Uniqueness in determination of the fractional order in the TFDE using one measurement

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Abstract: This article deals with an inverse problem of identifying the fractional order in the 1D time fractional diffusion equation (TFDE in short) using the measurement at one space-time point. Based on the expression of the solution to the forward problem, the inverse problem is transformed to a nonlinear algebraic equation. By choosing suitable initial values and the measured point, the nonlinear equation has a unique solution by the monotonicity of the Mittag-Lellfer function. Theoretical testifications are presented to demonstrate the unique solvability of the inverse problem.

Keywords: Time fractional diffusion equation (TFDE); inverse problem of fractional order; Mittag-Lellfer function; nonlinear algebraic equation; conditional uniqueness

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1 Introduction

Many diffusion processes in nature and engineering, such as contaminants transport in the soil, oil flow in porous media, long distance transport of pollutants in
the groundwater, etc., are referred to as anomalous diffusion, where the particle plume spreads slower or faster than predicted by the classical integer-order diffusion model. In recent three decades, fractional diffusion equations have been found to be efficient mathematical models to describe some anomalous diffusion phenomena (see [1, 6, 17, 35, 43, 48], e.g.). The TFDE model, that is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative, is given as

\[ \partial_t^\alpha u - D\Delta u = 0, \quad (x, t) \in \Omega_T, \]

where \( \Omega_T = \Omega \times (0, T) \) with \( T > 0, \Omega \subset \mathbb{R}^N \) is an open bounded domain with smooth boundary, and \( u = u(x, t) \) denotes the state variable at space point \( x \) and time \( t \), \( D > 0 \) is the diffusion coefficient, \( \alpha \in (0, 1) \) is the fractional order, and the fractional derivative \( \partial_t^\alpha u \) is defined by Caputo’s definition:

\[ \partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\alpha}. \]

See, e.g., Podlubny [38] and Kilbas et al [24] for the definitions and properties of the fractional derivatives.

There are quite a lot of researches on equation (1.1) from numerics, and some works from theoretical aspects, we refer to [8, 11, 12, 13, 16, 33, 34, 36, 39] for the theoretical analysis on the solution of the forward problem, and also see the monographs [38], [24], [25] and [14]. On the other hand, we refer to [2, 10, 15, 21, 22, 29, 31, 32, 37, 42, 44] for the researches on inverse problems arising from the TFDEs. As we know the fractional order in a fractional model is a key parameter to characterize the heavy-tail subdiffusion with memory effect. However, it is always unknown for real-life problems which leading to inverse fractional order problems. The inverse problems of identifying parameters including fractional orders have been studied during the last decade. We refer to Alimor and Ashurov [3], Ashurov and Umarov [4], Ashurov and Zumunov [5], Cheng et al [9], Hatano et al [18], Janno [19], Janno and Kinash [20], Jin and Kian [23], Li et al [26], Li et al [30], Li and Yamamoto [28], Sun et al [30], Tatar and Ulusoy [41], Yamamoto [45, 46], Yu et al [47], and so on.

It is noted that in the existing work on inverse fractional order problems, most of them were studied by using subdomain measurements or one-point measurements at \( t \in (0, T) \), or using subboundary data also at \( t \in (0, T) \) for arbitrary given \( T > 0 \). An interesting problem is can we determine the fractional orders in theory...
only using limited discretized data? Exactly speaking, there is one order \( \alpha \in (0, 1) \) in equation (1.1) which is unknown, can we determine it uniquely only using one measurement?

It is difficult to give an answer in theory for the above question, but the situation could be changed if having suitable conditions. Here we are concerned with the inverse problem of determining the fractional order \( \alpha \) in Eq. (1.1) using one measurement at one space-time point. By the eigenfunction expansion method, the solution of the forward problem is expressed by the Mittag-Leffler function, and the inverse problem is transformed to a nonlinear algebraic equation. By choosing suitable initial values and the measured point, the nonlinear equation can be solved uniquely by the monotonicity of the Mittag-Leffler function. The unique solvability of the inverse problem is testified by theoretical examples.

The rest of the paper is organized as follows.

In Section 2, some preliminaries on the Mittag-Leffler function, and solvability on a nonlinear algebraic equation are introduced. The forward problem and its solution, and the inverse problem of identifying the fractional order using one measurement are given in Section 3. In Section 4, a local well-posedness of the inverse problem is obtained by the theory of solution to the nonlinear algebraic equation, and a conditional uniqueness is proved by suitably choosing the initial value and the measured time, and numerical testifications are presented. Conclusions are given in Section 5.

2 Preliminaries

We give some basic facts on the Gamma function, the Mittag-Leffler function and their properties, and the theory of solving a nonlinear algebraic equation. We refer to the monographs \[14\] and \[27\] for details.

2.1 The Gamma function and Mittag-Leffler function

The Gamma function is an analytic continuation of the factorial function in the entire complex plane, which is defined by

\[
\Gamma(z) = \int_0^\infty x^z e^{-x} \, dx,
\]  \hspace{1cm} (2.1)
with \( \Re(z) > 0 \). For the Gamma function there hold the formula \( \Gamma(z + 1) = z \Gamma(z) \), and the Stirling’s approximate formula

\[
\Gamma(z + 1) \sim \sqrt{2\pi z} \ e^{-z} \ z^z, \quad z \to +\infty,
\]

and the derivative’s formula

\[
\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + z} \right),
\]

where \( \Gamma'(z) \) denotes the derivative of the Gamma function, and \( \gamma = 0.5772157 \ldots \) is the Euler constant. On the Gamma function, we have the following assertion.

**Lemma 1** For the parameter \( \alpha \in (0, 1) \), there holds

\[
\lim_{j \to \infty} \frac{\Gamma(\alpha j)}{\Gamma(\alpha j + \alpha)} = 0.
\]

**Proof** By (2.2) we have for large \( z > 0 \)

\[
\frac{\Gamma(\alpha j)}{\Gamma(\alpha j + \alpha)} \sim \frac{\sqrt{2\pi e^{-\alpha j} (\alpha j)^{\alpha j}}/\sqrt{\alpha j}}{\sqrt{2\pi e^{-\alpha (\alpha j + \alpha)^{\alpha j}}/(\alpha j + \alpha)}} (j \to \infty)
\]

\[
\sim e^{\alpha} \left( \frac{\alpha j}{\alpha j + \alpha} \right)^{\alpha j} \frac{1}{\alpha j + \alpha} \left( \frac{1}{1 + 1/j} \right)^\alpha \sqrt{1 + \frac{1}{j}} (j \to \infty).
\]

Thanks to \( \lim_{j \to \infty} \left( \frac{1}{1+1/j} \right)^j = e^{-1} \), we get

\[
\lim_{j \to \infty} \frac{\Gamma(\alpha j)}{\Gamma(\alpha j + \alpha)} = \lim_{j \to \infty} \left[ e^{\alpha} e^{-\alpha} \frac{1}{(\alpha j + \alpha)^\alpha} \sqrt{1 + \frac{1}{j}} \right] = \lim_{j \to \infty} \frac{1}{(\alpha j + \alpha)^\alpha} = 0.
\]

Following Mittag-Leffler’s classical definition, the one-parametric Mittag-Leffler function is defined by the power series

\[
E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + 1)}, \quad z \in \mathbb{C}, \ \alpha > 0.
\]

Obviously there is \( E_1(z) = e^z \) as \( \alpha = 1 \), i.e., \( E_\alpha(z) \) is a generalization of the exponential function \( e^z \).

On analytical properties of the Mittag-Leffler functions, we have the following assertions.
Lemma 2  (i) There exists a constant $c > 0$ such that $|E_\alpha(z)| \leq \frac{c}{1+|z|}$, $\alpha > 0$.

(ii) For real number $z \in \mathbb{R}$, the function $E_\alpha(z)$ is strictly monotone on $z > 0$, and decreasing on $z < 0$.

Lemma 3 Let $G(\alpha) = E_\alpha(-ct^\alpha)$ for $\alpha \in (0, 1)$ and given $0 < c, t < \infty$. There holds

$$G'(\alpha) = \sum_{j=1}^{\infty} (-c)^j \frac{t^{\alpha j} \ln(t) + \gamma_j}{\Gamma(\alpha j + 1)},$$

(2.8)

where $\gamma_j = \gamma + \frac{1}{\alpha j + 1} - \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+\alpha j+1} \right)$ according to (2.3).

Proof By (2.7), it is easy to deduce that the derivative $G'(\alpha)$ ($0 < \alpha < 1$) is expressed by (2.8), where $\gamma_j$ is given by (2.3), and we only need to prove the convergence of the series at the right-hand side of (2.8).

Denote

$$y_j = \frac{(-c)^j \frac{t^{\alpha j}}{\Gamma(\alpha j + 1)} \ln(t)}{\Gamma(\alpha j + 1)}, \quad \bar{y}_j = \frac{(-c)^j \frac{t^{\alpha j} \gamma_j}{\Gamma(\alpha j + 1)}}{\Gamma(\alpha j + 1)}, \quad j = 1, 2, \cdots,$$

(2.9)

and we have $G'(\alpha) = \sum_{j=1}^{\infty} (y_j + \bar{y}_j)$. For given $c$ and $t$ as parameters, there holds

$$\left| \frac{y_{j+1}}{y_j} \right| = \frac{c t^\alpha (j+1) \Gamma(\alpha j + 1)}{j \Gamma(\alpha j + \alpha + 1)} = \frac{c t^\alpha}{\Gamma(\alpha j + \alpha)} \frac{\Gamma(\alpha j + 1)}{\Gamma(\alpha j + \alpha + 1)},$$

(2.10)

here we utilize the formula $\Gamma(z+1) = z\Gamma(z)$. Then by Lemma 1 we get

$$\lim_{j \to \infty} \left| \frac{y_{j+1}}{y_j} \right| = 0,$$

(2.11)

by which the series $\sum_{j=1}^{\infty} y_j$ is convergent. Noting that

$$\lim_{j \to \infty} \frac{\gamma_{j+1}}{\gamma_j} = 1,$$

(2.12)

we deduce that $\sum_{j=1}^{\infty} \bar{y}_j$ is also convergent by the same arguments. So the derivative $G'(\alpha)$ is well-defined for $\alpha \in (0, 1)$. 

5
2.2 Solution to a nonlinear algebraic equation

Consider to solve a group of nonlinear algebraic equations

\[ F(x) = y, \quad (2.13) \]

where \( F: D \subset \mathbb{R}^N \rightarrow \mathbb{R}^N \) is continuous differential on \( D \), and \( D \) is an open bounded domain in \( \mathbb{R}^N \). On the local unique solvability of the equation \( (2.13) \), we give the following assertion \([27]\).

**Lemma 4** For the equation \( (2.13) \), assume that there exists a nonsingular matrix \( B \in \mathcal{L}(\mathbb{R}^N) \), and \( x^0 \in D \) with a closed sphere \( S_0 = \overline{S}(x^0, \delta) \) such that

\[ ||F(x) - F(y) - B(x - y)|| \leq \beta||x - y||, \quad \forall x, y \in S_0, \quad (2.14) \]

where \( 0 < \beta < ||B^{-1}||^{-1} \). Then \( F \) is a homeomorphism mapping from \( S_0 \) to \( F(S_0) \).

We now consider to solve a nonlinear algebraic equation in 1D case

\[ g(x) = y, \quad (2.15) \]

where \( g: D \subset \mathbb{R} \rightarrow \mathbb{R} \) is continuous differential on \( D \), and \( D \) is an open bounded interval in \( \mathbb{R} \). Based on Lemma 4 there holds

**Corollary 1** Suppose that the function \( g: D \subset \mathbb{R} \rightarrow \mathbb{R} \) is continuous differentiable at \( x_0 \in \text{int}(D) \), and \( g'(x_0) \neq 0 \), and \( V_0 = \{x : |x - x_0| \leq \delta\} \) for any \( \delta > 0 \) as a neighborhood of \( x_0 \). Then \( g \) is a local homeomorphism mapping at \( x_0 \), i.e., the equation \( g(x) = y \) has a unique solution for given \( y \in g(V_0) \). Moreover, the inverse function \( g^{-1}: g(V_0) \rightarrow V_0 \) is also continuous.

**Proof** Denote \( a = g'(x_0) \) and \( a \neq 0 \). By the definition of the derivative, we have

\[ \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} = a. \quad (2.16) \]

Thus \( \forall \varepsilon > 0 \) and \( \varepsilon < |a| \), there exists \( \delta > 0 \) such that

\[ |g(x) - g(x_0) - a(x - x_0)| \leq \varepsilon|x - x_0|, \quad \forall x, x_0 \in V_0. \quad (2.17) \]

Noting \( \varepsilon < |a| \), by Lemma 4 follows that the mapping \( g \) is a local homeomorphism from \( V_0 \) to \( g(V_0) \), and the equation \( g(x) = y \) has a unique solution for given \( y \in g(V_0) \) and the inverse function \( g^{-1} \) is continuous.
3 The forward problem and the inverse problem

3.1 The forward problem

Consider the homogeneous time-fractional diffusion equation in 1D case

$$\partial_t^\alpha u - Du_{xx} = 0, \quad (x,t) \in \Omega_T,$$

where $\Omega_T = \Omega \times (0,T)$ and $\Omega = (0,l)$ for $l > 0$. For the equation (3.1), given the homogeneous Dirichelet boundary condition

$$u(0,t) = u(l,t) = 0, \quad 0 < t \leq T,$$

and the initial distribution

$$u(x,0) = f(x), \quad x \in \Omega,$$

where $f(x) \in L^2(\Omega)$. A forward problem is composed by the equation (3.1) with the initial boundary value conditions (3.2)-(3.3). By using the eigenfunction expansion method, there exists a unique solution (see [39, 46] for instance):

$$u(x,t) \in C([0,T], L^2(\Omega)) \cap C((0,T], H^2(\Omega) \cap H^1_0(\Omega)),$$

and it can be expressed by the Mittag-Lellfer function given as

$$u(x,t) = \sum_{n=1}^{\infty} f_n \ E_\alpha (-D\lambda_n t^\alpha) \ \varphi_n(x),$$

where $f_n = \frac{2}{l} (f, \varphi_n)$, and $\lambda_n = \frac{n^2 \pi^2}{l^2}$, $\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right)$, and $E_\alpha(\cdot)$ is given by (2.7).

3.2 The inverse problem

Although we get the solution to the forward problem given by (3.5), it can not be put into practice if there are unknown parameters in the model. Actually, the fractional order $\alpha$ in Eq. (3.1) is an important index characterizing the anomalous diffusion with memory effect in time, but it cannot be measured which leads to inverse fractional order problems.

Suppose that the fractional order in Eq. (3.1) is unknown, and we are to determine it by limited data in time at one space point. Let $x_0 \in \Omega$ be fixed, and we have the measured data at time $t_k$ given as

$$u(x_0, t_k), \quad k = 1, 2, \cdots, K,$$
where $K \geq 1$. The inverse problem is to identify the order $\alpha \in (0, 1)$ using the data (3.6) based on the forward problem (3.1)-(3.3).

Denote $I = (0, 1)$, and $u(\alpha)(x, t)$ as the unique solution to the forward problem for any given $\alpha \in I$. An interesting problem is:

*Can we determine $\alpha \in I$ uniquely only using one measurement of $u(x_0, t_1)$ at $t_1 \in (0, T)$?*

Generally speaking, it is difficult to answer the above question in theory. Nevertheless, we can determine it in some special cases. Denote $d = u(x_0, t_1)$ as the measurement. Then noting the solution’s expression (3.5), we get a nonlinear algebraic equation:

$$F(\alpha) = u(\alpha)(x_0, t_1) = \sum_{n=1}^{\infty} f_n E_\alpha(-D\lambda_n t_1^\alpha) \varphi_n(x_0) = d. \quad (3.7)$$

As a result the inverse fractional order problem is transformed to solving of the nonlinear equation (3.7).

### 4 Unique solvability of the inverse problem

#### 4.1 Local well-posedness

We now prove the inverse problem is locally solvable and has unique solution for $\alpha \in I$ by choosing appropriate measured point and the measured time $t_1 > 0$. For that we need the existence of the derivative of the function $F(\alpha)$ on $\alpha \in I$, and some limitations for the initial value functions.

Without loss of generality, we set $l = \pi$. It is known that the eigenfunction system of the forward problem is $\{n^2, \sin(nx)\}$ for $n = 1, 2, \cdots$. For $f(x) \in L^2(0, \pi)$, assume that there exists a limited number set $\Sigma_N = \{n_1, n_2, \cdots, n_N\} \subset \mathbb{N}$ such that

$$(f(x), \sin(nx)) \begin{cases} \neq 0, & n \in \Sigma_N, \\ = 0, & \text{others}, \end{cases} \quad (4.1)$$

which means that there holds

$$f(x) = \sum_{n \in \Sigma_N} f_n \sin(nx), \quad (4.2)$$
where \(f_n = \frac{2}{\pi} (f(x), \sin(nx))\).

By (3.5) we have

\[
    u(x, t) = \sum_{n \in \Sigma_N} f_n E_\alpha (-D\lambda_n t^\alpha) \sin(nx), \quad 0 < t \leq T.
\]

(4.3)

Then the equation (3.7) reduces to

\[
    F(\alpha) = \sum_{n \in \Sigma_N} f_n E_\alpha (-D\lambda_n t_1^\alpha) \varphi_n(x_0) = d,
\]

(4.4)

and we get by using (2.8) in Lemma 3

\[
    F'(\alpha) = \sum_{n \in \Sigma_N} f_n \varphi_n(x_0) \sum_{j=1}^{\infty} \frac{(-1)^j j D^j \lambda_j t_1^{\alpha j}}{\Gamma(\alpha j + 1)} [\ln(t_1) + \gamma_j], \quad 0 < \alpha < 1.
\]

(4.5)

Henceforth, by using Corollary 1 there holds

Theorem 1 For suitable \(x_0 \in \Omega\) and \(t_1 > 0\), there exists \(\alpha_0 \in I\) such that \(F'(\alpha_0) \neq 0\), and then the equation (4.4) has a unique solution in a neighborhood of \(\alpha_0\), and the solution is continuously dependent upon the measured data locally, i.e., the inverse fractional order problem is of local well-posedness.

Proof By suitably choosing the measured point \(x_0\), we have \(\varphi_n(x_0) \neq 0\) for \(n \in \Sigma_N\), and then there exists one \(\alpha_0 \in I\) such that \(F'(\alpha_0) \neq 0\). Then by using Corollary 1, the nonlinear equation (4.4) has a unique solution at the neighborhood of \(\alpha_0\), i.e., the inverse fractional order problem is locally solvable. Moreover, noting the local continuity of the inverse function \(F^{-1}\), the inverse problem is of local stability.

Furthermore, we can get a conditional uniqueness under the nonnegative conditions for the initial values.

4.2 Conditional uniqueness

By Theorem 1 the nonlinear equation (4.4) is locally solvable for \(d = u(x_0, t_1)\) where \(0 < t_1 \leq T\) and \(x_0 \in \Omega\) be fixed. There are no global uniqueness for solving nonlinear equations in general case, but the situation could be changed if special conditions are overposed by which a conditional uniqueness is obtained. For convenience we also set \(\Omega = (0, \pi)\). There holds
**Theorem 2** Assume that the initial value function has the form of (4.2), and there hold $f_n \varphi_n(x) \geq 0$ for $n \in \Sigma_N$ and $x \in \Omega$. Then the fractional order can be uniquely determined by one measurement $u(x_0, t_1)$ if $\varphi_n(x_0) \neq 0$ and $t_1 > 0$.

**Proof** Under the given conditions, there are $\varphi_n(x) = \sin(nx)$ and $\lambda_n = n^2$ for $n = 1, 2, \cdots$. By (4.3) the solution of the forward problem is given as

$$u(\alpha)(x, t) = \sum_{n \in \Sigma_N} f_n E_\alpha(-Dn^2 t^\alpha) \sin(nx),$$

(4.6)

and we get the nonlinear equation on the order $\alpha \in (0, 1)$

$$F(\alpha) = \sum_{n \in \Sigma_N} E_\alpha(-Dn^2 t_1^\alpha) f_n \sin(nx_0) = d,$$

(4.7)

where $d = u(x_0, t_1)$ is the additional measurement.

By choosing $x_0 \in \Omega$ such that $\sin(nx_0) > 0$ for $n \in \Sigma_N$, and noting the condition $f_n \varphi_n(x) \geq 0$ for $n \in \Sigma_N$, we can deduce that the function $F(\alpha)$ is of strict monotonicity on $\alpha \in (0, 1)$ due to the strict monotonicity and positivity of the Mittag-Leffler function. Therefore the equation (4.7) has only one solution $\alpha^* \in I = (0, 1)$. The proof is completed.

### 4.3 Theoretical testification

#### 4.3.1 Example 1

In this example we set $D = 0.1$, and the initial function $f(x) = \frac{1}{2}\sin(2x)$ for $x \in [0, \pi]$, and we choose $x_0 = \frac{\pi}{4}$ as the measured point, and $t_1 = 2$ as the measured time, i.e., the additional measurement is $d = u(\frac{\pi}{4}, 2)$. Noting $n = 2$ and $f_n = \frac{1}{2}$, and $\sin(nx_0) = 1$, we have by (4.7)

$$F(\alpha) = \frac{1}{2} E_\alpha(-2^{1+\alpha}), \quad 0 < \alpha < 1,$$

(4.8)

and we need to solve the equation

$$F(\alpha) - d = 0,$$

(4.9)

using the additional data $u(x_0, t_1) = d$.

Let the exact order be $\alpha^{\text{exa}} = 0.75$, by which we get the additional data $d = 0.25818$ by solving the forward problem. In order to see the unique solvability of the
equation, we plot the continuous function $F(\alpha) - d$ on $\alpha \in [0, 1]$ in Figure 1, where $F(0)$ and $F(1)$ are defined by

$$
F(0) = \frac{1}{2} \sum_{k=0}^{\infty} \left(-\frac{2}{5}\right)^k, \quad F(1) = \frac{1}{2} e^{-\frac{4}{5}}.
$$

It can be seen clearly that the function $F(\alpha) - d$ is strictly monotone on $\alpha \in [0, 1]$, and the inverse order problem is of uniqueness.

![Figure 1](image)

Figure 1. The picture of $F(\alpha) - d$ on $\alpha \in [0, 1]$ in Ex. 1.

### 4.3.2 Example 2

We set $D = 0.05$, and the initial function $f(x) = 2 \sin(x) + \frac{1}{2} \sin(3x)$ for $x \in [0, \pi]$ in this example. For given $\alpha \in (0, 1)$, noting that $D = 0.05$, and $\lambda_1 = 1, \lambda_3 = 3$, the solution of the forward problem in this example is expressed as follows

$$
u(\alpha)(x, t) = 2 E_\alpha(-0.05t^\alpha) \sin(x) + \frac{1}{2} E_\alpha(-0.45t^\alpha) \sin(3x).
$$

For the inverse order problem, let the exact fractional order be $\alpha^{exa} = 0.5$. We choose $x_0 = \frac{\pi}{6}$ as the measured point, and $t_1 = 10$ as the measured time, i.e., the
additional measurement is \( d = u\left(\frac{x}{6}, 10\right) = 1.0112 \). We have the nonlinear equation combing with the additional data

\[
F(\alpha) = E_\alpha(-0.05 \cdot 10^\alpha) + \frac{1}{2} E_\alpha(-0.45 \cdot 10^\alpha) = d. \quad (4.12)
\]

As done in the above, the function \( F(\alpha) - d \) on \( \alpha \in [0, 1] \) are plotted in Figure 2.

![Figure 2. The picture of \( F(\alpha) - d \) on \( \alpha \in [0, 1] \) in Ex. 2.](image)

From Figure 2 it can be seen again that the equation \( F(\alpha) - d = 0 \) has a unique solution, and the inverse order problem is uniquely solvable by one measurement.

## 5 Conclusion

An inverse problem of identifying the fractional order in the 1D time fractional diffusion equation by using one measurement is investigated. Based on the expression of the solution to the forward problem, the inverse problem is reduced to a nonlinear algebraic equation on the fractional order, and the unique solvability can be obtained by the monotonicity and positivity of the Mittag-Lefler function of real variables under the positive conditions for the initial values and the measured point. Theoretical examples are given to illustrate the conditional uniqueness of the
inverse problem. It is noted that the derivative of the function $F(\alpha)$ on $\alpha \in (0,1)$ can be computed by (4.8) or (4.12) respectively, and some gradient-type iterative algorithms can be applied to solve the nonlinear equation for which we will give details in another work in the near future.

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