Is power-law inflation really attractive?

César A. Terrero-Escalante
Instituto de Física, UNAM, Apdo. Postal 20-364, 04510, México D.F., México.

It is argued that the order of the analytic expressions for the calculation of the primordial perturbations from inflation exerts a strong influence upon the results of the analysis of observables dynamics based on these expressions and, therefore, upon some predictions sometimes taken for granted as generic for the inflationary scenario.

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I. PRELIMINARIES

The idea about the energy density in the very early universe being dominated by the potential energy of a single real scalar field, commonly called inflaton, is strongly supported by the analysis of recent cosmological observations [1,2]. In this scenario density and metrics quantum fluctuations were stretched beyond the causal horizon due to the accelerated expansion produced by a negative pressure. Much after the inflationary period is ended these fluctuations reentered the causal horizon giving rise to perturbations in the gravitational potential with an almost scale-invariant spectrum. Depending on the exact time of reentering the causal universe these perturbations became the seeds for anisotropies in the cosmic microwave background (CMB) radiation or for the large scale structure formation. Therefore, measuring the spectra of CMB anisotropies and density distribution in the observable universe, the corresponding spectrum of primordial perturbations can be determined. Each inflaton potential characterizes different physics and, consequently, different spectra of fluctuations. This way, determining the primordial spectra can give us important hints about the physics in the very early universe.

Density (scalar) and metrics (tensorial) perturbations can be described by means of the spectral indices [1],

\[ \Delta \equiv \frac{n_S - 1}{2} = \frac{d \ln A_S}{d \ln k}, \]
\[ \delta \equiv \frac{n_T}{2} = \frac{d \ln A_T}{d \ln k}, \]

where the subscripts S and T stand for scalar and tensor modes respectively, \( A_S,T \) denotes the corresponding normalized spectral amplitudes and \( k = a H \) is the comoving wavenumber when the mode crosses the Hubble radius.

One of the very few models where closed analytic expressions of the perturbation spectra are available is power-law inflation, \( a(t) \propto t^n \) where \( p > 1 \) is a constant [3]. For this model the Hubble parameter, \( H \), and the potential, \( V \), as functions of the inflaton field, \( \phi \), are given by, \( H \propto \exp(-\sqrt{\kappa / 2p} \phi) \) and \( V \propto \exp(-\sqrt{2\kappa} \phi) \), where \( \kappa = 8\pi/m_{Pl}^2 \) is the Einstein constant and \( m_{Pl} \) is the Planck mass. This model has several attractive features concerning theory and observations. First of all, the exponential function is the limit of a Taylor series similar to the tree expansion often used for scalar field potentials arising in quantum field theory [4]. Second, the exponential potential is the only inflationary model yielding exact power-law spectra [4], \( A_S \propto k^{-3} \) and \( A_T \propto k^0 \). The deviation of these spectra amplitudes from scale-invariance (also known as the tilt of the spectra) is given by the constant spectral indices \( \delta = \Delta = -m/a + 1/p^2 + 1/p^3 + \cdots \leq 0 \). Since the spectra scale-dependence is predicted to be very weak, the power-law parametrization of the primordial spectra is used very often while analyzing the CMB data [3].

For more general potentials the spectra must be calculated numerically or by means of approximations based on the smallness of the horizon flow functions [1,3].

\[ \epsilon_{m+1} \equiv \frac{d \ln |\epsilon_m|}{dN}, \quad m \geq 0, \quad (3) \]
\[ \epsilon_0 \equiv \frac{d \ln A_{H}}{d \ln t}, \quad (4) \]
\[ \epsilon_1 \equiv \frac{d \ln d_{H}}{dN}. \quad (5) \]

where \( N \equiv \ln(a/a_i) \) is the number of e-folds since some initial time \( t_i \), and \( d_{H} \equiv d_{H}(t_i) \). According to (5),

\[ \epsilon_1 \equiv \frac{d \ln d_{H}}{dN}. \quad (5) \]

Inflation happens for \( \epsilon_1 < 1 \) (equivalent to \( \dot{a} > 0 \)) and \( \epsilon_1 > 0 \) from the weak energy condition (for a spatially flat universe). For \( m > 1 \), \( \epsilon_m \) may take any real value. Expressions (3) define a flow in the space \( \{\epsilon_m\} \). This flow is described by the equations of motion

\[ \epsilon_0 \epsilon_m = \frac{1}{d_{H}^{m+1}}. \quad (6) \]

According with Eqs. (3), for power-law inflation we have \( \epsilon_1 = 1/p = \text{constant} \) and \( \epsilon_m = 0 \) for \( m > 1 \).

So far, in the case of a slowly rolling inflaton, the more precise expressions for the spectral amplitudes were derived in Ref. [3]. (Rigourously speaking, the scalar amplitudes were derived by Stewart and Gong [4]. Using the

*Electronic address: ctererro@fis.cinvestav.mx
same procedure the tensorial ones were derived later by Leach et al. [5]). From these next-to-next-to-leading order expressions for the amplitudes and using definitions [4] and [5], the spectral indices are obtained,

\[
\Delta = - \epsilon_1 - \frac{1}{2} \epsilon_2 + \epsilon_2 - (C + \frac{3}{2} \epsilon_1 \epsilon_2 - \frac{C}{2} \epsilon_2 \epsilon_3 \\
- \epsilon_1^3 - (3C - \frac{\pi^2}{2} + \frac{17}{2}) \epsilon_1 \epsilon_2 \\
- \frac{3}{2} \epsilon_1 - \frac{C}{2} + \frac{17}{2} \epsilon_1 \epsilon_2 \\
- \frac{2}{2C + \frac{\pi^2}{2} + \frac{7}{2}} \epsilon_1 \epsilon_2 \epsilon_3 \\
- \frac{1 - \frac{\pi^2}{8}}{2} \epsilon_2 \epsilon_3 - \frac{C}{4} \epsilon_2 \epsilon_3 \epsilon_4 \\
- \frac{C}{4} - \frac{\pi^2}{48} \epsilon_2 \epsilon_3 \epsilon_4 ,
\]

\[
\delta = - \epsilon_1 - \epsilon_2 - (C + 1) \epsilon_1 \epsilon_2 \\
- \epsilon_1^3 - (3C - \frac{\pi^2}{2} + 8) \epsilon_1 \epsilon_2 \\
- \frac{C}{2} + C - \frac{\pi^2}{24} + \frac{1}{2} \epsilon_1 \epsilon_2 \\
- \frac{C}{2} + C - \frac{\pi^2}{24} + \frac{1}{2} \epsilon_1 \epsilon_2 \epsilon_3 .\n\]

It is easy to check that the limit of a very slowly rolling inflaton \( (\epsilon_m \to 0) \) coincides with power-law inflation. Since the current precision of the CMB data does not allow to discern any scale dependence for the spectral indices, it is commonly assumed that the underlying inflationary scenario must belong to the class of extreme slow-roll inflation. In this contribution it is shown how the qualitative analysis of the inflationary dynamics as described by the expressions used to calculate the primordial spectra strongly depends on the order of these expressions.

\section{II. DISCUSSION}

The leading order is recovered from Eqs. (4) and (5) by neglecting terms with order higher than the linear one,

\[
\Delta = - \epsilon_1 - \frac{\epsilon_2}{2} , \quad (9) \\
\delta = - \epsilon_1 . \quad (10)
\]

First of all, let us note that strictly speaking and according with Eqs. (4), neglecting terms like \( \epsilon_1 \epsilon_2 \) implies \( \epsilon_1 = \text{constant} \) and \( \epsilon_m = 0 \), leading to power-law inflation. However, in our approximation we will allow quadratic terms being neglected without necessarily neglecting any linear term. In other words, the horizon flow functions are assumed to be very weakly time-dependent. Using definitions (5) we can rewrite system (4) and (5) as,

\[
\dot{\delta} = \dot{\delta} - \Delta , \quad (11)
\]

where a circumflex accent denotes differentiation with respect to the variable \( \tau \), defined such that \( d\tau = d\ln H^2 \) and \( d\tau / dN = -2 \epsilon_1 \). As can be noted, power-law inflation is a trivial solution to this equation and, in fact, it is a fixed point of the dynamics described by this equation. Since this is a first order differential equation this fixed point can only be an attractor or a repeller. We have two observables we would like to trace for in order to make generic predictions for inflation, namely, \( \delta \) and \( \Delta \). Because the differentiation only involves the tensorial index, we are forced to make assumptions on the behavior of the scalar index or, at least, on the difference between the scalar and tensorial indices, taking into account that this difference is generally a function of time. Conditions upon \( (\delta - \Delta) \) can be translated into conditions upon \( \epsilon_2 \) which in terms of the inflaton potential and its derivatives with respect to \( \phi \) (denoted with prime) reads [4],

\[
\epsilon_2 \approx \frac{2}{\kappa} \left[ \frac{V'}{V} \right] - \frac{V''}{V} . \quad (12)
\]

Hence, power-law inflation will be an attractor of the dynamics described by Eq. (11) only if \( \epsilon_2 > 0 \), i.e., if \( V' > -V'' \) (note that \( d\tau / d\tau < 0 \)). This will be the case for any potential with \( V'' < 0 \) (in this paper we are considering the inflaton potential to be nonnegative). Examples of models with this kind of potential are the inverted quadratic model, the cosinus model, models with \( V \propto \phi^{-n} \) where \( n < m \) are some integers, etc. On the other hand, there are models with \( V'' > 0 \) and satisfying \( V'^2 > VV'' \). For instance, we have all the monomial models with \( V \propto \phi^p \) and \( p > 1 \) a real number. Indeed, the above listed models are amongst the most popular inflationary scenarios [4]. However, it is possible to find some other interesting models with \( V'' > 0 \) but such that \( V'^2 < VV'' \), i.e., scenarios where power-law inflation will be a repellor rather than an attractor. One of these models is, for instance, the hyperbolic cosinus having Taylor series quite close to the scalar field potential tree expansion with only even order terms [4]. This model provides a counterexample to the leading order analysis by Hoffmann and Turner [5] which yields power-law inflation as a generic attractor of the inflationary dynamics. The point here is that \( x'' \), where \( x \equiv V' / V \), assumed in Ref. [5] to be constant, for the hyperbolic cosinus model is equal to \( -2a^2 \sinh(ax) / \cosh^2(ax) \), and will not be approximately constant unless \( a \ll 1 \). For the next-to-leading order analysis we must keep only up to second order terms in Eqs. (4) and (5). After converting to differential equations in terms of \( \tau \) [4],

\[
2C \epsilon_1 \epsilon_1 - (2C + 3) \epsilon_1 \epsilon_1 - \epsilon_1 + \epsilon_1 + \Delta = 0 , \quad (13) \\
2(C + 1) \epsilon_1 \epsilon_1 - \epsilon_1 - \epsilon_1 - \delta = 0 , \quad (14)
\]

where \( C = -0.7296 \). Let us note here that this system of differential equations it is not a closed system. Even if we differentiate Eq. (14) with respect to \( \tau \) and substitute the result and Eq. (13) itself in Eq. (14), we still have
three independent variables to deal with. Information on the functional forms of one of the observables $\Delta$ and $\delta$ is needed again to describe the dynamics of $\epsilon_1$.

The dynamical analysis of Ref. [3] indicates that in the reduced phase spaces ($\Delta = $ constant) for the evolution of $\epsilon_1$ as described by Eq. (13), there exists a saddle point in the region where $\epsilon_1$ has interesting values and $\Delta < 0$. Thus, with respect to cosmic time, attractor–like behavior will be characteristic only of those trajectories that are very close to the unstable separatrices. Likewise, the saddle point acts as a repellor for those trajectories that are closer to the stable separatrices. For blue tilted spectra, $\Delta > 0$, there is not fixed point of any kind in the physically interesting range of $\epsilon_1$.

Thus, with respect to cosmic time, attractor–like behavior exists in both cases. These models provide further counterexamples to the leading order results in Ref. [8]. In particular, the eigenvalues of the corresponding Jacobian matrix have the form $\lambda_{\pm} = (g_2 \pm \sqrt{g_2^2 + 4g_1})/2$, where $g_{1,2}$ are functions of the roots of equation $\epsilon_1^3 + \epsilon_1^2 + \epsilon_1 + \delta = 0$. It can be shown that for $\delta < 0$ the eigenvalues will have opposite sign indicating the existence in the reduced phase space of a saddle point corresponding to $\epsilon_1 = $ constant. Thus, to next-to-next-to-leading order there is not more a constrain forcing the dynamics to be close to power-law inflation and this scenario becomes less attractive than to lower orders.

III. CONCLUSIONS

It was argued that, even if the power-law inflationary model has very attractive features from the theoretical and observational points of view, and it is the limit of scenarios with extremely slow rolling inflaton fields, it cannot be claimed that such a scenario must be expected to be an attractor of the inflationary dynamics driven by general potentials, particularly those that do not belong to the class describing a very slow rolling regime. In general, any claim about generic predictions for the inflationary observables dynamics must take into account the order of the expressions involved unless a non-dependence on the order is granted. This point concerns also the reliability of programmes for the inflaton potential reconstruction. The order to be used to calculate the primordial spectra will be fixed by the quality of the forthcoming cosmological observations, in particular by the capability of detecting any scale dependence of these spectra.

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References

[1] A. D. Linde, Particle Physics and Inflationary Cosmology, Chur: Harwood, 1990, pp. 362.; A. R. Liddle and D. H. Lyth, Cosmological Inflation and Large-Scale Structure, Cambridge: Univ. Press, 2000, pp. 400.
[2] W. Hu and S. Dodelson, astro-ph/0104114 (2001).
[3] F. Lucchin and S. Matarrese, Phys. Rev. D52, (1991) 1316.
[4] L. F. Abbott and M. B. Wise, Nucl. Phys. B 244, 541 (1984).
[5] D. J. Schwarz, C. A. Terrero-Escalante and A. A. Garcia, Phys. Lett. B517, (2001) 243. astro-ph/0106029.
[6] E.D. Stewart and J.O. Gong, Phys. Lett. B 510, (2001).
[7] S. M. Leach, A. R. Liddle, J. Martin and D. J. Schwarz, astro-ph/0202094 (2002).
[8] M. B. Hoffman and M. S. Turner, Phys. Rev. D 64, 023506 (2001).
[9] E. Ayon-Beato, A. Garcia, R. Mansilla and C. A. Terrero-Escalante, Phys. Rev. D62, (2000) 103513. astro-ph/0007477.
[10] E. Ayon-Beato, A. Garcia, R. Mansilla and C. A. Terrero-Escalante, in Proceedings of III DGFM-SMF Workshop on Gravitation and Mathematical Physics, León, México (2000), eds.: N. Bretón, O. Pimentel, and J. Socorro, astro-ph/0009355 (2000).
[11] C. A. Terrero-Escalante, E. Ayon-Beato and A. A. Garcia, Phys. Rev. D64, (2001) 023503. astro-ph/0101522.
[12] C. A. Terrero-Escalante and A. A. Garcia, Phys. Rev. D65, (2002) 023515. astro-ph/0108188.
[13] C. A. Terrero-Escalante, J. E. Lidsey and A. A. Garcia, Phys. Rev. D65 (2002) 083509. astro-ph/0111128.