Cones generated by random points on half-spheres and convex hulls of Poisson point processes

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Abstract

Let $U_1, U_2, \ldots$ be random points sampled uniformly and independently from the $d$-dimensional upper half-sphere. We show that, as $n \to \infty$, the $f$-vector of the $(d + 1)$-dimensional convex cone $C_n$ generated by $U_1, \ldots, U_n$ weakly converges to a certain limiting random vector, without any normalization. We also show convergence of all moments of the $f$-vector of $C_n$ and identify the limiting constants for the expectations. We prove that the expected Grassmann angles of $C_n$ can be expressed through the expected $f$-vector. This yields convergence of expected Grassmann angles and conic intrinsic volumes and answers thereby a question of Bárány, Hug, Reitzner and Schneider [Random points in halfspheres, Rand. Struct. Alg., 2017]. Our approach is based on the observation that the random cone $C_n$ weakly converges, after a suitable rescaling, to a random cone whose intersection with the tangent hyperplane of the half-sphere at its north pole is the convex hull of the Poisson point process with power-law intensity function proportional to $\|x\|^{-(d+\gamma)}$, where $\gamma = 1$. We compute the expected number of facets, the expected intrinsic volumes and the expected $T$-functional of this random convex hull for arbitrary $\gamma > 0$.

Keywords. Blaschke–Petkantschin formula, conic intrinsic volume, convex cone, convex hull, $f$-vector, random polytope, Poisson point process, spherical integral geometry.

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The study of random convex hulls has a long tradition in convex and stochastic geometry; see Chapter 8 in [32] as well as [16, 21, 29] for overviews. Motivated by Sylvester’s four-point problem, the modern development started with the works of Rényi and Sulanke [25, 26] on random polygons in the plane that are generated as convex hulls of random points uniformly distributed in a fixed convex set. Random polytopes of this type in general space dimensions \( d \geq 1 \) were studied intensively, for example by Bárány, Reitzner, Schütt [4, 24, 33], to name just a few. One of the functionals that attracted particular interest was the so-called \( f \)-vector, that is, the vector whose \( k \)-th entry is the number of \( k \)-dimensional faces, \( k \in \{0, 1, \ldots, d-1\} \). In particular, if a random polytope \( K_n \) is generated by \( n \geq d+1 \) independent random points that are uniformly distributed in a convex body \( K \subset \mathbb{R}^d \) with (sufficiently) smooth boundary, the expected number \( \mathbb{E} f_k(K_n) \) of \( k \)-dimensional faces of \( K_n \) asymptotically behaves like

\[
\mathbb{E} f_k(K_n) \sim c_{k,d} \Omega(K)n^{\frac{d-k}{d+1}},
\]

as \( n \to \infty \). Here, \( c_{k,d} \in (0, \infty) \) is a constant that only depends on \( k \) and \( d \), and \( \Omega(K) \) is the affine surface area of \( K \); see [24]. On the other hand, if \( K \) itself is a polytope, the expected number of \( k \)-dimensional faces of \( K_n \) grows like

\[
\mathbb{E} f_k(K_n) \sim c'_{k,d} T(K)(\log n)^{d-1},
\]

as \( n \to \infty \), with a different constant \( c'_{k,d} \in (0, \infty) \) and with \( T(K) \) being the number of towers of the polytope \( K \); see again [24].

Recently, Bárány, Hug, Reitzner and Schneider [5] investigated the \( f \)-vector, the spherical volume and some other quantities for the spherical convex hull of \( n \) uniformly distributed random points on the \( d \)-dimensional upper half-sphere. Among other results, they showed that the expected number of facets (i.e. \( (d-1) \)-dimensional faces) and the expected number of vertices and edges of such spherical random polytopes tend to finite constants, as \( n \to \infty \). This surprising result is the starting point for our work in which we consider the \( (d+1) \)-dimensional random convex cone generated by such random convex hulls; see Figure 2.1. Our first main result (Theorem 2.1) is a weak limit theorem for the sections of these random cones with the tangent hyperplane of the half-sphere at its north pole. We shall identify the limiting random polytope as the convex hull of a Poisson point process in the tangent hyperplane with a power-law intensity function. This in turn leads to limit theorems for the whole \( f \)-vector (Theorem 2.3 and Theorem 2.4) and the volume (Theorem 2.6) of the spherical convex hull on a half-sphere, which complements the findings in [5].

In addition, our weak limit theorem allows us to describe the expectation asymptotics of the conic intrinsic volumes (in fact, all three versions of them) of the induced random cone. This solves in an extended form a conjecture posed by Bárány, Hug, Reitzner and Schneider; see Section 9 in [5].
We also study separately the expected so-called $T$-functional of the convex hull of a general class of Poisson point processes in $\mathbb{R}^d$ with a power-law intensity function $\|x\|^{-(d+\gamma)}$; see Theorem 2.12. Here, $\gamma > 0$ is a parameter and $\|x\|$ is the Euclidean norm of $x$. In particular, we compute explicitly the expected volume (and, more generally, expected intrinsic volumes) and the expected number of facets of this random polytope, thus generalizing a two-dimensional result of Davis et al. [11].

The paper is structured as follows. In Section 2.1 we first rephrase the relevant results from [5] and introduce the random convex cones for which various limit theorems are presented in Sections 2.2 and 2.3. Convex hulls of Poisson point processes with a power-law intensity function are the content of Section 2.4. In order to keep the paper reasonably self-contained we have collected some background material needed in our arguments in Section 3. The proofs of our main results are contained in Sections 4, 5 and 6, while Section 7 collects some auxiliary lemmas.

2 Main results

2.1 Convex hulls on the half-sphere

We fix a dimension $d \geq 1$ and let $U_1, U_2, \ldots$ be independent random points distributed according to the uniform distribution on the $d$-dimensional upper half-sphere

$$S^d_+ := \{(x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1} : x_0^2 + x_1^2 + \ldots + x_d^2 = 1, x_0 \geq 0\}.$$ 

We are interested in the random convex cone in $\mathbb{R}^{d+1}$ defined as the positive hull of $U_1, \ldots, U_n$, $n \geq d+1$, that is

$$C_n = \text{pos}\{U_1, \ldots, U_n\} := \{\alpha_1 U_1 + \ldots + \alpha_n U_n : \alpha_1, \ldots, \alpha_n \geq 0\};$$

see Figure 2.1. As already discussed in the previous section, the random cone, or, more precisely, the random spherical polytope $C_n \cap S^d_+$, has been studied by Bárány et al. [5]. Some of their results concern the expected $f$-vector of $C_n$, that is, the expected number $E f_k(C_n)$ of $k$-dimensional faces of $C_n$, $k \in \{1, \ldots, d\}$. The $f$-vector of the cone $C_n$ is related to the $f$-vector of the spherical polytope $C_n \cap S^d_+$ by $f_k(C_n) = f_{k-1}(C_n \cap S^d_+)$. For our purposes, it is more convenient to work with cones rather than with spherical polytopes. By [5, Theorem 3.1] the expected number of facets $E f_d(C_n)$ of $C_n$ is explicitly given by

$$E f_d(C_n) = \frac{2 \omega_d}{\omega_{d+1}} \binom{n}{d} \int_0^\pi \left(1 - \frac{\alpha}{\pi}\right)^{n-d} \sin^{d-1} \alpha \, d\alpha. \quad (2.1)$$

Moreover, it has been shown in [5, Theorem 3.1] that

$$\lim_{n \to \infty} E f_d(C_n) = 2^{-d} d! \kappa_d^2. \quad (2.2)$$

where $\kappa_d$ denotes the volume of the $d$-dimensional unit ball, whereas $\omega_d$ is the $(d-1)$-dimensional Hausdorff measure (surface area) of the unit sphere $S^{d-1} \subset \mathbb{R}^d$, that is,

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \quad \text{and} \quad \omega_d = d \kappa_d = \frac{2 \pi^{d/2}}{\Gamma(\frac{d}{2})}. \quad (2.3)$$

Regarding the expected number of one-dimensional faces of $C_n$ (or, equivalently, vertices of $C_n \cap S^d_+$), [5, Theorem 7.1] says that

$$\lim_{n \to \infty} E f_1(C_n) = C(d) \pi^{d+1} \left(\frac{2}{\omega_{d+1}}\right)^{d+1} \omega_d$$

for a certain constant $C(d)$ given in form of a multiple integral; see [5, Equation (22)]. Let us also mention that cones generated by random points with uniform distribution on the whole sphere $S^d$ were studied by [10] and [17].
2.2 Weak convergence of the random cone and its consequences

2.2.1 The weak convergence theorem

In what follows, we shall present a weak limit theorem for the random cone $C_n$. It is clear that, for large $n$, the cone $C_n$ is close to the half-space $\{x_0 > 0\}$, so that in order to obtain a non-trivial limit for $C_n$ we need an appropriate rescaling. This is achieved by the linear operator $T_n : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ defined by

$$T_n(x_0, x_1, \ldots, x_d) := (nx_0, x_1, \ldots, x_d).$$

Let $H_1$ be the hyperplane $\{x_0 = 1\} \subset \mathbb{R}^{d+1}$. Note that $H_1$ is tangent to the half-sphere $\mathbb{S}^d_+$ at its north pole. Let $e_0$ be the unit vector $(1, 0, \ldots, 0) \in \mathbb{R}^{d+1}$ pointing to the north pole. We shall prove that the random convex polytope $(T_n C_n \cap H_1) - e_0$, which can be viewed as the “horizontal” section of the cone $T_n C_n$, converges in distribution on the space of compact convex subsets of $H_1 - e_0$ that we identify with $\mathbb{R}^d$; see Section 3 below for some background material on this notion of convergence.

To describe the limit, take some $\gamma > 0$, $c > 0$, and let $\Pi_{d,\gamma}^\circ(c)$ be a Poisson point process on $\mathbb{R}^d \setminus \{0\}$ whose intensity measure is absolutely continuous with respect to the Lebesgue measure and whose density function is given by

$$x \mapsto \frac{1}{\omega_{d+\gamma} \|x\|^{d+\gamma}}, \quad x \in \mathbb{R}^d \setminus \{0\},$$

(2.4)

where $\|x\|$ is the Euclidean norm of $x$; see Figure 2.2. Again, we refer to Section 3 for background material concerning Poisson point processes. Note that the number of points of $\Pi_{d,\gamma}(c)$ outside any ball centered at the origin having strictly positive radius is almost surely finite (because the intensity is integrable near $\infty$), while the number of points inside any such ball is infinite with probability one (because the integral of the intensity over such balls diverges). We denote by conv $\Pi_{d,\gamma}(c)$ the convex hull of all points of $\Pi_{d,\gamma}(c)$. Even though $\Pi_{d,\gamma}(c)$ almost surely consists of infinitely many points, the random convex set conv $\Pi_{d,\gamma}(c)$ turns out to be almost surely a polytope; see Corollary 4.2 below. The next theorem identifies the weak limit of the rescaled random polytopes $(T_n C_n \cap H_1) - e_0$ in terms of a Poisson point process of the type just discussed.

**Theorem 2.1.** As $n \to \infty$, the random polytopes $(T_n C_n \cap H_1) - e_0$ converge in distribution to conv $\Pi_{d,\gamma}(2)$ on the space of compact convex subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric.

Let us briefly explain the idea behind Theorem 2.1. Define the map $P : \mathbb{S}^d_+ \cap \{x_0 > 0\} \to \mathbb{R}^d$ by
the equality
\[ \mathcal{P}(x_0, x_1, \ldots, x_d) = \left( \frac{x_1}{x_0}, \ldots, \frac{x_d}{x_0} \right). \] (2.5)

The rays in directions \( U_1, \ldots, U_n \) intersect \( H_1 \) at the points \((1, \mathcal{P}(U_1)), \ldots, (1, \mathcal{P}(U_n))\). Therefore, the polytope \( C_n \cap H_1 - e_0 \) is the convex hull of \( \mathcal{P}(U_1), \ldots, \mathcal{P}(U_n) \). The next proposition describes the density according to which these points are distributed. The result is a consequence of \([6, \text{Proposition 4.2}]\) and, in a more general set-up, has been proved in the argument of \([7, \text{Theorem 7}]\).

**Proposition 2.2.** Let \((\xi_0, \ldots, \xi_d)\) be a random vector distributed uniformly on the half-sphere \( S^d_+ \). Then, the vector \( \mathcal{P}(\xi_0, \xi_1, \ldots, \xi_d) := (\xi_1/\xi_0, \ldots, \xi_d/\xi_0) \) has the following generalized Cauchy density

\[ x \mapsto \frac{2}{\omega_{d+1} \left( 1 + \|x\|^2 \right)^{\frac{d+1}{2}}}, \quad x \in \mathbb{R}^d. \]

Note that this density belongs to the class of beta'-distributions. Convex hulls of samples from these distributions were studied in \([18]\). In particular, the formula for the number of facets of this convex hull obtained in \([18, \text{Proposition 3.16}]\) contains (2.1) as a special case. Let us turn to the large \( n \) asymptotics. Since the above density is regularly varying at \( \infty \), see Lemma \([7, \text{Lemma 7.7}]\) in Section \([7]\), standard methods from extreme-value theory imply that the point process formed by the points \( \mathcal{P}(U_1)/n, \ldots, \mathcal{P}(U_n)/n \) converges weakly to the Poisson point process \( \Pi_{d,1}(2) \) in the space of locally-finite integer measures on \( \mathbb{R}^d \setminus \{0\} \) endowed with the vague topology. Using the continuous mapping theorem, we shall argue that the convex hull of \( \mathcal{P}(U_1)/n, \ldots, \mathcal{P}(U_n)/n \) converges weakly to the convex hull of the Poisson point process, thus proving Theorem 2.1.

### 2.2.2 Convergence of the \( f \)-vector

With the help of the continuous mapping theorem we shall now derive a number of consequences of Theorem 2.1. For a Euclidean or spherical \( d \)-dimensional polytope \( P \), we denote by \( f_k(P) \) the number of \( k \)-dimensional faces of \( P \), where \( k \in \{0, \ldots, d-1\} \). The collection \( \mathbf{f}(P) := (f_0(P), \ldots, f_{d-1}(P)) \) is the \( f \)-vector of \( P \). From Theorem 2.1 we shall derive the following result on the distributional convergence of the \( f \)-vector of the random spherical polytope \( C_n \cap S^d_+ \). We remind the reader that \( f_k(C_n \cap S^d_+) = f_{k+1}(C_n) \).
Theorem 2.3. As \( n \to \infty \), we have that
\[
\mathbf{f}(C_n \cap S^d_+) \xrightarrow{d} \mathbf{f}(\text{conv } \Pi_{d,1}(2)),
\]
where \( \xrightarrow{d} \) denotes convergence in distribution.

We shall argue also that the expected \( f \)-vector of the spherical random polytope \( C_n \cap S^d_+ \) converges to that of \( \text{conv } \Pi_{d,1}(2) \). Even more generally, we shall prove the convergence of moments of all orders. This generalizes the results from [5] discussed above and answers – in an extended form – a question raised in [5, Section 9]. Let us write \( \text{aff}\{x_1, \ldots, x_k\} \) for the affine hull of the points \( x_1, \ldots, x_k \).

Theorem 2.4. For every \( k \in \{1, \ldots, d\} \) and every \( m \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} E f^m_k(C_n) = \lim_{n \to \infty} E f^m_{k-1}(C_n \cap S^d_+) = E f^m_{k-1}(\text{conv } \Pi_{d,1}(2)).
\]

For \( m = 1 \) the limits of the expectations are
\[
\lim_{n \to \infty} E f_k(C_n) = \lim_{n \to \infty} E f_{k-1}(C_n \cap S^d_+) = E f_{k-1}(\text{conv } \Pi_{d,1}(2)) = \frac{2}{k!} B_{k,d},
\]
where \( B_{1,d}, \ldots, B_{d,d} \) are constants given by
\[
B_{k,d} = \frac{1}{2} \left( \frac{2}{\omega_{d+1}} \right)^k \int_{(\mathbb{R}^d)^k} \mathbb{P}(\Pi_{d,1}(2) \cap \text{aff}\{x_1, \ldots, x_k\} = \emptyset) \prod_{i=1}^k \frac{dx_i}{\|x_i\|^{d+1}} < \infty.
\]

Remark 2.5. We shall prove in Section 6.2 that
\[
B_{d,d} = (2\pi)^{d-1} \Gamma \left( \frac{d+1}{2} \right)^2.
\]

Together with Theorem 2.4 and Legendre’s duplication formula, this recovers Equation (4) of Bárany et al. [5] who proved that \( \lim_{n \to \infty} E f_d(C_n) = 2^{-d} d! \kappa_d^2 \). In Proposition 2.10 we shall compute the value of \( B_{2,d} \), yielding the formula
\[
\lim_{n \to \infty} E f_2(C_n) = B_{2,d} = \frac{1}{2} \left( \frac{d+1}{3} \right) \pi^2.
\]

2.2.3 Convergence of the solid angle

The next theorem deals with the solid angle of \( C_n \). Let \( \hat{\sigma} \) be the \( d \)-dimensional spherical Lebesgue measure on the unit sphere \( S^d \subset \mathbb{R}^{d+1} \) normalized such that \( \hat{\sigma}(S^d) = 1 \). The solid angle \( \alpha(C_n) \) of the convex cone \( C_n \) is defined by
\[
\alpha(C_n) := \hat{\sigma}(C_n \cap S^d).
\]

Clearly, we have that \( \alpha(C_n) \) almost surely converges to 1/2, as \( n \to \infty \). Theorem 7.1 in [5] provides a more delicate asymptotic result, namely
\[
\mathbb{E} \left( \frac{1}{2} - \alpha(C_n) \right) = C(d) \pi^{d+1} \left( \frac{2}{\omega_{d+1}} \right)^{d+1} \frac{\omega_d}{\omega_{d+1}} \frac{1}{n} + O(n^{-2}),
\]

as \( n \to \infty \), where \( C(d) \) is the same constant as in (2.3). The next theorem is a distributional counterpart to this formula.

Theorem 2.6. As \( n \to \infty \), we have that
\[
n \left( \frac{1}{2} - \alpha(C_n) \right) \xrightarrow{d} \frac{1}{\omega_{d+1}} \int_{\mathbb{R}^d \setminus \text{conv } \Pi_{d,1}(2)} \frac{dv}{\|v\|^{d+1}}.
\]
2.3 Conic intrinsic volumes

Next we consider the so-called conic intrinsic volumes of \( C_n \) or, equivalently, the spherical intrinsic volumes of \( C_n \cap S^n_+ \). In contrast to the classical intrinsic volumes in \( \mathbb{R}^d \) there exist several notions of conic intrinsic volumes in the literature; see [12] and, for equivalent formulations in the spherical setting, [13] and [32]. If \( C \subset \mathbb{R}^{d+1} \) is a polyhedral convex cone and \( x \in \mathbb{R}^{d+1} \) we let \( \Pi_C(x) \) be the metric projection of \( x \) onto \( C \), that is \( \Pi_C(x) \) is the uniquely determined point \( y \in C \) for which the squared Euclidean distance \( \|x - y\|^2 \) is minimal. If \( g \) is a standard Gaussian random vector in \( \mathbb{R}^{d+1} \) and \( F \subseteq C \) is a face of \( C \) with relative interior denoted by \( \text{relint}(F) \), we put \( v_F := \mathbb{P}(\Pi_C(g) \in \text{relint}(F)) \) and

\[
v_k(C) := \sum_{F \in \mathcal{F}_k(C)} v_F, \quad k \in \{0, 1, \ldots, d + 1\},
\]

where \( \mathcal{F}_k(C) \) is the set of all \( k \)-dimensional faces of \( C \). For convenience also define \( v_k(C) := 0 \) for \( k > d + 1 \). This is the \( k \)th conic intrinsic volume of \( C \). We notice that the conic intrinsic volumes of the upper halfspace \( H_{\text{up}} := \{x = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1} : x_0 \geq 0\} \) are given by \( v_k(H_{\text{up}}) = 0 \) if \( k \in \{0, 1, \ldots, d - 1\} \) and \( v_0(H_{\text{up}}) = v_{d+1}(H_{\text{up}}) = 1/2 \). If \( C \) is a \( k \)-dimensional linear subspace, then \( v_k(C) = 1 \), while all other conic intrinsic volumes vanish. Henceforth, we shall always exclude the case when \( C \) is linear subspace (since formulas (2.9) and (2.10) below are not valid in this case).

The solid angle \( h_k(C) := \frac{1}{2} \mathbb{P}(C \cap L \neq \{0\}) \), \( k \in \{0, 1, \ldots, d\} \) is equal to the \((k + 1)\)st Grassmann angle of \( C \) that has been introduced by Grünbaum [14]. In particular, the \((d + 1)\)st Grassmann angle \( h_{d+1}(C) \) coincides with the solid angle \( \alpha(C) \) studied above. Note also that all Grassmann angles \( h_1, \ldots, h_{d+1} \) of the upper halfspace \( H_{\text{up}} \) are equal to 1/2. The conic Crofton formula [11, Equation (2.10)] states that the conic intrinsic volumes and the Grassmann angles are related by

\[
h_{k+1}(C) = \sum_{i \geq 1 \atop i \text{ odd}} v_{k+i}(C). \tag{2.11}
\]

In the terminology of [12], the above sums (which are in fact finite) are called the half-tail functionals. For every cone \( C \) we have \( h_1(C) = 1/2 \) and we put \( h_0(C) = 1/2, h_{d+2}(C) = h_{d+3}(C) = \ldots = 0 \) in order to be consistent with (2.9).

Finally, we may consider the conic mean projection volumes defined for \( k \in \{0, 1, \ldots, d\} \) by

\[
w_k(C) := \frac{1}{k_{k+1}} \int_{G(d+1,k+1)} \text{Vol}_{k+1}(P_L(C) \cap \mathbb{B}^{d+1}) v_k(dL),
\]

where \( \text{Vol}_{k+1} \) stands for the Lebesgue measure in \( L \in G(d+1, k+1) \), \( P_L \) for the orthogonal projection onto \( L \) and \( \mathbb{B}^{d+1} \) for the \((d + 1)\)-dimensional unit ball. The conic mean projection volumes are related to the conic intrinsic volumes via what may be called the conic Kubota formula

\[
w_{k+1}(C) = \sum_{i=k+1}^{d+1} v_i(C) = h_{k+1}(C) + h_{k+2}(C). \tag{2.12}
\]
see Lemma 5.1. Thus, the conic mean projection volumes coincide with the tail functionals in the language of [2]. For the half-space \( H_{\text{up}} \) we have \( w_1(H_{\text{up}}) = \ldots = w_2(H_{\text{up}}) = 1 \) and \( w_{d+1}(H_{\text{up}}) = 1/2 \).

The next result relates the expected Grassmann angles of the random cone \( C_n \) to its expected \( f \)-vector.

**Theorem 2.7.** For all \( k \in \{1, \ldots, d\} \) we have

\[
2 \left( \frac{n + d + 1 - k}{d + 1 - k} \right) \left( \frac{1}{2} - \mathbb{E} h_{k+1}(C_n) \right) = \mathbb{E} f_{d-k+1}(C_{n+d+1-k}).
\]

The above formula should be compared to the well-known Efron identity [12] that states that for random points \( Q_1, Q_2, \ldots \) sampled uniformly and independently from a convex body \( K \subset \mathbb{R}^d \) and all \( n \geq d + 1 \) we have

\[
\frac{\mathbb{E} \text{Vol}_d \text{conv}\{Q_1, \ldots, Q_n\}}{\text{Vol}_d(K)} = 1 - \frac{\mathbb{E} f_0(\text{conv}\{Q_1, \ldots, Q_{n+1}\})}{n + 1}.
\]

Buchta [8] obtained an analogue of this identity for higher moments of the volume, but no identity relating the expected \( f \)-vector of random polytopes to their intrinsic volumes is known in the Euclidean case, to the best of our knowledge (however, we refer to [15, 30] for results in this direction for the zero cells of Poisson hyperplane tessellations).

Our next result identifies asymptotically the expected conic intrinsic volumes, the Grassmann angles and the conic mean projection volumes of the random cones \( C_n \). Note that this completely settles in an extended form the conjecture of Bárány et al. stated in [5, Section 9].

**Theorem 2.8.** For every \( k \in \{0, \ldots, d\} \) we have

\[
\lim_{n \to \infty} n^{d+1-k} \left( \frac{1}{2} - \mathbb{E} h_{k+1}(C_n) \right) = B_{d+1-k,d}, \tag{2.13}
\]

where \( B_{1,d}, \ldots, B_{d,d} \) are given by [2.6], and \( B_{d+1,d} = 0 \). Moreover, for all \( \ell, r \in \{0, \ldots, d-1\} \) we have

\[
\lim_{n \to \infty} n^{d-1-\ell} \mathbb{E} v_\ell(C_n) = B_{d-1-\ell,d}, \tag{2.14}
\]

\[
\lim_{n \to \infty} n^{d+1-r} (1 - \mathbb{E} w_{r+1}(C_n)) = B_{d+1-r,d}. \tag{2.15}
\]

**Remark 2.9.** Note that \( v_d(C_n) = h_d(C_n) \to 1/2 \) and \( v_{d+1}(C_n) = h_{d+1}(C_n) \to 1/2 \), as \( n \to \infty \), hence we have restricted ourselves to the conic intrinsic volumes \( v_\ell(C_n) \) of orders \( \ell \in \{0, \ldots, d-1\} \) in (2.14). Similarly, \( w_{d+1}(C_n) = h_{d+1}(C_n) \), hence we omitted the case \( r = d \) in (2.15).

**Proposition 2.10.** For all \( d \geq 2 \) we have

\[
B_{2,d} = \frac{1}{2} \binom{d+1}{3} \pi^2.
\]

**Proof.** For the expected surface area (i.e. \((d-1)\)-dimensional Hausdorff measure) of the spherical polytope \( C_n \cap S^d \), Bárány et al. [3] showed in their Theorem 5.1 that

\[
\mathbb{E} S(C_n \cap S^d) = \omega_d \left( 1 - \binom{d+1}{3} \pi^2 n^{-2} + O(n^{-3}) \right),
\]

where \( S(K) \) denotes the surface area of the spherical polytope \( K \). On the other hand, the relation \( 2\omega_d h_d(C_n) = 2\omega_d v_d(C_n) = S(C_n \cap S^d) \) and Theorem 2.8 with \( k = d-1 \) yield

\[
\mathbb{E} S(C_n \cap S^d) = \omega_d \left( 1 - 2B_{2,d} n^{-2} + o(n^{-2}) \right).
\]

Comparing both asymptotic relations, we obtain the required formula for \( B_{2,d} \).
Let us consider the special case $d = 2$, where $B_{2,2} = \frac{1}{2}\pi^2$ and hence

$$\lim_{n \to \infty} E f_0(C_n \cap S^2_+) = \lim_{n \to \infty} E f_1(C_n \cap S^2_+) = E f_0(\text{conv } \Pi_{2,1}(c)) = E f_1(\text{conv } \Pi_{2,1}(c)) = \frac{1}{2}\pi^2,$$

with $c > 0$ being arbitrary. For $d = 3$, the identities $B_{3,3} = 4\pi^2$ and $B_{2,3} = 2\pi^2$ (following from (2.7) and Proposition 2.10) combined with the Euler relation $f_0 - f_1 + f_2 = 2$ yield

$$\lim_{n \to \infty} (\mathbb{E} f_0(C_n \cap S^3_+), \mathbb{E} f_1(C_n \cap S^3_+), \mathbb{E} f_2(C_n \cap S^3_+)) = (\mathbb{E} f_0(\text{conv } \Pi_{3,1}(c)), \mathbb{E} f_1(\text{conv } \Pi_{3,1}(c)), \mathbb{E} f_2(\text{conv } \Pi_{3,1}(c))) = \left(2 + \frac{2}{3}\pi^2, 2\pi^2, \frac{4}{3}\pi^2\right).$$

**Remark 2.11.** Using the same methods as in the proof of Theorems 2.8 and 2.6 it is possible to prove the following distributional convergence for all $k \in \{0, 1, \ldots, d\}$:

$$n^{d+1-k} \left(\frac{1}{2} - h_{d+1}(C_n)\right) \sim \frac{1}{2} \left(\frac{2}{\omega_{d+1}}\right)^{d+1-k} \int_{(\mathbb{R}^d)^{d+1-k}} 1_{\{\Pi_{d+1}(2) \cap \text{aff} \{x_1, \ldots, x_{d+1-k}\} = \emptyset\}} \prod_{i=1}^{d+1-k} \frac{dx_i}{\|x_i\|^{d+1}},$$

as $n \to \infty$. Observe that since $h_{d+1}(C_n)$ coincides with the solid angle $\alpha(C_n)$, we recover Theorem 2.6 as a special case of this relation with $k = d$.

### 2.4 Convex hull of Poisson point process with power-law intensity

We are now going to state explicit formulae for the expected values of some functionals of the random polytopes $\text{conv } \Pi_{d,\gamma}(c)$ introduced in Section 2.2.1.

#### 2.4.1 Expectation of the $T$-functional

The results are most conveniently expressed in terms of the $T$-functional introduced by Wieacker [34]. For a polytope $P \subset \mathbb{R}^d$, $k \in \{0, 1, \ldots, d-1\}$ and for $a, b \geq 0$ it is defined by

$$T^{d,k}_{a,b}(P) = \sum_{F \in F_k(P)} \text{dist}^a(F) \text{Vol}^b(F),$$

where $\text{dist}(F)$ is the distance from the origin to the affine hull of the $k$-face $F$. The next theorem provides an explicit formula for the expected $T$-functional with $k = d-1$ of the random polytopes $\text{conv } \Pi_{d,\gamma}(c)$.

**Theorem 2.12.** For every $\gamma > 0$, $c > 0$ and all $a, b \geq 0$ such that $(\gamma - b)d + b - a > 0$ and $\gamma - b > 0$, we have that

$$\mathbb{E} T^{d,d-1}_{a,b}(\text{conv } \Pi_{d,\gamma}(c)) = \frac{c^d \omega_d}{\gamma d! \omega_{d+1}} \left(\frac{c}{\gamma \omega_{d+1}}\right)^{\frac{a-b+(b-\gamma)d}{\gamma}} \Gamma \left(\frac{(\gamma - b)d + b - a}{\gamma}\right) \times \frac{1}{((d-1)!)^b} \prod_{i=1}^{d-1} \frac{\Gamma \left(\frac{\gamma - b}{2} \cdot \frac{i+b+1}{2}\right)}{\Gamma \left(\frac{\gamma - b}{2} \cdot \frac{i+b}{2}\right)}.$$

If $(\gamma - b)d + b - a \leq 0$ or $\gamma - b \leq 0$, then the expectation equals $+\infty$.

Inserting special values for the parameters $a$ and $b$ leads to some interesting consequences.
2.4.2 Expected number of faces

Taking \( a = b = 0 \), and observing that almost surely

\[
T_{0,0}^{d,d-1}(\text{conv } \Pi_{d,\gamma}(c)) = f_{d-1}(\text{conv } \Pi_{d,\gamma}(c)),
\]

we obtain after simplification the following result for the mean number of facets of \( \text{conv } \Pi_{d,\gamma}(c) \).

**Corollary 2.13.** For every \( \gamma > 0 \) and \( c > 0 \), we have that

\[
\mathbb{E} f_{d-1}(\text{conv } \Pi_{d,\gamma}(c)) = \frac{2}{d} \gamma^{d-1} \pi \frac{d+1}{d} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \left(\frac{\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)}\right)^d,
\]

independently of the parameter \( c > 0 \).

**Remark 2.14.** All faces of the polytope \( \text{conv } \Pi_{d,\gamma}(c) \) are simplices with probability 1. The Dehn–Sommerville relation

\[
df_{d-1}(\text{conv } \Pi_{d,\gamma}(c)) = 2f_{d-2}(\text{conv } \Pi_{d,\gamma}(c))
\]

allows to compute the expected number of \((d-2)\)-faces of \( \text{conv } \Pi_{d,\gamma}(c) \), but computing the expected number of \( k \)-faces for general \( k \) remains an open problem.

In particular, for \( \gamma = 1 \) we obtain

\[
\mathbb{E} f_{d-1}(\text{conv } \Pi_{1,\gamma}(c)) = \frac{2\pi^{d-\frac{1}{2}} \Gamma\left(\frac{d+1}{2}\right)}{d \Gamma\left(\frac{d}{2}\right)} = \pi^{d-\frac{1}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right)}
\]

for all \( c > 0 \). Using Legendre’s duplication formula for the gamma function this can be rewritten as follows:

\[
\pi^{d-\frac{1}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right)} = \pi^{d-\frac{1}{2}} \frac{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(1 + \frac{d}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right) \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right)}} = \frac{d\pi^{d-\frac{1}{2}} \Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{d}{2}\right)}{\Gamma\left(1 + \frac{d}{2}\right)^2} = \frac{d\pi^{d-\frac{1}{2}} \Gamma\left(d\right) \sqrt{2\pi} 2^{-d+\frac{1}{2}}}{\Gamma\left(1 + \frac{d}{2}\right)^2} = 2^{-d} d^{\frac{1}{2}} k_{\frac{d}{2}}^2.
\]

This coincides with the limit in (2.2) and is consistent with Theorem 2.3. More generally, for any \( a \in [0, d) \) we have the explicit formula

\[
\mathbb{E} T_{a,0}^{d,d-1}(\text{conv } \Pi_{a,\gamma}(c)) = 2^{1-2a} c^a \left(\frac{\pi}{2}\right)^{d-a} \frac{\Gamma\left(d - a\right)}{\Gamma\left(1 + \frac{d}{2}\right) \Gamma\left(\frac{d}{2}\right)}.
\]

Another special case in which the formula from Corollary 2.13 simplifies is \( \gamma = 2 \). After simple transformations we obtain

\[
\mathbb{E} f_{d-1}(\text{conv } \Pi_{2,\gamma}(c)) = \left(\frac{2d}{d}\right).
\]

In dimension \( d = 2 \) this means that the expected number of edges (or vertices) of the convex hull of the Poisson point process with intensity \( \|x\|^{-4} \) in \( \mathbb{R}^2 \) is 6, a fact due to Rogers [28]. For \( d = 3 \) we obtain that the expected number of faces of the convex hull of the Poisson point process with intensity \( \|x\|^{-5} \) is 20. Since the faces are simplices a.s., the relation \( 3f_2 = 2f_1 \) holds, which together with the Euler relation \( f_0 - f_1 + f_2 = 2 \) yields that the expected number of edges (respectively, vertices) is 30 (respectively, 12). To summarize, the expected \( f \)-vector of \( \text{conv } \Pi_{3,2} \) is the same as the \( f \)-vector of the regular icosahedron.

Finally, observe that in the case \( d = 2 \) and for arbitrary \( \gamma > 0 \), Corollary 2.13 can be written as

\[
\mathbb{E} f_1(\text{conv } \Pi_{2,\gamma}(c)) = \mathbb{E} f_0(\text{conv } \Pi_{2,\gamma}(c)) = 4\pi \frac{B\left(\frac{1}{2}, \gamma + \frac{1}{2}\right)}{B^2\left(\frac{1}{2}, \gamma + \frac{1}{2}\right)}.
\]

where \( B \) denotes the Beta function. This formula is due to Davis et al. [11] Theorem 4.4]; see also Carnal [9] where a similar formula is derived for convex hulls of i.i.d. samples with spherically symmetric regularly varying distributions.
2.4.3 Expected volume
Let us compute the expected volume of $\text{conv} \Pi_{d,\gamma}(c)$. Since the origin is a.s. in the interior of $\text{conv} \Pi_{d,\gamma}(c)$, we have that

$$\text{Vol}_d(\text{conv} \Pi_{d,\gamma}(c)) = \frac{1}{d} T_{1,1}^{d,d-1}(\text{conv} \Pi_{d,\gamma}(c)),$$

which together with Theorem 2.12 leads to the following result for the mean volume of the convex hull of $\Pi_{d,\gamma}(c)$.

**Corollary 2.15.** For every $\gamma > 1$ and $c > 0$ we have that

$$\mathbb{E} \text{Vol}_d(\text{conv} \Pi_{d,\gamma}(c)) = \frac{c^d}{d!2^{d(1+\frac{c}{2})}} \left( \frac{\gamma}{\Gamma\left(\frac{\gamma+1}{2}\right)} \right)^{\frac{d(\gamma-1)}{\gamma}} \frac{\Gamma(1 + d - \frac{d}{\gamma})\Gamma(\frac{\gamma-1}{2})^d}{\Gamma(1 + \frac{d}{2})}.$$  

For $0 < \gamma \leq 1$ we have $\mathbb{E} \text{Vol}_d(\text{conv} \Pi_{d,\gamma}(c)) = +\infty.$

In the special case $\gamma = 2$ the formula becomes especially simple:

$$\mathbb{E} \text{Vol}_d(\text{conv} \Pi_{d,2}(c)) = \frac{1}{d!} \left( \frac{c}{2} \right)^{d/2}.$$  

2.4.4 Expected intrinsic volumes
We compute the expected values of the intrinsic volumes $V_k(\text{conv} \Pi_{d,\gamma}(c))$, $k \in \{0, \ldots, d\}$, of the random polytopes $\text{conv} \Pi_{d,\gamma}(c)$. We recall from [31] or [32, Eqn. (6.11) on page 222] that the intrinsic volume of degree $k \in \{0, 1, \ldots, d\}$ of a compact convex set $K \subset \mathbb{R}^d$ is given by

$$V_k(K) := \left( \frac{d!}{k!} \right) \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \int_{G(d,k)} \text{Vol}_k(P_L K) \nu_k(dL),$$

where, as above, $G(d,k)$ is the Grassmannian of all $k$-dimensional linear subspaces of $\mathbb{R}^d$ with the unique Haar probability measure $\nu_k$ and $P_L K$ is the orthogonal projection of $K$ onto $L$. For example $V_0(K) = 1_{\{K \neq \emptyset\}}$, $V_1(K)$ is a constant multiple of the mean width, $2V_{d-1}(K)$ is surface area and $V_d(K)$ is just the volume of $K$.

**Corollary 2.16.** For every $\gamma > 1$, $c > 0$ and $k \in \{1, \ldots, d\}$ we have that

$$\mathbb{E} V_k(\text{conv} \Pi_{d,\gamma}(c)) = \left( \frac{d!}{k!} \right) \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \frac{c^k}{2^{k(1+\frac{c}{2})}} \left( \frac{\gamma}{\Gamma\left(\frac{\gamma+1}{2}\right)} \right)^{\frac{k(\gamma-1)}{2}} \frac{\Gamma(1 + k - \frac{k}{\gamma})\Gamma(\frac{\gamma-1}{2})^k}{\Gamma(1 + \frac{k}{2})}.$$  

For $0 < \gamma \leq 1$ we have $\mathbb{E} V_k(\text{conv} \Pi_{d,1}(c)) = +\infty$ for all $k \in \{1, \ldots, d\}$.

2.4.5 Symmetric convex hulls
The symmetric convex hull $s\text{conv} \Pi$ of a point process $\Pi$ is defined as the convex hull of the points of the form $\pm x$, where $x$ is a point of $\Pi$. The next theorem evaluates the expected $T$-functional of $s\text{conv} \Pi_{d,\gamma}(c)$.

**Theorem 2.17.** For every $\gamma > 0$, $c > 0$ and all $a, b \geq 0$ such that $(\gamma - b)d + b - a > 0$ and $\gamma - b > 0$, we have that

$$\mathbb{E} T_{a,b}^{d,d-1}(s\text{conv} \Pi_{d,\gamma}(c)) = \mathbb{E} T_{a,b}^{d,d-1}(\text{conv} \Pi_{d,\gamma}(2c)).$$

It is now straightforward to state the formulae for the expected facet number, volume, and intrinsic volumes of the symmetric convex hull of $\Pi_{d,\gamma}(c)$.
3 Background material from stochastic geometry and theory of random measures

In order to keep the paper self-contained we collect in this section some background material that is used throughout the proofs.

3.1 Convergence of measures

Let $S$ be a locally compact metric space. We denote by $\mathcal{M}_S$ (respectively, $\mathcal{N}_S$) the space of locally finite (respectively, locally finite integer-valued) measures on $S$. We supply $\mathcal{M}_S$ and $\mathcal{N}_S$ with the topology of vague convergence and recall that a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{M}_S$ vaguely converges to a measure $\mu \in \mathcal{M}_S$ provided that

$$\lim_{n \to \infty} \int_S f(x) \mu_n(dx) = \int_S f(x) \mu(dx)$$

for all continuous functions $f : S \to [0, \infty)$ with compact support. We shall write $\mu_n \overset{v}{\to} \mu$ in such a case. It is known from [19, Lemma 15.7.4] that $\mathcal{N}_S$ is a vaguely closed subset of $\mathcal{M}_S$. The vague topology turns $\mathcal{M}_S$ and $\mathcal{N}_S$ into Polish spaces (see [19, Lemma 15.7.7]). A random measure (respectively, a point process) is a random variable, defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and taking values in $\mathcal{M}_S$ (respectively, $\mathcal{N}_S$). In this paper we denote by $\eta_n \overset{w}{\to} \eta$ the weak convergence of a sequence $(\eta_n)_{n\in\mathbb{N}}$ of random measures on $S$ to another random measure $\eta$, as $n \to \infty$.

3.2 Poisson point processes

Let $\mu$ be a locally finite measure on $S$ without atoms. A Poisson point process $\Pi$ on $S$ with intensity measure $\mu$ is a random variable defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ taking values in the measurable space $\mathcal{N}_S$ such that (i) $\Pi(B)$ is a Poisson random variable with mean $\mu(B)$ for each Borel set $B \subset S$ and (ii) the random variables $\Pi(B_1), \ldots, \Pi(B_n)$ are independent whenever the Borel sets $B_1, \ldots, B_n \subset S$ are pairwise disjoint. We remark that almost surely $\Pi$ can be represented as $\Pi = \sum_{i=1}^{\kappa} \delta_{x_i}$, with random points $x_1, x_2, \ldots \in S$ and a Poisson random variable $\kappa$ with mean $\mu(S)$ (which is interpreted as $+\infty$ if $\mu$ is not a finite measure). Here, $\delta_x$ stands for the unit mass at $x \in S$.

Let $k \in \mathbb{N}$ and denote by $\Pi^k$ the collection of $k$-tuples of distinct points charged by $\Pi$. It is a crucial fact that the Poisson point process $\Pi$ satisfies the multivariate Mecke equation

$$\mathbb{E} \sum_{(x_1, \ldots, x_k) \in \Pi^k} f(x_1, \ldots, x_k; \Pi) = \mathbb{E} \int_{S} \cdots \int_{S} f(x_1, \ldots, x_k; \Pi + \delta_{x_1} + \ldots + \delta_{x_k}) \mu(dx_1) \cdots \mu(dx_k)$$

(3.1)

for any non-negative measurable function $f : S^k \times \mathcal{N}_S \to \mathbb{R}$; see [32, Corollary 3.2.3]. Here, $\mathbb{E}$ denotes expectation (i.e. integration) with respect to $\mathbb{P}$.

3.3 Polytopes and cones

For a set $A \subset \mathbb{R}^d$, we denote by $\text{conv} A$ the convex hull of $A$. In particular, if $A$ is a finite set, $\text{conv} A$ is called a (convex) polytope. A face of a polytope (or a general closed convex set) $P \subset \mathbb{R}^d$ is the intersection of $P$ with one of its supporting hyperplanes (which are hyperplanes $H$ intersecting the boundary of $P$ and having the property that $P$ is entirely contained in one of the closed half-spaces bounded by $H$). If the affine hull of a face has dimension $k \in \{0, 1, \ldots, d-1\}$ we call it a $k$-face or a face of dimension $k$. By $\mathcal{F}_k(P)$ we denote the set of $k$-faces of a polytope $P$ and by $f_k(P)$ its cardinality. The $f$-vector $\mathbf{f}(P)$ of $P$ is given by $\mathbf{f}(P) := (f_0(P), \ldots, f_{d-1}(P))$. 

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A (polyhedral) cone is an intersection of finitely many closed half-spaces whose boundaries pass through the origin. If \( C \subset \mathbb{R}^d \) is a polyhedral cone, we denote by \( f_k(C) \) the number of \( k \)-dimensional faces of \( C \), for \( k \in \{0, 1, \ldots, d-1\} \).

### 3.4 Weak convergence of random compact convex sets

We denote by \( \mathcal{C}^d \) the space of compact subsets of \( \mathbb{R}^d \). The Hausdorff distance \( d_H \) on this space is defined as

\[
d_H(C, C') := \max \left\{ \min_{x \in C} \|x - y\|, \min_{x \in C'} \|x - y\| \right\}, \quad C, C' \in \mathcal{C}^d,
\]

where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^d \). We shall use the notation \( C_n \xrightarrow{d_H} C_0 \) to indicate that \( d_H(C_n, C_0) \to 0 \), as \( n \to \infty \), for a sequence \( (C_n)_{n \in \mathbb{N}_0} \subset \mathcal{C}^d \).

By \( \mathcal{K}^d \) we denote the space of compact convex subsets of \( \mathbb{R}^d \), which is a closed subspace of \( \mathcal{C}^d \) with respect to the Hausdorff distance. A random compact convex set is a random variable \( X \), defined on some probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \), which takes values in the measurable space \( \mathcal{K}^d \). It is known from [23, Theorem 7.8] that the distribution of such a random set is uniquely determined by its containment functional

\[
C_X(K) := \mathbb{P}(X \subseteq K), \quad K \in \mathcal{K}^d.
\]

Distributional convergence of a sequence \( (X_n)_{n \in \mathbb{N}} \) of random compact convex sets to another random compact convex set \( X_0 \) can be formulated in terms of the convergence of the containment functionals as follows. Namely, \( X_n \) converges in distribution to \( X_0 \) weakly on \( \mathcal{K}^d \), as \( n \to \infty \), if and only if \( \lim_{n \to \infty} C_{X_n}(K) = C_{X_0}(K) \) for all \( K \in \mathcal{K}^d \) for which \( C_{X_0}(K) = C_{X_0}(\text{int}(K)) \), where \( \text{int}(K) \) denotes the interior of \( K \); see [23, Theorem 7.12]. We shall indicate such convergence by \( X_n \xrightarrow{w} X_0 \) in this paper.

### 3.5 The affine Blaschke–Petkantschin formula

For \( k \in \{0, 1, \ldots, d\} \) we let \( G(d, k) \) and \( A(d, k) \) be the spaces of \( k \)-dimensional linear and affine subspaces of \( \mathbb{R}^d \), respectively. By \( \nu_k \) we denote the unique probability measure on \( G(d, k) \) which is invariant under the action of \( \text{SO}(d) \). The invariant measure \( \mu_k \) on \( A(d, k) \) is then given by

\[
\mu_k(\cdot) = \int_{G(d, k)} \int_{L^\perp} 1\{L + x \in \cdot\} \lambda_{L^\perp}(dx) \nu_k(dL),
\]

where \( \lambda_{L^\perp} \) denotes the Lebesgue measure on \( L^\perp \); see [32, pp. 168–169]. Similarly, we shall write \( \lambda_E \) for the Lebesgue measure on \( E \in A(d, k) \).

The affine Blaschke–Petkantschin formula is a so-called integral-geometric transformation formula and reads as follows; see [32, Theorem 7.2.7]. For any non-negative measurable function \( f : (\mathbb{R}^d)^{k+1} \to \mathbb{R} \) one has that

\[
\int_{(\mathbb{R}^d)^{k+1}} f(x_0, \ldots, x_k) \, d(x_0, \ldots, x_k) = b_{d,k}(k!)^{d-k} \int_{A(d, k)} \int_{E^{k+1}} f(x_0, \ldots, x_k) \Delta_k(x_0, \ldots, x_k)^{d-k-1} \lambda_E^{k+1}(d(x_0, \ldots, x_k)) \mu_k(dE),
\]

where the constant \( b_{d,k} \) is given by

\[
b_{d,k} = \frac{\omega_{d-k+1} \cdots \omega_d}{\omega_1 \cdots \omega_k}.
\]
4 Proofs: Weak limit theorems and convergence of moments

4.1 Continuity of functionals

Our next lemma is an essential ingredient in the proof of Theorem 2.1, Theorem 2.3 and Theorem 2.6. Let us recall that we denote by $\mathcal{N} := \mathcal{N}_{\mathbb{R}^d \cup \{\infty\} \setminus \{0\}}$ the space of locally finite integer-valued measures on $\mathbb{R}^d \cup \{\infty\} \setminus \{0\}$, where $\mathbb{R}^d \cup \{\infty\}$ is a one-point compactification of $\mathbb{R}^d$.

**Lemma 4.1.** Assume that $(\eta_n)_{n \in \mathbb{N}_0}$ is a sequence of deterministic measures in $\mathcal{N}$ and suppose that $\eta_n \xrightarrow{v} \eta_0$, as $n \to \infty$. Suppose further that $\eta_0$ satisfies $\eta_0(\{\infty\}) = 0$ and that the following two conditions are satisfied:

(a) $\eta_0(H_+) > 0$ for every open half-space $H_+ \subset \mathbb{R}^d$ such that $0 \in \partial H_+$,

(b) the atoms of $\eta_0$ are in general position, that is, no $k + 2$ atoms of $\eta_0$ lie in the same $k$-dimensional affine subspace for all $k = 1, \ldots, d - 1$.

Then, $\text{conv} \eta_0$ is a convex polytope. Moreover, as $n \to \infty$, we have the convergence

$$\text{conv} \eta_n \xrightarrow{d} \text{conv} \eta_0$$

on the space $\mathcal{K}^d$ as well as the convergence of the $f$-vectors

$$f(\text{conv} \eta_n) \longrightarrow f(\text{conv} \eta_0).$$

**Proof.** By the local finiteness of $\eta_0$ and since $\eta_0(\{\infty\}) = 0$, the set of atoms of $\eta_0$ is bounded. Hence $\text{conv} \eta_0$ is a compact convex set. We show that it is in fact a polytope. By the supporting hyperplane theorem (see [31, Chapter 1.3]), Assumption (a) implies that the origin $0$ is an interior point of $\text{conv} \eta_0$. Thus, there exists an open ball $B_r(0) \subset \text{conv} \eta_0$ with $r > 0$. Since $B_r(0)$ is open, the set $\mathbb{R}^d \cup \{\infty\} \setminus B_r(0)$ is compact and thus $\eta_0$ has only a finite number of atoms, say $A_1, \ldots, A_k$ outside of $B_r(0)$. We claim that

$$\text{conv} \eta_0 = \text{conv}\{A_1, \ldots, A_k\} \quad (4.1)$$

and, in particular, $\text{conv} \eta_0$ is a convex polytope. To prove (4.1), it suffices to show that $B_r(0) \subset \text{conv}\{A_1, \ldots, A_k\}$. Assume that $x \in B_r(0)$ but $x \notin \text{conv}\{A_1, \ldots, A_k\}$. By the separating hyperplane theorem (see again [31 Chapter 1.3]), there is an open half-space $G_+$ such that $x \notin G_+$ and $\text{conv}\{A_1, \ldots, A_k\} \subset G_+$. After applying an orthogonal transformation, we may assume that $G_+ = \{y \in \mathbb{R}^d : y_1 < a\}$, where $y_1$ is the first coordinate of $y \in \mathbb{R}^d$. Since $x \notin G_+$, its first coordinate satisfies $x_1 > a$, hence $a < r$. Now,

$$\text{conv} \eta_0 \subset \text{conv}\{A_1, \ldots, A_k \} \cup B_r(0) \subset \text{conv}(G_+ \cup B_r(0)) \subset \{y \in \mathbb{R}^d : y_1 \leq r\},$$

which is in contradiction with $B_{2r}(0) \subset \text{conv} \eta_0$. This proves (4.1).

By Proposition 3.13 in [27], the assumed vague convergence of $\eta_n$ to $\eta_0$, as $n \to \infty$, implies that for sufficiently large $n$, each $\eta_n$ has exactly $k$ atoms, say $\{A_1^{(n)}, \ldots, A_k^{(n)}\}$, in $\mathbb{R}^d \setminus B_r(0)$ and

$$\{A_1^{(n)}, \ldots, A_k^{(n)}\} \xrightarrow{d} \{A_1, \ldots, A_k\}, \quad (4.2)$$

as $n \to \infty$, on the space $\mathcal{C}^d$. Since the mapping $\text{conv} : \mathcal{C}^d \to \mathcal{C}^d$ is continuous with respect to the Hausdorff distance (see [32, Theorem 12.3.5]), we also have that

$$\text{conv}\{A_1^{(n)}, \ldots, A_k^{(n)}\} \xrightarrow{d} \text{conv}\{A_1, \ldots, A_k\},$$

as $n \to \infty$, on the space $\mathcal{C}^d$ as well as on the space $\mathcal{K}^d$. Now, since $B_{2r}(0) \subset \text{conv} \eta_0 = \text{conv}\{A_1, \ldots, A_k\}$, this yields that $B_r(0) \subset \text{conv}\{A_1^{(n)}, \ldots, A_k^{(n)}\}$ for large $n$ and therefore,

$$\text{conv} \eta_n = \text{conv}\{A_1^{(n)}, \ldots, A_k^{(n)}\}, \quad (4.3)$$
for all sufficiently large \(n\), which can be proved in the same way as (4.1).

Assumption (b) implies that the points of \(\{A_1, \ldots, A_k\}\) are in general position, which in conjunction with (4.2) yields that also the points of \(\{A_1^{(n)}, \ldots, A_k^{(n)}\}\) are in general position for sufficiently large \(n\). Therefore, (4.2) implies that for each \(k \in \{0, 1, \ldots, d - 1\}\) the number of \(k\)-dimensional faces of \(\text{conv}\{A_1^{(n)}, \ldots, A_k^{(n)}\}\) is the same as the number of \(k\)-dimensional faces of \(\text{conv}\{A_1, \ldots, A_k\}\) for all \(k \in \{0, \ldots, d - 1\}\) and large enough \(n\). This completes the proof of the lemma.

Since for each \(\gamma > 0\) and \(c > 0\), the Poisson point process \(\Pi_{d, \gamma}(c)\) is an element of the space \(N\), Lemma 4.1 yields the following result.

**Corollary 4.2.** For each \(\gamma > 0\) and \(c > 0\), \(\text{conv}\ \Pi_{d, \gamma}(c)\) is almost surely a convex polytope.

### 4.2 Proofs of weak limit theorems

We are now ready to prove Theorems 2.1, 2.3 and 2.6.

**Proof of Theorem 2.1.** Recall that the mapping \(P : S_d^+ \cap \{x_0 > 0\} \rightarrow \mathbb{R}^d\) was defined by the equality (2.5). For each \(i \in \{1, \ldots, n\}\) let \(\ell_i\) be the line in \(\mathbb{R}^{d+1}\) passing through the origin and the point \(U_i\). This line intersects the hyperplane \(H_1 := \{x_0 = 1\}\) at the point \((1, P(U_i) ) \in H_1\). This observation implies that

\[
C_n \cap H_1 = \text{conv}\{(1, P(U_i)) : i = 1, \ldots, n\}
\]

and, therefore,

\[
(T_n C_n \cap H_1) - e_0 = \text{conv}\{n^{-1} P(U_i) : i = 1, \ldots, n\}.
\]

Hence, it is enough to show that

\[
\text{conv}\{n^{-1} P(U_i) : i = 1, \ldots, n\} \xrightarrow{w} \text{conv} \Pi_{d,1}(2)
\]

on the space \(K^d\). But this follows from Lemma 4.1 above and the fact that

\[
\sum_{i=1}^{n} \delta_{P(U_i)/n} \xrightarrow{w} \Pi_{d,1}(2), \quad \text{as } n \rightarrow \infty.
\]

The latter convergence is a consequence of Lemma 7.7 below and Proposition 3.21 in [27]. The proof of Theorem 2.1 is thus complete.

**Remark 4.3.** For \(d = 1\) the convergence (4.5) also follows from Theorem 3.1 in [11].

**Proof of Theorem 2.3.** Let \(k \in \{1, \ldots, d\}\). From (4.4) we obtain the almost sure equality

\[
f_k(C_n) = f_{k-1}((T_n C_n \cap H_1) - e_0) = f_{k-1}(\text{conv}\{n^{-1} P(U_i) : i = 1, \ldots, n\})
\]

Lemma 4.1 in conjunction with convergence (4.6) yields the desired statement.

**Proof of Theorem 2.6.** We shall use the following alternative definition of the solid angle. For a convex cone \(C \subset \{x_0 \geq 0\} \subset \mathbb{R}^{d+1}\) the solid angle equals

\[
\alpha(C) = \frac{1}{2} \mathbb{P}(x \in C \cap S_d^+),
\]

where \(x\) is a random vector with the uniform distribution on the half-sphere \(S_d^+\). We have

\[
2n \left(1 - \alpha(C_n)\right) = n \left(1 - \mathbb{P}(x \in C_n \cap S_d^+|C_n)\right) = n \mathbb{P}(x \notin C_n \cap S_d^+|C_n),
\]

\[
\mathbb{P}(x \notin C_n \cap S_d^+|C_n)
\]

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where $\mathbf{x}$ is independent of $C_n$ and $\mathbb{P}(\cdot | \cdot)$ denotes conditional probability. Further,

$$n\mathbb{P}(\mathbf{x} \notin C_n \cap S^d_+ | C_n) = n\mathbb{P}(1(\mathbb{P}(\mathbf{x}) \notin C_n \cap H_1 | C_n)
= n\mathbb{P}(\mathbb{P}(\mathbf{x}) \notin \text{conv}\{\mathbb{P}(U_i) : i = 1, \ldots, n\} | U_1, \ldots, U_n)
= \mu_n(\mathbb{R}^d \setminus \text{conv}\{n^{-1}\mathbb{P}(U_i) : i = 1, \ldots, n\}),$$

where the measure $\mu_n$ is given by $\mu_n(\cdot) := n\mathbb{P}(n^{-1}\mathbb{P}(\mathbf{x}) \notin \cdot)$. According to Lemma 7.7, we have vague convergence on $\mathcal{M}_{\mathbb{R}^d,\{0\}}$ of the measures $\mu_n$ to $\nu$, as $n \to \infty$, where $\nu \in \mathcal{M}_{\mathbb{R}^d,\{0\}}$ is a measure with density (2.4) and with $\gamma = 1$ and $c = 2$. Lemma 7.8 below together with the distributional convergence (4.5) gives

$$\mu_n(\mathbb{R}^d \setminus \text{conv}\{n^{-1}\mathbb{P}(U_i) : i = 1, \ldots, n\}) \xrightarrow{d} \nu(\mathbb{R}^d \setminus \text{conv} \Pi_{d,1}(2)),$$

as $n \to \infty$, since $\nu$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$. The proof is complete. \qed

### 4.3 Convergence of moments: Proof of Theorem 2.4

In view of Theorem 2.3, we need to show that the sequence $(f^m_k(C_n))_{n \in \mathbb{N}}$ is uniformly integrable for every $k = 1, \ldots, d$ and $m \in \mathbb{N}$. This is equivalent to

$$\sup_{n \in \mathbb{N}} \mathbb{E} f^m_k(C_n) < \infty \quad (4.7)$$

for every $k = 1, \ldots, d$ and $m \in \mathbb{N}$, because (4.7) for a fixed $m$ implies uniform integrability of $(f^m_k(C_n))_{n \in \mathbb{N}}$ for $0 \leq \ell < m$. To prove (4.7), we note that for an arbitrary (spherical) polytope $P_n$ the number $f_k(P_n)$ of its $k$-dimensional faces satisfies

$$f_k(P_n) \leq \binom{f_0(P_n)}{k+1} \leq f^k_{0+1}(P_n), \quad k = 0, \ldots, d - 1.$$

From this observation it follows that (4.7) is equivalent to

$$\sup_{n \in \mathbb{N}} \mathbb{E} f^m_0(C_n \cap S^d_+) < \infty \quad (4.8)$$

for every $m \in \mathbb{N}$. Recall that $\mathcal{P} : S^d_+ \cap \{z_0 > 0\} \to \mathbb{R}^d$ is the map defined by (2.5). Clearly, $f_0(C_n \cap S^d_+)$ coincides with the number of vertices of the convex hull of $\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)$ in $\mathbb{R}^d$. Write

$$\mathbb{E} f^m_0(C_n \cap S^d_+) = \mathbb{E} \left[ \sum_{i=1}^n 1\{\mathcal{P}(U_i) \notin \text{conv}\{\mathcal{P}(U_j), j \neq i, j = 1, \ldots, n\}\} \right]^m
= \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \mathbb{P}(\mathcal{P}(U_{i_1}) \notin \text{conv}\{\mathcal{P}(U_j), j \neq i_k, j = 1, \ldots, n\}, k = 1, \ldots, m)
\leq \sum_{i_1=1}^n \cdots \sum_{i_m=1}^n \mathbb{P}(\mathcal{P}(U_{i_1}), \mathcal{P}(U_{i_2}), \ldots, \mathcal{P}(U_{i_m}) \notin \text{conv}\{\mathcal{P}(U_j), j \notin \{i_1, i_2, \ldots, i_m\}\}).$$

In view of this representation, the inequality (4.8) follows once we can show that

$$\mathbb{P}(\mathcal{P}(U_1), \mathcal{P}(U_2), \ldots, \mathcal{P}(U_k) \notin \text{conv}\{\mathcal{P}(U_{k+1}), \ldots, \mathcal{P}(U_n)\}) = O(n^{-k}),$$

as $n \to \infty$, for every fixed $k \in \mathbb{N}$ where a constant in the Landau term $O(\cdot)$ might depend on $k$. Denote by $K_n \subset \mathbb{R}^d$ the convex hull of the random points $\mathcal{P}(U_1), \ldots, \mathcal{P}(U_n)$. Fix $k \in \mathbb{N}$ and let $Y_1, Y_2, \ldots, Y_k$ be random variables independent and identically distributed according to the
Cauchy-type distribution described in Proposition 7.2. Assume also that $Y_1, \ldots, Y_k$ are independent of $K_n$. We are going to show that, as $n \to \infty$,

$$n^k \mathbb{P}(Y_1, \ldots, Y_k \notin K_n) = O(1).$$

Note that the left-hand side can be written as

$$n^k \mathbb{P}(Y_1, \ldots, Y_k \notin K_n) = n^k \mathbb{E}\left(\mathbb{P}^k(Y_1 \notin K_n|K_n)\right) = \mathbb{E}\left(\frac{2n}{\omega_{d+1} \int_{\mathbb{R}^d \setminus K_n} \frac{dx}{(1 + ||x||)^{\frac{d+1}{2}}}}\right)^k.$$

It suffices to show that

$$\mathbb{E}\left[\left(\frac{2n}{\omega_{d+1} \int_{\mathbb{R}^d \setminus K_n} \frac{dx}{(1 + ||x||)^{\frac{d+1}{2}}}}\right)^k \mathbb{1}_{\{0 \in K_n\}}\right] = O(1),$$

as $n \to \infty$, because $\mathbb{P}(0 \notin K_n) = O(e^{-cn})$ by Lemma 7.5. To bound the latter integral introduce the random variable

$$\theta_n := \min_{x \in \partial K_n} ||x||$$

and note that

$$\mathbb{E}\left[\left(\frac{2n}{\omega_{d+1} \int_{\mathbb{R}^d \setminus K_n} \frac{dx}{(1 + ||x||)^{\frac{d+1}{2}}}}\right)^k \mathbb{1}_{\{0 \in K_n\}}\right] \leq \mathbb{E}\left(\frac{2n}{\omega_{d+1} \int_{\mathbb{R}^d \setminus B_{\theta_n}(0)} \frac{dx}{(1 + ||x||)^{\frac{d+1}{2}}}}\right)^k,$$

where $B_{\theta_n}(0)$ is the ball of radius $\theta_n$ centered at the origin. From now on, for the sake of brevity, any constants only depending on $d$ and $k$ will be denoted by $c_1, c_2$ etc.

Passing to polar coordinates in the expression for the above expectation we obtain

$$I(n) := \mathbb{E}\left(\frac{2n}{\omega_{d+1} \int_{\mathbb{R}^d \setminus B_{\theta_n}(0)} \frac{dx}{(1 + ||x||)^{\frac{d+1}{2}}}}\right)^k = \mathbb{E}\left(c_1 n \int_{\theta_n}^{\infty} \frac{r^{d-1} dr}{(1 + r^2)^{\frac{d+1}{2}}}\right)^k.$$

Note that

$$\frac{r^{d-1}}{(1 + r^2)^{\frac{d+1}{2}}} \leq \frac{1}{\max\{r^2, 1\}}, \quad r > 0,$$

and therefore

$$\int_{\theta_n}^{\infty} \frac{r^{d-1} dr}{(1 + r^2)^{\frac{d+1}{2}}} \leq \int_{\theta_n}^{\infty} \frac{dr}{\max\{r^2, 1\}} = \begin{cases} 2 - \theta_n, & \theta_n \leq 1, \\ \frac{1}{\theta_n}, & \theta_n > 1. \end{cases}$$

Hence,

$$I(n) \leq 2^k c_1^k n^k \mathbb{P}(\theta_n < 1) + c_1^k \mathbb{E}\left(\frac{n}{\theta_n}\right)^k \mathbb{1}_{\{\theta_n \geq 1\}}$$

$$\leq 2^k c_1^k n^k \mathbb{P}(K_n \notin B_1(0)) + c_1^k \int_0^{\infty} \mathbb{P}\left(\left(\frac{n}{\theta_n}\right)^k \mathbb{1}_{\{\theta_n \geq 1\}} > x\right) dx$$

$$= 2^k c_1^k n^k \mathbb{P}(K_n \notin B_1(0)) + c_1^k \int_0^{n^k} \mathbb{P}\left(1 \leq \theta_n < n x^{-1/k}\right) dx$$

$$\leq 2^k c_1^k n^k \mathbb{P}(K_n \notin B_1(0)) + c_1^k \int_0^{n^k} \mathbb{P}(K_n \notin B_{nx^{-1/k}}(0)) dx$$

$$= 2^k c_1^k n^k \mathbb{P}\left(\frac{K_n}{n} \notin B_{n^{-1}}(0)\right) + c_1^k \int_0^{n^k} \mathbb{P}\left(\frac{K_n}{n} \notin B_{x^{-1/k}}(0)\right) dx.$$
Now we apply Lemma 7.5 to bound both summands to conclude that 

\[ I(n) \leq c_2 n^k \exp\{-c_3 n\} + c_2 \int_0^n \exp\left\{-\frac{1}{c_4 x^{-1/k} + c_5 n^{-1}}\right\} \, dx. \]

The first summand, clearly, converges to zero and it remains to show that the integral on the right-hand side is bounded in \( n \). If \( x \leq (c_4 c_5^{-1} n)^k \), then \( c_4 x^{-1/k} + c_5 n^{-1} \leq 2c_4 x^{-1/k} \) and we have

\[
\int_0^{(c_4 c_5^{-1} n)^k} \exp\left\{-\frac{1}{c_4 x^{-1/k} + c_5 n^{-1}}\right\} \, dx \leq \int_0^{(c_4 c_5^{-1} n)^k} \exp\left\{-\frac{1}{2c_4 x^{-1/k}}\right\} \, dx \\
\leq \int_0^\infty \exp\left\{-\frac{1}{2c_4 x^{-1/k}}\right\} \, dx < \infty.
\]

On the other hand, if \( x \in ((c_4 c_5^{-1} n)^k, n^k] \) (provided this interval is not empty), we have

\[
\int_{(c_4 c_5^{-1} n)^k}^{n^k} \exp\left\{-\frac{1}{c_4 x^{-1/k} + c_5 n^{-1}}\right\} \, dx \leq \int_{(c_4 c_5^{-1} n)^k}^{n^k} \exp\left\{-\frac{1}{c_5 n^{-1} + c_5 n^{-1}}\right\} \, dx = O(n^k e^{-n/(2c_4)}),
\]

as \( n \to \infty \). This completes the proof of the moment convergence.

The formula for the expectation \( \mathbb{E}\overline{f}_{k-1}(\text{conv } \Pi_{d,1}(2)) \) in Theorem 2.4 follows from the Mecke equation (3.1) applied with the function \( f(x_1, \ldots, x_k, \Pi) = 1_{\{(x_1, \ldots, x_k) \in \mathcal{F}_{k-1}(\text{conv } \Pi)\}} \). The proof of Theorem 2.4 is complete.

5 Proofs: Conic intrinsic volumes

In this section we prove Theorems 2.7 and 2.8. First of all, we prove the relationship (2.12) between the conic mean projection volumes and the conic intrinsic volumes.

**Lemma 5.1.** For \( k \in \{0, 1, \ldots, d\} \) and a cone \( C \subset \mathbb{R}^{d+1} \) we have that

\[
w_{k+1}(C) = \sum_{i=k+1}^{d+1} v_i(C).
\]

**Proof.** We let \( \mathcal{S}_k \) be the space of \( k \)-dimensional great subspheres of \( \mathbb{S}^d \), supplied with the unique rotation invariant Haar probability measure \( \tau_k \). For a spherically convex set \( K \subset \mathbb{S}^d \) and \( S \in \mathcal{S}_k \) we denote by \( K|S \) the spherical projection of \( K \) onto \( S \), see [32, p. 263]. The spherical mean projection volume of \( K \) is given by

\[
W_k(K) := \frac{1}{\omega_{k+1}} \int_{\mathcal{S}_k} \sigma_k(K|S) \tau_k(dS),
\]

where \( \sigma_k \) is the \( k \)-dimensional Lebesgue measure on \( S \in \mathcal{S}_k \). Putting \( C := \text{pos } K \) and using the fact that \( \tau_k \) is the probability distribution of \( L \cap \mathbb{S}^d \), where \( L \in G(d+1, k+1) \) is distributed according to the Haar measure \( \nu_{k+1} \), we obtain

\[
W_k(K) = \frac{1}{\omega_{k+1}} \int_{\mathcal{S}_k} \sigma_k(K|S) \tau_k(dS) = \frac{1}{\kappa_{k+1}} \int_{G(d+1, k+1)} \text{Vol}_{k+1}(P_L(C) \cap \mathbb{B}^{d+1}) \nu_{k+1}(dL).
\]

This leads to the equality \( W_k(K) = w_{k+1}(C) \). On the other hand, from [32, p. 263] we have the relationship

\[
W_k(K) = \sum_{i=k}^{d} v_i(K)
\]

with the spherical intrinsic volumes \( v_i(K) := v_{i+1}(C) \). This yields the required formula for \( w_{k+1}(C) \). \qed
We are interested in the expected Grassmann angle $E h_{k+1}(C_n) = \frac{1}{2} P(C_n \cap L \neq \{0\})$, where $L \in G(d+1, d+1-k)$ is a random subspace with distribution $\nu_{d+1-k}$, and $k \in \{1, \ldots, d\}$.

Proof of Theorem 2.7: We shall derive formulas for the expectations of Grassmann angles and the $f$-vectors of $C_n$ and then obtain Theorem 2.7 by comparing these formulas.

Step 1. We first prove the asymptotic formula for $E h_{k+1}(C_n)$.

Proof of Theorem 2.8: We first prove the asymptotic formula for $h_{k+1}$. For $k = 0$ the result is trivial since $h_1(C_n) = 1/2$, so let $k \in \{1, \ldots, d\}$. We use Theorem 2.7 together with Theorem 2.4 to obtain

\[ n^{d+1-k} \left( \frac{1}{2} - E h_{k+1}(C_n) \right) = \frac{1}{2} n^{d+1-k} \left( \frac{n + d + 1 - k}{d + 1 - k} \right)^{-1} E f_{d-k+1}(C_{n+d+1-k}) \to B_{d-k+1,d}, \]
as \( n \to \infty \). To deduce the result for the conic intrinsic volumes, recall (2.11) and note that it implies, for \( \ell \in \{0,1,\ldots,d-1\} \),

\[
\mathbb{E}v_\ell(C_n) = \mathbb{E}h_\ell(C_n) - \mathbb{E}h_{\ell+2}(C_n) = \left( \frac{1}{2} - \mathbb{E}h_{\ell+2}(C_n) \right) - \left( \frac{1}{2} - \mathbb{E}h_\ell(C_n) \right).
\]

So,

\[
\lim_{n \to \infty} n^{d-1-\ell} \mathbb{E}v_\ell(C_n) = \lim_{n \to \infty} n^{d-1-\ell} \left( \frac{1}{2} - \mathbb{E}h_{\ell+2}(C_n) \right) - \lim_{n \to \infty} n^{d-1-\ell} \left( \frac{1}{2} - \mathbb{E}h_\ell(C_n) \right).
\]

According to (5.2), the first limit equals \( B_{d-1-\ell,d} \), while the second one is 0 (indeed, the sequence goes to 0 like a constant multiple of \( n^{-1} \), as \( n \to \infty \)). Finally, the asymptotic formulas for the mean projection volumes can be deduced in a similar way from (2.12). Namely, for all \( r \in \{0,1,\ldots,d-1\} \) we have \( w_{r+1}(C_n) = h_{r+1}(C_n) + h_{r+2}(C_n) \), hence

\[
\lim_{n \to \infty} n^{d+1-r} (1 - \mathbb{E}w_{r+1}(C_n)) = \lim_{n \to \infty} n^{d+1-r} \left( \frac{1}{2} - \mathbb{E}v_{r+1}(C_n) \right) + \lim_{n \to \infty} n^{d+1-r} \left( \frac{1}{2} - \mathbb{E}v_{r+2}(C_n) \right).
\]

By (5.2), the first limit equals \( B_{d+1-r,d} \), whereas the second one is 0.

\[
\square
\]

6 Proofs: Functionals of the Poisson process

6.1 Invariance property

In our proof we shall use the following projection stability. It says that the projection of a Poisson point process with a power-law intensity measure as in (2.4) onto a linear subspace is again a Poisson point process of the same type within this subspace.

**Lemma 6.1.** Let \( \gamma > 0 \), \( c > 0 \) and \( k \in \{1, \ldots, d-1\} \). The orthogonal projection of \( \Pi_{d,\gamma}(c) \) onto any \( k \)-dimensional linear subspace \( L \) of \( \mathbb{R}^d \) has the same law as \( \Pi_{k,\gamma}(c) \), where we identify \( L \) with \( \mathbb{R}^k \).

**Proof.** First suppose that \( k = d-1 \). By rotational symmetry we may assume that we project onto the hyperplane \( \{x_1 = 0\} \). The intensity of the projected Poisson point process at \( (0,x_2,\ldots,x_d) \) with \( x_2^2 + \ldots + x_d^2 = a^2 \) equals

\[
\frac{c}{\omega_{d+\gamma}} \int_{-\infty}^{+\infty} \frac{dx_1}{(a^2 + x_1^2)^{\frac{d+\gamma}{2}}} = \frac{c}{\omega_{d+\gamma}} \int_{-\infty}^{+\infty} \frac{ady}{a^{d+\gamma}(1+y^2)^{\frac{d-\gamma}{2}}} = \frac{ca^{1-d-\gamma}}{\omega_{d+\gamma}} \int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^{\frac{d-\gamma}{2}}},
\]

where we used the change of variables \( y = x_1/a \). Applying the substitution \( y^2 = t \) the last integral equals

\[
\int_{-\infty}^{+\infty} \frac{dy}{(1+y^2)^{\frac{d-\gamma}{2}}} = \int_{0}^{\infty} t^{-\frac{1}{2}} \frac{dt}{(1+t)^{\frac{d+\gamma}{2}}} = \sqrt{\pi} \frac{\Gamma(d+\gamma-rac{1}{2})}{\Gamma(d+\gamma)},
\]

by definition of Euler’s beta function and its relationship to the gamma function. Hence, the intensity of the projected Poisson point process is

\[
\frac{ca^{1-d-\gamma}}{\omega_{d+\gamma}} \sqrt{\pi} \frac{\Gamma(d+\gamma-rac{1}{2})}{\Gamma(d+\gamma)} = \frac{c}{\omega_{d+\gamma-1}} \frac{1}{a^{d+\gamma-1}}
\]

by definition of \( \omega_{d+\gamma} \) and \( \omega_{d+\gamma-1} \). Arguing now inductively, we arrive at the desired claim.

\[
\square
\]
6.2 Expected $T$-functional: Proof of Theorem 2.12

We are now ready to prove Theorems 2.12, 2.17 and Corollary 2.16.

Proof of Theorem 2.12. To simplify the notation, we shall write $\Pi_{d,\gamma}$ for $\Pi_{d,\gamma}(c)$ in this proof and keep $c > 0$ fixed. Recall that $\text{conv} \Pi_{d,\gamma}$ denotes the convex hull of all points of the Poisson process $\Pi_{d,\gamma}$. By Corollary 4.2, $\text{conv} \Pi_{d,\gamma}$ is almost surely a convex polytope. Also recall that

$$T_{a,b}^{d,k}(\text{conv} \Pi_{d,\gamma}) = \sum_{F \in \mathcal{F}_{k}(\text{conv} \Pi_{d,\gamma})} \text{dist}^a(F) \text{Vol}_b(F).$$

Let us denote by $\Delta_{k-1}(x_1, \ldots, x_k)$ the $(k-1)$-dimensional volume of the simplex with vertices $x_1, \ldots, x_k \in \mathbb{R}^d$. We denote by $E = E(x_1, \ldots, x_k) \in A(d,k-1)$ the $(k-1)$-dimensional affine subspace spanned by the points $x_1, \ldots, x_k$. Let also $\text{dist}(E)$ be the distance from $E$ to the origin.

By the multivariate Mecke formula for Poisson point processes (3.1), we have

$$E T_{a,b}^{d,k-1}(\text{conv} \Pi_{d,\gamma}) = \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \Delta_{k-1}^b(x_1, \ldots, x_k) \text{dist}^a(E)$$

$$\times \mathbb{P} \left( \text{conv} \{x_1, \ldots, x_k\} \in \mathcal{F}_{k-1}(\text{conv} \bar{\Pi}_{d,\gamma}) \right) \prod_{i=1}^k \frac{c \, dx_i}{\omega_{d+\gamma}\|x_i\|^{d+\gamma}},$$

where $\bar{\Pi}_{d,\gamma} := \Pi_{d,\gamma} + \sum_{i=1}^k \delta_{x_i}$. Let $E^\perp$ be the orthogonal complement of $E$ and $P_{E^\perp}$ the orthogonal projection onto $E^\perp$. Note that $P_{E^\perp} x_1 = \ldots = P_{E^\perp} x_k$. Clearly, the simplex $\text{conv} \{x_1, \ldots, x_k\}$ is a $(k-1)$-dimensional face of $\Pi_{d,\gamma}$ if and only if $P_{E^\perp} x_1$ is not contained in $P_{E^\perp} \text{conv} \Pi_{d,\gamma}$. Define the non-absorption probability

$$p_{d,\gamma}(R) := \mathbb{P}(Re_1 \notin \text{conv} \Pi_{d,\gamma}), \quad R > 0,$$

where $e_1$ is any vector of unit length in $\mathbb{R}^d$. By Lemma 6.1, $P_{E^\perp} \Pi_{d,\gamma}$ has the same distribution as $\Pi_{d-k+1,\gamma}$, where we identify $E^\perp$ with $\mathbb{R}^{d-k+1}$. Hence,

$$E T_{a,b}^{d,k-1}(\text{conv} \Pi_{d,\gamma}) = \frac{1}{k!} \int_{(\mathbb{R}^d)^k} \Delta_{k-1}^b(x_1, \ldots, x_k) \text{dist}^a(E)$$

$$\times p_{d-k+1,\gamma}(\text{dist}(E)) \prod_{i=1}^k \frac{c \, dx_i}{\omega_{d+\gamma}\|x_i\|^{d+\gamma}}. \quad (6.2)$$

Next, we use the affine Blaschke–Petkantschin formula (3.3):

$$E T_{a,b}^{d,k-1}(\text{conv} \Pi_{d,\gamma}) = \frac{c^k((k-1))!d-k+1b_{d,k-1}}{k! \omega_{d+\gamma}^b} \int_{A(d,k-1)} \int_{E^k} \Delta_{k-1}^{b+d-k+1}(x_1, \ldots, x_k)$$

$$\times \text{dist}^a(E) p_{d-k+1,\gamma}(\text{dist}(E)) \prod_{i=1}^k \frac{1}{\|x_i\|^{d+\gamma}} \, d\lambda_E^b(x_1, \ldots, x_k) \mu_{k-1}(dE).$$

Writing

$$h(\text{dist}(E)) := \int_{E^k} \Delta_{k-1}^{b+d-k+1}(x_1, \ldots, x_k) \prod_{i=1}^k \frac{1}{\|x_i\|^{d+\gamma}} \, d\lambda_E^b(x_1, \ldots, x_k), \quad (6.3)$$

we arrive at

$$E T_{a,b}^{d,k-1}(\text{conv} \Pi_{d,\gamma}) = \frac{c^k((k-1))!d-k+1b_{d,k-1}}{k! \omega_{d+\gamma}^b} \int_{A(d,k-1)} \int_{E^k} \text{dist}^a(E) p_{d-k+1,\gamma}(\text{dist}(E)) h(\text{dist}(E)) \mu_{k-1}(dE).$$
Let $\beta := b + d - k + 1$. We compute

$$h(r) = \int_{(\mathbb{R}^{k-1})^k} \Delta^\beta_{k-1}(y_1, \ldots, y_k) \prod_{i=1}^k \frac{dy_i}{(r^2 + \|y_i\|^2)^{\frac{d+\gamma}{2}}}$$

$$= \int_{(\mathbb{R}^{k-1})^k} r^{(k-1)\beta} \Delta^\beta_{k-1}(z_1, \ldots, z_k) \prod_{i=1}^k \frac{r^{k-1}dz_i}{r^{d+\gamma}(1 + \|z_i\|^2)^{\frac{d+\gamma}{2}}}$$

$$= r^{(k-1)k-(d+\gamma)k+\beta(k-1)} \int_{(\mathbb{R}^{k-1})^k} \Delta^\beta_{k-1}(z_1, \ldots, z_k) \prod_{i=1}^k \frac{dz_i}{(1 + \|z_i\|^2)^{\frac{d+\gamma}{2}}},$$

where we have used the change of variables $y_i = rz_i$. Thus, the function $h$ satisfies the scaling property

$$h(r) = r^{(k-1)k-(d+\gamma)k+\beta(k-1)}h(1).$$

To compute the value of $h(1)$, let $Z_1, \ldots, Z_k$ be independent random variables on $\mathbb{R}^{k-1}$ with the so-called beta'-density $f(x)$ as in [18], that is,

$$f(x) = \frac{\omega_{d-k+1+\gamma}}{\omega_{d+\gamma}} (1 + \|x\|^2)^{-\frac{d+\gamma}{2}}, \quad x \in \mathbb{R}^{k-1}.$$ 

Recall that $\Delta_{k-1}(Z_1, \ldots, Z_k)$ is the volume of the simplex with vertices $Z_1, \ldots, Z_k$. Then, we can interpret $h(1)$ as follows

$$h(1) = \frac{\omega_{d+\gamma}^k}{\omega_{d-k+1+\gamma}^k} \mathbb{E} \Delta^\beta_{k-1}(Z_1, \ldots, Z_k).$$

The moments of $\Delta_{k-1}(Z_1, \ldots, Z_k)$ have been calculated by Miles [22, Eqn. (74)] and we have the explicit formula

$$\mathbb{E} \Delta^\beta_{k-1}(Z_1, \ldots, Z_k) = \frac{1}{((k-1)!)^\beta} \Gamma\left(\frac{d-k+1+\gamma}{2}\right) \Gamma\left(\frac{d-k+1+\beta+\gamma}{2}\right) \prod_{i=1}^{k-1} \Gamma\left(\frac{i+\beta}{2}\right)$$

provided that $d - k + 1 - \beta + \gamma > 0$. In fact, Miles stated his result for integer moments only, but it also holds for arbitrary moments as was argued in [18].

Let us consider the case $k = d$. Then $\beta = b + 1$ and the above formulae simplify to

$$h(r) = r^{(b+\gamma)d-b-1}h(1)$$

and

$$h(1) = \left(\frac{\omega_{d+\gamma}}{\omega_{1+\gamma}} \right)^d \frac{1}{((d-1)!)^{b+1}} \Gamma\left(\frac{\gamma-b+1}{2}\right) \Gamma\left(\frac{\gamma-b+d}{2}\right) \prod_{i=1}^{d-1} \Gamma\left(\frac{i+1+b}{2}\right)$$

provided that $\gamma - b > 0$. We also have $h(r) = +\infty$, $r > 0$, if $\gamma \leq b$. Since

$$b_{d,d-1} = \frac{\omega_d}{2} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)},$$

the above formulae yield

$$\mathbb{E} \mathbb{T}^{d,d-1}_{a,b}(\text{conv } \Pi_{d,\gamma}) = \int_{A(d,d-1)} \text{dist}^{\alpha}(E) p_{1,\gamma}(\text{dist}(E))h(\text{dist}(E)) \mu_{d-1}(dE).$$

Now, recalling the definition of $p_{1,\gamma}(R)$ from (6.1) we obtain

$$p_{1,\gamma}(R) = \mathbb{P}(R \notin \text{conv } \Pi_{1,\gamma}) = \mathbb{P}(\Pi_{1,\gamma}(R, \infty) = 0) = e^{-\frac{r}{\gamma+1}} \int_r^\infty \frac{dx}{x^{\frac{d+\gamma}{2}}} = e^{-\frac{r}{\gamma+1}} R^{-\gamma}.$$
Hence,

\[
\mathbb{E}T_{a,b}^{d,d-1}(\text{conv } \Pi_{d,\gamma}) = \frac{c^d(d-1)!\omega_d^d}{2d!\omega_d^{d+\gamma}}h(1) \\
\times \int_{A(d,d-1)} \text{dist}^{a-b-1+(b-\gamma)d}(E) e^{-\frac{x}{\omega_d^{d+\gamma}}} \text{dist}^{-\gamma}(E) \mu_{d-1}(dE).
\]

By the definition of the measure \( \mu_{d-1} \), we obtain

\[
\mathbb{E}T_{a,b}^{d,d-1}(\text{conv } \Pi_{d,\gamma}) = \frac{c^d(d-1)!\omega_d^d}{d!\omega_d^{d+\gamma}}h(1) \int_0^\infty x^{a-b-1+(b-\gamma)d} e^{-\frac{x}{\omega_d^{d+\gamma}}} dx.
\] \hspace{1cm} (6.7)

Evaluating the integral, we get

\[
\mathbb{E}T_{a,b}^{d,d-1}(\text{conv } \Pi_{d,\gamma}) = \frac{c^d(d-1)!\omega_d^d}{d!\omega_d^{d+\gamma}}h(1)\gamma^{-1} \left( \frac{c}{\gamma\omega_d^{d+\gamma}} \right)^{a-b+(b-\gamma)d} \Gamma \left( (\gamma-b)d+b-a \right)
\]

under the condition \((\gamma-b)d+b-a > 0\). Otherwise, the integral equals +\( \infty \). Applying formula (6.5) completes the proof. \( \square \)

**Proof of Corollary 2.16.** Lemma 6.1 implies that for any \( L \in G(d,k) \), the projected random polytope \( P_L \text{conv } \Pi_{d,\gamma} \) has the same distribution as \( \text{conv } \Pi_{k,\gamma} \) if we identify \( L \) with \( \mathbb{R}^k \). Using this together with the definition of intrinsic volumes and Fubini’s theorem we get

\[
\mathbb{E}V_k(\text{conv } \Pi_{d,\gamma}) = \binom{d}{k} \frac{k_d}{k_k k_{d-k}} \mathbb{E} \int_{G(d,k)} \text{Vol}_k(P_L \text{conv } \Pi_{d,\gamma}) \nu_k(dL)
\]

\[
= \binom{d}{k} \frac{k_d}{k_k k_{d-k}} \mathbb{E} \int_{G(d,k)} \text{Vol}_k(P_L \text{conv } \Pi_{d,\gamma}) \nu_k(dL)
\]

\[
= \binom{d}{k} \frac{k_d}{k_k k_{d-k}} \mathbb{E} \text{Vol}_k(\text{conv } \Pi_{k,\gamma}),
\]

since \( \nu_k \) is a probability measure. Now, Corollary 2.15 can be used to complete the proof. \( \square \)

**Proof of Theorem 2.17.** We keep the notation \( \Pi_{d,\gamma} \) for \( \Pi_{d,\gamma}(c) \). Recall that \( \text{conv } \Pi_{d,\gamma} \) denotes the convex hull of all points of the form \( z+e \), where \( z \) is a point of \( \Pi_{d,\gamma} \). By Corollary 4.2, \( \text{conv } \Pi_{d,\gamma} \) is a convex polytope a.s. Its \((k-1)\)-dimensional faces have the form \( \text{conv } \{e_1, x_1, \ldots, \varepsilon_k x_k\} \), where \( x_1, \ldots, x_k \) are distinct points from \( \Pi_{d,\gamma} \) and \( \varepsilon_1, \ldots, \varepsilon_k \in \{+1,-1\} \). Recalling that

\[
T_{a,b}^{d,k-1}(\text{conv } \Pi_{d,\gamma}) = \sum_{F \in F_{k-1}(\text{conv } \Pi_{d,\gamma})} \text{dist}^a(F) \text{Vol}^b_{k-1}(F)
\]

we can write

\[
\mathbb{E}T_{a,b}^{d,k-1}(\text{conv } \Pi_{d,\gamma}) = \frac{1}{k!} \mathbb{E} \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{+1,-1\}^k} \sum_{x_1, \ldots, x_k \in \Pi_{d,\gamma}} \text{dist}^a(\text{aff } \{e_1 x_1, \ldots, \varepsilon_k x_k\})
\]

\[
\times \Delta_{k-1}(e_1 x_1, \ldots, \varepsilon_k x_k) I\{\text{conv } e_1 x_1, \ldots, \varepsilon_k x_k \in F_{k-1}(\text{conv } \Pi_{d,\gamma})\}.
\]

Interchanging the expectation and the sum over \((\varepsilon_1, \ldots, \varepsilon_k)\) and using the Mecke formula (3.1), we obtain

\[
\mathbb{E}T_{a,b}^{d,k-1}(\text{conv } \Pi_{d,\gamma}) = \frac{1}{k!} \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{+1,-1\}^k} \mathbb{E} \int_{\mathbb{R}^d} \text{dist}^a(\text{aff } \{e_1 x_1, \ldots, \varepsilon_k x_k\})
\]

\[
\times \Delta_{k-1}(e_1 x_1, \ldots, \varepsilon_k x_k) I\{\text{conv } e_1 x_1, \ldots, \varepsilon_k x_k \in F_{k-1}(\text{conv } \Pi_{d,\gamma})\} \prod_{i=1}^k \frac{c}{\omega_d^{d+\gamma}||x_i||^{d+\gamma}},
\]

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where $\hat{\Pi}_{d,\gamma} = \Pi_{d,\gamma} + \delta x_1 + \ldots + \delta x_k$. Interchanging the integral and the expectation and noting that the expectation of an indicator function is the probability of the corresponding event, we get

$$
\mathbb{E} T_{a,b}^{d,k-1}(\text{sconv } \Pi_{d,\gamma}) = \frac{1}{k!} \sum_{(\varepsilon_1, \ldots, \varepsilon_k) \in \{+1, -1\}^k} \int_{(\mathbb{R}^d)^k} \text{dist}^a(\text{aff } \{\varepsilon_1 x_1, \ldots, \varepsilon_k x_k\})
\times \Delta_{k-1}^b(\varepsilon_1 x_1, \ldots, \varepsilon_k x_k) \mathbb{P}\left(\text{conv } \{\varepsilon_1 x_1, \ldots, \varepsilon_k x_k\} \in \mathcal{F}_{k-1}(\text{sconv } \hat{\Pi}_{d,\gamma}) \right) \prod_{i=1}^k \frac{c \, dx_i}{\omega_{d+\gamma} \|x_i\|^{d+\gamma}}.
$$

Now observe that

$$\text{sconv } \hat{\Pi}_{d,\gamma} = \text{sconv } \{\Pi_{d,\gamma} + \delta x_1 + \ldots + \delta x_k\} = \text{sconv } \{\Pi_{d,\gamma} + \delta x_1 + \ldots + \delta x_k\}.$$  

Noting that the integral remains invariant under the change of variables $\varepsilon_1 x_1 \mapsto x_1, \ldots, \varepsilon_k x_k \mapsto x_k$, we arrive at

$$
\mathbb{E} T_{a,b}^{d,k-1}(\text{sconv } \Pi_{d,\gamma}) = \frac{2^k}{k!} \int_{(\mathbb{R}^d)^k} \text{dist}^a(\text{aff } \{x_1, \ldots, x_k\})
\times \Delta_{k-1}^b(x_1, \ldots, x_k) \mathbb{P}\left(\text{conv } \{x_1, \ldots, x_k\} \in \mathcal{F}_{k-1}(\text{sconv } \hat{\Pi}_{d,\gamma}) \right) \prod_{i=1}^k \frac{c \, dx_i}{\omega_{d+\gamma} \|y_i\|^{d+\gamma}}.
$$

From now on we can argue exactly as in the proof of Theorem 2.12 but an additional factor of $2^k$ appears throughout and the non-absorption probability $p_{d,\gamma}(R)$ has to be replaced by its symmetrized version

$$q_{d,\gamma}(R) := \mathbb{P}(Re_1 \notin \text{sconv } \Pi_{d,\gamma}), \quad R > 0.$$  

In particular, in the special case $k = d$, we arrive at

$$
\mathbb{E} T_{a,b}^{d,d-1}(\text{sconv } \Pi_{d,\gamma}) = \frac{(2c)^d(d-1)! \omega_d}{2d! \omega_{d+\gamma}^d} \int_{A(d,d-1)} \text{dist}^a(E) q_{1,\gamma}(\text{dist}(E)) \mu_{d-1}(dE).
$$

The non-absorption probability can easily be calculated as follows:

$$q_{1,\gamma}(R) = \mathbb{P}(R \notin \text{sconv } \Pi_{1,\gamma}) = \mathbb{P}(\Pi_{1,\gamma}[R, \infty) = \Pi_{1,\gamma}(-\infty, -R] = 0)
= \mathbb{P}(\Pi_{1,\gamma}[R, \infty) = 0)^2 = e^{-\frac{2c}{\gamma+1} \int_R^\infty \frac{dx}{x^{\frac{d+\gamma}{\gamma}}} = e^{-\frac{2c}{\gamma+1} R^{-\gamma}}.
$$

By the definition of the measure $\mu_{d-1}$, we obtain

$$
\mathbb{E} T_{a,b}^{d,d-1}(\text{sconv } \Pi_{d,\gamma}) = \frac{(2c)^d(d-1)! \omega_d}{d! \omega_{d+\gamma}^d} h(1) \int_0^\infty x^{a-b-1+(\gamma) d} e^{-\frac{2c}{\gamma+1} x^{-\gamma}} \, dx, \quad (6.8)
$$

where $h(1)$ is given by (6.5). Now a comparison of (6.8) with (6.7) in the proof of Theorem 2.12 completes the proof.

**Proof of (2.7).** We compute the constant $B_{d,d}$. Using the Blaschke–Petkantschin formula (3.3) with $k = d - 1$ we see that

$$B_{d,d} = \frac{1}{2} \left(1 - \frac{2}{\omega_{d+1}}\right)^d \int_{A(d,d-1)} \int_{E^d} \mathbb{P}(\Pi_{d,1}(2) \cap E = \emptyset) \Delta_{d-1}(x_1, \ldots, x_d)
\times \prod_{i=1}^d \frac{dx_i}{\|x_i\|^{d+1}} \mu_{d-1}(dE).
$$

The probability has already been computed in (6.6):

$$\mathbb{P}(\Pi_{d,1}(2) \cap E = \emptyset) = e^{-\frac{1}{\gamma}}.$$
Suppose that for every choice of \( r > 0 \) denotes the distance of \( E \) to the origin. Thus, using the definition (6.5) of \( h(1) \) and the scaling relation (6.4) (with \( b = 0 \) and \( \gamma = 1 \)), we conclude that

\[
B_{d,d} = \left( \frac{2}{\omega_{d+1}} \right) \frac{d \omega_d}{2} (d-1)! h(1) \int_0^\infty e^{-\frac{1}{r^2}} r^{-(d+1)} \, dr
\]

\[
= \left( \frac{2}{\omega_{d+1}} \right) \frac{d \omega_d}{2} (d-1)! h(1) \pi^{d-1} \Gamma \left( \frac{d+1}{2} \right) = (2\pi)^{d-1} \Gamma \left( \frac{d+1}{2} \right)^2,
\]

where in the last step we have used Legendre’s duplication formula. This completes the proof. \( \square \)

7 Auxiliary lemmas

We collect here additional technical lemmas that have been used in the arguments in the previous sections.

**Lemma 7.1.** Suppose that for each \( (\varepsilon_1, \ldots, \varepsilon_d) \in \{-1, +1\}^d \) a point in \( \mathbb{R}^d \) is given whose coordinates have the same signs as \( \varepsilon_1, \ldots, \varepsilon_d \). Then, the convex hull of these \( 2^d \) points contains the origin.

**Proof.** We argue by induction over the dimension \( d \). The claim obviously holds for \( d = 1 \). Suppose it is true for dimension \( d - 1 \). Then we can take \( 2^{d-1} \) points corresponding to \( \varepsilon_1 = 1 \) and construct a convex combination \( a_+ \) of these points such that all coordinates of \( a_+ \) vanish except the first one (which is positive). Similarly, taking \( 2^{d-1} \) points corresponding to \( \varepsilon_1 = -1 \) we construct a convex combination \( a_- \) with negative first coordinate and all other coordinates being 0. Clearly, the origin can now be written as a convex combination of these two points \( a_+ \) and \( a_- \). \( \square \)

**Lemma 7.2.** For \( r \geq 0 \) and \( \varepsilon_2, \ldots, \varepsilon_d \in \{-1, +1\} \) define the set

\[
A_{\varepsilon_2, \ldots, \varepsilon_d}(r) := \{(z_1, \ldots, z_d) \in \mathbb{R}^d : z_1 > r, \varepsilon_2 z_2 > 0, \ldots, \varepsilon_d z_d > 0\}.
\]

Suppose that for every choice of \( (\varepsilon_2, \ldots, \varepsilon_d) \) a point in \( A_{\varepsilon_2, \ldots, \varepsilon_d}(r) \) and another point in \(-A_{\varepsilon_2, \ldots, \varepsilon_d}(0)\) are given. Then \((r, 0, \ldots, 0)\) can be represented as a convex combination of these points.

**Proof.** By Lemma 7.1, we can take all points in \( A_{\varepsilon_2, \ldots, \varepsilon_d}(r) \) or all points in \(-A_{\varepsilon_2, \ldots, \varepsilon_d}(0)\), respectively, corresponding to all choices of \( \varepsilon_2, \ldots, \varepsilon_d \) and construct a convex combination of these points such that all coordinates are zero except the first one (which is larger than \( r \) or smaller than 0, respectively). Obviously, there exists a convex combination of these two points which is equal to \((r, 0, \ldots, 0)\). \( \square \)

**Lemma 7.3.** Fix \( \varepsilon_2, \ldots, \varepsilon_d \in \{-1, +1\} \) and let \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \) be a random vector with Cauchy-type distribution as in Proposition 2.2. Then for all \( r > 0 \) and \( n \in \mathbb{N} \) we have

\[
\mathbb{P} \left( \frac{\xi}{n} \in A_{\varepsilon_2, \ldots, \varepsilon_d}(r) \right) \geq \frac{1}{\pi^{2d-1}} \frac{1}{rn + 1}.
\]

**Proof.** Every coordinate of \( \xi \) has a one-dimensional Cauchy distribution; see, e.g., Lemma 4.3(b) in [LN]. Hence,

\[
\mathbb{P} \left( \frac{\xi}{n} \in A_{\varepsilon_2, \ldots, \varepsilon_d}(r) \right) = \left( \frac{1}{2} \right)^{d-1} \mathbb{P}(\xi_1 > rn) = \left( \frac{1}{2} \right)^{d-1} \left( \frac{1}{2} - \frac{1}{\pi} \arctan(rn) \right) \geq \frac{1}{\pi^{2d-1}} \frac{1}{rn + 1},
\]

by the inequality \( \arctan(x) \leq \frac{x}{2} - \frac{1}{2\pi x} \) which holds for all \( x \geq 0 \). \( \square \)

**Lemma 7.4.** Let \( \xi^{(1)}, \ldots, \xi^{(n)} \in \mathbb{R}^d \) be independent random vectors with a Cauchy-type distribution as in Proposition 2.2. Then, there exist constants \( c_1, c_2 > 0 \) only depending on \( d \) such that, for all \( r > 0 \) and \( n \in \mathbb{N} \),

\[
\mathbb{P} \left( r e_1 \notin \text{conv} \left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \right) < c_1 \exp \left( - \frac{c_2}{r + \frac{1}{n}} \right).
\]
Proof. We let \( c_2 = \frac{1}{2^{2r+1}} \) be the constant from Lemma 7.3. Combining Lemma 7.2 with Lemma 7.3 yields
\[
\mathbb{P}\left( r e_1 \notin \text{conv}\left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \right) \leq 2^d \mathbb{P}\left( \left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \cap A_{+1,\ldots,+1}(r) = \emptyset \right)
\]
\[
= 2^d \left( 1 - \mathbb{P}\left( \frac{\xi^{(1)}}{n} \in A_{+1,\ldots,+1}(r) \right) \right)^n \leq 2^d \left( 1 - \frac{c_2}{rn+1} \right)^n \leq 2^d \exp\left( -\frac{c_2n}{rn+1} \right),
\]
where the last inequality follows since \( \log(1-x) \leq -x \) for \( x < 1 \). Putting \( c_1 := 2^d \) completes the proof. \( \square \)

**Lemma 7.5.** Let \( \xi^{(1)}, \ldots, \xi^{(n)} \in \mathbb{R}^d \) be as in Lemma 7.4. Then, there exist constants \( c_1, c_2, c_3 > 0 \) only depending on \( d \) such that, for all \( r > 0 \) and \( n \in \mathbb{N} \),
\[
\mathbb{P}\left( \text{conv}\left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \not\subset B_r(0) \right) < c_1 \exp\left( -\frac{1}{c_2 r^2 + \frac{c_3}{n}} \right).
\]

**Proof.** Let \( e_1, \ldots, e_d \) be the standard orthonormal basis of \( \mathbb{R}^d \). Pick a constant \( C(d) \) such that for all \( r > 0 \) the cross-polytope \( \text{conv}\{ \pm r C(d)e_j, j = 1, \ldots, d \} \) contains \( B_r(0) \). Then
\[
\mathbb{P}\left( \text{conv}\left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \not\subset B_r(0) \right) \leq \mathbb{P}\left( \varepsilon r C(d)e_j \notin \text{conv}\left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \text{ for some } j = 1, \ldots, d \text{ and } \varepsilon \in \{+1, -1\} \right)
\]
and
\[
\leq 2d \mathbb{P}\left( r C(d)e_1 \notin \text{conv}\left\{ \frac{\xi^{(1)}}{n}, \ldots, \frac{\xi^{(n)}}{n} \right\} \right).
\]
The claim now follows from Lemma 7.4 with \( r C(d) \) in place of \( r \). \( \square \)

**Lemma 7.6.** Let \( \mathbf{x} := (\xi_0, \xi_1, \ldots, \xi_d) \) be a random vector with the uniform distribution on the \( d \)-dimensional half-sphere \( S^d_+ \). Then \( \xi_0 \) has probability density
\[
t \mapsto \frac{2 \Gamma \left( \frac{d+1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{d}{2} \right)} (1 - t^2)^{\frac{d}{2} - 1}, \quad t \in [0, 1]. \tag{7.1}
\]

**Proof.** This follows from the slice integration formula for spheres [3, Corollary A.5], according to which the distribution function of \( \xi_0 \) equals
\[
2 \int_{S^{d-1}} 1_{\{x_0 < t\}} \, \bar{\sigma}(dx) = \frac{2 \omega_d}{\omega_{d+1}} \int_0^t (1 - x^2)^{\frac{d+1}{2}} \, dx, \quad t \in [0, 1],
\]
where \( \bar{\sigma} \) is the normalized spherical Lebesgue measure on \( S^{d-1} \). Differentiation with respect to \( t \) and the definitions of \( \omega_d \) and \( \omega_{d+1} \) yield (7.1). \( \square \)

Recall from (2.5) the definition of the mapping \( \mathcal{P} : S^d_+ \cap \{x_0 > 0\} \to \mathbb{R}^d \).

**Lemma 7.7.** Let \( \mathbf{x} := (\xi_0, \xi_1, \ldots, \xi_d) \) be a random vector with the uniform distribution on the \( d \)-dimensional half-sphere \( S^d_+ \). Then the distribution of the vector \( \mathbf{x}^0 := \mathcal{P}(\mathbf{x}) = (\xi_1/\xi_0, \ldots, \xi_d/\xi_0) \) is regularly varying in \( \mathbb{R}^d \) and we have the vague convergence
\[
n \mathbb{P}\left( n^{-1} \mathbf{x}^0 \in \cdot \right) \xrightarrow{n \to \infty} \nu(\cdot) \tag{7.2}
\]
on \( \mathcal{M}_{\mathbb{R}^d \setminus \{0\}} \), as \( n \to \infty \), where \( \nu \) is a measure on \( \mathbb{R}^d \setminus \{0\} \) with density (2.4) and with \( \gamma = 1 \) and \( c = 2 \).
Proof. From Proposition 2.2, we know that the distribution of $x^0$ is spherically symmetric in $\mathbb{R}^d$. Whence, (7.2) is equivalent to
\[
\lim_{n \to \infty} n \mathbb{P}(n^{-1}||x^0|| > r) = \nu(\{x \in \mathbb{R}^d : ||x|| > r\}) = \frac{2}{\omega_{d+1}} \int_{\{||x|| > r\}} \frac{dx}{||x||^{d+1}}
\]
for every $r > 0$. We have
\[
n \mathbb{P}(n^{-1}||x^0|| > r) = n \mathbb{P}(\xi_1^2 + \cdots + \xi_d^2 > n^2 r^2 \xi_0^2) = n \mathbb{P}(1 - \xi_0^2 > n^2 r^2 \xi_0^2)
\]
\[
= n \mathbb{P}(\xi_0 < (n^2 r^2 + 1)^{-1/2}) \to \frac{2 \Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} r,
\]
as $n \to \infty$, having utilized formula (7.1) in the last passage. It remains to verify that
\[
\frac{2}{\omega_{d+1}} \int_{\{||x|| > r\}} \frac{dx}{||x||^{d+1}} = \frac{2 \Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} r,
\]
This is done by transformation into spherical coordinates:
\[
\frac{2}{\omega_{d+1}} \int_{\{||x|| > r\}} \frac{dx}{||x||^{d+1}} = \frac{2 \omega_d}{\omega_{d+1}} \int_0^\infty \frac{dr}{r^2} = \frac{2 \omega_d}{\omega_{d+1}} \frac{1}{r} = \frac{2 \Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d}{2}\right)} r,
\]
where we used the definition of $\omega_d$. The proof is complete. \qed

The next lemma was used in the proof of Theorem 2.6.

Lemma 7.8. Assume that $(C_n)_{n \in \mathbb{N}_0}$ is a sequence of compact sets in $\mathbb{R}^d \cup \{\infty\} \setminus \{0\}$ such that $C_n \to C_0$, as $n \to \infty$. Assume further that $(\mu_n)_{n \in \mathbb{N}_0} \subset \mathcal{M}_{\mathbb{R}^d \setminus \{0\}}$ is a sequence such that $\mu_n \to \mu_0$, as $n \to \infty$. If $\mu_0(\partial C_0) = 0$, then $\mu_n(C_n) \to \mu_0(C_0)$, as $n \to \infty$.

Proof. Set $f_n(x) := \mathbb{1}_{\{x \in C_n\}}$ for $n \geq 0$. $B_n := \mathbb{R}^d \cup \{\infty\}$ for $n \in \mathbb{N}$ and $B_0 := \mathbb{R}^d \cup \{\infty\} \setminus \partial C_0$. The claim follows from Lemma 15.7.3 in [19]. \qed

We also need the following fact, which was used in the proofs of Theorems 2.4 and 2.8.

Lemma 7.9. Let $(X_n)_{n \in \mathbb{N}_0} \subset K^d$ be a sequence of random compact convex sets with $X_n \Rightarrow X_0$, as $n \to \infty$, and let $E$ be an affine subspace in $\mathbb{R}^d$ of dimension $k \in \{0, 1, \ldots, d-1\}$. Suppose that $\mathbb{P}(E \cap \partial X_n \neq \emptyset, E \cap \text{int}(X_n) = \emptyset) = 0$ for all $n \in \mathbb{N}_0$. Then, as $n \to \infty$,
\[
\mathbb{P}(X_n \cap E = \emptyset) \to \mathbb{P}(X_0 \cap E = \emptyset).
\]

Proof. Set $\mathcal{S}_1 := K^d$, $\mathcal{S}_2 := \{0, 1\}$ as well as $B_n := \{K \in K^d : E \cap \partial K \neq \emptyset, E \cap \text{int}(K) = \emptyset\}$ and $f_n : \mathcal{S}_1 \to \mathcal{S}_2, K \mapsto \mathbb{1}_{\{E \cap \partial K = \emptyset\}}$ for each $n \in \mathbb{N}_0$ (note that $B_n$ and $f_n$ are independent of $n$, but we decided to follow precisely the notation in [19] in order to simplify comparison). By assumption, $\mathbb{P}(X_n \in B_n) = 1$ for all $n \in \mathbb{N}_0$. Moreover, one easily checks that for a sequence $(K_n)_{n \in \mathbb{N}_0}$ with $K_n \in B_n$ for all $n \in \mathbb{N}_0$ such that $K_n \Rightarrow_0 K_0$ one has that $f_n(K_n) \to f_0(K_0)$, as $n \to \infty$. Now, [19, Lemma 15.4.2] implies that
\[
\mathbb{1}_{\{E \cap X_n = \emptyset\}} = f_n(X_n) \Rightarrow_0 f_0(X_0) = \mathbb{1}_{\{E \cap X_0 = \emptyset\}},
\]
as $n \to \infty$. Moreover, since the sequence $(f_n(X_n))_{n \in \mathbb{N}}$ of random variables only takes values in $\mathcal{S}_2 = \{0, 1\}$, it is uniformly integrable and we conclude that, as $n \to \infty$,
\[
\mathbb{P}(E \cap X_n = \emptyset) = \mathbb{E}(f_n(X_n)) \to \mathbb{E}(f_0(X_0)) = \mathbb{P}(E \cap X_0 = \emptyset),
\]
see [20, Lemma 4.11]. This completes the argument. \qed
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