Absence of Replica Symmetry Breaking in the Transverse and Longitudinal Random Field Ising Model

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Abstract It is proved that replica symmetry is not broken in the transverse and longitudinal random field Ising model. In this model, the variance of spin overlap of any component vanishes in any dimension almost everywhere in the coupling constant space in the infinite volume limit. The weak Fortuin–Kasteleyn–Ginibre property in this model and the Ghirlanda–Guerra identities in artificial models in a path integral representation based on the Lie–Trotter–Suzuki formula enable us to extend Chatterjee’s proof for the random field Ising model to the quantum model.

Keywords Transverse field Ising model · Quantum spin systems · Gaussian random field · The FKG inequality · The Ghirlanda–Guerra identities · The Lie–Trotter–Suzuki formula · Interpolation

1 Introduction

Replica symmetry breaking is known to be a non-trivial phenomenon in systems with quenched disorder. This phenomenon in mean field spin glass models has been studied deeply, since Talagrand proved the Parisi conjecture [23] for the Sherrington–Kirkpatrick (SK) model [25] in a mathematically rigorous manner [27]. When replica symmetry is broken, the observed value of an observable in a typical sample differs from its sample expectation with finite probability, even though all samples in the sample ensemble are synthesized using exactly the same method. Theoretical physicists and mathematicians have been seeking this phenomenon also in more realistic short range spin glass models, such as the Edwards–Anderson (EA) model [13], however, only a few rigorous results for the replica symmetry breaking have been obtained in low temperature region in short range systems. Nishimori and Sherrington showed that the replica symmetry breaking does not occur on the Nishi-
mori line located out of the spin glass phase in the EA model [20,21]. Recently, Chatterjee proved a remarkable theorem that replica symmetry is not broken in the Ising model with a longitudinal Gaussian random field in any dimension almost everywhere in the coupling constant space [3]. It was shown that the variance of overlap vanishes in the system with the Fortuin–Kasteleyn–Ginibre (FKG) property using the Ghirlanda–Guerra identities [1,15]. In the present paper, we extend his argument to quantum systems with the weak FKG property. This is a first rigorous result for replica symmetry breaking in quantum disordered systems with short range interactions.

2 Definitions and Main Result

We study disordered quantum spin systems on $d$-dimensional cubic lattice $V_L := [1, L]^d \cap \mathbb{Z}^d$ and their corresponding classical spin systems on $(d + 1)$-dimensional cubic lattice $W_{L,M} = V_L \times T_M$, where $T_M := [1, M] \cap \mathbb{Z}$ with positive integers $L$ and $M$. Let $B_L$ be a collection of interaction bonds which are translations of a pair of sites in $V_L$. One of the most important example is given by nearest neighbor bonds $B_L = \{\{x, y\} | x, y \in V_L, |x - y| = 1\}$. A spin operator $S^i_x$ ($i = 1, 2, 3$) at a site $x \in V_L$ on a Hilbert space $\mathcal{H} := \bigotimes_{x \in V_L} \mathcal{H}_x$ is defined by a tensor product of the Pauli matrix $\frac{1}{2}\sigma^i$ acting on $\mathcal{H}_x \simeq \mathbb{C}^2$ and unities. These operators are self-adjoint and satisfies the commutation relation

$$[S^1_x, S^2_y] = i \delta_{x,y} S^3_x, \quad [S^2_x, S^3_y] = i \delta_{x,y} S^1_x, \quad [S^3_x, S^1_y] = i \delta_{x,y} S^2_x,$$

and the spin at each site $x$ has a fixed magnitude

$$\sum_{j=1}^3 (S^j_x)^2 = \frac{3}{4} 1.$$

We study the following Hamiltonian

$$H_V(S, g) := A(S^1, g^1) + B(S^3, g^3),$$

consisting of non-commuting two terms $A$ and $B$ defined by

$$A(S^1, g^1) := -\sum_{x \in V_L} J_1 g^1_x S^1_x,$$

$$B(S^3, g^3) := -\sum_{\{x, y\} \in B_L} S^3_x S^3_y - \sum_{x \in V_L} (J_3 g^3_x + c) S^3_x,$$

where $(g^i_x)_{x \in V_L, i=1,3}$ are standard Gaussian i.i.d. random variables and $J_1, J_3, c \in \mathbb{R}$ are coupling constants.

Here, we define Gibbs state for the Hamiltonian. For a positive $\beta$, the partition function is defined by

$$Z_V(J, g) := \text{Tr} e^{-\beta H_V(S,g)}$$

where the trace is taken over the Hilbert space $\mathcal{H}$.

Let $f$ be an arbitrary function of spin operators $S^i_x$, $(x \in V_L, i = 1,2,3)$. The expectation of $f$ in the Gibbs state is given by

$$\langle f(S^i) \rangle = \frac{1}{Z_V(J, g)} \text{Tr} f(S^i) e^{-\beta H_V(S,g)}.$$

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Here, we introduce a fictitious time \( t \in [0, 1] \) and define a time evolution of operators with the Hamiltonian. Let \( O \) be an arbitrary self-adjoint operator, and we define an operator valued function \( O(t) \) of \( t \in [0, 1] \) by

\[
O(t) := e^{-tH} O e^{tH}.
\]  

(6)

Furthermore, we define the Duhamel expectation of time dependent operators \( O_1(t_1), \ldots, O_k(t_k) \) by

\[
(O_1, O_2, \ldots, O_k)_D := \int_{[0,1]^k} dt_1 \cdots dt_k \langle T [O_1(t_1) O_2(t_2) \cdots O_k(t_k)] \rangle,
\]

where the symbol \( T \) is a multilinear mapping of the chronological ordering. If we define a partition function with arbitrary self-adjoint operators \( O_0, O_1, \ldots, O_k \) and real numbers \( x_1, \ldots, x_k \)

\[
Z(x_1, \ldots, x_k) := \text{Tr} \exp \beta \left[ O_0 + \sum_{i=1}^k x_i O_i \right],
\]

the Duhamel expectation of \( k \) operators represents the \( k \)-th order derivative of the partition function [12,16,24]

\[
\beta^k (O_1, \ldots, O_k)_D = \frac{1}{Z} \frac{\partial^k Z}{\partial x_1 \cdots \partial x_k}.
\]

To study replica symmetry breaking, we consider \( n \) replicated spin model defined by the following Hamiltonian

\[
\sum_{\alpha=1}^n H_V (S^\alpha, g).
\]  

(7)

The overlap operator \( R_{i,\alpha,\beta}^i (i = 1, 2, 3) \) between different replicated spins is defined by

\[
R_{i,\alpha,\beta}^i = \frac{1}{|V_L|} \sum_{x \in V_L} S_x^i,\alpha S_x^i,\beta,
\]

for \( \alpha, \beta = 1, 2, \ldots, n \) and \( \alpha \neq \beta \).

It is well-known that quantum spin systems on a \( d \)-dimensional lattice can be represented as \((d+1)\)-dimensional classical Ising systems [26]. The Lie–Trotter–Suzuki formula for the Hamiltonian (1)

\[
e^{-\beta A - \beta B} = \lim_{M \to \infty} (e^{-\beta A/M} e^{-\beta B/M})^M
\]

and inserting \( M \) resolutions of unity in eigenstates of \( 2S_x^3 \) on \( \mathcal{H} \)

\[
1 = \sum_{\sigma \in \mathcal{S}_V} |\sigma\rangle_3 \langle \sigma|,
\]

(8)

where we define

\[
S_x^i |\sigma\rangle_i = \frac{\sigma_x}{2} |\sigma\rangle_i.
\]

\( \mathcal{S}_V := \{-1, 1\}^{V_L} \) is a set of eigenvalue configurations, enable us to represent the \( d \)-dimensional quantum spin system in the following \((d+1)\)-dimensional classical spin system

\[
Z_V (J, g) = \lim_{M \to \infty} C_W \sum_{\sigma \in \mathcal{S}_W} e^{-\beta H_W (\sigma, g)},
\]

(9)
where the summation is taken over spin configurations $\mathcal{S}_W := \{-1, 1\}^{W_{L,M}}$ on the $(d + 1)$-dimensional lattice $W_{L,M}$ and the factor $C_W$ is independent of spin configurations. In this representation, we impose the periodic boundary condition on spin configuration $\sigma_{x,t+M} = \sigma_{x,t}$ with respect to $t \in T_M$ and free boundary condition with respect to $x \in V_L$. For instance in the transverse field Ising model with longitudinal random field [9, 10, 12], the Hamiltonian is given by

$$H_W(\sigma, g) = -\sum_{t \in T_M} \left[ \frac{1}{4M} \sum_{\{x,y\} \in B_L} \sigma_{x,t} \sigma_{y,t} + \frac{1}{2M} \sum_{x \in V_L} (J_3 g_x^3 + c) \sigma_{x,t} + \sum_{x \in V_L} K_x \sigma_{x,t} \sigma_{x,t+1} \right],$$

(10)

where

$$\tanh \beta K_x = e^{-\beta J_1 g_x^1 / M},$$

and the factor is given by

$$C_W = \prod_{x \in V_L} \left| \frac{1}{2} \sinh \beta J_1 g_x^1 / M \right|^\frac{M}{2}.$$

(11)

To obtain our main result, we consider an artificial $(d + 1)$-dimensional random field Ising model with quenched i.i.d standard Gaussian random variables $(h^i_{x,t})_{(x,t) \in W_{L,M}, i = 1, 3}$ and arbitrary numbers $b_i, c \in \mathbb{R}$ for $i = 1, 3$. We define the following perturbed Hamiltonian

$$H(b_1, b_3, c, h_1, h_3) := -\sum_{t \in T_M} \left( \frac{1}{4M} \sum_{\{x,y\} \in B_L} \sigma_{x,t} \sigma_{y,t} + \sum_{x \in V_L} K_x \sigma_{x,t} \sigma_{x,t+1} \right)$$

$$-\frac{1}{2M} \sum_{(x,t) \in W_{L,M}} (J_3 g_x^3 + b_3 \sqrt{M} h_{x,t}^3 + c) \sigma_{x,t},$$

(12)

where

$$\tanh \beta K_{x,t} := e^{-\beta (J_1 g_x^1 + b_1 h_{x,t}^1) / M},$$

such that $H(0, 0, c, 0, 0)$ is identical to $H_W(\sigma, g)$ defined by (10). This model has operator representation

$$Z_V(b) = \lim_{M \to \infty} \text{Tr} \left( \prod_{t=1}^M (e^{\beta A_t / M} e^{B_t / M}) \right),$$

(13)

where

$$A_t(S^1, g^1, h^1) := -\sum_{x \in V_L} \left( J_1 g_x^1 + b_1 h_{x,t}^1 \right) S_x^1,$$

$$B_t(S^3, g^3, h^3) := -\sum_{\{x,y\} \in B_L} S_x^3 S_y^3 - \sum_{x \in V_L} \left( J_3 g_x^3 + b_3 \sqrt{M} h_{x,t}^3 + c \right) S_x^3.$$  

(14)

For lighter notation, we denote

$$H_1(b) := H(b, 0, c, h, 0), \quad H_3(b) := H(0, b, c, 0, h)$$

(15)

and define partition function $Z_i(b)$ by

$$Z_i(b) := \sum_{\sigma \in \mathcal{S}_W} e^{-\beta H_i(b)},$$
and define functions $\psi_{i,L}$ and $p_{i,L}$ by
\[
\psi_{i,L}(b) := \frac{1}{|V_L|} \log Z_i(b), \quad p_{i,L}(b) := E\psi_{i,L}(b),
\]
where a sample expectation $E$ denotes expectation over all random fields $(g^i_x)_{x \in V_L, i=1,3}$, $(h^i_{x,t})_{(x,t) \in W_L, i=1,3}$.

Note that
\[
Z_V = \lim_{M \to \infty} C_W Z_i(0),
\]
given by (12) and (15). Hereafter, $\langle f \rangle_{b,i}$ denotes the Gibbs expectation of a function $f : \mathcal{S}_W \to \mathbb{R}$ with the Hamiltonian $H_i(b)$
\[
\langle f(\sigma) \rangle_{b,i} := \frac{1}{Z_i(b)} \sum_{\sigma \in \mathcal{S}_W} f(\sigma) e^{-\beta H_i(b)},
\]
Note that for $i = 1, 3$
\[
\langle f(2S^3) \rangle = \lim_{M \to \infty} \langle f(\sigma) \rangle_{0,i}
\]
in the representations (12) and (15) for $H_i(b)$.

In the present paper, we obtain the following main theorem for the transverse and longitudinal random field Ising model.

**Theorem 1** Consider the transverse and longitudinal random field Ising model defined by the Hamiltonian (1) and its replicated model (7). Almost everywhere in the coupling constant space, the infinite volume limit
\[
\lim_{L \to \infty} E\langle R_{1,2}^i \rangle,
\]
exists for any $i = 1, 2, 3$ and the variance of the overlap operator calculated in the replica symmetric Gibbs state vanishes
\[
\lim_{L \to \infty} E\langle (R_{1,2}^i - E\langle R_{1,2}^i \rangle)^2 \rangle = 0. \tag{16}
\]

Theorem 1 shows that the overlap operator $R_{1,2}^i$ is self-averaging in this model. This implies that the observed value of the overlap operator $R_{1,2}^i$ converges in probability toward its Gibbs and sample expectation $E\langle R_{1,2}^i \rangle$. Since the replica symmetric Gibbs expectation of the overlap operator is spin glass order parameter, the phase diagram should be unique if the sample is synthesized in the same method.

There are two key techniques to prove Theorem 1: the weak FKG property of the transverse and longitudinal random field Ising model in the $(d + 1)$-dimensional representation and continuity of an artificial perturbative $(d + 1)$-dimensional model with the Ghirlanda–Guerra identities [1,15]. Since a straightforward extension of the Ghirlanda–Guerra identities to quantum systems is not sufficient to judge absence or appearance of replica symmetry breaking in quantum systems unlike the classical system [3], we utilize the classical Ghirlanda–Guerra identities in artificial models given by Hamiltonians (12) and (15). We prove that the expectation of the overlap operator is a continuous function of the perturbation parameter. These results enable us to prove Theorem 1 which shows absence of replica symmetry breaking in the transverse and longitudinal random field Ising model.
3 Proof

Here, we consider the perturbed model defined by the Hamiltonian (15) in \( d + 1 \) dimension. For this model, there are useful lemmas proved in the literature. Here we present them as Lemmas 1, 2, 3 and 4 without proofs. Lemma 1 is proved as in [2,8,11,18], Lemmas 2 and 3 are proved in [18,19], and Lemma 4 is proved in [14].

**Lemma 1** The following infinite volume limit independent of boundary conditions exists

\[ p_i(b) = \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}[\psi_{i,L}(b)]. \]

for each \((\beta, J_1, J_3, b, c) \in [0, \infty) \times \mathbb{R}^4\).

**Lemma 2** For any \((\beta, J_1, J_3, b, c) \in [0, \infty) \times \mathbb{R}^4\), there exists a positive number \(C\) independent of \(L\), such that the variance of \(\psi_L\) is bounded from the above as follows

\[ \lim_{M \to \infty} \mathbb{E}[\psi_{i,L}(b) - p_i,L(b)]^2 \leq \frac{C}{|V_L|}. \]

Here, we define two types of deviations of of an arbitrary operator \(O\) by

\[ \delta O := O - \langle O \rangle, \quad \Delta O := O - \mathbb{E}(O). \]

And, also define two types of deviations an arbitrary function \(f : \mathcal{A}_W \to \mathbb{R}\) by

\[ \delta f(\sigma) := f(\sigma) - \langle f(\sigma) \rangle_{b,i}, \quad \Delta f(\sigma) := f(\sigma) - \mathbb{E}(f(\sigma))_{b,i}. \]

Define an order parameter \(m^i_L\) by

\[ m^i_L := \frac{1}{|V_L|} \sum_{x \in V_L} S^i_x \]

and define the corresponding order parameter \(\mu^i_L\) by

\[ \mu^1_L := \frac{1}{4|V_L| M} \sum_{x \in V_L, t \in T_M} (1 - \sigma_{x,t} \sigma_{x,t+1}), \]

\[ \mu^3_L := \frac{1}{2|V_L| M} \sum_{x \in V_L, t \in T_M} \sigma_{x,t}. \]

**Lemma 3** For any \((\beta, J_1, J_3, b, c) \in [0, \infty) \times \mathbb{R}^4\) with \(\beta J_i \neq 0\), there exists a positive number \(C\) independent of \(L\), such that

\[ \lim_{M \to \infty} \mathbb{E}(\delta \mu_{L}^2)_{b,i} \leq \frac{C}{\beta J_i \sqrt{1/|V_L|}}. \]

Lemma 3 gives an upper bound of Duhamel product for \(b_i = 0\)

\[ \mathbb{E}(\delta m^i_L, \delta m^j_L)_{\text{D}} \leq \frac{C}{\beta J_j \sqrt{1/|V_L|}}. \]

We say that the system satisfies the weak Fortuin–Kasteleyn–Ginibre (FKG) condition, if the one point function \(\langle S^3_x \rangle\) is monotonically increasing function of \(g_y^3\) at any site \(y \in V_L\).
The weak FKG condition is equivalent to the positive semi-definiteness of truncated Duhamel function
\[
(S_3^x, S_3^y)_D - \langle S_3^x \rangle \langle S_3^y \rangle \geq 0,
\]
for any two sites \(x, y \in V_L\). The \(d\)-dimensional transverse and longitudinal random field Ising model satisfies weak FKG condition, because of the following lemma for the corresponding \((d+1)\)-dimensional classical model with positive semi-definite exchange interactions. To explain the FKG inequality, we define a partial order \(\leq\) over the set \(\mathcal{S}_W\) of spin configurations and generalized monotonicity for a function of spin configurations. For two spin configurations \(\sigma, \tau \in \mathcal{S}_W\), we denote \(\sigma \leq \tau\), if \(\sigma_x \leq \tau_x\) for all \(x \in W_{L,M}\). We say that a function \(f : \mathcal{S}_W \to \mathbb{R}\) is monotonically increasing in a general sense, if \(\sigma \leq \tau\) implies \(f(\sigma) \leq f(\tau)\). The following FKG inequality can be proved [14]. Note that the artificial Hamiltonians \(H_i(a, b)\) given by (12) and (15) satisfy the FKG condition as well. Therefore, the one point function \(\langle x_1, x_2, \ldots, x_d, \sigma \rangle\) is monotonically increasing function of \(h_{3,i}^x\).

**Lemma 4** Let \(f\) and \(g\) be monotonically increasing functions of spin configurations on \(W_{L,M}\) in a general sense. In the random field Ising model with positive semi definite exchange interactions, \(f\) and \(g\) satisfy the Fortuin–Kasteleyn–Ginibre inequality
\[
\langle f(\sigma); g(\sigma) \rangle_{b,i} \geq 0,
\]
where a truncated correlation function is defined by
\[
\langle f(\sigma); g(\sigma) \rangle_{b,i} := \langle f(\sigma)g(\sigma) \rangle_{b,i} - \langle f(\sigma) \rangle_{b,i} \langle g(\sigma) \rangle_{b,i}.
\]

**Lemma 5** For arbitrary sites \(w, x, y, z \in W_{L,M}\),
\[
\left| \langle \sigma_x \sigma_y; \sigma_w \sigma_z \rangle_{b,i} \right| \leq \langle (\sigma_x + \sigma_y); (\sigma_w + \sigma_z) \rangle_{b,i}
\]

**Proof** Define functions \(f_{\pm} : \mathcal{S}_W \times W_{L,M}^2 \to \mathbb{R}\) by
\[
f_{\pm}(\sigma, w, x) := (\sigma_w \pm 1)(1 \pm \sigma_x).
\]

For arbitrary fixed \(w, x, y, z \in W_{L,M}\), functions \(f_{\pm}(w, x)\) and \(f_{\pm}(y, z)\) of spin configurations are monotonically increasing in general sense. From the FKG inequality,
\[
\langle \sigma_x \sigma_y; \sigma_w \sigma_z \rangle_{b,i} + \langle (\sigma_x + \sigma_y); (\sigma_w + \sigma_z) \rangle_{b,i}
= \left( \langle f_{\pm}(\sigma, w, x); f_{\pm}(\sigma, y, z) \rangle_{b,i} + \langle f_{\pm}(\sigma, w, x); f_{\pm}(\sigma, y, z) \rangle_{b,i} \right) / 2 \geq 0,
\]
and also
\[
-\langle \sigma_x \sigma_y; \sigma_w \sigma_z \rangle_{b,i} + \langle (\sigma_x + \sigma_y); (\sigma_w + \sigma_z) \rangle_{b,i},
= \left( \langle f_{\pm}(\sigma, w, x); f_{\pm}(\sigma, y, z) \rangle_{b,i} + \langle f_{\pm}(\sigma, w, x); f_{\pm}(\sigma, y, z) \rangle_{b,i} \right) / 2 \geq 0.
\]

These inequalities give the inequality (23). \(\Box\)

Next we evaluate Gibbs expectation of functions of the overlap operators in the path integral representation with the Hamiltonian \(H_i(bc)\). In these representations, we denote
\[
\rho_{\alpha, \beta}^{S} := \frac{1}{16|V_L|M} \sum_{(x, s) \in W_{L,M}} (1 - \sigma_{x,s}^a \sigma_{x,s+1}^a) (1 - \sigma_{x,s}^b \sigma_{x,s+1}^b),
\]
\[
\rho_{\alpha, \beta}^{3} := \frac{1}{4|V_L|M} \sum_{(x, s) \in W_{L,M}} \sigma_{x,s}^a \sigma_{x,s}^b.
\]
for $\alpha \neq \beta$. Note that
\[ \langle \rho_{\alpha,\beta}^i \rho_{\gamma,\delta}^j \rangle_{0,i} = (R_{\alpha,\beta}^i, R_{\gamma,\delta}^j)_D, \]
for $\alpha \neq \beta, \gamma \neq \delta$ and for $i, j = 1, 3$.

**Lemma 6** In the model defined by the Hamiltonian (15) with $J_1 \neq 0$ and $J_3 \neq 0$, the following expectations calculated in the replica symmetric Gibbs state vanish
\[ \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E} \left[ \langle \rho_{1,2}^i \rangle_{b,i} - \langle \rho_{1,2}^i \rangle_{b,i} \right] = 0, \]
for any $i = 1, 3$.

**Proof** First, consider $i = 3$
\[
\mathbb{E} \left[ \langle \rho_{1,2}^3 \rangle_{b,3} - \langle \rho_{1,2}^3 \rangle_{b,3} \right] = \frac{1}{16|V_L|^2 M^2} \sum_{x,y \in W_{L,M}} \mathbb{E} \left[ \langle \sigma_x \sigma_y \rangle_{b,3} - \langle \sigma_x \rangle_{b,3} \langle \sigma_y \rangle_{b,3} \right] \\
\leq \frac{1}{8|V_L|^2 M^2} \sum_{x,y \in V_L} \mathbb{E} \left| \langle \sigma_x \sigma_y \rangle_{b,3} - \langle \sigma_x \rangle_{b,3} \langle \sigma_y \rangle_{b,3} \right| \\
= \frac{1}{8|V_L|^2 M^2} \sum_{x,y \in V_L} \mathbb{E} \langle \sigma_x ; \sigma_y \rangle_{b,3}. \tag{28}
\]
The final line is nonnegative because of the FKG inequality. Therefore,
\[ \lim_{M \to \infty} \mathbb{E} \left[ \langle \rho_{1,2}^3 \rangle_{b,3} - \langle \rho_{1,2}^3 \rangle_{b,3} \right] \leq \lim_{M \to \infty} \mathbb{E} \langle \delta \mu_{L}^3 \rangle_{b,3} \leq \frac{C}{\beta J_3 \sqrt{|V_L|}}, \tag{29} \]
where we have used Lemma 3.

For $i = 1$,
\[
\mathbb{E} \left[ \langle \rho_{1,2}^1 \rho_{1,2}^1 \rangle_{b,1} - \langle \rho_{1,2}^1 \rangle_{b,1} \right] \leq \mathbb{E} \lim_{M \to \infty} \frac{1}{128|V_L|^2 M^2} \sum_{x,y \in V_L} \sum_{s,t \in T_M} \mathbb{E} \left| \langle \sigma_{x,s} \sigma_{x,s+1} ; \sigma_{y,t} \sigma_{y,t+1} \rangle_{b,1} \right| \\
\leq \lim_{M \to \infty} \frac{1}{128|V_L|^2 M^2} \sum_{x,y \in V_L} \sum_{s,t \in T_M} \mathbb{E} \left| \langle \sigma_{x,s} + \sigma_{x,s+1} ; \sigma_{y,t} + \sigma_{y,t+1} \rangle_{b,1} \right| \\
\leq \lim_{M \to \infty} \frac{1}{32|V_L|^2 M^2} \sum_{x,y \in V_L} \sum_{s,t \in T_M} \mathbb{E} \langle \sigma_x ; \sigma_y \rangle_{b,1} \\
\leq \lim_{M \to \infty} \frac{1}{8|V_L|^2 M^2} \sum_{x,y \in V_L} \sum_{s,t \in T_M} \mathbb{E} \langle \sigma_x ; \sigma_y \rangle_{b,1} \\
\leq \lim_{M \to \infty} \frac{1}{8} \mathbb{E} \langle \delta \mu_{L}^3 \rangle_{b,1} \leq \frac{C'}{\beta J_3 \sqrt{|V_L|}}. \tag{30}
\]
We have used the inequality (23) in Lemmas 5 and 3.

These bounds give the limit. \hfill \Box
In the original model with \( b_1 = b_3 = 0 \), Lemma 6 implies
\[
\lim_{L \to \infty} \mathbb{E} \left[ (R_{i,1,2}^i, R_{i,1,2}^i)_{\mathcal{D}} - \langle R_{i,1,2}^i \rangle^2 \right] = 0,
\]
for any \( i = 1, 3 \).

Here we regard
\[
\psi_{i,L}(h) := \frac{1}{|V_L|} \log Z_i(b)
\]
as a function of disorder \( h = (h_w)_{w \in W_{L,M}} \). Let \( h \) and \( (h') \) be i.i.d. standard Gaussian random variables, and define square root interpolating random variables with \( v \in [0, 1] \) by
\[
\sqrt{v} h_w + \sqrt{1 - v} h'_w,
\]
for \( w \in W_{L,M} \). Then, we define a generating function \( \gamma_i(v) \) of the parameter \( v_i \in [0, 1] \) by
\[
\gamma_i(v) := \mathbb{E} [ \mathbb{E}' \psi_{i,L}(v) ]^2,
\]
where \( \mathbb{E} \) and \( \mathbb{E}' \) denote expectation in \( h \) and \( h' \), respectively. This generating function \( \gamma_i \) is a generalization of a function introduced by Chatterjee [5].

Lemma 7  
For any \((\beta, J_1, J_3, b, c) \in (0, \infty) \times \mathbb{R}^4\), any positive integer \( L \), any positive integer \( k \) and any \( v_0 \in (0, 1) \), an upper bound on the \( k \)-th order derivative of the function \( \gamma_i \) is given by
\[
\frac{d^k \gamma_i}{dv^k}(v_0) \leq \frac{(k - 1)! \beta^2 b^2}{(1 - v_0)^{k-1} 4|V_L|}. 
\]

For an arbitrary \( v \in [0, 1] \), the \( k \)-th order derivative of \( \gamma_i \) is represented in the following
\[
\frac{d^k \gamma_i}{dv^k}(v) = \sum_{w_1 \in W_{L,M}} \cdots \sum_{w_k \in W_{L,M}} \mathbb{E} \left[ \mathbb{E}' \psi_{i,L,w_1,\ldots,w_k}(\sqrt{v} h + \sqrt{1 - v} h') \right]^2.
\]

Here we denote
\[
\psi_{i,L,w_1,\ldots,w_k}(h) := \frac{\partial^k \psi_{i,L}(h)}{\partial h_{w_k} \cdots \partial h_{w_1}}.
\]

Proof  
We obtain the formula (36) with \( k \) times use of integration by parts. This implies non negativity of all coefficients of the Taylor series of the function \( \gamma_i(v) \) around any \( v = v_0 \in [0, 1] \). Then, \( k \)-th derivatives are monotonically increasing in \( v \). From Taylor’s theorem, there exists \( v_1 \in (v_0, 1) \) such that
\[
\gamma_i'(v) = \sum_{k=0}^{n-1} \frac{(v - v_0)^k}{k!} \gamma_i^{(k+1)}(v_0) + \frac{(v - v_0)^n}{n!} \gamma_i^{(n+1)}(v_1).
\]
Each term in this series is bounded from the above by
\[
\gamma_i'(1) = \frac{\beta^2 b^2}{4|V_L|^2 M} \sum_{w \in W_{L,M}} \mathbb{E} \langle \sigma_w \rangle_{b,i}^2 \leq \frac{\beta^2 b^2}{4|V_L|}.
\]
This completes the proof.  \( \square \)
We define a term of the energy density with random field
\[ h^1_L := \frac{1}{4|V_L|\sqrt{M}} \sum_{(x,t)\in W_{L,M}} \hat{h}_{x,t}(1 - \sigma_{x,t} \sigma_{x,t+1}), \tag{37} \]
\[ h^3_L := \frac{1}{2|V_L|\sqrt{M}} \sum_{(x,t)\in W_{L,M}} \hat{h}_{x,t} \sigma_{x,t}. \tag{38} \]

**Lemma 8** For any \( \beta b \neq 0 \), we have
\[ \mathbb{E}(\delta h^{i,2}_L)_{b,i} \leq \frac{C}{\beta^2 b^2 |V_L|} + \frac{C'}{|V_L|}, \tag{39} \]
where \( C \) and \( C' \) are positive constants independent of \( L \).

**Proof** For \( i = 3 \) integration by parts gives
\[ \mathbb{E}(\delta h^3_L)_{b,3} = \frac{1}{4|V_L|^2 M} \sum_{x,y\in W_{L,M}} \mathbb{E}h_x h_y \langle \sigma_x; \sigma_y \rangle_{b,3} \]
\[ = \frac{1}{4|V_L|^2 M} \left[ \sum_{x,y\in W_{L,M}} \mathbb{E}\frac{\partial^2}{\partial h_x \partial h_y} \langle \sigma_x'; \sigma_y \rangle_{b,3} + \sum_{x\in W_{L,M}} \mathbb{E}\langle \sigma_x; \sigma_x \rangle_{b,c,3} \right] \]
\[ \leq \frac{1}{|V_L|\beta^2 b^2} \sum_{x,y\in W_{L,M}} \mathbb{E}\frac{\partial^4 \psi_{3,L}}{\partial h_x^2 \partial h_y^2} + \frac{1}{4|V_L|} \]
\[ \leq \frac{1}{|V_L|\beta^2 b^2} \left[ |V_L|^2 M^2 \sum_{x,y\in W_{L,M}} \left( \mathbb{E}\frac{\partial^4 \psi_{3,L}}{\partial h_x^2 \partial h_y^2} \right)^2 + \frac{1}{4|V_L|} \right] \]
\[ \leq \frac{\sqrt{\gamma_3^{(4)}(0)}}{\beta^2 b^2} + \frac{1}{|V_L|} \leq \sqrt{\frac{3}{\beta^2 b^2 |V_L|}} + \frac{1}{4|V_L|}. \tag{40} \]

The bound for \( i = 1 \) is obtained in the same way. \( \square \)

**Lemma 9** For almost all \( b \in \mathbb{R} \), we have
\[ \frac{\partial p_i}{\partial b} = \lim_{L\to\infty} \lim_{M\to\infty} \beta \mathbb{E}(h^i_L)_{b,i} = \lim_{L\to\infty} \lim_{M\to\infty} \frac{\beta^2 b}{4} \left( 1 - \mathbb{E}(\rho^{i}_{1,2})_{b,i} \right), \tag{41} \]
for \( p_i(b) := \lim_{L\to\infty} \lim_{M\to\infty} p_{i,L}(b) \), and for \( b \neq 0 \),
\[ \lim_{L\to\infty} \lim_{M\to\infty} \mathbb{E}(\delta h^i_L)_{b,i} = 0. \tag{42} \]

**Proof** This can be shown in the standard convexity argument to obtain the Ghirlanda–Guerra identities in classical and quantum systems \([4,6,7,18,19,22,28]\). Note that \( \psi_{i,L}, p_{i,L} \) and \( p_i \) are convex functions of \( b \) and \( c \). To show the first equality (41), regard \( p_{i,L} \) and \( \psi_{i,L} \) as functions of \( b \) for lighter notation. By Lemma 7, we have
\[ \mathbb{E}\psi_{i,L}(b)^2 - p_{i,L}(b)^2 \leq \frac{C}{|V_L|}, \]
where $C$ is a positive number independent of $L$. Define the following functions

$$\begin{align*}
w_L(\varepsilon) &:= \frac{1}{\varepsilon} \left[ |\psi_{i,L}(b + \varepsilon) - p_{i,L}(b + \varepsilon)| + |\psi_{i,L}(b - \varepsilon) - p_{i,L}(b - \varepsilon)| \\
+ |\psi_{i,L}(b) - p_{i,L}(b)| \right] \\
e_L(\varepsilon) &:= \frac{1}{\varepsilon} \left[ |p_{i,L}(b + \varepsilon) - p_i(b + \varepsilon)| + |p_{i,L}(b - \varepsilon) - p_i(b - \varepsilon)| + |p_{i,L}(b) - p_i(b)| \right],
\end{align*}$$

for $\varepsilon > 0$. Note that the assumption on $\psi_{i,L}$ gives

$$\mathbb{E} w_L(\varepsilon) \leq \frac{3}{\varepsilon} \sqrt{\frac{C}{|V_L|}},$$

(43)

for any $\varepsilon > 0$. Since $\psi_{i,L}$, $p_{i,L}$ and $p_i$ are convex functions of $b$, we have

$$\begin{align*}
\frac{\partial \psi_{i,L}}{\partial b}(b) - \frac{\partial p_i}{\partial b}(b) &\leq \frac{1}{\varepsilon} \left[ \psi_{i,L}(b + \varepsilon) - \psi_{i,L}(b) \right] - \frac{\partial p_i}{\partial b}(b) \\
&\leq \frac{1}{\varepsilon} \left[ \psi_{i,L}(b + \varepsilon) - p_{i,L}(b + \varepsilon) + p_{i,L}(b + \varepsilon) - p_{i,L}(b) + p_{i,L}(b) - \psi_{i,L}(b) \right. \\
&\quad \left. - p_i(b + \varepsilon) + p_i(b + \varepsilon) + p_i(b) - p_i(b) \right] - \frac{\partial p_i}{\partial b}(b) \\
&\leq \frac{1}{\varepsilon} \left[ \psi_{i,L}(b + \varepsilon) - p_{i,L}(b + \varepsilon) \right] + \frac{1}{\varepsilon} \left[ p_i(b + \varepsilon) - p_i(b) \right] - \frac{\partial p_i}{\partial b}(b) \\
&\leq w_L(\varepsilon) + e_L(\varepsilon) + \frac{\partial p_i}{\partial b}(b + \varepsilon) - \frac{\partial p_i}{\partial b}(b).
\end{align*}$$

As in the same calculation, we have

$$\begin{align*}
\frac{\partial \psi_{i,L}}{\partial b}(b) - \frac{\partial p_i}{\partial b}(b) &\geq \frac{1}{\varepsilon} \left[ \psi_{i,L}(b) - \psi_{i,L}(b - \varepsilon) \right] - \frac{\partial p_i}{\partial b}(b) \\
&\geq -w_L(\varepsilon) - e_L(\varepsilon) + \frac{\partial p_i}{\partial b}(b - \varepsilon) - \frac{\partial p_i}{\partial b}(b).
\end{align*}$$

Then,

$$\mathbb{E} \left| \frac{\partial \psi_{i,L}}{\partial b}(b) - \frac{\partial p_i}{\partial b}(b) \right| \leq \frac{3}{\varepsilon} \sqrt{\frac{C}{|V_L|}} + e_L(\varepsilon) + \frac{\partial p_i}{\partial b}(b + \varepsilon) - \frac{\partial p_i}{\partial b}(b - \varepsilon).$$

Convergence of $p_{i,L}$ in the infinite volume limit implies

$$\lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E} \left| \beta \langle h_{L}^{i} \rangle_{b,i} - \frac{\partial p_i}{\partial b}(b) \right| \leq \frac{\partial p_i}{\partial b}(b + \varepsilon) - \frac{\partial p_i}{\partial b}(b - \varepsilon),$$

The right hand side vanishes, since the convex function $p_i(b)$ is continuously differentiable almost everywhere and $\varepsilon > 0$ is arbitrary. Jensen’s inequality gives

$$\lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E} \left| \beta \langle h_{L}^{i} \rangle_{b,i} - \frac{\partial p_i}{\partial b}(b) \right| = 0,$$

(44)

for almost all $b$. This leads the first equality (41). The equality (44) implies also

$$\lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E} |\langle \Delta h_{L}^{i} \rangle_{b,i}| = 0.$$
This and Lemma 8 enable us to obtain
\[ \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(|\Delta h^i_L|)_{b,i} = 0, \]

since
\[ \mathbb{E}(\Delta h^i_L)_{b,i} = \mathbb{E}(\Delta h^i_L + (\Delta h^i_L)_{b,i})_{b,i} \leq \mathbb{E}(\Delta h^i_L)_{b,i} + \mathbb{E}(\Delta h^i_L)_{b,i} \]
\[ \leq \sqrt{\mathbb{E}(\Delta h^i_L)^2} \, \mathbb{E}(\Delta h^i_L)_{b,i}. \]

Therefore the identities are obtained from the above as in the random field Ising model [3].

\[ \square \]

Note that Lemma 9 implies the existence of \( \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\rho_{1,2}^i)_{b,i} \) for \( b \neq 0 \).

**Lemma 10** Let \( f : \mathcal{S}^W \to \mathbb{R} \) be a bounded function of \( n \) replicated spin configurations. The Gibbs and sample expectations of \( f \) and spin overlap in the model defined by the Hamiltonian (12), satisfy the following identity for almost all \( b \in \mathbb{R} \)
\[ \lim_{L \to \infty} \lim_{M \to \infty} \left[ \mathbb{E}( f \rho_{1,n+1}^i)_{b,i} - \frac{1}{n} \mathbb{E}( f)_{b,i} \mathbb{E}(\rho^i_{1,2})_{b,c,i} - \frac{1}{n} \sum_{a=2}^n \mathbb{E}( f \rho^i_{1,a})_{b,i} \right] = 0, \] (45)
which provides the Ghirlanda–Guerra identities [1,15].

**Proof** From the identity (44) in Lemma 9,
\[ \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\Delta h^i_L f)_{b,i} = 0. \]
Calculating the right hand side gives the identity. \( \square \)

**Lemma 11** For almost all constant field \( c \in \mathbb{R} \), the expectation of the overlap in the infinite volume limit is continuous at \( b = 0 \)
\[ \lim_{b \to 0} \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\rho_{1,2}^i)_{b,i} = \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\rho_{1,2}^i)_{0,i}, \] (46)
\[ \lim_{b \to 0} \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\rho_{1,2}^i)^2_{b,i} = \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\rho_{1,2}^i)^2_{0,i}. \] (47)

**Proof** Evaluate the following partial derivative
\[ \frac{\partial}{\partial b} \mathbb{E}(\rho_{1,2}^i)_{b,3} \]
\[ = \frac{\beta^2 b}{8 |V_L|M^2} \sum_{v,w \in \mathcal{W}_L,M} \mathbb{E}(\sigma_v; \sigma_w)_{b,3} \left( (\sigma_v \sigma_w)_{b,3} - 3(\sigma_v)_{b,3}(\sigma_w)_{b,3} \right) \]
\[ \leq \frac{\beta^2 b}{8 |V_L|M^2} \sum_{v,w \in \mathcal{W}_L,M} \mathbb{E}(\sigma_v; \sigma_w)_{b,3} \left( (\sigma_v \sigma_w)_{b,3} - 3(\sigma_v)_{b,3}(\sigma_w)_{b,3} \right) \]
\[ \leq \frac{\beta^2 b}{2 |V_L|M^2} \sum_{v,w \in \mathcal{W}_L,M} \mathbb{E}(\sigma_v; \sigma_w)_{b,3} = \frac{\beta^2 b}{2 |V_L|M^2} \sum_{v,w \in \mathcal{W}_L,M} \mathbb{E}(\sigma_v; \sigma_w)_{b,3} \]
\[ \leq 2 \beta b \frac{\partial}{\partial c} \mathbb{E}(\mu^3_L)_{b,3} \] (48)
The FKG inequality has been used. This bound enables us to evaluate the following integral
\[
\int_{c_1}^{c_2} dc \left| \lim_{L \to \infty} \lim_{M \to \infty} \left[ \mathbb{E}(\rho^3_{1,2})_{b,3} - \mathbb{E}(\rho^3_{1,2})_{0,3} \right] \right|
\]
\[
= \lim_{L \to \infty} \int_{c_1}^{c_2} dc \left| \int_0^b db' \frac{\partial}{\partial b'} \lim_{M \to \infty} \mathbb{E}(\rho^3_{1,2})_{b',3} \right|
\]
\[
\leq \lim_{L \to \infty} 2\beta \int_{c_1}^{c_2} dc \int_0^b db' b' \frac{\partial}{\partial c} \lim_{M \to \infty} \mathbb{E}(\mu^3_{1,2})_{b',3}
\]
\[
= 2\beta \int_0^b db' \lim_{M \to \infty} \lim_{L \to \infty} \left[ \mathbb{E}(\mu^3_L)_{b',3,c=c_2} - \mathbb{E}(\mu^3_L)_{b',3,c=c_1} \right]. \tag{49}
\]

The boundedness of \( \mathbb{E}(\mu^3_{L})_{b,3} \) gives the limit
\[
\int_{c_1}^{c_2} dc \left| \lim_{L \to \infty} \lim_{M \to \infty} \left[ \mathbb{E}(\rho^3_{1,2})_{b,3} - \mathbb{E}(\rho^3_{1,2})_{0,3} \right] \right| = 0 \tag{50}
\]
for arbitrary \( c_1, c_2 \in \mathbb{R} \). Therefore, the integrand in the left hand side vanishes for almost all \( c \), and this implies the first equality (46) for \( i = 3 \).

For \( i = 1 \), evaluate the partial derivative
\[
\left| \frac{\partial}{\partial b} \mathbb{E}(\rho^1_{1,2})_{b,1} \right|
\]
\[
= \left| \frac{\beta^2 b}{64|V_L|^2 M^2} \sum_{x,y\in V_L,s,t\in T_M} \mathbb{E}(\sigma_{x,s}\sigma_{x,s+1}; \sigma_{y,t}\sigma_{y,t+1})_{b,1}(\sigma_{x,s}\sigma_{x,s+1}; \sigma_{y,t}\sigma_{y,t+1})_{b,1}
\]
\[
- 3(\sigma_{x,s}\sigma_{x,s+1})_{b,1}(\sigma_{y,t}\sigma_{y,t+1})_{b,1} + 2) \right|
\]
\[
\leq \frac{3\beta^2 b}{32|V_L|^2 M^2} \sum_{x,y\in V_L,s,t\in T_M} \mathbb{E}(\sigma_{x,s}\sigma_{x,s+1}; \sigma_{y,t}\sigma_{y,t+1})_{b,1}
\]
\[
\leq \frac{3\beta^2 b}{32|V_L|^2 M^2} \sum_{x,y\in V_L,s,t\in T_M} \mathbb{E}(\sigma_{x,s} + \sigma_{x,s+1}; (\sigma_{y,t} + \sigma_{y,t+1}))_{b,1}
\]
\[
\leq \frac{3\beta^2 b}{8|V_L|^2 M^2} \sum_{x,y\in W_{L,M}} \mathbb{E}(\sigma_{x}; \sigma_{y})_{b,1}
\]
\[
\leq \frac{3\beta b}{2} \frac{\partial}{\partial c} \mathbb{E}(\mu^3_L)_{b,1} \tag{51}
\]

The inequality (23) in Lemma 5 has been used. This bound and the same argument as for \( i = 3 \) give the first equality (46) for \( i = 1 \).

To show the second equality (47), the following representation obtained by the FKG inequality is useful
\[
\frac{\partial}{\partial b} \mathbb{E}(\rho^3_{1,2})_{b,3}^2 = \frac{\partial}{\partial b} \mathbb{E}\left( \frac{1}{4|V_L|^2 M} \sum_{w\in W_{L,M}} (\sigma_w)_{b,3}^2 \right)^2
\]
\[
= \frac{\beta^2 b}{16|V_L|^2 M^2} \sum_{x,y,z\in W_{L,M}} \mathbb{E}(\sigma_x; \sigma_z)_{b,3}(\sigma_x)_{b,3}(2\sigma_x; \sigma_z)_{b,3}(\sigma_y)_{b,3}
\]
\[
- 2(\sigma_x)_{b,3}(\sigma_y)_{b,3}(\sigma_z)_{b,3} + (\sigma_x)_{b,3}(\sigma_y; \sigma_z)_{b,3}
\]
\[
\frac{\beta^2 b}{2|V_L|M} \sum_{x,y,z \in W_{L,M}} \mathbb{E}\langle \sigma_x; \sigma_z \rangle_{b,3},
\]

\[
= 2\beta b \frac{\partial}{\partial c} \mathbb{E}(\mu_L^3)_{b,3},
\]

This bound and the boundedness of \( \mathbb{E}(\mu_L^3)_{b,3} \) enable us to prove the second equality (47) as well as the first one (46). The second equality (47) for \( i = 1 \) is proved by showing the bound

\[
\frac{\partial}{\partial b} \mathbb{E}(\rho_{1,2}^1)_{b,1} \leq \frac{5\beta b}{8} \frac{\partial}{\partial c} \mathbb{E}(\mu_L^3)_{b,1}.
\]

This bound and the boundedness of \( \mathbb{E}(\mu_L^3)_{b,1} \) enable us to prove the second equality (47) for \( i = 1 \), and this completes the proof. \( \square \)

**Proof of Theorem 1** Since \( S_x^3 |E \rangle \) is orthogonal to \( |E \rangle \) for an arbitrary eigenstate \( |E \rangle \) of the Hamiltonian, we obtain \( \langle S_x^3 \rangle = 0 \) and \( \langle S_x^3 S_y^3 \rangle = \delta_{x,y}/4. \) These imply

\[
\mathbb{E}(R_{1,2}^2) = 0, \quad \mathbb{E}(R_{1,2}^2) = \frac{1}{16|V_L|},
\]

then Theorem 1 is valid trivially for \( R_{1,2}^2. \) Therefore, we consider \( R_{a,\beta}^i \) for \( i = 1, 3. \) Since \( \lim_{L \to \infty} \lim_{M \to \infty} \mathbb{E}(\rho_{1,2}^i)_{b,i} \) exists by Lemma 9, this limit exists also for \( b = 0 \) by Lemma 11.

First, we use the Ghirlanda–Guerra identities for \( b \neq 0. \)

For \( n = 2 \) and \( f = \rho_{1,2}^i, \) the identity in Lemma 10

\[
\lim_{L \to \infty} \lim_{M \to \infty} \left[ 2\mathbb{E}(\rho_{1,2}^i \rho_{1,3})_{b,i} - (\mathbb{E}(\rho_{1,2}^i)_{b,i})^2 - \mathbb{E}(\rho_{1,2}^i)_{b,i} \right] = 0.
\]

(53)

For \( n = 3 \) and \( f = \rho_{2,3}^i, \) the identity in Lemma 10 gives

\[
\lim_{L \to \infty} \lim_{M \to \infty} \left[ 3\mathbb{E}(\rho_{2,3}^i \rho_{1,4})_{b,i} - (\mathbb{E}(\rho_{1,2}^i)_{b,i})^2 - \mathbb{E}(\rho_{2,3}^i)_{b,i} - \mathbb{E}(\rho_{2,3}^i \rho_{1,3})_{b,i} \right] = 0.
\]

(54)

These two identities and \( \langle \rho_{2,3}^i \rho_{1,4}^j \rangle_{b,i} = \langle \rho_{1,2}^i \rangle_{b,i}^2 \) and \( \langle \rho_{1,2}^i \rho_{1,3}^j \rangle_{b,i} = \langle \rho_{2,3}^i \rho_{1,2}^j \rangle_{b,i} = \langle \rho_{2,3}^i \rho_{1,3}^j \rangle_{b,i} \) in the replica symmetric Gibbs state imply

\[
2 \lim_{L \to \infty} \lim_{M \to \infty} \left[ \mathbb{E}(\rho_{1,2}^i)_{b,i} - (\mathbb{E}(\rho_{1,2}^i)_{b,i})^2 \right] = \lim_{L \to \infty} \lim_{M \to \infty} \left[ \mathbb{E}(\rho_{1,2}^i)_{b,i} - \mathbb{E}(\rho_{1,2}^i)_{b,i} \right]
\]

Since the right hand side vanishes in the above for any \( b \) because of Lemma 6, the left hand side vanishes for almost all \( b \neq 0. \) This fact and Lemma 11 imply that the left hand side vanishes also for \( b = 0. \) Then, (31) yields

\[
\lim_{L \to \infty} \left[ \mathbb{E}(R_{1,2}^i, R_{1,2}^j)_{D} - (\mathbb{E}(R_{1,2}^i))^2 \right]
\]

\[
= \lim_{L \to \infty} \left[ \mathbb{E}(R_{1,2}^i, R_{1,2}^j)_{D} - \mathbb{E}(R_{1,2}^i)^2 + \mathbb{E}(R_{1,2}^i)^2 - (\mathbb{E}(R_{1,2}^i))^2 \right]
\]

\[
= \lim_{L \to \infty} \lim_{M \to \infty} \left[ \mathbb{E}(\rho_{1,2}^i)_{0,i} - (\mathbb{E}(\rho_{1,2}^i)_{0,i})^2 \right] = 0.
\]
Harris’ inequality of the Bogolyubov type between the Duhamel product and the Gibbs expectation of the square of arbitrary self-adjoint operator $O$ \cite{Harris}

\[(O, O)_D \leq \langle O^2 \rangle \leq (O, O)_D + \frac{\beta}{12} \langle [O, [H, O]] \rangle, \tag{55}\]

enables us to obtain

\[\lim_{L \to \infty} \mathbb{E} (R_{1,2}^i, R_{1,2}^i)_D = \lim_{L \to \infty} \mathbb{E} (R_{1,2}^i)^2.\]

Therefore

\[\lim_{L \to \infty} \left[ \mathbb{E} (R_{1,2}^i)^2 - (\mathbb{E} (R_{1,2}^i))^2 \right] = 0. \tag{56}\]

This completes the proof of Theorem 1. \hfill $\square$

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