ON FREE ENERGIES OF THE ISING MODEL ON THE CAYLEY TREE

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Abstract. We present, for the Ising model on the Cayley tree, some explicit formulae of the free energies (and entropies) according to boundary conditions (b.c.). They include translation-invariant, periodic, Dobrushin-like b.c., as well as those corresponding to (recently discovered) weakly periodic Gibbs states. The later are defined through a partition of the tree that induces a 4-edge-coloring. We compute the density of each color.

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1. Introduction and definitions

On non-amenable graphs, not only Gibbs measures but also the free energy (and the entropy) depend on the boundary conditions.

The purpose of this paper is to study this dependence for one of the simplest such graph, the Cayley tree (Bethe lattice). Our analysis is restricted to the Ising model.

Let $\Gamma_k = (V,L)$ be the uniform Cayley tree, where each vertex has $k + 1$ neighbors with $V$ being the set of vertices and $L$ the set of edges.

On this tree, there is a natural distance to be denoted $d(x,y)$, being the number of nearest neighbor pairs of the minimal path between the vertices $x$ and $y$ (by path one means a collection of nearest neighbor pairs, two consecutive pairs sharing at least a given vertex).

The Ising model is defined by the formal Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y \rangle \subset V} \sigma(x)\sigma(y), \quad (1.1)$$

where the sum runs over nearest neighbor vertices $\langle x,y \rangle$ and the spins $\sigma(x)$ take values in the set $\Phi = \{+1,-1\}$.

For a fixed $x^0 \in V$, the root, let

$$W_n = \{x \in V : d(x,x^0) = n\}, \quad V_n = \{x \in V : d(x,x^0) \leq n\}$$

be respectively the sphere and the ball of radius $n$ with center at $x^0$, and for $x \in W_n$ let

$$S(x) = \{y \in W_{n+1} : d(y,x) = 1\},$$

be the set of direct successors of $x$. 

The (finite-dimensional) Gibbs distributions at inverse temperature $\beta = 1/T$ are defined by
\[
\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ \beta J \sum_{(x,y) \in V_n} \sigma(x)\sigma(y) + \sum_{x \in W_n} h_x \sigma(x) \right\},
\]
(1.2)

with partition functions given by
\[
Z_n \equiv Z_n(\beta, h) = \sum_{\sigma_n \in \Phi_{V_n}} \exp \left\{ \beta J \sum_{(x,y) \in V_n} \sigma(x)\sigma(y) + \sum_{x \in W_n} h_x \sigma(x) \right\}.
\]
(1.3)

Here
\[
h = \{h_x \in R, x \in V\}
\]
is a collection of real numbers that stands for (generalized) boundary condition.

The probability distributions (1.2) are said compatible if for all $\sigma_{n-1}$
\[
\sum_{\omega_n \in \Phi_{W_n}} \mu_n(\sigma_{n-1}, \omega_n) = \mu_{n-1}(\sigma_{n-1}).
\]
(1.4)

It is well known that this compatibility condition is satisfied if and only if for any $x \in V$ the following equation holds
\[
h_x = \sum_{y \in S(x)} f(h_y, \theta),
\]
(1.5)

where
\[
\theta = \tanh(\beta J), \quad f(h, \theta) = \text{arctanh}(\theta \tanh h).
\]
(1.6)

Namely, for any boundary condition satisfying the functional equation (1.5) there exists a unique Gibbs measure, the correspondence being one-to-one.

We will be interested in the dependence with respect to boundary conditions of the free energy defined as the limit
\[
F(\beta, h) = -\lim_{n \to \infty} \frac{1}{|V_n|} \ln Z_n(\beta, h),
\]
(1.7)

where $|\cdot|$ denotes hereafter the cardinality of a set.

A boundary condition satisfying (1.5) will be in the sequel called compatible.

The paper is organized as follows. Section 2 provides the first part of results: a general formula applied then to various known boundary conditions (translation-invariant, Bleher-Ganikhodjaev, Zachary, ART), and those about entropy. Periodic and weakly periodic cases are the subject of Section 3. A first appendix concerns the density of colors mentioned in the abstract. A second appendix provides a sufficient condition for the existence of the free energy in case of compatible boundary conditions.

2. General formula and first results

For compatible boundary conditions, the free energy is given by the formula
\[
F(\beta, h) = -\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x),
\]
(2.1)
where
\[ a(x) = \frac{1}{2\beta} \ln[4 \cosh(h_x - \beta J) \cosh(h_x + \beta J)]. \]

To see it, first notice that
\[ Z_n(\beta, h) = A_{n-1} Z_{n-1}(\beta, h), \quad (2.2) \]
where
\[ A_n = \prod_{x \in W_n} b(x) \text{ with } b(x) \text{ satisfying} \]
\[ \prod_{y \in S(x)} \sum_{u = \pm 1} \exp(\beta J \varepsilon u + uh_y) = b(x) \exp(\varepsilon h_x), \quad \varepsilon = \pm 1. \quad (2.3) \]

This formula used with both values of \( \varepsilon \) implies
\[ b(x) = \prod_{y \in S(x)} (4 \cosh(h_y - \beta J) \cosh(h_y + \beta J))^{1/2} = \exp \left( \beta \sum_{y \in S(x)} a(y) \right). \]

It is then enough to insert this formula into the recursive equation (2.2) to get by iteration
\[ Z_n(\beta, h) = \prod_{x \in V_{n-1}} b(x) \]
which gives (2.1).

Notice that
\[ F(\beta, h) = F(\beta, -h), \quad (2.4) \]
where \( -h = \{-h_x, x \in V\} \).

2.1. Translation-invariant boundary conditions. They correspond to constant functions, \( h_x = h \), in which case the condition (1.5) reads
\[ h = kf(h, \theta). \quad (2.5) \]

The equation (2.5) has a unique solution \( h = 0 \), if \( \theta \leq \theta_c = \frac{1}{k} \) and three distinct solutions \( h = 0, \pm h_* \) (\( h_* > 0 \)) when \( \theta > \theta_c \).

Let us denote by \( \mu_0, \mu_{\pm} \) the corresponding Gibbs measures and recall the following known results for the ferromagnetic Ising model \( (\theta \geq 0) \):

1. If \( \theta \leq \theta_c \), \( \mu_0 \) is unique and extreme.
2. If \( \theta > \theta_c \), \( \mu_- \) and \( \mu_+ \) are extreme.
3. \( \mu_0 \) is extreme if and only if \( \theta < \frac{1}{\sqrt{k}} \).

(see e.g. \[8\], \[6\], \[3\])

According to formula (2.1), the free energies of translation-invariant (TI) b.c. are given by:
\[ F_{TI}(\beta, 0) = -\frac{1}{\beta} \ln(2 \cosh(\beta J)). \quad (2.6) \]
\[ F_{TI}(\beta, h_*) = F_{TI}(\beta, -h_*) = -\frac{1}{2\beta} \ln[4 \cosh(\beta J - h_*) \cosh(\beta J + h_*)]. \quad (2.7) \]

Some particular plots are shown in Fig. 1.
In order to draw the free energy $F_{T_1}(\beta,h)$ as a function of $\beta$, we notice that the equation (2.5) gives

$$\beta(h) = \frac{1}{2J} \ln \frac{e^{(1+\frac{1}{k})2h} - 1}{e^{2h} - e^{\frac{2h}{k}}}$$  \hspace{1cm} (2.8)$$

and use the parametric representation defined by the following mapping:

$$h \rightarrow (\beta(h), F(h)), \text{ for } h \geq 0.$$  

The function $F(h)$ is defined by inserting (2.8) into (2.6) and (2.7).

2.2. \textbf{Bleher-Ganikhodjaev construction.} Here one consider the half tree. Namely the root $x^0$ has $k$ nearest neighbours. Consider an infinite path $\pi = \{x^0 = x_0 < x_1 < \ldots\}$ (the notation $x < y$ meaning that pathes from the root to $y$ go through $x$). Associate to this path a collection $h^\pi_x$ of numbers given by the condition

$$h^\pi_x = \begin{cases} -h_*, & \text{if } x \prec x_n, x \in W_n, \\ h_*, & \text{if } x_n \prec x, x \in W_n, \end{cases} \hspace{1cm} (2.9)$$
where \( x < x_n \) (resp. \( x_n < x \)) means that \( x \) is on the left (resp. right) from the path \( \pi \).

For any infinite path \( \pi \), the collection of numbers \( h^\pi \) satisfying relations (1.5) exists and is unique (see [2]).

A real number \( t = t(\pi) \), \( 0 \leq t \leq 1 \) can be assigned to the infinite path and the set \( h^{\pi(t)} \) is uniquely defined. The set of numbers \( h^{\pi(t)} \) being distinct for different \( t \in [0,1] \), it is also the case for the corresponding Gibbs measures. One thus obtains uncountable many Gibbs measures and they are extreme.

Since the solution \( h^\pi \) can differ from \( h_* \) or \( -h_* \) only on the path \( \pi \), it is thus obvious that the values \( h^\pi_x, x \in \pi \) do not contribute to the free energy. As a consequence, we get from the evenness (2.7), that the free energies of Bleher-Ganikhodjaev and translation-invariant boundary conditions coincide:

\[
F_{BG}(\beta,h^\pi) = F_{TI}(\beta,h_*) .
\] (2.10)

2.3. Zachary construction. This construction provides an (uncountable) set of distinct functions \( h^{(t)} \) satisfying condition (1.5) and parameterized by \( t \in (-h_*,h_*) \). It is assumed here that \( \theta > \theta_c \).

Take \( t \neq 0 \) and define the sequence \( (t_n)_{n \geq 0} \) recursively by

\[
t_n = k f(t_{n+1},\theta), \quad n \geq 0.
\] (2.11)

Since the function \( f \) is increasing and maps the interval \((-h_*,h_*)\) into itself, the definition of \( t_n \) make sense. Moreover one can see that \( \lim_{n \to \infty} t_n = 0 \) for each \( t_0 = t \).

Consider the function \( h_*(t) = t_n \) for all \( x \in W_n \). This function satisfies condition (1.5) for any \( t \) and by construction, distinct \( t \) assign distinct \( h^{(t)} \). The associated Gibbs measures are known to be extreme [11].

The corresponding free energies can be written as

\[
F_{Zach}(\beta,h^{(t)}) = -\frac{1}{2\beta} \lim_{n \to \infty} \frac{1}{|V_n|} \sum_{m=0}^{n} |W_m| \tilde{a}(t_m),
\]

where \( \tilde{a}(t) = \ln[4 \cosh(\beta J - t) \cosh(\beta J + t)] \).

By Stolz-Cesáro theorem (see e.g. [7]) applied to the sequences

\[
a_n = \sum_{m=0}^{n} |W_m| \tilde{a}(t_m), \quad b_n = |V_n|
\] (2.12)

one has

\[
\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{m=0}^{n} |W_m| \tilde{a}(t_m) = \lim_{n \to \infty} \frac{\sum_{m=0}^{n+1} |W_m| \tilde{a}(t_m) - \sum_{m=0}^{n} |W_m| \tilde{a}(t_m)}{|V_{n+1}| - |V_n|} = \lim_{n \to \infty} \frac{|W_{n+1}| \tilde{a}(t_{n+1})}{|W_{n+1}|} = \lim_{n \to \infty} \tilde{a}(t_{n+1}) = \tilde{a}(0).
\] (2.13)
As a consequence
\[ F_{\text{Zach}}(\beta, h^{(i)}) = F_{\text{TI}}(\beta, 0). \]  
(2.14)

2.4. ART construction. Let \( h \) be a boundary condition satisfying (1.5) on \( \Gamma^{k_0} \). For \( k \geq k_0 + 1 \) define the following boundary condition on \( \Gamma^k \):
\[ \tilde{h}_x = \begin{cases} 
  h_x, & \text{if } x \in V^{k_0} \\
  0, & \text{if } x \in V^k \setminus V^{k_0},
\end{cases} \]  
(2.15)

where \( V^k \) denote the set of vertices of \( \Gamma^k \). Namely, to each vertices of \( V^{k_0} \) one adds \( k - k_0 \) successors with vanishing value of the boundary condition. It is obvious the b.c. \( \tilde{h} \) satisfy the compatibility condition (1.5). In this way one constructs a new set of Gibbs measures that are extreme in the range \( 1/k_0 < \theta < 1/\sqrt{k} \).

For the corresponding free energy, we have
\[ F_{\text{ART}}(\beta, \tilde{h}) = \frac{1}{\beta} \lim_{n \to \infty} \frac{1}{|V_n^k|} \left\{ \left| V_n^k \right| - \left| V_n^{k_0} \right| \right\} \ln [2 \cosh (\beta J)] + \beta \sum_{x \in V_n^{k_0}} a(x), \]  
(2.16)

Since
\[ \lim_{n \to \infty} \frac{|V_n^{k_0}|}{|V_n^k|} = \frac{k - 1}{k_0 - 1} \lim_{n \to \infty} \frac{(k_0 + 1)k_0^n - 2}{(k + 1)k^n - 2} = 0, \]
by taking into account \( 0 \leq a(x) \leq C_b \), we get
\[ 0 \leq \sum_{x \in V_n^{k_0}} a(x) \leq |V_n^{k_0}| C_b. \]

As a consequence,
\[ \lim_{n \to \infty} \frac{1}{|V_n^k|} \sum_{x \in V_n^{k_0}} a(x) = 0, \]
so that
\[ F_{\text{ART}}(\beta, \tilde{h}) = \frac{1}{\beta} \ln [2 \cosh (\beta J)] = F_{\text{TI}}(\beta, 0). \]  
(2.17)

2.5. Entropy. To compute the entropy \( S(\beta, h) = -\frac{dF(\beta, h)}{dT} \), we first notice that equation (2.8) gives
\[ h'(\beta) = \frac{1}{\beta'(h)} = Jk \frac{\cosh(2h) - \cosh \left( \frac{2h}{k} \right)}{\sinh(2h) - k \sinh \left( \frac{2h}{k} \right)}. \]  
(2.18)

As a result of easy computations, we get the formula
\[ S(\beta, h) = \frac{1}{2} \ln \left[ 2 \cosh(2h) + 2 \cosh(2\beta J) \right] \\
+ \beta J \frac{k^2 \sinh(2h) \frac{\cosh(2h) - \cosh \left( \frac{2h}{k} \right)}{\sinh(2h) - k \sinh \left( \frac{2h}{k} \right)} + \sinh(2\beta J)}{\cosh(2h) + \cosh(2\beta J)}. \]  
(2.19)
and
\[ S(\beta,0) = \ln(2 \cosh(\beta J)) - \beta J \tanh(\beta J). \] (2.20)

Let us mention that in case \( k = 2 \), we get by solving equation (2.5) the following value of \( h^* \) as a function of the inverse temperature:
\[ \pm h^* = \frac{1}{2} \ln \left[ 2^{-1} \left( e^{4\beta J} - 2e^{2\beta J} - 1 \pm (e^{2\beta J} - 1)^2 \sqrt{(e^{2\beta J} + 1)(e^{2\beta J} - 3)} \right) \right]. \] (2.21)

The free energies and entropies then read
\[ F(\beta, h^*) = -\frac{1}{\beta} \left( \ln(e^{2\beta J} - 1) + \frac{1}{2} \ln(e^{-2\beta J} + 1) \right), \] (2.22)
\[ S(\beta, h^*) = \ln(2 \sinh(\beta J)) + \frac{1}{2} \ln(2 \cosh(\beta J)) - \beta J \frac{3 \cosh(2\beta J) + 1}{2 \sinh(2\beta J)}. \] (2.23)

3. Periodic and weakly periodic Gibbs measures

3.1. A group representation of the Cayley tree. Let \( G_k \) be a free product of \( k + 1 \) cyclic groups of the second order with generators \( a_1, a_2, \ldots, a_{k+1} \), respectively.

It is known that there exists a one-to-one correspondence between the set of vertices \( V \) of the Cayley tree \( \Gamma_k \) and the group \( G_k \).

To give this correspondence we fix an arbitrary element \( x_0 \in V \) and let it correspond to the unit element \( e \) of the group \( G_k \). Using \( a_1, \ldots, a_{k+1} \) we numerate nearest-neighbors of element \( e \), moving by positive direction (see Fig. 2). Now we shall give numeration of the nearest-neighbors of each \( a_i \), \( i = 1, \ldots, k + 1 \) by \( a_i a_j \), \( j = 1, \ldots, k + 1 \). Since all \( a_i \) have the common neighbor \( e \) we give to it \( a_1 a_i = a_i^2 = e \). Other neighbors are numerated starting from \( a_i a_i \) by the positive direction. We numerate the set of all nearest-neighbors of each \( a_i a_j \) by words \( a_i a_j a_q \), \( q = 1, \ldots, k + 1 \), starting from \( a_i a_j a_j = a_i \) by the positive direction. Iterating this argument one gets a one-to-one correspondence between the set of vertices \( V \) of the Cayley tree \( \Gamma_k \) and the group \( G_k \).

In the group \( G_k \), let us consider the left (right) shift transformations defined as follows. For \( g_0 \in G_k \), let us set
\[ T_g(h) = g h, \quad (T_g(h) = h g), \quad \text{for all} \ h \in G_k. \] (3.1)

The set of all left (right) shifts in \( G_k \) is isomorphic to the group \( G_k \).

3.2. Periodic boundary conditions. In this subsection, we consider periodic solutions of (1.5) and use the above group structure of the Cayley tree.

Definition 1. Let \( \tilde{G} \) be a normal subgroup of the group \( G_k \). The set \( h = \{ h_x : x \in G_k \} \) is said to be \( \tilde{G} \)-periodic if \( h_{yx} = h_x \) for any \( x \in G_k \) and \( y \in \tilde{G} \).

Let
\[ G^{(2)}_k = \{ x \in G_k : \text{the length of word} \ x \ \text{is even} \}. \]

Note that \( G^{(2)}_k \) is the set of even vertices (i.e. with even distance to the root). Consider the boundary conditions \( h^\pm \) and \( h^\mp \):
Figure 2. Some elements of group $G_2$ on Cayley tree of order two.

\[ h_x^\pm = -h_x^\mp = \begin{cases} h^*_x, & \text{if } x \in G_k^{(2)} \\ -h^*_x, & \text{if } x \in G_k \setminus G_k^{(2)}, \end{cases} \tag{3.2} \]

and denote by $\mu^{(\pm)}, \mu^{(\mp)}$ the corresponding Gibbs measures.

The $\hat{G}$-periodic solutions of equation (1.5) are either translation-invariant ($G_k$-periodic) or $G_k^{(2)}$-periodic (see 5), they are solutions to

\[ u = kf(v, \theta), \quad v = kf(u, \theta). \tag{3.3} \]

In the ferromagnetic case only translation invariant b.c. can be found. In the antiferromagnetic case ($\theta \leq 0$) the system (3.3) has a unique solution $h = 0$ if $\theta \geq -1/k$, and three distinct solutions $h = 0, h^\pm$ and $h^\mp$ if $\theta < -1/k$.

Let us also recall that for the antiferromagnetic Ising model:

1. If $\theta \geq -1/k$, $\mu_0$ is unique and extreme.
2. If $\theta < -1/k$, $\mu^{(\pm)}$ and $\mu^{(\mp)}$ are extreme.

see 6.

For periodic measures, we have

\[ F_{\text{Per}}(\beta, h^*_x) = -\frac{1}{2\beta} \ln[4 \cosh(\beta J + h^*_x) \cosh(\beta J - h^*_x)] = F_{\text{TI}}(\beta, h^*_x). \]

3.3. Weakly periodic Gibbs measures. Let $G_k/\hat{G}_k = \{H_1, ..., H_r\}$ be a factor group, where $\hat{G}_k$ is a normal subgroup of index $r \geq 1$. 
Definition 2. A set $h = \{h_x, x \in G_k\}$ is called weakly periodic, if $h_x = h_{ij}$, for any $x \in H_i, x_i \in H_j$, where $x_i$ denotes the ancestor of $x$.

Weakly periodic b.c. $h$ coincide with periodic ones if $h_x$ is independent of $x_i$.

Here, we will restrict ourself to the cases of index two and recall that any such subgroup has the form

$$H_A = \{ x \in G_k : \sum_{i \in A} \omega_x(a_i) - \text{even} \},$$

where $\emptyset \neq A \subseteq N_k = \{1, 2, \ldots, k+1\}$, and $\omega_x(a_i)$ is the number of $a_i$ in a word $x \in G_k$.

Let $\omega(h_i, x)$ correspond to solutions of system (3.5) on the invariant set $h = h_0, x \in H_i, x_i \in H_j$. Then, in view of (1.5), the $H_A$-weakly periodic b.c. has the form

$$h_x = \begin{cases} 
  h_1, & x \in H_0, x_i \in H_0, \\
  h_2, & x \in H_0, x_i \in H_1, \\
  h_3, & x \in H_1, x_i \in H_0, \\
  h_4, & x \in H_1, x_i \in H_1,
\end{cases}$$

(3.4)

where the $h_i$ satisfy the following equations:

$$\begin{cases} 
  h_1 = |A| f(h_3, \theta) + (k - |A|) f(h_1, \theta), \\
  h_2 = (|A| - 1) f(h_3, \theta) + (k + 1 - |A|) f(h_1, \theta), \\
  h_3 = (|A| - 1) f(h_2, \theta) + (k + 1 - |A|) f(h_4, \theta), \\
  h_4 = |A| f(h_2, \theta) + (k - |A|) f(h_4, \theta).
\end{cases}$$

(3.5)

For sake of simplicity, consider $k = 4$ and $|A| = 1$. In this case

$$H_0 = \{ x \in G_k : \omega_x(a_1) \text{ is even} \},$$

$$H_1 = \{ x \in G_k : \omega_x(a_1) \text{ is odd} \}.$$

These sets are shown in Fig. 3.

Let us recall the following results of [9]:

There exists a critical value $\alpha_{cr} \approx (0,1569)$ of $\alpha = e^{-2\beta J}$ such that:

1. If $\alpha > \alpha_{cr}$, there exists a unique weakly periodic state $\mu_0$.
2. If $\alpha = \alpha_{cr}$, there are three distinct weakly periodic states $\mu_0, \mu^-_1, \mu^+_1$.
3. If $0 \leq \alpha < \alpha_{cr}$, there are five distinct weakly periodic states $\mu_0, \mu^-_1, \mu^+_1, \mu^-_2, \mu^+_2$.

These measures correspond to solutions of system (3.5) on the invariant set $h_1 = -h_4, h_2 = -h_3$, that is to solutions of:

$$\begin{cases} 
  h_1 = 3 f(h_1, \theta) - f(h_2, \theta), \\
  h_2 = 4 f(h_1, \theta).
\end{cases}$$

(3.6)

More results about weakly periodic Gibbs measures can be found in [9], [10].
Denote
\[ A_n = \left| \{(x, y) \in L_n : x \in H_0, y = x \uparrow \in H_0\} \right|, \]
\[ B_n = \left| \{(x, y) \in L_n : x \in H_0, y = x \uparrow \in H_1\} \right|, \]
\[ C_n = \left| \{(x, y) \in L_n : x \in H_1, y = x \uparrow \in H_0\} \right|, \]
\[ D_n = \left| \{(x, y) \in L_n : x \in H_1, y = x \uparrow \in H_1\} \right|, \]

where \( L_n \) is the set of edges in \( V_n \).

Figure 3. The sets \( H_0 \) (black vertices) and \( H_1 \) (gray vertices).
For weakly periodic b.c. (3.6) we have
\[ F_{\text{WP}}(\beta, h) = -\frac{1}{2\beta} \lim_{n \to \infty} \frac{1}{|V_n|} \{(A_n + D_n) \ln[4 \cosh(\beta J - h_1) \cosh(\beta J + h_1)] + \}
\]
\[ (B_n + C_n) \ln[4 \cosh(\beta J - h_2) \cosh(\beta J + h_2)]\} . \]

By Proposition 1 of the appendix we obtain
\[ F_{\text{WP}}(\beta, h) = -\frac{1}{2\beta} \left\{ \frac{4}{5} \ln[4 \cosh(\beta J - h_1) \cosh(\beta J + h_1)] + \frac{1}{5} \ln[4 \cosh(\beta J - h_2) \cosh(\beta J + h_2)] \right\} \] (3.8)

In the case under consideration the proposition is straightforward. Indeed one immediately see, in view of definitions and Fig. 3, that \( (A_n + D_n)/(|V_n| - 1) = 4/5 \) and \( (B_n + C_n)/(|V_n| - 1) = 1/5 \) when \( n = 1 \); the induction is trivial.

The equation
\[ h_1 = 3f(h_1, \theta) - f(4f(h_1, \theta), \theta) \] (3.9)

that solves the system (3.6) (with \( h_2 = 4f(h_1, \theta) \)) can then be reduced to:
\[ \alpha^2 \xi^3 - \alpha \xi^2 - 2\alpha^2 \xi + \alpha + 1 = 0, \] (3.10)

which has two solutions \( \xi_1 \) and \( \xi_2 \) when \( 0 < \alpha < \alpha_{cr} \) (see [9]).

The free energy then reads:
\[ F_{\text{WP}}(\beta, h, h_*) = -\frac{1}{10\beta} \left\{ 4 \ln \left[ 2 \cosh(2\beta J) \right] + \frac{2\xi \cosh(2\beta J) - 4}{2 \cosh(2\beta J) - \xi} \ln[2 \cosh(2\beta J) + \xi^4 - 4\xi^2 + 2] \right\} . \]

The solution \( \xi_1 \) is given by
\[ \xi_1 = \frac{1}{3} \left( e^{2\beta} + \frac{2^{1/3} (6 + e^{4\beta})}{U} + 2^{-1/3} U \right), \]

where
\[ U = \left( -9e^{2\beta} - 27e^{4\beta} + 2e^{6\beta} + 3\sqrt{-96 - 39e^{4\beta} + 54e^{6\beta} + 69e^{8\beta} - 12e^{10\beta}} \right)^{1/3}. \]

The corresponding free energy reads
\[ F_{\text{WP}}(\beta) = -\frac{1}{10\beta} \ln \frac{(e^{-2\beta} + e^{2\beta})^8 \left( 2 + e^{-2\beta} + e^{2\beta} - 4 \left( e^{4\beta} \frac{W}{6} \right)^2 + \left( e^{4\beta} \frac{W}{6} \right)^4 \right)}{(e^{-2\beta} + e^{2\beta} - e^{4\beta} \frac{W}{6})^4}, \]

where
\[ W = \left( 2e^{-2\beta} + \frac{V}{U} + 2^{2/3} e^{-4\beta} U \right) \quad \text{with} \quad V = 2^{4/3} e^{-4\beta} \left( 6 + e^{4\beta} \right). \]

The plot is given in Fig. 4 from which we observe the strict inequalities
\[ F_{\text{TI}}(\beta, h_*) < F_{\text{WP}}(\beta, h) < F_{\text{TI}}(\beta, 0). \] (3.11)
in the range $\alpha \leq \alpha_{cr}$.

Figure 4. The free energies $F_{WP}(\beta)$ (dotted line) for $\alpha \leq \alpha_{cr}$ together with the previous free energies $F_{TI}(\beta, 0)$ (solid line), and $F_{TI}(\beta, h_*)$ (dashed line). Here $J = 1$ and $k = 4$.

The results for weakly periodic boundary conditions have to be compared with the inequalities given recently in [4]; the weakly periodic b.c. (3.6) corresponding to the so-called dimer covering in [4]. There, the inequalities are easier to catch with cluster expansion method in mind, the condition on the temperature is more restrictive, and the free energies cannot be expressed explicitly.

**Appendix: Density of edges in a ball**

In this appendix we consider a group representation of a Cayley tree and its partition with respect to an arbitrary subgroup of index two. This partition gives a 2-vertex-coloring on Cayley tree, which then gives 4-edge-coloring, say, colors $i = 1, 2, 3, 4$. We fix a root of the Cayley tree and give explicit formulas for number $A_{n,i}$ of edges with color $i$ in a ball $V_n$ of radius $n$ with the center at the root. Moreover, we compute the $\lim_{n \to \infty} (A_{n,i} / |V_n|)$ for each $i = 1, 2, 3, 4$. 
We will use the notation of Subsection 3.3 and let

\[ \alpha_n = |\{(x, y) \in L_n \setminus L_{n-1} : x \in H_0, y = x_1 \in H_0\}|. \]

\[ \beta_n = |\{(x, y) \in L_n \setminus L_{n-1} : x \in H_0, y = x_4 \in H_1\}|. \]

\[ \gamma_n = |\{(x, y) \in L_n \setminus L_{n-1} : x \in H_1, y = x_4 \in H_0\}|. \]

\[ \delta_n = |\{(x, y) \in L_n \setminus L_{n-1} : x \in H_1, y = x_1 \in H_1\}|, \]

for \( A = \{1, 2, 3, \ldots, j\} \) and \( 1 \leq j \leq k + 1 \).

Let \( M \) be the set of all unit balls with vertices in \( V \) and let \( S_1(x) \) denotes the set of all nearest neighbors of \( x \). For \( b \in M \) the center of \( b \) is denoted by \( c_b \).

**Lemma 1.** If \( c_b \in H_0 \), then

\[ |\{x \in S_1(c_b) : x \in H_1\}| = j, \quad |\{x \in S_1(c_b) : x \in H_0\}| = k - j + 1. \]

If \( c_b \in H_1 \), then

\[ |\{x \in S_1(c_b) : x \in H_1\}| = k - j + 1, \quad |\{x \in S_1(c_b) : x \in H_0\}| = j. \]

**Proof.** We have \( S_1(c_b) = \{c_ba_p : p = 1, 2, \ldots, k + 1\} \).

If \( c_b \in H_0 \), (the case \( c_b \in H_1 \) is similar) then \( c_ba_p \in H_1 \) for \( p = 1, 2, \ldots, j \) and \( c_ba_p \in H_0 \) for \( p = j + 1, j + 2, \ldots, k + 1 \), i.e. we have

\[ |\{x \in S_1(c_b) : x \in H_1\}| = j, \quad |\{x \in S_1(c_b) : x \in H_0\}| = k - j + 1. \]

Consider \( b = V_1 \in M \) with the center \( x^0 = e \in H_0 \), then in \( W_1 \) we have \( j \) vertices which belong to \( H_1 \), and \( k - j + 1 \) vertices which belong in \( H_0 \), consequently,

\[ \alpha_1 = k - j + 1, \quad \beta_1 = 0, \quad \gamma_1 = j, \quad \delta_1 = 0. \]

**Lemma 2.** For any \( n \in N \) the following recurrence system hold

\[
\begin{align*}
\alpha_{n+1} &= (k-j)\alpha_n + (k-j+1)\beta_n \\
\beta_{n+1} &= (j-1)\gamma_n + j\delta_n \\
\gamma_{n+1} &= (j-1)\beta_n + j\alpha_n \\
\delta_{n+1} &= (k-j)\delta_n + (k-j+1)\gamma_n,
\end{align*}
\]

with initial values \( \alpha_1 = k - j + 1, \beta_1 = 0, \gamma_1 = j, \delta_1 = 0. \)

**Proof.** By Lemma [1] an edge \( (x, y) \in L_n \setminus L_{n-1} \) with \( x \in H_0, y = x_4 \in H_0 \) has \( (k-j) \) neighbor edges \( (z, x) \in L_{n+1} \setminus L_n \) with \( z \in H_0, x = z_1 \in H_0 \). An edge \( (z, t) \in L_n \setminus L_{n-1} \) with \( z \in H_0, t = z_4 \in H_1 \) has \( (k-j+1) \) neighbor edges \( (u, z) \in L_{n+1} \setminus L_n \) with \( u \in H_0, z = u_4 \in H_0 \). Moreover, it is easy to see that only \( \alpha_n \) and \( \beta_n \) have contribution to \( \alpha_{n+1} \). Hence we have \( \alpha_{n+1} = (k-j)\alpha_n + (k-j+1)\beta_n \). Other equations of the system [A.1] can be obtained by a similar way. □
Remark 1. For $j = k + 1$ by Lemmas 1 and 2 we get $\alpha_n = \delta_n = 0$, for any $n \geq 1$ and

$$\beta_n = \begin{cases} 0, & \text{if } n = 2m - 1 \\ (k + 1)k^{2m - 1}, & \text{if } n = 2m \end{cases}, \quad m = 1, 2, \ldots$$

$$\gamma_n = \begin{cases} 0, & \text{if } n = 2m \\ (k + 1)k^{2(m-1)}, & \text{if } n = 2m - 1 \end{cases}, \quad m = 1, 2, \ldots$$

So in the sequel of this section we consider $j$ as $1 \leq j \leq k$.

Lemma 3. For $\alpha_n$ we have

$$\alpha_{n+2} = j(k - j + 1)|W_n| + (k - 2j)\alpha_{n+1} - (2j - 1)\alpha_n - k\alpha_{n-1}, \quad n \geq 2, \quad (A.2)$$

with initial values

$$\alpha_1 = k - j + 1, \quad \alpha_2 = (k - j)(k - j + 1), \quad \alpha_3 = ((k - 1)^2 + j(j - 1))(k - j + 1) \quad (A.3)$$

Proof. The initial values follow from Lemma 2.

By definitions of $\alpha_n, \beta_n, \gamma_n, \delta_n$ we have

$$\alpha_n + \beta_n + \gamma_n + \delta_n = |W_n| = k^{n-1}(k + 1), \quad n \geq 1. \quad (A.4)$$

From (A.1) we get

$$\beta_n = \frac{1}{k-j+1}(\alpha_{n+1} - (k - j)\alpha_n),$$

$$\gamma_n = \frac{j-1}{k-j+1}(\alpha_n - (k - j)\alpha_{n-1}) + j\alpha_{n-1}, \quad (A.5)$$

$$\delta_n = \frac{j}{2}(\beta_{n+1} - (j - 1)\gamma_n).$$

Substituting these values in (A.4) and then simplifying we get (A.2). $\square$

To find solution of (A.2) we denote

$$\alpha_n = q_n^2. \quad (A.6)$$

From (A.2) we get

$$k^n q_{n+2} = k^{n-1}(k + 1)(k - j + 1) + (k - 2j)k^{n-1}q_{n+1} - (2j - 1)k^{n-2}q_n - k^{n-2}q_{n-1},$$

dividing by $k^n$ we obtain

$$q_{n+2} = \frac{(k + 1)(k - j + 1)}{k} + \frac{k - 2j}{k}q_{n+1} - \frac{2j - 1}{k^2}q_n - \frac{1}{k^2}q_{n-1}, \quad (A.7)$$

with initial values

$$q_1 = k(k - j + 1), \quad q_2 = (k - j)(k - j + 1), \quad q_3 = \frac{((k - 1)^2 + j(j - 1))(k - j + 1)}{k} \quad (A.8)$$

In order to find solution to (A.7) first we rid $\frac{(k+1)(k-j+1)}{k}$ by denoting

$$q_n = p_n + \frac{k(k - j + 1)}{2}. \quad (A.9)$$
Substituting (A.9) to (A.7) we get

\[ p_{n+2} = \frac{k - 2j}{k} p_{n+1} - \frac{2j - 1}{k^2} p_n - \frac{1}{k^2} p_{n-1}, \]  

(A.10)

with

\[ p_1 = \frac{1}{2} k(k - j + 1), \quad p_2 = \frac{(k - 2j)(k - j + 1)}{2}, \quad p_3 = \frac{(k^2 - 4k + 2 + 2j^2 - 2j)(k - j + 1)}{2k}. \]  

(A.11)

The characteristic equation for (A.10) has the following form (setting \( p_n = \lambda^n \)):

\[ \lambda^3 - \frac{k - 2j}{k} \lambda^2 + \frac{2j - 1}{k^2} \lambda - \frac{1}{k^2} = 0, \]

which has solutions

\[ \lambda_1 = -\frac{1}{k}, \quad \lambda_{2,3} = \frac{k - 2j + 1 \pm \sqrt{(k - 2j)^2 - 2(k + 2j) + 1}}{2k}. \]  

(A.12)

Then the general solution to (A.10) is

\[ p_n = A_1 \lambda_1^n + A_2 \lambda_2^n + A_3 \lambda_3^n, \]

(A.13)

where the coefficients \( A_1, A_2, A_3 \) are determined by the initial conditions (A.11).

Using (A.9) and (A.6) we get

\[ \alpha_n = \frac{k - j + 1}{2} k^{n-1} + A_1 \cdot \frac{(-1)^n}{k^2} + A_2 \cdot \frac{A_2}{k^2} \cdot (k \lambda_2)^n + A_3 \cdot \frac{A_3}{k^2} \cdot (k \lambda_3)^n. \]  

(A.14)

Then using (A.5) and (A.14) one can find \( \beta_n, \gamma_n \) and \( \delta_n \).

We have

\[ A_n = \sum_{m=1}^{n} \alpha_m = \frac{k - j + 1}{2(k - 1)} (k^n - 1) + \frac{A_1}{2k^2} ((-1)^n - 1) + \frac{A_2}{k^2} (\lambda_2 k - 1) ((\lambda_2 k)^n - 1) + \frac{A_3}{k^2} (\lambda_3 k - 1) ((\lambda_3 k)^n - 1). \]  

(A.15)

**Proposition 1.** For any \( j = 1, \ldots, k \) and any fixed \( q = 0, 1, 2, \ldots \) we have

\[ \lim_{n \to \infty} \frac{\alpha_{n-q}}{|V_n|} = \lim_{n \to \infty} \frac{\delta_{n-q}}{|V_n|} = \frac{(k - 1)(k - j + 1)}{2(k + 1)k^q + 1}. \]

\[ \lim_{n \to \infty} \frac{\beta_{n-q}}{|V_n|} = \lim_{n \to \infty} \frac{\gamma_{n-q}}{|V_n|} = \frac{(k - 1)j}{2(k + 1)k^q + 1}. \]

\[ \lim_{n \to \infty} \frac{A_{n-q}}{|V_n|} = \lim_{n \to \infty} \frac{D_{n-q}}{|V_n|} = \frac{k - j + 1}{2(k + 1)k^q}. \]

\[ \lim_{n \to \infty} \frac{B_{n-q}}{|V_n|} = \lim_{n \to \infty} \frac{C_{n-q}}{|V_n|} = \frac{j}{2(k + 1)k^q}. \]
Proof. It is easy to check that \(|\lambda_2| < 1\) and \(|\lambda_3| < 1\), i.e.
\[
|k - 2j + 1 \pm \sqrt{(k-2j)^2 - 2(k+2j) + 1}| < 2k, \quad \text{for any} \quad 1 \leq j \leq k.
\]
Using these inequalities, formula (A.14) and (A.16) we get
\[
\lim_{n \to \infty} \frac{\alpha_{n-q}}{|V_n|} = \frac{(k-1)(k-j+1)}{2(k+1)k^{q+1}}.
\]
Now using this formula together with (A.5) we obtain
\[
\lim_{n \to \infty} \frac{\beta_{n-q}}{|V_n|} = \frac{k-j+1}{k-1} \left( \lim_{n \to \infty} \frac{\alpha_{n+1-q}}{|V_n|} - (k-j) \lim_{n \to \infty} \frac{\alpha_{n-q}}{|V_n|} \right) = \frac{j(k-1)}{2(k+1)k^{q+1}}.
\]
The formulae involving \(\gamma_n\) and \(\delta_n\) are obtained in a similar way.

By (A.15) and
\[
|V_n| = \frac{(k+1) \cdot k^{n-2}}{k-1}
\]
we get
\[
\lim_{n \to \infty} A_{n-q} = \frac{k-j+1}{2(k+1)k^q}.
\]
From (A.5) we get
\[
\mathcal{B}_n = \frac{1}{k-j+1} (A_n - \alpha_1 + \alpha_{n+1} - (k-j)A_n),
\]
\[
\mathcal{C}_n = \frac{j-1}{k-j+1} (A_n - (k-j)A_{n-1}) + jA_{n-1},
\]
\[
\mathcal{D}_n = \frac{1}{2} (\mathcal{B}_n - \beta_1 + \beta_{n+1} - (j-1)\mathcal{C}_n),
\]
that allows to prove the remaining formulæ. \(\square\)

Remark 2. By Proposition 4 it is clear that the values of \(A_1, A_2, A_3\) do not give any contribution to the equalities of the proposition. This is why we did not compute \(A_1, A_2, A_3\). But one can obtain the numbers by the initial conditions (A.11) for \(p_n\). For example, in the case \(k = 4, j = 1\) from (A.13) and (A.11) we have
\[
p_n = A_1(-\frac{1}{4})^n + A_2 \left( \frac{3 - \sqrt{7}i}{8} \right)^n + A_3 \left( \frac{3 + \sqrt{7}i}{8} \right)^n,
\]
\[
p_1 = 8, \quad p_2 = 4, \quad p_3 = 1.
\]
The initial conditions give
\[
\begin{align*}
8 &= -\frac{A_1}{4} + A_2 \frac{3-\sqrt{7}i}{8} + A_3 \frac{3+\sqrt{7}i}{8} \\
4 &= A_1 \frac{1}{16} + A_2 \frac{-3\sqrt{7}i}{32} + A_3 \frac{1+3\sqrt{7}i}{32} \\
1 &= -\frac{A_1}{4} + A_2 \frac{-9-5\sqrt{7}i}{128} + A_3 \frac{-9+5\sqrt{7}i}{128}
\end{align*}
\Rightarrow
\begin{align*}
A_1 &= 0 \\
A_2 &= 4 + \frac{20\sqrt{7}i}{7} \\
A_3 &= 4 - \frac{20\sqrt{7}i}{7}
\end{align*}
\]
Consequently, (for \(k = 4, j = 1\)) we have
\[
\alpha_n = 2 \cdot 4^{n-1} + \frac{1}{4} \left( 1 - \frac{5\sqrt{7}i}{7} \right) \left( \frac{3 + \sqrt{7}i}{2} \right)^n + \frac{1}{4} \left( 1 + \frac{5\sqrt{7}i}{7} \right) \left( \frac{3 - \sqrt{7}i}{2} \right)^n. \quad (A.18)
\]
Using \([A.5]\) and \([A.18]\) one can find \(\beta_n, \gamma_n\) and \(\delta_n\). Moreover, we have
\[
A_n = \sum_{m=1}^{n} \alpha_m = \frac{2(4^n - 1)}{3} + \frac{2\sqrt{7}i}{7} \cdot (3 - \sqrt{7}i)^n - (3 + \sqrt{7}i)^n.
\]

Note that \(\alpha_n\) and \(A_n\) are natural numbers for any \(n \geq 1\).

Remark 3. In the case \(j = k + 1\) by Remark 2 we get
\[
\lim_{n \to \infty} \frac{\alpha_n}{|V_n|} = \lim_{n \to \infty} \frac{\delta_n}{|V_n|} = \lim_{n \to \infty} \frac{A_n}{|V_n|} = \lim_{n \to \infty} \frac{D_n}{|V_n|} = 0.
\]
\[
\lim_{m \to \infty} \frac{\beta_{2m-1}}{|V_{2m-1}|} = \lim_{m \to \infty} \frac{\gamma_{2m-1}}{|V_{2m-1}|} = 0.
\]
\[
\lim_{m \to \infty} \frac{\beta_{2m}}{|V_{2m}|} = \lim_{m \to \infty} \frac{\gamma_{2m-1}}{|V_{2m-1}|} = \frac{k-1}{k}.
\]
\[
\lim_{m \to \infty} \frac{B_{2m}}{|V_{2m}|} = \lim_{m \to \infty} \frac{C_{2m-1}}{|V_{2m-1}|} = \frac{k}{k+1}.
\]
\[
\lim_{m \to \infty} \frac{B_{2m-1}}{|V_{2m-1}|} = \lim_{m \to \infty} \frac{C_{2m}}{|V_{2m}|} = \frac{1}{k+1}.
\]

APPENDIX: Existence of the free energy

As we have seen in the previous sections free energy exists for each known compatible boundary condition. We note that \(a(x)\) is bounded: \(\beta^{-1} \ln 2 \leq a(x) \leq C\beta\). Hence limit \((2.1)\) is also bounded. But the problem of convergence of \((2.1)\) is still open. Here we shall give some conditions on \(h\), under which the corresponding free energy exists.

Let \(\pi = \{x^0 = x_0 < x_1 < \ldots\}\) be an infinite path. A function \(h_x\) on the path \(\pi\) is called monotone non increasing (non decreasing) if \(h_{x_i} \geq h_{x_{i+1}}\), \((h_{x_i} \leq h_{x_{i+1}})\), \(i = 0, 1, 2, \ldots\).

Proposition 2. Let \(h = \{h_x, x \in V\}\) be a compatible b.c. If on any infinite path starting at \(x \in W_{n_0}\), \(|h_x|\) is monotone non increasing (non decreasing), then the corresponding free energy \(F(\beta, h)\) exists.

Proof. Using Stolz-Cesàro theorem we get
\[
\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} a(x) = \lim_{n \to \infty} A_n, \quad \text{with} \quad A_n = \frac{1}{|W_n|} \sum_{x \in W_n} a(x).
\]

We shall show that \(A_n\) is monotone for \(n > n_0\). We have
\[
A_{n-1} - A_n = \frac{1}{|W_n|} \left( k \sum_{x \in W_{n-1}} a(x) - \sum_{x \in W_n} a(x) \right) = \frac{1}{|W_n|} \sum_{x \in W_{n-1}} \left( k a(x) - \sum_{y \in S(x)} a(y) \right) = \frac{1}{|W_n|} \sum_{x \in W_{n-1}} \sum_{y \in S(x)} (a(x) - a(y)).
\]
By monotonicity of $|h_x|$ and by evenness of $a(x)$ we notice that $a(x) - a(y)$ does not change sign for all $x, y$ with $x < y$. Thus $A_n$ is monotone and since it is a bounded sequence it has a limit. □

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