Quintom fields from chiral anisotropic cosmology

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Abstract

In this paper we present an analysis of a chiral anisotropic cosmological scenario from the perspective of quintom fields. In this setup quintessence and phantom fields interact in a non-standard (chiral) way within an anisotropic Bianchi type I background. We present our examination from two fronts: classical and quantum approaches. In the classical program we find analytical solutions given by a particular choice of the emerged relevant parameters. Remarkably, we present an explanation of the “big-bang” singularity by means of a “big-bounce”. Moreover, isotropization is in fact reached as the time evolves. On the quantum counterpart the Wheeler–DeWitt equation is analytically solved for various instances given by the same parameter space from the classical study, and we also include the factor ordering Q. Having solutions in this scheme we compute the probability density, which is in effect damped as the volume...
function and the scalar fields evolve; and it also tends towards a flat FLRW framework when the factor ordering constant $Q \ll 0$. This result might indicate that for a fixed set of parameters, the anisotropies quantum-mechanically vanish for very small values of the parameter $Q$. Finally, classical and quantum solutions reduce to their flat FLRW counterparts when the anisotropies vanish.

**Keywords** Quintom fields · Chiral cosmology · Bianchi I · Exact solutions

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1 Introduction

The rather small deviation from isotropy observed in the cosmic microwave background (CMB) radiation [1] makes it plausible that at very early times the universe was indeed anisotropic, therefore prompting the introduction of anisotropic cosmological models to describe the evolution of the universe near the initial singularity [2, 3]. The Bianchi type I model is a natural choice for such a background given that its isotropic limit is the spatially flat Friedmann-Robertson-Lamâtre-Walker (FRLW) model (see, e.g., [4]). Indeed, the Bianchi type I model has been recently considered to explain the aforementioned tiny variations in the CMB by a number of researchers [5–8].

On the other hand, the multi-field cosmology paradigm has proven to be an effective framework to account (in a single model) for several important characteristics/ingredients of the universe, e.g., early acceleration (inflation) [9–25], dark matter [26–28], late acceleration [29–46]. With respect to early and late acceleration, the crossing of the phantom divide line is a most wanted feature in scalar field cosmology; it has been shown that this crossing cannot be achieved by considering a single scalar field/fluid (unless stability is not demanded) [47]. The standard quintom scenario [48] considers two scalar fields, a quintessence and a phantom, in order to realize such crossing in a simple way. As a byproduct, quintom fields allow (in particular cases) the avoidance of the initial singularity by means of a bounce [49] (see also the review [47]). In the conventional quintom scenario (and in ordinary multi-field cosmology)
the scalar fields interact in the following way:

\[ \mathcal{L}_\phi = \delta^{ab} g^{\mu\nu} \nabla_\mu \phi_a \nabla_\nu \phi_b + V(\phi_a, \phi_b). \]  

(1)

An incarnation of multi-field cosmology is the so called chiral-cosmology [50], in which the scalar fields define an “internal space” with a certain metric component \( m_{ab} \). They also interact in a non-standard manner within their kinetic terms, their couplings are governed by the metric \( m_{ab} \) (in short, we will replace \( \delta^{ab} \rightarrow m_{ab} \) in (1)). This metric can be seen as arising from casting a non-minimally coupled multi-scalar-tensor theory as General Relativity (i.e., in going from the Jordan frame to the Einstein frame) [51]. Non-minimal couplings are indeed required when considering the quantization of scalar fields in curved backgrounds [52], the use of non-canonical fields in (effective) descriptions of the early universe in Einstein’s general relativity is therefore theoretically consistent with standard quantum field theory.

In the present investigation we consider a Bianchi type I framework within a generalized quintom scenario, in which the scalar fields define a chiral space with a certain metric \( m_{ab} \) (so that the fields are not canonical). The exact classical solutions are obtained, then with them we show that the initial singularity is avoided by means of a bounce. Moreover, exact quantum solutions will show that the wave function of the universe is damped with respect to the average scale factor. Similar conclusions were made in the corresponding isotropic case [53]. In the remaining part of this introduction we proceed to describe the generalities of the chiral cosmology which we will be employing. We consider the following simple case of two scalar fields, a **quintessence** field \( \phi_1 \) and **phantom** field \( \phi_2 \) (with their corresponding potential terms) within the chiral cosmology paradigm [50, 53–57]

\[ \mathcal{L} = \sqrt{-g} \left( R + m^{ab} \xi_{ab}(\phi_c, g^{\mu\nu}) + C(\phi_c) \right), \]  

(2)

where \( R \) is the Ricci scalar, \( \xi_{ab}(\phi_c, g^{\mu\nu}) = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi_a \nabla_\nu \phi_b \) the kinetic energy, and \( C(\phi_c) = V(\phi_a, \phi_b) \) the corresponding scalar field potential, with \( m^{ab} \) a \( 2 \times 2 \) constant matrix; we consider the particular form \( m^{ab} = \left( \begin{array}{cc} 1 & m^{12} \\ m^{12} & -1 \end{array} \right) \). Thus, the Einstein-Klein-Gordon field equations are

\[ G_{\alpha\beta} = -\frac{1}{2} m^{ab} \left( \nabla_\alpha \phi_a \nabla_\beta \phi_b - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi_a \nabla_\nu \phi_b \right) + \frac{1}{2} g_{\alpha\beta} C(\phi_c), \]  

(3)

\[ m^{cb} \nabla_\nu \phi_b - \frac{\partial C(\phi_c)}{\partial \phi_c} = 0, \]  

(4)

where \( a, b, c = 1, 2 \). From (3) we read off the energy-momentum tensor for the scalar fields \( (\phi_1, \phi_2) \), as

\[ 8\pi G T_{\alpha\beta}(\phi_1, \phi_2) = -\frac{1}{2} m^{ab} \left( \nabla_\alpha \phi_a \nabla_\beta \phi_b - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi_a \nabla_\nu \phi_b \right). \]
and considering the analogy with a barotropic perfect fluid for the scalar fields,

\[ T_{\alpha\beta}(\phi_c) = (\rho + P)u_\alpha(\phi_c)u_\beta(\phi_c) + P g_{\alpha\beta}, \] 

we have that the pressure \( P \) and the energy density \( \rho \) of the scalar fields are

\[ P(\phi_c) = \frac{1}{2} m^{ab} \xi_{ab} - \frac{1}{2} C(\phi_c), \quad \rho(\phi_c) = \frac{1}{2} m^{ab} \xi_{ab} + \frac{1}{2} C(\phi_c), \]

the four-velocity becomes \( u_\alpha u_\beta = \nabla_\alpha \phi^a \nabla_\beta \phi^b \).

We will employ the scalar potential term \( C(\phi_c) = V_1(\phi_1) + V_2(\phi_2) = V_{01}e^{-\lambda_1 \phi_1} + V_{02}e^{-\lambda_2 \phi_2} \) (with \( \lambda_1, \lambda_2 \) non-negative) and the line element to be considered for this two-field cosmological model will be that of the anisotropic Bianchi type I model, which in Misner’s parameterization is given by

\[ ds^2 = -N^2 dt^2 + e^{2\Omega + 2\beta_+ + 2\sqrt{3}\beta_-} dx^2 + e^{2\Omega - 2\beta_+ - 2\sqrt{3}\beta_-} dy^2 + e^{2\Omega - 4\beta_+} dz^2, \]

where the scale factors are \( A = e^{\Omega + \beta_+ + \sqrt{3}\beta_-}, B = e^{\Omega + \beta_+ - \sqrt{3}\beta_-}, C = e^{\Omega - 2\beta_+} \), and \( (\beta_+, \beta_-) \) are the anisotropic parameters. Also \( (\Omega, \beta_+, \beta_-) \) are scalar functions depending on time, and \( N = N(t) \) is the lapse function. Plugging in (8) into (2) we obtain the following Lagrangian density (we eliminate the second time derivatives, previously)

\[ \mathcal{L} = e^{3\Omega} \left\{ 6 \left( \frac{\dot{\Omega}}{N} \right)^2 - 6 \left( \frac{\dot{\beta}_+}{N} \right)^2 - 6 \left( \frac{\dot{\beta}_-}{N} \right)^2 - \frac{\dot{(\phi_2)^2}}{2N} \right\} - \frac{m^{12}}{N} \phi_1 \phi_2 + \frac{(\phi_2)^2}{2N} + N (V_1(\phi_1) + V_2(\phi_2)) \].

Henceforth we will be utilizing the Lagrangian density (9) as starting point for our study. The document is organized as follows. Section 2 is devoted to set up the classical scheme via the Hamiltonian formalism, and to obtain exact classical solutions for several cases. In Sect. 3 the Wheeler–DeWitt equation is constructed considering a semi-general factor ordering, and exact quantum solutions are presented for various cases as well. Final remarks are stated in Sect. 4.

## 2 Classical scheme

In this section we present the classical solutions via the Hamiltonian formalism. We start with the momenta \( \Pi_q = \partial \mathcal{L} / \partial \dot{q} \) (with \( q^a = \Omega, \beta_+, \beta_-, \phi_1, \phi_2 \)), which are cal-
culated in the usual way, yielding
\[
\begin{align*}
\Pi_\Omega &= \frac{12}{N} e^{3\Omega} \dot{\Omega}, \\
\Pi_+ &= -\frac{12}{N} e^{3\Omega} \dot{\beta}_+, \\
\Pi_- &= -\frac{12}{N} e^{3\Omega} \dot{\beta}_-, \\
\Pi_{\phi_1} &= \frac{1}{N} e^{3\Omega} (-\dot{\phi}_1 - m_1^2 \dot{\phi}_2), \\
\Pi_{\phi_2} &= \frac{1}{N} e^{3\Omega} (\dot{\phi}_2 - m_1^2 \dot{\phi}_1),
\end{align*}
\]
\begin{align*}
\dot{\Omega} &= \frac{N}{12} e^{-3\Omega} \Pi_\Omega, \\
\dot{\beta}_+ &= -\frac{N}{12} e^{-3\Omega} \Pi_+, \\
\dot{\beta}_- &= -\frac{N}{12} e^{-3\Omega} \Pi_-, \\
\dot{\phi}_1 &= -e^{-3\Omega} \frac{N}{1 + (m_1^2)^2} \left( \Pi_{\phi_1} + m_1^2 \Pi_{\phi_2} \right), \\
\dot{\phi}_2 &= -e^{-3\Omega} \frac{N}{1 + (m_1^2)^2} \left( m_1^2 \Pi_{\phi_1} + \Pi_{\phi_2} \right),
\end{align*}
(10)

then the Lagrangian density (9) is rewritten in a canonical way, i.e. \( \mathcal{L}_{\text{canonical}} = \Pi_i q_i^1 - N \mathcal{H} \), so we arrive at the Hamiltonian density
\[
\mathcal{H} = \frac{e^{-3\Omega}}{24} \left\{ \frac{\Pi_\Omega^2}{\Delta} - \frac{\Pi_+^2}{\Delta} - \frac{\Pi_-^2}{\Delta} - 12 \frac{\Pi_{\phi_1}^2}{\Delta} - 12 m_1^2 \frac{\Pi_{\phi_1} \Pi_{\phi_2}}{\Delta} + 12 \frac{\Pi_{\phi_2}^2}{\Delta} \\
- 24 V_{01} e^{6\Omega - \lambda_1 \phi_1} - 24 V_{02} e^{6\Omega - \lambda_2 \phi_2} \right\},
\]
(11)

where \( \Delta = 1 + (m_1^2)^2 \). We now consider the canonical transformation \((\Omega, \phi_1, \phi_2, \beta_+, \beta_-) \leftrightarrow (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5)\)
\[
\begin{align*}
\xi_1 &= 6\Omega - \lambda_1 \phi_1, \\
\xi_2 &= 6\Omega - \lambda_2 \phi_2, \\
\xi_3 &= 6\Omega + \lambda_1 \phi_1 + \lambda_2 \phi_2, \\
\xi_4 &= \beta_+, \\
\xi_5 &= \beta_-,
\end{align*}
\]
\begin{align*}
\Omega &= \frac{\xi_1 + \xi_2 + \xi_3}{18}, \\
\phi_1 &= \frac{-2\xi_1 + \xi_2 + \xi_3}{3\lambda_1}, \\
\phi_2 &= \frac{\xi_1 - 2\xi_2 + \xi_3}{3\lambda_2}, \quad \beta_+ = \xi_4, \\
\beta_- = \xi_5,
\end{align*}
(12)

with the new conjugate momenta \((P_1, P_2, P_3, P_4, P_5)\) given by
\[
\begin{align*}
\Pi_\Omega &= 6P_1 + 6P_2 + 6P_3, \\
\Pi_{\phi_1} &= \lambda_1 (-P_1 + P_3), \\
\Pi_{\phi_2} &= \lambda_2 (-P_2 + P_3), \\
\Pi_+ &= P_4, \\
\Pi_- &= P_5.
\end{align*}
\]
(13)

Therefore the Hamiltonian density, in the gauge \( N = 24 e^{3\Omega} \), becomes
\[
\mathcal{H} = 12 \left( 3 - \Lambda_1 \right) P_1^2 + 12 \left( 3 + \Lambda_2 \right) P_2^2 + 12 \left( 3 - 2\Lambda_1 + \Lambda_2 - \Lambda_1 \right) P_3^2
\]
\[ +24 \left[ (3 + \Lambda_1 + \Lambda_{12}) P_1 + (3 + \Lambda_{12} - \Lambda_2) P_2 \right] P_3 \\
+24 \left( 3 - \Lambda_{12} \right) P_1 P_2 - P_4^2 - P_5^2 - 24 \left( V_{01} e^{\xi_1} + V_{02} e^{\xi_2} \right), \]

where \( \Lambda_1 = \lambda_1^2 / \Delta, \Lambda_2 = \lambda_2^2 / \Delta, \) and \( \Lambda_{12} = m^{12} \lambda_1 \lambda_2 / \Delta. \) Then, Hamilton’s equations read

\[
\begin{align*}
\dot{\xi}_1 &= 24 (3 - \Lambda_1) P_1 + 24 (3 - \Lambda_{12}) P_2 + 24 (3 + \Lambda_1 + \Lambda_{12}) P_3 \\
\dot{\xi}_2 &= 24 (3 + \Lambda_2) P_2 + 24 (3 - \Lambda_{12}) P_1 + 24 (3 - \Lambda_2 + \Lambda_{12}) P_3, \\
\dot{\xi}_3 &= 24 (3 + \Lambda_1 + \Lambda_{12}) P_1 + 24 (3 - \Lambda_2 + \Lambda_{12}) P_2 + 24 (3 + \Lambda_2 - \Lambda_1 - 2 \Lambda_{12}) P_3 \\
\dot{\xi}_4 &= -2 P_4, \\
\dot{\xi}_5 &= -2 P_5, \\
\dot{P}_1 &= 24 V_{01} e^{\xi_1}, \\
\dot{P}_2 &= 24 V_{02} e^{\xi_2}, \\
\dot{P}_3 &= 0, \\
\dot{P}_4 &= 0, \\
\dot{P}_5 &= 0,
\end{align*}
\]

(14)

Here straightforwardly one can set \( P_i = p_i = \text{constant}, \) with \( i = 3, 4, 5. \) Now we take the time derivative of \( \dot{\xi}_1 \) (first equation in (14)), then we combine it with \( \dot{P}_1, \) yielding

\[
\ddot{\xi}_1 = 576 V_{01} (3 - \Lambda_1) e^{\xi_1} + 576 V_{02} (3 - \Lambda_{12}) e^{\xi_2}.
\]

(15)

To find solutions of \( (\Omega, \beta_+, \beta_-, \phi_1, \phi_2) \) we introduce the transformation (12) in order to separate the set of equations coming from the Hamiltonian density (14), then we drop the mixed momenta by setting to zero their coefficients, therefore this procedure constraints the matrix element \( m^{12} \)

\[
m^{12} = \frac{\lambda_1 \lambda_2}{6} \left[ 1 \pm \sqrt{1 - \left( \frac{6}{\lambda_1 \lambda_2} \right)^2} \right],
\]

(16)

Moreover, we fix the second term in the square root of (16) to be a real number, and we consider \( \lambda_1 > 0, \lambda_2 > 0, \) hence yielding the relation \( \lambda_1 \lambda_2 \geq 6, \) which in turns ensures that \( m^{12} \) is always positive. Finally, the aforementioned simplifications yield the subsequent Hamilton equations

\[
\begin{align*}
\dot{\xi}_1 &= 24 \eta_1 P_1 + 24 (9 - |\eta_1|) P_3, \\
\dot{\xi}_2 &= 24 \eta_2 P_2 + 24 (9 - \eta_2) P_3, \\
\dot{\xi}_3 &= 24 (9 - |\eta_1|) P_1 + 24 (9 - \eta_2) P_2 + 24 (-9 + |\eta_1| + \eta_2) P_3, \\
\dot{\xi}_4 &= -2 P_4, \\
\dot{\xi}_5 &= -2 P_5, \\
\dot{P}_1 &= 24 V_{01} e^{\xi_1}, \\
\dot{P}_2 &= 24 V_{02} e^{\xi_2}, \\
\dot{P}_3 &= 0, \\
\dot{P}_4 &= 0, \\
\dot{P}_5 &= 0,
\end{align*}
\]

(17)

with \( \eta_1 = 3 - \Lambda_1 \) and \( \eta_2 = 3 + \Lambda_2. \) In the next segments we will compute analytical expression provided distinct combinations of \( (\lambda_1, \lambda_2). \)
2.1 Case $\lambda_1 \lambda_2 = 6$

For this set of values we can see that $m_{12}^2 = 1$, and $\Delta = 2$. We also set $\lambda_1 = 6/\lambda_2$, and restrict our results by fixing $\lambda_2 \neq \sqrt{6}$. We start taking the time derivative of $\dot{\xi}_1$ (from (17)), so we have a differential equation for the variable $\xi_1$,

$$
\ddot{\xi}_1 = 576\eta_1 V_{01} e^{\xi_1},
$$

which its solution has the form

$$
e^{\xi_1} = \frac{r_1^2}{288|\eta_1|V_{01}} \left\{ \begin{array}{l}
\text{Sech}^2 \left( r_1 t - q_1 \right), \quad \lambda_1 > \sqrt{6} \quad \text{at} \quad \eta_1 < 0.
\text{Csch}^2 \left( r_1 t - q_1 \right), \quad \lambda_1 < \sqrt{6} \quad \text{at} \quad \eta_1 > 0.
\end{array} \right.
$$

Note that this solution depends strongly on the value of $\lambda_1$. Moreover, $\dot{\xi}_2$ has the same functional structure as $\dot{\xi}_1$ when $\eta_1 > 0$, since $\eta_2 > 0$ for all $\lambda_2$, therefore its solution is

$$
e^{\xi_2} = \frac{r_2^2}{288\eta_2 V_{02}} \text{Csch}^2 \left( r_2 t - q_2 \right),
$$

where $r_i$ and $q_i$ (with $i = 1, 2$) are integration constants of both solutions (19) and (20). Given that two distinct solutions emerge due to $\lambda_1$ there are indeed two different scenarios: phantom and quintessence. Thus we will analyse both cases.

2.1.1 Phantom domination: solution when $\lambda_1 > \sqrt{6} \ (\eta_1 < 0)$, and $\lambda_2 < \sqrt{6}$.

We start with the solutions

$$
e^{\xi_1} = \frac{r_1^2}{288|\eta_1|V_{01}} \text{Sech}^2 \left( r_1 t - q_1 \right),
$$

$$
e^{\xi_2} = \frac{r_2^2}{288\eta_2 V_{02}} \text{Csch}^2 \left( r_2 t - q_2 \right),
$$

then we substitute them into Hamilton equations for the momenta (17), obtaining

$$
P_1 = p_1 + \frac{r_1}{12|\eta_1|} \text{Tanh} \left( r_1 t - q_1 \right),
$$

$$
P_2 = p_2 - \frac{r_2}{12\eta_2} \text{Coth} \left( r_2 t - q_2 \right),
$$

here $p_i$ ($i = 1, 2$) are integration constants. It can be easily verified that the Hamiltonian is identically zero when

$$
p_1 = \frac{|\eta_1| + 9}{|\eta_1|} p_3, \quad p_2 = \frac{\eta_2 - 9}{\eta_2} p_3,
$$

$$
p_3 = + \frac{1}{36} \sqrt{\frac{\eta_2 r_2^2 - |\eta_1| r_1^2 + 12|\eta_1|\eta_2 (p_4^2 + p_5^2)}{3 [|\eta_1|\eta_2 - 3|\eta_1| + 3\eta_2]}}.
$$
Thus the solutions for the $\xi_i$ coordinates become

$$
\xi_1 = \text{Ln} \left( \frac{r_1^2}{288|\eta_1|V_{01}} \right) + \text{Ln} \left[ \text{Sech}^2 \left( \frac{r_1}{t - q_1} \right) \right],
$$

$$
\xi_2 = \text{Ln} \left( \frac{r_2^2}{288\eta_2 V_{02}} \right) + \text{Ln} \left[ \text{Csch}^2 \left( \frac{r_2}{t - q_2} \right) \right],
$$

$$
\xi_3 = a_3 + 648 \frac{|\eta_1|\eta_2 - 3|\eta_1| + 3\eta_2}{|\eta_1|\eta_2} p_3 t + \frac{9 + |\eta_1|}{|\eta_1|} \text{Ln} \left[ \text{Cosh}^2 \left( \frac{r_1}{t - q_1} \right) \right]
+ \frac{\eta_2 - 9}{\eta_2} \text{Ln} \left[ \text{Sinh}^2 \left( \frac{r_2}{t - q_2} \right) \right],
$$

$$
\xi_4 = a_4 - 2p_4 t,
$$

$$
\xi_5 = a_5 - 2p_5 t,
$$

where $a_i$ ($i = 3, 4, 5$) stand as constants coming from integration. After applying the inverse canonical transformation we get the solutions in terms of the original variables $(\Omega, \phi_1, \phi_2, \beta_+, \beta_-)$,

$$
\Omega = \Omega_0 + \text{Ln} \left[ \text{Cosh}^{\beta_1} \left( \frac{r_1}{t - q_1} \right) \text{Csch}^{\beta_2} \left( \frac{r_2}{t - q_2} \right) \right] + 36 \frac{|\eta_1|\eta_2 - 3|\eta_1| + 3\eta_2}{|\eta_1|\eta_2} p_3 t,
$$

$$
\phi_1 = \phi_{10} + \text{Ln} \left[ \text{Cosh}^{\frac{2|\eta_1|+3}{|\eta_1|\eta_2}} \left( \frac{r_1}{t - q_1} \right) \text{Csch}^{\frac{6}{|\eta_1|\eta_2}} \left( \frac{r_2}{t - q_2} \right) \right] + 216 \frac{|\eta_1|\eta_2 - 3|\eta_1| + 3\eta_2}{\lambda_1 |\eta_1|\eta_2} p_3 t,
$$

$$
\phi_2 = \phi_{20} + \text{Ln} \left[ \text{Cosh}^{\frac{6}{|\eta_1|\eta_2}} \left( \frac{r_1}{t - q_1} \right) \text{Sinh}^{\frac{2(\eta_2 - 3)}{|\eta_1|\eta_2}} \left( \frac{r_2}{t - q_2} \right) \right] + 216 \frac{|\eta_1|\eta_2 - 3|\eta_1| + 3\eta_2}{\lambda_2 |\eta_1|\eta_2} p_3 t,
$$

$$
\beta_+ = a_4 - 2p_4 t,
$$

$$
\beta_- = a_5 - 2p_5 t,
$$

where $\beta_1 = 1/|\eta_1|$, $\beta_2 = 1/\eta_2$, and the constants $\Omega_0$, $\phi_{10}$, and $\phi_{20}$ are given by

$$
\Omega_0 = \text{Ln} \left[ \frac{r_1 r_2}{288\sqrt{|\eta_1|\eta_2 V_{01} V_{02}}} \right] + \frac{a_3}{18},
$$

$$
\phi_{10} = \text{Ln} \left[ \frac{12\sqrt{2r_2|\eta_1|V_{01}}}{r_2^2 \sqrt{\eta_2 V_{02}}} \right] + \frac{a_3}{3\lambda_1},
$$

$$
\phi_{20} = \text{Ln} \left[ \frac{12\sqrt{2r_1 \eta_2 V_{02}}}{r_2^2 \sqrt{|\eta_1| V_{01}}} \right] + \frac{a_3}{3\lambda_2}.
$$
This figure shows the time evolution of the volume function $V = V(t) = ABC$ and the Hubble parameter $H = H(t)$. We use arbitrary units of $V_{01} = 5.0$, $V_{02} = 10^{-5}$, $a_3 = -0.5$, $q_1 = q_2 = 0.1$, $r_1 = 16$, $r_2 = 0.5$ and $\lambda_1 = 4.3\sqrt{6}$, $p_4 = p_5 = 0.001$. Recall that $\lambda_2 = \sqrt{6}/\lambda_1$, and other constants depend on the aforementioned values.

Therefore the scale factors are

$$A(t) = \left[ \frac{r_1 r_2}{288\sqrt{|\eta_1|\eta_2 V_{01}V_{02}}} \right]^\frac{1}{2} e^{\frac{a_3 + a_4 + \sqrt{3}a_5}{18}} \cosh^{\beta_1} (r_1 t - q_1) \cosh^{\beta_2} (r_2 t - q_2) \times \exp \left\{ \frac{36 |\eta_1| \eta_2 - 3|\eta_1| + 3|\eta_2|}{|\eta_1| |\eta_2|} p_3 - 2p_4 - 2\sqrt{3}p_5 \right\} t, \quad (36)$$

$$B(t) = \left[ \frac{r_1 r_2}{288\sqrt{|\eta_1|\eta_2 V_{01}V_{02}}} \right]^\frac{1}{2} e^{\frac{a_3 + a_4 - \sqrt{3}a_5}{18}} \cosh^{\beta_1} (r_1 t - q_1) \cosh^{\beta_2} (r_2 t - q_2) \times \exp \left\{ \frac{36 |\eta_1| \eta_2 - 3|\eta_1| + 3|\eta_2|}{|\eta_1| |\eta_2|} p_3 - 2p_4 + 2\sqrt{3}p_5 \right\} t, \quad (37)$$

$$C(t) = \left[ \frac{r_1 r_2}{288\sqrt{|\eta_1|\eta_2 V_{01}V_{02}}} \right]^\frac{1}{2} e^{\frac{a_3 - 2a_4}{18}} \cosh^{\beta_1} (r_1 t - q_1) \cosh^{\beta_2} (r_2 t - q_2) \times \exp \left\{ \frac{36 |\eta_1| \eta_2 - 3|\eta_1| + 3|\eta_2|}{|\eta_1| |\eta_2|} p_3 + 4p_4 \right\} t, \quad (38)$$

and the volume function $V(t) = ABC = e^{3\Omega}$ becomes

$$V(t) = \left[ \frac{r_1 r_2}{288\sqrt{|\eta_1|\eta_2 V_{01}V_{02}}} \right]^\frac{1}{3} e^{\frac{a_3}{\tau}} \cosh^{3\beta_1} (r_1 t - q_1) \cosh^{3\beta_2} (r_2 t - q_2) \times \exp \left\{ \frac{108 |\eta_1| \eta_2 - 3|\eta_1| + 3|\eta_2|}{|\eta_1| |\eta_2|} p_3 t \right\}. \quad (39)$$

In Fig. 1 we can appreciate the evolution of the volume function $V$ and the Hubble parameter $H$, with respect to time. Note that this multi-field cosmological framework avoids the “big-bang” singularity by means of a “big-bounce”, and this behaviour is also evident due to the horizontal crossing of the Hubble parameter (at $H = 0$) where in fact corresponds at the time of the “big-bounce”. Indeed, this outcome has been already pointed out in [49]; however, authors studied a FLRW framework.
In this subsection there is another case left, when $\lambda_1 < \sqrt{6}$ and $\lambda_2 > \sqrt{6}$ (which corresponds to $\eta_1 > 0$); however, we have not included this scenario since the volume function decreases in time, hence becoming physically unfeasible. Nonetheless, we will continue examining the instance where $\lambda_1 = \lambda_2 = \sqrt{6}$.

### 2.2 Case $\lambda_1 = \lambda_2 = \sqrt{6}$

For these particular values we have that $\Lambda_1 = \Lambda_2 = 3$ with $\eta_1 = 0$ and $\eta_2 = 6$, then the Hamilton equations reduce to

\[
\begin{align*}
\dot{\xi}_1 &= 216p_3, \\
\dot{\xi}_2 &= 144P_2 + 72p_3, \\
\dot{\xi}_3 &= 216P_1 + 72P_2 - 72p_3, \\
\dot{\xi}_4 &= -2p_4, \\
\dot{\xi}_5 &= -2p_5,
\end{align*}
\]

(40)

Right away from $\dot{\xi}_1$, we have its solution

\[
\dot{\xi}_1 = a_1 + 216p_3t,
\]

(41)

here $a_1$ is an integration constant. Then, taking the time derivative of $\dot{\xi}_2$ results in

\[
\ddot{\xi}_2 = 3456V_{02}e^{\dot{\xi}_2},
\]

having a solution of the form

\[
\dot{\xi}_2 = \text{Ln} \left( \frac{\alpha_2^2}{1728V_{02}} \right) + \text{Ln} \left[ \text{Csch}^2(\alpha_2 t - \beta_2) \right].
\]

(42)

Now we know the functional form of $\dot{\xi}_1$ and $\dot{\xi}_2$, then we can compute the remaining momenta

\[
\begin{align*}
P_1(t) &= p_1 + \frac{V_{01}}{9p_3} e^{a_1 + 216p_3 t}, \\
P_2(t) &= p_2 - \frac{\alpha_2}{72} h(\alpha_2 t - \beta_2).
\end{align*}
\]

(43)

And for the rest of the variables $\xi_j$ we have

\[
\begin{align*}
\xi_3 &= a_3 + (216p_1 - 108p_3)t + \frac{V_{01}}{9p_3} e^{a_1 + 216p_3 t} + \text{Ln} \left[ \text{Csch} (\alpha_2 t - \beta_2) \right], \\
\xi_4 &= a_4 - 2p_4 t, \\
\xi_5 &= a_5 - 2p_5 t,
\end{align*}
\]

(44)
where $a_3$, $a_4$, $a_5$ are integration constants. By reinserting these momenta into the Hamiltonian density (13) leads to the constraints

$$p_2 = -\frac{p_3}{2}, \quad p_3 = 2p_1 \pm \frac{\sqrt{3}}{108} \sqrt{\alpha_2^2 + 15552 p_1^2 - 72(p_4^2 + p_5^2)}. \quad (45)$$

We then go back to our original variables $(\Omega, \beta_+, \beta_-, \phi_1, \phi_2)$, hence we have

$$\Omega(t) = \frac{a_1 + a_3}{18} + \ln \left[ \frac{\alpha_2}{24 \sqrt{3} V_0} \right]^{\frac{1}{\lambda_1}} + \ln \left[ \text{Csch}^{\frac{1}{2}} (\alpha_2 t - \beta_2) \right] + (12p_1 + 6p_3)t + \frac{V_{01}}{162p_3^3} e^{a_1 + 216p_3 t}, \quad (46)$$

$$\phi_1(t) = \frac{-2a_1 + a_3}{3\lambda_1} + \ln \left[ \frac{\alpha_2}{1728 V_0} \right]^{\frac{1}{\lambda_1}} + \frac{1}{\lambda_1} \left[ (72p_1 - 180p_3) t + \frac{V_{01}}{27p_3} e^{a_1 + 216p_3 t} + \ln [\text{Csch} (\alpha_2 t - \beta_2)] \right], \quad (47)$$

$$\phi_2(t) = \frac{a_1 + a_3}{3\lambda_2} + \ln \left[ \frac{\alpha_2}{1728 V_0} \right]^{\frac{1}{\lambda_2}} + \frac{1}{\lambda_2} \left[ (72p_1 + 36p_3) t + \frac{V_{01}}{27p_3} e^{a_1 + 216p_3 t} + \ln [\text{Sinh} (\alpha_2 t - \beta_2)] \right], \quad (48)$$

$$\beta_+ = a_4 - 2p_4 t, \quad (49)$$

$$\beta_- = a_5 - 2p_5 t, \quad (50)$$

therefore the scale factors become

$$A = \left[ \frac{\alpha_2}{24 \sqrt{3} V_0} \right]^{\frac{1}{\lambda_1}} e^{\frac{a_1 + a_3 + 18a_4 + 18 \sqrt{3}a_5}{18}} \text{Csch}^{\frac{1}{2}} (\alpha_2 t - \beta_2) \times \text{Exp} \left[ 2(6p_1 + 3p_3 - p_4 - \sqrt{3}p_5)t + \frac{V_{01}}{162p_3^3} e^{a_1 + 216p_3 t} \right], \quad (51)$$

$$B = \left[ \frac{\alpha_2}{24 \sqrt{3} V_0} \right]^{\frac{1}{\lambda_2}} e^{\frac{a_1 + a_3 + 18a_4 - 18 \sqrt{3}a_5}{18}} \text{Csch}^{\frac{1}{2}} (\alpha_2 t - \beta_2) \times \text{Exp} \left[ 2(6p_1 + 3p_3 - p_4 + \sqrt{3}p_5)t + \frac{V_{01}}{162p_3^3} e^{a_1 + 216p_3 t} \right], \quad (52)$$

$$C = \left[ \frac{\alpha_2}{24 \sqrt{3} V_0} \right]^{\frac{1}{\lambda_2}} e^{\frac{a_1 + a_3 - 36a_4}{18}} \text{Csch}^{\frac{1}{2}} (\alpha_2 t - \beta_2) \quad (53)$$

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Fig. 2 This figure shows the time evolution of the volume function $V = ABC$ and the Hubble parameter $H(t)$. We use arbitrary units, namely $\lambda_1 = \lambda_2 = \sqrt{6}$, $V_{01} = 1$, $V_{02} = 0.1$, $a_1 = a_3 = a_4 = a_5 = 1$, $\phi_1 = \beta_+ = \beta_- = 1$ and $\Psi_0 = 1$

$$\text{Exp} \left[ 2(6p_1 + 3p_3 + 2p_4)t + \frac{V_{01}}{162p_3^3} e^{a_1 + 216p_3t} \right],$$

and the volume function $V(t) = ABC$ is

$$V(t) = \left[ \frac{\alpha_2}{24\sqrt{3}V_{02}} \right]^{\frac{1}{3}} e^{-\frac{a_1 + a_3}{6}} \text{Csch}^{\frac{1}{2}} (\alpha_2 t - \beta_2) \times \text{Exp} \left[ 18(2p_1 + p_3)t + \frac{V_{01}}{54p_3^3} e^{a_1 + 216p_3t} \right]. \quad (54)$$

Figure 2 shows the time evolution of the volume function $V$ and the Hubble parameter $H$. At first glance $V$ exhibits only substantial growth; nonetheless, when zooming in a small bounce can be appreciated at a very short time scale. Thus $V$ circumvents again the “big-bang” singularity. Indeed, this bounce is more perceptible on the dynamical evolution of $H$, happening at the time $H = 0$.

We can measure the anisotropic density implementing the Misner’s parameterization

$$a = \Omega + \beta_+ \sqrt{3} \beta_-, \quad b = \Omega + \beta_+ - \sqrt{3} \beta_-, \quad c = \Omega - 2 \beta_+,$$

$$\Omega = \frac{1}{3} (a+b+c), \quad \beta_+ = \frac{1}{6} (a+b-2c), \quad \beta_- = \frac{\sqrt{3}}{6} (a-b). \quad (55)$$

The anisotropic and gravitational densities are defined by $\rho_{\text{anisotropic}} = (\dot{\beta}_+)^2 + (\dot{\beta}_-)^2$ and $\rho_\Omega = (\dot{\Omega})^2$, respectively. When the anisotropic-to-gravitational density rate goes to zero ($\rho_{\text{anisotropic}} / \rho_\Omega \rightarrow 0$) the spacetime becomes isotropic [60]. Remarkably, in all cases the anisotropic density is

$$\rho_{\text{anisotropic}} = 4(p_4^2 + p_3^2) = \text{constant}, \quad (56)$$

and since $\dot{\Omega}$ increases with time, isotropization is indeed reached eventually.
3 Quantum scheme

The quantum scheme is obtained by making the usual replacement \( \Pi q^\mu = -i\hbar \partial_q^\mu \) into the classical Hamiltonian density. Also, in order to consider different factor orderings among \( e^{-3\Omega} \) and \( \Pi_\Omega \), we take \( e^{-3\Omega} \Pi_\Omega^2 \rightarrow e^{-3\Omega} \left[ \Pi_\Omega^2 + Qi \hbar \Pi_\Omega \right] \) where \( Q \) is a real number that measures the ambiguity in the factor ordering. We therefore write the Hamiltonian (11)

\[
\mathcal{H} = \Pi_\Omega^2 + Qi \hbar \Pi_\Omega - \Pi_+^2 - \Pi_-^2 - 12\Lambda_2 \Pi_{\phi_1}^2 + 12\Lambda_1 \Pi_{\phi_2}^2 - 24\Lambda_0 \Pi_{\phi_1} \Pi_{\phi_2} - 24V_{01}e^{-\lambda_1 \phi_1 + 6\Omega} - 24V_{02}e^{-\lambda_2 \phi_2 + 6\Omega},
\]

where we have taken into account the constraint (16) of the matrix element \( m^{12} \), and we have fixed the gauge to \( N = 24e^{3\Omega} \). Once again we consider the canonical transformation (12) \( (\Omega, \phi_1, \phi_2, \beta_+, \beta_-) \leftrightarrow (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \), and the new momenta (13); hence we end up with

\[
\mathcal{H} = 12\eta_1 P_1^2 + 12\eta_2 P_2^2 + 12 (-9 + \eta_1 + \eta_2) P_3^2 + 24P_3 \left[ (9 - \eta_1) P_1 + (9 - \eta_2) P_2 \right]
+ 6Qi \hbar (P_1 + P_2 + P_3) - (P_4^2 + P_5^2) - 24 \left( V_{01}e^{\xi_1} + V_{02}e^{\xi_2} \right),
\]

here \( \eta_1 = 3 - \Lambda_1 \) and \( \eta_2 = 3 + \Lambda_2 \). Therefore the corresponding quantum Hamiltonian operator becomes

\[
\hat{\mathcal{H}} \Psi(\xi_j) = -12\hbar^2 \eta_1 \frac{\partial^2 \Psi}{\partial \xi_1^2} - 12\hbar^2 \eta_2 \frac{\partial^2 \Psi}{\partial \xi_2^2} - 12\hbar^2 (-9 + \eta_1 + \eta_2) \frac{\partial^2 \Psi}{\partial \xi_3^2}
- \hbar^2 24 \left[ (9 - \eta_1) \frac{\partial^2 \Psi}{\partial \xi_1 \partial \xi_3} + (9 - \eta_2) \frac{\partial^2 \Psi}{\partial \xi_2 \partial \xi_3} \right]
+ 6Qi \hbar \left( \frac{\partial \Psi}{\partial \xi_1} + \frac{\partial \Psi}{\partial \xi_2} + \frac{\partial \Psi}{\partial \xi_3} \right)
+ \hbar^2 \left( \frac{\partial^2 \Psi}{\partial \xi_4^2} + \frac{\partial^2 \Psi}{\partial \xi_5^2} \right) - 24 \left( V_{01}e^{\xi_1} + V_{02}e^{\xi_2} \right) \Psi = 0.
\]

Note that expression (59) is the Wheeler–DeWitt (WDW) equation. To find the wave function we propose the following ansatz \( \Psi = e^{(p_3 \xi_3 + p_4 \xi_4 + p_5 \xi_5)} B(\xi_1, \xi_2) \) with \( p_j = \text{constant}, \ j = 3, 4, 5; \) then the WDW equation can be separated as

\[
\mathcal{H} \Psi = -12\hbar^2 \eta_1 \frac{\partial^2 B}{\partial \xi_1^2} + 6\hbar^2 \left( Q - 4p_3 (9 - \eta_1) \right) \frac{\partial B}{\partial \xi_1}
+ 3\hbar^2 \left[ p_3 \left( Q - 2p_3 (9 - \eta_1 + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} - 8 \frac{V_{01}}{\hbar^2} e^{\xi_1} \right] B
- 12\hbar^2 \eta_2 \frac{\partial^2 B}{\partial \xi_2^2} + 6\hbar^2 \left( Q - 4p_3 (9 - \eta_2) \right) \frac{\partial B}{\partial \xi_2}.
\]
\[+3\hbar^2 \left[ p_3 \left( Q - 2p_3(9 - \eta_1 + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} - \frac{8}{\hbar^2} e^{\xi_2} \right] B = 0.\] 

(60)

Additionally we assume that \(B(\xi_1, \xi_2) = B_1(\xi_1)B_2(\xi_2)\), having;

\[-12\eta_1 \frac{d^2 B_1}{d\xi_1^2} + 6 \left( Q - 4p_3(9 - \eta_1) \right) \frac{dB_1}{d\xi_1} + 3 \left[ -\nu^2 + p_3 \left( Q - 2p_3(9 - \eta_1 + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} - \frac{8}{\hbar^2} e^{\xi_1} \right] B_1 = 0 \]

(61)

\[-12\eta_2 \frac{d^2 B_2}{d\xi_2^2} + 6 \left( Q - 4p_3(9 - \eta_2) \right) \frac{dB_2}{d\xi_2} + 3 \left[ \nu^2 + p_3 \left( Q - 2p_3(9 - \eta_1 + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} - \frac{8}{\hbar^2} e^{\xi_2} \right] B_2 = 0, \]

(62)

where for convenience we have written the separation constant as \(3\nu^2\). In the following sections we will show that quantum solutions can be divided into two classes, which will depend on combinations of \(\lambda_1, \lambda_2\). Moreover, in [61, 62] it has been shown that the best candidates for quantum solutions are wave functions that have a damping behavior with respect to the scale factor, since these allow to obtain good classical solutions when using the WKB approximation for any scenario in the evolution of our universe. Thus, the quantum analytical solutions to be presented in this paper will have this characteristic, featuring a damping behavior with respect to the average scale factor (cubic root of the isotropic volume \(V\)).

### 3.1 Quantum solution for \(\lambda_1 > \sqrt{6} (\eta_1 < 0), \text{ and } \lambda_2 < \sqrt{6}\).

We have to find solutions of \(B_1(\xi_1)\) and \(B_2(\xi_2)\) given the conditions \(\lambda_1 > \sqrt{6} (\eta_1 < 0)\), and \(\lambda_2 < \sqrt{6}\); however, we present only the explicit procedure for finding \(B_1(\xi_1)\). The same method is applied in order to obtain \(B_2\). Hence, the equation for \(B_1\) is

\[
\frac{d^2 B_1}{d\xi_1^2} + \frac{\left( Q - 4p_3(9 + |\eta_1|) \right) \frac{dB_1}{d\xi_1} + \frac{1}{4|\eta_1|} \left[ -\nu^2 + p_3 \left( Q - 2p_3(9 + |\eta_1| + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} - \frac{8}{\hbar^2} e^{\xi_1} \right]}{B_1} = 0.
\]

(63)
In fact this equation can be written as

\[ y'' + ay' + \left( be^\kappa x + c \right) y = 0, \]

and its solution is of the form \[ Y(x) = \operatorname{Exp} \left( -\frac{ax}{2} \right) Z_\rho \left( \frac{2\sqrt{b}}{\kappa} e^{\frac{x}{2}} \right), \] (64)

here \( Z_\rho \) are generic Bessel function with order \( \rho = \sqrt{a^2 - 4c/\kappa} \). If \( \sqrt{b} \) is real, \( Z_\rho \) are the ordinary Bessel function, otherwise the solution will be given by the modified Bessel function. Then we have the following relations

\[ a = \frac{Q - 4p_3(9 + |\eta_1|)}{2|\eta_1|}, \]
\[ b = -\frac{2V_{01}}{\hbar^2|\eta_1|}, \]
\[ c = \frac{1}{4|\eta_1|} \left[ -\nu^2 + p_3 \left( Q - 2p_3(9 + |\eta_1| + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} \right], \]
\[ \kappa = 1, \] (65)

since \( \sqrt{b} \) is imaginary, then the function \( Z_\rho \) becomes the modified Bessel function \( K_\rho \). Thus, \( B_1 \) is

\[ B_1 = \operatorname{Exp} \left[ -\frac{(Q - 4p_3(9 + |\eta_1|))}{4|\eta_1|} \xi_1 \right] K_{\rho_1} \left[ \frac{2}{\hbar} \sqrt{\frac{2V_{01}}{|\eta_1|}} e^{\frac{\xi_1}{2}} \right], \] (66)

with

\[ \rho_1 = \sqrt{\frac{\left( \frac{(Q - 4p_3(9 + |\eta_1|))}{2|\eta_1|} \right)^2}{\frac{1}{|\eta_1|}} - \left[ -\nu^2 + p_3 \left( Q - 2p_3(9 + |\eta_1| + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} \right]. \] (67)

Then for the solution of \( B_2 \) we have the following relations

\[ a = -\frac{6 \left( Q - 4p_3(9 - \eta_2) \right)}{2\eta_2}, \]
\[ b = \frac{2V_{02}}{\hbar^2 \eta_2}, \]
\[ c = -\frac{1}{4\eta_2} \left[ \nu^2 + p_3 \left( Q - 2p_3(9 + |\eta_1| + \eta_2) \right) + \frac{(p_4^2 + p_5^2)}{6} \right], \]
\[ \kappa = 1, \] (68)
in this case \( \sqrt{b} \) is real, then \( Z_\rho \) must be the ordinary Bessel function \( J_\rho \), therefore

\[
\mathcal{B}_2 = \exp \left[ \frac{(Q - 4p_3(9 - \eta_2))}{4\eta_2} \xi_2 \right] J_{\rho_2} \left[ \frac{2}{\hbar} \sqrt{\frac{2V_{02}}{\eta_2}} e^{\frac{\xi_2}{2}} \right],
\]

(69)

with

\[
\rho_2 = \sqrt{\left[ \frac{(Q - 4p_3(9 - \eta_2))}{2\eta_2} \right]^2 + \frac{1}{\eta_2} \left[ \nu^2 + p_3 (Q - 2p_3(9 + |\eta_1| + \eta_2)) + \frac{(p_3^2 + p_5^2)}{6} \right]}.
\]

(70)

Finally the wave function \( \Psi \) in the original variables becomes

\[
\Psi = \Psi_0 \nu^{2\alpha} \exp \left[ p_4 \beta_+ + p_5 \beta_- + \beta_1 \lambda_1 \phi_1 + \beta_2 \lambda_2 \phi_2 \right] \times K_{\rho_1} \left[ \frac{2}{\hbar} \sqrt{\frac{2V_{01}}{|\eta_1|}} \nu e^{-\frac{\lambda_1 \phi_1}{2}} \right] J_{\rho_2} \left[ \frac{2}{\hbar} \sqrt{\frac{2V_{02}}{\eta_2}} \nu e^{-\frac{\lambda_2 \phi_2}{2}} \right],
\]

(71)

where \( \Psi_0 \) is a normalization constant, the volume function \( V = ABC = e^{3\Omega} \), and

\[
\beta_1 = \frac{(Q - 4p_3(9 + |\eta_1|))}{4|\eta_1|} + p_3, \quad \beta_2 = -\frac{(Q - 4p_3(9 - \eta_2))}{4\eta_2} + p_3,
\]

\[
\alpha = -\frac{(Q - 4p_3(9 + |\eta_1|))}{|\eta_1|} + \frac{(Q - 4p_3(9 - \eta_2))}{\eta_2} + 4p_3.
\]

(72)

The behavior of the probability density \( |\Psi|^2 \) (from eq. (71)) in terms of the volume function \( V \) and the scalar field \( \phi_2 \) is presented in Fig. 3. Also, the evolution of \( |\Psi|^2 \) is shown for different values of the factor ordering parameter \( Q \); and actually, in all panels the probability density dies away as \( V \) and \( \phi_2 \) evolve. An expected outcome already reported in [39, 49, 53, 64, 65]. Moreover, when \( Q \ll 0 \) then \( |\Psi|^2 \) tends to behave similar to the isotropic case [49].

3.2 Quantum solution for \( \lambda_1 < \sqrt{6} \ (\eta_1 > 0) \), and \( \lambda_2 > \sqrt{6} \).

For this case equations for \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) are similar, therefore they have the same type of solution

\[
\mathcal{B}_1 = \exp \left[ \frac{(Q - 4p_3(9 - \eta_1))}{4\eta_1} \xi_1 \right] J_{\rho_1} \left[ \frac{2}{\hbar} \sqrt{\frac{2V_{01}}{\eta_1}} e^{\frac{\xi_1}{2}} \right],
\]

(73)

\[
\mathcal{B}_2 = \exp \left[ \frac{(Q - 4p_3(9 - \eta_2))}{4\eta_2} \xi_2 \right] J_{\rho_2} \left[ \frac{2}{\hbar} \sqrt{\frac{2V_{02}}{\eta_2}} e^{\frac{\xi_2}{2}} \right],
\]

(74)
Fig. 3 Phantom scenario. These figures show the probability density of the wave function $|\Psi|^2$ (from Eq. (71)) in terms of the volume function $V$ and the scalar field $\phi_2$ for various values of $Q = 2, 0$ (top panels from left to right), and $Q = -2, -15$ (bottom panels from left to right). We use arbitrary units of $\nu = 10, \lambda_1 = 4.3\sqrt{6}, \lambda_2 = 6/\lambda_1, V_{01} = 5, V_{02} = 10^{-5}, r_1 = 16, r_2 = 0.5, a_3 = -0.5, a_4 = a_5 = 1, \phi_1 = 1, \beta_+ = \beta_- = 1$. Other constants depend on the aforementioned values.

with

$$
\rho_1 = \sqrt{\left[ \frac{(Q - 4p_3(9 - \eta_1))}{4\eta_1} \right]^2 + \frac{-\nu^2 + p_3(Q - 2p_3(9 - \eta_1 + \eta_2)) + \frac{(p_4^2 + p_5^2)}{6}}{\eta_1}} \quad (75)
$$

$$
\rho_2 = \sqrt{\left[ \frac{(Q - 4p_3(9 - \eta_2))}{4\eta_2} \right]^2 + \frac{\nu^2 + p_3(Q - 2p_3(9 - \eta_1 + \eta_2)) + \frac{(p_4^2 + p_5^2)}{6}}{\eta_2}} \quad (76)
$$

The quantum solution of this set of parameters does not lead to a collapse of the probability density, thus the universe might be eternally quantum and the classical world never takes place.

3.3 Quantum solution for $\lambda_1 = \lambda_2 = \sqrt{6} \ (\eta_1 = 0)$, and $\eta_2 = 6$.

Now the equations for $B_1$ and $B_2$ are reduced to

$$
6(Q - 36p_3) \frac{dB_1}{d\xi_1} + 3 \left[ -\nu^2 + p_3(Q - 30p_3) + \frac{(p_4^2 + p_5^2)}{6} - 8 \frac{V_{01}}{\hbar^2} e^{\xi_1} \right] B_1 = 0
$$
\[-72 \frac{d^2 B_2}{d \xi_2^2} + 6 (Q - 12p_3) \frac{dB_2}{d \xi_2} + 3 \left[ v^2 + p_3 \left( Q - 30p_3 \right) + \frac{(p_3^2 + p_5^2)}{6} - 8 \frac{V_{02}}{\hbar^2} e^{\xi_2} \right] B_2 = 0, \quad (77)\]

and their corresponding solutions are

\[ B_1 = B_0 \exp \left[ \frac{-v^2 + p_3(Q - 30p_3) + \frac{p_4^2 + p_5^2}{6}}{2(36p_3 - Q)} \xi_1 + \frac{4V_{01}}{\hbar^2(Q - 36p_3)} e^{\xi_1} \right], \quad (79)\]

\[ B_2 = \exp \left[ \frac{Q - 12p_3}{24} \xi_2 \right] J_{\rho_2} \left[ \frac{2}{\hbar} \sqrt{\frac{V_{02}}{3}} e^{\frac{\xi_2}{2}} \right], \quad (80)\]

with

\[ \rho_2 = \sqrt{\left( \frac{Q - 12p_3}{12} \right)^2 + \frac{v^2 + p_3 \left( Q - 30p_3 \right) + \frac{(p_4^2 + p_5^2)}{6}}{6}}. \quad (81)\]

Thus, the wave function \( \Psi \) in the original variables becomes

\[ \Psi = \Psi_0 v^{2\alpha} \exp \left[ p_4 \beta_+ + p_5 \beta_- + \alpha_1 \lambda_1 \phi_1 + \alpha_2 \lambda_2 \phi_2 \right. \]

\[ \left. + \frac{4V_{01}}{\hbar^2(Q - 36p_3)} V^2 e^{-\lambda_1 \phi_1} \right] J_{\rho_2} \left[ \frac{2}{\hbar} \sqrt{\frac{V_{02}}{3}} V e^{-\frac{\lambda_2}{2} \phi_2} \right], \quad (82)\]

where \( \Psi_0 \) is a normalization constant, the volume function \( V = ABC = e^{3 \Omega} \), and

\[ \alpha_1 = - \frac{-v^2 + p_3(Q - 30p_3) + \frac{p_4^2 + p_5^2}{6}}{2(36p_3 - Q)} + p_3, \quad \alpha_2 = - \frac{Q - 12p_3}{24} + p_3 \]

\[ \alpha = - \frac{-v^2 + p_3(Q - 30p_3) + \frac{p_4^2 + p_5^2}{6}}{2(36p_3 - Q)} + p_3 + \frac{Q - 12p_3}{24}. \quad (83)\]

Figure 4 shows once again a damped \( |\Psi|^2 \) (from eq. (82)) due to evolution of \( V \) and the scalar field \( \phi_2 \). Now \( Q \) compresses the length over the axis on which the scalar field unfolds as time goes by, which in turns delays the progression of the probability density, hence retarding as well the accelerated expansion.

### 4 Final remarks

In this work we have performed a detailed analysis of a chiral anisotropic cosmological model from the perspective of quintom fields. The configuration of our proposal consisted of two fields interact in a non-standard (chiral) way: one quintessence and one
phantom; evolving within an anisotropic Bianchi type I background. Both a classical description and its quantum counterpart were presented.

In the classical scenario we find analytical solutions given a particular choice of the emerged relevant parameters. We highlight two cases. The first one when $\lambda_1 \lambda_2 = \sqrt{6}$, where we have selected the phantom domination instance at $\lambda_1 > \sqrt{6} \left( \eta_1 < 0 \right)$ and $\lambda_2 < \sqrt{6}$; and the second one for $\lambda_1 = \lambda_2 = \sqrt{6}$. Then in Figs. 1 and 2 we presented the time evolution of the volume $V$ and the Hubble $H$ functions. Notably in both scenarios the “big-bang” singularity is avoided via a “big-bounce”, yet $V$ grows very rapidly from there. Moreover, the horizontal crossing of the Hubble parameter (at $H = 0$) happens at the time of the “big-bounce”, thus reasserting this result. Indeed, this outcome has been already pointed out in a FLRW framework [49]. Also, we showed that isotropization is in fact reached as the time evolves for the two examples.

In the quantum scheme, the WDW equation is constructed and analytically solved for various instances given by the same parameter space from the classical study. First in Fig. 3 we presented four different examples due to $Q$ of the evolution of the probability density $|\Psi|^2$ in terms of $V$ and $\phi_2$. All cases display the expected damped behavior as $V$ and $\phi_2$ evolve. Remarkably, when $Q \ll 0$ the probability density tends to resemble to the isotropic case [49]. This result might indicate that for a fixed set of parameters, the anisotropies quantum-mechanically vanish for very small values of the parameter $Q$. Besides, this upshot (at least for small initial anisotropies) was already reported in [53]. In the second example (Fig. 4) the probability density also dies away as $V$ and $\phi_2$ evolve. However, this time $Q$ compresses the length over the axis on which the scalar field unfolds as time goes by, which in turns delays the progression of the probability density, hence retarding as well the accelerated expansion.

Finally, classical and quantum solutions reduce to their flat FLRW counterparts when the anisotropies vanish.

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References

1. Martinez-Gonzalez, E., Sanz, J.L.: Astron. Astrophys. 300, 346 (1995)
2. Belinskii, V.A., Khalatnikov, I.M.: Sov. Phys. JETP 63, 1121 (1972)
3. Folomeev, V.N., Gurovich, V.: Ts.: Gen. Relativ. Gravit. 32(7), 1255 (2000)
4. Michael P. Ryan Jr., Lawrence C. Shepely, Homogeneous Relativistic Cosmologies (Princeton University Press, Princeton, 1975)
5. Amirhashchi, H.: Phys. Rev. D 97, 063515 (2018)
6. Amirhashchi, H., Amirhashchi, S.: Phys. Rev. D 99, 023516 (2019)
7. Akarsu, O., et al.: Phys. Rev. D 100, 023532 (2019)
8. Goswami, G.K et al.: Mod. Phys. Let. A 2050086 (2020). https://doi.org/10.1142/S0217733220500868
9. Sasaki, M., Stewart, E.D.: Prog. Theor. Phys. 95, 71–78 (1996). https://doi.org/10.1143/PTP.95.71
10. Liddle, A.R., Mazumdar, A., Schunck, F.E.: Phys. Rev. D 58, 061301 (1998). https://doi.org/10.1103/PhysRevD.58.061301
11. Rigopoulos, G.: Class. Quantum Gravit. 21, 1737–1754 (2004). https://doi.org/10.1088/0264-9381/21/7/002
12. Bassett, B.A., Tsujikawa, S., Wands, D.: Rev. Mod. Phys. 78, 537–589 (2006). https://doi.org/10.1103/RevModPhys.78.537
13. Wands, D.: Lect. Notes Phys. 738, 275–304 (2008). https://doi.org/10.1007/978-3-540-74353-8_8
14. Lalak, Z., Langlois, D., Pokorski, S., Turzynski, K.: JCAP 07, 014 (2007). https://doi.org/10.1088/1475-7516/2007/07/014
15. Ashoorioon, A., Firouzjahi, H., Sheikh-Jabbari, M.M.: JCAP 06, 018 (2009). https://doi.org/10.1088/1475-7516/2009/06/018
16. Achucarro, A., Gong, J.O., Hardeman, S., Palma, G.A., Patil, S.P.: Phys. Rev. D 84, 043502 (2011). https://doi.org/10.1103/PhysRevD.84.043502
17. Achucarro, A., Gong, J.O., Hardeman, S., Palma, G.A., Patil, S.P.: JCAP 01, 030 (2011). https://doi.org/10.1088/1475-7516/2011/01/030
18. Achucarro, A., Gong, J.O., Hardeman, S., Palma, G.A., Patil, S.P.: JHEP 05, 066 (2012). https://doi.org/10.1007/JHEP05(2012)066
19. Achucarro, A., Atal, V., Cespedes, S., Gong, J.O., Palma, G.A., Patil, S.P.: Phys. Rev. D 86, 121301 (2012). https://doi.org/10.1103/PhysRevD.86.121301
20. Pi, S., Sasaki, M.: JCAP 10, 051 (2012). https://doi.org/10.1088/1475-7516/2012/10/051
21. Renaux-Petel, S., Turzynski, K.: Phys. Rev. Lett. 117(14), 141301 (2016). https://doi.org/10.1103/PhysRevLett.117.141301
22. Brown, A.R.: Phys. Rev. Lett. 121(25), 251601 (2018). https://doi.org/10.1103/PhysRevLett.121.251601
23. Achucarro, A., Kallosh, R., Linde, A., Wang, D.G., Welling, Y.: JCAP 04, 028 (2018). https://doi.org/10.1088/1475-7516/2018/04/028
24. Achucarro, A., Palma, G.A.: JCAP 02, 041 (2019). https://doi.org/10.1088/1475-7516/2019/02/041
25. Aragam, V., Paban, S., Rosati, R.: JHEP 03, 009 (2021). https://doi.org/10.1007/JHEP03(2021)009
26. Hui, L., Barkana, R., Gruzinov, A.: Phys. Rev. Lett. 85, 1158–1161 (2000). https://doi.org/10.1103/PhysRevLett.85.1158
27. Hui, L., Ostriker, J.P., Tremaine, S., Witten, E.: Phys. Rev. D 95(4), 043541 (2017). https://doi.org/10.1103/PhysRevD.95.043541
28. Téllez-Tovar, L.O., Matos, T, Vázquez, J.A: arXiv:2112.09337 [astro-ph.CO]
29. Boyle, L.A., Caldwell, R.R., Kamionkowski, M.: Phys. Lett. B 545, 17–22 (2002). https://doi.org/10.1016/S0370-2693(02)02590-X
30. Kim, S.A., Liddle, A.R., Tsujikawa, S.: Phys. Rev. D 72, 043506 (2005). https://doi.org/10.1103/PhysRevD.72.043506
31. van de Bruck, C., Weller, J.M.: Phys. Rev. D 80, 123014 (2009). https://doi.org/10.1103/PhysRevD.80.123014
32. Beltran Jimenez, J, Santos, P, Mota, D.F: Phys. Lett. B 723, 7-14 (2013) https://doi.org/10.1016/j.physletb.2013.04.051
33. Vardanyan, V., Amendola, L.: Phys. Rev. D 92(2), 024009 (2015). https://doi.org/10.1103/PhysRevD.92.024009
34. Leithes, A., Malik, K.A., Mulryne, D.J., Nunes, N.J.: J. Phys. Rev. D 80, 123501 (2009). https://doi.org/10.1103/PhysRevD.80.123501
35. Cicoli, M., Dibitetto, G., Pedro, F.G.: Phys. Rev. D 101(10), 103522 (2020). https://doi.org/10.1103/PhysRevD.101.103524
36. Cicoli, M., Dibitetto, G., Pedro, F.G.: JHEP 10, 035 (2020). https://doi.org/10.1007/JHEP10(2020)035
37. Anguelova, L., Dumancic, J., Gass, R., Wijewardhana, L.C.R.: JCAP 03, 018 (2022). https://doi.org/10.1088/1475-7516/2022/03/018
38. Elizalde, E., Nojiri, S., Odintsov, S.D.: Phys. Lett. B 819, 136427 (2021). https://doi.org/10.1016/j.physletb.2021.136427
39. Fomin, I.V., Pozdeeva, E.O., Sami, M., Yu, S.: Vernov Phys. Rev. D 100, 063522 (2019)
40. Kaiser, D.I.: Phys. Rev. D 81, 084044 (2010)
41. Birrell, N.D., Davies, P.C.W.: Quantum Fields in Curved Space. Cambridge University Press, New York (1982)
42. Socorro, C., Pérez-Payán, R., Hernández-Jiménez, A., Espinoza-García, A., Díaz-Barrón, L.R.: Class. Quantum Gravit. 38(13), 13502 (2021). https://doi.org/10.1088/1361-6382/abfbed7
43. Orizal, A., Leon, G.: Eur. Phys. J. Plus 137(1), 165 (2022). https://doi.org/10.1140/epjp/s13360-022-03108-8
44. Cai, Y.F., Saridakis, E.N., Setare, M.R., Xia, J.Q.: Phys. Rep. 493, 1–60 (2010). https://doi.org/10.1016/j.physrep.2010.04.001
45. Feng, B., Wang, X.L., Zhang, X.M.: Phys. Lett. B 607, 35 (2005). https://doi.org/10.1016/j.physletb.2004.12.071
46. J. Socorro, S. Pérez-Payán, R. Hernández-Jiménez, A. Espinoza-García and L. R. Díaz-Barrón, “Quintom Fields from Chiral K-Essence Cosmology,” Universe 8(10), 548 (2022). https://doi.org/10.3390/universe8100548
47. Chervon, S.V.: Russ. Phys. J. 38, 539–543 (1995). https://doi.org/10.1007/BF00559313
48. Kaiser, D.I.: Phys. Rev. D 81, 084044 (2010)
49. Chervon, S.V.: Chiral Cosmological Models: Dark Sector Fields Description Quantum Matter 2(2), 71–82 (2013)
50. Chervon, S.V.: Vernov Phys. Rev. D 100, 063522 (2019)
58. Fomin, I.V., Chervon, S.V.: New Method of Exponential Potentials Reconstruction Based on Given Scale Factor in Phantonical Two-Field Models arXiv:2112.0935
59. Paliathanasis, A., Leon, G., Pan, S.: Exact solutions in chiral cosmology. Gen. Relativ. Gravit. 51, 106 (2019)
60. Socorro, J., Pimentel, L.O., Espinoza-García, A.: Adv. High Energy Phys. 2014, 805164 (2014). https://doi.org/10.1155/2014/805164
61. Hartle, J.B., Hawking, S.W.: Phys. Rev. D 28, 2960–2975 (1983). https://doi.org/10.1103/PhysRevD.28.2960
62. Hawking, S.W.: Nucl. Phys. B 239, 257 (1984). https://doi.org/10.1016/0550-3213(84)90093-2
63. Zaitsev, V.F., Polyanin, A.D.: Handbook of Exact Solutions for Ordinary Differential Equations. Taylor and Francis Editorial (2002)
64. Socorro, J., Núñez, O.E., Hernández-Jiménez, R.: Classical and quantum exact solutions for a FRW multiscalar field cosmology with an exponential potential driven inflation. Adv. Math. Phys. 2018, 346838 (2018)
65. Socorro, J., Núñez, O.E., Hernández-Jiménez, R.: Classical and quantum exact solutions for the anisotropic Bianchi type I in multi-scalar field cosmology with an exponential potential driven inflation. Phys. Lett. B 809, 135667 (2020)

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