On a functional satisfying a weak Palais-Smale condition

A. Azzollini

Abstract

In this paper we study a quasilinear elliptic problem whose functional satisfies a weak version of the well known Palais-Smale condition. An existence result is proved under general assumptions on the nonlinearities.

Introduction

The aim of this paper is to generalize a recent result obtained in [2] concerning the following quasilinear elliptic problem

\[
\begin{cases} 
- \nabla \cdot (|\nabla u|^2 \nabla u) + |u|^{\alpha-2}u = f(u), & x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} u(x) = 0
\end{cases}
\]

where \( N \geq 2, \alpha \) behaves like \( t^{q/2} \) for small \( t \) and \( t^{p/2} \) for large \( t \), \( 1 < p < q < \min\{p^*, N\} \), \( 1 < \alpha \leq p^* q'/p' \), being \( p^* = \frac{pN}{N-p} \) and \( p' \) and \( q' \) the conjugate exponents, respectively, of \( p \) and \( q \).

In [2] the authors have proved that if \( f(t) = |t|^{s-2}t \) grows as \( t \) goes to \( +\infty \) more than \( \max\{t^{\alpha-1}, t^{q-1}\} \) and less than \( t^{p^*-1} \) and \( \phi \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) satisfies

\( \Phi_1 \) \( \phi(0) = 0 \),

\( \Phi_2 \) there exists a positive constant \( c \) such that

\[
\begin{cases} 
ct^{\frac{s}{2}} \leq \phi(t), & \text{if } t \geq 1, \\
ct^{\frac{q}{2}} \leq \phi(t), & \text{if } 0 \leq t \leq 1
\end{cases}
\]
(Φ3) there exists a positive constant $C$ such that

\[
\begin{aligned}
\phi(t) &\leq Ct^\frac{p}{2}, & \text{if } t \geq 1, \\
\phi(t) &\leq Ct^\frac{q}{2}, & \text{if } 0 \leq t \leq 1,
\end{aligned}
\]

(Φ4) the map $t \mapsto \phi(t^2)$ is strictly convex,

(Φ5) there exists $0 < \mu < 1$ such that

\[
\phi'(t)t \leq \frac{s\mu}{2} \phi(t), \quad \text{for all } t \geq 0,
\]

then the problem possesses infinitely many solutions. As remarked in that paper, the same result can be obtained if we assume more general hypotheses on the nonlinearity $f$. In particular, apart from the local assumptions related with the behaviour at $0$ and at infinity, it is required the following global condition

\[
0 < \theta F(t) \leq f(t)t, \quad \text{for all } t > 0,
\]

where $\theta > \alpha$ and $F(t) = \int_0^t f(z) \, dz$. This assumption, known as the Ambrosetti-Rabinowitz condition, (AR) in short, is quite classical in the field of critical points theory and typically occurs when we try to prove the boundedness of the Palais-Smale sequences related with the functional of the action.

However, some papers have shown that there are many situations in which (AR) can be successfully bypassed. In [6], for instance, the equation

\[
\begin{aligned}
-\Delta u &= g(u) \\
u(x) &\to 0, \quad \text{as } |x| \to \infty,
\end{aligned}
\]

is solved without (AR) in two steps. First the authors reduce the problem to that of minimizing a constrained (bounded below) functional, obtaining a solution to the equation, up to a Lagrange multiplier. Then they exploit the behaviour of the equation with respect to the rescaling to make the Lagrange multiplier disappear.

More recently, in [13] and [11] it has been shown a method, named monotonicity trick, which exploits the differentiability a.e. of the monotone functions to get bounded Palais Smale sequences for functionals related with approximating equations.

If there is no problem of compactness, this method allows us to get a Palais Smale sequence constituted by solutions of approximating equations. Afterwards, getting some more information on the elements of the sequence
using the fact that they solve an equation, we could prove the boundedness of the Palais-Smale sequence. Usually, the additional information we look for is the well known Pohozaev identity, an equality satisfied by sufficiently regular solutions of elliptic equations in the divergence form.

In our situation, a different approach is required. Consider the problem

$$
\begin{align*}
-\nabla \cdot [\phi(|\nabla u|^2) \nabla u] &= g(u), \quad x \in \mathbb{R}^N, \\
u(x) &\to 0, \quad \text{as } |x| \to \infty,
\end{align*}
$$

where we will assume on \( g \) hypotheses similar to those in [6].

Observe that, since \( \phi' \) is not homogeneous, we cannot proceed as in [6]. On the other hand, also the monotonicity trick does not seem to be of use. Indeed, since no regularity result on the solutions of (3) is available, we cannot obtain a Pohozaev identity in a standard way. To overcome these difficulties, we use a result contained in [10], where an alternative way of approaching (2) is showed. The method presented consists in adding a dimension to the space where the problem is set, and constructing a Palais-Smale sequence for a suitable auxiliary functional defined in this new space. Such a technique, which we call the adding dimension technique, permits to get additional information on a Palais-Smale sequence of the original functional and, possibly, to prove it is bounded. We remark the fact that this method does not require any regularity assumption on the solutions of the equations.

It is worthy of note that, differently from the functional related with (2), the functional of the action associated with (3) will be defined on a particular Orlicz-Sobolev space. Treating with this space carries some more complications when we try to solve (3) with a nonlinearity modeled on that in [6]. To explain better what we mean, we list our assumptions on \( g \) and state the main result.

Suppose that \( g \) is a continuous function satisfying the following hypotheses

\begin{itemize}
  \item [(g1)] \(-\infty \leq \limsup_{s \to +\infty} g(s)/s^{p^*-1} \leq 0, \text{ with } p^* = pN/(N - p);\)
  \item [(g2)] \(-\infty < \liminf_{s \to 0^+} g(s)/s^{\alpha - 1} \leq \limsup_{s \to 0^+} g(s)/s^{\alpha - 1} = -m < 0, \text{ for } 1 < \alpha < p^*;\)
\end{itemize}

and the following Berestycki-Lions condition

\begin{itemize}
  \item [(BL)] there exists \( \zeta > 0 \) such that \( G(\zeta) := \int_0^\zeta g(s) \, ds > 0.\)
\end{itemize}

We remark that, reading (AR) as a growth condition on \( f \), it is not difficult to see that, if we set \( g(u) = -|u|^{\alpha-2}u + f(u) \), condition (AR) implies (BL). We will prove the following result
Theorem 0.1. If $N \in \mathbb{N}$ with $N \geq 2$, $1 < p < q < \min\{N, p^*\}$, $p^* N'/p' \leq \alpha \leq p^* q'/p'$, and $(\Phi 1-\Phi 4)$, $(g1),(g2)$, $(BL)$ hold, then problem (3) possesses at least a nonnegative radially symmetric solution.

Comparing the main result in [2] with ours, we note that the prize we have to pay to generalize the nonlinearity is a more restrictive assumption on $\alpha$, which we require is not too close to 1. This fact arises from a significant difference between the classical embedding results known for Sobolev spaces, and the embedding results available for the Orlicz-Sobolev space where we set our problem.

To clarify this point, we recall a well known fact. Consider $D(\mathbb{R}^N)$, the set of all $C^\infty$ function in $\mathbb{R}^N$ with a compact support and set $1 < p < N$. Define the norm $\|\cdot\|_{D^{1,p}}$ such that for all $u \in D$,

$$
\|u\|_{D^{1,p}} = \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}}
$$

and set

$$
D^{1,p}(\mathbb{R}^N) = \overline{D(\mathbb{R}^N)}^{\|\cdot\|_{D^{1,p}}}.
$$

(4)

It is well known that $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, so that, for any $u \in D^{1,p}(\mathbb{R}^N)$,

$$
\left( \int_{\mathbb{R}^N} |u|^{p^*} \, dx \right)^{\frac{1}{p^*}} \leq C \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.
$$

(5)

If $1 < \alpha$ and $\|\cdot\|_\alpha$ is the usual Lebesgue norm, we define

$$
\mathcal{V} = \overline{D(\mathbb{R}^N)}^{\|\cdot\|_{D^{1,p}}+\|\cdot\|_\alpha}.
$$

(6)

Of course, since $\mathcal{V}$ is continuously embedded in $D^{1,p}(\mathbb{R}^N)$, inequality (5) holds true for any $u \in \mathcal{V}$. Observe that, if $\phi(t) = t^\frac{p}{p^*}$, the space $\mathcal{V}$ would be a nice set to study problem (1).

If we want to proceed analogously in our situation, we have to construct a space $\mathcal{W}$ taking into account $(\Phi 2)$ and $(\Phi 3)$. We have to substitute the norm $\|\cdot\|_{D^{1,p}}$, with a Luxemburg norm to be computed on $\nabla u$. Assumptions $(\Phi 2)$ and $(\Phi 3)$ suggest to use the norm of the space $L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$, which we call $\|\cdot\|_{p,q}$ and to replace Sobolev space $D^{1,p}(\mathbb{R}^N)$ with the Orlicz-Sobolev

$$
D^{1,p,q}(\mathbb{R}^N) = \overline{D(\mathbb{R}^N)}^{\|\nabla \cdot\|_{p,q}}.
$$

Unfortunately, the analogy with the Sobolev spaces stops here. Indeed, since $D^{1,p}(\mathbb{R}^N)$ and $D^{1,q}(\mathbb{R}^N)$ are continuously embedded in $D^{1,p,q}(\mathbb{R}^N)$,
there is no continuous embedding of $\mathcal{D}^{1,p,q}(\mathbb{R}^N)$ in any Lebesgue space (see Remark 1.8).

However, in [2] it has been proved that, if we define the analogous of $\mathcal{V}$ in the following way

$$\mathcal{W} = \overline{\mathcal{D}(\mathbb{R}^N)}^{\|\nabla \cdot \|_{p,q} + \| \cdot \|_\alpha}, \quad \text{for } 1 < \alpha < p^* q'/p'$$

then the following inequality holds true

$$\|u\|_{p^*} \leq C(\|u\|_\alpha + \|\nabla u\|_{p,q}), \quad \text{for all } u \in \mathcal{W}. \quad (7)$$

From this, we deduce that $\mathcal{W} \hookrightarrow L^r(\mathbb{R}^N)$ for any $\alpha \leq r \leq p^*$, even if, differently from $\mathcal{V}$, it is not possible to control the $L^{p^*}(\mathbb{R}^N)$ norm just with the $L^{p}(\mathbb{R}^N) + L^{q}(\mathbb{R}^N)$ norm of the gradient. This fact translates to a technical difficulty in proving the Palais Smale condition. Precisely, we will show that the functional of the action satisfies a compactness condition weaker than the Palais-Smale if $\alpha$ is not too close to 1.

Finally, we point out that, since we do not require assumption (Φ5), our existence result holds for function $\phi$ more general than those treated in [2].

The paper is so organized: in section 1 we will introduce the functional setting, and the related properties we will use in our variational approach to the problem. For the most part, the results contained in this section are proved in [2] and [4] so we only recall them. In section 2 we will define a new weakened version of the Palais-Smale condition, and we will introduce the definition of a particular type of Palais-Smale sequences. Finally, in section 3, we will prove our main result by means of the adding dimension technique introduced in [10].

**Notation**

- $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{R}^N$ according to the case.
- If $r > 0$, we denote by $B_r$ the ball of center 0 and radius $r$.
- If $\Omega \subset \mathbb{R}^N$, then $\Omega^c = \mathbb{R}^N \setminus \Omega$.
- Everytime we consider a subset of $\mathbb{R}^N$, we assume it is measurable and we denote by $| \cdot |$ its measure.
- We denote by $\mathcal{D}$ the space of all functions in $C^\infty(\mathbb{R}^N)$ with compact support.
• If \( \Omega \subset \mathbb{R}^N \), \( \tau \geq 1 \) and \( m \in \mathbb{N}^* \), we denote by \( L^\tau(\Omega) \) the Lebesgue space \( L^\tau(\Omega, \mathbb{K}) \), by \( \| \cdot \|_{L^\tau(\Omega)} \) its norm (\( \| \cdot \|_\tau \) if \( \Omega = \mathbb{R}^N \)) and by \( W^{m,\tau}(\Omega) \) the usual Sobolev spaces.

• \( C \) and \( c \) will denote generic constants which would change from line to line.

1 The functional setting

This section is devoted to the construction of the functional setting.

As a first step, we have to recall some known facts on the sum of Lebesgue spaces.

**Definition 1.1.** Let \( 1 < p < q \) and \( \Omega \subset \mathbb{R}^N \). We denote with \( L^p(\Omega) + L^q(\Omega) \) the completion of \( C_\infty^c(\Omega, \mathbb{K}) \) in the norm

\[
\| u \|_{L^p(\Omega) + L^q(\Omega)} = \inf \left\{ \| u \|_p + \| w \|_q \mid v \in L^p(\Omega), w \in L^q(\Omega), u = v + w \right\}.
\]

We set \( \| u \|_{p,q} = \| u \|_{L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)} \).

In the next proposition we give a list of properties that will be useful in the rest of the paper. The reader can find the proofs in [2] and [4]

**Proposition 1.2.** Let \( \Omega \subset \mathbb{R}^N \), \( u \in L^p(\Omega) + L^q(\Omega) \) and \( \Lambda_u = \{ x \in \Omega \mid |u(x)| > 1 \} \). We have:

(i) if \( \Omega' \subset \Omega \) is such that \( |\Omega'| < +\infty \), then \( u \in L^p(\Omega') \);

(ii) if \( \Omega' \subset \Omega \) is such that \( u \in L^\infty(\Omega') \), then \( u \in L^q(\Omega') \);

(iii) \( |\Lambda_u| < +\infty \);

(iv) \( u \in L^p(\Lambda_u) \cap L^q(\Lambda^c_u) \);

(v) \( L^p(\Omega) + L^q(\Omega) \) is reflexive and \( (L^p(\Omega) + L^q(\Omega))^\prime = L^{p^\prime}(\Omega) \cap L^{q^\prime}(\Omega) \);

(vi) \( \| u \|_{L^p(\Omega) + L^q(\Omega)} \leq \max \{ \| u \|_{L^p(\Lambda_u)}, \| u \|_{L^q(\Lambda^c_u)} \} \);

(vii) if \( B \subset \Omega \), then \( \| u \|_{L^p(\Omega) + L^q(\Omega)} \leq \| u \|_{L^p(B) + L^q(B)} + \| u \|_{L^p(\Omega \setminus B) + L^q(\Omega \setminus B)} \).

We can now define the Orlicz-Sobolev space where we will study our problem.
Definition 1.3. We assume the following definition

\[ D^{1,p,q}(\mathbb{R}^N) = \overline{D(\mathbb{R}^N)}^{\| \cdot \|_{p,q}}. \]

Moreover, if \( \alpha > 1 \), we denote with \( \mathcal{W} \) the following space

\[ \mathcal{W} = \overline{D(\mathbb{R}^N)}^{\| \cdot \|}. \]

where \( \| \cdot \| = \| \cdot \|_\alpha + \| \nabla \cdot \|_{p,q} \).

We again refer to [2] for the proofs of the following propositions and theorems on the space \( \mathcal{W} \).

Proposition 1.4. \((\mathcal{W}, \| \cdot \|)\) is a reflexive Banach space.

Proposition 1.5. If \( u \in \mathcal{W} \), then it verifies the following inequality

\[ \| u \|_{p^*}^r \leq C \left( \| u \|_{p^*}^{r-1} + \| u \|_{\alpha}^{(t-1)} \right) \| \nabla u \|_{p,q} \] (9)

where \( C \) is a positive constant which does not depend on \( u \) and \( t > 1 \) satisfies the equality \( \frac{t}{t-1} = \frac{p(N-1)}{N(p-1)} \).

Theorem 1.6. If \( 1 < p < \min\{q, N\} \) and \( 1 < p^* \frac{q'}{p'} \) then, for every \( \alpha \in \left( 1, p^* \frac{q'}{p'} \right) \), the space \((\mathcal{W}, \| \cdot \|)\) is continuously embedded into \( L^{p^*}(\mathbb{R}^N) \).

Remark 1.7. By interpolation we have that \((\mathcal{W}, \| \cdot \|)\) is continuously embedded into \( L^{r}(\mathbb{R}^N) \) for any \( r \in [\alpha, p^*] \).

Remark 1.8. The normed space \((\mathcal{W}, \| \nabla \cdot \|_{p,q})\) is not continuously embedded in any Lebesgue space. Indeed, consider \( \varphi \in D(\mathbb{R}^N) \) and for any \( t > 0 \) set \( \varphi_t = t^{\frac{N}{p} - \frac{N}{q}} \varphi(\frac{\cdot}{t}) \). Of course for any \( t > 0 \) the function \( \varphi_t \in \mathcal{W} \) and we have that

\[ \int_{\mathbb{R}^N} |\nabla \varphi_t|^p \, dx = \int_{\mathbb{R}^N} |\nabla \varphi|^p \, dx \]

\[ \int_{\mathbb{R}^N} |\varphi_t|^r \, dx = t^{\frac{N(p-N)}{p} + N} \int_{\mathbb{R}^N} |\varphi|^r \, dx \]

Since \( \| \nabla \varphi_t \|_{p,q} \leq \| \nabla \varphi \|_p \), we deduce that the family \((\varphi_t)_{t>0}\) is bounded in \((\mathcal{W}, \| \nabla \cdot \|_{p,q})\). On the other hand, if \( r \neq p^* \), we can make the Lebesgue norm as large as we want just taking large \( t \), if \( 1 < r < p^* \) and small \( t \), if \( p^* < r \). So \((\mathcal{W}, \| \nabla \cdot \|_{p,q})\) does not embed in any \( L^r(\mathbb{R}^N) \), for \( 1 < r \neq p^* \). We see that \((\mathcal{W}, \| \nabla \cdot \|_{p,q})\) does not embed even in \( L^{p^*}(\mathbb{R}^N) \) just observing that, if for any \( s > 0 \) we set \( \varphi_s = s^{\frac{N}{q} - \frac{N}{p}} \varphi(\frac{\cdot}{s}) \), then

\[ \sup_{s>0} \| \nabla \varphi_s \|_{p,q} \leq \sup_{s>0} \| \nabla \varphi_s \|_q < +\infty, \quad \sup_{s>0} \| \varphi_s \|_{p^*} = +\infty. \]
Now we define the functional of the action related with our problem. For any \( u \in \mathcal{W} \) we set (from now on, we omit the symbol \( dx \) in the integration)

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2) - \int_{\mathbb{R}^N} G(u),
\]

where \( G : \mathbb{R} \to \mathbb{R} \) is defined as in assumption (BL). To make the functional well defined and \( C^1 \), we modify \( g \) according to the following two possibilities:

1st case: \( \liminf_{s \to +\infty} \frac{g(s)}{s^{p^* - 1}} = 0 \).

Then we define \( \tilde{g} = g_1 - g_2 \) where

\[
g_1(s) = \begin{cases} 
(g(s) + ms^{\alpha - 1})^+, & \text{if } s \geq 0, \\
0, & \text{if } s < 0,
\end{cases}
\]

and

\[
g_2(s) = \begin{cases} 
g_1(s) - g(s), & \text{if } s \geq 0, \\
-g_2(-s), & \text{if } s < 0,
\end{cases}
\]

2nd case: \( \liminf_{s \to +\infty} \frac{g(s)}{s^{p^* - 1}} < 0 \).

Then there exist \( \varepsilon > 0 \) and an increasing sequence of positive numbers \( (s_n)_n \) such that \( g(s_n) \leq -\varepsilon s_n^{p^* - 1} \). By continuity, certainly there exists \( s_0 > 0 \) such that \( g(s_0) + ms_0^{\alpha - 1} = 0 \). We set \( \tilde{g} = g_1 - g_2 \) where

\[
g_1(s) = \begin{cases} 
(g(s) + ms^{\alpha - 1})^+, & \text{if } s \in [0, s_0], \\
0, & \text{if } s \in [0, s_0]^c,
\end{cases}
\]

and

\[
g_2(s) = \begin{cases} 
g_1(s) - g(s), & \text{if } s \in [0, s_0], \\
ms^{\alpha - 1}, & \text{if } s_0 < s \\
-g_2(-s), & \text{if } s < 0.
\end{cases}
\]

Since \( \tilde{g} \) satisfies

\[
(g3) \; \lim_{s \to \infty} \frac{|\tilde{g}(s)|}{s^{p^* - 1}} = 0,
\]

by [4] and Theorem 1.6 we can prove \( J \) is well defined and \( C^1 \) in \( \mathcal{W} \) if we replace \( g \) with \( \tilde{g} \). On the other hand, we point out that, if \( u \in \mathcal{W} \) solved equation (3) with \( \tilde{g} \) in the place of \( g \), then \( 0 \leq u \) and, if the second case occurred, then we also would have \( u \leq s_0 \). As a consequence, we deduce that no loss of generality would arise supposing that \( g \) is defined as \( \tilde{g} \) and \((g3)\) holds.

Some simple computations show that for functions \( g_1 \) and \( g_2 \) the following properties hold
(i) \( g_1 \) and \( g_2 \) are nonnegative in \( \mathbb{R}_+ \),

(ii) \( g = g_1 - g_2 \),

(iii) \( \lim_{t \to \infty} \frac{g_1(t)}{|t|^{p^* - 1}} = 0 \), \( \lim_{t \to 0^+} \frac{g_1(t)}{t^{\alpha - 1}} = 0 \),

(iv) there exists a positive constant \( a \) such that \( at^{\alpha - 1} \leq g_2(t) \), for any \( t \in \mathbb{R}_+ \),

(v) for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that \( g_1(t) \leq \varepsilon g_2(t) + C_\varepsilon t^{p^* - 1} \), for any \( t \in \mathbb{R}_+ \).

Once we have set \( G_i(z) := \int_0^z g_i(s) \, ds > 0 \) for \( i = 1, 2 \), we have that the functional can be written

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2) + \int_{\mathbb{R}^N} G_2(u) - \int_{\mathbb{R}^N} G_1(u),
\]

In order to have compactness, we introduce a symmetry requirement on our space.

**Definition 1.9.** Let us denote with

\[
\mathcal{D}(\mathbb{R}^N)_{\text{rad}} = \{ u \in \mathcal{D}(\mathbb{R}^N) \mid u \text{ is radially symmetric} \},
\]

and let \( \mathcal{W}_r \) be the completion of \( \mathcal{D}(\mathbb{R}^N)_{\text{rad}} \) in the norm \( \| \cdot \| \), namely

\[
\mathcal{W}_r = \overline{\mathcal{D}(\mathbb{R}^N)_{\text{rad}}}^{\| \cdot \|}.
\]

**Remark 1.10.** Observe that if \( 1 \leq \alpha \leq p^* \), the set \( \mathcal{W}_r \) is included in \( L^s(\mathbb{R}^N) \) for any \( s \in [\alpha, p^*] \).

Indeed, take \( u \in \mathcal{W}_r \), and consider the set \( \Lambda_{\nabla u} \). Since \( \| \nabla u \|_{p,q} < +\infty \), certainly \( \nabla u \in L^p(\Lambda_{\nabla u}) \). On the other hand, since \( u \in L^\alpha(\mathbb{R}^N) \), we have that \( u \in L^\alpha(\Lambda_{\nabla u}) \). So, if we define \( E(\Omega) = \{ v \in L^\alpha(\Omega) \mid \nabla u \in L^p(\Omega) \} \), then \( u \in E(\Lambda_{\nabla u}) \).

By symmetry of \( u \), the set \( \Lambda_{\nabla u} \) has a smooth boundary so, by standard arguments (see for example [1]), there exists a continuous extension operator \( T : E(\Omega) \to E(\mathbb{R}^N) \). Then embedding theorems hold in the domain \( \Lambda_{\nabla u} \) so we deduce that \( u \in L^s(\Lambda_{\nabla u}) \) for any \( s \in [\alpha, p^*] \). Analogously \( u \in L^s(\Lambda_{\nabla u}) \), for any \( s \in [\alpha, q^*] \). Since \( p^* < q^* \), we conclude.

At the present stage of our knowledge, we do not know if, for \( p^* q^* < \alpha < p^* \), these embeddings are also continuous.

The following compactness result holds.
Theorem 1.11. If $1 < p < q < N$ and $1 < \alpha \leq p^* \frac{q}{p}$, then the functionals

$$u \in \mathcal{W}_r \mapsto \int_{\mathbb{R}^N} G_1(u)$$
$$u \in \mathcal{W}_r \mapsto \int_{\mathbb{R}^N} g_1(u)u$$

are weakly continuous.

**Proof**  We prove the weak continuity of the first functional. By Lemma 2.13 in [2], for any $u \in \mathcal{W}_r$,\

$$|u(x)| \leq \frac{C}{|x|^{\frac{N}{q}}} \|
abla u\|_{p,q}, \text{ for } |x| \geq 1. \quad (10)$$

Now, consider a sequence $(u_n)_n$ in $\mathcal{W}_r$ weakly convergent to $u_0$. By Theorem 2.11 in [2], $(u_n)_n$ possesses a subsequence strongly convergent to $u_0$ in $L^\tau(\mathbb{R}^N)$ for any $\tau \in ]\alpha, p^*[$. So, up to subsequences, we can assume that $(u_n)_n$ converges almost everywhere to $u_0$. Set $P(t) = G_1(t)$, $Q(t) = |t|^\alpha + |t|^{p^*}$, $v = G_1(u_0)$. By property (iii) of the function $g_1$, remark 1.7 and (10), we can apply the Strauss compactness Lemma in the version as it appears in [6] and conclude. In a similar way we prove the rest of the statement. $\square$

2  A weak Palais-Smale condition

As it is well known, the Palais-Smale condition is a compactness property related to a functional defined on a Banach space. It states as follows: let $I : E \to \mathbb{R}$ be a $C^1$ functional on the Banach space $E$ and $c \in \mathbb{R}$. If for any given $(x_n)_n$ in $E$ such that $I(x_n) \to c$ and $I'(x_n) \to 0$ there exists a converging subsequence of $(x_n)_n$, we say that $I$ satisfies the Palais Smale condition at the level $c$. Usually, in the calculus of variation, testing the Palais Smale condition consists in two steps: first we check if every Palais Smale sequence (namely a sequence verifying the previous assumptions) is bounded, second we handle with compact embedding theorems to prove strong convergence (up to a subsequence) in the Banach space. Many times it happens that the main problem in verifying Palais-Smale condition is related with the first step. In such cases, one tries to prove that the functional satisfies at
least a weakened version of the Palais-Smale condition, and look for the existence of at least one sequence to which that condition can be applied. In this direction, we recall, for example, the well-known Cerami version of the Palais-Smale condition (see [8]), and the problem in [5] where this condition is applied.

Here we introduce a new version of a weakened Palais-Smale condition.

**Definition 2.1.** Suppose that \( (E, \| \cdot \|_E) \) and \( (F, \| \cdot \|_F) \) are two Banach spaces such that \( (E, \| \cdot \|_E) \hookrightarrow (F, \| \cdot \|_F) \). A functional \( I \in C^1(E, \mathbb{R}) \) satisfies a weak Palais-Smale condition with respect to \( E \) and \( F \) if for any sequence \( (x_n)_n \) in \( E \) such that

1. \( I(x_n) \) is bounded,
2. \( I'(x_n) \to 0 \) in \( E' \),
3. \( (\|x_n\|_F)_n \) is bounded,

there exists a converging subsequence (in the topology of \( E \)).

**Remark 2.2.** Consider the functional of the action related with the equation

\[-\Delta u = g(u)\]

where \( g \) is as in [6]. After having produced a suitable modification of the function \( g \), we can see that finite energy solutions of the equation are critical points of

\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} G(u), \quad u \in H^1_0(\mathbb{R}^N), \]

being \( G \) a primitive of \( g \). The properties on \( g_1 \) and \( g_2 \) listed in (i) \ldots (v) hold, except that we have to replace \( \alpha \) with \( 2 \) and \( p^* \) with \( 2^* \).

We show that \( I \) verifies a weak Palais-Smale condition with respect to \( H^1_0(\mathbb{R}^N) \) and \( D^{1,2}_r(\mathbb{R}^N) \), each one provided with its natural norm. Indeed suppose \( (u_n)_n \) is a Palais Smale sequence for which \( (\|\nabla u\|_2)_n \) is bounded. Since \( I(u_n) \) is bounded, for a certain \( M > 0 \) and any \( \varepsilon \in ]0, 1[ \) there exists a suitable \( C_\varepsilon > 0 \) for which we have

\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} G_2(u_n) \leq \int_{\mathbb{R}^N} G_1(u_n) + M \\
\leq \varepsilon \int_{\mathbb{R}^N} G_2(u_n) + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{2^*} + M \\
\leq \varepsilon \int_{\mathbb{R}^N} G_2(u_n) + C_S C_\varepsilon \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{\frac{2^*}{2}} + M
\]
where \( C_{\frac{1}{S}} \) is the Sobolev constant for the embedding \( \mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N) \). Then

\[
\frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + (1 - \varepsilon) \int_{\mathbb{R}^N} G_2(u_n) \leq C_S C_\varepsilon \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{\frac{2^*}{2}} + M
\]

and, since \((\|
abla u_n\|_2)_n\) is bounded, we conclude that \((\|
abla u\|_2)_n\) is also bounded since we have

\[
a(1 - \varepsilon) \int_{\mathbb{R}^N} |u_n|^2 \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + (1 - \varepsilon) \int_{\mathbb{R}^N} G_2(u_n) \leq C_S C_\varepsilon \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \right)^{\frac{2^*}{2}} + M.
\]

At this point the arguments are quite standard: we extract a weakly convergent (in \( H^1 \)-norm) subsequence and we use radial symmetry of functions in our space and a Strauss compactness lemma to find a strong convergent sequence.

Set

\[
\mathcal{Y}_r = \mathcal{D}(\mathbb{R}^N)_{\text{rad}} \|\nabla\|_{p,q}.
\]

We have the following result

**Theorem 2.3.** Under the assumptions of Theorem 0.1, the functional \( J \) satisfies a weak Palais-Smale condition, with respect to \( \mathcal{W}_r \) and \( \mathcal{Y}_r \).

**Proof** Suppose \((u_n)_n\) is a sequence of functions in \( \mathcal{W}_r \) such that 1, 2 and 3 of definition 2.1 hold.
We first prove that the sequence is bounded. By computations analogous to those in remark 2.2, there exist \( M > 0 \) and \( C_1 > 0 \), such that

\[
\frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u_n|^2) + (1 - \varepsilon) \int_{\mathbb{R}^N} G_2(u_n) \leq C_1 \int_{\mathbb{R}^N} |u_n|^{p^*} + M.
\]

Now, if \((u_n)_n\) is bounded in the \( L^{p^*} \)-norm, we have concluded.
By (9) and 3 of definition 2.1,

\[
\|u_n\|_{p^*}^{t^*} \leq C_2 \left( \|u_n\|_{p^*}^{t^* - 1} + \|u_n\|_{\alpha}^{t^* - 1} \right)
\]

for some \( C_2 > 0 \). Suppose that \((\|u_n\|_{p^*})_n\) diverges (up to a subsequence).
Then, by (13), certainly there exists a constant \( C \) such that, definitely,

\[
\|u_n\|_{p^*} \leq C \|u_n\|_{\alpha}^{\frac{t^* - 1}{t^* - 1}}.
\]
Comparing (12) and (14), taking into account that \( t_{i-1} = \frac{p'}{N} \) and \( a\|u_n\|_\alpha^\alpha \leq \int_{\mathbb{R}^N} G_2(u_n) \), we have, for some positive constant \( C \),

\[
\|u_n\|_\alpha^\alpha \leq C\|u_n\|_{\alpha^{p'}}^\alpha
\]

and then, since \( \alpha > \frac{Np^*}{p'} \), the sequence \((\|u_n\|_\alpha)_n\) is bounded.

Therefore, by Proposition 1.4 and Theorem 1.11, there exists \( u_0 \in \mathcal{W}_r \) such that, up to subsequences,

\begin{align*}
  u_n &\rightharpoonup u_0, \quad \text{weakly in } \mathcal{W}_r, \quad (15) \\
  \int_{\mathbb{R}^N} G_1(u_n) &\to \int_{\mathbb{R}^N} G_1(u_0), \quad (16) \\
  \int_{\mathbb{R}^N} g_1(u_n)u_n &\to \int_{\mathbb{R}^N} g_1(u_0)u_0, \quad (17)
\end{align*}

and, by [2, Theorem 2.11],

\[ u_n \rightharpoonup u_0, \quad \text{a.e. in } \mathbb{R}^N. \]

From this point till the end, the proof follows the scheme of Proposition 3.3 in [2] and of Proposition 2 in [7], step 9.1c (see also [3, Lemma 3.5]). We point out only the key passages. By (15) and arguing as in [12, page 208], we have

\[
\nabla u_n \rightharpoonup \nabla u_0, \quad \text{weakly in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N),
\]

\[
   u_n \rightharpoonup u_0, \quad \text{weakly in } L^\alpha(\mathbb{R}^N). \quad (19)
\]

As in [3] we prove that for any \( z \in C_0^\infty(\mathbb{R}^N) \), we have

\[
\int_{\mathbb{R}^N} g_i(u_n)z \to \int_{\mathbb{R}^N} g_i(u_0)z \quad i = 1, 2. \quad (20)
\]

Set

\[
A_1(u) = \frac{1}{2} \int_{\mathbb{R}^N} \phi(|\nabla u|^2), \quad A_2(u) = \int_{\mathbb{R}^N} G_2(u), \quad B(u) = \int_{\mathbb{R}^N} G_1(u).
\]

By (20), and since \((u_n)_n\) is bounded in \( \mathcal{W}_r \), \( J'(u_n) \to 0 \) in \( \mathcal{W}'_r \) implies that \( A_1'(u_n) \to B'(u_0) - A_2'(u_0) \) in \( \mathcal{W}'_r \). By convexity, we have

\[
A_1(u_n) \leq A_1(u_0) + A_1'(u_n)[u_n - u_0]
\]

and then, passing to the limit,

\[
\limsup_{n} A_1(u_n) \leq A_1(u_0).
\]
Since, by weak lower semicontinuity of $A_1$ we also have
\[ A_1(u_0) \leq \liminf_n A_1(u_n), \]
we conclude that
\[ \lim_n A_1(u_n) = A_1(u_0). \]  \tag{21}

By (18) and (21), we can deduce (see [9])
\[ \nabla u_n \to \nabla u_0, \quad \text{in } L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N). \]

Moreover, since
\[ \lim_n \int_{\mathbb{R}^N} g_2(u_n) u_n = \lim_n \left( \int_{\mathbb{R}^N} g_1(u_n) u_n - \int_{\mathbb{R}^N} \phi'(|\nabla u_n|^2)|\nabla u_n|^2 \right) \]
\[= \int_{\mathbb{R}^N} g_1(u_0) u_0 - \int_{\mathbb{R}^N} \phi'(|\nabla u_0|^2)|\nabla u_0|^2 \]
\[= \int_{\mathbb{R}^N} g_2(u_0) u_0, \]
we are able also to prove that $u_n \to u_0$ in $L^\alpha(\mathbb{R}^N)$ and we conclude. \[\square\]

## 3 Proof of the main Theorem

In view of Lemma 2.3, we have just to find a level for which we can find a Palais-Smale sequence satisfying the boundedness assumption 3 in the Definition 2.1.

**Lemma 3.1.** The set
\[ \Gamma := \{ \gamma \in C([0, 1], W_\tau) \mid \gamma(0) = 0, I(\gamma(1)) < 0 \} \]
is nonempty.

**Proof** Starting from the function $z \in \mathcal{D}(\mathbb{R}^N)$ for which $\int_{\mathbb{R}^N} G(z) > 0$ (the existence of such a function is proved in [6]), the proof is standard. Indeed consider $z_l(\cdot) = z(\cdot/l)$ for a value of $l > 0$ to be established and compute
\[ J(z_l) \leq C_1 \int_{\Lambda_{Gz_l}} |\nabla z_l|^q + C_2 \int_{\Lambda_{Gz_l}} |\nabla z_l|^p - \int_{\mathbb{R}^N} G(z_l) \]
\[\leq C \int_{\mathbb{R}^N} |\nabla z_l|^p - \int_{\mathbb{R}^N} G(z_l) = C l^{N-p} \int_{\mathbb{R}^N} |\nabla z|^p - l^N \int_{\mathbb{R}^N} G(z). \]
We deduce that $J(z_l) < 0$ if $l$ is sufficiently large. At this point any continuous path connecting $0$ with $z_l$ is in $\Gamma$.

Set
\[
\gamma_{mp} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).
\] (22)

**Lemma 3.2.** The level $\gamma_{mp}$ is positive.

**Proof** Of course it is enough to verify the following geometrical mountain pass assumptions: there exist $\delta, \rho > 0$ such that

- $J(u) \geq \delta$, for all $u \in W_r$ such that $\|u\| = \rho$

- $J(u) \geq 0$, for all $u \in W_r$ such that $\|u\| \leq \rho$.

By ($\Phi_2$), (vi) of Proposition 1.2 and since $W \hookrightarrow L^{p^*}(\mathbb{R}^N)$, we have that, if $\|u\|$ is sufficiently small (note that $p < q$)

\[
J(u) \geq c_1 \int_{\Lambda_{V_u}} \|\nabla u\|^q + c_2 \int_{\Lambda_{V_u}} \|\nabla u\|^p + (1 - \varepsilon) \int_{\mathbb{R}^N} G_2(u) - C_\varepsilon \int_{\mathbb{R}^N} |u|^{p^*}
\]

\[
\geq c \max \left( \int_{\Lambda_{V_u}} \|\nabla u\|^q, \int_{\Lambda_{V_u}} \|\nabla u\|^p \right) + c \int_{\mathbb{R}^N} |u|^\alpha - C_\varepsilon \int_{\mathbb{R}^N} |u|^{p^*}
\]

\[
\geq c \left[ \|u\|_{p,q}^q + \|u\|_{\alpha}^\alpha - \|u\|_{p^*}^{p^*} \right]
\]

\[
\geq c \left[ \|u\|_{\max(\alpha,q)}^\alpha - \|u\|_{p^*}^{p^*} \right].
\]

Taking respectively $\|u\| = \rho$ or $\|u\| \leq \rho$ with $\rho > 0$ sufficiently small we conclude. \qed

We introduce the following auxiliary functional on the space $\mathbb{R} \times W_r$

\[
\tilde{J}(\theta, u) = \frac{e^{N\theta}}{2} \int_{\mathbb{R}^N} \phi(e^{-2\theta}|\nabla u|^2) - e^{N\theta} \int_{\mathbb{R}^N} G(u).
\]

In analogy with $\Gamma$ and $\gamma_{mp}$, we define

\[
\tilde{\Gamma} = \{ \tilde{\gamma} \in C([0,1], \mathbb{R} \times W_r) \mid \tilde{\gamma}(0) = (0,0), \tilde{J}(\tilde{\gamma}(1)) < 0 \}
\]

and

\[
\tilde{\gamma}_{mp} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}} \sup_{t \in [0,1]} \tilde{J}(\tilde{\gamma}(t)).
\]
Proposition 3.3. The functional $\tilde{J}$ verifies the geometrical assumptions of the mountain pass theorem, so that $\tilde{c}_{mp}$ is the mountain pass level. Moreover $c_{mp} = \tilde{c}_{mp}$.

Proof. We estimate the functional $\tilde{J}$. Since $\phi$ is increasing in $\mathbb{R}_+$, by similar computations as those in Lemma 3.2, for small $\|u\|$ we have:

$$\tilde{J}(\theta, u) \geq \frac{e^{N\theta}}{2} \int_{\mathbb{R}^N} \phi(e^{-2\theta}|\nabla u|^2) - e^{N\theta} \int_{\mathbb{R}^N} G(u) \tag{23}$$

$$\geq ce^{N\theta} \left[ e^{-q\theta} \|\nabla u\|_{p,q}^q + \|u\|_{\alpha}^\alpha - \|u\|_{p^*}^{p^*} \right] \tag{24}$$

$$\geq ce^{N\theta} \left[ e^{-q\theta} \|u\|_{\max\{\alpha, q\}}^\max\{\alpha, q\} - C\|u\|^{p^*} \right] \tag{25}$$

So we deduce that there exists $\delta > 0$ such that $\tilde{J}(\theta, u)$ is nonnegative if $\sqrt{\theta^2 + \|u\|^2} \leq \delta$, and it is positive for $\sqrt{\theta^2 + \|u\|^2} = \delta$.

As in Lemma 3.1 we can prove the existence of $(\tilde{\theta}, \tilde{u}) \in \mathbb{R} \times \mathcal{W}_r$ for which $\tilde{J}(\tilde{\theta}, \tilde{u}) < 0$.

Finally observe that, since for any $\gamma \in \Gamma$ we have that $\tilde{\gamma} = (0, \gamma) \in \tilde{\Gamma}$ and $J \circ \gamma = \tilde{J} \circ \tilde{\gamma}$, certainly $\tilde{c}_{mp} \leq c_{mp}$. Now, suppose $\tilde{\gamma}(\cdot) = (\theta(\cdot), \gamma(\cdot)) \in \tilde{\Gamma}$. Then if we set $\eta(t)(\cdot) = \gamma(t)(e^{-\theta(t)} \cdot)$, we have that $\eta \in \Gamma$ and $J \circ \eta = \tilde{J} \circ \tilde{\gamma}$.

So, since $c_{mp} \leq \tilde{c}_{mp}$, we conclude that the two values coincide. \hfill \Box

Now we are ready to prove the following fundamental result

Proposition 3.4. There exists a sequence $(u_n)_n$ in $\mathcal{W}_r$ satisfying 1, 2 and 3 in Definition 2.1, being $F = \mathcal{Y}_r$, $E = \mathcal{W}_r$ and $I = J$.

Proof. By standard arguments related with the Ekeland principle, as in [10] we can get a Palais Smale sequence $(\theta_n, u_n)_n$ for the functional $\tilde{J}$ at the level $\tilde{c}_{mp}$ such that $\theta_n \to 0$.

So, from $\tilde{J}(\theta_n, u_n) \to \tilde{c}_{mp}$, $\frac{\partial J}{\partial \theta}(\theta_n, u_n) \to 0$ in $\mathcal{W}_r'$ and $\frac{\partial J}{\partial \theta}(\theta_n, u_n) \to 0$, we respectively have

$$e^{N\theta_n} \left( \frac{1}{2} \int_{\mathbb{R}^N} \phi(e^{-2\theta_n}|\nabla u_n|^2) - \int_{\mathbb{R}^N} G(u_n) \right) \to \tilde{c}_{mp} \tag{26}$$

$$e^{N\theta_n} \left( \int_{\mathbb{R}^N} \phi'(e^{-2\theta_n}|\nabla u_n|^2)e^{-2\theta_n}|\nabla u_n|^2 - \int_{\mathbb{R}^N} g(u_n)u_n \right) = o_n(1)\|u_n\| \tag{27}$$

$$e^{N\theta_n} \left( \frac{N}{2} \int_{\mathbb{R}^N} \phi(e^{-2\theta_n}|\nabla u_n|^2) \right.$$

$$\left. - \int_{\mathbb{R}^N} \phi'(e^{-2\theta_n}|\nabla u_n|^2)e^{-2\theta_n}|\nabla u_n|^2 - N \int_{\mathbb{R}^N} G(u_n) \right) \to 0 \tag{28}$$
Comparing (26) with (28) we deduce the following inequality
\[ \frac{e^{N\theta_n}}{N} \int_{\mathbb{R}^N} \phi'(e^{-2\theta_n} \nabla u_n |^2) e^{-2\theta_n} |\nabla u_n|^2 \to \tilde{c}_{mp} \]
which is equivalent to
\[ \frac{1}{N} \int_{\mathbb{R}^N} \phi'(|\nabla \tilde{u}_n|^2) |\nabla \tilde{u}_n|^2 \to \tilde{c}_{mp} \]
where \( \tilde{u}_n(\cdot) = u_n(e^{-\theta_n} \cdot) \). By convexity, we know that \( 0 \leq \frac{1}{2} \phi(t^2) \leq \phi'(t^2) t^2 \), so from the previous inequality we deduce that \( (\int_{\mathbb{R}^N} \phi(|\nabla \tilde{u}_n|^2))_n \) is bounded. Assumption (\( \Phi 2 \)) and property (vi) in Proposition 1.2 imply that \((\tilde{u}_n)_n\) is bounded in \( \mathcal{Y}_r \). Finally observe that from (26) we have that \( J(\tilde{u}_n) \to \tilde{c}_{mp} \) and since \( \frac{\partial J}{\partial u_n}(\theta_n, u_n) \to 0 \), we have
\[ e^{N\theta_n} \left( \nabla \cdot \phi'(|\nabla \tilde{u}_n|^2) e^{-\theta_n} \nabla \tilde{u}_n + g(\tilde{u}_n) \right) \to 0 \quad \text{in } \mathcal{W}'_r. \] (29)
Taking into account that \( \theta_n \to 0 \), from (29) we deduce that \( J'(\tilde{u}_n) \to 0 \) in \( \mathcal{W}'_r \). Then \((\tilde{u}_n)_n\) satisfies 1, 2, and 3 of Definition 2.1. \( \square \)
We conclude with the proof of our main Theorem
\[ \text{Proof} \quad [\text{Proof of Theorem 0.1}] \text{ Let } (u_n)_n \text{ be a sequence as in Proposition 3.4. By Theorem 2.3, we can extract a subsequence, relabeled } (u_n)_n, \text{ strongly convergent to some } u_0 \in \mathcal{W}_r. \text{ Finally, it is enough to observe that, by Lemma 3.2 and Proposition 3.3, } u_0 \neq 0. \text{ Moreover } u_0 \geq 0 \text{ by definition of } g_1. \quad \square \]

References

[1] R. A. Adams, Sobolev Spaces, Boston, MA: Academic Press, (1975) ISBN 978-0-12-044150-1.

[2] A. Azzollini, P. d’Avenia, A. Pomponio, Quasilinear elliptic equations in \( \mathbb{R}^N \) via variational methods and Orlicz-Sobolev embeddings, Calc. Var. (to appear).

[3] A. Azzollini, A. Pomponio On the Schrödinger equation in \( \mathbb{R}^N \) under the effect of a general nonlinear term, Indiana Univ. Journal, 58, (2009), 1361-1378.
[4] M. Badiale, L. Pisani, S. Rolando, *Sum of weighted Lebesgue spaces and nonlinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl. 18, (2011), 369–405.

[5] P. Bartolo, V. Benci, D. Fortunato, *Abstract critical point theorem and applications to some nonlinear problems with “strong” resonance at infinity*, Nonlin. Anal. TMA, 7, (1983), 981–1012.

[6] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations. I. Existence of a ground state*, Arch. Rational Mech. Anal., 82, (1983), 313–345.

[7] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations. II. Existence of infinitely many solutions*, Arch. Rational Mech. Anal., 82, (1983), 347–375.

[8] G. Cerami, *Un criterio di esistenza per i punti critici su varietà illimitate*, Rc. Ist. lomb. Sci. Lett., 112, (1978), 332–336.

[9] T. D’Aprile, G. Siciliano, *Magnetostatic solutions for a semilinear perturbation of the Maxwell equations*, Adv. Differential Equations 16 (2011), 435–466.

[10] J. Hirata, N. Ikoma, K. Tanaka, *Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass and symmetric mountain pass approaches*, TMNA 35, (2010), 253–276.

[11] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$*, Proc. R. Soc. Edinb., Sect. A, Math., 129, (1999), 787–809.

[12] E.H. Lieb, M. Loss, *Analysis*, American Mathematical Society, Providence, RI, 2001.

[13] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv., 60, (1985), 558–581.