THE MASS POWER SPECTRUM IN QUINTESSENCE COSMOLOGICAL MODELS
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ABSTRACT
We present simple analytic approximations for the linear and fully evolved nonlinear mass power spectrum of matter density fluctuations for spatially flat cold dark matter (CDM) cosmological models with quintessence (Q). Quintessence is a time-evolving, spatially inhomogeneous energy component with negative pressure and an equation of state \( w < 0 \). It clusters gravitationally on large length scales but remains smooth like the cosmological constant on small length scales. We show that the clustering scale is determined by the Compton wavelength of the Q-field and derive a shape parameter, \( \Gamma_o \), to characterize the linear mass power spectrum. The growth of linear perturbations as functions of redshift, \( w_o \), and matter density, \( \Omega_m \), is also quantified. Calibrating to N-body simulations, we construct a simple extension of Ma’s 1998 formula that closely approximates the nonlinear power spectrum for a range of plausible QCDM models.

Subject headings: cosmology: theory — dark matter — large-scale structure of universe — methods: analytical

1. INTRODUCTION
Quintessence (Q) offers an alternative to the cosmological constant (\( \Lambda \)) as the missing energy in a spatially flat universe with a subcritical matter density \( \Omega_m \) (Caldwell, Dave, & Steinhardt 1998 and references therein). It is an energy component which, similar to \( \Lambda \), has negative pressure and therefore a negative \( w \) in the equation of state \( p = w \rho \). However, unlike \( \Lambda \), for which \( w = -1 \), quintessence is time evolving and spatially inhomogeneous, and \( w \) can have a range of values. The observational imprints of the quintessence therefore differ from those of the commonly studied cold dark matter with a nonzero \( \Lambda \) (LCDM) cosmology (e.g., Wang et al. 1999).

In this Letter, we study spatially flat QCDM models in which the cold dark matter and Q-field together make up the critical density (i.e., \( \Omega_m + \Omega_Q = 1 \)). The quintessence is modeled as a scalar field that evolves with a constant equation of state \( w \). It drives the cosmological expansion at late times, influencing the rate of growth of structure. Fluctuations in Q behave as an ultralight mass scalar field: on very large length scales the quintessence clusters gravitationally, thereby modifying the level of cosmic microwave background temperature anisotropy relative to the matter power spectrum amplitude (in addition to a late-time—integrated Sachs-Wolfe effect); on small length scales, fluctuations in Q disperse relativistically and the Q-field behaves as a smooth component.

We investigate the effects of the quintessence on the spectrum and time evolution of gravitational clustering in both the linear and nonlinear regimes. We propose simple, analytical fitting formulas for both the linear and fully evolved nonlinear power spectrum of matter density fluctuations in plausible QCDM models. For the linear power spectrum (§ 2), we introduce a simple parameter \( \Gamma_o \) derived from the Compton wavelength of the Q-field to characterize its shape. This parameter determines the length scale above which the Q-field can cluster gravitationally and is reminiscent of \( \Gamma \) derived from the free-streaming distance of hot neutrinos in cold+hot dark matter (C+HDM) models by Ma (1996). For the nonlinear power spectrum (§ 3), we examine the validity of the simple linear-to-nonlinear mapping technique that has been successfully developed for scale-free, CDM, C+HDM, and LCDM models (Hamilton et al. 1991; Jain, Mo, & White 1995; Peacock & Dodds 1996; Ma 1998). We present a simple extension of the analytical formula of Ma (1998) that closely approximates the QCDM nonlinear power spectrum computed from a set of N-body simulations.

The formulae presented in this Letter are essential for gaining physical insight into the effects of the quintessence on gravitational collapse and for performing rapid predictions of observable quantities in the linear as well as nonlinear regimes in plausible QCDM models.

2. LINEAR POWER SPECTRUM
We use the conventional form to express the linear power spectrum for the matter density perturbation \( \delta_m \) in QCDM models:

\[
P(k, a) = A_Q k^n T_Q^2(k) \left( \frac{g_{Q,0}}{g_{Q,0}(a=1)} \right)^2, \tag{1}
\]

where \( A_Q \) is a normalization, \( k \) is the wavenumber, \( n \) is the spectral index of the primordial adiabatic density perturbations, and \( T_Q \) is the transfer function which encapsulates modifications to the primordial power-law spectrum. The function \( g_Q \) is the linear growth suppression factor, \( g_0 = D/\alpha \), where \( D \) is the standard linear growth factor for the matter density field in QCDM models and \( g_{Q,0} = g_0(a=1) \) denotes its value at the present day with scale factor \( a = 1 \). We discuss each piece of equation (1) in turn.

First we examine the transfer function for the matter density field. To isolate the effects of quintessence, we find it convenient and illuminating to compare a pair of QCDM and LCDM models that have the same set of cosmological parameters and differ only in \( w \) (recall \( w = -1 \) for LCDM). We define the relative transfer function for a pair of such models to be \( T_{Q,s} = T_Q/T_s \). For \( T_s \), we follow the convention and set the arbitrary amplitude of \( T_s \) to unity as \( k \to 0 \). The form of \( T_s \) is well known, and various fitting formulae have been published

\[
T_s(k) = \frac{\delta_m(k)}{\delta_m(k=0)} = \frac{1}{\frac{2}{3}
\]
(e.g., Bardeen et al. 1986; Efstathiou, Bond, & White 1992; Sugiyama 1995). More complicated fits with higher accuracy have also been developed for higher baryon ratios ($\Omega_m/\Omega_b \approx 20\%$) and for the features due to baryonic oscillations and damping (e.g., Bunn & White 1997; Eisenstein & Hu 1998).

The transfer function $T_q$ for QCDM models resembles $T_\lambda$ for $\Lambda$CDM, but with one key difference. The linear matter density field, $\delta_m$, evolves according to the equation $\delta_m + 2H\delta_m = 4\pi G(\rho\rho_m + \delta\rho_0 + 3\delta\rho)$, where $H = \alpha/a$ and the dots denote differentiation with respect to proper time. On small length scales, the Q-field is smooth ($\delta\rho_0 = \delta\rho$, and we recover the familiar equation for the evolution of $\delta_m$ (Caldwell et al. 1998). On very large length scales, however, the Q-field clusters and contributes to the energy density and pressure perturbations. The result is a different growth rate for $\delta_m$ on large and small scales once the quintessence starts to dominate the cosmological energy density. We can determine the characteristic scale separating these two regimes by examining the linear equation for the Q-field: $\delta\dot{Q} + 3H\delta Q + (k^2 + V_{QQ})\delta Q = \delta_m[(1 + w_q)\rho_0]^{1/2}$ (where $V$ is the Q-field potential, $V_{QQ} \equiv d^2V/ldQ^2$, and $\rho_0 = Q^2/2 + V$). We see that $\delta Q$ itself behaves as a scalar field with an effective mass $(V_{QQ})^{1/2}$ and a Compton wavenumber of $k_c \sim (V_{QQ})^{1/2}$. On small length scales (i.e., $k \gg k_c$), the amplitude of $\delta Q$ and hence $\delta\rho_0$ is damped and does not enter the evolution equation for $\delta_m$. On large scales ($k \ll k_c$), $\delta Q$ grows, so the Q-field clusters and in turn affects the evolution of $\delta_m$.

The change in the behavior of $\delta_m$ near $k \sim (V_{QQ})^{1/2}$ as a result of differing Q-clustering properties is illustrated in Figure 1 for $T_{Q\lambda}$ versus $k$ for a range of $w_q$ and $\Omega_m$. We have chosen to normalize $T_{Q\lambda}$ to unity at the high-$k$ end because both the Q-field and the cosmological constant are spatially smooth on these scales. The clustering property of Q is reminiscent of the case of massive neutrinos in $\Lambda$CDM models, which cannot cluster appreciably below the neutrino free-streaming scale but can cluster with the same amplitude as the cold dark matter on large scales. Analogous to the shape parameter $\Gamma_\lambda$ that was introduced to model the neutrino streaming distances in $\Lambda$CDM models (Ma 1990), we introduce a new shape parameter $\Gamma_\rho$ here to characterize the feature in Figure 1 in QCDM models. For a constant equation of state, $w_q$, one can show that $V_{QQ} = 6\pi G(1 - w_q)[2\rho + p + w_q\rho]$, where $\rho$ and $p$ are the total energy density and pressure. We approximate

$$k_c = \Gamma_q h = 2\sqrt{V_{QQ}} = \frac{3H}{c} \sqrt{(1 - w_q)[2 + 2w_q - w_q\Omega_m(a)]},$$

and we use a simple ratio of polynomials to express the relative transfer function:

$$T_{Q\lambda}(k, a) = \frac{T_Q}{T_\lambda} = \frac{\alpha + a\gamma^2}{1 + a\gamma^2}, \quad q = \frac{k}{\Gamma_q h},$$

where $k$ is in units of Mpc$^{-1}$ and $\alpha$ is a scale-independent but time-dependent coefficient that quantifies the relative amplitude of the matter density field $\delta_m$ on large and small length scales.

We find $\alpha$ well approximated by

$$\alpha = (-w_q)' = \frac{\partial w_q}{\partial a},$$

$$s = (0.012 - 0.036w_q - 0.017/w_q)[1 - \Omega_{\Lambda}(a)]$$

$$+ (0.098 + 0.029w_q - 0.085/w_q) \ln \Omega_{\Lambda}(a),$$

where the matter density parameter is $\Omega_m(a) = \Omega_m(\Omega_m + (1 - \Omega_m)a^{-3w_q})$, which reaches the value $\Omega_m$ at the present-day $a = 1$. Figure 1 illustrates the close agreement (with errors $\lesssim 10\%$) between the approximations given by equations (2)–(4) and the exact results from numerical integrations of the Boltzmann equations.

Next we examine the linear growth suppression factor of the density field in equation (1). This function is well studied for $\Lambda$CDM models (Heath 1977; Lahav et al. 1991). An empirical fit is given by $g_\Lambda = 2.5\Omega_m(a)[\Omega_m(a)^{1/2} - 1 + \Omega_m(a) + [1 + \Omega_m(a)]^{-1}[(1 + (1 - \Omega_m(a))/70)]^{-1}$ and is accurate to $\sim 2\%$ for $0.1 \leq \Omega_m \leq 1$ (Carroll, Press, & Turner 1992). This formula unfortunately cannot be generalized to QCDM models by simply replacing $(1 - \Omega_m) \rightarrow (1 - \Omega_m)a^{-3w_q}$. Instead, we propose the following formula to approximate the ratio of the QCDM and $\Lambda$CDM growth factors:

$$g_{Q\Lambda} = \frac{g_Q}{g_\Lambda} = (-w_q)',$$

$$t = -(0.255 + 0.305w_q + 0.0027/w_q)[1 - \Omega_{\Lambda}(a)]$$

$$- (0.366 + 0.266w_q - 0.07/w_q) \ln \Omega_{\Lambda}(a).$$

FIG. 1.—Ratio of the transfer functions, $T_{Q\lambda} = T_Q/T_\lambda$, at the present day for six pairs of flat QCDM and $\Lambda$CDM models. The solid (for $\Omega_m = 0.4$) and dashed (for $\Omega_m = 0.6$) curves are computed from the Boltzmann integrations and illustrate the dependence on the matter density parameter $\Omega_m$. For a given $\Omega_m$ the three curves illustrate the dependence on the equation of state: $w_q = -1/3$, $-1$, and $-2/3$ from top down. The dotted curves show the analytic approximation given by eqs. (2)–(4). Note that $T_q$, deviates from unity only on very large length scale ($k \approx 0.01$ h Mpc$^{-1}$) above which the Q-field can cluster spatially.
accurate to 2% for 0.2 ≤ Ω_m ≤ 1 and −1 ≤ w_Q ≤ −0.2. Figure 2 illustrates the dependence of g_Q on time, w_Q, and Ω_m. The growth is evidently slower in models with less negative w_Q for fixed Ω_m. This is because the energy density in the Q-field dominates over that in matter at an increasingly earlier time as w_Q is varied from −1 to 0; the growth of gravitational collapse therefore ceases earlier and results in a smaller value for g_Q. (It is sometimes useful to study the instantaneous growth rate of δ_m, f ≡ d log δ_m/d log a. See Wang & Steinhardt 1998 for a fitting formula for f.)

The remaining component in equation (1) to be specified is the normalization A_Q. It can be chosen by fixing the value of σ_8, the rms linear mass fluctuation within a top hat of radius ∼ h^−1 Mpc or by fixing to COBE results. For the latter we follow Bunn & White (1997). In the case that the temperature anisotropy is due to primordial adiabatic density perturbations with spectral index n, we write A_Q = δ_Q^2(c/H_0)^n/(4π), where

\[ \delta_Q = 2 \times 10^{-5} \alpha_0^{-1} (\Omega_m)^{1/3} c_s \ln \Omega_m, \]

\[ \alpha_0 = \alpha(a = 1) \text{ of equation (4), } c_s = -0.789 \times \left[ \frac{\Omega_m^{0.0754} - 0.211 \ln \Omega_m}{\Omega_m^{0.138} - 0.0727} \right], \]

and c_s = -0.138, for −1 ≤ w_Q ≤ −0.2. In the case of tensor perturbations, the primordial amplitudes follow the inflationary relation \( A_T = 8(1 - n)A_s \). The ratio of b = 10 multipole moments, \( r_0 \approx 0.48(1 - n)[1 + 0.1(1 - n)(8 + 7w_Q)\Omega_m^2 + 3] \)

\[ \times (1 - Q_0/k)^{2(\omega_Q)^{2/3}}, \]

where \( g_{10}(y) = 0.18 + 0.84y^2 \) and \( x = 0.75[1 - 0.66w_Q + 1.66w_Q^2 - 0.5(1 + w_Q)^3] \). One can rescale A_Q by A_Q \( \rightarrow A_Q/(1 + r_0) \) to accommodate the effect of tensors on the normalization.

3. NONLINEAR MASS POWER SPECTRUM

In this section we examine if the simple linear to nonlinear mapping technique initiated by Hamilton et al. (1991) can be extended to QCDM models. The basic approach is to search for a simple expression for the function \( \Delta_m(k) = f(\Delta_m(k)) \) that relates the linear and nonlinear density variance \( \Delta(k) = 4\pi k^3 P(k) \). Note that \( \Delta_m \) and \( \Delta \) are evaluated at different wave-numbers, where \( k_0 = k(1 + \Delta_m)^{-1/3} \) corresponds to the precollapsed scale of k. The strategy is to combine analytical clustering properties in asymptotic regimes with fits to numerical simulation results. This recipe has been successfully developed for scale-free models with a power law P(k) (Hamilton et al. 1991; Jain et al. 1995), flat CDM and ΛCDM models (Jain et al. 1995; Peacock & Dodds 1996, hereafter PD96; Ma 1998, hereafter Ma98), and flat C+HDM models with massive neutrinos (Ma98).

We investigate if the PD96 and Ma98 formulas proposed for ΛCDM models can be easily extended to QCDM models. These two formulae incorporate the time dependence of the mapping in different ways, but they share the feature that the dependence on parameters Ω_m and Ω_Q enters only through the linear growth factor g. In order to test the application of this method to QCDM models, we have performed N-body simulations for three values of w_Q: −3/2, −5/2, and −7/2, each with several different realizations. These three values should be sufficient since extensive tests of w_Q = −1 (i.e., ΛCDM models) have already been carried out in PD96 and Ma98. We restrict our attention to w_Q < −5/2 and cosmological parameter ranges that are in accordance with observations (Wang & Steinhardt 1998; Wang et al. 1999). Specifically, (Ω_m, Ω_Q, Ω_b, h) = (0.4, 0.6, 0.047, 0.65) for the w_Q = −3/2 and −5/2 models and (Ω_m, Ω_b, Ω_b, h) = (0.45, 0.55, 0.047, 0.65) for the w_Q = −7/2 model. The N-body code used is a parallel version of the particle-particle particle-mesh algorithm (Bertschinger & Gelb 1991; Ferrer & Bertschinger 1994). Each simulation uses 128^3 particles in a box of comoving volume 10^3 Mpc^3. The Plummer softening length is 50 kpc comoving, which allows us to compute the nonlinear power spectrum in highly clustered regions with k ≤ 10 h Mpc^−1 and Δ_m ≤ 1000. Since the Q-field clusters only on scales much above the box size, the presence of the quintessence only affects the initial conditions and the evolution of the scale factor a.

Figure 3 compares the linear power spectrum and the fully evolved spectrum from both the N-body runs and the approximations of PD96 and Ma98. Five redshifts are shown for each of three QCDM models. Overall, we find that the PD96 formula works well at z = 0 when the factor g in their formula is set to g = g_m, which is the appropriate growth factor for the density field for QCDM models [eq. (5)]. At earlier times, however, the PD96 formula underestimates Δ_m at k ≳ 1 h Mpc^−1 in the w_Q = −3/2 and −5/2 models by up to 30%. We have attempted less physically motivated combinations of growth factors (e.g., g = α g_m and g_A) but did not find a way to make PD96 fit.
We find that the Ma98 formula,

\[ \frac{\Delta(k)}{\Delta(k_0)} = G \left( \frac{g_0}{g_{0,0}} \right), \]

\[ G(x) = \left[ 1 + \ln (1 + 0.5x) \right] \frac{1 + 0.02x^3 + c_1x^5/g_0^3}{1 + c_2x^{7.5}}, \]

(6)
can be easily extended to QCDM models. Specifically, we propose to keep \( c_1 = 1.08 \times 10^{-4} \) and \( c_2 = 2.10 \times 10^{-5} \) used for \( \Lambda \)CDM in Ma98, but adopt \( g = g_0 \), which is the appropriate QCDM growth factor (eq. [5]), and \( g_0 = |w_0|^{1/3} \) in QCDM, where \( g_{0,0} \equiv g_0(a = 1) \). As described in Ma98, the parameter \( \beta \) in equation (6) is introduced to approximate the power-law dependence of the effective spectral index \( n_{\text{eff}} + 3 \) in previous work on \( \sigma_q \) 

\[ \frac{d}{d \ln (n_{\text{eff}} + 3)} d \ln \sigma_q \propto \beta. \]

We find \( \beta = 0.83 \) an excellent approximation for all three QCDM models that we tested (see the lower right-hand panel of Fig. 3). Other panels of Figure 3 illustrate the close agreement (rms errors of \( \sim 10\% \)) between N-body results and equation (6).

4. SUMMARY

We have presented simple formulas to approximate both the linear and nonlinear power spectra for matter density perturbations in viable quintessence cosmological models with an equation of state \( -1 \leq w_0 \leq -\frac{1}{3} \). Equations (2), (3), and (4) together specify the ratio of the linear transfer functions \( T_q \) and \( T_q \) for the matter density field for a given pair of QCDM and \( \Lambda \)CDM models with the same cosmological parameters. Equation (5) specifies the ratio of the linear growth suppression factors \( g_q \) and \( g_q \) in QCDM and \( \Lambda \)CDM models. Equation (6) approximates the nonlinear mass power spectrum.

A key difference between gravitational clustering in QCDM and \( \Lambda \)CDM models is that it is spatially smooth on all length scales, whereas the Q-field can cluster above a certain length scale. We characterize this length scale by the shape parameter \( \Gamma_q \), and the familiar \( \Gamma \propto \Omega_\Lambda h \) that corresponds to the crossover from radiation- to matter-dominated era. For the QCDM models studied in this Letter (i.e., constant \( w_0 \)), the Compton wavelength of the Q-field is very large: \( k_0 \sim 0.001 \) to \( 0.01 \) h Mpc\(^{-1}\) (see Fig. 1). On scales of galaxy clusters and below, therefore, the linear QCDM power spectrum has identical shape as in the corresponding \( \Lambda \)CDM model and differs only in the overall amplitude by a factor of \( (A_q/A_\Lambda)(g_0/g_{0,0})^2 \). This realization should simplify comparisons between QCDM and \( \Lambda \)CDM models.

For the fully evolved nonlinear power spectrum, we find that PD96 works well at \( z = 0 \) but underestimates its amplitude by up to \( \sim 30\% \) at earlier times. The formula of Ma98, on the other hand, can be easily extended to approximate the QCDM nonlinear \( P(k) \) (with errors \( \lesssim 10\% \)) for \( w_0 \leq -\frac{1}{3} \) and redshift up to \( z \approx 4 \). Equation (6) summarizes this result.

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