SEMITORIC SYSTEMS OF NON-SIMPLE TYPE

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ABSTRACT. A semitoric integrable system $F = (J, H)$ on a symplectic 4–manifold is simple if each fiber of $J$ contains at most one focus-focus critical point. Simple semitoric systems were classified about ten years ago by Pelayo–Vũ Ngọc in terms of five invariants. In this paper we explain how the simplicity assumption can be removed from the classification by adapting the invariants.

1. Introduction

Let $(M, \omega)$ be a connected symplectic 4-manifold. Integrable systems $(M, \omega, F: M \to \mathbb{R}^2)$ with non-degenerate singularities can have critical points of six types: elliptic-regular, hyperbolic-regular, elliptic-elliptic, elliptic-hyperbolic, hyperbolic-hyperbolic, and focus-focus. In this paper we do not consider systems which have singularities of elliptic-hyperbolic, hyperbolic-hyperbolic, or hyperbolic-regular type. A critical point $p$ of $F$ is a focus-focus point if there are local coordinates $(x_1, \xi_1, x_2, \xi_2)$ such that

\[
\omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2, \quad p = (0,0,0,0), \quad \text{and}
\]

\[
(1.1) \quad F = (x_1\xi_2 - x_2\xi_1, x_1\xi_1 + x_2\xi_2).
\]

The local models for elliptic-regular and elliptic-elliptic points are, respectively,

- For elliptic-regular:
  
  \[
  F = \left(\frac{x_1^2 + \xi_1^2}{2}, \xi_2\right)
  \]

- For elliptic-elliptic:
  
  \[
  F = \left(\frac{x_1^2 + \xi_1^2}{2}, \frac{x_2^2 + \xi_2^2}{2}\right)
  \]

A focus-focus fiber is any fiber of $F$ which contains at least one focus-focus point and all critical points in the fiber are focus-focus points; topologically these fibers are 2-tori pinched once for each focus-focus point.

1.1. Semitoric systems of simple type. A rich class of systems $(M, \omega, F = (J, H))$ having these types of singularities are those called semitoric, which means that $J$ is a proper function whose Hamiltonian flow is $2\pi$-periodic and $F$ has only non-degenerate singularities of these three types (i.e. no hyperbolic components).

Definition 1.1. Two semitoric systems $(M_i, \omega_i, F_i = (J_i, H_i))$, $i \in \{1, 2\}$, are isomorphic if there exists a symplectomorphism $\phi: M_1 \to M_2$ such that $\phi^*(J_2, H_2) = (J_1, f(J_1, H_1))$ for some smooth function $f(x, y)$ such that $\frac{\partial f}{\partial y} > 0$ everywhere.

Definition 1.2. A semitoric system is simple if each fiber of $J$ (and hence of $F$) contains at most one focus-focus point.

So if $F$ is simple then the focus-focus fibers of $F$ are homeomorphic to once-pinched tori as in $F^{-1}(e_1)$ in Figure 4.

About ten years ago simple semitoric systems were classified up to isomorphisms by Pelayo-Vũ Ngọc in terms of five invariants [PVuN09, PVuN11a]: the number of focus-focus points, the Taylor series invariant of each focus-focus point, the polygon invariant, the height invariant of each focus-focus point, and the twisting-index invariant of each focus-focus point (for surveys see [PVuN11b, AH19]). Simplicity was assumed because at that time only critical fibers containing one focus-focus point were understood, thanks to Vũ Ngọc [VuN03].
1.2. Semitoric systems of non-simple type. Recently Pelayo and Tang [PT19] extended Vũ Ngoc’s result to fibers of $F$ containing any number of focus-focus points as in $F^{-1}(c_2)$ and $F^{-1}(c_3)$ in Figure 4. Of the five invariants the first four are relatively straightforward to apply in the non-simple case (replacing the Taylor series [VuN03] by a collection of Taylor series [PT19]), but there is one, the twisting-index invariant, whose construction does not immediately extend.

In the case of simple systems the twisting-index invariant assigns an integer to each focus-focus point. We will explain why in the non-simple case it must assign data equivalent to an integer to each entire fiber containing any number of focus-focus points. The construction of this more general twisting-index is mixed with the construction of the Taylor series invariants and because of this we package all invariants together into a single one: the complete semitoric invariant. The main result of the paper (Theorem 4.10) says that the complete semitoric invariant classifies semitoric systems, simple or not, up to isomorphisms.

While in some sense the inclusion of non-simple systems only represents a small extension of the Pelayo–Vũ Ngoc classification, non-simple semitoric systems form an important class of integrable systems for several reasons. First of all, they are the most natural systems to include in the extension of this classification. Perhaps more importantly, this class of systems may be a source of counterexamples to two related general principles in integrable systems. It is a motivating question in integrable systems to ask under what conditions can an integrable system be recovered from the affine structure on the base of the associated Lagrangian fibration (see for instance [BMMT18, Question 1.1 and Problem 1.2]). Also, it is often of interest to attempt to recover a classical integrable system from the joint spectra of the associated quantum integrable system (see Remark 4.17 and the references therein). The following question is open: is it possible for two non-isomorphic non-simple semitoric systems which have a multiply-pinched focus-focus fiber to have the same affine manifold as their base or have the same joint spectra of the associated quantum integrable system?

In order to construct the invariant and prove the classification result, we follow an adapted version of the methods of the original classification of simple semitoric systems [PVuN09, PVuN11a], making this paper a concise overview of techniques and strategies used to construct invariants and classify integrable systems in this case. The paper builds on a number of remarkable results or ideas by other authors, including Arnold, Atiyah, Dufour, Duistermaat, Eliasson, Guillemin, Miranda, Molino, Pelayo, Sternberg, Tang, Toulet, Vũ Ngoc, and Zung, to whom this paper owes much credit and to whose works we will refer later in the paper.

2. Preliminaries

2.1. Polygons via cutting at focus-focus points. Let $(M, \omega, F = (J, H))$ be a semitoric system and let $B = F(M)$.

Definition 2.1. Let $M_f \subset M$ be the set of focus-focus points of $F$, that is, the critical points of $F$ with local model given by expression (1.1). A focus-focus value is an element of $F(M_f)$.

By [VuN07, Corollary 5.10] $M_f$ is finite. Let $m_i, v_i, \lambda_i \in \mathbb{Z}_{\geq 0}$ respectively be the cardinalities of $M_f$, $F(M_f)$, and $J(M_f)$. Let $B_f \subset B$ denote the set of regular values of $F$.

Arrange the focus-focus values lexicographically by $(x, y) < (z, t)$ if and only if $x < z$ or both $x = z$ and $y < t$ and label them $\{c_1, \ldots, c_{v_f}\} = F(M_f)$. For $i \in \{1, \ldots, m_f\}$ let $m_i$ be the number of focus-focus points in $F^{-1}(c_i)$, so $\sum_{i=1}^{v_f} m_i = m_f$, see Figure 4.

Definition 2.2. For $b \in B$ each $\beta \in T_b^*\mathbb{R}^2$ determines a vector field $\mathcal{X}_\beta$ on $F^{-1}(b) \subset M$ via the equation $F^* \beta = -\omega(\mathcal{X}_\beta, \cdot)$, and we let $\beta \in T_b^*\mathbb{R}^2$ act on $F^{-1}(b)$ by flowing along $\mathcal{X}_\beta$ for time $2\pi$. For $b \in B$ let $2\pi \Lambda_b \subset T_b^*\mathbb{R}^2$ be the isotropy subgroup of the action and let $\Lambda = \bigcup_{b \in B} \Lambda_b$, then $(B, \Lambda)$ is an integral affine manifold with corners and nodes, as in [Dui80]. We call $(B, \Lambda)$ the base of $(M, \omega, F)$.

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Let $\Lambda_{\text{can}}$ be the usual integral affine structure on $\mathbb{R}^2$. The focus-focus values create monodromy in the integral affine structure of $(B, \Lambda)$ and obstruct any global affine map $(B, \Lambda) \to (\mathbb{R}^2, \Lambda_{\text{can}})$, but we can define a map which is affine when restricted to each vertical region between the focus-focus values as follows.

**Proposition 2.3.** Let $(M, \omega, F = (J, H))$ be a semitoric system. Arrange the elements of the set $J(M_t) = \{j_1, \ldots, j_{\lambda_f}\}$ in strictly increasing order, that is, $j_1 < \ldots < j_{\lambda_f}$. Define $j_0 = -\infty$ and $j_{\lambda_f + 1} = +\infty$. For each $a \in \{0, \ldots, \lambda_f\}$ let $I_a = \{(x, y) \in \mathbb{R}^2 \mid j_a < x < j_{a+1}\}$ and let $B_a = B_t \cap I_a$. Then there exists an injective, orientation preserving, continuous function $A: B \to \mathbb{R}^2$ with the following properties:

1. $A$ preserves the $x$-coordinate;
2. for every $a \in \{0, \ldots, \lambda_f\}$ the restriction $A|_{B_a}$ of $A$ to $B_a$ is smooth;
3. $(A|_{B_a})^*\Lambda_{\text{can}} = \Lambda$, where $\Lambda_{\text{can}}$ is the usual affine structure of $\mathbb{R}^2$.

**Proof.** For each $a \in \{0, \ldots, \lambda_f\}$, since $B_a$ is a simply connected subset of $B_t$ and the function $J$ is $2\pi$-periodic there exists an orientation preserving map $A_a: B_a \to \mathbb{R}^2$ which preserves the first coordinate and for which $A_a^*\Lambda_{\text{can}} = \Lambda$, unique up to vertical translation and composition with powers of

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$ 

Similar to [VuN07, Theorem 3.8], translating if needed the maps $A_a$, $a \in \{0, \ldots, \lambda_f\}$, can be combined to a single map $A: B \to \mathbb{R}^2$ which is the desired map. \qed

**Definition 2.4.** An injective, orientation preserving, continuous function $A: B \to \mathbb{R}^2$ is a choice of piecewise affine coordinates if it satisfies properties (1)–(3) from Proposition 2.3.

Also note if $A$ is a choice of piecewise affine coordinates then it preserves the lexicographic order. Let $\text{pr}_i: \mathbb{R}^2 \to \mathbb{R}, i \in \{1, 2\}$ be the projection onto the $i^{\text{th}}$ component and for $a \in \mathbb{R}$ let $\ell_a = \text{pr}_1^{-1}(j)$. 

**Definition 2.5.** For a finite set $j \subset \mathbb{R}$ let $G_j$ denote the vertical piecewise integral affine group, that is, the group of homeomorphisms $\rho: \mathbb{R}^2 \to \mathbb{R}^2$ which preserve the first component and for which $(\rho|_{\mathbb{R}^2 \setminus \bigcup_{j \in j} \ell_j})^*\Lambda_{\text{can}} = \Lambda_{\text{can}}$.

The following two results are now immediate.

**Proposition 2.6.** For $x \in \mathbb{R}$ denote by $1_x$ the function which is 1 when $x \geq 0$ and 0 otherwise, and for every $j, b \in \mathbb{R}$ define the homeomorphisms $t_j$ and $\eta_b$ of $\mathbb{R}^2$ by

$$t_j(x, y) = (x, y + (x - j))1_{(x - j)}, \quad \eta_b(x, y) = (x, y + b).$$

Then, $G_j$ is the Abelian group generated by $T$, $\{t_j\}_{j \in j}$, and $\{\eta_b\}_{b \in \mathbb{R}}$, and is thus canonically isomorphic to $\mathbb{Z}^{\lambda_f} \times \mathbb{R}$, where $\lambda_f$ is the cardinality of $j$.

**Lemma 2.7.** Let $(M, \omega, F)$ be a semitoric system with base $(B, \Lambda)$ and let $j = (j_1, \ldots, j_{\lambda_f})$ be the $\lambda_f$-tuple of images of the focus-focus points of $F$ under $J$. Then there is a choice of piecewise affine coordinates on $(B, \Lambda)$ which is unique up to left composition by an element of $G_j$.

Let $A$ be a choice of piecewise affine coordinates and, similarly to [VuN07, PVuN09], let $\Delta = A(B) \subset \text{Polyg}(\mathbb{R}^2)$, as in Figure 2, where Polyg$(\mathbb{R}^2)$ is the set of closed polygons in $\mathbb{R}^2$. For $i \in \{1, \ldots, v_f\}$ let $c_i = A(c_i)$.

For $a \in \{1, \ldots, \lambda_f\}$ let $s_a \in \mathbb{Z}_{>0}$ be the number of focus-focus values in the line $\ell_{ja}$ and let $r_a \in \mathbb{Z}_{>0}$ be the lowest index such that $c_{ra} \in \ell_{ja}$. Then $\{c_{ra}, \ldots, c_{ra + s_a - 1}\} = (A \circ F(M_t)) \cap \ell_{ja}$, and
write \( \tilde{c}_i = (\tilde{c}_i^1, \tilde{c}_i^2) \) for the components of \( \tilde{c}_i \). The vertical line \( \ell_{ja} \) is separated into \((s_a + 1)\) segments by the images under \( A \) of the focus-focus values,

\[
\ell_{ja}^\alpha = \{(x, y) \in \ell_{ja} \mid \tilde{c}_{ra+a-1}^2 < y < \tilde{c}_{ra+a}^2 \},
\]

for \( \alpha \in \{0, \ldots, s_a\} \), above taking \( \tilde{c}_{ra-1}^2 = -\infty \) and \( \tilde{c}_{ra+s_a}^2 = \infty \), so that \( \ell_{ja} = \bigcup_{\alpha=0}^{s_a} \ell_{ja}^\alpha \), see Figure 1.

Similarly to [VuN07, Theorem 3.8], we have the following.

**Proposition 2.8.** For each \( a \in \{1, \ldots, \lambda_f\} \) and \( \alpha \in \{0, \ldots, s_a\} \) there exists a \( \omega_a^{\alpha} \in \mathbb{Z} \) such that

\[
(2.3) \lim_{x \to j_a} dA(x, y) = T^{\omega_a^{\alpha}} \circ \lim_{x \to j_a} dA(x, y),
\]

for any \( y \) such that \( A(j_a, y) \in \ell_{ja}^\alpha \).

**Figure 1.** A neighborhood in \( \Delta \) of a single line \( \ell_{ja} \) which includes three marked points, so \( s_a = 3 \). The line \( \ell_{ja} \) is separated into four parts \( \ell_{ja}^0, \ell_{ja}^1, \ell_{ja}^2, \) and \( \ell_{ja}^3 \) by the marked points, and each is labeled to the left by the wall crossing index. Note that the multiplicity labels on the marked points determine the difference in wall-crossing index above and below them.

**Definition 2.9.** We call \( \omega_a^{\alpha} \) the wall-crossing index of the line segment \( \ell_{ja}^\alpha \) and we call \( \omega_a^{0}, \ldots, \omega_a^{s_a} \) the wall-crossing indices of the line \( \ell_{ja} \). Furthermore, we call \( \omega_a = \omega_a^{0} \in \mathbb{Z} \) the lower wall-crossing index associated to the line \( \ell_{ja} \) since it describes the wall-crossing along the lowest segment of \( \ell_{ja} \).

Due to the monodromy effect of focus-focus values on the affine structure of \( B_{\ell_{ja}} \), these integers are subject to \( \omega_a^{\alpha+1} - \omega_a^{\alpha} = m_{ra+\alpha} \). Hence the lower wall-crossing index \( \omega_a \) and the multiplicities of the focus-focus fibers (i.e. the number of focus-focus points in the fiber) determine the tuple \((\omega_a^{0}, \ldots, \omega_a^{s_a}) \in \mathbb{Z}^{s_a+1} \). This is illustrated in Figure 1. Let \( \text{Vert}(\mathbb{R}^2) = \{\ell_j \mid j \in \mathbb{R}\} \).

**Conclusion:** For each choice of piecewise affine coordinates \( A \) as in Definition 2.4 we have obtained:

- the polygon \( \Delta \in \text{Polyg}(\mathbb{R}^2) \) endowed with the lines \( \ell_{j_1}, \ldots, \ell_{j_{\lambda_f}} \in \text{Vert}(\mathbb{R}^2) \), each labeled with a lower wall-crossing index \( \omega_1, \ldots, \omega_{\lambda_f} \in \mathbb{Z} \) as in Definition 2.9, and

- the points \( \tilde{c}_1, \ldots, \tilde{c}_{\nu_f} \in \text{int}(\Delta) \cap \bigcup_{\alpha=1}^{\lambda_f} \ell_{ja} \) which are the images under \( A \) of \( \{c_1, \ldots, c_{\nu_f}\} = F(M_f) \), each labeled with a multiplicity \( m_1, \ldots, m_{\nu_f} \in \mathbb{Z}_{>0} \) which is the number of focus-focus points in the corresponding fiber.

See Figures 2 and 4.
**Remark 2.10.** The polygon $\Delta$ generalizes the notion of weighted polygon of complexity $\lambda_t$ in [PVuN09, Definition 4.4], which applied to simple systems.

**Remark 2.11.** The idea of cutting the base used above was applied in [Sym03] to almost toric systems, and in [VuN07, PVuN09, PVuN11a, PRVuN17, LFP19b] to semitoric systems.

2.2. **Taylor expansions at focus-focus points.** Consider a focus-focus value $c_i \in F(M_t)$ and let $(p_i^\mu)_{\mu \in \mathbb{Z}_{m_i}} \subset F^{-1}(c_i)$ be the tuple of focus-focus points in the fiber over $c_i$ for each $i \in \{1, \ldots, v_f\}$. Choose the points to be in order according to the direction of the flow of $H$, so the choice of numbering is unique up to cyclic permutation, which is why we take the index $\mu$ to be in $\mathbb{Z}_{m_i}$ as in [PT19].

By Eliasson’s linearization theorem for non-degenerate focus-focus points [Eli84, VNW13], near $p_i^\mu$ and $c_i$ there is an orientation preserving symplectomorphism

$$\varphi^{\mu}_i : (M, \omega, p_i^\mu) \to (\mathbb{R}^4, \omega_0, 0)$$

and an orientation preserving diffeomorphism $E^{\mu}_i : (B, c_i) \to (\mathbb{R}^2, 0)$ such that $q \circ \varphi^{\mu}_i = E^{\mu}_i \circ F$, where $q$ is the local model of a focus-focus point in $\mathbb{R}^3$, as in Equation (1.1). Furthermore, since $J$ is periodic we may assume that $E^{\mu}_i$ only shifts the first component, i.e. that

$$(2.4) \quad \text{pr}_1 \circ E^{\mu}_i(x, y) = x - j_a.$$

Let $A = (A^1, A^2)$ be a choice of piecewise affine coordinates as in Definition 2.4. In order to find invariants of $(M, \omega, F)$ that are well-defined up to isomorphisms, we compare the coordinates $A$ and $E^\mu_i$ in $U_i \setminus \ell_{j_a}$, where $U_i \subset B$ is a neighborhood of $c_i$ and $c_i \in \ell_{j_a}$. We may assume that $U_i \setminus \ell_{j_a}$ has two connected components, corresponding to $x > j_a$ and $x < j_a$, and that $c_i$ is the only focus-focus value in $U_i$.

Let $\log_c : \mathbb{C} \setminus i\mathbb{R}^+ \to \mathbb{C}$ be the determination of log with $\log_+ 1 = 0$ and branch cut at $i\mathbb{R}^+$. Then, for $c \in \mathbb{C} \setminus i\mathbb{R}$, let $K_+ : \mathbb{C} \setminus i\mathbb{R}^+ \to \mathbb{R}$ be given by

$$K_+(c) = -\Im(c \log_+ c - c).$$

Identifying $\mathbb{R}^2 \cong \mathbb{C}$ so $E^\mu_i : B \to \mathbb{C}$, let $\tilde{S}^i : U_i \setminus \ell_{j_a} \to \mathbb{R}$ be given by

$$(2.5) \quad \tilde{S}^i = 2\pi A^2 - \sum_{\mu \in \mathbb{Z}_{m_i}} (E^{\mu}_i)^* K_+ + 2\pi w^\alpha_a(j_a - x) \mathbbm{1}(j_a - x).$$

In the set $x > j_a$ this function takes the difference of the piecewise affine coordinates and the sum of the pull-backs of the function $K_+$. When $x \leq j_a$ the third term accounts for how the piecewise affine coordinates change passing through $\ell_{j_a}$.

**Lemma 2.12.** Suppose that $A, A'$ are two choices of piecewise affine coordinates as in Definition 2.4, so there exists some $\rho \in G_j$ such that $A' = \rho \circ A$, where $G_j$ is as in Definition 2.5. Let $w^\alpha_a$ denote the wall crossing indices relative to $A$ as in Definition 2.9, let $\tilde{S}^i$ be the function given in Equation (2.5), and let $(w^\alpha_a)'$ and $(\tilde{S}^i)'$ be those relative to $A'$. Then the values of $A'$, $(w^\alpha_a)'$, and $(\tilde{S}^i)'$ in terms of $A$, $w^\alpha_a$, and $\tilde{S}^i$ are as follows when $\rho$ is a generator of $G_j$:

| $\rho \in G_j$ | $A' = \rho \circ A$ | $(w^\alpha_a)'$ | $(\tilde{S}^i)'$ |
|----------------|---------------------|----------------|----------------|
| $T$            | $(A^1, A^2 + x)$    | $w^\alpha_a$   | $\tilde{S}^i + 2\pi x$ |
| $\tau_j, j \neq j_a$ | $(A^1, A^2 + (x - j) \mathbbm{1}(x - j))$ | $w^\alpha_a$ | $\tilde{S}^i + 2\pi(x - j) \mathbbm{1}(x - j)$ |
| $\tau_{j_a}$   | $(A^1, A^2 + (x - j_a) \mathbbm{1}(x - j_a))$ | $w^\alpha_a - 1$ | $\tilde{S}^i + 2\pi(x - j_a)$ |
| $\eta_0$       | $(A^1, A^2 + b)$    | $w^\alpha_a$   | $\tilde{S}^i + 2\pi b$ |
Remark 2.16 which satisfies the system of equations (2.8). where \( s \) and \( g \) completely determine the entire tuple (2.9) obtained from the invariant in Definition 2.14 via construction used above is analogous to the one in [PT19, pages 16–18], (semi-global neighborhood of the fiber (some authors also use the term \( F \) for this). Because the action of \( G_j \) described in Lemma 2.12 does not affect the smoothness of \( \tilde{S}^i \) around \( c_i \) and there is always a choice of \( \mathcal{A} \) such that \( w_\alpha^0 = 0 \), the proof is complete.

Lemma 2.13. \( \tilde{S}^i \) can be extended to a smooth function in a neighborhood of \( c_i \) in \( B \).

Proof. The lemma is true when \( w_\alpha^0 = 0 \) according to the proof of [PT19, Lemma 6.3]. Since, the action of \( G_j \) described in Lemma 2.12 does not affect the smoothness of \( \tilde{S}^i \) around \( c_i \) and there is always a choice of \( \mathcal{A} \) such that \( w_\alpha^0 = 0 \), the proof is complete.

Definition 2.14. We still use \( \tilde{S}^i \) to denote the smooth extension to a neighborhood of \( c_i \). Let \( X = dx, Y = dy \). Performing a Taylor expansion of \( \tilde{S}^i \) around the origin under coordinates \( E^i_\mu \), we get a power series

\[
\tilde{s}^i_\mu = \text{Taylor}_0 \left[ \left( (E^i_\mu)^{-1} \right)^* \tilde{S}^i \right] = \sum_{p,q=0}^{\infty} (\tilde{s}^i_\mu)^{(p,q)} X^p Y^q,
\]

the **action Taylor series** at \( p^i_\mu \), where \( \mu \in \mathbb{Z}_{m_i} \). Expanding the transition maps between coordinates \( E^i_\mu \) and \( E^i_\nu \), we get

\[
\bar{g}^i_{\mu,\nu} = \text{Taylor}_0 [ \text{pr}_2 \circ E^i_\nu \circ (E^i_\mu)^{-1} ] = \sum_{p,q=0}^{\infty} (\bar{g}^i_{\mu,\nu})^{(p,q)} X^p Y^q,
\]

the **transition Taylor series** from \( p^i_\mu \) to \( p^i_\nu \), where \( \mu, \nu \in \mathbb{Z}_{m_i} \).

Similar to [PT19, Theorem 6.4], we have the following.

**Lemma 2.15.** The series in Equations (2.6) and (2.7) are constrained by the following relations:

\[
\begin{align*}
(\bar{g}^i_{\mu,\nu})^{(0,1)} &> 0, \\
\bar{s}^i_\mu(X,Y) &= \tilde{s}^i_\mu(X, \bar{g}^i_{\mu,\nu}(X,Y)), \\
\bar{g}^i_{\mu,\nu}(X,Y) &= Y, \\
\bar{g}^i_{\mu,\sigma}(X,Y) &= \tilde{g}^i_{\mu,\nu}(X, \bar{g}^i_{\nu,\sigma}(X,Y)),
\end{align*}
\]

for \( \mu, \nu, \sigma \in \mathbb{Z}_{m_i} \).

Let \( \mathbb{R}[[X,Y]] \) denote the set of Taylor series in two variables with real coefficients and \( \mathbb{R}_0[[X,Y]] \) be the subset of those which have zero constant term. In [PT19, Theorem 1.1] it was shown that the semi-local model \( \mathcal{F}^{-1}(c_i) \) is determined up to semi-local isomorphisms by a set \( (\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu})_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \), where \( \bar{s}^i_\mu \subseteq \mathbb{R}_0[[X,Y]]/\left(2\pi XZ\right) \) and \( \bar{g}^i_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \), and conversely that the semi-local model determines \( (\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu})_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \) up to cyclic reordering of the indices. By **semi-local** we mean in a neighborhood of the fiber (some authors also use the term **semi-global** for this). Because the construction used above is analogous to the one in [PT19, pages 16–18], \( (\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu})_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \) can be obtained from the invariant in Definition 2.14 via

\[
(\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu}) = \left( \tilde{s}^i_\mu - (\bar{s}^i_\mu)^{(0,0)} + 2\pi XZ, \bar{g}^i_{\mu,\nu} \right)
\]

for each \( \mu, \nu \in \mathbb{Z}_{m_i} \).

**Remark 2.16.** Because of the relations in (2.8), the elements of the tuple \( (\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu})_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \) are not independent. In particular, similarly to [PT19, Corollary 1.2], \( \bar{s}^i_0 \) and \( (\bar{g}^i_{\mu,\mu+1})_{\mu} \subseteq \mathbb{R}_0[[X,Y]] \) completely determine the entire tuple \( (\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu})_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \). That is, given any choice of \( \bar{s}^i_0 \in \mathbb{R}[[X,Y]] \) and \( \bar{g}^i_0, \ldots, \bar{g}^i_{m_i-2, m_i-1} \in \mathbb{R}_0[[X,Y]] \) there is exactly one possible choice of tuple \( (\bar{s}^i_\mu, \bar{g}^i_{\mu,\nu})_{\mu,\nu} \subseteq \mathbb{R}_0[[X,Y]] \) which satisfies the system of equations (2.8).
Furthermore, changing the choice of piecewise affine coordinates $A$ preserves $g_{\mu,\nu}^i$. Define an action of $G_j$ on $\tilde{s}_i^j$ by specifying the action of the generators as

$$\rho(\tilde{s}_i^j) = \begin{cases} 
\tilde{s}_i^j + 2\pi X + 2\pi j_a, & \text{if } \rho = T, \\
\tilde{s}_i^j + 2\pi X + 2\pi (j_{a'} - j_a), & \text{if } \rho = t_{j_a'} \text{ and } a' \leq a, \\
\tilde{s}_i^j + 2\pi b, & \text{if } \rho = \eta_b,
\end{cases} \tag{2.10}$$

where $c_i \in \ell_{j_a}$.

**Lemma 2.17.** Let $\tilde{s}_i^j$ be the action Taylor series at the focus-focus point $p_{\mu}^i$ as in Definition 2.14 relative to a choice of piecewise affine coordinates $A$ and let $(\tilde{s}_i^j)'$ be the action Taylor series of $p_{\mu}^i$ relative to a choice of piecewise affine coordinates $A' = \rho \circ A$ where $\rho \in G_j$. Then $(\tilde{s}_i^j)' = \rho(\tilde{s}_i^j)$.

**Proof.** Lemma 2.12 explains how changing piecewise affine coordinates changes the function $\tilde{S}^i$ (as in Equation (2.5)). Combining this with Equations (2.4) and (2.6), which show how to obtain $\tilde{s}_i^j$ from $\tilde{S}^i$, describes how changing affine coordinates affects $\tilde{s}_i^j$, which is exactly the same as the definition of $\rho(\tilde{s}_i^j)$ from Equation (2.10). \hfill \□

The tuple of Taylor series invariants we have constructed depends on the choice of ordering for the focus-focus points in the given focus-focus fiber, which is only unique up to cyclic permutation. Let $[\tilde{s}_i^j, g_{\mu,\nu}^i]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ denote the orbit of $(\tilde{s}_i^j, g_{\mu,\nu}^i)_{\mu,\nu \in \mathbb{Z}_{m_i}}$ under the action of $\mathbb{Z}_{m_i}$ by $z \cdot (\tilde{s}_i^j, g_{\mu,\nu}^i)_{\mu,\nu \in \mathbb{Z}_{m_i}} = (\tilde{s}_i^j + z, g_{\mu+2\pi z,\nu}^i)_{\mu,\nu \in \mathbb{Z}_{m_i}}$ for $z \in \mathbb{Z}_{m_i}$, where the addition in the indices is modulo $m_i$. Also note that one element of the orbit $[\tilde{s}_i^j, g_{\mu,\nu}^i]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ satisfies the system of equations (2.8) if and only if all elements of the orbit satisfy those equations.

**Conclusion:** We have assigned to each critical value $c_i$ a tuple of Taylor series $[\tilde{s}_i^j, g_{\mu,\nu}^i]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ with $\tilde{s}_i^j \in \mathbb{R}[[X,Y]]$ and $g_{\mu,\nu}^i \in \mathbb{R}_0[[X,Y]]$ for each $i \in \{1, \ldots, v_f\}$ and $\mu, \nu \in \mathbb{Z}_{m_i}$. Moreover, the $g_{\mu,\nu}^i$ are independent of the choice of piecewise affine coordinates and when changing the choice of piecewise affine coordinates the $\tilde{s}_i^j$ change according to Equation (2.10) and Lemma 2.17.

**Remark 2.18.** In light of Equations (2.9)–(2.10) and Lemma 2.17, note that the choice of $A$ does not affect the part of $\tilde{s}_i^j$ which represents the series from [VuN03, PT19], as expected.

**Remark 2.19.** The twisting-index invariant (the fifth invariant in the Pelayo–Vũ Ngọc classification [PVuN09, PVuN11a]) does not appear as an independent piece of the complete semitoric invariant, since the data of the twisting-index invariant is now encoded in the $X$ coefficient of the action Taylor series $\tilde{s}_i^j$. For a discussion of the relationship between the twisting-index invariant and the complete semitoric invariant see Section 4.4.

### 3. The Complete Semitoric Invariant

In the previous section we constructed a 5-tuple

$$\tilde{i}(M, \omega, F) = \left( \Delta, (\ell_{j_a})_{a=1}^{v_f}, (w_a)_{a=1}^{v_f}, (\tilde{c}_i)_{i=1}^{v_f}, (m_i, [\tilde{s}_i^j, g_{\mu,\nu}^i]_{\mu,\nu \in \mathbb{Z}_{m_i}})_{i=1}^{v_f} \right) \tag{3.1}$$

starting from the system $(M, \omega, F)$, which depends on the choice of piecewise affine coordinates $A$ as in Definition 2.4, so it is not yet a symplectic invariant of $(M, \omega, F)$.

In order to define the complete semitoric invariant, of simple or non-simple systems, we start with the following definition, motivated from Section 2.
Definition 3.1. Let
\[ \mathcal{X} = \left\{ (m, [s_\mu, g_{\mu, \nu}]_{\mu, \nu} \in \mathbb{Z}_m) \mid m \in \mathbb{Z}_{>0} \text{ and } s_\mu \in \mathbb{R}[[X, Y]], g_{\mu, \nu} \in \mathbb{R}[[X, Y]]_0 \right\} \]
and for \( \lambda, \nu \in \mathbb{Z}_{\geq 0} \) let
\[ X_{\lambda, \nu} = \text{Polyg}(\mathbb{R}^2)^{\lambda_\nu} \times (\text{Vert}(\mathbb{R}^2))^{\lambda_\nu} \times (\mathbb{R}^2)^{\nu_\iota} \times (\mathcal{X})^v. \]
Let \( z = (z_0, \ldots, z_{\lambda_\nu}) \in \mathbb{Z}^{\lambda_\nu + 1} \) and \( b \in \mathbb{R} \). Let \( T, t_j, \) and \( \eta_b \) be as given in Equations (2.1) and (2.2) and let these operators act on \( s_i^{b_j} \) as in Definition (2.10). By \( t_j^{b_j} \) we simply mean the composition of \( t_j \) with itself \( z_a \) times. Define the action \( (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \times X_{\lambda, \nu_\iota} \rightarrow X_{\lambda, \nu_\iota} \) of \( \mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R} \) on \( X_{\lambda, \nu_\iota} \) by (3.3)
\[ (z, b) \cdot \left( \begin{array}{c} \Delta, \\
(\ell_{j})_{a=1}^{\lambda_\nu}, \\
(w_{a})_{a=1}^{\lambda_\nu}, \\
(c_{i_{\nu_\iota}})_{i_{\nu_\iota}=1}^{\nu_\iota}, \\
(m_{i}, [s_{\mu}, g_{\mu, \nu}]_{\mu, \nu} \in \mathbb{Z}_m)_{i_{\nu_\iota}=1}^{\nu_\iota} \end{array} \right) = \left( \begin{array}{c} \eta_b \circ t_{j_1}^{z_1} \circ \cdots \circ t_{j_{\lambda_\nu}}^{z_{\lambda_\nu}} \circ T^{z_0}(\Delta), \\
(\ell_{j})_{a=1}^{\lambda_\nu}, \\
(w_{a} - z_a)^{\lambda_\nu}_{a=1}, \\
(m_{i}, [s_{\mu}, g_{\mu, \nu}]_{\mu, \nu} \in \mathbb{Z}_m)_{i_{\nu_\iota}=1}^{\nu_\iota} \end{array} \right). \]

It is straightforward to check that Equation (3.3) actually defines a group action. The construction of \( \tilde{\mathcal{I}}(M, \omega, F) \) in Equation (3.1) is unique up to the choice of piecewise affine coordinates \( \mathcal{A} \) as in Definition 2.4, which is unique up to the action of \( G_{\mathcal{A}} \) as in Definition 2.5, which is isomorphic to \( \mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R} \). Since we have taken the quotient by precisely this symmetry (see Lemmas 2.12 and 2.17) we have:

**Proposition 3.2.** The assignment of \( (M, \omega, F) \mapsto (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \cdot \tilde{\mathcal{I}}(M, \omega, F) \) is a well-defined function which has as its domain the set of all semitoric systems \( \mathcal{M} \) and as its codomain the quotient space \( \prod_{\lambda, \nu \in \mathbb{Z}_{>0}} \left( X_{\lambda, \nu_\iota} / (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \right). \)

**Definition 3.3.** The complete semitoric invariant of \( (M, \omega, F) \) is the \( (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \)-orbit of \( \tilde{\mathcal{I}}(M, \omega, F) \) from Equation (3.1), see Figure 2.

**Remark 3.4.** Definition 3.3 generalizes to non-simple semitoric systems the Pelayo–Vũ Ngọc invariants as given by [PVuN09, Definition 6.1]. We discuss this in Remark 4.12.

The following is an immediate consequence of the definitions.

**Lemma 3.5.** Let \( (M_1, \omega_1, F_1) \) and \( (M_2, \omega_2, F_2) \) be semitoric systems. If \( (M_1, \omega_1, F_1) \) and \( (M_2, \omega_2, F_2) \) are isomorphic then they have the same complete semitoric invariant.

**Remark 3.6.** Lemma 3.5 is equivalent to the fact that the function \( (M, \omega, F) \mapsto (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \cdot \tilde{\mathcal{I}}(M, \omega, F) \) induces a well-defined function \( i : \mathcal{M} / \sim \rightarrow \prod_{\lambda, \nu \in \mathbb{Z}_{>0}} \left( X_{\lambda, \nu_\iota} / (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \right) \)
\[ [(M, \omega, F)] \mapsto (\mathbb{Z}^{\lambda_\nu + 1} \times \mathbb{R}) \cdot \tilde{\mathcal{I}}(M, \omega, F), \]
where \( [(M, \omega, F)] \) denotes the isomorphism class of \( (M, \omega, F) \).

**4. Classification**

In this section we explain how to remove the simplicity assumption in the classification of semitoric systems in [PVuN09, PVuN11a], leading us to a classification which applies to both the simple and the non-simple cases, formulated in terms of the complete semitoric invariant of Definition 3.3.
Figure 2. A representative of the complete semitoric invariant of Definition 3.3 with $\lambda_f = 3$ and $v_f = 4$. Each marked point $\tilde{c}_i$ is indicated with an $\times$ and each vertical line $\ell_{j_a}$ is indicated with a dashed line. The integral lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ is also shown. In this example the vertices of the polygon are all on lattice points, but this is not true in general. The marked points are each labeled with their multiplicity $m_i$ and each segment of each vertical line is marked with its wall-crossing index to the left. The lower wall crossing indices for this example are $w_1 = -2$, $w_2 = 1$, and $w_3 = -4$. Not shown is the Taylor series label $[\tilde{s}_i^{\mu}, g_i^{\mu,\nu}]_{\mu,\nu \in \mathbb{Z}m_i}$ on each marked point for $i \in \{1, 2, 3, 4\}$.

4.1. Uniqueness.

**Proposition 4.1.** Let $(M_1, \omega_1, F_1)$ and $(M_2, \omega_2, F_2)$ be semitoric systems. Then $(M_1, \omega_1, F_1)$ and $(M_2, \omega_2, F_2)$ are isomorphic if and only if they have the same complete semitoric invariant as in Definition 3.3.

**Proof.** The implication from left to right is Lemma 3.5. For the implication from right to left we follow [PVuN09, pages 588–596] and only prove statements which are not analogous to those statements therein. The proof in [PVuN09] is split into three steps, we start with Step 1: first reduction. Assume that $(M_1, \omega_1, F_1)$ and $(M_2, \omega_2, F_2)$ are semitoric systems which have the same complete semitoric invariant:

\[
\left(\mathbb{Z}^{\lambda_f+1} \times \mathbb{R}\right) \cdot \left(\Delta, (\ell_{j_a})_{a=1}^{\lambda_f}, (w_a)_{a=1}^{\lambda_f}, (\tilde{c}_i)_{i=1}^{v_f}, \left( m_i, [\tilde{s}_i^{\mu}, g_i^{\mu,\nu}]_{\mu,\nu \in \mathbb{Z}m_i} \right)_{i=1}^{v_f} \right) \in X_{\lambda_f, v_f}/(\mathbb{Z}^{\lambda_f+1} \times \mathbb{R}).
\]

Recall that different representatives of the complete semitoric invariant correspond to different choices of piecewise affine coordinates as in Definition 2.4. First, we choose the same representative of the complete semitoric invariant for each system, so in particular as in the conclusion of Section 2.1 they have the same polygons $\Delta$ and the same ordered tuple of lower wall-crossing indices (as in Definition 2.9). This means that for $i \in \{1, 2\}$ there exist piecewise affine coordinates as in Definition 2.4 for each system which have the same image $\Delta$, we denote them by $A_i = (A_i^1, A_i^2): F_i(M_i) \rightarrow \Delta$ (in [PVuN09] $A_i$ is denoted instead by $g_i^{-1}$, but in this paper we use $g_i^{\mu,\nu}$ in the Taylor series following [PT19]).

Define $h = A_2^{-1} \circ A_1$. We wish to replace $F_2$ by $\tilde{F}_2 = F_2 \circ h$ so that $\text{Image}(\tilde{F}_2) = \text{Image}(F_1)$. In order for $\tilde{F}_2$ to be semitoric and isomorphic to $F_2$ the crucial point is to show that $h(x, y) = (x, f(x, y))$ for some smooth function $f$. By [VuN07, Theorem 3.8] $h$ has this form but a priori $f$ is not smooth.
If the systems are simple an argument in [PVuN09] shows that the fact that $F_1$ and $F_2$ have the same invariants (there are five invariants [PVuN09, Definition 6.1]) implies that $h$ is smooth [PVuN09, Claim 7.1]. The argument is unchanged away from the focus-focus values, so the proof of $h$ being a diffeomorphism can be referred to [PVuN09] except for the smoothness near a focus-focus value $c_i$, which we explain next.

By the fact that the transition Taylor series of Equation (2.7) are the same for the two systems, the local diffeomorphisms $(E_i^1)_1$ and $(E_i^2)_2$ (from Section 2.2) can be chosen to be equal. Considering Equation (2.6), the fact that the action Taylor series are the same for the two systems, and the fact that $(E_i^2)_2 = (E_i^2)_1$ with [PT19, Lemma 4.5], we conclude that $A_{2}^{2}$ and $A_{1}^{2}$ are equal up to a flat function. Finally, by [PT19, Lemma 6.8] and [VuN03, Lemma 5.1] on $\alpha_2 = dA_{2}^{2}$ and $\alpha_2' = dA_{2}^{2}$, we obtain the smoothness of $h$, which completes Step 1 from [PVuN09]. In [PVuN09] the authors also discuss the necessity that the two systems have equal twisting-index, which is not something we need to consider in our case since that information is now encoded in the new Taylor series.

In Step 2 of [PVuN09] it is proven that the semitoric systems $F_1$ and $F_2$ can be intertwined by symplectomorphisms using [VuN03, Theorem 2.1] on the preimages $F_i^{-1}(\Omega_\alpha)$, $i \in \{1, 2\}$, $\alpha \in I$, where the collection of sets $\Omega_\alpha$ is a convenient covering of the common base $F_1(M_1) = F_2(M_2)$. The sets of the covering are defined in such a way that they are of four types: 1) contain no critical points of the $F_i$, 2) contain critical points of rank 1 but not rank 0, 3) contain a critical point of rank 0 of elliptic type; 4) contain a critical point of rank 0 of focus-focus type. In our case the construction of the covering $\{\Omega_\alpha\}_{\alpha \in I}$ is identical to [PVuN09], as well as how to construct the symplectomorphisms $\varphi_\alpha, \alpha \in I$ such that $F_1 = F_2 \circ \varphi_\alpha$ in cases 1), 2) and 3). For case 4) the symplectomorphism $\varphi_\alpha$ can be constructed as follows: instead of using [VuN03, Theorem 2.1], which gives a semi-local normal form for fibers containing exactly one focus-focus point, we use [PT19, Theorem 1.1], which gives a semi-local normal form for fibers which contain any finite number of focus-focus points, with Equation (2.9), which shows how to extract the invariant from [PT19] from the complete semitoric invariant.

The proof in [PVuN09] concludes with Step 3 in which it is proven how to glue symplectically the semi-local symplectomorphisms in order to produce a global symplectomorphism $\varphi: M_1 \to M_2$. This step is unchanged in our case, since the existence of multiple focus-focus points in the same fiber does not play a role in the proof: only the local symplectomorphisms constructed in Step 2 are needed.

Remark 4.2. Proposition 4.1 has two implications. The implication from left to right was discussed in Remark 3.6. The other implication is equivalent to saying that the map $i$ discussed in Remark 3.6 is injective.

4.2. Existence. Recall that a vertex $v$ of a polygon in $\mathbb{R}^2$ is smooth if the polygon is convex in a neighborhood of $v$ and inwards pointing normal vectors to two edges meeting at $v$ can be chosen so that they span the integral lattice $\mathbb{Z}^2$. Also, given a polygon $\Delta$ we call the set $\{(x, y) \in \Delta \mid y \leq y' \text{ for all } y' \text{ such that } (x, y') \in \Delta\}$ the lower boundary of $\Delta$ and we call the set $\{(x, y) \in \Delta \mid y \geq y' \text{ for all } y' \text{ such that } (x, y') \in \Delta\}$ the upper boundary of $\Delta$, see Figure 3.

![Figure 3. The upper and lower boundaries of a non-compact polygon $\Delta$.](image_url)
Definition 4.3. Let $X_{\lambda_f, v_f}$ be as in Equation (3.2). A complete semitoric ingredient is any element $\mathcal{I}$ of the set $\prod_{\lambda_f, v_f \geq 0} (X_{\lambda_f, v_f}/(\mathbb{Z}^{\lambda_f + 1} \times \mathbb{R}))$ such that if $\mathcal{I} \in X_{\lambda_f, v_f}/(\mathbb{Z}^{\lambda_f + 1} \times \mathbb{R})$ is of the form

$$\mathcal{I} = (\mathbb{Z}^{\lambda_f + 1} \times \mathbb{R}) \cdot (\Delta, (\ell_{ja})_{a=1}^M, (w_a)_{a=1}^M, (\tilde{c}_i)_{i=1}^v, (m_i, [g^i_{\mu, \nu}]_{\mu, \nu \in \mathbb{Z}_{m_i}})_{i=1}^v)$$

then the following properties are satisfied:

1. $v_f = \lambda_f = 0$ or $v_f \geq \lambda_f \geq 1$;
2. $\Delta \cap \ell_j$ is compact for all $j \in \mathbb{R}$, where $\ell_j$ is as in Definition 2.5;
3. the entries of the $v_f$-tuple $(\tilde{c}_i)_{i=1}^v$ are distinct, ordered lexicographically, and contained in $\text{int}(\Delta) \cap \left(\bigcup_{a=1}^{\lambda_f} \ell_{ja}\right)$, and moreover $\ell_{ja} \cap \{\tilde{c}_i\}_{i=1}^v \neq \emptyset$ for all $a \in \{1, \ldots, \lambda_f\}$;
4. every vertex of $\Delta$ in $\Delta \setminus \left(\bigcup_{a=1}^{\lambda_f} \ell_{ja}\right)$ is smooth;
5. for every $a \in \{1, \ldots, \lambda_f\}$ if $P \in \partial \Delta \cap \ell_{ja}$ then:
   a. if $P$ is in the lower boundary of $\Delta$ then $t_{ja}^{-w_a}(\Delta)$ either has no vertex at $Q = t_{ja}^{-w_a}(P)$ or the vertex at $Q$ is smooth;
   b. if $P$ is in the upper boundary of $\Delta$ then $t_{ja}^{-(w_a + \tilde{m}_{ja})}(\Delta)$ either has no vertex at $Q = t_{ja}^{-(w_a + \tilde{m}_{ja})}(P)$ or the vertex at $Q$ is smooth, where $\tilde{m}_{ja} = \sum_{i, i \in \ell_{ja}} m_i$;
6. if for every $i \in \{1, \ldots, v_f\}$ we let $\tilde{c}_i = (\tilde{c}_1^i, \tilde{c}_2^i)$, then $(\tilde{g}^i_{\mu})^{(0,0)} = 2\pi \tilde{c}_1^i$ for all $\mu \in \mathbb{Z}_{m_i}$;
7. for every $i \in \{1, \ldots, v_f\}$ the tuple $[g^i_{\mu, \nu}]_{\mu, \nu \in \mathbb{Z}_{m_i}}$ satisfies the conditions in Equation (2.8).

In view of Definition 4.3 we denote by $X$ the set of complete semitoric ingredients, which is a proper subset of $\prod_{\lambda_f, v_f \geq 0} (X_{\lambda_f, v_f}/(\mathbb{Z}^{\lambda_f + 1} \times \mathbb{R}))$.

Remark 4.4. Definition 4.3 generalizes to non-simple semitoric systems the Pelayo–Vũ Ngo semitoric list of ingredients given by [PVuN11a, Definition 4.5].

An example of a complete semitoric ingredient appears in Figure 2.

Proposition 4.5. Let $(M, \omega, F)$ be a semitoric system. Then $(\mathbb{Z}^{\lambda_f + 1} \times \mathbb{R}) \cdot \tilde{i}(M, \omega, F)$, where $\tilde{i}(M, \omega, F)$ is given in Equation (3.1), satisfies conditions (1)-(7) in Definition 4.3.

Proof. Item (1) holds since the cardinality of $F(M_f)$ is $v_f$ and the cardinality of $\text{pr}_1(F(M_f))$ is $\lambda_f$. Let $A = (A^1, A^2)$ be the choice of piecewise affine coordinates as in Definition 2.4 such that $\Delta = A(B)$.

Item (2) holds because $J = A^1 \circ F$ is proper.

Item (3) holds because the $\tilde{c}_i$ are obtained as the images under $A$ of the focus-focus values, which lie in $\text{int}(\Delta)$, and $j_1, \ldots, j_{\lambda_f}$ are defined as the elements of $\text{pr}_1(\{\tilde{c}_1, \ldots, \tilde{c}_{v_f}\}) = \text{pr}_1(\{c_1, \ldots, \tilde{c}_{v_f}\})$.

Item (4) is immediate since $A \circ F: M \to \mathbb{R}^2$ is a toric momentum map away from $F^{-1}(\bigcup_{a=1}^{\lambda_f} \ell_{ja})$.

Similarly, Item (5) follows from the fact that if the wall-crossing index is zero for some segment of $\ell_{ja}$ then the piecewise affine coordinates are smoothly continued across that region of the wall, so again $A \circ F$ is locally a toric momentum map. The polygons $t_{ja}^{-w_a}(\Delta)$ and $t_{ja}^{-(w_a + \tilde{m}_{ja})}(\Delta)$ considered in the two parts of Item (5) are formed by choosing different piecewise affine coordinates for which the wall-crossing index near $t_{ja}^{-w_a}(P)$, respectively $t_{ja}^{-(w_a + \tilde{m}_{ja})}(P)$, is zero.

Item (6) holds because $(\tilde{g}^i_{\mu})^{(0,0)} = \tilde{S}^i(c_i) = 2\pi A^2(c_i) = 2\pi \tilde{c}_1^i$ by Equation (2.5) and the fact that $\tilde{c}_i = A(c_i)$.

Item (7) follows from [PT19, Theorem 6.4].

The following extends [PVuN11a, Theorem 4.6] to the non-simple case.

Proposition 4.6. Given a complete semitoric ingredient $\mathcal{I}$ as in Definition 4.3 there exists a semitoric system $(M, \omega, F)$ such that the complete semitoric invariant of $(M, \omega, F)$ is $\mathcal{I}$. 

\[\square\]
Proof. Given a complete semitoric ingredient $I$ choose a representative such that $w_a = 0$ for $a \in \{1, \ldots, \lambda_I\}$ so that
\[
I = (\mathbb{Z}^\lambda_I + 1 \times \mathbb{R}) \cdot \left( \Delta, (\ell_{ja})_{a=1}^{\lambda_I}, (0)_{a=1}^{\lambda_I}, (\tilde{c}_i)_{i=1}^\nu, \left( m_i, [\tilde{s}_i, \tilde{g}_{\mu, \nu}]_{\mu, \nu \in \mathbb{Z}_{m_i}} \right)_{i=1}^\nu \right).
\]
Note that such a choice of representative always exists because the action of $\mathbb{Z}^\lambda_I + 1 \times \mathbb{R}$ can be used to make the tuple of lower wall-crossing indices take any desired value, as seen in Equation (3.3).

Now we continue as in the proof of [PVuN11a, Theorem 4.6], which proceeds by gluing together the semi-local models of the fibers of $F$, essentially constructing $(M, \omega, F)$ backwards starting from $\Delta$ and using the semi-local models and symplectic gluing to construct $(\hat{M}, \omega$) and a map $\mu: M \to \Delta$ which will represent $F \circ A$ for some choice of piecewise affine coordinates $A$. The proof of [PVuN11a, Theorem 4.6] is split into four stages and we will consider each separately.

In the preliminary stage (a convenient covering) and first stage (away from the cuts) of the proof of [PVuN11a, Theorem 4.6] one constructs a convenient covering \{\{$\Omega_\alpha$\}$_{\alpha \in I}$ of $\Delta$. In [PVuN11a], following [VuN07], the polygon invariant is constructed by choosing rays in the momentum map image known as cuts which go either up or down from each focus-focus point, and then finding a toric momentum map on the manifold with preimages under $F$ of these cuts removed. In the first stage one restricts to the subcovering \{\{$\Omega_\alpha$\}$_{\alpha \in I}$ of sets which do not intersect the cuts, and for each of these constructs a local symplectic model $M_\alpha$ and an integrable system $F_\alpha : M_\alpha \to \Omega_\alpha$. In the language of the present paper, we replace the cuts referred to above by $\ell^{\text{nonzero}}$, where $\ell^{\text{nonzero}}$ the union of the regions of the lines $\ell_{ja}$, $a \in \{1, \ldots, \lambda_I\}$, which have non-zero wall-crossing index, as in Definition 2.9. Since we have chosen a representative for which the all of the lower wall-crossing indices are zero we have that $\Delta \setminus \ell^{\text{nonzero}}$ is connected. After making this choice, the remainder of the first stage continues exactly as in [PVuN11a, pages 113–116]. Using the general symplectic gluing theorem [PVuN11a, Theorem 3.11] these integrable models can be symplectically glued together in order to produce an integrable system $F_I': M_I' \to \bigcup_{\alpha \in I'} \Omega_\alpha$.

In the second stage (attaching focus-focus fibrations) of [PVuN11a, pages 116–118] it is explained how to symplectically glue the semi-local models in a neighborhood of the focus-focus points containing exactly one focus-focus point to the model $F_I': M_I' \to \bigcup_{\alpha \in I'} \Omega_\alpha$ of Step 1, to produce a proper map $F_{I''}: M_{I''} \to \bigcup_{\alpha \in I''} \Omega_\alpha$ on the symplectic manifold $M_{I''}$, which is a smooth toric momentum map away from the pre-images of the cuts. This same construction can be done for non-simple semitoric systems taking into account the following: fix some $i \in \{1, \ldots, v_I\}$ and consider the marked point $\tilde{c}_i = (\tilde{c}_{i}^1, \tilde{c}_{i}^2) \in \Delta$. Using [PT19, Theorem 1.1] construct the semi-local model $(M_i, \omega_i, F_i)$ over a neighborhood of the origin in $\mathbb{R}^2$ using the invariant $(\tilde{s}_i, \tilde{g}_{\mu, \nu})_{\mu, \nu \in \mathbb{Z}_{m_i}}$ obtained from $I$ as in Equation (2.9), well-defined up to cyclic reordering of the indices. Let $B_i = F_i(M_i)$ be the base and let $(p_{\mu i})_{\mu \in \mathbb{Z}_{m_i}}$ be the tuple of focus-focus points of $F_i$, where we assume that $F_i(p_{\mu i}^0) = 0$ for all $\mu \in \mathbb{Z}_{m_i}$. For each $\mu \in \mathbb{Z}_{m_i}$, there is a symplectomorphism $\varphi_{\mu i}^0: (M_i, \omega_i, p_{\mu i}^0) \to (\mathbb{R}^4, \omega_0, 0)$ and a diffeomorphism $E_{\mu i}: (B_i, 0) \to (\mathbb{R}^2, 0)$ such that $\varphi_{\mu i} = E_{\mu i} \circ F_i$. Let $\tilde{S}_i: B_i \to \mathbb{R}$ be a smooth function such that Equation (2.6) holds for one choice of $\mu$, and due to the relations (2.8) it thus holds for all choices of $\mu$. In order to obtain a system with the desired Taylor series invariants define

\[
\begin{align*}
A_i = (A_i^1, A_i^2): B_i \to \mathbb{R}^2
\end{align*}
\]

\[
(4.1)
\]

where $\tilde{c}_i$ is the $\alpha^\text{th}$ marked point on the line $\ell_{\tilde{c}_i}^1$, counting up from the bottom, and notice that $A_i$ is invariant under cyclic reordering of the indices in the Taylor series. Now we use $A_i \circ F_i: M_i \to \Delta$ to place $B_i$ into a neighborhood of $\tilde{c}_i$ in the polygon $\Delta$, and we perform gluing as in [PVuN11a, pages 116–118]. Here note that $A_i \circ F_i$ replaces the map $R_\alpha \circ g_i \circ F_i$ from [PVuN11a, page 117], in which $g_i$ is a smooth diffeomorphism of $\mathbb{R}^2$ analogous to $A_i$ and the map $R_\alpha$ was used to account for the
twisting-index, which in the present proof is already accounted for in the piecewise affine coordinates \( \mathcal{A} \), since the information of the twisting-index is included in the new Taylor series \( \tilde{S}_\mu, \tilde{g}_{\mu, \nu} |_{\mu, \nu \in z_m} \).

From Equation (4.1) it follows that the the Taylor series obtained from the constructed system will be the desired one, since isolating \( \tilde{S}^I \) in Equation (4.1) yields the definition of the desired action Taylor series as in Equation (2.5).

In the third stage (filling in the gaps) of the proof one considers the open sets \( \Omega_\alpha \) in the covering which are above the cuts and and includes them into the previous gluing data using symplectic gluing in order to obtain a symplectic manifold and and a proper map \( \mu: M \rightarrow \bigcup_{\alpha \in I} \Omega_\alpha \) with image \( \Delta \). This map \( \mu \) is a proper toric smooth momentum map only away from the cuts, and in the fourth stage (recovering smoothness) the authors show how to smoothen \( \mu \). In the case of non-simple semitoric systems these final two stages proceed exactly as in [PVuN11a], using different choices of representative for \( I \) in order to make the wall-crossing index of the vertical lines \( \ell_j, a \in \{1, \ldots, \lambda_f\} \), equal to zero around the remaining points to be glued in, which are the points on the lines \( \ell_{ja} \) which do not already have zero wall-crossing index in the representative of \( I \) we started with.

\[ I = (Z^{\lambda_f + 1} \times \mathbb{R}) \cdot \left( \frac{\langle \Delta, (\ell_{ja})_{a=1}^{\lambda_f} \rangle_{a=1}^{\lambda_f}, (w_a)_{a=1}^{\lambda_f}, (c_i)_{i=1}^{\lambda_f}, \left( m_i, [\tilde{S}_\mu, \tilde{g}_{\mu, \nu}] \right)_{\mu, \nu \in z_m} \right)_{i=1}^{\lambda_f} \]

is convex, but the conditions on the vertices (Items (4) and (5)) imply that the polygon associated to a representative is convex if \( w_0 \leq 0 \) and \( w_a \geq 0 \) for \( 1 \leq a \leq \lambda_f \), where \( s_a \) is the number of focus-focus values in line \( \ell_{ja} \). For instance, the polygon associated to any representative for which \( w_0 = \cdots = w_a = 0 \) is convex.

Remark 4.7. Let \( I \) be as in Definition 4.3. Not every polygon \( \Delta \) such that

\[ \mathcal{I} = (Z^{\lambda_f + 1} \times \mathbb{R}) \cdot \left( \frac{\langle \Delta, (\ell_{ja})_{a=1}^{\lambda_f} \rangle_{a=1}^{\lambda_f}, (w_a)_{a=1}^{\lambda_f}, (c_i)_{i=1}^{\lambda_f}, \left( m_i, [\tilde{S}_\mu, \tilde{g}_{\mu, \nu}] \right)_{\mu, \nu \in z_m} \right)_{i=1}^{\lambda_f} \]
is convex, but the conditions on the vertices (Items (4) and (5)) imply that the polygon associated to a representative is convex if \( w_0 \leq 0 \) and \( w_a \geq 0 \) for \( 1 \leq a \leq \lambda_f \), where \( s_a \) is the number of focus-focus values in line \( \ell_{ja} \). For instance, the polygon associated to any representative for which \( w_0 = \cdots = w_a = 0 \) is convex.

Remark 4.8. In [PVuN11a] the authors describe hidden and fake corners of the polygon, which represent the two possible cases in Item (5) above. A vertex which occurs on a line \( \ell_{ja} \) is a fake corner if there is no vertex there after changing the piecewise affine coordinates so that the adjacent wall-crossing index is zero, and such a vertex is a hidden corner if there is a smooth vertex remaining after changing to the appropriate coordinates. In Figure 2 the bottom right vertex on the line \( \ell_{ja} \) is hidden, since the slope of the bottom boundary changes by 5 even though the adjacent wall-crossing index is only \(-4\), and the rest of the vertices on the lines \( \ell_{j1}, \ell_{j2}, \) and \( \ell_{j3} \) are fake corners, since the changes in slope correspond to the adjacent wall-crossing indices.

Remark 4.9. Proposition 4.6 says that the injective map \( i: \mathcal{M}/\sim \rightarrow \mathbf{X} \) discussed in Remarks 3.6 and 4.2 is also surjective.

4.3. Classification. The following classification generalizes the Pelayo–Vũ Ngọc classification of simple semitoric systems [PVuN09, PVuN11a] by allowing the fibers of \( J \) (and hence of \( F \)) to have multiple focus-focus points per fiber. This includes fibers such as \( F^{-1}(c_2) \) and \( F^{-1}(c_3) \) in Figure 4. The proof follows from Propositions 3.2, 4.1, and 4.6.

Theorem 4.10. For each complete semitoric ingredient as in Definition 4.3 there exists a semitoric integrable system with that as its complete semitoric invariant as in Definition 3.3. Moreover, two semitoric systems are isomorphic if and only if they have the same complete semitoric invariant.

Remark 4.11. Most of the proof of Theorem 4.10 was already contained [PVuN09, PVuN11a]. In the present paper we have understood how the symplectic invariants in Definition 2.14 (the Taylor series invariants) constructed in [PT19] relate to the original construction of the twisting-index invariant. We have seen that they can be naturally packaged together into a single invariant which mixes the information of both original invariants (Section 2.2). We have also seen that the multipinched fibers change the affine structure induced (Section 2.1), and that our analogue of the “polygonal invariant” of the Pelayo–Vũ Ngọc classification (the third invariant) may no longer be convex (Remark 4.7), as
Figure 4. Focus-focus fibers of a semitoric system $F: M \to \mathbb{R}^2$ where $M$ is a symplectic 4-manifold. The piecewise affine coordinates $A$ (as in Definition 2.4) map the momentum map image onto a polygon. The system shown has $v_f = 3$, $\lambda_f = 3$, $m_1 = 1$, $m_2 = 5$, and $m_3 = 7$.

Illustrated in Figure 2. This polygonal invariant was the complete invariant of the classification of compact toric systems due to Atiyah–Guillemin–Sternberg–Delzant [Ati82, GS82, Del88], because compact toric systems cannot have focus-focus points; see [Pel17, GS05] for an expository account. For the case of non-compact toric systems see [KL15].

Remark 4.12. In the case that $(M, \omega, F)$ is a simple semitoric system the five original invariants (1)–(5) from [PVuN09, PVuN11a] can be obtained from the complete semitoric invariant given in Definition 3.3. (1) The number of focus-focus points is equal to $\lambda_f = v_f = m_f$. (2) From the Taylor series labels on each focus-focus value in the complete semitoric invariant one can extract the Taylor series invariant from [PT19] via Equation (2.9), which in the simple case is determined by a single series $\tilde{s}_i^0$ for each focus-focus value $c_i$. The relationship between this series and the Taylor series invariant $\langle S_i \rangle^\infty$ from [PVuN09] is

$$s_i^0(X, Y) = \langle S_i \rangle^\infty(Y, X) + \frac{\pi}{2} X \pmod{2\pi X}$$

where the addition of $(\pi/2)X$ is due to a change in convention between [PT19] and [VuN03], as discussed in [PT19, Remark 6.2]. (3) The semitoric polygon invariant in [PVuN09] is obtained by taking the image of a toric momentum map defined on the compliment in $M$ of the preimages under $F$ of rays which start at each focus-focus value and go either up or down, these are known as “cuts” in the base space. These polygons correspond to the subset of images of piecewise affine coordinates.
\[ \mathcal{A} = (A^1, A^2) \] (as in Definition 2.4) such that for each \( a \in \{1, \ldots, \lambda_i\} \) the lower wall-crossing index \( w_a \) satisfies either \( w_a = 0 \) (corresponding to an upwards cut) or \( w_a = -1 \) (corresponding to a downwards cut). (4) The height invariant \( h^i \) of the focus-focus value \( c_i = (c^1_i, c^2_i) \) is the distance from the marked point \( c_i \) to the bottom of the corresponding polygon, obtained by

\[
h^i = \frac{1}{2\pi} \left( \mathcal{g}_0 \right)^{(0,0)} - \min_{\ell | \ell | \Delta} A^2,
\]

but \( h^i \) does not depend on the choice of \( \mathcal{A} \) by Lemma 2.17. (5) Finally, the twisting-index invariant \( k^i_{\text{classical}} \) was originally defined in [PVuN09] by comparing \( \mathcal{A} \circ F \) with a local preferred momentum map, and is essentially the integer part of \( \frac{1}{2\pi} (\mathcal{g}_0)^{(1,0)} \), but again there is a shift by \((\pi/2)X\), so the twisting-index invariant of \( c_i \) is given as

\[
k^i_{\text{classical}} = \left\lfloor \frac{1}{2\pi} \left( (\mathcal{g}_0)^{(1,0)} - \frac{\pi}{2} \right) \right\rfloor + \frac{\epsilon_i - 1}{2}
\]

where \( \lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z} \) is the usual floor function and \( \epsilon_i = +1 \) if the cut at \( c_i \) is upward and \( \epsilon_i = -1 \) if the cut at \( c_i \) is downward. Note that this integer label on each \( c_i \) does depend on the choice of piecewise affine coordinates, since changing piecewise affine coordinates can shift the coefficient of \( X \) in \( \mathcal{g}_0 \) by an integer multiple of \( 2\pi \), as is seen in Lemma 2.17. The last term of \( k^i_{\text{classical}} \) is there so that it is preserved under a change in cut direction at \( c_i \) in [PVuN09] the dependence of the preferred momentum map on the cut direction was designed so that this would hold. For further discussion of the twisting index see Section 4.4.

Remark 4.13. Semitoric systems appear in physics, see for instance [JC63, SZ99, BD15].

Remark 4.14. It would be interesting to extend the classification to systems having “hyperbolic triangles” as in [DP16] and [LFP19a, Section 6.6].

Remark 4.15. Multipinched tori appear in mirror symmetry [GW97] (we thank Mark Gross for discussions). To get Lagrangian torus fibrations in mirror symmetry one can start with a K3 surface with an elliptic fibration and by hyperkähler rotation turn it into a special Lagrangian fibration, the

\[
\lfloor \cdot \rfloor
\]

where

\[
\mathcal{R} \rightarrow \mathcal{Z}
\]

is a toric variety (and toric varieties on simply connected, see [Dan78] and also see [DP09] for an

\[
\mathcal{A} \leftarrow (A^1, A^2)
\]

such that for each \( a \in \{1, \ldots, \lambda_i\} \) the lower wall-crossing index \( w_a \) satisfies either \( w_a = 0 \) (corresponding to an upwards cut) or \( w_a = -1 \) (corresponding to a downwards cut). (4) The height invariant \( h^i \) of the focus-focus value \( c_i = (c^1_i, c^2_i) \) is the distance from the marked point \( c_i \) to the bottom of the corresponding polygon, obtained by

\[
h^i = \frac{1}{2\pi} \left( \mathcal{g}_0 \right)^{(0,0)} - \min_{\ell | \ell | \Delta} A^2,
\]

but \( h^i \) does not depend on the choice of \( \mathcal{A} \) by Lemma 2.17. (5) Finally, the twisting-index invariant \( k^i_{\text{classical}} \) was originally defined in [PVuN09] by comparing \( \mathcal{A} \circ F \) with a local preferred momentum map, and is essentially the integer part of \( \frac{1}{2\pi} (\mathcal{g}_0)^{(1,0)} \), but again there is a shift by \((\pi/2)X\), so the twisting-index invariant of \( c_i \) is given as

\[
k^i_{\text{classical}} = \left\lfloor \frac{1}{2\pi} \left( (\mathcal{g}_0)^{(1,0)} - \frac{\pi}{2} \right) \right\rfloor + \frac{\epsilon_i - 1}{2}
\]

where \( \lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z} \) is the usual floor function and \( \epsilon_i = +1 \) if the cut at \( c_i \) is upward and \( \epsilon_i = -1 \) if the cut at \( c_i \) is downward. Note that this integer label on each \( c_i \) does depend on the choice of piecewise affine coordinates, since changing piecewise affine coordinates can shift the coefficient of \( X \) in \( \mathcal{g}_0 \) by an integer multiple of \( 2\pi \), as is seen in Lemma 2.17. The last term of \( k^i_{\text{classical}} \) is there so that it is preserved under a change in cut direction at \( c_i \) in [PVuN09] the dependence of the preferred momentum map on the cut direction was designed so that this would hold. For further discussion of the twisting index see Section 4.4.

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Remark 4.15. Multipinched tori appear in mirror symmetry [GW97] (we thank Mark Gross for discussions). To get Lagrangian torus fibrations in mirror symmetry one can start with a K3 surface with an elliptic fibration and by hyperkähler rotation turn it into a special Lagrangian fibration, the singular fibers of which can include the multipinched tori. Toric fibrations with singularities are important in the context of mirror symmetry and algebraic geometry [GS03, GS06, GS10, GS11], and symplectic geometry [LS10, Zun96, RWZ18]. Also, in [Via14, Via16, Via17] Vianna uses almost toric fibrations, nodal trades, and nodal slides (as in [Sym03]) to construct infinitely many non-Hamiltonian isotopic Lagrangian tori in \( \mathbb{C}P^2 \) and monotone del Pezzo surfaces.

Remark 4.16. One can think of a symplectic 4-manifold as the phase space of a mechanical system, and an action or integrable system on it as an additional symmetry. One may view Theorem 4.10 as a symplectic classification of Hamiltonian \((S^1 \times \mathbb{R})\)-actions on symplectic 4-manifolds (under the constraints on the types of singularities which can occur in semitoric systems). The symplectic classification of Hamiltonian \((\mathbb{R} \times \mathbb{R})\)-actions is expected to be difficult and essentially corresponds to classifying integrable systems with two degrees of freedom on symplectic 4-manifolds. The one degree of freedom case appeared in [DMT94]. On the other hand, there also exists a symplectic classification of symplectic \((S^1 \times S^1)\)-actions on 4-manifolds [Pel10]. For the case of Hamiltonian completely integrable torus actions on orbifolds, see [LT97] which extends the manifold case from [Del88]. The existence of these symmetries can have implications on the topology and geometry of \( M \). For instance, Karshon [Kar99] proved that if \( M \) is a compact symplectic 4-manifold which admits a Hamiltonian \( S^1 \)-action then \( M \) is Kähler (and hence there are strong constraints on the topology of \( M \), such as the fundamental group). In fact, the proof of Delzant’s theorem [Del88] shows that if \( M \) is a 2n-dimensional symplectic manifold which admits a Hamiltonian \( n \)-torus action then \( M \) is a toric variety (and toric varieties on simply connected, see [Dan78] and also see [DP09] for an
explicit construction of the toric variety with charts). Recently, there has been interest on extending Delzant’s classification to log-symplectic manifolds [GMP14, GLPR17].

Finally, one natural follow up problem to the classification of this paper is to apply the result to study the structure of the moduli space that semitoric systems form. In the case of simple systems this was done in [Pal17] and in the case of toric systems in [PPRS14]. Having fibers with multiple focus-focus points per fiber makes the problem even more interesting, as one needs to face the question of deformations between fibers which include different numbers of focus-focus points. The article [LFP19a] contains a number of results concerning the behavior of families under variations of parameters.

Remark 4.17. Theorem 4.10 includes the data of the twisting index inside of a Taylor series, giving a possible strategy to detect the twisting index from the semiclassical spectrum of a non-simple semitoric system. In the case of simple quantum semitoric systems it is known that one can recover all symplectic invariants from the semiclassical spectrum, with the possible exception of the twisting-index [LFPVuN16]. The Pelayo–Vũ Ngọc conjecture [PVVuN1a, Conjecture 9.1] states that the twisting index can also be recovered. In the case of toric systems on compact 2n-dimensional manifolds, n ≥ 1, this was proved first in [CPVuN13] (in this case the only invariant is an n-dimensional polytope) and later an alternative proof was given in [PPVuN14]. A complete semi-local result in a neighborhood of a critical fiber containing exactly one focus-focus point appeared in [PVVuN12].

4.4. Remarks on the twisting-index invariant. Let $c_i$ be a focus-focus value. Considering Equation (2.9) we see that given the semi-local invariant $[s_i^j, g_{i\nu \mu}]_{\nu, \mu \in \mathbb{Z}}$ (as in [PT19]) of the fiber $F^{-1}(c_i)$ this determines $[s_i^j, g_{i\nu \mu}]_{\nu, \mu \in \mathbb{Z}}$ a priori up to adding an integer multiple of $2\pi X$ to each action Taylor series $\tilde{s}_0, \ldots, \tilde{s}_{m_i-1}$. In fact, in this section we will see that given $[s_i^j, g_{i\nu \mu}]_{\nu, \mu \in \mathbb{Z}}$ the set of possible choices of $[s_i^j, g_{i\nu \mu}]_{\nu, \mu \in \mathbb{Z}}$ is naturally isomorphic to $\mathbb{Z}$, corresponding to the choice of twisting-index data for the fiber.

We now directly generalize the twisting-index and height invariants from [PVVuN09] to the non-simple case.

Definition 4.18. Write $c_i = (c_1^i, c_2^i)$, let $(p_{i\mu}^j)_{\mu \in \mathbb{Z}}$ be the tuple of focus-focus points in $F^{-1}(c_i)$, and fix a choice of piecewise affine coordinates $A = (A^1, A^2)$ as in Definition 2.4. We define the twisting-index invariant of $p_{i\mu}^j$ relative to $A$ by

\begin{equation}
  k_{i\mu}^j = \left\lfloor \frac{1}{2\pi} (\tilde{s}_\mu^i)^{(1,0)} \right\rfloor
\end{equation}

and the height invariant of $c_i$ by

\begin{equation}
  h^i = \frac{1}{2\pi} (\tilde{s}_0^i)^{(0,0)} - \min_{\ell_{i1} \cap \Delta} A^2.
\end{equation}

Note that $k_{i\mu}^j$ depends on the choice of $A$ but $h^i$ does not (by Lemma 2.17), and note that $\tilde{s}_0^i$ can be replaced by any $\tilde{s}_\mu^i$, $\mu \in \mathbb{Z}$, in Equation (4.3) without changing the value of $h^i$. In [PVVuN09] the twisting-index is defined by comparing the toric momentum map $A \circ F$ to a local preferred momentum map, which is nearly equivalent to the definition given in Equation (4.2) but differs by a shift due to a slight change of convention between [VuN03] and [PT19], see Remark 4.20. Let $\Psi: \mathbb{R}_0[[X, Y]]/(2\pi XZ) \to \mathbb{R}[[X, Y]]$ be the right inverse of the map $s \mapsto s - (0,0) + 2\pi XZ$ determined by the requirement that $(\Psi(s))^{(1,0)} \in [0, 2\pi)$ for all $s \in \mathbb{R}_0[[X, Y]]/(2\pi XZ)$. Then

\begin{equation}
  \tilde{s}_\mu^i = \Psi(s_{\mu, i}^j) + 2\pi k_{i\mu}^j X + 2\pi h^i
\end{equation}

which follows immediately from the definitions of $k_{i\mu}^j$ and $h^i$. 

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The following proposition explains how in some cases only partial data can be enough to completely recover the semi-local invariants and twisting indices.

Proposition 4.19. Let $c_i$ be a focus-focus value such that $F^{-1}(c_i)$ contains $m_i \in \mathbb{Z}_{>0}$ focus-focus points denoted $(p_{i}^{\mu})_{\mu \in \mathbb{Z}_{m_i}}$. Let $\mathcal{A}$ be a choice of piecewise affine coordinates as in Definition 2.4, and let $k_{\mu}^{i}$ denote the twisting-index of $p_{i}^{\mu}$ relative to $\mathcal{A}$ as in Definition 4.18. Let $[s_{\mu}^{i}, g_{\mu,\nu}^{i}]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ denote the Taylor series invariant as in Definition 2.14 via Equation 2.9. Then $k_{\mu}^{0}, s_{\mu}^{0},$ and $(g_{\mu,\mu+1}^{i})_{\mu \in \mathbb{Z}_{m_i}}$ determine the entire tuple $(k_{\mu}^{i}, s_{\mu}^{i}, g_{\mu,\nu}^{i})_{\mu,\nu \in \mathbb{Z}_{m_i}}$. That is, given $k_{0}^{0}, s_{0}^{0},$ and $(g_{0,\mu+1}^{i})_{\mu \in \mathbb{Z}_{m_i}}$ there is exactly one possible way to extend to $(k_{\mu}^{i}, s_{\mu}^{i}, g_{\mu,\nu}^{i})_{\mu,\nu \in \mathbb{Z}_{m_i}}$ under the assumptions above. In particular, the semi-local invariant $(s_{\mu}^{i}, g_{\mu,\nu}^{i})_{\mu,\nu \in \mathbb{Z}_{m_i}}$ and the data of a single twisting-index $k_{0}^{i}$ determine the other twisting-indices $k_{1}, \ldots, k_{m_i-1}$.

Proof. Using Equation (4.4), $s_{0}^{0}$ and $k_{0}^{0}$ determine all terms in $s_{\mu}^{0}$ except the constant term. Similarly to [PT19, Corollary 1.2], the relations in Equation (2.8) imply that $s_{0}^{0} - (s_{\mu}^{0})^{(0,0)} = \psi_{\mu}$, i.e. the non-constant terms of $s_{0}^{0}$ and $(g_{0,\mu+1}^{i})_{\mu \in \mathbb{Z}_{m_i}}$ completely determine the entire set $(s_{\mu}^{0} - (s_{\mu}^{0})^{(0,0)}, g_{\mu,\nu}^{i})_{\mu,\nu \in \mathbb{Z}_{m_i}}$. In turn each $(s_{\mu}^{0})^{(0,0)}$ can be used to determine $k_{\mu}^{0}$ for $\mu \in \mathbb{Z}_{m_i} \setminus \{0\}$ by Equation (4.2) and $s_{\mu}^{0}$ can be recovered from the non-constant terms of $s_{\mu}^{0}$ by Equation (2.9).

In Proposition 4.19 we use $(s_{\mu}^{i}, g_{\mu,\nu}^{i})_{\mu,\nu \in \mathbb{Z}_{m_i}}$ instead of $[s_{\mu}^{i}, g_{\mu,\nu}^{i}]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ so we can specify a single twisting index $k_{0}^{i}$, but the result also implies that $[s_{\mu}^{i}, g_{\mu,\nu}^{i}]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ and any single twisting index, assigned to one of the elements of $[s_{\mu}^{i}, g_{\mu,\nu}^{i}]_{\mu,\nu \in \mathbb{Z}_{m_i}}$, determine the remaining twisting indices.

Remark 4.20. Alternately, one can take the definition of the twisting-index to be given by

$$k_{\mu,\text{alternate}} = \left[ \frac{1}{2\pi} \left( (s_{\mu}^{0})^{(1,0)} - \frac{\pi}{2} \right) \right]$$

to agree with the original definition by Pelayo–Vũ Ngọc [PVuN09, Definition 5.9] in the case of simple systems with upward cuts, in which case all of the results in this section hold verbatim after redefining $\Psi$ to have the property that $(\Psi(s))^{(1,0)} = \left[ \frac{\pi}{2}, \frac{5\pi}{2} \right]$. Again, this difference is due to the shift by $\frac{\pi}{2} \mathcal{X}$ in the definition of the Taylor series between [PT19] and [VuN03], see [PT19, Remark 6.2].

Remark 4.21. Proposition 4.19 and Equation (4.4) imply that for each focus-focus fiber $F^{-1}(c_i)$ the data encoded in $[s_{\mu}^{i}, g_{\mu,\nu}^{i}]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ is equivalent to the data of the semi-local invariant $[s_{\mu}^{i}, g_{\mu,\nu}^{i}]_{\mu,\nu \in \mathbb{Z}_{m_i}}$ from [PT19], the height invariant $h^i$ from Equation (4.3), and additionally one integer, the twisting-index $k_{0}^{i}$ of any one of the focus-focus points in the fiber.

4.5. Example. An explicit example of a compact semitoric system which includes a twice-pinched torus can be obtained by certain choices of parameters for the system described in Hohloch–Palmer [HP18], which is a generalization of the coupled angular momentum system, see for instance [LFP19b, ADH18a].

Consider $M = \mathbb{S}^2 \times \mathbb{S}^2$ with coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ inherited from the usual inclusion $\mathbb{S}^2 \subset \mathbb{R}^3$ and symplectic form $\omega = -(R_1 \omega_{2d} \oplus R_2 \omega_{2d})$ for some parameters $R_1, R_2 > 0$, where $\omega_{2d}$ is the usual area form on the sphere giving it area $2\pi$. The parameters $R_1, R_2$ represent the radii of the spheres and in [HP18] it was assumed that $R_1 < R_2$ since taking $R_1 = R_2$ can produce non-simple semitoric systems or even systems which include degenerate singularities (depending on the other parameters). Using parameters $R_1 = R_2 = 1$ and $s_2 = s_1$, the semitoric system given in [HP18, Theorem 1.2] is $(\mathbb{S}^2 \times \mathbb{S}^2, -(\omega_{2d} \oplus \omega_{2d}), F_{s_1} = (J, H_{s_1}))$ where

\begin{equation}
\begin{aligned}
J &= z_1 + z_2, \\
H_{s_1} &= (1 - s_1)^2 z_1 + s_1^2 z_2 + 2s_1(1 - s_1)(x_1y_1 + x_2y_2),
\end{aligned}
\end{equation}
If $s_1 = \frac{1}{2} + \varepsilon$ for small $\varepsilon > 0$ there are two once-pinched fibers.

If $s_1 = \frac{1}{2}$ there is one twice-pinched fiber.

Figure 5. The polygon in a representative of the complete semitoric invariant associated to the system given in Equation (4.5) for $s_1$ close to $\frac{1}{2}$ and $s_1 = \frac{1}{2}$. The wall-crossing indices as in Definition 2.9 are indicated to the left of the line segments they label, and shown above the marked points are the corresponding focus-focus fibers. The arrows indicate that the focus-focus fibers are mapped to the marked points by $A \circ F$, where $A$ is the choice of piecewise affine coordinates (as in Definition 2.4) associated to the polygon in the figure.

and $s_1 \in [0, 1]$ remains a free parameter. For $s_1$ in a neighborhood of $\frac{1}{2}$ this is a semitoric system with two focus-focus points which occur at $(0, 0, \pm 1, 0, 0, \mp 1)$. If $s_1 = \frac{1}{2}$ then both of these points are in the same fiber $F^{-1}(0, 0)$, which is thus a twice-pinched torus. Taking $s_1 = \frac{1}{2} + \varepsilon$, with $\varepsilon \neq 0$ sufficiently small, produces a semitoric system which has two focus-focus points which are in different fibers of $F$, but nevertheless both focus-focus points lie in the same fiber of $J$, so it still does not satisfy the simplicity condition. Figure 5 shows the polygon and focus-focus fibers in each case. The semitoric polygon can be determined from the single representative shown in the figure, which has vertices at $(-2, -1)$, $(0, 1)$, $(2, 1)$, and $(0, -1)$ in both cases and has wall-crossing indices labeled. Computing the other invariants is more difficult, see for instance the techniques used in [ADH18a, ADH18b]. It would be interesting to compare the Taylor series invariants for the two separate focus-focus fibers in Figure 5a to the Taylor series invariants for the twice-pinched focus-focus fiber in Figure 5b.

Acknowledgements. Á. Pelayo thanks the Departamento de Álgebra, Geometría y Topología of the Universidad Complutense de Madrid for the excellent hospitality during a visit in September 2019, during which this paper was completed. J. Palmer thanks Rutgers University for supporting travel related to this paper, and the AMS and the Simons Foundation which also funded a visit to the second author through an AMS-Simons travel grant. X. Tang thanks Cornell University and University of Toronto where this paper was completed.

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