BOUND STATES BY A PSEUDOSCALAR COULOMB POTENTIAL IN ONE-PLUS-ONE DIMENSIONS

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Abstract

The Dirac equation is solved for a pseudoscalar Coulomb potential in a two-dimensional world. An infinite sequence of bounded solutions are obtained. These results are in sharp contrast with those ones obtained in 3+1 dimensions where no bound-state solutions are found. Next the general two-dimensional problem for pseudoscalar power-law potentials is addressed consenting us to conclude that a nonsingular potential leads to bounded solutions. The behaviour of the upper and lower components of the Dirac spinor for a confining linear potential nonconserving- as well as conserving-parity, even if the potential is unbounded from below, is discussed in some detail.
The Coulomb potential of a point electric charge in a 1+1 dimension, considered as the time component of a Lorentz vector, is linear and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [1]-[2] and in the Thirring-Schwinger model [3]. It is frustrating that, due to the tunneling effect (Klein’s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [4]-[5]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [6]-[7]. Recently it was incorrectly concluded that even in this case there is solely one bound state [8]. Later, the proper solutions for this last problem were found [9]-[11]. However, it is well known from the quarkonium phenomenology in the real 3+1 dimensional world that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, e.g., [12]). The mixed vector-scalar linear potential in 1+1 dimensions was recently considered [13]. There it was found that there are analytical bound-state solutions on condition that the scalar component of the potential is of sufficient strength compared to the vector component ($|V_s| \geq |V_t|$). As a by-product, that approach also showed that there exist relativistic confining potentials providing no bound-state solutions in the nonrelativistic limit. Although the discussion was confined to the vector-scalar mixing, the inclusion of a pseudoscalar potential could also be allowed.

In a recent paper, McKeon and Van Leeuwen [14] considered a nonconserving-parity pseudoscalar Coulomb (NCPPC) potential ($V = \lambda/r$) in 3+1 dimensions and concluded that there are no bounded solutions for the reason that the different parity eigenstates mix. Furthermore, they asserted that the absence of bound states in this system confuses the role of the $\pi$-meson in the binding of nucleons. Such an intriguing conclusion sets the stage for the analyses by other sorts of pseudoscalar potentials. A natural question to ask is if the absence of bounded solutions by a NCPPC potential is a characteristic feature of the four-dimensional world. With this in mind, we approach in the present paper the less realistic Dirac equation in one-plus-one dimensions with a NCPPC potential ($V = m\omega|x|$).

The two-dimensional Dirac equation can be obtained from the four-dimensional one with the mixture of spherically symmetric scalar, vector and anomalous magnetic-like (tensor) interactions. If we limit the fermion to move in the $x$-direction ($p_y = p_z = 0$) the four-dimensional Dirac equation decomposes into two equivalent two-dimensional equations with 2-component
spinors and 2×2 matrices \[15\]. Then, there results that the scalar and vector interactions preserve their Lorentz structures whereas the anomalous magnetic interaction turns out to be a pseudoscalar interaction. Furthermore, in the 1+1 world there is no angular momentum so that the spin is absent. Therefore, the 1+1 dimensional Dirac equation allow us to explore the physical consequences of the negative-energy states in a mathematically simpler and more physically transparent way. The confinement of fermions by a pure conserving-parity pseudoscalar double-step potential \[16\] and their scattering by a pure nonconserving-parity pseudoscalar step potential \[17\] have already been analyzed in the literature providing the opportunity to find some quite interesting results. Indeed, the two-dimensional version of the anomalous magnetic-like interaction linear in the radial coordinate, christened by Moshinsky and Szczepaniak \[18\] as Dirac oscillator, has also received attention. Nogami and Toyama \[19\], Toyama et al. \[20\] and Toyama and Nogami \[21\] studied the behaviour of wave packets under the influence of that conserving-parity potential whereas Szmytkowski and Gruchowski \[22\] proved the completeness of the eigenfunctions. More recently Pacheco et al. \[23\] studied some thermodynamics properties of the 1+1 dimensional Dirac oscillator.

Let us begin by presenting the Dirac equation in 1+1 dimensions. In the presence of a time-independent potential the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass \(m\) reads

\[
\mathcal{H}\Psi = E\Psi
\]

(1)

\[
\mathcal{H} = c\alpha p + \beta mc^2 + V
\]

(2)

where \(E\) is the energy of the fermion, \(c\) is the velocity of light and \(p\) is the momentum operator. \(\alpha\) and \(\beta\) are Hermitian square matrices satisfying the relations \(\alpha^2 = \beta^2 = 1\), \(\{\alpha, \beta\} = 0\). From the last two relations it steams that both \(\alpha\) and \(\beta\) are traceless and have eigenvalues equal to ±1, so that one can conclude that \(\alpha\) and \(\beta\) are even-dimensional matrices. One can choose the 2×2 Pauli matrices satisfying the same algebra as \(\alpha\) and \(\beta\), resulting in a 2-component spinor \(\Psi\). The positive definite function \(|\Psi|^2 = \Psi^\dagger\Psi\), satisfying a continuity equation, is interpreted as a probability position density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states \[21\]. We use \(\alpha = \sigma_1\) and \(\beta = \sigma_3\). For the potential
matrix we consider

\[ \mathcal{V} = 1V_t + \beta V_s + \alpha V_e + \beta \gamma^5 V_p \]  

(3)

where 1 stands for the \(2 \times 2\) identity matrix and \(\beta \gamma^5 = \sigma_2\). This is the most general combination of Lorentz structures for the potential matrix because there are only four linearly independent \(2 \times 2\) matrices. The subscripts for the terms of potential denote their properties under a Lorentz transformation: \(t\) and \(e\) for the time and space components of the 2-vector potential, \(s\) and \(p\) for the scalar and pseudoscalar terms, respectively. It is worth to note that the Dirac equation is covariant under \(x \rightarrow -x\) if \(V_e(x)\) and \(V_p(x)\) change sign whereas \(V_t(x)\) and \(V_s(x)\) remain the same. This is because the parity operator \(P = \exp(i\varepsilon)P_0\sigma_3\), where \(\varepsilon\) is a constant phase and \(P_0\) changes \(x\) into \(-x\), changes sign of \(\alpha\) and \(\beta \gamma^5\) but not of 1 and \(\beta\).

Defining the spinor \(\psi\) as

\[ \psi = \exp \left( \frac{i}{\hbar} \Lambda \right) \Psi \]  

(4)

where

\[ \Lambda(x) = \int_x^x dx' \frac{V_e(x')}{c} \]  

(5)

the space component of the vector potential is gauged away

\[ \left( p + \frac{V_e}{c} \right) \Psi = \exp \left( \frac{i}{\hbar} \Lambda \right) p\psi \]  

(6)

so that the time-independent Dirac equation can be rewritten as follows:

\[ H\psi = E\psi \]  

(7)

\[ H = \sigma_1 cp + \sigma_2 V_p + \sigma_3 \left( mc^2 + V_s \right) + 1V_t \]  

(8)

showing that the space component of a vector potential only contributes to change the spinors by a local phase factor.

Provided that the spinor is written in terms of the upper and the lower components

\[ \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \]  

(9)

the Dirac equation decomposes into:
\[
(V_t - E + V_s + mc^2) \phi(x) = i\hbar c \chi'(x) + iV_p \chi(x) \quad (10)
\]

\[
(V_t - E - V_s - mc^2) \chi(x) = i\hbar c \phi'(x) - iV_p \phi(x)
\]

where the prime denotes differentiation with respect to \( x \). In terms of \( \phi \) and \( \chi \) the spinor is normalized as \( \int_{-\infty}^{+\infty} dx (|\phi|^2 + |\chi|^2) = 1 \), so that \( \phi \) and \( \chi \) are square integrable functions. It is clear from the pair of coupled first-order differential equations (10) that both \( \phi \) and \( \chi \) must be discontinuous wherever the potential undergoes an infinite jump and have opposite parities if the Dirac equation is covariant under \( x \to -x \). In the nonrelativistic approximation (potential energies small compared to the rest mass) Eq. (10) loses all the matrix structure and becomes

\[
\chi = \frac{p}{2mc} \phi \quad (11)
\]

\[
\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_t + V_s + \frac{V_p^2}{2mc^2} + \frac{\hbar V_p'}{2mc}\right) \phi = (E - mc^2) \phi \quad (12)
\]

Eq. (11) shows that \( \chi \) if of order \( v/c << 1 \) relative to \( \phi \) and Eq. (12) shows that \( \phi \) obeys the Schrödinger equation without distinguishing the contributions of vector and scalar potentials (at this point the author digresses to make his apologies for mentioning in former papers (10)-[17] that the pseudoscalar potential does not present any contributions in the nonrelativistic limit). It is remarkable that the Dirac equation with a nonvector potential, or a vector potential contaminated with some scalar or pseudoscalar coupling, is not invariant under \( V \to V + \text{const.} \), this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas nonvector contaminants couple to the mass of the fermion. Therefore, if there is any nonvector coupling the absolute values of the energy will have physical significance and the freedom to choose a zero-energy will be lost. This last statement remains truthfully in the nonrelativistic limit if one considers that such a contaminant is a pseudoscalar potential.

Now, let us choose \( V_t = V_s = 0 \) and the intrinsically relativistic NCPPC potential \( V_p = m\omega c|x| \), where \( \omega \) is a real parameter. Although \( \omega < 0 \) gives rise to a potential unbounded from below, the possibility of such a sort of potential to bind fermions is already noticeable in the nonrelativistic limit of the
theory (see Eq. (12)), where the pseudoscalar linear potential, whether it is a conserving- or a nonconserving-parity potential, manifests itself effectively as a quadratic potential. Defining

$$\eta = \sqrt{\frac{2m|\omega|}{\hbar}} x$$

$$\nu = \frac{E^2 - m^2c^4}{2\hbar|\omega|mc^2} - \frac{1 + s(\omega)}{2}$$

where $s(\omega)$ stands for the sign function, the Dirac equation (10) turns into the Schrödinger-like differential equations

$$\begin{cases}
-d^2\phi\bigg/\bigg(d\eta\bigg)^2 + \frac{\eta^2}{4} \phi = (\nu + 1/2) \phi, & \eta > 0 \\
[\nu + 1/2 + s(\omega)] \phi, & \eta < 0
\end{cases}$$

$$\begin{cases}
-d^2\chi\bigg/\bigg(d\eta\bigg)^2 + \frac{\eta^2}{4} \chi = [\nu + 1/2 + s(\omega)] \chi, & \eta > 0 \\
(\nu + 1/2) \chi, & \eta < 0
\end{cases}$$

The second-order differential equations (15) have the form

$$y''(z) - \left(\frac{z^2}{4} + a\right) y(z) = 0,$$

whose solution is a parabolic cylinder function [25]. The solutions $D_{-a-1/2}(z)$ and $D_{-a-1/2}(-z)$ are linearly independent unless $n = -a - 1/2$ is a nonnegative integer. In that special circumstance $D_n(z)$ has the peculiar property that $D_n(-z) = (-1)^n D_n(z)$, and it is proportional to $\exp(-z^2/4) H_n(z/\sqrt{2})$, where $H_n(z)$ is a Hermite polynomial. The solutions of (15) do not exhibit this parity property so that we should not expect nonnegative integer values for $\nu$. The physically acceptable solutions for bound states must vanish in the asymptotic region $|\eta| \to \infty$ and are expressed as

$$\begin{align*}
\phi &= \theta(-\eta)C^{(-)}D_{\nu+s(\omega)}(-\eta) + \theta(+\eta)C^{(+)}D_\nu(\eta) \\
\chi &= \theta(-\eta)D^{(-)}D_\nu(-\eta) + \theta(+\eta)D^{(+)}D_{\nu+s(\omega)}(\eta)
\end{align*}$$

(17)
where $C^{(\pm)}$ and $D^{(\pm)}$ are normalization constants and $\theta(\eta)$ is the Heaviside function. Substituting the solutions (17) into the Dirac equation (10) and making use of the recurrence formulas

$$\frac{d}{dz} D_\nu(z) - \frac{z}{2} D_\nu(z) + D_{\nu+1}(z) = 0$$

(18)

$$\frac{d}{dz} D_\nu(z) + \frac{z}{2} D_\nu(z) - \nu D_{\nu-1}(z) = 0$$

one has as a result

$$\left[ \frac{C^{(+)}}{D^{(+)}} \right]^2 = \frac{E + mc^2}{E - mc^2} \times \left\{ \begin{array}{l l} - (\nu + 1), & \omega > 0 \\ -1/\nu, & \omega < 0 \end{array} \right.$$ \hspace{1cm} (19)

$$\left[ \frac{C^{(-)}}{D^{(-)}} \right]^2 = \frac{E + mc^2}{E - mc^2} \times \left\{ \begin{array}{l l} -1/ (\nu + 1), & \omega > 0 \\ -\nu, & \omega < 0 \end{array} \right.$$ \hspace{1cm} (19)

The continuity of the wavefunctions (17) at $\eta = 0$ furnishes

$$\frac{C^{(+)}}{D^{(+)}} = \frac{C^{(-)}}{D^{(-)}} \left[ \frac{D_{\nu+\delta(\omega)}(0)}{D_\nu(0)} \right]^2$$ \hspace{1cm} (20)

Together, (19) and (20) lead to the quantization condition

$$D_{\nu+(1+\delta(\omega))/2}(0) \pm \sqrt{\pm [\nu + (1 + s(\omega))/2]} D_{\nu-(1-\delta(\omega))/2}(0) = 0$$ \hspace{1cm} (21)

This last result combined with (21) shows that the use of the minus sign inside the radical demands that $-\rho < \nu < -1$ for $\omega > 0$, and $-\rho < \nu < 0$ for $\omega < 0$, here $\rho$ stands for $1 + m^2c^4/2\hbar|\omega|mc^2$. If, on the other side, one uses the plus sign then $\nu > -1$ for $\omega > 0$, and $\nu > 0$ for $\omega < 0$.

The numerical computation of (21), which can be done easily with a symbolic algebra program, reveals no solutions for $\nu < -1$. On the other side, for $\nu > -1$ an infinite sequence of allowed values of $\nu$ are found. The twenty lowest states for $\omega > 0$ are given in Table 1. The allowed values of $\nu$ for $\omega < 0$ are those ones for $\omega > 0$ added by the unit. By inspection of Table 1 one sees that for $\nu \to \infty$ their values show a tendency to half-integer
numbers. The energy levels are obtained by inserting those allowed values of \( \nu \) in (14):

\[
E = \pm \sqrt{m^2c^4 + 2\hbar|\omega|mc^2\{\nu + [1 + s(\omega)]/2\}}
\]  

(22)

One should realize that the energy levels are symmetrical about \( E = 0 \). It means that the potential couples to the positive-energy component of the spinor in the same way it couples to the negative-energy component. In other words, this sort of potential couples to the mass of the fermion instead of its charge so that there is no atmosphere for the production of particle-antiparticle pairs. No matter the intensity of the coupling parameter \( (\omega) \), the positive- and the negative-energy solutions never meet. There is always an energy gap greater or equal to \( 2mc^2 \), thus there is no room for transitions from positive-to negative-energy solutions. This all means that Klein´s paradox does not come to the scenario.

It is noticeable in (17) a somewhat left-right symmetry involving \( \phi \) and \( \chi \), such a symmetry is not exact inasmuch as \( C^{(\pm)} \neq D^{(\mp)} \). The upper and lower components of the spinor have the same number of zeros, or nodes, and as a consequence of such a quasi-symmetry the zeros of \( \phi \) and \( \chi \) exhibit, of course, an exact left-right symmetry. In whatever manner Eq. (19)-(20), and (21) of course, are invariant under the change \( E \to -E \) with the proviso that the normalization constants transforms as \( |C^{(\pm)}| \leftrightarrow |D^{(\mp)}| \), hence one can conclude that \( |\phi(\pm \eta)| \leftrightarrow |\chi(\mp \eta)| \) in such a way that \( |\psi(\pm \eta)| \to |\psi(\mp \eta)| \). If one recalls that the allowed values of \( \nu \) for \( \omega > 0 \) must be replaced by \( \nu + 1 \) for \( \omega < 0 \), a quasi-symmetry \( |\phi(\pm \eta)| \to |\phi(\mp \eta)| \) and \( |\chi(\pm \eta)| \to |\chi(\mp \eta)| \) can also be observed in (17) under the transformation \( \omega \to -\omega \). This symmetry is not exact because neither \( |C^{(\pm)}| \to |C^{(\pm)}| \) nor \( |D^{(\pm)}| \to |D^{(\mp)}| \). Figures 1–4 illustrate the behaviour of \( |\phi|^2, |\chi|^2 \) and \( |\psi|^2 = |\phi|^2 + |\chi|^2 \) for the ground-state contemplating all the possibilities of signs of \( E \) and \( \omega \). Comparison of these Figures shows that \( |\psi| \) tends to concentrate at the left (right) region when \( E \) and \( \omega \) have equal (different) signs. It is also evident that \( |\phi| \) is larger (smaller) than \( |\chi| \) when \( E > 0 \) (\( E < 0 \)). Note that \( |\phi| \) always tends to concentrate at the left when \( \omega > 0 \) (\( \omega < 0 \)). Figures 5 and Figure 6 illustrate the same things as Figure 1 for the first- and second-excited-states. Note from all these Figures that the probability position density has no nodes and the number of nodes of \( \phi \), which is equal to the one of \( \chi \), increases with \( \nu \). Furthermore, \( |\phi| \) is larger (smaller) than \( |\chi| \) when \( E > 0 \) (\( E < 0 \)), independently of the sign of \( \omega \), as might be expected.
In addition to their intrinsic importance, the above conclusions render a contrast to the result found in [14]. There are bound-state solutions for fermions interacting by a pseudoscalar Coulomb potential in 1+1 dimensions, to tell the truth there is confinement, notwithstanding the spinor is not an eigenfunction of the parity operator. Therefore, the quadri-dimensional version of this problem requires clarification for the absence of bound solutions. One might ponder that the underlying reason is the way the spinors are affected by the behavior of the potentials at the origin as well as at infinity because this is the radical difference between the potentials in those two dissimilar worlds. In order to clarify this point let us consider (10) for a general pseudoscalar potential. Those equations involve the coupling between the upper and lower components of the spinor. This coupling can be formally eliminated when these equations are written as second-order differential equations:

\[-\frac{\hbar^2}{2m} \frac{d^2 \Phi}{dx^2} + V_{\text{eff}}^\pm \Phi = E_{\text{eff}} \Phi (23)\]

where \(\Phi\) refers to \(\phi\) or \(\chi\) and

\[E_{\text{eff}} = \frac{E^2 - m^2 c^4}{2mc^2}\]

\[V_{\text{eff}}^\pm = \frac{V_p^2}{2mc^2} \pm \frac{\hbar V_p'}{2mc}, \quad \text{+ for } \phi \quad - \text{ for } \chi\]

these last results show that the solution for this class of problem consists in searching for bounded solutions for two Schrödinger equations. It should not be forgotten, though, that the equations for \(\phi\) or \(\chi\) are not indeed independent because the effective eigenvalue, \(E_{\text{eff}}\), appears in both equations. Therefore, one has to search for bound-state solutions for \(V_{\text{eff}}^\pm\) with a common eigenvalue. Now let us consider a nonconserving-parity pseudoscalar potential in the form \(V = \mu |x|^\delta\), then the effective potential becomes

\[V_{\text{eff}}^\pm = \frac{\mu^2}{2mc^2} |x|^{2\delta} \pm s(x) \frac{\hbar \mu \delta}{2mc} |x|^\delta \quad (25)\]

Firstly let us consider the case \(\mu > 0\). When \(\delta > 0\) the effective potential goes to infinity as \(|x| \to \infty\) and it is finite at the origin for \(\delta \geq 1\) whereas for
\[ 0 < \delta < 1 \] it is has a singularity given by \( \pm s(x)\mu \delta / 2mc|x|^{1-\delta} \), implying for \( x > 0 \) in a potential-well structure for \( V_{\text{eff}}^+ \) and an attractive potential less singular than \(-1/8mx^2\) for \( V_{\text{eff}}^- \). The roles of \( V_{\text{eff}}^+ \) and \( V_{\text{eff}}^- \) are changed for \( x < 0 \). Therefore, for \( \delta > 0 \) the power potential leads to effective potentials fulfilling the conditions to furnish discrete spectra. On the other hand, when \( \delta < 0 \) the effective potential vanishes as \( |x| \to \infty \) and \( V_{\text{eff}}^- (V_{\text{eff}}^+) \) is always repulsive at the origin for \( x > 0 \) (\( x < 0 \)) in such a way that it is repulsive everywhere. Therefore, for \( \delta < 0 \) the power-law potential does not lead to bound-state solutions. Note that these conclusions, either for \( \delta > 0 \) or \( \delta < 0 \), are independent of the sign of \( \mu \) on the condition that one has to change \( V_{\text{eff}}^\pm \) to \( V_{\text{eff}}^\mp \) if one changes the sign of \( \mu \). If one considers the conserving-parity potential \( V = \mu x^\delta \) one obtains

\[
V_{\text{eff}}^\pm = \frac{\mu^2}{2mc^2}x^{2\delta} \pm \frac{\hbar \mu \delta}{2mc}x^{\delta-1}
\]  

(26)

and a similar analyzes also permits us to conclude that only when \( \delta > 0 \) one could expect to find bounded solutions.

Now then, we return to focus our attention on the linear potential. Both cases, if nonconserving- or conserving-parity, present potential-well structures doing what is required to supply purely discrete spectra with infinite sequences of eigenvalues, in other words they are confining potentials, as it does every pseudoscalar power-law potential with \( \delta > 0 \). The nonconserving-parity potential \( V = m\omega c|x| \) gives rise to the effective potential \( V_{\text{eff}}^\pm = m\omega^2x^2/2 \pm s(x)\hbar \omega/2 \), an asymmetric potential with a discontinuity at \( x = 0 \) equal to \( \hbar \omega \). Now one can conceive why \( |\psi| \) and \( |\phi| \) are more concentrated at the left region in the case \( E > 0 \) and \( \omega > 0 \). This behaviour is precisely what one would expect for such an asymmetric potential. On the other hand, the change \( \omega \to -\omega \) results in \( V_{\text{eff}}^\pm \to V_{\text{eff}}^\mp \) and one should expect that \( |\psi| \) and \( |\phi| \) be more concentrated at the right side. In addition, when the energy is sufficiently large (\( \nu \) large) compared to \( \hbar \omega \) the discontinuity of the effective potential at the origin becomes unimportant and the fermion effectively undergo the influence of the average effective potential \((m\omega^2x^2/2)\) maintaining their levels (in the sense of \( E_{\text{eff}} \)) nearly equally spaced. This interpretation is reinforced by observing Figures 1, 5 and 6, there one can see the propensity of \( |\psi| \) to be closer to the origin for larger energies. As a matter of fact, a numerical calculation of the expectation value of \( \eta \) (with \( m = \omega = c = \hbar = 1 \)) furnishes \(-0.505, -0.121 \) and \(-0.084 \) for the ground-, first-excited and second-excited-states, respectively.

\[ 9 \]
The conserving-parity potential $V = m\omega cx$ (the two-dimensional generalized Dirac oscillator) is escorted by the effective potential $V_{\text{eff}}^\pm = m\omega^2 x^2/2 \pm \hbar \omega/2$, which in addition to be continuous at the origin is even under $x \rightarrow -x$.

In this case Eq. (15) for $\eta < 0$ must be disregarded and the equations valid for $\eta > 0$ must also be true for $\eta < 0$, so the upper and lower components of the Dirac spinor are given by

$$\phi = \theta(-\eta) C(-\eta) D_\nu(-\eta) + \theta(+\eta) C(+\eta) D_\nu(\eta)$$

$$\chi = \theta(-\eta) D(-\eta) D_{\nu+\nu(\omega)}(-\eta) + \theta(+\eta) D(+\eta) D_{\nu+\nu(\omega)}(\eta)$$

and the eigenvalues are even expressed by (22), nevertheless the boundary conditions implies that $C(+\eta) = C(-\eta)$ and $D(+\eta) = D(-\eta)$, with

$$\left[\frac{C(+\eta)}{D(+\eta)}\right]^2 = \frac{E + mc^2}{E - mc^2} \times \begin{cases} -(\nu + 1), & \omega > 0 \\ -1/\nu, & \omega < 0 \end{cases}$$

In addition, the boundary conditions implies, for $\omega > 0$, that $D_\nu(0) = 0$ with allowed values $\nu = 1, 3, 5, \ldots$ and $D'_\nu(0) = 0$ with allowed values $\nu = 0, 2, 4, \ldots$. In this case $D_\nu(-\eta) = (-1)^\nu D_\nu(\eta)$, and $D_\nu(\eta)$ is proportional to $\exp(-\eta^2/4) H_\nu(\eta/\sqrt{2})$, where $H_\nu(\eta)$ is a Hermite polynomial (for $\omega < 0$ one has to change the allowed values of $\nu$ by $\nu + 1$). The upper and lower components of the spinor have definite and opposite parities under $\eta \rightarrow -\eta$ and, since the Hermite polynomial $H_\nu(\eta)$ has $\nu$ different zeros, the number of nodes of $\chi$ is the number of nodes of $\phi$ plus (minus) one for $\omega > 0$ ($\omega < 0$). This fact can be better understood by keeping in mind that $V_{\text{eff}}^\pm = V_{\text{eff}} = \pm \hbar \omega$. The energy levels (in the sense of $E_{\text{eff}}$) are always equally spaced because $\phi$ as much as $\chi$ are affected by continuous and symmetric harmonic potentials. The symmetry already remarked for the potential $V = m\omega c|x|$ translates into $|C(+\eta)| \leftrightarrow |D(+\eta)|$ and $|\phi(\eta)| \leftrightarrow |\chi(\eta)|$, under the change $E \rightarrow -E$, in such a manner that $|\psi|$ is invariant. Also a quasi-symmetry is present under $\omega \rightarrow -\omega$: $|\phi(\eta)| \leftrightarrow |\chi(\eta)|$. In this case the symmetry is not perfect because neither $|C(+\eta)| \rightarrow |D(+\eta)|$ nor $|D(+\eta)| \rightarrow |C(+\eta)|$. All in all, $|\phi|$ is larger (smaller) than $|\chi|$ when $E > 0$ ($E < 0$), independently of the sign of $\omega$.

For short, in one-plus-one dimensions, we have succeed in searching bound-state solutions for the Coulomb potential as well as for the generalized Dirac oscillator. These problems are somehow similar as long as they are affected by effective harmonic potentials and they, in fact, possess the same formal
expression for the eigenvalues. Nonetheless, the quantum numbers and their related eigenfunctions differ because those different problems are subject to distinct boundary conditions.

As stated in the third paragraph of this work, the anomalous magnetic-like interaction in the four-dimensional world turns into a pseudoscalar interaction in the two-dimensional world. The anomalous magnetic interaction has the form $-i\mu\beta\vec{\alpha} \cdot \vec{\nabla} \phi(r)$, where $\mu$ is the anomalous magnetic moment in units of the Bohr magneton and $\phi$ is the electric potential, i.e., the time component of a vector potential [24]. In one-plus-one dimensions the anomalous magnetic interaction turns into $\sigma_2\mu\phi'$, then one might suppose that the pseudoscalar Coulomb potential is due to an electric potential proportional to $s(x)x^2/2$. The oddness of the electric potential (under $x \rightarrow -x$) does not present problem to the confinement of a fermion because its effective mass, an $x$-dependent mass which always increases as the fermion goes away from the origin, depends on $(\phi')^2$, an result independent of the sign of $s(x)$ [16]-[17].

A word should be said about the role of the $\pi$-meson field. The Lagrangian density describing the pion-nucleon interaction $\mathcal{L} = -i\lambda \bar{\psi}\gamma_5 \psi \pi$ is parity-invariant because the $\pi$-meson is a pseudoscalar field, i.e., $\pi(\vec{r},t) \rightarrow -\pi(-\vec{r},t)$ under the parity transformation. Nonetheless, the interaction matrix term present in the Dirac equation as written by McKeon and Van Leeuwen [14], $i\beta\gamma_5 V(r)$, supposed to be due to the $\pi$-meson field is not parity-invariant due to the reason that the potential function, $V(r)$, is parity-invariant but the potential matrix is not. This argument exposes the inadequacy of the interaction term in the Dirac equation as proposed in [14]. Therefore, any conundrum about the role of the $\pi$-meson should be consistently presented by taking into account a quintessential parity-invariant potential. Furthermore, in order to correspond to the physical reality one should be aware that the massive $\pi$-meson field gives rise to a Yukawa potential instead of a Coulomb potential.

The methodology of effective potentials prompted in this paper might be extended for the four-dimensional world. Such a methodology seems more difficult there because the upper and lower components of the Dirac spinor involve not only radial functions but also angular functions (the two-component spinor spherical harmonics). Probably, we have to deal with four coupled second-order differential equations. This task of carry the Dirac equation through equivalent Sturm-Liouville problems is under way and will be reported in due time.
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References

[1] S. Coleman, R. Jackiw and L. Susskind, Ann. Phys. (N.Y.) 93 (1975) 267.
[2] S. Coleman, Ann. Phys. (N.Y.) 101 (1976) 239.
[3] J. Fröhlich and E. Seiler, Helv. Phys. Acta 49 (1976) 889.
[4] A.Z. Capri and R. Ferrari, Can. J. Phys. 63 (1985) 1029.
[5] H. Galić, Am. J. Phys. 56 (1988) 312.
[6] G. ’t Hooft, Nucl. Phys. B 75 (1974) 461.
[7] J. Kogut and L. Susskind, Phys. Rev. D 9 (1974) 3501.
[8] R.S. Bhalerao and B. Ram, Am. J. Phys. 69 (2001) 817.
[9] A. S. de Castro, Am. J. Phys. 70 (2002) 450.
[10] R.M. Cavalcanti, Am. J. Phys. 70 (2002) 451.
[11] J.R. Hiller, Am. J. Phys. 70 (2002) 522.
[12] W. Lucha, F.F. Schöberl and D. Gromes, Phys. Rep. 200 (1991) 127 and references therein.
[13] A.S. de Castro, Phys. Lett. A 305 (2002) 100.
[14] D.G.C. McKeon and G. Van Leeuwen, Mod. Phys. Lett. A 17 (2002) 1961.
[15] P. Strange, Relativistic Quantum Mechanics, Cambridge University Press, Cambridge, 1998.
[16] A.S. de Castro and W.G. Pereira, Phys. Lett. A 308 (2003) 131.
[17] A.S. de Castro, Phys. Lett. A 309 (2003) 340.
[18] M. Moshinsky and A. Szczepaniak, J. Phys. A 22 (1989) L817.
[19] Y. Nogami and F.M. Toyama, Can. J. Phys. 74 (1996) 114.
[20] F.M. Toyama, Y. Nogami and F.A.B. Coutinho, J. Phys. A 30 (1997) 2585.

[21] F.M. Toyama and Y. Nogami, Phys. Rev. A 59 (1999) 1056.

[22] R. Szmytkowski and M. Gruchowski, J. Phys. A 34 (2001) 4991.

[23] M.H. Pacheco, R. Landim and C.A.S. Almeida, Phys. Lett. A 311 (2003) 93.

[24] B. Thaller, The Dirac Equation (Springer-Verlag, Berlin, 1992).

[25] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, Toronto, 1965).
Figure captions

Figure 1 - $|\phi|^2$ (full thin line), $|\chi|^2$ (dashed line) and $|\phi|^2 + |\chi|^2$ (full thick line), corresponding to positive-ground-state energy for the potential $V = m\omega c|x|$ with $\omega > 0$ and $m = |\omega| = c = \hbar = 1$.

Figure 2 - The same as in Figure 1 with $\omega < 0$.

Figure 3 - The same as in Figure 1 corresponding to negative-ground-state energy ($\omega > 0$).

Figure 4 - The same as in Figure 1 corresponding to negative-ground-state energy with $\omega < 0$.

Figure 5 - The same as in Figure 1 corresponding to positive-first-excited-state energy ($\omega > 0$).

Figure 6 - The same as in Figure 1 corresponding to positive-second-excited-state energy ($\omega > 0$).
Table 1: The lowest allowed values of $\nu$, for $\omega > 0$, such that Eq. (21) is satisfied.

| $\nu$ for plus sign in front of the radical | $\nu$ for minus sign in front of the radical |
|--------------------------------------------|---------------------------------------------|
| -0.654541                                  | 0.5485571                                   |
| 1.468573                                   | 2.522295                                    |
| 3.482395                                   | 4.514350                                    |
| 5.487785                                   | 6.510563                                    |
| 7.490650                                   | 8.508353                                    |
| 9.492428                                   | 10.506906                                   |
| 11.493638                                  | 12.505886                                   |
| 13.494514                                  | 14.505129                                   |
| 15.495179                                  | 16.504543                                   |
| 17.495700                                  | 18.504078                                   |
