Abstract. We obtain necessary and sufficient conditions for the $L^p$-boundedness of the operator with positive Bergman kernel in some homogeneous Siegel domains of $\mathbb{C}^n$. The key tool used here is the Okikiolu test, which finally leads to better result than the Schur test as it has been used so far.

1. Introduction

Let $D$ be a homogeneous Siegel domain of $\mathbb{C}^n$ of type II, $dv$ the Lebesgue measure defined in $\mathbb{C}^n$. We denote by $P$ the Bergman projector i.e., the orthogonal projector of the Hilbert space $L^2(D,dv)$ onto its closed subspace $A^2(D,dv)$ consisting of holomorphic functions on $D$. It is well-known that $P$ is an integral operator defined on $L^p(D,dv)$ whose kernel $B(.,.)$, called the Bergman kernel, is the reproducing kernel of $A^2(D,dv)$. In this paper, we are interested in values of $p$ for which the integral operator denoted $P^+$, whose kernel is $|B(.,.)|$, extends as a bounded operator in $L^p(D,dv)$.

In fact, in [11], the author determined a range of values $1 \leq p, q \leq \infty$ for which the weighted Bergman projector $P_s$ extends as a bounded operator on the weighted Lebesgue mixed normed space $L^{p,q}(T_\Omega, \Delta^s(y)\,dxdy)$; here, $\Delta^s(y) = \Delta_1^{-s_1}(y)\Delta_2^{-s_2}(y)\cdots\Delta_r^{-s_r}(y)$, the $\Delta_j$’s are the principal minors of the symmetric cone according to the terminology of [12] and $T_\Omega$ is a tube domain over the symmetric cone $\Omega$. In the process of obtaining this result, he used the Schur Lemma and found sufficient conditions for the boundedness of the operator $P_s^+$ and did not mention anything about the sharpness of these conditions. Note that the Schur test functions here are the generalized power of the delta function. Later on, in the same spirit, in [17], authors considered the tube
domain over homogeneous cones, symmetric or not. They found sufficient conditions for $P^+_s$ and necessary conditions and realised that they are not the same as in classical case. Moreover, the admissible range for the weights were not the best possible. However, these conditions coincide under some restrictions on the weights and the choice of the cone (the Vinberg cone for example see for instance [8]). It was then normal to ask ourselves why these conditions do not coincide. In one of their recent papers [16], authors considered a family of Bergman-type operators on tube domain $T_\Omega$ over a symmetric cone $\Omega$. The method they used to obtain sharp off-diagonal estimates for this family of operators led to a necessary and sufficient condition for boundedness of the operator $P^+$ as in the classical case. Instead of the Schur Lemma, it is the Okikiolu test that was used. They made it very clear that in this Okikiolu test that generalises the Schur test, the use of generalised power of the delta function were not necessary. This observation is what motivated us to reinvestigate the $L^p$-boundedness of the operator $P^+$ in homogeneous Siegel domains, symmetric or not, in the line of works done in [17] and [15].

2. Description of homogeneous cones and homogeneous Siegel domains of type II

In this section, we recall the description of a homogeneous cone within the framework of $T$-algebras. Next, we introduce homogeneous Siegel domains of type II and state our main results.

2.1. Homogeneous cones. We use the same notations as in [10] and [17]. We denote by $\mathcal{U}$ a (real) matrix algebra of rank $r$ with canonical decomposition

$$\mathcal{U} = \bigoplus_{1 \leq i, j \leq r} \mathcal{U}_{ij}$$

such that $\mathcal{U}_{ij} \mathcal{U}_{jk} \subset \mathcal{U}_{ik}$ and $\mathcal{U}_{ij} \mathcal{U}_{lk} = \{0\}$ if $j \neq l$. We assume that $\mathcal{U}$ has a structure of $T$-algebra (in the sense of [20]) in which an involution is given by $x \mapsto x^\ast$. This structure implies that the subspaces $\mathcal{U}_{ij}$ satisfy: $\mathcal{U}_{ii} = \mathbb{R}c_i$ where $c_i^2 = c_i$ and $\dim \mathcal{U}_{ij} = n_{ij} = n_{ji}$. Also, the matrix

$$e = \sum_{j=1}^r c_j$$

is a unit element for the algebra $\mathcal{U}$.

Let $\rho$ be the unique isomorphism from $\mathcal{U}_{ii}$ onto $\mathbb{R}$ with $\rho(c_i) = 1$ for all $i = 1, \ldots, r$. We shall consider the subalgebra

$$\mathcal{T} = \bigoplus_{1 \leq i \leq j \leq r} \mathcal{U}_{ij}$$

of $\mathcal{U}$ consisting of upper triangular matrices and let

$$H = \{ t \in \mathcal{T} : \rho(t_{ii}) > 0, \ i = 1, \ldots, r \}$$
be the subgroup of upper triangular matrices whose diagonal elements are positive.

Denote by $V$ the vector space of "Hermitian matrices" in $\mathcal{U}$

$$V = \{ x \in \mathcal{U} : x^* = x \}.$$ 

If we set

$$n_i = \sum_{j=1}^{i-1} n_{ji}, \quad m_i = \sum_{j=i+1}^{r} n_{ij},$$

then

$$\text{(1)} \quad \dim V = n = r + \sum_{i=1}^{r} m_i = r + \sum_{i=1}^{r} n_i.$$ 

The vector space $V$ becomes a Euclidean space with the inner product

$$(x|y) = tr (xy^*)$$

where

$$tr (x) = \sum_{i=1}^{r} \rho(x_{ii}).$$

Next we define

$$\Omega = \{ ss^* : s \in H \}.$$ 

By a theorem of Vinberg ([20, p. 384]), $\Omega$ is an open convex homogeneous cone containing no entire straight lines, in which the group $H$ acts simply transitively via the transformations

$$\pi(w) : uu^* \mapsto \pi(w)[uu^*] = (wu)(u^*w^*) \quad (w, u \in H).$$

Thus, to every element $y \in \Omega$ corresponds a unique $t \in H$ such that

$$y = \pi(t)[e].$$

Like in [17], we shall adopt the notation:

$$t \cdot e = \pi(t)[e].$$

We shall assume that $\Omega$ is irreducible, and hence rank $(\Omega) = r$. All homogeneous convex cones can be constructed in this way ([20, p. 397]).

As in [17], we denote by $Q_j$ the fundamental rational functions in $\Omega$ given by

$$Q_j(y) = \rho(t_{jj})^2, \quad \text{when } y = t \cdot e \in \Omega.$$ 

We consider the matrix algebra $\mathcal{U}'$ which differs from $\mathcal{U}$ only on its grading, in the sense that

$$\mathcal{U}'_{ij} = \mathcal{U}_{r+1-i, r+1-j} \quad (i, j = 1, \ldots, r).$$

It is proved in [20] that $\mathcal{U}'$ is also a $T$-algebra and $V' = V$ where $V'$ is the subspace of $\mathcal{U}'$ consisting of Hermitian matrices. We define accordingly its subalgebra

$$\mathcal{T}' = \bigoplus_{1 \leq i \leq j \leq r} \mathcal{U}'_{ij}.$$
of $\mathcal{U}$ consisting of lower triangular matrices and the subgroup $H'$ of $T'$ whose diagonal elements are positive. We have

$$T' = \{ t^* : t \in T \} \text{ and } H' = H^* = \{ t^* : t \in H \}.$$ 

The corresponding homogeneous cone coincides with the dual cone of $\Omega$, namely

$$\Omega^* = \{ \xi \in V' : (x|\xi) > 0, \ \forall x \in \overline{\Omega} \setminus \{0\} \}.$$ 

One also has

$$\Omega^* = \{ t^* t : t \in H \}.$$ 

(See [20, p. 390]).

For $\xi = t^* t \in \Omega^*$, we shall define

$$Q^*_j(\xi) = \rho(t^*_{jj}).$$

The group $H'$ acts simply transitively on the cone $\Omega^*$ via the transformations

$$\pi(w^*) : u^* u \mapsto (w^* u^*)(uw) \quad (w^*, u^* \in H').$$

We write

$$t^* \cdot e = \pi(t^*)[e] \quad (t^* \in H').$$

We have the following identity.

$$(4) \quad Q^*_j(t^* \cdot e) = Q_j(t \cdot e).$$

We write $\tau = (\tau_1, \tau_2, ..., \tau_r) \in \mathbb{R}^r$ and we identify a real number $\beta$ with the vector $(\beta, ..., \beta) \in \mathbb{R}^r$. For any $t \in H$, we have for $j = 1, ..., r$

$$(5) \quad Q_j(\pi(t)[x]) = Q_j(t \cdot e)Q_j(x).$$

Therefore, for any $t \in H$,

$$Q^r(\pi(t)[x]) = det \pi(t)Q^r(x)$$

since

$$det \pi(t) = Q^r(t \cdot e).$$
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(See [20, p. 388]). The above properties are also valid if we replace \( Q_j \) by \( Q_j^* \) and \( x \in \Omega \) by \( \xi \in \Omega^* \). In particular, for all \( \xi \in \Omega^* \) and \( t^* \in H' \), we have for \( j = 1, \ldots, r \)

\[
Q_j^*(\pi(t^*[\xi])) = Q_j^*(t^* \cdot e) Q_j^*(\xi).
\]

3. SOME USEFUL RESULTS IN A CONVEX HOMOGENEOUS CONE

In this section, we recall some important facts about homogeneous cones such as the Riemannian structure that yields an isometry between the cone and its dual and the Whitney decomposition of the cone. Most of these results have been established in [4], [8] and [17].

3.1. The Riemannian structure \( \Omega \) and its dual. Following [17], let \( d \) and \( d^* \) denote the Riemannian distances in \( \Omega \) and \( \Omega^* \) which are invariant under the action of \( G(\Omega) \) and \( G(\Omega^*) \) respectively, i.e.

\[
d(gx, gy) = d(x, y) \quad \text{and} \quad d_*(g^* \xi, g^* \eta) = d_*(\xi, \eta) \quad \forall g \in G(\Omega).
\]

Recall from [17] (see also [12, Chapter I]) that there is a bijection from \( \Omega \) to \( \Omega^* \) given by

\[
x = t \cdot e \in \Omega \mapsto x' = t^{*-1} \cdot e \in \Omega^*,
\]

such that \( x'' = x \). This is an isometry for the Riemannian distances [8]

\[
d_*(x', y') = d(x, y).
\]

We also use that [17, page 489]

\[
Q_j(x) Q_j^*(x') = 1, \quad \forall x \in \Omega, \ j = 1, \ldots, r.
\]

3.2. The invariant measure on \( \Omega \) and the Whitney decomposition. Since we have also this identification \( \Omega^* \equiv H' \cdot e \), we deduce from (5), that the measure

\[
dm_*(\xi) = (Q^*)^{-\tau}(\xi) d\xi
\]

is \( H \)-invariant on \( \Omega \) (resp. \( H' \)-invariant on \( \Omega^* \)).

**Lemma 3.1.** Given \( \lambda > 0 \), there is a constant \( C = C(\lambda) > 0 \) such that:

i) if \( d(y, t) \leq \lambda \) then \( \frac{1}{C} \leq \frac{Q_j(y)}{Q_j(t)} \leq C \) for all \( j = 1, \ldots, r \) and \( y \) and \( t \in \Omega \);

ii) if \( d_*(\xi, \eta) \leq \lambda \) then \( \frac{1}{C} \leq \frac{Q_j^*(\xi)}{Q_j^*(\eta)} \leq C \) for all \( j = 1, \ldots, r \) and \( \xi, \eta \in \Omega^* \).

Let \( \lambda > 0 \), \( y \in \Omega \) (resp. \( \xi \in \Omega^* \)) and \( d \) (resp. \( d_* \)) the \( G(\Omega) \)-invariant (resp. \( G(\Omega^*) \)-invariant ) distance defined in \( \Omega \) (resp. \( \Omega^* \)). We denote by

\[
B_\lambda(y) = \{ x \in \Omega : d(y, x) < \lambda \}
\]

and

\[
B_\lambda^*(\xi) = \{ \eta \in \Omega^* : d_*(\eta, \xi) < \lambda \}
\]
the $d$-ball (resp. $d_*$-ball) centered at the point $y$ (resp. $\xi$) with the radius $\lambda$.

We give now the Whitney decomposition of the cone $\Omega$, which is obtained, for instance, as in [8, Lemma 3.5].

**Lemma 3.2.** There exists a sequence $\{y_j\}_j$ of points of $\Omega$ such that the following three properties hold:

1. the balls $B_1(y_j)$ are pairwise disjoint;
2. the balls $B_1(\nu(y_j))$ form a covering of $\Omega$;
3. there is an integer $N = N(\Omega)$ such that every $y \in \Omega$ belongs to at most $N$ balls $B_1(y_j)$.

**Remark 3.3.** This lemma is also true for the dual cone $\Omega^*$.

**Definition 3.4.** A sequence $\{y_j\}_j$ in $\Omega$ as in Lemma 3.2 is called a lattice of $\Omega$. In this case, the sequence $\{y'_j\}_j$ is also a lattice in $\Omega^*$, called the dual lattice.

The family $\{B_1(y_j)\}_j$ (resp. $\{B_1^*(y'_j)\}_j$) is called the Whitney decomposition of the cone $\Omega$ (resp. $\Omega^*$).

We will need the following results whose proofs can be found in [17].

**Lemma 3.5.** Let $y_0 \in \Omega$, $\xi_0 \in \Omega^*$; then

$$|B_\lambda(y_0)| = C_\lambda Q^\tau(y_0) \quad \text{and} \quad |B_\lambda^*(\xi_0)| = C_\lambda(Q^*)^\tau(\xi_0).$$

**Proposition 3.6.** Let $y_0 \in \Omega$, $\xi_0 \in \Omega^*$. There is a constant $\gamma = \gamma(\Omega, \Omega^*) \geq 1$ such that

$$\frac{1}{\gamma} \leq \frac{y|x|}{(y_0|x_0)} \leq \gamma, \quad \forall y \in B_1(y_0), \xi \in B_1^*(\xi_0).$$

**Corollary 3.7.** Let $y_0 \in \Omega$. There is a constant $\gamma > 0$ such that

$$\frac{1}{\gamma} \leq (y|x|) \leq \gamma, \quad \forall y \in B_1(y_0), \xi \in B_1^*(\xi_0).$$

The following results hold if $\Omega$ is substituted by $\Omega^*$, provided the roles of $m_j$ and $n_j$ are reversed.

**Corollary 3.8.** [17] Let $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_1}{2}$, $j = 1, \ldots, r$. Then

$$\int_{\Omega} e^{-Q^{\nu-\gamma}(y)}dy = \Gamma_{\Omega}(\nu)(Q^{*})^{-\nu}(\xi), \quad \xi \in \Omega^*,$$

where $\Gamma_{\Omega}(\nu)$ denotes the gamma integral [13] in the cone $\Omega$.

**Remark 3.9.** Using the corollary above, one can extend $Q^*$ as an analytic function in $V' + i\Omega^*$. More precisely, if $\zeta \in V' + i\Omega^*$ and $\nu_j > \frac{m_j}{2}$, $j = 1, \ldots, r$, we set

$$\int_{\Omega} e^{i(\zeta|y|)}Q^{\nu-\gamma}(y)dy = \Gamma_{\Omega}(\nu)(Q^{*})^{-\nu}\left(\frac{\zeta}{i}\right).$$
LEMMA 3.10. [17, Lemma 4.19] Let \( v \in \Omega \) and \( \mu, \nu \in \mathbb{R}^r \). The integral
\[
\int_{\Omega} Q^\mu(y + v)Q^\nu(y)dy
\]
converges if for every \( j = 1, \ldots, r \) we have \( \nu_j > \frac{m_j}{2} \) and \( \mu_j + \nu_j < -\frac{n_j}{2} \). In this case, we have
\[
\int_{\Omega} Q^\mu(y + v)Q^\nu(y)dy = C_{\mu,\nu}Q^{\mu+\nu}(v).
\]

LEMMA 3.11. [17, Lemma 4.20] Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \). The integral
\[
\int_{\Omega} \left| Q^{-\alpha}\left(\frac{x + iy}{i}\right)\right| dx \quad (y \in \Omega)
\]
converges if and only if \( \alpha_j > 1 + n_j + \frac{m_j}{2} \), \( j = 1, \ldots, r \). In this case, there is a positive constant \( c_\alpha \) such that
\[
\int_{\Omega} \left| Q^{-\alpha}\left(\frac{x + iy}{i}\right)\right| dx = c_\alpha Q^{-\alpha+\tau}(y).
\]

LEMMA 3.12. [17, Lemma 4.21] Let \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r \) and \( 0 < \lambda < \frac{1}{4} \). There is a constant \( C_\alpha \) such that for all \( y \in \Omega \), \( \|y\| < \lambda \),
\[
\int_{\{x \in V : \|x\| < 1\}} \left| Q^{-\alpha}\left(\frac{x + iy}{i}\right)\right| dx \geq C_\alpha Q^{-\alpha+\tau}(y).
\]

3.3. Homogeneous Siegel domains of type II. Let \( V^\mathbb{C} = V + iV \) be the complexification of \( V \). Then each element of \( V^\mathbb{C} \) is identified with a vector in \( \mathbb{C}^n \). The coordinates of a point \( z \in \mathbb{C}^n \) are arranged in the form
\[
z = (z_{11}, z_2, z_{22}, \ldots, z_r, z_{rr})
\]
where
\[
z_j = (z_{1j}, \ldots, z_{j-1,j}), \quad j = 2, \ldots, r
\]
and
\[
z_{jj} \in \mathbb{C}, \quad z_{ij} = (z_{ij}^{(1)}, \ldots, z_{ij}^{(n_{ij})}) \in \mathbb{C}^{n_{ij}}, \quad 1 \leq i < j \leq r.
\]
For all \( j = 1, \ldots, r \) we denote \( e_{jj} = z \), where \( z_{jj} = 1 \) and the other coordinates are equal to zero and we denote
\[
e = \sum_{j=1}^{r} e_{jj} = (1, 0, 1, \ldots, 0, 1).
\]

Let \( m \in \mathbb{N} \). For each row vector \( u \in \mathbb{C}^m \), we denote \( u' \) the transpose of \( u \). Given \( m \times m \) Hermitian matrices \( \tilde{H}_1, \ldots, \tilde{H}_n \), we define a \( \Omega \)-Hermitian, homogeneous form \( F : \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}^n \) as
\[
F(u, v) = (u\tilde{H}_1v', \ldots, u\tilde{H}_nv'), \quad (u, v) \in \mathbb{C}^m \times \mathbb{C}^m
\]
such that
(i) \( F(u, u) \in \overline{\Omega} \).
(ii) $F(u, u) = 0$ if and only if $u = 0$;
(iii) for every $t \in H$, there exists $\tilde{t} \in GL(m, \mathbb{C})$ such that $t \cdot F(u, u) = F(\tilde{t}u, \tilde{t}u)$.

The point set
\[
D(\Omega, F) = \{(z, u) \in \mathbb{C}^n \times \mathbb{C}^m : \Re m z - F(u, u) \in \Omega\}
\]
in $\mathbb{C}^{n+m}$ is called a Siegel domain of type II associated to the open convex homogeneous cone $\Omega$ and to the $\Omega-$Hermitian, homogeneous form $F$. Recall that if $m = 0$, the domain $D$ is a tube type Siegel domain or a homogeneous Siegel domain of type I, associated with the cone $\Omega$, or the tube domain over the homogeneous cone $\Omega$, considered by the authors of [17].

Using (11), we write
\[
F(u, u) = (F_{11}(u, u), F_{22}(u, u), F_{i2}(u, u), ..., F_r(u, u), F_{rr}(u, u))
\]
where for $i = 1, ..., r$ and $j = 2, ..., r$,
\[
F_{ii}(u, u) = u\bar{H}_{ii}u', \quad F_{ij}(u, u) = u\bar{H}_{ij}u' = (F_{1j}(u, u), ..., F_{ij-1,j}(u, u))
\]
and for $1 \leq i < j \leq r$ and $\lambda = 1, ..., n_{ij}$,
\[
F_{ij}(u, u) = (F^{(1)}_{ij}(u, u), ..., F^{(n_{ij})}_{ij}(u, u)), \quad F^{(\lambda)}_{ij}(u, u) = u\bar{H}^{(\lambda)}_{ij}u'.
\]
The space $\mathbb{C}^m$ decomposes into the direct sum of subspaces $\mathbb{C}^{b_1} \oplus ... \oplus \mathbb{C}^{b_r}$ on which are concentrated the Hermitian forms $F_{ij}$, that is, with appropriate coordinates, we have for $i = 1, ..., r$,
\[
\tilde{H}_{ii} = \text{diag}(0_{(b_1)}, ..., 0_{(b_{i-1})}, I_{(b_i)}, 0_{(b_{i+1})}, ..., 0_{(b_r)})
\]
where $0_{(b_k)}$ and $I_{(b_k)}$ denote respectively the null matrix and the identity matrix of the vector space $\mathbb{C}^{b_k}$ for all $k = 1, ..., r$. (See for instance [21, pp. 127-129].)

In the sequel, we denote $b$ the vector
\[
b = (b_1, ..., b_r) \in \mathbb{N}^r.
\]
and we denote $dv$ the Lebesgue measure in $\mathbb{C}^m$. Let $\nu = (\nu_1, ..., \nu_r) \in \mathbb{R}^r$. For all $(x + iy, u) \in D$, we shall consider the measure
\[
dV_\nu(x + iy, u) = Q^{\nu - \frac{b}{2} - \tau}(y - F(u, u)) dx dy dv(u);
\]
with the convention that
\[
dV_\nu(y, u) = Q^{\nu - \frac{b}{2} - \tau}(y - F(u, u)) dy dv(u).
\]
We denote by $L_p^\nu(D)$, $1 \leq p \leq \infty$, the Lebesgue space $L_p(D, dV_\nu(z, u))$. The weighted Bergman space $A_p^\nu(D)$ is the (closed) subspace of $L_p^\nu(D)$ consisting of holomorphic functions. In order to have a non-trivial subspace, we take $\nu = (\nu_1, ..., \nu_r) \in \mathbb{R}^r$ such that $\nu_i > \frac{m_i + b_i}{2}$, $i = 1, ..., r$. (See [15].)
LEMMA 3.13 ([13], page 58). For all $\xi \in \Omega^*$, \begin{equation} \int_{\mathbb{C}^m} e^{-(F(u,v)\xi)} dv(u) = C(Q^*)^{-b}(\xi). \end{equation}

4. STATEMENT OF RESULTS

The orthogonal projector of the Hilbert space $L^2_{\nu}(D)$ on its closed subspace $A^2_{\nu}(D)$ is the weighted Bergman projector $P_{\nu}$. We recall that $P_{\nu}$ is defined by the integral \begin{equation} P_{\nu} f(z, u) = \int_D B_{\nu}((z, u), (w, v)) f(w, v) dV_{\nu}(w, v), \quad (z, u) \in D, \end{equation}

where for a suitable constant $d_{\nu,b},$ \begin{equation} B_{\nu}((z, u), (w, v)) = d_{\nu,b} Q^{-\nu - \frac{1}{2} - \tau} \left( \frac{z - \bar{w}}{2i} - F(u, v) \right) \end{equation}
is the weighted Bergman kernel i.e., the reproducing kernel of $A^2_{\nu}(D)$. (See [9, Proposition II.5].) The scalar product $\langle \cdot, \cdot \rangle_{\nu}$ is given by $\langle f, g \rangle_{\nu} = \int_D f(z, u) g(z, u) dV_{\nu}(z, u)$.

Let us now introduce mixed norm spaces. For $1 \leq p \leq \infty$ and $1 \leq q < \infty$, let $L^p_{\nu,q}(D)$ be the space of measurable functions on $D$ such that \begin{equation} \||f||_{L^p_{\nu,q}(D)} := \left( \int_{\mathbb{C}^m} \int_{\Omega^*+F(u,v)} \left( \int_{\nu} |f(x + iy, u)|^p dx \right)^{\frac{q}{p}} dV_{\nu}(y, u) \right)^{\frac{1}{q}} \end{equation}
is finite (with obvious modification if $p = \infty$). As before, we call $A^p_{\nu,q}(D)$ the (closed) subspace of $L^p_{\nu,q}(D)$ consisting of holomorphic functions. Note that for $p = q$, the Lebesgue mixed norm space $L^p_{\nu,q}(D)$ coincides with the Lebesgue space $L^p_{\nu}(D)$ and the mixed norm Bergman space $A^p_{\nu,q}(D)$ coincides with the Bergman space $A^p_{\nu}(D)$. The unweighted case corresponds to $\nu = \tau + \frac{1}{2}$.

Following [9], we shall denote $L^2_{(1,\nu)}(\Omega^* \times \mathbb{C}^m)$ the Hilbert space of functions $g : \Omega^* \times \mathbb{C}^m \to \mathbb{C}$ such that:

i) for all compact subset $K_1$ of $\mathbb{C}^n$ contained in $\Omega^*$ and for all compact subset $K_2$ of $\mathbb{C}^m$, the mapping $u \mapsto g(\cdot, u)$ is holomorphic on $K_2$ with values in $L^2(K_1, -\nu)$, where \begin{equation} L^2(K_1, -\nu) = \{ f : K_1 \to \mathbb{C} : \int_{K_1} |f(\xi)|^2 (Q^*)^{-\nu + \frac{1}{2}} d\xi < \infty \}; \end{equation}

ii) the function $g \in L^2(\Omega^* \times \mathbb{C}^m), (Q^*)^{-\nu + \frac{1}{2}}(\xi) e^{-2(F(u,v)\xi)} d\xi dv(u)$. We then define by \begin{equation} Lg(z, u) = (2\pi)^{-\frac{1}{2}} \int_{\Omega^*} e^{i(z|\xi)} g(\xi, u) d\xi \end{equation}
the "Laplace transform" of any function \( g \in L^2_{(\nu)}(\Omega \times \mathbb{C}^m) \). Now, we recall the Plancherel-Gindikin result found in \cite{9, Theorem II.2} which is a generalization of the Paley-Wiener Theorem \cite{17, Theorem 5.1}.

**Theorem 4.1.** Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) \in \mathbb{R}^r \) with \( \nu_j > \frac{m_j + 1}{2} \), \( j = 1, \ldots, r \). A function \( G \) belongs to \( A^2_{\nu}(D) \) if and only if \( G = Lg \), with \( g \in L^2_{(\nu)}(\Omega^* \times \mathbb{C}^m) \). Moreover there is a positive constant \( c_{\nu,b} \) such that

\[
\|G\|_{A^2_{\nu}(D)}^2 = e_{\nu,b} \|g\|_{L^2_{(\nu)}(\Omega^* \times \mathbb{C}^m)}^2.
\]

In this paper, we exhibit an example of convex homogeneous non symmetric cone for which the sufficient conditions obtained so far using Schur's Lemma are not optimal. Certainly the choice of the Schur test functions made so far need an improvement.

4.1. An example of a homogeneous cone. Consider the vector space \( E \) of matrices of the form

\[
x = \begin{pmatrix}
x_1 I_2 & x_4 e & \xi \\
x_4 e' & x_2 & x_6 \\
\xi' & x_6 & x_3
\end{pmatrix}
\]

where \( I_2 \) is the 2 \( \times \) 2 identity matrix, \( e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2 \) and \( x_k \in \mathbb{R}, \, k = 1, \ldots, 6 \). Here we denote \( X' \) the transpose of the vector \( X \).

Then \( E \subset \text{Sym}(4, \mathbb{R}) \) and the set

\[
\Omega_E = \{ x \in E : x \text{ is positive definite} \}
\]

is an irreducible, homogeneous and not self dual cone of rank 3, isomorphic to its dual cone. (See for instance \cite{14}). We have the following concerning \( \Omega_E : n_{12} = n_{23} = 1, \, n_{13} = 2 \) and \( m_1 = 3, \, m_2 = 1, \, m_3 = 0 \) and \( n_1 = 0, \, n_2 = 1, \, n_3 = 3 \). Thus \( \tau = \left( \frac{3}{1}, \frac{2}{3}, \frac{5}{3} \right) \).

Let \( \nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3 \) such that \( \nu_1 > \frac{3}{2}, \, \nu_2 > \frac{1}{2}, \, \nu_3 > 0 \). In \cite{17}, was established that the operator \( P^+_{\nu} \) with positive Bergman kernel in \( T_{\Omega_E} \) is bounded on \( L^p_{\nu}(T_{\Omega_E}) \) if \( a_{\nu} < q < a_{\nu} \) with

\[
a_{\nu} = 1 + \min_{1 \leq j \leq 3} \frac{\nu_j - m_j}{m_j} \frac{1}{n_j} = \min \left( 2\nu_2, \frac{2}{3}\nu_3 + 1 \right).
\]

The authors also give, without details, these necessary conditions for boundedness of the operator \( P^+_{\nu} \) with positive Bergman kernel in \( T_{\Omega_E} \): if \( P^+_{\nu} \) is bounded on \( L^p_{\nu}(T_{\Omega_E}) \) then \( c'_{\nu} < q < c_{\nu} \) with

\[
c_{\nu} = 1 + \min_{1 \leq j \leq 3} \frac{\nu_j}{n_j} = \min \left( 2\nu_2 + 1, \frac{2}{3}\nu_3 + 1 \right).
\]

It was observed that in the classical case \cite{4} for symmetric cones and for the Vinberg cone and its dual \cite{8}, these two conditions coincide. We were looking for an example of homogeneous cone for which the necessary conditions are not sufficient. We have found one with the
cone $\Omega_E$. As a matter of fact, if we take $\nu = \nu_0 = (2, \frac{2}{3}, 1)$, these conditions becomes

$$a_{\nu_0} = \min\left(\frac{4}{3}, \frac{5}{3}\right) = \frac{4}{3}, \quad c_{\nu_0} = \min\left(\frac{7}{3}, \frac{5}{3}\right) = \frac{5}{3}.$$  

This shows clearly that the necessary condition is not sufficient for $\Omega_E$. We shall then proceed following the idea developed by authors of [16] to find new sufficient conditions which are optimal. This will therefore complete the results of [17].

4.2. Results. The key result of this paper is the following:

**Theorem 4.2.** Let $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j + b_j}{2}$, $j = 1, \ldots, r$.

i) Assume that $P^+_\nu : L^p_{\nu} (D) \to L^p_{\mu} (D)$ is bounded. Then $\frac{1}{q} (|\nu| + \frac{3}{2} |b|) = \frac{1}{s} (|\mu| + \frac{3}{2} |b|) + |b|$ and for all $j = 1, \ldots, r$, we have

$$\begin{cases} 
1 < q < 1 + \frac{\nu_j + b_j}{2} \\
\frac{m_j}{2} + \frac{b_j}{2} < \mu_j < s (\nu_j + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2}.
\end{cases}$$

ii) Let $j = 1, \ldots, r$. Assume that

$$\begin{cases} 
\frac{1}{q} (\nu_j + \frac{b_j}{2}) = \frac{1}{s} (\mu_j + \frac{b_j}{2}) \\
1 < q < 1 + \frac{\nu_j + b_j}{2} \\
\frac{m_j}{2} + \frac{b_j}{2} < \mu_j < s (\nu_j + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2}.
\end{cases}$$

Then $P^+_\nu : L^p_{\nu} (D) \to L^p_{\mu} (D)$ is bounded.

As a corollary, we have this result which is better than [15, Theorem 2.1]

**Corollary 4.3.** Let $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$, $\mu = (\mu_1, \mu_2, \cdots, \mu_r) \in \mathbb{R}^r$ such that $\nu_j, \mu_j > \frac{m_j + b_j}{2}$, $j = 1, \ldots, r$.

i) Assume that $P^+_\nu : L^p_{\nu} (D) \to L^p_{\mu} (D)$ is bounded. Then $|\nu| = |\mu| + q |b|$ and

$$\max_{1 \leq j \leq r} \left(1, \frac{\mu_j + \frac{b_j}{2} + \frac{n_j}{2}}{\nu_j + \frac{b_j}{2}}\right) < q < 1 + \min_{1 \leq j \leq r} \frac{\nu_j + \frac{b_j}{2}}{\frac{n_j}{2}}.$$  

ii) Assume that $\nu_j = \mu_j$, $j = 1, \ldots, r$ and

$$1 + \max_{1 \leq j \leq r} \frac{\frac{n_j}{2}}{\nu_j + \frac{b_j}{2}} < q < 1 + \min_{1 \leq j \leq r} \frac{\nu_j + \frac{b_j}{2}}{\frac{n_j}{2}}.$$  

Then $P^+_\nu : L^p_{\nu} (D) \to L^p_{\nu} (D)$ is bounded.

Also, we have this result that improves [17, Theorem 3.1].
COROLLARY 4.4. Let $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j}{n_j}$, $j = 1, \ldots, r$. Let $\Omega$ be a convex open homogeneous cone. The operator $P_\nu^+$ is bounded from $L^p,q(\nu,T)$ into itself if and only if

$$1 + \max_{1 \leq j \leq r} \frac{n_j/2}{\nu_j} < q < 1 + \min_{1 \leq j \leq r} \frac{\nu_j}{n_j/2}.$$ 

Moreover, we establish the following result that gives necessary conditions for $L^2,q(\nu,D)$ estimates for the Bergman projector as was announced in [17, Section 8] for Siegel domains of type I.

**THEOREM 4.5.** Let $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j + b_j}{n_j}$, $j = 1, \ldots, r$. If the weighted Bergman projection $P_\nu$ is bounded on $L^2,q(\nu,D)$ then

$$2 \left(1 + \min_{1 \leq j \leq r} \frac{\nu_j + b_j}{n_j} \right)^{1/2} < q < 2 \left(1 + \min_{1 \leq j \leq r} \frac{\nu_j + b_j}{n_j} \right)^{1/2}.$$

Note that in the case of tube domains over symmetric cones symmetric cones, which Debertol considered, the necessary condition above of the $L^2,q(T_\Omega)$-boundedness of the weighted Bergman projector $P_\nu$ has been left open. As for the sufficient condition, so far, the one obtained in [15, Theorem 2.2] is not yet improved. Nevertheless, if we set

$$q_\nu = 1 + \min_{1 \leq j \leq r} \frac{\nu_j + b_j}{n_j}$$
and $$\overline{q}_\nu = 1 + \min_{1 \leq j \leq r} \frac{\nu_j + b_j}{n_j},$$
then by interpolating [15, Theorem 2.2] and part ii) of Corollary 4.3, we improve [15, Theorem 2.3] as follows:

**THEOREM 4.6.** Let $\nu = (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{R}^r$ such that $\nu_j > \frac{m_j + b_j}{n_j}$, $j = 1, \ldots, r$. The weighted Bergman projection $P_\nu$ extends as a bounded operator on $L^2,q(\nu,D)$ whenever

$$\left(\overline{q}_\nu - \frac{q_\nu}{q_\nu} + 2\right)^{1/2} < q < \overline{q}_\nu - \frac{q_\nu}{q_\nu} + 2.$$

This work is divided in 8 sections. In Section 4, we state our results; in Section 5, in Section 5, we introduce a family of integral operators associated to the Bergman projector. in Section 6, we prove off-diagonal estimates for the operator $S_{\alpha,\beta,\gamma}$ and the proofs of the off-diagonal estimates for the family of Bergman-type operators $T_{\alpha,\beta,\gamma}$ are given in Section 7. In the last section, we give the proof of Theorem 4.5.

5. INTEGRAL OPERATORS ASSOCIATED TO THE BERGMAN PROJECTOR

Now, as in [16], we consider the question of off-diagonal estimates for a family of operators generalizing the Bergman projection. Given vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \mathbb{R}^r$. 


We note that the boundedness of $T$ and $T^+$ implies the boundedness of $T$ and the reverse is not necessarily true. It is clear that estimates obtained for this family of operators imply corresponding estimates for the weighted Bergman projection since $P_\nu = T_{0,\nu - \frac{1}{2} - \tau,\nu}$ for all $\nu_j > \frac{m_j + b_j}{2}$, $j = 1, \ldots, r$. The $L^p_q(T_\Omega)$-boundedness of this family of operators has been considered in [5] for the case $T_{0,\nu - \frac{1}{2},\nu}$ and in [2] for $T_{0,\nu - \frac{1}{2} - \tau,\nu + \alpha}$ when $\Omega$ is the light cone. Also, the second author in [19, Theorem 1.1] considered this family and obtained $L^p_q(T_\Omega)$ optimal results for the operator $T^+$. In these works, was used the Schur Lemma with test functions which are generalized powers of the determinant function. Moreover, authors of [7] extended some results of [6] to the case of homogeneous Siegel domains of $\mathbb{C}^n$ of type II. They obtained sufficient conditions for the operator $T^+_{\alpha,\nu - \tau,\mu + \alpha}$ in the vectorial weighted case, using the Schur Lemma. As was indicated in the introduction of [15] and explicitly presented in [17], it is not yet known in general, with the techniques developed so far whether these sufficient conditions are necessary. However, with some restrictions on the weight or the choice of the cone (the Vinberg cone for example [8]), these sufficient conditions are necessary.

In order to find the necessary and sufficient conditions for boundedness of the family of operators $T^+$ on $D$, we shall establish connections between $T^+$ and a family $S$ of operators on the set $U = \{(y, u) : u \in \mathbb{C}^m, y \in \Omega + F(u, u)\}$.

5.1. Relationship between the family $T^+$ defined on $D$ and the family $S$ defined on $U$. We follow the scheme of [15, Proof of Theorem 2.1]. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r$ be such that $\mu_j > \frac{m_j + b_j}{2}$, $j = 1, \ldots, r$. Recall that $U = \{(t, u) : u \in \mathbb{C}^m, t \in \Omega + F(u, u)\}$. We define $L^q_{\mu}(U)$ as the space of all $g : U \to \mathbb{C}$ with norm given by

$$
\|g\|_{L^q_{\mu}(U)}^q = \int_{\mathbb{C}^m} \int_{\Omega + F(u, u)} |g(y, u)|^q Q^{\mu - \frac{1}{2} - \tau}(y - F(u, u))dydv(u)
$$

$$
= \int_{\mathbb{C}^m} \int_{\Omega} |g(y + F(u, u), u)|^q Q^{\mu - \frac{1}{2} - \tau}(y)dydv(u).
$$
We will need the following result.

**Proposition 5.1.** [15, Proposition 5.2] Let $u \in \mathbb{C}^m$; $y \in \Omega + F(u, u)$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r$, the integral

$$I_\lambda(y, u) = \int_{\mathbb{C}^m} Q^{-\lambda}(y + F(s, s) - 2\text{Re} F(u, s))dv(s)$$

converges if $\lambda_j - b_j > \frac{n_j}{2}$, $j = 1, \ldots, r$. In this case, there is a positive constant $C_\lambda$ such that

$$I_\lambda(y, u) = C_\lambda Q^{-\lambda+b}(y - F(u, u)).$$

Next, we shall use the following notation: for all $y \in \Omega$, $u \in \mathbb{C}^m$,

$$f_{y,u}(x) = f(x + iy, u), \; x \in V.$$ 

Thus, for $f \in L_p^a(D)$, using Minkowski’s inequality for integrals, Young’s inequality and Lemma 3.11, we get

$$\|T^{+}_{\alpha,\beta,\gamma}f\|_{L_q^a(D)} \leq C\|S_{\alpha,\beta,\gamma}g\|_{L_p^a(U)}$$

where for $v \in \mathbb{C}^m$ and $t \in \Omega + F(v, v)$,

$$g(t, v) = \|f_{t,v}\|_p$$

and $S_{\alpha,\beta,\gamma} := S$ is the integral operator with positive kernel defined on $L_p^a(U)$

$$Sg(y, u) = Q^\alpha(y - F(u, u)) \times \int_{U} Q^{-\gamma - \frac{n}{2}}(y - 2\text{Re} F(u, v) + x)g(x, v)Q^\beta(x - F(v, v))dxdv(v).$$

**6. Off-diagonal estimates for the family $S_{\alpha,\beta,\gamma}$**

In this section, we obtain necessary and sufficient conditions for the boundedness of the family of integral operators with positive kernel $S := S_{\alpha,\beta,\gamma}$ defined on $L_p^a(U)$ by (21). Recall that

$$U = \{(y, u) : u \in \mathbb{C}^m, \; y \in \Omega + F(u, u)\}.$$ 

This operator has already been considered by the author of [16] when $U = \Omega$ is a symmetric cone. Following their idea, we shall prove here, an off-diagonal estimate of this operator. To this effect, we shall use the following lemma which is one of the key tool needed to establish our result. It is actually an adapted version of the Okikiolu result [18] which generalises [1, Lemma 3.3].

**Lemma 6.1.** Let $p, q$ be positive numbers such that $1 < p \leq q$. Let $\ell \in \mathbb{N}$. For every $j = 1, \ldots, \ell$, consider the complex-valued measurable
functions $k_j(x, y)$ on $X \times Y$ and suppose there exist $0 < t_j \leq 1$, measurable functions $\varphi_{1j} : Y \to (0, \infty)$, $\varphi_{2j} : X \to (0, \infty)$ and nonnegative constants $M_{1j}$, $M_{2j}$ such that

\begin{equation}
\int_Y \prod_{j=1}^\ell |k_j(x, y)|^{t_jp'} \varphi_{1j}^{p'}(y)d\mu(y) \leq M_2^p \varphi_2^p(x) \ a.e \ on \ X; \tag{22}
\end{equation}

\begin{equation}
\int_X \prod_{j=1}^\ell |k_j(x, y)|^{(1-t_j)q} \varphi_{2j}^q(x)d\nu(x) \leq M_1^q \varphi_1^q(y) \ a.e \ on \ Y \tag{23}
\end{equation}

where for $k = 1, 2$,

$$\varphi_k(z) = \prod_{j=1}^\ell \varphi_{kj}(z) \quad \text{and} \quad M_k = \prod_{j=1}^\ell M_{kj}.$$ 

Then the operator $T : L^p(X, d\mu) \to L^q(Y, d\nu)$ given by

$$Tf(x) = \int_Y f(y) \prod_{j=1}^\ell |k_j(x, y)|d\mu(y)$$

is bounded and we have

$$\|Tf\|_{L^q(Y, d\nu)} \leq M_1 M_2 \|f\|_{L^p(X, d\mu)}.$$

**Proof.** Let

$$K(x, y) = \prod_{j=1}^\ell k_j(x, y)$$

and for $t = (t_1, t_2, \ldots, t_\ell)$ define

$$K(x, y)^t = \prod_{j=1}^\ell k_j^{t_j}(x, y).$$

So

$$Tf(x) = \int_X f(y)|K(x, y)|d\mu(y).$$

Then for all $x \in Y$, using Hölder’s inequality, we write

$$|Tf(x)| \leq \int_X |f(y)||K(x, y)|d\mu(y)$$

$$= \int_X [||K(x, y)^t|\varphi_1(y)||f(y)||K(x, y)^{1-t}||\varphi_1^{-1}(y)||d\mu(y)$$

$$\leq \left[ \int_X |K(x, y)|^{t'p'}|\varphi_{1j}^{p'}(y)d\mu(y) \right]^{\frac{1}{p'}} \left[ \int_X |f(y)|^{p}|K(x, y)^{(1-t)p}||\varphi_1^{-p}(y)d\mu(y) \right]^{\frac{1}{p}}.$$
But from (22),
\[
\int_X |K(x, y)^{t_p'}| \varphi^p_1(y) d\mu(y) = \int_X \prod_{j=1}^{\ell} |k_j(x, y)|^{t_p'} \varphi^p_1(y) d\mu(y) \leq M_2^p \varphi^p_2(x);
\]
thus
\[
|Tf(x)| \leq M_2 \varphi_2(x) \left( \int_X |f(y)|^p |K(x, y)^{(1-t)p}| \varphi_1^{-p}(y) d\mu(y) \right)^{\frac{1}{p}}.
\]

Using Minkowski’s inequality for integrals, we write
\[
\|Tf\|_{L^p(Y,d\nu)} \leq M_2 \left[ \int_Y \varphi^p_2(x) \left( \int_X |f(y)|^p |K(x, y)^{(1-t)p}| \varphi_1^{-p}(y) d\mu(y) \right)^{\frac{2}{p}} d\nu(x) \right]^\frac{1}{2}
\leq M_2 \left[ \int_Y \left( \int_X \varphi^p_2(x) |f(y)|^p |K(x, y)^{(1-t)p}| \varphi_1^{-p}(y) d\mu(y) \right)^{\frac{2}{p}} d\nu(x) \right]^\frac{1}{2} \times \frac{1}{p}
\leq M_2 \left[ \int_X \varphi_1^{-p}(y) |f(y)|^p \left( \int_Y \varphi^p_2(x) |K(x, y)^{(1-t)q} d\nu(x) \right)^{\frac{2}{p}} d\mu(y) \right]^\frac{1}{2}.
\]

But from (23)
\[
\int_Y \varphi^p_2(x) |K(x, y)^{(1-t)q}| d\nu(x) = \int_Y \prod_{j=1}^{\ell} |k_j(x, y)|^{(1-t)q} \varphi^p_2(x) d\nu(x) \leq M_1^q \varphi^q_1(y);
\]

Thus
\[
\|Tf\|_{L^p(Y,d\nu)} \leq M_2 \left[ \int_X \varphi_1^{-p}(y) |f(y)|^p M_1^q \varphi_1^{-q}(y) d\mu(y) \right]^\frac{1}{q} = M_2 M_1\|f\|_{L^p(X,d\mu)}.
\]

6.1. **Sufficiency for the boundedness of** $S_{\alpha,\beta,\gamma}$. The following result provides the right relations between the parameters under which the operators $S_{\alpha,\beta,\gamma} := S$ are bounded.

**Theorem 6.2.** Let $\Omega$ be an open convex homogeneous cone. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r), \mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r$ and $1 < q \leq s < \infty$, such that $\frac{\nu_j + \frac{b_j}{2}}{s} + \frac{\mu_j + \frac{b_j}{2}}{s} > 0, j = 1, \ldots, r$. Assume that the parameters satisfy for all $j = 1, \ldots, r$, the following
\[
(24) \quad \gamma_j = \alpha_j + \beta_j + \tau_j + \frac{b_j}{2} - \frac{\nu_j + \frac{b_j}{2}}{q} + \frac{\mu_j + \frac{b_j}{s}}{s},
\]
and
\[
(25) \quad \left\{ \begin{array}{l}
q \left( \beta_j - \gamma_j + \tau_j + \frac{\nu_j}{2} + \frac{b_j}{2} \right) - \frac{\nu_j}{2} - \frac{b_j}{2} < \nu_j, \\
q(\beta_j + \tau_j - \frac{\nu_j}{2}) + \frac{\nu_j}{2} + \frac{b_j}{2} > \nu_j,
\end{array} \right.
\]
and
\[ \frac{m_j}{2} - s\alpha_j + \frac{b_j}{2} < \mu_j < s(\gamma_j - \alpha_j + \frac{b_j}{2}) - n_j + \frac{b_j}{2}. \]

Then the operator \( S : L^q(U, \mathbb{C}) \to L^s(U, \mathbb{C}) \) are bounded.

**Proof.** Assume
\[ \gamma_j = \alpha_j + \beta_j + \tau_j + \frac{b_j}{2} - \frac{\nu_j + \frac{b_j}{q}}{q} + \frac{\mu_j + \frac{b_j}{s}}{s}. \]

and let
\[ \omega_j = \alpha_j + \beta_j - \gamma_j - \nu_j + \tau_j = -\left( \frac{\nu_j + \frac{b_j}{q'}}{q'} + \frac{\mu_j + \frac{b_j}{s}}{s} \right) < 0. \]

From the first inequality of (25), we have \( q(\beta_j - \gamma_j + \tau_j + \frac{n_j}{2} + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2} < \nu_j \) which is equivalent to \( \frac{\nu_j + \frac{n_j}{2} + \frac{b_j}{2}}{q} > \beta_j - \gamma_j + \tau_j + \frac{n_j}{2} + \frac{b_j}{2}. \)

Multiplying this last inequality by \( \omega_j < 0 \) yields \( \frac{\nu_j + \frac{n_j}{2} + \frac{b_j}{2}}{q'} \omega_j < (\beta_j - \gamma_j + \tau_j + \frac{n_j}{2} + \frac{b_j}{2})(\mu_j + \frac{b_j}{2}). \)

This is equivalent to
\[ \frac{\nu_j + \frac{n_j}{2} + \frac{b_j}{2}}{q'} \omega_j - \frac{\beta_j - \gamma_j - \nu_j + \tau_j}{q'}(\nu_j + \frac{b_j}{2}) > \frac{\beta_j - \gamma_j - \nu_j + \tau_j}{s}(\mu_j + \frac{b_j}{2}). \]

(27)

From the second inequality of (25), we have \( \nu_j < p(\beta_j + \tau_j - \frac{m_j}{2}) + \frac{m_j}{2} + \frac{b_j}{2} \) which is equivalent to \( \frac{\nu_j - \frac{m_j}{2} - \frac{b_j}{2}}{q} < \beta_j + \tau_j - \frac{m_j}{2} \) Hence, \( \nu_j + \frac{m_j}{2} - \frac{b_j}{2} - \frac{1}{q}(\nu_j - \frac{m_j}{2} - \frac{b_j}{2}) < \beta_j + \tau_j - \frac{m_j}{2} \) so that \( \beta_j - \nu_j + \tau_j + \frac{b_j}{2} + \frac{1}{q}(\nu_j - \frac{m_j}{2} - \frac{b_j}{2}) > 0. \) Multiplying this last inequality by \( \omega_j \) yields \( (\beta_j - \nu_j + \tau_j + \frac{b_j}{2})\omega_j + \frac{1}{q}(\nu_j - \frac{m_j}{2} - \frac{b_j}{2})\omega_j < 0, \) i.e.
\[ \frac{\nu - \frac{m_j}{2} - \frac{b_j}{2}}{q'} \omega_j - \frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{q'}(\nu_j + \frac{b_j}{2}) < \frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{s}(\mu_j + \frac{b_j}{2}). \]

(28)

From the left inequality of (26), we have \( \frac{m_j}{2} - s\alpha_j + \frac{b_j}{2} < \mu_j \) which is equivalent to \( -\frac{\nu_j - \frac{m_j}{2} - \frac{b_j}{2}}{s} - \alpha_j < 0. \) Multiplying this last inequality
by \( \omega_j < 0 \) yields 

\[
\frac{\mu_j - \frac{m_j}{s} - \frac{b_j}{s}}{\omega_j} < \frac{\alpha_j}{s}(\mu_j + \frac{b_j}{2})< \frac{\alpha_j}{q'}(\nu_j + \frac{b_j}{2}).
\]

From the right inequality of (26), we have \( \mu_j < s(\gamma_j - \alpha_j + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2} \) which is equivalent to \( \frac{\mu_j + \frac{n_j}{2} + \frac{b_j}{2}}{s} < \gamma_j - \alpha_j + \frac{b_j}{2} \). Multiplying this last inequality by \( \omega_j \) yields 

\[
\frac{\mu_j + \frac{n_j}{2} + \frac{b_j}{2}}{s} \omega_j + \frac{\gamma_j - \alpha_j + \frac{b_j}{2}}{s}(\mu_j + \frac{b_j}{2}) > -\frac{\gamma_j - \alpha_j + \frac{b_j}{2}}{q'}(\nu_j + \frac{b_j}{2}).
\]

(30)

The inequalities (28), (29), (27) and (30) yield the existence of real numbers \( u_j \) and \( v_j \) such that

\[
\begin{cases}
\frac{\nu_j - \frac{m_j}{q'} + \frac{b_j}{q'}}{\omega_j} - \frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{q'}(\nu_j + \frac{b_j}{2}) < u_j \omega_j + \frac{(\beta_j - \nu_j + \tau_j + \frac{b_j}{2})(\nu_j - u_j)}{s} < \frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{s}(\mu_j + \frac{b_j}{2}); \\
\frac{\mu_j - \frac{m_j}{s} - \frac{b_j}{s}}{\omega_j} - \frac{\alpha_j}{s}(\mu_j + \frac{b_j}{2}) < v_j \omega_j + \frac{\alpha_j(u_j - v_j)}{s} < \frac{\alpha_j}{q'}(\nu_j + \frac{b_j}{2})
\end{cases}
\]

and

\[
\begin{cases}
\frac{\nu_j + \frac{n_j}{2} + \frac{b_j}{2}}{\omega_j} - \frac{\beta_j - \gamma_j - \nu_j + \tau_j}{q'}(\nu_j + \frac{b_j}{2}) > u_j \omega_j + \frac{(\beta_j - \gamma_j - \nu_j + \tau_j)(\nu_j - u_j)}{s} > \frac{\beta_j - \gamma_j - \nu_j + \tau_j}{s}(\mu_j + \frac{b_j}{2}); \\
\frac{\mu_j + \frac{n_j}{2} + \frac{b_j}{2}}{s} \omega_j + \frac{\gamma_j - \alpha_j + \frac{b_j}{2}}{s}(\mu_j + \frac{b_j}{2}) > v_j \omega_j + \frac{(\gamma_j - \alpha_j + \frac{b_j}{2})(\nu_j - u_j)}{s} > -\frac{\gamma_j - \alpha_j + \frac{b_j}{2}}{q'}(\nu_j + \frac{b_j}{2}).
\end{cases}
\]

(32)

Now, (31) is equivalent to

\[
\begin{cases}
-\frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{\omega_j} \left[ -\frac{\mu_j + \frac{b_j}{2}}{s} - u_j + v_j \right] < u_j < \frac{\nu_j - \frac{m_j}{q'} + \frac{b_j}{q'}}{\omega_j} + \frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{s} \left[ -\nu_j + \frac{b_j}{q'} + u_j - v_j \right]; \\
-\frac{\alpha_j}{\omega_j} \left[ -\frac{\nu_j + \frac{b_j}{2}}{q'} + u_j - v_j \right] < v_j < \frac{\nu_j - \frac{m_j}{s} + \frac{b_j}{s}}{\omega_j} + \frac{\alpha_j}{\omega_j} \left[ -\mu_j + \frac{b_j}{s} - u_j + v_j \right]
\end{cases}
\]

(33)
and (32) is equivalent to
\[
\begin{align*}
&\left\{ \frac{\nu_j + \frac{b_j}{q'}}{q'} + \frac{\beta_j - \gamma_j - \nu_j + \tau_j}{\omega_j} \left[ \frac{\nu_j + b_j}{q'} + u_j - v_j \right] \right. \\
&\left. < u_j < \frac{\beta_j - \gamma_j + \nu_j + \tau_j}{\omega_j} \left[ -\frac{\mu_j + \frac{b_j}{s}}{s} - u_j + v_j \right] ; \right.
\end{align*}
\]
(34)
\[
\begin{align*}
&\left\{ \frac{\mu_j + \frac{a_j}{s} + \frac{b_j}{q}}{s} - \frac{\gamma_j - \omega_j + \frac{b_j}{s}}{\omega_j} \left[ -\frac{\nu_j + b_j}{s} - u_j + v_j \right] \\
&< v_j < \frac{\gamma_j - \alpha_j + \frac{b_j}{s}}{\omega_j} \left[ -\frac{\nu_j + b_j}{s} + u_j - v_j \right] .
\end{align*}
\]
Let
\[
t_j = \frac{-\nu_j + \frac{b_j}{q'}}{q'} + \frac{u_j - v_j}{\omega_j}
\]
then
\[
1 - t_j = \frac{-\frac{\mu_j + \frac{b_j}{s}}{s} - u_j + v_j}{\omega_j}.
\]
Since \(\omega_j < 0\), we choose \(u_j\) and \(v_j\) such that \(0 < v_j - u_j < \frac{\mu_j + \frac{b_j}{s}}{s}\). Thus, we have \(0 < t_j < 1\). Therefore (33) and (34) become
\[
\begin{align*}
&\left\{ -(\beta_j - \nu_j + \tau_j + \frac{b_j}{2})(1 - t_j) < u_j < \frac{\nu_j + \frac{a_j}{s}}{s} - \frac{\beta_j - \nu_j + \tau_j + \frac{b_j}{2}}{s}t_j \\
&0 < -\alpha_j t_j < v_j < \frac{\mu_j + \frac{a_j}{s} - \frac{b_j}{s}}{s} + \alpha_j (1 - t_j)
\end{align*}
\]
and
\[
\begin{align*}
&\left\{ \frac{\nu_j + \frac{a_j}{s} + \frac{b_j}{q'}}{q'} + (\beta_j - \gamma_j - \nu_j + \tau_j) t_j < u_j < -(\beta_j - \gamma_j + \nu_j + \tau_j)(1 - t_j) \\
&0 < (\gamma_j - \alpha_j + \frac{b_j}{2})(1 - t_j) < v_j < (\gamma_j - \alpha_j + \frac{b_j}{2})t_j.
\end{align*}
\]
respectively.

We shall now use the adapted version of Okikiolu test i.e. Lemma 6.1 to conclude. To this effect, we observe that the kernel of the operator \(S : L^p(U) \to L^q(U)\) with respect to the measure \(Q^{\nu - \frac{\beta}{2} - \tau}(x - F(u, v))dx dv(v)\) is given by
\[
K((y, u); (x, v)) = Q^\alpha(y - F(u, u))Q^{-\gamma - \frac{\beta}{2}}(y - 2\Re(u, v) + x)Q^{\beta - \nu + \frac{\beta}{2} + \tau}(x - F(u, v))
\]
\[
= \prod_{j=1}^r k_j((y, u); (x, v))
\]
with
\[
k_j((y, u); (x, v)) = Q^\alpha_j(y - F(u, u))Q^{-\gamma - \frac{\beta}{2}}(y - 2\Re(u, v) + x)Q^{\beta - \nu + \frac{\beta}{2} + \tau}(x - F(u, v)).
\]
Consider the positive functions
\[ \varphi_1(x, v) = Q^{-u}(x - F(v, v)) = \prod_{j=1}^{r} Q_{j}^{-u_j}(x - F(v, v)) \]
and
\[ \varphi_2(y, u) = Q^{-v}(y - F(u, u)) = \prod_{j=1}^{r} Q_{j}^{-v_j}(y - F(u, u)). \]

Then
\[
I_1 = \int_{U} K((y, u); (x, v))^{tq'} \varphi_1(x, v)^{q' Q^{\nu - \frac{b}{2} - \tau}}(x - F(v, v)) dv \]
\[ = \int_{C^m} \int_{\Omega + F(v, v)} K((y, u); (x, v))^{tq'} \varphi_1(x, v)^{q' Q^{\nu - \frac{b}{2} - \tau}}(x - F(v, v)) dv \]
\[ = \int_{C^m} \int_{\Omega} K((y, u); (x + F(v, v), v))^{tq'} \varphi_1(x + F(v, v), v)^{q' Q^{\nu - \frac{b}{2} - \tau}}(x) dv \]
\[ = Q^{tq' \alpha}(y - F(u, u)) \int_{C^m} J_1(y, u, v) dv \]

where
\[ J_1(y, u, v) = \int_{\Omega} Q^{-(\gamma + \frac{b}{2}) tq'}(y - 2\Re F(u, v) + F(v, v) + x) Q^{tq' (\beta - \nu + \frac{b}{2} + \tau) - q' u_j + v_j - \frac{b_j}{2} > m_j} \]

The integral \( J_1(y, u, v) \) converges because from the right inequality in (35) involving \( u_j \), we have
\[
t_j q' (\beta_j - \nu_j + \frac{b_j}{2} + \tau_j) - q' u_j + v_j - \frac{b_j}{2} > \frac{m_j}{2}
\]
and the inequality
\[
- t_j q' (\gamma_j + \frac{b_j}{2}) + t_j q' (\beta_j - \nu_j + \frac{b_j}{2} + \tau_j) - q' u_j + v_j - \frac{b_j}{2} < -\frac{n_j}{2}
\]
is satisfied because the left inequality in (36) involving \( u_j \) is satisfied. It follows using Lemma 3.10 that
\[ J_1(y, u, v) = C Q^{(-\gamma + \beta - \nu + \tau) t q' - q' u_j + \nu_j - \frac{b_j}{2} > \frac{n_j}{2}}(y - 2\Re F(u, v) + F(v, v)). \]

Thus
\[ I_1 = C Q^{tq' \alpha}(y - F(u, u)) \times \int_{C^m} Q^{(-\gamma + \beta - \nu + \tau) t q' - q' u_j + \nu_j - \frac{b_j}{2} > \frac{n_j}{2}}(y - 2\Re F(u, v) + F(v, v)) dv \]
and using Proposition 5.1, the integral above converges because \((\gamma_j - \beta_j + \nu_j - \tau_j)t_j q' + q' u_j - \nu_j + \frac{b_j}{2} - b_j > \frac{n_j}{2}\) which is exactly the left
inequality in (36) involving \( u_j \). It follows that
\[
I_1 = C_1 Q^{q'(\alpha - \gamma + \frac{b}{2} + r - \frac{1}{2})} (y - F(u, u))
\]
\[
= C_1 Q^{q' - q'' + \frac{b}{2}} (y - F(u, u)) = C_1 Q^{-q''} (y - F(u, u))
\]
since for every \( j = 1, \ldots, r \), we have
\[
t_j q' \omega_j - q' u_j + \nu_j + \frac{b_j}{2} = q' \left( -\frac{\nu_j + \frac{b_j}{2}}{q'} + u_j - \nu_j \right) - q' u_j + \nu_j + \frac{b_j}{2} = -q' \nu_j.
\]
On the other hand,
\[
I_2 = \int U K((y, u); (x, v)) (1-t) s \varphi_2(y, u) \varphi_2^\ast(x, v) Q^{\mu - \frac{b}{2} - \tau} (y - F(u, u)) dy dv(u)
\]
\[
= \int_{\mathbb{C}^m} \int_{\Omega + F(u, u)} K((y, u); (x, v)) (1-t) s \varphi_2(y, u) \varphi_2^\ast(x, v) Q^{\mu - \frac{b}{2} - \tau} (y - F(u, u)) dy dv(u)
\]
\[
= \int_{\mathbb{C}^m} \int_{\Omega} K((y, u); (y + F(u, u), u)) (1-t) s \varphi_2(y + F(u, u), u) \varphi_2^\ast(y) Q^{\mu - \frac{b}{2} - \tau} (y) dy dv(u)
\]
\[
= Q^{(1-t) s (\beta - \nu + \frac{b}{2} + r - \frac{1}{2})} (x - F(v, v)) \int_{\mathbb{C}^m} J_2(x, v, u) dv(u)
\]
where
\[
J_2(x, v, u) = \int_{\Omega} Q^{-\gamma + \frac{b}{2} + (1-t) s (\omega v + \nu - \frac{b}{2})} (y - 2 \text{Re} F(u, v) + F(u, u) + x) Q^{(1-t) s (\alpha - \nu + \mu - \frac{b}{2} - \tau)} (y) dy.
\]
The integral \( J_2(x, v, u) \) converges because from the right inequality in (35) involving \( \nu_j \), we have
\[
(1-t) s \alpha_j - sv_j + \mu_j - \frac{b_j}{2} > \frac{m_j}{2}
\]
and the inequality
\[
-(1-t) s (\gamma_j + \frac{b_j}{2}) + (1-t) s \alpha_j - sv_j + \mu_j - \frac{b_j}{2} < -\frac{m_j}{2}
\]
is satisfied because the left inequality in (36) involving \( v_j \) is satisfied. It follows from Lemma 3.10 again that
\[
J_2(x, v, u) = C Q^{(-\gamma + \frac{b}{2} + \alpha)(1-t) s - sv + \mu - \frac{b}{2}} (-2 \text{Re} F(u, v) + F(u, u) + x).
\]
Thus
\[
I_2 = C Q^{(1-t) s (\beta - \nu + \frac{b}{2} + r - \frac{1}{2})} (x - F(v, v)) \times
\]
\[
\int_{\mathbb{C}^m} Q^{-\gamma + \frac{b}{2} + (1-t) s (\omega v + \nu - \frac{b}{2})} (-2 \text{Re} F(u, v) + F(u, u) + x) dv(u)
\]
and using Proposition 5.1, the integral converges \( (\gamma_j + \frac{b_j}{2} - \alpha_j)(1-t) s + sv_j - \mu_j + \frac{b_j}{2} - b_j > \frac{n_j}{2} \) which is the left inequality in (36) involving \( v_j \).
It follows that
\[
I_2 = C_2 Q^{(\beta - \nu + \tau - \gamma + \alpha)(1 - t)x - sv + \mu + \frac{3b}{2} + \frac{3|b|}{2}}(x - F(v, v)) = C_2 Q^{(1 - t)\omega - sv + \mu + \frac{3b}{2} + \frac{3|b|}{2}}(x - F(v, v)) = C_2 \varphi_1(x, v)^s
\]
since for every \( j = 1, \ldots, r \), we have
\[
(1 - t_j)\omega_j - sv_j + \mu_j + \frac{b_j}{2} = - (\mu_j + \frac{b_j}{2}) - sv_j - sv_j + \mu_j + \frac{b_j}{2} = - sv_j.
\]
Thus by Lemma 6.1, we conclude that \( S : L^p_\nu(U) \rightarrow L^q_\mu(U) \) is bounded. \( \square \)

6.2. Necessity for the boundedness of \( S_{\alpha, \beta, \gamma} \).

**Theorem 6.3.** Let \( \nu, \mu \in \mathbb{R}^r \) and \( 1 < q \leq s < \infty \). If the operator \( S \) is bounded from \( L^p_\nu(U) \) into \( L^q_\mu(U) \), then the parameters satisfy the conditions (25), (26) and
\[
|\gamma| = |\alpha| + |\beta| + |\tau| + \frac{3|b|}{2} - \frac{|\nu| + \frac{3|b|}{2}}{q} + \frac{|\mu| + \frac{3|b|}{2}}{s}.
\]

**Proof.** We start with the proof of the homogeneity condition (37), that is,
\[
|\gamma| = |\alpha| + |\beta| + |\tau| + \frac{3|b|}{2} - \frac{|\nu| + \frac{3|b|}{2}}{q} + \frac{|\mu| + \frac{3|b|}{2}}{s}.
\]
For this, we recall that the fundamental compound function \( Q_j \) are homogeneous of degree 1. Let \( R > 0 \). To any \( f \in L^p_\nu(U) \), we associate the function \( f_R \) defined by \( f_R(y, u) = f(Ry, Ru) \). One easily checks that
\[
\|f_R\|_{L^p_\nu(U)} = R^{-|\nu| + \frac{3|b|}{q}}\|f\|_{L^p_\nu(U)}.
\]
An easy change of variable combined with the homogeneity of the function \( Q \) provides
\[
S(f_R)(y, u) = R^{-n - \frac{3|b|}{2} + |\gamma| - |\beta| - |\alpha|} SF(Ry, Ru).
\]
It follows using again the homogeneity of the compound function \( Q \) that
\[
\|Sf_R\|_{L^p_\nu(U)} = R^{-n - \frac{3|b|}{2} + |\gamma| - |\beta| - |\alpha| - \frac{|\nu| + \frac{3|b|}{2}}{q}}\|SF\|_{L^p_\nu(U)}.
\]
From the boundedness of the operator \( S \), we have that there is a constant \( C \) such that for any \( f \in L^p_\nu(U) \),
\[
R^{-n - \frac{3|b|}{2} + |\gamma| - |\beta| - |\alpha| - \frac{|\nu| + \frac{3|b|}{2}}{q}}\|SF\|_{L^p_\nu(U)} = \|Sf_R\|_{L^p_\nu(U)} \leq C\|f\|_{L^p_\nu(U)} = CR^{-\frac{|\nu| + \frac{3|b|}{q}}{q}}\|f\|_{L^p_\nu(U)},
\]
that is,
\[ R^{-n-\frac{3|b|}{2}+|\gamma|-|\beta|-|\alpha|-\frac{|\mu|+\frac{3|b|}{2}}{s} + \frac{|\nu|+\frac{3|b|}{2}}{q}} \|Sf\|_{L^q(U)} \leq C \|f\|_{L^q(U)}. \]

As the latter holds for every \( f \in L^q(U) \) and as \( R \) was taken arbitrary, we should necessarily have
\[ -n - \frac{3|b|}{2} + |\gamma| - |\beta| - |\alpha| - \frac{|\mu|+\frac{3|b|}{2}}{s} + \frac{|\nu|+\frac{3|b|}{2}}{q} = 0, \]
which leads to \( |\gamma| = |\tau| + \frac{3|b|}{2} + |\beta| + |\alpha| + \frac{|\mu|+\frac{3|b|}{2}}{s} - \frac{|\nu|+\frac{3|b|}{2}}{q} \). This proves (37).

We shall take advantage of the description of the homogeneous cone \( \Omega \) within the framework of \( T \)-algebra. We denote by \( e \) the identity element in \( V \) and by \( E(0,1) \) the Euclidean ball of \( \mathbb{C}^n \) centered at the origin with radius 1. Let \( g = \chi_{E(0,1) \times B(e,1)} \), where \( B(e,1) \) is the Bergman ball in \( \Omega \) about \( e \) with radius 1.

Following [19], we have
\[
Sg(y,u) = Q^\alpha(y-F(u,u)) \times \int_{E(0,1)} \int_{E(0,1)} Q^{-\gamma-\beta}(y-2\Re F(u,v) + F(v,v) + x)Q^\beta(x)dx dv(v) \
\simeq Q^\alpha(y-F(u,u)) \int_{E(0,1)} Q^{-\gamma-\frac{\beta}{2}}(y-2\Re F(u,v) + F(v,v) + e)dv(v)
\]

From the continuity of the function \( v \mapsto Q^{-\gamma-\frac{\beta}{2}}(y-2\Re F(u,v) + F(v,v) + e) \) on the compact set \( E(0,1) \), we write
\[
Q^\alpha(y-F(u,u))Q^{-\gamma-\frac{\beta}{2}}(y-2\Re F(u,v) + F(v,v) + e) \lesssim Sg(y,u)
\]
for some \( v_0 \in \overline{E}(0,1) \). It follows that if \( S \) is bounded from \( L^q(U, Q^{\nu-\frac{\beta}{2}-\tau}(y-F(u,u))dy dv(u)) \) to \( L^s(U, Q^{\mu-\frac{\beta}{2}-\tau}(y-F(u,u))dy dv(u)) \), then the function \( (y,u) \mapsto Q^\alpha(y-F(u,u))Q^{-\gamma-\frac{\beta}{2}}(y-2\Re F(u,v) + F(v,v) + e) \) is in \( L^s(U, Q^{\mu-\frac{\beta}{2}-\tau}(y-F(u,u))dy dv(u)) \), which means that the integral
\[
\int_{\mathbb{C}^m} \int_{\Omega} Q^{\alpha+\mu-\frac{\beta}{2}-\tau}(y)Q^{-\gamma-\frac{\beta}{2}}(y-2\Re F(u,v) + F(u,u) + F(v,v) + e)dy dv(u)
\]
converges. Integrating with respect to \( y \), we see from Lemma 3.10 that we should have
\[
s\alpha_j + \mu_j - \frac{b_j}{2} > \frac{m_j}{2} \quad \text{and} \quad -s\gamma_j - \frac{b_j}{2} + s\alpha_j + \mu_j - \frac{b_j}{2} < -\frac{n_j}{2}.
\]
That is,
\[
\frac{m_j}{2} + \frac{b_j}{2} - s\alpha_j < \mu_j < s(\gamma_j - \alpha_j + \frac{b_j}{2}) - \frac{n_j}{2} + \frac{b_j}{2}.
\]
In this case,
\[ \int_{\Omega} Q^{\alpha+\mu-\frac{b}{2} - \gamma} (y)Q^{-s_\gamma - s_\delta}(y - 2\Re F(u, v_0) + F(u, u) + F(v_0, v_0) + e)dy = CQ^{\alpha+\mu-\frac{b}{2} - \gamma - s_\delta} (e + F(v_0, v_0) - 2\Re F(u, v_0) + F(u, u)). \]

Integrating now on \( \mathbb{C}^m \) with respect to \( u \), we see, thanks to Proposition 5.1 that the integral
\[ \int_{\mathbb{C}^m} Q^{\alpha+\mu-\frac{b}{2} - \gamma - s_\delta} (e + F(v_0, v_0) - 2\Re F(u, v_0) + F(u, u))dv(u) \]
converges if \( \mu_j < s(\gamma_j - \alpha_j + \frac{b_j}{2}) - \frac{m_j}{2} - \frac{b_j}{2} \); combining with conditions (38), we get condition (26).

To prove the necessity of the condition (25), we proceed by duality. We have that the boundedness of \( S \) is bounded from \( L^q(U, Q^{\nu - \frac{b}{2} - \gamma})(y - F(u, u))dydv(u) \) to \( L^s(U, Q^{\mu - \frac{b}{2} - \gamma}(y - F(u, u))dydv(u) \) is equivalent to the boundedness of the adjoint \( S^* \) of \( S \) from \( L^s(U, Q^{\mu + \nu - \frac{b}{2} - \gamma}(y - F(u, u))dydv(u) \) to \( L^q(U, Q^{\nu - \frac{b}{2} - \gamma}(y - F(u, u))dydv(u) \). We note that \( S^* \) is given by
\[ S^* g(x, v) = Q^{\beta - \nu + \frac{b}{2} + \tau}(x - F(v, v)) \times \int_U Q^{-\tau - \frac{b}{2}}(x - 2\Re F(u, v) + y)g(y, u)Q^{\nu - \frac{b}{2} - \tau}(y)dydv(u). \]

Proceeding as above, we obtain that the function
\[ (x, v) \mapsto Q^{\beta - \nu + \frac{b}{2} + \tau}(x - F(v, v))Q^{-\tau - \frac{b}{2}}(x - 2\Re F(u_0, v) + F(u_0, u_0) + e) \]
must belong to \( L^q(U, Q^{\nu - \frac{b}{2} - \gamma}(x - F(v, v))dx dv(v) \); here \( u_0 \in \overline{E}(0, 1) \).

That is, the integral
\[ \int_{\mathbb{C}^m} \int_{\Omega} Q^{q'(\beta - \nu + \frac{b}{2} + \tau) + \nu - \frac{b}{2} - \tau}(x - F(v, v))Q^{q'(-\gamma - \frac{b}{2})} Q^{q'(-\gamma - \frac{b}{2})}(x - 2\Re F(u_0, v) + F(u_0, u_0) + e)dx dv(v) \]
converges.

Integrating with respect to \( x \) and using again Lemma 3.10, we see that we must have \( (\beta_j - \nu_j + \frac{b_j}{2} + \tau_j)q' + \nu_j - \frac{b_j}{2} > \frac{m_j}{2} \) and \(-q' \gamma_j - q' \frac{b_j}{2} + (\beta_j - \nu_j + \tau_j + \frac{b_j}{2})q' + \nu_j - \frac{b_j}{2} < -\frac{m_j}{2} \), which is equivalent to
\[ \begin{cases} \nu_j < q(\beta_j + \tau_j - \frac{m_j}{2}) + \frac{m_j}{2} + \frac{b_j}{2} \\ \nu_j > q(\beta_j - \gamma_j + \tau_j - \frac{b_j}{2} + \frac{m_j}{2}) - \frac{m_j}{2} - \frac{b_j}{2} \end{cases} \]
(39)

In this case, we get
\[ \int_{\Omega} Q^{q'(\beta - \nu + \tau) + \nu - \frac{b}{2} - \gamma}(x - F(v, v))Q^{q'(-\gamma - \frac{b}{2})} Q^{q'(-\gamma - \frac{b}{2})}(x - 2\Re F(u_0, v) + F(u_0, u_0) + e)dx = CQ^{q'(\beta - \nu + \gamma - \tau) + \nu - \frac{b}{2}}(e - 2\Re F(u_0, v) + F(u_0, u_0)). \]
Integrating now on $\mathbb{C}^m$ with respect to $v$, we see, thanks to Proposition 5.1 that the integral
\[ \int_{\mathbb{C}^m} Q'^{(\beta - \nu + \tau - \gamma + \nu - \frac{b}{2})} (e - 2\Re F(u_0, v) + F(u_0, u_0)) \, dv(v) \]
converges if $\nu_j > q(\beta_j - \gamma_j + \tau_j + \frac{n_j}{2} + \frac{b_j}{2}) - \frac{b_j}{2} - \frac{n_j}{2}$; combining with conditions (39) we get condition (25).

This completes the proof of the lemma.

\[ \square \]

7. Off-diagonal estimates for the family $T_{a,\beta,\gamma}^+$ of Positive Bergman-type operators

In this section, we shall take advantage on the relationships between the family of Bergman-type operators $T^+ := T_{a,\beta,\gamma}^+$ on $D$ with the family $S := S_{a,\beta,\gamma}$ and the results obtained in the previous section to find necessary and sufficient conditions for the boundedness of the family $T^+$ in mixed norm case.

7.1. Sufficiency. Here, we follow for example the proofs of [19, Theorem 1.1] and some techniques in [15]. We have the following:

**Theorem 7.1.** Let $\alpha, \beta, \gamma \in \mathbb{R}$. For $\nu = (\nu_1, \nu_2, \ldots, \nu_r), \mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathbb{R}^r$, $1 < p < \infty$ and $1 < q \leq s < \infty$ such that $\frac{\nu_j + \frac{b_j}{q}}{q} + \frac{\mu_j + \frac{b_j}{s}}{s} > 0$, $j = 1, \ldots, r$. Assume that the parameters satisfy for all $j = 1, \ldots, r$,

\[
\gamma_j = \alpha_j + \beta_j + \tau_j + \frac{b_j}{2} - \frac{\nu_j + \frac{b_j}{q}}{q} + \frac{\mu_j + \frac{b_j}{s}}{s},
\]

and

\[
\begin{cases}
q(\beta_j - \gamma_j + \tau_j + \frac{n_j}{2} + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2} < \nu_j, \\
q(\beta_j + \tau_j - \frac{m_j}{2}) + \frac{m_j}{2} + \frac{b_j}{2} > \nu_j,
\end{cases}
\]

and

\[
\frac{m_j}{2} - s\alpha_j + \frac{b_j}{2} < \mu_j < s(\gamma_j - \alpha_j + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2}.
\]

Then the operator $T_{a,\beta,\gamma}^+$ is bounded from $L^p_{\nu} (D)$ to $L^p_{\mu} (D)$.

**Proof.** The result follows from inequality (20) and Theorem 6.2. \[ \square \]

7.2. Necessity. In this section, we find necessary conditions for boundedness of the family of Positive Bergman-type operators. We shall establish this through a connection between the $T^+$ and the operator on the cone $S$. 
\textbf{Theorem 7.2.} Suppose $1 < q \leq s < \infty$ and suppose that the operator $T^+_{a,\beta,\gamma}$ is bounded from $L^{p,q}_\nu(D)$ to $L^{p,s}_\mu(D)$. Then the parameters satisfy the following

\begin{equation}
|\gamma| = |\alpha| + |\beta| + |\tau| + \frac{3|b|}{2} - \frac{|\nu| + \frac{3|b|}{2}}{q} + \frac{|\mu| + \frac{3|b|}{2}}{s}.
\end{equation}

and for all $j = 1, \ldots, r$, we have

\begin{equation}
\begin{cases}
q(\beta_j - \gamma_j + \tau_j + \frac{n_j}{2} + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2} < \nu_j, \\
q(\beta_j + \tau_j - \frac{m_j}{2}) + \frac{m_j}{2} + \frac{b_j}{2} > \nu_j,
\end{cases}
\end{equation}

and

\begin{equation}
\frac{m_j}{2} - s\alpha_j + \frac{b_j}{2} < \mu_j < s(\gamma_j - \alpha_j + \frac{b_j}{2}) - \frac{n_j}{2} - \frac{b_j}{2}.
\end{equation}

\textit{Proof.} Let $\lambda \in \mathbb{R}$ such that $0 < \lambda < \frac{1}{4}$ and $g$ the function defined on $D$ by $g(x + iy, u) = \chi_{\{|x|<1\}}(x)k(y, u)$ where $k \in L^p_\nu(U)$ is positive with support in $W = \{(y, u) \in U : |y| \leq \frac{\lambda}{2}, |u| \leq 1\}$. By Lemma 3.12, there is a constant $C$ such that for all $(y, u) \in W$,

$$T^+g(x + iy, u) \geq CSk(y, u).$$

It follows from our hypothesis that

$$\int_W (Sk(y, u))^s Q^{\mu - \frac{s}{2} - \tau}(y - F(u, u))dydv(u)$$

\begin{align*}
&= c \int_W \left( \int_{|x|<1} (Sk(y, u))^p dx \right)^{\frac{s}{p}} Q^{\mu - \frac{s}{2} - \tau}(y - F(u, u))dydv(u) \\
&\leq cC^{-s} \int_W \left( \int_{|x|<1} |T^+g(x + iy, u)|^p dx \right)^{\frac{s}{p}} Q^{\mu - \frac{s}{2} - \tau}(y - F(u, u))dydv(u) \\
&\leq c(b^{-s}||T^+g||_{L^{p,s}_\nu(D)}^s) \leq c'C^{-s}||g||_{L^{p,q}_\nu(D)}^s.
\end{align*}

Let $N \in \mathbb{N}^*$. The above reasoning remains true if $W$ is replaced by the Euclidean ball $B(O, N)$ in $\mathbb{R}^n \times \mathbb{C}^m \equiv \mathbb{R}^{n+2m}$ centered at the origin.

Since, continuous functions with compact support are dense in $L^p_\nu(U)$, so by letting $N$ tend to infinity, we conclude by the Lebesgue monotone Theorem that

$$\int_U (Sk(y, u))^s Q^{\mu - \frac{s}{2} - \tau}(y - F(u, u))dydv(u)$$

\begin{align*}
&\leq C||T^+g||_{L^{p,s}_\nu(D)}^s \leq C' \int_U (k(y, u))^s Q^{\mu - \frac{s}{2} - \tau}(y - F(u, u))dydv(u).
\end{align*}

Thus, if $T^+$ is bounded from $L^{p,q}_\nu(D)$ to $L^{p,s}_\mu(D)$ then $S$ is bounded from $L^p_\nu(U)$ to $L^s_\mu(U)$; hence from Theorem 6.3, conditions (43), (44) and (45) hold.

\hfill $\square$
As a special case of Theorem 7.1, we have the following which extends the diagonal case result \((q = s)\) [4, Theorem 4.3] (see also [19, Corollary 3.6]). This gives optimal indices for the boundedness of \(P_\nu^+\) as compared to [17, Theorem 3.1].

**Corollary 7.3.** Let \(1 < p < \infty, 1 < q \leq s < \infty,\) and assume that \(\nu_j, \mu_j > \frac{m_j}{2}, j = 1, \ldots, r.\) Let \(\Omega\) be an open convex homogeneous cone.

- Assume that the operator \(P_\nu^+\) is bounded from \(L_{p,q}^\nu(T)\) to \(L_{p,s}^\mu(T)\).

Then we have \(\frac{|\nu|}{q} = \frac{|\mu|}{s}\) and

\[
1 + \max_{1 \leq j \leq r} \frac{n_j/2}{\mu_j} < q < 1 + \min_{1 \leq j \leq r} \frac{\nu_j}{n_j/2}.
\]

- The operator \(P_\nu^+\) is bounded from \(L_{p,q}^\nu(T)\) to \(L_{p,s}^\mu(T)\) if \(\frac{\nu_j}{q} = \frac{\mu_j}{s}, j = 1, \ldots, r\) and

\[
1 + \max_{1 \leq j \leq r} \frac{n_j/2}{\mu_j} < q < 1 + \min_{1 \leq j \leq r} \frac{\nu_j}{n_j/2}.
\]

**Proof.** It suffices to take in Theorem 7.2 and Theorem 7.1 the parameters \(\alpha_i = 0, \beta_i = \nu_i - \frac{b_i}{2} - \tau_i\) and \(\gamma_i = \nu_i\) and \(m = 0\) i.e. \(b_i = 0,\) for all \(i = 1, \ldots, r.\) \(\square\)

The proof of Corollary 4.2 is just the diagonal case \(q = s\) of Corollary 7.3.

### 8. Proof of Theorem 4.5

In this section, we will use the Plancherel-Gindikin Theorem (Theorem 4.1) to prove that the "Laplace transform" is an isomorphism between \(A_{p,q}^2(D)\) and the space \(\beta_{g}^0(\Omega^* \times \mathbb{C}^m)\) to be defined below. This will lead us to the proof of Theorem 4.5. We shall then obtain the proof of Theorem 4.6 by interpolation. The results here are the analogues of those in the papers [3], [4], [8] and [17]. We will give only statements of the proofs that emphasize differences.

In the sequel, we consider the following disjoint covering of the cone \(\Omega^*\)

\[
E_1^* = B_1^*, \quad E_j^* = B_j^* \setminus \bigcup_{k=1}^{j-1} B_k^*, \quad j = 2, \ldots.
\]

where \(B_j^* = B_j^*(y_j')\) and \(\{y_j'\}\) is the dual of the lattice \(\{y_j\}\). We have \(\Omega^* = \bigcup_j E_j^*\) and

\[
|E_j^*| \sim |B_j^*| \sim (Q^*)^\tau(y_j').
\]

We shall need the following definitions:
DEFINITION 8.1. Let \( j \in \mathbb{N}^* \). We say that a function \( g \) belongs to \( L^{2,q}(E^*_j \times \mathbb{C}^m) \) if \( g \) is measurable and satisfies
\[
\|g\|_{L^{2,q}(E^*_j \times \mathbb{C}^m)}^q := \int_{\mathbb{C}^m} \left( \int_{E^*_j} e^{-2(F(u,\xi)|\xi|)} |g(\xi, u)|^2 d\xi \right)^{\frac{q}{2}} dv(u)
\]
is finite.

DEFINITION 8.2. Let \( q \geq 1 \) and \( \{\xi_j\} \) a lattice in \( \Omega^* \). We denote by \( \beta^q_{\nu}(\Omega^* \times \mathbb{C}^m) \) the space of all functions \( g \in L^{2,q}(E^*_j \times \mathbb{C}^m) \) so that
\[
\|g\|_{\beta^q_{\nu}(\Omega^* \times \mathbb{C}^m)}^q := \sum_j (Q^*)^{-\nu + \frac{1}{2}}(\xi_j) \|g\|_{L^{2,q}(E^*_j \times \mathbb{C}^m)}^q
\]
is finite.

We say that a sequence \( \{\lambda_j\}_j \) belongs to \( l^q_{\nu} \) if it satisfies
\[
\sum_j |\lambda_j|^q (Q^*)^{-\nu + \frac{1}{2}}(\xi_j) < +\infty.
\]

LEMMA 8.3. The space \( \beta^q_{\nu}(\Omega^* \times \mathbb{C}^m) \) is a Banach space.

Proof:
Just remark that \( \beta^q_{\nu}(\Omega^* \times \mathbb{C}^m) = l^q_{\nu}(L^{2,q}(E^*_j \times \mathbb{C}^m)) \).

REMARK 8.4. Let \( \{a_j\}_j \) a positive sequence. Then
\[
(\sum_j a_j)^{\delta} \leq \sum_j a_j^{\delta} \quad \text{if} \quad 0 < \delta \leq 1
\]
and
\[
\sum_j a_j^{\delta} \leq \left( \sum_j a_j \right)^{\delta} \quad \text{if} \quad \delta \geq 1.
\]

LEMMA 8.5. For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \) and \( \beta \in \mathbb{R} \), the integral
\[
I_{\alpha,\beta} = \int_{\Omega^*} Q^*(\xi)^\alpha (1 + |\log Q^*_1(\xi)|)^\beta e^{-(\xi|e)} d\xi
\]
is finite if and only if one of the following two conditions is satisfied:

i) \( \alpha_r > -1 \) and \( \alpha_j > -\frac{m_j}{2} - 1, \ j = 1, 2, \ldots, r - 1; \)

ii) \( \beta < -1; \alpha_r = -1 \) and \( \alpha_j > -\frac{m_j}{2} - 1, \ j = 1, 2, \ldots, r - 1. \)

Proof:
We use this change of variable: \( \xi = u^* u \in \Omega^* \). Therefore, we have
\[
d\xi = 2^r \left( \prod_{j=1}^{r-1} u_{jj}^{1+m_j} \right) u_{rr} du; \ Q^*(\xi) = \prod_{j=1}^r u_{jj}^2 \quad \text{(with} \ Q^*_1(\xi) = u_{rr}^2) \ \text{and}
\]
\[
(\xi|e) = \sum_{j=1}^r u_{jj}^2 \quad \text{with} \ u_{jj} > 0 \ \text{for all} \ j = 1, \ldots, r. \ \text{It follows that}
\]
\[
I_{\alpha,\beta} = 2^r J_{\alpha} K_{\alpha,\beta}
\]
where

\[ J_\alpha = \prod_{j=1}^{r-1} \left( \int_0^{+\infty} u_{jj}^{2\alpha_j+1+m_j} e^{-u_{jj}^2} du_{jj} \right) \]

and

\[ K_{\alpha\beta} = \int_0^{+\infty} u_{rr}^{2\alpha_{rr}+1} (1 + |\log u_{rr}|)^\beta e^{-u_{rr}^2} du_{rr}. \]

Hence, \( J_\alpha \) is finite if and only if \( \alpha_j > -\frac{m_j}{2} - 1, \ j = 1, 2, \ldots, r - 1; \) while \( K_{\alpha\beta} \) is finite if and only if either \( \alpha_{rr} > -1 \) or both \( \alpha_{rr} = -1 \) and \( \beta < -1. \)

**Theorem 8.6.** For \( q \geq 2 \left( 1 + \min_{1 \leq j \leq r} \frac{\nu_j + b_j}{2} \right) \), there is a function \( g \in \beta_q^0(\Omega^* \times \mathbb{C}^m) \) such that \( \mathcal{L}g \) does not belong to \( L^{2,q}(D) \).

**Proof:**
Let \( q = 2 \left( 1 + \min_{1 \leq j \leq r} \frac{\nu_j + b_j}{2} \right) \); we will and find a positive function \( g \) on \( \Omega^* \times \mathbb{C}^m \) such that \( \|g\|_{\beta_q^0(\Omega^* \times \mathbb{C}^m)} < +\infty \) but

\[ I(y, u) = \int_{\Omega^*} |g(\xi)|^2 e^{-2(\xi - F(u, u))|\xi|} d\xi = \infty \]

for all \( y \in \Omega + F(u, u) \) and \( u \in \mathbb{C}^m \).

Take

\[ g(\xi, u) = e^{-(\xi|e)^2} Q^*(\xi)\alpha (1 + |\log Q_1^*(\xi)|)^{-\frac{1}{2}} \]

with \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{r-1}, -\frac{1}{2}) \) such that \( (2\alpha_j + \tau_j)\frac{q}{2} > \nu_j + \frac{n+1}{2} + \frac{b_j}{2}, \ j = 1, 2, \ldots, r - 1. \)

By Plancherel formula, we have the following

\[ \|\mathcal{L}g\|_{L^{2,q}_\nu(D)}^q = \int_{\mathbb{C}^m} \int_{\Omega + F(u, u)} I(y, u)^{\frac{q}{2}} Q^{\nu - \frac{b}{2} - \tau} (y - F(u, u)) dy dv(u); \]

in particular, \( I(e, 0) = \int_{\Omega^*} Q^*(\xi)^{2\alpha} (1 + |\log Q_1^*(\xi)|)^{-1} e^{-2(\xi|e)d\xi}. \) According to Lemma 8.5, this integral is not finite. This shows that \( \mathcal{L}g \notin L^{2,q}_{\nu}(D) \). However, from Lemma 3.1, Hölder inequality and (iii)
of Lemma 3.2,

\[ \|g\|_{\beta^q_2(\Omega^* \times \mathbb{C}^m)}^q = \sum_j Q^*(\xi_j)^{-\nu + \frac{1}{r}} \int_{\mathbb{C}^m} \left( \int_{E_j^*} e^{-2F(u, u)} |g(\xi, u)|^2 d\xi \right)^{\frac{r}{2}} dv(u) \]

\[ = \sum_j Q^*(\xi_j)^{-\nu + \frac{1}{r}} \int_{\mathbb{C}^m} \left( \int_{E_j^*} e^{-2(\xi | e + F(u, u))} Q^*(\xi)^{2\alpha} (1 + |\log Q_1^*(\xi)|)^{-1} d\xi \right)^{\frac{r}{2}} dv(u) \]

\[ \leq C \sum_j \int_{\mathbb{C}^m} \left( \int_{E_j^*} e^{-2(\xi | e + F(u, u))} Q^*(\xi)^{2\alpha + \frac{1}{r}} (1 + |\log Q_1^*(\xi)|)^{-1} d\xi \right)^{\frac{r}{2}} dv(u) \]

\[ \leq Cq \int_{\Omega^*} e^{-q(\xi | e) Q^*(\xi)(2\alpha + \tau)^{-\frac{r}{2}} - \nu + \frac{1}{r} - \tau (1 + |\log Q_1^*(\xi)|)^{-\frac{r}{2}} \left( \int_{\mathbb{C}^m} e^{-q(\xi | F(u, u))} dv(u) \right) d\xi \]

\[ \leq Cq \int_{\Omega^*} e^{-q(\xi | e) Q^*(\xi)(2\alpha + \tau)^{-\frac{r}{2}} - \nu + \frac{1}{r} - \tau (1 + |\log Q_1^*(\xi)|)^{-\frac{r}{2}} d\xi; \]

the last inequality follows from Lemma 3.13. According to Lemma 8.5, observing that \( m_r = 0 \) by definition, the last integral above is convergent; so the function \( g \in \mathcal{B}^q_0(\Omega^*) \) whenever its Laplace transform doesn’t belong to \( L^2_\nu(D) \) where \( q = 2 \left( 1 + \frac{\nu + \frac{1}{r}}{2} \right) \geq 2 \left( 1 + \min_{1 \leq j \leq r} \frac{\nu_j + \frac{1}{r}}{2} \right) \). □

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