Cooper pairing at large $N$ in a 2-dimensional model

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Abstract

We study a 2-dimensional model of fermi fields $\psi$ that is closely related to the Gross-Neveu model, and show that to leading order in $\frac{1}{N}$ a $\langle \psi \psi \rangle$ condensate forms. This effect is independent of the chemical potential, a peculiarity that we expect to be specific to 2 dimensions. We also expect the condensate to be unstable against corrections at higher orders in $\frac{1}{N}$. We compute the Green’s functions associated with the composite $\psi\psi$, and show that the fermion acquires a Majorana mass proportional to the gap, and that a massless Goldstone pole appears.
Recently several papers have appeared [1-3] dealing with the properties of QCD at high density. The basic procedure is to approximate QCD by a direct four-quark interaction term, justifying this either by appeal to instanton effects or to one-gluon exchange. In the presence of a chemical potential, this theory admits a condensate of quark-quark pairs, very similar to the Cooper pairs that are well-known in the BCS theory of superconductivity. This phenomenon, which has been dubbed "color superconductivity," may or may not be accessible to experiments on heavy-ion collisions that will be performed over the next few years.

In this paper we shall examine similar phenomena in the context of a one-plus-one dimensional model that is a close relative to the Gross-Neveu (GN) model [4]. From its inception, it has been recognized that the GN model exhibits many of the same features as QCD, such as asymptotic freedom and spontaneous chiral symmetry breaking. Moreover, the four-fermi interaction is the model - there is no need to regard it as an approximation to an underlying gauge theory. Unlike in higher dimensions, in two dimensions this interaction is renormalizable, and we shall find, just as in the usual GN model, that coupling constant renormalization removes all the divergences that we shall encounter. Another advantage is that by judiciously introducing a flavor index $i$, $1 \leq i \leq N$, one can insure that the mean-field approximation that is commonly used in analyzing the condensate in QCD is in the case of the 2-dimensional model justified as the leading contribution in powers of $1/N$.

There are, however, a couple of peculiarities associated with two dimensions that make this GN-like model qualitatively different from QCD. The first is the Coleman-Mermin-Wagner theorem [5], which forbids spontaneous symmetry breaking of a continuous symmetry. Whereas in the original GN model the broken symmetry is a discrete one, and hence not in conflict with the theorem, in this case the formation of a $\langle \psi \psi \rangle$ condensate breaks fermion number, a continuous symmetry. The same problem arises in the chiral GN model [4], where the symmetry is continuous, and in a variety of other two-dimensional models where the spontaneous breaking of a continuous symmetry is predicted in leading order in $1/N$. This means that instabilities must arise in higher order that vitiate the prediction of a condensate. However, as Witten has pointed out [6], the $1/N$ expansion may still be an excellent guide to the physics of the model, except for the formation of the condensate (what happens in these models is that the condensate "almost" forms, in the sense that the pair-pair correlation function decays in the infrared only like a power instead of exponentially, and the power vanishes as $N \to \infty$).
The second peculiarity is, as we shall show below, the chemical potential has nothing to do with the formation of the condensate. In higher dimensions the chemical potential is crucial, because it gives rise to the Fermi surface at which the gap equation has an infrared singularity as the gap goes to zero. It is this feature that insures that the gap equation will have a solution for arbitrarily weak coupling. In two dimensions, however, the Fermi surface has dimension zero, and the infrared singularity exists whether or not there is a chemical potential. In fact, the gap equation turns out to be completely independent of the chemical potential. This behavior will be exhibited explicitly below.

The model we consider is defined by the following Lagrangian:

$$L = \bar{\psi}^{(i)} i \partial \psi^{(i)} + 2g^2 \bar{\psi}^{(i)} \gamma_5 \psi^{(j)} \bar{\psi}^{(i)} \gamma_5 \psi^{(j)}.$$(1)

$\psi^{(i)}$ is a two-component spinor with a flavor index that takes on $N$ values. Repeated flavor indices in eqn. (1) are summed. Because of the unconventional arrangement of flavor indices in the second term, the model does not have $SU(N)$ symmetry, but it does possess $O(N)$ symmetry, $\psi^{(i)} \rightarrow \vartheta^{ij} \psi^{(j)}$ where $\vartheta^{ij}$ is a real $N \times N$ orthogonal matrix. The Lagrangian (1) also has a $U(1)$ symmetry, which we shall find is broken by a $\langle \psi \psi \rangle$ condensate, whereas the $O(N)$ symmetry is kept intact.

Our representation for the $\gamma$-matrices is: $\gamma^0 = \sigma_1$; $\gamma^1 = -i \sigma_2$; $\gamma^5 = \sigma_3$, and it is then easy to check that

$$\bar{\psi}^{(i)} \gamma_5 \psi^{(j)} \bar{\psi}^{(i)} \gamma_5 \psi^{(j)} = -\frac{1}{2} (\epsilon_{\alpha\beta} \psi^{(i)}_{\alpha} \psi^{(i)}_{\beta})(\epsilon_{\gamma\delta} \psi^{(j)}_{\gamma} \psi^{(j)}_{\delta}).$$ (2)

Following the usual Hubbard-Stratonovich procedure, in the form introduced by Coleman [7], we add to $L$ the term

$$- \frac{1}{g^2}(B^\dagger - g^2 \epsilon_{\alpha\beta} \psi^{(i)}_{\alpha} \psi^{(i)}_{\beta})(B + g^2 \epsilon_{\gamma\delta} \psi^{(j)}_{\gamma} \psi^{(j)}_{\delta})$$ (3)

which does not affect the physics because $B$ and $B^\dagger$ are simply auxiliary fields. We then have

$$L = \bar{\psi}^{(i)} (i \partial - \mu \gamma^0) \psi^{(i)} - \frac{1}{g^2} B^\dagger B + B \epsilon_{\alpha\beta} \psi^{(i)}_{\alpha} \psi^{(i)}_{\beta} - B^\dagger \epsilon_{\alpha\beta} \psi^{(i)}_{\alpha} \psi^{(i)}_{\beta}$$ (4)
where we have also introduced a chemical potential $\mu$. In anticipation of taking the large $N$ limit, $N \to \infty$ with $\lambda = g^2 N$ fixed, we rewrite the second term as $-\frac{N}{\lambda} B^\dagger B$.

Thus the classical (or tree-level) term in $V_{\text{eff}}(B^\dagger B)$ is of order $N$. As we perform a perturbation expansion, we observe that additional factors of $N$ arise in 2 ways: (i) the $B - B^\dagger$ propagator is proportional to $1/N$; and (ii) each closed fermion loop gives a factor of $N$ from summing on the flavor index. If we examine the computation of higher-loop corrections to the effective potential (i.e. the summation over all one-particle-irreducible diagrams with zero-momentum $B$ and $B^\dagger$ external legs) we see that the one-loop term is of order $N$ (one fermion loop and no $B - B^\dagger$ propagator) whereas anything else is of order $N^0$ or lower. Hence the leading contribution is just what we get from keeping the tree and one-loop graphs. Moreover, in a path-integral approach, one first integrates out the fermions. Then, because the exponent is proportional to $N$, one can employ the stationary phase approximation in the integral over $B$ and $B^\dagger$ to evaluate the integrand at the solution of the equations

$$\frac{\partial V}{\partial B} = \frac{\partial V}{\partial B^\dagger} = 0 .$$

(5)

The task of integrating out the fermions is complicated slightly by the $\psi \psi$ and $\psi^\dagger \psi^\dagger$ terms. We observe that

$$\int D\psi e^{i(\psi, M \psi)} = det^{1/2} M ,$$

(6)

where $M$ is any anti-symmetric matrix. We write the fermion part of our Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\psi^\dagger A \psi - \psi A^T \psi^\dagger) + \psi^\dagger B \psi^\dagger + \psi B^\dagger \psi$$

(7)

where

$$A = (i \partial_0 + i \sigma_3 \partial_x - \mu)_{\alpha\beta} \delta^{ij} ;$$

(8a)

$$A^T = (-i \partial_0 - i \sigma_3 \partial_x - \mu)_{\alpha\beta} \delta^{ij} ;$$

(8b)

$$B = B_{\epsilon_{\alpha\beta}} \delta^{ij} = i B(\sigma_2)_{\alpha\beta} \delta^{ij} ;$$

(8c)

$$B^\dagger = -i B^\dagger(\sigma_2)_{\alpha\beta} \delta^{ij} .$$

(8d)
Now we perform a translation,

$$\psi = \chi + \alpha \psi^\dagger$$  \hspace{1cm}(9a)

$$= \chi + \psi^\dagger \alpha^T$$  \hspace{1cm}(9b)

where $\alpha = \frac{1}{2} (B^\dagger)^{-1} A^T$; this factorizes the $\chi$ and $\psi^\dagger$ path integrals into the product of two path integrals, and after some manipulations and discarding an overall factor that is independent of $B$ and $B^\dagger$, we obtain

$$e^{i\Gamma_{eff}^{(1)}(B,B^\dagger)} = det^{1/2} [1 + 4A^{-1}B(A^T)^{-1}B^\dagger]$$  \hspace{1cm}(10)

where $\Gamma_{eff}^{(1)}$ denotes the one-loop contribution to the effective action.

Note that the matrix $A^{-1}$ is given by

$$A^{-1}(x,y) = -\int \frac{d^2k}{(2\pi)^2} \frac{[k_0 - \sigma_3 k_1 + \mu]_{\alpha\beta} \delta^{ij} e^{ik \cdot (x-y)}}{(k_0 + \mu + i\epsilon \text{ sgn}k_0)^2 - k_1^2}$$  \hspace{1cm}(11)

where $k \cdot x = k_0 x^0 + k_1 x^1$, and the $i\epsilon$ prescription is introduced in the proper way to take account of the chemical potential. $(A^T)^{-1}$ is the same expression but with $x$ and $y$ interchanged.

To obtain the effective potential, we take $B$ and $B^\dagger$ to be constant. It is then convenient to write everything in momentum space, and after some algebra, using $\Gamma_{eff} = -V_{eff} \int d^2x$, we obtain

$$V_{eff} = \frac{NB^\dagger B}{\lambda} + \frac{iN}{2} \int \frac{d^2k}{(2\pi)^2}$$

$$tr ln \left[ 1 - \frac{4B^\dagger B}{[(k_0 + \mu + i\epsilon \text{ sgn}k_0) + k_1 \sigma_3] [(k_0 - \mu + i\epsilon \text{ sgn}k_0) - k_1 \sigma_3]} \right]$$  \hspace{1cm}(12)

where the trace is over the spinor indices only. Setting $B^\dagger B = M/4$ and $\kappa = 4\lambda$, and observing that $V_{eff}(M = 0) = 0$, we can write

$$\frac{1}{N} V_{eff} = \frac{1}{N} \int_0^M dM' \frac{dV}{dM'},$$  \hspace{1cm}(13)
where
\[
\frac{1}{N} \frac{dV}{dM} = \frac{1}{\kappa} + \frac{i}{2} \int \frac{d^2k}{(2\pi)^2} \text{tr} \left[ \frac{1}{M - k_0^2 + (k_1\sigma_3 + \mu)^2 - i\epsilon} \right].
\]  
(14)

Here the trace is just summation over $\sigma_3 = \pm 1$.

We note that this integral is logarithmically divergent. We shall deal with this by renormalizing $\kappa$; but first we shall do the $k_0$ integral. Let $M + (k_1\sigma_3 + \mu)^2 = \omega^2$ with $\omega > 0$. We have
\[
I = \int_{-\infty}^{\infty} dk_0 \frac{1}{k_0 - \omega^2 + i\epsilon} = \int_{-\infty}^{\infty} dk_0 \frac{1}{(k_0 - \omega + i\epsilon)(k_0 + \omega - i\epsilon)}
\]
\[
= -\frac{\pi i}{\omega}
\]
(15)

so
\[
\frac{1}{N} \frac{dV}{dM} = \frac{1}{\kappa} - \frac{1}{8\pi} \int_{-\infty}^{\infty} dk_1 \text{tr} \left( \frac{1}{\omega} \right).
\]  
(16)

We renormalize by requiring that at $\mu = 0$,
\[
\frac{1}{N} \frac{\partial^2 V_{eff}}{\partial B \partial B^\dagger} \bigg|_{B^\dagger B = M_0/4} = \frac{4}{\kappa_R}.
\]  
(17)

which is the same as
\[
\frac{1}{\kappa_R} = \frac{1}{N} \left[ \frac{\partial V}{\partial M} + M_0^2 \frac{\partial^2 V}{\partial M^2} \right] \bigg|_{M_0}.
\]  
(18)

The solution to this is
\[
\frac{1}{\kappa} = \frac{1}{\kappa_R} + \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{dk_1}{\omega_0} + \delta X
\]  
(19)

where $\omega_0^2 = k_1^2 + M_0$, and where for our choice of renormalization prescription, $\delta X = -\frac{1}{4\pi}$. Some other choice of prescription would yield a different pure number for $\delta X$.

We then have
\[
\frac{1}{N} \frac{dV}{dM} = \frac{1}{\kappa R} - \frac{1}{8\pi} \int_{-\infty}^{\infty} dk_1 tr \left[ \frac{1}{\omega} - \frac{1}{\omega_0} \right] + \delta X .
\] (20)

The gap equation is just the statement that \( \frac{dV}{dM} \) vanishes:

\[
\delta X + \frac{1}{\kappa R} = \frac{1}{8\pi} \int_{-\infty}^{\infty} dk_1 tr \left[ \frac{1}{\omega} - \frac{1}{\omega_0} \right] .
\] (21)

Note that \( tr \frac{1}{\omega} = \frac{1}{\sqrt{M+(k_1+\mu)^2}} + \frac{1}{\sqrt{M+(k_1-\mu)^2}} \) which is even in \( k_1 \). Therefore

\[
\delta X + \frac{1}{\kappa R} = \frac{1}{4\pi} J , \quad \text{where } J \text{ is the integral}
\] (22)

\[
J = \int_{0}^{\infty} dk \left[ \frac{1}{\sqrt{M+(k+\mu)^2}} + \frac{1}{\sqrt{M+(k-\mu)^2}} - \frac{2}{\sqrt{M_0+k^2}} \right] .
\] (23)

If we evaluate \( J \), we can find the particular \( M \) that obeys eqn. (22), thereby solving the gap equation. We can also obtain the more general expression

\[
\frac{1}{N} \frac{dV}{dM} = \frac{1}{\kappa R} - \frac{1}{4\pi} J + \delta X
\]

as a function of \( M \), and integrate it to obtain \( V(M) \) via eqn. (13).

We find, after some mild computational exertions, that

\[
J(M) = -ln \frac{M}{M_0}
\] (24)

and

\[
\frac{1}{N} V(M) = \left( \frac{1}{\kappa R} + \delta X \right) M + \frac{M}{4\pi} (lnM/M_0 - 1) + \vartheta \left( \frac{1}{N} \right) .
\] (25)

As advertised, these expressions are independent of \( \mu \). The solution to the gap equation for our choice of \( \delta X \) is

\[
M = M_0 e^{(1-4\pi/\kappa R)} .
\] (26)
There is no critical lower bound to $\kappa_R$ below which no solution exists. We see from eqn. (23) that this is due to the fact that as $M \to 0$, the expression for $J$ diverges logarithmically. This infrared singularity is present in 2 dimensions independent of the value of $\mu$. In higher dimensions, we expect this singularity to be present at the Fermi surface, $k = |\mu|$, and to disappear as $\mu \to 0$.

We see from eqn. (26) that as $M_0$ is increased for fixed $M$, $\kappa_R$ becomes smaller. This is an indication that the coupling $\kappa_R$ is asymptotically free, just as in the original GN model. In fact, it is not hard to show from the renormalization condition (19) with cutoff $\Lambda$ that the beta function has the form

$$\beta(\kappa) = -\frac{\kappa^2}{2\pi},$$

so that the gap, eq. (26), obeys $(2M_0 \frac{\partial}{\partial M_0} - \beta(\kappa_R) \frac{\partial}{\partial \kappa_R})M = 0$.

As a consequence of the Coleman-Mermin-Wagner theorem we expect that terms which are higher order in $\frac{1}{N}$ will destabilize the leading order result, i.e. will give rise to contributions that dominate the ones we have found for sufficiently large $M$.

We may also [4], [8] compute the Green’s functions associated with the fields $B$ and $B^\dagger$. To do this, we need the effective action, not just the effective potential. This can be read off from equation (10):

$$\Gamma_{\text{eff}} = \int d^4x \left( -\frac{4N}{\kappa} B^\dagger(x)B(x) - \frac{i}{2} \text{Tr} \ln \left[ 1 + 4A^{-1}B\tilde{A}^{-1}B^\dagger \right] \right). \quad (27)$$

Here we have defined $\tilde{A} = \sigma_2 A^T \sigma_2$. Because the gap equation is independent of $\mu$, we shall simplify our task somewhat by setting $\mu = 0$; then $A^T = -A$, and

$$A = i\partial_0 + i\sigma_3 \partial_1 \quad (28)$$

$$\tilde{A} = -i\partial_0 + i\sigma_3 \partial_1. \quad (29)$$

Note that $A\tilde{A} = \tilde{A}A = \partial_0^2 - \partial_1^2$.

To obtain the desired Green’s functions, we perform the following sequence of steps:

(a) We write

$$B = B_0 + B' \quad (30a)$$
\[ B^\dagger = B_0 + B'^\dagger. \] (30b)

where, as above, \( B_0 \) is a solution of the gap equation (\( M = 4B_0^2 \)).

(b) We expand \( \Gamma_{\text{eff}} \) to second order in \( B' \) and \( B'^\dagger \). The linear terms in \( B' \) and \( B'^\dagger \) will cancel because of the gap equation. The coefficients of the quadratic terms will be the inverses of the Green’s functions that we seek.

(c) We observe that by introducing the real and imaginary parts of \( B' \): \( B' = \phi_1 + i\phi_2 \), \( B'^\dagger = \phi_1 - i\phi_2 \), the off-diagonal terms will disappear; i.e. there will be no mixed \( \phi_1\phi_2 \) terms. In fact, what we find is:

\[
\Gamma_{\text{eff}} \simeq -4N \int d^2xd^2y\{\phi_1(x)\phi_1(y)\int \frac{d^2p}{(2\pi)^2} e^{-ip \cdot (x-y)} \left[ \frac{1}{\kappa} - \frac{i}{2} \Phi_+(p) \right] + \phi_2(x)\phi_2(y)\int \frac{d^2p}{(2\pi)^2} e^{-ip \cdot (x-y)} \left[ \frac{1}{\kappa} - \frac{i}{2} \Phi_-(p) \right] \}
\] (31)

where

\[
\Phi_\pm(p) = 2 \int \frac{d^2\kappa}{(2\pi)^2} \frac{[\kappa_0(p_0 + \kappa_0) - \kappa_1(p_1 + \kappa_1) \pm M]}{[-\kappa_0^2 + \kappa_1^2 + M] [-\kappa_0^2 + \kappa_1^2 + M]}. \] (32)

Now \( \Phi_\pm(p) \) is logarithmically divergent, but so is \( \frac{1}{\kappa} \), and using the gap equation it is easy to see that the divergence cancels, along with all residual dependence on \( \kappa_R \) and \( \delta X \).

(d) The integrals defining \( \Phi_\pm(p) \) can be done explicitly, with the result that

\[
G_{11}(p) \equiv \frac{1}{\kappa} - \frac{i}{2} \Phi_+(p) = \frac{1}{4\pi\beta} \log\left[ \frac{1 + \beta}{1 - \beta} \right] \] (33)

\[
G_{22}(p) \equiv \frac{1}{\kappa} - \frac{i}{2} \Phi_-(p) = \frac{\beta}{4\pi} \log\left[ \frac{1 + \beta}{1 - \beta} \right]. \] (34)

Here \( \beta = \sqrt{\frac{p^2}{p^2 - 4M}} \), and \( p^2 = p_0^2 - p_1^2 \).

We see that both \( G_{11} \) and \( G_{22} \) give rise to a branch point at \( p^2 = 4M \), and become complex for \( p^2 > 4M \), whereas they are real for \( p^2 < 4M \). Furthermore, \( G_{22}(p) \) has a simple zero in \( p^2 \) at \( p^2 = 0 \), which means the corresponding Green’s function has a pole.
Together, these results suggest that the fermion acquires a mass $m_F = \sqrt{M}$, while the pole at $p^2 = 0$ is evidence for the existence of a would-be Goldstone boson that reflects the condensation of $\langle \psi \psi \rangle$.

In this note we have analyzed a 2-dimensional model that exhibits the formation of Cooper pairs in leading order in $\frac{1}{N}$. This condensation occurs for all values of the coupling (as long as the bare coupling is positive) for any value of the chemical potential $\mu$, including $\mu = 0$. The coupling itself is asymptotically free. At $\mu = 0$, we have computed the two-point functions associated with the composite fields $\psi \psi$ and $\psi^\dagger \psi^\dagger$, and have found that $\psi$ acquires a Majorana mass $2B_0$, where $B_0 = \langle \psi \psi \rangle$. We also find evidence for a massless pole, which indicates the spontaneous breaking of fermion number at large $N$. However, we expect that the $\langle \psi \psi \rangle$ condensate will be unstable against higher order corrections in $\frac{1}{N}$, so as not to violate the Coleman-Mermin-Wagner theorem. Bearing this in mind, we speculate that our $GN$-like model at large $N$ could serve as a theoretical laboratory for a one-dimensional superconductor [9].

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