$L^p$-BOUNDEDNESS OF PSEUDO DIFFERENTIAL OPERATORS ON RANK ONE RIEMANNIAN SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. The aim of this paper is to study $L^p$-boundedness property of the pseudo differential operator associated with a symbol, on rank one Riemannian symmetric spaces of noncompact type, where the symbol satisfies Hörmander-type conditions near infinity.

1. Introduction

A differential operator $p(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) \frac{\partial^n}{\partial x^n}$ can be represented as

$$p(x, D) f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} p(x, \xi) \mathcal{F} f(\xi) \, d\xi,$$

where $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$ and $\mathcal{F} f$ is the Fourier transform of $f$ defined by

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx.$$

Let $a(x, \xi)$ be a general function on $\mathbb{R}^n \times \mathbb{R}^n$ not necessarily a polynomial in the $\xi$ variable. Consider the operator $a(x, D)$ defined by

$$a(x, D) f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi) \mathcal{F} f(\xi) \, d\xi.$$

This operator $a(x, D)$ is called pseudo differential operator associated with the ‘symbol’ $a(x, \xi)$. Pseudo differential operators have many applications in the theory of partial differential operators. If the symbol $a(x, \xi)$ is independent in $x$ variable, say $a(x, \xi) = m(\xi)$, then the associated operator

$$m(D) f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} m(\xi) \mathcal{F} f(\xi) \, d\xi,$$

is called multiplier operator. Boundedness of the multiplier operator is well studied on $\mathbb{R}^n$.

Let $m : \mathbb{R}^n \to \mathbb{C}$ satisfies the following Hörmander-Mihlin differential inequalities:

$$\left| \frac{d^j}{d\xi^j} m(\xi) \right| \leq A_j |\xi|^{-j},$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and for all $0 \leq j \leq \left[ \frac{n}{2} \right] + 1$. Then the multiplier operator $m(D)$ is a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

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Now let $G$ be a real rank one noncompact connected semisimple Lie group with finite center. Let $K$ be a maximal compact subgroup of $G$, and let $X = G/K$ be the associated symmetric space. Corresponding to a multiplier function $m$, the associated multiplier operator $T_m$ on $X$ is defined by

$$T_m(f)(g) = \int_{\mathbb{R}} \int_K m(\lambda) \tilde{f}(\lambda, k) e^{-(i\lambda + \rho)H(g^{-1}k)} |c(\lambda)|^{-2} d\lambda, \quad g \in G,$$

where $\tilde{f}$ is the (Helgason) Fourier transform of $f$ and $c(\lambda)$ is the Harish-Chandra $c$-function. The boundedness of the multiplier operator on the symmetric space has been studied by Clerc and Stein [4], Stanton and Tomas [22], Anker and Lohoué [2], Taylor [25], Anker [1] and Giulini, Mauceri, Meda [12]. Let us elaborate. The pioneer work was done in [5]. It was observed there that if $T_m$ to be a continuous operator on $L^p(G/K)$, then $m$ is necessarily holomorphic and bounded inside the strip $S_p^0$, where

$$S_p^0 = \left\{ \lambda \in \mathbb{C} \mid |\Im \lambda| \leq \frac{2}{p} - 1 \right\} \quad \text{for } p \in [1, \infty].$$

Conversely the authors in [5] also gave a sufficient conditions when $G$ is complex. Similar results were obtained later when $G$ is real rank one [22] and when $G$ is a normal real form [2]. In 1990, Anker ([1]) improved and generalized previous results of [5], [22], [2], and [25] by proving the following multiplier theorem on $X = G/K$:

**Theorem 1.1.** (Anker) Let $1 < p < \infty$, $v = \lfloor \frac{1}{p} - \frac{1}{2} \rfloor$ and $N = \lfloor v \dim X \rfloor + 1$. Let $m : \mathbb{R} \to \mathbb{C}$ extends to an even holomorphic function on $S_p^0$, $\frac{\partial^i}{\partial \lambda^i} m(i = 0, 1, \ldots, N)$ extends continuously to $S_p$ and satisfies

$$\sup_{\lambda \in S_p} (1 + |\lambda|)^{-i} \left| \frac{\partial^i}{\partial \lambda^i} m(\lambda) \right| < \infty,$$

for $i = 0, 1, \ldots, N$. Then the associated multiplier operator $T_m$ is a bounded operator on $L^p(X)$.

For technical reasons, it was necessary for the author to assume regularity assumptions on the boundary. Nonetheless, Anker mentioned that it should be possible to allow $m$ to have a singularity at the boundary points $\pm i\rho_p := i \frac{2}{p} - 1 \rho$; and $m$ will be still a $L^p$ multiplier on $X$. Later Ionescu [16] improved the theorem above by replacing the continuity of the multiplier $m$ on the boundary with that of singularity condition at $\pm i\rho_p$.

**Theorem 1.2.** (Ionescu) Let $1 < p < \infty$. Let $m : \mathbb{R} \to \mathbb{C}$ extends to an even holomorphic function on $S_p^0$ and satisfies

$$\left| \frac{\partial^j}{\partial \lambda^j} m(\lambda) \right| \leq A_j \left[ |\lambda + i\rho_p|^{-j} + |\lambda - i\rho_p|^{-j} \right], \lambda \in S_p,$$

for $j = 0, 1, \ldots, \left\lfloor \frac{\dim X + 1}{2} \right\rfloor + 1$. Then the associated multiplier operator $T_m$ is a bounded operator on $L^p(X)$.

The boundedness of multiplier operator for fixed $K$-types on $\text{SL}(2, \mathbb{R})$ has also been studied in [21]. Now one is naturally led to the following questions: what happens if we replace the multiplier $m(\lambda)$ by a symbol $\sigma(x, \lambda)$? and if the corresponding operator associated with the symbol should be bounded, then what are the conditions $\sigma(x, \lambda)$ should satisfy? In this
paper we introduce and study the boundedness of pseudo differential operators in the setting of noncompact symmetric spaces of rank one. In preparation for the statement of our main results, let us recall the analogous theory of pseudo differential operator in Euclidean setting and also let us introduce some more notations:

Let \( S^m \) be the symbols defined to be the set of all smooth functions \( a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \) satisfies,

\[
|\partial^\beta_\xi \partial^\alpha_x a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\alpha|}, \tag{1.3}
\]

for all \( x, \xi \in \mathbb{R}^n \) and for all multi indices \( \alpha \) and \( \beta \). Then we have the following theorem [23]:

**Theorem 1.3.** (23 Chap VI, Theorem 1) Let \( a \in S^0 \) and \( a(x, D) \) be the pseudo differential operator associated with the symbol \( a \). Then \( a(x, D) \) extends to a bounded operator on \( L^p(\mathbb{R}^n) \) to itself, for \( 1 < p < \infty \).

Moreover, a generalized version of this theorem is true for a family of symbols \( \{ a_\tau : \tau \in \Lambda \} \) on \( \mathbb{R}^n \):

**Theorem 1.4.** Suppose \( \{ a_\tau : \tau \in \Lambda \} \) be a family of symbols in \( S^0 \) such that

\[
|\partial^\beta_\xi \partial^\alpha_x a_\tau(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha|}, \tag{1.4}
\]

where \( C_{\alpha, \beta} \) does not depend on \( \tau \). Then for \( 1 < p < \infty \), there is a constant \( C_p > 0 \) independent of \( \tau \in \Lambda \), such that

\[
\|a_\tau(x, D)(f)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.5}
\]

for all \( \tau \in \Lambda \) and \( f \in L^p(\mathbb{R}^n) \).

Proof of this theorem (Theorem 1.4) is exactly similar to Theorem 1.3 and therefore we omit the proof.

**Remark 1.5.** We would like to mention that for pseudo differential operator one cannot weaken the regularity assumption on the symbol \( a(x, \xi) \), having singularity near \( \xi = 0 \). Particularly, if the symbol \( a \) satisfies the following simpler looking condition

\[
|\partial^\beta_\xi \partial^\alpha_x a(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{-|\alpha|}, \tag{1.6}
\]

then the corresponding operator may not be bounded. We refer the interested reader to the discussion [23 page 267].

Now suppose \( \sigma : X \times \mathbb{C} \to \mathbb{C} \) be a suitable function. We define the pseudo differential operator associated with the symbol \( \sigma \) by

\[
\Psi_\sigma f(x) = \int_{\mathbb{R}} \int_{K} \sigma(x, \lambda) \tilde{f}(\lambda, k) e^{-(i\lambda + \rho)H(x^{-1}k)} |c(\lambda)|^{-2} d\lambda dk, \tag{1.7}
\]

for any smooth compactly supported function \( f \) on \( X \). Our aim of this paper is to study boundedness of the pseudo differential operator \( \Psi_\sigma (\Psi DO) \) on rank one symmetric spaces. We divide the study into two cases: when \( p = 2 \) and when \( p \neq 2 \). First we state our result for the case when \( p \neq 2 \).

**Theorem 1.6.** Suppose \( p \in (1, 2) \cup (2, \infty) \) and \( X = G/K \). Let \( \sigma : X \times \mathbb{C} \to \mathbb{C} \) be a smooth function such that for each \( x \in X, \lambda \mapsto \sigma(x, \lambda) \) is an even holomorphic function on \( S^0_p \) and satisfies the differential inequalities:

\[
\left| \frac{\partial^\beta \partial^\alpha}{\partial s^\beta \partial \lambda^\alpha} \sigma(xa_s, \lambda) \right| \leq C_{\alpha, \beta} (1 + |\lambda|)^{-\alpha}, \tag{1.8}
\]

where \( s \) and \( \lambda \) are coordinates on \( X \) and \( \mathbb{C} \), respectively.
for all \( \alpha = 0, 1, \ldots, \left\lfloor \frac{\text{dim} X + 1}{2} \right\rfloor + 1; \beta = 0, 1, 2 \); for all \( x \in G, s \in \mathbb{R} \) and \( \lambda \in S_p \). Then the operator \( \Psi_\sigma \) extends to a bounded operator on \( L^p(\mathbb{X}) \) to itself.

**Remark 1.7.**

(1) Let us compare our result with the Euclidean case. As mentioned before in multiplier case, the holomorphic extension property of the symbol \( \sigma \) is a new and necessary condition for the pseudo differential operator \( \Psi_\sigma \) to be bounded on \( L^p(G/K) \). It is the boundary behavior of \( \sigma \) which is similar to the Euclidean case. Here we have considered Hörmander-type conditions near infinity.

(2) When \( \sigma \) is a multiplier, then one can write \( \Psi_\sigma \) as a convolution operator with a \( K \)-biinvariant kernel. This is the fundamental difference between the multiplier case and our situation. Particularly, in multiplier theory the author in [16] used a transference theorem, which is comparable to the Herz majorizing principle. For higher rank case the author used the same principle to estimate the \( L^p \) norm of multiplier operator (see [17, Lemma 4.3]). But in our case one cannot apply the methods above directly, to find the \( L^p \) norm estimate of \( \Psi_\sigma \).

(3) We also have established a connection between the \( L^p \) boundedness of (the local part of) pseudo differential operator on \( G/K \) with bounded Euclidean pseudo differential operators (see §4). From the Euclidean counterpart, we derived the derivative condition on the “\( x \)” variable of \( \sigma(x, \lambda) \).

Next we consider the case when \( p = 2 \). For \( p = 2 \), the boundedness of the multiplier operator follows easily due to Plancherel theorem in both Euclidean spaces and symmetric spaces. But in the same way we could not prove \( L^2 \) boundedness for the pseudo differential operator associated with a symbol. In this case we get only partial results, in the \( K \)-biinvariant setting. Let \( \sigma : \mathbb{X} \times \mathbb{R} \to \mathbb{C} \) be a smooth function such that \( x \mapsto \sigma(x, \lambda) \) is a \( K \)-biinvariant function and let \( \Psi_\sigma \) be the “radial” pseudo differential operator associated with \( \sigma \). That is,

\[
\Psi_\sigma(f)(x) = \int_{\mathbb{R}} \sigma(x, \lambda) \hat{f}(\lambda) \phi_\lambda(x)|c(\lambda)|^{-2}d\lambda,
\]

for \( f \in C^\infty_c(G//K) \), where \( \hat{f} \) is the spherical Fourier transform of the \( K \)-biinvariant function \( f \) and \( \phi_\lambda \) is the elementary spherical function. Suppose \( \text{dim} \mathbb{X} = d = m_1 + m_2 + 1 \), then we have the following theorem:

**Theorem 1.8.** Let \( \sigma : \mathbb{X} \times \mathbb{R} \to \mathbb{C} \) be a smooth function such that \( x \mapsto \sigma(x, \lambda) \) is \( K \)-biinvariant and

\[ \sigma(x, \lambda) = \sigma(x, -\lambda) \] for all \( \lambda \in \mathbb{R} \).

Assume that \( \sigma \) satisfies the following differential inequalities:

\[
\left| \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(a_t, \lambda) \right| \leq C_{\alpha, \beta}(1 + |\lambda|)^{-\alpha-d-1},
\]

for all \( \lambda, t \in \mathbb{R} \) and \( \alpha, \beta = 0, 1, 2 \). Then the operator \( \Psi_\sigma \) extends to a bounded operator from \( L^2(G//K) \cap L^p(G//K) \) to \( L^2(G//K) \) for any \( 1 \leq p < 2 \).

We improve this result in the following subcases:
Theorem 1.9. Let $\frac{3}{2} < p < 2$ and $X = G/K$. Let $\sigma : X \times \mathbb{R} \to \mathbb{C}$ be a smooth function such that $x \mapsto \sigma(x, \lambda)$ is $K$-biinvariant and satisfies

$$\left| \frac{\partial^\beta}{\partial s^\beta} \frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(x, \lambda) \right| \leq C_{\alpha, \beta} (1 + |\lambda|)^{-\alpha-d-1},$$

for all $\alpha = 0, 1, \ldots, \dim X + 2$; $\beta = 0, 1, 2$; for all $x \in G, s \in \mathbb{R}$ and $\lambda \in S_p$. Then the operator $\Psi_\sigma$ extends to a bounded operator from $L^p(G//K)$ to $L^2(G//K)$.

We have the following drawbacks in the last two theorems:

1. We had to take more decay condition on the symbol $\sigma$ than the required $S^0$ condition.
2. Also we did not get the required $L^2 - L^2$ bound.

All of these drawbacks are due to technical difficulties. We could not use direct semisimple theory to the proof; instead, we prove the theorems using boundedness of Euclidean pseudo differential operator. We overcome these drawbacks in complex symmetric spaces (in the $K$-biinvariant case). The following theorem is an analogue of Calderón and Vaillancourt theorem [18, Theorem 5.1], where the assumption on the symbol is weaker than usual $S^0$ condition.

Theorem 1.10. Let $X$ be an arbitrary rank complex symmetric space. Let $\sigma : X \times a^* \to \mathbb{C}$ be a $C^\infty$ function, such that $x \mapsto \sigma(x, \lambda)$ is $K$-biinvariant, $\lambda \mapsto \sigma(x, \lambda)$ is $W$-invariant and satisfies the following inequalities:

$$\left| \frac{\partial^\beta}{\partial H^\beta} \frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(H, \lambda) \right| \leq C_{\alpha, \beta}$$

for all $\lambda \in a^*$, $H \in a$ and for all multi index $\alpha, \beta$ with $|\alpha|, |\beta| \leq \lfloor \text{rank of } X/2 \rfloor + 1$. Then the pseudo differential operator $\Psi_\sigma$ extends to a bounded operator on $L^2(G//K)$.

For the boundedness of the pseudo differential operators on nilpotent Lie groups and stratified Lie groups, we refer to [3, 9, 10] and references there in.

Layout of this paper is as follows: Section 2 sets the harmonic analysis background on semisimple Lie groups and symmetric spaces. In Section 3, we will write $\Psi_\sigma$ as singular integral operators. We will prove Theorem 1.6 in Sections 4 and 5. We prove Theorem 1.8, Theorem 1.9 and Theorem 1.10 in Section 6. In Appendix, we state Coifman-Weiss transference theorem.

2. Preliminaries

In this section, we describe the necessary preliminaries regarding semisimple Lie groups and harmonic analysis on Riemannian symmetric spaces. These are standard and can be found, for example, in [11, 13, 14]. To make the article self-contained, we shall gather only those results used throughout this paper. Let $G$ be a noncompact connected semisimple real rank one Lie group with finite center, with its Lie algebra $\mathfrak{g}$. Let $\theta$ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the associated Cartan decomposition. Let $K = \exp \mathfrak{k}$ be a maximal compact subgroup of $G$ and let $X = G/K$ be an associated symmetric space with origin $0 = \{K\}$. Let $a$ be a maximal abelian subspace of $\mathfrak{p}$. Since the group $G$ is of real rank one, $\dim a = 1$. Let $\Sigma$ be the set of nonzero roots of the pair $(\mathfrak{g}, a)$, and let $W$ be the associated Weyl group. For rank one case, it is well known that either $\Sigma = \{-\alpha, \alpha\}$ or $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$, where $\alpha$ is a positive root and the Weyl group $W$ associated to $\Sigma$ is $\{\text{Id}, \text{Id}\}$, where $\text{Id}$ is the identity.
operator. Let $a^+ = \{ H \in a : \alpha(H) > 0 \}$ be a positive Weyl chamber, and let $\Sigma^+$ be the corresponding set of positive roots. In our case, $\Sigma^+ = \{ \alpha \}$ or $\{ \alpha, 2\alpha \}$. For any root $\beta \in \Sigma$, let $g_\beta$ be the root space associated to $\beta$. Let 

$$n = \sum_{\beta \in \Sigma^+} g_\beta,$$

and let 

$$\mathfrak{n} = \theta(n).$$

Also let 

$$N = \exp n, \quad \overline{N} = \exp \mathfrak{n}.$$ 

The group $G$ has an Iwasawa decomposition 

$$G = K(\exp a)N,$$

and a Cartan decomposition 

$$G = K(\exp a^+)K.$$

This decomposition is unique. For each $g \in G$, we denote $H(g) \in a$ and $g^+ \in \mathfrak{a}^+$ are the unique elements such that 

$$g = k \exp H(g)n, k \in K, n \in N,$$

and 

$$g = k_1 \exp(g^+)k_2, k_1, k_2 \in K.$$ 

We also have another Iwasawa decomposition 

$$G = \overline{N}(\exp a)K.$$ 

Let $H_0$ be the unique element in $a$ such that $\alpha(H_0) = 1$ and through this we identify $a$ with $\mathbb{R}$ as $t \leftrightarrow tH_0$ and $a_+ = \{ H \in a : \alpha(H) > 0 \}$ is identified with the set of positive real numbers. We also identify $a^*$ and its complexification $a^*_\mathbb{C}$ with $\mathbb{R}$ and $\mathbb{C}$ respectively by $t \leftrightarrow t\alpha$ and $z \leftrightarrow z\alpha, t \in \mathbb{R}, z \in \mathbb{C}$. Let $A = \exp a = \{ a_t := \exp(tH_0) : t \in \mathbb{R} \}$ and $A^+ = \{ a_t : t > 0 \}$. Let $m_1 = \dim g_\alpha$ and $m_2 = \dim g_{2\alpha}$ where $g_\alpha$ and $g_{2\alpha}$ are the root spaces corresponding to $\alpha$ and $2\alpha$. As usual then $\rho = \frac{1}{2}(m_1 + 2m_2)\alpha$ denotes the half sum of the positive roots. By abuse of notation we will denote $\rho(H_0) = \frac{1}{2}(m_1 + 2m_2)$ by $\rho$.

Let $dg, dk, dn$ and $d\mathfrak{n}$ be the Haar measures on the groups $G, K, N$ and $\mathfrak{n}$ respectively. We normalize $dk$ such that $\int_K dk = 1$. We have the following integral formulae corresponding to the Iwasawa and Cartan decomposition respectively, which holds for any integrable function $f$:

$$\int_G f(g)dg = \int_{\overline{N}} \int_{\mathbb{R}} \int_K f(\mathfrak{n}a_tk)e^{2\rho t}dkdt d\mathfrak{n},$$

and

$$\int_G f(g)dg = \int_K \int_{\mathbb{R}^+} \int_K f(k_1a_tk_2)\Delta(t) dk_1 dt dk_2.$$ 

where $\Delta(t) = (2\sinh t)^{m_1 + m_2}(2\cosh t)^{m_2}$. 

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2.1. **Iwasawa meets Cartan:** We will use the Iwasawa decomposition of $G = \mathcal{N}AK$ and its connection with the Cartan decomposition. This idea was also used by [24] and later in this direction Ionescu proved the following:

**Lemma 2.1.** ([17], Lemma 3) If $\overline{\nu} \in \overline{\mathcal{N}}$ and $r \geq 0$, then

$$[\overline{\nu}a_r]^+ = r + H(\overline{\nu}) + E(\overline{\nu}, r), \quad (2.5)$$

where

$$0 \leq E(\overline{\nu}, r) \leq 2e^{-2r}. \quad (2.6)$$

Let $P(\overline{\nu}) = e^{-\rho(H(\overline{\nu}))}$; then for any $\epsilon_0 > 0$ we have (see [17], (3.11))

$$\int_{\overline{\mathcal{N}}} P(\overline{\nu})^{1+\epsilon_0} d\overline{\nu} = C_{\epsilon_0} < \infty. \quad (2.7)$$

Also we have (see [14], Theorem 6.1, Chapter II, p. 180)

$$e^{2\alpha(H(\overline{\nu}))} = \left[1 + c_0 |X|^2 \right] + 4c_0 |Y|^2,$$

where $X$ and $Y$ are the coordinates of $\overline{\nu}$ in $\overline{\mathcal{N}}$ corresponding to the root spaces $g_{-\alpha}$ and $g_{-2\alpha}$, and $c_0 = \frac{1}{4}(m_1 + m_{2\alpha})$. Therefore,

$$H(\overline{\nu}) \geq 0, \quad \text{for all } \overline{\nu} \in \overline{\mathcal{N}}. \quad (2.8)$$

We recall the abelian group $A$ acts as a dilation on $\overline{\mathcal{N}}$ by the mapping

$$n \rightarrow an^{-1} \in \overline{\mathcal{N}}. \quad (2.9)$$

Moreover if $\delta_r$ be a dilation of $\overline{\mathcal{N}}$, defined by

$$\delta_r(\overline{n}) := \exp(rH_0)\overline{n} \exp(-rH_0),$$

then the following is true

$$\int_{\overline{\mathcal{N}}} h(\delta_r(\overline{n})) d\overline{n} = e^{2\rho r} \int_{\overline{\mathcal{N}}} h(\overline{n}) d\overline{n}, \quad (2.10)$$

for any integrable function $h$ on $\overline{\mathcal{N}}$.

2.2. **Fourier transform.** For a sufficiently nice function $f$ on $\mathbb{X}$, its (Helgason) Fourier transform $\tilde{f}$ is a function defined on $\mathbb{C} \times K$ given by

$$\tilde{f}(\lambda, k) = \int_{G} f(g) e^{i(\lambda - \rho)H(g^{-1}k)} dg, \quad \lambda \in \mathbb{C}, \quad k \in K, \quad (2.11)$$

whenever the integral exists ([14], p. 199]).

It is known that if $f \in L^1(\mathbb{X})$ then $\tilde{f}(\lambda, k)$ is a continuous function of $\lambda \in \mathbb{R}$, for almost every $k \in K$. If in addition $\tilde{f} \in L^1(\mathbb{R} \times K, |c(\lambda)|^{-2} d\lambda dk)$ then the following Fourier inversion holds,

$$f(gK) = |W|^{-1} \int_{\mathbb{R} \times K} \tilde{f}(\lambda, k) e^{-i(\lambda + \rho)H(g^{-1}k)} |c(\lambda)|^{-2} d\lambda dk, \quad (2.12)$$

for almost every $gK \in \mathbb{X}$ ([14], Chapter III, Theorem 1.8, Theorem 1.9]), where $c(\lambda)$ is the Harish Chandra's $c$-function given by

$$c(\lambda) = \frac{2^{\rho-i\lambda} \Gamma(\frac{m_1 + m_{2\lambda} + 1}{2}) \Gamma(i\lambda)}{\Gamma(\frac{\rho + i\lambda}{2}) \Gamma(\frac{m_1 + 2}{4} + \frac{i\lambda}{2})}.$$
We denote the set of all \( K \)-bi-invariant functions and are joint eigenfunctions of all \( L \) operator. They are parametrized by \( \lambda \) for any \( \lambda \in \mathbb{C} \), the elementary spherical function \( \phi \) has the following integral representation
\[
\phi(x) = \int e^{-i(\lambda+\rho)H(x)k} \, dk \quad \text{for all } x \in G.
\]

The spherical transform \( \hat{f} \) of a suitable \( K \)-bi-invariant function \( f \) is defined by the formula:
\[
\hat{f}(\lambda) = \int_{G} f(x)\phi(x^{-1}) \, dx.
\]

It is easy to check that for suitable \( K \)-bi-invariant function \( f \) on \( G \), its (Helgason) Fourier transform \( \hat{f} \) reduces to the spherical transform \( \hat{f} \).

We now list down some well known properties of the elementary spherical functions which are important for us (Prop 3.1.4 and Chapter 4, §4.6, Lemma 1.18, p. 221).

1. \( \phi(g) \) is \( K \)-bi-invariant in \( g \in G \), \( \phi = \phi_{-\lambda}, \phi_{\lambda}(g) = \phi_{\lambda}(g^{-1}) \).
2. \( \phi(g) \) is \( C^\infty \) in \( g \in G \) and holomorphic in \( \lambda \in \mathbb{C} \).
3. The following inequality holds:
\[
e^{-\rho t} \leq \phi_0(a_t) \leq (1 + |t|) \, e^{-\rho t}, \quad t \geq 0.
\]
4. \( |\phi(x)| \leq 1 \) for all \( x \in G \) if and only if \( \lambda \in S_1 = \{ \lambda \in \mathbb{C} \mid |\Im \lambda| \leq \rho \} \).
5. For all \( \lambda \in \mathbb{R} \) we have
\[
|\phi(\lambda)(g)| \leq \phi_0(g) \leq 1.
\]
6. The function \( \phi_{\lambda} \) satisfies the following identity (Chapter 4, Lemma 4.4, p. 418):
\[
\phi_{\lambda}(y^{-1}x) = \int_{K} e^{-(i\lambda+\rho)H(x^{-1})k} e^{(i\lambda-\rho)H(y^{-1}k)} \, dk,
\]
for all \( x, y \in G \) and \( \lambda \in \mathbb{R} \).

Let \( C_c^\infty(G//K) \) be the set of all \( C^\infty \) compactly supported \( K \)-bi-invariant functions on \( G \). Also let \( PW(\mathbb{C}) \) be the set of all entire functions \( h : \mathbb{C} \to \mathbb{C} \) such that \( h \) is of exponential type \( T \) for some \( T > 0 \), that is, for each \( N \in \mathbb{N} \),
\[
\sup_{\lambda \in \mathbb{C}} (1 + |\lambda|)^N |h(\lambda)| e^{-T|\Im \lambda|} < \infty
\]
and let \( PW(\mathbb{C})_e \) be the set of all even functions in \( PW(\mathbb{C}) \).
Then we have the following Paley-Wiener theorem:

**Theorem 2.2.** The function $f \mapsto \hat{f}$ is a topological isomorphism between $C_c^\infty(G//K)$ and $PW(\mathbb{C})_e$.

The Abel transform of a suitable $K$-biinvariant function is defined by
\[ \mathcal{A}(f)(t) = e^{\rho t} \int_{\mathbb{N}} f(\varpi a_t) d\varpi, \]
whenever the integral exists. It satisfies the following slice projection theorem:
\[ \mathcal{F} \mathcal{A}(f)(\lambda) = \hat{f}(\lambda), \quad (2.15) \]
for all $\lambda \in \mathbb{C}$, for which both side exists. From this, it follows that for a $K$-biinvariant function $f$,
\[ \mathcal{A}(f)(t) = \mathcal{A}(f)(-t), \quad (2.16) \]
for all $t \in \mathbb{C}$ for which Abel transform of $f$ exists. We need the following mapping property of the Abel transform [20]:

**Theorem 2.3** (S. K. Ray and R.P. Sarkar). Suppose $1 \leq p < 2$ and $\gamma_p = \frac{2}{p} - 1$. Then
\[ \left( \int_{\mathbb{R}} |\mathcal{A}(f)(a_t)|^r e^\rho |a_t| dt \right)^{\frac{1}{r}} \leq C \|f\|_{L^p(X)} \text{ for } 1 \leq r < \frac{1}{\gamma_p}, 0 < \beta < \gamma_p r. \quad (2.17) \]

Putting $r = p$ in (2.17) for $1 < p < 2$ we get
\[ \left( \int_{\mathbb{R}} |\mathcal{A}(f)(a_t)|^p dt \right)^{\frac{1}{p}} \leq C \|f\|_{L^p(X)}. \]

2.3. Spherical function. In our study of $L^p$ boundedness of $\Psi_\sigma$, representation of spherical function $\phi_\lambda$ is crucial. Although, such formulations of spherical function are well known and ubiquitous in the literature, for completeness we briefly recall the explicit expression of $\phi_\lambda(a_t)$ for small and large values of $t$.

Let $J_{\mu}(z)$ be the Bessel functions of first kind and
\[ J_\mu(z) = \frac{J_{\mu}(z)}{z^\mu} \Gamma(\mu + \frac{1}{2}) \Gamma(\frac{1}{2}) 2^{\mu-1}, \]
and
\[ c_0 = \pi^{\frac{1}{2}} 2^{\frac{m+1}{2} - 2} \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})}. \]

We have the following expansion of $\phi_\lambda$ near origin:

**Theorem 2.4.** ([22] Theorem 2.1) There exist $R_0 > 1$ and $R_1 > 1$ such that for any $t$ with $0 \leq t \leq R_0$ and any $M \geq 0$,
\[ \phi_\lambda(a_t) = c_0 \left[ \frac{t^{d-1}}{\Delta(t)} \right]^{\frac{1}{2}} \sum_{m=0}^{\infty} t^{2m} a_m(t) J_{(d-2)/2+m}(\lambda t) + E_{M+1}(\lambda t), \]

where $\Delta(t) = t^{d-1}$. 
where

\[ a_0(t) = 1 \]
\[ |a_m(t)| \leq CR_1^m \]
\[ |E_{M+1}(\lambda t)| \leq C_M t^{2(M+1)} \quad \text{if } |\lambda t| \leq 1 \]
\[ \leq C_M t^{2(M+1)} (\lambda t)^{-((d-1)/2+M+1)} \quad \text{if } |\lambda t| > 1. \]

Next we have the following expansion of \( \phi_\lambda \) away from origin.

**Proposition 2.5.** ([17, Prop. A1]) If \( t \geq 1/10 \) and \( N \in \mathbb{N} \), then \( \phi_\lambda(a_t) \) can be written in the following form

\[ \phi_\lambda(a_t) = e^{-\rho t} \left( e^{i\lambda t c(\lambda)} (1 + a(\lambda, t)) + e^{-i\lambda t c(-\lambda)} (1 + a(-\lambda, t)) \right), \]  

(2.18)

where the function \( a(\lambda, t) \) satisfies the following inequalities,

\[ \left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \frac{\partial}{\partial t} a(\lambda, t) \right| \leq C(1 + |\Re \lambda|)^{-\alpha}, \]  

(2.19)

for all integers \( \alpha \in [0, N] \), \( l \in \{0, 1\} \), and for all \( \lambda \) in the region \( 0 \leq \Im \lambda \leq \rho + 1/10 \).

The following two lemmas give the estimates of \( c(-\lambda)^{-1} \) and \( |c(-\lambda)|^{-2} \), along with their derivatives.

**Lemma 2.6.** ([16 App. A]) The function \( \lambda \to c(-\lambda)^{-1} \) is holomorphic inside the region \( \Im \lambda \geq 0 \) and satisfies the estimates

\[ \left| \frac{d^\alpha}{d \lambda^\alpha} c(-\lambda)^{-1} \right| \leq C_\alpha (1 + |\lambda|)^{d-1-\alpha}, \]  

(2.20)

for all integers \( \alpha = 0, 1, \ldots N \) and for all \( \lambda \) with \( 0 \leq \Im \lambda \leq \rho \).

**Lemma 2.7.** ([22 Lemma 4.2]) We have

\[ \left| \frac{d^\alpha}{d \lambda^\alpha} |c(-\lambda)|^{-2} \right| \leq C_\alpha (1 + |\lambda|)^{d-1-\alpha}, \]  

(2.21)

for all \( \lambda \in \mathbb{R} \) and \( \alpha \in [0, N] \).

Later we will require the following lemma.

**Lemma 2.8.** Let \( \sigma \) satisfies (1.8) and \( N \in \mathbb{N} \), then the following is true for all \( j \in [0, N] \),

\[ \lim_{\epsilon \to 0} \left| \frac{\partial^j}{\partial \lambda^j} \left( \frac{\sigma(x, \lambda) e^{-\epsilon \lambda^2}}{c(-\lambda)} \right) \right| \leq \frac{C_j}{(1 + |\lambda|)^{j-((d-1)/2)}}, \]  

for all \( \lambda \) with \( 0 \leq \Im \lambda \leq \rho \),

where \( C_j \)'s are independent of \( x \in \mathbb{R} \).

**Proof.** Let \( 0 < \epsilon \leq 1 \). By a simple calculation we have,

\[ |\lambda^n e^{-\epsilon \lambda^2}| \leq \left( \left( \frac{n}{2} \right)^{\frac{n}{2}} e^{-\epsilon \frac{n}{2}} \right) e^{-\frac{n}{2}} \text{ for all } \lambda \in \mathbb{R}, n \in \mathbb{N}. \]  

(2.22)
Then for $j \in [0, N]$ and $x \in \mathbb{X}$,

$$\left| \frac{\partial^j}{\partial \lambda^j} \left( \frac{\sigma(x, \lambda) e^{-c \lambda^2}}{c(-\lambda)} \right) \right| \leq C \sum_{t=0}^{j} \left| \frac{\partial^t}{\partial \lambda^t} \left( \sigma(x, \lambda) e^{-c \lambda^2} \right) \right| \left| \frac{\partial^{j-t}}{\partial \lambda^{j-t} c(-\lambda)^{-1}} \right| .$$

Now we apply (1.8), (2.20) and (2.22) repeatedly to get,

$$\sum_{t=0}^{j} \left| \frac{\partial^t}{\partial \lambda^t} \left( \sigma(x, \lambda) e^{-c \lambda^2} \right) \right| \left| \frac{\partial^{j-t}}{\partial \lambda^{j-t} c(-\lambda)^{-1}} \right| \leq \frac{1}{(1 + \lambda)^{j - \frac{3}{2}}} \left( 1 + \epsilon \frac{1}{2} (1 + |\lambda|) + \cdots + \epsilon \frac{1}{2} (1 + |\lambda|)^{j} \right).$$

By sending $\epsilon \to 0$ both side, we get our lemma. 

\[ \square \]

3. Singular integral realization of $\Psi DO$

We now study the integral representations of the pseudo differnetail operator $\Psi_\sigma$. By substituting the definition of Helgason Fourier transform (2.11) in (1.7), it transforms to

$$\Psi_\sigma f(x) = \int_{\mathbb{R}} \int_{K} \sigma(x, \lambda) \hat{f}(\lambda, k) e^{-(i\lambda + \rho) H(x^{-1}k)} |c(\lambda)|^{-2} d\lambda dk$$

$$= \int_{G} \int_{\mathbb{R}} \int_{K} f(y) \sigma(x, \lambda) e^{-(i\lambda + \rho) H(x^{-1}k)} e^{(i\lambda - \rho) H(y^{-1}k)} |c(\lambda)|^{-2} dk d\lambda dy.$$

Then using a formula of the spherical function (2.14), we have

$$\Psi_\sigma f(x) = \int_{G} f(y) K(x, y) dy, \quad (3.1)$$

where

$$K(x, y) = \int_{\mathbb{R}} \sigma(x, \lambda) \phi_{\lambda}(y^{-1}x) |c(\lambda)|^{-2} d\lambda. \quad (3.2)$$

**Decomposition of the $\Psi DO$:** Due to distinct behavior of spherical function near the origin and away from it, we now split $\Psi_\sigma$ into two parts. The idea of such decomposition employed here was first introduced by Clerc and Stein [5], where they proved multiplier theorem for complex $G$. Suppose $\eta : \mathbb{R} \to [0, 1]$ be a smooth even function supported on $[1, \infty)$ such that $\eta(t) = 1$ if $|t| \geq 2$. Let $\eta^*(t) = 1 - \eta(t)$. Using the Cartan decomposition, we extend the functions $\eta$ and $\eta^*$ to $K$-bi-invariant functions on $G$, by

$$\eta(x) = \eta(x^+) \quad \text{for all } x \in G. \quad (3.3)$$

Now we decompose the operator $\Psi_\sigma$ as a sum of local and global parts as:

$$\Psi_\sigma f(x) = \Psi_\sigma^{loc} f(x) + \Psi_\sigma^{glo} f(x), \quad (3.4)$$

where

$$\Psi_\sigma^{loc} f(x) = \int_{G} f(y) \eta^*(y^{-1}x) K(x, y) dy, \quad (3.5)$$

and

$$\Psi_\sigma^{glo} f(x) = \int_{G} f(y) \eta(y^{-1}x) K(x, y) dy. \quad (3.6)$$

In the upcoming two sections, we separately deal with the operators $\Psi_\sigma^{loc}$ and $\Psi_\sigma^{glo}$.
4. Analysis on the local part of $\Psi_\sigma$

In the multiplier case, one can write the local part of the multiplier operator as convolution with a compactly supported $K$-biinvariant function $K_m$ (say). Next they relate the convolution with $K_m$ to Euclidean multiplier and then the boundedness of multiplier on $G/K$ follows from Marcinkiewicz multiplier theorem. Here, we point out a fundamental difference between the multiplier case and our situation. In the case of multiplier operator, writing it as a convolution operator plays a crucial role. However in pseudo differential operator, one cannot use the theory of multiplier, due to an extra variable $x$ in the symbol $\sigma(x, \lambda)$. To prove the boundedness of $\Psi^\text{loc}_\sigma$, we used a generalized transference principle by Coifman-Weiss. This principle helped us to establish a connection between the $L^p$ boundedness of $\Psi^\text{loc}_\sigma$ on $G/K$ with Euclidean pseudo differential operator. We have the following result for the local part of $\Psi_\sigma$.

**Theorem 4.1.** Suppose $p \in (1, \infty)$ and $\sigma$ satisfies the properties of Theorem 1.6. Then there is a constant $C > 0$, such that

$$
\|\Psi^\text{loc}_\sigma f\|_{L^p(\mathbb{X})} \leq C \|f\|_{L^p(\mathbb{X})}, \quad \text{for all } f \in L^p(\mathbb{X}).
$$

(4.1)

Proof of the theorem above will be a consequence of the following lemma. In fact next lemma allow us to apply Coifman-Weiss transference principle (see (7.3)) in our setting. We will postpone the proof of following lemma in the next subsection. Assuming this, we complete the proof of Theorem 4.1.

**Lemma 4.2.** For $z \in G$, let $\mathcal{R}_sz := za_s$, $s \in \mathbb{R}$. Then we have the following inequality

$$
\left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} K(\mathcal{R}_sz, \mathcal{R}_{s-t}\mathcal{R}_sz) \eta^\circ(a_t)|\Delta(t)|g(s-t)dt \right|^p ds \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}} |g(t)|^p dt \right)^{\frac{1}{p}},
$$

(4.2)

for some $C > 0$ is independent of $z \in G$, and $g \in L^p(\mathbb{R})$.

**Proof of Theorem 4.1 assuming Lemma 4.2.** We will proceed with integral formula corresponding to Cartan decomposition followed by Minkowski’s integral inequality to obtain

$$
\left( \int_{\mathbb{X}} \left| \int_{\mathbb{X}} |\Psi^\text{loc}_\sigma f(x)|^p dx \right| \right)^{\frac{1}{p}} = \left( \int_{G} \left| \int_{G} |f(y)| \eta^\circ(y^{-1}x)K(x, y)dy \right|^p dx \right)^{\frac{1}{p}}
$$

$$
= \left( \int_{G} \left| \int_{G} |f(xy)| \eta^\circ(y^{-1})K(x, xy)dy \right|^p dx \right)^{\frac{1}{p}}
$$

$$
= \left( \int_{G} \left| \int_{K} \int_{\mathbb{R}} f(xka_t)\eta^\circ((ka_t)^{-1})K(x, xka_t)|\Delta(t)|dkdt \right|^p dx \right)^{\frac{1}{p}}
$$

$$
\leq \int_{K} \left( \int_{G} \left| \int_{\mathbb{R}} f(xka_t)\eta^\circ(a_t)K(x, xka_t)|\Delta(t)|dt \right|^p dx \right)^{\frac{1}{p}} dk
$$

$$
= \left( \int_{G} \left| \int_{\mathbb{R}} f(za_t)\eta^\circ(a_t)K(z, za_t)|\Delta(t)|dt \right|^p dz \right)^{\frac{1}{p}}
$$

$$
= \left( \int_{G} \left| \int_{\mathbb{R}} \mathcal{R}_tf(z)\eta^\circ(a_t)K(z, za_t)|\Delta(t)|dt \right|^p dz \right)^{\frac{1}{p}},
$$

(4.3)
where \( R \) is the representation of \( \mathbb{R} \), acting on the functions of \( G \), by

\[
R_{t}f(x) = f(xa_{t}).
\]

Now we apply the generalized Coifman-Weiss transference principle (see Appendix (7.2)) and Lemma 4.2 to conclude the theorem.

4.1. **Proof of Lemma 4.2.** Before proceeding further, we remark that we assume that \( \sigma(\cdot, \lambda) \) has rapid decay as \( \lambda \rightarrow \infty \). To be precise, we assume \( \sigma(\cdot, \lambda) \) is multiplied with a factor of the form \( e^{-\epsilon\lambda^2} \), where \( 0 < \epsilon \leq 1 \). Our estimates are uniform in \( \epsilon \). Once we prove suitable uniform estimates, standard limiting arguments allow us to pass to the general case.

Let \( K_{1} : G \times \mathbb{R}^{+} \rightarrow \mathbb{C} \) be a function defined by

\[
K_{1}(z, a_{t}) = \eta^{0}(a_{t})\Delta(t)K(z, za_{t})
\]

\[
= \eta^{0}(a_{t})\Delta(t) \int_{\mathbb{R}} \sigma(z, \lambda)\phi_{\lambda}(a_{t})|c(\lambda)|^{-2}d\lambda.
\]

We will prove Lemma 4.2 in the following steps:

**Step I : Separation of the Kernel \( K_{1} \).**

Suppose \( \Phi \) is a smooth even function on \( \mathbb{R} \) with \( 0 \leq \Phi \leq 1; \Phi = 1 \) when \( |\lambda| > 2; \Phi(\lambda) = 0 \) when \( |\lambda| < 1 \). Then we claim

\[
K_{1}(za_{s}, a_{t}) = K_{0}(za_{s}, t) + \zeta(za_{s}, t),
\]

satisfying the followings:

(i) There is some constant \( C > 0 \), such that

\[
|\zeta(z, t)| \leq C \zeta_{0}(t) \in L^{1}(\mathbb{R}), \text{ for all } z \in G,
\]

and

(ii) \( K_{0}(za_{s}, t) = C\eta^{0}(a_{t}) \left( \frac{t^{d-1}}{\Delta(t)} \right)^{1/2} \int_{0}^{\infty} \Phi(\lambda)\sigma(za_{s}, \lambda)J_{(d-2)/2}(\lambda t)|c(\lambda)|^{-2}d\lambda \).

The proof of the above claim will be based on the local expansion of \( \phi_{\lambda} \) from Theorem 2.4 and idea of [22]. First we extend the result [22, Proposition 4.1] to a more general setting.

**Proposition 4.3.** Let \( p : G \times \mathbb{R}^{+} \rightarrow \mathbb{C} \) be a smooth function satisfying the estimates

\[
\partial_{\lambda}^{\alpha}p(z, 0) = 0 \quad \text{when } 0 \leq \alpha \leq \left[ \frac{d+1}{2} \right] + 1, \quad (4.5)
\]

\[
|\partial_{\lambda}^{\alpha}p(z, \lambda)| \leq C_{\alpha}(1 + |\lambda|)^{-\alpha} \quad 0 \leq \alpha \leq \left[ \frac{d+1}{2} \right] + 1,
\]

for all \( z \in G \) and the constants \( C_{\alpha} \)'s are independent of \( z \in G \). Then there exists functions \( e_{z}, e_{0} \in L^{1}(G//K) \) such that

\[
\hat{p}(z, a_{t}) \eta^{0}(a_{t}) := \eta^{0}(a_{t}) \int_{0}^{\infty} p(z, \lambda)\phi_{\lambda}(a_{t})|c(\lambda)|^{-2}d\lambda
\]

\[
= C\eta^{0}(a_{t}) \left( \frac{t^{d-1}}{\Delta(t)} \right)^{1/2} \int_{0}^{\infty} p(z, \lambda)J_{(d-2)/2}(\lambda t)|c(\lambda)|^{-2}d\lambda + e_{z}(t), \quad (4.6)
\]
and
\[ |e_z(t)| \leq C e_0(t) \text{ for all } z \in G. \] (4.7)

**Proof of Proposition 4.3.** Applying Theorem 2.4 with \( M \) chosen to be \( N > (d + 1)/2 \) and defining
\[ e_z(t) = C \eta^\circ (a_t) \left( \frac{t^{d-1}}{\Delta(t)} \right)^{\frac{1}{2}} \sum_{m=0}^{N} t^{2m} a_m(t) \int_{0}^{\infty} p(z, \lambda) J_{(d-2)/2}(\lambda t) |c(\lambda)|^{-2} d\lambda \]
\[ + \eta^\circ (a_t) \int_{0}^{\infty} p(z, \lambda) E_{N+1}(\lambda t) |c(\lambda)|^{-2} d\lambda, \] (4.8)
we get (4.6). Now one can complete the proof following the technique of [5, Proposition 4.1]. Indeed one needs to utilize the fact that the derivatives of \( p(z, \lambda) \) have bounds independent of \( z \) and this will eventually lead us to (4.7).

Now coming back to the proof of our claim (4.3), we can write
\[ K_1(za_s, a_t) = \eta^\circ (a_t) \Delta(t) \int_{\mathbb{R}} \Phi(\lambda) \sigma(za_s, \lambda) \phi(\lambda, a_t) |c(\lambda)|^{-2} d\lambda \]
\[ + \eta^\circ (a_t) \Delta(t) \int_{\mathbb{R}} (1 - \Phi(\lambda)) \sigma(za_s, \lambda) \phi(\lambda, a_t) |c(\lambda)|^{-2} d\lambda. \]
We observe that \( \Phi(\lambda) \sigma(za_s, \lambda) \) satisfy the hypotheses of Proposition 4.3 so we choose
\[ \zeta(za_s, t) = \Delta(t) e_{za_s}(t) + \eta^\circ (a_t) \int_{\mathbb{R}} (1 - \Phi(\lambda)) \sigma(za_s, \lambda) \phi(\lambda, a_t) |c(\lambda)|^{-2} d\lambda. \]
Since \( |e_{za_s}(t)| \leq C e_0(t) \) and \( e_0 \in L^1(G//K) \), so \( \Delta(t) e_{za_s}(t) \in L^1(\mathbb{R}) \). The second term of \( \zeta(za_s, t) \) is bounded by \( \eta^\circ \cdot \int_{0}^{\infty} |\sigma(za_s, \lambda)| |c(\lambda)|^{-2} d\lambda \leq C \eta^\circ, \) (\( C \) is independent of \( z \)). Therefore the function \( \zeta \) satisfy the inequality (4.4), this also concludes our first step (4.3).

**Step II:** Connection with Euclidean pseudo differential operator.
Following **Step I**, it is clear that to establish Lemma 4.2 we need to prove \( K_0 \) satisfies (4.2). We will do that by showing \( \{K_0(za_s, t) : z \in G\} \) are kernels of Euclidean pseudo differential operators corresponding to a family of symbols \( \{a_z(s, y) : z \in G\} \), belonging to the symbol class \( \mathcal{S}^0 \). In fact, we will prove the following
\[ \left| \partial_x^\alpha \partial_y^\beta \int_{-\infty}^{\infty} e^{-2\pi i xy} K_0(za_s, x) dx \right| \leq C_{\alpha, \beta}(1 + |y|)^{-\alpha} \quad \alpha, \beta = 0, 1, 2, \] (4.9)
for all \( z \in G \). Assuming the inequality above, we will complete the proof of Lemma 4.2 below.

**Proof of Lemma 4.2 assuming (4.9):** For each \( z \in G \), we define
\[ a_z(s, y) := \int_{-\infty}^{\infty} e^{-2\pi i xy} K_0(za_s, x) dx, \quad s, y \in \mathbb{R}. \]
We observe that the constants \( C_{\alpha, \beta} \) in (4.9) are independent of \( z \in G \). Hence from (4.9) it follows that the family of symbols \( \{a_z(s, y) : z \in G\} \) satisfies the hypothesis of Theorem 4.1 and hence (4.2) holds, which also concludes the Lemma 4.2. \[ \square \]
Now it only remains to prove that $\mathcal{K}_0$ satisfies (4.9). We recall the definition of $\mathcal{K}_0$ from (4.3).

$$\mathcal{K}_0(z_{a_s}, t) = C\eta^0(a_t) \left( \frac{t^{d-1}}{\Delta(t)} \right)^{1/2} \int_0^\infty \Phi(\lambda)\sigma(z_{a_s}, \lambda)\mathcal{J}_{(d-2)/2}(\lambda t)|c(\lambda)|^{-2}d\lambda. \quad (4.10)$$

Next we will consider separately the cases $d$ odd and $d$ even.

When $d$ is even, we write

$$\mathcal{J}_{(d-2)/2} = C(z^{-1}\partial_z)^{(d-2)/2}\mathcal{J}_0(z),$$

and

$$\mathcal{J}_0(\lambda t) = \frac{2}{\pi} \int_\lambda^\infty (\mu^2 - \lambda^2)^{-1/2} \sin \mu t d\mu.$$

Let $q(z, \lambda) := (\partial_\lambda \cdot (1/\lambda))^{(d-2)/2}(\Phi(\lambda)\sigma(z, \lambda)|c(\lambda)|^{-2})$. Then using integration by parts we have,

$$\mathcal{K}_0(z_{a_s}, t) = C\eta^0(a_t) \left[ \frac{\Delta(t)}{t^{d-1}} \right]^{1/2} t \int_R q(z_{a_s}, \lambda)\mathcal{J}_0(\lambda t)d\lambda$$

$$= C\eta^0(a_t) \left[ \frac{\Delta(t)}{t^{d-1}} \right]^{1/2} t \int_R \sin \mu t \int_0^\mu q(z_{a_s}, \lambda)(\mu^2 - \lambda^2)^{-1/2}d\lambda d\mu$$

$$= C\eta^0(a_t) \left[ \frac{\Delta(t)}{t^{d-1}} \right]^{1/2} \int_R \cos \mu t \frac{d}{d\mu} \int_0^\mu q(z_{a_s}, \lambda)(\mu^2 - \lambda^2)^{-1/2}d\lambda d\mu.$$

Let us define

$$g(z, \mu) = \int_0^\mu (\mu^2 - \lambda^2)^{-1/2}q(z, \lambda) d\lambda, \quad z \in G, \mu \geq 0$$

$$= \frac{1}{2} \int_{-1}^1 (1 - \lambda^2)^{-1/2}q(z, \lambda\mu) d\lambda \quad z \in G, \mu > 0. \quad (4.11)$$

and

$$h(z, \mu) = \left( \frac{\partial}{\partial \mu} g \right)(z, |\mu|), \quad \mu \in \mathbb{R}.$$

Then we obtain

$$\mathcal{K}_0(z_{a_s}, t) = -C\eta^0(a_t) \left[ \frac{\Delta(t)}{t^{d-1}} \right]^{1/2} \int_0^\infty \frac{d}{d\mu} g(z_{a_s}, \mu) \cos \mu t d\mu$$

$$= -C\eta^0(a_t) \left[ \frac{\Delta(t)}{t^{d-1}} \right]^{1/2} \int_{-\infty}^\infty h(z_{a_s}, \mu)e^{i\mu t} d\mu.$$

Let $\nu(t) = -C\eta^0(a_t) \left[ \frac{\Delta(t)}{t^{d-1}} \right]^{1/2}, \quad t \in \mathbb{R}$. Then

$$\mathcal{F}(\mathcal{K}_0(z_{a_s}, \cdot))(\mu) = (\mathcal{F}(\nu) \ast \mathbb{R} h(z_{a_s}, \cdot))(\mu). \quad (4.12)$$

First we claim that

$$\sup_{\mu \in \mathbb{R}} \left( |\partial_\mu^\beta h(z_{a_s}, \mu)| + |(1 + \mu)\partial_\mu^\beta \partial_\mu h(z_{a_s}, \mu)| \right) < C_{\beta} \quad \text{for all} \ \beta, \quad (4.13)$$
where the constants $C_\beta$ are independent of $z \in G$. From (4.11) we have

$$h(za_s, \mu) = \frac{1}{2} \int_{-1}^{1} (1 - \lambda^2)^{-1/2} \frac{\partial}{\partial(\lambda \mu)} q(z, \lambda \mu) \, d\lambda \quad \text{for } \mu > 0.$$ 

Then using (1.8) and (2.7) we get

$$\left| \frac{\partial^\beta}{\partial s^\beta} \frac{\partial^l}{\partial \lambda^l} h(za_s, \lambda) \right| \leq C_\beta,l (1 + |\lambda|)^{1-l} \quad \text{for all } \beta, l. \quad (4.14)$$

Hence

$$\left| \frac{\partial^\beta}{\partial s^\beta} h(za_s, \mu) \right| \leq C_\beta \int_{0}^{1} (1 - \lambda^2)^{-1/2} \lambda \, d\lambda \leq C_\beta,$$

Also,

$$\left| (1 + \mu) \frac{\partial^\beta}{\partial s^\beta} \frac{\partial}{\partial \mu} h(za_s, \mu) \right| \leq C_\beta \int_{0}^{1} (1 - \lambda^2)^{-1/2} \frac{\lambda^2}{(1 + |\lambda\mu|)} \, d\lambda + C_\beta |\mu| \int_{0}^{1} (1 - \lambda^2)^{-1/2} \frac{\lambda^2}{(1 + |\lambda\mu|)} \, d\lambda \leq C_\beta \int_{0}^{1} (1 - \lambda^2)^{-1/2} \lambda^2 d\lambda + C_\beta |\mu| \int_{0}^{1} (1 - \lambda^2)^{-1/2} \frac{\lambda}{|\mu|} \, d\lambda \leq C_\beta.$$ 

Similarly we can prove that

$$\left| (1 + \mu)^\alpha \frac{\partial^\beta}{\partial s^\beta} \frac{\partial^\gamma}{\partial \mu^\gamma} h(za_s, \mu) \right| \leq C_{\alpha,\beta} \quad \text{for all } \alpha, \beta.$$

Therefore, from (4.12), we conclude that $K_0$ satisfies (4.9) using the fact that $F(\nu)$ is a Schwartz function.

When $d$ is odd, we write

$$J_{(d-2)/2}(z) = C(z^{-1} \partial_z)^{(d-1)/2}(\cos z).$$

After integration by parts, we see that it suffices to prove $(\partial_{\lambda'}(1/\lambda))^{(d-1)/2}(\Phi(\lambda)\sigma(za_s, t)|c(\lambda)|^{-2})$ satisfies (4.9), which follows from the estimates (2.21) on $|c(\lambda)|^{-2}$ and the hypothesis on $\sigma$.

5. Global analysis of the pseudo differential operator $\Psi_\sigma$

In the previous section, we saw how $K_0$ behaved like a kernel of Euclidean pseudo differential operators, and using the transference method we got a bound for $L^p$ operator norm of the local part of $\Psi\sigma$. But the analysis on the global part changes drastically as the local and global behavior of $\phi_\lambda$ entirely different. In multiplier theory the author in [16] used a transference theorem for convolution operator which is comparable to the Herz majorizing principle. For higher rank case the author used the same principle to estimate the $L^p$ norm of multiplier operator (see [17, Lemma 4.3]). Clearly such tools are not applicable in our setting.

In this section we will prove the $L^p$ boundedness of $\Psi^{\text{ glo}}_\sigma$. Here we will use the expansion of spherical function away from the origin (Proposition 2.5), and will see the global analysis of $\Psi_\sigma$ has no Euclidean analogue. The broad strokes of our approach in this section follow that of [16]. Also Theorem 1.6 will be derived as a consequence of following theorem and
Theorem 4.1. Before stating the main result in this section, let us recall the definition of \(\Psi_\sigma^{\text{glo}}\), the global part of the operator \(\Psi_\sigma\) from (3.6),

\[
\Psi_\sigma^{\text{glo}} f(x) = \int_{\mathcal{X}} f(y) \eta(y^{-1}x) K(x, y) \, dy,
\]

where

\[
K(x, y) = \int_{\mathbb{R}} \sigma(x, \lambda) \phi_\lambda(y^{-1}x)|c(\lambda)|^{-2} \, d\lambda.
\]

Then we have the following:

**Theorem 5.1.** Suppose \(p \in (1, 2) \cup (2, \infty)\). Then there is a constant \(C > 0\) such that

\[
\|\Psi_\sigma^{\text{glo}} f\|_{L^p(\mathcal{X})} \leq C\|f\|_{L^p(\mathcal{X})},
\]

for all \(f \in L^p(\mathcal{X})\).

Suppose \(\chi^+\) and \(\chi^-\) be the characteristic functions of the intervals \([0, \infty)\) and \((-\infty, 0)\) respectively. Next we define the following operators on the \(\overline{N}A\) group,

\[
\mathcal{E}^- (\mathfrak{m} a_t) = \int_{\mathfrak{m} \mathbb{R}} \int_{\mathbb{R}} f(\mathfrak{m} a_s) \eta(\delta_{-s}(\mathfrak{m}^{-1} \mathfrak{m}) a_{t-s}) \left( \int_{\mathbb{R}} \sigma(\mathfrak{m} a_t, \lambda) \phi_\lambda(\delta_{-s}(\mathfrak{m}^{-1} \mathfrak{m}) a_{t-s}) |c(\lambda)|^{-2} \, d\lambda \right) \cdot \chi^- (t-s) e^{2ps} |c(\lambda)|^{-2} \, d\lambda \, ds \, d\mathfrak{m},
\]

and

\[
\mathcal{E}^+ (\mathfrak{m} a_t) = \int_{\mathfrak{m} \mathbb{R}} \int_{\mathbb{R}} f(\mathfrak{m} a_s) \eta(\delta_{-s}(\mathfrak{m}^{-1} \mathfrak{m}) a_{t-s}) \left( \int_{\mathbb{R}} \sigma(\mathfrak{m} a_t, \lambda) \phi_\lambda(\delta_{-s}(\mathfrak{m}^{-1} \mathfrak{m}) a_{t-s}) |c(\lambda)|^{-2} \, d\lambda \right) \cdot \chi^+ (t-s) e^{2ps} |c(\lambda)|^{-2} \, d\lambda \, ds \, d\mathfrak{m}.
\]

Equipped with the expression of above, we aim to prove the \(L^p\) norm estimates of \(\mathcal{E}^\pm\) in \(\overline{N}A\) group. The reason for this is that,

\[
\Psi_\sigma^{\text{glo}} f(\cdot) = \mathcal{E}^- f(\cdot) + \mathcal{E}^+ f(\cdot), \quad \text{in the \(\overline{N}A\) group},
\]

and so the \(L^p(\overline{N}A)\) operator norm of \(\mathcal{E}^- + \mathcal{E}^+\) dominates the operator norm of \(\Psi_\sigma^{\text{glo}}\), \(\|\Psi_\sigma^{\text{glo}}\|_{L^p(\overline{N}A)}\), which in turn equals to \(\|\Psi_\sigma^{\text{glo}}\|_{L^p(\mathcal{X})}\). Thus the required estimate of \(\Psi_\sigma^{\text{glo}}\) of desired form (5.2) will follow, if we show that \(\mathcal{E}^\pm\) maps \(L^p(\overline{N}A)\) boundedly to itself with operator norm bounded by constant. So in the reminder of this section we focus on the boundedness of \(\mathcal{E}^\pm\).

Before proceeding further we recall that an we may assume that \(\sigma(\cdot, \lambda)\) has exponential decay as \(\lambda \to \infty\). In fact, we assume \(\sigma(\cdot, \lambda)\) is multiplied with a factor of the form \(e^{-\epsilon \lambda^2}\), where \(0 < \epsilon \leq 1\), and then sending \(\epsilon\) towards zero, we will get the required estimates. Let us summarize this discussion by writing: suppose \(f, h\) be two compactly supported functions on the group \(\overline{N}A\), and let us denote

\[
\langle \mathcal{E}^\pm f, h \rangle := \int_{\mathfrak{m} \mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathfrak{m} \mathbb{R}} \int_{\mathbb{R}} f(\mathfrak{m} a_s) h(\mathfrak{m} a_t) \eta(\delta_{-s}(\mathfrak{m}^{-1} \mathfrak{m}) a_{t-s}) \sigma(\mathfrak{m} a_t, \lambda) \phi_\lambda(\delta_{-s}(\mathfrak{m}^{-1} \mathfrak{m}) a_{t-s}) \cdot \chi^\pm (t-s) e^{2ps} e^{-\epsilon \lambda^2} |c(\lambda)|^{-2} \, d\lambda \, dt \, ds \, d\mathfrak{m} \, d\mathfrak{m}.
\]
Then we aim to prove
\[
\lim_{\epsilon \to 0} \left| \langle \mathcal{E}_\epsilon^+, f, h \rangle \right| \leq C \|f\|_{L^p(\mathbb{N} \lambda)} \|h\|_{L^{p'}(\mathbb{N} \lambda)}, \tag{5.7}
\]
for some constant \( C > 0 \) and \( p' = p/(p-1) \) is the conjugate exponent of \( p \). Now we proceed to prove (5.7). By a change of variable (5.6) transforms to,
\[
\langle \mathcal{E}_\epsilon^+, f, h \rangle = \int_{\mathbb{N}} \int_{\mathbb{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(m a_\mu) h(m \nu a_{t}) \eta(\delta_{s}(\tau) a_{t-s}) \sigma(m \nu a_{t}, \lambda) \phi_{\lambda}(\delta_{s}(\tau) a_{t-s}) \cdot \chi^{\pm}(t-s) e^{2\rho(s-r)} e^{-\epsilon \lambda^2} |c(\lambda)|^{-2} d\lambda dt ds d\mu \, d\nu \, d\tau. \tag{5.8}
\]

5.1. \( L^p \) operator norm estimate of \( \mathcal{E}^- \).

First we prove the estimate (5.7) for \( \mathcal{E}^- \). Here we will see how the holomorphic extension property of \( \sigma(\cdot, \lambda) \) corresponds to an exponential decay. By writing \( t = s - r \) and using (2.10), we get from (5.17)
\[
\langle \mathcal{E}_\epsilon^-, f, h \rangle = \int_{\mathbb{N}} \int_{\mathbb{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(m a_\mu) h(m \nu a_{s-r}) \eta(\nu a_{s-r}) \sigma(m \nu a_{s-r}, \lambda) \cdot \phi_{\lambda}(\nu a_{s-r}) e^{2\rho(s-r)} e^{-\epsilon \lambda^2} |c(\lambda)|^{-2} d\lambda ds d\mu \, d\nu \, d\tau. \tag{5.9}
\]

Now we will use Harish-chandra expansion of the spherical function \( \phi_{\lambda} \). So we substitute the expansion (2.18) into (5.9) and notice the integrand is \( W \)-invariant. One has
\[
\langle \mathcal{E}_\epsilon^-, f, h \rangle = 2 \int_{s=0}^{\infty} \int_{\mathbb{N}} \int_{\mathbb{N}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(m a_\mu) h(m \nu a_{s-r}) \eta(\nu a_{s-r}) \sigma(m \nu a_{s-r}, \lambda) \cdot e^{2\rho(s-r)} e^{(i \lambda - \rho) |\nu a_{s-r}|} |c(\lambda)|^{-2} d\lambda ds d\mu \, d\nu \, d\tau \tag{5.10}
\]

We recall the function \( \lambda \to a(\lambda, \cdot) \) satisfy favorable symbol-type estimate (only for rank one case). Moreover, by Leibniz rule one has for all \( j \in [0, N] \)
\[
\lim_{\epsilon \to 0} \left| \frac{\partial^j}{\partial \lambda^j} \left( \frac{\sigma(m \nu a, \lambda) a(\lambda, r) e^{-\epsilon \lambda^2}}{c(-\lambda)} \right) \right| \leq \frac{C_j}{(1 + |\lambda|)^{j-(d-1)/2}} \tag{5.11}
\]
for all \( \lambda \in \mathbb{C} \) with \( 0 \leq 3 \lambda \leq \rho_p, r \geq 1 \) and \( m \nu a \in \overline{N} A \). Therefore the required estimate of \( \langle \mathcal{E}_\epsilon^-, f, h \rangle \), induced by the error term \( a(\lambda, [\nu a_{s-r}]^+) \), follows similarly as of \( \langle \mathcal{E}_0^-, f, h \rangle \). So we only need to prove the lemma below.

Lemma 5.2. We have
\[
\lim_{\epsilon \to 0} \left| \langle \mathcal{E}_0^-, f, h \rangle \right| \leq C \|f\|_{L^p(\mathbb{N} \lambda)} \|h\|_{L^{p'}(\mathbb{N} \lambda)}. \tag{5.12}
\]
Remark 5.3.  (1) We observe that, we have not used the holomorphic property of the symbol yet. In this lemma we will start using the holomorphic condition, and it will be clear how it corresponds to an exponential decay.

(2) We will need the explicit expression of $[\bar{w}a_r]^+$, which is only available for $r \geq 0$, so we will use the evenness of Abel transform, whenever is $r$ is negative.

Proof of Lemma 5.2 We begin by moving the integration with respect to $\lambda$ in (5.10) from $\mathbb{R}$ to the line $\mathbb{R} + i(\rho_p - \rho/|\bar{w}a_r|)^+$, to get

$$
\langle \mathbf{e}_{0,\epsilon} f, \mathbf{h} \rangle = C \int_{r=0}^{\infty} \int_{\mathbb{N}} \int_{\mathbb{R}} \int \tilde{f}(m a_s) \mathcal{h}(\Im \delta_s(w)) a_{s-r}) \eta(\bar{w}a_r)
\cdot \int_{\mathbb{R}} \varphi \left( \tilde{m} \delta_s(w) a_{s-r}, \lambda + i(\rho_p - \frac{\rho}{|\bar{w}a_r|^+}) \right) e^{i\lambda \left( \frac{2p}{p} + \frac{\rho}{|\bar{w}a_r|^+} \right)} \eta |\bar{w}a_r| d\lambda \epsilon^2 d\epsilon d\rho_p d\rho d|\bar{w}a_r| d|\bar{w}a_r| dr;
$$

where

$$
\varphi(x, \lambda) := \frac{\sigma(x, \lambda)}{c(-\lambda)} e^{-\epsilon^2}.
$$

In order to effectively harness the oscillation of $e^{i\lambda |\bar{w}a_r|^+}$, we use integration by parts $\ell (> d+\frac{1}{2})$ times to obtain,

$$
\int_{\mathbb{R}} \varphi \left( \tilde{m} \delta_s(w) a_{s-r}, \lambda + i(\rho_p - \frac{\rho}{|\bar{w}a_r|^+}) \right) e^{i\lambda |\bar{w}a_r|^+} d\lambda
= C \int_{\mathbb{R}} \frac{\partial^{\ell}}{\partial \lambda^{\ell}} \varphi \left( \tilde{m} \delta_s(w) a_{s-r}, \lambda + i(\rho_p - \frac{\rho}{|\bar{w}a_r|^+}) \right) \frac{e^{i\lambda |\bar{w}a_r|^+}}{(\lambda |\bar{w}a_r|^+)^{\ell}} d\lambda.
$$

Now we define

$$
\mathcal{F}(s) = \left[ \int_{\mathbb{N}} |\tilde{f}(m a_s)|^p \, dm \right]^{\frac{1}{p}}
$$

and

$$
\mathcal{H}(s) = \left[ \int_{\mathbb{N}} |\mathcal{h}(m a_s)|^p \, dm \right]^{\frac{1}{p}}.
$$

After applying the Hölder’s inequality, substituting (5.15) gives us the absolute value of $\langle \mathbf{e}_{0,\epsilon} f, \mathbf{h} \rangle$ dominated by

$$
C \int_{r=0}^{\infty} \int_{\mathbb{R}} \mathcal{F}(s-r) \mathcal{H}(s-r) \int_{\mathbb{N}} \eta(\bar{w}a_r) e^{i\lambda \left( \frac{2p}{p} + \frac{1}{|\bar{w}a_r|^+} \right)} |\bar{w}a_r|^+ d\lambda \epsilon^2 d\epsilon d\rho_p d\rho d|\bar{w}a_r| d|\bar{w}a_r| dr
$$

where in the last line we invoked the evenness of Abel transform to change $[\bar{w}a_r]^+$ to $[\bar{w}a_r]^+$. We recall $\eta$ is a function supported in $[1, \infty)$. Using this, Lemma 2.8 and substituting the
expression of \([wa_r]^+\) \textcolor{red}{(2.5)}, yields the following

\[
\lim_{\epsilon \to 0} \left| \langle \mathbf{E}_{0,\epsilon}^+ f, h \rangle \right| 
\leq C \int_{r=0}^{1} \left( \int_{\mathbb{R}} \mathcal{F}(s) e^{2\rho(s-r)} \mathcal{F}(s-r) e^{2\rho(s-r)} ds \right) \int_{\mathbb{R}} e^{-\frac{2}{p} \rho(H(\mathcal{W}))} d\mathcal{W} \int_{\mathbb{R}} \frac{1}{(1 + |\lambda|)^{1-\frac{d}{2}}} d\lambda dr
\]

\[
+ \int_{r=1}^{\infty} \left( \int_{\mathbb{R}} \mathcal{F}(s) e^{2\rho(s-r)} \mathcal{F}(s-r) e^{2\rho(s-r)} ds \right) \int_{\mathbb{R}} e^{-\frac{2}{p} \rho(H(\mathcal{W}))} d\mathcal{W} \int_{\mathbb{R}} \frac{1}{(1 + |\lambda|)^{1-\frac{d}{2}}} d\lambda \frac{e^{(\frac{2}{p} - \frac{2}{p}) \rho(r)}}{r^d} dr.
\]

Finally we can reach \textcolor{red}{(5.12)} simply by an application of Hölder’s inequality and \textcolor{red}{(2.7)}.

\[\square\]

5.2. \textit{Lp} operator norm estimate of \(\mathbf{E}^+\).

It remains to prove the operator norm estimate of \(\mathbf{E}^+\). Recalling the definition of \(\langle \mathbf{E}_{\epsilon}^+ f, h \rangle\) from \textcolor{red}{(5.17)}, and after some changes of variables we have

\[
\langle \mathbf{E}_{\epsilon}^+ f, h \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(ma_s) h(m \nabla \alpha_t) \eta(\delta(m) \alpha_t) \rho(m \nabla \alpha_t, \lambda) \phi(m \nabla \alpha_t, \lambda) \chi(m \nabla \alpha_t, \lambda) \chi(m \nabla \alpha_t, \lambda)
\]

\[
\cdot \chi^+(t-s) e^{-\gamma^2 |c(\lambda)|^2} d\lambda dt ds d\mathcal{W} d\mathcal{M}
\]

\[
= \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(ma_s) h(m \nabla \alpha_t) \eta(m \nabla \alpha_t) \rho(m \nabla \alpha_t, \lambda) \phi(m \nabla \alpha_t, \lambda)
\]

\[
\cdot e^{2\rho(s+r)} e^{-\gamma^2 |c(\lambda)|^2} d\lambda d\mathcal{W} d\mathcal{M} ds dr.
\]

We substitute Harish-chandra series expansion of \(\phi_\lambda\) \textcolor{red}{(2.18)} into the expression above to get,

\[
\langle \mathbf{E}_{\epsilon}^+ f, h \rangle = \langle \mathbf{E}_{0,\epsilon}^+ f, h \rangle + \langle \mathbf{E}_{\epsilon,\epsilon}^+ f, h \rangle,
\]

where

\[
\langle \mathbf{E}_{0,\epsilon}^+ f, h \rangle = \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(ma_s) h(m \nabla \alpha_t) \eta(m \nabla \alpha_t) \rho(m \nabla \alpha_t, \lambda) \phi(m \nabla \alpha_t, \lambda)
\]

\[
\cdot e^{2\rho(s+r)} e^{(\lambda-\rho)[m \nabla \alpha_t]^2} d\lambda ds d\mathcal{W} d\mathcal{M} dr,
\]

\[
\langle \mathbf{E}_{\epsilon,\epsilon}^+ f, h \rangle = \int_{r=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f(ma_s) h(m \nabla \alpha_t) \eta(m \nabla \alpha_t) \rho(m \nabla \alpha_t, \lambda) \phi(m \nabla \alpha_t, \lambda)
\]

\[
\cdot e^{2\rho(s+r)} e^{(\lambda-\rho-2)[m \nabla \alpha_t]^2} a(\lambda, m \nabla \alpha_t)^2 d\lambda ds d\mathcal{W} d\mathcal{M} dr.
\]

Now from \textcolor{red}{(5.11)} and the discussion proceeding it, it’s clear \textcolor{red}{(5.7)} will follow from the lemma below. The following lemma will also complete the proof of Theorem \textcolor{red}{5.1} for \(p \in (1, 2)\).

\textbf{Lemma 5.4.} We have

\[
\lim_{\epsilon \to 0} \left| \langle \mathbf{E}_{0,\epsilon}^+ f, h \rangle \right| \leq C \|f\|_{L^p(\mathcal{W})} \|h\|_{L^{p'}(\mathcal{W})}.
\]

\[\textcolor{red}{(5.21)}\]
Proof. By moving the integration with respect to \( \lambda \) in (5.19) from \( \mathbb{R} \) to the line \( \mathbb{R} + i(\rho_p - \rho/|w a_r|^+) \), it follows

\[
\langle e_{0,\epsilon}^+, f, h \rangle = \int_{r=0}^{\infty} \int_{\mathbb{N}} \int_{\mathbb{R}} f(m a_s) h(m \delta_s(w) a_{s+r}) \eta(w a_r) \nonumber \cdot \left( \int_{\mathbb{R}} \vartheta_s(m \delta_s(w) a_{s+r}, \lambda + i(\rho_p - \frac{\rho}{|w a_r|^+})) e^{(i \lambda - \frac{2\rho}{p} - \frac{|w a_r|^+}{|w a_r|^+}) X} d\lambda \right) e^{2\rho(s+r)} ds m \bar{w} dr.
\]

Now by using integration by parts and Lemma 2.8, we get

\[
\lim_{\epsilon \to 0} \left| \langle e_{0,\epsilon}^+, f, h \rangle \right| \leq C \int_{\mathbb{R}} \left( \frac{d+1}{\mathbb{N}} \right) d\lambda \left( 1 + |\lambda| \right)^{d-\frac{d+1}{2}}.
\]

By taking \( l > \frac{d+1}{2} \), and substituting the expression \( |w a_r|^+ \), it follows

\[
\lim_{\epsilon \to 0} \left| \langle e_{0,\epsilon}^+, f, h \rangle \right| \leq C \|f\|_{L^N(\mathbb{N})} \|h\|_{L^N(\mathbb{N})}.
\]

In the last inequality, we used Hölder’s inequality along with (2.7). This completes the proof of the lemma and consequently the proof of Theorem 1.6.

\[\square\]

Remark 5.5. (1) The proof of Lemma 5.2 and 5.4 for \( p > 2 \) case is similar in spirit. Indeed one needs to utilize the fact that \( |(2/p - 1)| = (1 - 2/p') \) for all \( p > 2 \). Then those two lemmas (Lemma 5.4 and 5.2) will give us Theorem 1.6 and since Theorem 1.4 is true for all \( p \in (1, \infty) \), we will finally have Theorem 1.6 for all \( p \in (1, 2) \cup (2, \infty) \).

(2) We observe from the proof of the global part, it follows that if we assume only

\[
\left| \frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(x, \lambda) \right| \leq C_\alpha (1 + |\lambda|)^{-\alpha}, \tag{5.22}
\]

for all \( \alpha = 0, 1, \ldots \left\lceil \frac{d+1}{2} \right\rceil + 1, x \in G \) and \( \lambda \in S_p \), in addition with holomorphy condition on \( S_p^\circ \), etc. then the operator \( \Psi^{\text{lo}}_\sigma \) extends to a bounded operator on \( L^p(\mathbb{X}) \) to itself. That is, we do not need derivative condition on \( x \) variable. The derivative condition on \( x \) variable is required only to conclude the boundedness of the local part \( \Psi_\sigma^{\text{lo}} \) which follows from the boundedness of Euclidean pseudo differential operators.

(3) It is a natural question and perhaps an interesting problem to ask for \( L^p \) boundedness of \( \Psi_\sigma \) on \( \mathbb{X} \) if \( \sigma \) is allowed to have singularities on boundary. We recall that on \( \mathbb{R}^n \) this is not possible.
(4) Suppose $\sigma : \mathbb{X} \times \mathbb{C} \to \mathbb{C}$ be a smooth function such that for each $x \in \mathbb{X}$, $\lambda \mapsto \sigma(x, \lambda)$ is an even, holomorphic function on $S_p^\prime$ and satisfies (1.8) for some $p \in (1, 2)$, that is

$$\left| \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\alpha}{\partial \lambda^\alpha} \sigma(xa_s, \lambda) \right| \leq C_{\alpha, \beta}(1 + |\lambda|)^{-\alpha},$$

(5.23)

for all $\alpha = 0, 1, \ldots, \left\lfloor \frac{d+1}{2} \right\rfloor + 1$, $\beta = 0, 1, 2$, $x \in G$, $s \in \mathbb{R}$ and $\lambda \in S_p$ for some $p \in (1, 2)$. Then the operator $\Psi_\sigma$ extends to a bounded operator on $L^p(\mathbb{X})$ to itself and on $L^p(\mathbb{X})$ to itself. By interpolation, the operator $\Psi_\sigma$ is bounded on $L^2(\mathbb{X})$ to itself. Therefore, if $\lambda \to \sigma(x, \lambda)$ is holomorphic and satisfy the inequalities (5.22) on a strip, then $\Psi_\sigma$ is bounded on $L^2(\mathbb{X})$. But the condition on the symbol $\sigma$ above is not natural for the $L^2$ boundedness of $\Psi_\sigma$. In the next section, we prove the $L^2$ boundedness of $\Psi_\sigma$ in the $K$-biinvariant setting, assuming only smoothness condition (without holomorphicity) of the symbol $\sigma$ on the real line.

6. $L^2$ boundedness for symmetric spaces in $K$-biinvariant cases

In this section we shall prove the $L^2$ boundedness of pseudo differential operator associated with a symbol $\sigma$ in the $K$-biinvariant setting. We recall that $\mathbb{X} = G/K$ is a rank one symmetric space with $\dim \mathbb{X} = d$.

**Proof of the Theorem [1.8]**: Let $\eta : [0, \infty) \to [0, 1]$ be a smooth cut off function supported in $[1, \infty)$ and equal to 1 on $[2, \infty)$. Using Proposition (2.5), (2.15) and evenness of $\lambda \mapsto \sigma(\cdot, \lambda)$ we write,

$$\eta(t)\Psi_\sigma f(a_t)e^{it\lambda} = 2 \int_\mathbb{R} \eta(t) \frac{\sigma(t, \lambda)}{c(-\lambda)} e^{it\lambda} \hat{f}(\lambda) d\lambda + 2 \int_\mathbb{R} \eta(t) \frac{\sigma(t, \lambda) a(t, \lambda)}{c(-\lambda)} e^{it\lambda} \hat{f}(\lambda) d\lambda$$

$$= 2 \int_\mathbb{R} \eta(t) \frac{\sigma(t, \lambda)}{c(-\lambda)} e^{it\lambda} \mathcal{F}A f(\lambda) d\lambda + 2 \int_\mathbb{R} \eta(t) \frac{\sigma(t, \lambda) a(t, \lambda)}{c(-\lambda)} e^{it\lambda} \mathcal{F}A f(\lambda) d\lambda$$

$$= \tilde{\Psi}_{a_1}(\mathcal{F}A)(t) + \tilde{\Psi}_{a_2}(\mathcal{F}A)(t),$$

where

$$a_1(t, \lambda) = 2\eta(t) \frac{\sigma(t, \lambda)}{c(-\lambda)}$$

and

$$a_2(t, \lambda) = 2\eta(t) \frac{\sigma(t, \lambda) a(t, \lambda)}{c(-\lambda)},$$

(6.1)

and $\tilde{\Psi}_{a_1}, \tilde{\Psi}_{a_2}$ are the Euclidean pseudo differential operator associated with symbols $a_1, a_2$ respectively. Now from (1.10), (2.19) and (2.20) it follows that $a_1, a_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ are smooth functions and satisfies the following inequalities:

$$\left| \frac{\partial^\beta}{\partial t^\beta} \frac{\partial^\alpha}{\partial \lambda^\alpha} a_i(t, \lambda) \right| \leq C_{\alpha, \beta}(1 + |\lambda|)^{-\alpha},$$

(6.2)

for all $\lambda \in \mathbb{R}$, $\alpha, \beta \in \mathbb{Z}^+$ and $i = 1, 2$. Using the results of pseudo differential operators in Euclidean spaces (Theorem 1.3), it follows that $\tilde{\Psi}_{a_i}$ extends to a bounded operator from $L^2(\mathbb{R})$ to itself. Therefore for $f \in C_c^\infty(G//K)$,

$$\left( \int_2^\infty |\Psi_\sigma(f)(a_t)|^2 e^{2\mu t} dt \right)^{\frac{1}{2}} \leq \left( \int_1^\infty |\eta(t)\Psi_\sigma(f)(a_t)|^2 e^{2\mu t} dt \right)^{\frac{1}{2}} \leq C \|\mathcal{F}A f\|_{L^2(\mathbb{R})}.$$  

(6.3)
Now we choose $M > 0$, such that $|c(\lambda)|^{-2} \geq 1$ for all $|\lambda| \geq M$. Then using the Euclidean Plancherel theorem and (2.13), we have

$$
\|Af\|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} |\hat{f}(\lambda)|^2 \, d\lambda = 2 \int_0^M |\hat{f}(\lambda)|^2 \, d\lambda + 2 \int_M^\infty |\hat{f}(\lambda)|^2 \, d\lambda
\leq C\|f\|_{L^p(\mathbb{R})}^2 + \int_M^\infty |\hat{f}(\lambda)|^2 |c(\lambda)|^{-2} \, d\lambda
\leq C \left(\|f\|_{L^p(\mathbb{R})}^2 + \|f\|_{L^2(\mathbb{R})}^2\right),
$$

where we used the fact

$$
|\hat{f}(\lambda)| \leq \|f\|_{L^p(\mathbb{R})} \|\phi_0\|_{L^{p'}(\mathbb{R})},
$$

for all $\lambda \in \mathbb{R}$ (see [19, Proposition 2.1]). Hence we get

$$
\|Af\|_{L^2(\mathbb{R})} \leq C \left(\|f\|_{L^p(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}\right). \tag{6.4}
$$

Therefore from (6.3) we have,

$$
\left(\int_2^\infty \|\Psi(\sigma)(f)(a_t)\|^2 e^{2\rho t} \, dt\right)^{1/2} \leq C \left(\|f\|_{L^p(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}\right). \tag{6.5}
$$

Next from (1.9), (1.10) and Lemma 2.21 we get

$$
|\Psi(\sigma)(f)(a_t)| \leq \|f\|_{L^p(\mathbb{R})} \|\phi_0\|_{L^{p'}(\mathbb{R})} \int_\mathbb{R} |\sigma(t, \lambda)| |c(\lambda)|^{-2} \, d\lambda \leq C\|f\|_{L^p(\mathbb{R})}. \tag{6.6}
$$

Therefore from (6.5) and (6.6) we get

$$
\|\Psi f\|_{L^2(\mathbb{R})} \leq C \left(\|f\|_{L^p(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}\right) \quad \text{for any } 1 \leq p < 2.
$$

\[\square\]

**Proof of Theorem 1.9:** Let $\eta : [0, \infty) \to [0, 1]$ be a smooth cut off function supported in $[1, \infty)$ and equal to 1 on $[2, \infty)$. Suppose $2 < q < 3$. Then from (1.9), (2.13) and Proposition 2.5 we have

$$
\eta(t)\Psi(\sigma)(f)(a_t)e^{2\rho t} = e^{(\frac{2}{q}-1)\rho t} \int_\mathbb{R} \frac{2\eta(t)\sigma(t, \lambda)}{c(-\lambda)} e^{i\lambda t} F Af(\lambda) \, d\lambda
+ e^{(\frac{2}{q}-1)\rho t} \int_\mathbb{R} \frac{2\eta(t)\sigma(t, \lambda)e^{-2\rho t}(t, \lambda)}{c(-\lambda)} e^{i\lambda t} F Af(\lambda) \, d\lambda \tag{6.7}
$$

$$
= e^{(\frac{2}{q}-1)\rho t} \tilde{\Psi}_{a_1}(Af)(t) + e^{(\frac{2}{q}-1)\rho t} \tilde{\Psi}_{a_2}(Af)(t),
$$

where $a_1, a_2$ are defined in (6.1) and $\tilde{\Psi}_{a_1}, \tilde{\Psi}_{a_2}$ are the Euclidean pseudo differential operator associated with symbols $a_1, a_2$ respectively. Since $q > 2$, from above it follows that

$$
\left(\int_1^\infty |\eta(t)\Psi(\sigma)(f)(a_t)|^q e^{2\rho t} \, dt\right)^{\frac{1}{q}} \leq \|\tilde{\Psi}_{a_1}(Af)\|_{L^q(\mathbb{R})} + \|\tilde{\Psi}_{a_2}(Af)\|_{L^q(\mathbb{R})}
$$

Applying the result for pseudo differential operator in Euclidean case (Theorem 1.3) on $\tilde{\Psi}_{a_1}$ and $\tilde{\Psi}_{a_2}$ we get $\tilde{\Psi}_{a_i} : L^q(\mathbb{R}) \to L^q(\mathbb{R})$ are bounded for $i = 1, 2$, and so

$$
\left(\int_2^\infty |\Psi(\sigma)(f)(a_t)|^q e^{2\rho t} \, dt\right)^{\frac{1}{q}} \leq \left(\int_1^\infty |\eta(t)\Psi(\sigma)(f)(a_t)|^q e^{2\rho t} \, dt\right)^{\frac{1}{q}} \leq C\|Af\|_{L^q(\mathbb{R})} \tag{6.8}
$$
Since $2 < q < 3$, we put $r = q$ and $p = q'$ in (2.17), to get

$$
\left( \int_{\mathbb{R}} |A f(a_t)|^q dt \right)^{\frac{1}{q}} \leq C \|f\|_{L^{q'}(X)}. \tag{6.9}
$$

Therefore from (6.8) and (6.9) we get,

$$
\left( \int_2^{\infty} |\Psi_\sigma(f)(a_t)|^q e^{2\rho t} dt \right)^{\frac{1}{q}} \leq C \|f\|_{L^{q'}(X)}. \tag{6.10}
$$

Also from (1.9), (1.10) and (2.21) we have,

$$
|\Psi_\sigma(f)(a_t)| \leq \|f\|_{L^{q'}(X)} \|\phi_0\|_{L^q(X)} \int_{\mathbb{R}} |\sigma(t, \lambda)| |c(\lambda)|^{-2} d\lambda \leq C_p \|f\|_{L^{q'}(X)}.
$$

This implies

$$
\left( \int_0^2 |\Psi_\sigma(f)(a_t)|^q e^{2\rho t} dt \right)^{\frac{1}{q}} \leq C \|f\|_{L^{q'}(X)}. \tag{6.11}
$$

Finally from (6.10) and (6.11) we have for $2 < q < 3$,

$$
\|\Psi_\sigma(f)\|_{L^q(X)} \leq C \|f\|_{L^{q'}(X)}. \tag{6.12}
$$

Now if we take $3/2 < p < 2$, then $p = q'$ for some $2 < q < 3$. So from (6.12) we have,

$$
\|\Psi_\sigma(f)\|_{L^{q'}(X)} \leq C \|f\|_{L^p(X)}. \tag{6.13}
$$

Also from Theorem 1.6 we have

$$
\|\Psi_\sigma(f)\|_{L^p(X)} \leq C \|f\|_{L^{q'}(X)}. \tag{6.14}
$$

Thus by interpolation it follows that for all $r$ with $p \leq r \leq p'$ there is a $C > 0$ such that

$$
\|\Psi_\sigma(f)\|_{L^r(X)} \leq C \|f\|_{L^p(X)}. \tag{6.15}
$$

In particular $r = 2$ gives,,

$$
\|\Psi_\sigma(f)\|_{L^2(X)} \leq C \|f\|_{L^{p'}(X)}. \tag{6.16}
$$

□

From the proof of this theorem the following corollary follows:

**Corollary 6.1.** Let $\frac{3}{2} < p < 2$ and $\sigma$ a symbol satisfies condition of Theorem 1.9. Then

$$
\|\Psi_\sigma(f)\|_{L^{q'}(X)} \leq C \|f\|_{L^p(X)},
$$

for $p \leq r \leq p'$.

In the proof of last two theorems, we had to multiply the smooth cut off function $\eta$ (in first step) because we have the estimate of $a(t, \lambda)$ when $t$ is away from origin (see Proposition 2.5).

**Complex symmetric space:** Let $X$ be an arbitrary rank complex symmetric space. The elementary spherical functions on complex symmetric space $X$ are given by the expression

$$
\phi_\lambda(H) = c(\lambda) \sum_{s \in W} \det(s)e^{-is\lambda(H)} \phi(H), \quad \text{for } H \in \mathfrak{a}, \lambda \in \mathfrak{a}^*, \tag{6.17}
$$

where $c(\lambda)$ is the constant coefficient of the spherical function $\phi_\lambda$.
where \( \phi(H) \) is defined by the formula
\[
\phi(H) = \sum_{s \in W} \det(s)e^{is\phi(H)}, \quad \text{for } H \in \mathfrak{a},
\]
(see [8, p. 907 and p. 910]). It is known that the Haar measure on \( \mathfrak{a} \) corresponding to the polar decomposition of \( G \) is given by \( \phi(H)^2dH \) for \( H \in \mathfrak{a} \).

Thus the spherical transform of a suitable \( K \)-biinvariant function \( f \) is given by
\[
\hat{f}(\lambda) = \int_\mathfrak{a} f(H)\phi(\lambda(H))\phi(H)^2dH, \quad \text{for } \lambda \in \mathfrak{a}^*.
\]

We define the \( K \)-biinvariant pseudo differential operator \( \Psi_\sigma \) associated to \( \sigma \) by,
\[
\Psi_\sigma(f)(H) = \int_{\mathfrak{a}^*} \hat{f}(\lambda)\sigma(H, \lambda)\phi(\lambda(H))|c(\lambda)|^{-2}d\lambda.
\]

Now we shall prove Theorem 1.10:

**Proof of the Theorem 1.10:** From (6.19) and (6.17) we have
\[
\hat{f}(\lambda) = c(\lambda) \sum_{s \in W} \det(s) \int_\mathfrak{a} f(H)\phi(\lambda(H))e^{-is\lambda(H)}dH
\]
\[
= c(\lambda) \sum_{s \in W} \det(s) \int_\mathfrak{a} f(H)\phi(H)e^{-is\lambda(H)}dH
\]
\[
= c(\lambda) \sum_{s \in W} \det(s)\mathcal{F}g(s\lambda), \quad \text{for } \lambda \in \mathfrak{a}^*,
\]
where the function \( g \) on \( \mathfrak{a} \) is defined as
\[
g(H) = f(H)\phi(H), \quad \text{for } H \in \mathfrak{a},
\]
and \( \mathcal{F}g \) denotes the Euclidean Fourier transform of \( g \). Since \( \phi \) satisfies
\[
\phi(sH) = \det(s)\phi(H),
\]
for \( H \in \mathfrak{a}, s \in W \), we have
\[
g(sH) = \det(s)g(H), \quad \text{for } H \in \mathfrak{a}, s \in W.
\]
Therefore,
\[
\mathcal{F}g(s\lambda) = \det(s)\mathcal{F}g(\lambda), \quad \text{for } \lambda \in \mathfrak{a}^*, s \in W.
\]
Consequently the spherical transform of \( f \) can be written as
\[
\hat{f}(\lambda) = c(\lambda)|W|\mathcal{F}g(\lambda), \quad \text{for } \lambda \in \mathfrak{a}^*.
\]
Now from the definition (6.20) of $\Psi_\sigma$ we get,

$$\Psi_\sigma(f)(H) = \int_{a^*} \hat{f}(\lambda) \sigma(H, \lambda) \phi(\lambda)|c(\lambda)|^{-2}d\lambda$$

$$= \int_{a^*} \hat{f}(\lambda) \sigma(H, \lambda) \phi(\lambda)|c(\lambda)|^{-2}d\lambda$$

$$= |W| \int_{a^*} Fg(\lambda) c(\lambda)(H, \lambda) c(-\lambda) \sum_{s \in W} \det(s) e^{is\lambda(H)} \phi(H)|c(\lambda)|^{-2}d\lambda$$

$$= \frac{|W|}{\phi(H)} \sum_{s \in W} \det(s) \int_{a^*} Fg(\lambda) \sigma(H, \lambda) e^{is\lambda(H)}d\lambda.$$

Now by the change of variable $\lambda \mapsto s^{-1}\lambda$ and (6.22) we get

$$\phi(H)\Psi_\sigma(f)(H) = |W|^2 \int_{a^*} Fg(\lambda) \sigma(H, \lambda) e^{i\lambda(H)}d\lambda.$$

That is, that

$$\Psi_\sigma(f)(H) \phi(H) = |W|^2 \widetilde{\Psi}_\sigma(g)(H), \quad (6.24)$$

where $\widetilde{T}_\sigma$ is the Euclidean pseudo differential operator associated with the symbol $\sigma(H, \lambda)$. Using Calderón and Vaillancourt theorem [18, Theorem 5.1], we have $\widetilde{T}_\sigma : L^2 \to L^2$ bounded. Therefore we have

$$\int_a |\Psi_\sigma(f)(H)|^2 \phi(H)^2 dH = |W|^4 \int_a |\widetilde{\Psi}_\sigma(g)(H)|^2 dH \leq C \int_a |g(H)|^2 dH = \int_a |f(H)|^2 \phi(H)^2 dH.$$

This shows that

$$\|\Psi_\sigma(f)\|_{L^2(G//K)} \leq C \|f\|_{L^2(G//K)}.$$

This completes the proof. \[\square\]

7. APPENDIX

7.1. Coifman-Weiss transference method. ([6], [7]): Suppose $G$ is a locally compact group satisfying the following property: Given a compact subset $B$ of $G$ and $\epsilon > 0$, there exists an open neighborhood $V$ of identity having finite measure such that

$$\mu(VB^{-1}) \leq 1 + \epsilon,$$

where $\mu$ is a left Haar measure. Any locally compact abelian group satisfies this property. Let $\mathcal{R}$ be a representation of $G$ acting on functions on a $\sigma$ finite measure space $\mathcal{X}$ satisfying, for some $p \in [1, \infty]$

$$\int_{\mathcal{X}} |(\mathcal{R}_u f)(x)|^p d\mu(x) \leq C \int_{\mathcal{X}} |f(x)|^p d\mu(x), \quad (7.1)$$

where $C$ is independent of $f \in L^p(\mathcal{X})$ and $u \in G$. Consider the transformation of the following form

$$\mathcal{T} f(x) = \int_G \mathcal{R}(x, R_u x, u) f(R_u x) d\mu(u), \quad (7.2)$$
where $\mathcal{R}(x, y, u)$ is a measurable function on $X \times X \times G$, which is zero if $u$ does not belong to a compact set $B \subset G$. Moreover for each $x \in X$, the kernel $K_x(u, v) = K(\mathcal{R}_ux, \mathcal{R}_x^{-1}u, u) = K(\mathcal{R}_v x, \mathcal{R}_x^{-1}v, u)$ satisfies

$$\left( \int_G \left| \int_G K_x(v, u) g(u^{-1}v) d\mu(u) \right|^p d\mu(v) \right)^{1/p} \leq A \left( \int_G |g(u)|^p d\mu(u) \right)^{1/p},$$

where $A$ is independent of $x \in X$ and $g \in L^p(G)$. Then $T$ is a bounded operator on $L^p(X)$ with norm not exceeding $A$ ((7, (2.7))):

$$\left( \int_X |Tf(x)|^p dx \right)^{1/p} \leq A \left( \int_X |f(x)|^p dx \right)^{1/p}.$$  \hspace{1cm} (7.4)

8. Open Problem

It will be interesting to get an exact analogue of $L^2$ boundedness of pseudo differential operators on rank one symmetric spaces and also $L^p$ boundedness of pseudo differential operators on (arbitrary rank) symmetric spaces of noncompact type. This will be taken as our future project.

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