Consistency of the Shannon entropy in quantum experiments

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The consistency of the Shannon entropy, when applied to outcomes of quantum experiments, is analysed. It is shown that the Shannon entropy is fully consistent and its properties are never violated in quantum settings, but attention must be paid to logical and experimental contexts. This last remark is shown to apply regardless of the quantum or classical nature of the experiments.

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I. INTRODUCTION

Since its introduction in 1948, the Shannon entropy [1] has played a central rôle in all branches of communication theory, where it allows a precise and operational definition of useful concepts like transmission rate and channel capacity, and has found many important applications in other branches of science as well [2, 3]. Shannon introduced it to quantify the amount of “uncertainty” or “choice” present in a probability distribution for a set of messages or symbols. In this regard, his entropy possesses some mathematical properties [4, 5, 6, 7, 8, 9] that correspond to intuitive properties expected from a measure of uncertainty.

In a recent paper [10], however, Brukner and Zeilinger analyse the application of the Shannon entropy to the statistical outcomes of two simple quantum-mechanical thought-experiments and of a non-quantum-mechanical, or ‘classical’, one, and conclude that two particular properties of the Shannon entropy are violated in quantum experiments while holding in classical ones.

This conclusion poses serious limitations when we wish to use the Shannon entropy, because of its particular properties, in quantum-mechanical problems. It would be therefore reasonable to ascertain in which sense or to what extent (if any) the claimed violations occur.

Deep criticisms of the conclusions of Ref. [10] have already appeared in the literature, mainly Hall’s [11] and Timpson’s [12, 13]. These interesting criticisms are mainly of a theoretical (and partly philosophical) nature, and so do not explain whether or how the concrete and interesting thought-experiments given in Ref. [10] really represent a violation of the Shannon entropy’s properties.

The purpose of the present paper is more pragmatic: to reanalyse the experiments presented in Ref. [10] and to show by explicit calculation that they present in fact no violation of any property of the Shannon entropy.

We shall see that the seeming “violations” appear only because the authors of Ref. [10] tacitly shift between different contexts (different experimental arrangements), thus misapplying Shannon’s formulae. In order to explain what is meant here by ‘context’, and to make the last point clearer, let us consider the following example:

In another Wonderland, Alice is acquainted with two Cheshire Cats: a Frumious one and a very Mome one. The Frumious Cheshire Cat always appears with its tail \((T)\) first, and then with the rest \((R)\) of its body (head included), and in these apparitions Alice always observes first only that \(T\) = 2 cm, and then that \(R\) = 3 cm, hence

\[
T + R = 5 \text{ cm}.
\]

(1)

The Mome Cheshire Cat, instead, always appear with its rest \((R)\) first, and then comes the tail \((T)\), so that, when meeting it, Alice can first observe that \(R\) = 30 cm, and then that \(T\) = 20 cm; hence

\[
R + T = 50 \text{ cm}.
\]

(2)

Looking at Eqs. (1) and (2), Alice concludes:

\[
T + R \neq R + T,
\]

(3)
or: “the commutative property of addition is violated for Cheshire Cats”.

Is Alice’s conclusion about the commutative property correct? No. First of all, we see that the content of Eq. (1) is really just \(2 \text{ cm} + 3 \text{ cm} = 3 \text{ cm} + 2 \text{ cm} = 5 \text{ cm}\), and analogously for Eq. (2), hence the commutative property is in fact clearly satisfied in those equations.

On the other hand, what Alice means by Eq. (3) is just \(2 \text{ cm} + 3 \text{ cm} \neq 30 \text{ cm} + 20 \text{ cm}\), so that this equation has little to do with the commutative property of addition. In fact Eq. (3), as it is written, makes simply no sense, because the various mathematical symbols therein are inconsistently used: they have different values on the right- than on the left-hand side.

This comes about because Alice tacitly shifts from a given context, where the symbols ‘\(T\)’ and ‘\(R\)’ refer to the Frumious Cheshire Cat and have some values, to a different context,

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where the same symbols refer to the Mome Cheshire Cat and have different values — but the commutative property of addition is not meant to apply that way. Related to this is the fact that Alice seems to attribute a temporal meaning to the order in which the summands appear in the additions $1$–$3$, but we know that the order of terms in an addition has no temporal meaning. If Alice wishes to express temporal details, she can do so by, e.g., appending subscripts to the symbols, like $T_r$, $R_r$, etc.; in any case she cannot burden the addition operation with meanings that the latter does not have.

Hence, Alice’s incorrect conclusion stems from her using, inconsistently, the same symbols $T$ and $R$ for two different and incompatible logical and experimental contexts. She could have used, e.g., $T_{\text{Mome},r}$, $R_{\text{Frum},r}$, etc., instead, writing her findings as

$$T_{\text{Mome},r} + R_{\text{Mome},r'} = R_{\text{Mome},r'} + T_{\text{Mome},r}$$

$$\neq T_{\text{Frum},r} + R_{\text{Frum},r'} = R_{\text{Frum},r'} + T_{\text{Frum},r'},$$

(4)

where all symbols are consistently and informatively used and it is clear that no violations of any mathematical property occur.

In nuce, an analogous faux pas is made in Ref. [10], although less apparent, because there instead of the addition we have more complex mathematical functions — probabilities and entropies —, and quantum experiments take the place of the example’s Cheshire-Cats experiments. In the mentioned paper, the authors compare probabilities and Shannon entropies pertaining to different contexts (different experimental arrangements); thus their equations do not pertain to the properties they meant to discuss. To my knowledge, this simple point has hitherto never been noticed in the literature, and will be shown here by explicit calculation.

We shall also investigate how the peculiar reasoning in Ref. [10] comes about, and find that it stems from an improper temporal interpretation of the logical symbols ‘|’ and ‘∧’. This point has not been noticed in the literature either.2

It is also clear that such kinds of seeming “violations”, arising through mathematical and logical misapplication, can then easily be made to appear not only in quantum experiments, but in classical ones as well, and this will also be shown by simple examples.

For these purposes, some notation and definitions for probabilities and the Shannon entropy will first be introduced, where the rôle of the context — e.g., the given experimental arrangement — is emphasized. Then, after a discussion of the two questioned properties of the entropy, the experiments of Ref. [10] will be presented and re-analysed. Finally, two “counter-experiments” will be presented.

II. DEFINITIONS AND NOTATION

We shall work with probability theory as extended logic [7, 16, 17], so we denote by $P(a|l)$ the probability of the proposition $a$ given the prior knowledge $l$. This prior knowledge, also called (prior) data or prior information, represents the context the proposition refers to. It is useful to write explicitly the prior knowledge — a usage advocated by Keynes [18], Jeffreys [17, 19], Koopman [20, 21, 22], Cox [7, 23, 24], Good [25], and Jaynes [16] — because the meaning and the probability of a proposition always depend on a given context. Compare, e.g., $P(\text{“Tomorrow it will snow”} | \text{“We are in Stockholm and it is winter”})$ and $P(\text{“Tomorrow it will snow”} | \text{“We are in Rome and it is summer”}).$3 Even more, the probability of a proposition can be undefined in a given context (i.e., that proposition is meaningless in that context): consider, e.g., $P(\text{“My daughter’s name is Kristina”} | \text{“I have no children”}).$ Common usage tends to omit the context and writes $P(a)$ instead of $P(a|l)$, but this usage can in some cases — especially when the context is variable, i.e., several different contexts are considered in a discussion — lead to ambiguities, as will be shown.

In particular, a proposition can represent an outcome of an experiment or observation that was, is, or will be, performed. The context in this case consists of all the details of the experimental arrangement and of the theoretical model of the latter which are necessary to assign a probability to that outcome. This is true for both “classical” and “quantum” experiments.

The importance of the context is also evident when considering a set of propositions

$$A \equiv \{a_i : i = 1, \ldots, n_a\}$$

such that, in the context $l$, they are mutually exclusive and exhaustive (i.e., one and only one of them is true). Indeed, it is clear that the propositions may be mutually exclusive and exhaustive in some context while not being such in another. For example, the propositions (“A red ball is drawn at the first draw”, “A wooden ball is drawn at the first draw”) are mutually exclusive and exhaustive in the context “Balls are drawn from an urn containing one red plastic ball and one blue

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2 In fact, even Timpson [12, p. 15] seems to attach an improper temporal sense to the logical concept of ‘joint probability’.

3 The relevance of prior knowledge is nicely expressed in the following brief dialogue, which I could not resist quoting, between Holmes and Watson [26], taking place after the two listened to a client’s statement:

“I think that I shall have a whisky and soda and a cigar after all this cross-questioning. I had formed my conclusions as to the case before our client came into the room.”

“My dear Holmes!”

“[…]My whole examination served to turn my conjecture into a certainty. Circumstantial evidence is occasionally very convincing, as when you find a trout in the milk, to quote Thoreau’s example.”

“But I have heard all that you have heard.”

“Without, however, the knowledge of pre-existing cases which serves me so well.”
wooden ball", but are neither mutually exclusive nor exhaustive in the context "Balls are drawn from an urn containing one red wooden ball, one red plastic ball, and one blue plastic ball", and make no sense in the context "Cards are drawn from a deck".

In the following, propositions which are mutually exclusive and exhaustive in a given context will be called *alternatives*, and their set a *set of alternatives*. Such a set may represent the possible, mutually exclusive outcomes in a given experiment.

The probability distribution for a set of alternatives \( \{ a_i : i = 1, \ldots, n_a \} \) in the context \( l \) will be denoted by
\[
P(A | l) \overset{\text{def}}{=} \{ P(a_i | l) : i = 1, \ldots, n_a \},
\]
and has, of course, the usual properties
\[
P(a_i \land a_i' | l) = 0 \quad \text{for } i \neq i',
\]
\[
\sum_i P(a_i | l) = 1,
\]
where the conjunction \( \lor \) (sometimes also called 'logical product') of \( a \) and \( b \) is denoted by \( a \land b \).

The Shannon entropy \( H \) is a function of the probability distribution for a given set of alternatives \( A \) in a given context \( l \). It is defined as
\[
H(P(A | l)) \overset{\text{def}}{=} -K \sum_i P(a_i | l) \ln P(a_i | l), \quad K > 0,
\]
with the usual conventions for the units in which it is expressed and the positive constant \( K \), and the limiting procedure when vanishing probabilities are present.\(^5\) The Shannon entropy can be considered to quantify the amount of "uncertainty" associated with a probability distribution.\(^6\)

Given two sets of alternatives \( A \) and \( B \) in the context \( l \), the set of all conjunctions of the propositions \( \{ a_i \} \) with the propositions \( \{ b_j \} \),
\[
A \land B \overset{\text{def}}{=} \{ a_i \land b_j : i = 1, \ldots, n_a; j = 1, \ldots, n_b \},
\]
is also a set of alternatives in \( l \), as follows from the rules of probability theory; it will be called a *composite set*.\(^7\) Thus the composite probability distribution
\[
P(A \land B | l) \overset{\text{def}}{=} \{ P(a_i \land b_j | l) : i = 1, \ldots, n_a; j = 1, \ldots, n_b \}
\]
satisfies the properties
\[
P((a_i \land b_j) \land (a_i' \land b_j') | l) = 0 \quad \text{for } i \neq i' \text{ or } j \neq j',
\]
\[
\sum_{ij} P(a_i \land b_j | l) = 1.
\]

The probability distributions for the sets of alternatives \( A, B, \) and \( A \land B \) in the context \( l \) are related by the following standard probability rules (see e.g. Jaynes \[16, \text{Ch. 2}\], Jeffreys \[17, \text{Ch. 1}\], Cox \[19, \text{Ch. 1}\])
\[
P(a_i | l) = \sum_j P(a_i \land b_j | l),
\]
\[
P(b_j | l) = \sum_i P(a_i \land b_j | l)
\]
(marginal probabilities), and
\[
P(a_i \land b_j | l) = P(a_i | b_j \land l) P(b_j | l) = P(b_j | a_i \land l) P(a_i | l)
\]
(product rule), from which
\[
P(a_i | b_j \land l) = \frac{P(a_i \land b_j | l)}{P(b_j | l)} \quad \text{if } P(b_j | l) \neq 0,
\]
\[
P(b_j | a_i \land l) = \frac{P(a_i \land b_j | l)}{P(a_i | l)} \quad \text{if } P(a_i | l) \neq 0
\]
(Bayes’ rule) follow.

All the probability distributions for the sets of alternatives \( A, B, \) and \( A \land B \) have associated Shannon entropies. It is also possible to define the *conditional entropy* of the distribution for \( B \) relative to the distribution for \( A \), as follows:
\[
H[P(B | A \land l)] \overset{\text{def}}{=} \sum_i P(a_i | l) H[P(B | a_i \land l)]
\]
\[
= -K \sum_i P(a_i | l) \sum_j P(b_j | a_i \land l) \ln P(b_j | a_i \land l).
\]

An analogous definition is given for \( H[P(A | B \land l)] \).

### III. Properties of the Shannon Entropy and Quantum Experiments

The various Shannon entropies presented in the previous section possess several mathematical properties \[11, 12, 13, 14, 15\] which have a fairly intuitive meaning when the Shannon entropy is interpreted as a measure of "uncertainty". We shall focus our attention on two of these properties which, it is claimed \[10\], are violated in quantum experiments.

Consider the probability distribution for a composite set of alternatives \( A \land B \) in a context \( l \). The entropy \( H[P(B | l)] \) and the conditional entropy \( H[P(B | A \land l)] \) satisfy
\[
H[P(B | l)] \geq H[P(B | A \land l)].
\]
This property (also called concavity since it arises from the concavity of the function $-x \ln x$), intuitively states that the uncertainty of the probability distribution for $B$ in the context $l$ decreases or remains the same, on average, when the context is “updated” because one of the $[a_i]$ is known to be true. An intuitive picture can be given as follows. Imagine that someone performs an experiment (represented by $l$) consisting in two observations (represented by the sets $A$ and $B$), and writes the results of the experiment in an observation record, say, on a piece of paper denoted by ‘$l$’, under the headings ‘$A$’ and ‘$B$’. Now suppose that we know all the details of the experiment except for the outcomes: we have not yet taken a look at the record and are uncertain about what result is written under ‘$B$’. If we now read the result written under ‘$A$’, our uncertainty about the result under ‘$B$’ will decrease or remain the same on average (i.e., in most, though not all, cases), since the former result can give us some clues about the latter.

The second property (strong additivity \[ H[P(A \land B | l)] = H[P(A | l)] + H[P(B | A \land l)] \] (16b)) reads

$$H[P(A \land B | l)] = H[P(A | l)] + H[P(B | A \land l)],$$

and its intuitive meaning is that the uncertainty of the probability distribution for the composite set of alternatives $A \land B$ is given by the sum of the uncertainty of the probability distribution for $A$ and the average uncertainty of the updated probability distribution for $B$ if one of the $[a_i]$ is known to be true, or vice versa. Using the intuitive picture already proposed, we are initially uncertain about both outcomes written on the record ‘$l$’. This uncertainty will first decrease as we read the outcome for ‘$A$’, and then disappear completely when we read the outcome for ‘$B$’ (given that we do not forget what we have read under ‘$A$’). Equivalently, the total uncertainty will first decrease and then disappear as we read first the outcome under ‘$B$’ and then the one under ‘$A$’.

Three remarks are appropriate here. The first is that the above are mathematical, not physical, properties: they are not experimentally observed regularities, but rather follow mathematically, through the properties of basic arithmetical functions like the logarithm, once we put some numbers (the probabilities $P(a_1 | l)$, $P(b_2 | l)$, $P(b_2 | a_1 \land l)$, etc.) into the formulae \[ \text{9} \] and \[ \text{15} \]. In particular, they can be neither proven nor disproved by experiment, but only correctly or incorrectly applied. It is for this reason that one suspects that the seeming violations found in Ref. \[ \text{10} \] are only caused by an incorrect application of the formulae \[ \text{16} \]; we shall see that this is the case, because in the mentioned paper some probabilities are used in the right-hand sides of Eqs. \[ \text{16} \], and (numerically) different probabilities in the left-hand sides.

This leads us to the second remark: the above properties hold when all the probabilities in question refer to the same context; otherwise, they are not guaranteed to hold. This is because if a specific set of alternatives, in two different contexts, has two different sets of probabilities, then it will yield, in general, two different entropies as well. Hence we cannot expect the formulae \[ \text{16} \] to hold if we use, in the left-hand sides, the probabilities relative to a given experiment, and, in the right-hand sides, those relative to a different experiment. This remark amounts simply to saying that the expression ‘$x+y = y+x$’ holds only if we give $x$ and $y$ the same value on both sides (cf. Alice’s example in the Introduction). The (somewhat pedantic) notation $H[P(A | l)]$ and $H[P(B | A \land l)]$, instead of the more common $H(A)$ and $H(B | A)$, is used here to stress this point.

The third remark (cf. also Koopman \[ \text{22} \]) is that an expression like ‘$B | A$’ does not imply that “the observation represented by $A$ is performed before the one represented by $B$’; nor does ‘$A \land B$’ mean that the two observations are carried out “simultaneously”. The temporal order of the observations is formally contained in the context $l$, and the symbols ‘$\land$’ and ‘$\lor$’ have a logical, not temporal, meaning. One must be careful not to confuse logical concepts with mathematical objects or physical procedures, even when there may be some kind of relationship amongst them; this kind of confusion is strictly related to what Jaynes called “the mind projection fallacy” \[ \text{14 55 56}].

Consider, e.g., the following situation: an urn contains a red, a green, and a yellow ball, and we draw two balls without replacement (this is our context $l_0$). Consider the propositions $a \supseteq \text{‘Red at the first draw’}$, and $b \supseteq \text{‘Green at the second draw’}$. Now, suppose that we also know that the second draw yields ‘green’; what is the probability that the first draw yields ‘red’? The answer, to be found in any probability-theory textbook, is $P(a | b \land l_0) = P(a \land b | l_0)/P(b | l_0) = (1/3 \times 1/2)/(1/3) = 1/2$. We see that the expression ‘$a | b$’ does not mean that $b$ precedes $a$ (nor does it mean that $b$ “causes” $a$); it just expresses a logical connexion. We can also ask: what is the probability of obtaining first ‘red’ and then ‘green’? The answer is again standard: $P(a \land b | l_0) = 1/6$. We see that the expression ‘$a \land b$’ does not mean that $a$ and $b$ happen simultaneously; it just expresses a logical connexion.\[ \text{8} \]

Note that Shannon himself, in his paper, does not attribute any temporal meaning to the conditional or joint entropies, but only the proper logical one. He uses, e.g. \[ \text{11} \], \[ \text{11} \] and \[ \text{12} \], the expressions $H[P(X | Y \land l)]$ and $H[P(X \land Y | l)]$ (which in his notation are ‘$H_x(x')$’ and ‘$H(x,y)$’ \[ \text{6} \]) where ‘$Y$’ represents the received signal, while ‘$X$’ represents the signal source — i.e., where $X$ actually precedes $Y$.\[ \text{9} \] We may also remark that Shannon’s entropy formulae for channel characteristics are not directly dependent on the actual physical details of the system(s) constituting the channel, which can be classi-

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\(8\) We can change the example by considering, instead, the order in which three identically prepared radioactive nuclei, in three different laboratories, will decay. The analysis and the conclusion will nevertheless be the same. ‘Quantumness’ plays no rôle here.

\(9\) From Shannon’s paper: “First there is the entropy $H(x)$ of the source or of the input to the channel [...] The entropy of the output of the channel, i.e., the received signal, will be denoted by $H(y)$. [...] The joint entropy of input and output will be $H(x,y)$. Finally there are two conditional entropies $H_x(y)$ and $H_y(x)$, the entropy of the output when the input is known and conversely. Among these quantities we have the relations $H(x,y) = H(x) + H_y(x)$ \[ \text{1} \] \[ \text{11} \].
cal, quantum, or even made of pegasi and unifauns. Only their statistical properties are directly relevant.

Hence, the conjunction \( a_i \land b_j \equiv b_j \land a_i \) (and \( B \land A \equiv A \land B \)) is commutative even if the matrix product of two operators which can in some way be associated with these propositions is not.

We shall see that these remarks have indeed a bearing on the analysis presented in Ref. [10], where the validity of the formulae (16) was analysed in two quantum experiments and a classical one, and it was found that, seemingly, Eqs. (16) did not hold in the quantum case, while they still held in the classical one. These experiments will be now presented using first their original notation of Ref. [10] (indicated by the presence of quotation marks), which makes no reference to the context, and then re-analysed using the expanded notation introduced above.

A. First quantum experiment

The first quantum experiment [10, Fig. 4] runs as follows. Suppose we send a vertically polarised photon through a horizontal polarisation filter. The set of alternatives \( A \) refers to the photon’s coming out of the filter, with \( a_{\text{out}} \equiv “\text{The photon comes out of the horizontal filter}” \), \( b_{\text{not-out}} \equiv “\text{The photon does not come out of the horizontal filter}” \). Since we are sure about \( b_{\text{not-out}} \), i.e., “\( P(b_{\text{not-out}}) = 1 \)”, we have that

\[
H(B) = 0.
\]

Thus we find

\[
0 = H(B) < H(B | A) \quad (\text{seemingly})
\]

and property (16a) is seemingly violated.

Only seemingly, though. The fact that a diagonal filter was considered in the reasoning leading to Eq. (18) but not in the reasoning leading to Eq. (17), makes us doubt whether the comparison of the entropies “\( H(B) \)” and “\( H(B | A) \)” really corresponds to Eq. (16a); these entropies, in fact, apparently refer to two different experimental arrangements — i.e., to two different contexts. It becomes evident that this is indeed the case if we proceed by calculating numerically and explicitly all the probabilities first, and then the entropies, keeping the context in view.

Let us consider again the first experimental set-up, which will be denoted by \( l_{\text{one}} \), where only one, horizontal, filter is present. For the set \( B \), we have of course that, using our notation,

\[
P(b_{\text{out}} | l_{\text{one}}) = 0, \quad P(b_{\text{not-out}} | l_{\text{one}}) = 1, \quad (20)
\]

so that the Shannon entropy is

\[
H(P(B | l_{\text{one}})) = H(0, 1) = 0 \text{ bit.} \quad (21)
\]

At this point, asking for the probabilities for the set of alternatives \( A \), we realise that the propositions \( a_{\text{out}} \equiv “\text{The photon comes out of the diagonal filter}” \) and \( a_{\text{not-out}} \equiv “\text{The photon does not come out of the diagonal filter}” \) make no sense here, since no diagonal filter is present; consequently, there exist no entropies like \( H(P(A | l_{\text{one}})) \) or \( H(P(B | A \land l_{\text{one}})) \). So in this experimental set-up it is not even meaningful to consider the property (16a).

When we insert a diagonal filter before the horizontal one, we have a new, different experimental arrangement, which will be denoted by \( l_{\text{two}} \). In this new set-up it does make sense to speak of both sets \( A \) and \( B \). Quantum mechanics yields the following probabilities:

\[
P(a_{\text{out}} | l_{\text{two}}) = \frac{1}{2}, \quad P(a_{\text{not-out}} | l_{\text{two}}) = \frac{1}{2}. \quad (22)
\]

\[
P(b_{\text{out}} | a_{\text{out}} \land l_{\text{two}}) = \frac{1}{2}, \quad P(b_{\text{not-out}} | a_{\text{out}} \land l_{\text{two}}) = \frac{1}{2}. \quad (23)
\]

\[
P(b_{\text{out}} | a_{\text{not-out}} \land l_{\text{two}}) = 0, \quad P(b_{\text{not-out}} | a_{\text{not-out}} \land l_{\text{two}}) = 1. \quad (24)
\]

whence, by the product rule, it follows that

\[
P(b_{\text{out}} | l_{\text{two}}) = \frac{1}{2}, \quad P(b_{\text{not-out}} | l_{\text{two}}) = \frac{1}{2}. \quad (25)
\]

At this point we notice that the probabilities for the set \( B \), Eqs. (25), differ numerically from those calculated in the previous set-up, Eqs. (21). This difference forces us to take note of the difference of the set-ups \( l_{\text{one}} \) and \( l_{\text{two}} \); if we ignored this, inconsistencies would arise already for the probabilities, even before computing any entropy.

The Shannon entropy for the probability distribution for the set \( B \) appropriate in this context is readily calculated:

\[
H(P(B | l_{\text{two}})) = H(\frac{1}{2}, \frac{1}{2}) = 0.81 \text{ bit,} \quad (26)
\]

In the original formulation of the example [10], the authors denote “by \( A \) and \( B \) the properties of the photon to have polarization at +45° and horizontal polarization, respectively”; so that \( A \) should perhaps be defined as “The photon has diagonal (45°) polarisation”, “The photon has no diagonal (45°) polarisation”, and \( B \) analogously. However, there are problems with these propositions. If the photon is absorbed by the diagonal filter, then it makes no sense to say that the photon has no diagonal polarisation, since the photon is no longer present (note that this problem has nothing to do with the non-existence of properties of a system before an observation: the point is that, if no photon is present, then it does not make sense to speak about its properties anyway). For this reason the alternative propositions \( a_{\text{out}}, b_{\text{not-out}}, \ldots \) have been used here; however, this has not affected the point of the experiment in Ref. [10] — namely, the seeming violation of property (16a).
and we see that, in fact, it differs from the one in the first set-up: thus, in the first quantum example of Ref. [10] the expression "H(B)" is inconsistently and ambiguously used.

The conditional entropy relative to A is:

$$H[P(B \mid A \land l_{\rm two})] = 0.5 \text{ bit},$$

(27)

and, as a consequence,

$$0.81 \text{ bit} \approx H[P(B \mid l_{\rm one})] \geq H[P(B \mid A \land l_{\rm two})] = 0.5 \text{ bit},$$

(28)

in accord with the property [16a].

So no violations of the property [16a] are found here. Equation [19], from Ref. [10], is simply incorrect, and the seeming "violations" that arose from it disappear at once if we write that equation more correctly as

$$H[P(B \mid l_{\rm one})] < H[P(B \mid A \land l_{\rm two})],$$

(29)

where we see that the left-hand side refers to the set-up of one, whereas the right-hand side refers to the different set-up of two, so that the comparison is between entropies relative to different experiments, and the equation does not concern property [16b]. This fact escaped the attention of the authors of Ref. [10], partly because the contexts were not explicitly written, and partly because the probabilities were not explicitly calculated (so that the authors did not notice that the probabilities for B had two numerically different sets of values). The authors also seem to interpret the expression "H(B)" as implying that no observation must precede B, and the expression "H(B \mid A)" as implying that the observation relative to B must be preceded by the one relative to A. As already remarked, this needs not be the case.

It should be noted that the experimental arrangements of one and two are incompatible, also in the formal sense that their conjunction of one \& two is false. We could erroneously see two as a "more detailed" description of one, equivalent, for example, to the conjunction of one and the proposition "Moreover, a diagonal polarisation filter is present between the photon source and the horizontal filter". But this is not the case: two states that nothing is present between the source and the horizontal filter; if it had been otherwise, and one had left open the possibility that something unknown could be between source and filter (a linear or circular polarisation filter, or a mirror, or an opaque screen, or something else), then we should have assigned a different state to the photon reaching the horizontal filter, and the calculation of the probabilities would have been very different [37].

**B. Second quantum experiment**

The second experiment [10, Fig. 5] is as follows. We send a spin-1/2 particle with spin up along the z axis through a Stern-Gerlach apparatus aligned along the axis \( z \) that lies in the xz plane and forms an angle \( \alpha \) with the z axis. Let us denote by A the set \{a\text{up}, a\text{down}\} with \( a\text{up} \equiv \) "The particle comes out with spin up along \( a \)" and \( a\text{down} \equiv \) "The particle comes out with spin down along \( a \)." We have the following probabilities, in the notation of Ref. [10]:

$$P(a_{\text{up}}) = \cos^2 \frac{\alpha}{2}, \quad P(a_{\text{down}}) = \sin^2 \frac{\alpha}{2},$$

(30)

and a corresponding Shannon entropy which amounts to

$$H(A) = H(\cos^2 \frac{\alpha}{2}, \sin^2 \frac{\alpha}{2}).$$

(31)

The particle then proceeds to a second Stern-Gerlach apparatus aligned along the x axis; let us denote the corresponding set of alternatives by \( B = \{b_{\text{up}}, b_{\text{down}}\} \), where \( b_{\text{up}} \) and \( b_{\text{down}} \) are defined analogously to \( a_{\text{up}} \) and \( a_{\text{down}} \) above. The conditional probabilities for B relative to the outcomes for A are:

$$P(b_{\text{up}} \mid a_{\text{up}}) = \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(32)

$$P(b_{\text{down}} \mid a_{\text{up}}) = \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(33)

$$P(b_{\text{up}} \mid a_{\text{down}}) = \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(34)

$$P(b_{\text{down}} \mid a_{\text{down}}) = \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(35)

The sum of the entropies thus far calculated is

$$H(A) + H(B \mid A) = 2 H(\cos^2 \frac{\alpha}{2}, \sin^2 \frac{\alpha}{2}).$$

(36)

Now we suppose to exchange the two Stern-Gerlach apparatus, putting the one along \( x \) before the one along \( a \). We then find the following probabilities for B:

$$P(b_{\text{up}}) = \frac{1}{2}, \quad P(b_{\text{down}}) = \frac{1}{2},$$

(37)

with the associated Shannon entropy

$$H(B) = H(\frac{1}{2}, \frac{1}{2}).$$

(38)

The conditional probabilities, given by quantum theory, for the set A relative to the outcomes for B are:

$$P(a_{\text{up}} \mid b_{\text{up}}) = \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(39)

$$P(a_{\text{down}} \mid b_{\text{up}}) = \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(40)

$$P(a_{\text{up}} \mid b_{\text{down}}) = \cos^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(41)

$$P(a_{\text{down}} \mid b_{\text{down}}) = \sin^2 \left( \frac{\alpha}{2} - \frac{\pi}{4} \right),$$

(42)

and together with the probabilities [36] they lead to the conditional entropy

$$H(A \mid B) = \sin^2 \frac{\alpha}{2} \times H(\sin^2 \frac{\alpha}{2}, \cos^2 \frac{\alpha}{2})$$

$$+ \cos^2 \frac{\alpha}{2} \times H(\cos^2 \frac{\alpha}{2}, \sin^2 \frac{\alpha}{2}),$$

(43)

$$= H(\cos^2 \frac{\alpha}{2}, \sin^2 \frac{\alpha}{2}).$$

(44)
The sum of the entropies \[37\] and \[40\] now yields

\[
H(B) + H(A | B) = H\left(\frac{1}{2}, \frac{1}{2}\right) + H\left(\cos^2 \frac{\alpha}{2}, \sin^2 \frac{\alpha}{2}\right),
\]
but this is in general (e.g. for \(\alpha = \pi/4\)) different from the sum \[35\], and thus we find that, in general,

\[
H(A) + H(B | A) \neq H(B) + H(A | B)
\]
(simultaneously), \[42\] in seeming contradiction with the property \[16b\].

However, we notice that two different relative positions of the Stern-Gerlach apparatus were considered, in order to arrive at Eqs. \[35\] and \[41\] respectively. This makes the seeming contradiction above just an artifact produced, again, by the comparison of Shannon entropies relative to two different experimental arrangements, analogously to what happened in the experiment with photons previously discussed. Also in this case, this is shown by an explicit calculation of all the probabilities relative to the experiments.

In the first set-up, which can be denoted by \(m\), we send the spin-1/2 particle to the Stern-Gerlach apparatus oriented along \(a\) (associated to the set of alternatives \(A\)), which is in turn placed before the one oriented along \(x\) (associated to the set \(B\)). Basic quantum-mechanical rules yield the following probabilities (in our notation):

\[
\begin{align*}
P(a_{\text{up}} | m) &= \cos^2 \frac{\alpha}{2}, & P(a_{\text{down}} | m) &= \sin^2 \frac{\alpha}{2}, \\
P(b_{\text{up}} | a_{\text{up}} \land m) &= \cos^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right), \\
P(b_{\text{down}} | a_{\text{up}} \land m) &= \sin^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right), \\
P(b_{\text{up}} | a_{\text{down}} \land m) &= \sin^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right), \\
P(b_{\text{down}} | a_{\text{down}} \land m) &= \cos^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right),
\end{align*}
\]
and these are sufficient to calculate, by the product rule \[14b\], the joint probabilities

\[
\begin{align*}
p_{ab} &\equiv P(a_{\text{up}} \land b_{\text{up}} | m) = \cos^2 \frac{\alpha}{2} \cdot \cos^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right), \\
p_{ab} &\equiv P(a_{\text{up}} \land b_{\text{down}} | m) = \cos^2 \frac{\alpha}{2} \cdot \sin^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right), \\
p_{ab} &\equiv P(a_{\text{down}} \land b_{\text{up}} | m) = \sin^2 \frac{\alpha}{2} \cdot \cos^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right), \\
p_{ab} &\equiv P(a_{\text{down}} \land b_{\text{down}} | m) = \sin^2 \frac{\alpha}{2} \cdot \sin^2 \left(\frac{\alpha}{2} - \frac{\pi}{4}\right).
\end{align*}
\]

The formulae above show clearly that, as remarked in \[11\] it does make sense to consider the composite probability of the outcomes of the two temporarily separated observations, as a composite probability needs have nothing to do with the fact that the observations are performed “simultaneously” or not.

From the joint probabilities, we can compute all probabilities involved in this set-up. Applying the marginal probability rule \[14g\] and Bayes’ rule \[14h\] we find:

\[
\begin{align*}
P(a_{\text{up}} | m) &= p_{ab} + p_{\bar{a}b}, & P(a_{\text{down}} | m) &= p_{\bar{a}b} + p_{\bar{a}\bar{b}}, \\
P(b_{\text{up}} | m) &= p_{ab} + p_{\bar{a}b}, & P(b_{\text{down}} | m) &= p_{ab} + p_{\bar{a}\bar{b}}, \\
P(a_{\text{up}} | b_{\text{up}} \land m) &= \frac{p_{ab}}{p_{ab} + p_{\bar{a}b}}, & P(a_{\text{down}} | b_{\text{up}} \land m) &= \frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}}, \\
P(a_{\text{down}} | b_{\text{down}} \land m) &= \frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}}, & P(b_{\text{down}} | b_{\text{down}} \land m) &= \frac{p_{ab}}{p_{ab} + p_{\bar{a}b}}.
\end{align*}
\]
Here the probabilities \[45\] have been re-written in terms of the joint probabilities.

A look at Eqs. \[47\] and \[48\] shows that Bayes’ rule also applies in “quantum experiments”, and provides a counter-example to the incorrect statement that the expression “\(a_{\text{up}} \mid b_{\text{up}}\)” would mean “\(b_{\text{up}}\) precedes \(a_{\text{up}}\)”.

We can proceed to calculate the Shannon entropies

\[
\begin{align*}
H[P(A | m)] &= -K[(p_{ab} + p_{\bar{a}b}) \ln(p_{ab} + p_{\bar{a}b}) + (p_{ab} + p_{\bar{a}\bar{b}}) \ln(p_{ab} + p_{\bar{a}\bar{b}})], \\
H[P(B | m)] &= -K[(p_{ab} + p_{\bar{a}b}) \ln(p_{ab} + p_{\bar{a}b}) + (p_{ab} + p_{\bar{a}\bar{b}}) \ln(p_{ab} + p_{\bar{a}\bar{b}})].
\end{align*}
\]

as well as the conditional entropies

\[
\begin{align*}
H[P(B | A \land m)] &= K(p_{ab} + p_{\bar{a}b}) \left(\frac{p_{ab}}{p_{ab} + p_{\bar{a}b}} \ln \frac{p_{ab}}{p_{ab} + p_{\bar{a}b}} - \frac{p_{ab}}{p_{ab} + p_{\bar{a}\bar{b}}} \ln \frac{p_{ab}}{p_{ab} + p_{\bar{a}\bar{b}}}\right) \\
&\quad + K(p_{\bar{a}b} + p_{\bar{a}\bar{b}}) \left(-\frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}} \ln \frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}} - \frac{p_{\bar{a}b}}{p_{\bar{a}b} + p_{\bar{a}\bar{b}}} \ln \frac{p_{\bar{a}b}}{p_{\bar{a}b} + p_{\bar{a}\bar{b}}}\right) \\
&= K[-p_{ab} \ln p_{ab} - p_{\bar{a}b} \ln p_{\bar{a}b} + p_{ab} \ln p_{\bar{a}b} - p_{\bar{a}b} \ln p_{\bar{a}\bar{b}} + p_{ab} \ln p_{\bar{a}\bar{b}} + (p_{ab} + p_{\bar{a}b}) \ln(p_{ab} + p_{\bar{a}b}) + (p_{ab} + p_{\bar{a}\bar{b}}) \ln(p_{ab} + p_{\bar{a}\bar{b}})].
\end{align*}
\]
\[ H[P(A \mid B \land m)] = K(p_{ab} + p_{\bar{a}b})(-\frac{p_{ab}}{p_{ab} + p_{\bar{a}b}} \ln \frac{p_{ab}}{p_{ab} + p_{\bar{a}b}} - \frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}} \ln \frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}}) \\
+ K(p_{ab} + p_{\bar{a}b})(-\frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}} \ln \frac{p_{\bar{a}b}}{p_{ab} + p_{\bar{a}b}} - \frac{p_{ab}}{p_{ab} + p_{\bar{a}b}} \ln \frac{p_{ab}}{p_{ab} + p_{\bar{a}b}}) \\
= K[-p_{ab} \ln p_{ab} - p_{\bar{a}b} \ln p_{\bar{a}b} - p_{ab} \ln p_{ab} - p_{\bar{a}b} \ln p_{\bar{a}b} + (p_{ab} + p_{\bar{a}b}) \ln(p_{ab} + p_{\bar{a}b})], \tag{54} \]

where the expressions have been simplified making use of the additivity property of the logarithm.

Finally, from Eqs. (51) and (52), (53), and (54), we find

\[ H[P(A \mid m)] + H[P(B \mid A \land m)] = H[P(B \mid m)] + H[P(A \mid B \land m)] \]
\[ = -K(p_{ab} \ln p_{ab} + p_{\bar{a}b} \ln p_{\bar{a}b} + p_{ab} \ln p_{ab} + p_{\bar{a}b} \ln p_{\bar{a}b}) \]
\[ \equiv H[P(A \land B \mid m)], \tag{55} \]

whence we see that the property \((16b)\) is satisfied (we find, e.g., that \(H[P(A \mid m)] + H[P(B \mid A \land m)] \equiv H[P(B \mid m)] + H[P(A \mid B \land m)] \equiv H[P(A \land B \mid m)] \approx 1.20 \text{ bit, for } \alpha = \pi/4\).

We note that the way in which Eq. (55) has been found does not depend on the numerical values of the probabilities \(\{p_{ij}\}\), but only on the additivity property of the logarithm; so the calculations above can be seen as a mathematical proof of the property \((16b)\) for the special case of sets with only two alternatives.

If we invert the positions of the two Stern-Gerlach apparatus, placing the one oriented along \(x\) (associated to the set \(B\)) before the one oriented along \(a\) (associated to the set \(A\)), we then realise a new, different experimental arrangement, which can be denoted by \(m_{\text{inv}}\). The probabilities for the sets of alternatives \(A\) and \(B\) will thus differ from those in \(m\): we have indeed

\[ P(b_{\text{up}} \mid m_{\text{inv}}) = \frac{1}{2}, \quad P(b_{\text{down}} \mid m_{\text{inv}}) = \frac{1}{2}, \tag{56a} \]
\[ P(a_{\text{up}} \mid b_{\text{up}} \land m_{\text{inv}}) = \sin^2(\frac{\pi}{4} - \frac{\alpha}{2}), \]
\[ P(a_{\text{down}} \mid b_{\text{up}} \land m_{\text{inv}}) = \cos^2(\frac{\pi}{4} - \frac{\alpha}{2}), \tag{56b} \]
\[ P(a_{\text{up}} \mid b_{\text{down}} \land m_{\text{inv}}) = \cos^2(\frac{\pi}{4} - \frac{\alpha}{2}), \]
\[ P(a_{\text{down}} \mid b_{\text{down}} \land m_{\text{inv}}) = \sin^2(\frac{\pi}{4} - \frac{\alpha}{2}). \tag{56c} \]

It is clear that, from this point on, we can proceed as in the analysis of the first set-up, obtaining

\[ H[P(A \mid m_{\text{inv}})] + H[P(B \mid A \land m_{\text{inv}})] \]
\[ = H[P(B \mid m_{\text{inv}})] + H[P(A \mid B \land m_{\text{inv}})] \]
\[ = -K(p'_{ab} \ln p'_{ab} + p'_{ab} \ln p'_{ab} + p'_{ab} \ln p'_{ab} + p'_{ab} \ln p'_{ab}), \tag{57} \]
\[ \equiv H[P(A \land B \mid m_{\text{inv}})] \]

where the \(p'_{ij}\) are the values of the joint probabilities \(P(a_i \land b_j \mid m_{\text{inv}})\), different, in general, from the \(p_{ij}\). In any case, property \((16b)\) is again satisfied in the new context (we have, e.g., \(H[P(A \mid m_{\text{inv}})] + H[P(B \mid A \land m_{\text{inv}})] \equiv H[P(B \mid m_{\text{inv}})] + H[P(A \mid B \land m_{\text{inv}})] \equiv H[P(A \land B \mid m_{\text{inv}})] \approx 1.60 \text{ bit, for } \alpha = \pi/4\).

Thus we have found no inconsistencies in this second experiment either: Equation (42), from Ref. [10], is simply incorrect, and can more correctly be written as

\[ H[P(A \mid m)] + H[P(B \mid A \land m)] \]
\[ \neq H[P(B \mid m_{\text{inv}})] + H[P(A \mid B \land m_{\text{inv}})], \tag{58} \]

or equivalently and more briefly as

\[ H[P(A \land B \mid m)] \neq H[P(A \land B \mid m_{\text{inv}})], \tag{59} \]

and its content is that the Shannon entropies in the experiment \(m\) are in general different from those in the different experiment \(m_{\text{inv}}\). This does not surprise us, since the experiments have different set-ups.

The fact that Eq. (42) does not pertain to property \((16b)\) is not noticed in Ref. [10], again because the contexts are not kept explicit in the notation. Moreover, the expression “\(H(B \mid A)\)” is considered there as implying that the observation corresponding to \(A\) is performed before the one corresponding to \(B\):\footnote{cf. the discussion of the formula \(H(A) + H(B \mid A) = H(B) + H(A \mid B)\) in Ref. [10] §III.] it is for this reason that, in order to calculate “\(H(B \mid A)\)”, the authors consider the experiment in which the observation corresponding to \(A\) is performed before the one corresponding to \(B\), but then, in order to calculate “\(H(A \mid B)\)”, they feel compelled to change the order of the observations — with the only effect of changing the whole problem and all probabilities instead! But, as already remarked, the conditional symbol ‘\(\mid\)’ does not have that meaning. The point is that the temporal order of acquisition of knowledge about two physical events does not necessarily correspond to the temporal order in which these events occur.

C. Classical experiment

Together with the two quantum experiments, the authors of Ref. [10] also present an example of a classical experiment
in which, they claim, the property (16b) is not violated. It is useful to re-analyse this example as well, in order to show that, in fact, it is not an instance of confirmation of the property (16b), because it, too, involves two different experimental arrangements.

The idea [10, Fig. 3] is as follows. We fill a box with four balls of different colours (black and white) and compositions (plastic and wood). There are two black plastic balls, one white plastic ball, one white wooden ball. We shake the box, draw a ball blindfold, and consider the set \( A \) of the ball’s being black or white. If the ball is black, then we put all black balls in a new box, draw a new ball from this box, and consider the set \( B \) of the ball’s being plastic or wooden. We proceed analogously if the first drawn ball was white instead.\(^{12}\) Let us denote the set-up just described by \( n \).

The probabilities of first drawing a black or a white ball are respectively \( P(\alpha_{\text{black}} | n) = 1/2 \) and \( P(\alpha_{\text{white}} | n) = 1/2 \) and thus their Shannon entropy is

\[
H[P(A \mid n)] = H[\frac{1}{2}, \frac{1}{2}] = 1 \text{ bit}. \tag{60}
\]

The conditional probabilities of the second drawn ball’s being plastic or wooden, given the outcome of the first observation, are

\[
P(b_{\text{plastic}} \mid \alpha_{\text{black}} \wedge n) = 1, \quad P(b_{\text{wood}} \mid \alpha_{\text{black}} \wedge n) = 0, \tag{61}
\]

if the first result was ‘black’, and

\[
P(b_{\text{plastic}} \mid \alpha_{\text{white}} \wedge n) = \frac{1}{2}, \quad P(b_{\text{wood}} \mid \alpha_{\text{white}} \wedge n) = \frac{1}{2}, \tag{62}
\]

if it was ‘white’. From these probabilities the following Shannon conditional entropy can be computed:

\[
H[P(B \mid A \wedge n)] = \frac{1}{2} H(1, 0) + \frac{1}{2} H(\frac{1}{2}, \frac{1}{2})
\]

\[
= \frac{1}{2} \times 0 \text{ bit} + \frac{1}{2} \times 1 \text{ bit} = 0.5 \text{ bit}. \tag{63}
\]

Combining Eqs. (60) and (63) we obtain

\[
H[P(A \mid n)] + H[P(B \mid A \wedge n)] = 1.5 \text{ bit}. \tag{64}
\]

Now we suppose to make the observations in inverse order instead. We shake the initial box, draw a ball, consider first the set \( B \) of the ball’s being plastic or wooden. Depending on the outcome we fill a new box either with the plastic or the wooden balls, and draw a new ball; then we consider the set \( A \) of the new ball. Let us denote this new experiment by \( n_{\text{inv}} \). It is clear that \( n \) and \( n_{\text{inv}} \) are really different experiments, for the following reason: in \( n \), between the two draws, the second box contains black or white balls; while in \( n_{\text{inv}} \) it contains plastic or wooden balls.

The probabilities for the set \( B \) this time are \( P(b_{\text{plastic}} \mid n_{\text{inv}}) = \frac{1}{3} \) and \( P(b_{\text{wood}} \mid n_{\text{inv}}) = \frac{1}{3} \), with an entropy

\[
H[P(B \mid n_{\text{inv}})] = H[\frac{1}{3}, \frac{1}{3}] \approx 0.81 \text{ bit}, \tag{65}
\]

while the conditional probabilities for \( A \) are

\[
P(\alpha_{\text{black}} \mid b_{\text{plastic}} \wedge n_{\text{inv}}) = \frac{2}{3}, \quad P(\alpha_{\text{white}} \mid b_{\text{plastic}} \wedge n_{\text{inv}}) = \frac{1}{3}, \tag{66}
\]

if the first result was ‘plastic’, and

\[
P(\alpha_{\text{black}} \mid b_{\text{wood}} \wedge n_{\text{inv}}) = 0, \quad P(\alpha_{\text{white}} \mid b_{\text{wood}} \wedge n_{\text{inv}}) = 1, \tag{67}
\]

if it was ‘wooden’. The conditional entropy is

\[
H[P(A \mid B \wedge n_{\text{inv}})] &= \frac{2}{3} H[\frac{2}{3}, \frac{1}{3}] + \frac{1}{3} H(0, 1)
\]

\[
& \approx \frac{2}{3} \times 0.92 \text{ bit} + \frac{1}{3} \times 0 \text{ bit} 
\]

\[
& \approx 0.69 \text{ bit}, \tag{68}
\]

and adding this time Eqs. (65) and (68) we find

\[
H[P(B \mid n_{\text{inv}})] + H[P(A \mid B \wedge n_{\text{inv}})] = 1.5 \text{ bit}. \tag{69}
\]

We see that the entropies (64) and (69) are equal,

\[
H[P(A \mid n_{\text{inv}})] + H[P(B \mid A \wedge n_{\text{inv}})] = H[P(B \mid n_{\text{inv}})] + H[P(A \mid B \wedge n_{\text{inv}})], \tag{70}
\]

which is equivalent to

\[
H[P(A \wedge B \mid n)] = H[P(A \wedge B \mid n_{\text{inv}})], \tag{71}
\]

but it is clear that this is not the statement of property (16b), because the right- and left-hand sides of this equation refer to two different experiments. The content of the equality above is only that the Shannon entropies for the probability distributions for the composite set of alternatives \( A \wedge B \) are equal in the two experiments \( n \) and \( n_{\text{inv}} \).

**IV. THE RÔLE OF “QUANTUMNESS”**

We have shown thus far that the quantum experiments in Ref. [10] did not involve any violation of the Shannon entropy’s properties. However, we may still imagine someone raising the following argument:

“Very well, the properties (16) are not violated in any experiment. But one notices that the equality for different contexts

\[
H[P(A \wedge B \mid n)] = H[P(A \wedge B \mid n_{\text{inv}})], \tag{71}
\]

\[
H[P(A \wedge B \mid n)] = H[P(A \wedge B \mid n_{\text{inv}})], \tag{71}
\]
The Reader should not, too simply, identify the "system" here with some one of §III C, where it is found instead that

\[ H[P(A \land B | m)] \neq H[P(A \land B | m_{inv})]. \]

From this particular case, one can see that in classical experiments the Shannon entropy remains the same if the temporal order of observations is changed, whereas in quantum experiments the entropy changes together with the change in temporal order. This phenomenon is thus a peculiarity of the quantum nature of the experiments — a sort of 'quantum-context-dependence' of the Shannon entropy.\(^1\)

But this argument has no validity, of course. The Shannon entropy is always "context dependent", and this comes from the fact that probabilities are always context dependent, in both classical and quantum experiments. We can further and illustrate this fact by means of two more experiments, which will also serve as counter-examples of the ones already discussed.

A. First counter-example

The first counter-example is a modification, based on the examples presented by Kirkpatrick \(^{38,39}\) of the experiment with the balls discussed in §III C \(^3\). The balls are in addition big or small as well now: there are one big black plastic ball, one small white plastic ball, one small white wooden ball, and one small white wooden ball.

Initially, we prepare the box so that it contains only all small balls. Then we shake the box, draw a ball blindfold, and consider the set \( A \equiv \{ a_{black}, a_{white} \} \) for the ball’s being black or white. If the drawn ball is black, then we prepare the box so that it contains only all black balls (also the big black one that was initially not in the box), draw a new ball from this box, and consider the set \( B \equiv \{ b_{plastic}, b_{wood} \} \) for this ball’s being plastic or wooden. We proceed analogously if the first drawn ball was white.\(^4\) It is easy to see that, in the set-up just described, denoted by \( k \), we have the following probabilities:

\[
\begin{align*}
P(a_{black} | k) &= \frac{3}{7}, & P(a_{white} | k) &= \frac{4}{7}, \\
P(b_{plastic} | a_{black} \land k) &= 1, & P(b_{wood} | a_{black} \land k) &= 0, \\
P(b_{plastic} | a_{white} \land k) &= \frac{2}{7}, & P(b_{wood} | a_{white} \land k) &= \frac{5}{7}.
\end{align*}
\]

The Shannon entropies are

\[
\begin{align*}
H[P(A | k)] &= H\left(\frac{3}{7}, \frac{4}{7}\right) \approx 0.92 \text{ bit}, \\
H[P(B | A \land k)] &= \frac{1}{7} H(1,0) + \frac{6}{7} H\left(\frac{1}{2}, \frac{1}{2}\right) \\
&= \frac{1}{7} \times 0 \text{ bit} + \frac{6}{7} \times 1 \text{ bit} \approx 0.67 \text{ bit},
\end{align*}
\]

and their sum is

\[
H[P(A | k)] + H[P(B | A \land k)] \\
\equiv H[P(B | k)] + H[P(A | B \land k)] \\
\equiv H[P(A \land B | k)] \approx 1.58 \text{ bit.}
\]

Now let us consider the set-up \( k_{inv} \), in which the observations are made in reverse order, but with the same general procedure. The probabilities are then:

\[
\begin{align*}
P(b_{plastic} | k_{inv}) &= \frac{2}{7}, & P(b_{wood} | k_{inv}) &= \frac{5}{7}, \\
P(a_{black} | b_{plastic} \land k_{inv}) &= \frac{2}{7}, & P(a_{white} | b_{plastic} \land k_{inv}) &= \frac{5}{7}, \\
P(a_{black} | b_{wood} \land k_{inv}) &= 0, & P(a_{white} | b_{wood} \land k_{inv}) &= 1.
\end{align*}
\]

These lead to the entropies

\[
\begin{align*}
H[P(B | k_{inv})] &= H\left(\frac{2}{7}, \frac{5}{7}\right) \approx 0.92 \text{ bit,} \\
H[P(A | B \land k_{inv})] &= \frac{2}{7} H\left(\frac{2}{3}, \frac{1}{3}\right) + \frac{5}{7} H(0,1) \\
&\approx \frac{2}{7} \times 0.92 \text{ bit} + \frac{5}{7} \times 0 \text{ bit} \approx 0.61 \text{ bit,}
\end{align*}
\]

and the sum

\[
H[P(B | k_{inv})] + H[P(A | B \land k_{inv})] \\
\equiv H[P(A | k_{inv})] + H[P(B | A \land k_{inv})] \\
\equiv H[P(A \land B | k_{inv})] \approx 1.53 \text{ bit.}
\]

Comparing Eqs. (77) and (85) we find

\[
H[P(A \land B | k)] \neq H[P(A \land B | k_{inv})].
\]

Hence, for this experiment, of a clearly classical nature, we obtain different statistics and entropies depending on the order in which we observe colour and composition.\(^5\)

Note, in any case, that in each of the two set-ups — Eqs. (77) and (85) — the properties (16) are always satisfied, just as in the quantum experiments.

\(^3\) Cf. also the toy models discussed by Hardy \(^{40}\) and Spekkens \(^{41}\).

\(^4\) The Reader should not, too simply, identify the "system" here with some specific group of balls. It is rather associated with a variable collection of balls, in analogy with a classical open system associated with variable number (and species) of particles.

\(^5\) We may note, incidentally, that it has long been known that the thermodynamic entropy of a classical thermodynamic system in a non-equilibrium state depends on its complete previous history \(^{42}\).
B. Second counter-example

The second counter-example is of a quantum-mechanical nature. It runs precisely like the experiment with spin-1/2 particles discussed in §III B except that now the particle has initially spin up, not along the z axis, but along the axis b that bisects the angle $\alpha\hat{x}$ (i.e., $b$ lies in the $xaz$ plane and forms an angle $\beta \equiv \pi/4 - \alpha/2$ with both the $x$ and $a$ axes; remember that $\alpha$ is the angle $\alpha\hat{a}$). The analysis of this experiment proceeds completely along the lines of the re-analysis of §III B if we introduce the contexts $q$ and $q_{inv}$, and change Eqs. (56) with

$$P(a_{up} | q) = \cos^2 \frac{\beta}{2}, \quad P(a_{down} | q) = \sin^2 \frac{\beta}{2}, \quad (87a)$$

$$P(b_{up} | a_{up} \wedge q) = \cos^2 \beta, \quad (87b)$$

$$P(b_{down} | a_{up} \wedge q) = \sin^2 \beta, \quad (87c)$$

and Eqs. (56) with

$$P(b_{up} | q_{inv}) = \cos^2 \frac{\beta}{2}, \quad P(b_{down} | q_{inv}) = \sin^2 \frac{\beta}{2}, \quad (88a)$$

$$P(a_{up} | b_{up} \wedge q_{inv}) = \sin^2 \beta, \quad (88b)$$

$$P(a_{down} | b_{up} \wedge q_{inv}) = \cos^2 \beta, \quad (88c)$$

$$P(b_{up} | b_{down} \wedge q_{inv}) = \cos^2 \beta, \quad P(b_{down} | b_{down} \wedge q_{inv}) = \sin^2 \beta. \quad (88d)$$

It is obvious that this leads to the equalities $P(a_i \wedge b_j | q) = P(a_i \wedge b_j | q_{inv})$ and eventually to the equality

$$H[P(A \wedge B | q)] = H[P(A \wedge B | q_{inv})], \quad (89)$$

exactly as it happened in the experiment with the balls of §III C (for $\alpha = \pi/4$, e.g., we have $H[P(A \wedge B | q)] \approx 0.83$ bit).

But here the experiment is a purely quantum one: we see indeed that the observables do not commute here, the initial state is pure, and its density matrix is not diagonal in either of the observables’ bases. Compare this result with the discussion in Ref. [43].

V. CONCLUSIONS

We have shown that the properties of the Shannon entropy are not violated in quantum experiments, contrary to the conclusions of Ref. [10]. In that paper, an idiosyncratic temporal interpretation of the conditional symbol ‘|’ and of the conjunction symbol ‘\&’ leads the authors to change experimental set-ups in order to calculate various entropies. As a result, they compare entropies relative to different experimental arrangements instead, and do not notice that the probabilities are also different, and so their results (Eqs. (12) and (22) in this paper), besides not being formally correctly written, do not pertain to the properties of the Shannon entropy (Eqs. (16)), which refer to a single, well-defined experiment and hold unconditionally.

The peculiar results arising from the comparison of entropies relative to different experimental contexts can thus appear or not appear in any kind of experiments, classical as well as quantum; this has been shown by means of two counter-examples: a classical one, in which a change in the temporal order of the experiment leads to a change in entropy values, and a genuinely quantum one, in which the same temporal change leads to no entropy changes.

A conclusion is that Bohr’s dictum ought to be observed also in mathematical notation, and not only in the analysis of quantum phenomena, but in the analysis of classical phenomena as well, because probabilities — and, consequently, Shannon entropies — always depend on “the whole experimental arrangement” [44] taken into account.

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