Thermodynamic Concept of Neutron Separation Energy

V.P. Maslov *

Abstract

In the paper, we propose a new approach to the mathematical description of the separation of a neutron from the atom’s nucleus on the basis of the formalisms of tropical mathematics and nonstandard analysis. In studying the behavior of individual nucleons of the atom’s nucleus, instead of the ordinary approach involving the Einstein formula relating mass and energy, we use the thermodynamical approach to the disintegration of the nucleus. This approach allows to obtain a previously unknown general expression for the energy needed to separate a neutron from its atom’s nucleus, provided we know the de Broglie wavelength and the volume of the nucleus.

Introduction

At the present time, problems which arose when quantum mechanics arose have returned to the frontline of current research. For example, the Einstein–Podolsky–Rosen paradox [1], the de Broglie wave-particle dualism [2], and other problems. Modern mathematics has moved forward since the appearance of these problems, and has led to the solution of part of them. In particular, this can be said of nonstandard analysis and the dequantization procedure related to tropical mathematics [3]. In the present paper, we solve the problem of calculating the specific energy of the disintegration of a Bose particle.

The creation of contemporary wave mechanics begins with de Broglie’s note “Ondes et Quanta”, presented in 1923. In it, de Broglie studied the motion of electrons along a closed orbit and showed that the requirement that the phases agree leads to the Bohr–Sommerfeld quantization condition, i.e., to the quantization of the angular momentum (see [4]–[5], as well as [6]–[7] and [8]–[9]). Developing his ideas about waves, as related to particles, in 1927 de Broglie presented his theory of double solution [2], which led to the wave-particle dualism, still topical today.

De Broglie came to the conclusion that the presence of a continuous wave is related to the appearance of an extra term in the expression for the Lagrangian particle; this term can be regarded as a small addendum to the potential energy (compare formula (6) below).

We shall consider the case when the number of molecules in a gas is small. We will look at the neutrons and protons (nucleons) that constitute the atomic nucleus of molecules from the thermodynamical point of view. In particular, we shall make use of the de Broglie wavelength, which determines the size of the wave packet constituting the given quantum particle.

*National Research University Higher School of Economics, Moscow, 123458, Russia; Moscow State University, Physics Department, Moscow, 119234, Russia;
The Hartrey–Fock equation describing the weak interaction near the intersection of the wave packets allows to write out the self-consistent relation for the value of the potential that suffices to keep the nucleon in the nucleus and impedes the disintegration of the latter. Our approach consists in applying thermodynamical methods, related to the de Broglie wavelength, together with mathematical methods in number theory and in nonstandard analysis in order to compute the energy needed to separate the neutron from the atom’s nucleus.

According to wave-particle duality, the corpuscular or wave-like character of a particle may be determined by a quantitative parameter – the de Broglie wavelength. If the de Broglie wavelength is relatively large, then the particle is a wave packet, i.e., it is quantum. In particular, such quantum particles in nuclear physics are called bosons and fermions.

Under the condition that we know the de Broglie wavelength and the volume of the nucleus, it is possible to determine the energy required for the neutron to separate from the nucleus and the latter is transformed from a Bose particle to a Fermi particle or vice versa. This energy is usually calculated by means of the defect of mass, using the formula relating mass to energy discovered by Einstein. We will do this differently, namely, by using the de Broglie wavelength, we will determine whether the particle is quantum, and if so, we will find the level of energy at which the neutron is torn away from the atomic nucleus, thereby changing its spin. If the number of nucleons is even, then the atomic nucleus is a Bose particle. When one nucleon is torn away from such a nucleus, it becomes Fermi particle with nonzero spin.

### a. Bose statistics, Fermi statistics in Hogan–Watson diagrams and in Gentile statistics

The behavior of Bose particles and Fermi particles is described by the Bose–Einstein and Fermi–Dirac distributions respectively.

Let \( a = e^{\mu/T} \) be the activity (\( \mu \) being the chemical potential, \( T \) being the temperature). Let \( D \) be the number of degrees of freedom (the dimension) and let \( s = D/2 \). The total energy of all \( N \) particles is denoted by \( E \).

The Bose–Einstein distribution in terms of the polylogarithm has the form

\[
\text{Li}_s(a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t/a - 1} dt. \tag{1}
\]

The Fermi–Dirac distribution is written as

\[-\text{Li}_s(-a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t/a + 1} dt. \tag{2}\]

In these formulas, \( \text{Li}(\cdot) \) is the polylogarithm function, \( \Gamma \) is the Euler gamma-function, and \( t \) is time.

Usually, in physics, the Bose–Einstein and the Fermi–Dirac distributions are defined with the help of Gentile statistics \[10\]. Gentile statistics contains Bose–Einstein and the Fermi–Dirac statistics as particular cases. Gentile statistics contains an additional constant \( K \) that indicates the maximal number of particles located at a fixed level of energy. In particular, for \( K = 1 \), the distribution of Gentile statistics coincides with the distribution of the Fermi–Dirac statistics, so that the corresponding formulas coincide in form with \[2\]. In Gentile statistics, we always have \( K \geq 1 \).

For an ideal gas of dimension \( D \) obeying the Gentile statistics, i.e., in the case when there are no more than \( K \) particles (\( K \) being an integer) at each energy level, the total number of
particles $N$ is known:

$$N = \frac{V}{\Lambda^2\pi} \left( \text{Li}_s(a) - \frac{1}{(K + 1)^{s-1}} \text{Li}_s(a^{K+1}) \right), \quad (3)$$

where $V$ is the volume, $\Lambda$ is the de Broglie wavelength.

In our approach, unlike the standard Gentile statistics, we set $K = 0$. Note that in thermodynamics $N$ means the number of particles. In the present paper, we do not consider molecules, we only consider nuclei, i.e., we are in nuclear physics. In that sense, we can say that in our model the number of particles $N$ is equal to zero.

The consideration of the $\Omega$-potential (compare [11]–[13]) corresponding to Gentile statistics allows us to describe in more detail the passage from particles of the atomic nucleus of a Bose gas to those of a Fermi gas. In this situation, by analogy with the $\Omega$-potential considered by Landau and Lifshits [14], this allows to calculate the total energy of this passage.

We consider quantum particles, each of which is a wave packet [15]. These wave packets are characterized by their de Broglie wavelength $\Lambda$.

One can see that when the activity changes sign, the distributions (1) and (2) also change sign. This corresponds to the passage from negative pressures to positive ones. This picture naturally arises in the Van-der-Waals formulas [16]–[17]. Thus, the Bose particles and the Fermi particles are positioned in different parts of the $P,Z$-diagram of Hogan and Watson (in it, $P$ is the pressure, $Z = PV/NT$ is the compressibility factor, $V$, the volume, $N$, the number of particles, $T$, the temperature): the Bose particles are in the positive domain, while the Fermi particles are in the negative one.

We will use this technique to compute the specific energy required for the passage of particles of an ideal Bose gas to particles of a Fermi gas. We can consider that the Fermi gas is obtained when the activity changes sign. This process is described by the passage from formula (1) to formula (2).

b. Gentile statistics

Usually, in physics, the Bose–Einstein and the Fermi–Dirac decompositions are defined with the help of Gentile statistics [10]. Gentile statistics contains Bose–Einstein and the Fermi–Dirac statistics as particular cases. Gentile statistics contains an additional constant $K$ that indicates the maximal number of particles located at a fixed level of energy. In particular, for $K = 1$, the distribution of Gentile statistics coincides with the distribution of the Fermi–Dirac statistics, so that the corresponding formulas coincide in form with (2). In Gentile statistics, we always have $K \geq 1$.

The consideration of the $\Omega$-potential (compare [11]–[13]) corresponding to Gentile statistics [10] allows us to describe in more detail the passage from particles of the atomic nucleus of a Bose gas to those of a Fermi gas. In this situation, by analogy with the $\Omega$-potential considered by Landau and Lifshits [14], this allows to calculate the total energy of this passage.

We will use this technique to compute the specific energy required for the passage of particles of an ideal Bose gas to particles of a Fermi gas. We can consider that the Fermi gas is obtained when the activity changes sign. This process is described by the passage from formula (1) to formula (2).

c. Notation

Let us introduce the notation that will allow us to determine the energy in dimensionless form.
Let \( v = \Lambda^{2s} \). This quantity has the dimension of volume in \( 2s \)-dimensional space. Let 
\[
e = \frac{2\pi m}{\hbar^2} v^{\frac{1}{2}} .
\]
This quantity has the dimension of energy.

Now let us introduce the dimensionless quantities \( \mathcal{E} = E/e \) for the energy and \( \mathcal{V} = V/v \) for the volume. Note that the quantity \( \mathcal{V}^{1/D} \) is the ratio of the characteristic linear size of the system \( V^{1/D} \) to the de Broglie wavelength \( \Lambda \).

Usually one denotes by \( N_i \) the number of particles located at the \( i \)-th level of energy. It is customary to consider that in the case of a Fermi gas there can be no more than one particle at a fixed level, while for a Bose gas the number of particles \( N_i \) at a given level can be as large as we wish. We shall consider the Gentile statistics \([10]\), in which at each energy level there is no more than \( K \) particles. In other words, the number of particles at any level of energy cannot be greater than \( K \).

The maximal number of particles at a given energy level occurs when the activity \( a \) is maximal, i.e., at the point \( a = 1 \). Since \( \sum_{i=1}^{M} N_i = N \), for any Bose system we obviously have \( N_i \leq N \). Therefore, for a Bose system, we have \( K \leq N \). In Gentile statistics, \( K \) takes integer values.

### Results

We will assume that \( K = N \) in an infinitely small neighborhood of \([N]\), where \([N]\) is the integer part of the number \( N \).

The set of all points infinitely close to the number \([N]\) is called the Leibnitz differential \([18]\) in nonstandard analysis as developed by Robinson (see \([19]–[20]\)); the Leibnitz differential can be understood as the length of an elementary infinitely small segment (the monad). A differential is an arbitrary infinitely small increment of the variable.

In the positive domain of the \(-P,Z\)-diagram (Bose particles case) monads, i.e. fractional numbers, vary in the interval between 0 and 1. In the negative domain (Fermi particles case) they lie in the interval between -1 and 0. Hence, on passing from Bose particles to Fermi particles monads take all values from 0 to \([1]\).

Denote by \( x^p \) the difference \( N - [N] \), i.e., \( N - [N] = x^p < 0 \) (\( x > 0 \) corresponds to bosons in the \(-P,Z\)-diagram). We note that \( p \) is an arbitrary number, including monads, hyperreals or nonstandard reals, fractal numbers, etc. Hence the equations corresponding to the energy under consideration do not depend on \( p \).

This allows one to draw an analogy between integers in the standard number theory and their generalization in the abstract analytic number theory (see \([21]\)). This problem was studied in detail in \([22]–[23]\). This corresponds to the generality of computed “specific” separation energy of neutrons from the atomic nucleus. This approach allows us to assume that if one neutron separates from a boson nucleus, then its spin was equal to zero, and if one neutron separates from a fermion nucleus, then its spin becomes equal to zero after separation. In this sense, the methods of tropical geometry can be transferred to the mathematical analysis (see \([3]\)).

We shall search for the expansion in powers of \( x \) up to \( O(x^{2p}) \), which will imply that \( N \sim [N] \).

The self consistent relation for \( x \) in a neighborhood of \([N]\) is of the form:

\[
[N] + x^p = \frac{V/\Lambda^{2s}}{\Gamma(s)} \int_0^\infty \left( \frac{1}{e^{\xi/a} - 1} - \frac{[N] + x^p + 1}{e^{([N]+x^p+1)\xi/a} - 1} \right) \xi^{s-1} d\xi ,
\]  

\( 4 \)
The following thermodynamical formula for the energy is known:

\[ E = s \frac{(V/\Lambda^2)^{s+1}}{\Gamma(s+1)} \int_0^\infty \left( \frac{1}{e^{\xi/a} - 1} - \frac{[N] + x^p + 1}{e^{([N]+x^p+1)\xi/a} - 1} \right) \xi^s d\xi, \] 

(5)

Let us perform the same manipulations for \( x > 0 \). Then the term at the first power of \( x \) will be negative. It corresponds to a Fermi system.

Therefore, in our approach, unlike the standard Gentile statistics, we also set \( K = 0 \) and only consider the case \([N] = 0\). To the numbers \( N = K = 0 \), we apply nonstandard analysis as well as the technique of Gentile statistics [10].

Using the technique of nonstandard analysis, we add to the integer \( x \) the technique of Gentile statistics [10].

Let us expand the right-hand side of equation (4) in powers of small values of \( x \), neglecting terms of degree 3\( p \) or higher:

\[ x^p = \frac{V}{\Lambda^2} \frac{x^p}{2} \left( (s - 1) \log(a) \log(a) - (s - 1) \left( (s \log(a) - 2 \log(a) \log(a) \log(a) \right) + ... \right. \] 

(6)

In particular, for \( s = 3/2 \), has a unique solution \( a_0 \) which depends on \( \frac{V}{\Lambda^2} \) and \( s \).

The value of \( \log(a) \), where \( a = e^\mu/T \), corresponds to the total energy of passage, in particular in the three-dimensional case (\( s = 3/2 \)).

Equation (8) for sufficiently large values of \( V/\Lambda^2 \) has a unique solution \( a_0 \leq 1 \), which depends on \( V/\Lambda^2 \) and \( s \).

The value of the activity \( a \) for a known temperature \( T \) gives the corresponding value of the chemical potential \( \mu \):

\[ \mu = T \log(a) \leq 0. \]

(10)

In particular, for \( a = a_0 \), the higher the temperature \( T \), the smaller is \( a_0 \) and the larger becomes the corresponding value of \( |\mu_0| \). Thus, as the temperature grows, the point of passage \( \mu_0 \) approaches the point \( \mu = -\infty \), at which the pressure \( P \) changes sign.

Let \( a_0 = 1 \) and assume that we know the mass \( m \) of one nucleon and the volume \( V \) of the nucleus. Then equation (8) with the value \( \Lambda = \sqrt{\frac{2\pi R^2}{mT}} \) of the de Broglie wavelength taken into account can be regarded as an equation for the unknown \( T \).
Let us call critical the temperature that arises for \( a_0 = 1 \), i.e., as \( \mu_0 \to 0 \). Denote this temperature by \( T_s \).

From (8) for \( s > 1 \) we obtain

\[
T_s = \frac{2\pi \hbar^2}{m(V(s - 1)\zeta(s))^{1/s}},
\]

(11)

where \( \zeta(\cdot) \) is the Riemann zeta function.

Since the temperature \( T_s \) is the lowest one in the whole range of variation of \( \mu_0 \) (which is the ray \((-\infty, 0]\)), we will call the ratio \( T/T_s \) the regularized temperature and denote it by \( T_{\text{reg}} \). The change of temperature can be measured by \( T_{\text{reg}} \).

Expanding the energy (5) in powers of \( x \) to the first degree inclusive, we obtain:

\[
\mathcal{E}_{sp}(dx)^p = 2s\left(\frac{V}{\Lambda^{2s}}\right)^{\frac{1}{s+1}}\left(s \text{Li}_{s+1}(a_0) - \log(a_0) \text{Li}_s(a_0)\right)(dx)^p.
\]

(12)

The value of \( a_0 \) can be computed from (8) and (9).

Thus we have calculated the specific energy needed to separate a neutron from an atomic nucleus when one neutron leaves the nucleus, provided the volume of the nucleus and the de Broglie wavelength are known.

Let us consider the case of parastatistics with infinitely small \( K \) and \( N \) equal to each other.

In the case of parastatistics, we have the following relations, in which the first term in parentheses gives the distribution for Bose particles, and the second term, the parastatistical correction:

\[
E = \frac{V}{\lambda^D}T(\gamma + 1)(\text{Li}_{2+\gamma}(a) - \frac{1}{(K + 1)^{\gamma+1}}\text{Li}_{2+\gamma}(a^{K+1})),
\]

(13)

\[
N = \frac{V}{\lambda^D}(\text{Li}_{1+\gamma}(a) - \frac{1}{(K + 1)^{\gamma}}\text{Li}_{1+\gamma}(a^{K+1})),
\]

(14)

where \( \text{Li}_\gamma(\cdot) \) is the polylogarithm function, \( a = e^{\mu/T} \) is the activity (\( \mu \) being the chemical potential, \( T \) being the temperature), \( \gamma = D/2 - 1 \), \( D \) is the number of degrees of freedom (the dimension), \( \lambda = \sqrt{\frac{2\pi \hbar^2}{mT}} \) is the de Broglie wavelength, where \( \hbar \) is the Planck constant, \( m \) the mass of one particle.

If in the intermediate region between Bose particles and Fermi particles there is a certain number of nucleons, then physicists refer to it as a nuclear halo. The region corresponding to the difference of pressure \( P = 0 \) and to an infinitely small sequence \( \{P_K\} \to 0 \) constitutes the nuclear halo. The passage from the Bose-type region to the Fermi-type region occurs through the nuclear halo, which contains the value of the pressure \( P = 0 \). Above we have denoted by \( a_0 \) the maximal value of the activity \( a \) for Bose particles as \( N \to 0 \). The quantity \( a_0 \) indicates the maximal value of the activity at which the decomposition of bosons into fermion occurs.

For an ideal gas of dimension \( D = 3 \), relations (13), (14) become

\[
N = \frac{V}{\lambda^3}(\text{Li}_{3/2}(a) - \frac{1}{(K + 1)^{3/2}}\text{Li}_{3/2}(a^{K+1})),
\]

(15)

\[
E = \frac{3V}{2\lambda^3}T(\text{Li}_{5/2}(a) - \frac{1}{(K + 1)^{3/2}}\text{Li}_{5/2}(a^{K+1})).
\]

(16)
The expansion of the summand $\frac{1}{(K+1)^{1/2}} \text{Li}_{3/2}(a^{K+1})$ from formula (15) in small values of $K$ has the form:

$$\frac{1}{(K+1)^{1/2}} \text{Li}_{3/2}(a^{K+1}) = \text{Li}_{3/2}(a) - \left( K(\text{Li}_{3/2}(a)/2 - \log(a) \text{Li}_{1/2}(a)) + O(K^2) \right).$$  \hspace{1cm} (17)

Let $B = V/\lambda^3 > 0$. Then equation (15) for small $K$ acquires the form:

$$N = BK\left( \frac{1}{2} \text{Li}_{3/2}(a) - \log(a) \text{Li}_{1/2}(a) \right) + O(K^2).$$  \hspace{1cm} (18)

Dividing both sides of (18) by $N$ and taking the limit as $K \to 0$, yields an expression for $a_0$, i.e., the value of $a$ for which $K = N = 0$:

$$\frac{1}{2} \text{Li}_{3/2}(a_0) - \log(a_0) \text{Li}_{1/2}(a_0) - B^{-1} = 0.$$  \hspace{1cm} (19)

Equation (19) in the case of an arbitrary coefficient $\gamma = D/2 - 1$ instead of $1/2$, after similar arguments, acquires the form:

$$\gamma \text{Li}_{\gamma+1}(a_0) - \log(a_0) \text{Li}_{\gamma}(a_0) - \frac{\lambda^{2(\gamma+1)}}{V} = 0.$$  \hspace{1cm} (20)

Equation (20) has a unique solution $a_0 > 0$ that depends on $B$ and $\gamma$.

In the case $K = N$, equation (15) acquires the form

$$N = B(\text{Li}_{3/2}(a) - \frac{1}{(N+1)^{1/2}} \text{Li}_{3/2}(a^{N+1})).$$  \hspace{1cm} (21)

This equation obviously has the solution $N \equiv 0$ for any $a \geq 0$. However, for $a > a_0$, it has one more nonnegative solution $N(a)$. This can be verified by constructing the graphs of the right-hand and left-hand sides of (21) as a function of $a$ for an arbitrary fixed $N > 0$. The right-hand side of the equation is zero for $a = 0$ and monotonically grows for $a \in (0, \infty)$, while the left hand side is a constant that does not depend on $a$.

Substituting the obtained relation $N(a)$ in formula (16), we can find the dependence $E(a)$, and with it the pressure $P(a)$, by using the relation $E = (\gamma + 1)PV$.

In our considerations $K$ is an infinitely small number. Thus we are not dealing with the Fermi statistics or the Bose statistics, but with a parastatistics of a new type, which can be called a Bose-like statistics.

Let us substitute the obtained relation into the graph of the compressibility factor $Z = PV/(NT)$ as a function of $P$ (this graph is known as the Houten–Watson diagram).

For Fermi statistics in the case $D = 3$, we have the relations

$$N = -\frac{V}{\lambda^3} \text{Li}_{3/2}(-a),$$  \hspace{1cm} (22)

$$E = -\frac{3V}{2\lambda^3} T \text{Li}_{5/2}(-a).$$  \hspace{1cm} (23)

Let us call the curve on the P-Z diagram, constructed according to formulas (22)–(23) of Fermi statistics, the Fermi branch. The pressure $P$, as well as the number of particles $N$, on the Fermi branch is positive.

In the Bose-like region, the boson consists of two fermions of the same mass, while in the Fermi-like region the pair of fermions differ in mass. As the temperature grows, the
Table 1: Results for various isotopes

| isotope            | $B_{nExp}$, MeV | $a_0$    | $\mu_0$, MeV |
|--------------------|-----------------|----------|--------------|
| carbon-12          | 18.722          | 0.00219147 | -114.638     |
| oxygen-16          | 15.664          | 0.0012928  | -104.18      |
| neon-20            | 16.9            | 0.00059614 | -125.483     |
| silicon-30         | 10.61           | 0.00041606 | -82.5955     |
| argon-40           | 9.9             | 0.000207475| -83.9569     |
| chrome-50          | 13.001          | 0.0000704527 | -124.297   |
| chrome-52          | 12.039          | 0.0000718161 | -114.869   |
| ferrum-54          | 13.378          | 0.0000542755 | -131.391   |
| nickel-60          | 11.388          | 0.0000529799 | -112.122   |
| copper-65          | 9.911           | 0.0000534668 | -97.489    |
| zinc-66            | 11.059          | 0.0000427369 | -111.258   |
| germanium-70       | 11.534          | 0.0000338915 | -118.712   |
| selen-76           | 11.155          | 0.0000285583 | -116.721   |
| cripton-84         | 10.521          | 0.0000238875 | -111.966   |
| cripton-85         | 7.121           | 0.0000440699 | -71.4217   |
| molobdene-100      | 8.291           | 0.0000219127 | -88.949    |
| molibdene-101      | 5.399           | 0.0000433015 | -54.2455   |
| indium-115         | 9.037           | 0.000012972  | -101.691    |
| ksenon-132         | 8.937           | 9.06854 × 10^{-6} | -103.765   |
| neodim-150         | 7.381           | 8.75154 × 10^{-6} | -85.9612   |
| lutetium-175       | 7.667           | 5.40851 × 10^{-6} | -92.9818   |
| mercury-199        | 6.664           | 4.7946 × 10^{-6} | -81.6208   |
| plumbum-208        | 7.369           | 3.61133 × 10^{-6} | -92.3441   |
| radium-226         | 6.397           | 3.62971 × 10^{-6} | -80.1311   |
| plutonium-229      | 6.762           | 3.2006 × 10^{-6} | -85.554    |
| californium-250    | 6.625           | 2.60987 × 10^{-6} | -85.1724   |
| laurencium-261     | 6.792           | 2.23144 × 10^{-6} | -88.3834   |

difference in mass increases and the fermion with the smaller mass disappears. One fermion remains.

The table presents values of $a_0$ and $\mu_0 = T \log a_0$ for various isotopes. The value of $a_0$ in the case of separation can be found by means of formula (19), taking into account the expression of the de Broglie wavelength $\lambda$ in terms of the volume $V$ of the nucleus, its temperature $T$ and its mass $m$. The volume of the nucleus is taken to be that of a ball of radius $r_0 = A^{1/3}1.2 × 10^{-15} m^3$. The temperature $T$ of the nucleus expressed in energy units is taken equal to the energy of separation of a neutron $B_{nExp}$ (obtained from the database CDFE), since it is equal to the excitation of the nucleus.

We have obtained an equation from which we can find the value of $a_0$, and can determine the temperature $T$ at which this value is attained.

Let us consider the ratio of the compressibility factor $Z = PV/(NT)$ to the pressure $P$, i.e., $Z/P$.

On any isotherm $T$ is constant. The volume $V$ we consider to be constant, $N = K$ (see (21)) is an infinitely small number and therefore $1/N$ is an infinitely large number. Since the values of $V$ and $T$ are constant on isotherms and $P$ decreases, the ratio $Z/P$ becomes a constant, depending only on $V$ and $T$, divided by the infinitely small number $N$.

Let $\{P_K\}$ be an infinitely small sequence, coinciding with the infinitely small quantity
Figure 1: Dependence of the compressibility factor $Z$ on the pressure $P$, expressed in the units MeV/fm$^3$ for argon-40, copper-65, and molobdene-100. The continuous line represents the line $Z = 1$. The hashed lines show isotherms of the Bose branch, constructed according to formulas (15)–(16). The temperature is equal to the energy needed for the separation of the neutron $B_{nExp}$ (see Table I). The corresponding value of $a_0$ is given in Table I.
Fig. 1 shows the dependence of the compressibility factor on the pressure $P$, expressed in the units MeV/fm$^3$ for argon-40, copper-65, molybden-100. The dashed lines are the isotherms of the Bose branch constructed by means of formulas (15)–(16). The temperature is equal to the extremal value of the separation energy of the neutron $B_{n,\text{Exp}}$, indicated in table 1.

To each value of $a_0$ there corresponds a definite value of the temperature $T$. In turn, to each value of $T$ there corresponds an isotherm on the Hougen–Watson diagram. These isotherms lie in the negative quadrant. The temperature characterizing the isotherm becomes smaller as the point $a_0$ becomes nearer to $Z = 1$.

Thus we can say that the Van-der-Waals isotherms are in a sense opposite to the isotherms of nuclear matter shown in Fig. 1.

This shows that the chemical potential $\mu$ at $P = 0$ does not become equal to minus infinity and so the axis $Z$ at $P = 0$ is not the boundary between two unrelated structures. Since the value of $|\mu_0|$ is very large but not infinite between the values of the infinitely small quantities $\{P_K\}$ and the region obeying the Fermi–Dirac distribution, there is a narrow “halo” dividing the Bose region from the Fermi region.

We do not consider the problem of proving that the isotherms on Fig. 1 exist. We give an approximate solution of the obtained equations (see similar approaches in papers [24], [25]), in which all the isotherms corresponding to different values of $a_0$ are approximately constructed.

**Conclusion**

The separation of one neutron corresponds to the transition of the nucleus of a Bose gas molecule into the nucleus of a Fermi gas molecule, and conversely.

The passage from particles of a Bose gas to those of a Fermi gas occurs in its roughest form when the activity $a$ changes sign. In this passage, the activity $a$ becomes equal to zero, while the chemical potential becomes equal to minus infinity. In this process one nucleon leaves the nucleus.

We considered the behavior of the Bose–Einstein distribution in some neighborhood of the point $a = 0$ and showed that the separation of a neutron from the nucleus occurs at the point $a = a_0$, which is not zero. Then, using the analog of Gentile statistics for $K = 0$, we calculated the value of the nonstandard specific energy that was needed to separate the nucleon from a particle of the Bose gas. Although Gentile statistics was previously used for a number of particles greater than 1, the application of nonstandard analysis (Leibnitz differential or monad) and infinitesimal quantities allowed the author to generalize the relations of Gentile statistics to the case of a small number of Bose particles for $N = K = 0$.

The notion of wave packet means that the particle is not point-like, it is diffuse. This process depends on the de Broglie wavelength of the wave packets.

The main result of the present paper consists in the determination of the value of a parameter $a_0$ that allows us to construct an antipode of sorts of the Hougen–Watson P-Z diagram for nuclear matter. We have shown that, knowing the values of $a_0$, we can construct all the isotherms on the P-Z diagram.

The Bohr model of the nucleus of an atom is similar to the model of liquids. In the author’s papers [17], [26], it was shown that a liquid complying with the Van-der-Waals model may be approximated by the Bose–Einstein distribution for negative pressures and by the Fermi–Dirac distribution for positive ones. For a pressure $P < 0$ the liquid dilates. The dilation of liquids at negative pressures was already noticed by Huygens. It turns out
that in the region of positive pressure the Fermi–Dirac distribution describes the picture of a Van-der-Waals gas sufficiently well. This connection indirectly confirms the analogy between nuclear matter and liquids.

References

[1] Bell, J. S. On the Einstein Podolsky Rosen paradox. *Physics* **1** (3), 198–200 (1964).

[2] Broglie, de L. Wave mechanics and the atomic structure of matter and radiation. *J. Phys. Radium* **8** (5), 225–241 (1927).

[3] Litvinov, G. L. The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. *Contemp. Math.*, Vol. 377: *Idempotent Mathematics and Mathematical Physics* (Amer. Math. Soc., Providence, RI, 2005).

[4] Tyurin, N.A. Universal Maslov class of a Bohr-Sommerfeld Lagrangian embedding into a pseudo-Einstein manifold. *Theoretical and Mathematical Physics* **150** (2), 278–287 (2007).

[5] Esina, A.I. & Shafarevich, A.I. Analogs of Bohr-Sommerfeld-Maslov quantization conditions on Riemann surfaces and spectral series of nonself-adjoint operators. *Russian Journal of Mathematical Physics* **20** (2), 172–181 (2013).

[6] Yoshioka, A. Maslov’s Quantization Conditions for the Bound States of the Hydrogen Atom. *Tokyo J. Math.* **09** (2), 415–437 (1986).

[7] Czyż, On geometric quantization and its connection with the Maslov theory. *Reports of Math. Physics* **15** (1), 57–97 (1979).

[8] Barilari, D. & Antonio Lerario, A. Geometry of Maslov cycles. *Geometric Control Theory and Sub-Riemannian Geometry*. Springer INdAM Series, v.5, 15-35 — (Springer, 2014).

[9] Barge, J. & Ghys, E. Cocycles d’Euler et de Maslov. *Mathematische Annalen* **294** (1), 235–265 (1992).

[10] Dai, W.-S.& Xie, M. Gentile statistics with a large maximum occupation number. *Annals of Physics* **309**, 295–305 (2004).

[11] Smoczyk, K. Prescribing the Maslov form of Lagrangian immersions. *Geom. Dedicata* **91**, 59–69 (2002).

[12] Borrelli, V. Maslov form and J-volume of totally real immersions. *J. Geom. Phys.* **25** (3-4), 271–290 (1998).

[13] Vaisman, I. Conormal bundles with vanishing Maslov form. *Monatsh. Math.* **109** (4), 305-310. (1990).

[14] L. D. Landau and E. M. Lifshits, *Statistical Physics* (Nauka, Moscow, 1964) [in Russian].

[15] Littlejohn, R.G. Cyclic evolution in quantum mechanics and the phases of Bohr-Sommerfeld and Maslov. *Phys Rev Lett.* **61**(19), 2159–2162 (1988).
[16] Maslov, V.P. Probability Distribution for a Hard Liquid. *Math. Notes* **97** (6) 909–918 (2015).

[17] Maslov, V. P. Locally ideal liquid. *Russian J. Math. Phys.*, **22** (3), 361–373 (2015).

[18] Shchepin, E. V. The Leibniz differential and the Perron–Stieltjes integral. *J. Math. Sci.* **233** (1), 157–171 (2018).

[19] Robinson, A. *Non-standard analysis* (North-Holland Publishing Co., Amsterdam, 1966).

[20] Kanovei, V.V. & Reeken, M. *Nonstandard Analysis, Axiomatically* (Springer, 2004).

[21] Postnikov, A. G. *Introduction to Analytic Number Theory* (Nauka, Moscow, 1971).

[22] Maslov, V.P. & Nazaikinskii, V. E. Conjugate variables in analytic number theory. Phase space and Lagrangian manifolds. *Math. Notes* **100** (3), 421–428 (2016).

[23] Maslov, V.P. & Dobrokhotov, S. Yu. &. Nazaikinskii, V.E. Volume and entropy in abstract analytic number theory and thermodynamics. *Math. Notes* **100** (6), 828–834 (2016).

[24] Bruno, A. D. Self-similar solutions and power geometry *Russian Mathematical Surveys* **55** (1), 1–42 (2000).

[25] Weinstein, A. The Maslov Gerbe, *Letters in Mathematical Physics* **69** (1), 3-9 (2004).

[26] Maslov, V.P. Statistics corresponding to classical thermodynamics. Construction of isotherms, *Russian J. Math. Phys.*, **22** (1), 53–67 (2015).