Change Detection of Markov Kernels with Unknown Post Change Kernel using Maximum Mean Discrepancy

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Abstract

In this paper, we develop a new change detection algorithm for detecting a change in the Markov kernel over a metric space in which the post-change kernel is unknown. Under the assumption that the pre- and post-change Markov kernel is geometrically ergodic, we derive an upper bound on the mean delay and a lower bound on the mean time between false alarms.

1. Introduction

Change detection algorithm is an important statistical tool that takes a stream of observations as input and detects whether or not the statistical properties of an underlying system (generating the observations) has changed or not. This algorithm forms the bedrock of fault detection and attack detection in dynamical systems and anomaly detection in signal processing and industrial control system. We propose here a new online change detection algorithm for Markov chains over metric spaces and present its performance guarantees.

Two-sample testing is a specific type of change detection algorithm that determines whether two groups of samples come from the same distribution. Kernel based two-sample testing was developed in Gretton et al. (2012b) – which is advantageous in completely data-driven situations, especially when the underlying space is high dimensional. The testing works by projecting the underlying probability distributions of different groups of samples into a reproducing kernel Hilbert space (RKHS) through an injective mapping and measuring the distance between projections using the norm defined for that RKHS.

A seminal work on RKHS embedding of probability measures can be found in Sriperumbudur et al. (2010), which provides crucial inspiration and theoretical foundation for our work. MMD-based change detection methods have received significant attention in the past decade Li et al. (2015) Flynn and Yoo (2019) Li et al. (2019) Zaremba et al. (2013). The test statistics of these methods are built around the empirical estimation of the MMD. The two major online change detection framework are cumulative sum (CUSUM) type and (scanning) window type. The kernel CUSUM method introduced in Flynn and Yoo (2019) is an adaptation of the well-established CUSUM statistics Page (1954) by substituting the distance function from the estimated KL-divergence to the estimated MMD.

The performance guarantees for these change detection methods are given in the form of bounds on the mean time between false alarm (MTBFA), also known as average run length (ARL), and the mean delay (MD). Upper bound for MTBFA and lower bound for MD are obtained using standard concentration inequalities for supermartingales in Flynn and Yoo (2019). The window type approach presented in Li et al. (2019) employs a specially designed sliding window framework to reduce computation complexity during the estimation of MMD. The performance bounds are derived using a sophisticated change-of-measure
technique. However, for samples generated from Markov chains, which are at best $\beta$-mixing when the Markov kernel is uniformly ergodic, the analysis in the aforementioned studies breaks down and motivates us to search for a new set of tools.

Change detection for Markov chains has been studied earlier under stringent assumptions. In Xian et al. (2016), the authors adapt the CUSUM statistics for uniformly ergodic Markov chains over finite state spaces. The upper bound on MTBFA is obtain via characterizing the limiting distribution of the test statistic, and the lower bound on MD is derived by applying the Markov chain version of the Hoeffding’s inequality Glynn and Ormoneit (2002). Although our setting has the general state space, the Hoeffding’s inequality for uniformly ergodic Markov chain is also a crucial step in our performance analysis.

**Our Contribution:** Inspired by the existing approaches, we devised an online change detection algorithm for uniformly ergodic Markov Chains with general state space. The primary contributions of this paper are summarized below:

- The kernel based online change detection algorithm we propose works with weakly dependent samples generated from uniformly ergodic Markov chains. This is a relaxation over the independent sample assumption imposed by current kernel based online change detection methods.

- The algorithm we propose addresses the change detection problem when both pre- and post-change Markov kernels are unknown. This is an improvement over the method in Xian et al. (2016) which require the knowledge of the pre-change Markov chain transition matrix.

- The Markov chains we works have general state space i.e. metric space, which makes our method superior than existing method that assumes finite state space. This advantage allows for greater application potential in dynamic systems, financial engineering, and other areas that produce high-dimensional or continuous data.

- Our method can detect a change in the transition kernel of the Markov chain, not just a change in the invariant distribution. Let $P$ be a Markov kernel with invariant measure $\pi_P$. One can construct another Markov kernel $Q \neq P$ with invariant measure $\pi_Q = \pi_P$. Hence, testing for the invariant distribution for the Markov chain is insufficient. Our algorithm detects the change in the product measure $\pi_P \otimes P$, which ensures that any subtle change in the Markov kernel can be detected.

The rest of the paper is organized as follows. In Section 2, we review the definition and preliminary results on MMD. In Section 3, we formally state our problem setup. We present the test statistics and the main results deriving the bounds on MTBFA and MD in Section 4. The numerical simulations can be found in Section 5.

2. Preliminaries

In this section, we introduce the notion of RKHS, kernel mean embedding, maximum mean discrepancy (MMD), and weak dependence. The key reference for the terminology used this section is Sriperumbudur et al. (2010). Consider in a metric space $(\mathcal{X}, \mathcal{X})$. Consider a positive definite kernel function $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ (see Sriperumbudur et al. (2010) for definition
of positive definite kernels). For kernel \( k \), there exist an reproducing kernel Hilbert space (RKHS) \( \mathcal{H}_k \) of functions equipped with an inner product \( \langle \cdot, \cdot \rangle_{\mathcal{H}_k} \). For any function \( f \in \mathcal{H}_k \), \( f(x) = \langle f, k(x, \cdot) \rangle_{\mathcal{H}_k} \) due to the reproducing property.

We assume the kernel is bounded and without loss of generality we set the bound to be 1, i.e. \( \sup_{x,y \in X} |k(x, y)| \leq 1 \). Let \( \mathcal{P} \) be a measure in the space of probability measures over \( X \), denoted by \( \mathcal{P}(X) \). The kernel mean embedding \( \mu_{P} \) is defined as \( \mu_{P}(\cdot) := \mathbb{E}_{X \sim P}[k(X, \cdot)] \). This mapping from \( \mathcal{P} \) to \( \mathcal{H}_k \) is injective if and only if the kernel \( k \) is characteristic.

The maximum mean discrepancy (MMD) between two measures \( P, Q \in \mathcal{P}(X) \) is defined as

\[
\gamma = \sup_{f \in \mathcal{F}} |\mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{X \sim Q}[f(X)]|,
\]

where \( \mathcal{F} \) is some class of functions. When \( \mathcal{F} \) is chosen to be the unit ball in the RKHS \( \mathcal{H}_k \) and under the assumption that the kernel function \( k \) is bounded, the MMD can also be written as the distance between the embedding \( \gamma_k(P, Q) = \|\mu_P - \mu_Q\|_{\mathcal{H}_k} \). Note that \( \gamma_k \) is a pseudometric in when a general selection of the kernel function, since it fails to satisfy \( \gamma_k(P, Q) = 0 \iff P = Q \). The kernel \( k \) being characteristic is a necessary and sufficient condition for \( \gamma_k \) to be a metric. The square MMD is more commonly used in statistical hypothesis testing:

\[
\gamma_k^2(P, Q) = \mathbb{E}_{X, X' \sim P}[k(X, X')] - 2\mathbb{E}_{X \sim P, Y \sim Q}[k(X, Y)] + \mathbb{E}_{Y, Y' \sim Q}[k(Y, Y')].
\]

Several criteria for characteristic kernels are provided in Sriperumbudur et al. (2010). When the kernel is translation invariant, i.e. \( k(x, y) = \psi(x - y) \) and defined on \( \mathbb{R}^d \), \( k \) is characteristic if and only if the support of its Fourier transformation is \( \mathbb{R}^d \) Sriperumbudur et al. (2010). For a comprehensive characterisation of kernels \( k \) such that \( \gamma_k \) is a metric, please refer to Table 1 in Muandet et al. (2016). Gaussian kernel is frequently used as the characteristic kernel for MMD:

\[
k(x, y) = \psi(x - y) = \exp\left(-\frac{\|x - y\|^2}{2\sigma^2}\right), \quad x, y \in \mathbb{R}^d, \sigma > 0.
\]

### 3. Problem Formulation

Consider a Markov chain \( (X_i)_{i \in \mathbb{N}} \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in a metric space \((X, \mathcal{X})\). Let \( P : \mathcal{X} \to \mathcal{P}(\mathcal{X}) \) denote the transition kernel of the Markov chain. At some unknown time \( \tau \in \mathbb{N} \), the Markov chain immediately switches its transition kernel to \( Q (P \neq Q) \). Both the pre-change kernel \( P \) and the post-change kernel \( Q \) are assumed to be uniformly ergodic and they satisfy the Doeblin’s condition, which is defined below.

**Assumption 1 (Doeblin’s Condition)** A Markov kernel \( P \) satisfies Doeblin condition if there exist a probability measure \( \phi \) on \( \mathcal{X} \), \( \lambda > 0 \), and some integer \( l \geq 1 \) such that \( P^l(x, \cdot) \geq \lambda \phi(\cdot) \), for all \( x \in \mathcal{X} \).

The Doeblin’s condition is equivalent to uniform ergodicity for aperiodic and irreducible Markov chains. We refer the reader to Theorem 16.2.3 in Meyn and Tweedie (2012). Note the the Doeblin’s coefficients for \( P \) and \( Q \) do not need to be the same.

The goal is to design a stopping time \( T \) to detect the change point \( \tau \) as soon as it happens by regularly checking the statistical properties of the trajectory. To properly
define a distribution for samples generated from the pre- and post-change Markov chains, we now introduce the general RKHS embedding of Markov chains.

3.1. RKHS Embedding of Markov Kernels

Consider a measurable space \((\mathcal{X}, \mathcal{X})\) and let \(P : \mathcal{X} \to \mathcal{P}(\mathcal{X})\) be a Markov kernel. Assume that the kernel \(P\) admits an invariant probability measure \(\pi : \mathcal{X} \to [0, 1]\). Define a probability measure \(F_P\) on measurable space \((\mathcal{X} \times \mathcal{X}, \mathcal{X} \otimes \mathcal{X}) := (\mathcal{Z}, \mathcal{Z})\) as

\[
F_P(A \times B) = \pi \otimes P(A \times B) = \int_A \pi(dx)P(x, B) \quad \text{for } A, B \in \mathcal{X}.
\]

(3)

Let \(k : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}\) be a characteristic kernel function and \(\mathcal{H}_k\) be the corresponding RKHS. Then the RKHS embedding of \(F_P\) is written as \(\mu_P(\cdot) = E_{z \sim F_P} k(z, \cdot)\). In the following lemma, we identify a condition under which \(P \neq Q\) implies \(F_P \neq F_Q\).

**Lemma 1** Let \(P\) and \(Q\) be two Markov kernels with invariant distributions \(\pi_P\) and \(\pi_Q\), respectively. If \(P(x, \cdot) \neq Q(x, \cdot)\) on a set of \(x\) with positive \(\pi_P\)-measure, then \(F_P \neq F_Q\).

**Proof** Please refer to Appendix A.

Denote \((\bar{X}_i)_{i \in \mathbb{N}} = (X_i, X_{i+1})_{i \in \mathbb{N}}\) as the second order Markov chain with respect to the original Markov chain \((X_i)_{i \in \mathbb{N}}\). In the following lemma, we show that the second order Markov chain also satisfies the Doeblin’s condition.

**Lemma 2** Let \((X_i)_{i \in \mathbb{N}}\) be a Markov chain satisfying Doeblin’s condition, and \((\bar{X}_i)_{i \in \mathbb{N}} = (X_i, X_{i+1})_{i \in \mathbb{N}}\) be the second order Markov chain. Then, \((\bar{X}_i)_{i \in \mathbb{N}}\) also satisfies Doeblin’s condition.

**Proof** Please refer to Appendix B.

Let \(F_P\) and \(F_Q\) be defined the same as in (3) for \(P\) and \(Q\) respectively. The sequential hypothesis testing at time \(t\) can be constructed as

\[
\mathcal{H}_0 : \bar{X}_i \sim F_P, \ \forall i < t; \quad \mathcal{H}_1 : \exists 1 \leq \tau < t : \bar{X}_i \sim \begin{cases} F_Q, & i > \tau, \\ F_P, & \text{otherwise}. \end{cases}
\]

The stopping time \(T\) can be then interpreted as the first time the null hypothesis \(\mathcal{H}_0\) is rejected. To evaluate the detection performance of \(T\), consider the mean time between false alarms (MTFBA) and mean delay (MD) as follows.

\[
\text{MTFBA}(T) = E_{\tau}[T - \tau|T > \tau], \quad \text{MD}(T) = E_{\tau=\infty}[T - \tau|T > \tau].
\]

(4)

To distinguish \(F_P\) and \(F_Q\), we adopt maximum mean discrepancy (MMD) \(\gamma_k(F_P, F_Q)\) as the distance measure between two distributions. In the next section, we discuss how to use dependent samples to estimate \(\gamma_k(F_P, F_Q)\) and the corresponding error bounds induced by the dependence. For the remainder of the paper, we assume that the kernel function \(k\) is characteristic such that \(\gamma_k\) is a metric for probability measures.
3.2. MMD Estimation for Dependent Observations

The unbiased and biased estimators for MMD with corresponding error bounds are provided in Gretton et al. (2012a) under i.i.d assumption. The unbiased estimator can center around the true MMD with finite samples. However, the same estimator loses its unbiasedness with finite dependent samples. In Chérief-Abdellatif and Alquier (2022), an error bound is presented for the empirical mean estimator with samples satisfying the weak dependence condition in Assumption 2. We adopt their results for the MMD estimator of $\gamma_k(F_P,F_Q)$ and the estimation error bounds under both null and alternative hypothesis.

Let $(X_i)_{i=1}^{n+1}$ and $(Y_i)_{i=1}^{m+1}$ be two independent set of samples generated from kernels $P$ and $Q$ respectively. We have $(\hat{X}_i)_{i=1}^{n}$ and $(\hat{Y}_i)_{i=1}^{m}$ as the second order Markov chains with kernels $\tilde{P} := P^{\otimes 2}$ and $\tilde{Q} := Q^{\otimes 2}$. Assume all samples are obtained from the stabilized Markov chains, i.e., $\hat{X}_i \sim F_P$ and $\hat{Y}_i \sim F_Q$. Then $\gamma_k(\hat{F}_P,\hat{F}_Q)$ is an estimator of $\gamma_k(F_P,F_Q)$, where $\hat{F}_P = (1/n)\sum_{i=1}^{n} \delta_{\hat{X}_i}$ is the empirical measure of $F_P$ and $\delta_{\hat{X}_i}$ is the Dirac measure of sample $\hat{X}_i$. $\hat{F}_Q$ is defined similarly for $F_Q$.

Now, we introduce the following two weak dependence conditions that are originally proposed in Chérief-Abdellatif and Alquier (2022).

**Definition 3** Define for any $t \in \mathbb{N}$,

$$\rho_t = \left| \mathbb{E}(k(X_t, \cdot) - \pi^0 k, k(X_0, \cdot) - \pi^0 k)_{\mathcal{H}} \right| . \quad (5)$$

**Assumption 2** There exists a $\Sigma < +\infty$ such that $\sum_{t=1}^{n} \rho_t \leq \Sigma$ for $\forall n \in \mathbb{N}$, where $n$ is the length of the sample trajectory.

Note that with Assumption 1 imposed on the Markov chain, Assumption 2 is automatically satisfied for Markov chain $\{X_t\}_t$ satisfying the Doeblin’s condition.

Assume that the kernel function can be written as $k(x,y) = F(||x-y||)$, where $F(a) = \int_{\mathbb{R}} f(x)dx$, for some non-negative continuous function $f$ such that $\int_{\mathbb{R}} f(x)dx = 1$. Due to Proposition 4.4 in Chérief-Abdellatif and Alquier (2022), the coefficient defined in Definition 3 is bounded by $\rho_t \leq 4(1 - \lambda)^{t-1}$. For a Gaussian kernel $k$, the above condition is satisfied with the choice of $f(x) = \exp(-x/\sigma^2)/\sigma^2$. Therefore, Assumption 2 is satisfied with

$$\Sigma = \frac{4}{(1 - \lambda)(1 - (1 - \lambda)^{1/2})} .$$

Now given Assumption 2 is satisfied, the MMD estimator using empirical distributions is consistent for dependent observations as we show below.

**Lemma 4** (Consistency) Let $X = \{X_i\}_{i=1}^{n}$ be a Markov chain generated from Markov kernel $P$ which satisfies Assumption 2 with $\Sigma_X$. Let $\hat{P}_n = (1/n)\sum_{i=1}^{n} \delta_{X_i}$ be the empirical measure. Similarly $Y = \{Y_j\}_{j=1}^{m}$ is generated using Markov kernel $Q$. We define $\Sigma_Y,\hat{Q}_n$ analogously as above. Given $k$ is characteristic, we have

$$\left| \mathbb{E}\left[ \gamma_k(\hat{P}_n,\hat{Q}_n) \right] - \gamma_k(P,Q) \right| \leq c_{X,Y}(n) ,$$

where $c_{X,Y}(n) = \sqrt{1 + 2\Sigma_X n} + \sqrt{1 + 2\Sigma_Y n}$. 


Proof This result is a direct consequence of Lemma 7.1 in Chérief-Abdellatif and Alquier (2022).

We are now ready to present our test statistics for the hypothesis testing problem setup above. We further conduct its performance analysis to obtain a lower bound on the MTBFA and an upper bound on MD.

4. Main Results

In this section, we introduce a CUSUM-typed online test statistics based on MMD norm, and we give the theoretic guarantees on MTBFA and MD when applied to Markovian observations. The Kernel CUSUM algorithm was propose for independent samples in Flynn and Yoo (2019). We adopt the their test statistics for change detection in Markov chains. Since the independent assumption no long exists under our setting, it requires a completely different approach to derive the theoretical bounds for MTBFA and MD.

We first assume the availability of previous observations of the system under pre-change kernel. The length of the observed sample path is sufficiently long. Let \( X^{m+1} = (X_t)_{t=1}^{m+1} \) denote the past \( m+1 \) observations and \( \tilde{X}^t = (X_t, X_{t+1}) \). Let \( Y_t \) be the current system observation at time \( t \geq 0 \) whose kernel is subject to change from \( P \) to \( Q \), and \( \tilde{Y}^t = (Y_t, Y_{t+1}) \). To estimate MMD from samples, we also need to maintain a buffer \( B^r_t = \{Y_i\}_{t-r}^{t} \) of size \( r \in \mathbb{N} \). At time \( t \), the buffer always contains the most recent \( r+1 \) observation, so that we can have \( r \) concatenated states \( \{\tilde{Y}^i\}_{i=t-r}^{t} \).

Now, let us introduce the test statistics and the stopping rule by first defining the following function \( s : \mathbb{X}^r \to \mathbb{R} \),

\[
s(B^r_t) = \sqrt{\frac{1}{r^2} \sum_{1 \leq i,j \leq r} k(\tilde{Y}_i, \tilde{Y}_j) + \frac{1}{m^2} \sum_{1 \leq i,j \leq m} k(\tilde{X}_i, \tilde{X}_j) - \frac{2}{m^r} \sum_{i=1}^{r} \sum_{j=1}^{m} k(\tilde{Y}_i, \tilde{X}_j) - c_{X, \tilde{Y}}(m, r)}
\]

(6)

where \( c_{X, \tilde{Y}}(m, r) = \sqrt{\frac{1+\Sigma_{\tilde{X}}}{m}} + \sqrt{\frac{1+\Sigma_{\tilde{Y}}}{r}} \) is the estimation error bound defined in Lemma 4 and \( \Sigma_{\tilde{X}}, \Sigma_{\tilde{Y}} \) are defined in Assumption 2. Note that when the past observation and buffer have fixed length \( m \) and \( r \), \( c_{X, \tilde{Y}}(m, r) \) is a constant. Let the cumulative sum of the function \( s \) from \( t = k \) to \( n \) be \( S_{k:n} \), where \( k \geq 1 \) and \( n \geq k+1 \),

\[
S_{k:k} = s(B^r_k), \quad S_{k:n} = S_{k,n-1} + s(B^r_n).
\]

(7)

Define the test statistics \( \hat{S}_n \) and the corresponding stopping rule with threshold \( b \geq 0 \) and minimum sample \( M \in \mathbb{N} \) as:

\[
\hat{S}_n = \max_{1 \leq k \leq n-M} S_{k:n},
\]

(8)

\[
T(b, M) = \inf \left\{ n \in \mathbb{N} : \hat{S}_n \geq b \right\}.
\]

(9)

Let \( T \) be the stopping time as defined in Equation (9). For fixed sample size \( m \) under pre-change kernel \( P \) and a sliding window of size \( r \) under post-change kernel \( Q \), we have the following result on the detection performance, as defined in (4).
Theorem 5 (MTBFA Lower Bound) Suppose that Markov kernel $P$ satisfies Doeblin’s condition and $Q = P$. Denote $\tilde{P} = P^{\otimes r+1}$. There exists $\alpha_1 > 0$ such that for $b$ sufficiently large, we have

$$MTBFA(T(b, M)) \geq M - 1 + (b - \alpha_1)(1 + o(1)).$$

Proof Please refer to Appendix C.

Theorem 6 (MD Upper Bound) Suppose that $P$ is the pre-change kernel and $Q$ is the post-change kernel and they both satisfy Doeblin’s condition. Assume that MMD kernel $k$ is characteristic and $\|k\|_\infty \leq 1$. Then, there exists $\alpha > 0$ such that

$$MD(T(b, M)) \leq \max \left\{ M, \frac{b + \alpha}{D_r(P, Q)} \right\} (1 + o(1))$$

where $D_r(P, Q)$ is defined in (15).

Proof Please refer to Appendix D.

5. Numerical Simulations

In this section, we evaluate the performance of the kernel CUSUM change detector through numerical simulations. We use auto-regressive (AR) processes with an abrupt change in the noise statistics to mimic Markov chain with a change in the transition kernel. Let $A$ be a square matrix with spectral radius less than 1 and $\left\{ \omega_n \right\}_{n \in \mathbb{N}}$ be a sequence of i.i.d. samples drawn from some distribution. Two scenarios of change are simulated in our experiment: 1) change in the variance of $\omega_n$; 2) change in the mean of $\omega_n$. The state of the AR process $X_n$ evolves as

$$X_{n+1} = AX_n + \omega_n$$

We use the average of three Gaussian kernels with $\sigma = 10^{-1}, 1, 10$ to capture values of different scales. We set $X_n \in \mathbb{R}^4$ throughout all simulations. The system matrix $A$ is chosen with a spectral radius of 0.95. The pre-change system observations (reference data) are generated with $\omega_n$ iid drawn from a zero mean Gaussian random vector with an identity covariance matrix multiplied by 0.1. The following modification was done to the distribution of $\omega_n$.

1. Change in variance: after change point the covariance becomes identity matrix multiplied by 0.2, while the mean of the Gaussian vector stays zero;

2. Change in mean: after change point the mean of the Gaussian vector becomes 0.05, while the covariance stays the same as pre-change distribution;

Before each experiment, we calibrate the parameter $c_{\tilde{X}, \tilde{Y}}(m, r)$ by picking the appropriate value such that the MMD update term $s(B_r)$ are negative at each time step.

In Figures 1 and 2, we see that the kernel CUSUM change detector detects the change as expected. In both scenarios, the MMD update term turns to positive and the test statistics starts to grow shortly after the change point. Overall, the proposed kernel CUSUM change detector for Markov chain is able to detect the change when the problem lacks distributional assumption for both pre- and post-change Markov kernel.
6. Conclusion

In this study, we proposed a MMD based kernel CUSUM change detector for Markov chains. The advantages of our method includes: 1) completely data-driven with no assumption on both pre- and post-change Markov kernels; 2) works well with high dimensional data and compatible with general state space Markov chains. We derived a lower bound on the mean time between false alarm and an upper bound on the mean delay for the proposed algorithm. We further present two representative numerical simulations of change detection in autoregressive processes is presented to demonstrate the effectiveness of our method.

In the future, we will apply this change detection method for developing new dynamic watermarking algorithms for general nonlinear systems.

Appendix A. Proof of Lemma 1

There can be two cases: $\pi_P \neq \pi_Q$ and $\pi_P = \pi_Q$. In case $\pi_P \neq \pi_Q$, then it is obvious that $\pi_P \otimes P \neq \pi_Q \otimes Q$ since the marginal of the two measures are different. In the other case, if $\pi_P = \pi_Q$, then pick a closed set $A \subset X$ such that $P(x, \cdot) \neq Q(x, \cdot)$ for all $x \in A$ and $\pi_P(A) > 0$. Let $B_x, x \in A$ be any closed set such that $P(x, B_x) > Q(x, B_x)$, which exists due to Jordan-Hahn decomposition theorem. Consider the set $C = \bigcup_{x \in A} \{x\} \times B_x$ and note that $C$ is a measurable set since it is a closed set.

We claim that any section $\overline{C}_x := \{y : (x, y) \in \overline{C}\}$ is equal to $B_x$ for all $x \in A$. Contrary to the claim, suppose that $\overline{C}_x \neq B_x$ for some $\bar{x} \in A$ and pick a point $\bar{y} \in \overline{C}_x \setminus B_x$. Pick a sequence $\{(\bar{x}, \bar{y}_n)\}_{n \in \mathbb{N}} \subset C$ such that $(\bar{x}, \bar{y}_n) \to (\bar{x}, \bar{y}) \in C$. This implies that $\{\bar{y}_n\}_{n \in \mathbb{N}} \subset B_x$
and $\bar{y}_n \to \bar{y}$. Since $B_x$ is a closed set, we have $\bar{y} \in B_x$, which contradicts our assumption that $\overline{C}_x \neq B_x$. Thus, $\overline{C}_x = B_x$ for all $x \in A$.

From Proposition 3.3.2 in Bogachev (2007), we conclude that the functions $x \mapsto P(x, B_x)$ and $x \mapsto Q(x, B_x)$ are measurable functions since they are composition of measurable functions. Thus, we conclude that

$$\pi_P \otimes P(\overline{C}) = \int_A P(x, B_x) \pi_P(dx) > \int_A Q(x, B_x) \pi_P(dx) = \pi_Q \otimes Q(\overline{C}) = 0.$$ 

Thus, $\pi_P \otimes P \neq \pi_Q \otimes Q$ and the proof is complete.

**Appendix B. Proof of Lemma 2**

Given $(X_t)_{t \in \mathbb{N}}$ is uniformly ergodic with $(\Omega, \mathcal{F}, \mathbb{P})$, there exist a $V_X(x) \geq 1$ such that

$$PV_X(x) - V_X(x) \leq -\beta V_X(x) + b1_C(x),$$

for some $\beta > 0$, $b < \infty$, and some petite set $C$. As defined in Section 5.2 of Meyn and Tweedie (2012), there exists a $m > 0$, and a non-trivial measure $\nu_m$ on $\mathcal{B}(\Omega)$ such that for all $x \in C$, $A \in \mathcal{B}(\Omega)$, $P^m(x, A) \geq \nu_m(A)$. Now for the second order system $\tilde{X}_t = (X_t, X_{t+1})$, let $\tilde{V}_X(\tilde{X}_t) = V_X(X_{t+1}) \geq 1$, we have

$$P^{\otimes 2}V_X(\tilde{X}_t) - V_X(\tilde{X}_t) = PV_X(X_{t+1}) - V_X(X_{t+1}) \leq -(1 + \beta)V_X(X_{t+1}) + b1_C(X_{t+1}) \leq -(1 + \beta)V_X(\tilde{X}_t) + b1_{C_0 \times C}(\tilde{X}_t).$$

for some $C_0 \subseteq \mathcal{B}(\Omega)$ such that $P(C_0, C) > 0$. Next we need to show the petiteness of $C_0 \times C$ in the second order system. Let $\tilde{\nu}_{m+1}(C_1, C_2) = P(C_1, C_2)\nu_m(C_2)$, then for all $(x_1, x_2) \in C_0 \times C_1$, and $(A_1, A_2) \in \mathcal{B}(\Omega \times \Omega)$ we have

$$P^{\otimes 2, m+1}((x_1, x_2), (A_1, A_2)) = P^m(x_2, A_1)P(A_1, A_2) \geq \nu_m(A_1)P(A_1, A_2) = \tilde{\nu}_{m+1}(A_1, A_2).$$

Therefore by the same drift condition for the second order system, $(\tilde{X}_t)_{t \in \mathbb{N}}$ satisfies Doeblin’s condition.

**Appendix C. Proof of Theorem 5**

Obtaining MTBFA and MD in Appendix and relies on the concentration inequalities adopted for Markov Chains, especially the Hoeffding’s inequality for uniformly ergodic Markov Chains. These results are employed in our analysis to provide probabilistic bounds on the deviation of the test statistics from its mean.

**Theorem 7 (Hoeffding’s Inequality)** Suppose that Markov chain $\{X_t, t \geq 0\}$ is uniformly ergodic and let function $f : \mathbb{X} \to \mathbb{R}$ with $\|f\| = \sup_{x \in \mathbb{X}} f(x) < \infty$. Then, for any $\epsilon > 0$ and $n > \mu/\epsilon$, we have

$$\mathbb{P}(\|S_n - \mathbb{E}[S_n]\| \geq n\epsilon) \leq 2 \exp \left\{ -\frac{(n\epsilon - \mu)^2}{n\mu^2} \right\},$$

where $S_n = \sum_{t=1}^n f(X_t)$, $\mu = 2(l + 1)\|f\|/\lambda$, and $l, \lambda$ are described in Assumption 1.
For threshold \( b \geq 0 \), minimum sample \( M \), and stopping rule \( T(b, M) \), we have,
\[
\mathbb{E}_\infty [T(b, M)] = \sum_{C=1}^{\infty} \mathbb{P}_\infty \{T(b, M) > C\} = M - 1 + \sum_{C=M+1}^{\infty} \mathbb{P}_\infty \{T(b, M) > C\}.
\] (11)

Due to Lemma 2, \((B_t^r)_{t \geq r}\) is also a uniformly ergodic Markov chain with transition kernel \( \tilde{P} \triangleq P^{\otimes r+1} \) and invariant measure \( \pi_P \otimes P^{\otimes r} \). For \( t > r \), \( s(B_t^r) \) is a biased estimation of \( \gamma_k(\pi_P \otimes P^{\otimes r-1}, \pi_P \otimes P^{\otimes r-1}) = 0 \) minus the error bound in Lemma 4. Thus, we have
\[
0 \geq \mathbb{E}[s(B_t^r)] \geq -2c_{X,Y}(r, m).
\]
Let \( l_{\tilde{P}} \) and \( \lambda_{\tilde{P}} \) be the coefficients in Assumption 1 satisfied by Markov kernel \( \tilde{P} \). Let \( \alpha_1 := 2(l_{\tilde{P}} + 1)/\lambda_{\tilde{P}} \). Apply Theorem 7 to the following probability
when \( n > (b - \alpha_1)/\mathbb{E}_\infty [s(B_t^r)] \),
\[
\mathbb{P}_\infty (S_{1:n} > b) = \mathbb{P}_\infty \left( \sum_{t=0}^{n} s(B_t^r) > b \right) = \mathbb{P} \left( \sum_{t=0}^{n} s(B_t^r) - n\mathbb{E}_\infty [s(B_t^r)] > b - n\mathbb{E}_\infty [s(B_t^r)] \right)
\leq \exp \left( -2 \frac{(b - n\mathbb{E}_\infty [s(B_t^r)] - \alpha_1)^2}{n \sigma_1^2} \right) \leq \exp \left( -2 \frac{(b - \alpha_1)^2}{n \sigma_1^2} \right).
\] (12)

Expanding the right hand side of \ref{eq:11} and apply the inequality in \ref{eq:12}. After rearranging the terms, we arrive at
\[
\mathbb{P}_\infty \{T(b, M) > C\} = \sum_{c=M+1}^{\infty} (1 - \mathbb{P}_\infty \{T(b, M) \leq V\})
= \sum_{C=M+1}^{\infty} \left( 1 - \mathbb{P}_\infty \left\{ \bigcup_{n=M+1}^{C} \{T(b, M) = n\} \right\} \right)
\geq \sum_{C=M+1}^{L} \left( 1 - \mathbb{P}_\infty \left\{ \bigcup_{n=M+1}^{C} \bigcup_{k=1}^{n-M} \{S_{k:n} \geq b\} \right\} \right)
\geq \sum_{C=M+1}^{L} \left( 1 - \sum_{n=M+1}^{C} \sum_{k=1}^{n-M} \mathbb{P}_\infty \{S_{k:n} \geq b\} \right)
= \sum_{C=1}^{L-M} \left( 1 - \sum_{l=1}^{C} \sum_{k=1}^{l} \mathbb{P}_\infty \{S_{k:l+M} \geq b\} \right)
\geq \sum_{C=1}^{L-M} \left\{ 1 - \frac{1}{2} C(C - 1) \exp \left( -2 \frac{(b - \alpha_1)^2}{L \sigma_1^2} \right) \right\},
\] (13)

where \( L > M \). By choosing \( L \) as the solution of \( L = M + \exp \left( \frac{(b - \alpha_1)^2}{L \sigma_1^2} \right) \), it follows \( L \log L = (b - \alpha_1)^2(1 + o(1)) \). Applying the inequality on Lambert \( W \) function from Theorem 2.7 in \citet{Hoorfar and Hassani 2008}, we have \( L \leq \frac{(b - \alpha_1)^2}{\log(b - \alpha_1)} (1 + o(1)) \) when \( b \) is large. Plug into right hand side of \ref{eq:13}, we have for large \( b \),
\[
\sum_{C=M+1}^{\infty} \mathbb{P}_\infty \{T(b, M) > C\} \geq L - M - \frac{1}{6} (L - M)^3 \exp \left( -2 \frac{(b - \alpha_1)^2}{L \sigma_1^2} \right)
\geq \frac{5}{6} \exp \left( \frac{(b - \alpha_1)^2}{L \sigma_1^2} \right) \geq (b - \alpha_1)(1 + o(1))
\] (14)

Plugging this expression back in \ref{eq:11}, we get the desired bound.
Appendix D. Proof of Theorem 6

Since $P$ and $Q$ satisfy Assumption 2, so does the $\tilde{P}$ and $\tilde{Q}$ with weak dependent coefficient $\Sigma_{\tilde{P}}$ and $\Sigma_{\tilde{Q}}$ defined in (3.2). Note that for any $t > r$, $s(B_r^t)$ defined in (6) is an estimator of the MMD between $\tilde{P}$ and $\tilde{Q}$, using $m$ samples from $\tilde{P}$ and $r$ samples from $\tilde{Q}$. By Lemma 4, $s(B_r^t)$ is consistent such that

$$E[s(B_r^t)] \geq \gamma_k(\tilde{P}, \tilde{Q}) - 2c_{X,Y}(r, m) = D_r(P, Q),$$  \hspace{1cm} (15)$$

where $c_{X,Y}(r, m) = \sqrt{\frac{1 + 2\Sigma_{\tilde{P}}}{r}} + \sqrt{\frac{1 + 2\Sigma_{\tilde{Q}}}{m}}$. Assume $\tau = 0$, and for any $n > n_0 = \max\left\{ M, \frac{b + \alpha}{D_r(P, Q)} \right\}$, where $\alpha = 2m\tilde{Q}/\lambda_{\tilde{Q}}$, we have

$$P(S_{0:n} < b) = P\left( \sum_{t=0}^{n} s(B_r^t) < b \right) = P\left( \sum_{t=0}^{n} s(B_r^t) - nE[s(B_r^t)] < -(nE[s(B_r^t)] - b) \right) \leq \exp\left( -2\frac{(nE[s(B_r^t)] - \alpha - b)^2}{n\alpha^2} \right) \leq \exp\left( -2\frac{(nD_r(P, Q) - \alpha - b)^2}{n\alpha^2} \right)$$

where the first inequality follows from Theorem 7. The rest of the proof follows from Equation (14) and (15) in Xian et al. (2016), where $I(Q, P)$ is replaced with $D_r(P, Q)$. 

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