Research Article

Comparing Numerical Methods for Solving Time-Fractional Reaction-Diffusion Equations

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Multivariate Padé approximation (MPA) is applied to numerically approximate the solutions of time-fractional reaction-diffusion equations, and the numerical results are compared with solutions obtained by the generalized differential transform method (GDTM). The fractional derivatives are described in the Caputo sense. Two illustrative examples are given to demonstrate the effectiveness of the multivariate Padé approximation (MPA). The results reveal that the multivariate Padé approximation (MPA) is very effective and convenient for solving time-fractional reaction-diffusion equations.

1. Introduction

The fractional calculus and fractional differential equations have recently become increasingly important topics in the literature of engineering, science, and applied mathematics. Application areas include viscoelasticity, electromagnetics, heat conduction, control theory, and diffusion [1–4]. Reaction-diffusion equations are commonly used to model the growth and spreading of biological species. A fractional reaction-diffusion equation (FRDE) can be derived from a continuous-time random walk model when the transport is dispersive [5] or a continuous-time random walk model with temporal memory and sources [6]. The topic has received a great deal of attention recently, for example, in systems biology [7], chemistry, and biochemistry applications [8].

One of the time-fractional reaction-diffusion equations is the time-fractional Fisher equation. It was originally proposed by Fisher [9] as a model for the spatial and temporal propagation of a virile gene in an infinite medium. It is encountered in chemical kinetics [10], flame propagation [11], autocatalytic chemical reaction [12], nuclear reactor theory [13], neurophysiology [14], and branching Brownian motion process [15].
Another time-fractional reaction-diffusion equation is the time-fractional Fitzhugh-Nagumo equation. It is an important nonlinear reaction-diffusion equation and usually used to model the transmission of nerve impulses [16, 17]; it is also used in circuit theory, biology, and the area of population genetics [18] as mathematical models.

The generalized differential transform method (GDTM) was presented by [19–21]. This method is based on differential transform method (DTM) [22–25]; the DTM introduces a promising approach for many applications in various domains of science. By using the DTM, a truncated series solution is obtained. This series solution does not exhibit the real behaviors of the problem but gives a good approximation to the true solution in a very small region. Odibat et al. [26] proposed a reliable algorithm of the DTM. The new algorithm accelerates the convergence of the series solution over a large region and improves the accuracy of the DTM. The validity of the modified technique is varied through illustrative examples of Lotka-Volterra, Chen, and Lorenz systems. The generalized differential transform method (GDTM) has been applied to differential equations of fractional order in [19–21, 27].

In the literature, the univariate Padé approximation has been used to obtain approximate solutions of fractional order [28, 29]. So the objective of the this paper is to show the application of the multivariate Padé approximation (MPA) to provide approximate solutions for time-fractional diffusion-reaction equations and to make comparison with the generalized differential transform method (GDTM).

2. Multivariate Padé Approximation

The principles and theory of the multivariate Padé approximation and its applicability for various of differential equations are given in [30–40]. Consider the bivariate function \( f(x, y) \) with Taylor series development

\[
f(x, y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j
\]

around the origin. We know that a solution of univariate Padé approximation problem for

\[
f(x) = \sum_{i=0}^{\infty} c_i x^i
\]

is given by

\[
p(x) = \frac{m \sum_{i=0}^{m-1} c_i x^i \cdots \sum_{i=0}^{m-n} c_i x^i}{c_{m+1} x^m \cdots c_{m+1-n}}
\]

and

\[
q(x) = \frac{1 x \cdots x^n}{c_{m+1} \cdots c_{m+1-n}}
\]

(2.3)
Let us now multiply $j$th row in $p(x)$ and $q(x)$ by $x^{j+m-1} (j = 2, \ldots, n + 1)$ and afterwards divide $j$th column in $p(x)$ and $q(x)$ by $x^{j-1} (j = 2, \ldots, n + 1)$. This results in a multiplication of numerator and denominator by $x^{mn}$. Having done so, we get

\[
p(x) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j - \sum_{i=m+1}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j - \sum_{i=0}^{m} \sum_{j=m+1}^{n} c_{ij} x^i y^j}{\sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j - \sum_{i=m+1}^{m} \sum_{j=0}^{n} c_{ij} x^i y^j - \sum_{i=0}^{m} \sum_{j=m+1}^{n} c_{ij} x^i y^j}
\]

(2.4)

if ($D = \text{det} D_{m,n} \neq 0$).

This quotient of determinants can also immediately be written down for a bivariate function $f(x, y)$. The sum $\sum_{k=0}^{m} c_k x^k$ will be replaced with $k$th partial sum of the Taylor series development of $f(x, y)$ and the expression $c_k x^k$ by an expression that contains all the terms of degree $k$ in $f(x, y)$. Hereby, a bivariate term $c_{ij} x^i y^j$ is said to be of degree $i + j$. If we define

\[
p(x, y) = \begin{vmatrix}
\sum_{i+j=0}^{m} c_{ij} x^i y^j & \sum_{i+j=1}^{m} c_{ij} x^i y^j & \cdots & \sum_{i+j=m}^{m} c_{ij} x^i y^j \\
\sum_{i+j=0}^{m} c_{ij} x^i y^j & \sum_{i+j=1}^{m} c_{ij} x^i y^j & \cdots & \sum_{i+j=m}^{m} c_{ij} x^i y^j \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{i+j=0}^{m} c_{ij} x^i y^j & \sum_{i+j=1}^{m} c_{ij} x^i y^j & \cdots & \sum_{i+j=m}^{m} c_{ij} x^i y^j
\end{vmatrix}
\]

(2.5)

\[
q(x, y) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{vmatrix}
\]

then it is easy to see that $p(x, y)$ and $q(x, y)$ are of the form

\[
p(x, y) = \sum_{i+j=mn}^{mn+m} a_{ij} x^i y^j, \quad q(x, y) = \sum_{i+j=mn}^{mn+n} b_{ij} x^i y^j.
\]

(2.6)

We know that $p(x, y)$ and $q(x, y)$ are called Padé equations [30]. So the multivariate Padé approximant of order $(m, n)$ for $f(x, y)$ is defined as,

\[
r_{m,n}(x, y) = \frac{p(x, y)}{q(x, y)}.
\]

(2.7)
3. Generalized Differential Transform Method

The fractional derivatives are described in the Caputo sense which are defined in [41] as

\[
D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t) dt, \tag{3.1}
\]

for \( m - 1 < \alpha \leq m, m \in \mathbb{N}, x > 0 \); for \( m \) to be the smallest integer that exceeds \( \alpha \), the Caputo time-fractional derivative operator of order \( \alpha > 0 \) is defined as

\[
D^\alpha t u(x,t) = \frac{\partial^m u(x,t)}{\partial t^m}, \quad \text{for } \alpha = m \in \mathbb{N}. \tag{3.2}
\]

The basic definitions and fundamental operations of generalized differential transform method are defined in [19–21] as follows.

Definition 3.1. The generalized differential transform of the function \( u(x,y) \) is given as follows:

\[
U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} \left( (D^\alpha_{x_0})^k (D^\beta_{y_0})^h \right)_{(x_0,y_0)}, \tag{3.3}
\]

where \((D^\alpha_{x_0})^k = D^\alpha_{x_0} \cdot D^\alpha_{x_0} \cdots D^\alpha_{x_0}\).

Definition 3.2. The generalized differential inverse transform of \( U_{\alpha,\beta}(k,h) \) is defined as follows:

\[
u(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h) (x-x_0)^{k\alpha} (y-y_0)^{h\beta}. \tag{3.4}\]

The fundamental operations of generalized differential transform method are listed in Table 1 (see [19–21]).

4. Numerical Experiments

In this section, two methods, GDTM and MPA, will be illustrated by two examples, the time-fractional Fisher equation and the time-fractional FitzHugh-Nagumo equation. All the numerical results are calculated by using the software Maple12. The general model of reaction-diffusion equations is

\[
\frac{\partial^\alpha u}{\partial t^\alpha} = D \frac{\partial^2 u}{\partial x^2} + f(u), \quad 0 < \alpha \leq 1, \ t > 0, \ x \in \mathbb{R}, \tag{4.1}
\]

where \( D \) is the diffusion coefficient, and \( f(u) \) is a nonlinear function representing reaction kinetics.
Table 1: The operations of the GDTM.

| Original functions | Transformed functions |
|--------------------|-----------------------|
| \( u(x, y) = v(x, y) \pm w(x, y) \) | \( U_{\alpha, \beta}(k, h) = V_{\alpha, \beta}(k, h) \pm W_{\alpha, \beta}(k, h) \) |
| \( u(x, y) = \lambda v(x, y) \) | \( U_{\alpha, \beta}(k, h) = \lambda V_{\alpha, \beta}(k, h) \) |
| \( u(x, y) = D_x^\alpha v(x, y) \) | \( U_{\alpha, \beta}(k, h) = \frac{\Gamma(\alpha(k + 1) + 1)}{\Gamma(\alpha k + 1)} V_{\alpha, \beta}(k + 1, h), \ 0 < \alpha \leq 1 \) |
| \( u(x, y) = D_y^\beta v(x, y) \) | \( U_{\alpha, \beta}(k, h) = \frac{\Gamma(\beta(h + 1) + 1)}{\Gamma(\beta h + 1)} V_{\alpha, \beta}(k, h + 1), \ 0 < \beta \leq 1 \) |
| \( u(x, y) = D_x^\alpha D_y^\beta v(x, y) \) | \( U_{\alpha, \beta}(k, h) = \frac{\Gamma(\alpha(k + 1) + 1)\Gamma(\beta(h + 1) + 1)}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} V_{\alpha, \beta}(k + 1, h + 1), \ 0 < \alpha, \beta \leq 1 \) |
| \( u(x, y) = D_x^\gamma v(x, y) \) | \( U_{\alpha, \beta}(k, h) = \frac{\Gamma(\alpha + \gamma + 1)}{\Gamma(\alpha k + 1)} V_{\alpha, \beta}\left(\frac{k + \frac{\gamma}{\alpha}, h}{h}\right), \ m - 1 < \gamma \leq 1 \) |
| \( u(x, y) = D_x^\gamma D_y^\delta v(x, y) \) | \( U_{\alpha, \beta}(k, h) = \frac{\Gamma(\alpha + \gamma + 1)\Gamma(\beta h + \delta + 1)}{\Gamma(\alpha k + 1)\Gamma(\beta h + 1)} V_{\alpha, \beta}\left(\frac{k + \frac{\gamma}{\alpha}, h + \frac{\delta}{\beta}}{h}\right) \) |
| \( u(x, y) = (x - x_0)^\eta (x - x_0)^\eta \) | \( U_{\alpha, \beta}(k, h) = \delta(k - m)\delta(h - m) \) |
| \( u(x, y) = v(x, y)w(x, y) \) | \( U_{\alpha, \beta}(k, h) = \sum_{r=0}^{k-h} \sum_{s=0}^{h} V_{\alpha, \beta}(r, h - s) W_{\alpha, \beta}(k - r, s) \) |
| \( u(x, y) = v(x, y)w(x, y)q(x, y) \) | \( U_{\alpha, \beta}(k, h) = \sum_{r=0}^{k-h} \sum_{l=0}^{k-r} \sum_{s=0}^{h} \sum_{p=0}^{s} V_{\alpha, \beta}(r, h - s - p) W_{\alpha, \beta}(t, s) Q_{\alpha, \beta}(k - r - t, p) \) |
Example 4.1. Let us consider (4.1) with \( f(u) = 6u(1 - u) \), then we have the time-fractional Fisher equation [27]

\[
D^\alpha_t u = D^\alpha_x u + 6u(1 - u), \quad 0 < \alpha \leq 1, \quad t > 0, \quad x \in \mathcal{R},
\]

subject to the initial condition

\[
u(x, 0) = \frac{1}{(1 + e^x)^2}.
\]

Selecting \( \beta = 1 \) and applying the generalized differential transform of (4.2), using the related definitions in Table 1, Rida et al. [27] solved as it follows:

\[
\frac{\Gamma(\alpha(h + 1) + 1)}{\Gamma(\alpha h + 1)} U_{\alpha,1}(k, h + 1)
= (k + 1)(k + 2)U_{\alpha,1}(k + 2, h) + 6U_{\alpha,1}(k, h) - 6 \sum_{r=0}^{k} \sum_{s=0}^{h} U_{\alpha,1}(r, h - s)U_{\alpha,1}(k - r, s),
\]

that is,

\[
U_{\alpha,1}(k, h + 1)
= \frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)} \left[ (k+1)(k+2)U_{\alpha,1}(k+2, h) + 6U_{\alpha,1}(k, h) - 6 \sum_{r=0}^{k} \sum_{s=0}^{h} U_{\alpha,1}(r, h - s)U_{\alpha,1}(k - r, s) \right].
\]

By equating the series form of (4.3) with (3.4), the initial transformation coefficients \( U_{\alpha,1}(k, 0) \), \( k = 0, 1, 2, \ldots \) can be obtained as follows:

\[
U_{\alpha,1}(0, 0) = \frac{1}{4}, \quad U_{\alpha,1}(1, 0) = -\frac{1}{4}, \quad U_{\alpha,1}(2, 0) = \frac{1}{16}, \quad U_{\alpha,1}(3, 0) = \frac{1}{48}, \quad U_{\alpha,1}(4, 0) = -\frac{1}{96}.
\]

By applying (4.6) into (4.5), some values of \( U_{\alpha,1}(k, h) \) can be obtained as given in Table 1. Consequent substitution of all \( U_{\alpha,1}(k, h) \) into (3.4) and after some manipulations, the series from solutions of (4.2) and (4.3) has been obtained in [27] as

\[
u(x, t) = \left( \frac{1}{4} + \frac{5}{4 \Gamma(\alpha + 1)} t^\alpha + \frac{25}{8 \Gamma(2\alpha + 1)} t^{2\alpha} + \cdots \right)
+ \left( \frac{1}{4} - \frac{5}{8 \Gamma(\alpha + 1)} t^\alpha + \frac{25}{8 \Gamma(2\alpha + 1)} t^{2\alpha} + \cdots \right) x
+ \left( \frac{1}{16} - \frac{5}{16 \Gamma(\alpha + 1)} t^\alpha - \frac{25}{8 \Gamma(2\alpha + 1)} t^{2\alpha} + \cdots \right) x^2
+ \left( \frac{1}{48} - \frac{5}{24 \Gamma(\alpha + 1)} t^\alpha - \frac{25}{24 \Gamma(2\alpha + 1)} t^{2\alpha} + \cdots \right) x^3
+ \left( -\frac{1}{96} + \frac{5}{96 \Gamma(\alpha + 1)} t^\alpha + \frac{425}{384 \Gamma(2\alpha + 1)} t^{2\alpha} + \cdots \right) x^4.
\]
The exact solution of

\[ u(x, t) \]

and let

\[ \text{Fisher equation} \]

\[ x, t \]

can be written in the form:

\[
\begin{align*}
    u(x, t) & = \left( \frac{1}{4} - \frac{1}{4} x + \frac{1}{48} x^3 - \frac{1}{96} x^4 + \cdots \right) + \left( \frac{5}{4} - \frac{5}{8} x - \frac{5}{16} x^2 - \frac{5}{24} x^3 + \frac{5}{96} x^4 + \cdots \right) \frac{t^\alpha}{\Gamma(\alpha + 1)} \\
    & + \left( \frac{25}{8} + \frac{25}{8} x^2 - \frac{25}{24} x^3 + \frac{425}{384} x^4 + \cdots \right) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}.
\end{align*}
\]

(4.8)

The exact solution of (4.2), for the special case \( \alpha = 1.0 \), is given in \([27]\) as

\[
    u(x, t) = \frac{1}{(1 + e^{x - s})^2}.
\]

(4.9)

We have the generalized differential transform method solution for the time-fractional Fisher equation (4.2) (when \( \alpha = 1.0 \)) as

\[
\begin{align*}
    u(x, t) & = 0.2500000000 - 0.2500000000 x - 0.06250000000 x^2 + 0.0208333333 x^3 \\
    & - 0.0104166667 x^4 + (1.250000000 - 0.6250000000 x - 0.3125000000 x^2 \\
    & - 0.2083333333 x^3 + 0.0520833333 x^4) t \\
    & + 0.5000000000(3.125000000 + 3.125000000 x - 3.125000000 x^2 \\
    & - 1.401666667 x^3 + 1.106770833 x^4) t^2, \\
    & = 0.2500000000 - 0.2500000000 x - 0.06250000000 x^2 + 0.0208333333 x^3 \\
    & - 0.0104166667 x^4 + 1.250000000 t - 0.62500000000 x t - 0.3125000000 x^2 t \\
    & - 0.2083333333 x^3 t + 0.0520833333 x^4 t + 1.5625000000 t^2 + 1.5625000000 x t^2 \\
    & - 1.5625000000 x^2 t^2 - 0.5208333335 x^3 t^2 + 0.5533854165 x^4 t^2,
\end{align*}
\]

(4.10)

and let

\[
\begin{align*}
    A & = 0.2500000000 - 0.2500000000 x - 0.06250000000 x^2 + 0.0208333333 x^3 \\
    & - 0.0104166667 x^4 + 1.25000000000 t - 0.62500000000 x t - 0.3125000000 x^2 t \\
    & - 0.2083333333 x^3 t + 1.56250000000 t^2 + 1.5625000000 x t^2 - 1.5625000000 x^2 t^2, \\
    B & = 0.2500000000 - 0.2500000000 x - 0.06250000000 x^2 + 0.0208333333 x^3 \\
    & + 1.25000000000 t - 0.62500000000 x t - 0.3125000000 x^2 t + 1.5625000000 t^2 \\
    & + 1.5625000000 x t^2,
\end{align*}
\]
\[ C = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 \\
+ 1.2500000000t - 0.6250000000xt + 1.5625000000t^2. \]
\[(4.12)\]

Then let us calculate the approximate solution of (4.10) for \( m = 4 \) and \( n = 2 \) by using multivariate Padé approximation. To obtain multivariate Padé equations of (4.10) for \( m = 4 \) and \( n = 2 \), we use (2.5). By using (2.5), we obtain

\[
p(x,t) = -0.5533854165x^4 \left(-0.0001225490198x^5t - 0.03063725491t^5x^4 \\
+ 0.3082873774t^4x^3 + 0.02037377449t^3x^4 + 0.003604983663t^2x^5 \\
+ 0.00004901960791x^5 - 0.00001225490198x^6 - 1.470588235t^4 \\
- 0.3676470590t^4x^2 + 0.002757352939t^3x^6 - 0.04289215684t^2x^3 \\
+ 0.09803921566t^3x^2 - 0.0001633986928t^5x^6 - 0.2573529412t^3x \\
+ 0.1608455885t^4x + 0.0001225490196x^7t^2 - 0.01072303921x^8t^4 \\
- 0.06318933824t^5x^5 - 0.001914828416t^5x^6 + 0.00004084967320x^8t^2 \\
- 0.002323321562x^7t^3 + 0.02221200981x^8t^4 - 0.00002042483661x^9t^2 \\
- 0.0006382761434x^8t^3 + 0.007531658495x^7t^4 \\
- 0.4084967326 \times 10^{-5}x^7 + 0.2042483663 \times 10^{-5}x^8 - 9.191176472t^6 \\
+ 5.840226718t^6x^2 - 9.334788603t^6x - 7.352941178t^5 \\
+ 4.049862133t^5x^3 - 2.202052696t^5x^3 + 1.953125001t^5x \\
+ 0.3498391544t^4x^4 + 0.05895118467t^3x^5 + 0.002024611930t^2x^6 \\
+ 0.0001021241832tx^7 - 0.00004901960791x^4 - 0.006587009808t^2x^4 \\
+ 0.02205882354tx^3 + 0.001470588237tx^4 - 0.03431372552t^2x^2 \\
- 0.001960784315t^3),
\]
\[ q(x,t) = \begin{vmatrix} 1 & 1 & 1 \\ 0.0520833333x^4t - 0.5533854165x^2t^2 & \mathcal{A} & \mathcal{B} \\ 0.5533854165x^2t^2 & 0.0520833333x^4t - 0.5533854165x^2t^2 & \mathcal{A} \end{vmatrix} \]

\[ = 0.5533854165x^4 \left( 1.029411764t^3 + 5.882352942t^4 + 0.091911764t^4 x \right) \]

- 0.001960784314t^2x^3 + 0.01960784314t^3x^4 + 0.0208333333t^3x^5

+ 0.01531862746t^2x^4 + 0.1102941176t^3x^3 + 2.052696079t^4x^2

+ 0.007843137258tx^3 + 0.1372549020t^2x^2 + 0.009803921564t^3x^3

- 0.0490196080t^3x^2 + 0.001960784315x^4 + 0.009803921572tx^4 \right), \]

(4.13)

where \( \mathcal{A} \) denotes \(-0.01041666667x^4 - 0.2083333333x^3t - 1.5625000000x^2t^2\), and \( \mathcal{B} \) denotes \(0.0208333333x^3 - 0.3125000000x^2t + 1.5625000000xt^2\). So the multivariate Padé approximation is of order (4, 2) for (4.10), that is,

\[ [4, 2]_{(x,t)} = \left( -0.0001225490198x^5 - 0.03063725491t^6x^4 \right) \]

+ 0.3082873774t^4x^3 + 0.2037377449t^3x^4 + 0.003604983663t^2x^5

+ 0.00004901960791x^5 - 0.0001225490198x^6 - 1.470588235t^4

- 0.3676470590t^4x^2 + 0.02757352939t^3x^3 - 0.04289215684t^3x^3

+ 0.09803921566t^2x^2 - 0.000163986928tx^6 - 0.2573529412t^3x

+ 0.1608455885t^4x + 0.0001225490196x^7t^2 - 0.01072303921x^5t^4

- 0.06318933824t^5x^3 - 0.001914828416t^6x^6 + 0.0000408967320x^8t^2

- 0.00232325162x^7t^3 + 0.02221200981x^6t^4 - 0.00002042483661x^9t^2

- 0.0006382761434x^8t^3 + 0.007531658495x^7t^4 - 0.4084967326 \times 10^{-5}x^7

+ 0.2042483663 \times 10^{-5}x^8 - 9.191176472t^6 + 5.840226718t^6x^2 - 9.334788603t^6x

- 7.352941178t^5 + 4.049862133t^5x - 2.202052696t^5x^2 + 1.953125001t^6x

+ 0.3498391544t^4x^4 + 0.05895118467t^3x^3 + 0.002024611930t^2x^6

+ 0.0001021241832tx^7 - 0.00004901960791x^4 - 0.006587009808t^2x^4

+ 0.02205882354t^2x^3 + 0.001470588237tx^4 - 0.03431372552t^2x^2

- 0.001960784315tx^3) / (1.029411764t^3x + 5.882352942t^4 + 0.091911764t^4x)
The generalized differential transform method gives the solution for the time-fractional Fisher equation (4.2) (when $\alpha = 0.5$) which is given by

\[
 u(x, t) = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 + 0.0208333333x^3 - 0.0104166667x^4 \\
+ 1.128379167(1.2500000000 - 0.6250000000x - 0.3125000000x^2 - 0.2083333333x^3 + 0.0520833333x^4)t^{0.5} \\
+ (3.1250000000 + 3.1250000000x - 3.1250000000x^2 - 1.40166667x^3 + 1.1067708333x^4)t.
\]

(4.15)

For simplicity, let $t^{1/2} = a$, then

\[
 u(x, t) = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 \\
+ 0.0208333333x^3 - 0.0104166667x^4 \\
+ 1.128379167(1.2500000000 - 0.6250000000x - 0.3125000000x^2 - 0.2083333333x^3 + 0.0520833333x^4)a \\
+ (3.1250000000 + 3.1250000000x - 3.1250000000x^2 - 1.40166667x^3 + 1.1067708333x^4)a^2, \\
= 0.2500000000 - 0.2500000000x - 0.06250000000x^2 + 0.0208333333x^3 \\
- 0.0104166667x^4 + 1.410473959a - 0.7052369794ax - 0.3526184897ax^2 \\
- 0.2350789931ax^3 + 0.05876974828ax^4 + 3.1250000000a^2 + 3.1250000000a^2x \\
- 3.1250000000a^2x^2 - 1.40166667a^2x^3 + 1.1067708333a^2x^4,
\]

(4.16)

and let

\[
 E = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 + 0.0208333333x^3 \\
- 0.0104166667x^4 + 1.410473959a - 0.7052369794ax - 0.3526184897ax^2 \\
- 0.2350789931ax^3 + 3.1250000000a^2 + 3.1250000000a^2x - 3.1250000000a^2x^2,
\]
\[ F = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 + 0.0208333333x^3 + 1.410473959a - 0.7052369794ax - 0.3526184897ax^2 + 3.125000000a^2 + 3.125000000a^2x, \]
\[ G = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 + 1.410473959a - 0.7052369794ax + 3.125000000a^2. \]

(4.18)

Then, using (2.5) to calculate the multivariate Padé equations for (4.16), we get

\[ p(x, a) \]
\[ = \begin{vmatrix}
    E & F & G \\
    0.05876974828ax^4 - 1.401666667a^2x^3 & 0.05876974828ax^4 - 1.401666667a^2x^3 & 0.05876974828ax^4 - 1.401666667a^2x^3 \\
    1.106770833a^2x^4 & 0.05876974828ax^4 - 1.401666667a^2x^3 & 0.05876974828ax^4 - 1.401666667a^2x^3 \\
\end{vmatrix} \]
\[ = -1.106770833x^4(1.724963655a^4 + 0.00055331270431ax^4 + 0.000024509803928x^6 - 0.2042483661 \times 10^{-5} \times 2\times 10^{-5} \times 10^{-5} \times a + 0.101214131 \times 10^{-5} + 0.02621254569a^2x^3 - 0.005445232711a^2x^4 - 0.03677161152a^2x^2 - 0.0008296905640ax^3 - 0.2074226410a^3x + 0.000104497505a^2x^3 + 0.07657717919a^3x^4 + 0.0659779402a^4x^3 - 0.1247554646a^3x^3 - 1.014826625a^4x^2 + 0.1 \times 10^{-12}x^5 + 0.1019438376a^3x^2 - 0.00006914088034ax^6 - 2.941176471a^4 + 36.76470589a^6 + 23.3609687a^6x^2 - 37.33915441a^6x - 16.59381128a^5 + 9.204379692a^5x^3 - 2.830454788a^5x^2 + 5.444844332a^5x + 0.3449658071a^4x^4 + 0.0704202320a^3x^5 + 0.001477816147a^2x^6 + 0.00004609392028ax^7 - 0.00002450980395x^4). \]

(4.19)
where $C$ is $-0.0104166667x^4 - 0.2350789931ax^3 - 3.125000000a^2x + 0.0208333333x - 0.3526184979a^2x^2 - 0.3125000000a^3x$ recalling that $t^{1/2} = a$, we get multivariate Padé approximation of order $(4,2)$ for (4.15), that is,

$$ [4,2]_{(x,t)} = -(1.724963655t^2x + 0.00053331270431\sqrt{t}x^4 + 0.00002450980395x^5 $$

$$ - 0.6127450986 \times 10^{-5}x^6 - 0.2042483661 \times 10^{-5}x^7 + 0.1021241831 \times 10^{-5}x^8 $$

$$ + 0.02621254569tx^3 - 0.005445232711tx^4 - 0.03677161152tx^2 $$

$$ - 0.0008296905640\sqrt{t}x^3 - 0.2074226410t^{3/2}x + 0.000104497505tx^5 $$

$$ + 0.07657717919t^{3/2}x^4 + 0.0659779402t^2x^3 - 0.1247554646t^{3/2}x^3 $$

$$ - 1.014826625t^2x^2 + 0.1 \times 10^{-12}x^5\sqrt{t} + 0.1019438376t^{3/2}x^2 $$

$$ - 0.00006914088034\sqrt{t}x^5 - 2.941176471f^2 - 36.764705891t^3 $$

$$ + 23.3609687t^3x^2 - 37.33915441t^3x - 16.59381128t^{5/2} + 9.204379692t^{3/2}x^3 $$

$$ - 2.830454788t^{3/2}x^2 + 5.44844332t^{5/2}x + 0.3449658071t^2x^4 $$

$$ + 0.07042023202t^{3/2}x^5 + 0.001477816147tx^6 $$

$$ + 0.00004609392028\sqrt{t}x^7 - 0.0002450980395x^4) $$

$$ / (11.76470589t^2 + 0.003318762259\sqrt{t}x^3 + 0.00009803921577x^4 $$

$$ + 0.1470864461tx^2 + 0.02351215238tx^3 $$

$$ + 0.01353735183tx^4 + 0.1244535845t^{3/2}x^3 + 4.105392158t^2x^2 $$

$$ - 0.4079311941t^{3/2}x^2 + 0.183823528t^2x + 0.0005531270429\sqrt{t}x^4 $$

$$ + 0.8296905634t^{3/2}x). $$

(4.20)

The generalized differential transform method gives the solution for the time-fractional Fisher equation (4.2) (when $\alpha = 0.75$) which is given by

$$ u(x,t) = 0.2500000000 - 0.2500000000x - 0.06250000000x^2 + 0.0208333333x^3 $$

$$ - 0.0104166667x^4 + 1.088065252(1.250000000 - 0.6250000000x - 0.3125000000x^2 $$

$$ - 0.2083333333x^3 + 0.0520833333x^4)t^{0.75} $$

$$ + 0.7522527782(3.125000000 + 3.125000000x $$

$$ - 3.125000000x^2 - 1.401666667x^3 + 1.106770833x^4)t^{1.50} $$

(4.21)
For simplicity, let \( t^{1/4} = a \), then

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 0.2500000000 - 0.2500000000x - 0.6250000000x^2 + 0.0208333333x^3 \\
&- 0.01041666667x^4 + 1.360081565a^3 - 0.6800407825a^3x - 0.3400203912a^3x^2 \\
&- 0.22668002608a^3x^3 + 0.5667006520a^3x^4 + 2.350789932a^6 + 2.350789932a^6x \\
&- 2.350789932a^6x^2 - 0.7835966442a^6x^3 + 0.8325714340a^6x^4,
\end{align*}
\]

(4.22)

\[
\begin{align*}
\frac{\partial u}{\partial a} &= 0.2500000000 - 0.2500000000x - 0.6250000000x^2 + 0.0208333333x^3 \\
&- 0.01041666667x^4 + 1.360081565a^3 - 0.6800407825a^3x - 0.3400203912a^3x^2 \\
&- 0.22668002608a^3x^3 + 0.5667006520a^3x^4 + 2.350789932a^6 + 2.350789932a^6x \\
&- 2.350789932a^6x^2 - 0.7835966442a^6x^3 + 0.8325714340a^6x^4,
\end{align*}
\]

(4.23)

and let

\[
\begin{align*}
H &= 0.2500000000 - 0.2500000000x - 0.6250000000x^2 + 0.0208333333x^3 \\
&- 0.01041666667x^4 + 1.360081565a^3 - 0.6800407825a^3x - 0.3400203912a^3x^2 \\
&- 0.22668002608a^3x^3 + 0.5667006520a^3x^4 + 2.350789932a^6 + 2.350789932a^6x, \\
K &= 0.25000000000 - 0.2500000000x - 0.6250000000x^2 + 0.0208333333x^3 \\
&- 0.01041666667x^4 + 1.360081565a^3 - 0.6800407825a^3x - 0.3400203912a^3x^2 \\
&- 0.22668002608a^3x^3 + 2.350789932a^6, \\
L &= 0.2500000000 - 0.2500000000x - 0.6250000000x^2 + 0.0208333333x^3 \\
&- 0.01041666667x^4 + 1.360081565a^3 - 0.6800407825a^3x - 0.3400203912a^3x^2.
\end{align*}
\]

(4.24)

Then, using (2.5) to calculate the multivariate Padé equations for (4.23), we get

\[
p(x, a) = \begin{vmatrix}
H & K & L \\
-2.350789932a^6x^2 & \xi & \eta \\
-0.7835966442a^6x^3 & -2.350789932a^6x^2 & \xi
\end{vmatrix} \\
= 1.842071102x^2(0.55703900549a^6x^4 + 14.10473959a^{12} - 0.9467234329a^6x^3) \\
- 0.00001816058365x^{10} + 0.00003632116729x^9 + 0.0004358540080a^6
\]

(4.25)
\[-0.0004358540080x^7 + 0.0001089635019x^8 + 0.07834715868a^3x^4\]
\[-0.03616022708a^3x^3 + 1.5000000000a^6 + 0.8750000002a^2x^2\]
\[-0.9999999998a^6x - 0.04520028386a^3x^5 + 2.040122348a^9x^2\]
\[+ 18.80631945a^{12}x + 0.000987995000a^3x^{10} - 5.100305867a^9x^3\]
\[+ 0.00387786414a^3x^6 - 0.001848360442a^3x^8 - 0.0006463106887a^3x^9\]
\[+ 0.00534335889a^3x^7 - 0.05054699319a^6x^7 + 0.0297127611a^6x^5\]
\[-0.07855231035a^6x^6 - 1.416751630a^9x^4 - 1.360081565a^9x\]
\[+ 8.160489388a^9)\]

\[q(x, a) = \begin{vmatrix} 1 & 1 & 1 \\ -2.350789932a^6x^2 & \mathcal{E} & \mathcal{F} \\ -0.7835966442a^6x^3 & -2.350789932a^6x^2 & \mathcal{E} \end{vmatrix} = 1.842071102x^2( -0.1446409084a^3x^3 + 0.1687477264a^3x^4 + 5.999999998a^6\]
\[+ 1.999999999a^6x + 3.999999999a^6x^2 + 0.001743416031x^6\]
\[+ 0.02410681806a^3x^5)\]

(4.25)

where \(\mathcal{E}\) denotes \(0.05667006520a^3x^4 + 2.350789932a^6x\), and \(\mathcal{F}\) denotes \(2.350789932a^6 - 0.22668002608a^3x^3\); recalling that \(t^{1/4} = a\), we get multivariate Padé approximation of order \(7, 2\) for (4.21), that is,

\([7, 2]_{(x,t)} = (0.55703900549t^{3/2}x^4 + 14.10473959t^3 - 0.9467234329t^{3/2}x^3\]
\[-0.0000181058365x^{10} + 0.0003632116729x^9 + 0.0004358540080x^6\]
\[-0.0004358540080x^7 + 0.0001089635019x^8 + 0.07834715868t^{3/4}x^4\]
\[0.03616022708t^{3/4}x^3 + 1.500000000t^{3/2} + 0.8750000002t^{3/2}x^2\]
\[-0.9999999998t^{3/2}x - 0.04520028386t^{3/4}x^5 + 2.040122348t^{9/4}x^2\]
\[+ 18.80631945t^3x + 0.000987995000t^{3/4}x^{10} - 5.100305867t^{9/4}x^3\]
\[+ 0.00387786414t^{3/4}x^6 - 0.001848360442t^{3/4}x^8 - 0.0006463106887t^{3/4}x^9\]
\[+ 0.00534335889t^{3/4}x^7 - 0.5054699319t^{3/2}x^7 + 0.0297127611t^{3/2}x^5\]
the multivariate Padé approximation which we know the exact solution \( u \) subject to the initial condition \( Rida et al. /bracketleftmath ISRN Mathematical Analysis 15 \)

\[
\text{Taking the generalized differential of } \alpha \text{ of the Fitzhugh-Nagumo equation Let us consider Example 4.2.}
\]

\[
\Gamma \begin{pmatrix} \alpha(h + 1) + 1 \\ \Gamma(\alpha h + 1) \end{pmatrix} U_{a,1}(k, h + 1) 
= (k + 1)(k + 2)U_{a,1}(k + 2, h) - \mu U_{a,1}(k, h) + (1 + \mu) \sum_{r=0}^k \sum_{s=0}^h U_{a,1}(r, h - s)U_{a,1}(k - r, s) 
- \sum_{r=0}^k \sum_{t=0}^{k-r} \sum_{s=0}^{h-s} U_{a,1}(r, h - s - p)U_{a,1}(t, s)U_{a,1}(k - r - t, p),
\]

(4.29)
**Table 2:** Numerical values when $\alpha = 0.50$, $\alpha = 0.75$, and $\alpha = 1.0$ for Example 4.1.

| $x$  | $t$  | $\delta_{GDTM}$ | $\delta_{MPA}$ | $\delta_{GDTM}$ | $\delta_{MPA}$ | $\delta_{GDTM}$ | $\delta_{MPA}$ | $\delta_{Exact}$ |
|------|------|-----------------|-----------------|------------------|-----------------|------------------|-----------------|-----------------|
| 0.01 | 0.01 | 0.4194042447    | 0.4194042447    | 0.2926737579     | 0.2926737582    | 0.2601012529     | 0.2601012532    | 0.2600986403    |
| 0.02 | 0.02 | 0.5062062188    | 0.5062062185    | 0.3234070082     | 0.3234070086    | 0.2704098802     | 0.2704098802    | 0.2703889140    |
| 0.03 | 0.03 | 0.5796147904    | 0.5796147915    | 0.3516746983     | 0.3516747025    | 0.2809328587     | 0.2809328609    | 0.2808618961    |
| 0.04 | 0.04 | 0.6462358324    | 0.6462358347    | 0.3787958658     | 0.3787958838    | 0.2916767275     | 0.2916767370    | 0.2915080826    |
| 0.05 | 0.05 | 0.7086276674    | 0.7086276723    | 0.4053118054     | 0.4053118748    | 0.3026475836     | 0.3026476112    | 0.3023174246    |
| 0.06 | 0.06 | 0.7680978547    | 0.7680978705    | 0.4315082304     | 0.4315084185    | 0.3138510763     | 0.3138511466    | 0.3132793692    |
| 0.07 | 0.07 | 0.8254176805    | 0.8254177184    | 0.4575570408     | 0.4575574788    | 0.3252924053     | 0.3252925577    | 0.3243829010    |
| 0.08 | 0.08 | 0.8810841195    | 0.8810842025    | 0.4835702540     | 0.4835711634    | 0.3369763157     | 0.3369766126    | 0.3356165892    |
| 0.09 | 0.09 | 0.9354369921    | 0.9354371578    | 0.5096247442     | 0.5096264889    | 0.3489070959     | 0.3489076324    | 0.3469686330    |
| 0.1  | 0.1  | 0.9887186126    | 0.9887189207    | 0.5357751408     | 0.5357782276    | 0.3610885742     | 0.3610894836    | 0.3584266914    |
that is,

\[ U_{\alpha,1}(k, h + 1) = \frac{\Gamma(a(h + 1) + 1)}{\Gamma(a h + 1)} \left[ (k + 1)(k + 2)U_{\alpha,1}(k + 2, h) - \mu U_{\alpha,1}(k, h) \right. \\
+ (1 + \mu) \sum_{r=0}^{k} \sum_{s=0}^{h-r} U_{\alpha,1}(r, h - s)U_{\alpha,1}(k - r, s) \\
\left. - \sum_{r=0}^{k} \sum_{s=0}^{h-r} \sum_{p=0}^{h-s} U_{\alpha,1}(r, h - s - p)U_{\alpha,1}(t, s)U_{\alpha,1}(k - r - t, p) \right]. \] (4.30)

By equating the series form of (4.28) with (3.4), the initial transformation coefficients \( U_{\alpha,1}(k, 0), k = 0, 1, 2, \ldots \) can be obtained as follows:

\[ U_{\alpha,1}(0, 0) = \frac{1}{2}, \quad U_{\alpha,1}(1, 0) = -\frac{1}{4\sqrt{2}}, \quad U_{\alpha,1}(2, 0) = 0, \]
\[ U_{\alpha,1}(3, 0) = -\frac{1}{96}, \quad U_{\alpha,1}(4, 0) = 0. \] (4.31)

By applying (4.31) into (4.30), some values of \( U_{\alpha,1}(k, h) \) can be obtained as given in Table 1. Consequent substitution of all \( U_{\alpha,1}(k, h) \) into (3.4) and after some manipulations, the series from solutions of (4.27) and (4.28) has been obtained in [27] as:

\[ u(x, t) = \left( \frac{1}{2} - \frac{1 - 2\mu}{8\Gamma(a + 1)} t^a + \frac{(1 - 2\mu)^2}{8\Gamma(2a + 1)} t^{2a} + \cdots \right) + \left( -\frac{1}{4\sqrt{2}} - \frac{(1 - 2\mu)^2}{32\sqrt{2}\Gamma(2a + 1)} t^{2a} + \cdots \right)x \\
+ \left( \frac{1 - 2\mu}{64\Gamma(a + 1)} t^a + \frac{(1 - 2\mu)^2}{64\Gamma(2a + 1)} t^{2a} + \cdots \right)x^2 + \left( -\frac{1}{96\sqrt{2}} + \frac{(1 - 2\mu)^2}{192\sqrt{2}\Gamma(2a + 1)} t^{2a} + \cdots \right)x^3 \\
+ \left( \frac{1 - 2\mu}{768\Gamma(a + 1)} t^a - \frac{(1 - 2\mu)^2}{768\Gamma(2a + 1)} t^{2a} + \cdots \right)x^4. \] (4.32)

\( u(x, t) \) can be written in the form:

\[ u(x, t) = \left( \frac{1}{2} - \frac{1}{4\sqrt{2}} x - \frac{1}{96\sqrt{2}} x^3 + \frac{1}{1920\sqrt{2}} x^5 + \cdots \right) \\
+ \frac{1 - 2\mu}{2} \left( \frac{1}{4} - \frac{1}{32} x^2 + \frac{1}{384} x^4 - \frac{17}{92160} x^6 + \cdots \right) \frac{t^a}{\Gamma(a + 1)} \\
- \left( \frac{(1 - 2\mu)^2}{2} \right) \left( \frac{1}{2} + \frac{1}{8\sqrt{2}} x - \frac{1}{16} x^2 - \frac{1}{48\sqrt{2}} x^3 + \frac{1}{192} x^4 + \cdots \right) \frac{t^{2a}}{\Gamma(2a + 1)}. \] (4.33)
The exact solution of (4.27), for the special case $\alpha = 1.0$, is given in [27]

$$u(x, t) = \frac{1}{1 + e^{(1/\sqrt{2})(x + ((1-2\mu)/\sqrt{2})t)}}$$  \hspace{1cm} (4.34)

We have the generalized differential transform method solution for the time-fractional Fitzhugh-Nagumo equation (4.27) (when $\alpha = 1.0$ and $\mu = 0.45$) as

$$u(x, t) = 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5$$
$$+ 0.05000000000(0.25000000000 - 0.031250000000x^2 + 0.00260416667x^4$$
$$- 0.0001844618056x^6)t$$
$$- 0.00125000000(0.50000000000 - 0.08838834762x$$
$$- 0.06250000000x^2 - 0.0173139128x^3 + 0.00520833333x^4)t^2$$

(4.35)

$$= 0.5000000000 - 0.1767766952x - 0.007365695635x^3$$
$$+ 0.0003682847818x^5 + 0.01250000000t - 0.0015625000000tx^2$$
$$+ 0.000130208884dx^4 - 0.9223090280 \times 10^{-5}tx^6 - 0.0006250000000t^2$$
$$+ 0.0001104854345t^2x + 0.0000781250000t^2x^2 + 0.00001841423910t^2x^3$$
$$- 0.6510416666 \times 10^{-5}t^2x^4,$$

and let

$$M = 0.5000000000 - 0.1767766952x - 0.007365695635x^3$$
$$+ 0.0003682847818x^5 + 0.01250000000t - 0.0015625000000tx^2$$
$$+ 0.000130208884dx^4 - 0.0006250000000t^2 + 0.0001104854345t^2x$$
$$+ 0.0000781250000t^2x^2 + 0.00001841423910t^2x^3,$$

$$N = 0.5000000000 - 0.1767766952x - 0.007365695635x^3$$
$$+ 0.01250000000t - 0.0015625000000tx^2 - 0.0006250000000t^2$$
$$+ 0.0001104854345t^2x + 0.0000781250000t^2x^2,$$

$$R = 0.5000000000 - 0.1767766952x - 0.007365695635x^3$$
$$+ 0.01250000000t - 0.0015625000000tx^2 - 0.0006250000000t^2$$
$$+ 0.0001104854345t^2x.$$  \hspace{1cm} (4.37)
Then let us calculate the approximate solution of (4.35) for \( m = 5 \) and \( n = 2 \) by using multivariate Padé approximation. To obtain multivariate Padé equations of (4.35) for \( m = 5 \) and \( n = 2 \), we use (2.5). By using (2.5), We obtain

\[
p(x, t) = M_N R_{\mathcal{G}} = \begin{vmatrix} M & N & R \\ -0.6510416666 \times 10^{-5} t^2 x^4 & 0.00007812500000 t^2 x^2 \\ -0.9223090280 \times 10^{-5} t x^6 & -0.6510416666 \times 10^{-5} t^2 x^4 \end{vmatrix} \\
= 0.6004616067 \times 10^{-10} x^6 (-399.3073584 x^5 + 1.529411773 t x^4 - 25.29411764 t x^6 \\
- 49.91341983 t^2 x^3 + 254.1176472 t^2 x^2 - 254.1176470 t x^4 \\
+ 1129.411764 x^4 + 7.058823529 t^4 + 0.4656862748 t x^8 \\
+ 798.6147174 t x^3 - 8.8180375507 t^2 x^5 - 6.705882349 t^3 x^2 \\
- 1.455882353 t^4 x^4 - 0.2911764705 t^4 x^2 - 0.2870021640 t^4 x^3 \\
+ 0.0136409419 t^4 x^4 + 28.28427124 t x^5 - 0.1176470588 t^2 x^6 \\
+ 0.5407287154 t^3 x^3 + 0.01143849208 t^3 x^5 + 39.93073589 t^3 x \\
- 0.4991341962 t^4 x + 0.004939348840 t^6 x^3 + 0.07916666674 t^6 x^5 \\
+ 0.02066727542 t^6 x^5 + 0.002683823532 t^6 x^4 + 0.8318903301 t^6 x^9 \\
+ 0.2010401634 t^6 x^7 - 16.63780660 x^7 - 0.008823529412 t^6 \\
+ 0.00081923549111 t^6 x^7 + 0.0003119588732 t^6 x + 0.1764705882 t^5 \\
+ 0.001585790946 t^5 x^3 - 0.00573529412 t^5 x^2 - 0.02495670995 t^5 x),
\]

\[
q(x, t) = \begin{vmatrix} 1 & 1 & 1 \\ -0.6510416666 \times 10^{-5} t^2 x^4 & 0.00007812500000 t^2 x^2 \\ -0.9223090280 \times 10^{-5} t x^6 & -0.6510416666 \times 10^{-5} t^2 x^4 \end{vmatrix} \\
= 0.6004616067 \times 10^{-10} x^6 (14.11764706 t^4 + 2.11764706 t^3 x^2 + 0.7058823527 t^4 x^2 \\
+ 508.2352942 t^2 x^2 + 79.86147180 t^3 x + 1.996536794 t^4 x \\
+ 2258.823528 x^4 + 1597.229435 x^3 t + 39.93073587 t^2 x^3 \\
+ 56.56854248 t x^5 + 20.00000001 t^2 x^4 + 2.828427126 t^3 x^3),
\]

(4.38)
where $G$ denotes $0.0003682847818x^5 + 0.000130208884fx^4 + 0.00001841423910t^2x^3$. So the multivariate Padé approximation is of order $(5, 2)$ for (4.35), that is,

$$
[5, 2]_{(x,t)} = (-399.3073584x^5 + 1.529411773t^2x^4 - 25.29411764tx^6 - 49.91341983t^2x^3
+ 254.1176472t^2x^2 - 254.1176470tx^4 + 1129.411764x^4 + 7.058823529t^4
+ 0.4656862748tx^8 + 798.6147174t^3x^3 - 8.8180375507t^2x^5 - 6.705882349t^3x^2
- 1.455882353t^3x^4 - 0.291176705t^4x^2 - 0.2870021640t^4x^3 + 0.01360294119t^4x^4
+ 28.28427124tx^5 - 0.1176470588t^2x^6 + 0.5407287154t^3x^3 + 0.01143849208t^3x^5
+ 39.93073589t^3x - 0.4991341962t^4x + 0.0004939348840t^6x^3 + 0.07916666674t^3x^6
+ 0.02066727542t^4x^5 + 0.002683823532t^4x^4 + 0.8318903301x^9 + 0.2010401634t^4x^7
- 16.63780660x^7 - 0.008823529412t^6 + 0.0008823529411t^6x^2 + 0.0003119588732t^6x
+ 0.1764705882t^5 + 0.001585790946t^5x^3 - 0.005735294120t^5x^2
- 0.02495670995t^5x)

/ (14.11764706t^4 + 2.11764706t^3x^2 + 0.7058823527t^4x^2
+ 508.2352942t^2x^2 + 79.86147180t^3x + 1.996536794t^4x + 2258.823528x^4
+ 1597.229435x^3t + 39.93073587t^2x^3 + 56.56854248t^3x^5 + 20.00000000t^2x^4
+ 2.828427126t^4x^3).

(4.39)

We have the generalized differential transform method solution for the time-fractional Fitzhugh-Nagumo equation (4.27) (when $\alpha = 0.50$ and $\mu = 0.45$) as

$$
u(x,t) = 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5
+ 0.05641895835(0.2500000000 - 0.03125000000x^2 + 0.002604166667x^4
- 0.0001844618056x^6)t^{0.5}
- 0.00250000000(0.5000000000 - 0.00838834762x - 0.06250000000x^2
- 0.01473139128x^3 + 0.005208333333x^4)t
= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5
+ 0.01410473959t^{0.5} - 0.001763092448t^{0.5}x^2 + 0.0001469243707t^{0.5}x^4
- 0.0000140714293t^{0.5}x^6 - 0.00125000000t + 0.0002209708690tx
+ 0.0001562500000tx^2 + 0.00003682847820tx^3 - 0.00001302083333tx^4.

(4.41)
For simplicity, let \( t^{1/2} = a \), then

\[
u(x, a) = 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
+ 0.01410473959a - 0.001763092448ax^2 + 0.0001469243707ax^4 \\
- 0.0001040714293ax^6 - 0.001250000000a^2 + 0.0002209708690a^2x \\
+ 0.001562500000a^2x^2 + 0.00003682847820a^2x^3 - 0.0001302083333a^2x^4,
\]

and let

\[
S = 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
+ 0.01410473959a - 0.001763092448ax^2 + 0.0001469243707ax^4 - 0.001250000000a^2 \\
+ 0.0002209708690a^2x + 0.0001562500000a^2x^2 + 0.00003682847820a^2x^3, \\
T = 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.01410473959a \\
- 0.001763092448ax^20.001250000000a^2 + 0.0002209708690a^2x + 0.0001562500000a^2x^2, \\
V = 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.01410473959a \\
- 0.001763092448ax^2 - 0.001250000000a^2 + 0.0002209708690a^2x.
\]

Then, using (2.5) to calculate the multivariate Padé equations for (4.42), we get,

\[
p(x, a) = \begin{vmatrix}
S & T & V \\
\mathcal{L} & \mathcal{L} & \mathcal{L}
\end{vmatrix} = 0.1355096735 \times 10^{-9}x^6( - 176.9384663x^3 + 500.4575575x^4 + 14.1421362ax^5 \\
- 0.1042619911a^2x^6 - 127.0588234ax^4 - 12.64705882ax^6 \\
+ 179.7418057a^2x^7 - 34.59022553a^2x^8 - 1.864967352ax^4 \\
+ 0.01143849208a^3x^5 + 39.93073588ax^3 - 1.527651730a^4x \\
+ 0.5407287176a^3x^8 + 12.511438894a^4 + 0.2328431370ax^5 \\
+ 0.3686218050x^9 - 7.372436097x^7 - 5.430474480a^2x^5 \\
- 7.98840797a^3x^3 - 1.386412603a^3x^4 - 0.4194856542a^4x^2 \\
- 0.4711921255a^4x^3 + 0.02008511870a^4x^4 - 399.3073586ax^3 \\
+ 0.0011750953574a^5x^3 + 0.07716273155a^2x^6 + 0.02800876865a^4x^5 \\
+ 0.005367647059a^5x^4 + 0.1536097286a^2x^7 - 0.04991341984a^5x \\
- 0.03127859737a^6 + 0.003127859737a^6x^2 + 0.01147058823a^5x^2 \\
+ 0.3529411765a^6 + 0.0031751881a^6x^3 - 0.01147058823a^5x^3).
\]
q(x, a) = \begin{bmatrix} 1 & 1 & 1 \\ -0.0000130208333a^2x^4 & \mathcal{L}_{a} & 0.0001562500000a^2x^2 \\ -0.00001040714293ax^6 & \mathcal{L} & -0.0000130208333a^2x^4 \end{bmatrix}

= 0.1355096735 \times 10^{-9} x^6(25.02287789a^4 + 2.11764705a^3x^2 + 1.2511438894a^4x^2
+ 359.4836114a^2x^2 + 79.86147175a^3x + 3.538769329a^4x
+ 1000.915115x^4 + 798.6147171x^3a + 35.38769327a^2x^3
+ 28.28427125x^5a + 11.28379167a^2x^4 + 2.828427126a^3x^3),

(4.44)

where \mathcal{L} denotes 0.0003682847818x^3 + 0.001469243707ax^4 + 0.0003682847820a^2x^3, recalling that \( t^{1/2} = a \), we get multivariate Padé approximation of order (5, 2) for (4.40), that is,

\[ [5, 2]_{(x,a)} = \left( -176.9384663x^3 + 500.4575575x^4 + 14.14213562\sqrt{t}x^5 \\
- 0.1042619911tx^6 - 127.0588234\sqrt{t}x^4 - 12.64705882\sqrt{t}x^6 + 179.7418057tx^2 \\
- 34.59022553tx^3 - 1.864967528tx^4 + 0.01143849208t^{3/2}x^5 + 39.93073588t^{3/2}x \\
- 1.527651730t^2x + 0.5407287176t^{3/2}x^3 + 12.51143894t^2 + 0.2328431370\sqrt{tx^8} \\
+ 0.3686218050a^9 - 7.372436097x^7 - 5.430474480tx^5 - 7.988400799t^{3/2}x^2 \\
- 1.386412603t^{3/2}x^4 - 0.4194856542t^2x^2 - 0.4711921255t^2x^3 + 0.02008511870t^2x^4 \\
+ 399.3073586\sqrt{tx^3} + 0.0011750953574t^3x^3 + 0.07716273155t^{3/2}x^6 \\
+ 0.0280087665t^2x^2 + 0.005367647059t^{5/2}x^4 + 0.1536097286tx^7 \\
- 0.04991341984t^{5/2}x - 0.03127859737t^3x^3 + 0.003127859737t^3x^2 \\
+ 0.01147058823t^{5/2}x^2 + 0.3529411765t^{5/2} \\
+ 0.003171581881t^{5/2}x^3 - 0.01147058823t^{5/2}x^2 \right) \]

/ (25.02287789t^2 + 2.11764705t^{3/2}x^2 + 1.2511438894t^2x^2 + 359.4836114tx^2 \\
+ 79.86147175t^{3/2}x + 3.538769329t^2x + 1000.915115x^4 + 798.6147171x^3\sqrt{t} \\
+ 35.38769327tx^3 + 28.28427125x^5\sqrt{t} + 11.28379167tx^4 + 2.828427126t^{3/2}x^3).

(4.45)
We have the generalized differential transform method solution for the time-fractional Fitzhugh-Nagumo equation (4.27) (when $\alpha = 0.75$ and $\mu = 0.45$) as

\[
\begin{align*}
    u(x, t) &= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
    &\quad + 0.05440326260(0.2500000000 - 0.03125000000x^2 \\
    &\quad \quad + 0.00260416667x^4 - 0.0001844618056x^6)t^{0.75} \\
    &\quad - 0.001880631946(0.5000000000 - 0.08838834762x - 0.06250000000x^2 \\
    &\quad \quad - 0.01473139128x^3 + 0.005208333333x^4)t^{1.50} \\
    &= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
    &\quad + 0.013600815654.75 - 0.0017001019564.75x^2 + 0.00014167516304.75x^4 \\
    &\quad - 0.00010035324054.75x^6 - 0.00094031597304.75 + 0.0016622595024.75x \\
    &\quad + 0.00011753949664.75x^2 + 0.000027704325054.75x^3 - 0.9794958051 \times 10^{-5}t^{1.50}x^4. \\
\end{align*}
\]  

(4.46)

For simplicity, let $t^{1/4} = a$, then

\[
\begin{align*}
    u(x, a) &= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
    &\quad + 0.01360081565a^3 - 0.001700101956a^3x^2 + 0.0001416751630a^3x^4 \\
    &\quad - 0.0001003532405a^3x^6 - 0.0009403159730a^6 + 0.001662259502a^6x \\
    &\quad + 0.0001175394966a^6x^2 + 0.00002770432505a^6x^3 - 0.9794958051 \times 10^{-5}a^6x^4, \\
\end{align*}
\]  

(4.47)

and let

\[
\begin{align*}
    Y &= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
    &\quad + 0.01360081565a^3 - 0.001700101956a^3x^2 + 0.0001416751630a^3x^4 - 0.0009403159730a^6 \\
    &\quad + 0.001662259502a^6x + 0.0001175394966a^6x^2, \\
    Z &= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
    &\quad + 0.01360081565a^3 - 0.001700101956a^3x^2 + 0.0001416751630a^3x^4 - 0.0009403159730a^6 \\
    &\quad + 0.001662259502a^6x, \\
    I &= 0.5000000000 - 0.1767766952x - 0.007365695635x^3 + 0.0003682847818x^5 \\
    &\quad + 0.01360081565a^3 - 0.001700101956a^3x^2 - 0.0009403159730a^6. \\
\end{align*}
\]  

(4.49)
Then, using (2.5) to calculate the multivariate Padé equations for (4.48), we get

\[
p(x, a) = \frac{Y \partial}{\partial x} - 0.9794958051 \times 10^{-5} a^6 x^4 + 0.0001175394966 a^6 x^2 - 0.0001175394966 a^6 x^2 \\
= -0.9794958051 \times 10^{-5} x^4 \left(-0.00007257591929 x^6 - 0.514079428 \times 10^{-5} x^8 \\
+ 0.2227383885 \times 10^{-7} x^{11} + 0.00002462745270 a^3 x^5 \\
+ 0.0001152063889 a^3 x^3 - 0.3786552603 \times 10^{-8} x^{13} \\
+ 0.2886689512 \times 10^{-5} x^9 + 0.00002565946232 x^7 \\
- 0.00003010597218 a^3 x^3 + 0.1477000393 \times 10^{-5} a^6 x^5 \\
- 0.2263460486 \times 10^{-6} a^3 x^8 + 0.2840147227 \times 10^{-6} a^6 x^6 \\
+ 0.7826520955 \times 10^{-8} a^3 x^9 - 0.4382354311 \times 10^{-8} a^6 x^8 \\
- 0.1065755353 \times 10^{-5} a^9 x^2 - 0.1064464005 \times 10^{-5} a^9 x^2 \\
+ 0.6782425517 \times 10^{-5} a^9 x - 0.00001278906419 a^9 \\
- 0.1601429235 \times 10^{-6} a^6 x^7 + 0.00004469028926 a^6 x^3 \\
- 0.0000370665784 a^3 x^6 - 0.3384074604 \times 10^{-5} a^6 x^4 \\
- 0.1565304191 \times 10^{-6} a^3 x^7 + 0.4788518010 \times 10^{-7} a^9 x^5 \\
+ 0.2181467984 \times 10^{-6} a^9 x^4 + 0.178226356 \times 10^{-7} a^3 x^{10} \\
+ 0.884194129 \times 10^{-6} a^{12} + 0.1565760439 \times 10^{-6} a^{12} x^2 \\
- 0.6252196644 \times 10^{-5} a^{12} x - 0.0004701579863 a^6 \\
- 0.0001861042030 a^6 x^2 + 0.0004155648754 a^6 x) a^6 \right)
\]

\[
q(x, a) = \frac{1}{\partial} - 0.9794958051 \times 10^{-5} a^6 x^4 + 0.0001175394966 a^6 x^2 - 0.0001175394966 a^6 x^2 \\
= -0.9794958051 \times 10^{-5} x^4 \left(0.0099403159722 a^6 - 0.0002304127778 a^3 x^3 \\
- 0.0004986778508 a^6 x + 0.0001451518386 x^6 \\
- 0.0000212512744 a^3 x^4 - 0.00005676836553 a^3 x^3 \\
+ 0.00001028158857 x^8 + 0.0001958991611 a^6 x^3) a^6 ,
\]
where $\mathcal{D}$ denotes $0.0002770432505a^6x^3 - 0.000100332405a^3x^6$, and $\mathcal{D}$ denotes $0.001662259502a^6x + 0.0001416751630a^3x^4$; recalling that $t^{1/4} = a$ we get multivariate Padé approximation of order (8,2) for (4.46), that is,

$$
[8,2]_{(x,t)} = -( -0.0007257591929x^6 - 0.514079428 \times 10^{-2}x^8 + 0.22227383885 \times 10^{-1}x^{11}
+ 0.00002462745270t^{3/4}x^5 + 0.001152063889t^{3/4}x^3 - 0.3786552603 \times 10^{-6}x^{13}
+ 0.2886689512 \times 10^{-3}x^9 + 0.00002565946232x^7 - 0.00003010597218t^{3/4}x^4
+ 0.1477000393 \times 10^{-5}t^{3/2}x^5 - 0.2263460486 \times 10^{-6}t^{3/4}x^8 + 0.2840147227 \times 10^{-6}t^{3/2}x^6
+ 0.7826520955 \times 10^{-8}t^{3/4}x^9 - 0.4382354311 \times 10^{-8}t^{3/2}x^8 - 0.1065755353 \times 10^{-8}t^{9/4}x^2
- 0.1064464005 \times 10^{-5}t^{9/4}x^3 + 0.6782425517 \times 10^{-5}t^{9/4}x - 0.0001278906419t^{9/4}
- 0.1601429235 \times 10^{-6}t^{3/2}x^7 + 0.00004469028926t^{3/2}x^3 - 0.00001370665784t^{3/4}x^6
- 0.3384074604 \times 10^{-5}t^{3/2}x^4 - 0.1565304191 \times 10^{-6}t^{3/4}x^7 + 0.4788518010 \times 10^{-7}t^{9/4}x^5
+ 0.2181467984 \times 10^{-6}t^{9/4}x^4 + 0.1782226356 \times 10^{-7}t^{3/4}x^{10} + 0.884194129 \times 10^{-6}t^3
+ 0.1565760439 \times 10^{-6}t^{3}x^2 - 0.6252196644 \times 10^{-5}t^3x - 0.0004701579863t^{3/2}
- 0.0001861042030t^{3/2}x^2 + 0.0004155648754t^{3/2}x)
/ (0.009403159722t^{3/2} - 0.0002304127778t^{3/4}x^3
- 0.0004986778508t^{3/2}x + 0.0001451518386x^6
- 0.0000212512744t^{3/4}x^4 - 0.00005676836553t^{3/4}x^5
+ 0.00001028158857x^8 + 0.0001958991611t^{3/2}x^3).$$

(4.51)

As it is presented above, we obtained multivariate Padé approximations of the generalized differential transform method solution of the time-fractional Fitzhugh-Nagumo equation (4.27) for values of $\alpha = 1.0$, $\alpha = 0.50$, and $\alpha = 0.75$. Table 3 shows the approximate solutions for (4.27) obtained for different values of $\alpha$ using the generalized differential transform method (GDTM) and the multivariate Padé approximation (MPA). The values of $\alpha = 1.0$ are the only case for which we know the exact solution $u(x,t) = 1/(1 + e^{(1/\sqrt{I}(x^2 + (1-2\mu)/\sqrt{I})))}$, and the results of multivariate Padé approximation (MPA) are in excellent agreement with the exact solution and those obtained by the generalized differential transform method (GDTM).
Table 3: Numerical values when $\alpha = 0.50$, $\alpha = 0.75$, $\alpha = 1.0$, and $\mu = 0.45$ for Example 4.2.

| $x$     | $t$     | $\alpha = 0.50, \mu = 0.45$ | $\alpha = 0.75, \mu = 0.45$ | $\alpha = 1.0, \mu = 0.45$ |
|---------|---------|----------------------------|----------------------------|----------------------------|
|         |         | $u_{GDTM}$                  | $u_{MPA}$                  | $u_{GDTM}$                  | $u_{MPA}$                  | $u_{GDTM}$                  | $u_{MPA}$                  | $u_{Exact}$                |
| 0.001   | 0.001   | 0.5002680044                | 0.5002680043               | 0.4998996766                | 0.4998996769               | 0.4998357227                | 0.4998357226               | 0.4998107234               |
| 0.002   | 0.002   | 0.5002747302                | 0.5002747301               | 0.499774909                 | 0.499774911                | 0.4996714440                | 0.4996714440               | 0.4996214466               |
| 0.003   | 0.003   | 0.5002384692                | 0.5002384695               | 0.4996438587                | 0.4996438587               | 0.4995071641                | 0.4995071642               | 0.4994321701               |
| 0.004   | 0.004   | 0.5001799565                | 0.5001799565               | 0.4995089812                | 0.4995089813               | 0.4993428826                | 0.4993428826               | 0.4992428939               |
| 0.005   | 0.005   | 0.5001072237                | 0.5001072235               | 0.4993715189                | 0.4993715192               | 0.4991785998                | 0.4991785991               | 0.4990536177               |
| 0.006   | 0.006   | 0.5000243897                | 0.5000243893               | 0.4992321101                | 0.4992321104               | 0.4990143154                | 0.4990143158               | 0.4988643418               |
| 0.007   | 0.007   | 0.4999339015                | 0.4999339013               | 0.4990911545                | 0.4990911544               | 0.4988500295                | 0.4988500293               | 0.4986750662               |
| 0.008   | 0.008   | 0.4998373530                | 0.4998373532               | 0.4989489244                | 0.4989489247               | 0.4986857419                | 0.4986857418               | 0.4984857911               |
| 0.009   | 0.009   | 0.4997385159                | 0.499738518                | 0.4988056161                | 0.4988056163               | 0.4985214527                | 0.4985214532               | 0.4982965163               |
| 0.01    | 0.01    | 0.4996302043                | 0.4996302048               | 0.4986613772                | 0.4986613775               | 0.4983571616                | 0.4983571620               | 0.4981072420               |
5. Conclusion

By comparison with the generalized differential transform method (GDTM), the fundamental goal of this work has been to construct an approximate solution for time-fractional reaction-diffusion equations by using multivariate Padé approximation. The goal has been achieved by using the multivariate Padé approximation (MPA) and the generalized differential transform method (GDTM). The present work shows the validity and great potential of the multivariate Padé approximation for solving time-fractional reaction-diffusion equations from the numerical results. For the values of $\alpha = 1.0$ in Example 4.1 and for the values of $\alpha = 1.0$ in Example 4.2, numerical results obtained using the multivariate Padé approximation (MPA) and the generalized differential transform method (GDTM) are in excellent agreement with exact solutions and each other. For the values of $\alpha = 0.50$, $\alpha = 0.75$, in Example 4.1 and for the values of $\alpha = 0.50$, $\alpha = 0.75$ in Example 4.2, numerical results show that the results of multivariate Padé approximation are in excellent agreement with those results obtained by the generalized differential transform method (GDTM). The basic idea described in this paper is expected to be further employed to solve other similar problems in fractional calculus.

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