A dichotomy result for closed characteristics on compact star-shaped hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

In this paper, we prove that if all closed characteristics on a compact non-degenerate star-shaped hypersurface $\Sigma$ in $\mathbb{R}^{2n}$ are elliptic, then either there exist exactly $n$ geometrically distinct closed characteristics, or there exist infinitely many geometrically distinct closed characteristics.

Key words: Closed characteristic, star-shaped hypersurface, elliptic, Maslov-type index.

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1 Introduction and main results

Let $\Sigma$ be a $C^3$ compact hypersurface in $\mathbb{R}^{2n}$ strictly star-shaped with respect to the origin, i.e., the tangent hyperplane at any $x \in \Sigma$ does not intersect the origin. We denote the set of all such hypersurfaces by $\mathcal{H}_{st}(2n)$, and denote by $\mathcal{H}_{conv}(2n)$ the subset of $\mathcal{H}_{st}(2n)$ which consists of all strictly convex hypersurfaces. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

$$
\begin{cases}
\dot{y} = J N_\Sigma(y), \\
y(\tau) = y(0),
\end{cases}
(1.1)
$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, $I_n$ is the identity matrix in $\mathbb{R}^n$, $\tau > 0$, $N_\Sigma(y)$ is the outward normal vector of $\Sigma$ at $y$ normalized by the condition $N_\Sigma(y) \cdot y = 1$. Here $a \cdot b$ denotes the standard

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inner product of \(a, b \in \mathbb{R}^{2n}\). A closed characteristic \((\tau, y)\) is \textit{prime}, if \(\tau\) is the minimal period of \(y\). Two closed characteristics \((\tau, y)\) and \((\sigma, z)\) are \textit{geometrically distinct}, if \(y(\mathbb{R}) \neq z(\mathbb{R})\). We denote by \(\mathcal{T}(\Sigma)\) the set of geometrically distinct closed characteristics \((\tau, y)\) on \(\Sigma \in \mathcal{H}_{st}(2n)\). A closed characteristic \((\tau, y)\) is \textit{non-degenerate} if 1 is a Floquet multiplier of \(y\) of precisely algebraic multiplicity 2; \textit{hyperbolic} if 1 is a double Floquet multiplier of it and all the other Floquet multipliers are not on \(U = \{z \in \mathbb{C} \mid |z| = 1\}\), i.e., the unit circle in the complex plane; \textit{elliptic} if all the Floquet multipliers of \(y\) are on \(U\); \textit{irrationally elliptic} if 1 is its double Floquet multipliers, and the other \((2n - 2)\) locating on the unit circle with rotation angles being irrational multiples of \(\pi\). We call a \(\Sigma \in \mathcal{H}_{st}(2n)\) \textit{non-degenerate} if all the closed characteristics on \(\Sigma\) together with all of their iterations are non-degenerate.

There is a long standing conjecture on the number of closed characteristics on compact convex hypersurfaces in \(\mathbb{R}^{2n}\):

\[
\#\mathcal{T}(\Sigma) \geq n, \quad \forall \Sigma \in \mathcal{H}_{con}(2n).
\]

In 1978, P. Rabinowitz in [Rab] proved \(\#\mathcal{T}(\Sigma) \geq 1\) for any \(\Sigma \in \mathcal{H}_{st}(2n)\) and A. Weinstein in [Wei] proved \(\#\mathcal{T}(\Sigma) \geq 1\) for any \(\Sigma \in \mathcal{H}_{con}(2n)\) independently. When \(n \geq 2\), in 1987-1988, I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A. Szulkin (cf. [EkL], [EkH], [Szu]) proved

\[
\#\mathcal{T}(\Sigma) \geq 2, \quad \forall \Sigma \in \mathcal{H}_{con}(2n).
\]

In [HWZ1] of 1998, H. Hofer, K. Wysocki, and E. Zehnder proved \(\#\mathcal{T}(\Sigma) = 2\) or \(\infty\) holds for every \(\Sigma \in \mathcal{H}_{con}(4)\). In [LoZ] of 2002, Y. Long and C. Zhu further proved

\[
\#\mathcal{T}(\Sigma) \geq \left\lfloor \frac{n}{2} \right\rfloor + 1, \quad \forall \Sigma \in \mathcal{H}_{con}(2n).
\]

In particular, if all the prime closed characteristics on \(\Sigma\) are non-degenerate, then \(\#\mathcal{T}(\Sigma) \geq n\), cf. Theorem 1.1 and Corollary 1.1 of [LoZ]. In [WHL] of 2007, W. Wang, X. Hu and Y. Long proved \(\#\mathcal{T}(\Sigma) \geq 3\) for every \(\Sigma \in \mathcal{H}_{con}(6)\). In [Wan1] of 2016, W. Wang proved \(\#\mathcal{T}(\Sigma) \geq \left\lfloor \frac{n+1}{2} \right\rfloor + 1\) for every \(\Sigma \in \mathcal{H}_{con}(2n)\). In [Wan2] of 2016, W. Wang proved \(\#\mathcal{T}(\Sigma) \geq 4\) for every \(\Sigma \in \mathcal{H}_{con}(8)\).

Note that every contact form supporting the standard contact structure on \(M = S^{2n-1}\) arises from embeddings of \(M\) into \(\mathbb{R}^{2n}\) as a strictly star-shaped hypersurface enclosing the origin, it is conjectured that in fact the conjecture (1.2) holds for any \(\Sigma \in \mathcal{H}_{st}(2n)\). For star-shaped case, [Gir] of 1984 and [BLMR] of 1985 show that \(\#\mathcal{T}(\Sigma) \geq n\) for \(\Sigma \in \mathcal{H}_{st}(2n)\) under some pinching conditions. In [HuL] of 2002, X. Hu and Y. Long proved that \(\#\mathcal{T}(\Sigma) \geq 2\) for any non-degenerate \(\Sigma \in \mathcal{H}_{st}(2n)\). In [HWZ2] of 2003, H. Hofer, K. Wysocki, and E. Zehnder proved that \(\#\mathcal{T}(\Sigma) = 2\) or \(\infty\) holds for every non-degenerate \(\Sigma \in \mathcal{H}_{st}(4)\) provided that all stable and unstable manifolds of the hyperbolic closed orbits on \(\Sigma\) intersect transversally, and recently this transversal condition was removed in an important paper by D. Cristofaro-Gardiner, M. Hutchings and D. Pomerleano.
In [CGH] of 2016, D. Cristofaro-Gardiner and M. Hutchings proved that \( \#T(\Sigma) \geq 2 \) for every contact manifold \( \Sigma \) of dimension three. Later various proofs of this result for star-shaped hypersurfaces have been given in [GHHM], [LLo1] and [GiG].

Since the appearance of [HWZ1] and [HWZ2], Hofer, among others, has popularized in many talks the following much more stronger conjecture than (1.2), cf., Conjecture 1.5 of [WHL] and Conjecture 1.1 of [LLo3]:

**Conjecture 1.1.** For every integer \( n \geq 2 \), there holds

\[
\{ \#T(\Sigma) \mid \Sigma \in \mathcal{H}_{st}(2n) \} = \{ n \} \cup \{ +\infty \}.
\]

A typical example is the following weakly non-resonant ellipsoid:

\[
E_n(r) = \left\{ z = (x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n} \mid \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2 + y_i^2}{r_i^2} = 1 \right\},
\]

where \( r_i > 0 \) for \( 1 \leq i \leq n \), we have \( E_n(r) \in \mathcal{H}_{st}(2n) \). In this case, the corresponding Hamiltonian system is linear and all the solutions of (1.1) can be computed explicitly. It is easy to verify that \( \#T(E_n(r)) = n \), and all the closed characteristics on \( E_n(r) \) are irrationally elliptic whenever \( \frac{x_i}{r_j} \notin \mathbb{Q} \) for all \( i \neq j \), and \( \#T(E_n(r)) = +\infty \) otherwise.

Motivated by the dichotomy results of [HWZ1] and [HWZ2], the detailed information about the “exceptional” case of hypersurfaces with exactly two closed characteristics arouses many interest. In [Lon1] of 2000, Long proved that \( \Sigma \in \mathcal{H}_{con}(4) \) and \( \#T(\Sigma) = 2 \) imply that both of the closed characteristics must be elliptic. In [WHL] of 2007, W. Wang, X. Hu and Y. Long proved further that \( \Sigma \in \mathcal{H}_{con}(4) \) and \( \#T(\Sigma) = 2 \) imply that both of the closed characteristics must be irrationally elliptic. In [LLo2] of 2015 and [LLo3] of 2017, H. Liu and Y. Long proved that the existence of exactly two closed characteristics on \( \Sigma \in \mathcal{H}_{st}(4) \) implies that both of them must be irrationally elliptic provided that \( \Sigma \) is symmetric with respect to the origin. Recently, D. Cristofaro-Gardiner, U.L. Hryniewicz, M. Hutchings and H. Liu [CGHHL] removed this symmetric condition and proved a more general result that if there exist exactly two closed orbits on a contact manifold of dimension three, then both orbits are irrationally elliptic and this manifold is diffeomorphic to the three-sphere or a lens space.

In [Vit2] of 1989, C. Viterbo proved the existence of infinitely many closed characteristics on a compact star-shaped hypersurface \( \Sigma \) in \( \mathbb{R}^{4n} \), if all closed characteristics on \( \Sigma \) are hyperbolic. This result has also been obtained by U. L. Hryniewicz and L. Macarini in Corollary 1.8 of [HM].

In this paper, motivated by the above results, we prove Conjecture 1.1 provided that all closed characteristics on \( \Sigma \in \mathcal{H}_{st}(2n) \) are non-degenerate and elliptic.

**Theorem 1.2.** If all closed characteristics on a compact non-degenerate star-shaped hypersurface \( \Sigma \) in \( \mathbb{R}^{2n} \) are elliptic, then either there exist exactly \( n \) geometrically distinct closed characteristics, or there exist infinitely many geometrically distinct closed characteristics.
In this paper, let $N, N_0, Z, Q, R, C$ and $R^+$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, complex numbers and positive real numbers respectively. We define the function $[a] = \max\{k \in Z \mid k \leq a\}$, $\{a\} = a - [a]$, $E(a) = \min\{k \in Z \mid k \geq a\}$ and $\varphi(a) = E(a) - [a]$. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $R^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard $L^2$ inner product and $L^2$ norm.

For technical reasons we further modify the Hamiltonian $\tilde{H}_a$, and define the new Hamiltonian $\tilde{H}_a(x) = a\varphi_a(j(x))$ via Lemma 2.2 and Lemma 2.4 in [LLW]. The precise dependence of $\varphi_a$ on $a$ is explained in Remark 2.3 of [LLW].

For technical reasons we further modify the Hamiltonian $\tilde{H}_a$, and define the new Hamiltonian function $H_a$ via Proposition 2.5 of [LLW] and consider the fixed period problem

$$\dot{x}(t) = JH'_a(x(t)), \quad x(0) = x(T). \quad (2.1)$$

Then $H_a \in C^3(R^{2n} \setminus \{0\}, R) \cap C^1(R^{2n}, R)$. Solutions of (2.1) are $x \equiv 0$ and $x = \rho y(\tau t/T)$ with $\frac{\varphi'_\rho(\rho)}{\rho} = \frac{\tau}{aT}$, where $(\tau, y)$ is a solution of (1.1). In particular, non-zero solutions of (2.1) are in one to one correspondence with solutions of (1.1) with period $\tau < aT$.

For any $a > \hat{a}$, we can choose some large constant $K = K(a)$ such that

$$H_{a,K}(x) = H_a(x) + \frac{1}{2}K|x|^2 \quad (2.2)$$

is a strictly convex function, that is,

$$\langle \nabla H_{a,K}(x) - \nabla H_{a,K}(y), x - y \rangle \geq \frac{\epsilon}{2}|x - y|^2, \quad (2.3)$$
for all \( x, y \in \mathbb{R}^{2n} \), and some positive \( \epsilon \). Let \( H^*_{a,K} \) be the Fenchel dual of \( H_{a,K} \) defined by

\[
H^*_{a,K}(y) = \sup\{ x \cdot y - H_{a,K}(x) \mid x \in \mathbb{R}^{2n} \}.
\]

The dual action functional on \( X = W^{1,2}(\mathbb{R}/TZ, \mathbb{R}^{2n}) \) is defined by

\[
F_{a,K}(x) = \int_0^T \left[ \frac{1}{2}(J\dot{x} - Kx, x) + H^*_{a,K}(-J\dot{x} + Kx) \right] dt. \tag{2.4}
\]

Then \( F_{a,K} \in C^{1,1}(X, \mathbb{R}) \) and for \( KT \not\in 2\pi \mathbb{Z} \), \( F_{a,K} \) satisfies the Palais-Smale condition and \( x \) is a critical point of \( F_{a,K} \) if and only if it is a solution of (2.1). Moreover, \( F_{a,K}(x_a) < 0 \) and it is independent of \( K \) for every critical point \( x_a \neq 0 \) of \( F_{a,K} \).

When \( KT \not\in 2\pi \mathbb{Z} \), the map \( x \mapsto -J\dot{x} + Kx \) is a Hilbert space isomorphism between \( X = W^{1,2}(\mathbb{R}/(TZ); \mathbb{R}^{2n}) \) and \( E = L^2(\mathbb{R}/(TZ), \mathbb{R}^{2n}) \). We denote its inverse by \( M_K \) and the functional

\[
\Psi_{a,K}(u) = \int_0^T \left[ -\frac{1}{2}(M_K u, u) + H^*_{a,K}(u) \right] dt, \quad \forall u \in E. \tag{2.5}
\]

Then \( x \in X \) is a critical point of \( F_{a,K} \) if and only if \( u = -J\dot{x} + Kx \) is a critical point of \( \Psi_{a,K} \).

Suppose \( u \) is a nonzero critical point of \( \Psi_{a,K} \). Then the formal Hessian of \( \Psi_{a,K} \) at \( u \) is defined by

\[
Q_{a,K}(v) = \int_0^T (-M_K v \cdot u + H''_{a,K}(u)v \cdot v) dt, \tag{2.6}
\]

which defines an orthogonal splitting \( E = E_- \oplus E_0 \oplus E_+ \) of \( E \) into negative, zero and positive subspaces. The index and nullity of \( u \) are defined by \( i_K(u) = \dim E_- \) and \( \nu_K(u) = \dim E_0 \) respectively. Similarly, we define the index and nullity of \( x = M_K u \) for \( F_{a,K} \), we denote them by \( i_K(x) \) and \( \nu_K(x) \). Then we have

\[
i_K(u) = i_K(x), \quad \nu_K(u) = \nu_K(x), \tag{2.7}
\]

which follow from the definitions (2.4) and (2.5). The following important formula was proved in Lemma 6.4 of [Vit2]:

\[
i_K(x) = 2n([KT/2\pi] + 1) + \nu^v(x) \equiv d(K) + \nu^v(x), \tag{2.8}
\]

where the Viterbo index \( \nu^v(x) \) does not depend on \( K \), but only on \( H_a \).

By the proof of Proposition 2 of [Vit1], we have that \( v \in E \) belongs to the null space of \( Q_{a,K} \) if and only if \( z = M_K v \) is a solution of the linearized system

\[
\dot{z}(t) = JH''_{a,K}(x(t))z(t). \tag{2.9}
\]

Thus the nullity in (2.7) is independent of \( K \), which we denote by \( \nu^v(x) \equiv \nu_K(u) = \nu_K(x) \).
By Proposition 2.11 of [LLW], the index \(i^v(x)\) and nullity \(\nu^v(x)\) coincide with those defined for the Hamiltonian \(H(x) = j(x)^\alpha\) for all \(x \in \mathbb{R}^{2n}\) and some \(\alpha \in (1, 2)\). Especially \(1 \leq \nu^v(x) \leq 2n - 1\) always holds.

For every closed characteristic \((\tau, y)\) on \(\Sigma\), let \(aT > \tau\) and choose \(\varphi_a\) as above. Determine \(\rho\) uniquely by \(\frac{\varphi_a'(\rho)}{\rho} = \frac{\tau}{aT}\). Let \(x = \rho y(\frac{\tau}{aT})\). Then we define the index \(i(\tau, y)\) and nullity \(\nu(\tau, y)\) of \((\tau, y)\) by

\[
i(\tau, y) = i^v(x), \quad \nu(\tau, y) = \nu^v(x).
\]

Then the mean index of \((\tau, y)\) is defined by

\[
i(\tau, y) = \lim_{m \to \infty} \frac{i(m\tau, y)}{m}.
\]

Note that by Proposition 2.11 of [LLW], the index and nullity are well defined and are independent of the choice of \(a\). For a closed characteristic \((\tau, y)\) on \(\Sigma\), we simply denote by \(y^m \equiv (m\tau, y)\) the \(m\)-th iteration of \(y\) for \(m \in \mathbb{N}\).

We have a natural \(S^1\)-action on \(X\) or \(E\) defined by

\[
\theta \cdot u(t) = u(\theta + t), \quad \forall \theta \in S^1, \ t \in \mathbb{R}.
\]

Clearly both of \(F_{a,K}\) and \(\Psi_{a,K}\) are \(S^1\)-invariant. For any \(\kappa \in \mathbb{R}\), we denote by

\[
\Lambda_{a,K}^\kappa = \{u \in L^2(\mathbb{R}/(T\mathbb{Z}); \mathbb{R}^{2n}) \mid \Psi_{a,K}(u) \leq \kappa\}
\]

\[
X_{a,K}^\kappa = \{x \in W^{1,2}(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^{2n}) \mid F_{a,K}(x) \leq \kappa\}.
\]

For a critical point \(u\) of \(\Psi_{a,K}\) and the corresponding \(x = M_Ku\) of \(F_{a,K}\), let

\[
\Lambda_{a,K}(u) = \Lambda_{a,K}^{\Psi_{a,K}(u)} = \{w \in L^2(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^{2n}) \mid \Psi_{a,K}(w) \leq \Psi_{a,K}(u)\},
\]

\[
X_{a,K}(x) = X_{a,K}^{F_{a,K}(x)} = \{y \in W^{1,2}(\mathbb{R}/(T\mathbb{Z}), \mathbb{R}^{2n}) \mid F_{a,K}(y) \leq F_{a,K}(x)\}.
\]

Clearly, both sets are \(S^1\)-invariant. Denote by \(\text{crit}(\Psi_{a,K})\) the set of critical points of \(\Psi_{a,K}\). Because \(\Psi_{a,K}\) is \(S^1\)-invariant, \(S^1 \cdot u\) becomes a critical orbit if \(u \in \text{crit}(\Psi_{a,K})\). Note that by the condition \((F)\), the number of critical orbits of \(\Psi_{a,K}\) is finite. Hence as usual we can make the following definition.

**Definition 2.1.** Suppose \(u\) is a nonzero critical point of \(\Psi_{a,K}\), and \(\mathcal{N}\) is an \(S^1\)-invariant open neighborhood of \(S^1 \cdot u\) such that \(\text{crit}(\Psi_{a,K}) \cap (\Lambda_{a,K}(u) \cap \mathcal{N}) = S^1 \cdot u\). Then the \(S^1\)-critical module of \(S^1 \cdot u\) is defined by

\[
C^{S^1}_{a,K}(\Psi_{a,K}, S^1 \cdot u) = H_q((\Lambda_{a,K}(u) \cap \mathcal{N})_{S^1}, ((\Lambda_{a,K}(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}).
\]

Similarly, we define the \(S^1\)-critical module \(C^{S^1}_{a,K}(F_{a,K}, S^1 \cdot x)\) of \(S^1 \cdot x\) for \(F_{a,K}\).
We fix $a$ and let $u_K \neq 0$ be a critical point of $\Psi_{a,K}$ with multiplicity $\text{mul}(u_K) = m$, that is, $u_K$ corresponds to a closed characteristic $(\tau, y) \subset \Sigma$ with $(\tau, y)$ being $m$-iteration of some prime closed characteristic. Precisely, we have $u_K = -J\dot{x} + Kx$ with $x$ being a solution of (2.1) and $x = \rho y(t\tau)$ with $\rho'(\rho) = -\frac{T}{\rho}$. Moreover, $(\tau, y)$ is a closed characteristic on $\Sigma$ with minimal period $\frac{\tau}{m}$. For any $p \in \mathbb{N}$ satisfying $p\tau < aT$, we choose $K$ such that $pK \notin \frac{2\pi}{\tau} \mathbb{Z}$, then the $p$th iteration $u^p_{pK}$ of $u_K$ is given by $-J\dot{x}^p + pKx^p$, where $x^p$ is the unique solution of (2.1) corresponding to $(p\tau, y)$ and is a critical point of $F_{a,pK}$, that is, $u^p_{pK}$ is the critical point of $\Psi_{a,pK}$ corresponding to $x^p$.

**Lemma 2.2.** (cf. Proposition 4.2 and Remark 4.4 of [LLW]) If $u^p_{pK}$ is non-degenerate, i.e., $\nu^p_{pK}(u^p_{pK}) = 1$, let $\beta(x^p) = (-1)^{i_K(u^p_{pK})-i_K(u_K)} = (-1)^{i^p(x^p)-i(x)}$, then

$$C_{S^1,q-d(pK)+d(K)}(F_{a,K}, S^1 \cdot x^p) = C_{S^1,q}(F_{a,pK}, S^1 \cdot x^p) = C_{S^1,q}(\Psi_{a,pK}, S^1 \cdot u^p_{pK})$$

$$= \begin{cases} Q, & \text{if } q = i_{pK}(u^p_{pK}), \text{ and } \beta(x^p) = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.10)$$

**Theorem 2.3.** (cf. Theorem 1.1 of [LLW] and Theorem 1.2 of [Vit2]) Suppose that $\Sigma \in \mathcal{H}_d(2n)$ satisfying $\#T(\Sigma) < +\infty$. Denote by $\{(\tau_j, y_j)\}_{1 \leq j \leq k}$ all the geometrically distinct prime closed characteristics. Then the following identities hold

$$\sum_{1 \leq j \leq k \atop i(y_j) > 0} \frac{\hat{x}(y_j)}{i(y_j)} = \frac{1}{2}, \quad \sum_{1 \leq j \leq k \atop i(y_j) < 0} \frac{\hat{x}(y_j)}{i(y_j)} = 0, \quad (2.11)$$

where $\hat{x}(y) \in Q$ is the average Euler characteristic given by Definition 4.8 and Remark 4.9 of [LLW].

In particular, if all $y^m$’s are non-degenerate for $m \geq 1$, then

$$\hat{x}(y) = \begin{cases} (-1)^{i(y)}, & \text{if } i(y^2) - i(y) \in 2\mathbb{Z}, \\ \frac{(-1)^{i(y)}}{2}, & \text{otherwise.} \end{cases} \quad (2.12)$$

Let $F_{a,K}$ be a functional defined by (2.4) for some $a, K \in \mathbb{R}$ large enough and let $\epsilon > 0$ be small enough such that $[-\epsilon, 0)$ contains no critical values of $F_{a,K}$. For $b$ large enough, The normalized Morse series of $F_{a,K}$ in $X^{-\epsilon} \setminus X^{-b}$ is defined, as usual, by

$$M_a(t) = \sum_{q \geq 0, 1 \leq j \leq p} \dim C_{S^1,q}(F_{a,K}, S^1 \cdot v_j)t^{q-d(K)}, \quad (2.13)$$

where we denote by $\{S^1 \cdot v_1, \ldots, S^1 \cdot v_p\}$ the critical orbits of $F_{a,K}$ with critical values less than $-\epsilon$. The Poincaré series of $H_{S^1,*}(X, X^{-\epsilon})$ is $t^{d(K)}Q_a(t)$, according to Theorem 5.1 of [LLW], if we set $Q_a(t) = \sum_{k \in \mathbb{Z}} q_k t^k$, then

$$q_k = 0, \quad \forall k \in \hat{I},$$
where $I$ is an interval of $\mathbb{Z}$ such that $I \cap [i(\tau, y), i(\tau, y) + \nu(\tau, y) - 1] = \emptyset$ for all closed characteristics $(\tau, y)$ on $\Sigma$ with $\tau \geq aT$. Then by Section 6 of [LLW], we have

$$M_a(t) - \frac{1}{1-t^2} + Q_a(t) = (1 + t)U_a(t),$$

where $U_a(t) = \sum_{i \in \mathbb{Z}} u_it^i$ is a Laurent series with nonnegative coefficients. If there is no closed characteristic with $i = 0$, then

$$M(t) - \frac{1}{1-t^2} = (1 + t)U(t),$$

where $M(t) = \sum_{p \in \mathbb{Z}} M_pt^p$ denotes $M_a(t)$ as $a$ tends to infinity. In addition, we also denote by $b_p$ the coefficient of $t^p$ of $\frac{1}{1-t^2} = \sum_{p \in \mathbb{Z}} b_pt^p$, i.e. there holds $b_p = 1$, $\forall p \in 2\mathbb{N}_0$ and $b_p = 0$, $\forall p \notin 2\mathbb{N}_0$.

3 The Maslov-type index theory of symplectic paths

In [Lon1] of 1999, Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result he further established the precise iteration formulae of indices of symplectic paths in [Lon2] of 2000.

As in [Lon2], denote by

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \text{for } \lambda = \pm 1, \ b \in \mathbb{R},$$

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{for } \lambda \in \mathbb{R} \setminus \{0, \pm 1\},$$

$$R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi),$$

$$N_2(e^{\theta \sqrt{-1}}, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \text{for } \theta \in (0, \pi) \cup (\pi, 2\pi) \text{ and }$$

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ with } b_j \in \mathbb{R}, \quad \text{and } b_2 \neq b_3.$$}

Here $N_2(e^{\theta \sqrt{-1}}, B)$ is non-trivial if $(b_2 - b_3) \sin \theta < 0$, and trivial if $(b_2 - b_3) \sin \theta > 0$.

As in [Lon2], the $\circ$-sum (direct sum) of any two real matrices is defined by

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i \times 2i} \circ \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j \times 2j} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$}

For every $M \in \text{Sp}(2n)$, the homotopy set $\Omega(M)$ of $M$ in $\text{Sp}(2n)$ is defined by

$$\Omega(M) = \{N \in \text{Sp}(2n) | \sigma(N) \cap U = \sigma(M) \cap U \equiv \Gamma \text{ and } \nu_\omega(N) = \nu_\omega(M) \forall \omega \in \Gamma\}.$$
where \( \sigma(M) \) denotes the spectrum of \( M \), \( \nu_\omega(M) \equiv \dim \ker_{\mathbb{C}}(M - \omega I) \) for \( \omega \in U \). The component \( \Omega^0(M) \) of \( P \) in \( \text{Sp}(2n) \) is defined by the path connected component of \( \Omega(M) \) containing \( M \).

**Lemma 3.1.** (cf. [Lon2], Lemma 9.1.5 and List 9.1.12 of [Lon3]) For \( M \in \text{Sp}(2n) \) and \( \omega \in U \), the splitting number \( S^\pm_M(\omega) \) (cf. Definition 9.1.4 of [Lon3]) satisfies

\[
S^\pm_M(\omega) = 0, \quad \text{if } \omega \notin \sigma(M).
\]

\[
S^+_M(1, a) = \begin{cases} 1, & \text{if } a \geq 0, \\ 0, & \text{if } a < 0. \end{cases}
\]

For any \( M_i \in \text{Sp}(2n) \) with \( i = 0 \) and \( 1 \), there holds

\[
S^+_M(\omega) = S^+_M(\omega) + S^+_M(\omega), \quad \forall \omega \in U.
\]

For every \( \gamma \in \mathcal{P}_\tau(2n) \equiv \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \} \), we extend \( \gamma(t) \) to \( t \in [0, m\tau] \) for every \( m \in \mathbb{N} \) by

\[
\gamma^m(t) = \gamma(t - j\tau)\gamma^{(\tau)} \quad \forall j\tau \leq t \leq (j + 1)\tau \quad \text{and} \quad j = 0, 1, \ldots, m - 1,
\]

as in P.114 of [Lon1]. As in [LoZ] and [Lon3], we denote the Maslov-type indices of \( \gamma^m \) by \((i(\gamma, m), \nu(\gamma, m))\).

The following is the precise index iteration formulae for symplectic paths, which is due to Y. Long (cf. Chapter 8 of [Lon3] or Theorems 6.5 and 6.7 of [LoZ]).

**Theorem 3.2.** Let \( \gamma \in \mathcal{P}_\tau(2n) \). Then there exists a path \( f \in C([0, 1], \Omega^0(\gamma(\tau))) \) such that \( f(0) = \gamma(\tau) \) and

\[
f(1) = N_1(1, 1)^{p-} \circ I_{2p_0} \circ N_1(1, -1)^{p+} \circ (-I_{2q_0}) \circ N_1(1, -1)^{q-} \circ N_2(\omega_1, u_1) \circ \cdots \circ N_2(\omega_{r_*}, u_{r_*}) \circ M_0
\]

where \( N_2(\omega, u) \)s are non-trivial and \( N_2(\lambda, v) \)s are trivial basic normal forms; \( \sigma(M_0) \cap U = \emptyset \); \( p-, p_0, p+, q-, q_0, q_+ \), \( r \), \( r_* \) and \( r_0 \) are non-negative integers; \( \omega_j = e^{\sqrt{-1}\alpha_j} \), \( \lambda_j = e^{\sqrt{-1}\beta_j} \); \( \theta_j \), \( \alpha_j \), \( \beta_j \) \( \in (0, \pi) \cup (\pi, 2\pi) \); these integers and real numbers are uniquely determined by \( \gamma(\tau) \). Then using the functions defined in \((1.2)\), we have

\[
i(\gamma, m) = m(i(\gamma, 1) + p_- + p_0 - r) + 2 \sum_{j=1}^{r} E \left( \frac{m\theta_j}{2\pi} \right) - r - p_- - p_0
\]

\[
- \frac{1 + (-1)^m}{2} (q_0 + q_+) + 2 \sum_{j=1}^{r} \varphi \left( \frac{m\alpha_j}{2\pi} \right) - r_*
\]

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\[ \nu(\gamma, m) = \nu(\gamma, 1) + \frac{1 + (-1)^m}{2}(q_- + 2q_0 + q_+) + 2(r + r_s + r_0) \]

\[ -2 \left( \sum_{j=1}^{r} \varphi \left( \frac{m\theta_j}{2\pi} \right) + \sum_{j=1}^{r_s} \varphi \left( \frac{m\alpha_j}{2\pi} \right) + \sum_{j=1}^{r_0} \varphi \left( \frac{m\beta_j}{2\pi} \right) \right), \]

\[ \hat{i}(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi}, \]

where \( N_1(1, \pm 1) = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \) \( N_1(-1, \pm 1) = \begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}, \) \( R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \), \( N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix} \) with some \( \theta \in (0, \pi) \cup (\pi, 2\pi) \) and \( b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \), such that \((b_2 - b_3) \sin \theta > 0, \) if \( N_2(\omega, b) \) is trivial; \((b_2 - b_3) \sin \theta < 0, \) if \( N_2(\omega, b) \) is non-trivial. We have \( i(\gamma, 1) \) is odd if \( f(1) = N_1(1, 1), I_2, N_1(-1, 1), -I_2, N_1(-1, -1) \) and \( R(\theta); \) \( i(\gamma, 1) \) is even if \( f(1) = N_1(1, -1) \) and \( N_2(\omega, b); \) \( i(\gamma, 1) \) can be any integer if \( \sigma(f(1)) \cap U = \emptyset. \)

The common index jump theorem (cf. Theorem 4.3 of [LoZ]) for symplectic paths established by Long and Zhu in 2002 has become one of the main tools to study the multiplicity and stability problems of closed solution orbits in Hamiltonian and symplectic dynamics. Recently, the following enhanced version of it has been obtained by Duan, Long and Wang in [DLW], which will play an important role in the proof in Section 4.

**Theorem 3.3.** (cf. Theorem 3.5 of [DLW]) \((\text{The enhanced common index jump theorem for symplectic paths})\) Let \( \gamma_k \in \mathcal{P}_{\gamma_k}(2n) \) for \( k = 1, \cdots, q \) be a finite collection of symplectic paths. Let \( M_k = \gamma_k(\tau_k). \) We extend \( \gamma_k \) to \([0, +\infty)\) by (3.1) inductively. Suppose

\[ \hat{i}(\gamma_k, 1) > 0, \quad \forall k = 1, \cdots, q. \]

Then for every integer \( \bar{m} \in \mathbb{N}, \) there exist infinitely many \((q + 1)\)-tuples \((N, m_1, \cdots, m_q) \in \mathbb{N}^{q+1}\) such that for all \( 1 \leq k \leq q \) and \( 1 \leq m \leq \bar{m}, \) there holds

\[ \nu(\gamma_k, 2m_k - m) = \nu(\gamma_k, 2m_k + m) = \nu(\gamma_k, m), \]

\[ i(\gamma_k, 2m_k + m) = 2N + i(\gamma_k, m), \]

\[ i(\gamma_k, 2m_k - m) = 2N - i(\gamma_k, m) - 2(S^+_{M_k}(1) + Q_k(m)), \]

\[ i(\gamma_k, 2m_k) = 2N - (S^+_{M_k}(1) + C(M_k) - 2\Delta), \]

where \( C(M_k) = \sum_{0 < \theta < 2\pi} S^-_{M_k}(e^{\vartheta \cdot \theta}) \) and

\[ \Delta_k = \sum_{0 < \theta < 2\pi} S^-_{M_k}(e^{\vartheta \cdot \theta}), \quad Q_k(m) = \sum_{e^{\vartheta \cdot \theta} \in \sigma(M_k)} S^-_{M_k}(e^{\vartheta \cdot \theta}). \]

The following is the relation between Viterbo index and Maslov-type index of symplectic path.
Theorem 3.4. (cf. Theorem 2.1 of [HuL] and Theorem 6.1 of [LLo2]) Suppose \( \Sigma \in H_{st}(2n) \) and \((\tau, y) \in T(\Sigma)\). Then we have
\[
i(y^m) \equiv i(m \tau, y) = i(y, m) - n, \quad \nu(y^m) \equiv \nu(m \tau, y) = \nu(y, m), \quad \forall m \in \mathbb{N},
\]
where \( i(y^m) \) and \( \nu(y^m) \) are the index and nullity of \((m \tau, y)\) defined in Section 2, \( i(y, m) \) and \( \nu(y, m) \) are the Maslov-type index and nullity of \((m \tau, y)\) (cf. Section 5.4 of [Lon2]). In particular, we have \( \hat{i}(\tau, y) = \hat{i}(y, 1) \), where \( \hat{i}(\tau, y) \) is given in Section 2, \( \hat{i}(y, 1) \) is the mean Maslov-type index (cf. Definition 8.1 of [Lon3]). Hence we denote it simply by \( \hat{i}(y) \).

4 Proof of Theorem 1.2

In order to prove Theorem 1.2, we make the following assumption

\[(ECC) \text{ Suppose that all prime closed characteristics on a compact non-degenerate star-shaped hypersurface } \Sigma \text{ in } \mathbb{R}^{2n} \text{ are elliptic, and the total number of distinct closed characteristics on } \Sigma \text{ is finite, denoted by } \{(\tau_k, y_k)\}_{k=1}^q.\]

We denote by \( \gamma_k \equiv \gamma_{y_k} \) the associated symplectic path of \((\tau_k, y_k)\) for \(1 \leq k \leq q\) in the assumption \((ECC)\). Then by Lemma 3.3 of [HuL], there exists \( P_k \in Sp(2n) \) and \( U_k \in Sp(2n - 2) \) such that
\[
M_k \equiv \gamma_k(\tau_k) = P_k^{-1}(N_1(1, 1) \circ U_k)P_k, \quad 1 \leq k \leq q,
\]
where every \( U_k \) has the following form:
\[
U_k = R(\theta_k^1) \circ \cdots \circ R(\theta_k^r_k) \\
\circ N_2(e^{\alpha_j^k \sqrt{-1}}, A_1^k) \circ \cdots \circ N_2(e^{\alpha_j^k \sqrt{-1}}, A_j^k) \circ N_2(e^{\beta_j^k \sqrt{-1}}, B_1^k) \circ \cdots \circ N_2(e^{\beta_j^k \sqrt{-1}}, B_j^k),
\]
where \( \frac{\theta_k^j}{2\pi} \in (0, 1) \backslash \mathbb{Q} \) for \(1 \leq j \leq r_k^1, \frac{\alpha_k^j}{2\pi} \in (0, 1) \backslash \mathbb{Q} \) for \(1 \leq j \leq r_k^1, \frac{\beta_k^j}{2\pi} \in (0, 1) \backslash \mathbb{Q} \) for \(1 \leq j \leq r_k^1, \]
and
\[
r_k + 2r_k^1 + 2r_k^0 = n - 1. \tag{4.2}
\]
By Lemma 3.1 and Theorem 3.2 we obtain the index iteration formula of \( y_k^m \)
\[
i(y_k, m) = m(i(y_k, 1) + 1 - r_k) + 2 \sum_{j=1}^{r_k} \left\lfloor \frac{m \theta_k^j}{2\pi} \right\rfloor + r_k - 1, \quad \forall 1 \leq k \leq q, \quad m \geq 1. \tag{4.3}
\]
Therefore by Theorem 3.4 it yields
\[
i(y_k^m) = m(i(y_k) + n + 1 - r_k) + 2 \sum_{j=1}^{r_k} \left\lfloor \frac{m \theta_k^j}{2\pi} \right\rfloor + r_k - n - 1, \quad \forall 1 \leq k \leq q, \quad m \geq 1. \tag{4.4}
\]
which implies
\[
\hat{i}(y_k, 1) = \hat{i}(y_k) = i(y_k) + n + 1 - r_k + \sum_{j=1}^{r_k} \frac{\theta_j^k}{\pi}, \quad \forall \ 1 \leq k \leq q, \quad (4.5)
\]

**Claim 1.** For any $1 \leq k \leq q$ and $m \geq 1$, there holds $i(y_k^m) \in 2\mathbb{Z}$.

In fact, firstly by Theorem 3.2, every path $\gamma \in \mathcal{P}_r(2)$ with its end matrix homotopic to $N_1(1, 1)$ and $R(\theta)$ has odd index $i(\gamma, 1)$, and every path $\gamma \in \mathcal{P}_r(4)$ with its end matrix homotopic to $N_2(\omega, B)$ has even indices $i(\gamma, 1)$. Then by (4.1) and the homotopy invariance and the symplectic additivity of Maslov-type indices (cf. Theorem 6.2.7 of [Lon3]), for every closed characteristic $y_k, 1 \leq k \leq q$, $i(y_k, 1)$ has the same parity as $n$. Thus by Theorem 3.4, it yields
\[
i(y_k) = i(y_k, 1) - n = 0 \pmod{2}, \quad \forall \ 1 \leq k \leq q. \quad (4.6)
\]

By (4.2), (4.4) and (4.6), we get
\[
i(y_k^{m+1}) - i(y_k^m) = (i(y_k) + n + 1 - r_k) + 2 \sum_{j=1}^{r_k} \left[ \frac{(m + 1)\theta_j^k}{2\pi} \right] - 2 \sum_{j=1}^{r_k} \left[ \frac{m\theta_j^k}{2\pi} \right] = i(y_k) + (n + 1) + [2r_k^+ + 2r_k^- - (n - 1)] \pmod{2} = 0 \pmod{2}, \quad \forall \ 1 \leq k \leq q, \quad m \geq 1. \quad (4.7)
\]

Now (4.6) and (4.7) complete the proof of Claim 1.

**Claim 2.** Under the assumption (ECC), there holds $\hat{i}(y_k, 1) = \hat{i}(y_k) > 0, \forall \ 1 \leq k \leq q$.

At first we prove that $\hat{i}(y_k) \geq 0, \forall \ 1 \leq k \leq q$. In fact, assume that there exists some $q_0 \geq 1$ such that $\hat{i}(y_k) < 0$ for $1 \leq k \leq q_0$ and $\hat{i}(y_k) \geq 0$ for $q_0 + 1 \leq k \leq q$, then by the second identity in (2.11) of Theorem 2.3, we have
\[
\sum_{1 \leq k \leq q_0} \frac{\hat{\chi}(y_k)}{i(y_k)} = \sum_{1 \leq k \leq q_0} \frac{1}{i(y_k)} = 0, \quad (4.8)
\]
where, in the first equality, $\hat{\chi}(y_k) = (-1)^i(y_k) = 1$ by (2.12) and Claim 1. This is a contradiction.

Now we prove $\hat{i}(y_k) \neq 0, \forall \ 1 \leq k \leq q$. Otherwise, we assume that there exists some $q_0 \geq 1$ such that $\hat{i}(y_k) = 0$ for $1 \leq k \leq q_0$ and $\hat{i}(y_k) > 0$ for $q_0 + 1 \leq k \leq q$. By (4.5) we have
\[
\sum_{j=1}^{r_k} \frac{m\theta_j^k}{2\pi} = \frac{m}{2} (r_k - i(y_k) - n - 1), \quad \forall \ 1 \leq k \leq q_0, \quad m \geq 1, \quad (4.9)
\]
which implies
\[
m(r_k - i(y_k) - n - 1) - 2r_k \leq 2 \sum_{j=1}^{r_k} \left[ \frac{m\theta_j^k}{2\pi} \right] \leq m(r_k - i(y_k) - n - 1), \quad \forall \ 1 \leq k \leq q_0, \quad m \geq 1. \quad (4.10)
\]
Therefore, by (4.4), (4.10) and \( r_k \leq n - 1 \), it yields

\[-2n \leq -r_k - n - 1 \leq i(y_k^{m}) \leq r_k - n - 1 \leq -2, \quad \forall \ 1 \leq k \leq q_0, \ m \geq 1, \quad (4.11)\]

i.e., \( \{ i(y_k^{m}) \mid 1 \leq k \leq q_0, \ m \in \mathbb{N} \} \) is a bounded set. So, for each \( 1 \leq k \leq q_0 \), it follows from Claim 1 that there exists an even integer \( 2l_k \in [-2n, -2] \) such that there exist infinitely many \( m_k \in \mathbb{N} \) satisfying

\[ i(y_k^{m_k}) = 2l_k, \quad \forall \ 1 \leq k \leq q_0. \quad (4.12) \]

Now we can follow an argument in section 9 of [Vit2], for reader’s conveniences we sketch the ideas below.

For large enough \( a \in \mathbb{R} \), by Claim 1 and (4.12), all the closed characteristics \( y_k^{m} \) for \( 1 \leq k \leq q \) with period larger than \( aT \) will have their index: either (i) equal to \( 2l_k \), or (ii) different from \( 2l_k + 1 \) and \( 2l_k - 1 \). Let \( X^- (a, K) = \{ x \in X \mid F_{a,K} (x) < 0 \} \) as defined in Section 7 of [Vit2]. For arbitrarily large enough \( a, a' \) with \( a < a' \), consider the exact sequence of the triple \( (X, X^- (a', K), X^- (a, K)) \) as follows

\[ \rightarrow H_{S^1, d(K) + 2l_1} (X, X^- (a, K)) \xrightarrow{i_*} H_{S^1, d(K) + 2l_1} (X, X^- (a', K)) \xrightarrow{\partial_*} H_{S^1, d(K) + 2l_1 - 1} (X^- (a', K), X^- (a, K)) \rightarrow . \quad (4.13) \]

We claim the homomorphism \( i_* \) in (4.13) is nonzero. In fact, on one hand, since for large enough \( a < a' \), there is no any closed characteristic with index \( 2l_k - 1 \) and period locating between \( aT \) and \( a'T \), so by (7.25) of [Vit2] and its proof, there holds \( H_{S^1, d(K) + 2l_1 - 1} (X^- (a', K), X^- (a, K)) = 0 \), which, together with (4.13), implies that the homomorphism \( i_* \) is surjective.

On the other hand, there holds \( H_{S^1, d(K) + 2l_1} (X, X^- (a', K)) \neq 0 \). In fact, since when \( a \) increases, by (4.12) we always meet infinitely many closed characteristics of type (i) due to the existence of at least \( y_1 \) with \( i(y_1) = 0 \). Such closed characteristics \( y_1^{m_1} \) either contribute to \( H_{S^1, d(K) + 2l_1} (X, X^- (a, K)) \) or kill the homology class of dimension \( d(K) + 2l_1 + 1 \). But the latter is zero, because there is no any closed characteristic with index \( d(K) + 2l_1 + 1 \) by Claim 1, so infinitely many closed characteristics of type (i) must actually contribute to \( H_{S^1, d(K) + 2l_1} (X, X^- (a, K)) \), which is then nonzero. By the arbitrariness of \( a \) and \( a' \), we obtain \( H_{S^1, d(K) + 2l_1} (X, X^- (a', K)) \neq 0 \).

In summary, it is proved that \( i_* \) is surjective and \( H_{S^1, d(K) + 2l_1} (X, X^- (a', K)) \neq 0 \). So \( i_* \) is nonzero. However, in Step 2 of the proof of Theorem 7.1 in [Vit2], it is shown that \( i_* \) in (4.13) is zero. This contradiction completes the proof of Claim 2.

**Claim 3.** Under the assumption (ECC), there holds \( M_{2j} = b_{2j} = 1, M_{2j+1} = b_{2j+1} = 0, \forall \ j \in \mathbb{N}_0, \) and \( M_p = b_p = 0, \forall \ p \leq -1. \)
In fact, by (2.14) and \( b_p = 0, \forall p \leq -1 \), we have the following Morse inequalities
\[
M_p \geq b_p, \hspace{1cm} (4.14)
\]
\[
M_p - M_{p-1} + \cdots + (-1)^p M_0 + \cdots \geq b_p - b_{p-1} + \cdots + (-1)^p b_0, \hspace{0.5cm} \forall p \in \mathbb{Z}. \hspace{1cm} (4.15)
\]
Note that the set \( \{ i(y_k^m) | 1 \leq k \leq q, m \in \mathbb{N} \} \) is bounded below since the mean index \( \hat{i}(y_k) > 0 \) for any \( 1 \leq k \leq q \) by Claim 2. So the alternative sum \( (-1)^p \sum_{j=-\infty}^p (-1)^j M_j \) in the left side of (4.15) is a finite sum by (2.13)-(2.14) and Lemma 2.2.

Since Claim 1 implies \( i(y_k^m) - i(y_k) \in 2\mathbb{Z}, \forall m \geq 1 \) and \( 1 \leq k \leq q \), by Lemma 2.2 we have
\[
M_{2j} = \sum_{k=1}^q \# \{ m \geq 1 | i(y_k^m) = 2j \}, \hspace{0.5cm} M_{2j+1} = 0, \hspace{0.5cm} \forall j \in \mathbb{Z}. \hspace{1cm} (4.16)
\]
Thus it follows from \( M_{2j+1} = b_{2j+1} = 0 \) and (4.15) with \( p = 2j + 1 \) that
\[
M_{2j} - M_{2j-1} + \cdots - M_1 + M_0 - \cdots \leq b_{2j} - b_{2j-1} + \cdots - b_1 + b_0, \hspace{1cm} (4.17)
\]
which, together with (4.15) for \( p = 2j \), implies that the equality in (4.17) holds. Therefore by an induction argument we obtain
\[
M_p = b_p, \hspace{0.5cm} \forall p \in \mathbb{Z}, \hspace{1cm} (4.18)
\]
which, together with \( b_{2j} = 1, b_{2j+1} = 0 \) for \( j \in \mathbb{N}_0 \) and \( b_p = 0 \) for \( p \leq -1 \), yields Claim 3.

**Claim 4.** Under the assumptions (ECC), there holds \( i(y_k) \geq 0 \) and \( i(y_k^{m+1}) \geq i(y_k^m) + 2, \forall 1 \leq k \leq q, \forall m \geq 1 \).

In fact, assume that there exist some \( 1 \leq k_0 \leq q \) such that \( i(y_{k_0}) \leq -2 \), then \( M_{i(y_{k_0})} \geq 1 \) by Lemma 2.2, which contradicts to \( M_{i(y_{k_0})} = 0 \) in Claim 3 since \( i(y_{k_0}) \in 2\mathbb{Z} \). Thus \( i(y_k) \geq 0, \forall 1 \leq k \leq q, m \geq 1 \). Now by (4.2) and (4.4) we have
\[
i(y_k^{m+1}) - i(y_k^m) = (i(y_k) + n + 1 - r_k) + 2 \sum_{j=1}^{r_k} \left[ \frac{(m+1)\theta_j^k}{2\pi} \right] - 2 \sum_{j=1}^{r_k} \left[ \frac{m\theta_j^k}{2\pi} \right]
\geq (n + 1) - r_k
\geq 2, \hspace{0.5cm} \forall 1 \leq k \leq q, \hspace{0.5cm} m \geq 1, \hspace{1cm} (4.19)
\]
where we use \( i(y_k) \geq 0 \) in the first inequality, and \( r_k \leq n - 1 \) in the second inequality.

**Proof of Theorem 1.2.**

By the assumption (ECC) and Claim 2, we have \( \hat{i}(y_k) = \hat{i}(y_k, 1) > 0, \forall 1 \leq k \leq q \). So for some fixed \( \bar{m} \geq 1 \), it follows from Theorem 3.3 that there exist infinitely many \( (q + 1) \)-tuples \( (N, m_1, \cdots, m_q) \in \mathbb{N}^{q+1} \) such that for any \( 1 \leq k \leq q \), there holds
\[
i(y_k, 2m_k - m) = 2N - 2 - i(y_k, m), \hspace{0.5cm} 1 \leq m \leq \bar{m}, \hspace{1cm} (4.20)
i(y_k, 2m_k) = 2N - 1 - C(M_k) + 2\Delta_k, \hspace{1cm} (4.21)
i(y_k, 2m_k + m) = 2N + i(y_k, m), \hspace{0.5cm} 1 \leq m \leq \bar{m}, \hspace{1cm} (4.22)
\]
where, note that $S_{M_k}^+(1) = 1, Q_k(m) = 0, \forall m \geq 1$ by (4.1)-(4.2).

Then by (4.20)-(4.22) and Theorem 3.4, we obtain

\[
\begin{align*}
  i(y_{2m}^k) &= 2N - 2n - 2 - i(y_m^k), \quad \forall 1 \leq m \leq \bar{m}, \quad (4.23) \\
  i(y_{2m}^k) &= 2N - C(M_k) + 2\Delta_k - n - 1, \quad (4.24) \\
  i(y_{2m+k}^k) &= 2N + i(y_m^k), \quad \forall 1 \leq m \leq \bar{m}. \quad (4.25)
\end{align*}
\]

Furthermore, by (4.23)-(4.25) and Claim 4, we have

\[
\begin{align*}
  i(y_m^k) \leq i(y_{2m}^k) &\leq 2N - 2n - 2, \quad \forall 1 \leq m \leq 2m_k - 1, 1 \leq k \leq q, \quad (4.26) \\
  i(y_m^k) \geq i(y_{2m+1}^k) &\geq 2N, \quad \forall m \geq 2m_k + 1, 1 \leq k \leq q, \quad (4.27) \\
  i(y_{2m}^k) &= 2N - C(M_k) + 2\Delta_k - n - 1 \in [2N - 2n, 2N - 2], \quad 1 \leq k \leq q, \quad (4.28)
\end{align*}
\]

where we use $-(n - 1) \leq 2\Delta_k - C(M_k) \leq n - 1$ in the last estimate.

On one hand, by (4.26)-(4.28) and Lemma 2.2, we know that the sum $\sum_{p=2N-2n}^{2N-2} M_p$ are exactly contributed by $y_{2m}^k$ with $1 \leq k \leq q$, i.e., there holds

\[
\sum_{p=2N-2n}^{2N-2} M_p = q. \quad (4.29)
\]

On the other hand, it follows from Claim 3 that

\[
\sum_{p=2N-2n}^{2N-2} M_p = \sum_{p=2N-2n}^{2N-2} b_p = n. \quad (4.30)
\]

Therefore equalities (4.29) and (4.30) complete the proof of Theorem 1.2.

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