Modification of Convex Ends to Cylindrical and Symplectic Foliations

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Abstract

We show that the natural symplectic structure on the Milnor fiber of an isolated singularity in complex three variables whose link fibers over the circle can be modified into one which is cylindrical at the end. As a consequence we see that the foliation of codimension one on $S^5$ which is adapted to the Milnor open book of $S^5$ associated with such a singularity admits a leafwise symplectic structure. The modification also enables us to construct certain closed symplectic 4-manifolds.

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0 Introduction

In the previous work \[Mi2\] the author constructed leafwise symplectic structures on foliations adapted to the Milnor open books for the simple elliptic singularities $\tilde{E}_6, \tilde{E}_7,$ and $\tilde{E}_8$. The foliation adapted to $\tilde{E}_6$ is so called Lawson's foliation (see \[L\]) which is known to be the first foliation of codimension one constructed on the five sphere. In this paper we refine the construction mainly from the point of view of open book decomposition and give a sufficient condition not only for Milnor open book but also for general open books supporting contact structures so that the adapted foliation exists and admits a leafwise symplectic structure. As a consequence, for isolated hypersurface singularities of three complex variables, Lawson type (adapted) foliations of the Milnor open books of cusp singularities also admit leafwise symplectic structures, as well as simple elliptic ones on which we have already worked.

The notion of leafwise symplectic foliation or foliation with leafwise symplectic structure is essentially equivalent to that of regular Poisson structure and is defined as follows.

**Definition 0.1** (Leafwise symplectic structure. See \[Mi2\]) For a smoothly foliated manifold $(M, F)$, a leafwise symplectic structure is a smooth 2-form $\omega$ on $M$ which restricts to a symplectic form of each leaf of $F$.

Often it is regarded as a smooth section of 2nd exterior power of the cotangent bundle to the foliation, i.e., an element of $\Gamma^\infty(\bigwedge^2 T^*F)$ whose restriction to each leaf is symplectic on the leaf.

**Remark 0.2** The one in the sense of the first definition restricts to one of the second definition, the one in the sense of second definition always admits a lift to one in the sense of the first definition. Therefore in this article we do not distinct the two and shift from one to the other without mention.

The first definition has an importance if we would like to impose on the leafwise symplectic form $\omega$ to be closed as a 2-form on the ambient manifold $M$. For a foliation of codimension one with a globally closed leafwise symplectic form, Donaldson’s theory of approximately holomorphic geometry is applicable and we find very fine structures (see Martínez-Torres \[Mt\]). On the spheres a compact leaf is an apparent obstruction.

The second definition is completely equivalent to the notion of regular Poisson structures, i.e., Poisson structures all of whose symplectic leaves have the same dimension. From this point of view our work is paraphrased as constructions of regular Poisson structures.

In Section 1 we review the basic materials concerning singularities, including the Milnor fibrations and open books of isolated singularities, cusp and simple elliptic singularities. The associated foliations (often called
Lawson type foliations) with open books and their relations with the contact structures are reviewed in Section 2.

The key of the construction of leafwise symplectic structure on a Lawson type foliation is the modification of symplectic structure of a Milnor fiber into one with cylindrical end. The natural symplectic structure of Milnor fiber which is inherited from the Euclidean space has a convex end. In order to construct a leafwise symplectic structure on an associated foliation, the fibers should spiral around a compact symplectic leaf, and accordingly the fibers is required to admit a symplectically periodic end. For Milnor fibers of simple elliptic singularities and cusp singularities we show that the modification is possible in a rather simple way (Section 3, Theorem 3.1).

The method of modifying symplectic structures in Section 3 yields some further applications. We give some constructions of symplectic structures in Section 4 or of b-symplectic structures in Section 5 on certain closed 4-manifolds. A closed symplectic 4-manifold is obtained by gluing two copies of the Milnor fibers of certain cusp singularities (Construction 4.2), while b-symplectic structures are obtained on the doubles of the Milnor fiber of simple elliptic or cusp singularities. This construction seems to have some relation with recent study of homological mirror symmetry due to Hacking and Keating ([K], [HK]). A closer look at these closed symplectic 4-manifolds is passed to a forthcoming paper [K2M2].

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1 Milnor open book and associated symplectic structure on pages

1.1 Milnor fibration and open book decomposition

First we recall the construction of exact symplectic open book decomposition from a Milnor fibration of an isolated singularity. Let $f(z_0, ..., z_n)$ be a polynomial or holomorphic function in $n + 1$ complex variables and assume that the origin is an isolated critical point $\{f = 0\}$.

Let $S^2_{\rho_0}$ denote the sphere in $\mathbb{C}^n + 1$ of radius $\rho_0 > 0$ with center at the origin. The most classical Milnor fibration is defined as $S^2_{\rho_0} \setminus f^{-1}(0) \rightarrow S^1$ by taking $\arg f(z) = f(z)/|f(z)|$ for $z \in S^2_{\rho_0} \setminus f^{-1}(0)$. Then for a small enough $\varepsilon_0 > 0$, the restriction of this fibration to a compact manifold $M = S^2_{\rho_0} \cap \{z = (z_0, ..., z_n); |f(z)| \geq \varepsilon_0\}$ with boundary is also called the Milnor fibration of the singularity. $U = S^2_{\rho} \setminus M$ is a tubular neigh-
borhood of the link \( N = f^{-1}(0) \cap S^{2n+1}_r \) in the sphere \( S^{2n+1}_r \). As the normal bundle of the link \( N \) is trivialized by the value of \( f \), \( U \) is diffeomorphic to the product \( N \times \text{int} D^2_{\varepsilon_0} \) where \( D^2_{\varepsilon_0} = \{ w \in \mathbb{C}; |w| \leq \varepsilon_0 \} \). The common boundary \( W = \partial \overline{U} = \partial M \) is diffeomorphic to \( S^1 \times N \). The decomposition \( S^{2n+1}_r = \overline{U} \cup M \) together with the fibration \( \varphi = f/|f| : M \rightarrow S^1 \) is the Milnor open book decomposition of the sphere \( S^{2n+1} \) associated with the isolated singularity. The binding is nothing but the link \( N \).

If we choose \( \rho_0 > 0 \) small enough carefully, the real-hypersurface \( \{ \arg f = \theta \} \) for each \( \theta \in S^1 \) is transverse to \( S^{2n+1}_{\rho'} \setminus f^{-1}(0) \) for \( 0 < \rho' \leq \rho_0 \) so that the gradient flow of \( \rho^2 = \sum_{j=0}^n |z_j|^2 \) on the real-hypersurface \( \{ \arg f = \theta \} \) gives rise to a diffeomorphism between \( F_w = \{ f(z) = w \} \cap D^{2(n+1)}_{\rho_0} \) and \( \varphi^{-1}(\theta) \subset M \subset S^{2n+1}_{\rho_0} \) for \( w = \varepsilon_0 e^{i\theta} \). Therefore the fibration \( \varphi : M \rightarrow S^1 \) can be identified with \( \bigcup_{|w| = \varepsilon_0} F_w \rightarrow S^1 \), which is also called a Milnor fibration. In this article, we mainly use this formulation of Milnor fibration and thus it is also denoted by \( \varphi : M \rightarrow S^1 \). When the absolute value \( |w| \) is well-understood or a Milnor fiber \( F_w \) is regarded as a page of the open book, \( F_w \) is also denoted by \( F_{\theta} \) where \( \theta = \arg w \).

### 1.2 Exact symplectic structures on Milnor pages and open book decompositions

The canonical symplectic structure \( \omega_0 = \sum_{j=0}^n dx_j \wedge dy_j \) on \( \mathbb{C}^{n+1} \) with its canonical primitive (or a Liouville form) \( \lambda_0 = \frac{1}{2} \sum_{j=0}^n (x_j dy_j - y_j dx_j) \) on \( \mathbb{C}^{n+1} \) restricts to each Milnor fiber \( F_w \) to be an exact symplectic structure. We slightly modify this structure by isotopying the embedding \( F_w \hookrightarrow \mathbb{C}^{n+1} \)’s.

Let us identify the compact region \( CM = \{ |f| \leq \varepsilon_0 \} \cap (D^{2(n+1)}_{\rho_0/2} \setminus \text{int} D^{2(n+1)}_{\rho_0/2}) \subset \mathbb{C}^{n+1} \) with the product \( [\rho_0/2, \rho_0] \times D^2_{\varepsilon_0} \times N \) as follows. The first coordinate \( \rho \) is the distance from the origin, the second is the value of \( f \) in \( D^2_{\varepsilon_0} \subset \mathbb{C} \). Fixing the projection to \( N \) is not very important but it is better to fix it.

For example, first, on \( \rho = \rho_0 \), namely on \( \overline{U} \subset S^{2n+1}_{\rho_0} \), by the exponential map in \( S^{2n+1}_r \) from \( N \) we can define a projection to \( N \). Then, for each \( \rho' \in [\rho_0/2, \rho_0] \) the \( CM \cap \{ \rho = \rho' \} \), applying the same construction on \( S^{2n+1}_{\rho'} \), we obtain a smooth projection \( CM \rightarrow N \). For the second step we can also rely on the gradient flow of the function \( \rho \) on each \( f^{-1}(w) \) \( (w \in D^2_{\varepsilon_0}) \) to extend the projection to \( N \). (Also in the place of the gradient flow, the Liouville vector field on each \( F_w = f^{-1}(w) \) is available.) This gives a smooth identification of \( CM \) with \( [\rho_0/2, \rho_0] \times D^2_{\varepsilon_0} \times N \) and we regard \( (\rho, w, n) \in [\rho_0/2, \rho_0] \times D^2_{\varepsilon_0} \times N \) as the product coordinate of \( CM \).
Next take a smooth function $\psi(\rho)$ on $[\rho_0/2, \rho_0]$ satisfying

- $\psi(\rho) \equiv 1$, $\rho \in \left(\frac{\rho_0}{2}, \frac{2\rho_0}{3}\right)$,
- $0 \leq \psi(\rho) \leq 1$, $\rho \in \left[\frac{2\rho_0}{3}, \frac{5\rho_0}{6}\right]$,
- $\psi(\rho) \equiv 0$, $\rho \in \left[\frac{5\rho_0}{6}, \rho_0\right]$.

With these preparations we show the following.

**Proposition 1.1** For sufficiently small $\varepsilon_0$ and for any $w \in \mathbb{C}$ with $|w| = \varepsilon_0$, the identical embedding of $F_w$ admits an isotopy through symplectic embeddings with respect to $\omega_0$ in such a way that on the part $\rho \leq \rho_0/2$ the isotopy is trivial, through the isotopy $\rho$ is preserved, and the part $F_w \cup \{\rho \geq \frac{5\rho_0}{6}\}$ is embedded onto the corresponding part of $f^{-1}(0)$. Moreover these family of isotopies are chosen to be smooth on the parameter $\theta \in S^1$ and whole through the isotopies the projection to $N$ is stationary.

**Proof.** In the product coordinate, the hypersurface $F_w \cap \{\rho_0/2 \leq \rho \leq \rho_0\}$ is considered to be a graph of the constant function $w$ from $[\rho_0/2, \rho_0] \times N$ to $D^2_{\varepsilon_0}$.

As the modification of the identical embedding of $F_w$, take the function $\psi(\rho)w$ instead of the constant function $w$ and consider its graph. Let $s \in [0, 1]$ be the parameter for the isotopy and consider the graphs of the smooth map

$$(\rho, n, w, s) \mapsto ((1 - s) + s\psi(\rho))w$$

from $[\rho_0/2, \rho_0] \times N \times D^2_{\varepsilon_0} \times [0, 1]$ to $D^2_{\varepsilon_0}$, where $(w, s) \in D^2_{\varepsilon_0} \times [0, 1]$ are regarded as parameters so that the graphs are considered to be drawn in $[\rho_0/2, \rho_0] \times N \times D^2_{\varepsilon_0}$. For any $(w, s)$, on $[\rho_0/2, \frac{2\rho_0}{3}]$ and $[\frac{5\rho_0}{6}, \rho_0]$ the graph is symplectic in $\mathbb{C}^{n+1}$. Also note that if $s = 0$ or $w = 0$, the graphs are symplectic. From the compactness we can see easily that there exists a positive $\varepsilon > 0$ such that whenever $|w| \leq \varepsilon$ regardless to $s \in [0, 1]$ the graphs is symplectic. To complete the proof take $w = \varepsilon_0 e^{i\theta}$ for $\theta \in S^1$. \[\Box\]

Let $\Phi : M = \cup_{\theta \in S^1} F_w \cap D^2_{\rho_0(n+1)} \rightarrow \mathbb{C}^{n+1}$ be the map which is identity on $M \cap D^2_{\rho_0(n+1)}$ and is the above one with $w = \varepsilon_0 e^{i\theta}$ and $s = 1$. On each Milnor page, we take the restriction of $\Phi^* \omega_0$ and $\Phi^* \lambda_0$ which define a new exact symplectic structure on each Milnor page.

Along the image of $\Phi$ take the symplectic orthonormal plane field to $\Phi(F_w)$. It defines a symplectic parallel transport from the Milnor page $F_w \cap D^2_{\rho_0(n+1)}$, namely $\theta = 0$, through the pages $f^{-1}(\varepsilon e^{i\theta}) \cap D^2_{\rho_0(n+1)} (0 \leq \theta \leq 2\pi)$ to $F_w \cap D^2_{\rho_0(n+1)}$, $\theta = 2\pi$, which gives rise to a symplectic monodromy. It is clearly stationary on $\frac{5\rho_0}{6} \leq \rho \leq \rho_0$ so that the monodromy is identity around the boundary.
Each page has the identical symplectically convex end, which also coincides with the end of \( f^{-1}(0) \cap D^{2(n+1)}_{\rho_0} \). The boundary \( N = \partial \), which is also called the binding of the open book or also the link of the singularity, is naturally equipped with a standard contact structure \( \xi_N = \ker \lambda_0|_N \).

**Definition 1.2** (Exact symplectic open book decomposition) An open book decomposition of a \((2n + 1)\)-dimensional manifold \( M \) is an *exact symplectic open book decomposition* if each pages admits exact symplectic structure with contact type boundary (or equivalently a globally convex symplectic structure (see [EG])) and the monodromy preserves the symplectic structure and is identity in some neighborhood of the boundary.

Then there exists a contact structure \( \xi \) on \( M \), which is unique up to isotopy, such that the the binding is a contact submanifold which naturally coincides with one given as the boundary of the pages and \( \xi \) is almost tangent to the pages. It is also said that the open book is supporting the contact structure \( \xi \). For more details on these notions, see e.g., [BCS].

**Corollary 1.3** With a Milnor open book of an isolated singularity, an exact symplectic open book decomposition is associated.

In our construction of symplectic foliations, we need a less strict class of open books.

**Definition 1.4** (Convex symplectic open book decomposition) An open book decomposition of a \((2n + 1)\)-dimensional manifold \( M \) is a *convex symplectic open book decomposition* if each pages admits a symplectic structure with contact type boundary and the monodromy preserves the symplectic structure and is identity in some neighborhood of the boundary. Compared with the exact symplectic open books the condition is loosen from the exactness to not necessarily global convexity of the symplectic structures on pages.

The binding \( N \) of the open book which is regarded as the common boundary of all pages naturally admits a contact structure as the boundary of convex symplectic structure. The symplectic structure on a neighborhood of the boundary of a page is of the form of symplectization of the contact structure on \( N \).

### 1.3 Simple elliptic and cusp singularities

The singularities with which we are mainly concerned are the following isolated hypersurface singularities at the origin \( 0 = (0, 0, 0) \) in three com-
plex variables \((x, y, z) \in \mathbb{C}^3\).

\[
\tilde{E}_6 : \quad x^3 + y^3 + z^3 + kxyz, \quad k^3 \neq -27 \\
\tilde{E}_7 : \quad x^2 + y^4 + z^4 + kxyz, \quad k^4 \neq 64, \\
\tilde{E}_8 : \quad x^2 + y^3 + z^6 + kxyz, \quad k^6 \neq 432.
\]

These singularities are called simple elliptic singularities. Usually an simple elliptic singularity is defined as a normal surface singularity whose minimal resolution locus \(E\) is a smooth elliptic curve. The self-intersection \(E \cdot E\) is equal to \(-3, -2,\) and to \(-1\) respectively for \(\tilde{E}_6, \tilde{E}_7,\) and for \(\tilde{E}_8\). The negative of \(E \cdot E\) is called the index of the simple elliptic singularity. It is known that the simple elliptic singularities which are realized as hypersurface singularities are analytically equivalent to one of the above three (Saito \([S]\)). The restriction on \(k\) is to ensure that the singularity is isolated, and \(k\) can be zero. Even though the different \(k\)'s define analytically different singularities, they are, from smooth topological point of view, diffeomorphic to each other as the space of \(k\) is connected. Therefore in this article we do not pay particular attention to the value of \(k\).

The following polynomial also defines an isolated singularity at the origin, which is called a \textit{cusp} singularity:

\[
T_{pqr} : x^p + y^q + z^r + kxyz, \quad k \neq 0, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.
\]

In these cases, \(k\) should just avoid 0, then for different \(k\)'s they are linearly equivalent. For \(k = 0\) above polynomials define singularities of a different class. A cusp singularity is also usually defined as a normal surface singularity whose minimal resolution locus is a rational curve with a node or a certain cycle of smooth rational curves. See, e.g., \([H]\). It is also know that a cusp singularity is analytically realizable as an isolated hypersurface singularity if and only if it is analytically equivalent to one of the above \(T_{pqr}\)-singularities for \(1/p + 1/q + 1/r = 1\) (Karras \([K]\)).

The above three simple elliptic singularities can ‘almost’ be considered as \(T_{pqr}\)-singularities for \(1/p + 1/q + 1/r = 1\) with an attention to the value of \(k\). The link of these singularities are known to be \(T^2\)-bundles over the circle with monodromy

\[
A_{p,q,r} = \begin{pmatrix}
  r - 1 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  q - 1 & -1 \\
  1 & 0
\end{pmatrix} \begin{pmatrix}
  p - 1 & -1 \\
  1 & 0
\end{pmatrix}.
\]

See \([L]\), \([N]\), and \([K]\). For simple elliptic singularities, \(i.e., \quad (p,q,r) = (3,3,3), (2,4,4),\) and \((2,3,6), \quad A_{p,q,r}'s\) are conjugate to \(\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\) respectively, and they are unipotent, so that the links are nil 3-manifolds and also are considered to be the circle bundles over \(T^2\) with
euler numbers $-3, -2, \text{ and } -1$ respectively (see [Mi2]). For cusp singularities, i.e., $1/p + 1/q + 1/r < 1$, as $\text{tr} \ A_{p,q,r} = 2 + pqr(1 - (1/p + 1/q + 1/r))$, the monodromy is hyperbolic and the link is a solv 3-manifold.

2 Adapted foliations of open book decompositions

In this section, we review and discuss two criterions due to A. Mori ([Mo]) for open books supporting contact structures to be nicely foliated. The first one is on the binding so that the open book admits an adapted foliation and the second is on the contact structure of the binding so that the ambient contact structure converges to the foliation. Surprisingly these conditions give rise to leafwise symplectic foliations adapted to Milnor fibrations which we can associate starting from isolated singularities in complex three variables.

2.1 Mori’s criterions on open books

With an open book decomposition on a 3-manifold it is easy to construct an associated foliation of codimension one, which is called a spinnable foliation (cf. [KM$^3$]).

In general, foliating smoothly the exterior of the binding $N$ (which is denoted by $M$ in our context) is always easy. It suffices to make the pages winding around the boundary. However, foliating the tubular neighborhood of the binding is not trivial. Thurston’s $h$-principle [Th] tells us that for an open book of an odd dimensional closed manifold there always exists a foliation which has the boundary of the tubular neighborhood of the binding as a compact leaf, while from the $h$-principle we do not have enough control on the resultant foliation.

**Theorem 2.1** (Mori’s criterion-1, [Mo], [L]) Let $M^m$ be a closed smooth manifold which is given an open book decomposition. If the binding $N^{m-2}$ of the open book admits a Riemannian foliation $G$ of codimension one, then its pull-back to $U \cong N \times \text{int}D^2$ can be turbulized around the boundary $W = S^1 \times N$ so that $\overline{U} \cong N \times D^2$ is smoothly foliated and the boundary $W$ is a compact leaf. As a conclusion there exists a foliation $\mathcal{F}$ adapted to the open book decomposition.

This is a slight generalization of Lawson’s construction ([L]) of foliation adapted to an open book, which just assumes that the binding fibers over the circle. Then we can foliate the tubular neighborhood of the binding by so called turbulization so as to have the boundary as a compact leaf. It is easy to choose the holonomy around this compact leaf so that after pasting the pages we obtain a smooth foliation. From the point of view of
Mori’s criterion, Lawson’s construction adopts the bundle foliation which the fibration over the circle determines as the Riemannian foliation on the binding to start with. Such a foliation which is associated with such an open book is called a foliation of Lawson type or an adapted foliation. In this article, if we start from Mori’s criterion with a Riemannian foliation, we call the resultant foliation a generalized Lawson type foliation.

As is well known, if the binding admits a Riemannian foliation of codimension one, a slight perturbation of the defining closed 1-form among the non-singular closed 1-forms so as to represent a rational first cohomology class, we obtain a Riemannian foliation determined by a fibration over the circle (Tischler [11]). Therefore as the restriction of the topology of binding, Mori’s criterion is equivalent to Lawson’s one, however, classes of foliations on the binding and of resultant foliations are wider.

Now we explain the turbulization. First let us just assume the fibering of the binding $N$ over the circle $S^1 = \{ x \in \mathbb{R}/2\pi \mathbb{Z} \}$.

Also fix a normal coordinate $w \in D^2_{\varepsilon_0} \subset \mathbb{C}$ of the fiber as the normal bundle to $N \subset U$ in the tubular neighborhood $\overline{U}$ and put $r = |w|$. Let $\psi(r)$ be a smooth function on $[0, \varepsilon_0]$ satisfying
\begin{itemize}
  \item $\psi(r) \equiv 1$, $r \in [0, \varepsilon_0/2]$,  
  \item $\psi'(r) < 0$, $r \in (\varepsilon_0/2, \varepsilon_0)$,  
  \item $\psi(\varepsilon_0) = 0$,  
  \item $\psi(r)$ is flat to 0 at $\varepsilon_0$.
\end{itemize}

Then take
$$\alpha_U = \psi dx + (1 - \psi) dr$$
as the defining 1-form of the desired foliation on $\overline{U}$.

**Remark 2.2** Consider the solid 3-torus $S^1 \times D^2_{\varepsilon_0}$ with the coordinate $(x, w)$ and set $r = |w|$ as above. Then the 1-form $\alpha$ as above defines a so called Reeb component, i.e., a typical foliation on the solid torus. Two copies of Reeb components are glued together to form a Reeb foliation on $S^3$. The foliation on $\overline{U}$ constructed above can be regarded as the pull back of the Reeb component (or its open neighborhood) by the projection $p: \overline{U} \to S^1$ ($p(n, w) = (x(p), w)$) with the same monodromy and the fiber as those of $N \to S^1$.

For a generalized Lawson type foliation, first take a defining closed 1-form $\alpha_N$ of the Riemannian foliation on $N$ and take a Tischler fibration to $S^1$. Then the place of $dx$ use $\alpha_N$ and take
$$\alpha_U = \psi \alpha_N + (1 - \psi) dr.$$ 
Then it suffices to obtain a smooth foliation which is extendable even to the other side of the boundary as a trivial foliation defined by $dr$.

The reason why the foliation with which we start should be Riemannian is as follows. If we trivially extend the defining 1-form $\alpha_N$ of the
foliation on \( N \) to \( \overline{U} = D^2 \times N \) it also defines a foliation on \( \partial U \). Due to Rosenberg and Thurston [RT], for the smooth turbulization which can be extended to the other side as a trivial foliation, it should be Riemannian.

The fibration of the binding over the circle works not only for foliating the tubular neighborhood of binding but also plays an important role for existence of the leafwise symplectic structures.

**Theorem 2.3** (Mori’s criterion-2, [Mo]) Let \( \xi \) be a contact structure on a closed manifold \( M \). In the above criterion, i.e., the existence of a Riemannian foliation on the binding \( N^3 \), we also assume that there exists a contact 1-form \( \alpha \) defining \( \xi \) which is adapted to the supporting open book, the contact form \( \alpha|_N \) of the restricted contact structure \( \xi|_N \) has the Reeb vector field is tangent to \( G \). Then there exists a one-parameter family \( \xi_t \) of contact structures, which is an isotopic deformation of \( \xi \), converges to the Lawson type foliation \( \mathcal{F} \) as hyperplane field.

**Remark 2.4** This criterion applies to the Milnor open books of all simple elliptic and cusp singularities of hypersurfaces. However, the exact 2-forms \( d\alpha \)’s do not converge to the leafwise symplectic structures which are constructed in this article. The compact leaf is an apparent obstruction.

For the simple elliptic singularities, the natural contact structure obtained as the convex boundary of the Milnor page or the singular surface admits a Reeb vector field which is tangent to the circle fiber if the link is considered as a circle bundle over \( T^2 \). The circle fiber is contained in the \( T^2 \) fiber of the fibration to the circle. See [Mi2].

In the cusp case, a natural contact structure on the link which is a solv manifold admits a Reeb vector field tangent to one of the two eigen directions of the monodromy in each \( T^2 \) fiber. See [Ka] for this.

Mori [Mo] constructed a family of open books of \( S^4 \times S^1 \) supporting contact structures whose bindings are nil 3-manifolds, where any of \( -\ell \) for \( \ell \geq 4 \) are realized as the euler number of the link as circle bundle over \( T^2 \).

### 2.2 Leafwise symplectic structure around the binding

In this subsection we explain the existence of leafwise symplectic structures on the tubular neighborhood of the binding.

Surprisingly we will see the fibering of binding over the circle works further until the existence of the leafwise symplectic structures if we start from isolated singularities of complex three variables.

In this subsection we assume

(o) the manifold \( M^{2n+1} \) is closed and oriented,
(i) \( M \) admits an open book satisfying with a fixed fibration \( p_N : N^{2n-1} \to S^1 = \{ x \in \mathbb{R}/2\pi \mathbb{Z} \} \) of the binding over the circle with the fiber \( \Sigma^{2n-2} \) and the monodromy \( \varphi_{\Sigma} \rhd \Sigma \).

(ii) the fiber \( \Sigma \) is equipped with a symplectic structure \( \omega_{\Sigma} \) and the monodromy \( \varphi_{\Sigma} \) preserves \( \omega_{\Sigma} \),

and let the fibration \( p_N : N \to S^1 \) trivially extend to the closed tubular neighborhood as \( p : \overline{U} = N^{2n-1} \times D_{\varepsilon_0}^2 \to S^1 \times D_{\varepsilon_0}^2 \).

**Remark 2.5** To obtain a leafwise symplectic structure on a Lawson type foliation, we need that the compact leaf \( \partial U = S^1 \times N \) admits a symplectic structure. For this, the conditions (i) and (ii) are sufficient, because we can take \( \omega_{\partial U} = dx \wedge d\theta + \tilde{\omega}_{\Sigma} \) where \( \theta = \arg w \) \( (w \in \partial D_{\varepsilon_0}^2 \subset \mathbb{C}) \), where \( \tilde{\omega}_{\Sigma} \) is an extension of \( \omega_{\Sigma} \) on a fiber to the total space of the fibration as a closed 2-form, which we have thanks to the condition (ii).

For \( n = 2 \), (i) implies (ii) because any orientation preserving mapping class of a closed oriented surface contains an area preserving representative. Moreover, (i) is also necessary for the symplectic structure on \( \partial U \) due to Friedl-Vidussi [FV].

Moreover, if we start from an isolated singularity of three complex variables \( (n = 2) \), simple elliptic and cusp singularities are exactly those whose Milnor open book satisfy the condition (ii) (Neumann [N]). For a simple elliptic singularity \( \partial U \) is diffeomorphic to a so called Kodaira-Thurston nil-manifold which is also known as a Kodaira’s primary surface.

**Theorem 2.6** (Friedl-Vidussi, [FV]) A closed oriented 4-manifold \( W^4 = S^1 \times N^3 \) admits a symplectic structure if and only if \( N \) fibers over the circle.

In the case the symplectic structure on \( W \) is not necessarily in the above form which we adopt in this article.

**Theorem 2.7** (Neumann, [N]) The link of an isolated singularity of three complex variables is Seifert fibered or a graph manifold. If it fibers over the circle, it should be a nil 3-manifold or a solv 3-manifold, i.e., a torus bundle over the circle, and the singularity is a simple elliptic one or a cusp.

**Construction 2.8** (Leafwise symplectic structures on \( \overline{U} \), [Mi2]) We adopt the notations in the above remark. It is easy to take a leafwise symplectic structure \( \omega_R \) on the Reeb component so that on the toral leaf \( (\theta, x) \in S^1 \times S^1 \) it restricts to \( d\theta \wedge dx \). As the Lawson type foliation is the pull back of the Reeb component by the projection \( p \), which is a trivial extension of \( p : N \to S^1 \) to the \( D_{\varepsilon_0}^2 \) factor, we also have an extension of \( \tilde{\omega}_{\Sigma} \) which is denoted the same. Then, for any non-zero constants \( a \) and \( b \),

\[
\omega_{\overline{U}} = a(d\theta \wedge dx) + b(\tilde{\omega}_{\Sigma})
\]

is a leafwise symplectic form which restricts to the boundary as desired above.
3 Leafwise symplectic structures on adapted foliations

We state the main results in this article and give their proofs.

**Theorem 3.1** Let $M$ be a closed oriented manifold of dimension 5 equipped with a convex symplectic open book decomposition. Assume the following conditions.

1. The binding $N$ admits a fibration $p_N : N \to S^1$ over the circle. Consequently, there is a closed 2-form $\tilde{\omega}_\Sigma$ on $N$ it is restricted to an area form on each fiber of $p_N$.
2. The contact structure $\xi_N$ which the convex open book defines on $N$ admits a Reeb vector field which is tangent to the fibers of $p_N$. (Mori’s criterion-2)
3. The cohomology class $[\tilde{\omega}_\Sigma]$ is in the image of the restriction map $H^2(F; \mathbb{R}) \to H^2(N; \mathbb{R})$ in the real cohomology of the Milnor fiber $F$ to its boundary $N \cong \partial F$. This is equivalent to that the fundamental class $[\Sigma]$ of the fiber is non-trivial in $H^2(F; \mathbb{R})$.

Then the open book admits a leafwise symplectic foliation of Lawson type.

**Remark 3.2** We can formulate the theorem in a slightly generalized way, by Mori’s criterion-1 with a Riemannian foliation in the place of the foliation by the fibration $p_N$ in the conditions (1) and (2). In that case, the arguments in Subsection 2.2 is fixed exactly as in Remark 2.2. For the arguments below we need to further assume that $\tilde{\omega}_\Sigma \wedge \alpha_G$ is a volume form of $N$ and to replace $dx$ with $\alpha_G$, where $\alpha_G$ is a closed 1-form on $N$ which defines the Riemannian foliation $G$.

If we try to generalize these constructions to the Milnor open books of higher dimensional isolated singularities, for four or more complex variables, unfortunately the Milnor open book never satisfies not only (3) but also (1). That is because the Milnor fiber is 2-connected and the link is 1-connected.

**Corollary 3.3** Any isolated hypersurface singularity of three complex variables which is simple elliptic or cusp has the Milnor open book of $S^5$ which admits leafwise symplectic foliation of Lawson type.

**Proof** of Corollary 3.3 For simple elliptic and cusp singularities of three complex variables, the condition (1) and the first half of (3) hold as explained in 1.3. The condition (2) is, as mentioned in Remark 2.4 observed in [Mi2] for the simple elliptic case and in [Ka] for the cusp case. Therefore we need to check the condition (3). In [Mi2] it is proved by a Meyer-Vietoris argument in the simple elliptic case, while including that case it is
easily shown from the 1-connectedness of the Milnor fiber $F$ is in this dimension \([\mathcal{M}]\), a part of the cohomology long exact sequence for the pair \((F, N = \partial F)\) and the Poincare duality of \((F, N)\) as follows.

\[
H^2(F; \mathbb{R}) \to H^2(N; \mathbb{R}) \to H^3(F, N; \mathbb{R}) \cong H_1(F; \mathbb{R}) = 0.
\]

**Proof** of Theorem 3.1 goes along the three steps. The first two are preparatory, and in the third step we construct symplectic forms on the pages which are naturally glued smoothly to the leafwise symplectic structure on the closed tubular neighborhood $\overline{U}$. The construction is similar to that in [Mi2].

**Step 1.** We put a better coordinates on the ends of pages by a standard procedure in symplectic geometry. Let $(F_\theta, \omega_\theta)$ be the exact symplectic pages. The tubular neighborhoods of the boundaries of the pages are identified by the monodromy as symplectic manifolds so that we can assume they all are of the form $(1 - \epsilon_1, 1] \times N, d(q_1 \alpha_1))$ of a part of symplectization of a contact structure $\xi = \ker \alpha_1$ on the binding $N = \partial F_\theta$, where $\epsilon_1 > 0$ is some constant and $q_1$ is the coordinate of $(1 - \epsilon_1, 1]$. Once extend the end to $q_1 < \infty$ with the symplectic form $d(q_1 \alpha_1)$ and take a contact form $\alpha_N$ for $\xi = \ker \alpha_1$ which satisfies the condition (4). Then, if necessary by taking a constant multiplication, $\alpha_N$ can be regarded as the graph of the function $q_1 = \alpha_N/\alpha_1$ on $N$ with values in $[1, c]$ for some $c \geq 1$. Putting $\phi a N = q_1 \alpha_1$ and taking $\phi = (\alpha_1/\alpha_N)q_1$ as the new coordinate function in the place of $q_1$, the symplectic form is $d(\phi \alpha_N)$. Remark that in the case of Milnor open book $q_1 = r^2$.

**Step 2.** From the conditions (1) and (2) the binding $N$ is equipped with a closed 2-form $\tilde{\omega}_\Sigma$ which restricts to a symplectic form on each fiber $\Sigma_x$. The preimage $r_\theta^{-1}(\tilde{\omega}_\Sigma) \subset H^2(F; \mathbb{R})$ is non-vacant affine space by the condition (3) where $r_\theta : H^2(F; \mathbb{R}) \to H^2(N; \mathbb{R})$ is the restriction map, even though the monodromy of the open book might act, it is not difficult to find a family of closed 2-forms $\kappa_\theta$ on each page $F_\theta$ satisfying the following conditions:

(a) On the end $q \geq 1 \kappa_\theta = \tilde{\omega}_\Sigma$, where by abuse of notation $\tilde{\omega}_\Sigma$ also denotes the pull back of $\tilde{\omega}_\Sigma$ by the projection $[1, \infty) \times N \to N$.

(b) On $\bigcup_{\theta \in S^1} F_\theta \subset M, \kappa = \{\kappa_\theta\}_{\theta \in S^1}$ is a smooth family of pagewise closed 2-forms.

**Step 3.** First remark that there is a constant $b > 0$ so that $||\omega_\theta^2|| - b||\omega_\theta|| \cdot ||\kappa_\theta|| > 0$ at any point of $F_\theta$. Consequently, $\omega_1 = \omega_\theta + b\kappa_\theta$ is symplectic. For this we have to check $\omega_1^2 > 0$ on $\bigcup_{\theta \in S^1} F_\theta \setminus \{q > 1\}$ and on $\{q \geq 1\}$. On $\bigcup_{\theta \in S^1} F_\theta \setminus \{q > 1\} \cup \{q = 1\}$ the condition is open for $b$ and of course $b = 0$ satisfies it. Therefore there exists $b_0 > 0$ such that $0 < b < b_0$
satisfies this open condition. On \( \{ q \geq 1 \} \), for a product type Riemannian metric on \([1, \infty) \times N\) where we assume \( ||dq|| = 1\), once we can verify \( ||dq \wedge \alpha_N \wedge d\alpha_N|| > b \cdot ||dq \wedge \alpha_N \wedge \kappa_\theta||\) on \( \{ q = 1 \} \), the same inequality holds on \( \{ q \geq 1 \} \). Remark that on \( \{ q \geq 1 \} \) \( \kappa_\theta = \tilde{\omega}_\Sigma \). This inequality implies \( \omega_1^2 > 0 \) on \( \{ q \geq 1 \} \) because we have \((d\alpha_N)^2 = d\alpha_N \wedge \tilde{\omega}_\Sigma = 0\). We prepare four positive constants

\[
\text{a} = \min_{n \in N} ||\alpha_N \wedge d\alpha_N||_{n}, \quad \text{C} = \max_{n \in N} ||\alpha_N \wedge \tilde{\omega}_\Sigma||_{n}, \\
\text{A} = \max_{n \in N} ||\alpha_N \wedge d\alpha_N||_{n}, \quad \text{m} = \min_{n \in N} ||dx \wedge \tilde{\omega}_\Sigma||_{n}.
\]

We take \( b > 0 \) satisfying \( b < a / C \) and \( b \leq b_0 \). This is not only for that \( \omega_1 \) is symplectic but also for the following arguments.

Now we deform the symplectic structures on the pages \( F_\theta \), only on their ends which are identical independently of \( \theta \), we omit \( \theta \) from the description. On the end \([1, \infty) \times N\) the symplectic form is \( \omega = d(\rho \alpha_N) \). As in [Mi2] for some smooth functions \( K(\rho) \) and \( L(\rho) \), consider the following 2-form

\[
\omega_E = d(K(\rho) \wedge \alpha_N) + b\tilde{\omega}_\Sigma + L(\rho)d\rho \wedge dx
\]
on \( F \cap \{ q \geq 1 \} \). Let us take \( K(\rho) \) which satisfies the following conditions.

\[
\begin{align*}
\text{(K-1)} & \quad K(\rho) = \rho & 1 \leq q \leq 3, \\
\text{(K-2)} & \quad 0 \leq K'(\rho) \leq 1 & 3 \leq q < 4, \\
\text{(K-3)} & \quad -1 \leq K'(\rho) \leq 0 & 4 \leq q \leq 8, \\
\text{(K-4)} & \quad K(\rho) = 0 & 8 \leq q.
\end{align*}
\]

Then, we take \( L(\rho) \) which satisfies the following conditions.

\[
\begin{align*}
\text{(L-1)} & \quad L(\rho) = 0 & 1 \leq q \leq 2, \\
\text{(L-2)} & \quad 0 < L'(\rho) \leq 1 & 2 \leq q < 3, \\
\text{(L-3)} & \quad L(\rho) \equiv a & 3 \leq q,
\end{align*}
\]

where \( a \) is any constant satisfying \( a > \frac{2A + bC}{bm} \).

On the end \([8, \infty) \times N\) the 2-form \( \omega_E \) is a cylindrical symplectic form \( adq \wedge dx + b\tilde{\omega}_\Sigma \) as desired, because in making the ends of pages winding around the boundary \( \partial U = S^1 \times N \ni (\theta, n) \) of the tubular neighborhood of the binding \( N \), we take \( \theta = q \) (mod \( 2\pi \)).

We have to check that \( \omega_E \) is symplectic on \([1, 8] \times N\). The most important property of this construction is that at any point we have

\[
d(K(\rho) \wedge \alpha_N) \wedge (d\rho \wedge dx) = 0 \quad \cdots \quad (\ast).
\]

This follows from the condition (2) (Mori’s criterion-2) which implies \( d\alpha_N \) is divisible by \( dx \) as both of them annihilate the Reeb vector field of \( \xi_N \), and just because \( d(K(\rho)\alpha_N) = K'(\rho)d\rho \wedge \alpha_N + K(\rho)d\alpha_N \). Therefore we have

\[
\omega_E^2 = (d(K(\rho)\alpha_N))^2 + 2b(d(K(\rho)\alpha_N)) \wedge \tilde{\omega}_\Sigma + 2bL(\rho)d\rho \wedge dx \wedge \tilde{\omega}_\Sigma
\]

\[
= 2d\rho \wedge (K'(\rho)K(\rho)\alpha_N \wedge d\alpha_N + bK'(\rho)\alpha_N \wedge \tilde{\omega}_\Sigma + bL(\rho)dx \wedge \tilde{\omega}_\Sigma)
\]

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and we also remark that $|K'(Q)K(Q)| < 4$ holds.

- On $[1, 2] \times X$, $\omega_p = \omega_1$ is symplectic.
- On $[2, 3] \times N$, still $d(K(\rho) \wedge \alpha_N) + b\tilde{\omega}_\Sigma = \omega_1$ holds and thus the sum of the first two terms is $\omega_1^2 > 0$ and the third term is also positive.
- On $[3, 8] \times N$, the first two terms might be negative, while the third term is already big enough to conclude $\omega_2^2 > 0$.

On each page $F_{\theta}$, $\omega_{\theta} + b\kappa_{\theta}$ extends to the end to be $\omega_E$ as a symplectic form $\omega_{p\theta}$ and we obtain a leafwise symplectic structure $\omega_p = \{\omega_{p\theta}\}$ on $\bigcup_{\theta \in S^1} F_{\theta}$. Remark here that the subscript $*_{\theta}$ indicates the ‘page number’.

This completes Step 3.

As is already remarked above, the end $[8, \infty) \times N$ with the cylindrical (i.e., $\theta$-translation invariant) symplectic form $d\alpha \wedge dx + b\tilde{\omega}_\Sigma$ is exactly the covering of $\partial U = S^1 \times N$ with $d\theta \wedge dx + b\tilde{\omega}_\Sigma$ by $\theta = \theta \pmod{2\pi}$, and naturally after the tubulization, by putting $\omega_U$ on $U$ and $\omega_p$ on $M \setminus U$ we naturally obtain a smooth leafwise symplectic structure after the tubulization (see [Mi2] for more detail).

Remark 3.4 For higher dimensional generalization, we need not only to assume (ii) in Subsection 2.2 but to put a stronger condition than (2) in order to achieve a good modification of the symplectic structures on the pages. We do not know good examples nor geometric way to state the extra condition which we need.

If we take $\omega_E$ in the same way, e.g., in the case $n = 3$, in computing $\omega_3^3$, we would have terms $bKLdq \wedge dx \wedge d\alpha_N \wedge \omega_\Sigma$ or $bKK'dq \wedge \alpha_N \wedge d\alpha_N \wedge \omega_\Sigma$ of which we do not know the positivity and thus they should be compared with $bLdq \wedge dx \wedge \tilde{\omega}_\Sigma^2$. In the case of general dimensions there might appear terms

$b^n-i-1K'dq \wedge \alpha_N \wedge (d\alpha_N)^{n-i-1} \wedge \tilde{\omega}_\Sigma^i$

or

$b^n-i-1K'dq \wedge (d\alpha_N)^{n-i-1} \wedge \tilde{\omega}_\Sigma^i$

for $i = 1, 2, \cdots, n - 2$ and should be compared with $b^{n-i-1}L \wedge dq \wedge dx \wedge \tilde{\omega}_\Sigma^{n-i}$. As we have to take $b$ small enough, we rather have to assume that the above terms vanish. Mori’s criterion-2 kills the term of the above form of $i = 0$ and in the case of $d = 2$ we do not have other terms. So we would like to put the following condition

$(d\alpha_N)^{n-i-1} \wedge \tilde{\omega}_\Sigma^i = 0$ on $N$, \hspace{1cm} i = 1, 2, \cdots, n - 2.$

Then all the arguments work exactly in the same way as in the case $d = 2$ and we obtain a leafwise symplectic structure on the associated foliation with the open book.
4 A certain closed symplectic 4-manifold

In this section, we present two constructions of closed symplectic manifolds as application of the construction in the previous section.

4.1 Gluing two Milnor fibers

The first application of the end-cylindrical symplectic structure is a construction of a particular closed symplectic 4-manifold, which has similar properties as K3-surfaces, starting from a particular cusp singularity, $T_{2,3,7}$ or $T_{4,4,4}$.

**Proposition 4.1** For the cusp singularity $T_{2,3,7}$ or $T_{4,4,4}$, the monodromy of the link of such a singularity is given as $A_{p,q,r} = \begin{pmatrix} r-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p-1 & -1 \\ 1 & 0 \end{pmatrix}$ as mentioned in Subsection. Therefore we have $A_{2,3,7} = \begin{pmatrix} 5 & -11 \\ 1 & -2 \end{pmatrix}$ and $A_{4,4,4} = \begin{pmatrix} 21 & -8 \\ 8 & -3 \end{pmatrix}$ and they are conjugate to their own inverses $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$. As they are unimodular and symmetric, conjugate to their own inverses $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 5 & -8 \\ -8 & 13 \end{pmatrix}$ respectively.

**Construction 4.2** Let us take $T_{2,3,7}$ and one of its Milnor fiber, e.g., $F_0$ with the end-cylindrical symplectic structure $\omega_{\mathcal{F}o}$ constructed in the proof of Theorem 3.1 and consider the restriction $\omega(1) = \omega_{\mathcal{F}o}|_{F(1)}$ to the truncated Milnor fiber $F(1) = F_0 \cap \{ q \leq 10 \}$. Let $(F(2), \omega(2))$ be the identical copy of $(F(1), \omega(1))$. We glue them along their boundaries. The boundary $N$ is the suspension of $T^2$ by the monodromy matrix $A_{2,3,7} \in SL(2; \mathbb{Z})$, we can assume that the closed 2-form $\tilde{\omega}_{T^2}$ (here the fiber $\Sigma$ is $T^2$) is the ‘pull-back’ by the suspension of the standard area form of $T^2$. The fact that the monodromy is conjugate to its inverse in $SL(2; \mathbb{Z})$ implies that even if we invert the orientation of the base circle, still it is isomorphic to the original as oriented $T^2$-bundle. We identify the boundary $N(2) = \partial F(2)$ with the $N(1) = \partial F(1)$ by a map $\Phi : N(2) \to N(1)$ in this way, namely, the $T^2$-fiber orientation is preserved, but the orientation of the base circle is reversed as $x(1) \circ \Phi = -x(2) \in S^1 = \mathbb{R}/2\pi \mathbb{Z}$. Here $x(i)$ denotes the $x$-coordinate which we have been fixed in previous sections on $N(i) = N$. Similar notations are used here for other coordinates. Therefore we have $\Phi^* dx(1) = -dx(2)$ while $\Phi^* \tilde{\omega}_{T^2} = \tilde{\omega}_{T^2}$, and $\Phi : N(2) \to N(1)$ is orientation reversing. Now in order to glue $F(1)$ and $F(2)$, around the boundaries...
take the identification of $q$-coordinate as $q(2) = 20 - q(1)$. In this way not only without changing the orientation as 4-manifolds but also without changing the symplectic structures around the boundaries, we can glue two copies $F(1)$ and $F(2)$ to obtain a closed symplectic manifold $W_{2,3,7}$, because we have $\Phi^*\tau_{T^2} = \tau_{T^2}$ and $dq(1) \wedge dx(1) = (-dq(2)) \wedge (-dx(2)) = dq(2) \wedge dx(2)$.

Remark that this construction is not taking the ‘double’. Of course also to $T_{4,4,4}$ the same construction applies to obtain a closed symplectic 4-manifold $W_{4,4,4}$.

In both cases of $T_{2,3,7}$ and $T_{4,4,4}$ the Milnor fiber is simply connected and has Euler characteristic 12. In fact it is not difficult to compute the Milnor number $\mu(T_{p,q,r}) = p + q + r - 1$ (see [Ga], [K2M2]). Therefore the resultant 4-manifolds $W_{2,3,7}$ and $W_{4,4,4}$ are simply connected and have the Euler characteristic 24. Also their Milnor lattice (the intersection form) is also computed (also see [Ga], [K2M2]), and from these informations we see the integral cohomology ring of $W_{2,3,7}$ and the rational cohomology ring of $W_{4,4,4}$ coincide with those of a K3-surface respectively. In particular, we see $W_{2,3,7}$ is homeomorphic to a K3-surface. These cohomology computations and the arguments in 4-dimensional differential topology are informed to the author by M. Ue. Therefore it is strongly expected that they are diffeomorphic to a K3-surface. This strongly motivates the forthcoming paper [K2M2] which in fact achieved it.

4.2 Circular Example

In this subsection, we slightly generalize the method in Step 3 of the proof of Theorem 3.1 namely a modification of symplectic structure on the end, and construct symplectic structures on $W = S^1 \times N$. Here we present a construction for a Riemannian foliation on $N$ (see Remark 3.2). We start with the following objects.

- $N$: a $(2n - 1)$-dimensional closed oriented manifold,
- $p_N : N \to S^1 = \{ x \in \mathbb{R} / 2\pi \mathbb{Z} \}$: a smooth fibration,
- $\tilde{\omega}_\Sigma$: a closed 2-form on $N$ which restricts to a symplectic form on each fiber of $p_N$,
- $\xi_N$: a positive contact structure on $N$,
- $\alpha_N$: a contact 1-form defining $\xi_N$,
- $\mathcal{G}$: a Riemannian foliation of codimension one on $N$,
- $\alpha_\mathcal{G}$: a non-singular 1-form which defines $\mathcal{G}$.

We assume Mori’s criterion 2 for the Reeb vector field of $\xi_N$ and $\mathcal{G}$. We also assume the following condition.

$\text{(*)} : (\tilde{\omega}_\Sigma)^{n-1} \wedge \alpha_\mathcal{G}$ is a volume form on $N$.

In the case that the Riemannian foliation $\mathcal{G}$ is chosen to be the foliation
given by the fibration \( p_N \), this condition is automatically satisfied (see Remark \[3.2\]). For the case \( n \geq 3 \) we further assume not only the condition (ii) in Subsection \[2.2\] for a Tischler fibration \( N \to S^1 \) associated with the Riemannian foliation \( \mathcal{G} \) but the codition in Remark \[3.4\] with \( d \).

In the case of \( n = 2 \), for all the nil 3-manifolds \( N = Nil^3(-l) \) with \( (l \in \mathbb{N}) \) and all the solv manifolds \( N = \text{Sol}_A \) with hyperbolic monodoromy \( A \in \text{SL}(2; \mathbb{Z}) \), namely, \( \operatorname{tr} A \geq 3 \), any of them admits a contact structure \( \xi_N \) and a contact 1-form \( \alpha_N \) whose Reeb vector field is tangent to the fibers of the fibration \( p_N : N \to S^1 \). In these cases the condition is satisfied by taking the foliation by fibration as the Riemannian foliation \( \mathcal{G} \).

Following the construction of symplectic form \( \omega_E \) on the end of Milnor fiber in the proof of \[3.1\] let us consider the closed 2-form

\[
\omega = d(K(\theta)\alpha_N) + \tilde{\omega}_\Sigma + L(\theta)d\theta \wedge \alpha_G
\]
determined by smooth functions \( K(\theta) \) and \( L(\theta) \) on \( S^1 \) and consider when

\[
w^n = (K'(\theta)d\theta \wedge \alpha_N + K(\theta)d\alpha_N + \tilde{\omega}_\Sigma + L(\theta)d\theta \wedge \alpha_G)^n
\]
gives an everywhere positive volume form. In the cohomology we have

\[
[w] = [\tilde{\omega}_\Sigma] + \left( \int_{S^1} L(\theta) \, d\theta \right) [d\theta \wedge \alpha_G] \in H^2(W; \mathbb{R}).
\]

Because of Mori’s criterion-2 the term \( d\theta \wedge \alpha_G \wedge (d\alpha_N)^{n-1} \) vanishes. Then as we have an extra condition that we assume for the case \( n \geq 3 \), \( w^n \) has only three terms as follows.

\[
w^n = K'(\theta)K(\theta)^{n-1}d\theta \wedge \alpha_N \wedge (d\alpha_N)^{n-1} + L(\theta)d\theta \wedge \alpha_G \wedge \tilde{\omega}_\Sigma^{n-1}
\]
\[+ K'(\theta)d\theta \wedge \alpha_N \wedge \tilde{\omega}_\Sigma^{n-1}.
\]

We have a wide variety for the choices of \( K(\theta) \) and \( L(\theta) \) which make \( \omega \) symplectic. For any choice of \( K(\theta) \), let us consider the ratio of the sum of the first and the third terms against \( d\theta \wedge \alpha_G \wedge \tilde{\omega}_\Sigma^{n-1} \) as follows

\[
\lambda(\theta) = -\min_{n \in \mathbb{N}} \frac{K'(\theta)\alpha_N \wedge \left( K(\theta)^{n-1}(d\alpha_N)^{n-1} + \tilde{\omega}_\Sigma^{n-1}\right)}{\alpha_G \wedge \tilde{\omega}_\Sigma^{n-1}}(\theta, n)
\]

where the ratio is taken as the that of \( 2n - 1 \) forms on \( N \), abbreviating \( d\theta \) in common. Then apparently \( \omega \) is symplectic if and only if \( L(\theta) \) satisfies \( L(\theta) > \lambda(\theta) \) for any \( \theta \in S^1 \).

Remark that the choice of \( K(\theta) \) and that of \( L(\theta) \) can be made quite independently. For example, such a choice of \( K(\theta) \) and \( L(\theta) \) is possible that \( K(\theta) \equiv 0 \) on some intervals while \( L(\theta) \equiv 0 \) on some other intervals.

**Theorem 4.3** For a closed 3-dimensional solv or nil manifold \( N, W = S^1 \times N \) admits a symplectic form which looks like a co-symplectic form (see \[5.1\]) on some intervals in \( S^1 \) and like a slight deformation of a symplectization of the canonical contact structure on \( N \) on some other intervals.
5 Other applications

5.1 \(b\)-Symplectic structures

As another application of the construction of end-cylindrical symplectic structure, we remark in this section that certain \(b\)-symplectic manifolds are easily constructed.

A \(b\)-symplectic structure is a Poisson structure (Poisson bi-vector field) \(\Pi \in C^\infty(W; \wedge^2 TW)\) on a \(2n\)-dimensional manifold \(W\) such that the graph of \(\wedge^n \Pi\) is transverse to the zero section in \(C^\infty(W; \wedge^n TW)\). Therefore the singular locus \(Z = \{w \in W | \wedge^n \Pi(w) = 0\}\) is a smooth hypersurface of \(W\). This notion was introduced by Melrose in [Me] and recently is drawing more attentions from Poisson geometry. For more details, see e.g., [GMP].

For a \(b\)-symplectic manifold, a Darboux theorem holds around the singular locus \(Z\) in the following a way. For a point in \(Z\) there exists a coordinate neighborhood \((x_1, y_1, \ldots, x_n, y_n)\) so that the Poisson bi-vector \(\Pi\) is expressed as

\[
\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} x_i \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}
\]

where \(Z = \{x_1 = 0\}\). This is equivalent to say that the corresponding symplectic form \(\omega\) has the expression

\[
\omega = \frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.
\]

If we allow the degeneration (or the divergence) along the codimension one singular locus \(Z\) to be the \(\ell\)-th order \(\ell \in \mathbb{N}\) in transverse direction, it is called a \(b^\ell\)-symplectic structure and has a local expression replacing the coefficient \(x_1^\pm 1\) above with \(x_1^\pm \ell\) (\(\pm\) correspond to Poisson/symplectic formulations).

Let \(W\) be the double of the Milnor page of one of simple elliptic or cusp singularities. Namely, take two copies of a compact Milnor page \(F_\theta\) with boundary and paste them along the boundaries with identity to obtain \(W = -F_\theta \cup_\partial F_\theta\).

**Theorem 5.1**

1) \(W\) admits a \(b\)-symplectic structure whose singular locus is \(\partial F_\theta = N\).
2) Exactly the same holds for \(b^\ell\)-symplectic structures for odd \(\ell \in \mathbb{N}\).
3) For any even \(\ell \in \mathbb{N}\), \(W_{2,3,7}\) and \(W_{4,4,4}\) in Subsection 4.1 admit \(b^\ell\)-symplectic structures whose singular loci are \(\partial F(i) = N\).

Now our purpose in this section is a consequence of the following rather general remark.

**Proposition 5.2** Let \((W_\pm, \Omega_\pm)\) be a pair of \(2n\)-dimensional symplectic manifolds with boundary which have the identical collar neighborhoods of the boundaries as follows.
Then, the manifold admits a cosymplectic structure. From the normal form theorem due to Gotay \cite{Go},

\begin{align*}
\Omega &= dt \wedge P_N^*\alpha_N + P_N^*\Omega_N \\
\text{where } P_N \text{ denotes the projection } [0,2) \times N \to N \text{ and } t \text{ denotes the coordinate on } [0,2).
\end{align*}

Then the manifold $W = W_\pm \cup \partial W_- = \partial W_+$ obtained by identifying the boundaries admits a $b$-symplectic structure whose singular locus is $\partial W_- = \partial W_+$.

**Remark 5.3** 1) The condition around the boundary is also said that the boundary admits a cosymplectic structure. More precisely, if a compact boundary $N$ (or a compact closed hypersurface) of a symplectic manifold $(W,\Omega)$ admits a closed one form $\alpha_N$ with $\alpha_N \wedge \Omega_N^{n-1}$ a volume form of $N$, then $N$ is called of coymplectic type, and the pair $(\alpha_N, \Omega_N)$ is called a cosymplectic structure. From the normal form theorem due to Gotay \cite{Go}, $N$ has a collar neighborhood where $\Omega$ is described exactly as in the above Proposition. For our application we can directly arrange the situation in a stronger form as in the proposition.

2) For $\ell \geq 2$, the claim on $b^\ell$-symplectic structures is almost direct. Even for $\ell = 0$ the claim is nothing but Construction $4.2$

3) The assumption on such a closed 1-form $\alpha_N$ is nothing but Mori’s criterion-1.

**Proof of Proposition 5.2** Let $t_{\pm} \in [0,2)$ denote the each coordinate for $W_{\pm}$ corresponding to $t$. For the resultant manifold $W$, let $\tau \in (-2,2)$ be the corresponding coordinate and we construct $W$ by gluing $W_{\pm}$ by defining $\tau = -t_-$ on $(-2,0]$ and $\tau = t_+$ on $(-2,0]$. The two symplectic forms $\Omega_{\pm} = \Omega$ on $W_{\pm}$ do not coincide at $\tau = 0$ because they are indicated as $\Omega_{\pm} = \pm dt \wedge P_N^*\alpha_N + P_N^*\Omega_N$. Remark that the orientations of $W_{\pm}$ are opposite to each other in $W$. We do not need to orient $W$ but if we define it as that of $W_+$, the original orientation of $W_-$ is negative.

In order to obtain a $b$-symplectic structure whose singular locus is $N \times \{0\}$, Take a smooth positive function $p(\tau)$ on $(-2,2)$ satisfying

\begin{align*}
\text{(i)} & \quad p(\tau) = 1/t \quad \tau \in (-1/2,0) \cup (0,1/2), \\
\text{(ii)} & \quad p(\tau) \equiv -1 \quad \tau \in (-2,-3/2), \\
\text{(iii)} & \quad p(\tau) \equiv 1 \quad \tau \in (3/2,2), \\
\text{(iv)} & \quad p'(\tau) < -0 \quad \tau \in (-3/2,-1/2) \cup (1/2,3/2).
\end{align*}

Then replace $\Omega$ with $\Omega_b = p(t)dt \wedge P_N^*(\alpha_N) + P_N^*\Omega_N$ on $(-2,2)$ and the collar neighborhoods and put them on $W_i$’s. They naturally give rise to a $b$-symplectic structure on $W$. 

[5.2]
5.2 Leafwise symplectic foliations

Taking the double is useful in constructing regular leafwise symplectic foliations of codimension one. Because it concerns only the cylindrical end of the symplectic manifold, let us start only with the cylindrical end of the setting of Proposition 5.2 and follow some notations there.

Construction 5.4 Let $W_a = (a, \infty) \times N$ ($a$ has no particular meaning) be a $2n$-dimensional symplectic manifold with a symplectic form

$$\Omega = dt \wedge P_N^*a_N + P_N^*\Omega_N$$

as in Proposition 5.2 (2). We construct a symplectic foliation of codimension one on $M' = (-2, 2) \times N \times S^1$. Take a smooth function $\varphi(\tau)$ on $(-2, 2)$ satisfying

(i) $\varphi(\tau) \equiv 0$ ($\tau \leq -1$),

(ii) $\varphi(\tau) \equiv \pi$ ($\tau \geq 1$),

(iii) $\varphi(0) = \frac{\pi}{2}$,

(iv) $\varphi'(\tau) \geq 0$ ($-1 < \tau < 1$),

(v) $\varphi(\tau) \equiv \frac{\pi}{4}$ ($-\frac{2}{3} < \tau < -\frac{1}{3}$),

(vi) $\varphi(\tau) \equiv \frac{3\pi}{4}$ ($\frac{1}{3} < \tau < \frac{2}{3}$),

and consider the hyperplane field $\tau F' = \ker a'$ defined by the 1-form $a' = \cos \varphi(\tau)d\theta - \sin \varphi(\tau)d\tau$ where $\tau$ and $\theta$ are the coordinates of $(-2, 2)$ and $S^1$ respectively and extend them to. Apparently the hyperplane field is integrable and defines a foliation $F'$ of codimension one which is tangent to $\tau = \varphi'$ in $\tau\theta$-plane. $M'$ is covered by three open sets

$M_- = \{-2 < \tau < -1/3\}$,

$M_0 = \{-2/3 < \tau < 2/3\}$, and

$M_+ = \{1/3 < \tau < 2\}$.

We define the leafwise symplectic form $\Omega'$ on $(M', F')$ by restricting the 2-forms

$$\Omega_- = d\tau \wedge P_N^*a_N + P_N^*\Omega_N \quad \text{on } M_-,$$

$$\Omega_0 = d\theta \wedge P_N^*a_N + P_N^*\Omega_N \quad \text{on } M_0,$$

$$\Omega_+ = -d\tau \wedge P_N^*a_N + P_N^*\Omega_N \quad \text{on } M_+$$

to the leaves. Thanks to the conditions (v) and (vi), on $M_- \cap M_0$ and on $M_0 \cap M_+$ the two restrictions coincide to each other. Remark that the restriction of $(M', F', \Omega')$ to $M' \cap \{\tau < 0\}$ and $M' \cap \{\tau > 0\}$ are both isomorphic to $(W_a \times S^1, F = \{W_a \times \{\theta \in S^1\}\})$, $\Omega = dt \wedge P_N^*a_N + P_N^*\Omega_N$ as leafwise symplectic foliations.

As we have seen, we can modify the symplectic structures of the Milnor fibers of the simple elliptic or cusp singularities of hypersurface type so as to have such ends.

Theorem 5.5 Let $F_\theta$ be the Milnor fiber of one of $T_{p,q,r}$ singularities for $1/p + 1/q + 1/r < 1$ or $\tilde{E}_k$ for $k = 6, 7, 8$, and $W = -F_\theta \cup_{\partial F_\theta} F_\theta$ be its double. Then, $W \times S^1$ admits a leafwise symplectic foliation of codimension one.
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