SIMPLE CLOSED GEODESICS ON HYPERELLiptic
TRANSLATION SURFACES AND CLASSIFICATION THEOREM
FOR TRANSLATION SURFACES IN $H^\text{hyp}(4)$

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Abstract. In this paper, we give the maximum of the numbers $n$ such that we can take $n$ simple closed geodesics without singularities that are disjoint to each other for translation surfaces in the hyperelliptic components $H^\text{hyp}(2g-2)$ and $H^\text{hyp}(g-1, g-1)$. The maximum is different from that of the case of hyperbolic surfaces. We also give a classification theorem for translation surfaces in $H^\text{hyp}(4)$ with respect to their Euclidean structures.

1. Introduction

A translation surface $(X, \omega)$ is a Riemann surface $X$ together with a holomorphic 1-form $\omega$ on $X$. The 1-form $\omega$ induces a singular Euclidean structure on $X$. We can also construct a Euclidean structure on a compact Riemann surface $X$ from a holomorphic quadratic differential $q$ on $X$. The pair $(X, q)$ is called a flat surface.

Translation surfaces and flat surfaces are important for Teichmüller theory. The Teichmüller space $T_g$ is the space of equivalence classes of marked Riemann surfaces of genus $g$. The Teichmüller space $T_g$ has a complete metric called Teichmüller metric. By the uniformization theorem, Teichmüller space $T_g$ is also regarded as the space of equivalence classes of marked hyperbolic structures on a surface of genus $g$. From this, Teichmüller spaces are studied through the hyperbolic geometry on surfaces. However, deformations of Riemann surfaces along geodesics on the Teichmüller space $T_g$ are difficult to understand from hyperbolic structures. The Teichmüller theorem claims that Teichmüller geodesics on $T_g$ are represented by the deformation of translation surfaces and flat surfaces.

The moduli space $H_g$ of translation surfaces of genus $g$ is stratified by the orders of zeros of 1-forms. Fix $g \geq 2$. For positive integers $k_1, \ldots, k_m$ satisfying $k_1 \leq \cdots \leq k_m$ and $k_1 + \cdots + k_m = 2g - 2$, the stratum $H(k_1, \ldots, k_m)$ consists of all equivalence classes of pairs $(X, \omega)$ such that $\omega$ has exactly $m$ zeros of orders $k_1, \ldots, k_m$. Kontsevich-Zorich [KZ03] proved that $H(k_1, \ldots, k_m)$ has at most 3 connected components. The stratum $H(2g-2)$ and $H(g-1, g-1)$ have connected components called hyperelliptic components. We denote them by $H^\text{hyp}(2g-2)$ and $H^\text{hyp}(g-1, g-1)$, respectively. A translation surface $(X, \omega) \in H(2g-2)$ (resp. $H(g-1, g-1)$) is an element of $H^\text{hyp}(2g-2)$ (resp. $H^\text{hyp}(g-1, g-1)$) if and only if $X$ has the hyperelliptic involution $\tau$ and it satisfies $\tau^* \omega = -\omega$. We call translation surfaces in $H^\text{hyp}(2g-2) \cup H^\text{hyp}(g-1, g-1)$ hyperelliptic translation surfaces. In the case of genus 2, we have $H(2) = H^\text{hyp}(2)$ and $H(1, 1) = H^\text{hyp}(1, 1)$.

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In this paper, our main interests are hyperelliptic translation surfaces and their simple closed geodesics. On a translation surface, a closed geodesic may pass through some singularities. We call simple closed geodesics without singularities regular. One of the main problems in this paper is to determine the maximum of the numbers \( n \in \mathbb{N} \) such that there exist regular closed geodesics \( \gamma_1, \ldots, \gamma_n \) which are disjoint and not homotopic to each other on every hyperelliptic translation surface \((X, \omega)\). We denote the maximum number by \( N(X, \omega) \). In the case of hyperbolic surfaces of genus \( g \geq 2 \), the maximum number of pairwise disjoint simple closed geodesics is always \( 3g - 3 \). It is given by pants curves and the maximum number depends only on the genus \( g \). However, the maximal numbers depend on their strata for translation surfaces. Nguyen [Ngu17, Theorem 2.8] investigated the case of genus 2. If \((X, \omega) \in \mathcal{H}(2)\), then we have \( N(X, \omega) = 2 \). If \((X, \omega) \in \mathcal{H}(1, 1)\), then we have \( N(X, \omega) = 3 \). One of our main theorems is a generalization of this result to general genus.

**Theorem 2.15.** Let \( g \geq 2 \). If \((X, \omega) \in \mathcal{H}^{hyp}(2g - 2)\), then we have \( N(X, \omega) = g \). If \((X, \omega) \in \mathcal{H}^{hyp}(g - 1, g - 1)\), then we have \( N(X, \omega) = g + 1 \).

We can easily show by a topological argument that \( N(X, \omega) \leq g \) for \((X, \omega) \in \mathcal{H}^{hyp}(2g - 2)\) and \( N(X, \omega) \leq g + 1 \) for \((X, \omega) \in \mathcal{H}^{hyp}(g - 1, g - 1)\). To prove the converse inequality, we give \( g \) or \( g + 1 \) regular closed geodesics which are disjoint to each other. This is done by constructing a good triangulation of the quotient flat surface \((Y, q)\) of \((X, \omega)\) by the hyperelliptic involution. The flat surface \((Y, q)\) is of genus 0 and \( q \) is a holomorphic quadratic differential on \( Y \) that has only one zero and some simple poles. The triangles are “restricted triangles” defined as in Definition 3.6. All vertices of this triangulation are the unique zero of \( q \) and all poles of \( q \) are midpoints of some edges of triangles. This triangulation gives us saddle connections such that they connect poles and they are disjoint to each other. The preimages of the saddle connections are regular closed geodesics which we want.

By investigating triangulations of \((X, \omega) \in \mathcal{H}^{hyp}(4)\) by “restricted triangles”, we obtain the following theorem. Here, \( St_5 \) is a translation surface in \( \mathcal{H}^{hyp}(4) \) constructed from five squares as in Figure 1 and \( \text{GL}(2, \mathbb{R}) \cdot St_5 \) is its \( \text{GL}(2, \mathbb{R}) \)-orbit. A simple cylinder is a Euclidean cylinder in a translation surface that is foliated by parallel regular closed geodesics and each of whose boundary components passes the singular point only once.

**Theorem 4.2.** Let \((X, \omega) \in \mathcal{H}^{hyp}(4)\) and \( \tau \) the hyperelliptic involution of \((X, \omega)\). If \((X, \omega) \not\in \text{GL}(2, \mathbb{R}) \cdot St_5\), then there exist disjoint simple cylinders \( C_1, C_2, C_3 \) each of which is invariant under \( \tau \). If \((X, \omega) \in \text{GL}(2, \mathbb{R}) \cdot St_5\), then \((X, \omega)\) has at most two disjoint simple cylinders.

The following corollary is a classification theorem for translation surfaces in \( \mathcal{H}^{hyp}(4) \) with respect to the Euclidean structure on translation surfaces. Every translation surface \((X, \omega) \in \mathcal{H}^{hyp}(4) \setminus \text{GL}(2, \mathbb{R}) \cdot St_5\) and its Euclidean structure are realized by gluing three Euclidean cylinders with a flat surface of genus 0.

**Corollary 4.3** (Classification Theorem for translation surfaces in \( \mathcal{H}^{hyp}(4) \)). Let \((X, \omega) \in \mathcal{H}^{hyp}(4) \setminus \text{GL}(2, \mathbb{R}) \cdot St_5\). Then, there exists a flat surface \((X_0, q)\) of genus 0 such that \( q \) has a zero \( p_0 \) of order 2 and six simple poles \( p_1, p_2, \ldots, p_6 \), an involution \( \tau_0 : X_0 \to X_0 \), saddle connections \( s_1, s_2, s_3 \), and Euclidean cylinders \( C_1, C_2, C_3 \) satisfying the following (see Figure 17):
(1) $\tau_0^* q = q$.
(2) $s_i$ is a saddle connection of $(X_0, q)$ connecting $p_0$ and $p_i$ for $i = 1, 2, 3$.
(3) $\tau_0(s_i)$ connects $p_0$ and $p_{i+3}$ for $i = 1, 2, 3$.
(4) two of each $s_1, s_2, s_3, \tau_0(s_1), \tau_0(s_2)$, and $\tau_0(s_3)$ intersect only at $p_0$.
(5) the circumference of $C_i$ equals $2|s_i|$ for $i = 1, 2, 3$, and
(6) $(X, \omega)$ is obtained by cutting $(X_0, q)$ along the saddle connections $s_1, s_2, s_3, \tau_0(s_1), \tau_0(s_2), \tau_0(s_3)$ and gluing $C_i$ to the slits $s_i$ and $\tau_0(s_i)$ for all $i = 1, 2, 3$.

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2. Preliminaries

Through this paper, we assume that Riemann surfaces are always analytically finite. For an analytically finite Riemann surface $X$, we denote by $\overline{X}$ the compact Riemann surface $X$ with the punctures filled.

2.1. Holomorphic 1-forms and their moduli $\mathcal{H}_g$.

Definition 2.1 (Holomorphic 1-forms). Let $X$ be a Riemann surface. A holomorphic 1-form $\omega$ on $X$ is a tensor whose restriction to every local coordinate neighborhood $(U, z)$ has the form $fdz$, where $f$ is a holomorphic function on $U$.

Let $\omega$ be a holomorphic 1-form on a Riemann surface $X$. Choose local coordinate neighborhoods $(U, z), (V, w)$ with $U \cap V \neq \emptyset$. If $\omega = fdz$ on $(U, z)$ and $\omega = gdw$ on $(V, w)$, we have

$$f(z) \left( \frac{dz}{dw} \right) = g(w).$$

Thus, if $p_0 \in U \cap V$ is a zero of $f$ of order $n$ then it is also a zero of $g$ of order $n$. Now, zeros of $\omega$ and their orders are well-defined. We denote by $\text{Sing}(\omega)$ the set of all zeros of $\omega$. Let $\text{ord}_\omega(p)$ denote the order of a zero $p$ of $\omega$. A holomorphic 1-form on $X$ whose restriction to every local coordinate neighborhood is constant function 0 is denoted by 0. By the Riemann-Roch formula, we have the following.

Proposition 2.2. Let $X$ be a compact Riemann surface of genus $g \geq 2$ and $\omega \neq 0$ a holomorphic 1-form on $X$. Then we have

$$\sum_{p \in \text{Sing}(\omega)} \text{ord}_\omega(p) = 2g - 2.$$

Fix $g \geq 2$. Let $\mathcal{M}_g$ be the moduli space of compact Riemann surfaces of genus $g$. For a compact Riemann surface $X$, we denote by $\Omega(X)$ the vector space of all holomorphic 1-forms on $X$. We set $\Omega^*(X) = \Omega(X) - \{0\}$. Then, the moduli space of pairs $(X, \omega)$ is defined by

$$\mathcal{H}_g = \{ (X, \omega) : X \in \mathcal{M}_g, \omega \in \Omega^*(X) \} / \sim .$$

Where, for pairs $(X, \omega)$ and $(Y, \omega')$, the relation $(X, \omega) \sim (Y, \omega')$ holds if there exists a conformal map $f : X \to Y$ such that $f^* \omega' = \omega$. The moduli space $\mathcal{H}_g$ is a complex algebraic orbifold of dimension $4g - 3$. This is fibered over the moduli space $\mathcal{M}_g$ with the fiber over each $[X] \in \mathcal{M}_g$ equals $\Omega^*(X)/\text{Aut}(X)$. 
The orbifold $\mathcal{H}_g$ is stratified by the orders of zeros of 1-forms. Let $k_1, \ldots, k_m$ be positive integers whose sum is $2g - 2$ and which satisfy $k_1 \leq k_2 \leq \cdots \leq k_m$. The subspace $\mathcal{H}(k_1, \ldots, k_m)$ of $\mathcal{H}_g$ consists of all equivalence classes of pairs $(X, \omega)$ such that $\omega$ has exactly $m$ zeros of orders $k_1, \ldots, k_m$. Then we have
\[
\mathcal{H}_g = \bigcup_{0 < k_1 \leq k_2 \leq \cdots \leq k_m \quad k_1 + \cdots + k_m = 2g - 2} \mathcal{H}(k_1, \ldots, k_m).
\]
Each $\mathcal{H}(k_1, \ldots, k_m)$ is a stratum of $\mathcal{H}_g$. It is known that the dimension of $\mathcal{H}(k_1, \ldots, k_m)$ is $2g + m - 1$ (see [Mas82], [Vee82] and [Vee90]).

**Example 2.3.** The orbifold $\mathcal{H}_2$ has two strata $\mathcal{H}(2)$, and $\mathcal{H}(1, 1)$. The orbifold $\mathcal{H}_3$ has five strata $\mathcal{H}(4), \mathcal{H}(2, 2), \mathcal{H}(1, 3), \mathcal{H}(1, 1, 2)$, and $\mathcal{H}(1, 1, 1, 1)$.

In the rest of this section, we study connected components of strata of $\mathcal{H}_g$. In this paper, our interests are hyperelliptic components.

**Definition 2.4.** Let $g \geq 2$. Let $\mathcal{H}^{hyp}(2g - 2)$ be the subset of $\mathcal{H}(2g - 2)$ consisting of all $(X, \omega)$ such that $X$ has the hyperelliptic involution $\tau$ and $\omega$ satisfies $\tau^* \omega = -\omega$. Let $\mathcal{H}^{hyp}(g - 1, g - 1)$ be the subset of $\mathcal{H}(g - 1, g - 1)$ consisting of all $(X, \omega)$ such that $X$ has the hyperelliptic involution $\tau$ and $\omega$ satisfies $\tau^* \omega = -\omega$.

**Remark 1.** For the case of genus 2, we have $\mathcal{H}(2) = \mathcal{H}^{hyp}(2)$ and $\mathcal{H}(1, 1) = \mathcal{H}^{hyp}(1, 1)$.

The connected components of the strata are completely classified by Kontsevich and Zorich. See [KZ03] for the definition of $\mathcal{H}^{even}$, $\mathcal{H}^{odd}$, and $\mathcal{H}^{monhyp}$.

**Theorem 2.5** ([KZ03], Theorem 2). In the case of genus two, two strata $\mathcal{H}(2)$ and $\mathcal{H}(1, 1)$ are connected. In the case of genus three, $\mathcal{H}(4)$ has two connected components $\mathcal{H}^{hyp}(4)$ and $\mathcal{H}^{odd}(4)$. The stratum $\mathcal{H}(2, 2)$ has two connected components $\mathcal{H}^{hyp}(2, 2)$ and $\mathcal{H}^{odd}(2, 2)$. The other strata are connected.

**Theorem 2.6** ([KZ03], Theorem 1). Let $g \geq 4$. We have the following.

- The stratum $\mathcal{H}(2g - 2)$ has three connected components $\mathcal{H}^{hyp}(2g - 2)$, $\mathcal{H}^{even}(2g - 2)$, and $\mathcal{H}^{odd}(2g - 2)$.
- If $g$ is odd, the stratum $\mathcal{H}(g - 1, g - 1)$ has three connected components $\mathcal{H}^{hyp}(g - 1, g - 1)$, $\mathcal{H}^{even}(g - 1, g - 1)$, and $\mathcal{H}^{odd}(g - 1, g - 1)$.
- If $g$ is even, the stratum $\mathcal{H}(g - 1, g - 1)$ has two connected components $\mathcal{H}^{hyp}(g - 1, g - 1)$ and $\mathcal{H}^{monhyp}(g - 1, g - 1)$.
- All other strata which is of the form $\mathcal{H}(2k_1, \ldots, 2k_n)$ ($k_1, \ldots, k_n \in \mathbb{N}$) has two connected components $\mathcal{H}^{even}(2k_1, \ldots, 2k_n)$ and $\mathcal{H}^{odd}(2k_1, \ldots, 2k_n)$.
- The others are connected.

For $g \geq 2$, we call $\mathcal{H}^{hyp}(2g - 2)$ and $\mathcal{H}^{hyp}(g - 1, g - 1)$ hyperelliptic components. If $(X, \omega) \in \mathcal{H}^{hyp}(2g - 2)$, the unique zero of $\omega$ is fixed by the hyperelliptic involution of $X$. If $(X, \omega) \in \mathcal{H}^{hyp}(g - 1, g - 1)$, two zeros of $\omega$ are permuted by the hyperelliptic involution of $X$.

2.2. Meromorphic quadratic differentials.

**Definition 2.7** (Meromorphic quadratic differential). Let $X$ be a Riemann surface. A meromorphic quadratic differential $q$ on $X$ is a tensor whose restriction to every local coordinate neighborhood $(U, z)$ has the form $fdz^2$, where $f$ is a meromorphic
function on $U$. For a meromorphic quadratic differential $q$ on $X$, we denote by $|q|$ the 
 differential 2-form on $X$ whose restriction to every local coordinate neighborhood 
 $(U, z)$ is $|f|dx dy$ if the restriction of $q$ to $(U, z)$ is $fdz^2$. Here, $x$ is the real part of 
 $z$ and $y$ is the imaginary part of $z$. The $L^1$-norm $||q||$ of a meromorphic quadratic 
 differential $q$ on $X$ is defined by $||q|| = \int_X |q|$.

We assume that meromorphic quadratic differentials always have finite $L^1$-norms. 
By this assumption, meromorphic quadratic differential $q$ may have poles of order 
at most 1. Thus, the poles of $q$ are simple poles if they exist. The set of all zeros 
and poles of $q$ is denoted by $\text{Sing}(q)$.

Let $X$ be a compact Riemann surface of genus $g$. Let $q$ be a meromorphic 
quadratic differential on a Riemann surface $X$. The order of a zero $p$ of $q$ is denoted 
by $\text{ord}_q(p)$. If $p$ is a simple pole of $q$, we set $\text{ord}_q(p) = -1$. For other point $p'$ 
of $X$, we set $\text{ord}_q(p') = 0$. By the Riemann-Roch formula, we have 
$$\sum_{p\in X} \text{ord}_q(p) = 4g - 4.$$ 

2.3. Translation surfaces and Flat surfaces.

**Definition 2.8** (Translation surface and flat surface). A translation surface $(X, \omega)$ 
is a pair of a Riemann surface $X$ and a non-zero holomorphic 1-form $\omega$ on $X$. The 
points of $\text{Sing}(\omega)$ are called singular points of $(X, \omega)$. A flat surface $(X, q)$ is a pair 
of a Riemann surface $X$ and a non-zero meromorphic quadratic differential $q$ on $X$. 
The points of $\text{Sing}(q)$ are called singular points of $(X, q)$.

Let $(X, \omega)$ be a flat surface. The holomorphic 1-form $\omega$ gives a Euclidean structure 
on $X$. If $p_0 \in X$ is not a zero of $\omega$, we may choose a neighborhood $U$ of $p_0$ 
such that the map 
$$z(p) = \int_{p_0}^p \omega : U \to \mathbb{C}$$
is a chart of $X$. The collection of such charts $u = \{(U, z)\}$ is an atlas on $X - \text{Sing}(\omega)$. 
The transition functions of this atlas are always of the forms $w = z + (\text{const.})$. 
Thus, $u$ is a singular Euclidean structure on $(X, \omega)$. If $p_0$ is a zero of $\omega$ of order $n$ 
$(n = 1, 2, \ldots)$, there exists a chart $(U, \zeta)$ of $X$ around $p_0$ such that $\omega$ is represented 
as $\omega = \zeta^n d\zeta$. Then, we have a chart 
$$z(p) = \int_{p_0}^p \omega = \int_{p_0}^p \zeta^n d\zeta = \frac{1}{n + 1} \zeta(p)^{n+1}$$
around $p_0$. With respect to this chart, the angle around $p_0$ is $2(n + 1)\pi$.

Let $(X, q)$ be a translation surface. An atlas $u$ on $X - \text{Sing}(q)$ is also given 
by $q$. For each $p_0 \in X - \text{Sing}(q)$, we can choose a neighborhood $U$ such that the 
integration 
$$z(p) = \int_{p_0}^p \sqrt{q} : U \to \mathbb{C}$$
is well-defined and gives a chart. The collection of such charts $u = \{(U, z)\}$ is an atlas on $X - \text{Sing}(q)$ whose transition functions are of the forms $w = \pm z + (\text{const.})$. 
If $p_0$ is a zero of $q$ or a pole of $q$ with $\text{ord}_q(p) = n$ $(n = -1, 1, 2, \ldots)$, there exists 
a chart $(U, \zeta)$ of $X$ around $p_0$ such that $q$ is represented as $q = \zeta^n d\zeta^2$. Then, we have a chart 
$$z(p) = \int_{p_0}^p \sqrt{q} = \int_{p_0}^p \zeta^n d\zeta = \frac{2}{n + 2} \zeta(p)^{n+2}.$$
around \( p_0 \). This implies that the angle around \( p_0 \) with respect to this chart is \((n + 2)\pi\).

**Remark 2.** By the definition of \( \mathcal{H}_g \), the space \( \mathcal{H}_g \) is also regarded as the moduli space of translation surfaces of genus \( g \).

### 2.4. Regular closed geodesics and curve complexes.

We define some terminologies for translation surfaces which are also defined for flat surfaces as well.

**Definition 2.9** (Saddle connection). A saddle connection is a geodesic segment on a translation surface \((X, \omega)\) joining two singular points or a singular point to itself and containing no singular points in its interior.

For any hyperbolic surface, every homotopy class of simple closed curves contains a unique geodesic. This is not true for translation surfaces and flat surfaces.

**Proposition 2.10** ([Str84], Theorem 14.3). Let \((X, \omega)\) be a translation surface. Let \( \gamma \) be a simple closed curve on \((X, \omega)\). Then, one of the following holds.

- The homotopy class of \( \gamma \) contains a unique closed geodesic of \((X, \omega)\). Moreover, the geodesic is a concatenation of saddle connections. The angles between consecutive saddle connections are at least \( \pi \) on both sides.
- The homotopy class of \( \gamma \) contains infinitely many simple closed geodesics containing no singular points. They are parallel to each other and the union of all such geodesics forms an open Euclidean cylinder \( C_\gamma \). Each of the boundary components of \( C_\gamma \) is a concatenation of saddle connections that are parallel to \( \gamma \).

**Definition 2.11** (Regular closed geodesic). A simple closed geodesic \( \gamma \) on a translation surface that contains no singular points is called regular.

Let \((X, \omega)\) be a translation surface. We denote by \( N(X, \omega) \) the maximum of the numbers \( n \in \mathbb{N} \) such that there exist regular closed geodesics \( \gamma_1, \ldots, \gamma_n \) on \((X, \omega)\) which are disjoint and not homotopic to each other. For a hyperbolic surface of genus \( g \) with \( n \) punctures, the maximum number of disjoint closed geodesics is always \( 3g - 3 + n \). Hence, the maximum numbers depend only on the topological types of surfaces. However, for translation surfaces, it is not true.

**Theorem 2.12** ([Ngu17], Theorem 2.8). Let \((X, \omega)\) be a translation surface of genus 2. If \((X, \omega) \in \mathcal{H}(2)\), then \( N(X, \omega) = 2 \). If \((X, \omega) \in \mathcal{H}(1, 1)\), then \( N(X, \omega) = 3 \).

Theorem 2.12 is also described in terms of curve complex in [Ngu17]. The curve complex \( C(S) \) for a surface \( S \) of finite type is a simplicial complex such that the vertices are the free homotopy classes of essential simple closed curves on \( S \) and \( k + 1 \) vertices span a simplex if and only if they have pairwise disjoint representatives for each \( k \geq 0 \). The curve complexes for translation surfaces are analogies of this.

**Definition 2.13** (Curve complex for translation surface \((X, \omega)\)). The curve complex \( C_{\text{cyl}} = C_{\text{cyl}}(X, \omega) \) for a translation surface \((X, \omega)\) is a simplicial complex such that the vertices are regular closed geodesics on \((X, \omega)\) and \( k + 1 \) vertices span a simplex if and only if they are pairwise disjoint for each \( k \geq 0 \).

**Theorem 2.14** ([Ngu17], Theorem A). Let \((X, \omega)\) be a translation surface of genus 2. If \((X, \omega) \in \mathcal{H}(2)\), then \( C_{\text{cyl}}(X, \omega) \) contains no 2-simplexes. If \((X, \omega) \in \mathcal{H}(1, 1)\), then \( C_{\text{cyl}}(X, \omega) \) contains some 2-simplexes.
One of our main results is a generalization of these theorems to the case of higher genus case.

**Theorem 2.15.** Let $g \geq 2$. If $(X, \omega) \in \mathcal{H}^{hyp}(2g - 2)$, then $N(X, \omega) = g$. If $(X, \omega) \in \mathcal{H}^{hyp}(g - 1, g - 1)$, then $N(X, \omega) = g + 1$.

**Corollary 2.16.** Let $g \geq 2$. If $(X, \omega) \in \mathcal{H}^{hyp}(2g - 2)$, then $C_{\text{cyl}}(X, \omega)$ contains some $(g - 1)$-simplexes but contains no $g$-simplexes. If $(X, \omega) \in \mathcal{H}^{hyp}(g - 1, g - 1)$, then $C_{\text{cyl}}(X, \omega)$ contains some $g$-simplexes but contains no $(g + 1)$-simplexes.

2.5. SL(2,\mathbb{R})-orbits of translation surfaces and Veech groups. Let $g \geq 2$. There exists an action of SL(2,\mathbb{R}) on $\mathcal{H}_g$ which leaves each stratum invariant. Let $A \in \text{SL}(2, \mathbb{R})$ and $(X, \omega) \in \mathcal{H}_g$. Then, $\omega$ induces an Euclidean structure $u = \{(U, z)\} \in X$. We regard $A$ as a linear map and compose it to each chart of $u$. Then, the atlas $A \circ u = \{(U, A \circ z)\} \in \text{a Euclidean structure on the surface } X$. We consider it as a new conformal structure on $X$. Then, we have a holomorphic 1-form $A \circ \omega$ on $X$ whose restriction to each chart of $(U, w) \in A \circ u$ is $dw$. The 1-forms $\omega$ and $A \circ \omega$ are in the same stratum. The SL(2,\mathbb{R})-orbit of a translation surface $(X, \omega)$ is defined by

$$\text{SL}(2, \mathbb{R}) \cdot (X, \omega) = \{(X, A \circ \omega) : A \in \text{GL}(2, \mathbb{R})\}.$$ 

**Definition 2.17** (Veech group). The Veech group $\Gamma(X, \omega)$ of $(X, \omega)$ is the subgroup of all $A \in \text{SL}(2, \mathbb{R})$ which leaves $(X, \omega)$ invariant. That is, $\Gamma(X, \omega)$ is written as

$$\Gamma(X, \omega) = \{A \in \text{SL}(2, \mathbb{R}) : (X, A \circ \omega) = (X, \omega)\}.$$

**Theorem 2.18** ([Vee89], Proposition 2.7). *The Veech group $\Gamma(X, \omega)$ of a translation surface $(X, \omega)$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$.***

Veech groups are important to understand translation surfaces.

**Definition 2.19** (periodic direction). Let $(X, \omega)$ be a translation surface. A direction $\theta \in [0, \pi)$ is called periodic if every $z \in X$ is contained in a regular closed geodesic of direction $\theta$ or a saddle connection of direction $\theta$.

**Theorem 2.20** (Veech’s dichotomy Theorem, [Vee89], Propositions 2.4, 2.10, and 2.11). *Let $(X, \omega)$ be a translation surface whose Veech group $\Gamma(X, \omega)$ is a lattice in $\text{SL}(2, \mathbb{R})$ (i.e., a subgroup of finite covolume). For each direction $\theta \in [0, \pi)$, one of the following holds:

1. $\theta$ is periodic,
2. every geodesic with direction $\theta$ is dense in $X$.

Moreover, in case (1), there exists a parabolic element $A \in \Gamma(X, \omega)$ which has a unique eigenvector of direction $\theta$. If $\theta = 0$, the fixed point of $A$ as a Möbius transformation is $\infty$. If $\theta \neq 0$, the fixed point of $A$ as a Möbius transformation is $\cot \theta$.

We denote by $\text{Peri}(X, \omega)$ the subset of $[0, \pi)$ consisting of all periodic directions of $(X, \omega)$. Let us define the action of $\text{SL}(2, \mathbb{R})$ on $[0, \pi)$. For any $A \in \text{SL}(2, \mathbb{R})$ and $\theta, \theta' \in [0, \pi)$, we assume that the equation $A \cdot \theta = \theta'$ holds if and only if

$$A \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = t \begin{bmatrix} \cos \theta' \\ \sin \theta' \end{bmatrix}$$

holds for some $t \in \mathbb{R}$. With respect to this action, the Veech group $\Gamma(X, \omega)$ preserves $\text{Peri}(X, \omega)$. Moreover, we have the following by Theorem 2.20.
Proposition 2.21. Let \( f : [0, \pi) \to \mathbb{R} \) be a function such that

\[
f(\theta) = \begin{cases} 
\cot \theta & (\theta \in (0, \pi)) \\
\infty & (\theta = 0).
\end{cases}
\]

Let \((X, \omega)\) be a translation surface whose Veech group \( \Gamma(X, \omega) \) is a lattice in \( \text{SL}(2, \mathbb{R}) \). Then, the set \( f(\text{Peri}(X, \omega)) \) coincides with the set of all fixed points of parabolic elements of \( \Gamma(X, \omega) \). Moreover, there exists a bijection from \( \text{Peri}(X, \omega) / \Gamma(X, \omega) \) to \( f(\text{Peri}(X, \omega))/\Gamma(X, \omega) \). Here \( \Gamma(X, \omega) \) acts on \( \text{Peri}(X, \omega) \) as linear maps and on \( f(\text{Peri}(X, \omega)) \) as Möbius transformations.

Proof. Let \( \theta, \theta' \in \text{Peri}(X, \omega) \). Assume that there exists \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(X, \omega) \) satisfying \( A \cdot \theta = \theta' \). Then the equation

\[
A \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = t \begin{bmatrix} \cos \theta' \\ \sin \theta' \end{bmatrix}
\]

holds for some \( t \in \mathbb{R} \). This implies the equation

\[
\frac{af(\theta) + b}{cf(\theta) + d} = f(\theta') = f(A \cdot \theta)
\]

holds. Therefore, the function \( f \) induces a surjective map from \( \text{Peri}(X, \omega) / \Gamma(X, \omega) \) to \( f(\text{Peri}(X, \omega))/\Gamma(X, \omega) \). By following this argument in reverse, we can also conclude that this map is injective.

Next, we show that the set \( f(\text{Peri}(X, \omega)) \) coincides with the set of all fixed points of parabolic elements of \( \Gamma(X, \omega) \). By Theorem 2.20, there exists a parabolic element \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(X, \omega) \) which has a unique eigenvector of direction \( \theta \). By the same way as above

\[
\frac{af(\theta) + b}{cf(\theta) + d} = f(\theta)
\]

holds. Therefore, \( f(\theta) \) is a fixed point of a parabolic element \( A \in \Gamma(X, \omega) \). The converse is also proved by following this argument in reverse. \( \square \)

Let \((X, \omega)\) be a translation surface whose Veech group \( \Gamma(X, \omega) \) is a lattice in \( \text{SL}(2, \mathbb{R}) \). Assume that \( \theta \in [0, \pi) \) is periodic. By Theorem 2.20, the union of all regular closed geodesics of direction \( \theta \) consists of some open Euclidean cylinders \( C_1, \ldots, C_n \). Each boundary component of the cylinders consists of saddle connections of direction \( \theta \). The family of the cylinders \( \{C_1, \ldots, C_n\} \) is called the cylinder decomposition for \( \theta \).

Proposition 2.22. Let \((X, \omega)\) be a translation surface and \( u = \{(U, z)\} \) the atlas of \((X, \omega)\). Let \( \{C_1, \ldots, C_m\} \) and \( \{C'_1, \ldots, C'_n\} \) be cylinder decompositions for periodic directions \( \theta \) and \( \theta' \) of \((X, \omega)\), respectively. Suppose that there exists \( A \in \Gamma(X, \omega) \) which maps the direction \( \theta \) to \( \theta' \). Then, we have \( m = n \) and there exists a homeomorphism \( h : (X, \omega) \to (X, \omega) \) satisfying the following:

1. \( \{h(C_1), \ldots, h(C_m)\} = \{C'_1, \ldots, C'_n\} \) and
2. for any \((U, z)\) and \((V, w) \in u\) with \( h(U) \subset V \), the composition \( w \circ h \circ z^{-1} \) is an affine map whose derivative is \( A \).
Proof. Let \( H_i \) be the height of the cylinder \( C_i \) and \( \widetilde{C}_i \) the \( \left\lfloor \frac{1}{2} H_i \right\rfloor \)-neighborhood of the line \((\sin \theta) x - (\cos \theta) y = 0 \) in \( \mathbb{C} \) for each \( i = 1, 2, \ldots, m \). Then \( \widetilde{C}_i \) is the universal covering of \( C_i \). The linear map \( \tilde{h}_i : \widetilde{C}_i \to \tilde{h}_i(C_i) \); \( z \mapsto Az \) is projected to a homeomorphism \( h_i : C_i \to h_i(C_i) \). Gluing the cylinders \( h_1(C_1), \ldots, h_m(C_m) \) in the same way as the construction of \((X, \omega)\) from \( C_1, \ldots, C_m \), we obtain a flat surface \((X, A \circ \omega)\). Since \( A \in \Gamma(X, \omega) \), we have \((X, A \circ \omega) = (X, \omega)\). This implies that \( \{ h_1(C_1), h_2(C_2), \ldots, h_m(C_m) \} \) coincides with the cylinder decomposition \( \{ C'_1, \ldots, C'_n \} \). Thus, we have \( m = n \). Now, we can define a homeomorphism \( h : (X, \omega) \to (X, \omega) \) such that \( h |_{C_i} = h_i \) for all \( i = 1, 2, \ldots, n \) and satisfies the condition (2). \( \square \)

We finish this section after giving an example of Veech groups.

**Definition 2.23 ([Sch06]).** The translation surface \( St_5 \in \mathcal{H}^{byp}(4) \) is constructed from 5 squares as in Figure 1 and the 1-form \( dz \) on it with the same labels glued.

![Figure 1. The translation surface \( St_5 \).](image)

**Remark 3.** The action of the hyperelliptic involution \( \tau \) on \( St_5 \) is seen by cutting \( St_5 \) into five squares and rotating them by \( \pi \) rotation about the centers. We can glue the resulting squares in the same way as \( St_5 \). Then we again have \( St_5 \). The centers of the squares correspond to the fixed points of \( \tau \). The other fixed points are the unique singular point and the points corresponding to the midpoints of sides labeled by \( a \) and \( 3 \) as in Figure 1.

**Example 2.24.** The Veech group \( \Gamma(St_5) \) is calculated by Schmithüsen ([Sch06, Proposition 13]). The Veech group \( \Gamma(St_5) \) is described as

\[
\Gamma(St_5) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z}) : a + c \text{ and } b + d \text{ are odd} \right\}.
\]

Set \( T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \) and \( R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). By using the Reidemeister-Schreier method (see [Sch04]), we can see that

\[
\Gamma(St_5) = \left\langle R, T^2, (TR)T(TR)^{-1} \right\rangle
\]

and the coset representatives of \( \Gamma(St_5) \) in \( \text{SL}(2, \mathbb{R}) \) is \( \{ I, T, TR \} \). The quotient \( \mathbb{H}/\Gamma(St_5) \) is an orbifold of genus 0 and has two cusps and only one cone point of order 2. We observe the action of homeomorphisms corresponding to generators \( R, T^2, \) and \( (TR)T(TR)^{-1} \) as in Proposition 2.22. Let \( P \) be the union of 5 squares. The action of \( R \) is seen by rotating \( P \) by \( \frac{\pi}{2} \) rotation and cutting it into five squares and pasting them so that the form is the same as \( P \) (see Figure 2). To see the action of \( T^2 \), we cut \( St_5 \) along the horizontal sides of squares. Then \( St_5 \) is decomposed...
into 3 cylinders. On the cylinder of area 1, the matrix \( T^2 \) acts as the square of the right-hand Dehn twist along the core curve. The action of \( T^2 \) onto the other cylinders is the right-hand Dehn twists along the core curves (see Figure 3). The actions induce an action of \( T^2 \) on \( \text{St}_5 \) since each boundary component of the cylinders is pointwise fixed by the actions. The action of \( (TR)T(TR)^{-1} \) is difficult to see. Let \( S \in \text{SL}(2, \mathbb{R}) \) be a \( \frac{\pi}{4} \)-rotation. By computation, we have \( (TR)T(TR)^{-1} = ST^2S^{-1} \). This implies that \( (TR)T(TR)^{-1} \) preserves the direction \( \frac{\pi}{4} \). Let us cut \( P \) along the diagonals of direction \( \frac{\pi}{4} \) and glue them as in Figure 4. Let \( P' \) be the resulting parallelogram. Regard the bottom vertex of \( P' \) as the origin. We act the linear map \( (TR)T(TR)^{-1} = ST^2S^{-1} \) to \( P' \). Then, the permutation of the labels of the upper side given by \( (TR)T(TR)^{-1} \) is a cyclic permutation \( \sigma = (e \ d \ c \ b \ a) \). The labels of the lower side of \( P' \) in invariant under \( (TR)T(TR)^{-1} \). Next, by identifying the sides with label \( f \), the translation \( z \mapsto z + 2(1+i) \) on \( P' \) corresponds to the permutation \( \tau = (a \ d \ b \ e \ c) \) for the labels of the upper side and the permutation \( \mu = (a \ c \ e \ b \ d) \) for the labels of the lower side. Therefore, the composition of the two actions corresponds to the permutation \( \tau \sigma = (a \ c \ e \ b \ d) \) for the labels of the upper side and the permutation \( \mu = (a \ c \ e \ b \ d) \) for the labels of the lower side. As \( \tau \sigma = \mu \), the composition of the two actions is well-defined on the translation surface \( \text{St}_5 \).

**Figure 2.** The action of \( R \) onto \( \text{St}_5 \).

**Figure 3.** The action of \( T \) onto \( \text{St}_5 \).

Observing the actions of elements of \( \Gamma(\text{St}_5) \), we have the following which is used for Lemma 4.5.

**Proposition 2.25.** Let \( \tau \) be the hyperelliptic involution of \( \text{St}_5 \). Let \( p_1 \) and \( p_2 \) be fixed points of \( \tau \) which are the images of the midpoints of sides with labels \( a \) and \( 3 \), respectively. For any \( A \in \Gamma(\text{St}_5) \), the action of \( A \) onto \( \text{St}_5 \) preserves the set \( \{p_1, p_2\} \).
Proof. We can easily see that the generators $R, T^2, (TR)T(TR)^{-1}$ of $\Gamma(St_5)$ preserve the set $\{p_1, p_2\}$ by observing the above example. Moreover, $p_1$ and $p_2$ are fixed points of $\tau$. Thus, we have the claim. \hfill \qed

3. Proof of Theorem 2.15

In this section, we prove Theorem 2.15. Let $g \geq 2$. Let $(X, \omega)$ be a translation surface in $H^{hyp}(2g - 2) \cup H^{hyp}(g - 1, g - 1)$ with the hyperelliptic involution $\tau$.

3.1. Upper estimates of $N(X, \omega)$.

Proposition 3.1. If $(X, \omega) \in H^{hyp}(2g - 2)$, then we have $N(X, \omega) \leq g$. If $(X, \omega) \in H^{hyp}(g - 1, g - 1)$, then we have $N(X, \omega) \leq g + 1$.

Proof. Assume that $(X, \omega) \in H^{hyp}(2g - 2)$. If $N(X, \omega) \geq g + 1$, then there exist disjoint regular closed geodesics $\gamma_1, \ldots, \gamma_{g+1}$ in $(X, \omega)$ that are not homotopic to each other. Since $(X, \omega)$ is a surface of genus $g$, $X - (\gamma_1 \cup \cdots \cup \gamma_{g+1})$ has at least two connected components. Since $(X, \omega)$ has only one singular point, one of the connected components $X'$ is a translation surface that has no singular points. Taking a double of $X'$, we obtain a compact translation surface without singular points. Then, it must be a torus. This implies that $X'$ is a Euclidean cylinder. Now, the boundary components of $X'$ are regular closed geodesics that are homotopic to each other. This is a contradiction. Hence, we have $N(X, \omega) \leq g$.

Next, we assume that $(X, \omega) \in H^{hyp}(g - 1, g - 1)$. If $N(X, \omega) \geq g + 2$, then there exist disjoint regular closed geodesics $\gamma_1, \ldots, \gamma_{g+2}$ in $(X, \omega)$ that are not homotopic to each other. Since $(X, \omega)$ is a surface of genus $g$, $X - (\gamma_1 \cup \cdots \cup \gamma_{g+1})$ has at least three connected components. Since $(X, \omega)$ has only two singular points, one of the connected components $X'$ is a translation surface that has no singular points. By the same argument as above, this is a contradiction. Hence, we have $N(X, \omega) \leq g + 1$. \hfill \qed

3.2. The idea of lower estimates of $N(X, \omega)$. Hereafter, we prove the inequality:

\begin{equation}
N(X, \omega) \geq \begin{cases} 
g & \text{if } (X, \omega) \in H^{hyp}(2g - 2) 
g + 1 & \text{if } (X, \omega) \in H^{hyp}(g - 1, g - 1)
\end{cases}
\end{equation}
To do this, we consider the quotient of \((X, \omega) \in \mathcal{H}^{hyp}(2g-2) \cup \mathcal{H}^{hyp}(g-1, g-1)\) by its hyperelliptic involution \(\tau\). By definition, we have \(\tau^* \omega = -\omega\). Thus, the holomorphic quadratic differential \(\omega^2\) induces a meromorphic quadratic differential \(q\) on \(Y = X/\langle \tau \rangle\) via the natural projection \(\varphi : X \to Y\). Then we have the following.

**Proposition 3.2.** The Riemann surface \(Y = X/\langle \tau \rangle\) is a compact surface of genus 0. If \((X, \omega) \in \mathcal{H}^{hyp}(2g-2)\), the quadratic differential \(q\) has a unique zero of order \(2g-3\) and \(2g+1\) simple poles. If \((X, \omega) \in \mathcal{H}^{hyp}(g-1, g-1)\), the quadratic differential \(q\) has a unique zero of order \(2g-2\) and \(2g+2\) simple poles.

**Proof.** Since \(\tau\) is the hyperelliptic involution, \(\tau\) has \(2g+2\) fixed points, and \(Y = X/\langle \tau \rangle\) is a compact surface of genus 0. Suppose that \((X, \omega) \in \mathcal{H}^{hyp}(2g-2)\). Then the unique zero of \(\omega\) is a fixed point of \(\tau\). As the angle around \(p_0\) on \((X, \omega)\) is \((4g-2)\pi\), the angle around \(\varphi(p_0)\) on \((Y, q)\) is \((2g-1)\pi\). Therefore, \(\varphi(p_0)\) is a zero of \(q\) of order \(2g-3\). The angle of every other fixed point \(p\) of \(\tau\) is \(2\pi\). Thus, the angle around \(\varphi(p)\) on \((Y, q)\) is \(\pi\). This means that the point is a simple pole of \(q\). Every point of \(X \setminus \text{Fix}(\tau)\) is mapped to a point on \(Y\) that is not a zero nor a pole of \(q\). Next, we assume that \((X, \omega) \in \mathcal{H}^{hyp}(g-1, g-1)\). Then, all fixed points of \(\tau\) are mapped to simple poles of \(q\). Two zeros of \(\omega\) are mapped to a point on \((Y, q)\) that is zero of \(q\) of order \(2g-2\). Every point of \(X \setminus \text{Fix}(\tau)\) is neither mapped to a zero nor a pole of \(q\). \(\square\)

The inequality (1) is proved by giving disjoint regular simple closed geodesics \(\gamma_1, \ldots, \gamma_g\) (resp. \(\gamma_1, \ldots, \gamma_{g+1}\)) in \((X, \omega) \in \mathcal{H}^{hyp}(2g-2)\) (resp. \((X, \omega) \in \mathcal{H}^{hyp}(g-1, g-1)\)) each of which is invariant under the hyperelliptic involution \(\tau\) of \((X, \omega)\). Let \(\gamma\) be a regular closed geodesic in \((X, \omega)\) that is invariant under \(\tau\). Since every orientation reversing isometry of \(S^1\) has two fixed points, the image \(\varphi(\gamma)\) is a saddle connection on \((Y, q)\) that connects distinct simple poles of \(q\). Conversely, if \(s\) is such a saddle connection on \((Y, q)\), the preimage \(\varphi^{-1}(s)\) is a regular closed geodesic on \((X, \omega)\) that is invariant under \(\tau\). Therefore, we find such saddle connections on \((Y, q)\) that are disjoint to each other. To do this, we give a triangulation of \((Y, q)\) by “restricted triangles” defined by Definition 3.6. All vertices of this triangulation are the zero of \(q\) and all poles of \(q\) are midpoints of some edges.

#### 3.3. Geometry of flat surfaces of genus 0

In this subsection, we give some properties of flat surfaces of genus 0. Let \((Y, q)\) be a flat surface of genus 0. Note that \((Y, q)\) must have some simple poles. Let \(s' : [0, 1] \to (Y, q)\) be a geodesic segment connecting a singular point to a simple pole of \(q\). Then

\[
s(t) = \begin{cases} 
  s'(2t) & (0 \leq t \leq \frac{1}{2}) \\
  s'(2 - 2t) & (\frac{1}{2} \leq t \leq 1)
\end{cases}
\]

is a closed curve on \((Y, q)\) such that its image is a segment.

**Definition 3.3** (Returning geodesic). We call a closed curve \(s\) as above a *returning geodesic*. That is, a returning geodesic \(s : [0, 1] \to (Y, q)\) is a closed curve from a singular point of \((Y, q)\) to itself such that \(s\left(\frac{1}{2}\right)\) is a simple pole of \(q\), \(s|_{[0, \frac{1}{2}]}\) is a saddle connection, and \(s(t) = s(1-t)\) for all \(t \in [0, 1]\).

The next lemma is important for our proof.

**Lemma 3.4.** Let \((Y, q)\) be a flat surface of genus 0. Let \(s : [0, 1] \to (Y, q)\) be a curve satisfying the following:
• the absolute value of the derivative $|s'|$ is constant,
• $s(0)$ is a singular point and $s\left(\frac{1}{2}\right)$ is a simple pole,
• $s|_{[0, \frac{1}{2}]}$ is a local isometry such that $s\left(0, \frac{1}{2}\right)$ contains no singular points,
• $s|_{[\frac{1}{2}, 1]}$ is a local isometry and
• there exists $\varepsilon > 0$ such that $s(1 - t) = s(t)$ for all $t \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$.

Then $s(t) = s(1 - t)$ for all $t \in [0, 1]$. Especially, $s\left(\frac{1}{2}, 1\right)$ contains no singular points and $s(1)$ coincides with the singular point $s(0)$.

Proof. Since $s|_{[0, \frac{1}{2}]}$ and $s|_{[\frac{1}{2}, 1]}$ are local isometries and $s\left(\frac{1}{2}\right)$ is a simple pole, $s$ is a returning geodesic. Thus, we obtain the claim. \hfill \Box

**Proposition 3.5.** Let $(Y, q)$ be a flat surface of genus 0. Let $s : [0, 1] \rightarrow (Y, q)$ be a returning geodesic. Given a segment $s_0$ in $\mathbb{C}$ such that $s_0$ is parallel to $s$ and the length $|s_0|$ equals the length of $s$. Choose a non-zero vector $v$ that is not parallel to $s_0$. Then there exists an orientation preserving local isometric embedding $\rho_0 : D \rightarrow (Y, q)$ satisfying the following:

1. $D$ is an open parallelogram such that one of the sides is $s_0$ and another side is parallel to $v$.
2. $\rho_0(D)$ contains no singular points,
3. $\rho_0$ can be extended to an immersion $\rho_0 : \overline{D} \rightarrow (Y, q)$,
4. $s_0$ is mapped to $s$ via $\rho_0$,
5. the other side that is parallel to $s_0$ is mapped to an open segment containing a singular point.

Proof. For sufficiently small $\varepsilon > 0$, there exists an immersion $\rho_\varepsilon : D_\varepsilon \rightarrow (Y, q)$ satisfying the same conditions as (1), (2), (3), (4) and the distance between $s_0$ and its opposite side is $\varepsilon$. We show that there exists the maximum of $\varepsilon > 0$ such that the above $\rho_\varepsilon$ exists. If not, there exists a local isometric immersion from an open band with infinite area into $(Y, q)$. Since the area of $(Y, q)$ is finite, the image of this embedding must be a cylinder. Since the angle around $s\left(\frac{1}{2}\right)$ is $\pi$, this is a contradiction. Let $\varepsilon_0 > 0$ be the maximal of $\varepsilon > 0$ such that the above $\rho_\varepsilon$ exists. Then $\rho_0 = \rho_{\varepsilon_0}$ also satisfies the condition (5). \hfill \Box

**Definition 3.6** $(s_0, v)$-restricted triangle and left (right) strongly $(s_0, v)$-restricted triangle. Let $\Delta$ be a Euclidean triangle in $\mathbb{C}$ with a side $s_0$. Let $v$ be a unit vector that is not parallel to $s_0$. Assume that $P$ is the vertex of $\Delta$ that is opposite to $s_0$.

1. The triangle $\Delta$ is a $(s_0, v)$-restricted triangle if the Euclidean line $l$ that passes through $P$ and is of direction $v$ intersects with $s_0$.
2. A left (resp. right) side of a $(s_0, v)$-restricted triangle $\Delta$ is the side of $\Delta$ that is at the left (resp. right) when we put $\Delta$ so that $s_0$ is horizontal and $P$ is above $s_0$.
3. A left (resp. right) strongly $(s_0, v)$-restricted triangle $\Delta$ is a $(s_0, v)$-restricted triangle whose left (resp. right) side is not parallel to $v$.

We prove the following theorem.

**Theorem 3.7.** Let $(Y, q)$ be a flat surface of genus 0 such that $q$ has a unique zero $p_0$. Let $s$ be a returning geodesic on $(Y, q)$ that starts from the point $p_0$. Let $v$ be a unit vector that is not parallel to $s$. Then there exists a (closed) triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ and an orientation preserving immersion $\rho : \Delta \rightarrow (Y, q)$ satisfying the following:
(1) $s_0$ is parallel to $s$ and $\Delta$ is a left (resp. right) strongly $(s_0, v)$-restricted triangle,

(2) $\rho|_{\text{Int}(\Delta)}$ is a local isometric embedding,

(3) $\rho(\text{Int}(\Delta))$ contains no singular points,

(4) every vertex of $\Delta$ is mapped to $p_0$,

(5) $s = \rho \circ s_0$ and

(6) $\rho \circ s_i$ is a saddle connection or a returning geodesic for each $i = 1, 2$.

We prove Theorem 3.7 in the case where $\Delta$ is a left strongly $(s_0, v)$-restricted triangle. The right strongly $(s_0, v)$-restricted triangle case is proved in the same way. Theorem 3.7 is proved by the following five Lemmas.

Let $(Y, q)$ be a flat surface of genus 0 such that $q$ has a unique zero $p_0$. Let $s$ be a returning geodesic on $(Y, q)$ that starts from the point $p_0$. Given $s_0$, $v$, $D$, and $\rho_0 : D \to (Y, q)$ as in Proposition 3.5. Then we have $\rho_0 \circ s_0 = s$. We label all other sides of $D$ by $a, b, c$ so that the counterclockwise order of the sides is $s_0, c, a, b$. We label the vertices of $D$ by $P_1, P_2, P_3, P_4$ as in Figure 5. We also give the label $O$ to the midpoint of $s_0$.

We define the following terminologies.

- For a point $P \in \mathbb{C}$, we denote by $r_P$ the point reflection in $P$.
- A point $P \in a$ is a left point of $a$ if $P_4 P < \frac{1}{2} |a|$.
- A point $P \in a$ is a right point of $a$ if $P_4 P > \frac{1}{2} |a|$.
- A point $P \in b$ is a lower point of $b$ if $P_3 P < \frac{1}{2} |b|$.
- A point $P \in b$ is an upper point of $b$ if $P_3 P > \frac{1}{2} |b|$.
- A point $P \in c$ is a lower point of $c$ if $P_1 P < \frac{1}{2} |c|$.
- A point $P \in c$ is an upper point of $c$ if $P_1 P > \frac{1}{2} |c|$.

Lemma 3.8. If the interior of the side $a$ contains a point $P$ that is mapped via $\rho_0$ to $p_0$, there exists a triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ satisfying all conditions as in Theorem 3.7.

Proof. Setting $\Delta = \Delta P_1 P_2 P$ and $\rho = \rho_0|_{\Delta}$, we obtain the claim. \hfill \Box

Lemma 3.9. Assume that the interior of the side $a$ does not contain a point that is mapped via $\rho_0$ to $p_0$ and $\rho_0(c - \{P_2\})$ contains a singular point. Then, there exist a triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ satisfying all conditions as in Theorem 3.7.

Proof. Let $P$ be the point in $c - \{P_2\}$ such that $P$ is closest to $P_2$ in the points of $c - \{P_2\}$ that are mapped to singular points of $(Y, q)$. If $\rho_0(P) = p_0$, then $\Delta = \Delta P_1 P_2 P$ and $\rho = \rho_0|_{\Delta}$ satisfies all conditions as in Theorem 3.7. Hereafter, we assume that $\rho_0(P)$ is a pole of $q$. 
Case (i) Suppose that $P$ is a lower point or the midpoint of $c$. By Lemma 3.4, $Q = r_P(P_2) \in c$ is a point whose image via $\rho_0$ is $p_0$ (see Figure 6). Moreover, there exist no points whose images are singular points between $P$ and $Q$. Setting $\Delta = \Delta P_1 P_2 Q$ and $\rho = \rho_0|_{\Delta}$, $\Delta$ and $\rho$ satisfy the conditions as in Theorem 3.7.

![Figure 6. Case (i).](image-url)

Case (ii) Suppose that $P$ is an upper point of $c$. By assumption, there exists a point $R \in \text{Int}(a)$ whose image is a simple pole of $q$. Then, $R$ is not a right point of $a$. If $R$ is a right point of $a$, by Lemma 3.4, $r_R \circ r_P(P_2)$ is an interior point of $D$. However, $r_R \circ r_P(P_2)$ is mapped to $p_0$. This is a contradiction. If $R$ is the midpoint of $a$, we set $\Delta = \triangle P_1 P_2 r_P(P_2)$ and

$$\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta \cap D) \\
\rho_0 \circ r_R(z) & (z \in \Delta - D)
\end{cases}$$

(see Figure 7). Then, $\Delta$ and $\rho$ satisfy the conditions as in Theorem 3.7.

![Figure 7. Case (ii) with the assumption that $R$ is the midpoint of $a$.](image-url)

Finally, suppose that $R$ is a left point of $a$. Let $E = \Delta P_1 P_2 r_P(P_2) - D$ and $c'$ the side of $E$ that is parallel to $c$ (see Figure 8). If $\rho_0$ can be extended to a local isometry from $\text{Int}(D \cup E)$ to $(Y, q)$, then $\Delta = \Delta P_1 P_2 r_P(P_2)$ and $\rho = \rho_0|_{\Delta}$ satisfy the conditions as in Theorem 3.7. We now assume that $\rho_0$ cannot be extended to $\text{Int}(D \cup E)$ as an isometry. Let $\theta_0 = \angle P_1 P_2$ and $\theta_1 = \angle r_P(P_2) P_1 P_2$. For each $\theta$ ($\theta_0 < \theta < \theta_1$), let $l_\theta$ be the segment starting from $P_1$ to a point of $c'$ such that the angle between $P_1 P_2$ and $l_\theta$ is $\theta$. By assumption, there exists the maximal $\Theta$ such that $\rho_0$ can be extended to a local isometry from $\text{Int}(D \cup \bigcup_{\theta_0 < \theta < \Theta} l_\theta)$ to $(Y, q)$. Then, $l_\Theta$ contains points that are mapped to singular points. Let $S$ be the point that is closest to $s_0$ in the points. We show that $S$ is mapped to the zero $p_0$ of $q$. Suppose that $S$ is mapped to a pole of $q$. Let $T_1$ be the intersection of segments $P_1 r_s(P_1)$ and $P_2 r_P(P_2)$. Let $T_2$ be the point in $P_1 r_S(P_1)$ such that $r_P(T_2)$ is the
intersection of $P_1P_2$ and $r_p(T_1R_S(P_1))$. We define a curve $s : P_1r_S(P_1) \to (Y, q)$ by

$$s(z) = \begin{cases} 
\rho_0(z) & (z \in P_1T_1) \\
\rho_0 \circ r_p(z) & (z \in T_1T_2) \\
\rho_0 \circ r_{P0} \circ r_p(z) & (z \in T_2r_S(P_1)) 
\end{cases}.$$ 

By Lemma 3.4, we have $\rho_0 \circ r_{P0} \circ r_p(z_1) = \rho_0$. However, we have $r_{P0} \circ r_p(z_1) \in D$. This is a contradiction. Therefore, $S$ is mapped to $p_0$. Setting $\Delta = \Delta P_1P_2S$ and $\rho = \rho_0|\Delta$, then $\Delta$ and $\rho$ satisfy the conditions as in Theorem 3.7.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Case (ii) with the assumption that $R$ is the left point of $a$}
\end{figure}

**Lemma 3.10.** Assume that the interior of the side $a$ does not contain a point that is mapped to $p_0$, the midpoint $R$ of $a$ is mapped to a pole of $q$ and $\rho_0(c - \{P_2\})$ contains no singular points. Then, there exist a triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ satisfying all conditions as in Theorem 3.7.

**Proof.** Let $P$ be the point in $b$ such that $P$ is farthest from $P_1$ in the points of $b$ that are mapped to singular points of $(Y, q)$. Note that $P$ may possibly coincide with $P_1$. We first prove that $P$ is mapped to $p_0$. Suppose that $P$ is mapped to a pole of $q$. If $P$ is a lower point or the midpoint of $b$, by Lemma 3.4, $r_p(P_1) \in b$ is mapped to a singular point. This contradicts the definition of $P$. Assume that $P$ is an upper point of $b$. We show that $P$ is mapped to $p_0$. Suppose that $P$ is mapped to a pole of $q$. Let $P'$ be the point that is closest to $P$ in the points of $P_1P$ corresponding to singular points and $P' \neq P$. Let $s$ be a curve $s : P'r_P(P') \to (Y, q)$ defined by

$$s(z) = \begin{cases} 
\rho_0(z) & (z \in P'P_1) \\
\rho_0 \circ r_{P0}(z) & (z \in P'r_P(P')) 
\end{cases}.$$ 

By Lemma 3.4, we have $s(r_P(P')) = s(P')$. Hence, $s(r_P(P')) = \rho_0 \circ r_{P0}(r_P(P')) = \rho_0(r_{P0}r_P(P'))$ is a singular point. However, $r_{P0}r_P(P')$ is in $c - \{P_2\}$. This is a contradiction. Therefore, $P$ is mapped to $p_0$. Set $\Delta = \Delta P_1P_2r_{P0}(P)$ and

$$\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta \cap D) \\
\rho_0 \circ r_{P0}(z) & (z \in \Delta - D) 
\end{cases}$$

(see Figure 9). Then, $\Delta$ and $\rho$ satisfy the conditions.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Proof of Lemma 3.10}
\end{figure}

**Lemma 3.11.** Assume that the interior of the side $a$ does not contain a point that is mapped to $p_0$, the left point $R$ of $a$ is mapped to a pole of $q$, and $\rho_0(c - \{P_2\})$ contains no singular points. Then, there exist a triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$...
and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ satisfying all conditions as in Theorem 3.7.

Proof. Let $P$ be the point in $b$ such that $P$ is farthest from $P_1$ in the points of $b$ that are mapped to singular points of $(Y, q)$. Then, $P$ is mapped to $p_0$ by the same argument as in the proof of Lemma 3.10. We make the same argument as the last argument of the proof of Lemma 3.9. Let $E = \Delta P_1 P_2 R(P) - D$ and $b' = r_{R}(P P_4)$ (see Figure 10). If $\rho_0$ can be extended to a local isometry from $\text{Int}(D \cup E)$ to $(Y, q)$, then $\Delta = \Delta P_1 P_2 R(P)$ and $\rho = \rho_0|_\Delta$ satisfy the conditions as in Theorem 3.7. We now assume that $\rho_0$ cannot be extended to $\text{Int}(D \cup E)$ as a local isometry. Let $\theta_0 = \angle P_1 P_2 R(P_4)$ and $\theta_1 = \angle P_1 P_2 R(P)$. For each $\theta (\theta_0 < \theta < \theta_1)$, let $l_\theta$ be the segment starting from $P_2$ to a point of $b'$ such that the angle between $P_1 P_2$ and $l_\theta$ is $\theta$. By assumption, there exists the maximal $\Theta$ such that $\rho_0$ can be extended to a local isometry from $D' = \text{Int} \left( D \cup \bigcup_{\theta_0 < \theta < \Theta} l_\theta \right)$ to $(Y, q)$. Then, $l_\Theta$ contains points that are mapped to singular points except for $P_2$. Let $S$ be the point that is closest to $s_0$ in the points. If $S$ is mapped to a pole of $q$, $r_O \circ r_{R} \circ r_S(P_2)$ is mapped to $p_0$. However, we have $r_O \circ r_{R} \circ r_S(P_2) \in D$. This is a contradiction. Therefore, $S$ is mapped to $p_0$. Setting $\Delta = \Delta P_1 P_2 S$ and

$$\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta \cap \overline{D'}) \\
\rho_0 \circ r_{R}(z) & (z \in \Delta - \overline{D'}) 
\end{cases},$$

$\Delta$ and $\rho$ satisfy the conditions as in Theorem 3.7. \[\square\]

Lemma 3.12. Assume that the interior of the side $a$ does not contain a point that is mapped to $p_0$, a right point $R$ of $a$ is mapped to a pole of $q$, and $\rho_0(c - \{P_2\})$ contains no singular points. Then, there exist a triangle $\Delta$ in $C$ with sides $s_0, s_1, s_2$ and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ satisfying all conditions as in Theorem 3.7.
Proof. The proof is the same as the last argument of the proof of Lemma 3.9. Let $E = \triangle P_1 P_2 r_{R}(P_2) - D$ and $c' = r_{R}(c)$ (see Figure 11). If $\rho_0$ can be extended to a local isometry from $\text{Int}(D \cup E)$ to $(Y, q)$, $\Delta = \triangle P_1 P_2 r_{R}(P_2)$ and $\rho = \rho_0|_{\Delta}$ satisfy the conditions as in Theorem 3.7. We now assume that $\rho_0$ cannot be extended to $\text{Int}(D \cup E)$ as a local isometry. Let $\theta_0 = \angle P_2 P_1 r_{R}(P_3)$ and $\theta_1 = \angle P_2 P_1 r_{R}(P_2)$. For each $\theta (\theta_0 < \theta < \theta_1)$, let $l_0$ be the segment starting from $P_2$ to a point of $c'$ such that the angle between $P_1 P_2$ and $l_0$ is $\theta$. By assumption, there exists the maximal $\Theta$ such that $\rho_0$ can be extended to a local isometry from $D' = \text{Int}(D \cup \bigcup_{\theta_0 < \theta < \Theta} l_\theta)$ to $(Y, q)$. Then, $l_\Theta$ contains points that are mapped to singular points. Let $S$ be the point that is closest to $s_0$ in the points. If $S$ is mapped to a pole of $q$, then $r_O \circ r_{R} \circ r_{S}(P_1)$ is mapped to $p_0$. However, we have $r_O \circ r_{R} \circ r_{S}(P_1) \in D$. This is a contradiction. Therefore, $S$ is mapped to $p_0$. Setting $\Delta = \triangle P_1 P_2 S$ and

$$\rho(z) = \begin{cases} \rho_0(z) & (z \in \Delta \cap \overline{D'}) \\ \rho_0 \circ r_{R}(z) & (z \in \Delta - \overline{D'}) \end{cases},$$

$\Delta$ and $\rho$ satisfy the conditions as in Theorem 3.7. □

By Theorem 3.7, we obtain the following.

**Theorem 3.13.** Let $(Y, q)$ be a flat surface of genus 0 such that $q$ has a unique zero $p_0$. Suppose that $s$ is a returning geodesic on $(Y, q)$ that starts from the point $p_0$. Then there exists a triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ in counterclockwise order and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ satisfying the following:

1. $\rho|_{\text{Int}(\Delta)}$ is a local isometric embedding,
2. $\rho|_{\text{Int}(\Delta)}$ contains no singular points,
3. every vertex of $\Delta$ is mapped to $p_0$,
4. $s = \rho \circ s_0$ and
5. $\rho \circ s_1$ is a returning geodesic.

**Remark 4.** In the condition (6) of Theorem 3.13, we may replace $\rho \circ s_1$ to $\rho \circ s_2$.

**Proof.** Let $v_0$ be a unit vector that is not parallel to $s$. By Theorem 3.7, there exists a triangle $\Delta_0$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ in counterclockwise order and an orientation preserving immersion $\rho_0 : \Delta_0 \to (Y, q)$ satisfying the following:

1. $\Delta_0$ is a $(s_0, v_0)$-restricted triangle,
2. $\rho_0|_{\text{Int}(\Delta_0)}$ is a local isometric embedding,
3. $\rho_0|_{\text{Int}(\Delta_0)}$ contains no singular points,
(0-4) every vertex of $\Delta_0$ is mapped to $p_0$,
(0-5) $s = \rho_0 \circ s_0$, and
(0-6) $\rho_0 \circ s_i$ is a saddle connection or a returning geodesic for each $i = 1, 2$.
If $s_1$ is mapped via $\rho_0$ to a returning geodesic, then we are done. Now we assume
that $s_1$ is mapped via $\rho_0$ to a saddle connection. Since $(Y, q)$ is a surface of genus 0,
every simple closed curve on $(Y, q)$ divides $(Y, q)$. Thus, $Y - \rho_0(\Delta_0)$ has at most two
connected components. Let $Y_1$ be the connected component of $Y - \rho_0(\Delta_0)$ whose
boundary is $\rho_0 \circ s_1$. Let $q_1$ be the restriction of $q$ onto $Y_1$. After adjusting the
parameter of $s_1$ so that $(s_1)'$ is constant, we identify $\rho_0 \circ s_1(t)$ with $\rho_1 \circ s_1(1 - t)$ for
all $t \in [0, 1]$ on the boundary of $Y_1$. Then $Y_1$ is a compact surface of genus 0 and $q_1$
is a meromorphic quadratic differential such that $\rho_0 \circ s_1$ is a returning geodesic on
the flat surface $(Y_1, q_1)$.

(1-1) $\Delta_1$ is a right strongly $(s_1, v)$-restricted triangle,
(1-2) $\Delta_0 \cap \Delta_1 = s_1$,
(1-3) $\rho_1|_{\text{Int}(\Delta_1)}$ is a local isometric embedding,
(1-4) $\rho_1(\text{Int}(\Delta_1))$ contains no singular points,
(1-5) every vertex of $\Delta_1$ is mapped to $p_0$,
(1-6) $\rho_0 \circ s_2 = \rho_1 \circ s_2$, and
(1-7) $\rho_1 \circ s_i$ is a saddle connection or a returning geodesic for each $i = 3, 4$
(see Figure 12).

If $s_3$ is mapped via $\rho_1$ to a returning geodesic, then let $\Delta$ be the triangle spanned
by $s_0$ and $s_3$. By construction, $\Delta$ is contained in $\Delta_0 \cup \Delta_1$. Regarding $\rho_1$ as a map
from $\Delta_1$ to $(Y, q)$, the map $\rho : \Delta \to (Y, q)$ defined by
$$
\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta_0) \\
\rho_1(z) & (z \in \Delta_1)
\end{cases}
$$
is well-defined. The triangle $\Delta$ and the map $\rho$ satisfy all conditions as in Theorem
3.13. Hereafter we assume that $s_3$ is not mapped via $\rho_1$ to a returning geodesic.
Let $Y_2$ be the connected component of $Y_1 - \rho_1(\Delta_1)$ whose boundary is $\rho_1 \circ s_3$. We
sew the boundary of $Y_2$ as we do for $Y_1$. Then $Y_2$ is a compact surface of genus 0
and $q_1$ induces a meromorphic quadratic differential $q_2$ on $Y_2$ such that $\rho_1 \circ s_3$ is a
returning geodesic on the flat surface \((Y_2, q_2)\). We apply Proposition 3.7 to \(\rho_1 \circ s_3\) and the vector \(v\). Then, there exists a triangle \(\Delta_2\) in \(\mathbb{C}\) with sides \(s_3, s_5, s_6\) in counterclockwise order and an orientation preserving immersion \(\rho_2 : \Delta_2 \to (Y_2, q_2)\) satisfying the following:

(2-1) \(\Delta_2\) is a right strongly \((s_3, v)\)-restricted triangle,
(2-2) \(\Delta_1 \cap \Delta_2 = s_3\),
(2-3) \(\rho_2|_{\text{Int}(\Delta_2)}\) is a local isometric embedding,
(2-4) \(\rho_2(\text{Int}(\Delta_2))\) contains no singular points,
(2-5) every vertex of \(\Delta_2\) is mapped to \(p_0\),
(2-6) \(\rho_1 \circ s_3 = \rho_2 \circ s_3\), and
(2-7) \(\rho_2 \circ s_i\) is a saddle connection or a returning geodesic for each \(i = 5, 6\) (see Figure 13).

If \(s_5\) is mapped via \(\rho_2\) to a returning geodesic, then let \(\Delta\) be the triangle spanned by \(s_0\) and \(s_5\). By construction, \(\Delta\) is contained in \(\Delta_0 \cup \Delta_1 \cup \Delta_2\). Regarding \(\rho_1\) and \(\rho_2\) as a map to \((Y, q)\), the map \(\rho : \Delta \to (Y, q)\) defined by

\[
\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta_0) \\
\rho_1(z) & (z \in \Delta_1) \\
\rho_2(z) & (z \in \Delta_2)
\end{cases}
\]

is well-defined. The triangle \(\Delta\) and the map \(\rho\) satisfy all conditions as in Theorem 3.13. If \(s_5\) is not mapped via \(\rho_1\) to a returning geodesic, then we make the same argument as above by using the vector \(v\). Since the angle around \(p_0\) is finite, this process must stop by finitely many steps. At the end of this process, we have triangles \(\Delta_k\) with sides \(s_{2k-1}, s_{2k+1}, s_{2k+2}\) \((k = 1, \ldots, n)\) and orientation preserving immersions \(\rho_k : \Delta_k \to (Y_k, q_k)\) \((k = 1, \ldots, n)\) satisfying the following conditions:

(3-1) \(\Delta_k\) is a right strongly \((s_{2k-1}, v)\)-restricted triangle,
(3-2) \(\Delta_{k-1} \cap \Delta_k = s_{2k-1}\),
(3-3) \(\rho_k|_{\text{Int}(\Delta_k)}\) is a local isometric embedding,
(3-4) \(\rho_k(\text{Int}(\Delta_k))\) contains no singular points,
(3-5) every vertex of \(\Delta_k\) is mapped to \(p_0\),
(3-6) \(\rho_{k-1} \circ s_{2k-1} = \rho_k \circ s_{2k-1}\),
(3-7) \(\rho_k \circ s_{2k+1}\) is a saddle connection for each \(i = 1, \ldots, n - 1\), and
(3-8) $\rho_n \circ s_{2n+1}$ is a returning geodesic

Let $\Delta$ be the triangle spanned by $s_0$ and $s_{2n+1}$. By construction, $\Delta$ is contained in $\Delta_0 \cup \Delta_1 \cup \ldots \cup \Delta_n$. The map $\rho : \Delta \to (Y, q)$ defined by

$$\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta_0) \\
\rho_1(z) & (z \in \Delta_1) \\
\vdots \\
\rho_n(z) & (z \in \Delta_n)
\end{cases}$$

is well-defined. Now $\Delta$ and $\rho$ satisfy all conditions as in Theorem 3.13. \qed

3.4. Proof of the inequality (1). In this subsection, we prove Theorem 2.15. The upper estimates of $N(X, \omega)$ are given by Proposition 3.1. We give the lower estimates in Proposition 3.16.

Lemma 3.14. Let $g \geq 2$. Let $(Y, q)$ be a flat surface of genus $0$ that has a unique zero $p_0$ of order $2g-3$. Fix a simple pole $p_1$ of $q$ and a returning geodesic $s_0$ on $(Y, q)$ that passes through $p_1$. Then there exist saddle connections $\delta_1, \ldots, \delta_g$ satisfying the following conditions:

- the endpoints of $\delta_1, \ldots, \delta_g$ are poles of $q$ and
- $\delta_1, \ldots, \delta_g$, and $s_0$ are disjoint to each other.

Proof. We prove it by induction on $g$. If $g = 2$ then $p_0$ is a unique zero of $q$ of order 1. By Theorem 3.13, there exists a triangle $\Delta_0$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ in counterclockwise order and an orientation preserving immersion $\rho_0 : \Delta_0 \to (Y, q)$ satisfying the following conditions:

(0-1) $\rho_0|_{\text{Int}(\Delta_0)}$ is a local isometric embedding,
(0-2) $\rho_0|_{\text{Int}(\Delta_0)}$ contains no singular points,
(0-3) every vertex of $\Delta_0$ is mapped to $p_0$,
(0-4) $s = \rho \circ s_0$, and
(0-5) $\rho_0 \circ s_1$ is a returning geodesic.

Let $P_0$ be the vertex of $\Delta_0$ that is opposite to $s_1$ and $P_1$ the midpoint of $s_1$. The surface $Y_1 = Y - \rho_0(\Delta_0)$ is of genus 0 and has a boundary $\rho_0 \circ s_2$. Let $q_1$ be the restriction of $q$ onto $Y_1$. After adjusting the parameter of $s_2$ so that $(s_2)'$ is constant, we identify $\rho_0 \circ s_2(t)$ with $\rho_0 \circ s_2(1-t)$ for all $t \in [0, 1]$ on the boundary of $Y_1$. Then $Y_1$ is a compact surface of genus 0 and $q_1$ is a meromorphic quadratic differential such that $\rho_0 \circ s_2$ is a returning geodesic on the flat surface $(Y_1, q_1)$. Let $v$ be a unit vector that is parallel to the segment $P_0P_1$. By Theorem 3.7, there exists a triangle $\Delta_1$ in $\mathbb{C}$ with sides $s_2, s_3, s_4$ and an orientation preserving immersion $\rho_1 : \Delta_1 \to (Y_1, q_1)$ satisfying the following conditions (see Figure 14):

(1-1) $\Delta_1$ is a left strongly $(s_2, v)$-restricted triangle,
(1-2) $\Delta_0 \cap \Delta_1 = s_2$,
(1-3) $\rho_1|_{\text{Int}(\Delta_1)}$ is a local isometric embedding,
(1-4) $\rho_1|_{\text{Int}(\Delta_1)}$ contains no singular points,
(1-5) every vertex of $\Delta_1$ is mapped to $p_0$,
(1-6) $\rho_1 \circ s_2 = \rho_0 \circ s_2$, and
(1-7) $\rho_1 \circ s_i$ is a saddle connection or a returning geodesic for each $i = 3, 4$.

If both $\rho_1 \circ s_3$ and $\rho_1 \circ s_4$ are returning geodesics, we have $Y = \rho_0(\Delta_0) \cup \rho_1(\Delta_1)$. This contradicts the assumption that $q$ has a zero $p_0$. Thus one of $\rho_1 \circ s_3$ and $\rho_1 \circ s_4$
Lemma 3.14. Let $\rho$ be a local isometric embedding, $\rho$ is a saddle connection, we can apply the same argument. The surface $Y_2 = Y_1 - \rho(\Delta_1)$ is of genus 0 and has a boundary $\rho \circ s_3$. Let $q_2$ be the restriction of $q_1$ onto $Y_2$. We sew the boundary of $Y_2$ as we do for $Y_1$. Then $Y_2$ is a compact surface of genus 0 and $q_2$ is a meromorphic quadratic differential such that $\rho \circ s_3$ is a returning geodesic on the flat surface $(Y_2, q_2)$. Let $v'$ be a unit vector that is not parallel to $s_3$. By Theorem 3.7, there exists a triangle $\Delta_2$ in $\mathbb{C}$ with sides $s_3, s_5, s_6$ and an orientation preserving immersion $\rho_2 : \Delta_2 \to (Y_2, q_2)$ satisfying the following conditions:

(2-1) $\Delta_2$ is a left strongly $(s_3, v)$-restricted triangle,
(2-2) $\Delta_1 \cap \Delta_2 = s_3$,
(2-3) $\rho_2|_{\text{Int}(\Delta_2)}$ is a local isometric embedding,
(2-4) $\rho_2(\text{Int}(\Delta_2))$ contains no singular points,
(2-5) every vertex of $\Delta_2$ is mapped to $p_0$,
(2-6) $\rho_2 \circ s_3 = \rho_1 \circ s_3$, and
(2-7) $\rho_2 \circ s_i$ is a saddle connection or a returning geodesic for each $i = 5, 6$.

Since the angle around $p_0$ in $(Y, q)$ is $3\pi$ and all vertices of $\Delta_1$ are mapped to $p_0$ by $\rho_i$ for each $i = 0, 1, 2$, we have $Y = \rho_0(\Delta_0) \cup \rho_1(\Delta_1) \cup \rho_2(\Delta_2)$. Thus $\rho_1 \circ s_4$, $\rho_2 \circ s_5$, and $\rho_2 \circ s_6$ are returning geodesics. Set $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$ and define the map $\rho : \Delta \to (Y, q)$ by

$$
\rho(z) = \begin{cases} 
\rho_0(z) & (z \in \Delta_0) \\
\rho_1(z) & (z \in \Delta_1) \\
\rho_2(z) & (z \in \Delta_2).
\end{cases}
$$

Let $P_i$ be the midpoint of the side $s_i$ for $i = 4, 5, 6$ (see Figure 15). Then $\delta_1 = \rho(P_1P_2)$ and $\delta_2 = \rho(P_3P_6)$ are saddle connections satisfying all conditions as in Lemma 3.14.

Next, we assume that Lemma 3.14 is true for all flat surfaces of genus 0 that have a unique zero $p_0$ of odd order less than $2g - 3$. Let $(Y, q)$ be a flat surface of genus 0 that has a unique zero $p_0$ of order $2g - 3$. Let $s_0$ be a returning geodesic on $(Y, q)$. We can construct triangles $\Delta_1$, $\Delta_2$, immersions $\rho_1 : \Delta_1 \to (Y, q)$ and $\rho_2 : \Delta_2 \to (Y, q)$ that are the same as in the case of $q = 2$. We use the same labels for the sides and vertices. If $s_i \in \{s_3, s_4\}$ corresponds to a returning geodesic, then the segment connecting the midpoint of $s_i$ and $P_i$ corresponds to a saddle connection connecting poles of $(Y, q)$. We denote the saddle connection by $\delta_1$. Since $(Y, q)$ has

\[\text{Figure 14. The triangles } \Delta_0 \text{ and } \Delta_1\]
2g - 3 poles, the side s_{7-i} is a saddle connection. In the same way as in the case of g = 2, We can regard Y_2 = Y - (\rho_1(\Delta_1) \cup \rho_2(\Delta_2)) as a compact surface of genus 0 without boundary and q_2 = q|_{Y_2} as a meromorphic quadratic differential with 2g - 5 poles. Moreover, \rho(s_{7-i}) is a returning geodesic on (Y_2, q_2). By assumption, we can find saddle connections \delta_2, \ldots, \delta_g in (Y_2, q_2) satisfying all conditions as in Lemma 3.14. Regarding them as saddle connections in (Y, q), \delta_1, \ldots, \delta_g are saddle connections in (Y, q). By assumption, we can find saddle connections \delta_{2}, \ldots, \delta_{g} in (Y_2, q_2) satisfying all conditions as in Lemma 3.14. Next, assume that both s_3 and s_4 are saddle connections. Then Y - (\rho_1(\Delta_1) \cup \rho_2(\Delta_2)) has two connected components. Since (Y, q) has an odd number of poles, one of the components contains an odd number of poles (before sewing its boundary) and the other contains an even number of poles. Hereafter, we say a saddle connection s' is odd if the connected component of Y - s' that does not contain s_0 has an odd number of poles. We delete the labels s_3 and s_4 and construct triangles \Delta_2, \Delta_3, \ldots and immersions \rho_i : \Delta_i \to (Y_i, \rho_i) (i = 2, 3, \ldots) by the following process (see Figure 16):

(1) Set i = 1.
(2) Let s_{i+2} be the side of \Delta_i such that \rho_i(s_{i+2}) is odd and P_{i+1} the vertex of \Delta_i that is opposite to s_{i+1}. Let v_i be a unit vector that is parallel to the segment P_i P_{i+1}. Set Y_{i+1} = Y - \bigcup_{k=0}^{i} \rho_i(\Delta_i) and regard it as a closed surface of genus 0 without boundary by sewing \rho_i(s_{i+2}). Set q_{i+1} = q|_{Y_{i+1}}.
(3) By applying Theorem 3.7, let \Delta_{i+1} be a triangle in \mathbb{C} with s_{i+2} as a side and \rho_{i+1} : \Delta_{i+1} \to (Y_{i+1}, q_{i+1}) an orientation preserving immersion satisfying the following conditions:
   (i) \Delta_{i+1} is a left strongly (s_{i+2}, v_i)-restricted triangle,
   (ii) \Delta_{i+1} \cap \Delta_i = s_{i+2},
   (iii) \rho_{i+1}|_{\text{int}(\Delta_{i+1})} is a local isometric embedding,
   (iv) \rho_{i+1}(\text{int}(\Delta_{i+1})) contains no singular points,
   (v) every vertex of \Delta_{i+1} is mapped to p_0,
   (vi) \rho_{i+1} \circ s_{i+2} = \rho_i \circ s_{i+2}, and
   (vii) other sides of \Delta_{i+1} than s_{i+2} are mapped via \rho_{i+1} saddle connections or returning geodesics.
(4) If other sides of \Delta_{i+1} than s_{i+2} are both mapped via \rho_{i+1} saddle connections, then we replace i to i + 1 and repeat the process from (2). If not, we stop this process.
Figure 16. The process to construct triangles $\Delta_2, \Delta_3, \ldots$

Note that the segment $P_1 P_{i+1}$ ($i = 2, 3, \ldots$) is contained in $\bigcup_{k=0}^{i} \Delta_i$. This process stops by finitely many steps since the angle around $p_0$ is finite. Assume that we obtain triangles $\Delta_2, \Delta_3, \ldots, \Delta_n$ and immersions $\rho_i : \Delta_i \to (Y_i, \rho_i)$ ($i = 2, 3, \ldots, n$) satisfying the above conditions as the result of the process. Then $\Delta_n$ has a side $s_{n+1}$ that is mapped via $\rho_n$ a returning geodesic. Let $P_{n+1}$ be the midpoint of $s_{n+1}$. By construction, the segment $P_1 P_{n+1}$ is in $\bigcup_{k=0}^{n} \Delta_k$. The union

$$\delta_1 = \bigcup_{k=0}^{n} \rho_k (P_1 P_{n+1} \cap \Delta_k)$$

is a saddle connection in $(Y, q)$ that is disjoint from $s_0$. Let $W_1, W_2, \ldots, W_n$ be the connected components of $Y - \bigcup_{k=0}^{n} \rho_k(\Delta_k)$. We set $q_i = q|_{W_i}$ for each $i = 1, 2, \ldots, n$. The flat surface $(W_i, q_i)$ is of genus 0 and has a boundary. Denote by $2g_i$ the number of poles of $(W_i, q_i)$. Sewing the boundary, $(W_i, q_i)$ is a flat surface that has $2g_i + 1$ poles and the boundary is now a returning geodesic, say $s_i'$. By the assumption of the induction, we can find saddle connections $\delta_{i, 1}, \delta_{i, 2}, \ldots, \delta_{i, g_i}$ satisfying the following:

- the endpoints of $\delta_{i, 1}, \delta_{i, 2}, \ldots, \delta_{i, g_i}$ are poles of $q_i$ and
- $\delta_{i, 1}, \delta_{i, 2}, \ldots, \delta_{i, g_i}, s_i'$ are disjoint to each other.

Since $\sum_{i=1}^{n} g_i = g - 1$, the collection of curves

$$\{\delta_1\} \cup \{\delta_{i, j} : i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, g_i\}\}$$

contains $g$ saddle connections. The saddle connections satisfy all conditions in Lemma 3.14.

Next we prove the case that corresponds to the case of $H_{\text{hyp}}(g - 1, g - 1)$.

**Lemma 3.15.** Let $g \geq 2$. Let $(Y, q)$ be a flat surface of genus 0 that has a unique zero $p_0$ of order $2g - 2$. Then there exist saddle connections $\delta_1, \ldots, \delta_{g+1}$ such that each of whose end points are poles and they are disjoint to each other.

**Proof.** Given a returning geodesic $s$ on $(Y, q)$. By Theorem 3.13, there exist a triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ in counterclockwise order and an orientation preserving immersion $\rho : \Delta \to (Y, q)$ that satisfy the same conditions as in Theorem 3.13. We denote by $\delta_{g+1}$ be the saddle connection in $\rho(\Delta)$ that connects two poles.
We set \( Y' = Y - \rho(\Delta) \) and \( q' = q|_{Y'} \). Then the surface \((Y', q')\) is of genus 0 and has a boundary. Sewing the boundary, \((Y', q')\) is a flat surface that has \(2g + 1\) poles. By Lemma 3.14, there exist saddle connections \(\delta_1, \ldots, \delta_g\) satisfying the following conditions:

- the endpoints of \(\delta_1, \ldots, \delta_g\) are poles of \(q\) and
- \(\delta_1, \ldots, \delta_g\) and \(\sigma_0\) are disjoint to each other.

Regarding them as saddle connections on \((Y, q)\), the saddle connections \(\delta_1, \ldots, \delta_{g+1}\) satisfy all conditions as desired. \(\square\)

By Lemma 3.14 and Lemma 3.15, we have the following.

**Proposition 3.16.** Let \(g \geq 2\).

1. If \((X, \omega) \in H_{hyp}^{2g-2}\), then there exist disjoint regular closed geodesics \(\gamma_1, \ldots, \gamma_g\) on \((X, \omega)\) that are invariant under \(\tau\) and are not homotopic to each other. Moreover, we have \(N(X, \omega) \geq g\).

2. If \((X, \omega) \in H_{hyp}(g-1, g-1)\), then there exist disjoint regular closed geodesics \(\gamma_1, \ldots, \gamma_{g+1}\) on \((X, \omega)\) that are invariant under \(\tau\) and are not homotopic to each other. Moreover, we have \(N(X, \omega) \geq g + 1\).

**Proof.** Assume that \((X, \omega) \in H_{hyp}^{2g-2}\) and \(\tau\) is the hyperelliptic involution of \((X, \omega)\). Let \((Y, q)\) be a flat surface of genus 0 that is the quotient \((X, \omega)\) by \(\tau\). Then \((Y, q)\) has a unique zero \(p_0\) of order \(2g-3\). By Lemma 3.14, there exist disjoint saddle connections \(\delta_1, \ldots, \delta_g\) each of which connects distinct poles. Let \(\gamma_i\) be the regular closed geodesic in \((X, \omega)\) that is projected to \(\delta_i\) by the natural projection from \((X, \omega)\) to \((Y, q)\) for each \(i = 1, \ldots, g\). Then the regular closed geodesics \(\gamma_1, \ldots, \gamma_g\) are disjoint to each other. Therefore, we have \(N(X, \omega) \geq g\). The proof of the statement of (2) is also done in the same way. \(\square\)

4. **Classification theorem for translation surfaces in \(H_{hyp}(4)\)**

The classification theorem for compact surfaces claims that every oriented closed surface of genus \(g \geq 1\) is topologically constructed from a sphere by gluing \(g\) cylinders. In this section, we prove the same kind of classification theorem for almost all translation surfaces in \(H_{hyp}(4)\) with respect to their structures as translation surfaces.

**Definition 4.1** (Maximal cylinder and simple cylinder). Let \((X, \omega)\) be a translation surface and \(\gamma\) a regular closed geodesic on \((X, \omega)\). A **maximal cylinder** for \(\gamma\) is the union of all regular closed geodesics that are homotopic to \(\gamma\) (see Proposition 2.10). A **simple cylinder** is a maximal cylinder each of whose boundary components is only one saddle connection.

**Theorem 4.2.** Let \((X, \omega) \in H_{hyp}(4)\) and \(\tau\) the hyperelliptic involution of \((X, \omega)\). If \((X, \omega) \notin GL(2, \mathbb{R}) \cdot St_5\), then there exist disjoint simple cylinders \(C_1, C_2, C_3\) each of which is invariant under \(\tau\). If \((X, \omega) \in GL(2, \mathbb{R}) \cdot St_5\), then \((X, \omega)\) has at most two disjoint simple cylinders.

As corollaries of this theorem, we have the following.

**Corollary 4.3** (Classification theorem for translation surfaces in \(H_{hyp}(4)\)). Let \((X, \omega) \in H_{hyp}(4) \setminus GL(2, \mathbb{R}) \cdot St_5\). Then, there exists a flat surface \((X_0, q)\) of genus 0 such that \(q\) has a zero \(p_0\) of order 2 and six simple poles \(p_1, p_2, \ldots, p_6\),
an involution $\tau_0 : X_0 \to X_0$, saddle connections $s_1, s_2, s_3$ and Euclidean cylinders $C_1, C_2, C_3$ satisfying the following (see Figure 17):

1. $\tau_0^* q = q$,
2. $s_i$ is a saddle connection of $(X_0, q)$ connecting $p_0$ and $p_i$ for $i = 1, 2, 3$,
3. $\tau_0(s_i)$ connects $p_0$ and $p_{i+3}$ for $i = 1, 2, 3$,
4. two of each $s_1, s_2, s_3, \tau_0(s_1), \tau_0(s_2)$, and $\tau_0(s_3)$ intersect only at $p_0$,
5. the circumference of $C_i$ equals $2|s_i|$ for $i = 1, 2, 3$, and
6. $(X, \omega)$ is obtained by cutting $(X_0, q)$ along the saddle connections $s_1, s_2, s_3, \tau_0(s_1), \tau_0(s_2), \tau_0(s_3)$ and gluing $C_i$ to the slits $s_i$ and $\tau_0(s_i)$ for all $i = 1, 2, 3$.

**Figure 17.** The flat surface $(X_0, q)$ and cylinders $C_1, C_2$, and $C_3$.

**Proof.** Let $(X, \omega) \in \mathcal{H}_{\text{hyp}}(4) \setminus \text{GL}(2, \mathbb{R}) \cdot S_{t5}$. By Theorem 4.2, $(X, \omega)$ has disjoint simple cylinders $C_1, C_2, C_3$ each of which is invariant under $\tau$. Removing the cylinders from $(X, \omega)$ and sewing the 6 boundaries respectively, the resulting surface $X_0$ is a Riemann surface of genus 0 and has a quadratic differential $q$ induced by $\omega^2$. The hyperelliptic involution $\tau$ of $(X, \omega)$ induces an involution $\tau_0 : X_0 \to X_0$. Let $s_i$ be the sewed boundary of $(X_0, q)$. The boundary component $s_i$ is also a boundary component of $C_i$ for each $i = 1, 2, 3$. For each $i = 1, 2, 3$, the other boundary component of $C_i$ is $\tau(s_i)$ since $C_i$ is invariant under $\tau$ and $\tau^* \omega = -\omega$. Therefore, $\tau_0$ and $s_1, s_2, s_3$ satisfy the conditions as in the claim.

**Corollary 4.4.** Let $(X, \omega) \in \mathcal{H}_{\text{hyp}}(4) \setminus \text{GL}(2, \mathbb{R}) \cdot S_{t5}$. Then, $(X, \omega)$ is constructed from a center-symmetric hexagon $H$ and 3 parallelograms $P_1, P_2, P_3$ by gluing them as in Figure 18.

**Proof.** Let $(X, \omega) \in \mathcal{H}_{\text{hyp}}(4) \setminus \text{GL}(2, \mathbb{R}) \cdot S_{t5}$. By Corollary 4.3, the hyperelliptic translation surface $(X, \omega)$ is constructed from a flat surface $(X_0, q)$ that has a unique zero $p_0$ and an involution $\tau_0$ by gluing Euclidean cylinders $C_1, C_2, C_3$ along slits $s_1, s_2, s_3, \tau_0(s_1), \tau_0(s_2), \tau_0(s_3)$. We cut $X_0$ along the slits. Since $\tau_0$ is an involution, the resulting surface is a center-symmetric hexagon. We also cut each cylinder $C_i$ ($i = 1, 2, 3$) along a segment connecting the points corresponding to $p_0$ on its boundary components. Then, the resulting surfaces are parallelograms.
Hereafter, we show Theorem 4.2. First, we prove the last part of Theorem 4.2.

**Lemma 4.5.** If $\omega \in \text{GL}(2, \mathbb{R}) \cdot \text{St}_5$, then $\omega$ has at most two disjoint simple cylinders.

**Proof.** We may assume that $(X, \omega) = \text{St}_5$. Then, the Veech group $\Gamma(\text{St}_5)$ is a lattice in $\text{SL}(2, \mathbb{R})$ and the quotient $\mathbb{H}/\Gamma(\text{St}_5)$ has two punctures (see Example 2.24). Let $p_1$ and $p_2$ be fixed points of the hyperelliptic involution that are the images of the midpoints of sides with labels $a$ and $3$ as in Figure 1, respectively. The cylinder decomposition for direction $\pi/4$ has only one cylinder that is not a simple cylinder. The cylinder decomposition for direction $0$ has three cylinders. Only one of the cylinders is a simple cylinder. Let $C_0$ denote this simple cylinder. Let $C$ be a simple cylinder on $(X, \omega)$. By Theorem 2.20, the cylinder $C$ is contained in a cylinder decomposition for some direction $\theta$. Since $\text{Peri}(X, \omega)/\Gamma(X, \omega) = \{[0], [\pi/4]\}$, there exists $A \in \Gamma(X, \omega)$ that maps the direction $0$ to $\theta$. By Proposition 2.22 and Proposition 2.25, the cylinder $C$ must contain $p_1$ or $p_2$. This means that $(X, \omega)$ has at most two disjoint simple cylinders. \qed

Next, we prove the first claim of Theorem 4.2. Suppose that $(X, \omega) \in \mathcal{H}^{\text{hyp}}(4)$ and $\tau$ is the hyperelliptic involution of $(X, \omega)$. We assume that $(X, \omega)$ has no three disjoint simple cylinders each of which is invariant under $\tau$. We will show that $(X, \omega) \in \mathcal{H}^{\text{hyp}}(4) \cdot \text{St}_5$. To do this, we set $Y = X/\langle \tau \rangle$. Let $q$ be the holomorphic quadratic differential on $Y$ induced by $\omega^2$ via the natural projection $\varphi : X \to Y$. Then the genus of $Y$ is 0 and the quadratic differential $q$ has a unique zero $p_0$ of order 3 and 7 simple poles. As we do in Section 3, we construct the union of some triangles whose sides correspond to saddle connections or returning geodesics of $(Y, q)$ and vertices correspond to $p_0$. The following remark is important.

**Remark 5.** In the construction of triangles, as we do in Section 3, the last triangle has two sides corresponding to returning geodesics of $(Y, q)$. The preimage of the triangle via $\varphi$ is a simple cylinder that is invariant under $\tau$.

Moreover, we use the following lemma.

**Lemma 4.6.** Let $(Y_0, q_0)$ be a flat surface of genus 0 such that $q_0$ has no zero and 4 simple poles $p_0, p_1, p_2, p_3$. Let $s$ be a saddle connection on $(Y, q)$ connecting $p_0$ and $p_1$. Let $v$ be a unit vector that is not parallel to $s$. Then there exists a (closed)
triangle $\Delta$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ in counterclockwise order and an orientation preserving immersion $\rho : \Delta \rightarrow (Y, q)$ satisfying the following:

1. $s_0$ is parallel to $s$ and $\Delta$ is a left (resp. right) strongly $(s_0, v)$-restricted triangle,
2. $\rho|_{\text{Int}(\Delta)}$ is a local isometric embedding,
3. $\rho(\text{Int}(\Delta))$ contains no singular points,
4. every vertex of $\Delta$ is mapped to $p_0$,
5. $s = \rho \circ s_0$, and
6. the preimage of $\{p_2, p_3\}$ coincides with the set of midpoints of $s_1$ and $s_2$.

**Proof.** We prove in the case where $\Delta$ is a left strongly $(s_0, v)$-restricted triangle. Cut $(Y_0, q_0)$ along $s$. Then $(Y_0, q_0)$ is a flat surface with a geodesic boundary $s$. There exists an Euclidean cylinder $C$ in $(Y_0, q_0)$ one of whose boundary components is this geodesic boundary $s$. We can extend $C$ so that the other boundary component contains $p_2$ or $p_3$. Assume that this boundary component $C$ contains only $p_2$. Then $Y_0 - C$ is not empty. However, it contradicts the fact that the angle around $p_2$ is $\pi$. Therefore, this boundary component contains both $p_2$ and $p_3$. Moreover, $Y_0$ is obtained from $C$ by sewing each boundary component. We cut $C$ along the segment from $p_0$ that is parallel to $v$. Then the resulting surface $P$ is regarded as a paralleloegram in $\mathbb{C}$ with sides that are parallel to $s_0$ and there exists a natural map $\rho_0 : P \rightarrow (Y_0, q)$. Let $s$ be a side of $P$ that is mapped to $s_0$ via $\rho_0$. Let $a$ be the side of $P$ that is opposite to $s$. We label the other side of $P$ with $b$ and $c$ so that the counterclockwise order of sides is $s, c, a, b$. Let $P_0$ be the vertex of $P$ that is the intersection of $s$ and $b$ and $P_1$ the vertex of $P$ that is the intersection of $s$ and $c$. Then $\rho_0 \circ a$ is a saddle connection connecting $p_2$ and $p_3$. The sets $\rho_0^{-1}(p_2) - b$ and $\rho_0^{-1}(p_3) - b$ contain only one point. We set $\rho_0^{-1}(p_i) - b = \{P_i\}$ for $i = 2, 3$. We may assume without loss of generality that $P_0 P_2 < P_0 P_3$. Let $P_1$ be the intersection of the lines $P_0 P_2$ and $P_1 P_3$. Then $\Delta = \Delta P_0 P_1 P_3$ is a left strongly $(s_0, v)$-restricted triangle. We set $s_1 = P_1 P_3$ and $s_2 = P_0 P_2$. Let $l$ be the line that passes through $P_4$ and is parallel to $v$ (see Figure 19). If $P_3 \notin c$, then the line $l$ divides $\Delta - \text{Int}(P)$ into two components $\Delta_1$ and $\Delta_2$. We assume that $\Delta_1$ contains $P_2$. Then the map $\rho : \Delta \rightarrow (Y_0, q_0)$ defined by

$$
\rho(z) = \begin{cases} 
\rho_0(z) & (z \in P) \\
\rho_0 \circ r_{P_2}(z) & (z \in \Delta_1) \\
\rho_0 \circ r_{P_3}(z) & (z \in \Delta_2) 
\end{cases}
$$

satisfies all conditions as in the claim. Here, $r_{P_2}$ and $r_{P_3}$ are point reflections in $P_2$ and $P_3$, respectively. If $P_3 \in c$, then $P_2$ is the midpoint of $a$ and $l$ coincides with the line $P_1 P_3$. Then the map $\rho : \Delta \rightarrow (Y_0, q_0)$ defined by

$$
\rho(z) = \begin{cases} 
\rho_0(z) & (z \in P) \\
\rho_0 \circ r_{P_2}(z) & (z \in \Delta - P) 
\end{cases}
$$

satisfies all conditions as in the claim. \hfill \square

For the proof of Theorem 4.2, we will construct triangles $\Delta_0, \Delta_1, \Delta_2, \ldots$, and their sides $s_0, s_1, s_2, \ldots$. Hereafter, $Q_i$ denotes the midpoint of the side $s_i$ for $i = 1, 2, \ldots$. Given a returning geodesic $s$ on $(Y, q)$. By Theorem 3.13, there exists a triangle $\Delta_0$ in $\mathbb{C}$ with sides $s_0, s_1, s_2$ in counterclockwise order and an orientation preserving immersion $\rho_0 : \Delta_0 \rightarrow (Y, q)$ satisfying all conditions as in the claim.
of Theorem 3.13. The preimage $C_1 = \varphi^{-1}(\rho_0(\Delta_0))$ is a simple cylinder that is invariant under $\tau$. The surface $Y_1 = Y - \rho_0(\Delta_0)$ is of genus 0 and has a boundary $\rho_0 \circ s_2$. Sewing the boundary, $Y_1$ is a closed surface of genus 0 and $q$ induces a meromorphic quadratic differential $q_1$ on $Y_1$ such that $\rho \circ s_2$ is a returning geodesic on the flat surface $(Y_1, q_1)$. Applying Theorem 3.13, there exists a triangle $\Delta_1$ in $\mathbb{C}$ with sides $s_2, s_3, s_4$ in counterclockwise order and an orientation preserving immersion $\rho_1 : \Delta_1 \to (Y_1, q_1)$ satisfying the following:

\begin{enumerate}
  \item $\rho_1|_{\text{Int}(\Delta_1)}$ is a local isometric embedding,
  \item $\Delta_1 \cap \Delta_0 = s_2$,
  \item $\rho_1|_{\text{Int}(\Delta_1)}$ contains no singular points,
  \item every vertex of $\Delta_1$ is mapped to $p_0$,
  \item $\rho_1 \circ s_2 = \rho_0 \circ s_2$, and
  \item $\rho_1 \circ s_4$ is a returning geodesic.
\end{enumerate}

Rename $A : (Y, q)$ to $(Y, q)$ for some $A \in \text{GL}(2, \mathbb{R})$ if necessary, we may assume that $s_2$ is horizontal, $s_3$ is vertical, $|s_2| = |s_3|$, and $\Delta_1$ is above $\Delta_0$ (see Figure 20). The surface $Y_2 = Y_1 - \rho_1(\Delta_1)$ is of genus 0 and has a boundary $\rho_1 \circ s_3$. Sewing the boundary, $Y_2$ is a closed surface of genus 0 and $q_1$ induces a meromorphic quadratic differential $q_2$ on $Y_2$ such that $\rho_1 \circ s_3$ is a returning geodesic on the flat surface $(Y_2, q_2)$. Applying Theorem 3.7 with the vector $v = (1, 0)$, there exists a triangle $\Delta_2$ in $\mathbb{C}$ with sides $s_3, s_5, s_6$ in counterclockwise order and an orientation preserving immersion $\rho_2 : \Delta_2 \to (Y_2, q_2)$ satisfying the following:

\begin{enumerate}
  \item $\rho_2$ is a right strongly $(s_3, v)$-restricted triangle,
(2-2) \( \rho_2|_{\text{Int}(\Delta_2)} \) is a local isometric embedding,
(2-3) \( \Delta_2 \cap \Delta_1 = s_3 \),
(2-4) \( \rho_2(\text{Int}(\Delta_2)) \) contains no singular points,
(2-5) every vertex of \( \Delta_2 \) is mapped to \( p_0 \),
(2-6) \( \rho_2 \circ s_3 = \rho_1 \circ s_3 \), and
(2-7) \( \rho_2 \circ s_i \) is a saddle connection or a returning geodesic for each \( i = 5, 6 \).

If both \( \rho_2 \circ s_5 \) and \( \rho_2 \circ s_6 \) are saddle connections, \( Y_2 - \rho_2(\Delta_2) \) has two connected components. By Remark 5, we can find triangles whose preimages via \( \varphi \) are simple cylinders that are invariant under \( \tau \) in the connected components. This contradicts the assumption that \((Y, q)\) has no three disjoint simple cylinders each of which is invariant under \( \tau \). Since \( q \) has 7 simple poles, only one of \( \rho_2 \circ s_5 \) and \( \rho_2 \circ s_6 \) is a saddle connection. Moreover, by Remark 5, we can find a triangle in \( Y_2 - \rho_2(\Delta_2) \) whose preimage via \( \varphi \) is a simple cylinder that is invariant under \( \tau \). If \( \rho_2 \circ s_6 \) is a returning geodesic, the segment \( Q_4Q_6 \) is mapped via \( \rho_1 \) and \( \rho_2 \) to a saddle connection in \((Y, q)\) whose preimage via \( \varphi \) is a core curve of a simple cylinder that is invariant under \( \tau \). This contradicts the assumption that \((Y, q)\) has no three disjoint simple cylinders each of which is invariant under \( \tau \). Thus, \( \rho_2 \circ s_5 \) is a returning geodesic. The segment \( Q_4Q_5 \) cannot correspond to a saddle connection in \((Y, q)\) whose preimage via \( \varphi \) is a core curve of a simple cylinder. This implies that \( Q_4Q_5 \) and \( s_6 \) are horizontal (see Figure 21). The surface \( Y_3 = Y_2 - \rho_2(\Delta_2) \) is

![Figure 21. The triangles \( \Delta_0, \Delta_1, \) and \( \Delta_2 \).](image)

of genus 0 and has a boundary \( \rho_3 \circ s_6 \). Sewing the boundary, \( Y_3 \) is a closed surface of genus 0 and \( q_2 \) induces a meromorphic quadratic differential \( q_3 \) on \( Y_3 \) such that \( \rho_2 \circ s_6 \) is a returning geodesic on the flat surface \((Y_3, q_3)\). Applying Theorem 3.7 with the vector \( v_1 = (0, 1) \), there exists a triangle \( \Delta_3 \) in \( \mathbb{C} \) with sides \( s_6, s_7, s_8 \) in counterclockwise order and an orientation preserving immersion \( \rho_3: \Delta_3 \to (Y_3, q_3) \) satisfying the following:

(3-1) \( \Delta_3 \) is a left strongly \((s_6, v_1)\)-restricted triangle,
(3-2) \( \rho_3|_{\text{Int}(\Delta_3)} \) is a local isometric embedding,
(3-3) \( \Delta_3 \cap \Delta_1 = s_6 \),
(3-4) \( \rho_3(\text{Int}(\Delta_3)) \) contains no singular points,
(3-5) every vertex of \( \Delta_3 \) is mapped to \( p_0 \),
(3-6) \( \rho_3 \circ s_6 = \rho_2 \circ s_6 \), and
(3-7) \( \rho_3 \circ s_i \) is a saddle connection or a returning geodesic for each \( i = 7, 8 \).

By the same argument as above, we see that \( \rho_3 \circ s_8 \) is a returning geodesic, \( \rho_3 \circ s_7 \) is a vertical saddle connection, and \( s_7 \) is vertical (see Figure 22). Assume that \( |s_6| < |s_2| \). Let \( v_2 \) be the unit vector that is parallel to \( s_4 \) from top to bottom.
Then we reconstruct $\Delta_0$ so that $\Delta_0$ is a right strongly $(s_2, v_2)$-restricted triangle. Then $Q_0Q_4$ and $Q_1Q_5$ are mapped via $\rho_0$, $\rho_1$, and $\rho_2$ to saddle connections in $(Y, q)$ whose preimages via $\varphi$ are core curves of simple cylinders that are invariant under $\tau$. Moreover, by Remark 5, we can find a triangle in $Y_2 - \rho_2(\Delta_2)$ whose preimage via $\varphi$ is a simple cylinder that is invariant under $\tau$. This contradicts the assumption that $(Y, q)$ has no three disjoint simple cylinders each of which is invariant under $\tau$.

Thus, we have $|s_6| \geq |s_2|$. Next, if $|s_7| < |s_3|$, then the segment $Q_4Q_8$ is mapped via $\rho_1$, $\rho_2$, and $\rho_3$ to a saddle connection in $(Y, q)$ whose preimage via $\varphi$ is a core curve of a simple cylinder that is invariant under $\tau$. Moreover, by Remark 5, we can find a triangle in $Y_3 - \rho_3(\Delta_3)$ whose preimage via $\varphi$ is a simple cylinder that is invariant under $\tau$. This contradicts the assumption that $(Y, q)$ has no three disjoint simple cylinders each of which is invariant under $\tau$. Thus, we have $|s_7| \geq |s_3|$. The surface $Y_4 = Y_3 - \rho_3(\Delta_3)$ is of genus 0 and has a boundary $\rho_3 \circ s_7$. Sewing the boundary of $Y_4$, $Y_4$ is a closed surface of genus 0 and $q_1$ induces a meromorphic quadratic differential $q_1$ on $Y_4$ such that $\rho_3 \circ s_7$ is a returning geodesic on the flat surface $(Y_4, q_1)$. Let $v_3$ be the unit vector that is parallel to $s_5$ from bottom to top. Applying Lemma 4.6 with the vector $v_3$, there exists a triangle $\Delta_i'$ in $\mathbb{C}$ with sides $s_7, s_9, s_{10}'$ in counterclockwise order and an orientation preserving immersion $\rho_i' : \Delta_i' \rightarrow (Y_4, q_4)$ satisfying the following:

- (4-1)' $\Delta_i'$ is a right strongly $(s_7, v_3)$-restricted triangle,
- (4-2)' $\rho_i'|_{\text{Int}(\Delta_i')}$ is a local isometric embedding,
- (4-3)' $\Delta_i' \cap \Delta_3 = s_7$,
- (4-4)' $\rho_i'(\text{Int}(\Delta_i'))$ contains no singular points,
- (4-5)' every vertex of $\Delta_i'$ is mapped to $p_0$,
- (4-6)' $\rho_i' \circ s_7 = \rho_3 \circ s_7$, and
- (4-7)' $\rho_i' \circ s_i'$ is a returning geodesic for each $i = 9, 10$.

Let $Q_i'$ be the midpoint if $s_i'$ for $i = 9, 10$. Then $Q_5Q_8'$ is mapped via $\rho_1$, $\rho_2$, $\rho_3$, and $\rho_4'$ to a saddle connection in $(Y, q)$ whose preimage via $\varphi$ is a core curve of a simple cylinder that is invariant under $\tau$. By the assumption that $(Y, q)$ has no three disjoint simple cylinders each of which is invariant under $\tau$, $s_8$ must be parallel to $s_{10}'$. If not, $Q_8Q_{10}'$ is mapped via $\rho_3$ and $\rho_4'$ to a saddle connection in
(Y, q) whose preimage via ϕ is a core curve of a simple cylinder that is invariant under τ. Therefore, we have |s_7| = |s_3|. Moreover, if |s_6| > |s_2|, then the segment Q_4Q_8 is mapped via ρ_1, ρ_2, and ρ_3 to a saddle connection in (Y, q) whose preimage via ϕ is a core curve of a simple cylinder that is invariant under τ. This contradicts the assumption that (Y, q) has no three disjoint simple cylinders each of which is invariant under τ. Thus, we have |s_6| = |s_2|. Again, applying Lemma 4.6 with the vector v_3, there exists a triangle Δ_4 in C with sides s_7, s_9, s_10 in counterclockwise order and an orientation preserving immersion ρ_4 : Δ_4 → (Y_4, q_4) satisfying the following:

- 4-1) Δ_4 is a left strongly (s_7, v_3)-restricted triangle,
- 4-2) ρ_4|_{Int(Δ_4)} is a local isometric embedding,
- 4-3) Δ_4 ∩ Δ_3 = s_7,
- 4-4) ρ_4|_{Int(Δ_4)} contains no singular points,
- 4-5) every vertex of Δ_4 is mapped to p_0,
- 4-6) ρ_4 ∘ s_7 = ρ_3 ∘ s_7, and
- 4-7) ρ_4 ∘ s_i is a returning geodesic for each i = 9, 10.

Then Q_8Q_10 is mapped via ρ_2 and ρ_4 to a saddle connection in (Y, q) whose preimage via ϕ is a core curve of a simple cylinder that is invariant under τ. Thus, the side s_9 must be parallel to s_5. Moreover, the segment Q_4Q_9 must not correspond to a core curve of a simple cylinder that is invariant under τ. This implies that s_10 is horizontal. Now, Δ_1, Δ_2, Δ_3, and Δ_4 are all equilateral right triangles. By the same argument as above, we can reconstruct Δ_0 to be an equilateral right triangles equilateral right triangles with a vertical side s_0. Therefore, (Y, q) is the flat surface constructed from 5 right triangles equilateral right triangles as in Figure 23 so that the midpoints of the sides s_0, s_1, s_4, s_5, s_8, s_9, and s_10 correspond to simple poles of (Y, q). This implies that (X, ω) coincides with St_5.

![Figure 23. The triangles Δ_0, Δ_1, Δ_2, Δ_3, and Δ_4 that construct (Y, q).](image)

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