Relation Between the Resonance and the Scattering Matrix in the Massless Spin-Boson Model

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Abstract: We establish the precise relation between the integral kernel of the scattering matrix and the resonance in the massless Spin-Boson model which describes the interaction of a two-level quantum system with a second-quantized scalar field. For this purpose, we derive an explicit formula for the two-body scattering matrix. We impose an ultraviolet cut-off and assume a slightly less singular behavior of the boson form factor of the relativistic scalar field but no infrared cut-off. The purpose of this work is to bring together scattering and resonance theory and arrive at a similar result as provided by Simon (Ann Math Sect Ser 97(2):247–274, 1973), where it was shown that the singularities of the meromorphic continuation of the integral kernel of the scattering matrix are located precisely at the resonance energies. The corresponding problem has been open in quantum field theory ever since. To the best of our knowledge, the presented formula provides the first rigorous connection between resonance and scattering theory in the sense of (Simon 1973) in a model of quantum field theory.

1. Introduction

In this paper, we analyze the massless Spin-Boson model which is a non-trivial model of quantum field theory. It can be seen as a model of a two-level atom interacting with its second-quantized scalar field, and hence, provides a widely employed model for quantum optics which gives insights into scattering processes between photons and atoms. The unperturbed energies of the two-level atom shall be denoted by real numbers \( 0 = e_0 < e_1 \). It is well-known that after switching on the interaction with a massless scalar field, which may induce transitions between the atom levels, the free ground state energy \( e_0 \) is shifted to the interacting ground state energy \( \lambda_0 \) while the free excited state with energy \( e_1 \) turns into a resonance with complex “energy” \( \lambda_1 \).

One of the main mathematical difficulties in the study of the massless Spin-Boson model is the absence of a spectral gap which does not allow a straight-forward application of regular perturbation theory. Several techniques have been developed to overcome this
difficulty. There are two methods that rigorously address this problem: The so-called renormalization group (see e.g. [2,6,8–11,15,21,26,27,38]) which was the first one used to construct resonances in models of quantum field theory, and furthermore, the so-called multiscale method which was developed in [4,5,33,34] and also successfully applied in various models of quantum field theory. In both cases, a family of spectrally dilated Hamiltonians is analyzed since this allows for complex eigenvalues. Our work draws from the results obtained in a previous article [14] which is build on the latter technique mentioned above. Beyond the construction, we obtained several spectral estimates and analyticity properties in [14] which are crucial ingredients for this work.

In addition to the resonance theory, also the scattering theory is well-established in various models of quantum field theory, e.g., in [16,22–25], and in particular in the massless Spin-Boson model, e.g., in [12,17–20]. The purpose of this work is to bring these two well-developed fields together and to arrive at a similar result as provided by Simon in [39]. Therein, it was shown that the singularities of the meromorphic continuation of the integral kernel of the scattering matrix are located precisely at the resonance energies. To the best of our knowledge, this question has not yet been addressed in models of quantum field theory, which is most probably due to the fact that quantum field models involve new subtleties as compared to the quantum mechanical ones. These can however be addressed with the recently developed methods of multiscale analysis and spectral renormalization (while we rely on the former in this work). We provide a representation of the scattering matrix in terms of an expectation value of the resolvent of a spectrally dilated Hamiltonian; see Theorem 2.2 below. The relation of the scattering matrix and the resonance can then be read of this formula; see Eqs. (2.3) and (2.4) below. Loosely put, our results imply that, for the photon momenta $|k'|$ in a neighborhood of $\text{Re} \lambda_1 - \lambda_0$, the leading order (in $g$ for small $g$) of the integral kernel of the transition matrix fulfills

$$|T(k, k')|^2 \sim \frac{E_1^2 g^4}{(|k'| + \lambda_0 - \text{Re} \lambda_1)^2 + g^4 E_1^2},$$

(1.1)

where we define

$$E_1 := g^{-2} \text{Im} \lambda_1,$$

(1.2)

and it turns out that there are constant numbers $E_I < 0$, $a > 0$ and a uniformly bounded function $\Delta \equiv \Delta(g)$ such that $E_1 = E_I + g^a \Delta$. Heuristically, for an experiment in which a two-level atom is irradiated with monochromatic incoming light quanta of momentum $k' \in \mathbb{R}^3$, the relation (1.1) states that the intensity of the outgoing light quanta with momentum $k \in \mathbb{R}^3$ is proportional to $|T(k, k')|^2$, which is given as a Lorentzian function with maximum at $|k'| = \text{Re} \lambda_1 - \lambda_0$ and width $2 \text{Im} \lambda_1$. This relation is already found as folklore knowledge in physics text-books. In this work we give a rigorous derivation in the model at hand. On the other hand, the relation between the imaginary value of the resonance and the decay rate of the unstable excited state was established rigorously in several articles [1,13,29,37].

In [7], a rigorous mathematical justification of Bohr’s frequency condition is derived, using an expansion of the scattering amplitudes with respect to powers the fine-structure constant for the Pauli-Fierz model. In particular, they calculate the leading order term and provide an algorithm for computing the other terms. In [12], the photoelectric effect is studied for a model of an atom with a single bound state, coupled to the quantized electromagnetic field. In their work, they use similar techniques for estimating time evolution as the ones presented in this manuscript.
1.1. The Spin-Boson model. In this section we introduce the considered model and state preliminary definitions and well-known tools and facts from which we start our analysis. If the reader is already familiar with the introductory Sections 1.1 until 1.2 of [14], these subsections can be skipped. The notation is identical and these subsections are only given for the purpose of self-containedness.

The non-interacting Spin-Boson Hamiltonian is defined as

\[ H_0 := K + H_f, \quad K := \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix}, \quad H_f := \int d^3k \omega(k)a(k)^*a(k). \tag{1.3} \]

We regard \( K \) as an idealized free Hamiltonian of a two-level atom. As already stated in the introduction, its two energy levels are denoted by the real numbers \( 0 = e_0 < e_1 \). Furthermore, \( H_f \) denotes the free Hamiltonian of a massless scalar field having dispersion relation \( \omega(k) = |k| \), and \( a, a^* \) are the annihilation and creation operators on the standard Fock space which will be defined in (1.12) and (1.13) below. In the following we will sometimes call \( K \) the atomic part, and \( H_f \) the free field part of the Hamiltonian. The sum of the free two-level atom Hamiltonian \( K \) and the free field Hamiltonian \( H_f \) will simply be referred to as the “free Hamiltonian” \( H_0 \).

The interaction term reads

\[ V := \sigma_1 \otimes (a(f) + a(f)^*), \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.4} \]

where the boson form factor is given by

\[ f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad k \mapsto e^{-\frac{k^2}{\Lambda^2}|k|^{-\frac{1}{2}} + \mu}. \tag{1.5} \]

Note that the relativistic form factor of a scalar field should rather be \( f(k) = (2\pi)^{-\frac{3}{2}}(2|k|)^{-\frac{1}{2}} \), which however renders the model ill-defined due to the fact that such an \( f \) would not be square integrable. This is referred to as ultraviolet divergence. In our case, the gaussian factor in (1.5) acts as an ultraviolet cut-off for \( \Lambda > 0 \) being the ultraviolet cut-off parameter and in addition the fixed number

\[ \mu \in (0, 1/2) \tag{1.6} \]

implies a regularization of the infrared singularity at \( k = 0 \) which is a technical assumption chosen for this work to keep the proofs more tractable. With additional work, one can also treat the case \( \mu = 0 \) with methods described in [3]. The missing factor of \( 2^{-\frac{1}{2}}(2\pi)^{-\frac{3}{2}} \) will be absorbed in the coupling constant \( g \) in our notation. Note that the form factor \( f \) only depends on the radial part of \( k \). To emphasize this, we often write \( f(k) \equiv f(|k|) \).

The full Spin-Boson Hamiltonian is then defined as

\[ H := H_0 + gV \tag{1.7} \]

for some coupling constant \( g > 0 \) on the Hilbert space

\[ \mathcal{H} := \mathcal{K} \otimes \mathcal{F}[\hbar], \quad \mathcal{K} := \mathbb{C}^2, \tag{1.8} \]

where

\[ \mathcal{F}[\hbar] := \bigoplus_{n=0}^{\infty} \mathcal{F}_n[\hbar], \quad \mathcal{F}_n[\hbar] := \hbar^{\otimes n}, \quad \hbar := L^2(\mathbb{R}^3, \mathbb{C}) \tag{1.9} \]
denotes the standard bosonic Fock space, and superscript \( \odot n \) denotes the \( n \)-th symmetric tensor product, where by convention \( \mathcal{H}^{\odot 0} \equiv \mathbb{C} \). Note that we identify \( K = K \otimes 1_{\mathcal{H}[h]} \) and \( H_f = 1_K \otimes H_f \) in our notation (see Remark 1.2 below).

Due to the direct sum, an element \( \Psi \in \mathcal{F}[h] \) can be represented as a family \( (\psi^{(n)})_{n \in \mathbb{N}_0} \) of wave functions \( \psi^{(n)} \in \mathcal{H}^{\odot n} \). The state \( \Psi \) with \( \psi^{(0)} = 1 \) and \( \psi^{(n)} = 0 \) for all \( n \geq 1 \) is called the vacuum and is denoted by

\[
\Omega := (1, 0, 0, \ldots) \in \mathcal{F}[h].
\]

We define

\[
\mathcal{F}_0 := \left\{ \Psi = (\psi^{(n)})_{n \in \mathbb{N}_0} \in \mathcal{F}[h] \mid \exists N \in \mathbb{N}_0 : \psi^{(n)} = 0 \right\},
\]

where \( \mathcal{S}(\mathbb{R}^{3n}, \mathbb{C}) \) denotes the Schwartz space of infinitely differentiable functions with rapid decay.

Then, for any \( h \in \mathfrak{h} \), we define the operator \( a(h) : \mathcal{F}_0 \to \mathcal{F}_0 \) by

\[
(a(h)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n + 1} \int d^3 k \, h(k) \psi^{(n+1)}(k, k_1, \ldots, k_n)
\]

and \( a(h)\Omega = 0 \). The operator \( a(h) \) is closable and, using a slight abuse of notation, we denote its closure by the same symbol \( a(h) \) in the following. The operator \( a(h) \) is called the annihilation operator. The creation operator is defined as the adjoint of \( a(h) \) and we denote it by \( a(h)^* \). For \( \Psi = (\psi^{(n)})_{n \in \mathbb{N}_0} \in \mathcal{F}_0 \), we find that

\[
(a(h)^*\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h(k_i) \psi^{(n-1)}(k_1, \ldots, \tilde{k}_i, \ldots, k_n),
\]

where the notation \( \tilde{\cdot} \) means that the corresponding variable is omitted.

Occasionally, we shall also use the physics notation and define the point-wise creation and annihilation operators. The action of the latter in the \( n \) boson sector is to be understood as:

\[
(a(k)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n + 1} \psi^{(n+1)}(k, k_1, \ldots, k_n),
\]

for \( \Psi = (\psi^{(n)})_{n \in \mathbb{N}_0} \in \mathcal{F}_0 \). The operator \( a(k) \) is not closable. The point-wise creation operator \( a(k)^* \) is only defined as a quadratic form on \( \mathcal{F}_0 \) in the following sense:

\[
\langle \Phi, a(k)^*\Psi \rangle = \langle a(k)\Phi, \Psi \rangle, \quad \forall \Phi, \Psi \in \mathcal{F}_0.
\]

Moreover, we define quadratic forms:

\[
\mathcal{F}_0 \times \mathcal{F}_0 \to \mathbb{C}, \quad (\Phi, \Psi) \mapsto \int d^3 k \, h(k) \langle \Phi, a(k)\Psi \rangle,
\]

and

\[
\mathcal{F}_0 \times \mathcal{F}_0 \to \mathbb{C}, \quad (\Phi, \Psi) \mapsto \int d^3 k \, h(k) \langle \Phi, a(k)^*\Psi \rangle.
\]
It is not difficult to see that these quantities are equal to \( \langle \Phi, a(h) \Psi \rangle \) and \( \langle \Phi, a(h)^* \Psi \rangle \), respectively. The point-wise creation operator \( a(k)^* \) is not defined as an operator but, formally, we can express it in the following way:

\[
(a(k)^* \Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta^{(3)}(k - k_i) \psi^{(n-1)}(k_1, \ldots, \tilde{k}_i, \ldots, k_n). \tag{1.18}
\]

This is the usual formula that physicists use. Here, \( \delta \) denotes the Dirac’s delta tempered distribution acting on the Schwartz space of test functions. Note that \( a \) and \( a^* \) fulfill the canonical commutation relations:

\[
\forall h, l \in \mathfrak{h}, \quad [a(h), a^*(l)] = \langle h, l \rangle_2, \quad [a(h), a(l)] = 0, \quad [a^*(h), a^*(l)] = 0. \tag{1.19}
\]

Let us recall some well-known facts about the introduced model. Clearly, \( K \) is self-adjoint on \( \mathcal{K} \) and its spectrum consists of two eigenvalues \( e_0 \) and \( e_1 \). The corresponding eigenvectors are

\[
\varphi_0 = (0, 1)^T \quad \text{and} \quad \varphi_1 = (1, 0)^T \quad \text{with} \quad K\varphi_i = e_i\varphi_i, \quad i = 0, 1. \tag{1.20}
\]

Moreover, \( H_f \) is self-adjoint on its natural domain \( \mathcal{D}(H_f) \subset \mathcal{F}[\mathfrak{h}] \) and its spectrum \( \sigma(H_f) = [0, \infty) \) is absolutely continuous (see [36]). Consequently, the spectrum of \( H_0 \) is given by \( \sigma(H_0) = [e_0, \infty) \), and \( e_0, e_1 \) are eigenvalues embedded in the absolutely continuous part of the spectrum of \( H_0 \) (see [35]).

Finally, also the self-adjointness of the full Hamiltonian \( H \) is well-known (see, e.g., [31]) and it can be shown using the standard estimate in Lemma A.1.

**Proposition 1.1.** The operator \( gV \) is relatively bounded by \( H_0 \) with infinitesimal bound and, consequently, \( H \) is self-adjoint and bounded below on the domain

\[
\mathcal{D}(H) = \mathcal{D}(H_0) = \mathcal{K} \otimes \mathcal{D}(H_f), \tag{1.21}
\]

i.e., there is a constant \( b \in \mathbb{R} \) such that

\[
b \leq H. \tag{1.22}
\]

**Remark 1.2.** In this work we omit spelling out identity operators whenever unambiguous. For every vector spaces \( V_1, V_2 \) and operators \( A_1 \) and \( A_2 \) defined on \( V_1 \) and \( V_2 \), respectively, we identify

\[
A_1 \equiv A_1 \otimes 1_{V_2}, \quad A_2 \equiv 1_{V_1} \otimes A_2. \tag{1.23}
\]

In order to simplify our notation further, and whenever unambiguous, we do not utilize specific notations for every inner product or norm that we employ.
1.2. Access to the resonance: complex dilation. It is known (e.g., [31]) that the only eigenvalue in the spectrum of $H$ is

$$\lambda_0 := \inf \sigma(H)$$

while the rest of the spectrum is absolutely continuous. This implies that there is no stable excited state in the massless Spin-Boson model. Heuristically, the reason for this is that the atomic energy of the excited state $e_1$ turns into what can be seen as a complex “energy” $\lambda_1$ with strictly negative imaginary part once the interaction is switched on (see e.g. [4,5]). This complex energy $\lambda_1$ is referred to as resonance energy and its imaginary part is responsible for the decay of the excited state (see e.g. [1,29]).

Note that the ground state $\Psi_{\lambda_0}$ of $H$ corresponding to ground state energy $\lambda_0$, i.e.,

$$H \Psi_{\lambda_0} = \lambda_0 \Psi_{\lambda_0},$$

has already been constructed, e.g., in [31, Theorem 1], [28, Theorem 1] and [3, Theorem 3.5]. Since $H$ on $\mathcal{H}$ is a self-adjoint operator, $\lambda_1$ should rather be thought of as a complex eigenvalue of $H$ on a bigger space than $\mathcal{H}$. This prevents us from being able to calculate the resonance energy directly by regular perturbation theory on $\mathcal{H}$. The standard way to nevertheless get access to such a resonance without leaving the underlying Hilbert space is the method of complex dilation which will be introduced next. We start by defining a family of unitary operators on $\mathcal{H}$ indexed by $\theta \in \mathbb{R}$.

**Definition 1.3.** For $\theta \in \mathbb{R}$, we define the unitary transformation

$$u_\theta : \mathfrak{h} \to \mathfrak{h}, \quad \psi(k) \mapsto e^{-\theta |k|^2/2} \psi(e^{-\theta} k).$$

Similarly, we define its canonical lift $U_\theta : \mathcal{F}[\mathfrak{h}] \to \mathcal{F}[\mathfrak{h}]$ by the lift condition $U_\theta a(h)^* U_\theta^{-1} = a(u_\theta h)^*$, $h \in \mathfrak{h}$, and $U_\theta \Omega = \Omega$. This defines $U_\theta$ uniquely up to a phase which we choose to equal one. With slight abuse of notation, we also denote $1_{\mathbb{K}} \otimes U_\theta$ on $\mathcal{H}$ by the same symbol $U_\theta$.

Thereby, we define the family of transformed Hamiltonians, for $\theta \in \mathbb{R}$,

$$H^\theta := U_\theta H U_\theta^{-1} = K + H^\theta_f + g V^\theta,$$

where

$$H^\theta_f := \int d^3 k \, \omega^\theta(k) a^*(k) a(k), \quad V^\theta := \sigma_1 \otimes \left( a(f^\theta) + a(f^\theta)^* \right)$$

and

$$\omega^\theta(k) := e^{-\theta |k|}, \quad f^\theta : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad k \mapsto e^{\frac{\theta}{2} (1+\mu)} e^{-\frac{\theta^2 k^2}{\Lambda^2}} |k|^{-\frac{1}{2}+\mu}. \quad (1.29)$$

Equations (1.29), (1.28) and the right hand side of Eq. (1.27) can be defined for complex $\theta$. If $|\theta|$ is small enough, $K + H^\theta_f + g V^\theta$ is a closed (non self-adjoint) operator. However, the middle term in Eq. (1.27) is not necessarily correct because although $U_\theta$ can be defined for complex $\theta$, it turns out to be an unbounded operator, and $U_\theta H U_\theta^{-1}$ might not be densely defined.

We say that $\Psi$ is an analytic vector if the map $\theta \mapsto \Psi^\theta := U_\theta \Psi$ has an analytic continuation from an open connected set in the real line to a (connected) domain in the complex plane. In general we will not specify their domains of analyticity (it will
be clear from the context). It is well-known that there is a dense set of entire vectors (they are analytic in \(C\)). This result has been proven in a variety of similar models, for example, in [4,32]. For the sake of completeness, we provide a proof in “Appendix B”. Furthermore, we define the open disc

\[
D(x, r) := \{ z \in C : |z - x| < r \} \quad x \in C, r > 0,
\]

and note that for \(\theta \in D(0, \pi/16)\) we have

\[
\left\| V^{\theta} (H_0 + 1)^{-1/2} \right\| \leq \| f^{\theta} \|_2 + 2 \left\| f^{\theta} / \sqrt{\omega} \right\|_2
\]

which is guaranteed by the standard estimate \((A.4)\) given in “Appendix A”, since \((1.29)\) together with the special choice \(\theta \in D(0, \pi/16)\) imply that \( f^{\theta}, f^{\theta} / \sqrt{\omega} \in \mathfrak{h}\). Hence, for \(\theta \in D(0, \pi/16)\) the operators \(H^{\theta}\) are densely defined and closed. Moreover, the analytic properties of this family of operators in \(g\) and \(\theta\) are known:

**Lemma 1.4.** The family \(\{H^{\theta}\}_{\theta \in \mathbb{R}}\) of unitary equivalent, self-adjoint operators with \(D(H^{\theta}) = D(H)\) extends to an analytic family of type A for \(\theta \in D(0, \pi/16)\).

The above result was proven for the Pauli-Fierz model in [4, Theorem 4.4], and with small effort that proof can be adapted to our setting.

**Lemma 1.5.** Let \(\theta \in \mathbb{C}\). Then, \(\sigma(H_0^{\theta}) = \{ e_i + e^{-\theta} r : r \geq 0, i = 0, 1 \}\).

We provide a proof in “Appendix B”. For sufficiently small coupling constants and for \(\theta \in \mathcal{S}\), where \(\mathcal{S}\) is the subset of the complex plane defined in Eq. (3.2) below, it has been shown that \(H^{\theta}\) has two non-degenerate eigenvalues \(\lambda_0^{\theta}\) and \(\lambda_1^{\theta}\) with corresponding rank one projectors denoted by \(P_0^{\theta}\) and \(P_1^{\theta}\), respectively; see, e.g., [14, Proposition 2.1]. Note that there the \(\theta\)-dependence was omitted in the notation. For convenience of the reader, we make it explicit in this paper. The corresponding dilated eigenstates can, therefore, be written as

\[
\Psi_{{\lambda}_i}^{\theta} := P_{{\lambda}_i}^{\theta} \psi_i \otimes \Omega, \quad i = 0, 1,
\]

where the eigenstates \(\psi_i\) of the free atomic system are given in \((1.20)\), and \(\Omega\) is the bosonic vacuum defined in \((1.10)\). In our notation \(\Psi_{{\lambda}_i}^{\theta}\) is not necessarily normalized. We know from [14, Theorem 2.3] that the eigenvalues \(\lambda_i^{\theta}\) are independent of \(\theta\) as long as \(\theta\) belongs to the set \(\mathcal{S}\), and therefore, we suppress it in our notation writing \(\lambda_i^{\theta} \equiv \lambda_i\). Note that this is not true for the eigenstates \(\Psi_{{\lambda}_i}^{\theta}\). In [14] (as well as in Eq. (3.2) below) we choose an open connected set \(\mathcal{S}\) that does not include 0 (the imaginary parts of the points in this set are bounded from below by a fixed positive constant). We chose such a set in order to have a single set \(\mathcal{S}\) for the cases \(i = 0\) and \(i = 1\), because we want to keep our notation as simple as possible (otherwise a two cases formulation would propagate all over our papers). However, the fact that 0 is not contained in \(\mathcal{S}\) is only necessary for the case \(i = 1\) (the resonance - due to the self-adjointness of \(H\) the state \(\Psi_{{\lambda}_1}^{\theta}\) can not even exist for \(\theta = 0\)). For the case \(i = 0\) (the ground state) we can choose instead a connected open set containing 0. In this set, it is still valid that \(\lambda_0^{\theta}\) does not depend on \(\theta\) and, therefore, it equals the ground state energy, and \(\Psi_{{\lambda}_0}^{\theta = 0} = \Psi_{\lambda_0}\) - as introduced above. This is explained in [14, Remark 2.4].
1.3. Scattering theory. Finally, we give a short review of scattering theory which will be necessary to state the main results in Sect. 2.

The first obstacle in formulating a scattering theory of a second-quantized system lies in the definition of the wave operators. Unlike in first-quantized quantum theory, where one defines the scattering operator to be \( S := \Omega^* \Omega \) with the wave operators \( \Omega \) given by the strong limits \( \Omega := \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} \), in quantum field theory, the corresponding wave operators do not exist in a straightforward sense. Instead, one establishes the existence of the asymptotic annihilation and creation operators first, which can then be used to define the wave operators.

**Definition 1.6** (Basic components of scattering theory). We denote the dense subspace of compactly supported, smooth, and complex-valued functions on \( \mathbb{R}^3 \setminus \{0\} \) by

\[
\mathfrak{h}_0 := C_c^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \subset \mathfrak{h}.
\]

Furthermore, we define the following objects:

(i) For \( h \in \mathfrak{h}_0 \) and \( \Psi \in \mathcal{K} \otimes \mathcal{D}(H_f^{1/2}) \), the asymptotic annihilation operators

\[
a_\pm(h)\Psi := \lim_{t \to \pm \infty} a_t(h)\Psi, \quad a_t(h) := e^{itH} a(h_t) e^{-itH}, \quad h_t(k) := h(k)e^{-it\omega(k)}.
\]

The existence of this limit is proven in Lemma 4.1 (i) below. Moreover, we define the asymptotic creation operators \( a^*_\pm(h) \) as the respective adjoints.

(ii) The asymptotic Hilbert spaces

\[
\mathcal{H}_\pm := \mathcal{K}_\pm \otimes \mathcal{F}[h] \quad \text{where} \quad \mathcal{K}_\pm := \{ \Psi \in \mathcal{H} : a_\pm(h)\Psi = 0 \quad \forall h \in \mathfrak{h}_0 \}.
\]

(iii) The wave operators

\[
\Omega_\pm : \mathcal{H}_\pm \to \mathcal{H}
\]

\[
\Omega_\pm \Psi \otimes a^*(h_1) \ldots a^*(h_n) \Omega := a^*_\pm(h_1) \ldots a^*_\pm(h_n) \Psi, \quad h_1, \ldots, h_n \in \mathfrak{h}_0, \quad \Psi \in \mathcal{K}_\pm.
\]

(iv) The scattering operator \( S := \Omega^*_+ \Omega_- \).

The limit operators \( a_\pm \) and \( a^*_\pm \) are called asymptotic outgoing/ingoing annihilation and creation operators. The existence of the limits in (1.34), their properties, especially that \( \Psi_{\lambda_0} \in \mathcal{K}_\pm \) and \( \Omega_\pm \) are well-defined, are well-known facts (see e.g. [12,16–20,22–25]). For the convenience of the reader, Lemma 4.1 collects all relevant facts and we provide simplified proofs for our setting in “Appendix C”. We can thus define the following two-body scattering matrix coefficients:

\[
S(h, l) = \| \Psi_{\lambda_0} \|^2 \langle a^*_+(h) \Psi_{\lambda_0}, a^*_-(l) \Psi_{\lambda_0} \rangle, \quad \forall h, l \in \mathfrak{h}_0.
\]

where the factor \( \| \Psi_{\lambda_0} \|^2 \) appears due to the fact that, as already mentioned above, in our notation, the ground state \( \Psi_{\lambda_0} \) is not necessarily normalized. In addition, it will be convenient to work with the corresponding two-body transition matrix coefficients given by

\[
T(h, l) = S(h, l) - \langle h, l \rangle_2 \quad \forall h, l \in \mathfrak{h}_0.
\]
These matrix coefficients carry a ready physical interpretation as transition amplitudes of the scattering process in which an incoming boson with wave function $l$ is scattered at the two-level atom into an outgoing boson with wave function $h$. Notice that the transition matrix coefficients of multi-photon processes can be defined likewise but in this work we focus on one-photon processes only.

It has been shown in [31] that the spectrum of $H$ contains only one eigenvalue $\lambda_0$ (and it is non-degenerate), namely the ground state energy, and the rest of the spectrum of $H$ is absolutely continuous. In case that asymptotic completeness holds, i.e.

$$K^{\pm} = \text{Ran}(\chi_\text{pp}(H)),$$

all one-boson processes are of the form (1.37). Here, $\text{Ran}(\chi_\text{pp}(H))$ denotes the states associated with pure points in the spectrum of $H$.

Asymptotic completeness has actually been proven in [17–19] for the Hamiltonian $H$ defined in (1.7), however, with coupling functions $f \in C^3_c(\mathbb{R}^3\setminus\{0\}, \mathbb{C})$, i.e., the functions that are three times continuously differentiable and have compact support. In our case, we need an analytic continuation of our Hamiltonian in order to study resonances. This implies that the coupling function $f$ cannot be compactly supported (see (1.5)), however it belongs to the Schwartz space. We expect asymptotic completeness also to hold in our case, although our results do not depend on it.

2. Main Result

We are now able to state our main results. The corresponding proofs will be provided in Sect. 4 after we review a list of necessary results of a previous work [14] in Sect. 3.

First, we state a definition that we use for our main result

**Definition 2.1.** Using solid angles $d\Sigma, d\Sigma'$, we define, for all $h, l \in \mathfrak{h}_0$,

$$G : \mathbb{R} \to \mathbb{C}, \quad r \mapsto G(r) := \begin{cases} \int d\Sigma d\Sigma' r^4 h(r, \Sigma) l(r, \Sigma') f(r)^2 & \text{for } r \geq 0 \\ 0 & \text{for } r < 0. \end{cases}$$

(2.1)

We recall the definition $E_1 = g^{-2} \text{Im} \lambda_1$ given in (1.2). It follows from Eqs. (3.11) and (3.12) below that $E_1 = E_{\bar{I}} + g^a \Delta$ where $a > 0$, $\Delta \equiv \Delta(g)$ is uniformly bounded and $E_{\bar{I}} < 0$ is the constant defined in (3.11). This implies that

$$E_1 \leq -c < 0,$$

(2.2)

for some constant $c$ that does not depend on $g$ (for small enough $g$).

Our main result provides a relation between the scattering matrix element and the complex dilated resolvent of the Hamiltonian.

**Theorem 2.2** (Scattering formula). There is a constant $g > 0$ such that for every $g \in (0, g]$, $\theta$ in the set $S$ defined in (3.2) below, and for all $h, l \in \mathfrak{h}_0$, the two-body transition matrix coefficients are given by

$$T(h, l) = T_P(h, l) + R(h, l),$$

(2.3)
where
\[ T_P(h, l) := 4\pi i g^2 \| \Psi_{\lambda_0} \|^{-2} \int dr \, G(r) \left( \frac{\Re \lambda_1 - \lambda_0}{(r + \lambda_0 - \lambda_1)(r - \lambda_0 + \lambda_1)} \right) \]
\[ = M \int dr \, G(r) \left( \frac{E_1 g^2}{(r + \lambda_0 - \Re \lambda_1 - ig^2 E_1)(r - \lambda_0 + \lambda_1)} \right), \tag{2.4} \]
and there is a constant \( C(h, l) \) (that does not depend on \( g \)) such that
\[ |R(h, l)| \leq C(h, l) g^3 |\log g|. \tag{2.5} \]

Here, we use the notation
\[ M := 4\pi i (\Re \lambda_1 - \lambda_0) E_1^{-1} \| \Psi_{\lambda_0} \|^{-2}. \tag{2.6} \]

\( T_P(h, l) \) is the leading term in terms of powers of \( g \) for small \( g \), and \( R(h, l) \) is regarded as the error term. This is justified by Remark 2.3 below.

Our proof permits us to find an explicit formula for the dependence of \( C(h, l) \) on \( h \) and \( l \), see Remark 4.16 below.

**Remark 2.3.** The scattering processes described by the transition matrix in (2.3) clearly depend on the incoming and outgoing photon states, \( l \) and \( h \). This is well understood from the physics as well as the mathematics perspectives. For example, it can be read from (2.1) that, if \( l \) is supported on a ball of radius \( t \) and \( h \) is supported on its complement, then the principal term \( T_P(h, l) \) vanishes and only higher order terms (with respect to powers of \( g \)) contribute to the scattering process. The quantity \( T_P(h, l) \) is the only one that might produce scattering processes of order \( g^2 \) since the remainder is of order \( g^3 |\log g| \). If an experiment is appropriately prepared, then such a scattering process will be observed and the term describing this is \( T_P(h, l) \). This justifies why we call it the leading order (or principal) term. In “Appendix D”, we give an example of a large class of functions \( h \) and \( l \) that make \( T_P(h, l) \) larger or equal than a strictly positive constant times \( g^2 \). In particular, we prove that this happens when the corresponding function \( G \) is positive and strictly positive at \( \Re \lambda_1 - \lambda_0 \).

**Remark 2.4.** By Eqs. (2.4) and (2.1), we can express the principal term \( T_P(h, l) \) in terms of an integral kernel:
\[ T_P(h, l) = \int d^3 k d^3 k' \frac{\bar{h}(k') l(k') \delta(|k| - |k'|)}{\| k \|} T_P(k, k'), \tag{2.7} \]
where
\[ T_P(k, k') = M f(k) f(k') \left( \frac{E_1 g^2}{(|k'| + \lambda_0 - \Re \lambda_1 - ig^2 E_1)(|k'| - \lambda_0 + \lambda_1)} \right). \tag{2.8} \]

Equation (2.7) is important, because it allows us to calculate the leading order of the scattering cross section. It is proportional to the modulus squared of \( T_P(k, k') \):
\[ |T_P(k, k')|^2 = \left( \frac{|M|^2 |f(k)|^2 |f(k')|^2}{|| k' | - \lambda_0 + \lambda_1 ||^2} \right) \frac{E_1^2 g^4}{(|k'| + \lambda_0 - \Re \lambda_1)^2 + g^4 E_1^2}. \tag{2.9} \]
For momenta $|k'|$ in a neighborhood of $\text{Re} \lambda_1 - \lambda_0$, the behavior in the expression above is dominated by the Lorentzian function. As expected, there is a maximum when the energy of the incoming photons is close to the difference of the resonance and the ground state energies of the system and the width of this peak is controlled by the imaginary part of the resonance $\text{Im} \lambda_1$.

Note that the Dirac’s delta distribution in (2.7) is to be understood as the expression in (2.4). Notice that (2.8) is not defined for $k = 0$ or $k' = 0$. However, since we take $h, l \in \mathcal{C}_c^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$, the expression (2.7) is well-defined. Similar distribution kernels in a related model have been studied in [7,12].

**Remark 2.5.** In this work we denote by $C$ any generic (indeterminate) constant that might change from line to line. This constants do not depend on the coupling constant and the auxiliary parameter $n$ introduced in Sect. 3.2.

### 3. Known Results on Spectral Properties and Resolvent Estimates

In this section we present results about the spectrum of the dilated Spin-Boson Hamiltonian and resolvent estimates proven in our previous paper [14]. Here, we do not repeat proofs but give precise references for them. We collect only properties and estimates for the model under consideration that are necessary for the proofs of our main theorems.

Throughout this paper we address the case of small coupling, i.e., we assume the coupling constant $g$ to be sufficiently small. The restrictions on the coupling constant only stem from the requirements needed to prove the results reviewed in this section, i.e., the ones considered in [14]. We do not explicitly specify how small the coupling constant must be but give precise references from which such bounds can be inferred. This issue is addressed by the next definition:

**Definition 3.1 (Coupling Constant).** Throughout this work we assume that $g \leq g_s$ where $0 < g_s$ satisfies Definition 4.3 and Eq. (5.58) in [14], the Fermi-golden rule (see Eqs. (3.12) and (3.13) below) and Eq. (3.28) below.

We denote the imaginary part of the dilation parameter $\theta$ by

$$\nu := \text{Im} \theta$$

and assume that $\theta$ belongs to the set

$$S := \{ \theta \in \mathbb{C} : -10^{-3} < \text{Re} \theta < 10^{-3} \text{ and } \nu < \text{Im} \theta < \pi / 16 \},$$

where $\nu \in (0, \pi / 16)$ is a fixed number (see [14, Definition 1.4]).

### 3.1. Spectral estimates.** We know from [14, Proposition 2.1] that the Hamiltonian $H^\theta$ has two eigenvalues $\lambda_0$ and $\lambda_1$ in small neighborhoods of $e_0$ and $e_1$, respectively. Loosely put, $e_0$ turns into the ground state $\lambda_0$ and $e_1$ turns into the resonance $\lambda_1$ once the interaction is tuned on. Both $\lambda_0$ and $\lambda_1$ do not depend on $\theta$ provided that $\theta \in S$ and in the case of $\lambda_0$ we can take $\theta$ in a neighborhood of 0 and, therefore, infer that $\lambda_0$ is real and gives the ground state energy. This is proven in [14, Theorem 2.3] and [14, Remark 2.4].

In [14, Theorem 2.7], we give a very sharp estimation of the location of the spectrum of $H^\theta$. We prove, among other things that, locally, in neighborhoods of $\lambda_0$ and $\lambda_1$, its
spectrum is contained in cones with vertices at \( \lambda_0 \) and \( \lambda_1 \). To make this statement more precise we need to introduce some more concepts and notation. There are two auxiliary parameters that play an important role in our constructions:

\[
\rho \in (0, 1), \quad \rho_0 \in (0, \min(1, e_1/4)),
\]

which also satisfy the conditions in (3.31) below. In order to specify the spectral properties of \( H^\theta \) we define some regions in the complex plane:

**Definition 3.2.** For fixed \( \theta \in \mathcal{S} \), we set \( \delta = e_1 - e_0 = e_1 \) and define the regions

\[
A := A_1 \cup A_2 \cup A_3,
\]

where

\[
A_1 := \{ z \in \mathbb{C} : \text{Re} z < e_0 - \delta/2 \}
\]

\[
A_2 := \left\{ z \in \mathbb{C} : \text{Im} z > \frac{1}{8} \delta \sin(\nu) \right\}
\]

\[
A_3 := \{ z \in \mathbb{C} : \text{Re} z > e_1 + \delta/2, \text{Im} z \geq -\sin(\nu/2) (\text{Re}(z) - (e_1 + \delta/2)) \}
\]

and for \( i = 0, 1 \), we define

\[
B_{i}^{(1)} := \left\{ z \in \mathbb{C} : |\text{Re} z - e_i| \leq 1/2 \delta, -1/2 \rho_1 \sin(\nu) \leq \text{Im} z \leq 1/8 \delta \sin(\nu) \right\}.
\]

These regions are depicted in Fig. 1.

For a fixed \( m \in \mathbb{N}, m \geq 4 \), we define the cone

\[
C_m(z) := \left\{ z + xe^{-i\alpha} : x \geq 0, |\alpha - \nu| \leq \nu/m \right\}.
\]

It follows from the induction scheme in [14, Section 4] that \( \lambda_i \in B_{i}^{(1)} \), and moreover, [14, Theorem 2.7] together with [14, Lemma 3.13] yields

\[
\sigma(H^\theta) \subset \mathbb{C} \left[ A \cup (B_{0}^{(1)} \setminus C_m(\lambda_0)) \cup (B_{1}^{(1)} \setminus C_m(\lambda_1)) \right].
\]

![Fig. 1. An illustration of the subsets of the complex plane introduced in Definition 3.2](image)
As we mention above, we have $\lambda_0 \in \mathbb{R}$. The imaginary part of $\lambda_1$ can be also estimated (see [14, Remark 2.2] – Fermi golden rule): Recalling (1.5), we define

$$E_I := -4\pi^2 (e_1 - e_0)^2 |f(e_1 - e_0)|^2.$$  \hspace{1cm} (3.11)

Then, for $g$ small enough, there are constants $C, a > 0$ such that

$$\left| \text{Im} \lambda_1 - g^2 E_I \right| \leq g^2 + aC.$$  \hspace{1cm} (3.12)

This implies that, for $g$ small enough, there is constant $c > 0$ such that

$$\text{Im} \lambda_1 < -g^2 c < 0.$$  \hspace{1cm} (3.13)

### 3.2. Auxiliary (infrared cut-off) Hamiltonians

Some of the bounds in Sect. 4 employ a certain approximation of the Hamiltonian $H^\theta$ by Hamiltonians with infrared cut-offs. The strategy will be the following: A mathematical expression that depends on $H^\theta$ is replaced by a corresponding one that depends on a particular infrared cut-off Hamiltonian. We then analyze the infrared cut-off expression and estimate the difference between both expressions. The construction of a sequence of infrared cut-off Hamiltonians $(H^{(n),\theta})$ such that, as $n$ tends to infinity, the cut-off is removed is called multiscale analysis. In [14], we present the full details of this method and derive several results. Here, we only use some of those results and only present the notation necessary to review this part of [14]. The infrared cut-off Hamiltonians $H^{(n),\theta}$ are parametrized by a sequence of numbers (see also (3.3) and (3.31))

$$\rho_n := \rho_0 \rho^n,$$  \hspace{1cm} (3.14)

where the Hamiltonians $H^{(n),\theta}$ are defined by

$$H^{(n),\theta} := K + H_f^{(n),\theta} + gV^{(n),\theta} =: H_0^{(n),\theta} + gV^{(n),\theta}$$  \hspace{1cm} (3.15)

$$H_f^{(n),\theta} := \int_{\mathbb{R}^3 \setminus B_{\rho_n}} d^3k \omega^\theta(k) a(k)a^*(k), \quad \omega^\theta(k) = e^{-\theta |k|}$$  \hspace{1cm} (3.16)

$$V^{(n),\theta} := \sigma_1 \otimes \int_{\mathbb{R}^3 \setminus B_{\rho_n}} d^3k \left( f^\theta(k) a(k) + f^\theta(k) a^*(k) \right),$$  \hspace{1cm} (3.17)

$$f^\theta : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}, \quad k \mapsto e^{-\theta(1+\mu)} e^{-e^{2\theta k^2 / \Lambda^2}} |k|^{-1+\mu},$$  \hspace{1cm} (3.18)

on the Hilbert space

$$\mathcal{H}^{(n)} := \mathcal{K} \otimes \mathcal{F}[\mathfrak{h}^{(n)}], \quad \mathfrak{h}^{(n)} := L^2(\mathbb{R}^3 \setminus B_{\rho_n}, \mathbb{C}), \quad B_{\rho_n} := \left\{ x \in \mathbb{R}^3 : |x| < \rho_n \right\}.$$  \hspace{1cm} (3.19)

Additionally, we define

$$\tilde{H}^{(n),\theta} := H_0^\theta + gV^{(n),\theta}$$  \hspace{1cm} (3.20)

and fix the Hilbert spaces

$$\mathfrak{h}^{(n,\infty)} := L^2(B_{\rho_n}) \quad \text{and} \quad \mathcal{F}[\mathfrak{h}^{(n,\infty)}].$$  \hspace{1cm} (3.21)
defined as in (1.9) with $\mathcal{H}(n,\infty)$ instead of $\mathcal{H}$, with vacuum states $\Omega(n,\infty)$ and corresponding orthogonal projections $P_{\Omega(n,\infty)}$. Note that $\mathcal{H} \equiv \mathcal{H}(n) \otimes \mathcal{F}[\mathcal{H}(n,\infty)]$.

In [14, Proposition 2.1] and [14, Theorem 4.5], we prove that, for each $n \in \mathbb{N}$, $H^{(n),\theta}$ has isolated eigenvalues $\lambda_i^{(n)}$ in certain neighborhoods of $e_i$, for $i \in \{0,1\}$, respectively. The fact that these eigenvalues are isolated permits us to define their corresponding Riesz projections which are denoted by

$$P_i^{(n),\theta} \equiv P_i = \lim_{n \to \infty} P_i^{(n),\theta} \otimes P_{\Omega(n,\infty)},$$

and that the latter is analytic with respect to $\theta$ (see [14, Theorem 2.3]). Furthermore, it follows from [14, Remark 5.11] that

$$\|P_i^{\theta} - P_i^{(n),\theta} \otimes P_{\Omega(n,\infty)}\| \leq 2 \frac{g}{\rho} \rho_n^{\mu/2} \leq \rho_n^{\mu/2}.$$  (3.24)

This together with [14, Lemma 3.6] implies that there is a constant $C$ such that

$$\|P_i^{\theta} - P_i^{(n),\theta} \otimes P_{\Omega(n,\infty)}\| \leq Cg,$$  (3.25)

and in addition, we know from [14, Lemma 4.7] that

$$\|P_i^{(n),\theta}\| \leq 3,$$  (3.26)

for every $n \in \mathbb{N}$. Finally, [14, Lemma 5.1] yields that for all $n \in \mathbb{N}$

$$|\lambda_i - \lambda_i^{(n)}| \leq 2g \rho_n^{1+\mu/2}.$$  (3.27)

This together with [14, Lemma 3.10], which states that there is a constant $C$ such that $|e_i - \lambda_i^{(1)}| < Cg$, proves that there is a constant $C$ such that, for every $n \in \mathbb{N}$ and for $g$ sufficiently small, we have

$$|\lambda_i^{(n)} - e_i| \leq Cg \leq 10^{-3}e_1, \quad |\lambda_i - e_i| \leq Cg \leq 10^{-3}e_1.$$  (3.28)

3.3. Resolvent estimates. In [14], we derive bounds for the resolvent of $H^\theta$ in $A \cup (B_0^{(1)} \setminus C_m(\lambda_0)) \cup (B_1^{(1)} \setminus C_m(\lambda_1))$, see (3.10). The region $A$ is far away from the spectrum, and therefore, resolvent estimates in this region are easy. In [14, Theorem 3.2], we prove that there is a constant $C$ such that

$$\left\| \frac{1}{H^\theta - \bar{z}} \right\| \leq C \frac{1}{|\bar{z} - e_1|}, \quad \forall \bar{z} \in A.$$  (3.29)

Resolvent estimates in the regions $B_0^{(1)} \setminus C_m(\lambda_0)$ and $B_1^{(1)} \setminus C_m(\lambda_1)$ are much more complicated because these regions share boundaries with the spectrum.
In [14, Theorem 5.5], we prove that, for \( i \in \{0, 1\} \), \( B_i^{(1)} \setminus C_m \left( \lambda_i^{(n)} + (1/4) \rho_n e^{-i\nu} \right) \) is contained in the resolvent set of \( H^{(n),\theta} \) and that there is a constant \( C \) such that
\[
\left\| \frac{1}{H^{(n),\theta} - z} - \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| \leq C \frac{n+1}{\text{dist}(z, C_m(\lambda_i^{(n)} + (1/4) \rho_n e^{-i\nu})))},
\] (3.30)
for every \( z \in B_i^{(1)} \setminus C_m \left( \lambda_i^{(n)} + (1/4) \rho_n e^{-i\nu} \right) \), where \( \tilde{P}_i^{(n),\theta} = 1 - P_i^{(n),\theta} \). Here, the symbol \( \text{dist} \) denotes the Euclidean distance in \( \mathbb{C} \).

In [14], we select the auxiliary numbers \( \rho \) and \( \rho_0 \) satisfying
\[
C_8 \rho \mu \leq 1,
\] (3.31)
and hence,
\[
C \rho^{1/(1+\mu/4)} \leq 1,
\] (3.32)
where
\[
\rho \leq 10^{-6} e_1.
\] (3.33)

Finally, we prove in [14, Theorem 5.9] that the set \( B_i^{(1)} \setminus C_m \left( \lambda_i^{(n)} - e^{-i\nu} \rho_n^{1+\mu/4} \right) \) is contained in the resolvent set of both \( H^\theta \) and \( \tilde{H}^{(n),\theta} \) and for all \( z \) in this set there is a constant \( C \) such that:
\[
\left\| \frac{1}{H^\theta - z} - \frac{1}{\tilde{H}^{(n),\theta} - z} \right\| \leq C \frac{n+1}{\rho_n^{1+\mu/4}},
\] (3.34)
where we use (3.31). Notice that Eq. (3.30) implies that there is a constant \( C \) such that
\[
\left\| \frac{1}{H^{(n),\theta} - z} - \frac{1}{\tilde{P}_i^{(n),\theta}} \right\| \leq C \frac{n+1}{\rho_n},
\] (3.35)
for every \( z \in B_i^{(1)} \setminus C_m(\lambda_i^{(n)}) \). Moreover, [14, Theorem 2.6] implies that there is a constant \( C \) such that
\[
\left\| \frac{1}{H^\theta - z} \right\| \leq C \frac{n+1}{\rho_n^{1+\mu/4}},
\] (3.36)
for every \( z \in B_i^{(1)} \setminus C_m(\lambda_i^{(n)} - \rho_n^{1+\mu/4} e^{-i\nu}) \) and
\[
\left\| \frac{1}{H^\theta - z} \right\| \leq C \frac{n+1}{\text{dist}(z, C_m(\lambda_i))},
\] (3.37)
for every \( z \in B_i^{(1)} \setminus C_m(\lambda_i - 2 \rho_n^{1+\mu/4} e^{-i\nu}) \).
4. Proof of the Main Result

In the remainder of this work we provide the proofs of the main result Theorem 2.2. This section has three parts: In Sect. 4.1, we derive a preliminary formula for the scattering matrix coefficients (see Theorem 4.3 below). This formula together with several technical ingredients provided in Sects. 4.2 and 4.3 will pave the way for the proofs of the main results given in Sect. 4.4.

4.1. Preliminary scattering formula. In Theorem 4.3 below we derive a preliminary formula for scattering processes with one incoming and outgoing asymptotic photon. A related formula was already employed in [30]. In order to derive it rigorously we need several properties of the asymptotic creation and annihilation operators. The necessary properties are collected in Lemma 4.1. They have already been proven for a range of models in several works [12,16–20,22–25]. For convenience of the reader we provide a self-contained proof of Lemma 4.1 in the “Appendix C”.

Lemma 4.1. Let $\Psi \in K \otimes D(H^{1/2}_f)$ and $h, l \in h_0$. The asymptotic creation and annihilation operators $a^+_\pm, a^-\pm$ defined in Definition 1.6 have the following properties:

(i) The limits $a^\#\pm(h)\Psi = \lim_{t \to \pm\infty} a^\#_t(h)\Psi$ exist, where $a^\#$ stands for $a$ or $a^*$.

(ii) The next equalities hold true:

\[
a_+(h)\Psi = a(h)\Psi - ig \int_0^\infty ds e^{isH} \langle h_s, f \rangle_2 \sigma_1 e^{-isH} \Psi,
\]

\[
a_-(h)\Psi = a(h)\Psi + ig \int_{-\infty}^0 ds e^{isH} \langle h_s, f \rangle_2 \sigma_1 e^{-isH} \Psi.
\]

We point out to the reader that the integrals above are convergent since it can be shown by integration by parts that there is constant $C$ such that $|\langle h_s, f \rangle_2| \leq C/(1 + s^2)$ for $s \in \mathbb{R}$ (see (C.7) below).

(iii) The following pull-through formula holds true:

\[
e^{-isH} a_-(h)^*\Psi = a_-(h_s)^* e^{-isH} \Psi.
\]

(iv) The equality $a_\pm(h)\Psi_{\lambda_0} = 0$ holds true, i.e., $\Psi_{\lambda_0} \in K^\perp$.

(v) The following commutation relation holds: $\langle a_\pm(h)^*\Psi_{\lambda_0}, a_\pm(l)^*\Psi_{\lambda_0} \rangle = \langle h, l \rangle_2 \| \Psi_{\lambda_0} \|^2$.

(vi) There is a finite constant $C(h) > 0$ such that for all $t \in \mathbb{R}$

\[
\| a_t(h)^* (H_f + 1)^{-1/2} \|, \| a_t(h)(H_f + 1)^{-1/2} \| \leq C(h).
\]

Definition 4.2. Let $S(\mathbb{R}, \mathbb{C})$ denote the Schwartz space of functions with rapid decay. For all $u \in S(\mathbb{R}, \mathbb{C})$, we define the Fourier transform of a function and its inverse

\[
\mathcal{F}[u](x) := \int_{\mathbb{R}} ds \, u(s) e^{-isx}, \quad \mathcal{F}^{-1}[u](x) := (2\pi)^{-1} \int_{\mathbb{R}} ds \, u(s) e^{isx}.
\]

Note the factor $(2\pi)^{-1}$ which is not the normalization factor of the standard definition of the inverse Fourier transform. However, it is convenient in our setting (see e.g. [35]).
**Theorem 4.3 (Preliminary Scattering Formula).** For \( h, l \in \mathfrak{h}_0 \), the two-body transition matrix coefficient \( T(h, l) \) defined in (1.38) fulfills

\[
T(h, l) = \lim_{t \to -\infty} \int d^3k d^3k' \frac{1}{H(k)l(k') \delta(\omega(k) - \omega(k'))} T_t(k, k')
\]  

(4.6)

for the integral kernel

\[
T_t(k, k') = -2\pi i g f(k) \| \Psi_{\lambda_0} \|^{-2} \langle \sigma_1 \Psi_{\lambda_0}, a_t(k')^* \Psi_{\lambda_0} \rangle.
\]

(4.7)

The integral in (4.6) is to be understood as

\[
T(h, l) = -2\pi i g \| \Psi_{\lambda_0} \|^{-2} \left\langle \sigma_1 \Psi_{\lambda_0}, a_- (W)^* \Psi_{\lambda_0} \right\rangle
\]

(4.8)

for \( W \in \mathfrak{h}_0 \) given by

\[
\mathbb{R}^3 \ni k \mapsto W(k) := |k|^2 l(k) \int d \Sigma \, \bar{h}(|k|, \Sigma) f(|k|, \Sigma)
\]

(4.9)

using spherical coordinates \( k = (|k|, \Sigma) \) with \( \Sigma \) being the solid angle.

**Proof.** Let \( h, l \in \mathfrak{h}_0 \). Thanks to Lemma 4.1 (i) and the fact that the ground state \( \Psi_{\lambda_0} \) lies in \( D(H) = \mathcal{K} \otimes D(H_f) \), c.f. [31, Theorem 1] and Proposition 1.1, the transmission matrix coefficient given in (1.38), i.e.,

\[
T(h, l) = S(h, l) - \langle h, l \rangle_2 = \| \Psi_{\lambda_0} \|^{-2} \langle a_+(h)^* \Psi_{\lambda_0}, a_- (l)^* \Psi_{\lambda_0} \rangle - \langle h, l \rangle_2
\]

(4.10)

is well-defined. Lemma 4.1 (iv) and (v) implies that

\[
(4.10) = \| \Psi_{\lambda_0} \|^{-2} \left\langle \left[ a_+(h)^* - a_- (h)^* \right] \Psi_{\lambda_0}, a_- (l)^* \Psi_{\lambda_0} \right\rangle.
\]

(4.11)

Using Lemma 4.1 (ii), we obtain

\[
(4.10) = -i g \| \Psi_{\lambda_0} \|^{-2} \int_{-\infty}^{\infty} ds \langle \Psi_{\lambda_0}, e^{isH} \sigma_1 e^{-isH} a_- (l)^* \Psi_{\lambda_0} \rangle \langle h_s, f \rangle_2.
\]

(4.12)

Finally, we use Lemma 4.1 (iii) to get

\[
(4.10) = -i g \| \Psi_{\lambda_0} \|^{-2} \int_{-\infty}^{\infty} ds \left\langle e^{-isH} \Psi_{\lambda_0}, \sigma_1 a_- (l_s)^* e^{-isH} \Psi_{\lambda_0} \right\rangle \langle h_s, f \rangle_2
\]

\[
= -i g \| \Psi_{\lambda_0} \|^{-2} \int_{-\infty}^{\infty} ds \langle \sigma_1 \Psi_{\lambda_0}, a_- (l_s)^* \Psi_{\lambda_0} \rangle \langle h_s, f \rangle_2.
\]

(4.13)

We insert the definition of the asymptotic creation operator in (1.34) to find

\[
(4.10) = -i g \| \Psi_{\lambda_0} \|^{-2} \int_{-\infty}^{\infty} ds \lim_{t \to -\infty} \langle \sigma_1 \Psi_{\lambda_0}, a_t (l_s)^* \Psi_{\lambda_0} \rangle \langle h_s, f \rangle_2.
\]

(4.14)

Next, it is possible to interchange the \( ds \) integral and the limit \( t \to -\infty \). This can be seen as follows. A two-fold partial integration implies that there is a constant \( C \) such that, for all \( s \in \mathbb{R} \), we get

\[
|\langle h_s, f \rangle_2| \leq C \frac{1}{1 + |s|^2}.
\]

(4.15)
By applying Lemma 4.1 (vi), we infer that there is a finite constant $C_{(4.16)}(l) > 0$ such that for all $s \in \mathbb{R}$

$$
|\langle \sigma_1 \Psi_{\lambda_0}, a_t(l_s)^* \Psi_{\lambda_0} \rangle| \leq \| \sigma_1 \Psi_{\lambda_0} \| \| a_t(l_s)^* (H_f + 1)^{-1/2} \| \| (H_f + 1)^{1/2} \Psi_{\lambda_0} \|
$$

$$
\leq C_{(4.16)}(l) \| \Psi_{\lambda_0} \| \| \Psi_{\lambda_0} \|_{H_f}
$$

(4.16)

holds true. Both estimates, (4.15) and (4.16), give an integrable bound of the $ds$-integrand in (4.14) that is uniform in $t$. Hence, by dominated convergence, we have the equality

$$
(4.10) = -ig \left\| \Psi_{\lambda_0} \right\|^2 \lim_{t \to -\infty} \int_{-\infty}^{\infty} ds \left\{ \sigma_1 \Psi_{\lambda_0}, a_t(l_s)^* \Psi_{\lambda_0} \right\} \langle h_s, f \rangle_2
$$

$$
= -ig \left\| \Psi_{\lambda_0} \right\|^2 \lim_{t \to -\infty} e^{-it_0} \int_{-\infty}^{\infty} ds \left\{ e^{-itH} \sigma_1 \Psi_{\lambda_0}, a(l_{s+t})^* \Psi_{\lambda_0} \right\} \langle h_s, f \rangle_2,
$$

(4.17)

where in the last step we have inserted definition (1.34) and exploited the ground state property (1.25).

In order to rewrite this integral in form of (4.6)–(4.7), or more precisely, (4.8)–(4.9), we shall use the following approximation argument. Let

$$
\mathcal{H}_0 := \mathcal{K} \otimes \mathcal{F}_{\text{fin}}[h_0]
$$

(4.18)

be the set of states with only finitely many bosons, i.e.,

$$
\mathcal{F}_{\text{fin}}[h_0] := \left\{ \Psi = (\psi^{(n)})_{n \in \mathbb{N}_0} \in \mathcal{F}[h] \mid \exists N \in \mathbb{N}_0 : \psi^{(n)} = 0 \forall n \geq N, \forall n \in \mathbb{N} : \psi^{(n)} \in C_c^\infty(\mathbb{R}^{3n} \setminus \{0\}, \mathbb{C}) \right\}.
$$

(4.19)

Note that $\mathcal{H}_0$ is a dense subset of $\mathcal{H}$ with respect to the norm in $\mathcal{H}$ and it is dense in the domain of $H_f$ with respect to the graph norm of the operator $H_f$ defined by $\| \cdot \|_{H_f} := \| H_f \| + \| \cdot \|$. Hence, for $t \in \mathbb{R}$, there are sequences $(\Psi_m)_{m \in \mathbb{N}}$, $(\Phi_m^t)_{m \in \mathbb{N}}$ in $\mathcal{H}_0$ with $\| \Psi_m - \Psi_{\lambda_0} \|_{H_f} \to 0$, as $m \to \infty$, and $\| \Phi_m^t - e^{-itH} \sigma_1 \Psi_{\lambda_0} \| \to 0$, as $m \to \infty$. Then, Lemma A.1, applied in the same fashion as in (4.16), implies that

$$
\lim_{m \to \infty} \langle \Phi_m^t, a(l_{s+t})^* \Psi_m \rangle = \langle e^{-itH} \sigma_1 \Psi_{\lambda_0}, a(l_{s+t})^* \Psi_{\lambda_0} \rangle,
$$

(4.20)

uniformly in $s$. Thanks to the bound (4.15), we may apply dominated convergence theorem to conclude that

$$
\lim_{m \to \infty} \int_{-\infty}^{\infty} ds \left\{ \Phi_m^t, a(l_{s+t})^* \Psi_m \right\} \langle h_s, f \rangle_2 = \int_{-\infty}^{\infty} ds \left\{ e^{-itH} \sigma_1 \Psi_{\lambda_0}, a(l_{s+t})^* \Psi_{\lambda_0} \right\} \langle h_s, f \rangle_2.
$$

(4.21)

Now, we study the integrals in the left hand side of Eq. (4.21). The advantage of the sequences $(\Psi_m)_{m \in \mathbb{N}}$, $(\Phi_m^t)_{m \in \mathbb{N}}$ is that they allow to use point-wise annihilation operators in the following manner:
\[
\int_{-\infty}^{\infty} ds \{ \Phi_{m}^{s}, a(l_{s+t})^{*} \Psi_{m} \} \langle h_{s}, f \rangle \quad (4.25)
\]
\[
= \int_{-\infty}^{\infty} ds \int d^{3} k e^{-is_{0}(k)} e^{-it_{0}(k)} l(k') \langle a(k') \Phi_{m}^{s}, \Psi_{m} \rangle \int d^{3} k h(k) f(k) e^{is_{0}(k)}
\]
\[
= \int_{-\infty}^{\infty} ds \left[ ( \int_{-\infty}^{\infty} dr \ e^{isr} \Theta(r) u(r) \right) \left( \int_{-\infty}^{\infty} dr' \ e^{-isr'} \Theta(r') v_{m}^{s}(r') \right) \right],
\]
where \( \Theta \) is the Heaviside function and we use spherical coordinates and the abbreviations
\[
u_{m}^{s}(r') := e^{-itr'} r' \int d\Sigma' l(r', \Sigma') \{ a(r', \Sigma') \Phi_{m}^{s}, \Psi_{m} \}.
\]
By definition, \( v_{m}^{s} \) and \( u \) belong to \( C_{c}^{\infty}(\mathbb{R} \setminus \{0\}) \) so that the integrals with respect to \( r \) and \( r' \) above can be regarded as Fourier transform, introduced in Definition 4.2, i.e.,
\[
(4.22) = \int_{-\infty}^{\infty} ds \begin{array}{c}
\mathcal{F} \left[ \Theta u \right](s) \mathcal{F} \left[ \Theta v_{m}^{s} \right](s)
\end{array}
\]
holds true. Plancherel’s identity yields for all \( t \in \mathbb{R} \)
\[
(4.22) = 2\pi \int_{-\infty}^{\infty} dr' \Theta r u \Theta v_{m}^{s}(r')
\]
\[
= 2\pi \int_{0}^{\infty} dr' r'^{2} \int d\Sigma h(r', \Sigma) f(r', \Sigma) e^{-itr'} r'^{2}
\]
\[
\int d\Sigma' l(r', \Sigma') \{ a(r', \Sigma') \Phi_{m}^{s}, \Psi_{m} \}
\]
\[
= 2\pi \{ a(W) \Phi_{m}^{s}, \Psi_{m} \} = 2\pi \{ \Phi_{m}^{s}, a(W)^{*} \Psi_{m} \}
\]
where we have used the definition of \( W \) in (4.9) and the definition (1.34), in particular, the notation \( W_{t}(k) = W(k) e^{-it_{0}(k)} \). Using Lemma A.1, applied in the same fashion as in (4.16), allows to carry out the limit \( m \to \infty \) which results in
\[
(4.21) = \lim_{m \to \infty} (4.24) = 2\pi \left\{ e^{-itH} \sigma_{1} \Psi_{\lambda_{0}}, a(W)^{*} \Psi_{\lambda_{0}} \right\}.
\]
This together with (4.17) and Lemma 4.1 guarantees
\[
(4.10) = -ig \left\| \Psi_{\lambda_{0}} \right\|^{-2} \lim_{t \to -\infty} e^{-it\lambda_{0}} 2\pi \left\{ e^{-itH} \sigma_{1} \Psi_{\lambda_{0}}, a(W)^{*} \Psi_{\lambda_{0}} \right\}
\]
\[
= -2\pi ig \left\| \Psi_{\lambda_{0}} \right\|^{-2} \lim_{t \to -\infty} \{ \sigma_{1} \Psi_{\lambda_{0}}, a_{-}(W)^{*} \Psi_{\lambda_{0}} \}
\]
\[
= -2\pi ig \left\| \Psi_{\lambda_{0}} \right\|^{-2} \{ \sigma_{1} \Psi_{\lambda_{0}}, a_{-}(W)^{*} \Psi_{\lambda_{0}} \},
\]
which concludes the proof. \( \square \)

4.2. Technical ingredients. Here, we derive some technical results which will be applied in the proof of the main results in Sect. 4.4. The statements in this section will mostly be formulated without motivation, however, their importance will become clear later in the proofs of the main results.
4.2.1. General results

**Lemma 4.4.** For \( n \in \mathbb{N} \) and \( \theta \in S \), we have

\[
P_0^{(n),\theta} \sigma_1 P_0^{(n),\theta} = 0. \tag{4.29}
\]

The statement has already been proven in [3, Lemma 2.1].

Next, we prove a representation formula of the evolution operator similar to the Laplace transform representation (see, e.g., [4]).

**Lemma 4.5.** For \( \epsilon > 0 \) and sufficiently large \( R > 0 \), we consider the concatenated contour \( \Gamma(\epsilon, R) := \Gamma_-(\epsilon, R) \cup \Gamma_c(\epsilon) \cup \Gamma_d(R) \) (see Fig. 2), where

\[
\Gamma_-(\epsilon, R) := [-R, \lambda_0 - \epsilon] \cup [\lambda_0 + \epsilon, R],
\]

\[
\Gamma_d(R) := \left\{ -R - u e^{i \frac{\pi}{4}} : u \geq 0 \right\} \cup \left\{ R + u e^{-i \frac{\pi}{4}} : u \geq 0 \right\},
\]

\[
\Gamma_c(\epsilon) := \left\{ \lambda_0 - \epsilon e^{-it} : t \in [0, \pi] \right\}. \tag{4.30}
\]

The orientations of the contours in (4.30) are given by the arrows depicted in Figure 2. Then, for all analytic vectors \( \phi, \psi \in \mathcal{H} \) (analytic in a connected domain containing 0) and \( t > 0 \) the following identity holds true:

\[
\langle \phi, e^{-itH} \psi \rangle = \frac{1}{2\pi i} \int_{\Gamma(\epsilon, R)} \mathrm{d}z \, e^{-izt} \left\langle \psi \theta, \left( H^{\theta} - z \right)^{-1} \phi \theta \right\rangle. \tag{4.31}
\]

**Proof.** Let \( t > 0 \) and \( \epsilon > 0 \). We define a contour \( \hat{\Gamma}(\epsilon) := \mathbb{R} + i \epsilon \) with a mathematical negative orientation if the contour were closed in the lower complex plane. As an application of the residue theorem closing the contour in the lower complex plane, we observe for all \( E \in \mathbb{R} \)

\[
\frac{1}{2\pi i} \int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z \, \frac{e^{-izt}}{it(E - z)^2} = e^{-itE} \tag{4.32}
\]

holds true. Thanks to the spectral theorem we may write for all \( \psi \in \mathcal{H} \)

\[
\langle \psi, e^{-itH} \psi \rangle = \int_{\sigma(H)} \langle \psi, dP E \psi \rangle e^{-itE} = \frac{1}{2\pi i} \int_{\sigma(H)} \int_{\hat{\Gamma}(\epsilon)} \mathrm{d}z \, \langle \psi, dP E \psi \rangle \frac{e^{-izt}}{it(E - z)^2}. \tag{4.33}
\]
Next, we may interchange the order of the integrals by the Fubini-Tonelli Theorem since the following integral is finite:

\[
\int_{\sigma(H)} \langle \psi, dP_E \psi \rangle \int_{\hat{\Gamma}(\epsilon)} dz \left| \frac{e^{-itz}}{it(E-z)^2} \right|
\]

\[
\leq \frac{e^{\delta \epsilon}}{t} \int_{\sigma(H)} \langle \psi, dP_E \psi \rangle \int_{-\infty}^{\infty} dx |x-i\epsilon|^{-2} < \infty.
\]

Hence, after the interchange we may apply the spectral theorem again to find

\[
(4.33) = \frac{1}{2\pi i} \int_{\hat{\Gamma}(\epsilon)} dz \int_{\sigma(H)} \langle \psi, dP_E \psi \rangle \frac{e^{-itz}}{it(E-z)^2}
\]

\[
= \frac{1}{2\pi i} \int_{\hat{\Gamma}(\epsilon)} dz \frac{e^{-itz}}{it} \left\langle \psi, \frac{1}{(H-z)^2} \psi \right\rangle.
\]

Exploiting the polarization identities we recover for all \(\psi, \phi \in \mathcal{H}\) the identity

\[
\left\langle \psi, e^{-itH} \phi \right\rangle = \frac{1}{2\pi i} \int_{\hat{\Gamma}(\epsilon)} dz \frac{e^{-itz}}{it} \left\langle \psi, \frac{1}{(H-z)^2} \phi \right\rangle.
\]

The fact that the family \(H^{\theta}\) is an analytic family of type A implies that the operator valued function

\[
\theta \mapsto \frac{1}{H^{\theta} - z}
\]

is analytic for all \(z\) in the resolvent set of \(H^{\theta}\). A detailed and self-contained exposition of this topic is presented in [14, Section 7]. It is straightforward to prove that for real \(\theta\)

\[
\frac{1}{H^{\theta} - z} = U^\theta \frac{1}{H - z} (U^\theta)^{-1}.
\]

For complex \(\theta\), however, this expression is not necessarily correct (due to a problem of domains of unbounded operators). Nevertheless, Eqs. (4.37) and (4.38) imply that the function

\[
\theta \mapsto \left\langle \psi^\theta, \frac{1}{(H^\theta - z)^2} \phi^\theta \right\rangle
\]

where \(\phi^\theta = U^\theta \phi\), \(\psi^\theta = U^\theta \psi\), is analytic and it coincides with \(\left\langle \psi, \frac{1}{(H-z)^2} \phi \right\rangle\) for real \(\theta\), because in this case \(U^\theta\) is unitary. Hence, we conclude that

\[
\left\langle \psi^\theta, \frac{1}{(H^\theta - z)^2} \phi^\theta \right\rangle = \left\langle \psi, \frac{1}{(H-z)^2} \phi \right\rangle
\]

for every \(\theta\) in a connected (open) domain containing 0 such that (4.39) is analytic in this domain. We obtain:

\[
(4.36) = \frac{1}{2\pi i} \int_{\hat{\Gamma}(\epsilon)} dz \frac{e^{-itz}}{it} \left\langle \psi^\theta, \frac{1}{(H^\theta - z)^2} \phi^\theta \right\rangle
\]
Equations (3.10) and (3.13) imply that the only spectral point of $H^\theta$ on the real line is $\lambda_0^\theta$ and all other spectral points have strictly negative imaginary part. Therefore, the operator valued function

$$A \cup \mathbb{C}^+ \ni z \mapsto \frac{1}{H^\theta - z},$$

(4.42)

where $\mathbb{C}^+ = \{x + iy | x \in \mathbb{R}, y > 0\}$, is analytic. Moreover, for $R \geq e_1 + \delta = 2e_1$, $\Gamma_d(R)$ is contained in the region $A$, and hence, it follows from (3.29) that there is a constant $C$ such that

$$\left\| \frac{1}{H^\theta - z} \right\| \leq \frac{C}{|z - e_1|} \forall z \in \Gamma_d.$$

(4.43)

Due to the analyticity, we may deform the integration contour from $\hat{\Gamma}(\epsilon)$ to $\Gamma(\epsilon, R)$ which gives:

$$\int_{\Gamma(\epsilon, R)} dz \frac{e^{-itz}}{it} \left\langle \bar{\psi}, \frac{1}{(H^\theta - z)^2} \phi^\theta \right\rangle.$$

(4.44)

Now we observe that the integrand on the right-hand side features an exponential decay for large $|\text{Re } z|$ thanks to the factor $e^{-itz}$ in the integrand and the definition of $\Gamma_d(\epsilon, R)$. In particular, the decay in $|z|$ provided by the resolvent, i.e., bound (4.43), is not necessary anymore to make the integral converge. We may therefore perform an integration by parts. Note that, for $z$ in $A \cup \mathbb{C}^+$, we have

$$\frac{d}{dz} \left( \psi^\theta \frac{1}{(H^\theta - z)^2} \phi^\theta \right) = \left( \psi^\theta \frac{1}{(H^\theta - z)^2} \phi^\theta \right)$$

(4.45)

which is implied by the resolvent identity

$$\left\langle \psi^\theta \frac{1}{(H^\theta - (z+u))} \phi^\theta \right\rangle - \left\langle \psi^\theta \frac{1}{(H^\theta - z)} \phi^\theta \right\rangle = \left\langle \psi^\theta \frac{1}{(H^\theta - (zv+u))} \phi^\theta \right\rangle.$$

(4.46)

Moreover, the boundary terms of the partial integration resulting from the piece-wise concatenation of contours, i.e., $\Gamma(\epsilon, R) = \Gamma_-(\epsilon, R) \cup \Gamma(\epsilon) \cup \Gamma_d(R)$, cancel and the ones at $|\text{Re } z| \to \infty$ vanish because of the exponential decay. In conclusion, the identity

$$\int_{\Gamma(\epsilon, R)} dz \frac{e^{-itz}}{it} \left\langle \bar{\psi}, \frac{1}{H^\theta - z} \phi^\theta \right\rangle$$

(4.47)

holds true which proves the claim. $\square$
4.3. Key ingredients. The next definition is motivated by a simple geometric argument which we give in the following for the convenience of the reader: take a cone of the form \( C_m(\lambda_0^{(n)} - xe^{-iv}), x > 0 \), where \( m \) is a fixed (arbitrary) number greater or equal than 4. Although \( m \) is arbitrary, our estimates and constants depend on it. The distance between the vertex of the cone and the intersection of the line \( \lambda_0^{(n)} - ix \sin(v) + \mathbb{R} \) with the cone is

\[
\sqrt{\frac{2x \sin(v)}{\tan((1-1/m)v)}^2 + (2x \sin(v))^2} \leq 4x \frac{\sin(v)}{\sin((1-1/m)v)} \leq 8x.
\]

To obtain the last inequality we use the sum of angles formula for \( \sin(v) \), writing \( v = (v - v/m) + v/m \). Then, we have that the distance between \( \lambda_0^{(n)} \) and the line segment described above is smaller than 8x.

**Definition 4.6.** For every \( n \in \mathbb{N} \), we define

\[
\epsilon_n := 20 \rho^{1+\mu/4}.
\]

It follows from (3.33) and (3.28) that for every \( n \in \mathbb{N} \)

\[
D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)}.
\]

The geometric argument given above together with \( |\lambda_0^{(n)} - \lambda_0| \leq 10^{-2} \rho^{1+\mu/2} \) (see Definition 3.1 and (3.27)) yields that, for all \( n \in \mathbb{N} \) and a fixed (arbitrary) \( m \geq 4 \),

\[
C_m(\lambda_0^{(n)} - 2\rho_n^{1+\mu/4} e^{-iv}) \cap \left( \overline{C^+} + \lambda_0^{(n)} - i2 \sin(v)\rho_n^{1+\mu/4} \right)
\]

\[
\subset D(\lambda_0, \epsilon_n) \subset D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)}
\]

and

\[
C_m(\lambda_0 - 2\rho_n^{1+\mu/4} e^{-iv}) \cap \left( \overline{C^+} + \lambda_0 - i2 \sin(v)\rho_n^{1+\mu/4} \right)
\]

\[
\subset D(\lambda_0, \epsilon_n) \subset D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)}.
\]

Note that (3.27) and the fact that \( \lambda_0 \in \mathbb{R} \) imply that

\[
\text{Im} \lambda_0^{(n)} - 2 \sin(v)\rho_n^{1+\mu/4} \leq 2g\rho_n^{1+\mu/2} - 2 \sin(v)\rho_n^{1+\mu/4} < 0, \quad \forall n \in \mathbb{N},
\]

for small enough \( g \) (see Definition 4.3 in [14]). Equation (4.50) implies that for every \( n \in \mathbb{N} \)

\[
\Gamma_c(\epsilon_n) \subset B_0^{(1)} \setminus C_m(\lambda_0^{(n)} - 2\rho_n^{1+\mu/4} e^{-iv}).
\]

**Lemma 4.7.** For all \( n \in \mathbb{N} \), a fixed (arbitrary) \( m \geq 4 \) and \( \theta \in \mathcal{S} \), there is a constant \( C \) (that depends on \( m \)) such that

\[
\left\| \frac{1}{H^{\theta} - z} \sigma_1 \Psi_{\lambda_0}^\theta \right\| \leq CC^{n+1} \frac{1}{\rho_n},
\]

for every \( z \in B_0^{(1)} \setminus C_m(\lambda_0^{(n)} - \rho_n^{1+\mu/4} e^{-iv}), \) and hence, for every \( z \in B_0^{(1)} \setminus C_m(\lambda_0 - 2\rho_n^{1+\mu/4} e^{-iv}), \) see [14, Theorem 5.10].
Proof. We take $z \in B_0^{(1)} \setminus C_m(\lambda_0^{(n)} - \rho_n^{1+\mu/4} e^{-i\nu})$ and recall the definition $\Psi_{\lambda_0}^{\theta} = P_0^{\theta} \varphi_0 \otimes \Omega$. Then, Eq. (3.24) yields

$$\| \Psi_{\lambda_0}^{\theta} - P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)} \varphi_0 \otimes \Omega \| \leq \rho_n^{\mu/2}. \quad (4.55)$$

This together with Eqs. (3.34), (3.36), (3.31) and (3.26) implies that there is a constant $C$ such that (we use a telescopic sum argument)

$$\left\| \frac{1}{H^\theta - z} \sigma_1 \Psi_{\lambda_0}^{\theta} - \frac{1}{H(n,\theta) - z} \sigma_1 P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)} \varphi_0 \otimes \Omega \right\| \leq \left\| \frac{1}{H^\theta - z} \right\| \| P_0^{(n),\theta} \| + \left\| \frac{1}{H^\theta - z} \right\| \| \Psi_{\lambda_0}^{\theta} - P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)} \varphi_0 \otimes \Omega \| \leq C C^{n+1} \frac{1}{\rho_n}. \quad (4.56)$$

The fact (see Remark (1.2)) that

$$(\frac{1}{\overline{H(n,\theta) - z}}) (P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)} \varphi_0 \otimes \Omega) = \left( (\frac{1}{H(n,\theta) - z}) \otimes P_{\Omega(n,\infty)} \right) P_0^{(n),\theta} \varphi_0 \otimes \Omega \quad (4.57)$$

guarantees that there is a constant $C$ such that

$$\left\| \frac{1}{H(n,\theta) - z} \sigma_1 P_0^{(n),\theta} \otimes P_{\Omega(n,\infty)} \varphi_0 \otimes \Omega \right\| \leq \left( \frac{P_0^{(n),\theta}}{H(n,\theta) - z} \right) \otimes P_{\Omega(n,\infty)} \| \leq C C^{n+1} \frac{1}{\rho_n}. \quad (4.58)$$

Here, we use Eqs. (3.35), (3.26) and Lemma 4.4. □

Remark 4.8. Set $c \equiv c_g := \text{Im} \lambda_1$. Notice that there is a strictly positive constant $c$ (independent of $g$) with $c_g \leq -g^2 c$, for small enough $g$ (see (3.13)). Then, for all real numbers $b > a$ and every $x \in \{0, 1\}$,

$$\int_a^b \frac{1}{|r - \lambda_1|^{1+x}} = \int_a^b \frac{1}{g^{2(1+x)} ((r - \text{Re} \lambda_1)/g^2)^2 + (c/g^2)^2(1+x)/2} \leq \frac{1}{g^{2x}} \int_a^b \frac{1}{(a-\text{Re} \lambda_1)/g^2} dy \left( y^2 + c^2(1+x)/2 \right)^{(1+x)/2} \leq \frac{1}{g^{2x}} \int_0^1 \frac{1}{e^{1+x}} \frac{1}{g^{2x}} \int_0^{1+|b-\text{Re} \lambda_1|/g^2+a-\text{Re} \lambda_1|/g^2} d\tau \frac{1}{\tau^{1+x}} \leq C \left\{ \begin{array}{ll} \frac{1}{g^x}, & \text{if } x = 1, \\ |\log(g)|, & \text{if } x = 0, \end{array} \right.$$  

where $C$ does not depend on $g$ (for small enough $g$).

Lemma 4.9. Set $\mathcal{L} := B_1^{(1)} \cap \mathbb{R}$. Then, for $g > 0$ sufficiently small and $\theta \in S$,

$$\int_{\mathcal{L}} d\tau \left\| \frac{1}{H^\theta - r} \sigma_1 \Psi_{\lambda_0}^{\theta} \right\| \leq C |\log g|, \quad (4.60)$$

where $C$ is a constant that does not depend on $g$.
Proof. Let $c$ be the constant introduced in (2.2), see Remark 4.8. The vertex of the cone $C_m(λ_1 - 2\rho_n^{1+\mu/4} e^{-iν})$ belongs to the lower (open) half space of the complex plane if

$$-g^2c + 2\rho_n^{1+\mu/4} \sin(ν) < 0.$$ 

This is fulfilled if

$$n > \log\left(\frac{g^2c}{2 \sin(ν)\rho_n^{1+\mu/4}}\right) \frac{1}{(1 + \mu/4) \log(ρ)}.$$ 

We fix $n_0 > 0$ to be the smallest integer number satisfying this inequality. Then

$$n_0 \leq \log\left(\frac{g^2c}{2 \sin(ν)\rho_n^{1+\mu/4}}\right) \frac{1}{(1 + \mu/4) \log(ρ)} + 1.$$ 

For such $n_0$, the cone $C_m(λ_1 - 2\rho_n^{1+\mu/4} e^{-iν})$ belongs to the lower (open) half space of the complex plane and, therefore, $L$ is contained in the complement of this cone. This allows us to use (3.37) and estimate

$$\|H^{θ} \sigma_1 \Psi_{λ_0}^{θ} - r\| \leq C n_0 + 1 \text{dist}(r, C_m(λ_1)).$$ 

Equation (4.61) implies that

$$C^{n_0} \leq C \exp\left[-\log\left(\frac{g^2c}{2 \sin(ν)\rho_n^{1+\mu/4}}\right) \frac{\log(C)}{(1 + \mu/4) \log(ρ)}\right] = C\left(\frac{2 \sin(ν)\rho_n^{1+\mu/4}}{g^2c}\right)^{\frac{\log(C)}{(1 + \mu/4) \log(ρ)}} \leq C g^{-i},$$ 

where we use that $C \rho_n^{1+(1+\mu/4)} \leq 1$, see (3.31). This together with (3.37) lead us to

$$\left\|\frac{1}{H^{θ} - r}\right\| \leq C \text{dist}(r, C_m(λ_1)). \quad ∀r \in L. \quad (4.64)$$

It is geometrically clear, because $\text{Im} λ_1 < -g^2c < 0$, see (3.13), that there is a constant $C$ (that depends on $ν$ and $m$, but not on $g$) such that, for every $r \in L$,

$$|r - λ_1| \leq C \text{dist}(r, C_m(λ_1)). \quad (4.65)$$

Eqs. (4.64) and (4.65) yield

$$\left\|\frac{1}{H^{θ} - r}\right\| \leq C g^{-i} \frac{1}{|r - λ_1|}. \quad (4.66)$$

Moreover, we observe from (3.25) that

$$\|P_1^{θ} \sigma_1 \Psi_{λ_0}^{θ}\| = \|(1 - P_1^{θ}) \sigma_1 P_0^{θ} \varphi_0 \otimes Ω\| \leq \|(1 - P_1^{θ}) \varphi_1 \otimes Ω\| + C g = C g. \quad (4.67)$$

Inserting $P_1^{θ} + P_1^{θ} = 1$ in the left hand side of (4.60) and using (4.66), we find

$$\left\|\frac{1}{H^{θ} - r} \sigma_1 \Psi_{λ_0}^{θ}\right\| \leq C \left(\frac{1}{|r - λ_1|} + g \left\|\frac{1}{H^{θ} - r}\right\|\right). \quad (4.68)$$
This together with \( \iota \in (0, 1) \) and (4.66) yields that
\[
\left\| \frac{1}{H^\theta - r} \sigma_1 \Psi_{\lambda_0}^\theta \right\| \leq C \frac{1}{|\lambda_1 - r|}. \tag{4.69}
\]
From (4.69) and Remark 4.8, we obtain
\[
\int_L dr \left\| \frac{1}{H^\theta - r} \sigma_1 \Psi_{\lambda_0}^\theta \right\| \leq C \int_L \frac{1}{|r - \lambda_1|} \leq C |\log g|. \tag{4.70}
\]

**Lemma 4.10.** For every bounded measurable function \( h \), there is a constant \( C \) such that for all natural numbers \( n \in \mathbb{N} \) and \( \theta \in \mathcal{S} \)
\[
\int \left( (B_0^{(1)} \cup B_1^{(1)}) \cap \mathbb{R} \right) \setminus (\lambda_0 - \epsilon_n, \lambda_0 + \epsilon_n) dz |h(z)| \left\| \frac{1}{H^\theta - z} \sigma_1 \Psi_{\lambda_0}^\theta \right\| \leq C |\log g| \|h\|_\infty, \tag{4.71}
\]
where \( \|h\|_\infty \) denotes the supremum of \( |h| \).

**Proof.** We set
\[
\tilde{h}(z) = |h(z)| \left\| \frac{1}{H^\theta - z} \sigma_1 \Psi_{\lambda_0}^\theta \right\|.
\]
For any natural number \( l \geq 2 \), we set \( I_l := B_0^{(1)} \cap [\lambda_0 - \epsilon_{l-1}, \lambda_0 + \epsilon_{l-1}] \setminus (\lambda_0 - \epsilon_l, \lambda_0 + \epsilon_l) \). We define \( I_1 := \mathbb{R} \cap B_0^{(1)} \setminus I_2 \). Then, we compute
\[
\sum_{l=1}^n \int_{I_l} dz \tilde{h}(z) = \sum_{l=1}^n \int_{I_l} dz \tilde{h}(z). \tag{4.73}
\]
Using Definition 4.6 and Lemma 4.7, we obtain that there is a constant \( C \) such that
\[
\sum_{l=1}^n \int_{I_l} dz \tilde{h}(z) \leq C \sum_{l=2}^n C^{l+1} \frac{\epsilon_{l-1}}{\rho_l} \|h\|_\infty + C \|h\|_\infty \leq C \rho_{\mu/4}^{l+1} \|h\|_\infty + C \|h\|_\infty \leq C \|h\|_\infty, \tag{4.74}
\]
where we use (3.31). Equation (4.74) and Lemma 4.9 imply (4.71).

**Lemma 4.11.** For every bounded measurable function \( h \), there is a constant \( C \) such that for all natural numbers \( n \in \mathbb{N} \) and \( \theta \in \mathcal{S} \)
\[
\int_{\Gamma_c(\epsilon_n)} dz |h(z)| \left\| \frac{1}{H^\theta - z} \sigma_1 \Psi_{\lambda_0}^\theta \right\| \leq \rho_n^{\mu/8} C \sup_{z \in \Gamma_c(\epsilon_n)} |h(z)|. \tag{4.75}
\]

**Proof.** This is a direct consequence of Lemma 4.7 and (3.31).

**Lemma 4.12.** There is a constant \( C \) such that (for \( s > 0 \))
\[
\left| \int_{\Gamma_d(R) \cup \Gamma_-(\epsilon_n, R) \setminus (B_0^{(1)} \cup B_1^{(1)}) \cap \mathbb{R}} dz e^{-isz} \left( \overline{p^\theta_1} \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right) \right| \leq C g^{-1/s}. \tag{4.76}
\]
Lemma 4.13. For real numbers $0 < q \leq e^{-1} < 1 < Q < \infty$ and $\theta \in \mathcal{S}$, the term (recall (2.1))

$$R_1(q, Q) := \int_q^Q ds \int dr G(r) e^{is(r+\lambda_0)} \times \int_{\Gamma(\epsilon_n, R)} dz e^{-isz} \left\{ \bar{P}_1 \sigma_1 \bar{\Psi}_{\lambda_0}, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\}$$

fulfills

$$|R_1(q, Q)| \leq C g \left( |\log(q)| + |\log(g)| \right).$$

Notice that $R_1(q, Q)$ does not depend on $\epsilon_n$ and $R$, because a change in $\epsilon_n$ and $R$ implies a change in the contour of integration of the analytic function above.

Proof. First, we recall that (see (4.78))

$$\left\| \bar{P}_1 \sigma_1 \bar{\Psi}_{\lambda_0} \right\| \leq C g.$$  

(4.81)

A two-fold integration by parts together with the fact $G \in C_c^\infty(\mathbb{R}\setminus\{0\}, \mathbb{C})$ (recall (2.1)) shows that there is a constant $C$ such that

$$\left| \int dr G(r) e^{is(r+\lambda_0)} \right| \leq \frac{C}{1 + s^2}, \quad \forall s \in \mathbb{R}.$$  

(4.82)

Lemmas 4.10, 4.11 and 4.12, an Eq. (4.81) imply that

$$|R_1(q, Q)| \leq C g \int_q^Q ds \frac{1}{s^2 + 1} \left( \frac{1}{s} + e^{\epsilon_n Q \rho_n^\mu/8} + |\log g| \right).$$  

(4.83)

Since $R_1(q, Q)$ does not depend on $n$, we can take $n$ to infinity and obtain the bound

$$|R_1(q, Q)| \leq C g \int_q^Q ds \frac{1}{s^2 + 1} \left( \frac{1}{s} + |\log g| \right).$$  

(4.84)
Equation (4.84) and the condition $0 < q < e^{-1}$ imply (4.80). Notice that

$$
\int_q^Q \frac{ds}{(s^2 + 1)s} \leq \int_q^1 \frac{1}{s} + \int_1^\infty \frac{1}{s^2 + 1} \leq C|\log(q)|,
$$

(4.85)

since $\int_1^\infty \frac{1}{s^2 + 1}$ is a constant and $|\log(q)| \geq 1$. □

**Lemma 4.14.** For real numbers $0 < q < 1 < Q < \infty$ and $\theta \in S$, the term

$$
P_1(q, Q) = \int_q^Q ds \int dr \ G(r)e^{is(r+\lambda_0)}
$$

(4.86)

$$
\times \int_{\Gamma(\epsilon, R)} dz \ e^{-isz} \left\{ P_1^{\bar{\theta}} \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, (H^\theta - z)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\}
$$

fulfills

$$
P_1(q, Q) = -2\pi \int dr \ G(r) \frac{1}{r + \lambda_0 - \lambda_1} \left\{ \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, P_1^{\bar{\theta}} \sigma_1 \Psi^\theta_{\lambda_0} \right\} + R_2(q, Q),
$$

(4.87)

where

$$
|R_2(q, Q)| \leq C \left( q + \frac{1}{Qq^2} \right).
$$

(4.88)

Notice that $P_1(q, Q)$ does not depend on $\epsilon_n$ and $R$, because a change in $\epsilon_n$ and $R$ implies a change in the contour of integration of the analytic function above.

**Proof.** We have that

$$
P_1(q, Q) = \int_q^Q ds \int dr \ G(r)e^{is(r+\lambda_0)} \left\{ \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, P_1^{\bar{\theta}} \sigma_1 \Psi^\theta_{\lambda_0} \right\} \int_{\Gamma(\epsilon, R)} dz \ e^{-isz} \frac{1}{\lambda_1 - z}.
$$

(4.89)

The residue theorem together with methods of complex analysis provides

$$
\int_{\Gamma(\epsilon, R)} dz \ e^{-isz} \frac{1}{\lambda_1 - z} = 2\pi i e^{-isz\lambda_1},
$$

(4.90)

and hence, we obtain

$$
P_1(q, Q) = 2\pi i \int_q^Q ds \int dr \ G(r)e^{is(r+\lambda_0-\lambda_1)} \left\{ \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, P_1^{\bar{\theta}} \sigma_1 \Psi^\theta_{\lambda_0} \right\}
$$

$$
= 2\pi \int dr \ G(r) \left( e^{iQ(r+\lambda_0-\lambda_1)} - e^{iq(r+\lambda_0-\lambda_1)} \right) \frac{1}{r + \lambda_0 - \lambda_1} \left\{ \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, P_1^{\bar{\theta}} \sigma_1 \Psi^\theta_{\lambda_0} \right\}
$$

$$
= -2\pi \int dr \ G(r) \frac{1}{r + \lambda_0 - \lambda_1} \left\{ \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, P_1^{\bar{\theta}} \sigma_1 \Psi^\theta_{\lambda_0} \right\} + r_1(Q) + r_2(q),
$$

(4.91)

where

$$
r_1(Q) := 2\pi \int dr \ G(r) e^{iQ(r+\lambda_0-\lambda_1)} \frac{1}{r + \lambda_0 - \lambda_1} \left\{ \sigma_1 \Psi^{\bar{\theta}}_{\lambda_0}, P_1^{\bar{\theta}} \sigma_1 \Psi^\theta_{\lambda_0} \right\}
$$

(4.92)
and
\[ r_2(q) := 2\pi \int dr \, G(r)(1 - e^{iq(r+\lambda_0-\lambda_1)}) \frac{1}{r + \lambda_0 - \lambda_1} \left( \sigma_1 \Psi_{\lambda_0}^\theta, P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta \right). \tag{4.93} \]

It follows from
\[ |1 - e^{iq(r+\lambda_0-\lambda_1)}| \leq |q(r + \lambda_0 - \lambda_1)|, \tag{4.94} \]
that there is a constant C such that \( |r_2(q)| \leq C q \). Applying the integration by parts formula in Eq. (4.92), we obtain a factor \( \frac{1}{Q} \) and the derivative of \( G(r) \frac{1}{r + \lambda_0 - \lambda_1} \). We obtain
\[
|r_1(Q)| \leq C \frac{1}{Q} \int dr \left( |G(r)| \frac{1}{|r + \lambda_0 - \lambda_1|^2} \right.
+ \left. \left| \frac{d}{dr} G(r) \right| \frac{1}{|r + \lambda_0 - \lambda_1|} \left( \sigma_1 \Psi_{\lambda_0}^\theta, P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta \right) \right|
\leq C \frac{1}{Q} \left( \frac{1}{g^2} + |\log(g)| \right) \leq C \frac{1}{Q g^2}, \tag{4.95} \]
where we use (4.59), with \( x = 1 \) and \( x = 0 \), and \( r + \lambda_0 \) instead of \( r \). \( \square \)

**Lemma 4.15.** For real numbers \( 0 < q < 1 < Q < \infty \) and \( \theta \in \mathcal{S} \), we define the term
\[
\tilde{P}_1(q, Q) := \int_q^Q ds \int dr \, G(r)e^{is(r-\lambda_0)}
\times \int_{\Gamma(\epsilon_n, R)} dz \, e^{isz} \left( P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right), \tag{4.96} \]
where \( \Gamma(\epsilon_n, R) \) is a positively oriented curve obtained by conjugating the elements of \( \Gamma(\epsilon_n, R) \). It follows that
\[
\tilde{P}_1(q, Q) = 2\pi \int dr \, G(r) \frac{1}{r - \lambda_0 + \lambda_1} \left( P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta, \sigma_1 \Psi_{\lambda_0}^\theta \right) + \tilde{R}_2(q, Q), \tag{4.97} \]
where
\[
|\tilde{R}_2(q, Q)| \leq C \left( q + \frac{1}{Q g^2} \right). \tag{4.98} \]

**Proof.** This proof is very similar to the proof of Lemma 4.14: We have that
\[
\tilde{P}_1(q, Q) = -2\pi i \int_q^Q ds \int dr \, G(r)e^{is(r-\lambda_0+\lambda_1)} \left( P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta, \sigma_1 \Psi_{\lambda_0}^\theta \right) \tag{4.99} \]
and hence, we infer
\[
P_1(q, Q) = -2\pi \int dr \, G(r) \left( e^{iQ(r-\lambda_0+\lambda_1)} - e^{iq(r-\lambda_0+\lambda_1)} \right)
\frac{1}{r - \lambda_0 + \lambda_1} \left( P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta, \sigma_1 \Psi_{\lambda_0}^\theta \right)
= 2\pi \int dr \, G(r) \frac{1}{r - \lambda_0 + \lambda_1} \left( P_1^\theta \sigma_1 \Psi_{\lambda_0}^\theta, \sigma_1 \Psi_{\lambda_0}^\theta \right) + \tilde{R}_2(q, Q). \tag{4.100} \]
We conclude the proof as in the proof of Lemma 4.14. \( \square \)
4.4. Proof of Theorem 2.2. In this section, we give the proof of the main theorem based on the previous results.

Proof of Theorem 2.2. Let \( h, l \in h_0; \) c.f. (1.33). Recall the definition of \( W \) given in (4.9) and the form factor \( f \) in (1.5). Since \( f \in C^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \) we find

\[
hf, lf, W \in h_0.
\]

Theorem 4.3, i.e., equation (4.8), together with Lemma 4.1 (iv) yields

\[
T(h, l) = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle a_-(W)\sigma_1\Psi_{\lambda_0}, \Psi_{\lambda_0}\rangle
\]

and furthermore, recalling \( \omega(k) = |k| \), equation (4.2) in Lemma 4.1 (ii) implies

\[
T(h, l) = 2\pi (ig)^2 \|\Psi_{\lambda_0}\|^{-2} \int_0^\infty ds \langle W_s, f \rangle_2 \left\{ e^{isH}\sigma_1 e^{-isH}, \sigma_1 \right\} \Psi_{\lambda_0}, \Psi_{\lambda_0} \rangle
\]

where we used the abbreviations

\[
T^{(j)} := T^{(j)}(0, \infty)
\]

for \( j = 1, 2 \) with

\[
T^{(1)}(q, Q) := \int_q^Q ds \int d^3k W(k) f(k) e^{is(|k|+\lambda_0)} \left\{ \sigma_1\Psi_{\lambda_0}, e^{-isH}\sigma_1\Psi_{\lambda_0} \right\}
\]

\[
= \int_q^Q ds \int dr G(r) e^{is(r+\lambda_0)} \left\{ \sigma_1\Psi_{\lambda_0}, e^{-isH}\sigma_1\Psi_{\lambda_0} \right\}
\]

and

\[
T^{(2)}(q, Q) := \int_q^Q ds \int dr G(r) e^{is(r-\lambda_0)} \left\{ \sigma_1\Psi_{\lambda_0}, e^{isH}\sigma_1\Psi_{\lambda_0} \right\}.
\]

We recall the definitions (4.9) and (2.1):

\[
W(k) = |k|^2 l(k) \int d\Sigma h(|k|, \Sigma) f(|k|, \Sigma),
\]

\[
G(r) = \int d\Sigma d\Sigma' r^4 h(r, \Sigma) l(r, \Sigma') f(r)^2.
\]
We start with analyzing the term $T^{(1)}(q, Q)$. Lemma 4.5 together with the identity $P_1^\bar{\tau} + P_1^{\bar{\tau}} = 1$ allows us to write this term as

$$T^{(1)}(q, Q) = \frac{1}{2\pi i} P_1(q, Q) + \frac{1}{2\pi i} R_1(q, Q)$$ \hfill (4.109)

for all $0 < q < Q < \infty$. Here, $P_1(q, Q)$ and $R_1(q, Q)$ are defined in (4.86) and (4.79), respectively. Moreover, Lemma 4.14 implies

$$T^{(1)}(q, Q) = -2\pi \frac{1}{2\pi i} \int dr \frac{1}{r + \lambda_0 - \lambda_1} \left\{ \sigma_1 \Psi_{\lambda_0}^\bar{\tau}, P_1^{\bar{\tau}} \sigma_1 \Psi_\lambda^\theta \right\} + \frac{1}{2\pi i} \left( R_1(q, Q) + R_2(q, Q) \right)$$ \hfill (4.110)

where Lemmas 4.13 and 4.14 provide the estimates:

$$|R_1(q, Q)| \leq C g (|\log(q)| + |\log(g)|), \quad |R_2(q, Q)| \leq C (q + \frac{1}{g^2 Q})$$ \hfill (4.111)

for $0 < q \leq e^{-1} < 1 < Q$. As explained in Lemmas 4.13 and 4.14, the terms $P_1(q, Q)$ and $R_1(q, Q)$ do not depend on $n$ and $R$ because both are given by contour integrals of analytic functions and a change of these parameters signifies a change in the contour of integration only. Taking the limit $Q$ to infinity and $q = g$, we obtain from Eqs. (4.108), (4.110) and (4.111):

$$T^{(1)}(0, \infty) = i \int dr \frac{1}{r + \lambda_0 - \lambda_1} \left\{ \sigma_1 \Psi_{\lambda_0}^\bar{\tau}, P_1^{\bar{\tau}} \sigma_1 \Psi_\lambda^\theta \right\} + R_3$$ \hfill (4.112)

and that there is a constant $C$ such that

$$|R_3| \leq C |\log(g)|.$$ \hfill (4.113)

The term $T^{(2)}(0, \infty)$ can be inferred by repeating the calculation with $\theta$ replaced by $\bar{\theta}$ and reflecting the path of integration $\Gamma'(e, R)$ on the real axis when applying Lemma 4.5. In this case one has to consider the Hamiltonian $H^{\bar{\tau}}$. Notice that in this case the factor $\frac{1}{2\pi i}$ in Eq. (4.31) is substituted by $-\frac{1}{2\pi i}$, which is produced from the change of orientation of the integration curve. Due to the similarity of the calculation, we omit the proof and only state the result (it follows from Lemma 4.15 and similar computations)

$$T^{(2)}(0, \infty) = -\frac{1}{2\pi i} 2\pi \int dr \frac{1}{r - \lambda_0 + \lambda_1} \left\{ \sigma_1 \Psi_{\lambda_0}^\theta, P_1^{\bar{\tau}} \sigma_1 \Psi_{\lambda_0}^\bar{\tau} \right\} + R_4$$ \hfill (4.114)

and that there is a constant $C$ such that

$$|R_4| \leq C |\log(g)|.$$ \hfill (4.115)

The identities (4.103), (4.112) and (4.114), together with (4.113), (4.115) and (3.25) imply

$$T(h, l) = 2\pi g^2 \| \Psi_{\lambda_0} \|^{-2} \left( T^{(1)} - T^{(2)} \right) + R$$ \hfill (4.116)

$$= 2\pi i g^2 \| \Psi_{\lambda_0} \|^{-2} \int dr \frac{1}{r + \lambda_0 - \lambda_1} - \frac{1}{r - \lambda_0 + \lambda_1} + R$$
\[ 4\pi g^2 \| \Psi_{\lambda_0} \|^{-2} \int dr \, G(r) \left( \frac{\text{Re} \, \lambda_1 - \lambda_0}{(r + \lambda_0 - \lambda_1)(r - \lambda_0 + \lambda_1)} \right) + R, \]

where \(|R| \leq C |\log(g)| \). \( \square \)

**Remark 4.16.** The constant \(C(h, l)\) in Theorem 2.2 depends on \(h\) and \(l\). From our methods, this dependence can be made explicit. However, for the sake of simplicity and clarity we do not present this analysis in this paper, but indicate instead how to do it. The key ingredients are Eqs. (4.82) and (4.93) (notice that (4.92) does not play a role because the corresponding term vanishes when \(Q\) tends to infinity). These terms give a contribution of the form

\[ C \int dr \left[ |G(r)| + \left| \frac{d}{dr} G(r) \right| + \left| \frac{d^2}{dr^2} G(r) \right| \right], \tag{4.117} \]

for a constant \(C\) that does not depend on \(h\) and \(l\). Moreover, with respect to (4.82), a minor change in the proof of Lemma 4.13 would make the second derivative term unnecessary because we have an extra factor of the form \(s^{-1}\) in (4.76). This is essentially the only necessary contribution that comes from \(h\) and \(l\). However, in order to simplify our final formula, we substituted the inner products in Eqs. (4.112) and (4.114) by the constant 1 (using (3.25)). This produces (explicit) error terms that contribute differently as (4.117), as we can see from our arguments below (4.115).

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**A. Standard Estimates**

In the following we shall use the well-known standard inequalities

\[
\| a(h) \Psi \| \leq \| h/\sqrt{\omega} \|_2 \| H_f^{1/2} \Psi \|, \\
\| a(h)\Psi \| \leq \| h/\sqrt{\omega} \|_2 \| H_f^{1/2} \Psi \| + \| h \|_2 \| \Psi \|
\]

which hold for all \(h, h/\sqrt{\omega} \in \mathfrak{h}\) and \(\Psi \in \mathcal{H}\) such that the left- and right-hand side are well-defined; see [40, Eq. (13.70)].

**Lemma A.1.** Let \(h, h/\sqrt{\omega} \in \mathfrak{h}\). Then, we have the following estimates:

\[
\| a(h)\Psi \| \leq \| h\|_2 + \| h/\sqrt{\omega} \|_2, \\
\| a(h)\Psi \| \leq \| h/\sqrt{\omega} \|_2, \\
\| V (H_f + 1)^{-1/2} \| \leq \| f \|_2 + 2 \| f/\sqrt{\omega} \|_2.
\]


Proof. Let $\Psi \in \mathcal{F} [\hbar]$ with $\| \Psi \|_{\mathcal{H}} = 1$. Applying (A.1) and the spectral theorem, we find

$$
\| a(h)^*(H_f + 1)^{-\frac{1}{2}} \Psi \| \leq \| h \|_2 \| (H_f + 1)^{-\frac{1}{2}} \Psi \| + \| h/\sqrt{\omega} \|_2 \| H_f^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}} \Psi \|
$$

$$
\leq \| h \|_2 + \| h/\sqrt{\omega} \|_2.
$$

(A.5)

The inequality (A.4) is implied by the boundedness of $\sigma_1$ and the triangle inequality:

$$
\| V (H_f + 1)^{-\frac{1}{2}} \| \leq \| \sigma_1 \otimes a(f) (H_f + 1)^{-\frac{1}{2}} \| + \| \sigma_1 \otimes a(f)^* (H_f + 1)^{-\frac{1}{2}} \|
$$

$$
\leq \| a(f) (H_f + 1)^{-\frac{1}{2}} \| + \| a(f)^* (H_f + 1)^{-\frac{1}{2}} \| \leq \| f \|_2 + 2 \| f/\sqrt{\omega} \|_2.
$$

(A.7)

This completes the proof. \Box

As preparation of the proof of Lemma 4.1 (in “Appendix C” below) we recall that the Hamiltonians $H$, c.f. (1.7), as well as $H_f$, c.f. (1.3), are self-adjoint on the common domain $D(H) = \mathcal{K} \otimes D(H_f)$ and bounded below by the constant $b \in \mathbb{R}$; c.f. Proposition 1.1 and (1.22). By spectral calculus we can define the operators $H_f^\frac{1}{2}, (H - b + 1)^{-\frac{1}{2}}$ and $(H_f + 1)^{-\frac{1}{2}}, (H - b + 1)^{-\frac{1}{2}}$ which are closed and densely defined and the latter two are even bounded by 1. For the proof Lemma 4.1 we shall need the following lemma.

**Lemma A.2.** The following operators are bounded:

$$
H_f^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}},
$$

(A.8)

$$
(H - b + 1)^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}}.
$$

(A.9)

**Proof.** Let $\Psi \in \mathcal{H}$ with $\| \Psi \| = 1$. The boundedness of (A.8) follows from the equality

$$
\| H_f^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}} \Psi \|^2 = \langle (H_f + 1)^{-\frac{1}{2}} \Psi, H_f (H_f + 1)^{-\frac{1}{2}} \Psi \rangle
$$

$$
= \langle (H_f + 1)^{-\frac{1}{2}} \Psi, (H - K - gV) (H - b + 1)^{-\frac{1}{2}} \Psi \rangle
$$

(A.10)

and the fact that $K$ is bounded by $|e_1|$ and that for all $\epsilon > 0$

$$
\| (H_f + 1)^{-\frac{1}{2}} \Psi, gV (H_f + 1)^{-\frac{1}{2}} \Psi \| \leq \| (H_f + 1)^{-\frac{1}{2}} \Psi \| \| gV (H_f + 1)^{-\frac{1}{2}} \Psi \|
$$

$$
\leq \frac{g}{\epsilon} \| f/\sqrt{\omega} \|_2 \epsilon \| H_f^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}} \Psi \| + \| f \|_2 \| \Psi \|
$$

$$
\leq \left( \frac{g}{\epsilon} \| f/\sqrt{\omega} \|_2 \right)^2 + \epsilon^2 \| H_f^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}} \Psi \|^2 + \| f \|_2^2
$$

(A.11)

holds, which is a consequence of (A.1). Choosing $0 < \epsilon < 1$ an explicit bound is

$$
\| H_f^\frac{1}{2} (H_f + 1)^{-\frac{1}{2}} \Psi \|^2 \leq \frac{1 + |e_1| + \left( \frac{g}{\epsilon} \| f/\sqrt{\omega} \|_2 \right)^2 + \| f \|_2^2}{1 - \epsilon^2} < \infty.
$$

(A.12)
The boundedness of \((A.9)\) is implied by
\[
\| (H - b + 1)^{1/2} (H f + 1)^{-1/2} \Psi \|^2 = (H f + 1)^{-1/2} (K + H_f + g V - b + 1)(H_f + 1)^{-1/2} \Psi
\]
and, again as a consequence of \((A.1)\),
\[
|\langle (H f + 1)^{-1/2} \Psi, g V (H f + 1)^{-1/2} \Psi \rangle| \leq 2 \| f / \sqrt{\omega} \|_2 \| H f \| (H f + 1)^{-1/2} \Psi \| + \| f \|_2
\]
\[
(A.14)
\]
\[
\leq \| f \|_2 + 2 \| f / \sqrt{\omega} \|_2.
\]
\[
\square
\]

**B. Proofs for Section 1.2**

It is well-known that there is a dense domain of analytic vectors; for example
\[
D = \{ \chi_{[-R, R]}(A) \Psi : \Psi \in \mathcal{H}, R > 0 \}
\]
with \(A\) being the generator of \(U_\theta\) and \(\chi\) the corresponding spectral projection (c.f. [4,32]).

**Proof of Lemma 1.5.** Let \(\theta \in \mathbb{C}\). Definition in \((1.3)\) implies that \(H^\theta_0 = K \otimes 1_{\mathcal{F}[h]} + 1_{\mathcal{K}} \otimes H^\theta_f\) is a sum of commuting self-adjoint operators and \(\sigma(K) = \{e_0, e_1\}\). As shown in [36], we have \(\sigma(H_f) = \mathbb{R}_0^+\) and it follows from the definition of \(H^\theta_f = e^{-\theta H_f}\) in \((1.28)\) that \(\sigma(H^\theta_f) = \{e^{-\theta r} : r \geq 0\}\). The claim then follows from the spectral theorem for two commuting normal operators. \(\square\)

**C. Asymptotic Creation/Annihilation Operators**

**Proof of Lemma 4.1.** Let \(h, l \in h_0\) and \(\Psi \in \mathcal{K} \otimes \mathcal{D}(H^1_f)\). Thanks to Lemma A.2 we have \(\mathcal{K} \otimes \mathcal{D}(H^1_f) = \mathcal{D}(H - b + 1)^{1/2}\). We prove claims (i)–(vi) separately:

(ii) The subspace of \(\mathcal{H}_0\), defined in \((4.18)\), is dense in the domain of \((H - b + 1)^{1/2}\) w.r.t. the graph norm \(\| \cdot \|_{(H - b + 1)^{1/2}}\) so that there is a sequence \((\Psi_n)_{n \in \mathbb{N}}\) in \(\mathcal{K} \otimes \mathcal{F}_{\text{fin}}[h_0]\) with \(\Psi_n \to \Psi\) in this norm as \(n \to \infty\). For all \(n \in \mathbb{N}\), the definition in \((1.34)\) together with the group properties \((e^{-itH})_{t \in \mathbb{R}}\), in particular, the strong continuous differentiability on \(D(H)\), justify

\[
a_t(h) \Psi_n = e^{itH} a(h_t) e^{-itH} = a(h) \Psi_n + \int_0^t ds \frac{d}{ds} e^{isH} a(h_s) e^{-isH} \Psi_n
\]
\[
= a(h) \Psi_n - ig \int_0^t ds \langle h_s, f \rangle_2 e^{isH} \sigma_1 e^{-isH} \Psi_n,
\]
where the last integrand was computed by observing the CCR (c.f. \((1.19)\))
\[
[V, a(h_s)] = \sigma_1 \otimes [a(f) + a(f)^* , a(h_s)] = -\sigma_1 \langle h_s, f \rangle_2.
\]
\[
(C.2)
\]
We may now take the limit \( n \to \infty \) of identity (C.1) and find
\[
a_t(h)\Psi = a(h)\Psi - ig \int_{0}^{t} ds \langle h_s, f \rangle_2 e^{isH} \sigma_1 e^{-isH} \Psi \tag{C.3}
\]
because of the following two ingredients: First, by definition (1.34), the standard estimate (A.1) and Lemma A.2, for all \( m \in \mathfrak{h}_0 \), there is a finite constant \( C_{(C.4)} \) such that
\[
\| a_t(m)(\Psi - \Psi_n) \| = \| a(m_t)(H - b + 1)^{-\frac{1}{2}} e^{-itH} (H - b + 1)^{\frac{1}{2}} (\Psi - \Psi_n) \|
\leq \| m / \sqrt{\omega} \|_2 \| H_f^{\frac{1}{2}} (H - b + 1)^{-\frac{1}{2}} \| \| (H - b + 1)^{\frac{1}{2}} (\Psi - \Psi_n) \|
= C_{(C.4)} \| \Psi - \Psi_n \|_{(H-b+1)^{1/2}}, \tag{C.4}
\]
and likewise
\[
\| a(m)(\Psi - \Psi_n) \| = \| a(m)(H - b + 1)^{-\frac{1}{2}} (H - b + 1)^{\frac{1}{2}} (\Psi - \Psi_n) \|
\leq \| m / \sqrt{\omega} \|_2 \| H_f^{\frac{1}{2}} (H - b + 1)^{-\frac{1}{2}} \| \| (H - b + 1)^{\frac{1}{2}} (\Psi - \Psi_n) \|
= C_{(C.4)} \| \Psi - \Psi_n \|_{(H-b+1)^{1/2}}. \tag{C.5}
\]
Second, the integrand in (C.1) is continuous in \( s \) and, for sufficiently large \( n \), fulfills an \( n \)-independent bound
\[
\| e^{isH} \sigma_1 e^{-isH} (\Psi - \Psi_n) \| \leq \| \sigma_1 \| \| \Psi - \Psi_n \| \leq 1 \tag{C.6}
\]
so dominated convergence can be applied to interchanging the integral and the \( n \to \infty \) limit to prove (C.3).

Finally, a stationary phase argument in \( \omega = |k| \) as well as the facts that \( h \in \mathfrak{h}_0 \) and \( f \in C^\infty(\mathbb{R}\setminus\{0\}) \), c.f. (1.5), provide the estimate
\[
\langle h_s, f \rangle = C \frac{1}{1 + |s|^2} \tag{C.7}
\]
for all \( s \in \mathbb{R} \), thanks to a two-fold partial integration. Hence, we may finally carry out the limit \( t \to \pm \infty \) to find
\[
a_{\pm}(h)\Psi = \lim_{t \to \pm \infty} a_t(h)\Psi = a(h)\Psi - ig \int_{0}^{\pm \infty} ds \langle h_s, f \rangle_2 e^{isH} \sigma_1 e^{-isH} \Psi \tag{C.8}
\]
as the indefinite integral exists thanks to (C.7) and the continuity of the integrand in \( s \). We omit the proof for the asymptotic creation operator \( a_{\pm}^* \) as the argument is almost the same.

(i) This follows from (ii).

(iii) Next, we calculate
\[
e^{-isH} a_-(h) \psi = \lim_{t \to -\infty} e^{-isH} e^{itH} a(h_t) \sigma_1 e^{-itH} \psi
= \lim_{t \to -\infty} e^{i(t-s)H} a(h_{(t-s)+}) \sigma_1 e^{-i(t-s)H} e^{-isH} \psi
= \lim_{t' \to -\infty} e^{it'H} a(h_{t'+s}) \sigma_1 e^{-it'H} e^{-isH} \psi = a_-(h_s) e^{-isH} \psi \tag{C.9}
\]
which proves the pull-through formula in (iii).
First, for all $t \in \mathbb{R}$ we observe

$$\|a_t(h)(\Psi_{\lambda_0} - \Psi_n)\| = \|e^{itH} a(h_t)e^{-itH}(\Psi_{\lambda_0} - \Psi_n)\| = \|a(h_t)\Psi_{\lambda_0}\|$$  \hspace{1cm} (C.10)

due to the ground state property in (1.25). Second, for $\Psi = \Psi_{\lambda_0} \in \mathcal{D}(H) \subset \mathcal{K} \otimes \mathcal{D}(H_f^{1/2})$, we employ the same sequence $(\Psi_n)_{n \in \mathbb{N}}$ as in (ii) to compute

$$\|a(h_t)\Psi_n\|^2 = \sum_{l \in \mathbb{N}} \sqrt{l+1} \int d^3 k_1 \ldots d^3 k_l \int d^3 k e^{i\omega(k)h(k)} \psi_n^{(l+1)}(k, k_1, \ldots, k_l)^2,$$

where we used the Fock vector representation $\Psi_n = (\psi_n^{(l)})_{l \in \mathbb{N}_0}$. We observe that $\Psi_n \in \mathcal{H}_0$ implies $\psi_n^{(l)} \in \mathcal{K} \otimes C^\infty_0(\mathbb{R}^3 \setminus \{0\})$ and, by definition of $\mathcal{H}_0$, c.f. (4.18), there is a constant $L$ such that $\psi_n^{(l)} = 0$ for $l > L$. A stationary phase argument in $\omega(k) = |k|$ and a partial integration in $k$ gives

$$\left| \int d^3 k e^{i\omega(k)h(k)} \psi_n^{(l+1)}(k, k_1, \ldots, k_l) \right| \leq \frac{1}{t} \int d^3 k |k|^{-2} |\partial_{|k|}(|k|^2 h(|k|, \Sigma))| \psi_n^{(l+1)}(|k|, \Sigma, |k_1|, \Sigma_1, \ldots, |k_l|, \Sigma_l)) \right| \leq \frac{1}{t} \sum_{0 \leq l < L} \sqrt{l+1} \int d^3 k_1 \ldots d^3 k_l$$

which converges to zero for $t \to \pm \infty$. In conclusion, for all $n \in \mathbb{R}$ we have

$$\lim_{t \to \pm \infty} a(h_t)\Psi_n = 0.$$  \hspace{1cm} (C.14)

Moreover, there is a $t$-independent, finite constant $C(C.15)(h)$ such that

$$\|a_t(h)(\Psi_{\lambda_0} - \Psi_n)\| = \|e^{itH} a(h_t)e^{-itH}(\Psi_{\lambda_0} - \Psi_n)\|$$

$$\leq \|h\|/\sqrt{\omega} \|H_f^{1/2}(H - b + 1)^{-1/2} \psi_n^{(l)}(H - b + 1)^{1/2} \| \Psi - \Psi_n\|_{(H-b+1)^{1/2}}$$

$$= C(C.15)(h)\|\Psi - \Psi_n\|_{(H-b+1)^{1/2}}$$  \hspace{1cm} (C.15)

and

$$\|a_{\pm}(h)\Psi_{\lambda_0}\| \leq \lim_{t \to \pm \infty} (\|a_t(h)(\Psi_{\lambda_0} - \Psi_n)\| + \|a_t(h)\Psi_n\|)$$

$$\leq C(C.15)(h)\|\Psi - \Psi_n\|_{(H-b+1)^{1/2}}$$  \hspace{1cm} (C.16)

holds true for all $n \in \mathbb{N}$, where we have use the standard inequalities (A.1), Lemma A.2 and (C.14). Taking the limit $n \to \infty$ proves the claim (iv).
(v) We consider the same sequence $(\Psi_n)_{n \in \mathbb{N}}$ as in (iv) and, for all $n \in \mathbb{N}$, we observe that, by (i) and definition in (1.34), it holds
\[
\langle a(h)^*_\pm \Psi_{\lambda_0}, a(l)^*_\pm \Psi_{\lambda_0} \rangle = \lim_{t \to \pm \infty} \langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_{\lambda_0} \rangle.
\] (C.17)

Furthermore, using the CCR in (1.19), we find for all $n \in \mathbb{N}$ that
\[
\langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_n \rangle = \langle \Psi_{\lambda_0}, a(h_t)a(l_t)^* \Psi_n \rangle
\]
\[
= \langle \Psi_{\lambda_0}, (a(l_t)^* a(h_t) + [a(h_t), a(l_t)^*]) \Psi_n \rangle
\]
\[
= \langle a(l_t) \Psi_{\lambda_0}, a(h_t) \Psi_n \rangle + \langle \Psi_{\lambda_0}, \Psi_n \rangle \langle h, l \rangle_2
\] (C.18)
holds. We may control the limit $n \to \infty$ of this identity by
\[
|\langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* (\Psi_{\lambda_0} - \Psi_n) \rangle| \leq \|a(h_t)^* \Psi_{\lambda_0}\| \|a(l_t)^* (\Psi_{\lambda_0} - \Psi_n)\|
\]
\[
\leq (\|h\|_2 + \|h/\sqrt{\omega}\|_2)\|\Psi_{\lambda_0}\|_{(H-b+1)^{1/2}}(\|l\|_2 + \|l/\sqrt{\omega}\|_2)\|\Psi_{\lambda_0} - \Psi_n\|_{(H-b+1)^{1/2}},
\] (C.19)
and likewise,
\[
|\langle a(l_t) \Psi_{\lambda_0}, a(h_t) (\Psi_{\lambda_0} - \Psi_n) \rangle| \leq \|a(l_t) \Psi_{\lambda_0}\| \|a(h_t) (\Psi_{\lambda_0} - \Psi_n)\|
\]
\[
\leq (\|l\|_2 + \|l/\sqrt{\omega}\|_2)\|\Psi_{\lambda_0}\|_{(H-b+1)^{1/2}}(\|h\|_2 + \|h/\sqrt{\omega}\|_2)\|\Psi_{\lambda_0} - \Psi_n\|_{(H-b+1)^{1/2}},
\] (C.20)
which are ensured by the standard estimates (A.1) and Lemma A.2. These bounds allow to take the limit $n \to \infty$ of identity (C.18) which yields
\[
\langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_{\lambda_0} \rangle = \langle a(l_t) \Psi_{\lambda_0}, a(h_t) \Psi_{\lambda_0} \rangle + \langle \Psi_{\lambda_0}, \Psi_{\lambda_0} \rangle \langle h, l \rangle_2.
\]
Finally, recalling (C.17) and exploiting (iv) that states $a_{\pm}^*(h) \Psi_{\lambda_0} = 0$, we find
\[
\langle a(h)^*_\pm \Psi_{\lambda_0}, a(l)^*_\pm \Psi_{\lambda_0} \rangle = \lim_{t \to \pm \infty} \langle a(h_t)^* \Psi_{\lambda_0}, a(l_t)^* \Psi_{\lambda_0} \rangle = \langle \Psi_{\lambda_0}, \Psi_{\lambda_0} \rangle \langle h, l \rangle_2
\] which concludes the proof of (v).

(vi) Let $t \in \mathbb{R}$. Thanks to the standard estimate (A.1), we find
\[
\|a_t(h)(H_f + 1)^{-\frac{1}{2}}\| = \|e^{itH}a(h_t)(H - b + 1)^{-\frac{1}{2}}e^{-itH}(H - b + 1)^{\frac{1}{2}}(H_f + 1)^{-\frac{1}{2}}\|
\]
\[
\leq \|a(h_t)(H - b + 1)^{-\frac{1}{2}}\| \|(H - b + 1)^{\frac{1}{2}}(H_f + 1)^{-\frac{1}{2}}\|
\]
\[
\leq \|h/\sqrt{\omega}\|_2 \|H_f^{\frac{1}{2}}(H - b + 1)^{-\frac{1}{2}}\| \|(H - b + 1)^{\frac{1}{2}}(H_f + 1)^{-\frac{1}{2}}\|.
\] (C.21)

Lemma A.2 ensures that the right-hand side of (C.21) is bounded by a finite constant $C(h)$ which depends only on $h$. This proves the first inequality of (vi). The proof of the second is omitted here as it is almost identical. □
D. The Principle Term $T_p(h, l)$

In the section, we prove that if $G \equiv G(h, l)$ is positive and strictly positive at $\Re \lambda_1 - \lambda_0$ then the absolute of the principal term $T_p(h, l)$ can be bounded by a strictly positive constant times $g^2$.

**Lemma D.1.** Suppose that $G \equiv G(h, l)$ is positive and strictly positive at $\Re \lambda_1 - \lambda_0$, then, for small enough $g$ (depending on $G$), there is a constant $C(h, l) > 0$ (independent of $g$) such that

$$|T_p(h, l)| \geq C(h, l)g^2.$$

**(D.1)**

**Proof.** We set

$$I := \int dr \frac{G(r)}{(r + \lambda_0 - \Re \lambda_1 - ig^2E_1)(r - \lambda_0 + \lambda_1)},$$

**(D.2)**

and take small enough $g$. Recalling (2.4), we observe that

$$T_p(h, l) = g^2 E_1 MI.$$

**(D.3)**

We recall from the discussion below Definition 2.1 that $E_1 = E_I + g^a \Delta$, where $a > 0$, $\Delta \equiv \Delta(g)$ is uniformly bounded and $E_I$ is a strictly negative constant that does not depend on $g$, see (3.11). Additionally, it follows from (3.25) together with $\|\varphi_0 \otimes \Omega\| = 1$ that $\|\Psi_{\lambda_0}\| \geq C > 0$, for some constant $C$ that does not depend on $g$. Moreover, we conclude from (3.28) that $\Re \lambda_1 - \lambda_0 \geq C > 0$ for some constant $C$ (independent of $g$). Consequently, (2.6) guarantees that there is a constant $C$ (independent of $g$) such that $|M| \geq C > 0$.

This together with (D.3) implies that it suffices to show that there is a constant $C(h, l) > 0$ such that

$$|I| \geq C(h, l),$$

**(D.4)**

in order to conclude (D.1).

For $\alpha \equiv \alpha_g := \Re \lambda_1 - \lambda_0$ and recalling (1.2), we observe

$$I = \int dr \frac{G(r)}{(r - \alpha - ig^2E_1)(r + \alpha - ig^2E_1)} = \int dr \frac{G(r) \left( r^2 - \alpha^2 - g^4E_1^2 + 2ig^2E_1r \right)}{(r^2 - \alpha^2 - g^4E_1^2)^2 + 4g^4E_1^2r^2}.$$

**(D.5)**

Let $c > 0$ be such that $G$ is supported in the complement of the ball or radius $c$ and center $0$. Then, we have

$$| \text{Im}(I) | \geq | E_1 | \int dr \frac{2g^2r}{(r^2 - \alpha^2 - g^4E_1^2)^2 + 4g^4E_1^2c^2}.$$  

**(D.6)**

Substituting $s = r^2$, yields

$$| \text{Im}(I) | \geq | E_1 | \int ds G(\sqrt{s}) \frac{g^2}{(s - \alpha^2 - g^4E_1^2)^2 + 4g^4E_1^2c^2}.$$  

**(D.7)**

Since $G(\alpha) \neq 0$, then for small enough $g$ there is a constant $r_0$, that does not depend on $g$ and a constant $C > 0$ (independent of $g$) such that $G(\sqrt{s}) \geq C$, for every $s \in$
\[ \alpha^2 + g^4 E_1^2 - r_0, -\alpha^2 + g^4 E_1^2 + r_0 \]. We apply the change of variables \( u = s - \alpha^2 - g^4 E_1^2 \) and obtain
\[
|\text{Im}(I)| \geq C|E_1| \int_{-r_0}^{r_0} \frac{g^2}{s^2 + 4g^4 E_1^2 c^2} ds.
\] (D.8)

Finally, we change to the variable \( \tau = s/g^2 \) to obtain:
\[
|\text{Im}(I)| \geq C|E_1| \int_{-r_0/g^2}^{r_0/g^2} \frac{1}{\tau^2 + 4E_1^2 c^2} d\tau \geq C|E_1|,
\] (D.9)
for small enough \( g \) (depending on \( G \)).  

**List of Main Notations**

In this section we provide of list of main notations and their place of definition used in this

| Symbol | Place of definition |
|--------|---------------------|
| \( E_1 \) | Below (1.1) |
| \( H_0, K, H_f \) | (1.3) |
| \( e_0, e_1 \) | Below (1.3) |
| \( \omega \) | Below (1.3) |
| \( V, \sigma_1 \) | (1.4) |
| \( f \) | (1.5) |
| \( \mu \) | (1.6) |
| \( H \) | (1.7) |
| \( g \) | Below (1.7), see also Definition 3.1, (3.31) and Definition 4.3 in [14] |
| \( \mathcal{H}, \mathcal{K} \) | (1.8) |
| \( \mathcal{F}[\mathfrak{h}], \mathfrak{h} \) | (1.9) |
| \( \odot \) | Below (1.9) |
| \( \Omega \) | (1.10) |
| \( \mathcal{F}_0 \) | (1.11) |
| \( S(\mathbb{R}^3, \mathbb{C}) \) | Below (1.11) |
| \( \alpha(h) \) | (1.12) |
| \( \alpha(h)^* \) | (1.13) |
| \( \alpha(k) \) | (1.14) |
| \( \alpha(k)^* \) | (1.15) |
| \( \psi_0, \psi_1 \) | (1.20) |
| \( D(\bullet) \) | Below (1.20) |
| \( \sigma(\bullet) \) | Below (1.20) |
| \( \theta, u_\theta, U_\theta \) | Definition 1.3 |
| \( H^\theta \) | (1.27) |
| \( H_f^\theta, V_\theta^\phi \) | (1.28) |
| \( \omega_\phi^\theta, j_\phi^\theta \) | (1.29) |
| \( D(\bullet, \bullet) \) | (1.30) |
| \( \lambda_0, \lambda_1 \) | Below Lemma 1.5 |
| \( \Psi_{\lambda_0}, \Psi_{\lambda_1} \) | (1.32) |
| \( b_0 \) | (1.33) |
| \( a_{\pm}(h) \) | (1.34) |
| \( a_{\pm}(h)^* \) | Below (1.34) |
| \( \mathcal{K}^\pm, \mathcal{H}^\pm \) | (1.35) |
| \( \Omega_{\pm} \) | (1.36) |
| \( S(h, l) \) | (1.37) |
\[ T(h, l) \]
\[ G \]
\[ \nu \]
\[ \mathcal{S} \]
\[ \mathcal{P} \]
\[ \rho_0, \rho \]
\[ A \]
\[ B_0(1), B_1(1) \]
\[ C_{m}(z) \]
\[ E_I \]
\[ \rho_n \]
\[ H^{(n), \theta} \]
\[ V^{(n), \theta} \]
\[ G^{(n)} \]
\[ \eta^{(n)} \]
\[ \Omega^{(n, \infty)}, P_{\Omega^{(n, \infty)}} \]
\[ l_0^{(n)}, l_1^{(n)} \]
\[ P_0^{(n), \theta}, P_1^{(n), \theta} \]
\[ P_0^0, P_1^0 \]
\[ C \]
\[ \tilde{C} \]
\[ W \]
\[ \Sigma \]
\[ \mathcal{H}_0 \]
\[ \mathcal{F}_{\text{fin}}[h_0] \]
\[ \| \cdot \| \star \]
\[ \Gamma(\epsilon, R) \]
\[ \Gamma_{-}(\epsilon, R) \]
\[ \Gamma_{d}(R) \]
\[ \Gamma_{c}(\epsilon) \]
\[ \epsilon_n \]
\[ R_1(q, Q) \]
\[ P_1(q, Q) \]
\[ P_1(q, Q) \]

References

1. Abou Salem, W.K., Faupin, J., Fröhlich, J., Sigal, I.M.: On the theory of resonances in non-relativistic quantum electrodynamics and related models. Adv. Appl. Math. 43, 201–230 (2009)
2. Bach, V., Ballesteros, M., Fröhlich, J.: Continuous renormalization group analysis of spectral problems in quantum field theory. J. Funct. Anal. 268(5), 749–823 (2015)
3. Bach, V., Ballesteros, M., Könenberg, M., Menrath, L.: Existence of ground state eigenvalues for the Spin-Boson model with critical infrared divergence and multiscale analysis. J. Math. Anal. Appl. 453(2), 773–797 (2017)
4. Bach, V., Ballesteros, M., Pizzo, A.: Existence and construction of resonances for atoms coupled to the quantized radiation field. ArXiv preprint: arXiv:1302.2829 (2013)
5. Bach, V., Ballesteros, M., Pizzo, A.: Existence and construction of resonances for atoms coupled to the quantized radiation field. Adv. Math. 314, 540–572 (2017)
6. Bach, V., Chen, T., Fröhlich, J., Sigal, I.M.: Smooth Feshbach map and operator-theoretic renormalization group methods. J. Funct. Anal. 203, 44–92 (2003)
7. Bach, V., Fröhlich, J., Pizzo, A.: An infrared-finite algorithm for Rayleigh scattering amplitudes, and Bohr’s frequency condition. Commun. Math. Phys. 274, 457–486 (2007)
8. Bach, V., Fröhlich, J., Sigal, I.M.: Mathematical theory of nonrelativistic matter and radiation. Lett. Math. Phys. 34(3), 183–201 (1995)
9. Bach, V., Fröhlich, J., Sigal, I.M.: Quantum electrodynamics of confined nonrelativistic particles. Adv. Math. 137(2), 299–395 (1998)
10. Bach, V., Fröhlich, J., Sigal, I.M.: Renormalization group analysis of spectral problems in quantum field theory. Adv. Math. 137(2), 205–298 (1998)
11. Bach, V., Fröhlich, J., Sigal, I.M.: Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. Commun. Math. Phys. 207(2), 249–290 (1999)
12. Bach, V., Klopp, F., Zenk, H.: Mathematical analysis of the photoelectric effect. Adv. Theor. Math. Phys. 5, 969–999 (2001)
13. Bach, V., Møller, J.S., Westrich, M.C.: Beyond the van Hove timescale (preprint in preparation)
14. Ballesteros, M., Deckert, D.-A., Hänle, F.: Analyticity of resonances and eigenvalues and spectral properties of the massless Spin-Boson model. arXiv:1801.04021 (2018)
15. Ballesteros, M., Faupin, J., Fröhlich, J., Schubnel, B.: Quantum electrodynamics of atomic resonances. Commun. Math. Phys. 337(2), 633–680 (2015)
16. Bony, J.-F., Faupin, J., Sigal, I.: Maximal velocity of photons in non-relativistic QED. Adv. Math. 231(5), 3054–3078 (2012)
17. De Roeck, W., Griesemer, M., Kupiainen, A.: Asymptotic completeness for the massless Spin-Boson model. Adv. Math. 268, 62–84 (2015)
18. De Roeck, W., Kupiainen, A.: Approach to ground state and time-independent photon bound for massless Spin-Boson model. Ann. Henri Poincaré 14(2), 253–311 (2013)
19. De Roeck, W., Kupiainen, A.: Minimal velocity estimates and soft mode bounds for the massless Spin-Boson model. Ann. Henri Poincaré 16(2), 365–404 (2015)
20. Dereziński, J., Gérard, C.: Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians. Rev. Math. Phys. 11(4), 383–450 (1999)
21. Faupin, J.: Resonances of the confined hydrogen atom and the Lamb–Dicke effect in non-relativistic qed. Ann. Henri Poincaré 9, 743–773 (2008)
22. Faupin, J., Sigal, I.M.: Minimal photon velocity bounds in non-relativistic quantum electrodynamics. J. Stat. Phys. 154(1–2), 58–90 (2014)
23. Faupin, J., Sigal, I.M.: On Rayleigh scattering in non-relativistic quantum electrodynamics. Commun. Math. Phys. 328(3), 1199–1254 (2014)
24. Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic completeness for Rayleigh scattering. Commun. Math. Phys. 252(1), 415–476 (2004)
25. Fröhlich, J., Griesemer, M., Schlein, B.: Asymptotic completeness for Compton scattering. Commun. Math. Phys. 252(1), 415–476 (2004)
26. Fröhlich, J., Griesemer, M., Sigal, I.M.: Spectral renormalization group. Rev. Math. Phys. 21, 511–548 (2009)
27. Griesemer, M., Hasler, D.: On the smooth Feshbach–Schur map. J. Funct. Anal. 254(9), 2329–2335 (2008)
28. Hasler, D., Herbst, I.: Ground states in the Spin Boson model. Ann. Henri Poincaré 12(4), 621–677 (2011)
29. Hasler, D., Herbst, I., Huber, M.: On the lifetime of quasi-stationary states in non-relativistic QED. Ann. Henri Poincaré 9(5), 1005–1028 (2008)
30. Hübner, M., Spohn, H.: Radiative decay: nonperturbative approaches. Rev. Math. Phys. 07(03), 363–387 (1995)
31. Hübner, M., Spohn, H.: Spectral properties of the Spin-Boson Hamiltonian. Ann. d’I.H.P Sect. A 64(2), 289–323 (1995)
32. Jakšić, V., Pillet, C. A.: On a model for quantum friction. i. Fermi’s golden rule and dynamics at zero temperature. Ann. de l’I.H.P Physique théorique 62(1), 47–68 (1995)
33. Pizzo, A.: One-particle (improper) states in Nelson’s massless model. Ann. Henri Poincaré 4, 439–86 (2003)
34. Pizzo, A.: Scattering of an infraparticle: the one particle sector in Nelson’s massless model. Ann. Henri Poincaré 6, 553–606 (2005)
35. Reed, M., Simon, B.: Methods of Modern Mathematical Physics I: Analysis of Operators. Academic Press, London (1978)
36. Reed, M., Simon, B.: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness. Academic Press, London (1978)
37. Salem, W.K.A., Fröhlich, J.: Adiabatic theorems for quantum resonances. Commun. Math. Phys. 273(3), 651–675 (2006)
38. Sigal, I.M.: Ground state and resonances in the standard model of the non-relativistic QED. J. Stat. Phys. 134(5–6), 899–939 (2009)
39. Simon, B.: Resonances in n-body quantum systems with dilatation analytic potentials and the foundations of time-dependent perturbation theory. Ann. Math. Sect. Ser. 97(2), 247–274 (1973)

40. Spohn, H.: Dynamics of Charged Particles and their Radiation Field, 1st edn. Cambridge University Press, Cambridge (2008)

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