Forced sliding mode control for chaotic systems synchronization

A. A. Kuz’menko

Abstract  Synchronization of chaotic systems is considered to be a common engineering problem. However, the proposed laws of synchronization control do not always provide robustness toward the parametric perturbations. The purpose of this article is to show the use of synergy-cybernetic approach for the construction of robust law for Arneodo chaotic systems synchronization. As the main method of design of robust control, the method of design of control with forced sliding mode of the synergetic control theory is considered. To illustrate the effectiveness of the proposed law, in this article it is compared with the classical sliding mode control and adaptive backstepping. The distinctive features of suggested robust control law are the more good compensation of parametric perturbations (better performance indexes—the root-mean-square error (RMSE), average absolute value (AVG) of error) without designing perturbation observers, the ability to exclude the chattering effect, less energy consuming and a simpler analysis of the stability of a closed-loop system. The study of the proposed control law and the change of its parameters and the place of parametric perturbation’s application is carried out. It is possible to significantly reduce the synchronization error and RMSE, as well as AVG of error by reducing some parameters, but that leads to an increase in control signal amplitude. The place of application of parametric disturbances (slave or master system) has no effect on the RMSE and AVG of error. Offered approach will allow a new consideration for the design of robust control laws for chaotic systems, taking into account the ideas of directed self-organization and robust control. It can be used for synchronization other chaotic systems.

Keywords  Chaos synchronization · Robust control · Synergetic control theory · Arneodo chaotic system · Sliding mode control · Forced sliding mode

Mathematics Subject Classification  93B35 · 93B52 · 93C30 · 93D21 · 34H10

1 Introduction

Chaotic system study is the field of modern nonlinear science that is extremely relevant and is of great interest to both theorists and practitioners. The literature review shows that these systems are widely used in the development of technical systems (oscillation generators, lasers, cryptosystems, secure data transfer systems, neural networks, and robot technical systems), biology, chemistry, ecology, medicine, economics, etc. [1–15].

The most frequently studied and implemented chaotic systems are the chaotic systems of Lorenz, Chua, Chen, Lu, Rössler, etc. Also, new types of chaotic systems are periodically proposed. This article considers the
The Arneodo chaotic system [4–6]. Note that, a model of the Arneodo system with a different type of the right part of the third equation of the system is presented in the literature, for example, in [16,17].

The common engineering problem of chaotic systems control is the synchronization problem—it is necessary to ensure the synchronous behavior of two mutually connected chaotic systems with similar or different chaotic attractors with different initial conditions [18]: “The chaos synchronization problem has the following feature: the trajectories of a slave system must tracks the trajectories of the master system in spite of both master and slave systems being different.” That is, taking into account the behavior of the master system, it is necessary to control the slave system in a way to provide for their synchronization.

Traditionally the problem of chaotic system synchronization is solved by the methods of active control [3,6,7], methods of adaptive control [8,15,17,19], method of design of the sliding mode control (SMC) [15,20–27], backstepping [5,9,17] and, etc. [10–14]. Furthermore, the developers are interested not only in solving the synchronization problem, but also the solving the problem of ensuring the robustness of the closed-loop system to perturbations. In the presented works, robustness is ensured by constructing an observer or SMC. But when constructing an observer, the complexity and dimension of the control system increases, and systems with SMC have a negative chattering effect. Thus, the main problem of this article is the problem of increasing robustness and eliminating a chattering effect when synchronizing two Arneodo chaotic systems. Also, unfortunately, there are no attempts in the literature to combine the methods of modern control theory with the ideas of synergetics to solve the tasks of chaotic systems synchronization.

The main purpose of this paper is to demonstrate the new approach of modern control theory to ensuring a system’s robustness and eliminating a chattering effect in case of synchronization of two Arneodo chaotic systems. This approach is the synergy-cybernetic approach proposed by professor A. A. Kolesnikov [28–30] and also known as the synergetic control theory (SCT). It guarantees the stability of the object’s motion toward the target attractors (invariant manifolds) due to the appropriate design of nonlinear control laws that ensure both the fulfillment of control objectives (invariants) and compensation of external and parametric perturbations.

The synergy-cybernetic approach has the main strengths:

- can possess the properties of order reduction and decoupling in the design procedure [31];
- directed self-organization [28];
- the design of scalar and vector control laws with the nonlinearity and high dimensionality of the control object’s model [28,32,33];
- includes methods for the synthesis of adaptive [34,35], robust [36] and SMC control laws [37–39];
- synergetic laws are well suited for digital control implementation [30,31].

It should be noted that A. Levant in [40] wrote that “…sampling noises, unaccounted-for fast dynamics of sensors and actuators, delays, discretization and hysteresis effects might cause…” chattering and also “Sliding-mode control chattering is caused by the high, theoretically infinite, frequency of control switching and reveals itself as high-frequency dangerous vibrations of the whole system.” Thus, this article deals with the problem of reducing or eliminating the chattering caused by sliding control, while saving the idea of SMC and the structure of the sliding law.

The novelty of this paper is to combine synergy-cybernetic approach (i.e., SCT) and SMC techniques (forced sliding mode (FSM)) in order to synchronize two Arneodo chaotic system and to ensure robustness and eliminating a chattering effect. As the main method of design of robust control, the method of design of control with FSM of the synergetic control theory is considered. The effectiveness of the proposed control law is illustrated by an example of comparison with the classical SMC. Obtained results have features: the more good compensation of parametric perturbations (better performance indexes—the root-mean-square error, average absolute value of error) without the design of perturbation observers, in the possibility of eliminating the chattering effect in SMC, in less energy-consuming and in a simpler analysis of the stability of a closed-loop system.

This work is a development of the work [38] in which the problem of synchronization of chaotic Sprott’s systems is considered. The rest of this paper is organized as follows. Section 2 provides object and control problem. Section 3 describes the procedure for designing the control law with FSM. In Sect. 4, synthesis example for two Arneodo chaotic system is demonstrated. Simulation
results and discussion are presented in Sect. 5. Finally, conclusion and future study are drawn in Sect. 6.

2 Problem statement

Arneodo chaotic system model [5,6]:

\[
\begin{align*}
\dot{x}_1 (t) &= x_2; \\
\dot{x}_2 (t) &= x_3; \\
\dot{x}_3 (t) &= ax_1 - bx_2 - x_3 - x_1^2,
\end{align*}
\] (1)

where \( x_i \) are the state variables; and \( a, b \) are the constant parameters.

Figure 1 shows the strange attractor of the Arneodo system (1) with nominal parameters \( a = 7.5, b = 3.8 \). The detailed information and analysis of Arneodo system is presented in [41,42]. The relationships for calculating the equilibria, Lyapunov Exponents and Kaplan–Yorke dimension for chaotic systems are given in [43].

In the problem of the chaotic systems synchronization, the mathematical model of a control object with two Arneodo systems includes a model of a master system in the form of (1) and a model of a slave system [5]:

\[
\begin{align*}
\dot{y}_1 (t) &= y_2; \\
\dot{y}_2 (t) &= y_3; \\
\dot{y}_3 (t) &= ay_1 - by_2 - y_3 - y_1^2 + u,
\end{align*}
\] (2)

where \( y_i \) are the state variables of the slave system and \( u \) is the control.

By introducing the new variables \( e_i (t) = y_i (t) - x_i (t), i = 1, 3 \) are the synchronization errors, we represent the combined dynamics of master and slave systems (1), (2) by the following system [5]:

\[
\begin{align*}
\dot{e}_1 (t) &= e_2; \\
\dot{e}_2 (t) &= e_3; \\
\dot{e}_3 (t) &= a e_1 - b e_2 - e_3 - (y_1 + x_1) e_1 + u.
\end{align*}
\] (3)

Then, for the system (3), the problem is set to construct the control law \( u \) that provides for asymptotical synchronization of chaotic systems (1) and (2), i.e., the near-zero errors of synchronization \( e_i (t) \to 0, i = 1, 3 \) and robustness to parametric perturbations acting on the master system (1). That is, in the slave system (2) the parameters \( a, b \) do not change and are equal to nominal values.

3 Method of control design with FSM

It is possible to speed up the process of sliding by organizing a new sliding along the surface of lower dimensionality, i.e., by organizing FSM [39]. Introduction to the FSM system not only accelerates the transient process, but also endows the system with the properties of invariance to changes of object parameters in a wide range and to external unmatched disturbances and also makes it easier to solve the problem of analyzing the stability of a closed-loop system [38].

If, after organizing the first sliding in the system, it was possible to achieve the required dynamic properties, then there is no need to introduce the next sliding into the system. If it was not possible, then a second sliding should be organized, etc., up to and including \( (n - 1) \) sliding, where \( n \) is the control object’s dimensionality. Thus, a series of successive transitions from one sliding surface to another is organized.

The description of the main SCT method was presented in [28–31]. Its stages are described in “Appendix A”. Here, we will describe in detail the modification of this method named as FSM. The model of the system has the form [38,39,44]:

\[
\begin{align*}
\dot{x}_j (t) &= f_j (x_1, \ldots, x_n) + d_{j+1} x_{j+1}, \quad j = 1, n - 1; \\
\dot{x}_n (t) &= f_n (x_1, \ldots, x_n) + u,
\end{align*}
\] (4)

where \( x = [x_1, \ldots, x_n]^T \) is the vector of state variables; \( \text{dim } x = n \times 1 \); \( u = u (x) \) is the scalar control; \( f_i (x_1, \ldots, x_n), i = 1, n \) are the continuous differentiable functions (in general, they are nonlinear); and \( d_j \) are the constant parameters of the system.
For the system (4), the problem of design a sliding control with an FSM is stated: it is required to define such a control in the function of the state variables of the object (4), which ensures the transfer of the representing point (RP) of the object from an arbitrary initial state (in a certain admissible region) to a given state determined by the desired invariant—the goal of control.

At the stage of the control design with FSM, we will always consider a manifold of the form that depends on the vector of state variables of this stage

$$\psi_k(x^{(k)}) = \sum_{j=1}^{n_k-1} \beta_{kj} \left| x_j \right| + |x_k| = 0.$$  \hfill (5)

For example, for first stage there is $x^{(k)} = x$.

The structure (5) includes the sliding surface

$$s_k = \sum_{j=1}^{n_k-1} \alpha_{kj} x_j + x_{nk} + u_k (x_1, \ldots, x_{nk-1}),$$  \hfill (6)

where $\alpha_{kj}$, $\beta_{kj}$ are the parameters, due to the choice of which the motion of the system (4) in the sliding mode is given the required dynamic properties; $n_k$ is the dimension of the system at the $k$th stage; $u_k (x_1, \ldots, x_{nk-1})$ is the continuous function, unknown at this stage, which plays the role of “internal” control for the decomposed system of the next $(k+1)$th stage. Note that in terms of SCT, an invariant manifold is an attractor.

Under the control $u = u(x)$ and due to $s_1 = 0$ ($k = 1$) from (6) the behavior of system (4) will be described by a system of lower dimensionality—$(k+1)$th stage’s a decomposed system:

$$\dot{x}_j(t) = f_j(x_1, \ldots, x_j) + d_{j+1}x_{j+1}, \quad j = 1, n-2;$$

$$\dot{x}_{n-1}(t) = f_{n-1}(x_1, \ldots, x_{n-1})$$

$$- d_n \sum_{j=1}^{n-1} \alpha_{1j} x_j - d_n u_1 (x_1, \ldots, x_{nk-1}).$$  \hfill (7)

At each $k$th stage of the design, the basic functional equation of the SCT is considered [28, 29, 37–39]:

$$T_k \ddot{\psi}_k(t) + \psi_k = 0,$$  \hfill (8)

where $\dot{\psi}_k(t) = \sum_{i=1}^{n_k} \frac{\partial \psi_k}{\partial x_i}(t); T_k > 0$ is the adjustable parameter.

Let us substitute into (8) the expression of the manifold (5) and its derivative obtained by virtue of the original equations of the object (4). From the obtained equation, we express the desired control

$$u = - \sum_{j=1}^{n-1} \left( \alpha_{1j} + \frac{\partial u_1}{\partial x_j} \right) (f_j(x_1, \ldots, x_n) + d_{j+1}x_{j+1})$$

$$- f_n (x_1, \ldots, x_n) -$$

$$- \left( \sum_{j=1}^{n-1} \beta j (f_j(x_1, \ldots, x_n) + d_{j+1}x_{j+1}) \right) \text{sign} x_j + \frac{1}{T_1} \psi_1 \right) \text{sign} s_1.$$

The detailed finding of the expression (9) is shown in “Appendix B.” Also, the expression (9) can be represented in the form of

$$u = u_{eq} + u_{SMC},$$

where $u_{eq} = - \sum_{j=1}^{n-1} \left( \alpha_{1j} + \frac{\partial u_1}{\partial x_j} \right) (f_j(x_1, \ldots, x_n) + d_{j+1}x_{j+1}) - f_n (x_1, \ldots, x_n)$ is the equivalent control;

$$u_{SMC} = - \left( \sum_{j=1}^{n-1} \beta j (f_j(x_1, \ldots, x_n) + d_{j+1}x_{j+1}) \right)$$

$$\text{sign} x_j + \frac{1}{T_1} \psi_1 \right) \text{sign} s_1 =$$

$$= - M(x) \text{sign} s_1$$

is the discontinuous control, in which $M(x)$ is the nonlinear function analogous to the large SMC gain [45].

Control (9) transits the RP systems (4) from an arbitrary initial state to the manifold $\psi_1 = 0$—to the attractor whose structure includes a sliding surface (6) [28, 29]. Since the motion relative to $\psi_1 = 0$ is asymptotically stable at $T_1 > 0$ according (8). This means that the RP inevitably falls on the submanifold (6), that is, on the sliding surface $s_1 = 0$.

At the second stage ($k = 2$), we design the control law, which provides the FSM along the selected sliding surface of the first stage (6). Also at this stage, we define the expression of the “internal” control $u_1 (x_1, \ldots, x_{n-1})$. If we repeat the procedure for the design of the first stage, then a sliding of the second order, or FSM, appears in the original system. In other
Forced sliding mode control

words, a new sliding is organized on the first-order sliding surface, decreasing the dimensionality of the original system by one. Thus, at the second stage of the design for the decomposed system (7), a manifold of the form of (5) is introduced. Similarly, based on functional equations of the form (8), we find the “internal” control \( u_1(x_1, \ldots, x_{n-1}) \).

The specified procedure of the second stage, if necessary, can be repeated \((n - 1)\) times until the order of the original system becomes equal to one, for example, of the form

\[
\dot{x}_1(t) = f_1(x_1) - d_2u_q(x_1) \quad (10)
\]
or until we provide the desired target invariant (or target invariants in case of vector control design).

The expression \( u_q(x_1) \) for the finishing decomposed system (10) is selected or found using the SCT methods or some other methods of modern control theory. Then, the obtained equation \( u_q(x_1) \) is substituted to the previous control \( u_{q-1}(x_1, x_2) \), which, in turn, is also substituted to \( u_{q-2}(x_1, x_2, x_3) \), and so on, up to the control \( u_1(x_1, \ldots, x_{n-1}) \), which is directly included in the control law (9).

Please note, that in order to exclude the differentiation of the function \( \text{sign} (\cdot) \) included in each “internal” control, it is recommended to replace it with one of the equivalent continuous functions, for example, \( \tanh (A \cdot) \) or \((2/\pi) \text{arctan} (A \cdot)\), where \( A > 0 \) is the large coefficient, and \( (\cdot) \) is the argument of the function \( \text{sign} (\cdot)\).

The conditions for the asymptotic stability of the closed-loop system in SCT [28, 29] consist of the stability conditions for functional equations of the form (8), that is, \( T_k > 0 \), and stability conditions for the finishing decomposed system, the dimensionality of which is significantly lower than the dimensionality of the original system (4), for example, (10). Thus, we see that the stability analysis is quite simple, since there is no need to analyze the entire initial system.

At each \( k \)-stage of design, also it is necessary to check the condition for the occurrence of the sliding mode [45]:

\[
s_k \cdot \dot{s}_k(t) < 0. \quad (11)
\]

To analyze the condition (11), we need take into account:

- there is expression for the sliding surface \( s_k \). It expression is (6);
- it is expedient to express the derivative of the sliding surface from the corresponding equation of the form (8)—we need to insert (5) into (8). In common case, we have

\[
\dot{s}_k = -\left( \sum_{j=1}^{n_k-1} \beta_{kj} \text{sign}(x_j) \dot{x}_j(t) + \frac{1}{T_k} \psi_k \right) \text{sign}(s_k).
\]

4 Design of the robust control law for Arneodo chaotic systems

At the first stage of the design of control \( u \) for the system (3), we define an invariant manifold of the form (5):

\[
\psi_1(x^{(1)}) = \beta_{11} |e_1| + \beta_{12} |e_2| + |s_1| = 0, \quad (12)
\]

where \( s_1 = \alpha_{11}e_1 + \alpha_{12}e_2 + e_3 + u_1(e_1, e_2), x^{(1)} = [e_1, e_2, e_3]^T \).

Let us substitute (12) into the functional SCT Eq. (8). From which we obtain, by virtue of the equations of the object (3), the control law:

\[
u = - (a - e_1) e_1 - \left( \alpha_{11} - b + \frac{\partial u_1}{\partial e_1} \right) e_2 - \left( \alpha_{12} - 1 + \frac{\partial u_1}{\partial e_2} \right) e_3 - \\
- \left( \beta_{11} e_1 \text{sign} e_1 + \beta_{12} e_3 \text{sign} e_2 + \frac{1}{T_1} \psi_1 \right) \text{sign} s_1. \quad (13)
\]

Under the action of control (13), the RP of system (3) falls into a neighborhood of the manifold (12). The motion along which, by virtue of \( s_1 = 0 \), is described by a decomposed system:

\[
\dot{e}_1(t) = e_2; \\
\dot{e}_2(t) = -\alpha_{11} e_1 - \alpha_{12} e_2 - u_1(e_1, e_2). \quad (14)
\]

To find the “internal” control \( u_1(e_1, e_2) \) for the system (14) at the second stage of the design, we similarly define a manifold of the form

\[
\psi_2(x^{(2)}) = \beta_{21} |e_1| + |s_2| = 0, \quad (15)
\]

where \( s_2 = \alpha_{21} e_1 + e_2, x^{(2)} = [e_1, e_2]^T \).
Substituting this expression into the functional equation (8), we obtain, by virtue of the equations of the object (14), the law of “internal” control:

\[ u_1 (e_1, e_2) = -\alpha_1 e_1 - (\alpha_{12} - \alpha_{21}) e_2 + \left( \beta_{21} e_2 \text{ sign } e_1 + \frac{1}{T_2} \psi \right) \text{ sign } s_2 \]

(16)

Thus, substituting (16) in (13), we get the final expression for the control. Then, the motion of the system (14) under the action of the control (16) will be described by the equation:

\[ \dot{e}_1 (t) = -\alpha_{21} e_1, \]

which is stable at \( \alpha_{21} > 0 \), and the choice of the value of this parameter can provide the desired dynamics of the transient process. To these conditions, we add the stability conditions for the equations (8): \( T_i > 0, i = 1, 2 \). Thus, the stability conditions for the closed-loop system (3), (13) are \( \alpha_{21} > 0, T_i > 0, i = 1, 2 \).

Let us check the condition (11) for the occurrence of a sliding mode for each stage of the design. At the second stage of the design, the second sliding surface \( s_2 = \alpha_{21} e_1 + e_2 \) enters in (15). Then, its derivative, expressed from the functional Eq. (8) for the manifold (15), has the form

\[ \dot{s}_2 (t) = -\left( \beta_{21} e_2 \text{ sign } e_1 + \frac{1}{T_2} \psi \right) \text{ sign } s_2. \]

Hence, it can be seen that the condition (11) for the second sliding surface has form

\[ -(\alpha_{21} e_1 + e_2) \left( \beta_{21} e_2 \text{ sign } e_1 + \frac{1}{T_2} \psi \right) \text{ sign } s_2 < 0 \]

and is satisfied at \( \alpha_{21} > 0, \beta_{21} > 0, T_2 > 0 \).

Similarly, taking into account (12), we obtain the expression for the first sliding surface of the first stage of design:

\[ s_1 = \alpha_{11} e_1 + \alpha_{12} e_2 + e_3 + u_1 (e_1, e_2) = e_3 + \left( \beta_{21} e_2 \text{ sign } e_1 + \frac{1}{T_2} \psi \right) \text{ sign } s_2 + \alpha_{12} e_2, \]

and from (8), taking into account (12), we express its derivative

\[ \dot{s}_1 (t) = -\left( \beta_{11} e_2 \text{ sign } e_1 + \beta_{12} e_3 \text{ sign } e_2 + \frac{1}{T_1} \psi \right) \text{ sign } s_1. \]

From this, it can be seen that the condition for the occurrence of the sliding mode (11) for the first sliding surface is directly provided by the choice of parameters \( \alpha_{12} > 0, \beta_{11} > 0, \beta_{12} > 0, \beta_{23} > 0, T_1 > 0, T_2 > 0 \).

Thus, the analysis of the fulfillment of the condition of occurrence of the sliding mode (11) in the proposed methods is quite simple.

5 Simulation and discussion

5.1 SMC and FSM simulation

To illustrate the effectiveness of the proposed control law (13), let us compare it with the control law that realizes classical SMC. It can be designed according to the SMC method, for example described in [25,45]. Applying this method, we get

\[ u = (y_1 + x_1) e_1 - \frac{(ac_1 + ac_3) e_1}{c_3} - \frac{(c_1 + ac_2 - bc_3) e_2}{c_3} - \frac{(c_2 + c_3 (\alpha - 1)) e_3}{\beta e_3} \text{ sign } (s), \]

(17)

with sliding surface

\[ S = c_1 e_1 + c_2 e_2 + c_3 e_3, \]

(18)

where \( \alpha, \beta, c_j \) are control law constant parameters.

It is possible to provide the desired eigenvalues \( p_{01} = p_{02} = p_{03} = p_0 < 0 \) of the state matrix of the system (3) obtained taking into account control law (17) by choosing the coefficients

\[ \alpha = -p_0; c_2 = -2c_1/p_0; c_3 = c_1/p_0^2. \]

In the simulation, we assume that the parameters of the control laws (13), (17) and the slave system (2) are unchanged and equal to the nominal: \( a = a_0 = 7.5, b = b_0 = 3.8 \), but the parameters of the master system (1) change as follows:

\[ a = a(t) = \begin{cases} a_0, & t \leq 20; \\ 1.5 a_0, & t > 20; \end{cases} \]
\[ b = b(t) = \begin{cases} b_0, & t \leq 20; \\ 1.5 b_0, & t > 20. \end{cases} \]

(19)

Closed-loop system simulation was carried out in Matlab R2021a with the initial conditions \( x_1(0) = 3; x_2(0) = 2; x_3(0) = 2; y_1(0) = 1; y_2(0) = -1; y_3(0) = 7 \).

In Figs. 2, 3, 4, 5, 6, the simulating results of the closed-loop system (3) with perturbations (19) are shown: by the red line with FSMC control law (13) with parameters \( T_1 = T_2 = 0.1; \beta_{11} = \beta_{12} = \beta_{21} = 1; \alpha_{11} = 3.8; \alpha_{12} = 1; \alpha_{21} = 40; A = 20 \); by the black line with SMC law (17), (18) with parameters
$$p_0 = -10, \ c_1 = 10, \ \beta = 0.5. \ \text{The parameters of the laws are selected so that the closed-loop system has sufficiently similar dynamic characteristics of transients and control amplitudes. The time interval from 0 to 15 s omits on Figs. 2, 3, 4, 5.}$$

Figs. 2, 3 show that up to $t = 20$ s, the synchronization errors are zero. And after $t = 20$ s, there is no synchronization—the errors are different from zero, and in the case of law (13), the errors are more small for similar amplitudes of controls.

In order to evaluate the effects of presented different controllers, two performance indexes are introduced:

- The root-mean-square error (RMSE): $RMSE = \sqrt{\frac{1}{n} \sum_{i=1}^{n} e_i^2}$;
- (AVG): $AVG = \frac{1}{n} \sum_{i=1}^{n} |e_i|$.

To illustrate the simulation results and confirm the advantages of the proposed control law (13) and SMC, the performance indexes are calculated, which are presented in Table 1.

Figure 7 shows the phase trajectory of the system (3) with the control law (13) and the sliding surface (12). And Fig. 8 shows the phase trajectory of the decomposed system (14) with the control law (16) and the sliding surface (15).

5.2 Simulation of FSM and adaptive backstepping

To illustrate the effectiveness of the proposed FSM control law (13), let us compare it with the control law that realizes adaptive backstepping (AB) from [5]:

$$u = -\left(\hat{a} (t) + 3 - y_1 - x_1\right) e_1 - \left(5 - \hat{b} (t)\right) e_2 - 2e_3,$$

(20)
Table 1 Performance indexes for parametric perturbations (19): SMC and FSM

| Error | Classical SMC (17) | Forced SMC (13) |
|-------|-------------------|----------------|
|       | RMSE A VG         | RMSE A VG      |
| $e_1$, for $t < 20$ | 0.2354 0.0592 | 1.17e−4 3.69e−5 |
| $e_1$, for $t \geq 20$ | 0.107 0.1 | 0.0097 0.0078 |
| $e_2$, for $t < 20$ | 0.2092 0.0648 | 6.38e−4 1.52e−4 |
| $e_2$, for $t \geq 20$ | 0.0235 0.0199 | 0.014 0.012 |
| $e_3$, for $t < 20$ | 0.1920 0.0640 | 0.0079 0.0034 |
| $e_3$, for $t \geq 20$ | 0.041 0.026 | 0.039 0.033 |

where $\hat{a}(t), \hat{b}(t)$ are the dynamic evaluations of the Arneodo system parameters formed by the observer:

$$\dot{a}(t) = (2e_1 + 2e_2 + e_3)e_1 + k_1(a - \hat{a}) ;$$

$$\dot{b}(t) = -(2e_1 + 2e_2 + e_3)e_2 + k_2(b - \hat{b}) .$$

(21)

Closed-loop system simulation was carried out in Matlab R2021a with the initial conditions $x_1(0) = 3$; $x_2(0) = 2$; $x_3(0) = 2$; $y_1(0) = 1$; $y_2(0) = -1$; $y_3(0) = 7$; $\hat{a}(0) = 7.5$; $\hat{b}(0) = 3.8$ and parametric perturbations (19). The parameters of the law (13) are the same as in the Sect. 5.1, and the parameters of the adaptive law (20), (21) are $k_1 = k_2 = 10^4$.

In Figs. 9, 10, 11, 12, the results of simulating the closed-loop system (3) with perturbations (19) are shown: the red line represents the simulation with FSMC control law (13); the black line represents the simulation with the law (20). The time interval from 0 to 15 s omits on Figs. 9, 10, 11, 12.

To illustrate the simulation results and confirm the advantages of the proposed control law (13) and AB, the performance indexes are calculated, which are presented in Table 2.

Increasing the parameters $k_1, k_2$ in (21) improves the quality of estimation of parameters $a$ and $b$, but also
Table 2  Performance indexes for parametric perturbations (19): FSM and AB

| Error             | AB (20)       | Forced SMC (13) |
|-------------------|---------------|-----------------|
|                   | RMSE | AVG | RMSE | AVG |
| $e_1$, for $t < 20$ | 0.4035 | 0.1077 | 1.17e$-4$ | 3.69e$-5$ |
| $e_1$, for $t \geq 20$ | 5.9572 | 4.99 | 0.0097 | 0.0078 |
| $e_2$, for $t < 20$ | 0.4301 | 0.1354 | 6.38e$-4$ | 1.52e$-4$ |
| $e_2$, for $t \geq 20$ | 6.5374 | 5.4814 | 0.014 | 0.012 |
| $e_3$, for $t < 20$ | 0.3516 | 0.1239 | 0.0079 | 0.0034 |
| $e_3$, for $t \geq 20$ | 13.42 | 11.49 | 0.039 | 0.033 |

Fig. 9  Synchronization error $e_1(t)$ of the combined dynamics of master and slave Arneodo system

Fig. 10  Synchronization error $e_2(t)$ of the combined dynamics of master and slave Arneodo system

Fig. 11  Synchronization error $e_3(t)$ of the combined dynamics of master and slave Arneodo system

Fig. 12  AB and FSM controls of the combined dynamics of master and slave Arneodo system

increases the magnitude of the synchronization error for $e_1$, $e_2$, $e_3$. It is illustrated in Figs. 13, 14: Case 1 is $k_1 = k_2 = 10^3$, Case 2 is $k_1 = k_2 = 10^4$. At the same time, the simulation results show that Cases 1 and 2 lead to almost identical values of RMSE and AVG.

5.3 Discussion

Thus, the presented simulation results demonstrate that the control law with FSM (13), in comparison with the law (17) and (20) with a similar speed and control amplitude, provides:
More accurate synchronization with parametric perturbations (19). It is clearly seen from the comparison of Figs. 2, 3, 4: the amplitudes of synchronization errors are 2–3 times smaller. Values of RMSE and AVG are significantly less (see Tables 1, 2). From the comparison of Figs. 9, 10, 11, this difference is even more seriously.

No chattering effect: considering the changes in the control graphs in Fig. 5 at an increased scale (Fig. 6), it can be seen that the control law (17), before the disturbance occurs, represents a high-frequency switching of the control signal with an amplitude of ±6.5. And the control law (13) represents insignificant irregular changes in the interval ±0.5. This suggests that the control law (13) is less energy consuming.

Additionally, some new simulations were performed and their results demonstrated the following:

As it was mentioned previously that in (8) \( T_T^k > 0 \) is the adjustable parameter. Increasing the parameters \( T_1, T_2 \) in (13) slows down the transients and also slightly reduces the amplitude of the control (13) and significantly increases synchronization errors and the values of RMSE and AVG. Accordingly, their reduction speeds up transients, slightly increases the amplitude of the control (13) and significantly reduces synchronization errors and the values of RMSE and AVG.

We obtain similar simulation results (Figs. 2, 3, 4, 5, 6 and Table 1) if perturbations (19) do not affect the system (1), but the system (2).

Similar simulation results and values of performance indexes are obtained if the parametric perturbations (19) are applied to the slave system (2).

6 Conclusion

The paper demonstrates the use of a synergy-cybernetic approach for the design of a control law with FSM synchronization of Arneodo chaotic systems under conditions of parametric disturbances. The article’s goal has been achieved, and the effectiveness of the proposed control law (13) is clearly demonstrated.

To summarize the results, it is necessary to underline the following advantages of the proposed approach:

More good compensation of parametric perturbations without design of perturbation observers;

Values of RMSE and AVG are significantly less than with classical SMC and AB;

Ability to eliminate the chattering effect without losing robustness;

In general, the sliding surface is set implicitly, since it may contain an unknown function \( u_k(x^{(k)}) \). The sliding surfaces of the classical method are always set explicitly, as a rule, in the form of a linear combination of state variables (or errors);

Control law (13) is less energy consuming, since the control amplitude is very small in the steady-state;

Simpler analysis of the closed-loop system’s stability, since the analysis is reduced to the analysis of the stability of the final decomposed system, usually of significantly smaller dimension, and the analysis of the stability of functional equations (8). In the classical method, the stability of a closed-loop system is analyzed for system with initial dimension.

The approach presented in this work will allow a new consideration for the design of robust control laws
Forced sliding mode control for chaotic systems, taking into account the ideas of directed self-organization and nonlinear robust control.

As part of the development of this work, the results obtained will be compared with other methods of modern control theory, and the application of this approach to other chaotic systems will be continued.

Data availability The author declares that all data supporting the findings of this study are available within the article. The model, control laws and their parameters are fully presented in the article. It is not difficult to perform their modeling in Matlab. Also data were generated in article are available on request from the author.

Declarations

Conflict of interest The author declares that he have no conflict of interest. The author has no relevant financial or non-financial interests to disclose. No funding was received to assist with the preparation of this manuscript.

Appendix A

The basic method of the SCT is the method of analytical designing aggregated regulators [28,29]. Let us consider its stages described in [30].

Suppose the system to be controlled is described by a set of nonlinear differential equations in the form

\[ \dot{x}(t) = f(x, u, t), \]

where \( x \) is the state vector, \( u \) is the control input vector and \( t \) is time.

Let us start by defining a macro-variable as a function of the state variables:

\[ \psi = \psi(x). \]

The control will force the system to operate on the manifold \( \psi = 0 \). The designer can select the characteristics of this macro-variable according to the control specifications (e.g., limitation of the control output, etc.). In a trivial case, the macro-variable can be a simple linear combination of the state variables. In a nontrivial case, it is complex nonlinear function.

The same process can be repeated, defining as many macro-variables as there are control channels.

The desired dynamic evolution of the macro-variable is

\[ T \dot{\psi}(t) + \psi(x) = 0, \quad T > 0, \]

where \( T \) is the design parameter specifying the convergence speed to the manifold specified by the macro-variable. The chain rule of differentiation (A.2) gives

\[ \dot{\psi} = \frac{d\psi}{dx} \dot{x}(t) = \frac{d\psi}{dx} f(x, u, t). \]

Combining (A.1), (A.3), and (A.4), we obtain

\[ T \frac{d\psi}{dx} f(x, u, t) + \psi(x) = 0. \]

The Eq. (A.5) is finally used to synthesize the control law \( u \) : we express it from this equation.

To summarize, each manifold introduces a new constraint on the state space domain and reduces the order of the system, improving the global stability.

The procedure summarized here can be easily implemented as a computer program for automatic synthesis of the control law.

Appendix B

Let us introduce derivative for module function:

\[ |\dot{x}| = \frac{d|x|}{dt} = \dot{x}(t) \text{ sign}(x). \]

According (B.6), we find derivative of \( \psi_1 \) (5):

\[ \dot{\psi}_1(t) = \sum_{j=1}^{n-1} \beta_{1j} \text{ sign}(x_j) \dot{x}_j(t) + \dot{s}_1 \text{ sign}(s_1), \]

and derivative of \( s_1 \) (6):

\[ \dot{s}_1 = \sum_{j=1}^{n-1} \alpha_{1j} \dot{x}_j(t) + \dot{x}_n(t) + \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t). \]
And now we can substitute (B.7) and (B.8) into (8) for \( k = 1 \):

\[
T_1 \left( \sum_{j=1}^{n-1} \beta_{1j} \text{sign}(x_j) \dot{x}_j(t) + \left[ \sum_{j=1}^{n-1} \alpha_{1j} \dot{x}_j(t) + \dot{x}_n(t) \right] \right) + \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t) \text{sign}(s_1) + \psi_1 = 0. 
\]

To simplify, we divide left part by \( T_1 \) and multiply by \( \text{sign}(s_1) \):

\[
\text{sign}(s_1) \left[ \sum_{j=1}^{n-1} \beta_{1j} \text{sign}(x_j) \dot{x}_j(t) + \sum_{j=1}^{n-1} \alpha_{1j} \dot{x}_j(t) + \dot{x}_n(t) \right] + \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t) + \frac{1}{T_1} \psi_1 \text{sign}(s_1) = 0. \tag{B.9}
\]

We substitute \( \dot{x}_j(t) \) and \( \dot{x}_n(t) \) from (4) in (B.9):

\[
\text{sign}(s_1) \left[ \sum_{j=1}^{n-1} \beta_{1j} \text{sign}(x_j) \left( f_j(x_1, \ldots, x_n) + d_{j+1}x_{j+1} \right) \right] + \sum_{j=1}^{n-1} \alpha_{1j} \left( f_j(x_1, \ldots, x_n) + d_{j+1}x_{j+1} \right) + f_n(x_1, \ldots, x_n) + u + \sum_{j=1}^{n-1} \frac{\partial u_1}{\partial x_j} \dot{x}_j(t) + \frac{1}{T_1} \psi_1 \text{sign}(s_1) = 0. \tag{B.10}
\]

As a result, from (B.10) we can get the expression (9).

References

1. Pecora, L., Carroll, T.: Synchronization in chaotic systems. Phys. Rev. Lett. 64, 821 (1990). https://doi.org/10.1103/PhysRevLett.64.821
2. Boccaletti, S., Kurths, J., Osipov, G., Valladares, D., Zhou, C.: The synchronization of chaotic systems. Phys. Rep. 366, 1 (2002). https://doi.org/10.1016/S0370-1573(02)00137-0
3. Agiza, H., Yassen, M.: Synchronization of Rossler and Chen chaotic dynamical systems using active control. Phys. Lett. Sect. A. General Atomic Solid State Phys. 278, 191 (2001). https://doi.org/10.1016/S0375-9601(00)00777-5
4. Arneodo, A., Coullet, P., Tresser, C.: Occurrence of strange attractors in three-dimensional Volterra equations. Phys. Lett. Sect. A 79(4), 259 (1980). https://doi.org/10.1016/0375-9601(80)90342-4
5. Sundarapandian, V.: In Global chaos synchronization of Arneodo chaotic system via backstepping controller design. ACM Int. Conf. Proc. Ser. (2012). https://doi.org/10.1145/2393216.2393217
6. Sundarapandian, V., Rasappan, S.: Hybrid synchronization of Arneodo and Rössler chaotic systems by active nonlinear control. In: Lecture Notes of the Institute for Computer Sciences, Social-Informatics and Telecommunications Engineering, LNICT, vol. 84(PART 1), p. 73 (2012). https://doi.org/10.1007/978-3-642-27299-8_8
7. Chen, H.K.: Global chaos synchronization of new chaotic systems via nonlinear control. Chaos Solitons Fractals 23, 1245 (2005). https://doi.org/10.1016/S0960-0779(04)00373-X
8. Feki, M.: An adaptive chaos synchronization scheme applied to secure communication. Chaos Solitons Fractals 18, 141 (2003). https://doi.org/10.1016/S0960-0779(02)00585-4
9. Tan, X., Zhang, J., Yang, Y.: Synchronizing chaotic systems using backstepping design. Chaos Solitons Fractals 16, 37 (2003). https://doi.org/10.1016/S0960-0779(02)00153-4
10. Kapitaniak, T.: Continuous control and synchronization in chaotic systems. Chaos Solitons Fractals 6, 237 (1995). https://doi.org/10.1016/0960-0779(95)80030-K
11. Cuomo, K., Oppenheim, A.: Circuit implementation of synchronized chaos with applications to communications. Phys. Rev. Lett. 71, 65 (1993). https://doi.org/10.1103/PhysRevLett.71.65
12. Kocarev, L., Parlitz, U.: General approach for chaotic synchronization with applications to communication. Phys. Rev. Lett. 74, 5028 (1995). https://doi.org/10.1103/PhysRevLett.74.5028
13. Rulkov, N., Sushchik, M., Tsimring, L., Abarbanel, H.: Generalized synchronization of chaos in directionally coupled chaotic systems. Phys. Rev. E 51, 980 (1995). https://doi.org/10.1103/PhysRevE.51.980
14. Bai, E.W., Lonngren, K.: Synchronization and control of chaotic systems. Chaos Solitons Fractals 10, 1571 (1999). https://doi.org/10.1016/S0960-0779(98)00204-5
15. Yau, H.T.: Design of adaptive sliding mode controller for chaos synchronization with uncertainties. Chaos Solitons Fractals 22, 341 (2004). https://doi.org/10.1016/j.chaos.2004.02.004
16. Lu, J.: Chaotic dynamics and synchronization of fractional-order Arneodo’s systems. Chaos Solitons Fractals 26, 1125 (2005). https://doi.org/10.1016/j.chaos.2005.02.023
17. Hua, C., Guan, X., Shi, P.: Adaptive feedback control for a class of chaotic systems. Chaos Solitons Fractals 23, 757 (2005). https://doi.org/10.1016/j.chaos.2004.05.042
18. Femat, R., Sólis-Pérales, G.: On the chaos synchronization phenomena. Phys. Lett. Sect. A. General Atomic Solid State Phys. 262(1), 50 (1999). https://doi.org/10.1016/S0375-9601(99)00667-2
19. Zhang, R., Yang, S.: Adaptive synchronization of fractional-order chaotic systems via a single driving variable. Nonlinear Dyn. 66(4), 831 (2011). https://doi.org/10.1007/s11071-011-9944-2
20. Zhang, L., Yan, Y.: Robust synchronization of two different uncertain fractional-order chaotic systems via adaptive sliding mode control. Nonlinear Dyn. 76(3), 1761 (2014). https://doi.org/10.1007/s11071-014-1244-1
21. Chen, D., Zhang, R., Ma, X., Liu, S.: Chaotic synchronization and anti-synchronization for a novel class of multiple chaotic systems via a sliding mode control scheme. Nonlinear Dyn. 69(1–2), 35 (2012). https://doi.org/10.1007/s11071-011-0244-7

22. Yu, X., Zhihong, M.: Fast terminal sliding-mode control design for nonlinear dynamical systems. IEEE Trans. Circuits Syst. I Fundam. Theory Appl. 49, 261 (2002). https://doi.org/10.1109/81.983876

23. Laghrouche, S., Plestan, F., Glumineau, A.: Higher order sliding mode control based on integral sliding mode control. Automatica 43, 531 (2007). https://doi.org/10.1016/j.automatica.2006.09.017

24. Konishi, K., Hirai, M., Kokame, H.: Sliding mode control for a class of chaotic systems. Phys. Lett. Sect. A 245(6), 511 (1998). https://doi.org/10.1016/S0375-9601(98)00439-3

25. Vaidyanathan, S., Volos, C.K., Pham, V.T.: Global chaos control of a novel nine-term chaotic system via sliding mode control. Stud. Comput. Intell. 576, 571 (2015). https://doi.org/10.1007/978-3-319-11173-5_21

26. Rodríguez, A., De León, J., Fridman, L.: Quasi-continuous high-order sliding-mode controllers for reduced-order chaos synchronization. Int. J. Non-Linear Mech. 43(9), 948 (2008). https://doi.org/10.1016/j.ijnonlinmek.2008.07.007

27. Ahmed, H., Salgado, I., Ríos, H.: Robust synchronization of master-slave chaotic systems using approximate model: An experimental study. ISA Trans. 73, 141 (2018). https://doi.org/10.1016/j.isatra.2018.01.009

28. Kolesnikov, A.: Introduction of synergetic control. In: Proceedings of the American Control Conference, pp. 3013–3016 (2014). https://doi.org/10.1109/ACC.2014.6859397

29. Kolesnikov, A., Veselov, G., Popov, A., Dougal, R. et al., Synergetic approach to the modeling of power electronic systems. In: IEEE Workshop on Computers in Power Electronics, pp. 259–262 (2000)

30. Santi, E., Monti, A., Li, D., Proddutur, K., Dougal, R.: Synergetic control for power electronics applications: a comparison with the sliding mode approach. J. Circuits Syst. Comput. 13(4), 737 (2004). https://doi.org/10.1142/S0218126604001520

31. Rastegar, S., Araújo, R., Sadati, J., Mendes, J.: A novel robust control scheme for LTV systems using output integral discrete-time synergetic control theory. Eur. J. Control 34, 39 (2017). https://doi.org/10.1016/j.ejcon.2016.12.006

32. Rebai, A., Guesmi, K., Hemici, B.: Adaptive fuzzy synergetic control for nonlinear hysteretic systems. Nonlinear Dyn. 86(3), 1445 (2016). https://doi.org/10.1007/s11071-016-3088-3

33. Ni, J., Liu, C., Liu, K., Pang, X.: Variable speed synergetic control for chaotic oscillation in power system. Nonlinear Dyn. 78(1), 681 (2014). https://doi.org/10.1007/s11071-014-1468-0

34. Kuz’menko, A.: Nonlinear adaptive control of a turbogenerator. J. Comput. Syst. Sci. Int. 47(1), 103 (2008). https://doi.org/10.1134/s1064230708010139

35. Bouchama, Z., Essounbouli, N., Harmas, M., Hamzaoui, A., Saoudi, K.: Reaching phase free adaptive fuzzy synergetic power system stabilizer. Int. J. Electr. Power Energy Syst. 77, 43 (2016). https://doi.org/10.1016/j.ijeps.2015.11.017

36. Kuz’menko, A., Simitsyn, A., Mushenko, A.: The use of integral adaptation principle to increase the reliability of DFIG-Wind turbine power system. In: Proceedings of 2017 International Siberian Conference on Control and Communications, SIBCON-2017 (2017). https://doi.org/10.1109/SIBCON.2017.7998487

37. Kuz’menko, A.: Synchronous generator nonlinear excitation system: synergetic sliding mode control. In: Proceedings of 2015 International Siberian Conference on Control and Communications, SIBCON 2015 (2015). https://doi.org/10.1109/SIBCON.2015.7147112

38. Kuz’menko, A.: Synthesis of the sliding mode control law of synchronization of chaotic systems basing on sequential aggregate of invariant manifolds. In: Proceedings of 2019 3rd International Conference on Control in Technical Systems, CTS 2019, pp. 60–63 (2019). https://doi.org/10.1109/CTS48763.2019.8973371

39. Kolesnikov, A., Kuz’menko, A.: Forced sliding mode control: synergetic approach. In: Proceedings of 2020 2nd International Conference on Control Systems, Mathematical Modeling, Automation and Energy Efficiency, SUMMA 2020, pp. 36–40. (2020). https://doi.org/10.1109/SUMMA50634.2020.9280620

40. Levant, A.: Chattering analysis. IEEE Trans. Autom. Control 55(6), 1380 (2010). https://doi.org/10.1109/TAC.2010.2041973

41. Arneodo, A., Coullet, P., Spiegel, E., Tresser, C.: Phys. D Nonlinear Phenom. 14(3), 327 (1985). https://doi.org/10.1016/0167-2789(85)90093-4

42. Liu, Y., Li, Z., Cai, X., Ye, Y.: Local stability and Hopf bifurcation analysis of the Arneodo’s system. Appl. Mech. Mater. 130–134, 2550 (2012). https://doi.org/10.4028/www.scientific.net/AMM.130–134.2550

43. Zambrano-Serrano, E., Anzo-Hernández, A.: A novel antimonotic hyperjerk system: analysis, synchronization and circuit design. Phys. D Nonlinear Phenom. (2021). https://doi.org/10.1016/j.physd.2021.132927

44. Kuz’menko, A.: Synthesis of synergetic sliding mode control law of synchronous generator basing on sequential aggregate of invariant manifolds. In: Proceedings of 2019 International Russian Automation Conference, RusAutoCon 2019 (2019). https://doi.org/10.1109/RUSAUTOCON.2019.8867666

45. Utkin, V., Guldner, J., Shi, J.: Sliding Mode Control in Electromechanical Systems. Sliding Mode Control in Electromechanical Systems, 2nd edn. CRC Press, Boca Raton (2009)