Chromomagnetic Dipole-Operator Corrections in $\bar{B} \to X_s \gamma$ at $\mathcal{O}(\beta_0 \alpha_s^2)$

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We calculate the fermionic corrections to the photon-energy spectrum of $\bar{B} \to X_s \gamma$ which arise from the self-interference of the chromomagnetic dipole operator $Q_8$ at next-to-next-to-leading order by applying naive non-abelianization. The resulting $\mathcal{O}(\beta_0 \alpha_s^2)$ correction to the $\bar{B} \to X_s \gamma$ branching ratio amounts to a relative shift of $+0.12\% \,(+0.27\%)$ for a photon-energy cut of 1.6 GeV (1.0 GeV).

We also comment on the potential size of resummation and non-perturbative effects related to $Q_8$.

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I. INTRODUCTION

The inclusive radiative $B$-meson decay $\bar{B} \to X_s \gamma$ represents the “standard candle” of quark-flavor physics. It tests the electroweak structure of the underlying theory and provides information on the couplings and masses of heavy virtual particles appearing as intermediate states in and beyond the Standard Model (SM). See [1] for a concise overview.

The present experimental world average for a photon-energy cut of $E_\gamma > E_0$ with $E_0 = 1.6$ GeV in the $\bar{B}$-meson rest frame reads [2]

$$\mathcal{B}(\bar{B} \to X_s \gamma)_{\text{exp}}^{E_\gamma > 1.6 \text{ GeV}} = (3.55 \pm 0.24 \pm 0.09) \cdot 10^{-4}. \quad (1)$$

The quoted value includes various measurements from CLEO, BaBar, and Belle [3] and has a total error of below 8%, which consists of a combined statistical and systematic error as well as a systematic uncertainty due to the shape function.

In order to make full use of the available data, the SM calculation of $\bar{B} \to X_s \gamma$ should be performed with similar or better precision. This goal can only be achieved with dedicated calculations of next-to-next-to-leading order (NNLO) QCD effects in renormalization-group improved perturbation theory. Considerable effort has gone into such computations. The necessary two- and three-loop matching was performed in [4] and [5], while the mixing at three and four loops was calculated in [6] and [7]. The two-loop matrix element including bremsstrahlung corrections of the photonic dipole operator $Q_7$ was found in [8], confirmed in [9], and extended to include the full charm-quark mass dependence in [10]. The three-loop matrix elements of the current-current operators $Q_{1,2}$ were derived in [11] within the large-$\beta_0$ approximation. A calculation that goes beyond this approximation employs an interpolation in the charm-quark mass [12]. Contributions involving a massive quark-loop insertion into the gluon propagator of the three-loop $Q_{1,2}$ matrix elements are also known [13]. Calculations of other missing NNLO pieces, such as the $(Q_7, Q_8)$ interference were recently completed [14]. Further details on the status of the NNLO corrections to the branching ratio of $\bar{B} \to X_s \gamma$ can be found in [15].

Combining the results listed above, it was possible to obtain the first estimate of the $\bar{B} \to X_s \gamma$ branching ratio at NNLO. For $E_0 = 1.6$ GeV the result of the improved SM evaluation is given by [12, 16, 17, 18]

$$\mathcal{B}(\bar{B} \to X_s \gamma)_{\text{SM}}^{E_\gamma > 1.6 \text{ GeV}} = (3.15 \pm 0.23) \cdot 10^{-4}, \quad (2)$$

where the individual uncertainties from non-perturbative corrections (5%), parametric dependences (3%), higher-order perturbative effects (3%), and the interpolation in the charm-quark mass (3%) have been added in quadrature to obtain the total error. More details on the phenomenological NNLO analysis including the list of input parameters can be found in [12]. A systematic study of hadronic effects that cannot be described using a local operator product expansion has been recently carried out in [17, 18]. This analysis puts the naive estimate of the size of non-local power corrections in [12, 16] on firm theoretical grounds, and at the same time indicates that a further reduction of the theoretical uncertainty plaguing [2] below 5% would require a theoretical breakthrough.

Besides the branching ratio also the $\bar{B} \to X_s \gamma$ photon-energy spectrum is of theoretical interest and phenomenological relevance [19]. While close to the physical endpoint $E_\gamma = m_B/2$ the spectrum is dominated by the $(Q_{1,2}, Q_7)$ and $(Q_7, Q_7)$ contributions, the $(Q_8, Q_8)$ interference is numerically the most important one for $E_\gamma \lesssim 1.1$ GeV, because it involves a soft singularity $1/E_\gamma$ related to photon bremsstrahlung. The theoretical description of the $(Q_8, Q_8)$ part of the spectrum has a simple, but important feature, that is associated with the photon having a hadronic substructure, and manifests itself in the appearance of collinear singularities in the

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[1] Several NNLO corrections (see [12, 13, 14] and partly [15]) that were calculated after the publication of [14, 16] are not included in the central value of [12], but remain within the quoted perturbative higher-order uncertainty of 3%.
perturbative result of the fixed-order calculation. The leading contribution of the \((Q_8, Q_8)\) interference to the photon-energy spectrum in \(b \rightarrow X_s^{\text{part}} \gamma\) has been known for some time \([20]\). This contribution is suppressed by a single power of \(\alpha_s\) with the respect to the leading \((Q_7, Q_7)\) interference, and therefore is part of the next-to-leading order (NLO) corrections to the spectrum.

The \((Q_{1,2}, Q_{1,2}), (Q_{1,2}, Q_7),\) and \((Q_7, Q_8)\) corrections to the photon-energy spectrum were calculated in the large-\(\beta_0\) approximation, \(i.e.,\) including terms of order \(\beta^2/\alpha_s^2\) through naive non-abelianization \([21]\), already in \([22]\). However, in that work neither the \((Q_{1,2}, Q_8)\) nor the \((Q_8, Q_8)\) interference was considered. In this article, we close this gap partly by calculating the corrections to the photon-energy spectrum originating from the self-interference of the chromomagnetic dipole operator \(Q_8\) at \(O(\beta_0 \alpha_s^2)\). A calculation of the \((Q_{1,2}, Q_8)\) interference,

\[
\Gamma(b \rightarrow X_s^{\text{part}} \gamma)_E, > E_0 = \frac{6\alpha_{em}}{\pi} \left| \frac{V_{tb}^* V_{tb}}{V_{ub}} \right|^2 \Gamma(b \rightarrow X_s^{\text{part}} l \bar{\nu}) \sum_{i,j=1}^8 C_i^\text{eff} (\mu_b) C_j^\text{eff} (\mu_b) K_{ij} (E_0),
\]

where \(\alpha_{em} = \alpha_{em}(0) = 1/137.036\), \(V_{tb}\) are the relevant Cabibbo-Kobayashi-Maskawa matrix elements, and \(C_i^\text{eff}(\mu_b)\) denote the effective Wilson coefficients defined as in \([12]\).

In the following, we will present analytic formulas for the \(O(\beta_0 \alpha_s^2)\) corrections to \(K_{88}(E_0)\). This function describes the self-interference of the chromomagnetic dipole operator

\[
Q_8 = \frac{g}{16\pi^2} m_b(\mu) (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a,
\]

where \(g\) is the strong-coupling constant, \(m_b(\mu)\) denotes the running \(\overline{\text{MS}}\) mass of the bottom quark, \(q_{L,R}\) are left- and right-chiral quark fields, \(G_{\mu\nu}^a\) is the gluonic field strength tensor, and \(T^a\) are the color generators normalized to \(\text{Tr}(T^a T^b) = T_F \delta^{ab}\) with \(T_F = 1/2\).

Including QCD corrections up to NNLO, the coefficient \(K_{88}(E_0)\) can be written (in a notation following closely the one adopted in \([12]\)) as follows:

\[
K_{88}(E_0) = \sum_{n=1}^2 \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^n K_{88}^{(n)}(E_0).
\]

In agreement with \([20, 23]\), we find for \(K_{88}^{(1)}(E_0)\) the analytic expression

\[
K_{88}^{(1)}(E_0) = \frac{4}{27} \left\{ -\ln \frac{m_b^2}{m_s^2} \left[ \delta(2 + \delta) + 4 \ln \delta \right] + 4 \text{Li}_2 \delta - \frac{2\pi^2}{3} - 8 \ln \delta - \frac{2\delta^3}{3} + 3\delta^2 + 7\delta \right\}.
\]

where \(\delta = 1 - 2E_0/m_b\) and \(\bar{\delta} = 1 - \delta = 2E_0/m_b\). As expected, the NLO function \(K_{88}^{(1)}(E_0)\) is logarithmically divergent for both \(\delta \rightarrow 0\) (soft singularity) as well as

\[
m_s \rightarrow 0\) (collinear singularity). Notice that terms suppressed by positive powers of the ratio \(m_s/m_b\) have been neglected in \([6]\).

The NNLO function \(K_{88}^{(2)}(E_0)\) receives both fermionic and purely gluonic contributions. The former corrections arise from the Feynman diagrams shown in Figs. \([1, 2]\) and \([3]\). Since in the large-\(\beta_0\) approximation one con-
The coefficient $K_{88}^{(2,N_L)}(E_0)$ introduced in (8) describes the contribution of the graphs in Fig. 1 involving a single massless quark flavor. It can be written as

$$K_{88}^{(2,N_L)}(E_0) = \frac{4}{3} T_F \left[ \int_\delta^1 dz F_{88}^{(2,N_L)}(E_0) - K_{88}^{(1)}(E_0)L_b \right],$$

where $L_b = \ln (\mu_0^2/m_b^2)$, $z = 2E_\gamma/m_b$, and

$$F_{88}^{(2,N_L)} = \frac{8}{27} \left\{ -\ln \frac{m_s^2}{m_b^2} \left[ \frac{36-41z+13z^2+2z^3}{6z} - \frac{1+z^2}{z} \ln \frac{1+z^2}{z} \right] + \frac{1+z^2}{z} \left[ \ln^2 \frac{1+z^2}{z} - \frac{\pi^2}{3} \right] \right\},$$

with $\bar{z} = 1 - z$. The latter expression is the main analytic result of our paper. Similar to (6) also (10) contains a collinear divergence, which we have regulated by keeping a non-vanishing strange-quark mass. Again terms suppressed by positive powers of $m_s/m_b$ have been neglected in the function $F_{88}^{(2,N_L)}$.

The function $K_{88}^{(2,\text{rem})}(E_0)$ entering (7) encodes the $O(\alpha_s^2)$ contributions to the $(Q_8, Q_8)$ interference that are beyond the large-$\beta_0$ approximation. It takes the form

$$K_{88}^{(2,\text{rem})}(E_0) = \frac{33}{2} K_{88}^{(2,N_L)}(E_0) + \sum_{q=u,d,s} K_{88}^{(2,q,\gamma)}(E_0) + \sum_{q=c,b} K_{88}^{(2,q,M)}(E_0) + K_{88}^{(2,q)}(E_0),$$

The calculation of these contributions is beyond the scope...
pair production is not included in $b \to X_{\text{part}}^{\gamma}$ by definition, while $b\bar{b}$ pair production is kinematically forbidden. Thus, $K_{88}^{(2,q,\gamma)}(E_0)$ is non-vanishing for light quarks with $q = u, d, s$ only.

The function $K_{88}^{(2,q,M)}(E_0)$ originates from Fig.3 and vanishes for massless quarks ($q = u, d, s$) in dimensional regularization due to the appearance of scaleless integrals. Its analytic form for $q = c, b$ can be obtained by multiplying the NLO coefficient $K_{88}^{(1)}(E_0)$ by a renormalized one-loop vacuum-polarization function at zero-momentum transfer. Explicitly we find

$$K_{88}^{(2,q)}(E_0) = -\frac{4}{3} T_F K_{88}^{(1)}(E_0) L_q ,$$

where $L_q = \ln (\mu_0^2/m_c^2)$. Notice finally that the effects of the diagrams in Fig.3 can also been taken into account through gluon wave-function renormalization in the NLO graphs.

It is also straightforward to derive an expression for the function $K_{88}^{(2,\text{rem})}(E_0)$ in the large-$m_c$ limit. The latter enters the calculation of the $B \to X_{\gamma}^{\gamma}$ branching ratio via an interpolation in the charm-quark mass $m_c$. In agreement with that paper, we obtain the expression

$$K_{88}^{(2,\text{rem})}(E_0) = \left( -\frac{50}{3} + \frac{8\pi^2}{3} - \frac{2}{3} L_c \right) K_{88}^{(1)}(E_0) + X_{88}^{(2,\text{rem})}(E_0).$$

III. CALCULATIONAL TECHNIQUE

In order to calculate the $O(\beta_0 \alpha_s^2)$ corrections to the partonic $b \to X_{\text{part}}^{\gamma}$ cut rate arising from the self-interference $(Q_8, Q_8)$ we have employed the optical theorem. In particular, we have exploited the one-to-one correspondence between the interferences among diagrams contributing to the process $b \to s\gamma q\bar{q}$ and the physical cuts of three-loop bottom-quark self-energy diagrams. As can be seen by glancing at Fig.1 we are interested in diagrams in which the chromomagnetic dipole operator $Q_8$ appears on both sides of the cut. The contribution of a specific physical cut to the imaginary parts of the corresponding bottom-quark self-energy diagrams is thereby evaluated by means of the Cutkosky rules. See [12] for more detailed discussions.

We have evaluated the relevant four-particle cuts in two different ways. First, by a direct computation of the light-quark contributions using the set-up previously employed in the NNLO calculation of the $(Q_7, Q_7)$ contributions [9, 11], and, second, by performing the NLO calculation of the $(Q_8, Q_8)$ contribution with a fictitious gluon mass which allows us to obtain the sought $O(\alpha_s^2)$ contributions from a dispersion integral over the gluon virtuality [23]. For a recent detailed review of this technique we refer to [20]. In both cases, the reduction to master integrals via the Laporta algorithm [27] has been carried out keeping a non-vanishing strange-quark mass.

\footnote{This method was also used in the calculation of the $O(\beta_0 \alpha_s^2)$ corrections to the photon-energy spectrum of the $(Q_{1,2}, Q_{1,2})$, $(Q_{1,2}, Q_7)$, and $(Q_7, Q_8)$ terms [22]. We verified the correctness of the $(Q_7, Q_8)$ contribution given in the aforementioned article.}
FIG. 3: Three-particle cuts of the irreducible bottom-quark self-energy diagrams with a massive charm- and bottom-quark bubble contributing to the $b \to s\gamma g$ transition at $\mathcal{O}(\alpha_s^4)$. Left-right reflected diagrams are not shown. The second and third diagrams give rise to collinear logarithms in $(m_b^2/m_s^2)$.

to regulate the residual collinear divergences. All the other infrared (IR) or ultraviolet divergences, appearing in intermediate stages of the calculation, have been regulated dimensionally in $d = 4 - 2 \epsilon$ dimensions. The master integrals have been evaluated analytically both by direct integration over the phase space and by employing the differential equation method [28]. Throughout the calculation of the master integrals, we have neglected terms suppressed by positive powers of the ratio $m_s/m_b$. The agreement of the results obtained by the two methods serves as a powerful check of our calculation.

\[ \Delta B(\bar{B} \to X_s \gamma)\big|_{E_\gamma > E_0} = B(\bar{B} \to X_s \ell \bar{\nu})_{\text{exp}} - B(\bar{B} \to X_s \ell \bar{\nu})_{\text{SM} \big|_{E_\gamma > E_0}} = \frac{\alpha \alpha_s}{\pi C} \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \left| C_s^{\text{eff}(0)}(\mu_b) \right|^2 \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 K^{(2, \beta_0)}(E_0), \]  

IV. NUMERICAL ANALYSIS

In the following, we will investigate the numerical size of the $\mathcal{O}(\alpha_s^2)$ contributions to the $(Q_8, Q_8)$ interference at the level of the branching ratio of $\bar{B} \to X_s \gamma$. In order to simplify the comparison with the existing literature, we will adopt the conventions, the notations, and the numerical input parameters employed in [12]. Specifically, we will use $\mu_b = 2.5 \text{ GeV}$, $m_b = 4.68 \text{ GeV}$, $m_b/m_s = 50$, $m_s = 105.7 \text{ GeV}$, and $\mu_s = 500 \text{ GeV}$. The central value of the SM prediction for $\alpha_s(2.5 \text{ GeV}) = 0.271$, $\alpha_s(5 \text{ GeV}) = 0.240$, and $\alpha_s(10 \text{ GeV}) = 0.209$. These numbers imply that after naive non-abelianization the term $[9]$ constitutes a correction of almost 50% with respect to the $\mathcal{O}(\alpha_s)$ contributions.

As we have already mentioned, in the $(Q_8, Q_8)$ interference also the corrections which are not part of the large-$\beta_0$ approximation (such as the four-particle cut diagrams in Fig. [2]) involve collinear logarithms associated with photon fragmentation of $b \to sg$. Sufficiently far away from the endpoint of the photon-energy spectrum, the resulting IR-sensitive terms can be subtracted and absorbed into non-perturbative photon-fragmentation functions [24], which obey perturbative evolution (Dokshitzer-Gribov-Lipatov-Altarasi-Parisi or DGLAP) equations with non-perturbative initial distributions to be extracted from experiment.\(^3\) While a com-

\(^3\) In fact, in [23] only purely perturbative corrections are included.\(^4\) In the endpoint region the non-perturbative physics associated with the $(Q_8, Q_8)$ interference is encoded in a complicated sub-
The resummation of the collinear logarithms appearing in the $K_{88}^{(2,N_L)}(E_0)$ corrections is achieved by convoluting the hard function $C_{s,N_L}^{N_L}(x)$, that describes the process $b \to sq\bar{q}$ for fixed energy $x$ of the strange quark, with the universal strange-quark-to-photon fragmentation function $D_{s\to\gamma}(x)$. Explicitly, we find that the result of the resummation of the collinear logarithm in $K_{88}(E_0)$ takes the form\(^5\)

$$
\tilde{E}_{88}^{(2,N_L)} = \frac{2\pi}{\alpha_{em}} \int_x^{1} \frac{dx}{x} C_{s,N_L}^{N_L}(x) D_{s\to\gamma}\left(\frac{z}{x}\right),
$$

(15)

with

$$
C_{s,N_L}^{N_L}(x) = -\frac{8}{3} \left(\frac{10}{3} \delta(\bar{x}) - \left[\frac{1}{\bar{x}}\right]_+ + 1 + x - \frac{x^2}{2}\right),
$$

(16)

where $\bar{x} = 1 - x$ and $[1/\bar{x}]_+$ denotes the usual plus distribution. A factorization formula similar to the one given in (15) can also be derived for the complete $O(\alpha_s^2)$ correction to $K_{88}(E_0)$ in the collinear limit.

\(^5\) In the absence of QCD, the expression for the strange-quark-to-photon fragmentation function is given by $D_{s\to\gamma}(x) = \alpha_{em} Q_s^2/(2\pi) (1 + \bar{x}^2)/x \ln(\mu_s^2/\mu_r^2)$ with $Q_s = -1/3$, $\mu_s \approx m_s$, and $\mu_r \approx m_s \approx \Lambda_{QCD}$. Substituting this result into (15), one recovers the terms in (10) that are singular in the limit $m_s \to 0$.

The full photon-fragmentation functions $D_{i\to\gamma}(x)$ with $i = s, g$ are sums of perturbative and non-perturbative components. While the former are fully calculable in QCD, the latter have to be modeled. Following [22], which the interested reader should consult for further details, we will employ a vector-meson dominance model and assume that quarks and gluons first fragment into vector mesons which then turn into photons. We begin our discussion by studying the impact of the anomalous parts of the photon-fragmentation functions, i.e., the components encoding the perturbative evolution as described by the inhomogeneous DGLAP equations. Comparing the resummed with the fixed-order $O(\beta_0 \alpha_s^2)$ result, as indicated by the dashed and solid lines in the left panel of Fig. 4, respectively, we infer that the resummation of collinear logarithms decreases the obtained results. Numerically, we find a relative change of $+0.05\% (+0.12\%)$ for $E_0 = 1.6$ GeV ($E_0 = 1.0$ GeV), which implies that the resummation suppresses the considered correction by more than a factor of 2. We also mention that for photon-energy cuts around 1.6 GeV the resummation of collinear logarithms appearing in the $O(\beta_0 \alpha_s^2)$ correction can be effectively described by choosing $m_b/m_s = 14$ in the analytic expression (10).

We now turn our attention to the non-perturbative contributions related to the photon fragmentation from $b \to sq$. These corrections turn out to be potentially larger than the resummation effects. This is illustrated...
by the right panel in Fig. 4 which displays the relative change in $B(B \to X_{s\gamma})_{E_x > E_0}^{\nu_B}$ arising from the sum of the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\beta_0 \alpha_s^2)$ corrections to $(Q_8, Q_8)$, including the anomalous parts of $D_{i\to\gamma}(x)$ only (solid line) and employing the full photon-fragmentation functions with two different non-perturbative initial conditions (dashed and dotted lines). In each case, we have subtracted the fixed-order $\mathcal{O}(\alpha_s)$ corrections [6] from our results, since these effects are already a part of the SM prediction [24]. We see again that the choice $m_b/m_s = 50$, adopted throughout the recent literature on $B \to X_{s\gamma}$, tends to overestimate the effects of resumming collinear logarithms. Numerically, we find relative shifts of $-0.07\%$ and $-0.23\%$ for $E_0 = 1.6$ GeV and $E_0 = 1.0$ GeV, respectively. After incorporating on top of the anomalous also the non-perturbative components of $D_{i\to\gamma}(x)$, we obtain instead corrections of $-0.05\%$ and $-0.04\%$ or $-0.04\%$ and $0.37\%$. The former (latter) numbers correspond to set I (II) of the full photon-fragmentation functions $D_{i\to\gamma}(x)$ determined in [22]. We recall that while the initial conditions of the quark-to-photon fragmentation functions are well constrained by $e^+e^-$ data, the one of the gluon-to-photon fragmentation function is not. Compared to set I, the gluon-to-photon fragmentation function of set II is significantly larger, in particular, for small $x$. Since the function $D_{g\to\gamma}(x)$ enters the resummation of collinear logarithms in $(Q_8, Q_8)$ at $\mathcal{O}(\alpha_s)$ via [24]

$$R_{\delta}(E_0) = \frac{2\pi}{\alpha_{em}} \int_{\delta}^{1} dz \frac{8}{3} [D_{\delta\to\gamma}(z) + D_{g\to\gamma}(z)], \quad (17)$$

this results in larger shifts for set II than for set I.

In conclusion, our study of non-perturbative effects related to photon fragmentation seems to indicate, first, that the size of hadronic effects associated to the interference of $(Q_8, Q_8)$ should not shift the central value of $\beta_0$ by more than $+1\%$ and, second, that setting $m_b/m_s = 50$ in the term $\ln (m_b^2/m_s^2)$ entering the fixed-order result allows one to capture most of the numerical effect. A recent much more detailed study [17] finds slightly larger non-perturbative effects of $[-0.3, +1.9]\%$ related to the self-interference of the chromomagnetic dipole operator $Q_8$. While a straightforward comparison of this result with ours is difficult, given the very different nature of the used framework, the fact that the two calculations result in numbers in the same ballpark gives us further confidence that hadronic contributions in $B \to X_{s\gamma}$ related to $(Q_8, Q_8)$ indeed represent a minor effect.

V. CONCLUSIONS

In this work we have calculated the NNLO corrections to the $b \to X_{s}^{\text{part.}\gamma}$ photon-energy spectrum in the large-$\beta_0$ approximation that arise from self-interference contribution of the chromomagnetic dipole operator $Q_8$. The contributions from $(Q_8, Q_8)$ are known to be numerically subleading at NLO for the photon-energy cut currently employed in the measurements of the $B \to X_{s\gamma}$ branching ratio. We find that this trend continues at NNLO and that the calculated $\mathcal{O}(\beta_0 \alpha_s^2)$ corrections have only a marginal impact on the $B \to X_{s\gamma}$ branching ratio, amounting to a relative shift of a few permille. However, corrections to the spectrum arising from the $(Q_8, Q_8)$ interference are theoretically interesting in their own right, since they are known to be logarithmically divergent in the limit of vanishing photon energy, and because they contain collinear singularities that are associated with the intrinsic hadronic component of the photon. Concerning the latter issue, we have shown that non-perturbative effects in $(Q_8, Q_8)$ due to photon fragmentation from $b \to s\gamma$ presumably constitute an effect of below a percent only. Our results can be readily incorporated in the SM calculation of the $B \to X_{s\gamma}$ branching ratio. While a total non-perturbative uncertainty of about 5% will affect the SM prediction for the branching ratio for some time to come, it is still mandatory to update the available NNLO estimate by including all the $\mathcal{O}(\alpha_s^2)$ corrections which were calculated in the past four years, with the aim of reducing as much as possible the residual perturbative uncertainty. The calculation presented here, constitutes a necessary ingredient to achieve this goal.

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