**Research Article**

**Distance signless Laplacian eigenvalues, diameter, and clique number**

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**Abstract**

Let $G$ be a connected graph of order $n$. Let $\text{Diag}(\text{Tr})$ be the diagonal matrix of vertex transmissions and let $\mathcal{D}(G)$ be the distance matrix of $G$. The distance signless Laplacian matrix of $G$ is defined as $\mathcal{D}_s(G) = \text{Diag}(\text{Tr}) + \mathcal{D}(G)$ and the eigenvalues of $\mathcal{D}_s(G)$ are called the distance signless Laplacian eigenvalues of $G$. Let $\mathcal{D}_s(G) \geq \mathcal{D}_s(G) \geq \cdots \geq \mathcal{D}_s(G)$ be the distance signless Laplacian eigenvalues of $G$. The largest eigenvalue $\mathcal{D}_s(G)$ is called the distance signless Laplacian spectral radius. We obtain a lower bound for $\mathcal{D}_s(G)$ in terms of the diameter and order of $G$. With a given interval $I$, denote by $m_{\mathcal{D}_s(G)}(I)$ the number of distance signless Laplacian eigenvalues of $G$ which lie in $I$. For a given interval $I$, we also obtain several bounds on $m_{\mathcal{D}_s(G)}(I)$ in terms of various structural parameters of the graph $G$, including diameter and clique number.

**Keywords:** distance matrix; distance signless Laplacian matrix; spectral radius; diameter; clique number.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The order and size of $G$ are $|V(G)| = n$ and $|E(G)| = m$, respectively. The degree of a vertex $v$, denoted by $d_G(v)$, is the number of edges incident to the vertex $v$. In $G$, $N_G(v)$ is the set of all vertices which are adjacent to $v$. Further, $K_n$ denotes the complete graph on $n$ vertices. In a graph $G$, the subset $M \subseteq V(G)$ is called an independent set if no two vertices of $M$ are adjacent. A clique is a complete subgraph of a given graph $G$. The cardinality of the maximum clique is called the clique number of $G$ and is denoted by $\omega$. A vertex $u \in V(G)$ is called a pendant vertex if $d_G(u) = 1$. For other standard definitions, we refer the reader to [6, 11].

For $v_i, v_j \in V(G)$, the distance between $v_i$ and $v_j$, denoted by $d_G(v_i, v_j)$, is the length of a shortest path between $v_i$ and $v_j$. The diameter $d$ (or $\text{d}(G)$) of a graph $G$ is the maximum distance between any two vertices of $G$. The distance matrix of $G$, denoted by $\mathcal{D}(G)$, is defined as $\mathcal{D}(G) = (d_{ij})_{v_i, v_j \in V(G)}$. The transmission $Tr_G(v_i)$ (we will write $Tr(v_i)$ if the graph $G$ is understood) of a vertex $v_i$ is defined as the sum of the distances from $v_i$ to all other vertices in $G$, that is,

$$Tr_G(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j).$$

Let $Tr(G) = \text{diag}(Tr(v_1), Tr(v_2), \ldots, Tr(v_n))$ be the diagonal matrix of vertex transmissions of $G$. For a connected graph $G$, Aouchiche and Hansen [4] defined the distance Laplacian matrix of $G$ as $\mathcal{D}^L(G) = \text{Diag}(\text{Tr}) - \mathcal{D}(G)$ (or simply $\mathcal{D}^L$) and the distance signless Laplacian matrix as $\mathcal{D}_s(G) = Tr(G) + \mathcal{D}(G)$ (or simply $\mathcal{D}_s$). The eigenvalues of $\mathcal{D}_s(G)$ are called the distance signless Laplacian eigenvalues of $G$. Clearly, $\mathcal{D}_s(G)$ is a real symmetric matrix. We denote its eigenvalues by $\mathcal{D}_s(G)$’s and order them as $\mathcal{D}_s(G) \geq \mathcal{D}_s(G) \geq \cdots \geq \mathcal{D}_s(G)$. The largest eigenvalue $\mathcal{D}_s(G)$ is called the distance signless Laplacian spectral radius. Recent work on distance Laplacian matrix can be seen in [13, 14]. For more work done on distance signless Laplacian matrix of a graph $G$, we refer the reader to [1–3, 7–9, 12, 15–19]. If the graph $G$ is understood, we may write $\mathcal{D}_s$ in place of $\mathcal{D}_s(G)$ and refer the distance signless Laplacian eigenvalues as $\mathcal{D}_s$ eigenvalues. Let $m_{\mathcal{D}_s(G)}(I)$ be the number of distance signless Laplacian eigenvalues of $G$ that lie in the interval $I$. Also, let $m_{\mathcal{D}_s(G)}(\mathcal{D}_s(G))$ be the multiplicity of the distance signless Laplacian eigenvalue $\mathcal{D}_s(G)$.

In this paper, we obtain a lower bound for the distance signless Laplacian spectral radius of the graph $G$ in terms of diameter $d$ and order $n$. We show that the number of distance signless Laplacian eigenvalues in the interval $[n - 2, dn]$ is at least $d + 1$, where $d$ is the diameter of the graph $G$. We also obtain a lower bound for the number of distance signless Laplacian eigenvalues which fall in the interval $(n - 2, 2n - 2)$, in terms of the order $n$ and the number of vertices having

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degree \( n - 1 \). Moreover, we show that the number of distance signless Laplacian eigenvalues in the interval \([n - 2, 2n - \omega - 2]\) is at most \( n - \omega + 2 \), where \( n \) is the order and \( \omega \) is the clique number of the graph \( G \).

2. Distribution of distance signless Laplacian eigenvalues

We require the following lemmas to prove our main results.

Lemma 2.1. [5] Let \( G \) be a connected graph on \( n \geq 3 \) vertices. Then, \( \partial_1^Q(G) \geq \partial_1^Q(K_n) = 2n - 2 \) and \( \partial_i^Q(G) \geq \partial_i^Q(K_n) = n - 2 \) for all \( 2 \leq i \leq n \).

A particular case of the well known \( \min - \max \) theorem is the following result.

Lemma 2.2. [20] If \( N \) is a symmetric \( n \times n \) matrix with eigenvalues \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \), then for any \( x \in \mathbb{R}^n \) (\( x \neq 0 \)), we have

\[
\mu_1 \geq \frac{x^T N x}{x^T x},
\]

where the equality holds if and only if \( x \) is an eigenvector of \( N \) corresponding to the largest eigenvalue \( \mu_1 \).

Lemma 2.3. [10] Let \( M = (m_{ij}) \) be a \( n \times n \) complex matrix having \( l_1, l_2, \ldots, l_p \) as its distinct eigenvalues. Then,

\[
\{l_1, l_2, \ldots, l_p\} \subset \bigcup_{i=1}^{n} \left\{ z : |z - m_{ii}| \leq \sum_{j \neq i} |m_{ij}| \right\}.
\]

If we apply Lemma 2.3 for the distance signless Laplacian matrix of a graph \( G \) with \( n \) vertices, we get

\[
\partial_1^T(G) \leq 2\text{Tr}_{\text{max}}
\]  

(1)

Theorem 2.1 (Cauchy Interlacing Theorem). Let \( M \) be a real symmetric matrix of order \( n \), and let \( A \) be a principal submatrix of \( M \) with order \( s \leq n \). Then

\[
\lambda_i(M) \geq \lambda_i(A) \geq \lambda_{i+n-s}(M) \quad (1 \leq i \leq s).
\]

In the following theorem, we give the lower bound for the distance signless Laplacian spectral radius of the graph \( G \) in terms of diameter \( d \) and order \( n \).

Theorem 2.2. Let \( G \) be a connected graph on \( n \) vertices having diameter \( d \). Then

\[
\partial_1^Q(G) \geq \frac{2n + d(d + 1) - 2}{2}.
\]

Proof. Let \( P_{d+1} : v_1v_2 \ldots v_{d+1} \) be a diametral path in \( G \) such that \( d_G(v_1, v_{d+1}) = d \). Consider the \( n \)-vector

\[
y = (y_1, y_2, \ldots, y_{d-1}, y_d, y_{d+1}, \ldots, y_n)^T
\]

defined by

\[
y_i = \begin{cases} 
\frac{1}{\sqrt{2}} & \text{if } i = 1, d + 1 \\
0 & \text{otherwise.} 
\end{cases}
\]

By Lemma 2.2, we have

\[
\partial_1^Q(G) \geq \frac{y^T D^Q y}{y^T y} = \frac{\text{Tr}(v_1) + \text{Tr}(v_{d+1})}{2} + d_G(v_1, v_{d+1}).
\]  

(2)

Now, we have

\[
\text{Tr}(v_1) + \text{Tr}(v_{d+1}) \geq 2(1 + 2 + \cdots + d) + 2(n - d - 1) = d(d + 1) + 2(n - d - 1)
\]

On substituting the above inequality in Inequality (2), we get

\[
\partial_1^Q(G) \geq \frac{d(d + 1) + 2(n - d - 1)}{2} + d = \frac{2n + d(d + 1) - 2}{2}.
\]

\( \square \)

The next result shows that the number of distance signless Laplacian eigenvalues in the interval \([n - 2, dn]\) is at least \( d + 1 \), where \( d \) is the diameter of the graph \( G \).
**Theorem 2.3.** Let $G$ be a connected graph on $n \geq 3$ vertices having diameter $d$, then

$$m_{D^Q(G)}[n-2,dn] \geq d + 1.$$  

**Proof.** We consider the principal submatrix, say $M$, corresponding to the vertices $v_1, v_2, \ldots, v_{d+1}$ which belong to the induced path $P_{d+1}$ in the distance signless Laplacian matrix of $G$. Clearly,

$$Tr(v_i) \leq 1 + 2 + \ldots + d + d(n - d - 1) = \frac{d(2n - d - 1)}{2},$$

for all $i = 1, 2, \ldots, d + 1$. Also, the sum of the off diagonal elements of any row of $M$ is less than or equal to $d(d + 1)/2$. Using Lemma 2.3, we conclude that the maximum eigenvalue of $M$ is at most $dn$. Using Lemma 2.1 and Theorem 2.1, we see there are at least $d + 1$ distance signless Laplacian eigenvalues of $G$ which are greater than or equal to $n - 2$ and less than or equal to $dn$, that is

$$m_{D^Q(G)}[n-2,dn] \geq d + 1.$$

\[\square\]

An immediate consequence of Theorem 2.3 is the following result.

**Corollary 2.1.** Let $G$ be a connected graph on $n \geq 3$ vertices having diameter $d$. If $dn < 2Tr_{max}$, then

$$m_{D^Q(G)}(dn,2Tr_{max}) \leq n - d - 1.$$  

**Proof.** Since $dn < 2Tr_{max}$, by Lemma 2.1 and Inequality (1), we have

$$m_{D^Q(G)}[n-2,dn] + m_{D^Q(G)}(dn,2Tr_{max}) = n.$$

Thus, using Theorem 2.3, we get

$$m_{D^Q(G)}(dn,2Tr_{max}) \leq n - d - 1.$$  

\[\square\]

For proving the next result, we need the following lemma which can be found in [5].

**Lemma 2.4.** Let $G$ be a connected graph with $n$ vertices. If $K = \{v_1, v_2, \ldots, v_p\}$ is a clique of $G$ such that $N_G(v_i) - K = N_G(v_j) - K$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\partial = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\partial - 1$ is an eigenvalue of $D^Q(G)$ with multiplicity at least $p - 1$.

Now, we obtain a lower bound for the number of distance signless Laplacian eigenvalues which fall in the interval $(n - 2, 2n - 2)$, in terms of the order $n$ and the number of vertices having degree $n - 1$.

**Theorem 2.4.** Let $G$ be a connected graph on $n$ vertices. If $m_d = |\{u \in V(G) : d_G(u) = n - 1\}|$, where $1 \leq m_d \leq n$, then

$$m_{D^Q(G)}(n-2,2n-2) \leq n - m_d.$$

Equality holds when $m_d = n$, that is, $G \cong K_n$.

**Proof.** We consider the following two cases.

**Case 1.** Let $m_d = n$, that is, $G \cong K_n$. By Lemma 2.1, we see that the equality holds.

**Case 2.** Let $1 \leq m_d \leq n - 1$. Since $G$ contains $m_d$ vertices of degree $n - 1$, therefore, $G$ contains a clique, say $S$, of size $m_d$. Let $S = \{v_1, v_2, \ldots, v_{m_d}\}$. Clearly,

$$n - 1 = Tr(v_1) = Tr(v_2) = \cdots = Tr(v_{m_d}).$$

By Lemma 2.4, we observe that $n - 2$ is a distance signless Laplacian eigenvalue of $G$ with multiplicity at least $m_d - 1$. Also, we know that the distance signless Laplacian matrix corresponding to any connected graph $H$ is symmetric, positive and irreducible. Therefore, by the Perron-Frobenius Theorem, $\partial^Q_1(H - uv) > \partial^Q_1(H)$ whenever $uv \in E(H)$ and $H - uv$ is connected. As $m_d \leq n - 1$, therefore, $G \not\cong K_n$. Thus, from the above information $\partial^Q_1(G) > \partial^Q_1(K_n) = 2n - 2$. Hence,

$$m_{D^Q(G)}(n-2,2n-2) \leq n - (m_d - 1) - 1 = n - m_d.$$  

\[\square\]
The following lemma is used in proving Theorem 2.5.

**Lemma 2.5.** [5] Let $G$ be a graph with $n$ vertices. If $K = \{v_1, v_2, \ldots, v_p\}$ is an independent set of $G$ such that $N_G(v_i) = N_G(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$, then $\delta = Tr(v_i) = Tr(v_j)$ for all $i, j \in \{1, 2, \ldots, p\}$ and $\delta - 2$ is an eigenvalue of $D^Ω(G)$ with multiplicity at least $p - 1$.

The next result shows that the number of distance signless Laplacian eigenvalues in the interval $[n - 2, 2n - 4]$ is at most $n - p + 1$, where $n \geq 3$ is the order of $G$ and $p$ is the number of pendant vertices adjacent to common neighbour.

**Theorem 2.5.** Let $G$ be a connected graph of order $n \geq 3$. If $S = \{v_1, v_2, \ldots, v_p\} \subseteq V(G)$, where $|S| = p \leq n - 1$, is the set of pendant vertices such that every vertex in $S$ has the same neighbourhood in $V(G) \setminus S$, then

$$m_{D^Q(G)}([n - 2, 2n - 4]) \leq n - p + 1.$$  

**Proof.** Clearly all the vertices in $S$ form an independent set. Since all the vertices in $S$ are adjacent to same vertex, therefore, all the vertices of $S$ have the same transmission. Now, for any $v_i$ ($i = 1, 2, \ldots, p$) of $S$, we have

$$T = Tr(v_i) \geq 2(p - 1) + 1 + 2(n - p - 1) = 2n - 3.$$  

From Lemma 2.5, there are at least $p - 1$ distance signless Laplacian eigenvalues of $G$ which are equal to $T - 1$. From above we have $T - 1 \geq 2n - 3 - 1 = 2n - 4$. Thus, there are at least $p - 1$ distance signless Laplacian eigenvalues of $G$ which are greater than or equal to $2n - 4$. Using Lemma 2.1, we get $m_{D^Q(G)}([n - 2, 2n - 4]) \leq n - p + 1$. $\square$

Next, we show that the number of distance signless Laplacian eigenvalues in the interval $[n - 2, 2n - \omega - 2]$ is at most $n - \omega + 2$, where $n$ is the order and $\omega$ is the clique number of the graph $G$.

**Theorem 2.6.** Let $G$ be a connected graph of order $n$ having clique number $\omega \leq n - 1$. If only one vertex of the corresponding maximum clique is adjacent to the vertices outside of the clique, then

$$m_{D^Q(G)}([n - 2, 2n - \omega - 2]) \leq n - \omega + 2.$$  

**Proof.** Let $S = \{v_1, v_2, \ldots, v_w\}$ be the set of vertices of the maximum clique such that $v_w$ is the only vertex having neighbours outside of $S$. Clearly, the set of vertices $N = \{v_1, v_2, \ldots, v_w-1\}$ also form a clique such that every vertex of $N$ is adjacent to $v_w$ only outside of $N$. It is easy to see that all the vertices belonging to $N$ have the same transmission. For any $v_i \in N$, $i = 1, 2, \ldots, \omega - 1$, we have

$$T = Tr(v_i) \geq \omega - 1 + 2(n - \omega) = 2n - \omega - 1.$$  

Using Lemma 2.4, we observe that $T - 1$ is a distance signless Laplacian eigenvalue of $G$ of multiplicity at least $\omega - 2$. From Inequality (3), we have $T - 1 \geq 2n - 1 - \omega - 1 = 2n - \omega - 2$. So there are at least $\omega - 2$ distance signless Laplacian eigenvalues of $G$ which are greater than or equal to $2n - \omega - 2$. From Inequality (1), we get $m_{D^Q(G)}([2n - \omega - 2, 2Tr_{max}]) \geq \omega - 2$. Thus, by the above observation and Lemma 2.1, we have $m_{D^Q(G)}([n - 2, 2n - \omega - 2]) \leq n - \omega + 2$, which completes the proof. $\square$

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