ON APPROXIMATION CONSTANTS FOR LIOUVILLE NUMBERS

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Abstract. We investigate the Diophantine approximation constants \(\lambda_{k,j}(\zeta), \hat{\lambda}_{k,j}(\zeta)\) for \(1 \leq j \leq k + 1\) corresponding to the classical one-parameter successive minima problem related to the simultaneous approximation of \((\zeta, \zeta^2, \ldots, \zeta^k)\). Liouville numbers are defined by \(\lambda_{1,1}(\zeta) = \infty\), which means that \(\|\zeta x\| < x^{-\nu}\) has an integer solution \(x \neq 0\) for any \(\nu > 0\), where \(\|\cdot\|\) denotes the distance to the closest integer. For a special class of Liouville numbers including the famous representative \(\sum_{n \geq 1} 10^{-n!}\) and numbers in the Cantor set, we explicitly determine all approximation constants \(\lambda_{k,j}(\zeta), \hat{\lambda}_{k,j}(\zeta)\) simultaneously for all \(k \geq 1\).

Supported by the Austrian Science Fund FWF grant P24828.

Keywords: successive minima, Liouville numbers, Diophantine approximation
Math Subject Classification 2010: 11J13, 11J82

1. Definition and basic properties of \(\lambda_{k,j}, \hat{\lambda}_{k,j}\)

1.1. One dimensional approximation. We begin with the definition of the irrationality exponent. For a real number \(\zeta\) it is denoted by \(\mu(\zeta)\) and defined as the supremum of all \(\eta \geq 0\) such that

\[
\left| \zeta - \frac{y}{x} \right| \leq x^{-\eta}
\]

has infinitely many solutions \((x, y) \in \mathbb{Z}^2\). By Dirichlet’s Theorem, Corollary 2 in [25], \(\mu(\zeta) \geq 2\) for all \(\zeta \in \mathbb{R}\). Capelli showed showed that actually \(\mu(\zeta) = 2\) for Lebesgue almost all \(\zeta \in \mathbb{R}\), which was generalized first by Khinchin and later by Beresnevich, Dickinson and Velani [1]. Roth’s Theorem [10] asserts \(\mu(\zeta) = 2\) for all algebraic irrational reals \(\zeta\). Irrational numbers with \(\mu(\zeta) = \infty\) are called Liouville numbers. Liouville’s first example of a transcendental number

\[
L := \sum_{n \geq 1} 10^{-n!} = 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + \cdots
\]

is obviously a Liouville number, as for large \(n\) putting \(x = 10^{n!}\) and \(y = \sum_{j \leq n} 10^{j!}\) the left hand side in (1) is sufficiently small, and clearly the base 10 representation of \(L\) is not periodic, so \(L \notin \mathbb{Q}\).
1.2. Simultaneous approximation: The general case. Generalizing this concept for simultaneous approximation of \( \zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^k \), for \( 1 \leq j \leq k + 1 \) define \( \lambda_{k,j} \) resp. \( \hat{\lambda}_{k,j} \) the supremum of all \( \eta \in \mathbb{R} \) such that the system

\[
|\eta| \leq X \tag{3}
\]
\[
\max_{1 \leq j \leq k} |\zeta_j x - y_j| \leq X^{-\eta} \tag{4}
\]

has (at least) \( j \) linearly independent solutions \((x, y_1, y_2, \ldots, y_k) \in \mathbb{Z}^k \) for infinitely many \( X \) resp. for all \( X \geq X_0 \). Note that for \( k = 1 \), \( \zeta = \zeta_1 \) the identity \( \lambda_{1,1}(\zeta) + 1 = \mu(\zeta) \) holds. We should mention that the classical notation, used for example in [20], for \( \lambda_{k,1}(\zeta) \) resp. \( \hat{\lambda}_{k,1}(\zeta) \) is \( \omega \) resp. \( \hat{\omega} \), where the vector \( \zeta \) is considered fixed and its length is omitted in the notation. There is no real standard notation for the generalization \( j \geq 2 \), in [16] the constants are denoted \( \omega_j \) resp. \( \hat{\omega}_j \). We use our notation in order to relate to the successive power case will introduce in Section 1.3.

Clearly, for all \( 1 \leq j \leq k + 1 \) and \( \zeta \in \mathbb{R}^k \)

\[
\lambda_{k,j}(\zeta) \geq \lambda_{k,2}(\zeta) \geq \cdots \geq \lambda_{k,k+1}(\zeta) \geq 0, \tag{5}
\]
\[
\hat{\lambda}_{k,j}(\zeta) \geq \hat{\lambda}_{k,2}(\zeta) \geq \cdots \geq \hat{\lambda}_{k,k+1}(\zeta) \geq 0, \tag{6}
\]

and

\[
\lambda_{k,j}(\zeta) \geq \hat{\lambda}_{k,j}(\zeta), \quad 1 \leq j \leq k + 1
\]

hold. Furthermore, a generalization of Dirichlet’s Theorem states that we have

\[
\lambda_{k,1}(\zeta) \geq \hat{\lambda}_{k,1}(\zeta) \geq \frac{1}{k} \tag{7}
\]

for all \( \zeta \in \mathbb{R}^k \). This can be found on page 19, Chapter 2.4 in [25]. Again, due to Khinchin there is actually equality in both inequalities for almost all \( \zeta \in \mathbb{R}^k \), and for all together with \( 1 \) \( \mathbb{Q} \)-linearly independent algebraic \( \zeta \in \overline{\mathbb{Q}}^k \) by Schmidt’s Subspace Theorem [19]. Moreover, defining

\[
\chi_{k,1} = \chi_{k,2} = 1/k, \quad \chi_{k,3} = \chi_{k,4} = \cdots = \chi_{k,k+1} = 0, \quad \phi_{k,1} = 1/k, \quad \phi_{k,2} = \phi_{k,3} = \cdots = \phi_{k,k+1} = 0, \tag{8}
\]

for any \( \zeta \) that is \( \mathbb{Q} \)-linearly independent together with \( 1 \), (14)-(19) in [16] (where \( \omega_j \) corresponds to the present \( \lambda_{k,j}(\zeta) \)) translates into

\[
\chi_{k,j} \leq \lambda_{k,j}(\zeta) \leq \frac{1}{j-1}, \quad 1 \leq j \leq k + 1, \tag{9}
\]
\[
\phi_{k,j} \leq \hat{\lambda}_{k,j}(\zeta) \leq \frac{1}{j}, \quad 1 \leq j \leq k, \tag{10}
\]
\[
\phi_{k,k+1} \leq \hat{\lambda}_{k,k+1}(\zeta) \leq \frac{1}{k}. \tag{11}
\]

Individually, these bounds are best possible, as deduced in Corollary 9 in [16]. In fact, the assumption of \( \mathbb{Q} \)-linear independence is required only in (10). Several other restrictions involving the combined spectrum of the quantities \( \lambda_{k,j} \) resp. \( \hat{\lambda}_{k,j} \) are known, mostly
inferred by Minkowski’s second lattice point Theorem [9], see for instance [20], [21]. See also the Lemmas 2.1, 2.2 in Section 2.

For recent progress in the simultaneous approximation problem [3], (4) in this general case of $\mathbb{Q}$-linearity independence of $\zeta \in \mathbb{R}^k$ together with 1, see [14].

1.3. Simultaneous approximation: The successive power case. An interesting heavily studied special case is that $\zeta$ consists of the first successive powers of a real number $\zeta$, i.e. $\zeta = (\zeta, \zeta^2, \ldots, \zeta^k)$. We will mostly deal with this case in the sequel, so for simplicity for any pair of positive integers $k, j$ with $1 \leq j \leq k + 1$ we will write

$$\lambda_{k,j}(\zeta) := \lambda_{k,j}(\zeta, \zeta^2, \ldots, \zeta^k), \quad \hat{\lambda}_{k,j}(\zeta) := \hat{\lambda}_{k,j}(\zeta, \zeta^2, \ldots, \zeta^k).$$

The assumption of $\mathbb{Q}$-linearity independence of $\zeta \in \mathbb{R}^k$ together with 1 has the natural interpretation that $\zeta$ is not algebraic of degree $\leq k$.

Lebesgue almost all $\zeta$ satisfy $\lambda_{k,j}(\zeta) = \hat{\lambda}_{k,j}(\zeta) = 1/k$ for all pairs $j, k$ with $1 \leq j \leq k + 1$. This was established by Sprindžuk [23] for $j = 1$, combination with Minkowski’s second lattice point Theorem readily yields the general assertion. Generalizations of this metric result for $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k)$ belonging to a wider class of algebraic curves in $\mathbb{R}^k$ can be found in [7]. Under the restriction $\zeta = (\zeta, \zeta^2, \ldots, \zeta^k)$, the bounds in (9)-(11) are in general not optimal, in particular those related to the uniform approximation constants (with the "hat") differ vastly. For the uniform approximation constant related to the first successive minimum, there are sharper upper bounds than the bound 1 arising from (10) known, due to Davenport, Schmidt and Laurent [8]. They prove that for $\zeta$ not algebraic of degree $\leq \lceil k/2 \rceil$, we have

$$\hat{\lambda}_{k,1}(\zeta) \leq \left(\frac{k}{2}\right)^{-1}. \tag{12}$$

Here as usual $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to $\alpha \in \mathbb{R}$. We will briefly discuss upper bounds for $\hat{\lambda}_{k,1}(\zeta)$ further in Section 3.2. The spectrum of $\lambda_{k,1}(\zeta)$ among all real $\zeta$ in dependence of $k$ has been studied as well. Bugeaud proved that it contains $[1, \infty]$ in Theorem 2 in [3]. The natural conjecture already quoted in [3] is the following.

Conjecture 1.1. Let $k$ be a positive integer. The spectrum of $\lambda_{k,1}(\zeta)$ among all real $\zeta$ not algebraic of degree $\leq k$ is $[1/k, \infty]$.

Apart from $k = 1$ this is only known in case $k = 2$ by Beresnevich, Dickinson, Vaughan and Velani [2, 23]. See also [3] for further questions and results in this manner.

It follows from their definition that for a fixed $\zeta$ and $j \geq 1$ the quantities $\lambda_{k,j}(\zeta), \hat{\lambda}_{k,j}(\zeta)$ are non-increasing as $k$ increases, i.e. (with $\lambda_{0,1}(\zeta) := \infty, \hat{\lambda}_{0,1}(\zeta) := \infty$)

$$\lambda_{j-1,j}(\zeta) \geq \lambda_{j,j}(\zeta) \geq \lambda_{j+1,j}(\zeta) \geq \cdots, \quad j \geq 1, \tag{13}$$

$$\hat{\lambda}_{j-1,j}(\zeta) \geq \hat{\lambda}_{j,j}(\zeta) \geq \hat{\lambda}_{j+1,j}(\zeta) \geq \cdots, \quad j \geq 1. \tag{14}$$

Moreover, $\lambda_{k,j}(\zeta) \geq \hat{\lambda}_{k,j}(\zeta)$ holds as in the general context of arbitrary $\zeta \in \mathbb{R}^k$. For $j = 1$, Lemma 1 in [3] of Bugeaud states a result somehow reverse to (13), which in our notation says the following.
Theorem 1.2 (Bugeaud). For any positive integers $k$ and $n$, and any transcendental real number $\zeta$ we have
\[
\lambda_{kn,1}(\zeta) \geq \frac{\lambda_{n,1}(\zeta) - k + 1}{k}.
\]

Reverse inequalities in case of $\lambda_{kn,1}(\zeta) > 1$ were established by the author \[18\]. If we put $n = 1$ in Theorem \[1.2\] and let $\zeta$ be a Liouville number, we obtain what is Corollary 2 in \[3\].

Corollary 1.3 (Bugeaud). Let $\zeta$ be an irrational real number. We have $\lambda_{k,1}(\zeta) = \infty$ for all positive integers $k$ if and only if $\lambda_{1,1}(\zeta) = \infty$.

We shall utilize Corollary \[1.3\] in Section \[4\].

2. Introductory results for Liouville numbers

Parts of the proofs of the main results in Section \[4\] can be generalized, so we will prepend the more general versions in form of two Lemmas \[2.1\] \[2.2\]. The proofs of those Lemmas are basically a consequence of Minkowski’s second lattice point Theorem, hidden in the results from \[20\], \[21\] we utilize.

We introduce the functions $\psi_{k,j}$ and the derived quantities $\underline{\psi}_{k,j}$, $\overline{\psi}_{k,j}$ defined in \[20\] with a slightly different notation (subindex $k$ omitted). For fixed $\zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^k$ and a real parameter $Q > 0$ define $\psi_{k,j}(Q)$ for $1 \leq j \leq k + 1$ as the supremum of all values of $\nu \in \mathbb{R}$ such that the system
\[
|x| \leq Q^{1+\nu},
\]
\[
\max_{1 \leq j \leq k} |\zeta_j x - y_j| \leq Q^{-1/k+\nu}
\]
has $j$ linearly independent solutions $(x, y_1, \ldots, y_k) \in \mathbb{Z}^{k+1}$. Further let
\[
\underline{\psi}_{k,j} = \liminf_{Q \to \infty} \psi_{k,j}(Q), \quad \overline{\psi}_{k,j} = \limsup_{Q \to \infty} \psi_{k,j}(Q).
\]
The quantities obviously have the properties $\underline{\psi}_{k,j} \leq \psi_{k,j} \leq \overline{\psi}_{k,j}$ and
\[
-1 \leq \psi_{k,1} \leq \psi_{k,2} \leq \cdots \leq \psi_{k,k+1} \leq 1/k,
\]
\[
-1 \leq \overline{\psi}_{k,1} \leq \overline{\psi}_{k,2} \leq \cdots \leq \overline{\psi}_{k,k+1} \leq 1/k.
\]
Minkowski’s first lattice point Theorem (or Dirichlet’s Theorem) further implies
\[
\overline{\psi}_{k,1} \leq 0.
\]
Moreover, the generalization (13) in \[16\] of (1.9) in \[20\] for arbitrary $j$, shows that $\underline{\psi}_{k,j}$ and $\lambda_{k,j} = \lambda_{k,j}(\zeta)$ such as $\overline{\psi}_{k,j}$ and $\hat{\lambda}_{k,j} = \hat{\lambda}_{k,j}(\zeta)$ are closely connected. In our notation the correspondence is given by
\[
(1 + \lambda_{k,j})(1 + \psi_{k,j}) = (1 + \hat{\lambda}_{k,j})(1 + \overline{\psi}_{k,j}) = \frac{k + 1}{k}, \quad 1 \leq j \leq k + 1.
\]
We will use (19) frequently in the proof of the following Lemmas 2.1, 2.2. Before we state and prove them, we should mention that the functions \( \psi_{k,j}(Q) \) and thus the derived values \( \hat{\psi}_{k,j}, \psi_{k,j} \) have a natural geometric interpretation as successive minima of special convex bodies with respect to a lattice. However, we will not explicitly use the geometric view for our approaches and refer to [20] for details.

**Lemma 2.1.** For any positive integer \( k \) and any \( \zeta \in \mathbb{R}^k \) we have the equivalence

\[
\lambda_{k,1}(\zeta) = \infty \iff \hat{\lambda}_{k,2}(\zeta) = \hat{\lambda}_{k,3}(\zeta) = \cdots = \hat{\lambda}_{k,k+1}(\zeta) = 0.
\]

**Proof.** Actually, the author basically deduces the left to right implication of (20) in the last part of the proof of Theorem 1 in [16] with the aid of the quantities \( \hat{\psi}_{k,j}, \psi_{k,j} \). We want to prove it again rigorously, however. With (19) one can reinterpret \( \psi_{k,1} = -1 \). Together with inequality (1.6a) with \( i = 1 \) in [20], which is in our notation

\[
\psi_{k,1} - \psi_{k,2} - \psi_{k,3} - \cdots - \psi_{k,k+1} \geq 0,
\]

and noting \( \psi_j \leq 1/k \) for all \( 1 \leq j \leq k+1 \) by (17), we conclude \( \psi_{k,2} = \cdots = \psi_{k,k+1} = 1/k \). Again reinterpreting this using (19), the right hand side of (20) follows. For the other direction in (20), we use (1.10) in [21] for \( j = 1 \), which is in our notation \( \psi_{k,1} + k\psi_{k,2} \leq 0 \). In view of this and (19) with \( j = 2 \), the right hand side of (20) first implies \( \psi_{k,1} = -1 \) and further again with (19) with \( j = 1 \) finally \( \lambda_{k,1}(\zeta) = \infty \). \( \square \)

**Lemma 2.2.** Let \( k \) be a positive integer and \( \zeta \in \mathbb{R}^k \) arbitrary. Then

\[
\hat{\lambda}_{k,1}(\zeta) = \frac{1}{k} \iff \lambda_{k,2}(\zeta) \geq \lambda_{k,3}(\zeta) \geq \cdots \geq \lambda_{k,k+1}(\zeta) = \frac{1}{k}.
\]

**Proof.** First note that by (5), it remains to show the equivalence of the left hand side to the last equality on the right hand side. The implication left to right of the lemma is a consequence of (1.10) in [21], which in the notation of the present Section 2 states

\[
j \psi_{k,1} + (k + 1 - j) \psi_{k,j+1} \leq 0, \quad 0 \leq j \leq k.
\]

By (19) we know \( \hat{\lambda}_{k,1}(\zeta) = 1/k \) is equivalent to \( \psi_{k,1} = 0 \), hence (21) implies \( \psi_{k,j} \leq 0 \) for all \( 1 \leq j \leq k+1 \), and again with (19) we conclude \( \lambda_{k,k+1}(\zeta) \geq 1/k \). The reverse inequality \( \lambda_{k,k+1}(\zeta) \leq 1/k \) is by (19) equivalent to \( \psi_{k,k+1} \geq 0 \), but (1.13) in [21] and (18) indeed imply

\[
\psi_{k,k+1} \geq -\frac{1}{k} \psi_{k,1} \geq 0.
\]

We turn to the implication right to left. Suppose \( \lambda_{k,k+1}(\zeta) = 1/k \), which by (19) is equivalent to \( \psi_{k,k+1} = 0 \). By (19), the assertion \( \hat{\lambda}_{k,1}(\zeta) = 1/k \) is equivalent to \( \psi_{k,1} = 0 \). This is again a consequence of (22). \( \square \)
3. Inequalities involving \( \lambda_{k,1}, \hat{\lambda}_{k,1} \)

3.1. Connections between \( \lambda_{k,1}(\zeta), \hat{\lambda}_{k,1}(\zeta) \) in the general case. In this section we quote results on the connection between \( \hat{\lambda}_{k,1}(\zeta) \) and \( \lambda_{k,1}(\zeta) \), mostly due to Summerer and Schmidt. They are not ingredient of the main results in Section 4 but are considered to empathize similarities and differences between the general case and the successive power case we will deal with in the subsequent Section 3.2.

So let us briefly return to the general setting of \( \zeta = (\zeta_1, \ldots, \zeta_k) \in \mathbb{R}^k \) that is \( \mathbb{Q} \)-linearly independent together with 1, but not necessarily of the form \( (\zeta, \zeta^2, \ldots, \zeta^k) \). An inequality linking \( \lambda_{k,1}(\zeta) \) and \( \hat{\lambda}_{k,1}(\zeta) \) is

\[
\lambda_{k,1}(\zeta) \geq \frac{\hat{\lambda}_{k,1}(\zeta)^2 + (k-2)\hat{\lambda}_{k,1}(\zeta)}{(k-1)(1-\hat{\lambda}_{k,1}(\zeta))},
\]

due to Summerer and Schmidt [21]. This improves Jarník’s inequality [5]

\[
\frac{\hat{\lambda}_{k,1}(\zeta)^2}{1-\hat{\lambda}_{k,1}(\zeta)} \leq \lambda_{k,1}(\zeta),
\]

which is also valid for any \( \zeta \in \mathbb{R}^k \) linearly independent over \( \mathbb{Q} \) together with 1. The estimate (23) is still not best possible for \( k \geq 3 \), but close to optimal. More precisely, a conjecture of Summerer and Schmidt suggests that the minimal discrepancy between \( \lambda_{k,1}(\zeta) \) and \( \hat{\lambda}_{k,1}(\zeta) \) are obtained for \( \zeta \) that lead to what they call regular graphs. In any dimension \( k \), they are determined by one parameter, for instance a prescribed value of \( \psi_{k,1} \in [-1,0] \) (or equivalently \( \lambda_{k,1}(\zeta) \in [1/k, \infty] \), see [19]). The geometric shape of regular graphs is not important for our purposes, so we refer to Chapter 3 in [22]. The existence of any regular graph in arbitrary dimension was recently proved in Theorem 1.2 in [15], which also implies (23) is not optimal for \( k \geq 3 \). For \( \zeta \) inducing a regular graph, an estimate in the spirit of (23), which is believed to be optimal, is given by an implicit polynomial equation. This conjecture of optimality was established recently in the case \( k = 3 \), by a combination of results of Roy with results of Summerer and Schmidt [22]. For the regular graphs in dimension \( k = 3 \), we have equality in the inequality of Theorem 1 in [22], which is

\[
\lambda_{3,1}(\zeta) \geq \hat{\lambda}_{3,1}(\zeta) \cdot \frac{\hat{\lambda}_{3,1}(\zeta) + \sqrt{4\hat{\lambda}_{3,1}(\zeta) - 3\hat{\lambda}_{3,1}(\zeta)^2}}{2(1-\hat{\lambda}_{3,1}(\zeta))},
\]

for any \( \zeta \in \mathbb{R}^3 \) linearly independent over \( \mathbb{Q} \) together with 1. Concerning bounds in case of the regular graph in arbitrary dimension, where it seems hopeless to look for explicit formulas like the equality case in (24), Remark 7 to Proposition 5 in [16] states

\[
\frac{\hat{\lambda}_{k,1}(\zeta)^2}{1-\hat{\lambda}_{k,1}(\zeta)} \leq \lambda_{k,1}(\zeta) < \frac{\hat{\lambda}_{k,1}(\zeta)}{1-\hat{\lambda}_{k,1}(\zeta)}.
\]
The left hand side inequality in (25) can be improved via replacement by (23), as discussed above. Moreover, it is mentioned in [16] that as \( \lambda_{k,1}(\zeta) \) tends to infinity we have that \( \lambda_{k,1}(\zeta) + 1 - \lambda_{k,1}(\hat{\zeta})/\hat{\lambda}_{k,1}(\hat{\zeta}) \) tends to 0 in the regular graph case. This shows the right inequality in (25) is rather accurate for regular graphs, at least for large values of \( \lambda_{k,1}(\zeta) \).

Roughly speaking, the results mentioned in this Section 3.1 show that \( \lambda_{k,1}(\zeta) \) cannot be arbitrarily close to \( \hat{\lambda}_{k,1}(\hat{\zeta}) \) and improves the trivial inequality \( \lambda_{k,1}(\zeta) \geq \hat{\lambda}_{k,1}(\hat{\zeta}) \). Concerning reverse inequalities, note that the most extreme case, \( \lambda_{k,1}(\zeta) = \infty \) and \( \hat{\lambda}_{k,1}(\hat{\zeta}) = 1/k \) simultaneously, is possible even in the special case \( \zeta = (\zeta, \zeta^2, \ldots, \zeta^k) \), as we will establish in Theorem 4.2. See also Remark 4.4.

3.2. The successive power case. In case of \( \zeta = (\zeta, \zeta^2, \ldots, \zeta^k) \), the restriction (12) shows that for not too small values of \( \lambda_{k,1}(\zeta) \), there are much more stringent restrictions for the joint spectrum of \( (\lambda_{k,1}(\zeta), \hat{\lambda}_{k,1}(\hat{\zeta})) \) than those in the general \( Q \)-linearly independent case discussed in Section 3.1. Another result that affirms this is the following Theorem 1.18 in [18].

Theorem 3.1. Let \( \zeta \in \mathbb{R} \setminus \mathbb{Q} \). For any positive integer \( k \), we have

\[
\hat{\lambda}_{k,1}(\zeta) \leq \max \left\{ \frac{1}{\lambda_{1,1}(\zeta)}, \frac{1}{k} \right\}.
\]

We are again interested in the consequences for Liouville numbers. The following corollary is basically contained in Corollary 4.5 in [18].

Corollary 3.2. Let \( k \) be a positive integer. Let \( \zeta \in \mathbb{R} \setminus \mathbb{Q} \) such that \( \lambda_{1,1}(\zeta) \geq k \). Then

\[
\hat{\lambda}_{k,1}(\zeta) = \frac{1}{k}.
\]

In particular, if \( \zeta \) is a Liouville number, then \( \hat{\lambda}_{k,1}(\zeta) = 1/k \) simultaneously for all \( k \).

Proof. Combination of (7) and Theorem 3.1.

We should mention that for \( k = 2 \), a regular graph is induced the successive power case for extremal numbers \( \zeta \) with the values \( \lambda_{2,1}(\zeta) = 1, \hat{\lambda}_{2,1}(\zeta) = \gamma := (\sqrt{5} - 1)/2 \), see [11]. Moreover, \( \gamma \) is indeed the maximum value of \( \hat{\lambda}_{2,1}(\zeta) \) among all \( \zeta \) not algebraic of degree \( \leq 2 \). Note that (12) for \( k = 2 \) would only yield the trivial upper bound 1 that holds for any \( \zeta \) under the usual linear independence assumption, see (11). For \( k = 3 \), the bound 1/2 from (12) was improved as well by Roy [12], however the best value remains unknown.

4. All constants \( \lambda_{k,j}, \hat{\lambda}_{k,j} \) for Liouville numbers

4.1. The general case. The following theorem is readily inferred by the aid of results we have established so far. Its concern is to determine/bound the classic approximation constants for arbitrary Liouville numbers.
Theorem 4.1. Let $\zeta$ be a Liouville number. For any positive integer $k$, we have

\begin{align*}
\lambda_{k,1}(\zeta) &= \infty, \\
\frac{1}{k} \leq \lambda_{k,j}(\zeta) &\leq \frac{1}{j-1}, \quad 2 \leq j \leq k+1, \\
\hat{\lambda}_{k,1}(\zeta) &= \frac{1}{k}, \\
\hat{\lambda}_{k,j}(\zeta) &= 0, \quad 2 \leq j \leq k+1.
\end{align*}

Proof. Corollary 1.3 yields (26) and Lemma 2.1 implies (29). The upper bounds in (27) are due to (9), since the $\mathbb{Q}$-linearly independence condition is certainly satisfied as Liouville numbers are transcendental. The lower bounds in (27) are due to Lemma 2.2. Finally Corollary 3.2 gives (28). \hfill \Box

4.2. A special class of Liouville numbers. We provide a class of Liouville numbers $\zeta$, for which all approximation constants $\lambda_{k,j}(\zeta), \hat{\lambda}_{k,j}(\zeta)$ can be determined explicitly simultaneously for all $k$. The weakest assumption on $\zeta$ for our methods to work is basically the property that in the binary expansion (or some other base) of $\zeta$ there are two consecutive large gaps between ones (non-zero entries) infinitely often. This will be the assumption in Theorem 4.2. Specializations and slight generalizations are given subsequently. We will denote constants depending only on $k$ with $C(k)$ throughout the proof of Theorem 4.2. Moreover, we denote by $\|\alpha\|$ the distance from $\alpha \in \mathbb{R}$ to the nearest integer.

Theorem 4.2. Let $\zeta = \sum_{l \geq 1} 2^{-a_l}$ for a strictly increasing sequence $(a_l)_{l \geq 1}$ that satisfies

$$\limsup_{l \to \infty} \min \left\{ \frac{a_{l+1}}{a_l}, \frac{a_{l+2}}{a_{l+1}} \right\} = \infty,$$

which is in particular true if

$$\lim_{l \to \infty} \frac{a_{l+1}}{a_l} = \infty.$$

Then for any positive integer $k$ we have (with the convention $1/0 = \infty$)

$$\lambda_{k,j}(\zeta) = \frac{1}{j-1}, \quad 1 \leq j \leq k+1,$$

$$\hat{\lambda}_{k,1}(\zeta) = \frac{1}{k},$$

$$\hat{\lambda}_{k,j}(\zeta) = 0, \quad 2 \leq j \leq k+1.$$

Proof. Let $k$ be fixed. It is easily checked that $\zeta$ as in the theorem is a Liouville number. By Theorem 4.1, it suffices to prove

$$\lambda_{k,j}(\zeta) \geq \frac{1}{j-1}, \quad 2 \leq j \leq k+1.$$

To prove this, we construct $k+1$ sequences of approximation vectors

$$(x_1^{1,n}, y_1^{1,n}, \ldots, y_k^{1,n})_{n \geq 1}, (x_1^{2,n}, y_1^{2,n}, \ldots, y_k^{2,n})_{n \geq 1}, \ldots, (x_1^{k+1,n}, y_1^{k+1,n}, \ldots, y_k^{k+1,n})_{n \geq 1}$$
that are linearly independent for every fixed \( n \geq 1 \), and for any fixed \( 1 \leq j \leq k + 1 \) a sequence \( (X_{j,n})_{n \geq 1} \) with the property that for any \( \epsilon > 0 \) and large enough \( n \geq n(\epsilon) \)

\[
|x^{i,n}| \leq X_{j,n}, \quad 1 \leq i \leq j, \tag{31}
\]

\[
\max_{1 \leq i \leq k} |x^{i,n}_t - y^{i,n}_t| \leq X_{j,n}^{1+\epsilon}, \quad 1 \leq i \leq j. \tag{32}
\]

Recalling the definition of \( \lambda_{k,j}(\zeta) \) via (3), (4), this yields (30).

By the assumption of the theorem there exists a subsequence \( (b_n)_{n \geq 1} = (a_{l_n})_{n \geq 1} \) of \( (a_l)_{l \geq 1} \) with the property

\[
\lim_{n \to \infty} \frac{a_{l_n+1}}{a_{l_n}} = \lim_{l \to \infty} \frac{a_{l+2}}{a_{l+1}} = \infty. \tag{33}
\]

We claim a suitable choice is given by

\[
X_{j,n} := 2^{(j-1)a_{n+1}+kb_n} = 2^{(j-1)a_{n+1}+ka_n}, \quad n \geq 1, \quad 1 \leq j \leq k + 1, \tag{34}
\]

and corresponding approximations vectors

\[
x^{i,n} := 2^{(i-1)a_{n+1}+ka_n}, \quad n \geq 1, \quad 1 \leq i \leq j, \tag{35}
\]

\[
y^{i,n}_t := \lfloor \zeta^t x^{i,n} \rfloor, \quad n \geq 1, \quad 1 \leq i \leq j, \quad 1 \leq t \leq k. \tag{36}
\]

By construction (31) holds. We turn to prove (32). Let \( S_N \) be the partial sums of \( \zeta \), i.e.

\[
S_N := \sum_{l=1}^{N} 2^{-a_l}, \quad N \geq 1.
\]

It is convenient and instructive to deal with the case \( j = 1 \) not mentioned in (30) separately at first. Put \( N := l_n \) for arbitrary \( n \), which we consider sufficiently large. Expanding \( S^t_N \) by Binomial Theorem yields

\[
S^t_N x^{1,n} = S^t_N 2^{ka_n} \subset \mathbb{Z}, \quad 1 \leq t \leq k, \tag{37}
\]

since every summand is an integer. On the other hand \( S_N < \zeta < S_N + 2^{a_{n+1}} < 1 \), thus for a constant \( C(t) \) depending on \( t \) only for all \( n \) (and \( N = l_n \))

\[
0 \leq \zeta^t - S^t_N \leq (S_N + 2^{-a_{n+1}})^t - S^t_N \leq C(t) 2^{-a_{n+1}}. \tag{38}
\]

Hence \( 1 \leq t \leq k \) implies for all \( n \) (and \( N = l_n \))

\[
0 \leq \zeta^t - S^t_N \leq C_0(k) 2^{-a_{n+1}}. \tag{39}
\]

Combination of (37), (39) yields that \( S^t_N \) is the nearest integer to \( \zeta^t \), and

\[
||\zeta^t x^{1,n}|| = (\zeta^t - S^t_N) x^{1,n} \leq C_0(k) 2^{ka_n-a_{n+1}}, \quad 1 \leq t \leq k.
\]

Thus definition (36) yields

\[
\max_{1 \leq i \leq k} \left| \zeta^t x^{1,n}_t - y^{1,n}_t \right| \leq C_0(k) 2^{ka_n-a_{n+1}}, \quad 1 \leq t \leq k.
\]

By \( X^{1,n} = x^{1,n} = 2^{ka_n} \), see (34) for \( j = 1 \), and (33) we conclude

\[
\infty = \lim_{n \to \infty} \frac{a_{l_n+1}}{a_{l_n}} = \lim_{n \to \infty} \frac{a_{l_n+1} - ka_n}{ka_n} = \lim_{n \to \infty} \log_{X^{1,n}} \max_{1 \leq t \leq k} |\zeta^t x^{1,n}_t - y^{1,n}_t|,
\]
so recalling (32) indeed \( \lambda_{k,1}(\zeta) = \infty \) (which we already knew by Theorem 4.1).

Now let \( j \geq 2 \). For any pair \( 1 \leq t \leq k, 1 \leq i \leq k + 1 \) define \( T_{N+1,t}^i \) as the sum of those expressions of \( S_{N+1}^t \) after expanding, in which the factor \( 2^{-a_{N+1}} \) appears at most \((i - 1)\) times. Since \( S_{N+1}^t = S_N + 2^{-a_{N+1}} \), standard combinatorial identities yield

\[
T_{N+1,t}^i = \min\{t,i-1\} \sum_{\tau=0}^{t} \left( \begin{array}{c} t \\ \tau \end{array} \right) S_N^{t-\tau} 2^{-\tau a_{N+1}}, \quad 1 \leq t \leq k, 1 \leq i \leq k + 1.
\]

(41)

In particular \( T_{N+1,t}^1 = S_N^t \) for \( 1 \leq t \leq k \), but we won’t relate to this fact. From (41) it is easy to see

\[
T_{N+1,t}^{i+1} > T_{N+1,t}^i, \quad 1 \leq t \leq k,
\]

(42)

\[
T_{N+1,t}^{i+2} = T_{N+1,t}^{i+1} = \cdots = T_{N+1,t}^{k+1}, \quad 1 \leq t \leq k.
\]

(43)

Further the definition of \( T_{N+1,t}^i \) and that of \( x^{i,n} \) yield with \( N := l_n \)

\[
x^{i,n}T_{N+1,t}^i \in \mathbb{Z}, \quad 1 \leq t \leq k, 1 \leq i \leq k + 1,
\]

(44)

since every summand is an integer. On the other hand, similar to (33) in the case \( j = 1 \), we conclude

\[
0 < S_{N+1}^t - T_{N+1,t}^i \leq C_1(k) 2^{-i a_{n+1}},
\]

(45)

as well as

\[
0 < \zeta^t - S_{N+1}^t < C_2(k) 2^{-a_{n+2}}.
\]

(46)

By (33) the gap between \( a_{n+1} \) and \( a_{n+2} \) is large too, such that we have \( 2^{-a_{n+2}} = o(2^{-ka_{n+1}}) \) as \( n \to \infty \), so the error term in (45) is dominant. (Effectively we need \( 2^{-a_{n+2}} = O(2^{-ka_{n+1}}) \) in the sequel, for which \( a_{n+2} \notin [a_{n+1}, (k+1)a_{n+1}] \) is sufficient, see also Corollary 4.5.)

So by (45), (46) and triangular inequality again we infer \( x^{i,n}T_{N+1,t}^i \) is the nearest integer to \( \zeta^t x^{i,n} \), and

\[
\| \zeta^t x^{i,n} \| = (\zeta^t - T_{N+1,t}^i) x^{i,n} \leq C_3(k) 2^{-i a_{n+1}} 2^{(i-1)a_{n+1}+ka_{n+1}} = C_3(k) 2^{ka_{n+1}a_{n+1}}, \quad 1 \leq t \leq k.
\]

(47)

Thus for sufficiently large \( n \) and \( N = l_n \), by definition (36)

\[
y^{i,n}_t = x^{i,n}T_{N+1,t}^i, \quad 1 \leq t \leq k, 1 \leq i \leq k + 1
\]

(48)

\[
| \zeta^t x^{i,n} - y^{i,n}_t | \leq C_3(k) 2^{ka_{n+1}}, \quad 1 \leq t \leq k, 1 \leq i \leq k + 1.
\]

Due to (33), (34), (48) for any \( \epsilon > 0 \) and \( n \geq n_0(\epsilon) \) we have for any \( 1 \leq i \leq j \)

\[
- \log_{x^{i,n}} \left( \max_{1 \leq i \leq k} | \zeta^t x^{i,n} - y^{i,n}_t | \right) \geq - \frac{\log(C_3(k) 2^{ka_{n+1}a_{n+1}})}{\log 2^{(j-1)a_{n+1}+ka_{n+1}}} = - C_4(k) + ka_{n+1}a_{n+1} + (j-1)a_{n+1} + ka_{n+1}.
\]

Recalling (33) implies \( a_{n+1} = o(a_{n+1}) \) as \( n \to \infty \), the right hand side quotient tends to \( 1/(j-1) \) for any \( 1 \leq j \leq k + 1 \), which implies (32).
It remains to prove the linear independence of the constructed approximation vectors for all \( n \geq 1 \). This is equivalent to the regularity of the matrices \( V_n \in \mathbb{Z}^{(k+1) \times (k+1)} \), whose \( j \)-th row is the \( j \)-th approximation vector. By (17) and with \( N := l_n \) we can write

\[
V_n = \begin{pmatrix}
    x^{1,n} & T^1_{N+1,1}x^{1,n} & T^1_{N+1,2}x^{1,n} & \cdots & T^1_{N+1,k}x^{1,n} \\
x^{2,n} & T^2_{N+1,1}x^{2,n} & T^2_{N+1,2}x^{2,n} & \cdots & T^2_{N+1,k}x^{2,n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
x^{k,n} & T^k_{N+1,1}x^{k,n} & T^k_{N+1,2}x^{k,n} & \cdots & T^k_{N+1,k}x^{k,n} \\
x^{k+1,n} & T^{k+1}_{N+1,1}x^{k+1,n} & T^{k+1}_{N+1,2}x^{k+1,n} & \cdots & T^{k+1}_{N+1,k}x^{k+1,n}
\end{pmatrix}
\]

Beginning with \( h = k \) until \( h = 1 \) we perform the following row rearrangements: We subtract the \((x^{h+1,n}/x^{h,n})\)-th multiple of the \( h \)-th row from the \((h+1)\)-th row. After this procedure, by (43) we end up with a upper triangular matrix, whose diagonal entries do not vanish in view of (42). Hence \( V_n \) is indeed regular for all \( n \geq 1 \).

\[\square\]

**Remark 4.3.** For \( j > 1 \) the quantities \( \lambda_{k,j}(\zeta) \) need not equal the upper bound \( 1/(j-1) \) for any Liouville number \( \zeta \) (not of the form as in Theorem 4.2). This will most likely be wrong for many Liouville numbers that are normal to any base \( b \geq 2 \), whose existence was established in [4]. The following Theorem 4.10 suggests that there should be equality for some Liouville numbers normal to any base, though, because for generic choices of the \( q_i \) in Theorem 4.10 there is no reason for the resulting Liouville number not to be normal in any base.

**Remark 4.4.** The approximation functions \( \psi_{k,j}(Q) \) related to \((\zeta, \zeta^2, \ldots, \zeta^k)\) with \( \zeta \) as in Theorem 4.2 all take their maximum possible range, that is \((-1,0)\) for \( j = 1 \) and \(((j-k-1)/k, 1/k)\) for \( 2 \leq j \leq k+1 \), in any interval \((Q_0, \infty)\). This case was already constructed in Corollary 3 in [16] for \( \zeta \in \mathbb{R}^k \) linearly independent over \( \mathbb{Q} \) together with 1, however here is an example for the special case \( \zeta = (\zeta, \zeta^2, \ldots, \zeta^k) \).

The proof of Theorem 4.2 shows that for fixed \( k \), the assumptions on the consecutive large gaps between \( a_l, a_{l+1} \) and \( a_{l+1}, a_{l+2} \) of Theorem 4.2 can be weakened.

**Corollary 4.5.** Let \( k \) be a positive integer and \( \zeta \in [0,1) \) with expansion \( \zeta = \sum_{l \geq 1} 2^{-a_l} \) in base 2 as in Theorem 4.2. Assume there exists a subsequence \((a_{n_l})_{n \geq 1}\) of \((a_l)_{l \geq 1}\) with the property that for all \( C > 0 \) there exists \( n_0 = n_0(C) \) such that

\[
\frac{a_{n_l+1}}{a_{n_l}} > C, \quad n \geq n_0
\]

\[
\frac{a_{n_l+2}}{a_{n_l+1}} > k + 1, \quad n \geq n_0.
\]

Then the quantities \( \lambda_{k,j}(\zeta), \hat{\lambda}_{k,j}(\zeta) \) are given as in Theorem 4.2.

Obviously, the choice of the base 2 in Theorem 4.2 and Corollary 4.5 has no significance. We point out that clearly Theorem 4.2 applies to the Liouville-number \( L \) in [2], as well as some numbers in the Cantor middle third set, which can be defined as all numbers in \([0,1]\) that can be written without 1 in the 3-adic expansion.
Corollary 4.6. Let $L$ be as in (2). Then for any positive integer $k$ and $1 \leq j \leq k + 1$ the quantities $\lambda_{k,j}(L), \hat{\lambda}_{k,1}(L)$ are given as in Theorem 4.2.

Corollary 4.7. There exist numbers $\zeta$ in the Cantor middle third set whose approximation constants $\lambda_{k,j}(\zeta), \hat{\lambda}_{k,j}(\zeta)$ are given as in Theorem 4.2.

Proof. Choosing the base 3 instead of 2 and replacing $\zeta$ by $2\zeta$, which as rational transformation does not affect the quantities $\lambda_{k,j}, \hat{\lambda}_{k,j}$, we obviously obtain an element of the Cantor middle third set. \qed

Recall $\chi_{k,j}$ from (8). Theorem 4.2, Corollary 4.7 and the bounds of $\lambda_{k,j}(\zeta)$ from (9) suggest the following generalizations of Conjecture 1.1.

Conjecture 4.8 (Weak). Let $k$ be a positive integer and $1 \leq j \leq k + 1$. The (individual) spectrum of $\lambda_{k,j}(\zeta)$ among all real $\zeta$ not algebraic of degree $\leq k$ is $[\chi_{k,j}, 1/(j - 1)]$.

Conjecture 4.9 (Strong). Let $k$ be a positive integer and $1 \leq j \leq k + 1$. The (individual) spectrum of $\lambda_{k,j}(\zeta)$ among irrational $\zeta$ in the Cantor set is $[\chi_{k,j}, 1/(j - 1)]$.

Note again that the analogue of the Conjecture for the uniform constants $\hat{\lambda}_{k,j}(\zeta)$ with the bounds from (10), (11) fails heavily due to (12).

Theorem 4.2 and Corollary 4.5 can readily be generalized in a way that the resulting numbers $\zeta$ do not necessarily have a biased representation in some base.

Theorem 4.10. Let

$$\zeta = \sum_{l \geq 1} \frac{1}{q_l}$$

with integers $q_l$, such that $q_l|q_{l+1}$ for all $l \geq 1$. Further suppose that for a positive integer $k$ there exists a subsequence $(q_{l_n})_{n \geq 1}$ of $(q_l)_{l \geq 1}$ with the property that for all $C > 0$ there exists $n_0 = n_0(C)$ such that

$$\log q_{l_n+1} \log q_{l_n} > C, \quad n \geq n_0$$

$$\log q_{l_n+2} \log q_{l_n+1} > k + 1, \quad n \geq n_0.$$
small rearrangements. A concrete example is changing the base from 2 to 3 and replacing each $\zeta_j$ by $2\zeta_j$ in Corollary 7 in [16].

Eventually, we want to point out that the results in Section 4 have an equivalent interpretation in a well-studied linear form problem. Define $w_{k,j}(\zeta)$ resp. $\hat{w}_{k,j}(\zeta)$ as the supremum of real $\nu$ such that the system

$$|x_0| \leq X, \quad |x_0 + \zeta x_1 + \cdots + \zeta x_k| \leq X^{-\nu}$$

has $j$ linearly independent solutions $(x_0, \ldots, x_k) \in \mathbb{Z}^{k+1}$ for arbitrarily large $X$ resp. for all sufficiently large $X$. Then, with the convention $1/0 := \infty, 1/\infty := 0$, the identities

$$w_{k,j}(\zeta) = \frac{1}{\lambda_{k,k+2-j}(\zeta)}, \quad \hat{w}_{k,j}(\zeta) = \frac{1}{\lambda_{k,k+2-j}(\zeta)}$$

hold for $1 \leq j \leq k + 1$, see (1.24) in [17]. Thus, the constants $w_{k,j}(\zeta), \hat{w}_{k,j}(\zeta)$ can be readily determined for Liouville numbers $\zeta$ involved in Section 4.

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