General model of quantum key distribution

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Abstract

A general mathematical framework for quantum key distribution based on the concepts of quantum channel and Turing machine is suggested. The security for its special case is proved. The assumption is that the adversary can perform only individual (in essence, classical) attacks. For this case an advantage of quantum key distribution over classical one is shown.

1 Introduction

In a number of papers several concrete quantum key distribution protocols were suggested, most of them are based on the BB84 protocol [1]. Its security is considered, e.g., in [2–7]. Practical realizations of quantum key distribution protocols are described, e.g., in [6, 8]. However, formal mathematically rigorous general approach to quantum key distribution is lacking at present. For recent discussions of these problems see [9–11].

In this paper we suggest a rather general mathematical model of quantum key distribution. We also prove the security for its certain special case. The security is proved on the assumption that the adversary can perform only individual (in essence, classical) attacks. For this case an advantage of the quantum key distribution over classical one is shown.

2 Notations

\(\mathcal{H}\) is a Hilbert space, \(\mathcal{S}(\mathcal{H})\) is a convex set of quantum states (density operators) on \(\mathcal{H}\). Let \(\mathcal{H}_A\) and \(\mathcal{H}_B\) be a pair of Hilbert spaces. A channel \(\Theta\) from \(\mathcal{H}_A\) to \(\mathcal{H}_B\) is an affine map from \(\mathcal{S}(\mathcal{H}_A)\) to \(\mathcal{S}(\mathcal{H}_B)\) such that its linear extension has a completely positive conjugate map [12,13]. The sequence of channels

\[\Theta^n : \mathcal{S}(\mathcal{H}_A^\otimes n) \to \mathcal{S}(\mathcal{H}_B^\otimes n), \quad n \in \mathbb{N}\]

is associated with the channel \(\Theta\) by the formula

\[\Theta^n(\rho_1 \otimes \ldots \otimes \rho_n) = \Theta(\rho_1) \otimes \ldots \otimes \Theta(\rho_n), \quad \rho_i \in \mathcal{S}(\mathcal{H}_A), \quad i = 1, \ldots, n.\]

If \(\mathcal{A}\) and \(\mathcal{B}\) are finite sets, \(\mathcal{P}(\mathcal{A})\) and \(\mathcal{P}(\mathcal{B})\) are probability distributions on \(\mathcal{A}\) and \(\mathcal{B}\), then an affine map \(V : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B})\) specifies a classical channel. If \(P \in \mathcal{P}(\mathcal{A})\), then \(I(P, V)\) denotes the Shannon mutual information between input and output of the channel \(V\) if \(P\) is a distribution on input of the channel \(V\). A channel \(V\) can be also specified by a mapping of the corresponding random values. If \(X\) and \(Y\) are random values, then \(I(X; Y)\) denotes the mutual information between them. By \(V^n : \mathcal{P}(\mathcal{A}^n) \to \mathcal{P}(\mathcal{B}^n)\) denote the discrete memoryless channel corresponding to the channel \(V\). An
affine map $\Xi : P(A) \to S(\mathcal{H})$ can be specified by a function $\xi : A \to S(\mathcal{H})$. Let $\rho \in S(\mathcal{H})$ be a quantum state. Then the von Neumann entropy is defined by the formula

$$H(\rho) = - \text{Tr} \rho \log \rho.$$ 

Let $A$ be a finite set, $P \in P(A)$ a distribution, $\mathcal{H}$ a Hilbert space, $\xi : A \to S(\mathcal{H})$ a function. We define

$$C(\xi) = \max_P \left[ H \left( \sum_{a \in A} P(a) \xi(a) \right) - \sum_{a \in A} P(a) H(\xi(a)) \right].$$

If $\{M(b)\}$ is a positive operator-valued measure (POVM), i.e., an observable, on $\mathcal{H}$, then the formula

$$P(b) = \text{Tr} M(b) \rho, \quad \rho \in S(\mathcal{H})$$

specifies an affine map (a channel) from $S(\mathcal{H})$ to $P(B)$. The space of POVM on $\mathcal{H}$ taking values on $B$ denotes by $\mathcal{M}(\mathcal{H}; B)$. An observable from

$$\mathcal{M}(\mathcal{H}; B)^{\otimes n} \subset \mathcal{M}(\mathcal{H}^{\otimes n}; B^n)$$

we will call a factorized observable on the space $\mathcal{H}^{\otimes n}$ taking values on $B^n$. An observable that doesn’t belong to this class we will call an entangled observable. By

$$B^A \circ \mathcal{M}(\mathcal{H}; A) \subset \mathcal{M}(\mathcal{H}; B)$$

denote the class of observables of the form

$$\{F(b) = \sum_{a \in f^{-1}(b)} M(a)\}_{b \in B}, \quad \{M(a)\}_{a \in A} \in \mathcal{M}(\mathcal{H}; A),$$

where $f$ is an element of the set $B^A$ of functions from $B$ to $A$.

### 3 General model of quantum key distribution

In this section a definition of general mathematical model of quantum key distribution is outlined. We consider the following problem of key distribution. Two parties, Alice and Bob, want to get a pair of keys (one key for Alice and another one for Bob) using communication channels. A realization of a certain random value on a finite set $\mathcal{K}$, or this random value itself is regarded as a key. If Alice’s and Bob’s keys are identical with high probability, and Eve’s information about the keys is negligibly small, then the problem of key distribution is considered to be solved with some security degree.

We will model the parties using extended Turing machines, which form the model of classical computers interacting with external devices and communicating with each other by classical and quantum channels.

Let $(A, Q_A, \tau_A)$ be a Turing machine describing Alice, where $A$ is an alphabet, $Q_A$ is a set of states with additional chosen elements, and $\tau_A$ is a transition function. Further, let $(A, Q_B, \tau_B)$ and $(A, Q_E, \tau_E)$ be Turing machines describing Bob and Eve accordingly.

**Definition 1.** A system of quantum key distribution is a triple of objects (extended Turing machines)

$$(\text{ETM}_A, \text{ETM}_B, \text{ETM}_E),$$

where

$$\text{ETM}_A = (A, Q_A, \tau_A, P_A, \xi_A),$$
ETM_B = (\mathcal{A}, \mathcal{Q}_B, \tau_B, P_B, \{\xi_B^{(n)}\}_{n \in \mathbb{N}}),
ETM_E = (\mathcal{A}, \mathcal{Q}_E, \tau_E, P_E, \{\xi_E^{(n)}\}_{n \in \mathbb{N}}, \mu_E).

Here \(P_A, P_B, P_E\) are probability distributions on \(A^+ = \bigcup_{i=1}^{\infty} A\),

\[\xi_A : A^+ \rightarrow S_A \subset \bigcup_{i=1}^{\infty} S(\mathcal{H}_{A}^{\otimes i}),\]

\[\xi_B^{(n)} : A^+ \rightarrow \mathcal{M}_B^{(n)} \subset \mathcal{M}(\mathcal{H}_B^{\otimes n}; A^+),\]

\[\xi_E^{(n)} : A^+ \rightarrow \Omega^{(n)},\]

\[\mu_E : A^+ \rightarrow M_E \subset \mathcal{M}(\mathcal{H}_E; A^+),\]

where \(\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E\) are Hilbert spaces, \(\Omega^{(n)} = \{\Theta_i^{(n)}\}_{i \in I}\),

\[\Theta_i^{(n)} : S(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_E) \rightarrow S(\mathcal{H}_B^{\otimes n} \otimes \mathcal{H}_E).\]

Upon the functions \(\xi_A, \xi_B^{(n)}, \xi_E^{(n)}, \mu_E\) are imposed some restrictions. They will be presented in the further paper.

Interaction of the parties by quantum and classical channels is realized by the use of the additional chosen elements in \(\mathcal{Q}_A, \mathcal{Q}_B, \mathcal{Q}_E\) and the functions introduced above.

4 Special model of quantum key distribution

In this paper we will consider only a special case of the described model.

**Definition 2.** A system \(G\) of quantum key distribution is a family of the following objects:

\[G = \left(\mathcal{K}, \mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E, \Theta, \{q^{(n)}\}_{n \in \mathbb{N}}, \{M_B^{(n)}\}_{n \in \mathbb{N}}, \{M_E^{(n)}\}_{n \in \mathbb{N}}\right).\]  \hspace{1cm} (1)

Here \(\mathcal{K}\) is a finite set (a set of keys), \(\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E\) are Hilbert spaces,

\[\Theta : S(\mathcal{H}_A) \rightarrow S(\mathcal{H}_B \otimes \mathcal{H}_E)\]

is a channel. The functions

\[q^{(n)} : \mathcal{K} \rightarrow S(\mathcal{H}_A^{\otimes n})\]

specify channels \(Q^{(n)}\),

\[M_B^{(n)} \in \mathcal{M}(\mathcal{H}_B^{\otimes n}; \mathcal{K}), \quad M_E^{(n)} \subset \mathcal{M}(\mathcal{H}_E^{\otimes n}; \mathcal{K}).\]

For any \(n \in \mathbb{N}\) and \(M_E^{(n)} \in \mathcal{M}_E^{(n)}\) we define the channel

\[\Lambda_n = (M_B^{(n)} \otimes M_E^{(n)}) \circ \Theta^n \circ q^{(n)}\]

with the input alphabet \(\mathcal{K}\) and the output alphabet \(\mathcal{K}^2\).

Let \(K_A\) denote a random value uniformly distributed on \(\mathcal{K}\) (a key). We denote \(K_B\) and \(K_E\) random variables taking values on \(\mathcal{K}\) and related to \(K_A\) by the channel \(\Lambda_n\) for some \(n\), i.e.,

\[(K_B, K_E) = \Lambda_n(K_A).\]

The schematic view of the quantum key distribution process is presented on Fig. 1.
The proposed model is a quantum analogue of the classical key distribution model considered in [14], where the following theorem is proved.

**Theorem 1.** Let $\mathcal{K}, \mathcal{A}, \mathcal{B}, \mathcal{E}$ be finite sets, and a pair of channels

$$V : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{B}), \quad W : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{E})$$

is given. Further, let

$$\max_{P \in \mathcal{P}(\mathcal{A})} [I(P, V) - I(P, W)] > 0. \quad (2)$$

Then for any $\alpha, \beta \in (0, 1)$ and any sufficiently large $n \in \mathbb{N}$ there exists a channel (a random coder)

$$F_A : \mathcal{P}(\mathcal{K}) \to \mathcal{P}(\mathcal{A}^n)$$

and a function (a decoder)

$$f_B : \mathcal{B}^n \to \mathcal{K}$$

such that for any function (a decoder)

$$f_E : \mathcal{E}^n \to \mathcal{K}$$

we have:

1) $\Pr[K_A = K_B] \geq \alpha$,

2) $I(K_A; K_E) \leq 1 - \beta$,

where

$$K_B = f_B \circ V^n \circ F_A(K_A), \quad K_E = f_E \circ W^n \circ F_A(K_A).$$

If for any $P$ the condition

$$I(P, V) \geq I(P, W)$$

is true, then condition (2) is not only sufficient, but also necessary for existing of $F_A$ and $f_B$ with the specified properties for sufficiently large $n$.

So, condition (2) can be considered as a condition of possibility of classical key distribution with an arbitrary security degree. A pair of numbers $(\alpha, \beta)$ is regarded as a security degree.

Now we formulate our main theorem.

**Theorem 2.** Let us fix $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_E, \Theta$ in the model $G$ of quantum key distribution $G$. For any $n \in \mathbb{N}$ we put

$$\mathcal{M}^{(n)}_E = \mathcal{K}^E \circ \mathcal{M}(\mathcal{H}_E; \mathcal{E})^\otimes n,$$

where $\mathcal{E}$ is a finite set.

Suppose there exist a finite set $\mathcal{A}$ and a channel $\Xi$ specified by a function $\xi : \mathcal{A} \to \mathcal{S}(\mathcal{H}_A)$ that obey the following property:

$$C(\Theta_B \circ \xi) > C_1(\Theta_E \circ \xi), \quad (3)$$
where $\Theta_B = \text{Tr}_H \Theta$, $\Theta_E = \text{Tr}_B \Theta$,

$$C_1(\Theta_E \circ \xi) = \max_{P \in \mathcal{P}(A), M \in \mathcal{M}(H_E; K)} I(P, M \circ \Theta_E \circ \Xi).$$

Then for any $\alpha, \beta \in (0, 1)$ and any sufficiently large $n \in \mathbb{N}$ there exist a channel (a random coder)

$$F_A : \mathcal{P}(K) \rightarrow \mathcal{P}(A^n)$$

and an observable

$$M_B \in \mathcal{M}(H_B \otimes^n; K)$$

such that $\forall M_E^{(n)} \in \mathcal{M}_E^{(n)}$ the random values $K_A$, $K_B$ and $K_E$, where $(K_B, K_E) = \Lambda_n(K_A)$, obey the properties:

1) $\Pr[K_A = K_B] \geq \alpha$,
2) $I(K_A; K_E) \leq 1 - \beta$.

Here

$\Lambda_n(K_A) = (M_B^{(n)} \otimes M_E^{(n)}) \circ \Theta^n \circ Q^{(n)}$,

$Q^{(n)} = \Xi^n \circ F_A$.

Outline of proof. Let us denote

$$C_k(\Theta_B^{(k)} \circ \xi^{(k)}) = \max_{P \in \mathcal{P}(A), M \in \mathcal{M}(H_B \otimes^k; K)} I(P, M \circ \Theta_B^{(k)} \circ \Xi^k),$$

where

$$\xi^{(k)} : \mathcal{A}^k \rightarrow \mathcal{S}(H_A)$$

is the function that specifies the memoryless channel $\Xi^k$ corresponding to the channel $\Xi$, i.e.,

$$\xi^{(k)}(a_1, \ldots, a_k) = \xi(a_1) \otimes \ldots \otimes \xi(a_k).$$

As [13]

$$C(\Theta_B \circ \xi) = \lim_{k \rightarrow \infty} \frac{1}{k} C_k(\Theta_B^{(k)} \circ \xi^{(k)}),$$

then, in view of [3],

$$\exists n C_n(\Theta_B^{(n)} \circ \xi^{(n)}) > nC_1(\Theta_E \circ \xi).$$

Since for any $k$, in view of the conditions on the adversary’s measurements class,

$$\max_{P \in \mathcal{P}(A^n), M \in \mathcal{M}(H_E; \mathcal{X}) \otimes k} I(P, M \circ \Theta_E^{(n)} \circ \xi^{(k)}) = kC_1(\Theta_B \circ \xi),$$

this implies that

$$\max_{P \in \mathcal{P}(A^n)} I(P, M \circ \Theta_B^{(n)} \circ \xi^{(n)}) > \max_{P \in \mathcal{P}(A^n)} I(P, M \circ \Theta_E^{(n)} \circ \xi^{(n)})$$

Then, since the adversary can perform only individual measurements with the subsequent classical processing, the conditions of Theorem [1] are satisfied.
5 On advantages of quantum key distribution over classical one

Thus, it is possible that for a single use of a quantum channel the classical condition \[2\] of key distribution is not satisfied, but with multiple repetitions Bob uses the essentially quantum property, entangled measurements, in contrast to the adversary Eve, who performs in essence classical attacks – factorized measurements. As a result, Bob achieves advantage over Eve and the subsequent key distribution according to the classical scheme becomes possible.

For demonstration of this possibility we give an example of the quantum key distribution system, in which at every run of the quantum channel Eve receive the same state as Bob, i.e., there is an ideal eavesdropping. However, as a consequence of the effect of entangled measurements applying by Bob with the multiple use of the channel, he gets more information than Eve, and the key distribution becomes possible. The effect of entangled measurements is caused by the use of nonorthogonal states.

**Example.** Let \( \mathcal{H}_A = \mathbb{C}^2 \otimes \mathbb{C}^2 \), \( \mathcal{H}_B = \mathcal{H}_E = \mathbb{C}^2 \), \( \Theta \) is an identical map. \( A = \{0, 1\} \),

\[
\xi(0) = (|\varphi\rangle\langle\varphi|)^{\otimes 2},
\]

\[
\xi(1) = (|\psi\rangle\langle\psi|)^{\otimes 2},
\]

where

\[
|\varphi\rangle, |\psi\rangle \in \mathbb{C}^2, \quad \langle\varphi|\psi\rangle \neq 0.
\]

If for \( P \in \mathcal{P}(A) \) the condition \( P(0) = 0 \) or \( P(1) = 0 \) is satisfied, then

\[
\forall M_E \in \mathcal{M}(\mathcal{H}_E; \mathcal{E})
\]

we have

\[
I(P, M_E \circ \Theta_E \circ \Xi) = 0,
\]

so

\[
\max_P I(P, M_E \circ \Theta_E \circ \Xi)
\]

is achieved when \( P(0), P(1) > 0 \). Then, due to the fact that the operators \( P(0)\xi(0) \) and \( P(1)\xi(1) \) don’t commute, condition \[3\] is satisfied. So, by Theorem \[2\] the key distribution is possible.

There aren’t analogues in classical cryptography for the given effect when the key distribution is possible even under conditions of ideal eavesdropping. In this sense one can say about advantages of quantum cryptography over classical one.

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