Hurwitz rational functions

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Abstract

A generalization of Hurwitz stable polynomials to real rational functions is considered. We establish an analogue of the Hurwitz stability criterion for rational functions and introduce a new type of determinants that can be treated as a generalization of the Hurwitz determinants.

Introduction

It is well known that the problem of stability of a linear difference or differential system with constant coefficients reduces to the question of locating the zeroes of its characteristic polynomial in the left half-plane of the complex plane. One of the most famous results from stability theory is the Hurwitz theorem, which expresses stability of a real polynomial in terms of its coefficients [5, 6, 3] (see also [1, 7]). Namely, the Hurwitz theorem states that a real polynomial has all its zeroes in the open left half-plane if and only if some determinants constructed with the coefficients of the polynomial are positive (see Theorem 1.5). Those determinants are now called the Hurwitz determinants due to Adolf Hurwitz who introduced them in [5].

In the present work we investigate a class of real rational functions satisfying the Hurwitz conditions: the Hurwitz determinants constructed with the coefficients of the Laurent series at \( \infty \) of a real rational function are positive up to the order \( n \), where \( n \) is the sum of degrees of the numerator and denominator of the rational function. This is a generalization of the class of real Hurwitz stable polynomials. It turns out that such class of rational functions is characterized by location of poles and zeroes: all zeroes lie in the open left half-plane of the complex plane while all poles lie in the open right half-plane (Theorem 2.2). Finally, we express the Hurwitz determinants of real rational functions in terms of the coefficients of the numerator and denominator (Lemma 2.4). Thereby we introduce a new type of determinants and describe the class of real rational functions satisfying Hurwitz conditions in terms of coefficients of their numerator and denominator (Theorem 2.5). As well as in case of polynomials this class of rational functions is characterized by positivity of those determinants. Some simple properties of the new type of determinants that can be treated as a generalization of the Hurwitz determinants are considered.

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1 Hurwitz polynomials

Consider a real polynomial
\[ p(z) \overset{\text{def}}{=} a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \quad a_1, \ldots, a_n \in \mathbb{R}, \quad a_0 > 0. \] (1.1)

Throughout the paper we use the following notation
\[ l \overset{\text{def}}{=} \left\lfloor \frac{n}{2} \right\rfloor, \] (1.2)
where \( n = \deg p \), and \([\rho]\) denotes the largest integer not exceeding \( \rho \).

The polynomial \( p \) can always be represented as follows
\[ p(z) = p_0(z^2) + z p_1(z^2), \]
where \( p_0 \) and \( p_1 \) are the even and odd parts of the polynomial, respectively. Introduce the following function:
\[ \Phi(u) \overset{\text{def}}{=} \frac{p_1(u)}{p_0(u)}. \] (1.3)

**Definition 1.1.** We call \( \Phi \) the function associated with the polynomial \( p \).

Associate with the polynomial \( p \) the following determinants:
\[ \Delta_j(p) \overset{\text{def}}{=} \begin{vmatrix} a_1 & a_3 & a_5 & \ldots & a_{2j-1} \\ a_0 & a_2 & a_4 & \ldots & a_{2j-2} \\ 0 & a_1 & a_3 & \ldots & a_{2j-3} \\ 0 & a_0 & a_2 & a_4 & \ldots & a_{2j-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_j \end{vmatrix}, \quad j = 1, \ldots, n, \] (1.4)
where we set \( a_i \equiv 0 \) for \( i > n \).

**Definition 1.2.** The determinants \( \Delta_j(p) \), \( j = 1, \ldots, n \), are called the Hurwitz determinants or the Hurwitz minors of the polynomial \( p \).

Suppose that \( \deg p_0 \geq \deg p_1 \) and expand the function \( \Phi \) into its Laurent series at \( \infty \):
\[ \Phi(u) = \frac{p_1(u)}{p_0(u)} = s_{-1} + \frac{s_0}{u} + \frac{s_1}{u^2} + \frac{s_2}{u^3} + \frac{s_3}{u^4} + \cdots, \] (1.5)
where \( s_{-1} \neq 0 \) if \( \deg p_0 = \deg p_1 \), and \( s_{-1} = 0 \) if \( \deg p_0 > \deg p_1 \).

For a given infinite sequence \( (s_j)_{j=0}^{\infty} \), consider the determinants
\[ D_j(\Phi) \overset{\text{def}}{=} \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{j-1} \\ s_1 & s_2 & s_3 & \ldots & s_j \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{j-1} & s_j & s_{j+1} & \ldots & s_{2j-2} \end{vmatrix}, \quad j = 1, 2, 3, \ldots. \] (1.6)
These determinants are referred to as the Hankel minors or Hankel determinants.

Together with the determinants (1.3) we consider one more sequence of Hankel determinants.
\[ \widehat{D}_j(\Phi) \overset{\text{def}}{=} \begin{vmatrix} s_1 & s_2 & s_3 & \ldots & s_j \\ s_2 & s_3 & s_4 & \ldots & s_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_j & s_{j+1} & s_{j+2} & \ldots & s_{2j-1} \end{vmatrix}, \quad j = 1, 2, 3, \ldots \] (1.7)

It is very well known [5] [3] (see also [7]) that there are relations between the determinants \( D_j(\Phi) \), \( \widehat{D}_j(\Phi) \) and the Hurwitz minors \( \Delta_j(p) \):
1) If \( n = 2l \), then
\[
\Delta_{2j-1}(p) = a_0^{2j-1} D_j(\Phi), \quad j = 1, 2, \ldots, l; \tag{1.8}
\]
\[
\Delta_{2j}(p) = (-1)^j a_0^{2j} \hat{D}_j(\Phi),
\]
where \( \hat{D}_0(\Phi) \equiv 1 \).

2) If \( n = 2l + 1 \), then
\[
\Delta_{2j}(p) = \left( \frac{a_0}{s-1} \right)^{2j} D_j(\Phi), \quad j = 0, 1, \ldots, l;
\]
\[
\Delta_{2j+1}(p) = (-1)^j \left( \frac{a_0}{s-1} \right)^{2j+1} \hat{D}_j(\Phi),
\]
where \( \hat{D}_0(\Phi) \equiv 1 \).

Here \( l \) is defined in (1.2).

It is also well known \([5,3]\) (see also \([4]\)) that the number of poles of the function \( \Phi \) equals the order of the last non-zero minor \( D_j(\Phi) \). Since the function \( \Phi \) has at most \( l \) poles, we have
\[
D_j(\Phi) = \hat{D}_j(\Phi) = 0, \quad j > l.
\tag{1.10}
\]
Thus, in the sequel, we deal only with the determinants \( D_j(\Phi) \), \( \hat{D}_j(\Phi) \) of order at most \( l \).

**Definition 1.3.** The polynomial \( p \) defined in (1.1) is called Hurwitz or Hurwitz stable if all its zeroes lie in the open left half-plane of the complex plane.

The following criterion of Hurwitz stability of a real polynomial was (implicitly) established in \([5]\) (see also \([3,1,7]\)).

**Theorem 1.4.** Let a real polynomial \( p \) be defined by (1.1). The following conditions are equivalent:

1) the polynomial \( p \) is Hurwitz stable;

2) the following hold
\[
s_{-1} > 0 \quad \text{for} \quad n = 2l + 1,
\]
\[
D_j(\Phi) > 0, \quad j = 1, \ldots, l,
\]
\[
(-1)^j \hat{D}_j(\Phi) > 0, \quad j = 1, \ldots, l,
\]
where \( l = \left[ \frac{n}{2} \right] \).

This theorem together with formulas (1.8)–(1.9) imply the following theorem which is very well known as the Hurwitz criterion of polynomial stability.

**Theorem 1.5 (Hurwitz \([5]\)).** A real polynomial \( p \) of degree \( n \) as in (1.1) is Hurwitz stable if and only if all Hurwitz determinants \( \Delta_j(p) \) are positive:
\[
\Delta_1(p) > 0, \Delta_2(p) > 0, \ldots, \Delta_n(p) > 0; \tag{1.12}
\]

## 2 Hurwitz rational function

Consider a rational function
\[
R(z) \overset{def}{=} \frac{h(z)}{g(z)} = t_0 z^{r-m} + t_1 z^{r-m-1} + t_2 z^{r-m-2} + \ldots,
\tag{2.1}
\]
where \( h \) and \( g \) are real polynomials
\[
h(z) \overset{def}{=} b_0 z^r + b_1 z^{r-1} + \ldots + b_{r-1} z + b_r, \quad b_0, b_1, \ldots, b_r \in \mathbb{R}, \ b_0 > 0;
\]
\[
g(z) \overset{def}{=} c_0 z^m + c_1 z^{m-1} + \ldots + c_{m-1} z + c_m, \quad c_0, c_1, \ldots, c_m \in \mathbb{R}, \ c_0 > 0.
\tag{2.2}
\]
Here \( r, m \in \mathbb{N} \cup \{0\}, \ r+m > 0 \). Assume that the polynomials \( p \) and \( q \) are coprime. The number \( n = r+m \) is called the order of the function \( R \).
**Definition 2.1.** The real rational function $R$ is called Hurwitz rational function if both polynomials $h(z)$ and $g(-z)$ are Hurwitz stable.\(^1\)

It turns out that one can establish a criterion for real rational functions to be Hurwitz similar to the Hurwitz criterion of polynomial stability.

**Theorem 2.2.** A real rational function $R(z)$ of the form (2.1)–(2.2) is Hurwitz if and only if the leading principal minors $\Delta_k(R)$ of the infinite Hurwitz matrix

$$
\mathcal{H}(R) \overset{df}{=} \begin{pmatrix}
  t_1 & t_3 & t_5 & t_7 & t_9 & \ldots \\
  t_0 & t_2 & t_4 & t_6 & t_8 & \ldots \\
  0 & t_1 & t_3 & t_5 & t_7 & \ldots \\
  0 & t_0 & t_2 & t_4 & t_6 & \ldots \\
  0 & 0 & t_1 & t_3 & t_5 & \ldots \\
  0 & 0 & t_0 & t_2 & t_4 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

are positive up to the order $n$ (inclusive):

$$
\Delta_1(R) > 0, \quad \Delta_2(R) > 0, \ldots, \quad \Delta_n(R) > 0.
$$

**Proof.** Consider the following auxiliary polynomial

$$
P(z) \overset{df}{=} (-1)^m h(z)g(-z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n, \quad a_0 = b_0 c_0 > 0. \tag{2.4}
$$

By definition, the function $R$ is Hurwitz if and only if the polynomial $P$ is Hurwitz stable.

Consider the rational function $\Phi$ associated with the polynomial $P$:

$$
\Phi(u) = \frac{P_1(u)}{P_0(u)} = s_{-1} + \frac{s_0}{u} + \frac{s_1}{u^2} + \frac{s_2}{u^3} + \ldots, \tag{2.5}
$$

where the polynomials $P_0(u), P_1(u)$ are the even and odd parts of the polynomial $P$, respectively. Since $2P_0(z^2) = P(z) + P(-z)$ and $2zP_0(z^2) = P(z) - P(-z)$, we have

$$
z \Phi(z^2) = \frac{P(z) - P(-z)}{P(z) + P(-z)} = \frac{b(z)g(-z) - h(-z)g(z)}{b(z)g(z) + h(-z)g(z)} = \frac{R(z) - R(-z)}{R(z) + R(-z)}.
$$

Hence we obtain

$$
\Phi(u) = \frac{R_1(u)}{R_0(u)} = \frac{P_1(u)}{P_0(u)}, \tag{2.6}
$$

where

$$
R_0(z^2) = \frac{R(z) + R(-z)}{2},
$$

$$
R_1(z^2) = \frac{R(z) - R(-z)}{2z}
$$

are the even and odd parts of the function $R$, respectively. Thus, for $n = 2l + 1$, we have

$$
\Phi(u) = \frac{t_0 + t_2 u^{-1} + t_4 u^{-2} + \ldots}{t_1 + t_3 u^{-1} + t_5 u^{-2} + \ldots} = s_{-1} + \frac{s_0}{u} + \frac{s_1}{u^2} + \frac{s_2}{u^3} + \ldots,
$$

and for $n = 2l$, we have

$$
\Phi(u) = \frac{t_1 u^{-1} + t_3 u^{-2} + t_5 u^{-3} + \ldots}{t_0 + t_2 u^{-1} + t_4 u^{-2} + \ldots} = s_{-1} + \frac{s_0}{u} + \frac{s_1}{u^2} + \frac{s_2}{u^3} + \ldots
$$

We recall that $n = r + m$ is the order of the rational function $R$. Now note that the formulæ (1.8)–(1.9) are formal, so following verbatim the proof of the formulæ (1.8)–(1.9) (see, for instance, [7] [4]) one can establish the following relationships

\(^1\)Or one of them is Hurwitz stable while the second one is a constant.
1) for \( n = 2l \),
\[
\Delta_{2j-1}(R) = t_0^{2j-1} D_j(\Phi), \quad \Delta_{2j}(R) = t_0^{2j} \hat{D}_j(\Phi) \quad j = 1, 2, \ldots, l,
\]  
(2.7)

2) for \( n = 2l + 1 \),
\[
\Delta_{2j}(R) = \left( \frac{t_0}{s-1} \right)^{2j} D_j(\Phi), \quad j = 1, 2, \ldots, l,
\]
\[
\Delta_{2j+1}(R) = (-1)^j \left( \frac{t_0}{s-1} \right)^{2j+1} \hat{D}_j(\Phi), \quad j = 0, 1, \ldots, l.
\]  
(2.8)

where the Hurwitz minors \( \Delta_j(R) \) are defined as follows

\[
\Delta_j(R) \overset{\text{def}}{=} \begin{vmatrix}
  t_1 & t_2 & t_3 & \cdots & t_{2j-1} \\
  t_0 & t_2 & t_4 & \cdots & t_{2j-2} \\
  0 & t_1 & t_3 & \cdots & t_{2j-3} \\
  0 & t_0 & t_2 & \cdots & t_{2j-4} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & t_j
\end{vmatrix}
\]  
(2.9)

By Theorem 1.3 the polynomial \( P \) is Hurwitz stable if and only if the minors \( D_j(\Phi) \) and \( \hat{D}_j(\Phi) \) satisfy the inequalities (1.11). Now since the polynomial \( P \) is Hurwitz stable if and only if \( R \) is a Hurwitz rational function, the formulæ (2.7)–(2.8) and Theorem 1.4 imply the assertion of the theorem. □

Note that the leading principal minors \( \Delta_j(R) \) of order more than \( n \) equal zero regardless whether the function \( R \) being Hurwitz or not:
\[
\Delta_{n+1}(R) = \Delta_{n+2}(R) = \Delta_{n+3}(R) = \cdots = 0.
\]  
(2.10)

It follows from (1.10) and from the formulæ (2.7)–(2.8) which are obviously valid for \( j > l \).

**Remark 2.3.** It is easy to see that \( a_0 = b_0 c_0 \) and \( t_0 = \frac{b_0}{c_0} \). So the formulæ (1.8)–(1.9) and the formulæ (2.7)–(2.8) imply the equalities
\[
\Delta_j(P) = c_0^{2j} \Delta_j(R), \quad j = 1, 2, \ldots
\]  
(2.11)

This equalities verify (2.10), since \( \Delta_j(P) = 0 \) for \( j > n \).

Recall now that one of necessary conditions for the polynomial \( p \) defined in (1.1) to be Hurwitz stable is the positivity of its coefficients \( a_j > 0, \ j = 0, 1, \ldots, n \). This is the so-called Stodola theorem [5, 3]. It turns out that positivity of the coefficients \( t_j \) in (2.1) is not necessary condition for the Hurwitzness of the function \( R \). Indeed, if \( R = \frac{h}{g} \) is Hurwitz, then the polynomial \( g \) has all zeroes in the open right half-plane, so it has coefficients of different signs. Therefore, it is easy to find polynomials \( h \) and \( g \) such that \( R \) has both negative and positive coefficients. For example, the following function is Hurwitz, but its Laurent series at infinity has positive, negative and zero coefficients:

\[
F(z) = \frac{z^2 + z + 1}{z^2 - z + 1} = t_0 + t_1 z^{-1} + t_2 z^{-2} + t_3 z^3 + \ldots,
\]

where \( t_0 = 1 \) and \( t_{3j-2} = t_{3j-1} = (-1)^{(j-1)/2}, \quad t_{3j} = 0 \) for \( j = 1, 2, \ldots \)

Finally, we find a connection between the coefficients of the polynomials \( h \) and \( g \) and the Hurwitz determinants \( \Delta_j(R) \). This gives us a criterion of Hurwitzness of a real rational function in terms of the coefficients of its numerator and denominator.
Let again the function \( R \) be defined by (2.11) - (2.2). Introduce the following determinants of order \( 2j \), \( j = 1, 2, \ldots:\)

\[
\Omega_{2j}(h, g) \overset{\text{def}}{=} \left| \begin{array}{ccccccccc} c_0 & c_1 & c_2 & \cdots & c_{j-1} & c_j & \cdots & c_{2j-3} & c_{2j-2} & c_{2j-1} \\ b_0 & b_1 & b_2 & \cdots & b_{j-1} & b_j & \cdots & b_{2j-3} & b_{2j-2} & b_{2j-1} \\ 0 & 0 & c_0 & c_1 & c_2 & \cdots & c_{j-3} & c_{j-2} & c_{2j-5} & c_{2j-4} & c_{2j-3} \\ 0 & b_0 & b_1 & b_2 & \cdots & b_{j-2} & b_{j-1} & \cdots & b_{2j-4} & b_{2j-3} & b_{2j-2} \\ 0 & 0 & 0 & 0 & c_0 & \cdots & c_{j-5} & c_{j-4} & \cdots & c_{2j-7} & c_{2j-6} & c_{2j-5} \\ 0 & 0 & b_0 & b_1 & b_2 & \cdots & b_{j-3} & b_{j-2} & \cdots & b_{2j-5} & b_{2j-4} & b_{2j-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{j-2} & b_{j-1} & b_j \\ \end{array} \right|. \tag{2.12}
\]

Here we set \( b_k := 0 \) for \( k > r \), and \( c_j := 0 \) for \( j > m \).

**Lemma 2.4.** For any \( j = 1, 2, \ldots, \)

\[
\Omega_{2j}(h, g) = c_0^{2j} \Delta_j(R), \tag{2.13}
\]

where \( \Delta_j(R) \) are the Hurwitz determinants of the function \( R \) defined in (2.9).

**Proof.** First interchange the rows of the determinant (2.12) to obtain

\[
\Omega_{2j}(h, g) = \left| \begin{array}{cccccccc} c_0 & c_1 & c_2 & \cdots & c_{j-1} & c_j & \cdots & c_{2j-3} & c_{2j-2} & c_{2j-1} \\ 0 & 0 & c_0 & c_1 & c_2 & \cdots & c_{j-3} & c_{j-2} & c_{2j-4} & c_{2j-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \sqrt{0} & \sqrt{b_1} & \cdots & b_{j-1} & b_j \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & b_0 & b_1 & b_2 & \cdots & b_{j-2} & b_{j-1} & \cdots & b_{2j-3} & b_{2j-2} \\ b_0 & b_1 & b_2 & \cdots & b_{j-1} & b_j & \cdots & b_{2j-2} & b_{2j-1} \\ \end{array} \right|. \tag{2.14}
\]

This does not change the sign of \( \Omega_{2j}(h, g) \). In fact, lower the \( j \)th and \((j + 1)\)st row to their initial positions in (2.12). This will require an even number of transpositions. The next pair of rows will then meet, then lowering operation will be applied to them, and so on \((j - 1)\) times until we obtain the initial determinant (2.12).

Now we rearrange the columns of the determinant (2.14) in the following way: 1st, 3d, \ldots, \((2j - 1)\)st, 2nd, 4th, \ldots, 2\(j\)th. To move the 3d column to the second place, we need one transposition. To move the 5th column to the third place, we need two transpositions, and so on. At last, to move the \((2j - 1)\)st column to the \(j\)th place, we need \((j - 1)\) transpositions. During all those transpositions the resulting determinant will change its sign \(\sum_{i=1}^{j-1} i = \frac{j(j-1)}{2} \) times. Next we interchange the \((j + 1)\)st row with the 2\(j\)th row, the \((j + 2)\)nd row with the \((2j - 1)\)st row, and so on. This will also require \(\frac{j(j-1)}{2} \) transpositions. Thus, the resulting determinant has the same sign as the initial determinant, so we have

\[
\Omega_{2j}(h, g) = \left| \begin{array}{ccccccccc} c_0 & c_2 & c_4 & \cdots & c_{2j-2} & c_1 & c_3 & \cdots & c_{2j-1} \\ 0 & 0 & c_0 & c_2 & \cdots & c_{2j-4} & 0 & c_1 & \cdots & c_{2j-3} \\ 0 & 0 & 0 & \cdots & c_{2j-6} & 0 & 0 & c_1 & \cdots & c_{2j-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_0 & 0 & 0 & 0 & \cdots & c_1 \\ b_0 & b_2 & b_4 & \cdots & b_{2j-2} & b_1 & b_3 & \cdots & b_{2j-1} \\ 0 & b_1 & b_3 & \cdots & b_{2j-3} & b_0 & b_2 & \cdots & b_{2j-2} \\ 0 & b_0 & b_2 & \cdots & b_{2j-4} & 0 & b_1 & b_3 & \cdots & b_{2j-3} \\ 0 & 0 & b_1 & \cdots & b_{2j-5} & 0 & b_0 & b_2 & \cdots & b_{2j-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{j-1} & 0 & 0 & 0 & \cdots & b_j \\ \end{array} \right|. \tag{2.15}
\]

\(^2\)They are marked by \(\sqrt{}\) in (2.14).
Now from each $(j + 2i - 1)$st row, $i = 1, 2, \ldots, \left\lfloor \frac{j + 1}{2} \right\rfloor$, we subtract rows $i$th, $(i + 1)$st, \ldots, $j$th multiplied by $t_0$, $t_2$, \ldots,$t_{2(j-i)-2}$, respectively. Then, from each $(j + 2i)$th row, $i = 1, 2, \ldots, \left\lfloor \frac{j}{2} \right\rfloor$, subtract rows $(i + 1)$st, $(i + 2)$nd, \ldots, $j$th multiplied by $t_1$, $t_3$, \ldots,$t_{2(j-i)-1}$, respectively. As a result, we have a determinant, which can be represented in a block form:

$$\Omega_{2j}(h, g) = \begin{vmatrix} C_0 & C_1 \\ \tilde{A} & \tilde{A} \end{vmatrix}$$

(2.16)

where the upper triangular matrices $C_0$ and $C_1$ have the forms

$$C_0 = \begin{pmatrix} c_0 & c_2 & c_4 & \cdots & c_{2j-2} \\ 0 & c_0 & c_2 & \cdots & c_{2j-4} \\ 0 & 0 & c_0 & \cdots & c_{2j-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} c_1 & c_3 & c_5 & \cdots & c_{2j-1} \\ 0 & c_1 & c_3 & \cdots & c_{2j-3} \\ 0 & 0 & c_1 & \cdots & c_{2j-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 \end{pmatrix}$$

Note that $|C_0| = c_0^j > 0$.

In order to describe the matrices $\tilde{A}$ and $\hat{A}$, we establish a connection between the coefficients $b_i$s, $c_i$s and $t_i$s. Multiplying (2.14) by the denominator and equating coefficients, we get

$$b_k = \sum_{i=0}^{k} c_{k-i} t_i, \quad k = 0, 1, 2, \ldots$$

(2.17)

Taking into account those formulæ, we obtain for the entries of the matrix $\tilde{A}$ to have the form:

$$\tilde{a}_{2i-1,k} = \begin{cases} 0 & \text{if } k < i + 1, \\ b_{2(k-i)} - \sum_{q=0}^{k-i} c_{2(k-i-q)} t_{2q} = \sum_{q=1}^{k-i} c_{2(k-i-q)+1} t_{2q-1} & \text{if } k \geq i + 1, \end{cases}$$

where $i = 1, 2, \ldots, \left\lfloor \frac{j + 1}{2} \right\rfloor$.

(2.18)

$$\tilde{a}_{2i,k} = \begin{cases} 0 & \text{if } k < i + 1, \\ b_{2(k-i)-1} - \sum_{q=1}^{k-i} c_{2(k-i-q)} t_{2q-1} = \sum_{q=0}^{k-i-1} c_{2(k-i-q)-1} t_{2q} & \text{if } k \geq i + 1, \end{cases}$$

where $i = 1, 2, \ldots, \left\lfloor \frac{j}{2} \right\rfloor$.

From these formulæ it follows that the matrix $\tilde{A}$ can be represented as a product of two matrices:

$$\tilde{A} = \begin{pmatrix} 0 & t_1 & t_3 & \cdots & t_{2j-3} \\ 0 & t_0 & t_2 & \cdots & t_{2j-4} \\ 0 & 0 & t_1 & \cdots & t_{2j-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & t_{j-2} \end{pmatrix} \begin{pmatrix} c_1 & c_3 & c_5 & \cdots & c_{2j-1} \\ 0 & c_1 & c_3 & \cdots & c_{2j-3} \\ 0 & 0 & c_1 & \cdots & c_{2j-5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_1 \end{pmatrix}$$

(2.19)

Analogously, by (2.17), for the entries of the matrix $\hat{A}$, we have

$$\hat{a}_{2i-1,k} = \begin{cases} 0 & \text{if } k < i, \\ b_{2(k-i)+1} - \sum_{q=0}^{k-i} c_{2(k-i-q)+1} t_{2q+1} = \sum_{q=0}^{k-i} c_{2(k-i-q)} t_{2q+1} & \text{if } k \geq i, \end{cases}$$

where $i = 1, 2, \ldots, \left\lfloor \frac{j+1}{2} \right\rfloor$.
where \( i = 1, 2, \ldots, \left\lfloor \frac{j}{2} \right\rfloor \).

Therefore, the matrix \( \hat{A} \) can be represented as follows

\[
\hat{A} = \begin{pmatrix}
  t_1 & t_3 & t_5 & \cdots & t_{2j-1} \\
  t_0 & t_2 & t_4 & \cdots & t_{2j-2} \\
  0 & t_1 & t_3 & \cdots & t_{2j-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & t_j
\end{pmatrix} \begin{pmatrix}
  c_0 & c_2 & c_4 & \cdots & c_{2j-2} \\
  0 & c_0 & c_2 & \cdots & c_{2j-4} \\
  0 & 0 & c_0 & \cdots & c_{2j-6} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & c_0
\end{pmatrix} =: \Lambda_j C_0 \tag{2.20}
\]

It is clear that \( \Lambda_j \) is the \( j \times j \) leading principal submatrix of the matrix \( \mathcal{H}(R) \) defined in (2.13). Taking (2.20) into account, we multiply both parts of the equality (2.16) by the following determinant

\[
\begin{vmatrix}
  E_j & 0 \\
  0 & C_0^{-1}
\end{vmatrix} = c_0^{-j} > 0,
\]

where \( E_j \) is the \( j \times j \) identity matrix, to obtain

\[
c_0^{-j} \Omega_{2j}(h, g) = \begin{vmatrix}
  C_0 & C_1 C_0^{-1} \\
  \hat{A} & \Lambda_j
\end{vmatrix},
\]

or in the entry-wise from:

\[
c_0^{-j} \Omega_{2j}(h, g) = \begin{vmatrix}
  c_0 & c_2 & c_4 & \cdots & c_{2j-2} & \frac{c_1}{c_0} & \cdots & \cdots & \cdots & \cdots \\
  0 & c_0 & c_2 & \cdots & c_{2j-4} & 0 & \frac{c_1}{c_0} & \cdots & \cdots & \cdots \\
  0 & 0 & c_0 & \cdots & c_{2j-6} & 0 & 0 & \frac{c_1}{c_0} & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & c_0 & 0 & 0 & 0 & \cdots & \frac{c_1}{c_0} \\
  0 & c_1 t_1 & c_3 t_1 + c_1 t_3 & \cdots & \cdots & t_1 & t_3 & t_5 & \cdots & t_{2j-1} \\
  0 & c_1 t_0 & c_3 t_0 + c_1 t_2 & \cdots & \cdots & t_0 & t_2 & t_4 & \cdots & t_{2j-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & c_1 t_1 & \cdots & \cdots & 0 & t_1 & t_3 & \cdots & t_{2j-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & t_j
\end{vmatrix} \tag{2.21}
\]

The formulae (2.18)–(2.21) show that all the columns of the matrix \( \hat{A} \) are linear combinations of the columns of the matrix \( \Lambda_j \). Consequently, we are able to eliminate the entries in the lower left corner of the determinant (2.21). To do this, from the 2nd column we subtract the \( (j+1) \)st column multiplied by \( c_1 \). Then from the 3 column we subtract the \( (j+1) \)st column multiplied by \( c_1 \) and the \( (j+2) \)nd column multiplied by \( c_3 \), and so on. As a result, we obtain the determinant \( c_0^{-j} \Omega_{2j}(h, g) \) to have the following
form

\[ e_0^{-1} \Omega_2(h, g) = \begin{vmatrix} c_0 & \cdots & \cdots & c_1 & \cdots & \cdots & \cdots \\ c_0 & \cdots & \cdots & 0 & c_1 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & c_0 & 0 & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & t_1 & t_3 & \cdots & t_{2j-1} \\ 0 & \cdots & \cdots & 0 & t_0 & t_2 & \cdots & t_{2j-2} \\ 0 & \cdots & \cdots & 0 & 0 & t_1 & t_3 & \cdots & t_{2j-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & t_j \end{vmatrix} = e_0^j |\Lambda_j| \]

Since \( |\Lambda_j| = \Delta_j(R) \), we get

\[ \Omega_2(h, g) = e_0^{2j} \Delta_j(R), \]

as required.

Now Lemma 2.4 and Theorem 2.7 imply the following criterion of Hurwitzness of real rational functions in terms of the coefficients of their numerator and denominator.

**Theorem 2.5.** A real rational function \( R \) defined in (2.1)–(2.2) is Hurwitz if and only if the inequalities

\[ \Omega_2(h, g) > 0, \ \Omega_4(h, g) > 0, \ldots, \ \Omega_{2n}(h, g) > 0 \]

hold.

From (2.10) and (2.13) it also follows that

\[ \Omega_{2n+2}(h, g) = \Omega_{2n+4}(h, g) = \Omega_{2n+6}(h, g) = \cdots = 0. \]

Lemma 2.4 and the formulæ (2.11) imply the following corollary.

**Corollary 2.6.** Let the polynomials \( h \) and \( g \) be defined in (2.2). Then the Hurwitz minors of the polynomial \( P(z) = (-1)^n h(z) g(-z) \) satisfy the equalities:

\[ \Delta_j(P) = \Omega_{2j}(h, g), \quad j = 1, 2, \ldots, n, \]

where \( n = \deg P = \deg h + \deg g \).

Consider now the rational function

\[ F(z) = \frac{(-1)^n}{R(-z)}, \quad (2.22) \]

where \( R \) is defined in (2.1)–(2.2), and \( n = r + m \) is order of \( R \).

It is clear that the function \( F \) is Hurwitz if and only if the function \( R \) is Hurwitz. This fact can be verified by the following relationship between the Hurwitz minors of the functions \( F \) and \( R \).

**Theorem 2.7.** Let the function \( R \) be defined in (2.1)–(2.2), and let the function \( F \) be defined in (2.22). Then

\[ \Delta_j(R) = t_0^{2j} \Delta_j(F), \quad j = 1, 2, \ldots \quad (2.23) \]

**Proof.** Let \( F(z) = \frac{f(z)}{q(z)} \), where \( f(z) = (-1)^n g(-z) \), and \( q(z) = (-1)^r h(-z) \). Consider the polynomial \( Q(z) = (-1)^r q(-z) f(z) \), which is analogous to (2.2). We have

\[ Q(z) = (-1)^r q(-z) f(z) = (-1)^r (-1)^n h(z) (-1)^m g(-z) = (-1)^m h(z) g(-z) = P(z). \]

Now the formulæ (2.11) imply

\[ e_0^{2j} \Delta_j(R) = \Delta_j(P) = \Delta_j(Q) = b_0^{2j} \Delta_j(F), \quad j = 1, 2, \ldots, \]

that is exactly (2.23), since \( t_0 = \frac{b_0}{c_0} \).
**Corollary 2.8.** Let the polynomials \( h \) and \( g \) are defined in (2.2). Then

\[
\Omega_{2j}(h, g) = \Omega_{2j}(f, q), \quad j = 1, 2, \ldots,
\]

where \( f(z) = (-1)^m g(-z), \) and \( q(z) = (-1)^r h(-z). \)

Finally, note that if one of the polynomials \( h \) and \( g \) is a constant, the determinants \( \Omega_{2j}(h, g) \) become Hurwitz determinants (up to a constant factor) of the polynomial, which is not a constant. This follows from the formulæ (2.11).

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