CANCELLATION FOR INCLUSIONS OF C*-ALGEBRAS OF
FINITE DEPTH

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Abstract. Let \( 1 \in A \subset B \) be a pair of C*-algebras with common unit. We prove that if \( E : B \to A \) is a conditional expectation with index-finite type and a quasi-basis of \( n \) elements, then the topological stable rank satisfies
\[
\text{tsr}(B) \leq \text{tsr}(A) + n - 1.
\]
As an application we show that if an inclusion \( 1 \in A \subset B \) of unital C*-algebras has index-finite type and finite depth, and \( A \) is a simple unital C*-algebra with \( \text{tsr}(A) = 1 \) and Property (SP), then \( B \) has cancellation. In particular, if \( \alpha \) an action of a finite group \( G \) on \( A \), then the crossed product \( A \rtimes \alpha G \) has cancellation. For outer actions of \( \mathbb{Z} \), we obtain cancellation for \( A \rtimes \alpha \mathbb{Z} \) under the additional condition that \( \alpha_* = \text{id} \) on \( K_0(A) \). Examples are given.

1. Introduction

For two projections \( p, q \) in a C*-algebra, we write \( p \sim q \) if they are Murray-von Neumann equivalent. A C*-algebra \( A \) is said to have cancellation of projections if whenever \( p, q, r \in A \) are projections with \( p \perp r, q \perp r \), and \( p + r \sim q + r \), then \( p \sim q \).

For a unital C*-algebra \( A \), if the topological stable rank \( \text{tsr}(A) \) of \( A \) satisfies \( \text{tsr}(A) = 1 \), then \( A \) has cancellation (Proposition 6.5.1 of [3]). For a stably finite simple C*-algebra \( A \), it has been a long standing open question, settled negatively in [35], whether cancellation implies \( \text{tsr}(A) = 1 \). The construction uses Villadsen’s techniques [30]. The example is also separable and nuclear.

Let \( 1 \in A \subset B \) be a unital inclusion of C*-algebras with index-finite type and with finite depth. In particular, \( B \) could be a crossed product \( A \rtimes \alpha G \) of a unital C*-algebra by a finite group. Our main result, Theorem 4.6, is that if \( A \) is simple, has topological stable rank 1, and satisfies Property (SP) (every hereditary C*-subalgebra contains a nonzero projection), then \( B \) has cancellation.

As a corollary, suppose that \( A \) is a simple unital C*-algebra with tracial topological rank zero (\( \text{TR}(A) = 0 \); Definition 3.6.2 of [22]), and \( \alpha : G \to \text{Aut}(A) \) is an action of a finite group \( G \) on \( A \). Then \( A \rtimes \alpha G \) has cancellation. Examples for \( A \) include all simple unital AH-algebras with real rank zero and slow dimension.

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growth. Recently, the third author ([26], [27]) has proved that if in addition \( \alpha \) has the tracial Rokhlin property, then \( \text{TR}(A \rtimes_\alpha G) = 0 \). It follows that, in this case, 
\[
\text{tsr}(A \rtimes_\alpha G) = 1.
\]
The result of this paper shows that no conditions on the action are needed for cancellation.

As an intermediate result, we prove in Section 2 that if \( 1 \in A \subseteq B \) is a unital inclusion of \( \mathrm{C}^* \)-algebras, and if there is a faithful conditional expectation \( E : B \rightarrow A \) with index-finite type and a quasi-basis of \( A \) below, then
\[
\text{tsr}(B) \leq \text{tsr}(A) + n - 1
\]
In particular, this applies if \( B = A \rtimes_\alpha G \) and \( \text{card}(G) = n \).

Another important ingredient is a result of Blackadar (Theorem A below): if \( B \) is a simple \( \mathrm{C}^* \)-algebra, and \( P \) is a set of nonzero projections in \( B \) containing, in a suitable sense, arbitrarily small projections, such that \( \sup_{b \in P} \text{tsr}(pBp) < \infty \), then \( B \) has cancellation.

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2. Topological stable rank

For a unital \( \mathrm{C}^* \)-algebra \( A \), recall that the topological stable rank \( \text{tsr}(A) \) of \( A \) is defined to be the least integer \( n \) such that the set \( \text{Lg}_n(A) \) of all \( n \)-tuples \((a_1, a_2, \ldots, a_n) \in A^n\) which generate \( A \) as a left ideal is dense in \( A^n \). (See Definition 1.4 of [31].) The topological stable rank of a nonunital \( \mathrm{C}^* \)-algebra is defined to be that of its smallest unitization. Note that \( \text{tsr}(A) = 1 \) is equivalent to density of the set of invertible elements in \( A \). Furthermore, \( \text{tsr}(A) = 1 \) implies that \( \text{tsr}(A \otimes M_n) = 1 \) for all \( n \), and that \( \text{tsr}(A \otimes K) = 1 \), where \( K \) is the algebra of compact operators on a separable infinite dimensional Hilbert space. Conversely, if \( \text{tsr}(A \otimes M_n) = 1 \) for some \( n \), or if \( \text{tsr}(A \otimes K) = 1 \), then \( \text{tsr}(A) = 1 \). (See Theorems 3.3 and 3.6 of [31].) Simple AH algebras with slow dimension growth have topological stable rank one (Theorem 1 of [7]), as do irrational rotation algebras ([30]). If \( A \) is unital and \( \text{tsr}(A) = 1 \), the \( A \) has cancellation (Proposition 6.5.1 of [3]). It follows immediately that \( A \) is stably finite in the sense that no matrix algebra \( M_n(A) \) has an infinite projection.

As in [38] (see Definition 1.2.1 and the following discussion there), if \( 1 \in A \subseteq B \) is a pair of \( \mathrm{C}^* \)-algebras with a common unit, then a conditional expectation \( E : B \rightarrow A \) is a positive \( A \)-bimodule map of norm one. Following Definition 1.2.2 and Lemma 2.1.6 of [38], if \( E \) is faithful (a standing assumption in [38]; see the discussion after Definition 1.2.1 there), a quasi-basis for \( E \) is a finite family \((u_1, u_1^*), (u_2, u_2^*), \ldots, (u_n, u_n^*)\) of elements of \( B \times B \) such that
\[
b = \sum_{j=1}^{n} u_j E(u_j^* b) = \sum_{j=1}^{n} E(b u_j) u_j^*
\]
for all \( b \in B \), the expectation \( E \) has index-finite type if \( E \) has a quasi-basis, and the index of \( E \) is then defined by \( \text{Index}(E) = \sum_{j=1}^{n} u_j u_j^* \). By Proposition 1.2.8 and Lemma 2.3.1 of [38], the index is a positive invertible central element of \( B \) that does not depend on the choice of the quasi-basis. In particular, if \( 1 \in A \subseteq B \) is a pair of simple unital \( \mathrm{C}^* \)-algebras, then \( \text{Index}(E) \) is a positive scalar. By abuse of language, we will say that \( 1 \in A \subseteq B \) has index-finite type if there is a faithful conditional expectation \( E : B \rightarrow A \) with index-finite type.
The following example is standard. (See Lemma 3.1 of [24] for a good deal more.)

**Example 2.1.** Let $A$ be a unital C*-algebra, let $G$ be a finite group, and let \( \alpha : G \to \text{Aut}(A) \) be an action of $G$ on $A$. For $g \in G$, let \( u_g \in A \rtimes_G G \) be the standard unitary in the crossed product, implementing \( \alpha_g \). Then the function \( E : A \rtimes_G G \to A \), given by \( E(\sum_{g \in G} a_g u_g) = a_1 \), is a conditional expectation with index-finite type, \( \langle (u_g, u_g^*) \rangle_{g \in G} \) is a quasi-basis for $E$, and \( \text{Index}(E) = \text{card}(G) \cdot 1_{A \rtimes_G G} \).

**Theorem 2.2.** Let $B$ be a unital C*-algebra, let $A \subset B$ be a unital subalgebra, let \( E : B \to A \) be a faithful conditional expectation with index-finite type, and let \( \langle (v_k, v_k^*) \rangle_{1 \leq k \leq n} \) be a quasi-basis for $E$. Then \( \text{tsr}(B) \leq \text{tsr}(A) + n - 1 \).

**Proof.** Set \( m = \text{tsr}(A) - 1 \). We prove that \( L_{g_{m+n}}(B) \) is dense in \( B^{m+n} \). Let \( b_1, b_2, \ldots, b_{m+n} \in B \), and let \( \varepsilon > 0 \). Write

\[
b_j = \sum_{k=1}^{n} a_{j,k} v_k
\]

for \( 1 \leq j \leq m + n \), with all \( a_{j,k} \in A \).

We will work with various sizes of nonsquare matrices over $A$ and over $B$. We regard all of them as elements of the C*-algebra $B \otimes K$ by placing each one in the upper left corner of an infinite matrix, and taking all remaining entries of the infinite matrix to be zero. Multiplication of matrices of compatible sizes thus becomes the usual multiplication in $B \otimes K$, and all the usual properties of multiplication in C*-algebras are valid. We also write \( 1_l \) for the \( l \times l \) identity matrix, which, according to the convention just made, is a projection in $B \otimes K$.

Set

\[
a = \begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,2} \\
& \ddots & \ddots & \vdots \\
a_{m+n,1} & a_{m+n,2} & \cdots & a_{m+n,n}
\end{pmatrix}
\quad \text{and} \quad
v = \begin{pmatrix}
v_1 \\
v_2 \\
& \ddots \\
v_n
\end{pmatrix}.
\]

This gives

\[
av = \begin{pmatrix}
b_1 \\
b_2 \\
& \ddots \\
b_{m+n}
\end{pmatrix}.
\]

According to Definition 6.2 and Lemma 6.3 of [31], the C*-algebra $A$ satisfies the property \( L_{m(n)} \), so that there exists

\[
x = \begin{pmatrix}
x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\
x_{2,1} & x_{2,2} & \cdots & x_{2,2} \\
& \ddots & \ddots & \vdots \\
x_{m+n,1} & x_{m+n,2} & \cdots & x_{m+n,n}
\end{pmatrix},
\]

with \( x_{j,k} \in A \) for \( 1 \leq j \leq m + n \) and \( 1 \leq k \leq n \), such that \( \|x - a\| < \varepsilon/\|v\| \) and \( x \) is left invertible. This last condition means that there is a \( n \times (m + n) \) matrix \( z \), with entries in $A$, such that \( zx = 1_n \). Then \( 1_n = x^* z^* z x \leq \|z^* z\| x^* x \). Thus, with \( \delta = \|z^* z\|^{-1} \), we have \( x^* x \geq \delta \cdot 1_n \).
Define
\[ y_j = \sum_{k=1}^{n} x_{j,k}v_k \in B \]
for \( 1 \leq j \leq m + n \). Then
\[ xv = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+n} \end{pmatrix}. \]
We therefore get the following relation, which by our convention is really an inequality between matrices in \( B \otimes K \) whose only nonzero entries are in the 1,1 position:
\[ \sum_{j=1}^{m+n} y_j^*y_j = v^*x^*xv \geq \delta v^*v = \delta \sum_{k=1}^{n} v_k^*v_k = \delta \cdot \text{Index}(E). \]
The inequality is still correct when interpreted in \( B \). Since \( \text{Index}(E) \) is a positive invertible element of \( B \), it follows that \( \sum_{j=1}^{m+n} y_j^*y_j \) is invertible in \( B \). Therefore \((y_1, y_2, \ldots, y_{m+n}) \in \text{Lg}_{m+n}(B)\). We have \( ||xv - av|| \leq ||x - a||v|| \leq \varepsilon \), so that \( ||y_k - b_k|| < \varepsilon \) for \( 1 \leq j \leq m + n \). This proves that \( \text{Lg}_{m+n}(B) \) is dense in \( B^{m+n} \).

Using Theorem 2.2 we can sharpen the estimate of Proposition 5.3 of [24].

**Corollary 2.3.** Let \( B \) be a unital \( C^* \)-algebra, let \( A \subset B \) be a unital subalgebra, let \( E: B \to A \) be a faithful conditional expectation with index-finite type, and let \((v_k, v_k^*)\) be a quasi-basis for \( E \). Then
\[ \text{tsr}(A) \leq n \cdot \text{tsr}(B) + n^2 - 2n + 1. \]

**Proof.** In the proof of Proposition 5.3 of [24], substitute the estimate of Theorem 2.2 for the estimate of Corollary 2.6 of [24].

**Theorem 2.4.** Let \( A \) be a \( C^* \)-algebra, and let \( \alpha: G \to \text{Aut}(A) \) be an action of a finite group \( G \) on \( A \). Then \( \text{tsr}(A \rtimes_\alpha G) \leq \text{tsr}(A) + \text{card}(G) - 1 \).

**Proof.** The unital case follows from Theorem 2.2 and Example 2.1. For the nonunital case, let \( A^+ \) be the unitization of \( A \), and observe that \( A \rtimes_\alpha G \) is an ideal in \( A^+ \rtimes_\alpha G \). Using, in order, Theorem 4.4 of [31], the unital case, and Definition 1.4 of [31], we get
\[ \text{tsr}(A \rtimes_\alpha G) \leq \text{tsr}(A^+ \rtimes_\alpha G) \leq \text{tsr}(A^+) + \text{card}(G) - 1 = \text{tsr}(A) + \text{card}(G) - 1, \]
as desired.

**Remark 2.5.** As pointed out in Example 8.2.1 of [5], Theorems 4.3 and 7.1 of [31] can be used to show that for any action \( \alpha: \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(A) \), one has \( \text{tsr}(A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}) \leq \text{tsr}(A) + 1 \). The point is that \( A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z} \) is a quotient of \( A \rtimes_\alpha \mathbb{Z} \). The argument actually works for any finite cyclic group. This estimate is the same as that of Theorem 2.4 for \( \mathbb{Z}/2\mathbb{Z} \), and better for other cyclic groups.

**Remark 2.6.** The estimate in Theorem 2.4 is the best possible of its form. There is a (nonsimple) unital \( C^* \)-algebra \( A \) with \( \text{tsr}(A) = 1 \) and an action \( \alpha: \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(A) \) such that \( \text{tsr}(A \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}) = 2 \). See Example 8.2.1 of [5].
3. Inclusions of \( C^* \)-algebras of finite depth

The notion of finite depth for subfactors is well known. (See, for example, Section 4.6 of [13].) The basic properties of inclusions of \( C^* \)-algebras with finite depth are similar, but have not appeared in the literature. They do not differ greatly from the subfactor case, but, in the interest of completeness, we give proofs here.

**Definition 3.1.** Let \( 1 \in A \subset B \) be an inclusion of unital \( C^* \)-algebras with a conditional expectation \( E : B \to A \) with index-finite type. Set \( B_0 = A, B_1 = B, \) and \( E_1 = E. \) Recall the \( C^* \)-algebra version of the basic construction (Definition 2.2.10 of [38], where it is called the \( C^* \) basic construction). We inductively define \( e_k = e_{B_{k-1}} \) and \( B_{k+1} = C^*(B_k, e_k), \) the Jones projection and \( C^* \)-algebra for the basic construction applied to \( E_k : B_k \to B_{k-1}, \) and take \( E_{k+1} : B_{k+1} \to B_k \) to be the dual conditional expectation \( E_{B_k} \) of Definition 2.3.3 of [38]. This gives the tower of iterated basic constructions

\[
B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_k \subset \cdots,
\]

with \( B_0 = A \) and \( B_1 = B. \) It follows from Proposition 2.10.11 of [38] that this tower does not depend on the choice of \( E.\)

We then say that the inclusion \( A \subset B \) has finite depth if there is \( n \in \mathbb{N} \) such that \( (A' \cap B_n)e_n(A' \cap B_n) = A' \cap B_{n+1} \). We call the least such \( n \) the depth of the inclusion.

**Definition 3.2.** Let \( 1 \in A \subset B \) be an inclusion of unital \( C^* \)-algebras with a conditional expectation \( E : B \to A \) with index-finite type. We say that \( E \) is pseudominimal if \( \text{Index}(E) \) is a scalar multiple of \( 1_A \) and \( E( cb) = E(bc) \) for all \( c \in A' \cap B \) and \( b \in B. \)

When \( A \) and \( B \) have trivial centers, Lemma 3.11 of [17] shows that a minimal conditional expectation is pseudominimal.

The following is an extended version of Lemma 3.11 of [17].

**Lemma 3.3.** Let \( 1 \in A \subset B \) be an inclusion of \( C^* \)-algebras, with conditional expectation \( E : B \to A \) with index-finite type. Suppose that \( A \) is simple. Then there exists a pseudominimal conditional expectation \( F : B \to A \) with index-finite type.

**Proof.** Using Theorem 3.3 of [15] (also see Lemma 2.2 and Remark 2.4(3) of [15]), there exist central projections \( p_1, p_2, \ldots, p_n \in B \) such that \( Bp_j \) is simple for \( 1 \leq j \leq n \) and \( B = \bigoplus Bp_j. \) We then have unital inclusions \( p_jAp_j \subset p_jBp_j = Bp_j \) for each \( j. \) Note that \( p_jBp_j = Bp_j \) is a simple ideal in \( B. \) However, \( p_jAp_j \) is not a subalgebra of \( A, \) only of \( B. \) In fact, since \( A \) is simple, \( \varphi_j(a) = ap_j \) defines an isomorphism from \( A \) to \( p_jAp_j. \)

There are conditional expectations \( E_j : p_jBp_j \to p_jAp_j \) given by \( E_j(b) = p_jE(b)p_j \) for \( b \in p_jBp_j. \) If \( ((u_k, v_k^*)_{1 \leq k \leq m}) \) is a quasi-basis for \( E, \) then \( ((p_ju_k, (p_ju_k)^*)_{1 \leq k \leq m}) \) is a quasi-basis for \( E_j. \) Thus, the unital inclusion \( p_j \in p_jAp_j \subset p_jBp_j \) has index-finite type. Following Proposition 2 of [18], let \( F_j : p_jBp_j \to p_jAp_j \) be the minimal conditional expectation from \( p_jBp_j \) onto \( p_jAp_j \) (which automatically also has index-finite type). Its index can be written as \( \text{Index}(F_j) = \lambda_j p_j \) because it is in the center of the simple \( C^* \)-algebra \( p_jBp_j. \) Let \( ((v_{j,l}, v_{j,l}^*)_{1 \leq l \leq m_j}) \) be a quasi-basis
for $F_j$. Choose $\beta_1, \beta_2, \ldots, \beta_n > 0$ such that
\[
\sum_{j=1}^{n} \beta_j = 1 \quad \text{and} \quad \beta_1^{-1} \lambda_1 = \beta_2^{-1} \lambda_2 = \cdots = \beta_n^{-1} \lambda_n.
\]
Define $F : B \to A$ by
\[
F(b) = \sum_{j=1}^{n} \beta_j \varphi_j^{-1}(F_j(p_j b)).
\]
Then $F$ is a conditional expectation.

Set $w_{j,l} = \beta_j^{-1/2} v_{j,l}$. We claim that $((w_{j,l}, w_{j,l}^*))_{1 \leq j \leq n, 1 \leq l \leq m_j}$ is a quasi-basis for $F$. First note that for $a \in p_j A p_j$ we have $v_{j,l} \varphi_j^{-1}(a) = v_{j,l} a$ and $\varphi_j^{-1}(a) v_{j,l} = a v_{j,l}$, since $v_{j,l} \in p_j B p_j$. Then for $b \in B$ we have, using $p_k v_{j,l} = \delta_{j,k} v_{j,l}$ at the second step,
\[
\sum_{j=1}^{n} \sum_{l=1}^{m_j} w_{j,l} F(w_{j,l}^* b) = \sum_{j=1}^{n} \sum_{l=1}^{m_j} \beta_j^{-1} v_{j,l} \sum_{k=1}^{n} \beta_k \varphi_k^{-1}(F_k(p_k v_{j,l}^* b))
\]
\[
= \sum_{j=1}^{n} \sum_{l=1}^{m_j} v_{j,l} \varphi_j^{-1}(F_j(v_{j,l}^* b))
\]
\[
= \sum_{j=1}^{n} \sum_{l=1}^{m_j} v_{j,l} F_j(v_{j,l}^* b) p_j = \sum_{j=1}^{n} b p_j = b.
\]
The proof that $\sum_{j=1}^{n} \sum_{l=1}^{m_j} F(b w_{j,l}) w_{j,l}^* = b$ is similar.

Now we check the conditions in the definition of pseudominimality. By construction, we have
\[
\text{Index}(F) = \sum_{j=1}^{n} \sum_{l=1}^{m_j} w_{j,l} w_{j,l}^* = \sum_{j=1}^{n} \beta_j^{-1} \lambda_j p_j,
\]
which is a scalar by the choice of the $\beta_j$. For the commutation relation, let $c \in A' \cap B$ and $b \in B$. Then $p_j c \in (p_j A p_j)' \cap (p_j B p_j)$, so minimality of $F_j$, Lemma 3.11 of [17], and centrality of $p_j$, imply $F_j(p_j c) = F_j(p_j b)$, Thus
\[
F(bc) = \sum_{j=1}^{n} \beta_j \varphi_j^{-1}(F_j(p_j bc)) = \sum_{j=1}^{n} \beta_j \varphi_j^{-1}(F_j(p_j cb)) = F(cb).
\]
This completes the proof. \(\blacksquare\)

The next proposition was proved in the II$_1$ factor case by Pimsner and Popa in Theorem 2.6 of [28].

**Proposition 3.4.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a faithful conditional expectation $E : B \to A$ with index-finite type such that $\text{Index}(E) \in A$. Using the notation of Definition 3.1, for $n \geq 1$ we can identify $B_{2n}$ as the basic construction for $A \subset B_n$ and the conditional expectation $F = E_1 \circ E_2 \circ \cdots \circ E_n$.

**Proof.** As in the notation of Definition 3.1 let $e_k$ be the Jones projection for the inclusion of $B_{k-1}$ in $B_k$, so that $B_{k+1} = C^*(B_k, e_k)$. Further, set $z = \text{Index}(E)$. By Lemma 2.3.1 and Proposition 2.3.4 of [28], and induction, $z$ is a positive invertible element which is in the center of $B_n$ for all $n \geq 0$.\(\blacksquare\)
Following the beginning of Section 2 of [28] (but noting that our indexing conventions differ), define \( g_n^{(k)} \in B'_n \cap B_{2n+k} \) by
\[
g_n^{(k)} = (e_{n+k}e_{n+k-1}\cdots e_{k+1})(e_{n+k}e_{n+k-1}\cdots e_{k+2})\cdots (e_{2n+k-1}e_{2n+k-2}\cdots e_{n+k}).
\]
(Take \( g_0^{(k)} = 1 \).) Then set \( f_n^{(k)} = z^{n(n-1)/2}g_n^{(k)} \). Using, say, Definition 2.3.3, Propositions 2.1.1, 2.3.2, and 2.3.4, and Lemma 2.3.5 of [38], we have the usual relations among the projections \( e_k \):

1. \( e_k e_k = E_k(x)e_k \) for \( x \in B_k \).
2. \( e_k e_{k+1} = z^{-1}e_k \).
3. \( e_k e_l = e_l e_k \) when \( |k-l| > 1 \).

Since \( z \) is a positive, invertible, and central, the arguments of Section 2 of [28] (through the calculations in the proof of Theorem 2.6 there) go through (replacing scalars by powers of \( z \)), and imply that \( f_n = f_n^{(0)} \) is a projection, and (using our indexing) that \( E_{2n+k}(g_n^{(k)}) = z^{-n}g_{n-1}^{(k+1)} \) for all \( k \) and \( n \). It follows that \( E_{n+1} \circ E_{n+2} \circ \cdots \circ E_{2n}(f_n) = z^{-n} \).

We now want to calculate \( f_n x f_n \) for \( x \in B_n \). Since \( f_n^* = f_n \), we start out as follows, using \( e_n x e_n = E_n(x)e_n \) at the first step and

\[
E_n(x) \in B_{n-1} \subset B_{2n} \cap \{e_{n+1}, \ldots, e_{2n-2}, e_{2n-1}\}
\]
at the second step:

\[
(3e_{n-1}e_{n-2}\cdots e_n)x(e_n\cdots e_{2n-2}e_{n-1}) = (e_{n-1}e_{n-2}\cdots e_{n+2})E_n(x)e_{n+1}(e_{n+2}\cdots e_{2n-2}e_{n-1})
\]

\[
= E_n(x)(e_{n-1}e_{n-2}e_{n-3}(e_{n+2}e_{n+1}e_{n+2})(e_{n+3}\cdots e_{2n-2}e_{n-1})
\]

\[
= E_n(x)(e_{n-1}e_{n-2}e_{n-3}(e_{n+2}e_{n+1}e_{n+2})(e_{n+3}\cdots e_{2n-2}e_{n-1})
\]

Iterating the last two steps gives

\[
(3e_{n-1}e_{n-2}\cdots e_n)x(e_n\cdots e_{2n-2}e_{n-1}) = z^{-(n-1)}E_n(x)e_{2n-1}.
\]

Set \( x_{n-1} = z^{-(n-1)}E_n(x) \in B_{n-1} \). We now calculate

\[
(3e_{n-2}e_{n-3}\cdots e_{n-1})x_{n-1}(e_{n-1}\cdots e_{2n-4}e_{2n-3})
\]

by a similar method. First, use the commutation relation [3] to write this expression as

\[
e_{2n-2}[(3e_{n-2}e_{n-3}\cdots e_{n-1})x_{n-1}(e_{n-1}\cdots e_{2n-4}e_{2n-3})]e_{2n-1}e_{2n-2}.
\]

The method used above shows that the term in brackets is equal to

\[
z^{-(n-2)}E_{n-1}(x_{n-1})e_{2n-3},
\]

from which it follows, using the fact that \( e_{2n-2} \) commutes with \( B_{n-2} \) at the second step, that

\[
(3e_{n-2}e_{n-3}\cdots e_{n-1})x_{n-1}(e_{n-1}\cdots e_{2n-2}e_{n-2})
\]

\[
= e_{2n-2}[(3e_{n-2}e_{n-3}\cdots e_{n-1})x_{n-1}(e_{n-1}\cdots e_{2n-2}e_{n-2})]
\]

\[
= z^{-(n-2)}E_{n-1}(x_{n-1})(e_{2n-2}e_{2n-3})(e_{2n-1}e_{2n-2}).
\]

Proceeding inductively, and putting in the appropriate power of \( z \), we finally arrive at

\[
f_n x f_n = (E_1 \circ E_2 \circ \cdots \circ E_n)(x)f_n.
\]
This result implies in particular that the map \( x \mapsto x f_n \) is injective on \( B_0 \). Proposition 2.2.11 of \cite{38} therefore implies that the subalgebra \( C^*(B_n, f_n) \subset B_{2n} \) is the basic construction for \( B_0 \subset B_n \).

Using \( f_n^{(k)} \) in place of \( f_n = f_n^{(0)} \), we obtain analogous results, and, in particular, for every \( k \), the subalgebra \( C^*(B_{n+k}, f_n^{(k)}) \subset B_{2n+k} \) is the basic construction for \( B_k \subset B_{n+k} \).

We now prove by induction on \( n \) that \( C^*(B_{n+k}, f_n^{(k)}) = B_{2n+k} \) for all \( k \). This is true by hypothesis for \( n = 1 \), so suppose it is known for \( n - 1 \). The inclusion \( B_k \subset B_{k+1} \), with conditional expectation \( E_{k+1} \), has index-finite type by Proposition 1.6.6 and Definition 2.3.3 of \cite{38}. Let \( (v_1, v_1^*), (v_2, v_2^*), \ldots, (v_r, v_r^*) \) be a quasi-basis for this inclusion. Then

\[
\sum_{j=1}^r v_j e_{k+1} v_j^* = \sum_{j=1}^r E_{k+1}(v_j) v_j^* = 1,
\]

and \( v_j \) commutes with \( e_l \) for \( l > k + 1 \), so

\[
\sum_{j=1}^r v_j g_n^{(k)} v_j^* = (e_{n+k} e_{n+k-1} \cdots e_{k+2})(e_{n+k+1} e_{n+k} \cdots e_{k+2}) \cdots (e_{2n+k-1} e_{2n+k-2} \cdots e_{n+k}).
\]

(The difference from the definition of \( g_n^{(k)} \) is that here the last term in the first set of parentheses is missing.) Now

\[
(e_{n+k} e_{n+k-1} \cdots e_{k+2})(e_{n+k+1} e_{n+k} \cdots e_{k+2}) = (e_{n+k} e_{n+k-1} \cdots e_{k+3})(e_{n+k+1} e_{n+k} \cdots e_{k+4}) e_{k+2}
\]

\[
= z^{-1} (e_{n+k} e_{n+k-1} \cdots e_{k+3})(e_{n+k+1} e_{n+k+2} \cdots e_{k+4} e_{k+2}
\]

\[
= z^{-1} (e_{n+k} e_{n+k-1} \cdots e_{k+3} e_{k+2})(e_{n+k+1} e_{n+k} \cdots e_{k+4}).
\]

Next, we combine the second factor above with the third factor in the original expression:

\[
(e_{n+k+1} e_{n+k} \cdots e_{k+4})(e_{n+k+2} e_{n+k+1} \cdots e_{k+3}) = (e_{n+k+1} e_{n+k} \cdots e_{k+5})(e_{n+k+2} e_{n+k+1} \cdots e_{k+6}) e_{k+3}
\]

\[
= z^{-1} (e_{n+k+1} e_{n+k} \cdots e_{k+5})(e_{n+k+2} e_{n+k+1} \cdots e_{k+6} e_{k+3}
\]

\[
= z^{-1} (e_{n+k+1} e_{n+k} \cdots e_{k+5} e_{k+4} e_{k+3})(e_{n+k+2} e_{n+k+1} \cdots e_{k+6}).
\]

We proceed similarly through the rest of the factors. After \( l \) steps, the factor in position \( l + 1 \) is \( e_{n+k+l+1} e_{n+k+l-1} \cdots e_{k+2l+2} \). After \( n - 1 \) steps, and using the definitions of \( g_n^{(k)} \) and \( f_n^{(k)} \), we get

\[
\sum_{j=1}^r v_j g_n^{(k)} v_j^* = z^{-(n-1)} g_n^{-1}(k+1) \quad \text{and} \quad \sum_{j=1}^r v_j f_n^{(k)} v_j^* = f_n^{(k+1)}.
\]

Thus \( f_n^{(k+1)} \in C^*(B_{n+k}, f_n^{(k)}) \), so, using the induction hypothesis,

\[
B_{2n+k-1} = C^*(B_{n+k}, f_n^{(k+1)}) \subset C^*(B_{n+k}, f_n^{(k)}).
\]

In particular, \( e_l \in C^*(B_{n+k}, f_n^{(k)}) \) for \( l \leq 2n + k - 2 \). It remains to show that \( e_{2n+k-1} \in C^*(B_{n+k}, f_n^{(k)}) \), which takes some work.
To start, we claim that, for any \( m \) and \( j \), we have

\[
\left[ e_{j+1} e_{j+2} \cdots e_{m+j-2} e_{m+j-1} \right] g_{m}^{(j)} \left[ e_{m+j-1} e_{m+j-2} \cdots e_{j+2} e_{j+1} \right] = z^{-2(m-1)} g_{m-1}^{(j+2)} e_{j+1}.
\]

Let \( b \) be the left hand side of this equation. To prove the equation, start by using the relation \((2)\) a total of \( m - 1 \) times to show that the product of the first factor above and the first factor in the expression for \( g_{m}^{(j)} \) is \( z^{-(m-1)} e_{j+1} \). Then use the relation \((3)\) to commute \( e_{j+1} \) past all but the last term in the second factor in the expression for \( g_{m}^{(j)} \), getting

\[
b = z^{-(m-1)} (e_{m+j+1} e_{m+j} \cdots e_{j+3}) e_{j+1} e_{j+2} (e_{m+j+2} e_{m+j+1} \cdots e_{j+3}) \\
\cdots (e_{2m+j-1} e_{2m+j-2} \cdots e_{m+j+1}) (e_{j+1} e_{j+2} \cdots e_{m+j}) (e_{m+j-1} e_{m+j-2} \cdots e_{j+1}).
\]

For the same reason, one can commute \( e_{j+1} e_{j+2} \) past all but the last term in the next factor, then \( e_{j+1} e_{j+2} e_{j+3} \) past all but the last term in the factor after that, etc. This gives

\[
b = z^{-(m-1)} (e_{m+j+1} e_{m+j} \cdots e_{j+3}) (e_{m+j+2} e_{m+j+1} \cdots e_{j+4}) \\
\cdots (e_{2m+j-1} e_{2m+j-2} \cdots e_{m+j+1}) (e_{j+1} e_{j+2} \cdots e_{m+j}) (e_{m+j-1} e_{m+j-2} \cdots e_{j+1}).
\]

By repeated application of \((2)\), the terms in square brackets combine to give \( z^{-(m-1)} e_{j+1} \), and the other terms are by definition \( g_{m-1}^{(j+2)} \). This completes the proof of the claim.

Next, we claim that if \( m \geq 1 \) and a C*-algebra \( C \subset B_N \), for large \( N \), contains \( B_{m+j} \) and \( g_{m}^{(j)} \), then \( C \) also contains \( g_{m-1}^{(j)} \). To see this, choose a quasi-basis for \( E_{j+1} \): \( B_{j+1} \rightarrow B_j \), say \( (w_{l}, w_{l}^*) \) for \( 1 \leq l \leq r \), with \( w_{1}, w_{2}, \ldots, w_{r} \in B_{j+1} \subset C \). Again letting \( b \) be the left hand side of the equation in the previous claim, it follows directly from the formula that \( b \in C \). Therefore also \( \sum_{l=1}^{r} w_{l} b w_{l}^* \subset C \). Also, \( \sum_{l=1}^{r} w_{l} c_{j+1} w_{l}^* = 1 \) (apply the defining equation of a quasi-basis to the element 1), and \( g_{m-1}^{(j+2)} \) commutes with each \( w_{l} \) (because \( g_{m-1}^{(j+2)} \) is a product of the projections \( e_{j+3}, e_{j+4}, \ldots, e_{2m+j-1} \)). Using the previous claim, we therefore get

\[
\sum_{l=1}^{r} w_{l} b w_{l}^* = z^{-2(m-1)} \sum_{l=1}^{r} w_{l} g_{m-1}^{(j+2)} e_{j+1} w_{l}^* \\
= z^{-2(m-1)} g_{m-1}^{(j+2)} \sum_{l=1}^{r} w_{l} e_{j+1} w_{l}^* = z^{-2(m-1)} g_{m-1}^{(j+2)}.
\]

Since \( z^{-2(m-1)} \) is invertible, the claim follows.

Take \( C = C^* (B_{n+k}, f_{n}^{(k)}) \). Recall that \( g_{n}^{(k)} \) is a power of \( z \) times \( f_{n}^{(k)} \), and that we proved that \( B_{2n+k-1} \subset C \). Apply the previous claim, first with \( m = n \) and \( j = k \), to get \( g_{n-1}^{(k+2)} \in C \); then with \( m = n-1 \) and \( j = k+2 \), to get \( g_{n-2}^{(k+4)} \in C \); etc.; finally, with \( m = 2 \) and \( j = 2n+k-4 \), to get \( g_{1}^{(2n+k-2)} \in C \). Since \( g_{1}^{(2n+k-2)} = e_{2n+k-1} \), it follows that \( B_{2n+k} \subset C \). Thus \( C^* (B_{n+k}, f_{n}^{(k)}) = B_{2n+k} \). This completes the induction step.

Taking \( k = 0 \), we get \( C^* (B_{n}, f_{n}) = B_{2n} \), as desired. This completes the proof.
Remark 3.5. Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a faithful conditional expectation $E: B \to A$ with index-finite type. For right Hilbert $B$-modules $X$ and $Y$, we denote by $\text{Hom}_B(X,Y)$ the set of $B$-linear maps $T$ from $X$ to $Y$ which are bounded in the sense that $\|T\xi, T\xi\| \leq M\|\xi, \xi\|_B$ for some positive $M$ and which have adjoints $T^*$ with respect to the $B$-valued inner products. We put $\text{End}_B(X) = \text{Hom}_B(X,X)$. We use similar notation $\text{AHom}_B(X,Y)$ and $\text{End}(X)$ for left Hilbert $A$-modules $X$ and $Y$. Let $\text{Hom}_B(X,Y) = \text{AHom}(X,Y) \cap \text{Hom}_B(X,Y)$ for Hilbert $A-B$ bimodules, etc. As discussed after Lemma 1.10 of [17], for a Hilbert $B-B$ bimodule $X$, we can make the following identifications, in which $A'$ is interpreted as those maps commuting with the left action of $A$, and similarly for $B'$:

$$A' \cap \text{End}_B(X) = A\text{End}_B(X) \quad \text{and} \quad B' \cap \text{End}_B(X) = B\text{End}_B(X).$$

We now interpret $B_l$ as a Hilbert module over $A$ using the composition $E_1 \circ E_2 \circ \cdots \circ E_l$ of conditional expectations, and similarly over $B$ omitting the term $E_1$. (See the example after Lemma 1.10 of [17].) The basic construction $B_2 = C^*(B, e_1)$ for an inclusion $A \subset B$ of index-finite type can then be identified as $\text{End}_A(B)$, with the Jones projection $e_1 = e_A$ being the conditional expectation $E$, regarded as an operator on $B$. (See Definition 2.1.2, Proposition 1.3.3, and Definition 2.2.10 of [35].)

If $\text{Index}(E) \in A$, Proposition 3.4 then identifies $\text{End}_A(B_l)$ with $B_{2l}$ and $\text{End}_B(B_{l+1})$ with $B_{2l+1}$ as Hilbert $B-B$ bimodules. This gives

$$A' \cap B_{2l} = A\text{End}_A(B_l), \quad A' \cap B_{2l+1} = A\text{End}_B(B_{l+1}),$$

$$B' \cap B_{2l} = B\text{End}_A(B_l), \quad B' \cap B_{2l+2} = B\text{End}_B(B_{l+1})$$

for all $l$. Note that these are all finite dimensional. (Use Propositions 2.3.4 and 2.7.3 of [35].)

Lemma 3.6. Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a conditional expectation $E: B \to A$ with index-finite type. Then the basic construction $C^*(B, e)$ is equal to $B\text{e}B$.

Proof. Let $((u_j, u_j^*))_{1 \leq j \leq n}$ be a quasi-basis for $E$. Let $y \in C^*(B, e)$. Identifying $C^*(B, e)$ with $\text{End}_A(B)$ as in Remark 3.4, $y$ corresponds to a right $A$-module endomorphism $T$ of $B$. Then for $x \in B$ we have $T(x) = \sum_{j=1}^n T(u_j E(u_j^* x))$. Since $T(u_j) \in B$, this formula shows that $y = \sum_{j=1}^n T(u_j) e u_j^* \in B\text{e}B$.

The following is a C* version of Lemma 1.2 of [29].

Lemma 3.7. Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a conditional expectation $E: B \to A$ with index-finite type. Following Definition 3.4 but with different notation, let $e$ be the Jones projection for this inclusion, and let $C^*(B, e)$ be the basic construction. Then for every $x \in C^*(B, e)$ there is a unique $b \in B$ such that $be = xe$. Moreover, if $D \subset A$ and $x \in D' \cap C^*(B, e)$, then $b \in D' \cap B$.

Proof. We first claim that if $b, c \in B$ satisfy $bc = ce$, then $b = c$. It suffices to consider the case $c = 0$. If $be = 0$, then $E(\text{b}^*b)e = e\text{b}^*be = 0$. Identifying $C^*(B, e)$ with $\text{End}_A(B)$ as in Remark 3.5 we see that in fact $E(\text{b}^*b) = 0$, which implies $b = 0$ because $E$ is faithful. This proves the claim.

Uniqueness in the statement of the lemma follows.
Let $F: C^*(B, e) \to B$ be the dual conditional expectation (again, following Definition 3.1). We claim that if $x \in C^*(B, e)$ then $b = \text{Index}(E)F(xe)$ satisfies the conclusion of the lemma. By Lemma 3.8 it suffices to consider $x = y_1y_2$ with $y_1, y_2 \in B$. Then $xe = y_1E(y_2)e$, so the formula of Proposition 2.3.2 of [28] gives $F(xe) = \text{Index}(E)^{-1}y_1E(y_2)$. Therefore

$$be = \text{Index}(E)[\text{Index}(E)^{-1}y_1E(y_2)]e = xe.$$ 

This proves the claim.

It remains only to show that if $x \in D' \cap C^*(B, e)$, then $b \in D' \cap B$. Let $a \in D$. Then $a \in A$, so $a$ commutes with $e$ and $x$, and we get

$$abe = aexe = xeae = bea = bae.$$ 

The first claim in the proof now implies that $ab = ba$. This completes the proof. \[\]

**Corollary 3.8.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a conditional expectation $E: B \to A$ with index-finite type. Adopt the notation of Definition 3.1. Then for $k \geq 1$ we have

$$(A' \cap B_k)e_k(A' \cap B_k) = (A' \cap B_{k+1})e_k(A' \cap B_{k+1}).$$

**Proof.** Fix $k$. Proposition 2.3.4 of [28] implies that $E_k: B_k \to B_{k-1}$ has index-finite type. We may therefore apply Lemma 3.8 to the inclusion of $B_{k-1}$ in $B_k$, getting $(A' \cap B_k)e_k = (A' \cap B_{k+1})e_k$. Taking adjoints and applying the same lemma again, we obtain the conclusion. \[\]

If $A$ is any unital C*-algebra, we let $Z(A)$ denote the center of $A$.

**Lemma 3.9.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with a conditional expectation $E: B \to A$ with index-finite type. Assume that $Z(A)$ is finite dimensional. Adopt the notation of Definition 3.1. Then for $k \geq 0$ we have

$$\dim_{C}(Z(A' \cap B_k)) \leq \dim_{C}(Z(A' \cap B_{k+2})),$$

with equality if and only if $e_{k+1}$ is full in $A' \cap B_{k+2}$.

**Proof.** Using Propositions 2.3.4 and 2.7.3 of [28], and induction, we see that $B_k \cap B_{k+1}$ is finite dimensional whenever $0 \leq m \leq n$. Since $e_{k+1}[A' \cap B_{k+2}]e_{k+1}$ is a corner in $A' \cap B_{k+2}$, it therefore suffices to prove that

$$\dim_{C}(Z(A' \cap B_k)) = \dim_{C}(Z(e_{k+1}[A' \cap B_{k+2}]e_{k+1})).$$

Since $e_{k+1}$ commutes with everything in $B_k$, the set $(A' \cap B_k)e_{k+1}$ is a C*-algebra, and Lemma 3.7 further implies that $x \mapsto xe_{k+1}$ is an isomorphism $A' \cap B_k \to (A' \cap B_k)e_{k+1}$. Using the commutation relation at the first step,

$$(A' \cap B_k)e_{k+1} = E_k(A' \cap B_{k+1})e_{k+1} = e_{k+1}(A' \cap B_{k+1})e_{k+1}$$

at the second step, Corollary 3.8 at the third step, and $e_{k+1} \in A' \cap B_{k+2}$ at the fourth step, we further get

$$(A' \cap B_k)e_{k+1} = e_{k+1}(A' \cap B_{k+1})e_{k+1}(A' \cap B_k)e_{k+1}$$

$$= e_{k+1}[A' \cap B_{k+1}]e_{k+1} = e_{k+1}[A' \cap B_{k+1}]e_{k+1} = e_{k+1}(A' \cap B_{k+2})e_{k+1}.$$ 

Thus $A' \cap B_k \cong e_{k+1}(A' \cap B_{k+2})e_{k+1}$. In particular, their centers have the same dimension. \[\]
The following proposition is a C* analog of parts of Theorem 4.6.3 of [13].

**Proposition 3.10.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras, and let $E: B \to A$ be a conditional expectation with index-finite type such that $\operatorname{Index}(E) \in A$. Assume that $Z(A)$ is finite dimensional. Let $B_0 = A, B_1 = B, B_3, B_4, \ldots$ and $e_1, e_2, \ldots$ be as in Definition 3.1. Then the following conditions are equivalent:

1. There is $k_0 \in \mathbb{N}$ such that $(A' \cap B_{k_0})e_{k_0}(A' \cap B_{k_0}) = A' \cap B_{k_0+1}$. (That is, $E$ has finite depth in the sense of Definition 3.1)
2. There is $k_0 \in \mathbb{N}$ such that $(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$ for all $k \geq k_0$.
3. There is $L \in \mathbb{N}$ such that $\dim(Z(A' \cap B_{2k})) \leq L$ for $k \geq 1$.
4. There is $M \in \mathbb{N}$ such that $\dim(Z(A' \cap B_{2k+1})) \leq M$ for $k \geq 1$.
5. There is $L \in \mathbb{N}$ such that, for any $k \geq 1$, the number of equivalence classes of irreducible $A$-Hilbert bimodules which appear in $B_k$ is at most $L$.
6. There is $M \in \mathbb{N}$ such that, for any $k \geq 1$, the number of equivalence classes of irreducible $A$-$B$ Hilbert bimodules which appear in $B_k$ is at most $M$.

Moreover, the number $k_0$ in (2) can be chosen to be the depth of the inclusion, as in Definition 3.1.

**Proof.** Assume (1), for some $k_0$. Using (1) at the second step, $e_{k_0}e_{k_0+1}e_{k_0} = \operatorname{Index}(E)^{-1}e_{k_0}$ (with $\operatorname{Index}(E)^{-1} \in A' \cap B_{k_0+1}$ and invertible) at the third step, and Corollary 3.8 at the fifth step, we get

$$1 \in A' \cap B_{k_0+1} = (A' \cap B_{k_0+1})e_{k_0}(A' \cap B_{k_0+1}) = (A' \cap B_{k_0+1})e_{k_0}e_{k_0+1}e_{k_0}(A' \cap B_{k_0+1}) \\ \subset (A' \cap B_{k_0+1})e_{k_0+1}(A' \cap B_{k_0+1}) = (A' \cap B_{k_0+2})e_{k_0+1}(A' \cap B_{k_0+2}).$$

Since the last expression is an ideal in $A' \cap B_{k_0+2}$, we see by looking at the second last expression that

$$(A' \cap B_{k_0+1})e_{k_0+1}(A' \cap B_{k_0+1}) = A' \cap B_{k_0+2}.$$

Repeating this, we get

$$(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$$

for all $k \geq k_0$. This is (2), and also proves the last statement.

Assuming (2), Corollary 3.8 and Lemma 3.9 imply that

$$\dim(Z(A' \cap B_{k+2})) = \dim(Z(A' \cap B_k))$$

for $k \geq k_0$. This gives both (3) and (4).

We now prove that (3) implies (1). The proof that (4) implies (1) is the same, and is omitted. So suppose that $L = \sup_{k \geq 1} \dim(Z(A' \cap B_{2k}))$ is finite. Lemma 3.9 implies that $k \mapsto \dim(Z(A' \cap B_{2k}))$ is nondecreasing. So there exists $k$ such that $\dim(Z(A' \cap B_{2k})) = \dim(Z(A' \cap B_{2k+2}))$. Now Lemma 3.9 implies that $(A' \cap B_{2k+1})e_{2k+1}(A' \cap B_{2k+1}) = A' \cap B_{2k+1}$. This proves (1) with $k_0 = 2k+1$.

The equivalence of (3) and (5) follows by using Remark 3.3 to see that $\dim(Z(A' \cap B_{2l}))$ and $\dim(Z(A' \cap B_{2l-2}))$ are the numbers of equivalence classes of irreducible Hilbert $A$-$A$ bimodules which appear in $B_l$ and $B_{l-1}$ respectively. The equivalence of (4) and (6) is similar.

**Proposition 3.11.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras, and let $E: B \to A$ be a conditional expectation with index-finite type and such that
Index(\(E\)) \(\in A\). Assume that \(Z(A)\) is finite dimensional. Let \(A \subset B \subset B_2\) be the basic construction. If \(A \subset B\) has finite depth, then so does \(B \subset B_2\).

**Proof.** We will prove that \(A' \cap B_{2l+1} \cong B' \cap B_{2l+2}\) for \(l \geq 0\). By Remark 4.3, we have \(A' \cap B_{2l+1} = \mathcal{A} \text{End}_B(B_{l+1})\) and \(B' \cap B_{2l+2} = \mathcal{B} \text{End}_A(B_{l+1})\). Define a map \(\Phi\) from \(\mathcal{A} \text{End}_B(B_{l+1})\) to \(\mathcal{B} \text{End}_A(B_{l+1})\) by \(\Phi(T)(x) = (T(x^*))^*\) for \(T \in \mathcal{A} \text{End}_B(B_{l+1})\) and \(x \in B_{l+1}\). It is obvious that \(\Phi\) is an isomorphism from \(A' \cap B_{2l+1}\) to \(B' \cap B_{2l+2}\).

So Condition \(3\) of Proposition 3.10 for the inclusion \(A \subset B\) implies Condition \(4\) of Proposition 3.11 for the inclusion \(B \subset B_2\). \(\blacksquare\)

**Proposition 3.12.** Let \(1 \in A \subset B\) be an inclusion of simple unital C*-algebras, with a conditional expectation \(E\) \(E: B \to A\) with index-finite type. The inclusion \(A \subset B\) has finite depth in the sense of Definition 4.4 if and only if it has finite depth in the sense of Definition 4.5 of [15].

**Proof.** We use the notation of Definition 3.4.

It follows from Proposition 2.10.11 of [38] that Condition \(3\) of Proposition 3.10 does not depend on the choice of the conditional expectation, as long as it has index-finite type. By Lemma 3.3 we may therefore assume that Index(\(E\)) is a scalar, so that Proposition 3.10 applies.

Let \(\iota: A \otimes K \rightharpoonup B \otimes K\) be the inclusion. For any sector \(\eta \in \text{Sect}(B, A)\), in the sense of Section 4 of [15], let \(N(\eta)\) denote the number of distinct irreducible sectors in the decomposition of \(\eta\) of Lemma 4.1 of [15]. We prove the following:

\[
N([[(\tau)^k]]) = \dim_{\mathcal{C}}(Z(B' \cap B_{2k+1})), \quad N([[(\iota)^k]]) = \dim_{\mathcal{C}}(Z(A' \cap B_{2k+1})),
\]

\[
N([[(\gamma)^2]]) = \dim_{\mathcal{C}}(Z(A' \cap B_{2k})), \quad \text{and} \quad N([[(\gamma)^3]]) = \dim_{\mathcal{C}}(Z(B' \cap B_{2k+2})).
\]

The result will then follow from Proposition 3.10 and Proposition 3.11.

First, let \(\gamma: B \otimes K \rightharpoonup B \otimes K\) be the canonical endomorphism of Lemma 4.2 and Remark 4.3 of [15]. Then Remark 4.3 of [15] gives the downward tower of basic constructions

\[
(B \otimes K) \supset (A \otimes K) \supset \gamma(B \otimes K) \supset \gamma(A \otimes K) \supset \gamma^2(B \otimes K) \supset \gamma^2(A \otimes K) \supset \cdots.
\]

Reverse the segment of this tower ending at \(\gamma^{k+1}(A \otimes K)\). Using \(\gamma = \iota\), we can identify the result as

\[
A \otimes K \leftarrow B \otimes K \leftarrow A \otimes K \leftarrow \cdots \leftarrow A \otimes K \leftarrow B \otimes K,
\]

with all inclusions \(A \otimes K \leftarrow B \otimes K\) given by \(\iota\) and all inclusions \(B \otimes K \leftarrow A \otimes K\) given by \(\gamma\). Since the basic construction is preserved by tensoring with \(K\), this tower is isomorphic to

\[
A \otimes K \subset B \otimes K \subset B_2 \otimes K \subset \cdots \subset B_{2k+1} \otimes K \subset B_{2k+2} \otimes K.
\]

So, for example,

\[
[(\tau)^k\iota](B \otimes K)' \cap M(A \otimes K) \cong (B \otimes K)' \cap M(B_{2k+2} \otimes K).
\]

Remark 2.12 of [15] shows that the right hand side is isomorphic to \(B' \cap B_{2k+2}\). It is clear from the proof of Lemma 4.1 of [15] that if \(C\) and \(D\) are any simple stable \(\sigma\)-unital C*-algebras, and if \(\rho: C \to D\) defines a sector, then \(N([\rho]) = \dim_{\mathcal{C}}(\rho(C)' \cap M(D))\). The relation \(N([[(\tau)^k\iota]]) = \dim_{\mathcal{C}}(Z(B' \cap B_{2k+2}))\) now follows.

The other three relations to be proved follow similarly. \(\blacksquare\)
We will need two results about the cut-down $p \in pAp \subset pBp$ of an inclusion $1 \in A \subset B$ by a projection $p \in A' \cap B$. (Note that $pAp$ is usually not contained in $A$, because $p$ need not be in $A$.) These are proved in the setting of factors in Remark 2.6 of [1].

The next proposition looks very much like Corollary 4.2 of [24], but differs in that here the projection $p$ is in $A' \cap B$ rather than $A$.

**Proposition 3.13.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with index-finite type, and suppose $A$ is simple. If $p \in A' \cap B$ is a nonzero projection, then the inclusion $pAp \subset pBp$ also has index-finite type.

**Proof.** Following Lemma 3.3 let $E : B \to A$ be a pseudominimal conditional expectation. Then $E(p)$ is a nonzero element in $A \cap A' = C$, so $E(p) = \lambda \cdot 1$ for some $\lambda \in (0, \infty)$. Define a map $F$ from $pBp$ onto $pAp$ by $F(x) = \lambda^{-1}E(x)p$ for $x \in pBp$. It is easy to see that $F$ is a conditional expectation from $pBp$ onto $pAp$. Let $((u_j, u_j^*))_{1 \leq j \leq n}$ be a quasi-basis for $E$. We claim that $((\lambda^{1/2}\mu_j, \lambda^{1/2}\mu_j^*))_{1 \leq j \leq n}$ is a quasi-basis for $F$. For any $x \in B$, we have $E(px) = E(xp)$ by Definition 3.2. Using this and $p \in A'$ at the third step, for any element $x \in pBp \subset B$ we have

$$x = pxp = \sum_{j=1}^{n} p\mu_j E(u_j^*px)p = \sum_{j=1}^{n} p\mu_j E(pu_j^*px)p = \sum_{j=1}^{n} \lambda^{1/2}p\mu_j F(\lambda^{1/2}p\mu_j^*px).$$

A similar argument proves that

$$x = \sum_{j=1}^{n} F(x\lambda^{1/2}\mu_j)p\lambda^{1/2}p\mu_j^*p.$$

This proves the claim. The existence of a quasi-basis implies that $F$ has index-finite type.

**Proposition 3.14.** Let $1 \in A \subset B$ be an inclusion of unital C*-algebras with index-finite type and finite depth, and suppose $A$ is simple. If $p$ is a nonzero projection in $A' \cap B$, then the inclusion $pAp \subset pBp$ also has finite depth.

**Proof.** Lemma 3.3 provides a conditional expectation $E : B \to A$ with index-finite type and such that $\text{Index}(E)$ is a scalar. Let

$$A \subset B \subset B_2 \subset \cdots \subset B_k \subset \cdots \quad \text{and} \quad pAp \subset pBp \subset C_2 \subset \cdots \subset C_k \subset \cdots$$

be the towers of iterated basic constructions.

Let $R_p$ be the projection in $\text{End}_A(B) = B_2$ (see Remark 3.3 for this identification) defined by $R_p(b) = bp$ for $b \in B$. Then $R_p \in B' \cap B_2$, because the embedding of $B$ in $\text{End}_A(B)$ is via left multiplications. In particular $pR_p = R_p p$. Let $p_1$ be the projection in $A' \cap B_2$ defined by $p_1 = pR_p$, that is, as an operator in $\text{End}_A(B)$, we have $p_1(b) = pBp$ for $b \in B$. Define an isomorphism $\varphi$ from $p_1B_2p_1$ onto $\text{End}_{B_2}(pBp)$ by $\varphi(p_1xp_1) = xp_1xp_1|_{pBp}$. Then for $x, b \in B$, we have $(p_1xp_1)(pxp) = pxp(px)$, so $\varphi(p_1B_2p_1) = pBp$ and $\varphi(p_1Ap_1) = pAp$. Thus we have an isomorphism of inclusions

$$(pAp \subset pBp \subset C_2) \cong (p_1Ap_1 \subset p_1B_2p_1 \subset p_1B_2p_1).$$

By Proposition 3.3 we can identify $\text{End}_A(B_{2l-1})$ with $B_{2l}$ for any $l \geq 2$. The argument of the previous paragraph therefore gives projections $p_l = p_{l-1}R_{p_{l-1}} \subset A' \cap B_{2l}$ such that

$$p_l(x) = p_{l-1}xp_{l-1}$$
for $x \in B_2^{1-1}$, and isomorphisms of inclusions
\[(pAp \subset C_2^{1-1} \subset C_2) \cong (p_1Ap_1 \subset p_1B_2^{1-1}, p_1 \subset p_1B_2^{1-1}).\]
Since $A \subset B$ has finite depth, Proposition 3.10 provides a constant $M$ such that $\dim (Z(A' \cap B_2)) \leq M$ for $l \geq 2$. Therefore we have
\[\dim (Z((pAp)' \cap C_2)) = \dim (Z((p_1Ap_1)' \cap p_1B_2^{1-1})) \leq \dim (Z(A' \cap B_2)) \leq M.\]
Since $(\dim (Z((pAp)' \cap C_{2n})) \in n \in \mathbb{N}$ is nondecreasing (by Lemma 3.9), we have
\[\dim (Z((pAp)' \cap C_{2n})) \leq M\]
for $n \in \mathbb{N}$. The algebra $pAp$ is simple (being isomorphic to $A$), so Proposition 3.13 and Lemma 3.3 provide a conditional expectation $F: pBp \to pAp$ with index-finite type and such that $\text{Index}(F)$ is a scalar. Now Proposition 3.10 implies that $pAp \subset pBp$ has finite depth.\[\]

4. Cancellation for inclusions

In this section we prove a cancellation theorem for inclusions of simple C*-algebras with index-finite type. We need the following modification of Blackadar’s cancellation theorem in [2]. (Also see Theorem 4.2.2 of [4].) Since that theorem is itself a modification of an argument of Rieffel [32], we give a detailed proof for the reader’s convenience. It is based on an argument of Goodearl [12], which is given here with his permission.

**Theorem 4.1** (Blackadar [2]). Let $A$ be a simple C*-algebra. Let $P \subset M_\infty(A)$ be a set of nonzero projections with the following two properties:

1. For every nonzero projection $q \in M_\infty(A)$, there exists $p \in P$ such that $2[p] \leq [q]$ in $K_0(A)$.
2. $\sup_{p \in P} \text{tsr}(pM_\infty(A)p) < \infty$.

Then the projections in $M_\infty(A)$ satisfy cancellation.

In the arguments leading up to the proof, we will tacitly identify projections with their Murray-von Neumann equivalence classes. For example, if $p, q \in M_\infty(A)$ are projections, we will let $p \oplus q$ stand for any specific projection in $M_\infty(A)$ with the appropriate Murray-von Neumann equivalence class, without saying which one. We further let $n \ast p$ stand for the direct sum of $n$ copies of $p$. We use $\sim$ for Murray-von Neumann equivalence, and we write $p \preceq q$ when there exists $r$ such that $p \sim r \leq q$.

We will need the following result of Warfield, which we restate here for C*-algebras and in terms of projections.

**Theorem 4.2.** Let $A$ be a C*-algebra, let $p, q, r \in M_\infty(A)$ be projections, and let $n \in \mathbb{N}$. Assume that
\[p \oplus r \sim q \oplus r, \quad \text{tsr}(rM_\infty(A)r) \leq n, \quad \text{and} \quad n \ast r \preceq p.\]
Then $p \sim q$.

**Proof.** Using the Bass stable rank $\text{Bsr}(A)$ in place of $\text{tsr}(A)$, and using modules in place of projections, this is Theorem 2.1 of [32], which is really just a combination of Theorems 1.2 and 1.6 of [37]. But $\text{Bsr}(A) \leq \text{tsr}(A)$ by Corollary 2.4 of [31]. (In fact, for any unital C*-algebra $A$, we have $\text{Bsr}(A) = \text{tsr}(A)$ by [14].)\[\]

We further need the following well known lemma, whose proof we omit.
Lemma 4.3. Let $A$ be a simple C*-algebra, and let $p, q \in \mathcal{M}_\infty(A)$ be projections with $q \neq 0$. Then there exists $l \in \mathbb{N}$ such that $p \not\prec l \cdot q$.

Proof of Theorem 4.1. Set $n = \sup_{p \in P} \text{tsr}(p \mathcal{M}_\infty(A)p)$. By iterating Condition (1) of the hypothesis, we find that for every $m \in \mathbb{N}$ and every nonzero projection $q \in \mathcal{M}_\infty(A)$, there exists $p \in P$ such that $2^m[p] \leq [q]$ in $K_0(A)$. In particular, for every nonzero projection $q \in \mathcal{M}_\infty(A)$, there exists $p \in P$ such that $n[p] \leq [q]$ in $K_0(A)$.

We now claim that for every nonzero projection $q \in \mathcal{M}_\infty(A)$, there exists $p \in P$ such that $n \cdot p \not\prec q$. To prove the claim, given $q$, choose $p \in P$ such that $n[p] \leq [q]$ in $K_0(A)$. By the definition of the order on $K_0(A)$, there are projections $r, s \in \mathcal{M}_\infty(A)$ such that $n \cdot p \oplus r \oplus s \sim q \oplus s$. Lemma 4.3 provides $l$ such that $s \not\prec l \cdot p$, whence

\[ n \cdot p \oplus r \oplus l \cdot p \sim q \oplus l \cdot p. \]

Let $l_0$ be the smallest nonnegative integer $l$ such that (1) holds. This completes the proof of the claim.

Now we can prove the theorem. Let $p, q, r \in \mathcal{M}_\infty(A)$ satisfy $p \oplus r \sim q \oplus r$. Use the claim to choose $e \in P$ such that $n \cdot e \not\prec p$. Thus, there is a projection $s \in \mathcal{M}_\infty(A)$ such that $n \cdot e \oplus s \sim p$. We then have $n \cdot e \oplus s \oplus r \sim q \oplus r$. Lemma 4.3 provides $l$ such that $r \not\prec l \cdot e$, whence

\[ n \cdot e \oplus s \oplus l \cdot e \sim q \oplus l \cdot e. \]

As before, let $l_0$ be the least possible value of $l$ in (2), and if $l_0 > 0$ apply Theorem 4.2 with $(n + l_0 - 1) \cdot e \oplus s$ in place of $p$, with $q \oplus (l_0 - 1) \cdot e$ in place of $q$, and with $e$ in place of $r$, getting

\[ n \cdot e \oplus s \oplus (l_0 - 1) \cdot e \sim q \oplus (l_0 - 1) \cdot e. \]

This is a contradiction, whence $l_0 = 0$, and $p \sim n \cdot e \oplus q \sim q$. □

We need several observations about the topological stable rank of inclusions of C*-algebras to prove cancellation for crossed products. Recall that an inclusion $1 \in A \subset B$ of unital C*-algebras is called irreducible if $A' \cap B = \mathbb{C}1$. (See Example 3.14 of [13].) The irreducible case of the following lemma is essentially contained in Theorem 2.1 of [24].

Lemma 4.4. Let $1 \in A \subset B$ be an inclusion of simple unital C*-algebras with index-finite type and finite depth. Suppose that $A$ has property (SP). Then for any nonzero projection $q \in B$ there exists a nonzero projection $p \in A$ such that $p \not\prec q$.

Proof. The conclusion does not depend on the choice of the conditional expectation, so use Lemma 3.3 to choose $E: B \to A$ with index-finite type such that $\text{Index}(E)$ is a scalar.

We first assume that $1 \in A \subset B$ is irreducible. Let $q \in B$ be a nonzero projection. Since the inclusion $1 \in A \subset B$ is irreducible and has index-finite type and finite depth, Proposition 3.12 and Corollary 7.6 of [15] (also see Remark 2.4(3) of [15])
imply the following outerness condition on $E$: for any $x \in B$ and for any nonzero hereditary C*-subalgebra $C$ of $A$, 
\[
\inf \{ \|c(x - E(x))c\| : c \in C_+, \|c\| = 1 \} = 0.
\]
This condition is equivalent to outerness in the sense of Definition 2.2 of [23]. By Theorem 2.1 and the proof of Corollary 2.3 of [23], there is a nonzero projection $p \in A$ such that $p \lesssim q$.

Now we consider the general case. Since the inclusion has index-finite type, the relative commutant $A' \cap B$ is finite dimensional. (See Proposition 2.7.3 of [38].) The notation $C_B(A)$ there is the relative commutant, defined in the statement of Proposition 1.2.9 of [38].) Thus
\[
A' \cap B \cong \bigoplus_{l=1}^{n} M_{k(l)}(\mathbb{C})
\]
for suitable $n$ and $k(1), k(2), \ldots, k(n)$. For $1 \leq l \leq n$, let $\{ e_{l,j}^{(l)} \}_{1 \leq i,j \leq k(l)}$ be a system of matrix units for the summand $M_{k(l)}$. Each inclusion $e_{l,j}^{(l)} A = e_{l,j}^{(l)} A e_{l,j}^{(l)} \subset e_{l,j}^{(l)} B e_{l,j}^{(l)}$ is irreducible, and by Propositions 3.13 and 3.14 has index-finite type and finite depth. (Note: $e_{l,j}^{(l)} A e_{l,j}^{(l)}$ is usually not contained in $A$.) Set $k = \sum_{l=1}^{n} k(l)$.

Let $q \in B$ be a nonzero projection. Theorem 5.1 of [23] implies that $B$ has Property (SP). Since $B$ is simple, by Lemma 3.2 of [21], there is a projection $q_0 \in B$ such that, in the notation introduced before Theorem 1.2, we have $k * q_0 \lesssim q$. By Lemma 3.5.6 of [22], there is a nonzero projection $q_j^{(l)} \leq e_{j,j}^{(l)}$ such that $q_j^{(l)} \lesssim q_0$. The irreducible case provides nonzero projections
\[
r_j^{(l)} \in e_{j,j}^{(l)} A e_{j,j}^{(l)} \quad \text{and} \quad s_j^{(l)} \in q_j^{(l)} e_{j,j}^{(l)} B e_{j,j}^{(l)} q_j^{(l)} = q_j^{(l)} B q_j^{(l)}
\]
such that $r_j^{(l)} \sim s_j^{(l)}$. Thus $r_j^{(l)} \lesssim q_0$. Since $A$ is simple, the map $A \to e_{j,j}^{(l)} A$ is bijective, so there exists a projection $p_j^{(l)} \in A$, necessarily nonzero, such that $r_j^{(l)} = e_{j,j}^{(l)} p_j^{(l)}$.

Since $A$ is simple and has Property (SP), repeated application of Lemma 1.8 of [27] provides a nonzero projection $p \in A$ such that $p \lesssim p_j^{(l)}$ for all $l$ and $j$. Then we have
\[
p = \sum_{l=1}^{n} \sum_{j=1}^{k(l)} p e_{j,j}^{(l)} = \sum_{l=1}^{n} \sum_{j=1}^{k(l)} p_j^{(l)} e_{j,j}^{(l)} = \sum_{l=1}^{n} \sum_{j=1}^{k(l)} r_j^{(l)} \lesssim k * q_0 \lesssim q.
\]
This completes the proof. \qed

**Proposition 4.5.** Let $1 \in A \subset B$ be an inclusion of simple unital C*-algebras with index-finite type and finite depth. Suppose that $\text{tsr}(A) = 1$ and $A$ has Property (SP). Then $B$ has cancellation.

**Proof.** Let the notation be as in Definition 3.1 and assume that $A \subset B$ has depth $m$. Choose $k \in \mathbb{N}$ such that $k$ is odd and $k \geq m$. Let $P$ be the set of all nonzero projections in $A$. We claim that, as a subset of $B_k$, this set satisfies the conditions of Theorem 4.1.

The C*-algebra $B_k$ is stably isomorphic to $A$. (In [38], see Proposition 1.3.4 and the argument preceding Lemma 3.3.4.) Continuing by induction, and because $k + 1$ is even, we find that $B_l$ is stably isomorphic to $A$ when $l$ is even, and to $B$ when $l$ is
odd. Thus every $B_l$ is simple. Moreover, $B_{k+1}$ is stably isomorphic to $A$ and $B_k$ is stably isomorphic to $B$. Theorem 6.4 of [31] therefore gives $\text{tsr}(B_{k+1}) = 1$. Also, by Proposition 1.6.6 of [38] and induction, every inclusion $B_l \subset B_{l+1}$ has index-finite type.

We claim that Condition (2) of Theorem 4.1 is satisfied. Proposition 3.10 implies $(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$.

By Proposition 4.4 of [24], there are $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_n \in A' \cap B_{k+1}$ such that, for every $p \in P$, the family $\{(pu_j, u_j^*p)\}_{1 \leq j \leq n}$ is a quasi-basis for the conditional expectation $F_p = E_k|_{pB_{k+1}p}$ from $pB_{k+1}p$ onto $pB_kp$. For any such projection $p$, using Corollary 2.3 at the first step and $\text{tsr}(B_{k+1}) = 1$ and Theorem 4.5 of [6] at the second step, we get

$$\text{tsr}(pB_kp) \leq n \cdot \text{tsr}(pB_{k+1}p) + n^2 - 2n + 1 = n^2 - n + 1.$$ 

This proves the claim.

We next prove by induction on $l$ that $B_l$ has Property (SP) and that for every nonzero projection $q \in B_l$ there is a nonzero projection $p \in A$ such that $p \preceq q$. The case $l = 0$, corresponding to $B_1 = A$, is immediate. So suppose the result is known for $l$, and let $q \in B_{l+1}$ be a nonzero projection. The inclusion $B_l \subset B_{l+1}$ has finite depth by Proposition 3.11. Both $B_l$ and $B_{l+1}$ are simple, so $B_{l+1}$ has Property (SP) by Theorem 5.1 of [24]. Moreover, Lemma 3.5.6 then provides a nonzero projection $p_0 \in B_l$ such that $p_0 \preceq q$, and the induction hypothesis provides a nonzero projection $p \in A$ such that $p \preceq p_0$. This completes the induction.

To prove Condition (1) for $P$ as a subset of $B_k$, let $q \in M_\infty(B_k)$ be a nonzero projection. Since $B_k$ has Property (SP), Lemma 3.5.6 of [22] provides a nonzero projection $q_0 \in B_k$ such that $q_0 \preceq q$. The previous paragraph provides a nonzero projection $p_0 \in A$ such that $p_0 \preceq q_0$. Since $A$ has Property (SP), by Lemma 3.5.7 of [22] there exist orthogonal Murray-von Neumann equivalent nonzero projections $p_1, p_2 \in A$ with $p_1, p_2 \leq p_0$. This completes the proof of Condition (1).

We now conclude from Theorem 4.1 that $B_k$ has cancellation. Since $B$ is stably isomorphic to $B_k$, so does $B$.}

Now we drop the requirement that the larger algebra be simple. The following is our main theorem.

**Theorem 4.6.** Let $1 \in A \subset B$ be an inclusion of unital $C^*$-algebras of index-finite type and with finite depth. Suppose that $A$ is simple, $\text{tsr}(A) = 1$, and $A$ has Property (SP). Then $B$ has cancellation.

**Proof.** Since $1 \in A \subset B$ has index-finite type, results of [13] (Theorem 3.3, Definition 2.1, and Remark 2.4(3) there) provide projections $z_1, z_2, \ldots, z_k$ in the center of $B$ such that each $Bz_j$ is simple and

$$B = Bz_1 \oplus Bz_2 \oplus \cdots \oplus Bz_k.$$ 

By Propositions 6.13 and 6.14 each inclusion $z_j \in z_jAz_j \subset Bz_j$ has index-finite type and finite depth. By Proposition 4.5 each $Bz_j$ has cancellation. Hence $B$ has cancellation.

Using an observation by Blackadar and Handelman [8] we can sometimes determine $\text{tsr}(B)$. Recall that a unital $C^*$-algebra $A$ has real rank zero (see Theorem 2.6
of \([10]\)) if every selfadjoint element in \(A\) can be approximated arbitrarily closely by selfadjoint elements with finite spectrum.

**Corollary 4.7.** Let \(1 \in A \subset B\) be a pair of unital C*-algebras of index-finite type and with finite depth. Suppose that \(A\) is simple with \(\text{tsr}(A) = 1\) and Property (SP), and that \(B\) has real rank zero. Then \(\text{tsr}(B) = 1\).

**Proof.** The algebra \(B\) has cancellation by Theorem 4.6. Since \(B\) has real rank zero, \(B\) has Property (HP) by Theorem 2.6 of \([10]\). Therefore Theorem III.2.4 of \([8]\) implies \(\text{tsr}(B) = 1\). \(\Box\)

5. Cancellation for crossed products

In this section, we apply the results of the previous section to crossed products by finite groups, in particular generalizing Theorem 5.4 of \([16]\). For comparison we also give a result on crossed products by \(\mathbb{Z}\).

The following result should be compared with Question 8.2.3 of \([5]\); see Remark 5.7.

**Corollary 5.1.** Let \(A\) be an infinite dimensional simple unital C*-algebra, let \(G\) be a finite group, and let \(\alpha\) be an action of \(G\) on \(A\). Suppose that \(\text{tsr}(A) = 1\) and \(A\) has Property (SP). Then \(A \rtimes \alpha G\) has cancellation. Moreover, if \(A \rtimes \alpha G\) has real rank zero, then \(\text{tsr}(A \rtimes \alpha G) = 1\).

**Proof.** Take \(B = A \rtimes \alpha G\) in Theorem 4.6 and Corollary 4.7. The finite depth condition is satisfied by Lemma 3.1 of \([24]\). \(\Box\)

As an application of Corollary 5.1 we get an interesting result when the inclusion \(A \subset B\) has index 2, but does not necessarily have finite depth.

**Proposition 5.2.** Let \(A\) be an infinite dimensional simple unital C*-algebra with \(\text{tsr}(A) = 1\) and Property (SP). Suppose that the inclusion \(1 \in A \subset B\) has index 2. Then \(B\) has cancellation.

**Proof.** By Lemma 2.1.3 of \([20]\), there is an action \(\beta: \mathbb{Z}/2\mathbb{Z} \to B\) such that the basic construction \(C^*(B, e_A)\) is isomorphic to \(B \rtimes \beta \mathbb{Z}/2\mathbb{Z}\). Proposition 1.3.4 and the discussion before Lemma 3.3.4 of \([38]\) imply that \(C^*(B, e_A)\) is stably isomorphic to \(A\). Therefore \(C^*(B, e_A)\) is a simple unital C*-algebra with Property (SP), and \(\text{tsr}(C^*(B, e_A)) = 1\).

Let \(\hat{\beta}: \mathbb{Z}/2\mathbb{Z} \to B \rtimes \beta \mathbb{Z}/2\mathbb{Z}\) be the dual action. By Takai duality \((33)\) we have

\[
(B \rtimes \beta \mathbb{Z}/2\mathbb{Z}) \rtimes \hat{\beta} \mathbb{Z}/2\mathbb{Z} \cong M_2(B).
\]

Hence \(B\) has cancellation by Corollary 5.1. \(\Box\)

Let \(\alpha \in \text{Aut}(A)\) be an automorphism of a C*-algebra \(A\). There is no conditional expectation of index-finite type from the crossed product \(A \rtimes \alpha \mathbb{Z}\) onto \(A\). Nevertheless, we have the following result.

**Theorem 5.3.** Let \(A\) be a simple unital C*-algebra with \(\text{tsr}(A) = 1\) and Property (SP). Let \(\alpha \in \text{Aut}(A)\) generate an outer action of \(\mathbb{Z}\) on \(A\) (that is, \(\alpha^n\) is outer for every \(n \neq 0\)), such that \(\alpha_* = \text{id}\) on \(K_0(A)\). Then \(A \rtimes \alpha \mathbb{Z}\) has cancellation.
Proof. Let $P$ be the set of all nonzero projections in $M_\infty(A)$, regarded as a subset of $M_\infty(A \rtimes_\alpha \mathbb{Z})$. We claim that $P$ satisfies the conditions in Theorem 5.1.

For Condition (1), let $q \in M_\infty(A \rtimes_\alpha \mathbb{Z})$ be a nonzero projection. Use Theorem 4.2 of [10] to find $e \in P$ such that $e \preceq q$, and Proposition 3.5.6 of [22], to find $p \in P$ such that $2[p] \leq [q]$.

For Condition (2), let $p \in P$. Assume $p \in M_n(A)$, and let $\alpha$ also denote the corresponding automorphism of $M_n(A)$. Then $\alpha_*([p]) = [p]$ because $\alpha_* = \text{id}$. Since $\text{tsr}(A) = 1$, we can find a unitary $u \in M_n(A)$ such that $\alpha(p) = u^*pu$. Define $\beta \in \text{Aut}(M_n(A))$ by $\beta(a) = ua(a)u^*$. Then we calculate as follows, in which the last two isomorphisms come from the proof of Theorem 2.8.3(5) in [25]:

$$pM_\infty(A \rtimes_\alpha \mathbb{Z})p = p(M_n(A) \rtimes_\alpha \mathbb{Z})p \cong p(M_n(A) \rtimes_\beta \mathbb{Z})p \cong pM_n(A)p \rtimes_\beta \mathbb{Z}.$$

Using Theorem 7.1 of [31], we therefore get

$$\text{tsr}(pM_\infty(A \rtimes_\alpha \mathbb{Z})p) \leq \text{tsr}(pM_n(A)p) + 1 = 2.$$ 

Thus $\sup_{p \in P} \text{tsr}(pM_\infty(A \rtimes_\alpha \mathbb{Z})p) \leq 2$.

Now Theorem 5.1 implies that $A \rtimes_\alpha \mathbb{Z}$ has cancellation.

**Corollary 5.4.** Assume the hypotheses of Theorem 5.3. If in addition $A \rtimes_\alpha \mathbb{Z}$ has real rank zero, then $\text{tsr}(A \rtimes_\alpha \mathbb{Z}) = 1$.

**Proof.** The proof is the same as that of Corollary 5.1. □

**Example 5.5.** Let $A$ be a UHF algebra or an irrational rotation algebra. Let $\alpha \in \text{Aut}(A)$ generate an outer action of $\mathbb{Z}$. Then $A \rtimes_\alpha \mathbb{Z}$ has cancellation. This follows from Theorem 5.3 because $K_0(A)$ has no nontrivial order preserving automorphisms.

**Remark 5.6.** Let $A$ be a simple unital $\mathcal{AT}$-algebra with real rank zero and a unique tracial state. Let $\alpha \in \text{Aut}(A)$ be approximately inner and generate an outer action of $\mathbb{Z}$. Then the hypotheses of Theorem 5.3 are satisfied. In particular, by Corollary 5.4, if $A \rtimes_\alpha \mathbb{Z}$ has real rank zero then $\text{tsr}(A \rtimes_\alpha \mathbb{Z}) = 1$. In fact, under these hypotheses on $A$ and $\alpha$, Kishimoto proved [19] that the following are equivalent:

1. $A \rtimes_\alpha \mathbb{Z}$ has real rank zero.
2. $\alpha^m$ is uniformly outer for every $m \neq 0$.
3. $\alpha$ has the Rokhlin property.

Theorem 5.3 suggests that, in this situation, one might deduce $\text{tsr}(A \rtimes_\alpha \mathbb{Z}) = 1$ from a weaker condition on $\alpha$ than the Rokhlin property. We point out that Theorem 1.2 of [9] provides many examples of actions of $\mathbb{Z}$ on the $2^\infty$ UHF algebra which do not have the Rokhlin property and such that the crossed product is a simple $\mathcal{AT}$ algebra with real rank one (not zero), but of course with stable rank one.

**Remark 5.7.** Let $\alpha : G \to \text{Aut}(A)$ be an action of a discrete group $G$ on a unital $C^*$-algebra $A$. Taking the crossed product $A \rtimes_\alpha G$ can increase the topological stable rank if $G$ is finite and $A$ is not simple (see Example 8.2.1 of [5]) or if $G$ is infinite and $A$ is simple (see Example 8.2.2 of [5]). Blackadar asked, in Question 8.2.3 of [5], whether the crossed product of an AF algebra by a finite group has topological stable rank one. This question remains open, even for simple AF algebras and $\mathbb{Z}/2\mathbb{Z}$. We have seen in Corollary 5.1 that if $A$ is a simple unital $C^*$-algebra with $\text{TR}(A) = 0$ and $G$ is finite, then $A \rtimes_\alpha G$ has cancellation. It often happens that cancellation for a simple unital $C^*$-algebra $B$ implies $\text{tsr}(B) = 1$, for example if $B$
has real rank zero. However, a crossed product of a simple AF algebra by a finite group may have nonzero real rank (Example 9 of [11]), and cancellation for a simple unital C*-algebra $A$ does not imply $\text{tsr}(A) = 1$ ([35]).

**References**

[1] D. Bisch, *On the structure of finite depth subfactors*, pages 175–194 in: *Algebraic Methods in Operator Theory*, Birkhäuser Boston, Boston MA, 1994.

[2] B. Blackadar, *A stable cancellation theorem for simple C*-algebras*, Appendix to: *The cancellation theorem for projective modules over irrational rotation C*-algebras* [M. A. Rieffel, Proc. London Math. Soc. (3) 47(1983), 285–302], Proc. London Math. Soc. (3) 47(1983), 303–305.

[3] B. Blackadar, *K-Theory for Operator Algebras*, MSRI Publication Series 5, Springer-Verlag, New York, Heidelberg, Berlin, Tokyo, 1986.

[4] B. Blackadar, *Comparison theory for simple C*-algebras*, pages 21–54 in: *Operator Algebras and Applications*, D. E. Evans and M. Takesaki (eds.) (London Math. Soc. Lecture Notes Series no. 135), Cambridge University Press, Cambridge, New York, 1988.

[5] B. Blackadar, *Symmetries of the CAR algebra*, Ann. Math. (2) 131(1990), 589–623.

[6] B. Blackadar, *The stable rank of full corners in C*-algebras*, Proc. Amer. Math. Soc. 132(2004), 2945–2950.

[7] B. Blackadar, M. Dădărlat, and M. Rørdam, *The real rank of inductive limit C*-algebras*, Math. Scand. 69(1991), 211–216.

[8] B. Blackadar and D. Handelman, *Dimension functions and traces on C*-algebras*, J. Funct. Anal. 45(1982), 297–340.

[9] O. Bratteli, D. E. Evans, and A. Kishimoto, *The Rohlin property for quasi-free automorphisms of the Fermion algebra*, Proc. London Math. Soc. 71(1995), 675–694.

[10] L. G. Brown and G. K. Pedersen, *C*-algebras of real rank zero, J. Funct. Anal. 99(1991), 131–149.

[11] G. A. Elliott, *A classification of certain simple C*-algebras*, pages 373–385 in: *Quantum and Non-Commutative Analysis*, H. Araki etc. (eds.), Kluwer, Dordrecht, 1993.

[12] K. Goodearl, private communication.

[13] F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Mathematical Sciences Research Institute Publications 14, Springer-Verlag, New York, 1989.

[14] R. H. Herman and L. N. Vaserstein, *The stable range of C*-algebras*, Invent. Math. 77(1984), 553–555.

[15] M. Izumi, *Inclusions of simple C*-algebras*, J. reine angew. Math. 547(2002), 97–138.

[16] J. A. Jeong and H. Osaka, *Extremally rich C*-crossed products and the cancellation property*, J. Austral. Math. Soc. (Series A) 64(1998), 285–301.

[17] T. Kajiwara and Y. Watatani, *Jones index theory by Hilbert C*-bimodules and K-theory*, Trans. Amer. Math. Soc. 352(2000), 3429–3472.

[18] S. Kawakami and Y. Watatani, *The multiplicity of the minimal index of simple C*-algebras*, Proc. Amer. Math. Soc. 123(1995), 2809–2813.

[19] A. Kishimoto, *Automorphisms of AT algebras with the Rohlin property*, J. Operator Theory 40(1998), 277–294.

[20] K. Kodaka and T. Teruya, *Involutive equivalence bimodules and inclusions of C*-algebras with Watatani index 2*, J. Operator Theory, to appear.

[21] H. Lin, *Tracially AF C*-algebras*, Trans. Amer. Math. Soc. 353(2001), 693–722.

[22] H. Lin, *An Introduction to the Classification of Amenable C*-algebras*, World Scientific, River Edge NJ, 2001.

[23] H. Osaka, *SP-property for a pair of C*-algebras*, J. Operator Theory 46(2001), 159–171.

[24] H. Osaka and T. Teruya, *Topological stable rank of inclusions of unital C*-algebras*, International J. Math. 17(2006), 19–34.

[25] N. C. Phillips, *Equivariant K-Theory and Freeness of Group Actions on C*-Algebras*, Springer-Verlag Lecture Notes in Math. no. 1274, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1987.

[26] N. C. Phillips, *Crossed products by finite cyclic group actions with the tracial Rohlin property*, unpublished preprint (arXiv: math.OA/0306410).
[27] N. C. Phillips, The tracial Rokhlin property for actions of finite groups on $C^*$-algebras, preprint (arXiv: math.OA/0609782).

[28] M. Pimsner and S. Popa, Iterating the basic construction, Trans. Amer. Math. Soc. 310 (1988), 127–133.

[29] M. Pimsner and S. Popa, Entropy and index for subfactors, Ann. Sci. École Norm. Sup. (4) 19 (1986), 57–106.

[30] I. F. Putnam, The invertible elements are dense in the irrational rotation $C^*$-algebras, J. reine angew. Math. 410 (1990), 160–166.

[31] M. A. Rieffel, Dimension and stable rank in the $K$-theory of $C^*$-algebras, Proc. London Math. Soc. (3) 46 (1983), 301–333.

[32] M. A. Rieffel, The cancellation theorem for projective modules over irrational rotation $C^*$-algebras, Proc. London Math. Soc. (3) 47 (1983), 285–302.

[33] H. Takai, On a duality for crossed products of $C^*$-algebras, J. Funct. Anal. 19 (1975), 25–39.

[34] T. Teruya, Index for von Neumann algebras with finite dimensional centers, Publ. RIMS Kyoto Univ. 28 (1992), 437–453.

[35] A. S. Toms, Cancellation does not imply stable rank one, preprint (arXiv: math.OA/0509107).

[36] J. Villadsen, On the stable rank of simple $C^*$-algebras, J. Amer. Math. Soc. 12 (1999), 1091–1102.

[37] R. B. Warfield, Jr., Cancellation of modules and groups and stable range of endomorphism rings, Pacific J. Math. 91 (1980), 457–485.

[38] Y. Watatani, Index for $C^*$-subalgebras, Mem. Amer. Math. Soc. 83 (1990), no. 424.

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