FRACTIONAL ORLICZ-SOBOLEV EXTENSION/IMBEDDING ON AHLFORS $n$-REGULAR DOMAINS

TIAN LIANG

Abstract In this paper we build up a criteria for fractional Orlicz-Sobolev extension and imbedding domains on Ahlfors $n$-regular domains.

1. Introduction

The study of extension/imbeddings of function spaces (including Sobolev, BMO, Besov and Triebel-Lizorkin spaces) and their applications in harmonic analysis, potential theory and partial differential equations have attracted a lot attentions; see for example [11, 9, 10, 12, 23, 3, 13, 19, 24, 25, 7, 20, 21, 22, 30].

In this paper, we are interested in the fractional Orlicz-Sobolev extension. Let $\phi$ be a Young function, that is, $\phi \in C((0, \infty))$ is convex, $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. For any $\beta > 0$ and domain $\Omega \subset \mathbb{R}^n$, define the fractional Orlicz-Sobolev spaces $\dot{W}^{\beta, \phi}(\Omega)$ as the space of all $u \in L^1_{\text{loc}}(\Omega)$ whose (semi-)norm

$$\|u\|_{\dot{W}^{\beta, \phi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \phi\left(\frac{|u(x) - u(y)|}{\lambda}\right) \frac{dxdy}{|x-y|^{n+\beta}} \leq 1 \right\}$$

is finite. Modulo constant functions, $\dot{W}^{\beta, \phi}(\Omega)$ is a Banach space. If $\phi(t) = t^p$ with $p \geq 1$, then $\dot{W}^{\beta, \phi}(\Omega) = \dot{W}^{\beta/p, p}(\Omega)$. Here $\dot{W}^{s, p}(\Omega)$ with $s > 0$ and $p \geq 1$ is the fractional Sobolev space, that is, the collection of all $u \in L^1_{\text{loc}}(\Omega)$ with

$$\|u\|_{\dot{W}^{s, p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{1/p} < \infty.$$ 

To guarantee the nontriviality of $\dot{W}^{\beta, \phi}(\Omega)$, we always assume

$$\tag{1.1} C_\beta := \sup_{t>0} \frac{t^\beta}{\phi(t)} \int_0^1 \phi(s) \frac{ds}{s^\beta} < \infty.$$ 

Indeed, (1.1) implies that $C_\beta^\infty(\Omega) \subset \dot{W}^{\beta, \phi}(\Omega)$; see Lemma 2.2. Moreover, (1.1) is optimal to guarantee the nontriviality of $\dot{W}^{\beta, \phi}(\Omega)$ in the sense that if $\phi(t) = t^p$ with $p \geq 1$, then $\dot{W}^{\beta, \phi}(\Omega) = \dot{W}^{\beta/p, p}(\Omega)$ is nontrivial if and only if $p > \beta$ (see [5]), and if and only if $C_\beta < \infty$. Besides of $\phi(t) = t^p$ with $p \geq 1$ and $p > \beta$, we refer to Remark 2.1 for more Young functions satisfying (1.1), in particular, including $\phi(t) = t^p [\ln(1+t)]^\alpha$ with $p > \beta$, $p \geq 1$ and $\alpha \geq 1$. We remark that under (1.1), $\dot{W}^{\beta, \phi}$ has fractional smoothness strictly less than 1.

The main purpose of this paper is to build up the following criteria for fractional Orlicz-Sobolev $\dot{W}^{\beta, \phi}$-extension and -imbedding domains when $\beta \in (0, n) \cup (n, \infty)$, which generalize the corresponding results for fractional Sobolev spaces (see [11, 12, 20, 21, 30]). We also note that the case $\beta = n$ has already been considered in [26].
Theorem 1.1. Let $\beta \in (0, n) \cup (n, \infty)$ and $\phi$ be a Young function satisfying (1.1). Suppose that $\phi$ is doubling, that is, there exists a constant $K > 1$ such that $\phi(2t) \leq K\phi(t)$ for all $t > 0$. For any domain $\Omega \subset \mathbb{R}^n$, the following are equivalent:

(i) $\Omega$ is Ahlfors $n$-regular, that is, there exists a constant $\theta > 0$ such that $|B(x, r) \cap \Omega| \geq \theta r^n \quad \forall x \in \Omega, 0 < r < 2 \text{diam} \Omega$.

(ii) $\Omega$ is a $W^{\beta, \phi}$-extension domain, that is, any function $u \in W^{\beta, \phi}(\mathbb{R}^n)$ can be extended to a function $\tilde{u} \in W^{\beta, \phi}(\Omega)$ in a continuous and linear way.

(iii) $\Omega$ is a $W^{\beta, \phi}$-imbedding domain, that is,

(a) when $0 < \beta < n$, there exists a constant $C = C(\beta, n, \phi) > 0$ such that

\begin{equation}
\inf_{c \in \mathbb{R}} \|u - c\|_{L^{\phi/(n-\beta)}(\Omega)} \leq C\|u\|_{W^{\beta, \phi}(\Omega)} \quad \forall u \in \dot{W}^{\beta, \phi}(\Omega);
\end{equation}

(b) when $\beta > n$, there exists a constant $C = C(\beta, n, \phi) > 0$ such that for each $u \in \dot{W}^{\beta, \phi}(\Omega)$, we can find a function $\tilde{u}$ with $\tilde{u} = u$ almost everywhere and

\begin{equation}
|\tilde{u}(x) - \tilde{u}(y)| \leq C\phi^{-1}(|x - y|^\beta)\|u\|_{W^{\beta, \phi}(\Omega)} \quad \forall x, y \in \Omega.
\end{equation}

Above we denote by $L^{\phi}(\Omega)$ the Orlicz space, that is, the collection of all $u \in L^1_{\text{loc}}(\Omega)$ whose norm

\[\|u\|_{L^{\phi}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \phi \left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\} < \infty.\]

This paper is organized as follows. In Section 2, we recall some properties of Young functions and show the nontriviality of $W^{\beta, \phi}$ under (1.1). The proofs of (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii) and (iii)$\Rightarrow$(i) of Theorem 1.1 are given in Section 4, 3, 5 separately.

Note that (i)$\Rightarrow$(ii) follows from the following Theorem 1.2, where the doubling condition on $\phi$ is not needed. Theorem 1.2 will be proved by an argument similar to the case $\beta = n$ as given in [26], but, due to some technical differences caused by $\beta \neq n$, we give the details in Section 4 for reader’s convenience.

Theorem 1.2. Let $\beta \in (0, n) \cup (n, \infty)$ and $\phi$ be a Young function satisfying (1.1). If $\Omega \subset \mathbb{R}^n$ is an Ahlfors $n$-regular domain, then $\Omega$ is a $W^{\beta, \phi}$-extension domain.

The proof of (ii)$\Rightarrow$(iii) is given in Section 3. When $\beta > n$, we use a $(1, \phi)_B$-Poincaré inequality proved in Lemma 3.1; see Section 3.1. When $\beta \in (0, n)$, the proof relies on the following $(\phi^{n/(n-\beta)}, \phi)_B$-Poincaré inequality.

Theorem 1.3. Let $\beta \in (0, n)$ and $\phi$ be a Young function satisfying (1.1). Suppose that $\phi$ is doubling. Then there exists a constant $C = C(\beta, n, \phi) > 0$ such that

\[\inf_{c \in \mathbb{R}} \|u - c\|_{L^{\phi/(n-\beta)}(B)} \leq C\|u\|_{W^{\beta, \phi}(B)} \quad \forall u \in \dot{W}^{\beta, \phi}(B)\]

whenever $B$ is a ball of $\mathbb{R}^n$ or $B = \mathbb{R}^n$.

The $(\phi^{n/(n-\beta)}, \phi)_B$-Poincaré inequality is a self-improvement of the $(1, \phi)_B$-Poincaré inequality (see Lemma 3.1). One may wish to prove Theorem 1.3 via some known self-improvement approach from harmonic analysis. But since here Orlicz norm and fractional derivative are involved, the proof would be very complicated. Instead, as motivated by [17], by building up a local version of the known geometric inequality

\[\int_{\mathbb{R}^n \setminus E} \frac{dy}{|x - y|^{n+\beta}} \geq C(n, \beta)|E|^{-\beta/n} \quad \text{whenever } |E| < \infty,
\]
using some ideas from [17] and also the median value, we give a direct proof of Theorem 1.3. See Section 3.2 for details.

To prove (iii) $\Rightarrow$ (i) of Theorem 1.1, the imbedding assumption allows us to calculate the $\|u\|_{W^{1,\phi}(\Omega)}$ norm of some test functions. Using these and the doubling property of $\phi$, and following the idea from [7] (see also [8, 30]), we conclude that $\Omega$ is Ahlfors $n$-regular.

Notation used in the following is standard. Denote by $C$ some constant which may vary from line to line but is independent of the main parameters. The constant $C(X, Y, \ldots)$ depends only on the parameters $X, Y, \ldots$; while the constant with subscripts would not change in different occurrences, like $C_1$. The symbol $A \lesssim B$ means that $A \leq CB$. For any locally integrable function $u$ and measurable set $X$ with $|X| > 0$, we denote by $u_X$ the average of $u$ on $X$, namely $u_X = \frac{1}{|X|} \int_X u \, dx$. We use $d(x, E)$ to describe the Euclidean distance from $x$ to a set $E$.

2. Preliminaries

The following properties of Young functions are well-known, but for the convenience of the reader, we give the proof.

**Lemma 2.1.** Let $\phi$ be a Young function.

(i) Then $\phi$ is continuous, strictly increasing and $\lim_{t \to \infty} \phi(t) = \infty$.

(ii) The inverse $\phi^{-1}$ of $\phi$ is continuous, strictly increasing, $\phi^{-1}(0) = 0$ and $\lim_{t \to \infty} \phi^{-1}(t) = \infty$. Moreover, $\phi^{-1}$ is concave; in particular, $\phi^{-1}(2x) \leq 2\phi^{-1}(x)$ for all $x > 0$.

(iii) If $\phi$ is doubling with some constants $K > 1$, then $\phi^{-1}(tx) \leq t^{1/(K-1)} \phi^{-1}(x)$ for all $x > 0$ and $t \in [0, 1]$.

**Proof.** (i) Since $\phi$ is convex, we have $\phi(t) \geq t \phi(1)$ for all $t \geq 1$ and hence $\lim_{t \to \infty} \phi(t) = \infty$. Also note that $\phi'(t) \geq 0$ for almost all $t \in [0, \infty)$ and $\phi'$ is increasing. If $\phi$ is not strictly increasing, we must have $\phi(t) = \phi(t+s)$ for some $t \geq 0$ and $s > 0$. Thus $\phi' = 0$ almost everywhere in $[t, t+s]$ and hence in $[0, t+s]$. This implies that $\phi = 0$ in $[0, t+s]$, which contradicts with $\phi > 0$ in $(0, \infty)$.

(ii) From (i) we see easily that the inverse $\phi^{-1}$ is continuous, strictly increasing, $\phi^{-1}(0) = 0$ and $\lim_{t \to \infty} \phi^{-1}(t) = \infty$. Moreover, for any $x, y \geq 0$, from the convexity of $\phi$ it follows that for any $\lambda \in [0, 1]$, $\phi^{-1}(\lambda x + (1-\lambda)y) = \phi^{-1}(\lambda \phi^{-1}(x) + (1-\lambda)\phi^{-1}(y)) \geq \phi^{-1}(\lambda \phi^{-1}(x) + (1-\lambda)\phi^{-1}(y)) = \lambda \phi^{-1}(x) + (1-\lambda)\phi^{-1}(y)$.

Thus $\phi^{-1}$ is concave. Furthermore, thanks to $\phi^{-1}(0) = 0$ and the concavity, we get $\phi^{-1}(2x) \leq 2\phi^{-1}(x)$ for all $x > 0$.

(iii) Since $\phi'$ is increasing, we have $\phi(2t) - \phi(t) = \int_t^{2t} \phi'(s) \, ds \geq \phi'(t)t \quad \forall t > 0$.

By the doubling property of $\phi$, we have $\phi(2t) - \phi(t) \leq (K-1)\phi(t)$ and hence, $(K-1)\phi(t) \geq \phi'(t)t$, that is, $(\ln \phi)'(t) = \frac{\phi'(t)}{\phi(t)} \leq \frac{K-1}{t}$ for almost all $t > 0$.

Thus for $t \in (0, 1]$ we have $\ln \left( \frac{\phi(x)}{\phi(tx)} \right) = \int_{tx}^{x} (\ln \phi)'(s) \, ds \leq \int_{tx}^{x} \frac{K-1}{s} \, ds = \ln \left( \frac{1}{t^{K-1}} \right)$,
which gives that
\[ t^{K-1} \phi(x) \leq \phi(tx). \]
For any \( t > 0 \), \( \phi^{-1}(\phi(t)) = t \) implies \((\phi^{-1})'(\phi(t))\phi'(t) = 1\) for almost all \( t > 0 \), that is,
\[ (\phi^{-1})'(\phi(t)) = \frac{1}{\phi'(t)} \geq \frac{1}{t} \frac{1}{\phi(t) K - 1}. \]
We then have
\[ (\phi^{-1})'(s) \geq \frac{1}{s K - 1}, \]
that is, \((\ln \phi^{-1})'(s) \geq \frac{1}{s(K - 1)}\) for almost all \( s > 0 \).
Thus for \( t \in (0, 1] \) we have
\[
\ln \left( \frac{\phi^{-1}(x)}{\phi^{-1}(tx)} \right) = \int_{tx}^{x} (\ln \phi^{-1})'(s) ds \geq \int_{tx}^{x} \frac{1}{s(K - 1)} ds = \ln \left( \frac{1}{t^{1/(K-1)}} \right),
\]
which gives that
\[ t^{1/(K-1)} \phi^{-1}(x) \geq \phi^{-1}(tx) \]
as desired. \( \Box \)

**Lemma 2.2.** Let \( \beta > 0 \) and \( \phi \) be a Young function satisfying (1.1). For any domain \( \Omega \subset \mathbb{R}^n \), \( C^1_c(\Omega) \subset W^{\beta, \phi}(\Omega) \).

**Proof.** Given any \( u \in C^1_c(\Omega) \), assume \( L = ||u||_{L^\infty(\Omega)} + ||Du||_{L^\infty(\Omega)} > 0 \) and choose a domain \( W \subset \Omega \) such that \( V = \text{supp} u \in W \subset \Omega \). Then
\[
H := \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(z) - u(w)|}{\lambda} \right) \frac{dzdw}{|z - w|^\beta} \leq \int_{W} \int_{W} \phi \left( \frac{|z - w|}{\lambda L} \right) \frac{dzdw}{|z - w|^\beta} + 2 \int_{\Omega} \int_{V} \phi \left( \frac{L}{\lambda} \right) \frac{dzwdw}{|z - w|^\beta}.
\]
By (1.1), we have
\[
\int_{W} \int_{W} \phi \left( \frac{|z - w|}{\lambda L} \right) \frac{dzdw}{|z - w|^\beta} \leq \int_{V} \int_{B(\omega;2|\text{diam} W|)} \phi \left( \frac{|z - w|}{\lambda L} \right) \frac{dz}{|z - w|^\beta} dw = n\omega_n \int_{W} \int_{0}^{2|\text{diam} W|} \phi \left( \frac{t}{\lambda L} \right) \frac{dt}{t^\beta} dw
\]
\[
= n\omega_n |W| \left( \frac{L}{\lambda} \right)^\beta \int_{0}^{2|\text{diam} W|/\lambda} \phi(s) \frac{ds}{s^{\beta+1}}
\]
\[
= n\omega_n C_\beta |W| 2^{-\beta} |\text{diam} W|^{-\beta} \phi \left( \frac{2L |\text{diam} W|}{\lambda} \right).
\]
Moreover,
\[
2 \int_{\Omega} \int_{V} \phi \left( \frac{1}{\lambda L} \right) \frac{dzdw}{|z - w|^\beta} \leq 2\phi \left( \frac{L}{\lambda} \right) \int_{V} \int_{\Omega \setminus (V, W^C)} \frac{dwdz}{|z - w|^\beta} \leq 2\omega_n \phi \left( \frac{L}{\lambda} \right) |V| \text{dist} (V, W^C)^{-\beta}.
\]
Letting \( \lambda \) large enough and using the convexity of \( \phi \), we have \( H \leq 1 \). That is, \( u \in W^{\beta, \phi}(\Omega) \) as desired. \( \Box \)

**Remark 2.1.** Let \( \phi(t) = t^p \psi(t) \) be a Young function with \( p \geq 1 \). If \( p > \beta \) and there exists a constant \( C > 0 \) such that \( \psi(s) \leq C \psi(t) \) for all \( s \leq t \), then \( \phi \) satisfies the condition (1.1), that is, \( C_\beta < \infty \). Indeed, for any \( t > 0 \),
\[
\frac{t^\beta}{t^p \psi(t)} \int_{0}^{t} \frac{t^p \psi(s)}{s^\beta} \frac{ds}{s} \leq Ct^{\beta - p} \int_{0}^{t} s^{p-\beta-1} ds \leq \frac{C}{p - \beta}.
\]
Below are some typical examples of Young functions $\phi$ satisfying (1.1).

(i) $\phi(t) = t^p[\ln(1 + t)]^q$ with $p > \beta$, $p \geq 1$ and $\alpha \geq 1$.

(ii) $\phi(t) = \max \{t^p, t^{p+\beta}\}$ where $p \geq 1$, $p > \beta$ and $\delta > 0$.

(iii) $\phi(t) = t^p e^{ct^\delta}$ with $p > \beta$, $p \geq 1$, $c > 0$ and $\alpha > 0$.

(iv) $\phi(t) = e^{ct^p} - \sum_{j=0}^{\lfloor n/\alpha \rfloor} (ct^p)^j / j!$ where $p \geq 1$, $c > 0$ with $\alpha > 0$, where $\lfloor n/\alpha \rfloor$ is the integer less than or equal to $n/\alpha$.

Note that the Young functions given in (i) and (ii) further satisfy the doubling property, but the Young functions given in (iii) and (iv) do not.

3. Proofs of Theorem 1.3 and (ii)$\Rightarrow$(iii) of Theorem 1.1

In Section 3.1 we prove (ii)$\Rightarrow$(iii) of Theorem 1.1 when $\beta \in (n, \infty)$. In Section 3.2 we prove Lemma 3.1 and then (ii)$\Rightarrow$(iii) of Theorem 1.1 when $\beta \in (0, n)$. Below, we always denote by $\omega_n$ the $(n - 1)$-dimensional Lebesgue measure of the unit sphere $S^{n-1}$.

3.1. Case $\beta \in (n, \infty)$. First, we have the following $(1, \phi)_\beta$-Poincaré inequality.

**Lemma 3.1.** Let $\beta > 0$ and $\phi$ be a Young function satisfying (1.1). For any ball $B = B(z, r) \subset \mathbb{R}^n$ and $u \in \dot{W}^{\beta, \phi}(B)$, we have

$$
\int_B |u(x) - u_B| \, dx \leq \phi^{-1}(\frac{2^{\beta+n}r^{\beta-n}}{\omega_n^2}) \|u\|_{\dot{W}^{\beta, \phi}(B)}.
$$

**Proof:** Let $u \in \dot{W}^{\beta, \phi}(B)$. For any ball $B \subset \Omega$ and $\lambda > \|u\|_{\dot{W}^{\beta, \phi}(\Omega)}$, applying Jensen’s inequality, we know

$$
\phi\left(\frac{\int_B |u(x) - u_B| \, dx}{\lambda}\right) \leq \int_B \phi\left(\frac{|u(x) - u(y)|}{\lambda}\right) \, dy \, dx
$$

$$
\leq 2^{\beta+n} \omega_n^2 r^{\beta-n} \int_B \int_B \phi\left(\frac{|u(x) - u(y)|}{\lambda}\right) \, dy \, dx \frac{1}{|x - y|^{n+\beta}}
$$

$$
\leq 2^{\beta+n} \omega_n^2 r^{\beta-n},
$$

that is,

$$
\int_B |u(x) - u_B| \, dx \leq \lambda \phi^{-1}(2^{\beta+n} \omega_n^2 r^{\beta-n}).
$$

Letting $\lambda \to \|u\|_{\dot{W}^{\beta, \phi}(B)}$, we obtain

$$
\int_B |u(x) - u_B| \, dx \leq \phi^{-1}(2^{\beta+n} \omega_n^2 r^{\beta-n}) \|u\|_{\dot{W}^{\beta, \phi}(B)}
$$

as desired. $\square$

Applying Lemma 3.1 and Lemma 2.1, we obtain the following imbedding.

**Lemma 3.2.** Let $\beta > n$ and $\phi$ be a Young function satisfying (1.1) and the doubling property with constant $K$. There exists a positive constant $C(\beta, n)$ depending only on $n$ and $\beta$ such that a for all $u \in \dot{W}^{\beta, \phi}(\mathbb{R}^n)$, we can find a continuous function $\hat{u} \in \dot{W}^{\beta, \phi}(\mathbb{R}^n)$ such that $\hat{u} = u$ a. e. and

$$
|\hat{u}(x) - \hat{u}(y)| \leq C(\beta, n)\phi^{-1}(|x - y|^{\beta-n}) \|u\|_{\dot{W}^{\beta, \phi}(\mathbb{R}^n)} \quad \forall x, y \in \mathbb{R}^n.
$$

Proof. Let $u \in \dot{W}^{\beta,\phi}(\mathbb{R}^n)$. We first show that

\begin{equation}
|u(x) - u(y)| \leq C(\beta, n)\phi^{-1}\left(|x - y|^{\beta-n}\right)\|u\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)}
\end{equation}

for all Lebesgue points $x, y$ of $u$. Write

\[ |u(x) - u(y)| \leq |u(x) - u_{B(x, 2|x-y|)}| + |u_{B(x, 2|x-y|)} - u(y)|. \]

By Lemma 3.1,

\[ |u(x) - u_{B(x, 2|x-y|)}| \leq \sum_{i=0}^{\infty} |u_{B(x, 2^{-i+1}|x-y|)} - u_{B(x, 2^{-i}|x-y|)}| \]

\[ \leq 2^n \sum_{i=0}^{\infty} \int_{B(x, 2^{-i+1}|x-y|)} |u(z) - u_{B(x, 2^{-i}|x-y|)}| \, dz \]

\[ \leq 2^n \sum_{i=0}^{\infty} \phi^{-1}\left(2^{\beta+n}\omega_n^2 \left(2^{-i}|x-y|^{\beta-n}\right)\right)\|u\|_{\dot{W}^{\beta,\phi}(B(x, 2^{-i+1}|x-y|))} \]

\[ \leq 2^n \sum_{i=0}^{\infty} \phi^{-1}\left(2^{\beta+n}\omega_n^2 \left(2^{-i}|x-y|^{\beta-n}\right)\right)\|u\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)}. \]

Thanks to Lemma 2.1(iii), we have

\[ \phi^{-1}\left(2^{\beta+n}\omega_n^2 \left(2^{-i}|x-y|^{\beta-n}\right)\right) \leq 2^{-i(\beta-n)/(K-1)}\phi^{-1}\left(2^{\beta+n}\omega_n^2 |x-y|^{\beta-n}\right) \leq 2^{-i(\beta-n)/(K-1)}2^{\beta+n}\omega_n^2 \phi^{-1}\left(|x-y|^{\beta-n}\right). \]

Thus

\[ |u(x) - u_{B(x, 2|x-y|)}| \leq \sum_{i=0}^{\infty} 2^{-i(\beta-n)K}2^{\beta+n}\omega_n^2 \phi^{-1}\left(|x-y|^{\beta-n}\right)\|u\|_{\dot{W}^{\beta,\phi}(B)} \leq C(\beta, n)\phi^{-1}\left(|x-y|^{\beta-n}\right)\|u\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)}. \]

Similarly,

\[ |u(y) - u_{B(x, 2|x-y|)}| \leq C(\beta, n)\phi^{-1}\left(|x-y|^{\beta-n}\right)\|u\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)}. \]

Next, let $\hat{u}(x) := \lim_{r \to 0} u_{B(x,r)}$ for all $x \in \mathbb{R}^n$. Note that $\hat{u}$ is well-defined. Indeed, for any $0 < r < s$, by (3.2), we have

\[ |u_{B(x,r)} - u_{B(x,s)}| \leq \int_{B(x,s)} \int_{B(x,r)} |u(z) - u(w)| \, dz \, dw \leq C(\beta, n)\phi^{-1}\left((r + s)^{\beta-n}\right)\|u\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)}, \]

which together with the continuity of $\phi^{-1}$ and $\phi^{-1}(0) = 0$ implies the existence of $\hat{u}(x)$. Due to the Lebesgue differentiation theorem, we know $\hat{u} = u$ almost everywhere, and hence, $\hat{u} \in \dot{W}^{\beta,\phi}(\mathbb{R}^n)$. Moreover, $\hat{u}$ is continuous; indeed, by (3.2)

\[ |\hat{u}(x) - \hat{u}(y)| = \lim_{r \to 0} |u_{B(x,r)} - u_{B(y,r)}| \leq C(\beta, n)\phi^{-1}\left((r + s)^{\beta-n}\right)\|u\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)}. \]

This completes the proof of Lemma 3.2. \qed

Proof of (ii) $\Rightarrow$ (iii) Theorem 1.1: case $\beta \in (n, \infty)$. Since $\Omega$ is a $\dot{W}^{\beta,\phi}$-extension domain, for every $u \in \dot{W}^{\beta,\phi}(\Omega)$, we can find a $\tilde{u} \in \dot{W}^{\beta,\phi}(\mathbb{R}^n)$ such that $\tilde{u} = u$ in $\Omega$ and $\|\tilde{u}\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)} \leq C\|u\|_{\dot{W}^{\beta,\phi}(\Omega)}$. 
Using Lemma 3.2, there exists continuous function $\hat{u} \in \dot{W}^{\beta,\phi}(\mathbb{R}^n)$ such that $\hat{u} = \check{u}$ almost everywhere in $\mathbb{R}^n$ and for all $x, y \in \mathbb{R}^n$,

$$|\hat{u}(x) - \hat{u}(y)| \leq C(\beta, n)\phi^{-1}(|x - y|^{\beta - n})\|\hat{u}\|_{\dot{W}^{\beta,\phi}(\mathbb{R}^n)} \leq \phi^{-1}(|x - y|^{\beta - n})\|u\|_{\dot{W}^{\beta,\phi}(\Omega)}.$$

Therefore we have $\hat{u} = u$ almost everywhere in $\Omega$ and (1.3) holds. □

3.2. Case $\beta \in (0, n)$. In order to prove Theorem 1.3, we need the following geometry inequality.

**Lemma 3.3.** Let $\beta \in (0, n)$. Then there exists a constant $C(n, \beta)$ depending only on $n$ and $\beta$ such that for any ball $B \subset \mathbb{R}^n$ and $x \in B$, we have

$$\int_{B \setminus E} \frac{dy}{|x - y|^{n + \beta}} \geq C(n, \beta)|E|^{-\beta/n} \text{ whenever } E \subset B \text{ and } 0 < |E| < \frac{1}{2}|B|.$$

**Proof.** Let $\rho \in (0, 2 \text{ diam } B)$ such that $|E| = |B \cap (B(x, \rho))| \leq \omega_n \rho^n$. Then

$$|(B \setminus E) \cap B(x, \rho)| = |B \cap (B(x, \rho))| - |E \cap B(x, \rho)| = |E| - |E \cap B(x, \rho)| = |E| - |E \cap B(x, \rho)| = |E \cap B^c(x, \rho)|.$$

Hence

$$\int_{B \setminus E} \frac{dy}{|x - y|^{n + \beta}} = \int_{(B \setminus E) \cap B(x, \rho)} \frac{dy}{|x - y|^{n + \beta}} + \int_{((B \setminus E) \cap B^c(x, \rho))} \frac{dy}{|x - y|^{n + \beta}}$$

$$\geq \int_{(B \setminus E) \cap B(x, \rho)} \frac{dy}{\rho^{n + \beta}} + \int_{((B \setminus E) \cap B^c(x, \rho))} \frac{dy}{|x - y|^{n + \beta}}$$

$$= \frac{|(B \setminus E) \cap B(x, \rho)|}{\rho^{n + \beta}} + \int_{((B \setminus E) \cap B^c(x, \rho))} \frac{dy}{|x - y|^{n + \beta}}$$

$$\geq \frac{|E \cap B^c(x, \rho)|}{\rho^{n + \beta}} + \int_{((B \setminus E) \cap B^c(x, \rho))} \frac{dy}{|x - y|^{n + \beta}}$$

$$\geq \int_{B^c(x, \rho) \setminus B} \frac{dy}{|x - y|^{n + \beta}}.$$

If $B \cap (B(x, 3\rho) \setminus B(x, 2\rho)) = 0$, then $B \cap B(x, 2\rho) = B$. Thus

$$|B^c(x, \rho) \cap B| = |B \cap (B(x, 3\rho) \setminus B(x, 2\rho))| = |B| - |B \cap B(x, \rho)| > |E|.$$ 

Moreover, there exists $\theta > 0$ independent of $B$ such that $|E| = |B \cap B(x, \rho)| \geq \theta \omega_n \rho^n$. So we have

$$\int_{B^c(x, \rho) \setminus B} \frac{dy}{|x - y|^{n + \beta}} = \int_{B^c(x, 2\rho) \setminus B(x, \rho)} \frac{dy}{|x - y|^{n + \beta}} \geq \frac{|E|^{-\beta/n}}{\omega_n \rho^n (\omega_n \rho)^{1 + \beta/2}}.$$

If $B \cap (B(x, 3\rho) \setminus B(x, 2\rho)) \neq 0$, then there exists a point $z \in B \cap (B(x, 3\rho) \setminus B(x, 2\rho))$ such that

$$B \cap (B(z, \rho) \subset B \cap (B(x, 4\rho) \setminus B(x, \rho)) \subset B \setminus B(x, \rho)$$

and

$$|B \cap (B(x, 4\rho) \setminus B(x, \rho))| \geq |B(z, \rho) \cap B| \geq \frac{\theta}{2} \omega_n \rho^n.$$

Therefore,

$$\int_{B^c(x, \rho) \setminus B} \frac{dy}{|x - y|^{n + \beta}} \geq \int_{B^c(x, 4\rho) \setminus B(x, \rho)} \frac{dy}{|x - y|^{n + \beta}} \geq \frac{|E|^{-\beta/n}}{\omega_n \rho^n (\omega_n \rho)^{1 + \beta/2}}.$$

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This completes the proof of Lemma 3.3. □

**Lemma 3.4.** Let $\beta \in (0, n)$ and $\phi$ be a Young function satisfying (1.1) and the doubling property with a constant $K$. There exists a positive constant $C(n, \beta, K)$ depending only on $n, \beta$ and $K$ such that for any ball $B$ and $u \in W^{\phi, \beta}(B)$, we have

$$||u - u_B||_{L^\phi(n-\beta)(B)} \leq C(n, \beta, K)||u||_{W^{\phi, \beta}(B)}.$$  

**Proof.** Without loss of generality, we may assume $u \in L^\infty(B)$. Indeed, let

$$u_N := \max \{\min\{u(x), N\}, -N\} \quad \forall x \in B.$$  

By Lebesgue’s convergence theorem, we have $\lim_{N \to \infty} ||u_N||_{L^\phi(n-\beta)(B)} = ||u||_{L^\phi(n-\beta)(B)}$ and $\lim_{N \to \infty} ||u_N||_{W^{\phi, \beta}(B)} = ||u||_{W^{\phi, \beta}(B)}$. If $||u_N - (u_N)_B||_{L^\phi(n-\beta)(B)} \leq C||u_N||_{W^{\phi, \beta}(B)}$ holds for all $N$, sending $N \to \infty$ and noting $(u_N)_B \to u_B$, we have desired result.

Set the median value

$$m_u(B) := \inf \left\{c \in \mathbb{R} : ||x \in B : u(x) - c > 0|| \leq \frac{1}{2}|B| \right\}.$$  

Then

$$||x \in B : u(x) - m_u(B) > 0|| \leq \frac{1}{2}|B| \quad \text{and} \quad ||x \in B : u(x) - m_u(B) < 0|| \leq \frac{1}{2}|B|.$$  

Write $u_+ = [u - m_u(B)]_{X_{u \geq m_u(B)}}$ and $u_- = [-u - m_u(B)]_{X_{u \leq m_u(B)}}$. We know $u - m_u(B) = u_+ - u_-$. Note that

$$||u - u_B||_{L^\phi(n-\beta)(B)} \leq 2||u - m_u(B)||_{L^\phi(n-\beta)(B)},$$  

and

$$||u - m_u(B)||_{L^\phi(n-\beta)(B)} \leq C(||u_+||_{L^\phi(n-\beta)(B)} + ||u_-||_{L^\phi(n-\beta)(B)}).$$  

Since

$$||u(x) - u(y)|| = ||u(x) - m_u(B)|| - ||u(y) - m_u(B)|| = ||u_+(x) - u_+(y)|| + ||u_-(x) - u_-(y)|| \quad \forall x, y \in B,$$

to get $||u - u_B||_{L^\phi(n-\beta)(B)} \leq C||u||_{W^{\phi, \beta}(B)}$, it suffices to show $||u_+||_{L^\phi(n-\beta)(B)} \leq C||u_+||_{W^{\phi, \beta}(B)}$. Below we only prove this for $u_+$; the proof of $u_-$ is similar. It suffices to find a constant $C(n, \beta, K) \geq 1$ such that for any $\lambda > 4C(n, \beta, K)||u||_{W^{\phi, \beta}(\mathbb{R}^n)}$,

$$\int_B \phi^{\phi(n-\beta)} \left( \frac{|u_+(x)|}{\lambda} \right) dx \leq 1.$$  

To see (3.4), for $k \in \mathbb{Z}$, define

$$A_k := \{x \in B : u(x)_+ > 2^k\} \quad \text{and} \quad D_k := A_k \setminus A_{k+1} = \{x \in B : 2^k < u_+(x) \leq 2^{k+1}\}.$$  

Write $a_k := |A_k|$ and $d_k := |D_k|$. For any $\lambda > 0$, we have

$$T := \int_B \int_B \phi \left( \frac{|u_+(x) - u_+(y)|}{\lambda} \right) \frac{dxdy}{|x - y|^{n+\beta}} \geq 2 \sum_{i \in \mathbb{Z}} \sum_{j \leq i-2} \int_{D_i} \int_{D_j} \phi \left( \frac{|u_+(x) - u_+(y)|}{\lambda} \right) \frac{dxdy}{|x - y|^{n+\beta}}.$$  

Note that

$$|u_+(x) - u_+(y)| \geq 2^j - 2^{j+1} \geq 2^i - 2^{i-1} = 2^{i-1}$$

whenever $x \in D_i$ and $j \in \mathbb{Z}$ with $j \leq i - 2$. So by Lemma 3.3, we have

$$T \geq 2 \sum_{i \in \mathbb{Z}} \sum_{j \leq i-2} \int_{D_i} \int_{D_j} \phi \left( \frac{2^{i-1}}{\lambda} \right) \frac{dy}{|x - y|^{n+\beta}} dx$$
Next we show that

\[
2S \geq \sum_{i \in \mathbb{Z}, a_i \neq 0} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_i.
\]

Since \( d_i = a_i - \sum_{l \geq i+1} d_l \), one has

\[
S = \sum_{i \in \mathbb{Z}, a_i \neq 0} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_i - \sum_{i \in \mathbb{Z}, a_i \neq 0} \sum_{l \geq i+1} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_l.
\]

For \( l \geq i+1 \), we have \( a_{l-1} \leq a_{i-1} \), in particular, \( a_{i-1} \neq 0 \) implies \( a_{i-1} \neq 0 \). Thus by the convexity of \( \phi \),

\[
\sum_{i \in \mathbb{Z}, a_i \neq 0} \sum_{l \geq i+1} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_l \leq \sum_{i \in \mathbb{Z}, a_i \neq 0} \sum_{l \geq i-1} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_i,
\]

\[
\leq \sum_{i \in \mathbb{Z}, a_i \neq 0} \sum_{l \geq i-1} 2^{i-l} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_i,
\]

\[
\leq \sum_{i \in \mathbb{Z}, a_i \neq 0} \sum_{l \geq i-1} 2^{i-l} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_i
\]

\[
\leq \sum_{i \in \mathbb{Z}, a_i \neq 0} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} d_i = S,
\]

from which (3.6) follows.

By (3.6), one has

\[
T \geq C(n, \beta) \sum_{i \in \mathbb{Z}, a_i \neq 0} \phi \left( \frac{2^{i-1}}{\lambda} \right) a_i^{-\beta/n} a_i = C(n, \beta) \sum_{i \in \mathbb{Z}, a_i \neq 0} \phi \left( \frac{2^k}{\lambda} \right) a_i^{-\beta/n} a_{k+1}.
\]

Note that, using the H"{o}lder inequality and \( 0 < \beta < n \), noting \( a_k = 0 \) implies \( a_l = 0 \) for all \( l \geq k \), we have

\[
\sum_{k \in \mathbb{Z}} a_k^{1-\beta/n} \phi \left( \frac{2^k}{\lambda} \right) = \sum_{k \in \mathbb{Z}, a_k \neq 0} a_k^{1-\beta/n} \phi \left( \frac{2^k}{\lambda} \right) + \sum_{k \in \mathbb{Z}, a_k = 0} a_k^{1-\beta/n} \phi \left( \frac{2^k}{\lambda} \right)
\]

\[
\leq \left[ \sum_{k \in \mathbb{Z}, a_k \neq 0} a_k^{1-\beta/n} \phi \left( \frac{2^k}{\lambda} \right) \right] \left[ \sum_{k \in \mathbb{Z}, a_k \neq 0} a_k a_k^{-\beta/n} \phi \left( \frac{2^k}{\lambda} \right) \right]^{1-\beta/n}
\]

\[
\leq C(n, \beta) T^{1-\beta/n} \left[ \sum_{k \in \mathbb{Z}, a_k \neq 0} a_k^{1-\beta/n} \phi \left( \frac{2^k}{\lambda} \right) \right]^{\beta/n}.
\]
By the doubling property of $\phi$, one has

$$\sum_{k \in \mathbb{Z}, a_k \neq 0} a_k^{-\beta/n} \phi \left( \frac{2^k}{\lambda} \right) \leq K \sum_{k \in \mathbb{Z}} a_{k+1}^{-\beta/n} \phi \left( \frac{2^{k+1}}{\lambda} \right).$$

From this one concludes that

$$T \geq C(n, \beta, K) \sum_{k \in \mathbb{Z}} a_k^{-\beta/n} \phi \left( \frac{2^k}{\lambda} \right).$$

On the other hand,

$$\int_B \phi^{n/(n-\beta)} \left( \frac{|u_\ast(x)|}{4\lambda} \right) \, dx \leq \sum_{k \in \mathbb{Z}} \int_{B_k} \phi^{n/(n-\beta)} \left( \frac{2^{k-1}}{\lambda} \right) \, dx \leq \sum_{k \in \mathbb{Z}} \phi^{n/(n-\beta)} \left( \frac{2^{k-1}}{\lambda} \right) a_k.$$

Since $n/(n-\beta) \geq 1$, we obtain

$$\int_B \phi^{n/(n-\beta)} \left( \frac{|u_\ast(x)|}{4\lambda} \right) \, dx \leq \left[ \sum_{k \in \mathbb{Z}} \phi \left( \frac{2^{k-1}}{\lambda} \right) a_k^{-\beta/n} \right]^{n/(n-\beta)} \leq [C(n, \beta, K)]^{n/(n-\beta)}.$$

Up to considering $C(n, \beta, K) + 1$, assume that $C(n, \beta, K) \geq 1$. If $\lambda > 4C(n, \beta, K)||u||_{W^{\beta,4}(\mathbb{R}^n)}$, by Lemma 2.1, we have

$$\int_B \phi^{n/(n-\beta)} \left( \frac{|u_\ast(x)|}{4\lambda} \right) \, dx \leq \left[ \int_B \int_B \phi \left( \frac{|u_\ast(x) - u_\ast(y)|}{\lambda/4} \right) \frac{dxdy}{|x-y|^{n+\beta}} \right]^{\frac{\lambda}{4\beta}} \leq 1,$$

which gives (3.4).

**Proof of Theorem 1.3.** Lemma 3.4 gives Lemma 1.3 for any ball $B$ of $\mathbb{R}^n$. We still need to consider the case $B = \mathbb{R}^n$. Given any $u \in W^{\beta,4}(\mathbb{R}^n)$, applying Lemma 3.4, we have

$$||u - u_\ast||_{L^{\phi(n-\beta)}(B)} \leq C||u||_{W^{\beta,4}(B)} \leq ||u||_{W^{\beta,4}(\mathbb{R}^n)}, \quad \forall \text{ ball } B \subset \mathbb{R}^n.$$

For any $k \in \mathbb{Z}$, using Jensen’s inequality, one obtains

$$\phi^{n/(n-\beta)} \left( \frac{|u_{B(0,2^{k-1})} - u_{B(0,2^k)}|}{\lambda} \right) \leq \int_{B(0,2^{k-1})} \phi^{n/(n-\beta)} \left( \frac{|u(z) - u_{B(0,2^k)}|}{\lambda} \right) \, dz \leq \frac{1}{2^n} \int_{B(0,2^k)} \phi^{n/(n-\beta)} \left( \frac{|u(z) - u_{B(0,2^k)}|}{\lambda} \right) \, dz.$$

By Lemma 2.1, we get

$$|u_{B(0,2^{k-1})} - u_{B(0,2^k)}| \leq \phi^{-1}(2^{-k(n-\beta)})||u||_{L^{\phi(n-\beta)}(B(0,2^k))} \leq 2^{\frac{k(n-\beta)}{n-\beta}} ||u||_{W^{\beta,4}(\mathbb{R}^n)}.$$

This implies that $u_{B(0,2^k)}$ converges to some $c \in \mathbb{R}^n$ as $k \to \infty$ and

$$|u_{B(0,2^k)} - c| \leq \sum_{l \geq k} |u_{B(0,2^l)} - u_{B(0,2^{l+1})}| \leq \sum_{l \geq k} 2^{\frac{k(n-\beta)}{n-\beta}} ||u||_{W^{\beta,4}(\mathbb{R}^n)} \leq 2^{\frac{k(n-\beta)}{n-\beta}} ||u||_{W^{\beta,4}(\mathbb{R}^n)}.$$

Therefore,

$$||u - c||_{L^{\phi(n-\beta)}(B(0,2^{k+1}))} \leq ||u - u_{B(0,2^k)}||_{L^{\phi(n-\beta)}(B(0,2^{k+1}))} + ||u_{B(0,2^k)} - c||_{L^{\phi(n-\beta)}(B(0,2^{k+1}))} \leq ||u||_{W^{\beta,4}(\mathbb{R}^n)}.$$
Letting \( k \to \infty \), we get \( \inf_{c \in \mathbb{R}} ||u - c||_{L^{p_0(n,n^{-p})}((\mathbb{R}^n)^c)} \leq C||u||_{\dot{W}^{p,q}(\mathbb{R}^n)} \). This completes the proof of Theorem 1.3. \( \square \)

**Proof of (ii) \( \Rightarrow \) (iii) Theorem 1.1:** case \( \beta \in (0, n) \). Since \( \Omega \) is a \( W^{\beta,q}(\Omega) \)-extension domain, for any \( u \in \dot{W}^{\beta,q}(\mathbb{R}^n) \), we can find a \( \tilde{u} \in W^{\beta,q}(\mathbb{R}^n) \) such that \( \tilde{u} = u \) in \( \Omega \) and \( ||\tilde{u}||_{W^{\beta,q}(\Omega)} \leq C||u||_{\dot{W}^{\beta,q}(\Omega)} \). If \( \beta < n \), applying Theorem 1.3, we know

\[
\inf_{c \in \mathbb{R}} ||u - c||_{L^{p_0(n,n^{-p})}(\mathbb{R}^n)^c} \leq \inf_{c \in \mathbb{R}} ||\tilde{u} - c||_{L^{p_0(n,n^{-p})}((\mathbb{R}^n)^c)} \leq C||\tilde{u}||_{\dot{W}^{\beta,q}(\mathbb{R}^n)} \leq ||u||_{\dot{W}^{\beta,q}(\Omega)}
\]

as desired. \( \square \)

4. Proofs of Theorem 1.2 and (i) \( \Rightarrow \) (ii) of Theorem 1.1

Since (i) \( \Rightarrow \) (ii) of Theorem 1.1 follows from Theorem 1.2, below we only need to prove Theorem 1.2. To this end, we recall the properties of Ahlfors \( n \)-regular domains in Section 4.1. The proof of Theorem 1.1 is given in Section 4.2.

4.1. **Some basic properties of Ahlfors \( n \)-regular domains.** Let \( \Omega \) be an Ahlfors \( n \)-regular domain, and write \( U := \mathbb{R}^n \setminus \overline{\Omega} \). Observe that \( |\partial \Omega| = 0 \); see [7] and also [30]. Without loss of generality, we assume \( U \neq \emptyset \). Moreover, \( \text{diam} \Omega = \infty \) if and only if \( |\Omega| = \infty \); see [26].

It is well known that \( U \) admits a Whitney decomposition \( \mathcal{W} \).

**Lemma 4.1.** There exists a collection \( \mathcal{W} = \{Q_i\}_{i \in \mathbb{N}} \) of (closed) cubes satisfying

(i) \( U = \bigcup_{i \in \mathbb{N}} Q_i, \) and \( Q_i \cap Q_j = \emptyset \) for all \( i, k \in \mathbb{N} \) with \( i \neq k; \)
(ii) \( \sqrt{n}l(Q_k) \leq \text{dist}(Q_k, \partial \Omega) \leq 4 \sqrt{n}l(Q_k); \)
(iii) \( \frac{1}{2}l(Q_k) \leq l(Q_i) \leq 4l(Q_k) \) whenever \( Q_k \cap Q_i \neq \emptyset. \)

For any \( Q \in \mathcal{W} \), denote the neighbour cubes of \( Q \) in \( \mathcal{W} \) by

\[ N(Q) = \{ P \in \mathcal{W}, P \cap Q \neq \emptyset \}. \]

There exists an integer \( \gamma_0 \) depending only on \( n \) such that for all \( Q \in \mathcal{W}, \)

\[
\#N(Q) \leq \gamma_0.
\]

Moreover, for any \( Q \in \mathcal{W} \), we have

\[
\frac{1}{|Q|} \int_U \chi_{\mathring{Q}}(x) \, dx \leq 4^n \gamma_0.
\]

Indeed,

\[
\frac{1}{|Q|} \int_U \chi_{\mathring{Q}}(x) \, dx \leq \sum_{P \in \mathcal{W}} \frac{1}{|Q|} \int_P \chi_{\mathring{Q}}(x) \, dx.
\]

Since \( P \cap Q = \emptyset \) implies that \( P \subset N(Q) \) and \( l_0 \leq 4l_P \), by \( \#N(Q) \leq \gamma_0 \), we arrive at

\[
\frac{1}{|Q|} \int_U \chi_{\mathring{Q}}(x) \, dx \leq \sum_{P \in N(Q)} \frac{|P|}{|Q|} \leq 4^n \gamma_0
\]

as desired.

For \( \varepsilon > 0 \), set

\[ \mathcal{W}_\varepsilon := \{ Q \in \mathcal{W} : l_0 < \frac{1}{\varepsilon} \text{diam} \Omega \}. \]

Obviously, \( \mathcal{W} = \mathcal{W}_\varepsilon \) for any \( \varepsilon > 0 \) if \( \text{diam} \Omega = \infty \), and \( \mathcal{W}_\varepsilon \subseteq \mathcal{W} \) for any \( \varepsilon > 0 \) if \( \text{diam} \Omega < \infty \).
For each \( Q := Q(x_Q, l_Q) \in \mathcal{W}_\epsilon \) and let \( x_Q^* \in \Omega \) be a point nearest to \( x_Q \) on \( \overline{\Omega} \). By Lemma 4.1 (ii), we have
\[
\overline{Q}^* := Q(x_Q^*, l_Q) \subset 10 \sqrt{n}Q.
\]
Furthermore, write a reflecting ”cubes” of \( Q \) as
\[
\overline{Q}^{\epsilon} := (\epsilon Q^* \cap \Omega) \setminus \left( \bigcup \{ \epsilon P^* : P \in \mathcal{R}^\epsilon_Q \} \right).
\]
The following lemma says that if \( \epsilon \) is small enough, then the reflecting ”cubes” of \( \mathcal{W}_\epsilon \) enjoy the following fine properties.
Recall that the reflecting cubes was constructed in [21].

**Lemma 4.2.** let \( \epsilon_0 = (\theta/2\gamma_0)^{1/n}/(30 \sqrt{n}) \) and write \( Q^* = \overline{Q}^{\epsilon_0} \) for each \( Q \in \mathcal{W}_\epsilon \),

(i) \( Q^* \subset (10Q) \cap \Omega \) for all \( Q \in \mathcal{W}_\epsilon \);
(ii) \( |Q| \leq \gamma_1 |Q^*| \) whenever \( Q \in \mathcal{W}_\epsilon \);
(iii) \( \sum_{Q \in \mathcal{W}_\epsilon} \chi_{Q^*} \leq \gamma_2 \).

Above \( \gamma_1 \) and \( \gamma_2 \) are positive constants depending only on \( n \) and \( \theta \).

If \( \Omega \) is bounded, we let \( Q^* = \Omega \) as the reflected ”cube” of any cube \( Q \in \mathcal{W}_\epsilon \setminus \mathcal{W}_\epsilon \neq \emptyset \). Write
\[
\mathcal{W}_\epsilon^{(k)} = \{ Q \in N(P) : P \in \mathcal{W}_\epsilon^{(k-1)} \} \quad \forall k \geq 1,
\]
where \( \mathcal{W}_\epsilon^{(0)} = \mathcal{W}_\epsilon \). Namely, \( \mathcal{W}_\epsilon^{(k)} \) is the \( k \)-th neighbors of \( \mathcal{W}_\epsilon \). Meanwhile, we also write
\[
\mathcal{V}^{(k)} := \bigcup \{ x \in Q; Q \in \mathcal{W}_\epsilon^{(k)} \} \quad \forall k \geq 0.
\]
Since \( Q^* = \Omega \) for \( Q \notin \mathcal{W}_\epsilon \), applying Lemma 4.2 (iii), we have
\[
\sum_{Q \in \mathcal{W}_\epsilon^{(k)}} \chi_{Q^*} \leq \sum_{Q \in \mathcal{W}_\epsilon^{(k)}} \chi_{Q^*} + \#(\mathcal{W}_\epsilon^{(k)} \setminus \mathcal{W}_\epsilon^{(0)}) \chi_{\Omega} \leq [\gamma_2 + \#(\mathcal{W}_\epsilon^{(k)} \setminus \mathcal{W}_\epsilon^{(0)})] \chi_{\Omega} \quad \forall k \geq 1.
\]
For \( Q \in \mathcal{W}_\epsilon^{(k)} \setminus \mathcal{W}_\epsilon^{(0)} \), observe that \( l_Q \geq \tfrac{1}{\epsilon_0} \text{diam} \Omega \) and \( l_Q \leq 4^k l_P \leq \tfrac{4^k}{\epsilon_0} \text{diam} \Omega \) for some \( P \in \mathcal{W}_\epsilon^{(0)} \). Thus, by Lemma 4.1 (ii), we have
\[
Q \subset Q(\bar{x}, \text{diam} \Omega + 8 \sqrt{n} \tfrac{4^k}{\epsilon_0} \text{diam} \Omega)
\]
for any fixed \( \bar{x} \in \Omega \), and hence
\[
\#(\mathcal{W}_\epsilon^{(k)} \setminus \mathcal{W}_\epsilon^{(0)}) \leq (1 + 8 \sqrt{n} \tfrac{4^k}{\epsilon_0})^n \epsilon_0^n \leq (\epsilon_0 + 4^{k+2} \sqrt{n})^n.
\]
This yields that
\[
\sum_{Q \in \mathcal{W}_\epsilon^{(k)}} \chi_{Q^*} \leq \gamma_2 + (\epsilon_0 + 4^{k+2} \sqrt{n})^n \quad \forall k \geq 1.
\]

Associated to \( \mathcal{W}_\epsilon \), one has the following partition of unit of \( U \).

**Lemma 4.3.** There exists a family \( \{ \varphi_Q : Q \in \mathcal{W}_\epsilon \} \) of functions such that

(i) for each \( Q \in \mathcal{W}_\epsilon \), \( 0 \leq \varphi_Q \in C^\infty_0 (\overline{\Omega} \setminus Q) \);
(ii) for each \( Q \in \mathcal{W}_\epsilon \), \( |\nabla \varphi_Q| \leq L/\ell_Q \);
(iii) \( \sum_{Q \in \mathcal{W}_\epsilon} \varphi = \chi_U \).
4.2. Proof of Theorem 1.2. Let \( \Omega \) be an Ahlfors \( n \)-regular domain. To obtain Theorem 1.2, it suffices to prove the existence of a bounded linear operator \( E : \dot{W}^{\beta, \phi}(\Omega) \to \dot{W}^{\beta, \phi}(\mathbb{R}^n) \) such that \( Eu|_{\Omega} = u \) for all \( u \in \dot{W}^{\beta, \phi}(\Omega) \). The linear operator \( E \) is given as follows: for any \( u \in \dot{W}^{\beta, \phi}(\Omega) \) define

\[
Eu(x) = \begin{cases} 
  u(x), & x \in \Omega, \\
  0, & x \in \partial \Omega, \\
  \sum_{Q \in W} \varphi_Q(x)u_Q, & x \in U.
\end{cases}
\]

Obviously, \( Eu|_{\Omega} = u \) on \( \Omega \).

To prove the boundedness of \( E \), we just show that there exists a constant \( M > 0 \) such that for all \( \lambda > M \|u\|_{\dot{W}^{\beta, \phi}(\Omega)} \),

\[
H(\lambda) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dydx}{|x-y|^{n+\beta}} \leq 1.
\]

If \( \|u\|_{\dot{W}^{\beta, \phi}(\Omega)} = 0 \), then \( u \) and hence \( Eu \) must be a constant function essentially. So we may assume that \( \|u\|_{\dot{W}^{\beta, \phi}(\Omega)} > 0 \); and moreover, we further assume that \( \|u\|_{\dot{W}^{\beta, \phi}(\Omega)} = 1 \) on account of the linearity of \( E \).

For \( \lambda > 0 \), write

\[
H(\lambda) = \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\lambda} \right) \frac{dydx}{|x-y|^{n+\beta}} + 2 \int_{U} \int_{\Omega} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dydx}{|x-y|^{n+\beta}}
\]

\[
= H_1(\lambda) + H_2(\lambda) + H_3(\lambda).
\]

It suffices to find constants \( M_i \geq 1 \) depending only on \( n, \theta \) and \( \phi \) such that \( H_i(\lambda) \leq \frac{1}{4} \) whenever \( \lambda \geq M_i \) for \( i = 1, 2, 3 \). In fact, by taking \( M = M_1 + M_2 + M_3 \), we have \( H(\lambda) \leq 1 \) whenever \( \lambda \geq M \).

Firstly, let \( M_1 = 4 \). Then for \( \lambda > 4 \), by the convexity of \( \phi \) and \( \|u\|_{\dot{W}^{\beta, \phi}(\Omega)} = 1 \), we have

\[
H_1(\lambda) \leq \frac{1}{4} \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(x) - u(y)|}{\lambda/4} \right) \frac{dydx}{|x-y|^{2n}} \leq \frac{1}{4}.
\]

In order to find \( M_2 \) and \( M_3 \), we think about two cases: \( \text{diam} \ \Omega = \infty \) and \( \text{diam} \ \Omega < \infty \).

Case \( \text{diam} \ \Omega = \infty \). To find \( M_2 \), for any \( x \in U \) and \( y \in \Omega \), by Lemma 4.3(iii), one has

\[
Eu(x) - u(y) = \sum_{Q \in W} \varphi_Q(x)[u_Q - u(y)],
\]

and hence, using the convexity of \( \phi \) and Jensen’s inequality,

\[
\phi \left( \frac{|Eu(x) - u(y)|}{\lambda} \right) \leq \phi \left( \sum_{Q \in W} \varphi_Q(x) \frac{|u_Q - u(y)|}{\lambda} \right)
\]

\[
\leq \sum_{Q \in W} \varphi_Q(x) \phi \left( \int_{Q^*} \frac{|u(z) - u(y)|}{\lambda} dz \right) \leq \sum_{Q \in W} \varphi_Q(x) \int_{Q^*} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) dz.
\]

If \( \varphi_Q(x) \neq 0 \), then \( x \in \frac{12}{16}Q \). For \( z \in Q^* \), by \( Q^* \subset 10 \sqrt{n}Q \), we have \( |x - z| \leq 20nl(Q) \). If \( |x - y| \geq d(x, \Omega) \geq l(Q) \), we know \( |x - z| \leq 20n|x - y| \), that is,

\[
|y - z| \leq |x - y| + |x - z| \leq 21n|x - y|.
\]
Thus
\[ \int_{\Omega} \phi \left( \frac{|Eu(x) - u(y)|}{\lambda} \right) \frac{dy}{|x - y|^{\beta + n}} \leq (21n)^{\beta + n} \sum_{Q \in \mathcal{W}} \varphi_Q(x) \int_{\Omega} \int_{Q} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz dy}{|z - y|^{\beta + n}}. \]

By Lemma 4.2 (ii), we get
\[ H_2(\lambda) \leq 2(21n)^{\beta + n} \int_{U} \sum_{Q \in \mathcal{W}} \varphi_Q(x) \int_{Q} \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz dy}{|y - z|^{\beta + n}} dx \]
\[ \leq 2\gamma_1(21n)^{\beta + n} \sum_{Q \in \mathcal{W}} \left( \frac{1}{|Q|} \int_{U} \varphi_Q(x) dx \right) \int_{\Omega} \int_{Q} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz dy}{|y - z|^{\beta + n}}. \]

For \( \varphi_Q \leq \chi_{\ast \Omega} \) as given in Lemma 4.3, by (4.2) we have
\[ \frac{1}{|Q|} \int_{U} \varphi_Q(x) dx \leq \frac{1}{|Q|} \int_{U} \chi_{\ast \Omega}(x) dx \leq 4^n \gamma_0, \]
which implies that
\[ (4.5) \quad H_2(\lambda) \leq 2\gamma_1 4^n \gamma_0 (21n)^{\beta + n} \sum_{Q \in \mathcal{W}} \int_{\Omega} \int_{Q} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz dy}{|y - z|^{\beta + n}}. \]

By \( \sum_{Q \in \mathcal{W}} \chi_{Q} \leq \gamma_2 \) (see Lemma 4.2 (iii)), we obtain
\[ H_2(\lambda) \leq 2\gamma_1 4^n \gamma_0 \gamma_2 (21n)^{\beta + n} \int_{\Omega} \int \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz dy}{|y - z|^{\beta + n}}. \]

Let \( M_2 = 8\gamma_1 4^n \gamma_0 \gamma_2 (21n)^{\beta + n} \). By the convexity of \( \phi \) again, \( \lambda > M_2 \) gives \( H_2(\lambda) \leq \frac{1}{4} \).

To find \( M_3 \), for each \( x \in U \), set
\[ X_1(x) := \left\{ y \in U : |x - y| \geq \frac{1}{132n} \max\{d(x, \Omega), d(y, \Omega)\} \right\} \quad \text{and} \quad X_2(x) := U \setminus X_1(x). \]

Write
\[ H_3(\lambda) = \int_{U} \int_{X_1(x)} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dy dx}{|x - y|^{\beta + n}} + \int_{U} \int_{X_2(x)} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dy dx}{|x - y|^{\beta + n}} =: H_{31}(\lambda) + H_{32}(\lambda) \]

Below, we will find \( M_{3i} \) so that if \( \lambda > M_{3i} \), then \( H_{3i} \leq \frac{1}{8} \) for \( i = 1, 2 \). Note that letting \( M_3 = \max\{M_{31}, M_{32}\} \), for \( \lambda > M_3 \), we have \( H_3(\lambda) \leq \frac{1}{4} \) as desired.

To find \( M_{31} \), for \( x \in U \) and \( y \in X_1(x) \), thanks to
\[ \sum_{Q \in \mathcal{W}} \varphi_Q(x) = \sum_{P \in \mathcal{W}} \varphi_P(y) = 1, \]
we obtain
\[ Eu(x) - Eu(y) = \sum_{P \in \mathcal{W}} \sum_{Q \in \mathcal{W}} \varphi_Q(x) \varphi_P(y) [u_Q - u_P] = \sum_{P \in \mathcal{W}} \sum_{Q \in \mathcal{W}} \varphi_Q(x) \varphi_P(y) \int_{Q} \int_{P} [u(z) - u(w)] dz dw. \]
Again, applying the convexity of $\phi$ and Jensen’s inequality, one gets

$$\phi\left(\frac{|Eu(x) - Eu(y)|}{\lambda}\right) \leq \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{P}} \varphi_Q(x) \varphi_P(y) \phi\left(\int_Q \int_P \frac{|u(z) - u(w)|}{\lambda} dz dw\right)$$

$$\leq \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{P}} \varphi_Q(x) \varphi_P(y) \int_Q \int_P \phi\left(\frac{|u(z) - u(w)|}{\lambda}\right) dw dz.$$ 

For $x \in Q$ and $z \in Q^*$, $Q^* \subset 10 \sqrt{n} Q$ so that $|x - z| \leq 10n l_Q \leq 10n d(x, \Omega)$. Similarly, for $y \in P$, and $w \in P^*$, we have $|y - w| \leq 10n d(y, \Omega)$ as well. Since $y \in X_1(x)$ with $132n|x - y| \geq \max\{d(x, \Omega), d(y, \Omega)\}$, we further know

$$|z - w| \leq |x - z| + |x - y| + |y - w| \leq 2641n|x - y|.$$ 

As a consequence,

$$H_{31}(\lambda) \leq (2641n)^{\beta + n} \int_U \int_{X_1(x)} \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{P}} \varphi_Q(x) \varphi_P(y) \int_Q \int_P \phi\left(\frac{|u(z) - u(w)|}{\lambda}\right) \frac{dw dz}{|z - w|^\beta + n} dy dx.$$ 

By $|Q| \leq \gamma_1 |Q^*|$ and $|P| \leq \gamma_1 |P^*|$ (see Lemma 4.2 (ii)), we have

$$H_{31}(\lambda) \leq (2641n)^{\beta + n} \gamma_1^2 \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{P}} \left(\frac{1}{|Q|} \int_U \varphi_Q(x) dx\right) \left(\frac{1}{|P|} \int_U \varphi_P(y) dy\right) \int_Q \int_P \phi\left(\frac{|u(z) - u(w)|}{\lambda}\right) \frac{dw dz}{|z - w|^\beta + n}.$$ 

Applying Lemma 4.3 and (4.2) again, we have

$$\frac{1}{|Q|} \int_U \varphi_Q(x) dx \frac{1}{|P|} \int_U \varphi_P(y) dy \leq (4^\alpha \gamma_0)^2.$$ 

Thus

$$H_{31}(\lambda) \leq (2641n)^{\beta + n} \gamma_1^2 (4^\alpha \gamma_0)^2 \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{P}} \int_Q \int_P \phi\left(\frac{|u(z) - u(w)|}{\lambda}\right) \frac{dw dz}{|z - w|^\beta + n}.$$ 

Observing $\sum_{Q \in \mathcal{W}} \chi_Q = \gamma_2$ given by Lemma 4.2 (iii), we arrive at

$$H_{31}(\lambda) \leq (2641n)^{\beta + n} \gamma_1^2 \gamma_2^2 (4^\alpha \gamma_0)^2 \sum_{Q \in \mathcal{W}} \sum_{P \in \mathcal{P}} \int_Q \int_P \phi\left(\frac{|u(z) - u(w)|}{\lambda}\right) \frac{dw dz}{|z - w|^\beta + n}.$$ 

Taking $M_{31} = 8(2641n)^{\beta + n} \gamma_1^2 \gamma_2^2 (4^\alpha \gamma_0)^2$, if $\lambda > M_{31}$, by the convexity of $\phi$ once more, we have $H_{31}(\lambda) \leq \frac{1}{8}$. 

To find $M_{32}$, write

$$H_{32}(\lambda) = \int_U \sum_{P \in \mathcal{P}} \int_{P \cap X_2(x)} \phi\left(\frac{|Eu(x) - Eu(y)|}{\lambda}\right) \frac{dy}{|x - y|^\alpha \beta} dx.$$ 

By $\sum_{Q \in \mathcal{W}} (\varphi_Q(x) - \varphi_Q(y)) = 0$, for any $x \in U$ and $y \in X_2(x) \cap P$, we have

$$Eu(x) - Eu(y) = \sum_{Q \in \mathcal{W}} (\varphi_Q(x) - \varphi_Q(y)) [u_{Q^*} - u_P].$$ 

Furthermore, by $|\nabla \varphi_Q| \leq L/l_Q$, we obtain

$$|Eu(x) - Eu(y)| \leq L \sum_{Q \in \mathcal{W}} \frac{|x - y|}{l_Q} [\chi_{\# \varphi_Q}(x) + \chi_{\# \varphi_Q}(y)] [u_{Q^*} - u_P].$$
Since \( y \in X_2(x) \) with \( |x - y| \leq \frac{1}{132n} \max \{ d(x, \Omega), d(y, \Omega) \} \), taking \( \bar{y} \in \bar{\Omega} \) with \( |y - \bar{y}| = d(y, \Omega) \), we obtain

\[
d(x, \Omega) \leq |x - \bar{y}| \leq |x - y| + |y - \bar{y}| \leq \frac{1}{132n}d(x, \Omega) + \frac{1 + 132n}{132n}d(y, \Omega),
\]

which leads to \( d(x, \Omega) \leq \frac{132n + 1}{132n - 1}d(y, \Omega) \). Similarly, we have \( d(y, \Omega) \leq \frac{132n + 1}{132n - 1}d(x, \Omega) \). Therefore,

\[
|x - y| \leq \frac{1}{132n + 1} \frac{132n + 1}{132n - 1}d(x, \Omega).
\]

For \( y \in \frac{17}{16} Q \), we know \( Q \in \mathcal{N}(P) \) and

\[
d(y, \Omega) \leq d(y, Q) + \max_{a \in Q} d(a, \Omega) \leq \frac{1}{16} \sqrt{n}l_Q + 4 \sqrt{n}l_Q \leq \frac{65}{16} \sqrt{n}l_Q.
\]

This implies

\[
|x - y| \leq \frac{1}{132n + 1} \frac{132n + 1}{132n - 1} \times \frac{65}{16} \sqrt{n}l_Q \leq \frac{1}{132} \sqrt{n}l_Q,
\]

and hence \( x \in \frac{9}{8} Q \), that is,

\[
\chi_{\frac{9}{8} Q}(y) \leq \chi_{\frac{9}{8} Q}(x).
\]

Similarly, if \( x \in \frac{17}{16} Q \), we also have \( y \in \frac{9}{8} Q \) and \( Q \in \mathcal{N}(P) \). We may further write

\[
|Eu(x) - Eu(y)| \leq 2L \sum_{Q \in \mathcal{N}(P)} \frac{|x - y|}{l_Q} \chi_{\frac{9}{8} Q}(x)|u_Q - u_P|.
\]

By \( \sum_{Q \in \mathcal{W}} \chi_{\frac{9}{8} Q}(x) \leq \gamma_0 \) and the convexity of \( \phi \), we have

\[
\phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \leq \frac{1}{\gamma_0} \sum_{Q \in \mathcal{N}(P)} \chi_{\frac{9}{8} Q}(x) \phi \left( \frac{|x - y|}{l_Q} \frac{|u_Q - u_P|}{\lambda/2L\gamma_0} \right)
\]

and hence

\[
H_{32}(\lambda) \leq \frac{1}{\gamma_0} \int_{U} \sum_{P \in \mathcal{F}} \int_{P \cap X_2(x)} \sum_{Q \in \mathcal{N}(P)} \chi_{\frac{9}{8} Q}(x) \phi \left( \frac{|x - y|}{l_Q} \frac{|u_Q - u_P|}{\lambda/2L\gamma_0} \right) dy dx \leq \frac{1}{\gamma_0} \int_{U} \sum_{P \in \mathcal{F}} \sum_{Q \in \mathcal{N}(P)} \chi_{\frac{9}{8} Q}(x) \int_{P \cap X_2(x)} \phi \left( \frac{|x - y|}{l_Q} \frac{|u_Q - u_P|}{\lambda/2L\gamma_0} \right) dy dx.
\]

Note that for \( x \in \frac{9}{8} Q \) and \( y \in P \cap X_2(x) \), together with \( d(x, \Omega) \leq 4 \sqrt{n}l_Q \), we have

\[
|x - y| \leq \frac{1}{132n + 1} \frac{132n + 1}{132n - 1}d(x, \Omega) \leq l_Q.
\]

By the condition (1.1), we get

\[
\int_{P \cap X_2(x)} \phi \left( \frac{|x - y|}{l_Q} \frac{|u_Q - u_P|}{\lambda/2L\gamma_0} \right) dy \leq n\omega_n \int_0^{l_Q} \phi \left( \frac{t}{l_Q} \frac{|u_Q - u_P|}{\lambda/2L\gamma_0} \right) dt \leq C_\beta \omega_n \int_0^{l_Q} \phi \left( \frac{|u_Q - u_P|}{\lambda/2L\gamma_0} \right) dt.
\]
Therefore,

\[ H_{32}(\lambda) \leq nC_\beta \frac{1}{\gamma_0} \omega_n \sum_{P \in \mathcal{W}} \sum_{Q \in N(P)} (l_Q)^{\beta - \gamma_0} \int_U \phi \left( \frac{|u_Q - u_{P^*}|}{4\lambda 2\gamma_0} \right) dx \]

\[ \leq nC_\beta \frac{1}{\gamma_0} \omega_n \sum_{P \in \mathcal{W}} \sum_{Q \in N(P)} \left( \frac{1}{|Q|} \right) \int_U \phi \left( \frac{|u_Q - u_{P^*}|}{\lambda 2\gamma_0} \right) (l_Q)^{\gamma_0} \phi \left( \frac{|u_Q - u_{P^*}|}{\lambda 2\gamma_0} \right) \]

\[ \leq nC_\beta \omega_n 4^{n} \sum_{P \in \mathcal{W}} \sum_{Q \in N(P)} (l_Q)^{\gamma_0} \phi \left( \frac{|u_Q - u_{P^*}|}{\lambda 2\gamma_0} \right). \]

For any \( P \in \mathcal{W} \) and \( Q \in N(P) \), using Jessen’s inequality, one has

\[ \phi \left( \frac{|u_Q - u_{P^*}|}{\lambda 2\gamma_0} \right) \leq \int_{Q^*} \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\lambda 2\gamma_0} \right) \, dz \, dw. \]

Observe that \( z \in P^* \subset 10 \sqrt{n}P \) and \( w \in Q^* \subset 10 \sqrt{n}Q \) provides

\[ |z - w| \leq 10n(l_Q + l_P) \leq 50n \min\{l_Q, l_P\}. \]

If \( n - \beta < 0 \), then \( (l_Q)^{\beta - n} \leq (50n)^{\beta - n} \). Since \( |Q| \leq \gamma_1 |Q^*| \) and \( |P| \leq \gamma_1 |P^*| \), this implies that

\[ |z - w|^{2n} \leq (50n)^2 (\gamma_1)^2 |Q^*||P^*|. \]

Hence

\[ \phi \left( \frac{|u_Q - u_{P^*}|}{\lambda 2\gamma_0} \right) \leq (50n)^{\beta - n} \left( \gamma_1 \right)^2 \int_{Q^*} \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\lambda 2\gamma_0} \right) \, dz \, dw \]

\[ |z - w|^{n+\beta}. \]

If \( n - \beta > 0 \), \( |Q| \leq \gamma_1 |Q^*| \) implies \( (l_Q)^{\beta - n} \leq (\gamma_1 |Q^*|)^{1/n - \beta/n} \). Since \( |Q| \leq \gamma_1 |Q^*| \) and \( |P| \leq \gamma_1 |P^*| \), this implies that

\[ |z - w|^{n+\beta} \leq (50n)^{\beta - n} (\gamma_1)^{1/n} |Q^*||P^*|. \]

Therefore,

\[ \phi \left( \frac{|u_Q - u_{P^*}|}{\lambda 2\gamma_0} \right) \leq (50n)^{\beta - n} (\gamma_1)^2 \int_{Q^*} \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\lambda 2\gamma_0} \right) \, dz \, dw \]

\[ |z - w|^{n+\beta}. \]

We conclude that

\[ H_{32}(\lambda) \leq nC_\beta \omega_n 4^{n} (50n)^{\beta - n} (\gamma_1)^2 \sum_{P \in \mathcal{W}} \sum_{Q \in N(P)} \int_{Q^*} \int_{P^*} \phi \left( \frac{|u(z) - u(w)|}{\lambda 2\gamma_0} \right) \, dz \, dw \]

\[ |z - w|^{n+\beta}. \]

Applying \( \sum_{Q \in \mathcal{W}} \chi_{Q^*} \leq \gamma_2 \) again, we get

\[ H_{32}(\lambda) \leq nC_\beta \omega_n 4^{n} (50n)^{\beta - n} (\gamma_1)^2 (\gamma_2)^2 \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(z) - u(w)|}{\lambda 2\gamma_0} \right) \, dz \, dw \]

\[ |z - w|^{n+\beta}. \]

Taking \( M_3 = 8L\gamma_0 nC_\beta \omega_n 4^{n} (50n)^{\beta - n} (\gamma_1)^2 (\gamma_2)^2 \), if \( \lambda > M_3 \), we have \( H_{32}(\lambda) \leq \frac{1}{8} \) as desired.

**Case** \( \text{diam} \Omega < \infty \). To find \( M_2 \), write \( H_2(\lambda) \) as

\[ H_2(\lambda) = \int_{\Omega} \int_{\Omega} \phi \left( \frac{|Eu(x) - u(y)|}{\lambda} \right) \, dy \, dx \]

\[ + \int_{U \setminus \Omega} \int_{\Omega} \phi \left( \frac{|u_{Q^*} - u(y)|}{\lambda} \right) \, dy \, dx := H_{21}(\lambda) + H_{22}(\lambda). \]

It suffices to find \( M_{2i} \) such that \( H_{2i}(\lambda) \leq \frac{1}{8} \) for \( i = 1, 2 \). Note that letting \( M_2 = \max\{M_{21}, M_{22}\} \), for \( \lambda > M_2 \), we have \( H_2(\lambda) \leq \frac{1}{8} \) as desired.
For any $x \in U \setminus V^{(2)}$, $x$ belongs to some $Q \in \mathcal{W} \setminus \mathcal{W}^{(2)}_{e_0}$. Hence $N(Q) \cap \mathcal{W}_{e_0} = \emptyset$ and $P^* = \Omega$ for all $P \in N(Q)$. By $\sum_{P \in N(Q)} \varphi_P(x) = \sum_{P \in \mathcal{W}} \varphi_P(x) = 1$, we have

$$Eu(x) = \sum_{P \in \mathcal{W}} \varphi_P(x) u_{P^*} = \sum_{P \in N(Q)} \varphi_P(x) u_{P^*} = u_{\Omega}.$$ 

To find $M_{22}$, using Jensen’s inequality, we obtain

$$H_{22}(\lambda) \leq \int_{U \setminus V^{(2)}} \int_{\Omega} \left[ \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \right] \frac{dy}{|x - y|^{n+\beta}} dz \frac{dx}{|y|^{n+\beta}} \leq \int_{\Omega} \frac{|\text{diam } \Omega|^{n+\beta}}{|\Omega|^2} \int_{U \setminus V^{(2)}} \frac{dx}{|x - y|^{n+\beta}} \left[ \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \right] \frac{dz}{|z - y|^{n+\beta}} dy.$$ 

Clearly, for $y \in \Omega$, letting $Q \in \mathcal{W} \setminus \mathcal{W}_{e_0}$ and $x \in Q$, one has

$$|x - y| \geq d(x, \Omega) \geq l_Q \geq \frac{1}{\epsilon_0} \text{diam } \Omega.$$ 

Furthermore, since $\Omega$ is an Ahlfors $n$-regular domain, we have $|\Omega| \geq \theta |\text{diam } \Omega|^n$. This yields

$$\frac{|\text{diam } \Omega|^{n+\beta}}{|\Omega|^2} \int_{U \setminus V^{(2)}} \frac{dx}{|x - y|^{n+\beta}} \leq \frac{1}{\theta} |\text{diam } \Omega|^\beta \int_{|x - y| > \frac{1}{\epsilon_0} \text{diam } \Omega} \frac{dx}{|x - y|^{n+\beta}}$$

$$\leq \frac{n}{\theta} |\text{diam } \Omega|^\beta \omega_n \int_{\frac{1}{\epsilon_0} \text{diam } \Omega}^{\infty} \frac{1}{r^{n+1}} dr \leq \omega_n \frac{\epsilon_0^\beta}{\theta \beta}.$$ 

We then conclude

$$H_{22}(\lambda) \leq \omega_n \frac{\epsilon_0^\beta}{\theta \beta} \int_{\Omega} \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz}{|y - z|^{n+\beta}}.$$ 

Letting $M_{22} = 8 \omega_n \frac{\epsilon_0^\beta}{\theta \beta}$, by the convexity of $\phi$, for $\lambda > M_{22}$ we have $H_{22}(\lambda) \leq \frac{1}{8}$. 

To find $M_{21}$, note that for $x \in V^{(2)}$,

$$\sum_{Q \in \mathcal{W}} \varphi_Q(x) = \sum_{Q \in \mathcal{W}^{(2)}_{e_0}} \varphi_Q(x) = 1.$$ 

By the same argument as $H_2(\lambda)$ in the case $\text{diam } \Omega = \infty$, one has

$$H_{21}(\lambda) \leq 2 \gamma_1 4^n \gamma_0 (21n)^{n+\beta} \sum_{Q \in \mathcal{W}^{(2)}_{e_0}} \int_{Q} \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz}{|y - z|^{n+\beta}}.$$ 

Moreover,

$$\sum_{Q \in \mathcal{W}^{(2)}_{e_0}} \chi_Q \leq \gamma_2 + (\epsilon_0 + 64 \sqrt{n})^n,$$ 

and hence,

$$H_{21}(\lambda) \leq 2 \gamma_1 4^n \gamma_0 [\gamma_2 + (\epsilon_0 + 64 \sqrt{n})^n] (21n)^{n+\beta} \sum_{Q \in \mathcal{W}^{(2)}_{e_0}} \int_{Q} \int_{\Omega} \phi \left( \frac{|u(z) - u(y)|}{\lambda} \right) \frac{dz}{|y - z|^{n+\beta}}.$$ 

Letting $M_{21} = 16 \gamma_1 4^n \gamma_0 [\gamma_2 + (\epsilon_0 + 64 \sqrt{n})^n] (21n)^{n+\beta}$, by the convexity of $\phi$ again, if $\lambda > M_{21}$, we have $H_{21}(\lambda) \leq \frac{1}{8}$ as desired.
To find $M_3$, note that
\[ U \times U \subset [V^{(3)} \times V^{(3)}] \cup [V^{(2)} \times (U \setminus V^{(3)})] \cup [(U \setminus V^{(3)}) \times V^{(2)}] \cup [(U \setminus V^{(2)}) \times (U \setminus V^{(2)})]. \]
Write
\[
H_3(\lambda) = \int_{V^{(3)}} \int_{V^{(3)}} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dydx}{|x-y|^{\alpha + \beta}} + 2\int_{V^{(2)}} \int_{U \setminus V^{(3)}} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dydx}{|x-y|^{\alpha + \beta}} + \int_{U \setminus V^{(2)}} \int_{U \setminus V^{(2)}} \phi \left( \frac{|Eu(x) - Eu(y)|}{\lambda} \right) \frac{dydx}{|x-y|^{\alpha + \beta}} =: H_{31}(\lambda) + H_{32}(\lambda) + H_{33}(\lambda).
\]
Observe that $Eu(x) = Eu(y) = u_\Omega$ for $x, y \in U \setminus V$, this implies $H_{33}(\lambda) = 0$. It only suffices to find $M_3$ such that $H_{3i}(\lambda) \leq \frac{1}{8}$ for all $\lambda > M_3 (i = 1, 2)$.

For $H_{31}(\lambda)$, similarly to $H_{3}(\lambda)$ in the case $\text{diam } \Omega = \infty$, taking $M_3$ as $M_3$ with $\gamma_2$ replaced by $\gamma_2 + (\epsilon_0 + 4^2 \sqrt{n}) \omega_n$, we can prove that if $\lambda \geq M_{31}$, then $H_{31}(\lambda) \leq \frac{1}{8}$. Here we omit the details.

For $H_{32}(\lambda)$, note that for $y \in U \setminus V^{(3)}$, by $Eu(y) = u_\Omega$, we have
\[
H_{32}(\lambda) = \int_{V^{(2)}} \int_{U \setminus V^{(3)}} \phi \left( \frac{|Eu(x) - u_\Omega|}{\lambda} \right) \frac{dydx}{|x-y|^{\alpha + \beta}}.
\]
By Jessen’s inequality, one has
\[
H_{32}(\lambda) \leq \int_{V^{(2)}} \int_{U \setminus V^{(3)}} \frac{dy}{|x-y|^{\alpha + \beta}} \int_{\Omega} \phi \left( \frac{|Eu(x) - u_\Omega|}{\lambda} \right) dx dz.
\]
For any $x \in V^{(2)}$ and $y \in U \setminus V^{(3)}$, if $Q \in \mathcal{W}_{\epsilon_0}^{(3)} \setminus \mathcal{W}_{\epsilon_0}^{(2)}$ and $y \in Q$, then $|x - y| \geq l(Q) \geq \frac{1}{\epsilon_0} \text{diam } \Omega$. Thus
\[
\int_{U \setminus V^{(3)}} \frac{dy}{|x-y|^{\alpha + \beta}} \leq \frac{n\epsilon_0 \omega_n}{\beta} (\text{diam } \Omega)^{-\beta}.
\]
By $|\Omega| \geq \theta \text{diam } \Omega$, one has
\[
H_{32}(\lambda) \leq \frac{n\epsilon_0 \omega_n}{\theta \beta} (\text{diam } \Omega)^{-(\beta + n)} \int_{V^{(2)}} \int_{\Omega} \phi \left( \frac{|Eu(x) - u_\Omega|}{\lambda} \right) dx dz.
\]
For any $x \in V^{(2)}$, there exists a $P_i \in \mathcal{W}_{\epsilon_0}^{(i)}$ such that $x \in P_i$ and $P_i \in N(P_{i-1})$ for $i = 1, 2$. Together with $l(P_0) \leq \frac{1}{\epsilon_0} \text{diam } \Omega$ and Lemma 4.1, we know $l(P_2) \leq 4^2 \frac{1}{\epsilon_0} \text{diam } \Omega$. Hence for $y \in \Omega$,
\[
|x - y| \leq \text{dist}(x, \Omega) + \text{diam } \Omega \leq \text{diam } P_2 + \text{dist}(P_2, \Omega) + \text{diam } \Omega \leq 4^4 \frac{1}{\epsilon_0} \sqrt{n} \text{diam } \Omega.
\]
This yields
\[
H_{32}(\lambda) \leq \frac{n\epsilon_0 \omega_n}{\theta \beta} (4^4 \frac{1}{\epsilon_0} \sqrt{n})^{\alpha + \beta} \int_{V^{(2)}} \int_{\Omega} \phi \left( \frac{|Eu(x) - u_\Omega|}{\lambda} \right) dx dz \leq \frac{4^4 (\alpha + \beta) n \omega_n}{\theta \beta \epsilon_0^{\alpha + \beta}} H_{21}(\lambda).
\]
Letting $M_{32} = \frac{4^4 (\alpha + \beta) n \omega_n}{\theta \beta \epsilon_0^{\alpha + \beta}} M_{21}$, if $\lambda > M_{32}$, we have $H_{32}(\lambda) \leq \frac{1}{8}$ as desired. Then completes the proof of Theorem 1.2.
5. Proof of (iii)⇒(i) of Theorem 1.1

To prove (iii)⇒(i) of Theorem 1.1, we need the following estimates for test functions. Below we write \( B_{\Omega}(x, r) = B(x, r) \cap \Omega \). For \( x \in \Omega \) and \( 0 < r < t < \text{diam} \Omega \), set

\[
u_{x,r,t}(z) = \begin{cases} \frac{1}{t-|x-z|} & z \in B_{\Omega}(x, r) \\ \frac{1}{t-r} & z \in B_{\Omega}(x, t) \setminus B_{\Omega}(x, r) \\ 0 & z \in \Omega \setminus B_{\Omega}(x, t) \end{cases}.
\]

Then proof of Lemma 5.1 is similar to Lemma 5.1 of [26]. For reader’s convenience, we give the details.

**Lemma 5.1.** Let \( \beta > 0 \) and \( \phi \) be a Young function satisfying (1.1). For all \( \beta > 0 \), \( x \in \Omega \) and \( 0 < r < t < \text{diam} \Omega \), we have \( u_{x,r,t} \in W^{\beta,\phi}(\Omega) \) with

\[
\|u_{x,r,t}\|_{W^{\beta,\phi}(\Omega)} \leq C \left[ \phi^{-1} \left( \frac{(t-r)^\beta}{|B_{\Omega}(x,t)|} \right) \right]^{-1}.
\]

**Proof.** Write

\[
\int_{\Omega} \int_{\Omega} \phi \left( \frac{|u_{x,r,t}(z) - u_{x,r,t}(w)|}{\lambda} \right) \frac{dwdz}{|z-w|^{n+\beta}} = \int_{B_{\Omega}(x,t)} \int_{B_{\Omega}(x,t)} \phi \left( \frac{|u_{x,r,t}(z) - u_{x,r,t}(w)|}{\lambda} \right) \frac{dwdz}{|z-w|^{n+\beta}}
\]

\[
+ \int_{B_{\Omega}(x,t)} \int_{B_{\Omega}(x,t)} \phi \left( \frac{|u_{x,r,t}(z)|}{\lambda} \right) \frac{dwdz}{|z-w|^{n+\beta}} = H_1(\lambda) + H_2(\lambda).
\]

Clearly,

\[
H_2(\lambda) \leq \int_{B_{\Omega}(x,t) \setminus \Omega} \int_{B_{\Omega}(x,t) \setminus \Omega} \frac{dw}{|z-w|^{n+\beta}} \leq \int_{B_{\Omega}(x,t) \setminus \Omega} \int_{B_{\Omega}(x,t) \setminus \Omega} \frac{dw}{|z-w|^{n+\beta}} \leq \frac{n}{\beta} \omega_n(t - |z - x|)^{-\beta}.
\]

This induces

\[
H_2(\lambda) \leq \int_{B_{\Omega}(x,t) \setminus \Omega} \int_{B_{\Omega}(x,t) \setminus \Omega} \frac{n}{\beta} \omega_n(t - |z - x|)^{-\beta} dz + \int_{B_{\Omega}(x,t) \setminus \Omega} \frac{1}{\lambda} \frac{n}{\beta} \omega_n(t - |z - x|)^{-\beta} dz
\]

\[
\leq \frac{n}{\beta} \omega_n(\beta + 1) \frac{|B_{\Omega}(x,t)|}{(t-r)^{\beta}} \left[ \sup_{s \in (0,1]} \phi \left( \frac{s}{\lambda} \right) \frac{1}{s^{\beta}} + \phi \left( \frac{1}{\lambda} \right) \right].
\]

Moreover,

\[
\sup_{s \in (2^{-j-1}, 2^{-j+1})} \phi \left( \frac{s}{\lambda} \right) \frac{1}{s^{\beta}} \leq \frac{2^\beta}{\beta} \int_{2^{-j}}^{2^{-j+1}} \phi \left( \frac{s}{\lambda} \right) \frac{ds}{s^{\beta+1}},
\]

which leads to

\[
\sup_{s \in (0,1]} \phi \left( \frac{s}{\lambda} \right) \frac{1}{s^{\beta}} \leq \frac{2^\beta}{\beta} \int_{0}^{1} \phi \left( \frac{s}{\lambda} \right) \frac{ds}{s^{\beta+1}} \leq \frac{4^\beta}{\beta} \int_{0}^{1} \phi \left( \frac{2s}{\lambda} \right) \frac{ds}{s^{\beta+1}} \leq \frac{4^\beta}{\beta} \phi \left( \frac{2}{\lambda} \right).
\]

Therefore,

\[
H_2 \leq \left( \frac{4^\beta}{\beta} + 1 \right) \frac{n}{\beta} \omega_n(\beta + 1) \frac{|B_{\Omega}(x,t)|}{(t-r)^{\beta}} \phi \left( \frac{2}{\lambda} \right).
\]
If \( \lambda = M[\phi^{-1}\left(\frac{(t-r)^{\beta}}{|B_{\Omega}(x,t)|}\right)]^{-1} \) and \( M \geq 4 \left(\frac{\phi}{\beta} + 1\right)^{\alpha} \omega_n(\beta + 1) \), we have \( H_2(\lambda) \leq \frac{1}{2} \).

Moreover,

\[
H_1(\lambda) \leq \int_{B_{\Omega}(x,t)} \int_{B_{\Omega}(w,t-r)} \phi\left(\frac{|z-w|}{\lambda(t-r)}\right) \frac{dz}{|z-w|^{n+\beta}} dw + \int_{B_{\Omega}(x,t)} \int_{B_{\Omega}(w,2r) \setminus B_{\Omega}(w,t-r)} \phi\left(\frac{1}{\lambda}\right) \frac{dz}{|z-w|^{n+\beta}} dw.
\]

Observe that

\[
\int_{B_{\Omega}(w,t-r)} \phi\left(\frac{|z-w|}{\lambda(t-r)}\right) \frac{dz}{|z-w|^{n+\beta}} \leq n\omega_n \int_0^{t-r} \phi\left(\frac{s}{\lambda(t-r)}\right) \frac{ds}{s^{\beta+1}} \leq n\omega_n(t-r)^{-\beta} \int_0^1 \phi\left(\frac{s}{\lambda}\right) \frac{ds}{s^{\beta+1}}.
\]

Applying the condition (1.1), we get

\[
\int_{B_{\Omega}(w,t-r)} \phi\left(\frac{|z-w|}{\lambda(t-r)}\right) \frac{dz}{|z-w|^{n+\beta}} \leq n\omega_n(t-r)^{-\beta} \phi\left(\frac{1}{\lambda}\right).
\]

On the other hand,

\[
\int_{B_{\Omega}(w,2r) \setminus B_{\Omega}(w,t-r)} \frac{dz}{|z-w|^{n+\beta}} \leq \int_{\mathbb{R}^n \setminus B_{\Omega}(w,t-r)} \frac{dz}{|z-w|^{n+\beta}} = \frac{n}{\beta} \omega_n(t-r)^{-\beta}.
\]

Hence

\[
H_1(\lambda) \leq \frac{n}{\beta} \omega_n C_\beta(\beta + 1) \frac{|B_{\Omega}(x,t)|}{(t-r)^{\beta}} \phi\left(\frac{1}{\lambda}\right).
\]

If \( \lambda = M[\phi^{-1}\left(\frac{(t-r)^{\beta}}{|B_{\Omega}(x,t)|}\right)]^{-1} \) and \( M \geq 2 \left(\frac{\phi}{\beta} + 1\right) \omega_n C_\beta \), we have \( H_1(\lambda) \leq \frac{1}{2} \).

\[
\square
\]

We are ready to prove (iii) \( \Rightarrow \) (i) of Theorem 1.1.

**Proof of (iii) \( \Rightarrow \) (i) of Theorem 1.1.** Below we consider two cases: \( \beta > n \) and \( 0 < \beta < n \).

**Case \( \beta > n \).** Let \( \Omega \) be a \( \mathcal{W}^{\beta,\phi} \) imbedding domain. For any continuous function \( u \in \mathcal{W}^{\beta,\phi}(\Omega) \), we have

\[
|u(x) - u(y)| \leq C \phi^{-1}\left(|x - y|^{\beta-n}\right) \|u\|_{\mathcal{W}^{\beta,\phi}(\Omega)} \quad \text{for almost all } x, y \in \Omega.
\]

Given any \( x \in \Omega \) and \( 0 < r < t < \text{diam} \, \Omega \), let \( u = u_{s,r,t} \) be as in as Lemma 5.1. Applying Lemma 5.1, we have

\[
|u(x) - u(y)| \leq C \phi^{-1}\left(|t-r|^{\beta-n}\right) \phi^{-1}\left(\frac{(t-r)^{\beta}}{|B_{\Omega}(x,t)|}\right)^{-1}.
\]

Without loss of generality, we may assume \( C \geq 1 \).

On the other hand, let \( r = t/2 \). For \( y \in B_{\Omega}(x, t + t/2) \setminus B_{\Omega}(x, t) \), we have \( |u(x) - u(y)| = 1 \). Thus

\[
\phi^{-1}\left(\frac{(t/2)^{\beta}}{|B_{\Omega}(x,t)|}\right) \leq C \phi^{-1}\left(|t/2|^{\beta-n}\right).
\]

By the doubling property of \( \phi \), we have

\[
\frac{(t/2)^{\beta}}{|B_{\Omega}(x,t)|} = \phi\left[\phi^{-1}\left(\frac{(t/2)^{\beta}}{|B_{\Omega}(x,t)|}\right)\right] \leq \phi\left[C \phi^{-1}\left(t/2^{\beta-n}\right)\right] \leq C K \phi\left[\phi^{-1}\left(t^{\beta-n}\right)\right] \leq C K t^{\beta-n},
\]

namely, \( t^n \leq 2^\beta C K |B_{\Omega}(x,t)| \) as desired.
Case $0 < \beta < n$. Given any $0 < t < \text{diam } \Omega$, let $b_0 = 1$ and $b_j \in (0, 1)$ for $j \in N$ such that

$$
|B(x, b_j t) \cap \Omega| = 2^{-j}|B(x, b_{j-1} t) \cap \Omega| = 2^{-j}|B(x, t) \cap \Omega|.
$$

For each $j \geq 1$, let $u_{x,b_{j+1} t, b_j t}$ as Lemma 5.1. Note that $u_{x,b_{j+1} t, b_j t} - c \geq \frac{1}{2}$ either in $B_\Omega(x, b_{j+1} t)$ or in $\Omega \setminus B_\Omega(x, b_j t)$. For $j \geq 1$, it implies $B_\Omega(x, b_{j-1} t) \setminus B_\Omega(x, b_j t) \subset \Omega \setminus B_\Omega(x, b_j t)$ and

$$
|\Omega \setminus B_\Omega(x, b_j t)| \geq |B_\Omega(x, b_{j-1} t) \setminus B_\Omega(x, b_j t)| = |B_\Omega(x, b_j t)| = 2|B_\Omega(x, b_{j+1} t)|.
$$

Hence

$$
\int_\Omega \phi_n^{\alpha/(\alpha-\beta)} \left( \frac{|u_{x,b_{j+1} t, b_j t}(z) - c|}{\lambda} \right) dz \geq \int_{B_\Omega(x, b_{j-1} t)} \phi_n^{\alpha/(\alpha-\beta)} \left( \frac{1}{2 \lambda} \right) dz \geq |B_\Omega(x, b_{j+1} t)| \phi_n^{\alpha/(\alpha-\beta)} \left( \frac{1}{2 \lambda} \right),
$$

that is, for $j \geq 1$,

$$
\inf_{c \in \mathbb{R}} \|u_{x,b_{j+1} t, b_j t} - c\|_{L^{p_0/(\alpha-\beta)}(\Omega)} \geq 2 \left[ \phi^{-1} \left( \frac{1}{|B_\Omega(x, b_{j+1} t)|^{1-\beta/n}} \right) \right]^{-1}.
$$

On the other hand, by (5.1) and Lemma 5.1 one has

$$
\inf_{c \in \mathbb{R}} \|u_{x,b_{j+1} t, b_j t} - c\|_{L^{p_0/(\alpha-\beta)}(\Omega)} \leq C \|u_{x,b_{j+1} t, b_j t}\|_{W^{\beta,n}(\Omega)} \leq C \left[ \phi^{-1} \left( \frac{(b_j t - b_{j+1} t)\beta}{|B_\Omega(x, b_{j+1} t)|} \right) \right]^{-1}.
$$

Thus we conclude that

$$
\phi^{-1} \left( \frac{(b_j t - b_{j+1} t)\beta}{|B_\Omega(x, b_{j+1} t)|} \right) \leq C \phi^{-1} \left( \frac{1}{|B_\Omega(x, b_{j+1} t)|^{1-\beta/n}} \right).
$$

Without loss of generality, we may assume $C \geq 1$. Applying Lemma 2.1, we know

$$
\frac{(b_j t - b_{j+1} t)\beta}{|B_\Omega(x, b_{j+1} t)|} = \phi \left[ \phi^{-1} \left( \frac{(b_j t - b_{j+1} t)\beta}{|B_\Omega(x, b_{j+1} t)|} \right) \right] \leq \phi \left[ C \phi^{-1} \left( \frac{1}{|B_\Omega(x, b_{j+1} t)|^{1-\beta/n}} \right) \right] \leq C^{-1} \phi^{-1} \left( \frac{1}{|B_\Omega(x, b_{j+1} t)|^{1-\beta/n}} \right).
$$

Therefore,

$$
(b_j t - b_{j+1} t)\beta \leq C^{-1} 2^{1-\beta/n} |B_\Omega(x, b_j t)|^{\beta/n} \leq C^{-1} 2^{1-\beta/n(j+1)} |B_\Omega(x, t)|^{\beta/n}.
$$

Since $b_j \to 0$ as $j \to \infty$, we have

$$
b_1 t = \sum_{j \geq 1} (b_j t - b_{j+1} t) \leq \sum_{j \geq 1} C^{-1} 2^{1-\beta/n(j+1)} |B_\Omega(x, t)|^{\beta/n} \leq |B_\Omega(x, t)|^{1/n}.
$$

Applying an argument similar to in [7], both $b_1 \geq 1/10$ and $b_1 \leq 1/10$ satisfy $|B_\Omega(x, t)| \geq C r^n$ as desired. \hfill \Box

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DEPARTMENT OF MATHEMATICS, BEIHANG UNIVERSITY, BEIJING 100191, P.R. CHINA

E-mail address: liangtian@buaa.edu.cn