A weighted identity for stochastic partial differential operators and its applications *

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Abstract

In this paper, a pointwise weighted identity for some stochastic partial differential operators (with complex principal parts) is established. This identity presents a unified approach in studying the controllability, observability and inverse problems for some deterministic/stochastic partial differential equations. Based on this identity, one can deduce all the known Carleman estimates and observability results, for some deterministic partial differential equations, stochastic heat equations, stochastic Schrödinger equations and stochastic transport equations. Meanwhile, as its new application, we study an inverse problem for linear stochastic complex Ginzburg-Landau equations.

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1 Introduction

This paper is devoted to a pointwise weighted identity for a class of stochastic partial differential operators. Based on this identity, one can derive global Carleman estimates for deterministic/stochastic partial differential equations of different type. This idea first came from the Russian literature [20], in which some unique continuation results were established, based on suitable Carleman estimates. Carleman estimates were originally introduced by T. Carleman in 1939. They were energy estimates with exponential weights, and established in [6] to prove a strong unique continuation property for some elliptic equations in dimension two. Up to now, Carleman estimates have become a powerful tool in studying deterministic/stochastic partial differential equations, and the related control and inverse problems. For example, this type of weighted energy estimates was used to study the unique continuation
property of partial differential equations ([13]), the uniqueness and stability of Cauchy problems ([3, 5, 11]), inverse problems ([2, 12, 14]) and the controllability ([7, 9, 24, 27, 28]), respectively.

Roughly speaking, a Carleman estimate for the Laplacian operator is an inequality of the form:

$$|e^{\lambda \phi} f|_{L^2(G)} \leq C |e^{\lambda \phi} \Delta f|_{L^2(G)}, \quad \text{for any } f \in C^2_0(G),$$

where $G$ is a nonempty open subset of $\mathbb{R}^n$ with a smooth boundary, $\phi$ is a suitable weighted function, and $C$ is a constant, independent of the parameter $\lambda$ which may tend to $+\infty$.

In what follows, we give two simple examples to introduce the basic idea of establishing Carleman estimates.

**Example 1. The stability of an ordinary differential system**

Consider the following ordinary differential system:

\[
\begin{aligned}
\dot{x}(t) &= a(t)x(t) \quad t \in (0, T), \\
x(0) &= x_0,
\end{aligned}
\]  

(1.1)

where $x_0 \in \mathbb{R}^n$ and $a(\cdot) \in L^\infty(0, T; \mathbb{R}^{n \times n})$. Then for any $\lambda > 0$, by the first equation of (1.1), we have that

\[
\frac{d}{dt} (e^{-\lambda t} |x(t)|^2) = -\lambda e^{-\lambda t} |x(t)|^2 + 2 e^{-\lambda t} x(t) \cdot \dot{x}(t) \leq [2|a(t)|_{\mathbb{R}^{n \times n}} - \lambda] e^{-\lambda t} |x(t)|^2.  
\]  

(1.2)

Choosing a sufficiently large $\lambda$, one can obtain that

$$|x(t)| \leq e^{\lambda t} |x_0| \leq e^{\lambda T} |x_0|, \quad \forall \; t \in [0, T].$$

The key of this proof for the stability of (1.1) is the following identity:

\[
2 e^{-\lambda t} x(t) \cdot \dot{x}(t) = \frac{d}{dt} (e^{-\lambda t} |x(t)|^2) + \lambda e^{-\lambda t} |x(t)|^2. 
\]  

(1.3)

(1.3) can be viewed as a pointwise weighted identity for the principal operator $\dot{x}(t)$ of (1.1). After multiplied by a multiplier $2 e^{-\lambda t} x(t)$, the principal operator is rewritten as a sum of a “divergence” term $\frac{d}{dt} (e^{-\lambda t} |x(t)|^2)$ and an “energy” term $\lambda e^{-\lambda t} |x(t)|^2$. By choosing a sufficiently large parameter $\lambda$, the undesired lower order term $2|a(t)|_{\mathbb{R}^{n \times n}} e^{-\lambda t} |x(t)|^2$ with respect to $\lambda$ can be absorbed.

**Example 2. A Carleman estimate for first order differential operators**

For any fixed $\gamma_0 \in C(\overline{G})$ and $\gamma \in [C^1(\overline{G})]^n$, consider the following first order differential operator:

$$\mathcal{L}(x, D)u = \gamma \cdot \nabla u + \gamma_0 u, \quad \forall \; x \in \overline{G}.$$  

(1.4)

Set

$$\phi(x) = |x - x_0|^2, \quad \text{for some } x_0 \in \mathbb{R}^n.$$  

(1.5)

Then, we have the following known Carleman estimate for the operator (1.4).
Lemma 1.1 Assume that for $x_0 \in \mathbb{R}^n \setminus \overline{G}$ and a positive constant $c_0$,

$$\gamma(x) \cdot (x - x_0) \leq -c_0, \quad \text{in } \overline{G}. \tag{1.6}$$

Then there exist constants $\lambda^* > 0$ and $C > 0$, so that for any $\lambda \geq \lambda^*$,

$$\lambda \int_G e^{2\lambda \phi} u^2 dx \leq C \int_G e^{2\lambda \phi} |\mathcal{L}(x, D)u|^2 dx, \tag{1.7}$$

for any $u \in C_0^1(G)$.

Proof of Lemma 1.1. For any $\lambda > 0$, put

$$\ell(x) = \lambda \phi(x) \quad \text{and} \quad \theta = e^{\lambda \phi},$$

where $\phi$ is given by (1.5). Then by (1.5), it is easy to check that

$$(\theta^2 u) \gamma \cdot \nabla u = \theta^2 \gamma \cdot \nabla \left( \frac{1}{2} u^2 \right) = \text{div} \left( \frac{1}{2} \theta^2 u^2 \gamma \right) - \theta^2 \left[ \frac{1}{2} \text{div} \gamma + 2\lambda \gamma \cdot (x - x_0) \right] u^2. \tag{1.8}$$

This implies that there exists a constant $C_1 > 0$ such that

$$(\theta^2 u) \mathcal{L}(x, D)u = \text{div} \left( \frac{1}{2} \theta^2 u^2 \gamma \right) - \theta^2 \left[ \frac{1}{2} \text{div} \gamma + 2\lambda \gamma \cdot (x - x_0) - \gamma_0 \right] u^2 \geq \text{div} \left( \frac{1}{2} \theta^2 u^2 \gamma \right) - \theta^2 \left[ 2\lambda \gamma \cdot (x - x_0) + C_1 \right] u^2.$$

By (1.6), integrating the above inequality in $G$ and choosing a sufficiently large $\lambda$, we can get the desired estimate (1.7).

The key of this proof of Lemma 1.1 is the identity (1.8). It can be viewed as a pointwise weighted identity for the principal operator $\gamma \cdot \nabla u$ of $\mathcal{L}(x, D)$. After multiplied by a multiplier $\theta^2 u$, the principal operator is rewritten as a sum of a “divergence” term $\text{div} \left( \frac{1}{2} \theta^2 u^2 \gamma \right)$ and an “energy” term $-\theta^2 \left[ \frac{1}{2} \text{div} \gamma + 2\lambda \gamma \cdot (x - x_0) \right] u^2$. By choosing a sufficiently large parameter $\lambda$, the undesired lower order term $\frac{1}{2} \text{div} \gamma$ with respect to $\lambda$ can be absorbed.

From the above two examples, one can find that the key of proving Carleman estimates is to establish a suitable pointwise weighted identity for principal operators of differential equations. Notice that in [7], a pointwise weighted identity for the following deterministic partial differential operator was established:

$$Lw = (\alpha + i\beta) w_t + \sum_{j,k=1}^n (a^j_k w_{x_j}) x_k,$$

where $\alpha, \beta$ and $a^{j,k}(\cdot)$ ($k, j = 1, 2, \cdots, n$) are suitable real-valued functions, and $i$ is the imaginary unit. This identity presented a unified approach of deducing global Carleman estimates for many deterministic partial differential equations of different type. A natural problem is whether one can get the counterpart for stochastic partial differential equations. As far as we
know, there exist few works on global Carleman estimates for stochastic partial differential equations. We refer to [17, 18, 27] for some known results in this respect. However, there is not any known Carleman estimate for general stochastic partial differential operators with complex principal parts. In this paper, we mainly present a pointwise weighted identity for the following stochastic partial differential operator:

$$Lw = a_0 dw - (a + ib) \sum_{j,k=1}^n (a^{jk}w_{x_j})_{x_k} dt + b_0 \cdot \nabla w dt,$$

where \(a_0, a, b \in \mathbb{R}\) and \(b_0 \in \mathbb{R}^n\). The operator \(L\) may include some deterministic/stochastic partial differential operators of different type. Based on a pointwise weighted identity for this operator, we develop a unified approach of establishing global Carleman estimates for stochastic heat equations, stochastic Schrödinger equations, stochastic transport equations and linear stochastic complex Ginzburg-Landau equations. As applications of this identity, one can also study some inverse problems of these different stochastic partial differential equations.

In the deterministic case, in order to establish a pointwise weighted identity of the operator \(L\) in [7], the operator \((\alpha + i\beta)w_t\) was divided into \(\alpha w_t\) and \(i\beta w_t\). Then the product of them was estimated. However, in the stochastic case, the method does not work. Therefore, in this paper we adopt a new way to prove our pointwise weighted identity for stochastic partial differential operators, different from that in [7].

The rest of this paper is organized as follows. In Section 2, a pointwise weighted identity for some stochastic partial differential operators is established. Section 3 is devoted to its applications in control problems for deterministic/stochastic partial differential equations. As its another application, in Section 4, an inverse problem for linear stochastic complex Ginzburg-Landau equations is studied. Finally, Appendix A is given to prove a Carleman estimate for stochastic heat equations.

## 2 A pointwise weighted identity for stochastic partial differential operators

Let \(T > 0\) and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})\) be a complete filtered probability space, on which a one-dimensional standard Brownian motion \(\{B(t)\}_{t \geq 0}\) is defined such that \(\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration generated by \(B(\cdot)\), augmented by all the \(\mathcal{P}\)-null sets in \(\mathcal{F}\). Also, for any complex number \(c\), we denote by \(\overline{c}\), \(\text{Re}\ c\) and \(\text{Im}\ c\), its complex conjugate, real part and imaginary part, respectively.

For any \(a_0, a, b \in \mathbb{R}\), \(a^{jk} = a^{kj} \in L^2_{\mathbb{F}}(\Omega; C^t([0,T]; W^{2,\infty}(\mathbb{R}^n; \mathbb{R})))\) \((j, k = 1, \cdots, n)\) and \(b_0 = (b_0^1, \cdots, b_0^n) \in \mathbb{R}^n\), we define the following complex stochastic partial differential operator:

$$\mathcal{L}w = a_0 dw - (a + ib) \sum_{j,k=1}^n (a^{jk}w_{x_j})_{x_k} dt + b_0 \cdot \nabla w dt.$$  \hspace{1cm} (2.1)\]

This section is devoted to establishing a pointwise weighted identity for the operator \(\mathcal{L}\). To begin with, we introduce the following assumptions:
1. If $a \neq 0$, then $b_0 \cdot \nabla w$ is a lower order term for the operator $\mathcal{L}$. In this case, without loss of generality, we assume that $b_0 = 0$.

2. If $a = 0$ and $a_0, b \neq 0$, then $\mathcal{L}$ is a second-order stochastic Schrödinger operator. In this case, we assume that $b_0 = 0$.

3. If $a = b = 0$, we assume that $a_0 \neq 0$ and $b_0 \neq 0$. In this case, $\mathcal{L}$ is a first order stochastic transport operator.

The main result of this paper is stated as follows.

**Theorem 2.1** Assume that the assumptions 1-3 hold. Let $\ell \in C^3(\mathbb{R}^{n+1}; \mathbb{R})$, $\Phi \in C^1(\mathbb{R}^{n+1}; \mathbb{C})$ and $w$ be an $H^2(\mathbb{R}^n; \mathbb{C})$-valued continuous semimartingale. Set $\theta = e^{\ell}$ and $z = \theta w$. Then for a.e. $(x,t) \in \mathbb{R}^{n+1}$ and $\mathcal{P}$-a.s. $\omega \in \Omega$, one has the following pointwise weighted identity:

$$
2 \text{Re} (\theta \overline{I_1} \mathcal{L} w) = 2|I_1|^2 dt + dM + \sum_{k=1}^n V^k_{x_k} + B|z|^2 dt + \sum_{j,k=1}^n D^{jk} z_{x_j} \overline{z}_{x_k} dt
$$

$$
+ 2 \sum_{j=1}^n \left\{ \text{Re} \left[ \left( a E^j + \Phi b_0^j \right) \overline{z}_{x_j} \right] + b \text{Im} \left( F^j z \overline{z}_{x_j} \right) \right\} dt - aa_0 \sum_{j,k=1}^n a^{jk} d_{x_j} d_{\overline{z}_{x_k}}
$$

$$
- b_0 \cdot \nabla \left[ (a_0 \ell_t + b_0 \cdot \nabla \ell)|z|^2 \right] dt + a_0(aA + a_0 \ell_t + b_0 \cdot \nabla \ell)|dz|^2,
$$

$$
- 2a_0 b \sum_{j,k=1}^n a^{jk} \ell_{x_k} \text{Im} \left( dz d \overline{z}_{x_j} \right) + 2a_0 \left[ b \sum_{j,k=1}^n (a^{jk} \ell_{x_k})_{x_j} \text{Im} \left( zd \overline{z} \right) + \text{Re} \left( \Phi zd \overline{z} \right) \right],
$$

where

$$
\begin{align*}
A &= \sum_{j,k=1}^n \left[ a^{jk} \ell_{x_j} \ell_{x_k} - (a^{jk} \ell_{x_j})_{x_k} \right], \quad \Lambda = \sum_{j,k=1}^n (a^{jk} z_{x_j})_{x_k} + A z, \\
I_1 &= -a \Lambda + 2ib \sum_{j,k=1}^n a^{jk} \ell_{x_j} z_{x_k} + (\Phi - a_0 \ell_t - b_0 \cdot \nabla \ell) z,
\end{align*}
$$

(2.3)
\begin{align*}
B &= 2(a^2 + b^2) \sum_{j,k=1}^{n} (Aa^{jk} \ell_{x_j} x_k) + a_0 A_t + 2aA \Re \Phi - 2bA \Im \Phi \\
-2\Re \left[ \Phi (\overline{\Phi} - a_0 \ell_t - b_0 \cdot \nabla \ell) \right] + a_0 \left[ a_0 \ell_{tt} + (b_0 \cdot \nabla \ell)_{tt} \right] + b_0 \cdot \nabla (a_0 \ell_t + b_0 \cdot \nabla \ell), \\
D^{jk} &= -aa_0 a^{jk} + 2bA \Re \Phi a^{jk} - 2aA \Re \Phi a^{jk} \\
+2(a^2 + b^2) \sum_{j',k'=1}^{n} \left[ a^{jk'} (a^{j'k} \ell_{x_{j'}} x_{k'}) + a^{jk'} (a^{j'j} \ell_{x_{j'}} x_{k'}) - (a^{jk} a^{j'k} \ell_{x_{j'}} x_{k'}) \right], \\
M &= -aa_0 A |z|^2 + a_0 \sum_{j,k=1}^{n} a^{jk} \left[ a z_{x_j} \overline{z}_{x_k} + 2b \ell_{x_j} \Im (\overline{z}_{x_k} z) \right] - a_0 (a_0 \ell_t + b_0 \cdot \nabla \ell) |z|^2 , \\
V^k &= -2aa_0 \sum_{j=1}^{n} a^{jk} \Re (z_{x_j} \overline{a} z) - 2a_0 b \sum_{j=1}^{n} a^{jk} \ell_{x_j} \Im (z d \overline{z}) - 2A(a^2 + b^2) \sum_{j=1}^{n} a^{jk} \ell_{x_j} |z|^2 dt \\
+2a \sum_{j=1}^{n} a^{jk} \Re (\overline{z}_{x_j} \Phi) dt + 2b \sum_{j=1}^{n} a^{jk} \Im \left[ z_{x_j} (\overline{\Phi} - a_0 \ell_t) \overline{z} \right] dt \\
+2(a^2 + b^2) \sum_{j,j',k'=1}^{n} \left[ a^{jk} a^{j'k'} \ell_{x_j} \overline{z}_{x_{j'}} z_{x_{k'}} - a^{jk'} a^{j'k} \ell_{x_j} \overline{z}_{x_{j'}} (z_{x_{j'}} \overline{z}_{x_{k'}} + \overline{z}_{x_{j'}} z_{x_{k'}}) \right] dt, \\
E^j &= \sum_{k=1}^{n} a^{jk} \left[ 2\ell_{x_k} (\overline{\Phi} - a_0 \ell_t) - \overline{\Phi}_{x_k} \right], \\
F^j &= \sum_{k=1}^{n} \left[ a^{jk} (\Phi - a_0 \ell_t)_{x_k} - a_0 (a^{jk} \ell_{x_k})_{t} - 2a^{jk} \ell_{x_k} \Phi \right].
\end{align*}

**Remark 2.1** The pointwise weighted identity (2.2) is quite useful in deriving global Carleman estimates for the deterministic/stochastic partial differential operator (2.1). The advantage of Carleman inequalities derived by the identity (2.2) is that one can give an explicit estimate on constants (in Carleman estimates). This is crucial in studying nonlinear controllability and observability problems.

**Remark 2.2** The key point of proving the identity (2.2) is to multiply “the principal operator \( \mathcal{L} \)” by a weighted multiplier \( \theta_1 \). One can rewrite this product as a sum of “divergence” terms, “energy” terms and some lower order terms. Also, all terms in the right side of the sign of equality in (2.2) are real-valued functions. By choosing a suitable auxiliary function \( \Phi \) and a weighted function \( \theta \), one can derive global Carleman estimates for some deterministic/stochastic partial differential operators of different type.

**Remark 2.3** If choosing different coefficients in (2.1), one can get deterministic/stochastic partial differential operators of different type. For example, suppose that \( (a^{jk})_{1 \leq j,k \leq n} \) is a uniformly positive definite matrix and \( a_0 = 1 \). If \( a = 0 \) and \( b \neq 0 \), \( \mathcal{L} \) is a stochastic Schrödinger operator. If \( a \neq 0 \), \( \mathcal{L} \) is a linear stochastic complex Ginzburg-Landau operator. If \( a \neq 0 \), \( b = 0 \) and all functions are real-valued, \( \mathcal{L} \) is a stochastic heat operator. If \( a = b = 0 \) and \( b_0 \neq 0 \), \( \mathcal{L} \) is a stochastic transport operator. Also, if all functions (in the above operators)
are independent of sample points, then one can get a deterministic Schrödinger operator, a
deterministic linear complex Ginzburg-Landau operator, a deterministic heat operator and
a deterministic transport operator, respectively. In the following sections, we use the point-
wise weighted identity (2.2) to derive global Carleman estimates for the above determinis-
tic/stochastic partial differential operators. Moreover, it is applied to study inverse problems
of linear stochastic complex Ginzburg-Landau equations.

Proof of Theorem 2.1. The whole proof is divided into four steps.

Step 1. Set $\theta = e^\ell$ and $z = \theta w$. Then it is easy to show that

$$\theta Lw = a_0 \theta d(\theta^{-1} z) - \theta(a + ib) \sum_{j,k=1}^{n} [a^{jk}(\theta^{-1} z)_{x_j}]_{x_k} dt + \theta b_0 \cdot \nabla (\theta^{-1} z) dt = I_1 dt + I_2,$$

where $I_1$ is given in (2.3) and

$$I_2 = a_0 dz - ib \Lambda dt + 2a \sum_{j,k=1}^{n} a^{jk} \ell x_j z_{x_k} dt + b_0 \cdot \nabla z dt - \Phi z dt.$$

Therefore,

$$2\Re (\theta T_1 Lw) = \theta(T_1 Lw + I_1 \overline{Lw}) = 2|I_1|^2 dt + 2\Re (T_1 I_2). \quad (2.4)$$

Step 2. Let us compute “$2\Re (T_1 I_2)$”. By the assumptions 1-3, it is easy to find that

$$a b_0 = 0 \quad \text{and} \quad b b_0 = 0. \quad (2.5)$$

Recalling the definitions of $I_1$ and $I_2$, by (2.5) and a short calculation, we have that

$$2\Re (T_1 I_2)$$

$$= -2aa_0 \Re (\overline{\Lambda} dz) - 4(a^2 + b^2) \Re \sum_{j,k=1}^{n} a^{jk} \ell x_j (z_{x_k} \overline{\Lambda}) dt + 2a \Re (\overline{\Phi} \overline{\Lambda} z) dt$$

$$+ 4a b \sum_{j,k=1}^{n} a^{jk} \ell x_j \Im (z_{x_k} dz) + 4b \sum_{j,k=1}^{n} a^{jk} \ell x_j \Im (\overline{\Phi} \overline{z} z_{x_k}) dt$$

$$+ 2b \Im [(\overline{\Phi} - a_0 \ell_t) z \Lambda] dt + 4a \sum_{j,k=1}^{n} a^{jk} \ell x_j \Re [(\overline{\Phi} - a_0 \ell_t) z z_{x_k}] dt$$

$$+ 2\Re \left[ (\overline{\Phi} - a_0 \ell_t - b_0 \cdot \nabla \ell) z (a_0 dz + b_0 \cdot \nabla z dt) \right]$$

$$- 2\Re \left[ \Phi (\overline{\Phi} - a_0 \ell_t - b_0 \cdot \nabla \ell) \right] |z|^2 dt. \quad (2.6)$$

Step 3. Now we compute every term in the right side of the sign of equality in (2.6),

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respectively. By (2.3), we find that
\[-2aa_0 \text{Re}(\overline{\Lambda}dz) = -a a_0 (\overline{\Lambda}dz + \Lambda d\overline{z})\]
\[= -a a_0 \sum_{j,k=1}^{n} \left[(a^j k z_{x_j})_{x_k} d\overline{z} + (a^j k z_{x_j})_{x_k} dz \right] - a a_0 A(z d\overline{z} + \overline{z}dz)\]
\[= -a a_0 \sum_{j,k=1}^{n} (a^j k z_{x_j} dz + a^j k z_{x_j} d\overline{z})_{x_k} + \sum_{j,k=1}^{n} d(a a_0 a^j k z_{x_j} \overline{z}_{x_k}) - a a_0 \sum_{j,k=1}^{n} a^j k z_{x_j} \overline{z}_{x_k} dt \]
\[= -a a_0 \sum_{j,k=1}^{n} a^j k dz_{x_j} d\overline{z}_{x_k} - d(a a_0 A|z|^2) + a a_0 A|z|^2 dt + a a_0 A|dz|^2.\]

Further,
\[-4(a^2 + b^2) \text{Re} \sum_{j,k=1}^{n} a^j k \ell_{x_j} (z_{x_k} \overline{\Lambda}) dt = -2(a^2 + b^2) \sum_{j,k=1}^{n} a^j k \ell_{x_j} (\overline{z}_{x_k} + z_{x_k} \overline{\Lambda}) dt\]
\[= -2(a^2 + b^2) \sum_{j,k=1}^{n} a^j k \ell_{x_j} \left[\overline{z}_{x_k} \sum_{j',k'=1}^{n} (a^j k' z_{x_{j'}})_{x_{k'}} + z_{x_k} \sum_{j',k'=1}^{n} (a^j k' z_{x_{j'}})_{x_{k'}} \right] dt\]
\[= -2(a^2 + b^2) \sum_{j,k=1}^{n} (a^j k z_{x_j} |z|^2)_{x_k} dt + 2(a^2 + b^2) \sum_{j,k=1}^{n} (a a^j k \ell_{x_j})_{x_k} |z|^2 dt \]
\[= -2(a^2 + b^2) \sum_{j,k=1}^{n} \left[a^j k \ell_{x_j} a^j k' (z_{x_j} \overline{z}_{x_k} + \overline{z}_{x_j} z_{x_k}) \right]_{x_k}, dt\]
\[+ 2(a^2 + b^2) \sum_{j,k,j',k'=1}^{n} a^j k' (a^j k \ell_{x_j})_{x_k} (z_{x_j} \overline{z}_{x_k} + \overline{z}_{x_j} z_{x_k}) dt\]
\[+ 2(a^2 + b^2) \sum_{j,k,j',k'=1}^{n} \left[(a^j k \ell_{x_j} a^j k' z_{x_j} \overline{z}_{x_k})_{x_k} - (a^j k a^j k' \ell_{x_j})_{x_k} z_{x_j} \overline{z}_{x_k} \right] dt.\]

Notice that in the above derivation, we use the following identity:
\[2 \sum_{j,k,j',k'=1}^{n} a^j k a^j k' \ell_{x_j} (z_{x_j} \overline{z}_{x_k} + \overline{z}_{x_j} z_{x_k}) dt \]
\[= \sum_{j,k,j',k'=1}^{n} \left[a^j k a^j k' \ell_{x_j} (z_{x_j} \overline{z}_{x_k} + \overline{z}_{x_j} z_{x_k}) \right]_{x_k} - (a^j k a^j k' \ell_{x_j})_{x_k} (z_{x_j} \overline{z}_{x_k} + \overline{z}_{x_j} z_{x_k}) \right] dt\]
\[= 2 \sum_{j,k,j',k'=1}^{n} \left[(a^j k a^j k' \ell_{x_j} z_{x_j} \overline{z}_{x_k})_{x_k} - (a^j k a^j k' \ell_{x_j})_{x_k} z_{x_j} \overline{z}_{x_k} \right] dt.\]
Further,

\[ 2a \text{Re}(\Phi \overline{\alpha} z) dt = 2a \sum_{j,k=1}^{n} \text{Re} \left([a^{jk} \overline{x}_j] x_k \Phi z \right) dt + 2a \text{Re} \Phi |z|^2 dt \]

\[ = 2a \sum_{j,k=1}^{n} \text{Re} \left(a^{jk} \overline{x}_j \Phi z \right) dt - 2a \text{Re} \Phi \sum_{j,k=1}^{n} a^{jk} x_j \overline{x}_k dt \]

\[ -2a \sum_{j,k=1}^{n} \text{Re} \left(a^{jk} \Phi x_k z_j \overline{x}_j \right) dt + 2a \text{Re} \Phi |z|^2 dt. \]  \hspace{1cm} (2.9)

Note that for any \( k = 1, \cdots, n, \)

\[ \text{Im} (\overline{x}_k dz) = \text{Im} \left[d(\overline{x}_k z) - (zd\overline{z})_x - d\overline{x}_k dz + z_k d\overline{z}\right] = -\text{Im} (z_k d\overline{z}). \]  \hspace{1cm} (2.10)

Therefore, we get that

\[ 4a_0 b \sum_{j,k=1}^{n} a^{jk} \ell_j \text{Im} (\overline{x}_k dz) = 2a_0 b \sum_{j,k=1}^{n} a^{jk} \ell_j \text{Im} \left[d(\overline{x}_k z) - (zd\overline{z})_x - d\overline{x}_k dz\right] \]

\[ = 2a_0 b \sum_{j,k=1}^{n} \left\{d[a^{jk} \ell_j \text{Im} (\overline{x}_k z)] - [a^{jk} \ell_j \text{Im} (zd\overline{z})]_x\right\} \]  \hspace{1cm} (2.11)

\[ -2a_0 b \sum_{j,k=1}^{n} [(a^{jk} \ell_j)_x \text{Im} (\overline{x}_k z) dt - (a^{jk} \ell_j) x_k \text{Im} (zd\overline{z}) + a^{jk} \ell_j \text{Im} (dzd\overline{x}_k)]. \]

Further,

\[ 2b \text{Im} \left[[\overline{\Phi} - a_0 \ell_t] z\right] dt = 2b \sum_{j,k=1}^{n} \text{Im} \left[(a^{jk} x_j (\overline{\Phi} - a_0 \ell_t) \overline{z})_x \right] dt - 2b A \text{Im} \Phi |z|^2 dt \]

\[ = 2b \sum_{j,k=1}^{n} \text{Im} \left[a^{jk} x_j (\overline{\Phi} - a_0 \ell_t) \overline{z} \right] dt + 2b \text{Im} \Phi \sum_{j,k=1}^{n} a^{jk} x_j \overline{x}_k dt \]  \hspace{1cm} (2.12)

\[ -2b \sum_{j,k=1}^{n} a^{jk} \text{Im} \left[(\overline{\Phi} - a_0 \ell_t) x_k z_j \overline{z} \right] dt - 2b A \text{Im} \Phi |z|^2 dt. \]

**Step 4.** Let us compute \( 2 \text{Re} \left[[\Phi - a_0 \ell - b_0 \cdot \nabla \ell] z(a_0 dz + b_0 \cdot \nabla dt) \right]. \) Notice that

\[ 2 \text{Re} \left[[\Phi - a_0 \ell - b_0 \cdot \nabla \ell] z(a_0 dz + b_0 \cdot \nabla dt) \right] = 2 \text{Re} \left[[\Phi z(a_0 dz + b_0 \cdot \nabla dt)] \right] \]

\[ - (a_0 \ell + b_0 \cdot \nabla \ell) \left[a_0 d(|z|^2) - a_0 |dz|^2 + b_0 \cdot \nabla (|z|^2) dt \right]. \]  \hspace{1cm} (2.13)

Further,

\[ -(a_0 \ell + b_0 \cdot \nabla \ell) \left[a_0 d(|z|^2) - a_0 |dz|^2 + b_0 \cdot \nabla (|z|^2) dt \right] \]

\[-d[a_0(a_0 \ell + b_0 \cdot \nabla \ell)|z|^2 + a_0[a_0 \ell + (b_0 \cdot \nabla \ell)]|z|^2 dt + a_0(a_0 \ell + b_0 \cdot \nabla \ell)|dz|^2 \]  \hspace{1cm} (2.14)

\[-b_0 \cdot \nabla \left[(a_0 \ell + b_0 \cdot \nabla \ell)|z|^2 \right] dt + b_0 \cdot \nabla \left[(a_0 \ell + b_0 \cdot \nabla \ell) \right]|z|^2 dt. \]
Combining (2.6)-(2.14) with (2.4), we can get the desired identity (2.2).

\section{Applications in control problems for some deterministic/stochastic partial differential equations}

In this section, we give some concrete applications of Theorem 2.1 in deriving some known global Carleman estimates for some deterministic/stochastic partial differential equations. Based on these estimates, one can study the controllability and observability of deterministic/stochastic partial differential equations.

\subsection{A pointwise weighted identity for deterministic partial differential operators}

In [7], a pointwise weighted identity was established for the following deterministic partial differential operator:

\[ L = (\alpha + \beta i) \partial_t + \sum_{j,k=1}^{n} \partial_{x_k}(a^{jk}\partial_{x_j}), \]

with two real-valued functions \( \alpha \) and \( \beta \). Based on this identity, a universal approach of proving Carleman estimates was established to deduce the controllability/observability results for parabolic equations, hyperbolic equations, Schrödinger equations, plate equations and linear complex Ginzburg-Landau equations.

In this subsection, starting from Theorem 2.1, one can obtain the known weighted identity for deterministic partial differential operators in [7]. Indeed, as a consequence of Theorem 2.1, we have the following pointwise weighted identity.

**Corollary 3.1** Suppose that \( a^{jk} = a^{kj} \in C^2(\mathbb{R}^n) \) \((j, k = 1, 2, \ldots, n)\), \( \ell \in C^3(\mathbb{R}^n) \), \( \Phi \in C^1(\mathbb{R}^n) \), \( y \in C^2(\mathbb{R}^n) \) and all functions in (2.1) are real-valued. Set \( a_0 = b = 0, a = -1, b_0 = 0, \theta = e^\ell \) and \( z = \theta y \). Then

\[ 2\theta I_1 \sum_{j,k=1}^{n} (a^{jk}y_{x_j}y_{x_k})_{x_k} = 2|I_1|^2 + \sum_{k=1}^{n} V^k_{x_k} + B|z|^2 + \sum_{j,k=1}^{n} D^{jk}z_{x_j}z_{x_k} \]

\[ -2 \sum_{j,k=1}^{n} a^{jk}(2\ell_{x_k}\Phi - \Phi_{x_k})zz_{x_j}, \]  

where \( I_1 = \Lambda + \Phi z \) with \( \Lambda \) being given by (2.3),

\[ V^k = -2A \sum_{j=1}^{n} a^{jk} \ell_{x_j}|z|^2 - 2\Phi z \sum_{j=1}^{n} a^{jk} \]

\[ + 2 \sum_{j,j',k'=1}^{n} \left[ a^{jk}a^{jk'}\ell_{x_j}z_{x_j}z_{x_{k'}} - a^{jk}a^{jk'}\ell_{x_j}(z_{x_{j'}}z_{x_{k'}} + z_{x_{j'}}z_{x_{k'}}) \right], \]
and

\[
\begin{align*}
B &= 2 \sum_{j,k=1}^{n} (Aa^{jk}t_{x_j})x_k - 2A\Phi - 2|\Phi|^2, \\
D^{jk} &= 2\Phi a^{jk} + 2 \sum_{j',k'=1}^{n} \left[ 2a^{jk'}(a^{j'k}t_{x_{j'}})x_{k'} - (a^{jk}a^{j'k'}t_{x_{j'}})x_{k'} \right].
\end{align*}
\]

If we choose \(\alpha = \beta = 0\) in (7), the identity (3.1) in Corollary 3.1 is exactly the same as [7, Theorem 2.1]. Meanwhile, Corollary 3.1 is very similar to [8, Theorem 4.1], both imply weighted identities for elliptic operators. Comparing two results to each other, we find that the only difference is low-order terms. Therefore, it does not influence the derivation of global Carleman estimates for elliptic operators of second order. Furthermore, by Corollary 3.1, one also can deduce a weighted identity for deterministic hyperbolic operators.

If \(a_0 \neq 0\) and \(b_0 = 0\), Theorem 2.1 implies a weighted identity for deterministic parabolic operators, deterministic Schrödinger operator and deterministic linear complex Ginzburg-Landau operators, respectively.

If \(a = b = 0\), \(a_0 \neq 0\) and \(b_0 \neq 0\), Theorem 2.1 implies a weighted identity for deterministic transport operators.

### 3.2 A pointwise weighted identity for stochastic transport operators

In Theorem 2.1, assume that all functions are real-valued. If we choose \(a = b = 0\) and \(a_0 = 1\), then we have the following weighted identity for the stochastic transport operator:

\[
Lw = dw + b_0 \cdot \nabla w dt.
\]

Define \(S^{n-1} \triangleq \{ x \in \mathbb{R}^n : |x|_{\mathbb{R}^n} = 1 \} \). Then, we have the following pointwise weighted identity for the stochastic trasport operator.

**Corollary 3.2** Suppose that \(\ell \in C^3(\mathbb{R}^{n+1})\) and \(y\) is an \(H^1(\mathbb{R}^n) \times L^2(S^{n-1})\)-valued continuous semi-martingale. Set \(a = b = \Phi = 0, a_0 = 1, b_0 \neq 0, \theta = e^\ell\) and \(z = \theta y\). Then

\[
2\theta I_1(dy + b_0 \cdot \nabla y dt) = 2|I_1|^2 dt - d \left[ (\ell_t + b_0 \cdot \nabla \ell)|z|^2 \right] + B|z|^2 dt
\]

\[
- b_0 \cdot \nabla \left[ (\ell_t + b_0 \cdot \nabla \ell)|z|^2 \right] + (\ell_t + b_0 \cdot \nabla \ell)|dz|^2,
\]

where

\[
\begin{align*}
I_1 &= -(\ell_t + b_0 \cdot \nabla \ell)z, \\
B &= \ell_{tt} + (b_0 \cdot \nabla \ell)_t + b_0 \cdot \nabla (\ell_t + b_0 \cdot \nabla \ell).
\end{align*}
\]

Corollary 3.2 is exactly the same as [19, Proposition 2.1]. As we seen in [19], the identity (3.2) plays a key role in the study of observability/controllability problems for stochastic transport equations.
3.3 A global Carleman estimate for backward stochastic heat operators

As another application of Theorem 2.1, one can obtain global Carleman estimates for general forward and backward linear stochastic parabolic operators. For simplicity, in this subsection, we only consider backward stochastic heat operators. Notice that our pointwise weighted identity is different from [25, Theorem 3.1]. But, starting from this identity, we still can obtain the desired global Carleman estimate for backward stochastic heat equations (which was presented in [25]).

Let $G$ be a nonempty bounded domain in $\mathbb{R}^n$ with a boundary $\Gamma$ of class $C^4$. Put $Q = G \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. Assume that all functions are real-valued in this subsection. Based on the identity (2.1), we derive a global Carleman estimate for the following backward stochastic heat equation:

$$
\begin{align*}
\left\{ \begin{array}{ll}
 dy + \Delta y dt &= f dt + Y dw(t) & \text{in } Q, \\
 y &= 0 & \text{on } \Sigma, \\
 y(T) &= y_T & \text{in } G,
\end{array} \right.
\end{align*}
$$

where $f \in L^2_F(0, T; L^2(G))$ and $y_T \in L^2(\Omega, F_T, \mathcal{P}; L^2(G))$.

First, introduce some auxiliary functions. It is well known that ([9]), there exists a function $\psi \in C^4(G)$ such that $\psi(x) > 0$, in $G$; $\psi(x) = 0$, on $\Gamma$; and $|\nabla \psi(x)| > 0$, in $G \setminus G_1$, where $G_0$ and $G_1$ are two any given nonempty open subsets of $G$ such that $G_1 \subseteq G_0$. For any fixed integer $k \geq 1$, and positive parameters $\mu$ and $\lambda$, write

$$
\gamma(t) = \frac{1}{t^k(T - t)^k}, \quad \varphi(x, t) = \frac{e^{\mu \psi(x)}}{t^k(T - t)^k}, \quad \alpha(x, t) = \frac{e^{\mu \psi(x)} - e^{2\mu |\psi|_{C(\overline{G})}}}{t^k(T - t)^k} \quad \text{and} \quad \theta = e^{\lambda \alpha}.
$$

In the sequel, for any $k \in \mathbb{N}$, we denote by $O(\mu^k)$ a function of order $\mu^k$, for sufficiently large $\mu$; and by $O_{\mu}(\lambda^k)$ a function of order $\lambda^k$ for fixed $\mu$ and sufficiently large $\lambda$.

Next, based on (2.2), we have the following inequality for the backward stochastic heat operator:

$$
\mathcal{L}y = dy + \Delta y dt.
$$

**Lemma 3.1** Let $z = \theta y$ and $\ell = \lambda \alpha$. Then any solution $(y, Y) \in L^2_F(0, T; H^1_0(G)) \times L^2_F(0, T; L^2(G))$ of the equation (3.3) satisfies

$$
\mathbb{E} \int_Q 2\theta I_1 \mathcal{L}y dx \\
\geq \mathbb{E} \int_Q 2|I_1|^2 dx dt \\
+ \mathbb{E} \int_Q 2\lambda \mu^3 \varphi^3 |\nabla \psi|^4 z^2 dx dt + \mathbb{E} \int_Q 2\lambda \mu^2 \varphi |\nabla z|^2 |\nabla \psi|^2 dx dt \\
+ \mathbb{E} \int_Q \lambda^3 \varphi^3 O(\mu^3) + O_{\mu}(\lambda^2) \varphi^3 |z|^2 dx dt + \mathbb{E} \int_Q O(\mu) \lambda \varphi |\nabla z|^2 dx dt \\
+ \mathbb{E} \int_Q \theta^2 O(\lambda^2) \mu^2 \varphi^2 |Y|^2 dx dt,
$$

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where \( I_1 = \Delta z + (|\nabla \ell|^2 + \Delta \ell - \ell_1)z \).

For the readers’ convenience, we give a proof of Lemma 3.1 in Appendix A. By (3.4), proceeding exactly the same analysis as [25, Theorem 6.1], one can obtain the following global Carleman estimate for the equation (3.3).

**Theorem 3.1** There exists a positive constant \( \mu_0 \), depending only on \( n, G, G_0 \) and \( T \), such that for any \( \mu \geq \mu_0 \), one can find two positive constants \( \lambda_0 = \lambda_0(\mu) \) and \( C = C(\mu) \), so that for any \( f \in L^2_F(0,T;L^2(G)) \) and \( y_T \in L^2(\Omega,F_T,P;L^2(G)) \), any solution \((y,Y) \in L^2_F(0,T;H^1_0(G)) \times L^2_F(0,T;L^2(G))\) of the equation (3.3) satisfies

\[
\mathbb{E} \int_Q \theta^2 \left( \lambda^3 \gamma^3 y^2 + \lambda \gamma |\nabla y|^2 \right) \, dx \, dt \\
\leq C \left( \mathbb{E} \int_0^T \int_{G_0} \theta^2 \lambda^3 \gamma^3 y^2 \, dx \, dt + \mathbb{E} \int_Q \theta^2 f^2 \, dx \, dt + \mathbb{E} \int_Q \theta^2 \lambda^2 \gamma^2 Y^2 \, dx \, dt \right),
\]

for any \( \lambda \geq \lambda_0 \).

By this global Carleman estimate in Theorem 3.1, one can study the observability (resp. controllability) for backward (resp. forward) stochastic heat equations.

### 3.4 A global Carleman estimate for stochastic Schrödinger operators

In this subsection, based on the weighted identity (2.2), we derive a global Carleman estimate for stochastic Schrödinger operators. In (2.1), we choose \( a_0 = 1, b = 1, a = 0, b_0 = 0 \) and \((a^{jk})_{1 \leq j,k \leq n} = I_n\). Then \( \mathcal{L} \) is the following stochastic Schrödinger operator:

\[
\mathcal{L}w = dw - i\Delta wdt,
\]

and \( I_1 = 2i\nabla \ell \cdot \nabla z + (\Phi - \ell_1)z \) (with \( z = \theta w \)).

Notice that in [17], a weighted identity was derived, in order to establish a global Carleman estimate for the stochastic Schrödinger operator:

\[
Pv = idv + \Delta vdt.
\]

In [17], write \( u = \theta v \). Then

\[
\tilde{I}_1 = -i\ell t u - 2\nabla \ell \cdot \nabla u + \Psi u,
\]

where \( \Psi \) is a suitable auxiliary function.

If we set \( w = iv \) and \( \Phi = -i\Psi \), then it is easy to check that \( z = iu, I_1 = \tilde{I}_1 \) and \( \mathcal{L}w = Pv \). Therefore, based on (2.2), we can get the same pointwise weighted identity as that in [17], and a global Carleman estimate for stochastic Schrödinger operators.
4 Applications in inverse problems for linear stochastic complex Ginzburg-Landau equations

As another application of Theorem 2.1, in this section, we prove a uniqueness result for inverse problems of linear stochastic complex Ginzburg-Landau equations.

4.1 Main results

The deterministic complex Ginzburg-Landau equation was introduced by Ginzburg and Landau in 1950 ([10]). This kind of complex partial differential equations can describe a phase transition in the theory of superconductivity. In the last decades, a lot of stochastic versions of Ginzburg-Landau equations were studied. We refer to [4, 16] and the references therein for some known results.

Consider the following linear stochastic complex Ginzburg-Landau equation:

\[
\begin{aligned}
dw - (1 + ib) \Delta w dt &= (a_1 \cdot \nabla w + a_2 w) dt + a_3 w dB(t) & \quad & \text{in } Q, \\
w &= 0 & \quad & \text{on } \Sigma, \\
w(0) &= w_0 & \quad & \text{in } G,
\end{aligned}
\]

where \( b \in \mathbb{R}, a_1 \in L_F^\infty(0, T; L^\infty(G; \mathbb{C}^n)), a_2 \in L_F^\infty(0, T; L^\infty(G; \mathbb{C})), a_3 \in L_F^\infty(0, T; W^{1, \infty}(G; \mathbb{C})) \) and \( w_0 \in L^2(G; \mathbb{C}) \).

We first recall the definition of weak solutions of the equation (4.1).

**Definition 4.1** We call \( w \in L^2_F(\Omega; C([0, T]; L^2(G; \mathbb{C}))) \cap L^2_F((0, T); H^1_0(G; \mathbb{C})) \) is a weak solution of the equation (4.1), if for any \( t \in [0, T] \) and any \( p \in H^1_0(G) \), it holds that

\[
\begin{aligned}
\int_G w(t, x) \overline{p}(x) dx &= \int_G w_0(0) \overline{p}(x) dx \\
= \int_0^t \int_G \left\{ -(1 + ib) \nabla w(s, x) \cdot \nabla \overline{p}(x) + \left[ a_1(s, x) \cdot \nabla w(s, x) + a_2(s, x) w(s, x) \right] \overline{p}(x) \right\} dx ds \\
&\quad + \int_0^t \int_G a_3(s, x) w(s, x) \overline{p}(x) dx dB(s), & \mathcal{P}\text{-a.s.}
\end{aligned}
\]

Also, set

\[
r \triangleq 1 + |a_1|^2_{L_F^\infty(0, T; L^\infty(G; \mathbb{C}^n))} + |a_2|^2_{L_F^\infty(0, T; L^\infty(G; \mathbb{C}))} + |a_3|^2_{L_F^\infty(0, T; W^{1, \infty}(G; \mathbb{C}))}. \tag{4.2}
\]

Then we have the following well-posedness result for the equation (4.1), whose proof can be found in [22, Chapter 6].

**Lemma 4.1** For any \( w_0 \in L^2(G; \mathbb{C}) \), there exists a unique weak solution \( w \) of the equation (4.1). Moreover,

\[
|w|_{L^2_F(\Omega; C([0, T]; L^2(G; \mathbb{C}))))} + |w|_{L^2_F((0, T); H^1_0(G; \mathbb{C}))} \leq C r |w_0|_{L^2(G; \mathbb{C})}.
\]

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In this section, we are concerned with the following inverse problem: for \( t_0 \in [0, T) \), determine \( w(t_0, \cdot), \mathcal{P}\text{-a.s.} \) from \( w(T, \cdot) \). The main result can be stated as follows.

**Theorem 4.1** Let \( t_0 \in [0, T) \). Then there exist constants \( \tau \in (0, 1) \) and \( C > 0 \) such that

\[
|w(t_0)|_{L^2(\Omega, \mathcal{F}_t; \mathcal{P}, L^2(G; \mathcal{G}))} \leq C|w|_{L^2(0, T; L^2(G; \mathcal{G}))}^{1-\tau} \|w(T)\|_{L^2(\Omega, \mathcal{F}_T; \mathcal{P}, H^1(G; \mathcal{G}))},
\]

for any solution \( w \) of the equation (4.1).

As a consequence of Theorem 4.1, we can get the following backward uniqueness for the equation (4.1).

**Corollary 4.1** Assume that \( w \) is a weak solution of the equation (4.1). If \( w(T) = 0 \) in \( G, \mathcal{P}\text{-a.s.} \), then \( w(t) = 0 \) in \( G, \mathcal{P}\text{-a.s.} \) for all \( t \in [0, T] \).

In [7, 23], some global Carleman estimates for deterministic complex Ginzburg-Landau equations were established, respectively. However, as far as we know, there are no published papers addressing global Carleman estimates for stochastic complex Ginzburg-Landau equations. In the following, we derive a suitable Carleman estimate for a linear stochastic complex Ginzburg-Landau operator. Based on this result, we can study the uniqueness of this inverse problem.

### 4.2 A Carleman estimate for linear stochastic complex Ginzburg-Landau operators

In this subsection, we establish a Carleman estimate for the following linear stochastic complex Ginzburg-Landau equation:

\[
\begin{cases}
    dw - (\mu + ib)\Delta wdt = fdt + gdB(t) & \text{in } Q, \\
    w = 0 & \text{on } \Sigma, \\
    w(0) = w_0 & \text{in } G,
\end{cases}
\]

(4.4)

where \( f \in L^2(0, T; L^2(G; \mathcal{G})) \) and \( g \in L^2(0, T; H^1(G; \mathcal{G})) \).

First, we establish a pointwise weighted identity, which is a consequence of Theorem 2.1.

**Lemma 4.2** Under the assumptions of Theorem 2.1, for a parameter \( \mu \geq 1 \), choose \( \varphi(t) = e^{3\mu t}, \ell = \mu \varphi, \theta = e^\ell, z = \theta w \) and \( \Phi = -\mu \). Then, it holds that

\[
2\text{Re} \left[ \theta T_1 \left( dw - (\mu + ib)\Delta wdt \right) \right] = 2|I_1|^2 dt + d(|\nabla z|^2 - 3\mu^2 \varphi |z|^2) + \sum_{k=1}^n V_x^k + \mu^2 (3\mu \varphi - 2)|z|^2 dt
\]

\[
+ 2\mu |\nabla z|^2 dt - |\nabla dz|^2 - 2\mu \text{Re} (\bar{z} dz) + 3\mu^2 \varphi |dz|^2,
\]

(4.5)

where

\[
\begin{cases}
    I_1 = -\Delta z - (\mu + 3\mu^2 \varphi)z, \\
    V^k = -2\text{Re} (z_{x_k} \bar{z} + \mu \bar{z}_{x_k} z dt) - 2b(\mu + 3\mu^2 \varphi) \text{Im} (z_{x_k} \bar{z}) dt.
\end{cases}
\]

(4.6)
Proof. In Theorem 2.1, we choose \( a_0 = a = 1, b_0 = 0, (a^{jk})_{n \times n} = I_n, \varphi(t) = e^{\mu t}, \ell = \mu \varphi \) and \( \Phi = -\mu \). Then after a simple calculation, we can get the desired result (4.5). 

Based on Lemma 4.2, we have the following Carleman estimate for (4.4).

**Theorem 4.2** Let \( \delta \in [0, T) \). Then for any \( \mu \geq 2 \), one can find a constant \( C = C(\mu) > 0 \) so that

\[
\mu E \int_0^T \int_G \theta^2 |\nabla w|^2 \, dx dt + \mu^3 E \int_0^T \int_G \theta^2 |w|^2 \, dx dt \\
\leq C \left\{ E \int_G \left[ |\theta(\delta) \nabla w(\delta)|^2 + \mu^2 \varphi(\delta) \theta(\delta) |w(\delta)|^2 + \mu^2 \varphi(T) |\theta(T) w(T)|^2 \right] \, dx \right. \\
+ E \int_0^T \int_G (1 + \varphi) \theta^2 (|f|^2 + \mu^2 |g|^2 + |\nabla g|^2) \, dx dt \right\}.
\]

for any solution \( w \) of the equation (4.4).

**Proof.** Integrating the identity (4.5) in \([\delta, T] \times G\) for \( \delta \in [0, T) \), and taking mathematical expectation, by (4.4) and \( z|_\Sigma = 0 \), we have that

\[
2E \int_0^T \int_G |I_1|^2 \, dx dt + E \int_0^T \int_G d(|\nabla z|^2 - 3\mu^2 \varphi |z|^2) \, dx + \mu^2 E \int_0^T \int_G (3\mu \varphi - 2)|z|^2 \, dx dt \\
+ 2E \int_0^T \int_G \mu |\nabla z|^2 \, dx dt - E \int_0^T \int_G \left[ |\nabla z|^2 + 2\mu \Re(z \overline{dz}) - 3\mu^2 \varphi dz \overline{dz} \right] \, dx \\
= 2E \int_0^T \int_G \Re \left[ \theta T_1 (f \, dt + g \, dB) \right] \, dx \\
\leq 2E \int_0^T \int_G |I_1|^2 \, dx dt + 2E \int_0^T \int_G |\theta f|^2 \, dx dt.
\]

It is easy to check that

\[
- E \int_0^T \int_G d(|\nabla z|^2 - 3\mu^2 \varphi |z|^2) \, dx \leq C E \int_G \left[ |\nabla z(\delta)|^2 + \mu^2 \varphi(T) |z(T)|^2 \right] \, dx.
\]

On the other hand, noting that \( 2 \Re(z \overline{dz}) = z \overline{dz} + \overline{z} dz = d(|z|^2) - |dz|^2 \), we obtain that

\[
E \int_\delta^T \int_G \left[ |\nabla z(\delta)|^2 - 3\mu^2 \varphi dz \overline{dz} + 2\mu \Re(z \overline{dz}) \right] \, dx \\
\leq C E \int_\delta^T \int_G \theta^2 \left[ |\nabla g|^2 + \mu^2 (1 + \varphi) \theta^2 |g|^2 \right] \, dx dt + E \int_G \left| z(\delta) \right|^2 \, dx.
\]

By (4.8)-(4.10), we have that

\[
E \int_\delta^T \int_G 3\mu^2 \left( (\mu \varphi - 1) |z|^2 + 2\mu |\nabla z|^2 \right) \, dx dt \\
\leq C E \int_G \left[ |\nabla z(\delta)|^2 + \mu |z(\delta)|^2 + \mu^2 \varphi(T) |z(T)|^2 \right] \, dx \\
+ C E \int_\delta^T \int_G (1 + \varphi) \theta^2 (|f|^2 + \mu^2 |g|^2 + |\nabla g|^2) \, dx dt.
\]

Taking \( \mu_0 = 2 \) and noting that \( \varphi = e^{3\mu t} > 1 \), we obtain that \( \mu_0 \varphi - 1 > \frac{\mu_0}{2} \). Therefore, by (4.11) and \( z = \theta w \), one can get the desired inequality (4.7). 

\[\square\]
4.3 Proof of Theorem 4.1

This subsection is devoted to a proof of Theorem 4.1. We borrow some ideas from [18].

**Proof of Theorem 4.1.** The proof is divided into two steps.

**Step 1.** For any \( t_0 \in (0, T) \), we choose \( t_1 \) and \( t_2 \) satisfying that \( 0 < t_1 < t_2 < t_0 \). Let \( \rho \in C^\infty(\mathbb{R}; [0, 1]) \) be a function such that

\[
\rho = \begin{cases} 
1, & t \geq t_2, \\
0, & t \leq t_1. 
\end{cases} \tag{4.12}
\]

Put \( h = \rho w \). Then by the equation (4.1), \( h \) satisfies that

\[
\begin{align*}
\frac{dh}{dt} - (1 + ib)\Delta h + [(a_1, \nabla h) + a_2 h + \rho \varepsilon w]dt + a_3 h dB(t) & \quad \text{in } Q, \\
\int_\Sigma \varepsilon h = 0 & \quad \text{on } \Sigma, \\
\int_G h = 0 & \quad \text{in } G. 
\end{align*} \tag{4.13}
\]

Applying Theorem 4.2 (with \( \delta = 0 \)) to the equation (4.13), we can find a \( \mu_1 > 2 \) such that for any \( \mu \geq \mu_1 \),

\[
\mu \mathbb{E} \int_0^T \int_G \rho^2 |\nabla h|^2 dx dt + \mu^3 \mathbb{E} \int_0^T \int_G \varepsilon^2 |h|^2 dx dt \leq C \mathbb{E} \left\{ \varepsilon^2 (T) \int_G \left[ |\nabla h(T)|^2 + \mu^2 \varepsilon^2 |h(T)|^2 \right] dx \right. + \left. \int_Q \varepsilon^2 |\rho(t)w|^2 dx dt \right\}. \tag{4.14}
\]

Noting that \( \varepsilon = e^{\mu \varepsilon^2 t} \) is an increasing function of \( t \), by (4.12), we have that

\[
\mathbb{E} \int_Q \varepsilon^2 |\rho(t)w|^2 dx dt \leq C \mathbb{E} \int_1^t \int_G \varepsilon^2 |w|^2 dx dt \leq C \varepsilon^2 (t) |h|_{L^2(0,T;L^2(G;\mathbb{R}))}^2. \tag{4.15}
\]

Therefore, combining (4.14) and (4.15), we get that

\[
\mu^2 (t_0) \mathbb{E} \int_0^T \int_G |\nabla h|^2 dx dt + \mu^3 \varepsilon^2 (t_0) \mathbb{E} \int_0^T \int_G \varepsilon^2 |h|^2 dx dt \leq \mu \mathbb{E} \int_0^T \int_G \varepsilon^2 |\nabla h|^2 dx dt + \mu^3 \varepsilon^2 \int_0^T \int_G \varepsilon^2 |h|^2 dx dt \leq C \varepsilon^2 (T) \mathbb{E} \int_G \left[ |\nabla h(T)|^2 + \mu^2 \varepsilon^2 |h(T)|^2 \right] dx + C \varepsilon^2 (t_2) |w|^2_{L^2(0,T;L^2(G;\mathbb{R}))}. \tag{4.16}
\]

By (4.16) and noting that \( h = \rho w \), we obtain that

\[
\mu \mathbb{E} \int_0^T \int_G |\nabla h|^2 dx dt + \mu^3 \varepsilon \mathbb{E} \int_0^T \int_G \varepsilon^2 |h|^2 dx dt \leq C \varepsilon^{-2} (t_0) \varepsilon^2 (t_2) |w|^2_{L^2(0,T;L^2(G;\mathbb{R}))} + C \varepsilon^2 (T) \mathbb{E} \int_G \left[ |\nabla w(T)|^2 + \mu^2 \varepsilon^2 |w(T)|^2 \right] dx. \tag{4.17}
\]

**Step 2.** Let us estimate \( \mu \mathbb{E} \int_G |w(t_0)|^2 dx \).
By (4.1) and (4.2), it is easy to check that

\[
\mathbb{E} \int_G |w(t_0)|^2 dx - \mathbb{E} \int_G |w(T)|^2 dx \\
= -\mathbb{E} \int_t^T \int_G [w dw + \nabla w |dw|^2] dx \\
= 2 \int_t^T \int_G |\nabla w|^2 - \mathbb{E} \int_t^T \int_G [w(a_1, \nabla w) + \nabla(a_1, \nabla w) + 2a_2 |w|^2 + |a_3 w|^2] dx dt \\
\leq C \int_t^T \int_G |\nabla w|^2 dx dt + C r \mathbb{E} \int_t^T \int_G |w|^2 dx dt.
\]

Combining (4.17)-(4.18), we find that

\[
\mathbb{E} \int_G |w(t_0)|^2 dx \leq C \int_t^T \int_G |\nabla w|^2 + C r \mathbb{E} \int_t^T \int_G |w|^2 dx dt.
\]

(4.18)

Note that \( t_2 < t_0 \). We choose a \( \mu > 1 \) as a minimizer of the right hand side in the inequality (4.19). Then it follows that

\[
\mathbb{E} \int_G |w(t_0)|^2 dx \leq C \int_t^T \int_G |\nabla w|^2 + C r \mathbb{E} \int_t^T \int_G |w|^2 dx dt.
\]

(4.19)

This completes the proof of Theorem 4.1.

\[\square\]

5 Appendix A

Proof of Lemma 3.1. In (2.1), choose \( a_0 = 1, a = -1, b = 0, b_0 = 0 \) and \( (a^{jk})_{1 \leq j, k \leq n} = I_n \). Then by (2.2), we obtain that

\[
2\theta I_1 L y \\
= 2|I_1|^2 dt + dM + \sum_{k=1}^n V_{x_k}^k + B |z|^2 dt + \sum_{j,k=1}^n D^{jk} z_{x_j} z_{x_k} dt \\
+ 2 \sum_{j=1}^n \left( E^j z_{x_j} \right) dt + \sum_{j=1}^n |dz_{x_j}|^2 + (-A + \ell_i) |dz|^2 + 2 \Phi zdz,
\]

(5.1)
where

\[
\begin{aligned}
A &= |\nabla \ell|^2 - \Delta \ell, \quad \Lambda = \Delta z + A z, \quad I_1 = \Lambda + (\Phi - \ell)z, \\
B &= 2 \sum_{j=1}^{n} (A \ell_{x_j})_{x_j} - A_t - 2A\Phi - 2(\Phi^2 - \ell_t\Phi) + \ell_{tt}, \\
D^{jk} &= 2\Phi \delta_k^j + 4 \ell_{x_jx_k} - 2\Delta \ell \delta_k^j \quad \text{with} \quad \delta_k^j = \begin{cases} 1 & j = k, \\ 0 & j \neq k, \end{cases} \\
M &= A|z|^2 - |\nabla z|^2 - \ell_t|z|^2, \\
V^k &= 2z_{x_k}dz - 2A\ell_{x_k}|z|^2dt - 2z_{x_k}\Phi zd\tau + 2|\nabla z|^2\ell_{x_k}dt - 4\nabla \ell \cdot \nabla zz_{x_k}dt, \\
E^j &= 2\ell_{x_j}(\Phi - \ell_t) - \Phi_{x_j}.
\end{aligned}
\]

Also, set $\Phi = 2\Delta \ell$. Then it is easy to check that for any $j, k = 1, \cdots, n,$

\[
\ell_{x_j} = \lambda \mu \varphi \psi_{x_j}, \quad \ell_{x_jx_k} = \lambda \mu \varphi \psi_{x_jx_k} + \lambda \mu^2 \varphi \psi_{x_j} \psi_{x_k}, \quad \ell_{tt} = O(\mu^{\lambda}) \varphi^3,
\]

\[
A = \lambda^2 \mu^2 \varphi^2 |\nabla \psi|^2 - \lambda^2 \mu \varphi |\nabla \psi|^2 - \lambda \mu \varphi \Delta \psi = \lambda^2 \mu^2 \varphi^2 |\nabla \psi|^2 + O(\lambda) \mu^2 \varphi,
\]

and for any $k = 1, 2, \cdots, n,

\[
A_{x_k} = 2\lambda^2 \mu^3 \varphi^2 |\nabla \psi|^2 \psi_{x_k} + O(\lambda^2) \mu^2 \varphi^2 + O(\lambda) \mu^3 \varphi.
\]

In the following, we estimate every term in the right side of sign of equality in (5.1).

**Step 1.** First, notice that

\[
B = 2\nabla A \cdot \nabla \ell - 2A\Delta \ell - A_t + \ell_{tt} - 8(\Delta \ell)^2 + 4\Delta \ell \ell_t.
\]

Further,

\[
2\nabla A \cdot \nabla \ell = 2\lambda \mu \varphi \nabla A \cdot \nabla \psi
\]

\[
= 2 \sum_{j=1}^{n} \left[ 2\lambda^2 \mu^3 \varphi^2 |\nabla \psi|^2 \psi_{x_j} + O(\mu^2) \lambda^2 \varphi^2 + O(\lambda) \mu^3 \varphi \right] \cdot \lambda \mu \varphi \psi_{x_j} + O(\lambda^2) \mu^4 \varphi^2.
\]

Further,

\[
-2A\Delta \ell = -2 \left[ \lambda^2 \mu^2 \varphi^2 |\nabla \psi|^2 + O(\lambda) \mu^2 \varphi \right] \left[ \lambda \mu \varphi |\nabla \psi|^2 + O(\lambda) \mu \varphi \right] = -2\lambda^3 \mu^4 \varphi^3 |\nabla \psi|^4 + O(\mu^3) \lambda^3 \varphi^3 + O(\lambda^2) \mu^4 \varphi^2.
\]

Further,

\[
-8(\Delta \ell)^2 + 4\Delta \ell \ell_t
\]

\[
= O(\lambda^2) \mu^4 \varphi^2 + O(\lambda) \mu^2 \varphi \cdot \lambda e^{2|\varphi| O(\lambda)} \varphi^2 = O(\lambda^2) \mu^4 \varphi^2 + O(\lambda^2) \varphi^3.
\]
Combining (5.3)-(5.5) with (5.2), we get that

\[ B = 2\lambda^3\mu^4\varphi^3|\nabla\psi|^4 + \mathcal{O}(\mu^3)\lambda^3\varphi^3 + \mathcal{O}_\mu(\lambda^2)\varphi^3. \]  

(5.6)

**Step 2.** Noticing that \( z = 0 \) on \( \Sigma \), we have that for any \( k = 1, 2, \ldots, n, \)

\[
V^k|_{\Sigma} = 2\left(\nabla z)^2 \ell_{x_k} - 2\nabla z \cdot \nabla \ell_{x_k}\right) dt|_{\Sigma}
= 2\left(\lambda \mu \varphi \left| \frac{\partial z}{\partial \nu} \right|^2 \phi_{x_k} - 2\lambda \mu \varphi \left| \frac{\partial \phi}{\partial \nu} \right| \nu_k \right) dt|_{\Sigma},
\]

where \( \nu = (\nu_1, \ldots, \nu_n) \) denotes the unit outer normal vector on \( \Gamma \). Therefore,

\[
\sum_{k=1}^{n} V^k \cdot \nu_k = -2\lambda \mu \varphi \left| \frac{\partial z}{\partial \nu} \right| \left| \frac{\partial \phi}{\partial \nu} \right| \leq 0. \tag{5.7}
\]

**Step 3.** By the definitions of \( D^{jk} \) \( (j, k = 1, \ldots, n) \) and \( \Phi \), we have

\[
D^{jk} = 2\Delta \ell \delta^j_k + 4\ell_{x_j} \ell_{x_k}
= 2\left(\lambda \mu \varphi \Delta \phi + \lambda \mu^2 \varphi |\nabla \phi|^2 \right) \delta^j_k + 4\lambda \mu \varphi \phi_{x_j} \phi_{x_k} + 4\lambda \mu^2 \varphi \phi_{x_j} \phi_{x_k}
= \mathcal{O}(\mu)\lambda \varphi + 2\lambda \mu^2 \varphi |\nabla \phi|^2 \delta^j_k + 4\lambda \mu^2 \varphi \phi_{x_j} \phi_{x_k}.
\]

It follows that

\[
\sum_{j,k=1}^{n} D^{jk} z_{x_j} z_{x_k} = 2\lambda \mu^2 \varphi |\nabla \phi|^2 |\nabla z|^2 + \mathcal{O}(\mu)\lambda \varphi |\nabla z|^2 + 4\lambda \mu^2 \varphi |\nabla \phi| |\nabla \phi| \cdot |\nabla z|^2. \tag{5.8}
\]

**Step 4.** By the definitions of \( E^j \) \( (j = 1, 2, \ldots, n) \), we have that

\[
2 \sum_{j=1}^{n} \left( E^j z_{x_j} \right) dt = 2 \sum_{j=1}^{n} \left( 4\ell_{x_j} \Delta \ell - 2\ell_{x_j} \ell_t - 2\Delta \ell_{x_j} \right) z_{x_j} dt
= 4 \sum_{j=1}^{n} \ell_{x_j} \Delta \ell |z|^2 dt - 2 \sum_{j=1}^{n} \ell_{x_j} \ell_t |z|^2 dt + \mathcal{O}(\lambda)\mu^3 \varphi |z||\nabla z| dt
= \sum_{j=1}^{n} \left( 4\ell_{x_j} \Delta \ell \ell_{x_j} |z|^2 - 2\ell_{x_j} \ell_t |z|^2 \right) dt
- 4 \sum_{j=1}^{n} \ell_{x_j} \Delta \ell \ell_{x_j} z^2 dt + 2 \sum_{j=1}^{n} \ell_{x_j} \ell_t |z|^2 dt + \mathcal{O}(\lambda)\mu^3 \varphi |z||\nabla z| dt
= \sum_{j=1}^{n} \left( 4\ell_{x_j} \Delta \ell |z|^2 - 2\ell_{x_j} \ell_t |z|^2 \right) dt + \mathcal{O}(\lambda)\mu^4 \varphi^2 |z|^2 dt + \mathcal{O}_\mu(\lambda^2)\varphi^3 |z|^2 dt + \mathcal{O}(\lambda)\mu^3 \varphi |z||\nabla z| dt.
\]

Therefore,

\[
2\mathbb{E} \int_Q \sum_{j=1}^{n} \left( E^j z_{x_j} \right) dx\, dt = \mathbb{E} \int_Q \left[ \mathcal{O}(\lambda^2)\mu^4 \varphi^2 + \mathcal{O}_\mu(\lambda^2)\varphi^3 \right] |z|^2 \, dz\, dt + \mathbb{E} \int_Q \mathcal{O}(\lambda)\mu^3 \varphi^2 |z||\nabla z| dx\, dt. \tag{5.9}
\]

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Step 5. By the first equation of (3.3), we find that

$$\mathbb{E} \int_Q (-A + \ell_t)|dz|^2 dx = \mathbb{E} \int_Q \left[\mathcal{O}(\lambda^2)\mu^2\varphi^2 + \mathcal{O}_\mu(\lambda)\varphi^2\right] \theta^2 |Y|^2 dx dt.$$ 

Also, notice that

$$2\Phi z dz = 4\Delta \ell z dz = 2\Delta \ell [d(z^2) - (dz)^2]$$

$$= 2d(\Delta \ell z^2) - 2\Delta \ell (dz)^2 dt - 2\Delta \ell (dz)^2.$$ 

This implies that

$$\mathbb{E} \int_Q 2\Phi z dz dx = \mathbb{E} \int_Q \left[\mathcal{O}(\lambda)\mu^2\varphi^2|z|^2 + \mathcal{O}(\lambda)\mu^2\varphi\theta^2|Y|^2\right] dx dt. \quad (5.10)$$

Combining (5.6)-(5.10) with (5.1), one can get the desired inequality (3.4). 

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