Heavy-quarkonium creation and annihilation with \( \mathcal{O}(\alpha_s^3 \ln \alpha_s) \) accuracy

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Abstract

We calculate the \( \mathcal{O}(\alpha_s^3 \ln \alpha_s) \) contributions to the heavy-quarkonium production and annihilation rates. Our result sheds new light on the structure of the high-order perturbative corrections and opens a new perspective for a high-precision theoretical analysis. We also determine the three-loop anomalous dimensions of the nonrelativistic vector and pseudoscalar currents.

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The theoretical study of nonrelativistic heavy-quark-antiquark systems is among the earliest applications of perturbative quantum chromodynamics (QCD) [1] and has by now become a classical problem. Its applications to bottomonium [2] and top-antitop [3] physics entirely rely on the first principles of QCD. These systems allow for a model-independent perturbative treatment. Nonperturbative effects [4] are well under control for the top-antitop system and, at least within the sum-rule approach, also for bottomonium. This makes heavy-quark-antiquark systems an ideal laboratory to determine fundamental parameters of QCD, such as the strong-coupling constant \( \alpha_s \) and the heavy-quark masses \( m_q \). The bottom-quark mass \( m_b \) is of particular interest, in view of current and future \( B \)-physics experiments. In the observables employed to extract the Cabibbo-Kobayashi-Maskawa matrix elements and to gain deeper insight in the nature of CP violation, \( m_b \) enters as a crucial input parameter [5]. Thus, precise knowledge of \( m_b \) is essential for the interpretation of the experimental data. On the other hand, the top-quark mass \( m_t \) is one of the key parameters in the precision tests of the standard model of the electroweak interactions and in the search for new physics at a future \( e^+e^- \) linear collider. Furthermore, the study of \( tt \) threshold production should even allow us to probe Higgs-boson-induced effects [6]. Besides its phenomenological importance, the heavy-quarkonium system is also very interesting from the theoretical point of view because it possesses a highly sophisticated multiscale dynamics and its study demands the full power of the effective-field-theory approach. Equipped with reliable perturbative results and experimental data on heavy-quarkonium observables, one can test the effects and structure of the nonperturbative QCD vacuum.

The binding energy of the bound state and the value of its wave function at the origin are among the characteristics of the heavy-quarkonium system that are of primary phenomenological interest. The former determines the mass of the bound-state resonance, while the latter controls its production and annihilation rates. Recently, the heavy-quarkonium spectrum has been computed through \( \mathcal{O}(\alpha_s^5m_q) \) [7,8] including the third-order correction to the Coulomb approximation. On the other hand, as for the wave function at the origin, a complete result is so far only available through \( \mathcal{O}(\alpha_s^2) \) [9]. The \( \mathcal{O}(\alpha_s^2) \) correction has turned out to be so sizeable that the feasibility of an accurate perturbative analysis was challenged [10], and it appears indispensable to gain full control over the next order. Only the double-logarithmic third-order correction, of \( \mathcal{O}(\alpha_s^3\ln^2\alpha_s) \), is available so far [11]. In this Letter, we take the next step and calculate the single-logarithmic \( \mathcal{O}(\alpha_s^3\ln\alpha_s) \) correction. As a by-product of our analysis, we obtain the three-loop anomalous dimensions of the nonrelativistic vector and pseudoscalar currents, which constitute central ingredients for the renormalization-group improvement of the effective theory of nonrelativistic QCD (NRQCD) [12–14]. The main results are given by Eqs. (6), (7), and (10). As for the calculation, we follow the general approach of Ref. [7] (see also Ref. [15]). It is based on the nonrelativistic effective-theory concept [16] in its potential-NRQCD (pNRQCD) incarnation [17] implemented with the threshold-expansion technique [18].

Let us focus on two examples of paramount phenomenological relevance: the leptonic decays of the \( \Upsilon(1S) \) resonance and the threshold production of top quark-antiquark pairs.
in $e^+e^-$ annihilation. Both processes are essentially photon mediated and thus governed by the electromagnetic quark current $j_\mu = q^\gamma_\mu q$. Within the effective theory, $j_\mu$ has the following decomposition in terms of operators constructed from the nonrelativistic quark and antiquark two-component Pauli spinors $\psi$ and $\chi$ [16]:

$$j_i = c_v(\mu)\psi_i\sigma_i\chi + \frac{d_v(\mu)}{6m_q^2}\psi_i\sigma_iD^2\chi + \ldots,$$

where $\mu$ is the renormalization scale, $D$ are the space components of the gauge-covariant derivative involving the gluon fields, and the ellipsis stands for operators of higher mass dimension. The Wilson coefficients $c_v(\mu)$ and $d_v(\mu)$ may be evaluated as series in $\alpha_s(\mu)$ and represent the contributions from the hard modes (where energy and three-momentum scale like $m_q$) that have been integrated out. They are computed in full QCD for on-shell on-threshold external (anti)quark fields and are logarithmic functions of $\mu/m_q$. Also integrating out the soft (energy and three-momentum scale like $m_qv$, where $v$ is the heavy-quark velocity) modes and the potential (energy scales like $m_qv^2$, while three-momentum scales like $m_qv^2$) gluons yields the effective Hamiltonian of pNRQCD, which contains the potential (anti)quarks and the ultrasoft (energy and three-momentum scale like $m_qv^2$) gluons as active particles. The dynamics of the nonrelativistic potential heavy-quark-antiquark pair in pNRQCD is governed by the corresponding effective Schrödinger equation and its multipole interactions with the ultrasoft gluons. The effective-theory expression for the partial decay width of $\Upsilon(1S) \to l^+l^-$ reads [9]

$$\Gamma_1 = \Gamma_1^{LO}\rho_1\left[c_v^2(m_b) + \frac{C_F^2\alpha_s^2}{12}c_v(m_b)(d_v(m_b) + 3) + \ldots\right],$$

with $\Gamma_1^{LO} = 4\pi N_cQ_q^2\alpha_s^2|\psi_1^C(0)|^2/(3m_b^2)$ and $\rho_1 = |\psi_1(0)|^2/|\psi_1^C(0)|^2$, where $N_c = 3$, $Q_q$ is the fractional electric charge of quark $q$, $\alpha$ is Sommerfeld’s fine-structure constant, $\psi_1(x)$ is the ground-state wave function as computed in pNRQCD, and $\psi_1^C(x)$ is the Coulomb solution, which incorporates the leading binding effects and about which the perturbative expansion of $\psi_1(x)$ is constructed. For arbitrary principal quantum number $n$, we have $|\psi_n^C(0)|^2 = C_F^2\alpha_s^3m_q^3/(8\pi n^3)$, where $C_F = (N_c^2 - 1)/(2N_c)$. Here and in the following, $\alpha_s(\mu)$ is to be evaluated at the soft normalization scale $\mu_s = C_F\alpha_s(\mu_s)m_q$ whenever its argument is omitted. Nonperturbative contributions to Eq. (2) are ignored. The leading one, due to the gluon condensate of the vacuum, may be found in Ref. [19]. It is quite sizeable and out of control for higher resonances. A reliable quantitative estimate of the nonperturbative contributions to Eq. (2) can only be obtained through lattice simulations. On the other hand, to keep the nonperturbative effects under control, one can employ nonrelativistic $\Upsilon$ sum rules [2] based on the global-duality concept.

In the top-quark case, the nonperturbative effects are negligible. However, the effect of the top-quark total decay width $\Gamma_t$ has to be properly taken into account [3], as it is relatively large and smears out the Coulomb-like resonances below threshold. The NNLO\textsuperscript{1} analysis of the cross section [10] shows that only the ground-state pole gives rise

\textsuperscript{1}In the effective-theory framework, one has two expansion parameters, $\alpha_s$ and $v$, and the corrections
to a prominent resonance. The value of the normalized cross section \( R = \sigma(e^+e^- \rightarrow t\bar{t})/\sigma(e^+e^- \rightarrow \mu^+\mu^-) \) at the resonance energy is dominated by the contribution from the would-be toponium ground-state, which is of the form

\[
R_1 = R_1^{\text{LO}} \rho_1 \left[ c_v^2(m_t) + \frac{C_F^2 \alpha_s^2}{12} c_v(m_b) (d_v(m_b) + 3) + \ldots \right], \tag{3}
\]

with \( R_1^{\text{LO}} = 6\pi N_c Q_1^2 |\psi_1^C(0)|^2/(m_t^2 \Gamma_t) \). The contributions from the higher Coulomb-like poles and the continuum are not included in Eq. (3), and we postpone the complete analysis to a future publication. It is understood that \( \alpha \) appearing in \( \Gamma_1^{\text{LO}} \) and \( R_1^{\text{LO}} \) is to be evaluated at the mass scale of the respective resonance.

Starting from \( \mathcal{O} (\alpha_s^2) \), \( c_v(\mu) \) is infrared (IR) divergent. This divergence arises in the process of scale separation and is canceled against the ultraviolet (UV) one of the effective-theory result for the wave function at the origin. In our approach, dimensional regularization with \( d = 4 - 2\varepsilon \) space-time dimensions is used to handle the divergences, and the formal expressions derived from the Feynman rules of the effective theory are understood in the sense of the threshold expansion. This formulation of effective theory possesses two crucial virtues: the absence of additional regulator scales and the automatic matching of the contributions from different scales. For convenience, we subtract the IR and UV poles in \( c \) according to the modified minimal-subtraction (MS) prescription and set \( \mu = m_q \), so that \( c_v(m_q) \) is devoid of logarithms. The latter is known through \( \mathcal{O} (\alpha_s^2) \) and reads [20]

\[
c_v(m_q) = 1 - \frac{\alpha_s(m_q)}{\pi} 2C_F + \left( \frac{\alpha_s(m_q)}{\pi} \right)^2 \left[ -\frac{151}{72} \right.
+ \frac{89\pi^2}{144} - \frac{5\pi^2}{6} \ln 2 - \frac{13}{4} \zeta(3) \right] C_A C_F + \left( \frac{23}{8} - \frac{79\pi^2}{36} \right) C_F T_F
+ \frac{\pi^2}{2} \ln 2 - \frac{1}{2} \zeta(3) \right] C_F^2 + \left( \frac{22}{9} - \frac{2\pi^2}{9} \right) C_F T_F

+ \frac{11}{18} C_F T_F n_t \right] + \ldots, \tag{4}
\]

where \( \alpha_s \) is renormalized in the MS scheme, \( C_A = N_c, T_F = 1/2, n_t \) is the number of light-quark flavors, and \( \zeta(x) \) is Riemann’s \( \zeta \) function with value \( \zeta(3) = 1.202057 \ldots \). To the order considered, we have \( d_v(m_q) = 1 \).

The corrections to \( |\psi_1^C(0)|^2 \) read

\[
\rho_1 = 1 + \frac{\alpha_s}{\pi} \left[ \left( 4 - \frac{2\pi^2}{3} \right) \beta_0 + \frac{3}{4} a_1 \right]
+ \left( \frac{\alpha_s}{\pi} \right)^2 \left\{ -C_A C_F + \left( -2 + \frac{2}{3} S(S+1) \right) C_F^2 \right\}
\]

are classified according to the total power of \( \alpha_s \) and \( v \) as leading order (LO), next-to-leading order (NLO), NNLO, N^3LO, etc.
The origin of the logarithmic corrections is the presence of several scales in the threshold problem. A logarithmic integral between different scales yields a term proportional to $\ln v$, which becomes $\ln \alpha_s$ for bound states that are approximately Coulombic, so that $v \propto \alpha_s$. In effective-theory calculations, the scale defining the upper (lower) limit of a logarithmic integral is set to infinity (zero), which induces a UV (IR) divergence. Thus, the logarithmic corrections can be identified with the effective-theory singularities, which dramatically simplifies the calculation. Our analysis proceeds along the lines of Ref. [23] (see also Ref. [24]), where similar corrections have been considered for the QED bound-state of positronium.

In the calculation, we employ the NLO effective Hamiltonian derived in Ref. [7] and take into account the retardation effects due to the chromoelectric dipole interaction of the heavy-quark-antiquark pair with the dynamical ultrasoft gluons studied in Refs. [7,25]. Our result reads

$$\times \pi^2 \ln(C_F \alpha_s) + \left(-\frac{5\pi^2}{3} + 20\zeta(3) + \frac{\pi^4}{9}\right)\beta_0^2$$

$$+ \left(4 - \frac{2\pi^2}{3}\right)\beta_1 + \left(\frac{5}{2} - \frac{2\pi^2}{3}\right)\beta_{01} + \frac{3}{16}a_1^2 + \frac{3}{16}a_2$$

$$+ \frac{9\pi^2}{4}C_A C_F + \left(\frac{33\pi^2}{8} - \frac{13\pi^2}{9}S(S + 1)\right)C_F^2\right\}$$

$$+ \frac{\alpha^3}{\pi} \left(\left[-2C_A C_F + \left(-4 + \frac{4}{3}S(S + 1)\right)C_F^2\right]\beta_0$$

$$- \frac{2}{3}C_A^2 C_F + \left(-\frac{41}{12} + \frac{7}{12}S(S + 1)\right)C_A C_F^2 - \frac{3}{2}C_F^3\right]$$

$$\times \ln^2(C_F \alpha_s) + C_1 \ln(C_F \alpha_s) + \ldots\right\} + \ldots,$$  \hspace{1cm} (5)

where $\beta_i$ is the $(i+1)$-loop coefficient of the QCD $\beta$ function ($\beta_0 = 11C_A/12 - T_F n_f/3, \ldots$) and $a_i$ parameterizes the $i$-loop correction to the Coulomb potential ($a_1 = 31C_A/9 - 20T_F n_f/9, \ldots$ [21]). For the processes under consideration, the total spin of the quark-antiquark pair is $S = 1$. Nevertheless, we retain the full $S$ dependence, so that our result is also applicable to processes with $S = 0$, such as the decay $\eta_b \rightarrow \gamma \gamma$ or the production process $\gamma \gamma \rightarrow t\bar{t}$ at a future high-energy photon collider, which is dominated by the $S$ wave [22]. The corrections through $O(\alpha_s^2)$ have been derived in Ref. [9] for arbitrary $n$. The $O(\alpha_s^3 \ln^2 \alpha_s)$ correction has been obtained in Ref. [11]. In this Letter, we present the $O(\alpha_s^3 \ln \alpha_s)$ correction by specifying the missing coefficient $C_1$ in Eq. (5).

The origin of the logarithmic corrections is the presence of several scales in the threshold problem. A logarithmic integral between different scales yields a term proportional to $\ln v$, which becomes $\ln \alpha_s$ for bound states that are approximately Coulombic, so that $v \propto \alpha_s$. In effective-theory calculations, the scale defining the upper (lower) limit of a logarithmic integral is set to infinity (zero), which induces a UV (IR) divergence. Thus, the logarithmic corrections can be identified with the effective-theory singularities, which dramatically simplifies the calculation. Our analysis proceeds along the lines of Ref. [23] (see also Ref. [24]), where similar corrections have been considered for the QED bound-state of positronium.

In the calculation, we employ the NLO effective Hamiltonian derived in Ref. [7] and take into account the retardation effects due to the chromoelectric dipole interaction of the heavy-quark-antiquark pair with the dynamical ultrasoft gluons studied in Refs. [7,25]. Our result reads

$$C_1 = \left[-\frac{3}{2} + \frac{2\pi^2}{3}\right]C_A C_F + \left(\frac{4\pi^2}{3} - \left(-\frac{10}{9} + \frac{4\pi^2}{9}\right)\right)$$

$$\times S(S + 1)C_F^2\right]\beta_0 + \left[\left[-\frac{3}{4}C_A C_F + \left(-\frac{9}{4} + \frac{2}{3}S(S + 1)\right)\right]$$

$$\times C_F^2\right]a_1 + \left[\left[\frac{19}{108}S(S + 1)\right]C_A C_F^2 + \left(-\frac{35}{18} + 8\ln 2 - \frac{1}{3}\right)\right]$$

$$+ \ldots,$$
\[
\times S(S + 1) C_F^2 + \left(-\frac{32}{15} + 2 \ln 2 + (1 - \ln 2) S(S + 1)\right) \times C_F^2 T_F + \frac{49}{36} C_A C_F T_F n_l + \left(\frac{8}{9} - \frac{10}{27} S(S + 1)\right) C_F^2 T_F n_l.
\]  

(6)

For the analysis of Υ sum rules and \(t\bar{t}\) threshold production, one needs the extension of this result to arbitrary \(n\), which, leaving aside the trivial \(n\) dependence of \(|\psi^C_n(0)|^2\), is given by

\[
C_n = C_1 + \left[-2 C_A C_F + \left(-4 + \frac{4}{3} S(S + 1)\right) C_F^2\right] \beta_0
\times \left(\ln n + \Psi_1(n + 1) - 2 n \Psi_2(n) - 3 + \gamma_E + \frac{\pi^2}{3} + \frac{2}{n}\right)
+ \left[\frac{4}{3} C^2_A C_F + \left(\frac{41}{6} - \frac{7}{6} S(S + 1)\right) C_A C_F^2 + 3 C_F^3\right] \left(\ln n - \Psi_1(n) - 1 - \gamma_E + \frac{1}{n}\right)
+ \left(\frac{5}{3} - \frac{5}{3 n^2}\right) C_A C_F^2,
\]

(7)

where \(\Psi_n(x) = d^n \ln \Gamma(x)/dx^n\), \(\Gamma(x)\) is Euler’s \(\Gamma\) function, and \(\gamma_E = 0.577216\ldots\) is Euler’s constant.

The structure of the IR singularities in the Wilson coefficients can be read off from the UV-singular part of the effective-theory result for the wave function at the origin. In this way, we find the \(\mu\) dependence of \(c_v(\mu)\) to be

\[
c_v^2(\mu) = c_v^2(m_q) \left\{1 + \alpha^2_s(\mu) \gamma_v^{(2)} \ln \frac{\mu^2}{m_q^2} + \frac{\alpha^3(\mu)}{\pi}\right. \\
\times \left[\left(-\frac{3}{2} \beta_0 \gamma_v^{(2)} + \gamma_v^{(3)}\right) \ln^2 \frac{\mu^2}{m_q^2} + \left(-2 \beta_0 \gamma_v^{(2)}
+ \frac{4}{3} \gamma_v^{(3)} + \gamma_v^{(3)}\right) \ln \frac{\mu^2}{m_q^2} + \ldots\right] + \ldots,\]

(8)

where the two- [20] and three-loop anomalous dimensions of the nonrelativistic vector current are

\[
\gamma_v^{(2)} = -\frac{1}{2} C_A C_F + \left(-1 + \frac{1}{3} S(S + 1)\right) C_F^2,
\]

(9)

\[
\gamma_v^{(3)} = -\frac{1}{8} C^2_A C_F + \left(-\frac{29}{32} + \frac{7}{32} S(S + 1)\right) C_A C_F^2 - \frac{5}{16} C_F^3,
\]

\[
\gamma_v^{(3)} = \left[-\frac{1}{2} C_A C_F + \left(2 - \frac{11}{9} S(S + 1)\right) C_F^2\right] \beta_0
+ \left(-\frac{3}{8} + \frac{1}{12} S(S + 1)\right) C_F^2 a_1 - \left(\frac{53}{72} + \frac{3}{2} \ln 2\right) C_A C_F
+ \left(-\frac{107}{36} - \frac{3}{2} \ln 2 + \frac{373}{432} S(S + 1)\right) C_A C_F^2 + \left(-\frac{7}{8} + \frac{3}{2} \ln 2 + \frac{113}{48}\right) C_F^3
\]
\[ +3 \ln 2 - \frac{1}{4} S(S + 1) \right] C_F^3 + \left( -\frac{8}{5} + \frac{3}{2} \ln 2 + \left( -\frac{3}{4} - \frac{3}{4} \ln 2 \right) \right) \times S(S + 1) \right] C_F^2 T_F + \frac{49}{72} C_A C_F T_F n_l + \left( \frac{4}{9} - \frac{5}{27} \right) \times S(S + 1) \right] C_F^2 T_F n_l, \]

with \( S = 1 \). We retain the full \( S \) dependence in Eqs. (9) and (10) because, for \( S = 0 \), they give the anomalous dimension of the nonrelativistic pseudoscalar current \( \psi \gamma^\mu \chi \), which is relevant for two-photon processes. Note that the \( S = 0 \) result for \( C_1 \) and \( \gamma^{(3)}_0 \) corresponds to the \( d \)-dimensional spinor algebra in a regularization scheme adopted in [26] for the calculation of the two-loop hard corrections to the two-photon heavy quarkonium production and annihilation.

Let us now explore the numerical significance of our results. Putting everything together and substituting constants by their approximate numerical values, Eq. (2) becomes

\[
\Gamma_1 \approx \Gamma_1^{\text{LO}} \left( 1 - 1.70 \alpha_s(m_b) - 7.98 \alpha_s^2(m_b) + \ldots \right) \times \left( 1 - 0.30 \alpha_s - 5.19 \alpha_s^2 \ln \alpha_s + 17.16 \alpha_s^2 \right. \\
- 14.38 \alpha_s^3 \ln^2 \alpha_s - 0.165 \alpha_s^3 \ln \alpha_s + \ldots \right). \tag{11}
\]

Note that the NNLO contribution due to the second term in the square brackets of Eq. (2) is included in the third factor on the right-hand side of Eq. (11), as the corresponding appearance of \( \alpha_s \) is of nonrelativistic origin and normalized at the soft scale. Evaluating this using \( \alpha_s(M_Z) = 0.1185 \) and \( m_b = 5.3 \text{ GeV} \) [8], we obtain

\[
\Gamma_1 \approx \Gamma_1^{\text{LO}}(1 - 0.449_{\text{NLO}} + 1.771_{\text{NNLO}} - 0.766_{N^3\text{LO}'} + \ldots). \tag{12}
\]

where only the logarithmic \( N^3\text{LO} \) terms are retained, which is indicated by the prime on the subscript. A similar analysis of Eq. (3) with \( m_t = 174.3 \text{ GeV} \) yields

\[
R_1 \approx R_1^{\text{LO}}(1 - 0.244_{\text{NLO}} + 0.438_{\text{NNLO}} - 0.196_{N^3\text{LO}'} + \ldots). \tag{13}
\]

Without the \( O(\alpha_s^3 \ln \alpha_s) \) term, the \( N^3\text{LO} \) contributions in Eqs. (12) and (13) read \(-0.560\) and \(-0.148\), respectively. We learn the following: (i) while the coefficients in Eq. (11) and the analogous series for \( R_1 \) sharply increase in magnitude as we pass from NLO to NNLO, this disquieting trend discontinues as we move on to \( N^3\text{LO} \); (ii) the coefficients of the known NNLO and \( N^3\text{LO} \) terms are typically of order 10. These observations suggest that the magnitude of the coefficient of the missing non-logarithmic \( O(\alpha_s^3) \) term is unlikely to exceed this characteristic benchmark by far. This term would then be expected to yield corrections of order 25% and 3% to \( \Gamma_1 \) and \( R_1 \), respectively. This provides us with an estimate of the residual uncertainty of our approximation as far as the perturbative QCD corrections are concerned. Moreover, the absence of a rapid growth of the coefficients along with the alternating-sign character of the series suggest that the higher-order corrections are likely to be below these estimates. These observations can be
substantiated by investigating the scale dependence of $\Gamma_1$ and $R_1$. In fact, the shifts in these quantities due to a variation of $\mu_s$ by a factor of 2 are reduced from 50% and 13% to 19% and 9%, respectively, as we pass from NNLO to $N^3$LO. The latter values are in the same ballpark as the theoretical uncertainties estimated above. This renders a reliable perturbative description of heavy-quarkonium production and annihilation feasible.

To conclude, we computed the logarithmically enhanced $N^3$LO corrections to the heavy-quarkonium production and annihilation rates. Our results provide a useful hint on the general structure of the high-order corrections and open a new perspective for the theoretical analysis. Together with the $\mathcal{O}(\alpha_s^3 m_q)$ result for the heavy-quarkonium spectrum [7,8], they constitute central ingredients for the high-precision analysis of $\Upsilon$ sum rules and $t\bar{t}$ threshold production in $e^+e^-$ and $\gamma\gamma$ scattering. Calculation of the remaining non-logarithmic $N^3$LO term appears to be mandatory for reducing the theoretical uncertainty further. Another challenging problem is to complete the resummation of the next-to-next-to-leading logarithms [13]. Our derivation of the three-loop anomalous dimension of the nonrelativistic vector current marks a major step in this direction.

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**Note added:**
In a recent paper [27] the infrared structure of the three-loop corrections to the heavy quark pair production current was analyzed in the effective theory framework. On the basis of this investigation the next-to-next-to leading logarithmic corrections to the heavy quarkonium threshold production were partially resummed. By expanding this result in $\alpha_s$ the $\mathcal{O}(\alpha_s^3 \ln \alpha_s)$ correction to the heavy-quarkonium production and annihilation rates can be obtained. The result announced in [27] for the $S = 1$ spin-one process agrees with our result for $C_1$, Eq. (6).

**References**

[1] T. Appelquist and H.D. Politzer, Phys. Rev. Lett. **34**, 43 (1975).

[2] V.A. Novikov, L.B. Okun, M.A. Shifman, A.I. Vainshtein, M.B. Voloshin, and V.I. Zakharov, Phys. Rev. Lett. **38**, 626 (1977); **38**, 791(E) (1977); Phys. Rep. C **41**, 1 (1978).

[3] V.S. Fadin and V.A. Khoze, Pis’ma Zh. Eksp. Teor. Fiz. **46**, 417 (1987) [JETP Lett. **46**, 525 (1987)].

[4] M.B. Voloshin, Nucl. Phys. **B154**, 365 (1979); Yad. Fiz. **36**, 247 (1982) [Sov. J. Nucl. Phys. **36**, 143 (1982)]; H. Leutwyler, Phys. Lett. **98B**, 447 (1981).

[5] I.Y. Bigi, M.A. Shifman, and N. Ural'tsev. Annu. Rev. Nucl. Part. Sci. **47**, 591 (1997).
[6] M.J. Strassler and M.E. Peskin, Phys. Rev. D 43, 1500 (1991).

[7] B.A. Kniehl, A.A. Penin, V.A. Smirnov, and M. Steinhauser, Phys. Rev. D 65, 091503(R) (2002); Nucl. Phys. B635, 357 (2002).

[8] A.A. Penin and M. Steinhauser, Phys. Lett. B 538, 335 (2002).

[9] J.H. Kühn, A.A. Penin, and A.A. Pivovarov, Nucl. Phys. B534, 356 (1998); A.A. Penin and A.A. Pivovarov, Phys. Lett. B 435, 413 (1998); Nucl. Phys. B549, 217 (1999); B550, 375 (1999); K. Melnikov and A. Yelkhovsky, Phys. Rev. D 59, 114009 (1999).

[10] A.H. Hoang, M. Beneke, K. Melnikov, T. Nagano, A. Ota, A.A. Penin, A.A. Pivovarov, A. Signer, V.A. Smirnov, Y. Sumino, T. Teubner, O. Yakovlev, and A. Yelkhovsky, Eur. Phys. J.direct C 3, 1 (2000).

[11] B.A. Kniehl and A.A. Penin, Nucl. Phys. B577, 197 (2000); A.V. Manohar and I.W. Stewart, Phys. Rev. D 63, 054004 (2001).

[12] M.E. Luke, A.V. Manohar, and I.Z. Rothstein, Phys. Rev. D 61, 074025 (2000).

[13] A.H. Hoang, A.V. Manohar, I.W. Stewart, and T. Teubner, Phys. Rev. Lett. 86, 1951 (2001); Phys. Rev. D 65, 014014 (2002).

[14] A. Pineda, Phys. Rev. D 65, 074007 (2002); Phys. Rev. D 66, 054022 (2002).

[15] A. Pineda and J. Soto, Phys. Lett. B 420, 391 (1998); Phys. Rev. D 59, 016005 (1999); A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A 59, 4316 (1999); M. Beneke, A. Signer, and V.A. Smirnov, Phys. Lett. B 454 (1999) 137.

[16] W.E. Caswell and G.P. Lepage, Phys. Lett. B 167, 437 (1986); G.T. Bodwin, E. Braaten, and G.P. Lepage, Phys. Rev. D 51, 1125 (1995); 55, 5853(E) (1997).

[17] A. Pineda and J. Soto, Nucl. Phys. B (Proc. Suppl.) 64, 428 (1998); N. Brambilla, A. Pineda, J. Soto, and A. Vairo, Nucl. Phys. B566, 275 (2000).

[18] M. Beneke and V.A. Smirnov, Nucl. Phys. B522, 321 (1998); V.A. Smirnov, Applied Asymptotic Expansions in Momenta and Masses, (Springer-Verlag, Heidelberg, 2001).

[19] S. Titard and F.J. Yndurain, Phys. Rev. D 51, 6348 (1995); A. Pineda, Nucl. Phys. B494, 213 (1997).

[20] A. Czarnecki and K. Melnikov, Phys. Rev. Lett. 80, 2531 (1998); M. Beneke, A. Signer, and V.A. Smirnov, ibid. 80, 2535 (1998).

[21] Y. Schröder, Phys. Lett. B 447, 321 (1999) and references cited therein.
[22] A.A. Penin and A.A. Pivovarov, Nucl. Phys. B550, 375 (1999); Yad. Fiz. 64, 323 (2001) [Phys. Atom. Nucl. 64, 275 (2001)]; A. Czarnecki and K. Melnikov, Phys. Rev. D65, 051501 (2002).

[23] B.A. Kniehl and A.A. Penin, Phys. Rev. Lett. 85, 1210 (2000); 85, 3065(E) (2000); 85, 5094 (2000).

[24] R.J. Hill and G.P. Lepage, Phys. Rev. D 62, 111301 (2000); K. Melnikov and A. Yelkhovsky, ibid. 62, 116003 (2000); Phys. Rev. Lett. 86, 1498 (2001); A.A. Penin, Nucl. Phys. B (Proc. Suppl.) 96, 418 (2001); R.J. Hill, Phys. Rev. Lett. 86, 3280 (2001).

[25] B.A. Kniehl and A.A. Penin, Nucl. Phys. B563, 200 (1999).

[26] A. Czarnecki and K. Melnikov, Phys. Lett. B 519, 212 (2001).

[27] A. Hoang, Report No. MPP-2003-38 and hep-ph/0307376.