Potential-model calculation of an order-$v^2$
nonrelativistic QCD matrix element

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Abstract

We present two methods for computing dimensionally-regulated NRQCD heavy-quarkonium
matrix elements that are related to the second derivative of the heavy-quarkonium wave function
at the origin. The first method makes use of a hard-cutoff regulator as an intermediate step
and requires knowledge only of the heavy-quarkonium wave function. It involves a significant
cancellation that is an obstacle to achieving high numerical accuracy. The second method is more
direct and yields a result that is identical to the Gremm-Kapustin relation, but it is limited to use
in potential models. It can be generalized to the computation of matrix elements of higher order in
the heavy-quark velocity and can be used to resum the contributions to decay and production rates
that are associated with those matrix elements. We apply these methods to the Cornell potential
model and compute a matrix element for the $J/\psi$ state that appears in the leading relativistic
correction to the production and decay of that state through the color-singlet quark-antiquark
channel.

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I. INTRODUCTION

In the effective field theory nonrelativistic quantum chromodynamics (NRQCD), the leading relativistic corrections to $S$-wave heavy-quarkonium decay and production processes in the color-singlet quark-antiquark channel are proportional to matrix elements that are related to the second derivative of the quarkonium wave function at the origin. These matrix elements are inherently nonperturbative in nature. Their importance in phenomenological calculations has led to a number of attempts to determine their values.

Even before the introduction of the NRQCD approach for quarkonium decay and production [1], these matrix elements appeared in phenomenological studies of charmonium decays [2]. Owing to uncertainties that arise from the uncertainty in the charm-quark mass $m_c$ and from uncalculated terms of higher order in the quantum chromodynamic (QCD) strong coupling $\alpha_s$, such phenomenological determinations have not led to accurate values for the matrix elements. There also have been attempts to determine the matrix elements in lattice calculations [3]. In this case, large uncertainties arise because there is a substantial cancellation that occurs when one converts from lattice to continuum dimensionally-regulated matrix elements. In principle, one can determine these matrix elements approximately by making use of the Gremm-Kapustin relation [4], which expresses the matrix elements in terms of the quarkonium and heavy-quark masses. (See Ref. [5] for an example of this approach.) Unfortunately, this method is plagued by large uncertainties in $m_c$. In both the lattice and Gremm-Kapustin approaches, the uncertainties are so large that even the signs of the matrix elements are in some doubt.

A further difficulty that complicates the calculation of the matrix elements that are related to the second derivative of the wave function at the origin is that they contain a linear ultraviolet (UV) divergence, and, hence, must be regulated. Dimensional regularization of these matrix elements is particularly useful because it is consistent with existing calculations of quarkonium decay and production rates at relative order $\alpha_s$ and $\alpha_s^2$.

In this paper, we present two methods for calculating the second derivative of the wave function at the origin. In the first method, we initially use a hard-cut off regulator, with cutoff $\Lambda$, to define the relevant matrix element. Then we compute the difference between the hard-cut off regularization and dimensional regularization in perturbation theory. We subtract this difference from the hard-cut off matrix element. There remains a dependence
of the matrix element on $\Lambda$ that falls as $1/\Lambda$ in the limit $\Lambda \to \infty$. That dependence can be removed by calculating at a number of values of $\Lambda$ and extrapolating to $\Lambda = \infty$. However, in this method, it is difficult to achieve high numerical accuracy because large, cancelling $\Lambda$-dependent contributions appear. The second method that we present bypasses the hard-cutoff step, but it is applicable only to potential models for the wave function. It yields a result that is identical to the Gremm-Kapustin relation [4] for the potential model. That result can be used to resum certain contributions of higher order in $v$ to amplitudes that are computed in NRQCD.

Having established a formal procedure for computing the relevant matrix elements, we carry out an explicit computation for the $J/\psi$ (or $\eta_c$) states in the Cornell potential model. We do not distinguish between the $J/\psi$ and $\eta_c$ wave functions, which differ only in corrections of relative order $v^2$, where $v$ is the velocity of the heavy quark or antiquark in the quarkonium. ($v^2 \approx 0.3$ in charmonium and $v^2 \approx 0.1$ in bottomonium.) In principle, if we know the static heavy-quark-antiquark ($Q\bar{Q}$) potential exactly, then we can calculate the quarkonium wave function of the leading $Q\bar{Q}$ quarkonium Fock state up to corrections of relative order $v^2$. Existing lattice data for the static $Q\bar{Q}$ potential yield values for the string tension. We examine values for the parameters in the Cornell potential that bracket the lattice values for the string tension.

In our numerical calculations, the results from our two approaches agree well and give a value for the second derivative of the wave function at the origin that is in agreement with expectations from the $v$-scaling rules of NRQCD [1]. The largest uncertainties in our calculation are of relative order $v^2$. Therefore, our determination of the second derivative of the wave function at the origin is the most accurate to date and should be useful for phenomenological studies of quarkonium production and decay.

The remainder of this paper is organized as follows: In Sec. II, we give a brief description of the Cornell potential model. In Sec. III, we discuss the NRQCD matrix elements that are relevant to this work. Section IV contains a description of the hard-cutoff regulator. In Sec. V, we explain how we compute the difference between hard-cutoff regularization and dimensional regularization. The direct method of calculation, which bypasses the hard-cutoff step, is discussed in Sec. VII, along with its application to resummation of contributions of higher order in $v$. In Sec. VI, we decompose the difference between a hard-cutoff matrix element and a dimensionally-regulated matrix element into sums of short-distance coefficients.
times dimensionally-regulated matrix elements. Such a decomposition is used in existing calculations of the difference between a lattice matrix element and a dimensionally-regulated matrix element. Sec. VIII contains our numerical results and a discussion of them. We summarize our results in Sec. IX. In Appendix A, we illustrate our methods for the case of a pure Coulomb potential, and, in Appendix B, we list some integrals that are useful in our analyses.

II. POTENTIAL MODEL

We compute matrix elements for the $J/\psi$ or $\eta_c$ states using a potential model. In this model, we neglect the effects of the heavy-quark spin, which are suppressed as $v^2$. Therefore, we do not distinguish between the $J/\psi$ and $\eta_c$ wave functions or matrix elements. We note that, if we knew the heavy-quark potential exactly, then we could calculate the heavy-quarkonium wave function in a potential model up to corrections of relative order $v^2$ (Ref. [6]). We make use of the Cornell potential model of Ref. [7]. For appropriate choices of parameters, the Cornell potential provides a reasonably good fit to heavy-quark potentials that are measured in lattice calculations.\footnote{For a recent review that discusses heavy-quark potentials from lattice measurements, see Ref. [8].}

Now we summarize the methods that we use to constrain the parameters of the Cornell potential and to solve the Schrödinger equation. We refer the reader to Ref. [9] for further details.

The Cornell potential [7] is given by

$$V(r) = -\frac{\kappa}{r} + \frac{r}{a^2},$$

(1)

where the parameters $\kappa$ and $a$ determine the strength of Coulomb and linear potentials, respectively. For a color-singlet $QQ$ pair, the Coulomb-strength parameter $\kappa$ can be expressed in terms of an effective strong coupling $\alpha_s$ as

$$\kappa = \alpha_s C_F,$$

(2)

where $C_F = 4/3$. The parameter $a$ is related to the string tension $\sigma$ as

$$\sigma = \frac{1}{a^2},$$

(3)
Following Ref. [7], we replace \( \kappa \) and the binding energy \( \epsilon_B \) by dimensionless parameters \( \lambda \) and \( \zeta \):

\[
\kappa = (ma)^{-\frac{2}{3}} \lambda, \quad (4a)
\]

\[
\epsilon_B = m(ma)^{-\frac{4}{3}} \zeta. \quad (4b)
\]

For a given value of \( \lambda \), we fix the heavy-quark mass \( m \) and the parameter \( a \) in the Cornell potential as follows. First, we require that the energies that result from the solutions to the Schrödinger equation match the measured values of the difference of the \( J/\psi \) and \( \psi(2S) \) masses. We use \( M_{J/\psi} = 3.096916 \) GeV and \( M_{\psi(2S)} = 3.686093 \) GeV. Second, we require that the wave function at the origin matches a value that is derived from the measured value of the leptonic width of the \( J/\psi \) and the perturbative formula

\[
\Gamma[J/\psi \to \ell^+ \ell^-] = \frac{4\pi e_c^2 \alpha^2}{m_c^2} |\psi(0)|^2 \left(1 - \frac{8}{3} \frac{\alpha_s}{\pi} \right)^2. \quad (5)
\]

Here, \( \psi(0) \) is the wave function at the origin and \( e_c = 2/3 \) is the fractional electric charge of the charm quark. In Ref. [5], \( \psi(0) \) is estimated by using the the formula (5) at both leading order in \( \alpha_s \) (LO) and next-to-leading order in \( \alpha_s \) (NLO). The results are

\[
\psi(0) = \begin{cases} 
0.18619 \text{ GeV}^{3/2} & \text{(LO)} \\
0.23629 \text{ GeV}^{3/2} & \text{(NLO)},
\end{cases} \quad (6)
\]

where, for convenience, we have taken \( \psi(0) \) to be positive and real. In order to estimate the effects of the uncertainty in \( \psi(0) \), we carry out our calculations for both the LO and NLO values of \( \psi(0) \). For a more detailed discussion of the determination of \( m \) and \( a \), see Ref. [9].

Values of \( m, a \), and the scaled energies \( \zeta \) of the \( 1S \) and \( 2S \) states for various values of \( \lambda \) are shown in Tables I and II, along with values for \( \alpha_s \) from Eq. (2), \( \sigma \) from Eq. (3), and \( \gamma_C \) from Eq. (12). Table I contains values of the potential-model parameters that correspond to the LO value of \( \psi(0) \), while Table II contains those that correspond to the NLO value of \( \psi(0) \).

Lattice measurements of the heavy-quark potential yield values for effective coupling \( \alpha_s \) of 0.22 in the quenched case and approximately 0.26 in the unquenched case [8]. A lattice measurement of the string tension \( K = \sigma \) (Ref. [10]) gives \( K a_L^2 = 0.0114(2) \) at a lattice coupling \( \beta = 6.5 \), where \( a_L \) is the lattice spacing. Lattice calculations of the hadron spectrum at \( \beta = 6.5 \) yield values for \( 1/a_L \) of 3.962(127) GeV (Refs. [11, 12]) and
TABLE I: Potential-model parameters and derived quantities as a function of the strength $\lambda$ of the Coulomb potential. The definitions of the parameters and derived quantities are given in the text. The parameters are computed using the inputs $M_{J/\psi} = 3.096916$ GeV, $M_{\psi(2S)} = 3.686093$ GeV, and the LO value $\psi(0) = 0.18619$ GeV$^{3/2}$, as is described in the text.

| $\lambda$ | 0   | 0.2 | 0.4 | 0.6 | 0.7 | 0.8  | 1.0 | 1.2 | 1.4 |
|------------|-----|-----|-----|-----|-----|------|-----|-----|-----|
| $\zeta_{10}$ | 2.33811 | 2.16732 | 1.98850 | 1.80107 | 1.60441 | 1.39788 | 1.18084 | 0.95264 |
| $\zeta_{20}$ | 4.07890 | 3.97017 | 3.85003 | 3.72747 | 3.66528 | 3.60249 | 3.47510 | 3.34529 | 3.21307 |
| $m$ (GeV) | 1.70670 | 1.51548 | 1.35120 | 1.21003 | 1.14710 | 1.10877 | 0.98458 | 0.89501 | 0.81796 |
| $a$ (GeV$^{-1}$) | 1.97932 | 2.08520 | 2.19805 | 2.31833 | 2.38139 | 2.44648 | 2.58295 | 2.72816 | 2.88253 |
| $\sqrt{\sigma}$ (GeV) | 0.50522 | 0.47957 | 0.45495 | 0.43134 | 0.41992 | 0.40875 | 0.38715 | 0.36655 | 0.34692 |
| $\sigma$ (GeV$^2$) | 0.25525 | 0.22999 | 0.20698 | 0.18606 | 0.17633 | 0.16708 | 0.14989 | 0.13436 | 0.12035 |
| $\alpha_s$ | 0.06966 | 0.14519 | 0.22624 | 0.26866 | 0.31225 | 0.40255 | 0.49634 | 0.59272 |
| $\gamma_C$ (GeV) | 0.07037 | 0.13079 | 0.18250 | 0.21570 | 0.26423 | 0.29615 | 0.32322 |

3.811(59) GeV (Refs. [11, 13]). These yield values of the string tension of $K = 0.1790 \pm 0.0119$ and $K = 0.1656 \pm 0.0059$, respectively.

Comparing the results of these lattice measurements with the LO parameters in Table I, we see that the values of the string tension at $\lambda = 0.7$ and 0.8 span the range of lattice results.
for the string tension, while the values of \( \alpha_s \) at \( \lambda = 0.6 \) and 0.7 span the range of lattice results for \( \alpha_s \). Comparing the results of the lattice measurements with the NLO parameters in Table II, we see that the values of the string tension at \( \lambda = 1.0 \) and 1.1 span the range of lattice results for the string tension. However, the values of \( \alpha_s \) at \( \lambda = 0.9, 1.0, \) and 1.1 are all larger than the lattice values. It is not clear whether this discrepancy between the lattice and NLO potential-model values for \( \alpha_s \) arises from the use of an inaccurate value for \( \psi(0) \), from effects due to the running of \( \alpha_s \), which are not taken into account in the fits to the lattice data, or from the absence of corrections of relative order \( \alpha_s^2 \) in the Cornell potential model. However, we note that the NLO values for \( \alpha_s \) at \( \lambda = 0.9, 1.0, \) and 1.1 do not differ greatly from the value of the running \( \alpha_s \) at the scale of the heavy-quark momentum \( m_c \).

Finally, we mention that we obtain the \( J/\psi \) wave function by expressing the radial part of the Schrödinger equation as a difference equation, which we integrate numerically. See Ref. [9] for details.

### III. NRQCD MATRIX ELEMENTS

In this section we describe the NRQCD matrix elements that are relevant to our calculation.

In the rest frame of an \( S \)-wave heavy quarkonium \( H \) in a spin-singlet (\( ^1S_0 \)) or spin-triplet (\( ^3S_1 \)) state, one can express the wave function at the origin of the leading \( Q\bar{Q} \) Fock state in terms of the following color-singlet NRQCD matrix elements [1]:

\[
\psi(0) = \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}(p) = \frac{1}{\sqrt{2N_c}} \langle 0 | \chi^\dagger \psi | H(^1S_0) \rangle, \quad (7a)
\]

\[
\epsilon \psi(0) = \epsilon \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}(p) = \frac{1}{\sqrt{2N_c}} \langle 0 | \chi^\dagger \sigma \psi | H(^3S_1) \rangle. \quad (7b)
\]

Here \( \psi \) and \( \chi^\dagger \) are Pauli spinor fields that annihilate a quark and an antiquark, respectively. The bilinear operators involving \( \psi^\dagger \) and \( \chi \) are evaluated at zero space-time position. \( \sigma \) is a Pauli matrix, and \( \epsilon \) is the quarkonium polarization vector. \( \tilde{\psi}(p) \) is the momentum-space wave function for the leading \( Q(p)\bar{Q}(-p) \) Fock state in the quarkonium. The wave function is, of course, gauge dependent. Throughout this paper, we work in the Coulomb gauge. The normalization factor \( 1/\sqrt{2N_c} \) accounts for the traces in the SU(2)-spin and SU(3)-color spaces. In equating the wave functions for the spin-singlet and spin-triplet cases, we are ignoring effects of relative order \( \alpha_s^2 \).
Relativistic corrections to the production and decay rates for a heavy quarkonium involve matrix elements that are related to the second derivative of the wave function at the origin:

\[ \psi^{(2)}(0) \equiv \int \frac{d^3p}{(2\pi)^3} p^2 \tilde{\psi}(p) = \frac{1}{\sqrt{2N_c}} \langle 0| \chi^\dagger (-\nabla^2)^2 \psi | H^{(1S_0)} \rangle, \quad (8a) \]

\[ \epsilon \psi^{(2)}(0) \equiv \epsilon \int \frac{d^3p}{(2\pi)^3} p^2 \tilde{\psi}(p) = \frac{1}{\sqrt{2N_c}} \langle 0| \chi^\dagger \sigma (-\nabla^2) \psi | H^{(3S_1)} \rangle. \quad (8b) \]

Usually, these operator matrix elements are written in terms of the covariant derivative \( D \) (Ref. [1]), rather than \( \nabla \). However, in the Coulomb gauge, the difference between the \( D \) and \( \nabla \) is suppressed as \( v \) (Ref. [1]).

The quantity \( \psi^{(2)}(0) \) is the focus of this paper. It is common in phenomenology to make use of a parameter

\[ \langle \nu^2 \rangle = \frac{\psi^{(2)}(0)}{m_c^2 \psi(0)}, \quad (9) \]

where \( m_c \) is the charm-quark pole mass, which we distinguish from the parameter \( m \) that appears in our potential model. Note that \( \psi^{(2)}(0) \) is different from the expectation value of \( p^2 \):

\[ \psi^{(2)}(0) \neq \int \frac{d^3p}{(2\pi)^3} p^2 \tilde{\psi}^*(p) \tilde{\psi}(p). \quad (10) \]

Let us investigate the ultraviolet behavior of the matrix elements in Eq. (8). At large momentum \( |p| \), the interaction of the \( \bar{Q}Q \) pair in QCD is dominated by the Coulomb potential. In this limit, the bound-state wave function approaches the pure Coulomb wave function:

\[ \tilde{\psi}(p) \sim \frac{1}{(p^2 + \gamma_C^2)^2}. \quad (11) \]

Here \( \gamma_C \) is a parameter that is related to the binding energy of Coulomb interaction:

\[ \gamma_C = \frac{1}{2} \alpha_s C_F m, \quad (12) \]

where \( \alpha_s \) is the effective strong coupling, \( C_F = 4/3 \), and \( m \) is the quark mass. Substituting Eq. (11) into Eq. (8), we see that the matrix elements in Eq. (8) have a linear ultraviolet divergence. Hence, in order to define them, we must impose a regulator.

**IV. HARD-CUTOFF REGULATOR**

Our ultimate goal is to regulate the matrix elements in Eq. (8) using dimensional regularization. However, as an intermediate step, we impose a hard-cutoff regulator. In principle,
the methods that we use could be employed in lattice calculations of the matrix elements, in which the lattice supplies the hard-cutoff regulator. However, for the purposes of the present work, we make use of a simple, analytic UV regulator in momentum space:

\[
\psi^{(2)}_{\Lambda}(0) = \int \frac{d^3 p}{(2\pi)^3} \frac{\Lambda^2}{p^2 + \Lambda^2} p^2 \tilde{\psi}(p),
\]  

(13)

where \( \Lambda \) is the cutoff.

We introduce the Fourier transform of the wave function to coordinate space:

\[
\psi(x) = \int \frac{d^3 p}{(2\pi)^3} e^{ip \cdot x} \tilde{\psi}(p).
\]  

(14)

For \( S \)-wave states, we can write \( \psi(x) = R(r)/\sqrt{4\pi} \), where \( r = |x| \) and \( R(r) \) is the radial wave function. Substituting Eq. (14) into Eq. (13), we can carry out the angular integration over \( p \) to obtain an expression in coordinate space:

\[
\psi^{(2)}_{\Lambda}(0) = \frac{\Lambda^2}{\sqrt{4\pi}} \left[ R(0) - \Lambda^2 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \right].
\]  

(15)

V. DIFFERENCE BETWEEN HARD-CUTOFF AND DIMENSIONAL REGULARIZATION

Now we work out the difference between the hard-cutoff matrix element and the dimensionally-regulated matrix element

\[
\Delta \psi^{(2)}(0) \equiv \psi^{(2)}_{\Lambda}(0) - \psi^{(2)}_{\text{DR,}\Lambda}(0).
\]  

(16)

Because \( \Delta \psi^{(2)}(0) \) involves a change in the ultraviolet cutoff, it is sensitive only to the high-momentum part of the \( p \) integration in the momentum-space definition of \( \psi^{(2)}(0) \) in Eq. (8). Therefore, we can compute \( \Delta \psi^{(2)}(0) \) in perturbation theory. Here, we carry out the computation in lowest (one-loop) order.\(^2\) We note, however, that the perturbative calculation of \( \Delta \psi^{(2)}(0) \) does not contain all of the corrections that behave as powers of \( 1/\Lambda \) to the

\(^2\) One could, in principle, compute corrections of higher order in \( \alpha_s \). Ultimately, the series of these corrections diverges, owing to the renormalon ambiguity that appears in dimensionally-regulated matrix elements. That ambiguity is canceled by a corresponding ambiguity in the NRQCD short-distance coefficients, provided that one computes to the same loop order in both the matrix elements and the short-distance coefficients. See Ref. [14] for a discussion of this point.

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FIG. 1: Feynman diagram corresponding to $\tilde{I}^{(2)}(p)$, which is the one-loop renormalization of the operators in Eq. (8). The solid lines represent the heavy quark and antiquark, and the dotted line represents the potential between them.

The difference between $\psi^{(2)}(0)$ and $\psi^{(2)}_{DR,A}(0)$. Therefore, it is necessary to take the limit $\Lambda \to \infty$ in order to remove the effects of such power corrections.

We begin by using the Bethe-Salpeter equation (or, equivalently, the Schrödinger equation) to expose one explicit loop in the wave function. Then, we have

$$\psi^{(2)}(0) = \int \frac{d^3 p}{(2\pi)^3} \tilde{I}^{(2)}(p) \tilde{\psi}(p),$$

where $\tilde{I}^{(2)}(p)$ is the quantity that is represented by the Feynman diagram in Fig. 1. In Fig. 1, the dotted line represents the potential. Since the loop integral is dominated by large momenta, we can approximate the potential by the Coulomb part.

We emphasize that we can always make this approximation in QCD, irrespective of the use of a potential model. Asymptotic freedom allows one to evaluate the high-momentum loop in Fig. 1 in perturbation theory. Then, in the Coulomb gauge, the Coulomb-gluon interactions with the heavy quark and antiquark give the result of leading order in $v$, while the transverse-gluon interactions are suppressed as $v^2$ (Ref. [1]). In the language of NRQCD, $\tilde{I}^{(2)}(p)$ is the perturbative one-loop renormalization of the operators in Eq. (8) in leading order in $\alpha_s$ and $v$.

Note that in $\tilde{I}^{(2)}(p)$ we retain the binding energy of the heavy quarkonium

$$\epsilon_B = -\gamma_B^2/m.$$
We assign momenta such that the $Q$ and $\bar{Q}$ each carry half of the binding energy in the rest frame of the heavy quarkonium.

$\Delta \psi^{(2)}(0)$ is given by

$$\Delta \psi^{(2)}(0) = \int \frac{d^3 p}{(2\pi)^3} \Delta \tilde{I}^{(2)}(p) \bar{\psi}(p),$$

where

$$\Delta \tilde{I}^{(2)}(p) = \tilde{I}^{(2)}_\Lambda(p) - \tilde{I}^{(2)}_{\text{DR}}(p).$$

Writing out the expression for the diagram in Fig. 1 for the cases of a hard cutoff and dimensional regularization, we obtain

$$\Delta \tilde{I}^{(2)}_B(p) = 4\pi\alpha_s C_F m \int \frac{d^4 k}{(2\pi)^4} \frac{(k + p)^2}{k^2} \left[ \frac{i}{k_0 - \frac{\gamma_B^2}{2m} - \frac{(k + p)^2}{2m} + i\epsilon} \right] \times \left[ -k_0 - \frac{\gamma_B^2}{2m} - \frac{(k + p)^2}{2m} + i\epsilon \right] \left[ \frac{\Lambda^2}{(k + p)^2 + \Lambda^2} - 1_{\text{DR}} \right].$$  \hspace{1cm} (21)

The first term in the numerator brackets in Eq. (21) corresponds to the hard cutoff, while the second term “$1_{\text{DR}}$” corresponds to dimensional regularization. The meaning of the “$1_{\text{DR}}$” is that it is unity unless it multiplies a scaleless integral (which vanishes in dimensional regularization), in which case it is zero. The subscript “$B$” in $\Delta \tilde{I}^{(2)}_B(p)$ indicates that we have retained the dependence on the binding energy $-\gamma_B^2/2m$. Since the integral in Eq. (21) is dominated by large momenta, we expect the final result to be insensitive to $\gamma_B$. However, in discussions that occur later in this paper, it is illuminating to retain the full $\gamma_B$ dependence.

The integral over the loop energy $k_0$ in Eq. (21) can be carried out by using the residue theorem. The result is

$$\Delta \tilde{I}^{(2)}_B(p) = 4\pi\alpha_s C_F m \int \frac{d^3 k}{(2\pi)^3} \frac{(k + p)^2}{k^2} \left[ \frac{\Lambda^2}{(k + p)^2 + \Lambda^2} - 1_{\text{DR}} \right].$$ \hspace{1cm} (22)

(Note that we also could have reached this expression directly from the momentum-space Schrödinger equation.) In the second term in brackets in Eq. (22), we discard a scaleless integral of the form

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1_{\text{DR}}}{k^2}.$$ \hspace{1cm} (23)

In the first term in brackets in Eq. (22), we rewrite the numerator $(k + p)^2$ as $[(k + p)^2 + \gamma_B^2] - \gamma_B^2$ and partial-fraction the $-\gamma_B^2$ term. The result of these manipulations is

$$\Delta \tilde{I}^{(2)}_B(p) = \frac{4\pi\alpha_s C_F m}{\Lambda^2 - \gamma_B^2} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} \left[ \frac{\Lambda^4}{(k + p)^2 + \Lambda^2} - \frac{\gamma_B^4}{(k + p)^2 + \gamma_B^2} \right].$$ \hspace{1cm} (24)
Evaluation of the integral in Eq. (24) is straightforward. The result is

$$ \Delta \tilde{I}_B^{(2)}(p) = \frac{\alpha_s C_F m}{|p| (\Lambda^2 - \gamma_B^2)} \left[ \Lambda^4 \arctan \left( \frac{|p|}{\Lambda} \right) - \gamma_B^4 \arctan \left( \frac{|p|}{\gamma_B} \right) \right] \left( \Lambda^2 - \gamma_B^2 \right). $$

(25)

As expected, $\tilde{I}_B^{(2)}(p)$ is insensitive to $\gamma_B$. Neglecting terms of higher order in $\gamma_B^2/\Lambda^2$, we obtain

$$ \Delta \tilde{I}_B^{(2)}(p) \equiv \lim_{\gamma_B \to 0} \Delta \tilde{I}_B^{(2)}(p) = \frac{\alpha_s C_F m}{|p|} \Lambda^2 \arctan \left( \frac{|p|}{\Lambda} \right), $$

(26)

where the subscript “NB” indicates that we have neglected the binding energy by dropping contributions of higher order in $\gamma_B^2/\Lambda^2$. (Note that $\arctan(|p|/\gamma_B)$ is well behaved as $\gamma_B \to 0$ and bounded over the entire range of its argument.)

In our numerical analyses, we make use of $\Delta \tilde{I}_B^{(2)}(p)$, rather than $\Delta \tilde{I}_B^{(2)}(p)$, and consistently neglect the binding energy in short-distance (high-momentum) quantities. Of course, the binding-energy dependence in the wave function is retained fully.

In our numerical analyses, we solve the Schrödinger equation in coordinate space rather than in momentum space. The Fourier transformation of the coordinate-space wave function to momentum space involves an oscillating integrand and, hence, is difficult to evaluate numerically. Therefore, it is convenient to evaluate $\Delta \psi^{(2)}(0)$ in coordinate space:

$$ \Delta \psi^{(2)}(0) = \int d^3x \Delta I^{(2)}(x) \psi(x), $$

(27)

where

$$ \Delta I^{(2)}(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \Delta \tilde{I}^{(2)}(p) $$

(28)

is the Fourier transform of $\Delta \tilde{I}^{(2)}(p)$. It is a simple matter to evaluate $\Delta I^{(2)}(x)$ analytically. The results are

$$ \Delta I_B^{(2)}(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \Delta \tilde{I}_B^{(2)}(p) = \frac{\alpha_s C_F m}{4\pi r^2 (\Lambda^2 - \gamma_B^2)} \left( \Lambda^4 e^{-\Lambda r} - \gamma_B^4 e^{-\gamma_B r} \right), $$

(29a)

$$ \Delta I_{NB}^{(2)}(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \Delta \tilde{I}_{NB}^{(2)}(p) = \frac{\alpha_s C_F m \Lambda^2}{4\pi r^2} e^{-\Lambda r}. $$

(29b)

Substituting $\Delta I^{(2)}(x)$ into Eq. (27), we obtain

$$ \Delta \psi_B^{(2)}(0) = \frac{\alpha_s C_F m}{\Lambda^2 - \gamma_B^2} \int_0^\infty \left( \Lambda^4 e^{-\Lambda r} - \gamma_B^4 e^{-\gamma_B r} \right) \frac{R(r)}{\sqrt{4\pi}} \, dr, $$

(30a)

$$ \Delta \psi_{NB}^{(2)}(0) = \alpha_s C_F m \Lambda^2 \int_0^\infty e^{-\Lambda r} R(r) \frac{1}{\sqrt{4\pi}} \, dr. $$

(30b)
Then, we obtain the dimensionally-regulated matrix element by computing

\[ \psi_{\text{DR,}\Lambda}^{(2)}(0) = \psi^{(2)}_{\Lambda}(0) - \Delta\psi^{(2)}_{\text{NB}}(0) \]  

and taking the limit \( \Lambda \to \infty \) in order to remove uncompensated power corrections that vanish as \( 1/\Lambda \):

\[ \psi^{(2)}_{\text{DR}}(0) = \lim_{\Lambda \to \infty} \psi^{(2)}_{\text{DR,}\Lambda}(0). \]  

\[ (32) \]

**VI. DECOMPOSITION OF \( \Delta\psi^{(2)}(0) \)**

It is illuminating to decompose \( \Delta\psi^{(2)}(0) \) into products of NRQCD matrix elements times short-distance coefficients. In existing lattice calculations [3], the difference between matrix elements in hard-cutoff regularization and those in dimensional regularization is expressed in terms of such a decomposition. We decompose \( \Delta\psi^{(2)}(0) \) by applying the method of regions [15] to the integration over the wave-function momentum \( p \) in Eq. (19). We take for \( \Delta\tilde{I}^{(2)}(p) \) the no-binding expression \( \Delta\tilde{I}^{(2)}_{\text{NB}}(p) \), which is given in Eq. (26).

First, we extract the leading term, which is obtained from the \( |p| \) part of \( \Delta\tilde{I}^{(2)}_{\text{NB}}(p) \), namely, \( \alpha_s C_F m \Lambda \). It yields a contribution to \( \Delta\psi^{(2)}(0) \) that is

\[ \Delta\psi^{(2)}_1(0) = \alpha_s C_F m \Lambda \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}(p) = \alpha_s C_F m \Lambda \psi(0) = 2\gamma_C \Lambda \psi(0). \]  

\[ (33) \]

We identify this as a one-loop short-distance coefficient times the matrix element of leading order in \( v \), namely, \( \psi(0) \). \( \Delta\psi^{(2)}_1(0) \) is linearly divergent in the limit \( \Lambda \to \infty \) and cancels the linear divergence in the hard-cutoff matrix element \( \psi^{(2)}_{\Lambda}(0) \), resulting in a matrix element \( \psi^{(2)}_{\text{DR}}(0) \), which, according to the NRQCD \( v \)-scaling rules, is of order \( (mv)^2 \psi(0) \). The \( v \) scaling of this quantity is verified by the result of an analytic calculation of \( \psi^{(2)}_{\text{DR}}(0) \) for the case of a pure Coulomb potential in Appendix A.

Next we examine the remainder of \( \Delta\tilde{I}^{(2)}_{\text{NB}}(p) \), namely, \( \alpha_s C_F m \Lambda [(\Lambda/|p|) \arctan(|p|/\Lambda) - 1] \). We decompose the loop integration over \( p \) into regions of small \( |p| \), in which \( |p| \ll \Lambda \), and large \( |p| \), in which \( |p| \sim \Lambda \).

In the small-\( |p| \) region, we expand in powers of \( |p| \). The resulting contributions, which we call \( \Delta\psi^{(2)}_2(0) \), have the form

\[ \Delta\psi^{(2)}_2(0) = \alpha_s C_F m \int_{\text{DR}} \frac{d^3p}{(2\pi)^3} \tilde{\psi}(p) \sum_{n=3}^{\infty} c_n \frac{|p|^{n-1}}{\Lambda^{n-2}}, \]  

\[ (34) \]
where $c_n$ is the coefficient of $(|p|/\Lambda)^n$ in the power-series expansion of $\arctan(|p|/\Lambda)$, and the subscript DR on the integral indicates that any UV divergence is to be regulated dimensionally. $\Delta \psi^{(2)}_3(0)$ corresponds to one-loop short-distance coefficients times dimensionally-regulated NRQCD matrix elements of higher order in $v$. The contributions to $\Delta \psi^{(2)}_3(0)$ are of order $\alpha_s m (mv)^{n-1}/\Lambda^{n-2}$ and, hence, are suppressed relative to $\psi^{(2)}_{DR}(0)$ by at least one power of $m/\Lambda$ as $\Lambda \to \infty$.

In the large-|$p$| region, we approximate $\tilde{\psi}(p)$ by its asymptotic form at large $|p|$, which amounts to neglecting corrections of higher order in $\alpha_s$ and $mv/\Lambda$. The asymptotic form of $\tilde{\psi}(p)$ is obtained by using the Bethe-Salpeter equation to expose an explicit loop in the wave function, as in the derivation of Eq. (22). The result is

$$\tilde{\psi}(p) \sim \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(k) \left( \frac{8\pi \gamma_C}{p - k} \right)^2 \sim \frac{8\pi \gamma_C \psi(0)}{p^4} \equiv \tilde{\psi}_{asy}(p).$$

(35)

Then, we have for the contribution in the large-|$p$| region

$$\Delta \psi^{(2)}_3(0) = \alpha_s C_F m \Lambda \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{asy}(p) \left( \frac{\Lambda}{|p|} \arctan \left( \frac{|p|}{\Lambda} \right) - 1 \right)$$

$$= -\left(1/2\right) \alpha_s C_F m^2 \psi(0) = -2 \gamma_C^2 \psi(0).$$

(36)

$\Delta \psi^{(2)}_3(0)$ has the interpretation of a two-loop short-distance coefficient times the lowest-order NRQCD matrix element.

Although the contribution $\Delta \psi^{(2)}_3(0)$ is suppressed as $\alpha_s/\Lambda$ relative to $\Delta \psi^{(2)}_1(0)$, $\Delta \psi^{(2)}_3(0)$ can be important numerically because, in $\psi^{(2)}_{\Lambda,DR}(0)$, $\Delta \psi^{(2)}_1(0)$ is canceled by contributions from $\psi^{(2)}_{\Lambda}(0)$, leaving a small remainder. As we have remarked, $\psi^{(2)}_{DR}(0)$ itself is nominally of order $(mv)^2 \psi(0)$. Therefore, $\Delta \psi^{(2)}_3(0)$ is suppressed only as $\alpha_s^2/v^2$ relative to $\psi^{(2)}_{DR}(0)$. According to the $v$-scaling rules of NRQCD [1], $\alpha_s$ is of order $v^2$, and, hence, there is a slight suppression. It turns out that, for the Cornell potential, the numerical value of $\Delta \psi^{(2)}_3(0)$ is about $-40\%$ of $\psi^{(2)}_{DR}(0)$. This suggests that two-loop corrections to the short-distance coefficients could be important numerically in converting results of lattice calculations of $\psi^{(2)}(0)$ to continuum regularization. In the case of a pure Coulomb potential, $\alpha_s$ is of order $v$, and, so, there is no suppression of $\Delta \psi^{(2)}_3(0)$ relative to $\psi^{(2)}_{DR}(0)$ at all. We show in Appendix A, $\psi^{(2)}_{DR}(0)$ satisfies the Gremm-Kapustin relation in the case of a pure Coulomb potential, while $\psi^{(2)}_{DR}(0) - \Delta \psi^{(2)}_3(0)$ does not.
VII. DIRECT METHOD OF CALCULATION OF $\psi^{(2)}_{\text{DR}}(0)$

In the method of calculation that we have outlined, it is necessary to take the limit of the quantity $\psi^{(2)}_{\text{DR},0}(0)$ as $\Lambda$ goes to infinity. $\psi^{(2)}_{\text{DR},0}(0)$ consists of a difference of $\Lambda$-dependent terms that grow approximately linearly with $\Lambda$. If one computes at a large enough value of $\Lambda$ to be near the asymptotic value of $\psi^{(2)}_{\text{DR},0}(0)$, then there is a substantial cancellation between these terms. For example, for the NLO parameters at $\lambda = 1.0$, the ratio $\psi^{(2)}_{\Lambda}(0)/\psi^{(2)}_{\text{DR},0}(0)$ takes on the values 18, 34, and 50 when $\Lambda/m$ equals 10, 20, and 30, respectively. In numerical calculations, the cancellation in $\psi^{(2)}_{\text{DR},0}(0)$ is a significant obstacle to achieving good accuracy. Therefore, it is desirable to have a direct method of computation of $\psi^{(2)}_{\text{DR}}(0)$ that does not pass through an intermediate hard-cutoff step. In this section, we present such a method. We also show how it leads to a simple formula for the resummation of a class of corrections of higher order in $v$.

A. Method

We begin again by exposing one loop in the matrix element, as in Fig. 1. Now, however, since we are computing the matrix element itself, not a difference of matrix elements for different regulators, the loop in Fig. 1 is not dominated by large momenta. Therefore, we must retain the complete Cornell potential in the corresponding expression. Repeating the steps that lead to Eq. (22), but for the complete Cornell potential, and using Eq. (17), we obtain for the dimensionally-regulated matrix element

$$\psi^{(2)}_{\text{DR}}(0) = -m \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{k^2 + \gamma_B^2 + i\epsilon} \int \frac{d^3p}{(2\pi)^3} \tilde{V}(k - p) \tilde{\psi}(p).$$

(37)

It turns out that, in the case of the Cornell potential, the binding energy is positive. Therefore, it is convenient to define a quantity $\tilde{\gamma}_B$ as

$$\epsilon_B = -\frac{\gamma_B^2}{m} = \frac{\tilde{\gamma}_B^2}{m} > 0.$$ 

(38)

From the $i\epsilon$ prescription in Eq. (37), we see that we can analytically continue from $\gamma_B$ positive to $\gamma_B$ positive imaginary. Hence, we make the identification $\gamma_B = i\tilde{\gamma}_B$, where $\tilde{\gamma}_B$ is real and positive. Discarding the contribution of the scaleless integral

$$\int \frac{d^3k}{(2\pi)^3} \tilde{V}(k - p) 1_{\text{DR}} = \int \frac{d^3k}{(2\pi)^3} \tilde{V}(k) 1_{\text{DR}}$$

(39)
in Eq. (37), we have
\[
\psi_{DR}^{(2)}(0) = -m \tilde{\gamma}_B^2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} \frac{\tilde{V}(k-p)\tilde{\psi}(p)}{k^2 - \tilde{\gamma}_B^2 + i\epsilon}.
\] (40)

Now we note that \(\tilde{\psi}(p)\) satisfies the momentum-space Schrödinger equation
\[
(k^2 - \tilde{\gamma}_B^2)\tilde{\psi}(k) = -\int \frac{d^3p}{(2\pi)^3} m\tilde{V}(k-p)\tilde{\psi}(p).
\] (41)

Therefore, Eq. (40) can be written as
\[
\psi_{DR}^{(2)}(0) = \tilde{\gamma}_B^2 \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(k) = \tilde{\gamma}_B^2 \psi(0).
\] (42)

The Gremm-Kapustin relation [4] follows from the NRQCD equations of motion at leading order nontrivial in \(v\). It states that
\[
\langle v^2 \rangle = \frac{\psi^{(2)}(0)}{m_c^2 \psi(0)} = \frac{\epsilon_B}{m_c} + O(v^4).
\] (43)

Using the definition of \(\tilde{\gamma}_B^2\) in Eq. (38), we see that our result in Eq. (42) is precisely the Gremm-Kapustin relation for the potential model with mass \(m = m_c\). This is not surprising, since a potential model can be derived from NRQCD at leading order in \(v\) (Ref. [6]) and, hence, implicitly respects the NRQCD equations of motion at leading order in \(v\).

B. Resummation

We note that the result in Eq. (42) can easily be generalized to the case of NRQCD operator matrix elements involving additional powers of \(\nabla^2\), such as
\[
\psi^{(2n)}(0) \equiv \int \frac{d^3p}{(2\pi)^3} P^{2n} \tilde{\psi}(p) = \frac{1}{\sqrt{2N_c}} \langle 0|\chi^\dagger (-\nabla^2)^n \psi|H^{(1)S_0}\rangle,
\] (44a)
\[
\epsilon \psi^{(2n)}(0) \equiv \epsilon \int \frac{d^3p}{(2\pi)^3} P^{2n} \tilde{\psi}(p) = \frac{1}{\sqrt{2N_c}} \langle 0|\chi^\dagger \sigma (-\nabla^2)^n \psi|H^{(3)S_1}\rangle,
\] (44b)

where \(n\) is an integer. As we have remarked earlier, such operator matrix elements are usually written in terms of the covariant derivative \(D\) (Ref. [1]), rather than \(\nabla\). However, in the Coulomb gauge, the difference between \(D\) and \(\nabla\) is suppressed as \(v\) (Ref. [1]).

Repeating the steps that lead to Eq. (37), we obtain
\[
\psi_{DR}^{(2n)}(0) = -m \int \frac{d^3k}{(2\pi)^3} \frac{k^{2n} \psi_{1DR}^{(2n)}}{k^2 - \tilde{\gamma}_B^2 + i\epsilon} \int \frac{d^3p}{(2\pi)^3} \tilde{V}(k-p)\tilde{\psi}(p).
\] (45)
In Eq. (45), we write
\[
\frac{k^{2n}}{k^2 - \bar{\gamma}_B^2 + i\epsilon} = k^{2n-2} + \bar{\gamma}_B^2 k^{2n-4} + \cdots + \frac{\bar{\gamma}_B^{2n}}{k^2 - \bar{\gamma}_B^2 + i\epsilon}
\] (46)
and discard scaleless integrals of the form
\[
\int \frac{d^3k}{(2\pi)^3} \tilde{V}(k - p) k^{2n} 1_{DR} = \int \frac{d^3k}{(2\pi)^3} \tilde{V}(k + p) k^{2n} 1_{DR},
\] (47)
where \(n'\) is an integer. The result is
\[
\psi^{(2n)}_{DR}(0) = -m \bar{\gamma}_B^{2n} \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{\tilde{V}(k - p) \tilde{\psi}(p)}{k^2 - \bar{\gamma}_B^2 + i\epsilon} = \bar{\gamma}_B^{2n} \psi(0),
\] (48)
where we have used the momentum-space Schrödinger equation [Eq. (41)]. The expression in Eq. (48) can be thought of as a generalized Gremm-Kapustin relation that holds in a potential model.

Now, suppose that there are contributions to a decay or production amplitude, computed in NRQCD, of the form
\[
A^{(2n)} = \frac{1}{n!} H^{(2n)}(0) \psi^{(2n)}_{DR}(0).
\] (49)
Here, \((1/n!)H^{(2n)}(0)\) is an NRQCD short-distance coefficient that is defined by
\[
H^{(2n)}(0) = \left( \frac{\partial}{\partial p^2} \right)^n H(p^2) \bigg|_{p=0},
\] (50)
where \(H^{(2n)}(p^2)\) is the hard-scattering amplitude for the process, evaluated in the quarkonium rest frame at \(Q\) or \(\bar{Q}\) momentum squared \(p^2\). We can use Eq. (48) to resum the contributions \(A^{(2n)}\):
\[
\sum_n A^{(2n)} = \sum_n \frac{1}{n!} H^{(2n)}(0) \psi^{(2n)}_{DR}(0) = \sum_n \frac{1}{n!} H^{(2n)}(0) \bar{\gamma}_B^{2n} \psi(0) = H(\bar{\gamma}_B^2) \psi(0). \] (51)

VIII. NUMERICAL RESULTS AND DISCUSSION

We now use the hard-cutoff method of Secs. IV and V and the direct method of Sec. VII to compute \(\psi^{(2)}_{DR}(0)\).

We begin with the hard-cutoff method. We substitute the 1S-state coordinate-space radial wave function that we obtain by integrating the Schrödinger equation numerically into Eqs. (15) and (30b) and carry out the integrations over \(r\) numerically. We then use Eq. (31) to compute \(\psi^{(2)}_{DR,A}(0)\). The results are shown in Fig. 2. As can be seen, \(\psi^{(2)}_{A}(0)\) grows
FIG. 2: \( \psi^{(2)}_\Lambda(0) \) and \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \equiv \psi^{(2)}_\Lambda(0) - \Delta\psi^{(2)}_{\text{NB}}(0) \) as a function of \( \Lambda/m \). The left figure corresponds to the LO potential-model parameters of Table I for \( \lambda = 0.6, 0.7, \) and \( 0.8 \). The right figure corresponds to the NLO potential-model parameters of Table II for \( \lambda = 0.9, 1.0, \) and \( 1.1 \). In each figure, the three curves that are nearly linear are \( 0.1 \times \psi^{(2)}_\Lambda(0) \), and the three curves that reach a plateau at large \( \Lambda/m \) are \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \).

Nearly linearly with \( \Lambda \), as is expected from the linear ultraviolet divergence that it contains. \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \) is the result of a substantial cancellation between \( \psi^{(2)}_\Lambda(0) \) and \( \Delta\psi^{(2)}_{\text{NB}}(0) \), at the level of about one part in 50 at \( \Lambda/m = 30 \) for the NLO parameters. From Fig. 2, it is apparent that \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \) reaches a plateau at large \( \Lambda \). \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \) deviates from its asymptotic value by an amount that is of order \( 1/\Lambda \). It is the value of \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \) at the plateau that corresponds to the dimensionally-regulated matrix element [Eq. (32)]. A fit of the LO-parameter results for \( \psi^{(2)}_{\text{DR,}\Lambda}(0) \) at \( \lambda = 0.6, 0.7, \) and \( 0.8 \) to a constant plus a term that is proportional to \( 1/\Lambda \) yields \( \psi^{(2)}_{\text{DR,}\Lambda}(0) = 0.123 \text{ GeV}^{7/2}, 0.108 \text{ GeV}^{7/2}, \) and \( 0.095 \text{ GeV}^{7/2} \), respectively. A similar fit to the NLO-parameter results at \( \lambda = 0.9, 1.0, \) and \( 1.1 \) yields \( \psi^{(2)}_{\text{DR,}\Lambda}(0) = 0.144 \text{ GeV}^{7/2}, 0.124 \text{ GeV}^{7/2}, \) and \( 0.107 \text{ GeV}^{7/2} \), respectively.

Next we consider the direct method for calculating \( \psi^{(2)}_{\text{DR}}(0) \). Substituting the LO potential-model parameters into Eq. (42), we obtain \( \psi^{(2)}_{\text{DR}}(0) = 0.124 \text{ GeV}^{7/2}, 0.109 \text{ GeV}^{7/2}, \) and \( 0.096 \text{ GeV}^{7/2} \) at \( \lambda = 0.6, 0.7, \) and \( 0.8 \), respectively. The results for the NLO-parameters are \( \psi^{(2)}_{\text{DR}}(0) = 0.146 \text{ GeV}^{7/2}, 0.127 \text{ GeV}^{7/2}, \) and \( 0.109 \text{ GeV}^{7/2} \) at \( \lambda = 0.9, 1.0, \) and \( 1.1 \), respectively. These results for \( \psi^{(2)}_{\text{DR}}(0) \) are in good agreement with those from the extrapolation of
$\psi^{(2)}_{\text{DR},\Lambda}(0)$ to $\Lambda = \infty$ in the hard-cutoff method.

We consider the NLO value for $\psi(0)$ to be slightly more reliable than the LO value. Therefore, we use the NLO parameters to compute our central value for $\psi^{(2)}_{\text{DR}}(0)$. We use the LO parameters to give an indication of the uncertainty in $\psi^{(2)}_{\text{DR}}(0)$ that arises from the uncertainty in $\psi(0)$. As we have already discussed in Sec. II, the potential-model values of the string tension that derive from the NLO parameters for $\lambda = 1.0$ and $1.1$ span the range of lattice values for the string tension. The potential-model values of $\alpha_s$ that derive from the NLO parameters for $\lambda = 1.0$ and $1.1$ are somewhat larger than the lattice values for the fixed parameter $\alpha_s$, but are compatible with the value of the running $\alpha_s$ at the scale of the heavy-quark momentum $m_c v$. We consider the lattice value of the string tension to be more relevant than the lattice value of the fixed parameter $\alpha_s$, since the latter does not take into account the running of $\alpha_s$ in QCD. Therefore, we determine the central value of $\psi^{(2)}_{\text{DR}}(0)$ by taking the average of the values of $\psi^{(2)}_{\text{DR}}(0)$ for the NLO parameters at $\lambda = 1.0$ and $1.1$. We take the difference of these values as the uncertainty that is attributable to the uncertainty in the potential-model parameters. We take the difference between the average of the values of $\psi^{(2)}_{\text{DR}}(0)$ for the NLO parameters at $\lambda = 1.0$ and $1.1$ and the average of the values of $\psi^{(2)}_{\text{DR}}(0)$ for the LO parameters at $\lambda = 0.7$ and $0.8$ to be the uncertainty that is attributable to the uncertainty in $\psi(0)$. We add these uncertainties in quadrature. We include an additional 30% uncertainty to account for the fact that our potential model neglects terms of relative order $v^2 \approx 0.3$. Using the direct-method values for $\psi^{(2)}_{\text{DR}}(0)$, we obtain

$$\psi^{(2)}_{\text{DR}}(0) = 0.118 \pm 0.024 \pm 0.035 \text{ GeV}^{7/2}, \quad (52)$$

where the second uncertainty arises from the $v^2$ error. This is the dominant source of error.

The effects of the uncertainty in the value of $\psi(0)$ tend to cancel in the ratio $\psi^{(2)}_{\text{DR}}(0)/\psi(0)$. We compute this ratio for the LO-parameter and NLO-parameter direct-method results and use the same method for determining the central value and the uncertainties that we described above for $\psi^{(2)}_{\text{DR}}(0)$. The result is

$$\psi^{(2)}_{\text{DR}}(0)/\psi(0) = m_c^2 v^2 = 0.50 \pm 0.09 \pm 0.15 \text{ GeV}^2, \quad (53)$$

where, again, the second uncertainty arises from the $v^2$ error. Taking $m_c = 1.4$ GeV in Eq. (53), we have $\langle v^2 \rangle = 0.25 \pm 0.05 \pm 0.08$, which is in good agreement with expectations from the NRQCD $v$-scaling rules.
We can now address the numerical importance of the two-loop contribution to $\Delta \psi^{(2)}(0)$ that is given in Eq. (36). Taking the average of the values of $\gamma_C$ for $\lambda = 1.0$ and 1.1 in Table II and using the NLO value $\psi(0) = 0.23629 \text{GeV}^{3/2}$, we find that $\Delta \psi^{(2)}_3(0) = -0.048 \text{GeV}^{7/2}$. This is about $-40\%$ of the value of $\psi^{(2)}_{\text{DR}}(0)$ in Eq. (52).

The $v$-scaling rules of NRQCD state that the hard-cutoff matrix element $\psi^{(2)}_\Lambda(0)$, evaluated at $\Lambda \sim m v$, should be of order $m_c^2 v^2 \psi(0)$. Taking $m_c = 1.4 \text{GeV}$, $v^2 = 0.3$, and $\psi(0) = 0.23629 \text{GeV}^{3/2}$, we obtain $m_c^2 v^2 \psi(0) \approx 0.14 \text{GeV}^{7/2}$, which is in reasonably good agreement with the value of $\psi^{(2)}_\Lambda(0)$ at $\Lambda = m_c v \approx m$ for the NLO results at $\lambda = 1.0$ and 1.1 in Fig. 2.

In lattice determinations of $\psi^{(2)}_{\text{DR},\Lambda}(0)$, the lattice ultraviolet cutoff, which is of order $\pi$ divided by the lattice spacing, corresponds approximately to the hard cutoff $\Lambda$. Existing lattice determinations of $\psi^{(2)}_{\text{DR},\Lambda}(0)$ (Ref. [3]) have been carried out in the vicinity of $\Lambda = m_c$. This value of $\Lambda$ is at the boundary of the region in which asymptotic freedom allows one to evaluate quantities in QCD in perturbation theory. However, as can be seen from Fig. 2, $\Lambda = m_c$ is far from the region in which $\psi^{(2)}_{\text{DR},\Lambda}(0)$ approaches its asymptotic value. Apparently, at $\Lambda = m_c$, power corrections of order $m_c v/\Lambda$ are still important. Therefore, we expect that the lattice determinations of $\psi^{(2)}_{\text{DR}}(0)$ contain large $1/\Lambda$ errors.

IX. SUMMARY

In this paper we have presented two methods for calculating NRQCD matrix elements that are proportional to $\psi^{(2)}(0)$, the negative of the second derivative of the wave function at the origin. These matrix elements enter into the relativistic corrections for the decay and production of heavy-quarkonium states. The matrix elements are linearly ultraviolet divergent, and, hence, they must be regulated. We compute the matrix elements in dimensional regularization, since that is the regularization that is used most commonly in phenomenology.

One method that we have presented makes use of a hard-cutoff regulator as an intermediate step, then employs perturbation theory to compute the difference between the hard-cutoff matrix element and a dimensionally-regulated matrix element. This method is quite general, in that it requires knowledge only of the $Q\bar{Q}$ wave function in the quarkonium state. In principle, it could be applied to wave functions that are determined directly by lattice meth-
ods. It involves computing cancelling quantities in the limit in which the hard cutoff is taken to infinity. This method has the disadvantage that the limiting procedure and cancellation make it difficult to achieve adequate numerical accuracy. In the case of a pure Coulomb potential, we have obtained analytic expressions for the quantities that enter into the hard-cutoff method. The result for the matrix elements agrees with the Gremm-Kapustin relation in that case.

The second method that we have presented involves a computation of the matrix elements directly in dimensional regularization. The method is specific to potential models. It yields the Gremm-Kapustin relation for the matrix elements. This is not surprising, since the Gremm-Kapustin relation is based on the NRQCD equations of motion at leading-order in $v$, and the potential model respects those equations of motion. This method is easily generalized to the computation of matrix elements that are proportional to higher powers of $\nabla^2$ acting on the wave function at the origin. We have used the results for such matrix elements to write a formula that resums certain contributions of higher order in $v$ to NRQCD amplitudes.

We have used both the direct method and the hard-cutoff method to evaluate the dimensionally-regulated quantity $\psi^{(2)}_{\text{DR}}(0)$ for the $J/\psi$ (or the $\eta_c$) in the Cornell potential model. Since the Cornell potential model contains no spin dependence, we do not distinguish between the $J/\psi$ and $\eta_c$ matrix elements. If the potential itself were exact, then the potential model would reproduce QCD up to corrections of relative order $v^2$. Existing lattice measurements of the static $Q\bar{Q}$ potentials yield values for the string tension. In order to estimate errors from our choice of potential, we have carried out computations for sets of Cornell-potential parameters that bracket the lattice values of the string tension.

The two methods for computing $\psi^{(2)}_{\text{DR}}(0)$ yield results that agree well numerically. Our final result, including estimates of uncertainties, is given in Eq. (52). The first error estimate arises from the uncertainty in the potential-model parameters and from the uncertainty in the value of the wave function at the origin that is obtained from the leptonic width of the $J/\psi$. The second error estimate accounts for a 30% uncertainty that arises from the fact that we have neglected corrections of relative order $v^2$. This is the dominant source of error. The effects of the uncertainty in the wave function at the origin tend to cancel in the ratio of matrix elements $\psi^{(2)}_{\text{DR}}(0)/\psi(0)$ in Eq. (53). From the ratio in Eq. (53), we estimate that $\langle v^2 \rangle = 0.25 \pm 0.05 \pm 0.08$, which is in good agreement with the $v$-scaling rules of NRQCD.
From our analysis, it is clear that, in the hard-cutoff method, there are large corrections of order $m_c v / \Lambda$, even at cutoffs $\Lambda \approx m_c$. This implies that existing lattice computations of $\psi_{\text{DR}}^{(2)}(0)$ (Ref. [3]) contain large errors that arise from such power corrections.

We also have identified a large contribution to the difference between the dimensionally-regulated matrix element $\psi_{\text{DR}}^{(2)}(0)$ and the hard-cutoff matrix element $\psi_\Lambda^{(2)}(0)$ that arises from a two-loop contribution to a particular short-distance coefficient [Eq. (36)]. That contribution is suppressed only as $\alpha_s^2 / v^2$ relative to $\psi_{\text{DR}}^{(2)}(0)$ and is about $-40\%$ of $\psi_{\text{DR}}^{(2)}(0)$ in the case of the $J/\psi$. Therefore, the two-loop contribution to this short-distance coefficient may be important numerically when one converts a lattice-regulated value of $\psi^{(2)}(0)$ to a dimensionally-regulated value.

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APPENDIX A: TEST OF THE METHOD WITH A COULOMB WAVE FUNCTION

In this appendix, we apply our methods for calculating $\psi_{\text{DR}}^{(2)}(0)$ to the case of the wave function for a pure Coulomb potential. In this case, we can evaluate all of the relevant expressions analytically. Ultimately, we compare our results with the prediction of the Gremm-Kapustin relation.

Coulomb wave functions in coordinate space and momentum space are

$$\psi_C(x) = \psi_C(0) e^{-\gamma_C r},$$

$$\tilde{\psi}_C(p) = \int d^3 x e^{-i p \cdot x} \psi_C(x) = 8 \pi \psi_C(0) \frac{\gamma_C}{(p^2 + \gamma_C^2)^2}, \quad (A1a)$$

$$\tilde{\psi}_C(p) = \int d^3 x e^{-i p \cdot x} \psi_C(x) = 8 \pi \psi_C(0) \frac{\gamma_C}{(p^2 + \gamma_C^2)^2}, \quad (A1b)$$
where $\gamma_C$ is defined in Eq. (12). The corresponding binding energy is

$$\epsilon_B = -\frac{\gamma_C^2}{m}. \quad (A2)$$

It then follows from Eq. (18) that

$$\gamma_B = \gamma_C. \quad (A3)$$

We first carry out the calculation of $\psi^{(2)}_{\text{DR}}(0)$ in the hard-cutoff method. Substituting the Coulomb wave function into either Eq. (13) or Eq. (15), we obtain the hard-cutoff matrix element $\psi^{(2)}_\Lambda(0)$:

$$\psi^{(2)}_\Lambda(0) = \Lambda^2 \psi_C(0) \left[1 - \frac{\Lambda^2}{(\Lambda + \gamma_C)^2}\right]. \quad (A4)$$

The calculation of $\psi^{(2)}_\Lambda(0)$ can be carried out by using Eqs. (27) and (29a). For the difference between the hard-cutoff and dimensionally-regulated matrix elements, we have

$$\Delta \psi^{(2)}_B(0) = \psi_C(0) \frac{2\gamma_C}{\Lambda^2 - \gamma_B^2} \left(\frac{\Lambda^4}{\Lambda + \gamma_C} - \frac{\gamma_B^4}{\gamma_B + \gamma_C}\right), \quad (A5)$$

where we have used the definition (12) to replace the prefactor $\alpha s C_F m$ with $2\gamma_C$. In the limit $\gamma_B \to 0$, Eq. (A5) becomes

$$\Delta \psi^{(2)}_{NB}(0) = \Lambda^2 \psi_C(0) \frac{2\gamma_C}{\Lambda + \gamma_C}. \quad (A6)$$

It follows from Eq. (31) that $\psi^{(2)}_{\text{DR},\Lambda}(0)$ is given by

$$\psi^{(2)}_{\text{DR},\Lambda}(0) = \psi^{(2)}_\Lambda(0) - \Delta \psi^{(2)}_{NB}(0) = -\gamma_C^2 \psi_C(0) \times \left(\frac{\Lambda}{\Lambda + \gamma_C}\right)^2. \quad (A7)$$

Taking the limit $\Lambda \to \infty$, we obtain the dimensionally-regulated matrix element $\psi^{(2)}_{\text{DR}}(0)$:

$$\psi^{(2)}_{\text{DR}}(0) = \lim_{\Lambda \to \infty} \psi^{(2)}_{\text{DR},\Lambda}(0) = -\gamma_C^2 \psi_C(0). \quad (A8)$$

We note that the deviations from the asymptotic value go as $\gamma_C/\Lambda \sim mv/\Lambda$ as $\Lambda \to \infty$. Incidentally, had we retained the effects of the binding energy by using $\Delta \psi^{(2)}_B(0)$ instead of $\Delta \psi^{(2)}_{NB}(0)$, then we would have obtained

$$\psi^{(2)}_{\text{DR},\Lambda}(0) = \psi^{(2)}_\Lambda(0) - \Delta \psi^{(2)}_B(0) = -\gamma_C^2 \psi_C(0), \quad (A9)$$

where we have made use of the Coulomb-potential relation $\gamma_B = \gamma_C$. In this case, the $\Lambda$ dependence would have vanished. However, this simplification is a special property of the
pure Coulomb case that arises from the fact that the Coulomb-gluon exchange in $\Delta \psi_B^{(2)}(0)$ is, in this special case, an interaction of the complete potential.

The result in Eq. (A8) agrees with the result from the direct method in Eq. (42) and with the Gremm-Kapustin relation (43). We note that, had we failed to include the two-loop correction to $\Delta \psi^{(2)}(0)$ in Eq. (36), then we would have obtained $\psi^{(2)}_{\text{DR}}(0) = -3\gamma_C^2 \psi_C(0)$, which does not agree with the Gremm-Kapustin relation.

**APPENDIX B: INTEGRALS**

Here we record some integrals that are useful in deriving the expressions in this paper.

1. **Loop integrals**

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{1}{(k-p)^2(k^2 + \Lambda^2 + i\epsilon)} = \frac{i}{8\pi|p|} \log \left( \frac{\Lambda - i|p|}{\Lambda + i|p|} \right) = \frac{1}{4\pi|p|} \arctan \left( \frac{|p|}{\Lambda} \right), \tag{B1a}
\]

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{1}{(k-p)^2(k^2 + \Lambda^2 + i\epsilon)^2} = \frac{1}{8\pi\Lambda(p^2 + \Lambda^2)}, \tag{B1b}
\]

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{\arctan \left( |k|/\Lambda \right)}{|k|(k^2 + \Lambda^2 + i\epsilon)^2} = \frac{1}{16\pi\Lambda^2}. \tag{B1c}
\]

Analytically continuing $\Lambda$ to $i\Lambda$, we obtain the following formulas:

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{1}{(k-p)^2(k^2 - \Lambda^2 + i\epsilon)} = \frac{1}{4\pi|p|} \arctan \left( \frac{|p|}{i\Lambda} \right) = -\frac{i}{4\pi|p|} \tanh^{-1} \left( \frac{|p|}{\Lambda} \right), \tag{B2a}
\]

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{1}{(k-p)^2(k^2 - \Lambda^2 + i\epsilon)^2} = \frac{i}{8\pi\Lambda(p^2 - \Lambda^2)}, \tag{B2b}
\]

\[
\int \frac{d^3 k}{(2\pi)^3} \frac{\arctan \left( -i|k|/\Lambda \right)}{|k|(k^2 - \Lambda^2 + i\epsilon)^2} = -\frac{1}{16\pi\Lambda^2}. \tag{B2c}
\]
2. Fourier Transformation

\[ \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \frac{\arctan(|p|/\Lambda)}{|p|} = \frac{e^{-\Lambda r}}{4\pi r^2}, \]  

\( (B3a) \)

\[ \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \frac{1}{(p^2 + \Lambda^2)} = \frac{e^{-\Lambda r}}{4\pi r}, \]  

\( (B3b) \)

\[ \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \frac{1}{(p^2 + \Lambda^2)^2} = \frac{e^{-\Lambda r}}{8\pi \Lambda}, \]  

\( (B3c) \)

where \( r \equiv |x| \).

[1] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D 51, 1125 (1995); 55, 5853(E) (1997) [arXiv:hep-ph/9407339].
[2] W. Kwong, P. B. Mackenzie, R. Rosenfeld, and J. L. Rosner, Phys. Rev. D 37, 3210 (1988).
[3] G. T. Bodwin, S. Kim, and D. K. Sinclair, Nucl. Phys. B, Proc. Suppl. 34, 434 (1994); 42, 306 (1995) [arXiv:hep-lat/9412011]; G. T. Bodwin, D. K. Sinclair, and S. Kim, Phys. Rev. Lett. 77, 2376 (1996) [arXiv:hep-lat/9605023]; Int. J. Mod. Phys. A 12, 4019 (1997) [arXiv:hep-ph/9609371]; Phys. Rev. D 65, 054504 (2002) [arXiv:hep-lat/0107011].
[4] M. Gremm and A. Kapustin, Phys. Lett. B 407, 323 (1997) [arXiv:hep-ph/9701353].
[5] E. Braaten and J. Lee, Phys. Rev. D 67, 054007 (2003); 72, 099901(E) (2005) [arXiv:hep-ph/0211085].
[6] N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. B566, 275 (2000) [arXiv:hep-ph/9907240].
[7] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D 17, 3090 (1978); 21, 313(E) (1980).
[8] G. S. Bali, Phys. Rep. 343, 1 (2001) [arXiv:hep-ph/0001312].
[9] G.T. Bodwin, D. Kang, and J. Lee, arXiv:hep-ph/0603185.
[10] S. P. Booth, D. S. Henty, A. Hulsebos, A. C. Irving, C. Michael, and P. W. Stephenson (UKQCD Collaboration), Phys. Lett. B 294, 385 (1992) [arXiv:hep-lat/9209008].
[11] R. Gupta and T. Bhattacharya, Phys. Rev. D 55, 7203 (1997) [arXiv:hep-lat/9605039].
[12] S. Kim and D. K. Sinclair, Phys. Rev. D 48, 4408 (1993).
[13] S. Kim and S. Ohta, Nucl. Phys. B, Proc. Suppl. 53, 199 (1997) [arXiv:hep-lat/9609023].
[14] G. T. Bodwin and Y. Q. Chen, Phys. Rev. D 60, 054008 (1999) [arXiv:hep-ph/9807492].

[15] M. Beneke and V. A. Smirnov, Nucl. Phys. B522, 321 (1998) [arXiv:hep-ph/9711391].