(Super)integrability from coalgebra symmetry: formalism and applications

A Ballesteros¹, A Blasco¹, F J Herranz¹, F Musso¹ and O Ragnisco²

¹ Departamento de Física, Universidad de Burgos, E-09001 Burgos, Spain
² Dipartimento di Fisica, Università di Roma Tre and Instituto Nazionale di Fisica Nucleare sezione di Roma Tre, Via Vasca Navale 84, I-00146 Roma, Italy

E-mail: angelb@ubu.es, ablasco@ubu.es, fjherranz@ubu.es, fmusso@ubu.es, ragnisco@fis.uniroma3.it

Abstract. The coalgebra approach to the construction of classical integrable systems from Poisson coalgebras is reviewed, and the essential role played by symplectic realizations in this framework is emphasized. Many examples of Hamiltonians with either undeformed or $q$-deformed coalgebra symmetry are given, and their Liouville superintegrability is discussed. Among them, (quasi-maximally) superintegrable systems on $N$-dimensional curved spaces of nonconstant curvature are analysed in detail. Further generalizations of the coalgebra approach that make use of comodule and loop algebras are presented. The generalization of such a coalgebra symmetry framework to quantum mechanical systems is straightforward.

1. Introduction

Poisson coalgebras are Poisson algebras endowed with a compatible coproduct structure. The aim of this contribution is to provide a self-contained review of a recently introduced symmetry approach in which Poisson coalgebras play an essential role as ‘hidden’ dynamical symmetries underlying the (super)integrability properties of a wide class of $N$-dimensional (ND) classical Hamiltonian systems. Within this construction, once a symplectic realization of the coalgebra is given, their generators play the role of dynamical symmetries of the Hamiltonian—which is written as a function of them—while the coproduct map (coalgebra structure) is used to ‘propagate’ the integrability to any arbitrary dimension.

From this so-called coalgebra approach, many well-known (super)integrable systems have been recovered, and some integrable deformations for them as well as new ND integrable systems have also been obtained (see [1]–[8] and references therein). As a remarkable application, this framework has been recently used to introduce integrable Hamiltonians describing geodesic flows on spaces with either constant or nonconstant curvature, and (super)integrable potential terms preserving the coalgebra symmetry can also be considered on such spaces [9]–[16].

Although in this contribution we shall concentrate on classical mechanical systems obtained from (commutative) Poisson coalgebras, we stress that all the (super)integrability properties of the coalgebra symmetric systems that we are going to describe are preserved at the quantum mechanical level by using the corresponding (noncommutative) operator coalgebras. In this way, the coalgebra approach has also been used, for instance, to solve in [17, 18] the quantum Calogero–Gaudin system [19], as well as supersymmetric generalizations of this model and of its
q-deformations, which have been constructed by starting from the underlying supersymmetric coalgebra structures [20, 21].

The paper is organized as follows. In section 2 we present in a complete and self-contained way the general coalgebra approach to integrable Hamiltonians, that will be illustrated in section 3 by applying it to three relevant Poisson coalgebras, namely \( \mathfrak{sl}(2, \mathbb{R}) \), its non-standard quantum deformation, \( \mathfrak{sl}_q(2, \mathbb{R}) \) and the two-photon Poisson-coalgebra \( h_6 \). Many physically interesting Hamiltonian systems will be explicitly constructed on these and other coalgebras in sections 4, 5, 6 and 7. Finally, some possible generalizations of the coalgebra formalism will be sketched in section 8.

2. Hamiltonian systems on Poisson coalgebras

First of all, let us fix the terminology concerning (Liouville) integrability and superintegrability (see, for instance, [22]). An \( N \)-dimensional Hamiltonian \( H^{(N)} \) is called completely integrable if there exists a set of \( (N - 1) \) globally defined and functionally independent constants of the motion in involution that Poisson-commute with \( H^{(N)} \). The Hamiltonian will be called maximally superintegrable (MS) if there exists a set of \( (2N - 2) \) globally defined functionally independent constants of the motion Poisson-commuting with \( H^{(N)} \); among them, at least two different subsets of \( (N - 1) \) constants in involution can be found. Finally, a Hamiltonian system will be called quasi-maximally superintegrable (QMS) if it has \( (2N - 3) \) independent integrals with the abovementioned properties.

We also recall that a coalgebra \((A, \Delta)\) is a (unital, associative) algebra \( A \) endowed with a coproduct map [23, 24]:

\[
\Delta : A \rightarrow A \otimes A
\]

which is coassociative

\[(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta\]

that is, the following diagram is commutative:

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\Delta} & A \\
\downarrow \Delta & & \downarrow \Delta \otimes \text{id} \\
A \otimes A & \xrightarrow{\Delta \otimes \text{id}} & A \otimes A \otimes A \\
\end{array}
\]

Due to the coassociativity property, the comultiplication \( \Delta \) provides a ‘two-fold way’ for the definition of the objects on \( A \otimes A \otimes A \) that, as we shall explain in section 2.2, will be deeply connected with superintegrability properties.

Let us now summarize the general construction of [2]. Let \((A, \Delta)\) be a Poisson coalgebra with \( l \) generators \( X_i \) (\( i = 1, \ldots, l \)), and \( r \) functionally independent Casimir functions \( C_j(X_1, \ldots, X_l) \) (with \( j = 1, \ldots, r \)). The coassociative coproduct \( \Delta \equiv \Delta^{(2)} \) is a Poisson map with respect to the usual Poisson bracket on \( A \otimes A \):

\[
\{X_i \otimes X_j, X_r \otimes X_s\}_{A \otimes A} = \{X_i, X_r\}_A \otimes X_j X_s + X_i X_r \otimes \{X_j, X_s\}_A.
\]

Then, the \( m \)-th coproduct map \( \Delta^{(m)}(X_i) \)

\[
\Delta^{(m)} : A \rightarrow A \otimes A \otimes \ldots \otimes A
\]
Table 1. Functions obtained by applying the coproduct map.

|      | $X_1$ | $X_2$ | ... | $X_1$ | $C_1$ | $C_2$ | ... | $C_r$ |
|------|------|------|-----|------|------|------|-----|------|
| $\Delta^{(2)}(X_1)$ | $\Delta^{(2)}(X_2)$ | ... | $\Delta^{(2)}(X_1)$ | $\Delta^{(2)}(C_1)$ | $\Delta^{(2)}(C_2)$ | ... | $\Delta^{(2)}(C_r)$ |
| $\Delta^{(3)}(X_1)$ | $\Delta^{(3)}(X_2)$ | ... | $\Delta^{(3)}(X_1)$ | $\Delta^{(3)}(C_1)$ | $\Delta^{(3)}(C_2)$ | ... | $\Delta^{(3)}(C_r)$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $\Delta^{(m)}(X_1)$ | $\Delta^{(m)}(X_2)$ | ... | $\Delta^{(m)}(X_1)$ | $\Delta^{(m)}(C_1)$ | $\Delta^{(m)}(C_2)$ | ... | $\Delta^{(m)}(C_r)$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... |
| $\Delta^{(N)}(X_1)$ | $\Delta^{(N)}(X_2)$ | ... | $\Delta^{(N)}(X_1)$ | $\Delta^{(N)}(C_1)$ | $\Delta^{(N)}(C_2)$ | ... | $\Delta^{(N)}(C_r)$ |

can be defined by applying recursively the coproduct $\Delta^{(2)}$ in the form

$$\Delta^{(m)} := (\text{id} \otimes \text{id} \otimes \ldots \otimes \text{id} \otimes \Delta^{(2)}) \circ \Delta^{(m-1)}.$$  \hspace{1cm} (2)

Such an induction ensures that $\Delta^{(m)}$ is also a Poisson map.

In this way, we can construct the set of functions shown in table 1. From them, given a smooth function $H(X_1,\ldots,X_l)$, the $N$-sites Hamiltonian is defined as the $N$-th coproduct of $H$:

$$H^{(N)} := \Delta^{(N)}(H(X_1,\ldots,X_l)) = H(\Delta^{(N)}(X_1),\ldots,\Delta^{(N)}(X_l)).$$  \hspace{1cm} (3)

From [2] it can be proven that the set of $r \cdot N$ functions ($m = 1,\ldots,N; j = 1,\ldots,r$)

$$C_j^{(m)} := \Delta^{(m)}(C_j(X_1,\ldots,X_l)) = C_j(\Delta^{(m)}(X_1),\ldots,\Delta^{(m)}(X_l)),$$  \hspace{1cm} (4)

such that $C_j^{(1)} = C_j$, Poisson-commute with the Hamiltonian

$$\left\{C_j^{(m)}, H^{(N)} \right\}_{A \otimes A \otimes \ldots \otimes A} = 0$$  \hspace{1cm} (5)

and all these constants are, by construction, in involution:

$$\left\{C_i^{(m)}, C_j^{(n)} \right\}_{A \otimes A \otimes \ldots \otimes A} = 0 \quad m, n = 1,\ldots,N \quad i, j = 1,\ldots,r.$$  \hspace{1cm} (6)

This construction can be applied to any Poisson coalgebra. But there are two relevant families of Poisson coalgebras that will constitute the core of the integrable systems presented in this paper:

- **Lie–Poisson algebras.** The Poisson analogue of any Lie algebra with generators $X_i$ ($i = 1,\ldots,l$) is a coalgebra when endowed with the (primitive) coproduct map [25]

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i \quad \Delta(1) = 1 \otimes 1.$$  \hspace{1cm} (7)

Equivalently, this means that Lie algebras are cocommutative Hopf algebras.

- **Poisson analogues of quantum algebras and groups** [23, 24, 25]. These are also (deformed) coalgebras $(A_z, \Delta_z)$, where $z$ is the quantum deformation parameter ($q = e^z$), and the deformed coproduct gives rise to a noncocommutative Hopf algebra. The ‘classical’ limit $z \to 0$ (or, equivalently, $q \to 1$) provides the corresponding Lie–Poisson coalgebra. In fact, these $q$-Poisson coalgebras can be written as quadratic Poisson structures on certain (dual) Lie–Poisson groups, that are invariant with respect to the group multiplication (which is nothing but a coproduct map on the dual, see [6]).
Some remarks are in order.

- In general, we cannot say \textit{a priori} that $H^{(N)}$ is a completely integrable Hamiltonian system because [8]: (i) We have to determine the number of degrees of freedom of $H^{(N)}$ by choosing a explicit symplectic realization of the coalgebra $(A, \Delta)$. (ii) Once this number is fixed, we have to check whether the number of independent invariants extracted from $C_j^{(m)}$ (under such a specific symplectic realization) is enough to guarantee complete integrability.

- However, if the coalgebra symmetry provides the complete integrability for $H^{(N)}$, the integrals of the motion can always be obtained in arbitrary dimension $N$ in an explicit form. Hence the coalgebra symmetry arises as a unified approach to integrability since we obtain families of ND systems that share a very large common set of integrals of the motion. Furthermore, in this case $H^{(N)}$ will probably be superintegrable (to some extent) [7].

- In the case of $q$-deformations of Poisson coalgebras, the Hamiltonians obtained through the coalgebra approach are integrable deformations of the $z = 0$ ($q = 1$) cases in such a manner that different quantum algebras give rise to different integrable deformations. Moreover, this fact can provide some relevant information concerning the geometric/physical interpretation of the deformation parameter.

- This construction holds for noncommutative coalgebras as well. Thus, quantum mechanical systems can also be constructed (although ordering problems have to be fixed).

Before entering into explicit examples of the coalgebra construction, two important and general aspects have to be more deeply analysed in order to provide a global overview on the subject: they are the role of the symplectic realizations (which has been thoroughly studied in [8]) and the superintegrability features of the coalgebra symmetry (which were presented for the first time in [7]).

\subsection*{2.1. Symplectic realizations and complete integrability}

Let $(A, \Delta)$ a Poisson coalgebra with $r$ Casimir functions $C_j (j = 1, \ldots, r)$. A symplectic leaf of $A$ (which is always even-dimensional) will be denoted by $A_{(k_1, k_2, \ldots, k_r)}$, where the leaf is characterized by a given set of constant values $(k_1, k_2, \ldots, k_r)$ for the Casimirs.

An \textit{s-dimensional symplectic realization} $D$ for $A_{(k_1, k_2, \ldots, k_r)}$ is given (locally) in terms of $s$ pairs $(q_i, p_i)$ of canonical Darboux variables

$$ D : x \rightarrow x(q_1, p_1, q_2, p_2, \ldots, q_s, p_s) $$

where $x$ is any point on $A_{(k_1, k_2, \ldots, k_r)}$. We remark that, in principle, different symplectic leaves $A_{(k_1, k_2, \ldots, k_r, i)}$ can be chosen for each copy $i$ of $A$ within $A \otimes A \otimes \ldots \otimes A$.

In particular, if we consider symplectic realizations with the same $s$ for all the $N$ sites in the tensor product $A \otimes A \otimes \ldots \otimes A$, the Hamiltonian (3)

$$ H^{(N)} = (D \otimes D \ldots \otimes D)(\Delta^{(N)}(\mathcal{H})) $$

turns out to be a function of $N \cdot s$ pairs of canonical variables, i.e., it defines a system with $N \cdot s$ degrees of freedom. Then for each nonlinear Casimir $C_j$, we get at most $(N - 1)$ integrals coming from its $m$-th coproducts (4) written under the $m$-th tensor product of the symplectic realization $D$ (8):

$$ C_j^{(m)} = (D \otimes D \ldots m \otimes D)(\Delta^{(m)}(C_j)) $$

where $m = 2, \ldots, N$; notice that $C_j^{(1)} = D(C_j) = k_j$. Note that under symplectic realizations $m$-th coproducts of linear Casimirs always give just numerical constants.
Thus, if we have $R$ nonlinear Casimirs we find a maximum possible number of integrals in involution given by

$$(N - 1) \cdot R.$$ 

In order to get complete integrability, we should have that

$$N \cdot s - 1 \leq (N - 1) \cdot R$$

thus we need that the chosen symplectic realization $D$ of $A$ fulfils (for any $N$)

$$s \leq R - \frac{R - 1}{N}. \quad (11)$$

Consequently, the necessary condition for complete integrability connects the dimension of the symplectic realization and the number of nonlinear Casimir functions through the conditions [8]:

- $s = 1$ for coalgebras with $R = 1$.
- $s < R$ for coalgebras with $R > 1$.

Let us now consider a particular type of symplectic realizations whose dimension is fixed by the dimension $l$ of the Poisson coalgebra and its number $r$ of Casimir functions. We shall call ‘generic’ to the symplectic realization with maximal dimension $s_m$ given by

$$s_m = \frac{l - r}{2}. \quad (12)$$

Hence, the integrability condition for the generic symplectic realization is

$$s_m = \frac{l - r}{2} \leq R - \frac{R - 1}{N}$$

which leads to the final expression

$$l \leq (2R + r) - \frac{2}{N} (R - 1). \quad (13)$$

Therefore, complete integrability for the generic symplectic realization can be achieved if [8]:

- $l \leq 2 + r$ for coalgebras with $R = 1$.
- $l < 2R + r$ for coalgebras with $R > 1$.

As a byproduct of the above results, if we consider simple Lie algebras, with rank $r = R$, we find that for coalgebras with $R = 1$, their dimension must be $l \leq 3$, and for coalgebras with $R > 1$, then we get the condition $l < 3R$. This, in turn, shows that only simple Lie algebras of rank 1 can provide complete integrable systems in the generic symplectic realization, whilst all higher-rank ones do not fulfill the condition.

Another family of interesting cases is given by Lie coalgebras in which the generic symplectic realization has $s_m = 1$, since the coalgebra symmetry always satisfies the integrability condition provided that $R \geq 1$. This possibility covers many non-simple Lie algebras fulfilling $l - r = 2$. 


symmetry will be, in principle, superintegrable of ‘right’ integrals such that and the dimensionality of the chosen symplectic realizations can be applied to the additional considered corresponding symplectic realization. 

However, if we label from 1 to \(N\) the sites of the chain \(A \otimes A \otimes \ldots \otimes A\), lower dimensional \(m\)th-coproducts (with \(2 < m < N\)) will be ‘different’ in the sense that the ‘left’ coproducts \(\Delta^{(m)}(R)\) will contain objects living on the tensor product space \(1 \otimes 2 \otimes \cdots \otimes m\), whilst the ‘right’ coproducts \(\Delta^{(m)}_R\) will be defined on the sites \((N - m + 1) \otimes (N - m + 2) \otimes \cdots \otimes N\).

Therefore, the coalgebra symmetry of a given Hamiltonian gives rise to two ‘pyramidal’ sets of \(r \cdot N\) integrals of the motion in involution that Poisson-commute with \(H^{(N)}\), under the corresponding sympletic realization \(D\). Namely, both sets are just the ‘left’ integrals above considered \(C^{(m)}(\Delta)\) together with the ‘right’ ones given by

\[
C_{j,m} := \Delta^{(m)}_R(C_j(X_1, \ldots, X_l)) = C_j(\Delta^{(m)}_R(X_1), \ldots, \Delta^{(m)}_R(X_l))
\]

such that \(C^{(N)}_j \equiv C_{j,(N)}\), while both \(C^{(1)}_j\) and \(C_{j,(1)}\) are constants.

The very same arguments discussed in section 2.1 on the connection between integrability and the dimensionality of the chosen symplectic realizations can be applied to the additional set of ‘right’ integrals \(C_{j,(m)}\). In this way, a completely integrable Hamiltonian \(H^{(N)}\) with coalgebra symmetry will be, in principle, superintegrable.

For the sake of simplicity let us assume that we are dealing with a Poisson coalgebra with \(R = 1\), that is, with \(N\) degrees of freedom and with a single nonlinear Casimir. Therefore if we

---

**Table 2.** Coalgebra symmetry and QMS: the \((2N - 3)\) integrals coming from a Casimir \(C\).

| ‘Left’ set of \((N - 1)\) integrals \(C^{(m)}\) in involution | Tensor product space |
|---|---|
| \(C^{(2)} \equiv \Delta^{(2)}(C)\) | \(1 \otimes 2\) |
| \(C^{(3)} \equiv \Delta^{(3)}(C)\) | \(1 \otimes 2 \otimes 3\) |
| \(\vdots\) | \(\vdots\) |
| \(C^{(m)} \equiv \Delta^{(m)}(C)\) | \(1 \otimes 2 \otimes \cdots \otimes m\) |
| \(\vdots\) | \(\vdots\) |
| \(C^{(N)} \equiv \Delta^{(N)}(C)\) | \(1 \otimes 2 \otimes \cdots \otimes (N - 1) \otimes N\) |

| ‘Right’ set of \((N - 1)\) integrals \(C^{(m)}\) in involution | Tensor product space |
|---|---|
| \(C^{(2)} \equiv \Delta^{(2)}_R(C)\) | \((N - 1) \otimes N\) |
| \(C^{(3)} \equiv \Delta^{(3)}_R(C)\) | \((N - 2) \otimes (N - 1) \otimes N\) |
| \(\vdots\) | \(\vdots\) |
| \(C^{(m)} \equiv \Delta^{(m)}_R(C)\) | \((N - m + 1) \otimes (N - m + 2) \otimes \cdots \otimes N\) |
| \(\vdots\) | \(\vdots\) |
| \(C^{(N)} = C^{(N)}_j \equiv \Delta^{(N)}_R(C)\) | \(1 \otimes 2 \otimes \cdots \otimes (N - 1) \otimes N\) |
The Casimir function for this coalgebra is defined by the following Lie–Poisson brackets and comultiplication map:

A one-particle symplectic realization of this coalgebra is given by

3. Three Poisson coalgebras
We stress that the coalgebra approach to complete integrability is completely general and constructive for any Poisson coalgebra endowed with a suitable symplectic realization. In order to illustrate the above ideas, we will mainly consider in this contribution the following three Poisson coalgebras: (i) \( \mathfrak{sl}(2, \mathbb{R}) \), (ii) its non-standard \( q \)-deformation \( \mathfrak{sl}_2(2, \mathbb{R}) \) and (iii) the (two-photon) algebra \( h_6 \). With them a bunch of physically interesting ND Hamiltonians can be obtained, and some of them will be explicitly presented in sections 4, 5 and 6, respectively.

3.1. The \( \mathfrak{sl}(2, \mathbb{R}) \) Lie–Poisson coalgebra
This coalgebra is defined by the following Lie–Poisson brackets and comultiplication map:

\[
\{ J_3, J_+ \} = 2J_+ \quad \{ J_3, J_- \} = -2J_- \quad \{ J_-, J_+ \} = 4J_3 
\]

\[
\Delta(1) = 1 \otimes 1 \quad \Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i \quad i = +, -, 3.
\]

The Casimir function for \( \mathfrak{sl}(2, \mathbb{R}) \) reads

\[
C = J_+J_- - J_3^2.
\]

A one-particle symplectic realization of this coalgebra is given by

\[
D(J_-) = q_1^2 \quad D(J_+) = p_1^2 + \frac{b_1}{q_1^2} \quad D(J_3) = q_1p_1,
\]

where \( \{ q_1, p_1 \} = 1 \). Note that, under this realization, \( C^{(1)} = D(C) = b_1 \). Hence, according to the notation and results presented in section 2.1 we are dealing with a Poisson coalgebra with dimension \( l = 3 \) and with a single nonlinear Casimir. Thus \( r = R = 1 \) and this implies that (20) is the generic symplectic realization with \( s \equiv s_m = (l - r)/2 = 1 \). The corresponding \( N \)-particle symplectic realization of \( \mathfrak{sl}(2, \mathbb{R}) \), living on \( \mathfrak{sl}(2, \mathbb{R}) \otimes \ldots \otimes \mathfrak{sl}(2, \mathbb{R}) \), is obtained by applying

\[
J_3^{(N)} = (D \otimes D \otimes \ldots \otimes D)(\Delta^{(N)}(J_i))
\]

which gives \([3, 12]\)

\[
J_3^{(N)} = \sum_{i=1}^{N} q_i^2 \equiv q^2 
\]

\[
J_3^{(N)} = \sum_{i=1}^{N} q_ip_i \equiv q \cdot p
\]
\[ J_i^{(N)} = \sum_{i=1}^{N} \left( p_i^2 + \frac{b_i}{q_i} \right) = p_i^2 + \sum_{i=1}^{N} \frac{b_i}{q_i}, \]  

(21)

where \( b_i \) are \( N \) arbitrary real parameters. This means that the \( N \)-particle generators (21) fulfill the commutation rules (17) with respect to the \( N \) canonical Poisson bracket.

Next, since \( s_m = 1 \) and \( R = 1 \) we are in the case of a Poisson coalgebra endowed with the \( (2N - 3) \) integrals displayed in table 2; these turn out to be [7, 12]:

\[ C^{(m)} = \sum_{1 \leq i < j} I_{ij} + \sum_{i=1}^{m} b_i \quad C_{(m)} = \sum_{N-m+1 \leq i < j} I_{ij} + \sum_{i=N-m+1}^{N} b_i \]  

(22)

where \( m = 2, \ldots, N \) and

\[ I_{ij} = (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_j^2}{q_i} + b_j \frac{q_i^2}{q_j} \right) \]  

(23)

are the \( b_i \)-generalization of the square of the ‘angular momentum’ generators \( J_{ij} = q_i p_j - q_j p_i \) which span an \( \mathfrak{so}(N) \) Lie–Poisson algebra. As a consequence of the coalgebra symmetry, the generators (21) Poisson commute with these \( (2N - 3) \) functions. Therefore, any arbitrary function \( H \) defined as

\[ H^{(N)} = H \left( J_+^{(N)}, J_-^{(N)}, J_3^{(N)} \right) = H \left( q_i^2, p_i^2 + \sum_{i=1}^{N} \frac{b_i}{q_i}, q \cdot p \right) \]  

(24)

gives rise to an \( N \)D QMS Hamiltonian system which is always endowed, at least, with the \( (2N - 3) \) integrals (22).

3.2. \( q \)-Poisson coalgebras: the \( \mathfrak{sl}_2(\mathbb{R}) \) case

The Poisson analogues of quantum algebras and groups [23, 24] are also (deformed) coalgebras \( (A, \Delta) \) \( (q = e^z) \), which means that any function of the generators of a given ‘quantum’ Poisson algebra (with deformed Casimir elements \( C_{ij} \)) will provide a deformation of the Hamiltonian generated by the undeformed coalgebra. Such a \( q \)-coalgebraic deformation will preserve, by construction, the (super)integrability properties of the system defined on the undeformed Lie–Poisson coalgebra. Therefore, \( q \)-deformations can be understood in this context as the algebraic machinery suitable for generating integrable deformations of Hamiltonian systems.

In particular, let us focus on the non-standard \( \mathfrak{sl}_2(\mathbb{R}) \) Poisson coalgebra defined by the following (deformed) Poisson brackets and coproduct map (see [3, 26]):

\[ \{ J_3, J_+ \} = 2J_+ \cosh zJ_- \quad \{ J_3, J_- \} = -2 \frac{\sinh zJ_-}{z} \quad \{ J_-, J_+ \} = 4J_3 \]  

(25)

\[ \Delta_z(1) = 1 \otimes 1 \quad \Delta_z(J_-) = J_- \otimes 1 + 1 \otimes J_- \]  \[ \Delta_z(J_i) = J_i \otimes e^{zJ_-} + e^{-zJ_-} \otimes J_i \]  

(\( i = +, 3 \)).

The Casimir function for \( \mathfrak{sl}_2(\mathbb{R}) \) reads

\[ C_z = \frac{\sinh zJ_-}{z} J_+ - J_3^2. \]  

(27)

A one-particle (deformed) symplectic realization of \( \mathfrak{sl}_2(\mathbb{R}) \) is:

\[ D_z(J_-) = q_i^2 \quad D_z(J_+) = \frac{\sinh zq_i^2}{zq_i^2} p_i^2 + \frac{zq_i}{\sinh zq_i} \quad D_z(J_3) = \frac{\sinh zq_i^2}{zq_i^2} q_ip_i \]  

(28)
such that $C_0^{(1)} = D_z(C_z) = b_1$. Hence we are dealing again with a coalgebra with $l = 3$, $r = R = 1$ and $s_m = 1$. The $N$th-coproduct of (26) through the one-particle representation (28) gives rise to an $N$-particle symplectic realization on $\mathfrak{sl}_2(2, \mathbb{R}) \otimes \ldots \otimes \mathfrak{sl}_2(2, \mathbb{R})$ through

$$J^{(N)}_3 = (D_z \otimes D_z \otimes \ldots \otimes D_z) (\Delta_z^{(N)}(J_3)),$$

namely

$$J^{(N)}_+ = \sum_{i=1}^N \left( \frac{\sinh z q_i^2}{q_i^2} p_i^2 + \frac{z b_i}{\sinh z q_i^2} \right) \exp \left\{ -z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^N q_l^2 \right\} = \tilde{p}_z^2$$

$$J^{(N)}_- = \sum_{i=1}^N \left( \frac{\sinh z q_i^2}{z q_i^2} p_i^2 \right) \exp \left\{ -z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^N q_l^2 \right\} = (\mathbf{q} \cdot \mathbf{p})_z$$

where the $b_i$'s are again $N$ arbitrary real parameters that label the representation on each ‘lattice’ site.

In this case the $(2N - 3)$ functions written in table 2 are explicitly given by [3, 7]:

$$C_z^{(m)} = \sum_{1 \leq i < j}^m I_{ij}^* \exp \left\{ -2z \sum_{k=1}^{i-1} q_k^2 - z q_i^2 + z q_j^2 + 2z \sum_{l=j+1}^m q_l^2 \right\}$$

$$+ \sum_{i=1}^m b_i \exp \left\{ -2z \sum_{k=1}^{i-1} q_k^2 + 2z \sum_{l=i+1}^m q_l^2 \right\}$$

$$C_{z,(m)} = \sum_{N-m+1 \leq i < j}^N I_{ij}^* \exp \left\{ -2z \sum_{k=N-m+1}^{i-1} q_k^2 - z q_i^2 + z q_j^2 + 2z \sum_{l=i+1}^N q_l^2 \right\}$$

$$+ \sum_{i=N-m+1}^N b_i \exp \left\{ -2z \sum_{k=N-m+1}^{i-1} q_k^2 + 2z \sum_{l=i+1}^N q_l^2 \right\}$$

(30)

where $m = 2, \ldots, N$ and

$$I_{ij}^* = \frac{\sinh z q_i^2}{z q_i^2} \frac{\sinh z q_j^2}{z q_j^2} (q_i p_j - q_j p_i)^2 + \left( b_i \frac{\sinh z q_j^2}{\sinh z q_i^2} + b_j \frac{\sinh z q_i^2}{\sinh z q_j^2} \right).$$

Consequently, any smooth function $H_z$ defined on the $N$-particle symplectic realization (29) of the generators of $\mathfrak{sl}_2(2, \mathbb{R})$ in the form

$$H_z^{(N)} = H_z \left( J_-^{(N)}, J_+^{(N)}, J_3^{(N)} \right) = H_z \left( \mathbf{q}^2, \tilde{p}_z^2, (\mathbf{q} \cdot \mathbf{p})_z \right)$$

(32)

defines a QMS Hamiltonian system. We stress that all the choices for $H_z$ share the ‘universal’ set of $(2N - 3)$ constants of motion given by (30).

Clearly, the non-deformed limit $z \rightarrow 0$ of all the above expressions gives rise to the ones corresponding to the $\mathfrak{sl}(2, \mathbb{R})$-coalgebra presented in section 3.1, which shows that quantum algebras may provide (super)integrable generalizations of non-deformed Hamiltonian systems.

We also remark that this quantum deformation has been interpreted in [3] as an algebraic way to introduce long-range interactions in the underlying underformed systems due to the exponentials of the type $\exp(z \sum_j q_j^2)$ coming from the deformed symplectic realization (29) (compare with (21) and (24)).
3.3. The two-photon Lie–Poisson coalgebra

The third coalgebra that we explicitly consider also satisfies the necessary condition (11) for complete integrability. Nevertheless, we shall use this example to show that the latter is not a sufficient condition, thus making evident the essential role played by the symplectic realization of the chosen coalgebra.

The two-photon coalgebra \((h_6, \Delta)\) is spanned by six generators \(\{K, A_+, A_-, B_+, B_-, M\}\) with Lie–Poisson brackets given by [27]

\[
\begin{align*}
\{K, A_+\} &= A_+ & \{K, A_-\} &= -A_- & \{A_-, A_+\} &= M \\
\{K, B_+\} &= 2B_+ & \{K, B_-\} &= -2B_- & \{B_-, B_+\} &= 4K + 2M \\
\{A_+, B_-\} &= -2A_- & \{A_+, B_+\} &= 0 & \{M, \cdot\} &= 0 \\
\{A_-, B_+\} &= 2A_+ & \{A_-, B_-\} &= 0
\end{align*}
\]

(33)

together with the usual nondeformed coproduct map given by (7). This is just the Poisson analogue of the (non-simple) \(h_6\) Lie algebra which has been widely applied in quantum optics [28, 29]. The \(h_6\) algebra is isomorphic to the \((1 + 1)D\) Schrödinger algebra [30] and the following embedding of the Heisenberg–Weyl \(h_3\), harmonic oscillator \(h_4\) and two-photon coalgebras can be easily identified:

\[
(h_3, \Delta) \subset (h_4, \Delta) \subset (h_6, \Delta) \quad h_3 = \langle A_-, A_+, M \rangle \quad h_4 = \langle K, A_-, A_+, M \rangle.
\]

The \(h_6\)-coalgebra is endowed with two Casimir operators [31], namely

\[
C_1 = M \quad C_2 = (MB_+ - A_+^2)(MB_- - A_-^2) - (MK - A_-A_+ + M^2/2)^2.
\]

(34)

Let us now introduce the one-particle symplectic realization of \(h_6\) given by [16, 27]

\[
\begin{align*}
D(A_+) &= \lambda_1 p_1 & D(A_-) &= \lambda_1 q_1 & D(K) &= q_1 p_1 - \frac{1}{2} \lambda_1^2 \\
D(B_+) &= p_1^2 & D(B_-) &= q_1^2 & D(M) &= \lambda_1^2
\end{align*}
\]

(35)

where \(\lambda_1\) is a non-vanishing constant that labels the realization such that \(C_1^{(1)} = D(C_1) = \lambda_1^2\) and \(C_2^{(1)} = D(C_2) = 0\). Notice that if \(\lambda_1 = 0\) the \(h_6\)-coalgebra reduces to the \(\mathfrak{sl}(2, \mathbb{R}) = \langle B_-, B_+, K \rangle\) one (20) with \(b_1 = 0\). In fact, for any \(\lambda_1 \neq 0\), \(\mathfrak{gl}(2, \mathbb{R}) = \langle B_-, B_+, K, M \rangle\) is also a sub-coalgebra of \(h_6\).

Therefore, by taking into account section 2.1 we are now dealing with a coalgebra with \(l = 6, r = 2, R = 1\) \((C_1 = M\) is linear and leads to trivial integrals) and the chosen symplectic realization has \(s = 1\). This means that the necessary condition for integrability (11) is fulfilled since \(s = 1 \equiv R = 1\), although note that the symplectic realization is not a generic one (12) since \(s_m = 1 \neq \frac{l^2}{4R^2} = 2\).

By making use of the coproduct map it is immediate to show that the \(N\)-particle symplectic realization on \(h_6 \otimes \ldots \otimes h_6\) reads

\[
\begin{align*}
A_+^{(N)} &= \sum_{i=1}^{N} \lambda_i p_i \equiv \mathbf{p} & A_-^{(N)} &= \sum_{i=1}^{N} \lambda_i q_i \equiv \mathbf{q} & M^{(N)} &= \sum_{i=1}^{N} \lambda_i^2 \equiv \lambda^2 \\
B_+^{(N)} &= \sum_{i=1}^{N} p_i^2 \equiv \mathbf{p}^2 & B_-^{(N)} &= \sum_{i=1}^{N} q_i^2 \equiv \mathbf{q}^2 & K &= \sum_{i=1}^{N} \left( q_ip_i - \frac{\lambda_i^2}{2} \right) \equiv \mathbf{q} \cdot \mathbf{p} - \frac{\lambda^2}{2}
\end{align*}
\]

(36)

where \(\lambda = (\lambda_1, \ldots, \lambda_N)\). From this viewpoint it is clear that the \(h_6\)-coalgebra may give rise to a generalization of the integrable systems with \(\mathfrak{sl}(2, \mathbb{R})\)-coalgebra symmetry, provided that all \(b_i = 0\) but, as we shall show in what follows, one has to pay, in principle, the price of losing
complete integrability. We stress that the $h_6$ Poisson algebra was formerly considered in [32] in the framework of some 3D integrable systems, but only the additional coalgebra structure here presented makes possible to consider $h_6$ as a useful symmetry in ND systems.

We can now compute the $(2N - 3)$ integrals of motion displayed in table 2 and coming from the nonlinear Casimir

$$C = C_2/C_1 = MB_+B_- - B_+A_2^2 - B_-A^2_+ - M(K + M/2)^2 + 2A_- A_+(K + M/2).$$

(37)

A straightforward computation shows that $C^{(2)} = C^{(2)} = 0$ and the remaining ones read

$$C^{(m)} = \sum_{1 \leq i < j < k}^m (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2$$

$$C^{(m)} = \sum_{N-m+1 \leq i < j < k}^{N} (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2$$

(38)

where $m = 3, \ldots, N$. This means that any smooth function

$$H^{(N)} = \mathcal{H}\left(K^{(N)}, B^{(N)}_+, B^{(N)}_-, A^{(N)}_-, A^{(N)}_+, M^{(N)}\right)$$

$$= \mathcal{H}\left(q \cdot p - \frac{1}{2} \lambda^2, q^2, p^2, \lambda \cdot q, \lambda \cdot p, \lambda^2\right)$$

(39)

Poisson commutes with the $2N - 5$ functions (38) but now the sets $\{C^{(3)}, \ldots, C^{(N)}\}$ and $\{C^{(3)}, \ldots, C^{(N)}\}$ are only formed by $N - 2$ integrals in involution so that, in general, $H^{(N)}$ does not provide a completely integrable system. Since there is only one constant left, say $I$, to obtain complete integrability we have called these systems quasi-integrable ones [16]. Nevertheless, this initial failure of the coalgebra approach can be circumvented by making use of the rich subalgebra structure of $h_6$ and, for some particular choices of $H^{(N)}$, such a remaining integral can be obtained as follows (see [16] for a complete discussion on the subject).

Firstly, in order to ensure the existence of $I$ for any dimension $N$, we shall assume that this additional integral is also $h_6$-coalgable invariant, which means that it can be written as a function

$$I = I(K, B_+, B_-, A_+, A_-, M)$$

(40)

where the $h_6$ generators are written in their $N$-particle symplectic realization (36). In this way, if $I$ is functionally independent with respect to both the $h_6$ Casimir (37) and the Hamiltonian $\mathcal{H}$, the coalgebra symmetry guarantees –by construction– the involutivity of $I$ with respect to the $(N-2)$ ‘left’ integrals $C^{(m)}$ ($m = 3, \ldots, N$) and its functional independence with respect to them. And the very same result holds for the $(N-3)$ ‘right’ integrals $C^{(m)}$ where $m = 3, \ldots, N - 1$ (we recall that $C^{(N)} = C^{(N)}$). Therefore, if $I$ can be finally found then the corresponding Hamiltonian is not only integrable but furthermore superintegrable with $2N - 4$ functionally independent constants of motion (one less than a QMS system).

In particular, we can consider two different situations in which the existence of $I$ is guaranteed by construction:

- **Subalgebra integrability.** If the Hamiltonian $\mathcal{H}$ is defined within a subalgebra of $h_6$ that has a nonlinear Casimir invariant, the $N$-particle realization of the Casimir of the subalgebra provides automatically the integral $I$.

- **Generator integrability.** Now, let us choose a given generator $X$ of $h_6$. If we look for all the generators $X_j$ ($j = 1, \ldots, n$) commuting with $X$ and we also look for all the subalgebras
$g_k$ ($k = 1, \ldots, t$) containing $X$ as generator, then the Hamiltonian constructed through any function of the type

$$\mathcal{H}_X = \mathcal{H}_X (C_{g_1}, \ldots, C_{g_t}, X, X_1, \ldots, X_n),$$

where $C_{g_k}$ is the Casimir function of the subalgebra $g_k$, is such that

$$\{\mathcal{H}_X, X\} = 0.$$ (42)

Moreover, the $N$-th particle symplectic realization of both $X$ and $\mathcal{H}_X$ will Poisson-commute with the two sets of integrals $C^{(m)}$ and $C_{(m)}$ (38). Under such hypotheses, $\mathcal{H}_X$ is completely integrable since the $N$-th particle symplectic realization of the generator $X$ is just the additional constant of motion $I$. Since we have five relevant generators $\{K, B_+, B_-, A_+, A_-\}$ this procedure will give rise to five families of completely integrable systems that have been studied in detail in [16]. As we shall see in section 6, $h_6$-coalgebra systems include natural Hamiltonians as well as static electromagnetic fields and curved geodesic flows.

But even in the case that a given Hamiltonian does not fit within the two previous cases, the search for the remaining integral $I$ –in case it does exist– can also be performed by using direct methods that can be computerized. In fact, the additional integral $I$ can be searched among $h_6$ functions with cubic or higher dependence on the momenta (note that all the integrals that we have presented so far are, at most, quadratic in the momenta). Indeed, some particular solutions to this problem have recently been found and lead to new completely integrable $ND$ Hamiltonians (see [16, 33]).

Finally, it could also happen that for a certain $\mathcal{H}$ the additional integral $I$ does exist, but it cannot be written as a function (40) of the $h_6$ generators (i.e., $I$ is not coalgebra-invariant). This implies that the explicit form for $I$ has to be found for each dimension, and –hopefully– a generic $ND$ expression can be found inductively. Nevertheless, due to the physical interest of $h_6$ invariant systems (see section 6) this possibility is worth to be explored with the aid of symbolic computation packages.

4. QMS Hamiltonians with $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry

In the sequel we will show the potentialities of the coalgebra framework by summarizing some of the $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetric Hamiltonians of the form (24) that have been so far described in the literature. Some of them were already known but many others have been constructed for the first time by making use of the constructive approach detailed in section 3.1. Since we are always dealing with $ND$ systems and for the sake of simplicity of the notation, from now on we shall drop the index ‘$(N)$’ in both the $ND$ symplectic realization and the Hamiltonians.

4.1. Evans systems

The following generalization of the motion of a classical particle on an $ND$ Euclidean space $\mathbb{E}^N$ under a spherically symmetric potential is $\mathfrak{sl}(2, \mathbb{R})$-coalgebra invariant [3]

$$H = \frac{1}{2} J_+ + \mathcal{F} (J_-) = \frac{1}{2} \mathbf{P}^2 + \mathcal{F} (\mathbf{q}^2) + \sum_{i=1}^{N} \frac{b_i}{2q_i^2}$$ (43)

where $\mathcal{F}$ is a smooth function and the terms depending on the $b_i$’s constants are just additional ‘centrifugal barriers’ coming from a non-zero symplectic realization of $\mathfrak{sl}(2, \mathbb{R})$. In fact, the $\mathfrak{sl}(2, \mathbb{R})$-coalgebra provides a common set of (right and left) integrals of the motion (22) for any choice of $\mathcal{F}$ [12]. Note that the $ND$ Smorodinsky–Winternitz Hamiltonian [34] is recovered when $\mathcal{F}(\mathbf{q}^2) = \omega^2 \mathbf{q}^2 / 2$, and the Kepler–Coulomb potential corresponds to $\mathcal{F}(\mathbf{q}^2) = -k/|\mathbf{q}|$, where $\omega$ and $k$ are real constants (and $|\mathbf{q}| = \sqrt{\mathbf{q}^2}$). Therefore, both outstanding MS systems do have coalgebra symmetry.
4.2. Superintegrable electromagnetic field Hamiltonians

Certain velocity-dependent potentials on $\mathbb{R}^N$ giving rise to superintegrable electromagnetic field Hamiltonians [35, 36] can also be obtained in this framework. The most general example of this class is given by [12]:

$$
H = \frac{1}{2}J_+ - eJ_3G(J_-) + eF(J_-) + \frac{1}{2}p^2 - e(q \cdot p)G(q^2) + eF(q^2) + \sum_{i=1}^{N} \frac{b_i}{2q_i^2}
$$

(44)

where $e$ is the electric charge, while $G$ and $F$ are smooth functions.

When $N = 3$, the (time-independent) scalar $\psi$ and vector $A$ potentials read

$$
\psi(q) = F(q^2) - e\frac{q^2G^2(q^2)}{2} + \frac{3}{2}\sum_{i=1}^{3} \frac{b_i}{2eq_i^2}, \quad A(q) = qG(q^2).
$$

Then the electric $E = -\nabla \psi$ and magnetic $H = \nabla \times A$ fields turn out to be

$$
E = (eG^2 + 2eG^2G' - 2F') q + \frac{1}{e} \frac{b_1}{q_1}, \frac{b_2}{q_2}, \frac{b_3}{q_3}
$$

$$
H = 0
$$

(45)

where $G'$ and $F'$ are the derivatives with respect to the variable $q^2$. This kind of construction can also be applied to obtain certain $ND$ Fokker–Planck Hamiltonians (see [37] and references therein).

4.3. Free motion on Riemannian spaces of constant curvature

The kinetic energy $T$ of a particle on the $ND$ Euclidean space $\mathbb{E}^N$ is just given by the generator $J_+$ in the symplectic realization (21) with all $b_i = 0$:

$$
H \equiv T = \frac{1}{2}J_+ = \frac{1}{2}p^2.
$$

(45)

Surprisingly enough, the kinetic energy on $ND$ Riemannian spaces with constant sectional curvature $\kappa$ can also be expressed in Hamiltonian form as a function of the $ND$ symplectic realization of the $\mathfrak{sl}(2,\mathbb{R})$ generators (21). In fact, this can be done in two different ways [12]:

$$
H^P \equiv T^P = \frac{1}{2}(1 + \kappa J_-)^2 J_+ = \frac{1}{2}(1 + \kappa q^2)^2 p^2
$$

$$
H^B \equiv T^B = \frac{1}{2}(1 + \kappa J_-) (J_+ + \kappa J_3^2) = \frac{1}{2}(1 + \kappa q^2) (p^2 + \kappa (q \cdot p)^2).
$$

(46)

In these expressions the function $T^P$ is just the kinetic energy for a free particle on the spherical $S^N$ ($\kappa > 0$) and hyperbolic $H^N$ ($\kappa < 0$) spaces when Poincaré coordinates $q$ and their canonical momenta $p$ (coming from a stereographic projection in $\mathbb{R}^{N+1}$ [38]) are used. On the other hand $T^B$ corresponds to Beltrami coordinates and momenta (central projection). The canonical transformation between both sets of phase space variables can be found in [39].

We can immediately conclude that, by construction, both Hamiltonians are QMS ones since they are $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetric. In other words, we can say that the $ND$ Riemannian spaces with constant curvature are $\mathfrak{sl}(2,\mathbb{R})$-coalgebra spaces.
4.4. Superintegrable potentials on spaces with constant curvature
Without breaking the \(\mathfrak{s}(2, \mathbb{R})\)-coalgebra symmetry, QMS potentials on constant curvature spaces can now be constructed by adding some suitable functions depending on \(J_-\) to (46) and also by considering arbitrary centrifugal terms that come from symplectic realizations of the \(J_+\) generator with generic \(b_i\)'s. With all these ingredients, the full \(\mathfrak{s}(2, \mathbb{R})\) Hamiltonian will be of the form

\[
H = T (J_-, J_+, J_3) + \mathcal{U} (J_-).
\]  

(47)

The QMS systems obtained in this way are just the curved counterpart of the Euclidean systems, and by making use of the curvature parameter \(\kappa\) we can simultaneously describe the spherical \(S^N (\kappa > 0)\), hyperbolic \(H^N (\kappa < 0)\) and Euclidean \(E^N (\kappa = 0)\) cases.

We remark that special choices for \(\mathcal{U}\) lead to the many interesting QMS systems on constant curvature spaces, that can always be expressed in both Poincaré and Beltrami phase space coordinates. In what follows we write three types of examples of this kind [12] which share the same set of constants of the motion (22) although the geometric meaning of the canonical coordinates and momenta can be different for each example.

4.4.1. Curved Evans systems
The ND constant curvature generalization of the generic 3D Euclidean system with radial symmetry [40] is given by

\[
H^P = T^P + \mathcal{U} \left( \frac{4J_-}{(1 - \kappa J_-)^2} \right) = \frac{1}{2} (1 + \kappa q^2)^2 p^2 + \mathcal{U} \left( \frac{4q^2}{1 - \kappa q^2} \right) + (1 + \kappa q^2)^2 \sum_{i=1}^{N} \frac{b_i}{2q_i^2}
\]

\[
H^B = T^B + \mathcal{U} (J_-) = \frac{1}{2} (1 + \kappa q^2) \left( p^2 + \kappa (q \cdot p)^2 \right) + \mathcal{U} (q^2) + (1 + \kappa q^2) \sum_{i=1}^{N} \frac{b_i}{2q_i^2}
\]  

(48)

where \(\mathcal{U}\) is the smooth function that gives the central potential and its specific dependence in terms of \(J_-\) corresponds to the square of the radial distance in each coordinate system [12]. Hence, the Hamiltonians (48) are the curved generalization of the (flat) Evans systems on \(E^N\) (43), which are recovered under the limit (contraction) \(\kappa \to 0\).

4.4.2. The curved Smorodinsky–Winternitz system
This well-known Hamiltonian [41, 42, 43, 44, 45, 46] is just the Higgs oscillator [47, 48] (that comes from the function \(\mathcal{U}\) in (48)) plus the corresponding \(N\) centrifugal terms:

\[
H^P = T^P + \frac{\omega^2 J_-}{2(1 - \kappa J_-)^2} = \frac{1}{2} (1 + \kappa q^2)^2 p^2 + \frac{\omega^2 q^2}{2(1 - \kappa q^2)^2} \sum_{i=1}^{N} \frac{b_i}{2q_i^2}
\]

\[
H^B = T^B + \frac{1}{2} \omega^2 J_- = \frac{1}{2} (1 + \kappa q^2) \left( p^2 + \kappa (q \cdot p)^2 \right) + \frac{1}{2} \omega^2 q^2 \sum_{i=1}^{N} \frac{b_i}{2q_i^2}
\]  

(49)

This is a MS Hamiltonian whose integrals of the motion are quadratic in the momenta. The only integral of the motion that does not come from the coalgebra symmetry has been explicitly written, in this variables, in [12]. Clearly the limit \(\kappa \to 0\) of both expressions (49) gives rise to the superposition of the \(N\)D isotropic harmonic oscillator with \(N\) centrifugal terms in Cartesian coordinates:

\[
H = \frac{1}{2} J_+ + \frac{1}{2} \omega^2 J_- = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 q^2 \sum_{i=1}^{N} \frac{b_i}{2q_i^2}.
\]
4.4.3. The curved generalized Kepler–Coulomb system The Kepler–Coulomb potential on constant curvature spaces [42, 43, 44, 49, 50, 51, 52, 53, 54] with coupling constant $k$ and $N$ centrifugal terms is given by

$$H^P = T^P - k \left( \frac{J^-}{(1 - \kappa J^-)^2} \right)^{-1/2} = \frac{1}{2}(1 + \kappa q^2)^2 p^2 - k \frac{(1 - \kappa q^2)}{|q|} + (1 + \kappa q^2)^2 \sum_{i=1}^{N} \frac{b_i}{2q_i^2}$$

$$H^B = T^B - k J_+^{-1/2} = \frac{1}{2}(1 + \kappa q^2) (p^2 + \kappa (q \cdot p)^2) - \frac{k}{|q|} + (1 + \kappa q^2)^2 \sum_{i=1}^{N} \frac{b_i}{2q_i^2}. \quad (50)$$

This has recently been shown to be a MS system [39, 55]. When, at least, one of the centrifugal terms vanishes ($b_i = 0$) the remaining constant of the motion is quadratic in the momenta [12], whilst when all the constants $b_i \neq 0$, the additional integral is quartic in the momenta and has been recently presented in [39]. We recall that the MS nature of the Euclidean case

$$H = \frac{1}{2} J_+ - k J_+^{-1/2} = \frac{1}{2} p^2 - \frac{k}{|q|} + \sum_{i=1}^{N} \frac{b_i}{2q_i^2}$$

was proven in [55], and it can be recovered from our formalism when $\kappa = 0$.

4.5. Free motion on spherically symmetric spaces of nonconstant curvature

With the previous examples in mind, it is easy to realize that any metric of the type

$$ds^2 = f(|q|)^2 dq^2 \quad (51)$$

leads to the free Hamiltonian given by

$$H = T = \frac{J_+}{2 f(\sqrt{J_+})} = \frac{p^2}{2 f(|q|)^2} \quad (52)$$

provided that all $b_i = 0$. Thus this system determines the geodesic motion on the ND spherically symmetric space (51) with conformal factor $f(|q|)$; this is generically a Riemannian space of nonconstant curvature. In particular, its scalar curvature turns out to be

$$R = -(N - 1) \frac{2f''(|q|) + 2(N - 1)|q|^{-1}f'(|q|) + (N - 2)f'(|q|)^2}{f(|q|)^2} \quad (53)$$

where $f'(|q|) = df/d|q|$ and $f''(|q|) = d^2 f/d|q|^2$.

Therefore, since the Hamiltonian (52) corresponds to an $\mathfrak{sl}(2, \mathbb{R})$-coalgebra space then its geodesic flows define a QMS system for any choice of $f(|q|)$.

Clearly, the geodesic flows on the Riemannian spaces of constant curvature described in section 4.3 are particular examples of the family (52) corresponding to set $f(|q|) = (1 + \kappa q^2)^{-1}$ in Poincaré coordinates. Let us now describe other physically relevant examples with coalgebra symmetry.

4.5.1. Darboux spaces Four interesting examples of this class of spherically symmetric spaces with nonconstant curvature are the ND generalizations of the four 2D Darboux spaces introduced in [13]. We recall that Darboux surfaces are the only 2D spaces with nonconstant curvature for which there exists two functionally independent quadratic Killing tensors, i.e., the geodesic motion on these 2D spaces is (quadratically) MS. There are only four spaces of this
Table 3. Examples of spherically symmetric spaces and their corresponding QMS free Hamiltonian with $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetry.

| Space          | Metric $d\mathbf{s}^2$                                      | Hamiltonian $T = T(J_-, J_+)$ |
|----------------|------------------------------------------------------------|--------------------------------|
| Darboux I      | $\ln|q|d\mathbf{q}^2$                                      | $\frac{J_+J_-}{2\ln \sqrt{J_-}}$ |
| Darboux II     | $1 + \ln^2|q| \frac{d\mathbf{q}^2}{q^2\ln^2|q|}$          | $\frac{J_+^2J_-}{2(1 + \ln^2 \sqrt{J_-})}$ |
| Darboux IIIa   | $1 + |q| \frac{d\mathbf{q}^2}{q^4}$                       | $\frac{J_+}{2(1 + \sqrt{J_-})}$    |
| Darboux IIIb   | $(k + q^2) \frac{d\mathbf{q}^2}{q^4}$                   | $\frac{J_+}{2(k + J_-)}$           |
| Darboux IV     | $a + \cos(\ln|q|) \frac{d\mathbf{q}^2}{q^4\sin^2(\ln|q|)}$ | $\frac{J_+ \sin^2(\ln \sqrt{J_-})J_+}{2(a + \cos(\ln \sqrt{J_-}))}$ |
| Multifold Kepler| $(a + b|q|\frac{\frac{1}{2}}{q^2\frac{\frac{1}{2}}}) \frac{d\mathbf{q}^2}{q^4}$ | $\frac{J_+^{1\frac{1}{2}}J_-}{2(a + b\sqrt{J_-})}$ |
| Taub-NUT       | $(4m + |q|) \frac{d\mathbf{q}^2}{|q|}$                    | $\frac{\sqrt{J_-}J_+}{2(4m + \sqrt{J_-})}$ |

type, that were characterized by Koenigs in a note included in the famous Darboux treatise [56]. It turns out that, by using appropriate charts, the four 2D Darboux metrics [57, 58, 59, 60] can be rewritten in the form (51). Therefore, ND spherically symmetric generalizations of such spaces can be obtained by expressing their kinetic energy Hamiltonians with $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetry. The four ND Darboux metrics [13] are explicitly written in table 3 (where $a$ and $k$ are real constants).

Once again, we have that the $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetry ensures that the geodesic flows on all these spaces are QMS Hamiltonian systems. In fact, one more independent integral of the motion is expected to exist for all of them, since all these spaces are presumably MS in any dimension. Such an additional integral has already been found for the Darboux metric of type III [14].

4.5.2. Iwai–Katayama spaces These are the ND counterpart of the 3D spaces underlying the so called ‘multifold Kepler’ systems introduced in [61, 62], which depend on two real constants, $a$ and $b$, as well on a rational parameter $\nu$ as shown in table 3. The physical interest of these systems relies on the fact that they are generalizations of the Taub-NUT metric which is recovered as the particular case with $\nu = 1$, $a = 4m$ and $b = 1$. We stress that in the 3D case the multifold Kepler Hamiltonians have been shown to be MS [63], but the additional integral of motion is not, in general, quadratic in the momenta.

4.5.3. Potentials We stress that centrifugal terms and central potentials can directly be added to the free system (52) by considering again an ND symplectic realization with arbitrary $b_i$’s plus a function $U(\sqrt{J_-})$ as

$$H = \frac{J_+}{2f(\sqrt{J_-})} + U(\sqrt{J_-}) = \frac{P^2}{2f(|q|^2)} + U(|q|) + \frac{1}{2f(|q|^2)2} \sum_{i=1}^{N} b_i \frac{1}{q_i^2}. \quad (54)$$
Several examples, such as oscillator and Kepler–Coulomb potentials on spherically symmetric spaces, can be found in [15].

5. QMS Hamiltonians with $\mathfrak{sl}_2(\mathbb{R})$-coalgebra symmetry

A generalization of the construction presented in the previous section can be obtained by making use of the non-standard quantum deformation of $\mathfrak{sl}(2, \mathbb{R})$ described in section 3.2 as the dynamical symmetry for the Hamiltonians corresponding to geodesic motion. In this case, the spaces so obtained are, in general, of nonconstant curvature, and the latter depends on the deformation parameter $z$.

5.1. Free motion

In general, we can consider an infinite family of QMS geodesic flows with $\mathfrak{sl}_2(\mathbb{R})$-coalgebra symmetry (sharing the set of integrals (30)) through the family of Hamiltonians

$$H_z = \frac{1}{2} J_+ g(zJ_-) = \frac{1}{2} g(z^2) \sum_{i=1}^N \frac{\sinh z q_i^2}{z q_i^2} p_i^2 \exp \left\{ -z \sum_{k=1}^{i-1} q_k^2 + z \sum_{l=i+1}^N q_l^2 \right\}$$ (55)

where $g$ is a smooth function such that $\lim_{z \to 0} g(zJ_-) = 1$, so that $\lim_{z \to 0} H_z = \frac{1}{2} p^2$. This, in turn, means that $H_z$ is a QMS deformation of the free Euclidean motion which defines a geodesic flow on an ND $\mathfrak{sl}_2(\mathbb{R})$-coalgebra space with metric given by:

$$ds^2 = \frac{1}{g(z^2)} \sum_{i=1}^N \frac{2zq_i^2}{\sinh zq_i^2} dq_i^2 \exp \left\{ z \sum_{k=1}^{i-1} q_k^2 - z \sum_{l=i+1}^N q_l^2 \right\}.$$ (56)

For $N = 2$, the Gaussian curvature of the space turns out to be [9]

$$K(x) = z \left( g'(x) \cosh x + \left( g''(x) - g(x) - g'^2(x) / g(x) \right) \sinh x \right)$$ (57)

where $x \equiv zJ_- = z(q_1^2 + q_2^2) \equiv zq^2$, $g' = \frac{dg(x)}{dx}$ and $g'' = \frac{d^2g(x)}{dx^2}$. For $N = 3$, the scalar curvature reads [64]

$$R(x) = z \left( 6g'(x) \cosh x + \left( 4g''(x) - 5g(x) - 5g'^2(x) / g(x) \right) \sinh x \right)$$ (58)

where $x \equiv zJ_- = z(q_1^2 + q_2^2 + q_3^2) \equiv zq^2$.

Now we briefly comment on some specific $\mathfrak{sl}_2(\mathbb{R})$-coalgebra spaces that arise for simple choices of the function $g$ (see [9, 11, 64]). For each of them we write the Gaussian (57) and scalar (58) curvatures corresponding to $N = 2, 3$.

- The simplest choice is to set $g(zJ_-) = 1$. The curvatures reduce to

$$K = -z \sinh (zq^2) \quad R = -5z \sinh (zq^2)$$

that is, they are nonconstant and negative so the space is of hyperbolic type.

- Another possibility, which generalizes the above one, is to take $g(zJ_-) = \exp(azJ_-)$ where $a$ is a real constant. The curvatures are given by

$$K = ze^{azq^2} \left( a \cosh (zq^2) - \sinh (zq^2) \right) \quad R = ze^{azq^2} \left( 6a \cosh (zq^2) - (5 + a^2) \sinh (zq^2) \right).$$

It is worthy to remark that the very special cases with $a = \pm 1$ gives $K = \pm z$ and $R = \pm 6z$, so this choice for $g$ includes the three classical Riemannian spaces of constant curvature (see section 4.3), for which the role of the sectional curvature $\kappa$ is now played by the real deformation parameter $z$. 


• As a third example we consider \( g(zJ_-) = \cosh(zJ_-)^b \) where \( b \) is a real constant. This yields

\[
K = z \left( \cosh(zq^2) \right)^{b-2} \sinh(zq^2) \left( b + (b - 1) \cosh^2(zq^2) \right) \\
R = -z \left( \cosh(zq^2) \right)^{b-2} \sinh(zq^2) \left( 5(10b) \cosh^2(zq^2) + b(4 + b) \sinh^2(zq^2) \right).
\]

When \( b = 1 \), the expressions are rather simplified as \( K = z \tanh(zq^2) \) and \( R = 5K \).

We stress that the geodesic motion of a classical particle on any of these spaces will have as integrals of the motion the functions (30).

5.2. Potentials
We can also introduce more general \( \mathcal{N} \)D QMS Hamiltonians based on \( \mathfrak{sl}_z(2, \mathbb{R}) \) (29) by considering symplectic realizations with arbitrary \( b_i \)'s (thus giving rise to ‘deformed centrifugal terms’) and by adding functions depending on \( J_- \). In particular, we have already studied the Hamiltonians for the explicit 2D and 3D construction (see [10] and [64], respectively). In this case the \( \mathcal{N} \)D Hamiltonian reads

\[
H_z = \frac{1}{2} J_+ g(zJ_-) + \mathcal{U}(zJ_-) \\
= \frac{1}{2} g(zq^2) \sum_{i=1}^{N} \left( \frac{\sinh{zq^2_i}}{zq^2_i} p^2_i + \frac{zb_i}{\sinh{zq^2_i}} \right) \exp \left\{ -z \sum_{i=1}^{N} q^2_i + z \sum_{l=i+1}^{N} q^2_l \right\} + \mathcal{U}(zq^2) \tag{59}
\]

where the smooth functions \( g \) and \( \mathcal{U} \) are such that

\[
\lim_{z \to 0} \mathcal{U}(zJ_-) = \mathcal{V}(J_-) \quad \lim_{z \to 0} g(zJ_-) = 1.
\]

This condition means that the non-deformed/flat limit for these Hamiltonians is

\[
\lim_{z \to 0} H_z = \frac{1}{2} p^2 + \mathcal{V}(q^2) + \sum_{i=1}^{N} \frac{b_i}{2q^2_i},
\]

which are just the Evans systems (43). So in this way we could define QMS analogues of the Smorodinsky–Winternitz and Kepler potentials [64] on the \( \mathfrak{sl}_z(2, \mathbb{R}) \)-coalgebra spaces.

In any case, a common feature of interacting systems with \( \mathfrak{sl}_z(2, \mathbb{R}) \)-coalgebra symmetry is the long-range nature of the ‘interaction terms’ depending on the coordinates and appearing in (59).

6. Quasi-integrable Hamiltonians with \( \mathfrak{h}_6 \)-coalgebra symmetry
We present three families of Hamiltonians with the underlying \( \mathfrak{h}_6 \)-coalgebra symmetry described in section 3.3. We recall that all of them are endowed with the \( 2\mathcal{N} - 5 \) integrals (38) but, in principle, they are only quasi-integrable since only \( \mathcal{N} - 2 \) integrals are in involution. Nevertheless, for some specific choices of the function \( H \) the subalgebra structure of \( \mathfrak{h}_6 \) has allowed us to obtain the remaining integral by following the procedure already described in section 3.3. We omit here the explicit expressions for the families of integrable cases, that are fully described in [16].

6.1. Natural systems
The Hamiltonian

\[
H = \frac{1}{2} B_+ + \mathcal{F} (A_-, B_-) \tag{60}
\]
where $F$ is a function playing the role of a potential, gives rise to the following quasi-integrable system on $E^N$:

$$H = \frac{1}{2}p^2 + F(\lambda \cdot q, q^2).$$

(61)

Notice that central potentials arise if $F$ does not depend on $A_-$; in the case with generic $F(A_-, B_-)$, spherical symmetry is broken and QMS is, in principle, reduced to quasi-integrability.

6.2. Electromagnetic Hamiltonians

The most general $h_6$-Hamiltonian including linear terms in the momenta is given by

$$H = \frac{1}{2}B^2 + KF(A_-, B_-) + A_+G(A_-, B_-) + \mathcal{R}(A_-, B_-)$$

(62)

where $F$, $G$ and $\mathcal{R}$ are smooth functions. This means that

$$H = \frac{1}{2}p^2 + \left(\lambda \cdot p - \frac{\lambda^2}{2}\right)F(\lambda \cdot q, q^2) + (\lambda \cdot p)G(\lambda \cdot q, q^2) + \mathcal{R}(\lambda \cdot q, q^2).$$

(63)

In 3D, this Hamiltonian describes the motion of a particle on $E^3$ under the action of a static magnetic field with vector and scalar potentials given by

$$A_i = -\frac{q_i}{e}F(A_-, B_-) - \frac{\lambda_i}{e}G(A_-, B_-) \quad i = 1, 2, 3$$

$$\psi = \frac{1}{e}\mathcal{R}(A_-, B_-) - \frac{1}{2e}M F(A_-, B_-)$$

$$- \frac{1}{2e} \left[ B_- F(A_-, B_-)^2 + 2A_- F(A_-, B_-)G(A_-, B_-) + MG(A_-, B_-)^2 \right]$$

(64)

where $e$ is the electric charge. The $h_6$ quasi-integrability implies that, for any choice of the functions $F$, $G$ and $\mathcal{R}$, the Hamiltonian (63) commutes with the function

$$C^{(3)} = (\lambda_1(p_2q_3 - p_3q_2) + \lambda_2(p_3q_1 - p_1q_3) + \lambda_3(p_1q_2 - p_2q_1))^2.$$ 

Note that these systems can be thought of as a generalization of the electromagnetic Hamiltonians presented in section 4.2 and coming from the $\mathfrak{sl}(2,\mathbb{R})$-coalgebra symmetry (provided that all $b_i = 0$).

6.3. Geodesic flow Hamiltonians

A third family of relevant systems is given by $ND$ Hamiltonians of the type

$$H = \sum_{i,j=1}^{N} g^{ij}(q_1, \ldots, q_N) p_i p_j$$

that are obtained by considering

$$H = B_+F(A_-, B_-) + A_+^2G(A_-, B_-)$$

$$+ \left( K + \frac{M}{2} \right)^2 \mathcal{R}(A_-, B_-) + \left( K + \frac{M}{2} \right) A_+S(A_-, B_-)$$

(65)
since for any choice of the functions $F$, $G$, $R$ and $S$ we obtain a Hamiltonian which is a quadratic homogeneous function in the momenta; namely,

$$H = p^2 F(\lambda \cdot q, q^2) + (\lambda \cdot p)^2 G(\lambda \cdot q, q^2) + (q \cdot p)^2 R(\lambda \cdot q, q^2) + (q \cdot p)(\lambda \cdot p) S(\lambda \cdot q, q^2).$$ (66)

The specific form of the metric $g^{ij}$ is so determined by $F$, $G$, $R$ and $S$ which, in general, give rise to an $ND$ space of nonconstant curvature. In any case, the set of constants of motion (38) is universal and does not depend on the specific choice of the functions in the Hamiltonian.

Moreover, additional potentials on these $h_6$-coalgebra spaces can be naturally considered by adding functions such as, e.g., $U(A_-, B_-)$ to the free Hamiltonian (65). In this way the natural Euclidean systems (60) can be generalized to the curved spaces defined through (65) without breaking the quasi-integrability of the free Hamiltonian.

7. Integrable Hamiltonians with other coalgebra symmetries

For the sake of completeness, we briefly present some integrable Hamiltonians which are based on different coalgebras to the above considered. We recall that a systematic study of Lie–Poisson coalgebras with dimensions 3, 4 and 5 has been performed in [8], and some more examples can be found in [4, 5, 6].

7.1. The Calogero–Gaudin system

The Calogero–Gaudin (CG) Hamiltonian [65, 66]

$$H = \sum_{1 \leq i < j} 2 p_i p_j (1 - \cos(q_i - q_j))$$ (67)

was proven in [1, 2] to have an underlying $\mathfrak{so}(2, 1)$-coalgebra symmetry and its constants of the motion were identified with the coproducts of the $\mathfrak{so}(2, 1)$ Casimir under a certain symplectic realization. The generalized CG system

$$H = \left( \sum_{i=1}^{N} p_i \right)^2 + \frac{1}{2} \sum_{i,j=1}^{N} p_i p_j \left( (\kappa_i + \kappa_j) \cos(q_i - q_j) - (\kappa_i - \kappa_j) \cos(q_i + q_j) \right)$$ (68)

where $\kappa_i$ are free parameters, was also proven to be completely integrable in [2].

We stress that, in general, the modification of the chosen symplectic realization drastically changes the ‘shape’ of the Hamiltonian through an associated canonical transformation. For instance, by using the Gelfan’d–Dyson symplectic map the very same CG system reads [5]

$$H = \sum_{1 \leq i < j} \left\{ - p_i p_j (q_i - q_j)^2 - b (p_i - p_j) (q_i - q_j) \right\} + \frac{b^2}{4} N^2$$ (69)

where $b$ is a constant. On the other hand, the $N = 2$ rational Calogero–Moser Hamiltonian [67] has also been proven to have $\mathfrak{sl}(2, \mathbb{R})$-coalgebra symmetry [5].

7.2. An integrable deformation of the CG system from $\mathfrak{so}_+ (2, 1)$

As we have pointed out previously, given a Hamiltonian with certain coalgebra symmetry any quantum deformation of the underlying coalgebra provides an integrable deformation of the
initial Hamiltonian. This was explicitly shown for the first time by constructing the following deformed CG system [1]:

\[ H_z = \sum_{1 \leq i < j}^N 2 \pi_i \pi_j (1 - \cos(q_i - q_j)) \]  

(70)

where the non-local deformations of the momenta are

\[ \pi_k = 2 \frac{\sinh(\frac{z p_k}{2})}{z} \prod_{i=1}^{k-1} e^{-\frac{z p_i}{2}} \prod_{j=k+1}^N e^{\frac{z p_j}{2}}. \]

The corresponding constants of the motion come from the (deformed) coproduct of the (deformed) Casimir of \( \mathfrak{so}(2,1) \) (the standard deformation of \( \mathfrak{so}(2,1) \)) and, as expected, in the limit \( z \rightarrow 0 \) we recover the ‘classical’ CG system given in (67). We remark that the quantum mechanical version of this deformed CG system has been explicitly solved [17, 18] and a ‘twisted’ version of the Gaudin magnet has already been introduced [68] through a quantum deformation of the \( \mathfrak{gl}(2,\mathbb{R}) \)-coalgebra.

7.3. Systems defined on the harmonic oscillator coalgebra

The Hamiltonian [69]

\[ H = (\lambda + \mu) \sum_{i=1}^N p_i + 2 \mu \sum_{i<j}^N \sqrt{p_i p_j} \cosh(q_i - q_j) \]  

(71)

where \( \lambda \) and \( \mu \) are real constants, can also be proven to be coalgebra invariant under the harmonic oscillator coalgebra \( (h_4, \Delta) \). The quantum version of this Hamiltonian has also been solved [70] and the system turns out to be equivalent to a chain of coupled oscillators [70, 71].

7.4. Ruijsenaars–Schneider-like systems from a quantum Poincaré coalgebra

Another interesting example of coalgebra-invariant system is the following analogue [2] of the Ruijsenaars–Schneider model [72]:

\[ H_z = \sum_{i=1}^N \cosh \theta_i \exp \left( -\frac{z}{2} \sum_{j=1}^{i-1} q_j \right) + \frac{z}{2} \left( \sum_{k=i+1}^N q_k \right) \]  

(72)

where \( (q_i, \theta_i) \) are canonically conjugate variables such that \( \{ q_i, \theta_j \} = \delta_{ij} \). This completely integrable Hamiltonian was obtained by using the Poisson analogue of the quantum deformation of the (1+1)D Poincaré algebra introduced in [73].

8. Generalizations of coalgebra symmetry

So far we have shown how the coalgebra formalism provides an algebraic approximation to the integrability properties of Hamiltonian systems defined on a chain of \( N \) copies, \( A \otimes A \otimes \cdots \otimes A \), of a given coalgebra \( A \). Amongst the possible generalizations of this approach, two of them have already been analysed. The first one deals with integrable systems defined on the \( N \)-chain \( V \otimes A \otimes A \otimes \cdots \otimes A \) in which \( V \) is an \( A \)-comodule algebra [4, 74]. The second extension of the formalism deals with the so-called loop coproducts [75], that generalize in terms of loop algebras all the algebraic machinery presented in this paper. As a concluding section of this contribution, we briefly sketch both approaches.
8.1. Comodule algebra symmetry

We recall that a (right) coaction of a coalgebra \((A, \Delta)\) on a vector space \(V\) is a linear map \(\phi : V \to V \otimes A\) such that the following diagram is commutative:

\[
\begin{array}{c}
\phi \\
\downarrow \phi \otimes \text{id} \\
V \otimes A \\
\downarrow \phi \\
V \otimes A \\
\end{array}
\]

\[
\begin{array}{c}
\phi \otimes \text{id} \\
\downarrow \Delta \\
V \otimes A \otimes \cdots \otimes A \\
\end{array}
\]

If \(V\) is an algebra, we shall say that \(V\) is an \(A\)-comodule algebra if the coaction \(\phi\) is a homomorphism with respect to the product on the algebra \(V\):

\[\phi(ab) = \phi(a) \phi(b) \quad \forall a, b \in V.\]

Moreover, if \(V\) is a Poisson algebra and

\[\phi(\{a, b\}) = \{\phi(a), \phi(b)\} \quad \forall a, b \in V\]

we will say that \(V\) is a Poisson \(A\)-comodule algebra.

It is straightforward to realize that this comodule structure can be used to mimic the coalgebra construction (see [4, 74]) in order to define completely integrable systems on \(V \otimes A \otimes \cdots \otimes A\). However, in this case the superintegrability is lost since ‘right’ integrals do not exist (the left-right symmetry of the chain of algebras has been broken).

Clearly, \(A\) itself is an \(A\)-comodule algebra, with the coproduct \(\Delta\) playing the role of the coaction \(\phi\), so that the comodule symmetry is indeed a generalization of the coalgebra one. In [74] some examples of classical and quantum integrable systems with comodule symmetry have been presented. We illustrate here one of them. As we said in section 3.3, the two photon algebra \(h_6 = \langle K, A_+, A_-, B_+, B_-, M \rangle\) with Poisson brackets (33) contains the \(\mathfrak{gl}(2, \mathbb{R}) = \langle B_-, B_+, K, M \rangle\) subalgebra. Moreover, it is easy to realize that \(\mathfrak{gl}(2, \mathbb{R})\) is a (deformed) \(h_6\)-comodule algebra with the coaction \(\phi\) defined by:

\[
\begin{align*}
\phi(M) &= 1 \otimes M + M \otimes 1 \\
\phi(K) &= 1 \otimes K + K \otimes \frac{1}{1 - \sigma A_-} + M \otimes \frac{\sigma A_-}{1 - \sigma A_-} \\
\phi(B_+) &= 1 \otimes B_+ + B_+ \otimes \frac{1}{(1 - \sigma A_-)^2} - 2\sigma K \otimes \frac{A_+}{1 - \sigma A_-} - \sigma^2 K^2 \otimes \frac{M}{(1 - \sigma A_-)^2} \\
\phi(B_-) &= 1 \otimes A_- + A_- \otimes 1 - \sigma A_- \otimes A_- \\
\end{align*}
\]

where \(\sigma\) is a deformation parameter. If we consider the following symplectic realization of \(h_6\)

\[
\begin{align*}
D(B_+) &= p_1^2 \\
D(B_-) &= p_1^2 \\
D(K) &= -p_1q_1 + \frac{\lambda^2}{2} \\
D(M) &= -\lambda_1^2 \\
D(A_+) &= -\lambda_1q_1 \\
D(A_-) &= -\lambda_1p_1 \\
\end{align*}
\]
and we take as the Hamiltonian on $\mathfrak{gl}(2, \mathbb{R})$ the function

$$H = \frac{1}{2} (B_+ + B_-)$$

the symplectic realization $D$ will give us the 1D harmonic oscillator:

$$H^{(1)} := D(H) = \frac{p_1^2}{2} + \frac{q_1^2}{2}.$$ 

Now, by using the coaction map (73) we can define a Hamiltonian function on $\mathfrak{gl}(2, \mathbb{R}) \otimes h_6$:

$$\phi(H) = \frac{1}{2} (\phi(B_+) + \phi(B_-)).$$

This Hamiltonian can be expressed in terms of canonical coordinates by taking the symplectic realization $D \otimes D$ of (73). It reads:

$$H^{(2)}_\sigma = (D \otimes D)(\phi(H)) = \frac{1}{2} (p_1^2 + p_2^2) + \frac{q_1^2}{2} + \frac{q_2^2}{2(1 + \sigma \lambda_2 p_2)^2} + \sigma \lambda_2 \left( \frac{p_1^2 p_2 + q_2 (\lambda_2^2 - 2q_1 p_1)}{2(1 + \sigma \lambda_2 p_2)} \right) + \sigma^2 \lambda_2^2 \left( \frac{1}{2} \frac{p_1^2 p_2 + (\lambda_2^2 - 2q_1 p_1)^2}{8(1 + \sigma \lambda_2 p_2)} \right).$$

This Hamiltonian is just an integrable deformation of the 2D isotropic oscillator, since $\lim_{\sigma \to 0} H^{(2)}_\sigma = \frac{1}{2} (p_1^2 + p_2^2) + \frac{q_1^2 + q_2^2}{2}$. The integral of motion in involution with $H^{(2)}_\sigma$ is just obtained as the coaction of the $\mathfrak{gl}(2, \mathbb{R})$ Casimir:

$$C^{(2)}_\sigma = (D \otimes D)(\phi(C)) = - \frac{2(p_2q_1 - p_1q_2) + \sigma p_1(2p_1q_1 - 4p_2q_2 - \lambda_2^2) - \sigma^2 \lambda_2^2 p_1 p_2(-2p_1q_1 + 2p_2q_2 + \lambda_2^2)}{16(1 + \sigma \lambda_2 p_2)^2}.$$ 

As expected, the limit $\sigma \to 0$ of (74) is just $-(p_2q_1 - p_1q_2)^2/4$. Obviously, further iterations of the coaction map would provide the corresponding integrable deformation of the isotropic oscillator in an arbitrary dimension. Nevertheless, the explicit form of such a Hamiltonian is quite involved due to the form of the coaction mapping $\phi$.

8.2. Loop coproducts

In [75] a further generalization of the coalgebra approach is proposed by weakening the Poisson homomorphism property:

$$\Delta^{(m)}(\{a, b\}) = \left\{ \Delta^{(m)}(a), \Delta^{(m)}(b) \right\}.$$ 

Here we recall how the generalization works for the standard coalgebra structure (7) associated with a Lie–Poisson algebra.

Let $A$ be a Poisson algebra with generators $\{X^\alpha\}, \alpha = 1, \ldots, l = \dim(A)$, with Poisson brackets given by

$$\{X^\alpha, X^\beta\} = F^{\alpha\beta}(X) \quad \vec{X} = (X^1, \ldots, X^l)$$

and with $r$ Casimirs functions $C_j, j = 1, \ldots, r$. Let us assume that one succeeds in finding a set of $m$ maps depending on a parameter $\lambda$:

$$\Delta^{(k)}_\lambda : A \to A \otimes A \otimes \ldots \otimes A \quad k = 1, \ldots, m$$

(75)
such that for the generators of $A$ the following relations hold:

\[
\{\Delta^{(i)}_\lambda(X^\alpha), \Delta^{(k)}_\mu(X^\beta)\} = f^{ik}(\lambda, \mu) F^{\alpha\beta}(\Delta^{(i)}_\lambda(\vec{X})) \\
\{\Delta^{(k)}_\lambda(X^\alpha), \Delta^{(k)}_\mu(X^\beta)\} = f^{k}(\lambda, \mu) F^{\alpha\beta}(\Delta^{(k)}_\lambda(\vec{X})) + g^{k}(\lambda, \mu) F^{\alpha\beta}(\Delta^{(k)}_\mu(\vec{X}))
\]

(76) \hspace{1cm} (77)

where $f^{ik}, f^k, g^k$ are some functions only depending on the parameters $\lambda$ and $\mu$. If the maps $\Delta^{(k)}_\lambda$ are defined on any smooth function $h$ of the generators of $A$ as:

\[
\Delta^{(k)}_\lambda(h(X^1, \ldots, X^i)) = h(\Delta^{(k)}(X^1), \ldots, \Delta^{(k)}(X^i)),
\]

then it can be proven that the following relations are satisfied [75]:

\[
\{\Delta^{(i)}_\lambda(C_j), \Delta^{(k)}_\mu(X^\beta)\} = 0 \quad k > i \\
\{\Delta^{(i)}_\lambda(C_j), \Delta^{(k)}_\mu(C_n)\} = 0.
\]

(78) \hspace{1cm} (79)

This result can be interpreted as a generalization of the coalgebra approach. If $A$ is a Lie–Poisson algebra with commutation rules

\[
\{X^\alpha, X^\beta\} = C^{\alpha\beta}_\gamma X^\gamma
\]

and equipped with the standard coproduct (7), then

\[
\{\Delta^{(i)}(X^\alpha), \Delta^{(k)}(X^\beta)\} = C^{\alpha\beta}_\gamma \Delta^{(i)}(X^\gamma) \quad k \geq i,
\]

i.e., this is of the form (76) and (77) with

\[
f^{ik} = f^k = 1 \quad g^k = 0 \quad F^{\alpha\beta}(\Delta^{(i)}(\vec{X})) = C^{\alpha\beta}_\gamma \Delta^{(i)}(X^\gamma).
\]

Equations (78) and (79), in turn, generalize equations (5) and (6). Consequently, the maps (75) can be properly called ‘loop coproducts’.

In [75] it is shown that the loop coproducts given by

\[
\Delta^{(k)}_\lambda(X^\alpha) = \Delta^{(k-1)}_\lambda(X^\alpha) \frac{X^\alpha_k}{\lambda - \epsilon} \quad k = 2, \ldots, N
\]

(80)

with $\epsilon$ an arbitrary constant parameter, satisfy equations (76) and (77). From equation (79) it follows that the residues of the functions $\Delta^{(k)}_\lambda(C_j)$ ($k = 2, \ldots, N$, $j = 1, \ldots, r$) are in involution. Furthermore this family of functions in involution contains the standard coproduct of the Casimirs $\Delta^{(k)}(C_j)$.

Finally, it can be shown that if the Lie–Poisson algebra $A$ is a simple one and

\[
\{I_1, \ldots, I_s\} \quad s = \frac{l - r}{2}
\]

(to be compared with (12)) define an integrable system on $A$, then the residues of the functions $\Delta^{(k)}_\lambda(C_j)$ ($k = 2, \ldots, N$, $j = 1, \ldots, r$) together with the functions $\Delta^{(N)}(I_1), \ldots, \Delta^{(N)}(I_s)$ define an integrable system on $A \otimes A \otimes \ldots \otimes A$. This last result solves for this particular case the integrability problem mentioned in section 2.1 for the case of simple Lie–Poisson algebras.
Acknowledgments

This work was partially supported by the Spanish Ministerio de Ciencia e Innovación under grant MTM2007-67389 (with EU-FEDER support), by the Junta de Castilla y León (Project GR224) and by INFN–CICyT.

References

[1] Ballesteros A, Corsetti M and Ragnisco O 1996 *Czech. J.Phys.* **46** 1153
[2] Ballesteros A and Ragnisco O 1998 *J. Phys. A: Math. Gen.* **31** 3791
[3] Ballesteros A and Herranz F J 1999 *J. Phys. A: Math. Gen.* **32** 8851
[4] Grabowski J, Marmo G and Michor P W 1999 *Mod. Phys. Lett. A* **14** 2109
[5] Ballesteros A and Ragnisco O 2002 *J. Math. Phys.* **43** 954
[6] Ballesteros A and Ragnisco O 2003 *J. Phys. A: Math. Gen.* **36** 10505
[7] Ballesteros A, Herranz F J, Musso F and Ragnisco O 2004 in *Superintegrability in Classical and Quantum Systems* (CRM Proc. and Lecture Notes vol. 37) ed P Tempesta et al (Providence, RI: AMS) p 1 arXiv:math-ph/0412067
[8] Ballesteros A and Blasco A 2008 *J. Phys. A: Math. Gen.* **41** 304028
[9] Ballesteros A, Herranz F J and Ragnisco O 2005 *Phys. Lett. B* **610** 107
[10] Ballesteros A, Herranz F J and Ragnisco O 2006 *J. Phys. A: Math. Gen.* **38** 7129
[11] Ragnisco O, Ballesteros A, Herranz F J and Musso F 2006 *SIGMA* **3** 026
[12] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2007 *Phys. Lett. B* **652** 376
[13] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2008 *Physica D* **237** 505
[14] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 *Ann. Phys.* **324** 1219
[15] Ballesteros A, Blasco A and Herranz F J 2009 *J. Phys. A: Math. Gen.* to appear
[16] Ballesteros A and Blasco A 2009 in preparation. arXiv:0903.2337
[17] Musso F and Ragnisco O 2000 *J. Math. Phys.* **41** 7386
[18] Musso F and Ragnisco O 2001 *J. Phys. A: Math. Gen.* **34** 2625
[19] Calogero F and Van Diejen J F 1996 *J. Math. Phys.* **37** 4243
[20] Breiderhoff J, Musso F and Ragnisco O 2005 *J. Math. Phys.* **46** 032109
[21] Musso F, Petrella M, Satta G and Ragnisco O 2005 *Nucl. Phys. B* **716** 543
[22] Arnold V I, Kozlov V V and Neishtadt A I 1997 *Mathematical aspects of classical and celestial mechanics* (Berlin: Springer)
[23] Drinfel’d V G 1987 in *Proc. Int. Congress of Math. (Berkeley 1986)* ed A V Gleason (Providence, RI: American Mathematical Society) p 798
[24] Charī V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[25] Tjin T 1992 *Int. J. Mod. Phys. A* **7** 6175
[26] Ohn C 1992 *Lett. Math. Phys.* **25** 85
[27] Ballesteros A and Herranz F J 2001 *J. Nonlin. Math. Phys.* **8** Suppl. 18
[28] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[29] Brit C 1996 *Ann. Phys.* **251** 180
[30] Ballesteros A, Herranz F J and Parashar P 1997 *J. Phys. A: Math. Gen.* **30** 8587
[31] Burdet G, Patera J, Perrin M and Winternitz P 1978 *J. Math. Phys.* **19** 1758
[32] Wojciechowski S 1983 *Phys. Lett. A* **96** 389
[33] Ballesteros A and Blasco A 2009 in preparation.
[34] Fris J, Mandrosov V, Smoròlnsky Y A, Uhlir M and Winternitz P 1965 *Phys. Lett.* **16** 354
[35] Dorizzi B, Grammaticos B, Ramani A and Winternitz P 1985 *J. Math. Phys.* **26** 3070
[36] McSween E and Winternitz P 1985 *J. Math. Phys.* **41** 2957
[37] Hietarinta J 1985 *J. Math. Phys.* **26** 1970
[38] Donoghovine B, Novikov S and Fomenko A 1982 *Géométrie Contemporaine, Méthodes et Applications First Part* (Moscow: MIR)
[39] Ballesteros A and Herranz F J 2009 *J. Phys. A: Math. Gen.* to appear arXiv:0903.2337
[40] Evans N W 1990 *Phys. Rev. A* **41** 5666
[41] Kalnins E G, Miller W Jr and Pogosyan G S 1997 *J. Math. Phys.* **38** 5416
[42] Rañada M F and Santander M 1999 *J. Math. Phys.* **40** 5026
[43] Kalnins E G, Miller W Jr and Pogosyan G S 2000 *J. Phys. A: Math. Gen.* **33** 6791
[44] Kalnins E G, Kress J M, Pogosyan G S and Miller W Jr 2001 *J. Phys. A: Math. Gen.* **34** 4705
[45] Ballesteros A, Herranz F J, Santander M and Sanz-Gil T 2003 *J. Phys. A: Math. Gen.* **36** L93
[46] Herranz F J, Ballesteros A, Santander M, Sanz-Gil T 2004 *Superintegrability in Classical and Quantum...*
Systems (CRM Proc. and Lecture Notes vol. 37) ed P Tempesta et al (Providence, RI: AMS) p 75

[47] Higgs P W 1979 J. Phys. A: Math. Gen. 12 309
[48] Leemon I 1979 J. Phys. A: Math. Gen. 12 489
[49] Schrödinger E 1940 Proc. R. Ir. Acad. A 46 9
[50] Kalnins E G, Miller W Jr and Pogosyan G S 2000 J. Math. Phys. 41 2629
[51] Nersessian A and Pogosyan G S 2001 Phys. Rev. A 63 020103
[52] Rodríguez M A and Winternitz P 2002 J. Math. Phys. 43 1309
[53] Kalnins E G, Williams G C, Miller W Jr and Pogosyan G S 2002 J. Phys. A: Math. Gen. 35 4755
[54] Herranz F J and Ballesteros A 2006 SIGMA 2 010
[55] Verrier P E and Evans N W 2008 J. Math. Phys. 49 022902
[56] Koenigs G 1972 in Leçons sur la théorie généralé des surfaces vol 4 ed G Darboux (New York: Chelsea) p 368

[57] Kalnins E G, Kress J M and Winternitz P 2002 J. Math. Phys. 43 970
[58] Kalnins E G, Kress J M, Miller W Jr and Winternitz P 2003 J. Math. Phys. 44 5811
[59] Grosche C, Pogosyan G S and Sissakian A N 2007 Phys. Part. Nucl. 38 299
[60] Grosche C, Pogosyan G S and Sissakian A N 2007 Phys. Part. Nucl. 38 525
[61] Iwai T and Katayama N 1994 J. Phys. A: Math. Gen. 27 3179
[62] Iwai T and Katayama N 1995 J. Math. Phys. 36 1790
[63] Ballesteros A, Enciso A, Herranz F J and Ragnisco O 2009 Comm. Math. Phys. doi: 10.1007/s00220-009-0793-5 arXiv:0810.0999
[64] Ballesteros A and Herranz F J 2008 arXiv: 0812.4124
[65] Calogero F 1995 Phys. Lett. A 201 306
[66] Gaudin M 1983 La Fonction d’Onde de Bethe (Paris: Masson)
[67] Calogero F 1971 J. Math. Phys. 12 419
[68] Ballesteros A 1999 Int. J. Mod. Phys. B 13 2903
[69] Calogero F 1995 J. Math. Phys. 36 9
[70] Calogero F and Van Diejen J F 1995 Phys. Lett. A 205 143
[71] Karimipour V 1998 J. Math. Phys. 39 913
[72] Ruijsenaars S N M and Schneider H 1986 Ann. Phys. 170 370
[73] Vaksman L and Korogodskii L I 1989 Sov. Math. Dokl. 39 173
[74] Ballesteros A, Musso F and Ragnisco O 2002 J. Phys. A: Math. Gen. 35 8197
[75] Musso F, Ballesteros A, and Ragnisco O in preparation