Pion and kaon condensation at zero temperature in three-flavor $\chi$PT at nonzero isospin and strange chemical potentials at next-to-leading order

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ABSTRACT: We consider three-flavor chiral perturbation theory ($\chi$PT) at zero temperature and nonzero isospin ($\mu_I$) and strange ($\mu_S$) chemical potentials. The effective potential is calculated to next-to-leading order (NLO) in the $\pi^\pm$-condensed phase, the $K^\pm$-condensed phase, and the $K^0/\bar{K}^0$-condensed phase. It is shown that the transitions from the vacuum phase to these phases are second order and take place when, $|\mu_I| = m_\pi$, $|\frac{1}{2}\mu_I + \mu_S| = m_K$, and $| -\frac{1}{2}\mu_I + \mu_S| = m_K$, respectively at tree level and remains unchanged at NLO. The transition between the two condensed phases is first order. The effective potential in the pion-condensed phase is independent of $\mu_S$ and in the kaon-condensed phases, it only depends on the combinations $\pm\frac{1}{2}\mu_I + \mu_S$ and not separately on $\mu_I$ and $\mu_S$. We calculate the pressure, isospin density and the equation of state in the pion-condensed phase and compare our results with recent (2 + 1)-flavor lattice QCD data. We find that the three-flavor $\chi$PT results are in good agreement with lattice QCD for $\mu_I < 200$ MeV, however for larger values $\chi$PT produces values for observables that are consistently above lattice results. For $\mu_I > 200$ MeV, the two-flavor results are in better agreement with lattice data. Finally, we consider the observables in the limit of very heavy $s$-quark, where they reduce to their two-flavor counterparts with renormalized couplings. The disagreement between the predictions of two and three flavor $\chi$PT can largely be explained by the differences in the experimental values of the low-energy constants.

KEYWORDS: Chiral Lagrangians, Lattice QCD, Phase Diagram of QCD

ArXiv ePrint: 1909.10575

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1 Introduction

Quantum chromodynamics (QCD), the theory of the strong force, is challenging to study due to its non-perturbative nature and the inability to use lattice QCD simulations in the phenomenologically most interesting regime, namely finite baryon density, due to the infamous fermion sign problem [1, 2] present in classical Monte Carlo algorithms. As such, except for asymptotically large baryon chemical potentials, where QCD is expected to be in a color-flavor-locked phase [3, 4] and can be studied due to asymptotic freedom, most of the phenomenologically relevant QCD phase diagram must be mapped out by other methods, e.g. low-energy effective models.
Recently, there has been renewed interest in a slightly different regime of QCD, one with a finite isospin chemical potential due to the possibility of a new form of compact stars known as pion stars, first discussed in ref. [5]. This type of compact object could form in regions with large densities of neutrinos, which in turn leads to the production of pions and their subsequent condensation [6]. These pions under weak equilibrium lead to stable pion stars, which may be electromagnetically neutralized by either electrons or muons, or both. They are expected to have radii and masses that are substantially larger than those of neutron stars [7]. Pion stars are also different from neutron stars in the sense that at $T = 0$ it is interactions that give rise to an (effective) equation of state, and not the statistics of its constituents.

QCD at finite isospin chemical potential was first studied by Son and Stephanov using chiral perturbation theory ($\chi$PT) [8–12] in their seminal paper [13]. In refs. [5, 14–19] one can find various applications of $\chi$PT including some partial next-to-leading order results. Since then finite isospin systems have been studied extensively in other versions of QCD including two-color and adjoint QCD [20, 21], in the NJL [22–34], in the quark-meson model [35–38], but also through lattice QCD, where it does not suffer from the fermion sign problem (except at finite magnetic fields [39, 40] due to the charge asymmetry of the up and down quarks). The first lattice QCD calculations of finite isospin QCD were done in refs. [41, 42] and a more recent, thorough analysis in refs. [43–45]. They find as expected from chiral perturbation theory calculations that at zero temperature there is a second order phase transition at an isospin chemical potential, $|\mu_I| = m_\pi$, which remains largely unaltered at finite temperatures up to approximately 170 MeV beyond which quarks become deconfined [15]. Similarly, with increasing isospin chemical potentials the quarks in the pions become more loosely bound and occur in a BCS phase though owing to the fact that this phase has the same order parameter as the BEC phase, there is no real phase transition, only a crossover transition, with the size of the pion condensate decreasing substantially within a narrow isospin window.

There have been a number of studies in recent years comparing (2 + 1) flavor lattice QCD results with both QCD models and effective theories. Recently, the NJL model (non-renormalizable) comparisons [33] were made that showed good agreement with the lattice while the quark-meson model [38] (which is renormalizable) largely agrees with the lattice. Furthermore, there have been other comparisons of lattice QCD with results from an effective field theory (and model-independent) description [18], which is valid for asymptotically large isospin chemical potentials [16], where the pions behave as a free Bose gas. A recent review can be found in ref. [46].

The focus of this work is to compare the results of three-flavor $\chi$PT at finite isospin density [47] with that of (2 + 1)-flavor lattice QCD of refs. [43–45]. We previously studied two-flavor $\chi$PT at next-to-leading order (NLO) [48] and found that the NLO results are in better agreement with lattice QCD than the tree-level results though the pressure, isospin density and energy density were all found to be consistently smaller than lattice QCD values. This is not entirely unexpected since the lattice QCD observables included the

\[^1\]The $|\mu_I| = \frac{1}{2}m_\pi$ conventions is also frequently found in the literature. See eq. (2.9).
effects of the sea strange quarks [49] while two-flavor χPT does not. As such, we extend our previous work in NLO two-flavor χPT to include the effect of the strange quarks by using three-flavor χPT at finite isospin chemical potential and find that the observables near the second phase transition is in good agreement with lattice QCD. As a natural extension of our finite isospin study, we also construct the NLO, one-loop effective potential to study the effects of the simultaneous presence of both the isospin and strange quark chemical potential.\footnote{Note that the “strange quark chemical potential” ($\mu_s$) is different from the “strange chemical potential” ($\mu_S$). We define them in eq. (2.10).} We find the second-order phase transition in the pion condensed phase remains at $|\mu_I| = m_\pi$ even with the inclusion of $\mu_S$ and NLO corrections.\footnote{This property is expected to hold to all orders in perturbation theory.} Similarly, the second order phase transition in the kaon condensed phases remains at $|\pm \frac{1}{2}\mu_I + \mu_S| = m_K$ where $m_K$ is the kaon mass. Furthermore the effective potential even in the presence of $\mu_S$ in the pion condensed phase only depends on $\mu_I$ and in the kaon condensed phases on the combination $|\pm \frac{1}{2}\mu_I + \mu_S|$ but not $\mu_I$ and $\mu_S$ separately.

The paper is organized as follows. In the next section, we discuss the Lagrangian of three-flavor chiral perturbation theory at finite isospin and strange chemical potentials at next-to-leading order in the low-energy expansion. In section 3, we review the ground state of the theory and fluctuations in the different phases. In section 4 the NLO effective potential in the three different phases of the theory is calculated. In section 5, we derive the pressure, isospin density, and equation of state in the pion-condensed phase. We also consider the large-$m_s$ limit, where it is shown that the observables in three-flavor χPT reduce to the two-flavor observables of ref. [48] with renormalized couplings. In section 6, we discuss the phase diagram in more detail and derive medium-dependent masses at tree level. We compare our results for the thermodynamic functions with recent lattice simulations.

## 2 χPT Lagrangian at $\mathcal{O}(p^4)$

In this section, we briefly discuss the symmetries of three-flavor QCD as well the chiral Lagrangian to next-to-leading order in the low-energy expansion and its renormalization. The three-flavor Lagrangian of QCD is

$$
\mathcal{L} = \bar{\psi} (i\slash{D} - m) \psi - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu},
$$

where $m = \text{diag}(m_u, m_d, m_s)$ is the quark mass matrix, $\slash{D} = \gamma^\mu \partial_\mu - ig \lambda^a A_\mu^a$ is the covariant derivative, $\lambda^a$ are the Gell-Mann matrices, $g$ is the strong coupling, $A_\mu^a$ is the gauge field, and $F_{\mu\nu}^a$ is the field-strength tensor. The global symmetry of massless three-flavor QCD is $SU(3)_L \times SU(3)_R \times U(1)_B$, which is spontaneously broken down to $SU(3)_V \times U(1)_B$ in the vacuum. For two degenerate light quarks, i.e. in the isospin limit the symmetry is $SU(2)_I \times U(1)_Y \times U(1)_B$, where $Y$ represents hypercharge. If $m_u \neq m_d$, this symmetry is reduced to $U(1)_{I_3} \times U(1)_Y \times U(1)_B$. If we add a chemical potential for each of the quarks, the symmetry is $U(1)_{I_3} \times U(1)_Y \times U(1)_B$, irrespective of the quark masses.

In the present paper, we consider three-flavor QCD with two degenerate light quarks. The chiral Lagrangian then describes the octet of pseudo-Goldstone bosons consisting of
the three pions ($\pi^\pm$ and $\pi^0$), the four kaons ($K^\pm$, $K^0$ and $\bar{K}^0$), and the eta ($\eta$). We begin with the chiral perturbation theory Lagrangian at $O(p^2)$ \[9\]

\[ \mathcal{L}_2 = \frac{f^2}{4} \text{Tr} \left[ \nabla_\mu \Sigma^\dagger \nabla^\mu \Sigma \right] + \frac{f^2}{4} \text{Tr} \left[ \chi^\dagger \Sigma + \chi \Sigma^\dagger \right], \]

where $f$ is the bare pion decay constant, $\chi = 2B_0M,$ and

\[ M = \text{diag}(m_u, m_d, m_s) \]

is the quark mass matrix, $\Sigma = U\Sigma_0 U^\dagger,$ where $U = \exp^{i\frac{\lambda_i \phi_i}{2f}},$ and $\Sigma = 1$ is the vacuum. Moreover, $\lambda_i$ ($i = 1, 2, \ldots, 8$) are the Gell-Mann matrices that satisfy $\text{Tr} \lambda_i \lambda_j = 2\delta_{ij}$ and $\phi_i$ are the fields that parametrize the Goldstone manifold. The covariant derivative at nonzero quark chemical potentials ($q = u, d, s$) is defined as follows

\[ \nabla_\mu \Sigma \equiv \partial_\mu \Sigma - i [v_\mu, \Sigma], \]

\[ \nabla_\mu \Sigma^\dagger \equiv \partial_\mu \Sigma^\dagger - i [v_\mu, \Sigma^\dagger], \]

where

\[ v_\mu = \delta_{\mu 0} \text{diag}(m_u, m_d, m_s), \]

We can also express $v_\mu$ in terms of the baryon, isospin and strangeness chemical potentials $\mu_B, \mu_I,$ and $\mu_S$ as

\[ v_\mu = \delta_{\mu 0} \text{diag} \left( \frac{1}{3} \mu_B + \frac{1}{2} \mu_I, \frac{1}{3} \mu_B - \frac{1}{2} \mu_I, \frac{1}{3} \mu_B - \mu_S \right). \]

where

\[ \mu_B = \frac{3}{2} (\mu_u + \mu_d), \]

\[ \mu_I = \mu_u - \mu_d, \]

\[ \mu_S = \frac{1}{2} (\mu_u + \mu_d - 2\mu_s). \]

This yields

\[ v_0 = \frac{1}{3} (\mu_B - \mu_S) \frac{1}{3} + \frac{1}{2} \mu_I \lambda_3 + \frac{1}{\sqrt{3}} \mu_S \lambda_8. \]

We note that the $\mu_B$-dependent term in eq. (2.11) commutes with $\Sigma$ and $\Sigma^\dagger$ in eqs. (2.4)–(2.5) and so the baryon chemical potential drops completely out of the chiral Lagrangian. This reflects the fact that we have only included the mesonic octet, which has zero baryonic charge. We therefore set $\mu_B = 0$ in the remainder of the paper.

2.1 Next-to-leading order Lagrangian

In order to perform calculations beyond tree level, we must go to next-to-leading order in the low-energy expansion and consider the terms that contribute to $\mathcal{L}$ at $O(p^4)$. There are \footnote{One factor of $\nabla_\mu$ counts one power of $p$ and one factor of $\chi$ counts two powers of $p$.}
twelve operators in $L_4$ \[10\], but only eight of them are relevant for the present calculations. They are

$$L_4 = L_1 \left( \text{Tr} \left[ \nabla_\mu \Sigma^\dagger \nabla_\mu \Sigma \right] \right)^2 + L_2 \text{Tr} \left[ \nabla_\mu \Sigma^\dagger \nabla_\mu \Sigma \right] \text{Tr} \left[ \nabla_\nu \Sigma^\dagger \nabla_\nu \Sigma \right] \right. $$

$$+ L_3 \text{Tr} \left[ \left( \nabla_\mu \Sigma^\dagger \nabla_\mu \Sigma \right) \left( \nabla_\nu \Sigma^\dagger \nabla_\nu \Sigma \right) \right] + L_4 \text{Tr} \left[ \nabla_\mu \Sigma^\dagger \nabla_\mu \Sigma \right] \text{Tr} \left( \chi^\dagger \Sigma + \chi \Sigma^\dagger \right)$$

$$+ L_5 \text{Tr} \left[ \left( \nabla_\mu \Sigma^\dagger \nabla_\mu \Sigma \right) \left( \chi^\dagger \Sigma + \chi \Sigma^\dagger \right) \right] + L_6 \left( \text{Tr} \left[ \chi^\dagger \Sigma + \chi \Sigma^\dagger \right] \right)^2$$

$$+ L_8 \text{Tr} \left[ \chi^\dagger \Sigma \chi^\dagger + \chi \Sigma^\dagger \chi \right] + H_2 \text{Tr} \left[ \chi^\dagger \chi \right].$$

(2.12)

where $L_i$ and $H_i$ are unrenormalized couplings. The relations between the bare and renormalized couplings $L_i^r(\Lambda)$ and $H_i^r(\Lambda)$ are

$$L_i = L_i^r(\Lambda) + \Gamma_i \lambda,$$

$$H_i = H_i^r(\Lambda) + \Delta_i \lambda,$$

(2.13)

(2.14)

where

$$\lambda = -\frac{\Lambda^{-2\epsilon}}{2(4\pi)^2} \left[ \frac{1}{\epsilon} + 1 \right].$$

(2.15)

Here $\Gamma_i$ and $\Delta_i$ are constants and $\Lambda$ is the renormalization scale in the modified minimal subtraction scheme $\overline{\text{MS}}$. The renormalized couplings satisfy the renormalization group equations

$$\Lambda \frac{d}{d\Lambda} L_i^r = -\frac{\Gamma_i}{(4\pi)^2}, \quad \Lambda \frac{d}{d\Lambda} H_i^r = -\frac{\Delta_i}{(4\pi)^2}.$$

(2.16)

These are obtained by differentiation of eqs. (2.13)–(2.14) noting that the bare parameters are independent of the scale $\Lambda$. The solutions are

$$L_i^r(\Lambda) = L_i^r(\Lambda_0) - \frac{\Gamma_i}{2(4\pi)^2} \log \frac{\Lambda^2}{\Lambda_0^2}, \quad H_i^r(\Lambda) = H_i^r(\Lambda_0) - \frac{\Delta_i}{2(4\pi)^2} \log \frac{\Lambda^2}{\Lambda_0^2},$$

(2.17)

where $\Lambda_0$ is a reference scale. We note that the contact term $H_2 \text{Tr}[\chi^\dagger \chi]$ gives a constant contribution to the effective potential which is the same in all phases. We keep it, however, since it is needed to show the scale independence of the final result for the effective potential.

In three-flavor QCD, the constants $\Gamma_i$ and $\Delta_i$ are

$$\Gamma_1 = \frac{3}{32}, \quad \Gamma_2 = \frac{3}{16}, \quad \Gamma_3 = 0, \quad \Gamma_4 = \frac{1}{8}, \quad \Gamma_5 = \frac{3}{8}, \quad \Gamma_6 = \frac{11}{144}, \quad \Gamma_8 = \frac{5}{48}, \quad \Delta_2 = \frac{5}{24}.$$

(2.18)

(2.19)

3 Ground state and fluctuations

In this section, we will discuss the phase structure of the theory as a function of the chemical potentials $\mu_1$ and $\mu_2$. We will also discuss how to parametrize the fluctuations above the ground state.

The most general SU(3) matrix for the ground state can be written as

$$\Sigma_\alpha = e^{i \alpha \delta \lambda},$$

(3.1)
where $\alpha$ is a rotation angle, $\hat{\phi}_i$ are variational parameters and a sum over the repeated index $i$ is implied. In order to ensure the normalization of the ground state, $\Sigma_\alpha \Sigma_\alpha^\dagger = 1$, the coefficients must satisfy $\sum_i \hat{\phi}_i^2 = 1$. However, depending on the chemical potentials, we expect that the ground state takes a certain form, i.e. that it is rotated in a specific way. For example, in the case $\mu_S = 0$, we expect pion condensation for $|\mu_I| > m_\pi$ [13] and that the two-flavor results carry over. We therefore briefly review the two-flavor case first.

Here the ground state can be written as
\begin{equation}
\Sigma_\alpha = e^{i\alpha \hat{\phi}_i \tau_i} = \cos \alpha + i \hat{\phi}_i \tau_i \sin \alpha ,
\end{equation}
where $\tau_i$ are the Pauli matrices and $\hat{\phi}_i$ are again variational parameters. The static part of the $\mathcal{O}(g^2)$ Hamiltonian $\mathcal{H}_2$ reads
\begin{equation}
\mathcal{H}_2^{\text{static}} = \frac{f^2}{4} \text{Tr}[v_0, \Sigma_\alpha][v_0, \Sigma_\alpha^\dagger] - \frac{f^2}{2} B_0 \text{Tr}[M \Sigma_\alpha + M \Sigma_\alpha^\dagger] ,
\end{equation}
where in the two-flavor case $v_0 = \frac{1}{2} \tau_3 \mu_I$, cf. eq. (2.11) and $M = \text{diag}(m_u, m_d) = \text{diag}(m, m)$. The first term in eq. (3.3) can be written as
\begin{equation}
\mathcal{H}_2^{\text{static}(a)} = \frac{f^2}{4} \text{Tr}[v_0, \Sigma_\alpha][v_0, \Sigma_\alpha^\dagger] = \frac{f^2}{8} \mu_I^2 \sin^2 \alpha(\hat{\phi}_1^2 + \hat{\phi}_2^2) .
\end{equation}
This form suggests that $\mathcal{H}_2^{\text{static}(a)}$ favors directions that anticommute with $\tau_3$ [13]. Substituting eq. (3.2) into eq. (3.4), this expectation is made explicit, $\mathcal{H}_2^{\text{static}(a)} = -\frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha(\hat{\phi}_1^2 + \hat{\phi}_2^2)$. Evaluating the other term in eq. (3.3), we find
\begin{equation}
\mathcal{H}_2^{\text{static}} = 2 f^2 B_0 m \cos \alpha - \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha(\hat{\phi}_1^2 + \hat{\phi}_2^2) .
\end{equation}
The first term favors $\alpha = 0$, i.e. the vacuum state $\Sigma_0 = 1$, and it is clear that there is competition between the two terms in eq. (3.5). We notice that the static energy only depends on $\hat{\phi}_1^2 + \hat{\phi}_2^2$, and it is minimized by setting $\hat{\phi}_3 = 0$. Without loss of generality and for later convenience, we can choose $\hat{\phi}_1 = 1$ and $\hat{\phi}_2 = 0$. The rotated vacuum eq. (3.2) can then be written as
\begin{equation}
\Sigma_\alpha = A_\alpha \Sigma_0 A_\alpha^\dagger ,
\end{equation}
where
\begin{equation}
A_\alpha = e^{i\frac{\alpha}{2} \tau_1} = \cos \frac{\alpha}{2} + i \tau_1 \sin \frac{\alpha}{2} .
\end{equation}
Minimizing eq. (3.5) with respect to $\alpha$, we find two phases, $\alpha = 0$ for $2 B_0 m < \mu_I^2$ and $\cos \alpha = \frac{2 B_0 m}{\mu_I^2}$ for $2 B_0 m > \mu_I^2$. The first phase is the vacuum phase and the second phase consists of a condensate of charged pions.

In analogy with the two-flavor case, we expect that pion condensation in the three-flavor case can be captured by writing eq. (3.1) as
\begin{equation}
\Sigma_\alpha^\pm = A_\alpha \Sigma_0 A_\alpha^\dagger ,
\end{equation}
\footnote{$\lambda_1$ plays the role of $\tau_1$ and $\lambda_2$ that of $\tau_2$. We are free to choose any linear combination of the two and we choose $\lambda_2$.}
\begin{align*}
\Sigma_\alpha^+ &= A_\alpha \Sigma_0 A_\alpha^\dagger ,
\Sigma_\alpha^- &= A_\alpha \Sigma_0 A_\alpha^\dagger ,
\end{align*}
where
\[
A_\alpha = e^{i\frac{\alpha}{2} \lambda_2} = \frac{1 + 2 \cos \frac{\alpha}{2}}{3} \mathbb{1} + i \lambda_2 \sin \frac{\alpha}{2} + \frac{\cos \frac{\alpha}{2} - 1}{\sqrt{3}} \lambda_8.
\]  
(3.9)

The rotated ground state can also be conveniently written as
\[
\Sigma_\alpha^\pm = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1
\end{pmatrix},
\]  
(3.10)

which shows that the rotation does not affect the $s$-quark. The symmetry breaking pattern
in this case is
\[
U(1)_{I_3} \times U(1)_Y \times U(1)_B \to U(1)_Y \times U(1)_B.
\]  
(3.11)

Since $U(1)_Q \not\subset U(1)_Y \times U(1)_B$, electric charge $Q$ is also broken and the system is both a
superfluid and a superconductor.

We next consider kaon condensation in three-flavor $\chi$PT. Depending on the values of $\mu_1$ and $\mu_s$, either the charged kaons or neutral kaons condense. If $|\frac{1}{2} \mu_1 + \mu_s| = |\mu_u - \mu_s| > m_K$, we expect either $K^+$ or $K^-$ to condense depending on the sign. If $|\frac{1}{2} \mu_1 + \mu_s| = |\mu_d - \mu_s| > m_K$, we expect $K^0$ or $\bar{K}^0$ to condense depending on the sign. In the case of charged kaon condensation, $\lambda_4$ and $\lambda_5$ replace $\lambda_1$ and $\lambda_2$, respectively, and without loss of generality we can write $\Sigma^K_\alpha = e^{i\frac{\alpha}{2} \lambda_5} \Sigma_0 e^{i\frac{\alpha}{2} \lambda_5}$. The rotated ground state takes the form
\[
\Sigma^K_\alpha = \frac{1 + 2 \cos \alpha}{3} \mathbb{1} + \frac{\cos \alpha - 1}{2\sqrt{3}} \left(\sqrt{3} \lambda_3 - \lambda_8\right) + i \lambda_5 \sin \alpha
\]
\[
= \begin{pmatrix}
\cos \alpha & 0 & \sin \alpha \\
0 & 1 & 0 \\
-\sin \alpha & 0 & \cos \alpha
\end{pmatrix}.
\]  
(3.12)

The symmetry-breaking pattern is
\[
U(1)_{I_3} \times U(1)_Y \times U(1)_B \to U(1)_Y \times U(1)_B.
\]  
(3.13)

Again, since $U(1)_Q \not\subset U(1)_Y \times U(1)_B$, electric charge is spontaneously broken and the
superfluid is also a superconductor.

Finally, in the case of neutral kaon condensation the rotated ground state is $\Sigma^K_0/\bar{K}^0 = e^{i\frac{\alpha}{2} \lambda_7} \Sigma_0 e^{i\frac{\alpha}{2} \lambda_7}$, or
\[
\Sigma^K_0/\bar{K}^0 = \frac{1 + 2 \cos \alpha}{3} \mathbb{1} + \frac{1 - \cos \alpha}{2\sqrt{3}} \left(\sqrt{3} \lambda_3 + \lambda_8\right) + i \lambda_7 \sin \alpha
\]
\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & \sin \alpha \\
0 & -\sin \alpha & \cos \alpha
\end{pmatrix}.
\]  
(3.14)

The symmetry-breaking pattern is now
\[
U(1)_{I_3} \times U(1)_Y \times U(1)_B \to U(1)_Q \times U(1)_B.
\]  
(3.15)

implying that the superfluid is not a superconductor.
While we have considered the possibility of a single species condensing, in principle it is possible for the ground state to have simultaneous condensation of multiple mesons. However, explicit calculations in refs. [46, 47] that include the possibility of multiple rotations into multiple condensed phases show that such phases are not the global minima except on the first order transition line. We discuss the line at the end of this section.

We now return to the evaluation of the static Hamiltonian \( \mathcal{H}_2 \). In the case of pion condensation, the static Hamiltonian reduces to

\[
\mathcal{H}_2 = -2f^2B_0m\cos\alpha - f^2B_0m_s - \frac{1}{2}f^2\mu^2_1 \sin^2\alpha .
\]

The minimum of the static Hamiltonian is

\[
\cos\alpha = 1 , \quad \mu^2_1 < 2B_0m
\]

\[
\cos\alpha = \frac{2B_0m}{\mu^2_1} , \quad \mu^2_1 > 2B_0m.
\]

The ground-state energy in the vacuum and pion-condensed phase is

\[
\mathcal{H}_2 = -f^2B_0(2m + m_s) , \quad \mu^2_1 < 2B_0m ,
\]

\[
\mathcal{H}_2 = -\left(\frac{2fB_0m}{\mu^2_1}\right)^2 - f^2B_0m_s - \frac{1}{2}f^2\mu^2_1 \left(1 - \frac{(2B_0m)^2}{\mu^2_1}\right) , \quad \mu^2_1 > 2B_0m .
\]

In the case of charged kaon condensation, the static Hamiltonian reduces to

\[
\mathcal{H}_2 = -f^2B_0m(1 + \cos\alpha) - f^2B_0m_s \cos\alpha - \frac{1}{2}f^2\left(\frac{1}{2}\mu_1 + \mu_s\right)^2 \sin^2\alpha .
\]

The minimum of the static Hamiltonian is

\[
\cos\alpha = 1 , \quad \left(\frac{1}{2}\mu_1 + \mu_s\right)^2 < B_0(m + m_s)
\]

\[
\cos\alpha = \frac{B_0(m + m_s)}{\left(\frac{1}{2}\mu_1 + \mu_s\right)^2} , \quad \left(\frac{1}{2}\mu_1 + \mu_s\right)^2 > B_0(m + m_s).
\]

The ground-state energy in the vacuum and the charged kaon-condensed phase is

\[
\mathcal{H}_2 = -f^2B_0(2m + m_s) , \quad \left(\frac{1}{2}\mu_1 + \mu_s\right)^2 < B_0(m + m_s) ,
\]

\[
\mathcal{H}_2 = -f^2B_0m - \frac{f^2B_0^2(m + m_s)^2}{\left(\frac{1}{2}\mu_1 + \mu_s\right)^2} - \frac{1}{2}f^2\left(\frac{1}{2}\mu_1 + \mu_s\right)^2 \left(1 - \frac{B_0^2(m + m_s)^2}{\left(\frac{1}{2}\mu_1 + \mu_s\right)^4}\right) ,
\]

\[
\left(\frac{1}{2}\mu_1 + \mu_s\right)^2 > B_0(m + m_s) .
\]

Finally, we consider the case of condensation of neutral kaons. The results for this phase can be obtained from the results of the phase of condensed charged kaons by the substitution \( \mu_I \to -\mu_I \) since \( -\frac{1}{2}\mu_I + \mu_S = \mu_d - \mu_s \). In order to find the global minimum, we must compare eqs. (3.20) and (3.25) in the region \( |\mu_I| > m_s \) and \( |\frac{1}{2}\mu_1 + \mu_s| > m_K \). The boundary
between the pion-condensed phase and the kaon-condensed phase is then given by equating these expressions. This yields

\[ - \frac{(\mu_I^2 - m^2_\pi)^2}{2\mu_I^2} = - \frac{[m^2_K - (\frac{1}{2}\mu_I + \mu_S)^2]^2}{2(\frac{1}{2}\mu_I + \mu_S)^2}, \quad (3.26) \]

or

\[ \pm \frac{1}{2}\mu_I + \mu_S = \frac{\mu_I^2 - m^2_\pi + \sqrt{(\mu_I^2 - m^2_\pi)^2 + 4\mu_I^2m^2_K}}{2\mu_I}, \quad (3.27) \]

where we used the tree-level relations \( m^2_\pi = 2B_0m \) and \( m^2_K = B_0(m + m_s) \). We will return to the phase diagram in the \( \mu_I - \mu_S \) plane in section 6.1.

### 3.1 Parametrizing fluctuations

Since we want to study the thermodynamics of the pion-condensed and kaon-condensed phases including leading-order quantum corrections, it is natural to expand the chiral perturbation theory Lagrangian around the relevant ground state. The Goldstone manifold as a consequence of chiral symmetry breaking is \( \text{SU(3)}_L \times \text{SU(3)}_R/\text{SU(3)}_V \). We will focus on the pion-condensed phase for simplicity. The remarks below also apply to the kaon-condensed phases. Following refs. [21, 48], we write

\[ \Sigma = L_\alpha \Sigma_\alpha R_\alpha^\dagger, \quad (3.28) \]

with

\[ L_\alpha = A_\alpha U A_\alpha^\dagger, \quad (3.29) \]
\[ R_\alpha = A_\alpha^\dagger U^\dagger A_\alpha. \quad (3.30) \]

We emphasize that the fluctuations parameterized by \( L_\alpha \) and \( R_\alpha \) around the ground state depend on \( \alpha \) since the broken generators (of QCD) need to be rotated appropriately as the condensed vacuum rotates with the angle \( \alpha \) [21]. In the present case, \( U \) is an \( \text{SU(3)} \) matrix that parameterizes the fluctuations around the vacuum,

\[ U = \exp \left( \frac{i\phi_\alpha \lambda_\alpha}{2f} \right). \quad (3.31) \]

With the parameterizations stated above, we get

\[ \Sigma = A_\alpha (U \Sigma_\alpha U) A_\alpha. \quad (3.32) \]

This parameterization not only produces the correct linear terms that vanish when evaluated at the minimum of the static Hamiltonian \( \mathcal{O}(p^2) \), the divergences of the one-loop vacuum diagrams also cancel using counterterms from the \( \mathcal{O}(p^4) \) Lagrangian. Furthermore, the parametrization produces a Lagrangian that is canonical in the fluctuations and has the correct limit when \( \alpha = 0 \), whereby

\[ \Sigma = U \Sigma_0 U = U^2 = \exp \left( \frac{i\phi_\alpha \lambda_\alpha}{f} \right), \quad (3.33) \]
as expected. If one expands the Lagrangian using the parametrization \( \Sigma = L \Sigma_R U = U \Sigma_A \Sigma_0 A \Sigma U \) instead of eq. \((3.28)\), the kinetic terms of the Lagrangian are non-canonical. By a field redefinition that depends on the chemical potentials, these terms can be made canonical. However, calculating the leading corrections to the tree-level potential, it can be shown that the ultraviolet divergences can be eliminated by renormalization only at the minimum of the classical potential.\(^6\) Thus one cannot find the minimum of the next-to-leading order effective potential as a function of \( \alpha \), showing that this parametrization is erroneous. Let us finally take a look at the rotated generators. To linear order in the \( \phi_i \), an infinitesimal fluctuation can be written as

\[
L_\alpha = \begin{pmatrix}
\cos \frac{\phi}{2} & \sin \frac{\phi}{2} & 0 \\
-\sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\cos \frac{\phi}{2} - \sin \frac{\phi}{2} & 0 & 0 \\
\sin \frac{\phi}{2} & \cos \frac{\phi}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(3.34)

Using the \((\text{anti})\)commutator relations of the Gell-Mann matrices, eq. \((3.34)\) takes the form

\[
L_\alpha = 1 + \frac{i \phi_1}{2 f} (\cos \alpha \lambda_1 + \sin \alpha \lambda_3) + \frac{i \phi_2}{2 f} \lambda_2 + \frac{i \phi_3}{2 f} (\cos \alpha \lambda_3 - \sin \alpha \lambda_1) \\
+ \frac{i \phi_4}{2 f} (\cos \alpha \lambda_4 - \sin \alpha \lambda_6) + \frac{i \phi_5}{2 f} (\cos \alpha \lambda_5 - \sin \alpha \lambda_7) + \frac{i \phi_6}{2 f} (\cos \alpha \lambda_6 + \sin \alpha \lambda_4) \\
+ \frac{i \phi_7}{2 f} (\cos \alpha \lambda_7 + \sin \alpha \lambda_5) + \frac{i \phi_8}{2 f} \lambda_8.
\]

(3.35)

The linear combinations \( \lambda'_i = (\cos \alpha \lambda_1 + \sin \alpha \lambda_3) \), \( \lambda'_2 = \lambda_2 \), etc can be thought of as rotated generators, some of them, however, only by half the angle. The rotated generators \( \lambda'_i \) satisfy the same \((\text{anti})\)commutation relations as do \( \lambda_i \). To all orders in \( \alpha \), we then have

\[
L_\alpha = \exp \left( \frac{i \phi_i \lambda'_i}{2 f} \right).
\]

(3.36)

### 3.2 Leading-order Lagrangian

Using the parameterization eq. \((3.32)\) discussed above, we can write down the Lagrangian in terms of the fields \( \phi_a \), which parametrizes the Goldstone manifold. The leading-order terms in the low-energy expansion are given by \( \mathcal{L}_2 \), which can be expanded as a power series in the fields

\[
\mathcal{L}_2 = \mathcal{L}_2^{\text{linear}} + \mathcal{L}_2^{\text{static}} + \mathcal{L}_2^{\text{quadratic}} + \cdots
\]

(3.37)

where the ellipses indicate terms that are cubic or higher order in the fields. We will carry out the expansion for the normal phase, the pion-condensed phase, and the charged kaon-condensed phase. Similar results can be obtained for the neutral kaon-condensed phase.

---

\(^6\) Renormalization of the effective potential is carried out by renormalizing the low-energy constants in the NLO static Lagrangian, see section 4.
3.2.1 Normal phase

In the normal phase, the different terms in eq. (3.37) are

\[ L^{\text{static}}_2 = f^2 B_0 (2m + m_s), \]
\[ L^{\text{linear}}_2 = 0, \]
\[ L^{\text{quadratic}}_2 = \frac{1}{2} \partial_\mu \phi_4 \partial^\mu \phi_4 - \frac{1}{2} (2B_0 m - \mu_I^2) (\phi_1^2 + \phi_2^2) - \frac{1}{2} (2B_0 m) \phi_3^2 \]
\[ - \frac{1}{2} \left[ B_0 (m + m_s) - \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \right] (\phi_4^2 + \phi_5^2 + \phi_6^2 + \phi_7^2) \]
\[ - \frac{B_0 (m + 2m_s)}{3} \phi_8^2 + \mu_I (\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) + \left( \frac{1}{2} \mu_I + \mu_S \right) (\phi_4 \partial_0 \phi_5 - \phi_5 \partial_0 \phi_4) \]
\[ + \left( -\frac{1}{2} \mu_I + \mu_S \right) (\phi_6 \partial_0 \phi_7 - \phi_7 \partial_0 \phi_6). \]

The inverse propagator is block diagonal and can be written as

\[ D^{-1} = \begin{pmatrix}
D^{-1}_{12} & 0 & 0 & 0 & 0 \\
0 & P^2 - m_3^2 & 0 & 0 & 0 \\
0 & 0 & D^{-1}_{45} & 0 & 0 \\
0 & 0 & 0 & D^{-1}_{67} & 0 \\
0 & 0 & 0 & 0 & P^2 - m_8^2
\end{pmatrix}, \]
\[ m_3^2 = 2B_0 m, \]
\[ m_8^2 = \frac{2B_0 (m + 2m_s)}{3}, \]

where \( P = (p_0, p) \) is the four-momentum and \( P^2 = p_0^2 - p^2 \). The submatrices are

\[ D^{-1}_{12} = \begin{pmatrix}
P^2 - m_1^2 & ip_0 m_{12} \\
-ip_0 m_{12} & P^2 - m_2^2
\end{pmatrix}, \]
\[ D^{-1}_{45} = \begin{pmatrix}
P^2 - m_4^2 & ip_0 m_{45} \\
-ip_0 m_{45} & P^2 - m_5^2
\end{pmatrix}, \]
\[ D^{-1}_{67} = \begin{pmatrix}
P^2 - m_6^2 & ip_0 m_{67} \\
-ip_0 m_{67} & P^2 - m_7^2
\end{pmatrix}. \]

The masses are

\[ m_1^2 = 2B_0 m - \mu_I^2, \]
\[ m_2^2 = m_1^2, \]
\[ m_{12} = 2\mu_I, \]
\[ m_4^2 = B_0 (m + m_s) - \left( \frac{1}{2} \mu_I + \mu_S \right)^2, \]
\[ m_5^2 = m_4^2, \]
\[ m_{45} = \mu_I + 2\mu_S, \]
\[ m_6^2 = m_4^2, \]
\[ m_7^2 = m_4^2, \]
\[ m_{67} = -\mu_I + 2\mu_S. \]
We get for the inverse propagator:

\[
D^{-1} = \begin{pmatrix}
    D_{12}^{-1} & 0 & 0 & 0 & 0 \\
    0 & p^2 - m_3^2 & 0 & 0 & 0 \\
    0 & 0 & D_{45}^{-1} & 0 & 0 \\
    0 & 0 & 0 & D_{67}^{-1} & 0 \\
    0 & 0 & 0 & 0 & p^2 - m_8^2 \\
\end{pmatrix},
\]

\[
m_3^2 = 2B_0 m \cos \alpha + \mu_I^2 \sin^2 \alpha, \quad (3.64)
\]

\[
m_8^2 = \frac{2B_0(m \cos \alpha + 2m_s)}{3}. \quad (3.65)
\]

The dispersion relations for the charges mesons are

\[
E_{\pi^\pm} = \sqrt{p^2 + 2B_0 m} \pm \mu_I = \sqrt{p^2 + m_{\pi,0}^2} \pm \mu_I, \quad (3.55)
\]

\[
E_{\pi^0} = \sqrt{p^2 + 2B_0 m} = \sqrt{p^2 + m_{\pi,0}^2}, \quad (3.56)
\]

\[
E_{K^\pm} = \sqrt{p^2 + B_0(m + m_s) + \left(\frac{1}{2} \mu_I + \mu_S\right)} = \sqrt{p^2 + m_{K,0}^2} \pm \left(\frac{1}{2} \mu_I + \mu_S\right), \quad (3.57)
\]

\[
E_{K^0} = \sqrt{p^2 + B_0(m + m_s) - \left(\frac{1}{2} \mu_I + \mu_S\right)} = \sqrt{p^2 + m_{K,0}^2} \pm \left(-\frac{1}{2} \mu_I + \mu_S\right), \quad (3.58)
\]

\[
E_\eta = \sqrt{p^2 + \frac{2}{3} B_0(m + 2m_s)} = \sqrt{p^2 + m_{\eta,0}^2}. \quad (3.59)
\]

The tree-level masses of the pions, kaons, and the \( \eta \) are then given by \( m_{\pi,0}^2 = 2B_0 m \), \( m_{K,0}^2 = B_0(m + m_s) \), and \( m_{\eta,0}^2 = \frac{2}{3} B_0(m + 2m_s) \).

### 3.2.2 Pion-condensed phase

In the pion-condensed phase, the different terms in eq. (3.37) are

\[
L_2^{\text{linear}} = f = f(-2B_0 m + \mu_I^2 \cos \alpha) \sin \alpha \phi_2 - f \mu_I \sin \alpha \partial_0 \phi_1, \quad (3.61)
\]

\[
L_2^{\text{quadratic}} = \frac{1}{2} \partial_\mu \phi_\alpha \partial^\mu \phi_\alpha - \frac{1}{2} (2B_0 m \cos \alpha - \mu_I^2 \cos^2 \alpha) \phi_1^2 - \frac{1}{2} (2B_0 m \cos \alpha - \mu_I^2 \cos^2 \alpha) \phi_2^2 - \frac{1}{2} (2B_0 m \cos \alpha + \mu_I^2 \sin^2 \alpha) \phi_3^2 - \frac{1}{2} \left(B_0(m \cos \alpha + m_s) - \frac{1}{4} \mu_I^2 \cos 2\alpha - \mu_I \mu_S \cos \alpha - \mu_S^2 \right) \left(\phi_4^2 + \phi_5^2\right) - \frac{1}{2} \left[B_0(m \cos \alpha + m_s) - \frac{1}{4} \mu_I^2 \cos 2\alpha + \mu_I \mu_S \cos \alpha - \mu_S^2 \right] \left(\phi_6^2 + \phi_7^2\right) - \frac{B_0(m \cos \alpha + 2m_s)}{3} \phi_8^2 + \mu_I \cos \phi_8 (\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1) + \left(\frac{1}{2} \mu_I \cos \alpha + \mu_S\right) (\phi_9 \partial_0 \phi_9 - \phi_9 \partial_0 \phi_9) \gtrless \quad (3.62)
\]

We get for the inverse propagator:
In the kaon-condensed phase, the different terms in eq. (3.37) are

\[ E_{\pi^0} = p^2 + m_{\pi}^2, \quad E_{\pi^+} = p^2 + \frac{1}{2} (m_{\pi}^2 + m_{\pi}^2 + m_{12}^2) + \frac{1}{2} \sqrt{4p^2m_{12}^2 + (m_{\pi}^2 + m_{\pi}^2 + m_{12}^2)^2 - 4m_{\pi}^2m_{12}^2}, \]

\[ E_{K^+} = p^2 + \frac{1}{2} (m_K^2 + m_{\pi}^2 + m_{45}^2) + \frac{1}{2} \sqrt{4p^2m_{45}^2 + (m_K^2 + m_{\pi}^2 + m_{45}^2)^2 - 4m_K^2m_{45}^2}, \]

\[ E_{K^0} = p^2 + \frac{1}{2} (m_K^2 + m_{\pi}^2 + m_{67}^2) + \frac{1}{2} \sqrt{4p^2m_{67}^2 + (m_K^2 + m_{\pi}^2 + m_{67}^2)^2 - 4m_K^2m_{67}^2}, \]

\[ E_{\eta^0} = p^2 + m_{\eta^0}^2. \]

**3.2.3 Charged kaon-condensed phase**

In the kaon-condensed phase, the different terms in eq. (3.37) are

\[ \mathcal{L}_{2\text{static}} = f^2B_0[m + (m + m_s)\cos \alpha] + \frac{1}{2} f^2 \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \]

\[ \mathcal{L}_{2\text{linear}} = f \left[ -B_0(m + m_s) + \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos \alpha \right] \sin \alpha f_4 \]

\[ -f \left( \frac{1}{2} \mu_I + \mu_S \right) \sin \alpha f_4 \phi_4 \]

\[ \mathcal{L}_{2\text{quadratic}} = \frac{1}{2} \partial_\mu \phi_4 \partial^\mu \phi_4 - \frac{1}{2} \left\{ \frac{1}{2} B_0 [3m - m_s + (m + m_s) \cos \alpha] \right. \]

\[ - \frac{1}{16} \left[ 3\mu_I - 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right]^2 + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \left( \phi_1^2 + \phi_2^2 \right) \]
\[-\frac{1}{2} \left\{ \frac{1}{2} B_0 [3m - m_s + (m + m_s) \cos \alpha] + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\} \phi_3^2 \]

\[-\frac{1}{2} \left\{ B_0 (m + m_s) \cos \alpha - \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos^2 \alpha \right\} \phi_4^2 \]

\[-\frac{1}{2} \left\{ B_0 (m + m_s) \cos \alpha - \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos 2\alpha \right\} \phi_5^2 \]

\[-\frac{1}{2} \left\{ \frac{1}{2} B_0 (m + m_s) (1 + \cos \alpha) - \frac{1}{16} \left[ -3\mu_I + 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right]^2 \right. \]

\[\left. + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\} \left( \phi_6^2 + \phi_7^2 \right) \]

\[-\frac{1}{2} \left\{ \left[ \frac{1}{6} B_0 (-m - 3m_s + 5(m + m_s) \cos \alpha) + \frac{3}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right] \right\} \phi_8^2 \]

\[-\left\{ \frac{1}{2\sqrt{3}} B_0 (m + m_s) (\cos \alpha - 1) + \frac{\sqrt{3}}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\} \phi_3 \phi_8 \]

\[\frac{1}{4} \left[ 3\mu_I - 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right] (\phi_1 \delta_0 \phi_2 - \phi_2 \delta_0 \phi_1) \]

\[+ \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha (\phi_3 \delta_0 \phi_5 - \phi_5 \delta_0 \phi_3) \]

\[+ \frac{1}{4} \left[ -3\mu_I + 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right] (\phi_6 \delta_0 \phi_7 - \phi_7 \delta_0 \phi_6) \].

The inverse propagator is block diagonal and can be written as

\[
D^{-1} = \begin{pmatrix}
D_{12}^{-1} & 0 & 0 & 0 \\
0 & D_{38}^{-1} & 0 & 0 \\
0 & 0 & D_{45}^{-1} & 0 \\
0 & 0 & 0 & D_{67}^{-1}
\end{pmatrix},
\]

where the submatrices are

\[
D_{12}^{-1} = \begin{pmatrix}
P^2 - m_{12}^2 & ip_{0m_{12}} \\
-ip_{0m_{12}} & P^2 - m_{12}^2
\end{pmatrix},
\]

\[
D_{38}^{-1} = \begin{pmatrix}
P^2 - m_{38}^2 & -ip_{0m_{38}} \\
-ip_{0m_{38}} & P^2 - m_{38}^2
\end{pmatrix},
\]

\[
D_{45}^{-1} = \begin{pmatrix}
P^2 - m_{45}^2 & ip_{0m_{45}} \\
-ip_{0m_{45}} & P^2 - m_{45}^2
\end{pmatrix},
\]

\[
D_{67}^{-1} = \begin{pmatrix}
P^2 - m_{67}^2 & ip_{0m_{67}} \\
-ip_{0m_{67}} & P^2 - m_{67}^2
\end{pmatrix}.
\]
The masses are

\[m_1^2 = \left\{ \frac{1}{2} B_0 [3m - m_s + (m + m_s) \cos \alpha] \right. \]
\[- \frac{1}{16} \left[ 3\mu_I - 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right]^2 \left. + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\}, \tag{3.88} \]

\[m_2^2 = m_1^2, \tag{3.89} \]

\[m_3^2 = \frac{1}{2} B_0 [3m - m_s + (m + m_s) \cos \alpha], \tag{3.90} \]

\[m_4^2 = \left\{ \frac{1}{2} B_0 (m + m_s) \cos \alpha + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\}, \tag{3.91} \]

\[m_5^2 = \left\{ B_0 (m + m_s) \cos \alpha - \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos^2 \alpha \right\}, \tag{3.92} \]

\[m_4^2 = \left\{ B_0 (m + m_s) \cos \alpha - \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos^2 \alpha \right\}, \tag{3.93} \]

\[m_4^2 = \left\{ \frac{1}{2} B_0 [3m - m_s + (m + m_s) \cos \alpha] - \frac{1}{16} \left[ -3\mu_I + 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right]^2 \right. \]
\[+ \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\}, \tag{3.95} \]

\[m_5^2 = m_6^2, \tag{3.96} \]

\[m_6^2 = \frac{1}{2} \left[ -3\mu_I + 2\mu_S + 2 \left( \frac{1}{2} \mu_I + \mu_S \right) \cos \alpha \right], \tag{3.97} \]

\[m_7^2 = \left\{ \frac{1}{6} B_0 (-m + 3m_s + 5(m + m_s) \cos \alpha) + \frac{3}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\}, \tag{3.98} \]

\[m_8^2 = \frac{1}{2\sqrt{3}} B_0 (m + m_s) (\cos \alpha - 1) + \frac{\sqrt{3}}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha. \tag{3.99} \]

The quasiparticle dispersion relations can be easily found and read

\[E_{\sigma_\alpha}^2 = p^2 + \frac{1}{2} (m^2 + m^2_3) + \frac{1}{2} \sqrt{(m^2_3 - m^2_8)^2 + 4m^4_{38}}, \tag{3.100} \]

\[E_{\pi^\pm}^2 = \frac{1}{2} [m^2 + m^2_3 + m^2_{12}] \pm \frac{1}{2} \sqrt{4p^2 m^2_{12} + (m^2 + m^2_3 + m^2_{12})^2 - 4m^2_{12} m^2_8}, \tag{3.101} \]

\[E_{K^\pm}^2 = \frac{1}{2} [m^2 + m^2_3 + m^2_{15}] \pm \frac{1}{2} \sqrt{4p^2 m^2_{15} + (m^2 + m^2_3 + m^2_{15})^2 - 4m^2_{15} m^2_8}, \tag{3.102} \]

\[E_{K^0}^2 = \frac{1}{2} [m^2 + m^2_3 + m^2_{67}] \pm \frac{1}{2} \sqrt{4p^2 m^2_{67} + (m^2 + m^2_3 + m^2_{67})^2 - 4m^2_6 m^2_7}, \tag{3.103} \]

\[E_{\eta^0}^2 = \frac{1}{2} (m^2 + m^2_3 - m^2_8) \pm \frac{1}{2} \sqrt{(m^2 - m^2_8)^2 + 4m^4_{38}}. \tag{3.104} \]

The linear terms in the condensed phases are given by eqs. (3.61) and (3.83). By differentiation with respect to \( \alpha \), it is straightforward to see that the terms vanish at the extremum.
of the corresponding static Lagrangian. To show this at NLO, requires the calculation of
the one-loop diagram that contribute to the one-point function, see ref. [48] for details.

4 Next-to-leading order effective potential

In this section, we calculate the NLO effective potential in the three different phases we
consider. At $\mathcal{O}(p^2)$, the contribution to the effective potential in each phase is given by
evaluating $-\mathcal{L}_2^{\text{static}}$ using $\Sigma_\alpha^\pm$, $\Sigma_K^\pm$, or $\Sigma_\alpha^{K^0}/K^0$. At $\mathcal{O}(p^4)$, there are two contributions
to the effective potential. The first is the Gaussian fluctuation about the ground state,
i.e. the standard one-loop contribution. The second is given by evaluating $-\mathcal{L}_4^{\text{static}}$, again
using $\Sigma_\alpha^\pm$, $\Sigma_K^\pm$, or $\Sigma_\alpha^{K^0}/K^0$. The one-loop contribution is ultraviolet divergent and needs
regularization. We regularize the ultraviolet divergences using dimensional regularization
in $d = 3 - 2\varepsilon$ dimensions. The divergences are cancelled by renormalizing the coupling
constants that multiply the operators in $\mathcal{L}_4$. The sum of the three contributions is the
complete effective potential to $\mathcal{O}(p^4)$ in $\chi$PT.

After going to Euclidean space, the one-loop contribution to the effective potential of
a free massive boson is given by

$$V_1 = \frac{1}{2} \int \log \left[ p^2 + m^2 \right] = \frac{1}{2} \int \frac{dp_0}{2\pi} \int_p \log \left[ p_0^2 + p^2 + m^2 \right], \tag{4.1}$$

where $m$ is the mass and the second integral is defined in $d = 3 - 2\varepsilon$ as dimensions

$$\int_p = \frac{\left(\epsilon \gamma \Lambda^2\right)}{4\pi} \int \frac{d^dp}{(2\pi)^d}, \tag{4.2}$$

and where $\Lambda$ is the renormalization scale associated with the modified minimal subtraction
scheme ($\overline{\text{MS}}$). Integrating over $P_0$, one finds

$$V_1 = \frac{1}{2} \int_p \sqrt{p^2 + m^2} = -\frac{m^4}{4(4\pi)^2} \left(\frac{\Lambda^2}{m^2}\right)^\varepsilon \left[\frac{1}{\varepsilon} + \frac{3}{2} + \mathcal{O}(\varepsilon)\right]. \tag{4.3}$$

4.1 Normal phase

The leading-order contribution to the effective potential is minus the static Lagrangian
given in eq. (3.38)

$$V_0 = -f^2 B_0(2m + m_s). \tag{4.4}$$

The one-loop contribution to the effective potential is

$$V_1 = \frac{1}{2} \int_p \left[ E_{\pi^+} + E_{\pi^-} + E_{\pi^0} + E_{K^+} + E_{K^-} + E_{K^0} + E_{\bar{K}^0} + E_{\rho^0} + E_{\pi^0} \right]. \tag{4.5}$$
where the particle energies are given by eqs. (3.55)–(3.59). Using eq. (4.3), we can write eq. (4.5) as

\[
V_1 = -\frac{3}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_{\pi,0}^2} \right) \right] [2B_0m]^2 \\
- \frac{1}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_{K,0}^2} \right) \right] [B_0(m + m_s)]^2 \\
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_{\eta,0}^2} \right) \right] \left[ \frac{2B_0(m + 2m_s)}{3} \right]^2.
\] (4.6)

The $\mathcal{O}(p^4)$ contribution from minus the static Lagrangian $\mathcal{L}_{4}^{\text{static}}$ is given by

\[
V_1^{\text{static}} = -16L_0 B_0^2 (2m + m_s)^2 - 8L_0 B_0^2 (2m^2 + m_s^2) - 4H_2 B_0^2 (2m^2 + m_s^2)
\] (4.7)

After renormalization, the effective potential is

\[
V_{\text{eff}} = -f^2 B_0 (2m + m_s) - 16L_0 B_0^2 (2m + m_s)^2 - 8L_0 B_0^2 (2m^2 + m_s^2) - 4H_2 B_0^2 (2m^2 + m_s^2)
\]

\[
- \left[ \frac{1}{(4\pi)^2} \left( \frac{37}{18} + 3 \log \frac{\Lambda^2}{m_{\pi,0}^2} + \log \frac{\Lambda^2}{m_{K,0}^2} + \frac{1}{9} \log \frac{\Lambda^2}{m_{\eta,0}^2} \right) \right] B_0^2 m^2 \\
- \left[ \frac{1}{(4\pi)^2} \left( \frac{11}{9} + 2 \log \frac{\Lambda^2}{m_{K,0}^2} + \frac{4}{9} \log \frac{\Lambda^2}{m_{\eta,0}^2} \right) \right] B_0^2 mm_s \\
- \left[ \frac{1}{(4\pi)^2} \left( \frac{13}{18} + \log \frac{\Lambda^2}{m_{K,0}^2} + \frac{4}{9} \log \frac{\Lambda^2}{m_{\eta,0}^2} \right) \right] B_0^2 m_s^2.
\] (4.8)

Using the renormalization group equations (2.17) for the couplings, we find that the effective potential is independent of the renormalization scale $\Lambda$. We note that the renormalized effective potential of eq. (4.8) is independent of the chemical potentials $\mu_I$ and $\mu_S$. This independence is a result that we expect will generalize at next-to-next-to-leading order (NNLO) and higher orders. This is due to a general argument, namely the Silver Blaze property, that shows the isospin independence of the eigenvalues of the Dirac operator at finite isospin density (in the normal phase) [55] and consequently the isospin independence of the partition function and resulting thermodynamic quantities. While the original proof in ref. [55] did not include the strange chemical potential, we expect that it generalizes to systems with both isospin and strange chemical potentials.

### 4.2 Pion-condensed phase

The tree-level contribution to the effective potential is minus the static Lagrangian given in eq. (3.60)

\[
V_0 = -f^2 B_0 (2m \cos \alpha + m_s) - \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha.
\] (4.9)

The one-loop effective potential is

\[
V_1 = \frac{1}{2} \int_p \left[ E_{\pi^+} + E_{\pi^-} + E_{\pi^0} + E_{K^+} + E_{K^-} + E_{K^0} + E_{\bar{K}^0} + E_{\eta^0} \right].
\] (4.10)
where the energies are given by eqs. (3.77)-(3.81). The integrals of $E_{\pi^0}$ and $E_{\eta^0}$ can be calculated analytically in dimensional regularization using eq. (4.3). The remaining contributions require a little more work. Let us consider the contribution from the charged pions. In order to eliminate the divergences, their dispersion relations are expanded in powers of $1/p$ as

$$E_{\pi^+} + E_{\pi^-} = 2p + \frac{2(m_1^2 + m_2^2) + m_{12}^2}{4p} - \frac{8(m_1^4 + m_2^4) + 4(m_1^2 + m_2^2)m_{12}^2 + m_{12}^4}{64p^3} + \ldots \quad (4.11)$$

To this order, the large-$p$ behavior in eq. (4.11) is the same as the sum $E_1 + E_2$, where $E_1 = \sqrt{p^2 + m_1^2 + \frac{1}{4}m_{12}^2}$ and $E_2 = \sqrt{p^2 + m_2^2 + \frac{1}{4}m_{12}^2}$. For later convenience we introduce the masses $\tilde{m}_1^2 = m_1^2 + \frac{1}{4}m_{12}^2 = 2B_0m_1 \cos \alpha$, $\tilde{m}_2^2 = m_2^2 + \frac{1}{4}m_{12}^2 = 2B_0m_2 \cos \alpha + \mu_1^2 \sin^2 \alpha = m_3^2$. The integral over $E_{\pi^+} + E_{\pi^-} - E_1 - E_2$ is convergent in the ultraviolet and the subtraction integrals of $E_1$ and $E_2$ can be done analytically in dimensional regularization. We can then write

$$V_{1,\pi^+} + V_{1,\pi^-} = V_{1,\pi^+}^{\text{div}} + V_{1,\pi^-}^{\text{div}} + V_{1,\pi^+}^{\text{fin}} + V_{1,\pi^-}^{\text{fin}} \quad (4.12)$$

where

$$V_{1,\pi^+}^{\text{div}} + V_{1,\pi^-}^{\text{div}} = \frac{1}{2} \int_p [E_1 + E_2], \quad (4.13)$$

$$V_{1,\pi^+}^{\text{fin}} + V_{1,\pi^-}^{\text{fin}} = \frac{1}{2} \int_p [E_{\pi^+} + E_{\pi^-} - E_1 - E_2]. \quad (4.14)$$

The contributions from the kaons can be calculated analytically as follows. Consider first the contribution from the charged kaon which is given by

$$V_{1,K^+} + V_{1,K^-} = \frac{1}{2} \int_p \log \left[ (P^2 + m_4^2)(P^2 + m_5^2) + p_0^2 m_{45}^2 \right], \quad (4.15)$$

which can be rewritten as

$$V_{1,K^+} + V_{1,K^-} = \frac{1}{2} \int_p \log \left\{ \left[ P^2 + \frac{1}{2}(m_4^2 + m_5^2) \right]^2 + p_0^2 m_{45}^2 - \frac{1}{4} (m_4^2 - m_5^2)^2 \right\}. \quad (4.16)$$

Since $m_4 = m_5$, the last term vanishes and the integrand can be factorized as

$$V_{1,K^+} + V_{1,K^-} = \frac{1}{2} \int_p \log \left[ \left( p_0 + \frac{im_{45}}{2} \right)^2 + p^2 + m_4^2 + \frac{1}{4} m_{45}^2 \right] \times \left[ \left( p_0 - \frac{im_{45}}{2} \right)^2 + p^2 + m_4^2 + \frac{1}{4} m_{45}^2 \right]. \quad (4.17)$$

Shifting integration variables in the two terms, $p_0 \to p_0 \mp \frac{im_{45}}{2}$, the integral simplifies to

$$V_{1,K^+} + V_{1,K^-} = \int_p \log \left[ P^2 + m_4^2 + \frac{1}{4} m_{45}^2 \right]. \quad (4.18)$$
The contribution from the neutral kaons is obtained simply by replacing \( m_4 \) by \( m_6 \) and \( m_{45} \) by \( m_{67} \). Since \( \tilde{m}_4^2 = m_4^2 \) and by defining \( \tilde{m}_4^2 = m_4^2 + \frac{1}{8} m_{45}^2 = m_6^2 + \frac{1}{4} m_{67}^2 = B_0 (m \cos \alpha + m_s) + \frac{1}{4} \mu_f^2 \sin^2 \alpha \), we can write the divergent part of the one-loop contribution as

\[
V_1^{\text{div}} = - \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_f^2} \right) \right] \left[ 2B_0 m \cos \alpha \right]^2
- \frac{1}{2(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_3^2} \right) \right] \left[ 2B_0 m \cos \alpha + \mu_f^2 \sin^2 \alpha \right]^2
- \frac{1}{(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_5^2} \right) \right] \left[ B_0 (m \cos \alpha + m_s) + \frac{1}{4} \mu_f^2 \sin^2 \alpha \right]^2
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_8^2} \right) \right] \left[ \frac{2B_0 (m \cos \alpha + 2m_s)}{3} \right]^2, \tag{4.19}
\]

The static part of the Lagrangian \( L_4 \) as a function of \( \alpha \) is

\[
V_1^{\text{static}} = -(4L_1 + 4L_2 + 2L_3) \mu_f^4 \sin^4 \alpha - 8L_4 B_0 (2m \cos \alpha + m_s) \mu_f^2 \sin^2 \alpha
- 8L_6 B_0 m \mu_f^2 \cos \alpha \sin^2 \alpha - 16L_8 B_0^2 (2m \cos \alpha + m_s)^2
- 8L_8 B_0^2 (2m^2 \cos 2\alpha + m_s^2) - 4H_2 B_0^2 (2m^2 + m_s^2). \tag{4.20}
\]

The renormalized one-loop effective potential \( V_{\text{eff}} = V_0 + V_1 + V_1^{\text{static}} \) is given by the sum of eqs. (4.9), (4.19), and (4.20) then reads

\[
V_{\text{eff}} = -f^2 B_0 (2m \cos \alpha + m_s) - \frac{1}{2} f^2 \mu_f^2 \sin^2 \alpha - (4L_1^r + 4L_2^r + 2L_3^r) \mu_f^4 \sin^4 \alpha
- 8L_4^r B_0 (2m \cos \alpha + m_s) \mu_f^2 \sin^2 \alpha - 8L_6^r B_0 m \mu_f^2 \cos \alpha \sin^2 \alpha
- 16L_8^r B_0^2 (2m \cos \alpha + m_s)^2 - 8L_8^r B_0^2 (2m^2 \cos 2\alpha + m_s^2) - 4H_2^r B_0^2 (2m^2 + m_s^2)
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{\Lambda^2}{m_f^2} \right) \right] \left[ 2B_0 m \cos \alpha \right]^2
- \frac{1}{2(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{\Lambda^2}{m_3^2} \right) \right] \left[ 2B_0 m \cos \alpha + \mu_f^2 \sin^2 \alpha \right]^2
- \frac{1}{(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{\Lambda^2}{m_5^2} \right) \right] \left[ B_0 (m \cos \alpha + m_s) + \frac{1}{4} \mu_f^2 \sin^2 \alpha \right]^2
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{\Lambda^2}{m_8^2} \right) \right] \left[ \frac{2B_0 (m \cos \alpha + 2m_s)}{3} \right]^2 + V_{1,\pi^+}^{\text{fin}} + V_{1,\pi^-}^{\text{fin}}. \tag{4.21}
\]

Again, it can be verified that the NLO effective potential is independent of the scale \( \Lambda \). It is also explicitly independent of the strangeness chemical potential \( \mu_S \).

### 4.3 Charged kaon-condensed phase

The tree-level contribution to the effective potential is

\[
V_0 = - f^2 B_0 \left[ m + (m + m_s) \cos \alpha \right] - \frac{1}{2} f^2 \left( \frac{1}{2} \mu_f + \mu_S \right)^2 \sin^2 \alpha. \tag{4.22}
\]
The one-loop effective potential is

\[
V_1 = \frac{1}{2} \int_p \left[ E_{\pi^+} + E_{\pi^-} + E_{\pi^0} + E_{K^+} + E_{K^-} + E_{K^0} + E_{\bar{K}^0} + E_{\eta^0} \right].
\]  

(4.23)

The contributions from \(\pi^\pm, K^\pm, K^0\) and \(\bar{K}^0\) can be treated as in the previous section and it is only the terms \(V_{1,K^\pm}\) that require a subtraction term. The relevant masses are defined as

\[
\tilde{m}_1^2 = m_1^2 + \frac{1}{4} m_{12}^2 = \frac{1}{2} B_0 [3m - m_s + (m + m_s) \cos \alpha] + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha,
\]  

(4.24)

\[
\tilde{m}_4^2 = m_4^2 + \frac{1}{4} m_{45}^2 = B_0 (m + m_s) \cos \alpha,
\]  

(4.25)

\[
\tilde{m}_5^2 = m_5^2 + \frac{1}{4} m_{45}^2 = B_0 (m + m_s) \cos \alpha + \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha,
\]  

(4.26)

\[
\tilde{m}_6^2 = m_6^2 + \frac{1}{4} m_{67}^2 = \frac{1}{2} B_0 (m + m_s)(1 + \cos \alpha) + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha.
\]  

(4.27)

The contribution from the mixed \(\pi^0\) and \(\eta^0\) is given by

\[
V_{1,\pi^0} + V_{1,\eta^0} = \frac{1}{2} \int_p \log \left[ (P^2 + m_3^2)(P^2 + m_8^2) - m_{38}^4 \right]
\]  

\[
= \frac{1}{2} \int_p \log \left[ P^2 + \tilde{m}_3^2 \right] + \log \left[ P^2 + \tilde{m}_8^2 \right],
\]

(4.28)

where the new masses are defined as

\[
\tilde{m}_{3,8}^2 = \frac{1}{2} \left[ m_3^2 + m_8^2 \pm \sqrt{(m_3^2 - m_8^2)^2 + 4m_{38}^4} \right].
\]  

(4.29)

This yields

\[
V_1 = -\frac{1}{2(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_1^2} \right) \right] \left\{ \frac{1}{2} B_0 [(3m - m_s + (m + m_s) \cos \alpha]
\]  

\[
+ \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\}^2
\]

\[
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_4^2} \right) \right] \left[ B_0^2 (m + m_s)^2 \cos^2 \alpha \right]
\]

\[
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_5^2} \right) \right] \left[ B_0 (m + m_s) \cos \alpha + \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right]^2
\]

\[
- \frac{1}{2(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_6^2} \right) \right] \left[ \frac{1}{2} B_0 (m + m_s)(1 + \cos \alpha) + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right]^2
\]

\[
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_3^2} \right) \right] \tilde{m}_3^2 - \frac{1}{4(4\pi)^2} \left[ \frac{1}{\epsilon} + \frac{3}{2} + \log \left( \frac{\Lambda^2}{m_8^2} \right) \right] \tilde{m}_8^2.
\]  

(4.30)
The static part of the Lagrangian $\mathcal{L}_4$ as a function of $\alpha$ is

$$V_{\text{static}}^{\text{static}} = -(4L_4 + 4L_2 + 2L_3) \left( \frac{1}{2} \mu_I + \mu_S \right)^4 \sin^4 \alpha$$

$$-8L_4B_0[m + (m + m_s) \cos \alpha] \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha$$

$$-4L_5B_0(m + m_s) \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos \alpha \sin^2 \alpha - 16L_6B_0^2[m + (m + m_s) \cos \alpha]^2$$

$$-4L_8B_0^2(3m^2 - 2mm_s + m_s^2 + (m + m_s)^2 \cos 2\alpha) - 4H_2B_0^2(2m^2 + m_s^2). \quad (4.31)$$

After renormalization, the effective potential is

$$V_{\text{eff}} = -f^2B_0 [m + (m + m_s) \cos \alpha] - \frac{1}{2}f^2 \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha$$

$$-(4L_4^* + 4L_2^* + 2L_3^*) \left( \frac{1}{2} \mu_I + \mu_S \right)^4 \sin^4 \alpha$$

$$-8L_4^*B_0[m + (m + m_s) \cos \alpha] \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha$$

$$-4L_5^*B_0(m + m_s) \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \cos \alpha \sin^2 \alpha - 16L_6B_0[3m - m_s + (m + m_s) \cos \alpha]^2$$

$$-4L_8B_0^2(3m^2 - 2mm_s + m_s^2 + (m + m_s)^2 \cos 2\alpha) - 4H_2B_0^2(2m^2 + m_s^2)$$

$$-\frac{1}{2(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{A^2}{m_I^4} \right) \right] \left\{ \frac{1}{2}B_0[(3m - m_s + (m + m_s) \cos \alpha] \right.$$  

$$\left. + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right\}^2$$

$$-\frac{1}{4(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{A^2}{m_I^4} \right) \right] \left[ B_0^2(m + m_s)^2 \cos^2 \alpha \right]$$

$$-\frac{1}{4(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{A^2}{m_I^4} \right) \right] \left[ B_0(m + m_s) \cos \alpha + \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right]^2 \quad (4.32)$$

$$-\frac{1}{2(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{A^2}{m_I^4} \right) \right] \left[ \frac{1}{2}B_0^2(m + m_s)^2(1 + \cos \alpha) + \frac{1}{4} \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \sin^2 \alpha \right]^2$$

$$-\frac{1}{2(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{A^2}{m_I^4} \right) \right] \tilde{m}_{4,5}^2 - \frac{1}{4(4\pi)^2} \left[ \frac{1}{2} + \log \left( \frac{A^2}{m_I^4} \right) \right] \tilde{m}_{4,5}^2 + V_{1,K^+}^{\text{fin}} + V_{1,K^-}^{\text{fin}},$$

where the subtraction terms and energies are defined by

$$V_{1,K^+}^{\text{fin}} + V_{1,K^-}^{\text{fin}} = \frac{1}{2} \int [E_{K^+} + E_{K^-} - E_4 - E_5], \quad (4.33)$$

$$E_{4,5} = \sqrt{p^2 + \tilde{m}_{4,5}^2}, \quad (4.34)$$

with $\tilde{m}_{4,5}$ given by eqs. (4.25)–(4.26).

The effective potential depends only on the combination $\frac{1}{2} \mu_I + \mu_S$ as is evident by inspection. Using the expressions for the running couplings, eq. (2.17), the scale dependence in the final results for the effective potential, eqs. (4.8), (4.21), and (4.32) cancels.
5 Thermodynamic functions

In this section, we derive the thermodynamic functions from the effective potential. We will focus on the pion-condensed phase since we are interested in comparing our results with lattice simulations.

5.1 Pion-condensed phase

The pressure $P$ is given by $-V_{\text{eff}}$. In the pion-condensed phase, we get from eq. (4.21)

$$P = f^2 B_0 (2m \cos \alpha + m_s) + \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha$$

$$+ 4L'_1 + 2L' + \frac{1}{16(4\pi)^2} \left[ 9 + 8\log \frac{\Lambda^2}{m_3^2} + \log \frac{\Lambda^2}{m_4^2} \right] \mu_I^2 \sin^4 \alpha$$

$$+ \left[ 8L'_5 + \frac{1}{2(4\pi)^2} \left( \frac{3}{2} + 4\log \frac{\Lambda^2}{m_3^2} - \log \frac{\Lambda^2}{m_4^2} \right) \right] B_0 (2m \cos \alpha + m_s) \mu_I^2 \sin^2 \alpha$$

$$+ \left[ 16L'_6 + 8L'_5 + 4H'_2 + \frac{1}{(4\pi)^2} \left[ 13 \log \frac{\Lambda^2}{m_4^2} + 4 \log \frac{\Lambda^2}{m_5^2} \right] \right] B_0^2 m_5^2$$

$$+ \left[ 64L'_6 + 16L'_5 + 8H'_2 + \frac{1}{(4\pi)^2} \left[ \frac{37}{18} + \log \frac{\Lambda^2}{m_4^2} + 2 \log \frac{\Lambda^2}{m_5^2} + \log \frac{\Lambda^2}{m_6^2} \right] \right] B_0^2 m_6^2 \cos^2 \alpha - [16L'_6 - 8H'_2] B_0^2 m_5^2 \sin^2 \alpha$$

$$- V_{1,\pi^+}^\text{fin} - V_{1,\pi^-}^\text{fin},$$

(5.1)

The isospin density is given by

$$n_I = - \frac{\partial V_{\text{eff}}}{\partial \mu_I}$$

$$= f^2 \mu_I \sin^2 \alpha + \left[ 16L'_1 + 16L'_5 + 8L'_3 + \frac{1}{4(4\pi)^2} \left( 8 \log \frac{\Lambda^2}{m_3^2} + \log \frac{\Lambda^2}{m_4^2} \right) \right] \mu_I^2 \sin^4 \alpha$$

$$+ \left[ 16L'_4 + \frac{1}{4(4\pi)^2} \log \frac{\Lambda^2}{m_4^2} \right] B_0 (2m \cos \alpha + m_s) \mu_I \sin^2 \alpha$$

$$+ \left[ 16L'_5 + \frac{1}{4(4\pi)^2} \left( 4 \log \frac{\Lambda^2}{m_5^2} - \log \frac{\Lambda^2}{m_4^2} \right) \right] B_0 m_5 \cos \alpha \sin^2 \alpha - \frac{\partial V_{1,\pi^+}^\text{fin}}{\partial \mu_I} - \frac{\partial V_{1,\pi^-}^\text{fin}}{\partial \mu_I}.$$ 

(5.2)

The energy density is given by

$$\epsilon = -P + \mu_i n_i.$$ 

where $n_i = \frac{\partial V_{\text{eff}}}{\partial \mu_i}$ is the charge density associated with the chemical potential $\mu_i$. In the pion-condensed phase it takes the following form

$$\epsilon = -P + \mu_I n_I.$$ 

(5.3)
since the effective potential is independent of $\mu_S = 0$ in this phase. Using eqs. (4.21) and (5.2), we find the following energy density

$$\epsilon = -f^2 B_0 (2m \cos \alpha + m_s) + \frac{1}{2} f^2 \mu_I \sin^2 \alpha$$

(5.5)

$$+ \left[ 12L_4^r + 12L_2^r + 6L_5^r + \frac{1}{(4\pi)^2} \left( -\frac{9}{32} + \frac{3}{2} \log \frac{\Lambda^2}{m_3^2} + \frac{3}{16} \log \frac{\Lambda^2}{m_4^2} \right) \right] \mu_I^4 \sin^4 \alpha$$

$$+ \left[ 8L_4^r + \frac{1}{(4\pi)^2} \left( \frac{1}{2} + \log \frac{\Lambda^2}{m_4^2} \right) \right] B_0 (2m \cos \alpha + m_s) \mu_I^2 \sin^2 \alpha$$

$$+ \left[ 8L_5^r + \frac{1}{(4\pi)^2} \left( -\frac{3}{2} + 4 \log \frac{\Lambda^2}{m_3^2} - \log \frac{\Lambda^2}{m_5^2} \right) \right] B_0 \mu_I^2 \cos \alpha \sin^2 \alpha$$

$$- 16L_6^r + 8L_5^r + 4H_2^r + \frac{1}{(4\pi)^2} \left( \frac{13}{18} + \log \frac{\Lambda^2}{m_4^2} + \frac{4}{9} \log \frac{\Lambda^2}{m_8^2} \right) \right] B_0^2 m_4^2$$

$$- \left[ 64L_6^r + \left( \frac{11}{9} + 2 \log \frac{\Lambda^2}{m_4^2} + \frac{4}{9} \log \frac{\Lambda^2}{m_8^2} \right) \right] B_0^2 m_8 m_s \cos \alpha$$

$$- \left[ 64L_6^r + 16L_8^r + 8H_2^r + \frac{1}{(4\pi)^2} \left( \frac{37}{18} + \log \frac{\Lambda^2}{m_4^2} + 2 \log \frac{\Lambda^2}{m_3^2} + \log \frac{\Lambda^2}{m_4^2} + \frac{1}{9} \log \frac{\Lambda^2}{m_8^2} \right) \right]$$

$$\times B_0^2 m_2 \cos^2 \alpha + [16L_8^r - 8H_2^r] B_0^2 m_2^2 \sin^2 \alpha + V_{1,\pi^+}^{\text{fin}} + V_{1,\pi^-}^{\text{fin}} - \mu_I^2 \frac{\partial V_{1,\pi^+}^{\text{fin}}}{\partial \mu_I} - \mu_I^2 \frac{\partial V_{1,\pi^-}^{\text{fin}}}{\partial \mu_I},$$

5.2 Large-$m_s$ limit

We are interested in the large-$m_s$ limit of our three-flavor results for thermodynamic quantities. In this limit, general effective field theory arguments tell us that the mesonic degrees of freedom containing the $s$-quark decouple. Thus one should recover the two-flavor results of ref. [48] with modified couplings. The modified couplings then contain the loop effects from integrating out kaons and the eta.

The one-loop expressions for the pion-decay constant and the light-quark condensate in the vacuum are given by [10]

$$f_\pi^2 = f^2 \left[ 1 + \left( 8L_4^r + 8L_5^r + \frac{2}{(4\pi)^2} \log \frac{\Lambda^2}{m_{\pi,0}^2} \right) \frac{m_{\pi,0}^2}{f^2} \right]$$

(5.6)

$$+ \left[ 16L_6^r + \frac{1}{(4\pi)^2} \log \frac{\Lambda^2}{m_{K,0}^2} \right] \frac{m_{K,0}^2}{f^2}$$

$$\langle \bar{\psi} \psi \rangle = -f^2 B_0 \left[ 1 + \left( 16L_6^r + 4L_8^r + 4H_2^r + \frac{3}{(2\pi)^2} \log \frac{\Lambda^2}{m_{\pi,0}^2} \right) \frac{m_{\pi,0}^2}{f^2} \right]

+ \left[ 32L_6^r + \frac{1}{(4\pi)^2} \log \frac{\Lambda^2}{m_{K,0}^2} \right] \frac{m_{K,0}^2}{f^2} + \frac{m_{\eta,0}^2}{6(4\pi)^2} \log \frac{\Lambda^2}{m_{\eta,0}^2}. \right].$$

(5.7)
The loop corrections involve pions, kaons, and etas. Integrating out the $s$-quark corresponds to setting $m = 0$ or ignoring the pionic loop corrections. This yields

$$f^2 = f^2 \left[ 1 + \left( 16L_4^r + \frac{1}{(4\pi)^2} \log \frac{\Lambda^2}{\tilde{m}_{K,0}^2} \right) \frac{\tilde{m}_{K,0}^2}{f^2} \right], \quad (5.8)$$

$$\langle \bar{\psi} \psi \rangle = -f^2 B_0 \left[ 1 + \left( 32L_6^r + \frac{1}{(4\pi)^2} \log \frac{\Lambda^2}{\tilde{m}_{K,0}^2} \right) \frac{\tilde{m}_{K,0}^2}{f^2} + \frac{\tilde{m}_{\eta,0}^2}{6(4\pi)^2 f^2} \log \frac{\Lambda^2}{\tilde{m}_{\eta,0}^2} \right], \quad (5.9)$$

where the masses are $\tilde{m}_{K,0}^2 = B_0 m_s$ and $\tilde{m}_{\eta,0}^2 = \frac{4B_0 m_s}{3}$. Defining $\tilde{B}_0$ via $\tilde{f} = f^2 \tilde{B}_0$ yields

$$\tilde{B}_0 = B_0 \left[ 1 - (16L_4^r - 32L_6^r) \frac{\tilde{m}_{K,0}^2}{f^2} + \frac{\tilde{m}_{\eta,0}^2}{6(4\pi)^2 f^2} \log \frac{\Lambda^2}{\tilde{m}_{\eta,0}^2} \right]. \quad (5.10)$$

Using the renormalization group equations for $L_4^r$ and $L_6^r$, one verifies that eqs. (5.8) and (5.10) are independent of the scale $\Lambda$. Moreover, in ref. [10], the authors derived the relations among the renormalized couplings in two- and three-flavor $\chi$PT. The relevant relations are

$$l_1^r = 4L_4^r + 2L_3^r + \frac{1}{48(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{K,0}^2} - 1 \right], \quad (5.11)$$

$$l_2^r = 4L_2^r + \frac{1}{24(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{K,0}^2} - 1 \right], \quad (5.12)$$

$$l_3^r = -8L_4^r - 4L_5^r + 16L_6^r + 8L_8^r + \frac{1}{36(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{\eta,0}^2} - 1 \right], \quad (5.13)$$

$$l_4^r = 8L_4^r + 4L_5^r + \frac{1}{4(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{K,0}^2} - 1 \right], \quad (5.14)$$

$$h_1^r = 8L_4^r + 4L_5^r - 4L_8^r + 2H_2^r + \frac{1}{4(4\pi)^2} \left[ \log \frac{\Lambda^2}{m_{K,0}^2} - 1 \right]. \quad (5.15)$$

The relations between the renormalized couplings $l_i^r, h_i^r$ and the low-energy constants $\tilde{l}_i, \tilde{h}_i$ in two-flavor $\chi$PT are

$$l_i^r(\Lambda) = \frac{\gamma_i}{2(4\pi)^2} \left[ \tilde{l}_i + \log \frac{2B_0 m_s}{\Lambda^2} \right], \quad h_i^r(\Lambda) = \frac{\delta_i}{2(4\pi)^2} \left[ \tilde{h}_i + \log \frac{2B_0 m_s}{\Lambda^2} \right], \quad (5.16)$$

where $\gamma_1 = \frac{1}{3}, \gamma_2 = \frac{2}{3}, \gamma_3 = -\frac{4}{3}, \gamma_4 = 2$, and $\delta_1 = 2$ [9]. Using the renormalization group equations for renormalized couplings, one finds that the $\Lambda$-dependence are the same on the left- and right-hand sides of eqs. (5.11)–(5.15).

The low-energy limit of eqs. (5.1), (5.2), and (5.5) are then obtained as follows. We expand them in powers of $1/m_s$, express the result using eqs. (5.8), and (5.10)–(5.16). The
pressure is

\[
P = 2f^2\tilde{B}_0 m \cos \alpha + \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha - \frac{4}{(4\pi)^2} \left[ -\tilde{h}_1 + \tilde{l}_1 \right] B_0^2 m^2 \\
+ \frac{1}{(4\pi)^2} \left[ \frac{3}{2} - \tilde{l}_3 + 4\tilde{l}_4 + \log \left( \frac{2\tilde{B}_0 m}{\tilde{m}_1} \right) + 2 \log \left( \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right) \right] B_0^2 m^2 \cos^2 \alpha \\
+ \frac{1}{(4\pi)^2} \left[ \frac{1}{2} - \tilde{l}_4 + \log \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right] 2B_0 \mu_I^2 \cos \alpha \sin^2 \alpha \\
+ \frac{1}{(4\pi)^2} \left[ \frac{1}{2} + \frac{1}{3} \tilde{l}_1 + \frac{2}{3} \tilde{l}_2 + \log \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right] \mu_I^4 \sin^4 \alpha - V_{1,\pi^+}^{\text{fin}} - V_{1,\pi^-}^{\text{fin}},
\]

the isospin density is

\[
n_I = \tilde{f}^2 \mu_I \sin^2 \alpha + \frac{2}{(4\pi)^2} \left[ \tilde{l}_4 + \log \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right] 2B_0 \mu_I \cos \alpha \sin^2 \alpha \\
+ \frac{2}{(4\pi)^2} \left[ \frac{1}{3} \tilde{l}_1 + \frac{2}{3} \tilde{l}_2 + \log \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right] \mu_I^2 \sin^4 \alpha - \frac{\partial V_{1,\pi^+}^{\text{fin}}}{\partial \mu_I} - \frac{\partial V_{1,\pi^-}^{\text{fin}}}{\partial \mu_I},
\]

and the energy density is

\[
\epsilon = -2f^2\tilde{B}_0 m \cos \alpha + \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha + \frac{4}{(4\pi)^2} \left[ -\tilde{h}_1 + \tilde{l}_1 \right] B_0^2 m^2 \\
- \frac{1}{(4\pi)^2} \left[ \frac{3}{2} - \tilde{l}_3 + 4\tilde{l}_4 + \log \left( \frac{2\tilde{B}_0 m}{\tilde{m}_1} \right) + 2 \log \left( \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right) \right] B_0^2 m^2 \cos^2 \alpha \\
- \frac{1}{(4\pi)^2} \left[ \frac{1}{2} - \tilde{l}_4 - \log \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right] 2B_0 \mu_I^2 \cos \alpha \sin^2 \alpha \\
- \frac{1}{(4\pi)^2} \left[ \frac{1}{2} - \tilde{l}_1 - 2\tilde{l}_2 - 3 \log \frac{2\tilde{B}_0 m}{\tilde{m}_2} \right] \mu_I^4 \sin^4 \alpha + V_{1,\pi^+}^{\text{fin}} + V_{1,\pi^-}^{\text{fin}} \\
- \mu_I \frac{\partial V_{1,\pi^+}^{\text{fin}}}{\partial \mu_I} - \mu_I \frac{\partial V_{1,\pi^-}^{\text{fin}}}{\partial \mu_I}.
\]

Up to different notation \((2\tilde{B}_0 m \rightarrow m^2)\), the results for \(P, n_I, \) and \(\epsilon\) are of the same form as the two-flavor results derived in [48] with renormalized parameters \(\tilde{B}_0\) and \(\tilde{f}\). (In two-flavor \(\chi PT \) \(m\) is the tree level pion mass [48].)

6 Results and discussion

In this section, we study the (tree-level) quasiparticle masses, isospin density, pressure and the equation of state. In order to evaluate these quantities, we need the numerical values of the low-energy constants \((L_i)\) as well as the meson masses and decay constants. The low-energy constants have been determined experimentally, with the following values and uncertainties at the scale \(\mu = m_\rho\), where \(\Lambda^2 = 4\pi e^{-\gamma_E} \mu^2\) [59], where \(m_\rho\) is the mass of the
\[ L_1^{\rho} = (1.0 \pm 0.1) \times 10^{-3}, \quad L_2^{\rho} = (1.6 \pm 0.2) \times 10^{-3} \tag{6.1} \]
\[ L_3^{\rho} = (-3.8 \pm 0.3) \times 10^{-3}, \quad L_4^{\rho} = (0.0 \pm 0.3) \times 10^{-3} \tag{6.2} \]
\[ L_5^{\rho} = (1.2 \pm 0.1) \times 10^{-3}, \quad L_6^{\rho} = (0.0 \pm 0.4) \times 10^{-3} \tag{6.3} \]
\[ L_7^{\rho} = (-0.4 \pm 0.2) \times 10^{-3}, \quad L_8^{\rho} = (0.5 \pm 0.2) \times 10^{-3} \tag{6.4} \]

Since we are mainly interested in comparing our results to the predictions of the lattice simulations in refs. [43], we will use their values for the pion and kaon masses as well as the pion and kaon decay constants. With uncertainties, they are given by [58]

\[ m_\pi = 131 \pm 3 \text{MeV}, \quad m_K = 481 \pm 10 \text{MeV}, \tag{6.5} \]
\[ f_\pi = \frac{128 \pm 3}{\sqrt{2}} \text{MeV}, \quad f_K = \frac{150 \pm 3}{\sqrt{2}} \text{MeV}. \tag{6.6} \]

These uncertainties (in the masses and decay constants) arise due to lattice discretization errors and consequently differ slightly from their experimental values. Since we have three parameters in the Lagrangian, \( B_0 m, B_0 m_s, \) and \( f \), we need to pick three observables from the set above, and we choose \( m_\pi, m_K, \) and \( f_\pi. \)

The relevant meson masses and the pion decay constants at one-loop are given by eqs. (A.1), (A.2), and (A.3) in terms of the parameters \( B_0 m, B_0 m_s, \) and \( f \) at next-to-leading order. Using the lattice values given above, we can solve for \( B_0 m, B_0 m_s, \) and \( f. \)

This yields

\[ f^{\text{cen}} = 75.16 \text{MeV}, \quad f^{\text{low}} = 79.88 \text{MeV}, \quad f^{\text{high}} = 70.44 \text{MeV}, \tag{6.7} \]
\[ m_{\pi, \text{tree}}^{\text{low}} = 148.45 \text{MeV}, \quad m_{\pi, \text{tree}}^{\text{cen}} = 131.28 \text{MeV}, \quad m_{\pi, \text{tree}}^{\text{high}} = 115.93 \text{MeV}, \tag{6.8} \]
\[ m_{K, \text{tree}}^{\text{cen}} = 520.65 \text{MeV}, \quad m_{K, \text{tree}}^{\text{low}} = 617.35 \text{MeV}, \quad m_{K, \text{tree}}^{\text{high}} = 437.84 \text{MeV}, \tag{6.9} \]

where the subscripts indicate that the values correspond to the central, minimum, and maximum values of the low-energy constants. Using the one-loop chPT expression for the \( f_K, \) eq. (A.4), we find \( f_K = 113.9 \text{MeV} \) for the central values, which is off by approximately 7\% compared to the lattice value of \( f_K = \frac{150}{\sqrt{2}} = 106.1 \text{MeV}. \) The uncertainties in the LECs, \( L_i, \) the pion mass, \( m_\pi, \) the pion decay constant, \( f_\pi, \) and the kaon mass, \( m_K, \) lead to uncertainties in \( B_0 m, B_0 m_s, \) and \( f. \) These uncertainties are dominated by the uncertainties in the LECs with the uncertainty in the lattice parameters contributing the least. Additionally, it turns out that the lowest values of LECs calculated after including the LEC uncertainty leads to unphysical values of the \( \eta \) mass. As such we were forced to choose the lowest values of the LECs using 0.46 times the uncertainties leading to the results in eq. (6.7).

The thermodynamic quantities are functions of the effective potential evaluated at its minimum as a function of \( \alpha \) for given values of the isospin and strange chemical potentials. Hence, we must solve the equation

\[ \frac{\partial V_{\text{eff}}}{\partial \alpha} = 0. \tag{6.10} \]
In figure 1, we show the solution to eq. (6.10) as function of the isospin chemical potential $\mu_I$ and $\mu_S = 0$. The red curve is the tree-level result, while the blue curve is the one-loop result in two-flavor $\chi$PT, the green curve is the one-loop result in three-flavor $\chi$PT and the brown curve is the one-loop result in two-flavor $\chi$PT using three-flavor LECs. In section 6.3, we use $\alpha_{gs}$ to calculate the pressure, isospin density and the equation of state.

### 6.1 Phase diagram

We find that $\alpha_{gs}$ becomes non-zero when $|\mu_I| > m_\pi$. In order to show that the transition from the vacuum phase to the Bose-condensed phases occurs at a critical chemical potential equal to the physical pion mass, we expand the effective potential in a power series in $\alpha$ around $\alpha = 0$ up to order $\alpha^4$ to obtain an effective Landau-Ginzburg energy functional [53],

$$V_{\text{eff}}^{\text{LG}}(\alpha) = a_0 + a_2 \alpha^2 + a_4 \alpha^4 + O(\alpha^6).$$

As pointed out before, in the charged pion-condensed phase, $V_{\text{eff}}$ and therefore the coefficients are independent of $\mu_S$. Similarly, in the charged kaon-condensed phase, they only depend on the combination $\frac{1}{2} \mu_I + \mu_S$, and in the neutral kaon-condensed phase, only on the combination $-\frac{1}{2} \mu_I + \mu_S$. Using the expressions for the pion mass $m_\pi$ (A.1) and the pion-decay constant $f_\pi$, (A.3), it can be shown that in the pion-condensed phase (see ref. [48] for details)

$$a_2(\mu_I) = \frac{1}{2} f_\pi^2 \left[ m_\pi^2 - \mu_I^2 \right].$$

The critical isospin chemical potential $\mu_I^c$ is defined by the vanishing of $a_4(\mu_I)$, and eq. (6.12) shows that $|\mu_I^c| = m_\pi$. Moreover, using the techniques in ref. [53] it can be shown that $a_4(\mu_I^c) > 0$, implying that the transition from the vacuum phase to a pion-condensed phase is second order located at $\mu_I^c = \pm m_\pi$. If $a_4(\mu_I^c) < 0$, the transition is first order.

$$a_2 \left( \frac{1}{2} \mu_I + \mu_S \right) = \frac{1}{2} f_K^2 \left[ m_K^2 - \left( \frac{1}{2} \mu_I + \mu_S \right)^2 \right],$$
where \( m_K \) is the physical kaon mass, whose one-loop expression is given by eq. (A.2). The critical chemical potential is again given by the vanishing of \( a_2 \), i.e. \( \frac{4}{3} \mu_I + \mu_S = m_K \). The coefficient of the order \( \alpha^4 \) term can be shown to be positive when evaluated at \( \frac{1}{2} \mu_I + \mu_S = m_K \). This shows there is a second-order transition to a kaon-condensed phase at \( \frac{1}{2} \mu_I + \mu_S = \pm m_K \). For the transition to a neutral kaon-condensed phase, we have \( -\frac{1}{2} \mu_I + \mu_S = \pm m_K \).

While the transitions from the vacuum to either a pion-condensed phase or a kaon-condensed phase are second order, the transition between the two Bose-condensed phases is first order. At leading, this is straightforward to see. For example the pion and kaon condensates are given by

\[
\langle \pi^+ \rangle = 2 f^2 B_0 \sin \alpha = 2 f^2 B_0 \sqrt{1 - \frac{m^4}{\mu^2}}, \quad \mu_I > m_\pi \quad (6.14)
\]

\[
\langle K^+ \rangle = 2 f^2 B_0 \sin \alpha = 2 f^2 B_0 \sqrt{1 - \frac{m^4}{(\frac{1}{2} \mu_I + \mu_S)^2}}, \quad \frac{1}{2} \mu_I + \mu_S > m_K. \quad (6.15)
\]

For any \( \mu_I > m_\pi \) and \( \frac{1}{2} \mu_I + \mu_S > m_K \), these condensates jump discontinuously to zero as we cross the phase line. The transition line itself is given by the equality of the pressures in the two phases. While it is possible to find this line analytically at tree level as shown in eq. (3.27), in order to find the line at NLO, we need to compare the pressure in the pion and kaon condensed phases, which can only be done numerically. We have performed this calculation for the central values from eqs. (6.4) and (6.6). In figure 2 we show the phase diagram in the \( \mu_I-\mu_S \) plane with the first order transition line increasing to higher strange chemical potential for all values of the isospin chemical potential greater than the pion mass. The vacuum phase is in the region bounded by the straight lines \( \mu_I = \pm m_\pi \), \( \mu_S = \pm (\frac{1}{2} \mu_I + m_K) \), and \( \mu_S = \pm (-\frac{1}{2} \mu_I + m_K) \). The corners from where the first-order lines emerge are located at \( (\mu_I, \mu_S) = (\pm 131, \pm 415.5) \) MeV. The solid lines represent second-order transitions while the dashed line indicates the tree level first-order transition and the green dot dashed line indicates the NLO first-order transition. In the vacuum phase, the thermodynamic functions are independent of the isospin and strange chemical potentials. This is an example of the so-called Silver Blaze property [55].

### 6.2 Medium-dependent masses

In this subsection, we will briefly discuss the medium-dependent masses. We restrict ourselves to a leading-order calculation, i.e. we consider the tree-level dispersion relations evaluated at \( p^2 = 0 \). In the pion-condensed phase, they are given by eqs. (3.77)–(3.81). In the kaon-condensed phase, they are given by eqs. (3.100)–(3.104). In the left panel of figure 3, we show the medium-dependent masses as a function of the isospin chemical potential \( \mu_I \) for fixed strange chemical potential \( \mu_S = 200 \) MeV. For \( \mu_I = 0 \), we are in the normal phase, the pion masses take on their vacuum values, while the kaons are degenerate in pairs. The mass of \( \pi^+ \) decreases as we increase \( \mu_I \) and vanishes when \( \mu_I = m_\pi \) and enter the pion-condensed phase. At \( \mu_I = m_\pi \), the masses vary continuously reflecting the second-order nature of the transition. We also note that the mass of \( \eta^0 \) is independent of \( \mu_I \), which follows directly from eq. (3.81). Finally, for asymptotically large values of \( \mu_I \),
the kaons and pions are pairwise degenerate. In the right panel of figure 3, we show the medium-dependent masses as a function of isospin chemical potential $\mu_I$ for fixed strange chemical potential $\mu_S = 460$ MeV. At $\mu_I = 0$, we are in the vacuum phase. The kaons are again degenerate in pairs, the pions are also degenerate taking on their vacuum values. We enter the kaon-condensed phase at $\mu_I = 42$ MeV, which is a second-order transition. In this phase, $K^+$ is the Goldstone mode associated with the spontaneous breakdown of the U(1)-symmetry. As we increase the isospin chemical potential past approximately $\mu_I = 268$ MeV, we enter the pion-condensed phase. In this phase, $\pi^+$ is the Goldstone mode associated with the spontaneous breakdown of the U(1)$_I$-symmetry. This first-order nature of the transition can be seen by the jumps in the quasiparticle masses.

Finally, we also note that in the charged pion and kaon condensed phases the mass eigenstates do not coincide with the charge eigenstates \[56\]. It is easy to see using the form of the inverse propagators in eqs. (3.62) and (3.84) that in the condensed phases the mass eigenstates can be found using momentum-dependent rotations of the mass eigenstates. However, the pion and kaon charge eigenstates are the standard ones

$$\pi^+ = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \quad K^+ = \frac{\phi_4 + i\phi_5}{\sqrt{2}}.$$

They can be deduced using the canonical form of the quadratic, kinetic terms in eqs. (3.61) and (3.83) in the unbroken phase with $\alpha = 0$, which possesses a global U(1) symmetry. When gauged (using electromagnetic fields), the Lagrangian possesses a local U(1) (gauge) symmetry, which is broken by the pion condensed phase.
6.3 Pressure, isospin density, and equation of state

In this subsection, we discuss the pressure, the isospin density and the equation of state in the pion-condensed phase and compare our results to the (2 + 1)-flavor lattice QCD results of refs. [43–45]. We begin with figure 4, where we plot the pressure (divided by $m_\pi^4$) as a function of $\mu_I = m_\pi$. The pressure has been normalized to be zero in the normal vacuum, which also has a zero isospin density. As pions condense beginning at the critical isospin chemical potential, $\mu_I^c = m_\pi$, the pressure increases with increasing chemical potential and continues to increase monotonically, a feature that is consistent with results from lattice QCD. The pressure from two-flavor $\chi$PT is smaller than that from lattice QCD even when the uncertainties within the LECs, the pion mass and pion decay constant are taken into account. The range of pressures due to the uncertainties calculated within two-flavor $\chi$PT is represented by the blue band. We find that the uncertainty in the pion mass and the pion decay constant (as opposed to the uncertainty in the LECs) dominates the uncertainty in the pressure. On the other hand, the pressure from three-flavor $\chi$PT (shown in green), which includes the contribution from strange quarks unlike two-flavor $\chi$PT, overestimates the pressure. In figure 4, we use a dark green band to show the uncertainty in the pressure due to the uncertainties in the pion mass and the pion decay constant, and we use a light green band to represents the uncertainty in the pressure due to the LECs, the pion mass and the pion decay constant. The result shows that unlike in two-flavor $\chi$PT, the uncertainty in the pressure is dominated by the uncertainty in the LECs.

It is clear from figure 4 that the difference in pressure calculated in two-flavor $\chi$PT versus that calculated in three-flavor $\chi$PT is quite significant. The tree level pressure in two and three-flavor $\chi$PT is identical. Therefore, the difference arises through the NLO contribution to the pressure in two-flavor and three-flavor $\chi$PT. Since the NLO contribution is suppressed by a power of $1/(4\pi f_\pi)^2$, the difference in the two-flavor and three-flavor pressure seems unusually large. In order to explain the difference we have mapped three-flavor $\chi$PT by expanding the effective potential in the limit of large strange quark masses in subsection 5.2. After identifying the appropriate two-flavor LECs in terms of three-flavor LECs — see eqs. (5.11)–(5.15) — we find the appropriate two-flavor LECs that are...
consistent with the values of three-flavor LECs. They are

\[ \tilde{l}_1(N_f = 3) = 14.5, \quad \tilde{l}_2(N_f = 3) = 6.5, \quad \tilde{l}_3(N_f = 3) = 4.1, \quad \tilde{l}_4(N_f = 3) = 4.2, \]  

with \( \tilde{l}_i \) being defined in eq. (5.16). In order to contrast the above values with the two-flavor LECs, we state the LECs below where

\[ \tilde{l}_1(N_f = 2) = -0.4, \quad \tilde{l}_2(N_f = 2) = 4.3, \quad \tilde{l}_3(N_f = 2) = 2.9, \quad \tilde{l}_4(N_f = 2) = 4.4. \]  

We note that while \( \tilde{l}_4 \) looks quite similar in the two cases, \( \tilde{l}_2 \) and \( \tilde{l}_3 \) are somewhat different with the difference in \( \tilde{l}_1 \) being the most significant (they have opposite signs). We calculated the pressure in two-flavor \( \chi PT \) using the LECs found to be consistent with three-flavor \( \chi PT \) — we show this result in figure 4 in brown (dashed). The result shows that even the two-flavor \( \chi PT \) overestimates the pressure compared to that from 2+1 flavor lattice QCD. This analysis shows that the overestimation of the pressure is due to the values of the LECs of three-flavor \( \chi PT \), which also have large uncertainties compared to two-flavor LECs. As a secondary observation, we note that as the strange quark mass becomes lighter, the pressure increases in \( \chi PT \), particularly for larger isospin chemical potential.

In figure 5, we plot the isospin density (divided by \( m_\pi^2 \)) as a function of the normalized chemical potential, \( \mu_I/m_\pi \). The isospin density is zero in the vacuum phase and monotonically increases in the pion-condensed phase. The rate of increase decreases as the isospin chemical potential increases. The isospin density from three-flavor \( \chi PT \) is consistent with that of lattice QCD in the normal vacuum and near the critical isospin chemical potential up to approximately \( \mu_I = 1.4m_\pi \). For larger isospin chemical potentials, three-flavor \( \chi PT \) consistently overestimates the isospin density. This is unlike the result in two-flavor \( \chi PT \) which is in extremely good agreement with lattice QCD. The two-flavor \( \chi PT \) result using three-flavor LECs is plotted in brown and shows that the three-flavor \( \chi PT \) result is largely explained by the discrepancy in the values of the LECs in two-flavor and three-flavor \( \chi PT \).
Finally, in figure 6 we plot the equation of state: the energy density divided by $m_{\pi}^4$ is plotted against the pressure divided by $m_{\pi}^4$. Three-flavor $\chi$PT consistently overestimates the energy density for all pressures though up to $P/m_{\pi}^4 \approx 0.10$, the discrepancy is small. Two-flavor $\chi$PT, on the other hand, underestimates the energy density up to $P/m_{\pi}^4 \approx 0.2$ but is largely consistent for values above it. Using three-flavor LECs, we can show that most of the discrepancy between two-flavor and three-flavor $\chi$PT is due to the discrepancy between the two sets of LECs, a theme common to all observables we have calculated in this work. The two-flavor results using three-flavor LEC is shown using brown dashed lines. The result is consistent with three-flavor $\chi$PT for $P/m_{\pi}^4 \approx 0.2$ while above it the two-flavor $\chi$PT result gives larger values of energy density. It is also worth noting that for a given value of pressure the energy density decreases with decreasing strange quark masses.
Acknowledgments

The authors would like to thank B. Brandt, G. Endródi and S. Schmalzbauer for useful discussions as well as for providing the data points of ref. [7]. P.A. would like to acknowledge the hospitality of the Niels Bohr International Academy, where a portion of the work was done.

A Meson masses and decay constants

In order to show the second-order nature of the phase transition from the vacuum to a Bose-condensed phase at \( \mu_I^2 = m_\pi \), and \( \mu^c_I + \mu^c_S = m_K \) where \( m_\pi \) and \( m_K \) are the physical meson masses in the vacuum, we need to express them in terms of the parameters \( B_0 m_\pi \), \( B_0 m_s \), and \( f \) of the chiral Lagrangian. The pion and kaon masses are [10]

\[
m_\pi^2 = m_{\pi,0}^2 \left[ 1 - \left( 8L_4^r + 8L_5^r - 16L_6^r - 16L_8^r + \frac{1}{2(4\pi)^2} \log \frac{\Lambda^2}{m_{\pi,0}^2} \right) \frac{m_{\pi,0}^2}{f^2} \\
-(L_4^r - 2L_6^r) \frac{16m_{K,0}^2}{f^2} + \frac{m_{\eta,0}^2}{6(4\pi)^2 f^2} \log \frac{\Lambda^2}{m_{\eta,0}^2} \right],
\]

(A.1)

\[
m_K^2 = m_{K,0}^2 \left[ 1 - (L_4^r - 2L_6^r) \frac{8m_{\pi,0}^2}{f^2} - (2L_4^r + L_5^r - 4L_6^r - 2L_8^r) \frac{8m_{K,0}^2}{f^2} \\
- \frac{m_{\eta,0}^2}{3(4\pi)^2 f^2} \log \frac{\Lambda^2}{m_{\eta,0}^2} \right].
\]

(A.2)

The pion and kaon decay constants, \( f_\pi \) and \( f_K \) respectively, are [10]

\[
f_\pi^2 = f^2 \left[ 1 + \left( 8L_4^r + 8L_5^r + \frac{2}{(4\pi)^2} \log \frac{\Lambda^2}{m_{\pi,0}^2} \right) \frac{m_{\pi,0}^2}{f^2} \\
+ \frac{16L_4^r + \frac{1}{(4\pi)^2} \log \frac{\Lambda^2}{m_{K,0}^2}}{m_{K,0}^2} \right]
\]

(A.3)

\[
f_K^2 = f^2 \left[ 1 + \left( 12L_4^r + \frac{3}{4(4\pi)^2} \log \frac{\Lambda^2}{m_{\pi,0}^2} \right) \frac{m_{\pi,0}^2}{f^2} + \left( 8L_5^r + \frac{3}{2(4\pi)^2} \log \frac{\Lambda^2}{m_{K,0}^2} \right) \frac{m_{K,0}^2}{f^2} \\
+ \left( 12L_4^r + \frac{3}{4(4\pi)^2} \log \frac{\Lambda^2}{m_{\eta,0}^2} \right) \frac{m_{\eta,0}^2}{f^2} \right].
\]

(A.4)

Using the expressions for the renormalization group equations, eq. (2.16), it is straightforward to see that the \( \Lambda \)-dependence of the coupling cancels against the chiral logarithms in expressions for the masses and decay constants.

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