The theta divisor of the bidegree (2,2) threefold in $\mathbb{P}^2 \times \mathbb{P}^2$

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0. Introduction

(0.1). In this paper we apply a new approach to the study of the theta divisor of a standard conic bundle. As an example we examine the Verra threefold $T = T(2,2)$ – the divisor of bidegree $(2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. The threefold $T$ deserves a special attention because of the recent observation of A.Verra (see [Ve]) that the existence of two conic bundle structures on $T$ implies a new counterexample to the Torelli theorem for Prym varieties. Moreover, this counterexample is not related to the 4-gonal correspondence (see [Do]) of Donagi (which had covered all the known non-trivial counterexamples).

(0.2). Let $p : T \to \mathbb{P}^2$ be any of the two projections which make $T$ a standard conic bundle over $\mathbb{P}^2$. In section 3 we show that the existence of $p$ implies the existence of a special family $\mathcal{C}_\theta$ of curves on $T$, which is mapped (via the Abel-Jacobi map) onto a copy of the theta divisor $\Theta(T)$ (see Theorem 4.1). The general curve $C \in \mathcal{C}_\theta$ is an elliptic curve of bidegree $(3,6)$ which lies in a hyperplane section of $T$. By construction, the curves of $\mathcal{C}_\theta$ parameterize the minimal sections (resp. – the maximal subbundles of rank 1) of a special family of rank 2 vector bundles (resp. – ruled surfaces) over the space of plane cubics (see section 2, (3.1), [LN], [Se]). It turns out that $\mathcal{C}_\theta$ is generically a 2-sheeted covering of the family of effective divisors $\text{Supp } \Theta$ related to the Wirtinger description of the theta divisor (see sect. 2,3,4, and (1.2.2)). The last is used in the proof of Theorem 4.1: The Abel-Jacobi image of $\mathcal{C}_\theta$ is a copy of $\Theta(T)$. As a second application of the connection between $\mathcal{C}_\theta$ and $\Theta$ we prove that the general hyperplane section of $T(2,2)$ is a K3 surface, which is a double covering of $\mathbb{P}^2$ branched over a sextic with a vanishing theta null (see (5.2.1) and (5.3)).

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In section 6, we separate a 3-dimensional component $Z \subset \text{Sing} \, \Theta$ as an Abel-Jacobi image of the 6-dimensional family of elliptic sextics of bidegree $(3,3)$ on $T$. The connection between the geometry of $T$ and $\text{Sing} \, \Theta$, based on the study of the last family of curves, is used to prove the Torelli theorem for $T(2,2)$: The threefold $T \subset \mathbb{P}^8$ coincides with the intersection of the tangent cones of $\Theta$ in the points of $Z$ (see Theorem 6.6). Note that the Torelli theorem, stated in this form, is not a direct consequence of general facts about Prym varieties: one has also to see that the projectivized tangent cone $\text{Cone}_z$ ($z$ – a general point of $Z$) does not belong to the “trivial” component $D_6(W)$ of the determinantal locus $D_6(T)$ – see (6.3.3), (6.5.1), and the proof of Theorem (6.6).

In his paper [Ve], A.Verra proves that the two discriminant pairs, which correspond to the two conic bundle structures on $T(2,2)$, fits into the classical Dixon construction. In section 7 we show that, in case of a nodal $T(2,2)$, this Dixon correspondence can be represented as a composition of two 4-gonal correspondences of Donagi (see Corollary 7.3).

1. Preliminaries

(1.1). The bidegree $(2,2)$ divisor $T$ (see [Ve]).

(1.1.1). Let $\text{seg} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8$ be the Segre embedding, and let $W \subset \mathbb{P}^8$ be the image of $\text{seg}$. Let $p_1 : W \to \mathbb{P}^2$ and $p_2 : W \to \mathbb{P}^2$ be the canonical projections. Denote by $\mathcal{O}_W(m, n)$ the sheaf $p_1^*\mathcal{O}_{\mathbb{P}^2}(m) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(n)$, $m, n$ – integers.

(1.1.2). Let $T = T(2,2) \in | \mathcal{O}_W(2,2) |$, be a bidegree $(2,2)$ divisor on $W$, and let $p_i : T \to \mathbb{P}^2$ be the restrictions of $p_i$ on $T$.

(1.1.3). Let, moreover, $T = T(2,2)$ be smooth. Then $p_i$ defines a standard conic bundle structure on $T$, $i = 1, 2$. Let $i \in \{1, 2\}$ be fixed, and let $\Delta_i = \{ x \in \mathbb{P}^2 : \text{sing} \, p^{-1}(x) \neq \emptyset \} = \{ x \in \mathbb{P}^2 : p^{-1}(x) = l + l \text{ is a plane conic of rank 2 } \}$ be the discriminant curve of $p_i$. It is not hard to see that $\Delta_i$ is a smooth plane sextic ($T$ is smooth). Denote by $\tilde{\Delta}_i = \{ l - \text{ a line in } T : \exists x \in \Delta, s.t. l \text{ is a component of } p^{-1}(x) \}$ the double discriminant curve of $p_i$. With a probable abuse of the notation, we denote by $p_i : \tilde{\Delta}_i \to \Delta_i$ also the induced (unbranched) double covering. Let $\eta_i \in \text{Pic}^2 \Delta_i$ be the torsion-2 sheaf, which defines $p_i$.

(1.1.4). Let $J(T)$ be the intermediate jacobian of $T$, and let $P_i = P(\Delta_i, \eta_i)$ be the Prym
variety of \((\Delta_i, \eta_i)\). It is well-known, that \(J(T)\) and \(P_i\) are isomorphic as principally polarized abelian varieties (see[B1]).

In particular, \(\dim J(T) = \dim P_i = 9\). It follows immediately that \(P(\Delta_1, \eta_1)\) and \(P(\Delta_2, \eta_2)\) are isomorphic as p.p.a.v..

\(1.1.5\). In [Ve], A.Verra proves that the discriminant pairs \((\Delta_1, \eta_1)\) and \((\Delta_2, \eta_2)\) correspond to each other by the classical Dixon construction. Moreover, let \(P_6 = \{ (\Delta, \eta) : \Delta \text{ is a smooth plane curve of degree 6, and } \eta \text{ is a torsion-2 sheaf in } \text{Pic}(\Delta) \}\), and let \(p_6 : P_6 \rightarrow \mathcal{A}_9\) be the Prym map. Then \(\text{deg } p_6 = 2\), and \(p_6\) is branched along the locus of intermediate Jacobians of nodal quartic double solids, see [Ve]. The general fiber of \(p_6\) consists of a couple of pairs \(((\Delta_1, \eta_1), (\Delta_2, \eta_2))\), which arises from a bidegree \((2,2)\) divisor \(T\).

\(1.1.6\). We call the smooth bidegree \((2,2)\) divisor \(T \in W\) the Verra threefold.

\(1.2\). The intermediate jacobian \(J(T)\) as a Prym variety.

\(1.2.1\). Let \(\Theta\) be the divisor of the principal polarization (the theta divisor) of \(J(T)\). It follows from the preceding that we can identify \(\Theta\) and the theta divisor of the Prym variety \(P_i\), \(i = 1,2\). Then, because of the Wirtinger description of Prym varieties, we can describe \(J = J(T), \Theta, \text{ etc., only in terms of } \tilde{\Delta}_i \text{ and } \Delta_i\). As a direct corollary we obtain:

\(1.2.2\). Let \(p_i\) be fixed. Then:

The jacobian \(J(T)\) is isomorphic to \(P(\tilde{\Delta}_i, \Delta_i) = \{ \mathcal{L} \in \text{Pic}^{18}\tilde{\Delta}_i : Nm\mathcal{L} = \omega_{\Delta_i}, \text{ and } h^0(\mathcal{L}) \text{ even } \}\).

\(\Theta(T) \cong \Theta_i = \{ \mathcal{L} \in J(T) : h^0(\mathcal{L}) \geq 2 \}\).

There exists a subset in \(\text{Sing } \Theta\), which is isomorphic to the set of stable singularities of \(\Theta\), with respect to \(p_i\), i.e.:

\[\text{Sing}^\text{st}_i(\Theta) = \{ \mathcal{L} \in \Theta : h^0(\mathcal{L}) \geq 4 \}\]

similarly – for the exceptional singularities of \(\Theta\), w.r. to \(p_i\).

In section 6, we shall describe a component \(Z\) of \(\text{Sing } \Theta\), the points of which are stable w.r.to both \(p_1\) and \(p_2\).

\(1.3\). Minimal sections of ruled surfaces over elliptic curves

\(1.3.0\). Here we collect some facts about ruled surfaces (esp. – on elliptic curves), which will be used in section 2, see [H, ch.V.2], [LN], [Se].
(1.3.1). Let $C$ be a smooth curve. By definition, a ruled surface $p : S \to C$ is a surface which can be represented in the form $S = \mathbb{P}_C(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank 2 (a rank 2 vector bundle) over $C$. The representation $S = \mathbb{P}_C(\mathcal{E})$ is unique up to a multiplication by an invertible sheaf: $\mathbb{P}_C(\mathcal{E}) \sim \mathbb{P}_C(\mathcal{E} \otimes \mathcal{L})$, $\mathcal{L} \in \text{Pic}(C)$.

The bundle $\mathcal{E}$ is called normalized if $h^0(\mathcal{E}) \neq 0$, but $h^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for any invertible $\mathcal{L}$ such that $\deg \mathcal{L} < 0$.

(1.3.2). Obviously, any ruled surface $S$ has a representation $S = \mathbb{P}_C(\mathcal{E})$, for some normalized $\mathcal{E}$. Anyway, such a representation is, in general, far from unique (see e.g. [LN, Cor.3.2], or (1.3.4) – in case $g(C) = 1$).

Let $\mathcal{E}$ be normalized, and let $C_o \in |\mathcal{O}_{\mathbb{P}(\mathcal{E})/C}(1)|$ be a tautological section for $\mathcal{E}$. The invariant property of such a $C_o$ is that $C_o$ is a section of $p : S \to C$, for which the number $-e(C) = (C.C)_S$, $C$ – a section of $S$ is minimal, i.e. $C_o$ is a minimal section of $S$. The number $e = e(S) = -(C_o.C_o)$ is an invariant of $S$.

(1.3.3). The surface $S = \mathbb{P}(\mathcal{E})$ is called decomposable if $\mathcal{E}$ is decomposable (see e.g. [H, ch.V.2]). Otherwise, $S$ is called indecomposable (ibid).

(1.3.4). The cardinality of the set of minimal sections of $S$ closely depends on the decomposability of $S$, and on (the parity) of the invariant $e = e(S)$ (see [LN, Cor.3.2]).

In section 3, we shall use the description of these sets only in case $g(C) = 1$. All of the following can be found in [H, ch.V.2]:

(*) Minimal sections of a ruled surface over an elliptic curve.

Let $C$ be an elliptic curve, and let $S$ be a ruled surface on $C$. Then, one of the following alternatives is valid:

(1). $S$ is decomposable, $e = e(S) > 0$. The normalized sheaf $\mathcal{E}$ for $S$ is unique, and the minimal section $C_o$ is unique.

(2). $S$ is decomposable, $e = e(S) = 0$. In this case $S = \mathbb{P}_C(\mathcal{O} \oplus \varepsilon)$, $\deg(\varepsilon) = 0$, and either:

(a). $\varepsilon = \mathcal{O}_C$, $S = C \times \mathbb{P}^1$, and the set of minimal sections of $S$ is parameterized by the points of $\mathbb{P}^1$, or:

(b). $\varepsilon \neq \mathcal{O}_C$. Then the normalized sheaf $\mathcal{E}$ can be chosen in two ways: $\mathcal{E}^+ = \mathcal{E}$ and $\mathcal{E}^- = \mathbb{P}(\mathcal{O} \oplus (-\varepsilon))$, where $-\varepsilon = \varepsilon^{\otimes -1}$. Correspondingly, there are exactly two minimal
sections of $S$: $C^+$ and $C^-$ (each – the unique tautological section of the corresponding normalized bundle).

(3). $S$ is the unique indecomposable ruled surface, s.t. $e(S) = 0$. Then the normalized sheaf for $S$ is unique, and the corresponding minimal section is unique.

(4). $S$ is the unique indecomposable ruled surface, s.t. $e(S) = -1$. Then, the set

$\{ \mathcal{E} - normalized : S = P(\mathcal{E}) \}$ is parameterized by the points of the elliptic curve $C$. For any such $\mathcal{E}$, the minimal section $C_0(\mathcal{E})$ is unique.

2. Minimal sections of the canonical conic bundle surfaces

(2.0). Everywhere in this section the conic bundle structure $p : T \to \mathbb{P}^2$ is fixed; we let $p = p_1, \Delta = \Delta_1, \eta = \eta_1$, etc.

(2.1). The sets $\text{Supp} \Theta$ and $\text{Supp} P^-$.

(2.1.1). Let $p = p_1 : T \to \mathbb{P}^2$, etc., be as above. Let $Nm : \text{Pic}^{18} \tilde{\Delta} \to \text{Pic}^{18} \Delta$ be the norm map (see [ACGH, app.C]) Then $Nm^{-1}(\omega_\Delta)$ splits into two components:

$P^+ = \{ \mathcal{L} \in Nm^{-1}(\omega_\Delta) : h^0(\mathcal{L}) \text{ even} \}$ and $P^- = \text{the same, but } h^0(\mathcal{L}) \text{ odd.}$

(2.1.2). $\text{Supp} \Theta$ and $\text{Supp}(P^-)$.

The general sheaf $\mathcal{L} \in P^+$ is non-effective, i.e. the linear system of effective divisors $| \mathcal{L} |$ is empty. However, the subset of the effective sheaves $\mathcal{L} \in P^+$ is exactly the theta divisor $\Theta$ (see (1.2.2)). This gives a reason to define the set

$\text{Supp} \Theta := \{ L \in | \mathcal{L} | : \mathcal{L} \in \Theta \}$,

i.e., $\text{Supp} \Theta$ is the set of all effective divisors in the linear systems of the sheaves $\mathcal{L} \in \Theta$.

Similarly:

$\text{Supp}(P^-) := \{ L \in | \mathcal{L} | : \mathcal{L} \in P^- \}$ (all the sheaves $\mathcal{L} \in P^-$ are effective).

(2.1.3). The maps $p_+ : \text{Supp} \Theta \to | \mathcal{O}_{\mathbb{P}^2}(3) |$ and $p_- : \text{Supp}(P^-) \to | \mathcal{O}_{\mathbb{P}^2}(3) |$.

The set $\Theta \cup P^-$ coincides with the set of all “effective” sheaves in the preimage $Nm^{-1}(\omega_\Delta)$. Moreover, on the level of effective divisors, the map $Nm$ coincides with the usual projection $p_* : \text{Symm}^{18} \tilde{\Delta} \to \text{Symm}^{18} \Delta$. In particular, if $\mathcal{L} \in Nm^{-1}\omega$ is effective and $L \in | \mathcal{L} |$, then $\mathcal{O}(p_*L) = NmL = \omega_\Delta = \mathcal{O}_\Delta(3)$. Since $\text{deg} \Delta = 6$, the linear system $| \mathcal{O}_\Delta(3) |$ is isomorphic to $| \mathcal{O}_{\mathbb{P}^2}(3) |$. In particular, the effective divisor $p_*L \in \text{Symm}^{18} \Delta$ is a scheme intersection of
\[ \Delta \text{ and (a unique) plane cubic curve } C(L) = p_*(L). \] In particular, after composing with the corresponding restriction maps, the map \( p_* \) defines the maps:

\[
\begin{align*}
p^+_\ast &: \text{Supp } \Theta \to |O_{\mathbb{P}^2}(3)| \quad (\text{returning the set of all the plane cubics}) \\
p^-\ast &: \text{Supp } (P^-) \to |O_{\mathbb{P}^2}(3)|.
\end{align*}
\]

It is not hard to see (see e.g. [Sh, Lemma 3.20]) that:

1. the maps \( p^+_\ast \) and \( p^-\ast \) are surjective;
2. the general fibers of \( p^+_\ast \) and \( p^-\ast \) are finite.

In (2.2), we shall describe the general fibers of these two maps.

**2.2.** The canonical conic bundle surface \( S(C) \) and the preimage \( p^{-1}_*(C) \).

2.2.1. Let \( C \) be a sufficiently general plane cubic curve. In particular, \( C \) can be supposed to be smooth, and intersecting the discriminant sextic \( \Delta \) in 18 disjoint points \( x_1, \ldots, x_{18} \).

The non-minimal ruled surface \( S(C) = p^{-1}(C) \subset T = T(2, 2) \) is a standard conic bundle over the (elliptic) curve \( C \); the degenerate fibers of \( p : S(C) \to C \) are \( f_i = p^{-1}(x_i) = l_i + 7_i, i = 1, \ldots, 18 \).

We call a surface \( p^{-1}(C) \subset T \), \( C \) — any plane cubic (resp. \( C \) — a general plane cubic), a canonical conic bundle surface on \( T \) (resp. — a general c.c.b.s.) — w.r.t. \( p = p_1 \).

2.2.2. The set \( \Sigma(C) \).

Let the cubic \( C \) be as in (2.2.1), and let

\[ \Sigma(C) = \{ \sigma : \bigcup \{x_i\} \to \bigcup \{l_i, 7_i\} : \{\sigma(x_i) \in \{l_i, 7_i\}, i = 1, \ldots, 18\} \}
\]

be the set of “choice” maps for \( C \). We can define, in an obvious way, the two-argument signature function:

\[ sgn : \Sigma(C) \times \Sigma(C) \to \{+1, -1\} \text{ as follows:} \]

\[ sgn(\sigma', \sigma'') = +1, \text{ if } \#(\text{Image}(\sigma') \cap \text{Image}(\sigma'')) \in 2\mathbb{Z}, \]

otherwise \( sgn(\sigma', \sigma'') = -1. \)

2.2.3. The map \( L : \Sigma(C) \to p^{-1}_*(C) \).

Let \( \sigma \in \Sigma(C) \). The map \( \sigma \) defines the effective divisor \( L(\sigma) = \sigma(x_1) + \ldots + \sigma(x_{18}) \). Clearly

\[ L(\sigma) \in p^{-1}_*(C) = \text{Supp } \Theta \cup \text{Supp } P^- \.]
Lemma. Let $C$ be as above. Then

1. The preimage $p^{-1}_*(C)$ coincides with the union of the disjoint sets (each of cardinality $2^{17}$):

   \[ \Sigma_\Theta(C) = \{ \sigma \in \Sigma(C) : L(\sigma) \in \text{Supp } \Theta \} \] and

   \[ \Sigma_{P^-}(C) = \{ \sigma \in \Sigma(C) : L(\sigma) \in \text{Supp } P^- \} \].

2. $\text{sgn}(\sigma', \sigma'') = +1$ iff both $\sigma'$ and $\sigma''$ belong to one of these two sets; otherwise $\text{sgn} = -1$.

Proof. Let $L(\sigma')$ and $L(\sigma'')$ be two elements of $p^{-1}_*(C)$. The divisors $L(\sigma')$ and $L(\sigma'')$ are obtained from each other by a finite number of replacements of the type: $L \mapsto L + l - l$, where $l + l = p^{-1}(x)$ for some $x \in \Delta$. In this case $x \in \{ x_1, ..., x_{18} \}$, and $L(\sigma')$ and $L(\sigma'')$ are effective. Moreover, $L(\sigma')$ and $L(\sigma'')$ can be regarded as general elements of $\text{Supp } \Theta \cup \text{Supp } P^-$ (the cubic $C$ is general). Therefore, $h^0(L(\sigma'))$ and $h^0(L(\sigma''))$ can be only 1 or 2 (see e.g. [W]).

Now, the lemma is a direct consequence of the following statement:

(2.2.4)(*). Let $\tilde{\Delta}$ be a smooth curve with an involution $l \leftrightarrow \overline{l}$ without fixed points. Let $L$ be an effective (see (2.1)) invertible sheaf on $\tilde{\Delta}$, and let $l \in \tilde{\Delta}$. Then

\[ h^0(L) - h^0(L + l - \overline{l}) \in \{ +1, -1 \} \] (see e.g. [Sh, 3.14], where (*) has been proved under more general conditions). \textbf{q.e.d.}

(2.2.5). Let $U = \{ C - \text{a smooth plane cubic} : C \cap \Delta = 18 \text{ disjoint points} \}$, let $C \in U$, and let $\text{Supp } \Theta(C) = \{ L(\sigma) : \sigma \in \Sigma_\Theta(C) \}$. Let $\text{Supp } \Theta^U = \bigcup \{ \text{Supp } \Theta(C) : C \in U \}$. Then the algebraic set $\text{Supp } \Theta \in \text{Symm}^{18}(\tilde{\Delta})$ is, in an obvious way, a closure of the open subset $\text{Supp } \Theta^U$. The same (up-to replacing of the notation) is true also for $\text{Supp}(P^-)$ and $\text{Supp}(P^-)^U$.

(2.3). The global invariants $e(\Theta)$ and $e(P^-)$.

(2.3.1). The maps $\tilde{\sigma} : S(C) \to S(L(\sigma))$.

Let $C \in U$ be as in (2.2.5), and let $\sigma \in \Sigma(C)$ (see (2.2)). Up-to change of the involutive line ($l \leftrightarrow \text{a point of } \tilde{\Delta}$), we may assume that $L(\sigma) = l_1 + ... + l_{18}$. The lines $l_i$ and $\overline{l}_i$ are (-1)-curves on the non-minimal ruled surface $S(C) = p^{-1}(C)$. Therefore, $\sigma$ defines, in a unique way, a morphism $\tilde{\sigma} : S(C) \to S(L(\sigma))$, where $\tilde{\sigma}$ is the blow-down of the 18-tuple...
The invariants $e(\Theta)$ and $e(P^-)$.

Let $e(L(\sigma))$ be the invariant of the ruled surface $p(\sigma) : S(L(\sigma)) \to C$ (see (1.3.2)). This way, we define a map:

\[ \tilde{e} : \text{Supp} \Theta^U \cup \text{Supp}(P^-)^U \to \mathbb{Z}, \quad \tilde{e} : L(\sigma) \mapsto e(L(\sigma)), \] (see (2.2.5)).

It is standard that the map $\tilde{e}$ must take a constant value on some open subset of each of the components of its domain. Therefore, there exists a pair $e(\Theta), e(P^-)$, and a pair of (possibly smaller) open subsets $\text{Supp} \Theta^\text{op}$ and $\text{Supp}(P^-)^\text{op}$, such that $\tilde{e}(L) = e(\Theta)$ for any $L \in \text{Supp} \Theta^\text{op}$, $\tilde{e}(L) = e(P^-)$ for any $L \in \text{Supp}(P^-)^\text{op}$.

We shall find these two numbers.

(2.4). $e(\Theta) = 0, e(P^-) = -1$.

(2.4.1) Lemma. Let $e = e(\Theta)$ and $\varpi = e(P^-)$ be as in (2.3.2). Then $|e - \varpi| = 1$.

Proof. Let $C \in U$ (see (2.2.5)) be such that $L(\sigma) \in \text{Supp} \Theta^\text{op} \cup \text{Supp}(P^-)^\text{op}$, for any $\sigma \in \Sigma(C)$ (see (2.2),(2.3)). In particular, $e(S(L(\sigma))) = e$ for any $L(\sigma) \in \text{Supp} \Theta(C)$, and $e(S(L(\sigma))) = \varpi$ for any $L(\sigma) \in \text{Supp} P^-(C)$. Let $x_i, l_i, \tilde{l}_i$, etc. be as in (2.2),(2.3), and let $\sigma'$ and $\sigma''$ be such that $\sigma'(x_i) = \sigma''(x_i)$ for any $i$, except $i = j, j - \text{fixed}$. Then the minimal models $S' = S(L(\sigma'))$ and $S'' = S(L(\sigma''))$ are obtained from each other by a single elementary transformation $elm$, centered in the point $z_j = l_j \cap \tilde{l}_j \in p^{-1}(x_j)$. Now, (2.4.1) follows from the following:

(*). Sublemma (see [LN, Lemma 4.3], or [Se, Lemma 7]). Let $S' \to C$ and $S'' \to C$ be two ruled surfaces over the smooth base curve $C$, and let $S'' = elm_p(S'')$, where $elm_p$ denotes the elementary transformation of $S' - centered in the point $P \in S'$. Then:

(i). If no minimal section of $S'$ (see (1.3.4)) passes through $P$, then $e(S'') = e(S') - 1$.
(ii). If a minimal section of $S'$ passes through $P$, then $e(S'') = e(S') + 1$.

q.e.d.

(2.4.2) Lemma. $\varpi = -1$.

Proof. Assume the contrary, i.e. $\varpi \geq 0$.

Let $\mathcal{L} \in \Theta$ be general, and let $|\mathcal{L}| = \{L(t), t \in \mathbb{P}^1\}$ be the linear system of $\mathcal{L}$ (see (1.2.2)). Just as in (2.1.3), the pencil $\{L(t)\}$ defines the rational pencil of plane cubics
\( C(t) = C(L(t)) \). Since \( L \in \Theta \) is general, the general curve \( C(t) \) of the pencil \( \{C(t)\} \) is a smooth plane cubic, and the only degenerations of \( \{C(t)\} \) are a finite number of nodal plane cubics. Moreover, there is a finite number of plane cubics \( C(t) \in \{C(t)\} \), which are simply tangent to \( \Delta \). Anyway, there is an open subset \( V_L \subset P^1 \) such that \( L(\sigma) \in (\text{Supp } \Theta)^{op} \cup \text{Supp } (P^-)^{op} \) for any \( t \in V_L \) and for any \( \sigma \in \Sigma(C(t)) \) (see (2.2.2) – (2.2.4)).

Let \( L(t) = l_1(t) + \ldots + l_{18}(t), t \in V_L \), and let \( L_j(t) = L(t) + l_j(t) - l_j(t), j = 1, \ldots, 18 \).

Let \( \tau \) be different from \(-1\). Then \( \tau \geq 0 \), and any of the ruled surfaces \( S(L_j(t)) \) has only a finite number of minimal sections (see (1.3.4)). The case (1.3.4)(2.a) can be excluded, because of considerations involving the supposed general position. We shall prove that:

\textbf{(*) Sublemma.} If the bidegree (2,2) divisor \( T \) is general, then the general \( S(L), L \in \text{Supp } \Theta \cup \text{Supp } P^- \), cannot be of type (1.3.4)(2.a) or of type (1.3.4)(3).

\textbf{Proof.} On the one hand, since the general plane cubic \( C(t) \) is a general plane cubic, it is in a general position w.r. to the discriminant sextic \( \Delta \). The various minimal ruled models of the surface \( S(C) \) are obtained from each other by elementary transformations, related to the 18-tuple of degenerated fibers \( \{l_i + l_i = p^{-1}(x_i), i = 1, \ldots, 18\} \). On the other hand, we can fix for a moment the smooth plane cubic \( C \). Since the general plane sextic can be represented as a discriminant of (one of the conic bundle structures of) some bidegree (2,2) divisor \( T \) (see [Ve]), we can choose the 18-tuple \( \{x_1, \ldots, x_{18}\} \) without any closed restrictions. The rest repeats the proof of [Se, Lemma 12]. \textbf{q.e.d.}

So, the surface \( S(L_j(t)) \) has a finite number (one or two) of minimal sections (By assumption, the case \( \tau = -1 \) has been excluded). In particular, this assumption, together with (*), imply that the general \( S(L), L \in \text{Supp } P^- \) has one (or two) minimal sections. After taking the closure, one can define \( \overline{C} \) to be the family of all these minimal sections (related to the component \( P^- \)). The last and (*) imply (under the assumption \( \tau \geq 0 \) ) that the family \( \overline{C} \) is 9-dimensional. The general curve \( \overline{C} \in \overline{C} \) is mapped, via \( p = p_1 \), isomorphically onto a smooth plane cubic. Therefore, \( \deg(\overline{C}) = (3, d) \), where \( \deg \) is the bidegree map. The straightforward check, based on the normal bundle sequence for the triple \( \overline{C} \subset p^{-1}(p(\overline{C})) \subset T \), imply that the total degree of \( \overline{C} \in (the \ 9-dimensional \ family) \overline{C} \) can be 8 , or 9 (see also [ZR]). Therefore, \( d = 5, or 6 \).

Let \( \overline{C}_L \) be the completion (in \( \overline{C} \)) of the family of minimal sections of the surfaces \( S(L_j(t)) \)
(see above). By definition, the base of this family is an algebraic curve $B$. If the general $S(L), L \in \text{Supp } P^-$ has two (resp one) minimal sections, the base $B = B_L$ is, in a natural way, a 36-sheeted (resp. – a 18-sheeted) covering of the projective line. Let $\mathcal{S}_L$ be the union of the curves $C \in \mathcal{C}_L$. Clearly, $\mathcal{S}_L$ is an effective divisor on $T$ (the curves, which sweep $\mathcal{S}_L$ out, form an 1-dimensional algebraic family parameterized by the algebraic curve $B$).

The family $\mathcal{S}_L$ can be reducible, or not. The irreducible components of this surface correspond to the irreducible components of the base $B = B_L$. Let $B_o$ be one of these irreducible components, let $\mathcal{C}_o \rightarrow B_o$ be the corresponding irreducible family, and let $\mathcal{S}_o$ be the corresponding irreducible component of $\mathcal{S}_L$.

The surface $\mathcal{S}_o$ represents the element

$$cl(\mathcal{S}_o) \in \text{Pic}(T) = \mathbb{Z}.l + \mathbb{Z}.h, \text{ where } l = cl(p_1^* \mathcal{O}_{\mathbb{P}^2}(1)), h = cl(p_2^* \mathcal{O}_{\mathbb{P}^2}(1)), p_1 = p.$$  

Therefore $cl(\mathcal{S}_o) = al + bh$ for some integers $a, b$.

(**). Sublemma. The integers $a$ and $b$ are non-negative.

Proof. Let, for example, $b \leq 0$. Let $f$ be the general fiber of $p = p_1 : T \rightarrow \mathbb{P}^2$. Since $\mathcal{S}_o \in |al + bh|$ is effective and $f$ is not a fixed curve on $T$, the intersection number $(f.\mathcal{S}_o)_T$ must be non-negative. Therefore $0 \leq (f.(al + bh))_T = a(f.l)_T + b(f.h)_T = b(l^2.h)_T = 2b \leq 0$ – contradiction. **q.e.d.**

Let $\mathcal{C}_o = \{\mathcal{C}(\xi) : \xi \in B_o\}$. The curves $\mathcal{C}(\xi)$ belong to the same algebraic family. In particular, the integer $k = (\mathcal{C}(\xi_1).\mathcal{C}(\xi_2))_{\mathcal{S}_o} = (\mathcal{C}(\xi)^2)_{\mathcal{S}_o}$ is a (non-negative) constant, which does not depend on the particular choice of $\xi_1, \xi_2, \xi$. The general curve $\mathcal{C}(\xi) \subset \mathcal{S}_o$ is a smooth elliptic curve of bidegree $(3, d), d \in \{5, 6\}$. The formal adjunction takes place on the divisor $\mathcal{S}_o$ on $T$. In particular, $0 = \text{deg}(K_{\mathcal{C}(\xi)}) = (K_{\mathcal{S}_o} + \mathcal{C}(\xi)).\mathcal{C}(\xi) = ((a - 1)l + (b - 1)h).\mathcal{C}(\xi) + k = 3(a - 1) + d(b - 1) + k$, where $a, b, k \geq 0$ and $d \in \{5, 6\}$.

Case 1. $d = 5$. Then $3a + 5b + k = 8 \Rightarrow a = b = 1, k = 0$.

Case 2. $d = 6$. Then $3a + 6b + k = 9 \Rightarrow a = b = 1, k = 0$.

In both cases $cl(\mathcal{S}_o) = l + h$, i.e. $\mathcal{S}_o$ is a hyperplane section of $T$. Since $\mathcal{S}_o$ is irreducible, it is a surface of type $K3$ on $T$. (In fact, the general choice of the initial sheaf $\mathcal{L} \in \Theta$, and the check of the parameters, imply that $\mathcal{S}_o$ has to be a general hyperplane section of $T$.) The family $\mathcal{C}_o \rightarrow B_o$ is an irreducible family of curves on $\mathcal{S}_o$, and the general member of the family is an elliptic curve. Therefore, $\mathcal{C}_o$ is a rational pencil, i.e. $B_o \cong \mathbb{P}^1$. 

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Let $\overline{C}(\xi)$ be general. Then the plane cubic $C(\xi) = p(\overline{C}(\xi))$ belongs to the set $U$ (see (2.2.5)). In particular, $\overline{C}(\xi)$ is a minimal section of some minimal model of the surface $S(C(\xi)) = p^{-1}(C(\xi))$ (see (2.3.1)). (In fact, the corresponding minimal model must have two minimal sections – see e.g. (*).) Let $\overline{L}(\xi)$ be the corresponding effective divisor on $\tilde{\Delta}$ (ibid.) Clearly, $\overline{L}(\xi) \in \text{Supp} P^-$.

By taking the completion, we obtain the rational family 

\[ \{ \overline{L}(\xi) : \xi \in \mathbb{P}^1 \} \subset \text{Supp} P^- \]

In particular, all the $\overline{L}(\xi)$ belong to a same (nontrivial) linear system. As it follows from the definition, this system must be the linear system of some of the divisors $L_j(t)$ (see the beginning of the proof). Clearly, $L_j(t)$ represents the general element of $\text{Supp}(P^-)$ (the initial sheaf $\mathcal{L}$ is a general element of $\Theta$). This is a contradiction, since the linear system of the general $L \in \text{Supp} P^-$ is trivial (see [W]). Therefore, $\overline{e} = e(P^-) = -1$. Lemma (2.4.2) is proved.

(2.4.3) Corollary. $e = e(\Theta) = 0$, $\overline{e} = e(P^-) = -1$.

3. The family of minimal sections $\mathcal{C}_\theta$.

(3.1). Definition of $\mathcal{C}_\theta$.

Let $L \in \text{Supp} \Theta$ be general, and let $S(L)$ be the corresponding minimal model of $S(C(L))$ (see (2.3)). It follows from (2.4.1)-(2.4.3) that $e(S(L)) = e = 0$, and $S(L)$ is decomposable (see (2.4.2)(*), and (1.3.4)(2)). Let $C^+ = C^+(L)$ and $C^- = C^-(L)$ be the minimal sections of $S(L)$. By definition,

\[ \mathcal{C}_\theta = \{ \text{the closure of} \{ C^+(L), C^-(L) : L \in \text{Supp} \Theta \text{ is general} \} \}, \]

where the term “general” can be defined in an obvious way.

Clearly, $\text{dim}(\mathcal{C}_\theta) = 9$ (the family $\mathcal{C}_\theta$ is generically a finite covering of the 9-dimensional projective space of plane cubics). Since $p = p_1$ maps the general $C^+ \in \mathcal{C}_\theta$ isomorphically onto a plane cubic, $\text{deg}(C^+) = (3, d)$ for some integer $d$. We shall prove the following

(3.1.2) Proposition. Let $\mathcal{C}_\theta$ be the family of minimal sections defined in (3.1.1), and let $C^+$ be a general element of $\mathcal{C}_\theta$. Then:

(a). $\text{deg}(C^+) = (3, 6)$;
(b). $C^+$ lies on a hyperplane section $S_{C^+} \subset T$.

Proof. Let $L = L(0) \in \text{Supp} \Theta$ be general, and let $C^+(L) = C^+(0)$ and $C^-(L) = C^-(0)$ be the minimal sections of $S(L) = S(L(0))$. Just as in the proof of Lemma (2.4.2), we
can prove that: (1). The sections \( C^+(0) \) and \( C^-(0) \) can be included in rational families \( \{ C^+(t) : t \in \mathbb{P}^1 \} \) and \( \{ C^-(t) : t \in \mathbb{P}^1 \} \). (2). The enveloping surfaces \( S^+ = S_{C^+} \) and \( S^- = S_{C^-} \) of these two families are hyperplane sections of the threefold \( T \). This proves (b).

(*) Remark. The check of the parameters implies that the general hyperplane section of \( T \) carries such a pencil of minimal sections. Moreover, it is not hard to see that the general minimal section determines its enveloping hyperplane sections in a unique way – there is a unique hyperplane in \( \mathbb{P}^8 \) which contains the general curve \( C \in C_\theta \) (see also [T] and [C], where corresponding results are proved for the Reye sextics on the Quartic double solid).

The part (a) is a corollary from (b), and from standard arguments involving the normal bundle sequences for the triples \( C^+ \subset S(C) \subset T \) and \( C^+ \subset S^+ \subset T \) (similarly – for \( C^- \)).

q.e.d.

4. The Abel-Jacobi image of the family \( C_\theta \).

(4.1) Theorem. Let \( \Phi : C_\theta \rightarrow J(T) \) be the Abel-Jacobi map for the family \( C_\theta \) (see e.g. [C]). Then the image \( \Phi(C_\theta) \) is a copy of the theta divisor \( \Theta(T) \).

Proof. It follows from (3.1.1),(3.1.2) that the family \( C_\theta \) is generically a 2-sheeted covering of the family \( \text{Supp} \Theta : \) (Let \( L \in \text{Supp} \Theta \) be general. Then the fiber \( \xi^{-1}(L) \) of the covering \( \xi : C_\theta \rightarrow \text{Supp} \Theta \) is actually the pair \( C^+(L), C^-(L) \) – see above). It follows from the definition of \( \text{Supp} \Theta \) (see (2.1)) that the (Prym-)Abel-Jacobi map \( \Psi \) sends \( \text{Supp} \Theta \) onto the copy \( \Theta \) described in (1.2.2). In fact, \( \Psi \) contracts the linear systems \( |L|, L \in \Theta \), to the points \( L \).

Let \( L = L(0) \in \text{Supp} \Theta \) be general, and let \( C^+(L) = C^+(0) \) and \( C^-(L) = C^-(0) \) be the minimal sections of \( S(L) = S(0) \). Let \( \{ C^+(t) \} \) and \( \{ C^-(t) \} \) be the rational families defined as in the proof of (3.1.2), and let \( \{ L^+(t) \} \) and \( \{ L^-(t) \} \) be the corresponding pencils in \( \text{Supp} \Theta \). Since \( L^+(0) = L^-(0) = L(0) = L \), these two pencils coincide. Clearly, \( L^+(t) = L^-(t) = L(t) \), where \( \{ L(t) \} \) is the pencil defined by the linear system \( |L| = |O_\Delta(L)| \). The general \( C^+(t) \) and \( C^-(t) \) are the minimal sections of the corresponding surfaces \( S(L(t)) \). Therefore, the natural 2-sheeted covering \( \xi \) factors through general linear systems \( |L|, L \in \Theta \). Now, the only which is left, is to prove the following

(4.2) Lemma. Let \( L \in \text{Supp} \Theta \) be general, and let \( C^+ = C^+(L) \) and \( C^- = C^-(L) \) be the minimal sections of \( S(L) \). Then \( \Phi(C^+(L)) = \Phi(C^-(L)) \).
Proof of the Lemma. Let \( C = p(C^+) = p(C^-) \) be the isomorphic image of \( C^+ \) and \( C^- \), and let \( S(C) = p^{-1}(C) \subset T \). Let \( S(L) = P_C(\mathcal{O} \oplus \varepsilon) \) be as in (1.3.4)(2). By definition, \( C^+ \) is the tautological section of \( E^+ = \mathcal{O} \oplus \varepsilon \), and \( C^- \) is the tautological section of \( E^- = \mathcal{O} \oplus -\varepsilon \). Let \( p_L : S(L) \to C \) be the projection, and let \( \sim \) denotes the linear equivalence on the ruled surface \( S(L) \). It follows from the definitions of \( C^+ \) and \( C^- \) that \( C^+ \sim C^- + \varepsilon.fL \), where \( \varepsilon.fL := p^*_L(\varepsilon) \).

Let \( \delta(C) = \Delta \cap C \), and let \( Q \) be a point of \( C \). Since \( C \) is an elliptic curve and \( \varepsilon \in \text{Pic}^0(C) \), the sheaf \( \varepsilon \) can be represented in the form \( \varepsilon = \mathcal{O}_\Delta(P - Q) \), where \( P = P(\varepsilon) \) is determined uniquely by the choice of \( Q \) (the sheaf \( \varepsilon \) is fixed). Clearly, because of the continuous choice of the initial point \( Q \), we can always choose \( Q \) such that \( \{Q, P\} \cap \delta(C) = \emptyset \). Then \( C^+ \sim C^- + (P - Q).fL \), and \( fL \) can be also regarded as a fiber of the surface \( S(C) \subset T \) (equivalently \( -fL \) can be replaced by the general fiber \( f \) of the projection \( p : T \to \mathbb{P}^2 \)). Therefore \( \Phi(C^+) = \Phi(C^- + (P - Q).f) = \Phi(C^-) \), since the fibers \( P.f = p^{-1}(P) \) and \( Q.f = p^{-1}(Q) \) are rationally equivalent on \( T \). q.e.d.

The Theorem is proved.

5. Hyperplane sections of the Verra threefold, and plane sextics with vanishing theta-null.

(5.1) Proposition. Let \( T \) be a smooth bidegree \((2,2)\) divisor in \( \mathbb{P}^2 \times \mathbb{P}^2 \) (a Verra threefold), let \( T \subset \mathbb{P}^8 \) be the Segre embedding, and let \( p_i : T \to \mathbb{P}^2 \), \( i = 1, 2 \) \((p_1 = p)\) be the canonical projections. Let \( S \subset T \) be a general (esp. – irreducible) hyperplane section of \( T \).

Then:

1. \( p_i : S \to \mathbb{P}^2 \) is a double covering branched along a plane sextic \( D_i \) which is totally tangent to the discriminant sextic \( \Delta_i \), \( i = 1, 2 \).

2. \( D_i \) has a vanishing theta-null, \( i = 1, 2 \).

Proof. On the one hand, the family of hyperplane sections of \( T \) is naturally isomorphic to \((\mathbb{P}^8)^*\), and the general hyperplane section of \( T \) is a smooth K3 surface. On the other hand, the Abel-Jacobi map sends the 9-dimensional family \( C_\theta \) onto a copy of the 8-dimensional theta divisor \( \Theta = \Theta(T) \), and the general \( C \in C_\theta \) lies in a unique hyperplane section. On the other hand, the components of the general fiber of \( \Phi : C_\theta \to \Theta \) are (pairs of) pencils of minimal
sections, and the general pencil of this type is actually an elliptic pencil on some hyperplane section $S$ of $T$ (see (3.1.2)(*) and Theorem 4.1). A comparison between the dimensions, and standard arguments involving general positions, imply that the general hyperplane section of $T$ (which is a smooth $K3$ surface) carries a finite number of pencils of minimal sections. Obviously, this argument can be applied both: to $p = p_1$, and to $p_2$. Clearly, $p_i : S \rightarrow \mathbb{P}^2$ is a double covering. Since $S$ is a $K3$ surface, the branch locus $D_i$ of $p_i$ is a sextic curve, and it is not hard to see that $D_1$ is totally tangent to the discriminant sextic $\Delta_i$ ($\Rightarrow$ (1).

Fix for a moment $p_i = p_1$. Let $\{C(t) : t \in \mathbb{P}^1\}$ be one of the elliptic pencils of minimal sections (w.r. to $p_1$) on $S$. The elements of this pencil are curves of arithmetical genus 1 and of bidegree (3,6), and $p_1$ projects the pencil $\{C(t)\}$ onto a pencil $\{p_1(C(t))\}$ of plane cubics. Clearly, $p_1(C(t))$ is totally tangent to the branch locus $D_1$ of the double covering $p_1 : S \rightarrow \mathbb{P}^2$. Therefore, $\{p_1(C(t))\}$ defines a vanishing theta-null $\chi_1$ on $D_1$. Similarly – for $p_2$. ($\Rightarrow$ (2).

(5.2) Remarks.

(5.2.1) Prym-Canonical systems of $(\Delta_i, \eta_i)$, and sextics with vanishing theta-null.

Let $i \in \{1, 2\}$, and let $\eta_i$ be the torsion sheaf which defines the double covering $\Delta_i \rightarrow \Delta_i$. Since $D_i$ is totally tangent to $\Delta_i$, $D_i |_{\Delta_i} = 2.\delta_i$, for some effective divisor $\delta_i = \delta_i(S)$ on $\Delta_i$. It can be seen (see [Ve]) that $\delta_i$ belongs to the Prym-Canonical linear system $\omega_{\Delta_i}(\eta_i) = O_{\Delta_i}(3) \otimes \eta_i$. However, the effective divisor $\delta_i$ does not determine the totally tangent sextic (along $\delta_i$) in a unique way. In fact, any plane sextic (≠ $\Delta_i$) of the pencil $\{D_i(t)\}$, which is spanned on $D_i$ and $\Delta_i$, is totally tangent to $\Delta_i$ along $\delta_i(S)$. Proposition (5.1) tells that the branch sextic $D_i = D_i(S)$ “chooses”, among the curves of this pencil, a sextic with a vanishing theta-null.

(5.2.2) The two splittings.

In fact, the non-uniqueness of the minimal section of the general surface $S(L), L \in \text{Supp} \Theta$ (see e.g. (3.1)), implies that the family of plane cubics $\{p_1(C(t))\}$ (see the proof of (5.1)) defines a vanishing theta-null on at least two branch loci: the branch loci of the enveloping hyperplane sections $S^+$ and $S^-$ (see the proof of Proposition (3.1.2)).

A similar splitting takes place also on the general hyperplane section $S$. Let e.g. $\{C(t)\}$ be one of the pencils of minimal sections on the hyperplane section $S$ ; we let $p = p_1$. Let
\( \{ p(C(t)) \} \) be the corresponding rational pencil of plane cubics, and let \( \{ S(t) = p^{-1}(p(C(t))) \} \) be the pencil of canonical conic bundle surfaces above \( \{ p(C(t)) \} \). Let \( S \) be the enveloping hyperplane section of \( \{ C(t) \} \). Then the pencil \( \{ S \cap S(t) \} \) (of singular curves of arithmetical genus 10, and of bidegree \((6,12)\)) splits into two pencils: the pencil \( \{ C(t) \} \), and the conjugate pencil \( \{ \overline{C}(t) = S \cap S(t) - C(t) \} \). Obviously, \( \{ \overline{C}(t) \} \subset C_g \), and the two pencils \( \{ C(t) \} \) and \( \{ \overline{C}(t) \} \) (which lie on the same hyperplane section \( S \)) are projected onto the same pencil of plane cubics.

(5.2.3) The Double Flag, and the Dixon correspondence between the branched loci \( D_1(S) \) and \( D_2(S) \).

Let \( T, \Delta, \eta_i \), etc., be as usual. The identity between the Prym varieties \( P(\Delta_1, \eta_1) \) and \( P(\Delta_2, \eta_2) \) gives a counterexample to the Torelli theorem for Prym varieties ([Ve]). There exists another family of counterexamples to the Torelli theorem for Prym varieties, which can be regarded as a “hyperelliptic” degeneration of the family of bidegree \((2,2)\) divisors. This is the family of Double Flags (see below), and the general member of this family can be described as follows:

Let \( \text{seg} : \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^8 \) be the Segre embedding, and let \( W \) be the image of \( \text{seg} \). Then the general hyperplane section \( Y \) of \( W \) is a smooth Fano threefold, and any such a threefold is isomorphic to the Flag variety \( \mathbf{P}(T_{\mathbf{P}^2}) \) – the incidence correspondence between points and lines on the projective plane \( \mathbf{P}^2 \). By definition, a Double Flag is any double covering \( \pi : X \to Y \), branched along an intersection \( S \) of \( Y \) and a quadric. Clearly, the smooth Double Flag \( X \) is a smooth Fano threefold, and the maps \( \tilde{p}_i = p_i \circ \pi : X \to \mathbf{P}^2 \) \((i = 1, 2)\) define conic bundle structures on \( X \). The discriminant curve of \( \tilde{p}_i \) coincides with the branch locus \( D_i = D_i(S) \) of the 2-sheeted covering \( p_i : S \to \mathbf{P}^2 \), \( i = 1, 2 \). Obviously, the branch surface \( S \) is also a hyperplane section of a Verra threefold. In particular, the discriminant sextic \( D_i \) has a vanishing theta-null. Just as in the case of the Verra threefold, the conic bundle structure \( \tilde{p}_i \) defines, in a standard way, the torsion-2 sheaf \( \tilde{\eta}_i \) on \( D_i \). In [Ve] A.Verra proves that the discriminant pairs \( (\Delta_1, \eta_1) \) and \( (\Delta_2, \eta_2) \), for the bidegree \((2,2)\) divisor \( T \), are connected to each other by the classical Dixon correspondence. The same argument can be applied to the couple of discriminant pairs for the Double Flag. Remember that the discriminant sextics for the Double Flag are also branched loci, related to a hyperplane section of a Verra threefold.
Obviously, the general hyperplane section $S$ of $T$ is also a branch locus of a double covering of the Flag variety $Y$. Therefore, the branch loci $D_1(S)$ and $D_2(S)$ (defined by the general hyperplane section $S$ of the threefold $T$) can be included in discriminant pairs which are connected by the Dixon correspondence. In fact, the condition imposed by the existence of a theta-null, is a closed condition of codimension 1, on the 19-dimensional space of non-isomorphic plane sextics. The K3-surfaces, which are double coverings of $\mathbf{P}^2$, also form a 19-dimensional moduli space $\mathcal{M}$. A vanishing theta-null, imposed on the branch locus of such a K3-surface, separates a codim.1 subspace $\mathcal{M}_o$ of $\mathcal{M}$. Note also that the Double Flags are codim.1 degeneration of Verra threefolds, and the general plane sextic appears as one of the discriminant curves for some Verra threefold. This implies the following:

(5.3) Corollary. There exists a component (of maximal dimension) $\mathcal{M}'$ of the 18-dimensional parameter space $\mathcal{M}_o$ (of K3-surfaces which are double coverings of $\mathbf{P}^2$ branched along sextics with a vanishing theta-null), such that the general element $S$ of $\mathcal{M}'$ can be represented by two ways as a double covering of $\mathbf{P}^2$. Moreover, if $D_1$ and $D_2$ are the branch loci of these two coverings, then $D_1$ and $D_2$ can be included into discriminant pairs related to each other by the Dixon correspondence.

6. The family $\mathcal{D}$ of elliptic sextics of bidegree (3,3) on $T(2,2)$.

(6.0). In this section we prove that the Abel-Jacobi map sends the 6-dimensional family $\mathcal{D}$ of elliptic sextics on $T$ of bidegree (3,3) onto a 3-dimensional subvariety $Z \subset J(T)$ – isomorphic to a component of $\text{Sing } \Theta$.

(6.1). Definition of $\mathcal{D}$.

Let $C \subset T$ be a reduced and connected curve of arithmetical genus $p_a(C) = 1$, and of total degree $\lvert \text{deg}(C) \rvert = 6$. Here $\text{deg}$ is the usual bidegree map $\text{deg} : \{1\text{-cycles on } T\} \to \mathbf{Z} \oplus \mathbf{Z}$. If such a curve $C \subset T$ does exist, the bidegree $\text{deg}(C)$ can be one of the pairs $(2,4), (3,3), (4,2)$. We shall consider the middle case: $\text{deg}(C) = (3,3)$. By definition:

$\mathcal{D} = \{C - \text{ a reduced and connected curve on } T : p_a(C) = 1, \text{deg}(C) = (3,3)\}$.

We call $\mathcal{D}$ the family of elliptic sextics of bidegree (3,3) on $T$. 

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(6.2). The existence of \( C \in \mathcal{D} \).

We shall find at least one reduced and connected curve \( C \in \mathcal{D} \). The existence of such a curve makes it possible to apply general techniques (see e.g. \([ZR]\)) to obtain more information about the non-empty family \( \mathcal{D} \).

(6.2.1). Curves of small degree on the components of the reducible hyperplane sections of \( T \).

Let \( l \subset \mathbb{P}^2 \) be a general line, and let \( S_l = p^{-1}(l) \); here, as usual, \( p = p_1 \), etc. The following can be seen immediately:

(1). \( \text{deg}(S_l) = 6; K_{S_l} = -h \); here \( h \) is the restriction on \( S_l \) of the class \( h = cl(p^*O_{\mathbb{P}^2}(1)) \).

(2). Let \( q = p_2 \) be the 2-nd projection. Then \( q : S_l \to \mathbb{P}^2 \) is a double covering branched along a smooth quartic curve. In particular, \( S_l \) is isomorphic to a del Pezzo surface with \( K^2 = 2 \), and the projection \( q \) defines the anticanonical linear system on \( S_l \).

(3). Let \( \Delta = \Delta_1 \) be the discriminant sextic for \( p \), and let \( \Delta \cap l = \{\xi_1, ..., \xi_6\} \) (\( l \) is general). The projection \( p \) separates one of the various conic bundle structures on the del Pezzo surface \( S_l \). In particular, \( p : S_l \to l \) defines (in a non-unique way) a morphism \( \sigma : S_l \to \mathbb{P}^2 \), s.t.:

(i). \( \sigma \) is a composition of seven \( \sigma \)-processes \( \sigma_{x_i}, i = 0, ..., 6 \). Here \( x_0, ..., x_6 \) are 7 points in \( \mathbb{P}^2 \), in a general position w.r.to the plane curves of degree \( \leq 3 \) (see \([DPT]\)).

(ii). The pencil of conics \( p : S_l \to l \) is defined by the \( \sigma \)-preimage of the non-complete linear system \( |O_{\mathbb{P}^2}(1 - x_o)| \).

(iii). The degenerate fiber \( p^{-1}(\xi_i) = [x_i] + [x_o, x_i] \), where \( [x_i] \) is the exceptional curve of \( \sigma_i \), and \( [x_o, x_i] \) is the proper \( \sigma \)-preimage of the line \( <x_o, x_i> \), \( i = 1, ..., 6 \).

(4). The (-1)-curves \( [x_i] \) and \( [x_o, x_i] \), \( i = 1, ..., 6 \), are the only lines on the surface \( S_l \subset \mathbb{P}^5(l) = \text{Span}(S_l) \subset \mathbb{P}^8 \), and the embedding \( S_l \subset \mathbb{P}^5(l) \) is defined by the non-complete linear system \( |O_{\mathbb{P}^2}(4 - 2x_o - (x_1 + ... + x_6))| \).

The next lemma is a direct corollary of the properties (1),(2),(3),(4). We refer to the articles I – V of M.Demazure in \([DPT]\), which provide a comprehensive study of the del Pezzo surfaces.

(*) Lemma. Let \( S_l = p^{-1}(l) \) be general. Then, in the terms of (1)-(4):

(a). There exist exactly \( 2^5 = 32 \) morphisms \( S_l \to \mathbb{F}_1 = \mathbb{P}^1(O \oplus O(1)) \), and exactly 32 morphisms \( S_l \to \mathbb{F}_o = \mathbb{P}(O \oplus O(1)) \), in which \( l \) remains a base, and \( l_i = [x_i] \) and \( \bar{l}_i \),
i=1,...,6, are mapped to fibers or points.

(b). The 32 curves on $S_l$, which are mapped onto the exceptional curves of the 32 ruled surfaces $F_1$, are conics of bidegree (1,1). These 32 conics are:

(i). the (-1)-curve $[x_o]$;
(ii). the (-1)-curves $[x_o, x_i]$, $i=1,...,6$;
(iii). the (-1)-curves $[x_o, \ldots, \widehat{x_i}, \ldots, x_j, \ldots, x_6] := \text{(the proper preimage of the conic through the corresponding 5-tuple of points)}$;
(iv). the (-1)-curve $[2x_o, x_1, \ldots, x_6] := \text{(the proper preimage of the cubic through $x_1, \ldots, x_6$, which has a node in $x_o$)}$.

Moreover, these 32 curves are the only bidegree (1,1)-conics on $S_l$.

(c). The general curve $C$ on $S_l$, which is mapped onto a 0-section on $F_o \to \mathbb{P}^1 = l (!)$, is a twisted cubic on $S_l$ of bidegree (1,2). The 32 pencils of (1,2)-curves on $S_l$, which correspond to the 32 fibrations of type $F_1$, can be described explicitly – as in (b). Just as in (b), these 32 pencils of twisted cubics, are the only pencils of bidegree (1,2)-curves on $S_l$, and any bidegree (1,2)-curve on $S_l$ is an element of one of these pencils.

(**). Corollary.

(a). The family $C_{1,1}$ of conics on $T$ of bidegree (1,1) is 2-dimensional. (In fact, $C_{1,1}$ is, in a natural way, a 32-sheeted covering of $P^2$ – see above). There are finitely many conics on $T$ of bidegree (1,1), which pass through the general point of $T$.

(b). The family $C_{1,2}$ of twisted cubics on $T$ of bidegree (1,2) is 3-dimensional. If $z \in T$ is a general point, then the family $C_{1,2}(z)$ of these bidegree (1,2)-cubics on $T$ which pass through $z$ is an algebraic set of dimension 1.

Proof. Standard.

(6.2.2). The existence of a reduced curve $C \in \mathcal{D}$.

We shall find such a curve.

Let $l \subset \mathbb{P}^2$ be a general line, and let $C_{1,1} \subset T$ be one of the 32 conics of bidegree (1,1) “above” $l$ (see (6.1.1)(*)(b)). The 2-nd projection $q = p_2$ maps $C_{1,1}$ isomorphically onto a line $l' \subset \mathbb{P}^2$. The line $l'$ intersects the 2-nd discriminant sextic $\Delta_2$ in 6 points. Therefore, there are 6 lines on $T$ of bidegree (1,0), which intersect the curve $C_{1,1}$. Let $L_{1,0}$ be one of these lines, and let $m = p(L_{1,0})$. Let $x$ be a general point of $C_{1,1}$. Clearly, $x$ can be regarded as a
general point of $T$. Let $S(x)$ be the union of all the curves which belong to the 1-dimensional family $C_{1,2}(x)$ (see (6.2.1)(**)). $S(x) \subset T$ is a surface, and it is not hard to see that the general choice of $x \in C_{1,1}$ implies that $S(x)$ does not contain the line $L_{1,0}$. Therefore, $S(x)$ intersects $L_{1,0}$ in a finite number of points (different form the point $C_{1,1} \cap L_{1,0}$. Let $y \in L_{1,0}$ be one of these points. In particular, there exists a twisted cubic $C_{1,2} \in C_{1,2}$, which passes through $x$ and $y$. Obviously, the curve $C = C_{1,1} + L_{1,0} + C_{1,2}$ is reduced, and $deg(C) = (3,3)$.

The existence of $C \in D$ assures the non-emptiness of $D$. The general results in [ZR] imply that the family $D$ is at least 6-dimensional. It is not hard to see that the general curve $C \in D$ is smooth, and the projections $p_1$ and $p_2$ send $C$ isomorphically onto the smooth plane cubics $C_1$ and $C_2$. Let $S_{C_1} = p_1^{-1}(C_1)$ and $S_{C_2} = p_2^{-1}(C_2)$. Just as in sections 2, 3, 4, the curve $C$ defines, via intersection, the effective divisors $L_i(C)$ on $	ilde{\Delta}_i$, which belong to the linear system of some $\mathcal{L}_i \in Nm^{-1}(\omega_{\Delta_i})$, $i = 1, 2$.

According to the agreement, we fix $p = p_1$. Let $\Delta = \Delta_1$, $L(C) = L_1(C)$, etc., be the corresponding objects. Obviously, $C$ is a minimal section of the ruled surface $S(L(C))$ (see (2.3.1)), and the same arguments as in sect. 2, 3, 4 imply that $S(L(C))$ is indecomposable and $e(S(L(C))) \geq 3$. (The same normal bundle sequence argument gives $dim(D) = 6$). In particular, $C$ is the unique minimal section of the canonical surface $S(p(C)) = p^{-1}(p(C))$ (which corresponds to the unique minimal section of the ruled model $S(L(C))$).

The curve $C$ is not an element of the total family of minimal sections $\mathcal{C}_\theta$. This is caused by the special position of the cubics $p(C), C \in D$, which form a 6-dimensional subset of the 9-space of all the plane cubics. We collect the last in the following

**(6.2.3) Lemma.**

(i). $dimD = 6$;

(ii). The general curve $C \in D$ is a smooth sextic of bidegree $(3,3)$, and the projections $p_1 = p$ and $p_2$ map $C$ isomorphically onto plane cubics. In particular:

(iii). ($p = p_1$). Let $L(C) \in Symm^{18}(\tilde{\Delta})$ be, as usual, the effective divisor on $\tilde{\Delta}$, defined by the 18-tuple of lines of bidegree $(0,1)$ which intersect the curve $C$. Then $L(C) \in Supp \Theta \cup Supp P^\perp$. 

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(6.3). The family $\mathcal{D}$ and the quadrics of rank 6 through $T$.

(6.3.1). The elements of $\mathcal{D}$ as components of canonical curves on $T$.

Let $P^6$ be a subspace of $P^8 = \text{Span}(T)$ such that $\text{dim}(T \cap P^6) = 1$, and let $C(P^6) = T \cap P^6$. The curve $C(P^6) \subset T$ is a canonical curve of degree 12, and of arithmetical genus 7. Call such a curve $C$ a canonical curve on $T$. Obviously, all the canonical curves on $T$ are rationally equivalent, and the family of canonical curves on $T$ can be represented by the 14-dimensional Grassmann variety $G(7, 9) = G(6 : P^8)$.

The the curves of the family $\mathcal{D}$ are closely connected with the degenerations of the family of canonical curves on $T$. More precisely, let $C \in \mathcal{D}$ be general, let $P^5(C) = \text{Span}(C)$, and let $P^6(\mathcal{D}) = P^8 / P^5(C)$ be a 6-space through $P^5(C)$. Then the canonical curve $C(P^6)$ splits into two components: $C(P^6) = C + \tilde{C}$ where $\tilde{C} \in \mathcal{D}$, and $\delta(C, \tilde{C}) = \#(C \cap \tilde{C}) = 6$.

(6.3.2). The determinantal subvarieties of $I_2(W)$.

Let $W \subset P^8$ be the Segre image of $P^2 \times P^2$, and let $P^2 \times P^2 = P(E) \times P(F)$, where $E$ and $F$ are complex 3-spaces. Let $P^8 = P(E \otimes F)$. Then, as it follows from the definition of the Segre map, the elements of $W \subset P^8$ are in 1:1 correspondence with the $C^\ast$-classes of unitary tensor products $u \otimes v : u \in E, v \in F$. In particular, let $(x_i, e_i)$ and $(y_j, f_j)$ be any coordinate systems on $E$ and $F$. Then $(z_{ij} = x_i \cdot y_j, g_{ij} = e_i \otimes f_j)$ is a coordinate system on $E \otimes F$. Let $[z_{ij}]$ be the coordinate matrix, and let $I_2(W)$ be the projective space of quadrics in $P^8$ which pass through $W$. Then, in matrix coordinates $z_{ij}$, $I_2(W)$ is spanned on the 9 quadratic equations $\text{rank}([z_{ij}]) = 1$, i.e. $I_2(W)$ is a projective 8-space. Clearly, the choice of the coordinates $z_{ij}$ defines a linear isomorphism

$$\psi(z) : P^8 = P(E \otimes F) \cong I_2(W).$$

The linear map $\psi(z)$ sends the $C^\ast$-classes of the unit tensor products (i.e. the elements of $W$) to quadrics of rank 4. Moreover, any quadric of rank 4, which contains $W$, can be represented in this way. It is well-known (see e.g. [LV]) that the set

$$Sec(W) := \text{the closure of the union of all the bisecant lines of } W,$$

is a cubic hypersurface in $P^8$, i.e., $W$ is one of the four Severi varieties (ibid.). It follows from the definition of $\psi(z)$ that $\psi(z)$ sends the points of $Sec(W)$ to quadrics of rank 6. It is not hard to see that any quadric of rank 6 which contains $W$ can be represented (in a non-unique way) as an image of a point of $Sec(W)$. We collect these observations in the
following:

(\ast). Lemma. Let \( I_2(W) \) be the projective space of quadrics in \( P^8 \) which contain the fourfold \( W \), and let \( D_k(W) = D_k(I_2(W)) = (\text{the closure of}) \{ Q \in I_2(W) : \text{rank}(Q) = k \} \) be the \( k \)-th determinantal of \( I_2(W) \). Then \( D_k(W) \neq \emptyset \iff k \in \{4, 6, 9\} \). Moreover, there exists a linear isomorphism \( \psi : P^8 \to I_2(W) \) such that

\[
\psi(W) = D_4(W), \quad \psi(Sec(W)) = D_6(W).
\]

Proof. Let \([z_{ij}]\) be as above, and let \( \psi = \psi(z) \). Then the natural action of

\( \text{PGL}(E) \times \text{PGL}(F) \) on \( P^8 \), which does not change the rank of the \( 3 \times 3 \) matrix \([z]\), splits \( P^8 \) into 3 orbits – the fourfold \( W \), and the quasi-projective varieties \( Sec(W) - W \) and \( P^8 - Sec(W) \). The linear map \( \psi \) sends the closure of any of these orbits onto a determinantal subvariety of \( I_2(W) \), and the only which has to be seen is that these three determinantal subvarieties are \( D_4, D_6 \) and \( D_9 \).

(6.3.3). Quadrics of rank 6 related to the incidence correspondence \( \Sigma \subset D \times D \).

Let \( T \) be a general bigode (2,2) divisor, and let \( Q \subset P^8 \) be any quadric such that \( Q \cap W = T \). It follows from (6.3.2) that such a quadric \( Q \) is not unique – \( Q \) can be replaced by any quadric in the projective 9-space \( \text{Span}(Q, I_2(W)) \), i.e. \( Q \) is unique mod.\( I_2(W) \).

Let \( \Sigma \subset G(7,9) \) (see (6.3.1)) be the incidence correspondence

\[
\Sigma = (\text{the closure of}) \{ P^6 : C(P^6) = C + \tilde{C}, \text{ where } C, \tilde{C} \in D \}. \quad \text{The set } \Sigma \text{ can be regarded (up-to closed subsets of codim.> 1) also as an incidence correspondence } \Sigma \subset D \times D.
\]

Let \( P^6 \in \Sigma \) be general. The codim.2 subspace \( P^6 \subset P^8 \) intersects the fourfold \( W \) in an anticanonically embedded del Pezzo surface \( S(P^6) \) of degree 6, and the quadric \( Q \) intersects \( S(P^6) \) in a pair of elliptic sextics \( C + \tilde{C} \). Let \( P^5(C) \) and \( P^5(\tilde{C}) \) be, as in (6.3.1), the spans of \( C \) and \( \tilde{C} \), and let \( H \) and \( \tilde{H} \) be linear forms on \( P^8 \) such that \( (H)_o \cap P^6 = P^5(C) \), \( (\tilde{H})_o \cap P^6 = P^5(\tilde{C}) \). The splitting \( Q \cap S(P^6) = C + \tilde{C} \) implies:

\[
Q \big|_{P^6} = H.\tilde{H} \big|_{P^6} \text{ (mod. } I_2(S(P^6)) \text{) = the “restriction” of } I_2(W) \text{ on } P^6).
\]

(Here \( Q \) is the quadratic form of \( Q \), and we disregard the multiplication by a non-zero constant). Let \( P^6 = (H_1 = H_2 = 0) \) be any pair of linear equations which define the subspace \( P^6 \subset P^8 \). It follows from the preceding that \( Q \) can be represented in the form:

\[
Q = H.\tilde{H} + H_1.\tilde{H}_1 + H_2.\tilde{H}_2, \text{ mod.} I_2(W), \text{ where } \tilde{H}_1 \text{ and } \tilde{H}_2 \text{ are some linear forms. Clearly,}
\]
the quadric $H.\tilde{H} + H_1.\tilde{H}_1 + H_2.\tilde{H}_2$ does not belong to the set $D_6(W)$ (the restriction of this quadric to $P^6$ does not contain the surface $S(P^6)$). In particular, the quadric $Q$ in the definition of $T$ can be replaced by this quadric of rank 6.

(6.4). The Abel-Jacobi image $Z = \Phi(D)$.

Let $T \subset W, D$, etc., be as above, and let $\Phi: D \to J = J(T)$ be the Abel-Jacobi map for the family $D$. Let $C \in D$ be general, and let

$$\Phi^*: H^1(T, \Omega^2) \to H^1(N_{C/T} \otimes \omega_T)$$

be the codifferential of $\Phi$ in the point $C \in D$ (see e.g. [C]).

The space $H^1(T, \Omega^2)$ is naturally isomorphic to $H^0(J(T))$ – the cotangent space of $J(T)$ in a fixed point. The normal bundle sequence for the embedding $T \subset W$, and the formulae of Both and K"unneth imply the isomorphism:

$$\alpha: H^0(O_{P^8}(1)) \cong H^0(T, \mathcal{O}(1,1)) \to H^1(T, \Omega^2).$$

(see also [Ve]). In particular, the elements of $H^1(T, \Omega^2)$ can be regarded as linear forms on $P^8$.

The following proposition is an analogue of Lemma 4.6 in [Vo]:

(6.4.1). Proposition. Let $C \in D$ be general, let $P^5(C) = Span(C)$ be as in (6.3.1), let $P^6$ be any 6-space through $P^5(C)$, and let $Q = H.\tilde{H} + H_1.\tilde{H}_1 + H_2.\tilde{H}_2, mod.I_2(W)$ be any of the representations of the quadric $Q$ defined by the element $P^6 \in \Sigma$ (see (6.3.3)).

Let $\Phi^*$ and $\alpha$ be as above, and let $\Phi^*.\alpha$ be their composition. Then the subspace $\ker(\Phi^*.\alpha) \subset H^0(O_{P^8}(1))$ is spanned on the forms $H, \tilde{H}, H_1, \tilde{H}_1, H_2, \tilde{H}_2$.

Proof. see [Vo], the proof of Lemma 4.6. Note that the analogue of the family $D$, studied in [Vo], is the family of “halves” of canonical curves on the quartic double solid $B$ – the family of elliptic quartics on $B$ (see (6.3.1)).

(6.4.2). Corollary. Let $Z = \Phi(D)$ be the Abel-Jacobi image of the family of elliptic sextics on $T$ of bidegree (3,3). Then $\dim(Z) = 3$.

(6.4.3). The fiber $\Phi^{-1}(z)$.

Let $C \in D$ be general, let $z = \Phi(C)$, and let $\Phi^{-1}(z)_o$ be the connected component of $\Phi^{-1}(z)$ such that $C \in \Phi^{-1}(z)_o$. Let $P^5(C) = Span(C)$, and let $u, v$ be the local parameters in the (general) point $P^7 = P^7(0,0) \in P^2(C)^* = P^8/P^5(C)$. Let $S(u, v) = T \cap P^7(u, v)$. The surface $S(u, v)$ is a hyperplane section of $T$ which contains the (general) $C \in D$. Such a
surface $S(u, v)$ cannot be reducible. Otherwise $l_i = p_i(C) = p_i(S(u, v)), i = 1, 2$ will be lines (see (6.2.1), (6.2.3)). Therefore, $S(u, v)$ is a surface of type $K3$, and the elliptic curve $C = C(u, v; 0) \subset S(u, v)$ moves in a pencil \{\{C(u, v; t) \subset S(u, v)\}\}. Obviously, $C(u, v; t) \in \mathcal{D}$ and $\Phi(C(u, v; t)) = \Phi(C)$, since the curve $C(u, v; t)$ is a rational deformation of $C = C(u, v; 0)$. Therefore $C(u, v; t)$ is the general point of the irreducible 3-fold $\Phi^{-1}(z)_o$, where $z = \Phi(C)$.

Let $P^6 \supset P^5(C)$, and let $Q_o = H.\tilde{H} + H_1.\tilde{H}_1 + H_2.\tilde{H}_2$ be defined as in (6.3.3),(6.4); let $Q_o$ be also the quadric defined by the equation $Q_o = 0$. Let $\Lambda$ be the generator of $Q_o$ defined by the condition $P^5(C) \in \Lambda$. Obviously, the 5-space $P^5(u, v; t) = \text{Span}(C(u, v; t)) \in \Lambda$. The correspondence $C(u, v; t) \leftrightarrow P^5(u, v; t)$ can be completed to a map $\lambda^{-1} : P^3 \cong \Lambda \rightarrow \Phi^{-1}(z)_o$. It is easy to see that $\lambda^{-1}$ (hence $\lambda$) is an isomorphism.

(*) It follows from the preceding that the quadric $Q_o$ does not depend on the element $C(u, v; t) \in \Phi^{-1}(z)_o$. We write $Q_o = Q_o(C) = Q_o(C(u, v; t)) = Q_o(z)$.

(6.5). Proposition. Let $T \subset W$ be a general bidegree $(2, 2)$ divisor, let $p_i : T \rightarrow P^2, i = 1, 2$, be the projections, and let $\text{Sing}^i*(\Theta)$ be as in (1.2.2). Then:

(i). There exist canonically defined maps $\mathcal{L}_i : Z \rightarrow \mathcal{L}_i(Z) \subset P(\Delta_i, \Delta_i) \cong J(T)$, where $\mathcal{L}_i(Z)$ is a component of $\text{Sing}^i*(\Theta), i = 1, 2$.

(ii). Let $C \in \mathcal{D}$ be general, and let $z = \Phi(C)$. Then the quadric $Q_o(C) = Q_o(z)$ (see (6.4.3)(*)) coincides with the projectivized tangent cone $\text{Cone}_z$ of $\Theta$ in the point $z \in Z \subset \text{Sing}(\Theta)$.

Proof. Let $Q \subset P^8$ be any quadric such that $T = W \cap Q$, and let $I_2(T) = P(H^0(P^8, O(2 - T)))$ be the space of quadrics through $T$ (see also (6.3.2)). Clearly,

$I_2(T) \cong \text{Span}\{I_2(W), Q\} \cong P^9$.

Let $D_k(T) := \text{(the closure of)}\{P \in I_2(T) : \text{rank}(P) = k\}$ be the $k$-th determinantal locus in $I_2(T)$.

Let $k = 6$. Then (see (6.3.2)(*)) $D_6(T) \supset D_6(W) \cong \text{Sec}(W)$, and $Q_o(z)$ does not belong to $I_2(W) \supset D_6(W)$ (see (6.3.3)). Therefore, the rule $z \mapsto Q_o(z)$ defines a map $Q_o : Z \rightarrow D_6(T)$, and the image $Q_o(Z)$ is not a subset of $D_6(W)$.

(6.5.1) Lemma. $Q_o(Z)$ is a component of $D_6(T)$. Moreover, $D_6(T) = Q_o(Z) \cup D_6(W)$.
Proof of (6.5.1). Let \( Q_o \in D_6(T) - D_6(W) \), and let \( \Lambda \) be one of the two generators of the quadric \( Q_o \). Let \( P^5 \in \Lambda \). Then the set \( C(P^5) = T \cap P^5 = (W \cap Q) \cap P^5 = (W \cap Q_o) \cap P^5 = W \cap P^5 \) is a curve on \( T \). Moreover, as it follows from the elementary projective properties of the fourfold \( W \), \( C(P^5) \) is an elliptic curve of bidegree \((3,3)\), i.e. \( C \in D \). Clearly, \( Q_o = Q_o(C(P^5)) = Q_o(z) \), where \( z = \Phi(C(P^5)) \in Z \). \textbf{q.e.d.}

**6.5.2.** The differential \( dQ_o \) via the Gauss map of \( Z \).

It follows from the definition of \( Q_o(Z) \) that \( \dim(Q_o(Z)) \leq \dim(Z) = 3 \). Moreover, \( Q_o(Z) \) is a component of the determinantal locus \( D_6(T) \subset I_2(T) \cong P^9 \), and the general quadric \( Q \in I_2(T) \) has rank 9. It follows from the general properties of the determinantal varieties (see [F, ch.14]) that the components of \( D_6(T) = D_{9-3}(T) \) cannot be of codimension greater than \( (3 + 1)/2 = 6 \). Therefore, \( \dim(Q_o(Z)) = \dim(Z) = 3 \), and the map \( Q_o : Z \to Q_o(Z) \) is finite. In particular, the differential \( dQ_o : T_Z \to T_{Q_o(Z)} \) is a local isomorphism. Let \( P(dQ_o) : P(T_Z) \to P(T_{Q_o(Z)}) \) be the projectivization of \( dQ_o \). Let \( Gauss : Z \to G(3,9) = G(2 : P^8) \), \( z \mapsto [ \text{ the translate in } o \in J(T), \text{ of } ] \) the set of all 3-spaces in the tangent space of \( Z \subset J(T) \) in the point \( z \in Z \)

be the (rational) Gauss map, defined by the embedding \( Z \subset J(T) \). Let \( z \in Z \) be general. (In particular, \( z \) is a regular point of \( Z \).) It follows from Proposition (6.4.1) that the projective 2-space \( P^2(z) = vertex(Q_o(z)) \) can be identified with \( P(Gauss(z)) \subset P^8 = P(T_{J(T)} |_o) \). Since \( P(dQ_o) \) is a local isomorphism, we can identify the spaces \( P(T_{Q_o(Z)} |_z) \) and \( P^2(z) = vertex(Q_o) \) (see also [Ve, 4.29]).

6.5.3. Proof of (6.5)(i).

Let \( i \in \{1,2 \} \) be fixed, and let \( L_i : D \to Symm^{18}(\Delta_i) \) be the map defined in Lemma (6.2.3). (In (6.2.3), \( p = p_i \) and \( L(C) = L_i(C) \); the definition of \( L_2(C) \) is evident.) Let \( C_o \in D \) be general, and let \( z = \Phi(C_o) \). Let \( \Lambda \) be the generator of the quadric \( Q_o(z) \) defined by the condition \( P^5(C_o) \in \Lambda \). Let \( \Lambda_i = L_i(\Lambda) = \{ L_i(C) : C = T \cap P^5, P^5 \in \Lambda \} \subset Symm^{18}(\Delta_i) \).

Clearly, the map \( L_i : \Lambda \to \Lambda_i \) is an isomorphism. In particular, \( \Lambda_i \cong P^3 \), i.e., the set \( \Lambda_i \) is a rational subfamily of \( Supp \Theta_i \cup Supp P_i^- \) (see (6.2.3)(iii)). Here \( \Theta_i \) and \( P_i^- \) are the components of the effective part in \( Nm^{-1}(\omega_{\Delta_i}) \) — see e.g. [W]. Therefore, all the divisors
Let $L_i(C) \in \Lambda_i, C = C(\mathbb{P}^5), \mathbb{P}^5 \in \Lambda$ belong to the same linear system $| L_i(C_o) |$ on $\tilde{\Delta}_i$.

Let $L_i = L(L_i(C_o)) \in \text{Pic}^{18}(\tilde{\Delta}_i)$ be the invertible sheaf defined by the effective divisor $L_i(C_o) \in \text{Symm}^{18}(\tilde{\Delta}_i)$.

The rule $L_i(C_o) \mapsto L(L_i(C_o))$ defines a map $L : L_i(D) \rightarrow \text{Supp} \Theta_i \cup \text{Supp}(P^{-})_i$.

It follows from Corollary (6.4.2), and from the definition of the map $L\circ L_i$ that $\dim(L\circ L_i(D)) = \dim(Z) = 3$. So, we obtained a 3-dimensional subset $L\circ L_i(D)$ such that $h^0(L) \geq 4$, for any $L \in L \circ L_i(D)$; here we use the same symbol $L$ for the sheaf $L$ and for the map $L$.

The number $d = \min\{ \dim | L | : L \in L \circ L_i(D) \}$ is a constant throughout an open subset of $L \circ L_i(D)$.

\textbf{Lemma.} Let $d$ be as above. Then $d = 3$.

\textbf{Proof} of (*).

Let, e.g., $d = 4$. (The case $d \geq 5$ can be treated in a similar way.) Let $- : \tilde{\Delta}_i \rightarrow \tilde{\Delta}_i$ be the involution induced by the double covering $\tilde{\Delta}_i \rightarrow \Delta_i$, and let

$$W_i = \{ M = F \otimes \mathcal{O}(x - \bar{x}) : F \in L \circ L_i(D), x \in \tilde{\Delta} \}.$$  

It follows from the definition of $W_i$ that $W_i \subset Nm^{-1}(\omega_{\Delta_i})$, and $h^0(M) = 4$ for the general $M \in W_i$ (see e.g. [Sh, Lemma 3.14]).

Therefore $W_i$ is a 4-dimensional subset of $\text{Sing}^s_i(\Theta)$ (see (1.2.2)). However $\dim(\text{Sing}(\Theta)) = 3$ (see [Ve, Prop. 4.24]). Therefore $d$ cannot be 4. \textbf{q.e.d.}

\textbf{Remark.} The intermediate jacobian $J(T)$ is a Prym variety which arises from a double covering of a general plane sextic, in contrast to the intermediate jacobian $J(B)$ of the desingularized nodal quartic double solid $B$ – in which case the plane sextic $\Delta$ has a totally tangent conic. The existence of a totally tangent conic is a closed condition of codimension one, on the 19-dimensional moduli space of the plane sextics. Moreover, $\dim(\text{Sing}(\Theta(B))) = 4$ (see [De, 7]), in contrast to $\dim(\text{Sing}(\Theta(T))) = 3$. This, probably, once more explains why the Dixon correspondence, which can be identified with a bidegree $(2,2)$ divisor, cannot be applied for a discriminantal pair which comes from a nodal quartic double solid (see [Ve, 5]).

It follows from the preceding that the sheaf $L \circ L_i(C)$ does not depend on the particular choice of the curve $C \in \Phi^{-1}(z), z = \Phi(C) = \Phi(C_o)$. Therefore, the map $L \circ L_i : D \rightarrow J(T)$
factors through the Abel-Jacobi map $\Phi : \mathcal{D} \to \mathcal{Z}$. Denote by $\mathcal{L}_i : \mathcal{Z} \to \mathcal{L}_i(\mathcal{Z}) = \mathcal{L} \circ L_i(\mathcal{D})$ the quotient map. It follows from (*) that $\mathcal{L}_i(\mathcal{Z})$ is a 3-dimensional component of $\text{Sing}_{st}^i(\Theta)$. This proves (i).

(6.5.4). Proof of (6.5)(ii).

Let $\mathcal{L}_i(z), z = \Phi(C_0)$, etc., be as above. The sheaf $\mathcal{L}_i(z)$ is a stable singularity of $\Theta$, with respect to $p_i$. Therefore, the projectivized tangent cone $\text{Cone}_{\mathcal{L}_i(z)}$ of $\Theta$, in the point $\mathcal{L}_i(z)$, is a quadric which passes through the Prym-canonical image $\Delta_i^T$ of the discriminant curve $\Delta_i, i = 1, 2$ (see [Tju]). Here we use the following results, due to Verra (see [Ve, 3, and the proof of 4.21]):

(*) Let $s : \Delta_i \to T$ be the Steiner map, defined by the rule:

$s : x \mapsto \text{Sing}(p_i^{-1}(x)).$

Then the image $s(\Delta_i)$ coincides with the Prym-canonical curve $\Delta_i^T$.

(**). Let $Q \subset \mathbb{P}^8$ be a quadric which passes through the Steiner curves $s(\Delta_1)$ and $s(\Delta_2)$. Then $Q \supset T$.

It follows from the preceding, and from (*) and (**), that

(i). $\text{Cone}_z := \text{Cone}_{\mathcal{L}_1(z)} = \text{Cone}_{\mathcal{L}_2(z)}$;

(ii). $\text{Cone}_z \supset T$, i.e., $\text{Cone}_z \in \mathbf{I}_2(T)$.

It is well-known that $\text{Cone}_z$ is a quadric of rank 5 or 6 (see [K]), i.e. $\text{Cone}_z \in D_5(T) \cup D_6(T)$. It is not hard to see that if $T$ is general then $D_5(T) = \emptyset$. In fact, the general choice of the quadric $Q$, such that $W \cap Q = T$, implies that $\text{codim}(D_5(T) \subset \mathbf{I}_2(T)) = 10$, outside the fixed determinantal $D_4(T) = D_4(W) \cong W$ (see (6.3.2), and [F, ch. 14]). Therefore, $\text{rank}(\text{Cone}_z) = 6$. The maps $\mathcal{L}_1$ and $\mathcal{L}_2$ are local isomorphisms. Therefore the projective tangent spaces $\text{vertex}(Q_o(z)) = \mathbb{P}^2(z) = \mathbb{P}(T_Z \mid z)$ (see (6.4.1)), and $\mathbb{P}(T_{\mathcal{L}_i(z)} \mid \mathcal{L}_i(z)) = \text{vertex}(\text{Cone}_{\mathcal{L}_i(z)}) = \text{vertex}(\text{Cone}_z), i = 1, 2$ (see [Sh, 2.7 and 3.20]) can be identified.

It follows that the two quadrics: $\text{Cone}_z$ and $Q_o(z)$ have the same vertex $\mathbb{P}^2(z)$, and $\text{Cone}_z \supset T, Q_o(z) \supset T$. Moreover, $\text{Cone}_z$ and $Q_o(z)$ belong to the determinantal locus $D_6(T)$. An elementary projective consideration implies that these two quadrics must coincide. This proves (ii).

As a corollary we obtain:
(6.6) Theorem (The Torelli theorem for the Verra threefold).

Let $T = T(2,2)$ be a general smooth bidegree $(2,2)$ divisor in the Segre image $W$ of $\mathbb{P}^2 \times \mathbb{P}^2$, and let $(J(T), \Theta)$ be the principally polarized intermediate jacobian of $T$. Then there exists a component $Z$ of $\text{Sing} \Theta$ such that $\text{dim}(Z) = 3$, and $T$ coincides with the intersection of all the projectivized tangent cones of $\Theta$ in the regular points of $Z$.

Proof. It rests to be mentioned that the map $Q_o$ sends the set $Z = \Phi(D)$ onto the component $Q_o(Z)$ of $D_6(T)$ (see (6.3.3) and (6.5.1)), and that the space $I_2(T) \cong \mathbb{P}^9$ is spanned on the quadrics of the determinantal locus $D_6(T)$. Moreover, $D_6(T) = Q_o(Z) \cup D_6(W)$, and $\text{Span}(D_6(W))$ coincides with the proper subspace $I_2(T) \cong \mathbb{P}^8$ (see (6.3.2)(*) and (6.5.1)). Then, as it is not hard to see, $\text{Span}(Q_o(Z)) = I_2(T)$. Since the graded ideal $I_o(T) = \bigoplus I_d(T)$ of $T \subset \mathbb{P}^8$ is generated by the component $I_2(T)$, and $I_2(T) = \mathbb{P}(I_2(T))$, the quadrics of $Q_o(Z)$ (resp. – the projective tangent cones of $\Theta$ in the points of $Z$) cut the projective subvariety $T \subset \mathbb{P}^8$ out (see also [Vo, Prop.4.14]).

(6.7). Remarks.

(i). (see [Ve, 4.17]): Let $Z_T = \text{Sing}^{st}_1(\Theta) \cup \text{Sing}^{st}_2(\Theta)$. Then $Z \subset Z_T$ can be separated among the components of $Z_T$ by the numerical property:

$Z = \text{the union of all the irreducible components of } Z_T \text{ having not class } 12.\Theta^6/6!.$

(ii). Let $z \in Z$ be general, and let $C \in D$ be such that $\Phi(C) = z$. Let $Q_o(C) = Q_o(z) \in Q_o(Z) \subset D_6(T)$ be the rank 6 quadric attached to $z$, let $\Lambda$ be the generator of $Q_o(z)$ defined by $\mathbb{P}^5(C) \in \Lambda$, and let $\overline{\Lambda}$ be the complimentary generator of $Q_o(z)$. Let $\overline{\mathbb{P}^5} \in \Lambda$, let $\overline{C} = T \cap \overline{\mathbb{P}^5}$, and let $\overline{\tau} = \Phi(\overline{C})$ be the Abel-Jacobi image of the curve $\overline{C} \in D$. Obviously, $Q_o(\overline{C}) = Q_o(\overline{\tau}) = Q_o(z)$, i.e., the degree of the finite map $Q_o : Z \rightarrow Q_o(Z)$ is at least two. In fact, as it follows from the definition of the map $Q_o$, the only preimages of the quadric $Q_o(z)$ are the two points $z$ and $\overline{\tau}$, identified with the two generators $\Lambda$ and $\overline{\Lambda}$ of the quadric.

7. The nodal $T(2,2)$.

(7.1). The tetragonal triples of Donagi connected with the nodal $T(2,2)$.

Here we describe the two tetragonal triples which correspond to the 4-gonal systems on the two nodal discriminant sextics of the nodal $T(2,2)$ (see e.g. [Do]).

Let $T = W \cap Q$ has a simple node in the point $(z)_o = (x)_o \times (y)_o$. Let $p = p_1 : T \rightarrow \mathbb{P}^2$.
and \( q = p_2 : T \to \mathbb{P}^2 \) be the natural projections. Then the discriminant sextic \( \Delta_p \) of \( p \) (resp. \( \Delta_q \) of \( q \)) has a simple node in the point \( (x)_o \) (resp. \( - \) in the point \( (y)_o \)). Let \( \mathbb{P}_1 = | \mathcal{O}_{\mathbb{P}^2}(1-(x)_o) | \) be the plane pencil of lines through \( (x)_o \) (resp. \( - \mathbb{P}_q = | \mathcal{O}_{\mathbb{P}^2}(1-(y)_o) | \)).

Let \( p^{-1}((x)_o) = L + T \), \( q^{-1}((y)_o) = M + \overline{T} \). Clearly, \( L \cap T = M \cap \overline{T} = (z)_o \).

Let \( pr : T \to \mathbb{P}^7 \) be the rational projection through \( (z)_o \), and let \( T_+ \) be the image of \( T \). In particular, the proper images \( pr(L), pr(\overline{T}), pr(M) \), and \( pr(\overline{M}) \) are 4 isolated singular points of \( T_+ \) which lie on the exceptional quadric \( Q_0 \subset T_+ \) of \( pr \).

Let \( l \in \mathbb{P}_1 - \{ p(M), p(\overline{M}) \} \), \( m \in \mathbb{P}_2 - \{ q(L), q(\overline{L}) \} \), and let \( l \) and \( m \) be, otherwise, general. Let \( C(l, m) = p^{-1}(l) \cap q^{-1}(m) \cap T \). A straightforward check gives that \( C(l, m) \) is a curve of bidegree \((2,2)\) and of arithmetical genus one, which has a simple node in the point \((z)_o\). Let \( q(l, m) = pr(C(l, m)) \subset T_+ \) be the proper image of \( C(l, m) \). It follows that \( q(l, m) \) is a conic. Thus, \( T_+ \) is birational to a conic bundle \( s^+ : T^+ \to \mathbb{P}_1 \times \mathbb{P}_3 \). It is not hard to describe the birational morphism \( T_+ \to T^+ \); it is a composition of the blow-ups of the singular points \( pr(L), pr(\overline{L}), pr(M), pr(\overline{M}) \), followed by contracting of the four exceptional divisors along their rulings. Because of the complexity of the notation, caused by the additional exceptional sets, we shall work on \( T_+ \), disregarding the difference between \( T_+ \) and \( T^+ \); the statement will not change substantially, if we work on \( T^+ \).

The (birational) conic bundle structure \( \{ q(l, m) : l, m \in \mathbb{P}_1 \times \mathbb{P}_1 \} \) on \( T_+ \) determines (the non-trivial component of) the discriminant curve \( \Delta \subset \mathbb{P}_1 \times \mathbb{P}_1 \) of \( s_+ \). Clearly, \( \Delta \) is a smooth curve of bidegree \((4,4)\) on the quadric \( \mathbb{P}_1 \times \mathbb{P}_1 \).

Let \( l \in \mathbb{P}_1 \) be general. Then the surface \( S_4(l) = pr(p^{-1}(l)) \subset T_+ \) is an anticanonically embedded del Pezzo surface of degree 4. The map \( s_+ : S_4(l) \to [l] \times \mathbb{P}_1 \) defines a separated conic bundle structure on \( S_4(l) \), degenerated in the 4-tuple \( \Delta \cap ([l] \times \mathbb{P}_1) \). Let \( \tilde{\Delta} \) be the double discriminant curve for \( s_+ \); \( \tilde{\Delta} \) is isomorphic to the curve of components of the degenerated fibers over \( \Delta \).

Let \( g_p \subset \text{Symm}^4(\Delta) \) be the 4-gonal system on \( \Delta \) defined by the set of effective divisors \( \{ [l] \times \mathbb{P}_1 : l \in \mathbb{P}_1 \} \) (similarly – for \( g_q \)), and let \( s^+_*(g_p) = \{ L \in \text{Symm}^4(\tilde{\Delta}) : (s_+)_*(L) \in g_p \} \). Let \( l \in \mathbb{P}_1 \), and \( S_4(l) \) be as above. The set of sixteen (-1)-curves on the anticanonically embedded \( S_4(l) \) coincide with the set of lines on \( S_4(l) \). The map \( p = p_1 \) defines a splitting of this set into two “equal” parts: Eight of these curves come from the components of the
degenerated fibers $p^{-1}(x), (x) \in (l \cap \Delta_p) - (x)_o$, and the second 8-tuple is the set of these lines on $S_4(l)$ which are components of the four degenerated $s$-conics $s_+^{-1}(u), u \in \Delta \cap ([l] \times P^1_q)$. The lines from the second 8-tuple are components of fibers of the conic bundle structure $s_+$. However, the lines from the first 8-tuple are “sections” of $s_+$ — the map $s_+$ sends each of these lines isomorphically onto a line on the base quadric $P^1_p \times P^1_q$. Each of the lines of the first 8-tuple intersects exactly 4 lines of the second 8-tuple. Moreover, if $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is such a 4-tuple of lines (of the 2-nd system), then $(s_+)_*(\lambda_1 + ... + \lambda_4) = ([l] \times P^1_q) \cap \Delta$.

The last causes a splitting of the natural preimage $(s_+)^*(((l] \times P^1_q) \cap \Delta)$ of $((l] \times P^1_q) \cap \Delta$, in $Symm^4(\Delta)$, into the following two subsets — each of cardinality 8:

(1). The set $\tilde{\Delta}_p^+(l)$ = the set of 4-tuples defined by the intersections with the (projections of) the 8 components of the degenerated fibers $p^{-1}(x), (x) \in (l \cap \Delta_p) - (x)_o$;

(2). The complimentary set $\tilde{\Delta}_p^-(l) := (s_+)^*(((l] \times P^1_q) \cap \Delta) - \tilde{\Delta}_p^+(l)$.

Clearly, this splitting does not depend on the particular choice of the general line $l \in P^1_p$. Therefore, it defines a global splitting $(s_+)^*(g_p) = \tilde{\Delta}_p^+ \cup \tilde{\Delta}_p^-$. Obviously, the component $\tilde{\Delta}_p^+$ is isomorphic to the non-singular model of the double discriminant curve $\tilde{\Delta}_p$ for the projection $p : T \to P^2$.

There are naturally defined involutions $i_p^+: \tilde{\Delta}_p^+ \to \tilde{\Delta}_p^+$ and $i_p^-: \tilde{\Delta}_p^- \to \tilde{\Delta}_p^-$, defined by interchanging of the 4-tuple $\lambda_1, ..., \lambda_4$ by its completion $\overline{\lambda}_1, ..., \overline{\lambda}_4$. (By definition, $\lambda_i + \overline{\lambda}_i, i = 1, ..., 4$, are the four degenerated conics of $s_+: S_4(l) \to (l] \times P^1_q)$).

In fact, the 4-tuples $(\lambda_1, ..., \lambda_4) \in \tilde{\Delta}_p^+$ are (-1)-curves on $S_4(l)$; the same — for the complimentary 4-tuples. Denote by $S(\lambda_1, ..., \lambda_4)$ the ruled surface, which is defined by contraction of the complimentary 4-tuple $(\overline{\lambda}_1, ..., \overline{\lambda}_4)$. It follows from the definition of the elements of $\tilde{\Delta}_p^+$ ( = the existence of a secant (-1)-curve — see above) that $S(\lambda_1, ..., \lambda_4) \cong F_1$.

Similarly, the 4-tuples which belong to the component $\tilde{\Delta}_p^-$ correspond to the relatively minimal models $S(\overline{\lambda}_1, ..., \overline{\lambda}_4)$, of the surfaces $S_4(l)$, which are of type $F_0$ (i.e. — quadrics).

Let $\Delta^+_p = \tilde{\Delta}_p^+/i_p^+$ and $\Delta^-_p = \tilde{\Delta}_p^-/i_p^-$ be the quotient curves. Obviously, the natural 8-sheeted coverings $\tilde{\Delta}_p^+ \to P^1_p$ and $\tilde{\Delta}_p^- \to P^1_p$ define the 4-sheeted coverings (the 4-gonal systems): $g_p^+: \Delta^+_p \to P^1_p$ and $g_p^-: \Delta^-_p \to P^1_p$.

In fact, we restored the tetragonal construction of Donagi (see e.g.[Do]). Therefore, we proved the following (see the notation above):
(7.2). Proposition.

\{ (\tilde{\Delta}, \Delta), (\tilde{\Delta}^+_p, \Delta^+_p), (\tilde{\Delta}^-_p, \Delta^-_p) \} is a 4-gonal triple of Donagi (see [Do]). Moreover, \( \tilde{\Delta}^+_p \) is isomorphic to the smooth model of the nodal discriminant plane sextic \( \Delta_p \subset \mathbb{P}^2 \), and the involution \( i^+_p : \tilde{\Delta}^+_p \to \tilde{\Delta}^-_p \) is a desingularization of the involution \( i_p \) on \( \tilde{\Delta}_p \) (defined by the covering \( \tilde{\Delta}_p \to \Delta_p \)).

Clearly, the same is true also for the 4-gonal triple
\( \{ (\tilde{\Delta}, \Delta), (\tilde{\Delta}^+_q, \Delta^+_q), (\tilde{\Delta}^-_q, \Delta^-_q) \} \) of Donagi, which corresponds to the 4-gonal system \( g_q \) on the (4,4)-curve \( \Delta \). (Just like above, the curve \( \tilde{\Delta}^-_q \) is isomorphic to the smooth model of the double discriminant curve \( \tilde{\Delta}_q \) of \( q : T \to \mathbb{P}^2 \).)

(7.3). Corollary. Let \( T = T(2,2) \) be a general nodal bidegree (2,2) divisor, and let \( (\tilde{\Delta}_p, \Delta_p) \) and \( (\tilde{\Delta}_q, \Delta_q) \) be the discriminant pairs for the natural projections \( p : T \to \mathbb{P}^2 \) and \( q : T \to \mathbb{P}^2 \). Let \( (\tilde{\Delta}, \Delta) \) be the discriminant pair of the conic bundle structure \( s_+ : T^+ \to \mathbb{P}^1 \times \mathbb{P}^1 \) defined by the node of \( T \) (see above), and let \( g_p \) and \( g_q \) be the 4-gonal systems on the (4,4) curve \( \Delta \) defined by the rulings of the quadric \( \mathbb{P}^1 \times \mathbb{P}^1 \). Then \( g_p \) and \( g_q \) define the two tetragonal triples of Donagi:

\( \{ (\tilde{\Delta}, \Delta), (\tilde{\Delta}^+_p, \Delta^+_p), (\tilde{\Delta}^-_p, \Delta^-_p) \} \) and

\( \{ (\tilde{\Delta}, \Delta), (\tilde{\Delta}^+_q, \Delta^+_q), (\tilde{\Delta}^-_q, \Delta^-_q) \} \)

such that \( (\tilde{\Delta}^+_p, \Delta^+_p) \) is a desingularization of the nodal pair \( (\tilde{\Delta}_p, \Delta_p) \) (see above), and the pair \( (\tilde{\Delta}^+_q, \Delta^+_q) \) is a desingularization of \( (\tilde{\Delta}_q, \Delta_q) \). In other words, the Dixon correspondence between the discriminant pairs of the nodal bidegree (2,2) divisor \( T \) is a composition of two 4-gonal correspondences of Donagi.

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