Double Extended Cubic Peakon Equation

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Abstract

The Hamiltonian structure for the supersymmetric $N = 2$ Novikov equation is presented. The bosonic sector give us two-component generalization of the cubic peakon equation. The double extended: two-component and two-peakon Novikov equation is defined. The Bi-Hamiltonian structure for this extended system is constructed.

1 Introduction

Recently a family of equations of the form [1 - 7]

\[ u_t - u_{xxx} = \frac{1}{2} \left( - (b + 1)u^2 + 2uu_{xx} + (b - 1)u_x^2 \right)_x \]  

(1)

has been investigated in the literature.

When $b = 2$ Esq.(1) reduces to the Camassa - Holm

\[ u_t - u_{xxx} = \frac{1}{2} \left( - 3u^2 + 2uu_{xx} + u_x^2 \right)_x \]  

(2)

equation, which describes a special approximation of shallow water theory. This equation shares most of the important properties of an integrable system of KdV type, for example, the existence of Lax pair formalism, the Bi-Hamiltonian structure, the multi-solitons solutions. Moreover this equation admits peaked solitary wave solutions.

Degasperis and Procesi showed that the Esq. (1) is integrable also for the $b = 3$ case. The Degasperis - Procesi equation

\[ u_t - u_{xxx} = (-2u^2 + uu_{xx} + u_x^2)_x \]  

(3)
can be considered as a model for shallow-water dynamics also and found to be completely integrable. Similarly to the Camassa-Holm case the Degasperis-Procesi equation has the Lax pair and admits peakon dynamics also.

The peakon equation have been generalized to the so called cubic peakon equation by Novikov [8] and one of them is

\[ m_t + u^2 m_x + 3u u_x m = 0, \quad m = u - u_{xx}. \]

Hone and Wang [9] proposed a Lax representation for the equation and found the Bi-Hamiltonian structure and infinitely many conserved quantities.

The Camassa-Holm, Degasperis-Procesi and Novikov equations have been generalized to the multi-component case in different manners. For example the energy-dependent Schrödinger spectral problem for Camassa-Holm equation can be formulated with the help of Lax operator as

\[ \Psi_{xx} = \left( \frac{1}{4} - \lambda m \right) \Psi \]
\[ \Psi_t = -\left( \frac{1}{2} \lambda + u \right) \Psi_x + \frac{1}{2} u_x \Psi \]

Compatibility condition for the above system, yields two independent equations

\[ m_t = -2mu_x - m_x u, \quad m = u - u_{xx} \]  

Considering now the class of the two-component Schrödinger equation [10, 11]

\[ \Psi_{xx} = \left( \frac{1}{4} - \lambda m + \lambda^2 \rho^2 \right) \Psi \]

the compatibility condition \( \Psi_{x,t} = \Psi_{t,x} \), where \( \Psi_t \) is the same as in [5] yields equation

\[ \rho_t = -(u \rho)_t, \]
\[ m_t = -2mu_x - m_x u + \rho \rho_x \]

This equation is known as the two-component generalization of the Camassa-Holm equation.

On the other side the name multi-component generalization of Camassa-Holm equation [10, 13] have been used also to the equations, which are reduced, for each component, to the single Camassa-Holm equation. Therefore it is reasonable to introduce different names on different types of the extensions of the peakon equations. We will use the name the multi-peakon generalizations on the mentioned extension while for the other types the multi-component generalization.

A two-peakon generalization of the Novikov equation was constructed by Geng and Xue [7].

\[ m_t + 3u_x v m + u v m_x = 0, \]
\[ n_t + 3v_x u n + u v n_x = 0, \]
\[ m = u - u_{xx}, \quad n = v - v_{xx}. \]

They calculated the \( N \)-peakons and conserved quantities and found a Hamiltonian structure. The Bi-Hamiltonian structure have been found by Li and Liu [14].
Other types of the cubic peakon equation have been extended to the multi-peakon equation [15] also.

The Degasperis-Procesi equation have been generalized to the two-component [16] case using the supersymmetric technique but it appeared that it has one-Hamiltonian structure only. It is possible to obtain the two-component Camassa-Holm equation [17] using the supersymmetric approach.

The aim of the present paper is twofold. The first is to construct the two-component Novikov equation, the second is to builds its two-peakon extension. This type of equations we will name as double extended cubic peakon equations.

In order to defined two-component Novikov equation we use, similarly to the Degasperis-Procesi equation, the extended $N=2$ supersymmetric technique. The first step is to supersymmetrize the second Hamiltonian structure of the Novikov equation and next to obtain the supersymmetric analog of the Novikov equation. The so called bosonic sector of this supersymmetric extension gives us the two-component generalization of the Novikov equation which has one-Hamiltonian structure. As we show this equation could be obtained without using the supersymmetric methods, it is enough to explore the decompression technique [16]. The idea of decompression based on the utilising the larger Hamiltonian operator for which the verification of the Jacobi identity is easy. Next apply the Dirac reduction to this Hamiltonian operator and use this new operator to the construction of the equation of motion.

However we have been not able to defined the first Hamiltonian structure for the two-component Novikov equation. On the other side Li and Liu [14] have obtained the first Hamiltonian structure for the Novikov equation [1] using the Bi-Hamiltonian structure of the two-peakon generalization of the Novikov equation [9]. More precisely Li and Liu considered the operator $KL^{-1}K$ where $L$ and $K$ are the first and second Hamiltonian operator of the two-peakon Novikov equation respectively. They applied the Dirac reduction procedure to this operator and defined in such a manner the first Hamiltonian operator of the Novikov equation.

We apply the decompression technique to the two-peakon Novikov equation and as the result we obtained double extended Novikov equation, two-peakon and two-component. As we show this system is Bi-Hamiltonian system and can be reduced to the two-peakon Novikov equation. This twice generalized equation can not be reduced to the two-component Novikov equation and therefore we can not apply the approach proposed by Li and Liu for the generations of the first Hamiltonian structure for this system.

The paper is organised as follows. The first section contains the supersymmetric approach in which we also explain the decompression technique. In the second section we constructed the doubly extended Novikov equation and presented its Bi-Hamiltonian structure. The last section contains concluding remarks. In the appendix we presented main tricks used in the computations of the Jacobi identity for the second Hamiltonian operator of the doubly extended Novikov equation.
2 Supersymmetric Novikov Equation

We will use the extended $N = 2$ supersymmetric formalism which allows us to consider the supersymmetric analog of the second Hamiltonian of the Novikov equation.

Here we will use the supersymmetric algebra of super-derivatives where

$$D_1 = \frac{\partial}{\partial \theta_1} - \frac{1}{2} \theta_2 \partial_x, \quad D_2 = \frac{\partial}{\partial \theta_2} - \frac{1}{2} \theta_1 \partial_x,$$

$$\{D_1, D_2\} = -\partial_x, \quad D_3 = [D_1, D_2] = \partial_x + 2D_1 D_2, \quad D_1^2 = D_2^2 = 0$$

Now let us consider the following Hamiltonian operator

$$J = \left( \begin{array}{ccc}
-D_3 \partial_x & 2 \partial_x W & 2(D_1 W)D_2 + 2(D_2 W)D_1 \\
\partial_x V + V_x & 2(D_1 V)D_2 + 2(D_2 V)D_1 & 0
\end{array} \right)$$

where $W, V$ are the $N = 2$ supersymmetric bosonic functions.

The super functions $W, V$ can be thought as the $N = 2$ supermultiplets, for example $W(x, t, \theta_1, \theta_2) = w_0 + \theta_1 \chi_1 + \theta_2 \chi_2 + \theta_1 \theta_2 w_1$ where $w_0(x, t), w_1(x, t)$ are the classical functions, $\chi_1(x, t), \chi_2(x, t)$ are Grassman valued functions and $\theta_1, \theta_2$ are the Majorana spinors.

It is easy to check that this operator satisfy the Jacobi identity.

Let us briefly explain the standard Dirac reduction formula [18]. Let $U, V$ be two linear spaces with the coordinates $u$ and $v$. Let

$$P(u, v) = \left( \begin{array}{cc}
P_{u,u} & P_{u,v} \\
P_{v,u} & P_{v,v}
\end{array} \right)$$

be a Poisson tensor on $U \oplus V$. Assume that $P_{v,v}$ is invertible, then

$$P = P_{u,u} - P_{u,v} P_{v,v}^{-1} P_{v,u}$$

is a Poisson tensor on $U$.

Performing the Dirac reduction for $J$ when $W = 1$ and taking into an account that

$$(-D_3 + 2)^{-1} = (4 - \partial_{xx})^{-1}(2 + D_3)$$

we obtained new Hamiltonian operator

$$K = [\partial_x V + V_x + 2(D_1 V)D_2 + 2(D_2 V)D_1] \quad [4 \partial_x - \partial_{xxx})^{-1}(2 + D_3)$$

$$[\partial_x V + 2(D_1 V)D_2 + 2(D_2 V)D_1]$$

which satisfy the Jacobi identity.

Let us now parametrise the superfunction $V$ as $V = (1 - D_3)A$ where $A$ is a new supersymmetric function and consider the following equation

$$V_t = K \frac{\delta H}{\delta V}$$

Choosing $H = \frac{1}{2} \int dxd\theta_1 d\theta_2 VA$ we obtain

$$V_t = V_x A^2 + VA_x A + (D_2 V)(D_1 A^2) + (D_1 V)(D_2 A^2)$$
It is our supersymmetric extension of the Novikov equation. In order to have the connection with the classical Novikov equation let us compute the bosonic sector of

\[
V = v_0 + \theta_2 \theta_1 v_1, \quad A = u + \theta_2 \theta_1 a_1, \quad v_0 = u - 2a_1, \quad v_1 = a_1 - \frac{1}{2} u_{xx}, \quad a_1 = \frac{1}{2} (u - \rho), \quad v_1 = \frac{1}{2} (u - u_{xx} - \rho) = \frac{1}{2} (m - \rho).
\]

As the result we obtained

\[
\begin{align*}
\rho_t &= \rho_x u^2 + \rho u u_x, \\
m_t &= 3u_x u m + u^2 m_x - \rho(u \rho)_x.
\end{align*}
\]

It is the two-component generalization of the Novikov equation. This can be rewritten in the Hamiltonian form as

\[
\left( \begin{array}{c} \rho \\ m \end{array} \right)_t = \hat{K} \left( \begin{array}{c} \delta H \\ \delta m \end{array} \right)
\]

where \( \hat{K} \) is a bosonic part of the operator \( K \) and

\[
H = \frac{1}{2} \int dx (mu - \rho^2)
\]

\[
\hat{K} = \begin{pmatrix}
\rho^{-1} \partial \rho^2 \mathcal{L}^{-1} \rho^2 \partial \rho^{-1} & 3 \rho^{-1} \partial \rho^2 \mathcal{L}^{-1} m^{1/3} \partial m^{2/3} \\
3 m^{2/3} \partial \mathcal{L}^{-1} \rho^2 \partial \rho^{-1} & -\rho \partial \rho + 9 m^{2/3} \partial m^{1/3} \mathcal{L}^{-1} m^{1/3} \partial m^{2/3}
\end{pmatrix}
\]

where \( \mathcal{L} = \partial^3 - 4 \partial_x \).

Let us notice that \( \hat{K} \) operator is a Dirac reduced version of the following Hamiltonian operator

\[
\begin{pmatrix}
\partial_{xxx} - 4u \partial - 2u_x & \rho_x - \rho \partial_x & -m_x - 3m \partial \\
-\rho \partial_x - 2 \rho_x & 0 & 0 \\
-2m_x - 3m \partial_x & 0 & -\rho \partial_x \rho
\end{pmatrix}
\]

when \( u = 1 \). Now it is easy to verify that this operator satisfy the Jacobi identity and hence \( \hat{K} \) satisfy Jacobi identity as well. It is our decomposition idea.

We have been not able to find the first Hamiltonian structure for this two-component Novikov equation. On the other side Li and Liu have obtained the first Hamiltonian structure for the Novikov equation using the Bi-Hamiltonian structure of the two-component Novikov equation. In the next section we explain why this idea does not work in our case.

### 3 Double Extended Novikov Equation

Our aim is to construct the Bi-Hamiltonian structure for the two-component generalizations of the two-pekon Novikov equation [9]. To end this let us notice
that Li and Liu \[14\] defined the second Hamiltonian operator for the two-peakon generalization of the Novikov equation \[9\] as

\[
\frac{1}{2} \left( 3m\partial + 2m_x \; , \; 3n\partial + 2n_x \right)^T \hat{\mathcal{L}}^{-1} \left( 3m\partial + m_x \; , \; 3n\partial + n_x \right)
\]

This operator is a Dirac reduced version of the following operator

\[
\frac{1}{2} \left( \begin{array}{cccc}
-\partial_{xxx} + 4u\partial_x + 2u_x & 3m^{2/3}\partial_x m^{1/3} & 3n^{2/3}\partial_x n^{1/3} \\
3m^{1/3}\partial_x m^{2/3} & 3m\partial^{-1}m & -3m\partial^{-1}n \\
3n^{1/3}\partial_x n^{2/3} & -3n\partial^{-1}m & 3n\partial^{-1}n
\end{array} \right)
\]

when \(u = 1\) and satisfy the Jacobi identity.

We would like to generalize this operator to the higher dimensional case. The supersymmetric approach is less useful for this aim, because we do not know how to find the supersymmetric counterpart of the nonlocal part of the operator \[20\]. Therefore we consider the generalizations of the above operator to the five dimensional matrix operator as

\[
\mathcal{J} = \frac{1}{2} \left( \begin{array}{cccccc}
-\partial_{xxx} + 4u\partial_x + 2u_x & \rho_1^2\partial\rho_1^{-1} & 3m^{2/3}\partial_x m^{1/3} & \rho_2^2\partial_x\rho_2^{-1} & 3m_2^{2/3}\partial_x m_2^{1/3} \\
\rho_1^{-1}\partial\rho_1^2 & 3m_1^{1/3}\partial_x m_1^{2/3} & 0 & \mathcal{J}_{3,3} & 0 & \mathcal{J}_{3,5} \\
3m_1^{1/3}\partial_x m_1^{2/3} & \rho_2^{-1}\partial\rho_2^2 & 0 & 0 & 0 & 0 \\
3m_2^{1/3}\partial_x m_2^{2/3} & 0 & -\mathcal{J}_{3,5}^* & 0 & \mathcal{J}_{3,5}
\end{array} \right)
\]

where \(u, m_i, \rho_i, i = 1, 2\) are the function of \(t, x\) and

\[
\begin{align*}
\mathcal{J}_{3,3} &= \lambda_1 m_1 \partial^{-1} m_1 + \lambda_2 m_2 \partial^{-1} m_2 + \lambda_3 (m_1 \partial^{-1} m_2 + m_2 \partial^{-1} m_1) \\
&\quad + k_{1i} \rho_1 \partial \rho_1 + k_{2i} \rho_2 \partial \rho_2 + k_{3} (\rho_1 \partial \rho_2 + \rho_2 \partial \rho_1). \\
\mathcal{J}_{3,5} &= \lambda_4 m_1 \partial^{-1} m_1 + \lambda_5 m_2 \partial^{-1} m_2 + \lambda_6 m_1 \partial^{-1} m_2 + \lambda_7 m_2 \partial^{-1} m_1 \\
&\quad + k_{4i} \rho_2 \partial + k_{5i} \rho_1 \partial \rho_2 + k_{6i} \partial \rho_1 \rho_2 + k_{7i} \rho_2^2 \partial + k_{8i} \rho_1 \rho_2 \partial \\
&\quad + k_{9i} \rho_1 \partial \rho_1 + k_{10i} \rho_2 \partial \rho_2, \\
\mathcal{J}_{5,5} &= \lambda_8 m_1 \partial^{-1} m_1 + \lambda_9 m_2 \partial^{-1} m_2 + \lambda_{10} (m_1 \partial^{-1} m_2 + m_2 \partial^{-1} m_1) \\
&\quad + k_{11i} \partial \rho_1 + k_{12i} \partial \rho_2 + k_{13i} (\rho_1 \partial \rho_2 + \rho_2 \partial \rho_1).
\end{align*}
\]

and \(\lambda_i, k_i\) are an arbitrary constants.

We fixed some of the \(\lambda\) constants, before verifying the Jacobi identity, in such a way that the Dirac reduced version of the operator \(\mathcal{J}\) when \(u = 1\) \(\mathcal{K} = \mathcal{J}|_{u=1}\) produces the local equation of motion for

\[
\begin{pmatrix}
\rho_1 \\
m_1 \\
\rho_2 \\
m_2
\end{pmatrix}_t = \mathcal{K} 
\begin{pmatrix}
H_{1,\rho_1} \\
H_{1,m_1} \\
H_{1,\rho_2} \\
H_{1,m_2}
\end{pmatrix}
\]

\[
H_1 = \frac{1}{2} \int dx \left( m_1 u_1 + m_2 u_2 + 2\rho_1^2 + 2\rho_2^2 \right)
\]
This assumption gives us $\lambda_1 = \lambda_2 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_9 = \lambda_{10} = 0$.

As we checked, see appendix for the details, the Jacobi identity holds if

$$
\begin{align*}
J_{3,3} &= s_0 m_2 \partial^{-1} m_2 + \rho_1 \partial (s_1 \rho_1 + s_2 \rho_2) + \rho_2 \partial (s_2 \rho_1 + s_3 \rho_2) \\
J_{3,5} &= -s_0 m_2 \partial^{-1} m_1 + s_1 (\rho_1 \partial \rho_2 - \rho_2 \partial \rho_1) \\
J_{5,5} &= s_0 m_1 \partial^{-1} m_1 + \rho_1 \partial (s_1 \rho_1 + s_2 \rho_2) + \rho_2 \partial (s_2 \rho_1 + s_3 \rho_2)
\end{align*}
$$

where now $s_i$ are an arbitrary constants.

In order to fix the constants $s_i$ we postulate that the system (22) is the Bi-Hamiltonian. Our result is that for $s_0 = 3, s_4 = 1, s_1 = s_2 = s_3 = 0$ it is possible to construct the following Bi-Hamiltonian structure

$$
\begin{pmatrix}
\rho_1 \\
\rho_2 \\
\rho_3
\end{pmatrix}_t = \mathcal{L} \begin{pmatrix}
H_{0, \rho_1} \\
H_{0, \rho_2} \\
H_{0, m_2}
\end{pmatrix} = \mathcal{K} \begin{pmatrix}
H_{1, \rho_1} \\
H_{1, \rho_2} \\
H_{1, m_2}
\end{pmatrix}.
$$

(24)

where

$$
H_0 = -\int dx \left( m_1 (u_{2,x} u_1^2 - u_{1,x} u_1 u_2) + m_2 (u_{2,x} u_1 u_2 - u_{1,x} u_2^2) + \rho_1 \rho_2 (u_2 u_{2,x} + u_1 u_{1,x}) + \rho_2 \rho_1 (u_1^2 + u_2^2) \right)
$$

(25)

$$
\mathcal{L} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 - \partial_{xx} \\
-1 & 0 & 0 & 0 \\
0 & -1 + \partial_{xx} & 0 & 0
\end{pmatrix}
$$

(26)

$$
\mathcal{K} = -\frac{1}{2} \begin{pmatrix}
2 \rho_{1,x} + \rho_1 \partial & 2m_1 \partial & 2 \rho_{2,x} + \rho_2 \partial & 2m_2 \partial
\end{pmatrix} (\partial_{xxx} - 4 \partial_x)^{-1} \begin{pmatrix}
-\rho_{1,x} + \rho_1 \partial \\
\rho_{1,x} + 3m_1 \partial \\
-\rho_{2,x} + \rho_2 \partial \\
\rho_{2,x} + 3m_2 \partial
\end{pmatrix}^T + 
$$

(27)

The system of equation (23) becomes

$$
\begin{align*}
\rho_{1,t} &= \rho_{1,x}(u_1^2 + u_2^2) + \rho_1 (u_{1,x} u_1 + u_{2,x} u_2) \\
m_{1,t} &= [m_1 (u_1^2 + u_2^2)]_x + m_1 (u_{1,x} u_1 + u_{2,x} u_2) - 3m_2 (u_{2,x} u_1 - u_{1,x} u_2) + u_2 (\rho_2 \rho_{1,x} - \rho_1 \rho_{2,x}) \\
\rho_{2,t} &= \rho_{2,x}(u_1^2 + u_2^2) + \rho_2 (u_{1,x} u_1 + u_{2,x} u_2) \\
m_{2,t} &= [m_2 (u_1^2 + u_2^2)]_x + m_2 (u_{1,x} u_1 + u_{2,x} u_2) + 3m_1 (u_{2,x} u_1 - u_{1,x} u_2) + u_1 (\rho_1 \rho_{2,x} - \rho_2 \rho_{1,x})
\end{align*}
$$

(28)

\textbf{7}
It is our double extended cubic peakon equation.

We checked using the symbolic computer program that the Hamiltonian operators $\mathcal{K}$ and $\mathcal{L}$ are compatible, that is we verified that

$$\int dx \ f(\mathcal{K}'[\mathcal{L}g] + \mathcal{L}'[\mathcal{K}g])h + c.p. = 0 \quad (29)$$

where $f, g, h$ are the test function while $\mathcal{K}'[\mathcal{L}g]$ denotes the Gateaux derivative along $\mathcal{L}g$.

It is impossible to reduce double extended cubic peakon equation to the two-component Novikov equation. Notice that when $\rho_1 = \rho_2, m_1 = m_2$ we obtain the decoupled system of equation. It is a reason that the idea of Li and Liu does not work in our case.

Let us consider the following linear transformation of $m_1, m_2, u_1, u_2, \rho_1, \rho_2$

$$m_1 = i(n_1 - n_2)/2, \quad u_1 = i(v_1 - v_2)/2, \quad \rho_1 = ir_1 \quad (30)$$
$$m_2 = (n_1 + n_2)/2, \quad u_2 = (v_1 + v_2)/2, \quad \rho_2 = r_2$$

Under this transformation our equations are

$$r_{1,t} = \frac{1}{2} r_1 (v_1 v_2)_x + r_{1,x} v_1 v_2 \quad (31)$$
$$n_{1,t} = v_1 v_2 n_{1,x} + 3v_{1,x} v_2 n_1 + v_2 (r_{2} r_{1,x} - r_1 r_{2,x})$$
$$r_{2,t} = \frac{1}{2} r_2 (v_1 v_2)_x + r_{2,x} v_1 v_2$$
$$n_{2,t} = v_1 v_2 n_{2,x} + 3v_{2,x} v_1 n_2 - v_1 (r_2 r_{1,x} - r_1 r_{2,x})$$

When $r_1 = r_2 = 0$ then these equation are reduced to the equations considered by Geng and Xue [7].

4 Conclusion

In this paper we constructed the $N = 2$ supersymmetric Hamiltonian structure for the supersymmetric Novikov equation. The bosonic sector gives us the two-component generalization of the cubic peakon equation. Next we decompressed second Hamiltonian operator of the two-peakon equation to the five dimensional matrix operator. We checked the Jacobi identity for this operator and reduced this operator to the four dimensional matrix operator. This four dimensional matrix operator was used to the construction of the double extended, two-component and two-peakon Novikov equation. The first Hamiltonian structure have been defined also and thus Bi-Hamiltonian structure for this extended system was defined. Moreover these Hamiltonian operators are compatible. This doubly extended Novikov equation, up to our knowledge, is a new Bi-Hamiltonian system. From that reason it is interesting to study it in more details and it is tempting to check whether this system possess the Lax representation.
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5 Appendix

We used symbolic computer algebra technique for the verification of the Jacobi identity.

In order to prove that the operator $J$ satisfy the Jacobi identity we utilize the standard form of the Jacobi identity \[ \int dx A J J B C + p.c. = 0 \] (32)

where $A, B, C$ are the test vector functions as for example $A = (a_1, a_2, a_3, a_4, a_5)$ while $\ast$ denotes the Gateaux derivative along the vector $J B$.

This formula has three typical components.

The first component contains terms in which the integral operator appear twice, the second contains terms in which integral operator appear only once. The last third term does not contain the integral operators.

The first component is constructed as

$$\int dx m_i a_j \partial^{-1} m_k b_s \partial^{-1} m_r c_l + ....$$ (33)

This can be be transformed to

$$\int dx \ m_k b_s (\partial^{-1} m_i a_j)(\partial^{-1} m_r c_l) + ....$$ (34)

Introducing the notation

$$m_i a_j = Z(a, i, j)_x, \quad m_i b_j = Z(b, i, j)_x, \quad m_i c_j = Z(c, i, j)_x$$ (35)

the last formula transforms to

$$\int dx \ Z(b, k, s)_x Z(a, i, j) Z(c, r, l) + ...$$ (36)

Replacing $Z(a, i, j)_x$ by $\partial Z(a, i, j) - Z(a, i, j) \partial$ the first component turn to zero.

The second component is constructed as

$$\int dx \ (W_1 + W_2) \partial^{-1} m_i a_j + m_i a_j \partial^{-1} (V_1 + V_2) + ...$$ (37)
where $W_1$ or $V_1$ are the functions constructed out of \{\rho_1, \rho_2, \rho_{1,x}, \rho_{2,x} b_s, b_{s,x}, c_k, c_{k,x}\} while $W_2$ or $V_2$ are constructed out of \{\nu_i, \nu_{i,x}, c_k, c_{k,x}, b_s, b_{s,k}\} These terms we order as

$$
\int dx (W_1 + W_2) \partial^{-1} m_i a_j - (V_1 + V_2) \partial^{-1} m_i a_j + ...
$$

If we replace $a_k, x$ by $\partial a_k - a_k \partial$ and next $b_k, x$ as $\partial b_k - b_k \partial$ then the second component does not contain the integral operators.

We add just computed second component to the third component and this sum vanishes. This finish the proof.