FREE ABELIAN GROUP ACTIONS ON NORMAL PROJECTIVE VARIETIES:
SUB-MAXIMAL DYNAMICAL RANK CASE

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ABSTRACT. Let \(X\) be a normal projective variety of dimension \(n\) and \(G\) an abelian group of automorphisms such that all elements of \(G \setminus \{\text{id}\}\) are of positive entropy. Dinh and Sibony showed that \(G\) is actually free abelian of rank \(\leq n - 1\). The maximal rank case has been well understood by De-Qi Zhang. We aim to characterize the pair \((X, G)\) such that \(\text{rank} \ G = n - 2\).

1. INTRODUCTION

We work over the field \(\mathbb{C}\) of complex numbers. Let \(X\) be a normal projective variety. Denote by \(\text{NS}(X) := \text{Pic}(X)/\text{Pic}^0(X)\) the Néron–Severi group of \(X\), i.e., the finitely generated abelian group of Cartier divisors on \(X\) modulo algebraic equivalence. For a field \(F = \mathbb{Q}, \mathbb{R}\) or \(\mathbb{C}\), we denote by \(\text{NS}(X)_F\) the finite-dimensional \(F\)-vector space \(\text{NS}(X) \otimes_{\mathbb{Z}} F\). The first dynamical degree \(d_1(g)\) of an automorphism \(g \in \text{Aut}(X)\) is defined as the spectral radius of its natural pullback \(g^*\) on \(\text{NS}(X)_\mathbb{R}\), i.e.,

\[
d_1(g) := \rho(g^*|_{\text{NS}(X)_\mathbb{R}}) = \max \left\{|\lambda| : \lambda \text{ is an eigenvalue of } g^*|_{\text{NS}(X)_\mathbb{R}} \right\}.
\]

We say that \(g\) is of positive entropy if \(d_1(g) > 1\), otherwise it is of null entropy. For a subgroup \(G\) of the automorphism group \(\text{Aut}(X)\), we define the null-entropy subset of \(G\) as

\[
N(G) := \{g \in G : g \text{ is of null entropy, i.e., } d_1(g) = 1\}.
\]

We call \(G\) of positive entropy (resp. of null entropy), if \(N(G) = \{\text{id}\} \subseteq G\) (resp. \(N(G) = G\)). Indeed, when \(X\) is smooth and hence \(X(\mathbb{C})\) is a compact Kähler manifold, our positivity notion of entropy is compatible with the one in complex dynamics. We refer to [DS17, §4] and references therein for a comprehensive exposition on dynamical degrees, topological and algebraic entropies.

In [DS04], Dinh and Sibony proved that for any abelian subgroup \(G\) of \(\text{Aut}(X)\), if \(G\) is of positive entropy, then \(G\) is free abelian of rank \(\leq \dim X - 1\). This was subsequently extended by De-Qi Zhang [Zha09] to the solvable group case. We are thus interested in...
algebraic varieties admitting the action of free abelian groups of positive entropy. Therefore, it is meaningful for us to consider the following hypothesis.

**Hyp** \((n, r)\). \(X\) is a normal projective variety of dimension \(n\) and \(G \simeq \mathbb{Z}^r\) is a subgroup of \(\text{Aut}(X)\) with \(1 \leq r \leq n - 1\), such that \(G\) is of positive entropy, i.e., all elements of \(G \setminus \{\text{id}\}\) are of positive entropy.

Often, we shall call the above positive integer \(r\) the dynamical rank of \(G\) to emphasize that \(G\) is of positive entropy in the context of dynamics, not just being a free abelian group. See section 2.3 for a more general consideration on dynamical ranks.

In the last years, the maximal dynamical rank case \(r = n - 1\) has been intensively studied by De-Qi Zhang in his series papers (see e.g. [Zha09, Zha13, Zha16]), which extend the known surface case [Can99] to higher dimensions. We rephrase one of his main results as follows.

**Theorem 1.1** (cf. [Zha16, Theorems 1.1 and 2.4]). Let \((X, G)\) satisfy Hyp\((n, n - 1)\) with \(n \geq 3\). Suppose that \(X\) is not rationally connected, or \(X\) has only \(\mathbb{Q}\)-factorial Kawamata log terminal (klt) singularities and the canonical divisor \(K_X\) is pseudo-effective. Then after replacing \(G\) by a finite-index subgroup, the following assertions hold.

1. There is a birational map \(X \dasharrow Y = A/F\) such that the induced action of \(G\) on \(Y\) is biregular, where \(A\) is an abelian variety and \(F\) is a finite group whose action on \(A\) is free outside a finite subset of \(A\).
2. The canonical divisor of \(Y\) is \(\mathbb{Q}\)-linearly equivalent to zero, i.e., \(K_Y \sim_{\mathbb{Q}} 0\).
3. There is a faithful action of \(G\) on \(A\) such that \(A \longrightarrow A/F = Y\) is \(G\)-equivariant.
   Every \(G\)-periodic proper subvariety of \(Y\) or \(A\) is a point.

In this note, we aim to investigate the sub-maximal dynamical rank case \(r = n - 2\). Below is our main result.

**Theorem 1.2.** Let \((X, G)\) satisfy Hyp\((n, n - 2)\) with \(n \geq 3\). Then the Kodaira dimension \(\kappa(X)\) of \(X\) is at most one. Moreover, after replacing \(G\) by a finite-index subgroup, one of the following assertions holds.

1. Suppose that \(\kappa(X) = 1\). Let \(F\) be a very general fiber of the Iitaka fibration \(X \dasharrow B\) of \(X\), where \(\dim B = 1\). Then \(G\) descends to a trivial action on the base curve \(B\) and acts faithfully on \(F\) such that \(F\) is \(G\)-equivariantly birational to a K3 surface, an Enriques surface, or a \(\mathbb{Q}\)-abelian variety (see Definition 2.1).
2. Suppose that \(X\) has only klt singularities and \(K_X \equiv 0\). Then there exists a finite cover \(Y \longrightarrow X\), étale in codimension one, such that \(Y\) is \(G\)-equivariantly birational to a weak Calabi–Yau variety (see Definition 2.4), an abelian variety, or a product of a weak Calabi–Yau surface and an abelian variety.
3. Suppose that \(X\) is uniruled. Let \(\pi : X \longrightarrow Z\) be the special MRC fibration of \(X\) (see Definition 2.10). Then either \(X\) is rationally connected, or \(Z\) is birational to
a curve of genus $\geq 1$, a K3 surface, an Enriques surface, or a $Q$-abelian variety $A/F$, where $A$ is an abelian variety and $F$ is a finite group whose action on $A$ is free outside a finite subset of $A$. In particular, if $\dim Z \geq 3$, then there exists a finite cover $X' \to X$, étale in codimension one, such that the induced rational map $\pi': X' \to A$ is $G$-equivariantly birational to the MRC fibration of $X'$.

Remark 1.3. (1) If $X$ has only klt singularities with $\kappa(X) = 0$, the minimal model program (MMP) predicts that there exists a minimal model $X_m$ of $X$ so that $K_{X_m} \sim_{\mathbb{Q}} 0$. Modulo this, one still has to consider the induced birational (not necessarily biregular) action of $G$ on $X_m$, since unlike the maximal dynamical rank case (cf. [Zha16, Proposition 3.11]), it is not clear that we can run a $G$-equivariant MMP to get the minimal model.

(2) For a normal projective variety $X$, the following is well known:

$$X \text{ is rationally connected} \implies X \text{ is uniruled} \implies \kappa(X) = -\infty.$$ 

However, the implication "$\kappa(X) = -\infty \implies X \text{ is uniruled}" is unknown and turns out to be closely related to one of the most important conjectures in birational geometry, namely, the Non-vanishing conjecture (cf. [BCHM10, Conjecture 2.1]; see also [BDPP13, Conjecture 0.1]). This is the reason that we assume $X$ to be uniruled in Theorem 1.2(3).

2. Preliminaries

Throughout this section, unless otherwise stated, $X$ is a normal projective variety of dimension $n$ defined over $\mathbb{C}$.

We refer to Kollár–Mori [KM98] for the standard definitions, notation, and terminologies in birational geometry. For instance, see [KM98, Definitions 2.34 and 5.8] for the definitions of canonical, Kawamata log terminal (klt), rational, and log canonical (lc) singularities.

The Kodaira dimension $\kappa(W)$ of a smooth projective variety $W$ is defined as the Kodaira–Iitaka dimension $\kappa(W, K_W)$ of the canonical divisor $K_W$. The Kodaira dimension of a singular variety is defined to be the Kodaira dimension of any smooth model.

We say that $X$ is uniruled, if there is a dominant rational map $\mathbb{P}^1_C \times Y \to X$ with $\dim Y = n - 1$. We call $X$ rationally connected, in the sense of Campana [Cam92] and Kollár–Miyaoka–Mori [KMM92], if any two general points of $X$ can be connected by an irreducible rational curve on $X$; when $X$ is smooth, this is equivalent to saying that any two points of $X$ can be connected by an irreducible rational curve (cf. [Kol96, IV.3]).

For a uniruled projective variety $X$, there exists a maximal rationally connected fibration (MRC fibration for short) $\pi: X \to Y$, in the sense of Campana and Kollár–Miyaoka–Mori, such that $Y$ is a non-uniruled normal projective variety (cf. [GHS03]) and a general fiber of $\pi$ is rationally connected. This fibration is unique up to birational equivalence (cf. [Kol96, IV.5]).
Definition 2.1. A normal projective variety $X$ is called $Q$-abelian if there are an abelian variety $A$ and a finite surjective morphism $A \to X$ which is étale in codimension one.

In general, given a $G$-action on an algebraic variety $V$, i.e., there is a group homomorphism $G \to \text{Aut}(V)$, we denote by $G|_V$ the image of $G$ in $\text{Aut}(V)$. The action of $G$ on $V$ is faithful, if $G \to \text{Aut}(V)$ is injective.

Let $G$ be a subgroup of the automorphism group $\text{Aut}(X)$ of $X$. A rational map $\pi : X \to Y$ is called $G$-equivariant if the $G$-action on $X$ descends to a biregular (possibly non-faithful) action on $Y$. In other words, for each $g_X \in G$, there is an automorphism $g_Y$ of $Y$ such that $\pi \circ g_X = g_Y \circ \pi$. We hence denote by $G|_Y$ the image of $G$ in $\text{Aut}(Y)$.

2.1. Weak decomposition. The famous Bogomolov–Beauville decomposition theorem asserts that for any compact Kähler manifold with numerically trivial canonical bundle, there is a finite étale cover that can be decomposed as a product of a torus, Calabi–Yau manifolds, and irreducible holomorphic symplectic manifolds (cf. [Bea83]). Recently, this has been very successfully generalized to normal projective varieties with only klt singularities and numerically trivial canonical divisors by Höring and Peternell [HP19], based on the previous significant work by Druel [Dru18], Greb, Guenancia, Kebekus and Peternell [GKP16, GGK19]. However, in this note, instead of utilizing their strong decomposition theorem, we shall work on a weaker version due to Kawamata [Kaw85] and developed by Nakayama–Zhang [NZ10]; see Remark 3.5 for a brief explanation.

We begin with the definition of the so-called augmented irregularity. Note that the irregularity of normal projective varieties is generally not invariant under étale in codimension one covers.

Definition 2.2 (Augmented irregularity). Let $X$ be a normal projective variety. The irregularity of $X$ is defined by $q(X) := h^1(X, \mathcal{O}_X)$. The following is called the augmented irregularity of $X$ (which is a variant of the usual irregularity):

$$
\tilde{q}(X) := \sup \{ q(Y) : Y \to X \text{ is finite, surjective, and étale in codimension one} \}.
$$

Remark 2.3. (1) Let $X$ be a normal projective variety with only klt singularities such that $K_X \sim_{\mathbb{Q}} 0$. Then $\tilde{q}(X) \leq \dim X$. See e.g. [NZ10, Proposition 2.10].

(2) The augmented irregularity is invariant under étale in codimension one covers. Namely, if $Y \to X$ is étale in codimension one, then $\tilde{q}(Y) = \tilde{q}(X)$. Clearly, $\tilde{q}(Y) \leq \tilde{q}(X)$ by the definition. On the other hand, any two étale in codimension one covers of $X$ is dominated by a third one so that $\tilde{q}(Y) \geq \tilde{q}(X)$.

Definition 2.4 (Weak Calabi–Yau variety). A normal projective variety $X$ is called a weak Calabi–Yau variety, if

- $X$ has only canonical singularities,
- the canonical divisor $K_X \sim 0$, and
• the augmented irregularity $\tilde{q}(X) = 0$.

Remark 2.5. (1) Note that a two-dimensional weak Calabi–Yau variety is exactly a normal projective surface with du Val singularities such that its minimal resolution is a K3 surface and that there is no finite surjective morphism from any abelian surface.

(2) It is worth mentioning that those smooth Calabi–Yau threefolds of quotient type A or K in the sense of [OS01] are, however, not weak Calabi–Yau according to the above definition. See also [GGK19, §14.2]. It is a natural question whether the topological fundamental group $\pi_1(X)$ of a weak Calabi–Yau variety $X$ is finite; one can also ask a similar question for the étale fundamental group $\hat{\pi}_1(X_{\text{reg}})$ of the smooth locus $X_{\text{reg}}$ of $X$.

Lemma 2.6 (cf. [NZ10, Lemma 2.12]). Let $X$ be a normal projective variety with only klt singularities such that $K_X \sim_{\mathbb{Q}} 0$. Then there exists a finite morphism $\tau : X^{\text{alb}} \rightarrow X$ satisfying the following conditions, uniquely up to isomorphism over $X$:

1. $\tau$ is étale in codimension one.
2. $\tilde{q}(X) = q(X^{\text{alb}})$.
3. $\tau$ is Galois.
4. If $\tau' : X' \rightarrow X$ satisfies the conditions (1) and (2), then there exists a finite surjective morphism $\sigma : X' \rightarrow X^{\text{alb}}$, étale in codimension one, such that $\tau' = \tau \circ \sigma$.

The above Galois cover $\tau$ is called the Albanese closure of $X$ in codimension one by Nakayama and Zhang; a similar result for smooth projective varieties could be found in [Bea83]. Here, the significance is that we are able to lift the group actions to this covering space.

Lemma 2.7 (cf. [NZ10, Proposition 3.5]). Let $X$ be a normal projective variety with at most klt singularities such that $K_X \sim_{\mathbb{Q}} 0$, and $f$ an automorphism of $X$. Then there exist a morphism $\pi : \tilde{X} \rightarrow X$ from a normal projective variety $\tilde{X}$, an automorphism $\tilde{f}$ of $\tilde{X}$ such that the following conditions hold.

1. $\pi$ is finite surjective and étale in codimension one.
2. $\tilde{X}$ is isomorphic to the product variety $Z \times A$ for a weak Calabi–Yau variety $Z$ (see Definition 2.4) and an abelian variety $A$.
3. The dimension of $A$ equals the augmented irregularity $\tilde{q}(X)$ of $X$.
4. There are automorphisms $\tilde{f}_Z$ and $\tilde{f}_A$ of $Z$ and $A$, respectively, such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\pi} & \tilde{X} \\
\downarrow f & & \downarrow \tilde{f} \\
X & \xrightarrow{\pi} & \tilde{X} \\
\end{array}
\begin{array}{ccccccc}
& & \simeq & & Z \times A \\
& & \downarrow f_Z \times \tilde{f}_A & & \\
& & \simeq & & Z \times A \\
\end{array}
$$
Proof. For the convenience of the reader, we sketch their proof as follows. First, let us take the global index-one cover $X_1 \to X$, which is a finite surjective morphism and étale in codimension one, such that $X_1$ has only canonical singularities with $K_{X_1} \sim 0$ (cf. [KM98, Definition 5.19]). The uniqueness of the global index-one cover asserts that the automorphism $f$ can be lifted to an automorphism $f_1$ on $X_1$. So at the expense of replacing $(X, f)$ by $(X_1, f_1)$, we may assume that $X$ has only canonical singularities with $K_X \sim 0$.

Next, let $\tau: X_{\text{alb}} \to X$ be the Albanese closure of $X$ in codimension one, whose existence is guaranteed by Lemma 2.6. It thus follows from the uniqueness of $\tau$ that we can lift $f$ to an automorphism $f_{\text{alb}}$ on $X_{\text{alb}}$. More precisely, applying Lemma 2.6(4) to $f \circ \tau$, there exists a finite surjective morphism $f_{\text{alb}}: X_{\text{alb}} \to X_{\text{alb}}$ such that $f \circ \tau = \tau \circ f_{\text{alb}}$. Clearly, $f_{\text{alb}}$ is an automorphism since so is $f$. Therefore, replacing $(X, f)$ by $(X_{\text{alb}}, f_{\text{alb}})$ if necessary, we may assume further that $\tilde{q}(X) = q(X)$.

Note that the augmented irregularity is invariant under étale in codimension one covers; see e.g. Remark 2.3(2). Hence, the above $\tilde{q}(X)$ is indeed equal to the augmented irregularity of the original $X$, even though we have replaced our $X$ by new models.

Lastly, under the above assumptions, the Albanese morphism $\text{alb}_X: X \to A := \text{Alb}(X)$ turns out to be an étale fiber bundle, i.e., there is an isogeny $\phi: B \to A$ such that $X \times_A B \simeq Z \times B$, where $Z$ is a fiber of $\text{alb}_X$ (cf. [Kaw85, Theorem 8.3]). It is not hard to verify that $Z$ is a weak Calabi–Yau variety by its definition and also the definition of the augmented irregularity $\tilde{q}(X)$. Clearly, there is an induced automorphism of $A$ by the universal property of the Albanese morphism $\text{alb}_X$; denote it by $f_A$. Take an isogeny $\psi: A \to B$ further so that $\phi \circ \psi = [m_A]$ is just the multiplication-by-$m$ map on $A$ for some positive integer $m$. Then there is an automorphism $\tilde{f}_A$ of $A$ such that $[m_A] \circ \tilde{f}_A = f_A \circ [m_A]$. Consider a new fiber product $\tilde{X} := X \times_A A$ of $\text{alb}_X: X \to A$ and $[m_A]: A \to A$. Let $\pi: \tilde{X} \to X$ denote the finite étale cover induced from the first projection. Then $\tilde{X} \simeq Z \times A$ for the same fiber $Z$ of $\text{alb}_X$ as above. It is clear that those automorphisms $f_1, f_A$ and $\tilde{f}_A$ induce an automorphism $\tilde{f}$ on $\tilde{X}$ satisfying that $\pi \circ \tilde{f} = f \circ \pi$. Note that as a weak Calabi–Yau variety, $Z$ is nonruled and has only canonical (and hence rational by [KM98, Theorem 5.22]) singularities, and its augmented irregularity vanishes. It thus follows from Lemma 2.9 that the induced automorphism of $\tilde{f}$ on $Z \times A$ splits as $\tilde{f}_Z \times \tilde{f}_A$. In other words, we have the
following commutative diagram endowed with equivariant group actions:

Finally, in view of the Albanese morphism \( \text{alb}_X \), we see that \( \dim A = q(X) = \tilde{q}(X) \).

**Remark 2.8.** (1) In the above lemma, by Nakayama’s celebrated result on the Abundance conjecture in the Kodaira dimension zero case (cf. [Nak04, Corollary V.4.9]), we can replace the condition "\( K_X \sim_Q 0 \)" by "\( K_X \equiv 0 \)". When \( X \) has only canonical singularities, this was originally due to Kawamata; see [Kaw85, Theorem 8.2].

(2) For any subgroup \( G \leq \text{Aut}(X) \), the action of \( G \) on \( X \) extends to a faithful action on \( \tilde{X} \), denoted by \( \tilde{G} \), which then splits as a subgroup of \( G|_Z \times G|_A \) by the following lemma. Note that the action of \( G \) on \( X \) can be identified with a not necessarily faithful action of \( \tilde{G} \) on \( X \) (with finite kernel). If \( G \simeq \mathbb{Z}^r \) which is the case in this note, we can always apply [Zha13, Lemma 2.4] so that a finite-index subgroup of \( \tilde{G} \) also acts faithfully on \( X \).

Below is a simple variant of Nakayama and Zhang’s splitting criterion for automorphisms of certain product varieties.

**Lemma 2.9** (cf. [NZ10, Lemma 2.14]). Let \( Z \) be a nonruled normal projective variety with only rational singularities, and \( A \) an abelian variety. Suppose that \( q(Z) = 0 \). Then any automorphism \( f \) of \( Z \times A \) splits, i.e., there are suitable automorphisms \( f_Z \) and \( f_A \) of \( Z \) and \( A \), respectively, such that \( f = f_Z \times f_A \).

2.2. **Special MRC fibration.** In this subsection, we collect basic materials on the special MRC fibration introduced by Nakayama [Nak10].

**Definition 2.10** (Nakayama). Given a projective variety \( X \), a dominant rational map \( \pi : X \dashrightarrow Z \) is called the special MRC fibration of \( X \), if it satisfies the following conditions:

1. \( Z \) is a non-uniruled normal projective variety (cf. [GHS03]).
2. The graph \( \Gamma_\pi \subseteq X \times Z \) of \( \pi \) is equidimensional over \( Z \).
3. A general fiber of \( \Gamma_\pi \dashrightarrow Z \) is rationally connected.
4. If \( \pi' : X \dashrightarrow Z' \) is a dominant rational map satisfying (1)--(3), then there is a birational morphism \( \nu : Z' \dashrightarrow Z \) such that \( \pi = \nu \circ \pi' \).
The existence and the uniqueness (up to isomorphism) of the special MRC fibration is proved in [Nak10, Theorem 4.18]. One of the advantages of the special MRC is the descent property of endomorphisms (cf. [Nak10, Theorem 4.19]).

**Lemma 2.11.** Let \( \pi : X \to Z \) be the special MRC fibration, and \( G \leq \text{Aut}(X) \). Then \( G \) descends to a biregular action on \( Z \), denoted by \( G|_Z \). Moreover, there exist a birational morphism \( p : W \to X \) and an equidimensional surjective morphism \( q : W \to Z \) satisfying the following conditions:

1. \( W \) is a normal projective variety.
2. A general fiber of \( q \) is rationally connected.
3. Both \( p \) and \( q \) are \( G \)-equivariant.

**Proof.** By [Nak10, Theorem 4.19], \( G \) descends to a biregular action on \( Z \). We take \( W \) as the normalization of the graph \( \Gamma_\pi \) of \( \pi \) which admits a natural faithful \( G \)-action. Then (2) follows readily from Definition 2.10, while (3) the \( G \)-equivariance of \( \pi \). \( \square \)

**Lemma 2.12** (cf. [NZ10, Lemma 4.4]). With notation as in Lemma 2.11, let \( \theta_Z : Z' \to Z \) be a \( G|_Z \)-equivariant finite surjective morphism from a normal projective variety \( Z' \). Then there exist finite surjective morphisms \( \theta_X : X' \to X \) and \( \theta_W : W' \to W \), a birational morphism \( p' : W' \to X' \), and an equidimensional surjective morphism \( q' : W' \to Z' \) satisfying the following conditions:

1. Both \( X' \) and \( W' \) are normal projective varieties.
2. A general fiber of \( q' \) is rationally connected.
3. \( \pi' := q' \circ p'^{-1} \) is \( G \)-equivariantly birational to the MRC fibration of \( X' \).
4. In the commutative diagram below, every morphism or rational map other than \( \theta_Z \) is \( G \)-equivariant.

Moreover, if \( \theta_Z \) is étale in codimension one, then so are \( \theta_X \) and \( \theta_W \).

**Proof.** Let \( W' \) be the normalization of the fiber product \( W \times_Z Z' \). Denote by \( \theta_W : W' \to W \) and \( q' : W' \to Z' \) the morphisms induced from the first and second projections, respectively. Then \( q' \) is an equidimensional surjective morphism whose general fibers are rationally connected varieties and in particular irreducible, since so is \( q \). Here we use the fact that smooth rationally connected varieties are simply connected. This forces \( W' \) to be
irreducible and hence $W'$ is a normal projective variety. Clearly, the $G$-actions on $W$ and $Z'$ can be naturally extended to $W \times Z Z'$ and hence to $W'$, which is faithful since $G$ acts faithfully on $W$. Note that $Z'$ is non-uniruled since so is $Z$. It follows that $q'$ is $G$-equivariantly birational to the special MRC fibration of $W'$ by Definition 2.10. Taking the Stein factorization of the composite $p \circ \theta_W: W' \to W \to X$, we then have a birational morphism $p': W' \to X'$ and a finite morphism $\theta_X: X' \to X$ for a normal projective variety $X'$ such that $p \circ \theta_W = \theta_X \circ p'$; furthermore, the faithful $G$-actions on $W'$ and $X$ also induce a faithful $G$-action on $X'$. Since the notion of the MRC fibration is essentially birational in nature, $\pi' = q' \circ p'^{-1}$ is also $G$-equivariantly birational to the MRC fibration of $X'$. So all conditions (1)–(4) have been satisfied.

The last part follows from the fact that being étale is a local property stable under base change. □

2.3. Dynamical ranks. In this section, we shall consider the dynamical rank of group actions in a much more general setting. We first recall the following Tits alternative type theorem due to De-Qi Zhang [Zha09].

**Theorem 2.13** (cf. [Zha09]). Let $X$ be a normal projective variety of dimension $n \geq 2$ and $G$ a subgroup of $\text{Aut}(X)$. Then one of the following two assertions holds.

1. $G$ contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} \ast \mathbb{Z}$.
2. There is a finite-index subgroup $G_1$ of $G$ such that the induced group $G_1|_{\text{NS}(X)_{\mathbb{R}}}$ is solvable and $Z$-connected. Moreover, the null-entropy subset $N(G_1)$ of $G_1$ is a normal subgroup of $G_1$ and the quotient group $G_1/N(G_1)$ is free abelian of rank $r \leq n - 1$.

**Remark 2.14.** In general, the induced group $G|_{\text{NS}(X)_{\mathbb{R}}}$ of $G$ is called $Z$-connected if its Zariski closure in $\text{GL}(\text{NS}(X)_{\mathbb{C}})$ is connected with respect to the Zariski topology. Note that being $Z$-connected is only a technical condition for us to apply the theorem of Lie–Kolchin type for a cone in [KOZ09]. Actually, it is always satisfied by replacing the group with a finite-index subgroup (see e.g. [DHZ15, Remark 3.10]). We will frequently use this fact without mentioning it very precisely.

We also remark that in the second assertion of the above Theorem 2.13, the rank of $G_1/N(G_1)$ is independent of the choice of $G_1$. Hence, it makes sense to think of this as an invariant of $G$. We introduce the following notion of dynamical rank in a much broader sense.

**Definition 2.15** (Dynamical rank). Let $X$ be a normal projective variety of dimension $n$ and $G$ a subgroup of $\text{Aut}(X)$ such that $G|_{\text{NS}(X)_{\mathbb{R}}}$ is solvable. Then the rank of the free abelian group $G/N(G)$ is called the dynamical rank of $G$, and denoted by $\text{dr}(G)$.

As one may have noticed, we suppress the condition "$G|_{\text{NS}(X)_{\mathbb{R}}}$ is $Z$-connected". This does not affect the well-definedness of our dynamical rank according to Remark 2.14.
Sometimes, we may write $\text{dr}(G|_X)$ to emphasize that it is the dynamical rank of the group $G$ acting on $X$. Conventionally, the dynamical rank of a group of null entropy is always zero.

We first quote the following result which generalizes [Zha09, Lemma 2.10].

**Lemma 2.16** (cf. [Hu, Lemmas 4.1 and 4.3]). Let $\pi : X \to Y$ be a $G$-equivariant dominant rational map of normal projective varieties with $n = \dim X > \dim Y = m > 0$. Suppose that $G|_{\text{NS}(X)_R}$ is solvable. Then so is $G|_{\text{NS}(Y)_R}$, and we have

$$\text{dr}(G|_X) \leq \text{dr}(G|_Y) + n - m - 1.$$  

In particular, $\text{dr}(G|_X) = n - 2$ only if $\text{dr}(G|_Y) = m - 1$.

The lemma below asserts that our dynamical rank is actually a birational invariant. See also [Zha16, Lemma 3.1] for a similar treatment.

**Lemma 2.17** (cf. [Hu, Lemmas 4.2 and 4.4]). Let $\pi : X \to Y$ be a $G$-equivariant generically finite dominant rational map of normal projective varieties. Then after replacing $G$ by a finite-index subgroup, $G|_{\text{NS}(X)_R}$ is solvable if and only if so is $G|_{\text{NS}(Y)_R}$. Moreover, $\text{dr}(G|_X) = \text{dr}(G|_Y)$.

### 3. Proof of Theorem 1.2

The theorem will follow immediately from the following lemmas. Each one will correspond to one assertion of Theorem 1.2.

**Lemma 3.1.** Let $(X, G)$ satisfy $\text{Hyp}(n, n - 2)$ with $n \geq 3$. Suppose that the Kodaira dimension $\kappa(X)$ of $X$ is positive. Then $\kappa(X) = 1$ and there exists a dominant rational fibration $\phi : X \to B$ for some curve $B$ such that after replacing $G$ by a finite-index subgroup, the following assertions hold.

1. $G$ descends to a trivial action on the base curve $B$ of $\phi$.
2. Let $F$ be a very general fiber $F$ of $\phi$. Then the induced $G$-action on $F$ is faithful such that the pair $(F, G|_F)$ satisfies $\text{Hyp}(n - 1, n - 2)$. Moreover, $F$ is $G$-equivariantly birational to a K3 surface, an Enriques surface, or a $Q$-abelian variety (see Definition 2.1).

**Proof.** Let $\phi := \Phi_{|mK_X|} : X \to B \subseteq \mathbb{P}(H^0(X, mK_X))$ be the Iitaka fibration of $X$ with $B$ the image of $\Phi_{|mK_X|}$ for $m \gg 0$. It follows from the Deligne–Nakamura–Ueno theorem (cf. [Uen75, Theorem 14.10]) that $G$ descends to a finite group $G|_B$ acting on $B$ biregularly. Replacing $G$ by a finite-index subgroup, which does not change its dynamical rank, we may assume that $G|_B = \{\text{id}\}$. Further, replacing $X$ and $B$ by $G$-equivariant resolutions of indeterminacy locus of $\phi$ and singularities of $B$, we may also assume that $\phi$ is a regular morphism and $B$ is smooth, since by Lemma 2.17 the new pair $(X, G)$ still satisfies $\text{Hyp}(n, n - 2)$. If $\kappa(X) = n$, i.e., $X \to B$ is birational, then again thanks to Lemma 2.17, we have $n - 2 = \text{dr}(G|_X) = \text{dr}(G|_B) = 0$, a contradiction. So we may assume that
0 < \kappa(X) < n$, which yields that $\phi: X \to B$ is a non-trivial $G$-equivariant fibration. It thus follows from Lemma 2.16 that

$$n - 2 = \text{dr}(G|_X) \leq \text{dr}(G|_B) + n - \dim B - 1 = n - \kappa(X) - 1,$$

and hence $\kappa(X) = 1$ so that $B$ is a curve.

It remains to show the assertion (2). Since $G$ acts trivially on the base, $G$ acts naturally on the very general fiber $F$ of $\phi$. For any $g \in G$, let $g_F$ denote the induced automorphism of $g$ on $F$. By the product formula (cf. [DN11, Theorem 1.1]), the first dynamical degree $d_1(g_F)$ of $g_F$ equals $d_1(g)$ which is larger than 1 if $g \neq \text{id}$. Therefore, $G$ acts faithfully on $F$ so that we can identify $G$ with $G|_F \leq \text{Aut}(F)$ and $(F, G|_F)$ satisfies $\text{Hyp}(n - 1, n - 2)$.

Lastly, note that $F$, as a very general fiber of the Iitaka fibration, has Kodaira dimension zero and hence is not rationally connected. Then Zhang’s Theorem 1.1 yields that, up to replacing $G$ by a finite-index subgroup, $F$ is $G$-equivariantly birational to a $\mathbb{Q}$-abelian variety if $\dim F \geq 3$ or equivalently $n \geq 4$. On the other hand, if $\dim F = 2$, since it admits an automorphism of positive entropy, it is well known that our $F$ is either a K3 surface, an Enriques surface, or an abelian surface (cf. [Can99, Proposition 1]).

\textbf{Remark 3.2.} Using a similar proof of the above lemma, one can also show the following result. Let $(X, G)$ satisfy $\text{Hyp}(n, r)$ with $1 \leq r \leq n - 2$. If the Kodaira dimension $\kappa(X)$ of $X$ is positive, then $\kappa(X) \leq n - r - 1$ (this is actually not new; see [Zha09, Lemma 2.11]). Moreover, after replacing $G$ by a finite-index subgroup, we may assume that $G$ acts trivially on $B$ and naturally on the very general fiber $F$ of the Iitaka fibration $\phi: X \dashrightarrow B$ with $\dim F = n - \kappa(X)$. Better still, the product formula asserts that for each $g \in G \setminus \{\text{id}\}$, the restriction $g_F$ of $F$ is of positive entropy since so is $g$. Hence, the $G$-action on $F$ is faithful and $(F, G|_F)$ satisfies $\text{Hyp}(n - \kappa(X), r)$. In summary, we have the following reduction:

\[ \text{Hyp}(n, r) \text{ with } \kappa > 0 \rightsquigarrow \text{Hyp}(n', r) \text{ with } \kappa = 0 \text{ and } n' < n. \]

\textbf{Lemma 3.3.} Let $(X, G)$ satisfy $\text{Hyp}(n, n - 2)$ with $n \geq 3$. Suppose that $X$ has only klt singularities and $K_X \equiv 0$. Then after replacing $G$ by a finite-index subgroup, there exist a finite cover $Y \to X$, étale in codimension one, and a faithful $G$-action on $Y$ such that $Y$ is $G$-equivariantly birational to one of the following varieties:

1. an abelian variety $A$, where $(A, G|_A)$ satisfies $\text{Hyp}(n, n - 2)$;
2. a weak Calabi–Yau variety $Z$, where $(Z, G|_Z)$ satisfies $\text{Hyp}(n, n - 2)$;
3. a product of a weak Calabi–Yau surface $S$ and an abelian variety $A$, where $(S, G|_S)$ and $(A, G|_A)$ satisfy $\text{Hyp}(2, 1)$ and $\text{Hyp}(n - 2, n - 3)$, respectively.

\textbf{Proof.} It follows from Lemma 2.7 and Remark 2.8 that there is a finite cover $\tilde{X} \to X$, étale in codimension one, such that $\tilde{X} \cong Z \times A$ for a weak Calabi–Yau variety $Z$ and an abelian variety $A$ of dimension $\tilde{q}(X)$, the augmented irregularity of $\tilde{X}$; furthermore, the action of $G$ on $X$ extends to a faithful action of $\tilde{G}$ on $\tilde{X}$. Replacing $\tilde{G}$ by a finite-index subgroup, we may assume that $\tilde{G}$ also acts faithfully on $X$ and can be identified with a
finite-index subgroup of $G$ (cf. [Zha13, Lemma 2.4]). Therefore, after replacing $G$ by the above mentioned finite-index subgroup, we may assume that $(\tilde{X}, G)$ satisfies Hyp$(n, n - 2)$ by Lemma 2.17 since so does $(X, G)$. We hence have the following three cases to analyze.

Case 1. $\tilde{q}(X) = n$ and hence $\tilde{X} = A$ is an abelian variety. In this case, the pair $(A, G|_A)$ satisfies Hyp$(n, n - 2)$ and we just take $Y$ to be $A$.

Case 2. $\tilde{q}(X) = 0$ and hence $\tilde{X} = Z$ is a weak Calabi–Yau variety of dimension $n$. So the pair $(Z, G|_Z)$ also satisfies Hyp$(n, n - 2)$. We then choose $Y$ to be $Z$.

Case 3. $0 < \tilde{q}(X) < n$ so that $\tilde{X}$ is an actual product $Z \times A$ with each factor being positive-dimensional. According to Lemma 2.9, we denote by $G|_Z$ and $G|_A$ the induced group actions of $G$ on $Z$ and $A$, respectively; note that both are finitely generated abelian groups. It follows from Lemma 2.16 that $\dim(G|_Z) = \dim Z - 1 =: r_1$ and $\dim(G|_A) = \dim A - 1 =: r_2$. Applying [DS04, Theorem I] to the pair $(A, G|_A)$ yields that the null-entropy subgroup $N(G|_A)$ of $G|_A$ is finite. So, up to replacing $G|_A$ and hence $G$ by a finite-index subgroup, we may assume that $G|_A \simeq \mathbb{Z}^{r_2}$ is a free abelian group of positive entropy. Thanks to Lemma 2.17, the same argument applies to the $G|_Z$-equivariant resolution of $Z$. Thus we can assume that $G|_Z \simeq \mathbb{Z}^{r_1}$ is of positive entropy. In particular, $(Z, G|_Z)$ and $(A, G|_A)$ satisfy Hyp$(r_1 + 1, r_1)$ and Hyp$(r_2 + 1, r_2)$, respectively.

If $\dim Z = 2$ (i.e., $r_1 = 1$), then $Z$ is just a weak Calabi–Yau surface $S$. So in this case we take $Y$ to be $\tilde{X} \simeq S \times A$.

Let us consider the case when $\dim Z \geq 3$. Recall that as a weak Calabi–Yau variety (see Definition 2.4), $Z$ is not rationally connected and has only canonical singularities with $K_Z \sim 0$. So applying Theorem 1.1 to $(Z, G|_Z \simeq \mathbb{Z}^{r_1})$ asserts that, up to replacing $G|_Z$ and hence $G$ by a finite-index subgroup, $Z$ is birational to a Q-abelian variety $B/F$ such that the induced action of $G|_Z$ on $B/F$ is biregular, where $B$ is an abelian variety and $F$ is a finite group whose action on $B$ is free outside a finite subset of $B$; moreover, there is a faithful action of $G|_Z$ on $B$ such that $B \to B/F$ is $G|_Z$-equivariant. Clearly, the pair $(B, G|_B = G|_Z)$ satisfies Hyp$(r_1 + 1, r_1)$ since so does $(Z, G|_Z)$. Let $\tilde{Z}$ be the normalization of the fiber product $Z \times_{B/F} B$, which inherits a natural faithful $G|_Z$-action. Then the induced projection $\tilde{Z} \to Z$ is finite surjective and étale in codimension one. Also, $\tilde{Z} \to B$ is a $G|_Z$-equivariant birational map. This yields that $Y := \tilde{Z} \times A$ is $G$-equivariantly birational to the abelian variety $B \times A$, while $Y \to \tilde{X} \to X$ is still étale in codimension one. It is easy to see that $(B \times A, G = G|_B \times G|_A)$ also satisfies Hyp$(n, n - 2)$. We thus complete the proof of Lemma 3.3.

\[\qed\]

Remark 3.4. If $X$ is smooth, we are able to give a finer characterization as follows. Recall that for a projective manifold $X$ with numerically trivial canonical bundle, there exists a unique minimal splitting cover $\tilde{X}$ in the sense of Beauville (cf. [Bea83, §3]), of the form

$$A \times \prod V_i \times \prod X_j,$$
where $A$ is an abelian variety, the $V_i$ are (simply connected) Calabi–Yau manifolds and the $X_j$ are projective hyper-Kähler manifolds. As a consequence, any automorphism of $X$ extends to $\tilde{X}$ and then splits into pieces (up to permutations). More precisely, if $G \simeq \mathbb{Z}^{n-2}$ is a subgroup of $\text{Aut}(X)$ such that $G$ is of positive entropy, then there exists a group $\tilde{G}$ (the lifting of $G$) acting faithfully on $\tilde{X}$ such that $G = \tilde{G}/F$, where $F$ is the Galois group of the minimal splitting cover $\tilde{X} \to X$. Replacing $\tilde{G}$ by a finite-index subgroup, we may assume that $\tilde{G}$ also acts faithfully on $X$ (cf. [Zha13, Lemma 2.4]), both $(\tilde{X}, \tilde{G})$ and $(X, G)$ satisfy $\text{Hyp}(n, n-2)$; further, the group $\tilde{G}$ acting on $\tilde{X}$ splits as a subgroup of $\tilde{G}|_{A} \times \prod \tilde{G}|_{V_i} \times \prod \tilde{G}|_{X_j}$.

One can use the similar argument as in Lemma 3.3 to show that there are at most two factors. Moreover, it is well-known that $\text{dr}(\tilde{G}|_{X_j}) \leq 1$ (see e.g. [KOZ09, Theorem 4.6]) so that the $X_j$ are K3 surfaces. In summary, the covering space $\tilde{X}$ decomposes into a product of abelian varieties, Calabi–Yau manifolds, or K3 surfaces with at most two factors. Clearly, there are seven possibilities/classes.

Remark 3.5. Unfortunately, we are not able to deal with the singular case in an analogous way as in Remark 3.4, though we already have the Bogomolov–Beauville decomposition for minimal models with trivial canonical class due to Höring and Peternell (cf. [HP19, Theorem 1.5]). The reason for this is as follows. Let $X$ be a normal projective variety with at most klt singularities such that $K_X \equiv 0$. Let $\pi: \tilde{X} \to X$ be a finite cover, étale in codimension one, such that

$$\tilde{X} \simeq A \times \prod Y_j \times \prod Z_k,$$

where $A$ is an abelian variety, the $Y_j$ are (singular) Calabi–Yau varieties and the $Z_k$ are (singular) irreducible holomorphic symplectic varieties (see [GGK19, Definition 1.3]). Note that a compact Kähler manifold with numerically trivial canonical bundle has an almost abelian (aka abelian-by-finite) fundamental group. This fact is used to conclude the existence of the unique minimal splitting cover in [Bea83, §3] for the smooth case. However, in the general singular setup, as far as we can tell, the finiteness of fundamental groups of Calabi–Yau varieties is still unknown (see e.g. [GGK19, §13]). It is thus not clear to us that we can always lift the automorphisms of $X$ to some splitting cover $\tilde{X}$. The failure of the strategy of Remark 3.4 for general singular varieties forces us to work on the weak decomposition as we mentioned earlier at the beginning of section 2.1.

Lemma 3.6. Let $(X, G)$ satisfy $\text{Hyp}(n, n-2)$ with $n \geq 3$. Suppose that $X$ is uniruled but not rationally connected. Let $\pi: X \to Z$ be the special MRC fibration of $X$ (see Definition 2.10). Then one of the following assertions holds.

1. $Z$ is a curve of genus $\geq 1$.
2. $Z$ is a K3 surface, an Enriques surface, or an abelian surface such that $\text{dr}(G|_Z) = 1$.
3. $Z$ has dimension at least 3. Then after replacing $G$ by a finite-index subgroup, $Z$ is birational to a $Q$-abelian variety $A/F$ such that the induced action of $G|_Z$ on $A/F$ is
biregular, where $A$ is an abelian variety and $F$ is a finite group whose action on $A$ is free outside a finite subset of $A$; moreover, there is a faithful action of $G|_Z$ on $A$ such that the quotient map $A \rightarrow A/F$ is $G|_Z$-equivariant, and hence by Lemma 2.12 there exists a finite cover $X' \rightarrow X$, étale in codimension one, such that the induced map $\pi': X' \rightarrow A$ is $G$-equivariantly birational to the MRC fibration of $X'$.

Proof. Note that $Z$ has dimension at least one because $X$ is not rationally connected. By Lemma 2.11 or [Nak10, Theorem 4.19], $G$ descends to a biregular action $G|_Z$ on $Z$. Since $Z$ is non-uniruled (cf. [GHS03]), $\pi$ is a non-trivial $G$-equivariant rational fibration. It follows from Lemma 2.16 that $\dim(G|_Z) = \dim Z - 1$. Note that $Z$ is not rationally connected since it is non-uniruled. Therefore, if $\dim Z = 1$, then $Z$ is a curve of genus $\geq 1$. If $\dim Z = 2$, then $Z$ is either a K3 surface, an Enriques surface or an abelian surface (see e.g. [Can99, Proposition 1]). If $\dim Z \geq 3$, then by Theorem 1.1, up to replacing $G|_Z$ and hence $G$ by a finite-index subgroup, $Z$ is birational to a $Q$-abelian variety $A/F$ such that the induced action of $G|_Z$ on $A/F$ is biregular, where $A$ is an abelian variety and $F$ is a finite group whose action on $A$ is free outside a finite subset of $A$; moreover, the $G|_Z$-action on $A/F$ extends to a faithful action on $A$ such that $A \rightarrow A/F$ is also $G|_Z$-equivariant. Now, by Lemma 2.12, there exist a normal projective variety $X'$ and a finite cover $X' \rightarrow X$, étale in codimension one, such that the induced map $\pi': X' \rightarrow A$ is $G$-equivariantly birational to the MRC fibration of $X'$.

Proof of Theorem 1.2. It follows from Lemmas 3.1, 3.3 and 3.6.

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REFERENCES

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, Existence of minimal models for varieties of log general type, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468. MR2601039

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, J. Algebraic Geom. 22 (2013), no. 2, 201–248. MR3019449

[Bea83] Arnaud Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, Classification of algebraic and analytic manifolds (Katata, 1982), Progr. Math., vol. 39, Birkhäuser Boston, Boston, MA, 1983, pp. 1–26. MR728605

[Cam92] Frédéric Campana, Connexité rationnelle des variétés de Fano, Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 539–545. MR1191735
Serge Cantat, *Dynamique des automorphismes des surfaces projectives complexes*, C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), no. 10, 901–906. MR1689873 ↑ 2, 11, 14

Tien-Cuong Dinh, Fei Hu, and De-Qi Zhang, *Compact Kähler manifolds admitting large solvable groups of automorphisms*, Adv. Math. 281 (2015), 333–352. MR3366842 ↑ 9

Tien-Cuong Dinh and Viêt-Anh Nguyên, *Comparison of dynamical degrees for semi-conjugate meromorphic maps*, Comment. Math. Helv. 86 (2011), no. 4, 817–840. MR2851870 ↑ 11

Stéphane Druel, *A decomposition theorem for singular spaces with trivial canonical class of dimension at most five*, Invent. Math. 211 (2018), no. 1, 245–296. MR3734275 ↑ 4

Tien-Cuong Dinh and Nessim Sibony, *Groupes commutatifs d'automorphismes d'une variété kählérienne compacte*, Duke Math. J. 123 (2004), no. 2, 311–328. MR2066940 ↑ 1, 12

Tien-Cuong Dinh and De-Qi Zhang, *Compact Kähler manifolds admitting large solvable groups of automorphisms*, Adv. Math. 281 (2015), 333–352. MR3366842 ↑ 9

Tien-Cuong Dinh and Nessim Sibony, *Families of rationally connected varieties*, J. Amer. Math. Soc. 16 (2003), no. 1, 57–67. MR1937199 ↑ 3, 7, 14

Tien-Cuong Dinh, Stefan Kebekus, and Thomas Peternell, *Étale fundamental groups of Kawamata log terminal spaces, flat sheaves, and quotients of abelian varieties*, Duke Math. J. 165 (2016), no. 10, 1965–2004. MR3522654 ↑ 4

Andreas Höring and Thomas Peternell, *Algebraic integrability of foliations with numerically trivial canonical bundle*, Invent. Math. 216 (2019), no. 2, 395–419. MR3953506 ↑ 4, 13

Fei Hu, *A theorem of Tits type for automorphism groups of projective varieties in arbitrary characteristic (with an appendix by Tomohide Terasoma)*, to appear in Math. Ann., 30 pp., arXiv:1801.06555, DOI:10.1007/s00208-019-01812-9. ↑ 10

Yuujiro Kawamata, *Minimal models and the Kodaira dimension of algebraic fiber spaces*, J. Reine Angew. Math. 363 (1985), 1–46. MR814013 ↑ 4, 6, 7

János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR1658959 ↑ 3, 6

János Kollár, Yoichi Miyaoka, and Shigefumi Mori, *Rationally connected varieties*, J. Algebraic Geom. 1 (1992), no. 3, 429–448. MR1158625 ↑ 3

János Kollár, Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32, Springer-Verlag, Berlin, 1996. MR1440180 ↑ 3

JongHae Keum, Keiji Oguiso, and De-Qi Zhang, *Conjecture of Tits type for complex varieties and theorem of Lie-Kolchin type for a cone*, Math. Res. Lett. 16 (2009), no. 1, 133–148. MR2480567 ↑ 9, 13

Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004. MR2104208 ↑ 7

Noboru Nakayama and De-Qi Zhang, *Polarized endomorphisms of complex normal varieties*, Math. Ann. 346 (2010), no. 4, 991–1018. MR2587100 ↑ 4, 5, 7, 8
[OS01] Keiji Oguiso and Jun Sakurai, *Calabi-Yau threefolds of quotient type*, Asian J. Math. 5 (2001), no. 1, 43–77. MR1868164

[Uen75] Kenji Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, vol. 439, Springer-Verlag, Berlin-New York, 1975, Notes written in collaboration with P. Cherenack. MR0506253

[Zha09] De-Qi Zhang, *A theorem of Tits type for compact Kähler manifolds*, Invent. Math. 176 (2009), no. 3, 449–459. MR2501294

[Zha13] ———, *Algebraic varieties with automorphism groups of maximal rank*, Math. Ann. 355 (2013), no. 1, 131–146. MR3004578

[Zha16] ———, *n-dimensional projective varieties with the action of an abelian group of rank n − 1*, Trans. Amer. Math. Soc. 368 (2016), no. 12, 8849–8872. MR3551591

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