Cosmological scaling solutions in generalised Gauss-Bonnet gravity theories

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Abstract The conditions for the existence and stability of cosmological power-law scaling solutions are established when the Einstein-Hilbert action is modified by the inclusion of a function of the Gauss-Bonnet curvature invariant. The general form of the action that leads to such solutions is determined for the case where the universe is sourced by a barotropic perfect fluid. It is shown by employing an equivalence between the Gauss-Bonnet action and a scalar-tensor theory of gravity that the cosmological field equations can be written as a plane autonomous system. It is found that stable scaling solutions exist when the parameters of the model take appropriate values.

Keywords  Generalised gravity · Gauss-Bonnet · Scaling solutions · Cosmology

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1 Introduction

In recent years, there has been considerable interest in the possibility that Einstein’s theory of general relativity may become modified in high-curvature regimes and over large distance scales. This possibility has been motivated by a wealth of high redshift observations, which indicate that the universe is presently undergoing a phase of accelerated expansion [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]. (For a review, see, e.g. [16]). In many such studies, the Einstein-Hilbert action is modified by the introduction of terms involving higher-order curvature invariants. An important quadratic combination of such invariants, which is motivated by string theory, is given by the Gauss-Bonnet (GB) invariant

$$\mathcal{G} \equiv R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\tau}R_{\mu\nu\rho\tau}. \quad (1)$$

In four dimensions, the GB term is a topological invariant and introducing a term proportional to $\mathcal{G}$ into the Einstein-Hilbert action does not modify the dynamics. Recently, however, the cosmology of models based on a class of generalised theories with an action of the form

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2} + f(\mathcal{G}) \right) + S_m \quad (2)$$
has been considered, where \( f(G) \) is a differentiable function of \( G \) and \( S_m \) represents the matter action [17].

The purpose of the present paper is to investigate the existence and stability of cosmological power-law scaling solutions derived from theories of the type (2) in the presence of a perfect fluid matter source. Scaling (attractor) solutions play an important role in cosmology, since they enable the asymptotic behaviour and stability of a particular cosmological background to be determined. Moreover, they provide a framework for establishing the behaviour of more general cosmological solutions [32,33,34,35,36,37].

The structure of the paper is as follows. We begin in Section 2 by summarizing the derivation of the cosmological field equations by employing an equivalence between the action (2) and a corresponding action involving a self-interacting scalar field that is non-minimally coupled to gravity. We focus on the spatially flat and isotropic Friedmann-Lemaitre-Robertson-Walker (FLRW) universe and proceed in Section 3 to identify the most general form for the function \( f(G) \) that results in power-law (scaling) solutions when the matter source is a barotropic fluid with a constant equation of state parameter. Specifically, we find that scaling solutions may arise when \( f = \pm 2\sqrt{\alpha G} \), where \( \alpha \) is an arbitrary constant. We then show that for this form of the action, the field equations can be expressed as a plane autonomous system. This allows us to employ dynamical systems theory to investigate the stability of the vacuum and non-vacuum solutions and this is done in Sections 4 and 5, respectively. We find that scaling solutions, corresponding either to a stable node or a stable spiral node, can arise when the equation of state of the fluid and the parameter, \( \alpha \), satisfy appropriate conditions. We conclude with a discussion in Section 6. Units are chosen such that \( 8\pi G = c = 1 \).

2 Cosmological Field Equations

Action (2) may be expressed in an alternative form by introducing two auxiliary scalar fields \( \chi \) and \( \zeta \) such that [17,38,39,40]

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{2} + \zeta (G - \chi) + f(\chi) \right) + S_m. \tag{3}
\]

Varying Eq. (3) with respect to \( \zeta \) yields the constraint \( \chi = G \), thereby reproducing action (2). On the other hand, varying action (3) with respect to \( \chi \) implies that \( \zeta = F(\chi) \), where \( F(\chi) \equiv \partial f(\chi)/\partial \chi \), and substituting this condition back into Eq. (3) leads to

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{2} + F(\chi)(G - \chi) + f(\chi) \right) + S_m. \tag{4}
\]

It follows, therefore, that the action (2) is equivalent to the action [17,38]

\[
S = \int d^4x \sqrt{-g} \left( \frac{R}{2} - V(\phi) - h(\phi)G \right) + S_m, \tag{5}
\]

where the scalar field, \( \phi \), is defined implicitly by

\[
h(\phi) \equiv -F(G) \tag{6}
\]

for some function \( h(\phi) \) and has an effective self-interaction potential

\[
V(\phi) \equiv GF(G) - f(G), \tag{7}
\]

where \( F \equiv \partial f/\partial G \). Eq. (5) may be interpreted as an effective ‘scalar-tensor’ theory, where the scalar field has a vanishing kinetic term.

To study cosmological models based on action (2), one may proceed directly by varying the action to derive the field equations or, indirectly, by varying the equivalent action (5). We employ the latter approach in the present work in view of its potential simplicity. The field equations in this case take the form

\[
R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = T^{\mu\nu}_m + T^{\mu\nu}_G \tag{8}
\]
where $T^\mu_\nu_m$ is the energy-momentum tensor of the matter fields and $T^\mu_\nu_\phi$ denotes the effective energy-momentum tensor resulting from the scalar field, $\phi$, and the GB term. Since the GB term is a topological invariant in four dimensions, the standard field equations of GR are recovered when $h(\phi) = \text{constant}$. Consequently, only terms involving derivatives of $h(\phi)$ arise in the energy-momentum tensor, which is given by

$$T^\mu_\nu_\phi = -g^\mu_\nu V(\phi) - 2[\nabla^\mu \nabla^\nu h(\phi)]R + 4[\nabla^\beta \nabla^\gamma h(\phi)]R^\beta_\gamma + 4\nabla^\mu \nabla^\nu h(\phi) \nabla^\rho R^\rho_\mu + 4\nabla^\mu \nabla^\nu h(\phi) \nabla^\rho \nabla^\sigma R^\rho_\sigma - 4\nabla^\mu \nabla^\nu h(\phi) \nabla^\rho \nabla^\sigma R^\rho_\sigma + 4\nabla^\mu \nabla^\nu h(\phi) \nabla^\rho_\gamma R^\mu_\nu R^\rho_\gamma. \quad (9)$$

Finally, the equation of motion for the scalar field takes the form

$$V_\phi(\phi) + h_\phi(\phi) \mathcal{G} = 0, \quad (10)$$

where a comma denotes differentiation with respect to $\phi$.

Our aim is to study the dynamics of the isotropic and spatially flat FLRW universe sourced by a perfect barotropic fluid with an equation of state parameter, $w_m = p_m / \rho_m$, where $p_m$ and $\rho_m$ denote the pressure and energy density of the fluid, respectively. For this spacetime, the GB invariant is given by $\mathcal{G} = 4H^2 (H + H^2)$, where $H \equiv \dot{a} / a$ defines the Hubble parameter, $a$ represents the scale factor of the universe and a dot denotes differentiation with respect to cosmic time. The Friedmann and Raychaudhuri equations derived from Eqs. (8)-(9) for this background are then given by

$$3H^2 = V(\phi) + 24H^2 \dot{h} + \rho_m, \quad (11)$$

$$\left(2 \frac{H}{H^2} + 3 \right) H^2 = V(\phi) + 8H^2 \dot{h} + 16H^3 \dot{h} \left(1 + \frac{\dot{H}}{H^2} \right) - p_m, \quad (12)$$

respectively, and the scalar field equation (10) reduces to

$$V_\phi + 24h_\phi H^2 (H + H^2) = 0. \quad (13)$$

It proves convenient to interpret the GB gravitational terms on the right-hand side of the Friedmann equation (11) as an effective energy density, such that $\rho_\phi = T^\mu_\nu_\phi + V(\phi)$, where $T^\mu_\nu_\phi = 24H^3$ plays the role of a kinetic energy. It is then natural to introduce the dimensionless variables

$$y_1 \equiv \frac{V(\phi)}{3H^2}, \quad y_2 \equiv 8H \dot{h}, \quad (14)$$

and the fractional energy densities

$$\Omega_m \equiv \frac{\rho_m}{3H^2} = 1 - y_1 - y_2, \quad (15)$$

$$\Omega_\phi \equiv y_1 + y_2. \quad (16)$$

The background field equations (11)-(13) can then be expressed in terms of these variables such that

$$\frac{dy_1}{dN} = 2\varepsilon y_1 - (1 - \varepsilon)y_2, \quad (17)$$

$$\frac{dy_2}{dN} = -2\varepsilon + 3(1 - y_1) - (2 - \varepsilon)y_2 + 3w_m\Omega_m, \quad (18)$$

where $\varepsilon \equiv -\dot{H} / H^2$ and $N \equiv \ln a$. 
3 Cosmological Scaling Solutions

We wish to identify the class of GB theories that admit scaling solutions such that each of the terms in the Friedmann equation (11) scales at the same rate, \( H^2 \propto \rho_m \propto V(\phi) \propto T_G \). These conditions result in a power-law solution to Eqs. (11)-(13) of the form \( a \propto t^{1/\varepsilon} \), where \( \varepsilon = \text{constant} \). For such a scaling solution, it follows from Eq. (13) that

\[
V,_{\phi} = -\frac{1}{\alpha}V^2h,_{\phi}
\]

when \( \varepsilon \neq 1 \), where \( \alpha \) is a finite constant. Integrating Eq. (19) then implies that

\[
h = \frac{\alpha}{V} + \beta,
\]

where \( \beta \) is an arbitrary integration constant.

Relating the functions \( V(\phi) \) and \( h(\phi) \) in this way is equivalent to specifying the form of the GB function, \( f(G) \), via the definition given in Eq. (7). Indeed, substituting Eq. (20) into Eq. (7) results in the first-order, non-linear differential equation

\[
\left( G \frac{df}{dG} - f \right) \left( \frac{df}{dG} + \beta \right) = -\alpha.
\]

Eq. (21) is an example of Clairaut’s equation and may be solved in full generality by differentiating with respect to \( G \):

\[
\frac{d^2f}{dG^2} \left[ \left( \frac{df}{dG} + \beta \right)^2 - \frac{\alpha}{G} \right] = 0.
\]

Eq. (22) is trivially solved by \( f(G) = \alpha_0 + \alpha_1G \), where \( \alpha_i \) are constants. However, this simply corresponds to the introduction of a cosmological constant in the action (2) and is not physically interesting to the present discussion. (Recall that a contribution of the form \( f \propto \mathcal{G} \) is also uninteresting since the GB term is a topological invariant). On the other hand, a singular solution to Eq. (21) with no arbitrary constants can be found by setting the square bracketed term in Eq. (22) to zero and substituting the result into Eq. (21). We find that

\[
f(G) = \pm 2\sqrt{\alpha G},
\]

where we have specified \( \beta = 0 \) without loss of generality. Moreover, requiring the action (2) to be real implies that \( \alpha G > 0 \).

Eqs. (20) and (23) represent the necessary and sufficient conditions for the existence of power-law scaling solutions, where \( \varepsilon = \text{constant} \). More general solutions to the field equations, where \( \varepsilon \) is time-dependent, exist for this model. If the cosmological behaviour of the model (23) is to be determined, the coupled differential equations (17)-(18) must close. This implies that the parameter \( \varepsilon \) must be expressible as a function of \( y_1 \) and \( y_2 \) only. When Eq. (20) is satisfied, we find that

\[
\varepsilon = 1 - \frac{3}{8\alpha}y_1^2.
\]

Hence, substituting Eq. (24) into Eqs. (17)-(18) yields the plane autonomous system:

\[
\frac{dy_1}{dN} = 2y_1 - \frac{3}{4\alpha}y_1^3 - \frac{3}{8\alpha}y_1y_2,
\]

\[
\frac{dy_2}{dN} = 2(y_2 - 1) - \frac{3}{8\alpha}y_1^2y_2 + \frac{3}{4\alpha}y_1^2 + 3(1 + w_m)(1 - y_1 - y_2).
\]

Before concluding this section, it should be remarked that the equivalence between actions (2) and (5) does not apply for the special case \( \varepsilon = 1 \) (\( y_1 = 0 \)), corresponding to the coasting solution, \( a \propto t \). In this case, integration of Eq. (13) would yield \( V(\phi) = V_0 = \text{constant} \) and the solution to Eq. (7) would then be given by \( f(\mathcal{G}) = -V_0 + \gamma \mathcal{G} \) for some constant \( \gamma \). This disparity can be traced to the singular nature of the coasting
solution for the model (23). Specifically, the Friedmann equation derived directly from action (2) for this model is given by

$$3H^2 = \pm \frac{\sqrt{6\alpha}H^2(2H^3 - \dot{H})}{(H + H^3)^{3/2}} + \rho_m$$

(27)

and the term originating from the GB contribution is ill-defined when $\varepsilon = 1$ ($y_1 = 0$). Consequently, we do not consider this solution in the phase plane analyses of the following sections.

4 Vacuum solutions

In this Section, we consider vacuum solutions where $\Omega_m = 0$ and $y_1 = 1 - y_2$. The pair of equations (25)-(26) then reduces to the one-dimensional system

$$\frac{dy_1}{dN} = y_1 \left(2 - \frac{3}{8\alpha}y_1 - \frac{3}{8\alpha}y_1^2\right).$$

(28)

There exist two power-law solutions when $y_1 \neq 0$:

$$y_1 = -\frac{1}{2} \pm \frac{1}{6} \sqrt{9 + 192\alpha},$$

(29)

which we denote as $\gamma^{\pm}$, respectively. The reality of the fixed points requires that $\alpha \geq -9/192$. The power of the expansion can be expressed in terms of the effective equation of state parameter

$$w_{eff} \equiv -1 + \frac{2}{3} \varepsilon$$

(30)

such that $a(t) \propto t^{2/[3(1+w_{eff})]}$. It is determined by the value of the GB coupling parameter, $\alpha$, and substituting Eqs. (24) and (29) into Eq. (30) implies that

$$w_{eff} = \frac{1}{24\alpha} \left[-40\alpha - 3 \pm \sqrt{9 + 192\alpha}\right],$$

(31)

where the $+/-$ corresponds to the points $\gamma^{\pm}$, respectively. This dependency of the effective equation of state on the GB parameter is illustrated in Fig. 1. The solution $\gamma^+$ corresponds to an inflationary cosmology when $\alpha > 0$ and the exponential, de Sitter solution arises when $\alpha = 3/8$. The solution $\gamma^-$ is in a super-inflationary regime ($w_{eff} < -1$) for $\alpha > 0$. When $\alpha < 0$, the effective equation of state corresponds to that of an ultra-stiff fluid ($w_{eff} \geq 1$). Our results are in line with the recent conclusions of Ref. [45], where a study of the late-time cosmology based on the model $f(G) \propto -G^\alpha$ was made with the field equations derived directly from action (2).

The eigenvalues associated with the equilibrium points $\gamma^{\pm}$ are given by

$$\mu^\pm = -4 - \frac{3}{16\alpha} \pm \frac{1}{16\alpha} \sqrt{9 + 192\alpha}.$$  

(32)

The solution $\gamma^+$ is stable for $\alpha > -9/192$. The solution $\gamma^-$ is a stable point when $\alpha > 0$ and unstable for $-9/192 < \alpha < 0$. 


The stability of these vacuum solutions is altered when a matter source is introduced into the system and a second-order analysis, which is beyond the scope of the present work, analyse the stability of the equilibrium point for these particular choices of parameter values would require the de Sitter solution if $\alpha > 3/8$. The middle panel corresponds to $^{\prime\prime}$ when $\alpha < 0$ and in this regime $w_{eff} \geq 1$. The right-hand panel corresponds to $^{\prime}$ when $\alpha > 0$ and in this regime $w_{eff} < -1$.

5 Non-vacuum solutions

In this Section, we study the background dynamics of models based on GB theories of the type (23) in the presence of a perfect fluid. The vacuum solutions $^{\prime\prime}$ remain as equilibrium points of the autonomous system (25)-(26): 

$$(y_1, y_2) = \left( -\frac{1}{2} \pm \frac{1}{6} \sqrt{9 + 192\alpha}, \quad \frac{3}{2} \pm \frac{1}{6} \sqrt{9 + 192\alpha} \right).$$

(33)

In addition, there exist two scaling solutions, where $\Omega_m$ and $\Omega_{\text{eff}}$ are constants:

$$(y_1, y_2) = \left( \pm \frac{2\sqrt{-3\alpha(1 + 3w_m)}}{3}, \quad \pm \frac{12\alpha(1 + w_m)}{\sqrt{-3\alpha(1 + 3w_m)}} \right),$$

(34)

$$\Omega_m = 1 \pm \frac{2\sqrt{-3\alpha(1 + 3w_m)}}{3} \pm \frac{12\alpha(1 + w_m)}{\sqrt{-3\alpha(1 + 3w_m)}},$$

(35)

$$\Omega_{\text{eff}} = \pm \frac{2\sqrt{-3\alpha(1 + 3w_m)}}{3} \pm \frac{12\alpha(1 + w_m)}{\sqrt{-3\alpha(1 + 3w_m)}},$$

(36)

and $w_{eff} = w_m$. We denote these solutions by $^{\prime\prime\prime}$.

The eigenvalues associated with the equilibrium points $^{\prime\prime}$ are given by

$$\mu_1^\pm = -\frac{1}{32\alpha} \left[ 48\alpha(3 + w_m) + 9 \mp 3\sqrt{9 + 192\alpha} \right] + \lambda_1^\pm$$

(37)

$$\mu_2^\pm = -\frac{1}{32\alpha} \left[ 48\alpha(3 + w_m) + 9 \mp 3\sqrt{9 + 192\alpha} \right] - \lambda_1^\pm$$

(38)

$$\lambda_1^\pm = \frac{1}{32\alpha} \left[ 256\alpha^2(1 + 3w_m)^2 + 288\alpha(1 + w_m) + 18 \mp 32\alpha(1 + 3w_m)\sqrt{9 + 192\alpha} \mp 6\sqrt{9 + 192\alpha} \right]^{1/2}$$

(39)

The stability of these vacuum solutions is altered when a matter source is introduced into the system and depends on both the GB parameter, $\alpha$, and the perfect fluid equation of state, $w_m$. This dependency is illustrated in Fig. 3. The solid lines represent the regions where the nature of the equilibrium points changes as the parameter values are altered. The stability of $^{\prime\prime\prime}$ is determined by the sign of the GB parameter, $\alpha$. On the boundary distinguishing the nature of the fixed point $^{\prime\prime\prime}$, one of the eigenvalues $\mu_{1,2}^\pm$ vanishes. To analyse the stability of the equilibrium point for these particular choices of parameter values would require a second-order analysis, which is beyond the scope of the present work.
In this paper we have investigated the existence and stability of cosmological power-law scaling solutions sourced by a barotropic fluid when an appropriate function of the Gauss-Bonnett topological invariant is introduced into the Einstein-Hilbert action. It was found that the general class of such theories that admit power-law solutions is given by Eq. (23), i.e., \( f(\mathcal{R}) = \pm 2\sqrt{\alpha \mathcal{R}} \) for some constant coefficient, \( \alpha \). By exploiting an equivalence between generalized Gauss-Bonnet gravitational theories and a corresponding higher-order, scalar-tensor theory, it was further shown that the Friedmann equations for this class of model can be written in the form of a two-dimensional dynamical system. The stability of the equilibrium points for these solutions is illustrated in Fig. 3. The points are real in the region of parameter space, \( \alpha(1 + 3w_m) \leq 0 \). Furthermore, they are only physically meaningful if \( \Omega_m = 1 - y_1 - y_2 \geq 0 \). This results in a further restriction in the \((w_m, \alpha)\) plane after substitution of Eq. (35).

The top two panels of Fig. 3 correspond to the scaling solution \( \mathcal{R}^+ \) where \( y_1 > 0 \) and the bottom two panels correspond to \( \mathcal{R}^- \) where \( y_1 < 0 \). The point \( \mathcal{R}^+ \) is always a stable node or a stable spiral. The point \( \mathcal{R}^- \) is always a saddle. On the curve \( \Omega_m = 0 \), one of the eigenvalues of \( \mathcal{R}^\pm \) vanishes.

To illustrate the scaling dynamics, let us consider the specific case where \( (\alpha, w_m) = (0.05, -0.6) \). At this location in parameter space, there exist two equilibrium points \( \mathcal{R}^\pm \), the saddle point \( \mathcal{R}^+ \) and the stable node \( \mathcal{R}^- \). The basin of attraction for \( \mathcal{R}^+ \) is shown in Fig. 4. As a second example, we consider the case \( (\alpha, w_m) = (-0.005, -0.05) \), where there exist four equilibrium points: an unstable vacuum solution \( \mathcal{R}^- \), a saddle point \( \mathcal{R}^- \), a stable \( \mathcal{R}^+ \) and a stable spiral \( \mathcal{R}^+ \). The spiral nature of the point \( \mathcal{R}^+ \) is illustrated in the phase portrait of Fig. 5 where the initial conditions were specified to be \( \Omega_m = \Omega_g = 0.5 \).

6 Conclusion

In this paper we have investigated the existence and stability of cosmological power-law scaling solutions sourced by a barotropic fluid when an appropriate function of the Gauss-Bonnett topological invariant is introduced into the Einstein-Hilbert action. It was found that the general class of such theories that admit power-law solutions is given by Eq. (23), i.e., \( f(\mathcal{R}) = \pm 2\sqrt{\alpha \mathcal{R}} \) for some constant coefficient, \( \alpha \). By exploiting an equivalence between generalized Gauss-Bonnet gravitational theories and a corresponding higher-order, scalar-tensor theory, it was further shown that the Friedmann equations for this class of model can be written in the form of a two-dimensional dynamical system. The stability of the equilibrium points for these solutions is illustrated in Fig. 3. The points are real in the region of parameter space, \( \alpha(1 + 3w_m) \leq 0 \). Furthermore, they are only physically meaningful if \( \Omega_m = 1 - y_1 - y_2 \geq 0 \). This results in a further restriction in the \((w_m, \alpha)\) plane after substitution of Eq. (35).

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1 Note that the point \( \mathcal{R}^- \) also exists but this occurs in the region \( y_1 < 0 \). Stable scaling solutions arise only for \( y_1 > 0 \) and, since \( y_1 = 0 \) is a separatrix, a trajectory beginning in the region \( y_1 < 0 \) will not be able to reach \( \mathcal{R}^- \). We therefore choose the initial conditions in Fig. 4 such that \( y_1 > 0 \). This is equivalent to choosing the negative sign in Eq. (23).
both vacuum and non-vacuum models was established. In the former case, the GB parameter, $\alpha$, determines the effective equation of state parameter. For non-vacuum solutions, the nature of the critical points depends on both $\alpha$ and the fluid equation of state parameter, $w_m$. The regions of parameter space ($\alpha, w_m$) that admit stable non-vacuum scaling solutions were identified.

The models we have investigated do not admit a transition from a decelerating to an accelerating phase of cosmic expansion. However, our aim in this paper has been to focus on power-law solutions rather than develop a phenomenological model of generalized Gauss-Bonnet gravity as a candidate for dark energy. Power-law solutions are of interest since they can be regarded as approximations to more realistic models. In particular, phenomenological models could be constructed where the parameter $\alpha$ is given by some function of $G$ (or equivalently the scalar field $\phi$), such that $\alpha$ is slowly varying for much of the history of the universe, but at some epoch undergoes a change in sign. In principle, this could cause the universe to enter a phase of accelerated expansion. It would be interesting to develop specific models of this type, along the lines outlined in Ref. [27].

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Fig. 4 Illustrating the dynamics of the model (23) for the particular case where \((\alpha, w_m) = (0.05, -0.6)\). The left-hand panel depicts the phase space, where the straight line \(y_1 = 1 - y_2\) corresponds to the vacuum solution \(\Omega_m = 0\). The red dot represents the scaling fixed point \(S^+\). For the range of initial conditions chosen, all non-vacuum, physically acceptable solutions are attracted to \(S^+\). The right-hand panel depicts the evolution of the fractional energy densities of the perfect fluid, \(\Omega_m\), and the GB contribution, \(\Omega_G\), for the initial conditions \(\Omega_m = \Omega_G = 0\). It is seen that the fractional densities asymptote to constant values at late times, thus indicating that the solution is scaling.

Fig. 5 Illustrating the dynamics of the model (23) for the particular case where \((\alpha, w_m) = (-0.005, -0.05)\). The left-hand panel depicts the phase space for this scenario, whereas the right-hand panel depicts the evolution of the fractional energy densities \(\Omega_m\) and \(\Omega_G\). The initial conditions were chosen such that \(\Omega_m = \Omega_G = 0.5\). At late times, the fractional energy densities of the fluid and GB contribution tend to constant values.

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