Quaternary matroids are vf-safe

Robert Brijder\textsuperscript{1} and Hendrik Jan Hoogeboom\textsuperscript{2}

\textsuperscript{1} Hasselt University and Transnational University of Limburg, Belgium
\texttt{robert.brijder@uhasselt.be}
\textsuperscript{2} Leiden Institute of Advanced Computer Science,
Leiden University, The Netherlands
\texttt{hoogeboom@liacs.nl}

Abstract. Binary delta-matroids are closed under vertex flips, which consist of the natural operations of twist and loop complementation. In this note we provide an extension of this result from $GF(2)$ to $GF(4)$. As a consequence, quaternary matroids are “safe” under vertex flips (vf-safe for short). As an application, we find that the matroid of a bicycle space of a quaternary matroid is independent of the chosen representation. This extends a result of Vertigan [J. Comb. Theory B (1998)] concerning the bicycle dimension of quaternary matroids.

1 Preliminaries

1.1 Notation and terminology

For finite sets $U$ and $V$, a $U \times V$-matrix $A$ (over some field $\mathbb{F}$) is a matrix where the rows are indexed by $U$ and the columns by $V$, i.e., $A$ is formally a function $U \times V \to \mathbb{F}$. Hence, the order of the rows/columns is not fixed (i.e., interchanging rows or columns is mute). For $X \subseteq U$ and $Y \subseteq V$, the submatrix of $A$ induced by $X$ and $Y$ is denoted by $A[X,Y]$. We often abbreviate $A[X,X]$ by $A_X$. Let $A$ be a $V \times V$-matrix and $I_X$ for $X \subseteq V$ be the $V \times V$-matrix where the diagonal entries corresponding to $X$ are 1 and all other entries are 0. We abbreviate $A + I_X$ by $A + X$.

1.2 Principal pivot transform

Let $\alpha$ be an automorphism of a field $\mathbb{F}$. By abuse of notation, we extend $\alpha$ point-wise to vectors, matrices, and subspaces over $\mathbb{F}$. Hence for a $V \times V$-matrix $A = (a_{i,j})_{i,j \in V}$, we let $\alpha(A) = (\alpha(a_{i,j}))_{i,j \in V}$. Moreover, for subspace $L \subseteq \mathbb{F}^V$, we let $\alpha(L) = \{\alpha(v) \mid v \in L\}$.

Let $A$ be a $V \times V$-matrix (over an arbitrary field), and let $X \subseteq V$ be such that $A[X]$ is nonsingular, i.e., $\det A[X] \neq 0$. The principal pivot transform (or PPT for short) of $A$ on $X$, denoted by $A * X$, is defined as follows, see \cite{8}. If
\[ A = \begin{pmatrix} X & V \setminus X \\ \setminus & \setminus \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \]

then

\[ A \ast X = \begin{pmatrix} X & V \setminus X \\ \setminus & \setminus \end{pmatrix} \begin{pmatrix} P^{-1} & -P^{-1}Q \\ RP^{-1} & S - RP^{-1}Q \end{pmatrix}. \]

Matrix \((A \ast X) \setminus X = S - RP^{-1}Q\) is called the Schur complement of \(X\) in \(A\). Hence, \(A \ast X\) is defined iff \(A[X]\) is nonsingular.

It is easy to verify (by the above definition of PPT) that \(- (A \ast X)^T = (-A^T) \ast X\) for all \(X \subseteq V\) with \(A[X]\) nonsingular. As a consequence, if \(A\) is skew-symmetric, then \(A \ast X\) is skew-symmetric.

**Lemma 1.** Let \(A\) be a \(V \times V\)-matrix over some field \(\mathbb{F}\), and let \(\alpha\) be an automorphism of \(\mathbb{F}\). If \(X \subseteq V\) is such that \(A[X]\) is nonsingular, then \(\alpha(A \ast X) = \alpha(A) \ast X\).

**Proof.** Obviously, for a nonsingular matrix \(P\), \(PP^{-1} = I\), where \(I\) is the identity matrix (of suitable size). Therefore, \(I = \alpha(PP^{-1}) = \alpha(P)\alpha(P^{-1})\), and so

\[ \alpha(P^{-1}) = \alpha(P)^{-1}. \]

If \(A = \begin{pmatrix} X & V \setminus X \\ \setminus & \setminus \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \)

then in both cases we obtain:

\[ \begin{pmatrix} X & V \setminus X \\ \setminus & \setminus \end{pmatrix} \begin{pmatrix} \alpha(P)^{-1} & -\alpha(P)^{-1}\alpha(Q) \\ \alpha(R)\alpha(P)^{-1} & \alpha(S) - \alpha(R)\alpha(P)^{-1}\alpha(Q) \end{pmatrix}. \]

\[ \Box \]

**Proposition 1 ([8]).** Let \(A\) be a \(V \times V\)-matrix, and let \(X \subseteq V\) be such that \(A[X]\) is nonsingular. Then, for all \(Y \subseteq V\), \(\det((A \ast X)[Y]) = \det(A[X \Delta Y]) / \det(A[X])\). In particular, \((A \ast X)[Y]\) is nonsingular iff \(A[X \Delta Y]\) is nonsingular.

### 1.3 Pivot and loop complementation on set systems

A set system (over \(V\)) is a tuple \(M = (V, E)\) with \(V\) a finite set called the ground set and \(E \subseteq 2^V\) a family of subsets of \(V\). Set system \(M\) is called proper if \(E \neq \emptyset\).

We write simply \(Y \in M\) to denote \(Y \in E\). Let \(\max(E)\) be the family of maximal sets in \(E\) with respect to set inclusion, and let \(\max(M) = (V, \max(E))\) be the corresponding set system.

We define, for \(X \subseteq V\), pivot (also called twist) of \(M\) on \(X\), denoted by \(M \ast X\), as \((V, E \ast X)\), where \(E \ast X = \{ Y \Delta X \mid Y \in E \}\). In case \(X = \{ u \}\) is a singleton, we also write simply \(M \ast u\). Moreover, we define, for \(u \in V\), loop complementation of \(M\) on \(u\), denoted by \(M + u\), as \((V, E')\), where \(E' = E \Delta \{ X \cup \{ u \} \mid X \in E, u \notin X \}\). We assume left associativity of set system operations. Therefore, e.g., \(M + u \ast v\) denotes \((M + u) \ast v\). It has been shown in [8] that pivot \(* u\) and loop complementation \(+ u\) on a common element \(u \in V\) are involutions (i.e., of
order 2) that generate a group $F_u$ isomorphic to $S_3$, the group of permutations on 3 elements. In particular, we have $+u* u + u = *u + u* u$, which is the third involution (in addition to pivot and loop complementation), and is called the dual pivot, denoted by $\bar{\cdot}$. The elements of $F_u$ are called vertex flips. We have, e.g., $+ u * u = \bar{\cdot} u + u = * u \bar{\cdot} u$ and $* u + u = + u \bar{\cdot} u = \bar{\cdot} u + u$ for $u \in V$ (which are the two vertex flips in $F$ of order 3).

While on a single element the vertex flips behave as the group $S_3$, they commute when applied on different elements. Hence, e.g., $M * u + v = M + v * u$ and $M * u + v = M + v \bar{\cdot} u$ when $u \neq v$. Also, $M + u + v = M + v + u$ and thus we (may) write, for $X = \{u_1, u_2, \ldots, u_n\} \subseteq V, M + X$ to denote $M + u_1 \ldots + u_n$ (as the result is independent on the order in which the operations $+u_i$ are applied).

Similarly, we define $M \bar{\cdot} X$ for $X \subseteq V$.

One may explicitly define the sets in $M * V$, $M + V$, and $M \bar{\cdot} V$ as follows: $X \in M * V$ iff $V - X \in M$, and $X \in M + V$ iff $\{|Z \in M \mid Z \subseteq X\}$ is odd. Dually, $X \in M \bar{\cdot} V$ iff $\{|Z \in M \mid X \subseteq Z\}$ is odd. In particular $\emptyset \in M \bar{\cdot} V$ iff the number of sets in $M$ is odd.

Finally, it is observed in [3] that $\max(M) = \max(M \bar{\cdot} X)$ for all $X \subseteq V$.

We will often use the results of this subsection without explicit mention. Also we often simply denote the ground set of the set system under consideration by $V$.

### 1.4 $vf$-safe $\Delta$-matroids

We assume that the reader is familiar with the basic notions of matroids, see, e.g., [7].

A $\Delta$-matroid is a proper set system $M$ that satisfies the symmetric exchange axiom: For all $X, Y \in M$ and all $u \in X \Delta Y$, either $X \Delta \{u\} \in M$ or there is a $v \in X \Delta Y$ with $v \neq u$ such that $X \Delta \{u, v\} \in M$ [1]. Note that $\Delta$-matroids are closed under pivot, i.e., $M \bar{\cdot} X$ for $X \subseteq V$ is a $\Delta$-matroid when $M$ is a $\Delta$-matroid. If we assume a matroid $M$ is described by its basis, i.e., $M$ is the set system $(V, B)$ where $B$ is the set of bases of $M$, then it is shown in [2] Proposition 3 that a matroid $M$ is precisely a equicardinal $\Delta$-matroid. Hence $\Delta$-matroids form a generalization of matroids.

We say that a $\Delta$-matroid $M$ is vertex-flip-safe (or $vf$-safe for short) if for any sequence $\varphi$ of vertex flips (equivalently, pivots and loop complementations) over $V$ we have that $M \varphi$ is a $\Delta$-matroid. The family of $vf$-safe $\Delta$-matroids is minor closed [4]. We say that a family of $\Delta$-matroids is $vf$-closed if the family is closed under invertible vertex flips. There are (delta-)matroids that are not $vf$-safe, such as the 6-point line $U_{2,6}$, $P_6$, and the non-Fano $F_7^-$. In fact, they are excluded minors for the family of $vf$-safe $\Delta$-matroids.

Let $v \in \mathbb{F}^V$ be a vector. The support of $v$ is the set $X \subseteq V$ such that the entries of $X$ in $v$ are nonzero and entries of $V \setminus X$ in $v$ are zero. Let $L \subseteq \mathbb{F}^V$ be a subspace of $\mathbb{F}^V$. We denote by $M(L)$ the matroid with ground set $V$ such that for all $X \subseteq V$, $X$ is a circuit of $M(L)$ iff there is a $v \in L$ with support $X$ and $X$ is minimal with this property among the non-empty subsets of $V$. For a
Let $A$ be a $V \times V$-matrix. We denote by $M(A) = M(\ker(A))$, the matroid corresponding to the nullspace of $A$.

Let $A$ be a $V \times V$-matrix. We denote by $M_A$ the set system $(V, D)$ where $D = \{X \subseteq V \mid A[X] \text{ is nonsingular}\}$. It is shown in [1] (cf. Lemma 2) that $M_A$ is a $\Delta$-matroid if $A$ is skew-symmetric (i.e., $-A^T = A$). A $\Delta$-matroid $M$ is said to be representable over $\mathbb{F}$, if $M = M_A \ast X$ for some skew-symmetric $V \times V$-matrix $A$ and some $X \subseteq V$. It turns out that a matroid $M$ is representable in the $\Delta$-matroid sense iff $M$ is representable in the usual (matroid) sense. Moreover, if $A$ is skew-symmetric, then $\max(M_A)$ is a matroid represented by its bases and equal to $M(A)$ (this follows from the strong principal minor theorem [6]).

A $\Delta$-matroid is said to be binary if it is representable over $GF(2)$. It is shown in [3] that the family of binary $\Delta$-matroids is $\mathfrak{v}$-$\mathfrak{f}$-closed. Consequently, the class of $\mathfrak{v}$-$\mathfrak{f}$-safe $\Delta$-matroids contains the class of binary $\Delta$-matroids (and therefore also the class of binary matroids).

## 2 $\alpha$-symmetry and delta-matroids

Let $A$ be a $V \times V$-matrix over some field $\mathbb{F}$, and let $\alpha$ be an automorphism of $\mathbb{F}$. Then $A$ is called $\alpha$-symmetric if $\alpha(-A^T) = A$. Note that if $A$ is $\alpha$-symmetric, then $\alpha(\alpha(x)) = x$ for all elements $x$ of $A$. Thus $\alpha$ behaves as an involution on the elements of $A$. As a consequence, if $A$ is $\alpha$-symmetric, then $A^T$ is $\alpha$-symmetric. Also note that $A$ is id-symmetric with $\text{id}$ the identity automorphism iff $A$ is skew-symmetric.

If $A$ is $\alpha$-symmetric and $X \subseteq V$ is such that $A[X]$ is nonsingular, then $A \ast X$ is $\alpha$-symmetric. Indeed, $\alpha(-(A \ast X)^T) = \alpha((-A^T) \ast X) = \alpha(-A^T) \ast X = A \ast X$, where in the second equality we use Lemma 1.

The next result is a straightforward extension of a result of [1] (the original formulation restricts to the case $\alpha = \text{id}$).

**Lemma 2 ([1]).** Let $\alpha$ be an automorphism of some field $\mathbb{F}$, and let $A$ be a $\alpha$-symmetric $V \times V$-matrix over $\mathbb{F}$. Then $M_A$ is a $\Delta$-matroid.

**Proof.** Let $X, Y \in M_A$ and $x \in X \Delta Y$. If entry $A \ast X[\{x\}]$ is nonzero, then by Proposition 1 $X \Delta \{x\} \in M_A$ and we are done. Thus assume that $A \ast X[\{x\}]$ is zero. Since $A[Y]$ is nonsingular, $A \ast X[V \Delta Y]$ is nonsingular. Hence there is a $y \in X \Delta Y$ with entry $A \ast X[\{x\}, \{y\}]$ nonzero (note that $x \neq y$). Since $A \ast X$ is $\alpha$-symmetric, $A \ast X[\{x, y\}]$ is of the form

\[
\begin{pmatrix}
x & y \\
y & \alpha(-t_1) & t_2
\end{pmatrix}
\]

for some $t_1 \in \mathbb{F} \setminus \{0\}$ and $t_2 \in \mathbb{F}$. Thus $A \ast X[\{x, y\}]$ is nonsingular and $X \Delta \{x, y\} \in M_A$. \[\square\]

We say that a $\Delta$-matroid $M$ is $\alpha$-representable over $\mathbb{F}$, if $M = M_A \ast X$ for some $\alpha$-symmetric $V \times V$-matrix $A$ and $X \subseteq V$. Note that this is a natural extension of the notion of representable from [1] which coincides with id-representable.
A $V \times V$-matrix $A$ over $\mathbb{F}$ is called \textit{principally unimodular} (PU, for short) if for all $Y \subseteq V$, $\det(A[Y]) \in \{0, 1, -1\}$. Note that any $V \times V$-matrix over $GF(2)$ or $GF(3)$ is principally unimodular.

We now consider the field $GF(4)$. Let us denote the unique nontrivial automorphism of $GF(4)$ by $\text{inv}$. Note that $\text{inv}(x) = x^{-1}$ for all $x \in GF(4) \setminus \{0\}$, and thus $\text{inv}$ is an involution.

\textbf{Theorem 1.} Let $A$ be a inv-symmetric $V \times V$-matrix over $GF(4)$. Then $A$ is a principally unimodular.

\textit{Proof.} Recall that $1 = -1$ in $GF(4)$. We have $\det(A) = \det(\text{inv}(-A^T)) = \text{inv}(\det(-A^T)) = \text{inv}(\det(A))$. Thus $\det(A) \in \{0, 1\}$. \hfill $\square$

\textbf{Remark 1.} The proof of Theorem 1 essentially uses that the field $\mathbb{F}$ under consideration is of characteristic 2, i.e., $\mathbb{F} = GF(2^k)$ for some $k \geq 1$, and $\mathbb{F}$ has an automorphism $\alpha$ such that $\alpha$ is an involution and $\alpha$ has only trivial fixed points (the set of fixed points form $GF(2)$). The automorphisms $\alpha$ of $GF(2^k)$ are of the form $x \mapsto x^{\alpha l}$, with $1 \leq l \leq k$, and $\alpha$ is an involution when either $k = 1$ (and thus $l = 1$) or both $k$ is even and $l = k/2$. Moreover, for $l = k/2$ and $k$ even, the corresponding automorphism $\alpha$ has only trivial fixed points iff $l = 1$.

Consequently, the proof of Theorem 1 only works for $\alpha = \text{inv}$ and $\mathbb{F} = GF(4)$ (and, of course, $\alpha = \text{id}$ and $\mathbb{F} = GF(2)$).

The following result is a straightforward adaption of a result of [3].

\textbf{Proposition 2 (Theorem 8 of [3]).} Let $A$ be a principally unimodular $V \times V$-matrix over a field $\mathbb{F}$ of characteristic 2. Then, for all $X \subseteq V$, $\mathcal{M}_{A+X} = \mathcal{M}_A + X$.

\textit{Proof.} It suffices to show the result for $X = \{j\}$ with $j \in V$. Let $Z \subseteq V$. We compare $\det A[Z]$ with $\det(A + \{j\})[Z]$. First assume that $j \notin Z$. Then $A[Z] = (A + \{j\})[Z]$, thus $\det A[Z] = \det(A + \{j\})[Z]$. Now assume that $j \in Z$, which implies that $A[Z]$ and $(A + \{j\})[Z]$ differ in exactly one position: $(j,j)$. We may compute determinants by Laplace expansion over the $j$-th column, and summing minors. As $A[Z]$ and $(A + \{j\})[Z]$ differ at only the matrix-element $(j,j)$, these expansions differ only in the inclusion of minor $\det A[Z \setminus \{j\}]$. Thus $\det(A + \{j\})[Z] = \det A[Z] + \det A[Z \setminus \{j\}]$, and this computation is in $GF(2)$ as $A$ is PU and $\mathbb{F}$ of characteristic 2. From this the statement follows. \hfill $\square$

The following result is an adaption of [3] Theorem 8.2. ([3] Theorem 8.2] shows that the family of binary $\Delta$-matroids is vf-closed).

\textbf{Theorem 2.} The family of $\Delta$-matroids inv-representable over $GF(4)$ is vf-closed.

\textit{Proof.} Let $M$ be a $\Delta$-matroid inv-representable over $GF(4)$. Then $M = \mathcal{M}_A \ast X$ for some inv-symmetric $V \times V$-matrix $A$ over $GF(4)$ and $X \subseteq V$. Let $\varphi$ be a sequence of vertex flips over $V$. Let $W \in \mathcal{M}_A \ast X \varphi$, and consider now $\varphi' = \ast X \varphi \ast W$. By the $S^1_k$ group structure of vertex flips (see [3] Theorem 12)), $\varphi'$
can be put in the following normal form: \( M_{A \phi'} = M_A + Z_1 * Z_2 + Z_3 \) for some \( Z_1, Z_2, Z_3 \subseteq V \) with \( Z_1 \subseteq Z_2 \). By Theorem 1, \( A \) is PU. By Proposition 2, \( M_A + Z_1 = M_{A+Z_1} \). Thus \( M_A + Z_1 * Z_2 + Z_3 = M_{A+Z_1} * Z_2 + Z_3 \). By construction \( \emptyset \in M_{A \phi'} \). Hence we have \( \emptyset \in M_{A+Z_1} \). Therefore \( Z_2 \in M_{A+Z_1} \) and so \( A + Z_1 * Z_2 \) is defined. Consequently, \( A' = A + Z_1 * Z_2 + Z_3 \) is defined and \( M_{A \phi'} = M_{A'} \). Hence \( M_\phi = M_A * \phi = M_A * W \) and thus inv-symmetric matrix \( A' \) represents \( M_\phi \). Consequently, \( M_\phi \) a \( \Delta \)-matroid inv-representable over \( GF(4) \).

In contrast with Theorem 2, it is shown in [4] that there are \( \Delta \)-matroids id-representable over \( GF(4) \) that are not \( \phi \)-safe.

3 Quaternary matroids and bicycle matroids

Let \( M = (V, B) \) be a matroid representable over \( \mathbb{F} \), and described by its bases. Let \( B \) be a standard representation of \( M \) over \( \mathbb{F} \). Then \( B \) is equal to

\[
\begin{pmatrix}
X & V \setminus X \\
X & (I & S)
\end{pmatrix}
\]

for some \( X \in B \), where \( I \) is the identity matrix of suitable size. Let \( \alpha \) be an automorphism of \( \mathbb{F} \) that is an involution. We define \( R(B, \alpha) \) to be the \( \alpha \)-symmetric \( V \times V \)-matrix

\[
\begin{pmatrix}
X & V \setminus X \\
V \setminus X & \begin{pmatrix}
0 & S \\
\alpha(-S^T) & 0
\end{pmatrix}
\end{pmatrix}
\]

The next result is from [1].

**Proposition 3** (Theorem 4.4 of [1]). Let \( M \) be a matroid representable over \( \mathbb{F} \), \( B \) be a \( X \times V \)-matrix over \( \mathbb{F} \) that is a standard representation of \( M \). Then \( \mathcal{M}_A = M * X \) with \( A = R(B, \text{id}) \).

Note that if \( A = R(B, \alpha) \) and \( A' = R(B, \text{id}) \), then \( A[Y] \) is nonsingular iff \( A'[Y] \) is nonsingular for all \( Y \subseteq V \). Hence \( \mathcal{M}_A = \mathcal{M}_{A'} \).

Hence, by Proposition 3 if a matroid \( M \) is (id-)representable over \( \mathbb{F} \), then \( M \) is \( \alpha \)-representable for all automorphisms \( \alpha \) of \( \mathbb{F} \) that are involutions. Conversely, if \( M \) is \( \alpha \)-representable for all involutions \( \alpha \) of \( \mathbb{F} \), then by the exact same reasoning as the only-if direction of the proof of Theorem 4.4 of [1], we have that matroid \( M \) is representable over \( \mathbb{F} \). Consequently, a matroid \( M \) is representable over \( \mathbb{F} \) in the usual (matroid) sense iff \( M \) is \( \alpha \)-representable for some involution and automorphism \( \alpha \) of \( \mathbb{F} \) iff \( M \) is \( \alpha \)-representable for all automorphisms \( \alpha \) of \( \mathbb{F} \) that are involutions. Therefore, choosing \( \alpha = \text{id} \) may not necessarily be the most natural extension of the matroid notion of representability to \( \Delta \)-matroids. Indeed, in view of Theorem 2 and the remark below it, we argue that over \( GF(4) \), inv-representability is the most natural extension of the matroid notion of representability to \( \Delta \)-matroids.
In particular, by Proposition 3 every quaternary matroid is a $\Delta$-matroid invert-representable over $GF(4)$. Hence by Theorem 2 we have the following result, which was conjectured in [4].

**Corollary 1.** Every quaternary matroid is vf-safe.

In this paragraph use terminology of [9]. Let $L \subseteq F^V$ be a subspace of $F^V$, and let $\alpha$ be an automorphism of $F$. We define $bd(L, F, \alpha) = \dim(L \cap \alpha(L^\perp))$, where $L^\perp$ is the orthogonal subspace of $L$. If $|F| = q \in \{2, 3, 4\}$, then there is an automorphism $\alpha : F \to F$ with $\alpha(x) = x^{-1}$ for all $x \in F \setminus \{0\}$. In these cases, $bd(L, F, \alpha)$ is called the **bicycle dimension** of $L$ and we denote it by $bd(L, q)$.

Let $L$ be a subspace of $GF(4)^V$. In line with [9], we call $L \cap \text{inv}(L^\perp)$ the **bicycle space** of $L$, and denote it by $BC_L$. We know from [9] that the dimension of $BC_L$ is determined by $M(L)$. We now extend this result by showing that the matroid of $BC_L$ is determined by $M(L)$. Moreover we give an explicit formula for $M(BC_L)$.

Also, the proof of this result below is direct, and therefore not obtained as a consequence of an evaluation of the Tutte polynomial as in [9].

**Theorem 3.** Let $M$ be a quaternary matroid, and let $A$ be a representation of $M$ over $GF(4)$. Then the matroid $M(BC_{\ker(A)})$ is equal to $\max(M + V)$.

**Proof.** Let $A'$ be a standard representation of $M$ with $L = \ker(A') = \ker(A)$. Let $P = R(A', \text{inv})$. By Proposition 4, $\mathcal{M}_P \ast X = M$ for some $X \subseteq V$. By Theorem 1, $P$ is PU and by Proposition 2, $\mathcal{M}_P + V = \mathcal{M}_{P+V}$. Now, $A' = M + V[X, V]$. Recall from, e.g., [7] Proposition 2.2.23, that $\ker(I - S)^\perp = \ker(-S^T I)$. Thus, $\ker(P + V[V \setminus X, V]) = \text{inv}(L^\perp)$. Consequently, $\ker(P + V) = BC_L$. Now, $M(BC_L) = M(P + V) = \max(M_{P+V}) = \max(M_P + V) = \max(M * Y + V) = \max(M + V \ast Y) = \max(M + V)$. \hfill $\square$

While $BC_A$ and $BC_{A'}$ may differ when $A$ and $A'$ are different representations of $M$ over $GF(4)$, Theorem 3 shows that $M(BC_A) = M(BC_{A'})$.

A binary matroid $M$ has an odd number of bases iff the dimension of the bicycle space is zero—a result originally shown in [5] for the case where $M$ is a graphic matroid. Theorem 3 shows that this statement holds in general for quaternary matroids $M$.

**References**

1. A. Bouchet. Representability of $\Delta$-matroids. In *Proc. 6th Hungarian Colloquium of Combinatorics, Colloquia Mathematica Societatis János Bolyai*, volume 52, pages 167–182. North-Holland, 1987.
2. A. Bouchet. Coverings and delta-coverings. In E. Balas and J. Clausen, editors, *IPCO*, volume 920 of *Lecture Notes in Computer Science*, pages 228–243. Springer, 1995.
3. R. Brjider and H.J. Hoogeboom. The group structure of pivot and loop complementation on graphs and set systems. *European Journal of Combinatorics*, 32:1353–1367, 2011.
4. R. Brijder and H.J. Hoogeboom. Nullity and loop complementation for delta-matroids. To appear in SIAM Journal on Discrete Mathematics, preprint [arXiv:1010.4497], 2013.
5. W.-K. Chen. On vector spaces associated with a graph. SIAM Journal on Applied Mathematics, 20:526–529, 1971.
6. V. Kodiyalam, T.Y. Lam, and R.G. Swan. Determinantal ideals, Pfaffian ideals, and the principal minor theorem. In Noncommutative Rings, Group Rings, Diagram Algebras and Their Applications, pages 35–60. American Mathematical Society, 2008.
7. J.G. Oxley. Matroid theory, Second Edition. Oxford University Press, 2011.
8. A.W. Tucker. A combinatorial equivalence of matrices. In Combinatorial Analysis, Proceedings of Symposia in Applied Mathematics, volume X, pages 129–140. American Mathematical Society, 1960.
9. D. Vertigan. Bicycle dimension and special points of the Tutte polynomial. Journal of Combinatorial Theory, Series B, 74(2):378–396, 1998.