Approximation of A Class of Non-Zero-Sum Investment and Reinsurance Games for Regime-Switching Jump-Diffusion Models

Trang Bui,* Xiang Cheng,† Zhuo Jin,‡ George Yin§

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Abstract

This work develops an approximation procedure for a class of non-zero-sum stochastic differential investment and reinsurance games between two insurance companies. Both proportional reinsurance and excess-of-loss reinsurance policies are considered. We develop numerical algorithms to obtain the Nash equilibrium by adopting the Markov chain approximation methodology and applying the dynamical programming principle for the nonlinear integro-differential Hamilton-Jacobi-Isaacs (HJI) equations. Furthermore, we establish the convergence of the approximation sequences and the approximation to the value functions. Numerical examples are presented to illustrate the applicability of the algorithms.

Key Words. Stochastic control, non-zero-sum game, investment, reinsurance, regime switching.

*Department of Mathematics, Wayne State University, Detroit, MI 48202. Email: trang.bui@wayne.edu. The research was supported in part by the Air Force Office of Scientific Research under Grant FA9550-18-1-0268.
†Centre for Actuarial Studies, Department of Economics, The University of Melbourne, VIC 3010, Australia, cheng.x@unimelb.edu.au.
‡Centre for Actuarial Studies, Department of Economics, The University of Melbourne, VIC 3010, Australia, zjin@unimelb.edu.au.
§Department of Mathematics, Wayne State University, Detroit, MI 48202. Email: gyin@math.wayne.edu. The research was supported in part by the Air Force Office of Scientific Research under Grant FA9550-18-1-0268.
1 Introduction

Insurers tend to accumulate relatively large amounts of cash or cash equivalents through the written insurance portfolio. Investing the surplus in a financial market in order to pay future claims and to avoid financial ruin becomes a natural choice. In terms of financial performance the investment income allows significant pricing flexibility in underwriting to the insurers. The surplus is allowed to be invested in a financial market in continuous time.

On the other hand, reinsurance has been considered as an effective risk management tool for insurance companies to transfer their risk exposure to another commercial institution. The primary insurer pays the reinsurer a certain portion of the premiums. In return, the reinsurer is obliged to share the risk of large claims with the primary insurer. Proportional reinsurance and excess-of-loss reinsurance are two major types of reinsurance strategies. With proportional reinsurance, the reinsurance company covers a fixed percentage of losses. The fraction of risk shared by the reinsurance company is determined when the reinsurance contract is sold. The other type of reinsurance policy is nonproportional reinsurance. The most common nonproportional reinsurance policy is excess-of-loss reinsurance, where the primary insurance carrier (called cedent) will pay all of the claims up to a predetermined amount (termed retention level).

The optimal risk controls for an insurance corporation has been studied extensively since the classical collective risk model was introduced by Lundberg (1903). The insurance companies can reduce or eliminate the risk of loss by involving in a reinsurance program and reinvesting in the stock market. Particularly, advanced stochastic control theory and dynamic programming principle are widely used to design the optimal reinsurance and investment strategies of an insurance company in previous works. Asmussen et al. (2000) used diffusion approximation to find an optimal policy balancing the risk and expected profits for a financial corporation with excess-of-loss reinsurance. Choulli et al. (2001) investigated the case of excess-of-loss reinsurance for an insurance company facing a constant liability payment to maximize the expected present value of total future dividend pay-outs. Hald and Schimidli (2004) studied the optimal proportional reinsurance policy to maximize the adjustment coefficient of the ruin probability with the variance premium principle; see also Browne (1995), Chen et al. (2010), Zhang and Siu (2012), and Zhang et al. (2016).

Recently, the extension of optimal investment and reinsurance problem (Nash equilibrium) in the context of stochastic differential games including zero-sum games and non-zero-sum games has been developed rapidly; see the existence of the Nash equilibrium of non-zero-sum stochastic differential game with N players over infinite time horizon in Bensoussan and Frehse (2000). The existence of the Nash equilibrium of a non-zero-sum stochastic differential game between two insurance companies in Zeng (2010), Liu and Yiu (2013) studied a zero-sum stochastic differential reinsurance and investment game between two competing insurance companies under VaR constraints for the purpose of risk management. Bensoussan et al. (2014) investigated a class of non-zero sum stochastic differential game between two insurers by using the objectives of relative performance and obtained explicit solutions for optimal reinsurance and investment strategies. Chen and Shen (2017) formulated a stochastic Stackelberg differential reinsurance game between an insurer and a reinsurer, allowing them to consider the benefits of both parties in the reinsurance contracts. Under the criteria of maximizing the expected utilities of the players’ terminal surpluses, the Stackelberg equilibrium strategies and value functions are obtained by using the backward stochastic differential equation (BSDE) approach; see also Taksar and Zeng (2011), Pun and Wong (2016), and Yan et al. (2017).

Furthermore, people have recently realized that stochastic hybrid models have advantages to capture discrete movements (such as random environment, market trends, interest rates, business cycles, etc.) in the insurance market. For example, we can consider an insurance market with
two modes to represent the dynamic insurance cycle. Market mode 1 represents a “soft” market, where the investment return is high and the premium rate is low. While market mode 2 represents a “hard” market, where the investment return is low and the premium rate is high. In different markets, insurance companies adopt completely different strategies in investment and policy sales. The insurance company is more likely to expand its business and write more policies when market is in mode 1. The potential loss from the difference between premium income and claims are compensated by the high investment returns. The hybrid systems enable the consideration of the coexistence of continuous dynamics and discrete events in the systems. To reflect the hybrid feature, one of the recent trends is to use a finite state Markov process to describe the transitions among different regimes. The Markov-modulated switching systems are known as regime-switching systems. The formulation of regime-switching models is a more general and versatile framework to describe the complicated financial markets and their inherent uncertainty and randomness. Because the control strategies are affected by the asset prices on the stock market and economic trends change quickly, Markovian regime-switching processes were introduced widely to capture movements of random environment. In Wei et al. (2010), the optimal dividend and proportional reinsurance strategy under utility criteria are studied for the regime-switching compound Poisson model. Sotomayor and Cadenillas (2011) studies the optimal dividend problem in the regime-switching model when the dividend rates are bounded, unbounded, and when there are fixed costs and taxes corresponding to the dividend payments. Zhu (2014) studied the dividend optimization for a regime-switching diffusion model with restricted dividend rates. A comprehensive study of switching diffusions with “state-dependent” switching is in Yin and Zhu (2010). Bensoussan et al. (2014) provided closed-form Nash equilibrium for a mixed regime-switching Cramér-Lundberg diffusion approximation process; see also related works Bensoussan et al. (2012) and Bensoussan et al. (2017) for regime-switching models for real options and real options with competition. Jin et al. (2013) designed numerical methods for a zero-sum stochastic differential reinsurance game with regime-switching.

In this work, we are concerned with an insurance market including two insurance companies. The two competing insurance companies adopt optimal investment and reinsurance strategies to manage the insurance portfolios. The surplus process of each insurance company is subject to the randomness of the market. Following the work of Bensoussan et al. (2014), the randomness of the market is modelled by a continuous-time finite-state Markov chain and an independent market-index process. Nevertheless, we model the surplus process as a regime-switching jump-diffusion process, in lieu of a mixed regime-switching Cramér-Lundberg diffusion approximation process. This allows us to work with both proportional and excess-of-loss reinsurance policies. Equilibrium strategies are studied by solving a system of HJI (Hamilton-Jacobi-Isaacs) equations for the value functions of various players, derived from the principle of dynamic programming. Owing to the inclusion of the random switching environment and jump processes, the system of HJI equations becomes more complicated and closed-form solutions are virtually impossible to obtain. Thus, we adopt the Markov chain approximation method (MCAM) developed in Kushner and Dupuis (2001) to deal with a system of HJI (HJI) equations arising from the associated game problems. The convergence of the approximation sequence to the jump process and the convergence of the value function will be established. In the actual computation, we will use our approximation schemes for constant absolute risk aversion (CARA) insurers.

As far as the significance of the contributions is concerned, this paper reveals clearly the advantage in the following aspects. First, the problem of finding the optimal investment and reinsurance strategy between two insurance companies under different insurance market models involves a stochastic hybrid system. Closed-form solutions can only be obtained in special cases in Bensoussan et al. (2014) in a diffusion approximation model. Comparing with work in
Bensoussan et al. (2014), a more versatile jump-diffusion regime-switching non-zero-sum game model is formulated. Jin et al. (2013) only studies a zero-sum reinsurance game, where the state process is degenerated to one dimension. While in the non-zero-sum game of the current work, a high-dimension system of HJI equations is obtained, which adds much difficulty to construct the approximating Markov chain. Second, the aforementioned literature mainly considers proportional reinsurance strategies in the reinsurance games. Pun and Wong (2016) derives the Nash equilibrium of the robust non-zero-sum stochastic differential reinsurance game where the reinsurance strategies are proportional. Due to the unboundedness of the excess-of-loss strategies, it is very hard to develop the Nash equilibrium strategies for an reinsurance game. In addition to the investment, finding the explicit optimal game strategies are virtually impossible. To the best of our knowledge, this is the first paper to study the non-zero-sum investment and reinsurance games in a jump-diffusion regime-switching with nonproportional reinsurance strategies. Third, the reinsurance premium principle, which is a generalized nonlinear functional in our model, adds further difficulties to design the game strategies. Meng et al. (2013) and Meng et al. (2013) investigate the nonlinear premium principles in a simplified reinsurance game formulation. Our models considers the investment and reinsurance game in more complex formulation and presents the impact of the nonlinearity of the premium principle on investment and reinsurance strategies. Hence, the constructed numerical algorithm shows its advantage in finding the game strategies under our complex stochastic hybrid system.

The rest of the paper is organized as follows. A generalized formulation for surplus processes and the associated control problem are presented in Section 2. We design the numerical algorithm based on MCAM in Section 3. A directly constructed approximating Markov chain is constructed and is proved to be locally consistent with the original processes. Section 4 deals with the convergence of the approximated process and the value functions. Numerical examples are reported in Section 5 to illustrate the performance of the method. Section 6 concludes the paper with a few more remarks.

2 Formulation

Let $(Ω, F, F_t, P)$ be a complete filtered probability space, where the filtration $\{F_t\}_{t \geq 0}$ satisfies the usual condition and $F_t = F_t^\alpha \vee F_t^W$. We work with a finite horizon $[0, T]$, where $T < \infty$ is a positive real number. The processes $\{α(t)\}_{t \geq 0}$ and $\{W(t)\}_{t \geq 0} = \{W_Z(t), W_S(t)\}_{t \geq 0}$ are defined below. We use $K$ as a generic constant throughout the paper, whose value may change for different appearances.

2.1 Insurance Models

We are considering an insurance market including two competing insurance companies. Each of the two insurance companies adopts optimal investment and reinsurance strategies to manage the insurance portfolios. The surplus process of each insurance company is subject to the random fluctuation of the market. Following the work of Bensoussan et al. (2014), the randomness of the market is modelled by a continuous-time finite-state Markov chain and an independent market-index process.

To delineate the random economy environment and other random economic factors, we use a continuous-time Markov chain $α(t)$ taking values in a finite space $M = \{1, \ldots, m\}$. The states of economy are represented by the Markov chain $α(t)$. Let the continuous-time Markov chain $α(t)$ be
generated by \( Q = (q_{ij}) \in \mathbb{R}^{m \times m} \). That is,
\[
P\{\alpha(t + \delta) = j | \alpha(t) = i, \alpha(s), s \leq t\} = \begin{cases} 
q_{ij}\delta + o(\delta), & \text{if } j \neq i, \\
1 + q_{ii}\delta + o(\delta), & \text{if } j = i,
\end{cases}
\] (2.1)
where \( q_{ij} \geq 0 \) for \( i, j = 1, 2, \ldots, m \) with \( j \neq i \) and \( \sum_{j \in \mathcal{M}} q_{ij} = 0 \) for each \( i \in \mathcal{M} \).

Furthermore, we are considering the insurance portfolios in a financial market with a market index \( Z(t) \), whose prices satisfies
\[
dZ(t) = \mu_Z(t, Z(t))dt + \sigma_Z(t, Z(t))dW_Z(t),
\] (2.2)
where \( W_Z(t) \) is a standard Brownian motion. Denote by \( \{F_t^W\} \) the filtration generated by Brownian motion \( \{W_Z(t)\}_{t \geq 0} \). We note that \( Z(t) \) captures the dynamics of the financial market.

Following the classical Cramér-Lundberg process, we assume that \( \hat{X}_k(t), k \in \{1, 2\} \), the surplus of insurance company \( k \) without investment and reinsurance satisfies
\[
d\hat{X}_k(t) = c_k(\alpha(t), Z(t))dt - Y_k(t), \quad t \geq 0,
\] (2.3)
where \( \hat{X}_k(0) := \hat{x}_k \) is the initial surplus, \( c_k(\alpha(t), Z(t)) \) is the rate of premium, and \( Y_k(t) = \sum_{i=1}^{N_k(t)} A_i^k \)
is a compound Poisson process with the claim size \( A_i^k \) with \( \{A_i^k : i > 1\} \) being a sequence of positive, independent and identically distributed random variables.

In this paper, we consider a Poisson measure in lieu of the traditionally used Poisson process. Suppose \( \Theta \subset \mathbb{R}_+ \) is a compact set and the function \( q_k(\cdot) \) is the magnitude of the claim sizes.

\[
N_k(t, H) = \text{number of claims on } [0, t] \text{ with claim size taking values in } H \in \Theta,
\] (2.4)
counts the number of claims up to time \( t \), which is a Poisson counting process. For \( k = 1, 2 \), \( Y_k(t) \) is jump processes representing claims for each company with arrival rate \( \lambda_k \). Note that claim frequencies depends on the economy and financial market states. The function \( q_k(\alpha(t), Z(t), \rho_k) \) is assumed to be the magnitude of the claim sizes, where \( \rho_k \) has distribution \( \Pi_k(\cdot) \), and \( q_k(i, \cdot, \rho_k) \) is continuous for each \( \rho_k \) and each \( i \in \mathcal{M} \). At different regimes and financial market states, taking into consideration of random environment, the values of \( q_k(i, \cdot, \rho_k) \) could be much different. Then the Poisson measure \( N_k(\cdot, \cdot) \) has intensity \( \lambda_k dt \times \Pi_k(d\rho_k) \) where \( \Pi_k(d\rho_k) = f(\rho_k)d\rho_k \).

Let \( \nu_{k,n} \) denote the time of the \( n \)-th claim and \( \zeta_{k,n} = \nu_{k,n+1} - \nu_{k,n} \). Let \( \{\zeta_{k,v}, \rho_k, v \geq n\} \) be independent of \( \{X_k(s), \alpha(s), s \leq \nu_{k,n}, \zeta_{k,v}, ?_k, v \leq n\} \). Then the \( n \)-th claim term is \( q_k(\alpha(\nu_{k,n}), Z(\nu_{k,n}), \rho_k) \), and the claim amount of \( Y_k \) can be written as \( Y_k(t) = \sum_{\nu_{k,n} \leq t} q_k(\alpha(\nu_{k,n}), (\nu_{k,n}), \rho_k) \).

### 2.2 Reinsurance and Investment

Let \( a_k(t) \) be the \( \mathcal{F}_t \)-progressively measurable process valued in \( [0, 1] \), an exogenous retention level, which is a control chosen by the insurance company representing the reinsurance policy and \( g(a_k) \) is the reinsurance premium rate. Denote by \( A_k := \{a_k(t) : 0 \leq a_k(t) \leq 1, 0 \leq t \leq T\} \), the set of reinsurance strategies of insurer \( k \). Recall that \( A_i^k \) is the size of the \( i \)-th claim. Let \( A_i^k(a_k(t)) \) be the fraction of each claim paid by the primary insurance company. Then the aggregation claim amount paid by the primary insurance company is denoted as \( Y_k^{a_k}(t) \).
Remark 2.1. Note that both the claim frequencies and severities are depending on the Markov regimes and market index. It is a more general formulation compared with the work in Bensoussan et al. (2014), where only the claim frequencies depends on the Markov regimes and market index. Therefore, the diffusion approximation in Bensoussan et al. (2014) is simplified due to the state-independence of claim severity. Further, with the compound Poisson jumps, the surplus process forms a controlled jump-diffusion regime-switching process. We aim to find optimal reinsurance strategies under the jump-diffusion regime-switching process formulation numerically.

The insurance companies invest in both risk-free assets $S_0(t)$ and risky assets $S(t)$ with prices satisfying

$$
\begin{align*}
\frac{dS_0(t)}{S_0(t)} &= r(\alpha(t)) dt, \\
\frac{dS(t)}{S(t)} &= \mu_S(\alpha(t), Z(t)) dt + \sigma_S(\alpha(t), Z(t)) dW_S(t),
\end{align*}
$$

where $r(\alpha(t))$ and $\mu(\alpha(t), Z(t))$ are the return rates of the risk-free and risky assets, respectively; $\sigma_S(\alpha(t), Z(t))$ is the corresponding volatility; $W_S(t)$ is a standard Brownian motion independent of $W_Z(t)$. For $k = 1, 2$, the investment behavior of the insurer $k$ is modelled as a portfolio process $b_k(t)$, where $b_k(t)$ is invested in the risky asset $S(t)$. Let $B_k = \{b_k(t) : 0 \leq t \leq T\}$ denote the set of investment strategies of insurer $k$.

Combining the reinsurance and investment strategies, the surplus process of the insurance company $k$, denoted by $\bar{X}_k(t)$, follows

$$
\begin{align*}
\frac{d\bar{X}_k(t)}{\bar{X}_k(t)} &= \left\{ r(\alpha(t))\bar{X}_k(t) + b_k(t)[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_k(\alpha(t), Z(t)) - g(a_k(t)) \right\} dt \\
&\quad + b_k(t)\sigma_S(\alpha(t), Z(t)) dW_S(t) - dY_k^{a_k}(t),
\end{align*}
$$

where $Y_k^{a_k}(t) = \int_0^t \int_{\mathbb{R}_+} \tilde{q}_k(q_k, a_k) N_k(dt, d\rho_k)$ and $\tilde{q}_k(q_k, a_k)$ is the magnitude of the claim sizes with respect to the surplus process.

In this work, we model the competition of two insurance companies with investment and reinsurance schemes in finite time horizon using a game theoretic formulation. The performance of each company is measured by the relative performance of their surpluses against their competitor’s. Thus, the competition between the two companies formulates a game with two players, each of which can adjust its reinsurance strategies based on the competitor’s scheme. Let the relative surplus performance for insurance company $k$ be $X_k(t) := \bar{X}_k(t) - \kappa_k \bar{X}_I(t)$. Hence, $X_k(t)$ is governed by the following dynamic system

$$
\begin{align*}
dX_k(t) &= \sum_{i \in M} I_{\{\alpha(t) = i\}} \left\{ r(i)X_k(t) + (b_k(t) - \kappa_k b_I(t))[\mu_S(i, Z(t)) - r(i)] + c_k(i, Z(t)) - \kappa_k c_I(i, Z(t)) \\
&\quad - (g(a_k(t)) - \kappa_k g(a_I(t))) dt + (b_k(t) - \kappa_k b_I(t))\sigma_S(i, Z(t)) dW_S(t) \right\} - dY_k^{a_k}(t) + \kappa_k dY_I^{a_I}(t),
\end{align*}
$$

where

$$
Y_k^{a_k}(t) = \int_0^t \int_{\mathbb{R}_+} \tilde{q}_k(q_k, a_k) N_k(dt, d\rho_k).
$$
2.3 Proportional Reinsurance

We allow the insurance companies to continuously reinsure a fraction of its claim with the retention level $a_k \in [0,1]$, $k = 1, 2$. Note that $a_k$ is the exogenous retention level, and the control chosen by the insurance company for the reinsurance policy. Then $\hat{q}(q_k, a_k) = q_k(t) q_k(\alpha(t), Z(t), \rho_k)$. We have

$$Y_{t}^{ak}(t) = \sum_{i=1}^{N_k(t)} A_{i}^{k}(a_k) = \sum_{i=1}^{N_k(t)} a_k A_{i}^{k}.$$

Considering the proportional reinsurance strategies, for $k = 1, 2$. The relative surplus process of the insurance company $k$, under the reinsurance and investment, follows

$$dX_k(t) = \left\{ r(\alpha(t))X_k(t) + (b_k(t) - \kappa_k b_l(t))[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_k(\alpha(t), Z(t)) \\ -\kappa_k c_l(\alpha(t), Z(t)) - [g(\alpha(t)) - \kappa_k g(a_l(t))] \right\} dt + (b_k(t) - \kappa_k b_l(t)) \sigma_S(\alpha(t), Z(t)) dW_S(t) \\
- a_k(t) \int_{\mathbb{R}^+} q_k(\alpha(t), Z(t), \rho_k) N_l(dt, d\rho_k) + \kappa_k a_l(t) \int_{\mathbb{R}^+} q_l(\alpha(t), Z(t), \rho_l) N_l(dt, d\rho_l)
X_k(0) = \tilde{x}_k - \kappa_k \tilde{x}_l.
$$

(2.8)

2.4 Excess-of-loss Reinsurance

We allow the insurance companies to continuously reinsure its claim and pay all of the claims up to a pre-given level of amount (termed retention level). We still let $a_k$, $k = 1, 2$ be the retention level chosen by the insurance company to determine the reinsurance policy. We have that

$$Y_{t}^{ak}(t) = \sum_{i=1}^{N_k(t)} A_{i}^{k}(a_k) = \sum_{i=1}^{N_k(t)} (A_{i}^{k} \land a_k).$$

Then $\hat{q}_k(q_k, a_k) = q_k(\alpha(t), Z(t), \rho_k) \land a_k(t)$.

Considering the excess-of-loss reinsurance strategies, for $k = 1, 2$, the relative surplus process of the insurance company $k$, under the reinsurance control and investment, follows

$$dX_k(t) = \left\{ r(\alpha(t))X_k(t) + (b_k(t) - \kappa_k b_l(t))[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_k(\alpha(t), Z(t)) \\ -\kappa_k c_l(\alpha(t), Z(t)) - [g(\alpha(t)) - \kappa_k g(a_l(t))] \right\} dt + (b_k(t) - \kappa_k b_l(t)) \sigma_S(\alpha(t), Z(t)) dW_S(t) \\
- \int_{\mathbb{R}^+} (q_k(\alpha(t), Z(t), \rho_k) \land a_k) N_k(dt, d\rho_k) + \kappa_k \int_{\mathbb{R}^+} (q_l(\alpha(t), Z(t), \rho_l) \land a_l) N_l(dt, d\rho_l)
X_k(0) = \tilde{x}_k - \kappa_k \tilde{x}_l.
$$

(2.9)

2.5 Control Problem

For $k = 1, 2$, insurer $k$ has a utility function $U_k : \mathbb{R} \rightarrow \mathbb{R}$, where $U_k$ is assumed to be increasing, strictly concave, and satisfies Inada conditions, i.e.,

$$\partial_\alpha U_k(-\infty) = +\infty, \quad \partial_\alpha U_k(+\infty) = 0.$$

Following the work [Espinosa and Touzi (2013)], the insurer $k$ aims to maximize the expected utility of his relative performance at the terminal time $T$ by adopting a pair of investment and reinsurance
strategy $u_k = (a_k, b_k) \in A_k \times B_k$, denote $U_k \triangleq A_k \times B_k$. For an arbitrary pair of admissible control $u = (u_1, u_2) \in U \triangleq \mathcal{U}_1 \times \mathcal{U}_2$, the objective function is

$$J^k(t, x_k, z, i, u) = \mathbb{E}\left[U_k \left( (1 - \kappa_k)\bar{X}_k(T) + \kappa_k(\tilde{X}_k(T) - \bar{X}_l(T)) \right) \right] = \mathbb{E}\left[U_k \left( \tilde{X}_k(T) - \kappa_k\bar{X}_l(T) \right) \right],$$

for $k \neq l \in \{1, 2\}$. For $k = 1, 2$, $\kappa_k$ measures the sensitivity of insurer $k$ to the performance of his competitor.

The control $u_k = (a_k, b_k)$ with $k \in \{1, 2\}$ is said to be admissible if $a_k$ and $b_k$ satisfy

(i) $a_k(t), b_k(t)$ are nonnegative for any $t \geq 0$,

(ii) Both $a_k, b_k$ are adapted to $\mathcal{F}_t$.

(iii) $J^k(t, x_k, z, i, u) < \infty$ for any admissible pair $u_k = (a_k, b_k)$.

For $k = 1, 2$, let $\mathcal{B}(U_k \times [0, \infty))$ be the $\sigma-$ algebra of Borel subsets of $U_k \times [0, \infty)$. We use a relaxed control formulation; see Kushner and Dupuis (2001) for a definition and more discussions. Recall that an admissible relaxed control $m_k(\cdot)$ is a measure on $\mathcal{B}(U_k \times [0, \infty))$ such that $m_k(U_k \times [0, t)) = t$ for each $t \geq 0$. With the given probability space, we say that $m_k(\cdot)$ is an admissible relaxed (stochastic) control for $\mathcal{F}$, if $m_k(\cdot, \omega)$ is a deterministic relaxed control with probability one and if $m_k(A \times [0, t))$ is $\mathcal{F}_t$-adapted for all $A \in \mathcal{B}(U_k)$.

Given a relaxed control $m_k(\cdot)$ of $u_k(\cdot)$, we define the derivative $m_{t,k}(\cdot)$ such that

$$m_k(K) = \int_{U_k \times [0, \infty)} I_{\{(u_k, t) \in K\}} m_{t,k}(d\phi_k)dt$$

for all $K \in \mathcal{B}(U_k \times [0, \infty))$, and that for each $t$, $m_{t,k}(\cdot)$ is a measure on $\mathcal{B}(U_k)$ satisfying $m_{t,k}(U_k) = 1$.

For example, we can define $m_{t,k}(\cdot)$ in any convenient way for $t = 0$ and as the left-hand derivative for $t > 0$,

$$m_{t,k}(A) = \lim_{\varrho \to 0} \frac{m_k(A \times [t - \varrho, t])}{\varrho}, \quad \forall A \in \mathcal{B}(U_k).$$

Note that $m_k(d\phi_k dt) = m_{t,k}(d\phi_k)dt$. It is natural to define the relaxed control representation $m_k(\cdot)$ of $u_k(\cdot)$ by $m_k(A) = I_{\{(u_k(t) \in A) \}} \forall A \in \mathcal{B}(U_k)$. Define the relaxed control $m(\cdot) = (m_{1,1}(\cdot) \times m_{2,2}(\cdot))$ with derivative $m_t(\cdot) = m_{t,1}(\cdot) \times m_{t,2}(\cdot)$. Thus $m(\cdot)$ is a measure on the Borel sets of $U \times [0, \infty)$.

### 2.6 Nash Equilibrium

A Nash equilibrium $u^* = (u_1^*, u_2^*) \in U$ is achieved such that

$$\mathbb{E}[U_1(\bar{X}_1^{u_1^*}(T) - \kappa_1\bar{X}_2^{u_2^*}(T))] \leq \mathbb{E}[U_1(\bar{X}_1^{u_1^*}(T) - \kappa_1\bar{X}_2^{u_2^*}(T))],$$

$$\mathbb{E}[U_2(\bar{X}_2^{u_2^*}(T) - \kappa_2\bar{X}_1^{u_1^*}(T))] \leq \mathbb{E}[U_2(\bar{X}_2^{u_2^*}(T) - \kappa_2\bar{X}_1^{u_1^*}(T))].$$

(2.11)

For $\alpha(t) = i \in \mathcal{M}$, $Z(t) = z$, and $X_k(t) = x_k$, where $0 \leq t \leq T$ and $k \neq l \in \{1, 2\}$, the value function of insurance company $k$ follows

$$V^k(t, x_k, z, i) = \sup_{u_k \in H_k} \mathbb{E}[U_k(\bar{X}^{u_k}_{k}(T) - \kappa_k\bar{X}^{u_l}_{l}(T))],$$

(2.12)

where $V^k(\cdot, \cdot, \cdot, \cdot)$ is the value function in $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{M}$. 

8
To obtain the system of Hamilton-Jacobi-Bellman (HJB) equations, we assume the existence of optimal control. For an arbitrary \( u_k \in U_k \), \( \alpha(t) = i \in \mathcal{M} \), \( k \neq l \in \{1, 2\} \), and \( V^k(\cdot, \cdot, \cdot, i) \in C^2(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathcal{M}) \), define an integro-differential operator \( \mathcal{L}^{u_k, u_l} \) by

\[
\mathcal{L}^{u_k, u_l} V^k(t, x_k, z, i) = V^k_{x_k}(t, x_k, z, i) \left\{ r(i)x_k + (b_k - \kappa_k b_l)[\mu(i, z) - r(i)] + c_k(i, z) - \kappa_k c_l(i, z) - (g(a_k) - \kappa_k g(a_l)) \right\} + V^k_z(t, x_k, z, i)m(t, z) + \frac{1}{2}(b_k - \kappa_k b_l)^2 \sigma^2_S(i, z)V_{x_k x_k}(t, x_k, z, i) + \frac{1}{2} \sigma^2_Z(t, z)V_{z z}(t, x_k, z, i) - \lambda_k \int_{\mathbb{R}_+} [V^k(t, x_k - \rho, z, \rho) - V^k(t, x_k, z, i)]f(\rho_k)d\rho_k + \lambda_l \int_{\mathbb{R}_+} [V^k(t, x_k + \rho, z, \rho) - V^k(t, x_k, z, i)]f(\rho_l)d\rho_l + QV(t, x_k, z, i)(i),
\]

where

\[
QV(t, x_k, z, i)(i) = \sum_{j \neq i} q_{ij}(V(t, x_k, z, j) - V(t, x_k, z, i)).
\]

Formally, for \( k \neq l \in \{1, 2\} \), we conclude that \( V^k \) satisfies the following system of integro-differential HJI (Hamilton-Jacobi-Isaacs) equations: for each \( i \in \mathcal{M} \),

\[
\begin{cases}
V^k_t(t, x_k, z, i) + \sup_{u_k \in \mathcal{U}_k} \mathcal{L}^{u_k, u_l} V^k(t, x_k, z, i) = 0, \\
V^k(T, x_k, z, i) = U_k(x_k).
\end{cases}
\]

### 3 Numerical Algorithm

We begin by construction a discrete-time, finite-state, controlled Markov chain to approximate the controlled diffusion process with regime-switching in the absence of jumps with the dynamic system

\[
\begin{align*}
\frac{dX_k(t)}{dt} &= \sum_{i \in \mathcal{M}} I_{\{\alpha(t) = i\}} \left\{ r(i)X_k(t) + (b_k(t) - \kappa_k b_l(t))[\mu_S(i, Z(t)) - r(i)] + c_k(i, Z(t)) - \kappa_k c_l(i, Z(t)) - (g(a_k) + \kappa_k g(a_l)) \right\}, \\
\frac{dZ(t)}{dt} &= \mu_Z(t, Z(t))dt + \sigma_Z(t, Z(t))dW_Z(t) \\
X_k(0) &= \bar{x}_k - \kappa_k \bar{x}_l \\
Z(0) &= z_0.
\end{align*}
\]

Because the value function depends on both the state \( x \) and the time variable \( t \), two stepsizes are needed. That is, we need to discretize both the state and time. We use \( h > 0 \) as the stepsizes of the state and \( \delta > 0 \) as the stepsizes of the time. In fact, for any given \( T > 0 \), we use \( N = N(\delta) = \lceil T/\delta \rceil \).

Let \( e_i \) denote the unit vector in the \( i \)-th coordinate direction and \( \mathbb{R}^3_\delta \) denote the uniform \( h \)-grid on \( \mathbb{R}^3 \); i.e. \( \mathbb{R}^3_\delta = \{(x_1, x_2, z) : (x_1, x_2, z) = h(k_1 e_1 + k_2 e_2 + k_3 e_3); k_1, k_2, k_3 = 0, \pm 1, \pm 2, \ldots \} \). We use \( S_\delta = \mathbb{R}^3_\delta \) denote \( x = (x_1, x_2, z) \) and \( y = (y_1, y_2, z^*) \).

We can rewrite the system in the short form as the following:

\[
\frac{dX(t)}{dt} = \mu(X(t), \alpha(t), u(t))dt + \sigma(X(t), \alpha(t), u(t))dW(t),
\]

\[
\frac{dZ(t)}{dt} = \mu_Z(t, Z(t))dt + \sigma_Z(t, Z(t))dW_Z(t).
\]
where
\[
\mu(X(t), \alpha(t), u(t)) = \begin{pmatrix}
(r(\alpha(t))X_1(t) + (b_1(t) - \kappa_1b_2(t))[\mu_S(\alpha(t), Z(t)) - r(\alpha(t))] + c_1(\alpha(t), Z(t)) - \kappa_1c_2(\alpha(t), Z(t)) \\
-(g(a_1(t)) + \kappa_1g(a_2(t))) \\
(\mu_Z(t, Z(t))
\end{pmatrix}
\]
\[
\sigma(X(t), \alpha(t), u(t)) = \mathrm{diag}(b_1(t) - \kappa_1b_2(t)\sigma_S(\alpha(t), Z(t)), (b_2(t) - \kappa_2b_1(t))\sigma_S(\alpha(t), Z(t)), \sigma_Z(t, Z(t)))
\]
\[
W(t) = (W_S(t), W_S(t), W_Z(t))',
\]
where $A'$ is the transpose of $A$. Let $\{(s_n^h, \alpha_n^h, n < \infty\}$ be a controlled discrete-time Markov chain on $\mathbb{R}_+^n \times \mathcal{M}$ and denote by $p_D^{h\delta}(x, i, (y, j), n\delta|\phi)$ the transition probability from a state $(x, i)$ to another state $(y, j)$, for $\phi \in \mathcal{U}$. We use $u_n^{h\delta}$ to denote the random variable that is the control action for the chain at discrete time $n$ and $p_D^{h\delta}$ is so defined that the constructed Markov chain’s evolution well approximates the local behavior of the controlled regime-switching diffusion \cite{2, 7}.

For each $k = 1, 2$, we construct the transition probability $p_{Dk}^{h\delta}(x, i, (y, j), n\delta|\phi)$ which is associated with $J_k^k(t, x, i, \phi) = \mathbb{E}[U_k(X_k(T))]$ satisfying the followings:

\[
\begin{align*}
J_k^k(t, x, i, \phi) + C_{uk}u_jJ_k^k(t, x, i, \phi) &= 0, \\
J_k^k(T, x, i, \phi) &= U_k(x_k).
\end{align*}
\]

To figure out the form of $p_{Dk}^{h\delta}(x, i, (y, j)|\phi)$, we define a finite difference approximation to (3.3) as

\[
J_{x}^k(t, x, i, \phi) \to \frac{J_k^k(t + \delta, x, i, \phi) - J_k^k(t, x, i, \phi)}{\delta},
\]
\[
J_{x}^k(t, x, i, \phi) \to \frac{J_k^k(t, x + h\epsilon_k, i, \phi) - J_k^k(t, x, i, \phi)}{\epsilon_k},
\]
if $r(i)x_k + (b_k - \kappa_kb_l)[\mu_S(i, z) - r(i)] + c_k(i, z) - \kappa_kc_l(i, z) - (g(a_k) - \kappa_kg(a_l)) > 0$,
\[
J_{x}^k(t, x, i, \phi) \to \frac{J_k^k(t, x, i, \phi) - J_k^k(t, x - h\epsilon_k, i, \phi)}{\epsilon_k},
\]
if $r(i)x_k + (b_k - \kappa_kb_l)[\mu_S(i, z) - r(i)] + c_k(i, z) - \kappa_kc_l(i, z) - (g(a_k) - \kappa_kg(a_l)) < 0$,
\[
J_{x}^k(t, x, i, \phi) \to \frac{J_k^k(t, x + h\epsilon_3, i, \phi) - J_k^k(t, x, i, \phi)}{\epsilon_3},
\]
if $\mu_Z(t, z) > 0$,
\[
J_{x}^k(t, x, i, \phi) \to \frac{J_k^k(t, x, i, \phi) - J_k^k(t, x - h\epsilon_3, i, \phi)}{\epsilon_3},
\]
if $\mu_Z(t, z) < 0$,
\[
J_{x}^k(t, x, i, \phi) \to \frac{J_k^k(t, x + h\epsilon_3, i, \phi) - 2J^k(t, x, i, \phi) + J_k^k(t, x - h\epsilon_3, i, \phi)}{\epsilon_3^2}.
\]
To proceed, define

\[ p_{k,D}^{\delta}(x,i), (x \pm \delta h i, k), n \delta | \phi) = \frac{\delta}{n} \left\{ r(i)x_k + (b_k - \kappa \delta h l_j)[\mu S(i, z) - r(i)] + c_k(i, z) - \kappa \delta h l(i, z) \right\} \]

\[ - (g(a_k) - \kappa \delta h l(a_l)) \right\} + \frac{\delta}{2z^2}(b_k - \kappa \delta h l_j)^2 \sigma Z^2(i, z), \]

where \( K^+ = \max\{K, 0\} \) and \( K^- = \min\{-K, 0\} \). By choosing \( \delta \) and \( h \) appropriately, we can have \( p_{k,D}^{\delta}(x,i), (x,i), n \delta | \phi) \) given in (3.4) nonnegative. Thus, \( p^{\delta}(\cdot | \phi) \) are well-defined transition probability.

Next, we need to approximate the Poisson jumps for ensuring the local properties of claims for (2.7). We can rewrite the system in the matrix form as follow

\[ dX(t) = \mu(X(t), \alpha(t), u(t))dt + \sigma(X(t), \alpha(t), u(t))dB(t) + \frac{Y_1(t)e_1 + Y_2(t)e_2}{\delta}, \] (3.5)

where \( Y_k(t) \) is the jump process w.r.t the surplus process \( X_k(t) \), for \( k = 1, 2 \).

The relative surplus process \( X_k(t) \) is determined by two jump terms with the arriving rate \( \lambda_k \) and \( \lambda_l \), respectively. Denote by \( R_k(t) \) the difference of the two jumps. That is,

\[ R_k(t) = \int_{\mathbb{R}_+} \tilde{q}_k(q_k, a_k)N_k(dt, d\rho_k) - \kappa_k \int_{\mathbb{R}_+} \tilde{q}_l(q_l, a_l)N_l(dt, d\rho_l). \]

Since the difference of two Poisson processes is again a Poisson process, events in the new process \( R_k(t) \) will occur according to a Poisson process with the rate \( \lambda_k \) with \( \lambda = \lambda_k + \lambda_l \) and each event, independently, will be from the first jump process with probability \( \lambda_k / (\lambda_k + \lambda_l) \), yielding the generic claim size

\[ A_k = \begin{cases} A^k(a_k), & \text{with probability } \frac{\lambda_k}{\lambda_k + \lambda_l}; \\ -\kappa_k A^l(a_l), & \text{with probability } \frac{\lambda_l}{\lambda_k + \lambda_l}. \end{cases} \]

Suppose that the current state is \( \xi_n^{h, \delta} = x, \alpha_n^{h, \delta} = i, \) and control is \( u_n^{h, \delta} = \phi \). The next interpolation interval is determined by (3.4) and \( \bar{q}_k(q_k(i, z, \rho_k), a_k) \) is the nearest value of \( \bar{q}_k(q_k(i, z, \rho_k), a_k) \) so that \( \xi_{n+1}^{h, \delta} \in S_k \). Then \( |\bar{q}_k(q_k(i, z, \rho_k), a_k) - \bar{q}_k(q_k(i, z, \rho_k), a_k)| \rightarrow 0 \) as \( h \rightarrow 0 \), uniformly in \( x \). To present the claim terms, we determine the next case \( \xi_{n+1}^{h, \delta}, \alpha_{n+1}^{h, \delta} \) by noting:

1. No claims occur in \([n \delta, n \delta + \delta)\) with probability \( 1 - \lambda \delta + o(\delta) \), we determine \( \xi_{n+1}^{h, \delta}, \alpha_{n+1}^{h, \delta} \) by transition probability \( p_{k,D}^{h, \delta}(\cdot) \) as in (3.4).

2. There is a claim of the relative surplus process \( X_k(t) \) in \([n \delta, n \delta + \delta)\) with probability \( \lambda \delta + o(\delta) \), we determine \( \xi_{n+1}^{h, \delta}, \alpha_{n+1}^{h, \delta} \) by

\[ \xi_{n+1}^{h, \delta} = \xi_n^{h, \delta} - q_k(q_k(i, z, \rho_k), a_k) e_k, \quad \alpha_{n+1}^{h, \delta} = \alpha_n^{h, \delta}. \]
So, we define
\[ p^h_\delta((x,i),(y,j)|\phi) = (1 - \lambda \delta + o(\delta))p^h_\delta((x,i),(y,j)|\phi) + (\lambda \delta + o(\delta))\Pi_k\{\rho : \tilde{q}_h(i,z,\rho)e_k = x - y\}. \] (3.6)

**Definition 3.1.** A controlled Markov chain \( \{ (\xi^h_n, \alpha^h_n), n < \infty \} \) with the one-step transition probability \( p^h_\delta((x,i),(y,j)|\phi) \) is given by
\[ p^h_\delta((x,i),(y,j)|\phi) = p^h_1((x,i),(y,j)|\phi)I_{\{k=1\}} + p^h_2((x,i),(y,j)|\phi)I_{\{k=2\}}, \] (3.7)
where \( k \) is the index of the cost function.

The piecewise constant interpolations \( \xi^h_\delta(\cdot), \alpha^h_\delta(\cdot) \), and \( u^h_\delta(\cdot) \) are defined as
\[ \xi^h_\delta(t) = \xi^h_n, \quad \alpha^h_\delta(t) = \alpha^h_n, \quad u^h_\delta(t) = u^h_n \quad \text{for} \quad t \in [n\delta, n\delta + \delta) \] (3.8)
Use \( \mathbb{E}_{x,i,n}^\phi, \text{Var}_{x,i,n}^\phi, \) and \( P_{x,i,n}^\phi \) to denote the conditional expectation, variance, and marginal probability given \( \{\xi^h_n, \alpha^h_n, u^h_n, n \leq n, \xi^h_n = x, \alpha^h_n = i, u^h_n = \phi\} \), respectively. Define the difference
\[ \Delta \xi^h_n = \xi^h_{n+1} - \xi^h_n. \]

With the approximation of the Markov chain constructed above, we can obtain an approximation of the utility function as follows:
\[ J^k(\delta, x, i, u^h) = (1 - \lambda \delta + o(\delta)) \sum_{(y,j)} p^h_\delta((x,i),(y,j)|u^h)J^k(\delta, y, j, u^h) \]
\[ + (\lambda \delta + o(\delta)) \int_{\mathbb{R}_+} J^k(\delta, x_k - \tilde{q}_k(i,z,\rho_k), a^h_k, z, i, u^h)\Pi_k(d\rho_k) \]
\[ + (\lambda \delta + o(\delta)) \int_{\mathbb{R}_+} J^k(\delta, x_k + \kappa_k \tilde{q}_k(i,z,\rho_k), a^h_k, z, i, u^h)\Pi_k(d\rho_k). \]
(3.9)

Moreover,
\[ V^{h,\delta}(\delta, x, i) = \sup_{u^h} J^{k,\delta}(\delta, x, i, u^h). \] (3.10)

**Definition 3.2.** The sequence \( \{ (\xi^h_n, \alpha^h_n) \} \) is said to be **locally consistent**, if it

1. There is a transition probability \( p^h_\delta \) is **locally consistent** in the sense
\[ \mathbb{E}_{x,i,n}^\phi[\Delta_s^h_n] = \mu^h(\delta, x, i, \phi) + o(\delta), \]
\[ \text{Var}_{x,i,n}^\phi[\Delta_s^h_n] = \sigma^h(\delta, x, i, \phi) + o(\delta). \] (3.11)

where
\[ \mu^h(\delta, x, i, \phi) = \begin{pmatrix} r(i)x_1 + (b_1 - \kappa_1b_2)[\mu_S(i,z) - r(i)] + c_1(i,z) - \kappa_1c_2(i,z) - g(a_1) + \kappa_1g(a_2) \\ r(i)x_2 + (b_2 - \kappa_2b_1)[\mu_S(i,z) - r(i)] + c_2(i,z) - \kappa_2c_1(i,z) - g(a_2) + \kappa_2g(a_1) \\ \mu_z(t,z) \end{pmatrix} \]
\[ \sigma^h(\delta, x, i, \phi) = \text{diag}\left((b_1 - \kappa_1b_2)^2\sigma^2_z(i,z), (b_2 - \kappa_2b_1)^2\sigma^2_z(i,z), \sigma^2_z(t,z)\right). \]

2. The one-step transition probability \( p^h_\delta((x,i),(y,j)|\phi) \) for the chain can be represented in the factored form:
\[ p^h_\delta((x,i),(y,j)|\phi) = (1 - \lambda \delta + o(\delta))p^h_\delta((x,i),(y,j)|\phi) + (\lambda \delta + o(\delta))\Pi\{\rho : q_h(i,z,\rho) = x - y\} \]
4 Convergence of Numerical Approximation

4.1 Representations of Approximation Sequences

To proceed, we first show that the constructed Markov chain is locally consistent. This ensures that our approximation is reasonable in certain sense.

**Lemma 4.1.** The Markov chain $\xi_{n}^{h,\delta}$ with transition probabilities $p_{1}(\cdot)I_{k=1} + p_{2}(\cdot)I_{k=2}$ Using (3.4) and (3.7). It is easy to see that

$$
E_{x,i,n}[\Delta \xi_{n}^{h,\delta}] = E_{x,i,n}[\Delta \xi_{n}^{h,\delta}I_{k=1}] + E_{x,i,n}[\Delta \xi_{n}^{h,\delta}I_{k=2}]
$$

Likewise, we obtain $E_{x,i,n}[\Delta \xi_{n}^{h,\delta}I_{k}]$ and $\text{Var}_{x,i,n}[\Delta \xi_{n}^{h,\delta}]$.

Let $\xi_{n}^{h,\delta}(0) = x, a_{n}^{h,\delta}(0) = i$. Define the relaxed control representation $m_{h,\delta}^{k}(\cdot)$ by using its derivative $m_{h,\delta}^{k}(A) = I_{\{a_{n}^{h,\delta}(t)\}}(A)$. Let $H_{n}^{h,\delta}$ denote the event that $\xi_{n}^{h,\delta}, a_{n}^{h,\delta}$ is determined by the case of “no claim occurs” and use $T_{n}^{h,\delta}$ to denote the event of “one claim occurs”. Let $I_{H_{n}^{h,\delta}}$ and $I_{T_{n}^{h,\delta}}$ be corresponding indicator functions, respectively. Then $I_{H_{n}^{h,\delta}} + I_{T_{n}^{h,\delta}} = 1$ and we can write

$$
\xi_{n}^{h,\delta}(t) = \xi_{n}^{h,\delta}(0) + \sum_{r=0}^{[t/\delta]-1} [\Delta \xi_{r}^{h,\delta}I_{H_{r}^{h,\delta}} + \Delta \xi_{r}^{h,\delta}I_{T_{r}^{h,\delta}}]
$$

Define $F_{n}^{h,\delta}$ as the smallest $\sigma$-algebra generated by $\{\xi_{n}^{h,\delta}(s), a_{n}^{h,\delta}(s), m_{r}^{h,\delta}(s), H_{r}^{h,\delta}, s \leq t\}$ and $F_{t}^{h,\delta}$ as the smallest $\sigma$-algebra generated by $\{\xi_{n}^{h,\delta}(s), a_{n}^{h,\delta}(s), m_{r}^{h,\delta}(s), H_{r}^{h,\delta}(s), s \leq t\}$.

For $k, l = 1, 2$ and $k \neq l$, denote

$$
M_{h,\delta}(t) = \sum_{r=0}^{[t/\delta]-1} (\Delta \xi_{r}^{h,\delta} - E_{x,i}^{h,\delta}[\Delta \xi_{r}^{h,\delta}])I_{H_{r}^{h,\delta}}
$$

$$
Y_{k,l}^{h,\delta}(t) = - \sum_{r=0}^{[t/\delta]-1} [\Delta \xi_{r}^{h,\delta}]\epsilon_{k}I_{T_{r}^{h,\delta}}
$$

Then $M_{h,\delta}(t)$ is a martingale with respect to $F_{[t/\delta]}^{h,\delta}$. Now, we represent $M_{h,\delta}(t)$ similar to the
diffusion term in (2.7). Define \( W^h,\delta (\cdot) \) as
\[
W^h,\delta (t) = \sum_{r=0}^{|t/\delta| - 1} \left[ \sigma^h,\delta(x,\phi,i) - \mathbb{E}_{x,i}^{\phi,h,\delta} \Delta \xi^h,\delta_r I_{H^h,\delta} \right] = \int_0^t \left[ \sigma^h,\delta(\xi^h,\delta(s), \alpha^h,\delta(s), u^h,\delta(s)) \right]^{-1} dM^h,\delta (s).
\]

The local consistency leads to
\[
\sum_{r=0}^{|t/\delta| - 1} \mathbb{E}_{x,i}^{\phi,h,\delta} \Delta \xi^h,\delta_r I_{H^h,\delta} = \int_0^t \left[ \mu^h,\delta(\xi^h,\delta(s), \alpha^h,\delta(s), u^h,\delta(s)) \right] ds + \varepsilon^h,\delta (t).
\]

For each \( t, \mathbb{E}[\text{number of } r : t^h_r \leq t] = \lambda_k t \) as \( h, \delta \to 0 \). This implies that we can drop \( I_{H^h,\delta} \) with no effect on the above limit.

As a consequence, we can rewrite (4.1) as following:
\[
\begin{align*}
\xi^h,\delta (t) &= x + \int_0^t \int_U \mu^h,\delta(\xi^h,\delta(s), \alpha^h,\delta(s), \phi) m^h,\delta_s (d\phi) ds \\
&\quad + \int_0^t \int_U \sigma^h,\delta(\xi^h,\delta(s), \alpha^h,\delta(s), \phi) m^h,\delta_s (d\phi) dW^h,\delta (s) \\
&\quad + \left( \int_0^t \int_{\mathbb{R}_+} \int_U \tilde{q}^h_1 (q_1(\alpha^h,\delta(s), Z^h,\delta(s), \rho_1), a_1) m^h,\delta_s (d\phi) N_1^{h,\delta}(ds, d\rho_1) \\
&\quad - \int_0^t \int_{\mathbb{R}_+} \int_U \kappa_1 \tilde{q}_2 (q_2(\alpha^h,\delta(s), Z^h,\delta(s), \rho_2), a_2) m^h,\delta_s (d\phi) N_2^{h,\delta}(ds, d\rho_2) \right) e_1 \\
&\quad + \left( \int_0^t \int_{\mathbb{R}_+} \int_U \tilde{q}^h_1 (q_1(\alpha^h,\delta(s), Z^h,\delta(s), \rho_1), a_1) m^h,\delta_s (d\phi) N_1^{h,\delta}(ds, d\rho_1) \\
&\quad - \int_0^t \int_{\mathbb{R}_+} \int_U \kappa_2 \tilde{q}_2 (q_2(\alpha^h,\delta(s), Z^h,\delta(s), \rho_2), a_2) m^h,\delta_s (d\phi) N_2^{h,\delta}(ds, d\rho_2) \right) e_2 \\
&\quad + \varepsilon^h,\delta (t).
\end{align*}
\]

We can also rewrite (3.5) as
\[
X(t) = x + \int_0^t \int_U \mu(X(s), \alpha(s), \phi(s)) m_s (d\phi) ds + \int_0^t \int_U \sigma(X(s), \alpha(s), \phi) m_s (d\phi) dW(s) \\
+ \left( \int_0^t \int_{\mathbb{R}_+} \int_U \tilde{q}_1 (q_1(\alpha(s), Z(s), \rho_1), a_1) m_s (d\phi) N_1(ds, d\rho_1) \\
- \int_0^t \int_{\mathbb{R}_+} \int_U \kappa_1 \tilde{q}_2 (q_2(\alpha(s), Z(s), \rho_2), a_2) m_s (d\phi) N_2(ds, d\rho_2) \right) e_1 \\
+ \left( \int_0^t \int_{\mathbb{R}_+} \int_U \tilde{q}_2 (q_2(\alpha(s), Z(s), \rho_2), a_2) m_s (d\phi) N_2(ds, d\rho_2) \\
- \int_0^t \int_{\mathbb{R}_+} \int_U \kappa_2 \tilde{q}_1 (q_1(\alpha(s), Z(s), \rho_1), a_1) m_s (d\phi) N_1(ds, d\rho_1) \right) e_2.
\]
4.2 Convergence of Approximating Markov Chains

Lemma 4.2. Using the transition probability \( \{p^{h,\delta}(\cdot)\} \) defined in (3.7), the interpolated process of the constructed Markov chain \( \{\alpha^{h,\delta}(\cdot)\} \) converges weakly to \( \alpha(\cdot) \), the Markov chain with generator \( Q \).

Proof. The proof can be obtained similar to [Yin et al., 2003, Theorem 3.1]. The details are thus omitted.

Theorem 4.3. Let the approximating chain \( \{\xi_{n}^{h,\delta}, \alpha_{n}^{h,\delta}, n < \infty\} \) constructed with transition probabilities defined in (3.7) be locally consistent with (3.11), \( m^{h,\delta}(\cdot) \) be the relaxed control representation of \( \{u^{h,\delta}, n < \infty\} \), \( (\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot)) \) be the continuous-time interpolation defined in (3.8). Then \( \{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_{1}^{h,\delta}(\cdot, \cdot), N_{2}^{h,\delta}(\cdot, \cdot)\} \) is tight.

Proof. Note that \( \alpha^{h,\delta}(\cdot) \) is tight. It follows that for each \( \Delta > 0 \), each \( t > 0 \), and \( 0 < \tilde{t} \leq \Delta \), there is a random variable \( \gamma^{h,\delta}(\Delta) > 0 \) such that

\[
\mathbb{E}[W^{h,\delta}(t+\tilde{t}) - W^{h,\delta}(t)]^2 = \sum_{[t/\delta]} \mathbb{E}[\{(\sigma^{h,\delta}(x, \phi, i)\}^{-1}(\Delta \xi_{r}^{h,\delta} - \mathbb{E}^{\phi,h,\delta}_{x,i,r} \Delta \xi_{r}^{h,\delta})I H_{r}^{h,\delta})]^2 \geq \mathbb{E}[f^{h,\delta}(\Delta)]
\]

satisfying \( \lim_{\Delta \to 0} \lim_{\tilde{t} \to 0} \mathbb{E}[W^{h,\delta}(t)] = 0 \), which yields the tightness of \( W^{h,\delta}(\cdot) \). A similar argument leads to the tightness of \( M^{h,\delta}(\cdot) \). The sequence \( m^{h,\delta}(\cdot) \) is tight because of its compact range space. By virtue of [Kushner and Dupuis, 2001][Theorem 9.2.1], we obtain the tightness of \( \{N_{k}^{h,\delta}(\cdot), k = 1, 2\} \) since the mean number of claims on any bounded interval \([t, t+1]\) is bounded and

\[
\lim_{\Delta \to 0} \inf_{r} P\{\nu_{k,r+1}^{h,\delta} - \nu_{k,r}^{h,\delta} > \Delta |\nu_{k,r}^{h,\delta}\} = 1.
\]

This implies the tightness of \( \{R_{k}^{h,\delta}(\cdot), k = 1, 2\} \). As a consequence, \( \xi^{h,\delta}(\cdot) \) is tight and

\[
\{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_{1}^{h,\delta}(\cdot, \cdot), N_{2}^{h,\delta}(\cdot, \cdot)\}
\]

is tight.

Since \( \{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_{1}^{h,\delta}(\cdot, \cdot), N_{2}^{h,\delta}(\cdot, \cdot)\} \) is tight, the Prohorov’s theorem implies that it is sequentially compact. Thus we can extract a weakly convergent subsequence. Select such a convergent subsequence and still index the sequence by \( h, \delta \) for notational simplicity. We proceed to characterize the limit process.

Theorem 4.4. Let \( \{\xi(\cdot), \alpha(\cdot), m(\cdot), W(\cdot), N_{1}(\cdot, \cdot), N_{2}(\cdot, \cdot)\} \) be the limit of weakly convergent subsequence and \( \mathcal{F} \) be the \( \sigma \)-algebra generated by \( \{X(s), \alpha(s), m(s), W(s), N_{1}(s, \cdot), N_{2}(s, \cdot), s \leq t\} \). Then \( W(\cdot) \) is a standard \( \mathcal{F}_{t} \)-Brownian motion and \( N_{1}(\cdot, \cdot) \), \( N_{2}(\cdot, \cdot) \) are \( \mathcal{F}_{t} \)-Poisson measures, and \( m(\cdot) \) is an admissible relaxed control.

Proof. The proof is divided into several steps.

Step 1: By the Skorohod representation, with a slight abuse of notation, \( \{\xi^{h,\delta}(\cdot), \alpha^{h,\delta}(\cdot), m^{h,\delta}(\cdot), W^{h,\delta}(\cdot), N_{1}^{h,\delta}(\cdot, \cdot), N_{2}^{h,\delta}(\cdot, \cdot)\} \) converges to \( \{\xi(\cdot), \alpha(\cdot), m(\cdot), W(\cdot), N_{1}(\cdot, \cdot), N_{2}(\cdot, \cdot)\} \) w.p.1, and the convergence is uniform on any compact set.
To proceed, we first verify that $W(\cdot)$ is an $\mathcal{F}_t$-Brownian motion. For any real-valued and continuous function $\psi$, define

$$\langle \psi, m \rangle_t = \int_0^t \int_{\mathbb{R}} \psi(\phi, s) m_s(d\phi).$$

(4.7)

For any given $f(\cdot) \in C_0^2(\mathbb{R}^3)$ ($C^2$ function with compact support), consider an associate operator

$$\mathcal{L}_w f(w) = \frac{1}{2} \sum_{i=1}^3 \frac{\partial^2}{\partial w_i \partial w_i} f(w).$$

Let $t, \bar{t} > 0$ be given with $t + \bar{t} \leq T$, along with arbitrary positive integers $\kappa$ and $\bar{\kappa}$, arbitrary $t_i \leq t$ and continuous functions $\psi_j$ with $i \leq \kappa$ and $j \leq \bar{\kappa}$, any bounded and continuous function $h(\cdot)$, and arbitrary $f \in C^2_0(\mathbb{R}^3)$. Denote $\{\Gamma^\kappa_i, i \leq \kappa\}$ as a sequence of nondecreasing partition of $\mathbb{R}_+$ such that $\Pi(\partial \Gamma^\kappa_i) = 0$ for all $i, \kappa$, where $\partial \Gamma^\kappa_i$ is the boundary of the set $\Gamma^\kappa_i$. As $\kappa \to \infty$, let the diameter of the sets $\Gamma^\kappa_i$ go to zero.

By (4.3), $W^{h, \delta}(\cdot)$ is an $\mathcal{F}_t^{h, \delta}$-Brownian motion,

$$\mathbb{E} h(\xi^{h, \delta}(t), \alpha^{h, \delta}(t), W^{h, \delta}(t), (\psi_j, m^{h, \delta})_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \times \left( f(W^{h, \delta}(t + \bar{t}) - W^{h, \delta}(t)) - \int_t^{t+\bar{t}} \mathcal{L}_w f(W^{h, \delta}(s))ds \right) = 0.$$

By the weak convergence and the Skorohod representation, we may assume that $W^{h, \delta}(\cdot)$ converges to $W(\cdot)$ w.p.1, and hence as $\delta \to 0$,

$$\mathbb{E} h(\xi^{h, \delta}(t), \alpha^{h, \delta}(t), W^{h, \delta}(t), (\psi_j, m^{h, \delta})_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \left( f(W^{h, \delta}(t + \bar{t}) - f(W^{h, \delta}(t)) \right)$$

$$\to \mathbb{E} h(\xi(t), \alpha(t), W(t), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \left( f(W(t + \bar{t}) - f(W(t)) \right),$$

$$\mathbb{E} h(\xi^{h, \delta}(t), \alpha^{h, \delta}(t), W^{h, \delta}(t), (\psi_j, m^{h, \delta})_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \left( \int_t^{t+\bar{t}} \mathcal{L}_w f(W^{h, \delta}(s))ds \right)$$

$$\to \mathbb{E} h(\xi(t), \alpha(t), W(t), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \left( \int_t^{t+\bar{t}} \mathcal{L}_w f(W(s))ds \right).$$

Thus,

$$\mathbb{E} h(\xi(t), \alpha(t), W(t), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \left( f(W(t + \bar{t}) - f(W(t)) - \int_t^{t+\bar{t}} \mathcal{L}_w f(W(s))ds \right) = 0.$$

Moreover, consider the collection of random variables generated by $\{h(\xi(t), \alpha(t), W(t), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{\kappa})\}$. It follows that $f(W(t)) - f(W(0)) - \int_0^t \mathcal{L}_w f(W(s))ds$ is a martingale. By virtue of the Skorohod representation and the dominated convergence theorem together with (4.6), we have

$$\mathbb{E} h(\xi(t), \alpha(t), W(t), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{\kappa}) \left( (W(t + \delta) - W(t))^2 - \delta \right) = 0.$$

Moreover, the quadratic variation of the martingale $W(t)$ is $tI_3$ and $W(\cdot)$ is an $\mathcal{F}_t$-Brownian motion, where $I_3$ is an $3 \times 3$ identity matrix.

Step 2: We proceed to show that $N_k(\cdot, \cdot)$ is an $\mathcal{F}_t$-Poisson measure for each $k = 1, 2$. Let $\theta(\cdot)$ be a continuous function on $\mathbb{R}_+$ and define the process

$$\Theta_k = \int_0^t \int_{\mathbb{R}_+} \theta(\rho) N_k(ds, d\rho).$$
Using similar argument as in the proof of the Brownian motion above, if $f(\cdot) \in C^2_0(\mathbb{R}^2)$ then

$$
\mathbb{E}h(\xi(t), \alpha(t), W(t), (\psi_j, m)_{t_i}, N(t, \Gamma^{\kappa}_i), i \leq \kappa, j \leq \bar{\kappa})
\times \left[ f(\Theta_k(t + \tilde{t})) - f(\Theta_k(t)) - \lambda_k \int_{t + \tilde{t}}^{t + \tilde{t} + \bar{\kappa}} \int_{\mathbb{R}_+} [f(\Theta_k(s) + \theta(\rho)) - f(\Theta_k(s))] \Pi(d\rho) ds \right] = 0.
$$

This implies that $N_k(\cdot, \cdot)$ is an $\mathcal{F}_t$-Poisson measure for each $k = 1, 2$.

Step 3.1: We will use (4.1) for the rest of the proof. Note that $\mathbb{E}|e^{h, \delta}(t)| \to 0$ as $h, \delta \to 0$. Letting $h, \delta \to 0$ and using the Skorohod representation for (4.1), we have

$$
\int_0^t \int_{\mathcal{U}} \mu^{h, \delta}(\xi^{h, \delta}(s), \alpha^{h, \delta}(s), \phi)m^{h, \delta}(d\phi) ds - \int_0^t \int_{\mathcal{U}} \mu(\xi(s), \alpha(s), \phi)m^{h, \delta}(d\phi) ds \to 0,
$$

uniformly on any bounded time interval with probability one. On the other hand, the sequence $m^{h, \delta}(\cdot)$ converges in the compact-weak topology, thus, for any continuous and bounded function $\psi(\cdot)$ with compact support,

$$
\int_0^t \int_{\mathcal{U}} \psi(\phi, s)m^{h, \delta}(d\phi) ds \to \int_0^t \int_{\mathcal{U}} \psi(\phi, s)m(d\phi) ds
$$

as $h, \delta \to 0$.

By virtue of the Skorohod representation and the weak convergence, as $h, \delta \to 0$,

$$
\int_0^t \int_{\mathcal{U}} \mu^{h, \delta}(\xi^{h, \delta}(s), \alpha^{h, \delta}(s), \phi)m^{h, \delta}(d\phi) ds - \int_0^t \int_{\mathcal{U}} \mu(\xi(s), \alpha(s), \phi)m^{h, \delta}(d\phi) ds \to 0,
$$

uniformly in $t$ with probability one on any bounded interval.

Step 3.2: For any $t^1 \geq t$, $t^2 \geq 0$ with $t^1 + t^2 \leq T$, any $C^1_0$ function $f(\cdot)$ (functions that have compact support whose first partial derivative w.r.t. the time variable and the second partial derivatives w.r.t. the state variable $x$ are continuous), bounded and continuous function $h(\cdot)$, any positive integers $\kappa, \bar{\kappa}, t_i$, and any continuous function $\psi_j$ satisfying $t \leq t_i \leq t_1$ and $i \leq \kappa$, and $j \leq \bar{\kappa}$, the weak convergence and the Skorohod representation imply that

$$
\mathbb{E}h(\xi^{h, \delta}(t), \alpha^{h, \delta}(t), M^{h, \delta}(t), (\psi_j, m^{h, \delta})_{t_i} : i \leq \kappa, j \leq \bar{\kappa})
\times [f(t^1 + t^2, M^{h, \delta}(t^1 + t^2)) - f(t^1, M^{h, \delta}(t^1))]
\to \mathbb{E}h(\xi(t), \alpha(t), M(t), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{\kappa})[f(t^1 + t^2, M(t^1 + t^2)) - f(t^1, M(t^1))]
$$

with $h, \delta \to 0$. Choose a sequence $\{n^\delta\}$ such that $n^\delta \to \infty$ but $\Delta^\delta = \delta n^\delta \to 0$, then

$$
\mathbb{E}h(\xi^{h, \delta}(t), \alpha^{h, \delta}(t), M^{h, \delta}(t), (\psi_j, m^{h, \delta})_{t_i} : i \leq \kappa, j \leq \bar{\kappa})
\times [f(t^1 + t^2, M^{h, \delta}(t^1 + t^2)) - f(t^1, M^{h, \delta}(t^1))]
= \mathbb{E}h(\xi^{h, \delta}(t), \alpha^{h, \delta}(t), M^{h, \delta}(t), (\psi_j, m^{h, \delta})_{t_i} : i \leq \kappa, j \leq \bar{\kappa})
\times \left[ \sum_{\text{ln}^\delta = 0}^{(t^1 + t^2)/\delta - 1} \delta(\text{ln}^\delta + n^\delta), M^{h, \delta}(\delta(\text{ln}^\delta + n^\delta))
\right.
\left. - f(\delta \text{ln}^\delta, M^{h, \delta}(\delta(\text{ln}^\delta + n^\delta))) + f(\delta \text{ln}^\delta, M^{h, \delta}(\delta(\text{ln}^\delta + n^\delta))) - f(\delta \text{ln}^\delta, M^{\delta}(\delta \text{ln}^\delta)) \right].
$$

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Note that
\[
\sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \sum_{\ln^\delta + n^\delta = \ln^\delta + k \ln^\delta - 1}^{(t^1+t^2)/\delta} \left[ f(\delta(\ln^\delta + n^\delta), M^\delta(\ln^\delta + n^\delta)) - f(\ln^\delta, M^h,\delta(\ln^\delta + n^\delta)) \right]
\]
\[
= \sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \sum_{k = \ln^\delta}^{(t^1+t^2)/\delta} \left[ f(\delta(k+1), M^h,\delta(\ln^\delta + n^\delta)) - f(\delta k, M^h,\delta(\ln^\delta + n^\delta)) \right]
\]
\[
= \sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \left[ \frac{\partial f(\delta \ln^\delta, M^\delta(\ln^\delta + n^\delta))}{\partial s} \right] \Delta^\delta + o(1),
\]
where \( o(1) \to 0 \) in mean uniformly in \( t \) as \( \delta \to 0 \). Letting \( \delta \ln^\delta \to s \) as \( \delta \to 0 \), then \( \delta(\ln^\delta + n^\delta) \to s \) since \( \Delta^\delta = \delta n^\delta \to 0 \) as \( \delta \to 0 \). Then, by the weak convergence and the Skorohod representation, the continuity of \( h(\cdot) \), and the smoothness of \( f(\cdot) \) imply that
\[
\mathbb{E} h(\xi,\alpha,\delta(t_i), M(t_i), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{k}) \times \sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \left[ f(\delta(\ln^\delta + n^\delta), M^h,\delta(\ln^\delta + n^\delta)) - f(\ln^\delta, M^h,\delta(\ln^\delta + n^\delta)) \right]
\]
\[
\to \mathbb{E} h(\xi(t_i), \alpha(t_i), M(t_i), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{k}) \left[ \int_{t^1}^{t^1+t^2} \frac{\partial f(s, M(s))}{\partial s} ds \right] \quad \text{as } h, \delta \to 0.
\]
(4.11)
The last part of (4.10) can be seen as
\[
\sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \left[ f(\delta \ln^\delta, M^h,\delta(\ln^\delta + n^\delta)) - f(\ln^\delta, M^h,\delta(\ln^\delta)) \right]
\]
\[
= \sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \left[ \sum_{i=1}^{\ln^\delta + n^\delta - 1} \left\{ \frac{3}{2} \right\} f_{M_i,M_i}(\delta \ln^\delta, M^h,\delta(\ln^\delta)) \right]
\]
\[
\quad + \varepsilon^h,\delta(t^1 + t^2 - \varepsilon^h,\delta(t^1),
\]
where \( f_{M_i,M_i} \) denotes the second partial derivatives, \( M^h,\delta(\cdot) \) is the \( i \)-th component of \( M^h,\delta(\cdot) \) and \( \sup_{t \leq t^1 \leq T} |\overline{\varepsilon^h,\delta(t^1)}| \to 0 \) as \( h, \delta \to 0 \).

By (4.2) and the definition of \( \sigma^h,\delta(\cdot) \), we have
\[
\sum_{\ln^\delta = t^1/\delta}^{(t^1+t^2)/\delta} \left\{ \sum_{i=1}^{\ln^\delta + n^\delta - 1} \left\{ \frac{2}{2} \right\} f_{M_i,M_i}(\delta \ln^\delta, M^h,\delta(\ln^\delta)) \right\}
\]
\[
\to \int_{t^1}^{t^1+t^2} \int_\mathcal{U} \left[ H_M f(s, M(s)) \sigma(\xi(s), \alpha(s), \phi)[\sigma(\xi(s), \alpha(s), \phi)]^t \right] m_s(d\phi) ds,
\]
(4.12)
where \( H_M f(s, M(s)) \) is the Hessian matrix of \( f(\cdot) \) at time \( s \), \( Tr(\cdot) \) represents for the trace of a matrix.

Using (4.9)-(4.12), we have
\[
\mathbb{E} h(\xi(t_i), \alpha(t_i), M(t_i), (\psi_j, m)_{t_i} : i \leq \kappa, j \leq \bar{k}) \times \left[ f(t^1 + t^2, M(t^1 + t^2)) - f(t^1, M(t^1)) - \int_{t^1}^{t^1+t^2} \frac{\partial f(s, M(s))}{\partial s} ds \right]
\]
\[
- \int_{t^1}^{t^1+t^2} \int_\mathcal{U} \frac{1}{2} Tr \left( H_M f(s, M(s)) \sigma(\xi(s), \alpha(s), \phi)[\sigma(\xi(s), \alpha(s), \phi)]^t \right) m_s(d\phi) ds = 0.
\]
Therefore,
\[
\int_0^t \int_U \sigma^{h,\delta}(\xi(s),\alpha^{h,\delta}(s),\phi) m^h_{\delta}(d\phi)ds \rightarrow \int_0^t \int_U \sigma(\xi(s),\alpha(s),\phi)m_\delta(d\phi)ds,
\]
uniformly in \( t \) with probability one on any bounded interval.

Using the same arguments as in Step 2 and Step 3.2, we obtain the limits for the latter parts of (4.4). As a result, \( \xi(\cdot) \) is the solution of (4.5), which means \( \xi(s) = X(s) \) w.p.1 and \( m(\cdot) \) is an admissible relaxed control.

### 4.3 Convergence of the Cost and the Value Functions

Note that since \( U_k(\cdot) \) satisfies the Inada’s conditions. There exist positive real numbers \( K \) and \( k_0 \) such that \( |U_k(X_k)| \leq K(1 + |X_k|^{k_0}) \). We proceed to prove the following result.

**Theorem 4.5.** Suppose that the utility functions \( U_k(\cdot) \) has at most polynomial growth. Then the value functions \( V^{k, h, \delta}(t,x,i) \) converges to \( V^k(t,x,i) \) for \( k = 1,2 \), respectively, as \( h, \delta \to 0 \).

**Proof.** By Theorem 4.3 each sequence \( \{\xi^{h,\delta}(\cdot),\alpha^{h,\delta}(\cdot),m^{h,\delta}(\cdot),W^{h,\delta}(\cdot),N_1^{h,\delta}(\cdot,\cdot),N_2^{h,\delta}(\cdot,\cdot)\} \) has a weakly convergent subsequence with the limit \( \xi^{h,\delta}(\cdot) \) satisfying (4.5). Using the same notation as above and applying the Skorohod representation, the weak convergence, as \( h, \delta \to 0 \), \( J^{k, h, \delta}(t, x, i, m^{h,\delta}) \to J^k(t, x, i, m) \), for \( k = 1,2 \). The cost function is given by (3.9). Since \( V^k(t,x,i) \) is the maximizing expected utility, for any admissible control \( m(\cdot) \). \( J^k(t,x,i,m) \leq V^k(t,x,i) \), for \( k = 1,2 \). Let \( \tilde{m}^{h,\delta}(\cdot) \) be an optimal relaxed control for \( \{\xi^{h,\delta}(\cdot)\} \), which implies

\[
V^{k, h, \delta}(t,x,i) = J^{k, h, \delta}(t,x,i,\tilde{m}^{h,\delta}) = \sup_{m^{h,\delta}} J^{k, h, \delta}(t,x,i,m^{h,\delta}).
\]

Choose a subsequence \( \{\tilde{h}, \tilde{\delta}\} \) of \( \{h, \delta\} \) such that

\[
\lim_{h, \delta \to 0} V^{k, h, \delta}(t,x,i) = \lim_{\tilde{h}, \tilde{\delta} \to 0} V^{k, \tilde{h}, \tilde{\delta}}(t,x,i) = \lim_{\tilde{h}, \tilde{\delta} \to 0} J^{k, \tilde{h}, \tilde{\delta}}(t,x,i,\tilde{m}^{\tilde{h},\tilde{\delta}}).
\]

Without loss of generality, we may assume that \( \{\xi^{\tilde{h},\tilde{\delta}}(\cdot),\alpha^{\tilde{h},\tilde{\delta}}(\cdot),W^{\tilde{h},\tilde{\delta}}(\cdot),\tilde{m}^{\tilde{h},\tilde{\delta}}(\cdot),N_1^{\tilde{h},\tilde{\delta}}(\cdot,\cdot),N_2^{\tilde{h},\tilde{\delta}}(\cdot,\cdot)\} \) converges weakly to \( \{X(\cdot),\alpha(\cdot),W(\cdot),m(\cdot),N_1(\cdot,\cdot),N_2(\cdot,\cdot)\} \), where \( m(\cdot) \) is an admissible relaxed control. Then the weak convergence and the Skorohod representation leads to

\[
\lim_{h, \delta \to 0} \sup V^{k, \delta}(t,x,i) = \lim_{\tilde{h}, \tilde{\delta} \to 0} J^{k, \tilde{h}, \tilde{\delta}}(t,x,i,\tilde{m}^{\tilde{h},\tilde{\delta}}) = J^k(t,x,i,m) \leq V^k(t,x,i).
\]

We claim that \( \liminf_{\delta} V^{k, \delta}(t,x,i) \geq V^k(t,x,i) \).

Suppose that \( \overline{m}(\cdot) \) is an optimal control with Brownian motion \( W(\cdot) \) such that \( (X(\cdot) \) is the associated trajectory. By the chattering lemma (see [Yin et al. (2009)] and page 59-60 of [Kushner and Dupuis (2001)]), for any given \( \eta, \delta_\eta > 0 \), there is an \( \varepsilon > 0 \) and an ordinary control \( u^{\eta,\delta_\eta}(\cdot) \) that takes only finite many values, \( u^{\eta,\delta_\eta}(\cdot) \) is a constant in \((\varepsilon,\varepsilon+\varepsilon)\), \( \overline{m}^{\eta,\delta_\eta}(\cdot) \) is its relaxed control representation, and \( J^k(t,x,i,\overline{m}^{\eta,\delta_\eta}) \geq V^k(t,x,i) - \eta \). For each \( \eta, \delta_\eta > 0 \), and the corresponding \( \varepsilon > 0 \), consider an optimal control problem with piecewise constant on \((\varepsilon,\varepsilon+\varepsilon)\). We consider the process \( \{X^{\eta,\delta_\eta}(\varepsilon),\alpha^{\eta,\delta_\eta},m^{\eta,\delta_\eta}(\varepsilon),W^{\eta,\delta_\eta}(\varepsilon)\} \). Let \( \tilde{u}^{\eta,\delta_\eta}(\cdot) \) be the optimal control, \( \tilde{m}^{\eta,\delta_\eta}(\cdot) \) the relaxed control representation, and \( \tilde{X}^{\eta,\delta_\eta}(\cdot) \) the associated trajectory. Since \( \tilde{m}^{\eta,\delta_\eta}(\cdot) \) is the optimal control, \( J^k(t,x,i,\tilde{m}^{\eta,\delta_\eta}) \geq J^k(t,x,i,\tilde{m}^{\eta,\delta_\eta}) \geq V^k(t,x,i) - \eta \). Using the chattering lemma, we can approximate \( \tilde{m}^{\eta,\delta_\eta}(\cdot) \) by a sequence of \( m^{\eta,\delta_\eta}(\cdot) \). Then

\[
V^{k, \delta}(t,x,i) \geq J^{k, \delta}(t,x,i,\tilde{m}^{\eta,\delta_\eta}) \to J^k(t,x,i,\tilde{m}^{\eta,\delta_\eta}).
\]
Moreover,
\[
\liminf_{h,\delta \to 0} V^{k,h,\delta}(t, x, i) \geq \lim_{h,\delta \to 0} J^{k,h,\delta}(t, x, i, m^{h,\delta}) = J^k(t, x, i, \tilde{m}^{\eta,\delta}).
\]
Thus, \( \liminf_{h,\delta \to 0} V^{k,h,\delta}(t, x, i) \geq V^k(t, x, i) - \eta. \) The arbitrariness of \( \eta \) implies that \( \liminf_{\delta \to 0} V^{k,\delta}(t, x, i) \geq V^k(t, x, i) \), which completes the proof. \( \square \)

5 Numerical Examples

In this section, we present some numerical results for the case in which both insurance companies are constant absolute risk aversion (CARA) agents, i.e., each agent has an exponential utility function. More precisely, the utility function of each insurer has the form
\[
U_k(X_k) = -\frac{1}{\eta_k} \exp(-\eta_k X_k), \quad \text{for } \eta_k > 0, \ k = 1, 2.
\]

(5.1)

Based on the algorithm constructed above, we carry out the computation by valuing iterations in a backward manner by time.

1. Set \( t = T - \delta \) and \( J^{k,h,\delta}(T, x, i, u^{h,\delta}) = U_k(x_k) \), for each \( k = 1, 2 \).

2. By (3.9), we obtain
\[
J^{k,\delta}(t, x, i, u^{h,\delta}) = (1 - \lambda \delta) \sum_{(y,j)} P^{h,\delta}_{k,y}(x, i, (y,j)) u^{h,\delta}(t + \delta, y, j, u^{h,\delta})
\]
\[
+ \lambda \delta \int_{\mathbb{R}^+} J^{k,\delta}(t + \delta, x_k - \tilde{q}^{h}_{k}(q_k(i, z, \rho_k), a^{h,\delta}_{k}, z, i, u^{h,\delta}) \Pi_k(d\rho_k)
\]
\[
+ \lambda \delta \int_{\mathbb{R}^+} J^{k,\delta}(t + \delta, x_k + \kappa_k \tilde{q}^{h}_{l}(q_l(i, z, \rho_l), a^{h,\delta}_{l}, z, i, u^{h,\delta}) \Pi_l(d\rho_l)
\]

Find the pair \( \{\tilde{u}^{h,\delta}_{k}, k = 1, 2\} \) and record \( \hat{u}^{h,\delta}_{k}(t) = \tilde{u}^{h,\delta}_{k} \) satisfying that for any \( u^{h,\delta}_{k} \in U_k \):

\[
J^{1,h,\delta}(t, x, i, u^{1,h,\delta}, u^{2,h,\delta}) \leq J^{1,h,\delta}(t, x, i, \hat{u}^{1,h,\delta}_{k}, \hat{u}^{2,h,\delta}_{k})
\]
\[
J^{2,h,\delta}(t, x, i, u^{1,h,\delta}, u^{2,h,\delta}) \leq J^{2,h,\delta}(t, x, i, \hat{u}^{1,h,\delta}_{k}, \hat{u}^{2,h,\delta}_{k}).
\]

3. Let \( t = t - \delta \) and continue the procedure until \( t = 0 \). We consider the case in which the discrete event consists of two states, or equivalently, the Markov chain has two states with given claim size distributions. In addition, we assume that the claim size distributions are identical in each regime. By using the value iteration methods, we numerically solve the optimal control problems. The continuous-time Markov chain \( \alpha(t) \) representing the discrete event state has the generator \( Q = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} \) and takes values in \( \mathcal{M} = \{1, 2\} \).

The parameters of the utility function are \( \eta_1 = 17.0 \) and \( \eta_2 = 21.0 \) respectively. The sensitivities are \( \kappa_1 = 0.8 \) and \( \kappa_2 = 0.7 \). The claim severity of both players follows exponential distribution \( f(y) = \theta_k e^{-\theta_k y} \) with \( \theta_1 = 0.3 \) and \( \theta_2 = 0.2 \). To incorporate the difference between claim densities in different regimes, we assume arriving rates of Poisson jump are different. So is the setup for risk-free return, and premium income rate. The detail of setup is as follows in Table [I]
Table 1: Parameters values

| Regime | $c_1$ | $c_2$ | $\lambda_1$ | $\lambda_2$ |
|--------|-------|-------|-------------|-------------|
| 0.02   | 0.05  | 0.02  | 0.20        | 0.30        |
| 0.03   | 0.10  | 0.20  | 0.80        | 0.70        |

The reinsurance premium rates are computed from expectation premium principle as:

$$g_k (a_k) = (1 + l_k) (1 - a_k) \mathbb{E}[A_k],$$

where $l_1 = 1.1$ and $l_2 = 1.15$. Further, the volatility and the drift of the financial market index and the risky return rate are modeled respectively by:

$$\mu_S (i, Z) = 0.2iZ;$$
$$\sigma_S (i, Z) = 0.4iZ;$$
$$\mu_Z (t, Z) = 0.4(t + 1)Z;$$
$$\sigma_Z (t, Z) = 0.1(t + 1)Z.$$

State discretization follows $\delta = 0.04$ and $h = 0.2$. For reinsurance, we discretize the reinsurance rate into six levels uniformly located from 0 to 1. The investment amount is free from restrictions, and can vary from -3 to 3 with 0.2 increase. The trend of investment and reinsurance for the varying relative surplus of player one is plotted in Figure 1, and that for the second player is in Figure 2.

![Figure 1](image1.png)
![Figure 2](image2.png)

Figure 1: Controls for varying $X_1$ with $T = 0.08$, $Z = 1.01$, and $X_2 = 0$.

From Figure 1 we can observe that both players always hold a low proportion of claim, which is due to the fact that both players are very risk-averse. A big claim will not only reduce their relative surplus but also drive the surplus of their opponent side up. Precisely, in regime one, the proportion held by player two is 0, which is less than 0.2 of player one. This results from the claim
arriving rate of the player two is relatively much higher than that of player one. In regime two, considering the high premium income rate and the same expected claim amount, a small proportion of claim is affordable for both players.

In view of the investment part of Figure 1, player two’s investment amount is always no less than that of player one. Since the relative sensitivity of player one to player two is higher, and player two’s relative surplus is at a higher relative level initially, player two is willing to accept more risk for higher expected return in order to beat player one. Then, along with growing $X_1$, since player one’s condition is improved, he tends to bear less risk. Meanwhile, player two adopts conservative strategy as well, since he can lower the uncertainty and make use of his advantage that he owns a higher premium income rate. Because a higher market volatility in regime two will introduce more risk, both players choose the investment amount much closer to 0 to lower the uncertainty.

![Figure 2: Controls for varying $X_2$ with $T = 0.08$, $Z = 1.01$, and $X_1 = 1$.](image)

Similar results can be seen from Figure 2. For the reinsurance part, player two initially holds full proportion of a claim, since $X_2 = -1$ leaves player two in a relative bad situation compared to $X_1 = 1$. To change this situation, he chooses to bear lots of risk to reduce the loss from reinsurance premium. This is more obvious for regime 1, where the claim arriving rate is lower. Hence, in more risky scenarios where claim arriving rate is higher, players show relative risk averse and transfer more risks by reinsurance tools. For the investment part, we can see that player two hold higher positions in risky assets in his portfolio in both regimes, which is consistent with the observations in Figure 1.

6 Concluding Remarks

In this paper, we considered a non-zero-sum stochastic investment and reinsurance game between two insurance companies. Both proportional and non-proportional reinsurance contracts were considered. Although we are able to obtain the systems of HJI equations using dynamic programming
principle, solving the problem explicitly is virtually impossible. Thus we developed a numerical scheme using the Markov chain approximation method (MCAM) to solve the problem. Due to the complexity of the stochastic game formulation, even numerically solving the systems of HJI equations is much more difficult than that of the previous work in stochastic optimization problems. The difficulties arise from the following two aspects. (1) With complex nonlinear state processes, the formulated high-dimension problem adds much difficulties in building approximating Markov chain. (2) The curse of dimensionality makes a significant impact and slow down the computation due to the large numbers of control variables and the dimensions of the HJI systems. Although the paper was devoted to a problem arising in risk management and insurance fields, the game problem formulation and the numerical methods developed can be more widely used in various other control and game problems.

For our problem, the nature of the Markov chain approximation relies on building a high dimensional lattice of both driving state and control strategy to approximate the value functions under different control scenarios. The optimization on every state follows the same computing rule, leading to the possibility of using parallel acceleration techniques. The first option coming to our mind is to incorporate multi-thread programming techniques into our completed C++ MCAM template library, which enables us to reduce development time by reusing the algorithm architecture of single-thread library. The latest eighth generation Intel CPUs are equipped with six computation cores, which allow maximal twelve threads to run simultaneously. If we parallelize the algorithm using ten threads, we can enhance the time efficiency ten times. However, this is not enough to handle the computational complexity required for our problem. The high dimensionality requires the lattice to be very precise, thus obtaining accurate results relies on generating a large number of nodes. Ten times acceleration seems a big enhancement, but it can only allow us to explore $10^7 \approx 1.39$ times of $Z, X_1, X_2, a_1, a_2, b_1,$ and $b_2$, which is unable to meet requirements of the computational complexity.

GPU acceleration, e.g., CUDA, is another attractive choice here. Although the frequency of GPU core is much lower than that of CPU, the number of GPU cores is usually hundreds of times of the number of CPU cores, and this makes GPU more suitable for parallel computation. The tenth generation NVidia GPU owns more than two thousand CUDA cores, which make it an easy solution for solving the complexity issue of MCAM algorithm on common stochastic optimization problem, where the maximal or the minimal value on a state is acquired from repeatedly comparing the newly computed value function value against the temporary optimal value so far. However, focusing on MCAM algorithm on our high dimensional game problem, CUDA acceleration is of very limited use. Not like CPU memory, which can be easily more than 64GB, the capacity of GPU memory is usually less than 8GB. The equilibrium strategy is obtained by searching on the value function information stored for different values of the control strategy. As a result, this memory consumption will occur for every GPU thread, which will easily lead the aggregated memory consumed by MCAM algorithm to exceed the GPUs memory capacity. From the above considerations, it appears that using parallel programming techniques to high dimensional game problems needs a lot of more thinking and effort. Finding more efficient way for the numerical solution is our on-going work.

References
Asmussen, S., Høgaard, B., Taksar, M. (2000). Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation, *Finance and Stochastics*, 4.3: 299-324.
Bensoussan, A., Frehse, J. (2000). Stochastic games for N players, *Journal of Optimization Theory and Applications*, 105(3): 543-565.
Bensoussan, A., Hoe, S., Yan, Z., and Yin, G. (2017). Real options with competition and regime switching, *Math. Finance*, 27: 224-250.

Bensoussan, A., Sin, C. C., Yam, S. C. P. and Yang, H. (2014). A class of non-zero-sum stochastic differential investment and reinsurance games, *Automatica*, 50(8): 2025-2037.

Bensoussan, A., Z. Yan, Z., and Yin, G. (2012). Threshold-type policies for real options using regime-switching models, *SIAM J. Financial Math.*, 3: 667-689.

Browne, S. (1995). Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin. *Mathematics of Operations Research* 20(4), 937-958.

Chen, S., Li, Z. and Li, K. (2010). Optimal investment-reinsurance policy for an insurance company with VaR constraint. *Insurance: Mathematics and Economics* 47(2), 144-153.

Chen, L. and Shen, Y. (2017). On a new paradigm of optimal reinsurance: A stochastic Stackelberg differential game between an insurer and a reinsurer. *Working paper.*

Choulli, T., Taksar, M. and Zhou, X.Y. (2001). Excess-of loss reinsurance for a company with debt liability and constraints on risk reduction, *Quantitative Finance*, 1:573-596.

Elliott, R.J., Siu, T.K. (2011). A stochastic differential game for optimal investment of an insurer with regime switching, *Quantitative Finance*, 11(3): 1-16.

Espinosa, G., Touzi, N. (2015). Optimal investment under relative performance concerns, *Mathematical Finance*, 25(2):221-257.

Hald, M. and Schmidli, H. (2004) On the maximization of the adjustment coefficient under proportional reinsurance, *ASTIN Bulletin* 34: 75-83.

Liu, J. and Yiu, K. F. C. (2013). Optimal stochastic differential games with VaR constraints. *Discrete and Continuous Dynamical Systems Series B* 18(7), 1889-1907.

Lundberg, F. (1903) Approximerad Framställning av Sannolikhetstilfunktionen, Aterförsäkering av Kollektivrisken, Almqvist & Wiksell, Uppsala. Akad. Afhandling, Almqvist o. Wiksell, Uppsala.

Meng, H., Li, S. and Jin, Z. (2015). A reinsurance game between two insurance companies with nonlinear risk processes. *Insurance: Mathematics and Economics* 62, 91-97.

Meng, H., Siu, T. K. and Yang, H. (2013). Optimal dividends with debts and nonlinear insurance risk processes. *Insurance: Mathematics and Economics* 53, 110-121.

Jin, Z., Yin, G., and Wu, F. (2013). Optimal reinsurance strategies in regime-switching jump diffusion models: Stochastic differential game formulation and numerical methods, *Insurance: Mathematics and Economics*, 53:733-746.

Kushner, H. (1990). Weak Convergence Methods and Singularly Perturbed Stochastic Control and Filtering Problems, Birkhäuser, Boston, MA.

Kushner, H. and Dupuis, P. (2001). Numerical Methods for Stochastic Control Problems in Continuous Time, second ed, Stochastic Modelling and Applied Probability, vol.24. Springer, New York.

Pun, C. S., and Wong, H. Y. (2016). Robust non-zero-sum stochastic differential reinsurance game. *Insurance: Mathematics and Economics*, 68, 169-177.

Sotomayor, L. and Cadenillas, A. (2011) Classical, singular, and impulse stochastic control for the optimal dividend policy when there is regime switching. *Insurance: Mathematics and Economics*, 48(3), 344–354

Taksar, M. and Zeng, X. (2011). Optimal non-proportional reinsurance control and stochastic differential games. *Insurance: Mathematics and Economics* 48(1), 64-71.

Wei, J., Yang, H. and Wang, R. (2010). Classical and impulse control for the optimization of dividend and proportional reinsurance policies with regime switching. *Journal of Optimization Theory and Applications*, 147(2).
Yan, M., Peng, F. and Zhang, S. (2017). A reinsurance and investment game between two insurance companies with the different opinions about some extra information. *Insurance: Mathematics and Economics* **75**, 58-70.

Yin, G., Jin, H., and Jin. Z., Numerical methods for portfolio selection with bounded constrains, *Journal of Computational and Applied Mathematics*, 233(2009), 564-581.

Yin, G., Zhang, Q., and Badowski, G. (2003). Discrete-time singularly perturbed Markov chains: aggregation, occupation measures, and switching diffusion limit, *Advances in Applied Probability*, 35, 449-476.

Yin, G. and Zhu, C. (2010). *Hybrid Switching Diffusions: Properties and Applications*. Springer, New York.

Zeng, X. (2010). Stochastic differential reinsurance games, *Journal of Applied Probability*, 47(2): 335-349.

Zhang, X., Meng, H. and Zeng, Y. (2016). Optimal investment and reinsurance strategies for insurers with generalized mean-variance premium principle and no-short selling. *Insurance: Mathematics and Economics* **67**, 125-132.

Zhang, X. and Siu, T. K. (2012). On optimal proportional reinsurance and investment in a Markovian regime-switching economy. *Acta Mathematica Sinica (English Series)* **28**, 67-82.

Zhu, J. (2014). Dividend optimization for a regime-switching diffusion model with restricted dividend rates. *ASTIN Bulletin*, 44:459–494.