Robust Model Predictive Control with State Estimation under Set-Membership Uncertainty

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Abstract—Robust design of autonomous systems under uncertainty is an important and challenging problem. In practice, state disturbances and state estimation error may cause a controller to fail to accomplish its task or even to stabilize the system. In this paper, we propose a novel tube-based robust model predictive control design based on set-membership state estimation for constrained systems with unknown but bounded state and output disturbances. With more accurate and efficient error bounding methods, the proposed controller ensures robust constraint satisfaction, guarantees recursive feasibility and stability, generates less conservative behavior than the previous work, and scales well to high-dimensional robotic applications. Our simulated experiments on three different systems (double integrator, robotic walker, and quadrotor) demonstrate that the proposed controller is robust against a larger range of disturbances and results in superior closed-loop performance compared to baseline methods.

I. INTRODUCTION

Model predictive control (MPC) is a feedback control technique based on the iterative solution of an optimization problem \cite{Mayne2000}. By using a predictive model of the system and the current state information, MPC plans the optimal control sequence based on the cost function. The system executes the first control action in the optimal sequence and the procedure repeats at the next sampling time. MPC has received considerable attention over the last decades due to its ability to handle multi-variable systems and hard constraints on states and control actions \cite{Bemporad2002}.

In practice, however, two important issues arise in MPC design: (i) actual system states are not available as feedback signals, leading to the necessity of state estimation, and (ii) measurements and the model used for prediction are uncertain (e.g., due to unmodeled dynamics and disturbances) \cite{Chisci2000}. A simple way to apply MPC on uncertain systems is by employing ‘certainty equivalence’, i.e., designing a deterministic/nominal MPC with the state estimates in replacement of the actual states. However, the stability of such closed-loop uncertain system cannot be ensured by simply combining a stable estimator with the nominal MPC \cite{Mayne2000b,Mayne2002}.

To overcome this problem, extensive research has focused on robust MPC design that takes the state estimation error directly into account, dating back to the early 2000s. Bemporad and Garulli \cite{Bemporad2002} combined predictive control and the minimum volume parallelotopic state estimation to achieve robust constraint satisfaction for uncertain systems. Chisci and Zappa \cite{Chisci2000} adopted the Kalman filter as the state estimator to design the robust controller. Mayne et al. \cite{Mayne2000,Mayne2002} constructed the robust output feedback MPC as a combination of a stable Luenberger observer, whose estimation error is bounded by an invariant set, and a tube-based predictive controller which ensures that all possible realizations of the state trajectory lie in an uncertainty tube. In recent years, Le et al. \cite{Le2013} realized the state estimation using a zonotopic set-membership estimation, then a tube-based MPC is designed in a similar manner to \cite{Mayne2002}. The output feedback MPC developed by Qiu et al. \cite{Qiu2012} employed an ellipsoidal state estimation and incorporated the system constraints into LMI conditions such that the control actions can be computed by using semi-definite programming.

However, these approaches may suffer from high computational complexity \cite{Chisci2000,Le2013}, rely on probabilistic assumptions on disturbances which are hard to validate \cite{Zapp1998}, generate conservative closed-loop behavior under large disturbances \cite{Mayne2000,Mayne2002}, or lacks guarantees on recursive feasibility and stability \cite{Bemporad2002}. Furthermore, all the above methods were only validated on two-dimensional systems and the generalization of these methods to high-dimensional systems remains unclear.

In robust MPC literature, bounding sets for disturbances are often described as boxes or polytopes due to the resulting simplicity of bounding errors for controllers \cite{Chisci2000,Mayne2002,Le2013}. Through real experiments, however, ellipsoids turn out to be a more suitable description of uncertainty for noises (Figure 1). As a result, we adopt ellipsoidal set-membership state estimation, which has another advantage of lower computational complexity than parallelotopic state estimation in terms of updating the state uncertainty sets \cite{Geng2019,Geng2020,Geng2021,Geng2022}.

We consider the robust output feedback problem for constrained linear systems subject to state and measurement disturbances. The design procedure of our controller in the

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Fig. 1: Real disturbances in different systems turn out to lie in ellipsoidal bounding sets. \textit{Left} \cite{Geng2019}: Measurements for eight different test points from Decawave indoor positioning system. Blue dots indicate the measurement results and black dots indicate the places of anchors. \textit{Right}: RTK GPS position measurements on a static Polaris GEM autonomous vehicle \cite{Driggs-Campbell2016}. The measurements have been normalized to have zero mean.
paper is as follows: (i) the set-membership state estimation error is bounded by a precomputable set (Section III-A); (ii) the control error is subsequently bounded by a precomputable invariant set (Section III-B); and (iii) by using the bounding sets of the estimation and control error, the ‘cross-section’ of the trajectory tube can be computed and the ‘center’ of the tube (the nominal state) can then be optimized based on the cost function while ensuring that the entire tube satisfies all the constraints on the original system (Section IV). Since all possible realizations of the state trajectory lie in the trajectory tube, the constraint satisfaction can be guaranteed for all admissible disturbance sequences.

Using the novel state decomposition method and the simplicity of the computation of error bounding sets, the proposed controller ensures robust constraint satisfaction, guarantees recursive feasibility and stability, has comparable computational complexity to nominal MPC, and scales well to high-dimensional systems. Moreover, the theory and the simulation examples both indicate that the resulting controller generates less conservative closed-loop behavior and adapts to larger range of disturbances compared to existing methods [6], [9]. Although we focus on linear systems in this paper, the extension to the control of a class of nonlinear systems using the proposed controller is also achievable by applying state feedback linearization [19]–[21].

Nomenclature: In the following sections, $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$. A set $\mathcal{U} \subset \mathbb{R}^n$ is a $C$ set if it is compact, convex, and contains the origin in its non-empty interior. Let $\rho(A)$ denote the spectral radius of a given matrix $A \in \mathbb{R}^{n \times n}$. Given two sets $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^n$, Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} := \{u + v \mid u \in \mathcal{U}, v \in \mathcal{V}\}$ and Minkowski difference by $\mathcal{U} \ominus \mathcal{V} := \{x \mid x + \mathcal{V} \subset \mathcal{U}\}$. Hereafter, the maximization of a vector-valued function is to be performed elementwise.

II. Problem Formulation and Background

In the following section, we introduce the robust optimal control problem of constrained linear systems and describe the set-membership state estimation algorithm.

A. Robust Constrained Optimal Control

We consider the following uncertain discrete-time linear time-invariant system:

$$
\begin{align*}
x_{t+1} &= Ax_t + Bu_t + Dw_t, \quad x_0 = x_S, \\
y_t &= Cx_t + v_t,
\end{align*}
$$

(1)

where $x_t \in \mathbb{R}^n$ is the system state at time $t$, $x_S$ is an initial state, $u_t \in \mathbb{R}^m$ is the control action, $w_t \in \mathbb{R}^q$ is an unknown state disturbance, $y_t \in \mathbb{R}^p$ is the measured output, $v_t \in \mathbb{R}^p$ is an unknown output disturbance, and $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times q}$ are known matrices, where the couple $(A, B)$ is assumed to be controllable and $(A, C)$ observable. At each time $t$, the additive state and output disturbances $w_t$ and $v_t$ are only known to the extent that they lie, respectively, in the $C$ sets $\mathcal{W} \subset \mathbb{R}^q$ and $\mathcal{V} \subset \mathbb{R}^p$: $(w_t, v_t) \in \mathcal{W} \times \mathcal{V}, \quad \forall t \in \mathbb{N}$. System (1) is subject to the following mixed constraints on the states and control actions:

$$
Fx_t + Gu_t \leq f, \quad (2)
$$

where $(F, G, f) \in \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times m} \times \mathbb{R}^d$ are known matrices. In order to counteract the disturbances, the control action is a feedback policy $u_t(\cdot)$ with state estimate $\hat{x}_t \in \mathbb{R}^n$ as the feedback signal, where $u_t(\hat{x}_t) \in \mathbb{R}^m$.

To evaluate the predicted performance of uncertain systems, we consider the nominal cost corresponding to the case of no model uncertainty, i.e., $u_t = 0, \forall t \in \mathbb{N}$. Our goal is to design a controller that solves the following infinite horizon robust constrained optimal control problem:

$$
V^*_\infty(\hat{x}_0) = \min_{u(\cdot)} \sum_{t=0}^{\infty} q(s_t, u_t(s_t))
$$

(3)

subject to:

$$
\begin{align*}
x_{t+1} &= Ax_t + Bu_t(s_t) + Dw_t, \quad s_0 = \hat{x}_0, \\
x_{t+1} &= Ax_t + Bu_t(\hat{x}_t) + Dw_t + v_t = x_S, \\
Fx_t + Gu_t(\hat{x}_t) &\leq f, \quad \forall v_t \in \mathcal{W}, \forall u_t \in \mathcal{V}, \\
t &= 0, 1, 2, \ldots,
\end{align*}
$$

where $u(\cdot) := \{u_0(\cdot), u_1(\cdot), \ldots\}$ is the sequence of feedback policies, $s_t \in \mathbb{R}^n$ is the nominal state, and $q : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}_+$ is a positive definite stage cost. Note that only the state estimate rather than the true state is available to the controller. The output disturbance $v_t$ implicitly affects the system through the feedback control $u_t(\hat{x}_t)$ as the state estimate $\hat{x}_t$ is a function of the output $y_t$ at time $t$. The key challenge here is to design a robust controller such that the constraint (2) is satisfied for all admissible disturbance sequences $w := \{w_0, w_1, \ldots\}$ and $v := \{v_0, v_1, \ldots\}$.

The optimization problem (3) is intractable because: (i) the optimization is performed over the infinite-dimensional space of all feedback policies; and (ii) the horizon is infinite. In this paper, we adopt control policy approximation to convert the infinite horizon robust optimal control problem to a tractable finite horizon problem and apply the receding horizon control strategy to control the system [13], [22].

B. Set-Membership State Estimation

In order to estimate the state, a recursive set-membership state estimation algorithm is employed [16], [23]. Hereafter, we shall assume that the uncertainty sets $\mathcal{W}$ and $\mathcal{V}$ are described as ellipsoids:

$$
\begin{align*}
w_t &\in \mathcal{W} := \{w \mid w^TQ^{-1}w \leq 1\}, \quad \forall t \in \mathbb{N}, \\
v_t &\in \mathcal{V} := \{v \mid v^TR^{-1}v \leq 1\}, \quad \forall t \in \mathbb{N},
\end{align*}
$$

(4a)

(4b)

where $Q$ and $R$ are known positive definite matrices. Moreover, the initial state is bounded by a given ellipsoid:

$$
x_0 \in \mathcal{X}_0 := \{x \mid (x - \hat{x}_0)^TP^{-1}(x - \hat{x}_0) \leq 1\},
$$

(5)

where $\hat{x}_0$ is a given initial guess of $x_0$ and $\Psi$ is a known positive definite matrix. For the sake of completeness, we outline the discrete time set-membership state estimation algorithm developed by Bertsekas and Rhodes [16].

Recursive Set-Membership State Estimation: Given the system (1), the disturbance bounds (4), and the initial state
bound (5), a bounding set $X_{t|t}$ to the set of all possible states $x_t$ at time $t$ given the outputs observed up to time $t$ can be described as an ellipsoid:

$$X_{t|t} = \{ x \mid (x - \hat{x}_t)^T P_{t|t}^{-1} (x - \hat{x}_t) \leq 1 - \delta_t^2 \}, \quad (6)$$

where the positive definite matrix $P_{t|t}$ is recursively given by the equations:

$$P_{t+1|t+1} = [(1 - \rho)P_{t|t}^{-1} + \rho C^T R^{-1} C]^{-1},$$
$$P_{t+1|t} = (1 - \beta)^{-1} A P_{t|t} A^T + \beta^{-1} D Q D^T, \quad (7)$$
$$P_{0|0} = \Psi.$$

The estimate $\hat{x}_t$ evolves according to:

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t + \rho P_{t+1|t+1} C^T (y_{t+1} - C(\hat{A}\hat{x}_t + Bu_t)),$$

with $\hat{x}_0$ as the initial condition and the non-negative real number $\delta_t$ is given by the equation:

$$\delta_{t+1}^2 = (1 - \beta)(1 - \rho)\delta_t^2 + (y_{t+1} - C(\hat{A}\hat{x}_t + Bu_t))[(1 - \rho)\delta_{t+1}^2 C^T + \rho^{-1} R^{-1}]^{-1}(y_{t+1} - C(\hat{A}\hat{x}_t + Bu_t)), \quad (9)$$

where $\beta$, $\rho$ are parameters with $0 < \beta < 1$ and $0 < \rho < 1$.

We point out two desirable properties of the state estimation algorithm given by (6-9).

1. The matrix $P_{t|t}$ does not depend on the outputs along the trajectory, and hence can be precomputed.
2. In cases of time-invariant systems with $(A, B)$ controllable and $(A, C)$ observable, the solution to (7) goes to a steady state as time goes to infinity, i.e., $P_{t|t} \to P_{\infty}$ as $t \to \infty$. We refer to [16] for detailed proof.

The above two properties will be helpful to compute bounds for the estimation error and subsequently the control error while planning the optimal control actions.

Note that the time indices used in Section III-A and Section III-B do not need to match in practice. For example, the state estimation can start running before control is applied, in which case the initial state in control could be the steady state in state estimation.

III. BOUNDING ESTIMATION AND CONTROL ERRORS

In this section, we extend the set-membership state estimation algorithm (Section II-B) to compute bounds for the estimation error over the prediction horizon and adopt the state decomposition method to compute bounds for the control error.

A. Bounding the Estimation Error

We present the following proposition bounding the estimation error $\varepsilon_t := \hat{x}_t - x_t$ without prior knowledge of the subsequent system outputs.

**Proposition III.1.** Consider the set-membership state estimation (6-9) associated with the system (7). At time $t$, it is guaranteed that the estimation error in the next $k$ steps $\varepsilon_{t+k} \in E_{t+k|t}$ :\n
$$\{ \varepsilon \mid \varepsilon^T P_{t+k|t}^{-1} \varepsilon \leq 1 - (1 - \beta)^k(1 - \rho)^k \delta_t^2 \}$$

for all $t \in \mathbb{N}$, all $k \in \mathbb{N}$, and all admissible disturbance sequences $w$ and $v$, where $E_{t+k|t}$ is the bounding ellipsoid of the estimation error at time $t + k$ based on the information available at time $t$.

**Proof.** Given the information available at time $t$, we first prove by induction that $\delta_{t+k}^2 \geq (1 - \beta)^k(1 - \rho)^k \delta^2_t$ for all $k \in \mathbb{N}$. Clearly, $k = 0$ satisfies the inequality. Assume that $\delta_{t+k}^2 \geq (1 - \beta)^k(1 - \rho)^k \delta^2_t$. From (9), we then have:

$$\delta_{t+k+1}^2 = (1 - \beta)(1 - \rho)\delta_{t+k}^2 + (y_{t+k+1} - C(\hat{A}\hat{x}_{t+k} + Bu_{t+k})) [(1 - \rho)\delta_{t+k}^2 C^T + \rho^{-1} R^{-1}]^{-1}(y_{t+k+1} - C(\hat{A}\hat{x}_{t+k} + Bu_{t+k})) \geq (1 - \beta)(1 - \rho)\delta_{t+k}^2 \geq (1 - \beta)^{k+1}(1 - \rho)^k \delta^2_t,$$

by the fact that $(1 - \rho)^{-1} C P_{t+k+1|t+k} C^T + \rho^{-1} R^{-1}$ is positive definite and that $0 < \beta < 1$, $0 < \rho < 1$. Thus, we conclude that $\delta_{t+k}^2 \geq (1 - \beta)^k(1 - \rho)^k \delta^2_t$ for all $k \in \mathbb{N}$.

From (6), for all $x_{t+k} \in X_{t+k|t+k}$, we have:

$$(x_{t+k} - \hat{x}_{t+k})^T P_{t+k|t+k}^{-1} (x_{t+k} - \hat{x}_{t+k}) \leq 1 - \delta_{t+k}^2 \leq 1 - (1 - \beta)^k(1 - \rho)^k \delta^2_t.$$

By definition of the estimation error $\varepsilon_t$, we then have $\varepsilon_{t+k} \in E_{t+k|t}$, which completes the proof.

Note that the bounding set $E_{t+k|t}$ of the estimation error $\varepsilon_{t+k}$ can be precomputed at time $t$ for all $t \in \mathbb{N}$ and all $k \in \mathbb{N}$, regardless of the actual subsequent disturbances and control actions in the next $k$ steps. The precomputation of the bounds for the estimation error enables us to robustly plan the control actions over the prediction horizon. We now establish the time-invariant bounding set $E_{\infty}$ of the estimation error in steady state by the following corollary, which will be used to bound the control error (Section III-B).

**Corollary III.1.** In steady state, it is guaranteed that the estimation error $\varepsilon_t \in E_{\infty} := \{ \varepsilon \mid \varepsilon^T P_{\infty}^{-1} \varepsilon \leq 1 \}$, where $P_{\infty}$ is the steady-state solution of the equation (7).

**Proof.** Similar to the proof of Proposition III.1 from the equation (6) and the definition of the estimation error, we have $\varepsilon_{t+k}^T P_{t+k|t}^{-1} \varepsilon_{t+k} \leq 1 - \delta_t^2$ for all $t \in \mathbb{N}$. In steady state, we have $\varepsilon_{t+k}^T P_{\infty}^{-1} \varepsilon_{t+k} \leq 1 - \delta_t^2 \leq 1$ by the fact that $\delta^2_t \geq 0$, $\forall t \in \mathbb{N}$. ■

Note that the time-invariant bounding ellipsoid $E_{\infty}$ is also precomputable. We now present our method of computing bounds for the control error.

B. Bounding the Control Error

We consider a control law combining a feed-forward component, given by the tube-based model predictive controller, and a feedback component [6], [9]:

$$u_t(\hat{x}_t) = e_t + K(\hat{x}_t - s_t), \quad (10)$$

where $K \in \mathbb{R}^{m \times n}$ is a fixed feedback matrix satisfying $\rho(A + BK) < 1$. With this control law, we can decouple

\footnote{Detailed proof is provided in Appendix A}
the system dynamics (1) into a nominal state $s_t$ and an error state $e_t := x_t - s_t$. Note that $\dot{x}_t - s_t = e_t + \varepsilon_t$. Thus, the nominal and error state evolve, respectively, according to:

$$
s_{t+1} = As_t + Bc_t, \quad s_0 = \hat{x}_0 \tag{11}
$$

$$
e_{t+1} = AKe_t + Dw_t + BK\varepsilon_t, \quad e_0 = -\varepsilon_0, \tag{12}
$$

where $AK := A + BK$. Above, the nominal state $s_t$ is deterministic, and the disturbances only affect the error state $e_t$. Note that by definition $x_t = s_t + e_t$. Providing we can bound $e_t$, we can plan the nominal control sequence $c$ so that the actual state and control satisfy the original constraint (9).

As a comparison of the methods for bounding $x_t$, the previous work in robust feedback model predictive control [6], [9] first bounds the state estimate from the nominal state, and then bound the actual state from the state estimate. Our formulation does not suffer from the over-approximation introduced by such hierarchical structure in practice, thus leading to a less conservative robust controller.

By substituting (11) and (12) into (2), the original constraint on the actual state can be reformulated in terms of the nominal and error state:

$$
FS_t + GC_t + (F + GK)e_t + GK\varepsilon_t \leq f. \tag{13}
$$

The estimation error $\varepsilon_t$ has been bounded in Section III-A. Before computing bounds for the other uncertain state $e_t$, we make the following steady state assumption:

**Assumption III.1.** The recursive state estimation (6)–(9) achieves its steady state before control is applied.

The above assumption is a common setting in robust model predictive control [6]–[8], [24]. In practice, the steady-state operation of the estimation module before applying the actual control is also desirable and achievable. For example, motion capture systems or onboard sensors on robots are usually turned on before the control module for a long enough time to reach a better estimation accuracy. Hereafter, the time indices of variables are defined with respect to the start of control, and that the matrix $P_{t+k|t+k}$ in Proposition III.1 shall be replaced with $P_\infty$.

We recall the following standard definition of (robust) positively invariant set [25], [26]:

**Definition III.1.** A set $\Omega \subset \mathbb{R}^n$ is said positively invariant for the system $x_{t+1} = f(x_t)$ if $f(x_t) \in \Omega$ for all $x_t \in \Omega$. A set $\Omega \subset \mathbb{R}^n$ is said robust positively invariant for the system $x_{t+1} = f(x_t, w_t)$ and the constraint set $\mathcal{W}$ if $f(x_t, w_t) \in \Omega$ for all $x_t \in \Omega$ and all $w_t \in \mathcal{W}$.

The error state dynamics (12) can be rewritten in the form:

$$
e_{t+1} = AKe_t + \vartheta_t, \quad \vartheta_t := Dw_t + BK\varepsilon_t, \tag{14}
$$

where $\vartheta_t$ lies in the $C$ set $\Theta$ defined by:

$$
\Theta := D\mathcal{W} \oplus BK\mathcal{E}_\infty \tag{15}
$$

by the fact that $\varepsilon_t \in \mathcal{E}_\infty$ for all $t \in \mathbb{N}$ from Assumption III.1 and Corollary III.1. Since $\rho(AK) < 1$, there exists a $C$ set $\Theta$ that is finite time computable and robust positively invariant for the system (14) and the constraint set $\Theta$ [26]. We refer to [27, Chapter 3] for details on computing robust positively invariant sets. The following proposition bounds the error state $e_t$ into the future:

**Proposition III.2.** If the initial system and nominal state, $x_0$ and $s_0$, respectively, satisfy $e_0 = x_0 - s_0 \in \mathbb{S}$, then $e_t \in \mathbb{S}$ for all $t \in \mathbb{N}$, and all admissible disturbance sequences $w$ and $v$.

**Proof.** Follows immediately from the definition of robust positively invariant sets in Definition III.1.

In practice, we choose $s_0 = \hat{x}_0$. Thus, the initial condition of Proposition III.2 is equivalent to $-\varepsilon_0 \in \mathbb{S}$. Note that $\varepsilon_0 \in \mathbb{E}_{00}$ and that $E_{00}$ is symmetric about the origin. The initial condition is then guaranteed to be satisfied if $E_{00} \subset \mathbb{S}$. In contrast, the initial condition of the estimation error in the previous work [6] is a pure assumption, the satisfaction of which cannot be checked at the time of control.

With the two uncertain states $e_t$ and $\varepsilon_t$ bounded in the constraint (13), we are now ready to design the robust model predictive controller.

IV. ROBUST MPC WITH STATE ESTIMATION

In this section, we present three different robust model predictive controllers with set-membership (SM) state estimation, whose properties may be desirable for different systems. With the previously described propositions on bounds for the uncertain states, we can guarantee the satisfaction of the original constraint (2) by planning the nominal state under tighter constraints. Let $s_{t+k|t}$ be the predicted nominal state and $c_{t+k|t}$ be the optimal nominal control action into the future at time $t$. The three different approaches adopt different choices of $s_{t|t}$ when solving the optimal control problem at time $t$, and the differences are summarized as:

1) Safe SM-MPC: The nominal state $s_{t|t}$ is chosen to be $s_t$, which evolves according to the dynamics (1), i.e., $s_{t+1} = As_t + Bc_{t|t}$, $s_0 = \hat{x}_0$. The Safe SM-MPC guarantees recursive feasibility and stability and generalizes well to high-dimensional systems, but the closed-loop behavior is relatively more conservative than the following two controllers (still less conservative than the previous work [6]).

2) Aggressive SM-MPC: The nominal state $s_{t|t}$ is an optimization variable to be decided. The resulting controller exhibits less conservative behavior than the Safe SM-MPC, but lacks guarantees for feasibility and stability.

3) Switching SM-MPC: This approach switches between the above two controllers. The Switching SM-MPC inherits the recursive feasibility and stability from the Safe SM-MPC and less conservative closed-loop behavior from the Aggressive SM-MPC, but suffers from computational complexity of high-dimensional problems.

Note that the nominal state $s_{t+k|t}$ in all the above three approaches evolves according to the dynamics (11) within the prediction horizon of the optimal control problem at time $t$, i.e., $s_{t+k+1|t} = As_{t+k|t} + Bc_{t+k|t}$ with $s_{t|t}$ as the initial value.
A. Controller Design for Safe SM-MPC

Note that the original constraint \([2]\) is guaranteed to be satisfied if the inequality \([13]\) holds for all admissible error states \(e_t\) and estimation errors \(e_\hat{t}\). As a result, the Safe SM-MPC solves the following finite horizon constrained optimal control problem at each time \(t\):

\[
V_N^s(s_t) = \min_{c_{t:t+N-1}|t}} \sum_{k=0}^{N-1} q(s_{t+k}|t), c_{t+k}|t) + p(s_{t+N}|t)
\]

subject to

\[
s_{t+k+1}|t) = A_{s_{t+k}|t} + B_{c_{t+k}|t} F_{s_{t+k}|t} + G_{c_{t+k}|t} \leq f - (F + GK)e_t + GK e_{\hat{t}},
\]

\[
s_{t+k}|t) = s_{t}, \forall t \in \mathbb{S}, \forall e_{\hat{t}} + k \in \mathbb{E}_{t+k}|t)\]

(16)

where \(c_{t:t+N-1}|t := \{c_{t}|t, c_{t+1}|t, \ldots, c_{t+N-1}|t\}\) is the control sequence, \(s_t\) is the closed-loop nominal state evolving according to \(s_{t+1} = A_s s_t + B_{c_t}|t\) with the initial condition \(s_0 = x_0\). \(N\) is the prediction horizon, \(p(\cdot)\) is the terminal cost function, \(S_f\) is the terminal constraint set, \(S\) is the robust positively invariant set for the error state, and \(E_{t+k}|t\) is the bounding set for the estimation error at time \(t + k\) given the information available at time \(t\). Note that the computation of \(\max_{e_{\hat{t}} + k \in \mathbb{E}} (F + GK)e_{\hat{t}} + GK e_{\hat{t}}\) does not require the expensive Minkowski sum \(\mathbb{S}\) thus making the algorithm generalize better to high-dimensional systems than the previous work \([6], [9]\). Let \(\bar{f}\) be the tightened constraint considering the error state \(e_t\) and the estimation error \(e_\hat{t}\), i.e., \(\bar{f} := f - \max_{e_{\hat{t}} + k \in \mathbb{E}} (F + GK)e_{\hat{t}} + GK e_{\hat{t}}\). The positive definite terminal cost \(p(\cdot)\) and the terminal constraint set \(S_f\) satisfy the usual stabilizing conditions \([28], [29]\):

**Assumption IV.1.** \(A_K s_{t+N} \in S_f, (F + GK)s_{t+1} \leq f\), \(s_{t} \in S\).

**Assumption IV.2.** \(p(A_K s_{t+N}) - p(s_{t}) \leq -q(s_{t}, K s_{t})\), \(s_{t} \in S\).

The choices of the terminal cost and the terminal constraint set satisfying the above conditions are the same as those in the nominal MPC with the only difference in using the tightened constraint \(\bar{f}\) rather than the original constraint \(f\).

Upon solving (16) at time \(t\), the controller applies

\[
u_t(\hat{x}_t) = c_t^\star|t) + K(\hat{x}_t - s_t)
\]

(17)

to the system, where \(\hat{x}_t\) is the state estimate given by \([9]\), and \(c_t^\star|t)\) is the first element in the optimal nominal control sequence from (16). The controller (17) along with the state estimation \(\hat{x}_t\) forms the receding horizon control strategy for the system.

We summarize the control procedure in Algorithm 1. Note that step 5 requires only one optimization for \(k = 0\), the results for the other \(k\) values can be scaled from the case of \(k = 0\) based on the value of \(\beta\) and \(\rho\). In addition, step 5 can be precomputed before control if we relax the bounding set \(E_{t+k}|t)\) to \(E_{\infty}\), which leads to computing \(\max_{e_{\hat{t}} \in \mathbb{E}} (F + GK)e_{\hat{t}}\).

Algorithm 1 Safe SM-MPC

**Require:** State estimation parameters \(\beta, \rho \in (0,1)\), Stabilizing feedback matrix \(K\), Prediction horizon \(N\)

1: **Initialize** the nominal state \(s_0 \leftarrow \hat{x}_0\)
2: Compute \(\max_{e_{\hat{t}} \in \mathbb{E}} (F + GK)e_{\hat{t}}\) and \(\max_{e_{\hat{t}} \in \mathbb{E}} (F + GK)e_{\hat{t}}\)
3: Compute the constraint set \(S_f\) and the terminal cost \(p(\cdot)\)
4: **repeat** at each time \(t = 0, 1, 2\),
5: Compute \(\max_{e_{\hat{t}} \in \mathbb{E}} (F + GK)e_{\hat{t}}\), \(k = 0, \ldots, N-1\)
6: Obtain the state estimate \(\hat{x}_t\) from (8)
7: Solve problem (16) to obtain \(c_t^\star|t)\)
8: Apply \(u_t = c_t^\star|t) + K(\hat{x}_t - s_t)\) to the system
9: \(s_{t+1} \leftarrow A_{s_{t}} + B_{c_t}|t)\)
10: **until** convergence

We now establish the recursive feasibility and stability of Safe SM-MPC:

**Theorem IV.1.** Consider system (7) controlled by the Safe SM-MPC controller (16) and (17). Suppose that problem (16) is feasible at time \(t = 0\), then the Safe SM-MPC controller (16) and (17) is feasible for all \(t \geq 0\). Moreover, the set \(S\) is robustly asymptotically stable for the closed-loop system (1), (16) and (17).

**Proof.** The proof follows from standard MPC arguments. Assume that at time \(t\) the optimal control problem (16) is feasible and let \(\{s_{t+1}^{\star}, s_{t+2}^{\star}, \ldots, s_{t+N-1}^{\star}, s_{t+N}^{\star}\}\) and \(\{c_{t+1}^{\star}, c_{t+2}^{\star}, \ldots, c_{t+N-1}^{\star}, K s_{t+N}^{\star}\}\) be the optimal nominal trajectory and control sequence. At time \(t + 1\), we have:

\[
s_{t+1} = A_{s_{t}} + B_{c_t}|t) + A_{s_{t}} + B_{c_t}|t) = s_{t+1}^{\star} + b_{n+1}^{\star}.
\]

In addition, by the fact that \(\delta_{t+1}^{2} \geq (1 - \beta)(1 - \rho)\delta_{t}^{2}\) and Proposition III.1, we know that

\[
E_{t+1+k}|t+1 \subset E_{t+1+k}|t \subset E_{\infty}, \forall k \in \mathbb{N}.
\]

(18)

Thus, the nominal state trajectory

\[
\{s_{t+1}^{\star}, s_{t+2}^{\star}, \ldots, s_{t+N-1}^{\star}, s_{t+N}^{\star}\} K s_{t+N}^{\star}\}
\]

(19)

and the related control sequence

\[
\{c_{t+1}^{\star}, c_{t+2}^{\star}, \ldots, c_{t+N-1}^{\star}, K s_{t+N}^{\star}\}
\]

(20)

is a feasible solution to the problem (16) at time \(t + 1\) by Assumption IV.1 and the fact that \(s_{t+N}^{\star} \in S_f\). Therefore, we conclude by induction that the Safe SM-MPC controller (16) and (17) is feasible \(\forall t \geq 0\).

We next show that \(V_N^s(\cdot)\) is decreasing along the trajectory. Note that (19) and (20) is a suboptimal solution to the problem (16) at time \(t + 1\), therefore we have:

\[
V_N^s(s_{t+1}) \leq \sum_{k=1}^{N-1} q(s_{t+k}|t), c_{t+k}|t) + q(s_{t+k}|t), c_{t+k}|t) + p(A_{s_{t+k}|t}) K s_{t+N}^{\star}\}
\]

(21)

\[
V_N^s(s_{t}) - q(s_{t}, c_{t}|t) + p(s_{t}|t) + q(s_{t+k}|t), c_{t+k}|t) + p(A_{s_{t+k}|t}) K s_{t+N}^{\star}\}
\]

(22)

2Detailed explanation is provided in Appendix B.
Therefore we have:
\[ V_N^*(s_{t+1}) - V_N^*(s_t) \leq -q(s_t^*, c_t^*) \]
by Assumption 2 and the fact that \( s_{t+N|t}^* \in S_f \). Note that the stage cost \( q(\cdot) \) and the terminal cost \( p(\cdot) \) are both positive definite. Therefore, the optimal cost \( V_N^*(\cdot) \) is a decreasing Lyapunov function along the closed-loop trajectory, which implies that the nominal state \( s_t \) converges to the origin as \( t \to \infty \). Moreover, we know that \( x_t \in s_t \oplus S \) by the definition of the error state. Thus, we conclude that the set \( S \) is asymptotically stable for the closed-loop system. 

**B. Controller Design for Aggressive SM-MPC**

The Aggressive SM-MPC is inspired by some existing methods \([6], [9], [10]\), where the nominal state \( s_{t|t} \) is also a decision variable as the nominal control signal at time \( t \). The finite horizon constrained optimal control problem solved online is:

\[
V_N^*(\hat{x}_t) = \min_{s_{t|t}, c_{t+k|t}} \sum_{k=0}^{N-1} q(s_{t+k|t}, c_{t+k|t}) + p(s_{t+N|t})
\]

s.t. \( s_{t+k|t} = A s_{t+k|t} + B c_{t+k|t} \)

\[
F \hat{x}_t + G c_{t+k|t} \leq f - (F + G K) c_{t+k} - G K e_{t+k}, \quad \forall e_{t+k} \in S, \quad \forall \hat{x}_{t+k} \in \hat{S}_{t+k|t},
\]

\[
\hat{x}_t - s_{t|t} \in S_0 \ominus \hat{S}_{t|t}, \quad s_{t+N|t} \in S_f,
\]

\[
k = 0, 1, \ldots, N-1.
\]

(21)

The only difference between the problem (16) and (21) is the extra condition on \( s_{t|t} \) making the stability property not guaranteed. Moreover, the set \( S \) is robustly asymptotically stable for the closed-loop system.

**Algorithm 2 Switching SM-MPC**

**Require:** \( \beta, \rho \in (0, 1), K, N \)

1. **Initialize:** \( s_0 \leftarrow \hat{x}_0 \), \( V_{\text{prev}} \leftarrow +\infty \)
2. Compute the tightened constraint \( \bar{f} \)
3. Compute the constraint set \( S_f \) and the terminal cost \( p(\cdot) \)
4. **repeat** at each time \( t = 0, 1, 2, \ldots \)
5. Compute \( \max_{\varepsilon_{t+k} \in \hat{S}_{t+k|t}} G K e_{t+k}, k = 0, \ldots, N-1 \)
6. Obtain the state estimate \( \hat{x}_t \) from (8)
7. **try**
8. Solve problem (21) to obtain \( s_{t|t}^*, c_{t|t}^* \), and \( V_N^* \)
9. if \( V_N^* < V_{\text{prev}} \) then \( V_{\text{prev}} \leftarrow V_N, s_t \leftarrow s_t^* \)
10. **else** raise Exception
11. **end if**
12. except
13. Solve problem (16) to obtain \( c_{t|t}^* \)
14. **end try**
15. Apply \( u_t = c_{t|t}^* + K(\hat{x}_t - s_t) \) to the system
16. \( s_{t+1} \leftarrow A s_t + B c_{t|t}^* \)
17. **until** convergence

**Theorem IV.2.** Consider system (7) controlled by the Switching SM-MPC in Algorithm 2. Suppose that either problem (16) or (21) is feasible at time \( t = 0 \), then the Switching SM-MPC controller in Algorithm 2 is feasible for all \( t \geq 0 \). Moreover, the set \( S \) is robustly asymptotically stable for the closed-loop system.

**Proof.** Detailed proof is provided in Appendix C.

**V. SIMULATION EXAMPLES**

We evaluate the proposed controllers on three different applications: double integrator, robotic walker, and quadrotor. A comparison with LQR \([30]\), nominal MPC, and robust output feedback MPC \([6]\) is also provided. The code will be released upon publication. Hereafter, we refer to the classical results shown in \([6]\) as Mayne’s MPC for brevity. In the following subsections, the stage cost and terminal cost are defined by:

\[
q(s, c) := (1/2)[s^T Q s + c^T R c], \quad p(s) := (1/2)s^T P s,
\]

where \( \hat{P} \), \( Q \) and \( \hat{R} \) are positive definite matrices.

**A. Double Integrator**

We use the same example as in Mayne’s MPC paper to compare the performance of our controllers and the baselines. The model is a double integrator of the form:

\[
x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 & 1 \\ \end{bmatrix} u_t + w_t, \quad y_t = \begin{bmatrix} 1 & 1 \end{bmatrix} x_t + v_t,
\]

with additive disturbances \( (w_t, v_t) \in \mathbb{W} \times \mathbb{V} \) where \( \mathbb{W} := \{ w \in \mathbb{R}^2 \mid \|w\|_2 \leq 2 \sqrt{\lambda}, \lambda \in \mathbb{R}_+ \} \) and \( \mathbb{V} := \{ v \in \mathbb{R} \mid |v| \leq \mu, \mu \in \mathbb{R}_+ \} \). The state and control constraints are \( (x_t, u_t) \in \mathbb{X} \times \mathbb{U} \) where \( \mathbb{X} := \{ x \in \mathbb{R}^2 \mid x_1 \in [-50, 3], x_2 \in [-50, 3] \} \) and \( \mathbb{U} := \{ u \in \mathbb{R} \mid |u| \leq 3 \} \). The weighting matrices in the stage cost are \( \hat{Q} = I \) and \( \hat{R} = 0.01 \); the matrix \( \hat{P} \) in the terminal cost is the solution to the algebraic Riccati equation...
of the unconstrained optimal control problem for the nominal system; the feedback control matrix $K$ is the associated LQR controller; the output injection matrix $L$ used in Mayne’s MPC is set to be the same as in the original paper [6]. The terminal constraint set $S_f$ is the maximally positively invariant set [25] for system $s_{t+1} = (A + BK)s$ under the tighter constraint $\bar{f}$. The prediction horizon is $N = 15$. The optimal state estimation parameters $\beta$ and $\rho$ are determined offline by a grid search such that the trace of the matrix $P_\infty$ is minimized.

Figure 2 shows the closed-loop trajectory generated by the Safe SM-MPC, for an initial state $x_0 \in s_0 \oplus S$ where $s_0 = (-2.7, -7.8)^T$ and a random sequence of admissible state and output disturbances with $\lambda = 0.1$ and $\mu = 0.05$.\footnote{The tube shown in Figure 2 is the minimum bounding box for the true set $\{s_t \oplus S\}$, as the exact shape of $S$ cannot be explicitly computed for the ellipsoidal uncertainty sets.}

As shown by the dashed line in Figure 2, MPC can achieve faster convergence rate by making use of the space. In this example, the closer the controller gets to the constraint of $x_2 \leq 3$, the less conservative the closed-loop system is.

For each combination of $\lambda$ and $\mu$, we compare different methods over 100 runs with different realizations of admissible state and output disturbances. The performance is evaluated by the following three metrics:

1) # infeas.: the total number of infeasibility that occurs;
2) # steps: the average number of steps to convergence;
3) min D2C: the average minimum Distance from the closed-loop state to the Constraint of $x_2 \leq 3$.

The last two metrics are only computed over runs without the loss of feasibility. The initial state $x_0 = (-3.1, -8)^T$, and the convergence to the origin is declared once the norm of $x_t$ is less than $\max(\lambda, \mu)$. The common parameters across all methods are set to be the same. Here, LQR fails to satisfy the constraint for every single run, leading to the # infeas. being 100, thus is omitted in the comparison.

Table I and Table II summarize the results. As shown, the Aggressive SM-MPC and Switching SM-MPC generate the least conservative closed-loop behaviors, and the feasibility of the Safe SM-MPC and Switching SM-MPC is preserved over the full range of disturbances. By contrast, the nominal MPC fails due to the ignorance of disturbances in the model and the infeasibility of Mayne’s MPC comes from the conservatively tightened constraints under larger uncertainties. Note that the Safe SM-MPC also exhibits less conservative closed-loop behavior than Mayne’s MPC.

### B. Robotic Walker

We further validate our approach on a robotic walker [31], which can be modeled as a two-wheeled inverted pendulum with state $x = [r, \dot{r}, \theta, \dot{\theta}, \psi, \dot{\psi}]^T \in \mathbb{R}^6$ where $r$ is the linear displacement, $\theta$ is the pitch angle, and $\psi$ is the yaw angle. The inputs to the system, $u = [T_\theta, T_\psi]^T \in \mathbb{R}^2$, are the torques along the lateral and vertical axis respectively. We assume that the matrix $D$ in (1) is an identity matrix and the state disturbance lives in the bounding ellipsoid of the hyperrectangle defined by the points $[\pm 0.01, \pm 0.02, \pm \pi/600, \pm 0.01, \pm \pi/600, \pm 0.02]^T$. The output matrix $C$ is defined as an identity matrix as in [31], and the output disturbance lives in the axis-aligned bound-

---

**TABLE I**: Closed-loop performance of different controllers for double integrator with $\mu = 0.05$

| Method         | $\lambda = 0.05$ | $\lambda = 0.10$ | $\lambda = 0.15$ | $\lambda = 0.20$ |
|----------------|------------------|------------------|------------------|------------------|
|                | # infeas. | # steps | min D2C | # infeas. | # steps | min D2C | # infeas. | # steps | min D2C | # infeas. | # steps | min D2C |
| Nominal MPC    | 72      | 11.50  | 0.006  | 76      | 10.42  | 0.010  | 81      | 10.21  | 0.009  | 81      | 9.74   | 0.097  |
| Mayne’s MPC    | 0       | 16.22  | 0.588  | 0       | 14.19  | 0.882  | 100     | -      | -      | 100     | -      | -      |
| Safe SM-MPC    | 0       | 12.91  | 0.308  | 0       | 12.67  | 0.443  | 0       | 13.60  | 0.567  | 0       | 15.63  | 0.721  |
| Aggressive SM-MPC | 0 | 12.23  | 0.256  | 3       | 11.67  | 0.376  | 13      | 11.84  | 0.490  | 13      | 12.29  | 0.590  |
| Switching SM-MPC | 0   | 12.06  | 0.250  | 0       | 11.67  | 0.378  | 0       | 11.69  | 0.471  | 0       | 12.35  | 0.599  |

**TABLE II**: Closed-loop performance of different controllers for double integrator with $\lambda = 0.10$

| Method         | $\mu = 0.05$ | $\mu = 0.10$ | $\mu = 0.15$ | $\mu = 0.20$ |
|----------------|----------------|----------------|----------------|----------------|
|                | # infeas. | # steps | min D2C | # infeas. | # steps | min D2C | # infeas. | # steps | min D2C | # infeas. | # steps | min D2C |
| Nominal MPC    | 76      | 10.42  | 0.378  | 100     | -      | -      | 100     | -      | -      | 100     | -      | -      |
| Mayne’s MPC    | 0       | 14.19  | 0.882  | 100     | -      | -      | 100     | -      | -      | 100     | -      | -      |
| Safe SM-MPC    | 0       | 12.67  | 0.443  | 0       | 14.51  | 0.580  | 0       | 14.77  | 0.640  | 0       | 16.17  | 0.703  |
| Aggressive SM-MPC | 3   | 11.67  | 0.376  | 1       | 12.69  | 0.499  | 3       | 12.76  | 0.557  | 6       | 13.28  | 0.667  |
| Switching SM-MPC | 0   | 11.67  | 0.378  | 0       | 12.80  | 0.472  | 0       | 12.62  | 0.546  | 0       | 13.34  | 0.643  |
ing ellipsoid of the hyperrectangle formed by the points 
\[ [\pm 0.02, \pm 0.02, \pm \pi/600, \pm 0.01, \pm \pi/360, \pm 0.02]^T. \] The control task is to regulate the robot from the initial state \( x_0 = [5, 0, 0, 0, 0, 0, 0]^T \) to the origin while satisfying the constraint of \( (x,u) \in X \times U \) where \( X := \{ x \in \mathbb{R}^6 \mid |\dot{r}| \leq 4, |\theta| \leq \pi/9, |\psi| \leq 1 \} \) and \( U := \{ u \in \mathbb{R}^7 \mid T_0 \leq 8, |T_\psi| \leq 0.4 \} \).

The robotic walker model can be decoupled into two separate subsystems: (i) a pendulum system describing the linear displacement and rotation about the lateral axis and (ii) a rotation system describing the rotation about the vertical axis [32]. We apply controllers with identical common parameters to the two subsystems independently.

Figure 3 summarizes the results. As shown, the Safe SM-MPC robustly drives the system towards the origin under disturbances. LQR fails due to unmodeled constraints while the nominal MPC fails due to unmodeled disturbances. Mayne’s MPC, Aggressive SM-MPC and Switching SM-MPC fail to solve the optimal control problem in this example due to the expensive Minkowski sum in the computation of the robust positively invariant set.

### C. Quadrotor Dynamics

Finally to validate our approach scales well to high-dimensional systems, we apply the proposed controller to a quadrotor model with state \( x = [p, \dot{p}, R, \dot{R}, \Omega]^T \in \mathbb{R}^{12} \) where \( p = [p_x, p_y, p_z] \) is the position, \( R = [\phi, \theta, \psi] \) is the orientation, \( \dot{v} = [v_x, v_y, v_z] \) is the translational velocity, and \( \Omega = [\omega_x, \omega_y, \omega_z] \) is the angular velocity [33], [34]. The inputs to the model, \( u = [T, M_x, M_y, M_z]^T \in \mathbb{R}^4 \), are the total thrust and moment respectively. Detailed description of disturbances, state and input constraint, and the linearized model around the equilibrium state are provided in Appendix [D]. The regulation task is to drive the system from the initial state of \( p = [5, 0, 0]^T, R = [0, 0, 0]^T, v = [0, 0, 0]^T, \ \Omega = [0, 0, 0]^T \) to the origin.

Figure 4 shows part of the results. As in the robotic walker example, the Safe SM-MPC is the only controller that robustly drives the system to the origin while maintaining recursive feasibility and stability of the closed-loop system.

### VI. Conclusion

In conclusion, we presented a set of novel approaches for robust output feedback model predictive control of constrained linear systems in the presence of bounded state and output disturbances. Considering the nature of real-world disturbances and the requirement for low computational complexity, we adopt ellipsoidal set-membership state estimation which describes the uncertainty set of disturbances as ellipsoids rather than boxes or polytopes. The proposed controllers were evaluated in simulation on three different systems and were shown to improve closed-loop performance beyond baseline methods while ensuring robust constraint satisfaction, recursive feasibility and stability. Due to the construction of feasible nominal states, the Safe SM-MPC does not require explicit computation of robust positively invariant set, thus scaling well to high-dimensional systems.
**APPENDIX**

**A. Lemma for Proposition IV.2**

**Lemma.** The matrix \([(1 - \rho)^{-1} CP_{t+k+1|t+k} + \rho^{-1} R]^{-1}\) is positive definite.

**Proof.** Note that \(0 < \beta < 1\) and that the matrix \(Q \in \mathbb{R}^{n\times n}\) is positive definite [16]. From (7), for all \(t \in \mathbb{N}\) and all \(z \in \mathbb{R}^{n}\), we have:

\[
z^T P_{t+1|t} z = (1 - \beta)^{-1} (z^T A P_{t|t} (A^T z) + \beta^{-1} (z^T D) Q (D^T z)) \geq 0.
\]

Thus for all \(t \in \mathbb{N}\), all \(k \in \mathbb{N}\), and all \(z \in \mathbb{R}^{n}\), we have:

\[
(1 - \beta)^{-1} (z^T C) P_{t+k+1|t+k} (C^T z) + \rho^{-1} z^T R z > 0
\]

by the fact that \(R\) is positive definite and that \(0 < \beta < 1\). Therefore the matrix \([(1 - \rho)^{-1} CP_{t+k+1|t+k} + \rho^{-1} R]^{-1}\) is positive definite. It is known that the inverse of a positive definite matrix is also positive definite. Thus we conclude that the matrix \([(1 - \rho)^{-1} CP_{t+k+1|t+k} + \rho^{-1} R]^{-1}\) is positive definite, which gives the lemma. \(\blacksquare\)

**B. Note for the Computation of Constraint Tightening**

In this subsection, we provide implementation details on computing \(\max_{e \in \mathbb{S}} (F + G\varepsilon)\). The proposed algorithm is an extension of the method in [27, Chapter 3] to ellipsoidal uncertainty sets.

Let a pair of positive integers \(r_1, r_2\) and a pair of scalars \(0 < \alpha_1, \alpha_2 < 1\) satisfy:

\[
A^r_1 D \mathbb{W} \subset \alpha_1 D \mathbb{W}, \quad A^r_2 BK \mathbb{E} \subset \alpha_2 BK \mathbb{E} \subset \mathbb{R}^{n\times n}.
\]

It can be shown that the minimal robust positively invariant set approximation for \(e_1\) can be determined as [27, Chapter 3]:

\[
\mathbb{S} = \frac{1}{1 - \alpha_1} \bigoplus_{j=0}^{r_1-1} A^j_1 D \mathbb{W} \oplus \frac{1}{1 - \alpha_2} \bigoplus_{j=0}^{r_2-1} A^j_2 BK \mathbb{E}.
\]

Therefore, we have:

\[
\max_{e \in \mathbb{S}} (F + G\varepsilon) = \frac{1}{1 - \alpha_1} \max_{j=0}^{r_1-1} \max_{w_j \in \mathbb{W}} (F + G\varepsilon) A^j_1 D w_j
\]

\[
+ \frac{1}{1 - \alpha_2} \max_{j=0}^{r_2-1} \max_{e_j \in \mathbb{E} \subset \mathbb{R}^{n\times n}} (F + G\varepsilon) A^j_2 BK e_j,
\]

which is a convex optimization problem. In cases where \(D\) is full row rank, we know that \(A^r_1 D \mathbb{W} \subset \alpha_1 D \mathbb{W}\) if and only if \(D^T A^r_1 D w\) lies in the scaled ellipsoid \(\alpha_1 \mathbb{W}\). The same reasoning applies to cases where \(\theta := BK\) is full row rank. Therefore, the condition (22) is equivalent to:

\[
\max_{w \in \mathbb{W}} (D^T A^r_1 D w)^T Q^{-1} (D^T A^r_1 D w) \leq \alpha_1^2,
\]

\[
\max_{e \in \mathbb{E}} (\theta^T A^r_2 \theta e)^T P^{-1} (\theta^T A^r_2 \theta e) \leq \alpha_2^2.
\]

Given \(r_1, r_2\), the optimal value of \(\alpha_1, \alpha_2\) can be computed accordingly. More accurate approximation of the robust positively invariant set can be achieved with larger \(r_1, r_2\) and smaller \(\alpha_1, \alpha_2\). Although (23) and (24) are non-convex problems, the optimizer [35] always return the optimal solution efficiently in our experiments with non-zero initializations.

In cases where either \(D\) or \(\theta\) are rank-deficient, one can either (i) find a full-dimensional bounding ellipsoid of \(D \mathbb{W}\) or \(\theta \mathbb{E}\) and follow the proposed algorithm, or (ii) find a full-dimensional bounding polytope of \(D \mathbb{W}\) or \(\theta \mathbb{E}\) and follow the standard procedure in [27, Chapter 3]. We note that the computation of \(\max_{e \in \mathbb{S}} (F + G\varepsilon)\) described in this subsection does not require the expensive Minkowski sum, making the proposed algorithm generalize well to high-dimensional systems.

**C. Proof of Theorem IV.2**

**Proof.** Note that if Algorithm 2 solves problem (16) at time \(t\), it is guaranteed that there exists a feasible solution to problem (16) at time \(t + 1\) by Theorem IV.1. Therefore, we only have to show that the solution to problem (21) at time \(t\) provides a solution to problem (16) at time \(t + 1\). Assume that at time \(t\) the problem (21) is feasible and let \(\{s^*_t, s^*_t, \ldots, s^*_t, A^* K s^*_t\}\) and \(\{c^*_t, c^*_t, \ldots, c^*_t, K s^*_t\}\) be the optimal nominal trajectory and control sequence. With the same reasoning as in the proof of Theorem IV.1 we can show that the nominal state trajectory:

\[
\{s^*_{t+1|t}, s^*_{t+1|t}, \ldots, s^*_{t+N-1|t}, s^*_{t+N|t}, A^* K s^*_{t+N|t}\}
\]

and the related control sequence:

\[
\{c^*_{t+1|t}, c^*_{t+1|t}, \ldots, c^*_{t+N-1|t}, K s^*_{t+N|t}\}
\]

is a feasible solution for problem (16) at time \(t + 1\), which concludes the proof for recursive feasibility.

To prove stability, we only have to show that \(V^*_N\) is decreasing in the case where problem (21) is solved at time \(t\) and problem (16) is solved at time \(t + 1\). Again, with the same reasoning as in the proof of Theorem IV.1 we can show that \(V^*_N(s^*_{t+1}) - V^*_N(s^*_{t}) \leq -q(s^*_t, c^*_t), \) where \(V^*_N(s^*_{t+1})\) comes from problem (16) and \(V^*_N(s^*_{t})\) from problem (21).

The remaining proof is identical to that in Theorem IV.1 \(\blacksquare\)

**D. Details on Quadrotor Dynamics Example**

The quadrotor dynamics with 12 states, 4 inputs can be described as:

\[
\dot{\mathbf{p}} = \mathbf{v}, \quad \dot{\mathbf{v}} = m g e_3 - T \mathbf{R} e_3, \quad \dot{\mathbf{R}} = R \mathbf{\Omega}, \quad \dot{\mathbf{\Omega}} = M - \mathbf{\Omega} \times \mathbf{\Omega},
\]

where \(\mathbb{R}^3 \rightarrow SO(3)\) is the hat operator, \(m \in \mathbb{R}\) is the mass of the quadrotor, \(g \in \mathbb{R}\) is the gravitational force, and \(J := \text{diag}(J_x, J_y, J_z)\) is the moment of inertia matrix. The inertial property of the quadrotor model is adopted from [36, Chapter 16]. The model used for control is the linearized model of (25) around the equilibrium state where
To apply MPC controllers, a time discretization is used with $dt = 0.2s$. We assume that the matrix $D$ in (1) is an identity matrix and the state disturbance $w$ in (1) is an identity matrix and the state disturbance $w$ in (1) is a zero matrix. The output matrix $C$ is defined as an identity matrix as in (34), and the output disturbance lives in the axis-aligned minimum bounding ellipsoid of the hyperrectangle defined by the points $[-0.01, 0.01, \pm \pi/600, \pm \pi/600, \pm \pi/600, \pm 0.02, \pm 0.02, \pm 0.01, \pm 0.01]^{T}$. The output matrix $C$ is defined as an identity matrix as in (34), and the output disturbance lives in the axis-aligned minimum bounding ellipsoid of the hyperrectangle defined by the points $[-0.01, 0.01, \pm \pi/600, \pm \pi/600, \pm \pi/600, \pm 0.02, \pm 0.02, \pm 0.01, \pm 0.01]^{T}$.

The control task is to regulate the system from the initial state towards the origin while satisfying the constraint of $(x, u) \in X \times U$ where $X := \{x \in \mathbb{R}^{12} | |\phi| \leq \pi/9, |\theta| \leq \pi/9, |\psi| \leq \pi/9\}$ and $U := \{u \in \mathbb{R}^{4} | |T| \leq 5\}$.

(26)

REFERENCES

[1] A. Bemporad and M. Morari, "Robust model predictive control: A survey" in Robustness in Identification and Control. Springer, 1999, pp. 207–226.

[2] Q. Qiu, F. Yang, Y. Zhu, and E. Moussavinejad, "Output feedback model predictive control based on set-membership state estimation," IET Control Theory & Applications, vol. 14, no. 4, pp. 588–567, 2019.

[3] A. R. de Souza, D. Efimov, T. Raissi, and X. Ping, "Robust output feedback MPC: An interval-observer approach," in IEEE Conference on Decision and Control (CDC), 2020, pp. 2529–2534.

[4] A. N. Atassi and H. K. Khalil, "A separation principle for the stabilization of a class of nonlinear systems," IEEE Transactions on Automatic Control, vol. 44, no. 9, pp. 1672–1679, 2009.

[5] A. Teel and L. Praly, "Tools for semiglobal stabilization by partial state and output feedback," SIAM Journal on Control and Optimization, vol. 33, no. 5, pp. 1443–1488, 1995.

[6] D. Q. Mayne, S. V. Raković, R. Findeisen, and F. Allgöwer, "Robust output feedback model predictive control of constrained linear systems," Automatica, vol. 42, no. 1, pp. 1211–1222, 2006.

[7] A. Bemporad and A. Garulli, "Output-feedback predictive control of constrained linear systems via set-membership state estimation," International Journal of Control, vol. 73, no. 8, pp. 655–665, 2000.

[8] L. Chisci and G. Zappa, "Feasibility in predictive control of constrained linear systems: the output feedback case," International Journal of Robust and Nonlinear Control, vol. 12, no. 5, pp. 465–487, 2002.

[9] D. Q. Mayne, S. Raković, R. Findeisen, and F. Allgöwer, "Robust output feedback model predictive control of constrained linear systems: Time varying case," Automatica, vol. 45, no. 9, pp. 2082–2097, 2009.

[10] V. T. H. Le, C. Stoica, D. Dumur, T. Alamo, and E. F. Camacho, "Robust tube-based constrained predictive control via zonotopic set-membership estimation" in IEEE Conference on Decision and Control and European Control Conference, 2011, pp. 4580–4585.

[11] "Indoor localization ros wiki." [http://wiki.ros.org/indoor_localization], accessed: 2021-09-20.

[12] P. Du, Z. Huang, T. Liu, T. Ji, K. Xu, Q. Gao, H. Sibai, K. Driggs-Campbell, and S. Mitra, "Online monitoring for safe pedestrian-vehicle interactions," in IEEE International Conference on Intelligent Transportation Systems (ITSC), 2020, pp. 1–8.

[13] U. Rosolia, X. Zhang, and F. Borrelli, "Robust learning model predictive control for iterative tasks: Learning from experience," in IEEE Conference on Decision and Control (CDC), 2017, pp. 1157–1162.

[14] Y. Kim, X. Zhang, J. Guanetti, and F. Borrelli, "Robust model predictive control with adjacent uncertainty sets," IEEE Conference on Decision and Control (CDC), 2018, pp. 5176–5181.

[15] F. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," IEEE Transactions on Automatic Control, vol. 13, no. 1, pp. 22–28, 1968.

[16] D. Bertsekas and I. Rhodes, "Recursive state estimation for a set-membership description of uncertainty," IEEE Transactions on Automatic Control, vol. 16, no. 2, pp. 117–128, 1971.

[17] L. Chisci, A. Garulli, and G. Zappa, "Recursive state bounding by parallelotopes," Automatica, vol. 32, no. 7, pp. 1049–1055, 1996.

[18] A. Vicino and G. Zappa, "Sequential approximation of feasible parameter sets for identification with set membership uncertainty," IEEE Transactions on Automatic Control, vol. 41, no. 6, pp. 774–785, 1996.

[19] A. Delgado, C. Kambhampati, and K. Warwick, "Dynamic recurrent neural network for system identification and control," IEEE Proceedings-Control Theory and Applications, vol. 142, no. 4, pp. 307–314, 1995.

[20] J. Wu, A. Isidori, R. Lu, and H. K. Khalil, "Performance recovery of dynamic feedback-linearization methods for multivariable nonlinear systems," IEEE Transactions on Automatic Control, vol. 65, no. 4, pp. 1365–1380, 2019.

[21] J. de Jesús Rubio, "Robust feedback linearization for nonlinear processes control" ISA Transactions, vol. 74, pp. 135–164, 2018.

[22] M. Buyanurah, X. Zhang, U. Rosolia, and F. Borrelli, "Adaptive MPC for iterative tasks," in IEEE Conference on Decision and Control (CDC), 2018, pp. 6322–6327.

[23] D. P. Bertsekas, "Control of uncertain systems with a set-membership description of the uncertainty," Ph.D. dissertation, Massachusetts Institute of Technology, 1971.

[24] S. Raković, "Robust control of constrained discrete time systems: Characterization and implementation," Ph.D. dissertation, University of London, 2005.

[25] F. Blanchini, "Set invariance in control," Automatica, vol. 35, no. 11, pp. 1747–1767, 1999.

[26] S. V. Rakovic, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne, "Parameter approximations of the minimal robust positively invariant set," IEEE Transactions on Automatic Control, vol. 50, no. 3, pp. 406–410, 2005.

[27] B. Kouvaritakis and M. Cannon, "Model predictive control: Classical, robust and stochastic," Springer, 2015.

[28] F. Borrelli, A. Bemporad, and M. Morari, "Predictive control for linear and hybrid systems," Cambridge University Press, 2005.

[29] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, "Constrained model predictive control: Stability and optimality," Automatica, vol. 36, no. 6, pp. 789–814, 2000.

[30] D. Liberzon, "Calculation of variances and control: a concise introduction," Princeton university press, 2015.

[31] A. R. da Silva and F. C. Sup, "A robotic walker based on a two-wheeled inverted pendulum," Journal of Intelligent & Robotic Systems, vol. 86, no. 1, pp. 17–34, 2017.

[32] F. Grasser, A. D’arrigo, S. Colombi, and A. C. Rufer, "JOE: a mobile and stable robotic walker," in IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), 2017, pp. 107–114, 2002.

[33] D. D. Fan, A.-a. Agha-mohammadi, and E. A. Theodorou, "Deep learning tubes for tube MPC," arXiv preprint arXiv:2002.01587, 2020.

[34] P. Liu, J. Geng, Y. Li, Y. E. Bayiz, J. W. Langelaan, and Y. Cao, "Bio-inspired inverted landing strategy in a small aerial vehicle using policy gradient," in IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), 2020, pp. 7772–7777.

[35] J. A. Andersson, J. Gillis, G. Horn, J. B. Rawlings, and M. Diehl, "CasADi: a software framework for nonlinear optimization and optimal control," Mathematical Programming Computation, vol. 11, no. 1, pp. 1–36, 2019.

[36] K. P. Valavanis and G. J. Vachtsevanos, "Handbook of unmanned aerial vehicles," Springer, 2015, vol. 1.

$\mathbb{R}_{e} = [0, 0, 0]^{T}$ and $T_{e} = mg$: