Spectral Analysis for Some Multifractional Gaussian Processes

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Dedicated to the memory of Ya. Yu. Nikitin, our colleague and friend

Abstract. We study the small ball asymptotics problem in $L^2$ for two generalizations of the fractional Brownian motion with variable Hurst parameter. To this end, we perform a careful analysis of the singular value asymptotics for associated integral operators.

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1. INTRODUCTION

The spectral asymptotics for Gaussian processes have been intensively investigated in the last two decades; they are closely connected to the problem of small deviation asymptotics for such processes in the Hilbert norm. Namely, see [17], to obtain the logarithmic $L^2$-small ball asymptotics of a Gaussian process $X$, it is sufficient to know the one-term asymptotics of the eigenvalue counting function of its covariance operator.

In the most elaborate case of the so-called Green Gaussian processes, i.e., the processes whose covariance functions $G_X$ are the Green functions for ordinary differential operators (ODO), one can obtain even two-term asymptotics of the eigenvalues with the remainder estimate and thus manage the exact small deviation asymptotics (up to a constant). This approach was developed in [20, 18], see more references in [22].

The case of fractional Gaussian processes is more complicated. In the pioneering paper [8], the one-term spectral asymptotics was calculated for the fractional Brownian motion (FBM) $W^H$, i.e., the zero mean-value Gaussian process with the covariance function

$$G_{W^H}(x, y) = \frac{1}{2} \left( x^{2H} + y^{2H} - |x - y|^{2H} \right)$$

(here $H \in (0, 1)$ is the so-called Hurst index, the case $H = \frac{1}{2}$ corresponds to the standard Wiener process).

A more general approach was suggested in [21]. That approach is based on powerful theorems on spectral asymptotics of integral operators [3], see also [4, Appendix 7], and covers many fractional processes. Now the problem of logarithmic $L^2$-small ball asymptotics for such processes is also well studied. We mention also the breakthrough paper [9], where the two-term asymptotics of the eigenvalues with the remainder estimate was obtained for the FBM in the full range of the Hurst index. Some generalization of the seminal idea of [9] was given in [19], where the reader can also find an extensive bibliography.

In this paper, we consider some more sophisticated Gaussian processes.

The multifractional Brownian Motion (mBM) was introduced in [23] and [2] and was investigated in several papers, see, e.g., [11, 1], and [10]. There are some different definitions of mBM equivalent up to a multiplicative deterministic function. We choose the so-called harmonizable representation [2]

$$W^{H(\cdot)}(x) = C_\ast(H(x)) \int_{-\infty}^{\infty} \frac{e^{ix\xi} - 1}{|\xi[H(x)] + \frac{1}{2}|} dW(\xi),$$

(1)

where $W(\xi)$ is a conventional Wiener process and the functional Hurst parameter $H(x)$ satisfies the inequality $0 < H(x) < 1$. The choice of the normalizing factor

$$C_\ast(H) = \left( \frac{\Gamma(2H + 1) \sin(\pi H)}{2\pi} \right)^{\frac{1}{2}}$$

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ensures that $EX^2(1) = 1$.

A different process of the same structure is the multifractional Brownian Motion (mfBM) introduced in [24], see also [25]. It can be constructed as

$$X^{H(\cdot)}(x) = \int_0^x K(x,y,H(x)) \, dW(y),$$

where

$$K(x,y,H) = c_*(H)y^{\frac{1}{2}-H} \int_y^x (z-y)^{H-\frac{3}{2}} \, dz \, I_{[0,x]}(y),$$

while the variable Hurst parameter $H(x)$ satisfies $\frac{1}{2} < H(x) < 1$. The normalizing factor is defined by the formula

$$c_*(H) = \left( \frac{H(2H-1)\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)\Gamma(H-\frac{1}{2})} \right)^{\frac{1}{2}},$$

where $\Gamma$ is Euler’s gamma-function.

Both the processes (1) and (2) obviously have zero mean, and their covariance functions were derived in [1] and [25], respectively. For $H(x) = H = \text{const}$, they both coincide with the conventional FBM.

We derive one-term spectral asymptotics for the processes (1) and (2) under some regularity assumptions on the functional parameter $H(x)$. Then, using the results of [14], we obtain the logarithmic $L_2$-small ball asymptotics for these processes. Despite the fact that the behavior of covariances of mBM and mfBM is significantly different, it turns out that, under the assumption $H(x) > \frac{1}{2}$, these logarithmic asymptotics coincide.

The structure of our paper is as follows. In Section 2, we introduce operators associated with the processes under consideration and formulate the result concerning the asymptotics of their singular values. This result is proved in Section 3. Section 4 is devoted to $L_2$-small ball behavior of mBM and mfBM. Some auxiliary estimates and asymptotics of singular values of compact operators are collected in Appendix.

We use the letter $C$ to denote various positive constants. To indicate that $C$ depends on some parameters, we list them in parentheses: $C(\ldots)$.

2. OPERATORS ASSOCIATED WITH mBM AND mfBM

Here we define integral operators associated with the processes (1) and (2), see [16, Subsec. 3.2]:

$$\mathcal{T} : L_2(\mathbb{R}) \to L_2(0,1), \quad (\mathcal{T} f)(x) := c_*(H(x)) \int_{-\infty}^\infty \frac{e^{ix\xi} - 1}{|\xi|^{H(x)} + 2} f(\xi) \, d\xi,$$

$$\mathcal{S} : L_2(0,1) \to L_2(0,1), \quad (\mathcal{S} f)(x) := \int_0^x K(x,y,H(x)) f(y) \, dy$$

(5)

(the function $K$ is defined in (3)). It is easy to see that the covariance functions

$$\mathcal{G}_{W^{H(\cdot)}}(x,y) := EW^{H(\cdot)}(x)W^{H(\cdot)}(y), \quad \mathcal{G}_{X^{H(\cdot)}}(x,y) := EX^{H(\cdot)}(x)X^{H(\cdot)}(y)$$

are the kernels of the integral operators $\mathcal{T}\mathcal{T}^*$ and $\mathcal{S}\mathcal{S}^*$, respectively.

In what follows, we denote by $\{\lambda_k(\mathbb{K})\}$ the nonincreasing sequence of eigenvalues of a compact self-adjoint positive operator $\mathbb{K}$ in a Hilbert space $\mathcal{H}$, enumerated with multiplicities.

Recall that, for a compact operator $\mathbb{A} : \mathcal{H}_1 \to \mathcal{H}_2$, we have $\lambda_k(\mathbb{A}^*\mathbb{A}) = \lambda_k(\mathbb{A}\mathbb{A}^*)$, and the square roots of these eigenvalues $s_k(\mathbb{A}) := (\lambda_k(\mathbb{A}^*\mathbb{A}))^{\frac{1}{2}}$ are called the singular values of the operator $\mathbb{A}$.

We denote by $\mathcal{N}(t,\mathbb{A})$, $t > 0$, the distribution function of the singular values,

$$\mathcal{N}(t,\mathbb{A}) := \#\{k \mid s_k(\mathbb{A}) > t^{-1}\}.$$
Notice that the function \( t \mapsto \mathcal{N}(t, \mathcal{A}) \) is conceptually inverse to the function \( k \mapsto s_k^{-1}(\mathcal{A}) \). Thus, if the singular values have moderate speed of decay, then the one-term asymptotics of \( s_k(\mathcal{A}) \) as \( k \to \infty \) is uniquely defined by the one-term asymptotics of \( \mathcal{N}(t, \mathcal{A}) \) as \( t \to \infty \).

We suppose that the functional parameter \( H(x) \) is a Hölder continuous function. Then it turns out that the singular value asymptotics for the operators (4) and (5) heavily depend on the behavior of \( H(x) \) in a neighborhood of the set where it attains its minimal value. We set

\[
H_{\text{min}} := \min_{x \in [0,1]} H(x), \quad \mathcal{D} := \{ x \in [0,1] \mid H(x) = H_{\text{min}} \}.
\]

In what follows, we use the notation \( m = H_{\text{min}} + \frac{1}{2} \).

Next, we introduce the regularized distance to \( \mathcal{D} \), see, e.g., [15], which is a function \( d(x), x \in [0,1], \) such that \( d(x) \propto \text{dist}(x, \mathcal{D}) \) and

\[
d \in \mathcal{C}^\infty([0,1] \setminus \mathcal{D}), \quad |d^{(n)}(x)| \leq C(n)d^{1-n}(x), \quad x \in [0,1] \setminus \mathcal{D}, \quad n \in \mathbb{N}.
\]

We describe the behavior of \( H(x) \) in a neighborhood of \( \mathcal{D} \) under the following assumptions.

1. The function \( h(x) := H(x) - H_{\text{min}} \) is bounded by a power of the distance \( d(x) \):

\[
h(x) \leq C d^\kappa(x) \quad \text{for some } \kappa > 0.
\]

2. The function \( h(x) \) admits an asymptotic representation with the smooth leading term in a neighborhood of \( \mathcal{D} \). More precisely, \( h(x) = h_0(x) + h_1(x) \), where \( h_0, h_1 \in \mathcal{C}^\beta[0,1] \) with \( \beta > 0 \), and

\[
h_0 \in \mathcal{C}^\infty([0,1] \setminus \mathcal{D}), \quad |h_0^{(n)}(x)| \leq C(n)h_0(x)d^{-n}(x), \quad x \in [0,1] \setminus \mathcal{D},
\]

\[
h_1(x) = O(h_0^{1+\tau}(x)) \quad \text{as } h_0(x) \to 0 \quad \text{for some } \tau > 0.
\]

3. The measure of the set of small values for the function \( h \) is a regularly varying function of the level:

\[
\text{meas}(x \in [0,1] \mid 0 < h(x) < s) = s^\sigma \varphi(s^{-1}), \quad 0 < s < s_0,
\]

where \( \sigma, s_0 > 0 \) and \( \varphi \) is a slowly varying function (SVF), see [26].

**Theorem 1.** Let the functional parameter \( H(x) \) satisfy the assumptions 1–3. Assume in addition that

\[
\tau > \left\{ \begin{array}{ll}
\max\{0, \frac{1}{2}(1-\sigma)\}, & \text{for } \text{meas } \mathcal{D} > 0; \\
\sigma m + \frac{1}{2}(1-\sigma), & \text{for } \text{meas } \mathcal{D} = 0.
\end{array} \right.
\]

Then, as \( t \to \infty \),

\[
\mathcal{N}(t, T) = \frac{1}{\pi} \left( (2\pi)^{\frac{1}{2}} C_\varphi(H_{\text{min}}) \right)^{\frac{1}{2}} \int_0^1 t^{\frac{1}{2}(\nu+\frac{1}{2})} dx \left( 1 + O(\log^{-\nu} t) \right);
\]

\[
\mathcal{N}(t, S) = \frac{1}{\pi} \left( c_\varphi(H_{\text{min}}) \Gamma \left( H_{\text{min}} - \frac{1}{2} \right) \right)^{\frac{1}{2}} \int_0^1 t^{\frac{1}{2}(\nu+\frac{1}{2})} dx \left( 1 + O(\log^{-\nu} t) \right),
\]

where \( \nu = \nu(H_{\text{min}}, \sigma, \tau) > 0 \). If \( \text{meas } \mathcal{D} > 0 \), then \( \nu \) is an arbitrary exponent less than \( \frac{\tau - \frac{1}{2}(1-\sigma)}{m+1} \).

**Remark 1.** 1. Recall that the definition of the operator \( T \) admits \( 0 < H(x) < 1 \) while the definition of the operator \( S \) admits only \( \frac{1}{2} < H_{\text{min}} < 1 \). However, for \( H_{\text{min}} > \frac{1}{2} \), the asymptotic formulas (8) and (9) coincide since

\[
(2\pi)^{\frac{1}{2}} C_\varphi(H) = c_\varphi(H) \Gamma \left( H - \frac{1}{2} \right) = (\Gamma(2H+1) \sin(\pi H))^{\frac{1}{2}}.
\]

2. If \( h_1(x) \equiv 0 \), then formulas (8) and (9) are valid with \( O(t^{-r}) \) remainder estimate for some \( r > 0 \).

3. Applying the Laplace method to the integral in (8) and (9), we obtain, as \( t \to \infty \),

\[
\int_0^1 t^{\frac{1}{2}(\nu+\frac{1}{2})} dx = t^{\frac{1}{2}} (\text{meas } \mathcal{D} + m^{2\sigma} \Gamma(\sigma+1) (\log t)^{-\sigma} \varphi(\log t) (1 + o(1))).
\]

Notice that if \( \text{meas } \mathcal{D} > 0 \), then the main term of the asymptotics is purely power-law. If \( \sigma < \nu \) in addition, then we have even two-term asymptotics. In the case \( \nu \leq \sigma \), we obtain only one-term power asymptotics with logarithmic remainder term.

4. We stress that the asymptotics (10) does not change if we replace the function \( \varphi \) by an equivalent SVF. 
3. PROOF OF THEOREM 1

The idea of the proof is as follows. We separate the principal terms in the operators $T$, $S$. These terms are compact pseudodifferential operators of variable order. The singular values asymptotics for such operators is known, see [12] and [13]. Then, using asymptotic perturbation theory, we verify that the remainder terms do not influence the spectral asymptotics of the principal terms thus obtained.

3.1. Operator $T$ (mBM)

Since the symbol of a pseudodifferential operator must be smooth with respect to the dual variable $\xi$, we introduce an even smoother positive function $p(\xi)$ such that

\[ p(\xi) = |\xi| \quad \text{for} \quad |\xi| > 2; \quad p(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1. \]

Now we consider the pseudodifferential operator of variable order

\[ (A f)(x) := C_*(H(x)) \int_{-\infty}^{\infty} e^{ix\xi} p(\xi)^{-((H(x)+\frac{1}{2})} f(\xi) \, d\xi. \]

The singular value asymptotics for $A$ is given in part 2 of Corollary 1, see Section 5.1. Thus, formula (8) is ensured by part 3 in Proposition 1 if we prove the following lemma.

**Lemma 1.** The following estimate holds:

\[ \mathcal{N}(t, T - A) \leq C t^{\frac{1}{\mu}} - \mu, \quad t > 1, \]

for some $\mu > 0$.

**Proof.** The kernels of $T$ and $A$ have the same bounded multiplier $C_*(H(x))$ and, therefore, we are to estimate singular values for the operator with the kernel

\[ \frac{e^{ix\xi} - 1}{|\xi|^{(H(x)+\frac{1}{2})}} \frac{e^{ix\xi}}{p(\xi)^{H(x)+\frac{1}{2}}} = q_1(x, \xi) - q_2(x, \xi) + q_3(x, \xi) \]

\[ \zeta(\xi) - 1 \frac{e^{ix\xi}}{|\xi|^{(H(x)+\frac{1}{2})}} + \zeta(\xi) \frac{e^{ix\xi} - 1}{|\xi|^{(H(x)+\frac{1}{2})}}. \]

Here $\zeta(\xi)$ is a fixed even cut-off function,

\[ \zeta(\xi) = 0 \quad \text{for} \quad |\xi| \geq 3; \quad \zeta(\xi) = 1 \quad \text{for} \quad |\xi| \leq 2 \]

(notice that $p(\xi) = |\xi|$ for $\zeta(\xi) \neq 1$).

According to (12), we need to estimate singular values of the operator $Q_j = Q_1 - Q_2 + Q_3$, where $Q_j : L_2(\mathbb{R}) \to L_2(0, 1)$ are the integral operators with kernels $q_j(x, \xi)$, $j = 1, 2, 3$. Due to part 2 of Proposition 1, the estimate (11) follows from similar estimates for the operators $Q_j$.

The bound $\mathcal{N}(t, Q_1) \leq C(\varepsilon) t^{\frac{1}{\mu} + \varepsilon + \mu}$ with arbitrary $\varepsilon > 0$ and $\lambda < \beta$ follows from Lemma 3.

The kernel $q_2(x, \xi)$ is bounded in $x$ and smooth and compactly supported in $\xi$. Therefore, by Proposition 2, $\mathcal{N}(t, Q_2) \leq C(\varepsilon) t^\varepsilon$ for any $\varepsilon > 0$.

The kernel $q_3$ is singular at the point $\xi = 0$. We separate the principal part of this singularity

\[ q_3(x, \xi) = q_{3,0}(x, \xi) + q_{3,1}(x, \xi), \quad q_{3,0}(x, \xi) := \zeta(\xi) \frac{ix\xi}{|\xi|^{(H(x)+\frac{1}{2})}}. \]

Thus, we can write $Q_3 = Q_{3,0} + Q_{3,1}$.

The function $q_{3,1}(x, \cdot)$ is compactly supported and belongs to the Sobolev space $W_2^1(\mathbb{R})$ uniformly with respect to $x$. Therefore, Proposition 2 gives $\mathcal{N}(t, Q_1) = O(t^{\frac{1}{\mu}})$. The required estimate follows from the inequality $\frac{1}{\mu} > \frac{1}{2}$.

It remains to estimate singular values of the operator $Q_{3,0}$. Its kernel is nonzero only for $|\xi| < 3$, and we can consider $Q_{3,0}$ as an operator from $L_2(-3, 3)$ to $L_2(0, 1)$.

We introduce the isometry

\[ U : L_2(-3, 3) \to L_2(\mathbb{R}); \quad (U f)(z) := \sqrt{3} e^{-\frac{|z|}{3}} f(3 \text{sgn}(z) e^{-|z|}). \]

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Then singular values of $\mathbb{Q}_{3,0}$ coincide with those of $\mathbb{Q}_{3,0} U^{-1}: L_2(0, \infty) \to L_2(0, 1)$. Since the kernel $q_{3,0}$ is odd with respect to $\xi$, we have

$$
(\mathbb{Q}_{3,0} U^{-1} g)(x) = 3^{1-H(x)} i x \int_0^\infty \zeta(3e^{-z}) e^{-(1-H(x))z} (g(z) - g(-z)) \, dz.
$$

(13)

The function $1 - H(\cdot)$ belongs to $C^\beta[0, 1]$ and is bounded away from zero. Therefore, the integrand in (13) belongs to $C^\beta$ in $x$ and decays exponentially as $z \to \infty$. Lemma 3 yields the estimate $\mathcal{M}(t, \mathbb{Q}_{3,0}) \leq C(t) t^\varepsilon$ for any $\varepsilon > 0$.

Summing the obtained estimates, we arrive at (11).

3.2. The operator $\mathbb{S}$ (mfBM)

First, we change the variable $z = y(1 + w)$ in the integral (3) and rewrite the kernel of $\mathbb{S}$ in the form

$$
K(x, y, H) = c_s(H) y^{H-\frac{1}{2}} \int_0^w w^{H-\frac{1}{2}} (1 + w)^{H-\frac{1}{2}} dw I_{[0, 1]}(y).
$$

(14)

Next, we separate the principal homogeneous term in (14) and write

$$
K(x, y, H) = \overline{q}_1(x, y, H) + \overline{q}_2(x, y, H),
$$

where

$$
\overline{q}_1(x, y, H) := c_s(H) y^{H-\frac{1}{2}} \int_0^{\frac{x-y}{|y|}} w^{H-\frac{1}{2}} \, dw \chi_{[0, 1]}(y) = \frac{c_s(H)}{H-\frac{1}{2}} (x-y)^{H-\frac{1}{2}} \chi_{[0, 1]}(y);
$$

$$
\overline{q}_2(x, y, H) = c_s(H) y^{H-\frac{1}{2}} \Phi\left(\frac{x-y}{y}, H\right) \chi_{[0, 1]}(y),
$$

$$
\Phi(s, H) := \int_0^s w^{H-\frac{1}{2}} ((1 + w)^{H-\frac{1}{2}} - 1) \, dw.
$$

Since $x, y \in (0, 1)$, we can assume that $\overline{q}_1$ is multiplied by a cut-off function $\theta(x-y)$,

$$
\theta \in C^\infty(\mathbb{R}), \quad \theta(w) = 1 \text{ for } |w| \leq 1, \quad \theta(w) = 0 \text{ for } |w| \geq 2.
$$

(15)

Let $\overline{Q}_1$ and $\overline{Q}_2$ be operators in $L_2(0, 1)$ with kernels $\overline{q}_1(x, y, H(x))$ and $\overline{q}_2(x, y, H(x))$, respectively. We claim that $\overline{Q}_1$ is in fact a pseudodifferential operator of variable order. Indeed, we have

$$
(\overline{Q}_1 f)(x) = (2\pi)^{-1} c_s(H(x)) \int_{-\infty}^\infty e^{i(x-y)\xi} R\left(\xi, H(x) - \frac{1}{2}\right) f(y) \, dyd\xi,
$$

where

$$
R(\xi, \gamma) := \int_0^\infty e^{-iz\xi} \theta(z) \frac{z^\gamma}{\gamma} \, dz.
$$

For any $\gamma > -1, \gamma \neq 0$, we have $R(\cdot, \gamma) \in C^\infty(\mathbb{R})$. Moreover, up to a function of the Schwartz class $\mathcal{S}$, the function $R(\cdot, \gamma)$ coincides at infinity with the Fourier transform of $z^\gamma / \gamma$:

$$
R(\xi, \gamma) = \Gamma(\gamma)|\xi|^{-(1+\gamma)} \exp(i \text{sgn}(\xi)(1+\gamma)\pi/4) + O(|\xi|^{-n}) \text{ for any } n \in \mathbb{N}.
$$

Thus, $R(\xi, \gamma)$ is a classical symbol of order $-(1+\gamma)$. Therefore, $\overline{Q}_1$ can be regarded as a pseudodifferential operator of variable order, and the claim follows.

We define a pseudodifferential operator

$$
(\mathcal{A} f)(x) := (2\pi)^{-1} c_s(H(x)) \Gamma\left(H(x) - \frac{1}{2}\right) \int_{-\infty}^\infty e^{i(x-y)\xi} \left(\mathcal{P}(\xi)\right)^{-(m+h(x))} f(y) \, dyd\xi,
$$

where $m$ and $h(x)$ are functions of $x$.
where \( h(x) = H(x) - H_{\min} \) while \( \tilde{p}(\xi) \) is a smooth complex-valued function such that
\[
\tilde{p}(\xi) \neq 0; \quad \tilde{p}(\xi) = |\xi| \exp \left( i \text{sgn}(\xi) \pi/4 \right) \quad \text{for} \quad |\xi| > 1.
\]

The kernel of \( \tilde{Q}_1 - \tilde{A} \) is smooth in \( y \) and bounded in \( x \). By Proposition 2, we have \( \mathcal{N}(t, \tilde{Q}_1 - \tilde{A}) = O(t^r) \) as \( t \to \infty \) for any \( r > 0 \), and part 3 of Proposition 1 gives \( \mathcal{N}(t, \tilde{Q}_1) = \mathcal{N}(t, \tilde{A}) + O(t^r) \).

Part 1 in Corollary 1, see Section 5.1, gives the singular value asymptotics for the operator \( \tilde{A} \), and therefore, for the operator \( \tilde{Q}_1 \). Thus, formula (9) is ensured by part 3 of Proposition 1 if we prove the following lemma.

**Lemma 2.** The following estimate holds:
\[
\mathcal{N}(t, \tilde{Q}_2) \leq Ct^\frac{q}{2}, \quad t > 1.
\]

**Proof.** The kernel \( \tilde{q}_2 \) has singularities on the diagonal \( y = x \) and at the point \( y = 0 \), i.e., for \( s = \frac{x-y}{y} = 0 \) and \( s = \infty \), respectively. Let us consider the influence of these singularities on the singular value asymptotics separately.

Consider two functions
\[
\Phi_0(s, H) = \theta(s) \Phi(s, H), \quad \Phi_1(s, H) = (1 - \theta(s)) \Phi(s, H),
\]
where the cut-off function \( \theta \) is defined in (15), and denote by \( \tilde{q}_{2,j}, j = 0, 1 \), the operators in \( L_2(0, 1) \) with the kernels
\[
\tilde{q}_{2,j}(x, y, H(x)) = c_a(H(x)) y^{H(x) - \frac{1}{2}} \Phi_j \left( \frac{x-y}{y}, H(x) \right) \chi_{[0, 1]}(y).
\]

Since \( \tilde{Q}_2 = \tilde{Q}_{2,0} + \tilde{Q}_{2,1} \), the bound (16) follows from similar bounds for the operators \( \tilde{Q}_{2,j} \).

We begin with the operator \( \tilde{Q}_{2,0} \). The kernel \( \tilde{q}_{2,0}(x, y, H(x)) \) does not vanish for \( \frac{x}{y} < y < x \) only. Since \( \Phi(s, H) = O(s^{H+\frac{1}{2}}) \) and \( \partial_y \Phi(s, H) = O(s^{H-\frac{1}{2}}) \) as \( s \to 0 \), we obviously have the bound
\[
|\tilde{q}_{2,0}(x, y, H)| \leq C \left( \frac{x-y}{y} \right)^{H+\frac{1}{2}}; \quad \left| \partial_y \tilde{q}_{2,0}(x, y, H, H) \right| \leq C \left( \frac{(x-y)^{H-\frac{1}{2}}}{y} + \frac{(x-y)^{H+\frac{1}{2}}}{y^2} \right).
\]

Recall that the functional parameter satisfies \( H(x) > \frac{1}{2} \). Therefore, for a chosen \( x \), the function \( \tilde{q}_{2,0}(x, \cdot, H(x)) \) is \( C^1 \)-smooth. Moreover, the following bound holds:
\[
\|\tilde{q}_{2,0}(x, \cdot, H(x))\|_{L^2_w(0, 1)}^2 \leq C x^2 \int_{x/3}^x \left( \frac{(x-y)^{H(x)-\frac{1}{2}}}{y^2} \right)^2 dy \leq C x^{2H(x)-2},
\]
and thus, the integral
\[
\int_0^1 \|\tilde{q}_{2,0}(x, \cdot, H(x))\|_{L^2_w(0, 1)}^2 dx
\]
converges. Now Proposition 2 yields the bound
\[
\mathcal{N}(t, \tilde{Q}_{2,0}) \leq Ct^{\frac{q}{2}}, \quad t > 1.
\]

Further, the kernel \( \tilde{q}_{2,1}(x, y, H(x)) \) does not vanish only for \( 0 \leq y \leq x/2 \) and has a singularity at the point \( y = 0 \).

Similarly to the estimate for the operator \( \tilde{Q}_{3,0} \) in the previous subsection, we introduce the isometry
\[
\tilde{U} : L_2(0, 1) \to L_2(0, \infty); \quad (\tilde{U} f)(z) := e^{-\frac{z}{2}} f(e^{-z}).
\]

Then the singular values of the operator \( \tilde{Q}_{2,1} \) in \( L_2(0, 1) \) coincide with those of the operator
\[
\tilde{Q}_{2,1} \tilde{U}^{-1} : L_2(0, \infty) \to L_2(0, 1).
\]

Changing the variable, we see that the kernel of \( \tilde{Q}_{2,1} \tilde{U}^{-1} \) is
\[
r(x, z, H(x)) := c_a(H(x)) x^{H(x)} (1 + s)^{-H(x)} \Phi(s) s = xe^{-1}.
\]
The following bounds for \( n \in \mathbb{N} \cup \{0\} \) are obvious:
\[
((1 + s)\partial_s)^n \Phi_1(s, H) = O(s^{2H - 1}) \quad \text{as} \quad s \to \infty.
\]
Since \( \partial_z \bigl(g(xe^z - 1)\bigr) = (1 + s)\partial_s g(s) |_{s = xe^z - 1} \), for any \( n \in \mathbb{N} \cup \{0\} \), we obtain
\[
|\partial_z^n r(x, z, H(x))| \leq C(n)xe^{(1 + s)e^{-H(x)}(1 + s)^{-(1 - H(x))}) |_{s = xe^z - 1} \leq C(n)xe^{-(1 - H(x))z}
\]
(the inequality \((\ast)\) follows from \( H(x) > \frac{1}{2} \)).

The function \( 1 - H(\cdot) \) belongs to \( C^\beta[0, 1] \) and is bounded away from zero. Therefore, the kernel \( r(x, z, H(x)) \) belongs to \( C^\beta \) in \( x \) and decays exponentially as \( z \to \infty \). Lemma 3 yields the bound \( \mathcal{N}(t, \mathcal{Q}_{a, 1}) \leq C(\varepsilon)t^\varepsilon \) for any \( \varepsilon > 0 \).

Summing the estimates for \( \mathcal{Q}_{a, 0} \) and \( \mathcal{Q}_{a, 1} \), we arrive at (16). \( \square \)

4. SMALL BALL ASYMPTOTICS FOR mBM AND mfBM

As explained in the introduction, the one-term asymptotics of eigenvalues for covariance operator provides, under mild assumptions (see [17, Th. 1]), the logarithmic \( L_2 \)-small ball asymptotics for the corresponding process.

We begin with multifractional Brownian motion (1). The case meas \( D > 0 \) is in fact quite elementary. In this case, formula (8) and part 3 of Remark 1 give
\[
\mathcal{N}(t, \mathcal{T}) \sim \frac{\mathcal{C}}{\pi} t^{\frac{1}{m}} \quad \text{as} \quad t \to \infty,
\]
where
\[
\mathcal{C} = (\Gamma(2m) \sin(\pi H_{\min}))^{\frac{1}{m}} \text{meas } D.
\]
Since the function \( t \mapsto \mathcal{N}(t, \mathcal{T}) \) is inverse in essence to the function \( k \mapsto s_k^{-1}(\mathcal{T}) \), we have, as \( k \to \infty \),
\[
s_k(\mathcal{T}) \sim \left( \frac{\mathcal{C}}{\pi k} \right)^m \iff \lambda_k(\mathcal{T}^+) \sim \left( \frac{\mathcal{C}}{\pi k} \right)^{2m}.
\]
Notice that \( 2m > 1 \). Applying Proposition 2.1 in [21], we obtain
\[
\lim_{\varepsilon \to 0} \varepsilon^{\frac{2m - 1}{m - 1}} \log \mathbb{P}\{||W^{(1)}||_2 \leq \varepsilon\} = -\frac{2m - 1}{2} \left( \frac{\mathcal{C}}{2m \sin(\frac{\pi m}{2m})} \right)^{\frac{2m}{m - 1}}.
\]
Now we consider the case meas \( D = 0 \). Then formula (8) and part 3 of Remark 1 give
\[
\mathcal{N}(t, \mathcal{T}) \sim \frac{\bar{\mathcal{C}}}{\pi m^\sigma} (\log t)^{-\sigma} \varphi(\log t),
\]
where
\[
\bar{\mathcal{C}} = (\Gamma(2m) \sin(\pi H_{\min}))^{\frac{1}{m}} m^{2\sigma} \Gamma(\sigma + 1).
\]
Therefore, we have as \( k \to \infty \)
\[
s_k(\mathcal{T}) \sim \left( \frac{\bar{\mathcal{C}}}{\pi m^\sigma} \frac{\varphi(\log k)}{k \log^\sigma k} \right)^m \iff \lambda_k(\mathcal{T}^+) \sim \left( \frac{\bar{\mathcal{C}}}{\pi m^\sigma} \frac{\varphi(\log k)}{k \log^\sigma k} \right)^{2m}.
\]
Since \( \lambda_k \) is a sequence regularly varying with the index \( 2m > 1 \), we can apply [14, Th. 4.2], where a general situation was considered. A concretization of formula (4.5) in [14] for our case gives
\[
\lim_{\varepsilon \to 0} \varepsilon^{\frac{2m - 1}{m - 1}} \frac{m}{\varphi(\log \varepsilon)} \log \mathbb{P}\{||W^{(1)}||_2 \leq \varepsilon\} = -\frac{2m - 1}{2} \left( \frac{\bar{\mathcal{C}}}{2m \sin(\frac{\pi m}{2m})} \right)^{\frac{2m}{m - 1}}.
\]
Now we are able to formulate the final statement.

**Theorem 2.** Assume that the variable Hurst parameter \( 0 < H(x) < 1 \) satisfies the assumptions 1–3 of Theorem 1 with \( \tau \) subjected to condition (7). Then, for the mBM \( W^{(1)}(\cdot) \), the following relation holds as \( \varepsilon \to 0 \):
1. If \( \text{meas } D > 0 \), then

\[
\log P\{\|W^{H(\cdot)}\|_{L^2(0,1)} \leq \varepsilon \} \sim -\varepsilon \left( -\frac{1}{\sigma_{\min}} \right)
\times \frac{H_{\min} \text{meas } D}{(2H_{\min} + 1) \sin \left( \frac{\pi}{2H_{\min} + 1} \right)} \left( \Gamma(2H_{\min} + 1) \sin \left( \frac{\pi}{2H_{\min} + 1} \right) \right)^{\frac{2H_{\min} + 1}{2H_{\min}}}.
\]

(17)

2. If \( \text{meas } D = 0 \), then

\[
\log P\{\|W^{H(\cdot)}\|_{L^2(0,1)} \leq \varepsilon \} \sim -\varepsilon \left( -\frac{1}{\sigma_{\min}} \right)
\times H_{\min} \left( \frac{\Gamma(2H_{\min} + 1) \sin \left( \pi H_{\min} \right)}{(2H_{\min} + 1) \sin \left( \frac{\pi}{2H_{\min} + 1} \right)} \right)^{\frac{2H_{\min} + 1}{2H_{\min}}}.
\]

(18)

To illustrate the theorem, we give several examples. For simplicity only, we assume that \( H_{\min} = \frac{1}{4} \).

**Example 1.** Let \( H(x) = \frac{1}{4} + x - x_0 \) for \( 0 < x < 1 \), and \( \gamma > 0 \). Then we have \( D = [0, x_0] \), and formula (17) reads

\[
\log P\{\|W^{H(\cdot)}\|_{L^2(0,1)} \leq \varepsilon \} \sim -\frac{x_0^2}{8} \varepsilon^{-2}, \quad \text{as } \varepsilon \to 0.
\]

For \( x_0 = 1 \), we obtain the standard Wiener process on \([0, 1]\), and this result is well known. However, for \( x_0 < 1 \), even this simplest result seems to be new.

**Example 2.** Let \( H(x) = \frac{1}{4} + |x - x_0|^{\gamma} \) and \( \gamma > 0 \). In this case, we have purely power-like behavior of the measure of the small values set. Namely, formula (6) holds with

\[
\sigma = \frac{1}{\gamma}; \quad \varphi(s) \equiv 2 \quad \text{if } 0 < x < 1; \quad \varphi(s) \equiv 1 \quad \text{if } x = 0, 1.
\]

Therefore, formula (18) gives

\[
\log P\{\|W^{H(\cdot)}\|_{L^2(0,1)} \leq \varepsilon \} \sim -\tilde{C}(x_0) \Gamma^2 \left( 1 + \frac{1}{\gamma} \right) \cdot \left( \varepsilon \log \frac{1}{\varepsilon} \right)^{-2}, \quad \text{as } \varepsilon \to 0,
\]

(19)

where

\[
\tilde{C}(x_0) = 2^{-1 - \frac{4}{\gamma}} \quad \text{if } 0 < x < 1; \quad \tilde{C}(x_0) = 2^{-3 - \frac{4}{\gamma}} \quad \text{if } x = 0, 1.
\]

**Example 3.** Let \( H(x) = \frac{1}{4} + \min_{1 \leq k \leq N} |x - x_k|^{\gamma_k} \), with \( 0 \leq x_k \leq 1 \), \( \gamma_k > 0 \), \( k = 1, \ldots, N \). In this case, the function \( H \) attains its minimum at several points, but only the point(s) with the maximal \( \gamma_k \) affect the asymptotics. For instance, if \( \gamma_k = \gamma \) for \( k \leq n \), \( \gamma_k < \gamma \) for \( k > n \), and \( 0 < x_k < 1 \) for \( k \leq n \), then formula (19) holds with \( \tilde{C}(x_0) = 2^{-1 - \frac{4}{\gamma}} n^2 \).

**Example 4.** Let \( H(x) = \frac{1}{4} + |x - x_0|^{\gamma} \log^b (|x - x_0|^{-1}) \), with \( x_0 \in (0, 1) \), \( \gamma > 0 \), and \( b \in \mathbb{R} \). Then formula (6) holds with

\[
\sigma = \frac{1}{\gamma}; \quad \varphi(s^{-1}) = 2 \gamma \left( \log s^{-1} \right)^{-\frac{4}{\gamma}} (1 + o(1)) \quad \text{as } s \to 0.
\]

By part 4 of Remark 1, we can drop \( o(1) \) in the last relation, and formula (18) gives

\[
\log P\{\|W^{H(\cdot)}\|_{L^2(0,1)} \leq \varepsilon \} \sim -\frac{\gamma b}{2} \Gamma^2 \left( 1 + \frac{1}{\gamma} \right) \cdot \left( \varepsilon \log \frac{1}{\varepsilon} \right)^{-\frac{4}{\gamma}} \log \frac{1}{\varepsilon}^{-2}, \quad \text{as } \varepsilon \to 0.
\]

**Example 5.** Let \( H(x) = \frac{1}{4} + \text{dist}^{\gamma}(x, \mathcal{D}) \), where \( \mathcal{D} \) is the standard Cantor set. A tedious but simple calculation gives

\[
\text{meas } \{ x \in [0, 1] | 0 < h(x) < s \} = s^1 \frac{\log 2}{\log 3} \phi(\log s^{-1})
\]
for small $s > 0$, where $\phi$ is a periodic function. Therefore, in this case, the assumption 3 of Theorem 1 is not valid, and this case is not covered by our Theorem 2. We are planning to consider the corresponding class of processes in a forthcoming paper.

Now we turn to the multifractal Brownian motion (2). By part 1 of Remark 1, formulas (8) and (9) coincide for $H_{\min} > \frac{1}{2}$, and thus the logarithmic asymptotics for $X^{H(\cdot)}$ coincides with that of $W^{H(\cdot)}$.

**Theorem 3.** Assume that the variable Hurst parameter $\frac{1}{2} < H(x) < 1$ satisfies the assumptions 1–3 of Theorem 1 with $\tau$ subjected to condition (7). Then, for the mfBM $X^{H(\cdot)}$, the following relation holds as $\varepsilon \to 0$:

1. If $\text{meas } D > 0$, then
   \[
   \log P\{||X^{H(\cdot)}||_{L_2(0,1)} \leq \varepsilon\} \sim -\varepsilon^{-\frac{1}{H_{\min}}} \frac{H_{\min} \text{meas } D}{(2H_{\min} + 1) \sin \left(\frac{\pi}{2H_{\min} + 1}\right)} \left(\frac{\Gamma(2H_{\min} + 1) \sin(\pi H_{\min}) \text{meas } D}{(2H_{\min} + 1) \sin \left(\frac{\pi}{2H_{\min} + 1}\right)}\right)^{\frac{2H_{\min} + 1}{2H_{\min}}}
   \]

2. If $\text{meas } D = 0$, then
   \[
   \log P\{||X^{H(\cdot)}||_{L_2(0,1)} \leq \varepsilon\} \sim -\varepsilon^{-\frac{1}{H_{\min}}} \left(\frac{\varphi(\log \frac{1}{\varepsilon})}{\log \frac{1}{\varepsilon}}\right)^{\frac{2H_{\min} + 1}{2H_{\min}}}
   \times H_{\min} \left(\frac{\Gamma(2H_{\min} + 1) \sin(\pi H_{\min})}{(2H_{\min} + 1) \sin \left(\frac{\pi}{2H_{\min} + 1}\right)}\right)^{\frac{2H_{\min} + 1}{2H_{\min}}}.
   \]

**Remark 2.** For both of the processes (1) and (2), in the case of $\text{meas } D > 0$, we obtain a purely power-law asymptotics. Even in the case of $\sigma < \nu$, when Theorem 1 gives a two-term asymptotics, see part 3 in Remark 1, the estimate of the remainder term is not sufficient to obtain exact small ball asymptotics.

The asymptotic coefficient in this case depends on $\text{meas } D$ and, for $\text{meas } D = 1$, coincides with the classical result of Bronski [8], see also [21, Th. 3.1].

5. APPENDIX

5.1. Asymptotics of Singular Values for Pseudodifferential Operators of Variable Order

We consider a compact pseudodifferential operator $A: L_2(\mathbb{R}) \to L_2(0,1),$

\[
(A f)(x) := (2\pi)^{-1} a(h(x)) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-y)\xi} (p(\xi))^{-(m+h(x))} f(y) dy d\xi.
\]

(20)

Here $m > \frac{1}{2}$, and $h \in C^\beta[0,1]$ is a nonnegative function. The complex-valued nonvanishing function $p$ is an elliptic symbol in the Hormander class $S^0_{\rho,0}$, $0 < \rho \leq 1$, namely,

\[
p \in C^\infty(\mathbb{R}), \quad |p(\xi)| = v|\xi| (1 + O(|\xi|^{-\mu})) \quad \text{as } |\xi| \to \infty \quad \text{for some } \mu > 0;
\]

\[
|p^{(n)}(\xi)| \leq C(n) (1 + |\xi)|^{1-\rho n} \quad \text{for any } n \in \mathbb{N} \cup 0.
\]

The (complex-valued) multiplier $a \in C^\infty[0,\infty)$ satisfies $a(0) \neq 0$.

The order of decay of the symbol $p^{(n)}(\xi)^{-(m+h(x))}$ as $|\xi| \to \infty$, which can be interpreted as the local order of the operator $A$, depends on $x$. Thus, we say that $A$ is an operator of variable order.

Similarly to Section 2, we consider the set $D := \{x \in [0,1] \mid h(x) = 0\}$.

**Theorem 4.** Assume that conditions 1–3 of Theorem 1 are fulfilled. Assume in addition that

\[
\tau > \begin{cases} 
\max\{0, \frac{1}{2} (1 - \sigma)\} & \text{for } \text{meas } D > 0; \\
\sigma m + \frac{1}{2} (1 - \sigma) & \text{for } \text{meas } D = 0.
\end{cases}
\]
Then, as $t \to \infty$, 
\[ \mathcal{N}(t, \mathbb{A}) = \frac{1}{\pi \nu} (a(0))^{\frac{1}{\nu}} \int_{0}^{1} t^{m+1/\nu} \, dx \left( 1 + O(\log^{-\nu} t) \right). \tag{22} \]

Here $\nu$ is defined in (21), while $\nu = \nu(m, \sigma, \tau) > 0$. If $\text{meas} \, D > 0$, then $\nu$ is an arbitrary exponent less than $\frac{1}{m+1}$.

**Proof.** Assumptions 1–3 ensure that conditions of [13, Th. 1.3] are satisfied. In [13], the statement was proved for $a \equiv 1$, even in the multidimensional case. The proof runs without changes for the multiplier $a(h_0(x))$. To include the nonsmooth multiplier $a(h(x))$, one needs to study the operator with the symbol

\[ (a(b)(x) - a(h_0(x))) (p(\xi))^{-(m+h(x))}. \]

Repeating the argument of [13, Th. 5.1], we obtain the bound

\[ O(\log^{-\nu} t) \int_{0}^{1} t^{m+1/\nu} \, dx \]

for the counting function of the singular values of this operator. Part 4 of Proposition 1 completes the proof with regard to (10). \qed

**Remark 3.** 1. Theorem 4 in fact claims that Weyl’s spectral asymptotic formula

\[ \mathcal{N}(t, \mathbb{A}) \sim (2\pi)^{-\frac{1}{2}} \text{meas} \{ x, \xi \in (0, 1) \times \mathbb{R} : |a(h(x))p(\xi)^{-(m+h(x))}| > t^{-1} \} \]

is valid for the operator $\mathbb{A}$. The asymptotics (22) is just a concretization of Weyl’s formula for our problem with a bound for the remainder.

2. If $h_1(x) \equiv 0$, then formula (22) is valid with the bound $O(t^{-r})$ for the remainder for some $r > 0$.

**Corollary 1.** 1. One can regard the operator in formula (20) as an operator acting on $L_2(0, 1)$ by extending $u \in L_2(0, 1)$ by zero. Formula (22) remains valid in this case.

2. Since the Fourier transform is a unitary operator in $L_2(\mathbb{R})$, formula (22) holds also for the operator

\[ (\mathbb{A}f)(x) := (2\pi)^{-\frac{1}{2}} a(h(x)) \int_{-\infty}^{\infty} e^{ix\xi} (p(\xi))^{-(m+h(x))} f(\xi) \, d\xi. \]

5.2. Estimates for the Singular Values of an Integral Operator

We need the following results from asymptotic perturbation theory.

**Proposition 1.** 1. Let $\mathbb{A} : \mathcal{H}_1 \to \mathcal{H}_2$ and $\mathbb{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be compact operators. If $\mathcal{N}(t, \mathbb{A}) = O(t^{p_1})$ and $\mathcal{N}(t, \mathbb{B}) = O(t^{p_2})$ as $t \to \infty$, then $\mathcal{N}(t, \mathbb{A} \mathbb{B}) = O(t^p)$ with

\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \]

2. Let $\mathbb{A}, \mathbb{B} : \mathcal{H}_1 \to \mathcal{H}_2$ be compact operators. If $\mathcal{N}(t, \mathbb{A}) = O(t^p)$ and $\mathcal{N}(t, \mathbb{B}) = O(t^p)$ as $t \to \infty$, then $\mathcal{N}(t, \mathbb{A} + \mathbb{B}) = O(t^p)$.

3. Let $\mathbb{A}, \mathbb{B} : \mathcal{H}_1 \to \mathcal{H}_2$ be compact operators. If $\mathcal{N}(t, \mathbb{A}) = t^p V(t)$ as $t \to \infty$ with a slowly varying function $V$ and $\mathcal{N}(t, \mathbb{B}) = O(t^q)$ for some $q < p$, then

\[ \mathcal{N}(t, \mathbb{A} + \mathbb{B}) = t^p \left( V(t) + O(t^{-r}) \right), \quad \text{where} \quad r < \frac{p-q}{q+1}. \]

4. Let $\mathbb{A}, \mathbb{B} : \mathcal{H}_1 \to \mathcal{H}_2$ be compact operators. Assume that

\[ \mathcal{N}(t, \mathbb{A}) = t^p (\delta + \log^{-a} t V(\log t)) \quad \text{as} \quad t \to \infty \]

with $a > 0$ and a slowly varying function $V$ and $\mathcal{N}(t, \mathbb{B}) = O(t^p \log^{-b} t)$ for some $b > a$. Then

\[ \mathcal{N}(t, \mathbb{A} + \mathbb{B}) = t^p (\delta + \log^{-a} t V(\log t) + O(\log^{-r} t)), \]

where $r$ is arbitrary exponent such that

\[ r < \frac{b}{p+1} \quad \text{if} \quad \delta > 0; \quad r < \frac{ap+b}{p+1} \quad \text{if} \quad \delta = 0. \]
where the cut-off function \( W \) follows, its kernel. The results are given for \( d \)-dimensional domains, though we need only the case \( d = 1 \). In what follows, \( W_2 \) stands for the standard Sobolev–Slobodetskiï space; see, e.g., [27, Sec. 2.3.1].

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with Lipschitz boundary. We write \( \mathcal{N} \) the operators \( \mathcal{N} \circ \mathcal{R} \) coincide.

**Proposition 2.** (see [7, Sec. 11.8]). Let \( R(\cdot, y) \in W_2^2(\Omega) \) for a.e. \( y \in \mathbb{R}^d \), and let

\[
M^2 := \int_{\mathbb{R}^d} \| R(\cdot, y) \|_{W_2^2(\Omega)}^2 \, dy < \infty.
\]

Then the following bound holds:

\[
\mathcal{N}(t, \mathcal{R}) \leq C(\Omega) \left( M \| a \|_{L_\infty(\Omega)} \right)^p t^p, \quad \frac{1}{p} = \frac{1}{2} + \frac{\lambda}{d}.
\]

Now we take into account the decay of the kernel with respect to the second variable. Let \( \tilde{\mathcal{R}} : L_2(\mathbb{R}^d) \to L_2(\Omega), \quad (\tilde{\mathcal{R}} f)(x) := \int_{\mathbb{R}^d} \tilde{R}(x, \xi) f(\xi) \, d\xi. \)

**Lemma 3.** Let \( \tilde{R}(\cdot, \cdot) \in W_2^2(\Omega) \) for a.e. \( \xi \in \mathbb{R}^d \), and let

\[
\| \tilde{R}(\cdot, \xi) \|_{W_2^2(\Omega)} \leq M(\ell, \lambda) (1 + |\xi|)^{-\ell} \quad \text{for some } \ell > \frac{d}{2}.
\]

Assume also that \( \tilde{R}(x, \cdot) \in C(\mathbb{R}^d) \) and

\[
| \partial_\xi^{\alpha} \tilde{R}(x, \xi) | \leq C(\alpha)(1 + |\xi|)^{-(\ell + |\alpha|)}, \quad x \in \Omega, \, \xi \in \mathbb{R}^d,
\]

for some \( \rho > 0 \) and any multi-index \( \alpha. \)

Then

\[
\mathcal{N}(t, \tilde{\mathcal{R}}) \leq C(\ell, \lambda, \rho, \Omega) t^n \quad \text{for any } n > \frac{d}{\ell + \lambda}.
\]

**Proof.** Since the Fourier transform \( \mathcal{F} : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \) is unitary, it follows that the singular values of the operators \( \mathcal{R} \) and \( \mathcal{N} = \mathcal{R} \mathcal{F} \) coincide.

We split \( \mathcal{R} \) into two parts:

\[
\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1,
\]

with the kernels

\[
R_0(x, y) = \theta(|y|) R(x, y), \quad R_1(x, y) = (1 - \theta(|y|)) R(x, y),
\]

where the cut-off function \( \theta \) is defined in (15), while

\[
R(x, y) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \tilde{R}(x, \xi) e^{-iy\xi} \, d\xi, \quad x \in \Omega, \, y \in \mathbb{R}^d.
\]

Integrating by parts with respect to \( \xi \) and using the identity

\[
e^{-iy\xi} = (-\Delta)^N |y|^{-2N} e^{-iy\xi},
\]

we see that the function \( R_1(x, \cdot) \) belongs to the Schwartz class uniformly with respect to \( x \in \Omega \). By [5, Th. 4.8], this gives \( \mathcal{N}(t, \mathcal{R}_1) = O(t^n) \) for any \( \varepsilon > 0 \). Thus, part 2 of Proposition 1 shows that the bound for \( \mathcal{N}(t, \mathcal{R}) \) is governed by the bound for \( \mathcal{N}(t, \mathcal{R}_0) \).
Choose some $0 < l < \ell - \frac{d}{2}$ and write $\mathcal{R}_0$ in the form $\mathcal{R}_0 = \mathcal{R}_{0,0}\mathcal{R}_{0,1}$, where the kernels of $\mathcal{R}_{0,j}$, $j = 0, 1$, are

$$R_{0,0}(x, \xi) = (2\pi)^{-\frac{d}{2}}R(x, \xi)(1 + |\xi|^2)^{\frac{d}{2}}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^d;$$

$$R_{0,1}(\xi, y) = e^{-iy\xi}(1 + |\xi|^2)^{-\frac{1}{2}}\theta(|y|), \quad \xi, y \in \mathbb{R}^d.$$

For the operator $\mathcal{R}_{0,0}$ we have the bound $\mathcal{N}(t, \mathcal{R}_{0,0}) \leq C(\Omega)t^\frac{d}{2}$ (this is a particular case of the Rozenblum–Lieb–Cwikel bound; see, e.g., [5, Th. 6.5]). The bound (24) ensures inequality (23), and Proposition 2 yields the estimate $\mathcal{N}(t, \mathcal{R}_{0,0}) \leq C(\ell, \lambda, l)t^{\frac{d}{2}}, \quad \frac{1}{p} = \frac{1}{2} + \frac{\lambda}{d} + \frac{l}{d} < \frac{\ell + \lambda}{d}$, by part 1 of Proposition 1, we have

$$\mathcal{N}(t, \mathcal{R}_0) \leq C(\ell, \lambda, l, \Omega)t^\frac{d}{2},$$

and the statement follows. \qed

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**REFERENCES**

[1] A. Ayache, S. Cohen, and J. Lévy Véhel, “The Covariance Structure of Multifractional Brownian Motion, with Application to Long Range Dependence”, In: IEEE Int. Conf. on Acoustics, Speech, and Signal Processing, 6:2000 (2000), 3810–3813.

[2] A. Benassi, S. Cohen, and J. Istas, “Identifying the Multifractional Function of a Gaussian Process”, *Statist. Probab. Lett.*, 39 (1998), 337–345.

[3] M. S. Birman and M. Z. Solomyak, “Asymptotic Behavior of the Spectrum of Weakly Polar Integral Operators”, *Izv. AN SSSR. Ser. Mat.*, 34:6 (1970), 1143–1158; *Math. of the USSR-Izvestiya*, 4:5 (1970), 1151–1168.

[4] M. S. Birman and M. Z. Solomyak, “Quantitative Analysis in Sobolev Imbedding Theorems and Applications to Spectral Theory”, In: Proceed. of X Summer Mathematical School. Yu.A. Mitropol’skiy and A.F. Shestopal (Eds), 1974, 5–189; *AMS Translations, Series 2*, 114 (1980).

[5] M. Sh. Birman and M. Z. Solomyak, “Estimates of Singular Numbers of Integral Operators”, *Uspekhi Mat. Nauk*, 32:1 (1977), 17–84; *Russian Math. Surveys*, 32:1 (1977), 15–89.

[6] M. Sh. Birman and M. Z. Solomyak, “On the Negative Discrete Spectrum of a Periodic Elliptic Operator in a Waveguide-Type Domain, Perturbed by a Decaying Potential”, *J. Anal. Math.*, 83, 337–391.

[7] M. Sh. Birman and M. Z. Solomyak, “Spectral Theory of Self-Adjoint Operators in Hilbert Space”, 2nd ed., revised and extended. *Lan’*, St.Petersburg, 2010; *Mathematics and Its Applications. Soviet Series*, 5 (1987).

[8] J. C. Bronski, “Small Ball Constants and Tight Eigenvalue Asymptotics for Fractional Brownian Motions”, *J. Theoret. Probab.*, 16 (2003), 87–100.

[9] P. Chigansky and M. Kleptsyna, “Exact Asymptotics in Eigenproblems for Fractional Brownian Covariance Operators”, *Stochastic Process. Appl.*, 128:6 (2018), 2007–2059.

[10] J. F. Coeurjolly, “Identification of Multifractional Brownian Motion”, *Bernoulli*, 11:6 (2005), 987–1008.

[11] S. Cohen, “From Self-Similarity to Local Self-Similarity: the Estimation Problem”, In *Fractals: Theory and Applications in Engineering*. M. Dekking, J. Lévy Véhel, E. Lutton and C. Tricot (Eds). Springer Verlag, 1999, 3–16.

[12] A. I. Karol, “The Asymptotic Behavior of Singular Numbers of Compact Pseudodifferential Operators with Symbol Nonsmooth in Spatial Variables”, *Func. Anal. Appl.*, 53:4 (2019), 313–316.

[13] A. I. Karol, “The Singular Values of Compact Pseudodifferential Operators with Spatially Nonsmooth Symbols”, *Siberian Math. J.*, 61:4 (2020), 671–686.

[14] A. Karol, A. Nazarov, and Ya. Nikitin, “Small Ball Probabilities for Gaussian Random Fields and Tensor Products of Compact Operators”, *Trans. Amer. Math. Soc.*, 360:3 (2008), 1443–1474.

[15] G. Lieberman, “Regularized Distance and Its Applications”, *Pacific J. Math.*, 117:2 (1985), 329–352.

[16] M. A. Lifshits, *Lectures on Gaussian Processes*, Springer, New York, 2012.

[17] A. I. Nazarov, “Log-Level Comparison Principle for Small Ball Probabilities”, *Statist. Probab. Lett.*, 79:4 (2009), 481–486.

[18] A. I. Nazarov, “Exact $L_2$-Small Ball Asymptotics of Gaussian Processes and the Spectrum of Boundary-Value Problems”, *J. Theoret. Probab.*, 22:3 (2009), 640–665.

[19] A. I. Nazarov, “Spectral Asymptotics for a Class of Integro-Differential Equations Arising in the Theory of Fractional Gaussian Processes, DOI 10.1142/S0219199720500492”, *Commun. Contemp. Math.*, 2020, 1–25.

[20] A. I. Nazarov and Ya. Yu. Nikitin, “Exact Small Ball Behavior of Integrated Gaussian Processes under $L_2$-Norm and Spectral Asymptotics of Boundary Value Problems”, *Probab. Theory Related Fields*, 129:4 (2004), 469–494.
[21] A. I. Nazarov and Ya. Yu. Nikitin, “Logarithmic $L_2$-Small Ball Asymptotics for Some Fractional Gaussian Processes”, Teor. Veroyatnost. i Primenen., 49:4 (2004), 695–711; Theor. Probab. Appl., 49:4 (2005), 645–658.

[22] A. I. Nazarov and Ya. Yu. Nikitin, “On Small Deviation Asymptotics in $L_2$ of Some Mixed Gaussian Processes”, Mathematics, 6:4 (2018), 1–9.

[23] R.-F. Peltier and J. Lévy Véhel, “Multifractional Brownian Motion: Definition and Preliminary Results”, Inria Research Report No. 2645, 1995.

[24] K. V. Ral’chenko and G. M. Shevchenko, “Path Properties of Multifractal Brownian Motion”, Teor. Ìmovir. Mat. Stat., 2009, No. 480, 106–116; Theor. Prob. and Math. Stat., 2010, No. 480, 119–130.

[25] J. Ryvkina, “Fractional Brownian Motion with Variable Hurst Parameter: Definition and Properties”, J. Theoret. Probab., 28:3 (2015), 866–891.

[26] E. Seneta, “Regularly Varying Functions”, Lecture Notes in Math., 508 (1976).

[27] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, Deutscher Verlag Wissensch., Berlin, 1978.