Evolution of physical observables and entropy in laser process studied on the basis of Kraus-form solution of laser’s master equation

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Though laser physics began at 1960s, its entropy evolution has not been touched until very recently the Kraus-form solution of laser’s master equation is reported (Ann. Phys. 334 (2013)). We study the new physics based on the discovery in this paper. We analyze time evolution of physical observables and quantum optical properties in the laser process with arbitrary initial states, such as the photon number, the second degree of coherence, etc. The evolution of entropy of these states is also studied. Our results well conform with the known behavior of laser, which confirms that the master equation describes laser suitably, and the Kraus-form operator solution is correct, elegant and useful.

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\section{I. INTRODUCTION}

The birth of laser, opening up the research area of quantum optics in 1960s, is one of the greatest inventions of mankind. Quantum mechanically, the stable laser above some threshold is described by a coherent state. However, as a thermo object, the entropy evolution of a laser process has not been touched until very recently we evaluated it after solving time evolution master equation of the laser\textsuperscript{[4]}

\begin{equation}
\frac{d\rho(t)}{dt} = g[2a^\dagger \rho(t)a - aa^\dagger \rho(t) - \rho(t)aa^\dagger] + \kappa [2\rho(t)a^\dagger - a^\dagger \rho(t) - \rho(t)a^\dagger a],
\end{equation}

and obtained its solution in the form of infinite sum (or named Kraus form solution)

\begin{equation}
\rho(t) = \sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger,
\end{equation}

\(M_{ij}\) is usually called Kraus operator\textsuperscript{[5,6]}. In Eqn. \textsuperscript{[1]} \(g\) and \(\kappa\) represent the cavity gain and loss respectively, \(a^\dagger\) and \(a\) are photon creation and annihilation operator. For Eqn. \textsuperscript{[2]} we have used the entangled state representation\textsuperscript{[7]} to derive

\begin{equation}
M_{ij} = \sqrt{T_{1i}T_{2j}} e^{a^\dagger a} \ln T_{2} \rho_0 a^\dagger \rho_0 a,
\end{equation}

and

\begin{equation}
T_1 = \frac{1 - e^{-2(\kappa-g)t}}{\kappa - ge^{-2(\kappa-g)t}}, \quad T_2 = \frac{(\kappa - g)e^{-2(\kappa-g)t}}{\kappa - ge^{-2(\kappa-g)t}}.
\end{equation}

It is noticeable that they are not independent of each other, in fact

\begin{equation}
T_3 = 1 - gT_1, \quad T_2^2/T_3 = 1 - \kappa T_1.
\end{equation}

One can check the trace conservative law

\begin{equation}
\sum_{i,j=0}^{\infty} M_{ij}^\dagger M_{ij} = 1.
\end{equation}

Since laser’s evolution and properties are very important we must answer the important question: does the solution in Eqs. \textsuperscript{[2,4]} can indeed reflect laser channel’s physical properties? In this work, we shall study the evolution of some important physical quantities in laser process such as the photon number, the second degree of coherence, for laser with arbitrary initial states. Since entropy involves both thermodynamic and informative properties, it is important to explore the evolution of entropy of these states. As one can see shortly later, our results well conform with the behavior of laser, which confirms that our solution Eqs. \textsuperscript{[2,4]} is correct, elegant and useful.

\section{II. THE EVOLUTION OF \(\rho\)}

In terms of the number state \(|m\rangle = \frac{a^m}{\sqrt{m!}} \langle 0|\) in Fock space, and
\[a\langle m\rangle = \sqrt{m \langle m-1\rangle} , \ a^\dagger \langle m\rangle = \sqrt{m+1 \langle m+1\rangle} .\]

For arbitrary initial state \(\rho_0 = \sum_{m,n=0}^{\infty} \rho_{m,n} \langle m\rangle \langle n|\), using Eqs. \textsuperscript{[2,4]} we have

\begin{equation}
\rho(t) = \sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger
= \sum_{i,j=0}^{\infty} \sum_{m,n=0}^{\infty} T_{1i}^{m+i} T_{2j}^{m+j} \frac{m!n!(m+i+j)!(n+i+j)!}{(m-i)!n!(m-n+i)!} \times \rho_{m,n} \langle m-i+j\rangle \langle n+i+j|.
\end{equation}
We now discuss \( \rho(t) \) for different working conditions of laser as follows. If \( \kappa > g \), gain is less than loss, then from Eqn. (1) we know

\[
T_1 = \frac{1}{\kappa} + O \left( e^{-2(\kappa-g)t} \right), \quad (8)
\]

\[
T_2 = \frac{\kappa - g}{\kappa} e^{-(\kappa-g)t} + O \left( e^{-3(\kappa-g)t} \right),
\]

\[
T_3 = \frac{\kappa - g}{\kappa} + O \left( e^{-2(\kappa-g)t} \right).
\]

Therefore only those terms with \( m' = n' = 0 \) in Eqn. (1) will contribute to \( \rho(+\infty) \) (long time limit)

\[
\rho(+\infty) = \sum_{i,j=0}^{\infty} T_3 (+\infty) \kappa^i g^j T_1^{i+j} (+\infty) \rho_{ii} |j\rangle \langle j|.
\]

The quantum state approaches to therm-equilibrium state with equivalent temperature \( T = \frac{\hbar c}{k_B \ln 2} \), where \( k_B \) is the Boltzmann constant. The properties of therm-equilibrium state are well-known to the physicists.

If \( \kappa < g \), then

\[
T_1 = \frac{1}{g} - \frac{g - \kappa}{g^2} e^{2(\kappa-g)t} + O \left( e^{3(\kappa-g)t} \right), \quad (10)
\]

\[
T_2 = \frac{g - \kappa}{g} e^{(\kappa-g)t} + O \left( e^{3(\kappa-g)t} \right),
\]

\[
T_3 = \frac{g - \kappa}{g} e^{2(\kappa-g)t} + O \left( e^{3(\kappa-g)t} \right).
\]

The overall factor \( T_3 \) of \( \rho(t) \) in Eqn. (1) is also exponentially small, therefore \( \rho(t) \) will not approach to any specific state when \( t \to +\infty \). More works are need to analysis the behavior of the laser when \( \kappa < g \).

III. EVOLUTION OF THE EXPECTATION OF PHYSICAL QUANTITIES IN LASER

The expectation of a physical quantity \( \hat{A} \) for a system described by density operator \( \rho \) is defined by \( \langle \hat{A} \rangle \equiv Tr (\hat{A} \rho) \). Since \( \rho \) evolves with time, \( \langle \hat{A} \rangle \) is a function of time too. For laser processes with different initial states \( \rho_0 \), it will be overwhelming to calculate \( \rho(t) = \sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger \) for each \( \rho_0 \). Noticing that

\[
\langle \hat{A} \rangle_t \equiv Tr \left[ \hat{A} \rho(t) \right] = Tr \left( \sum_{i,j=0}^{\infty} \hat{A} M_{ij} \rho_0 M_{ij}^\dagger \right), \quad (11)
\]

\[
= Tr \left( \sum_{i,j=0}^{\infty} M_{ij}^\dagger \hat{A} M_{ij} \rho_0 \right) = Tr \left( \hat{A}_t \rho_0 \right)
\]

where the evolving \( \hat{A} \) operator is defined as

\[
\hat{A}_t = \sum_{i,j=0}^{\infty} M_{ij}^\dagger \hat{A} M_{ij}, \quad (12)
\]

\[
= \sum_{i,j=0}^{\infty} T_3 \kappa^i g^j T_1^{i+j} a_i^\dagger a_j e^{\kappa t} T_2 e^{\kappa t} a_i a_j.
\]

Thus instead of calculating \( \rho(t) \) for each \( \rho_0 \), we just need to calculate the evolving \( \hat{A} \) operator \( \hat{A}_t \) for each \( \hat{A} \). Once we obtain \( \hat{A}_t \), we can calculate \( \langle \hat{A} \rangle_t \equiv \text{Tr} (\hat{A}_t \rho_0) \) for arbitrary initial state \( \rho_0 \) straightforwardly. Further, even when the operator \( \hat{A}(t) \) itself is time dependent, we can still define the evolving \( \hat{A}(t) \) operator \( \hat{A}(t) \equiv \sum_{i,j=0}^{\infty} M_{ij}^\dagger \hat{A}(t) M_{ij} \). And we still have \( \langle \hat{A}(t) \rangle_t = \text{Tr} \left[ \hat{A}(t) \rho(t) \right] = \text{Tr} \left( \hat{A}(t) \rho_0 \right) \).

IV. THE GENERATING FUNCTION OF THE EVOLVING \((a^\dagger a)^m\) OPERATORS

In this section we do some preparatory work for calculating the evolution of the expected photon number and the second degree of coherence in lasers.

Since

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \lambda^m (a^\dagger a)^m = e^{\lambda a^\dagger a}, \quad (13)
\]

we have

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \lambda^m (a^\dagger a)^m = \langle e^{\lambda a^\dagger a} \rangle_t. \quad (14)
\]

Once we obtain \( \langle e^{\lambda a^\dagger a} \rangle_t \), the evolving photon number operators \( \langle (a^\dagger a)^m \rangle_t \) can be calculated using the generating function \( \langle e^{\lambda a^\dagger a} \rangle_t \) (this is also named the cumulant expansion):

\[
\langle (a^\dagger a)^m \rangle_t = \frac{d^m}{d\lambda^m} \langle e^{\lambda a^\dagger a} \rangle_t |_{\lambda = 0}. \quad (15)
\]

Similarly, since

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \mu^m a^\dagger a \lambda^m =: e^{\lambda a^\dagger a} := \exp \left[ \ln (1 + \mu) a^\dagger a \right], \quad (16)
\]

we have

\[
\langle a^\dagger a \lambda^m \rangle_t = \lambda^m \frac{d^m}{d\mu^m} \langle e^{\ln (1 + \mu) a^\dagger a} \rangle_t |_{\mu = 0}. \quad (17)
\]

Now we employ the completeness relation of the coherent state \( |z\rangle = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \)

\[
\int \frac{d^2z}{\pi} |z\rangle \langle z| = \int \frac{d^2z}{\pi} e^{-|z|^2 + a^\dagger a \lambda z^* + z^* a^\dagger a} := 1 \quad (18)
\]
and the method of integration within an ordered product of operators, we can calculate the generating function $\left< e^{\lambda a^\dagger a} \right>_t$

\[
\left< e^{\lambda a^\dagger a} \right> = \sum_{i,j=0}^{\infty} \frac{T_3 T_3^*}{1-g_{T_1}(1+\mu)} e^{\lambda i a^\dagger a} e^{2(\ln T_2+\lambda) a^\dagger a} e^{2(\ln T_2+\lambda) a^\dagger a}
\]

\[
= \sum_{i,j=0}^{\infty} \frac{T_3 T_3^*}{1-g_{T_1}(1+\mu)} e^{\lambda i a^\dagger a} e^{(\lambda+2 \ln T_2+\lambda) a^\dagger a} a^\dagger a
\]

\[
= \int \frac{d\lambda}{\pi} e^{\lambda T_2^*} e^{\lambda T_2} e^{\lambda T_2^*} e^{\lambda T_2} e^{\lambda T_2^*} e^{\lambda T_2}
\]

\[
= \frac{T_3}{1-g_{T_1}(1+\mu)} \exp \left[ \left( \frac{T_2^* T_2}{1-g_{T_1}(1+\mu)} + \kappa T_1 - 1 \right) a^\dagger a \right]
\]

\[
(19)
\]

Note that the convergence of the integration over $\lambda$ demands that $1 - g_{T_1}(1+\mu) > 0$, $\lambda < -\ln (g_{T_1})$.

We can immediately write down

\[
\left< e^{\lambda a^\dagger a} \right>_t = \left< e^{\lambda a^\dagger a} \right>_0 e^{\kappa (T_1 T_2) + \kappa T_1 - 1} a^\dagger a
\]

Thus using Eqs. (17) (20) we have

\[
\left< a^\dagger a \right>_t = e^{2(\kappa - g)t} \left[ \left< a^\dagger a + \frac{g}{g - \kappa} \right> - \frac{g}{g - \kappa} \right]
\]

And if $g > \kappa$ we have further

\[
\left< a^\dagger a \right>_t \sim N g e^{2\delta(\kappa - g)t}
\]

as $t \to +\infty$, where $N$ is a linear combination of $\left< a^\dagger a \right>_0$ ($\delta \leq \delta$).

We have the following conclusions for arbitrary initial states:

If $g < \kappa$, damping is larger than pumping, then $\left< a^\dagger a \right>_t \sim \frac{g}{\kappa - g}$ as $t \to +\infty$, as one should expect for therm-equilibrium state with temperature $T = \frac{h c}{k_B T_0}$.

If $g > \kappa$, pumping is larger than damping, then for any initial states the system’s photon number $\left< a^\dagger a \right>_t$ will increase exponentially, $\left< a^\dagger a \right>_t \sim e^{2(\kappa - g)t} \left( \left< n_0 \right> + \frac{g}{g - \kappa} \right)$, as a laser should behave when it is working well.

Now we examine how the second degree of coherence $g^{(2)}_0 = \frac{\left< a^\dagger a \right>}{\left< a^\dagger a \right>^2}$ evolves into $g^{(2)}(t) = \frac{\left< a^\dagger a \right>}{\left< a^\dagger a \right>^2}$. According to Eqs. (21) (22) we have

\[
g^{(2)}(t) = \frac{\left< a^\dagger a \right>}{\left< a^\dagger a \right>^2} = 2 + \frac{g_0^{(2)} - 2}{1 + \chi^2}
\]

where

\[
\chi(t) = \frac{g T_1}{e^{2(\kappa - g)t} \left( \left< n_0 \right> + T_3 \right)}
\]

When $g \leq \kappa$, $\chi(t) = \frac{g T_1}{e^{2(\kappa - g)t} \left( \left< n_0 \right> + T_3 \right)}$, $\chi(t)$ = $+\infty$, therefore $g^{(2)}(\infty) = 2$ for any initial state $\rho_0$.

When $g > \kappa$, $\chi(\infty) = \frac{g T_1}{\left( \left< n_0 \right> + T_3 \right)} > 0$, $g^{(2)}(\infty) = 2 + \frac{g_0^{(2)} - 2}{1 + \chi(\infty)^2}$.

(27)

For arbitrary initial state $\rho_0$, we have

\[
\rho^{(2)}_0 = \frac{\left< n_0 \right> + T_3 - \left< n_0 \right>^2}{\left< n_0 \right>^2} \geq 1 - \frac{1}{\left< n_0 \right>^2},
\]

therefore

\[
\rho^{(2)}(\infty) = 2 + \frac{\left< n_0 \right> - 2}{1 + \chi(\infty)^2}
\]

(29)

\[
\geq 2 + \frac{1}{\left< n_0 \right>^2} - \frac{1}{\left< n_0 \right>^2} > 1,
\]

i.e., when a laser works in $g > \kappa$ region, the photons of the laser tend to be bunching.

V. THE EVOLUTION OF ENTROPY

In our preceding paper [5] for an initial state $\rho_0 = \left< z \right> \left< z \right|$, we have derived the exact entropy expression

\[
S(\rho_z(t)) = -k_B \left( \ln T_3 + \frac{g T_1}{1 - g T_1} \ln g T_1 \right),
\]

(30)

which is independent of the initial value $z$. Therefore for initial state $\rho_0 = \int \frac{d^2z}{\pi} P(z) \left< z \right> \left< z \right|$, with positive coefficient in its Glauber–Sudarshan P-representation (such quantum system has a classical analog), by the concavity of von Neumann entropy, we have the following inequality

\[
S(\rho(t)) = S \left[ \int \frac{d^2z}{\pi} P(z) \rho_z(t) \right]
\]

(31)

\[
\geq \int \frac{d^2z}{\pi} P(z) S(\rho_z(t))
\]

\[
= S(\rho(t)) \int \frac{d^2z}{\pi} P(z)
\]

\[
= -k_B \left( \ln T_3 + \frac{g T_1}{1 - g T_1} \ln g T_1 \right),
\]
which provides an estimation on the entropy.

Now we examine the evolution of the entropy for more general initial states.

According to Eqn. (22), when $\kappa > g$, the density operator will approach to $\rho(+) = \frac{\kappa}{g - \kappa} \sum_{j=0}^{\infty} (\frac{g}{\kappa})^j |j\rangle \langle j|$ for arbitrary initial state $\rho_0 = \sum_{m,n=0}^{\infty} \rho_{m,n} |m\rangle \langle n|$. The entropy will approach to

$$ S_\infty = -kB \sum_{j=0}^{\infty} \frac{\kappa - g}{\kappa} \left( \frac{g}{\kappa} \right)^j \ln \left[ \frac{\kappa - g}{\kappa} \left( \frac{g}{\kappa} \right)^j \right] \quad (32) $$

$$ = kB \left( \frac{g}{\kappa - g} \ln \frac{g}{\kappa} + \ln \frac{\kappa - g}{\kappa} \right). $$

In the case $\kappa < g$, we first consider initial states which are diagonal in photon-number representation, $\rho_0 = \sum_{m=0}^{\infty} \rho_{m,m} |m\rangle \langle m|$. We have

$$ \rho(t) = \sum_{i,j=0}^{\infty} M_{ij} \rho_0 M_{ij}^\dagger = \sum_{k=0}^{\infty} \rho_{kk} (t) |k\rangle \langle k|, \quad (33) $$

where

$$ \rho_{kk} (t) = \sum_{m=0}^{k} \frac{T_3 (gT_1)^k - m}{(k - m)!} T_2^m \frac{k!}{m!} f^{(m)} (\kappa T_1) \quad (34) $$

and

$$ f(x) = \sum_{i=0}^{\infty} \rho_{ii} x^i, \quad (35) $$

$$ f^{(m)} (x) = \sum_{i=0}^{\infty} \rho_{m+i,m+i}(m + i)! \frac{i^m}{i!} x^i = d^m dx^m f(x). $$

Correspondingly, the von Neumann entropy is

$$ -S/k_B = \sum_{k=0}^{\infty} \rho_{kk} (t) \ln \rho_{kk} (t) = I_1 + I_2, \quad (36) $$

where

$$ I_1 = \sum_{k=0}^{\infty} \rho_{kk} (t) \ln \left[ T_3 (gT_1)^k \right], \quad (37) $$

$$ I_2 = \sum_{k=0}^{\infty} \rho_{kk} (t) \ln \left[ \sum_{m=0}^{k} \frac{T_2^m}{(gT_1)^k} \frac{k! m!}{(k - m)!} f^{(m)} (\kappa T_1) \right]. $$

The first term in Eqn. (36) is

$$ I_1 = \sum_{k=0}^{\infty} \rho_{kk} (t) \ln T_3 + \sum_{k=0}^{\infty} k \rho_{kk} (t) \ln gT_1 $$

$$ = \ln T_3 + \langle \tilde{n} \rangle_0 \ln gT_1 $$

$$ = 2(\kappa - g) t + \ln \frac{g - \kappa}{g} - \left( 1 + \frac{g - \kappa}{g} \langle \tilde{n} \rangle_0 \right) + o(1) $$

when $t \to +\infty$. And $I_2$ is bounded below,

$$ I_2 \geq \sum_{k=0}^{\infty} \rho_{kk} (t) \ln f (\kappa T_1) = \ln f (\kappa T_1). \quad (39) $$

Noticing that $\rho_{ii} > 0$ and $f(1) = \sum_{i=0}^{\infty} \rho_{ii} = 1 < +\infty$, we see that the radius of convergence $R \geq 1$ for power series $f(\frac{\kappa}{g} T_1) = f(\kappa T_1 + y)$ converges for $0 < y \leq R - \kappa T_1$. Therefore each term $f^{(m)} (\kappa T_1) \frac{\kappa^m}{m! (k - m)!}$ is bounded with respect to $m$. We have

$$ \sum_{m=0}^{k} \frac{T_2^m}{(gT_1)^k} \frac{k!}{m!} f^{(m)} (\kappa T_1 + y) \leq f(\kappa T_1 + y) \left( 1 + \frac{T_2^2}{(gT_1)^2} \right)^k, $$

therefore the second term in Eqn. (36)

$$ I_2 \leq \sum_{k=0}^{\infty} \rho_{kk} (t) \ln \left[ f(\kappa T_1 + y) \left( 1 + \frac{T_2^2}{(gT_1)^2} \right)^k \right] $$

$$ = \ln f (\kappa T_1 + y) + \langle \tilde{n} \rangle_0 \ln \left( 1 + \frac{T_2^2}{(gT_1)^2} \right) $$

$$ \sim \ln f \left( \frac{\kappa}{g} + y \right) + \left( \frac{g - \kappa}{g} \langle \tilde{n} \rangle_0 + 1 \right) \frac{g - \kappa}{g} $$

as $t \to +\infty$, which is a finite number for $0 < y \leq R - \frac{\kappa}{g}$. Particularly, choosing $y = 1 - \frac{g}{\kappa}$ in Eqn. (41), we see that

$$ I_2 \lesssim \frac{g - \kappa}{g} \langle \tilde{n} \rangle_0 + 1. \quad (42) $$

Combining Eqn. (39, 42), we see $\ln f (\kappa T_1) \leq I_2 \leq \frac{g - \kappa}{g} \langle \tilde{n} \rangle_0 + 1$ is a finite number as $t \to +\infty$.

Finally from Eqs. (36), (39, 42), we have

$$ 2(\kappa - g) t + \ln \frac{g}{(g - \kappa)(g - \kappa) + 1} + \frac{g - \kappa}{g} \langle \tilde{n} \rangle_0 $$

$$ \gtrsim \frac{S/k_B}{2(\kappa - g) t + O(1)} \gtrsim \frac{2(\kappa - g) t + \ln \frac{g}{g - \kappa}}{2(\kappa - g) t + \frac{g - \kappa}{g}} $$

as $t \to +\infty$ for laser with initial state $\rho_0 = \sum_{m=0}^{\infty} \rho_{m,m} |m\rangle \langle m|$ and $g > \kappa$. The specific entropy

$$ \frac{S}{\langle \tilde{n} \rangle} \sim \frac{2kB (g - \kappa) t}{g - \kappa + \langle \tilde{n} \rangle_0} e^{-2(\kappa - g) t} $$

goes to zero exponentially.
For arbitrary initial states $\rho_0 = \sum_{m,n=0}^{\infty} \rho_{m,n} |m\rangle \langle n|$, we see from Eqn. (7) that the off-diagonal elements of $\rho(t)$ are exponentially small compared with diagonal elements as $t \to +\infty$ when $g > \kappa$, i.e., $\rho(t)$ tends to be diagonal for laser with arbitrary initial state. So we can well assume that $S = 2k_B (g - \kappa) t + O(1)$ as $t \to +\infty$ for laser with arbitrary initial state when $g > \kappa$.

This result affirms that when a laser is working properly $(g > \kappa)$, the entropy increases linearly with time, yet the expected number of photons increases much faster, therefore the specific entropy will goes to zero exponentially. The photons in the laser are highly coherent and bunching in this case.

VI. SUMMARY

In this paper, we analyze the laser process with arbitrary initial states, and obtain the evolution law of the photon number, the second degree of coherence and the entropy. If $\kappa > g$, then the photons in the laser will approach a therm-equilibrium state with equivalent temperature $T = \frac{k_B \ln \frac{\kappa}{g}}{\kappa - g}$. The expected photon number approaches to $\frac{\kappa}{\kappa - g}$, the second degree of coherence $g^{(2)}$ approaches to 2. The entropy approaches to $k_B \left( \frac{g}{\kappa - g} \ln \frac{\kappa}{g} + \ln \frac{\kappa}{\kappa - g} \right)$. If $g > \kappa$, then the photon number will increase exponentially, $\langle n \rangle = e^{2(g-\kappa)t} \left( \langle n \rangle_0 + \frac{g}{g-\kappa} - \frac{g}{g-\kappa} \right)$, and the second degree of coherence $g^{(2)}$ approaches to $2 + \frac{g^{(2)}_{0} - 2}{1 + \frac{g}{g-\kappa} \langle n \rangle_0} > 1$. For $\rho_0 = \sum_{m=0}^{\infty} \rho_{m,m} |m\rangle \langle m|$ in $g > \kappa$ case, we proved that the entropy will increase linearly, $S \sim 2k_B (g - \kappa) t$.

All these results conform with the known behavior of laser, this confirms that the master equation (1) describes laser’s behavior well, and the Kraus-form operator solution (2)-(4) is correct, elegant and useful.

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