Regimes of classical simulability for noisy Gaussian boson sampling

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As a promising candidate for exhibiting quantum computational supremacy, Gaussian boson sampling (GBS) is designed to exploit the ease of experimental preparation of Gaussian states. In this work, we establish sufficient conditions for efficient approximate simulation of GBS under the effect of errors such as photon losses and dark counts. We show that efficient simulation of GBS is possible if the per-photon transmission rate scales as the inverse of the square root of the average number of photons. As a consequence, for architectures where losses increase exponentially with circuit depth, our results place an upper bound on circuit size beyond which GBS becomes simulable. We also find that increasing the amount of input squeezing is helpful to evade our classical simulation algorithm, which suggests a potential route to fight against photon loss.

An important milestone in the field of quantum computation will be the demonstration of quantum computational supremacy [1, 2] – when a quantum device performs a computational task that is beyond the capabilities of classical computers. With this in mind, various restricted quantum computational models [3–8] were proposed for which there is evidence that efficient classical simulation is impossible, up to commonly-believed complexity-theoretic conjectures. Boson sampling (BS) [6], together with its variants [9, 10], are examples of such proposals which require simple quantum optical components, such as linear optics, photon detectors, and single-photon sources.

Gaussian boson sampling (GBS) [11] is a computational model in which the photon statistics is directly measured from a Gaussian state. Any pure zero-mean Gaussian state can be generated from a set of squeezed vacuum states followed by a passive linear-optical transformation. Accordingly, an arbitrary Gaussian boson sampling instance can be implemented, without loss of generality, by three steps: (i) deterministic preparation of \( K \) single-mode squeezed vacuum states, usually assumed to have same squeezing strength, by pumping a laser into a non-linear crystal; (ii) interference over an \( M \) mode interferometer, implemented by simple optical elements such as beam splitters and phase shifters [12, 13]; (iii) sampling of output photon statistics by an array of photon number resolving detectors. The deterministic sources, together with high generation probability and sampling rate, render GBS a highly efficient alternative to BS. Besides demonstrating quantum computational supremacy, GBS might have potential applications for specific computational problems such as dense subgraph searching [14, 15], perfect matching counting [16], graph isomorphism [17] and the simulation of vibrational spectra [18]. In addition, GBS was shown recently to be strongly connected to non-Gaussian probabilistic state engineering [19–21].

Because these near-term quantum devices do not benefit from fault-tolerant error correction, noise inevitably increases with the complexity and size of realistic experiments, limiting their correct functioning. Therefore, investigating the noise boundaries where near-term devices transition from quantum complexity to classical simulability is paramount for a demonstration of quantum computational supremacy. Numerous works have studied the effects of experimental noise on the robustness of BS, including partial photon distinguishability [22, 23], fabrication imperfections on linear-optical elements [24–26], losses [27–31], and detector dark counts [27]. However, no rigorous analysis has been carried out for GBS so far.

A common approach to investigate the power of imperfect devices is to show under which conditions, over the noise parameters, they become efficiently classically simulable. In that case, the device is not expected to provide any computational advantage. Rahimi-Keshari et al. first established a rigorous bound for exact classical simulability of BS under the effect of photon losses and detector dark counts [27]. However, the notion of exact simulability is very stringent, as a real-world experiment is incapable of producing samples from the exact theoretical distribution. Therefore, a more relevant approach is to focus on approximate classical simulation, i.e., algorithms that sample from a distribution sufficiently close to that generated by the quantum device.

In this manuscript, we describe an efficient classical algorithm for approximate simulation of GBS under the effect of imperfect sources, photon losses and dark counts. We investigate how the error in our simulation scales with the noise parameters, establishing boundaries beyond which GBS would be efficient simulable and comparing them with recent experiments.

Our classical algorithm works as follows. We begin by transforming a realistic model of losses and dark counts into a virtual model consisting of identical thermalized input squeezed states (each denoted by \( \sigma \)), followed by an ideal interferometer and an array of noisy detectors. We then consider the task of sampling from the output distribution obtained when the state \( \sigma \) is approximated by another Gaussian state \( \tau \). We use the theory of phase-space quasi-probability distributions (PQDs), described in Ref. [27], to obtain an efficient sampling algorithm that works exactly for a class of states \( \tau \). We then optimize over those states and quantum distance measures to obtain the closest approximation to \( \sigma \).

Quasi-probability distributions and Gaussian states.—The main ingredient of our results is the theory of PQDs for bosonic systems [27, 32]. Consider an \( M \)-mode bosonic system and let \( \vec{x} := (q_1, p_1, \ldots, q_M, p_M) \) denote the row vector
of its quadrature operators. They satisfy the commutation relations \( [\hat{x}_j, \hat{x}_k] = 2i\Omega_{jk} \), where \( \Omega = I_M \otimes (0^{1 \times 1}) \) and \( I_M \) is the \( M \times M \) identity matrix. The annihilation operators are given by \( \hat{a}_j = \frac{1}{2} (\hat{q}_j + i\hat{p}_j) \) and \( \hat{a} := (\hat{a}_1, \ldots, \hat{a}_M)^T \).

The \( t \)-ordered PQD \( \rho^{(t)} \) of an \( M \)-mode Hermitian operator \( \rho \) is defined as

\[
W^{(t)}(x) = \int d^{2M} x W^{(t)-}(n|x) W^{(t)}(x) ,
\]

where \( W^{(t)}(x) \) is the \( (t) \)-PQD of the pre-measurement output state and \( W^{(t)-}(n|x) \) is the \((-t)\)-PQD of the measurement operator. If both PQDs are positive and can be simulated efficiently for some \( t \), the device as a whole can be efficiently simulated by successively sampling from the chain of distributions given in Eq. (1).

For our result, the main calculations will be reduced to deal only with single-mode Gaussian states, so we briefly review them here. Generalizing to multi-mode Gaussian states is straightforward, and we refer the readers to Ref. [33] for a comprehensive review. A state \( \rho \) is Gaussian if it is fully characterized by its mean vector \( \vec{x}_\rho \) and its covariance matrix \( \rho_{\rho} \), which matrix elements are given by

\[
V_{\rho}^{j,k} = \frac{1}{2} \text{Tr} \{ \rho [\hat{x}_j - \hat{x}_\rho^j, \hat{x}_k - \hat{x}_\rho^k]_+ \} ,
\]

where \([\cdot, \cdot]_+\) is the anti-commutator. All distance measures we consider are minimized when two Gaussian states are displaced along the same direction by the same amount [34], so we set \( \vec{x}_\rho = 0 \) from now on without loss of generality.

For a single-mode Gaussian state, the \((t)\)-PQD is simply a Gaussian function:

\[
W^{(t)}(x) = \exp \left[ -\frac{1}{2} x^T (V_{\rho} - t I_2)^{-1} x \right] \overline{\det(V_{\rho} - t I_2)} .
\]

Equation (3) only holds when \( V_{\rho} - t I_2 \) is positive definite. Since the covariance matrix \( V_{\rho} \) is positive definite, there always exists \( t_{\rho}^+ \in [0, 1] \) such that, for \( t < t_{\rho}^+ \), the \((t)\)-PQD is a Gaussian function; for \( t = t_{\rho}^+ \), the \((t)\)-PQD has \( \delta \)-function singularities; and for \( t > t_{\rho}^+ \), the \((t)\)-PQD does not exist. A Gaussian state is called classical if its \( P \) function (i.e., for \( t = 1 \)), is well-behaved. We generalize this notion of classicality and say that \( \rho \) is \((t)\)-classical if \( V_{\rho} - t I_2 \) is positive definite, or equivalently, if its \((t)\)-PQD is non-singular (i.e., if \( t \leq t_{\rho}^+ \)). Finally, we define \( \rho^{(t)}_G \) as the set of \((t)\)-classical Gaussian states.

Another useful parameterization of Gaussian states is in terms of squeezed thermal states (STS) [35]. Any single-mode Gaussian state \( \rho \) can be written as

\[
\rho = S(\sigma, \phi) \rho_T S(\sigma, \phi) ,
\]

where \( \rho_T \) is a thermal state with average photon number \( \lambda_T \) and covariance matrix \( V_T = 2(\lambda_T + 1) I_2 \), and \( S(\sigma, \phi) = \exp \left[ \frac{1}{2} \lambda_T \sigma \hat{a} \hat{a} - \frac{1}{2} \lambda_T \sigma \right] \) is the squeezing operator with squeezing parameter \( \sigma \) and phase rotation \( \phi \). It is straightforward to show that a Gaussian state \( \rho \) is \((t)\)-classical for \( t < t_{\rho}^+ \), where [35]

\[
t_{\rho}^+ = e^{-2\sigma} (2\lambda_T + 1) .
\]

**Gaussian boson sampling and noise model.**—Any experimental implementation of GBS is imperfect (see Fig. 1). The squeezed light injected into the interferometer is usually noisy due to photon loss in its preparation [36], which we model as a thermal squeezing source with \( \sigma = \tau, \lambda_T = 0 \), and \( \phi = 0 \), followed by a lossy channel with transmission \( \eta_T \). Due to losses, the interferometer is described by a \( M \times M \) matrix \( A \) satisfying \( 0 < AA^T \leq I_M \) [37] which transforms the annihilation operators according to \( \hat{a} \rightarrow \hat{a}A + \sqrt{I_M - AA^T} \hat{1} \), where \( \hat{1} \) represent environment modes in vacuum states. The matrix \( A \) admits the singular value decomposition \( A = V D W \). The explicit form of \( A \) usually depends on the implementation and architecture [61]. For integrated platforms—a common choice due to their stability and high transmissivity—propagation losses along different optical paths are roughly the same and determined by the length of the chip. Therefore, it is a good approximation to assume that losses are uniform, i.e., \( D = \eta_I I_M \). Hence we can write \( A = \eta_I U \) with \( U = V W \). That is, the linear optical transformation can be simplified to \( M \) identical single-mode lossy channels, each with transmission \( \eta_I \), followed by an ideal unitary transformation \( U \) (see [28, 29] for a rigorous treatment). As a final simplification, we combine the source and interferometer losses into a single loss channel with transmission \( \eta := \eta_S \eta_I \) and encapsulate it into a mixed input Gaussian state \( \sigma \) (see Fig. 1(b)).

If the number of modes \( M \) is sufficiently large, i.e., \( M = O(K^2) \), and assuming constant squeezing, the probability of
detecting more than one photon at each output detector is negligible for a typical (Haar-random) unitary [38, 39]. In this no-collision regime, we can safely replace photon counting detectors by threshold detectors [27, 40]. Each threshold detector has sub-unity quantum efficiency denoted by $\eta_D$ and registers random dark counts with probability $p_D$. Following Ref. [27, 41] we describe these detectors by their POVM elements: $\Pi_0 = (1 - p_D) \sum_n (1 - \eta_D) \rho^n |n\rangle \langle n|$ and $\Pi_1 = I - \Pi_0$. We define $q_D = p_D/\eta_D$ as the figure of merit that characterizes the detectors.

With this noise model, the probability distribution at the output reads $P(n) = \text{Tr} \{ \rho_{\text{out}} \Pi_n \}$, with $n = (n_1, n_2, \ldots, n_M)$ and $n_i \in \{0, 1\}$, where
\[
\rho_{\text{out}} = U \left( \sigma^\otimes K \otimes |0\rangle \langle 0|^{(M-K)} \right) U^\dagger \tag{5}
\]
and $\sigma$ has covariance matrix $V_\sigma = \text{diag} \{a_+, a_-\}$ with $a_\pm = \eta e^{\pm 2\epsilon} + (1 - \eta)$. In what follows it will be useful to use the STS parameterization $\sigma = \text{PSTS}(s_\sigma, 0, n_\sigma)$ with [35, 38]
\[
\begin{align*}
    s_\sigma &= \frac{1}{4} \ln \frac{a_+}{a_-}, \\
    n_\sigma &= \frac{1}{2} (\sqrt{a_+ a_-} - 1). \tag{6,7}
\end{align*}
\]

**Exact simulation of noisy GBS.**—Following the algorithm sketched around Eq. (1) (see Ref. [27] for details), we now obtain a classical algorithm for exact simulation of GBS based on PQDs. This follows from the fact that the unitary $U$ does not alter the negativity of the input PQDs nor their efficient simulability (since it only induces a rotation in the phase space), and that the $(-t)$-PQDs of both $\Pi_0$ and $\Pi_1$ are non-negative for some $t \geq 1 - 2q_D$ [27]. Since $\sigma$ is $(t)$-classical for $t \leq t_\sigma^*$, we conclude that efficient exact simulation is possible when $t_\sigma^* > 1 - 2q_D$. Using Eqs. (4), (6), and (7) this can be written as
\[
\eta < q_D (1 + \coth t). \tag{8}
\]

Two remarks regarding Eq. (8) are in order. First, it shows that increasing squeezing can evade our classical simulation algorithm. Second, in the absence of dark counts ($p_D = 0$), there is no valid $\eta$ for which the above condition holds. This is consistent with the result that GBS should be hard to simulate exactly under the effect of only losses (i.e., no dark counts), as we discuss in the Supplemental Material [38]. This also underpins the fact that exact simulation is a very restrictive scenario, as one might expect that a sufficiently lossy GBS setup would be classically simulable, even for ideal detectors. This is what we prove next.

**Efficient approximate simulation of noisy GBS.**—Consider the set of $M$-mode states of the form
\[
\hat{\rho} = U \left( \tau^\otimes K \otimes |0\rangle \langle 0|^{(M-K)} \right) U^\dagger, \tag{9}
\]
for $\tau \in C_G^{(t)}$ and $t \in [1 - 2q_D, 1]$. The state $\hat{\rho}$ has the same structure as $\rho_{\text{out}}$, the output state of our noisy GBS given in Eq. (5), except that the thermalized state $\sigma$ was replaced by a $(t)$-classical Gaussian state $\tau$. Let $\tilde{P}(n) = \text{Tr} \{ \Pi_n \hat{\rho} \}$ be the photon distribution obtained by measuring $\hat{\rho}$ with noisy threshold detectors. From our previous discussion, sampling efficiently from $\tilde{P}(n)$ is possible by the classical algorithm described in Ref. [27]. Therefore, if $\tau$ is sufficiently close to $\sigma$, an algorithm that samples from $\tilde{P}(n)$ also gives a good approximation to $P(n)$.

To measure the distance between $\tau$ and $\sigma$ we choose the sandwiched Rényi relative entropy, denoted by $D_\alpha(\sigma || \tau)$, for $\alpha \in [\frac{1}{2}, 1)$ [42, 43]. The total variation distance between the outcome distributions, $D_T(\hat{P}, P) := \frac{1}{2} \sum_n |\hat{P}(n) - P(n)|$, can be bounded as follows:
\[
D_T(\hat{P}, P) \leq \sqrt{\frac{2}{\alpha} D_\alpha(\rho || \hat{P})} \leq \sqrt{\frac{2}{\alpha} D_2(\rho || \hat{P})}, \tag{10}
\]
where we used a generalized Pinkser’s inequality [44] and, in a slight abuse of notation, $D_\alpha(\rho || \hat{P})$ also denotes the Rényi divergence between two distributions [44]. The second inequality follows from the data-processing inequality under quantum measurements [42]. Since the sandwiched Rényi relative entropy is invariant under the action of the unitary transformation and is additive under tensor products [42], we have that $D_\alpha(\rho || \hat{P}) = K D_\alpha(\sigma || \tau)$. Using Eq. (10), the requirement that GBS can be simulated to within $\epsilon$ total variation distance leads to
\[
\frac{2}{\alpha} D_\alpha(\sigma || \tau) \leq \frac{\epsilon^2}{K}. \tag{11}
\]
We could optimize this bound over $\alpha \in [\frac{1}{2}, 1]$, since $D_\alpha$ is non-decreasing over $\alpha$ [42], we expect an optimal $\alpha^*$ exists. For simplicity we discuss only the case $\alpha = \frac{1}{2}$, which admits analytical calculations, but a numerical optimization over $\alpha$ can be found in the Supplemental Material [38]. In this case, we have $D_{\frac{1}{2}}(\sigma || \tau) = -\ln F(\sigma, \tau)$, where $F(\sigma, \tau) := \text{Tr} \{ \sqrt{\sqrt{\sigma} \tau \sqrt{\sigma}} \}$ is the quantum fidelity. A tighter bound is obtained if we further optimize $\tau$ over $C_G^{(t)}$ with $t \in [1 - 2q_D, 1]$, which gives
\[
-\ln \left[ F_{\max}(\eta, q_D) \right] \leq \frac{\epsilon^2}{4K}. \tag{12}
\]
Here $F_{\max}(\eta, q_D) := \max_{\tau \in C_G^{(t)}} F(\sigma, \tau)$.

The fidelity between two single-mode Gaussian states is given by [45]
\[
F(\sigma, \tau) = \frac{1}{\sqrt{\Delta + \Lambda - \sqrt{\Delta}}} , \tag{13}
\]
where $\Delta = \frac{1}{2} \det(V_\sigma + V_\tau)$ and $\Lambda = \frac{1}{2} (\det V_\sigma - 1)(\det V_\tau - 1)$. A straightforward calculation (see the Supplemental Material [38]) leads to
\[
\Delta = (n_\sigma - n_\tau)^2 + (2n_\sigma + 1)(2n_\tau + 1) \cosh^2(s_\sigma - s_\tau), \Lambda = 4n_\sigma(n_\sigma + 1)n_\tau(n_\tau + 1).
\]
Since $\tau$ is a $(t)$-classical Gaussian state we have $s_\tau \leq s_\sigma := \frac{1}{2} \ln \frac{2a_+ + 1}{t}$. After optimizing the quantum fidelity over the
values of \((s_r, n_r)\), we obtain that its maximum value is
\[
\text{sech} \left[ \Theta \left( s_r - \frac{1}{2} \ln \frac{n_r+1}{n_r} \right) \right],
\]
where \(\Theta\) is the step function. Finally, we can further optimize this over \(t \in [1 - 2q_D, 1]\). It is clear that the maximum value is achieved at \(t = 1 - 2q_D\). Using Eqs. (6)-(7) we finally write the maximum fidelity as
\[
F_{\text{max}}(\eta, q_D) = \text{sech} \left[ \frac{1}{2} \Theta \left( \ln \left( \frac{1 - 2q_D}{\eta e^{-2r} - 1} \right) \right) \right]. \quad (14)
\]
Plugging this into Eq. (12) we obtain our final condition for classical simulability of noisy GBS. Note setting \(\epsilon = 0\) recovers the corresponding condition for exact simulation given in Eq. (8), as expected.

**Asymptotic analysis.**—The combination of Eqs. (12) and (14) draws a boundary in the parameter space \((\eta, q_D, K, \epsilon)\). In Fig. 2 (a) and (b) we plot them in the \(\eta - K\) plane for several values of squeezing parameter \(r\) and detector quality \(q_D\). For noise parameters falling below the corresponding solid line, GBS can be efficiently simulated with error no more than \(\epsilon = 0.01\). We observe that the region where quantum computational supremacy has not been ruled out expands as we increase the quality of detectors or the amount of input squeezing, and the latter underpins the notion of squeezing as a non-classical resource. For bounds obtained by using general sandwiched Rényi relative entropies, our numerical simulation results in Fig. 2(c) and (d) suggest that the quantum fidelity (corresponding to \(\alpha = 1/2\)) is optimal, while quantum relative entropy (corresponding to \(\alpha \to 1\)) gives the worst bound. For more details see the Supplemental Material [38].

In the limit of \(K \to \infty\), the r.h.s. of Eq. (12) goes to zero and we recover the bound for exact simulation in Eq. (8). This is shown as the dashed lines in Fig. 2, which corresponds to \(\eta_{\infty} := q_D(1 + \coth r)\). Thus, one interpretation of our result is that it improves on exact sampling algorithms by providing tighter bounds for finite-size experiments.

It is interesting to combine Eqs. (14) and (12) and expand the result around \(\eta_{\infty}\) to obtain the following asymptotic simulability condition
\[
\eta < \eta_{\infty} + \frac{4\sqrt{2}(1 - 2q_D)}{1 - e^{-2r}} \frac{\epsilon}{\sqrt{k}} + \omega \left( \frac{1}{\sqrt{k}} \right).
\quad (15)
\]
In this sense, the second term can also be interpreted as a correction when we allow for errors in our classical simulation. Beyond that, however, it also gives a nontrivial bound even for perfect detectors (i.e., when \(\eta_{\infty} = 0\)). In that case, if we assume constant squeezing \(r\), we have average photon number given by \(\bar{N} = K \sinh^2 r\). This implies approximate classical simulability of GBS when the average number of surviving photons is less than \(O(\sqrt{N})\), which is the same scaling as the results of [28, 29] for standard BS.

In most linear-optical architectures, photon loss is defined by unit depth of the circuit, leading to an exponential decrease of transmission with circuit depth. Our results imply that GBS implemented on these platforms is rendered efficiently simulable by losses if the depth is linear in the number of modes. It is easy to see that Eq. (15) is always satisfied for circuits that have super-logarithmic depth in these architectures, which also happens for BS [28, 29]. In the other extreme, for planar circuits (i.e., with only nearest-neighbor beam splitters) of logarithmic depth we can construct a tensor network simulation that runs in quasi-polynomial time. This is analogous to a similar algorithm for BS [29]. The only difference is that we need to introduce an additional cutoff on the Hilbert space, for large photon numbers, that does not degrade too much the Gaussian state nor slows down the simulation. These two results, therefore, imply that quantum computational supremacy via GBS requires either novel architectures where losses do not scale exponentially with depth, or proposals that exploit shallow non-planar circuits [46].

**Implications for recent experiments.**—Motivated by its experimental advantages, recently several small-scale GBS experiments have been demonstrated [47–49]. It is interesting to analyze if those experiments satisfy our simulability condition for some error threshold \(\epsilon\). For instance, in Ref. [48], \(K = 4\) squeezed vacuum with \(r \sim 0.1\) are injected into a 12-mode random-walk circuit with overall transmission \(\eta = 0.088\) and detector efficiency \(\eta_D = 0.78\). For typical superconducting nanowire single photon detectors, the dark count rate is...
around $p_D = 10^{-4}$ [50]. Using these numbers, the experiment can be efficiently simulated by our sampling algorithm with error $0.056\%$. In the most recent GBS demonstration, $K = 6$ squeezed vacuums with $r \approx 0.3$ are coupled into a 12-mode interferometer implemented with bulk optics in free space [49]. With $\eta = 0.99$ [51], $\eta_D = 0.75$ [49] and assuming $p_D = 10^{-4}$, the error in our simulation would be of $18\%$.

Conclusion.—In this work, we establish sufficient conditions for efficient approximate simulation. We show that efficient classical simulation of GBS becomes possible for sufficiently high losses, i.e., if the transmission probability scales as the inverse of the square root of the average number of photons. For a model of losses where transmission decays exponentially with the depth of the circuit, GBS becomes classically simulable in the same regime as BS. Our results also suggest that increasing the input squeezing might be helpful to evade classical simulability, highlighting the benefits and flexibility of GBS relative to other approaches.

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The validity of assuming uniform propagation loss varies, depending on specific implementations. For spatial encoding, like in free-space and integrated platforms, photon loss in the interferometer is dominated by propagation loss, and uniform loss is usually a good approximation. For temporal encoding \cite{52, 53}, the dominant losses arise within the implementation of each beam splitter. Therefore, in this case, the loss is non-uniform in general \cite{28} and depends on specific decomposition scheme \cite{12, 13} being used.
## Supplemental materials

### I. NON-COLLISION REGIME FOR GBS

For standard boson sampling with $N$ single-photon inputs and $M$ modes, it was shown [6, 39] that the probability of detecting collision events is bounded as follows:

$$\langle P_{\text{collision}} \rangle_U \leq \frac{8N^2}{M}, \quad (16)$$

where the average is over Haar-random unitaries. The proof of Eq. (16) relies on the fact that Haar-random unitaries map any $N$-photon, $M$-mode state onto the maximally mixed state (its density matrix is given by the identity on the corresponding Hilbert space).

For GBS, the input state has indefinite photon number. Specifically, the probability of generating $S$ photon pairs is given by [11]

$$F(S) = \left( \frac{K^2 + S - 1}{S} \right) \text{sech}^2(r) \tanh^S(r). \quad (17)$$

Therefore, from Eq. (16), the probability of detecting collision events at the output of GBS satisfies

$$\langle P_{\text{collision}} \rangle_U \leq \sum_{S=0}^{\infty} F(S) \left[ \frac{8(2S)^2}{M} \right] = \frac{32}{M} \langle S^2 \rangle_F. \quad (18)$$

$F(S)$ in Eq. (17) is a negative binomial (or Pascal) distribution. In the large $K$ limit, $F(S)$ converges to Gaussian distribution with mean value $\frac{K^2}{2} \sinh^2 r$ and variance $\frac{K^2}{2} \sinh^2 r \cosh^2 r$, which gives $\langle S^2 \rangle_F = O(K^2)$. Therefore, we also expect no-collision outputs in GBS to dominate whenever

$$M = O(K^2). \quad (19)$$

### II. EXACT LOSSY GBS IS HARD

In this section we give evidence that exact classical simulation of a lossy GBS device cannot be efficient, unless the polynomial hierarchy collapses to its third level. The post-selection based argument we use is standard and was used to prove similar claims for many different restricted models of quantum computation (see e.g. [5, 6]), so we only detail the parts of the argument that pertain to GBS. The construction we use is directly inspired by the scattershot boson sampling model [54, 55], though our purpose is different, as we are interested especially in the effect of losses and the complexity of the model.

For this proof, we assume that losses are uniform within the interferometer $U$, and so we can move all losses to the end (this is a standard assumption that is a good approximation for e.g. integrated photonic devices, but was also shown to hold under more general conditions [28]). In contrast to the results in the main paper, here we also ignore all other sources of losses. This is an important caveat, but can be justified as losses in photon sources and detectors are effectively constant, whereas losses inside the interferometer $U$ scale with its depth (which, for boson sampling, also typically scales with the number of photons). Therefore, photon loss within the linear optical network is the main scalability bottleneck. We leave it as an open question whether this caveat can be eliminated.

**Theorem 1** If there is an efficient classical algorithm to sample from the output distribution of a lossy Gaussian boson sampling instance exactly (or up to multiplicative error), then the polynomial hierarchy collapses to its third level.

**Proof.** Consider the following lossy GBS setup. We prepare $2K$ identical SMSV states with identical squeezing parameter $r$. These states are input, in pairs, into 50:50 beam splitters, generating two-mode squeezed vacuum (TMSV) states of the form

$$\text{sech}(r) \sum_{S=0}^{\infty} \tanh^n r |n, n\rangle. \quad (20)$$

For each TMSV state we couple one mode directly to a number-resolving detector (these are the heralding registers H), whereas the other half are sent into the lossy interferometer $U$ (which may also require some additional vacuum inputs). The detectors at the output of $U$ are called the boson sampling registers R. This entire setup is shown in Fig. 3.

We now run this device, post-selecting on outcomes that satisfy two properties:
(i) Exactly one photon is observed in each of the heralding modes H, and

(ii) There are exactly \( n \) photons in total in the R registers.

The two properties above guarantee that, in every event accepted by the post-selection, exactly one photon was injected into each non-vacuum input of \( U \) [due to the form of Eq. (20)], and no photons were lost within \( U \). Therefore, the resulting conditional probability distribution is the same as an ideal boson sampling instance with single-photon inputs and interferometer \( U \).

Now note that standard boson sampling, when augmented with the power of post-selection, can perform universal quantum computation [6]. Thus, by choosing the interferometer \( U \) properly, the same is true for the device of Fig. 3. From this it immediately follows that, by a standard argument (see e.g. [5]), there can be no efficient classical algorithm to simulate the output distribution of the lossy GBS device exactly (or up to multiplicative error), otherwise the polynomial hierarchy collapses to its third level.

It is a well-accepted complexity-theoretic conjecture that the polynomial hierarchy is infinite, and so Theorem 1 can be taken as evidence that an efficient classical algorithm which exactly simulates a lossy GBS device does not exist. Note that the theorem did not require any assumptions on the strength of either losses or squeezing, so it holds for any squeezing parameter \( r > 0 \) and any interferometer transmissivity \( \eta > 0 \). Interestingly, if we replace \( U \) by the construction described in [56], Theorem 1 also proves that GBS (lossy or not) is hard to simulate even if the entire linear-optical sector of Fig. 3 has only five layers of (long-range) beam splitters.

Just like previous similar results [6, 27], Theorem 1 is not too relevant in a realistic scenario. The requirement of simulating the output distribution exactly (or with multiplicative error) is too strict, since a realistic device with experimental imperfections is not simulating the idealized device to that precision either. At best, Theorem 1 places bounds on how far a proposed efficient classical algorithm can be extended (for example, it shows that any exact simulation of lossy GBS based on the algorithm of [27] can only be efficient in the presence of dark counts).

III. MAPPING SQUEEZED LOSSY STATES TO SQUEEZED THERMAL STATES

Any zero-mean single-mode Gaussian state can be decomposed into a squeezed thermal state \( \rho = S(s, \phi_\rho) p_T^{n_\rho} S^\dagger(s, \phi_\rho) \). Here \( p_T^{n_\rho} \) is a thermal state with average photon number \( n_\rho \), and its covariance matrix is given by \((2n_\rho + 1)I_2\). \( S(r, \phi) \) is the Stoler squeezing operator,

\[
S(s, \phi) = \exp \left[ \frac{1}{2}se^{i\phi}a^\dagger a^2 - \frac{1}{2}se^{-i\phi}a^2 \right].
\]

Its transformation on the quadratures \( x \) and \( p \) is given by a symplectic matrix

\[
\begin{pmatrix}
cosh r + \cos \phi \sinh r & -\sin \phi \sinh r \\
-\sin \phi \sinh r & \cosh r - \cos \phi \sinh r
\end{pmatrix}.
\]
The covariance matrix of $\rho$ can be written as

$$V_\rho = (2n_\rho + 1) \begin{pmatrix} \cosh 2s_\rho - \cos \phi_\rho \sinh 2s_\rho & -\sin \phi_\rho \sinh 2s_\rho \\ -\sin \phi_\rho \sinh 2s_\rho & \cosh s_\rho + \cos \phi_\rho \sinh 2s_\rho \end{pmatrix}.$$  \hfill (23)

For the lossy squeezed state $\sigma$ with $V_\sigma = \text{diag} \{a_+, a_-\}$ where $a_\pm = \eta^\pm 2r + (1 - \eta)$, we can easily solve $s_\sigma$ and $n_\sigma$ by directly comparing $V_\sigma$ with Eq. (23), which gives

$$s_\sigma = \frac{1}{4} \ln \frac{a_+}{a_-},$$  \hfill (24)

$$n_\sigma = \frac{1}{2} (\sqrt{a_+ a_-} - 1).$$  \hfill (25)

IV. SUFFICIENT CONDITIONS DERIVED BY USING QUANTUM FIDELITY

In the main text we derive the following bound by using quantum fidelity,

$$- \ln |F_{\text{max}}(\eta, q_D)| \leq \frac{\epsilon^2}{4K},$$  \hfill (26)

where

$$F_{\text{max}}(\eta, q_D) := \max_{t \in [1 - 2q_D, 1]} \max_{\tau \in C_G^{(t)}} F(\sigma, \tau).$$  \hfill (27)

We explicitly calculate this quantity in what follows.

We start by computing the quantum fidelity between the lossy squeezed state $\sigma$ and a $(t)$-classical Gaussian state $\tau$. The fidelity between two single-mode Gaussian states is given by [37]

$$F(\sigma, \tau) = \frac{1}{\sqrt{\Delta + \Lambda - \sqrt{\Lambda}}},$$  \hfill (28)

where $\Delta = \frac{1}{4} \det(V_\sigma + V_\tau)$ and $\Lambda = \frac{1}{4} (\det V_\sigma - 1)(\det V_\tau - 1)$. To optimize Eq. (28) we make use of the STS parameterization for $\sigma$ and $\tau$. The fidelity is minimized when the squeezing axes are aligned [35], so we can set $\phi_\tau = 0$ for simplicity. Straightforward calculations then lead to

$$\Delta = (n_\sigma - n_\tau)^2 + (2n_\sigma + 1)(2n_\tau + 1) \cosh^2 (s_\sigma - s_\tau),$$

$$\Lambda = 4n_\sigma (n_\sigma + 1)n_\tau (n_\tau + 1).$$  \hfill (29)

(30)

Since $\tau$ is a $(t)$-classical Gaussian state, using Eq. (4) in the main text we have $s_\tau \leq s_0 := \frac{1}{2} \ln \frac{2n_{\sigma - 1}}{t}$. The task is then to find the point $(s^*_\sigma, n^*_\tau)$ that maximizes the quantum fidelity subject to that constraint. Note from Eq. (28) that the fidelity monotonically decreases with $|s_\sigma - s_\tau|$. So its optimization has two regimes.

First, when $s_\sigma \leq s_0$, a maximum of the fidelity is reached at $s^*_\sigma = s_\sigma$ and $n^*_\tau = n_\tau$, which gives $F(s^*_\sigma, n^*_\tau) = 1$. This corresponds to the case when $\sigma \in C_G^{(t)}$, i.e., $\sigma$ itself is a $(t)$-classical Gaussian state. This regime reproduces the previous result for exact simulation of GBS.

Second, when $s_\sigma \geq s_0$, the fidelity is maximized at $s^*_\sigma = s_0$. Substitute $s_0$ into Eqs. (28)-(30), we have a function of $n_\tau$ to optimize. It follows that its maximum is reached at $n^*_\tau = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 2r \sinh(2s_\tau)} \exp(2s_\tau)$, where $s_\tau = \frac{1}{2} \ln(2n_\sigma + 1)$. The corresponding maximum fidelity is $F(s^*_\sigma, n^*_\tau) = \text{sech} \left(s_\sigma - \frac{1}{2} \ln \frac{2n_{s_\sigma - 1}}{t} \right)$.

Combining both regimes we write the maximum fidelity compactly as $\text{sech} \left[ \Theta \left( s_\sigma - \frac{1}{2} \ln \frac{2n_{s_\sigma - 1}}{t} \right) \right]$, where $\Theta$ is the step function. We now need to further optimize this over $t \in [1 - 2q_D, 1]$. It is clear that the maximum value is achieved at $t = 1 - 2q_D$. Using Eqs. (24)-(25) we finally write the maximum fidelity as

$$F_{\text{max}}(\eta, q_D) = \text{sech} \left[ \frac{1}{2} \Theta \left( \ln \left( \frac{1 - 2q_D}{\eta e^{-2r} + 1 - \eta} \right) \right) \right].$$  \hfill (31)

Plugging this into Eq. (26) we obtain our final sufficient condition for classical simulability of noisy GBS. Notice that if we set $\epsilon = 0$ we recover the corresponding condition for exact simulation given in Eq. (8) in the main text, as expected.
V. Optimizing the Simulability Condition Using Sandwiched Rényi Relative Entropy

The additive error of our approximate classical algorithm of GBS is upper bounded by using sandwiched Rényi relative entropy, which is formally defined as follows for two quantum states $\rho_1$ and $\rho_2$ [42, 43],

$$D_\alpha(\rho_1 \| \rho_2) = \frac{1}{\alpha - 1} \ln \Tr \left\{ \left( \frac{1-\alpha}{2} \rho_1 \rho_2 \rho_1 \rho_2 \right)^{1-\alpha} \right\},$$

(32)

for $\alpha \in (0, 1) \cup (1, \infty)$. Some of its properties which are useful for us is listed below:

- **Unitary invariance:** Given any unitary $U$, $D_\alpha(U\rho_1 U^\dagger \| U\rho_2 U^\dagger) = D_\alpha(\rho_1 \| \rho_2)$ for $\alpha \in \left[\frac{1}{2}, 1\right) \cup (1, \infty)$ [42].

- **Additivity:** $D_\alpha(\rho_1 \otimes w_1 \| \rho_2 \otimes w_2) = D_\alpha(\rho_1 \| w_1) + D_\alpha(\rho_2 \| w_2)$ for $\alpha \in \left[\frac{1}{2}, 1\right) \cup (1, \infty)$ [42].

- **Data processing:** For any completely positive trace-preserving map $\mathcal{E}$, we have

$$D_\alpha(\rho_1 \| \rho_2) \geq D_\alpha(\mathcal{E}(\rho_1) \| \mathcal{E}(\rho_2))$$

(33)

for $\alpha \in \left[\frac{1}{2}, 1\right] \cup (1, \infty)$ [58, 59].

- **Monotonicity:** $D_\alpha(\rho_1 \| \rho_2) \geq D_{\alpha'}(\rho_1 \| \rho_2)$ for $\infty > \alpha \geq \alpha' > 0$ [42, 59].

Other than above properties of a valid quantum distance measure, sandwiched Rényi relative entropy includes well-known quantum distance measures as its special cases. Specifically, we have [42, 43]

$$D_{\frac{1}{2}}(\rho_1 \| \rho_2) = -\ln F(\rho_1, \rho_2),$$

(34)

$$\lim_{\alpha \to 1} D_\alpha(\rho_1 \| \rho_2) = D(\rho_1 \| \rho_2) := \Tr \{ \rho_1 (\ln \rho_1 - \ln \rho_2) \},$$

(35)

$$\lim_{\alpha \to \infty} D_\alpha(\rho_1 \| \rho_2) := D_{\max}(\rho_1 \| \rho_2) = \inf \{ \lambda \in \mathbb{R} : \rho_1 \leq e^\lambda \rho_2 \},$$

(36)

which correspond to the logarithm of quantum fidelity, quantum relative entropy and max-relative entropy, respectively.

Another essential ingredient in our derivation of sufficient conditions is the generalized Pinsker’s inequality [44]:

$$D_T(P, Q) \leq \sqrt{\frac{2}{\alpha} D_\alpha(P \| Q)}$$

(37)

for $\alpha \in (0, 1]$ and $D_\alpha(P \| Q)$ is the Rényi divergence between two distributions,

$$D_\alpha(P \| Q) = \frac{1}{1-\alpha} \ln \sum_i p_i^\alpha q_i^{1-\alpha}.$$  

(38)

Notice that it is only proved for $\alpha \in (0, 1]$, which is the reason why we have to restrict our optimization over $\alpha \in \left[\frac{1}{2}, 1\right)$. In the main text, by using aforementioned properties, we derive the following sufficient conditions for efficient simulation of GBS:

$$\frac{2}{\alpha} D_\alpha^{\text{min}}(\eta, q_D) \leq \frac{\epsilon^2}{K}, \quad \alpha \in \left[\frac{1}{2}, 1\right],$$

(39)

where $D_\alpha^{\text{min}}(\eta, q_D)$ is the $\alpha$-order Rényi relative entropy minimized over all permitted $(t)$-classical Gaussian states:

$$D_\alpha^{\text{min}}(\eta, q_D) := \min_{t \in [1-2q_D, 1]} \min_{\sigma \in \mathcal{C}_t^{\text{cl}}} D_\alpha(\sigma \| \tau).$$

(40)

Since $D_\alpha$ is non-decreasing over $\alpha$, the l.h.s of Eq. (39) is expected to reach it’s minimum at some $\alpha^*$. To optimize over $\alpha$ we first try to calculate $D_\alpha^{\text{min}}(\eta, q_D)$ for fixed $\alpha$.

To facilitate our calculation we first define $Q(\sigma \| \tau)$ by

$$D_\alpha(\sigma \| \tau) := \frac{1}{\alpha - 1} \ln Q_\alpha(\sigma \| \tau),$$

(41)

$$Q_\alpha(\sigma \| \tau) = \Tr \left\{ \left( \tau^{\frac{1-\alpha}{2}} \sigma^\frac{1+\alpha}{2} \right)^\alpha \right\}.$$  

(42)
FIG. 4: Here each point is obtained by numerically minimize $\frac{2}{\alpha} D_{\alpha}(\sigma \| \tau)$ over $\tau \in C_{G}^{(1)}$ and over $t \in [1 - 2q_{D}, 1]$, assuming the landscape given in Eq. (48)-(49). The tightest bound happens at $\alpha = \frac{1}{2}$.

From Ref. [34, Theorem 21], we have the following expression for $Q_{\alpha}(\sigma \| \tau)$ between two single-mode Gaussian states with zero means:

$$Q_{\alpha}(\sigma \| \tau) = \frac{1}{Z_{\sigma}^{\alpha} Z_{\tau}^{1-\alpha}} \sqrt{\det [(V_{\xi,\alpha} + i\Omega)]/2},$$

where

$$Z_{\sigma} = \sqrt{\det [(V_{\sigma} + i\Omega)/2]},$$

$$V_{\xi,\alpha} = \frac{(I_2 + (V_{\xi}\Omega)^{-1})^a + (I_2 - (V_{\xi}\Omega)^{-1})^a}{(I_2 + (V_{\xi}\Omega)^{-1})^a - (I_2 - (V_{\xi}\Omega)^{-1})^a},$$

$$V_{\xi} = V_{\sigma} - \sqrt{I_2 + (V_{\xi}\Omega)^{-2} V_{\xi}(V_{\tau,\alpha} + V_{\sigma})^{-1} V_{\tau} \sqrt{I_2 + (\Omega V_{\sigma})^{-2}},}$$

$$\beta = \frac{1 - \alpha}{\alpha}. \quad (47)$$

From our optimization for $\alpha = 1/2$, we expect the same landscape for general $\alpha$-order sandwiched Rényi relative entropy: it is minimized at

$$\phi_{\tau} = \phi_{\sigma}, \quad (48)$$

$$s_{\tau} = \begin{cases} s_{\sigma}, & \text{for } s_{\sigma} < \frac{1}{2} \ln \frac{2n_t + 1}{t}, \\ \frac{1}{2} \ln \frac{2n_t + 1}{t} & \text{for } s_{\sigma} \geq \frac{1}{2} \ln \frac{2n_t + 1}{t}. \end{cases} \quad (49)$$

This will give us a function of $n_{\tau}$ to minimize. The expression of $D_{\alpha}$ is too complicated to analytically show that above assumption is true. However, it can be verified analytically for $\alpha = 1$, when Rényi relative entropy reduces to quantum relative entropy [60].

For fixed $\alpha$, we obtain $D_{\alpha}^{\text{min}}(\eta, q_{D})$ by using Eq. (48) and then numerically minimizing over $n_{\tau}$. As shown in the Fig. 2 in the main text, We find out that the l.h.s of Eq. (39) is minimized at $\alpha = \frac{1}{2}$. That is when $D_{\alpha}(\sigma, \tau) = - \ln F(\sigma, \tau)$. Therefore, the bound we calculate in the main text by using quantum fidelity is the tightest one we can get. To give an explicit example, in the figure above we plot $\frac{2}{\alpha} D_{\alpha}^{\text{min}}$ against $\alpha$ for $\eta = 0.1, r = 0.11, q_{D} = 0$. 

\begin{align*}
\text{FIG. 4: Here each point is obtained by numerically minimize} \quad & \frac{2}{\alpha} D_{\alpha}(\sigma \| \tau) \quad \text{over} \quad \tau \in C_{G}^{(1)} \quad \text{and over} \quad t \in [1 - 2q_{D}, 1], \\
\text{assuming the landscape} \quad & \text{given in Eq. (48)-(49). The tightest bound happens at} \quad \alpha = \frac{1}{2}. \\
\text{From Ref. [34, Theorem 21], we have the following expression for} \quad & Q_{\alpha}(\sigma \| \tau) \quad \text{between two single-mode Gaussian states with zero means:} \\
\quad & Q_{\alpha}(\sigma \| \tau) = \frac{1}{Z_{\sigma}^{\alpha} Z_{\tau}^{1-\alpha}} \sqrt{\det [(V_{\xi,\alpha} + i\Omega)]/2},
\end{align*}