SYNCHROTRON AND INVERSE-COMPTON EMISSIONS FROM PAIRS FORMED IN GRB AFTERGLOWS (ANALYTICAL TREATMENT)

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ABSTRACT

We calculate the synchrotron and inverse-Compton emissions from pairs formed in gamma-ray burst (GRB) afterglows from high-energy photons (above 100 MeV), assuming a power-law photon spectrum \( C_\nu \propto \nu^{-2} \) and considering only the pairs generated from primary high-energy photons. The essential properties of these pairs (number, minimal energy, cooling energy, distribution with energy) and of their emission (peak flux, spectral breaks, spectral slope) are set by the observables GeV fluence \( \Phi(t) = Ft \) and spectrum, and by the Lorentz factor, \( \Gamma \), and magnetic field, \( B \), of the source of high-energy photons, at observer time, \( t \). Optical and X-ray pseudo light curves, \( F_{\nu}(t) \), are calculated for the given \( B \); proper synchrotron self-Compton light curves are calculated by setting the dynamics \( \Gamma' = 20 \) of the high-energy photon source to be that of a decelerating, relativistic shock. It is found that the emission from pairs can accommodate the flux and decays of the optical flashes measured during the prompt (GRB) phase, but it decays faster than the X-ray plateaus observed during the delayed (afterglow) phase. The brightest pair optical emission is obtained for \( 100 < \Gamma < 500 \), and depends mostly on the GeV fluence, being independent of the source redshift. Emission from pairs formed during the GRB phase offers an alternate explanation to reverse-shock optical flashes. These two models may be distinguished based on their corresponding flux decay index–spectral slope relations, different correlations with the Large Area Telescope fluence, or through modeling of the afterglow multiwavelength data.

Key words: methods: analytical – radiation mechanisms: non-thermal – relativistic processes – shock waves

Online-only material: color figures

1. INTRODUCTION

The first Fermi–Large Area Telescope (LAT) gamma-ray burst (GRB) catalog (Ackermann et al. 2013) identifies a “temporally extended” emission at 100 MeV–10 GeV for eight bursts, with four other having LAT detections well after the end of the Fermi–Gamma-ray Burst Monitor (GBM) prompt phase. The LAT emission of those eight afterglows peaks at 10–20 s after trigger, having a fluence \( \Phi = 10^{-5.1} \text{ erg cm}^{-2} \), followed by a flux decay \( \nu F_\nu \propto t^{-2.1 \pm 0.3} \) until up to 1 ks, with a photon spectrum of \( C_\nu \propto \nu^{-2.1 \pm 0.3} \). The energetic output of those LAT afterglows, \( E_{100\text{MeV}} = 10^{51.1} \text{ erg} \), is 10%–100% of their GRB output (at \( \sim 1 \text{ MeV} \)).

The condition of optical thinness to pair formation for the highest-energy LAT photons yields lower limits on the Lorentz factor, \( \Gamma \), of their source. Assuming that the prompt LAT emission has same origin as the GBM burst and using the burst variability timescale to determine the source radius, it was inferred that \( \Gamma_{\text{grb}} > 200–1000 \) (e.g., Abdo et al. 2009). The general lack of pair-formation signatures in the light curves and spectra of LAT afterglows indicates that the afterglow source also has \( \Gamma_{\text{ag}} > 200 \) (Panaitescu et al. 2014), with a detection bias against afterglows with \( \Gamma \lesssim 75 \), for which pair-formation attenuates the intrinsic afterglow emission too much and the emergent afterglow is too dim to be detected by LAT.

In this work, we calculate the synchrotron and inverse-Compton emission from pairs formed in LAT afterglows, in a simplified setup. The temporal and spectral properties of LAT afterglows being consistent with those at lower photon energies (optical and X-ray), indicates that the LAT emission arises in the forward-shock driven by the GRB ejecta into the circumburst medium (Kumar & Barniol Duran 2009). An important simplification is that we consider only the pairs formed behind the forward shock, whose energy distribution is set primarily by the spectrum of the high-energy photons, and ignore the pairs formed ahead of the forward shock, whose energy is set by their shock acceleration (given that half of the emitted photons travel ahead of the forward shock, it follows that a comparable number of pairs form behind and ahead of the shock). For ease of calculating the number of pairs, we assume that the single power-law spectrum of the high-energy LAT photons extends well below 100 MeV and above 100 GeV, and that it has a spectral index–2 (in photon number). Furthermore, we consider only the pairs formed from high-energy photons, for which the photon front is optically thick to pair formation. Another simplification made is that the threshold energy for pair formation is set only by the relativistic collimation of the seed LAT photons, i.e., we ignore the scattering/decollimation of the high-energy photons by the already formed pairs.

Section 2 calculates the number of pairs with the above approximations and their minimal energy in the shock frame; Section 3 presents the calculation of the spectral breaks of the pair emission, the regions in the \( nb - \Gamma \) corresponding to various orderings of the spectral breaks being identified in Section 4. The calculation of the received synchrotron self-Compton emission, taking into account synchrotron self-absorption, radiative cooling, first inverse-Compton scattering if the optical thickness to scattering by pairs is \( \tau < 1 \), and higher orders of inverse-Compton if \( \tau > 1 \), is presented in Section 5. Optical and X-ray pair light curves are discussed in Section 6.

2. NUMBER OF PAIRS AND THEIR DISTRIBUTION WITH ENERGY

A photon of observer-frame energy \( \varepsilon \text{ MeV} \) forms a pair when interacting with another photon with energy above the
source-frame threshold
$$\epsilon_i(\epsilon) = \frac{4 \Gamma^2 (m_e c^2)^2}{(z + 1) \epsilon} = 35 Z^{-1} \frac{\Gamma^2}{\epsilon}\text{ MeV},$$
with $\Gamma$, the Lorentz factor of the source that produced both photons, and using the notations
$$X_n = \frac{X(cgs)}{10^n}, \quad Z \equiv \frac{z + 1}{3}. \quad (2)$$
For an afterglow fluence of $\Phi$ at 0.1–10 GeV, with a photon-number distribution with energy $C_\epsilon \propto \epsilon^{-2}$, the number of photons with energy above $\epsilon$ is
$$N_\gamma(>\epsilon) = \frac{4 \pi d_l^2 \Phi}{z + 1} 4.6 \epsilon = 1.14 \times 10^{56} Z^3 \frac{\Phi}{\epsilon_8}, \quad (3)$$
where $d_l = 5.10^{27}(z + 1)^2$ cm is the luminosity distance for redshift $z$. Thus, the number of photons above the threshold energy $\epsilon_i$ is
$$N_\gamma(>\epsilon_i(\epsilon)) = 3.3 \times 10^{56} Z^2 \frac{\Phi}{\Gamma_2^2} \epsilon_8. \quad (4)$$
A relativistic source moving at constant $\Gamma$ has a radius of
$$R = \frac{4}{5} \frac{ct}{z + 1} \Gamma^2 \quad (5)$$
at observer-frame time, $t$, the factor of four-thirds corresponding to photons emitted from the “edge” of the source, i.e., from the fluid flowing at an angle $\Gamma^{-1}$ relative to the direction toward the observer. Thus, the optical thickness to pair formation for a $\epsilon$ photon is
$$\tau_{\gamma \gamma}(\epsilon) = \frac{\sigma_{\gamma \gamma}}{4 \pi R^2} = 1.8 Z^6 \frac{\Phi}{\Gamma_2^2} \epsilon_8, \quad (6)$$
where $\sigma_{\gamma \gamma} = 0.18 \sigma_0$ is the pair-formation cross-section averaged over the $\epsilon^{-3}$ photon distribution and $\sigma_0$ is the lepton scattering cross-section. Thus, the afterglow photon front is optically thick to pair formation ($\tau_{\gamma \gamma} > 1$) for photons of energy above
$$\epsilon_\pm = 56 Z^{-6} \Gamma_2^6 \frac{\Phi}{\Gamma_2^2} \epsilon_8. \quad (7)$$
We approximate the number of pairs formed as that of photons with energy $\epsilon > \epsilon_\pm$ also $\epsilon_\pm$. A fraction $\tau_{\gamma \gamma}(\epsilon) < 1$ of the photons with $\epsilon < \epsilon_\pm$ also forms pairs; these pairs are roughly a factor of $\ln(\epsilon_\pm/\epsilon_m) > 1$ more numerous than those for which $\tau_{\gamma \gamma}(\epsilon) > 1$, $\epsilon_m$ being the peak energy of the $\nu \nu$ LAT spectrum. This approximation is made for two reasons. One is to avoid carrying unknown observables—$\epsilon_m$ and the spectral slope below that peak—in the following calculations, the other is to work with a single power-law pair distribution with energy. The ensuing underestimation of the true pair number increases with $\Gamma$ because $\epsilon_\pm \propto \Gamma^0$. However, for high Lorentz factors, pairs are cooling much faster than they are created and only the pairs formed from photons with $\epsilon > \epsilon_\pm$ radiate synchrotron emission at the frequencies of interest (optical and X-rays).

Given that each photon with energy above $\epsilon_\pm$ yields two leptons (an electron and a positron), it follows that the number of leptons formed is
$$N = 2 N_\gamma(>\epsilon_\pm) = 1.34 \times 10^{56} Z^8 \frac{\Phi}{\Gamma_2^2} \epsilon_8. \quad (8)$$

The above results hold when the pair-formation threshold energy for $\epsilon_\pm$ photons
$$\epsilon_i(\epsilon_\pm) = 21 Z^3 \frac{\Phi}{\Gamma_2^2} \epsilon_8 \text{ MeV} \quad (9)$$
is below $\epsilon_\pm$, i.e., when there are enough absorbing photons above $\epsilon_i(\epsilon_\pm)$. The condition $\epsilon_i(\epsilon_\pm) = \epsilon_\pm$ defines a Lorentz factor
$$\Gamma_c \equiv 91 Z^2 \frac{\Phi}{\epsilon_8} \Gamma_2^{0.4}, \quad (10)$$
such that $\epsilon_i(\epsilon_\pm) < \epsilon_\pm$ for $\Gamma \geq \Gamma_c$. For $\Gamma < \Gamma_c$, we have $\epsilon_i(\epsilon_\pm) > \epsilon_\pm$, i.e., there are fewer photons above $\epsilon_i(\epsilon_\pm)$ than above $\epsilon_\pm$ and, consequently, not all photons above $\epsilon_\pm$ can form pairs, even though $\tau_{\gamma \gamma}(\epsilon_\pm) > 1$. In this case, the energy $\epsilon_\pm$ above which all photons form pairs is given by $\epsilon_i(\epsilon_\pm) = \epsilon_\pm$. Then, Equation (1) leads to
$$\epsilon_\pm = \epsilon_i(\epsilon_\pm) = 2 \frac{\Gamma m_e c^2}{z + 1} = 34 Z^{-1} \Gamma_2 \text{ MeV} \quad (\Gamma < \Gamma_c). \quad (11)$$
Because the $\epsilon_\pm$ form pairs mostly with other $\epsilon_\pm$ photons, the number of leptons formed is just the number of photons with energy above $\epsilon_\pm$:
$$N = N_\gamma(>\epsilon_\pm) = 1.11 \times 10^{56} Z^3 \frac{\Phi}{\Gamma_2} \quad (\Gamma < \Gamma_c) \quad (12)$$
using Equation (3).

Most pairs form at threshold, with a typical lab-frame energy $\gamma m_e c^2 = (z + 1)[\epsilon + \epsilon_i(\epsilon)]/2$, thus
$$\gamma m_e c^2 \simeq (z + 1) \left[ \frac{\epsilon_i(\epsilon)}{\gamma_i} \frac{\epsilon}{\gamma_i} \frac{\epsilon}{2} \frac{\epsilon_\pm}{\gamma_\pm} \right] \quad (13)$$
This approximate one-to-one correspondence between the absorbed photon energy and the electron–positron pair energy implies that the distribution with energy of pairs is that of the high-energy photons
$$\frac{dN}{d\gamma_i} (\gamma_i) \propto \gamma_i^{-2}. \quad (14)$$

The above pair distribution with energy holds above a shock-frame energy, $\gamma_i'$, which can be determined from the pair lab-frame energy, $\gamma_i$, corresponding to the minimum energy, $(z + 1)\epsilon_\pm$, above which all photons form pairs, and from the angle, $\theta_\pm$, at which the pairs move (in the lab frame) relative to the radial direction of shock’s motion. Taking into account that the high-energy photons are collimated (by the relativistic motion of their source, the shock) within an angle $\theta_i \simeq \Gamma^{-1}$ of the radial direction, it follows that (1) the center of momentum of the colliding photons moves at angle $\theta_m \sim \Gamma^{-1}$ relative to the radial direction and (2) the pairs emerge at a typical angle $\theta_{out} = (\epsilon_i(\epsilon_\pm)1/2)[\Gamma(\epsilon + \epsilon_i(\epsilon)/2)]$ relative to the direction of motion of the center of momentum. In the lab frame, the emerging pairs move at an angle $\theta_\pm \simeq \max(\theta_m, \theta_{out})$ relative to the shock’s direction of motion.

For $\Gamma < \Gamma_c$, we have $\epsilon_\pm = \epsilon_i(\epsilon_\pm)$, thus $\theta_{out} = 1/2 (\Gamma)$, from which $\theta_\pm \simeq \theta_m \simeq \Gamma^{-1}$, which implies that, in the shock-frame (moving outward at Lorentz factor, $\Gamma$), the minimal pair energy is $\gamma_i' \simeq \gamma_i / \Gamma$. From Equations (11) and (13), $\gamma_i m_e c^2 = (z + 1)\epsilon_\pm = 2 \Gamma m_e c^2$, thus
$$\gamma_i' = 2 \quad (\Gamma < \Gamma_c). \quad (15)$$
For $\Gamma \gg \Gamma_c$, $\gamma_{\pm} \gg \gamma_i(\epsilon_{\pm})$, thus $\theta_{\text{out}} \simeq (\epsilon_i/\epsilon_{\pm})^{1/2}/\Gamma \ll 1$, from where $\theta_{\pm} \simeq \theta_{\text{cm}} \simeq 1/\Gamma$, which implies that $\gamma' \simeq \gamma_i/\Gamma$, as for $\Gamma < \Gamma_c$. From Equation (13), $\gamma_i m_e c^2 = (z + 1) \epsilon_{\pm}/2$, which, together with Equation (7), leads to

$$\gamma' = \frac{(z + 1) \epsilon_{\pm}}{2 \Gamma m_e c^2} = 1.6 \times 10^3 \frac{\Gamma^2 \epsilon_i}{\Phi_{\gamma}} = \left(\frac{\Gamma}{\Gamma_c}\right)^5 \left(\Gamma \gg \Gamma_c\right). \quad (16)$$

### 3. SYNCHROTRON AND INVERSE-COMPTON SPECTRAL CHARACTERISTICS

For a power-law distribution with the energy of the radiating particles, the synchrotron spectrum is a sequence of power laws with breaks at frequencies

$$v_{a,c} = \frac{e \gamma_i^2 B \Gamma}{2 m_e c} \frac{1}{z + 1} = 3 \times 10^8 \frac{\gamma_i^2 B \Gamma}{c} \text{ Hz}, \quad (17)$$

where $\gamma$ is the pair random Lorentz factor in the shock’s frame (prime notation dropped), $v_i$ is the injection frequency corresponding to the $\gamma_i$ (Equations (15) and (16)). $v_a$ is the synchrotron self-absorption frequency and $v_c$ is the cooling frequency; both are calculated below.

$B$ is the magnetic field in the forward shock, parameterized by the fraction $b$ of the post-shock energy density $\epsilon_{\parallel} = \Gamma_n m_p c^2$ that it contains ($u_{\parallel} = B^2/8\pi n$), with $n = 4 \Gamma n$, post-shock proton density, and $n$, the external medium proton density at the location $R(t)$ of the forward shock. Thus,

$$B = (32 \pi b R^2 n m_p c^2)^{1/2} = 39 (nb)^{1/2} \frac{\Gamma_2 G}{2} \quad (18)$$

The synchrotron and inverse-Compton emissions depend on $n$ and $b$ only through the product $nb$.

The continuous creation of pairs in the shocked medium, with the power-law distribution given in Equation (14), and their radiative cooling leads to an effective pair distribution with energy

$$\frac{dN}{d\gamma} \propto \begin{cases} \gamma^{-2} & (\gamma_p < \gamma < \gamma_b), \\ \gamma^{-3} & (\gamma_b < \gamma) \end{cases} \quad (19)$$

with $\gamma_p \equiv \min(\gamma_i, \gamma_c)$, $\gamma_b \equiv \max(\gamma_i, \gamma_c)$,

where $\gamma_c$ is the pair cooling Lorentz factor, defined as the energy to which a pair cools on the dynamical timescale, $t'_d = R/c(\Gamma) = (4/3)R[t/(c + 1)]$. The pair number, $N$, and minimum Lorentz factor, $\gamma_c$, change on a dynamical timescale.

Pairs created with an energy $\gamma_i m_e c^2$ in the shock-frame cool radiatively at power $P'(\gamma) = (4/3)\sigma_e c B^2 \gamma^5/8\pi(Y + 1)^2 \equiv c \gamma^5$, where $Y$ is the Compton parameter (the inverse-Compton to synchrotron power ratio). Integrating the equation for pair cooling, $c \gamma^2 = -d(\gamma m_e c^2)/dt'$, one obtains that a pair of high initial energy reaches a random Lorentz factor $\gamma(t') = m_e c^2/c t'$ after a time, $t'$, since its creation. Therefore, the cooling Lorentz factor is $\gamma_c \equiv \gamma(t'_d) = m_e c^2/c t'_d$,

$$\gamma_c = \frac{9 \pi}{2} \left(\frac{z + 1}{\theta}\right) \frac{1}{\sigma_e} \frac{1}{\Gamma_2 B^2(Y + 1)} \quad (21)$$

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with $\Theta(x) = 0$ for $x < 0$, $\Theta(x) = 1$ for $x > 0$, and $\gamma_a$ is the random Lorentz factor of the pairs that radiate at the synchrotron self-absorption frequency $v_a$ (see below). Equation (22) approximates the effect of synchrotron self-absorption by switching off synchrotron cooling when $\gamma_c$ has decreased to $\gamma_a$. After that, the radiative cooling continues only through inverse-Compton scatterings. This means that, if $\gamma_c$ calculated from Equation (22) for $\gamma < 1$ and $\Theta = 0$ is larger than $\gamma_a$, then the correct cooling Lorentz factor is $\gamma_c \simeq \gamma_a$.

From Equation (6.53) of Rybicki & Lightman (1979) for the synchrotron self-absorption coefficient for a power-law distribution of particles, it can be shown that the pair optical thickness to self-absorption at the peak energy, $\tau_p = \min(v_i, v_c)$, of the intrinsic synchrotron spectrum, $F_\nu$, is

$$\tau_p = \frac{5 \pi}{\sigma_e B Y_p} = 3.6 \times 10^{15} \frac{\tau}{B Y_p}, \quad (23)$$

where

$$\tau = \frac{\sigma_e N}{4 \pi R^2} \quad (24)$$

is the pair optical thickness to photon scattering. For the pair distribution given in Equation (19), the optical thickness to self-absorption at frequency $\nu$ is

$$\tau_a(\nu) = \begin{cases} \frac{\gamma_p \nu_p^{3/5}}{\nu} & (\nu < \nu_p) \\ \frac{\gamma_p \nu_p^{3/5}}{\nu} & (\nu_p < \nu < \nu_b) \\ \frac{\gamma_p \nu_p^{3/5}}{\nu_b^{7/2}} & (\nu_b < \nu) \end{cases} \quad (25)$$

where $\nu_b = \max(v_p, v_c)$. From here, the self-absorption frequency $\nu_a$ defined by $\tau_a(\nu_a) = 1$ has a corresponding pair Lorentz factor, $\gamma_a$, given by

$$\nu_a = \begin{cases} \frac{\gamma_p \nu_p^{3/5}}{\nu} & (\gamma_p < \gamma_a) \\ \frac{\gamma_p \nu_p^{3/5}}{\nu} & (\gamma_p < \gamma_a < \gamma_b) \\ \frac{\gamma_p \nu_p^{3/5}}{\nu} & (\gamma_b < \gamma_a) \end{cases} \quad (26)$$

The Compton parameter, $Y$, is the ratio of the pairs energy output in inverse-Compton emission to that in synchrotron, hence $Y = P_{ic}/P_{ic} = \gamma_{\pm}^2/\gamma_{\pm}$, where $\gamma_{\pm}$ is the energy density of the synchrotron photons received by a scattering lepton. Synchrotron self-absorption reduces $\gamma_{\pm}$, because photons of energy less than $h\nu_{ic}$ are absorbed before being scattered. However, that reduction is not substantial for the particle distribution given in Equation (19) because most of the synchrotron energetic output is at frequencies above $v_a$, for which the pairs medium is transparent (to self-absorption); (1) for $v_a < v_{\max}$ with $v_{\max} \equiv \max(v_a, v_c)$, the synchrotron output above $v_a$ is $F_{\nu} \propto \nu^{5/2} / \nu < v_{\max}$ and $max \nu F_{\nu} \propto \nu^{0}$ at $v_{\max} < v$, thus all the synchrotron output is above $v_a$; (2) for $v_{\max} < v_a$, the synchrotron output is $F_{\nu} \propto \nu^{1/2}$ at $v < v_{\max}$ and $max \nu F_{\nu} \propto \nu^{0}$ at $v < v_{\max}$, hence the reduction of the synchrotron output due to self-absorption is a factor $\ln(\nu_{ic}/\nu_{ic})/\ln(\nu_{ic}/\nu_{max})$ with $\nu_{ic}$ the high-energy end of the otherwise diverging $F_{\nu}(\nu_{max}) \propto \nu^{-1}$ synchrotron spectrum, thus the reduction is the order of unity for $\nu_{ic} \gg v_a$.

For a single photon scattering, the ratio $u_{\nu_{ic}}/u_{\nu}$ is the product of $(4/3)Y^2$ (the average increase in photon energy due to scattering) and the fraction $\min(\gamma, 1)$ of photons that are
upsattered, $\gamma' \simeq n Y$, for the pair distribution of Equation (19), thus

$$Y_1 = \frac{4}{3} \gamma Y_r \min(\tau, 1) \tag{27}$$

for the first scattering.

In the lab frame, the relative velocity between a photon (moving at $c$) and the pair front (moving at Lorentz factor $\Gamma$) is $v_r = c/(2\Gamma^2)$, thus the photon crosses the pair front of geometrical thickness $\Delta = R/(2\Gamma^2)$ in a time $t_d = \Delta/v_r = R/c = t_d$. This means that, for $\tau < 1$, only the first scattering takes place within a dynamical timescale, $t_d$, and higher order scatterings occur on a longer timescale. Because the effective pair distribution with energy is that resulting from pair creation and cooling over one $t_d$, higher order scatterings (taking longer than $t_d$) are ignored for $\tau < 1$. However, higher order scatterings should be considered for $\tau > 1$ because, in that case, the time between scatterings is less than $t_d$.

4. REGIONS IN THE ($nb-\gamma$) PARAMETER SPACE

To obtain the pair synchrotron self-Compton emission at some observing frequency $\nu$, one must first calculate the spectral breaks of the previous section. The calculation of the injection break $Y_i$ is trivial, the analytical expression of the cooling break $Y_c$ depends on whether the Compton parameter, $\gamma_c$, is below or above unity, that of $Y$ depends on whether the pair optical thickness $\tau$ is above/below unity. In general, the self-absorption break, $Y_a$, is not needed for the synchrotron emission, as the optical thickness to self-absorption $\tau_a(\nu)$ suffices. However, the location of $v_a$ relative to $v_c$ is useful for the calculation of $Y_a$ when $\gamma < 1$ (see Equation (22)) and is required for determining the upsattered absorption break of the inverse-Compton spectrum.

Given the observables $\Phi, t, \tau$, and $z$, all quantities needed depend on only two parameters: $nb$ and $\Gamma$. The conditions $\tau = 1$, $Y = 1$, and $\gamma = 1$ define lines in this $nb$–$\Gamma$ plane while the equality of two spectral breaks defines boundaries. Expressions for these lines and boundaries are derived below and are useful for a correct calculation of the synchrotron/inverse-Compton spectral breaks and of the peak flux for each spectral component.

The first separation of the $nb$–$\Gamma$ plane is provided by $\Gamma = \Gamma_c$, (Equation (10)), across which $N$ changes from Equations (8) to (12) and $\gamma_c$ from Equations (15) to (16). A similarly simple separation is provided by the condition $\tau = 1$. Using Equations (5), (8), (12), and (24), the pair optical thickness to photon scattering is

$$\tau = \left\{ \begin{array}{ll} 3.3 \epsilon^{3/4} \Phi^{-5/2} \epsilon^{-2} \gamma^{-5} & \Gamma < \Gamma_c \\ 4.0 \epsilon^{10} \Phi^{-4} \epsilon^{-1} \gamma^{-10} & \Gamma_c < \Gamma \end{array} \right. \tag{28}$$

which, after using Equation (10), can be written as

$$\tau = \left\{ \begin{array}{ll} \sigma_e \sqrt{\epsilon} \left( \frac{\Gamma_c / \Gamma}{\Gamma_c / \Gamma} \right)^5 & \Gamma < \Gamma_c \\ 2 \frac{\sigma_e}{\epsilon} \left( \frac{\Gamma_c / \Gamma}{\Gamma_c / \Gamma} \right)^{10} & \Gamma_c < \Gamma \end{array} \right. \tag{29}$$

This implies that $\tau < 1$ for $\Gamma > \Gamma_c$ where

$$\Gamma_c \equiv \left( \frac{2 \sigma_e}{\epsilon} \right)^{1/10} \Gamma_c = 1.27 \Gamma_c = 115 \epsilon Z \frac{\Phi_{0.5}^{1/4}}{\epsilon_{10}^{1/4}} \tag{30}$$

The extrapolations of the $\Gamma < \Gamma_c$ and $\Gamma > \Gamma_c$ branches of Equations (16) and (29) intersect at $\Gamma_c = 2^{1/5} \Gamma_c$ for both equations. For that reason, we approximate them by

$$\tau = \left\{ \begin{array}{ll} \frac{\sigma_e}{\epsilon} \left( \frac{\Gamma_c / \Gamma}{\Gamma_c / \Gamma} \right)^5 & \Gamma < \Gamma_c \\ 2 \frac{\sigma_e}{\epsilon} \left( \frac{\Gamma_c / \Gamma}{\Gamma_c / \Gamma} \right)^{10} & \Gamma_c < \Gamma \end{array} \right. \tag{31}$$

$$\gamma_a = \left\{ \begin{array}{ll} \frac{2}{\epsilon} \left( \frac{\Gamma_c / \Gamma}{\Gamma_c / \Gamma} \right)^5 & \Gamma < \Gamma_c \\ 2 \left( \frac{\Gamma_c / \Gamma}{\Gamma_c / \Gamma} \right)^{10} & \Gamma_c < \Gamma \end{array} \right. \tag{32}$$

The range $\Gamma_c < \Gamma < \Gamma_c$, where $\gamma > 2$ and $\tau > 1$, is narrow, its 10% extent in $\Gamma$ corresponding to a 30% increase in observer time, for a shock decelerating as $\Gamma \propto t^{-3/5}$ (homogeneous external medium), i.e., it lasts less than one dynamical timescale.

4.1. Optically Thin Pairs: $\Gamma > \Gamma_c$

Equation (27) is simply

$$Y = \frac{4}{3} \gamma Y_r \tau = (Y_1) \tag{33}$$

retaining only the first scattering, together with Equation (22) leading to

$$Y(Y + 1) = \frac{(nb)_{Y}}{nb} , \quad (nb)_{Y} \equiv 10^{4} \epsilon^{0} \Phi_{-5}^{0.5} \Gamma_{1}^{0.28} \Gamma_{2}^{0.8} \tag{34}$$

The line $nb = (nb)_{Y}$ separates the $nb$–$\Gamma$-plane in two regions, with $Y < 1$ for $nb > (nb)_{Y}$ and $Y > 1$ for $nb < (nb)_{Y}$ (see Figure 1).

4.1.1. $nb < (nb)_{Y}$ (iC Cooling)

In this case $Y > 1$ and Equations (22) and (34) yield

$$\gamma_a = 11 \epsilon^{2} \left( \frac{t_{1}^{2}}{(nb)_{Y}} \right)^{1/2} , \quad Y \simeq \left( \frac{(nb)_{Y}}{nb} \right)^{1/2} \tag{35}$$

From Equations (16) and (35), the condition $\gamma_a = \gamma$ defines the boundary

$$(nb)_{a} \equiv 47 \epsilon^{0} \Phi_{-5}^{0.5} \Gamma_{2}^{0.8} \Gamma_{1}^{0.28} \tag{36}$$

so that $\gamma < \gamma_a$ for $nb < (nb)_{a}$. From Equations (34) and (36), it follows that

$$\frac{(nb)_{a}}{(nb)_{Y}} = \frac{9 \epsilon^{2}}{64} \left( \frac{\sigma_{e}}{\sigma_{c}} \right)^{2} = \frac{1}{219} \tag{37}$$

thus $(nb)_{a}$ is just a shift of $(nb)_{Y}$.

For $nb < (nb)_{a}$, we have $\gamma < \gamma_a$ and the condition $\gamma_a = \gamma_a$ is equivalent to $\tau_c = (nb)_{a} = 1$ which, after using Equations (16), (23), (28), is satisfied on the boundary

$$(nb)_{a} \equiv 0.014 \epsilon^{0} \Phi_{-5}^{0.5} \Gamma_{2}^{0.8} \Gamma_{1}^{0.28} \tag{38}$$

with $\gamma < \gamma_a$ for $nb < (nb)_{a}$. In this regime, $\gamma_a \equiv \gamma_{a1/6}$, using Equations (16), (23), (28), and (35), $\gamma_a = \gamma_a$ is satisfied on the boundary

$$(nb)_{a} \equiv 0.011 \epsilon^{0} \Phi_{-5}^{0.5} \Gamma_{2}^{0.8} \Gamma_{1}^{0.28} \tag{39}$$

with $\gamma < \gamma_a$ if $nb < (nb)_{a}$. 

For \( nb > (nb)_{\text{c1}} \), we have \( \gamma_c < \gamma_i \) and the \( \gamma_a = \gamma_c \) boundary is defined by \( \tau_p \equiv \tau_c(\gamma_c) = 1 \) which, together with Equations (23), (28), and (35), yield

\[
(nb)_{ac} \equiv 8.8 \times 10^{-10} z^{-0.25} \Phi_{-5}^{2.7},
\]

with \( \gamma_a > \gamma_c \) for \( nb > (nb)_{ac} \). In this case, \( \gamma_a = \gamma_c \tau_p^{1/6} \), using Equations (16), (23), (28), and (35), it follows that \( \gamma_a = \gamma_i \) on the boundary

\[
(nb)_{ai} \equiv 200 \times 38^{\Phi_{-5}^{1.5}} t_{20}^{-40},
\]

such that \( \gamma_i < \gamma_a \) if \( nb < (nb)_{ai} \).

4.1.2. \( (nb)_T < nb < (nb)_{ac} \) (sy Cooling)

In this case, \( Y < 1 \); for \( \gamma_e < \gamma_c \), pairs cool mostly through synchrotron emission and Equations (22) and (34) yield

\[
\gamma_c \simeq 1.15 \frac{z}{(nb)_T t_{20}^{3}}, \quad Y = \frac{(nb)_T}{nb} \left( \tau < 1, Y < 1, \gamma_c < \gamma_a \right).
\]

Because \( (nb)_T > (nb)_{c1} \), we have \( \gamma_c < \gamma_i \), and the condition \( \gamma_a = \gamma_c \) is equivalent to \( \tau_p \equiv \tau_c(\gamma_c) = 1 \) which, using Equations (23), (28), and (42), is satisfied on the boundary

\[
(nb)_{ai/c} \equiv 0.19 \times 38^{\Phi_{-5}^{0.44}} t_{20}^{-0.89},
\]

4.1.3. \( nb > \max[(nb)_T, (nb)_{aci}] \)

For \( nb > (nb)_{aci} \), the \( \gamma_c < \gamma_a \) regime occurs; without synchrotron cooling, \( \gamma_c \propto Y^{-1} \), and the expressions for \( Y \) and \( \gamma_c \) are the same as in Equation (35) for \( Y > 1 \), although \( Y < 1 \) now. Consequently, the \( \gamma_a = \gamma_i \) and \( \gamma_a = \gamma_c \) boundaries are the same as for \( Y > 1 \); \( (nb)_{ai} \) and \( (nb)_{ac} \), respectively.

In contrast with that case, a new region appears now, defined by \( (nb)_{ai/c} < nb < (nb)_{ac} \), for which \( \gamma_c \) of Equation (35) does not satisfy the \( \gamma_c < \gamma_a \) condition. This is the case where inverse-Compton cooling, operating alone after the epoch \( t_{ac} < t' \), when the cooled pair Lorentz factor \( \gamma(t_{ac}') \) has reached \( \gamma_c(t_{ac}) \), does not decrease \( \gamma(t') \) significantly until \( t' \). In this case, we impose \( \gamma_c \equiv \gamma(t_{ac}) \) and, thus

\[
\gamma_c = \gamma_a = \left( \frac{5e\gamma}{\sigma_c B} \right)^{1/5} = 5.2 \times 10^2 \left( \frac{\Phi_{-5}}{(nb)^{1/2} t_{20}^{1/5}} \right)^{1/5}.
\]

4.1.4. \( nb > (nb)_{c1} \) or \( nb > \max[(nb)_{c2}, (nb)_{aci}] \)

In some of the regions identified above, radiative cooling can be strong enough that \( \gamma(t') = 1 \) at some time \( t' < t' \). In this case, \( \gamma_c \equiv \gamma(t_{ac}') = 1 \), and \( \gamma_a \) and \( Y \) are those given by Equations (26) and (33) with \( \gamma_c = 1 \). From Equations (35), (42), and (44), the condition \( \gamma_c = 1 \) defines three lines:

\[
(nb)_{c1} \equiv 1.27 \times 10^{-4} \frac{t_{10}}{\Phi_{-5}^{2.5}} t_{20}^{-3.5}.
\]

\[
(nb)_{c2} \equiv 3.6 \times 10^{-20} \Phi_{-5}^{2.5} t_{20}^{-22.2}.
\]

\[
(nb)_{c3} \equiv 0.043 \frac{Z}{t_{10}} \Gamma_{5}^{-3.5}.
\]

such that \( \gamma_c = 1 \) if \( nb > (nb)_{c1} \) or \( nb > \max[(nb)_{c2}, (nb)_{c3}] \).

The pair cooling to \( \gamma_c = 1 \) means that a pair of high initial energy radiatively loses all of its energy over a dynamical
timescale. From the equation for radiative cooling, \( \gamma(t') = m_e c^2 / c t' \), it follows that the time for complete cooling is \( t'_c(\gamma \sim 1) \approx m_e c^2 / c \). When \( t'_c(\gamma \sim 1) < t'_c \), a fraction \( f_b = t'_c(\gamma \sim 1) / t'_c \) of the pairs created in the last dynamical timescale are hot (\( \gamma > 1 \)) and radiate synchrotron and upscatter to absorb that emission, and a fraction \( 1 - f_b \) are cold (\( \gamma = 1 \)), scatter photons without a significant change in energy and absorb the emission below their characteristic synchrotron frequency. Noting that \( t'_c(\gamma \sim 1) / t'_c = m_e c^2 / (c t'_c) = \gamma_c \), with \( \gamma_c \) the cooling Lorentz factor of Equation (22), we find that

\[
f_b = \gamma_c^{\text{(eq 22)}} = \frac{0.144 Z}{(nb)_{s,1} \Gamma_{-3}^2} \left( Y + \Theta(1 - \gamma_c) \right).
\] (48)

The parameters of interest—\( \gamma_c, Y \) and peak flux \( F_p \propto N \)—are those of Equations (26) and (27) with \( \gamma_c = 1 \) and for a number of hot pairs, \( N_b \), or optical thickness, \( \tau_b \), satisfying \( N_b / N = \tau_b / \tau = f_b \).

A first consequence of \( \tau_b = f_b \tau = \gamma_c \) for \( \gamma_c = 1 \) is that \( Y = (4/3) \gamma \Gamma \approx (4/3) \gamma \gamma_c \) for \( \gamma_c = 1 \), thus the Compton parameter given in Equation (33) applies whether pairs cool completely \( f_b < 1 \) or not \( f_b = 1 \) during one dynamical timescale. That entails the conclusions that, for \( \gamma_c = 1 \), \( Y = 1 \) on the \( (nb)_Y \) line given in Equation (35) and that \( Y \) is as in Equation (35) for \( \gamma_c < \gamma_c \) and as in Equation (42) for \( \gamma_c > \gamma_c \).

Owing to the accumulation of pairs at \( \gamma_c = 1 \), the synchrotron self-absorption thickness satisfies \( \tau_a (\gamma < \gamma_c) \propto N \) (all pairs absorb synchrotron emission below \( \gamma_c \)) and \( \tau_a (\gamma > \gamma_c) \propto N_b \) (only hot pairs absorb above \( \gamma_c \)), yielding a discontinuity of \( \tau_a (\gamma) \) at \( \gamma_c \) and a slight ambiguity on the boundary between the \( \gamma_c < \gamma < \gamma_i \) region, as follows. The \( \gamma_a = \gamma_c \) boundary of the \( \gamma_c < \gamma_a \) region satisfies \( \tau_a (\gamma) = (5e \gamma a / \sigma B)^{1/6} = 1 \), with \( \tau_b = f_b \tau \propto \tau / Y \). Using Equations (28), (35), and (48), we find that \( \gamma_a = \gamma_c \) along the boundary

\[
(nb)_{av} \approx 4.4 \times 10^5 \frac{\Phi_0^{1/5} \gamma_c^{1/3}}{\Gamma^2 \gamma_c^{1/3}}.
\] (49)

The \( \gamma_a = \gamma_c \) boundary of the \( \gamma_c < \gamma_a \) region satisfies \( \tau_a (\gamma_c) = (5e \gamma a / \sigma B)^{1/6} = 1 \), leading to the same condition as for \( \gamma_c = 1 \) for the \( \gamma_a = \gamma_c < \gamma_i \) region (see Figure 1); therefore, the \( (nb)_{av} \) line of Equation (46) is also a \( \gamma_a = \gamma_c \) boundary.

For \( \gamma_c < \gamma_a < \gamma_i \), Equations (23) and (26) give \( \gamma_a = \gamma_a (\gamma_a, \gamma_c) = (5e \gamma a / \sigma B)^{1/6} \), then, \( \tau_b = \tau_c \), and \( \gamma_c \) in Equation (42), extending into the \( \gamma_c = 1 \) region.

### 4.1.5. Klein–Nishina Scattering

The Compton parameter, \( Y \), of Equation (33) is for inverse-Compton scatterings in the Thomson regime. The pair synchrotron flux is \( \nu F_{\nu} \propto \nu^{1/2} \) for \( \nu_p < \nu < \nu_b \) and flat above \( \nu_b \) for a \( C_{\nu} \propto \nu^{-2} \) spectrum of the high-energy photons that form pairs. However, for a typical LAT spectrum, which is slightly softer than \( \nu^{-2} \), the synchrotron flux \( \nu F_{\nu} \) peaks at \( \nu_b \). For \( nb > (nb)_{av} \), we have \( \nu_i < \nu_b \); hence, \( \nu F_{\nu} \) peaks at \( \nu_i \). Pairs of energy, \( \gamma_i \), scatter synchrotron photons at the \( \nu_i \) peak of \( \nu F_{\nu} \) in the Klein–Nishina (KN) regime if \( \gamma (\nu_i) = \gamma_c (\nu_i / 2m_e c^2) \Gamma \gamma_i^2 B > m_e c^2 \). Staying in the \( \gamma_c < \gamma_i \) case, the KN regime will reduce the Compton parameter significantly if the \( \gamma_i \) pairs satisfy the above inequality, i.e., if \( \gamma_i > \gamma_{KN} = (2 m_e c^2 / h\nu B)^{1/3} \). Using Equations (16) and (18), the condition \( \gamma_i > \gamma_{KN} \) becomes \( nb > (nb)_{KN} \) where

\[
(nb)_{KN} \equiv 0.29 \times 30 \frac{\Phi_6}{l_{12}^2} \Gamma_{-3}^{-2}.
\] (50)

For \( nb > 10^{-5} \), we have \( (nb)_{KN} > (nb)_{av} \) (Figure 1), thus \( nb > (nb)_{KN} \) implies \( nb > (nb)_{av} \) and the derivation of Equation (50) is self-consistent. The KN scattering effect on the calculation of the Compton parameter of Equation (33) is as follows. For \( nb < (nb)_{KN} \), most pairs, being between \( \gamma_c \) and \( \gamma_i \), scatter the synchrotron input in Thomson regime and the KN effect is negligible. For \( nb > (nb)_{KN} \), pairs above an energy \( \gamma = \gamma_{KN} \) scatter photons from the KN regime and the Compton Y is reduced; however, \( Y = 1 \) is very likely because \( nb > (nb)_{KN} \) implies \( nb > (nb)_{Y} \) (Figure 1), hence that reduction of \( Y \) by the KN effect is largely inconsequential.

For \( nb < 10^{-5} \), we have \( \gamma_{KN} < \gamma_c < \gamma_i \); in this case, all pairs upscatter in the KN regime, the \( \nu F_{\nu} \) photons, and also the \( \gamma_i \) synchrotron photons (where \( \nu F_{\nu} \) peaks); thus, Equation (33) significantly overestimates the true Compton parameter, leading to an overestimation of the inverse-Compton flux and an underestimation of \( \gamma_i \propto Y^{-1} \). The latter leaves unchanged the synchrotron flux below \( \nu_c \), but underestimates the synchrotron flux above \( \nu_i \). For the rest of this paper, we will avoid the \( nb < 10^{-5} \) region, so that the KN effect can be ignored.

### 4.2. Optically Thick Pairs: \( \Gamma < \Gamma_{\tau} \)

Over one dynamical timescale, a photon undergoes \( \tau \) scatterings and diffuses a distance \( \Delta l' = \sqrt{\Delta l} \), in the pair-shell comoving frame, where \( l' = \Delta / \tau \) is the mean free path between scatterings. Thus, over one dynamical timescale, the observer receives photons from a geometrical depth, \( \Delta l' = \Delta / \sqrt{\Delta l} \), corresponding to a scattering optical depth \( \sqrt{\Delta l} \).

A synchrotron photon undergoes \( \tau^2 \) scatterings before escaping the pair medium. In the lab frame, if the pair medium were stationary, the photon mean free path between scatterings would be \( l = \Delta / \tau \), with \( \Delta = R / (2 \Gamma^2) \) the geometrical thickness of the pair front. Because the pair medium moves at Lorentz factor, \( \Gamma \), the lab-frame photon-pair relative velocity is \( \nu_r = c / (2 \Gamma^2) \), hence each scattering takes a time \( \tau_s = 2 \Gamma / c \). A photon starting from a geometrical depth \( \tau^2 = \tau^2_{av} \), corresponding to a scattering optical depth \( \tau \), will diffuse through the pair medium a distance \( \Delta l' = \sqrt{\Delta l} \), measured relative to the forward direction after \( n_s \) scatterings, and will exit the pair medium in the direction toward the observer when \( l' = l'_{av} \), i.e., after \( n_s = (x / l')^2 = \tau^2_{av} \), scatterings, in a lab-frame time \( t_{av} = n_s \tau = \tau^2_{av} \tau / \tau \). Therefore, only photons up to an optical thickness depth \( \sqrt{\Delta l} \) cross the pair shell within one dynamical timescale, and these photons undergo up to \( \tau \) scatterings.

Because we approximate the effective pair distribution with energy as that established by injection and cooling over \( t_d \), we will consider only \( \tau \) scatterings for the calculation of inverse-Compton parameter, cooling, and emission. Hence, the pair cooling, \( \gamma_c \), is that of Equation (22) with the Compton parameter corresponding to \( \tau \) scatterings: \( \gamma = Y_1 + \cdots + Y_{\tau} \). As long as scatterings occur in the Thomson regime, the Compton parameter of \( \tau \)th scattering is \( Y_1 = Y_1' \), where \( Y_1 = (4/3) \gamma_1 \gamma_c \) is the Compton parameter of the first scattering for \( \gamma_c > 1 \). Because \( Y_1 > 1 \), we can approximate \( Y_1 + \cdots + Y_{\tau} = \cdots \)
\((Y_1^{t+1} - 1)/(Y_1 - 1) \simeq Y_1^{t}\), thus
\[
Y = Y_1^{t}, \quad Y_1 = \frac{4}{\gamma} \gamma_i \gamma_c \quad (\tau > 1, \gamma_c > 1).
\] (51)

Adding Equation (22) with \(Y > 1\), we find that
\[
\gamma_c = \left(\frac{3}{4\gamma_i}\right)^{\tau} \left(\frac{1150}{(nb)\Gamma_2^T}\right)^{1/(r+1)} \quad (\tau > 1, \gamma_c > 1).
\] (52)

After \(i\) scatterings, a synchrotron photon of initial energy \(h\nu\) reaches an energy \((\gamma_i^{t+1})^{r+1}h\nu\), where \(\gamma_i^{t+1}\) is the average \(\gamma^2\) for pairs. The \(i\)th scattering occurs at the end of the Thomson regime if, in the pair frame, the \((i - 1)\)th scattered photon is below \(m_e c^2\) (Thomson scattering) and the \(i\)th scattered photon is above \(m_e c^2\) (Klein–Nishina scattering): \(\gamma_i^{(t)} \gamma^{-1} h\nu < m_e c^2 < \gamma_i^{(t+1)} \gamma^{-1} h\nu\), where \(\gamma_i = (\gamma_i^{(t)})^{1/2}\). Thus the \(i\)th scattering is bordering Thomson regime if \((\gamma_i < \gamma_c) < 1/2\) or \((\gamma_i > \gamma_c) > 1/2\), and we approximate the number of Thomson scatterings \(\tau_i\) by \((\gamma_i \gamma_c)^{r/2} h\nu = m_e c^2\).

The synchrotron photon of energy \(h\nu\) considered here should be that where most of the synchrotron output \(\nu F_\nu\) lies, which is \(\max(v_i, v_c)\), leading to
\[
\max(\gamma_i^2, \gamma_c^2) = \gamma_c^{\gamma_c} = \frac{2 m_e^2 \gamma_c^2}{h\nu} = \frac{3.6 \times 10^{11}}{(nb) \Gamma_2^T}. \quad (53)
\]

Substituting here \(\gamma_c\) from Equation (52) and using Equations (31) and (32), one can determine the \(nb - \Gamma\) region where \(\tau < n_T\), i.e., for which all \(\tau\) scatterings occurring within one dynamical timescale are in the Thomson regime. The result is that, for the \(nb\) range shown in Figure 1, all \(\tau\) scatterings occur in the Thomson regime, and that result can be illustrated easier if we use the more stringent condition that \(v_i\) photons are upscattered by \(\gamma_c\) pairs. Then, from Equation (53) with \(\gamma_i\) substituted by \(\gamma_c\) and with \(\gamma_c\) from Equation (52), it follows that \(\tau < n_T\) is equivalent to \(nb > (nb)_{KN}\) with
\[
(nb)_{KN} = 2.4 \times 10^{-4} \left[\left(\frac{3}{4\gamma_i}\right)^{\tau} \left(\frac{1150}{(nb)\Gamma_2^T}\right)^{1/3}\right]. \quad (54)
\]

Taking into account the dependence on \(\Gamma\) and \(\gamma_i\), we find that \((nb)_{KN}\) above rises with \(\Gamma\), reaching a maximum value at \(\Gamma_c\), where \(\tau = 1\) and \(\gamma_i = 3.3\), hence that maximum value is \((nb)_{KN}(\Gamma_c) \simeq 2.10^{-5}\). Thus, the \(nb\) range shown in Figure 1 satisfies \(nb > (nb)_{KN}\) and all \(\tau\) scatterings occur in the Thomson regime.

4.2.1. \(\gamma_i > 1\)

From Equation (52), it follows that \(\gamma_i = \gamma_i\) along the boundary
\[
(nb)_{ic} = \left(\frac{3}{4\gamma_i}\right)^{\tau} \left(\frac{1150}{\gamma_i t_i \Gamma_2^T}\right) \quad (55)
\]

that \(\gamma_i = \gamma_i(= \gamma_i t_p^{1/6})\) for
\[
(nb)_{ic} = \left[3.9 \left(\frac{1150}{(390\gamma_i)^{2}} \frac{t_i^{(2-3/2)} \Gamma_2^{1-2}}{\Phi_{5}^{(3/2)}}\right)^{12/(11-r)} \right] \quad (56)
\]

(this boundary exists only for \(\tau < 11\), i.e., \(\Gamma > \Gamma_c\)), and that \(\gamma_i = 1\) on
\[
(nb)_{ic} = \left(\frac{3}{4\gamma_i}\right)^{\tau} \left(\frac{1150}{\gamma_i t_i \Gamma_2^T}\right) = \gamma_i^{\gamma_i+(nb)_{ic}} \quad (57)
\]

with \(\gamma_i\) given in Equation (32). These results become simpler for \(\Gamma < \Gamma_c\), where \(\gamma_i = 2\) and the \((nb)_{ic}\) boundary is just a shift of \((nb)_{ic}\).

4.2.2. \(\gamma_c = 1\)

For \(nb > (nb)_{ic}\), only a fraction
\[
f_h = \frac{1150 \Xi_t}{(nb)_{ic} t_i \Gamma_2^T}, \quad Y = \sum_{i=1}^{\text{max}(1, i)} Y_1, \quad Y_1 = \frac{4}{\gamma} \gamma_i \min(1, \tau_i) \quad (58)
\]

of all pairs upscatter synchrotron photons. This result is Equation (48) for \(\tau_i = f_h \tau\) upscatterings (a synchrotron photon undergoes \(\tau\) scatterings in one dynamical timescale, out of which a fraction \(f_h < 1\) are upscatterings by hot pairs) and for \(\gamma_i > \gamma_c = 1\) (as Figure 1 shows for \(\Gamma < \Gamma_c\)). To find \(f_h, \tau_i\), and \(Y\), Equations (58) with \(\tau_i = f_h \tau\) must be solved.

For \(\tau_i > 1\), we have \(Y_1 > 1\) and \(Y = \sum_{i=1}^{\text{max}(1, i)} Y_1 \simeq Y_1\), thus \(\tau_i\) satisfies
\[
(\frac{4}{3\gamma_i})^{\tau_i} \tau_i = \frac{1150 \Xi_t}{(nb)_{ic} t_i \Gamma_2^T}, \quad Y = (\frac{4}{3\gamma_i})^{\tau_i} \quad (\tau_i > 1, \gamma_i = 1). \quad (59)
\]

From here, \(\tau_i\) can be determined numerically, a rough approximation being \(\tau_i \simeq \log(1150 \Xi_t/(nb)_{ic} t_i \Gamma_2^T)/\log(4/3\gamma_i)\).

From Equation (59), the line corresponding to \(\tau_i = 1\) is
\[
(nb)_{ic} = \frac{870 \Xi_t}{t_i \Gamma_2^T}, \quad Y = \frac{2200 \Xi_t^5}{t_i} \Gamma_2^{18} \Gamma^{18} \Gamma_c \Gamma < \Gamma \Gamma_c \quad (60)
\]

For \(nb > (nb)_{ic}\), the hot pairs are optically thick (\(\tau_i > 1\)) to photon scattering while for \(nb < (nb)_{ic}\), they are optically thin (\(\tau_i < 1\)) despite that \(\tau_i > 1\).

For \(\tau_i < 1\), at most one upscattering occurs within a dynamical timescale, thus \(Y = Y_1 = (4/3\gamma_i)_{ic} \tau_i\), and
\[
\tau_i = (\frac{(nb)_{ic}}{nb})^{1/2}, \quad Y = (\frac{4}{3\gamma_i})_{ic} \tau_i \quad (\tau_i < 1, \gamma_i = 1) \quad (61)
\]

which implies that \(Y > 1\) for \(nb < (nb)_{ic}\) with
\[
(nb)_{ic} = \left(\frac{4}{3\gamma_i}\right)^2 (nb)_{ic} = 1540 \Xi_t^{\gamma_i \tau_i} \Gamma_2^{18} \Gamma_c (nb)^{\gamma_i/(4\gamma_i)} \quad (62)
\]

with the last equality resulting from Equations (31) and (32). Thus, the \(Y = 1 - (nb)_{ic}\) line given in Equation (34) for \(\Gamma > \Gamma_c\) extends in the \(\Gamma < \Gamma_c\) region. We also note that \(Y = (4/3\gamma_i)_{ic} \tau_i\) implies that, for \(\Gamma < \Gamma_c\) (i.e., for \(\gamma_i = 2\)), the \(\tau_i = 1\) line is a shift of the \(Y = 1\) line.

5. Pair Emission

5.1. Synchrotron

The intrinsic synchrotron spectrum peaks at \(\nu_p = \min(v_i, v_c)\), where the flux density is
\[
F_p = \frac{z + 1}{4\pi d^2} \frac{e^3}{m_e c^2} N B \Gamma = 6.8 \Xi_t^{1/2} \frac{\Phi_{5}^{(1/2)}}{t_i \Gamma_2^{1/2}} \quad \text{Jy} (\Gamma > \Gamma_c) \quad (63)
\]
with \( N_b = N \) if \( \gamma_c > 1 \) and \( N_b = f_b N \) if \( \gamma_c = 1 \). From Equations (17), (18), (16), (35) and (42), the injection and cooling frequencies scale as

\[
v_i \propto n^{1/2} Z^{-11/2} \Gamma^{12} \Phi^{-2} t^4, \quad v_c \propto Z^{-3} n^{1/2} \Gamma^{-4} t^{-2} \quad Y < 1
\]

\[
Z^{-5} n^{-1/2} \Gamma^{4} \Phi^{-1} t \quad Y > 1,
\]

(64)

For the pair distribution with energy given in Equation (19), the spectrum of the unabsorbed synchrotron emission is

\[
F_{\nu}^{(o)} = F_p \left( \frac{v}{v_p} \right)^{1/3} \quad (v < v_p)
\]

\[
= \left( \frac{v_p}{v} \right)^{1/2} \quad (v_p < v < v_b)
\]

\[
= \left( \frac{v}{v_p} \right)^{1/2} \left( \frac{v_b}{v} \right) \quad (v_b < v),
\]

(65)

where \( v_p = \min(v_i, v_c) \) and \( v_b = \max(v_i, v_c) \). For an LAT photon spectrum \( C_{\nu} \propto \nu^{-2} \), the corresponding pair synchrotron \( \nu F_{\nu} \) spectrum is flat above \( v_b \), but peaks at \( v_b \) for softer LAT photon spectra.

1. \( \Gamma > \Gamma_r \). Then \( \tau < 1 \) and the received/emergent synchrotron flux is that of Equation (65) corrected only for self-absorption:

\[
F_c = \frac{F_{\nu}^{(o)}}{\max[1, \tau_a(v)]} \quad (\tau < 1),
\]

(66)

where \( \tau_a(v) \) is the synchrotron self-absorption optical thickness (Equation (25)) at observing frequency \( v \), for all pairs (if \( \gamma_c > 1 \) or if \( \gamma_c = 1 \) and \( v < v_c \)) or only for hot pairs (if \( \gamma_c = 1 \) and \( v > v_c \)). Equation (66) simply means that the observer receives emission from the entire pair medium at frequencies where \( \tau_a(v) < 1 \) and from a layer of optical depth (to self-absorption) unity if \( \tau_a(v) > 1 \).

Substituting Equations (63) and (64) in (65), we arrive at the synchrotron light curves listed in Table 1 for \( v > v_a, \gamma_c > 1 \), and for two types of ambient medium: homogeneous, for which \( n = \) const. and the shock deceleration is given by

\[
\Gamma^2 n R^3 = \Gamma^2 n (\Gamma^2 t)^3 \quad \text{const, hence } \Gamma \propto t^{-3/8}; \text{wind-like, where } n \propto t^{-2}; \text{thus } \Gamma \propto t^{-1/4}, R \propto t^{1/2}, \text{and } n \propto t^{-1}.
\]

Table 1 illustrates the unsurprising correlation of the pair flux with the fluence, \( \Phi \), of the high-energy photons that form the pairs. For the same distribution of leptons with energy, the synchrotron emission always decays faster than the forward-shock's (taking into account that \( \Phi \propto t^{-1.3 \pm 0.3} \)). Compared with the synchrotron emission from the reverse shock, after that shock crossed the ejecta shell and electrons adiabatically cool, the pair emission decay is steeper at \( v < v_p \) and slower at \( v > v_b \).

2. \( \Gamma < \Gamma_r \) and \( nb > (nb)_r \). In this case, \( \tau > 1, \gamma_c = 1 \), the scattering optical thickness of hot pairs is \( \tau_h < 1 \), i.e., most scatterings are on cold pairs and leave the synchrotron photon energy unchanged. Scatterings by cold leptons increase the optical thickness to self-absorption to an effective value, \( \tau_a = \sqrt{\tau_a(\tau_a + \tau)} \). The observer receives emission from a layer of geometrical thickness, \( l'_s \), corresponding to one self-absorption optical thickness, \( l'_s = \Delta/\tau_a \). Because we take into account only photons that escape the pair medium in one dynamical timescale (on which the number of pairs changes), i.e., only photons from a scattering optical depth \( \sqrt{\tau} \), the received synchrotron flux is

\[
F_v = \frac{F_{\nu}^{(o)}}{\max[1, \tau_a(v)]} \quad \tilde{\tau}_a(v) \equiv \sqrt{\tau_a(v)[\tau_a(v) + \tau]}
\]

(67)

or

\[
F_v = F_{\nu}^{(o)} \left( \frac{\tau_{a}^{1/2}}{(\tau_a t)^{-1/2}} \right) \quad (1 < \tau_a < \tau)
\]

\[
= \tau_a^{-1} \quad (\tau_a < 1),
\]

(68)

3. \( \Gamma < \Gamma_r \) and \( nb > (nb)_r \). In this case, \( \tau > 1, \gamma_c = 1 \), and \( \tau_h > 1 \). Because the upscattering of a synchrotron photon by hot pairs means the “destruction” of the synchrotron photon (which will be counted as an inverse-Compton photon), upscatterings reduce the emergent

| type of medium | \( F(v < v_i < v) \) | \( F(v < v_i < v) \) | \( F(v < v_i < v) \) | \( F(v_i, v_i < v) \) |
|---------------|-----------------|-----------------|-----------------|-----------------|
| homogenous    | \( \Phi^{1/3} t^{-1/3} \) | \( \Phi^{1/3} t^{-1/3} \) | \( \Phi^{1/3} t^{-1/3} \) | \( \Phi^{1/3} t^{-1/3} \) |
| \( n \propto -2 \) window | \( \Phi^{1/3} t^{-5/3} \) | \( \Phi^{1/3} t^{-5/3} \) | \( \Phi^{1/3} t^{-5/3} \) | \( \Phi^{1/3} t^{-5/3} \) |

Notes. \( \Phi \) is the fluence at time, \( \tau \), of the high-energy photons that form pairs, \( Y \) is the Compton parameter, and \( v_i \) and \( v_c \) are the injection and cooling frequencies.

The synchrotron spectrum is given in Equation (65).

\[
\nu F_{\nu} \propto \nu^{-1/3} \left[ \frac{\tau}{\nu} \right]^{1/3} \quad (1 < \tau_a < \tau)
\]

\[
= \frac{1}{\nu} \quad (\tau_a < 1)
\]

\[
= \frac{1}{\nu} \quad (\tau_a = 1)
\]

\[
= \frac{1}{\nu} \quad (\tau_a > 1)
\]
syncrotron flux in a fashion similar to self-absorption

\[ F_v = \frac{F_v^{(o)}}{\max[\sqrt{\tau}, \tilde{\tau}_v(v), \tilde{\tau}_b]} \quad (\tau > 1) \]  \hspace{1cm} (69)

where \( \tilde{\tau}_v = \sqrt{\tau} \tilde{\tau}_v \), \( \tilde{\tau}_v(v) \equiv \sqrt{\tau}(v)(\tilde{\tau}_v(v) + \tau_v) \).  \hspace{1cm} (70)

The self-absorption frequency \( \tilde{\nu}_v \) defined by \( \tilde{\tau}_v(\tilde{\nu}_v) = 1 \) satisfies \( \tau_v + \tilde{\tau}_v(\tilde{\nu}_v) = 1 \), which implies that \( \tau_v(\tilde{\nu}_v) \approx 1 \). Adding that \( \tau_v(v) \) decreases with \( v \), this means that \( \tilde{\tau}_v > \tilde{\tau}_v(\tilde{\nu}_v) \) as \( \tilde{\tau}_v \) is the self-absorption frequency without scatterings, defined by \( \tau_v(\tilde{\nu}_v) = 1 \). Obviously, scatterings by cold leptons (without changing the energy of the synchrotron photon) increase the self-absorption frequency. For \( \tau_v < \tau_v(\tilde{\nu}_v) \), we have \( \tau_v(\tilde{\nu}_v) \approx 1 \), which \( \tilde{\tau}_v(\tilde{\nu}_v) \approx \tilde{\tau}_v(\tilde{\nu}_v) \approx 1 \), which can be solved easily; the solution is that of Equation (26), but with \( \tau_v, \tau_p \) instead of \( \tau_r, \tau_p \). The boundaries involving \( \tilde{\tau}_v \), shown in Figure 1, remain unchanged for \( \tau > 1 \) because \( \gamma_v > 1 \), hence \( \tau_v = 0 \) (there are not any cold pairs).

4. \( \Gamma < \Gamma_r \) and \( \nu < (\nu_b) \). In this case, \( \gamma_v > 1 \), \( \tau_v = 0 \), and \( \tilde{\tau}_v = \tau > 1 \), thus

\[ F_v = \frac{F_v^{(o)}}{\max[\tau, \tilde{\tau}_v(v), \tilde{\tau}_b, \tau/3]} \]  \hspace{1cm} (71)

valid for any \( \tau \) and \( \tilde{\tau}_v \). The last term in the denominator accounts for the formation of pairs at the diffusing photon. As discussed before, a photon starting from a scattering optical depth \( \tau_v \), will undergo \( \nu_v^2 \) scatterings before it diffuses over a distance \( L_d = \tau_v d/v \) (in the comoving-frame) and exits the pair shell. These scatterings take a time \( \delta t' = \tau_v^2 d/v \), hence the photon diffuses at average speed \( v' = L_d/\delta t' = c/\nu_v \), relative to the shocked fluid. For the photon to really escape the pair medium, that diffusion speed should exceed the outer edge of the pair medium, \( \nu_v^{(o)} \), of the outer edge of the pair medium relative to the shocked fluid. When the pair medium is optically thick, most pairs form within the shocked fluid that produced the pair-forming photons, whose outer edge is the forward-shock, which moves at speed \( \nu_v^{(f)} = c/3 \) relative to the post-shock fluid. Thus, the condition \( v' > \nu_v^{(f)} \) implies that only photons from an optical depth of \( \tau_v < 3 \) escape the pair medium.

5.2. Inverse-Compton

5.2.1. \( \tau < 1 \) (\( \Gamma > \Gamma_r \))

Only one upscattering occurs during one dynamical timescale, hence \( \Gamma = \Gamma_r = (4/3)\gamma_v \gamma_r \). The upscattering of a fraction, \( \tau \), of synchrotron photons implies that the peak of the first inverse-Compton spectrum is

\[ F_p^{(1)} = \tau_h F_p^{(y)} \]  \hspace{1cm} (73)

with \( F_p^{(y)} \) the flux at the peak of the emergent synchrotron spectrum:

\[ F_p^{(y)} = F_p \left\{ \begin{array}{ll} 1 & (v_a < v_p) \\ (v_p/v_a)^{1/2} & (v_p < v < v_b) \\ (v_b/v_a)^{1/2} & (v < v_b) \end{array} \right. \]  \hspace{1cm} (74)

with \( F_p \) the peak flux of the intrinsic synchrotron spectrum, given in Equation (63), \( v_a \) being the self-absorption frequency, \( v_p = \min(v_1, v_r) \) is the peak energy of the \( F_p \) synchrotron spectrum, and \( v_b = \max(v_r, v_c) \) is the peak of the \( v F_p \) synchrotron spectrum. The emergent synchrotron spectrum peaks at \( \max(v_p, v_a) \).

The spectrum of the first inverse-Compton emission has breaks at

\[ v_a^{(1)} = \frac{4}{3} \gamma_p^2 v_a, \quad v_p^{(1)} = \frac{4}{3} \gamma_p^2 \max(v_p, v_a), \quad v_b^{(1)} = \frac{4}{3} \gamma_p^2 \max(v_b, v_a). \]  \hspace{1cm} (75)

The shape of the intrinsic inverse-Compton spectrum is the same as for the synchrotron spectrum (Equation (65)), but upscattering of the self-absorbed synchrotron emission \( F_v \propto \nu^{1/2}/\nu_b(v) \propto \nu^2 \) yields a softer emergent inverse-Compton spectrum \( F_v^{(1)} \propto \nu \), that spectrum being

\[ F(v) = F_p \left\{ \begin{array}{ll} (v/v_a^{(1)})^{3/2} & (v < v_a^{(1)}) \\ \nu/v_b^{(1)} & (v_a^{(1)} < v < v_p^{(1)}) \\ (v_p^{(1)}/v_b)^{1/2} & (v < v_p^{(1)}) \\ \nu/v_b^{(1)} & (v < v_p^{(1)}) \end{array} \right. \]

\[ v_a^{(1)}/\nu \quad \nu/v_b^{(1)} \]  \hspace{1cm} (76)

5.2.2. \( \tau > 1 \) (\( \Gamma < \Gamma_r \)).

If the effective optical thickness to scattering by hot pairs \( \tilde{\tau}_h = \sqrt{\tau_h \tau} < 1 \), then the above equations for \( \tau < 1 \) apply, but with the self-absorption frequency \( \tilde{\nu}_h \) accounting for scatterings by cold pairs instead of \( \tilde{\nu}_v \).

For \( \tilde{\tau}_h > 1 \), the cooling \( \gamma_c \) and Compton \( Y \) are those calculated in Section 4.2 for \( \tilde{\tau}_h \) upscatterings that a synchrotron photon, starting from optical depth, \( \sqrt{\tau} \), undergoes during one dynamical timescale. Taking into account than only photons from a scattering optical depth \( \tau_v < 3 \) catch up with the forward shock, and that such photons undergo up to nine scatterings before escaping the pair medium, out of which \( 9 \tau_h/\tau \) are upscatterings, it follows that the maximal inverse-Compton order to be considered is \( \tau_c = \min(\tau_h, 9 \tau_h/\tau) \). The emergent upscattered emission is the sum of \( \tau_c \max \) inverse-Compton components peaking at progressively higher energies. After \( i \) upscatterings, the scattered photon has diffused, on average, a distance \( \sqrt{\tau_i} \), where \( \ell_i' = L_i/\tau_i \) is the photon free mean path between upscatterings. Therefore, most of the \( \ell \) inverse-Compton emission arises from upscatterings of seed photons produced by a layer located at upscattering optical depth \( \sqrt{\tau_i} - 1 - \sqrt{\tau_i} \), which we will call the "\( \ell \)th iC layer." With
increasing distance from it, inner and outer layers yield a lesser and lesser contribution to the \(i\)th inverse-Compton emission.

For ease of calculations and accounting for the seed synchrotron photons, we assume that the \(i\)th inverse-Compton emission arises from upscatterings of synchrotron photons produced only by the \(i\)th IC layer. Given that all seed photons from \(i > 1\) layers are upscattered, this one-to-one correspondence between inverse-Compton order and optical depth should entail that all the synchrotron photons produced by the \(i\)th IC layer become \(i\)th inverse-Compton photons, thus the peak flux of the \(i\)th inverse-Compton spectrum is equal to the peak flux of the synchrotron spectrum for the \(i\)th IC layer:

\[
F^{(i)}_p = \sqrt{i - \sqrt{i - 1}} F^{(sy)}_p \quad [ i = 1 - \min(t_h, 9\tau_h/\tau)] \quad (77)
\]

where \(F^{(sy)}_p\) is the flux at the peak of the synchrotron spectrum for the entire pair medium, given in Equation (74), but with \(\tilde{v}_a\) corresponding to self-absorption only in the \(i\)th IC layer:

\[
\tilde{\tau}^{(i)}_a(\tilde{v}_a) = 1, \quad \tilde{\tau}^{(i)}_a = \sqrt{\tilde{\tau}^{(i)}_a(\tau^{(i)}_c + \tau^{(i)}_a)} \quad (78)
\]

where \(\tau^{(i)}_a \equiv (\sqrt{i} - \sqrt{i - 1})\tau_e/\tau_b\) is the optical thickness to scattering by cold pairs in the \(i\)th IC layer (although the \(i\)th IC layer is optically thin to upscatterings by hot pairs, it is not necessarily thin to scatterings by cold pairs), and with \(\tau^{(i)}_a(v)\) as given in Equation (25) but for the scattering optical thickness \(\tau^{(i)}_a \equiv (\sqrt{i} - \sqrt{i - 1})\tau_e/\tilde{\tau}_b\) of the \(i\)th IC layer.

The flux of the \(i\)th inverse Compton emission is as given in Equation (76), but with \(F^{(i)}_p\) instead of \(F^{(1)}_p\) and with spectral breaks at

\[
\nu^{(i)}_a \equiv \left(\frac{4}{3} \nu^2_p\right)^i \tilde{v}_a \quad (79)
\]

\[

\nu^{(i)}_b \equiv \left(\frac{4}{3} \nu^2_p\right)^i \max(\nu_p, \tilde{v}_a). \quad (81)
\]

To summarize the accounting of photons for \(\tau > 1\): the outermost layer of one upscattering optical depth \((i = 1)\) yields the synchrotron emission and the first inverse-Compton scattering, layers of geometrical depth \((\sqrt{i} - 1)/(\Delta \tilde{\tau}_b)\) produce all the seed photons for the \(i\)th inverse-Compton emission, but we ignore the inverse-Compton emission of order higher than \(t_h\), produced by pairs at geometrical depth larger than \((\sqrt{\tilde{\tau}} - \sqrt{i - 1})/\tilde{\tau}_b = \Delta \sqrt{i}/\tau\) because that emission is trapped in the pair medium for longer than one dynamical timescale, on which change the number of pairs and their distribution with energy.

6. OPTICAL AND X-RAY LIGHT CURVES

The formalism presented thus far allows the calculation of the optical and X-ray synchrotron self-Compton flux from pairs at the observer time, \(t\), when LAT measured a fluence \(\Phi(t)\), for an assumed source Lorentz factor, \(\Gamma\), and a shock magnetic field, \(B\), (parameterized and tied to \(\Gamma\) by the \(nb\) parameter).

The “pseudo light curves” of Figure 2 show the synchrotron and inverse-Compton flux at 2 eV (optical) and 100 keV (hard X-ray) as functions of the source Lorentz factor \(\Gamma\) at \(t = 10\) s (during the prompt emission phase), for an LAT fluence \(\Phi = 10^{-5} \text{ erg cm}^{-2}\), and a few values for the magnetic field parameter, \(nb\). The brightest optical flux is obtained when the peak frequency \(\nu_p\) of the synchrotron spectrum is close to the optical range. We note that the peak flux of Equation (63) means magnitude, \(V = 6.6\), comparable to that shown in Figure 2 for \(nb = 1\) and with the brightest optical counterpart (emission during the prompt/burst phase) ever observed (GRB 080319B,
reached peak magnitude $V = 5.3$ at 20 s after trigger—Greco et al. (2009).

The 10 keV–1 MeV fluence of Fermi-GBM bursts is about ten times larger than the 100 MeV–10 GeV fluence of the corresponding LAT afterglows, thus the average MeV flux of a 10 s burst is $F_{\text{MeV}} \sim 10^{-3}$ erg cm$^{-2}$ s$^{-1}$. Figure 2 shows that, during the prompt phase, the pair emission at 100 keV is dimmer by about two orders of magnitude than the burst.

Figure 3 shows proper pair-emission optical and soft X-ray light curves for a source deceleration corresponding to a blast wave interacting with a homogeneous ambient medium (uniform $n$)—$\Gamma \propto t^{-3/8}$—a slowly decreasing high-energy fluence $\Phi$, a constant magnetic field parameter, $nb$, and starting from various initial Lorentz factors $\Gamma_0$ at $t_0 = 10$ s. The shock deceleration corresponds to moving from right to left, on a horizontal line in Figure 1. The pair flux is calculated at each time (s) $t$ corresponding to a shock deceleration $\Gamma = \Gamma(t/t_0)^{-3/8}$, and a magnetic field parameter $A_s$, $b = 10^{-2}$. Light-curve breaks originate from a spectral break crossing the optical (“a = o” for absorption, “i = o” for injection), from the occurrence of the ($\nu_s = \gamma_c$, $Y < 1$) regime (“a = c”), or from an inverse-Compton to synchrotron transition (“iC $\rightarrow$ sy,” in the X-ray).

(A color version of this figure is available in the online journal.)

Figure 4. Same as Figure 3, but for a source deceleration $\Gamma = \Gamma_0(t/t_0)^{-1/4}$ corresponding to a wind-like medium of proton density $n(R) = 3 \times 10^{35} A_s R^{-2}$, and a magnetic field parameter $A_s$, $b = 10^{-2}$. Light-curve breaks originate from a spectral break crossing the optical (“a = o” for absorption, “i = o” for injection), from the occurrence of the ($\nu_s = \gamma_c$, $Y = 1$) regime (“a = c”), or from an inverse-Compton to synchrotron transition (“iC $\rightarrow$ sy,” in the X-ray).

(A color version of this figure is available in the online journal.)
Figure 5. Left panel: distribution (number of events in $\Delta R = 1$ bin, relative to maximum) of pair optical flash magnitude $R$ at $t = 10$ s, for three fixed GeV fluences $\Phi$, (corresponding to a bright, dim, and undetectable LAT emission) and for a GeV fluence distributed uniformly in log space (solid line). The source magnetic field parameter, $\nu F_{\nu}$, at time, $t$, were assumed to be uniformly distributed in log space, spanning the range $10^{-6}$ to $10^{4}$ erg cm$^{-3}$ and $10^{2}$–$10^{4}$, respectively (the shape of the distributions and its peak location depend on the assumed range and distribution of these two parameters, but the brightest end magnitude depends only on the fluence). Right panel: brightest pair optical flash flux, $R_{\text{min}}$ (i.e., the bright end of the distributions shown in the left panel), as a function of fluence $\Phi$, with the two parameters $(\Gamma, n b)$ left free. The $R_{\text{min}}$ is reached for $\nu_{c} < \nu_{p} < \nu_{i}$, $Y > 1$, $\tau < 1$, and when the self-absorption frequency is in the optical. For this ordering of synchrotron break frequencies, $R_{\text{min}}$ is independent of the burst redshift and very weakly dependent on the epoch $t$, thus the upper limit, $R_{\text{min}}$, on the optical flash flux shown here is a robust prediction for the synchrotron emission from pairs.

(A color version of this figure is available in the online journal.)

$vF_{\nu}$ peak frequency $v_b = \max(v_i, v_c)$ yields a light-curve break.

For a typical LAT light curve, $\Phi(t) = Ft \propto t^{-1/3}$, the pair flux has a power-law decay $F_{\nu} \propto t^{-\alpha}$ with $\alpha \in (1.0, 1.4)$ for a homogeneous medium and $\alpha \in (1.2, 2.0)$ for a wind, the latter range being more compatible with the observations of GRB optical counterparts ( flashes): $\alpha = 1.8$ for GRB 990123 (Akerlof et al. 1999), $\alpha = 2.0$ for GRB 061126 (Perley et al. 2008), $\alpha = 2.5$ for GRB 080319B (Wozniak et al. 2009), $\alpha = 1.7$ for GRB 130427A (Vestrand et al. 2014). Thus, a wind-like medium is favored when interpreting the optical flashes as emission from pairs; still, we note that steeper decays of the pair emission result for a fluence, $\Phi$, decreasing faster than $t^{-1/3}$.

While Figures 3 and 4 show that the X-ray flux from pairs at 0.1–10 ks can be compatible with that measured for Swift X-ray afterglows—I$\nu = 10^{-11} – 10^{-10}$ erg cm$^{-2}$ s$^{-1}$ (O’Brien et al. 2006)—the above pair light-curve decays are too steep compared with X-ray plateau measurements, $\alpha \in (0.2, 0.8)$; thus, only the faster-decaying plateaus can be accommodated by pair emission, provided that the fluence, $\Phi$, is nearly constant and that the ambient medium is homogeneous.

6.1. Brightest Optical Flash from Pairs

Figure 5 shows the maximal optical flux from pairs at $t = 10$ s (i.e., during the prompt emission phase), for a range of high-energy fluence, obtained by searching a reasonable range of the ($\Gamma, n b$) parameter space: $\Gamma \in (10, 10^4)$ and $n b \in (10^{-5}, 10^3)$. As illustrated in Figure 2, one expects that $\Gamma \in (100, 1000)$ for the brightest optical flash, because too low Lorentz factors lead to optically thick pairs and less radiation escapes in a dynamical timescale, while too high Lorentz factors increase the pair-formation threshold energy, leading to the formation of fewer pairs, and reducing the flux produced by pairs. As suggested by Figure 2, Figure 5 shows that the maximal optical synchrotron emission from pairs is larger than that from inverse-Compton, at any fluence.

The important result shown in Figure 5 is the existence of an upper limit, $R_{\text{min}}$, on the brightness of the optical flash from pairs, which offers a test of this model: any flash brighter than shown in Figure 5, for the measured LAT fluence, cannot originate from pairs. In more detail, the brightest observed LAT afterglows ($\Phi \sim 10^{-6}$ erg cm$^{-2}$) could yield optical flashes as bright as $R = 5$, the dimmest LAT afterglows ($\Phi \sim 10^{-6}$ erg cm$^{-2}$) may produce $R = 9$ optical flashes, with reasonably bright flashes originating from the pairs produced in GRB afterglows that are not detectable by LAT above 100 MeV.

We note that the existence of $R_{\text{min}}$ is not a immediate consequence of Equation (63), which leaves the possibility of a brighter optical flash than shown in Figure 5, if the peak frequency $v_p$ of the synchrotron spectrum fell in the optical and if it were not self-absorbed. Instead, the brightest optical flash from pairs shown in Figure 5 corresponds to $v_p$, being below optical ($v_{i}$); more exactly, $v_c < v_{i} = v_p = v_{i}$ (and $Y > 1$, $\tau < 1$) is satisfied everywhere on the $R_{\text{min}}$ line, a condition also fulfilled by the brightest two “peaks” shown in Figure 2 (left panel).

The surprising aspect of the brightest optical flash shown in Figure 5 is that $R_{\text{min}}(\Phi)$ is independent of the burst redshift, $z$. Perhaps a moderate dependence of $R_{\text{min}}$ on redshift is expected because $R_{\text{min}}$ is calculated for a fixed fluence, $\Phi$, thus a higher $z$ implies a larger afterglow output above 100 MeV, a larger number of formed pairs, and a larger synchrotron luminosity that compensates for the larger luminosity distance. That $R_{\text{min}}(\Phi)$ is $z$-independent can be proven in the following way. From
Equations (63)–(65), it follows that, for \( v_c < v_a < v_o < v_i \), the optical flux satisfies above self-absorption is

\[
F_o = F_p \left( \frac{v_o}{v_p} \right)^{1/2} \propto \frac{Z^3/2(n_b)^{1/4}}{\Gamma^2 T^2} \propto \Gamma^{-2}. \tag{82}
\]

For \( v_c < v_a < v_i \), Equations (17) and (23)–(26) lead to

\[
\gamma_a = \gamma_c t_p^{1/6} \propto \left( \frac{\Gamma Y}{B} \right)^{1/6}, \quad v_o \propto Z^{-5/6} \frac{\Phi^{1/2}(n_b)^{1/6}}{\Gamma^2 T^2}. \tag{83}
\]

From Equations (25) and (63)–(66), the optical flux below \( v_o \) is

\[
F_o = F_p \left( \frac{v_o}{v_p} \right)^{1/2} \left( \frac{v_o}{v_f} \right)^{5/2} \propto Z^{-5/2} \frac{\Gamma^2 t_i^2 \Phi_0(n_b)^{1/4}}{\Phi^4(n_b)^{1/4}}. \tag{84}
\]

Defining \( \Gamma_p \) by \( v_p(\Gamma_p) = v_o \), it follows from Equation (83) that (1) \( v_o < \gamma_c \), for \( \Gamma < \Gamma_p \), hence \( F_o \propto \Gamma^2 \) (Equation (84)), the optical flux increasing with \( \Gamma \), and (2) \( v_o < v_o \) for \( \Gamma > \Gamma_p \), hence \( F_o \propto \Gamma^{-2} \) (Equation (82)), the optical flux decreasing with \( \Gamma \). Consequently, the optical flux is maximal for \( \Gamma = \Gamma_p \), with \( \Gamma_p \propto Z^{5/4} \Phi^{1/8}(n_b)^{1/2} \left( 1 - \gamma_i / \gamma_p \right)^{1/4} \) following from the defining condition \( v_p(\Gamma_p) = v_o \). Substituting \( \Gamma_p \) in either Equation (82) or (84), the maximal optical flux satisfies

\[
F_0^{\text{max}} \equiv F_0(\Gamma_p) \propto Z^0(n_b)^{0} \Phi^{1/4} t_i^{1/4}. \tag{85}
\]

being independent of redshift and also of the magnetic field parameter \( n_b \) (meaning that, for any \( n_b \) that allows \( v_c < v_o \), \( v_o < v_i \), \( y > 1 \), and \( \tau < 1 \), there is a \( \Gamma_p \) that maximizes the optical flux to the same value \( F_0(\Phi, t_i) \)). The coefficient missing from Equation (85) can be determined by carrying the coefficients of all equations involved in its derivation. From Figure 5, the maximal optical flux of Equation (85) is

\[
R_{\text{min}} = 8.7 - 2.5 \log \left( \frac{F_0^{\text{max}}}{Jy} \right) = 7.0 - \frac{15}{8} \log \Phi - 5 \frac{5}{8} \log t_i, \tag{86}
\]

For GRB 130427A, the optical flash peaked at \( R \approx 7.4 \) at 10–20 s after trigger (Wren et al. 2013), when the LAT 0.1–100 GeV fluence was \( \Phi \approx 4 \times 10^{-5} \) erg cm\(^{-2}\) (Tam et al. 2013). The upper limit given in Equation (86), \( R_{\text{min}} \approx 5.8 \), is brighter than measured; thus, a pair origin for the optical flash of GRB 130427A is not ruled out.

6.2. Caveats

Our assumption that the power-law spectrum of the high-energy photons (measured by LAT at 100 MeV–10 GeV) extends well outside that range could lead to an overestimation of the emission from pairs in two ways.

Equations (17) and (18) show that the synchrotron emission at photon energy \( h \nu \) is produced by pairs of shock-frame energy \( y(\nu) = 155 \left( Z h \nu / 1 \text{eV} \right)^{1/2}(n_b)^{-1/2} T^{1/2} \). Pairs of this energy are formed from photons of observer-frame energy \( e(\nu) \approx 2 \Gamma m_e c^2 (z + 1) \) is \( 5.3 \left( Z h \nu / 1 \text{eV} \right)^{1/2} Z^{-1/2} (n_b)^{-1/2} \text{GeV} \). Thus, optical synchrotron emission requires pairs formed from seed photons of \( \epsilon \approx 10 \text{ GeV} \), which have been occasionally detected by LAT, 1 keV synchrotron emission requires photons of \( \epsilon \approx 200 \text{ GeV} \) (the highest-energy photon detected by LAT had \( \sim 100 \text{ GeV} \), for the \( z = 0.3 \) GRB 130427A—Fan et al. 2013, Tam et al. 2013), while 100 keV synchrotron emission requires photons of energy \( \epsilon \approx 3 \text{ TeV} \).

At the other end, if the high-energy spectrum has a break not far below 100 MeV, the assumption of a single power law overestimates the number of target photons for the \( \epsilon(\nu) \) photon, which leads to an overestimation of the optical thickness to pair formation and of the number of \( y(\nu) \) pairs that are formed (if the true optical thickness \( \tau_{\gamma y}(\epsilon(\nu)) \) is below unity). From Equation (1), the pair-formation threshold energy for a \( \epsilon(\nu) \) photon is \( \epsilon(\nu) = 0.22 h \nu / 1 \text{eV} \left( Z^{-3/2} (n_b)^{1/4} T^{-2} \right) \text{MeV} \), which, for both optical and X-ray photons, is well below the lower edge of the LAT window.

Not accounting for the decollimation introduced by the scatting of the seed photons on pairs leads to an underestimation of the true number of pairs. An estimate of the importance of that pair cascade can be obtained by first noting that most pairs are formed by the more numerous, lower energy photons above the threshold for pair formation, i.e., by the \( \sim 1 \) MeV photons of the LAT spectrum extrapolation. For a flat \( \nu F_\nu \) LAT spectrum, the flux of the 1 MeV photons is comparable to the LAT fluence, hence the 1 MeV photons should have an energy output of \( 10^{51.5} \) erg. For a \( 10^{51} \) erg burst lasting for 10 s, the outflow pair loading through a cascade process was shown (Kumar & Panaitescu 2004) to produce a significant pair enrichment up to a radius of \( R_{\text{max}} = 5 \times 10^{15} \) cm. For an observer-frame time, \( t \), this radius corresponds to source Lorentz factor, \( \Gamma_{\text{max}} = [(z + 1) R_{\text{max}} / c t]^{1/2} \approx 220 (Z / t_i)^{1/2} \). Thus, ignoring the pair-cascade process implies an underestimation of the pair number for a source with \( \Gamma < \Gamma_{\text{max}} = 1.5(2^{1/2} Z^{1/2} T^{-1/2})^{-1} \Gamma \), i.e., the pair cascade is effective mostly when the pair medium produced by the unscattered seed photons is optically thick (\( \tau > 1 \)). As illustrated in Figure 2, a larger number of pairs and the associated larger optical thickness, \( \tau \), leads to a dimmer emergent emission, hence the pair cascade would reduce even more the pair emission.

7. CONCLUSIONS

The above investigation of the broadband emission from pairs shows that the synchrotron emission from pairs formed from >100 MeV afterglow photons can accommodate the brightest optical counterparts (flashes) that were observed (in a few cases) during the prompt (GRB) phase, with the fast decay of optical flash pointing to a wind-like circumburst medium or to a faster decaying fluence of the high-energy photons. The inverse-Compton emission from pairs may yield an afterglow brightening, as seldom seen in optical afterglow light curves.

A brighter Fermi-LAT afterglow implies more pairs that can form and, thus, a brighter synchrotron and inverse-Compton emission from those pairs. The light-curve scalings given in Table 1 quantify the positive correlation between the synchrotron pair flux and the high-energy photon fluence, \( \Phi \). In addition to the 100 MeV fluence, the pair emission depends also on the source Lorentz factor, \( \Gamma \), and on the magnetic field in the pair medium. A very high Lorentz factor (\( \Gamma \approx 1000 \)) raises the threshold energy for pair formation, reduces the number of pairs and, implicitly, their emission. A very low Lorentz factor (\( \Gamma \approx 50 \)) leads to a larger number of pairs, an optically thick pair medium, which traps the emission and, consequently, yields a dim pair flux. Leaving free the source Lorentz factor, \( \Gamma \), and the magnetic field parameter, \( n_b \), we find that the brightest optical flash produced by pairs satisfies Equation (86), which gives an upper limit to that optical flash that depends mostly on the high-energy fluence, is weakly dependent on the epoch of observation, and, most importantly, is independent of the source redshift. We note that the brightest optical flash from pairs for
a given LAT fluence is obtained when the self-absorption frequency is in the optical, thus the intrinsic synchrotron optical spectrum of the brightest optical flash from pairs should be flat.

Given that a powerful source of high-energy photons is needed to produce enough pairs that can account for optical flashes/counterparts and that the source of high energy photons is, most likely, the forward shock (Kumar & Barniol Duran 2009), the multiwavelength data of GRB afterglows at early times should be interpreted/data modeled as the sum of synchrotron and inverse-Compton emissions from the forward shock, the reverse shock, and from pairs, at least in those cases where LAT measures a bright high-energy afterglow. To calculate the pair emission requires three parameters: the magnetic field parameter, \( n_b \) (which is also constrained by fitting the multiwavelength afterglow data with the reverse/forward-shock emission), the LAT afterglow fluence, \( \Phi \) (which is the blast-wave emission), and the source Lorentz factor, \( \Gamma \) (which is constrained directly by the double-shock emission fits at the deceleration time, and, indirectly, through the ratio of blast-wave energy to external density, \( E/n \), after deceleration onset). Thus, the pair emission does not entail any new parameters. We note that lower limits on \( \Gamma \) during the burst phase (predeceleration) can be obtained from the photon–photon opacity of the prompt LAT emission (e.g., Abdo et al. 2009).

The high LAT fluence of GRB 130427A (Ackermann et al. 2014), its bright optical counterpart (Vestrand et al. 2014), and broadband coverage would make it a good candidate for such a study. The low circumburst medium density, inferred by modeling its 10 s–10 d radio, optical, X-ray, and 100 MeV measurements, with synchrotron emission from the reverse and forward shocks (Laskar et al. 2013; Panaitescu et al. 2013; Perley et al. 2014), implies a very high afterglow Lorentz factor (\( \Gamma_0 \sim 800 \) at 10 s) and, consequently, a low number of formed pairs, leading to an optical emission from pairs that is well below the bright optical flash of GRB 130427A.

The pair emission discussed here offers an alternate explanation (to the reverse shock) for GRB optical counterparts. Both “mechanisms” can yield bright optical flashes with a fast decay after the peak. The pair optical flux should be correlated with the GeV contemporaneous fluence, but that feature may exist also for reverse-shock flashes, if most of the GeV flux arises from the reverse shock. The post-peak decay of the optical pair flux, given in Table 1 for a \( \varepsilon^{-2} \) LAT spectrum, can be generalized and used to test a pair origin for the optical flash, although a continuous reverse shock (i.e., not one that ceases at the optical peak time, followed by adiabatic cooling of ejecta electrons) may yield a similar decay index—spectral slope closure relation. Perhaps, modeling of the broadband afterglow data will be the best way to disentangle the pair emission and reverse-shock emissions.

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