CORRELATION AND SIMPSON CONVERSION IN $2 \times 2 \times 2$ CONTINGENCY TABLES

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Abstract. We study a generalisation of Simpson reversal (also known as Simpson’s paradox or the Yule-Simpson effect) to $2 \times 2 \times 2$ contingency tables and characterise the cases for which it can and cannot occur with two combinatorial-geometric lemmas. We also present a conjecture based on some computational experiments on the expected likelihood of such events.

1. Introduction

Simpson reversal is a phenomenon in statistics in which a common trend among different groups disappears, or even reverses, when the groups are combined. More precisely, if $A_1$, $A_2$, and $B$ are random events such that

$$P(A_1 \cap A_2 | B) > P(A_1 | B)P(A_2 | B) \quad \text{and} \quad P(A_1 \cap A_2 | \overline{B}) > P(A_1 | \overline{B})P(A_2 | \overline{B}),$$

where $\overline{B}$ denotes the complement of the random event $B$, then it is possible for

$$P(A_1 \cap A_2) < P(A_1)P(A_2).$$

Real-world examples of the paradox are well-documented, for instance in [2] and [8], but it has also been observed and studied in several applications from biology, such as [5] and [9].

Example 1.1. A concrete example of Simpson reversal is illustrated by the voting results for (the Senate version) of the Civil Rights Act of 1964 in the United States House of Representatives. The votes are listed below broken down according to political party (Democrats and Republicans) and region of the country (Northern, Southern, and all states), as presented in [4].

|          | Yes | No |
|----------|-----|----|
| Democrats| 144 |  8 |
| Republicans | 137 | 24 |

Table 1. House of Representatives voting results for the Civil Rights Act of 1964 according to political party among Northern states (left), Southern states (middle), or all states (right).

From these tables, one can observe that a higher percentage of Democrats voted in favour of the bill in both the Northern and the Southern states (95% and 9% of the Democrats compared to 85% and 0% of the Republicans, respectively), but a higher percentage of Republicans voted in favour of the bill overall (80% of the Republicans compared to 63% of the Democrats). This instance of Simpson reversal can be explained by the fact that the relationship between party and vote on the bill
was significantly affected by the regions corresponding to the voters. Indeed, most of
the Southern representatives at that time were Democrats and the vast majority of
negative votes came from that region.

While the existence of a Simpson reversal is surprising – even paradoxical – at first
glance, there are several geometric interpretations that illustrate why the phenomenon
is not only possible, but a relatively common occurrence. A well known geometric
interpretation of Simpson reversal is as follows: Suppose \( v_1, v_2, w_1, \) and \( w_2 \)
are vectors in \( \mathbb{R}^2 \) based at the origin such that the slope of \( v_i \) is greater than the slope
of \( w_i \) for \( i \) equal to 1 and 2. Then it is possible that the slope of \( v_1 + v_2 \) is less than
the slope of \( w_1 + w_2 \) as shown in Figure 1.

![Figure 1. A geometric illustration of Simpson reversal.](attachment:image1.png)

Another interpretation of Simpson reversal is that if \( A \) and \( B \) are \( 2 \times 2 \) invertible
matrices with entries in \( \mathbb{R}_+ \), then it is possible for
\[
\text{sign} \left( \det(A) \right) = \text{sign} \left( \det(B) \right) = -\text{sign} \left( \det(A + B) \right).
\]
This can be seen geometrically via triangulations of a square: Any generic real-valued
function \( f : P_0 \to \mathbb{R} \) on the vertices \( P_0 \) of a convex polygon \( P \) induces a unique
triangulation on \( P \) by taking the convex hull of the set
\[
\{ (p_1, p_2, f((p_1, p_2)) ) \mid (p_1, p_2) \in P_0 \} \subseteq \mathbb{R}^3
\]
and projecting its upper envelope onto \( P \). This process is nicely explained in [3] and
illustrated for a case where \( P \) is a square in Figure 2.

![Figure 2. Positive correlation (right) and negative correlation induces
different triangulations of the underlying square.](attachment:image2.png)

If we identify a \( 2 \times 2 \) contingency table consisting of values \( F_{00}, F_{10}, F_{01}, F_{11} \) with
the function on the unit square in \( \mathbb{R}^2 \) that sends \( (x, y) \) to \( F_{xy} \), we can associate a
triangulation to every generic \( 2 \times 2 \) contingency table using the process described
above. Since a $2 \times 2$ contingency table with values $F_{00}, F_{10}, F_{01}, F_{11}$ is positively correlated if $F_{00} \cdot F_{11} > F_{01} \cdot F_{10}$ and negatively correlated if the inequality is reversed, correlations in $2 \times 2$ contingency tables can be encoded in the triangulations of the square they induce. For a contingency table with $F_{00} \cdot F_{11} = F_{01} \cdot F_{10}$, the variables are independent and the corresponding tetrahedron is degenerate, so we do not get a triangulation. One can think of that as an intermediate state.

It is sometimes easier to work with linear relations, so we introduce the notation $f_{xy} = \ln F_{xy}$ in which case positive and negative correlations are indicated by whether the quantity $w := f_{00} + f_{11} - f_{01} - f_{10}$ is positive or negative. Thus, a pair of contingency tables, $F$ and $G$, give rise to a Simpson reversal when $f_{xy} = \ln F_{xy}$ and $g_{xy} = \ln G_{xy}$ induce the same triangulation of the square, but $\ln(F_{xy} + G_{xy})$ induces the other triangulation (there are only two).

With these interpretations, the existence of Simpson reversals is not so surprising. In fact, from the perspective of causality, see [7], it is a rather natural notion.

The remainder of this paper is structured as follows: In Section 2, we propose a generalisation of Simpson reversal for $2 \times 2 \times 2$ contingency tables, which we call Simpson conversion, that involves an arrangement of hyperplanes in $\mathbb{R}^8$ and the set of triangulations of the 3-dimensional cube. We describe the relationship between the linear forms defining said hyperplane arrangement and the triangulations in Section 3. Section 4 contains the main results of the paper, in which we characterise the 3-dimensional analog of Simpson’s reversals for $2 \times 2 \times 2$ contingency tables. We conclude the paper with a conjecture on the frequency of Simpson conversion and some observations based on computational experiments in Section 5.

2. Correlations in Three Dimensions

The instance of Simpson reversal in the House of Representatives vote on the Civil Rights Bill of 1964 presented in Example 1.1 (taken from [4]) can also be observed in the Senate vote on the same bill, the results of which are listed below appended to the data in Table 1.

| Northern States | Yes | No |
|-----------------|-----|----|
| House Democrats | 144 | 8  |
| Republicans     | 137 | 24 |
| Senate Democrats | 45  | 1  |
| Republicans     | 27  | 5  |

Table 2. Voting results for the Civil Rights Act of 1964 according to legislative chamber and political party among Northern states (left), Southern states (middle), or all states (right).

With this, one might be interested in studying the various relationships between party, vote on the bill, and chamber of congress simultaneously by combining the two $2 \times 2$ contingency tables corresponding to each chamber of congress over each region.
into a $2 \times 2 \times 2$ contingency table. Understanding how observed correlations change when the data from different regions are combined becomes much more complicated than in the 2-dimensional case since there are many more notions of correlation for a 3-way contingency table. Indeed, a $2 \times 2 \times 2$ contingency table can exhibit mutual (all variables dependent on each other), marginal (two variables are dependent ignoring the third), and conditional (two variables are dependent given the third) correlations, and these three types of correlation are distinct from one another in the sense that dependencies of one type do not necessarily imply dependencies of the other types.

More precisely, if $A$, $B$, and $C$ are random events, then there are eight distinct relations arising from mutual dependencies of the form

$$P(X \cap Y \cap Z) \neq P(X)P(Y)P(Z);$$

three distinct relations arising from marginal dependencies of the form

$$P(X \cap Y) \neq P(X)P(Y);$$

and six distinct relations arising from conditional dependencies of the form

$$P(X \cap Y \mid Z) \neq P(X \mid Z)P(Y \mid Z)$$

where $X$, $Y$, and $Z$ are distinct events chosen from $A$ or $\overline{A}$, $B$ or $\overline{B}$, and $C$ or $\overline{C}$. Note that there are only three and six distinct relations from marginal and conditional dependencies, respectively, since

$$P(X \cap Y) \neq P(X)P(Y) \iff P(X \cap \overline{Y}) \neq P(X)P(\overline{Y}).$$

We leave it to the reader to check that all of the marginal and mutual, but only two of the conditional correlations in Table 2 exhibit Simpson reversals.

For a $2 \times 2 \times 2$ contingency table with values $F_{xyz}$, these 17 dependencies give linear relations on the variables $f_{xyz} = \ln F_{xyz}$, which have a natural correspondence with the set of linear relations arising from the 74 triangulations of the 3-dimensional cube. Just as a generic $2 \times 2$ contingency table induces a triangulation of the square, a generic $2 \times 2 \times 2$ contingency table induces a triangulation of the cube (into tetrahedra) via a projection of the upper envelope of the convex hull of the points $(x, y, z, F_{xyz})$ in $\mathbb{R}^4$. Beerenwinkel, Pachter, and Sturmfels [1] showed that the contingency tables corresponding to each triangulation of the cube are determined by 20 linear relations, which we list in Appendix A. The conditional correlations correspond to the forms labeled $a$ through $f$; the marginal correlations correspond to the sums $g + h$, $i + j$, and $k + l$; and the mutual correlations correspond to the forms labeled $m$ through $t$. The triangulations of the cube are listed in Appendix B, organised into six types according to symmetry – every triangulation of a given type can be obtained from any other triangulation of that type via a rotation or reflection. We have used the same notation as in [1] for easier comparison.

3. THE CORRESPONDENCE BETWEEN LINEAR FORMS AND TRIANGULATIONS

In this section, we describe the algebro-geometric correspondence between the linear forms in Appendix A and the triangulations of the 3-dimensional cube in Appendix B: One can think of the positive quadrant of $\mathbb{R}^8$ being divided into 74 regions by the 20 hyperplanes associated to the linear forms, where each region corresponds to a unique triangulation of the cube. The signs of the forms $a$-$f$ correspond to the diagonals on the six 2-dimensional squares that make up the surface of the cube and the signs of the forms $g$-$l$ correspond to flipping the hyperdiagonal of the cube.
within the six rectangles formed by opposite pairs of edges in the cube (these six rectangles each slice the cube into two triangular prisms), and finally the forms m-t correspond to whether or not the hyperdiagonal is present when passing between the triangulations of Types I and II.

**Example 3.1.** Consider the triangulation labeled 3 in Appendix B shown below:

![Triangulation Diagram](image)

This triangulation consists of six tetrahedra, namely \{000, 001, 010, 100\}, \{011, 001, 010, 100\}, \{011, 001, 111, 100\}, \{011, 010, 111, 100\}, and \{110, 010, 111, 100\}, and \{110, 010, 111, 100\}, where the notation $xyz$ means $(x, y, z)$.

The sequence of letters $b, d, -e, -t$ below the cube indicates this triangulation is associated to the contingency tables $F$ satisfying the relations

$$
0 < b := f_{001} + f_{111} - f_{011} - f_{010},
$$

$$
0 < d := f_{010} + f_{111} - f_{110} - f_{011},
$$

$$
0 > e := f_{000} + f_{011} - f_{010} - f_{001}, \text{ and}
$$

$$
0 > t := f_{010} + f_{001} + f_{111} - f_{100} - 2f_{011},
$$

from Appendix A or, equivalently,

$$
\frac{F_{001}F_{111}}{F_{011}F_{010}} > 1, \quad \frac{F_{010}F_{111}}{F_{110}F_{011}} > 1,
$$

$$
\frac{F_{000}F_{011}}{F_{010}F_{001}} < 1, \quad \frac{F_{010}F_{001}F_{111}}{F_{100}F_{011}^2} < 1.
$$

The value on the other 16 linear forms follows from these 4.

Notice that the vertices 001, 100, 010, and 111 in Triangulation 3 are each incident with three 2-dimensional (blue) diagonals and the remaining four vertices are not incident with any 2-dimensional diagonals. Such vertices will play an important role in the next section, so we will give them a name.

**Definition 3.2.** For a given triangulation of the cube,

- a vertex is called **full** and marked with a filled circle in Appendix B if it is incident with all three 2-dimensional diagonals belonging to the square faces that contain it;
- a vertex is called **empty** and marked with an empty circle in Appendix B if it is not incident with any of the three 2-dimensional diagonals belonging to the square faces that contain it.

Not every triangulation has full or empty vertices. For instance, one can see in Appendix B that the triangulations of Type IV do not have any full vertices and the triangulations of Type VI do not have any empty vertices.

We conclude this section with some additional observations on how to interpret the types of correlations that correspond to a certain triangulation: In the triangulations of Type I and II, the diagonal edges in each pair of opposite faces have opposite directions, which means that the correlation of any two variables in the corresponding contingency tables are dependent on the value of the third. On the contrary, the diagonal edges in each pair of opposite faces in the triangulations of Type IV and VI have the same direction, which means that the correlation of any two variables in the corresponding contingency tables are not dependent on the value of the third.
4. Simpson Conversion in Three Dimensions

This is the main section of the paper where we consider the question: If a pair of $2 \times 2 \times 2$ contingency tables correspond to the same triangulation of the cube, is it possible that their (component-wise) sum induces a different triangulation? Just as in the case of $2 \times 2$ contingency tables, it is not difficult to find an example which answers this question in the affirmative. Unlike the case of $2 \times 2$ contingency tables, however, there are many different ways in which these instances can arise – instead of reversing from one (of two) triangulations to the other, it is possible to convert from one triangulation of a cube to a subset of the other $73$ – we call these instances Simpson conversions. Our main theorem is a classification of all pairs of triangulations, $A$ and $B$, for which there exists a Simpson conversion from $A$ to $B$. We proceed with some essential lemmas for the classification.

4.1. Setup and Essential Lemmas. For each $x \in \{0, 1\}$, we define

$$\mathcal{T} = \begin{cases} 1 & x = 0, \\ 0 & x = 1. \end{cases}$$

We denote by $F_x$ the value of the function $F : \{0, 1\}^k \to \mathbb{R}_+$ and input $v \in \{0, 1\}^k$, i.e. the entry in the corresponding contingency table. We start with a lemma giving relations that follow if we have Simpson reversal happening in two dimensions.

**Lemma 4.1.** Let $F, G : \{0, 1\}^2 \to \mathbb{R}_+$ and $(x, y) \in \{0, 1\}^2$. If $F_{xy}F_{\overline{x}\overline{y}} < F_{xy}F_{\overline{x}\overline{y}}$, $G_{xy}G_{\overline{x}\overline{y}} < G_{xy}G_{\overline{x}\overline{y}}$, and

$$(F_{xy} + G_{xy})(F_{\overline{x}\overline{y}} + G_{\overline{x}\overline{y}}) > (F_{xy} + G_{xy})(F_{\overline{x}\overline{y}} + G_{\overline{x}\overline{y}}),$$

then one of the following pairs of inequalities must hold

$$F_{xy}G_{xy} > F_{xy}G_{\overline{x}\overline{y}} \text{ and } F_{xy}G_{xy} < F_{xy}G_{\overline{x}\overline{y}} \text{ or}$$

$$F_{xy}G_{xy} < F_{xy}G_{\overline{x}\overline{y}} \text{ and } F_{xy}G_{xy} > F_{xy}G_{\overline{x}\overline{y}}.$$

**Proof.** It suffices to prove this for the case where $x = y = 0$. Suppose $F_{10}F_{01} < F_{00}F_{11}$, $G_{10}G_{01} < G_{00}G_{11}$, and $(F_{10} + G_{10})(F_{01} + G_{01}) > (F_{00} + G_{00})(F_{11} + G_{11})$. Expanding the products in the third inequality and multiplying each side by $F_{00}G_{00}$, we get that

$$F_{00}F_{10}F_{01}G_{00} + F_{00}F_{10}G_{00}G_{01} + F_{00}F_{01}G_{00}G_{10} + F_{00}G_{00}G_{10}G_{01} > F_{00}F_{00}F_{11}G_{00} + F_{00}F_{00}G_{00}G_{11} + F_{00}G_{11}G_{00}G_{00} + F_{00}G_{00}G_{00}G_{11}.$$

Applying the first and second inequalities to the first and fourth summands, we get that

$$F_{00}F_{10}G_{00}G_{01} + F_{00}F_{01}G_{00}G_{10} > F_{00}F_{00}G_{00}G_{11} + F_{00}F_{11}G_{00}G_{00},$$

and hence $(F_{10}G_{00} - F_{00}G_{10})(F_{00}G_{01} - F_{01}G_{00}) > 0$. The desired result follows immediately. \hfill $\square$

The next lemma says that if two $2 \times 2 \times 2$ contingency tables, $F$ and $G$, both have a full vertex at $xyz$, then the sum $F + G$ cannot have an empty vertex at $xyz$. That is, if all three 2-dimensional diagonals are incident to vertex $xyz$ in both $F$ and $G$, then they cannot all flip in the sum $F + G$. 

Lemma 4.2. Let \( F, G : \mathbb{F}_2^3 \to \mathbb{R}_+ \) and \((x, y, z) \in \mathbb{F}_2^3\). If each of the following six inequalities hold

\[
F_{xyz}F_{x\bar{y}z} < F_{xyz}F_{\bar{x}y\bar{z}}, \quad G_{xyz}G_{x\bar{y}z} < G_{xyz}G_{\bar{x}y\bar{z}}, \\
F_{xyz}F_{x\bar{y}z} < F_{xyz}F_{\bar{x}y\bar{z}}, \quad G_{xyz}G_{x\bar{y}z} < G_{xyz}G_{\bar{x}y\bar{z}}, \\
F_{x\bar{y}z}F_{\bar{x}y\bar{z}} < F_{xyz}F_{\bar{x}y\bar{z}}, \quad G_{x\bar{y}z}G_{\bar{x}y\bar{z}} < G_{xyz}G_{\bar{x}y\bar{z}},
\]

then it is not possible for all three of the following three inequalities to hold

\[
(F_{xyz} + G_{xyz})(F_{x\bar{y}z} + G_{x\bar{y}z}) > (F_{xyz} + G_{xyz})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}), \\
(F_{xyz} + G_{xyz})(F_{x\bar{y}z} + G_{x\bar{y}z}) > (F_{xyz} + G_{xyz})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}), \\
(F_{x\bar{y}z} + G_{x\bar{y}z})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}) > (F_{xyz} + G_{xyz})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}).
\]

Proof. If all nine inequalities hold, then by Lemma 4.1 we get the three logical conclusions

\[
F_{xyg}G_{xyz} > F_{xyz}G_{xyg} \quad \text{and} \quad F_{xyg}G_{xyz} < F_{xyz}G_{xyg} \quad \text{or} \\
F_{xyg}G_{xyz} < F_{xyz}G_{xyg} \quad \text{and} \quad F_{xyg}G_{xyz} > F_{xyz}G_{xyg}, \quad (1)
\]

\[
F_{xyg}G_{xyz} > F_{xyz}G_{xyg} \quad \text{and} \quad F_{xyg}G_{xyz} < F_{xyz}G_{xyg} \quad \text{or} \\
F_{xyg}G_{xyz} < F_{xyz}G_{xyg} \quad \text{and} \quad F_{xyg}G_{xyz} > F_{xyz}G_{xyg}, \quad (2)
\]

\[
F_{xyg}G_{xyz} > F_{xyz}G_{xyg} \quad \text{and} \quad F_{xyg}G_{xyz} < F_{xyz}G_{xyg} \quad \text{or} \\
F_{xyg}G_{xyz} < F_{xyz}G_{xyg} \quad \text{and} \quad F_{xyg}G_{xyz} > F_{xyz}G_{xyg}. \quad (3)
\]

Note that if the first conjunction in (1) is true, then the first conjunction of (3) cannot hold, so the second conjunction of (3) must be true. This in turn implies that the second conjunction of (2) is true, which is a direct contradiction with the first conjunction of (1). We get a similar contradiction if we assume the second conjunction of (1) to be true. \(\square\)

There is a parity argument in the proof of Lemma 4.2 that relies on the assumption that there are three diagonals emanating from the vertex \(xyz\). This can be extended to the case where there is exactly one diagonal emanation from \(xyz\). In particular, if a table \(F\) has exactly one diagonal incident with vertex \(xyz\) and another table \(G\) also has only that same diagonal incident with vertex \(xyz\), then the sum \(F + G\) cannot have all three diagonals on the sides incident with \(xyz\) different from those in \(F\) and \(G\). This is the content of the next lemma.

Lemma 4.3. Let \( F, G : \mathbb{F}_2^3 \to \mathbb{R}_+ \) and \((x, y, z) \in \mathbb{F}_2^3\). If each of the following six inequalities hold

\[
F_{xyz}F_{x\bar{y}z} < F_{xyz}F_{\bar{x}y\bar{z}}, \quad G_{xyz}G_{x\bar{y}z} < G_{xyz}G_{\bar{x}y\bar{z}}, \\
F_{xyz}F_{x\bar{y}z} > F_{xyz}F_{\bar{x}y\bar{z}}, \quad G_{xyz}G_{x\bar{y}z} > G_{xyz}G_{\bar{x}y\bar{z}}, \\
F_{x\bar{y}z}F_{\bar{x}y\bar{z}} > F_{xyz}F_{\bar{x}y\bar{z}}, \quad G_{x\bar{y}z}G_{\bar{x}y\bar{z}} > G_{xyz}G_{\bar{x}y\bar{z}},
\]

then it is not possible for all three of the following three inequalities to hold

\[
(F_{xyz} + G_{xyz})(F_{x\bar{y}z} + G_{x\bar{y}z}) > (F_{xyz} + G_{xyz})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}), \\
(F_{xyz} + G_{xyz})(F_{x\bar{y}z} + G_{x\bar{y}z}) > (F_{xyz} + G_{xyz})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}), \\
(F_{x\bar{y}z} + G_{x\bar{y}z})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}) > (F_{xyz} + G_{xyz})(F_{\bar{x}y\bar{z}} + G_{\bar{x}y\bar{z}}).
\]

Proof. Very similar to proof of Lemma 4.2. \(\square\)
4.2. Infeasible 3-Dimensional Simpson Conversions. We are now ready to describe the pairs of triangulations, \(A\) and \(B\), of the cube for which it is not possible for the sum of two \(2 \times 2 \times 2\) contingency tables associated to Triangulation \(A\) to be associated to Triangulation \(B\). For brevity, we will only list the examples for pairs of triangulations up to symmetry. There are \(74^2 = 5476\) ordered pairs of triangulations, but only 167 up to symmetry. Among the 167 equivalence classes of ordered pairs, 55 satisfy the hypotheses of Lemmas 4.2 and 4.3 and therefore cannot give rise to Simpson conversions. These (equivalence classes of) pairs are listed in Table 3.

| Triangulation A | Infeasible Triangulations B                          |
|----------------|------------------------------------------------------|
| 1              | 2, 7, 20, 35, 50                                     |
| 3              | 2, 7, 8, 20, 21, 23, 35, 36, 41, 50, 52, 62          |
| 11             | 2, 7, 8, 9, 22, 23, 24, 25, 30, 40, 41, 42, 43, 54, 55, 61, 62, 63, 64 |
| 35             | 2, 7, 8, 10, 14, 23, 27, 33, 41, 46, 55, 62, 70      |
| 71             | 1, 3, 4, 14, 41, 55                                  |

Table 3. List of pairs for which two contingency tables corresponding to Triangulation \(A\) cannot have a sum with Triangulation \(B\).

4.3. Feasible 3-Dimensional Simpson Conversions. For the remaining 112 (equivalence classes of) pairs of triangulations, we randomly sampled the space of contingency tables and checked which triangulations arose from sums of pairs corresponding to each of the triangulations \(A\). We found specific instances for all 112 cases, which means Lemmas 4.2 and 4.3 characterise all the cases in which 3-dimensional Simpson conversions cannot occur. The feasible 3-dimensional Simpson conversions (up to symmetry) are listed in Table 4.

| Triangulation A | Feasible Triangulations B                          |
|----------------|----------------------------------------------------|
| 1              | 1, 3, 4, 11, 47, 71                                |
| 3              | 1, 3, 4, 5, 9, 11, 12, 14, 47, 48, 49, 71, 72      |
| 11             | 1, 3, 4, 5, 11, 12, 13, 14, 15, 16, 19, 20, 21, 35, 36, 37, 47, 48, 49, 50, 52, 56, 57, 58, 71, 72, 73 |
| 35             | 1, 3, 4, 6, 11, 12, 14, 17, 18, 28, 29, 35, 36, 38, 39, 41, 44, 46, 47, 48, 51, 53, 54, 65, 67, 71, 72, 73 |
| 47             | 1, 3, 4, 6, 11, 12, 17, 18, 20, 21, 26, 28, 29, 32, 35, 36, 38, 39, 44, 47, 48, 50, 51, 52, 53, 59, 60, 65, 67, 68, 71, 72, 74 |
| 71             | 11, 12, 35, 36, 47, 48, 71, 72                      |

Table 4. List of pairs for which two contingency tables corresponding to Triangulation \(A\) can have a sum with Triangulation \(B\).

The Python script used to generate the data in Tables 3 and 4 is available at: https://www.mattstamps.com/simpson/supplementary.zip

Pulling everything together and interpreting the information in Tables 3 and 4, we have established the following results.
Theorem 4.4. If two $2 \times 2 \times 2$ contingency tables, $F$ and $G$, induce the same triangulation of the cube, then their sum $F + G$ can induce any triangulation of the cube not ruled out by Lemmas 4.2 and 4.3.

Corollary 4.5. It is not possible for the sum of two $2 \times 2 \times 2$ contingency tables associated to a triangulation of Type I or II to induce a triangulation of Type IV.

Corollary 4.6. It is not possible for the sum of two $2 \times 2 \times 2$ contingency tables associated to a triangulation of Type VI to induce a triangulation of Type I or II.

5. Other Computations & Observations

We conclude this paper with a conjecture and some observations from additional computational experiments.

5.1. Frequency of Simpson Conversion. In addition to there being many possible ways in which Simpson conversions can occur, Simpson conversions appear to occur somewhat frequently. For the 2-dimensional case, Pavlides and Perlman [6] showed that the probability of a Simpson reversal occurring is $1/60$. We ran a simulation to approximate the analogous value for a 3-dimensional Simpson conversion to produce the following conjecture:

Conjecture 5.1. The probability that a 3-dimensional Simpson conversion occurs is $4/25$.

We obtained this by uniformly sampling $2 \times 2 \times 2$ contingency tables from the standard simplex in $\mathbb{R}^8$. In particular, we generated a million pairs of contingency tables corresponding to common triangulations and recorded the percentage that exhibited Simpson conversions. We repeated this experiment a hundred times, which resulted in a 99% confidence interval of $0.16 \pm 0.001$. The Python script for the experiment is available at the website listed Section 4.3.

5.2. Generalized Simpson Conversions. We also considered the following, more general, version of the main question from Section 4: For which triangulations $A$, $B$, and $C$ is it possible for the sum of a contingency table corresponding to Triangulation $A$ and a contingency table corresponding to Triangulation $B$ to correspond to Triangulation $C$? Just as before, it is not difficult to find examples of such triples (for instance, in Table 2, the contingency tables for the Northern, Southern, and all states correspond to Triangulations 19, 30, and 35, respectively), but Lemmas 4.2 and 4.3 still imply that it is not possible to find such examples of every triple of triangulations. The $\binom{74}{2} \cdot 74 = 199874$ triples of triangulations can be partitioned into 4655 equivalence classes based on the symmetries of the cube. Of those 4655 equivalence classes, 351 cannot occur because of Lemmas 4.2 and 4.3. Using random search, we have found specific instances for 4287 of the remaining 4304 cases, leaving 17 triples unaccounted for. Since it is not clear whether these remaining triples are infeasible (for reasons other than Lemmas 4.2 and 4.3) or simply very rare, we are continuing to search for specific instances and maintaining an up-to-date spreadsheet of known instances with the supplementary documents at:

https://www.mattstamps.com/simpson/feasible_triples.csv
ACKNOWLEDGEMENTS

We are grateful to Lior Pachter for asking this question and to Soo Go for several helpful suggestions that increased the efficiency and speed of our computations. This work was supported in part by grant 621-2014-4780 from the Swedish Science Council and National Science Foundation Grant #1159206.

APPENDIX A: LINEAR FORMS

The following list of linear forms, taken from the paper [1], are used to determine the triangulation of the cube corresponding to each $2 \times 2 \times 2$ contingency table.

\begin{align*}
  a & := f_{000} + f_{110} - f_{010} - f_{100} \\
  b & := f_{001} + f_{111} - f_{011} - f_{101} \\
  c & := f_{000} + f_{101} - f_{001} - f_{100} \\
  d & := f_{010} + f_{111} - f_{110} - f_{011} \\
  e & := f_{000} + f_{011} - f_{010} - f_{001} \\
  f & := f_{100} + f_{111} - f_{101} - f_{110} \\
  g & := f_{000} + f_{111} - f_{011} - f_{100} \\
  h & := f_{001} + f_{110} - f_{010} - f_{101} \\
  i & := f_{000} + f_{111} - f_{010} - f_{110} \\
  j & := f_{001} + f_{110} - f_{011} - f_{100} \\
  k & := f_{000} + f_{111} - f_{001} - f_{110} \\
  l & := f_{010} + f_{101} - f_{011} - f_{100} \\
  m & := f_{001} + f_{010} + f_{100} - f_{111} - 2f_{000} \\
  n & := f_{110} + f_{101} + f_{011} - f_{000} - 2f_{111} \\
  o & := f_{100} + f_{010} + f_{111} - f_{001} - 2f_{110} \\
  p & := f_{011} + f_{101} + f_{000} - f_{110} - 2f_{001} \\
  q & := f_{001} + f_{100} + f_{111} - f_{010} - 2f_{101} \\
  r & := f_{110} + f_{011} + f_{000} - f_{101} - 2f_{010} \\
  s & := f_{101} + f_{110} + f_{000} - f_{011} - 2f_{100} \\
  t & := f_{010} + f_{001} + f_{111} - f_{100} - 2f_{011}
\end{align*}

APPENDIX B: TRIANGULATIONS

The following chart comprises the 74 triangulations of the 3-dimensional cube. The numbering is taken from the paper [1] for convenience, but we have chosen to list them in a slightly different order to enhance some similarities. The diagonals on the surface of the cube are blue and the hyperdiagonals are red.
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