Scaling behavior of improvement and renormalization constants

Tanmoy Bhattacharya\textsuperscript{a}, Rajan Gupta\textsuperscript{a}, Weonjong Lee\textsuperscript{a}, Stephen Sharpe\textsuperscript{b}

\textsuperscript{a}MS B-285, Los Alamos National Lab, Los Alamos, New Mexico 87545, USA
\textsuperscript{b}Physics Department, University of Washington, Seattle, Washington 98195, USA

This talk summarizes results for all the scale independent renormalization constants for bilinear currents ($Z_A$, $Z_V$, and $Z_S/Z_P$), the improvement constants ($c_A$, $c_V$, and $c_T$), the quark mass dependence of $Z_O$, and the coefficients of the equation of motion operators for $O(a)$ improved lattice QCD. Using data at $\beta = 6.0, 6.2$ and 6.4 we study the scaling behavior of these quantities and quantify residual discretization errors.

The use of axial and vector Ward identities has proven to be a very efficient and reliable way of extracting the improvement and renormalization constants for the $O(a)$ improved fermion action. The methodology, references to previous calculations, and the notation we use are given in \[\text{[1]}\].

The new features of our calculation summarized here are: new determinations of $c_A$ including $O(m^2a^2)$ corrections and using non-zero momentum correlators; improved chiral extrapolations in the extraction of $Z_A^0$, $c_T$ and $b_P - b_A$; and a quantitative comparison of the scaling behavior of the differences between our results and those of the ALPHA collaboration \[\text{[2,2]}\] and 1-loop perturbation theory. Results are summarized in Table 1 and will be presented in detail in \[\text{[4]}\].

The first feature we discuss is the need for including an $O(m^2a^2)$ term in the extrapolation of $c_A$ to the chiral limit. Data at $\beta = 6.4$ is shown in Fig. 1. Table 1 gives results from both the linear and the preferred quadratic fit. Our results show a weak dependence of $c_A$ on $\beta$ in the range $6.0 - 6.4$, unlike that found by the ALPHA collaboration, but consistent with the recent results by Collins et al. \[\text{[5]}\].

The second new feature is the demonstration that consistent estimates for $c_A$ are obtained from correlators with zero and non-zero momentum once additional $O(p^2a^2)$ errors are accounted for. A plot of $c_A$ versus $(12p^2/\pi)^2$ at $\beta = 6.4$ is shown in Fig. 2. We find that a linear extrapolation to $p = 0$ yields results consistent with those obtained using zero momentum correlators, and with a slope of expected magnitude.

The second point concerns the chiral extrapolation for $Z_A^0$, $c_T$, $b_P - b_A$. Our estimates presented in \[\text{[4]}\] were based on constant fits as these quantities are not expected to have $O(ma)$ corrections if the theory is fully improved to $O(a)$. We now advocate using results of linear extrapolation (marked with an asterisk in Table 1) as our data show a dependence on $m$. Such behavior can be explained by $O(a\Lambda_{QCD} ma)$ corrections which can arise as a result of using a mass-dependent $c_A$ in intermediate stages of the calculations. To show the size of this effect, we give both estimates

\[\text{Figure 1. Comparison of linear and quadratic extrapolation of } c_A \text{ to the chiral limit}\]
Table 1
The first error in LANL estimates is statistical, and the second, where present, corresponds to the difference between using 2-point and 3-point discretization of the derivative in extraction of \(c_A\). Asterisks mark values which include \(O(ma)\) corrections in the chiral extrapolations.

| \(\beta = 6.0\) | \(\beta = 6.2\) | \(\beta = 6.4\) |
|-----------------|--|------------------|
| \(c_{SW}\) | LANL | ALPHA | P. Th. | LANL | ALPHA | P. Th. | LANL | ALPHA | P. Th. |
| 1.769 | 1.769 | 1.521 | 1.614 | 1.614 | 1.481 | 1.526 | 1.526 | 1.449 |
| \(Z_V^0\) | +0.770(1) | +0.789(6) | +0.810 | +0.787(4) | +0.792(4) | +0.821 | +0.802(1) | +0.832(6) | +0.830 |
| \(Z_A^0\) | +0.807(2)(8) | +0.7906(94) | +0.829 | +0.818(2)(5) | +0.807(8)(2) | +0.839 | +0.827(1)(4) | +0.827(8)(1) | +0.847 |
| \(Z_A^0*\) | +0.802(2)(8) | +0.815(2)(5) | +0.822(1)(4) | +0.884(3)(1) | +0.901(2)(5) |
| \(Z_V^0/Z_Z^0\) | +0.842(5)(1) | N.A. | +0.956 | N.A. | +0.959 | N.A. | +0.962 |
| \(c_A\) | -0.037(4)(8) | -0.083(5) | -0.013 | -0.032(3)(6) | -0.038(4) | -0.012 | -0.029(2)(4) | -0.025(2) | -0.011 |
| \(c_{A*}\) | -0.038(4) | -0.033(3) | -0.032(3) | -0.009(2)(1) | -0.026 | -0.08(1)(2) | -0.13(5) |
| \(c_V\) | -0.107(17)(4) | -0.32(T) | -0.028 | -0.09(2)(1) | -0.21(7) | -0.026 | -0.13(5) |
| \(c_T\) | +0.063(7)(29) | N.A. | +0.020 | +0.051(7)(17) | N.A. | +0.019 | +0.041(3)(23) |
| \(c_T^*\) | +0.0767(10) | +0.059(8) | +0.051(4) | +0.051(4) |
| \(b_V\) | +1.43(1)(4) | N.A. | +1.106 | +1.30(1)(1) | N.A. | +1.099 | +1.24(1)(1) | N.A. | +1.093 |
| \(b_V\) | +1.52(1) | +1.54(2) | +1.274 | +1.42(1) | +1.41(2) | +1.255 | +1.39(1) | +1.36(3) | +1.239 |
| \(b_A - b_V\) | -0.263(3)(4) | N.A. | -0.002 | -0.11(3)(4) | N.A. | -0.002 | -0.09(1)(1) | N.A. | -0.002 |
| \(b_A - b_V\) | -0.243(3)(4) | N.A. | -0.002 | -0.11(3)(4) | N.A. | -0.002 | -0.08(1)(1) | N.A. | -0.002 |
| \(b_P - b_S\) | -0.064(3)(3) | N.A. | -0.066 | -0.09(2)(1) | N.A. | -0.062 | -0.09(10)(1) | N.A. | -0.059 |
| \(b_P - b_A\) | -0.074(5) | N.A. | +0.002 | -0.09(3)(3) | N.A. | +0.001 | -0.12(2)(5) | N.A. | +0.001 |
| \(b_P - b_A^*\) | -0.08(30) | +0.03(10) | -0.02(4) | |
| \(\Delta Z_V^0\) | -55a | -(464a)^3 | } \(\Delta Z_A^0\) = -(181a) + (763a)^2, \(\Delta c_A\) = -(367a) + (669a)^2, where \(a\) is in units of \((\text{MeV}^{-1})\) and has values 1/210, 1/2910 and 1/3850 at the three \(\beta\). Considering that the expected size of the terms is \(O(a_{QCD})^n\), all the coefficients look reasonable, however, the errors in them are large. We make the following observations:

- The difference in \(Z_V^0\) is dominated by the \(O(a^2)\) term.
- The errors in the coefficients for \(Z_A^0\), and \(c_V\) are >100%. This is not surprising since the combined error at each \(\beta\) is approximately equal to the difference.
- The coefficients in the fit for \(\Delta c_A\) have reasonable errors, however the fit is dominated...
corrections dominate the differences in the fits. Over this range of $\Delta$, the leading residual discretization error in $Z^0_A$, $O(\alpha_s^2)$, corrections. The results are

$$
\Delta Z^0_A = -(158a)^2 - (1.4\alpha_s)^2
$$

$$
\Delta Z_V = (197a)^2 - (1.4\alpha_s)^2
$$

$$
\Delta Z_V^0/Z_S^0 = -(502a)^2 - (1.8\alpha_s)^2
$$

$$
\Delta c_A = -(13a) - (1.3\alpha_s)^2
$$

$$
\Delta c_V = -(51a) - (1.7\alpha_s)^2
$$

$$
\Delta c_T = (94a) + (0.8\alpha_s)^2
$$

$$
\Delta b_V = (930a) - (2.6\alpha_s)^2
$$

$$
\Delta b_V = (429a) + (1.5\alpha_s)^2
$$

where $a$, expressed in MeV$^{-1}$, $\alpha_s = g^2/(4\pi u_0^4)$ is the tadpole improved coupling with values 0.1340, 0.1255 and 0.1183 at the three $\beta$, and $u_0$ is the plaquette$^{1/4}$. The errors in the other $b$ are too large to allow any meaningful fits.

The errors in the coefficients for the three $\Delta Z$'s are reasonably small, providing some confidence in the fits. Over this range of $\beta$, the perturbative corrections dominate the differences in $Z^0_A$ and $Z^0_V$, whereas in $Z^0_V/Z^0_S$, the two corrections are comparable.

The errors in the coefficients for the three $\Delta c$'s are large. Even though the coefficients are of the size expected, it is important to note that the non-perturbative estimates are 2 - 4 times the perturbative values.

Both corrections are large in $\Delta b_V$ and $\Delta b_V$, with the discretization error being the larger of the two.

Overall, these fits, since they are based on data at just three $\beta$ values with $1/a$ between 2.1 and 3.86 GeV and since we have ascribed no errors to $a$ or $\alpha_s$, should be considered indicative and qualitative and certainly not sufficient to draw precise conclusions. This is why we refrain from quoting errors in the fits.

Finally, in Table 2 we present results for the coefficients of the equation of motion operators. Estimates at $\beta = 6.0$ are poor, but become reasonably precise at $\beta = 6.2$ and 6.4. We find that except for $c_P$, the corrections to the tree level value $c^0 = 1$ are large.

### Table 2

| $\beta$ | 6.0(f) | 6.0(b) | 6.2 | 6.4 |
|---------|--------|--------|-----|-----|
| $c_V + c_P$ | 2.82(15) | +2.68(19) | 2.62(8) | 2.44(4) |
| $c_A + c_P$ | 2.43(24) | +2.12(31) | 2.43(14) | 2.27(6) |
| $2c_P$ | 0.88(97) | -0.65(57) | 1.82(24) | 1.85(8) |
| $c_S + c_P$ | 2.44(13) | +2.40(13) | 2.40(7) | 2.27(4) |
| $c_T + c_P$ | 2.40(18) | +2.27(20) | 2.42(9) | 2.28(5) |
| $c_V$ | 2.38(50) | +3.00(37) | 1.72(16) | 1.52(4) |
| $c_A$ | 1.99(56) | +2.45(46) | 1.53(20) | 1.35(6) |
| $c_P$ | 0.44(49) | -0.33(29) | 0.91(12) | 0.93(4) |
| $c_S$ | 2.00(48) | +2.72(33) | 1.49(14) | 1.35(4) |
| $c_T$ | 1.96(49) | +2.60(38) | 1.51(15) | 1.36(4) |

### REFERENCES

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