Homoclinic Bifurcation of a Quadratic Family of Real Functions with Two Parameters

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Abstract

In this work the homoclinic bifurcation of the family

\[ H = \{ h_{a,b}(x) = ax^2 + b : a \in \mathbb{R}/\{0\}, b \in \mathbb{R} \} \]

is studied. We proved that this family has a homoclinic tangency associated to \( x = 0 \) of \( P_1 \) for \( b = \frac{-2}{a} \). Also we proved that \( W^u(P_1) \) does not intersect the backward orbit of \( P_1 \) for \( b > \frac{-2}{a} \), but has intersection for \( b < \frac{-2}{a} \) with \( a > 0 \). So \( H \) has this type of the bifurcation.

Subject Areas

Dynamical System

Keywords

Local Unstable Set, Unstable Set, Homoclinic Point, Homoclinic Orbit, Non-Degenerate, Homoclinic Tangency, Homoclinic Bifurcation

1. Introduction

There are various definitions for the homoclinic bifurcation. In the sense of Devaney, the homoclinic bifurcation is a global type of bifurcations, that is, this type of bifurcation is a collection of local and simple types of bifurcations [1] (like, period-doubling and saddle-node of bifurcation [2]).

According to [3] [4] [5] we have another definition of the homoclinic bifurcation via the notions of the unstable sets of a repelling periodic point (fixed point) and the intersection of this set with the backward orbits of this point.

The purpose of this work is to prove the family

\[ H = \{ h_{a,b}(x) = ax^2 + b : a \in \mathbb{R}/\{0\}, b \in \mathbb{R} \} \]

has homoclinic bifurcation at \( b = \frac{-2}{a} \).
following the second definition.

2. Definitions and Basic Concepts

2.1. Definition 1: [6]
A fixed point \( P \) is said to be **expanding** for a map \( f \), if there exists a neighborhood \( U(P) \) such that \( |f'(x)| > 1 \) for any \( x \in U(P) \).

The neighborhood in the previous definition is exactly the **local unstable set**.

2.2. Definition 2: [7]
Let \( P \) be a repelling fixed point for a function \( f: I \rightarrow I \) on a compact interval \( I \subset \mathbb{R} \), then there is an open interval about \( P \) on which \( f \) is one-to-one and satisfies the **expansion** property. \( |f(x) - f(P)| > |x - P|, \forall x \in I \) where \( x \neq P \).

The interval in the previous definition is exactly the **unstable set of** \( P \).

2.3. Definition 3: [8]
Let \( P \) is fixed point and \( f'(P) > 1 \) for a map \( f: \mathbb{R} \rightarrow \mathbb{R} \). A point \( q \) is called **homoclinic point** to \( P \) if \( q \in w^{u}_{loc} (P) \) and there exists \( n > 0 \) such that \( f^n(q) = P \).

2.4. Definition 4: [9]
The union of the forward orbit of \( q \) with a suitable sequence of preimage of \( q \) is called the **homoclinic orbit of** \( P \). That is \( O(q) = \{ P, \ldots, q_{-2}, q_{-1}, q, q_1, q_2, \ldots, q_m = P \} \) where \( q_{i+1} = f(q_i) \) for \( i \leq m-1 \), \( q_m = P \) and \( \lim_{i \to \infty} q_i = P \).

2.5. Definition 5: [10]
The critical \( x \) point is **non-degenerate** if \( f''(x) \neq 0 \). The critical point \( x \) is **degenerate** if \( f''(x) = 0 \).

2.6. Definition 6: [11]
Let \( f \) be a smooth map on \( I \subset \mathbb{R} \), and let \( p \) be a hyperbolic fixed point for the map \( f \). If \( W^u(p) \) intersects the backward orbit of \( p \) at a nondegenerate critical point \( x_{cr} \) of \( f \), then \( x_{cr} \) is called a **point of homoclinic tangency associated to** \( p \).

2.7. Definition 7: [3]
Let \( f_\varphi \) be a smooth map on \( I \subset \mathbb{R} \), and let \( p \) be a hyperbolic fixed point for a map \( f_\varphi \). We say that \( f_\varphi \) has **homoclinic bifurcation associated to** \( p \) at \( \varphi = \hat{\varphi} \) if:

1) For \( \varphi < \hat{\varphi} \ (\varphi > \hat{\varphi}) \), \( W^u(p) \) and the backward orbit of \( p \) has no intersect.
2) For \( \varphi = \hat{\varphi} \), \( f_\varphi \) has a point of homoclinic tangency \( x_{cr} \) associated to \( p \).
3) For \( \varphi > \hat{\varphi} \ (\varphi < \hat{\varphi}) \), the intersection of \( W^u(p) \) with the backward orbit of \( p \) is nonempty.
3. Homoclinic Bifurcation of the Family

\[ H = \left\{ h_{a,b}(x) = ax^2 + b : a \in \mathbb{R}/\{0\}, b \in \mathbb{R} \right\} \]

In this section, we show that the family \( H \) has a point of homoclinic tangency associated to \( P_1 \) at \( b = \frac{-2}{a} \), and \( H \) has a homoclinic bifurcation.

We need the following results proved in [12].

At the first, the fixed points of \( h_{a,b}(x) \) are

\[ P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a}, \quad P_2 = \frac{1 - \sqrt{1 - 4ab}}{2a}. \]

**a) Proposition:**

For \( h_{a,b}(x) \in H \) with \( a > 0 \) the local unstable set of the fixed point \( P_1 \) is

\[ w^u(P_1) = \left\{ \frac{1}{2|a|}, \infty \right\}. \]

**b) Lemma:**

For \( h_{a,b}(x) \in H \), \( h_{a,b}^{-1}(P_1) = \pm \sqrt{\frac{P_1 - b}{a}} = \mp P_1 \) where \( P_1 > b \) for \( a > 0 \).

**c) Theorem:**

For \( h_{a,b}(x) \in H \) with \( a > 0 \), the unstable set of the fixed point \( P_1 \) is

\[ w^u(P_1) = \left\{ \frac{1}{|a|}, -P_1, \infty \right\}. \]

**d) Remark:** [13]

The local unstable set of the fixed point \( P_2 \) is \( w^u(P_2) = \left\{ -\infty, -\frac{1}{2|a|} \right\} \), and the unstable set of the fixed point \( P_2 \) is \( w^u(P_2) = \left\{ -\infty, -\frac{1}{|a|}, -P_2 \right\} \). In this work we will omit the result about \( P_2 \) because \( h_{a,b}'(P_2) < -1 \), when \( b < \frac{-3}{4a} \) for \( a > 0 \), \( b > \frac{-3}{4a} \) for \( a < 0 \). Thus we will not care for the fixed point \( P_2 \). (See definition (2.3)).

**e) Remark:**

For \( h_{a,b}(x) \in H \), \( h_{a,b}^{-2}(P_1) = \mp \sqrt{\frac{-P_1 - b}{a}}. \)

**f) Proposition:**

For \( h_{a,b} \in H \), if \( b < \frac{-(5 + 2\sqrt{5})}{4a} \) with \( a > 0 \), then the second preimage of the fixed point \( P_1 \) belongs to the local unstable set of \( P_1 \).

**g) Proposition:**

For \( h_{a,b} \in H \), if \( \frac{-(5 + 2\sqrt{5})}{4a} \leq b \leq \frac{-2}{a} \) with \( a > 0 \), then the third preimage
of the fixed point \( P_1 \) belongs to the local unstable set of \( P_1 \).

**h) Theorem:**

For the family \( H = \{ h_{a,b}(x) = ax^2 + b : a > 0 \} \), there exist homoclinic points to the fixed point \( P_1 \) whenever \( b \leq \frac{-2}{a} \). Moreover \( h_{a,b}^3(P_1) = q_{1,1}, h_{a,b}^3(P_1) = q_{2,1} \) are the first homoclinic points for \( b < \frac{(5 + 2\sqrt{5})}{4a}, \frac{-(5 + 2\sqrt{5})}{4a} \leq b \leq \frac{-2}{a} \) respectively.

**i) Example:**

For \( h_{a,b}(x) = x^2 - 6 \), a homoclinic orbit of a homoclinic point \( \sqrt{5} \) is: \( O(\sqrt{5}) = \{3, -3, \sqrt{5}, \ldots, 3\} \).

**The main result:**

### 3.1. Lemma 1

For \( h_{a,b}(x) = ax^2 + b \) with \( a \in \mathbb{R}/\{0\} \), the critical point of \( h_{a,b}(x) \) is 0, and it is a non-degenerate critical point.

**Proof:**

Clearly that the critical point of \( h_{a,b}(x) \) is zero.

Since \( a \neq 0 \), then

\[
h^*_a(x) = 2a \neq 0.
\]

So \( h_{a,b}(x) \) has a non-degenerate critical point at \( x = 0 \). ■

### 3.2. Lemma 2

If \( b = 0 \) of \( h_{a,b}(x) \in H \) with \( a \in \mathbb{R}/\{0\} \), then the backward orbit of the repelling fixed point \( P_1 \) is undefined in \( \mathbb{R} \).

**Proof:**

\( h_{a,0}(x) = ax^2 \), clearly \( P_1 = \frac{1}{a} \).

Now the first preimage of \( h_{a,0}(x) \) is

\[
h^{-1}_{a,0}(x) = \pm \sqrt[3]{x a}, \text{ where } x > 0 \text{ for } a > 0.
\]

By Lemma (3-b), we have

\[
h^{-1}_{a,0} \left( \frac{1}{a} \right) = \mp \sqrt[3]{\frac{1}{a^2}} = \mp \frac{1}{a} = \mp P_1.
\]

But \( P_1 \) is a fixed point, and \( -P_1 = \left( -\frac{1}{a} \right) w_{\text{loc}}(P_1) = \left( \frac{1}{2a}, \infty \right) \), see Proposition (3-a).

By Remark (3-e) we have

\[
h_{a,0}^2(P_1) = \mp \sqrt[3]{\frac{-P_1}{a}} = \mp \sqrt[3]{\frac{-1}{a^2}} \notin \mathbb{R},
\]
Therefore $h_{ab}^n(P_1)$ are undefined in $\mathbb{R}$ with $n \geq 2$.
Thus the backward orbit of the repelling fixed point $P_1$ is undefined in $\mathbb{R}$.

### 3.3. Theorem 1

For the family $h_{ab}(x) = ax^2 + b$, $0$ belong to the backward orbit of $P_1$ whenever $b = \frac{-2}{a}$ with $a \in \mathbb{R}/\{0\}$, and the backward orbit of $P_1$ is:

$$h_{a,b}^{-n}(P_1) = \left\{ \frac{2}{a} - \frac{2}{a}, 0, \frac{\sqrt{4a^2 - 4ab}}{a}, \ldots \right\}.$$

#### Proof:

We test the values of $n$ which makes $h_{a,b}^{-n}(P_1) = 0$.

By Lemma (3-b), $h_{a,b}^{-1}(P_1) = \pm P_1$.

So $h_{a,b}^{-1}(P_1) \neq 0$.

Now suppose that $h_{a,b}^{-2}(P_1) = 0$, by Remark (3-e) then $h_{a,b}^{-2}(P_1) = \mp \sqrt{\frac{-P_1 - b}{a}} = 0$, thus

$$-\frac{P_1 - b}{a} = 0 \quad \Rightarrow \quad -P_1 - b = 0 \quad \Rightarrow \quad P_1 = b.$$

Since the fixed point $P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a}$, therefore

$$\frac{1 + \sqrt{1 - 4ab}}{2a} = -b,$$

then

$$1 + \sqrt{1 - 4ab} = -2ab \quad \Rightarrow \quad \sqrt{1 - 4ab} = -2ab - 1 \quad \Rightarrow \quad 1 - 4ab = 4a^2b^2 + 4ab + 1$$

$$4a^2b^2 + 8ab + 1 = 0,$$

which implies

$$4ab(ab + 2) = 0,$$

then either $b = 0$, but by the above Lemma (3.2) the backward orbit of $P_1$ is undefined, so we omit this case.

Or $ab + 2 = 0$, thus

$$b = \frac{-2}{a}.$$

Now, $P_1 = \frac{2}{a}$ and to find the backward orbit of $P_1$, we consider

$$h_{a,b}^{-1,2}(x) = \pm \frac{\sqrt{ax + 2}}{a}.$$
By Lemma (3-b) \( h_{\frac{-1}{2}}^{-1}(P) = \pm P \), then

\[
h_{\frac{-1}{2}}^{-1}\left(\frac{2}{a}\right) = \pm \frac{2}{a},
\]

But \( \pm \frac{2}{a} \) is a fixed point, therefore

\[
h_{\frac{-1}{2}}^{-1}\left(\frac{2}{a}\right) = -\frac{2}{a}.
\]

So

\[
h_{\frac{-2}{a}}^{-2}\left(\frac{2}{a}\right) = \frac{\sqrt{a(0) + 2}}{a} = \frac{\sqrt{2}}{a},
\]

and so on.

Therefore the backward orbit of \( P_1 = \frac{2}{a} \) is:

\[
h_{\frac{-n}{a}}^{-n}\left(P_1 = \frac{2}{a}\right) = \left\{2, -2, 0, \frac{\sqrt{2}}{a}, \cdots \right\}.
\]

3.4. Example

For \( h_{-2}(x) = x^2 - 2 \), 0 belongs to the backward orbit of \( P_1 = 2 \) (Figure 1), and the backward orbit of \( P_1 \) is \( h_{-2}^{-n}(2) = \left\{2, -2, 0, \sqrt{2}, \cdots, 2\right\} \).

Figure 1. For \( h_{-2}(x) = x^2 - 2 \), the backward orbit of \( P_1 = 2 \).
3.5. Theorem 2

If \( b > \frac{-2}{a} \) for \( h_{a,b}(x) \in H \) with \( a > 0 \), then there is no intersection of the backward orbit with the unstable set of \( P_1 \).

**Proof:**

The backward orbit of \( P_1 \)

By Lemma (3-b) \( h_{a,b}^{-1}(P_1) = \pm P_1 \), since \( +P_1 \) is a fixed point, then we consider \( h_{a,b}^{-1}(P_1) = -P_1 \).

By Remark (3-e), \( h_{a,b}^{-2}(P_1) = \mp \sqrt{-\frac{P_1 - b}{a}} \).

If \( -P_1 > b \), then by Theorem (3-h),
\[ b \leq \frac{-2}{a} \]
which is a contradiction with \( b > \frac{-2}{a} \). Therefore
\[ -P_1 < b \], which implies
\[ h_{a,b}^{-2}(P_1) \notin \mathbb{R} \].

So \( h_{a,b}^{-2}(P_1) \) are undefined in \( \mathbb{R} \) with \( n \geq 2 \).

Thus the backward orbit of \( P_1 \) is undefined.

So the intersection of \( W^u(P_1) \) with the backward orbit of \( P_1 \) is also undefined.

\[ \Box \]

3.6. Theorem 3

If \( b = \frac{-2}{a} \) for \( h_{a,b}(x) \in H \) with \( a > 0 \), then \( h_{a,b}^{-2}(P_1) \) has a point of homoclinic tangency at 0 associated to \( P_1 \).

**Proof:**

By Theorem (3.3), \( h_{a,b}^{-2}(P_1) = \left\{ \frac{2}{a}, \frac{2}{a}, \ldots \right\} \).

By Theorem (3-c), \( W^u(P_1) = \left\{ \frac{1}{a} - P_1, \infty \right\} \), then
\[ W^u\left( P_1 = \frac{2}{a} \right) = \left\{ \frac{1}{a} - \frac{2}{a}, \infty \right\} \], i.e.
\[ W^u\left( P_1 = \frac{2}{a} \right) = \left\{ -1, \infty \right\} \].

Now
\[ h_{a,b}^{-2}(P_1 = \frac{2}{a}) \] intersects \( W^u\left( P_1 = \frac{2}{a} \right) \) at 0.

By Lemma (3.1) 0 is a non-degenerate critical point. So \( h_{a,b}^{-2}(P_1 = \frac{2}{a}) \) has a point of homoclinic tangency at 0 associated to \( P_1 \).

\[ \Box \]

3.7. Theorem 4

If \( b < \frac{-2}{a} \) for \( h_{a,b}(x) \in H \) with \( a > 0 \), then the backward orbit of \( P_1 \) crosses the unstable set \( W^u(P_1) \).
Proof:
First consider the backward orbit of $P_1$.
By Lemma (3-b) $h_1^{-1}(P_1) = \pm P_1$.
But $+ P_1$ is a fixed point, therefore we consider
$$h_1^{-1}(P_1) = -P_1.$$ 
By Remark (3-e), $h_{a,b}^{-2}(P_1) = \pm \sqrt{\frac{-P_1 - b}{a}}$.
Since $b < -\frac{2}{a}$, then by Theorem (3-h)
$$h_{a,b}^{-2}(P_1) \in \mathbb{R}.$$ 
Let $h_{a,b}^{-2}(P_1) = q_{1,1}$, $h_{a,b}^{-3}(P_1) = q_{2,1}$.
By Proposition (3-f), if $\frac{-5 + 2\sqrt{5}}{4a} < b < -\frac{2}{a}$, then $q_1,1 \in W_{loc}^u(P_1)$.
By Proposition (3-g), if $-\frac{5 + 2\sqrt{5}}{4a} \leq b < -\frac{2}{a}$, then $q_{2,1} \in W_{loc}^u(P_1)$.

Now since the local unstable set of the repelling fixed point contained in the unstable set of the repelling fixed point. Therefore
$$h_{a,b}^{-2}(P_1) \cap W^u(P_1) \neq \emptyset.$$ 

Following examples explain the cases for $b > -\frac{2}{a}$, $b = -\frac{2}{a}$ and $b < -\frac{2}{a}$ (with $a > 0$) respectively.

3.8. Example 1

For $h_{-1}(x) = x^2 - 1$, we have no intersection of the backward orbit of $P_1$ with the unstable set of $P_1$.

Solution:
Consider the fixed point of $h_{-1}(x)$ is $P_1 = \frac{1 + \sqrt{5}}{2}$, and
$$h_{-1}^{-1}(x) = \pm \sqrt{x + 1}.$$ 
The backward orbit of $P_1 = \frac{1 + \sqrt{5}}{2}$
$$h_{-1}^{-1}\left(\frac{1 + \sqrt{5}}{2}\right) = \pm \frac{1 + \sqrt{5}}{2},$$ where $+ \frac{1 + \sqrt{5}}{2}$ is a fixed point, therefore we consider
$$h_{-1}^{-1}\left(\frac{1 + \sqrt{5}}{2}\right) = -\frac{1 + \sqrt{5}}{2}. Now$$
$$h_{-1}^{-2}\left(\frac{1 + \sqrt{5}}{2}\right) = \mp \sqrt{\frac{1 + \sqrt{5}}{2}} + 1 \not\in \mathbb{R}. $$
So $h_{-1}^{-2}\left(\frac{1 + \sqrt{5}}{2}\right)$ are undefined in $\mathbb{R}$ with $n \geq 2$. 

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Thus the backward orbit of $P_1$ is undefined.

So the intersection of $W^u\left(\frac{1+\sqrt{5}}{2}\right)$ with the backward orbit of $P_1$ is also undefined. ■

3.9. Example 2

For $h_{_{-2}}(x) = x^2 - 2$, then $h_{_{-2}}$ has a point of tangency at 0 associated to $P_1$.

Solution:

Consider the fixed point of $h_{_{-2}}(x)$ is $P_1 = 2$.

By Example (3.4), The backward orbit of $P_1 = 2$ is

$h_{_{-2}}^u(2) = \{2, -2, 0, \sqrt{2}, ..., 2\}$.

On the other hand, the unstable set of $P_1 = 2$ is $\mathcal{W}^u(2) = (-1, \infty)$, (see Theorem (3-c)). Now $h_{_{-2}}^u(2)$ intersects $\mathcal{W}^u(2)$ at 0.

By Lemma (3.1), 0 is a non-degenerate critical point. So $h_{_{-2}}$ has a point of tangency at 0 associated to $P_1$. ■

3.10. Example 3

For $h_{_{-6}}(x) = x^2 - 6$, the backward orbit of $P_1$ crosses the unstable set $\mathcal{W}^u(P_1)$.

Solution:

First consider the fixed point $P_1 = 3$.

The backward orbit of 3 is:

$h_{_{-6}}^u(3) = \{3, -3, \sqrt{3}, ..., 3\}$ (see Example (3-i)), with $h_{_{-6}}^u(3) = h_{_{-2}}^u(\sqrt{3})$, and $h_{_{-6}}^u(3) = \sqrt{3}$.

Since $\sqrt{3}$ is a homoclinic point of $P_1 = 3$, then

$\sqrt{3} \in \mathcal{W}^u_{_{loc}}(3)$.

Now since the local unstable set of the repelling fixed point $P_1 = 3$ contained in the unstable set of the repelling fixed point $P_1 = 3$. Therefore

$h_{_{-6}}^u(3) \cap \mathcal{W}^u(3) \neq \emptyset$. ■

Note, the main theorem in the work:

3.11. Theorem 5

$h_{_{\alpha, \beta}}(x) = ax^2 + b, a > 0$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{_{\alpha, \beta}}$, $P_1 = \frac{1+\sqrt{4ab}}{2a}$, at $b = \frac{-2}{a}$.

Proof:

1) For $b > \frac{-2}{a}$, by Theorem (3.5) the intersection of the backward orbit of $P_1$ and the unstable set of $P_1$ is undefined.

2) For $b = \frac{-2}{a}$, by Theorem (3.6) $h_{_{\alpha, \beta}}$ has a point of homoclinic tangency
associated to $P_1$ at $x = 0$.

3) For $b < \frac{-2}{a}$, by Theorem (3.7) the backward orbit of $P_1$ crosses the unstable set of $P_1$, $W^u(P_1)$.

Therefore $h_{a,b}$ has a homoclinic bifurcation associated to $P_1$ at $b = \frac{-2}{a}$. □

3.12. Example

$h_{a,b}(x) = x^2 - 2$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{a,b}$, $P_1 = 2$, at $b = -2$.

$h_{a,b}(x)$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{a,b}$, $P_1 = 2$, at $b = -2$. See examples (3.8), (3.9), (3.10).

3.13. Remark

For $a < 0$, we have same results which proved above for $a > 0$. In fact, we can prove in similar ways, that: $h_{a,b}(x) = ax^2 + b, a < 0$ has a homoclinic bifurcation associated to the repelling fixed point of $h_{a,b}$, $P_1 = \frac{1 + \sqrt{1 - 4ab}}{2a}$, at $b = \frac{-2}{a}$.

4. Conclusion

We conclude that the family $H = \{h_{a,b}(x) = ax^2 + b : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \}$ has homoclinic tangency associated to $P_1$ at the critical point $x = 0$. Also for $b > \frac{-2}{a}$ we have no intersection between the backward orbit of $P_1$ and the unstable set of $P_1$, and the backward orbit of $P_1$ crosses the unstable set of $P_1$ for $b < \frac{-2}{a}$. So we have homoclinic bifurcation at $b = \frac{-2}{a}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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