Linear Volterra backward stochastic differential equations

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Abstract

We present an explicit solution triplet \((Y, Z, K)\) to the backward stochastic Volterra integral equation (BSVIE) of linear type, driven by a Brownian motion and a compensated Poisson random measure. The process \(Y\) is expressed by an integral whose kernel is explicitly given. The processes \(Z\) and \(K\) are expressed by Hida-Malliavin derivatives involving \(Y\).

1 Introduction and main theorem

Backward stochastic Volterra integral equations (BSVIEs) were introduced to solve stochastic optimal control problems for controlled Volterra type systems. Yong (see e.g. \cite{8}, \cite{9}) gives a systematic study, including the existence and uniqueness of the solution of general nonlinear equations. In general the classical backward stochastic differential equations are hard to solve, and those of Volterra type are even worse. In this paper we aim to present an explicit solution formula for linear BSVIEs.

To present our main result we start with a basic probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with a filtration \(\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}\) satisfying the usual condition. Let \((B(t), 0 \leq t \leq T)\) be a

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one dimensional Brownian motion and let \( \{N(t, A), 0 \leq t \leq T, A \in \mathcal{B}(\mathbb{R})\} \) be an independent Poisson random measure with Lévy measure \( \nu \). Both the Brownian motion \( B \) and Poisson random measure \( N \) are adapted to the filtration \( \mathbb{F} \). Let \( \{F(t), 0 \leq t \leq T\} \) be a given stochastic process, not necessarily adapted.

Let \( (\Phi(t, s), 0 \leq t < s \leq T) \) and \((\xi(s), \beta(s, \zeta); 0 \leq s \leq T, \zeta \in \mathbb{R})\) be given measurable functions of \( t, s, \) and \( \zeta \), with values in \( \mathbb{R} \). For simplicity we assume that these functions are bounded, and we assume that there exists \( \epsilon > 0 \) such that \( \beta(s, \zeta) \geq -1 + \epsilon \) for all \( s, \zeta \). We consider the following linear backward stochastic Volterra integral equations in the unknown process triplet \((Y(t), Z(t, s), K(t, s, \zeta))\):

\[
Y(t) = F(t) + \int_t^T \left[ \Phi(t, s)Y(s) + \xi(s)Z(t, s) + \int_\mathbb{R} \beta(s, \zeta)K(t, s, \zeta)\nu(d\zeta) \right] ds \\
- \int_t^T Z(t, s)dB(s) - \int_t^T \int_\mathbb{R} K(t, s, \zeta)\tilde{N}(ds, d\zeta),
\]

where \( 0 \leq t \leq T \) and \( \tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt \) is the compensated Poisson random measure. We want to find three processes \((Y(t), Z(t, s), K(t, s, \zeta))\) which are adapted (meaning that \( Y(s), Z(t, s), K(t, s, \zeta) \in \mathcal{F}_s \) for any \( s \in [t, T] \) and any \( \zeta \in \mathbb{R} \)) such that the above equation (1.1) is satisfied.

To find such a solution, we first try to get rid of the unknowns \( Z(t, s) \) and \( K(t, s, \zeta) \) inside the first integral in (1.1). To this end, we define the probability measure \( Q \) by

\[
dQ = M(T)dP \text{ on } \mathcal{F}_T,
\]

where

\[
M(t) := \exp \left( \int_0^t \xi(s)dB(s) - \frac{1}{2} \int_0^t \xi^2(s)ds + \int_0^t \int_\mathbb{R} \ln(1 + \beta(s, \zeta))\tilde{N}(ds, d\zeta) \\
+ \int_0^t \int_\mathbb{R} \{\ln(1 + \beta(s, \zeta)) - \beta(s, \zeta)\}\nu(d\zeta)ds \right), \quad 0 \leq t \leq T.
\]

Then (see e.g. [7] Ch.1) under the probability measure \( Q \) the process

\[
B_Q(t) := B(t) - \int_0^t \xi(s)ds, \quad 0 \leq t \leq T.
\]

is a Brownian motion, and the random measure

\[
\tilde{N}_Q(dt, d\zeta) := \tilde{N}(dt, d\zeta) - \beta(t, \zeta)\nu(d\zeta)dt
\]

is the \( Q \)-compensated Poisson random measure of \( N(\cdot, \cdot) \), in the sense that the process

\[
\tilde{N}_Q(t) := \int_0^t \int_\mathbb{R} \gamma(s, \zeta)\tilde{N}_Q(ds, d\zeta)
\]
is a local $\mathbb{Q}$-martingale, for all predictable processes $\gamma(t, \zeta)$ such that
\[
\int_0^T \int_{\mathbb{R}} \gamma^2(t, \zeta)\beta^2(t, \zeta)\nu(d\zeta)dt < \infty. \tag{1.6}
\]
We also introduce, for $0 \leq t \leq r \leq T$,
\[
\Phi^{(1)}(t, r) = \Phi(t, r), \quad \Phi^{(2)}(t, r) = \int_t^r \Phi(t, s)\Phi(s, r)ds
\]
and inductively
\[
\Phi^{(n)}(t, r) = \int_t^r \Phi^{(n-1)}(t, s)\Phi(s, r)ds, \quad n = 3, 4, \cdots. \tag{1.7}
\]

Remark 1.1. Note that if $|\Phi(t, r)| \leq C$ (constant) for all $t, r$, then by induction
\[
|\Phi^{(n)}(t, r)| \leq \frac{C^nT^n}{n!} \tag{1.8}
\]
for all $t, r, n$. Hence,
\[
\sum_{n=1}^{\infty} |\Phi^{(n)}(t, r)| < \infty \tag{1.9}
\]
for all $t, r$.

With these notations, we can state our main theorem of this paper as follows:

**Theorem 1.2.** Put
\[
\Psi(t, r) = \sum_{n=1}^{\infty} \Phi^{(n)}(t, r). \tag{1.10}
\]

Then we have the following explicit form of the solution triplet:

(i) The $Y$ component of the solution triplet is given by
\[
Y(t) = \mathbb{E}_Q \left[ F(t) \mid \mathcal{F}_t \right] + \int_t^T \mathbb{E}_Q \left[ F(r) \mid \mathcal{F}_t \right] dr = \mathbb{E}_Q \left[ F(t) + \int_t^T \Psi(t, r)F(r)dr \mid \mathcal{F}_t \right]. \tag{1.11}
\]

(i) The $Z$ and $K$ components of the solution triplet are given by the following: Define
\[
U(t) = F(t) + \int_t^T \Phi(t, r)Y(r)dr - Y(t); \quad 0 \leq t \leq T. \tag{1.12}
\]
Then $Z(t, s)$ and $K(t, s, \zeta)$ can be expressed by the Hida-Malliavin derivatives $D_s$ and $D_{s, \zeta}$ with respect to $B$ and $N$, respectively, as follows:

$$Z(t, s) = \mathbb{E}_Q[D_s U(t) - U(t) \int_s^T D_s \xi(r) dB_Q(r)|\mathcal{F}_s]; \quad 0 \leq t \leq s \leq T \tag{1.13}$$

and

$$K(t, s, \zeta) = \mathbb{E}_Q[U(t)(\tilde{H}_s - 1) + \tilde{H}_s D_{s, \zeta} U(t)|\mathcal{F}_s]; \quad 0 \leq t \leq s \leq T, \tag{1.14}$$

where

$$\tilde{H}_s = \exp \left[ \int_0^s \int_{\mathbb{R}} \left[ D_{s, \beta}(r, x) + \log(1 - \frac{D_{s, \beta}(r, x)}{1 - \beta(r, x)}) \right] \nu(dx)dr \right. + \left. \int_0^s \int_{\mathbb{R}} \log(1 - \frac{D_{s, \beta}(r, x)}{1 - \beta(r, x)}) \tilde{N}_Q(dr, dx) \right]. \tag{1.15}$$

**Proof.** With the processes $B_Q$ and $\tilde{N}_Q$ defined in (1.4)-(1.5) we can eliminate the unknowns $Z(t, s)$ and $K(t, s, \zeta)$ inside the first integral in (1.1). More precisely, we can rewrite equation (1.1) as

$$Y(t) = F(t) + \int_t^T \Phi(t, s) Y(s) ds - \int_t^T Z(t, s) dB_Q(s) - \int_t^T \int_{\mathbb{R}} K(t, s, \zeta) \tilde{N}_Q(ds, d\zeta), \tag{1.16}$$

where $0 \leq t \leq T$. Taking the conditional $Q$-expectation on $F_t$, we get

$$Y(t) = \mathbb{E}_Q \left[ F(t) + \int_t^T \Phi(t, s) Y(s) ds | \mathcal{F}_t \right] = \tilde{F}(t, t) + \int_t^T \Phi(t, s) \mathbb{E}_Q \left[ Y(s) | \mathcal{F}_t \right] ds, \quad 0 \leq t \leq T. \tag{1.17}$$

Here, and in what follows, we denote

$$\tilde{F}(t, s) = \mathbb{E}_Q \left[ F(t) | \mathcal{F}_t \right]. \tag{1.18}$$

Fix $r \in [0, t]$. Taking the conditional $Q$-expectation on $\mathcal{F}_r$ of (1.17), we get

$$\mathbb{E}_Q \left[ Y(t) | \mathcal{F}_r \right] = \tilde{F}(t, r) + \int_t^T \Phi(t, s) \mathbb{E}_Q \left[ Y(s) | \mathcal{F}_r \right] ds, \quad r \leq t \leq T \tag{1.19}$$

Denote

$$\tilde{Y}(s) = \mathbb{E}_Q \left[ Y(s) | \mathcal{F}_r \right], \quad r \leq s \leq T.$$

Then the above equation can be written as

$$\tilde{Y}(t) = \tilde{F}(t, r) + \int_t^T \Phi(t, s) \tilde{Y}(s) ds, \quad r \leq t \leq T.$$
Substituting \( \tilde{Y}(s) = \tilde{F}(s, r) + \int_s^T \Phi(s, u) \tilde{Y}(u)du \) in the above equation, we obtain

\[
\tilde{Y}(t) = \tilde{F}(t, r) + \int_t^T \Phi(t, s) \left\{ \tilde{F}(s, r) + \int_s^T \Phi(s, u) \tilde{Y}(u)du \right\} ds
\]

\[= \tilde{F}(t, r) + \int_t^T \Phi(t, s) \tilde{F}(s, r)ds + \int_t^T \Phi^{(2)}(t, u) \tilde{Y}(u)duds, \quad r \leq t \leq T,
\]

By repeatedly using the above argument, we get

\[
\tilde{Y}(t) = \tilde{F}(t, r) + \sum_{n=1}^{\infty} \int_t^T \Phi^{(n)}(t, u) \tilde{F}(u, r)du
\]

\[= \tilde{F}(t, r) + \int_t^T \Psi(t, u) \tilde{F}(u, r)du,
\]

where \( \Psi \) is defined by (1.10). Now substituting \( \mathbb{E}_Q(Y(s)|\mathcal{F}_t) = \tilde{Y}(s) \) (with \( r = t \)) into (1.17) we obtain part (i) of the theorem.

It remains to prove (1.13)-(1.14). By (1.16) we have

\[
U(t) = \int_t^T Z(t, s)dB_Q(s) + \int_t^T \int_{\mathbb{R}} K(t, s, \zeta) \tilde{N}_Q(ds, d\zeta); \quad 0 \leq t \leq s \leq T.
\]

(1.21)

Note that by the Clark-Ocone formula under change of measure (see \([5], [6]\)), extended to \( L^2(\mathbb{Q}, \mathcal{F}_T) \) as in \([1]\), we get

\[
Z(t, s) = \mathbb{E}_Q[D_sU(t) - U(t) \int_s^T \int_{\mathbb{R}} D_s \zeta(r)dB_Q(r)|\mathcal{F}_s]; \quad t \leq s \leq T
\]

(1.22)

and

\[
K(t, s, \zeta) = \mathbb{E}_Q[U(t)(\tilde{H}_s - 1) + \tilde{H}_s D_s \zeta U(t)|\mathcal{F}_s]; \quad t \leq s \leq T
\]

(1.23)

where

\[
\tilde{H}_s = \exp \left[ \int_0^s \int_{\mathbb{R}} [D_{s, x} \beta(r, x) + \log(1 - D_{s, x} \beta(r, x))] \nu(dx)dr + \int_0^s \int_{\mathbb{R}} \log(1 - D_{s, x} \beta(r, x)) \tilde{N}_Q(dr, dx) \right],
\]

(1.24)

as claimed.

\[\square\]

2 Application to smoothness of the solution triplet

It is of interest to study when the solution components \( Z(t, s), K(t, s, \zeta) \) are smooth \( (C^1) \) with respect to \( t \). Such smoothness properties are important in the study of optimal control (see e.g. \([3]\)). It is also important in the numerical solutions (see e.g. \([4]\) and references therein). Using the explicit form of the solution triplet (Theorem 1.2) we can give sufficient conditions for such smoothness in the linear case.
Theorem 2.1. Assume that $\xi, \beta$ are deterministic and that $F(t)$ and $\Phi(t, s)$ are $C^1$ with respect to $t$ satisfying

$$
E_Q \left[ \int_0^T \left( \int_t^T \left\{ F^2(t) + \Phi^2(t, s) + \left( \frac{dF(t)}{dt} \right)^2 + \left( \frac{\partial \Phi(t, s)}{\partial t} \right)^2 \right\} ds \right] dt < \infty. \tag{2.1}
$$

Then, for $t < s \leq T$,

$$
Z(t, s) = E_Q[D_s F(t)] + \int_t^T D_s \phi(t, r) Y(r)dr|F_s], \tag{2.2}
$$

$$
K(t, s, \zeta) = E_Q[D_{s, \zeta} F(t)] + \int_t^T D_{s, \zeta} \phi(t, r) Y(r)dr|F_s]. \tag{2.3}
$$

In particular, we have

$$
E_Q \left[ \int_0^T \left( \int_t^T \left( \frac{\partial Z(t, s)}{\partial t} \right)^2 ds \right) dt + \int_0^T \left( \int_t^T \int_\mathbb{R} \left( \frac{\partial K(t, s, \zeta)}{\partial t} \right)^2 \nu(\text{d}\zeta) ds \right) dt \right] < \infty. \tag{2.4}
$$

Proof. Since $Y(t)$ is $\mathcal{F}_t$-measurable, we get that $D_s Y(t) = D_{s, \zeta} Y(t) = 0$ for all $s > t$. Hence by (1.12)

$$
E_Q[D_s U(t)|\mathcal{F}_s] = E_Q[D_s F(t)] + \int_t^T D_s \phi(t, r) Y(r)dr|\mathcal{F}_s] \tag{2.5}
$$

and

$$
E_Q[D_{s, \zeta} U(t)|\mathcal{F}_s] = E_Q[D_{s, \zeta} F(t)] + \int_t^T D_{s, \zeta} \phi(t, r) Y(r)dr|\mathcal{F}_s]. \tag{2.6}
$$

Then the result follows from (1.13) and (1.14). \qed

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