Abstract

Deconfinement and screening of higher-representation sources in finite-temperature SU(2) lattice gauge theory is investigated by both analytical and numerical means. The effective Polyakov-line action at strong coupling is simulated by an efficient cluster-updating Monte Carlo algorithm for the case of $d = 4$ dimensions. The results compare very favourably with an improved mean-field solution. The limit $d \to \infty$ of the SU(2) theory is shown to be highly singular as far as critical behaviour is concerned. In that limit the leading amplitudes of higher representation Polyakov lines vanish at strong coupling, and subleading exponents become dominant. Each of the higher-representation sources then effectively carry with them their own critical exponents.
For non-Abelian gauge theories one can define a static potential between matter sources transforming as arbitrary irreducible representations of the gauge group, from now on taken to be $SU(N)$. This static potential between two infinitely heavy sources is believed to depend crucially on the manner in which the chosen representation behaves under transformations restricted to the center $Z(N)$ of the gauge group. Representations that are insensitive to $Z(N)$ transformations should yield a screened potential, while those sensitive to these transformations should yield a confining potential. This is the standard picture of confinement and screening in non-Abelian gauge theories, dating back almost twenty years (see, e.g., ref. [1]).

With numerical simulations these ideas can be tested in the context of lattice gauge theories. Indeed, one can go further and investigate in more details the dynamics behind these screening and confining mechanisms. In particular, one can measure the relevant distance scales, those that separate “short distances” (essentially perturbative physics, on account of asymptotic freedom) from “long distances” (a confining string between two irreducible sources transforming non-trivially under $Z(N)$, a screened potential between those transforming trivially). The dynamics of the intermediate region has been found to be, in many respects, quite rich. It has been observed that even representations transforming trivially under the center group could feel a linearly rising potential (with a slope different from the string tension of the fundamental representation) at intermediate distances [2]. At a certain range of distance scales all representations appear to carry with them their own dynamics. In the limit of an infinite number of colours, factorization is sufficient to show that all irreducible representations are confined by a linearly rising potential if the fundamental representation is, with string tensions that depend on the representations [3]. Essentially, the intermediate distance region in which a non-zero string tension exists for all representations grows with $N$, the number of colours, reaching infinity as $N \to \infty$.

These results ought to have some bearing on the physics of finite-temperature gauge theories as well. Indeed, some very simple numerical simulations have shown that just as, for example, adjoint sources may feel a linearly rising potential at intermediate distances, such sources appear to “deconfine” at precisely the same critical temperature $T_c$ at which fundamental sources deconfine [4]. Since, for finite $N$, adjoint sources are not genuinely confined out to arbitrarily large distances this is of course only to be understood at the qualitative level.

For gauge theories with continuous deconfinement phase transitions one obvious question concerns the critical behavior of the Polyakov lines corresponding to sources of arbitrary representations. Again, representations transforming trivially under $Z(N)$ should not be able to serve as order parameters for the phase transitions, while those sensitive to $Z(N)$ should, since it is precisely a global $Z(N)$ symmetry which is broken at the deconfinement phase transition [1]. Universality arguments would, for such continuous phase transitions, place the finite-temperature $SU(N)$ gauge theory in the universality class of globally $Z(N)$ invariant spin systems with short-range interactions [4]. The analogue of the spin operator would be the Polyakov line in the fundamental representation. What about Polyakov lines corresponding to traces taken in higher representations of $SU(N)$? There appears to be no room for independent exponents for these higher representations from the spin-system

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1 Even representations transforming non-trivially under $Z(N)$ could in principle deconfine at temperatures different from the temperature at which the fundamental representation deconfines. This has not been observed, however [3].

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point of view, since there is simply no obvious analogue of the “internal” $SU(N)$ degrees of freedom in the $Z(N)$ fixed-point language. This would indicate that all higher representations sensitive to the center group should be equivalent order parameters with exactly the same critical behavior as the fundamental representation. Physically, this would also be in agreement with the screening argument according to which all higher representations sensitive to $Z(N)$ eventually, for large enough distances, should be colour screened as far down as possible (i.e. down to the charge in the fundamental representation). On the other hand, since strings with representation dependent string tensions do form at some intermediate distance scales for all representations, it is not quite obvious how much of the screening mechanism will be observable at standard present-day lattice sizes used for Monte Carlo simulations. The clear change in behavior of the adjoint Polyakov line at the deconfinement “phase transition” (there is of course no genuine transition in a finite volume) as measured on small lattices is already one indication that there may be difficulties with numerical investigations of this problem. With the same level of statistics (and the same lattice sizes) that are used routinely to confirm the universality arguments based on the fundamental Polyakov line, a surprisingly different behaviour was found for the higher representations of $SU(2)$ lattice gauge theory in (3+1) dimensions. For the continuous deconfinement transitions of $SU(2)$ and $SU(3)$ lattice gauge theories in (2+1) dimensions, see ref. [10, 11]). These numerical simulations indicated that sources of higher representations that were sensitive to $Z(N)$ would correspond to different magnetization exponents, one exponent for each representation. But these results could all be criticized on the grounds that they also seemed to indicate critical behavior for Polyakov lines that simply could not be order parameters for the transition, those of transforming trivially under $Z(N)$. Indeed, in a Monte Carlo study of (3+1)-dimensional $SU(2)$ lattice gauge by Kiskis the expected behavior (adjoint Polyakov line non-vanishing across the transition, the isospin 3/2 Polyakov line behaving like the fundamental) was eventually extracted very close to the finite-volume “critical point” $T_c$. Some of the numerical difficulties involved are discussed in ref. [13].

It is important to realize that all of these issues can be addressed even in the strong-coupling region of the lattice theory. In fact, in this regime the universality arguments are even strengthened (since the effective Polyakov-line interactions can be shown explicitly to be short-ranged), and the question of the critical behavior of higher-representation sources near the phase transition point is as meaningful in the strong-coupling regime as near the continuum limit. The advantage of going to the strong-coupling regime is of course that the question here can be studied in a much simplified setting which still captures all the essentials. The leading-order effective Polyakov-line action reads, with $Tr_1 W$ indicating the trace in the fundamental representation:

$$S_{\text{eff}}[W] = \frac{1}{2} J \sum_{x,j} \{ Tr_1 W(x) Tr_1 W^\dagger(x + j) + Tr_1 W^\dagger(x) Tr_1 W(x + j) \} .$$

Here the sum on $j$ runs over nearest neighbours. The effective coupling $J$ is related to the gauge coupling $g$ and $N_\tau$, the number of time-like links in the compactified temporal direction. To lowest order, for $SU(2)$, it is

$$J(g, N_\tau) = \left( \frac{I_2(4/g^2)}{I_1(4/g^2)} \right)^{N_\tau} ,$$

\[2\] For a recent discussion of the value of the adjoint Polyakov line at the phase transition point, see also ref. [14].
with $I_n$ indicating the $n$th order modified Bessel function. For $SU(2)$ we will use a notation in which $Tr_n W$ means the trace taken in the representation of isospin $n/2$. $Tr_2 W$ is thus the trace in the adjoint representation, etc. Higher orders in the expansion (1) (and corrections to the effective coupling (2)) can be computed in a systematic expansion [17], but we will not need these corrections for the present purpose. The effective action (1) becomes asymptotically exact in the strong coupling limit.

For what follows, it is useful to write the effective Polyakov-line action (1) for $SU(2)$ in terms of a new variable $\Phi(x) \equiv \frac{1}{2} Tr_1 W(x)$. The partition function then takes the following form:

$$Z = \int_{-\infty}^{\infty} \left[ \oplus \right] \exp \left[ \Delta J \sum_{s=1}^{\infty} \oplus(s) \oplus(s + 1) + \sum_{s} \tilde{V}[\oplus^s] \right],$$

(3)

with a local potential $\tilde{V}[\Phi^2] = \frac{1}{2} \ln [1 - \Phi(x)^2]$.

There are two simple limiting cases in which the effective Polyakov-line action (1) can be solved exactly. One is the large-$N$ limit [18], where the deconfinement phase transition turns out to be of first order (in agreement with large-$N$ reduction arguments based directly on the full gauge theory [19], and where universality arguments hence cannot be addressed. The other exactly solvable case is the mean-field limit in which $d \to \infty$, with $d$ being the number of spatial dimensions [8]. In this limit one finds a genuine second-order critical point for the gauge group $SU(2)$ at a critical coupling $J_c \to 0$ as $d \to \infty$. For $J > J_c$ all higher-representation expectation values $\langle Tr_n W \rangle$ are non-zero [8]:

$$\langle Tr_n W \rangle = (n + 1) \frac{I_{n+1}(2a)}{I_1(2a)},$$

(4)

with $a = 2dJ\langle Tr_1 W \rangle$ being the self-consistent mean-field solution for the fundamental representation. Surprisingly, they all display non-trivial critical behavior close to $J_c$:

$$\langle Tr_n W \rangle \sim (J - J_c)^{\beta_n},$$

(5)

where $\beta_n = n/2$. For the fundamental representation this just corresponds to the mean-field Ising magnetization exponent $\beta_1 = \beta = 1/2$, in complete agreement with the universality arguments. For the higher representations this new critical behavior is highly unexpected. In this unphysical but exactly solvable limit all standard screening arguments appear to break down, and we are seeing new behaviour which is not predicted by universality.\[4\]

This $d \to \infty$ result (5) is disturbing on several counts. It is normally assumed that the relevant $Z(2)$ spin system universality class to which the $SU(2)$ finite-$T$ phase transition should belong (if continuous) would display “classical” mean field exponents all the way down from $d = \infty$ to the upper critical dimension $d_u$ (in this case with $d_u = 4$, the critical behaviour being modified by logarithmic corrections just at $d = d_u$). At a first glance this might seem to indicate that the non-trivial behaviour (5) for all representations should

\[3\] Still, the large-$N$ solution does display a number of interesting features such as the simultaneous deconfinement of all higher representations at the transition temperature $T_c$, independently of whether these transformations transform trivially under the center symmetry (in this case $U(1)$) or not. All representations of the Polyakov line are hence equally good order parameters in this special case, and all display a discontinuous jump at the phase transition.

\[4\] One also finds separate exponents $\delta_n = 3/n$ for the behaviour of $\langle Tr_n W \rangle \sim h^{1/\delta_n}$ at $J = J_c$ in a small magnetic field coupled to $Tr_1 W$.\[4\]
remain valid for all \( d > 4 \) (\( d = 4 \) just being the limiting case), with non-trivial critical scaling even for representations of integer isospin, and with new critical exponents for all isospin half-integer representations as well. The first conclusion simply cannot be correct (and we will show explicitly below why the argument is invalid), because at strong coupling one can compute, for example, the adjoint Polyakov line and see that to first non-trivial order in \( 1/g^2 \) it is non-zero. This result should be valid at least up to the phase transition point, if this critical point lies sufficiently deep inside the strong-coupling region \([4]\). What about the odd-\( n \) representations? Could it be that they display new non-trivial critical behaviour of the kind (5) even at \( d = 4 \)? To investigate this question we have first performed a Monte Carlo simulation of the effective Polyakov-line action for \( SU(2) \) in \( d = 4 \) spatial dimensions. Again, the advantages of using directly this effective action instead of the full \( SU(2) \) lattice gauge theory are enormous from a numerical point of view (extracting the critical indices by conventional means is notoriously difficult for lattice gauge theories due to the large lattices and high statistics required), and at the same time the question we are addressing is as urgent in the context of the effective action (1) as in the full \( SU(2) \) gauge theory.

We have simulated the model numerically using a cluster algorithm similar to that proposed in ref. \([\text{20}]\). The algorithm consists of two parts. First there is a single cluster update \([\text{21}]\) of the sign of the field \( \Phi \). The delete probability is given by

\[
p_d = \exp(-4J(\Phi(x)\Phi(x+j) + |\Phi(x)\Phi(x+j)|))
\]

Then, since the cluster update is not ergodic, we have supplemented it with a standard Metropolis update that allows changes in the absolute value of the field \( \Phi(x) \). The ratio of the number of single cluster updates to the number of Metropolis sweeps is a free parameter of the algorithm. We have fixed it by the following rule of thumb: for every Metropolis sweep one performs approximately (lattice volume)/(average cluster size) single cluster updates. The step size of the Metropolis algorithm was chosen such that the acceptance rate was approximately 1/2. Ben-Av et al. \([\text{22}]\) implemented such a combination of the single cluster update and a local heat-bath update for the \( N_{\tau} = 1 \) finite temperature \( SU(2) \) gauge theory in \( 3 + 1 \) dimensions. Critical slowing down was drastically reduced compared to the local update procedure.

We have determined the critical coupling of the model using the fourth order cumulant \( U_1 = 1 - \langle m^4 \rangle / (3 \langle m^2 \rangle^2) \) \([\text{23}]\), where \( m \) is the single lattice average over \( \Phi \), \( i.e. \) the Polyakov line in the fundamental representation. We have first simulated lattices of sizes \( L = 4, 6, 8, 12 \) and 16 at \( 4J_0 = 0.55 \), which was our guess for the critical coupling after some preliminary simulations with low statistics. We have then measured 20,000 times, with one measurement being performed after two Metropolis sweeps plus the corresponding number of single cluster updates. The integrated autocorrelation time of \( m \) was \( \tau_m \approx 1.7 \) in units of measurement steps for all our lattices sizes. We have subsequently computed the fourth order cumulant \( U_1 \) in a neighbourhood of the actual simulation coupling, by reweighting \( \langle m^2 \rangle \) and \( \langle m^4 \rangle \) to the correct Boltzmann weight

\[
\langle m^n \rangle(J) = \frac{\sum_i m^n_i \exp((-J + J_0)\tilde{S})}{\sum_i \exp((-J + J_0)\tilde{S})},
\]

where \( \tilde{S} \) is defined by \( S = JS + \sum_x \tilde{V} \). The resulting curves are plotted in Fig. 1. The crossings of the cumulant provide estimates of the critical coupling \( J_c \). The error should vanish like \( L^{1/\nu} \).
Crossings between lines corresponding to \( L = 4 \) and 6, 6 and 8, 8 and 12, and, finally, between \( L = 12 \) and 16 occur at \( 4J_{\text{cross}} = 0.5507(8), 0.5509(5), 0.5511(3), \) and \( 0.5507(2) \), respectively. We take the crossings of the Binder cumulant for \( L = 12 \) and \( L = 16 \) lattices as our best estimate of the critical coupling, i.e., \( 4J_c = 0.5507(2) \). The limiting value of this crossing is fixed within a given universality class, in this case expected to be the one of the 4-d Ising model. Brezin and Zinn-Justin \[25\] have argued that the effective potential for dimensions \( d \geq 4 \) at the critical point is given by just a \( \phi^4 \) term. It follows that the fourth-order cumulant should take the value \( U = 1 - r_2/3 = 0.27052 \ldots \) with \( r_2 = \Gamma(1/4)^4/(8\pi^2) = 2.1884396 \ldots \) at the critical point. The values we find for the fourth-order cumulant of the the 4-d Ising model on similar and larger lattices that are consistent with our numerical result. Also Binder et al. \[26\] estimate a value of the critical cumulant of the 5-d Ising model that is consistent with the above numbers. In their theoretical discussion Binder et al. allow for a finite mass term in the effective potential at the critical point.

Other universal quantities can be extracted from the fourth-order Binder cumulant. Its derivative should scale like \( dU/dJ \sim L^{1/\nu} \) at the critical coupling. A fit of the data according to this equation leads to \( \nu = 0.490(6) \), which is consistent within two standard deviations with the 4-d Ising value \( \nu = 0.5 \).

Next, consider the fourth-order cumulant \( U_n \) for higher representations. Fig. 2 shows the results for \( n = 2 \). With increasing lattice sizes the cumulant converges toward 2/3 for couplings both below and above the critical point. The value \( U = 2/3 \) signals a finite expectation value of the observable. We hence clearly see that the Polyakov line in the adjoint representation is not an order parameter. It stays finite in both phases. The fourth-order cumulant for the \( n = 3 \) representation takes values close to 2/3 in the broken phase and values close to zero in the high temperature phase; it behaves as an order parameter. But the curves do not display crossings close to the critical coupling predicted from the cumulant for the fundamental representation. The curve for \( L = 16 \) comes close to that of the fundamental representation, so we might expect that for still larger lattices the cumulant for the \( n = 3 \) representation converges towards the fundamental ones, and that the crossings can then be observed at the critical coupling \( J_c \). The data for the higher representation were so affected by errors that no reliable results for the cumulants could be extracted.

It is instructive to look at the finite-size behaviour of the Polyakov lines of different representations at the critical point. Values for \( n = 1, 2, 3, 4 \) and 5 are given in table 1. The numbers are obtained from reweighting the 4\( J_0 = 0.55 \) simulations. Asymptotically the value of the odd representations should scale down with increasing lattice size like \( \langle Tr_n W \rangle \sim L^{-\beta/\nu} = L^{-1} \), while the values for the even representations should converge to a finite value. It appears that the values for the even representations indeed stabilize at a (very small) finite value. The behaviour of the magnetization of the odd representations are best visualized by looking at \( \langle Tr_n W \rangle L \), which in the asymptotic scaling regime should give a constant. For \( n = 1 \) this behaviour is nicely seen in the data, for \( n = 3 \) there appears to be a stabilization for the largest lattice sizes, while for \( n = 5 \) no sign of stabilization is visible on the lattices we have considered. It thus appears that the higher the representation, the larger lattices are required to see the correct infinite-volume behaviour. Physically, this also
makes sense in the light of screening considerations. The higher the representation, the more screening is required to reproduce the behaviour of the fundamental representation, and the larger distances one needs to probe in order to see this.

We have also simulated the model for various $J > J_c$ on lattices of sizes up to $L = 16$. Here we have typically performed 10,000 measurements per simulation, the aim being an approximate determination of the critical exponents $\beta_n$ directly from the Monte Carlo measurements of the different representations. We have not attempted to fit our data to an ansatz including corrections to scaling, but have instead defined a $J$-dependent “effective” exponent $\beta_n^{\text{eff}}$ by

$$\beta_n^{\text{eff}} = (J - J_c) \frac{d \langle Tr_n W \rangle / dJ}{\langle Tr_n W \rangle}.$$  

The derivative of $\langle Tr_n W \rangle$ with respect to $J$ was computed from the relation

$$\frac{d}{dJ} \langle Tr_n W \rangle = \langle (Tr_n W) \cdot \tilde{S} \rangle - \langle Tr_n W \rangle \langle \tilde{S} \rangle.$$  

The final results are presented in fig. 4 for the odd representations, and in fig. 5 for the even ones. We have carefully checked the dependence of the result on the lattice size and included only values which were consistent on the two largest lattice sizes considered. It turn out that for the coupling close to $J_c$ even the $L = 16$ lattice was not sufficient to give a stable result for the $n = 5$ representation.\footnote{The curves plotted in these two figures are improved mean-field predictions. See below.}

Figs. 4 and 5 demonstrate fairly convincingly that the odd representations converge toward an effective $\beta_n^{\text{eff}} = 1/2$ independent of $n$, while the even representations converge toward $\beta_n^{\text{eff}} = 0$ (as expected if these representations remain finite at $J_c$). But the plots also reveal an interesting phenomenon for larger values of $(J - J_c)/J_c$: the effective $J$-dependent exponents $\beta_n^{\text{eff}}$ quickly reach a regime of couplings where they are essentially equally spaced, growing linearly with $n$. Although they never actually reach the mean-field prediction (5), they get quite close, and they certainly obey the rule $\beta_n^{\text{eff}} \sim n \cdot \beta_1^{\text{eff}}$ to surprisingly high accuracy. This is just as for the original observations in the full $(3+1)$-dimensional $SU(2)$ gauge theory \[8\]. It appears that this approximate linear relation between the $\beta_n$’s, when measured not too close to the critical point, can be viewed as the “remnant” of the $d = \infty$ solution. It is then only very close to the critical point the behaviour changes, and the single critical exponent $\beta$ emerges for the odd representations, while the even representations run smoothly across the transition point. We can estimate this narrow window in the original gauge coupling $4/g^2$ by using the relation (2). In the case of $N_r = 2$ the transition occurs at $4/g_2^2 = 1.6424(4)$. In order to obtain $\beta^{\text{eff}} < 0.625$ (i.e. 25% above the correct value $\beta = 0.5$) for the $n=3$ representation we would have to take $4/g^2 < 1.66$.

While these results may have clarified the situation in the $d=4$ theory, we are still left with the surprising $d=\infty$ results where mean field theory is believed to be exact. How can they be explained? Consider the representation of the effective Polyakov-line action given in eq. (3). This is a $Z(2)$-invariant effective scalar field theory in $d$ dimensions, as expected on general grounds. But it is a very particular effective scalar theory, one that embodies the underlying $SU(2)$ structure (in the restrictions on the integration interval of $\Phi (x)$, and in the very special form of the local potential $V[\Phi^2]$, which reflects the Haar measure for $SU(2)$).
Since we at this point wish to focus on the \( d = \infty \) results, we can restrict ourselves to “classical” mean-field considerations. It is instructive \([10]\) to generalize the partition function above to an arbitrary local potential \( V[\Phi^2] \) and relax the limitation on the integrations over \( \Phi(x) \) to be in the interval \([-1, 1] \). The \( d = \infty \) solution is then found by considering the single-site partition function

\[
Z_{SS} = \int_{-\infty}^{\infty} \left[ \sum \exp \left[ \sum + V[\Phi] \right] \right],
\]

where \( v = 4dJ\langle \Phi \rangle \) will be determined by the self-consistency solution. Clearly, for \( n \) being any non-negative integer, \( \langle \Phi^{2n+1} \rangle = 0 \) unless the global \( Z(2) \) symmetry is spontaneously broken. Call the critical coupling at which this occurs \( J_c \). If the phase transition is continuous, \( \langle \Phi \rangle \) will be small just above \( J_c \), and it is meaningful to expand in \( v \) (no matter how large \( d \) is taken, once fixed). The result is, for the expectation values of the first two non-trivial mean-field moments of \( \Phi \) \([10]\):

\[
\langle \Phi^2 \rangle = \langle \Phi^2 \rangle_0 + \frac{1}{2} \left[ \langle \Phi^4 \rangle_0 - \left( \langle \Phi^2 \rangle_0 \right)^2 \right] v^2 + \ldots
\]

\[
\langle \Phi^3 \rangle = \langle \Phi^4 \rangle_0 v + \frac{1}{2} \left[ \frac{1}{3} \langle \Phi^6 \rangle_0 - \langle \Phi^2 \rangle_0 \langle \Phi^4 \rangle_0 \right] v^3 + \ldots,
\]

where the subscript “0” indicates the (constant) expectation value in the unbroken phase \( J < J_c \). Higher moments can be worked out analogously, by expanding both the partition function \( Z_{SS} \) and the unweighted averages in powers of \( v \). Using the recursion relation \( \chi_{n+1} = \chi_n \chi_1 - \chi_{n-1} \) for \( SU(2) \) characters, we find the general \( d = \infty \) predictions \([10]\):

\[
\langle Tr_2 W \rangle = \left[ 4 \langle \Phi^2 \rangle_0 - 1 \right] + 2 \left[ \langle \Phi^4 \rangle_0 - \left( \langle \Phi^2 \rangle_0 \right)^2 \right] v^2 + \ldots
\]

\[
= A_2 + B_2 v^2 + \ldots
\]

\[
\langle Tr_3 W \rangle = \left[ 8 \langle \Phi^4 \rangle_0 - 4 \langle \Phi^2 \rangle_0 \right] v + 4 \left[ \frac{1}{3} \langle \Phi^6 \rangle_0 - \langle \Phi^2 \rangle_0 \langle \Phi^4 \rangle_0 + \frac{1}{2} \langle \Phi^2 \rangle_0 \right] v^3 + \ldots
\]

\[
= A_3 v + B_3 v^3 + \ldots,
\]

where \( A_2, B_2, A_3 \) and \( B_3 \) are (non-universal) constants. This shows the behaviour expected from universality arguments. The adjoint Polyakov line will remain non-vanishing across the phase transition at \( J_c \) (and is hence not an order parameter), and the isospin-3/2 representation scales near \( J_c \) as \( v \), i.e., as the fundamental representation. But if we take the particular potential \( \tilde{V}[\Phi^2] \) of eq. (3), and restrict the integration over \( \Phi \) to the interval \([-1, 1] \), then devious cancellations occur. One finds \( \langle \Phi^2 \rangle_0 = 1/4 \) and \( \langle \Phi^4 \rangle_0 = 1/8 \), leading to

\[
\langle Tr_2 W \rangle = \frac{1}{8} v^2 + \ldots = 2d^2 J^2 \langle \Phi \rangle^2 + \ldots
\]

\[
\langle Tr_3 W \rangle = \left[ \frac{4}{3} \langle \Phi^6 \rangle_0 + \frac{3}{8} \right] v^3 + \ldots.
\]

It is thus suddenly the non-leading terms in the general expansion of the Polyakov lines that become important, due to the amplitudes of the leading terms vanishing in this limit. The cancellations required for this phenomenon are actually simple consequences of the orthogonality relations for \( SU(2) \) characters, as follows if one performs the mean field calculation directly in \( SU(2) \) language \([3]\). They occur similarly for all higher representations, leading, of course, eventually to the general \( d=\infty \) solution (5).
It is interesting to compare this result with a general renormalization-group analysis by Kiskis [12]. Not being restricted to \( d = \infty \), the results of Kiskis can be summarized by

\[
\langle Tr_2 W \rangle = a_2 + b_2 t^{1-\alpha} + \ldots \\
\langle Tr_3 W \rangle = a_3 t^\beta + b_3 t^{1+\beta} + \ldots ,
\]

where \( \alpha \) is the usual specific heat critical exponent, and \( t \) is the reduced temperature near the critical point at \( T_c \). In the \( d = \infty \) limit, these are precisely of the form (12) above provided we make the identifications

\[
1 - \alpha = 2\beta , \quad 1 + \beta = 3\beta
\]

(assuming that the coefficients \( B_2, B_3, b_2 \) and \( b_3 \) are all non-zero). Solving these equations, we find \( \beta = 1/2 \) and \( \alpha = 0 \), the \( d = \infty \) Ising model exponents. With the above qualification, the identities (15) appear to be new scaling relations for the \( Z(2) \) fixed point at \( d = \infty \), imposed by the combined restrictions of mean field theory and the renormalization group.

We are now in a better position to understand the \( d = \infty \) results. As shown above, the appearance of new exponents for each of the odd-\( n \) representations in the limit \( d = \infty \) is due to very delicate cancellations that make the amplitudes of the leading terms in the expansion close to the critical point vanish. Although the same mechanism is responsible for the fact that also even-\( n \) representations display non-trivial critical behaviour in the \( d = \infty \) theory, that phenomenon is of course far more difficult to understand from the point of view of physics. The even-\( n \) Polyakov-line representations simply ought not to be order parameters for the deconfinement transition, even in the \( d = \infty \) limit, since such sources should be screened both above and below the critical point. The resolution of this apparent paradox lies in the fact that the critical coupling \( J_c \) actually vanishes (like \( 1/d \)) when \( d \to \infty \), as follows directly from the mean-field solution (4).

In terms of the gauge coupling \( g \) this entails, for fixed \( N \), \( g \to \infty \). Although this makes the strong-coupling effective Lagrangian analysis more and more accurate, it also pushes the confinement/deconfinement phase transition right to the extreme limit \( g = \infty \) where all sources are “confined” (\( \langle Tr_n W \rangle = 0 \) for all \( n \) at \( g = \infty \) in the full gauge theory simply as a consequence of the orthogonality property of the group characters). It is for this simple reason that the mean-field solution, correctly, predicts critical behaviour for all representations of \( SU(2) \).

The limit \( d = \infty \) of finite-temperature gauge theories is thus in many respects highly singular. This, together with the Monte Carlo data presented above for the \( d = 4 \) \( SU(2) \) theory, indicates that the usual assumption of \( d = \infty \) exponents being valid down to the upper critical dimension \( d_u \) simply fails in this case. Can we understand the singular nature of the \( d = \infty \) limit in an analytical way? As explained above, there are actually no reasons to doubt that mean field theory predicts the \( d = \infty \) behaviour correctly. The only resolution would then be that any finite dimensionality \( d \) should correspond to radically different behaviour close to the critical point, \( i.e., \) that \( 1/d \)-corrections discontinuously should alter the critical indices. To see whether this is the case, we have considered a slightly improved mean-field solution of the same effective Polyakov-line action (1). (This improvement appears to be equivalent to what is known as the Bethe-approximation, see, \( e.g., \) ref. [27].) Consider a system that consists of \( 2d+1 \) sites, a central site (C) and its \( 2d \) nearest neighbours (O).

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\(^6\)This behaviour is not an artifact of the mean-field solution. It can be checked to hold as well in the exact solution of the \( N = \infty \) theory [18].
The remaining $2d-1$ sites of the O-sites are replaced by an external field $W$, which here is the analogue of the conventional mean field. Hence the partition function of this system is given by

$$Z = \int_{-1}^{+1} d\Phi_C \sqrt{1 - \Phi_C^2} \prod_O \int_{-1}^{+1} d\Phi_O \sqrt{1 - \Phi_O^2} \exp(4J(\Phi_C + W)\Phi_O)$$

The integration over the $\phi_O$ fields leads to

$$Z \propto \int_{-1}^{+1} d\Phi_C \sqrt{1 - \Phi_C^2} \left[ I_1(4J(\Phi_C + W)) \right]^{2d}$$

The remaining one-dimensional integration we have performed numerically. In order to fix the external field $W$ we require that the magnetization of the fundamental representation is equal for the central site (C) and its neighbours (O). We solved this condition numerically using the Newton method. Expectation values are evaluated on the central site.

One can readily check that this improved mean field theory coincides with the conventional mean field theory in the limit $d \to \infty$. It is, however, expected to be more accurate for finite values of $d$, especially for non-universal quantities. We have solved the self-consistency equations for $d = 4, 8, 16$ and 32. The value found for the critical coupling $J_c = 0.5352\ldots$ in 4-d deviates from the Monte Carlo value by only 2.8%, while standard mean field theory is off by 9.2%. But a more striking consequence of the improvement is seen in the behaviour of the even-$n$ representations, which are now non-vanishing for all values of the coupling $J \neq 0$. The numerical values at or below $J_c$, however, decrease rapidly with $d$. For the adjoint representation it is reduced by a factor of approximately 2 at $J_c$ when one doubles the dimension, while for the $n = 4$ representation it drops by almost a factor of 4. In this fashion the present solution matches the usual mean-field results in the limit $d = \infty$.

Since such an apparently minor improvement of the mean-field method produces this drastic change in behaviour for the even-$n$ representations, it is worthwhile to understand it better. In fact, almost any improvement of the $d = \infty$ mean-field results are bound to give qualitatively the right behaviour for the even-$n$ representations. We will here show that, except for the mean-field limit $d = \infty$, the expectation values of Polyakov lines in even-$n$ representations will always be positive for all finite $d$ and $J$. Note that we can rewrite the expectation value $\langle Tr_n W \rangle$ as

$$\langle Tr_n W \rangle = \int dSP(S)M_n(S)$$

where $M_n(S)$ is the conditional expectation value of $Tr_n W$ at a site C for fixed sum $S$ of the fields $\Phi_O$ at the neighboring sites O. $P(S)$ is the probability distribution of the sum of the fields $\Phi_O$. $M_n(S)$ is given by

$$M_n(S) = \frac{\int_{-1}^{+1} d\Phi_C \exp\left[ 4JS\Phi_C + \hat{V}[\Phi_C^2] \right] Tr_n W}{\int_{-1}^{+1} d\Phi_C \exp\left[ 4JS\Phi_C + \hat{V}[\Phi_C^2] \right]}$$

We have already computed $M_n(S)$, since it is of precisely the same form as the mean-field solution of the model: we just have to take $a = 4JS$ as argument in eq. (5). For $n$ even $M_n(S)$ is a even function, since odd (modified) Bessel-functions are odd functions. The only real zero of the function is given at $S = 0$. For all other real arguments $M_n(S)$ is strictly positive.
Next consider the conditional probability distributions $p(\Phi_{NO}, S)$, where $\Phi_{NO}$ are the fields on the neighbouring sites of the $O$ sites. Note that the computation of $p(\Phi_{NO}, S)$ requires only the integration of the $\Phi_O$ fields. One reads from the definition that $p(\Phi_{NO}, S) > 0$ for $|S| < 2d$ for any configuration of the $\Phi_{NO}$ with $|\Phi_{NO}| \leq 1$. Hence we also have $P(S) > 0$ for $|S| < 2d$.

Taking the properties of $M_n(S)$ with $n$ even and $P(S)$ as discussed above $\langle Tr_n W \rangle$ for $n$ even has to be strictly larger than zero. Mean-field theory gives zero for $J \leq J_c$ since $P(S)$ is replaced by $\delta(S)$. Any improvement in the mean-field solution that provides a smooth function for $P(S)$ will remove the mean-field pathologies that lead to vanishing expectation values for the even-$n$ representations at and below $J_c$. Of course, in the limiting case $d=\infty$, the exact $P(S)$ is a $\delta$-function at $J=J_c$, and the naive mean-field results are exact.

With the improved mean field theory we can finally make a much more accurate comparison with our $d=4$ Monte Carlo results. In figs. 4 and 5 we have thus plotted (as smooth curves) the corresponding predictions for the $J$-dependent effective exponents $\beta_{n}^{eff}$ defined as in eq. (8). Qualitatively the behaviour of the Monte Carlo data is quite well reproduced. Since it is clear that the conventional mean field results must be reproduced as $d \to \infty$, it is interesting to see what happens with these $\beta_{n}^{eff}$-exponents as $d$ is increased. In fig. 6 we show for $n=2$ how the lowest-order mean-field behaviour is recovered as $d$ grows. One sees that the window close to $J_c$ where $\beta_{2}^{eff}$ turns over and starts deviating from the $d=\infty$ result $\beta_{2} = 1$ gets more and more narrow as $d$ is increased. This behaviour is not restricted to the adjoint representation. In fig. 7 we show the results for all $\beta_{n}^{eff}$ up to $n=5$ in $d=32$ dimensions. The linear spacing of effective exponents is seen throughout, except for an extremely narrow interval close to $J_c$.

Clearly, as $d \to \infty$ the window in which the conventional results are reproduced shrinks to zero. In fact, one can easily estimate from the improved mean-field solution (16) that this window decreases in size as $1/d$, eventually disappearing at $d=\infty$. In the more conventional language, this is the point at which the amplitudes of the leading terms in the expansion for the Polyakov lines vanish.

Acknowledgment: One of us (PHD) would like to thank Jeff Greensite for helpful discussions. We would like to thank Hans Gerd Evertz for drawing our attention to ref. [22].

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Table 1: Magnetizations at the critical coupling $4J_c = 0.5507$ for various representations. The number in the first bracket gives the statistical error at the given coupling, while the number in the second gives the uncertainty due to the error of the critical coupling.

| $L$ | $\langle Tr_1 W \rangle$ | $\langle Tr_2 W \rangle$ | $\langle Tr_3 W \rangle$ | $\langle Tr_4 W \rangle$ | $\langle Tr_5 W \rangle$ |
|-----|-----------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 4   | 0.354(2)(1)                 | 0.1605(14)(6)               | 0.0659(4)(2)                | 0.0101(6)(1)                | 0.0543(3)(0)                |
| 6   | 0.249(2)(2)                 | 0.1348(7)(6)                | 0.0344(3)(2)                | 0.00802(30)(6)              | 0.02369(14)(1)              |
| 8   | 0.194(2)(3)                 | 0.1255(4)(7)                | 0.0230(2)(3)                | 0.00693(14)(6)              | 0.01321(7)(1)               |
| 12  | 0.129(1)(5)                 | 0.1177(2)(6)                | 0.0136(1)(4)                | 0.00592(8)(6)               | 0.00593(3)(1)               |
| 16  | 0.100(1)(7)                 | 0.1157(1)(7)                | 0.0102(1)(6)                | 0.00592(4)(6)               | 0.00333(2)(1)               |

Table 2: Improved mean field theory: $J_c$, and $m_j$ evaluated at $J_c$.

| $d$ | $4J_c$               | $\langle Tr_2 W \rangle$ | $\langle Tr_4 W \rangle$ |
|-----|----------------------|-----------------------------|-----------------------------|
| 4   | 0.5352319055         | 0.07382995665               | 0.00254915596               |
| 8   | 0.2582840308         | 0.03387515014               | 0.00055551163               |
| 16  | 0.1270085358         | 0.01625711255               | 0.00013006341               |
| 32  | 0.0629953883         | 0.00796783664               | 0.00003149409               |

Figure Captions

1.) The fourth-order Binder cumulant for the fundamental representation. The crossing determines our best estimate of $J_c$.
2.) Same as fig.1, but for the adjoint source. The convergence toward $U_2 = 2/3$ indicates that the expectation value of the Polyakov line in the adjoint representation is non-zero throughout.
3.) The fourth-order Binder cumulant for the $n = 3$ source. Although the behaviour is radically different from that of the adjoint source, it still does not show the pattern of fig.1 for the smaller lattices.
4.) The effective magnetization exponents (8) for odd-$n$. The $n=1$ and $n=3$ representations nicely appear to converge toward the Ising value of $\beta = 0.5$, while the $n=5$ representation only shows the same trend. The drawn curves refer to the improved mean-field solution discussed at the end of the paper.
5.) Same as fig. 4, but for the even-$n$ representations. The effective exponent appears to converge toward 0, as expected if these magnetizations remain non-zero at the critical point.
6.) The way the mean-field solution $\beta_2 = 1.0$ is recovered in the limit $d \to \infty$. The region in $(J-J_c)/J_c$ where $\beta_2^{eff}$ eventually turns to zero shrinks as $d$ grows, disappearing in the limit $d \to \infty$.
7.) The linear spacing of $\beta_n^{eff}$, here for $d=32$. 
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9404008v2
This figure "fig2-1.png" is available in "png" format from:

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Fig. 1
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Fig. 2
This figure "fig1-3.png" is available in "png" format from:

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This figure "fig1-4.png" is available in "png" format from:

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Fig. 4

The figure shows a plot of \( \beta_{\text{eff}} \) against \( (J - J_C)/J_C \). There are curves for different values of \( n \) indicated as follows:

- \( \diamond \) for \( n = 1 \)
- \( \circ \) for \( n = 3 \)
- \( \square \) for \( n = 5 \)

The data points are shown with error bars, and the curves represent the theoretical predictions.
This figure "fig1-5.png" is available in "png" format from:

http://arxiv.org/ps/hep-lat/9404008v2
Fig. 5
Fig. 6
n = 2

\[ \beta_{\text{eff}} \]

\[ \frac{(J - J_C)}{J_C} \]

- \( d = 32 \)
- \( d = 16 \)
- \( d = 8 \)
- \( d = 4 \)
Fig. 7
d = 32

\[ \frac{\beta_{\text{eff}}}{J} = \frac{J - J_C}{J_C} \]

- \( n = 1 \)
- \( n = 2 \)
- \( n = 3 \)
- \( n = 4 \)
- \( n = 5 \)

\[ (J - J_C)/J_C \]