A Model Theoretic Proof of Szemerédi’s
Theorem

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Abstract

We present a short proof of Szemerédi’s Theorem using a dynamical system enriched by ideas from model theory. The resulting proof contains features reminiscent of proofs based on both ergodic theory and on hypergraph regularity.

1 Introduction

Szemerédi’s Theorem states:

**Theorem 1.** For any $\delta > 0$ and any $k$, there is an $n$ such that whenever $N \geq n$, $A \subseteq [1, N]$, and $|A| \geq \delta N$, there exists an $a$ and a $d$ such that

$$a, a + d, a + 2d, \ldots, a + (k - 1)d \in A.$$ 

Szemerédi’s original proof [14] used graph theoretic methods, in particular the Szemerédi Regularity Lemma [15]. Shortly after, Furstenberg gave a different proof [5, 4], based on a correspondence argument which translates the problem into one in ergodic theory. Beginning with a new proof given by Gowers [7], a number of new proofs have been developed in the last decade. (Tao counts a total of roughly sixteen different proofs [19].)

Hrushovski has recently used a stronger correspondence-type argument [9] to make progress on a similar combinatorial problem (the so-called non-commutative Freiman conjecture). In this paper, we use Hrushovski’s method to give a short proof of Szemerédi’s theorem.

The proof here bears a similarity to proofs based on hypergraph regularity, such as [8, 12, 13, 16]: in particular the proof is very similar to the infinitary regularity-like arguments introduced by Tao [17] and used by Austin to prove both Szemerédi’s Theorem [2] and generalizations [1]. Indeed, this proof was inspired by noticing that the use of “wide types” (countable intersections of definable sets of positive measure) in Hrushovski’s arguments was analogous to the use of the regularity lemma in finitary arguments. (In fact, Hrushovski essentially sketches a proof of the $k = 3$ case of Szemerédi’s Theorem in [9]; however his arguments depend on stability theoretic methods which don’t seem to generalize to higher $k$. This seems related to the fact that stability implies 3-amalgamation, but not 4-amalgamation.)
The methods here are also reminiscent of those used by Tao to prove the “diagonal ergodic theorem” [18], and especially to our infinitary reformulation of that proof [20]. This paper might shed light on the connection between that method and the technique of “pleasant extensions” used by Austin [3].

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2 A Correspondence Principal

Suppose that Szemerédi’s Theorem fails; that is, for some \( \delta > 0 \) and every \( n \), there is an \( A_n \subseteq [1, n] \) with \( |A_n| \geq \delta n \) such that \( A_n \) contains no \( k \)-term arithmetic progression. We must first make a technical adjustment: we view the sets \( A_n \) as subsets of the group \([1, 2n + 1]\). Then \( |A_n| \geq \delta n/2 - \epsilon \) where \( \epsilon \to 0 \) as \( n \to \infty \), and \( A_n \) contains no \( k \)-term arithmetic progressions in this group.

We extend the language of groups with a predicate symbol \( A \) and the following additional class of formulas:

- Whenever \( \alpha_1, \ldots, \alpha_k, \gamma \) is a sequence of rationals, \( \vec{x} \) a tuple, and \( B_1(\vec{x}, \vec{y}_1), \ldots, B_k(\vec{x}, \vec{y}_k) \) a sequence of formulas, \( \int \sum_{i \leq k} \alpha_i \cdot B_i dm(\vec{x}) > \gamma \) is a formula with free variables \( \vec{y}_1, \ldots, \vec{y}_k \).

We let \( ([1, 2n + 1], A_n) \) be models, interpreting the symbol \( A \) by \( A_n \) and

\[
\left[ \int \sum_{i \leq k} \alpha_i \cdot \phi_i dm(\vec{x}) > \gamma \right] (\vec{p}_1, \ldots, \vec{p}_k) \iff \sum_{i \leq k} \alpha_i \left| \left\{ \vec{x} \mid \phi_i(\vec{x}, \vec{p}_i) \right\} \right| > \gamma
\]

We write \( m_{\vec{x}}(\phi) \) as an abbreviation for \( \int 1 \cdot \phi dm(\vec{x}) \), and sometimes omit \( \vec{x} \) when it is clear from context.

Form an ultraproduct of those groups \([1, 2n + 1]\) such that \( 2n + 1 \) is prime. We obtain a model \((G, +, A)\). By transfer, the formula

\[ \exists a, d (a \in A \land a + d \in A \land \cdots a + d + \cdots + d \in A) \]

is false.

Observe that for any countable set \( M \) and any \( n \), the sets of \( n \)-tuples definable with parameters from \( M \) form an algebra of internal sets, and using the Loeb measure construction, we may extend the internal counting measure on this model to a measure \( \mu^n \) on the \( \sigma \)-algebra of Borel sets generated from these definable sets (for basic facts about this construction, see [6]). The measures \( \mu^n \) satisfy Fubini’s Theorem [10, 11]: that is,

\[
\int f d\mu^n = \int \int f d\mu^n d\mu^n_1
\]

\[1\]Without this modification, \( A_n \) might contain no arithmetic progressions as a set of integers, but contain arithmetic progressions as a subset of the group, since the group contains progressions which “wrap around”: for instance, 3, 10, 4 is an arithmetic progression in \([1, 13]\), but does not correspond to an arithmetic progression in the integers. Expanding the group by adding a “dead zone” disjoint from \( A_n \) is a standard way of avoiding this problem.
where \( n_0 + n_1 = n \).

**Lemma 2.** If \((G, +, A) \models m(\phi) > \gamma\) then \( \mu(\{ \vec{x} \mid \phi(\vec{x}) \}) \geq \gamma \). If \((G, +, A) \models \neg m(\phi) > \gamma\) then \( \mu(\{ \vec{x} \mid \phi(\vec{x}) \}) \leq \gamma \).

**Proof.** For the first part, since \((G, +, A) \models m(\phi) > \gamma\), for almost every \( n \), \(((1, 2n + 1), +, A_{2n+1}) \models m(\phi) > \gamma\), and therefore

\[
\frac{|\{ \vec{x} \mid \phi(\vec{x}) \}|}{(2n + 1)|\vec{x}|} > \gamma
\]

holds in \(((1, 2n + 1), A_{2n+1})\). (Here \( \phi \) may contain parameters.)

But since this holds for almost every \( n \), by transfer

\[
\frac{|\{ \vec{x} \mid \phi(\vec{x}) \}|}{(2n + 1)|\vec{x}|} > \gamma
\]

holds in \((G, +, A)\) where the inequality is between nonstandard rational numbers. This means that

\[
\mu(\{ \vec{x} \mid \phi(\vec{x}) \}) = \text{st}(\frac{|\{ \vec{x} \mid \phi(\vec{x}) \}|}{(2n + 1)|\vec{x}|}) \geq \gamma.
\]

For the second part, suppose \((G, +, A) \models \neg m(\phi) > \gamma\). Then for almost every \( n \), \(((1, 2n + 1), +, A_{2n+1}) \models \neg m(\phi) > \gamma\), and therefore

\[
\frac{|\{ \vec{x} \mid \phi(\vec{x}) \}|}{(2n + 1)|\vec{x}|} \leq \gamma
\]

holds in \(((1, 2n + 1), A_{2n+1})\). But since this holds for almost every \( n \), by transfer,

\[
\frac{|\{ \vec{x} \mid \phi(\vec{x}) \}|}{(2n + 1)|\vec{x}|} \leq \gamma
\]

holds in \((G, +, A)\), so also

\[
\mu(\{ \vec{x} \mid \phi(\vec{x}) \}) = \text{st}(\frac{|\{ \vec{x} \mid \phi(\vec{x}) \}|}{(2n + 1)|\vec{x}|}) \leq \gamma.
\]

\( \square \)

Since both \( m \) and \( \mu \) are additive, this extends immediately to the corresponding integrals.

As a notational convenience, let us write \( \vec{x} \) for a sequence \( x_1, \ldots, x_k \), and \( \vec{x}_i \) for the sequence \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \) and \( \vec{x}_{1:k} \) for the sequence \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \).

Observe that for any \( b \), the function \( a \mapsto a + b \) is a definable bijection, as is \( a \mapsto k \cdot a \) for any integer \( k \). For \( r \leq k \), we define formulas \( A_r \) on \( x_{1:k} \). When \( r < k \), we define

\[
A_r(x_{1:k}) : \Leftrightarrow \sum_{i < k, i \neq r} i \cdot x_i + r \cdot \left( x_k - \sum_{i < k, i \neq r} x_i \right) \in A.
\]

3
We define
\[ A_k(x^k) := \sum_{i<k} i \cdot x_i \in A. \]

Note that for any \( x^k_{k-1,k}, \mu(\{ x_{k-1} \mid x_k \in A_k \}) = \mu(A) \), so by the Fubini property of these measures, \( \mu^{k-1}(A_k) = \mu(A) \).

Define \( \sigma(\vec{a}^k) := \sum_{i<k} x_i \); it is easy to see that for any \( \vec{a}^k \), the function \( a_i \mapsto \sigma(\vec{a}^k) \) is a bijection. Whenever \( A_k(\vec{a}^k) \), also \( A_i(\vec{a}^k, \sigma(\vec{a}^k)) \) for each \( i < k \). In particular, if we define \( \hat{A} := \{ \vec{a}^k \mid A_k(\vec{a}^k) \land \forall i < k A_i(\vec{a}^k, \sigma(\vec{a}^k)) \} \), we have \( \hat{A} = A_k \), and so \( \mu^{k-1}(\hat{A}) = \mu(A) > 0 \).

In the next section, we will show that, under these conditions, \( \mu^k(\bigcap_{i<k} A_i) > 0 \). First, however, we show that this is enough to prove Szemerédi’s Theorem. If \( \mu^k(\bigcap_{i<k} A_i) > 0 \), we may find \( \vec{a} \in \bigcap_{i<k} A_i \) such that \( a_k \neq \sum_{i<k} a_i \). Setting \( a := \sum_{i<k} i \cdot a_i \) and \( d := a_k - \sum_{i<k} a_i \), we have \( a + id \in A \) for \( i \in [0, k-1] \). Therefore \( a, a + d, \ldots, a + (k-1)d \) is a \( k \)-term arithmetic progression in \( A \).

This contradicts the construction of the model, which in turn means that the initial assumption that the sets \( A_N \subseteq [1, N] \) exist must fail. Therefore for every \( \delta \), there is an \( N \) such that for every \( A_N \subseteq [1, N] \) with \( |A_N| \geq \delta n \), \( A_N \) contains an arithmetic progression of length \( k \).

### 3 Amalgamation

**Definition 3.** Let \( M \subseteq G \) be a set, let \( n \) be a positive integer, and let \( I \subseteq [1, n] \) be given. We define \( B^n_i(M) \) to be the Boolean algebra of subsets of \( G^n \) of the form
\[ \{(x_1, \ldots, x_n) \mid \phi(\{x_i\}_{i \in I})\} \]
where \( \phi \) is a formula with parameters from \( M \).

We write \( \binom{S}{k} \) for the collection of subsets of \( S \) with cardinality \( k \). When \( k \leq n \), we define \( B^n_{<k}(M) = \bigvee_{I \subseteq \binom{[1,n]}{k}} B^{n,I}(M) \).

We also define \( B^n_{<I}(M) = \bigvee_{J \subseteq \binom{[1,n]}{I}} (M) \).

If \( \mathcal{B} \) is any Boolean algebra, we write \( \mathcal{B}^\sigma \) for the \( \sigma \)-algebra generated by \( \mathcal{B} \).

We equate formulas with the sets they define, so we will also speak of \( \mathcal{B} \) as being a Boolean algebra of formulas.

**Lemma 4.** Let \( \mathcal{B}, \mathcal{B}_0, \mathcal{B}_1 \) be Boolean algebras with \( \mathcal{B} \subseteq \mathcal{B}_0 \cap \mathcal{B}_1 \). Suppose there is an elementary submodel \( M \) such that:

- Every parameter in every formula in \( \mathcal{B}_1 \) belongs to \( M \),
- If \( \phi(\vec{x}, \vec{a}) \in \mathcal{B}_0, \vec{b} \in M, \) and \( |\vec{a}| = |\vec{b}| \), then \( \phi(\vec{x}, \vec{b}) \in \mathcal{B} \).

Then for any \( f \in L^2(\mathcal{B}^\sigma), ||E(f \mid \mathcal{B}^\sigma) - E(f \mid \mathcal{B}_0^\sigma)|| = 0 \).

An illustrating case is when \( \mathcal{B} = B_{n,k}(M), \mathcal{B}_0 = B_{n,k}(M \cup N) \), and \( \mathcal{B}_1 = B_{n,k+1}(M) \).
Proof. Suppose not. Then setting \( \epsilon := ||f - E(f \mid B^c)||_{L^2} \) and \( \delta := ||f - E(f \mid D^c_0)||_{L^2} \), we must have \( \delta < \epsilon \). For some \( \beta_1, \ldots, \beta_m \) and \( A_1(\vec{x}, \vec{b}_1), \ldots, A_m(\vec{x}, \vec{b}_m) \in B_1 \) with each \( \vec{b}_i \in M \), we have \( ||f - \sum_{i \leq m} \beta_i \chi_{A_i}(\vec{x}, \vec{b}_i) ||_{L^2} < (\epsilon - \delta)/4 \).

Since \( ||f - E(f \mid B^c_0)|| = \delta \), there are \( \alpha_1, \ldots, \alpha_n \) and \( D_1(\vec{x}, \vec{a}_1), \ldots, D_n(\vec{x}, \vec{a}_n) \in D_0 \) with \( ||f - \sum_{i \leq n} \alpha_i \chi_{D_i}(\vec{x}, \vec{b}_i) ||_{L^2} < \epsilon - (\epsilon - \delta)/2 \), and therefore

\[
\| \sum_{i \leq m} \beta_i \chi_{A_i}(\vec{x}, \vec{b}_i) - \sum_{i \leq n} \alpha_i \chi_{D_i}(\vec{x}, \vec{b}_i) \|_{L^2} < \epsilon - 3(\epsilon - \delta)/4.
\]

This means the formula

\[
\exists \vec{y}_1, \ldots, \vec{y}_m - (\| \sum_{i \leq m} \beta_i \chi_{A_i}(\vec{x}, \vec{b}_i) - \sum_{i \leq n} \alpha_i \chi_{D_i}(\vec{x}, \vec{b}_i) \|^2_{L^2} > [\epsilon - 3(\epsilon - \delta)/4]^2)
\]

is satisfied (where, to view this as a formula, we expand the norm into an integral of sums of definable formulas).

This is a formula with parameters from \( M \), so by the elementarity of \( M \), there are witnesses \( \vec{a}_1', \ldots, \vec{a}_n' \) in \( M \) satisfying:

\[
\| \sum_{i \leq m} \beta_i \chi_{A_i}(\vec{x}, \vec{b}_i) - \sum_{i \leq n} \alpha_i \chi_{D_i}(\vec{x}, \vec{a}_i') \|_{L^2} \leq \epsilon - 3(\epsilon - \delta)/4.
\]

It follows that \( ||f - \sum_{i \leq n} \alpha_i \chi_{D_i}(\vec{x}, \vec{a}_i') ||_{L^2} < \epsilon \), and since \( \sum_{i \leq n} \alpha_i \chi_{D_i}(\vec{x}, \vec{a}_i') \) is measurable with respect to \( D^c \), this contradicts the assumption that \( ||f - E(f \mid D^c) ||_{L^2} = \epsilon \).

\[\Box\]

**Theorem 5.** Let \( m \leq k \) and suppose that for each each \( I \subseteq \{1, n\} \) with \( |I| = k \), we have a set \( A_I \in B^c_{n, I}(M) \), and suppose there is a \( \delta > 0 \) such that whenever \( B_I \in B_{n, I}(M) \) and \( \mu^n(A_I \setminus B_I) < \delta \) for all \( I \in \{1, n\} \), \( \cap_{I \in \{1, n\}} B_I \) is non-empty. Then \( \mu^n(\cap_{I \in \{1, n\}} A_I) > 0 \).

**Proof.** We proceed by main induction on \( k \). When \( k = 1 \), the claim is trivial: we must have \( \mu(A_I) > 0 \) for all \( I \), since otherwise we could take \( B_I = \emptyset \); then \( \mu(\cap A_I) = \prod \mu(A_I) > 0 \). So we assume that \( k > 1 \) and that whenever \( B_I \in B_{n, I}(M) \) and \( \mu^n(A_I \setminus B_I) < \delta \) for all \( I \), \( \cap_{I \in \{1, n\}} B_I \) is non-empty. Throughout this proof, the variable \( I \) ranges over \( \{1, n\} \).

**Claim 1.** For any \( I_0 \),

\[
\int (\chi_{A_{I_0}} - E(\chi_{A_{I_0}} \mid B^c_{n, < I_0}(M))) \prod_{I \neq I_0} \chi_{A_I} d\mu^n = 0.
\]

**Proof.** When \( k = n \), this is trivial since \( \prod_{I \neq I_0} \chi_{A_I} \) is an empty product, and therefore equal to 1.
If \( k < n \), we have

\[
\int (\chi_{A_{i_0}} - E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M))) \prod_{I \neq I_0} \chi_{A_I} d\mu^n
=\int (\chi_{A_{i_0}} - E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M))) \prod_{I \neq I_0} \chi_{A_I} d\mu^k(\{x_i\}_{i \in I_0}) d\mu^{n-k}(\{x_i\}_{i \notin I_0}).
\]

Observe that for any choice of \( \{a_i\}_{i \notin I_0}, \mathcal{B}_{n,<I_0}(M), \mathcal{B}_{n,<I_0}(M \cup \{a_i\}_{i \notin I_0}), \mathcal{B}_{n,I_0}(M) \) satisfy the preceding lemma, so

\[
\|E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M)) - E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M \cup \{a_i\}_{i \notin I_0})\|_{L^2} = 0.
\]

The function \( \prod_{I \neq I_0} \chi_{A_i}(\{x_i\}_{i \in I_0}, \{a_i\}_{i \notin I_0}) \) is measurable with respect to \( \mathcal{B}^*_{n,<I_0}(M \cup \{a_i\}) \). Combining these two facts, we have

\[
\int (\chi_{A_{i_0}} - E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M))) \prod_{I \neq I_0} \chi_{A_I} d\mu^k(\{x_i\}_{i \in I_0})
=\int (\chi_{A_{i_0}} - E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M \cup \{a_i\}_{i \notin I_0})) \prod_{I \neq I_0} \chi_{A_I} d\mu^k(\{x_i\})
=0
\]

Since this holds for any \( \{a_i\}_{i \notin I_0} \), the claim follows by integrating over all choices of \( \{a_i\} \).

\[\square\]

**Claim 2.** For any \( I_0 \), there is an \( A'_{i_0} \in \mathcal{B}^*_{n,<I_0}(M) \) such that:

- Whenever \( B_I \in \mathcal{B}_{n,1}(M) \) for each \( I, \mu^n(A_I \setminus B_I) < \delta \) for each \( I \neq I_0 \), and \( \mu^n(A'_{i_0} \setminus B_{I_0}) < \delta, \bigcap_{I \in \{1, \ldots, n\}} B_I \) is non-empty, and

- If \( \mu^n(A'_{i_0} \cap \bigcap_{I \neq I_0} A_I) > 0, \mu^n(\bigcap A_I) > 0 \).

**Proof.** Define \( A'_{i_0} := \{x_i \mid E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M))(x_i) > 0\}. \) If \( \mu^n(A'_{i_0} \cap \bigcap_{I \neq I_0} A_I) > 0 \) then we have

\[
\int E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M)) \prod_{I \neq I_0} \chi_{A_I} d\mu^n > 0
\]

and by the previous claim, this implies that \( \mu(\bigcap A_I) > 0 \).

Suppose that for each \( I, B_I \in \mathcal{B}_{n,1}(M) \) with \( \mu^n(A_I \setminus B_I) < \delta \) for \( I \neq I_0 \) and \( \mu^n(A'_{i_0} \setminus B_{I_0}) < \delta \). Since

\[
\mu^n(A_{i_0} \setminus A'_{i_0}) = \int \chi_{A_{i_0}}(1 - \chi_{A'_{i_0}}) d\mu^n = \int E(\chi_{A_{i_0}} \mid \mathcal{B}^*_{n,<I_0}(M))(1 - \chi_{A'_{i_0}}) d\mu^n = 0,
\]

we have \( \mu^n(A_{i_0} \setminus B_{I_0}) < \delta \) as well, and therefore \( \bigcap B_I \) is non-empty.

\[\square\]
By applying the previous claim to each \( I \in \binom{[1,n]}{k} \), we may assume for the rest of the proof that for each \( I, A_I \in \mathcal{B}_{n,k-1}(M) \).

Fix some finite algebra \( \mathcal{B} \subseteq \mathcal{B}_{n,k-1}(M) \) so that for every \( I \), \( ||\chi_{A_I} - E(\chi_{A_I} \mid \mathcal{B})||_{L^2} < \sqrt{\frac{\delta}{2(k+1)}} \) (such a \( \mathcal{B} \) exists because there are finitely many \( I \) and each \( A_I \) is \( \mathcal{B}_{n,k-1}(M) \)-measurable). For each \( I \), set \( A_I^* = \{ \bar{a}_I \mid E(\chi_{A_I} \mid \mathcal{B})(\bar{a}) > \frac{\delta}{(k+1)} \} \).

**Claim 3.** For each \( I, \mu(A_I \setminus A_I^*) \leq \delta/2 \)

*Proof.* \( A_I \setminus A_I^* \) is the set of points such that \( \chi_{A_I} - E(\chi_{A_I} \mid \mathcal{B}) > \frac{\delta}{(k+1)} \).

By Chebyshev’s inequality, the measure of this set is at most

\[
\left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) \frac{1}{(k+1)} \int (\chi_{A_I} - E(\chi_{A_I} \mid \mathcal{B}))^2 d\mu = \left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) \frac{1}{(k+1)} ||\chi_{A_I} - E(\chi_{A_I} \mid \mathcal{B})||_{L^2}^2 < \frac{\delta}{2} \tag{\text{Chebyshev’s inequality}}
\]

**Claim 4.** \( \mu(\bigcap_I A_I) \geq \mu(\bigcap_I A_I^*) / \left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) + 1 \).

*Proof.* For each \( I_0 \),

\[
\mu((A_{I_0}^* \setminus A_{I_0}) \cap \bigcap_{I \neq I_0} A_I^*) = \int \chi_{A_{I_0}^*} (1 - \chi_{A_{I_0}}) \prod_{I \neq I_0} \chi_{A_I^*} d\mu^n
= \int \chi_{A_{I_0}^*} (1 - E(\chi_{A_{I_0}} \mid \mathcal{B})) \prod_{I \neq I_0} \chi_{A_I^*} d\mu^n
\leq \frac{1}{\left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) + 1} \int \prod_{I} \chi_{A_I^*} d\mu^n
= \frac{1}{\left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) + 1} \mu(\bigcap I A_I^*)
\]

But then

\[
\mu(\bigcap_I A_I^* \setminus \bigcap_I A_I) \leq \sum_{I_0} \mu((A_{I_0}^* \setminus A_{I_0}) \cap \bigcap_{I \neq I_0} A_I^*) \leq \frac{\left( \begin{array}{c}
\frac{n}{k}
\end{array} \right)}{\left( \begin{array}{c}
\frac{n}{k}
\end{array} \right) + 1} \mu(\bigcap_I A_I^*).
\]

Each \( A_I^* \) may be written in the form \( \bigcup_{i \leq r_I} A_{I,i}^* \) where \( A_{I,i}^* = \bigcap_{J \in \binom{[1,n]}{k-1}} A_{I,i,J}^* \) and \( A_{I,i,J}^* \) is an element of \( \mathcal{B}_{n,j}(M) \). We may assume that if \( i \neq i' \) then \( A_{I,i}^* \cap A_{I,i'}^* = \emptyset \).

We have

\[
\mu(\bigcap_I A_I^*) = \mu(\bigcup_{I \subseteq [1,r_I]} \bigcap_{J \in \binom{[1,n]}{k-1}} A_{I,i,J}^*)
\]

For each \( \bar{I} \in \prod_{I \subseteq [1,r_I]} \), let \( D_{\bar{I}} = \bigcap_I \bigcap_{J \in \binom{[1,n]}{k-1}} A_{I,i,J}^* \). Each \( A_{I,i,J}^* \) is an element of \( \mathcal{B}_{n,j}(M) \), so we may group the components and write \( D_{\bar{I}} = \bigcap_{J \in \binom{[1,n]}{k-1}} D_{I,J} \). For each \( \bar{J} \in \prod_{I \subseteq [1,r_I]} \), the set \( D_{\bar{J}} = \bigcap_{I \supseteq J} A_{I,i,J}^* \).
Suppose, for a contradiction, that \( \mu(\bigcap I^*_{A}) = 0 \). Then for every \( \vec{i} \in \prod_{I}[1,r_I] \), \( \mu(D_{\vec{i}}) = \mu(\bigcap I^*_{D_{\vec{i},I}}) = 0 \). By the inductive hypothesis, for each \( \gamma > 0 \), there is a collection \( B_{\vec{i},J} \in B_{n,J}(M) \) such that \( \mu(D_{\vec{i},J} \setminus B_{\vec{i},J}) < \gamma \) and \( \bigcap J B_{\vec{i},J} = 0 \). In particular, this holds with \( \gamma = \frac{\delta}{2(k-1)(\prod_{I}r_I)(\max_I r_I)} \).

For each \( I, i \leq r_I, J \subset I \), define

\[
B^*_{i,J} = \bigcap_{\vec{i},i,J} \left( B^*_{i,J} \cup \bigcup_{I' \supseteq J, I' \neq I} A^*_{I',i,J} \right).
\]

**Claim 5.** \( \mu(A_{i,J}^* \setminus B^*_{i,J}) \leq \frac{\delta}{2(k-1)(\max_I r_I)} \).

**Proof.** Observe that if \( x \in A_{i,J}^* \setminus B^*_{i,J} \), then for some \( \vec{i} \) with \( i_{\vec{i}} = i \), \( x \notin B^*_{i,J} \cup \bigcup_{I' \supseteq J, I' \neq I} A^*_{I',i,J} \). This means \( x \notin B^*_{i,J} \) and \( x \in \bigcap_{I' \supseteq J} A^*_{I',i,J} = D_{\vec{i},J} \). So

\[
\mu(A_{i,J}^* \setminus B^*_{i,J}) \leq \sum_{\vec{i} \in \prod_{I}[1,r_I]} \mu(D_{\vec{i},J} \setminus B_{\vec{i},J}) \leq \frac{\delta}{2(k-1)(\max_I r_I)}.
\]

Define \( B^*_I = \bigcup_{i \leq r_I} \bigcap_{J} B^*_{i,J} \).

**Claim 6.** \( \mu(A_I \setminus B^*_I) < \delta \).

**Proof.** Since \( \mu(A_I \setminus A^*_I) < \delta/2 \), it suffices to show that \( \mu(A_I \setminus B^*_I) < \delta/2 \).

\[
\mu(A_I \setminus \bigcup B^*_{i,J}) = \mu\left( \bigcup_{i \leq r_I} \left( A^*_{i,J} \setminus B^*_{i,J} \right) \right) \\
\leq \sum_{i \leq r_I} \mu\left( \bigcap_j A^*_{i,J} \setminus B^*_{i,J} \right) \\
\leq \sum_{i \leq r_I} \sum_j \mu(A^*_{i,J} \setminus B^*_{i,J}) \\
\leq r_I \cdot \left( \begin{array}{c} k \\ k-1 \end{array} \right) \cdot \frac{\delta}{2(k-1)(\max_I r_I)} \\
\leq \delta/2
\]

**Claim 7.** \( \bigcap_I B^*_I \subseteq \bigcup_{i,J} B_{i,J} \).
Proof. Suppose $x \in \bigcap_I B^*_I = \bigcap_I \bigcup_{i \leq r_I} \bigcap_J B^*_{I,i,J}$. Then for each $I$, there is an $i_I \leq r_I$ such that $x \in \bigcap_J B^*_{I,i_I,J}$. Since $B^*_{I,i,J} \subseteq A^*_{I,i,J}$, for each $I$ and $J \subset I$, $x \in A^*_{I,i,J}$.

For any $J$, let $I \supset J$. Then

$$x \in B^*_{I,i_I,J} = A^*_{I,i_I,J} \cap \bigcap_{i_I' = i_I} \left[ B^*_{I,I,i_I'} \cup \bigcup_{I' \supset J, I' \neq I} A^*_{I',i_I',J} \right].$$

In particular, $x \in \left[ B^*_{I,i_I,J} \cup \bigcup_{I' \supset J, I' \neq I} (A^*_{I',i_I',J}) \right]$ for the particular $i_I$ we have chosen.

Since $x \in A^*_{I,i_I,J}$ for each $I' \supset J$, it must be that $x \in B^*_{I,i_I,J}$. This holds for any $J$, so $x \in \bigcap_J B^*_{I,i_I,J}$.

From our assumption, $\bigcap_I B^*_I$ is non-empty, and therefore there is some $i_I$ such that $\bigcap_J B^*_{I,i_I,J}$. But this leads to a contradiction, so it must be that $\mu(\bigcap_I A^*_I) > 0$, and therefore, as we have shown, $\mu(\bigcap_I A^*_I) \geq \frac{1}{(1 + 1)} \mu(\bigcap_I A^*_I) > 0$. $\square$

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