Neutrino masses and mixing in $A_4$ models with three Higgs doublets

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We study neutrino masses and mixing in the context of flavor models with $A_4$ symmetry, three scalar doublets in the triplet representation, and three lepton families. We show that there is no representation assignment that yields a dimension-5 mass operator consistent with experiment. We then consider a type-I seesaw with three heavy right-handed neutrinos, explaining in detail why it fails, and allowing us to show that agreement with the present neutrino oscillation data can be recovered with the inclusion of dimension-3 heavy neutrino mass terms that break softly the $A_4$ symmetry.

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I. INTRODUCTION

One can attempt to explain the structure of neutrino masses and mixing by building full theories subject to discrete symmetries (for recent reviews, see e.g. Refs. 11–14). As an alternative, one may consider effective operators in the low-energy limit and their remnant symmetries, without recourse to the full theory 15. In either case, it is important to ensure that the vacuum structure chosen for the scalar sector of the theory does indeed correspond to a global (and not merely a local) minimum. Recently 16, the global minima of theories with three Higgs doublets, $\Phi_k$ ($k = 1, 2, 3$), in triplet representations of $A_4$ 17 or $S_4$ have been identified. It turns out that the allowed alignments for the vacuum expectation values (VEVs) in $A_4$ are 16

$$\begin{align*}
v(1, 0, 0), \\
v(1, 1, 1), \\
v(\pm 1, \eta, \eta^*) \text{ with } \eta = e^{i\pi/3}, \\
v(1, e^{i\alpha}, 0) \text{ with any phase } \alpha.
\end{align*}$$

Permutations of these VEVs are still global minima; other solutions of the stationarity conditions are not. The quark sector of such theories was explored by us in Ref. 8. It was shown that, at tree level, there is no consistent theory with only three families of standard quark fields that can explain the fact that the quark masses and the Cabibbo-Kobayashi-Maskawa CP-violating phase are nonvanishing.

In this article, we extend the analysis of Ref. 8 to the neutrino sector, assuming that neutrino masses are generated through a low-energy mass operator in an effective theory. We consider models with three Higgs doublets $\Phi_k$ in a triplet representation of $A_4$, so that the only possible VEV structures are those in Eq. 1.

In Sec. I1 we recall the features that apply to the charged-lepton Yukawa matrices in $A_4$ 8. Then, in Sec. I1 we turn to the neutrino sector. In Sec. I1 A we address the question of whether it is possible to get a consistent picture in an effective theory with the following particle content: three scalar doublets in a triplet representation of $A_4$, three left-handed lepton doublets $L_L$, and three right-handed charged leptons $\ell_R$ in any representation of $A_4$. We conclude that, although an $A_4$-invariant dimension-5 effective operator $(L_L \Phi)(L_L \Phi)$ can always be built, it is not possible to assign suitable $A_4$ representations to the fields in order to obtain viable (nondegenerate and nonvanishing) mass spectra, with the VEV alignments given in Eq. 1 and neutrino masses arising solely from the effective operator. As a result, in such a minimal $A_4$ framework, none of the standard seesaw mechanisms, usually invoked to give small masses to neutrinos, is consistent with the experimental data. With the aim of identifying what features need to be corrected when building a complete viable model, in Sec. I1 B we discuss in detail type-I (type-III) seesaw models 18 that contain a minimal particle content, namely, three generations of left-handed lepton doublets and right-handed charged lepton singlets, and three right-handed neutrino singlets $\nu_R$ (fermion triplets $\Sigma_R$). Since, at the Lagrangian level, the $A_4$ flavor group structure is the same for both seesaw cases, so are the conclusions. Clearly, the above minimal setup is not sufficient to build a consistent $A_4$ flavor model that leads to nonzero nondegenerate charged lepton and neutrino masses, and to the correct leptonic mixing. The detailed discussion presented in Sec. I1 B is used in Sec. IV to uncover examples where a soft breaking of the $A_4$ symmetry is enough to provide consistency with experiment. Our conclusions are briefly summarized in Sec. V.

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1 We do not consider here the type-II seesaw scenario since it involves triplet scalars, thus making, in general, the identification of the global minima of the scalar potential extremely hard.
II. \textit{A}_4 AND CHARGED LEPTONS

The \textit{A}_4 group has four irreducible representations: three singlets \textbf{1}, \textbf{1}', \textbf{1}'' and one triplet \textbf{3}. Here we adopt the basis used in Ref. \cite{8} for the generators of the group, and place the three Higgs doublets in the triplet representation, $\Phi \sim \textbf{3}$. The conclusions reached in Table II of Ref. \cite{8} for the down-type quarks also hold for charged leptons. In particular, requiring nonvanishing nondegenerate charged lepton masses forces the VEV alignments to be $v(1,1,1)$ or $v(\pm 1, \eta, \eta^*)$, restricting the possible representation assignments to the cases listed in Table II.

| $L_L$ | $\ell_R$ |
|-------|-------|
| 3     | 3     |
| (1,1,1) | 3     |

\textbf{Table I}. Possible representations of the left-handed lepton doublet ($L_L$) and right-handed charged-lepton singlets ($\ell_R$), which lead to nonvanishing nondegenerate charged lepton masses, when the three Higgs doublets are in a triplet representation \textbf{3}. The case $L_L \sim 3$, $\ell_R \sim 3$ leads to nonrealistic charged-lepton masses.

Let us consider in detail the case $L_L \sim 3$, $\ell_R \sim 3$. We may parametrize the charged lepton mass matrix as

$$m_\ell = \begin{pmatrix} 0 & b e^{i\beta} v_3 & a e^{i\alpha} v_2 \\ a e^{i\alpha} v_3 & 0 & b e^{i\beta} v_1 \\ b e^{i\beta} v_2 & a e^{i\alpha} v_1 & 0 \end{pmatrix}. \quad (2)$$

Taking the VEV alignment $(1,1,1)$ or $(\pm 1, \eta, \eta^*)$\footnote{Henceforth, we assume without loss of generality that $v = 1$.} the eigenvalues of $m_\ell m_\ell^\dagger$ are

$$a^2 + b^2 + 2ab \cos(\alpha - \beta),$$
$$a^2 + b^2 - 2ab \cos(\alpha - \beta \pm \pi/3), \quad (3)$$

which coincide with the squared masses of the charged leptons. Although the eigenvalues depend on three parameters, one can see that they cannot be used to fit the hierarchical structure of the charged lepton masses. Indeed, it is easy to show that

$$3(a^2 + b^2) = m_\nu^2 + m_\mu^2 + m_\tau^2,$$
$$3ab = \sqrt{m_\mu^4 + m_\tau^4 + m_\nu^4 - m_\mu^2 m_\tau^2 - m_\mu^2 m_\nu^2 - m_\nu^2 m_\tau^2}. \quad (4)$$

Substituting for the experimental masses $\cite{9}$, we find that $(a - b)^2 \simeq -1.05 \text{ GeV}^2$. Thus, in this case, the charged lepton masses cannot be fitted.

III. \textit{A}_4 AND MAJORANA NEUTRINOS

If neutrinos were Dirac particles, with masses arising from Yukawa couplings with $\Phi_k$, our conclusions would be the same as those reached in Ref. \cite{8} for the up-type quarks. Clearly, in that case, the existence of one massless neutrino or the lack of leptonic CP violation would not contradict current experiments $\cite{10}$. In this work, however, we focus on Majorana neutrinos whose masses are given by some seesaw mechanism or else are generated effectively by integrating out unspecified heavy degrees of freedom.

A. Effective dimension-5 operator for neutrino masses

There is only one dimension-5 operator made out of the SM fields which respects the SM gauge symmetry $\cite{11}$,

$$\mathcal{L}_{\text{eff}} = \left( L_L \tilde{\Phi} \right) K \left( \tilde{\Phi}^\dagger L_L^\dagger \right) + \text{H.c.}, \quad (5)$$

where $K$ is a complex symmetric matrix, and $\tilde{\Phi} = i\sigma_2 \Phi$. Since we are only interested in the group structure, we will only refer to $L_L L_L$ and $\tilde{\Phi} \Phi$.

If $L_L \sim 3$, we can form the symmetric bilinears

$$(L_L \otimes L_L)_1 = L_L^2 + L_L^{3} + L_L^{3}, \quad (6)$$
$$(L_L \otimes L_L)_1' = L_L^2 + \omega^2 L_L^{3} + \omega L_L^{3}, \quad (7)$$
$$(L_L \otimes L_L)_1'' = L_L^2 + \omega L_L^{3} + \omega^2 L_L^{3}, \quad (8)$$
$$(L_L \otimes L_L)_3 = 2(L_2 L_3, L_3 L_1, L_1 L_2), \quad (9)$$

where $\omega = e^{2i\pi/3}$. Similar combinations hold for $\tilde{\Phi} \Phi$. Forming all combinations that can lead to a singlet, we find

$$\mathcal{L}_{\text{eff}} = \lambda_1 (L_L^2 + L_L^{3} + L_L^{3})(\tilde{\Phi}^2 + \tilde{\Phi}^2 + \tilde{\Phi}^2) + \lambda_2 (L_L^2 + \omega^2 L_L^{3} + \omega L_L^{3})(\tilde{\Phi}^2 + \tilde{\Phi}^2 + \omega^2 \tilde{\Phi}^2)$$
$$+ \lambda_3 (L_L^2 + \omega L_L^{3} + \omega^2 L_L^{3})(\tilde{\Phi}^2 + \tilde{\Phi}^2 + \omega \tilde{\Phi}^2) + \lambda_4 (L_2 L_3 \tilde{\Phi}^2 \Phi^2 + L_3 L_1 \tilde{\Phi}^2 \Phi^2 + L_1 L_2 \tilde{\Phi}^2 \Phi^2) \quad (10)$$

where $\lambda_1 \ldots \lambda_4$ are complex parameters.

After spontaneous symmetry breaking, the fields get VEVs ($\Phi_k = v_k$, and the elements of the symmetric matrix $K$ become

$$k_{11} = \lambda_1 (v_1^2 + v_2^2 + v_3^2) + \lambda_2 (v_1^2 + \omega v_2^2 + \omega^2 v_3^2)$$
$$+ \lambda_3 (v_1^2 + \omega^2 v_2^2 + \omega v_3^2),$$
$$k_{22} = \lambda_1 (v_1^2 + v_2^2 + v_3^2) + \lambda_3 (v_1^2 + \omega^2 v_2^2 + \omega v_3^2),$$
$$k_{33} = \lambda_1 (v_1^2 + v_2^2 + v_3^2) + \lambda_2 (v_1^2 + v_2^2 + \omega v_3^2),$$
$$k_{12} = \frac{1}{2} v_1^* v_2^* \lambda_4, \quad k_{13} = \frac{1}{2} v_1^* v_3^* \lambda_4, \quad k_{23} = \frac{1}{2} v_2^* v_3^* \lambda_4 \quad (11)$$

Only the VEV alignments $(1,1,1)$ and $(\pm 1, \eta, \eta^*)$ lead to nonvanishing charged lepton masses. With $(1,1,1)$, we
meaning that \( KK^\dagger \) has a doubly degenerate eigenvalue 
\( 3|\alpha_1|^2 \) and a third eigenvalue \( 3|\alpha_2|^2 \). Similarly, for the VEV \( \pm \eta, \eta^* \), the relevant VEV combinations are 0, 3, and 0, respectively, with 
\[
K = 3 \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_2
\end{pmatrix}.
\]
Again, \( KK^\dagger \) has a doubly degenerate eigenvalue \( 3|\alpha_2|^2 \) and a third eigenvalue \( 3|\alpha_1|^2 \). As a result, all cases are ruled out.

### B. Type-I and type-III seesaw

We consider first the type-I \([12]\) seesaw mechanism with \( n_\nu = 3 \) right-handed neutrino fields. The relevant Lagrangian is

\[
- \mathcal{L}_1 = \bar{L}_L \sum_{k=1}^3 Y_{\ell,k} \Phi_k \ell_R + \bar{\ell}_L \sum_{k=1}^3 Y_{\nu,k}^* \Phi_k \nu_R + \frac{1}{2} \bar{\nu}_R M_R \nu_R + \text{H.c.},
\]

where \( L_L = (\ell_L, \nu_L)^T \), \( \ell_R \), and \( \nu_R \) are vectors in the three-dimensional generation spaces of left-handed doublets, right-handed charged lepton singlets, and right-handed neutrino singlets, respectively. For each scalar doublet \( \Phi_k \) there is a charged lepton Yukawa matrix \( Y_{\ell,k} \), and a neutrino Yukawa matrix \( Y_{\nu,k} \). After the spontaneous symmetry breaking we obtain the mass terms

\[
- \mathcal{L}_1 = \bar{\ell}_L m_{\ell} \ell_R + \bar{\nu}_L m_D \nu_R + \frac{1}{2} \bar{\nu}_R M_R \nu_R + \text{H.c.},
\]

and \( M_R \) is a symmetric matrix. To correctly reproduce the light neutrino masses, the eigenvalues of \( M_R \) should be much larger than \(|v_1|^2 + |v_2|^2 + |v_3|^2|^{1/2} \). Integrating out the heavy right-handed Majorana fields, the low-energy effective Lagrangian becomes

\[
- \mathcal{L}_{\text{eff}} = \bar{\ell}_L m_{\ell} \ell_R + \frac{1}{2} \bar{\nu}_L^T M_{\nu} \nu_L + \text{H.c.},
\]

and the light neutrinos acquire an effective mass

\[
m_{\nu} = -m_D M_R^{-1} m_D^T.
\]

In the basis where the charged lepton mass matrix is diagonal,

\[
m_{\ell} = \text{diag} (m_e, m_\mu, m_\tau),
\]

the neutrino mass matrix \( m_\nu \) is diagonalized by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) \([13]\) leptonic mixing matrix \( U \) as

\[
U^T m_\nu U = \text{diag} (m_1, m_2, m_3),
\]

where \( m_i \) are the light neutrino masses.

We will try to assign the lepton fields to \( A_4 \) representations, subject to the following constraints:

1. The matrix \( M_R \) corresponding to the heavy Majorana fields cannot have a zero eigenvalue;

2. The charged lepton masses cannot vanish or be degenerate.

The first condition forces the right-handed neutrino fields to be in one of the following three representations. One can have \( \nu_R \sim (1, 1, 1) \) and

\[
M_R = \begin{pmatrix}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times
\end{pmatrix},
\]

where \( \times \) represents an independent complex entry. Alternatively, one can have \( \nu_R \sim (1, 1', 1'') \) and

\[
M_R = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \times \\
0 & \times & 0
\end{pmatrix}.
\]
Finally, if $\nu_R \sim 3$, then

$$M_R = M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $M$ is an arbitrary complex number. This corresponds to degenerate heavy neutrinos. Other combinations are ruled out by our first requirement.

Before proceeding, let us look back at Eq. (19). Because $\det M_R \neq 0$, the existence (or absence) of massless light neutrinos depends on the nature of $m_D$. If $\det m_D = 0$ (det $m_D \neq 0$), then $\det m_\nu = 0$ (det $m_\nu \neq 0$). As a result, we will consider the constraints coming from charged leptons in each of these two cases, separately.

Notice that, regardless of the $\nu_R$ representation, no case with both $L_L$ and $\ell_R$ in singlet representations is possible because $\Phi \sim 3$, leading to $m_\nu = 0$. Similarly, the cases where both $L_L$ and $\nu_R$ are in singlet representations are excluded because they lead to $\det m_D = 0$ and, through Eq. (19), to $m_\nu = 0$. Finally, as shown in Sec. II cases when both $L_L$ and $\ell_R$ are in the triplet representation of $A_4$ are not possible. The remaining cases are listed in Table II.

| Case | $L_L$ | $\nu_R$ | $\ell_R$ | Neutrino masses |
|------|-------|---------|---------|----------------|
| i)   | 3     | 3       | (1, 1', 1'') | 2 degenerate  |
| ii)  | (1, 1', 1'') | 3 | 3 | 2 degenerate  |
| iii) | 3     | (1, 1', 1'') | (1, 1', 1') | 2 degenerate  |
| iv)  | 3     | (1, 1, 1)  | (1, 1', 1'') | 2 massless    |

TABLE II. Possible representations of the left-handed lepton doublet ($L_L$), the right-handed neutrino singlets ($\nu_R$), and right-handed charged lepton singlets ($\ell_R$), when the three Higgs doublets are in a triplet representation 3.

### 1. Nonvanishing Neutrino Masses

We start by looking at the cases in which $\nu_R \sim 3$. From Table II one concludes that there are two possibilities which may lead to nonvanishing neutrino and charged lepton masses: i) $L_L \sim 3$, $\nu_R \sim 3$, $\ell_R \sim (1, 1', 1'')$ and ii) $L_L \sim (1, 1', 1'')$, $\nu_R \sim 3$, $\ell_R \sim 3$.

Let us consider in detail case i). Then, Eq. (24) holds and we may parametrize

$$m_\ell = \begin{pmatrix} a e^{i \alpha} v_1 & b e^{i \beta} v_1 & c e^{i \gamma} v_1 \\ a e^{i \alpha} v_2 & b e^{i \beta} v_2 & c e^{i \gamma} v_2 \\ a e^{i \alpha} v_3 & b e^{i \beta} v_3 & c e^{i \gamma} v_3 \end{pmatrix},$$

and

$$m_D = \begin{pmatrix} 0 & f e^{i \varphi} v_3 & d e^{i \delta} v_2 \\ d e^{i \delta} v_3 & 0 & f e^{i \varphi} v_1 \\ f e^{i \varphi} v_2 & d e^{i \delta} v_1 & 0 \end{pmatrix}.$$
Writing
\[ M_R = \begin{pmatrix} r_1 e^{i\sigma_1} & 0 & 0 \\ 0 & 0 & r_2 e^{i\sigma_2} \\ 0 & r_2 e^{i\sigma_2} & 0 \end{pmatrix}, \]
(33)
we can use Eq. (19) to determine \( m_\nu \). The expression is long, but the eigenvalues of \( m_\nu m_\nu^\dagger \) are simply given by \((3d^2/r_1)^2\) and \((3f^2/r_2)^2\), with the latter twice degenerate. Thus, we also get two degenerate light neutrinos, so this case is also excluded.

2. Vanishing neutrino masses

We now turn to the possibility that \( m_D \), and thus \( m_\nu \), have determinants equal to zero, with at most one massless light neutrino. The only case consistent with realistic charged lepton masses is iv) \( L_L \sim 3, \nu_R \sim (1,1,1), \ell_R \sim (1,1,1') \), provided that the VEV alignment is \((1,1,1)\) or \((1,\eta,\eta')\).

Let us show that this possibility is also inconsistent with experiment. In this case, \( m_\ell \) has the same matrix structure as case i), while the Dirac neutrino mass matrix has the form
\[
m_D = \begin{pmatrix} d e^{i\delta} v^*_1 f e^{i\xi} v^*_1 & g e^{i\xi} v^*_1 \\ d e^{i\delta} v^*_2 f e^{i\xi} v^*_2 & g e^{i\xi} v^*_2 \\ d e^{i\delta} v^*_3 f e^{i\xi} v^*_3 & g e^{i\xi} v^*_3 \end{pmatrix}. \tag{34}\]

Taking the VEV \((1,1,1)\), we find that
\[
m_D = V_L D_D V_R^\dagger, \tag{35}\]
where \( D_D = \sqrt{3} e^{i\xi} R_3 \text{diag}(0,0,1) \),
\[
V_L = \begin{pmatrix} 1 \sqrt{2} & 1 \sqrt{6} & 1 \sqrt{3} \\ 0 & -\sqrt{\frac{3}{2}} & \frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}, \tag{36}\]
\[
V_R = \begin{pmatrix} -\frac{\sqrt{2}}{2} e^{-i(\delta-\xi)} & \frac{f}{\sqrt{2} R_3} e^{i(\xi-\epsilon-\xi)} & \frac{f}{\sqrt{2} R_3} e^{-i(\delta-\epsilon)} \\ 0 & -\frac{f}{\sqrt{2} R_3} e^{i(\epsilon-\xi)} & \frac{f}{\sqrt{2} R_3} e^{-i(\delta-\epsilon)} \\ \frac{f}{\sqrt{2} R_3} & \frac{f}{R_2 R_3} e^{i(\delta+\epsilon-2\xi)} & \frac{f}{R_3} \end{pmatrix}, \tag{37}\]
\[
R_2 = (d^2 + g^2)^{1/2} \quad \text{and} \quad R_3 = (d^2 + f^2 + g^2)^{1/2}. \]

Here, \( V_L \) and \( V_R \) are the unitary matrices that diagonalize the Hermitian matrices \( m_D m_D^\dagger \) and \( m_D^\dagger m_D \), respectively. Using Eq. (19), we get
\[
m_\nu = -V_L D_D X D_D V_L^T = -e^{2i\xi} R_3^T X_{33} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{38}\]
where \( X = V_R^\dagger M_R^{-1} V_R^\dagger \).

From Eq. (38), we find the eigenvalues of \( m_\nu m_\nu^\dagger \) to be \( m_1^2 = m_2^2 = 0 \) and \( m_3^2 = 9 R_3^2 \left| X_{33} \right|^2 \). Since there are two massless neutrinos, this case is ruled out. For the VEV alignment \((1,\eta,\eta')\), the intermediate steps get more involved, but the eigenvalues of \( m_\nu m_\nu^\dagger \) have the same expressions, and therefore this possibility is also ruled out.

Before concluding this section, let us comment on the type-III seesaw mechanism \([13]\). In the type-III seesaw framework, instead of three right-handed singlet neutrino fields, one adds three Majorana neutrinos, \( \Sigma_R \), in the triplet representation of the gauge group \( SU(2)_L \),
\[
\Sigma_R = \begin{pmatrix} \Sigma_i^0 / \sqrt{2} \\ \Sigma_i^- \\ -\Sigma_i^0 / \sqrt{2} \end{pmatrix}, \quad i = 1,2,3. \tag{39}\]
The relevant Lagrangian is very similar to Eq. (15) for a type-I seesaw:
\[
-\mathcal{L}_{III} = \bar{L}_L \sum_{k=1}^3 Y_{L,k} \Phi_k \ell_R + \bar{L}_L \sum_{k=1}^3 Y_{L,k}^\dagger \bar{\Phi}_k \Sigma_R + \frac{1}{2} \Sigma_R \sum_{i,j=1}^3 (M_\Sigma)_{ij} \text{Tr} (\Sigma_{iR}^C \Sigma_{jR}) + \text{H.c.} \tag{40}\]
The effective light neutrino mass matrix acquires the same seesaw structure as Eq. (19), with \( M_R \) replaced by \( M_\Sigma \). As a result, the analysis of flavor structures under the \( A_4 \) symmetry is the same as before, and all the conclusions hold. In particular, Table III applies, with the obvious replacement \( \nu_R \rightarrow \Sigma_R \).

IV. SOFTLY BROKEN \( A_4 \) SYMMETRY

We now consider the possibility that the effective operator is not invariant under \( A_4 \). This situation is well behaved as long as we guarantee that the non-invariance comes, at the UV level, from terms that do not spoil renormalizability. We therefore assume that \( A_4 \) is broken softly by dimension-3 terms contributing to the right-handed neutrino mass matrix \( M_R \) \([13]\).

We start again from the cases listed in Table I and analyzed in Sec. III B. Let us consider first case iv) of Table I where \( \nu_R \sim (1,1,1) \), and \( M_R \) is, according to Eq. (22), the most general \( 3 \times 3 \) symmetric complex matrix. Including soft-breaking terms does not alter this feature. Since Eq. (38) leads to two massless eigenvalues in \( m_\nu m_\nu^\dagger \), we conclude that case iv) is not viable, even after the inclusion of soft-breaking terms.

Next, we show that the remaining cases listed in Table II lead to viable fits of the current experimental neutrino data after the inclusion of soft-breaking terms in \( M_R \).

We start with case i). The charged lepton and Dirac neutrino mass matrices are given by Eqs. (25) and (26),
respectively. For simplicity, we study the VEV alignment $(1, 1, 1)$—the results for $(\pm 1, \eta, \eta^*)$ will be equivalent. We can always change the basis of $L_L$, corresponding to multiplying the matrices $m_\nu$ and $m_D$ on the left by the same unitary matrix. In this case, the matrices $m_\nu^L m_\nu^L$ and $m_D^L m_D^L$ are diagonalized by the same matrix,

$$V_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}. \quad (41)$$

Indeed, multiplying the mass matrices on the left by $V_\omega^L$, we obtain in the new basis

$$m_\ell = \sqrt{3} \text{diag} (a e^{i\alpha}, b e^{i\beta}, c e^{i\gamma}),$$

$$m_D = D_D V_R^\dagger,$$  

where

$$V_R^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega & 1 \\ \omega^2 & \omega^2 & 1 \end{pmatrix}, \quad (43)$$

$$D_D = \text{diag} (d e^{i\delta} + f e^{i\epsilon}, d e^{i\delta} + \omega^2 f e^{i\epsilon}, d e^{i\delta} + \omega f e^{i\epsilon}).$$

The PMNS mixing matrix and the light neutrino masses are then obtained from the diagonalization of $m_\nu$, [cf. Eqs. (19) and (21)].

We have randomly generated matrices $m_D$ and $M_R$ satisfying the current experimental data, via a procedure described in detail in Appendix B. In particular, we define in Eq. (24) a figure of merit $\sigma$ probing how much the matrix $M_R$ differs from its form in the exact $A_4$ limit, given in the present case by Eq. (24). Values of $\sigma \sim 1$ correspond to soft-breaking terms of the order of the $A_4$-symmetric terms. Smaller values of $\sigma$ correspond to cases where the terms that break the symmetry are perturbative; that is, they are about 1 order of magnitude smaller than the terms which preserve $A_4$.

Our aim is to show that there exist viable solutions where the deviations from Eq. (24) are perturbative. This becomes clear from Fig. 1 which shows $\sigma$ as a function of $m_1$ (in eV). We notice several features: a) one can produce fits with values of $m_1$ in almost the whole range attempted (from 0 to 0.2 eV), except for very small values of $m_1$; b) solutions with $\sigma < 0.2$ can be found, meaning that the soft-breaking terms are perturbative; c) smaller values of $\sigma$ tend to prefer values for $m_1$ around $0.05 \text{ eV}$.

In our simulation, smaller values of $\sigma$ have no correlation with the measured quantities—namely, the neutrino mixing angles $\theta_{12}, \theta_{13}, \theta_{23}$, and the mass-squared differences $\Delta m^2_{21} = m^2_2 - m^2_1$, or $\Delta m^2_{23} = m^2_3 - m^2_2$. Although $\sigma < 0.2$ implies values of the Dirac phase $\delta_D$ close to $\pm \pi/2$, all values of $\delta_D$ are possible if one allows $\sigma < 1$. In contrast, even a loose cut of $\sigma < 1$, forces the Majorana phases $\alpha_M \sim 0$ and $\beta_M \sim \pi$ (recall that the phases are defined mod $2\pi$). This is illustrated in Fig. 2.

Next, we consider case ii), where $L_L \sim (1, 1', 1'')$, $\nu_R \sim 3$, $\ell_R \sim 3$, so that Eqs. (29) and (30) hold for the charged lepton mass matrix $m_\ell$ and the Dirac neutrino mass matrix $m_D$, respectively. We choose the VEV alignment $(1, 1, 1)$. This case is interesting because $m_\ell$ is diagonalized exclusively through a unitary transformation on the right-handed fields $\ell_R$, implying that the PMNS matrix arises exclusively from the diagonalization of $m_\nu$. The eigenvalues of $m_\ell^L m_\ell^L$ from Eq. (29) are $3a^2$, $3b^2$, and $3c^2$. Thus, we take $a = m_e/\sqrt{3}$, $b = m_\mu/\sqrt{3}$, and $c = m_\tau/\sqrt{3}$, and consider $\alpha = \beta = \gamma = 0$.

We will now assume that $A_4$ is softly broken in the right-handed neutrino sector, such that the form of $M_R$ in Eq. (24) is altered. As before, we follow the fit procedure described in Appendix B. Figure 3 shows $\sigma$ as a function of $s^2_{23} = \sin^2 \theta_{23}$. We notice that all values of $s^2_{23}$ are possible, but that smaller values of $\sigma$ show some preference for smaller values of $s^2_{23}$. There is no such correlation with $\theta_{12}, \theta_{13}, m_1, \Delta m^2_{21}, \text{ or } \Delta m^2_{31}$. The
CP-violating phases $\delta_D$, $\alpha_M$, and $\beta_M$ exhibit the same behavior as in case i).

It is interesting to look at a numerical example in detail. We consider the symmetric matrix $M_R = M Y_R$, with $M = 10^{14}$ GeV and
\[
(Y_R)_{11} = 0.6622 + 7.7769 i,
(Y_R)_{12} = -0.4304 + 0.5404 i,
(Y_R)_{13} = -0.0490 - 0.4532 i,
(Y_R)_{22} = 5.0726 + 5.9385 i,
(Y_R)_{23} = 0.1819 + 0.1479 i,
(Y_R)_{33} = 6.5927 + 4.2281 i.
\]
For use in the matrix $m_D$ of Eq. (30), we take
\[
d e^{i \delta} = 0.7663 + 0.2958 i,
fe^{i \beta} = 0.7516 - 0.0537 i,
g e^{i \gamma} = -0.7477 - 0.5013 i,
\]
and $v_1 = v_2 = v_3 = 246$ GeV. Upon diagonalization of the neutrino mass matrix $m_\nu$ given by Eq. (19), we find a nearly degenerate normal neutrino mass spectrum with $m_1 \approx 0.1545$ eV, $\Delta m^2_{21} = 7.46 \times 10^{-5}$ eV$^2$ and $\Delta m^2_{31} = 2.55 \times 10^{-3}$ eV$^2$, which are well within the 1σ ranges given in Ref. [10]. Next, we equate the diagonalizing matrix $U$ [see Eq. (21)] with the PMNS neutrino mixing matrix written in the standard PDG form [3]. This leads to $\sin^2 \theta_{12} = 0.336$, $\sin^2 \theta_{13} = 0.613$, and $\sin^2 \theta_{13} = 0.0247$. According to Ref. [10], the first value is close to its 1σ upper bound, while the other two coincide with the best-fit values. Finally, the as-yet-unmeasured CP-violating phases turn out to be $\delta = 0.53\pi$, $\alpha_M = 0.28\pi$, and $\beta_M = 1.08\pi$.

In the above numerical example, the magnitudes of the entries in $M_R$ are
\[
|M_R| = 7.83 \times 10^{14} \text{GeV} \times \begin{pmatrix}
0.997 & 0.088 & 0.058 \\
0.088 & 0.997 & 0.030 \\
0.058 & 0.030 & 1.000
\end{pmatrix}. \quad (46)
\]
This shows that the current neutrino oscillation data can be easily fitted in flavor models of the type discussed in this article, if the $A_4$ symmetry is softly broken in the heavy Majorana neutrino mass matrix $M_R$ by coefficients that deviate perturbatively (of the order of 10% or less) from the leading $A_4$-symmetric terms. Notice also that in the above example a normal (quasidegenerate) neutrino mass spectrum is obtained. Numerical examples that lead to an inverted neutrino mass hierarchy, and simultaneously to low-energy neutrino parameters in agreement with the present 1σ ranges, can be equally constructed.

Finally, we turn to case iii), where $L_L \sim 3$, $\nu_R \sim (1, 1', 1''')$, $\ell_R \sim (1, 1', 1''')$. The charged lepton mass matrix is given in Eq. (25), while the Dirac neutrino mass matrix $m_D$ is given by Eq. (32). As before, we concentrate on the VEV $(1, 1, 1)$. Using Eq. (41), we multiply the matrices $m_\ell$ and $m_D$ by $V_D^\dagger$, obtaining, in the new basis,
\[
m_\ell = \sqrt{3} \text{ diag } (a e^{i \alpha}, b e^{i \beta}, c e^{i \gamma}),
m_D = \sqrt{3} \text{ diag } (d e^{i \delta}, f e^{i \epsilon}, g e^{i \xi}). \quad (47)
\]
The numerical results for this case exhibit many similarities with case ii). In particular, there are no fundamental constraints on the observables, except for $\delta_D \sim \pm \pi/2$, $\alpha_M \sim 0$, and $\beta_M \sim \pi$.

V. CONCLUSIONS

In this paper we have studied the possibility of generating the neutrino masses and mixing in the context of models with three scalar doublets in the triplet representation of the $A_4$ group and three lepton families. We have shown that none of the possible VEV alignments that correspond to a global minimum of the scalar potential yields phenomenologically viable charged lepton and neutrino mass matrices. In particular, there is no representation assignment that leads to a dimension-5 neutrino mass operator consistent with the present oscillation data. This in turn implies that, in this minimal $A_4$ construction, the canonical type-I (type-III) seesaw mechanism is not consistent with experiment. Notice that, from the point of view of the low-energy effective operator, this conclusion holds for any number of right-handed singlet (triplet) neutrinos, since the dimension-5 operator is the same regardless of the number of heavy fields. Furthermore, since $A_4$ is a subgroup of $S_4$, our conclusions also remain valid in flavor models based on the latter group.

In the context of a type-I seesaw mechanism with three heavy right-handed neutrinos, we have analyzed in detail what happens when the $A_4$-symmetric Lagrangian is enlarged by adding soft breaking, through dimension-3 right-handed neutrino mass terms. We find three cases where this framework can be implemented in good agreement with neutrino oscillation data. We have also
pointed out that this is possible perturbatively, i.e., by keeping the soft-breaking terms much smaller than the $A_4$-symmetric terms of the right-handed neutrino mass matrix.

At this point, it is worthwhile to comment on other possibilities considered in the literature. Several studies propose various renormalizable extensions of the $A_4$ three-Higgs-doublet model, including always additional fields as well as new symmetries. For example, Ma and Rajasekaran [17] discuss a very simple extension, where an additional scalar doublet $\eta$ is added to case i). This scalar will be the one responsible for the Dirac neutrino mass term. Therefore, an additional $Z_2$ is added in order to forbid $\Phi$ and $\eta$ to interchange sectors. This model still needs some soft-breaking terms to split the degeneracy in the light neutrino spectrum. There are also models using two scalar $A_4$ triplets, one for each sector. In most models these are flavon fields, which lead to a non-renormalizable theory [18, 19]. Renormalizable models with two scalar triplets of $A_4$ will lead to a six-Higgs-doublet model, with a large increase in the number of parameters. In Ref. [20] a case iii) is studied, but with a Higgs triplet for each sector, and with an additional $Z_2$ symmetry. However, the vacuum alignments utilized — $(v_1, v_2, v_3)$ and $v(1, 1, -2)$ — are not shown to be the absolute minimum. To our knowledge, the work is the first to fully study all possibilities consistent with one Higgs $A_4$ triplet, for which the global minima of the scalar potential have been recently identified.

The analysis performed in this paper dealt with the most general case of three-Higgs-doublet models with the scalars in a triplet of $A_4$. This study can serve as a starting point for more elaborate models. Nevertheless, we emphasize that the scalar potential in such new extensions has to be fully analyzed, in order to ensure that the vacua utilized can indeed be global minima. In general, this is not a trivial task.

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Appendix A: Couplings of $L_LL_L$ when $L_L$ is in a singlet representation of $A_4$

Let us consider the group structure of the combination $L_LL_L$, when $L_L$ is an $A_4$ singlet. We concentrate on the group constraints, ignoring any spinor or $SU(2)_L$ characteristics. We also note that, in all cases of interest to us, the $L_LL_L$ coupling matrix must be symmetric. Disregarding irrelevant permutations, we must consider the cases

\[
(1, 1, 1), \quad (1, 1, 1'), \\
(1, 1', 1'), \quad (1, 1, 1''), \\
(1, 1'', 1''), \quad (1, 1', 1''), \\
(1', 1', 1'), \quad (1', 1', 1''), \\
(1', 1'', 1''), \quad (1'', 1', 1'').
\]  

(A1)

We can combine the couplings of $L_LL_L$ with some other group structure in a $3 \times 3$ matrix. To explain our notation, we will use the example of $L_L \sim (1, 1, 1)$. We construct the matrix of all field products

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
\]

(A2)

where a matrix element 1 means that there is at that matrix position an arbitrary complex entry, if we are coupling $L_LL_L$ to some other group structure transforming like 1 of $A_4$, and zero otherwise. For example, the scalar combination $(\Phi\Phi) \sim 1$ has couplings to all bilinears of $L_LL_L$ when $L_L \sim (1, 1, 1)$, while a $(\Phi\Phi) \sim 1'$ will couple to none.

As a further example, consider $L_L \sim (1, 1, 1')$. The corresponding matrix is

\[
\begin{pmatrix}
1 & 1 & 1' \\
1 & 1 & 1' \\
1' & 1' & 1''
\end{pmatrix}.
\]

(A3)

This means that a $(\Phi\Phi) \sim 1$ will only introduce couplings in the upper-left $2 \times 2$ submatrix, and a $(\Phi\Phi) \sim 1''$ will only have couplings to $(L_LL_3 + L_3L_4)$ with $i = 1, 2$, while $(\Phi\Phi) \sim 1'$ would only couple to $L_3L_3$. For simplicity, we write $L_L = (L_1, L_2, L_3)$. We also recall that $1' \otimes 1'' = 1$.

The remaining possibilities in Eq. (A1) lead to

\[
\begin{pmatrix}
1 & 1' & 1' \\
1' & 1'' & 1'' \\
1'' & 1'' & 1''
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 1'' \\
1'' & 1'' & 1'' \\
1'' & 1'' & 1''
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 1'' \\
1 & 1 & 1'' \\
1 & 1 & 1''
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1'' & 1'' \\
1 & 1'' & 1'' \\
1 & 1'' & 1''
\end{pmatrix}, \quad \begin{pmatrix}
1'' & 1 & 1 \\
1 & 1'' & 1 \\
1 & 1'' & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1' & 1' \\
1 & 1' & 1' \\
1 & 1' & 1'
\end{pmatrix},
\]

(A4) \hspace{1cm} (A5) \hspace{1cm} (A6) \hspace{1cm} (A7)

respectively.
Appendix B: The fit procedure

In this appendix we present the fit procedure adopted in Sec. IV for the cases in which \( A_4 \) follows the Particle Data Group notation \([9]\), and contains the Majorana phases. Recent constraints on the mixing angles and phases of the PMNS matrix can be found in Refs. \([16, 21]\). The Majorana phases \( \alpha_M \) and \( \beta_M \) are unconstrained, while interpretations differ about constraints on the Dirac phase \( \delta_D \). We leave \( \delta_D \) free, and follow Ref. \([16]\) for the ranges of the other parameters.

In the cases of interest to us, it is easy to diagonalize \( m_\ell \). In the basis where \( m_\ell \) is diagonal, Eqs. \((20)\) and \((21)\) hold. Using the type-I seesaw relation \((19)\) and Eq. \((21)\), we find

\[
M_R = -m_D^T U \text{diag}(m_1^{-1}, m_2^{-1}, m_3^{-1}) U^T m_D. \tag{B6}
\]

We perform our fits in the following fashion:

1. We use random values for \( s_{12}, s_{13}, s_{23}, \Delta m^2_{21}, \) and \( \Delta m^2_{31}, \) within the 1\( \sigma \) intervals found in Ref. \([16]\), corresponding to the normal hierarchy.
2. We generate random values for \( \delta_D, \alpha_M, \beta_M, \) and for \( m_1, \) keeping the latter between 0 and 0.2 eV.
3. We generate random values for the theoretical parameters in the matrix \( m_D, \) written in the basis where \( m_\ell \) is diagonal.
4. We obtain \( M_R \) from Eq. \((B6)\)—by construction, \( m_D \) and \( M_R \) are consistent with the experimental observations.
5. We define a figure of merit \( \sigma, \) which measures the difference between the form of \( M_R \) obtained from Eq. \((B6)\) and that predicted in Eqs. \((23)\) or \((24)\) when the symmetry is exact.

6. To keep the soft breaking perturbative, only cases where \( \sigma \) is smaller than some reference value are maintained.

The only cases where soft breaking of \( A_4 \) allows the type-I seesaw mechanism to fit the experimental data are cases i), ii), and iii) in Table \([11]\). In the first two cases, \( v_R \sim 3, \) and \( M_R \) has, in the exact \( A_4 \) limit, the form in Eq. \((24)\). We wish to keep deviations from this form somewhat small. To be precise, we define

\[
\sigma = \sum_{i,j=1}^3 |\Sigma_{ij}|^2, \tag{B7}
\]

with

\[
\Sigma = \frac{M_R}{|(M_R)_{11}|} - \text{diag}(1, 1, 1), \tag{B8}
\]

where \( M_R \) comes from Eq. \((B6)\) in the fit explained above. We only keep points where \( \sigma < 1. \)

In case iii), \( v_R \sim (1, 1', 1''), \) and \( M_R \) has the form in Eq. \((23)\) in the exact \( A_4 \) limit. For this case we define

\[
\sigma = \frac{|(M_R)_{12}|^2 + |(M_R)_{13}|^2 + |(M_R)_{22}|^2 + |(M_R)_{33}|^2}{\min \left( |(M_R)_{11}|^2, |(M_R)_{23}|^2 \right)} \tag{B9}
\]

and only keep points where \( \sigma < 1. \)

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