let $G'' = A''_1 \times A''_2 \times A''_3$ be the complex Lie group considered above, and let $\exp: g''_0 \cong \text{Lie}(G'') \longrightarrow G''$ be the exponential map: then consider $x_{2e_i}(c) := \exp(c X_{2e_i})$, $h'_{2e_i}(c) := \exp(c H'_{2e_i})$, $x_{-2e_i}(c) := \exp(c X_{-2e_i})$ and $h''(c) := \exp(c H'_h)$ for all $c \in \mathbb{C}$. It is clear that $G''$ is generated by the set

$$
\Gamma''_0 := \left\{ x_{2e_i}(c) \mid c \in \mathbb{C} \right\}
$$

(actually the $h''(c)$’s might be discarded, but we choose to keep them); therefore, looking at $G''$ as a supergroup, thought of as a group-valued functor $G'' : (\text{Wsalg})_C \longrightarrow (\text{grps})$, every abstract group $G''(A)$ of its $A$–points — for $A \in (\text{Wsalg})_C$ — is generated by the set

$$
\Gamma''_1(A) := \left\{ x_{2e_i}(a), h''_{2e_i}(a), h''(a), x_{-2e_i}(a) \mid a \in A_0 \right\}
$$

(5.3)

According to [2.4.3] the group $G''_\sigma(A) := G_{\sigma''}(A)$ is generated by $G''(A)$ and all elements of the form $x_{\pm\theta}(\eta) := (1+\eta X_{\pm\theta})$ or $x_{\pm\bar{\beta}}(\eta) := (1+\eta X_{\pm\bar{\beta}})$ with $\eta \in A_1$ and $i \in \{1,2,3\}$ — as now the chosen $\mathbb{C}$–basis of $g''_1$ is $\{Y_i\}_{i \in \mathbb{I}} = \{X_{\pm\theta}, X_{\pm\bar{\beta}}\}_{i=1,2,3}$; the set of all the latter is denoted $\Gamma''_1(A) := \left\{ x_{\pm\theta}(\eta), x_{\pm\bar{\beta}}(\eta) \mid \eta \in A_1 \right\}$ — coinciding with $\Gamma''_1(A)$ in [5.2.1].

From the recipe in [2.4.3] and the fact that $G''(A)$ is generated by $\Gamma''_1(A)$, with a slight modification of the relations in [2.4.3] we can find the following full set of relations among generators of $G''_\sigma(A)$:

1.

$$
1 = 1, \quad g' \cdot g'' = g'_{\sigma''} g'' \quad (\forall \; g', g'' \in G''(A))
$$

$$
h'_{2e_i}(a) x_{\pm\beta}(\eta) h''_{2e_i}(a)^{-1} = x_{\pm\beta}(c^{\pm(-1)^{i,j} a \eta})
$$
with 

$$\{i, j, k\} = \{1, 2, 3\}.$$

5.3.2. **Singular specializations of the supergroup(s) $G^\sigma_\sigma$.** One sees easily that for the supergroups $G^\sigma_\sigma$ we have

$$G^\sigma_\sigma$$ is simple (as a Lie supergroup)

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$ such that $\sigma_i \neq 0$ for all $i \in \{1, 2, 3\}$.

This follows from the presentation of $G^\sigma_\sigma$ in [5.3.1] above, and also as a consequence of the relation $\text{Lie}(G^\sigma_\sigma) = g^\sigma_\sigma(\sigma)_C = g^\sigma_\sigma$ along with Proposition 3.1.4.

Things change, instead, for “singular values” of the parameter $\sigma$. Before seeing it, we need to introduce some further objects of interest.

For every $i \in \{1, 2, 3\}$ and $A \in (\text{Wsalg})_C$, define in $G^\sigma_\sigma(A)$ the subgroups

$$B^\sigma_i(A) := \left\{ h_\sigma(a), h^\sigma_{2e_i}(a_j), x_\alpha(c), x_\beta(\eta) \right\}_{a_j \in A_i, \eta \in A_i}$$

and

$$K^\sigma_i(A) := \left\{ x_{2e_i}(a_+), x_{-2e_i}(a_-) \right\}_{a_+ \in A_i, a_- \in A_i}$$

that overall — as $A$ ranges in $(\text{Wsalg})_C$ — define Lie subsupergroups $B^\sigma_i$ and $K^\sigma_i$ of $G^\sigma_\sigma$.

The following result then tells us how $G^\sigma_\sigma$ looks like in the singular cases:

**Theorem 5.3.1.** Given $\sigma \in V$, let $G^\sigma_\sigma$ and its Lie subsupergroups $B^\sigma_i$ and $K^\sigma_i$ be as above.

(1) If $\sigma_i = 0$ and $\sigma_j \neq 0 \neq \sigma_k$ for $\{i, j, k\} = \{1, 2, 3\}$, then there exists a split short exact sequence

$$1 \rightarrow K^\sigma_i \rightarrow G^\sigma_\sigma \rightarrow B^\sigma_i \rightarrow 1$$
so that \( G''_\sigma \cong B''_\sigma \times K''_\sigma \), and a second short exact sequence

\[
1 \longrightarrow \text{PSL}(2|2) \longrightarrow B''_\sigma \longrightarrow \left\{ \frac{h'_{2\varepsilon_i}(a)}{a \in A_0} \right\} \longrightarrow 1
\]

(2) If \( \sigma_h = 0 \) for all \( h \in \{1,2,3\} \), i.e. \( \sigma = 0 \), then there exists a first short exact sequence

\[
1 \longrightarrow K''_1 \times K''_2 \times K''_3 \longrightarrow G''_{\sigma=0} \longrightarrow B'' \longrightarrow 1
\]

where

\[
B'' := \left\{ \frac{h'_{2\varepsilon_i}(a)}{a \in A_0, \eta \in A_1} \right\}
\]

and a second short exact sequence

\[
1 \longrightarrow \left\{ \frac{x_{\beta}(\eta)}{\eta \in A_1} \right\} \longrightarrow B'' \longrightarrow \left\{ \frac{h'_{2\varepsilon_i}(a)}{a \in A_0} \right\} \longrightarrow 1
\]

where \( \left\{ \frac{x_{\beta}(\eta)}{\eta \in A_1} \right\} \) is isomorphic to \( \mathbb{A}^0 \) — the (totally odd) complex affine supergroup of superdimension \((0|8)\) — and \( \left\{ \frac{h'_{2\varepsilon_i}(a)}{a \in A_0} \right\} \) is isomorphic to \( \mathbb{T}^3 \) — the (totally even) \(3\)-dimensional complex torus.

**Proof.** Like for Theorem \[5.1.1\] one can deduce the claim from the presentation of \( G''_\sigma \) in \[5.3.1\] or also from the relation \( \text{Lie}(G''_\sigma) = \mathfrak{g}''(\sigma) \) along with Theorem \[4.3.1\].

### 5.4. Lie supergroups from contractions: the family of the \( \hat{G}_\sigma \)'s.

Given \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \in V \), following \[4.4\] we fix the element \( \tau := x_1 x_2 x_3 \in \mathbb{C}[x] \) and the ideal \( I = I_\sigma := \left\{ x_i - \sigma_i \right\}_{i=1,2,3} \), and we consider the corresponding complex Lie algebra \( \hat{\mathfrak{g}}(\sigma) \), with \( \hat{\mathfrak{g}}(\sigma)_0 \) and \( \hat{\mathfrak{g}}(\sigma)_1 \) as its even and odd part, respectively. With a slight abuse of notation, for any element \( Z \in \hat{\mathfrak{g}}(x)_c \) we denote again by \( Z \) its corresponding coset in \( \hat{\mathfrak{g}}(\sigma)_c = \mathbb{C}[\sigma] \otimes_{\mathbb{C}[x]} \hat{\mathfrak{g}}(x)_c \cong \hat{\mathfrak{g}}(x)_c / I_\sigma \hat{\mathfrak{g}}(x)_c \) (see \[4.4\] for notation). By construction, \( \hat{\mathfrak{g}}(\sigma)_c \) admits as \( \mathbb{C} \)-basis the set

\[
\hat{B} := \left\{ X_\alpha, H_{2\varepsilon_i} | \alpha \in \Delta, i \in \{1,2,3\} \right\} \cup \left\{ \hat{X}_\beta := \tau X_\beta | \beta \in \Delta_1 \right\}
\]

We consider also \( \hat{a}_i := a_i \) (\( := \mathbb{C} X_{2\varepsilon_i} \oplus \mathbb{C} H_{2\varepsilon_i} \oplus \mathbb{C} X_{-2\varepsilon_i} \)) for all \( 1 = 1, 2, 3 \), that all are Lie subalgebras of \( \hat{\mathfrak{g}}_0 \), with \( \hat{a}_i \cong \mathfrak{sl}_2 \) when \( \sigma_i \neq 0 \) and \( \hat{a}_i \cong \mathbb{C}^{2\times 3} \) — the \(3\)-dimensional Abelian Lie algebra — if \( \sigma_i = 0 \) (see also \[5.1\]).

Recalling the construction of \( G_\sigma \) in \[5.1\] for each \( i \in \{1,2,3\} \) we set \( \hat{A}_i := A_i \) (isomorphic to either \( SL_2 \) or \( \mathbb{C} \times \mathbb{C}^* \times \mathbb{C} \) depending on \( \sigma_i \neq 0 \) or \( \sigma_i = 0 \)) and \( \hat{G} := \mathbb{C}^3 \otimes_{\mathbb{C}} \hat{A}_i = G \), a complex Lie group such that \( \text{Lie}(\hat{G}) = \left( \hat{\mathfrak{g}}(\sigma)_c \right)_0 \). Just like in \[5.1\] the adjoint action of \( \hat{\mathfrak{g}}(\sigma)_c \) onto \( \hat{\mathfrak{g}}(\sigma)_c \) integrates to a Lie group action of \( \hat{G} \) onto \( \hat{\mathfrak{g}}(\sigma)_c \); endowed with this action, the pair \( \hat{P}_\sigma := (\hat{G}, \hat{\mathfrak{g}}(\sigma)_c) \) is a super Harish-Chandra pair (cf. \[2.4.1\]), by construction. Eventually, we can define

\[
\hat{G}_\sigma := G_{\hat{P}_\sigma}
\]

to be the complex Lie supergroup associated with \( \hat{P}_\sigma \) following \[2.4.3\].
5.4.1. A presentation of $\hat{G}_\sigma$. To describe the supergroups $\hat{G}_\sigma$, we provide hereafter an explicit presentation by generators and relations of all the abstract groups $\hat{G}_\sigma(A)$, with $A \in (\text{Wsalg})_C$. To begin with, let $\exp : \hat{g}_0 \cong \text{Lie}(\hat{G}) \to \hat{G}$ be the exponential map. Like we did in [5.1.1] for the supergroup $G_h$, inside each subgroup $A$, we consider

$$x_{2z_i}(c) := \exp (c X_{2z_i}), \quad h_{2z_i}(c) := \exp (c H_{2z_i}), \quad x_{-2z_i}(c) := \exp (c X_{-2z_i})$$

for every $c \in C$; then $\hat{I}_i := \{ x_{2z_i}(c), h_{2z_i}(c), x_{-2z_i}(c) \}_{c \in C}$ is a generating set for $\hat{A}_i$; also, we consider elements $h_\theta(c) := \exp (c H_\theta)$ for all $c \in C$. It follows that the complex Lie group $\hat{G} = \hat{A}_1 \times \hat{A}_2 \times \hat{A}_3$ is generated by

$$\hat{I}_0 := \{ x_{2z_i}(c), h_{2z_i}(c), h_\theta(c), x_{-2z_i}(c) \}_{c \in C}^{i \in \{1,2,3\}}$$

(we could drop the $h_\theta(c)$'s, but we prefer to keep them among the generators).

When we think of $\hat{G}$ as a (totally even) supergroup, looking at it as a group-valued functor $\hat{G} : (\text{Wsalg})_C \to (\text{grps})$, the abstract group $G(A)$ of its $A$–points — for $A \in (\text{Wsalg})_C$ — is generated by the set

$$\hat{I}_0(A) := \{ x_{2z_i}(a), h_{2z_i}(a), h_\theta(a), x_{-2z_i}(a) \}_{a \in A_0}^{i \in \{1,2,3\}} \quad (5.4)$$

In fact, we would better stress that, by construction (cf. [5.1]), we have an obvious isomorphism $\hat{G} \cong G$ (see [5.1.1] for the definition of $G$) as complex Lie groups.

To generate the group $\hat{G}_\sigma(A) := G_{\hat{g}_\sigma}(A)$ applying the recipe in [2.4.3] we fix in $(\hat{g}(\sigma))_C$ the $C$–basis $\{ Y_i \}_{i \in I}$ as $\{ \hat{X}_\beta : \tau X_\beta \big| \beta \in \Delta_1 = \{ \pm \theta, \pm \beta_1, \pm \beta_2, \pm \beta_3 \} \}$. Thus, besides the generating elements from $G(A)$, we take as generators also those of the set

$$\hat{I}_1(A) := \{ \hat{x}_{\pm \theta}(\eta) := (1 + \eta \hat{X}_{\mp \theta}), \hat{x}_{\pm \beta_i}(\eta) := (1 + \eta \hat{X}_{\mp \beta_i}) \}_{\eta \in A_1}^{i \in \{1,2,3\}}$$

Taking into account that $\hat{G}(A)$ is generated by $\hat{I}_0(A)$, we can modify the set of relations in [2.4.3] by letting $g_+ \in \hat{G}(A)$ range inside the set $\hat{I}_0(A)$: then we can find the following full set of relations (freely using notation $e^Z := \exp(Z)$):

$$1_{\hat{g}} = 1, \quad g' \cdot g'' = g' \hat{g} \hat{g}'' \quad (\forall g', g'' \in \hat{G}(A))$$

$$h_{2z_i}(a) \hat{x}_{\pm \beta_i}(\eta) h_{2z_i}(a)^{-1} = \hat{x}_{\pm \beta_i}(e^{\pm(-1/2) \sigma_i a \eta})$$

$$h_{2z_i}(a) \hat{x}_{\pm \theta}(\eta) h_{2z_i}(a)^{-1} = \hat{x}_{\pm \theta}(e^{\pm \sigma_i a \eta})$$

$$h_\theta(a) \hat{x}_{\pm \beta_i}(\eta) h_\theta(a)^{-1} = \hat{x}_{\pm \beta_i}(e^{\pm \sigma_1 \eta})$$

$$h_\theta(a) \hat{x}_{\pm \theta}(\eta) h_\theta(a)^{-1} = \hat{x}_{\pm \theta}(e^{\pm \theta \eta})$$

$$x_{2z_i}(a) \hat{x}_{\beta_i}(\eta) x_{2z_i}(a)^{-1} = \hat{x}_{\beta_i}(\eta) \hat{x}_{\beta_i}(e^{\pm \eta \beta_i \eta})$$

$$x_{2z_i}(a) \hat{x}_{\beta_i}(\eta) x_{2z_i}(a)^{-1} = \hat{x}_{\beta_i}(\eta) \hat{x}_{\beta_i}(e^{\pm \eta \beta_i \eta})$$

$$x_{-2z_i}(a) \hat{x}_{-\beta_i}(\eta) x_{-2z_i}(a)^{-1} = \hat{x}_{-\beta_i}(\eta) \hat{x}_{-\beta_i}(e^{\pm \eta \beta_i \eta})$$

$$x_{-2z_i}(a) \hat{x}_{-\beta_i}(\eta) x_{-2z_i}(a)^{-1} = \hat{x}_{-\beta_i}(\eta) \hat{x}_{-\beta_i}(e^{\pm \eta \beta_i \eta})$$

$$x_{2z_i}(a) \hat{x}_{\theta}(\eta) x_{2z_i}(a)^{-1} = \hat{x}_{\theta}(\eta) \hat{x}_{\theta}(e^{\pm \eta \theta \eta})$$

$$x_{2z_i}(a) \hat{x}_{\theta}(\eta) x_{2z_i}(a)^{-1} = \hat{x}_{\theta}(\eta) \hat{x}_{\theta}(e^{\pm \eta \theta \eta})$$

$$x_{-2z_i}(a) \hat{x}_{-\theta}(\eta) x_{-2z_i}(a)^{-1} = \hat{x}_{-\theta}(\eta) \hat{x}_{-\theta}(e^{\pm \eta \theta \eta})$$

$$x_{-2z_i}(a) \hat{x}_{-\theta}(\eta) x_{-2z_i}(a)^{-1} = \hat{x}_{-\theta}(\eta) \hat{x}_{-\theta}(e^{\pm \eta \theta \eta})$$
5.4.2. Singular specializations of the supergroup(s) $\hat{G}_\sigma$. The very construction of the supergroups $\hat{G}_\sigma$ implies that

$$\hat{G}_\sigma$$ is simple (as a Lie supergroup)

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$ such that $\sigma_i \neq 0$ for all $i \in \{1, 2, 3\}$.

This also follows from the presentation of $\hat{G}_\sigma$ in §5.4.1 above, or as a direct consequence of the relation $\text{Lie}(\hat{G}_\sigma) = \hat{g}(\sigma) = \hat{g}_\sigma$ and of the fact that $\hat{g}_\sigma \cong g_\sigma$ when $\sigma_i \neq 0$ for all $i$.

On the other hand, things change instead at “singular values” of $\sigma$, as we now show:

**Theorem 5.4.1.** Let $\sigma \in V := \mathbb{C}^3 \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$ as before. Assume $\sigma_1 \sigma_2 \sigma_3 = 0$.

1. We have $\hat{G}_\sigma \cong (\hat{G}_\sigma)_r \ltimes (\hat{G}_\sigma)_t$ where $(\hat{G}_\sigma)_r \cong \hat{A}_1 \times \hat{A}_2 \times \hat{A}_3$ and $(\hat{G}_\sigma)_t$ is the supersubgroup of $\hat{G}_\sigma$ generated by the $\hat{x}_{i\theta}$’s and the $\hat{x}_{\pm\beta}$’s (for all $i$).
2. If $\sigma_i = 0$ then $\hat{A}_i \subseteq \hat{G}_\sigma$ is a central Lie subgroup in $\hat{G}_\sigma$ with $\hat{A}_i \cong \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}$, otherwise $\hat{A}_i \cong \text{SL}_2$.
3. For $i \in \{1, 2, 3\}$, let $V_i$ be the 2-dimensional representation of $\hat{A}_i$ defined as follows:
   i) if $\sigma_i = 0$, then $V_i := \mathbb{C} \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation;
   ii) if $\sigma_i \neq 0$, then $V_i := \mathbb{C}$.
   Then $(\hat{G}_\sigma)_t \cong V_1 \otimes V_2 \otimes V_3$ as a $(\hat{G}_\sigma)_r$-module (in particular, it is Abelian).
4. For $\sigma = 0$ we have $(\hat{G}_\sigma)_r \cong (\mathbb{C} \times \mathbb{C}^* \times \mathbb{C})^3 \times (\mathbb{C} \oplus \mathbb{C})^{23}$, hence $(\hat{G}_\sigma)_r$ is Abelian.

**Proof.** As for the parallel results for $G_\sigma$, $G'_\sigma$ and $G''_\sigma$, we can deduce the claim from the presentation of $\hat{G}_\sigma$ in §5.4.1 or from the link $\text{Lie}(\hat{G}_\sigma) = \hat{g}(\sigma)$ along with Theorem 4.4.1. □

5.5. Lie supergroups from contractions: the family of the $\hat{G}'_\sigma$’s.

Given $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$, we follow again §4.4 and set $\tau := x_1 x_2 x_3 \in \mathbb{C}[x]$ and $I = I_\sigma := \{x_i - \sigma_i\}_{i=1,2,3}$; but now we consider the corresponding complex Lie algebra $\hat{g}'(\sigma)_c$, with $\hat{g}'(\sigma)_c^e$ and $\hat{g}'(\sigma)_c^o$ as its even and odd part, respectively (and we still make use of some abuse of notation as in §5.4). By construction, a $\mathbb{C}$-basis of $\hat{g}'(\sigma)_c$ is

$$\hat{B}' := \{X'_\alpha, H'_2, | \alpha \in \Delta, i \in \{1, 2, 3\}\} \cup \{\hat{X}'_\beta \mid \beta \in \Delta_1\}$$
Consider also \( \hat{a}_1' := a_1' := \mathbb{C}X_{2e_1} \oplus \mathbb{C}H_{2e_1} \oplus \mathbb{C}X_{-2e_1} \), cf. Remark 4.2.1 for all \( 1 = 1, 2, 3 \): all these are Lie subalgebras in \( \mathfrak{g}(\sigma)_c \), isomorphic to \( \mathfrak{s}t_2 \), and \( \left( \mathfrak{g}(\sigma)_c \right)_0 = \oplus_{\beta = 1}^{3} \hat{a}_1' \).

The faithful adjoint action of \( \left( \mathfrak{g}(\sigma)_c \right)_0 \) onto \( \mathfrak{g}(\sigma)_c \) gives a Lie algebra monomorphism \( \left( \mathfrak{g}(\sigma)_c \right)_0 \rightarrow \mathfrak{gl}(\mathfrak{g}(\sigma)_c) \), by which we identify \( \left( \mathfrak{g}(\sigma)_c \right)_0 \) with its image in \( \mathfrak{gl}(\mathfrak{g}(\sigma)_c) \). Then \( \exp : \mathfrak{gl}(\mathfrak{g}(\sigma)_c) \rightarrow \text{GL}(\mathfrak{g}(\sigma)_c) \) yields a Lie subgroup \( \hat{G} := \exp \left( \left( \mathfrak{g}(\sigma)_c \right)_0 \right) \) in \( \text{GL}(\mathfrak{g}(\sigma)_c) \) which faithfully acts onto \( \mathfrak{g}(\sigma)_c \) and is such that \( \text{Lie}(\hat{G}) = \left( \mathfrak{g}(\sigma)_c \right)_0 \). The pair \( \hat{P}_\sigma := (\hat{G}, \hat{g}(\sigma)_c) \) with this action then is a super Harish-Chandra pair (cf. 2.4.1).

As alternative method, we might also construct the super Harish-Chandra pair \( \hat{P}_\sigma \) via the same procedure, up to the obvious, minimal changes, adopted for \( \hat{P}_\sigma \) in §5.2; indeed, one can also do the converse, namely use the present method to construct \( \hat{P}_\sigma \) as well.

Once we have the super Harish-Chandra pair \( \hat{P}_\sigma \), it makes sense to define

\[
\hat{G}_\sigma := \text{G}_{\hat{P}_\sigma}
\]

that is the complex Lie supergroup associated with \( \hat{P}_\sigma \) after the recipe in §2.4.3.

5.5.1. A presentation of \( \hat{G}_\sigma \). We shall presently describe the supergroups \( \hat{G}_\sigma \) by means of an explicit presentation by generators and relations of all the abstract groups \( \text{G}_{\hat{P}_\sigma}(A) \), for all \( A \in (\text{Wsalg})_C \). To begin with, let \( \exp : \hat{G}_0 \cong \text{Lie}(\hat{G}) \rightarrow \hat{G} \) be the exponential map.

Just like for the supergroup \( \text{G}_\sigma \) in §5.1.1 inside each subgroup \( A_i \) we consider

\[
x'_{2e_1}(c) := \exp (c X'_{2e_1}), \quad h'_{2e_1}(c) := \exp (c H'_{2e_1}), \quad x'_{-2e_1}(c) := \exp (c X'_{-2e_1})
\]

for every \( c \in \mathbb{C} \); then \( \hat{I}_i := \left\{ x'_{2e_1}(c), h'_{2e_1}(c), x'_{-2e_1}(c) \right\}_{c \in \mathbb{C}} \) is a generating set for \( \hat{A}_i = A_i \); also, we consider elements \( h'_0(c) := \exp (c H'_0) \) for all \( c \in \mathbb{C} \). It follows that the complex Lie group \( \hat{G} = \hat{A}_1 \times \hat{A}_2 \times \hat{A}_3 \) is generated by

\[
\hat{I}_0 := \left\{ x'_{2e_1}(c), h'_{2e_1}(c), h'_0(c), x'_{-2e_1}(c) \right\}_{c \in \mathbb{C}}
\]

(as before, we could drop the \( h'_0(c) \)'s, but we prefer to keep them among the generators).

When thinking of \( \hat{G} \) as a (totally even) supergroup, considered as a group-valued functor \( \hat{G} : (\text{Wsalg})_C \rightarrow (\text{grps})_C \), the abstract group \( \hat{G}(A) \) of its \( A \)-points — for \( A \in (\text{Wsalg})_C \) — is generated by the set

\[
\hat{I}_0(A) := \left\{ x'_{2e_1}(a), h'_{2e_1}(a), h'_0(a), x'_{-2e_1}(a) \right\}_{a \in A_0}
\]

(5.5)

Indeed, we can also stress that, by construction (cf. §5.1), there exists an obvious isomorphism \( \hat{G} \cong \hat{G}' \) as complex Lie groups.

Now, to generate the group \( \hat{G}'_\sigma(A) := \text{G}_{\hat{P}_\sigma}(A) \) following the recipe in §2.4.3 we fix in \( \left( \mathfrak{g}(\sigma)_c \right)_1 \) the \( \mathbb{C} \)-basis \( \left\{ Y_i \right\}_{i \in I} = \left\{ \hat{X}_\beta := \tau X'_\beta \big| \beta \in \Delta_1 = \left\{ \pm \theta, \pm \beta_1, \pm \beta_2, \pm \beta_3 \right\} \right\} \). Then, beside the generating elements from \( \hat{G}'(A) \) we take as generators also those of the set

\[
\hat{I}_1(A) := \left\{ \hat{x}'_{\pm \theta}(\eta) := (1 + \eta \hat{X}'_{\pm \theta}), \hat{x}'_{\pm \beta_i}(\eta) := (1 + \eta \hat{X}'_{\pm \beta_i}) \right\}_{\eta \in A_1}^{i \in \left\{ 1, 2, 3 \right\}}
\]
Knowing that $\hat{G}'(A)$ is generated by $\hat{T}'_0(A)$, we can modify the set of relations in 2.4.3 by letting $g \in \hat{G}'(A)$ range inside $\hat{T}'_0(A)$; eventually, we can find the following full set of relations (freely using notation $e^Z := \exp(Z)$):

\[
1_{G'} = 1 \quad , \quad g' \cdot g'' = g' \hat{\circ} g'' \quad (\forall \; g', g'' \in \hat{G}'(A))
\]

\[
h'_{2e_i}(a) \bar{x}'_{\pm\beta_i}(\eta) \bar{h}'_{2e_i}(a)^{-1} = \bar{x}'_{\pm\beta_i}(e^{\pm a} \eta)
\]

\[
h'_{2e_i}(a) \bar{x}_{\pm\delta}(\eta) \bar{h}'_{2e_i}(a)^{-1} = \bar{x}_{\pm\delta}(e^{\pm a} \eta)
\]

\[
h'_{2e_i}(a) \bar{x}'_{\pm\beta_i}(\eta) h'_{2e_i}(a)^{-1} = \bar{x}'_{\pm\beta_i}(e^{\pm a} \eta), \quad h'_{2e_i}(a) \bar{x}'_{\pm\theta}(\eta) h'_0(a)^{-1} = \bar{x}'_{\pm\theta}(\eta)
\]

\[
x'_{2e_i}(a) \bar{x}'_{\pm\beta_i}(\eta) x'_{2e_i}(a)^{-1} = \bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\theta}(\delta_i a \eta)
\]

\[
x'_{2e_i}(a) \bar{x}'_{\pm\theta}(\eta) x'_{2e_i}(a)^{-1} = \bar{x}'_{\pm\theta}(\eta) \bar{x}'_{\pm\delta}(\delta_i a \eta)
\]

\[
x'_{2e_i}(a) \bar{x}'_{\mp\theta}(\eta) x'_{2e_i}(a)^{-1} = \bar{x}'_{\mp\theta}(\eta) \bar{x}'_{\pm\delta}(\delta_i a \eta)
\]

\[
x'_{2e_i}(a) \bar{x}'_{\pm\delta}(\eta) x'_{2e_i}(a)^{-1} = \bar{x}'_{\pm\delta}(\eta) \bar{x}'_{\pm\theta}(\delta_i a \eta)
\]

\[
\bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\beta_i}(\eta') = x'_{2e_i}(1 - \delta_i a \eta) \bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\beta_i}(\eta')
\]

\[
\bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\mp\beta_i}(\eta') = x'_{2e_i}(-1 - \delta_i a \eta) \bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\mp\beta_i}(\eta')
\]

\[
\bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\beta_i}(\eta') = x'_{2e_i}(\delta_i a \eta) \bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\beta_i}(\eta')
\]

\[
\bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\theta}(\eta) = h'_0(\tau_{\sigma} \eta) \bar{x}'_{\pm\theta}(\eta) \bar{x}'_{\pm\beta_i}(\eta)
\]

\[
\bar{x}'_{\pm\beta_i}(\eta) \bar{x}'_{\pm\theta}(\eta') = \bar{x}'_{\pm\theta}(\eta') \bar{x}'_{\pm\beta_i}(\eta') = \bar{x}'_{\pm\beta_i}(\eta + \eta')
\]

with $\{i, j, k\} \in \{1, 2, 3\}$.

5.5.2. Singular specializations of the supergroup(s) $\hat{G}'_{\sigma}$. From the very construction of the supergroups $\hat{G}'_{\sigma}$ we get

$\hat{G}'_{\sigma}$ is simple (as a Lie supergroup)

for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in V$ such that $\sigma_i \neq 0$ for all $i \in \{1, 2, 3\}$.

This follows from the presentation of $\hat{G}'_{\sigma}$ in 5.5.1 above, but also as direct consequence of the relation $\text{Lie}(\hat{G}'_{\sigma}) = \hat{g}'(\sigma) = \hat{g}'_{\sigma}$ and of $\hat{g}'_{\sigma} \cong \hat{g}'$ when $\sigma_i \neq 0$ for all $i$.

At “singular values” of $\sigma$ instead things are quite different. The precise claim is as follows:

**Theorem 5.5.1.** Let $\sigma \in V := \mathbb{C}^3 \cap \{\sigma_1 + \sigma_2 + \sigma_3 = 0\}$ as before. Assume $\sigma_1 \sigma_2 \sigma_3 = 0$, and let $(\hat{G}'_{\sigma})_{1}$ be the supersubgroup of $\hat{G}'_{\sigma}$ generated by the $\bar{x}'_{\pm\theta}$’s and the $\bar{x}'_{\pm\beta_i}$’s (for all $i$).
(1) \((\hat{G}'_\sigma)_1\) is normal in \(G'_\sigma\), and there exist isomorphisms \((\hat{G}'_\sigma)_\text{rd} \cong \text{SL}_2^{\times 3}\) and \((\hat{G}'_\sigma)_1 \cong \mathbb{R}^3_{i=1} \square_i\) (as a \((\hat{G}'_\sigma)_\text{rd}-\text{module})\); in particular, \((\hat{G}'_\sigma)_1\) is Abelian.

(2) There exists an isomorphism \((G'_\sigma) \cong \hat{G}'_\sigma \times (\hat{G}'_\sigma)_1\).

Proof. Much like the similar result for \(G'_\sigma\), we can deduce the present claim from the presentation of \((G'_\sigma)_1\) in §5.5.1 or from the relation \(\text{Lie}(G'_\sigma) = \hat{g}'(\sigma)\) along with Theorem 44.42 □

5.6. The integral case: \(G'_\sigma, G''_\sigma, \hat{G}'_\sigma, \text{and } \hat{G}'_\sigma\) as algebraic supergroups. In the integral case, i.e. when \(\sigma \in \mathbb{Z}^3\), the Lie supergroups we have introduced above are, in fact, complex algebraic supergroups: indeed, this follows as a consequence of an alternative presentation of them, that makes sense if and only if \(\sigma \in \mathbb{Z}^3\).

Let us look at \(G'_\sigma\), for some fixed \(\sigma \in \mathbb{Z}^3\). Consider the generating set \(\{\bar{1}\}\) for the groups \(G_0(A)\), and for each \(\alpha \in \{2 \varepsilon_1, 2 \varepsilon_2, 2 \varepsilon_3, \theta\}\) replace the generators \(h_\alpha(a) := \exp(a H_\alpha)\) — for all \(a \in A_0\) — therein with \(\tilde{h}_\alpha(a)\) — for all \(u \in U(A_0)\), the group of units of \(A_0\). Every such \(\tilde{h}_\alpha(a)\) is the toral element in \(G(A)\) whose adjoint action on \(g_\sigma\) is given by \(A(d(\tilde{h}_\alpha(u)))(X_\gamma) = u^\gamma H_\alpha X_\gamma\) for all \(\gamma \in \Delta\); note that this makes sense, since we have \(\gamma(H_\alpha) \in \mathbb{Z}\) just because \(\sigma \in \mathbb{Z}^3\). Now, the set

\[\tilde{\Gamma}_0(A) := \left\{ x_{2\gamma_i}(a), \tilde{h}_{2\gamma_i}(u), \tilde{h}_\theta(u), x_{-2\gamma_i}(a) \bigg| i \in \{1, 2, 3\}, a \in A_0, u \in U(A_0) \right\}\]

still generates \(G(A)\). A moment’s thought shows that \(G_\sigma(A)\) can be realized as the group generated by \(\tilde{\Gamma}(A) := G(A) \cup \Gamma_1(A)\) with the same relations as in §5.5.1 up to the following changes: all relations that involve no generators of type \(h_\alpha(a)\) are kept the same, while the others are replaced by the following ones (with \(\{i, j, k\} \in \{1, 2, 3\}\):

\[\tilde{h}_{2\gamma_i}(u) x_{\pm \beta_j}(\eta) \tilde{h}_{2\gamma_i}(u)^{-1} = x_{\pm \beta_j}(u^{\pm(1) - \delta_{i,j} \sigma_i} \eta)\]

\[\tilde{h}_{2\gamma_i}(u) x_\theta(\eta) \tilde{h}_{2\gamma_i}(u)^{-1} = x_\theta(u^{\pm \sigma_i} \eta)\]

\[\tilde{h}_\theta(u) x_{\pm \beta_j}(\eta) \tilde{h}_\theta(u)^{-1} = x_{\pm \beta_j}(u^{\pm \sigma_i} \eta), \quad \tilde{h}_\theta(u) x_\theta(\eta) \tilde{h}_\theta(u)^{-1} = x_\theta(\eta)\]

\[x_\theta(\eta_+) x_{-\theta}(\eta_-) = \tilde{h}_\theta(\eta_- \eta_+) x_{-\theta}(\eta_-) x_\theta(\eta_+)\]

In fact, the key point here is that if (and only if) \(\sigma \in \mathbb{Z}^3\), then all our construction does make sense in the framework of algebraic supergeometry, namely \(P_\sigma := \left(G, \sigma(g_\sigma)\right)\) is a super Harish-Chandra pair in the algebraic sense — like in [Ga2] — and \(G_\sigma := G_\sigma\) is nothing but the corresponding algebraic supergroup associated with \(P_\sigma\) trough the algebraic version of category equivalence in [2.13] — cf. [Ga2] again. If we present the groups \(G(A)\) using \(\tilde{\Gamma}_0(A)\) as generating set, we can also extend such a description — as \(\sigma \in \mathbb{Z}^3\) — to a presentation of the groups \(G_\sigma(A)\) as above.

Leaving details to the reader, the same analysis applies when we look at \(G''_\sigma\), \(G''_\sigma\), \(\hat{G}'_\sigma\), or \(\hat{G}'_\sigma\) instead of \(G_\sigma\): whenever \(\sigma \in \mathbb{Z}^3\), all of them are in fact complex algebraic supergroups.

5.7. A geometrical interpretation. In the previous discussion we considered five families of Lie supergroups indexed by the points of \(V\), namely \(\{G_\sigma\}_{\sigma \in V}\), \(\{G'_\sigma\}_{\sigma \in V}\), \(\{G''_\sigma\}_{\sigma \in V}\), \(\{\hat{G}_\sigma\}_{\sigma \in V}\) and \(\{\hat{G}'_\sigma\}_{\sigma \in V}\). Our analysis shows that these families have in common all the elements indexed by “general” points, i.e. elements \(\sigma \in V \setminus S\), where \(S := \bigcup_{i=1}^3 \{\sigma_i = 0\}\). On the other hand, these families are entirely different at all points in the “singular locus” \(S\).
In geometrical terms, each family forms a fibre space, say $L_{G[x]}$, $L_{G[x]'}$, $L_{G[x]''}$, $L_{\hat{G}[x]}$, respectively, over the base space $\text{Spec}(\mathbb{C}[x]) \cong V \cup \{\star\}$ ( $\cong \mathbb{A}^2_{\mathbb{C}} \cup \{\star\}$ ), whose fibres are Lie supergroups. Our result show that the fibres in the two fibre spaces do coincide at general points — where they are simple Lie supergroups — and do differ instead at singular points — where they are non-simple indeed.

As an outcome, loosely speaking we can say that our construction provides five different “completions” of the family $\{G_\sigma\}_{\sigma \in V \setminus S}$ of simple Lie supergroups, by adding — in five different ways (yet many others more can be made up) — some new non-simple extra elements.

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