Perturbation theory in radial quantization approach and the expectation values of exponential fields in sine-Gordon model

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Abstract

A perturbation theory for Massive Thirring Model (MTM) in radial quantization approach is developed. Investigation of the twisted sector in this theory allows us to calculate the vacuum expectation values of exponential fields \( \langle \exp ia\varphi(0) \rangle \) of the sine-Gordon theory in first order over Massive Thirring Models coupling constant. It appears that the apparent difficulty in radial quantization of massive theories, namely the explicite ”time” dependence of the Hamiltonian, may be successfully overcome. The result we have obtained agrees with the exact formula conjectured by Lukyanov and Zamolodchikov and coincides with the analogous calculations recently carried out in dual angular quantization approach by one of the authors.
1 Introduction

It is almost two decades the two dimensional exactly solvable models of QFT attract much attention because of their wide applications in the condensed matter physics and string theory. From the other hand, the knowledge of the exact solutions of such nontrivial interacting theories provide us with better understanding of the general concepts of QFT. While the complete on-shell solution (i.e. the mass spectrum and exact s-matrix) for many of 2d Integrable QFT’es (IQFT) are well known, the construction of form-factors and correlation functions are under current intensive investigations now.

The main subject of investigation in this paper is the most studied example of 2d IQFT the sine-Gordon model, which is defined by the action

$$S_{SG} = \int d^2x \left\{ \frac{1}{16\pi} \partial_\nu \varphi \partial^\nu \varphi + 2\mu \cos \beta \varphi \right\}$$  \hspace{1cm} (1.1)$$

where \(\varphi\) is a real Bose field. The spectrum of this model includes the soliton, anti soliton and some number (depended on the coupling constant \(\beta\)) of their bound states named breathers. The scattering in this model is factorized: the many-particle scattering process is reduced to the two-particle ones and the sets of two-momenta of incoming and outgoing particles are identical. It is well known since 1975 that the SG model is equivalent to the Massive Thirring model (MTM) \cite{2} with the action

$$S_{MTM} = \int d^2x \left\{ i \bar{\psi} \gamma^\nu \partial_\nu \psi - M \bar{\psi} \psi - \frac{g}{2} (\bar{\psi} \gamma^\nu \psi) (\bar{\psi} \gamma^\nu \psi) \right\},$$  \hspace{1cm} (1.2)$$

where \(\bar{\psi}, \psi\) are two component Dirac spinors. This equivalence assumes the identification of the fundamental (anti) Fermions of the action (1.2) with the (anti) solitons of sine-Gordon model and the following relations among parameters and currents, established by Coleman \cite{2}:

$$\frac{g}{\pi} = \frac{1}{2\beta^2} - 1; \quad J^\nu \equiv \bar{\psi} \gamma^\nu \psi = -\frac{\beta}{2\pi} \epsilon^{\nu\mu} \partial_\mu \varphi$$  \hspace{1cm} (1.3)$$

More recently Al.Zamolodchikov has obtained an exact relation between the soliton mass \(M\) and the parameter \(\mu\) in the action (1.1) \cite{3}:

$$\mu = \frac{\Gamma(\beta^2)}{\pi \sqrt{\pi} \Gamma \left( \frac{1+\xi}{2} \right)^2 \Gamma(\xi)} \left[ \frac{M \sqrt{\pi} \Gamma \left( \frac{1+\xi}{2} \right)}{2 \Gamma(\xi^2)} \right]^{2-2\beta^2},$$  \hspace{1cm} (1.4)$$

where

$$\xi = \frac{\beta^2}{1-\beta^2}.$$  \hspace{1cm} (1.5)$$
In this paper we consider the Vacuum Expectation Values (VEV) of exponential fields in the sine-Gordon model

\[ G_a = \langle \exp ia\varphi (0) \rangle, \quad (1.6) \]

where the exponential fields are normalized by the condition

\[ \langle e^{ia\varphi(x)} e^{-ia\varphi(y)} \rangle_{SG} \to |x - y|^{-4a^2} \quad \text{as} \quad |x - y| \to 0, \quad (1.7) \]

which emphasizes that the UV limit of this theory is governed by the \( c = 1 \) conformal free bosonic field.

For two special values of sine-Gordon coupling constant, namely for \( \beta \to 0 \) (semiclassical limit) and \( \beta^2 = 1/2 \) (free fermion case), this function admits a direct calculation. The authors of \([4]\) have used these special cases to guess the following expression for the expectation values \((1.6)\) for generic \( \beta^2 < 1 \) and \( |\text{Re}(a)| < 1/(2\beta) \)

\[ G_a = \left( \frac{m\Gamma \left( \frac{1+\xi}{2} \right) \Gamma \left( \frac{2-\xi}{2} \right)}{4\sqrt{\pi}} \right)^{2a^2} \times \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh^2 (2a\beta t)}{2 \sinh(\beta^2 t) \sinh t \cosh ((1 - \beta^2) t)} - 2a^2 e^{-2t} \right] \right\}. \quad (1.8) \]

In order to support the formula \((1.8)\), some extra arguments, based on the reflection relations with Liouville reflection amplitude have been presented in the subsequent papers \([5], [6]\), but there is no rigorous mathematical proof yet. The article \([8]\) provides another evidence supporting the Lukyanov-Zamolodchikov formula \((1.8)\), where its correctness has been checked in first order of MTM coupling constant \( g \), by us of perturbation theory in angular quantization approach.

In present paper we apply radial quantization to the same problem. The Hamiltonian of massive theories in the radial quantization approach has explicit time dependence \([11]\). It appeared that this apparent difficulty can be overcome. We believe that such perturbative calculations substantially increase the confidence in reflection relations method as whole, which appears to be very powerful tool for investigation of 2d Conformal Field Theory (CFT) and IQFT \([2], [4]\).

The paper is organized as follows. In section 2 we introduce the radial quantization of MTM. In section 3 we calculate the VEV \((1.6)\) at free fermion point \( g = 0 \). The calculation of VEV in the first order of perturbation theory is presented in section 4. Here a special attention has been paid to regularization procedure of the product of local fields at the coinciding points, which has some new features in comparison with the ordinary quantization in Cartesian coordinates. It appears that the Hankel-transform is a useful tool to carry out the calculations of section 4. The relevant mathematical details are presented in Appendix.
2 Radial Quantization of the Massive Thirring Model

In two dimensional space it is convenient to use the following representation of Dirac matrices
\[
\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = -i\sigma_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (2.1)
and denote the components of Dirac spinors as
\[
\psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \bar{\psi} \equiv \psi^\dagger \gamma^0.
\] (2.2)

In this notation the action (1.7) in Euclidean space takes the factorized form
\[
\mathcal{A}_{MTM} = \int d^2 z \left[ \bar{\psi}_R^\dagger \partial \psi_R + \bar{\psi}_L^\dagger \partial \psi_L - \frac{iM}{2} (\bar{\psi}_L \psi_R - \bar{\psi}_R \psi_L) + g \bar{\psi}_L^\dagger \psi_L \psi_R^\dagger \psi_R \right],
\] (2.3)

where \( z = x^2 + ix^1, \bar{z} = x^2 - ix^1 \) are the complex coordinates on the Euclidean plane, \( \partial \equiv \partial/\partial z, \bar{\partial} \equiv \partial/\partial \bar{z} \) and \( d^2z \equiv 2dx^1dx^2 \) is the volume element.

As we are interested in the VEV’s of local fields \( \langle e^{ia\phi(0)} \rangle \), which have rotational symmetry, it is natural to use the polar coordinates \( \eta, \theta \) defined by
\[
z = e^{\eta + i\theta}; \quad \bar{z} = e^{\eta - i\theta}
\] (2.4)
and interpret \( \eta, \theta \) as Euclidean time and space coordinates respectively.

Since the conformal dimensions of the Fermi fields \( \psi_L \) and \( \psi_R \) are \((1/2, 0)\) and \((0, 1/2)\), they behave under the conformal transformations (2.4) as
\[
\psi_L(z, \bar{z}) \rightarrow \left( \frac{\partial \xi}{\partial z} \right)^{1/2} \psi_L(\eta, \theta); \quad \psi_R(z, \bar{z}) \rightarrow \left( \frac{\partial \bar{\xi}}{\partial \bar{z}} \right)^{1/2} \psi_R(\eta, \theta),
\] (2.5)
where \( \xi = \eta + i\theta, \bar{\xi} = \eta - i\theta \). The same transformation lows hold for the fields \( \psi_{L,R}^\dagger \).

Thus in \((\eta, \theta)\) coordinates the action (2.3) becomes
\[
\mathcal{A}_{MTM} = \int_{\theta = 0}^{2\pi} d\theta \int_{\eta = -\infty}^{\infty} d\eta \left[ i\psi_R^\dagger (\partial_{\eta} - i\partial_{\theta}) \psi_L - i\psi_L^\dagger (\partial_{\theta} + i\partial_{\eta}) \psi_R - iMe^\eta (\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L) + 2g\psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right].
\] (2.6)
Treating the radial coordinate $\eta$ as a ”time”, we get the Hamiltonian

$$H = \int_0^{2\pi} d\theta \left[ \psi_L^\dagger i\partial_\theta \psi_L - \psi_R^\dagger i\partial_\theta \psi_R - iMe^\eta \left( \psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L \right) \right. + 2g\psi_L^\dagger \psi_L^\dagger \psi_R^\dagger \psi_R \right]. \quad (2.7)$$

The usual canonical quantization scheme will bring us to the standard ”equal time” anti-commutation relations

$$\left\{ \psi_L(\theta), \psi_L^\dagger(\theta') \right\} = \delta(\theta - \theta'), \quad \left\{ \psi_R(\theta), \psi_R^\dagger(\theta') \right\} = \delta(\theta - \theta'). \quad (2.8)$$

As usual, in order to develop perturbation theory one first has to solve the problem with unperturbed Hamiltonian (i.e. to ignore the last quartic term in (2.7)). We found it easier to handle this problem in Schrödinger picture, instead of more conventional in QFT Heisenberg or Interaction pictures. Thus our field operators $\psi_{L,R}$ will not depend on “time” $\eta$ and the state vectors will evolve according to the Schrödinger equation. Let us define the creation, annihilation operators $c_k^\dagger, d_k^\dagger, c_k, d_k$ through the Fourier mode decompositions

$$\psi_L(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathcal{N} - \frac{1}{2}} \left( c_k e^{-ik\theta} + d_k^\dagger e^{ik\theta} \right)$$

$$\psi_R(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathcal{N} - \frac{1}{2}} \left( c_{-k} e^{ik\theta} + d_{-k}^\dagger e^{-ik\theta} \right)$$

$$\psi_L^\dagger(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathcal{N} - \frac{1}{2}} \left( d_k^\dagger e^{-ik\theta} + c_k e^{ik\theta} \right)$$

$$\psi_R^\dagger(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathcal{N} - \frac{1}{2}} \left( d_{-k}^\dagger e^{ik\theta} + c_{-k} e^{-ik\theta} \right) \quad (2.9)$$

where all sums are taken over all positive half-integers ($\mathcal{N}$ is the set of positive integers).

As a consequence of equations (2.8) and (2.9), one can easily get the following anti-commutation relations for the operators $c_k, d_k, c_k^\dagger, d_k^\dagger$:

$$\{ c_k, c_l \} = \{ d_k, d_l \} = \{ c_k^\dagger, d_l^\dagger \} = 0,$$

$$\{ c_k, c_l^\dagger \} = \delta_{k,l}, \quad \{ d_k, d_l^\dagger \} = \delta_{k,l}, \quad k, l \in \mathcal{Z} - \frac{1}{2}, \quad (2.10)$$

where $\mathcal{Z}$ is the set of integers. As usual the Fock space (let us denote it $\mathcal{H}$) has the following basic vectors

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4
\[ \prod_{k \in \mathbb{Z} - \frac{1}{2}} \left( c_k^\dagger \right)^{n_k} \left( d_k^\dagger \right)^{\tilde{n}_k} |0\rangle, \]  

(2.11)

where \( n_k \in \{0, 1\} \) (\( \tilde{n}_k \in \{0, 1\} \)) are the occupation numbers of “particles” created by the operators \( c_k^\dagger \ (d_k^\dagger) \) out of bare vacuum \( |0\rangle \), which by definition satisfies the conditions

\[ c_k |0\rangle = d_k |0\rangle = 0; \quad k \in \mathbb{Z} - \frac{1}{2}, \]  

(2.12)

In terms of the creation, annihilation operators is the Hamiltonian (2.7) acquires the form

\[ H_0 = \sum_{k \in \mathbb{N} - \frac{1}{2}} \left[ k \left( c_k^\dagger c_k - d_k d_k^\dagger + c_{-k}^\dagger c_{-k} - d_{-k} d_{-k}^\dagger \right) \right. \]

\[ \left. -i M e^\eta \left( c_k^\dagger d_{-k} - d_{-k} c_k + d_k c_{-k} - c_{-k}^\dagger d_k^\dagger \right) \right] \]  

(2.13)

The evolution of arbitrary state \( |s\rangle \) along Euclidean time \( \eta \) caused by the Hamiltonian \( H_0 \) is given by the Schrödinger equation

\[ -r \frac{\partial}{\partial r} |s, r\rangle = H_0 |s, r\rangle. \]  

(2.14)

Here and henceforth we prefer to use \( r \equiv M e^\eta \) rather than \( \eta \). To find the general solution to the Schrödinger equation (2.14) let us denote, that the Hamiltonian \( H_0 \) has a factorized form

\[ H_0 = \sum_{k \in \mathbb{N} - \frac{1}{2}} \left( H_k^{(1)} + H_k^{(2)} \right), \]  

(2.15)

where the operator \( H_k^{(1)} \ (H_k^{(2)}) \) includes only \( c_k, d_{-k}, c_k^\dagger, d_{-k}^\dagger \) (\( d_k, c_{-k}, d_k^\dagger, c_{-k}^\dagger \)). This makes convenient to represent the Fock space \( \mathcal{H} \) as an infinite tensor product

\[ \mathcal{H} = \otimes_{k \in \mathbb{N} - 1/2} \left( \mathcal{H}_k^{(1)} \otimes \mathcal{H}_k^{(2)} \right), \]  

(2.16)

where \( \mathcal{H}_k^{(1)} \) and \( \mathcal{H}_k^{(2)} \) are four dimensional vector spaces with base vectors

\[
|0_k^{(1)}\rangle, \quad c_k^\dagger d_{-k}^\dagger |0_k^{(1)}\rangle, \quad (even \ sector) \\
\]

\[
c_k^\dagger |0_k^{(1)}\rangle, \quad d_{-k}^\dagger |0_k^{(1)}\rangle, \quad (odd \ sector)
\]

and

\[
|0_k^{(2)}\rangle, \quad d_k^\dagger c_{-k}^\dagger |0_k^{(2)}\rangle, \quad (even \ sector) \\
\]

\[
d_k^\dagger |0_k^{(2)}\rangle, \quad c_{-k}^\dagger |0_k^{(2)}\rangle, \quad (odd \ sector)
\]

(2.17)

(2.18)
respectively. The vectors \( |0_k^{(1)}\rangle \) and \( |0_k^{(2)}\rangle \) are defined by the conditions

\[
c_k |0_k^{(1)}\rangle = d_{-k} |0_k^{(1)}\rangle = 0, \\
d_k |0_k^{(2)}\rangle = c_{-k} |0_k^{(2)}\rangle = 0, \tag{2.19}
\]

for any \( k \in \mathbb{N} - 1/2 \). Note that the bare vacuum introduced earlier (see (2.12)) is equal to

\[
|0\rangle = \bigotimes_{k \in \mathbb{N} - 1/2} [ |0_k^{(1)}\rangle \otimes |0_k^{(2)}\rangle ]. \tag{2.20}
\]

The operator \( H_k^{(1)} \) (\( H_k^{(2)} \)) nontrivially acts only on the factor \( \mathcal{H}_k^{(1)} \) (\( \mathcal{H}_k^{(2)} \)) of the full Fock space \( \mathcal{H} \), hence we have reduced the initial QFT problem of infinitely many degrees of freedom to the simple quantum mechanical one, with four dimensional Hilbert space. A further simplification provides the observation, that the reduced Hamiltonians \( H_k^{(1)} \), \( H_k^{(2)} \) don’t mix even and odd sectors (see (2.17), (2.18)). The resulting Schrödinger equations in this reduced spaces take the form

\[
- r \frac{\partial}{\partial r} \left[ \alpha_k(r) + \beta_k(r) c_k^\dagger d_{-k}^\dagger \right] |0_k^{(1)}\rangle = H_k^{(1)} \left[ \alpha_k(r) + \beta_k(r) c_k^\dagger d_{-k}^\dagger \right] |0_k^{(1)}\rangle = \\
\left[ -k \alpha_k(r) + i r \beta_k(r) + (k \beta_k(r) - i r \alpha_k(r)) c_k^\dagger d_{-k}^\dagger \right] |0_k^{(1)}\rangle, \tag{2.21}
\]

and

\[
- r \frac{\partial}{\partial r} \left[ \gamma_k(r) c_k^\dagger + \delta_k(r) d_{-k}^\dagger \right] |0_k^{(1)}\rangle = H_k^{(1)} \left[ \gamma_k(r) c_k^\dagger + \delta_k(r) d_{-k}^\dagger \right] |0_k^{(1)}\rangle = 0. \tag{2.22}
\]

Evidently, to obtain the equations for the another sector with Hamiltonian \( H_k^{(2)} \), one simply has to change the upper indices (1) into (2) and make the substitutions \( c_k \rightarrow d_k \) and \( d_{-k} \rightarrow c_{-k} \). Thus, the differential equations for the unknown functions \( \alpha_k(r) \), \( \beta_k(r) \), \( \gamma_k(r) \), \( \delta_k(r) \) in both cases remain the same:

\[
\left( \frac{\partial}{\partial r} - \frac{k}{r} \right) \alpha_k(r) = -i \beta_k(r), \\
\left( \frac{\partial}{\partial r} + \frac{k}{r} \right) \beta_k(r) = i \alpha_k(r) \\
\frac{\partial}{\partial r} \gamma_k(r) = \frac{\partial}{\partial r} \delta_k(r) = 0. \tag{2.23}
\]

The second pare of these equations show, that in fact \( \gamma_k \) and \( \delta_k \) are constants, while the first pare reduces to the modified Bessel differential equation with general solution
\[ \alpha_k(r) = r^{\frac{1}{2}} \left( a_k I_{k - \frac{1}{2}}(r) + b_k K_{k - \frac{1}{2}}(r) \right), \]
\[ \beta_k(r) = i r^{\frac{1}{2}} \left( a_k I_{k + \frac{1}{2}}(r) - b_k K_{k + \frac{1}{2}}(r) \right). \] (2.24)

One should fix the constants \( a_k, b_k, \gamma_k \) and \( \delta_k \) imposing initial conditions at the arbitrary “time” \( r_0 \). For the further application let us write down explicit expressions with specified constants for two basic cases:

a) when the initial state coincides with \(|0^{(1)}_k\rangle\) or \(|0^{(2)}_k\rangle\):

\[ \alpha_k(r) = \sqrt{r r_0} \left( K_{k + \frac{1}{2}}(r_0) I_{k - \frac{1}{2}}(r) + I_{k + \frac{1}{2}}(r_0) K_{k - \frac{1}{2}}(r) \right), \]
\[ \beta_k(r) = i \sqrt{r r_0} \left( K_{k + \frac{1}{2}}(r_0) I_{k + \frac{1}{2}}(r) - I_{k + \frac{1}{2}}(r_0) K_{k + \frac{1}{2}}(r) \right). \] (2.25)

b) when the initial state coincides with \( i c_k^\dagger d_{-k}|0^{(1)}_k\rangle \) or \( i d_k^\dagger c_{-k}|0^{(2)}_k\rangle \):

\[ \alpha_k(r) = \sqrt{r r_0} \left( K_{k - \frac{1}{2}}(r_0) I_{k - \frac{1}{2}}(r) - I_{k - \frac{1}{2}}(r_0) K_{k - \frac{1}{2}}(r) \right), \]
\[ \beta_k(r) = i \sqrt{r r_0} \left( K_{k - \frac{1}{2}}(r_0) I_{k + \frac{1}{2}}(r) + I_{k - \frac{1}{2}}(r_0) K_{k + \frac{1}{2}}(r) \right). \] (2.26)

During the proof of the formulae (2.25)(2.26) we have the standard Wronskian identity for the modified Bessel functions [13]

\[ I_{k - \frac{1}{2}}(r) K_{k + \frac{1}{2}}(r) + I_{k + \frac{1}{2}}(r) K_{k - \frac{1}{2}}(r) = \frac{1}{r}. \] (2.27)

It is interesting to note, that due to explicit time dependence of the Hamiltonian, the system being initially at the ground state of that particular moment, after finite time of evolution will find himself in an excited state. Nevertheless, long time evolution of any state with non vanishing overlap with the ground state of the initial time, eventually approaches to the ground state of the infinite future

\[ |\infty\rangle \equiv \prod_{k \in \mathbb{N} - \frac{1}{2}} \frac{1}{2} \left( \left( 1 + i c_k^\dagger d_{-k} \right) \left( 1 + i d_k^\dagger c_{-k} \right) \right) |0\rangle. \] (2.28)

Evidently, at the small \( r \)'s (far past), the ground state approaches to the bare vacuum \(|0\rangle\). In particular, if \( r \gg 1 \) and \( r_0 \ll 1 \) (2.27) gives

\[ \alpha_k(r) \rightarrow \left( \frac{2}{r_0} \right)^k \frac{e^r}{\sqrt{4\pi}} \Gamma \left( k + \frac{1}{2} \right), \]
\[ \beta_k(r) \rightarrow \left( \frac{2}{r_0} \right)^k \frac{e^r}{\sqrt{4\pi}} \Gamma \left( k + \frac{1}{2} \right). \] (2.29)
3 The VEV’s of the Exponential Fields in Free Fermion Case

The VEV (1.6)

\[ G_a = \left\langle e^{ia\varphi} (0) \right\rangle = \frac{\int \mathcal{D}\varphi e^{ia\varphi} e^{-S_{SG}(\varphi)}}{\int \mathcal{D}\varphi e^{-S_{SG}(\varphi)}}, \]

with \( S_{SG} \) being the action (1.1), can be expressed alternatively in terms of the appropriately regularized (see below) Euclidean functional integral over the Dirac fermions

\[ G(a) = \frac{\int \mathcal{F}_a \left[ \mathcal{D}\psi \mathcal{D}\bar{\psi} \right] e^{-A_{MTM}}}{\int \mathcal{F}_0 \left[ \mathcal{D}\psi \mathcal{D}\bar{\psi} \right] e^{-A_{MTM}}}, \]

where \( A_{MTM} \) is the Euclidean action (2.6). The functional integral in the numerator of (3.31) is taken over the space \( \mathcal{F}_a \) of the twisted field configurations \( \psi(z, \bar{z}) \) and \( \bar{\psi}(z, \bar{z}) \), which acquire the phase

\[ \psi(z, \bar{z}) \rightarrow e^{i\frac{2\pi a}{\beta}} \psi(z, \bar{z}), \quad \bar{\psi}(z, \bar{z}) \rightarrow e^{-i\frac{2\pi a}{\beta}} \bar{\psi}(z, \bar{z}), \]

when continued analytically around the point \( z = 0 \) in anti-clockwise direction \( [4] \). This is due to the non-trivial monodromy of Dirac fields with respect to the exponential fields \( \exp ia\varphi(0) \). It is easy to see, that to ensure such twisted boundary conditions on Dirac fields, one has to shift Fourier mode indices as follows

\[ k \rightarrow k - \frac{a}{\beta}, \quad \text{in sector 1 (i.e. in } c_k, d_k \text{ sector)} \]
\[ k \rightarrow k + \frac{a}{\beta}, \quad \text{in sector 2 (i.e. in } d_k, c_k \text{ sector).} \]

For example, the Fourier decomposition of the field \( \psi_L(\theta) \) takes the form (cf. with the first line of (2.3))

\[ \psi_L(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{N} - \frac{1}{2}} \left( c_{k-\alpha} e^{-i(k-\alpha)\theta} + d^\dagger_{k-\alpha} e^{i(k-\alpha)\theta} \right), \]

where

\[ \alpha \equiv \frac{a}{\beta}. \]

With such shifts, all the results of the previous sector remain valid since we have never used the arithmetical properties of the Fourier mode indices.

In radial Hamiltonian formalism the regularized version of the functional integral (3.31) may be represented as

\[ G(a, r_0) = \frac{\langle \infty | S(r, r_0) | 0 \rangle_a}{\langle \infty | S(r, r_0) | 0 \rangle}, \]
where the matrix element of the evolution operator $S(r, r_0)$ in the numerator is taken in twisted sector (this is indicated by the lower index $a$). To regularize the expression, in (3.36) we have assumed, that the evolution begins at some small $r_0$. A simple Conformal Field Theory consideration\footnote{In the limit $r \to 0$ the action (2.6) describes the Massless Thiring Model, which is well known to be conformal invariant.}, which takes into account the fact, that the conformal dimension of the field $e^{i a \phi}$ is $a^2$, leads to

$$
G_0(a) = \lim_{r_0 \to 0} (r_0)^{-2a^2} G_0(a, r_0).
$$

(3.37)

In general case it is not known yet how to calculate the functional integral (3.31) or the matrix elements in (3.36) exactly. Below we’ll evaluate (3.36) at the free fermion point $g = 0$. As in this case we already know the time evolution of every constituent of the vacuum $|0\rangle$ (see eq. (2.20)) from the previous section, it is not difficult to pick up all the necessary factors from (2.29) with appropriate shifts of Fourier mode indices and obtain

$$
G_0(a, r_0) = \prod_{k \in \mathbb{N} - \frac{1}{2}} \left( \frac{2}{r_0} \right)^{k+\alpha} e^{\frac{r}{\sqrt{4\pi}}} \Gamma \left( k + \alpha + \frac{1}{2} \right) \prod_{l \in \mathbb{N} - \frac{1}{2}} \left( \frac{2}{r_0} \right)^{l-\alpha} e^{\frac{r}{\sqrt{4\pi}}} \Gamma \left( l - \alpha + \frac{1}{2} \right)
$$

(3.38)

$$
\lim_{r \to \infty} \frac{\prod_{k \in \mathbb{N} - \frac{1}{2}} \left( \frac{2}{r_0} \right)^{k+\alpha} e^{\frac{r}{\sqrt{4\pi}}} \Gamma \left( k + \alpha + \frac{1}{2} \right) \prod_{l \in \mathbb{N} - \frac{1}{2}} \left( \frac{2}{r_0} \right)^{l-\alpha} e^{\frac{r}{\sqrt{4\pi}}} \Gamma \left( l - \alpha + \frac{1}{2} \right)}{\prod_{k \in \mathbb{N} - \frac{1}{2}} \left( \frac{2}{r_0} \right)^{k} e^{\frac{r}{\sqrt{4\pi}}} \Gamma \left( k + \frac{1}{2} \right) \prod_{l \in \mathbb{N} - \frac{1}{2}} \left( \frac{2}{r_0} \right)^{l} e^{\frac{r}{\sqrt{4\pi}}} \Gamma \left( l + \frac{1}{2} \right)}
$$

where we have added a subscript 0 to $G$ in order to emphasize, that the free Fermion case $g = 0$ is considered. We have to be careful, when evaluating infinite products in (3.39) and treat the ill defined sums like $\sum_{i=0}^{\infty} (i \pm a)$ by means of Riemann $\zeta$-function regularization. Let us remind that

$$
\zeta (z, a) = \sum_{i=0}^{\infty} \frac{1}{(i + a)^z}
$$

(3.39)

and

$$
\zeta (-1, a) + \zeta (-1, -a) - 2 \zeta (-1, 0) = -a^2
$$

(3.40)

To carry out the remaining infinite products of $\Gamma$-functions (also divergent, if treated literally), it is convenient to use the integral representation

$$
\ln \Gamma (\nu) = \int_{0}^{\infty} \left[ \frac{e^{-\nu t} - e^{-t}}{1 - e^{-t}} + (\nu - 1) e^{-t} \right] \frac{dt}{t}.
$$

(3.41)

As a result we obtain simple geometric progressions, coming from the first term of the eq. (3.41) and contributions, coming from the second term, which can be
easily handled applying $\zeta$-function regularization once more. The final expression has the form

\[ G_0(a, r_0) = \left( \frac{r_0}{2} \right)^{a^2} \exp \int_0^\infty \left[ \frac{\sinh^2(\alpha t)}{\sinh^2 t} - \alpha^2 e^{-2t} \right] \frac{dt}{t} \]  

or, taking into account equations (3.37) and (3.35) with the free Fermion point value $\beta = 1/\sqrt{2}$

\[ G_0(a) = \left( \frac{M}{2} \right)^{2a^2} \exp \int_0^\infty \left[ \frac{\sinh^2(\sqrt{2}at)}{\sinh^2 t} - 2a^2 e^{-2t} \right] \frac{dt}{t}. \]  

This is in full agreement with the result, obtained by S.Lukyanov and A.Zamolodchikov in [4], using angular quantization technic.

## 4 VEV of the Exponential Field in the First Order of Perturbation Theory

In this section we calculate the VEV (1.6) in first order of the MTM’s coupling constant $g$. The perturbation is given by the last term of the Hamiltonian (2.7):

\[ \mathcal{H}_{int} = 2g \int_0^{2\pi} N \left( \Psi_L^+ \Psi_L \Psi_R^+ \Psi_R \right) d\theta, \]  

where we denoted by $N(\ldots)$ an appropriately regularized product of local fields at coinciding point. One has to chose such a regularization, which will not break the translational invariance of the theory if it is transformed back to the initial Euclidean coordinates $x^1, x^2$. The conventional normal ordering with respect to creation-annihilation operators fails to satisfy this condition, because of the non-trivial time dependence of physical vacuum. Instead, we should suppress all the contractions among fields inside the correct normal ordering symbol $N(\ldots)$

\[ N \left( \psi_L^+ \psi_L \psi_R^+ \psi_R \right) = \psi_L^+ \psi_L \psi_R^+ \psi_R - \left< \psi_L^+ \psi_L \right>_0 \psi_R^+ \psi_R - \left< \psi_R^+ \psi_R \right>_0 \psi_L^+ \psi_L + \left< \psi_L^+ \psi_L \right>_0 \psi_R^+ \psi_R + \left< \psi_R^+ \psi_R \right>_0 \psi_L^+ \psi_L - \left< \psi_L^+ \psi_L \right>_0 \left< \psi_R^+ \psi_R \right>_0 - \left< \psi_R^+ \psi_R \right>_0 \left< \psi_L^+ \psi_L \right>_0 \]

\[ =: \psi_L^+ \psi_L \psi_R^+ \psi_R : + \left< \psi_L^+ \psi_L \right>_0 \psi_R^+ \psi_R : - \left< \psi_R^+ \psi_R \right>_0 \psi_L^+ \psi_L : - \left< \psi_R^+ \psi_R \right>_0 \left< \psi_L^+ \psi_L \right>_0 + \left< \psi_L^+ \psi_L \right>_0 \left< \psi_R^+ \psi_R \right>_0, \]  

where the vacuum expectation value of any operator $X$ is defined by

\[ \left< X \right>_0 = \frac{\left< \infty | S(R, r) X S(r, r_0) | 0 \right>}{\left< \infty | S(R, r_0) | 0 \right>} \]  

(4.3)
with all matrix elements taken in untwisted sector. The first part of eq. (4.2) could be understood for example as a zero distance limit of corresponding point-split expression. In (4.3) a small initial time $r_0$ and a large final time $R_0$ are introduced in order to keep intermediate expressions finite. $R$ and $r_0$ eventually will be sent to 0 and $\infty$ correspondingly. Note also the explicit $r$ dependence of (4.3) and hence of (4.2), which reflects the inhomogeneity of “time” in our scheme of quantization. The standard time dependent perturbation theory in first order of the coupling constant $g$ gives

$$G(a, r_0) = \lim_{R \to \infty} \left( \langle \infty | S(R, r_0) | 0 \rangle_a + \int_{r_0}^{R} \langle \infty | S(R, r) H_{int} S(r, r_0) | 0 \rangle_a \frac{dr}{r} \right),$$

As we already have obtained explicit expressions for time evolution of states from various sectors of Fock space in section 2, it is not difficult to calculate the matrix element under the integral in eq. (4.4)

$$G(a, r_0) = \lim_{R \to \infty} \langle \infty | S(R, r_0) | 0 \rangle_a \{ 1 + \frac{g}{\pi} \sum_{k,l=0}^{\infty} \int_{r_0}^{R} rdr \left[ 2 I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} - I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} - I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} + I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} + I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} + I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} - I_{k+1-a} K_{k-a} I_{l+1+a} K_{l-a} \right] \},$$

with the pre factor $\langle \infty | S(R, r_0) | 0 \rangle_a$ given by eq. (3.42) (in (3.42) we have to insert $\beta = \frac{1}{\sqrt{2}}(1 - \frac{g}{2\pi} + o(g))$ and expand the resulting expression over $g$ up to linear term). The remaining calculation of integrals over the quartic Bessel functions is presented in the appendix. Using these results we obtain

$$G(a, r_0) = \left( \frac{r_0}{2} \right)^{a^2} \exp \left\{ \int_{0}^{\infty} \left( \frac{\sinh^2(\alpha t)}{\sinh^2 t} \right) - \alpha^2 e^{-2t} \right\} \times$$

$$\left\{ 1 + \frac{g}{2\pi} \left[ 2\alpha^2 \log \frac{r_0}{2} + \int_{0}^{\infty} \left( \frac{\alpha \sinh(2\alpha t)}{2 \sinh^2 t} - \frac{\alpha^2}{t} e^{-2t} \right) dt \right] + O(g^2) \right\} \times$$

$$\left\{ 1 + \frac{g}{\pi} \left[ \int_{0}^{\infty} \sum_{k,l=0}^{\infty} \left( \frac{8 \cosh^2 t \sinh^2 \alpha t - 4 \sinh^2 2\alpha t}{\sinh 2t} e^{-2(k+l+1)t} \right) dt \right] + O(g^2) \right\}.$$ 

Now performing summation over $k$ and $l$ in the third line of the eq. (4.6) we obtain logarithmically diverging at $t = 0$ integral. In fact, the same problem we have
faced when carrying out calculations exactly at the free-fermion point. Indeed the
the product in (3.39) diverges for large $k, l$, but we overcame this difficulty using
$\zeta$-function regularization inside the integral representation of $\Gamma$-function (3.41).
Here we'll not care of a similar appropriate regularization. Rather, noticing that
various regularization scheme will differ from each other by a term $\sim a^2$, and that
the coefficient of $-1/2a^2$ in the expansion of $\langle e^{i a \phi} \rangle$ is just the VEV $\langle \phi^2 \rangle$, which
has been calculated in [4] using standard Feynman diagram technic with result
(below $\gamma = 0.577216...$ is the Euler constant)

$$\langle \phi^2(0) \rangle = -4(1 + \gamma + \log(M/2)) + \frac{g}{\pi}(7\zeta(3) - 2) + O(g^2), \quad (4.7)$$

we simply cut the above mentioned integral over $t$ on the lower bound and equate
the undefined coefficient of $-1/2a^2$ to the one predicted by the eq.(4.7). The final
result is

$$\langle e^{i a \phi(0)} \rangle = \left(\frac{M}{2}\right)^{a^2} \exp\left\{\int_0^\infty \left(\frac{\sinh^2(\alpha t)}{\sinh^2 t} - 2 \alpha^2 e^{-2t} \right) \frac{dt}{t}\right\} \times \quad (4.8)$$

$$\left\{1 + \frac{g}{\pi} \left[\int_0^\infty \left(\frac{\alpha \sinh(2\alpha t)}{2 \sinh^2 t} - \frac{\sinh^2(\alpha t)}{\sinh^3 t} \right) dt - 2\alpha^2 \log 2 \right] + O(g^2) \right\},$$

which is in complete agreement with the Lukyanov-Zamolodchikov formula (1.8).

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Appendix

It appears that the Hankel-transforms are appropriate tools allowing us to per-
form the integration over $r$ in (1.3). Roughly speaking, in polar coordinates the
Hankel-transforms play the same role, as the ordinary Fourier-transforms in the
Cartesian one.

Let us briefly recall the main formulae concerning to the Hankel-transforms
(for details see [13] and references therein). The $\nu$-th order ($\nu > -1$) direct and
inverse Hankel-transforms of the function \( f(x) \) defined on \((0, \infty)\) are given by

\[
f(x) = \int_0^\infty J_\nu (sx) \tilde{f}_\nu (s) \, ds, \tag{A.1}
\]

\[
\tilde{f}_\nu (s) = \int_0^\infty J_\nu (sx) f(x) \, dx, \tag{A.2}
\]

where \( J_\nu \) is the Bessel function. In complete analogy with the case of Fourier-transform, it follows from (A.1), (A.2), that the “scalar product” of any two functions \( f(x), g(x) \) coincides with that of their images

\[
\int_0^\infty f(x) g(x) \, dx = \int_0^\infty \tilde{f}_\nu (s) \tilde{g}_\nu (s) \, ds. \tag{A.3}
\]

To use (A.3) for the calculation of the integral in eq.(4.5) we need to know Hankel images of the functions \( I_\nu(x) K_\mu(x) \) and \( I_{\nu+1}(x) K_\nu(x) \) which can be easily obtained from the general formula \( [13] \)

\[
K_{-\nu}(x) I_\mu(x) = \int_0^\infty J_{-\nu+\mu} (2x \sinh t) e^{-(\nu+\mu)t} dt, \tag{A.4}
\]

namely

\[
I_l(x) K_l(x) = \int_0^\infty J_0(xs) \frac{e^{-2lt(s)}}{s\sqrt{s^2 + 4}} ds, \tag{A.5}
\]

\[
I_{l+1}(x) K_l(x) = \int_0^\infty J_1(xs) \frac{e^{-(2l+1)t(s)}}{s\sqrt{s^2 + 4}} ds, \tag{A.6}
\]

where \( t(s) \) is defined by

\[
2 \sinh t = s, \quad dt = \frac{ds}{\sqrt{s^2 + 4}}. \tag{A.7}
\]

Though the direct application of eq. (A.3) to each term of (4.5) at first sight seems to be problematic due to the logarithmic divergence of the integral over at large \( r \), but nevertheless it leads to a correct result, because of mutual cancellation of these divergences by various terms.

\[\text{Since the functions we are dealing with are regular in the interval } (0, \infty), \text{ the only thing one has to care of is the convergence of integrals at the extreme points 0 and } \infty.\]
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