A family of weakly universal cellular automata in the hyperbolic plane with two states

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January 13, 2014

Abstract

In this paper, we construct a family of weakly universal cellular automaton for all grids \( \{p, 3\} \) of the hyperbolic plane for \( p \geq 13 \). The scheme is general for \( p \geq 17 \) and for \( 13 \leq p < 17 \), we give such a cellular automaton for \( p = 13 \), which is enough. Also, an important property of this family is that the set of cells of the cellular automaton which are subject to changes is actually a planar set. The problem for \( p < 13 \) for a truly planar construction is still open. The best result, for \( p = 7 \), is four states and was obtained by the same author.

Keywords: cellular automata, hyperbolic plane, tessellations, universality

1 Introduction

The first result about universality for cellular automata in a grid \( \{p, 3\} \) of the hyperbolic plane was obtained by the author and Y. Song, see [18]. The result was just constructed for \( p = 7 \), i.e. the grid \( \{7, 3\} \) called the heptagrid. As there is no grid of this family when \( p < 7 \), the construction in the heptagrid can easily be extended to the whole family. A bit later, the author obtained a significant improvement the heptagrid, with four states only. It is still the best result if we consider a truly planar cellular automata, which means that the cells which change during the execution of the cellular automaton constitute a planar graph with many cycles. In [13], the author has proved that there is a weakly universal cellular automaton with two states only in all tilings \( \{p, q\} \) of the hyperbolic plane. Now, the result was obtained by embedding the elementary cellular automaton rule 110 into the tiling. The problem is that this embedding produces a one-dimensional structure.
This paper is a first answer to this latter question about a planar weakly cellular automaton with two states actually living in the hyperbolic plane. In this paper, I construct such an automaton in infinitely many tilings of the hyperbolic plane. More precisely, I do this for all tilings \( \{p, 3\} \) when \( p \geq 13 \). The lower bound \( p = 13 \) leaves the question open for the cases when \( 7 \leq p < 13 \).

In this paper, as in many papers of the author alone or with coauthors, see [2, 17, 19, 12, 18], I use the railway model to define a weakly universal cellular automaton. As in [2], it is proved that using such a model, one can simulate a two-register machine, it is enough to implement such a simulation by a cellular automaton in the considered grid. This model was never used in the Euclidean plane for cellular automata, but is used for the same purpose in the hyperbolic plane.

In planar simulations of the railway model, especially when we try to reduce the number of states, the difficult point is to model the crossings. This point will be clear in the figures of Section 2 for readers who are not familiar with the model. The main features of this model are explained in Section 2. Note that in the 3D-space, this problem vanishes as crossings can be easily replaced by bridges. This is why simulations of the same model in the hyperbolic 3D-space have always been performed with less states than in the planar cases. This is also why the threshold of two states with a true spatial cellular automaton was reached by the author in a regular tiling of the hyperbolic 3D-space, see [15] before this paper. Now, it is interesting to notice that if a previous result in the hyperbolic 3D-space has taken benefit from an idea used in a result in the hyperbolic plane, this paper takes benefit from an idea used in [15]. However, as there is no crossing in the 3D-simulation, something new had to be found for the plane. Subsection 5.2 explains this new idea and Section 5 thoroughly describes the implementation.

As most often, a computer program was used here too in order to check the coherence of the set of rules, the cellular automaton obtained in the paper being rotation invariant. The rules and their construction are explained in Section 6. Traces of executions of various pieces of the simulation are given in Section 6.6. Section 7 gives a uniform rules for \( \{p, 3\} \) when \( p \geq 17 \). With this section, we shall reach the end of the proof of the following result:

**Theorem 1** (Margenstern) — There is a rotation invariant cellular automaton on all grids \( \{p, 3\} \) of the hyperbolic plane, with \( p \geq 13 \), which is weakly universal and which has two states. There is a uniform set of rules for the automaton when \( p \geq 17 \). The initial configuration of the automaton is infinite: it is ultimately periodic along two different rays of mid-points \( r_1 \) and \( r_2 \) of the tiling \( \{p, 3\} \) and finite in the complement of the parts attached to \( r_1 \) and \( r_2 \). The set of cells which change their state at least once during the computation is a planar graph with infinitely many cycles.

Section 3 remembers the reader with the main features of hyperbolic geometry which are needed in order to understand the implementation.

2 The railway circuit

As initially devised in [21] and then mentioned in [5, 2, 17, 18, 12], the circuit uses tracks represented by lines and quarters of circles and switches. There are
three kinds of switches: the **fixed**, the **memory** and the **flip-flop** switches. They are represented by the schemes given in Fig. 1.

![Figure 1](image)

**Figure 1** *The three kinds of switches. From left to right: fixed, flip-flop and memory switches.*

Note that a switch is an oriented structure: on one side, it has a single track $u$ and, on the the other side, it has two tracks $a$ and $b$. This defines two ways of crossing a switch. Call the way from $u$ to $a$ or $b$ active. Call the other way, from $a$ or $b$ to $u$ passive. The names comes from the fact that in a passive way, the switch plays no role on the trajectory of the locomotive. On the contrary, in an active crossing, the switch indicates which track between $a$ and $b$ will be followed by the locomotive after running on $u$: the new track is called the **selected** track.

As indicated by its name, the **fixed switch** is left unchanged by the passage of the locomotive. It always remains in the same position: when actively crossed by the locomotive, the switch always sends it onto the same track. The flip-flop switch is assumed to be crossed actively only. Now, after each crossing by the locomotive, it changes the selected track. The memory switch can be crossed by the locomotive actively and passively. In an active passage, the locomotive is sent onto the selected track. Now, the selected track is defined by the track of the last passive crossing by the locomotive. Of course, at initial time, the selected track is fixed.

![Figure 2](image)

**Figure 2** *The elementary circuit.*

With the help of these three kind of switches, we define an **elementary circuit** as in [21], which exactly contains one bit of information. The circuit is illustrated by Fig. 2 above. It can be remarked that the working of the circuit strongly depends on how the locomotive enters it. If the locomotive enters the circuit through $R$, it leaves the circuit through $E_1$ or $E_2$, depending on the selected track of the memory switch which stands near $R$. If the locomotive
enters through $W$, the application of the given definitions shows that the selected track at the switches near $R$ and $W$ are both changed: the switch at $W$ is a flip-flop which is changed by the very active passage of the locomotive and the switch at $R$ is a memory one which is changed because it is passively crossed by the locomotive and through the non-selected track. The just described actions of the locomotive correspond to a read and a write operation on the bit contained by the circuit which consists of the configurations of the switches at $R$ and at $L$. It is assumed that the write operation is triggered when we know that we have to change the bit which we wish to rewrite.

Figure 3 Working of the elementary circuit. Above: reading. Below: writing. To left, changing 0 into 1. To right, changing 1 into 0.

Figure 4 Here, we have two consecutive units of a register. A register contains infinitely many copies of units. Note the tracks $i$, $d$, $r$, $j_1$, and $j_2$. For incrementing, the locomotive arrives at a unit through $i$ and it leaves the unit through $r$. For decrementing, it arrives though $d$ and it leaves also through $r$ if decrementing the register was possible, otherwise, it leaves through $j_1$ or $j_2$.

Figure 4 illustrates this working. As mentioned in the caption, the unit can be used in two ways. The reading way enters the circuit through the switch $R$
and exits through the track $E_1$ or $E_2$ depending on what is read at $R$. The writing way enters the circuit through $W$ and exits through $R$. This is a special writing: it changes the content of what is found into the opposite value. This means that before writing we have to test what is in the unit. If we have to write the same thing, nothing has to be done. If we have to write the other value, then the enter through $W$ is appropriate.

The combination of such elementary circuits allows us to define more complex structures which are useful to control the motion of the locomotive through the circuit. As an example, Fig. 4 illustrates an implementation of a unit of a register.

Other parts of the needed circuitry are described in [5, 2]. The main idea in these different parts is to organize the circuit in possibly visiting several elementary circuits which represent the bits of a configuration which allow the whole system to remember the last visit of the locomotive. The use of this technique is needed for the following two operations.

![Diagram](image.png)

**Figure 5** An example of the implementation of a small program of a register machine. On the left-hand side of the figure, the part of the sequencer. It can be noticed how the tracks are attached to each instruction of the program. Note that there are four decrementing instructions for $W$: this is why a selector gathers the arriving tracks before sending the locomotive to the control of the register. On the way back, the locomotive is sent on the right track.

When the locomotive arrives to a register $R$, it arrives either to increment $R$ or to decrement it. As can be seen on Fig. 4 when the instruction is performed, the locomotive goes back from the register by the same track. Accordingly, we need somewhere to keep track of the fact whether the locomotive incremented $R$ or it decremented $R$. This is one type of control. The other control comes from the fact that several instructions usually apply to the same register. Again, when the locomotive goes back from $R$, in general it goes back to perform a new instruction which depends on the one it has just performed on $R$. Again this can be controlled by what we called the selector in [5, 2].

At last, the dispatching of the locomotive on the right track for the next
instruction is performed by the sequencer, a circuit whose main structure looks like its implementation in the classical models of cellular automata such as the game of life or the billiard ball model. The reader is referred to the already quoted papers for full details on the circuit. Remember that this implementation is performed in the Euclidean plane, as clear from Fig. 5 which illustrates the case of a few lines of a program of a register machine.

Now, before turning to the implementation in the hyperbolic plane, we provides the reader with the minimal properties of hyperbolic geometry for a better understanding of the paper.

3 Short introduction to hyperbolic geometry

Hyperbolic geometry appeared in the first half of the 19th century, in the last attempts to prove the famous parallel axiom of Euclid’s Elements from the remaining ones. Independently, Lobachevsky and Bolyai discovered a new geometry by assuming that in the plane, from a point out of a given line, there are at least two lines which are parallel to the given line. Later, models of the new geometry were found, in particular Poincaré’s model, which is the frame of all this study.

3.1 Poincaré’s disc model

In this model, the hyperbolic plane is the set of points which lie in a fixed open disc \( \mathcal{D} \) of the Euclidean plane. Let \( \mathcal{C} \) be the border of \( \mathcal{D} \). The lines of the hyperbolic plane in Poincaré’s disc model are either the trace of diametral lines or the trace of circles which are orthogonal to \( \mathcal{C} \), see Fig. 6. We say that the considered lines or circles support the hyperbolic line, simply line for short,

![Figure 6](image)

*Figure 6* The lines \( p \) and \( q \) are parallel to the line \( \ell \), with points at infinity \( P \) and \( Q \), on the border of the unit disc. The \( h \)-line \( m \) is non-secant with \( \ell \): it can be seen that there are infinitely many such lines.

when there is no ambiguity, \( h \)-line when it is needed to avoid it. Fig. 6 illustrates the notion of parallel and non-secant lines in this setting. The points of \( \mathcal{C} \) which do not belong to the hyperbolic plane are called points at infinity. If
$P$ is such a point, and if a the circle which supports a line $\ell$ of the model passes through $P$, we say that $\ell$ passes through $P$ and that $P$ is a point at infinity of $\ell$. Points at infinity play an important role in the hyperbolic plane. They define a kind of direction.

The angle between two $h$-lines are defined as the Euclidean angle between the tangents to their support. The reason for choosing the Poincaré’s model is that hyperbolic angles between $h$-lines are, in a natural way, the Euclidean angle between the corresponding supports. In particular, orthogonal circles support perpendicular $h$-lines.

An important difference between Euclidean and hyperbolic geometries is the notion of similarity. In Euclidean spaces, a figure can exist in different shapes. This is so familiar that most probably, the reader does need a precise definition to understand about what it is speaking. As an example, a square exists in infinitely many sizes: just change the length of the side. This is not at all the case in hyperbolic spaces. As an example, there is no square, there. But instead, there is regular convex pentagon with right angles between consecutive sides. Now, for this pentagon, there is a unique length for the side. This leads us to say that in the hyperbolic plane, a shape has a definite size. This does not mean, however, that there is not at all similarity in the hyperbolic plane. Something can be said about that but this is outside the scope of this paper.

### 3.2 The family of tilings \( \{p, 3\} \)

Remember that in the Euclidean plane and up to similarities, there are only three kinds of tilings based on the recursive replication of a regular polygon by reflection in its sides and of the images in their sides. In the hyperbolic plane, where the notion of similarity is not very meaningful, there are infinitely many such tilings. In this paper, we consider the family of tilings \( \{p, 3\}, p \geq 7 \). They are defined on the basis of a regular convex polygon with an angle of \( \frac{2\pi}{3} \) between consecutive angles. It is known that such a polygon exists in the hyperbolic plane starting from $p = 7$. It is also known that for all these values of $p$, it is possible to tile the plane by replicating the polygon in its sides and, recursively, by replicating the images in their sides. This provides a tiling: there is no overlap and no hole. As a consequence of the angle of the polygon, there are exactly three of them around any vertex. The smallest polygon of the family is the heptagon, defined with $p = 3$. It gives rise to the heptagrid \( \{7, 3\} \). Fig. 7 and 8 give an illustrative representation of this tiling. We refer the reader to [8] and to [7] for more details and references.

The left-hand side of Fig. 7 illustrates the heptagrid. But, besides the occurrence of a lot of symmetries, nothing can be grasped on the structure of the tiling from this mere picture. The right-hand side picture of Fig. 7 illustrates the main tool to make the structure visible. There, we can see two lines which we call mid-point lines as they join mid-points of edges of consecutive heptagons belonging to the tiling. On the figure, a half of each line is drawn with a thicker stroke. It is a ray issued from the common point of these lines: here, a mid-point of an edge of the central heptagon of the figure. We shall say a ray of mid-points. These two rays define an angle, and the set of tiles whose all mid-points of the edges fall inside the angle is called a sector.

Fig. 7 and 8 sketchily remember that the tiling is spanned by a generating
tree. In fact, as can be noticed on both the right-hand side of Fig. 7 and the left-hand side of Fig. 8, the set of tiles constituting a sector is spanned by a Fibonacci tree, see [8, 7] for references. The name of the tree comes from the fact that the number of nodes on a given level $n$ is $f_{2n+1}$, where $\{f_n\}$ denotes the Fibonacci sequence with $f_1 = 1, f_2 = 2$.

Now, as indicated in Fig. 8, seven sectors around a central tile allow us to exactly cover the hyperbolic plane with the heptagrid which is the tessellation obtained from the regular heptagon described above and easily seen on the figures.

![Figure 7](image)

**Figure 7** On the left: the tiling; on the right: the delimitation of the sectors which are spanned by a tree. Note the rays of mid-points. They are issued from the same point: a mid-point of an edge of the central cell of the figure.

In the left-hand side picture of Fig. 8 we represent the sectors in terms of tiles. The tiles are in bijection with the tree which is represented on the right-hand side part of the figure. This allows us to define the coordinates in a sector of the heptagrid, see [8]. We number the nodes of the tree, starting from the root and going on, level by level and, on each level, from the left to the right. Then, we represent each number in the basis defined by the quoted Fibonacci sequence, taking the maximal representation, see [4, 8].

![Figure 8](image)

**Figure 8** On the left: seven sectors around a central tile; on the right: the representations of the numbers attached to the nodes of the Fibonacci tree.
One of the reasons to use this system of coordinates is that from any cell, we can find out the coordinates of its neighbours in linear time with respect to the coordinate of the cell. Also in linear time from the coordinate of the cell, we can compute the path which goes from the central cell to the cell. These properties are established in [6, 8] and they rely on a particular property of the coordinates in the tree which allow to compute the coordinate of the father of a node in constant time from the coordinate of the node. In the paper, the coordinate of a cell is of the form $\nu(\sigma)$ where $\sigma$ is the number of the sector where the cell is and $\nu$ is its number in the Fibonacci tree which spans the sector.

What was said for the heptagrid can be extended to any tiling $\{p, 3\}$ with $p \geq 7$, see [8]. We give here the main properties which can be used in Section 5.

Consider $p$ as fixed with $p \geq 7$. Around a central cell, we can manage $p$ sectors defined by mid-point rays exactly as this was the case in the heptagrid, illustrated by Fig. 8. Here too, there is a tree $T$ in bijection with the tiles which have all their mid-points in the angle defined by the rays. This tree generalizes the one given in the right-hand side of Fig. 8. Let $u_n$ be the number of tiles which are on the level $n$ of the tree, with $u_0 = 1$ as level 0 is that of the root of the tree. We have $p-4$ sons for the root which defines $u_1 = p-4$ and further, $u_{n+2} = (p-2)u_{n+1} - u_n$. From this sequence, we can define a representation of the integers, see [1, 3, 8] as follows: each positive number $n$ can be written as $n = \sum_{i=1}^{k} a_iu_i$ with $a_i \in [0..p-3]$. In general, this representation is not unique, but it can be unique by requiring it to be the maximal one in size. We can make this restriction more clear as follows. Say that $T$ has two kind of nodes: white sons have $p-4$ sons and black sons have $p-5$ of them. The root is a white node. Now, there is a precise rule for the position of black and white nodes. Each node has exactly one black node and, when running through the sons of a node from left to right, the black node is the penultimate node. Number the nodes of the tree level by level, 1 being given to the root and then, increasingly on each level from left to right. Say that the maximal representation of $n$ as above defined is the coordinate of the node numbered with $n$. Then, we have that the black nodes are exactly those whose coordinate end in 0. Similarly, the coordinate of the black son of a node is obtained by appending one 0 to the coordinate of the node. This property allows us to define the path from a node to the root in linear time from the size of the coordinate of the node.

As we said, the nodes of $T$ are in bijection with the tiles of a sector. The bijection allows us to consider as coordinate of a tile the coordinate of the node associated to the tile by the bijection. Say that two tiles are neighbour of each other if and only if they share a common side. The main problem we have to solve is to compute the coordinates of the neighbour of a tile $\tau$ from the coordinate of $\tau$. We refer the reader to [8] for the result and the corresponding explanation. We display the results in Table 1 in Section 5.

Now, as the system of coordinates is fixed, we can turn to the application to the implementation of cellular automata on the heptagrid.
4 Cellular automata on the grids \(\{p, 3\}\)

A cellular automaton on a grid \(\{p, 3\}\) is defined by a local transition function which can be put in form of a table. Each row of the table defines a rule and the table has \(p+2\) columns numbered from 0 to \(p+1\), each entry of the table containing a state of the automaton. On each row, column 0 contains the state of the cell to which the rule applies. The rule applies because columns 1 to \(p\) contain the states of the neighbours of the cell defined in the following way. For the central cell, its neighbour 1 is fixed once and for all. For another cell, its neighbour 1 is its father. In all cases, the other neighbours are increasingly numbered from 2 to \(p\) while counter-clockwise turning around the cell starting from side 1. The representation mentioned in Section 3 allows to find the coordinates of the neighbours from that of the coordinate of the cell in linear time. As promised in that section, Table 1 indicates how to compute the coordinates of a cell \(\nu\) from the coordinate of \(\nu\). The list of states on a row, from column 0 to \(p\) is called the context of a rule. It is required that two different rules have different contexts. We say that the cellular automaton is deterministic. As there is a single row to which a rule can be applied to a given cell, the state of column \(p+1\) defines the new state of the cell. The local transition function is the function which transforms the state of a cell into its new one, also depending on the states of the neighbours as just mentioned. In Table 1 we introduce a function \(\chi(\nu)\) which indicates the lowest digit of the coordinate of \(\nu\). It also defines the notion of common ancestor of two cells.

An important case in the study of cellular automata is what are called rotation invariant cellular automata. To define this notion, we consider the following transformation on the rules. Say that the context of a rule is the rotated image of another one if and only if both contexts have the same state in column 0 and if one context is obtained from the other by a circular permutation on the contents of columns 1 to \(p\). Now, a cellular automaton is rotation invariant if and only if its table of transition \(T\) possesses the following properties:

- for each row \(\rho\) of \(T\), \(T\) also contains \(p-1\) rules exactly whose contexts are the rotated image of that of \(\rho\) and whose new state is that of \(\rho\);
- if \(\rho_1\) and \(\rho_2\) are two rules of \(T\) whose contexts are the rotated image of each other, then their column \(p+1\) contains the same state.

In the rest of the paper, sometimes we shall have to write the rules of the automaton for a precise situation. The rules can be written according to the following format:

\[\eta_0, \eta_1, \ldots, \eta_p \rightarrow \eta_0^1,\]

where \(\eta_0\) is the state of the cell, \(\eta_i\) the state of its neighbour \(i\) and \(\eta_0^1\) is its new state.

However, in tables and also in order to have a more compact notation, a rule will be written as a word. The above is rewritten as the following word:

\[\eta_0\eta_1\ldots\eta_p\eta_0^1,\]

using the same notations.

The name of rotation invariance comes from the fact that a rotation around a tile \(T\) leaving the tiling globally invariant is characterized by a circular permutation on the neighbours of \(T\) defined as above.

Note that the universal cellular automata devised in [2, 17, 18] are rotation invariant while the one of [16] is not. For the question of rotation invariance for
cellular automata on the heptagrid, we refer the reader to [9].

### Table 1 Computation of the coordinates of the neighbours of a node.

| $\chi(\nu)$ | $\chi(f+1)$ | neighbours |
|------------|-------------|-------------|
| 0          | $f, c_1+1, c_1+2, \ldots, c_1+p-4, c, c+1, c+2$ |             |
| 1          | $f, c_1+2, \ldots, c_1+p-3, c, c+1, c+2$ |             |
| 3,...,$p-3$ | $f, c_1+2, \ldots, c_1+p-3, c, c+1, f+1$ |             |
| 2          | $st(g) = 0: f, c_1+2, \ldots, c_1+p-3, c, c+1, c+2$ |             |
|            | $st(g) = 1: f, c_1+2, \ldots, c_1+p-3, c, c+1, f+1^\circ$ |             |
| 2          | $f, f-1, c_1+2, \ldots, c_1+p-3, c, c+1$ |             |
| 4,...,$p-3$ | $f, c_1+1, c_1+2, \ldots, c_1+p-3, c, c+1$ |             |
| 3          | $st(g_1) = 0: f, f-1, c_1+2, \ldots, c_1+p-3, c, c+1$ |             |
|            | $st(g_1) = 1: f, c_1+1, c_1+2, \ldots, c_1+p-3, c, c+1^\circ$ |             |
| 3,...,$p-3^*$ | $f, c_1+1, c_1+2, \ldots, c_1+p-3, c, c+1$ |             |

**Note.** Here, $\nu$ is the node, $f$ its father, $c$ its black son; $c_1$ is the black son of $\nu-1$. When $\chi(\nu) = 1$ or $\chi(nu) = 2$, define $h$ as the common ancestor of $\nu$ and $\nu+1$, when $\chi(\nu) = 1$, of $\nu-1$ and $\nu$, when $\chi(nu) = 2$; we denote by $g$ and $g_1$ the sons of $h$ on respectively the left-hand side, right-hand side branch issued from $h$.

$^*$ When $f$ is black, $\chi(\nu)$ ranges only in $[3..p-4]$.

$^\circ$ When $\nu$ is on the border, $c_1+1$ is to be considered on the left-hand side tree and $f+1$ is to be considered on the right-hand side tree.

Now, we can turn to the simulation of the railway circuit by a cellular automaton.

### 5 Implementation in the hyperbolic plane

In this section, Subsection 5.1 will first present the general features of the implementation of a railway circuit which are shared by all the simulations of the previous papers, see [2, 17, 19, 12, 18, 20], as well as the implementation of this paper. Then, in Subsection 5.2, we describe the main lines of the present implementation. Then, we develop the presentation thoroughly in Subsections 5.3, 5.4, and 5.5.

#### 5.1 The implementation of the railway circuit

In [2], the various elements of the circuit mentioned in [5] are implemented. In fact, the paper does not give an exact description of the implementation: it only gives the guidelines, but with enough details, so that an exact implementation is useless. In this paper, we take the same model, and we repeat the main lines of implementation mentioned in [2, 13]. So that we refer the reader to these papers for more precise details. Just to help him/her to have a better of view of the overall configuration, we refer the reader to Fig. 9. The figure provides
a simplified illustration of the implementation of the example given by Fig. 5 which takes place in the heptagrid.

If the reader carefully looks at the figure, he/she will notice that the tracks mostly follow branches of a Fibonacci tree and sets of nodes which are on the same level of the Fibonacci tree. In this implementation, we have to pay a very precise attention to this situation. We shall tune it a bit with the help of an intermediary structure. As it will be used for the initial configuration only, there is no need to translate this structure into the states of the automaton.

Figure 9 The implementation, on the heptagrid, of the example of Fig. 5. In sector 1, also overlapping onto sector 2, the sequencing of the instructions of the program of the register machine. In sector 3, we can see the first register and, in sector 5, the second one. For simplicity, the figure represents two registers only.

Note the instructions which arrive to the control of the register through tracks in the shape of an arc of circle. Also note the return from the controller of the register when decrementing a register fails, because its content was zero.

The intermediary structure which we shall use is illustrated in Fig. 10 in the case of the heptagrid. It consists in defining horizontal and vertical. Horizontals and verticals are a familiar way to define a rectangular grid. However, we can easily understand that we can distort the structure provided that the cycles defined by the rectangles are preserved. We can also alter the structure provided that we keep enough room for infinitely many zones of arbitrary sizes. In the hyperbolic plane, we have no rectangle but, in the tilings \( p,3 \) of the hyperbolic plane, we have a generic way to define a kind of horizontal: it is a set of tile whose complement in the plane has two components exactly and two of our horizontals either coincide or do not meet. Also, we can guarantee the existence of only half verticals: they are sets of tiles which follow a half-line and they cross infinitely many horizontals. Fig. 10 illustrates this notion. This is enough for our purpose: the intersections between verticals and horizontals
allow us to define quadrangle as large as needed to enclose the structures we have to define to implement the circuits described in Section 2.

**Figure 10** The definition of horizontals and verticals:

On the left-hand side, the coloration which allows to define the horizontals, which are drawn on the right-hand side. The verticals are represented by rays of yellow tiles. On the right-hand side picture, note that the common side of adjacent yellow tiles is not drawn: rays of yellow tiles appear as solid blocks of tiles.

Fig. 10 is also based on the tree represented in Fig. 8. Now, the colours of the picture have a meaning. Each tile defines the colours of its sons according to the following rules:

\[
G \rightarrow YBG, \quad Y \rightarrow YBG, \quad O \rightarrow YBO, \quad B \rightarrow BO,
\]

where \(G, Y, O\) and \(B\) have obvious meaning.

The horizontals are defined by the levels of the tree. Now, a level is delimited by the leftmost and the rightmost branches of the tree. In order to constitute an infinite horizontal, the levels are glued by considering an infinite sequence of growing trees whose roots are set along an infinite vertical. The union of these trees is the whole hyperbolic plane. Then the level \(n\) of a tree is the level \(n+1\) of the smallest tree of the sequence which contains it, see [8]. Now, each branch of any tree can be a vertical. We restrict the choice to the branches issued from a yellow tile and which consists of yellow tiles only, see Fig. 10.

It is important to notice that the fact that our verticals are rays only does not prevent them from constituting a grid with the horizontals we defined. We have the fact that in between two verticals starting from a horizontal \(\iota\), new verticals appear as we go down from a horizontal to the next one, starting from \(\iota\). We may ignore these new verticals if we do not need them. And so, these verticals with the piece of \(\iota\) and a piece of another deeper level at which we decide to stop constitute a figure which we may call a **quadrangle**. These quadrangles allow us to implement the pieces of circuitry described in Section 2.

For the registers, it is enough to display such quadrangles in such a way that the quadrangles have a side along the same yellow branch of a tree. This is enough to see that we can consider the setting of Fig. 9 as enough for our purpose.

This situation of the heptagrid can be transported to any grid \(\{p, 3\}\) with
$p \geq 7$. The same definition of the mid-point lines allows us to transport the constructions. The difference with the heptagrid is that in $\{p,3\}$, the larger $p$, the larger the number of sons for a node and so, the higher the exponential rate of the growth of a disc around a tile.

Later on, we shall illustrate our construction in the tiling $\{13,3\}$, but we shall not represent it in Poincaré’s model as we can see almost nothing in such a representation. Instead, we shall use a symbolic representation with circles: a circle will represent a 13-gon and a common side of two 13-gons will be defined by a tangential contact between the circles representing the 13-gons. A common vertex to three 13-gons will appear as a kind of triangle between the tangency points of three circles which are pairwise tangent. As an example, Fig. 11 represents a 13-gons with its neighbours.

![Figure 11](image)

**Figure 11** Representation of a cell in the tiling $\{13,3\}$ with its neighbours using circles. Note the role of the tangency points.

### 5.2 The scenario of the present implementation

The present scenario is based on three ideas. One appeared in [18]. The previous implementations represented the tracks as a linear structure on which the locomotive appeared, the structure being defined by a colour, different from that of the background. In [18], we introduced a new implementation of the tracks. This time, the track does not differ from the background: it is simply delimited by regularly dispatched milestones on both sides of the track. In this new representation, the locomotive was represented by a block of two contiguous cells, with two colours, different from that of the background. In [20], a new idea allowed us to establish the existence of a spatial two-state universal cellular automaton. With two states, it is no more possible to use two colours for the locomotive. Moreover, with a single cell, it is impossible to distinguish the front from the rear on the locomotive itself. This is why the locomotive was there reduced to a single cell, and it was now called a particle. The counterpart was to change the implementation of the tracks. It was defined by the fact that a track is now one-way. Accordingly, in the cases when the tracks have to be crossed by the locomotive in both directions, the corresponding tracks are implemented as two one-way track: one in one direction, the other, in the opposite direction. This raises a change in the switches which are illustrated by Fig. 12.
Indeed, as the flip-flop switch must be crossed actively only, it is implemented by one-way tracks only, see the red pattern of the memory switch in Fig. 12. Due to its definition, the fixed switch has a passive one-way switch, see Fig. 12 and an active one-way track with no switch at all.

![fixed switch](image1)

![memory switch](image2)

**Figure 12** The new switches. Note that the red part of the memory switch also implements the flip-flop switch.

On the contrary, the memory switch is the superposition of two switches: an active and a passive one. On one hand, the active switch is passive: it is in fact a **programmable** fixed switch. The selected track depends from the last passive passage of the particle. When it is fixed, the active switch is also fixed. On the other hand, the passive switch is in fact active: it may react to the passage of the particle. If the particle goes through the selected track, nothing happens. Now, if it arrives through the non-selected track, then the switch changes the selected track to a non-selected one and conversely. But the switch also acts actively: it triggers a signal sent to the passive part in order to change the selection also there.

Thanks to the bridges in the hyperbolic 3D space, this new idea was enough to go down to two states for a universal cellular automaton.

Now, for the plane, it is needed to find a new idea as crossings cannot be avoided. And so, for this implementation, we introduce a new pattern which we call a **round-about**: it incorporates a motorway pattern to the railway circuits. The pattern deals with the crossing of two one-way tracks, see Fig. 13.

![A](image3)

![B](image4)

**Figure 13** Left-hand side: scheme of the simple round-about. Right-hand side: its symbolization for later use.

Note that whether the particle arrives from A or from B to the round-about,
it has to exit at the second track meeting the round-about. And so we have to count up to 2. But we have two states only. The solution is the following: when the particle arrives at the round about, an additional particle is appended to it. This pair of consecutive particles goes on along the round-about. When it meets the first way going away or arriving to the round-about, the first particle vanishes and the second one follows the round about. This time, when a single particle arrives to a track which goes out from the round-about, the particle follows this track and so, it leaves the round-about on the right way.

It is now not very difficult to define the crossing of two two-way tracks, which is illustrated in Fig. 14.

![Figure 14](image.png)

**Figure 14** Left-hand side: the round-about for the crossing of two two-way tracks. Four one-way track round-abouts allow us to solve the problem.

Right-hand side: using the symbolic representation of the one-way round-about.

Note the symbolic representation used for the crossing of one-way tracks. We shall use it in our further representations of the switches as the memory switch will need them.

In the next subsection we thoroughly look at the configurations needed to obtain the expected behaviour of the particle. We first deal with the tracks in Subsection 5.3, then with the crossings in Subsection 5.4 and then with the switches in Subsection 5.5. This study will allow us to construct the rules, a process of which an account is given in Section 6.

### 5.3 The implementation of the tracks

From now on, we call tiles **cells**. Most cells are in a quiescent state which we also call the **blank**. In the following figures of the paper, it is represented by a light colour, not necessarily the same in order to facilitate the understanding of the configurations. In our setting we have another colour which we call black but which will have several dark or bright colours in the figures. Remember that cells are said to be neighbours if and only if they share a side.

As already mentioned, the track is here implemented with milestones as in [18, 20]. However, as the tracks must be one-way, and as the locomotive is a particle, represented by a single cell, the direction must be defined by the...
context visited by the particle: it cannot be deduced from the particle itself. We have met this problem in [20]. It was solved in a simple way which cannot be projected on the plane as, in most cases, the projection would map both ways on the single track.

In Fig. 15 we illustrate the solution in the case of the tiling \{13, 3\}. For most of the tiles, we have the configurations represented by the upper row of the figure. On one side we have the pattern for one direction while the other side illustrates the pattern for the other direction. A track consists in a sequence of such cells which in some sense follow a path of the tiling from one cell to another, see Fig. 16.

\[\text{Figure 15} \quad \text{Above: the most common cells of the tracks. To left: in one direction; to right: in the opposite direction.}\]

\[\text{Below: a cell which has necessarily two 'supporting' milestones, see Fig. 16. To left: in one direction; to right: in the opposite direction.}\]

\[\text{In all these patterns, I indicates the neighbour of the cell through which the particle enters the cell: this neighbour plays the role of \textit{input}. Similarly, O indicates the neighbour through which the particle leaves the cell: this other neighbour plays the role of \textit{output}.}\]

To go from a tile \(P\) to the tile \(Q\), we first take a shortest path \(\pi\) in the tiling going from \(P\) to \(Q\). If we consider the neighbours of the cells of the path which do not belong to the path, we can share these cells into two cells: one of them, say \(\tau_1\), is on one side of the path from \(P\) to \(Q\) while the cells of the other set \(\tau_2\) are on the other side of the path, see Fig. 16. Now, all cells of \(\pi\) are milestones. They correspond to the black cell which is in between the cells \(I\) and \(O\) in the first row of Fig. 15.

In Fig. 16 say that the cells of \(\tau_1\) go from \(P\) to \(Q\). We can see that the patterns of cells 1, 2 and 3 which belong to \(\tau_1\) correspond to the pattern defined by the left-hand side picture in the first row of Fig. 15. We can also see that if
we want that the particle goes from $Q$ to $P$ along the track $\tau_2$, we have to use
the pattern which is on the right-hand side of the first row of Fig. 15. This can
be seen by the small cells below cell 4 in Fig. 16.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure16.png}
\caption{Above A, B and C, the cells of the track $\tau_2$ going from $P$ to $Q$. The
additional patterns are indicated for cells 1, 2 and 3 of the track.
Below A, B and C, we have the track $\tau_1$ from $Q$ to $P$. The additional patterns are
indicate for cell 4 of $\tau_2$.}
\end{figure}

Now, in Fig. 16, we can see that a cell of $\tau_1$ has two consecutive milestones
which are not compatible with the already considered patterns. For these particular cells, we use the other patterns defined by the lower row of Fig. 15.

Accordingly, Fig. 16 illustrates how to define a general algorithm to define
a two-way track going from a given cell of the tiling $\{p, 3\}$ to another one. However, when a one-way track only is needed, we have some flexibility to tune
the path using the same method as is illustrated by Fig. 17.

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{figure17.png}
\caption{Here, a one way track going from $P$ to $Q$. Note that it crosses the initial
shortest path $\pi$ joining $P$ to $Q$. One cell of $\pi$ is used by the track and its neighbours
on $\pi$ are milestones.}
\end{figure}

As Fig. 17 suggests, for a one way track, it is possible to use one or more
cells of the shortest path to go from one given tile to another given one.

The rules attached to the corresponding motions of the particle are given in
Section 6.
5.4 The crossings

Now, we are in the position to study the crossing of two tracks. Thanks to what we have seen in Subsection 5.3, especially with Fig. 14, it is enough to look at the crossing of two one-way tracks. Following Fig. 13 and the principle defined for the tracks in Subsection 5.3, we may consider that the ring which materializes the round-about consists of cells which are around a single cell which call the core.

![Figure 18](image18.png)

**Figure 18** The general view of a round-about. We have four pairs of pink and pale orange cells, each one marking the branching of a way with the round-about.

![Figure 19](image19.png)

**Figure 19** Zoom on a branching of the round-about. The cell C is concerned with the arrival of a particle from a way abutting the round-about at the branching. The cell D is concerned by the exit or continuation of the particle arriving at a branching from the round-about.

We can see an illustration of the configuration on Fig. 18. The cells where the way meet with the round-about is called a branching. The cells in pink and pale orange colours mark the branchings. The figure illustrates the configuration as it is when the particle is away from it. In Fig. 19 we can see a zoom which focuses our attention on a branching and its particular surrounding. On the
zoomed image, the pink cell is called $C$ and the pale orange one is called $B$.

We shall look on three situations regarding the zoomed image:
- a particle arrives from a way which abuts the round-about at $C$
  through the cell $E$, the yellow neighbour of $C$;
- two contiguous particles arrive at $A$;
- a single particle arrive at $A$.

The second situation is a consequence of the first one: when the particle
arrives at $C$, $C$ appends a new particle to the visiting one. Consequently, at the
next branching, we have two consecutive particles: this is the second situation.
At this stage, the second particle is cancelled so that at the next branching,
a single particle arrives at $A$: this is the third situation. We can notice that
while turning around the round-about, starting from any arriving, the second
branching is the expected exit.

Let us look at what happens at each branching in these situations.

The arrival of the particle to the round-about is illustrated by Fig. 20 where
we have a zoom at the concerned branching.

![Figure 20](image-url)  

**Figure 20** Zoom on a branching when the particle arrives at the round-about. Note the creation of a second particle which is contiguous to the initial one on the tracks.

As can be seen in the figure, the particle crosses $E$ through which it arrives
at $C$ on the round-about. Number the neighbours of $C$ from 1 to 13, neighbour 1
being the milestone which is common to all the cells of the round-about and the
numbers being increasing while counter-clockwise turning around $C$. We shall
do the same fro $B$, so that we shall denote the neighbours of these cells by $C.i$
and $B.i$ respectively. The neighbour $C.2$ is $B$ and $E$ is $C.4$. Now, the occurrence
of the particle at $C$ triggers the change of $C.5$ which becomes white. When the
particle is at $C$, the cell can see that $C.4$ and $C.5$ are both white: it is the
signal for the creation of the additional particle. This means that while the first
particle is in $D$, $C$ is still black: $C$ returns to white the next time only. Later on,
both particles travel together along the round-about in the counter-clockwise
direction.

Accordingly, both particles arrive at the next branching of the round-about.
What now happens is illustrated by Fig. 21. Due to the number of surroundings
we have to define for the cells which play a particular role, we have to decide that
$F$ is an ordinary cell of the tracks. Note that $F$ is $B.10$ and that $B.11$, which
is a milestone for $F$, can see both $F$ and $B$. So that $B.11$ can detect whether there are one or two particles arriving at the branching from the round-about. We decide that $B.11$ does not change if there is a single particle: as $F$ is an ordinary cell of the tracks, the motion will go on and the particle will leave the round-about. If $B.11$ detects two particles, then it changes from black to white. We say that $B.11$ flashes. This flash destroys the two particles at the next time. But at the next time, $B.11$ returns to black and $B.12$ flashes as it has witnessed the flash of $B.11$. The flash of $B.12$ triggers a new particle in $C$.

In Fig. 21, we can see that $B.11$ flashes again after the flash of $B.12$: this is required by the rules which were devised globally, so that the flash is induced by a rule created for another context which also applies to this one by the rotation

**Figure 21** Zoom on a branching when two particle arrive at a branching from the round-about. Note that both particles are cancelled and that a new particle is created in $C$ which goes on along the round-about.

**Figure 22** Zoom on a branching when the particle arrives alone at the round-about. Note that it continues its motion on the tracks leaving the round-about.
effect. Once a particle is present in \( C \), it goes on its way along the round-about, until it meets the next branching. As already mentioned, in this case, when the single particle arrives at \( F \), \( B.11 \) detects that the particle is alone and so, it remains black. Accordingly, the particle goes on its way on the tracks leaving the round-about: this is exactly what was expected. Such a motion is illustrated by Fig. 22.

5.5 The switches

We can now turn to the description of the configurations needed by the switches. As in the case of the crossing, we first present the configuration when the particle is at large which we call the \textit{idle configuration}, see Figure 23. Later, we present several pictures illustrating the motion of the particle through the configuration with all the changes it induces with respect to the idle configuration. We shall successively look at the fixed switch, the flip-flop and the memory switch.

![Figure 23](image)

\textbf{Figure 23} The idle configurations of the switches: above, the fixed one and the flip-flop; below, the memory switch. For the flip-flop and for the memory switch, only the switch where the left-hand side is selected is illustrated.

The fixed switch and the flip-flop are simple switches: they both involve a single one-way switch. It is a passive switch in the case of the fixed switch, it
is an active switch in the case of the flip-flop. In the memory switch we have a more complex structure: we have two one-way switches and one of them has an influence on the other.

The one-way switches have the same structure: three paths arrive to a centre. In the case of the fixed switch, there is no consequence after the passive crossing. In the case of the flip-flop, the active crossing only is implemented but the crossing induces a change.

As in [18], the centre of the switch is a blank cell: it belongs to the tracks. But its milestones define a pattern which is very different from that of the cells of the tracks. Figure 23 show the patterns used to distinguish each kind of switch. In the fixed switch, the black cells around the centre of the switch are all milestones and the cells named $A$, $B$ and $C$ which represent the abutting tracks are ordinary cells of the tracks. In the flip-flop and in the memory switch we have new cells which we call sensors and controllers. We shall study them in Subsubsections 5.5.2 and 5.5.3.

In the figures illustrating the switches, we represent only one case: the case when the selected track is the left-hand side one.

Now, we shall consider that the particle arrives at a switch. Each case is examined in an appropriate Sub-subsection.

### 5.5.1 Fixed switches

The crossing of a fixed switch is illustrated by Fig. 25. It is a passive crossing where the selected track is not mentioned as in such a crossing, the side from where the particle comes does not matter. The matter is only for the selected track in the active crossing of the switch. Now, Fig. 12 suggests that we might choose the active track in such a way that the active tracks do not cross passive ones. This is not true. If we fix an order for the directions in two-way tracks, it may happen that the active tracks has to cross passive ones. This is not difficult to implement as in this case, the crossing involves a one-way crossing as illustrated by Fig. 24.

![Figure 24](image)

**Figure 24.** The implementation of a fixed switch for which the active way has to cross a passive one.

We note that in Fig. 24 the centre of the switch must stand at a large enough distance from the crossing: 6 cells from the centre of the switch to the centre of the round-about is a secure enough distance.

From Fig. 25 we note that all black cells around the centre of the fixed switch can be simple milestones. As we shall see in Section 6, the motion rules of the tracks will be enough to ensure that the crossing can be performed without additional marking: that of the central cell of the switch is enough. This will spare us a few patterns which will be useful to distinguish the sensors and
controllers of the flip-flop from those of the memory switch, remembering that in the latter switch we have two one-way ones which differ between each other.

![Diagrams showing the passive crossing of a fixed switch by the particle.](image1)

**Figure 25** *The passive crossing of a fixed switch by the particle.*

### 5.5.2 Flip-flop switches

We have seen the idle configuration of a flip-flop in Fig. 23. It also appears as the first picture in Fig. 26 which illustrates the motion of the particle through the flip-flop. Here two, we denote by $A$, $B$ and $C$ the cells of the tracks in contact with the centre of the switch. Here, it is important to specify the role of each cell. The cell $A$ is the cell from which the particle arrives at the switch. It leaves the switch either through the cell $B$ or through the cell $C$, depending on which is the selected track.

![Diagrams showing the active crossing of a flip-flop by the particle.](image2)

**Figure 26** *The active crossing of a flip-flop by the particle. Here, the selected track is the left-hand side one.*

In the configurations of Fig. 23 and Fig. 26, we mark the non-selected track instead of the selected one. The difference between $B$ and $C$ is that there is an additional black neighbour in the cell of the non-selected track: the lack of this mark indicates that the particle must enter this cell; the presence to this mark forbids the particle to enter the cell. This can be seen in the mentioned
figures: the mark is a small cell neighbouring the cell $D$. If $D.1$ is the central cell, the cells possibly supporting the marks are $D.3$ and $D.12$. This means that one of these two cells is black while the other is white. As $C$ is $D.2$ and $B$ is $D.13$, $D$ can see both $D.3$ and $C$ as well as both $D.12$ and $B$. The particle enters the side where the mark is missing. This means that $D$ is in position to detect the presence of the mark in the appropriate side. When this is the case, $D$ flashes and, at the next time, $D.3$ and $D.12$ both change their state. This automatically changes the side of the selected track as required. The last configuration of Fig. 26 is the idle one where the selected track is the righthand side one.

5.5.3 Memory switches

The idle configuration of a memory switch is very different from the already studied configurations of the fixed and the flip-flop switches as can already be noticed from Fig. 23. As this switch involves both a passive one-way switch and an active one-way switch, it might seem possible to simply take copies of the previous switches to perform the needed action. This is not possible for several reasons. First, the passive switch has to send a signal to the active one, which is a difference with the fixed switch. And so, the cell $Z$ of the passive memory switch cannot be the cell $D$ of the fixed switch. Moreover, the active switch of the memory switch is in fact passive: the selected track is fixed as long as there is no passive crossing on the other one-way switch. This means that the active switch should be viewed as a programmable one-way fixed switch. And so, in this case, the cell $D$ in the active one-way switch of the memory switch acts as a controller while $D.3$ and $D.11$ are flexible markers.

![Figure 27](image)

**Figure 27** The active crossing of a memory switch by the particle. Here, the selected track is the left-hand side one.

An active passage of the particle is illustrated by Fig. 27. We can see that nothing is changed in the markers $D.3$ and $D.11$. 

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Figure 28 The passive crossing of a memory switch by the particle through the selected track. Here, the selected track is the left-hand side one.

Figure 29 The passive crossing of a memory switch by the particle through the non-selected track. Here, the selected track is the left-hand side one. Note the change of selection when the particle leaves the cell. Note the delay of the change in the active one-way switch.

For what is the passive one-way switch, it has to detect whether a passage
occurs through the selected or through the non-selected tracks. This can be
detected by the cell \( Z \) which can see at the same time the cells \( X, Y, Z.3 \)
and \( Z.11 \). If \( X \) is black and \( Z.11 \) is white, or if \( Y \) is black and \( Z.3 \) is white, then
it is a passage through the selected track. No change of state occurs in \( Z.3 \) or
\( Z.11 \) and the particle leaves the switch through \( V \). This situation is illustrated
by Fig. 28.

If \( V \) and \( Z.11 \) are both black or if \( U \) and \( Z.3 \) are both black, then \( Z \) must
flash: this triggers the change of state in both \( Z.3 \) and \( Z.11 \) and it also sends
a particle to \( D \) through \( Z.8 \). This second particle travels through two crossings
to \( D.7 \). When \( D.7 \) is black, \( D \) flashes which triggers the change of state in both
\( D.3 \) and \( D.11 \). This is illustrated by Fig. 29. Note that the change of selection
occurs in the figure and also, note the delay of the transmission of the signal
from \( Z.8 \) to \( D.7 \). The actual delay can be made less than the one given in the
figure. The last configuration in Fig. 29 is the idle configuration of memory
switch where the selected track is the right-hand side one.

We can now turn to the rules: in Section 6 we look at the case of the tiling
\( \{13, 3\} \); in Section 7 we uniformly define the rules for the tilings \( \{p, 3\} \) when
\( p \geq 17 \).

6 The rules for a cellular automaton on \( \{13, 3\} \)

It is clear that the solution for \( p = 13 \) can be extended to the case \( \{p, 3\} \) for any
\( p \geq 13 \). However, the solution for \( p \geq 17 \) can be given a more simple expression
which allows us to check its correctness easily: this will be seen in Section 7.

The set of rules of the automaton are displayed as tables in Subsections 6.3
6.4 and 6.5. Here, we indicate how the rules where computed by a computer
program and how the rules are represented and dispatched in the table. After
a precision to Section 4 about the format of the rules, we follow the discussion
of Subsections 5.3, 5.4 and 5.5 in order to establish the rules for all possible
situations.

6.1 The format of the rules and rotation invariance

We remind that the format of the rules is the one introduced in Section 4. First,
we look at the rules in \( \{13, 3\} \) and then in \( \{p, 3\} \).

This means that a rule is presented as a word of the form:

\[ \eta_0 \eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \eta_8 \eta_9 \eta_{10} \eta_{11} \eta_{12} \eta_{13} \eta_{14} \]

We remind that \( \eta_0 \) is the current state of the cell, that \( \eta_i \) is the state of
the neighbour \( i \) and that \( \eta_{13} \) is the new state of the cell after the rule has been
applied to it.

In all this section and in the tables of Subsections 6.3 6.4 and 6.5, the states of the automaton are \( W \) and \( B \). We decide that \( W \) is the quiescent state,
which means that the rule \( WWWW \) is in the table. We shall also write \( W^{13} \) which will be called a condensed format. Similarly we decide that any
neighbourhood with less than three rules does not change the current state.
This induces the following rules:

\[ XW^{13}X, XWB^{12}X, XXB^{11}X, XBBW^{10}X, XBB^{9}X, XBB^{8}X, XBB^{7}X, XBB^{6}X, XBB^{5}X. \]
where $X \in \{B, W\}$. Also, a rule $XYX$, with $Y \in \{B, W\}^{13}$ means that both rules $XYW$ and $BYB$ are in the table. Note that the rules $XBw^kBw^kX$ and $XBw^kBw^kX$ are the same under the rotation invariance condition.

The just described rules are not listed in the tables of Subsections 6.3, 6.4, and 6.5 which contain all the other rules for $\{13, 3\}$ in a condensed notation.

For the general case, we shall denote the rules as $XZ$ where $X, Y \in \{W, B\}$ and $Z \in \{W, B\}^p$. As previously, the rules $XYB^{-1}YX$ and $Z/\pi X$, with $X$ as before and $Z \in \{W, B\}^p$ with $|Z|_s = 2$ are in the table, $|Z|_s$ being the number of occurrences of $B$ in $Z$. We say that $XZ$ is the context of the rule.

The rules for the case $\{13, 3\}$ were defined and checked with the help of a computer program. In particular, the program checked the rotation invariance of the rules. This was performed as described in [18, 14]. The principle is as follows: in a rotation invariant cellular automaton, if a rule $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$ belongs to the table of the automaton, the rule $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$ also belongs to the table for any circular permutation $\pi$ on the numbers from 1 up to 13. In this case, we say that $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$ is a rotated image of the rule $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$. Now, if we consider the contexts $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$ for all circular permutations $\pi$, there is a minimal one with respect to the lexicographic order. The rule $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$ is called the minimal rotated form of $\eta_0 \eta_1 \eta_2 \ldots \eta_{13} \eta_0$ for the $\pi$ which realizes the minimal rotated context. Now, it is clear that two rules are rotated images of each other if and only if they have the same minimal rotated form. From this, we easily deduce an algorithm for checking the coherence of any new rule introduced in an already coherent set of rules. The programming of the algorithm raises no difficulty. The computer program was written in ADA.

The program uploads the initial configuration of the crossings and of the switches from a file and puts the corresponding information into a table 0. In this table, each row represents a cell. The entries of the row indicate the coordinates of the neighbours of the cell as well as the states of the cell and of its neighbours. The program also contains a copy of table 0 with no state in the cells which we call table 1. The set of rules is in a file under an appropriate format, close to the one which was just depicted.

During the construction of the set of rules, the program works as follows.

First, the program reads the file of the rules which, initially contains the rule $W^{13}W$ which says that a cell in the quiescent state whose neighbours are all quiescent remains quiescent. The program scans the cells of the list one after the other. It takes the context $\kappa$ of the considered cell $c$ in table 0. Then, it compares $\kappa$ with the contexts of the rules of the file. If it finds a match, it copies the new state of the rule at the address of $c$ in table 1, under column 0. If it does not find a match, it asks for a new rule which the user writes into the file. To help the user, the program indicates the context of the cell. The user enters the new state only. Then the program resumes its computation: it reads again table 0 from the initial configuration and performs the computation as far as possible. If it can compute the new state of all cells of table 0, it completes table 1 by computing the new states of the neighbours of each cell. When this task is over, the program copies table 1 onto table 0; a new step of the computation of the cellular automaton can be processed. This cycle is repeated until no new rule is required and until the fixed in advance number of steps is reached.
Now, when a new rule is entered by the user on a cell $c$, it may happen that the new rule is in conflict with the previously entered rules. This happens when there is a rule $\eta$ whose context is a rotated form of the context of $c$, but the state suggested by the user is not the new state of the rule. In this case, the program stops with an error message which also displays the rule with which the program have found a mismatch. If the rule constructed on the context of the cell and the state indicated by the user is a rotated form of an already existing rule, it is appended to the set of rules.

When the program can be run without asking a new rule nor indicating any error, we know that the set of rules is computed.

In the display of the tables of Subsections 6.3, 6.4 and 6.5 we assume that the minimal rotated forms of the rules are pairwise distinct. This means that on the thirteen rotated forms which should be present for each rule, we keep only those needed by the tested configurations. We find 207 rules and most of them are rotation independent from the others.

### 6.2 Implementation of the configurations

For each configuration of the crossings and the switches, the number of cells to explore would be rather important and only a very few number of them is supposed to change during the computation. And so, the idea is to consider only the set of cells which are possibly changing. We have a two-dimensional table. The first column defines the set of cells under inspection by the program. This defines these sets as an ordered list $L$ from 1 to $|L|$, the number of elements of $L$. Each row corresponds to the same cell of this set. It contains the cell itself and the thirteen neighbours of the cell. For the cell and each neighbour, the corresponding entry contains two fields: $num$ and $state$. For a cell, considered as neighbour 0 of its row, $num$ is the number of the cell in $[1..|L|]$. For a neighbour, $num = 0$ if the neighbour never changes its state. Otherwise, the considered neighbour is a cell of the list, say whose number is $m$, so that in this case, $num = m$. This table allows us to easily apply the rules to any cell $c$. It is enough to compute the minimal form of the rule and also to compute the minimal context defined by the state of $c$ and those of its neighbours: it is enough to take them in the row associated to $c$. We look in the table after a rule such that the context of its minimal form matches the minimal context of the neighbours of $c$. When the rule is found, the new state of $c$ is found.

Before turning to the various parts of a configuration, we shall divide the rules which we shall define into two classes. First, we consider the conservative rules which are applied to a cell without changing its state. This is the case, in particular, for the cells of an idle configuration. The rules applied to each cell must not change the state. The other rules, which we call active, do change the state of the cells to which they are applied. This is in particular the case of the rules which apply to cells directly connected to the motion of the particle: we call them the motion rules.

We can now turn to the various configurations we have to study.

### 6.3 Rules for the tracks

Looking at Fig. 15, we first define its conservative rules. The milestones are applied the rule $BW^{13}B$ as well as the rule $BBW^{12}B$ and even the rule $BBWB^{10}B$.
for the case when the cell has two 'supporting' milestones. There are four
conservative rules for a cell of the tracks itself, as the number of pictures in
Fig. 15:

\[
\text{WBBWBWBW, WBBWBWBW, WBBWBWBW, WBBWBWBW}
\]

Indeed, the four patterns of Fig. 15 are rotationally different as already
noticed, so that the minimal forms of the rules are also different as we can now
see.

For the white neighbours of the cell and for the white neighbours of the
milestones, the rules are: \text{WBBW}^{13}, \text{WBBW}^{12} and \text{WBBW}^{11}.

Now, taking into account the definitions of the cells, we have the follo\-wing
motion rules, first for one direction and then, the two last lines, for the opposite
direction:

\[
\text{WB}^{3} \text{W}^{7} \text{BBW}, \quad \text{WB}^{4} \text{WB}^{7} \text{W}, \quad \text{WB}^{5} \text{WB}^{6} \text{W}
\]

Taking the same order for the non idle configurations, we can display them
as follows:

![Figure 30](image)

Figure 30 Illustration of the motion rules for the tracks. In the first and the last
pictures of each row, we have the idle configuration. It is assumed that I and O are
both cells of the tracks. Number the neighbours from the first block of two
consecutive black cells while turning around the cell c. Let c.1 be the leftmost cell in
this first block. The input of c is c.5 or c.6 in the first two rows, c.11 or c.13 in
the last two rows. Similarly, the output of c is c.3 in the first two rows, while it is
c.13 or c.3 in the last two rows.

Now, we have to append a few rules for the case when two contiguous par-
ticles travel on the tracks, which happens for instance in the tracks which runs
along a round-about.

$$BB^3W^7BBWB, BB^4W^6BBWB, BB^3WBBW^7B, BB^4WBBW^6B.$$ $BB^4WBB^{w}W, BB^5WBB^wW, BB^4W^7BBW, BB^5W^6BBW.$

In these situations, these cells are black and there is another black cell, either at the entrance of the particle. As mentioned in the caption of Fig. 30, the entrance of a cell $c$ of the tracks is $c.5$ or $c.6$ in one direction, while the output is $c.3$. In the other direction, the entrance is $c.11$ or $c.13$ while the output is $c.13$ or $c.3$ respectively. In the first case, the cell remains black, in the second one, it turns black to white.

It is not difficult to see that the above rules are in a minimal form, assuming that $B < W$. As an easy corollary, they are rotation independent.

The flexibility of the way with which we can define tracks shows us that there is no need of further simulations in order to check the correctness of these rules.

### 6.4 Rules for the crossings

This time, we deduce the rules from the illustrations given in Fig. 23 and Fig. 19. The figures already show that most of the conservative rules are induced by the rules of the form $XZX$ where $|Z| = 13$ and $|Z|_B \leq 2$.

Here, the active rules are those of the tracks, in particular $A$, $B$, $C$, $D$, $E$, $F$, $B.11$, $B.12$ and $C.5$, see Fig. 19 and Subsection 6.4. As $A$ and $D$ are cells of the tracks, the rules applying to them have already been seen in Subsection 6.3. Now, $B$ and $C$ have particular surroundings as they have a particular behaviour with respect to cells of the tracks. Also, $E$ and $F$ are cells of the tracks but at least one of their milestones is not permanently black: it may flash at a definite event. This means that the milestone becomes white at a certain time, turning back to black at the next time. Of course, $B.11$, $B.12$ and $C.5$ have to be studied separately.

In order to find out the rules, we indicate the neighbours of the cell according to what was said above and then, we explain the construction of the rules with the help of an appropriate table. The rows of the table follow the trajectory of the particle: the row below the row associated to time $t$ is associated to time $t+1$. We decide that the first row is always the conservative rule associated to the considered cell. It is considered that the last rule is followed by the conservative rule so that the conservative rule appears only once in the representation of a motion.

#### Table 2 Rules for a simple cell of the tracks

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| 0 | B | B | O | B | I | B | W | W | W | W | W | W | n |

| W | B | B | W | B | W | B | W | W | W | W | W | W |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| W | B | B | W | B | B | W | W | W | W | W | W | W |
| B | B | B | W | B | W | B | W | W | W | W | W | W |
| W | B | B | B | W | B | W | W | W | W | W | W | W |

As an example, Table 2 represents such a table for a simple cell of the tracks.
Accordingly, the table has four rows and we can see that the particle arriving near \( c \) visits \( c.5 \) at time \( t+1 \), visits \( c \) at time \( t+2 \) and then visits \( c.3 \) at time \( t+3 \). It is clear that at times \( t \) and \( t+3 \) the conservative rule applies. We shall follow this convention in all tables. Moreover, in all tables, we indicate in bold characters the particle as well as the flashes of the sensors and the controllers. Also, in order to better handle the rules, we give each of them the number it has in the program.

Our next table is the table devoted to the cell \( B \), see Table 3. The crossing of a cell requires in fact three times as previously. Here, however, the possible presence of a particle adds one more step and the flashes of two cells brings in two additional steps.

**Table 3 Rules for \( B \). The cell is concerned in three different situations which are clearly separated.**

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|---|---|
| A | F | B\(^{11}\) | B\(^{12}\) | C |

---

- **I- a particle arrives to the round-about from outside**

\[
\begin{array}{ccccccccc}
W & B & W & B & W & W & W & B & B & W & B & W & W & 9 \\
W & B & W & B & W & W & W & B & B & W & B & B & W & 13 \\
W & B & W & B & W & W & W & B & B & W & B & B & W & 13 \\
\end{array}
\]

- **II- two particles arrive from the round-about**

\[
\begin{array}{ccccccccc}
W & B & W & B & W & W & W & B & B & W & B & B & W & W & 9 \\
W & B & B & B & B & W & W & W & B & B & W & B & W & B & 10 \\
B & B & B & B & W & W & W & B & B & W & B & B & W & B & 11 \\
B & B & W & B & W & W & W & B & B & B & B & B & W & W & 12 \\
W & B & W & B & W & W & W & B & B & B & B & W & W & W & 19 \\
W & B & W & B & W & W & W & B & B & W & B & W & W & 8 \\
W & B & W & B & W & W & W & B & B & W & W & B & W & B & 14 \\
\end{array}
\]

- **III- a single particle arrives from the round-about**

\[
\begin{array}{ccccccccc}
W & B & W & B & W & W & W & B & B & W & B & B & W & W & 9 \\
W & B & B & B & W & W & W & B & B & B & B & W & B & B & 10 \\
B & B & W & B & W & W & W & B & B & W & B & B & W & W & 15 \\
W & B & W & B & W & W & W & B & B & B & B & W & B & W & 21 \\
\end{array}
\]

Note that in the part -II- of the table, the last time is triggered by the new particle created in \( C \) after the destruction of the two particles which passed through \( F \): as \( B \) and \( C \) are in contact, this rule is also needed.

We can see in Table 3 that the cell \( B \) is mainly concerned when one or two particles arrive to the branching from the round-about. It witnesses \( F \) and also the two sensors \( B.11 \) and \( B.12 \) which discriminate the distinction between
the two cases and rule the situation when a new particle is generated for \( C \).
Rule 12 can see that the conditions for the flashing of \( B.11 \) are present and
rule 19 witnesses the flash. As \( C \) can be seen from \( B \), \( B \) is also a witness of
the creation of the single particle which goes on on the round-about. This is
witnessed by rule 14. The flashes of \( B.11 \) and \( B.12 \) are clearly visible: rule 19
and 14 for \( B.11 \), rule 8 for \( B.12 \).

**Table 4**  
*Rules for \( C \).*  
The cell is concerned in three different situations which are clearly separated.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|---|---|
| B | B | B | E | C | D |   |   |   |   |   |    |    |    |   |

---

- **I-** a particle arrives to the round-about from outside

- **II-** two particles arrive to the round-about

- **III-** a single particle arrives to the round-about

We can see that the cell \( C \) is not very much concerned by the case when a
single particle arrives to the branching from the round-about. The cell is mainly
cconcerned when a particle arrives at the round-about from outside and when
two particles arrive to the branching from the round-about.

In the first case, we can see that two particles are created, rules 16 and 17.
Rule 16 transfers the particle from \( E \) to \( C \). Rule 17 creates a new particle as
it can see the flash of \( C.5 \). Next, rules 12 and 21 control the motion of two
particles together.

When two particles arrive from the round-about, \( C \) is concerned by the cre-
atation of the single particle. We note that rule 22 is applied twice: it simply
witnesses that two particles pass through \( B \). The flash of \( B.12 \) allows rule 18 to
create the single particle. Rules 20 and 21 act as simple motion rules adapted to the context of $C$.

Note that rule 19 appeared already in Table 3. Here, in Table 3 it is a conservative rule while in Table 3 it allows us to witness the flash of $B.11$. In fact, the two different forms of the rule are compatible because they are rotated forms of each other.

Table 5 Rules for $E$. The cell is concerned in two different situations which are clearly separated.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $B^{12}$ |   |   |   |   |   |   |   |   |   |   | 171 |   |   |   |
| $tr$ |   |   |   |   |   |   |   |   |   |   | 189 |   |   |   |
| $C^5$ |   |   |   |   |   |   |   |   |   |   | 171 |   |   |   |
| $C$  |   |   |   |   |   |   |   |   |   |   | 171 |   |   |   |

-I- a particle arrives to the round-about from outside

|   | $W$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $W$ | $B$ | $W$ | $W$ | $94$ |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|----|
|   | $W$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $B$ | $B$ | $W$ | $W$ | $95$ |
|   | $B$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $B$ | $B$ | $B$ | $W$ | $171$ |
|   | $W$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $W$ | $W$ | $W$ | $B$ | $96$ |
|   | $W$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $B$ | $B$ | $W$ | $W$ | $189$ |

-II- two particles arrive from the round-about

|   | $W$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $W$ | $B$ | $W$ | $W$ | $95$ |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|----|
|   | $W$ | $B$ | $W$ | $B$ | $W$ | $W$ | $W$ | $W$ | $B$ | $B$ | $B$ | $B$ | $B$ | $49$ |
|   | $W$ | $B$ | $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $B$ | $B$ | $B$ | $B$ | $96$ |

In Table 6 we can see that rule 91 is the conservative rule attached to the configuration of the neighbours around $E$.

When a particle arrives to the round-about from the tracks, it runs over $E$. In the caption of the table below the numbers of the neighbours, we can see that $E.11$ is a cell of the track arriving to $E$. In fact, $E$ has the configuration of an ordinary cell of the track and rule 94 is simply a rotated form of the first rule in Table 2. Now, rules 95 and 181 convey the particle to $E$ and then to $C$, see also rule 16 in Table 4. This makes $C.5$ flash which is witnessed by rule 199 which triggers the creation of the second particle in $C$. Rule 96 also witnesses the second particle in $C$.

In Table 6 the conservative rule is rule 2. This explains why we said that cell $F$ is almost a cell of the tracks. We said almost because one of its milestones is in fact a sensor: it is the cell $B.11$ which can see both $F$ and $B$ and thus, is able to distinguish the case when two particles arrive at $B$ from the case when a single one arrives at $B$. This is witnessed by rule 4 for the presence of two particles on the round-about, while rule 44 witnesses the flash of $B.11$. The second flash of $B.11$ is witnessed by the rule 47.

The second part of the table is devoted to the case when a single particle arrives at $B$. In this case, the rule 58 witnesses that there is a single particle. We can notice that in this case, as $B.11$ remains black, the cell $F$ behaves as an
ordinary cell of the tracks. Note that the rules are different from those related to rule 91. Indeed, in both cases we have rules for cells of the tracks but one cell is in one direction while the other cell is in the opposite one.

**Table 6** Rules for $F$. The cell is concerned in two different situations which are clearly separated.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $tr$ | B$^{11}$ | B |

- **II-** two particles arrive from the round-about

|   | W | B | B | W | B | W | W | W | W | W | W | W | W | W | W | W | 2 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
|   | W | B | B | W | B | B | W | W | W | W | W | W | W | W | 3 |
|   | B | B | B | W | B | B | W | W | W | W | W | W | W | B | 4 |
|   | B | B | B | W | W | B | W | W | W | W | W | W | W | 44 |
|   | W | B | B | W | B | B | W | W | W | W | W | W | W | 2 |
|   | W | B | B | W | W | W | W | W | W | W | W | W | W | 47 |

- **III-** a single particle arrives from the round-about

|   | W | B | B | W | B | W | W | W | W | W | W | W | W | W | W | W | 2 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
|   | W | B | B | W | B | B | W | W | W | W | W | W | W | B | 3 |
|   | B | B | B | W | B | B | W | W | W | W | W | W | W | W | 58 |
|   | W | B | B | B | B | W | W | W | W | W | W | W | W | 5 |

Table 7 is devoted to the rules needed by the cell $B.11$. This cell shares with $F$ the fact that it is not concerned by the arrival of a particle to the round-about from outside. In fact, as explained in Section 5, the role of this cell is to detect the occurrence of one or two particles to $F$. Rule 24 witnesses the arrival of the particle at $B$ and rule 26 can see that there is a particle at $F$ and a particle at $B$. This is why the rule makes $B.11$ flash. Rule 27 witnesses this flash and rule 167 witnesses the consequence of this flash: both the cancellation of the particles on the tracks which go around $B.11$ and the flash of $B.12$ caused by the previous flash of $B.11$. And the rule makes $B.11$ flash again. The reason is that rule 167 is also used in another context where it makes a black cell flash in the memory switch: see further Table 58.

When a single particle arrives at $B$, $B.11$ can see the particle travel on four cells of the tracks which are its neighbours, in consecutive positions: $B$, $F$ and the cells called $tr_1$ and $tr_2$ in the caption of the table under the numbers of the neighbours. We can see that the particle visits the four cells successively, the one after the other.

It is the point to note that the branching is different, depending on whether the branching concerns tracks which arrive at the round-about or tracks which leave the round-about. If the tracks arrive at the round-about, the arrival at the round-about is $C$ and the tracks pass through $E$. In this case, there is no need to define cells of the tracks after $tr_1$ and $tr_2$ which are needed by the
test for discriminating the number of arriving particles. On the contrary, when
the branching concerns tracks which leave the round-about, the neighbour of \(E\)
need not be a cell of the tracks and the tracks which leave the round-about goes
through \(B, F, tr_1\) and \(tr_2\).

Table 7 Rules for B11. The cell is concerned in two different situations which are
clearly separated.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | \(n\) |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| B | F | \(tr_1\) | \(tr_2\) | B \(^{12}\) |

- **II-** two particle arrive from the round-about

| B | W | W | W | W | B | B | B | B | W | B | B | W | B | B | 23 |
| B | B | W | W | W | B | B | B | B | W | B | B | W | B | B | 24 |
| B | B | B | W | W | B | B | B | B | W | B | B | W | B | B | 26 |
| W | W | B | B | W | B | B | B | W | B | B | W | B | B | 27 |
| B | W | W | W | W | B | B | B | B | W | B | B | W | B | W | 157 |
| W | W | W | W | W | B | B | B | W | B | B | W | B | W | B | 29 |

- **III-** a single particle arrives from the round-about

| B | W | W | W | W | B | B | B | B | W | B | B | W | B | B | 23 |
| B | B | W | W | W | B | B | B | B | W | B | B | W | B | B | 24 |
| B | W | B | W | W | B | B | B | B | W | B | B | W | B | B | 25 |
| B | W | W | B | W | W | B | B | B | W | B | B | W | B | B | 30 |
| B | W | W | W | B | B | B | B | W | B | B | W | B | B | 31 |

Table 8 deals with the cell B.12 and as this cell is a common neighbour of \(B\)
and \(C\), it is concerned by all the cases we have to consider for a crossing. And
so, we have the splitting into three cases already seen in Tables 4 and 5. The
table is a bit shorter than Table 4 for instance.

The conservative rule for the cell B.12 is rule 32. It is of course different from
that of B.11. We can notice that the pattern attached to the neighbourhood
of B.12 consists of almost black cells only for the cells which are invariant.

In the first part of the table, the particle arrives at the round-about through
the cell E. This is witnessed by rule 34 and rule 35 witnesses the presence of
the particle at C. The rule is applied twice as a second particle is created in C.

In the second part, the role of B.12 is important: it conveys the signal sent
by B.11 that two particles arrived to C, in order to trigger a new particle which,
from C, will go to the next branching. Rule 36 applies twice as two particles
arrive at B. Rule 37 recognizes the flash of B.11 and makes B.12 flash also.
When B.11 flashes again, the new particle is in C: this allows rule 38 to cancel
the flash of B.12 and to ignore the second flash of B.11.

In the third part, as B.11 always remains black, rule 36 simply witnesses
that the particle passed through B.
Table 8  Rules for B12. The cell is concerned in three different situations which are clearly separated.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| B | B | | | | | | | | | | | | | |

- I- a particle arrives to the round-about from outside

| B | W | B | B | B | W | B | B | B | B | W | W | B | 32 |
| B | W | B | B | B | W | B | B | B | B | W | B | B | 34 |
| B | W | B | B | B | W | B | B | B | B | W | B | B | 35 |
| B | W | B | B | B | W | B | B | B | B | W | B | B | 35 |

- II- two particles arrive from the round-about

| B | W | B | B | B | W | B | B | B | B | W | W | B | 32 |
| B | W | W | B | B | B | W | B | B | B | W | W | B | 36 |
| B | W | W | B | B | B | W | B | B | B | W | W | B | 36 |
| W | W | B | B | B | W | B | B | B | B | W | W | B | 33 |
| W | W | W | B | B | B | W | W | B | B | B | W | B | 38 |

- III- a single particle arrives from the round-about

| B | W | B | B | B | W | B | B | B | B | W | W | B | 32 |
| B | B | B | B | W | B | B | B | B | W | W | W | B | 36 |

Table 9 gives the rules for C.5 during a crossing. This time, the cell is a neighbour of C but it cannot see B neither B.12. And so, it is concerned by the arrival of the particle at the round-about from the tracks only and by the arrival of two particles at B. In the latter case, this triggers an ultimate creation of a particle in C, a feature that C.5 can detect. Consequently, the table has two parts.

In the first part of the table, we can see that the conservative rule is rule 39. It indicates a new pattern for the black neighbours which remain invariant. Rule 40 detects the arrival of the particle at the cell of the tracks which is just before E. Rule 41 detects the presence of the particle in E so that the rule makes C.5 flash when the particle is in C. This is witnessed by rule 42 which can see that the particle is in C and which makes C.5 turn back to black. Now, this flash of C.5 triggers the presence of a new particle in C. The particle which was seen by rule 42 is now further on the round-about so that the particle which is seen in C by rule 43 is the second one. The rule simply witnesses this fact and the state of C.5 is unchanged. The second particle follows the first one on their way to the next branching.

In the second part, the particle witnesses the creation of the particle in C: again rule 43 applies. The particle goes on its way to the next branching.
Table 9 Rules for C5. The cell is concerned in two different situations which are clearly separated.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| C | E | tr |   |   |   |   |   |   |   |    |    |    |    |   |

-I- a particle arrives to the round-about from outside

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B | W | W | W | B | B | B | W | B | B | W | W | B | B | 39 |
| B | W | W | B | B | B | W | B | B | W | B | B | W | B | 40 |
| B | W | B | W | B | B | B | W | B | B | W | B | B | W | 41 |
| W | B | W | W | B | B | B | W | B | B | W | B | B | W | 42 |
| B | B | W | W | B | B | B | W | B | B | W | B | W | W | 43 |

-II- two particles arrive from the round-about

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B | W | W | W | B | B | B | W | B | B | W | B | W | B | 39 |
| B | B | W | W | B | B | B | W | B | B | W | B | W | W | 43 |

Now we have seen that all rules of Tables 3, 4, 5, 6, 7, 8 and 9 realize the implementation of the scenario described in Subsection 5.2.

6.5 The rules for the switches

This subsection deals with the fixed switch and with the flip-flop. We first look at the first one, a very simple situation, and then at the second one, a bit more complex situation, involving many cells, as in the case of the crossings.

6.5.1 Rules for the fixed switch

Table 10 gives the rule for the cell B when the particle goes through B, see Fig.23 and Fig.25.

Table 10 Passive fixed switch. Rules for B. The particle comes from B.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| tr | O |   |   |   |   |   |   |   |   |    |    |    |    |   |

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| W | B | B | W | W | W | W | W | W | B | W | B | W | W | 94 |
| W | B | B | W | W | W | W | W | W | W | W | B | B | W | 95 |
| B | B | B | W | W | W | W | W | W | W | W | B | W | B | 171 |
| W | B | B | W | W | W | W | W | W | W | W | B | B | W | 96 |

We note that the conservative rule is rule 91: this rule is also the conservative rule of E in the crossings. The reason is that it is a cell of the tracks and that it
is oriented like $E$. The difference with $E$ is that here, the black cells of the idle configuration are milestones. They always stay black. The table simply involves ordinary motion rules. The rules for $C$, when the particle goes through $C$, are given by Table 11. This time the conservative rule is rule 2. Here too, we can see purely motion rules.

Table 11  
Passive fixed switch. Rules for $C$. The particle comes from $C$.

| $n$ | $O$ | $tr$ |
|-----|-----|------|
| 2   | W   | B B W B W B W W W W W W W W |
| 3   | W   | B B W B B B W W W W W W W B |
| 58  | B   | B B W B W B W W W W W W W W |
| 5   | W   | B B B B W B W W W W W W W W |

Table 12  
Passive fixed switch. Rules for $O$.

| $n$ | $A$ | $C$ | $B$ |
|-----|-----|-----|-----|
| 97  | W   | B   | W B B W B W B W W W W W W W |
| 101 | W   | B   | B B W B B W B W W W W W W B |
| 98  | B   | B   | B B W B B W B W W W W W W W |
| 99  | W   | B   | B B B B W B W W W W W W W W |

Table 12 gives the rules for the cell $O$ which is the central cell of the switch: the three one-way tracks meet there. It is better to consider that the tracks coming from $B$ and $C$ merge at $O$ from where they constitute a unique track leaving the switch through $A$. When the particle comes through $B$, rule 101 attracts the particle to $O$. Rule 98 witnesses that the particle leaves the cell and rule 99 confirms that the leaving happens through $A$. When the particle comes through $C$, this time rule 100 attracts the particle to $O$. Once it is in $O$, the same rules, 98 and 99 apply which witness the leaving of the particle from the switch through $A$.

Note that here, the conservative rule is rule 97 and that the pattern of the
neighbours is very different from the one which we have met up to now.

We do not give the rules for $A$: it is an ordinary cell of the tracks and the rules for it have already be given. In order to complete the set of rules, we have to check the rules for $B$ and $C$ when the particle crosses the switch from the other side. This is done by Table 13: we can see that the involved rules have already been used in Table 10 for rules 94 and 96 and in Table 11 for rules 2 and 5.

### Table 13 Passive fixed switch. Rules for B and C when they are silent.

| n | n |
|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| -I- at $B$ when the particle comes from $C$ |
| W | B | B | W | W | W | W | W | W | B | W | B | W | W |
| W | B | B | W | W | W | W | W | W | B | W | B | B | W |
| 94 | 96 |
| -II- at $C$ when the particle comes from $B$ |
| W | B | B | W | B | W | B | W | W | W | W | W | W | W |
| W | B | B | B | B | B | W | W | W | W | W | W | W | W |
| 2 | 5 |

6.5.2 Rules for the flip-flop

We now turn to the rules for the flip-flop, a more complex structure than that of the fixed switch. It relies on the figures we have seen in the study of this switch, see Fig. 23 and Fig. 26.

Remember that in this switch, outside the cells $A$, $B$, $C$ and $O$ already considered in the fixed switch, which are cells on the tracks, we have two sensors, $H$ and $K$ and a controller $D$. Here, $A$ is an ordinary cell of the track, so there is no need to give the concerned cells: we have already seen them. Now, for $B$ and $C$, they are not ordinary cells of the track as the sensors are necessary one of their neighbours. As there must be an exchange between the states of the sensor, the simplest way to implement this is the situation indicated in Fig. 23 and Fig. 26. This entails that here, $B$ and $C$ have a specific pattern. This is a fortiori the case for $D$, $H$ and $K$.

This time, we start with the rules for $O$ given by Table 14.

We can see that the conservative rule for $O$ is rule 51 which also differs from the configuration induced by rule 94 which is the idle configuration of the fixed switch.

The particle arrives by $A$ as witnessed by rule 53. The rule attracts the particle to $O$ and rule 52 witnesses that it leaves $O$. When appropriate, rule 54 shows that the particle leaves $O$ through $C$ and rule 55 that it leaves through $B$. The difference cannot be directly detected by $O$: the selection is performed by $B$ and $C$ themselves as we have seen in Subsubsection 5.5.2.
Table 14 Flip-flop. Rules for $O$.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n   |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| A | W | B | W | B | W | B | W | B | W | B | W | W | W | W | 51  |
| C | W | B | B | W | B | W | B | W | B | W | B | W | W | W | 53  |
| B | B | B | W | B | W | B | W | B | W | B | W | W | W | W | 52  |

-I- the particle goes to $C$

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n   |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
|   | W | B | W | B | W | B | W | B | W | B | W | W | W | W | 54  |

-II- the particle goes to $B$

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n   |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
|   | W | B | W | B | W | B | W | B | W | B | W | W | W | W | 55  |

Table 15 Flip-flop. Rules for $B$, the particle goes to $C$

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n   |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| tr | O | D | H |
|---|---|---|---|
|   | W | B | B | W | B | W | B | W | W | B | W | W | W | W | 66  |
|   | W | B | B | W | B | W | B | W | W | B | B | W | W | W | 67  |
|   | W | B | B | W | B | W | B | W | W | B | B | W | W | W | 66  |
|   | W | B | B | W | B | W | W | B | W | B | B | W | W | W | 63  |
|   | W | B | B | W | B | W | W | W | B | B | W | W | W | W | 61  |

Table 16 Flip-flop. Rules for $C$. The particle goes to $B$

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n   |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| tr | K | D | O |
|---|---|---|---|
|   | W | B | B | W | B | W | B | W | W | B | B | W | W | W | 66  |
|   | W | B | B | W | B | W | B | W | W | B | B | W | B | W | 76  |
|   | W | B | B | W | B | W | B | W | W | B | B | W | B | W | 66  |
|   | W | B | B | W | B | W | B | W | W | B | W | B | W | W | 72  |
|   | W | B | B | W | B | W | B | W | W | W | B | W | W | W | 70  |

Tables 15 and 17 show the rules for the cell $B$ while Tables 16 and 18 show the rules for the cell $C$. We shall first study what happens for a cell which is not on the selected path for the particle. This means that for this cell, the sensor
is black: \( H \) is black if \( B \) is not selected while \( K \) is black if \( C \) is not selected. Consequently, we first look at Tables 15 and 17.

We can see that the conservative rule is rule 66 for both \( B \) and \( C \). However, the place of the common neighbours, here \( O \) and \( D \) is not the same as clear from the tables. Rule 67 for \( B \) and rule 76 for \( C \) witness the occurrence of the particle at \( O \). Now, at the next time, the particle disappears: it went to \( C \) in Table 15 and to \( B \) in Table 16. This is why the conservative rule 66 applies in both cases. Now, as \( D \) has seen the particle in its appropriate place, it flashes, which is witnessed by rule 63 for \( B \) and rule 72 for \( C \). The flash of \( D \) make both \( H \) and \( K \) change their states. This is witnessed by rule 61 for \( B \) and rule 70 for \( C \). Note that these rules are the conservative rules for \( B \) and \( C \) respectively when the sensor is white.

**Table 17** Flip-flop. Rules for \( B \). The particle goes to \( B \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n  |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( tr \) | \( O \) | \( D \) | \( H \) |
| W B B W B W B W W B B W W W W 61 |
| W B B W B B W W B B W B B W 65 |
| B B B W B W B W W B B W W W 62 |
| W B B B W W W W B W B W W W 64 |
| W B B W B W B B W W W W W 66 |

**Table 18** Flip-flop. Rules for \( C \). The particle goes to \( C \)

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n  |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| \( tr \) | \( K \) | \( D \) | \( O \) |
| W B B W B W B W W B B W B W 70 |
| W B B W B W B W W W B B W B 74 |
| B B B W B W B W W W B B W W 71 |
| W B B B W B B W W W W W W W 73 |
| W B B W W W B W W B W B B W 66 |

This is what we can see in the first row of Tables 17 and Table 18. We are know in the case when \( B \) or \( C \) is on the selected tracks. As previously, the particle occurs in \( O \), which is witnessed by new rules, 65 and 74 respectively which make the particle enter the cell. Rules 62 and 71 make the particle leave \( B \) or \( C \), respectively, and to go on on the selected tracks. This is witnessed by rules 64 for \( B \) and 73 for \( C \) which also witness the flash of \( D \). Now, as \( H \) and \( K \) both change their states, which is witnessed by rule 66 in both cases, we find again the idle configuration of the previous tables with, of course, the same rule.

Now, we can turn to \( D \), the controller of the flip-flop, making the sensors change their states when the particle has taken the selected tracks. The table for the rules of this cell is Table 19.
When the selected track is $C$, $B$ respectively, see first, second part of Table 19 respectively, the conservative rule is rule 80, rule 78 respectively. Then rule 82, rule 83, respectively, witnesses the occurrence of the particle in $O$. Then rule 84, rule 85, respectively, witness the entrance of the particle on the right tracks, $C$ or $B$, respectively. Both rules make $D$ flash. Then, rule 81, rule 79, respectively, make the cell return to black. Then, rule 78, rule 80, witness the change of states in both $H$ and $K$. Accordingly, we have the conservative rules which we noted at the initial step.

**Table 19  Flip-flop. Rules for $D$.**

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| O |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| C |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| K |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| H |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| B |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |

- I- the particle goes to $C$

|   | W | W | W | B | W | W | B | W | B | W | B | B | W | W | B | 80 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|   | B | B | W | W | B | W | B | W | B | W | B | B | W | W | B | 82 |
|   | B | W | B | W | B | W | B | W | B | W | B | B | W | W | B | 84 |
|   | W | W | W | B | W | B | W | B | W | B | B | W | B | W | B | 81 |
|   | B | W | B | W | B | W | B | W | B | W | B | B | W | W | B | 78 |

- II- the particle goes to $B$

|   | W | W | B | B | W | W | B | W | W | B | 78 |
|---|---|---|---|---|---|---|---|---|---|---|
|   | B | B | W | B | W | W | B | W | B | B | 83 |
|   | B | W | B | W | B | W | B | W | W | B | B | 85 |
|   | W | W | B | B | W | W | B | W | W | B | B | 79 |
|   | B | W | W | W | B | W | B | W | B | W | B | B | W | W | B | 80 |

We conclude this subsection with the rules for $H$ and $K$, see Tables 20 and 21 respectively.

As we know from Subsections 5.2 and 5.5.2, these cells behave as markers of the non-selected tracks. If the marker is black, the access to its neighbour belonging to the path is forbidden for the particle. This what we have seen on Tables 15, 16. When the marker is white, the access to its neighbour belonging to the path is performed when the particle is in $O$. We have seen the appropriate rules in Tables 18 and 17. We also have seen in Table 19 that the flash of $D$ triggers the change of state in both $H$ and $K$. And so, we have to complete the set of rules by the rules especially dedicated to these cells.

In the first part of Table 20 the conservative rule is rule 87. As the state of $H$ is black, the particle cannot go to $B$, so that it goes to $BC$. Accordingly, the column devoted to $B$ remains with $W$ in this part of the table. Rule 90 witnesses the flash of $D$ so that it makes the state of $H$ change to white: rule 86 is the conservative rule associated to this state, as confirmed by the first row in the second part of the rules.

In the second part of Table 20 where the conservative rule is rule 86, as
the state of $H$ is white, the particle cannot go to $C$ as it goes to $B$ when it is present at $O$. The entrance to $B$ is witnessed by rule 88. Rule 89 detects the flash of $D$ making the state of $H$ change to black. Rule 87 keeps this situation as it is the conservative rule for $H$ when its state is black.

Table 20  
Flip-flop. Rules for $H$.

|   |   |   |   |   |   |   |   |   |   |   |  
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$  
| B | D |   |   |   |   |   |   |   |   |   |   |   |   |   |

-I- the particle goes to $C$

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B | W | B | B | B | B | B | W | W | W | W | W | B | 87 |
| B | W | W | B | B | B | B | B | W | W | W | W | W | W | 90 |
| W | W | B | B | B | B | B | W | W | W | W | W | W | W | 86 |

-II- the particle goes to $B$

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| W | W | B | B | B | B | B | B | W | W | W | W | W | W | 86 |
| W | B | B | B | B | B | B | W | W | W | W | W | W | W | 88 |
| W | W | W | B | B | B | B | B | W | W | W | W | W | B | 89 |
| B | W | B | B | B | B | B | B | W | W | W | W | W | B | 87 |

Table 21  
Flip-flop. Rules for $K$.

|   |   |   |   |   |   |   |   |   |   |   |  
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$  
| C | D |   |   |   |   |   |   |   |   |   |   |   |   |   |

-I- the particle goes to $C$

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| W | W | W | B | B | B | B | B | W | W | W | W | B | 91 |
| W | B | W | B | B | B | B | B | W | W | W | W | B | 93 |
| W | W | W | B | B | B | B | B | W | W | W | W | W | W | 89 |
| B | W | W | B | B | B | B | B | W | W | W | W | W | B | 92 |

-II- the particle goes to $B$

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B | W | W | B | B | B | B | B | W | W | W | W | B | B | 92 |
| B | W | W | B | B | B | B | B | W | W | W | W | W | W | 90 |
| W | W | W | B | B | B | B | B | W | W | W | W | W | W | 91 |

Similarly, Table 21 gives the rule for $K$ which is the marker of the selection/non-selection of the tracks whose initial cell is $C$.

We can see that the idle configuration is different from that of $H$. They
both have a block of five consecutive black cells which remain black. However, this block is not at the same place with respect with D while turning around the cell in the same direction and, in this direction, starting from the first black cell of the block.

When K is black, second part of Table 21 the conservative rule is rule 92. Then rule 90 detects the flash of D when it occurs. Note that as K cannot see O nor B, it knows that the particle entered the switch by the flash of D only. Rule 90 makes K turn to white and rule 91 keeps this new state as it is the corresponding conservative rule.

When K is white, first part of Table 21 we check that the conservative rule is rule 91. Now, rule 93 detects that the particle entered cell C. At the next time, rule 89 detects the flash of D, making the state of K change from white to black. Rule 92 keeps this state as it is the conservative rule for K when its state is black. This has previously been noted.

### 6.5.3 Rules for the memory switch

We arrive to the final set of rules which deal with the memory switch. The rules dedicated to this switch fall into two subsets: those needed for the active switch and those needed to the passive one.

#### The active memory switch

This subsection is parallel to Subsection 6.5.2. The main reason is that the configuration of the active memory switch looks like that of the flip-flop, see Fig. 23. This is also why we use the same names for the cells for which we show the rules. In fact, we use the same cells A and B and C for the tracks as in the flip-flop, which means with the same rules and a few additional ones.

First, we look at the rules for O, as those for A are simple motion rules for an ordinary cell of the tracks.

#### Table 22 Rules for O. The particle goes to C

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| A | W | B | W | B | W | W | B | W | B | W | W | B | W | 51 |
|   | W | B | B | B | W | W | B | W | B | W | W | W | B | 53 |
|   | B | B | W | B | W | B | W | W | W | B | W | W | W | 52 |

- I- the particle goes to C

| W | B | W | B | W | W | B | B | W | B | W | W | W | W | 54 |

- II- the particle goes to B

| W | B | W | B | W | W | B | B | B | B | W | W | W | W | 55 |
Table 22 gives us the rules and the reader may notice that this table contains exactly the same rules as those of Table 14 for the rules for O in a flip-flop. The reason is very simple: at this stage, as the configuration of O is here the same as its configuration in the flip-flop, the particle does not even know that it is in a memory switch. This will be come clear later, when the particle will go either to B or to C. Here, we take the same cells B and C as in the flip-flop switch, although there is a slight difference in the working in the active memory switch as we shall see a bit later. The reason is that D, which is also different, does not behave in the same way as in the flip-flop: D does not flash when the particle is in the exit of B or C. It flashes in a situation when B and C are both idle. However, as H can see only B and D and as K can see only D with C, we keep H and K the same as in the flip-flop.

Table 23 Active memory switch. Rules for B. The particle goes to C

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n  |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| tr | O | D | H |   |   |   |   |   |   |    |    |    |    |    |
| W | B | B | W | B | W | B | W | B | W | B | W | B | W | W | 66 |
| W | B | B | W | B | B | B | W | W | B | B | W | B | W | W | 67 |

Table 24 Active memory switch. Rules for C. The particle goes to B

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n  |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
|   | K | D | O | tr |   |   |   |   |   |    |    |    |    |    |
| W | B | B | W | W | B | B | W | B | W | B | W | B | W | W | 66 |
| W | B | B | W | W | B | B | W | B | B | B | B | B | B | W | 76 |

Compared with the rules of Tables 15 and 16 we can see that the behaviour of B and C for an active passage when they are not on the selected track is much simpler than in the case of the flip-flop where the flash of D is always triggered. Here, rule 66 is the conservative rule. Rules 67 and 76 simply witness the passage of the particle through O.

Now, as shown by Tables 25 and 26 the rules for B and C are a bit different from those of the cells with the same names in the flip-flop: here, B and C behave passively as D does not react to the passage of the particle through them. Consequently, here we have simple motion rules adapted to the pattern of the cells.

The conservation rules are now rules 61 and 70 for B and C respectively, as in the flip-flop as H and K are now white in both tables, of course applied at different times. Rules 65 and 74 attract the particle from O to B and C respectively, again as in the flip-flop. Also as in the flip-flop, rules 62 and 71 return the cell to white. Now, as there is no flash of D, the similarity with the flip-flop stops here. Rule 68 and 134 witness that the particle went to the next cell of the tracks and then, in both cases, we have the conservation rules, rules 61 and 70 respectively, which are not mentioned according to our convention. Let us remark that rules 68 and 134 were not used in the working of the flip-flop.
Table 25 *Active memory switch. Rules for B. The particle goes to B*

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
|   |   |   |   |   |   |   |   |   |   |   |    |    |    |    | tr |
| W | B | B | W | B | W | B | W | W | B | B | W | W | W | 61 |
| W | B | B | W | B | W | B | W | W | B | B | W | W | W | 65 |
| B | B | B | W | B | W | B | W | W | B | B | W | W | W | 62 |
| W | B | B | B | B | W | B | W | W | B | B | W | W | W | 68 |

Table 26 *Active memory switch. Rules for C. The particle goes to C*

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | tr |
| W | B | B | W | B | W | B | W | W | B | B | W | W | W | 70 |
| W | B | B | W | B | W | B | W | W | B | B | W | B | W | 74 |
| B | B | B | W | B | W | B | W | W | B | B | W | B | W | 71 |
| W | B | B | W | W | B | B | W | W | B | B | W | B | B | 134 |

We can check that the idle configuration of the neighbours of B and C are the same as those of their homonyms in the flip-flop.

Table 27 gives the rules for D and we can see that it strongly differs from the rules of Table 19 for the similar situation in the flip-flop. Also note that the pattern of the neighbours of D is different from what it is in the flip-flop. The reason is the difference of behaviour of the cell which is passive here.

Table 27 *Active memory switch. Rules for D.*

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | O |
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | C |
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | K |
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | f |
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | H |
|   |   |   |   |   |   |   |   |   |   |    |    |    |    |    | B |

—I- the particle goes to C

| B | W | W | W | B | W | B | B | B | B | B | B | W | B | 125 |
| B | B | W | W | B | W | B | B | B | B | B | W | B | 126 |
| B | W | B | W | B | W | B | B | B | B | B | W | B | 128 |

—II- the particle goes to B

| B | W | W | B | B | B | W | B | B | B | B | W | W | B | 124 |
| B | B | W | B | B | W | B | B | B | B | W | W | B | 127 |
| B | W | W | B | B | W | B | B | B | B | W | B | B | 129 |
In particular, we note the presence of a particular neighbour of $D$, namely $D_7$, denoted by $f$ in the table. This cell is usually white and it remains unchanged during an active crossing of the switch by the particle. We shall see what happens in the rules for the passive switch.

For what is an active passage, the rules of the table simply witness the passage of the particle. Rules 125 and 124 give the idle configuration which is different, depending on the states of $H$ and $K$ which are always different. We can notice that these states remain unchanged during the passage of the particle. Rules 126 and 127 witness the presence of the particle in $O$ and rules 128 and 129 witness its passage through $C$ and $B$ respectively.

Now, we turn to the rules for $H$ and $K$: the cells are the same as in the flip-flop. As there is no flash of $D$ at this stage, we have just the motion rules used in the flip-flop when $H$ or $K$ can see the particle and the conservative rules otherwise.

**Table 28** *Active memory switch. Rules for $H$.*

|   | B | D |
|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n  |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   | n  |
| -I- the particle goes to $C$ |
| B | W | B | B | B | B | B | W | W | W | W | W | W | B | 87 |
| -II- the particle goes to $B$ |
| W | W | B | B | B | B | B | B | W | W | W | W | W | W | 86 |
| W | B | B | B | B | B | B | B | W | W | W | W | W | B | 88 |

**Table 29** *Active memory switch. Rules for $K$.*

|   | C | D |
|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n  |
|   |   |   |   |   |   |   |   |   |   |   |   |   |   | n  |
| -I- the particle goes to $C$ |
| W | W | W | B | B | B | B | B | W | W | W | W | W | B | 91 |
| W | B | W | B | B | B | B | B | W | W | W | W | W | B | 93 |
| -II- the particle goes to $B$ |
| B | W | W | B | B | B | B | B | W | W | W | W | W | B | 92 |

The passive memory switch

We can now turn to the rules devoted to the passive memory switch. As
known from Fig.23 this switch looks a bit like the fixed switch: its cell T has the same neighbouring as the cell O in the fixed switch. Now, as here the switch has a very different behaviour, we change the names of the cells. As just mentioned, the central cell of the switch is the cell T. The tracks arriving to the switch arrive at the neighbours X and Y of T and in between X and Y we exactly have the neighbour Z of T. The cells I and J are placed as a common neighbour for B with Z and for C with Z respectively. They play the role of H and K in the flip-flop.

We first look at the rules for T, when the particle already arrived at the switch.

Table 30 Passive memory switch. Rules for T. The particle comes from X.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| V | W | B | B | W | B | W | B | W | B | W | W | W | n  |
| Y | W | B | B | W | B | W | B | W | B | W | W | W | 97 |
| Z | W | B | B | W | B | W | B | W | B | W | B | W | 101|
| X | B | B | B | W | B | W | W | B | W | B | B | W | 150|

-I- X was not selected

| B | B | B | W | B | W | W | B | W | B | W | W | W | 98 |

-II- X was selected

| B | B | B | W | B | W | W | B | W | B | W | W | W | 99 |

In Table 30 the particle arrives to T from X. Now, there are two cases, depending on whether X was or not on the selected track. The first row of the table gives the conservative rule for T: rule 97, exactly the same rule as for O in the fixed switch, see Table 12. Then, as in the fixed switch, rule 101 attracts the particle into T. But this time, rule 150 is applied to return the state of T from black to white because Z is flashing at this moment: X was not on the selected track. Next, rule 99 witnesses that the particle left T and is now in V.

If X was the selected track, there is no flash and so, the rules of the fixed switch apply, see Table 12: rule 101 again draws the particle to T, then rule 98 witnesses its migration to V, returning T to the white state. Again, rule 99 witnesses that the particle is now in V.

In both cases, later, the conservative rule 97 applies to T.

Table 31 gives the rules for T when the particle arrives there coming from Y. There are also two cases, as previously, depending on whether Y was or was not the selected track. The organization of Table 31 is parallel to that of Table 30. Note that the conservative rule is of course the same. The rule which attracts the particle to T is different: it is here rule 100 in both cases. Next, the rule
which turns $T$ back to white is rule 150 as previously if $Y$ was not selected because $Z$ is flashing. If $Y$ was selected, rule 98 applies as in the fixed switch. Then, in both cases, the conservative rule 97 again applies.

| Table 31 | Passive memory switch. Rules for $T$. The particle goes from $Y$. |
|----------|-------------------------------------------------------------|
|          | $V$ $Y$ $Z$ $X$                                             |
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 | $n$              |
| W B B W B W W B B W B W W W W | 97               |
| W B B W B W W B B B W B W B W | 100              |

- I- $Y$ was not selected

| B B B W B W W B W W W W W | 150 |

- II- $Y$ was selected

| B B B W B W W B W B W W W W | 98 |
| W B B B B W W B W B B B W W W | 99 |

Let us now turn to the cells $X$ and $Y$. We start with the cell $X$. Table 32 gives the rule when the particle comes from $Y$.

| Table 32 | Passive memory switch. Rules for $X$. The particle comes from $Y$. |
|----------|-------------------------------------------------------------|
|          | $T$ $Z$ $I$ $tr$                                           |
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 | $n$              |
| W W B W W W B B B B B W B B B W | 102 |
| W B W W W B B B B B W B B B W | 103 |
| W W B B W W B B B B B W B B W | 106 |

- I- $Y$ was not selected

| W W B B W B B B B B W B B B W | 106 |
| W B B B W B B B B B W B B W | 107 |

The conservative rules are rule 102 when $Y$ was not in the selected tracks and rule 106 when $Y$ was in the selected tracks. There is a difference as $I$ is not in the same state: $I$ is white when $Y$ is not selected and it is black when $Y$ is
selected. When $Y$ was selected, rule 107 is performed after rule 106, and rule 107 witnesses that the particle is now in $T$. Later, rule 106 is again in action, as the particle is no more visible. If $Y$ was not selected, when the particle is in $T$, $Z$ is flashing because the particle was just before in $Y$. Rule 103 witnesses both the occurrence of the particle in $T$ and the flash of $Z$. Later the particle is no more visible but rule 106 is the conservative rule for $X$ when $I$ is black as now the selected track is $Y$.

Table 33  Passive memory switch. Rules for $X$. The particle comes from $X$.

|   | T | Z | I | tr | n   |
|---|---|---|---|----|-----|
|---|---|---|---|----|-----|
| -I- $X$ was not selected |   |   |   |    |     |
| 0 | W | W | B | B | 106 |
| 1 | W | W | B | B |     |
| 2 | B | W | B | B |     |
| 3 | W | B | B | B | 108 |
| 4 | W | B | B | B |     |
| 5 | W | B | B | B |     |
| 6 | W | B | B | B |     |
| 7 | W | B | B | B |     |
| 8 | W | B | B | B |     |
| 9 | W | B | B | B |     |
| 10 | W | B | B | B |     |
| 11 | W | B | B | B |     |
| 12 | W | B | B | B |     |
| 13 | W | B | B | B |     |
| -II- $X$ was selected |   |   |   |    |     |
| 0 | W | W | B | W | 102 |
| 1 | W | W | B | W |     |
| 2 | B | W | B | B |     |
| 3 | W | B | B | B | 105 |
| 4 | W | B | B | B |     |
| 5 | W | B | B | B |     |
| 6 | W | B | B | B |     |
| 7 | W | B | B | B |     |
| 8 | W | B | B | B |     |
| 9 | W | B | B | B |     |
| 10 | W | B | B | B |     |
| 11 | W | B | B | B |     |
| 12 | W | B | B | B |     |
| 13 | W | B | B | B |     |

Table 33 gives the rules for $X$ when the particle comes from $X$. The situation is not completely symmetric with the situation of Table 32. In Table 32, we can remark that $X$ is always white, while in Table 33, there are two times when $X$ is black, one when $X$ is on the selected tracks and another when it is not.

When $X$ is on the selected tracks, the conservative rule is rule 102. Then we simply have motion rules adapted to $X$. Indeed, Rule 105 detects the occurrence of the particle on the next cell of the tracks after $X$. This makes $X$ change to black. Then, rule 112 changes $X$ back to white while rule 104 witnesses that the particle has reached $T$.

When $X$ is not on the selected tracks, the conservative rule is rule 106. This time, to the motion rules adapted to the configuration of $X$ in which $I$ is black, we have the flash of $Z$ which occurs when the particle is in $T$. Indeed, rule 108 detects the particle coming from the tracks and then $X$ turns to black. Rule 109 turns $X$ back to white and the particle will go to $T$. Rule 110 witnesses that the particle is indeed in $T$ and it also witnesses the flash of $Z$ as already mentioned. From this flash, the cell $I$ turned to white which is witnessed by rule 102, the other conservative rule for $X$, when $I$ is white.
Tables 34 and 35 repeat for $Y$ what has been done for $X$. In this regard, the rules are very parallel to those of Tables 30 and 33. The difference lies in the difference of the patterns of neighbours around $X$ and $Y$ as can be seen from the conservative rules of the tables. When the particle comes from $X$, $Y$ always remains white. If $X$ was on the selected tracks, we have the conservative rule, rule 115 and a witness that the particle passes through $T$, rule 120. When $X$ was not on the selected tracks, outside the conservative rule, rule 113, rule 114 can see both the particle in $T$ and the flash of $Z$. As $J$ changes its state after the

### Table 34 Passive memory switch. Rules for $Y$. The particle comes from $X$.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $T$ | $tr$ | $J$ | $Z$ |

- **I-** $X$ was not selected

| W | W | B | B | B | W | B | B | B | W | W | W | B | W | 113 |
| W | B | B | B | B | W | B | B | B | W | W | W | W | W | 114 |
| W | W | B | B | B | W | B | B | B | W | W | W | W | B | W | 115 |

- **II-** $X$ was selected

| W | W | B | B | B | W | B | B | B | W | W | W | B | B | W | 115 |
| W | B | B | B | B | W | B | B | B | W | W | W | B | B | W | 118 |

### Table 35 Passive memory switch. Rules for $Y$. The particle comes from $Y$.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $T$ | $tr$ | $J$ | $Z$ |

- **I-** $Y$ was not selected

| W | W | B | B | B | W | B | B | B | W | W | B | B | W | 115 |
| W | W | B | B | B | B | B | B | B | W | W | B | B | B | 119 |
| B | W | B | B | B | W | B | B | B | W | W | W | B | B | W | 120 |
| W | B | B | B | B | W | B | B | B | W | W | W | B | W | W | 121 |
| W | W | B | B | B | W | B | B | B | W | W | W | W | W | W | 113 |

- **II-** $Y$ was selected

| W | W | B | B | B | W | B | B | B | W | W | W | B | W | 113 |
| W | W | B | B | B | B | B | B | B | W | W | W | W | B | B | 117 |
| B | W | B | B | B | B | B | W | W | W | W | W | B | W | 122 |
| W | B | B | B | B | W | B | B | B | W | W | W | W | W | W | 116 |
flash, the conservative rule 115 is now in action.

When the particle comes from $Y$, we have pure motion rules when $Y$ is on the selected tracks, second part of Table 31. When $Y$ was not on the selected tracks, we have motion rules too but also rule 121 which witnesses both the passage of the particle in $T$ and the flash of $Z$. The latter event triggers the change of state of $J$, rule 113 which is the conservative rule when $J$ is white, which means that $Y$ is now again on the selected tracks.

Now, we turn to the rules for $Z$. Remember that $Z$ controls the arrival of the particle through the non-selected tracks. In case of this event, it flashes, which triggers both the change of state of the markers $I$ and $J$ and the sending of a signal to $D$, the controller of the active memory switch.

Tables 36 and 37 display the rules for $Z$.

Table 36 Passive memory switch. Rules for $Z$. The particle comes from $X$.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| $T$ | $Y$ | $J$ | $g$ | $I$ | $X$ |

--- X was not selected

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| B   | W   | W   | W   | W   | B   | B   | B   | B   | W   | B   | B   | W   | B   | 138 |
| B   | W   | W   | W   | W   | B   | B   | B   | B   | W   | B   | W   | B   | W   | 143 |
| W   | B   | W   | W   | W   | B   | B   | B   | B   | W   | B   | W   | B   | B   | 139 |
| B   | W   | W   | B   | W   | B   | B   | B   | B   | B   | W   | W   | B   | 146 |
| B   | W   | W   | B   | W   | B   | B   | B   | W   | B   | W   | W   | B   | 137 |

--- X was selected

|     |     |     |     |     |     |     |     |     |     |     |     |     |     |     |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| B   | W   | W   | B   | B   | B   | B   | B   | W   | B   | B   | B   | W   | W   | 137 |
| B   | W   | W   | B   | B   | B   | B   | B   | W   | B   | W   | B   | B   | 142 |
| B   | B   | W   | B   | W   | B   | B   | B   | B   | B   | W   | W   | W   | B   | 149 |

In Table 36 we look at the first part, when the particle come from $X$ in the case the corresponding tracks was not selected. The conservative rule is here rule 138. Rule 143 witnesses that the particle is in $X$ and so, the rule makes $Z$ turn to flash: it becomes white. At the next time, rule 139 restores the black state in $Z$ and witnesses that the particle is in $T$. Rule 146 witnesses the change of states in both $I$ and $J$ which was caused by the flash of $Z$ and the sending of a particle to $D$ through $g$ which is Z.8. As the particle is no more visible, a conservative rule is now in action: rule 146 which is attached to the configuration when $I$ is white and $J$ is black.

It is this configuration that we find in the first row of the second part of Table 36 with rule 137. The other rules witness the motion of the particle: rule 142 can see it in $X$, rule 149 can see it in $T$ and then rule 137 applies again. As $X$ is on the selected track, the passage of the particle does not cause any change.

We have a perfectly similar table for the case when the particle arrives to

53
the switch through $Y$: Table 37

The first part of the table deals with the case when $Y$ was not on the selected tracks. This time rule 137 is the conservative rule, as $I$ is white and $J$ is black. Rule 144 witnesses that the particle is in $Y$, making $Z$ flash. Rule 140 witnesses that it passed to $T$. Rule 147 witnesses that $I$ and $J$ changed their states and that a particle was sent through $g$ to $D$. Rule 138 applies as a conservative rule for the new states of $I$ and $J$.

Table 37 Passive memory switch. Rules for $Z$. The particle comes from $Y$.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T$ | $Y$ | $J$ | $g$ | $I$ | $X$ |

- I- passage through $Y$, $Y$ was not selected

| B | W | W | B | B | B | B | W | B | W | W | B | 137 |
| B | W | B | W | B | B | B | W | B | W | W | W | 144 |
| W | B | W | B | W | B | B | W | B | W | W | W | 140 |
| B | W | W | W | W | B | B | B | W | W | B | 147 |
| B | W | W | W | W | B | B | B | W | W | W | 138 |

- II- $Y$ was selected

| B | W | W | W | W | B | B | B | W | B | W | W | B | 138 |
| B | W | B | W | W | B | B | W | B | W | B | 141 |
| B | B | W | W | W | B | B | W | B | W | B | 148 |

The second part of Table 37 starts with the configuration rule associated to a white $J$ and a black $I$ as $Y$ is selected: rule 138 again. Then rule 141 can see the particle in $Y$, rule 148 can see it in $T$ and then rule 138 applies again.

Next, Tables 38 and 39 give the rules for $I$ and $J$ respectively.

Table 38 Passive memory switch. Rules for $I$.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | $Z$ |

- I- The particle comes from $X$, $X$ being not selected

| B | W | B | W | W | B | B | B | W | B | W | B | 153 |
| B | B | B | W | W | B | B | B | W | B | W | B | 155 |
| B | W | W | W | W | B | B | B | W | B | W | W | 157 |
| W | W | B | W | W | B | B | B | W | B | W | W | 152 |
Table 38 (continued) Passive memory switch. Rules for $I$. 

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | $Z$ |   |   |   |   |   |   |   |   |   |   |   |   |   |

- **II-** The particle comes from $X$, $X$ being selected

| W | W | B | W | W | B | B | B | B | W | B | B | W | W | W | 152 |
| W | B | B | W | W | B | B | B | B | W | B | B | W | W | W | 76  |

- **III-** The particle comes from $Y$, $Y$ being not selected

| W | W | B | W | W | B | B | B | B | W | B | B | W | W | W | 152 |
| W | W | W | W | W | B | B | B | B | W | B | B | W | W | B | 156 |
| B | W | B | W | W | B | B | B | B | W | B | B | W | W | B | 153 |

- **IV-** The particle comes from $Y$, $Y$ being selected

| B | W | B | W | W | B | B | B | B | W | B | B | W | W | B | 153 |

In Table 38 we have four cases first, depending on which side the particle arrives at the central cell and then, on whether the tracks was selected or not.

Indeed, as we can easily understand in the fourth part of the table, if the particle passes through $Y$ and if $Y$ is the selected tracks, $Z$ does not react and so, $I$ cannot see that a particle passed through the switch. This is why in this case we have a conservative rule only, here rule 153. If the $Y$ was on the selected tracks and if the particle passes through $Y$, then $Z$ flashes. This is why we have three instructions in the third part of the table: the conservative rule 152 corresponding to a black state in $J$, rule 156 which, seeing the flash of $Z$ makes $I$ turn to black and rule 153, the conservative rule for the new situation when $I$ is black.

Now, the fist two parts of Table 38 deal with the case when the particle passes through $X$. In the first part, the rules concern the case when $X$ was not selected. In this case, after the conservative rule 153 as $I$ is black, corresponding to the non selection of $X$, rule 155 witnesses that the particle is in $X$. Then rule 157 can see the flash of $Z$ and so, it turns the state of $I$ to white; from now on, $X$ is selected. At the next time, rule 152 applies as it is the conservative rule for the case when $I$ is white.

Table 39 is alike Table 38 with, this time the cell $J$ being concerned. The two first parts of the table look at which rules are used when the particle comes from $X$. As symmetrically in part IV of Table 38, $J$ is not concerned by a passage of the particle though $X$ when $X$ is on the selected tracks. This is why we have the single conservative rule 153 in part II. In part I of Table 39 where the conservative rule is rule 152, the cell $J$ is concerned by the passage through $X$: $Z$ flashes, as $X$ is not on the selected tracks. This flash is witnessed by rule 156 which makes $J$ take the black state. Then rule 153 applies as the
conservative rule for this new configuration of \( J \).

The last two parts of the table deal with the case when the particle passes through \( Y \). Then, the cell always witnesses the passage of the particle. When \( Y \) is not on the selected tracks, the conservative rule is rule 153. The passage of the particle through \( Y \) is witnessed by rule 161 and rule 157 witnesses the flash of \( Z \), making \( J \) take the white state. We now arrive to the configuration when \( J \) is white. The configuration is kept unchanged by rule 152.

In the case when \( Y \) is on the selected tracks, \( Z \) does not flash and so, \( J \) simply witnesses the passage of the particle through \( Y \): the conservative rule is 152 as just noticed and rule 160 witnesses the passage of \( X \).

| Table 39 Passive memory switch. Rules for \( J \). |
|-------------------------------------------------|
|-----------------------------------------------|
| \( Y \) | \( Z \) |
|        |        |
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 \( n \) |
|-------------------------------|
| \( n \) |

- **I-** The particle comes from \( X \), \( X \) being not selected

  \[
  \begin{array}{cccccccccccccc}
  W & W & W & B & B & B & B & W & B & B & W & W & W & B & W & 152 \\
  W & W & W & B & B & B & B & W & B & B & W & W & W & W & B & 156 \\
  B & W & W & B & B & B & B & W & B & B & W & W & W & B & B & 153 \\
  \end{array}
  \]

- **II-** The particle comes from \( X \), \( X \) being selected

  \[
  \begin{array}{cccccccccccccc}
  B & W & W & B & B & B & B & W & B & B & W & W & W & B & B & 153 \\
  \end{array}
  \]

- **III-** The particle comes from \( Y \), \( Y \) being not selected

  \[
  \begin{array}{cccccccccccccc}
  B & W & W & B & B & B & B & W & B & B & W & W & W & B & B & 153 \\
  B & B & W & B & B & B & B & W & B & B & W & W & W & W & B & 161 \\
  B & W & W & B & B & B & B & W & B & B & W & W & W & B & B & 157 \\
  W & W & W & B & B & B & B & W & B & B & W & W & W & B & B & 152 \\
  \end{array}
  \]

- **IV-** The particle comes from \( Y \), \( Y \) being selected

  \[
  \begin{array}{cccccccccccccc}
  W & W & W & B & B & B & B & W & B & B & W & W & W & B & W & 152 \\
  W & B & W & B & B & B & B & W & B & B & W & W & W & B & W & 160 \\
  \end{array}
  \]

At last, and not the least, we have noted that when the particle arrives to the passive switch through the non selected tracks, this makes \( Z \) flash. We have studied the consequences of this flash on \( I \) and \( J \) but we also have to look at the consequence on other cells. Indeed, the flash is also waited by \( Z.8 \), a neighbour of \( Z \) which is white but which becomes black when \( Z \) flashes.

When \( Z \) flashes, \( Z.8 \) which is called \( Z1 \) in the Table 40 transmits the signal, a particle, to \( D \) through a sequence of ordinary cells of the tracks. The last cell is \( D1 \), the neighbour \( D.7 \) of \( D \). When \( D \) can see that \( D1 \) is black, it flashes,
making both $H$ and $K$ change their state.

First, we look at Table 40, which gives the rules for $Z_1$ when the particle comes from a non selected track. Note that $Z_1$ remains white as long as $Z$ does not flash and $Z_1$ does not see another cell outside its neighbour on the track. And so, in this situation, the conservative rule 164 only is in action, first row of the table.

| Table 40 Passive memory switch. Rules for $Z_1$. |
|-----------------------------------------------|
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 | $n$ |
|-----------------------------------------------|
| $Z_{tr}$                                    |
| W B B W B B B B W B W B W                | 164 |
| W W B B B B B B W B W B W                | 165 |
| B B B W B B B B W W B W B W              | 166 |
| W B B W B B B B B B B W B W W             | 167 |

When $Z$ flashes, this is witnessed by rule 165, $Z_1$ becomes black and rule 166 returns its state to white. Next, Rule 167 witnesses that a particle is now on the tracks, on its route to $D_1$.

Table 41 tells us what happens there. As long as $D_1$ is white, the conservative rule 134 applies. When the particle sent by $Z$ arrives close to $D_1$, it is noticed by rule 135 which attracts the signal into $D_1$. Rule 136 returns the state of $D_1$ to white and rule 70 witnesses the flash of $D$.

| Table 41 Passive memory switch. Rules for $D_1$. |
|-----------------------------------------------|
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 | $n$ |
|-----------------------------------------------|
| $tr$                                        |
| W B B W W B B W W B W B W                | 134 |
| W B B W W B B W W B B B B              | 135 |
| B B B W W B B W W B W B W              | 136 |
| W B B W W B B B B W W B W             | 70  |

Table 42 Passive memory switch. Rules for $D$. The particle comes from $X$.

| Table 42 Passive memory switch. Rules for $D$. The particle comes from $X$. |
|-----------------------------------------------|
| 0 1 2 3 4 5 6 7 8 9 10 11 12 13 | $n$ |
|-----------------------------------------------|
| $O$ $C$ $K$ $D_1$ $H$ $B$             |
| B W W W B W B B B W B W B            | 125 |
| B W W W B W B B B W B W W W           | 130 |
| W W W W B W B B B W B W B W           | 131 |
| B W W B B W B W B W B W W B W         | 124 |

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When $D$ flashes, this acts upon its sensible neighbours: $H$, $K$, $B$ and $C$. So that we have to look at possibly additional rules for these cells. First, we look at the rules for $D$ indicated by Tables 42 and 43.

A quick look at Table 42 shows us that the rules are different. Indeed, the cell $D$ in the active memory switch has not the same neighbourhood as $D$ in the flip-flop as they do not behave the same way.

When the particle sent by $Z$ is not yet arrived at $D_1$, the conservative rule 125 applies if $Y$ is selected and the conservative rule 124 is applied if $X$ is selected. Still assuming that $X$ is selected, when the signal arrives at $D_1$, it is recognized by rule 37 which makes $D$ flash. Note that this rule is also used in the crossing for the cell $B$. Next, rule 133 returns the state of $D$ to black, and rule 125 witnesses the change of state in both $H$ and $K$. It is also the conservative rule when $Y$ is selected. If $Y$ is selected, the signal in $D_1$ is recognized by rule 130 which makes $D$ flash. Then, rule 131 returns the state of $D$ to black. Rule 124 witnesses the change of state in both $H$ and $K$.

### Table 43 Passive memory switch. Rules for $D$. The particle comes from $Y$.

|      | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| O C K | B | W | W | B | B | W | B | B | B | W | W | W | B |    |   |
| D1  |   | B | W | W | B | B | W | B | B | B | W | W | W |   |   |
| H B  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

When $D$ flashes, this acts upon its sensible neighbours: $H$, $K$, $B$ and $C$. So that we have to look at possibly additional rules for these cells. First, we look at the rules for $D$ indicated by Tables 42 and 43.

A quick look at Table 43 shows us that the rules are different. Indeed, the cell $D$ in the active memory switch has not the same neighbourhood as $D$ in the flip-flop as they do not behave the same way.

When the particle sent by $Z$ is not yet arrived at $D_1$, the conservative rule 125 applies if $Y$ is selected and the conservative rule 124 is applied if $X$ is selected. Still assuming that $X$ is selected, when the signal arrives at $D_1$, it is recognized by rule 37 which makes $D$ flash. Note that this rule is also used in the crossing for the cell $B$.12. Next, rule 133 returns the state of $D$ to black, and rule 125 witnesses the change of state in both $H$ and $K$. It is also the conservative rule when $Y$ is selected. If $Y$ is selected, the signal in $D_1$ is recognized by rule 130 which makes $D$ flash. Then, rule 131 returns the state of $D$ to black. Rule 124 witnesses the change of state in both $H$ and $K$.

### Table 44 Passive memory switch. Rules for $H$ when $D$ triggered by $Z$.

|      | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| B D  |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |

- **I**- The particle comes from $Y$

|      | B | W | B | B | B | B | B | B | B | W | W | W | W | W | B | 87 |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
|      | B | W | W | B | B | B | B | B | B | W | W | W | W | W | W | 90 |
|      | W | W | B | B | B | B | B | B | B | W | W | W | W | W | W | 86 |

- **II**- The particle comes from $X$

|      | W | W | B | B | B | B | B | B | B | W | W | W | W | W | W | 86 |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
|      | W | W | W | B | B | B | B | B | B | W | W | W | W | W | W | 89 |
|      | B | W | B | B | B | B | B | B | B | W | W | W | W | W | W | 87 |

Tables 43 and 44 display the rules for $H$ and $K$ when $D$ flashes after receiving the signal sent by $Z$.
The rules are close to those of the same cells in the flip-flop, see Tables 20 and 21 with this difference that rule 88 for \( H \) and rule 93 for \( K \) are not used here as the particle does not pass through \( B \) nor through \( C \). And so, when the particle went through a non selected \( X \), rules 87, 90 and 86 apply as in the flip-flop and when the particle went through a non selected \( Y \), rules 86, 89 and 87 apply. Note that rules 87 and 86 are conservative rules, when \( H \) is black and when it is white respectively.

**Table 45** Passive memory switch. Rules for \( K \) when \( D \) triggered by \( Z \).

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| C     |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| D     |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |

-\( \text{I}\)- The particle comes from \( Y \)

|       |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| ---   |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| B W W B B B B B W W W W W B B 92 |
| B W W B B B B B W W W W W W W W 90 |
| W W W B B B B B W W W W W W W W B B 91 |

-\( \text{II}\)- The particle comes from \( X \)

|       |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| ---   |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| W W W B B B B B W W W W W W W W 91 |
| W W W B B B B B W W W W W W W W 89 |
| B W W B B B B B W W W W W W W W B B 92 |

**Table 46** Passive memory switch. Rules for \( B \) when \( D \) triggered by \( Z \).

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | n |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|---|
| D     |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| H     |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |

-\( \text{I}\)- The particle comes from \( Y \)

|       |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| ---   |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| W B B W B W W B W W W B B W W W W 61 |
| W B B W B W W W W W W W B B W W W W 69 |
| W B B W B W W W W W W B B W W W W 66 |

-\( \text{II}\)- The particle comes from \( X \)

|       |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| ---   |   |   |   |   |   |   |   |   |   |   |    |    |    |    |   |
| W B B W B W W B W W B W W W W W W W B B 66 |
| W B B W B W W W W W W W W W W W W W B B 63 |
| W B B W B W W W W W W W W W W W W W B B 61 |

We note that for \( K \) we have also rules contained in those used for the cell \( K \) of the flip-flop, as mentioned previously. The conservative rules are here rule 92.
and 91 when $K$ is black and white respectively. Rules 90 and 89 recognize the flash of $D$ making $K$ change to white or black respectively.

We remain with the rules for $B$ and $C$.

The rules for $B$ are given in Table 46. The conservative rules 61 and 66 are those in action when $D$ does not flash and when no particle goes through $B$ nor $C$. These rules are those used in the flip-flop, see Tables 15 and 17. When the particle passed through $X$ when it was not on the selected tracks, which means that $H$ is black, the rules are a part of the rules of Table 15; there is no flash of $D$ in the situation we consider presently. Now, when the particle passed through $Y$ when it was not on the selected tracks, we have rule 69 which was not yet used. A similar remark holds for the rules for $C$ given in Table 47. The conservative rules are 66 and 70 as in the flip-flop, see Tables 18 and 19. When $K$ is black, the rules are a part of the rules given in Table 16 as no flash of $D$ occurs. However, when $K$ is white, we have a new rule, rule 77 which was not yet used.

### Table 47 Passive memory switch. Rules for C when D triggered by Z.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | $n$ |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| K | D |
|---|---|
| -I- The particle comes from $Y$ |
| W | B | B | W | W | B | B | W | B | B | W | W | 66 |
| W | B | B | W | W | B | B | W | B | W | B | W | 72 |
| W | B | B | W | W | B | B | W | W | W | B | B | W | 70 |
| -II- The particle comes from $X$ |
| W | B | B | W | W | B | B | W | W | B | W | W | 70 |
| W | B | B | W | W | B | B | W | W | W | B | W | 77 |
| W | B | B | W | W | B | B | W | B | B | W | W | 66 |

These new rules 69 and 77 witness the flash of $D$ in a situation which cannot occur during the crossing of a flip-flop by the particle. Indeed, when the flash of $D$ occurs in the flip-flop when $H$ is white, this means that the particle goes through $B$, the particle just left $B$ and is visible from $B$ as it is on the next cell of the tracks, see Table 15 at a moment where rule 64 applies. We have a symmetric situation for $C$ when $K$ is white and rule 73. Here, the next cell of the track is always empty at this moment, this is why the new rules 69 and 77 are needed.

### 6.6 Traces of execution

In this subsection, we complete the study of the rules with traces of execution by the program of the rules in all the configurations we have illustrated in Section 5.

Such traces could be obtained from the above tables. It is easier to tune
the computer program in order to execute this task too. This what we do now, following the same plan as in Section 6.6.

6.6.1 Crossings

Table 48 gives the trace of execution performed by the computer program for the crossing.

For displaying reasons, we could not keep the names $B.11$, $B.12$ and $C.5$ of the sensors controlling the correct execution of the branching at a round-about. We have used other names: $BF$, $BC$ and $CE$ respectively. Clearly, $BF$ can see both $B$ and $F$, $BC$ can see both $B$ and $C$ and $CE$ can see both $C$ and $E$.

Table 48 Trace of execution for the crossings.

|   | A | B | C | D | E | F | F1 | BF | BC | CE |
|---|---|---|---|---|---|---|----|----|----|----|
| 1 | W | W | W | W | W | W | W | B  | B  | B  |
| 2 | W | W | W | W | W | W | W | B  | B  | B  |
| 3 | W | W | W | W | B | W | W | B  | B  | B  |
| 4 | W | W | B | W | W | W | W | B  | B  | W  |
| 5 | W | W | B | B | W | W | W | B  | B  | B  |
| 6 | B | B | W | W | W | W | W | B  | B  | B  |
| 7 | B | B | W | W | W | W | W | B  | B  | B  |
| 8 | W | B | W | W | W | B | W | B  | B  | B  |
| 9 | W | W | W | W | W | B | B | W  | B  | B  |
|10 | W | W | W | W | W | W | W | B  | W  | B  |
|11 | W | W | B | W | W | W | W | B  | B  | B  |
|12 | W | W | W | B | W | W | W | B  | B  | B  |
|13 | B | W | W | W | W | W | W | B  | B  | B  |
|14 | W | B | W | W | W | W | W | B  | B  | B  |
|15 | W | W | W | W | W | B | W | B  | B  | B  |
|16 | W | W | W | W | W | B | W | B  | B  | B  |
|17 | W | W | W | W | W | W | W | B  | B  | B  |
|18 | W | W | W | W | W | W | W | B  | B  | B  |

Table 48 shows another particularity performed by the computer program. The table represents all the possible cases at a branching. Instead of simulating three distinct branchings placed around the round-about, the program used the same branching executing the following scenario. First, the particle arrives from outside to $E$: time 3 in the table. Then, it goes to $C$ and then to $D$. Now, in the program, the next neighbour of $D$ on the tracks is $A$. As an additional particle was created in $C$, when the first particle arrives at $A$ of the same branching, the situation is exactly the same as if it arrived at the next one. And so, when both particles are erased at time 10, see the table, a new one is created in $C$ at time 11. Now, this particle travels to $D$ and then again to $A$, again at the same branching. But the situation is exactly the same as if it would arrive at the next one. And so, at the second arrival from $A$, we can see that the single particle goes to $B$, then to $F$ and so on, without being destroyed.
6.6.2 Fixed switch

Table 49 gives two executions for the fixed switch. In one of them, the particle comes from B, in the other, it comes from C. As can be seen, in both cases, it leaves the switch through A.

Table 49 Traces of execution for the fixed switch. Left-hand side: the particle comes from B; right-hand side: it comes from C.

| aA | A | O | B | bB | C | bC | aA | A | O | B | bB | C | bC |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 1  | W | W | W | W | W | W | 1  | W | W | W | W | W | W | W |
| 2  | W | W | W | W | B | W | 2  | W | W | W | W | W | W | B |
| 3  | W | W | W | B | W | W | 3  | W | W | W | W | W | W | B |
| 4  | W | W | B | W | W | W | 4  | W | W | B | W | W | W | W |
| 5  | W | B | W | W | W | W | 5  | W | B | W | W | W | W | W |
| 6  | B | W | W | W | W | W | 6  | B | W | W | W | W | W | W |
| 7  | W | W | W | W | W | W | 7  | W | W | W | W | W | W | W |

6.6.3 Flip-flop

Table 50 gives two executions for the flip-flop. In the first one, the selected tracks is C, in the second one, it is B.

Table 50 Traces of execution for the flip-flop. Above: to C; below: to B.

| bA | A | O | B | aB | C | aC | D | H | K |
|----|----|----|----|----|----|----|----|----|----|
| 1  | W | W | W | W | W | W | W | B | B | W |
| 2  | W | W | W | W | W | W | W | B | B | W |
| 3  | B | W | W | W | W | W | W | B | B | W |
| 4  | W | B | W | W | W | W | W | B | B | W |
| 5  | W | B | W | W | W | W | W | B | B | W |
| 6  | W | W | W | W | B | W | W | B | B | W |
| 7  | W | W | W | W | W | B | W | B | B | W |
| 8  | W | W | W | W | W | W | B | W | B | B |

| bA | A | O | B | aB | C | aC | D | H | K |
|----|----|----|----|----|----|----|----|----|----|
| 1  | W | W | W | W | W | W | W | B | B | W |
| 2  | W | W | W | W | W | W | W | B | B | W |
| 3  | B | W | W | W | W | W | W | B | B | W |
| 4  | W | B | W | W | W | W | W | B | B | W |
| 5  | W | W | B | W | W | W | W | B | B | W |
| 6  | W | W | B | W | W | W | W | B | B | W |
| 7  | W | W | W | W | B | W | W | W | W | B |
| 8  | W | W | W | W | W | W | B | B | B | W |
In both cases, we can notice that $H$ and $K$ both change their states, at time 8 exactly.

### 6.6.4 Active memory switch

Table 51 gives the traces of execution for the active memory switch. As mentioned in Section 5, the switch looks like a passive programmable flip-flop. At this stage, the flip-flop mechanism exists but it is frozen as $D$ is not the same cell as in the flip-flop.

**Table 51 Traces of execution for the active memory switch. Above: to $C$; below: to $B$.**

|   | bA | A | O | B | aB | C | aC | D | H | K |
|---|----|---|---|---|----|---|----|---|---|---|
| 1 | W  | W | W | W | W  | W | W  | B | B | W |
| 2 | W  | W | W | W | W  | W | W  | B | B | W |
| 3 | W  | B | W | W | W  | W | W  | B | B | W |
| 4 | W  | B | W | W | W  | W | W  | B | B | W |
| 5 | W  | W | B | W | W  | W | W  | B | B | W |
| 6 | W  | W | W | W | B  | W | W  | B | B | W |
| 7 | W  | W | W | W | W  | W | B  | B | B | W |
| 8 | W  | W | W | W | W  | W | W  | B | B | W |

We notice that, in this execution, $H$ and $K$ are not affected by the passage of the particle, which is not the case in the flip-flop.

In the next subsubsection, devoted to the passive memory switch we shall see the action on the flip-flop mechanism of the active memory switch.

### 6.6.5 Passive memory switch

In the computer program, we simplified the execution of the situation when the particle arrives from the non selected tracks. We know that the flash of $Z$, the controller of the passive switch, triggers a signal to $D$, the controller of the active switch. This signal is performed as a particle which follows a path which looks like the usual tracks exactly. We call here locomotive the particle which runs over the circuit to simulate a computation. As we can see in Fig. 23 the path of the messenger particle crosses two other paths followed by the locomotive particle. And so there are two crossings. As we have previously checked that
the program correctly performs the crossings, it is useless to check it again. And so, we connected \( Z \) and \( D \) directly by a small path of five cells including \( Z_1 \) and \( D_1 \). One of these cells, \( M \), is represented in Tables 52 and 53.

Table 52. Traces of execution for the passive memory switch when the particle comes from \( X \). Above: \( X \) was not selected; below: \( X \) is selected.

|      |      |      |      |      |      |      |      |      |      |      |      |
|------|------|------|------|------|------|------|------|------|------|------|------|
|      | aV   | vO   | X    | bx   | Y    | bY   | Z    | I    | D    | H    | K    |
| 1    | W    | W    | W    | W    | W    | W    | B    | B    | B    | W    | W    | W    |
| 2    | W    | W    | W    | W    | W    | W    | B    | B    | B    | W    | W    | W    |
| 3    | W    | W    | W    | B    | W    | W    | B    | B    | B    | W    | W    | W    |
| 4    | W    | W    | W    | B    | W    | W    | B    | B    | B    | W    | W    | W    |
| 5    | W    | W    | B    | W    | W    | W    | W    | B    | B    | B    | W    | W    |
| 6    | W    | B    | W    | W    | W    | W    | B    | B    | B    | W    | B    | W    |
| 7    | B    | W    | W    | W    | W    | W    | B    | B    | B    | B    | W    | W    |
| 8    | W    | W    | W    | W    | W    | W    | B    | B    | B    | W    | B    | W    |
| 9    | W    | W    | W    | W    | W    | W    | B    | B    | B    | B    | W    | W    |
| 10   | W    | W    | W    | W    | W    | W    | B    | B    | B    | B    | W    | W    |
| 11   | W    | W    | W    | W    | W    | W    | B    | B    | B    | W    | W    | W    |
| 12   | W    | W    | W    | W    | W    | W    | B    | B    | B    | W    | B    | W    |
| 13   | W    | W    | W    | W    | W    | W    | B    | B    | W    | B    | B    | W    |
| 14   | W    | W    | W    | W    | W    | W    | B    | B    | B    | W    | B    | W    |
| 15   | W    | W    | W    | W    | W    | W    | B    | W    | B    | B    | W    | W    |

In both tables, we can notice that when the particle comes from the non-selected tracks, the message of \( Z \) to \( D \) is sent at time 6 as \( Z \) flashes at time 5, the locomotive particle being in \( X \) or \( Y \) at time 4. The messenger particle arrives in \( D_1 \) at time 10. This corresponds to the running over the path of five cells between \( Z \) and \( D \). Accordingly, \( D \) is able to flash at time 11. And so, at time 12, \( H \) and \( K \) have both changed their states. Accordingly, the passive and the active memory switch now indicate the same selected tracks.

The real length of the path between \( Z \) and \( D \) is much longer. Looking at Fig. 23, consider that the central tile of the tiling is placed at the central cell of the round-about \( R_0 \) which is the closest to both the active and the passive switch. We can imagine at least three cells between the neighbour of \( X \) on the tracks and the cell \( E \) of the branching to \( R_0 \). Consider \( R_{p} \), the round-about closer to the passive memory switch. We can imagine that its central cell is at three cells from the central cell of \( R_0 \). The path from \( Z_1 \) can go around the
immediate neighbours of $Z$ towards $X$. Then, it goes around the immediate neighbours of $X$ and, at a few cells from the tracks which leaves $X$. We may consider that the cells of the path are on the third level of the tree rooted at the central cell of $R_0$. Now, the number of cells on the third level of the tree rooted at the central cell of $R_0$ is 711. Now, the path between $R_p$ and $R_a$, the round-about closer to the active switch is still longer. This means that the simulation could not consider the real path and that the artifact we presented is a good solution for the simulation.

Table 53  Traces of execution for the passive memory switch when the particle comes from $Y$. Above: $Y$ was not selected; below: $Y$ is selected.

|   | a | v | w | x | b | y | z | i | j | d | h | k | z1 | m | d1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | W | W | W | W | W | B | B | B | W | W | W | W | W | W | W |
| 2 | W | W | W | W | W | W | B | B | B | W | W | W | W | W | W |
| 3 | W | W | W | W | B | W | B | B | B | W | W | W | W | W | W |
| 4 | W | W | B | W | W | W | W | W | W | B | B | B | B | W | W |
| 5 | W | B | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 6 | B | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 7 | B | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 8 | B | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 9 | W | B | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 10 | W | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 11 | W | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 12 | W | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 13 | W | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 14 | W | W | W | W | W | W | W | W | W | B | B | B | B | W | W |
| 15 | W | W | W | W | W | W | W | W | W | B | B | B | B | W | W |

7  The rules for $\{p, 3\}$, $p \geq 17$

As we completed the study of the case $= 13$, we can now turn to the study of the general case of $\{p, 3\}$, when $p \geq 17$.

We shall follow the scenario described in Subsection 5.2. There will be some changes as certain cells which were black in the case of $\{13, 3\}$ are now white and as for them, flashing means turning from white to black and then return to
white at the next step.

We shall follow the same plan as in the case of \{13, 3\}. We start by defining the patterns and the rules for the tracks. From that we can define the patterns and rules for the crossings and then for the switches: fixed, flip-flop and memory switch in this order.

Before turning to the detailed study of each case, we briefly indicate the general idea. Define the **context** of a cell as the pattern defined by the state of its neighbours as a word on \{B, W\}, up to circular permutations: considering the various rotated forms, we shall always take the smallest one in the order where B < W. We shall devise the patterns in the following conditions: in the patterns of the cells of the tracks, the context contains at most one block of two or three contiguous black states and no block with more contiguous black states. For the central cells of the switches, cells O or T in Section 5 there are exactly two blocks of three contiguous black cells and they do not contain blocks with more contiguous black states. There may be an additional black of two cells. The other cells have at least one block of four contiguous black states. This block of four black cells will be called the **anchor**. With the orientation which we may assume to be known by the cell, this gives a direction: there will be a first cell of the block while turning around the cell. This will allow us to use the cells which are after the four one of the block to encode a number which defines which kind of cell we have. As we have twenty kinds of cells outside those of the tracks and the central cells of the switches, we need five cells for encoding their number: we shall speak of the \(i^{th}\) cell of the encoding. As not all possible codes from \{O, 1\}\(^5\) are needed, we can avoid any other block of four contiguous black states. It will be possible to have one block of five contiguous cells: the four ones of the anchor and the fifth one being the first cell of the encoding. The first two cells of the encoding are \(W\) for crossings, they are \(BW\) in this order for the memory switch and \(WB\) in this order for the flip-flop. This block of cells defined by this just described encoding is the **type** of the cell. As outside \(O\) all the cells of a fixed switch are cells of the track, the encoding is \(W^5\) for the fixed switch. This requires that the last three cells of the encoding contain at least one black cell in the case of crossings. This constraint can be observed.

In what follows, the format of the rules will follow the general form given in Section 3 which we remind here for the convenience of the reader:

\[
\eta_0, \eta_1, \ldots, \eta_p \rightarrow \eta_1^0,
\]
where \(\eta_0\) is the state of the cell, \(\eta_i\) the state of its neighbour \(i\) and \(\eta_1^0\) is its new state. We assume that the rule is given in such a way that \(\eta_1, \ldots, \eta_p\) is the minimal form among the rotated forms of the rule obtained by circular permutation on \([1..p]\).

We shall systematically use the following format given as a word in \{B, W\}\(^*\):

\[
\eta_0\eta_1\ldots\eta_p\eta_1^0.
\]
Clearly, the context of the cell is \(\eta_1\ldots\eta_p\).

### 7.1 Patterns and rules for the tracks

We can now define the configurations and the rules for the tracks. The general constraints which we have defined force us to change the contexts of the cells of the tracks.

The context of an ordinary cell of the track will be \(BWWWWWB^k\) or \(BWB^kW^2BWWW\)
with $k = p - 8$. Note that these patterns cannot be deduced from each other by any circular permutation. The patterns are illustrated by the first row of Fig. 31 for the case when $p = 17$. The yellow cells are those through which the particle may enter the cell. The orange ones are those through which the particle exits from the cell. This possible choice gives more flexibility for the construction of the tracks near the switches.

Figure 31  The patterns for the cells of the tracks. Here, the illustration when $p = 17$. The patterns for ordinary cells are BBWWBW, to left, and BBWWBWBB, to right. In more intense colour: the favorite entrance and exit.

Figure 32  Sketch of the paths from $P$ to $Q$. The shortest path is given by the cells $A$, $B$ and $C$. We have given a part of the path around cell $A$. We give the context only for cells 1 and 2 in one direction, for cell 3 in the other one. Note the cell $a$ which has two consecutive black neighbours.

As in Section 5.3 we can go from one cell $P$ to any other one $Q$ by first fixing a shortest path $\pi$ from $P$ to $Q$ and then by going along the upper neighbours of $\pi$ in the direction form $P$ to $Q$ and by going along the lower neighbours of $\pi$ in the direction from $Q$ to $P$. Fig. 32 gives an attempt of illustration for such a path. We can see that even when $p = 17$ drawings become almost useless as details are very difficult to see: the neighbours of the cell $A$ can be seen but many neighbours of cell 1 or 2 are hardly visible, especially to distinguish clearly those which are black.

The figure allows us to notice that here two, we have the phenomenon already noticed in Section 5.3 a cell which belongs to the tracks constructed along a shortest path may have two contiguous black neighbours. We can see that the cell $a$ in Fig. 32 belongs to this case. We say that such a cell is a **corner**. For this situation, we need to adapt the previous neighbourhoods. This new contexts are illustrated by Fig. 33 and the corresponding contexts are: BBWWBWBB in one direction and BBWWBW in the other. In both cases, $k = p - 9$. 

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The patterns for the cells of the tracks having two contiguous black cells as ‘basic’ neighbours. Here, the illustration when $p = 17$. The patterns for these cells are $\text{BBWWBW}^k\text{BBWW}$, to left, and $\text{BBWWBW}^k\text{BBWW}$, to right.

Fig. 32 also shows us that in consecutive ordinary cells, the position of the entrance and the exit of the particle during the motion are around the black cell which is in between the two others, here the black cell around which the path is constructed. This position of the entrance and exit is the more used, it is called favorite in the caption of Fig. 31. Other positions for the entrance and exit are rare and used only to tune the approach or leaving of the particle around a switch or near a round-about.

The rules for the motion of the particle are easy. For ordinary cells they are:

\[
\begin{align*}
\text{WBBWWWBW}^k\text{BBWW}, & \quad \text{WBBWWWBW}^k\text{BBWW}, \\
\text{BBWWBWWWBW}^k\text{BBWW}, & \quad \text{BBWWBWWWBW}^k\text{BBWW}, \\
\text{BBBWBBWWWBW}^k\text{BBWW}, & \quad \text{BBBWBBWWWBW}^k\text{BBWW}. \\
\end{align*}
\]

(a)

and, for the opposite direction:

\[
\begin{align*}
\text{WBBWWWBW}^{k+1}\text{W}, & \quad \text{WBBWWWBW}^{k+1}\text{W}, \\
\text{BBWWBWWWBW}^{k+1}\text{W}, & \quad \text{BBWWBWWWBW}^{k+1}\text{W}, \\
\text{BBBWWWBW}^{k+1}\text{W}, & \quad \text{BBBWWWBW}^{k+1}\text{W}. \\
\end{align*}
\]

(b)

In both cases, we have that $k = p - 8$. Note that here, the rules are not exactly conforming to the requirement of minimality of context in the writing of the rules. The rule $\text{WBBWWWBW}^k\text{BBWW}$ is not written in its minimal form. It should be written: $\text{WBBWWWBW}^k\text{BBWW}$. Accordingly, the conforming presentation of the rules is:

\[
\begin{align*}
\text{WBBWWWBW}^{k+1}\text{W}, & \quad \text{WBBWWWBW}^{k+1}\text{W}, \\
\text{BBWWBWWWBW}^{k+1}\text{W}, & \quad \text{BBWWBWWWBW}^{k+1}\text{W}, \\
\text{BBBWBBWWWBW}^{k+1}\text{W}, & \quad \text{BBBWBBWWWBW}^{k+1}\text{W}. \\
\end{align*}
\]

(c)

\[
\begin{align*}
\text{WBBWWWBW}^{k+1}\text{W}, & \quad \text{WBBWWWBW}^{k+1}\text{W}, \\
\text{BBWWBWWWBW}^{k+1}\text{W}, & \quad \text{BBWWBWWWBW}^{k+1}\text{W}, \\
\text{BBBWWWBW}^{k+1}\text{W}, & \quad \text{BBBWWWBW}^{k+1}\text{W}. \\
\end{align*}
\]

(d)

We can note that in lines (a) and (b), the fourth rules have the same minimal form as the fourth rules in lines (c) and (d). The reason is that the corresponding contexts are symmetric.

Similarly, for the corners we get the following rules, this time in minimal form:

\[
\begin{align*}
\text{WBBWWWBW}^k\text{BBWW}, & \quad \text{WBBWWWBW}^k\text{BBWW}, \\
\text{BBBWWWBW}^k\text{BBWW}, & \quad \text{BBBWWWBW}^k\text{BBWW}, \\
\text{BBWWWBW}^k\text{BBWW}, & \quad \text{BBWWWBW}^k\text{BBWW}. \\
\end{align*}
\]

(e)

\[
\begin{align*}
\text{WBBWWWBW}^k\text{BBWW}, & \quad \text{WBBWWWBW}^k\text{BBWW}, \\
\text{BBBWWWBW}^k\text{BBWW}, & \quad \text{BBBWWWBW}^k\text{BBWW}. \\
\end{align*}
\]

(f)

and, here, we have $k = p - 9$. We have a similar remark as previously: the last two rules in lines (e) and (f) have the same minimal form due to a symmetric neighbourhood.
We conclude this subsection with an important remark. We have seen that, in the crossings, two consecutive particles may travel on the round-about. This requires new rules, namely the following ones:

For ordinary cells:

\[
\text{BBBWWWBW, BBBWBWWW, BBBWBWWW, BBBWBWWBWW, (g)}
\]

and for corners:

\[
\text{BBBWWWBW, BBBWWBW, BBBWBBWW, BBBWBBWWW, (h)}
\]

We remark that these new rules have the same contexts than several rules of lines \((c), (d), (e)\) and \((f)\). However, in these cases, the new state is always the same. So that the new rules are compatible with the others.

We also remark that in these new rules we assume the favorite entrance and exit in the cells: this will be the case in the round-about as we shall soon see.

### 7.2 Patterns and rules for the crossings

The configuration of the round about is basically similar to that of Fig. \[18\] when \(p = 13\). Fig. \[34\] illustrates the new configuration when \(p = 17\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure34.png}
\caption{The configuration of a round-about. The configurations are adapted to the new setting of Fig. \[31\]. The configurations of the cells \(B\) (orange) and \(C\) (pink) are very different from those of Fig. \[18\] for the cells with the same names.}
\end{figure}

In order to better explain the situation, we give the same names for the cells \(A, B, C, D, E\) and \(F\) as in the case when \(p = 13\). However, here, \(B.11\), \(B.12\) have a different meaning than in the case when \(p = 13\). We call \(BF\) the neighbour of \(B\) which can also see \(F\) and \(BC\) that which can also see \(C\). Although \(C.5\) has the same meaning here, we call it \(CE\).

We fix the patterns for each cell which are not tracks of the cell with the patterns defined in Subsection \[41\].

First, the black cell which is the core of the round-about, we have white neighbours and at most two consecutive black cells. We shall consider that rules for which the context contains at most two black cells are always conservative.

Remember that at the beginning of Section \[7\] we said that in the block of cells which encodes its type, we have two cells devoted to the identification of the
situation, which switch or crossing, while the three other cells are a numbering of the types in the considered situation.

In a round-about, the type of the cell is: $BBBBWWXYZ$ where $XYZ$ is given in Table 54 as well as the corresponding types.

**Table 54** The types of the cells of the round point which have patterns different from those of Subsection 7.1

|   | $B$ | $C$ | $BC$ | $BF$ | $CE$ |
|---|-----|-----|------|------|------|
|   | WW  | BB  | WW   | BB   | WW   |
|   | $BBBWWBWW$ | $BBBWWBBW$ | $BBBWWBWB$ | $BBBWWWWB$ | $BBBWWBWB$ |

Note that Table 54 contains five cells only. This means that the patterns of the others are among those described in Subsection 7.1.

The rules for $B$

From Fig. 54, we know that its context is $BBBBWWBWW$, with $a, b \geq 1$. The type is here $BBBBWWBWW$ as indicated by Table 54.

The rules are defined by each type of situation where $B$ is concerned. It's the always case: if a particle arrives from outside, it goes through $E$ and then $C$ so $B$ can see the particle passing through $C$. Also, it is concerned by the arrival of one or two particles from $A$, see Fig. 20 and the following ones. In the rules, when it is present, we indicate by a bold letter which black cell is the particle.

Consider the case when a particle is present at $A$. We have two rules:

$BBBBWWBWWW$  $BBBBWWBWBW$  $BBBBWWBWWB$  $BBBBWWBWBW$  $BBBBWWBWBW$  $(cr.B.a)$

The left-hand side one is the conservative rule of the cell. The right-hand side one makes the particle enter $B$. The bold $B$ indicates the particle. If we have a single particle, we have then:

$BBBBWWBWWW$  $BBBBWWBWB$  $BBBBWWBWWW$  $BBBBWWBWB$  $BBBBWWBWB$  $(cr.B.1b)$

The left-hand one makes the particle leave the cell. The second one witnesses that it entered $F$.

Now, at this point, as $BF$ checks the number of particles passing through $B$, it can see a particle in $F$ and none in $B$, so $BF$ remains the same. At the next time, as the particle left $F$, the conservative rule again applies.

Consider the case when two particles arrive at $B$. The first one is dealt with the rules of $(cr.B.a)$. But the rules of $(cr.B.1b)$ cannot apply as a particle is still present in $A$. We have the rules:

$BBBBWWBWWW$  $BBBBWWBWBWB$  $BBBBWWBWBWB$  $BBBBWWBWBWB$  $BBBBWWBWBWB$  $(cr.B.2b)$

The left-hand side rule can see that there is still a particle in $A$ and this second particle is admitted to $B$. The right-hand side rule deals with this second particle which will leave $B$ and it witnesses that the first particle is in $F$.

And so, at this time, one particle is in $B$ and the other is in $F$. The cell $BF$ can check this situation and so it flashes and then $BC$ flashes and then a particle appears in $C$. All these events are witnessed by $B$ thanks to the following rules in this order.
The rules for $C$

From Fig. 34, we know that its context is $\text{WBWWWBWWWb}$, with $a, b \geq 1$. The type is here $\text{BBBBWWWBW}$ as indicated by Table 54.

The rules are defined by each type of situation where $C$ is concerned. Now, $C$ is concerned in two cases only. When a particle arrives from outside, it goes through $E$ and then $C$ can see the particle passing through $C$. Also, when two particles arrive at $A$, eventually one particle is created in $C$ by the flash of $BC$. However, when a single particle arrives at $A$, $C$ just witnesses that it passes through $B$ and is no more concerned. But this witnessing was already required when two particles arrive.

First, consider the case when a particle arrives at $E$. We have first the conservative rule and then the rule which makes the particle pass from $E$ to $C$:

$$\text{WBWWWBWWWb WBWWWbWBWWWb}$$

$$(\text{cr.B.2c})$$

The next rule shows that $CE$ is flashing so that as one $B$ is leaving $C$ by the motion rules which apply to the ordinary cell $D$, a second $B$ is created in $C$.

The second rule in $(\text{cr.C.b})$ witnesses that the second $B$ leaves the cell while the first one can be seen in $D$. The last rule of $(\text{cr.C.b})$ witnesses that the second $B$ is in $D$.

$$\text{BBBBWWWBWb WBWWWbWBWWWb}$$

$$(\text{cr.C.b})$$

Consider now the case when two particles arrive at $A$. The presence of the first particle and then the second one in $B$ is witnessed by the first rule of $(\text{cr.C.c})$ which is applied twice. The second rule of $(\text{cr.C.c})$ is in action when $BC$ is flashing: this means that a particle must be created in $C$ which is the result of the application of the rule. The third rule of $(\text{cr.C.c})$ contributes to the move of the particle from $C$ to $D$ and witnesses the return of $BC$ to its normal black state. The rule which witnesses that the particle is now in $D$ is the last one of $(\text{cr.C.b})$.

$$\text{WBWWWBWWWb WBWWWbWBWWWb}$$

$$(\text{cr.C.c})$$

We remember that $E$ and $F$ are on the tracks and that they are ordinary cells. We simply note that when some sensors are flashing, there are only two black cells in the neighbourhood of the cell of the tracks. In such a case, the state of the cell is unchanged. Now, it may happen that the flash cancels a black cell and that a new one occurs as the particle going to enter or to leave the cell.

Two rules are in use in this case. The first one is in the neighbour $F_1$ of $F$ which is on the tracks. When $BF$ flashes, the two particles are still present, one in $F$ and one already in $F_1$. The particles are destroyed: as one milestone is missing for $F$ the presence of the particle in $F_1$ restores the configuration of a cell of the tracks so that the particle present at $F$ is erased by a motion rule of the tracks. Now, in $F_1$, the flash also cancels a milestone. But the presence of the particle in $F$ creates a black neighbour at an unusual place for a cell of the tracks. This also requires a new rule: this is the left-hand side rule in $(\text{cr.tr})$.

Another situation appears in $E$ where the creation of the particle in $C$ and the flash of $CE$ creates a neighbourhood with five black cells which has no counter
part in the motion rules of the track for ordinary cells. The appropriate rule is
the right-hand side one in (cr.tr).

\[
\text{BBWBBWWWWW}^a{}^b \text{W}^c \text{W} \quad \text{WBBWWBW}^a{}^b \text{W}^c \quad (\text{cr.tr})
\]

Now, we turn to the auxiliary cells which are not cells of the tracks.

**The rules for BF**

This cell is a milestone of \( F \) and also of \( F_1 \), as we have already noticed. The role of this cell is to check whether one or two particles arrive at \( B \). From Fig. 34, we know that its context is
\[
\text{W}^a \text{BBBBWWBBWWW}^b \text{WBWWWWW}^c \text{B}
\]
with \( a, b \geq 1 \). The type is here \( \text{BBBBWWBBW} \) as indicated by Table 54.

The conservative rule is the first one in (cr.BF.a). If a particle arrive in \( B \), it is noticed by the second rule of (cr.BF.a) and nothing happens as \( F \) is still white. And then, the particle goes to \( F \), which is witnessed by the third rule of (cr.BF.a). If the particle is alone, then the particle left \( F \) and is now in \( F_1 \) as witnessed by the fourth rule of (cr.BF.a).

\[
\text{BBBBWWBBW}^a{}^b \text{W}^c \text{WWW} \quad \text{BBBBWWBBW}^a{}^b \text{WWW} \quad (\text{cr.BF.a})
\]

If two particles are present, then after the application of the first two rules of (cr.BF.a), there is a particle in \( B \) and one in \( F \). At this moment, BF must flash, which is obtained by the first rule of (cr.BF.b). But, still at this time, the motion rules apply to other cells, \( F \) and \( F_1 \) in particular so now, both particles are now in \( F \) and in \( F_1 \). This situation is witnessed by the second rule of (cr.BF.b) which returns BF to black as BF flashes only one step. But, as \( F, F_1 \) and \( BC \) can now see the flash of \( F \), they all change their state to white, which is witnessed by the third rule of (cr.BF.b).

\[
\text{BBBBWWBBW}^a{}^b \text{WWW} \quad \text{BBBBWWBBW}^a{}^b \text{WWW} \quad (\text{cr.BF.b})
\]

**The rules for BC**

We know that the flash of BF is transmitted to \( C \) by BC. From Fig. 34, we know that the context of BC is
\[
\text{W}^a \text{BBBBWWBBWWW}^b \text{WBWWW}^c \text{B}
\]
with \( a, b \geq 1 \). The type is here \( \text{BBBBWWBBW} \) as indicated by Table 54.

The first rule of (cr.BC) is conservative. The second rule witnesses that a particle is in \( B \). It is applied twice of two particles arrive at \( B \). The third rule detects the flash of BF and so, it changes the state of BC to white. Now, the fourth rule restores the black state of BC as the flash is for one step only. The fifth rule witnesses the effect of the flash of BC and \( C \): a black state appeared both in \( C \) and \( BC \) after the flash of BC. After that, the conservative rule again applies.

\[
\text{BBBBWWBBW}^a{}^b \text{WWW}^c \text{B} \quad \text{BBBBWWBBW}^a{}^b \text{WWW}^c \text{B} \quad (\text{cr.BC})
\]

**The rules for CE**

When a particle arrives from outside to the round-about, it enters the round-about through \( C \) but a second particle has to accompany the initial one until
the next branching. The role of \( CE \) is to control and to perform this task. From Fig. 34 we know that its context is \( \text{BBBBBBWWWWBWWBWWB} \), with \( a, b \geq 1 \). The type is here \( \text{BBBBWWWWB} \) as indicated by Table 54.

The first rule of \((\text{cr}.CE)\) is the conservative rule. The second rule is triggered by the occurrence of the particle in \( E \). This is why the rule makes \( CE \) to flash by turning to black. The third instruction returns \( CE \) to white and it witnesses the presence of the particle in \( C \). The effect of the flash is a second particle in \( C \) which is witnessed by the fourth rule of \((\text{cr}.CE)\).

\[
\begin{align*}
\text{BBBBBBWWWWBWWBWWB} & \text{BBBBBBWWWWBWWBWWB} \\
\text{BBBBBBWWWWBWWBWWB} & \text{BBBBBBWWWWBWWBWWB}
\end{align*}
\]

\((\text{cr}.CE)\)

7.3 Patterns and rules for the fixed switch

In the fixed switch, all the cells around \( O \) which are on the tracks are ordinary cells. This can be checked on Fig. 35 which illustrates the idle configuration of the central cell of the switch when \( p = 17 \).

![Figure 35](image)

*Figure 35* The idle configuration of a fixed switch. The configuration is adapted from the new setting of Fig. 31. The cells \( B \) and \( C \) are in yellow. The cell \( A \) is in orange. We can check that they are cells of the tracks.

In these cells of the tracks, as indicated in Subsection 7.1, we take advantage on the flexibility given by the possible choice of the exact neighbour through which the particle enters or leaves the cell.

Accordingly, we only have to look at the rules for the cell \( O \). From Fig. 35 we can see that the context of the rules is \( \text{BBBBBBWWWWBWWBWWB} \), with \( a, b \geq 1 \). As mentioned at the beginning of Section 7 this configuration is characterized by the occurrence of two blocks of consecutive three black cells. Here, the shortest distance between the blocks is 4 cells.

The rules for \( O \) are simply motion rules as no sensor is present around the cell. They are given by formulas \((fx.O)\).

The rule of the first line of \((fx.O)\) is the conservative rule of \( O \) when the particle is far from the switch.
The second line gives the rules which allow the particle to enter the cell: in the left-hand side rule, the particle is in \( C \), in the right-hand side one, it is in \( B \). In the third line, the first rule changes the state of \( O \) from black to white: the particle is ejected from the cell. The second rule witnesses that the particle went to \( A \).

### 7.4 Patterns and rules for the flip-flop

The idle configuration of the flip-flop is illustrated by Fig. 36. We can notice that here too, we have two blocks of consecutive three black cells. However, here, the distance between the blocks is 5 cells.

![Figure 36](image)

**Figure 36** The idle configuration of a flip-flop. The configuration is adapted from the new setting of Fig. 31. The cells \( B \) and \( C \) are in yellow. The cell \( A \) is in orange. Only \( A \) is an ordinary cell of the tracks.

Indeed, the context is now \( \text{BBBBWB}^a \text{WB} \text{BBBBW}^b \text{WB} \). But the big difference is that outside \( A \) which is still an ordinary cell of the tracks, the cells \( B \) and \( C \) are no more ordinary cells of the tracks: they have sensors and a specific surrounding. Also, the cell \( D \), which is in between \( B \) and \( C \) is the controller of the switch and it has too a specific surrounding. Now, in between \( B \) and \( D \) as well as in between \( C \) and \( D \) there is a sensor, \( H \) and \( K \) respectively. These latter cells also have a complex neighbourhood. We know that the context of all these cells contain the pattern \( \text{BBBBWB} \) where \( \text{WB} \) characterizes the flip-flop. We now turn to the study of the rules of all these particular cells.

**The rules for \( B \)**

The context of \( B \) is defined by \( \text{WBBW}^a \text{BBBBWB}^b \text{WB} \text{BB} \) where \( a, b \geq 1 \). This is illustrated by Fig. 36 when \( p = 17 \).

The rules for \( B \) are given by \((f_f Ba)\). The first rule is the conservative rule.
Then, in the second line, the first rule witnesses the passage of the particle in the other tracks: this means that it is the case when B is not selected, which means that H is black. Still when B is not selected, the third rule of (ff.Ba) witnesses the flash of D caused by the particle after its passage through C, two steps later.

$$\text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH}$$

$$\text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH}$$

Next, the rules deals with the case when B is selected by the switch: this means that H is white. The first rule of (ff.Bb) is the conservative rule of this situation. The second rule can see the particle in O. As B is selected, it attracts the particle: the rule changes the state of B to black. The third rule returns the state of B to white as the particle goes further on the tracks. This is witnessed by the fourth rule which can see the particle in the next cell of the tracks. The same rule also witnesses the flash of D which occurs at this very moment.

$$\text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH}$$

The rules for D

The context of D is defined by \( \text{BBBBBBWBBBH} \) where \( a, b \geq 1 \). This is illustrated by Fig. 35 when \( p = 17 \).

The context of D is \( \text{BBBBBBWBBBH} \) when B is not selected and it is \( \text{BBBBBBWBBBH} \) when B is selected.

The rules of (ff.Da) are the conservative rules of D. There are two such rules depending on which tracks between B and C is the selected one.

$$\text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH}$$

The rules of the first line of (ff.Db) gives the rules which make D flash. In both cases, the particle occurs in the neighbour belonging to the selected tracks.

$$\text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH}$$

The rules of the second row of (ff.Db) bring back D to the black state as the flash is intended for one step only.

The rules for C

We turn to the rules for C which have a similar role as those for B. The number of the cell is BBW, so that it context contains the pattern BBWBBBW.

The context of the cell is \( \text{BBBBBBWBBBH} \) when C is not selected and \( \text{BBBBBBWBBBH} \) when C is selected, with \( a, b \geq 1 \). The rules for C are displayed in (ff.Ca) and (ff.Cb).

The first rule of (ff.Ca) is the conservative rule for C when the cell is not selected. Then, if a particle passes through O, it does not move to C, which is attested by the second rule: the state of C is not change. The third rule witnesses a flash of D which occurs when a particle crosses the switch as it will pass through B and this will trigger the flash of D.

$$\text{BBBBBBWBBBH} \quad \text{BBBBBBWBBBH}$$

The first rule in (ff.Cb) is the conservative rule when C is selected. Now,
the second rule shows that if the particle is present in $O$ it then passes to $C$. The third rule witnesses the flash of $D$ and it also turns back the state of $C$ to white. The fourth rule witnesses that the particle left the cell and is now in the neighbouring cell of the tracks.

\[
\begin{align*}
\text{WWBWBW} & \quad \text{WWWWW} \\
\text{BBBBBB} & \quad \text{WWWWW} \\
\text{BBBBBB} & \quad \text{WWWWW} \\
\text{BBBBBB} & \quad \text{WWWWW} \\
\end{align*}
\]

\[(ff.Cb)\]

The rules for $H$

Now, we arrive to the sensors of $D$: $H$ and $K$. These cells show the non-selected track and they change their state to the other one when $D$ flashes.

The cell $H$ has the number $WWB$, so that its context contains the pattern $BBBBWBWB$. The context of $H$ is $WBWBWBWBWBWB$, whatever the state of $H$.

The rules for $H$ are displayed in $(ff.H)$.

The first two rules are conservative: one for the white state, the other for the black one. The third rule witnesses the passage of the particle in $B$ when $H$ is white. The last two rules change the state of $H$ to its opposite one as $D$ is flashing in both cases.

\[
\begin{align*}
\text{WBWBWB} & \quad \text{WWWWW} \\
\text{BBBBBB} & \quad \text{BBBBBB} \\
\text{BBBBBB} & \quad \text{BBBBBB} \\
\text{BBBBBB} & \quad \text{BBBBBB} \\
\end{align*}
\]

\[(ff.H)\]

The rules for $K$

The number of the cell is $BWB$ and so its context contains the pattern $BBBBWBWB$. The context of $K$ is $WWBBBBBBBB$, with $a, b \geq 1$, whatever the state of $K$.

The structure of the rules is very similar to that of the rules for $H$. The first two rules are the conservative rules for $K$: one when it is white, the other when it is black.

Here too, the first two rules are conservative for $K$: one for the white state, the other for the black one. The third rule witnesses the passage of the particle in $C$. The last two rules trigger the change of state: from white to black and from black to white. In both cases, the rule witnesses the flashing of $D$.

\[
\begin{align*}
\text{WBWBWB} & \quad \text{WWWWW} \\
\text{BBBBBB} & \quad \text{BBBBBB} \\
\text{BBBBBB} & \quad \text{BBBBBB} \\
\text{BBBBBB} & \quad \text{BBBBBB} \\
\end{align*}
\]

\[(ff.K)\]

We can now turn to the memory switch.

7.5 Patterns and rules for the memory switch

The idle configurations of both parts of the memory switch are represented by Fig. 37 and 38 for the active switch and the passive one respectively. The figures illustrate the case when $p = 17$. As mentioned in the captions, we detail the neighbourhood of the cells which contribute to the working of the switch. However, $H$, $K$, $I$ and $J$ cannot be named as their representation is too small. They can be deduced from what we explain in the rules.

We first look at the rules for the active switch in Subsubsection 7.5.1. Then, in Subsubsection 7.5.2, we look at the rules for the passive memory switch. Remember that the names of the cells are very different in both cases so that
there will be no confusion between the cells of one one-way switch and those of
the other.

Figure 37 The idle configuration of the active memory switch. The configuration is
adapted from the new setting of Fig. 31. The cell A, the entrance, is in yellow. The
cells B and C, the exits, are in orange. The cell A only is an ordinary cell of the
tracks. Note D1, the neighbour in orange of D, waiting the signal of Z.

Figure 38 The idle configuration of the passive memory switch. The configuration
is adapted from the new setting of Fig. 31. The cells X and Y, the entries, are in
yellow. The cell V, the exit, is in orange. The cell V only is an ordinary cell of the
tracks. Among the neighbours of Z, note the one in orange: it is Z1 which leads to
the path to D1.

7.5.1 The active memory switch

As mentioned in the case when $p = 13$, we can reproduce the configurations of
the cells implied in the flip-flop for the active memory switch with one excep-
tion: the cell $D$ which has a different behaviour in the active memory switch. Accordingly, the rules which we have established for $B$, $C$, $O$, $H$ and $K$ are also in action here. We have just to give the rules for $D$. As this switch has some similarity with the flip-flop and as a single cell requires specific rules, the cell shares the pattern $BBBBWB$ with the cells of the flip-flop. However, it receives the number $WBB$ so that it characteristic pattern is $BBBBWBWB$. 

From Fig. 37 we can see that the context of $D$ is given by the word $WWWWBBBBWB$ when $C$ is the selected track and $WBBWBBBBBBWW$ when $B$ is the selected track. The big difference with $D$ in a flip-flop is that here, $D$ does not react to the passage of the particle through $B$ or $C$. The concerned rules do not change the state of $D$ which remains black. The flash of $D$ is here triggered by $D1$.

The rules of $(ma.Da)$ deal with the case when $C$ is selected. The first rule is the conservative rule. The second rule witnesses the particle is in $O$ and the third rule witnesses that the particle is in $C$. Later, the particle is gone on the tracks and so, the conservative rule again applies.

$$BBBBWBWBBW^aBW\quad (ma.Da)$$

$$BBBBWBWBBW^aBW!BW\quad (ma.Da)$$

The rules of $(ma.Db)$ are exactly parallel to those of $(mq.Da)$: they deal with the case when $B$ is selected. First, the conservative rule, then the witnessing of the particle at $O$ and then of its presence in $B$.

$$BBBBWBWBBWW^aBW\quad (ma.Db)$$

$$BBBBWBWBBWW^aBW!BW\quad (ma.Db)$$

The rules of $(ma.Dc)$ deal with the flash of $D1$. It may occur whatever the selected track is. In the first line, $B$ is selected. The first rule witnesses the flash of $D1$ so that the rule makes $D$ flash too. The second rule returns $D$ to the black state. In the second line, $C$ is selected. Again the first rule makes $D$ flash when the flash of $D1$ is detected. The second rule also returns $D$ to the black state. In both cases, the change of states in $H$ and $K$ is witnessed by the appropriate conservative rule, see $(ma.Da)$ and $(ma.Db)$.

$$BBBBWBWBBW^aBW!BW\quad (ma.Dc)$$

$$BBBBWBWBBW^aBW!BW\quad (ma.Dc)$$

### 7.5.2 The passive memory switch

In the memory switch, many cells have a specific surrounding. However, the central cell $O$ of the switch is the same as that of the fixed switch as the latter switch is also a one-way passive switch. Also, the cell $V$, the exit of $O$, is an ordinary cell of the tracks. So that We need not give its rules. But for the other cells, $X$, $Y$, $Z$, $I$, $J$, $Z1$ and $D1$, we have to look at the rules specifically. We not that here, the rules contain the pattern $BBBBB$ where $B$ characterizes the memory switch.

**The rules for $X$**

This cell receives the number $BW$ so that its characteristic pattern is the word $BBBBBBW$. From Fig. 35 we know that $WBBWBBBBBBWW$ is the context of $X$ when it is not selected and $WBBWBBBBBBWW$ when it is, where, in both cases, $a, b \geq 1$. 

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First, consider the case when $X$ is not selected. The first rule of $(mp.Xa)$ is the conservative rule corresponding to this configuration. The second rule detects the arrival of the particle from the tracks and it makes it enter $X$. The third rule returns $X$ to white. The fourth rule witnesses that the particle is now in $O$ and it also witnesses the flash of $Z$ as $Z$ has detected that a particle passed through non selected tracks.

Next, we look at the case when $X$ is selected. The first rule of $(mp.Xb)$ is the conservative rule of this situation. The next three rules are ordinary motion rules adapted to the context of $X$: the first rule attracts the particle into $X$, the second one returns the state of $X$ to white and the third one witnesses that the particle is in the neighbour of $X$ which is the next cell on the tracks. Note that here, as $X$ is selected, there is no flash of $Z$ which is always black.

A last rule is needed when $X$ is selected and when the particle passes through $Y$. Nothing happens in $X$, except when $Z$ flashes as the particle crosses $Y$ which was not selected. This flash is also noticed by $X$ as the particle is already in $O$, whence the rule $(mp.Xc)$.

The rules for $Y$

This cell receives the number $BBW$ so that its characteristic pattern is the word $BBBBBWBBW$. From Fig. 58, we know that $WWWBBBBBWBWWBB$ is the context of $Y$ when it is not selected and $WWWBBBBBWBWW$ when it is, where, in both cases, $a, b \geq 1$.

First, we look at the case when $Y$ is not selected. The concerned rules are given in $(mp.Ya)$. The first rule is the conservative rule of this case. The neighbour $J$ is black. The second rule attracts the particle into $Y$. The third rule returns the state of $Y$ to white. The fourth state witnesses that the particle is in $O$. It also witnesses that $Z$ flashes as $Z$ has seen that the particle went through the non selected tracks at the previous step.

Next, we look at the rules when $Y$ is selected which are displayed by $(mp.Yb)$. Again, the first rule is conservative, adapted to this configuration, where $J$ is white. The other rules are ordinary motion rules adapted to the context of $Y$. The second rule of $(mp.Yb)$ attracts the particle into $Y$. The third rule witnesses that the particle goes out from $Y$ as the state of $Y$ returns to white. The fourth rule witnesses that the particle is now in $O$.

This case still requires another rule: when the particle passes through $X$, as $X$ is not selected, this triggers the flash of $Z$ which is seen by $Y$ at the time when the particle is in $O$. Whence the rule of $(mp.Yc)$.
The rules for $Z$

This cell receives the number $WBW$ so that its characteristic pattern is the word $BBBBBBW$. From Fig. 38, we know that $BBBBBBW$ is the context of $Z$ when $Y$ is selected and $BBBBBBW$ is the context of $Z$ when $X$ is selected and, in both cases, $a, b \geq 1$.

First, we look at the case when the particle goes through the non-selected tracks. The rules are displayed in $(mp. ZY n)$ when the particle goes through $X$ and in $(mp. ZY n)$ when the particle goes through $Y$.

In both cases, the first rule is the conservative rule of the idle configuration for $Z$. The second rule witnesses that the particle goes through the non selected tracks: $X$ in $(mp. ZX n)$, $Y$ in $(mp. ZX n)$. accordingly, the rule makes $Z$ flash. The third rule restores the black state in $Z$ and it witnesses that the particle is now in $O$.

With $I$, we arrive to the sensors which contribute, with $Z$, to the management of the switch. This sensor is in between $X$ and $Z$ and it allows $Z$ to detect the passage of the particle on the non-selected tracks when $Y$ is selected.

As a cell of the passive memory switch, $I$ receives the number $BBB$, so that its context contains the pattern $BBBB$. whether the state of $I$ is white or black, the context of $I$ is always $BBBBW$. In $(mp. Ib)$, we can see the two corresponding conservative rules.

In $(mp. Ib)$ the rules witness the presence of the particle in $X$, whatever the state of $I$ which is unchanged.

At last, we know that $I$ changes its colour, depending on the flash of $Z$. The rules are given in $(mp. Ic)$: when $I$ is white it becomes black and when it is
black, it becomes white.

\[
\text{BBBBBBWBBW}^a \text{WWW}^a \text{WWW}^a \text{WWW}^a \text{W}^{a,b} \quad (mp.Jc)
\]

The rules for $J$

The cell $J$ plays with $Y$ and $Z$ the same role as $I$ plays with $X$ and $Z$. Its number is $\text{BWB}$, so that its context contains the pattern $\text{BBBBBBWBWB}$. Accordingly, the context of the cell is now $\text{WWW}^a \text{BBBBBBWBWB}^a \text{WWW}^a$ whether its state is white or black.

In $(mp.Ja)$, we have the conservative rules, one for the white state, the other for the black one.

\[
\text{BBBBBBWBBW}^a \text{WWW}^a \text{WWW}^a \text{WWW}^a \text{W}^{a,b} \quad (mp.Ja)
\]

In $(mp.Jb)$, we have the rules which detects the presence of the particle in $Y$, whatever the state of $J$.

\[
\text{BBBBBBWBBW}^a \text{WWW}^a \text{WWW}^a \text{WWW}^a \text{W}^{a,b} \quad (mp.Jb)
\]

In $(mp.Jc)$, we have the rules which manage the effect of the flash of $Z$. When $Z$ flashes, the particle is in $O$ but cannot be seen by $J$. One rule changes the state from black to white, the other from white to black.

\[
\text{BBBBBBWBBW}^a \text{WWW}^a \text{WWW}^a \text{WWW}^a \text{W}^{a,b} \quad (mp.Jc)
\]

The rules for $Z1$

When $Z$ flashes, it also sends a signal to the active memory switch in order to change the states of $H$ and $K$, both at the same time. This is performed by sending a particle along a path from $Z$ to $D$. This path crosses twice other paths thanks to roundabouts as we have seen in Subsections 5.2 and 5. Outside its first and its last cells and outside the roundabouts, the path consists of ordinary cells of the tracks and of corners. Its first cell is $Z1$, a neighbour of $Z$. Its last cell is $D1$, a neighbour of $D$. We presently see the rules of $Z1$.

As a cell of the passive memory switch, its number is $\text{WBB}$, so that its context contains the pattern $\text{BBBBBWBWB}$. The context itself is $\text{BBBBBWBWB}$. The rules are displayed in $(mp.Z1)$.

\[
\text{BBBBBBWBBW}^a \text{WWW}^a \text{WWW}^a \text{WWW}^a \text{W}^{a,b} \quad (mp.Z1)
\]

The rules for $Z1$ are displayed by $(mp.Z1)$. The first rule is conservative and the second one reacts to the flash of $Z$ by changing the state of $Z1$ from white to black. The third rule turns back the state of $Z1$ to white. Now, the motion rules of the neighbour of $Z1$ on the tracks has taken the particle emitted by $Z$: this is witnessed by the fourth rule which can see the particle in this neighbour.

The rules for $D1$

Just before reaching $D$ of the active memory switch, the particle sent by $Z$ passes through $D1$, the neighbour of $D$ which is on the path from $Z$ to $D$. And so, when the particle is in $D$, the next time $D$ flashes, see the rules in $(ma.Dc)$.

The cell $D1$ has the number $\text{BBB}$ so that its contexts contains the pattern $\text{BBBBBWBWB}$. The context is: $\text{BBB}^a \text{BBBBBWBWB}^a$, with $a, b \geq 1$. The rules are displayed in $(mp.D1)$.

We can see that the first rule is a conservative rule for $D1$. The second rule applies when $D1$ can see a particle in its neighbouring cell of the tracks: it
changes the state of $D1$ to black. The third rule restores the white state of $D1$. The fourth rule can witness the fact that $D$ is flashing as it has seen $D1$ black the time before.

\[ \text{BBBBBBWWWBBWWBWWWWW} \quad \text{BBBBBBWWWBBWWBWWWWW} \quad (mp.D1) \]

The effect of the flash of $D$ on $H$ and $K$ has been studied on the rules for these cells, namely in $(ff.H)$ and in $(ff.K)$. Now, this flash of $D$ requires a rule which was not present in those of $B$ and $C$ in the flip-flop. Indeed, when $B$ or $C$ can see the flash of $D$ in the flip-flop, the particle can also be seen by the cell, see the last rule of $(ff.Bb)$ and the last one of $(ff.Cb)$. Here, in the memory switch, when the flash of $Z$ eventually reaches $D$, there is no particle in the switch so that when the flash of $D$ occurs, there is no particle seen by $B$ nor $C$. And so, this requires the rules of $(ma.BC)$. The left-hand side rule applies to $B$ and the right-hand side one applies to $C$.

\[ \text{BBBBBBWWWBBWWBWWWWW} \quad \text{BBBBBBWWWBBWWBWWWWW} \quad (ma.BC) \]

With these last rules, we have examined all possible cases. Accordingly, this completes the proof of Theorem 1.

8 Conclusion

We have now reached the minimal number of states in order to get a universal cellular automaton on the heptagrid with a true planar cellular automaton. However, we have not exactly all possible tessellations of the hyperbolic plane. A few of them are missing in the \{p, 3\} family: the values of $p$ from 7 up to 12, which means 6 cases. Moreover, the question arises of what can be said for tilings \{p, 4\} which are tightly connected to the tilings \{p, 3\}: we now that \{p, 4\} and \{p+2, 3\} have the same spanning tree, see [8] where other references can also be found.

Accordingly, we remain with some hard work ahead.

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