A feasible adaptive refinement algorithm for linear semi-infinite optimization

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ABSTRACT

A numerical method is developed to solve linear semi-infinite programming problem (LSIP) in which the iterates produced by the algorithm are feasible for the original problem. This is achieved by constructing a sequence of standard linear programming problems with respect to the successive discretization of the index set such that the approximate regions are included in the original feasible region. The convergence of the approximate solutions to the solution of the original problem is proved and the associated optimal objective function values of the approximate problems are monotonically decreasing and converge to the optimal value of LSIP. An adaptive refinement procedure is designed to discretize the index set and update the constraints for the approximate problem. Numerical experiments demonstrate the performance of the proposed algorithm.

KEYWORDS

Linear semi-infinite optimization, feasible iteration, concavification, adaptive refinement

1. Introduction

Linear semi-infinite programming problem (LSIP) refers to the optimization problem with finitely many decision variables and infinitely many linear constraints associated with some parameters, which can be formulated as

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad c^\top x \\
\text{s.t.} & \quad a(y)^\top x + a_0(y) \geq 0 \quad \forall y \in Y, \\
& \quad x_i \geq 0, \ i = 1, 2, ..., n,
\end{align*}
\]

(LSIP)

where \( c \in \mathbb{R}^n, a(y) = [a_1(y), ..., a_n(y)]^\top \) and \( a_i : \mathbb{R}^m \mapsto \mathbb{R}, \) for \( i = 0, 1, ..., n, \) are real-valued coefficient functions, \( Y \subseteq \mathbb{R}^m \) is the index set. In this paper, we assume that \( Y = [a, b] \) is an interval with \( a < b. \) Denote by \( F \) the feasible set of (LSIP):

\[
F = \{x \in \mathbb{R}^n_+ \mid a(y)^\top x + a_0(y) \geq 0, \forall y \in Y\},
\]

where \( \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, 2, ..., n\}. \)
Linear semi-infinite programming has wide applications in economics, robust optimization and numerous engineering problems, etc. More details can be found in [1–3] and references therein.

Numerical methods have been proposed for solving linear semi-infinite programming problems such as discretization methods, local reduction methods and descent direction methods (See [4–7] for an overview of these methods). The main idea of discretization methods is to solve the following linear program

\[
\min_{x \in \mathbb{R}_+^n} f(x)
\]

\[
\text{s.t. } a(y)^\top x + a_0(y) \geq 0 \quad \forall y \in T,
\]

in which the original index set \( Y \) in \((\text{LSIP})\) is replaced by its finite subset \( T \). The iterates generated by the discretization methods converge to a solution of the original problem as the distance between \( T \) and \( Y \) tends to zero (see [2, 4, 8]). The reduction methods solve nonlinear equations by quasi-Newton method, which require the smoothing conditions on the functions defining the constraint [9]. The feasible descent direction methods generate a feasible direction based on the current iterate and achieve the next iterate by such a direction [10].

The purification methods proposed in [11, 12] generate a finite feasible sequence where the objective function value of each iterate is reduced. The method proposed in [11] requires that the feasible set of \((\text{LSIP})\) is locally polyhedral, and the method proposed in [12] requires that the coefficient functions \( a_i, i = 0, 1, ..., n \), are analytic.

Feasible iterative methods for nonlinear semi-infinite optimization problems have been developed via techniques of convexification or concavification etc [13–15]. These methods might be applicable to solve \((\text{LSIP})\) directly. However, they are not developed specifically for \((\text{LSIP})\). Computational time will be reduced if the algorithm can be adapted to linear case effectively.

In this paper, we develop a feasible iterative algorithm to solve \((\text{LSIP})\). The basic idea is to construct a sequence of standard linear optimization problems with respect to the discretized subsets of the index set such that the feasible region of each linear optimization problem is included in the feasible region of \((\text{LSIP})\). The proposed method consists of two stages. The first stage is based on the restriction of the semi-infinite constraint. The second stage is base on estimating the lower bound of the coefficient functions using concavification or interval method.

The rest of the paper is organized as follows. In section 2, we propose the methods to construct the inner approximate regions for the feasible region of \((\text{LSIP})\). Numerical method to solve the original linear semi-infinite programming problem is proposed in section 3. In section 4, we implement our algorithm to some numerical examples to show the performance of the method. At last, we conclude our paper in section 5.

2. Restriction of the lower level problem

The restriction of the lower level problem leads to inner approximation of the feasible region of \((\text{LSIP})\) and thus, to feasible iterates. Two-stage procedures are performed to achieve the restriction for \((\text{LSIP})\). At the first stage, we construct an uniform lower-bound function w.r.t decision variables for the function defining constraint in \((\text{LSIP})\). This step requires to solve a global optimization associated with coefficient functions over the index set. The second stage is to estimate the lower bound of the coefficient
functions over the index set rather than solving the optimization problems globally which significantly reduce the computational cost.

2.1. Construction of the lower-bound function

The semi-infinite constraint of (LSIP) can be reformulated as

$$\min_{y \in Y} \{a(y)^\top x + a_0(y)\} \geq 0.$$  \hspace{1cm} (1)

Since $a(y)^\top x = \sum_{i=1}^n a_i(y)x_i$, \((1)\) is equivalent to

$$\min_{y \in Y} \{\sum_{i=1}^n a_i(y)x_i + a_0(y)\} \geq 0.$$  \hspace{1cm} (2)

By exchanging the minimization and summation on the left side of the inequality, we obtain a new linear inequality

$$\sum_{i=1}^n \{\min_{y \in Y} a_i(y)\}x_i + \min_{y \in Y} a_0(y) \geq 0.$$  \hspace{1cm} (2)

Since the decision variables $x_i \geq 0$, $i = 1, 2, \ldots, n$, we have

$$\sum_{i=1}^n \{\min_{y \in Y} a_i(y)\}x_i + \min_{y \in Y} a_0(y) \leq \min_{y \in Y} \{\sum_{i=1}^n a_i(y)x_i + a_0(y)\}.$$

Thus, we obtain an uniform lower-bound function for $\min_{y \in Y} \{a(y)^\top x + a_0(y)\}$. And any point $x$ satisfying \((2)\) is a feasible point for LSIP. Let $\bar{F}$ be the feasible region defined by the inequality \((2)\), i.e.,

$$\bar{F} = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \{\min_{y \in Y} a_i(y)\}x_i + \min_{y \in Y} a_0(y) \geq 0\}.$$  \hspace{1cm} (2)

From above analysis, we conclude that $\bar{F} \subseteq F$.

The main difference between the original constraint \((1)\) and the restriction constraint \((2)\) is that the minimization is independent on the decision variable $x$ in the latter case. In order to compute $\bar{F}$, we need to solve a series of problems as follows:

$$\min_y a_i(y) \quad \text{s.t.} \quad y \in Y$$  \hspace{1cm} (3)

for $i = 0, 1, \ldots, n$. Based on $\bar{F}$, we can construct a linear program associated with one linear inequality constraint such that it has the same objective function as LSIP and any feasible point of the constructed problem is feasible for LSIP. Such a problem is defined as

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad x \in \bar{F}.$$  \hspace{1cm} (R-LSIP)
To characterize how well R-LSIP approximates LSIP, we can estimate the distance between \( g(x) = \min_{y \in Y} \{a(y)^T x + a_0(y)\} \) and \( \bar{g}(x) = \sum_{i=1}^{n} \{\min_{y \in Y} a_i(y)\} x_i + \min_{y \in Y} a_0(y) \) which have been used to define the constraints of LSIP and R-LSIP. Assume that each function \( a_i(y) \) is Lipschitz continuous on \( Y \), i.e., there exist some constant \( L_i \geq 0 \) such that \( |a_i(y) - a_i(z)| \leq L_i |y - z| \) holds for any \( y, z \in Y, i = 0, 1, 2, ..., n \).

By direct computation, we have

\[
|g(x) - \bar{g}(x)| \leq \left( \sum_{i=1}^{n} L_i x_i + L_0 \right) (b - a).
\]

It turns out that for any fixed \( x \), the error between \( g(x) \) and \( \bar{g}(x) \) is bounded linearly with respect to \( b - a \). Furthermore, if we assume that the decision variables are upper bounded (e.g., \( 0 \leq x_i \leq U_i \) for some constants \( U_i > 0, i = 0, 1, 2, ..., n \)), we have

\[
|g(x) - \bar{g}(x)| \leq \left( \sum_{i=1}^{n} L_i U_i + L_0 \right) (b - a).
\]

This indicates that the error between \( g(x) \) and \( \bar{g}(x) \) goes to zeros uniformly as \( |b - a| \) tends to zero. By dividing the index set \( Y = [a, b] \) into subintervals, one can construct a sequence of linear programs that approximate LSIP exhaustively as the size of the subdivision (formally defined in section 3.1) tends to zero. Given a subdivision, constructing R-LSIP on each subinterval requires to solve \([R-LSIP]\) globally which will become computationally expensive due to the increasing number of subintervals and non-convexity of the coefficient functions in general. In fact, it is not necessary to solve \([R-LSIP]\) exactly. In the next section, we will discuss how to estimate a good lower bound of \([R-LSIP]\) and use it to construct the feasible approximation problems for \([LSIP]\).

### 2.2. Construction of the inner approximation region

In order to guarantee that the feasible region \( \bar{F} \) derived from inequality (2) is an inner approximation of the feasible region of \([LSIP]\) optimization problem (3) needs to be solved globally. However, computing a lower bound for (3) is enough to generate a restriction problem of \([LSIP]\) in this section, we present two alternative approaches to approximate problem (3). The idea of the first approach comes from the techniques of interval methods [16, 21]. Given an interval \( Y = [a, b] \), the range of \( a_i(y) \) on \( Y \) is defined as \( R(a_i, Y) = \bar{R}_i^u, R_i^l = \{a_i(y) \mid y \in Y\} \). An interval function \( A_i(Y) = \bar{A}_i^u, A_i^l \) is called an inclusion function for \( a_i(y) \) on \( Y \) if \( R(a_i, Y) \subseteq A_i(Y) \). A natural inclusion function can be obtained by replacing the decision variable \( y \) in \( a_i(y) \) with the corresponding interval and computing the resulting expression using the rules of interval arithmetic [21]. In some special cases, the natural inclusion function is tight (i.e., \( R(a_i, Y) = A_i(Y) \)). However, in more general cases, the natural interval function overestimates the original range of \( a_i(y) \) on \( Y \) which implies that \( A_i^l < \min_{y \in Y} a_i(y) \). In such cases, the tightness of the inclusion can be measured by

\[
\max \{|R_i^l - A_i^l|, |R_i^u - A_i^u|\} \leq \gamma |b - a|^p \quad \text{and} \quad |A_i^l - A_i^u| \leq \delta |b - a|^p, \quad (4)
\]

where \( p \geq 1 \) is the convergence order, \( \gamma \geq 0 \) and \( \delta \geq 0 \) are constants which depend on the expression of \( a_i(y) \) and the interval \([a, b]\). By replacing \( \min_{y \in Y} a_i(y) \) in (2) with
$A_i^l$ for $i = 0, 1, \ldots, n$, we have a new linear inequality as follows

$$
\sum_{i=1}^{n} A_i^l x_i + A_0^l \geq 0. \tag{5}
$$

It is obvious that any $x$ satisfying (5) is a feasible point for [LSIP].

The second approach to estimate the lower bound of problem (2) is to construct a uniform lower bound function $\bar{a}_i(y)$ such that $\bar{a}_i(y) \leq a_i(y)$ holds for all $y \in Y$. In addition, we require that the optimal solution for

$$
\min_y \bar{a}_i(y) \quad \text{s.t.} \quad y \in Y
$$

is easy to be identified. Here, we construct a concave lower bound function for $a_i(y)$ by adding a negative quadratic term to it, i.e.,

$$
\bar{a}_i(y) = a_i(y) - \frac{\alpha_i}{2}(y - \frac{a + b}{2})^2,
$$

where $\alpha \geq 0$ is a parameter. It follows that $\bar{a}_i(y) \leq a_i(y) \forall y \in Y$. Furthermore, $\bar{a}_i(y)$ is twice continuously differentiable if and only if $a_i(y)$ is twice continuously differentiable and the second derivative of $\bar{a}_i(y)$ is $\bar{a}_i''(y) = a_i''(y) - \alpha_i$. Thus $\bar{a}_i(y)$ is concave on $Y$ if the parameter $\alpha_i$ satisfies $\alpha_i \geq \max_{y \in Y} a''_i(y)$. To sum up, we select the parameter $\alpha_i$ such that

$$
\alpha_i \geq \max\{0, \max_{y \in Y} a''_i(y)\}. \tag{6}
$$

This guarantees that $\bar{a}_i(y)$ is a lower bound concave function of $a_i(y)$ on the index set $Y$. The computation of $\alpha_i$ in (6) involves a global optimization. However, we can use any upper bound of the right hand side in (6). Such an upper bound can be obtained by interval methods proposed above. On the other hand, the distance between $\bar{a}_i(y)$ and $a_i(y)$ on $[a, b]$ is

$$
\max_{y \in Y} |a_i(y) - \bar{a}_i(y)| = \frac{\alpha_i}{8} (b - a)^2.
$$

Since $\bar{a}_j(y)$ is concave on $Y$, the minimizer of $\bar{a}_i(y)$ on $Y$ is attained on the boundary of $Y$ (see [22]), i.e., $\min_{y \in Y} \bar{a}_i(y) = \min\{\bar{a}_i(a), \bar{a}_i(b)\}$. By replacing $\min_{y \in Y} a_i(y)$ in (2) with $\min_{y \in Y} \bar{a}_i(y)$, we get the second type of restriction constraint as follows

$$
\sum_{i=1}^{n} \min\{\bar{a}_i(a), \bar{a}_i(b)\} x_i + \min\{\bar{a}_0(a), \bar{a}_0(b)\} \geq 0. \tag{7}
$$

The two approaches are distinct in the sense that the interval method requires mild assumptions on the coefficient function while the concave-function based method admits better approximation rate.
3. Numerical method

Based on the restriction approaches developed in the previous section, we are able to construct a sequence of approximations for (LSIP) by dividing the original index set into subsets successively and constructing linear optimization problems associated with restricted constraints on the subsets.

**Definition 3.1.** We call \( T = \{\tau_0, ..., \tau_N\} \) a subdivision of the interval \([a, b]\) if
\[
a = \tau_0 \leq \tau_1 \leq ... \leq \tau_N = b.
\]

Let \( Y_k = [\tau_{k-1}, \tau_k] \) for \( k = 1, 2, ..., N \), the length of \( Y_k \) is defined by \( |Y_k| = |\tau_k - \tau_{k-1}| \) and the length of the subdivision \( T \) is defined by \( |T| = \max_{1 \leq k \leq N} |Y_k| \). It follows that \( Y = \bigcup_{k=1}^{N} Y_k \).

The intuition behind the approximation of (LSIP) through subdivision comes from an observation that the original semi-infinite constraints in (LSIP)
\[
a(y)^\top x + a_0(y) \geq 0, \quad \forall y \in Y
\]
can be reformulated equivalently as finitely many semi-infinite constraints
\[
a(y)^\top x + a_0(y) \geq 0, \quad \forall y \in Y_k, k = 1, 2, ..., N.
\]

Given a subdivision, we can construct the approximate constraint on each subinterval and combine them together to formulate the inner-approximation of the original feasible region. The corresponding optimization problem provide a restriction of (LSIP). The solution of the approximate problem approach to the optimal solution of (LSIP) as the size of the subdivision tends to zero.

The two different approaches (e.g., interval method and Convexification method) were introduced in section 2 to construct the approximate region that lies inside of the original feasible region. This induces two different types of approximation problems when applied to a particular subdivision. We only describe main results for the first type (e.g., interval method) and focus on the convergence and algorithm for the second one.

We introduce the Slater condition and a lemma derived from it which will be used in the following part. We say Slater condition holds for (LSIP) if there exists a point \( \bar{x} \in \mathbb{R}_+^n \) such that
\[
a(y)^\top \bar{x} + a_0(y) > 0, \quad \forall y \in Y.
\]

Let \( F^o = \{x \in F \mid a(y)^\top \bar{x} + a_0(y) > 0, \quad \forall y \in Y\} \) be the set of all the Slater points in \( F \). It is shown that the feasible region \( F \) is exactly the closure of \( F^o \) under the Slater condition [4]. We present this result as a lemma and give a direct proof in the appendix.

**Lemma 3.2.** Assume that the Slater condition holds for (LSIP) and the index set \( Y \) is compact, then we have
\[
F = \text{cl}(F^o),
\]
where \( \text{cl}(F^o) \) represents the closure of the set \( F^o \).
3.1. Restriction based on interval method

Let $A_i(Y_k) = [A^l_{i,k}, A^u_{i,k}]$ be the inclusion function of $a_i(y)$ on $Y_k$. By estimating the lower bound for $\min_{y \in Y_k} a_i(y)$ via interval method, we can construct the following linear constraints

$$
\sum_{i=1}^n A^l_{i,k} x_i + A^l_{0,k} \geq 0, \quad k = 1, 2, ..., N,
$$

corresponding to the original constraints $a(y) \top x + a_0(y) \geq 0$, $\forall y \in Y_k$, $k = 1, 2, ..., N$. For simplicity, we reformulate the inequalities as

$$
A^T_T x + b^T_T \geq 0, \quad (8)
$$

where $A_T(i, k) = A^l_{i,k}$ and $b_T(k) = A^l_{0,k}$ for $i = 1, 2, ..., n, k = 1, 2, ..., N$. The approximation problem for (LSIP) in such case is formulated as

$$
\min_{x \in \mathbb{R}^n_+} c^T x \quad \text{s.t.} \quad A^T_T x + b_T \geq 0. \quad \text{R1-LSIP}(T)
$$

Following the analysis in section 2, we know that $\{x \in \mathbb{R}^n_+ \mid A^T_T x + b_T \geq 0\} \subseteq F$. Therefore, any feasible point of $\text{R1-LSIP}(T)$ is feasible for (LSIP) provided that the feasible region of $\text{R1-LSIP}(T)$ is non-empty. By solving $\text{R1-LSIP}(T)$ we can obtain a feasible approximate solution for (LSIP) and the corresponding optimal value of $\text{R1-LSIP}(T)$ provides an upper bound for the optimal value of (LSIP).

Let $F(T) = \{x \in \mathbb{R}^n_+ \mid A^T_T x + b_T \geq 0\}$ be the feasible region of $\text{R1-LSIP}(T)$. We say that $F(T)$ is consistent if $F(T) \neq \emptyset$. In this case, the corresponding problem $\text{R1-LSIP}(T)$ is called consistent. The following lemma shows that the approximate problem $\text{R1-LSIP}(T)$ is consistent for all $|T|$ small enough if Slater condition holds for (LSIP).

**Lemma 3.3.** Assume that the Slater condition holds for (LSIP) and the coefficient functions $a_i(y)$, $i = 0, 1, ..., n$, are Lipschitz continuous on $\bar{Y}$, then $F(T)$ is nonempty for all $|T|$ small enough.

In following theorem, we show that any accumulation point of the solutions of the approximate problems $\text{R1-LSIP}(T)$ is a solution to (LSIP) if the size of the subdivision tends to zero.

**Theorem 3.4.** Assume the Slater condition holds for (LSIP) and the level set $L(\bar{x}) = \{x \in F \mid c^T x \leq c^T \bar{x}\}$ is bounded ($\bar{x}$ is a Slater point). Let $\{T_k\}$ be a sequence of subdivisions of $Y$ such that $T_0$ is consistent and $\lim_{k \to \infty} |T_k| = 0$ with $T_k \subseteq T_{k+1}$. Let $x^*_k$ be a solution of $\text{R1-LSIP}(T_k)$. Then any accumulation point of the sequence $\{x^*_k\}$ is an optimal solution to (LSIP).

3.2. Restriction based on concavification

Given a subdivision $T = \{\tau_0, ..., \tau_N\}$ and $Y_k = [\tau_{k-1}, \tau_k], k = 1, 2, ..., N$, by applying concavification method in section 2 to each of the finitely many semi-infinite con-
strains
\[ a(y)^T x + a_0(y) \geq 0, \quad \forall y \in Y_k, k = 1, 2, ..., N, \]

we can construct the linear constraints as follows
\[
\sum_{i=1}^{n} \min\{\bar{a}_i(\tau_{k-1}), \bar{a}_i(\tau_k)\} x_i + \min\{\bar{a}_0(\tau_{k-1}), \bar{a}_0(\tau_k)\} \geq 0, \quad k = 1, 2, ..., N,
\]

where \( \bar{a}_i(\cdot) \) is the concavification function defined on \( Y_k \) when we calculate \( \bar{a}_i(\tau_{k-1}) \) or \( \bar{a}_i(\tau_k) \) (i.e., \( \bar{a}_i(y) = a_i(y) - \frac{\alpha_i k}{2} (y - \tau_k - \tau_{k-1})^2 \)). We rewrite the above inequalities as
\[
\bar{A}^T x + \bar{b}_T \geq 0,
\]

where \( \bar{A}_T(i, k) = \min\{\bar{a}_i(\tau_{k-1}), \bar{a}_i(\tau_k)\} \) and \( \bar{b}_T(k) = \min\{\bar{a}_0(\tau_{k-1}), \bar{a}_0(\tau_k)\} \). The corresponding approximate problem for (LSIP) is defined by
\[
\min_{x \in \mathbb{R}^n_+} c^T x \quad \text{s.t.} \quad \bar{A}^T x + \bar{b}_T \geq 0. \quad \text{R2-LSIP(T)}
\]

Let \( \bar{F}(T) = \{x \in \mathbb{R}^n_+ | \bar{A}^T x + \bar{b}_T \geq 0\} \) be the feasible set of the problem R2-LSIP(T). We can conclude that \( \bar{F}(T) \subseteq F \).

The approximate problem R2-LSIP(T) is similar to R1-LSIP(T) in the sense that both problems induce restrictions of (LSIP). Therefore, any feasible solution of R2-LSIP(T) is feasible for (LSIP) and the corresponding optimal value provide an upper bound for the optimal value of the problem (LSIP).

The following lemma shows that if the Slater condition holds for (LSIP), R2-LSIP(T) is consistent for all \(|T|\) small enough (e.g., \( \bar{F}(T) \neq \emptyset \)). Proof can be found in appendix.

**Lemma 3.5.** Assume the Slater condition holds for (LSIP) and \( a_i(y), i = 1, 2, ..., n, \) are twice continuously differentiable. Then R2-LSIP(T) is consistent for all \(|T|\) small enough.

In order to find a good approximate solution for (LSIP) R2-LSIP(T) need to be solved iteratively during which the subdivision will be refined. We present a particular strategy of the refinement here such that the approximate regions of R2-LSIP(T) are monotonically enlarging from the inside of the feasible region \( F \). Consequently, the corresponding optimal values of the approximation problems are monotonically decreasing and converge to the optimal value of the original linear semi-infinite problem. Note that such a refinement procedure can not guarantee the monotonic property when applied to solve R1-LSIP(T).

Let \( T = \{\tau_k | k = 0, 1, ..., N\} \) be a subdivision of the \( Y \). Assume \( Y_k = [\tau_{k-1}, \tau_k] \) is the subinterval to be refined. Denote by \( \tau_{k,1} \) and \( \tau_{k,2} \) the trisection points of \( Y_k \):
\[
\tau_{k,1} = \tau_{k-1} + \frac{1}{3}(\tau_k - \tau_{k-1}), \quad \tau_{k,2} = \tau_{k-1} + \frac{2}{3}(\tau_k - \tau_{k-1}).
\]
The constraint in $\text{R2-LSIP}(T)$ on the subset $Y_k$ is

$$\sum_{i=1}^{n} \left[ \min[\bar{a}_i(\tau_{k-1}), a_i(\tau_k)] x_i + \min[\bar{a}_0(\tau_{k-1}), \bar{a}_0(\tau_k)] \right] \geq 0, \quad (9)$$

where $\bar{a}_i(y) = a_i(y) - \frac{\alpha_{i,k}}{2}(y - \frac{\tau_{k-1} + \tau_k}{2})^2$ and parameter $\alpha_{i,k}$ is calculated in the manner of (6). The lower bounding functions on each subset after refinement are defined by

$$\bar{a}_i^1(y) = a_i(y) - \frac{\alpha_{i,k}^1}{2}(y - \frac{\tau_{k-1} + \tau_k}{2})^2, \quad y \in Y_{k,1} = [\tau_{k-1}, \tau_{k,1}],$$

$$\bar{a}_i^2(y) = a_i(y) - \frac{\alpha_{i,k}^2}{2}(y - \frac{\tau_{k,1} + \tau_{k,2}}{2})^2, \quad y \in Y_{k,2} = [\tau_{k,1}, \tau_{k,2}],$$

$$\bar{a}_i^3(y) = a_i(y) - \frac{\alpha_{i,k}^3}{2}(y - \frac{\tau_{k,2} + \tau_k}{2})^2, \quad y \in Y_{k,3} = [\tau_{k,2}, \tau_k],$$

where $\alpha_{i,k}^j, j = 1, 2, 3$ are selected such that $\alpha_{i,k}^j \geq \max\{0, \max_{y \in Y_{k,j}} \nabla^2 a_i(y)\}$ and $\alpha_{i,k}^j \leq \alpha_{i,k}$ for $j = 1, 2, 3$. The refined approximate region $\bar{F}(T \cup \{\tau_{k,1}, \tau_{k,2}\})$ is obtained by replacing the constraint (9) in $\bar{F}(T)$ with

$$\sum_{i=1}^{n} \left[ \min[\bar{a}_i^1(\tau_{k-1}), a_i(\tau_k)] x_i + \min[\bar{a}_0^1(\tau_{k-1}), \bar{a}_0(\tau_k)] \right] \geq 0,$$

$$\sum_{i=1}^{n} \left[ \min[\bar{a}_i^2(\tau_{k,1}), a_i(\tau_k)] x_i + \min[\bar{a}_0^2(\tau_{k,1}), \bar{a}_0(\tau_k)] \right] \geq 0,$$

$$\sum_{i=1}^{n} \left[ \min[\bar{a}_i^3(\tau_{k,2}), a_i(\tau_k)] x_i + \min[\bar{a}_0^3(\tau_{k,2}), \bar{a}_0(\tau_k)] \right] \geq 0.$$
We present in the following theorem the general convergence results for approximating (LSIP) via a sequence of restriction problems.

**Theorem 3.7.** Assume that the assumptions in Theorem 3.4 hold. Let \(\{T_k\}\) be a sequence of subdivisions of the index set \(Y\), which is obtained by trisection refinement recursively, such that \(T_0\) is consistent and \(\lim_{k \to \infty} |T_k| = 0\). Denote by \(x_k^*\) the optimal solution to R2-LSIP\((T_k)\). Then we have:

1. \(x_k^*\) is feasible for \((LSIP)\) and any accumulation point of the sequence \(\{x_k^*\}\) is a feasible solution to \((LSIP)\).
2. \(\{f(x_k^*) : f(x_k^*) = c^\top x_k^*\}\) is a decreasing sequence and \(v^* = \lim_{k \to \infty} f(x_k^*)\) is an optimal value to \((LSIP)\).

**Proof.** The proof of the first statement is similar to the proof in Theorem 3.4. From Lemma 3.6, we know that \(\bar{F}(T_k - 1) \subseteq \bar{F}(T_k)\) holds for \(k \in \mathbb{N}\) which implies that the sequence \(\{f(x_k^*)\}\) is decreasing. Since the level set \(L(\bar{x})\) is bounded, the sequence \(\{f(x_k^*)\}\) is bounded. Therefore, the limit of the sequence exists which is denoted by \(v^*\). From (1), we know that \(v^*\) is an optimal value to \((LSIP)\).

This completes our proof.

### 3.3. Adaptive refinement algorithm

In this section, we present a specific algorithm to solve \((LSIP)\). The algorithm is based on solving the approximate linear problems R2-LSIP\((T)\) (or R1-LSIP\((T)\)) for a given subdivision \(T\) and then refine the subdivision to improve the solution. The key idea of the algorithm is to select the candidate subsets in \(T\) to be refined in an adaptive manner rather than making the refinement exhaustively.

We introduce the optimality condition for \((LSIP)\) as follows before presenting the details of the algorithm. Given a point \(x \in F\), let \(A(x) = \{y \in Y \mid a(y)^\top x + a_0(y) = 0\}\) be the active index set for \((LSIP)\) at \(x\). If some constraint qualification (e.g., Slater condition) holds for \((LSIP)\), a feasible point \(x^* \in F\) is an optimal solution if and only if \(x^*\) satisfies the KKT systems (4), i.e.,

\[
\begin{align*}
c - \sum_{y \in A(x^*)} \lambda_y a(y) &= 0,
\end{align*}
\]

for some \(\lambda_y \geq 0, y \in A(x^*)\).

**Definition 3.8.** We say that \(x^* \in F\) is an \((\epsilon, \delta)\) optimal solution to \((LSIP)\) if there exist some indices \(y \in Y\) as well as \(\lambda_y \geq 0\) such that

\[
||c - \sum_{y \in A(x^*, \delta)} \lambda_y a(y)|| \leq \epsilon,
\]

where \(A(x^*, \delta) = \{y \in Y \mid 0 \leq a(y)^\top x^* + a_0(y) \leq \delta\}\).

To obtain a consistent subdivision in the first step of Algorithm 1, we apply the adaptive refinement algorithm to the following problem

\[
\begin{align*}
\min_{(x,z) \in \mathbb{R}_+^n \times \mathbb{R}} \quad z \quad \text{s.t.} \quad a(y)^\top x + a_0(y) &\geq z \quad \forall y \in Y \quad \text{LSIP}_0
\end{align*}
\]
Algorithm 1 (Adaptive Refinement Algorithm for LSIP)

S1. Find an initial subdivision \( T_0 \) such that R2-LSIP\( (T_0) \) is consistent. Choose an initial point \( x_0 \) and tolerances \( \epsilon \) and \( \delta \). Set \( k = 0 \).
S2. Solve R2-LSIP\( (T_k) \) to obtain a solution \( x^*_k \) and the active index set \( A(x^*_k) \).
S3. Terminate if \( x^*_k \) is an \((\epsilon, \delta)\) optimal solution to [LSIP]. Otherwise update \( T_{k+1} \) and \( F(T_{k+1}) \) by trisection refinement procedure for subintervals in \( T_k \) that correspond to \( A(x^*_k) \).
S4. Let \( k = k + 1 \) and go to step 2.

until a feasible solution \((x_0, z_0)\), with \( z_0 \geq 0 \), of the problem LSIP\( _0(T_0) \) is found for some subdivision \( T_0 \). The current subdivision \( T_0 \) is consistent and chosen as the initial subdivision of Algorithm 1. In addition, \( x_0 \) is feasible for the original problem and selected as the initial point for the algorithm.

The refinement procedure in the third step of the algorithm is taken as follows. In the \( k \)th iteration, each \([\tau^k_{i-1}, \tau^k_i]\) is divided into three equal length subsets for \( i \in A(x^*_k) \). New constraints are constructed on the subsets and used to update the constraint corresponding to \([\tau^k_{i-1}, \tau^k_i]\) for each index \( i \in A(x^*_k) \). Then we have \( F(T_{k+1}) \) and the associated approximation problem R2-LSIP\( (T_{k+1}) \).

**Theorem 3.9 (Convergence of Algorithm 1).** Assume the Slater condition holds for [LSIP] and the coefficient functions \( a_i(y) \), \( i = 0, 1, \ldots, n \), are twice continuously differentiable. Then Algorithm 1 terminates in finitely many iterations for any positive tolerances \( \epsilon \) and \( \delta \).

**Proof.** Let \( x^*_k \) be a solution to the approximate subproblem R2-LSIP\( (T_k) \) with \( T_k = \{\tau^k_j \mid j = 0, 1, \ldots, N_k\} \), there exists some \( \lambda^k_j \geq 0 \) for \( j \in A(x^*_k) \) such that

\[
c - \sum_{j \in A(x^*_k)} \lambda^k_j \min[a(\tau^k_{j-1}), a(\tau^k_j)] = 0, \tag{10}
\]

where \( A(x^*_k) = \{j \mid \min[a(\tau^k_{j-1}), a(\tau^k_j)]x^*_k + \min[a_0(\tau^k_{j-1}), a_0(\tau^k_j)] = 0\} \) is the active index set for R2-LSIP\( (T_k) \) at \( x^*_k \) and \( \min[a(\tau^k_{j-1}), a(\tau^k_j)] \) represents a vector in \( \mathbb{R}^n \) such that the \( i \)th element is defined by \( \min[\bar{a}_i(\tau^k_{j-1}), \bar{a}_i(\tau^k_j)] \). Since \( \bar{a}_i(y) = a_i(y) - \alpha^k y - \frac{\tau^k_{j-1} + \tau^k_j}{2} \) for \( y \in [\tau^k_{j-1}, \tau^k_j] \), we have

\[
\min[a(\tau^k_{j-1}), a(\tau^k_j)] = \min[a(\tau^k_{j-1}), a(\tau^k_j)] - \frac{1}{8} (\tau^k_j - \tau^k_{j-1})^2 \alpha^k,
\]

where \( \alpha^k = (\alpha^k_1, \alpha^k_2, \ldots, \alpha^k_n)^T \) is the parameter vector on the subset \([\tau^k_{j-1}, \tau^k_j]\) with all elements are uniformly bounded. On the other hand, since \( a_i(y) \) is twice continuously differentiable, there exists \( \tau^k_{j-1} \) such that

\[
a_i(\tau^k_j) = a_i(\tau^k_{j-1}) + a_i'(\tau^k_{j-1})(\tau^k_j - \tau^k_{j-1}), 1 \leq i \leq n
\]

which implies that \( \min[a(\tau^k_{j-1}), a(\tau^k_j)] = a(\tau^k_{j-1}) + (\tau^k_j - \tau^k_{j-1})\beta^k \) where \( \beta^k \in \mathbb{R}^n \) is
a constant vector (e.g., $\beta_i^k = a'_i(\tau_{j-1}^k)$ if $\min[a(\tau_{j-1}^k), a(\tau_j^k)] = a(\tau_j^k)$ and $\beta_i^k = 0$ otherwise). It follows that

$$\min[\bar{a}(\tau_{j-1}^k), a(\tau_j^k)] = a(\tau_{j-1}^k) + (r_j^k - r_{j-1}^k)\beta^k - \frac{1}{8}(r_j^k - r_{j-1}^k)^2 \alpha^k. \quad (11)$$

Substitute $\min[\bar{a}(\tau_{j-1}^k), a(\tau_j^k)]$ into (11) into (10) and $A(x_k^*)$, we can claim that it suffices to prove the lengths of all the subsets $[\tau_{j-1}^k, \tau_j^k]$ for $j \in A(x_k^*)$ converge to zeros as the iteration $k$ tends to infinity. From the algorithm, we know that in each iteration at least one subset $[\tau_{j-1}^k, \tau_j^k]$ is divided into three equal subintervals where the length of each subinterval is bounded above by $\frac{1}{3}(\tau_j^k - \tau_{j-1}^k) \leq \frac{1}{3}(b - a)$. For each integer $p \in \mathbb{N}$, at least one interval with its length bounded by $\leq \frac{1}{3^p}(b - a)$ is generated. Furthermore, all the subintervals $[\tau_{j-1}^k, \tau_j^k]$, $j \in A(x_k^*)$ are different for all $k \in \mathbb{N}$. Since for each $p \in \mathbb{N}$, only finitely subintervals with length greater than $\frac{1}{3^p}(b - a)$ exists. This implies that the lengths of the subsets $[\tau_{j-1}^k, \tau_j^k]$ for $j \in A(x_k^*), k \in \mathbb{N}$ must tend to zero.

We can conclude from Theorem 3.9 that if the tolerances $\epsilon$ and $\delta$ are decreasing to zero then any accumulation point of the sequence generated by Algorithm 1 is a solution to the original linear semi-infinite programming.

**Corollary 3.10.** Let the assumptions in Theorem 3.9 be satisfied and the tolerances $(\epsilon_k, \delta_k)$ are chosen such that $(\epsilon_k, \delta_k) \searrow (0, 0)$. If $x_k^*$ is an $(\epsilon_k, \delta_k)$ KKT point for $\text{(LSIP)}$ generated by Algorithm 1, then any accumulation point $x^*$ of the sequence $\{x_k^*\}$ is a solution to $\text{(LSIP)}$.

It follows from Corollary 3.10 the sequence $\{c^T x_k^*\}$ is monotonically decreasing to the optimal value of $\text{(LSIP)}$ as $k$ tends to infinity. In the implement of our algorithm, the termination criterion is set as

$$|c^T x_k^* - c^T x_{k-1}^*| \leq \epsilon.$$

The convergence of Algorithm 1 is also applicable to the case that the approximate problem $\text{R1-LSIP}(T_k)$ is used in the second step. The proof is similar to that in theorem 3.9 as we explained in appendix. However, we can not guarantee the sequence $\{c^T x_k^*\}$ is monotonically decreasing.

### 3.4. Remarks

The proposed algorithm can be applied to solve linear semi-infinite optimization problem with finitely many semi-infinite constraints and some extra linear constraints, i.e.,

$$\min_{x \in X} c^T x \quad \text{s.t.} \quad a_j^T(y) x + a_0^j(y) \geq 0, \forall y \in Y, j = 1, 2, ..., m,$$

where $X = \{x \in \mathbb{R}^n \mid Dx \geq d\}$ and $a_j^T(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$. In such a case, we split each decision variable $x_i$ into two non-negative variable $y_i \geq 0$ and $z_i \geq 0$ such that $x_i = y_i - z_i$, and then substitute $x_i$ into the above problem. Then the problem is reformulated as a linear semi-infinite programming problem with non-negative decision variables in which the Algorithm 1 can be applied to solve it. Such a technique is applied in the numerical experiments.
In the case that \( X = [X_l, X_u] \) is a box in \( \mathbb{R}^n \), we can set a new variable transformation as \( x = z + X_l \) in which \( z \geq 0 \). The advantage to reformulate the original problem in such a translation is that the dimension of the new variables is the same as that of the original decision variables.

4. Numerical experiments

We present the numerical experiments for a couple of optimization problems selected from the literature. The algorithm is implemented in Matlab 8.1 and the subproblem is solved by using \texttt{linprog} of Optimization Toolbox 6.3 with default tolerance and active set algorithm. All the following experiments were run on 3.2 GHz Intel(R) Core(TM) processor.

The computation of the bounds for the coefficient functions and the parameter \( \alpha \) in the second approach are obtained directly if the closed form bound exists. Otherwise, we use Matlab toolbox \texttt{Intlab} 6.0 [23] to obtain the corresponding bounding values.

The problems in the literature are listed as follows.

**Problem 1.**

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} i^{-1} x_i \\
\text{s.t.} & \quad \sum_{i=1}^{n} y^{-1} x_i \geq \tan(y), \quad \forall y \in [0, 1].
\end{align*}
\]

This problem is taken from [4] and also tested in [8] for \( n = 8 \). For \( 1 \leq n \leq 7 \), the problem has unique optimal solution and the Strong Slater condition holds. The problem for \( n = 8 \) is hard to solve and thus a good test of the performance for our algorithm.

**Problem 2.** This problem has same formulation as Problem 1 with \( n = 9 \) which is also tested in [8].

**Problem 3.**

\[
\begin{align*}
\min & \quad \sum_{i=1}^{8} i^{-1} x_i \\
\text{s.t.} & \quad \sum_{i=1}^{8} y^{-1} x_i \geq \frac{1}{2 - y}, \quad \forall y \in [0, 1].
\end{align*}
\]

This problem is taken from [19] and also tested in [8].

**Problem 4.**

\[
\begin{align*}
\min & \quad \sum_{i=1}^{7} i^{-1} x_i \\
\text{s.t.} & \quad \sum_{i=1}^{7} y^{-1} x_i \geq -\sum_{i=0}^{4} y^{2i}, \quad \forall y \in [0, 1].
\end{align*}
\]
Problem 5.

\[
\min \sum_{i=1}^{9} i^{-1} x_i \\
\text{s.t.} \sum_{i=1}^{9} 9^{-1} x_i \geq \frac{1}{1 + y^2}, \forall y \in [0, 1].
\]

Problem 4 and 5 are taken from [20] and also tested in [8].

The following problems, as noted in [8], arise in the design of finite impulse response (FIR) filters which are more computationally demanding than the previous ones (see, e.g., [7, 8]).

**Problem 6.**

\[
\min -\sum_{i=1}^{10} r_{2i-1} x_i \\
\text{s.t.} 2 \sum_{i=1}^{10} \cos((2i - 1)2\pi y) x_i \geq -1, \forall y \in [0, 0.5],
\]

where \( r_i = 0.95^i \).

**Problem 7.** This problem is formulated as Problem 6 where \( r_i = 2\rho \cos(\theta) r_{i-1} - \rho^2 r_{i-2} \) with \( \rho = 0.975, \theta = \pi/3, r_0 = 1, r_1 = 2\rho \cos(\theta)/(1 + \rho^2) \).

**Problem 8.** This problem is also formulated as Problem 6 where \( r_i = \frac{\sin(2\pi f_s i)}{2\pi f_s} \) with \( f_s = 0.225 \).

The numerical results are summarized in Table 1 where CPU Time is the time cost when the algorithm terminates, Objective Value represents the objective function value at the iteration point when the algorithm terminates, No of Iteration is the number of iterations when the algorithm terminates for each particular problem and Violation measures the feasibility of the solution \( x^* \) obtained by the algorithm which is defined by \( \min_{y \in Y} g(x^*, y) \) with \( Y = a : 10^{-6} : b \). We also list the numerical results for these problems by MATLAB toolbox `fseminf` as a reference. We can see that the algorithm proposed in this paper generates the feasible solutions for all the problems tested. This is coincide with the theoretical results. Furthermore, Algorithm 1 works well for the computational demanding problems 6-8. The solver `fseminf` is faster than our method, however the feasibility is not guaranteed for this kind of method.

**5. Conclusion**

A new numerical method for solving linear semi-infinite programming problems is proposed which guarantees that each iteration point is feasible for the original problem. The approach is based on a two-stage restriction of the original semi-infinite constraint. The first stage restriction allows us to consider semi-infinite constraint independently to the decision variables on the subsets of the index set. In the second stage, the lower bounds for the optimal values of the optimization problems associated with coefficient functions are estimated using two different approaches. The approximation error goes to zero as the size of the subdivisions tends to zero.

The approximate problems with finitely many linear constraints is constructed such
Table 1. Summary of numerical results for the proposed algorithm in this paper

| Problem | Approach 1 | Algorithm | CPU Time(sec) | Objective Value | No. of Iterations | Violation |
|---------|------------|-----------|---------------|-----------------|-------------------|-----------|
| Problem 1. | Approach 1 | 1.8382 | 0.6174 | 169 | 2.3558e-04 |
| Problem 1. | Approach 2 | 1.8910 | 0.6174 | 172 | 1.5835e-04 |
| Problem 1. | fseminf | 0.2109 | 0.6163 | 33 | -1.2710e-04 |
| Problem 2. | Approach 1 | 5.2691 | 0.6163 | 273 | 4.1441e-04 |
| Problem 2. | Approach 2 | 4.0928 | 0.6166 | 266 | 1.8372e-04 |
| Problem 2. | fseminf | 0.3188 | 0.6157 | 46 | -7.6194e-04 |
| Problem 3. | Approach 1 | 0.1646 | 0.6988 | 12 | 2.7969e-03 |
| Problem 3. | Approach 2 | 0.1538 | 0.6988 | 13 | 2.8014e-03 |
| Problem 3. | fseminf | 0.2387 | 0.6932 | 35 | -5.8802e-07 |
| Problem 4. | Approach 1 | 4.1606 | -1.7841 | 354 | 1.9689e-05 |
| Problem 4. | Approach 2 | 4.1928 | -1.7841 | 356 | 1.9646e-05 |
| Problem 4. | fseminf | 0.4974 | -1.7869 | 70 | -3.4649e-09 |
| Problem 5. | Approach 1 | 4.2124 | 0.7861 | 300 | 1.9829e-05 |
| Problem 5. | Approach 2 | 4.7892 | 0.7861 | 302 | 1.9243e-05 |
| Problem 5. | fseminf | 0.3642 | 0.7855 | 32 | -8.5507e-07 |
| Problem 6. | Approach 1 | 1.7290 | -0.4832 | 137 | 5.0697e-06 |
| Problem 6. | Approach 2 | 1.5302 | -0.4832 | 132 | 5.0914e-06 |
| Problem 6. | fseminf | 1.1476 | -0.4754 | 86 | -1.2219e-04 |
| Problem 7. | Approach 1 | 2.5183 | -0.4889 | 170 | 2.8510e-04 |
| Problem 7. | Approach 2 | 3.2521 | -0.4890 | 219 | 2.8861e-04 |
| Problem 7. | fseminf | 1.0480 | -0.4883 | 86 | -1.5211e-03 |
| Problem 8. | Approach 1 | 4.4262 | -0.4972 | 252 | 4.5808e-05 |
| Problem 8. | Approach 2 | 4.0216 | -0.4972 | 252 | 5.0055e-05 |
| Problem 8. | fseminf | 0.4324 | -0.4973 | 45 | -4.3322e-07 |

* Approach 1 represents Algorithm 1 with R1-LSIP and Approach 2 represents Algorithm 1 with R2-LSIP.
that the corresponding feasible regions are included in the feasible region of (LSIP). It follows that any feasible solution of the approximate problem is feasible for (LSIP) and the corresponding objective function value provide an upper bound for the optimal value of (LSIP). It is proved that the solutions of the approximate problems converge to that of the original problem. Also, the sequence of optimal values of the approximate problems converge to the optimal value of (LSIP) in a monotonic manner.

An adaptive refinement algorithm is developed to obtain an approximate solution to (LSIP) which is proved to terminate in finite iterations for arbitrarily given tolerances. Numerical results show that the algorithm works well in finding feasible solutions for (LSIP).

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6. Appendices

Proof of Lemma 3.2

Since the Slater condition holds, there exists a point $\bar{x} \in \mathbb{R}^n$ such that

$$a(y)^\top \bar{x} + a_0(y) > 0, \forall y \in Y.$$  

It has been shown in [5] that the boundary of $F$ is

$$\partial F = \{x \in F \mid \min_{y \in Y} \{a(y)^\top x + a_0(y)\} = 0\}.$$  

It follows that $F = F^o \cup \partial F$. The compactness of the index set $Y$ implies that the function $g(x) = \min_{y \in Y} \{a(y)^\top x + a_0(y)\}$ is continuous. Thus, $F$ is closed. It suffices to prove that

$$\partial F \subseteq \text{cl}(F^o).$$  

For any $\bar{x} \in \partial F$, we have $a(y)^\top \bar{x} + a_0(y) = 0$ for all $y \in A(\bar{x})$ with $A(\bar{x}) = \{y \in Y \mid a(y)^\top \bar{x} + a_0(y) = 0\}$. Then

$$a(y)^\top (\bar{x} - \tilde{x}) > 0, \forall y \in A(\tilde{x}).$$  

This indicates that for any $\tau > 0$, we have

$$a(y)^\top (\bar{x} + \tau(\bar{x} - \tilde{x})) + a_0(y) > 0, \forall y \in A(\tilde{x}).$$  

For a point $y \in Y$ and $y \notin A(\tilde{x})$, there holds that $a(y)^\top \tilde{x} + a_0(y) > 0$. Therefore, $a(y)^\top (\bar{x} + \tau(\bar{x} - \tilde{x})) + a_0(y) > 0$ for $\tau$ small enough. Since $Y$ is compact, we can chose a uniform $\tau$ such that $a(y)^\top (\bar{x} + \tau(\bar{x} - \tilde{x})) + a_0(y) > 0, \forall y \in Y$ for $\tau$ small enough. It follows that we can choose a sequence $\tau_k > 0$ with $\lim_{k \to \infty} \tau_k = 0$ such that

$$a(y)^\top (\bar{x} + \tau_k(\bar{x} - \tilde{x})) + a_0(y) > 0, \forall y \in Y, k \in \mathbb{N}.$$  

Hence $x_k = \bar{x} + \tau_k(\bar{x} - \tilde{x}) \in F^o$ and $\lim_{k \to \infty} x_k = \tilde{x}$ which implies that $\tilde{x} \in \text{cl}(F^o)$. 

This completes our proof.

Proof of Lemma 3.3
The Slater condition implies that there exists a point \( \bar{x} \in \mathbb{R}^n_+ \) such that
\[
a(y)\bar{x} + a_0(y) > 0, \forall y \in Y.
\]
Let \( T = \{ \tau_k \mid k = 0, 1, ..., N \} \), from [1] we know that for each \( Y_k = [\tau_{k-1}, \tau_k], k = 1, 2, ..., N \), there holds that
\[
\min_{y \in Y_k} a_i(y) - A_i^l, k \leq \gamma_i |Y_k|^p \leq \gamma_i |T|^p, i = 0, 1, ..., n, k = 1, 2, ..., N,
\]
with \( p \geq 1 \). By direct computation, we have
\[
\sum_{i=1}^{n} \left[ \min_{y \in Y_k} a_i(y) |\bar{x}_i| + \min_{y \in Y_k} a_0(y) \right] - \left[ \sum_{i=1}^{n} A_{i,k}^l |\bar{x}_i| + A_{0,k}^l \right] \leq \sum_{i=1}^{n} \gamma_i |\bar{x}_i + \gamma_0| |Y_k|^p.
\]
The Lipschitz continuity of \( a_i(y), i = 0, 1, ..., n \), implies that
\[
\min_{y \in Y_k} \sum_{i=1}^{n} a_i(y) |\bar{x}_i| + \min_{y \in Y_k} a_0(y) - \left[ \sum_{i=1}^{n} A_{i,k}^l |\bar{x}_i| + A_{0,k}^l \right] \leq \sum_{i=1}^{n} L_i |\bar{x}_i + L_0| |Y_k|.
\]
It follows from the last two inequalities that
\[
\sum_{i=1}^{n} A_{i,k}^l |\bar{x}_i| + A_{0,k}^l \geq \min_{y \in Y_k} \sum_{i=1}^{n} a_i(y) |\bar{x}_i| + a_0(y) - \left[ \sum_{i=1}^{n} \gamma_i |\bar{x}_i + \gamma_0| |Y_k|^p + \sum_{i=1}^{n} L_i |\bar{x}_i + L_0| |Y_k| \right],
\]
which implies that \( \sum_{i=1}^{n} A_{i,k}^l |\bar{x}_i| + A_{0,k}^l \geq 0, k = 1, 2, ..., N \), for \( |T| \) small enough. This implies that \( \bar{x} \) is a feasible point for the approximate region \( F(T) \).

This completes our proof.

**Proof of Theorem 3.4**

By the construction of the approximate regions, we know that \( F(T_k) \) is included in the original feasible set, i.e., \( F(T_k) \subseteq F \) for all \( k \in \mathbb{N} \). Hence, we have \( \{x^*_k \} \subseteq F \).

Let \( \bar{x} \) be any Slater point, we can conclude from Lemma 3.3 that \( \bar{x} \) is contained in \( F(T_k) \) for \( k \) large enough. Thus \( c^T x^*_k \leq c^T \bar{x} \) which indicates that \( x^*_k \in L(\bar{x}) \) for sufficient large \( k \). Since the level set \( L(\bar{x}) \) is compact, there exists at least an accumulation point \( x^* \) of the sequence \( \{x^*_k \} \). Assume without loss of generality that the sequence \( \{x^*_k \} \) itself converges to \( x^* \), i.e., \( \lim_{k \to \infty} x^*_k = x^* \). It suffices to prove that \( x^* \) is an optimal solution to \([LSIP]\). It is obvious that \( x^* \) is feasible for \([LSIP]\).

Let \( x_{opt} \) be an optimal solution to \([LSIP]\). If \( x_{opt} \in F^0 \), then \( x_{opt} \in F(T_k) \) for all \( k \) large enough. This indicates that \( f(x^*_k) \leq f(x_{opt}) \) for \( k \) large enough and thus
\[
f(x^*) = \lim_{k \to \infty} f(x^*_k) \leq f(x_{opt}),
\]
where \( f(x) = c^T x \). If \( x_{opt} \) lies on the boundary of the feasible set \( F \), there exists a sequence of the Slater points \( \{\bar{x}_j \mid \bar{x}_j \in F^0 \} \) such that \( \lim_{j \to \infty} \bar{x}_j = x_{opt} \). For each \( \bar{x}_j \in F^0 \) there exists at least an index \( k = k(j) \) such that \( \bar{x}_j \in F(T_k(j)) \) which
implies that \( f(x^*_k) \leq f(\bar{x}_j) \) for \( j \in \mathbb{N} \). Since \( \{x^*_k\} \) converges to \( x^* \) and \( \{x^*_k\} \) is a subsequence of \( \{x^*_k\} \), the sequence \( \{x^*_k\} \) is convergent and \( \lim_{j \to \infty} f(x^*_k(j)) = f(x^*) \). By the continuity of \( f \) we have

\[
 f(x^*) = \lim_{j \to \infty} f(x^*_k(j)) \leq \lim_{j \to \infty} f(\bar{x}_j) = f(x_{opt}).
\]

To sum up, we have \( x^* \in F \) and \( f(x^*) \leq f(x_{opt}) \).

This completes our proof.

**Proof of Lemma 3.5**

Let \( \bar{x} \in F \) be a Slater point, then we have

\[
 a(y)^\top \bar{x} + a_0(y) > 0, \ \forall y \in Y_k, k = 1, 2, ..., N.
\]

Since \( a_i(\cdot), i = 1, 2, ..., n \) are twice continuously differentiable, they are Lipschitz continuous, i.e., there exist a constant \( L \) such that

\[
 |a_i(y) - a_i(z)| \leq L|y - z|, \ \forall y, z \in Y.
\]

Let \( g_k(x) = \sum_{i=1}^{n} \min\{\bar{a}_i(\tau_k - 1), \bar{a}_i(\tau_k)\}x_i + \min\{\bar{a}_0(\tau_k - 1), \bar{a}_0(\tau_k)\} \), then we have

\[
 a(y)^\top \bar{x} + a_0(y) - g_k(\bar{x}) = \sum_{i=1}^{n} a_i(y)\bar{x}_i + a_0(y) - \sum_{i=1}^{n} \min\{\bar{a}_i(\tau_k - 1), \bar{a}_i(\tau_k)\} \bar{x}_i + \min\{\bar{a}_0(\tau_k - 1), \bar{a}_0(\tau_k)\}
\]

\[
 \leq (L \sum_{i=1}^{n} (\bar{x}_i + 1))|Y_k|, \ \forall y \in Y_k, k = 1, 2, ..., N.
\]

It follows that \( g_k(\bar{x}) \geq 0 \) if \( |T| \) is sufficiently small which implies \( \bar{x} \in \bar{F}(T) \).

This completes our proof.

**Convergence of Algorithm 1 for R1-LSIP**

Since \( x_k \) is a solution of R1-LSIP(\( T_k \)) for a consistent subdivision \( T_k = \{\tau_j^k \mid j = 0, 1, ..., N_k\} \) in the \( k \)th iteration, it must satisfy the KKT condition as follows:

\[
 c - \sum_{j \in A(x_k)} \lambda_j^k A_T(\cdot, j) = 0 \tag{12}
\]

where \( A(x^*_k) = \{j \mid A(\cdot, j)^\top x^*_k + b_T(j) = 0\} \) and \( A_T(i, j) = A^i_{t,j}, b_T(j) = A^0_{0,j} \) is the corresponding lower bound for \( a_i(y) \) and \( a_0(y) \) on \( [\tau_{j-1}^k, \tau_j^k] \). By (12) we know that for any \( \bar{\tau}_j^k \in [\tau_{j-1}^k, \tau_j^k] \) there holds that

\[
 |a_i(\bar{\tau}_j^k) - A_T(i, j)| \leq \gamma_i^n |\tau_j^k - \tau_{j-1}^k|^p, \ 0 \leq i \leq n, j \in A(x^*_k)
\]
where $\gamma_i^k, 1 \leq i \leq n, p \geq 1$ are constants. Thus, there exist some constants $0 \leq \beta_i^k \leq \gamma_i^k, i = 0, 1, 2, ..., n$ such that $A_{T_k}(i, j) = a_i(\bar{\tau}_j^k) + \beta_i^k|\tau_j^k - \tau_{j-1}^k|^p$. Substitute this into (12) and $A(x_k^*)$ we have
\[
c - \sum_{j \in A(x_k^*)} \lambda_j^k[a(\bar{\tau}_j^k) + (|\tau_j^k - \tau_{j-1}^k|^p)\beta_i^k] = 0,
\]
\[
A(x_k^*) = \{ j \mid [a(\bar{\tau}_j^k) + (|\tau_j^k - \tau_{j-1}^k|^p)\beta_i^k]x_k^* + a_0(\bar{\tau}_j^k) + (|\tau_j^k - \tau_{j-1}^k|^p)\beta_0^k = 0 \},
\]
where $a(\bar{\tau}_j^k) = (a_1(\bar{\tau}_j^k), a_2(\bar{\tau}_j^k), ..., a_n(\bar{\tau}_j^k))^T$. It follows that $x_k^*$ is a $(\epsilon, \delta)$ KKT point of (LSIP) if the lengths of the subsets $[\tau_{j-1}^k, \tau_j^k]$ for $j \in A(x_k^*)$ converge to zero as $k$ goes to infinity. This is true due to the similar argument in the proof of theorem 3.9.