A Note on the Orderability of Dehn Fillings of
the Manifold $v2503$

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April 18, 2019

Abstract

We show that Dehn filling on the manifold $v2503$ results in a non-orderable space for all rational slopes in the interval $(-\infty, -1)$. This is consistent with the L-space conjecture, which predicts that all fillings will result in a non-orderable space for this manifold.

1 Introduction

This paper studies the orderability of a certain 3-manifold in view of an outstanding conjectured relationship between orderability and L-spaces.

A left-ordering on a group $G$ is a total ordering $\prec$ on the elements of $G$ that is invariant under left-multiplication; that is, $g \prec h$ implies $fg \prec fh$ for all $f, g, h \in G$. A group is said to be left-orderable if it is nontrivial and admits a left ordering. A 3-manifold $M$ is called orderable if $\pi_1(M)$ is left-orderable.

If $M$ is a rational homology 3-sphere, then the rank of its Heegaard Floer homology is greater than or equal to the order of its first (integral) homology group. $M$ is called an L-space if equality holds; that is, if $\text{rk}(\widehat{HF}(M)) = |H_1(M; \mathbb{Z})|$.

This work is motivated by the following proposed connection between L-spaces and orderability, first conjectured by Boyer, Gordon, and Watson.

Conjecture 1 ([BGW13]). An irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable.
In \cite{BGW13}, this equivalence was shown to hold for all closed, connected, oriented, geometric three-manifolds that are non-hyperbolic.

If $M$ is a rational homology solid torus, then a framing of the boundary $(\mu, \lambda)$ is called a homological framing for $\partial M$ if $\lambda$ is (rationally) nullhomologous. Given a framing on $\partial M$ and a reduced fraction $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$, we denote the $\frac{p}{q}$ Dehn filling by $M(\frac{p}{q})$.

Culler and Dunfield \cite{CD16} have remarked that the cusped hyperbolic manifold $v2503$ has the property that every non-longitudinal Dehn filling is an L-space (the longitudinal filling is $S^1 \times S^2 \# \mathbb{RP}^3$). Thus, if Conjecture \ref{conjecture} holds, one would expect none of the Dehn fillings of $v2503$ to be orderable (the longitudinal filling is non-orderable as its fundamental group has torsion). To that end, we prove the following partial result:

\textbf{Theorem 1.} Let $M = v2503$. Then for a certain homological framing, $M(r)$ is not orderable for any rational slope $r \in (-\infty, -1)$.

\section*{Acknowledgements}

The author would like to thank Professors Zoltán Szabó and Peter Ozsváth for suggesting this problem as well as for providing feedback on drafts of this paper.

\section{Ordering}

We note the following useful facts, which hold for any left-ordered group $(G, \prec)$:

- For each $g \in G$, $1 \prec g \iff g^{-1} \prec 1$
- For all $a, b \in G$, $1 \prec a, b \Rightarrow 1 \prec ab$ and similarly $a, b \prec 1 \Rightarrow ab \prec 1$.

We also call any element $g$ of $G$ \textit{positive} whenever $1 \prec g$, and similarly, $g$ is said to be \textit{negative} if $g \prec 1$.

Let $M$ be a compact, connected, oriented irreducible 3-manifold with incompressible torus boundary, and let $(\mu, \lambda)$ be a framing for $\partial M$. In \cite{CW10}, Clay and Watson describe a criterion for obstructing left-orderability of Dehn fillings of $M$. One corollary of that criterion is:
Theorem 2 ([CW10]). Let \( \frac{p}{q}, \frac{p_0}{q_0}, \frac{p_1}{q_1} \) be rational numbers satisfying \( \frac{p}{q} \in (\frac{p_0}{q_0}, \frac{p_1}{q_1}) \) such that \( q, q_0, q_1 > 0 \) and \( p, p_0, p_1 < 0 \). Suppose that \( \pi_1(\partial M) \) is not sent to 1 by the quotient map \( \pi_1(M) \to \pi_1\left(M\left(\frac{p}{q}\right)\right) \) and that for each left ordering \( \prec \) of \( \pi_1(M) \), \( \mu^{p_0} \lambda^{q_0} \prec 1 \) implies \( \mu^{p_1} \lambda^{q_1} \prec 1 \). Then \( \pi_1\left(M\left(\frac{p}{q}\right)\right) \) is not left-orderable.

Remark. This is essentially Corollary 2.2 in [CW10] except in that paper, \( p, p_0, p_1 \) are all required to be positive; however, their proof works just as well assuming they are all negative instead. Alternatively, one can simply replace \( \mu \) with \( \mu^{-1} \) and apply their theorem directly, noting that the only necessary property of \( \mu \) and \( \lambda \) is that they generate \( \pi_1(\partial M) \).

3 The Manifold \( v2503 \)

Now let us turn our attention to the manifold named \( v2503 \) in the SnapPy census [CDGW], which we denote \( M \) for the rest of this section. It is also known as \( M_{2459} \) in the nomenclature of [CHW99]. \( M \) is a hyperbolic 3-manifold with one toroidal cusp, and \( M \) is also a rational homology solid torus. Indeed, SnapPy gives that \( H_1(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} \).

Fundamental Group

According to SnapPy, the fundamental group of \( M = v2503 \) has the following presentation:

\[
\pi_1(M) = \langle a, b | a^2b^{-2}ab^{-2}a^2ba^2baba^2b = 1 \rangle
\]  

(1)

In addition, SnapPy also gives that the “meridian” \( m \) and “longitude” \( l \) are:

\[
m = b^{-1}a^2ba^2
\]

\[
l = b^{-2}ab^{-2}ab^{-1}
\]

We follow the convention of Culler and Dunfield [CD16] for the homological framing. In particular, our homological meridian \( \mu \) and homological longitude \( \lambda \) correspond to \((0, 1)\) and \((-1, 0)\) respectively in SnapPy’s framing. That is:

\[
\mu = l = b^{-2}ab^{-2}ab^{-1}
\]

(2)

\[
\lambda = m^{-1} = a^{-2}b^{-1}a^{-2}b
\]

(3)
Notice that, by considering the abelianisation of (1), the generator $a$ corresponds to a generator of the torsion subgroup of $H_1(M; \mathbb{Z})$, whereas $b$ is a free generator. Moreover, $[\mu] = [a]^2[b]^{-5} \in H_1(M; \mathbb{Z})$ and $[\lambda] = [a]^{-4}$, and so $\lambda$ is rationally nullhomologous, which is consistent with its being a homological longitude.

For convenience, let us put:

$$x = b^{-2}a$$
$$y = ba^2$$

We record for later the following:

$$a^2x^2aba^2baba^2b = 1$$
$$\mu = x^2b^{-1}$$
$$\lambda = y^{-2}b^2$$
$$\lambda = baba^2ba^2b^{-2}ab^{-1} = bay^2xb^{-1}$$
$$\mu^{-1}\lambda = \lambda\mu^{-1} = bay^2x^{-1} = ya^{-1}y^2x^{-1}$$

Apart from (9), these are straightforward consequences of (1)–(5). To see why (9) holds, observe that the group relation in (1) can be rewritten as:

$$1 = a^2b^{-2}ab^{-1}(b^{-1}a^2ba^2)baba^2b$$
$$= a^2b^{-2}ab^{-1}(a^{-2}b^{-1}a^{-2}b)^{-1}baba^2b$$
$$= a^2b^{-2}ab^{-1}\lambda^{-1}baba^2b$$

where (3) was used in the last step to substitute for $\lambda$. Now the desired expression follows by isolating $\lambda$ in the equation above.

**Orderability constraints for $v2503$**

We now use the information about the fundamental group of $v2503$ to prove the following observations, which are the basic ingredients for the proof of the main theorem.

**Lemma 1.** Let $\prec$ be a left ordering of $\pi_1(v2503)$. If $\mu^{-1}\lambda \prec 1$ then $\mu^{-n}\lambda \prec 1$ for all $n \geq 1$. 

Proof. Suppose that $\mu^{-1}\lambda < 1$. There are four cases, depending on the signs of the generators $a$ and $b$.

Case I: $b < 1 < a$. In this case, $1 < \mu$ since, by (2), $\mu$ can be expressed as the product of positive terms. Hence, $\mu^{-1} < 1$ and so for each $n \geq 1$, $\mu^{-n}\lambda < 1$ as it is the product of negative terms.

Case II: $a, b < 1$. Notice that, by (6), it must hold that $1 < x$ for otherwise, $1$ would be expressed as the product of negative terms. Now by (7), we see that $\mu$ is the product of positive terms, and hence $1 < \mu$. As in Case I, we once again have $\mu^{-n}\lambda < 1$ for all $n \geq 1$.

Case III: $1 < a, b$. In this case, we see from (5) that $1 < y$ as $y$ is the product of positive terms. On the other hand, we have that $x < 1$ for otherwise, $1$ would be expressed as the product of positive terms in (6). But then, by (10), we see that $\mu^{-1}\lambda$ is expressed as a product of positive terms, contradicting the hypothesis that $\mu^{-1}\lambda < 1$. So this case cannot happen.

Case IV: $a < 1 < b$. In (4), we see $x$ expressed as the product of negative terms, and so $x < 1$. Now, by (10'), we conclude that $y < 1$ as otherwise, $\mu^{-1}\lambda$ would be the product of positive terms, contradicting the hypothesis that $\mu^{-1}\lambda < 1$. Now, by (8), $1 < \lambda$ because $\lambda$ is expressed as the product of positive terms. The hypothesis that $\mu^{-1}\lambda < 1$ implies, by invariance under left-multiplication, that $\lambda < \mu$. Hence, $1 < \mu$, but, from (2) we see $\mu$ as a product of positive elements, a contradiction. So this case, too, cannot happen.

Lemma 2. Let $r \in \mathbb{Q}$. If $\pi_1(\partial M)$ is sent to $1$ by the quotient map $\pi_1(M) \to \pi_1(M(r))$, then $M(r)$ is not orderable.

Proof. If the subgroup $\pi_1(\partial M)$ of $\pi_1(M)$ is sent to $1$ by the quotient map, then that map factors as: $\pi_1(M) \to \langle \pi_1(M) | \mu = 1, \lambda = 1 \rangle \to \pi_1(M(r))$. Let us examine the group $G = \langle \pi_1(M) | \mu = 1, \lambda = 1 \rangle$. By (7), (8), and (10), we see that the following relations hold in $G$:

\begin{align*}
  b &= x^2 \\
  b^2 &= y^2 \\
  x &= bay^2
\end{align*}

(11) \hspace{1cm} (12) \hspace{1cm} (13)

Notice further that (6) can be re-written as

\[ x^2ay^2a^{-1}bay^2 = 1 \]
This becomes, using (11) and (13):

\[ x a^{-1} y^2 = 1 \]

Using (11) and (12), this becomes:

\[ x a^{-1} x^4 = 1 \]

\[ a = x^5 \]

Hence, recalling (4) and (5), \( G \) has the following presentation:

\[ G = \langle a, b, x, y | a = x^5, b = x^2, x = b^{-2} a, y = ba^2, b^2 = y^2, x = bay^2 \rangle \]

This can be simplified to:

\[ G = \langle x, y | y = x^{12}, x^4 = y^2, x = x^7 y^2 \rangle \]

\[ = \langle x | 1 = x^{20}, 1 = x^{30} \rangle \]

\[ = \langle x | 1 = x^{10} \rangle \]

\[ \cong \mathbb{Z}/10\mathbb{Z} \]

Therefore, as \( \pi_1(M(r)) \) is the quotient of a finite group, it is finite as well, and hence not left-orderable (recall that, by convention, the trivial group is considered not left-orderable).

We are now ready to prove our main result.

**Proof of Theorem 1.** Let \( r \in \mathbb{Q} \cap (-\infty, -1) \). By Lemma 2, we may assume that \( \pi_1(\partial M) \) is not sent to 1 by the quotient map \( \pi_1(M) \to \pi_1(M(r)) \). Furthermore, as \( M \) is hyperbolic, it is irreducible and has incompressible torus boundary. Then, since \( r \in (-n, -1) \) for some integer \( n \geq 1 \), Lemma 1 together with Theorem 2 tells us that \( M(r) \) is not orderable, as required.

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