NONTRIVIAL TWISTED SUMS OF $c_0$ AND $C(K)$

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Abstract. We obtain a new class of compact Hausdorff spaces $K$ for which $c_0$ can be nontrivially twisted with $C(K)$.

1. Introduction

In this article, we present a broad new class of compact Hausdorff spaces $K$ such that there exists a nontrivial twisted sum of $c_0$ and $C(K)$, where $C(K)$ denotes the Banach space of continuous real-valued functions on $K$ endowed with the supremum norm. By a twisted sum of the Banach spaces $Y$ and $X$ we mean a short exact sequence $0 \to Y \to Z \to X \to 0$, where $Z$ is a Banach space and the maps are bounded linear operators. This twisted sum is called trivial if the exact sequence splits, i.e., if the map $Y \to Z$ admits a bounded linear left inverse (equivalently, if the map $Z \to X$ admits a bounded linear right inverse). In other words, the twisted sum is trivial if the range of the map $Y \to Z$ is complemented in $Z$; in this case, $Z \cong X \oplus Y$. As in [7], we denote by Ext($X,Y$) the set of equivalence classes of twisted sums of $Y$ and $X$ and we write Ext($X,Y$) = 0 if every such twisted sum is trivial.

Many problems in Banach space theory are related to the quest for conditions under which Ext($X,Y$) = 0. For instance, an equivalent statement for the classical Theorem of Sobczyk ([5, 13]) is that if $X$ is a separable Banach space, then Ext($X,c_0$) = 0 ([3, Proposition 3.2]). The converse of the latter statement clearly does not hold in general: for example, Ext($\ell_1(I),c_0$) = 0, since $\ell_1(I)$ is a projective Banach space. However, the following question remains open: is it true that Ext($C(K),c_0$) $\neq 0$ for any nonseparable $C(K)$ space? This problem was stated in [4, 5] and further studied in the recent article [6], in which the author proves that, under the continuum hypothesis (CH), the space Ext($C(K),c_0$) is nonzero for a nonmetrizable compact Hausdorff space $K$ of finite height. In addition to this result, everything else that is known about the problem is summarized in [6, Proposition 2], namely that Ext($C(K),c_0$) is nonzero for a $C(K)$ space under any one of the following assumptions:

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• $K$ is a nonmetrizable Eberlein compact space;
• $K$ is a Valdivia compact space which does not satisfy the countable chain condition (ccc);
• the weight of $K$ is equal to $\omega_1$ and the dual space of $C(K)$ is not weak*-separable;
• $K$ has the extension property (8) and it does not have ccc;
• $C(K)$ contains an isomorphic copy of $\ell_\infty$.

Note also that if $\text{Ext}(Y, c_0) \neq 0$ and $X$ contains a complemented isomorphic copy of $Y$, then $\text{Ext}(X, c_0) \neq 0$.

Here is an overview of the main results of this article. Theorem 2.3 gives a condition involving biorthogonal systems in a Banach space $X$ which implies that $\text{Ext}(X, c_0) \neq 0$. In the rest of Section 2, we discuss some of its implications when $X$ is of the form $C(K)$. It is proven that if $K$ contains a homeomorphic copy of $[0, \omega] \times [0, \varepsilon]$ or of $2^\varepsilon$, then $\text{Ext}(C(K), c_0)$ is nonzero, where $\varepsilon$ denotes the cardinality of the continuum. In Sections 3 and 4, we investigate the consequences of the results of Section 2 for Valdivia and Corson compacta. Recall that Valdivia compact spaces constitute a large superclass of Corson compact spaces closed under arbitrary products; moreover, every Eberlein compact is a Corson compact (see [11] for a survey on Valdivia compacta). Section 3 is devoted to the proof that, under CH, it holds that $\text{Ext}(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space $K$. The question of whether $\text{Ext}(C(K), c_0) \neq 0$ for an arbitrary nonmetrizable Valdivia compact space $K$ remains open (even under CH), but in Section 4 we solve some particular cases of this problem.

2. General results

Throughout the paper, the weight and the density character of a topological space $X$ are denoted, respectively, by $w(X)$ and $\text{dens}(X)$. Moreover, we always denote by $\chi_A$ the characteristic function of a set $A$ and by $|A|$ the cardinality of $A$. We start with a technical lemma which is the heart of the proof of Theorem 2.3. A family of sets $(A_i)_{i \in I}$ is said to be almost disjoint if each $A_i$ is infinite and $A_i \cap A_j$ is finite, for all $i, j \in I$ with $i \neq j$.

Lemma 2.1. There exists an almost disjoint family $(A_{n,\alpha})_{n \in \omega, \alpha \in \varepsilon}$ of subsets of $\omega$ satisfying the following property: for every family $(A'_{n,\alpha})_{n \in \omega, \alpha \in \varepsilon}$ with each $A'_{n,\alpha} \subset A_{n,\alpha}$ cofinite in $A_{n,\alpha}$, it holds that $\sup_{p \in \omega} |M_p| = +\infty$, where:

$$M_p = \{n \in \omega : p \in \bigcup_{\alpha \in \varepsilon} A'_{n,\alpha}\}.$$ 

Proof. We will obtain an almost disjoint family $(A_{n,\alpha})_{n \in \omega, \alpha \in \varepsilon}$ of subsets of $2^{<\omega}$ with the desired property, where $2^{<\omega} = \bigcup_{k \in \omega} 2^k$ denotes the set of finite sequences in $2 = \{0, 1\}$. For each $\epsilon \in 2^\omega$, we set:

$$A_\epsilon = \{\epsilon | k : k \in \omega\},$$

so that $(A_\epsilon)_{\epsilon \in 2^\omega}$ is an almost disjoint family of subsets of $2^{<\omega}$. Let $(B_\alpha)_{\alpha \in \varepsilon}$ be an enumeration of the uncountable Borel subsets of $2^\omega$. Recalling that
\(|B_\alpha| = c\) for all \(\alpha \in c\) (Theorem 13.6), one easily obtains by transfinite recursion a family \((\epsilon_{n,\alpha})_{n \in \omega, \alpha \in c}\) of pairwise distinct elements of \(2^\omega\) such that \(\epsilon_{n,\alpha} \in B_\alpha\), for all \(n \in \omega\), \(\alpha \in c\). Set \(A_{n,\alpha} = A_{\epsilon_{n,\alpha}}\) and let \((A'_{n,\alpha})_{n \in \omega, \alpha \in c}\) be as in the statement of the lemma. For \(n \in \omega\), denote by \(D_n\) the set of those \(\epsilon \in 2^\omega\) such that \(n \in M_p\) for all but finitely many \(p \in A_{\epsilon}\). Note that:

\[
D_n = \bigcup_{k_0 \in \omega} \bigcap_{k \geq k_0} \{ C_p : p \in 2^k \text{ with } n \in M_p \},
\]

where \(C_p\) denotes the clopen subset of \(2^\omega\) consisting of the extensions of \(p\).

The above equality implies that \(D_n\) is an \(F_\sigma\) (and, in particular, a Borel) subset of \(2^\omega\). We claim that the complement of \(D_n\) in \(2^\omega\) is countable. Namely, if it were uncountable, there would exist \(\alpha \in c\) with \(B_\alpha = 2^\omega \setminus D_n\). But, since \(n \in M_p\) for all \(p \in A'_{n,\alpha}\), we have that \(\epsilon_{n,\alpha} \in D_n\), contradicting the fact that \(\epsilon_{n,\alpha} \in B_\alpha\) and proving the claim. To conclude the proof of the lemma, note that for each \(n \geq 1\) the intersection \(\bigcap_{k < n} D_k\) is nonempty; for \(\epsilon \in \bigcap_{k < n} D_k\), we have that \(\{i : i < n\} \subset M_p\), for all but finitely many \(p \in A_{\epsilon}\). \(\square\)

Let \(X\) be a Banach space. Recall that a biorthogonal system in \(X\) is a family \((x_i, \gamma_i)_{i \in I}\) with \(x_i \in X\), \(\gamma_i \in X^*\), \(\gamma_i(x_j) = 1\) and \(\gamma_i(x_j) = 0\) for \(i \neq j\). The cardinality of the biorthogonal system \((x_i, \gamma_i)_{i \in I}\) is defined as the cardinality of \(I\).

**Definition 2.2.** Let \((x_i, \gamma_i)_{i \in I}\) be a biorthogonal system in a Banach space \(X\). We call \((x_i, \gamma_i)_{i \in I}\) bounded if \(\sup_{i \in I} \|x_i\| < +\infty\) and \(\sup_{i \in I} \|\gamma_i\| < +\infty\); weak*-null if \((\gamma_i)_{i \in I}\) is a weak*-null family, i.e., if \((\gamma_i(x))_{i \in I}\) is in \(c_0(I)\), for all \(x \in X\).

**Theorem 2.3.** Let \(X\) be a Banach space. Assume that there exist a weak*-null biorthogonal system \((x_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in c}\) in \(X\) and a constant \(C \geq 0\) such that:

\[
\left\| \sum_{i=1}^k x_{n_i, \alpha_i} \right\| \leq C,
\]

for all \(n_1, \ldots, n_k \in \omega\) pairwise distinct, all \(\alpha_1, \ldots, \alpha_k \in c\), and all \(k \geq 1\). Then \(\text{Ext}(X, c_0) \neq 0\).

**Proof.** By [7] Proposition 1.4.f], we have that \(\text{Ext}(X, c_0) = 0\) if and only if every bounded operator \(T : X \to \ell_\infty/c_0\) admits a lifting\(^1\), i.e., a bounded operator \(\tilde{T} : X \to \ell_\infty\) with \(T(x) = \tilde{T}(x) + c_0\), for all \(x \in X\). Let us then show that there exists an operator \(T : X \to \ell_\infty/c_0\) that does not admit a lifting. To this aim, let \((A_{n,\alpha})_{n \in \omega, \alpha \in c}\) be an almost disjoint family as in Lemma 2.1 and consider the unique isometric embedding \(S : c_0(\omega \times c) \to \ell_\infty/c_0\) such that \(S(\epsilon_{n,\alpha}) = \chi_{A_{n,\alpha}} + c_0\), where \((\epsilon_{n,\alpha})_{n \in \omega, \alpha \in c}\) denotes the canonical basis of \(c_0\). More concretely, a nontrivial twisted sum of \(c_0\) and \(X\) is obtained by considering the pull-back of the short exact sequence \(0 \to c_0 \to \ell_\infty \to \ell_\infty/c_0 \to 0\) by an operator \(T : X \to \ell_\infty/c_0\) that does not admit a lifting.
Denote by $\Gamma : X \to c_0(\omega \times \mathfrak{c})$ the bounded operator with coordinate functionals $(\gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ and set $T = S \circ \Gamma : X \to \ell_\infty/c_0$. Assuming by contradiction that there exists a lifting $\hat{T}$ of $T$ and denoting by $(\mu_p)_{p \in \omega}$ the sequence of coordinate functionals of $\hat{T}$, we have that the set:

$$A'_{n,\alpha} = \{ p \in A_{n,\alpha} : \mu_p(x_{n,\alpha}) \geq \frac{1}{2} \}$$

is cofinite in $A_{n,\alpha}$. It follows that for each $k \geq 1$, there exist $p \in \omega$, $n_1, \ldots, n_k \in \omega$ pairwise distinct, and $\alpha_1, \ldots, \alpha_k \in \mathfrak{c}$ such that $p \in A'_{n_i,\alpha_i}$, for $i = 1, \ldots, k$. Hence:

$$\frac{k}{2} \leq \mu_p \left( \sum_{i=1}^{k} x_{n_i,\alpha_i} \right) \leq C \| \hat{T} \|,$$

which yields a contradiction.

**Corollary 2.4.** Let $K$ be a compact Hausdorff space. Assume that there exists a bounded weak*-$\mathfrak{c}$-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in $C(K)$ such that $f_{n,\alpha} f_{m,\beta} = 0$, for all $n, m \in \omega$ with $n \neq m$ and all $\alpha, \beta \in \mathfrak{c}$. Then $\text{Ext}(C(K), c_0) \neq 0$. □

**Definition 2.5.** We say that a compact Hausdorff space $K$ satisfies property (\*) if there exist a sequence $(F_n)_{n \in \omega}$ of closed subsets of $K$ and a bounded weak*-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in $C(K)$ such that:

$$F_n \cap \bigcup_{m \neq n} F_m = \emptyset$$

(1)

and $\text{supp} f_{n,\alpha} \subset F_n$, for all $n \in \omega$ and all $\alpha \in \mathfrak{c}$, where $\text{supp} f_{n,\alpha}$ denotes the support of $f_{n,\alpha}$.

In what follows, we denote by $M(K)$ the space of finite countably-additive signed regular Borel measures on $K$, endowed with the total variation norm. We identify as usual the dual space of $C(K)$ with $M(K)$.

**Lemma 2.6.** Let $K$ be a compact Hausdorff space and $L$ be a closed subspace of $K$. If $L$ satisfies property (\*), then so does $K$.

**Proof.** Consider, as in Definition 2.5, a sequence $(F_n)_{n \in \omega}$ of closed subsets of $L$ and a bounded weak*-$\mathfrak{c}$-null biorthogonal system $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ in $C(L)$. By recursion on $n$, one easily obtains a sequence $(U_n)_{n \in \omega}$ of pairwise disjoint open subsets of $K$ with each $U_n$ containing $F_n$. Let $V_n$ be an open subset of $K$ with $F_n \subset V_n \subset \overline{V_n} \subset U_n$. Using Tietze’s Extension Theorem and Urysohn’s Lemma, we get a continuous extension $\tilde{f}_{n,\alpha}$ of $f_{n,\alpha}$ to $K$ with support contained in $\overline{V_n}$ and having the same norm as $f_{n,\alpha}$. To conclude the proof, let $\tilde{\gamma}_{n,\alpha} \in M(K)$ be the extension of $\gamma_{n,\alpha} \in M(L)$ that vanishes identically outside of $L$ and observe that $(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \mathfrak{c}}$ is a bounded weak*-null biorthogonal system in $C(K)$. □

As an immediate consequence of Lemma 2.6 and Corollary 2.4, we obtain the following result.
Theorem 2.7. If a compact Hausdorff space $L$ satisfies property $(\ast)$, then every compact Hausdorff space $K$ containing a homeomorphic copy of $L$ satisfies $\text{Ext}(C(K), c_0) \neq 0$. 

We now establish a few results which give sufficient conditions for a space $K$ to satisfy property $(\ast)$. Recall that, given a closed subset $F$ of a compact Hausdorff space $K$, an extension operator for $F$ in $K$ is a bounded operator $E : C(F) \to C(K)$ which is a right inverse for the restriction operator $C(K) \ni f \mapsto f|_F \in C(F)$. Note that $F$ admits an extension operator in $K$ if and only if the kernel

$$
C(K|F) = \{ f \in C(K) : f|_F = 0 \}
$$

of the restriction operator is complemented in $C(K)$. A point $x$ of a topological space $X$ is called a cluster point of a sequence $(S_n)_{n \in \omega}$ of subsets of $X$ if every neighborhood of $x$ intersects $S_n$ for infinitely many $n \in \omega$.

Lemma 2.8. Let $K$ be a compact Hausdorff space. Assume that there exist a sequence $(F_n)_{n \in \omega}$ of pairwise disjoint closed subsets of $K$ and a closed subset $F$ of $K$ satisfying the following conditions:

(a) $F$ admits an extension operator in $K$;

(b) every cluster point of $(F_n)_{n \in \omega}$ is in $F$ and $F_n \cap F = \emptyset$, for all $n \in \omega$;

(c) there exists a family $(f_{n,\alpha}, \gamma_{n,\alpha})_{n \in \omega, \alpha \in \xi}$, where $(f_{n,\alpha}, \gamma_{n,\alpha})_{\alpha \in \xi}$ is a weak*-null biorthogonal system in $C(F_n)$ for each $n \in \omega$ and

$$
\sup_{n \in \omega, \alpha \in \xi} \|f_{n,\alpha}\| < +\infty, \quad \sup_{n \in \omega, \alpha \in \xi} \|\gamma_{n,\alpha}\| < +\infty.
$$

Then $K$ satisfies property $(\ast)$.

Proof. From (b) and the fact that the $F_n$ are pairwise disjoint it follows that (1) holds. Let $(U_n)_{n \in \omega}$, $(V_n)_{n \in \omega}$, and $(\tilde{f}_{n,\alpha})_{n \in \omega, \alpha \in \xi}$ be as in the proof of Lemma 2.6, we assume also that $V_n \cap F = \emptyset$, for all $n \in \omega$. Let $\tilde{\gamma}_{n,\alpha} \in M(K)$ be the extension of $\gamma_{n,\alpha} \in M(F_n)$ that vanishes identically outside of $F_n$. We have that $(\tilde{f}_{n,\alpha}, \tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \xi}$ is a bounded biorthogonal system in $C(K)$ and that $(\tilde{\gamma}_{n,\alpha})_{\alpha \in \xi}$ is weak*-null for each $n$, though it is not true in general that the entire family $(\tilde{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \xi}$ is weak*-null. In order to take care of this problem, let $P : C(K) \to C(K|F)$ be a bounded projection and set $\hat{\gamma}_{n,\alpha} = \tilde{\gamma}_{n,\alpha} \circ P$. Since all $\tilde{f}_{n,\alpha}$ are in $C(K|F)$, we have that $(\hat{f}_{n,\alpha}, \hat{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \xi}$ is biorthogonal. To prove that $(\hat{\gamma}_{n,\alpha})_{n \in \omega, \alpha \in \xi}$ is weak*-null, note that (b) implies that $\lim_{n \to +\infty} \|f_{\hat{f}_{n}}\| = 0$, for all $f \in C(K|F)$. 

Corollary 2.9. Let $K$ be a compact Hausdorff space. If $C(K)$ admits a bounded weak*-null biorthogonal system of cardinality $\mathfrak{c}$, then the space $[0, \omega] \times K$ satisfies property $(\ast)$. In particular, $L \times K$ satisfies property $(\ast)$ for every compact Hausdorff space $L$ containing a nontrivial convergent sequence.

Proof. Take $F_n = \{n\} \times K$, $F = \{\omega\} \times K$, and use the fact that $F$ is a retract of $[0, \omega] \times K$ and thus admits an extension operator in $[0, \omega] \times K$. 

\[\square\]
Corollary 2.10. The spaces $[0, \omega] \times [0, c]$ and $2^c$ satisfy property $(\ast)$. In particular, a product of at least $c$ compact Hausdorff spaces with more than one point satisfies property $(\ast)$.

Proof. The family $(\chi_{[0,\alpha]}, \delta_\alpha - \delta_{\alpha+1})_{\alpha \in c}$ is a bounded weak*-null biorthogonal system in $C([0, c])$, where $\delta_\alpha \in M([0, c])$ denotes the probability measure with support $\{\alpha\}$. It follows from Corollary 2.9 that $[0, \omega] \times [0, c]$ satisfies property $(\ast)$. To see that $2^c$ also does, note that the map $[0, c] \ni \alpha \mapsto \chi_{\alpha} \in 2^c$ embeds $[0, c]$ into $2^c$, so that $2^c \cong 2^\omega \times 2^c$ contains a homeomorphic copy of $[0, \omega] \times [0, c]$. \qed

Recall that a projectional resolution of the identity (PRI) of a Banach space $X$ is a family $(P_\alpha)_{\omega \leq \alpha \leq \text{dens}(X)}$ of projection operators $P_\alpha : X \to X$ satisfying the following conditions:

- $\|P_\alpha\| = 1$, for $\omega \leq \alpha \leq \text{dens}(X)$;
- $P_{\text{dens}(X)}$ is the identity of $X$;
- $P_\alpha[X] \subset P_\beta[X]$ and $\text{Ker}(P_\beta) \subset \text{Ker}(P_\alpha)$, for $\omega \leq \alpha \leq \beta \leq \text{dens}(X)$;
- $P_\alpha(x) = \lim_{\beta < \alpha} P_\beta(x)$, for all $x \in X$, if $\omega < \alpha \leq \text{dens}(X)$ is a limit ordinal;
- $\text{dens}(P_\alpha[X]) \leq |\alpha|$, for $\omega \leq \alpha \leq \text{dens}(X)$.

We call the PRI strictly increasing if $P_\alpha[X]$ is a proper subspace of $P_\beta[X]$, for $\omega \leq \alpha < \beta \leq \text{dens}(X)$.

Corollary 2.11. Let $K$ and $L$ be compact Hausdorff spaces such that $L$ contains a nontrivial convergent sequence and $w(K) \geq c$. If $C(K)$ admits a strictly increasing PRI, then the space $L \times K$ satisfies property $(\ast)$.

Proof. This follows from Corollary 2.9 by observing that if a Banach space $X$ admits a strictly increasing PRI, then $X$ admits a weak*-null biorthogonal system $(x_\alpha, \gamma_\alpha)_{\omega \leq \alpha < \text{dens}(X)}$ with $\|x_\alpha\| = 1$ and $\|\gamma_\alpha\| \leq 2$, for all $\alpha$. Namely, pick a unit vector $x_\alpha$ in $P_{\alpha+1}[X] \cap \text{Ker}(P_\alpha)$ and set $\gamma_\alpha = \phi_\alpha \circ (P_{\alpha+1} - P_\alpha)$, where $\phi_\alpha \in X^*$ is a norm-one functional satisfying $\phi_\alpha(x_\alpha) = 1$. \qed

3. Nontrivial Twisted Sums for Corson Compacta

Let us recall some standard definitions. Given an index set $I$, we write $\Sigma(I) = \{x \in \mathbb{R}^I : \text{supp } x \text{ is countable}\}$, where the support $\text{supp } x$ of $x$ is defined by $\text{supp } x = \{i \in I : x_i \neq 0\}$. Given a compact Hausdorff space $K$, we call $A$ a $\Sigma$-subset of $K$ if there exist an index set $I$ and a continuous injection $\varphi : K \to \mathbb{R}^I$ such that $A = \varphi^{-1}[\Sigma(I)]$. The space $K$ is called a Valdivia compactum if it admits a dense $\Sigma$-subset and it is called a Corson compactum if $K$ is a $\Sigma$-subset of itself. This section will be dedicated to the proof of the following result.

Theorem 3.1. If $K$ is a Corson compact space with $w(K) \geq c$, then $\text{Ext}(C(K), c_0) \neq 0$. In particular, under CH, we have $\text{Ext}(C(K), c_0) \neq 0$ for any nonmetrizable Corson compact space $K$. 
The fact that Ext \((C(K), c_0) \neq 0\) for a Valdivia compact space \(K\) which does not have ccc is already known ([6, Proposition 2]). Our strategy for the proof of Theorem 3.1 is to use Lemma 2.8 to show that if \(K\) is a Corson compact space with \(w(K) \geq c\) having ccc, then \(K\) satisfies property (\(\ast\)). We start with a lemma that will be used as a tool for verifying the assumptions of Lemma 2.8. Recall that a closed subset of a topological space is called regular if it is the closure of an open set (equivalently, if it is the closure of its own interior). Obviously, a closed subset of a Corson compact space is again Corson and a regular closed subset of a Valdivia compact space is again Valdivia.

**Lemma 3.2.** Let \(K\) be a compact Hausdorff space and \(F\) be a closed non-open \(G_\delta\) subset of \(K\). Then there exists a sequence \((F_n)_{n \in \omega}\) of nonempty pairwise disjoint regular closed subsets of \(K\) such that condition (b) in the statement of Lemma 2.8 holds.

**Proof.** We can write \(F = \bigcap_{n \in \omega} V_n\), with each \(V_n\) open in \(K\) and \(V_{n+1}\) properly contained in \(V_n\). Set \(U_n = V_n \setminus \overline{V_{n+1}}\), so that all cluster points of \((U_n)_{n \in \omega}\) are in \(F\). To conclude the proof, let \(F_n\) be a nonempty regular closed set contained in \(U_n\). \(\square\)

Once we get the closed sets \((F_n)_{n \in \omega}\) from Lemma 3.2 we still have to verify the rest of the conditions in the statement of Lemma 2.8. First, we need an assumption ensuring that \(w(F_n) \geq c\), for all \(n\). To this aim, given a point \(x\) of a topological space \(X\), we define the weight of \(x\) in \(X\) by:

\[
w(x, X) = \min \{ w(V) : V \text{ neighborhood of } x \text{ in } X \}.
\]

Recall that if \(K\) is a Valdivia compact space, then \(C(K)\) admits a PRI ([15, Theorem 2]). Moreover, a trivial adaptation of the proof in [15] shows in fact that \(C(K)\) admits a strictly increasing PRI. Thus, by the argument in the proof of Corollary 2.11 \(C(K)\) admits a weak*-null biorthogonal system \((f_\alpha, \gamma_\alpha)_{\omega \leq \alpha < w(K)}\) such that \(\|f_\alpha\| \leq 1\) and \(\|\gamma_\alpha\| \leq 2\), for all \(\alpha\). The following result is now immediately obtained.

**Corollary 3.3.** Let \(K\) be a Valdivia compact space such that \(w(x, K) \geq c\), for all \(x \in K\). Assume that there exists a closed non-open \(G_\delta\) subset \(F\) admitting an extension operator in \(K\). Then \(K\) satisfies property (\(\ast\)). \(\square\)

Assuming that \(K\) has ccc, the next lemma allows us to reduce the proof of Theorem 3.1 to the case when \(w(x, K) \geq c\), for all \(x \in K\).

**Lemma 3.4.** Let \(K\) be a ccc Valdivia compact space and set:

\[H = \{ x \in K : w(x, K) \geq c \}\]

Then:

(a) \(H \neq \emptyset\), if \(w(K) \geq c\);  
(b) \(w(K \setminus \text{int}(H)) < c\), where \(\text{int}(H)\) denotes the interior of \(H\);  
(c) \(H\) is a regular closed subset of \(K\).
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(d) $w(x, H) \geq \mathfrak{c}$, for all $x \in H$.

Proof. If $H = \emptyset$, then $K$ can be covered by a finite number of open sets with weight less than $\mathfrak{c}$, so that $w(K) < \mathfrak{c}$. This proves (a). To prove (b), let $(U_i)_{i \in I}$ be maximal among antichains of open subsets of $K$ with weight less than $\mathfrak{c}$. Since $I$ is countable and $\mathfrak{c}$ has uncountable cofinality, we have that $U = \bigcup_{i \in I} U_i$ has weight less than $\mathfrak{c}$. From the maximality of $(U_i)_{i \in I}$, it follows that $K \setminus H \subset U$; then $K \setminus \text{int}(H) = K \setminus H \subset U$. To conclude the proof of (b), let us show that $w(U) < \mathfrak{c}$. Let $A$ be a dense $\Sigma$-subset of $K$ and let $D$ be a dense subset of $A \cap U$ with $|D| \leq w(U)$. Then $\overline{D}$ is homeomorphic to a subspace of $R^{w(U)}$, so that $w(U) = w(\overline{D}) \leq w(U) < \mathfrak{c}$.

To prove (c), note that $H$ is clearly closed; moreover, by (b), the open set $K \setminus \text{int}(H)$ has weight less than $\mathfrak{c}$ and hence it is contained in $K \setminus H$. Finally, to prove (d), let $V$ be a closed neighborhood in $K$ of some $x \in H$. By (b), we have $w(V \setminus H) < \mathfrak{c}$. Recall from [9, p. 26] that if a compact Hausdorff space is the union of not more than $\kappa$ subsets of weight not greater than $\kappa$, then the weight of the space is not greater than $\kappa$. Since $w(V) \geq \mathfrak{c}$, it follows from such result that $w(V \cap H) \geq \mathfrak{c}$.

Proof of Theorem 3.1. By Lemma 3.4, it suffices to prove that if $K$ is a nonempty Corson compact space such that $w(x, K) \geq \mathfrak{c}$ for all $x \in K$, then $K$ satisfies property $(\ast)$. Since a nonempty Corson compact space $K$ admits a $G_\delta$ point $x$ ([11, Theorem 3.3]), this fact follows from Corollary 3.3 with $F = \{x\}$.

Remark 3.5. It is known that under Martin’s Axiom (MA) and the negation of CH, every ccc Corson compact is metrizable ([11]). Thus, Theorem 3.1 implies that $\text{Ext}(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space $K$ under MA.

4. TOWARDS THE GENERAL VALDIVIA CASE

In this section we prove that $\text{Ext}(C(K), c_0) \neq 0$ for certain classes of nonmetrizable Valdivia compact spaces $K$ and we propose a strategy for dealing with the general problem. First, let us state some results which are immediate consequences of what we have done so far.

Proposition 4.1. If $K$ is a Valdivia compact space with $w(K) \geq \mathfrak{c}$ and $L$ is a compact Hausdorff space containing a nontrivial convergent sequence, then $L \times K$ satisfies property $(\ast)$.

Proof. As we have observed in Section 3 if $K$ is a Valdivia compact space, then $C(K)$ admits a strictly increasing PRI. The conclusion follows from Corollary 2.11.

Proposition 4.2. Let $K$ be a Valdivia compact space admitting a $G_\delta$ point $x$ with $w(x, K) \geq \mathfrak{c}$. Then $\text{Ext}(C(K), c_0) \neq 0$ and, if $K$ has ccc, then $K$ satisfies property $(\ast)$. 

Proof. If $H = \emptyset$, then $K$ can be covered by a finite number of open sets with weight less than $\mathfrak{c}$, so that $w(K) < \mathfrak{c}$. This proves (a). To prove (b), let $(U_i)_{i \in I}$ be maximal among antichains of open subsets of $K$ with weight less than $\mathfrak{c}$. Since $I$ is countable and $\mathfrak{c}$ has uncountable cofinality, we have that $U = \bigcup_{i \in I} U_i$ has weight less than $\mathfrak{c}$. From the maximality of $(U_i)_{i \in I}$, it follows that $K \setminus H \subset U$; then $K \setminus \text{int}(H) = K \setminus H \subset U$. To conclude the proof of (b), let us show that $w(U) < \mathfrak{c}$. Let $A$ be a dense $\Sigma$-subset of $K$ and let $D$ be a dense subset of $A \cap U$ with $|D| \leq w(U)$. Then $\overline{D}$ is homeomorphic to a subspace of $R^{w(U)}$, so that $w(U) = w(\overline{D}) \leq w(U) < \mathfrak{c}$.

To prove (c), note that $H$ is clearly closed; moreover, by (b), the open set $K \setminus \text{int}(H)$ has weight less than $\mathfrak{c}$ and hence it is contained in $K \setminus H$. Finally, to prove (d), let $V$ be a closed neighborhood in $K$ of some $x \in H$. By (b), we have $w(V \setminus H) < \mathfrak{c}$. Recall from [9, p. 26] that if a compact Hausdorff space is the union of not more than $\kappa$ subsets of weight not greater than $\kappa$, then the weight of the space is not greater than $\kappa$. Since $w(V) \geq \mathfrak{c}$, it follows from such result that $w(V \cap H) \geq \mathfrak{c}$.

Proof of Theorem 3.1. By Lemma 3.4, it suffices to prove that if $K$ is a nonempty Corson compact space such that $w(x, K) \geq \mathfrak{c}$ for all $x \in K$, then $K$ satisfies property $(\ast)$. Since a nonempty Corson compact space $K$ admits a $G_\delta$ point $x$ ([11, Theorem 3.3]), this fact follows from Corollary 3.3 with $F = \{x\}$.

Remark 3.5. It is known that under Martin’s Axiom (MA) and the negation of CH, every ccc Corson compact is metrizable ([11]). Thus, Theorem 3.1 implies that $\text{Ext}(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space $K$ under MA.

4. TOWARDS THE GENERAL VALDIVIA CASE

In this section we prove that $\text{Ext}(C(K), c_0) \neq 0$ for certain classes of nonmetrizable Valdivia compact spaces $K$ and we propose a strategy for dealing with the general problem. First, let us state some results which are immediate consequences of what we have done so far.

Proposition 4.1. If $K$ is a Valdivia compact space with $w(K) \geq \mathfrak{c}$ and $L$ is a compact Hausdorff space containing a nontrivial convergent sequence, then $L \times K$ satisfies property $(\ast)$.

Proof. As we have observed in Section 3 if $K$ is a Valdivia compact space, then $C(K)$ admits a strictly increasing PRI. The conclusion follows from Corollary 2.11.

Proposition 4.2. Let $K$ be a Valdivia compact space admitting a $G_\delta$ point $x$ with $w(x, K) \geq \mathfrak{c}$. Then $\text{Ext}(C(K), c_0) \neq 0$ and, if $K$ has ccc, then $K$ satisfies property $(\ast)$. 

Proof. If $H = \emptyset$, then $K$ can be covered by a finite number of open sets with weight less than $\mathfrak{c}$, so that $w(K) < \mathfrak{c}$. This proves (a). To prove (b), let $(U_i)_{i \in I}$ be maximal among antichains of open subsets of $K$ with weight less than $\mathfrak{c}$. Since $I$ is countable and $\mathfrak{c}$ has uncountable cofinality, we have that $U = \bigcup_{i \in I} U_i$ has weight less than $\mathfrak{c}$. From the maximality of $(U_i)_{i \in I}$, it follows that $K \setminus H \subset U$; then $K \setminus \text{int}(H) = K \setminus H \subset U$. To conclude the proof of (b), let us show that $w(U) < \mathfrak{c}$. Let $A$ be a dense $\Sigma$-subset of $K$ and let $D$ be a dense subset of $A \cap U$ with $|D| \leq w(U)$. Then $\overline{D}$ is homeomorphic to a subspace of $R^{w(U)}$, so that $w(U) = w(\overline{D}) \leq w(U) < \mathfrak{c}$.

To prove (c), note that $H$ is clearly closed; moreover, by (b), the open set $K \setminus \text{int}(H)$ has weight less than $\mathfrak{c}$ and hence it is contained in $K \setminus H$. Finally, to prove (d), let $V$ be a closed neighborhood in $K$ of some $x \in H$. By (b), we have $w(V \setminus H) < \mathfrak{c}$. Recall from [9, p. 26] that if a compact Hausdorff space is the union of not more than $\kappa$ subsets of weight not greater than $\kappa$, then the weight of the space is not greater than $\kappa$. Since $w(V) \geq \mathfrak{c}$, it follows from such result that $w(V \cap H) \geq \mathfrak{c}$.

Proof of Theorem 3.1. By Lemma 3.4, it suffices to prove that if $K$ is a nonempty Corson compact space such that $w(x, K) \geq \mathfrak{c}$ for all $x \in K$, then $K$ satisfies property $(\ast)$. Since a nonempty Corson compact space $K$ admits a $G_\delta$ point $x$ ([11, Theorem 3.3]), this fact follows from Corollary 3.3 with $F = \{x\}$.

Remark 3.5. It is known that under Martin’s Axiom (MA) and the negation of CH, every ccc Corson compact is metrizable ([11]). Thus, Theorem 3.1 implies that $\text{Ext}(C(K), c_0) \neq 0$ for every nonmetrizable Corson compact space $K$ under MA.
Proof. As mentioned before, the non-ccc case is already known. Assuming that $K$ has ccc, define $H$ as in Lemma 3.4 and conclude that $H$ satisfies property (*)&. 

Corollary 4.3. Let $K$ be a Valdivia compact space with $w(K) \geq c$ admitting a dense $\Sigma$-subset $A$ such that $K \setminus A$ is of first category. Then Ext($C(K), c_0$) $\neq 0$ and, if $K$ has ccc, then $K$ satisfies property (*)&.

Proof. By [11, Theorem 3.3], $K$ has a dense subset of $G_\delta$ points. Assuming that $K$ has ccc and defining $H$ as in Lemma 3.4 we obtain that $H$ contains a $G_\delta$ point of $K$, which implies that $K$ satisfies the assumptions of Proposition 4.2.

Now we investigate conditions under which a Valdivia compact space $K$ contains a homeomorphic copy of $[0, \omega] \times [0, c]$. Given an index set $I$ and a subset $J$ of $I$, we denote by $r_J : R^I \to R^I$ the map defined by setting $r_J(x)|J = x|_J$ and $r_J(x)|_{I \setminus J} \equiv 0$, for all $x \in R^I$. Following [2], given a subset $K$ of $R^I$, we say that $J \subset I$ is $K$-good if $r_J[K] \subset K$. In [2, Lemma 1.2], it is proven that if $K$ is a compact subset of $R^I$ and $\Sigma(I) \cap K$ is dense in $K$, then every infinite subset $J$ of $I$ is contained in a $K$-good set $J'$ with $|J| = |J'|$.

Proposition 4.4. Let $K$ be a Valdivia compact space admitting a dense $\Sigma$-subset $A$ such that some point of $K \setminus A$ is the limit of a nontrivial sequence in $K$. Then $K$ contains a homeomorphic copy of $[0, \omega] \times [0, \omega_1]$. In particular, assuming $CH$, we have that $K$ satisfies property (*)&.

Proof. We can obviously assume that $K$ is a compact subset of some $R^I$ and that $A = \Sigma(I) \cap K$. Since $A$ is sequentially closed, our hypothesis implies that there exists a continuous injective map $[0, \omega] \ni n \mapsto x_n \in K \setminus A$. Let $J$ be a countable subset of $I$ such that $x_n|J \neq x_m|J$, for all $n, m \in [0, \omega]$ with $n \neq m$. Using [2, Lemma 1.2] and transfinite recursion, one easily obtains a family $(J_\alpha)_{\alpha \leq \omega_1}$ of $K$-good subsets of $I$ satisfying the following conditions:

- $J_\alpha$ is countable, for $\alpha < \omega_1$;
- $J \subset J_0$;
- $J_\alpha \subset J_\beta$, for $0 \leq \alpha \leq \beta \leq \omega_1$;
- $J_\alpha = \bigcup_{\beta < \alpha} J_\beta$, for limit $\alpha \in [0, \omega_1]$;
- for all $n \in [0, \omega]$, the map $[0, \omega_1] \ni \alpha \mapsto J_\alpha \cap \text{supp } x_n$ is injective.

Given these conditions, it is readily checked that the map $[0, \omega] \times [0, \omega_1] \ni (n, \alpha) \mapsto r_{J_\alpha}(x_n) \in K$ is continuous and injective.

Remark 4.5. The following converse of Proposition 4.4 also holds: if $K$ is a Valdivia compact space containing a homeomorphic copy of $[0, \omega] \times [0, \omega_1]$, then $K \setminus A$ contains a nontrivial convergent sequence, for any dense $\Sigma$-subset $A$ of $K$. Namely, if $\phi : [0, \omega] \times [0, \omega_1] \to K$ is a continuous injection, then $\phi(n, \alpha) \in K \setminus A$ for some $\alpha \in [0, \omega_1]$, since $[0, \omega_1]$ is not Corson ([11, Example 1.10 (i)]). Moreover, the nontrivial sequence $(\phi(n, \alpha))_{n \in \omega}$ converges
to \( \phi(\omega, \alpha) \). One consequence of this observation is that if \( K \setminus A \) contains a nontrivial convergent sequence for some dense \( \Sigma \)-subset \( A \) of \( K \), then \( K \setminus A \) contains a nontrivial convergent sequence for any dense \( \Sigma \)-subset \( A \) of \( K \).

Remark 4.6. If a Valdivia compact space \( K \) admits two distinct dense \( \Sigma \)-subsets, then the assumption of Proposition 4.4 holds for \( K \). Namely, given dense \( \Sigma \)-subsets \( A \) and \( B \) of \( K \) and a point \( x \in A \setminus B \), then \( x \) is not isolated, since \( B \) is dense. Moreover, \( x \) is not isolated in \( A \), because \( A \) is dense. Finally, since \( A \) is a Fréchet–Urysohn space ([11, Lemma 1.6 (ii)]), \( x \) is the limit of a sequence in \( A \setminus \{ x \} \).

Finally, we observe that the validity of the following conjecture would imply, under CH, that \( \text{Ext}(C(K), c_0) \neq 0 \) for any nonmetrizable Valdivia compact space \( K \).

Conjecture. If \( K \) is a nonempty Valdivia compact space having \( ccc \), then either \( K \) has a \( G_\delta \) point or \( K \) admits a nontrivial convergent sequence in the complement of a dense \( \Sigma \)-subset.

To see that the conjecture implies the desired result, use Lemma 3.4 and Propositions 4.2 and 4.3, keeping in mind that a regular closed subset of a ccc space has ccc as well. The conjecture remains open, but in what follows we present an example showing that it is false if the assumption that \( K \) has ccc is removed.

Recall that a tree is a partially ordered set \( (T, \leq) \) such that, for all \( t \in T \), the set \( \{ s \in T : s < t \} \) is well-ordered. As in [14, p. 288], we define a compact Hausdorff space from a tree \( T \) by considering the subspace \( P(T) \) of \( 2^T \) consisting of all characteristic functions of paths of \( T \); by a path of \( T \) we mean a totally ordered subset \( A \) of \( T \) such that \( (\cdot, t) \subset A \), for all \( t \in A \). It is easy to see that \( P(T) \) is closed in \( 2^T \); we call it the path space of \( T \).

Denote by \( S(\omega_1) \) the set of countable successor ordinals and consider the tree \( T = \bigcup_{\alpha \in S(\omega_1)} \omega_1^\alpha \), partially ordered by inclusion. The path space \( P(T) \) is the image of the injective map \( \Lambda : \omega \ni \lambda \mapsto \chi_{A(\lambda)} \in 2^T \), where \( \Lambda = \bigcup_{\alpha \leq \omega_1} \omega_1^\alpha \) and \( A(\lambda) = \{ t \in T : t \subset \lambda \} \).

Proposition 4.7. If the tree \( T \) is defined as above, then its path space \( P(T) \) is a compact subspace of \( \mathbb{R}^T \) satisfying the following conditions:

(a) \( P(T) \cap \Sigma(T) \) is dense in \( P(T) \), so that \( P(T) \) is Valdivia;
(b) \( P(T) \) has no \( G_\delta \) points;
(c) no point of \( P(T) \setminus \Sigma(T) \) is the limit of a nontrivial sequence in \( P(T) \).

Proof. To prove (a), note that \( \chi_{A(\lambda)} = \lim_{\alpha < \omega_1} \chi_{A(\lambda^\alpha)} \) for all \( \lambda \in \omega_1^{\omega_1} \). Let us prove (b). Since \( P(T) \) is Valdivia, every \( G_\delta \) point of \( P(T) \) must be in \( \Sigma(T) \) ([11, Proposition 2.2 (3)])], i.e., it must be of the form \( \chi_{A(\lambda)} \) with \( \lambda \in \omega_1^\alpha, \alpha < \omega_1 \). To see that \( \chi_{A(\lambda)} \) cannot be a \( G_\delta \) point of \( P(T) \), it suffices to check that for any countable subset \( E \) of \( T \), there exists \( \mu \in \Lambda, \mu \neq \lambda \), such that \( \chi_{A(\lambda)} \) and \( \chi_{A(\mu)} \) are identical on \( E \). To this aim, simply take
\[ \mu = \lambda \cup \{ (\alpha, \beta) \}, \text{ with } \beta \in \omega \setminus \{ t(\alpha) : t \in E \text{ and } \alpha \in \text{dom}(t) \}. \]

Finally, to prove (c), let \( (\chi_{A(\lambda_n)})_{n \geq 1} \) be a sequence of pairwise distinct elements of \( P(T) \) converging to some \( \epsilon \in P(T) \) and note that the support of \( \epsilon \) must be contained in the countable set \( \bigcup_{n \neq m} (A(\lambda_n) \cap A(\lambda_m)) \).

It is easy to see that, for \( T \) defined as above, the space \( P(T) \) does not have ccc. Namely, setting \( U_t = \{ \epsilon \in P(T) : \epsilon(t) = 1 \} \) for \( t \in T \), we have that \( U_t \) is a nonempty open subset of \( P(T) \) and that \( U_t \cap U_s = \emptyset \), when \( t, s \in T \) are incomparable.

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