GLOBAL CLASSICAL SOLUTION TO THE CAUCHY PROBLEM OF 2D BARATROPIC COMPRESSIBLE NAVIER-STOKES SYSTEM WITH LARGE INITIAL DATA

JINGCHI HUANG AND CHAO WANG

Abstract. For periodic initial data with initial density, we establish the global existence and uniqueness of strong and classical solutions for the two-dimensional compressible Navier-Stokes equations with no restrictions on the size of initial data provided the shear viscosity is a positive constant and the bulk one is \( \lambda = \rho^\beta \) with \( \beta > 1 \).

Keywords: Compressible Navier-Stokes equations; global strong solutions; large initial data.

1. Introduction

We study the two-dimensional barotropic compressible Navier-Stokes equations which read as follows

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \\
\frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \otimes u) + \nabla P &= \mu \Delta u + \nabla((\mu + \lambda) \text{div} u),
\end{align*}
\]

where \( t \geq 0, x = (x_1, x_2) \in \mathbb{T}^2, \rho, u = (u_1, u_2) \) stand for the density and velocity of the fluid respectively, and the pressure \( P \) is given by

\[
P(\rho) = R\rho^\gamma, \quad \gamma > 1.
\]

The shear viscosity \( \mu \) and the bulk \( \lambda(\rho) = b\rho^\beta \) satisfy the following hypothesis:

\[
0 < \mu = \text{const}, \quad \lambda(\rho) = b\rho^\beta, \quad b > 0, \quad \beta > 0.
\]

In the sequel, we set \( R = b = 1 \) without losing any generality.

We consider the Cauchy problem with the given initial data \( \rho_0 \) and \( m_0 \), which are periodic with period 1 in each space direction \( x_i, i = 1, 2 \), i.e., functions defined on \( \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \). We require that

\[
\rho(x, 0) = \rho_0(x), \quad pu(x, 0) = m_0(x), \quad x \in \mathbb{T}^2.
\]

The compressible Navier-Stokes equations have a very long history. There is a huge literature concerning the theory of the weak solutions to (1.1). Hoff proved the global existence of weak solution for the discontinuous initial data with small energy in [16, 17]. For the large initial data, the global existence of weak solution was proved by Lions [24] for the isentropic Navier-Stokes equation, i.e. \( P = R\rho^\gamma \) for \( \gamma \geq \frac{9}{5} \). Jiang and Zhang [21, 22] proved the global existence of weak solution for any \( \gamma > 1 \) for the spherically symmetric or axisymmetric initial data. Inspired by the Jiang and Zhang’s work, Feireisl, Novotný and Petzeltová [15] improved Lions’s result to \( \gamma > \frac{3}{2} \). However, the question of the regularity and uniqueness of weak solutions is completely open even in the case of two dimensional space.

Compared to the weak solutions, the results on the strong solutions are much less. 1962, Nash proved the local existence and uniqueness of smooth solution of the system (1.1) for smooth initial data without vacuum in [26]. In a seminal paper [25], Matsumura and Nishida proved that the solution is global in time if the initial data is close to equilibrium. However, whether smooth solutions with large initial data blow up in finite time is an open problem. Xin [34] proved that smooth solution of the full compressible Navier-Stokes equations will blow up in finite time if the initial density has compact support. Recently, Sun, Wang and Zhang [30] showed that smooth
solution does not blow up if the upper bound of the density is bounded, see [31] for the heat-conductive flow.

For the global existence results in [3, 4, 5, 9, 25], the initial density is required to be close to a positive constant in $L^\infty$ norm, hence precluding the large oscillation of the density at any point. Recently, Fang and Zhang [12] proved the global existence and uniqueness of (1.1) for the initial density $\rho_0$ close to a positive constant in $L^2$ norm and $u_0$ small in $L^p$ norm for $p > 3$, hence allowing the density to have large oscillation on a set of small measure. Similar result has also been obtained by Huang, Li and Xin [20] for the initial data with vacuum, but a compatible condition is imposed on the initial data. Recently, Wang, Wang and Zhang [33] prove the global well-posedness for some classes of large initial data.

For the global well-posedness of the strong solutions with general large initial data, Vaigant and Kazhikhov [32] proved existence of global solutions for system (1.1)–(1.3) with $\beta > 3$. Recently, Huang and Li [18, 19] relax the restriction $\beta > 3$ to $\beta > \frac{4}{3}$. In this paper, we aim to relax this restriction to $\beta > 1$.

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote \[ \int f \, dx = \int_{T^2} f \, dx, \quad \bar{f} = \frac{1}{|T^2|} \int f \, dx. \]

For $1 \leq r \leq \infty$, we also denote the standard Lebesgue and Sobolev spaces as follows:
\[ L^r = L^r(T^2), \quad W^{s,r} = W^{s,r}(T^2), \quad H^s = W^{s,2}. \]

**Theorem 1.1.** Assume that

(1.5) \[ \beta > 1, \quad \gamma > 1, \]

and that the initial data $(\rho, u_0)$ satisfies that for some $q > 2$,

\[ 0 < \rho_0 \in W^{1,q}, \quad u_0 \in H^2. \]

Then, the system (1.1)–(1.4) has a unique global strong solution $(\rho, u)$ satisfying that

\[ \begin{cases} 
\rho \in C([0,T]; W^{1,q}), \quad \rho_t \in C([0,T]; L^2), \\
u \in L^2(0,T; H^3), \quad u \in C([0,T]; H^2), 
\end{cases} \]

for any $0 < T < \infty$.

**The organization of the paper.** In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to derivation of upper bound on the density which is the key to extend the local solution to all time. Based on the previous estimates, higher-order ones are established in Section 4. We also prove the blow-up criterion and the main result, Theorem 1.1, in this Section.

2. Preliminaries

The following well-known local existence theory, where the initial density is strictly away from vacuum, can be found in [28, 29].

**Lemma 2.1.** Assume that $(\rho_0, u_0)$ satisfies

(2.1) \[ \rho_0 \in W^{1,q}, \quad u_0 \in H^2, \quad \inf_{x \in T^2} \rho_0(x) > 0, \quad m_0 = \rho_0 u_0. \]

Then there are a small time $T > 0$ and a constant $C_0 > 0$ both depending only on $\|\rho_0\|_{H^2}, \|u_0\|_{H^2}$, and $\inf_{x \in T^2} \rho_0(x)$ such that there exists a unique strong solution $(\rho, u)$ to the problem (1.1)–(1.4) in $T^2 \times (0,T)$ satisfying

(2.2) \[ \begin{cases} 
\rho \in C([0,T]; W^{1,q}), \quad \rho_t \in C([0,T]; L^2), \\
u \in L^2(0,T; H^3), \quad u \in C([0,T]; H^2), 
\end{cases} \]
Lemma 2.2. There exists a positive constant $C$ depending only on $H^1(\mathbb{T}^2)$ such that every function $u \in H^1(\mathbb{T}^2)$ satisfies for $2 < p < \infty$,

\begin{equation}
\|u - \bar{u}\|_{L^p} \leq C p^{\frac{1}{2}} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^{1 - \frac{2}{p}}, \quad \|u\|_{L^p} \leq C p^{\frac{1}{2}} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^{1 - \frac{2}{p}}.
\end{equation}

Moreover, for $q > 2$, there exists some positive constant $C$ depending only on $q$ and $\mathbb{T}^2$ such that every function $v \in W^{1, q}(\mathbb{T}^2)$ satisfies

\begin{equation}
\|v\|_{L^q} \leq C \|v\|_{H^1} \log^{\frac{1}{2}}(1 + \|\nabla v\|_{L^q}) + C.
\end{equation}

The following Poincare type inequality can be found in [14].

Lemma 2.3. Let $u \in H^1(\mathbb{T}^2)$, and $\rho$ be a non-negative function such that

\begin{equation}
0 < M_1 \leq \int \rho \, dx, \quad \int \rho^\gamma \, dx \leq M_2,
\end{equation}

with $\gamma > 1$. There exists a positive constant $C$ depending only on $M_1$ and $M_2$ such that

\begin{equation}
\|u\|_{L^2}^2 \leq C \int \rho u^2 \, dx + C \|\nabla u\|_{L^2}^2.
\end{equation}

Then, we state the following Beale-Kato-Majda type inequality which was proved in [1] when $\text{div} \ u \equiv 0$ and will be used later to estimate $\|\nabla u\|_{L^\infty}$ and $\|\nabla \rho\|_{L^p}$.

Lemma 2.4. For $2 < p < \infty$, there is a constant $C(p)$ such that the following estimate holds for all $\nabla u \in W^{1, p}(\mathbb{T}^2)$,

\begin{equation}
\|\nabla u\|_{L^\infty} \leq C(\|\text{div} \ u\|_{L^\infty} + \|\text{rot} \ u\|_{L^\infty}) \log(e + \|\nabla^2 u\|_{L^p}) + C \|\nabla u\|_{L^2} + C.
\end{equation}

Next, let $\Delta^{-1}$ denote the Laplacian inverse with zero mean on $\mathbb{T}^2$ and $R_i$ be the usual Riesz transform on $\mathbb{T}^2$: $R_i = (-\Delta)^{-\frac{1}{2}} \partial_i$. Let $H^1(\mathbb{T}^2)$ and $\text{BMO}(\mathbb{T}^2)$ stand for the usual Hardy and BMO space:

\begin{align*}
H^1 &= \{ f \in L^1(\mathbb{T}^2) : \|f\|_{H^1} = \|f\|_{L^1} + \|R_1 f\|_{L^1} + \|R_2 f\|_{L^1} < \infty, \bar{f} = 0 \}, \\
\text{BMO} &= \{ f \in L^1_{\text{loc}}(\mathbb{T}^2) : \|f\|_{\text{BMO}} < \infty \},
\end{align*}

with

\begin{equation}
\|f\|_{\text{BMO}} = \sup_{x \in \mathbb{T}^2, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} \left| f(y) - \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(z) \, dz \right| \, dy,
\end{equation}

where $d$ is the diameter of $\mathbb{T}^2$, $\Omega_r(x) = \mathbb{T}^2 \cap B_r(x)$, and $B_r(x)$ is a ball with center $x$ and radius $r$. Given a function $b$, define the linear operator

\begin{equation}
[b, R_i R_j](f) \triangleq b R_i \circ R_j(f) - R_i \circ R_j(b f), \quad i, j = 1, 2.
\end{equation}

The following properties of the commutator $[b, R_i R_j](f)$, which are due to [7, 8] respectively, will be useful for our work.
Lemma 2.5. Let $b, f \in C^\infty(T^2)$. Then for $p \in (1, \infty)$, there is $C(p)$ such that

$$\| [b, R_iR_j](f) \|_{L^p} \leq C(p) \| b \|_{BMO} \| f \|_{L^p}. \quad (2.8)$$

Moreover, for $q_i \in (1, \infty)$ ($i = 1, 2, 3$) with $q_1^{-1} = q_2^{-1} + q_3^{-1}$, there is a $C$ depending only on $q_i$ such that

$$\| \nabla [b, R_iR_j](f) \|_{L^{q_1}} \leq C \| \nabla b \|_{L^{q_2}} \| f \|_{L^{q_3}}. \quad (2.9)$$

3. A Priori Estimates: Upper Bound of the Density

First, we have the following standard energy inequality.

Lemma 3.1. There exists a positive constant $C$ depending only on $\gamma, T$, $\| \rho_0 \|_{L^\gamma}$, and $\| \rho_0^{\frac{1}{2}}u_0 \|_{L^2}$ such that

$$\sup_{0 \leq t \leq T} \int (\rho |u|^2 + \rho^\gamma) \, dx + \int_0^T \left( \int (\mu |\nabla u|^2 + \lambda(\rho)(\text{div} \, u)^2) \, dx \right) \, dt \leq C. \quad (3.1)$$

Next, we state the $L^p$ estimate of the density due to Vaigant-Kazhikhov [32].

Lemma 3.2. Let $\beta > 1$, for any $1 < p < \infty$, there is a positive constant $C(T)$ depending only on $T, \mu, \beta, \gamma, \text{E}_0 \equiv \| \rho_0 \|_{L^\infty} + \| \rho_0^{\frac{1}{2}}u_0 \|_{L^2} + \| \nabla u_0 \|_{L^2}$ such that

$$\sup_{0 \leq t \leq T} \| \rho(\cdot,t) \|_{L^p} \leq C(T)p^2 \beta^{-1}. \quad (3.2)$$

To proceed, we denote by

$$\nabla^\perp = (\partial_2, -\partial_1), \quad \frac{D}{Dt} f = \dot{f} = f_t + u \cdot \nabla f,$$

where $\frac{D}{Dt} f$ is the material derivative of $f$. Let $G$ and $\omega$ be the effective viscous flux and the vorticity respectively as follows:

$$G \equiv (2\mu + \lambda(\rho)) \text{div} \, u - (P - \bar{P}), \quad \omega \equiv \nabla^\perp \cdot u = \partial_2 u_1 - \partial_1 u_2.$$

Then we rewrite the momentum equation (1.1) as

$$\rho \dot{u} = \nabla G + \mu \nabla^\perp \omega, \quad \text{which shows that } G \text{ solves}$$

$$\Delta G = \text{div}(\rho \dot{u}) = \partial_t (\text{div}(\rho u)) + \text{div} \, \text{div} (\rho u \otimes u).$$

This implies

$$G - \bar{G} + \frac{D}{Dt} \left( (-\Delta)^{-1} \text{div}(\rho u) \right) = F, \quad \text{where } F \text{ is a commutator defined by}$$

$$F \equiv \sum_{i,j=1}^2 [u_{ij}, R_iR_j](\rho u_j) = \sum_{i,j=1}^2 u_{ij}R_i \circ R_j(\rho u_j) - R_i \circ R_j(\rho u_i u_j). \quad (3.5)$$

The mass equation (1.1) leads to

$$-\text{div} \, u = \frac{1}{\rho} \frac{D}{Dt} \rho,$$

which combining with (3.4) gives that

$$\frac{D}{Dt} \varphi(\rho) + P = \frac{D}{Dt} \psi + \bar{P} - \bar{G} + F, \quad (3.6)$$
with
\[
\varphi(\rho) \triangleq 2\mu \log \rho + \beta^{-1}\rho^\gamma, \quad \psi \triangleq (-\Delta)^{\frac{1}{2}} \text{div}(\rho u) .
\]

**Lemma 3.3.** Assume that (1.5) holds. Then there is a constant \( C \) depending only on \( \mu, \beta, \gamma, T, \) and \( E_0 \) such that
\[
\sup_{0 \leq t \leq T} \log(e + A^2(t)) + \int_0^T \frac{B^2(t)}{e + A^2(t)} \, dt \leq CR_T^{1+\delta \beta}, \quad \delta \in (0, 1),
\]
where
\[
A^2(t) \triangleq \int \left( \omega^2(t) + \frac{G^2(t)}{2\mu + \lambda(\rho(t))} \right) \, dx, \quad B^2(t) \triangleq \int \rho(t)|\dot{u}(t)|^2 \, dx,
\]
and
\[
R_T \triangleq 1 + \sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty}.
\]

**Proof.** First, direct calculations show that
\[
\nabla^\perp \cdot \dot{u} = \frac{D}{Dt} \omega - (\partial_1 u \cdot \nabla)u_2 + (\partial_2 u \cdot \nabla)u_1 = \frac{D}{Dt} \omega + \omega \text{div} u,
\]
and that
\[
\text{div} \dot{u} = \frac{D}{Dt} \text{div} u + (\partial_1 u \cdot \nabla)u_1 + (\partial_2 u \cdot \nabla)u_2
\]
\[
= \frac{D}{Dt} \left( \frac{G}{2\mu + \lambda} + \frac{P - \bar{P}}{2\mu + \lambda} \right) - 2\nabla u_1 \cdot \nabla^\perp u_2 + (\text{div} u)^2.
\]

Multiplying (3.3) by \( 2\dot{u} \) and integrating the resulting equality over \( \mathbb{T}^2 \), we obtain after using (3.11) and (3.12) that
\[
\frac{d}{dt} A^2 + 2B^2
\]
\[
= - \int \omega^2 \text{div} u \, dx + 4 \int G \nabla u_1 \cdot \nabla^\perp u_2 \, dx - 2 \int G (\text{div} u)^2 \, dx
\]
\[
- \int \frac{(\beta - 1)\lambda - 2\mu}{(2\mu + \lambda)^2} G^2 \text{div} u \, dx + 2\beta \int \frac{(P - \bar{P})}{(2\mu + \lambda)^2} G \text{div} u \, dx
\]
\[
- 2\gamma \int \frac{P}{2\mu + \lambda} G \text{div} u \, dx + 2(\gamma - 1) \int P \text{div} u \, dx \int \frac{G}{2\mu + \lambda} \, dx
\]
\[
= \sum_{i=1}^7 I_i .
\]

Now, we estimate each \( I_i \) as follows:
First, we deduce from (3.3) to get that
\[
\triangle G = \text{div}(\rho \dot{u}), \quad \mu \triangle \omega = \nabla^\perp \cdot (\rho \dot{u}),
\]
and use with standard \( L^p \) estimate of elliptic equations to obtain that for any \( p \in (1, \infty) \),
\[
\|\nabla G\|_{L^p} + \|\nabla \omega\|_{L^p} \leq C(p, \mu) \|\rho \dot{u}\|_{L^p}.
\]

In particular, we have
\[
\|\nabla G\|_{L^2} + \|\nabla \omega\|_{L^2} \leq C(\mu) \|\rho \dot{u}\|_{L^2} \leq C(\mu)R_T^{\frac{1}{2}}B.
\]
This combining with (2.4) gives
\[
\|\omega\|_{L^4} \leq C \|\omega\|_{L^2}^{\frac{1}{2}} \|\nabla \omega\|_{L^2}^{\frac{1}{2}} \leq CR_T^{\frac{1}{2}}A^{\frac{1}{2}}B^{\frac{1}{2}},
\]
and use with standard
which leads to
\begin{equation}
|I_1| \leq C \|\omega\|_{L^4} \|\text{div} u\|_{L^2} \leq \varepsilon B^2 + C(\varepsilon) R_T \|\nabla u\|_{L^2}^2 A^2.
\end{equation}

Next, we will use an idea due to [10, 27] to estimate $I_2$. Noticing that
\[ \text{rot}\nabla u_1 = 0, \quad \text{div} \nabla u_2 = 0, \]
one can derives from [6] that
\[ \|\nabla u_1 \cdot \nabla u_2\|_{H^1} \leq C \|\nabla u\|_{L^2}^2. \]
Together with the fact that $B_{\mathcal{MO}}$ is the dual space of $\mathcal{H}^1$ (see [13]), we obtain
\begin{equation}
|I_2| \leq C \|G\|_{B_{\mathcal{MO}}} \|\nabla u_1 \cdot \nabla u_2\|_{H^1} \leq C \|\nabla G\|_{L^2} \|\nabla u\|_{L^2}^2
\end{equation}
\begin{equation}
\leq CR_T^\beta B \|\nabla u\|_{L^2} (1 + A) \leq \varepsilon B^2 + C(\varepsilon) R_T \|\nabla u\|_{L^2}^2 (1 + A^2),
\end{equation}
where in the third inequality we have use (3.15) and the following simple fact that for $t \in [0, T]$,
\begin{equation}
C^{-1} \|\nabla u(\cdot, t)\|_{L^2} - C A^2(t) \leq C R_T^\beta \|\nabla u(\cdot, t)\|_{L^2}^2 + C
\end{equation}
due to (3.2).

Next, Hölder’s inequality, (1.2) and (3.2) yield that for $\delta \in (0, 1),$
\begin{equation}
\sum_{i=3}^7 |I_i| \leq C \int \text{div} u \left( \|G\|_{L^2} + \frac{P}{2\mu + \lambda} + \frac{G^2}{2\mu + \lambda}\right) dx + C \int \text{div} u dx \int \frac{|G|}{2\mu + \lambda} dx
\end{equation}
\begin{equation}
\leq C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} P + C \|\nabla u\|_{L^2} G \|L^{2(2+\delta)}\| + C \|\nabla u\|_{L^2} \|P\|_{L^2} G \|L^{2(2+\delta)}\|
\end{equation}
\begin{equation}
\leq C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} G \|L^{2(2+\delta)}\|.
\end{equation}

Then, noticing that (3.9) gives
\begin{equation}
\|G\|_{L^2} \leq CR_T^\beta A,
\end{equation}
which together with the Hölder inequality, (2.4) and (3.15) yield that for $0 < \delta < 1,$
\begin{equation}
\|G\|_{L^2} \leq C \left( \frac{G}{\sqrt{2\mu + \lambda}} \right)^{1-\delta} \|G\|_{L^2} \leq C \frac{G}{\sqrt{2\mu + \lambda}} \|G\|_{L^2} \leq C \frac{G}{\sqrt{2\mu + \lambda}} \|G\|_{L^{2(2+\delta)}} \leq C A^{1-\delta} \|G\|_{L^2} \|\nabla G\|_{L^2} \leq C R_T^{-\delta} AB.
\end{equation}

Similarly, we have
\begin{equation}
\|G\|_{L^{2(2+\delta)}} \leq C \frac{\delta}{L^{2(2+\delta)}} \|\nabla G\|_{L^2} \|G\|_{L^{2(2+\delta)}} \leq C R_T^{1+\delta} A^{2+\delta} B^{2+\delta}.
\end{equation}

Putting (3.22), and (3.23) into (3.20) yields
\begin{equation}
\sum_{i=3}^7 |I_i| \leq C R_T^{1+\delta} \|\nabla u\|_{L^2} (AB + A^{2+\delta} B^{2+\delta})
\end{equation}
\begin{equation}
\leq CR_T^{1+\delta} \|\nabla u\|_{L^2} (AB + A + B)
\end{equation}
\begin{equation}
\leq \varepsilon B^2 + C(\varepsilon) R_T^{1+\delta} (1 + \|\nabla u\|_{L^2}) (1 + A^2).
\end{equation}

Finally, substituting (3.17), (3.18), and (3.24) into (3.13), choosing $\varepsilon$ sufficiently small and $\delta \in (0, 1)$, we obtain
\begin{equation}
\frac{d}{dt} A^2 + B^2 \leq CR_T^{1+\delta} (1 + \|\nabla u\|_{L^2}) (1 + A^2).
\end{equation}
Dividing this inequality by \(e + A^2\), and using (3.1), we reach (3.8) and finish the proof of Lemma 3.3.

The following \(L^p\) estimates of the momentum will play an important role in the estimate of the upper bound of the density.

**Lemma 3.4.** For any \(p > 2\), there exists a positive constant \(C\) depending only on \(p, \mu, \beta, \gamma, T\), and \(E_0\) such that

\[
\|\rho u\|_{L^p} \leq CR_T(e + \|\nabla u\|_{L^2}).
\]

Moreover, let \(\alpha = \frac{\mu^2}{2(\mu + 1)} R_T^n \in (0, \frac{1}{4}]\), then for any \(q > 3\) and \(\varepsilon > 0\) there exists a positive constant \(C_1\) depending only on \(q, \mu, \beta, \gamma, T\), and \(E_0\) such that

\[
\|\rho u\|_{L^q} \leq C_1 R_T^{\frac{1}{2} - \frac{1}{q} + \varepsilon} (e + \|\nabla u\|_{L^2})^{\frac{2 + \alpha}{q}} \log \frac{2 + \alpha + 4}{2\alpha}(e + \left(\frac{B^2}{e + A^2}\right)^{\frac{1}{2}}),
\]

where \(A, B\) are defined as (3.9).

**Proof.** (3.26) is the directly consequence from (2.4) and (2.6). We just focus on the proof of (3.27).

Multiplying (1.1) by \((2 + \alpha)|u|^\alpha u\), we get after integrating the resulting equation over \(\mathbb{T}^3\) that

\[
\frac{d}{dt} \int \rho |u|^{2+\alpha} dx + (2 + \alpha) \int |u|^\alpha (\mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2) dx
\]

\[
\leq (2 + \alpha) \alpha \int (\mu + \lambda) |\nabla u|^2 |u|^\alpha dx + C \int \rho |u|^\alpha |\nabla u| dx
\]

\[
\leq \frac{2 + \alpha}{2} \int (\mu + \lambda) |\nabla u|^2 |u|^\alpha dx + \left(\frac{2 + \alpha}{8(\mu + 1)} + \mu\right) \int |u|^\alpha |\nabla u|^2 dx
\]

\[
+ C \int \rho |u|^{2+\alpha} dx + C \int \rho^{(2+\alpha)} \gamma - \frac{\mu}{2} dx,
\]

which together with Gronwall’s inequality and (3.2) thus gives

\[
\sup_{0 \leq t \leq T} \int \rho |u|^{2+\alpha} dx \leq C.
\]

Then let \(s = 1 - \frac{2+\alpha}{q}\), it follows form Hölder’s inequality that

\[
\|\rho u\|_{L^q} \leq C \|\rho u\|_{L^\infty}^{\frac{2}{q+s}} \|\rho u\|_{L^\infty}^{\frac{s}{s}} \leq C R_T^{\frac{2}{1+s} - \frac{2}{q+s}} R_T^s \|\rho u\|_{L^\infty}^s = C R_T^{\frac{1}{2} - \frac{1}{q} + \varepsilon} \|\rho u\|_{L^\infty}^s.
\]

Using (2.5), (2.6), and (3.1), we have

\[
\|u\|_{L^\infty} \leq C \|u\|_{H^1} \log^\frac{\delta}{\lambda}(e + \|\nabla u\|_{L^2}) \leq C(1 + \|\nabla u\|_{L^2}) \log^\frac{\delta}{\lambda}(e + \|\nabla u\|_{L^4}).
\]

But from (3.16), (3.19), (3.22), and (3.2), we obtain that

\[
\|\nabla u\|_{L^4} \leq C(\|\text{div} u\|_{L^4} + \|\omega\|_{L^4}) \leq C \left(\frac{G + P - \bar{P}}{2\mu + \lambda}\right) \|\nabla u\|_{L^4} + CR_T^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}
\]

\[
\leq C \left(\frac{G^2}{2\mu + \lambda}\right) \|\nabla u\|_{L^2} + C + CR_T^{\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}} \leq CR_T^{\frac{1 + 2\delta}{4}} A^{\frac{1}{2}} B^{\frac{1}{2}}
\]

\[
\leq CR_T^{\frac{1 + 2\delta}{4}} \left(e + A^2\right)^{\frac{1}{2}} \left(\frac{B^2}{e + A^2}\right)^{\frac{1}{4}} \leq CR_T^{\frac{1 + 2\delta}{4}} (e + \|\nabla u\|_{L^2}) \left(\frac{B^2}{e + A^2}\right)^{\frac{1}{4}}.
\]

Substituting (3.30) and (3.31) into (3.29), we obtain (3.27), which completes the proof of Lemma 3.4.

Now we are in the position to prove the main result of this section.
Proposition 3.1. Under the conditions of Theorem 1.1, there is a constant $C$ depending only on $\mu, \beta, \gamma, T$, and $E_0$ such that

\begin{equation}
\sup_{0 \leq t \leq T} (\|\rho(t)\|_{L^\infty} + \|\nabla u(t)\|_{L^2}) + \int_0^T \int \rho |\dot{u}|^2 \, dx \, dt \leq C.
\end{equation}

Proof. First, it follows from (3.1) and (3.2) that

\begin{equation}
\|\rho u\|_{L^\infty} \leq C \|\rho\|_{L^\infty} \|\rho^\frac{1}{2} u\|_{L^2} \leq C,
\end{equation}

which together with (2.5), (3.8), (3.19), and (3.26) yields that for any $\varepsilon > 0$,

\begin{align*}
\|\psi\|_{L^\infty} &\leq C \|\psi\|_{H^1} \log \left( e + \|\nabla \psi\|_{L^3} \right) + C \\
&\leq C(\|\rho u\|_{L^\infty} \|\rho^\frac{1}{2} u\|_{L^2}) \log \left( e + \|\rho u\|_{L^3} \right) + C \\
&\leq CR_T^\frac{1}{2} \log \left( R_T (e + \|\nabla u\|_{L^2}) \right) + C \\
&\leq CR_T^{\frac{1}{2} + \varepsilon} \log \left( e + A^2 \right) + C \\
&\leq CR_T^{1 + \frac{\beta}{2} + \varepsilon}.
\end{align*}

Next, on one hand, we deal with the $\|F\|_{L^\infty}$. Taking $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ and Applying the (2.5), we have that

\begin{align}
\|F\|_{L^\infty} &\leq C \|F\|_{H^1} \log \left( e + \|\nabla F\|_{L^3} \right) + C \\
&\leq C \left( \|\nabla u\|_{L^2} \|\rho u\|_{L^2} + \|\nabla u\|_{L^5} \|\rho u\|_{L^P} \right) \log \left( e + \|\nabla u\|_{L^3} \|\rho u\|_{L^2} \right) + C \\
&\leq C \left( R_T^\frac{1}{2} \|\nabla u\|_{L^2} \|\rho^\frac{1}{2} u\|_{L^2} + \|\nabla u\|_{L^5} \|\rho u\|_{L^2} \right) \log \left( e + \|\nabla u\|_{L^3} \|\rho u\|_{L^2} \right) + C.
\end{align}

By the Hölder inequality and (3.31), we have that

\begin{align*}
\|\nabla u\|_{L^q} &\leq \|\nabla u\|_{L^2} \|\nabla u\|_{L^4}^{1 - \theta} \leq C \|\nabla u\|_{L^2} \left( R_T^{\frac{1 + \beta}{4} + \frac{\beta}{2}} \|\nabla u\|_{L^2} (\frac{B}{\varepsilon + A})^\frac{1}{2} + 1 \right)^{1 - \theta} \\
&\leq C(1 + \|\nabla u\|_{L^2}) \left( R_T^{\frac{1 + \beta}{4} + \frac{\beta}{2}} (\frac{B}{\varepsilon + A})^\frac{1}{2} + 1 \right)^{1 - \theta}.
\end{align*}

where $\frac{1}{q} = \frac{\theta}{2} + \frac{1 - \theta}{4}$.

Next, by (3.27), we have that for $s = 1 - \frac{2 + \alpha}{p}$,

\begin{equation}
\|\rho u\|_{L^p} \leq CR_T^\frac{1 - \theta}{p + \varepsilon} \|u\|_{H^1} \left( \log(1 + (\frac{B}{\varepsilon + A})^\frac{1}{2}) \right)^s.
\end{equation}

Combining above two estimates, we have that

\begin{equation}
\|\nabla u\|_{L^q} \|\rho u\|_{L^p} \leq CR_T^\frac{1 - \theta}{p + (1 - \theta)(\frac{1 + \beta}{4} + \frac{\beta}{2}) + \varepsilon} \left( 1 + \|u\|_{H^1} \right)^{1 + s + \varepsilon} (1 + (\frac{B}{\varepsilon + A})^\frac{1 - \theta}{2} + \varepsilon).
\end{equation}

By the definition of $s, \theta$, we have that

\begin{equation*}
s = 1 - (2 + \alpha) \frac{1 - \theta}{4},
\end{equation*}

which implies that when $0 < \theta < 1$

\begin{equation*}
2s < 1 + \theta.
\end{equation*}

Taking $\varepsilon$ small enough such that

\begin{equation*}
2s + 3\varepsilon \leq 1 + \theta.
\end{equation*}
Thus, we have
\[
\int_0^T \|F\|_{L^\infty} \, dt \leq C \left( R_{T}^{\frac{1}{\beta} + (1-\theta)(\frac{1+\beta \delta}{2}+\varepsilon)} \right)^{\frac{4}{\pi+\beta}} \int_0^T \|u\|^2_{H^1} \, dt + C \int_0^T \left( 1 + \frac{B}{e+A} \right)^2 \, dt.
\]
If we take \( \theta \) close to 1, we have that
\[
\int_0^T \|F\|_{L^\infty} \, dt + \|\psi\|_{L^\infty} \leq C(R_{T}^{1+\frac{\delta \beta}{2}+\varepsilon} + 1).
\]
Now, we are in the position to prove the upper bound of the density. Recalling the (3.6) and integrating to get that
\[
\begin{align*}
R_{T}^\beta &\leq C(R_{T}^{1+\frac{\delta \beta}{2}+\varepsilon} + 1) + C \int_0^T (1 + \|\rho\|^1_{L^2} \|\text{div}u\|_{L^2}) \, dt \\
&\leq C(R_{T}^{1+\frac{\delta \beta}{2}+\varepsilon} + 1).
\end{align*}
\]
where the constant depends on the initial data and the time \( T \). Thus, if we take \( \beta > 1, \delta \) and \( \varepsilon \) sufficiently small such that \( \beta > 1 + \frac{\delta \beta}{2} + \varepsilon \), we can get that
\[
R_T \leq C.
\]
By the inequality (3.25), we have that
\[
\sup_{0 \leq t \leq T} (e + A^2(t)) + \int_0^T B^2(t) \, dt \leq C.
\]
By the classic elliptic estimate, we get the desired result. \( \square \)

4. A Blow-up Criteria

In this section, we will establish a blow-up criterion for the (1.1)–(1.3) for all \( \beta > 1 \).

**Proposition 4.1.** Assume that \((\rho, u)\) is the strong solution of (1.1)–(1.3). Let \( T^* \) be a maximal existence time of the solution. If \( T^* < \infty \), then we have
\[
\limsup_{T \to T^*} \|\rho(x, t)\|_{L^\infty(0,T;L^\infty(\Omega))} = \infty.
\]

First, from the proof of the Lemma 3.1 and Lemma 3.3, we obtain

**Proposition 4.2.** There exists a constant \( C \) depending only on \( \mu, \beta, \gamma, T, \) and \( E_0, \|\rho\|_{L^\infty} \) such that
\[
\sup_{0 \leq t \leq T} \int (\rho|u|^2 + \rho^\gamma) (t) \, dx + \int_0^T \left( \mu|\nabla u|^2 + \lambda(\mu)(\text{div} u)^2 \right) \, dx \, dt \leq C,
\]
\[
\sup_{0 \leq t \leq T} (e + A^2(t)) + \int_0^T B^2(t) \, dt \leq C.
\]

Next, we have the second order estimates as following:

**Proposition 4.3.** There exists a constant \( C \) depending only on \( \mu, \beta, \gamma, T, E_0 \) and \( \|\rho\|_{L^\infty} \) such that
\[
\sup_{0 \leq t \leq T} \|\frac{1}{\beta} \dot{u}(t)\|_{L^2} + \int_0^T \|\nabla \dot{u}\|^2 \, dx \, dt \leq C.
\]
Proof. Take the material derivative $\frac{D}{Dt}$ on the both side of (1.1) to get,

$$
(\rho \dot{u})_t + \text{div}(\rho \dot{u}) = -\mu \triangle u - \partial_j((\mu + \lambda)\text{div}u)
$$

$$
= \mu \partial_t(- \partial_i u \cdot \nabla u_j + \text{div} \partial_i u_j) - \mu \text{div}(\partial_i u \partial_j u) - \partial_j[(\mu + \lambda) \partial_i u \cdot \nabla u_i - (\mu + (1 - \beta) \rho^\beta)(\text{div} u^2)]
$$

$$
- \text{div}(\partial_j(\mu + \lambda) \text{div} u) + (\gamma - 1) \partial_j(P \text{div} u) + \text{div}(P \partial_j u).
$$

Then, multiplying $\dot{u}$ on the both sides of (4.3), we get that

$$
\frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \mu \int |\nabla \dot{u}|^2 \, dx + \int (\mu + \lambda)(\text{div} \dot{u})^2 \, dx 
$$

$$
\leq \varepsilon \int |\nabla \dot{u}|^2 \, dx + C_\varepsilon \left( \|\nabla u\|_{L^1} + \|u\|_{L^2}^2 \right).
$$

Plugging (3.31) into the above inequality, we obtain

$$
\frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \mu \int |\nabla \dot{u}|^2 \, dx + \int (\mu + \lambda)(\text{div} \dot{u})^2 \, dx \leq C_\varepsilon (A^2 B^2 + \|\nabla u\|_{L^2}^2).
$$

Thus, by Proposition 4.3, the proof is completed. \qed

Next, we compute the higher order estimates for density:

**Proposition 4.4.** There exists a constant $C$ depending only on $\mu, \beta, \gamma, T,$ and $E_0, \|\rho\|_{L^\infty}$ such that

$$
\sup_{0 \leq t \leq T} (\|\rho\|_{W^{1,q}} + \|\nabla u\|_{H^1}^2) + \int_0^T \|\nabla^2 u\|_{L^2} \, dt \leq C,
$$

where $q \geq 2$.

**Proof.** The proof comes from [18, 19]. First, we denote $\Phi \triangleq (2\mu + \lambda(\rho)) \nabla \rho$, then from the density equation, we have

$$
\dot{\Phi} + (u \cdot \nabla) \Phi + (2\mu + \lambda(\rho)) \nabla u \cdot \nabla \rho + \rho \nabla (G + P) + \Phi \text{div} u = 0.
$$

Multiplying $|\Phi|^{q-2} \Phi$ on the both sides of the above equation and integrating by parts, we obtain

$$
\frac{d}{dt} \|\Phi\|_{L^q} \leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^q} + C \|\nabla G\|_{L^q}
$$

$$
\leq C(1 + \|\nabla u\|_{L^\infty}) \|\nabla \rho\|_{L^q} + C \|\rho \dot{u}\|_{L^q}.
$$

On one hand, recalling (3.14), for any $q > 2$ we get

$$
\|\nabla u\|_{L^\infty} \leq C(\|\text{div} u\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(1 + \|\nabla u\|_{L^q}) + C \|\nabla u\|_{L^2} + C
$$

$$
\leq C(1 + \|G\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(1 + \|\text{div} u\|_{L^q}) + C \|\nabla u\|_{L^2} + C
$$

$$
\leq C(1 + \|G\|_{L^\infty} + \|\omega\|_{L^\infty}) \log(1 + \|\text{div} u\|_{L^q}) + C \|\rho \dot{u}\|_{L^q} + C \|\nabla u\|_{L^2} + C
$$

$$
\leq C(1 + \|\rho \dot{u}\|_{L^q}) \log(1 + \|\text{div} u\|_{L^q}) \|\nabla u\|_{L^2} + C \|\nabla u\|_{L^2} + C
$$

$$
\leq C(1 + \|\rho \dot{u}\|_{L^q}) \log(1 + \|\nabla u\|_{L^q}) + C \|\nabla u\|_{L^2} + C
$$

On the other hand, by Lemma 2.3, we have that

$$
\|\dot{u}\|_{L^2} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2}),
$$

which implies that, by Proposition 4.4,

$$
\|\rho \dot{u}\|_{L^2(0, T; L^q)} \leq C.
$$
Plugging above estimate into (4.5) and applying the Gronwall’s inequality, we obtain
\[ \| \nabla \rho \|_{L^q} \leq C. \]

Thus, we have
\[ \| \nabla^2 u \|_{L^q} \leq C(\| \nabla \text{div} u \|_{L^q} + \| \nabla \omega \|_{L^q}) \leq C(1 + \| \nabla \rho \|_{L^q} + \| \rho \dot{u} \|_{L^q}), \]
which implies that
\[ \int_0^T \| \nabla^2 u \|_{L^q}^2 \, dt + \| \nabla^2 u \|_{L^2}^2 \leq C, \]
which completes the proof of Proposition 4.4.

**Proof of the Proposition 4.1:** Now we are in the position to prove Proposition 4.1. We prove it by the contradiction argument. Assume that \( T^* < \infty \) and
\[ \sup_{s \in [0, T^*)} \| \rho(s) \|_{L^\infty((0, T^*) \times \Omega)} < \infty. \]

Then, by Proposition (4.4), we have
\[ \sup_{0 \leq t \leq T^*} (\| \rho(t) \|_{W^{1,q}} + \| \nabla u(t) \|_{H^{1,1}}^2) + \int_0^{T^*} \| \nabla^2 u \|_{L^q}^2 \, dt \leq C. \]

By Lemma 2.1, we can extend the solution to \([0, T^* + \varepsilon]\) for some small data \( \varepsilon \). Thus, the Proposition 4.1 is proved.

**Proof of the Theorem 1.1:** We prove it by the contradiction argument. Assume that lifespan \( T^* < \infty \). Thus, by the Proposition 3.1, we have
\[ \sup_{s \in [0, T^*)} \| \rho(s) \|_{L^\infty((0, T^*) \times \Omega)} < \infty. \]

Then, applying Proposition 4.1, we can extend the solution which contradicts with the definition of the \( T^* \).

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