Entanglement swapping theory and beyond

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Abstract

We focus on the theoretical research of entanglement swapping and entanglement swapping chains of pure states and mixed states. We show the general theory of entanglement swapping, through which the final states after entanglement swapping can be obtained. We consider the entanglement swapping of 2-level maximally entangled states without limiting each subsystem to be in the same basis. We also consider the entanglement swapping between d-level maximally entangled states, which is realized by performing a joint measurement on the particles that contain the first particle of the selected entangled states and the particles without involving the first particle in other entangled states. For the entanglement swapping of general pure state, we generalize case of two bipartite general pure states to multi-state cases. Moreover, we propose the entanglement swapping between bipartite general pure states and maximally entangled states. For entanglement swapping chains, we propose the entanglement swapping chains for maximally entangled states, which is realized by performing measurements on multiple particles. We prove the results of entanglement swapping chains of general pure states and maximally entangled states. In addition, we study the entanglement swapping and entanglement swapping chains of mixed states, including X states and mixed maximally entangled states. What is more, we provide a new proof for our previous work [2022, Physica A, 585, 126400]. Finally, we propose a new algorithm for deriving entanglement swapping results, and verify the correctness of the algorithm by comparing it with the entanglement swapping of two Bell states.

Keywords: quantum entanglement, entanglement swapping, pure state, mixed state, maximally entangled state

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1 Introduction

The principles of quantum mechanics are widely believed to reveal the mysteries of nature more profoundly than any other scientific theory. Superposition, entanglement and entanglement swapping are peculiar and striking quantum mechanics phenomena [1, 2]. Suppose that there are several independent entangled states in which each state contains at least two particles, then two new entangled states can be created simultaneously by performing a joint measurement on the particles selected from each entangled state. This process is called entanglement swapping, which has attracted extensive attention since it was discovered [3, 4].

The original idea of entanglement swapping appeared in the quantum teleportation scheme proposed by Bennett et al. in 1992 [5]. Soon after, Zeilinger et al. formally proposed entanglement swapping and completed its experimental realization using parametric down-converters [6]. The basic idea of entanglement swapping is that Alice and Bob share an entangled state, and Bob shares another one with Charlie, beforehand; Then, Bob performs a Bell measurement on the two particles he holds, which eventually enables Alice and Charlie to share a new entangled state. Afterwards, Bose et al. [7] and Bouda et al. [8] generalized entanglement swapping to multi-qubit systems and multi-qudit systems, respectively. Hardy and Song proposed the entanglement swapping chains of d-level bipartite general pure states [9]. Later, Karimipour et al. considered the entanglement swapping between a multi-particle maximally entangled state and Bell states in d-level systems [10]. Entanglement swapping is not only an important way to verify the nonlocality of quantum mechanics, but also plays an important role in quantum computing and quantum information security [3, 11, 12]. A particularly attractive application is that it is the key to construct quantum repeaters, making it possible to construct large-scale quantum networks [3].

This paper mainly focuses on the theoretical research of entanglement swapping, including entanglement swapping and entanglement swapping chains for pure and mixed states. Firstly, we study entanglement swapping between 2-level maximally entangled states without limiting all the subsystems to be in the natural basis, which is enlightening for studying the entanglement swapping between graph states. Secondly, we study entanglement swapping between d-level maximally entangled states, including the case of performing a joint measurement on the selected particles containing the first particle in some entangled states, and the ones without the first particle in the remaining entangled states. Further, we give the formulas for two extreme entanglement swapping cases, that is, measuring the first part of the particles in all states and the last part of the particles in all states. For the entanglement swapping of general pure states, we consider the generalization of the entanglement swapping between two bipartite general pure states to multi-state case. Moreover, we propose the entanglement swapping between general pure states and maximally entangled states. On the entanglement swapping chains, we employ mathematical induction to provide a proof for the results of the entanglement swapping chains of general pure states. We propose entanglement swapping chains of multi-particle maximally entangled states, and also give a proof for the results using mathematical induction. In addition, we consider the entanglement swapping chains for mixed Bell states. What is more, we review and provide a neat proof for our work in Ref. [3]. Finally, we propose a new algorithm for deriving entanglement swapping results. Then, we demonstrate the algorithm by the entanglement swapping between two bipartite entangled states, and derive the results of entanglement swapping between two 2-level Bell states, which are consistent with those obtained through algebraic calculations.

The structure of the rest of this paper is as follows. Sec. 2 presents some basic theory of entanglement swapping. Sec. 3 introduces the measurement operators used for realizing entanglement swapping. Sec. 4 shows the entanglement swapping of pure states. Firstly, we describe the entanglement swapping between maximally entangled states for 2-level systems and d-level systems, respectively. Then, we describe the cases for general pure states. In Sec. 5, we depict the entanglement swapping of mixed states, including X states and mixed maximally entangled states. Sec. 6 shows the entanglement swapping chains for pure states and mixed maximally entangled states. Sec. 7 shows the new proof for the conclusion in Ref. [3]. Sec. 8 demonstrates a new algorithm for deriving entanglement swapping results. This paper is summarized in Sec. 9.

2 Basic theory of entanglement swapping

There are two ways to formulate entanglement swapping. One way is to formulate entanglement swapping directly from the definition of quantum measurements; Indeed, entanglement swapping is realized by measurements. Another way is a pure mathematical process, which can be briefly described as the expansion and combination of polynomials composed of vectors, permutation of vectors, and re-expansion of the polynomials. In what follows,
we would first like to give the basic principles of these two ways for pure states, and give a general theory of using density matrix to derive entanglement swapping results. Then, we give the general theory of entanglement swapping of mixed states from the counterpart of pure states. We will show that entanglement swapping results are determined by the permutation of particle order, that is, entanglement swapping caused by different permutation of particle order are not all the same. At the same time, we only consider, without loss of generality, the cases under the assumption that all subsystems in a entanglement swapping scheme are in the same dimension (we always assume that all subsystems are in $d$-level systems, unless we make a special statement), because the Hilbert space of any subsystem can be embedded in a larger one by local actions [13]: Indeed, a low dimensional system is a special form of a higher one.

\textbf{2.1 Basics of pure states entanglement swapping}

Before entering the topic, let us introduce two lemmas and provide brief proofs,

\textbf{Lemma 1} Suppose that there are $n$ arbitrary pure states containing $m_r$ particles each, denoted as $|\psi_r\rangle|_{r=1}^n$, then their tensor product, $\otimes_{r=1}^n |\psi_r\rangle$, can always be expressed in the following form:

$$\sum_i p_i \otimes_{j=1}^M |l_{ij}\rangle,$$

$$\sum_i |p_i|^2 = 1, \quad M = \sum_{r=1}^n m_r,$$

where $|l_{ij}\rangle$ are single-particle states.

\textbf{Proof} Suppose that $\{|e_1^r, e_2^r, \ldots, e_m^r\rangle|_{1,2,\ldots,m_r}\}$, where the subscripts $1, 2, \ldots, m_r$ denote the particles respectively, is an arbitrary orthonormal basis in the $\mathcal{H}^\otimes_m$ corresponding to $|\psi_r\rangle$, for example, the natural basis

$$\{ |e_1^r, e_2^r, \ldots, e_m^r\rangle|_{1,2,\ldots,m_r} | e_i^r = 0, 1, \ldots, d - 1, \quad i = 1, 2, \ldots, m_r \}.$$  

One can mark $|\psi_r\rangle$ by

$$|\psi_r\rangle := \sum_{e_1^r, e_2^r, \ldots, e_m^r} \lambda_{e_1^r, e_2^r, \ldots, e_m^r} |e_1^r e_2^r \ldots e_m^r\rangle,$$

$$\sum_{e_1^r, e_2^r, \ldots, e_m^r} |\lambda_{e_1^r, e_2^r, \ldots, e_m^r}|^2 = 1,$$

such that one can arrive at

$$\bigotimes_{r=1}^n |\psi_r\rangle = \mathcal{P} \sum_{e_1^1, e_2^1, \ldots, e_m^1} \sum_{e_1^2, e_2^2, \ldots, e_m^2} \cdots \sum_{e_1^n, e_2^n, \ldots, e_m^n} \prod_{r=1}^n \lambda_{e_1^r, e_2^r, \ldots, e_m^r} \bigotimes_{r=1}^n |e_1^r e_2^r \ldots e_m^r\rangle,$$

$$\mathcal{P} = \frac{1}{\sqrt{\sum_{e_1^1, e_2^1, \ldots, e_m^1} \sum_{e_1^2, e_2^2, \ldots, e_m^2} \cdots \sum_{e_1^n, e_2^n, \ldots, e_m^n} \prod_{r=1}^n |\lambda_{e_1^r, e_2^r, \ldots, e_m^r}|^2}}.$$  

\textbf{Here}, the symbol $\mathcal{P}$ is the added probability normalization factor (this symbol will be frequently used in this paper). \textbf{It can be seen that Eq. 4 is consistent with Eq. 1 in form. QED.} 

\textbf{Lemma 2} Suppose that there is a composite pure state $|\ell_1 \ell_2 \cdots \ell_m\rangle$ composed of $m$ single-particle states $|\ell_1\rangle$, $|\ell_2\rangle$, $|\ell_3\rangle$, $|\ell_m\rangle$, then $|S_k (|\ell_1 \ell_2 \cdots \ell_m\rangle)$, where $k = 1, 2, \ldots, m!$ and $S_k (|\ell_1 \ell_2 \cdots \ell_m\rangle)$ refers to an arbitrary permutation of the position of all single-particle states in $|\ell_1 \ell_2 \cdots \ell_m\rangle$, are all the same iff $\ell_i \equiv \ell_j \forall i, j \in \{1, 2, \ldots, m\}$.

\textbf{Proof} The necessity of Lemma 2 is obvious, thus one only need to prove the sufficiency. Since $|S_k (|\ell_1 \ell_2 \cdots \ell_m\rangle)$, $k = 1, 2, \ldots, m!$, are all the same implies that

$$|\ell_i S ((\ell_2 \ell_3 \cdots \ell_m) \setminus \{\ell_i\}) \rangle = |\ell_i S (|\ell_1 \ell_2 \cdots \ell_m\rangle \setminus \{\ell_i\}) \rangle, \quad \forall i, j \in \{1, 2, \ldots, m\},$$

\textbf{in which $S$ (·) represents all permutations, one can get $\ell_i \equiv \ell_j \forall i, j \in \{1, 2, \ldots, m\}$}. QED.
Let us now present the first basic principle for characterizing entanglement swapping.

**Theorem 1** Suppose that there are \(n\) pure states \([|\psi_i\rangle]_{i=1}^{n}\) containing \(m\) particles each, and that \(k\) particles in \([|\psi_i\rangle]\) are selected. According to Lemma 1, one can mark the joint state by

\[
|\Psi\rangle := \sum_i A_i \left| S(t_i t_{i2} \cdots t_{iK}) \right| R(j_1 j_{i2} \cdots j_{iM-K}) \right>,
\]

where \(|S(t_i t_{i2} \cdots t_{iK})\rangle\) and \(|R(j_1 j_{i2} \cdots j_{iM-K})\rangle\) are a permutation of the order of all the selected particles and remaining ones, respectively. Suppose that a measurement operator, selected from the measurement operator set \([|\Phi_j\rangle \langle \Phi_j|]\) constructed by the orthonormal basis \([|\Phi_j\rangle]\) in \(\mathcal{H}_{nk}\), is performed on the selected particles, such that a state in the basis can be obtained. That is, the measurement result is in one of \([|\Phi_j\rangle]\). With the basis, \(|S(t_i t_{i2} \cdots t_{iK})\rangle\) can be expressed as

\[
|S(t_i t_{i2} \cdots t_{iK})\rangle := \sum_j A_{ij} |\Phi_j\rangle,
\]

\[
\sum_j |A_{ij}|^2 = 1,
\]

such that entanglement swapping results, that is, the joint states of the measurement result and the state that the remaining particles collapse onto, are given by

\[
|\bar{\Psi}\rangle := \mathcal{P} \sum_i \sum_j A_{ij} |\Phi_j\rangle |R(j_1 j_{i2} \cdots j_{iM-K})\rangle,
\]

\[
\mathcal{P} = \frac{1}{\sqrt{\Sigma_i \Sigma_j |A_{ij}|^2}}.
\]

Suppose that the selected measurement operator is \(|\Phi_j\rangle \langle \Phi_j|\), then the remaining particles collapse onto the state

\[
|\bar{\Psi}\rangle := \mathcal{P} \sum_j A_{ij} |\Phi_j\rangle |R(j_1 j_{i2} \cdots j_{iM-K})\rangle,
\]

\[
\mathcal{P} = \frac{1}{\sqrt{\Sigma_j |A_{ij}|^2}}.
\]

The entanglement swapping result shown in Eq. 8 is the superposition of all possible states, while the entanglement swapping result \(|\Phi_j\rangle \otimes |\bar{\Psi}\rangle\) is one of these states. Lemma 1 shows that it is feasible to make such an assumption as Eq. 6. It is also feasible to make the hypothesis shown in Eq. 7, since the measurement operator set \([|\Phi_j\rangle \langle \Phi_j|]\) is constructed by the orthonormal basis \([|\Phi_j\rangle]\). It can be seen from Lemma 2 that the entanglement swapping results caused by different permutations of particles are different in a large probability, because the permutation only remains unchanged when \(t_{i1} \equiv t_{i2} \equiv \cdots \equiv t_{iK} = 0, 1, \ldots, d-1\) and \(j_{i1} \equiv j_{i2} \equiv \cdots \equiv j_{iK} = 0, 1, \ldots, d-1\) for given \(i\).

From the definition of quantum measurement [14], entanglement swapping results can also be obtained by

\[
A \otimes B = \frac{\langle M_m \otimes I | \Psi \rangle}{\sqrt{|\Psi\rangle \langle M_m \otimes I | \Psi \rangle}}.
\]

Let us provide more details about Eq. 10. Before doing this, let us introduce the following conclusion:

**Lemma 3** Suppose that \([|\epsilon^1_1\rangle, |\epsilon^1_2\rangle, \ldots, |\epsilon^m_1\rangle]\), are an arbitrary orthonormal basis in \(\mathcal{H}\), respectively, then the \(m\)-particle state set

\[
\{|\tilde{\epsilon}_1 \tilde{\epsilon}_2 \cdots \tilde{\epsilon}_m\rangle \left| \tilde{\epsilon}_r\rangle \in [|\epsilon^r_i\rangle], \right. \quad r = 1, 2, \ldots, m\},
\]

construct an orthonormal basis in \(\mathcal{H}^{\otimes m}\).
Proof \( |\hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_n\rangle\) is complete and normalized since
\[
\sum_{\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_n} |\hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_n\rangle \langle \hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_n| = \sum_{\hat{\psi}_1, \hat{\psi}_2, \ldots, \hat{\psi}_n} \bigotimes_{i=1}^{m} |\hat{\psi}_i\rangle \langle \hat{\psi}_i| = \bigotimes_{i} I = I,
\]
where \( I \) and \( \bar{I} \) are the identity matrix in \( \mathcal{H} \) and \( \mathcal{H}^{\text{out}} \), respectively. Due to
\[
\langle \hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_n| \hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_n\rangle = \prod_{i=1}^{m} \langle \hat{\psi}_i| \hat{\psi}_i\rangle = \prod_{i=1}^{m} \delta_{\psi_i\phi_i},
\]
Eq. 17 can be transformed into
\[
|\hat{\psi}_1 \hat{\psi}_2 \cdots \hat{\psi}_n\rangle = \prod_{i} I = I.
\]

As Theorem 1, let us mark
\[
|\phi\rangle := \sum_{j} \lambda_j |\mathcal{A}_j\rangle |\mathcal{B}_j\rangle,
\]
where \( \lambda_j \) is the identity matrix in \( I \) and \( \lambda_j \) are the same or orthogonal. QED.

According to Lemma 1, the state \(|\psi\rangle\) can be wrote in the same form as Eq. 6. For example, let us suppose that
\[
|\psi\rangle := \sum_{i} \lambda_i |\mathcal{A}_i\rangle |\mathcal{B}_i\rangle,
\]
such that the probability of getting the measurement result \( m \) is
\[
p(m) = \sqrt{\langle \psi | \left( M_m^o M_m \otimes I \right) |\psi\rangle} = \sqrt{\left( \sum_{i} \lambda_i |\mathcal{A}_i\rangle |\mathcal{B}_i\rangle \right)^\dagger \left( \sum_{i} \lambda_i |\mathcal{A}_i\rangle |\mathcal{B}_i\rangle \right)} = \sqrt{\sum_{i} |\lambda_i|^2 \langle \mathcal{A}_i | M_m^o M_m | \mathcal{A}_i \rangle \langle \mathcal{B}_i | \mathcal{B}_i \rangle} = \sum_{i} |\lambda_i|^2 \langle \mathcal{A}_i | M_m^o M_m | \mathcal{A}_i \rangle.
\]
(14)

Here, Lemma 2 is used in the derivation of the above equation. We can now arrive at a clearer form than Eq. 10 as follow:
\[
A \otimes B = \frac{\sum_{i} \lambda_j M_m |\mathcal{A}_i\rangle |\mathcal{B}_i\rangle}{\sqrt{\sum_{i} |\lambda_i|^2 \langle \mathcal{A}_i | M_m^o M_m | \mathcal{A}_i \rangle}}.
\]
(15)

Theorem 2 can also be described with density matrix, both of which are essentially the same.

**Theorem 3** Let \( \rho := |\psi\rangle \langle \psi| \), and mark the measurement result as \( \rho_A \) and the state that the remaining particles collapses onto as \( \rho_B \), then entanglement swapping results are given by
\[
\rho_A \otimes \rho_B = \frac{(M_m \otimes I) \rho (M_m \otimes I)^\dagger}{\text{Tr} \left( (M_m \otimes I)^\dagger (M_m \otimes I) \rho \right)} = \frac{(M_m \otimes I) \rho (M_m \otimes I)^\dagger}{\text{Tr} \left( (M_m \otimes I)^\dagger (M_m \otimes I) \rho \right)}.
\]
(16)

### 2.2 Basics of mixed states entanglement swapping

In what follows we consider basics for mixed states entanglement swapping. Suppose that \( \rho_{\text{mix}} \) is the mixture of the pure states \( \rho_i \) with the probabilities \( \lambda_i \), where \( \lambda_i := |\varphi_i\rangle \langle \varphi_i| \), such that
\[
\rho_{\text{mix}} = \sum_{i} \lambda_i |\varphi_i\rangle \langle \varphi_i|,
\]
(17)
\[
\sum_{i} \lambda_i = 1.
\]

As Theorem 1, let us mark \( |\varphi_i\rangle \) after swapping particles by
\[
|\tilde{\varphi}_i := \sum_{j} \lambda_j |S_{ij}\rangle |R_{ij}\rangle,
\]
(18)
such that Eq. 17 can be transformed into
\[
\tilde{\rho}_{\text{mix}} = \sum_{i,j,f} \lambda_i \lambda_j |S_{ij}\rangle \langle S_{ij}| \otimes |R_{ij}\rangle \langle R_{ij}|,
\]
(19)
As Eq. 7, let \( |S_{ij} \rangle := \sum_k A_{ijk} |\phi_k \rangle \), then we can get the entanglement swapping results as follows:

\[
\hat{\rho}_{\text{mix}} = P \sum_{i,j,k} A_i A_j A_{jk} |\psi_{ijk} \rangle \langle \psi_{ijk}| = \frac{1}{|\psi_{ijk} \rangle \langle \psi_{ijk}|} \sum_{i,j,k} A_i A_j A_{jk} |\psi_{ijk} \rangle \langle \psi_{ijk}|
\]

which means that if the measurement result is assumed to be \( |\phi_j \rangle \), then the remaining particles collapse onto

\[
P \sum_{i,j,k} A_i A_j A_{jk} |\psi_{ijk} \rangle \langle \psi_{ijk}| = \frac{1}{|\psi_{ijk} \rangle \langle \psi_{ijk}|} \sum_{i,j,k} A_i A_j A_{jk} |\psi_{ijk} \rangle \langle \psi_{ijk}|
\]

The mixed states entanglement swapping results can also be derived from Theorem 3. Substituting \( \hat{\rho}_{\text{mix}} \) into Eq. 16, we can arrive at

\[
\rho_A \otimes \rho_B = \frac{\sum_{i,j,k} A_i A_j A_{jk} M_m |S_{ij} \rangle \langle S_{ij}| M_m \otimes |R_{ij} \rangle \langle R_{ij}|}{\sum_{i,j,k} A_i |l_{ij} \rangle \langle l_{ij}| \text{Tr} (M_m M_m |S_{ij} \rangle \langle S_{ij}|)}
\]

### 3 Measurement operators

In this section, we will introduce the measurement operators that will be used frequently to realize entanglement swapping considered in this paper. Before doing this, let us introduce the d-level m-particle maximally entangled states \([8, 10]\),

\[
|\phi(u_1, u_2, \ldots, u_m) \rangle := \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \zeta^{iu_1} |l, l \oplus u_2, l \oplus u_3, \ldots, l \oplus u_m \rangle
\]

\[
u_1, u_2, \ldots, u_m = 0, 1, \ldots, d - 1,
\]

where \( \zeta = e^{2\pi i/d} \) and the symbol \( \oplus \) denotes addition modulo \( d \) throughout this paper. These states form a orthonormal basis in the d-level Hilbert space \( \mathcal{H}^{dm} \) since

\[
\langle \phi(u_1, u_2, \ldots, u_m) | \phi(u_1', u_2', \ldots, u_m') \rangle = \prod_{i=1}^{m} \delta_{u_i, u_i'},
\]

\[
\sum_{u_1, u_2, \ldots, u_m = 0}^{d-1} |\phi(u_1, u_2, \ldots, u_m) \rangle \langle \phi(u_1, u_2, \ldots, u_m)| = I,
\]

where \( \delta \) is Kronecker delta and \( I \) is the identity matrix in \( \mathcal{H}^{dm} \). From Eq. 23, one can get the following relations:

\[
|l_1, l_2, \ldots, l_m \rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \zeta^{l_i} |l, l \oplus l_1, l \oplus l_2, \ldots, l \oplus l_m \rangle
\]

which form a complete and orthonormal basis in \( \mathcal{H}^{d^2} \). Another special case is provided by limiting \( u_1, u_2, \ldots, u_m = 0, 1 \) in \( |\phi(u_1, u_2, \ldots, u_m) \rangle \), which is given by

\[
|\phi(u_1, u_2, \ldots, u_m) \rangle := \frac{1}{\sqrt{2}} ([0, u_2, u_3, \ldots, u_m] + (-1)^{u_1} [1, \bar{u}_2, \bar{u}_3, \ldots, \bar{u}_m]),
\]

\[
u_1, u_2, \ldots, u_m = 0, 1,
\]
where a bar over a bit indicates the logical negation of the bit throughout this paper (e.g., if \( u_2 = 0 \), \( \bar{u}_2 = 1 \), otherwise \( \bar{u}_2 = 0 \)). These states are often called multi-particle GHZ states. In this case, one can get

\[
|u_1, u_2, \ldots, u_m\rangle = \frac{1}{\sqrt{2}} \left( |\psi(0, u_2, u_3, \ldots, u_m)\rangle + (-1)^{u_1} |\psi(1, u_2, u_3, \ldots, u_m)\rangle \right).
\]

(28)

It is easy to check that

\[
|\psi(0, 0)\rangle \equiv |\phi(0, 0)\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad |\psi(0, 1)\rangle \equiv |\phi(0, 1)\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle),
\]

\[
|\psi(1, 0)\rangle \equiv |\phi(1, 0)\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \quad |\psi(1, 1)\rangle \equiv |\phi(1, 1)\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle),
\]

and that

\[
|\psi(0, 0, 0)\rangle \equiv |\phi(0, 0, 0)\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle), \quad |\psi(0, 0, 1)\rangle \equiv |\phi(0, 0, 1)\rangle = \frac{1}{\sqrt{2}} (|001\rangle + |110\rangle),
\]

\[
|\psi(0, 1, 0)\rangle \equiv |\phi(0, 1, 0)\rangle = \frac{1}{\sqrt{2}} (|010\rangle + |101\rangle), \quad |\psi(0, 1, 1)\rangle \equiv |\phi(0, 1, 1)\rangle = \frac{1}{\sqrt{2}} (|011\rangle + |100\rangle),
\]

\[
|\psi(1, 0, 0)\rangle \equiv |\phi(1, 0, 0)\rangle = \frac{1}{\sqrt{2}} (|000\rangle - |111\rangle), \quad |\psi(1, 0, 1)\rangle \equiv |\phi(1, 0, 1)\rangle = \frac{1}{\sqrt{2}} (|001\rangle - |110\rangle),
\]

\[
|\psi(1, 1, 0)\rangle \equiv |\phi(1, 1, 0)\rangle = \frac{1}{\sqrt{2}} (|010\rangle - |101\rangle), \quad |\psi(1, 1, 1)\rangle \equiv |\phi(1, 1, 1)\rangle = \frac{1}{\sqrt{2}} (|011\rangle - |100\rangle),
\]

(30)

which are the familiar Bell states and GHZ states, respectively.

Let us now introduce the measurement operators. With the d-level maximally entangled states introduced above, one can construct the measurement operators

\[
\overline{M} := |\phi\rangle \langle \phi|, \quad |\phi\rangle \in \{|\phi(u_1, u_2)\rangle\},
\]

\[
\overline{M} := |\overline{\phi}\rangle \langle \overline{\phi}|, \quad |\overline{\phi}\rangle \in \{|\phi(u_1, u_2, \ldots, u_m)\rangle\},
\]

(31)

and

\[
\overline{M} := |\overline{\psi}\rangle \langle \overline{\psi}|, \quad |\overline{\psi}\rangle \in \{|\phi(u_1, u_2)\rangle\},
\]

\[
\overline{M} := |\overline{\psi}\rangle \langle \overline{\psi}|, \quad |\overline{\psi}\rangle \in \{|\phi(u_1, u_2, \ldots, u_m)\rangle\},
\]

(32)

which can be used to realize the entanglement swapping of d-level entangled states and 2-level ones, respectively. Generally, \( \overline{M} \) and \( \overline{M} \) are called Bell measurement operator and multi-particle GHZ measurement operator, respectively.

Let us then introduce another class of maximally entangled states, which has the form

\[
|\varphi(w_1, w_2, \ldots, w_m)\rangle := \frac{1}{\sqrt{2}} \left( |e_1', w_2, w_3, \ldots, w_m\rangle + (-1)^{w_1} |e_2', \bar{w}_2, \bar{w}_3, \ldots, \bar{w}_m\rangle \right),
\]

(33)

\[
w_1 \in \{0, 1\}, \quad w_k \in \{|e_1'\rangle, |e_2'\rangle\}, \quad \bar{w}_k \in \{|e_1'\rangle, |e_2'\rangle\} \setminus \{w_k\}, \quad k = 2, 3, \ldots, m, \quad j = 1, 2, \ldots, m,
\]

where \( \{|e_1'\rangle, |e_2'\rangle\} \) is an arbitrary complete and orthonormal basis in \( \mathcal{H}_2 \). For simplicity, we call these states CAT states hereinafter. It can be seen that they are complete and orthonormal since we have

\[
\langle \varphi(w_1, w_2, \ldots, w_m) | \varphi(w_1', w_2', \ldots, w_m') \rangle = \frac{1}{4} (\langle e_1 | e_1' \rangle \prod_{i=2}^{m} \langle w_i | w_i' \rangle + (-1)^{w_1+w_1'} \langle e_2 | e_2' \rangle \prod_{i=2}^{m} \langle \bar{w}_i | \bar{w}_i' \rangle) = \prod_{i=1}^{m} \delta_{w_i,w_i'},
\]

(34)

and

\[
\sum_{w_1,w_2,\ldots,w_m} |\varphi(w_1, w_2, \ldots, w_m)\rangle \langle \varphi(w_1, w_1, w_2, \ldots, w_m) | = \sum_{w_1,w_2,\ldots,w_m} (|\varphi(0, w_2, \ldots, w_m)\rangle \langle \varphi(0, w_1, w_2, \ldots, w_m) | + |\varphi(1, w_2, \ldots, w_m)\rangle \langle \varphi(1, w_1, w_2, \ldots, w_m) |)
\]

(34)
In particular, one can construct the measurement operators
corresponding to the states

thus we can construct the measurement operators

where

between

are given by

level pure states,

consider special cases including the entanglement swapping of general pure states, which was studied by Hardy et
al. by proposing entanglement swapping chains [9], and the entanglement swapping of some maximally entangled
states such as Bell states. As mentioned before, since a low dimensional system can be embedded into a higher
one, we assume that all subsystems are in a natural basis with same dimensions. At the same time, we would only
like to consider the entanglement swapping of bipartite generalized pure states, because the set composed of all
possible states of each particle is equivalent. Considering that the 2-level system is a special form of the d-level
system and its importance in quantum information processing, we will devote a lot of attention to the formulation
of the entanglement swapping in 2-level systems.

4 Pure states entanglement swapping

From this section, we begin to study entanglement swapping. As shown above, due to entanglement swapping
results caused by different permutation of particle order are not all the same, we would only like to consider one
kind of permutation in an entanglement swapping scheme in this paper (unless a special statement is made). We
will start with the entanglement swapping of generalized pure states for any number of quantum systems, and then
consider special cases including the entanglement swapping of general pure states, which was studied by Hardy et
al. by proposing entanglement swapping chains [9], and the entanglement swapping of some maximally entangled
states such as Bell states. As mentioned before, since a low dimensional system can be embedded into a higher
one, we assume that all subsystems are in a natural basis with same dimensions. At the same time, we would only
like to consider the entanglement swapping of bipartite generalized pure states, because the set composed of all
possible states of each particle is equivalent. Considering that the 2-level system is a special form of the d-level
system and its importance in quantum information processing, we will devote a lot of attention to the formulation
of the entanglement swapping in 2-level systems.

4.1 Entanglement swapping for generalized pure states

Before entering the description of entanglement swapping between pure states, let us introduce the following d-
level pure states,

\[ |P_{\text{pure}}\rangle := \sum_{l_1, l_2=0}^{d-1} A_{l_1,l_2} |l_1,l_2\rangle, \]

and we call this kind of states generalized pure state for the convenience of addressing. The entanglement swapping
between \( n \) generalized pure states, denoted as \( \bigotimes_{r=1}^{2r} |P_{\text{pure}}\rangle \), is realized by performing the measurement operator \( \tilde{M} \) on the first
particle in each state, are given by

\[
\bigotimes_{r=1}^{2r} |P_{\text{pure}}\rangle \Rightarrow \sum_{l_1, l_2=0}^{d-1} \prod_{r=1}^{n} A_{l_1,l_2} \left| l_1', l_2' \right\rangle_{2r-1,2r},
\]

\[
= \sum_{l_1, l_2=0}^{d-1} \prod_{r=1}^{n} A_{l_1,l_2} \left| l_1', l_2' \right\rangle_{2r-1,2r} = \sum_{l_1, l_2=0}^{d-1} \prod_{r=1}^{n} A_{l_1,l_2} \left| l_1', l_2' \right\rangle_{2r-1,2r}.
\]

where

\[
P = \frac{1}{\sqrt{\sum_{l_1, l_2=0}^{d-1} \prod_{r=1}^{n} A_{l_1,l_2}^2}}.
\]
and the symbol $\Rightarrow$, which will be frequently used in this paper, represents the swapping of the particles. It can be seen that when the measurement result is $|\phi(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)\rangle$, the remaining particle collapse onto

$$
P = \frac{d}{\sqrt{\sum_{r=0}^{d-1} \lambda_r^4 \prod_{n=2}^d \lambda_r^{4(2^n - 1)}}}.
$$

When $n = 2$, Eq. 39 reduces to the entanglement swapping between two generalized pure states, which is given by

$$
\prod_{r=1}^2 |\mathcal{P}_{\text{pure}}\rangle_{2r-1,2r} \rightarrow \mathcal{P} = \frac{\lambda_1}{\sqrt{\sum_{r=1}^2 \lambda_1^4 \prod_{n=2}^d \lambda_n^{4(2^n - 1)}}}.
$$

As a simple special case of the entanglement swapping shown above, the entanglement swapping between two general pure states, which was considered in Ref. [9], can be expressed as

$$
\left|\mathcal{P}_1\right\rangle \otimes \left|\mathcal{P}_2\right\rangle_{1,2} \rightarrow \mathcal{P} = \frac{1}{\sqrt{\sum_{r=1}^2 \lambda_r^4 \prod_{n=2}^d \lambda_r^{4(2^n - 1)}}}.
$$

Let us now turn our attention to the entanglement swapping between 2-level systems. Indeed, by limiting $d = 2$ in Eq. 39 and Eq. 42, one can get, in a direct way, the formulas for the entanglement swapping between 2-level generalized pure states and the one between 2-level general pure states, respectively. In what follows we would like to take the entanglement swapping of general pure states as an example, as the derivation process shown in Ref. [3], to derive the entanglement swapping results through the direct expansion and combination of polynomials. Such a derivation process is more complicated than Eq. 42, but it can more specifically show the entangled states generated by entanglement swapping. Let us assume that there are $n$ two-level general pure states and mark them by

$$
\left|\mathcal{P}_{n}^{\prime}\right\rangle = A_r^1 |00\rangle_{2r-1,2r} + A_r^2 |11\rangle_{2r-1,2r},
$$

such that the entanglement swapping between them, realized by performing $\mathcal{M}$ on the first particle in each state, can be expressed as

$$
\prod_{r=1}^n |\mathcal{P}_{n}^{\prime}\rangle_{2r-1,2r} = \prod_{r=1}^n (A_r^1 |00\rangle_{2r-1,2r} + A_r^2 |11\rangle_{2r-1,2r})
\prod_{r=1}^{n-1} A_r^1 (|00\rangle_{2r-1,2r} + |11\rangle_{2r-2,2r})
\prod_{r=1}^{n-2} (|00\rangle_{2r-1,2r} + |11\rangle_{2r-2,2r})
\cdots
\prod_{r=1}^1 (|00\rangle_{2r-1,2r} + |11\rangle_{2r-2,2r}).
$$
As before, let us assume that there are \( m \) quantum states (see Eq. 23). As before, let us assume that there are \( m \) states. As before, let us assume that there are \( m \) states. In what follows, we focus on the research of the entanglement swapping between \( d \)-level maximally entangled states. As before, we will first consider the entanglement swapping between \( d \)-level systems, and then transition to the case of 2-level systems.

4.2 Entanglement swapping for maximally entangled states

In what follows, we present the entanglement swapping between two 2-level general pure states, presented in Ref. [9], can be easily derived and described by

\[
|\psi_{12}^H\rangle \otimes |\psi_{23}^H\rangle \rightarrow |\tilde{\phi}(0,0)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |00\rangle + |\lambda_2|^2 |11\rangle \right)_{24} + |\tilde{\phi}(0,1)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |01\rangle + |\lambda_2|^2 |10\rangle \right)_{24} + |\tilde{\phi}(1,0)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |00\rangle - |\lambda_2|^2 |11\rangle \right)_{24} + |\tilde{\phi}(1,1)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |01\rangle - |\lambda_2|^2 |10\rangle \right)_{24},
\]

where

\[
\mathcal{P} = \frac{1}{\sqrt{\sum_{\Sigma=1}^{2n-1} (|\tilde{\lambda}_\Sigma|^2 + |\tilde{\lambda}_{\Sigma+1}|^2)}},
\]

(46)

Then, the entanglement swapping between two 2-level general pure states, presented in Ref. [9], can be easily derived and described by

\[
|\psi_{12}^H\rangle \otimes |\psi_{23}^H\rangle \rightarrow |\tilde{\phi}(0,0)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |00\rangle + |\lambda_2|^2 |11\rangle \right)_{24} + |\tilde{\phi}(0,1)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |01\rangle + |\lambda_2|^2 |10\rangle \right)_{24} + |\tilde{\phi}(1,0)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |00\rangle - |\lambda_2|^2 |11\rangle \right)_{24} + |\tilde{\phi}(1,1)\rangle_{13} \otimes \frac{\mathcal{P}}{\sqrt{2}} \left( |\lambda_1|^2 |01\rangle - |\lambda_2|^2 |10\rangle \right)_{24},
\]

where

\[
\mathcal{P} = \frac{1}{\sqrt{\sum_{\Sigma=1}^{2n-1} (|\tilde{\lambda}_\Sigma|^2 + |\tilde{\lambda}_{\Sigma+1}|^2)}},
\]

(46)

Note here that \( |\tilde{\phi}(0,0)\rangle, |\tilde{\phi}(0,1)\rangle, |\tilde{\phi}(1,0)\rangle, |\tilde{\phi}(1,1)\rangle \) are Bell states (see Eq. 29).

4.2 Entanglement swapping for maximally entangled states

In what follows, we will first consider the entanglement swapping between \( d \)-level systems, and then transition to the case of 2-level systems.

For \( d \)-level systems, we will consider the entanglement swapping realized by performing joint measurement on the particles containing the first particle in some entangled states and the the particles without the first particles in the remaining entangled states. Then we turn our attention to special cases, including the entanglement swapping schemes achieved by measuring the particles containing the first particle in all states, and the scheme achieved by measuring the particles without containing the first particle of all states, which was studied in Ref. [8]. Several other special cases, such as the entanglement swapping between a \( d \)-level maximally entangled state and a \( d \)-level Bell state [10], and the entanglement swapping between \( d \)-level Bell states [8], will also be introduced. Finally, the entanglement swapping between generalized \( d \)-level GHZ states is presented.

For 2-level systems, we will consider the entanglement swapping between CAT states. These states are a variant of the multi-particle GHZ states. The main feature that distinguishes them from GHZ states is that not all subsystems are confined to the natural basis. In other words, all subsystems are on different bases, such as natural basis \([0], [1]\) and physical basis \([\pm], [\mp]\). We will then present the entanglement swapping for some other special 2-level maximally entangled states, such as cluster states.

4.2.1 Entanglement swapping of \( d \)-level maximally entangled states

Let us now enter into the consideration of the entanglement swapping between the multi-particle maximally entangled states (see Eq. 23). As before, let us assume that there are \( n \) multi-particle maximally entangled states containing \( m_1, m_2, \ldots, m_n \) particles each, and denote them as \( \left\{ |\phi(u_{r1}, u_{r2}, \ldots, u_{rn})\rangle \right\}_{r=1}^{n} \). Without losing generality, let
us select the first $k_r$ particles in $\left|\phi \left( u'_1, u'_2, \ldots, u'_m \right) \right>$ for $r = 1, 2, \ldots, t$, and the last $k_r$ particles in $\left|\phi \left( u'_1, u'_2, \ldots, u'_m \right) \right>$ for $r = t + 1, t + 2, \ldots, n$. Assuming that the measurement operator $\tilde{M}$ is performed on the selected particles, we have

$$\bigotimes_{r=1}^{n} \left|\phi \left( u'_1, u'_2, \ldots, u'_m \right) \right>$$

$$= \frac{1}{d^{m/2}} \sum_{l_1, l_2, \ldots, l_m=0}^{d-1} \xi_{\sum_{r=1}^{m} l_r}^{m} \left| l_1, l_2, \ldots, l_m \oplus l_1, l_2, \ldots, l_m \right>_{1,2,\ldots,m},$$

$$= \frac{1}{d^{m/2}} \sum_{l_1, l_2, \ldots, l_m=0}^{d-1} \xi_{\sum_{r=1}^{m} l_r}^{m} \left| l_1, l_2, \ldots, l_m \oplus l_1, l_2, \ldots, l_m \right> \bigotimes_{r=1}^{n} \left| l_r \oplus u'_{m-k_r+1}, l_r \oplus u'_{m-k_r+2}, \ldots, l_r \oplus u'_m \right>$$

$$= \frac{1}{d^{m/2}} \sum_{l_1, l_2, \ldots, l_m=0}^{d-1} \xi_{\sum_{r=1}^{m} l_r}^{m} \left| l_1, l_2, \ldots, l_m \oplus l_1, l_2, \ldots, l_m \right> \bigotimes_{r=1}^{n} \left| l_r \oplus u'_{m-k_r+1}, l_r \oplus u'_{m-k_r+2}, \ldots, l_r \oplus u'_m \right>$$

$$\bigotimes_{r=1}^{n} \left| \phi \left( v'_1, v'_2, \ldots, v'_m, v''_1, v''_2, \ldots, v''_m \right) \right>$$

(48)

where

$$v'_j = u'_1, \quad j = 2, 3, \ldots, k_1,$$

$$v'_j = v'_1 \oplus u'_j, \quad i = 2, 3, \ldots, t, \quad j = 2, 3, \ldots, k_i,$$

$$v'_j = v'_1 \oplus u'_{m-k_i+1} \oplus u'_{m-k_i+2} \ldots, v'_1 \oplus u'_{m-k_i+1} \oplus u'_{m-k_i+2} \ldots, v'_1 \oplus u'_m, \quad i = 2, 3, \ldots, k_i,$$

$$v'_j = \tilde{\phi} u'_j \oplus v'_1,$$

$$v''_j = u'_{k_i+j} \oplus u'_{k_i+j}, \quad j = 2, 3, \ldots, m_1 - k_i,$$

$$v''_j = v'_1 \oplus u'_{k_i+j} \oplus u'_{k_i+j}, \quad i = 1, 2, \ldots, t, \quad j = 1, 2, \ldots, m_i - k_i,$$

$$v''_j = v'_1 \oplus u'_{k_i+j} \oplus u'_{m-k_i+1}, \quad i = t + 1, t + 2, \ldots, n, \quad j = 2, 3, \ldots, m_i - k_i.$$

(49)

When $m = 2$, the above entanglement swapping scheme reduce to the case of d-level Bell states, which is given by

$$\bigotimes_{r=1}^{n} \left|\phi \left( u'_1, u'_2 \right) \right>$$

$$= \frac{1}{d^{m/2}} \sum_{l_1, l_2=0}^{d-1} \xi_{\sum_{r=1}^{2} l_r}^{2} \left| l_1, l_2 \right>_{1,2}$$

$$\Rightarrow \frac{1}{d^{m/2}} \sum_{l_1, l_2=0}^{d-1} \xi_{\sum_{r=1}^{2} l_r}^{2} \left| l_1, l_2, l_1 \oplus u''_1, l_2 \oplus u''_1, \ldots, l_n \oplus u''_1 \right>_{1,2,\ldots,n}$$

$$= \frac{1}{d^{m/2}} \sum_{l_1, l_2=0}^{d-1} \xi_{\sum_{r=1}^{2} l_r}^{2} \left| l_1, l_2, l_1 \oplus u''_1, l_2 \oplus u''_1, \ldots, l_n \oplus u''_1 \right> \bigotimes_{r=1}^{n} \left| \phi \left( v_1, v_2, v_3, v_4 \right) \right>$$

$$\bigotimes_{r=1}^{n} \left| \phi \left( v_1, v_2, v_3, v_4 \right) \right>$$

(50)

There are two extreme cases included in the entanglement swapping scheme presented in Eq. 48. One is to measure the first $k_r$ particles in $\left|\phi \left( u'_1, u'_2, \ldots, u'_m \right) \right>$, which has not been considered in the literature. The other is to measure the last $k_r$ particles in $\left|\phi \left( u'_1, u'_2, \ldots, u'_m \right) \right>$, which was proposed in Ref. [8] and organized in Ref. [3]. Let us first characterize the former. Suppose that the measurement operator $\tilde{M}$ is performed on the first $k_r$ particles in each state, then the entanglement swapping can be expressed as

$$\bigotimes_{r=1}^{n} \left|\phi \left( u'_1, u'_2, \ldots, u'_m \right) \right>$$
Assume that is performed on the first particle in each state, one can get
\[ b \]
From Eqs. 51 and 53, one can derive the formulas of the entanglement swapping between a d-level maximally entangled state and a d-level Bell state, which is proposed in Ref. [10]. Assume that the measurement operator \( \hat{M} \) is performed on the first particle in each state, one can get
\[ \phi(u_1, u_2, \ldots, u_m) \]
Assume that \( \hat{M} \) is performed on the two particles without involving the first particle in each state, one can get
\[ \phi(u_1, u_2, \ldots, u_m) \otimes \phi(u_1', u_2') \]
In Eqs. 55 and 56, the entanglement swapping between d-level Bell states can be derived as follows [8, 10].

\[
\frac{1}{d} \sum_{j_1,j_2=0}^{d-1} \phi(u_1 \oplus u_2^2 \oplus v_1, u_1 \oplus u_2 \oplus v_2) \left| \phi(u_1 \oplus u_2 \oplus u_2^2 \oplus v_2) \right\rangle \text{on one of the particles in a d-level multi-particle maximally entangled state and}
\]

Let us assume that there are \( n \) generalized GHZ states containing \( m_1, m_2, \ldots, m_n \) particles each, and mark them by \( |\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle \) respectively. Without losing generality, let us assume that the measurement operator \( \widetilde{M} \) is performed on the first \( k_i \) particles in \( |\psi_i\rangle \), then the entanglement swapping is given by

\[
\otimes_{r=1}^{n} |\psi_i\rangle = \frac{1}{d^{\frac{n}{2}}} \sum_{l_1, \ldots, l_n=0}^{d-1} \otimes_{r=1}^{n} |l_1, l_2, \ldots, l_n\rangle_{1,2,\ldots,m_i}
\]

\[
\Rightarrow \frac{1}{d^{\frac{n}{2}}} \sum_{l_1, \ldots, l_n=0}^{d-1} \otimes_{r=1}^{n} |l_1, l_2, \ldots, l_n\rangle_{1,2,\ldots,k_i} \otimes_{r=1}^{n} |l_{k_i+1}, \ldots, l_{k_i+k_r+2,\ldots,m_i}|
\]

\[
= \frac{1}{d^{\frac{n}{2}}} \sum_{v_1^1, v_2^1, \ldots, v_1^n, v_2^n=0}^{d-1} \left| \phi(v_1^1, v_2^1, \ldots, v_1^1, v_2^1, \ldots, v_1^n, v_2^n, \ldots, v_1^n) \right\rangle \otimes \left| \phi(d - v_1^1, v_2^1, \ldots, v_1^1, v_2^1, \ldots, v_1^n, v_2^n, \ldots, v_1^n) \right\rangle,
\]

where

\[
v_j^1 = 0, \quad j = 2, 3, \ldots, k_i,
\]

\[
v_1^1 = v_2^1 = \cdots = v_{k_i}^1, \quad i = 2, 3, \ldots, n.
\]

It is easy to derive the the entanglement swapping formula for two bipartite generalized GHZ states,

\[
\otimes_{r=1}^{2} |\psi_i\rangle_{2r-1,2r} \rightarrow \sum_{u,v=0}^{d-1} \left| \phi(u,v) \right\rangle_{1,3} \left| \phi(d - u, v) \right\rangle_{2,4},
\]

which is realized by performing the measurement operator \( \widehat{M} \) on the first particle in each state.

In what follows, we would like to review an interesting entanglement swapping scheme proposed in Ref. [10], which is realized by performing \( \widehat{M} \) on one of the particles in a d-level multi-particle maximally entangled state and one particle in one state of d-level Bell states in turn. We employ mathematical induction, which will be frequently adopted in what follows, to provide a proof for the entanglement swapping results.
Without losing generality, let us assume that there is a $d$-level \((n+1)\)-particle maximally entangled state and mark it by $|\phi(u_1^0, u_1^1, \ldots, u_1^n)\rangle$, and that there are $n$ $d$-level Bell states, denoted as $|\phi(u_1^0, u_2^1)\rangle, |\phi(u_2^0, u_2^2)\rangle, \ldots, |\phi(u_n^0, u_n^2)\rangle$, respectively. Let us assume that $\mathcal{M}$ is performed on the particle with the label $u_1^0$ in $|\phi(u_1^0, u_1^1, \ldots, u_1^n)\rangle$ and the one with the label $u_1^1$ in $|\phi(u_1^0, u_1^1, \ldots, u_1^n)\rangle$ for $r = 1, 2, \ldots, n$, and that the measurement results are $|\phi(v_1^1, v_2^1)\rangle, |\phi(v_1^2, v_2^2)\rangle, \ldots, |\phi(v_1^n, v_2^n)\rangle$, respectively, then the remaining particles are projected onto

$$
|\phi(u_1^0, u_1^1, u_2^0, u_2^1, u_2^2, v_1^0, v_1^1, v_1^2, v_2^0, v_2^1, v_2^2, \ldots, u_n^0, u_n^1, u_n^2, w_1^0, w_1^1, w_1^2, \ldots, w_n^0, w_n^1, w_n^2)\rangle,
$$

(61)

The conclusion can be demonstrated by mathematical induction as follows.

**Proof** From Eq. 56, the entanglement swapping results after the first measurement are given by

$$
\frac{1}{d} \sum_{v_1^1, v_1^2=0}^{d-1} \xi^{(d^2-1)(u_1^i-1)} \left|\left| \phi(u_1^0, u_1^1, u_1^0, u_1^0, u_1^2, u_1^1, u_1^1, u_1^0, u_1^2, u_2^0, u_2^1, u_2^2, \ldots, u_n^0, u_n^1, u_n^2, v_1^0, v_1^1, v_1^2, v_2^0, v_2^1, v_2^2, \ldots, v_n^0, v_n^1, v_n^2)\right|\right| \right| \phi(v_1^1, v_1^2)\rangle.
$$

(62)

The entanglement swapping results after the second measurement are given by

$$
\frac{1}{d} \sum_{l_1, l_2=0}^{d-1} \xi^{(d^2-1)(u_1^i-1)+1} \left|\left| l_1, l_2, l_1, l_2, l_1, u_1^0, u_1^0, u_1^2, u_1^1, u_1^1, u_1^0, u_1^2, u_2^0, u_2^1, u_2^2, \ldots, u_n^0, u_n^1, u_n^2, v_1^0, v_1^1, v_1^2, v_2^0, v_2^1, v_2^2, \ldots, v_n^0, v_n^1, v_n^2)\right|\right| \right| \phi(v_1^1, v_1^2, v_1^3)\rangle.
$$

(63)

It can be seen that the results meet Eq. 61. Let us now suppose that the entanglement swapping results through the \((n-1)\)-th measurement meet Eq. 61, such that the results after the final measurement are given by

$$
\frac{1}{d} \sum_{l_1, l_2=0}^{d-1} \xi^{(d^2-1)(u_1^i-1)+1} \left|\left| l_1, l_2, l_1, l_2, l_1, u_1^0, u_1^0, u_1^2, u_1^1, u_1^1, u_1^0, u_1^2, u_2^0, u_2^1, u_2^2, \ldots, u_n^0, u_n^1, u_n^2, v_1^0, v_1^1, v_1^2, v_2^0, v_2^1, v_2^2, \ldots, v_n^0, v_n^1, v_n^2)\right|\right| \right| \phi(v_1^1, v_1^2, v_1^3)\rangle.
$$

(64)

We would like to give a formula, which can directly prove Eq. 61 or derive the results shown in Eq. 61, rather than multiple calculations like the above proof. Such a formula is given by

$$
\left|\left| \phi(u_1^0, u_1^1, \ldots, u_1^n)\right|\right| = \frac{1}{d^{(n+1)/2}} \sum_{l_1, l_2=0}^{d-1} \xi^{\sum_{r=1}^{n}(u_1^i-1)} \left|\left| l_1, l_2, l_1, l_2, l_1, u_1^0, u_1^0, u_1^2, u_1^1, u_1^1, u_1^0, u_1^2, u_2^0, u_2^1, u_2^2, \ldots, u_n^0, u_n^1, u_n^2, v_1^0, v_1^1, v_1^2, v_2^0, v_2^1, v_2^2, \ldots, v_n^0, v_n^1, v_n^2)\right|\right| \right| \phi(v_1^1, v_1^2, v_1^3)\rangle.
$$

(65)
\[
\bigotimes_{r=1}^{n} |\phi(\nu'_1, \nu'_2)\rangle,
\]
where the coefficient can be obtained by multiplying the coefficients in the calculation results in stages, that is,
\[
\frac{1}{d^r} \sum_{\nu'_1, \nu'_2, \nu'_3, \nu'_4 = 0}^{d-1} \zeta \sum_{\nu'_1}^{(d^r-1)\nu'_1} |\nu'_1\rangle \zeta |\nu'_1\rangle \equiv \prod_{r=1}^{n} \frac{1}{d^r} \sum_{\nu'_1, \nu'_2 = 0}^{d-1} \zeta |\nu'_1\rangle |\nu'_2\rangle.
\]

### 4.2.2 Entanglement swapping of 2-level maximally entangled states

Below we study entanglement swapping for 2-level maximally entangled states. Let us start by introducing the entanglement swapping between two Bell states shown in Eq. 29, which is presented in Ref. [6]. Assuming that the measurement operator \(\hat{M}\) is performed on the first particle in each state, we can express the entanglement swapping between two Bell states as
\[
\bigotimes_{r=1}^{2} |\phi(\nu'_1, \nu'_2)\rangle_{2r-1, 2r} = \frac{1}{2} \left[ |0\nu'_10\nu'_2\rangle + (-1)^{\nu'_1} |0\nu'_11\bar{\nu}'_2\rangle + (-1)^{\nu'_2} |1\nu'_10\bar{\nu}'_2\rangle + (-1)^{\nu'_1+\nu'_2} |1\nu'_11\bar{\nu}'_2\rangle \right]_{1234}
\]
\[
= \frac{1}{2} \left[ |0\nu'_10\nu'_2\rangle + (-1)^{\nu'_1} |0\nu'_11\bar{\nu}'_2\rangle + (-1)^{\nu'_2} |1\nu'_10\bar{\nu}'_2\rangle + (-1)^{\nu'_1+\nu'_2} |1\nu'_11\bar{\nu}'_2\rangle \right]_{1234}
\]
\[
= \frac{1}{4} \left[ 1 + (-1)^{\nu'_1+\nu'_2} \left( |\nu'(0, 0)\rangle |\nu'(0, \nu'_1, \nu'_2, \nu'_3)\rangle \pm |\nu'(1, 0)\rangle |\nu'(1, \nu'_1, \nu'_2, \nu'_3)\rangle \right) + \frac{1}{4} \left[ 1 - (-1)^{\nu'_1+\nu'_2} \left( |\nu'(0, 0)\rangle |\nu'(0, \nu'_1, \nu'_2, \nu'_3)\rangle \pm |\nu'(1, 0)\rangle |\nu'(1, \nu'_1, \nu'_2, \nu'_3)\rangle \right) \right.
\]
\[
+ \frac{1}{4} \left[ (-1)^{\nu'_1} + (-1)^{\nu'_2} \left( |\nu'(0, 1)\rangle |\nu'(0, \nu'_1, \nu'_2, \nu'_3)\rangle \pm |\nu'(1, 1)\rangle |\nu'(1, \nu'_1, \nu'_2, \nu'_3)\rangle \right) \right]
\]
\[
+ \left. \frac{1}{4} \left[ (-1)^{\nu'_1} - (-1)^{\nu'_2} \left( |\nu'(1, 1)\rangle |\nu'(0, \nu'_1, \nu'_2, \nu'_3)\rangle \pm |\nu'(1, 1)\rangle |\nu'(1, \nu'_1, \nu'_2, \nu'_3)\rangle \right) \right] \right]
\]
\]
which is realized by performing \(\hat{M}\) on the first particle in each state.

Let us then consider the entanglement swapping between CAT states (see Eq. 33). Let us suppose that there are \(n\) CAT states and mark them by \(\left\{ |\varphi(\nu'_1, \nu'_2, \ldots, \nu'_{m})\rangle \right\}_{r=1}^{n}\), where \(\nu'_1 \in [0, 1]\). Without losing generality, let us suppose that the first \(k\) particles in \(\left| \varphi(\nu'_1, \nu'_2, \ldots, \nu'_{m})\right\rangle\) are selected, and that the measurement operator \(\hat{M}\) is performed on them. Let us number these particles \(1, 2, \ldots, K\) and the remaining particles \(1, 2, \ldots, R\), where \(K = \sum_{r=1}^{n} k\) and \(R = \sum_{r=1}^{n} m - K\) throughout this paper, such that we can arrive at
\[
\bigotimes_{r=1}^{n} |\varphi(\nu'_1, \nu'_2, \ldots, \nu'_{m})\rangle
\]
\[
= \bigotimes_{r=1}^{n} |\nu'_1, \nu'_2, \ldots, \nu'_{m}\rangle + (-1)^{n-1} \bigotimes_{r=1}^{n-1} |\nu'_1, \nu'_2, \ldots, \nu'_{m}\rangle |\nu'_1, \nu'_2, \nu'_3, \nu'_4, \ldots, \nu'_{m}\rangle
\]
\[
+ (-1)^{n-2} \bigotimes_{r=1}^{n-2} |\nu'_1, \nu'_2, \ldots, \nu'_{m}\rangle |\nu'_1, \nu'_2, \nu'_3, \nu'_4, \ldots, \nu'_{m-1}\rangle |\nu'_1, \nu'_2, \nu'_3, \ldots, \nu'_{m-1}\rangle |\nu'_1, \nu'_2, \nu'_3, \ldots, \nu'_{m-1}\rangle + \cdots
\]
... + \left( -1 \right)^{n-1} \left( \bigotimes_{r=1}^{n-1} \left| e_{1}^{r}, w_{2}^{r}, w_{3}^{r}, \ldots, w_{n}^{r} \right> \right) \bigotimes_{r=1}^{n} \left| w_{n+1}^{r}, w_{n+2}^{r}, \ldots, w_{m}^{r} \right>

+ \left( -1 \right)^{n-1} \left( \bigotimes_{r=1}^{n-1} \left| e_{1}^{r}, w_{2}^{r}, w_{3}^{r}, \ldots, w_{n}^{r} \right> \right) \bigotimes_{r=1}^{n} \left| w_{n+1}^{r}, w_{n+2}^{r}, \ldots, w_{m}^{r} \right>

+ \left( -1 \right)^{n-1} \left( \bigotimes_{r=1}^{n-1} \left| e_{1}^{r}, w_{2}^{r}, w_{3}^{r}, \ldots, w_{n}^{r} \right> \right) \bigotimes_{r=1}^{n} \left| w_{n+1}^{r}, w_{n+2}^{r}, \ldots, w_{m}^{r} \right>

= \sum_{j=1}^{2^{n-1}} \left[ \left( -1 \right)^{\nu_{j}} \left( \bigotimes_{i=2}^{K} \left| e_{1}^{i}, w_{2}^{i}, w_{3}^{i}, \ldots, w_{n}^{i} \right> \right) \bigotimes_{i=1}^{R} \left| e_{j}^{i} \right> \right] \bigotimes_{i=1}^{R} \left| \omega_{j}^{i} \right>^{{\text{\scriptsize{2^{n-1}}}+1}} \bigotimes_{i=1}^{R} \left| \beta_{j}^{i} \right>.

Note that some inessential coefficients are ignored in Eq. 69. For CAT states, if one sets \( \left\{ \left| e_{1}^{i} \right>, \left| e_{2}^{i} \right> \right\} = \left\{ \left| 0 \right>, \left| 1 \right> \right\} \) or \( \left\{ \left| + \right>, \left| - \right> \right\} \), where \( \left| \pm \right> = \frac{1}{\sqrt{2}} \left( \left| 0 \right> \pm \left| 1 \right> \right) \), then the entanglement swapping considered is enlightening for the entanglement swapping of graph states [15, 16]. Let us provide a simple case of the entanglement swapping of graph states, that is, the entanglement swapping between two four-particle cluster states, which has the form [15]

\(|C\rangle := \frac{1}{2} (|+; 0, +, 0 \rangle + |+; 0, -, 0 \rangle + |-; 1, -; 0 \rangle + \left. |-; 1, +, 1 \rangle \right). \quad (70)\)

Let us assume that the measurement operator \( \tilde{M} \) is performed on the first two particles in each state, then the
entanglement swapping between them can be expressed as

$$\left| C_{1,2,3,4} \otimes C_{1,2,3,4} \right> \rightarrow \frac{1}{4} \left( |\tilde{\phi}_1 \rangle \langle \tilde{\phi}_1 | + |\tilde{\phi}_2 \rangle \langle \tilde{\phi}_2 | + |\tilde{\phi}_3 \rangle \langle \tilde{\phi}_3 | + |\tilde{\phi}_4 \rangle \langle \tilde{\phi}_4 | \right) \left( |\tilde{\phi}_1 \rangle \langle \tilde{\phi}_1 | + |\tilde{\phi}_2 \rangle \langle \tilde{\phi}_2 | + |\tilde{\phi}_3 \rangle \langle \tilde{\phi}_3 | + |\tilde{\phi}_4 \rangle \langle \tilde{\phi}_4 | \right), \quad (71)$$

where the subscripts $1, 2, 3, 4$ indicate the particles in the two states, and

$$\left| \phi \right> \left( 0, 0, +, 0 \right) = \frac{1}{\sqrt{2}} \left( |+000-11-1 \rangle + |-000-11-1 \rangle \right) , \quad \left| \phi \right> \left( 1, 0, +, 0 \right) = \frac{1}{\sqrt{2}} \left( |+000-11-1 \rangle + |-000-11-1 \rangle \right),$$

$$\left| \phi \right> \left( 0, 0, -, 1 \right) = \frac{1}{\sqrt{2}} \left( |+000+110-1 \rangle - |-000+110-1 \rangle \right) , \quad \left| \phi \right> \left( 1, 0, +, 0 \right) = \frac{1}{\sqrt{2}} \left( |+000+110-1 \rangle - |-000+110-1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 0, -, 1 \right) = \frac{1}{\sqrt{2}} \left( |+000+110+1 \rangle + |-000+110+1 \rangle \right) , \quad \left| \phi \right> \left( 0, 0, +, 0 \right) = \frac{1}{\sqrt{2}} \left( |+000+110+1 \rangle + |-000+110+1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 0, +, 1 \right) = \frac{1}{\sqrt{2}} \left( |+000+110+1 \rangle + |-000+110+1 \rangle \right) , \quad \left| \phi \right> \left( 1, 0, -, 1 \right) = \frac{1}{\sqrt{2}} \left( |+000+110+1 \rangle + |-000+110+1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 0, +, 1 \right) = \frac{1}{\sqrt{2}} \left( |+000+110+1 \rangle + |-000+110+1 \rangle \right) , \quad \left| \phi \right> \left( 1, 0, -, 1 \right) = \frac{1}{\sqrt{2}} \left( |+000+110+1 \rangle + |-000+110+1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 1, +, 0 \right) = \frac{1}{\sqrt{2}} \left( |+1000-11-1 \rangle + |-1000-11-1 \rangle \right) , \quad \left| \phi \right> \left( 1, 1, +, 1 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 1, -, 0 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right) , \quad \left| \phi \right> \left( 1, 1, -, 1 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 1, +, 0 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right) , \quad \left| \phi \right> \left( 1, 1, +, 1 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right),$$

$$\left| \phi \right> \left( 1, 1, -, 0 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right) , \quad \left| \phi \right> \left( 1, 1, -, 1 \right) = \frac{1}{\sqrt{2}} \left( |+1000+11+1 \rangle + |-1000+11+1 \rangle \right).$$
where
\[
\mathcal{P} = \frac{1}{\sqrt{\sum_{l_1, \ldots, l_d} \sum_{l_1', \ldots, l_d'} \left| \sum_{j=0}^{d-1} \sum_{j'=0}^{d-1} \lambda_{l_j, l_{j'}} \prod_{r=2}^{m} \lambda_{l_{r}, l_{r}' \oplus 1} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \right|^2}}.
\]

In particular, if \( \widetilde{M} \) acts on the first particle in all states, the entanglement swapping result is given by
\[
\mathcal{P} \sum_{l_1, \ldots, l_d} \sum_{l_1', \ldots, l_d'} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \lambda_{l_j, l_{j'}, \ldots, l_d, l_{d'}} \prod_{r=2}^{m} \lambda_{l_{r}, l_{r}' \oplus 1} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \| l_1 \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \|$$(73)\]

where \( \mathcal{P} \) is the same as that in Eq. 73. If \( \widetilde{M} \) acts on the second particle in all states, the entanglement swapping result is given by
\[
\mathcal{P} \sum_{l_1, \ldots, l_d} \sum_{l_1', \ldots, l_d'} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \lambda_{l_j, l_{j'}, \ldots, l_d, l_{d'}} \prod_{r=2}^{m} \lambda_{l_{r}, l_{r}' \oplus 1} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \| l_1 \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \|$$(74)\]

where \( \mathcal{P} \) is also the same as that in Eq. 73. By the way, as mentioned earlier, selecting the first and the second particle in the generalized pure state are equivalent.

As a special case of entanglement swapping schemes shown in Eq. 73, the entanglement swapping between general pure states and bipartite maximally entangled states can be characterized as
\[
\begin{align*}
\mathcal{P} = & \frac{1}{\sqrt{\sum_{l_1, \ldots, l_d} \sum_{l_1', \ldots, l_d'} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \lambda_{l_j, l_{j'}, \ldots, l_d, l_{d'}} \prod_{r=2}^{m} \lambda_{l_{r}, l_{r}' \oplus 1} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \| l_1 \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \)}
\end{align*}
\]

Further, as a simple special case of Eq. 76, the entanglement swapping between a two-level general pure state and a two-level maximally entangled state, is given by
\[
\begin{align*}
\mathcal{P} = & \frac{1}{\sqrt{\sum_{l_1, \ldots, l_d} \sum_{l_1', \ldots, l_d'} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \lambda_{l_j, l_{j'}, \ldots, l_d, l_{d'}} \prod_{r=2}^{m} \lambda_{l_{r}, l_{r}' \oplus 1} \sum_{l_2 \in \{0, 1\}} \sum_{l_2' \in \{0, 1\}} \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \| l_1 \otimes \phi(v_{l_1}, v_{l_1'}, \ldots, v_{l_d}, v_{l_d'}) \)}
\end{align*}
\]

5 Mixed states entanglement swapping

In this section, we study entanglement swapping of mixed states. We will first consider X states entanglement swapping in d-level systems, which indeed include the 2-level system case studied in Refs. [17]. Then, we will entanglement swapping cases for mixed maximally entangled states, and formulate the entanglement swapping of mixed d-level Bell state, as the special cases.
5.1 Entanglement swapping of X states

The main characteristic of a X state is that non-zero elements only appear on the diagonals of its density matrix, including the major diagonal and minor one. An X state having the following form,

\[
\rho_{X_a} := \sum_{l_1, l_2, \ldots, l_m=0}^{d-1} \left( \alpha_{a,a} | l_1, l_2, \ldots, l_m \rangle \langle l_1, l_2, \ldots, l_m | + \alpha_{a,\bar{a}} | l_1, l_2, \ldots, l_m \rangle \langle d - 1 - l_1, d - 1 - l_2, \ldots, d - 1 - l_m | \right),
\]

where \(a\) and \(\bar{a}\) are the decimal representations of \(l_1 l_2 \cdots l_m\) and \((d - 1 - l_1)(d - 1 - l_2) \cdots (d - 1 - l_m)\), respectively. For example, suppose that \(d = 3\) and \(l_1 l_2 \cdots l_m = 210\), then \(a = 21\) and \(\bar{a} = 5\). As general pure states considered before, below we would only like to consider the entanglement swapping cases for the bipartite states

\[
\rho_{X_a} := \sum_{l_1, l_2=0}^{d-1} (\alpha_{a,a} | l_1, l_2 \rangle \langle l_1, l_2 | + \alpha_{a,\bar{a}} | l_1, l_2 \rangle \langle d - 1 - l_1, d - 1 - l_2 |).
\]

In the form of matrix, \(\rho_{X_a}\) can be expressed as:

\[
\begin{pmatrix}
\lambda_{0,0} & 0 & 0 & \cdots & 0 & 0 & \lambda_{0,d-1} \\
0 & \lambda_{1,1} & 0 & \cdots & 0 & \lambda_{1,d-2} & 0 \\
0 & 0 & \lambda_{2,2} & \cdots & \lambda_{2,d-3} & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \lambda_{d-2,1} & 0 & \cdots & \lambda_{d-2,d-3} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \lambda_{d-2,d-2} & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \lambda_{d-1,d-1} \\
\end{pmatrix}
\]

(80)

From \(\rho_{X_a}\), one can get a special case described in Ref. [17], the 2-level X states, which is given by

\[
\tilde{\rho}_{X_3} := \lambda_{0,0} | 0, 0 \rangle \langle 0, 0 | + \lambda_{0,3} | 0, 0 \rangle \langle 1, 1 | + \lambda_{1,1} | 0, 1 \rangle \langle 0, 1 | + \lambda_{1,2} | 0, 1 \rangle \langle 1, 0 | + \lambda_{2,2} | 1, 0 \rangle \langle 1, 0 | + \lambda_{2,1} | 1, 0 \rangle \langle 0, 1 | + \lambda_{3,3} | 1, 1 \rangle \langle 1, 1 | + \lambda_{3,0} | 1, 1 \rangle \langle 0, 0 |
\]

\[
\equiv \begin{pmatrix}
\lambda_{0,0} & 0 & 0 & \lambda_{0,3} \\
0 & \lambda_{1,1} & \lambda_{1,2} & 0 \\
0 & \lambda_{2,1} & \lambda_{2,2} & 0 \\
\lambda_{3,0} & 0 & 0 & \lambda_{3,3} \\
\end{pmatrix}.
\]

(81)

Now, let us start to formulate entanglement swapping for any number of X states in \(\rho_{X_a}\). Suppose that there are \(n\) X states, \(\rho_{X_1}, \rho_{X_2}, \ldots, \rho_{X_n}\). For clarity, let us mark them by

\[
\rho_{rX_a} := \sum_{l_1, l_2=0}^{d-1} (\alpha_{1,1} | r_1, r_2 \rangle \langle r_1, r_2 | + \alpha_{1,\bar{a}} | r_1, r_2 \rangle \langle d - 1 - r_1, d - 1 - r_2 |)
\]

(82)

where \(r = 1, 2, \ldots, n\). Let us suppose that the measurement operator \(\tilde{M}\) is performed on the second particle in each X states, then the entanglement swapping formula is given by

\[
\bigotimes_{r=1}^{n} \rho_{rX_a}
\]

\[
= \sum_{l_1, l_2, \ldots, l_n=0}^{d-1} \prod_{r=1}^{n} \lambda_{r,1} | l_1, l_2 \rangle \langle l_1, l_2 | (\prod_{r=1}^{n} \lambda_{r,\bar{a}}) | d - 1 - l_1, d - 1 - l_2 \rangle \langle d - 1 - l_1, d - 1 - l_2 | (a)
\]

\[
+ \sum_{l_1, l_2, \ldots, l_n=0}^{d-1} \prod_{r=1}^{n} \lambda_{r,1} | d - 1 - r_1 - d - 1 - r_2 \rangle \langle r_1, r_2 | (\prod_{r=1}^{n} \lambda_{r,\bar{a}}) | l_1, l_2 \rangle \langle d - 1 - l_1, d - 1 - l_2 | (b)
\]

\[
+ \sum_{l_1, l_2, \ldots, l_n=0}^{d-1} \prod_{r=1}^{n} \lambda_{r,1} | d - 1 - 1 - l_1 - d - 1 - l_2 \rangle \langle d - 1 - 1 - l_1, d - 1 - l_2 | (\prod_{r=1}^{n} \lambda_{r,\bar{a}}) | l_1, l_2 \rangle \langle d - 1 - l_1, d - 1 - l_2 |,
\]

\[
\times | l_1, l_2 \rangle \langle l_1, l_2 |, | l_1, l_2 \rangle \langle l_1, l_2 |, \ldots, | l_1, l_2 \rangle \langle l_1, l_2 |, | l_1, l_2 \rangle \langle l_1, l_2 |,
\]

\[
\ldots
\]

\[
\ldots
\]

\[
\ldots
\]
\[ dd_{1}^{d-1}, dd_{2}, \ldots, dd_{1}^{d-1}, dd_{2}, \ldots, dd_{1}^{d-1}, dd_{2}, \ldots, dd_{1}^{d-1}, dd_{2}, \ldots, dd_{1}^{d-1}, dd_{2}, \ldots, dd_{1}^{d-1}, dd_{2}, \ldots, dd_{1}^{d-1}, dd_{2}, \ldots, \quad (c) \]

\[ + \sum_{\ell_1, \ell_2, \ell_3, \ldots, \ell_n = 0}^{n=\frac{2}{n}} \prod_{i=1}^{n} A_{\ell_i, \ell_i} \prod_{j \neq (1, \ldots, n)} A_{\ell_j, \ell_j} \prod_{k \neq (1, \ldots, n)} A_{\ell_k, \ell_k} \prod_{l=1}^{d-1} A_{\ell_l, \ell_l} \|	heta_{d-1, d-1, \ell_l} \right| \quad (d) \]

\[ \Rightarrow \sum_{\ell_1, \ell_2, \ell_3, \ldots, \ell_n = 0}^{n=\frac{2}{n}} \prod_{i=1}^{n} A_{\ell_i, \ell_i} \prod_{j \neq (1, \ldots, n)} A_{\ell_j, \ell_j} \prod_{k \neq (1, \ldots, n)} A_{\ell_k, \ell_k} \prod_{l=1}^{d-1} A_{\ell_l, \ell_l} \|	heta_{d-1, d-1, \ell_l} \right| \quad (d) \]
where 

\[ P_1 = \frac{d}{T_2}, \quad P_2 = \frac{d}{P_2}, \quad P'_1 = \sum_{r_1^1, r_1^2, r_1^3 = 0}^{d-1} \left| \sum_{r_2 = 2}^{d-1} A[r_1^1, r_1^2] \prod_{r = 2}^{d-1} A[r_1^1, r_2^1] \otimes |r_1^1 \rangle \langle r_1^1 | \otimes \phi(v_1, v_2, \ldots, v_n) \right| \]

\[ + \sum_{r_1^1, r_1^2, r_1^3 = 0}^{d-1} \sum_{v_1, v_2, \ldots, v_n = 0}^{d-1} A[r_1^1, r_1^2] \prod_{r = 2}^{d-1} A[r_1^1, r_2^1] \otimes |r_1^1 \rangle \langle r_1^1 | \otimes \phi(v_1, v_2, \ldots, v_n) \]

\[ \otimes \phi(v_1^1, v_1^2, \ldots, v_1^n) \left\langle \phi(v_1, v_2, \ldots, v_n) \right\rangle, \text{ if } d \text{ is an even number,} \]

\[ P_2 = \frac{d}{T_2}, \quad P_2 = \frac{d}{P_2}, \quad P'_2 = \sum_{r_1^1, r_1^2, r_1^3 = 0}^{d-1} \left| \sum_{r_2 = 2}^{d-1} A[r_1^1, r_1^2] \prod_{r = 2}^{d-1} A[r_1^1, r_2^1] \otimes |r_1^1 \rangle \langle r_1^1 | \otimes \phi(v_1, v_2, \ldots, v_n) \right| \]

\[ + \sum_{r_1^1, r_1^2, r_1^3 = 0}^{d-1} \sum_{v_1, v_2, \ldots, v_n = 0}^{d-1} A[r_1^1, r_1^2] \prod_{r = 2}^{d-1} A[r_1^1, r_2^1] \otimes |r_1^1 \rangle \langle r_1^1 | \otimes \phi(v_1, v_2, \ldots, v_n) \]

\[ \otimes \phi(v_1^1, v_1^2, \ldots, v_1^n) \left\langle \phi(v_1, v_2, \ldots, v_n) \right\rangle \]

\[ \text{ (83)} \]
where

\[ \mathcal{P}_2 = \sum_{d=1}^{d-1} \sum_{v_i,v_2=0}^{d-1} A[v_i,v_2][v_i,v_2] \mathcal{P}_2 \left( [d-1,e_1,d-1-e_1] \right) \]

\[ \times \prod_{r=1}^{n} A[l_r,l_r] \left( [d-1,e_1,d-1-e_1] \right) \]

Note that in the above formula, Eqs. (a) and (b) represent the first and last terms of the expansion of \( \bigotimes_{r=1}^{n} \mathcal{P}_2 \), respectively, while (c) and (d) the remainder. In addition, the following parameter substitution for the orders of particles are made for the remaining items:

\[
\begin{align*}
&1 \quad 2 \quad 3 \quad \cdots \quad n \\
&k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_n
\end{align*}
\]

As a special case of the entanglement swapping shown above, let us give the formula for two X sates,

\[
\bigotimes_{r=1}^{3} \mathcal{P}_2
\]

\[
\mathcal{P}_1 = \frac{d}{d-1} \sum_{l_r,l_r=0}^{d-1} \sum_{v_i,v_2=0}^{d-1} A[l_r,l_r][v_i,v_2][v_i,v_2] \mathcal{P}_1 \left( [d-1,l_r,d-1-l_r] \right) \mathcal{P}_1 \left( [d-1,e_1,d-1-e_1] \right) \]

\[ \times \prod_{r=1}^{n} A[l_r,l_r] \left( [d-1,e_1,d-1-e_1] \right) \]

\[
\mathcal{P}_2 = \frac{d}{d-1} \sum_{l_r,l_r=0}^{d-1} \sum_{v_i,v_2=0}^{d-1} A[l_r,l_r][v_i,v_2][v_i,v_2] \mathcal{P}_2 \left( [d-1,l_r,d-1-l_r] \right) \mathcal{P}_2 \left( [d-1,e_1,d-1-e_1] \right) \]

\[ \times \prod_{r=1}^{n} A[l_r,l_r] \left( [d-1,e_1,d-1-e_1] \right) \]

if \( d \) is an even number,

\[
\mathcal{P}_3 \rightarrow \mathcal{P}_2
\]

\[
\mathcal{P}_3 = \frac{d}{d-1} \sum_{l_r,l_r=0}^{d-1} \sum_{v_i,v_2=0}^{d-1} A[l_r,l_r][v_i,v_2][v_i,v_2] \mathcal{P}_3 \left( [d-1,l_r,d-1-l_r] \right) \]

\[ \times \prod_{r=1}^{n} A[l_r,l_r] \left( [d-1,e_1,d-1-e_1] \right) \]

\[
\mathcal{P}_4 \rightarrow \mathcal{P}_2
\]

\[
\mathcal{P}_4 = \frac{d}{d-1} \sum_{l_r,l_r=0}^{d-1} \sum_{v_i,v_2=0}^{d-1} A[l_r,l_r][v_i,v_2][v_i,v_2] \mathcal{P}_4 \left( [d-1,l_r,d-1-l_r] \right) \]

\[ \times \prod_{r=1}^{n} A[l_r,l_r] \left( [d-1,e_1,d-1-e_1] \right) \]

\[ \mathcal{P}_5 \rightarrow \mathcal{P}_2 \]

\[
\mathcal{P}_5 = \frac{d}{d-1} \sum_{l_r,l_r=0}^{d-1} \sum_{v_i,v_2=0}^{d-1} A[l_r,l_r][v_i,v_2][v_i,v_2] \mathcal{P}_5 \left( [d-1,l_r,d-1-l_r] \right) \]

\[ \times \prod_{r=1}^{n} A[l_r,l_r] \left( [d-1,e_1,d-1-e_1] \right) \]

where

\[ \frac{\mathcal{P}_1}{\mathcal{P}_2} = \frac{d}{d-1}, \quad \frac{\mathcal{P}_2}{\mathcal{P}_2} = \frac{d}{d-1} \]
\[ \mathcal{P} = \sum_{d^{-1}} \left[ \sum_{d^{-1}} A[i,j] \left| \phi^{(1)} \right\rangle \check{\phi} \left( 0,0 \right) \right] \otimes \left\langle \phi^{(1)} \check{\phi} \left( 0,0 \right) \right\rangle, \]

Further, one can get the formula of entanglement swapping between two 2-level X states shown in Eq. 81, which was studied in Ref. [17],

\[ \sum_{r=1}^{2} \rho_{\text{X}}, \]

\[ \rightarrow \mathcal{P} = \frac{1}{2} \left[ \left( \lambda_{00} + \lambda_{11} \right) \left| 00 \right\rangle \left\langle 00 \right| + \left( \lambda_{03} + \lambda_{12} \right) \left| 00 \right\rangle \left\langle 11 \right| + \left( \lambda_{02} + \lambda_{13} \right) \left| 01 \right\rangle \left\langle 01 \right| + \left( \lambda_{01} + \lambda_{10} \right) \left| 01 \right\rangle \left\langle 10 \right| \right. \]

\[ + \left( \lambda_{20} + \lambda_{31} \right) \left| 10 \right\rangle \left\langle 00 \right| + \left( \lambda_{21} + \lambda_{30} \right) \left| 10 \right\rangle \left\langle 01 \right| + \left( \lambda_{23} + \lambda_{32} \right) \left| 10 \right\rangle \left\langle 10 \right| + \left( \lambda_{22} + \lambda_{33} \right) \left| 10 \right\rangle \left\langle 11 \right| \]

\[ \left. + \left( \lambda_{11} \right) \left| 01 \right\rangle \left\langle 01 \right| + \left( \lambda_{10} \right) \left| 01 \right\rangle \left\langle 11 \right| + \left( \lambda_{12} \right) \left| 01 \right\rangle \left\langle 10 \right| + \left( \lambda_{13} \right) \left| 01 \right\rangle \left\langle 11 \right| \right] \otimes \left[ \phi \left( 0,0 \right) \right] \left\langle \phi \left( 0,0 \right) \right\rangle, \]

5.2 Entanglement swapping of mixed maximally entangled states

In what follows we consider the entanglement swapping of the mixture of the maximally entangled states which has the form

\[ \rho (u_1, u_2, \ldots, u_m) = \sum_{u_1, u_2, \ldots, u_m=0}^{d^{-1}} \lambda_{u_1 u_2, \ldots, u_m} \left| \phi \left( u_1, u_2, \ldots, u_m \right) \right\rangle \left\langle \phi \left( u_1, u_2, \ldots, u_m \right) \right\rangle, \]
Let us assume that there are \( n \) such mixed states containing \( m_1, m_2, \ldots, m_n \) particles each, and denote them as \( \rho(u_1^1, u_2^1, \ldots, u_m^1), \rho(u_2^2, u_3^2, \ldots, u_m^2), \ldots, \rho(u_1^n, u_2^n, \ldots, u_m^n) \) in turn. As before, we will first consider the general entanglement swapping case realized by performing \( M \) on the first \( k \) particles in \( \rho(u_1^r, u_2^r, \ldots, u_m^r) \) for \( r = 1, 2, \ldots, t \) and the last \( k \) particles in \( \rho(u_1^r, u_2^r, \ldots, u_m^r) \) for \( r = t+1, t+2, \ldots, n \). We will then consider two extreme cases: measuring the first \( k \) particles in all mixed states and measuring the last \( k \) particles. Finally, from these cases, we will give the entanglement swapping formulas for mixed \( d \)-level Bell states.

Now, let us give the the general entanglement swapping formula, which can be expressed as

\[
\sum_{r=1}^{n} \rho(u_1^r, u_2^r, \ldots, u_m^r)
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
\cdots + \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]

\[
= \frac{1}{d^n} \sum_{u_1^1, u_2^1, \ldots, u_m^1 = 0}^{d-1} \cdots \sum_{u_1^n, u_2^n, \ldots, u_m^n = 0}^{d-1} \prod_{r=1}^{n} \lambda_{u_1^r, u_2^r, \ldots, u_m^r} \xi_{\sum_{r}(u_r - v_r)} \psi_{r, t}
\]
where $v_j$ satisfies Eq. 49 for $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n$, and

$$\mathcal{P} = \frac{1}{d^{\sum_{i=1}^{d-1} \sum_{\alpha_i=0}^{d-1} \sum_{\alpha_{d-1}=0}^{d-1} \cdots \sum_{\alpha_{d^n-1}=0}^{d-1} \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n}}}
$$

The first extreme entanglement swapping case mentioned above can be expressed as

$$\bigotimes_{r=1}^{n} \rho \left( u_1^r, u_2^r, \ldots, u_m^r \right)$$

$$= \frac{1}{d^n} \sum_{u_1^{r-1}, u_2^{r-1}, \ldots, u_m^{r-1}=0}^{d-1} \sum_{\alpha_1^{r-1}=0}^{d-1} \sum_{\alpha_2^{r-1}=0}^{d-1} \cdots \sum_{\alpha_{m-1}^{r-1}=0}^{d-1} \sum_{\alpha_m^{r-1}=0}^{d-1} \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n} \sum_{\alpha_{d^n}}^\sum \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n} \sum_{\alpha_{d^n}}^\sum \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n}$$

$$\bigotimes_{r=1}^{n} \left| l, l \otimes u_1^r, l \otimes u_1^r, \ldots, l \otimes u_m^r \right\rangle \langle l', l' \otimes u_1^r, l' \otimes u_1^r, \ldots, l' \otimes u_m^r |$$

$$= \frac{1}{d^n} \sum_{u_1^{r-1}, u_2^{r-1}, \ldots, u_m^{r-1}=0}^{d-1} \sum_{\alpha_1^{r-1}=0}^{d-1} \sum_{\alpha_2^{r-1}=0}^{d-1} \cdots \sum_{\alpha_{m-1}^{r-1}=0}^{d-1} \sum_{\alpha_m^{r-1}=0}^{d-1} \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n} \sum_{\alpha_{d^n}}^\sum \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n} \sum_{\alpha_{d^n}}^\sum \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n}$$

$$\bigotimes_{r=1}^{n} \left| l, l \otimes u_1^r, l \otimes u_1^r, \ldots, l \otimes u_m^r \right\rangle \langle l', l' \otimes u_1^r, l' \otimes u_1^r, \ldots, l' \otimes u_m^r |$$

$$= \mathcal{P} \sum_{r=1}^{n} \sum_{u_1^{r-1}, u_2^{r-1}, \ldots, u_m^{r-1}=0}^{d-1} \sum_{\alpha_1^{r-1}=0}^{d-1} \sum_{\alpha_2^{r-1}=0}^{d-1} \cdots \sum_{\alpha_{m-1}^{r-1}=0}^{d-1} \sum_{\alpha_m^{r-1}=0}^{d-1} \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n} \sum_{\alpha_{d^n}}^\sum \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n} \sum_{\alpha_{d^n}}^\sum \prod_{r=1}^{n} A_{\alpha_1^{r-1}}^{r-1} \alpha_{d^n}$$

$$\bigotimes_{r=1}^{n} \left| l, l \otimes u_1^r, l \otimes u_1^r, \ldots, l \otimes u_m^r \right\rangle \langle l', l' \otimes u_1^r, l' \otimes u_1^r, \ldots, l' \otimes u_m^r |$$

where $v_j$ satisfy Eq. 52 for $i = 1, 2, \ldots, n$ and $j = 2, 3, \ldots, n$, and $\mathcal{P}$ satisfy Eq. 89.
mixed maximally entangled states. For pure states entanglement swapping chains, as before, we will first consider generalized pure states, and then turn. Below we will formulate the entanglement swapping chains for pure states and mixed states, respectively.

Entanglement swapping chain is an application of the entanglement swapping between two entangled states. It is

\[ (2^{n-1})^2 \]

where all the \( v \) in the above three formulas are the same and satisfy Eq. 89.

As the special cases of the above entanglement swapping schemes, respectively, we can formulate the entanglement swapping of the mixed d-level Bell states as follows.

\[
\mathcal{P} \left\{ \rho(u'_1, u'_2) \right\} = \sum_{i=1}^{n} \left\{ \rho(u'_1, u'_2) \right\}
\]

\[
\mathcal{P} \left\{ \rho(u'_1, u'_2) \right\} = \sum_{i=1}^{n} \left\{ \rho(u'_1, u'_2) \right\}
\]

\[
\mathcal{P} \left\{ \rho(u'_1, u'_2) \right\} = \sum_{i=1}^{n} \left\{ \rho(u'_1, u'_2) \right\}
\]

where all the \( \mathcal{P}s \) in the above three formulas are the same and satisfy Eq. 89.

6 Entanglement swapping chains

Entanglement swapping chain is an application of the entanglement swapping between two entangled states. It is a key technology that can be used to construct chained quantum networks by performing measurements at all the intermediate locations of more than two entangled states [9]. Suppose that there are \( n \) entangled states, \( \{\rho_{2^n-1,2}^{n-1}\} \), where the subscripts \( (2r - 1, 2r) \) indicate two particles in each states, such that a entanglement swapping chain can be realized by performing the measurement operator \( \mathcal{M} \) on the particles \( (2, 3), (4, 5), \ldots, (2n - 2, 2n - 1) \) in turn. Below we will formulate the entanglement swapping chains for pure states and mixed states, respectively.

For pure states entanglement swapping chains, as before, we will first consider generalized pure states, and then characterize some special cases. For mixed states entanglement swapping chains, we would only like to consider mixed maximally entangled states.
6.1 Entanglement swapping chains for pure states

Let us begin with the formulation of the entanglement swapping chains of generalized pure states. As the entanglement swapping scheme shown in Eq. 39, let us assume that there are \( n \) generalized pure states, \( \{|\varphi_{\text{pure}}^r\rangle\}_{r=1}^n \), then performing \( \hat{M} \) on the particles \((2,3),(4,5), \ldots,(2n-2,2n-1)\) in turn and marking the measurement results as \( |\phi(v_{1}^{1}, v_{2}^{1})\rangle, |\phi(v_{1}^{2}, v_{2}^{2})\rangle, \ldots, |\phi(v_{1}^{n-1}, v_{2}^{n-1})\rangle \) respectively, will eventually make the particles 1 \& \( 2n \) collapse onto

\[
|\varphi_{\text{pure}}\rangle_{2n-1,2n} = \frac{1}{\sqrt{d^{n-1}}} \sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 \sum_{r=2}^{n} A_{l_1}^{r-1} A_{l_2}^{r-2} \zeta^{-\sum_{i=1}^{r-1}(\xi_{1}^{i}-\xi_{2}^{i})} |l_1, l_2\rangle_{1,2n} |\phi(v_{1}^{r-1}, v_{2}^{r-1})\rangle_{2n-1,2n}.
\]

(95)

This result can be proved as follows,

**Proof** The entanglement swapping result after the first measurement which is performed on the particles \((2,3)\), is given by

\[
|\varphi_{\text{pure}}^1\rangle_{2,3} = \frac{1}{\sqrt{d}} \sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 \sum_{r=2}^{n} A_{l_1}^{r-1} A_{l_2}^{r-2} \zeta^{-\sum_{i=1}^{r-1}(\xi_{1}^{i}-\xi_{2}^{i})} |l_1, l_2\rangle_{1,4} |\phi(v_{1}^{r-1}, v_{2}^{r-1})\rangle_{3,4}.
\]

(96)

After the second measurement performed on the particles \((4,5)\), one can arrive at

\[
|\varphi_{\text{pure}}^2\rangle_{4,5} = \frac{1}{\sqrt{d}} \sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 \sum_{r=2}^{n} A_{l_1}^{r-1} A_{l_2}^{r-2} \zeta^{-\sum_{i=1}^{r-1}(\xi_{1}^{i}-\xi_{2}^{i})} |l_1, l_2\rangle_{1,6} |\phi(v_{1}^{r-1}, v_{2}^{r-1})\rangle_{5,6}.
\]

(97)

where

\[
|\varphi\rangle = \frac{d}{\sqrt{\sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 |l_1, l_2\rangle_{1,4}}} = \frac{d}{\sqrt{\sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 |l_1, l_2\rangle_{1,6}}.
\]

It can be seen that after the entanglement swapping, both the states that remaining particles collapse onto meet Eq. 95. Assume that the entanglement swapping result after the \((n-2)\)-th measurement is consistent with Eq. 95, then the entanglement swapping result after the final measurement performed on the particles \((2n-2,2n-1)\) is given by

\[
|\varphi^n\rangle_{2n-2,2n-1} = \frac{d^{n-2}}{\sqrt{\sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 |l_1, l_2\rangle_{1,2n}}} = \frac{d^{n-2}}{\sqrt{\sum_{l_1, l_2, z_1 = 0}^{d-1} A_{l_1}^1 A_{l_2}^2 |l_1, l_2\rangle_{1,2n}}},
\]

(98)

from which one can get the state that is consistent with Eq. 95. QED.
We would like to employ mathematical induction, as above, to provide a proof for the formulas of the entanglement swapping chains between general pure states, which was proposed in Ref. [9]. Suppose that measurement results are \( \ket{\phi(v_1^1, v_2^1)}, \ket{\phi(v_1^2, v_2^2)}, \ldots, \ket{\phi(v_1^{n-1}, v_2^{n-1})} \) respectively, then the particles (1,2n) collapse onto

\[
\mathcal{P} \sum_{i=0}^{d-1} \frac{d-1}{\sqrt{d}} \prod_{r=1}^{n-1} A_{l_i \otimes \sum_{j=1}^{r} v_j^r} \sum_{i=1}^{d-1} \prod_{r=1}^{n-1} A_{l_i \otimes \sum_{j=1}^{r} v_j^r} \right| l_1, l_1 \otimes \sum_{r=1}^{n-1} v_j^r \rangle \right\rangle_{1,1,1,1,1}\]

Proof Via the first measurement, the entanglement swapping result is

\[
\mathcal{P} = \left[ \frac{d-1}{\sqrt{d}} \right]^2.
\]

After the second measurement, the entanglement swapping result is

\[
\mathcal{P} = \left[ \frac{d-1}{\sqrt{d}} \right]^2.
\]

It can be seen that after the entanglement swapping results via the first two measurements are consistent with Eq. 99. Let us now assume that the entanglement swapping result after the \((n-2)\)-th measurement meets Eq. 99, then the entanglement swapping via the final measurement is given by

\[
\mathcal{P} = \left[ \frac{d-1}{\sqrt{d}} \right]^2.
\]

where

\[
\mathcal{P} = \left[ \frac{d-1}{\sqrt{d}} \right]^2,
\]

It can be seen from the state the particles (1,2n) collapse onto is consistent with the state shown in Eq. 99. QED.

Let us now formulate entanglement swapping chains for maximally entangled states. Let us assume that there are \( n \) d-level (2m)-particle maximally entangled states, and mark them by \( \ket{\phi(u_1^1, u_2^1, \ldots, u_m^1)} \) respectively, where the subscripts \( (1, 3, 5, \ldots, 2n-1) \) and \( (2, 4, 6, \ldots, 2n) \) denote the first \( m \) particles and the last \( m \) particles respectively.
As above, assume that the measurement operator $\hat{M}$ is performed on the particles $(2, 3), (4, 5), \ldots, (2n - 2, 2n - 1)$ in turn, such that the particles $(1,2n)$ collapse onto

$$\frac{1}{d^{n-1}} \sum_{v_1^1, \ldots, v_{2n}^{2n} = 0}^{d-1} \sum_{i_1^1, \ldots, i_{2n}^{2n} = 0}^{d-1} \xi^{n-1}_{\Sigma_1}(u_i^{i_1}v_i^{i_2}) \sum_{m=1}^{2n} (u_m^{m_1}v_m^{m_2}) \left( \phi(\theta_{r=1}^{n-i}u_i^{i_1} \otimes \theta_{r=1}^{n-i}v_i^{i_2}) \right)_{1, 2m},$$

and the measurement results are

$$\left\{ \left( \phi(v_1^1, v_2^1, \ldots, v_{2m}^{2m}) \right)_{2m+1, 2m} \right\}_{r=1}^{n-1},$$

$$v^i_j = u^i_m, \ i = 2, 3, \ldots, m,$n

$$v^i_j = v^i_{m+1} \otimes u^i_{m+1} \otimes u^i_j, \ j = m + 2, m + 3, \ldots, 2m.$$  

**Proof** The entanglement swapping result after the first measurement is given by

$$\left| \phi(u_1^1, u_2^1, \ldots, u_{2m}^{2m}) \right|_{1, 2} \otimes \left| \phi(u_1^3, u_2^3, \ldots, u_{2m}^{2m}) \right|_{3, 4}$$

$$-\frac{1}{d} \sum_{v_{1,2n}^{2n} = 0}^{d-1} \xi^{n-1}_{\Sigma_1}(u_1^{1}, v_1^{1}) \left( \phi(u_1^1 \otimes u_1^3 \otimes v_1^1, u_2^1, u_3^1, \ldots, u_{m-1}^1, u_{m+1}^1 \otimes v_{m+1}^1, u_{m+2}^1 \otimes v_{m+1}^1, \ldots, u_{m-1}^1, u_{m+2}^1 \otimes v_{m+1}^1, \ldots, u_{m-1}^1, u_{m+2}^1 \otimes v_{m+1}^1) \right),$$

$$\left| \phi(u_1^1, v_1^1, \ldots, v_{2m}^{2m}) \right|_{3, 4} \otimes \left| \phi(v_1^1, v_2^1, \ldots, v_{2m}^{2m}) \right|_{3, 4}.$$  

where

$$v^i_1 = u^i_2, \ i = 2, 3, \ldots, m,$n

$$v^i_j = v^i_{m+1} \otimes u^i_{m+1} \otimes u^i_j, \ j = m + 2, m + 3, \ldots, 2m.$$  

The entanglement swapping result after the second measurement is given by

$$\frac{1}{d} \sum_{v_{1,2n}^{2n} = 0}^{d-1} \xi^{n-1}_{\Sigma_1}(u_1^{1}, v_1^{1}) \left( \phi(u_1^1 \otimes u_1^3 \otimes v_1^1, u_2^1, u_3^1, \ldots, u_{m-1}^1, u_{m+1}^1 \otimes v_{m+1}^1, u_{m+2}^1 \otimes v_{m+1}^1, \ldots, u_{m-1}^1, u_{m+2}^1 \otimes v_{m+1}^1, \ldots, u_{m-1}^1, u_{m+2}^1 \otimes v_{m+1}^1) \right),$$

$$\left| \phi(u_1^1, v_1^1, \ldots, v_{2m}^{2m}) \right|_{4, 5} \otimes \left| \phi(v_1^1, v_2^1, \ldots, v_{2m}^{2m}) \right|_{5, 6}.$$

where

$$v^i_1 = u^i_2, \ i = 2, 3, \ldots, m,$n

$$v^i_j = v^i_{m+1} \otimes u^i_{m+1} \otimes u^i_j, \ j = m + 2, m + 3, \ldots, 2m.$$  

It can be seen that the results meet Eq. 103. Let us now suppose that the entanglement swapping results after the $(n-1)$-th measurement meet Eq. 103, such that the result after the final measurement is given by

$$\frac{1}{d^{n-2}} \sum_{v_{1,2n}^{2n} = 0}^{d-1} \sum_{i_1^1, \ldots, i_{2n}^{2n} = 0}^{d-1} \xi^{n-2}_{\Sigma_1}(u_1^{i_1}v_1^{i_2}) \sum_{m=1}^{2n} (u_m^{i_1}v_m^{i_2}) \left( \phi(\theta_{r=1}^{n-i}u_i^{i_1} \otimes \theta_{r=1}^{n-i}v_i^{i_2}) \right)_{1, 2m-2} 

\otimes \phi(u_1^1, u_2^1, \ldots, u_{2m}^{2m})_{2m-1, 2m},$$

$$\left| \phi(u_1^1, u_2^1, \ldots, u_{2m}^{2m}) \right|_{2m-1, 2m}.$$
where

\[ \psi_i^{n-1} = u_i^p, \quad i = 2, 3, \ldots, m, \]
\[ \psi_j^{n-1} = \psi_{m+1}^{n-1} \oplus u_j^{n-1}, \quad j = m + 2, m + 3, \ldots, 2m. \]

It can be seen that the state of the particles (1,2n) is consistent with Eq. 103. QED.

As a special case, one can get the entanglement swapping chain for d-level Bell states from Eq. 103. Suppose that the measurement results are \( \left\{ \phi \left( \psi_1^j, \psi_2^j \right) \right\}_{2r+1,2r} \), then the particles (1,2n) collapse onto

\[ \frac{1}{d^{d-1}} \sum_{v_1^{n-1}, v_2^{n-1}, \ldots, v_{2m}^{n-1}=0} \zeta_{n-1}^{\psi_1^j, \psi_2^j} \sum_{l_1^{n-1}, l_2^{n-1}} \phi \left( l_1^{n-1} \psi_{m+1}^{n-1} \oplus l_2^{n-1}, \psi_{m+2}^{n-1}, \ldots, \psi_{2m}^{n-1} \right) \psi_{m+1}^{n-1} \psi_{m+2}^{n-1}, \ldots, \psi_{2m}^{n-1} \right)_{2r+1,2r} \]

\[ \cdot \phi \left( \psi_{m+1}^{n-1} \psi_{m+2}^{n-1}, \ldots, \psi_{2m}^{n-1} \right)_{2r+1,2r}. \]

We have characterized entanglement swapping chains, and provided proofs for corresponding results using mathematical induction. Obviously, it requires three steps for deriving the final entanglement swapping results. Below we would like to give a formula by which one can directly get the results of entanglement swapping chains. Let us first give the formula for the entanglement swapping chain of general pure states,

\[ \bigotimes_{r=1}^n \left| \phi_i^{r} \right\rangle_{2r-1,2r} \]

\[ = \bigotimes_{r=1}^n \left| \phi_i^{r} \right\rangle_{2r-1,2r} \]

\[ \bigotimes_{r=1}^n \left| \phi_i^{r} \right\rangle_{2r-1,2r} \]

\[ = \left( \bigotimes_{r=1}^n \left| \phi_i^{r} \right\rangle_{2r-1,2r} \right)_{2r+1,2r} \]

\[ = \frac{1}{\sqrt{d^{d-1} \prod_{j=1}^d \prod_{k=1}^d d_j^{\phi(j,k)}}} \left( \bigotimes_{r=1}^n \left| \phi_i^{r} \right\rangle_{2r-1,2r} \right). \]

It can be seen that when measurement results are \( \left| \phi \left( \psi_1^j, \psi_2^j \right) \right\rangle, \left| \phi \left( \psi_1^j, \psi_2^j \right) \right\rangle, \ldots, \left| \phi \left( \psi_{m+1}^{n-1}, \psi_{m+2}^{n-1} \right) \right\rangle \), the particles (1,2n) collapse onto the state shown in Eq. 95. Similarly, one can get the result of the entanglement swapping chain of general pure states through

\[ \bigotimes_{r=1}^n \left| \phi_i^{r} \right\rangle_{2r-1,2r} \]
Finally, the entanglement swapping chains for maximally entangled states can be characterized as

\[
\mathcal{P} = \frac{1}{\sqrt{\text{det}(\mathcal{M})}} \prod_{r=1}^{n-1} \Lambda_i \, \bigg| \prod_{r=1}^{n-1} \lambda_i \phi(v_1, v_2) \bigg|_{2r+1,2r},
\]

where

\[
\mathcal{P} = \frac{1}{\sqrt{\det(\mathcal{M})}} \prod_{r=1}^{n-1} \Lambda_i \, \bigg| \prod_{r=1}^{n-1} \lambda_i \phi(v_1, v_2) \bigg|_{2r+1,2r},
\]

Finally, the entanglement swapping chains for maximally entangled states can be characterized as

\[
\bigotimes_{r=1}^{n} \phi(u_1, u_2, \ldots, u_{2m}) \bigg|_{2r-1,2r},
\]

\[
= \frac{1}{d^{2m-1}} \sum_{u_1, u_2} \sum_{u_1', u_2'} \sum_{u_1''} \sum_{u_2''} \prod_{r=1}^{n} \Lambda_i \phi(u_1', u_2') \bigg| \prod_{r=1}^{n} \phi(v_1, v_2) \bigg|_{2r-1,2r},
\]

\[
= \frac{1}{d^{2m-1}} \sum_{u_1, u_2} \sum_{u_1', u_2'} \sum_{u_1''} \sum_{u_2''} \prod_{r=1}^{n} \Lambda_i \phi(u_1', u_2') \bigg| \prod_{r=1}^{n} \phi(v_1, v_2) \bigg|_{2r-1,2r},
\]

where

\[
v'_i = u'_i, i = 2, 3, \ldots, m, r = 1, 2, \ldots, n - 1,
\]

\[
v'_j = u'_{m+1} \oplus u'_j, j = m + 2, m + 3, \ldots, 2m, r = 1, 2, \ldots, n - 1.
\]

### 6.2 Entanglement swapping chains for mixed states

We would only like to consider the entanglement swapping chains for mixed Bell states. We can arrive at

\[
\bigotimes_{r=1}^{n} \phi(u_1, u_2)
\]

\[
= \frac{1}{d^n} \sum_{u_1, u_2} \sum_{u_1', u_2'} \sum_{u_1''} \sum_{u_2''} \prod_{r=1}^{n} \Lambda_i \phi(u_1', u_2') \bigg| \prod_{r=1}^{n} \phi(v_1, v_2) \bigg|_{2r-1,2r},
\]

\[
= \frac{1}{d^n} \sum_{u_1, u_2} \sum_{u_1', u_2'} \sum_{u_1''} \sum_{u_2''} \prod_{r=1}^{n} \Lambda_i \phi(u_1', u_2') \bigg| \prod_{r=1}^{n} \phi(v_1, v_2) \bigg|_{2r-1,2r},
\]

\[
= \frac{\mathcal{P}}{d^n} \sum_{u_1, u_2} \sum_{u_1', u_2'} \sum_{u_1''} \sum_{u_2''} \prod_{r=1}^{n} \Lambda_i \phi(u_1', u_2') \bigg| \prod_{r=1}^{n} \phi(v_1, v_2) \bigg|_{2r-1,2r},
\]

where

\[
\mathcal{P} = \frac{1}{d^n} \sum_{u_1, u_2} \sum_{u_1', u_2'} \sum_{u_1''} \sum_{u_2''} \prod_{r=1}^{n} \Lambda_i \phi(u_1', u_2') \bigg| \prod_{r=1}^{n} \phi(v_1, v_2) \bigg|_{2r-1,2r},
\]

### 7 Concise proof for our recent work

Recently, we have considered some interesting entanglement swapping schemes that can generate two identical entangled states, and demonstrated their applications in quantum information processing [3]. Indeed, the conclusion can be proved using Eq. 69, which is similar to the counterpart in Ref. [3]. In what follows, we would like to provide a more concise proof through Eq. 51. The conclusion, presented in Ref. [3], is given by
Theorem 4 Suppose that there are \( n \) 2-level entangled states, \(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\) containing \( 2m_1, 2m_2, \ldots, 2m_n \) particles each, are in one of the states \(|\phi(u_1, u_2, \ldots, u_m)|\rangle_{u_1, u_2, \ldots, u_m=0}\) (see Eq. 27). Suppose that the first or the last \( m_1, m_2, \ldots, m_n \) particles in \(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\) are selected respectively, and that the measurement operator \( \mathcal{M} \) is performed on them. Let us denote the measurement result as \(|\psi\rangle\), and the state that the remaining particles collapse onto as \(|\phi\rangle\). If the initial states \(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\) meet the following conditions:

1. The number of \(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\) in one of the states \(|\phi(1, u_2, \ldots, u_m)|\rangle_{u_1, u_2, \ldots, u_m=0}\) is even.

2. \( \forall r = 1, 2, \ldots, n \), \( \prod_{i=1}^{m_r} \delta_i(\sum_{i=1}^{m_r} u_i) + \prod_{i=1}^{m_r} \delta_i(\sum_{i=1}^{m_r} v_i) = 1 \), where the subscripts \( i \) represent the selected particles in \(|\phi_r\rangle\) and the subscripts \( m_r + i \) the remaining particles.

then \(|\psi\rangle\) and \(|\phi\rangle\) are the same.

Now we would like to give the proof from the formulas of the entanglement swapping between \( d \)-level maximally entangled states.

Proof Suppose that the first \( k_r \) particles in \(|\phi_r\rangle\) for \( r = 1, 2, \ldots, t \) and the last \( k_r \) particles in \(|\phi_r\rangle\) for \( r = t + 1, t + 2, \ldots, n \) are selected. From Eq. 48, after performing the measurement operator \( \mathcal{M} \) on the selected particles, we can get

\[
\bigotimes_{r=1}^{t} \left| \phi\left(u'_1, u'_2, \ldots, u'_{m_n}\right) \right> \rightarrow \sum_{v'_1, v'_2, \ldots, v'_{m_n} = 0}^{n-1} \xi_{v'_1,v'_2,\ldots,v'_n}^{(1)} v'_1 u'_1 v'_2 u'_2 v'_3 u'_3 \ldots v'_n u'_n \left| \phi\left(v'_1, v'_2, \ldots, v'_1, v'_n, v'_1, v'_2, \ldots, v'_n\right) \right> \otimes \left| \phi\left(v'_1, v'_2, \ldots, v'_1, v'_n, v'_1, v'_2, \ldots, v'_n\right) \right>
\]

(113)

where

\[
v'_j = u'_j, \quad j = 2, 3, \ldots, m_1,
\]

\[
v'_j = v'_j \oplus u'_j, \quad i = 2, 3, \ldots, t, \quad j = 2, 3, \ldots, m_1,
\]

\[
v'_j = v'_j \oplus u'_{m+1} \oplus u'_{m+j}, \quad i = t + 1, t + 2, \ldots, n, \quad j = 2, 3, \ldots, m_1,
\]

\[
v'_1 = v'_1 \oplus u'_1 \oplus v'_1,
\]

\[
v'_j = u'_{m+j} \oplus u'_{m+1}, \quad j = 2, 3, \ldots, m_1,
\]

\[
v'_j = v'_1 \oplus u'_{m+1} \oplus u'_{m+j}, \quad j = 1, 2, \ldots, t, \quad j = 1, 2, \ldots, m_1,
\]

\[
v'_j = v'_1 \oplus u'_{m+1} \oplus u'_{m+1}, \quad i = t + 1, t + 2, \ldots, n,
\]

\[
v'_j = v'_1 \oplus u'_{m+1} \oplus u'_{m+1}, \quad i = t + 1, t + 2, \ldots, n, \quad j = 2, 3, \ldots, m_1.
\]

(114)

The generation of two identical entangled states through entanglement swapping means that the following equations hold:

\[
\oplus_{u'_{m+j}} u'_{1} \oplus v'_{1} = v'_{1}, \quad (115a)
\]

\[
u'_1 \oplus u'_{m+1} \oplus u'_{m+1} = v'_1, \quad (115b)
\]

\[
u'_1 \oplus u'_{m+1} \oplus u'_{m+1} = v'_1, \quad i = 2, 3, \ldots, t,
\]

\[
u'_1 \oplus u'_{m+1} \oplus u'_{m+j} = v'_1 \oplus u'_j, \quad i = 2, 3, \ldots, t, \quad j = 2, 3, \ldots, m_1,
\]

\[
u'_1 \oplus u'_{m+1} \oplus u'_{m+1} = v'_1, \quad i = t + 1, t + 2, \ldots, n,
\]

\[
u'_1 \oplus u'_{m+1} \oplus u'_{m+1} \oplus u'_j = v'_1 \oplus u'_{m+1} \oplus u'_{m+1} \oplus u'_j, \quad i = t + 1, t + 2, \ldots, n, \quad j = 2, 3, \ldots, m_1.
\]

(115f)

From Eq. 115a, we have

\[
\oplus_{u'_{m+j}} u'_{1} = 0,
\]

(116)

which means that the number of \(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\) in one of the states \(|\phi(1, u_2, \ldots, u_m)|\rangle_{u_1, u_2, \ldots, u_m=0}\) is even. From Eqs. 115b-115f, we have

\[
u'_{m+j} = \begin{cases} u'_j, & i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m_1, \\ u'_j, & i = 1, 2, \ldots, n, \quad j = 2, 3, \ldots, m_1, \\ \end{cases}
\]

(117f)

which is consistent with the second condition in Theorem 4. QED.
The above conclusion can be further extended to the entanglement swapping between CAT states, which can be proved through Eq. 69. Moreover, it can be extended to general pure states, but we do not intend to do so, because such an extension may not be of great significance for finding applications in quantum information processing. For two extreme cases, that is, measuring the first half of the particles in all states, or the last half of the particles in all states, the proof can be presented similarly. We would like to give the proof for the latter case.

**Proof** From Eq. 53, the measurement results are given by

\[
\phi\left(v_1^1, v_1^2, \ldots, v_{m_1}^1, v_1^2, \ldots, v_{m_2}^2, \ldots, v_n^1, v_2^2, \ldots, v_{m_n}^n\right),
\]

where

\[
v_j^j = u_{m_j+1}^1 \otimes u_{m_j+1}^1, \quad j = 2, 3, \ldots, m_1,
\]

\[
v_i^i = v_i^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i, \quad i = 2, 3, \ldots, n, \quad j = 2, 3, \ldots, m_i.
\]

The remaining particles collapse onto

\[
\phi\left(\bigotimes_{i=1}^n u_i^i \otimes v_1^i, u_2^1, \ldots, u_{m_1}^1, v_1^2 \otimes u_{m_1+1}^1 \otimes u_{m_1+1}^1, v_1^2 \otimes u_{m_1+1}^1 \otimes u_{m_2+1}^2 \otimes u_2^2, \ldots, v_1^2 \otimes u_{m_1+1}^1 \otimes u_{m_2+1}^2 \otimes u_{m_2+1}^2, \ldots, v_1^2 \otimes u_{m_1+1}^1 \otimes u_{m_2+1}^2 \otimes u_{m_2+1}^2, v_1^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i, v_i^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i, \ldots, v_i^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i\right).
\]

Generating two identical entangled states through entanglement swapping implies that

\[
\bigotimes_{i=1}^n u_i^i \otimes v_1^i = v_1^i,
\]

\[
u_{m_j+1}^1 \otimes u_{m_j+1}^1 = u_j^j, \quad j = 2, 3, \ldots, m_1,
\]

\[
v_1^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i = v_i^i, \quad i = 2, 3, \ldots, n,
\]

\[
v_i^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i = v_i^i \otimes u_{m_i+1}^i \otimes u_{m_i+1}^i, \quad i = 2, 3, \ldots, m_i.
\]

From Eqs. 121 and 122, one can derive the equations as Eqs. 115a-115f. QED.

### 8 Discussion

In this section, we will introduce a novel algorithm for entanglement swapping, and demonstrate the algorithm through the entanglement swapping between two entangled states containing two particles each. We will verify the correctness of the algorithm through the entanglement swapping between two Bell states which is achieved by performing Bell measurements on the first particle in each Bell state.

Let us suppose that there is a quantum system and that the system is isolated all the time, i.e. there has never been any energy exchange between the system and the external environment. We can know from the law of conservation of energy that energy cannot be generated or disappeared out of thin air, so the total energy possessed by such a system must be zero, which means that the energy of the system must be divided into positive energy and negative energy, and these two types of energy are balanced (they are equal in quantity). Here, we might as well use \( |\varphi\rangle \) to represent the state of quantum system, and \(|\alpha\rangle\) and \(|\beta\rangle\) to represent the state of the positive and negative energy of the system, respectively, where \(|\varphi\rangle\) can be assumed to be of any dimension. In this way, the system can be represented as

\[
|\varphi\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle + |\beta\rangle).
\]

Due to the balance between the positive energy and negative energy, it is clear that \(|\beta\rangle = -|\alpha\rangle\), and the probability that \(|\varphi\rangle\) is in either the state \(|\alpha\rangle\) or the state \(|\beta\rangle\) is 50%. It can be seen that \(|\alpha\rangle\) and \(|\beta\rangle\) differ by a global phase, i.e. \(e^{i\pi}\), which however, does not have any observational effects, attributed to

\[
\langle \beta | M_m^* M_m | \beta \rangle = \langle \alpha | e^{-i\pi} M_m^* M_m e^{i\pi} | \alpha \rangle = \langle \alpha | M_m^* M_m | \alpha \rangle,
\]

where \(M_m\) represents a measurement operator and the subscript \(m\) represents a possible measurement result [14]. Therefore, all possible observation results obtained from observing the system \(|\varphi\rangle\) are only \(|\alpha\rangle\); In other words, the probability of obtaining \(|\alpha\rangle\) through observations is 100%.

It should be pointed out that \(|\varphi\rangle\) can be a system with only one subsystem (i.e., a single-particle system) or a system with multiple subsystems (i.e., a multi-particle system). Let us give a few familiar examples, if \(|\varphi\rangle\) is a
single-particle system, one can take \(|\alpha\rangle\) as one of \(|0\rangle\) and \(|1\rangle\), or as one of \(|+\rangle\) and \(|-\rangle\), where \(|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\) and \(|-\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)\), in which case one can establish a coordinate system as shown in Fig. 1. If \(|\psi\rangle\) is a multi-particle system, \(|\alpha\rangle\) can be taken as one of \(|00\rangle\), \(|01\rangle\), \(|10\rangle\) and \(|11\rangle\), or as one of the Bell states (see Eq. 29), such that two coordinate systems shown in Fig. 2 can be established.

Let us now assume that there is an independent quantum system with two subsystems, and assume that \(|\alpha\rangle\) is in one of the following two states:

\[
|\Psi^{\pm}\rangle_{1,2} = \frac{1}{\sqrt{2}} \left( |\bar{u}_1 u_2\rangle \pm |\bar{u}_1 \bar{u}_2\rangle \right),
\]

where the subscripts 1, 2 represent two subsystems, \(|\bar{u}_1 u_2\rangle\) and \(|\bar{u}_1 \bar{u}_2\rangle\) are two sets of orthonormal bases, such that \(|\bar{u}_1|u_1\rangle = |\bar{u}_2|u_2\rangle = 0\) and \(|u_1|\bar{u}_1\rangle = |u_2|\bar{u}_2\rangle = 1\). Table 1 shows all possible combinations of the states of the particles 1 and 2, where the states with global phase, \(-|u_1\rangle\), \(-|\bar{u}_1\rangle\), \(-|u_2\rangle\), \(-|\bar{u}_2\rangle\), can not ignored even though they do not have observational effects, and \(\overline{3}, \overline{4}\) can be considered as the measurement results from \(-|\Psi^{+}\rangle_{1,2}\).

In what follows, let us introduce the new algorithm for computing entanglement swapping. Without losing generality, let us consider the entanglement swapping between two systems each containing two subsystems. The algorithm is also applicable to multi-particle systems, since it is just the generalization of two-particle systems. Let us assume there are two two-particle systems that are independent of the external environment and each other. For simplicity, we assume that the two systems are in the state shown in Eq. 125 and select only the combinations numbered \(\overline{1}\) and \(\overline{2}\) in Table 1. Let us represent the two systems as \(|\Psi^{+}\rangle_{1,2}\) and \(|\Psi^{-}\rangle_{1,2}\) respectively,

\[
|\Psi^{+}\rangle_{1,2} = \frac{1}{\sqrt{2}} |u_1 u_2\rangle \pm \frac{1}{\sqrt{2}} |\bar{u}_1 \bar{u}_2\rangle,
\]

\[
|\Psi^{-}\rangle_{1,2} = \frac{1}{\sqrt{2}} |u_1 \bar{u}_2\rangle \pm \frac{1}{\sqrt{2}} |\bar{u}_1 u_2\rangle.
\]

Furthermore, without loss of generality, assuming that the first subsystem in both \(|\Psi^{+}\rangle_{1,2}\) and \(|\Psi^{-}\rangle_{1,2}\) is observed simultaneously, and the observation result is denoted as \(M_1\), while the state that the remaining subsystems collapse onto is denoted as \(M_2\). In addition, we use the symbol \(S\) to represent the combinations of the states of two subsystems. Let us discuss different scenarios in turn, including four combinations: \(\{ |\Psi^{+}\rangle_{1,2}, |\Psi^{+}\rangle_{1,2}\}\), \(\{ |\Psi^{+}\rangle_{1,2}, |\Psi^{-}\rangle_{1,2}\}\), \(\{ |\Psi^{-}\rangle_{1,2}, |\Psi^{-}\rangle_{1,2}\}\), and \(\{ |\Psi^{-}\rangle_{1,2}, |\Psi^{-}\rangle_{1,2}\}\). For the first case, we list all the possible states of
from which we can get the result of the entanglement swapping between a subsystem and the corresponding observation results in the four sub-tables in Table 2 (Note that for the sake of simplicity, unnecessary coefficients are ignored in the table).

Table 2: Subsystem states and observation results

| $S$ | $M_1$ | $M_2$ |
|-----|-------|-------|
| $(|\alpha_1\rangle, |\alpha_2\rangle)$ | $|\alpha_1\rangle |\alpha_1\rangle$ | $|\alpha_2\rangle |\alpha_2\rangle$ |
| $(|\bar{\alpha}_1\rangle, |\bar{\alpha}_2\rangle)$ | $|\bar{\alpha}_1\rangle |\bar{\alpha}_1\rangle$ | $|\bar{\alpha}_2\rangle |\bar{\alpha}_2\rangle$ |
| $(|\beta_1\rangle, |\beta_2\rangle)$ | $|\beta_1\rangle |\beta_1\rangle$ | $|\beta_2\rangle |\beta_2\rangle$ |
| $(|\bar{\beta}_1\rangle, |\bar{\beta}_2\rangle)$ | $|\bar{\beta}_1\rangle |\bar{\beta}_1\rangle$ | $|\bar{\beta}_2\rangle |\bar{\beta}_2\rangle$ |

From the four sub-tables, we can summarize the following corresponding relationships:

(a) \[ \begin{align*}
|a_1\beta_1\rangle + |\bar{\alpha}_1\bar{\beta}_1\rangle &\leftrightarrow |a_2\beta_2\rangle + |\bar{\alpha}_2\bar{\beta}_2\rangle, \\
|a_1\bar{\beta}_1\rangle + |\bar{\alpha}_1\beta_1\rangle &\leftrightarrow |a_2\bar{\beta}_2\rangle + |\bar{\alpha}_2\beta_2\rangle,
\end{align*} \]

(b) \[ \begin{align*}
|a_1\beta_1\rangle - |\bar{\alpha}_1\bar{\beta}_1\rangle &\leftrightarrow |a_2\beta_2\rangle - |\bar{\alpha}_2\bar{\beta}_2\rangle, \\
-|a_1\bar{\beta}_1\rangle + |\bar{\alpha}_1\beta_1\rangle &\leftrightarrow -|a_2\bar{\beta}_2\rangle + |\bar{\alpha}_2\beta_2\rangle,
\end{align*} \]

(c) \[ \begin{align*}
|a_1\beta_1\rangle - |\bar{\alpha}_1\bar{\beta}_1\rangle &\leftrightarrow |a_2\beta_2\rangle - |\bar{\alpha}_2\bar{\beta}_2\rangle, \\
|a_1\bar{\beta}_1\rangle - |\bar{\alpha}_1\beta_1\rangle &\leftrightarrow |a_2\bar{\beta}_2\rangle - |\bar{\alpha}_2\beta_2\rangle,
\end{align*} \]

(d) \[ \begin{align*}
|a_1\bar{\beta}_1\rangle + |\bar{\alpha}_1\beta_1\rangle &\leftrightarrow |a_2\bar{\beta}_2\rangle + |\bar{\alpha}_2\beta_2\rangle, \\
-|a_1\bar{\beta}_1\rangle - |\bar{\alpha}_1\beta_1\rangle &\leftrightarrow -|a_2\bar{\beta}_2\rangle - |\bar{\alpha}_2\beta_2\rangle,
\end{align*} \]

From which we can get the result of the entanglement swapping between $|\alpha^+\rangle_{1,2}$ and $|\beta^+\rangle_{1,2}$:

\[ |\alpha^+\rangle_{1,2} \otimes |\beta^+\rangle_{1,2} \rightarrow (|a_1\beta_1\rangle + |\bar{\alpha}_1\bar{\beta}_1\rangle) \otimes (|a_2\beta_2\rangle + |\bar{\alpha}_2\bar{\beta}_2\rangle) \]
In a similar way, we can further obtain the results for other entangled swapping cases, which are given by

\[ |\varphi^+\rangle_{1,2} \otimes |\bar{B}^-\rangle_{1,2} \rightarrow \left( |a_1\bar{b}_1\rangle + |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle + |\bar{a}_2b_2\rangle \right) \]

\[ + \left( |a_1\bar{b}_1\rangle - |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle - |\bar{a}_2b_2\rangle \right) \]

\[ + \left( |a_1\bar{b}_1\rangle - |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle - |\bar{a}_2b_2\rangle \right) \] (128)

Let us set \( a_1, b_1, a_2, b_2 \in \{0, 1\} \) in Eq. 128, then four cases of the entanglement swapping between two Bell states can be obtained, which are given by

\[ |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} \rightarrow |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^-\rangle_{1,2} \otimes |\varphi^+\rangle_{3,4} + |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^-\rangle_{1,2} \otimes |\varphi^+\rangle_{3,4} \]

\[ |\psi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} \rightarrow |\varphi^-\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^-\rangle_{1,2} \otimes |\varphi^+\rangle_{3,4} \]

\[ |\psi^-\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} \rightarrow |\varphi^-\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^+\rangle_{1,2} \otimes |\varphi^-\rangle_{3,4} + |\varphi^-\rangle_{1,2} \otimes |\varphi^+\rangle_{3,4} \]

\[ |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} \rightarrow |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} + |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} + |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} + |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} \] (129)

\[ |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} \rightarrow \left( |a_1\bar{b}_1\rangle + |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle - |\bar{a}_2b_2\rangle \right) \]

\[ + \left( |a_1\bar{b}_1\rangle - |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle + |\bar{a}_2b_2\rangle \right) \]

\[ + \left( |a_1\bar{b}_1\rangle - |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle + |\bar{a}_2b_2\rangle \right) \] (130)

\[ |\psi^-\rangle_{1,2} \otimes |\psi^-\rangle_{3,4} \rightarrow \left( |a_1\bar{b}_1\rangle + |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle - |\bar{a}_2b_2\rangle \right) \]

\[ - \left( |a_1\bar{b}_1\rangle + |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle - |\bar{a}_2b_2\rangle \right) \]

\[ - \left( |a_1\bar{b}_1\rangle - |\bar{a}_1b_1\rangle \right) \otimes \left( |a_2\bar{b}_2\rangle + |\bar{a}_2b_2\rangle \right) \] (131)

One can verify the correctness of Eqs.(132-135) through Eq. 67, which indicates that the proposed algorithm is feasible.
9 Conclusion

We have studied entanglement swapping for pure states and mixed states, and proposed basic theory for deriving entanglement swapping results. We have studied the entanglement swapping of 2-level the maximally entangled states by assuming that all the subsystems are in different bases. We have studied the entanglement swapping of multiple d-level maximally entangled states by assuming that measurement operators are performed on the selected particles that contain the first particle in some entangled states and the ones that do not contain the first particle in the remaining entangled states. We have considered the generalization of the entanglement swapping between bipartite general pure states in the multi-system case. We have provided a proof for the entanglement swapping chains of bipartite general pure states through mathematical induction. We have proposed the entanglement swapping chains of multi-particle maximally entangled states, and proposed the entanglement swapping between general pure states and maximally entangled states. We have also proposed entanglement swapping schemes and entanglement swapping chains for mixed states. We have reconsidered our recent work and provided a concise proof for the proposed conclusion. Last but not the least, we have proposed a new algorithm for obtaining entanglement swapping results and verified its correctness through the entanglement swapping of two Bell states. Our algorithm is not more convenient than the existing algebraic algorithms, but it is simpler and easier to understand.

Further research on multi-particle entanglement and entanglement swapping through the proposed algorithm can serve as future research work. The phenomenon of quantum superposition tells us that everything is in a superposition of existence and non-existence, and if we determine that something exists or does not exist, it is because we have observed it. We believe that our work in this paper can provide a new insight into quantum mechanics laws.

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