CHANNEL CAPACITIES VIA $p$-SUMMING NORMS

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Abstract. In this paper we show how the metric theory of tensor products developed by Grothendieck perfectly fits in the study of channel capacities, a central topic in Shannon’s information theory. Furthermore, in the last years Shannon’s theory has been generalized to the quantum setting to let the quantum information theory step in. In this paper we consider the classical capacity of quantum channels with restricted assisted entanglement. In particular these capacities include the classical capacity and the unlimited entanglement-assisted classical capacity of a quantum channel. To deal with the quantum case we will use the noncommutative version of $p$-summing maps. More precisely, we prove that the (product state) classical capacity of a quantum channel with restricted assisted entanglement can be expressed as the derivative of a completely $p$-summing norm.

1. Introduction

In the late 1940s Shannon single-handedly established the entire mathematical field of information theory in his famous paper A Mathematical Theory of Communication ([28]). Some groundbreaking ideas like the quantization of the information content of a message by the Shannon entropy, the concept of channel capacity or the schematic way to understand a communication system were presented in [28], laying down the pillars of the future research in the field. Being naturally modeled by a stochastic action, a noisy channel is defined as a (point-wise) positive linear map $\mathcal{N}: \mathbb{R}^n_A \to \mathbb{R}^n_B$ between the sender (Alice) and the receiver (Bob) which preserves probability distributions. In terms of notation, we will denote a channel by $\mathcal{N}: \ell^1 \to \ell^1$. Shannon defined the capacity of a channel as an asymptotic ratio:

$$\frac{\text{number of transmitted bits with an } \epsilon \to 0 \text{ error}}{\text{number of required uses of the channel in parallel}}.$$
One of the most important results presented in [28] is the so called **noisy channel coding theorem**, which states that for every noisy channel $\mathcal{N} : \ell_1^n \to \ell_1^n$ its capacity is given by

\[ C_c(\mathcal{N}) = \max_{P = (p(x))_x} H(X : Y), \quad (1.1) \]

where $H(X : Y)$ denotes the mutual information\(^3\) of an input distributions $P = (p(x))_x$ for $X$ and the corresponding induced distribution at the output of the channel $(\mathcal{N}(P))_y$.

Although our main Theorem 1.2 will be stated in a much more general context, it already uncovers a beautiful relation between Shannon information theory and $p$-summing maps when it is applied to classical channels. Indeed, it states that for every channel $\mathcal{N} : \ell_1^n \to \ell_1^n$ we have

\[ C_c(\mathcal{N}) = \frac{d}{dp} [\pi_q(\mathcal{N}^*)] \big|_{p=1}, \quad (1.2) \]

where $\pi_q(\mathcal{N}^*)$ denotes the $q$-summing norm of the map $\mathcal{N}^* : \ell_\infty^n \to \ell_\infty^n$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In the very last years, Shannon’s theory has been generalized to the quantum setting. In this new context, one replaces probability distributions by density operators: Semidefinite positive operators $\rho$ of trace one; so the natural space to work with is $S_{1^n}$ (the space of trace class operators). Then, we define a **quantum channel** as a completely positive and trace preserving linear map on $M_n$. Analogously to the classical case, we will denote a quantum channel by $\mathcal{N} : S_{1^n} \to S_{1^n}$.

Quantum information becomes particularly rich when we deal with bipartite states thanks to **quantum entanglement**. Entanglement is a fundamental resource in quantum information and quantum computation and it is not surprising that it plays a very important role in the study of channels. In particular, it can be seen that the capacity of a quantum channel can be increased if the sender and the receiver are allowed to use a shared entangled state in their protocols. In this work we will study the capacity of a quantum channel to transmit classical information; that is, the **classical capacity**. However, we can consider different classical capacities depending on the amount of shared entanglement allowed to Alice and Bob. Given a quantum channel $\mathcal{N} : S_{1^n} \to S_{1^n}$ we will call $d$-*restricted classical capacity of $\mathcal{N}$* to the classical capacity of the channel when Alice and Bob are

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\(^3\) $H(X : Y) = H(X) + H(Y) - H(X, Y)$, where $H$ represents the Shannon entropy.

\(^4\) $S_{1^n}$ if we are dealing with $n$ quantum bits or qubits.

\(^5\) The requirement of completely positivity is explained by the fact that our map must be a channel when we consider our system as a physical subsystem of an amplified one (with an environment) and we consider the map $1_{\mathcal{E}_{\text{env}}} \otimes \mathcal{N}$. 

allowed to use a $d$-dimensional entangled state per channel use in the protocol. In fact, our capacity is very closely related to the one studied in [29], where the author imposed the restriction on the entropy of entanglement per channel use. We will explain the connections between the two definitions in Section 5. Therefore, we define a family of capacities such that for the case $d = 1$ we recover the so called classical capacity of $\mathcal{N}$ (without entanglement), $C_c(\mathcal{N})$, and taking the supremum on $d \geq 1$ we obtain the so called (unlimited) entanglement-assisted classical capacity of $\mathcal{N}$, $C_E(\mathcal{N})$. This family of capacities can be defined within the following common ratio-expression:

$$
\lim_{\epsilon \to 0} \lim_{k \to \infty} \sup \left\{ \frac{m}{k} : \exists A, \exists B \text{ such that } \|id_{\mathbb{C}^m} - B \circ \mathcal{N}^k \circ A\| < \epsilon \right\}.
$$

Here $A$ and $B$ represent Alice’s encoder and Bob’s decoder channels respectively (which will depend on the kinds of resources they can use in their protocol) and $\mathcal{N}^k$ denotes the $k$ times uses of the channel in parallel. The reader will find a more extended explanation about the different classical capacities of a quantum channel in Section 5.

In order to compute the classical capacities of a quantum channel $\mathcal{N}$ one could expect to have an analogous result to Equation (1.1). However, the situation is more difficult in the case of quantum channels. A first approach to the problem consists of restricting the kinds of protocols that Alice and Bob can perform. We will talk about the product state version of a capacity when we impose that Alice (the sender) is not allowed to distribute one entangled state among more than one channel use. Following the same ideas as in [5] and [29] we will show:

**Theorem 1.1.** Given a quantum channel $\mathcal{N} : S_1^n \to S_1^n$, for any $1 \leq d \leq n$ we define

\begin{equation}
C^d(\mathcal{N}) := \sup \left\{ S\left( \sum_{i=1}^N \lambda_i (\mathcal{N} \circ \phi_i)((tr_{M_d} \otimes 1_{M_d})(\eta_i)) \right) \right. \\
+ \sum_{i=1}^N \lambda_i \left[ S\left( (1_{M_d} \otimes tr_{M_d})(\eta_i) \right) - S\left( (1_{M_d} \otimes (\mathcal{N} \circ \phi_i))(\eta_i) \right) \right] \}. 
\end{equation}

Here, $S(\rho) := -tr(\rho \log_2 \rho)$ denotes the von Neumann entropy of a quantum state $\rho$ and the supremum runs over all $N \in \mathbb{N}$, all probability distributions $(\lambda_i)_{i=1}^N$, and all families $(\phi_i)_{i=1}^N$, $(\eta_i)_{i=1}^N$, where $\phi_i : S_1^n \to S_1^n$ is a quantum channels and $\eta_i \in S_1^d \otimes S_1^d$ is a pure state for every $i = 1, \ldots, N$.

Then, $C^d(\mathcal{N})$ is the classical capacity of $\mathcal{N}$ with assisted entanglement when
a) Alice and Bob are restricted to protocols in which they start sharing a (pure) \(d\)-dimensional state per channel use.

b) The sender is not allowed to distribute one entangled state among more than once per channel use.

Theorem 1.1 reduces to Equation (1.1) when \(\mathcal{N}\) is a classical channel. On the other hand, it can be seen that in the case \(d = 1\) we recover the Holevo-Schumacher-Westmoreland’s Theorem, which describes the product state classical capacity (or Holevo capacity) of a quantum channel (see \[14, 27\]). Moreover, in the case \(d = n\) we recover the Bennett, Shor, Smolin, Thapliyal’s Theorem, which gives a formula to compute the entanglement-assisted classical capacity of a quantum channel (see \[5\]). The reader will find a brief introduction to these capacities in Section \(5\). In order to obtain the general capacities (rather than the product state version) one has to consider the corresponding regularization. It is not difficult to see that in this case the regularization is given by

\[
C_d^d(\mathcal{N})_{\text{reg}} = \sup_k \frac{C_{d^k}(\otimes^k \mathcal{N})}{k}.
\]

Recently Hastings solved a long-standing open question in quantum information theory by showing that \(C_1^1(\mathcal{N})_{\text{reg}} \neq C_1^1(\mathcal{N})\) for certain quantum channels (\[13\]). Hastings’ result shows that we do need to consider the regularization of \(C_1^1(\mathcal{N})\) to compute the classical capacity and it is not enough to consider the much easier formula given in \[14, 27\]. On the other hand, one can check the the formulae (1.1) and the one given in \[5\] to express the product state capacity of the unlimited entanglement-assisted classical capacity of a quantum channel are additive on channels: \(C(\mathcal{N}_1 \otimes \mathcal{N}_2) = C(\mathcal{N}_1) + C(\mathcal{N}_2)\). This is a crucial fact since in these cases, no regularization of the product state version of the corresponding capacities is required and, thus, those formulae describe the general capacities.

Theorem 1.1, which will be proved in Section 5, expresses mathematically the capacity of a channel, which was previously defined by means of concepts like protocols or many uses of the channel in parallel. The main result presented in this work shows a direct connection between the quantity \(C_d^d(\mathcal{N})\) and the theory of absolutely \(p\)-summing maps. Introduced first by Grothendieck in \[12\], the theory of \(p\)-summing maps was exhaustively studied by Pietsch (\[23\]) and Lindenstrauss and Pełczyński (\[19\]). In fact, it was in this last seminal work where the authors showed the extreme utility of \(p\)-summing maps in the study of many different problems in Banach space theory. We recommend the references.
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[7] and [9] for a complete study on the topic. The generalization of the theory of absolutely $p$-summing maps to the noncommutative setting was developed by Pisier by means of the so-called completely $p$-summing maps (see [25]). Even generalizing the definition of $p$-summing maps to the noncommutative setting is not obvious since it requires the concept of noncommutative vector valued $L_p$-spaces. However, absolutely $p$-summing maps admit another natural generalization to the so-called $(cb,p)$-summing maps, introduced by the first author (see [15]), which can be considered as a generalization to an intermediate setting between the Banach space case and the completely $p$-summing maps. In a complete general way we will consider here the $\ell_p(S^d_p)$-maps which include, in particular, the two previous definitions. Thanks to the factorization theorem proved by Pisier ([25, remark 5.11]) we have the following easy definition for maps defined on $M_n$:

$$\pi_{q,d}(T : M_n \rightarrow M_n) = \inf \left\{ \|1_{M_d} \otimes \tilde{T}\|_{M_d(S^d_p) \rightarrow M_d(M_n)} : T = \tilde{T} \circ M_{a,b} \right\},$$

where the infimum runs over all factorizations of $T$ with $a, b \in M_n$ verifying $\|a\|_{S^d_p} = \|b\|_{S^d_p} = 1$, $M_{a,b} : M_n \rightarrow S_n^p$ being the linear map defined by $M_{a,b}(x) = axb$ for every $x \in M_n$ and $\tilde{T} : S_n^p \rightarrow M_n$ being a linear map.

Our main result states as follows.

**Theorem 1.2.** Given a quantum channel $\mathcal{N} : S_1^n \rightarrow S_1^n$,

$$C^d(\mathcal{N}) = \frac{d}{dp} \left[ \pi_{q,d}(\mathcal{N}^*) \right]_{p=1},$$

(1.4)

where $\frac{1}{p} + \frac{1}{q} = 1$. Here, $\pi_{q,d}(\mathcal{N}^*)$ denotes the $\ell_q(S^d_p)$-summing norm of $\mathcal{N}^* : M_n \rightarrow M_n$.

As we mentioned before, Hastings’ result ([13]) says that we cannot avoid the regularization of $C^d(\mathcal{N})$ if $d = 1$. We will show that the additivity of $C^d$ has a particularly bad behavior for $1 < d < n$. Indeed, we will prove the following extreme non-additive result.

**Theorem 1.3.** There exists a channel $\mathcal{N} : S_1^{2n} \rightarrow S_1^{2n}$ such that

$$C^n(\mathcal{N} \otimes \mathcal{N}) \geq \frac{1}{3} \log_2 n + 2C^{\sqrt{n}}(\mathcal{N}),$$

where we use the symbol $\geq$ to denote inequality up to universal (additive) constants which do not depend on $n$.

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6 Actually, to have the equality (1.4) we must define $C^d(\mathcal{N})$ ([13]) by using the ln-entropy instead of using $\log_2$ as usually. However, since both definitions are the same up to a multiplicative factor, we could use the standard entropy $S$ in Definition ([13]) and we should then write (1.4) as $C^d(\mathcal{N}) = \frac{1}{\ln 2} \frac{d}{dp} \left[ \pi_{q,d}(\mathcal{N}^*) \right]_{p=1}$. 


Theorem 1.3 says that the general $d$-restricted capacity with $1 < d < n$ can be, in fact, very different from $C^d$ (the product state version). Nevertheless, we should emphasize that the nature of the non additivity of $C^d$ with $1 < d < n$ comes from the fact that one must change the entanglement dimension from $d$ to $d^2$ when one considers the tensor product of two channels. This makes the problem of additivity (so the regularization) completely different from the much deeper case $d = 1$.

The paper is organized as follows. In the first section we briefly introduce the notion of non-commutative $L_p$ spaces and $\ell^p(S_p)$-summing maps. Furthermore, we prove a modified version of Pisier’s theorem in order to have a more accurate result for the particular maps that we are considering in this work. In Section 3 we give the proof of our main result, Theorem 1.2 and we explain how to obtain the particular cases commented above. In Section 4 we explain why the $d$-restricted capacity is easier to compute when we deal with covariant channels and we use this fact to prove Theorem 1.3. Finally, in Section 5 we give an extended explanation of the restricted classical capacities of quantum channels and we state some of the most important results in the area. Moreover, we prove Theorem 1.1 and we also discuss the physical interpretation of the $C^d$ capacity and the connections with some previous results by Shor.

2. Pisier’s theorem for quantum channels

Following the metric theory of tensor product developed first by Grothendieck and subsequently by Pietsch, Lindenstrauss and Pelczynski in terms of $p$-summing maps, in Pisier introduced the notion of completely $p$-summing map between operator spaces. Pisier showed a satisfactory factorization theorem for these kinds of maps, analogous to the existing result in the commutative setting. In this section we will study such a factorization theorem when it is applied to completely positive maps and we will show that in this case one can get some extra properties in the statement of the theorem. Furthermore, in order to define our restricted capacities, we will need to consider the more general $\ell_p(S_p^d)$-summing maps. For the sake of completeness we will start with a brief introduction to noncommutative (vector valued) $L_p$-spaces and completely $p$-summing maps. Since we will restrict our work to finite dimensional C$^*$-algebras, we will focus on this setting. However, the theory of noncommutative $L_p$-spaces has been developed in a much more general context and most of the results can be stated in such a general framework. We recommend and for a complete study of the subject. Since the key point to define noncommutative
(vector valued) \(L_p\)-spaces is to consider operator spaces, we will assume the reader to be familiarized with them. We recommend [24] for the non familiar reader with the topic.

Given finite dimensional \(C^\ast\)-algebras \(A\) and \(B\), for any \(1 \leq p < \infty\) we denote by \(L_p(A)\) the Banach space \(A\) joint with the norm \(\|x\|_p = \text{tr}(|x|^p)^{\frac{1}{p}}\). In the case \(p = \infty\) we define \(L_\infty(A) = A\). In the particular case \(A = M_d\) we write \(L_p(M_d) = S^d_p\) for \(1 \leq p < \infty\) and \(L_\infty(M_d) = M_d\). Given a map \(T : L_q(A) \to L_p(B)\), we denote the operator norm by

\[
\|T\| = \sup_{A \in A} \frac{\|T(A)\|_p}{\|A\|_q}.
\]

The Banach space \(L_p(A)\) can be endowed with an operator space structure (o.s.s). Indeed, as a \(C^\ast\)-algebra, \(L_\infty(A) = A\) has a natural o.s.s. On the other hand, one can consider a natural o.s.s. on \(L_1(A)\) as a (pre) dual of \(A\). In the case \(1 < p < \infty\) we define an o.s.s. on \(L_p(A)\) by complex interpolation: \(L_p(A) = [A, L_1(A)]_{\frac{p}{p}}\). The definition of an o.s.s. on \(L_p(A)\) allows us to talk about the completely bounded norm of a map \(T : L_q(A) \to L_p(B)\),

\[
\|T\|_{cb} = \sup_{d \in \mathbb{N}} \left\|1_{M_d} \otimes T : M_d(L_q(A)) \to M_d(L_p(B))\right\|
\]

or, equivalently,

\[
(2.1) \quad \|T\|_{cb} = \sup_{d \in \mathbb{N}} \left(\sup_{Y} \frac{\|1_{M_d} \otimes T(Y)\|_{M_d(L_p(B))}}{\|Y\|_{M_d(L_q(A))}}\right).
\]

It can be checked (see [25] Proposition 1.7) that for a fixed \(d\) we have

\[
\|Y\|_{M_d(L_p(A))} = \sup_{A, B \in B_{2p}} \| (A \otimes 1_A)Y(B \otimes 1_A) \|_{L_p(M_d \otimes_{\text{min}} A)}.
\]

For our purpose we need to introduce the non-commutative vector valued \(L_p\)-spaces. Given an operator space \(X\), we define \(L_\infty(A, X)\) as \(A \otimes_{\text{min}} X\), where \(\text{min}\) denotes the minimal tensor norm in the category of operator spaces. On the other hand, Effros and Ruan introduced the space \(L_1(A, X)\) as the (operator) space \(L_1(A) \hat{\otimes} X\), where \(\hat{\otimes}\) denotes the projective operator space tensor norm. Then, using complex interpolation Pisier defined the non-commutative vector valued (operator) space \(L_p(A, X) = [L_\infty(A, X), L_1(A, X)]_{\frac{p}{p}}\) for any \(1 \leq p \leq \infty\). Furthermore, he proved that \(L_p(A, X)\) verifies the expected properties analogous to the commutative setting. In this work we will mainly deal with the case \(X = S^d_q\) for some \(1 \leq q \leq \infty\) and in many cases we will have \(A = M_n\). In this case, we will simplify notation by writing \(L_p(M_n, S^d_q) = S^p_n[S^d_q]\) and we will also denote \(L_p(M_n, X) =\)
$S_p^n[X]$ for any operator space $X$. It can be seen that, given $1 \leq p, q \leq \infty$ and defining $\frac{1}{r} = |\frac{1}{p} - \frac{1}{q}|$, we have

1. If $p \leq q$, 
$$\|X\|_{S_p^n[S_q]} = \inf \left\{ \|A\|_{S_p^n} \|Y\|_{S_q^n} \|B\|_{S_p^n} \right\},$$

where the infimum runs over all representations $X = (A \otimes 1_{M_d})Y(B \otimes 1_{M_d})$ with $A, B \in M_n$ and $Y \in M_n \otimes M_d$.

2. If $p \geq q$,
$$\|X\|_{S_p^n[S_q]} = \sup \left\{ \|A \otimes 1_{M_d}X(A \otimes 1_{M_d})\|_{S_q^n} : A, B \in B_{S_p^n} \right\}.$$

As an interesting application of this expression for the norm in $S_p^n[S_q]$ in [25, Theorem 1.5 and Lemma 1.7] Pisier showed that for a given map $T : S_q \to S_p$ we can compute its completely bounded norm as
$$\|T\|_{cb} = \sup_{d \in \mathbb{N}} \|1_{M_d} \otimes T : S_t^n[S_q] \to S_t^n[S_q]\|$$
for every $1 \leq t \leq \infty$. That is, we can replace $\infty$ in (2.1) with any $1 \leq t \leq \infty$ in order to compute the cb-norm.

**Remark 2.1.** It is known ([30], [3]) that if $T$ is completely positive we can compute $\|T : S_q \to S_p\|$ by restricting to positive elements $A \in S_q$. Moreover, in this case one can also consider positive elements $Y \geq 0$ to compute the cb-norm ([8, Section 3]). In particular, one can consider $A = B > 0$ in the expression (2.2) for $\|Y\|_{S_p^n[S_q]}$. According to this, if $X > 0$ and $q = 1$, (2.2) becomes
$$\|X\|_{S_p^n[S_1]} = \sup_{A > 0} \frac{\|(A \otimes 1_{M_d})X(A \otimes 1_{M_d})\|_{S_q^n}}{\|A\|_{2p'}} = \|(1_{M_n} \otimes tr_{M_d})(X)\|_p,$$
where $\frac{1}{p} + \frac{1}{p'} = 1$.

A linear map between operator spaces $T : E \to F$ is called completely $p$-summing if
$$\pi_p^0(T) := \pi_{p,\infty}(T) = \|1_{S_p} \otimes T : S_p \otimes_{\min} E \to S_p[F]\| < \infty.$$

Note that we can write, equivalently,
$$\pi_p^0(T) := \sup_d \pi_p^d(T),$$
where $\pi_p^d(T) = \|1_{M_d} \otimes T : S_p^d \otimes_{min} E \to S_p^d[F]\|$. This definition generalizes the absolutely $p$-summing maps defined in the Banach space category. In [25] Pisier proved that most of the properties of $p$-summing maps have an analogous statement in this noncommutative setting. In particular, as we mentioned before, it can be seen that the completely $p$-summing maps verify a satisfactory Pietsch factorization theorem (see [25, Theorem 5.1]).

The theory of completely $p$-summing maps becomes particularly nice when we consider the case $E = F = M_n$. Then, the definition of the completely $p$-summing norm of the map $T : M_n \to M_n$ can be stated as

$$\pi_p^o(T) := \sup_d \| (1_{M_d} \otimes T) \circ \text{flip} : M_n(S_p^d) \to S_p^d[M_n]\|,$$

where the flip operator is defined as $\text{flip}(a \otimes b) = b \otimes a$. Pietsch factorization theorem is particularly simple in this case and has a complete analogous statement to the commutative result. In particular, one can deduce

$$\pi_p^o(T : M_n \to M_n) = \pi_p^a(T : M_n \to M_n),$$

and

$$\pi_p^d(T : M_n \to M_n) = \sup \left\{ |tr(S \circ T)| : \pi_p^d(S : M_n \to M_n) \leq 1 \right\},$$

where $1 = \frac{1}{p} + \frac{1}{q}$ (see for instance [15]).

As we will explain later in detail, in order to consider a general family of restricted capacities we will have to deal with the completely $p$-summing norm of maps defined between finite dimensional $C^*$-algebras. Therefore, we will need to adapt Pisier’s factorization theorem for completely $p$-summing maps to our particular context. Actually, due to the fact that we will consider quantum channels, we will state such a factorization theorem for these particular maps obtaining some extra properties. In the following, we will write $\| \cdot \|_+$ to denote the corresponding norm $\| \cdot \|$ when we restrict to positive elements.

**Theorem 2.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras with $\dim(\mathcal{A}) < \infty$ and let $T : \mathcal{A} \to \mathcal{B}$ be a completely positive map. Then, the following assertions are equivalent:

a) $\pi_p^o(T) \leq C$.

b) $\|1_{S_p} \otimes T : S_p \otimes_{min} \mathcal{A} \to S_p[\mathcal{B}]\|_+ \leq C$. 

c) There exists a positive element $a \in \mathcal{A}$ verifying $\|a\|_{L_{2p}(\mathcal{A})} \leq 1$ such that for every $x \in S_p \otimes \text{min} \mathcal{A}$ we have
\[
\| (1_{S_p} \otimes T)(x) \|_{S_p[S]} \leq C \| (1_{S_p} \otimes a)x(1_{S_p} \otimes a) \|_{S_p[L_p(\mathcal{A})]}.
\]
d) There exist a positive element $a \in \mathcal{A}$ verifying $\|a\|_{L_{2p}(\mathcal{A})} \leq 1$ and a completely positive linear map $\alpha : L_p(\mathcal{A}) \to \mathcal{B}$ such that $T = \alpha \circ M_a$ and $\|\alpha\|_{cb} \leq C$.
Here $M_a : \mathcal{A} \to L_p(\mathcal{A})$ is the linear map defined by $M_a(x) = axa$ for every $x \in \mathcal{A}$.

Furthermore, $\pi_p^\alpha(T) = \inf \{ C : C$ verifies any of the above conditions $\}$.

The proof of Theorem 2.1 is based on a slight modification of the Hahn-Banach argument used in (25, Theorem 5.1) and a non trivial use of Cauchy-Schwartz inequality. Since Pisier’s theorem has not been stated for finite dimensional $\mathcal{C}^*$-algebras and the positivity is always tricky in these contexts we will give a sketch of the proof.

**Proof.** Let $\mathcal{A} = \bigoplus_{i=1}^N M_{k_i}$. In order to make notation easier, we will consider $k_i = k$ for every $i = 1, \cdots, N$ and we will write $\mathcal{A} = \ell^N_\infty(M_k)$. The proof for the general case is exactly the same up to notation.

The proof of $a) \Rightarrow b)$ is trivial. The implications $c) \Rightarrow d)$ and $d) \Rightarrow a)$ follow by standard arguments. So, we have to show $b) \Rightarrow c)$.

Now, by assumption, for every $x_1, \cdots, x_m$ positive elements in $S_p \otimes \ell^N_\infty(M_k)$ we have
\[
\sum_{j=1}^m \|(1_{S_p} \otimes T)(x_j)\|^p_{S_p[S]} \leq C^p \sup_i \sup_{a_i, b_i} \sum_{j=1}^m \|(1 \otimes a_i)x_j(i)(1 \otimes b_i)\|^p_{S_p(\ell^2_2 \otimes \ell^2_2)},
\]
where $x_j = (x_j(i))_{i=1}^N \in \ell^N_\infty(S_p \otimes M_k)$ for every $j = 1, \cdots, m$ and the supremum on the right hand side is taken over all $a_i$ and $b_i$ positive elements in the unit ball of $S^k_{2p}$ for every $i = 1, \cdots, N$. Furthermore, as a consequence of Hölder’s inequality, we deduce that for every $x_1, \cdots, x_m$ positive elements in $S_p \otimes \ell^N_\infty(M_k)$ we have
\[
\sum_{j=1}^m \|(1_{S_p} \otimes T)(x_j)\|^p_{S_p[S]} \leq C^p \sup_i \sup_{a_i} \sum_{j=1}^m \|(1_{S_p} \otimes a_i)x_j(i)(1_{S_p} \otimes a_i)\|^p_{S_p(\ell^2_2 \otimes \ell^2_2)},
\]
where the sup is taken over $a_i$ positive elements in the unit ball of $S^k_{2p}$.

Following a Hahn-Banach argument as in (25, Theorem 5.1) we can conclude the existence of a sequence of positive numbers $(\lambda_i)_{i=1}^N$ verifying $\sum_{i=1}^N \lambda_i = 1$ and a sequence of
positive elements $a_i$ in the unit ball of $S_{2p}^N$ such that

$$\| (1_{S_p} \otimes T)(x) \|_{S_p[B]}^p \leq C^p \sum_{i=1}^N \lambda_i^p \| (1 \otimes a_i) x(i)(1 \otimes a_i) \|_{S_p(\ell_2 \otimes \ell_2^p)}^p$$

for every positive element $x \in S_p \otimes \ell_\infty^N(M_k)$.

To simplify notation we will write $\tilde{T} = 1_{S_p} \otimes T$, $\tilde{a}_i = 1_{S_p} \otimes a_i \in B(\ell_2 \otimes \ell_2^k)$ and $A = \sum_{i=1}^N \lambda_i^k e_i \otimes a_i \in \ell_\infty^N \otimes B(\ell_2 \otimes \ell_2^k)$. Using this notation Equation (2.3) becomes

$$\| \tilde{T}(x) \|_{S_p[B]}^p \leq C^p \| A x A \|_{\ell_\infty^N(S_p(\ell_2 \otimes \ell_2^k))}^p$$

for every positive element $x \in S_p \otimes \ell_\infty^N(M_k)$. To finish the proof we will show that Inequality (2.4) holds for every $x \in S_p \otimes \ell_\infty^N(M_k)$.

First note that we can assume that $A$ is invertible. If this is not the case, we consider elements of the form $p_A x p_A$, where $p_A$ is the support projection of $A$ (since (2.4) is trivially satisfied otherwise) and we can follow the same argument. Then, it is not difficult to see that for every $x \in \ell_\infty^N \otimes B(\ell_2 \otimes \ell_2^k)$ we have

$$\| A x A \|_{\ell_\infty^N(S_p(\ell_2 \otimes \ell_2^k))} = \inf \left\{ \| y A \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))} , \| z A \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))} \right\},$$

where the infimum is taken over all possible decomposition $x = y^* z$. Indeed, inequality $\leq$ follows from Hölder’s inequality. To see that the infimum is attained we use the fact that we can write $A x A = x_1 x_2$ so that

$$\| A x A \|_{\ell_\infty^N(S_p(\ell_2 \otimes \ell_2^k))} = \| x_1 \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))} \| x_2 \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))}.$$ 

Therefore, if we define $y^* = A^{-1} x_1$ and $z = x_2 A^{-1}$, we have $y^* z = x$ and

$$\| y A \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))}, \| z A \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))} = \| A x A \|_{\ell_\infty^N(S_p(\ell_2 \otimes \ell_2^k))}.$$ 

On the other hand, a similar argument as in [8, Theorem 12] using Stinespring’s Theorem allows us to use the fact that $T$ is completely positive to conclude that

$$\| \tilde{T}(y^*) x z \|_{S_p[B]} \leq \| \tilde{T}(y^*) y \|_{S_p[B]} \| z^* \|_{S_p[B]} \| A \|_{\ell_\infty^N(S_p(\ell_2 \otimes \ell_2^k))}.$$ 

for every $y, z \in \ell_2^N(S_{2p}(\ell_2 \otimes \ell_2^k))$. Using Equations (2.4) and (2.6) we obtain

$$\| \tilde{T}(y^*) x z \|_{S_p[B]} \leq C \| y A \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))}, \| z A \|_{\ell_2^p(S_{2p}(\ell_2 \otimes \ell_2^k))}$$

for every $y, z \in \ell_2^N(S_{2p}(\ell_2 \otimes \ell_2^k))$. 

Therefore, we can use Equation (2.5) to conclude that for every $x \in S_p \otimes \ell_\infty^N(M_k)$ we have

$$\|\tilde{T}(x)\|_{S_\infty[\mathbb{R}]} \leq \inf_{x = y^*z} C\|yA\|_{\ell^N_2(s_{2p}(\ell_2^2 \otimes \ell_2^2))} \|zA\|_{\ell^N_2(s_{2p}(\ell_2^2 \otimes \ell_2^2))} = C\|AxA\|_{\ell^N_p(s_{2p}(\ell_2^2 \otimes \ell_2^2))}. $$

This proves c).

The final statement on the constant $C$ follows easily by standard arguments. □

The following corollary will be very important.

**Corollary 2.2.** Given a completely positive map $T : M_n \to \ell_\infty^N(M_k)$, we have that

$$\pi_q^0(T) = \sup \left\{ |\text{tr}(S \circ T)| : \pi_p^0(S : \ell_\infty^N(M_k) \to M_n) \leq 1 \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In particular,

$$\pi_q^0(T) = \left\| \text{flip} \circ (1_{\ell_\infty^N(s_k^1)} \otimes T^*) : \ell_\infty^N(s^k_1) \otimes [\ell_1^N(S^k_1)] \to S_1 \left[ \ell_\infty^N(s^k_1) \right] \right\|_\pi,$$

and the norm can be computed restricting to elements of the form $\rho = \sum_{i=1}^N \lambda_i e_i \otimes e_i \otimes M_{a_i}$ with $a_i \in S^k_2$ positive for every $i = 1, \ldots, N$. Here, we identify the tensor $M_{a_i} \in M_k \otimes M_k$ with the associated map $M_{a_i} : M_k \to M_k$.

**Proof.** It is known that for any $T : M_n \to \ell_\infty^N(M_k)$ we have

$$\pi_q^0(T) = \sup \left\{ |\text{tr}(S \circ T)| : \pi_p^0(S : \ell_\infty^N(M_k) \to M_n) \leq 1 \right\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ (see [15] for details). Therefore, we only have to prove inequality $\leq$ in Equation (2.7).

According to Theorem 2.1 for every $S : \ell_\infty^N(M_k) \to M_n$ with $\pi_p^0(S) \leq 1$ we can assume that $S$ factorizes as $S = \alpha \circ \beta$, where $\alpha : S_p \to M_n$ is completely contractive and completely positive and $\beta : \ell_\infty^N(M_k) \to S_p$ is an element in the unit ball of $S_p[\ell_1^N(S^k_1)]$. We will show that we can assume that $\beta$ is a positive element of $S_p[\ell_1^N(S^k_1)]$ (which is equivalent to say that $\beta : \ell_\infty^N(M_k) \to S_p$ is a completely positive map). To see this, we first note that, according to [15],

$$S_p[\ell_1^N(S^k_1)] = S_{2p}(B(\ell_2^2 \otimes_{\min} \ell_1^N(S^k_1))) S_{2p}.$$

Indeed, this is obtained from the fact that completely $p$-integral maps and completely $p$-summing maps coincide on $\ell_\infty^N(M_k)$. On the other hand, as a consequence of Wittstock’s factorization theorem (see [22] Theorem 8.4]) one can see that every element
$x \in B_{B(\ell_2)\otimes\min\ell_1^N(S_k^d)}$ can be written as $x = yz$ with $yy^* \in B_{B(\ell_2)\otimes\min\ell_1^N(S_k^d)}$ and $z^*z \in B_{B(\ell_2)\otimes\min\ell_1^N(S_k^d)}$. Therefore, we can assume that $\beta = a \cdot yz \cdot b$ where $a, b \in S_{2p}$ and $y$ and $z$ are as before. Since $\alpha$ is a positive element in the unit ball of $S_q \otimes\min M_n$ and $T$ is completely positive we know that $\hat{\alpha} = (1_{S_q} \otimes T)(\alpha) \in S_q[\ell_N^\infty(M_k)]$ is a positive element. Therefore, we use Cauchy-Schwartz inequality to conclude

$$|\text{tr}(S \circ T)| = |\langle \beta, \hat{\alpha} \rangle| = |\langle a \cdot yz \cdot b, \hat{\alpha} \rangle| \leq |\langle a \cdot yy^* \cdot a^*, \hat{\alpha} \rangle|^\frac{1}{2} |\langle b^* \cdot z^*z \cdot b, \hat{\alpha} \rangle|^\frac{1}{2}.$$ 

Since $a \cdot yy^* \cdot a^*$ and $b^* \cdot z^*z \cdot b$ are both positive elements in $S_p[\ell_1^N(S_k^d)]$, the right hand term of the last inequality is lower or equal than $\sup \{ |\langle \beta, \hat{\alpha} \rangle| : \beta \in B_{S_p[\ell_1^N(S_k^d)]}, \beta \geq 0 \}$. Therefore, we can assume that $\beta$ is completely positive, so $S = \alpha \circ \beta$ is completely positive too. Thus, we obtain (2.7).

Finally, Equation (2.8) follows directly from Theorem 2.1, Equation (2.7) and duality.

In this work we will need to consider the following generalization of completely $p$-summing maps. A linear map between operator spaces $T : E \to F$ is called $\ell_p(S_p^d)$-summing map if

$$\pi_{p,d}(T) := \left\| 1_{\ell_p(S_p^d)} \otimes T : \ell_p(S_p^d) \otimes\min E \to \ell_p(S_p^d)[F] \right\| < \infty.$$ 

The author should note the difference between notation $\pi_{p,d}(T)$ above and notation $\pi_{p,d}^d(T)$ introduced in page 8-9.

Note that the case $d = \infty$ above corresponds to the completely $p$-summing maps. On the other hand, the case $d = 1$ was first introduced by the first author and they are called $(p, cb)$-summing maps (see [15]). They can be considered as a generalization of the absolutely $p$-summing maps to an intermediate setting between the Banach space case and the completely $p$-summing maps. It can be seen that $\ell_p(S_p^d)$-summing maps verify a Pietsch factorization theorem analogous to the theorem for completely $p$-summing maps (see [25, remark 5.11]). Actually, following the proof of Theorem 2.1 step by step one can show an analogous version of the theorem for the $\ell_p(S_p^d)$-summing maps. In particular, as an application of Cauchy-Schwartz inequality one can conclude that, in order to compute the $\pi_{p,d}$ norm of a completely positive map $T$ between $C^*$-algebras, it suffices to consider positive elements in the definition of the norm. However, instead of developing a parallel theory for the $\ell_p(S_p^d)$-summing maps, we will consider the following point of view.
**Remark 2.2.** Given $M_n$, for every natural number $1 \leq d \leq n$ we consider the unit ball of the space of completely bounded maps from $M_n$ to $M_d$, $\mathcal{K} = B_{CB(M_n,M_d)}$. Then, we can define the map

$$j_d : M_n \to \ell_\infty(\mathcal{K}, M_d)$$

given by

$$j_d(A) = (\phi(A))_{\phi \in \mathcal{K}} \text{ for every } A \in M_n.$$ 

It is not difficult to see that the previous map $j_d$ defines a $d$-isometry. That is, the map

$$1_{M_d} \otimes j_d : M_d \otimes_{\min} M_n \to M_d \otimes_{\min} \ell_\infty(\mathcal{K}, M_d)$$

is an isometry. Therefore, it follows from the very definition of the $\pi_{p,d}$ norm that for a map $\mathcal{J} : M_n \to M_n$ we have

$$\pi_{p,d}(\mathcal{J}) = \pi_{p,d}(j_d \circ \mathcal{J}) = \pi_p^0(j_d \circ \mathcal{J}),$$

where the last equality comes from the Pietch factorization theorem and the fact that a map $T : X \to \ell_\infty(\mathcal{K}, M_d)$ verifies that $\|T\|_{cb} = \|T\|_d$.

When $\mathcal{J} : M_n \to M_n$ is completely positive we can actually consider the set

$$(2.11) \quad \mathcal{P} = CPU(M_n, M_d) := \{ T : M_n \to M_d, T \text{ is completely positive and unital} \}$$

to obtain an equation similar to (2.10). Indeed, if we consider the map $j_d : M_n \to \ell_\infty(\mathcal{P}, M_d)$, where $j_d$ is defined as above, one can easily check that this map is a $d$-isometry on positive elements (that is, when we restrict the map $1_{M_d} \otimes j_d$ to positive elements in $M_d \otimes_{\min} M_n$). Let us briefly explain this point. Given any positive element $A \in M_{dn}$, we must show that there exists a completely positive map $T : M_n \to M_d$ such that $\|A\| = \|(1_{M_d} \otimes T)(A)\|_{M_d}$. Now, since $A$ is positive there must exist a unit element $\xi \in \ell_2^d \otimes \ell_2^n$ such that $\|A\| = \langle \xi, A\xi \rangle$. On the other hand, since $\xi$ has rank $d$, we can find a projection $P : \ell_2^n \to \ell_2^d$ so that $\langle \xi, A\xi \rangle = \langle \eta, (1_{\ell_2^d} \otimes P)A(1_{\ell_2^d} \otimes P^*)\eta \rangle$ for a certain $\eta \in M_d$. Therefore, the completely positive and unital map $T : M_n \to M_d$ defined by $x \mapsto PxP^*$ verifies what we want.

By the nice behavior of the $p$-summing maps shown in Theorem 2.1 and the comments above, one can deduce that

$$\pi_{p,d}(\mathcal{T}) = \pi_{p,d}(j_d \circ \mathcal{T}) = \pi_p^0(j_d \circ \mathcal{T}).$$
We will need to give a more general definition of a quantum channel in order to consider also the case of infinite dimensional von Neumann algebras. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two complex Hilbert spaces and let us denote by $\mathcal{B}(\mathcal{H}_i)$ the von Neumann algebra of all bounded operators from $\mathcal{H}_i$ to $\mathcal{H}_i$ for $i = 1, 2$. Let us also denote $S_1(\mathcal{H}_i)$ the trace class of operators from $\mathcal{H}_i$ to $\mathcal{H}_i$ for $i = 1, 2$. We can define a quantum channel as a completely positive and trace preserving map $N : S_1(\mathcal{H}_1) \rightarrow S_1(\mathcal{H}_2)$. In this case, we say that we are describing the channel in the Schrödinger picture. On the other hand, for a given quantum channel we can consider the adjoint map to obtain a completely positive map $N^* : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$ which turns out to be unital. In this case, we say that we are working with the Heisenberg picture of the channel. Although in this work we are interested in quantum channels defined on finite dimensional von Neumann algebras, in order to study their capacities we will need to consider certain channels defined on the direct sum of infinitely many copies of finite dimensional matrix algebras $\bigoplus_{i=1}^{\infty} M_{n_i}$. More precisely, we will consider channels $N : \ell_1(I, S_1^q) \rightarrow S_1^{n_q}$ so that the adjoint is defined on $N^* : M_n \rightarrow \ell_\infty(I, M_d)$. Here, $I$ denotes an arbitrary index set. Since most of our results can be stated for general C$^*$-algebras $A$ and $B$, we will denote a general quantum channel by $N : L_1(A) \rightarrow L_1(B)$ or $N^* : B \rightarrow A$.

The following definition will be crucial in the rest of the work.

**Definition 2.1.** Given two C$^*$-algebras $A, B$ and a quantum channel $N : L_1(A) \rightarrow L_1(B)$, we define

$$\tilde{C}(N) = \lim_{q \to \infty} q \ln (\Pi_q^0(N^*)) .$$

The following two lemmas tell us about the soundness of the previous definition.

**Lemma 2.3.** Let $A$ and $B$ be two C$^*$-algebras and let $T : A \rightarrow B$ be a linear map such that $\|T\|_{cb} = 1$. The function $f_T : [1, \infty) \rightarrow \mathbb{R}$ defined by $f_T(q) = q \ln \pi_q^0(T)$ is non-negative and non-increasing. In particular,

$$\lim_{q \to \infty} q \ln \pi_q^0(N^*) = \inf_q q \ln \pi_q^0(N^*) \leq \pi_0^0(N^*) < \infty$$

is well defined for every quantum channel $N : \ell_1(I, S_1^q) \rightarrow S_1^{n_q}$.

**Proof.** On the one hand, the non negativity follows from the fact that $\pi_q^0(T) \geq \|T\|_{cb} = 1$ for every $q \geq 1$. In order to prove the second assertion let us consider $1 \leq q_0 < q < \infty$. Then, by using an interpolation argument one can show that for every $\alpha \in (0, 1)$ such that
\[ \frac{1}{q} = \frac{\alpha}{q_0} + \frac{1-\alpha}{\infty} \] we have \[ \pi_q^*(T) \leq \pi_{q_0}^*(T)^\alpha \pi_{\infty}^*(T)^{1-\alpha} = \pi_{q_0}^*(T)^\alpha. \] Therefore,

\[ q \ln \pi_q^*(T) = \ln (\pi_q^*(T)^q) \leq \ln (\pi_{q_0}^*(T)^{q_0}) = \ln (\pi_{q_0}^*(T)^{q_0}) + q_0 \ln \pi_{q_0}^*(T). \]

The last assertion follows immediately by the statement of the lemma and the fact that \( N^*: M_n \to \ell_\infty(I, M_d) \) has finite rank.

Lemma 2.4. Let \( f_T : [1, \infty) \to \mathbb{R}^+ \) be a function such that \( \lim_{q \to \infty} f(q) = 1 \). Then, \( \lim_{q \to \infty} q \ln f(q) \) exists if and only if \( \frac{d}{dp}[f(q)]_{p=1} := \lim_{p \to 1+} \frac{f(q)-1}{p-1} \) exists and in this case the limits are the same. Here, \( \frac{1}{p} + \frac{1}{q} = 1 \).

In particular, for any quantum channel \( N : \ell_1(I, S_1^d) \to S_1^n \) we have

\[ \tilde{\mathcal{C}}(N) = \frac{d}{dp}[\pi_q^*(N^*)]_{p=1}. \]

Proof. First note that \( \lim_{p \to 1+} \frac{f(q)-1}{p-1} = \lim_{q \to \infty} (q-1)(f(q)-1) \). Here, we have used that \( \frac{1}{p} + \frac{1}{q} = 1 \). Therefore, since \( \lim_{q \to \infty} f(q) = 1 \), the first assertion of the lemma follows from the fact that \( \lim_{q \to \infty} \frac{\ln(f(q))}{(f(q)-1)} = 1 \).

On the other hand, since \( N^* \) is a completely positive and unital map between \( C^* \)-algebras, we know that \( \|N^*\|_{cb} = 1 \). Then, by denoting \( f(q) = \pi_q^*(N^*) \) we have that \( \lim_{q \to \infty} f(q) = \lim_{q \to \infty} \pi_q^*(N^*) = \|N^*\|_{cb} = 1 \). Therefore, the second assertion of the lemma is a direct consequence of Lemma 2.3.

3. Main Theorem: Restricted capacities via \( \ell_p(S_p^d) \)-summing maps

In this section we will prove Theorem 1.2. In fact, this result will follow from a more general theorem:

Theorem 3.1. Let \( (N_i : S_1^d \to S_1^n)_{i \in I} \) be a family of quantum channels indexed in a set \( I \). Let us define the quantum channel

\[ N : \ell_1(I, S_1^d) \to S_1^n \]

by linearity with

\[ N(e_i \otimes \rho_i) = N_i(\rho_i) \quad \text{for every } \ i \in I. \]

Then,

\[ \tilde{\mathcal{C}}(N) = C_E((N_i)_i) := \sup \left\{ S \left( \sum_{i=1}^{N} \lambda_i (tr_{M_d} \otimes 1_{M_n})((1_{M_d} \otimes N_i)(\eta_i)) \right) \right\} \]
Lemma 3.2

\[ \sum_{i=1}^{N} \lambda_i \left[ S \left( (1_{M_d} \otimes tr_{M_n})(\phi_i) \right) - S \left( (1_{M_d} \otimes N_i)(\eta_i) \right) \right], \]

where the supremum runs over all \( N \in \mathbb{N} \), all probability distributions \( (\lambda_i)_{i=1}^{N} \) and all pure states \( \eta_i \in S_1^d \otimes S_1^d \).

Remark 3.1. Using that \( N_i \) is a quantum channel for every \( i \in I \) it is trivial to see that the right hand term in Theorem 3.1 can be written as

\[
\sup \left\{ S \left( \sum_{i=1}^{N} \lambda_i N_i((tr_{M_d} \otimes 1_{M_d})(\eta_i)) \right) + \sum_{i=1}^{N} \lambda_i \left[ S \left( (1_{M_d} \otimes tr_{M_d})(\eta_i) \right) - S \left( (1_{M_d} \otimes N_i)(\eta_i) \right) \right] \right\},
\]

where the supremum runs over all \( N \in \mathbb{N} \), all probability distributions \( (\lambda_i)_{i=1}^{N} \) and all pure states \( \eta_i \in S_1^d \otimes S_1^d \).

We will first show how to obtain Theorem 1.2 from Theorem 3.1.

Let \( N : S_1^n \to S_1^n \) be a quantum channel. Then, we define the new quantum channel \( \hat{N} : \ell_1(\mathcal{P}, S_1^d) \to S_1^n \), given by \( \hat{N}((\rho_\phi)_{\phi \in \mathcal{P}}) = \sum_{\phi \in \mathcal{P}} N(\phi^*(\rho_\phi)) \), where \( \mathcal{P} \) is defined as in (2.11). That is, \( \hat{N} \) is defined by the family of channels \( (N \circ \phi^* : S_1^d \to S_1^n)_{\phi \in \mathcal{P}} \). On the other hand, it is very easy to check that \( \hat{N}^* = j_d \circ N^* : M_n \to \ell_\infty(\mathcal{P}, S_1^d) \). According to Remark 2.2, we conclude that \( \pi_{p,d}(N^*) = \pi_{p,d}(\hat{N}^*) = \pi_{p,d}(\hat{N}^*) \). Therefore, Theorem 3.1 tells us that

\[
\lim_{q \to \infty} q \ln \pi_{q,d}(N^*) = \lim_{q \to \infty} q \ln \pi_{q,d}(\hat{N}^*) = \sup \left\{ S \left( \sum_{i=1}^{N} \lambda_i (\hat{N} \circ \phi_i)(N(\phi_i)) \right) + \sum_{i=1}^{N} \lambda_i \left[ S \left( (1_{M_d} \otimes tr_{M_d})(\eta_i) \right) - S \left( (1_{M_d} \otimes N(\phi_i))(\eta_i) \right) \right] \right\}.
\]

Here, the supremum runs over all \( N \in \mathbb{N} \), all probability distributions \( (\lambda_i)_{i=1}^{N} \), all pure states \( \eta_i \in S_1^d \otimes S_1^d \) and all quantum channels \( \phi_i : S_1^d \to S_1^n \). So we obtain Theorem 1.2.

The rest of the section will be devoted to proving Theorem 3.1. For the first inequality, \( \hat{C}(N) \leq C_E((N_i)_i) \), we will need the following well known lemma.

Lemma 3.2 ([18]). For every positive element in \( M_n \otimes M_m \) we have that

\[
\|x\|_{S_1^n[S_1^m]} \leq \left\| (1_{M_n} \otimes tr_{M_m})(x^p) \right\|_{S_1^n}^{\frac{1}{p}}.
\]
We will also use the following two well known results about the von Neumann entropy. The first one is about its continuity and the second one relates the von Neumann entropy of a state with its \( p \)-norm.

**Theorem 3.3.** \([2, \text{Theorem 1}]\) For all \( n \)-dimensional states \( \rho, \sigma \) we have

\[
|S(\rho) - S(\sigma)| \leq T \ln(n-1) + H(T, 1-T),
\]

where \( T = \frac{\|\rho - \sigma\|_1}{2} \) and \( H \) denotes the Shannon entropy.

In particular, given \( \epsilon > 0 \), there exists a \( \gamma = \gamma(\epsilon, n) > 0 \) such that \( T < \gamma \) implies \( |S(\rho) - S(\sigma)| < \epsilon \).

**Theorem 3.4.** The function \( F(\rho, p) = \frac{1-\|\rho\|_p}{p-1} \) is well defined for \( p \) positive with \( p \neq 1 \) and \( \rho \) a density matrix. It can be extended by continuity to \( p \in (0, \infty) \) and this extension verifies

\[
F(\rho, 1) = -\frac{d}{dp} \|\rho\|_p \bigg|_{p=1} = S(\rho).
\]

Moreover, if \( (\rho_p)_p \) is a net of density matrices of dimension \( n \) such that \( \lim_{p \to 1} \rho_p = \rho \), then

\[
\lim_{p \to 1} F(\rho_p, p) = S(\rho).
\]

**Proof.** The first statement is well known and very easy to check and the second one is a direct consequence of the fact that the convergence of \( F \) at \( p = 1 \) is uniform in \( \rho \) and the continuity of the von Neumann entropy. \qed

The following two remarks will be very useful:

**Remark 3.2.** For any real numbers \( \lambda \in (0, 1] \) and \( p \geq 1 \) we have \( \lambda - \lambda^p = \int_1^\lambda \lambda^q (-\ln \lambda) dq \).

Therefore,

\[
\lambda^p (-\ln \lambda)(p-1) \leq \lambda - \lambda^p \leq \lambda (-\ln \lambda)(p-1).
\]

Taking \( \mu = \lambda^p \in (0, 1] \) we obtain

\[
\mu(-\ln \mu^\frac{1}{p})(p-1) \leq \mu - \mu^\frac{1}{p} \leq \mu^\frac{1}{p}(-\ln \mu^\frac{1}{p})(p-1).
\]

**Remark 3.3.** We will restrict our study to quantum channels of the form \( \mathcal{N} : \mathcal{E}_1^N(S_1^d) \to S_1^n \), where \( \mathcal{N} \) is defined by a family of quantum channels \( (\mathcal{N}_i : S_1^d \to S_1^n)_{i=1}^N \) such that \( \mathcal{N}(\sum_{i=1}^N e_i \otimes \rho_i) = \sum_{i=1}^N \mathcal{N}_i(\rho_i) \). In this case, according to Corollary 2.2 and Lemma 2.4.
we can write
\[
\tilde{C}(N) = \lim_{p \to 1} \frac{1}{p-1} \left( \| \text{flip} \circ (1_{\ell_p^\infty(M_d)} \otimes N) : \ell_p^N(S_d^d) \rightarrow S_1^a[\ell_p^N(S_d^d)] \| - 1 \right).
\]
Moreover, according to Corollary 2.2, for a fixed \( p \) we know that the previous norm is attained on a positive element of the form \( \rho_p = \sum_{i=1}^N \lambda_i(p) e_i \otimes e_i \otimes M_{a_i}(p) \).

**Proof of inequality \( \leq \) in Theorem \( 3.1 \).** Let \( \epsilon > 0 \). We must find a \( \delta = \delta(\epsilon) > 0 \) such that
\[
\frac{\pi^0_q(N^*) - 1}{p - 1} \leq C_E((N_i)_i) + \epsilon,
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \). Using standard arguments, for every fixed \( p > 1 \) we can find an \( N = N(\epsilon, p) \in \mathbb{N} \) such that
\[
\left| \pi^0_q(N^*) - \pi^\ast_q(N_{1,s}(S_1^d)) \right| < (p - 1) \epsilon.
\]
Therefore, it suffices to prove that
\[
\frac{\pi^0_q(N^*) - 1}{p - 1} \leq C_E((N_i)_i) + \epsilon,
\]
where we consider \( N : \ell_1^{N(\epsilon, p)}(S_1^d) \rightarrow S_1^a \). Moreover, as we explained in Remark 3.3 for a fixed \( p > 1 \) we have
\[
\pi^0_q(N^*) = \sup_{\rho_p} \frac{\| \text{flip} \circ (1_{\ell_1^{N(\epsilon, p)}(M_d)} \otimes N)(\rho_p) \|_{S_1^a[\ell_1^{N(\epsilon, p)}(S_1^d)]}}{\| \rho_p \|_{\ell_1^{N(\epsilon, p)}(S_1^d)}[\ell_1^{N(\epsilon, p)}(S_1^d)]}
\]
and this supremum is attained on a positive element of the form
\[
\rho_p = \sum_{i=1}^{N(\epsilon, p)} \lambda_i(p) e_i \otimes e_i \otimes M_{a_i}(p).
\]
Assuming that \( \| \rho_p \|_{\ell_1^{N(\epsilon, p)^2}(S_1^d)} = 1 \) (otherwise we can normalize) we can write
\[
\rho_p = \sum_{i=1}^{N(\epsilon, p)} \beta_i(p) e_i \otimes e_i \otimes B_i(p)
\]
for certain positive numbers \( \{\beta_i(p)\}_{i=1}^{N(\epsilon, p)} \) verifying
\[
\sum_{i=1}^{N(\epsilon, p)} \beta_i(p)^p = 1.
\]
and where

\[(3.6) \quad \|B_i(p)\|_{S_1^\otimes} = 1\]

is the tensor associated to an operator of the form \(M_{b_i(p)}\) for every \(i = 1, \ldots, N(\epsilon, p)\). It is interesting to note that \(B_i(p)\) is a pure state associated to the element \(b_i(p) \in \ell_2^\otimes\) for every \(i = 1, \ldots, N(\epsilon, p)\).

Now, we will choose our \(\delta = \delta(\epsilon, n, d) > 0\) \((n \text{ and } d \text{ are fixed parameters in the problem})\) independently of \(p\), so that Equations (3.9), (3.11) and (3.14) are verified whenever \(p - 1 < \delta\). Note that Equations (3.9) and (3.11) depend on \(N = N(\epsilon, p)\). However, the crucial point here is that this dependence does not play any role once we have our normalization conditions (3.5) and (3.6). This is what makes possible to choose \(\delta\) independently of \(p\). Here, we will just explain how such a \(\delta\) can be chosen and we will not make the computations to give an explicit one. However, the reader can check that the following rough choice works:

\[(3.7) \quad \delta = \frac{\ln \left(1 + \frac{\gamma(\epsilon, n)\epsilon}{3n}\right)}{nd},\]

where \(\gamma(\epsilon, n)\) (which can be assumed to be lower than \(\epsilon\)) is the constant required in Theorem 3.3.

For \(p - 1 < \delta\), let us consider the corresponding element \(\rho_p = \sum_{i=1}^{N(\epsilon, p)} \beta_i(p)e_i \otimes e_i \otimes B_i(p)\). From this point on we will remove the dependence of \(p\) and \(\epsilon\) in \(N\), \(\beta_i\) and \(B_i\) for every \(i = 1, \ldots, N\). We will denote \(\tilde{B}_i = (1_{M_d} \otimes N_i)(B_i)\), which is a state in \(S_1^d \otimes S_1^n\) for every \(i = 1, \ldots, N\) and

\[\xi_p = \text{flip} \circ (1_{E_{S_1^d}}(M_d) \otimes N)(\rho_p) = \sum_{i=1}^{N} \beta_i e_i \otimes \text{flip}(\tilde{B}_i).\]

Now, using this notation we can write \(\frac{\pi_2^d(N^*) - 1}{p - 1}\) as

\[
\frac{1}{p - 1} \left( \frac{\|\xi_p\|_{S_1^d} \left[\ell_1^N(S_p^d)\right]}{\|\rho_p\|_{\ell_1^N(S_p^d)} \left[\ell_1^N(S_p^d)\right]} - 1 \right) = \frac{1}{\|\rho_p\|_{\ell_1^N(S_p^d)} \left[\ell_1^N(S_p^d)\right]} \left( \left\|\xi_p\|_{S_1^d} \left[\ell_1^N(S_p^d)\right] - \|\rho_p\|_{\ell_1^N(S_p^d)} \left[\ell_1^N(S_p^d)\right] \right\| \right) \left( \frac{1}{p - 1} \right)
\]

which can be written as

\[(3.8) \quad \frac{1}{\|\rho_p\|_{\ell_1^N(S_p^d)} \left[\ell_1^N(S_p^d)\right]} \left( \left\|\xi_p\|_{S_1^d} \left[\ell_1^N(S_p^d)\right] - \|\rho_p\|_{\ell_1^N(S_p^d)} \left[\ell_1^N(S_p^d)\right] \right\| \left( \frac{1}{p - 1} \right) + \|\xi_p\|_{\ell_1^N(S_p^d)} - 1 \right) + \frac{\|\xi_p\|_{\ell_1^N(S_p^d)} - 1}{p - 1} + \frac{1 - \|\rho_p\|_{\ell_1^N(S_p^d)} \left[\ell_1^N(S_p^d)\right]}{p - 1}.\]
Using Remark 2.1 we see that we can find a $\delta = \delta(\epsilon, \delta)$ (in particular, Equation (3.7)) such that $p - 1 < \delta$ guarantees

$$
\frac{1}{\|p\|_{\ell_p^f(S_p^d)}[\ell_p^f(S_p^d)\]} = \frac{1}{\left(\sum_{i=1}^{N} \beta_i^p \|(1_{M_d} \otimes tr_{M_d})(\tilde{B}_i)\|_{S_p}^{\frac{1}{p}}\right)} \leq 1 + \epsilon.
$$

Thus, we need to study the three terms in Equation (3.8). We will start showing that

$$
\frac{1}{\|p\|_{\ell_p^f(S_p^d)}[\ell_p^f(S_p^d)\]} - \|p\|_{\ell_p^f(S_p^d)} \leq 3\epsilon.
$$

In order to see this, let us define $\Lambda_p = \sum_{i=1}^{N} \beta_i^p (tr_{M_d} \otimes 1_{M_n})(\tilde{B}_i)$, which is a positive element in $M_n$. It is trivial that $tr(\Lambda_p) = \|p\|_{\ell_p^f(S_p^d)}[\ell_p^f(S_p^d)\]$. Moreover, we know that $\|\Lambda_p\| \leq tr(\Lambda_p) \leq 1$. Therefore, we can apply functional calculus and Remark 3.2 to conclude that

$$
\frac{1}{\|p\|_{\ell_p^f(S_p^d)}[\ell_p^f(S_p^d)\]} - \|p\|_{\ell_p^f(S_p^d)} \leq \|p\|_{\ell_p^f(S_p^d)} \leq 3\epsilon.
$$

On the other hand, according to Lemma 3.2 and taking into account the flip map in the definition of $\xi_p$ we have

$$
\|\xi_p\|_{\ell_p^f(S_p^d)} \leq tr_{M_n} \left(\sum_{i=1}^{N} \beta_i^p (tr_{M_d} \otimes 1_{M_n})(\tilde{B}_i)\right)^{\frac{1}{p}}.
$$

Thus,

$$
\frac{\|\xi_p\|_{\ell_p^f(S_p^d)} - \|p\|_{\ell_p^f(S_p^d)}}{p - 1} \leq \frac{\|p\|_{\ell_p^f(S_p^d)} - \|p\|_{\ell_p^f(S_p^d)}}{p - 1} \leq S(\Lambda_p^{\frac{1}{p}}),
$$

where we denote $S(\Lambda_p^{\frac{1}{p}}) = tr(\Lambda_p^{\frac{1}{p}}(- \ln \Lambda_p^{\frac{1}{p}}))$.

Now, using Theorem 3.3, Remark 3.2 and Equation (3.7) one can conclude

$$
S\left(\sum_{i=1}^{N} \beta_i^p (tr_{M_d} \otimes 1_{M_n})(\tilde{B}_i)\right) - S(\Lambda_p^{\frac{1}{p}}) < 3\epsilon.
$$

Thus, we obtain Equation (3.10).

In order to see Equation (3.11), let us denote $\Delta_p = \sum_{i=1}^{N} \beta_i^p (tr_{M_d} \otimes 1_{M_n})(\tilde{B}_i)$. Then, one can write

$$
\left|S(\Delta_p) - S(\Lambda_p^{\frac{1}{p}})\right| \leq \left|S(\Delta_p) - S\left(\frac{\Lambda_p^{\frac{1}{p}}}{\|\Lambda_p\|_{S_p^d}}\right)\right| + \left|S\left(\frac{\Lambda_p^{\frac{1}{p}}}{\|\Lambda_p\|_{S_p^d}}\right) - S(\Lambda_p^{\frac{1}{p}})\right|.
$$
Since $\Delta_p$ and $\frac{\Lambda_p}{\|\Lambda_p\|_{S^1}}$ are states in $M_n$, according to Theorem 3.3 in order to upper bound the first term in the above equation it suffices to control

\begin{equation}
\left\| \Delta_p - \frac{\Lambda_p}{\|\Lambda_p\|_{S^1}} \right\|_{S^1_1} \leq \left\| \Delta_p - \Lambda_p \right\|_{S^1_1} + \left\| \Lambda_p - \frac{\Lambda_p}{\|\Lambda_p\|_{S^1}} \right\|_{S^1_1}.
\end{equation}

Now, it is very easy to see that

\begin{equation}
\left\| \Delta_p - \Lambda_p \right\|_{S^1_1} \leq \sum_{i=1}^{N} \beta_p \|\tilde{B}_i - \tilde{B}_p\|_{S^1_1}.
\end{equation}

Using that $\tilde{B}_i \in M_d \otimes M_n$ and the normalization conditions (3.5) and (3.6), one can find a $\delta = \delta(\epsilon, n, d) > 0$ (as (3.7)) so that $p - 1 < \delta$ implies that $\left\| \Delta_p - \Lambda_p \right\|_{S^1_1} < \epsilon$. In order to control the second term in Equation (3.12), we realize again that $\Lambda_p$ and $\frac{\Lambda_p}{\|\Lambda_p\|_{S^1}}$ are commuting operators in $M_n$. Let us denote $(\mu_i)_{i=1}^n$ the eigenvalues of the state $\frac{\Lambda_p}{\|\Lambda_p\|_{S^1}}$.

Then, $\gamma_i = \|\Lambda_p\|_{S^1_1} \mu_i$, $i = 1, \cdots, n$, are the eigenvalues of $\Lambda_p$. Furthermore,

\begin{equation}
\left\| \Lambda_p - \frac{\Lambda_p}{\|\Lambda_p\|_{S^1}} \right\|_{S^1_1} = \sum_{i=1}^{n} |\mu_i - \gamma_i|.
\end{equation}

An easy application of Remark 3.2 shows that we can find a $\delta = \delta(\epsilon, n) > 0$ so that if $p - 1 < \delta$ we can assure that the previous sum is smaller than $\epsilon$.

Finally, in order to control the second term in Equation (3.12), we realize again that $\frac{\Lambda_p}{\|\Lambda_p\|_{S^1}}$ and $\frac{\Lambda_p}{\|\Lambda_p\|_{S^1}}$ are commuting operators in $M_n$. Again, if we call $\mu_i$ and $\gamma_i = \|\Lambda_p\|_{S^1_1} \mu_i$ with $i = 1, \cdots, n$, their corresponding eigenvalues, we have that

\begin{equation}
\left| S(\frac{\Lambda_p}{\|\Lambda_p\|_{S^1}}) - S(\Lambda_p) \right| = \left| \sum_{i=1}^{n} \mu_i (-\ln(\mu_i)) - \sum_{i=1}^{n} \gamma_i (-\ln(\gamma_i)) \right|
\end{equation}

\begin{equation}
\leq \left| 1 - \|\Lambda_p\|_{S^1_1} \sum_{i=1}^{n} \mu_i (-\ln(\mu_i)) + \|\Lambda_p\|_{S^1_1} \ln(\|\Lambda_p\|_{S^1}) \right|.
\end{equation}

Again, it is very easy to see that we can find a $\delta = \delta(\epsilon, n) > 0$ (as (3.7)) so that if $p - 1 < \delta$ we can assure that the previous value is smaller than $\epsilon$. This gives us Equation (3.11).
We can now study the other two terms in Equation (3.8). First, note that
\[
\frac{\|\xi\|^p}{p-1} + \frac{1 - \|\rho\|^p}{p-1} \leq \sum_{i=1}^N \beta_i^p \left( \frac{\|\tilde{B}_i\|^p}{p-1} + \frac{1 - \| (1_{M_d} \otimes tr_{M_d})(B_i) \|^p}{p-1} \right),
\]
where we have used that \( \sum_{i=1}^N \beta_i^p \| (1_{M_d} \otimes tr_{M_d})(B_i) \|^p \leq \left( \sum_{i=1}^N \beta_i^p \| (1_{M_d} \otimes tr_{M_d})(B_i) \|^p \right)^{1/p} \).

Let \((\alpha_j^i)_{j=1}^{dn}\) be the eigenvalues of the state \(\tilde{B}_i\) for every \(i = 1, \ldots, N\). Then, using Remark 3.2 once again we obtain
\[
(3.14) \quad \frac{\|\tilde{B}_i\|^p}{p-1} - 1 = \frac{\sum_{j=1}^{dn} (\alpha_j^i)^p - \alpha_j^i}{p-1} \leq \frac{1}{p-1} \sum_{j=1}^{dn} \alpha_j^i \ln \alpha_j^i + \epsilon = -S(\tilde{B}_i) + \epsilon.
\]
Here, for the first inequality we just have to impose that \(p - 1 < \delta\) for a certain \(\delta = \delta(\epsilon, n, d) > 0\) as in (3.7). One can check analogously that
\[
\frac{1 - \| (1_{M_d} \otimes tr_{M_d})(B_i) \|^p}{p-1} \leq S((1_{M_d} \otimes tr_{M_d})(B_i)).
\]
Therefore, we have shown that Equation (3.8) is upper bounded by
\[
(1 + \epsilon) \left\{ S \left( \sum_{i=1}^N \beta_i^p \left( tr_{M_d} \otimes 1_{M_n} \right) \left( (1_{M_d} \otimes N_i)(B_i) \right) \right) \right\}
+ \sum_{i=1}^N \beta_i^p \left[ S \left( (1_{M_d} \otimes tr_{M_d})(B_i) \right) - S \left( (1_{M_d} \otimes N_i)(B_i) \right) \right] + 4\epsilon.
\]
If we denote \(\lambda_i = \beta_i^p\) and \(\eta_i = B_i\) for every \(i = 1, \ldots, N\) we see that the previous expression is (up to the \(\epsilon\)’s) one of those appearing in the definition of \(C_E((N_i)_i)\). Using that \(d\) and \(n\) are fixed numbers and that \(S(\rho) \leq \log m\) for any state \(\rho \in S_1^m\), since the previous estimate holds for an arbitrary \(\epsilon > 0\), the statement of the proposition follows. \(\square\)

To prove the converse inequality, \(\tilde{C}(N) \geq C_E((N_i)_i)\), we will need the following two lemmas.

**Lemma 3.5.** Let \(A\) and \(B\) be finite dimensional \(C^*\)-algebras and let \(N : L_1(A) \rightarrow L_1(B)\) be a quantum channel such that \(N(1_A)\) has full support (that is, it is an invertible element of \(B\)). Let \(T : B \rightarrow A\) be a completely positive contraction such that there exists a positive element \(a\) in the unit ball of \(B\) verifying
\[
N^*(x) = T(axa)
\]
for every $x \in \mathcal{B}$. Then $a = 1_B$ and $N^* = T$.

**Proof.** Using that $\mathcal{N}$ is trace preserving we have

$$tr_A(N(1_A)) = tr_B(aT^*(1_A)a) = tr_B(T^*(1_A)\frac{1}{2}T^*(1_A)\frac{1}{2}a^2) \leq tr_B(T^*(1_A)) = tr_B(1_BT^*(1_A)) = tr_A(1_AT(1_B)) \leq tr_A(1_A),$$

where the inequality $tr_B(T^*(1_A)\frac{1}{2}T^*(1_A)\frac{1}{2}a^2) \leq tr_B(T^*(1_A))$ follows from Cauchy-Schwartz inequality. Now, the fact that $\mathcal{N}(1_A)$ has full support implies that $T^*(1_A)$ has also full support. Therefore, $a^2 = 1_B$. So, $a = 1_B$. □

**Remark 3.4.** Given a quantum channel $\mathcal{N} : L_1(\mathcal{A}) \to L_1(\mathcal{B})$ between finite dimensional C*-algebras, we can always assume that $\mathcal{N}(1_A)$ has full support. Otherwise, we consider the finite dimensional C*-algebra $\tilde{\mathcal{B}} = p\mathcal{B}p$, where $p$ is the support projection of $\mathcal{N}(1_A)$, and consider the new quantum channel $\mathcal{N} : L_1(\mathcal{A}) \to L_1(\mathcal{B})$.

**Lemma 3.6.** Let $(a(p))_p$ be a net of positive and invertible operators in $M_n$ verifying the following properties:

1) $\sup_p \|a(p)^{-1}\|_{M_n} \leq M$ for a certain positive constant $M$, and
2) $\lim_{p \to 1} \ln \|a(p)\|_{S^p_{1/p}} = 0$, where $\frac{1}{p} + \frac{1}{p^*} = 1$.

Then,

$$\liminf_{p \to 1+} \frac{tr(a(p)^{-1}\rho) - 1}{p - 1} \geq S(\rho)$$

for every density operator $\rho$.

**Proof.** According to Theorem 3.4 we have,

$$\lim_{p \to 1} \frac{\|\rho\|_{\frac{p}{p-1}} - 1}{p - 1} = S(\rho).$$

This implies

$$\lim_{p \to 1} \frac{\|\rho\|_{\frac{p}{p-1}} - 1}{p - 1} = S(\rho). \quad (3.15)$$

On the other hand, for $p > 1$ we can write

$$\frac{\|\rho\|_{\frac{p}{p-1}} - 1}{p - 1} = \frac{\|a(p)^{\frac{1}{2}}a(p)^{-\frac{1}{2}}pa(p)^{-\frac{1}{2}}a(p)^{\frac{1}{2}}\|_{\frac{p}{p-1}} - 1}{p - 1} \leq \frac{\|a(p)\|_{p^*} \|a(p)^{-\frac{1}{2}}pa(p)^{-\frac{1}{2}}\|_1 - 1}{p - 1}. \quad (3.16)$$
where we have used the generalized Holder’s inequality with \( \frac{1}{2p} = 1 + \frac{1}{p-1} = 1 + \frac{1}{p} \).

Since we have
\[
\frac{\|a(p)\|_{p'}}{p - 1} (\|a(p)\|_{p'} - \frac{1}{p - 1}) = \left( \frac{\|a(p)\|_{p'} - \frac{1}{p - 1}}{p - 1} \right) tr(a(p)^{-1} \rho) + \frac{tr(a(p)^{-1} \rho) - 1}{p - 1}
\]
for every \( p \), we conclude our proof from Equations (3.15) and (3.16) if we show
\[
\lim_{p \to 1} \left( \frac{\|a(p)\|_{p'} - \frac{1}{p - 1}}{p - 1} \right) tr(a(p)^{-1} \rho) = 0.
\]
To this end note that
\[
\lim_{p \to 1} \left| \frac{\|a(p)\|_{p'} - \frac{1}{p - 1}}{p - 1} \right| \leq M \lim_{p \to 1} \left| \frac{\|a(p)\|_{p'} - \frac{1}{p - 1}}{p - 1} \right| = M \lim_{p \to 1} \frac{p'}{p} \left( \|a(p)\|_{p'} - 1 \right)
\]
\[
= M \lim_{p \to 1} \frac{p'}{p} \left( e^{\frac{1}{p} \ln \|a(p)\|_{p'}^p} - 1 \right) = M \lim_{p \to 1} \frac{1}{p} \left( \ln \|a(p)\|_{p'}^p \right) = 0,
\]
where we have used that \( e^{\frac{1}{p} \ln \|a(p)\|_{p'}^p} \simeq 1 + \frac{1}{p} \ln \|a(p)\|_{p'}^p \) when \( p \) is close to 1. □

We are now ready to prove the second inequality in Theorem 3.1.

**Proof of inequality \( \geq \) in Theorem 3.1** Let \( \Upsilon = \{(\lambda_i)_{i=1}^N, (\eta_i)_{i=1}^N \} \) be an ensemble optimizing \( C_E((N_i)_i) \). We have to show that
\[
\lim_{q \to \infty} q \ln (\Pi_q^N(N^*)) \geq \sup \left\{ S \left( \sum_{i=1}^N \lambda_i (tr_{M_d} \otimes 1_{M_n})((1_{M_d} \otimes N_i)(\eta)) \right) \right.
\]
\[
\left. + \sum_{i=1}^N \lambda_i \left[ S \left( (1_{M_d} \otimes tr_{M_n})((1_{M_d} \otimes N_i)(\eta)) \right) - S \left( (1_{M_d} \otimes N_i)(\eta) \right) \right] \right\}.
\]
Clearly, it suffices to prove the previous inequality if we consider the new channel defined by restricting \( N \) to \( \ell_1^N(S_1^d) \). We will use the same notation \( N : \ell_1^N(S_1^d) \to S_1^n \) for the restricted channel. Moreover, as we explained in Remark 3.4 we can assume that \( N(1_{\ell_1^N(S_1^d)}) \) has full support.

According to Theorem 2.1 for every \( 1 < q \) we can consider an optimal factorization
\[
N^* = T_q M_{a(q)},
\]
where \( T_q : S_1^n \to \ell_\infty^N(M_d) \) is a completely positive map, \( M_{a(q)} : M_n \to S_1^n \) is the associated operator to certain positive element \( a(q) \in M_n \) and such that \( \|a(q)\|_{2q}^2 \|T_q\|_{cb} = \Pi_q^N(N^*) \).
Here, $1 = \frac{1}{p} + \frac{1}{q}$. Furthermore, by scaling we may assume
\[
\|a(q)\|_2^2 = \pi_q^\circ(N^*)^{\frac{1}{2}}; \quad \|T_q : S_q^n \to \ell_\infty^N(M_d)\|_{cb} = \pi_q^\circ(N^*)^{\frac{1}{p}}.
\]
Actually, the fact that $\mathcal{N}(1\ell_1^n(S_1^d)) = a(q)T_q^*(1\ell_1^n(S_1^d))a(q)$ has full support guarantees that $a(q)$ is also invertible for every $q$.

By continuity, we deduce that
\[
\lim_{q \to \infty} \|a(q)\|_2^2 = \lim_{q \to \infty} \pi_q^\circ(N^*) = \|N^*\|_{cb} = 1.
\]
On the other hand,
\[
\|T_q : M_n \to \ell_\infty^N(M_d)\|_{cb} \leq \|1_{M_n} : M_n \to S_q^n\|_{cb}\|T_q : S_q^n \to \ell_\infty^N(M_d)\|_{cb} \leq n^{\frac{1}{p}} \pi_q^\circ(N^*)^{\frac{1}{p}}.
\]
By a compactness argument we can assume that
\[
\lim_{q \to \infty} T_q = T : M_n \to \ell_\infty^N(M_d),
\]
where $T$ is a completely positive and completely contractive map. In the same way, we see that
\[
\|a(q)\| \leq \|a(q)\|_2^2 = \pi_q^\circ(N^*)^{\frac{1}{2p}};
\]
so we can assume that
\[
\lim_{q \to \infty} a(q) = a,
\]
where $a$ is a positive operator in $M_n$ verifying $0 \leq a \leq 1_{M_n}$.

It follows by construction that $\mathcal{N}^*(x) = T(axa)$ for every $x \in M_n$. Moreover, we can apply Lemma 3.5 to conclude that $a = 1_{M_n}$ and $T = N^*$. This implies, in particular, that $\lim_{q \to \infty} a(q)^{-1} = 1$. Considering a subnet we can assume that $\sup_q \|a(q)^{-1}\| \leq M$ for a positive constant $M$.

Now, Lemma 2.4 and Equation (3.17) allow us to write
\[
\lim_{q \to \infty} q \ln \left( \Pi_q^\circ(N^*) \right) = \lim_{q \to \infty} q \left( \ln (\|a(q)\|_2^2) + \ln (\|T_q : S_q^n \to \ell_\infty^N(M_d)\|_{cb}) \right)
= \lim_{q \to \infty} q \ln \|T_q\|_{cb} = \lim_{q \to \infty} q \ln \|T_q^* : \ell_1^N(S_1^d) \to S_p^n\|_{cb} = \lim_{p \to 1} \frac{\|T_q^*\|_{cb} - 1}{p - 1}.
\]
In order to simplify notation we will denote $T_p = T_q^* : \ell_1^N(S_1^d) \to S_p^n$. 

Now, note that

$$\lim_{p \to 1} \frac{\|T_p\|_{cb} - 1}{p - 1} \geq \lim_{p \to 1} \frac{\|1\epsilon_N(M_d) \otimes T_p : \epsilon_p^N(S_p^d) \Rightarrow \epsilon_p^N(S_p^d)[S_p^N] - 1}{p - 1}$$

$$\geq \lim_{p \to 1} \frac{1}{p - 1} \left( \frac{\|1 \otimes T_p\|_p - 1}{\|\epsilon_p^N(S_p^d)\|_{cb} - 1} \right),$$

where $\rho = \sum_{i=1}^N \lambda_i e_i \otimes e_i \otimes \eta_i \in \epsilon_1^N(S_1^d)$ is the state defined by the ensemble $\Upsilon$ that we have considered at the beginning of the proof.

Let us denote $\tilde{\xi}_p = (1\epsilon_N(M_d) \otimes T_p)(\rho) = \sum_{i=1}^N \lambda_i e_i \otimes (1M_d \otimes T_p^i)(\eta_i) \in \epsilon_p^N(S_p^{iN})$, where $T_p = (T^i : S^d_1 \to S_p^{iN})_{i=1}^N$.

The previous expression can be written as

$$\lim_{p \to 1} \frac{1}{p - 1} \left( \frac{\|\tilde{\xi}_p\|_p - 1}{\|\rho\|_{(p,1)} - 1} \right) = \lim_{p \to 1} \frac{1}{\|\rho\|_{(p,1)}} \left( \frac{\|\tilde{\xi}_p\|_p - 1 + 1 - (1 \otimes tr)(\rho)\|\epsilon_p^N(S_p^d)\|_{cb}}{p - 1} \right),$$

where we have used Equation (2.1) to write

$$\|\rho\|_{(p,1)} = \|(1 \otimes tr)(\rho)\|\epsilon_p^N(S_p^d) = \left\| \sum_{i=1}^N \lambda_i e_i \otimes (1M_d \otimes trM_d)(\eta_i) \right\|\epsilon_p^N(S_p^d).$$

The fact that $\rho$ is a state guarantees

$$\lim_{p \to 1} \|\rho\|_{(p,1)} = 1.$$

Actually, according to Theorem 3.3 and the definition of the von Neumann entropy this also implies that

$$\lim_{p \to 1} \frac{1 - \|(1 \otimes tr)(\rho)\|\epsilon_p^N(S_p^d)}{p - 1} = S\left( \sum_{i=1}^N \lambda_i e_i \otimes (1M_d \otimes trM_d)(\eta_i) \right)$$

$$= \sum_{i=1}^N \lambda_i S\left( (1M_d \otimes trM_d)(\eta_i) \right) + H\left( \lambda_i \right)_{i=1}^N.$$

In order to study the other term in Equation (3.18) we define the state $\xi_p = \frac{\tilde{\xi}_p}{\|\xi_p\|\epsilon_p^N(S_p^{iN})}$ and then write

$$\frac{\|\tilde{\xi}_p\|\epsilon_p^N(S_p^{iN}) - 1}{p - 1} = \|\tilde{\xi}_p\|_{(p,1)} \left( \frac{\|\xi_p\|_p - 1}{p - 1} \right) + \frac{\|\tilde{\xi}_p\| - 1}{p - 1}.$$
Now, according to our construction
\[
\lim_{p \to 1} \tilde{\xi}_p = \lim_{p \to 1} (1 \ell\infty(M_d) \otimes T_q^*)(\rho) = (1 \ell\infty(M_d) \otimes T^*)(\rho) = (1 \ell\infty(M_d) \otimes \mathcal{N})(\rho),
\]
and
\[
\lim_{p \to 1} \|\tilde{\xi}_p\|_{\ell^p(S^m)} = 1.
\] (3.21)

On the other hand, Theorem 3.4 says that
\[
\lim_{p \to 1} \|\xi_p\|_p = -S((1 \ell\infty(M_d) \otimes \mathcal{N})(\rho)) = -S\left(\sum_{i=1}^N \lambda_i e_i \otimes (1\mathcal{M} \otimes \mathcal{N}_i)(\eta_i)\right)
\]
\[
= -\sum_{i=1}^N \lambda_i S((1\mathcal{M} \otimes \mathcal{N}_i)(\eta_i)) - H((\lambda_i)_{i=1}^N).
\] (3.22)

Finally, (3.17) allows us to apply Lemma 3.6 to the net \((a(p)^2)_p\) to obtain
\[
\lim_{p \to 1} \|\xi_p\|_{\ell^p(S^m)} - 1 \geq -\sum_{i=1}^N \lambda_i S((1\mathcal{M} \otimes \mathcal{N}_i)(\eta_i)) - H((\lambda_i)_{i=1}^N)
\] (3.23)

Equations (3.21)-(3.23) state that
\[
\frac{\|\xi_p\|_{\ell^p(S^m)} - 1}{p - 1} \geq -\sum_{i=1}^N \lambda_i S((1\mathcal{M} \otimes \mathcal{N}_i)(\eta_i)) - H((\lambda_i)_{i=1}^N)
\]
\[
+ S\left(\sum_{i=1}^N \lambda_i \mathcal{N}_i((tr\mathcal{M} \otimes 1\mathcal{M})(\eta_i))\right).
\] (3.24)

Hence, Equations (3.19), (3.20) and (3.24) allow us to conclude that the expression in (3.18) is bigger than \(C_E(\mathcal{N}_j)\). This concludes the proof. \(\Box\)

**Remark 3.5.** Actually, we have shown that the states \(\eta_i\)'s and the probabilities \(\lambda_i\)'s in the expression
\[
\sup \left\{ S\left(\sum_{i=1}^N \lambda_i (\mathcal{N} \circ \phi_i)((tr\mathcal{M} \otimes 1\mathcal{M})(\eta_i))\right) + \sum_{i=1}^N \lambda_i S\left((1\mathcal{M} \otimes tr\mathcal{M})(\eta_i)\right) \right\}
\]
\[ -S\left( (1_{M_d} \otimes (N \circ \phi_i))(\eta_i) \right) \] 

in Theorem 3.1 are given by Theorem 2.1. This means that the factorization theorem tells us the objects that we have to use in order to attain the capacity of the channel. In particular, we have shown that considering pure states \( \eta_i \) in the expression (1.3) is not a restriction, but it covers the general case.

**Remark 3.6 (Classical channels).** As we pointed out in the introduction, when the channel \( \mathcal{N} \) is classical the definition of \( C^d(\mathcal{N}) \) coincides with \( C_c(\mathcal{N}) \) for every \( d \). On the other hand, it is very easy to see that \( \pi_p(T : \ell_\infty \to \ell_\infty) = \pi_p(T : \ell_\infty \to \ell_\infty) \) for every \( T : \ell_\infty \to \ell_\infty \). Thus, in this case we recover (1.2).

**Remark 3.7 (The cases \( d = 1 \) and \( d = n \)).** As we said before, given a quantum channel \( \mathcal{N} : S^n_1 \to S^n_1 \), \( C^1(\mathcal{N}) \) and \( C^n(\mathcal{N}) \) coincide, respectively, with the Holevo capacity and the unlimited entanglement-assisted classical capacity of \( \mathcal{N} \).

To see the first one, we just write the expression in Theorem 1.2 for \( d = 1 \) and we obtain

\[
C^1(\mathcal{N}) = \sup \left\{ S\left( \sum_{i=1}^{N} \lambda_i N(\xi_i) \right) - \sum_{i=1}^{N} \lambda_i S\left( N(\xi_i) \right) \right\},
\]

where the supremum runs over all \( N \in \mathbb{N} \), all probability distributions \( (\lambda_i)_{i=1}^{N} \) and all families \( (\xi_i)_{i=1}^{N} \), with \( \xi_i \) state in \( M_n \) for every \( i = 1, \ldots, N \). This is exactly the expression of the Holevo capacity of \( \mathcal{N} \) (see Theorem 5.1 in Section 5).

The key point to study the case \( d = n \) is to realize that we do not need to consider the embedding \( j_n : M_n \hookrightarrow \ell_\infty(\mathcal{P}, M_n) \). First of all, let us recall that \( \pi_q(\mathcal{N}^*) = \pi_q^0(\mathcal{N}^*) \), which follows from the definition of the norms (see Section 2). Then, using that \( j_n \) is a complete isometry on positive elements and the good behavior of \( \pi_q^0 \) with respect to positivity shown in Section 2 it is easy to see that

\[
\pi_q^0(j_n \circ \mathcal{N}^*) = \pi_q^0(\mathcal{N}^*) = \pi_{q,n}(\mathcal{N}^*)
\]

for every quantum channel \( \mathcal{N} : S^n_1 \to S^n_1 \). Therefore, in this case Theorem 1.2 is obtained from Theorem 3.1 applied on the single channel \( \mathcal{N} \) instead of on a family of infinitely many channels \( (\mathcal{N}_i)_i \). Then, we have

\[
C^n(\mathcal{N}) = \lim_{q \to \infty} q \ln \left( (\Pi_q^0((\mathcal{N}^*))) \right) = \sup \left\{ S\left( \mathcal{N}\left( (tr_{M_n} \otimes 1_{M_n})(\eta) \right) \right) + S\left( (1_{M_n} \otimes tr_{M_n})(\eta) \right) - S\left( (1_{M_n} \otimes \mathcal{N})(\eta) \right) \right\},
\]
where the supremum runs over all pure states $\eta \in S_1^n \otimes S_1^n$. This is exactly the expression of $C_E(N)$ (see Theorem 5.2 in Section 5).

4. COVARIANT CHANNELS AND NON ADDIVITY OF $C^d$

In this section we will discuss a particularly nice kind of quantum channels called covariant channels. We will see that the factorization theorem has a very simple form for these channels. As a direct consequence of this fact, we will show that there is an easy relation between the (unlimited) entanglement-assisted classical capacity $C_E$ of a covariant channel and the $cb$-$min$ entropy of a quantum channel introduced in [8]. In the second part of this section, we will use our results on covariant channels to prove Theorem 1.3. As we will explain in Section 5 a direct consequence of this theorem is that the product state capacity of the $d$-restricted capacity, $C^d$, does not coincide, in general, with its regularization version for $1 < d < n$.

**Definition 4.1.** Let $N : S_1^n \to S_1^n$ be a quantum channel. We will say that $N$ is a covariant channel if there exits a group of unitaries $G \subset U(n)$ verifying the following two properties:

1. $\int_G gxg^{-1}dg = \frac{1}{n^2}tr_{M_n}(x)1_{M_n}$ for every $x \in M_n$.
2. $N(gxg^{-1}) = gN(x)g^{-1}$ for every $x \in M_n$ and $g \in G$.

Here, $U(n)$ denotes the group of unitary $n \times n$ matrices.

The following result is an easy consequence of a Pisier’s version of the Wigner-Yanase-Dyson inequalities (see [25, Lemma 1.14]).

**Lemma 4.1.** Let $T : M_n \to M_n$ be a map such that there exits a group of unitaries $G \subset U(n)$ verifying that $T$ is covariant with respect to $G$ in the sense of Definition 4.1. Then, for any $1 \leq d \leq n$ we have

$$\pi_{p,d}(T) = n^{\frac{1}{d}p}\|T : S_p^m \to M_n\|_d.$$ 

In the case $d = 1$ we obtain $\pi_{p,cb}(T) = n^{\frac{1}{p}n}\|T : S_p^m \to M_n\|$ while for $d = n$ we have $\pi_{p}(T) = n^{\frac{1}{p}n}\|T : S_p^m \to M_n\|_{cb}$. 

**Proof.** To prove inequality $\leq$ just note that

$$\pi_{p,d}(T : M_n \to M_n) \leq \pi_{p,d}(1_{M_n} : M_n \to S_p^m)\|T : S_p^m \to M_n\|_d = n^{\frac{1}{d}p}\|T : S_p^m \to M_n\|_d.$$
For the converse inequality let us fix $1 \leq d \leq n$ and assume that $\pi_{p,d}(T) = 1$. We will conclude our proof if we show that $\|T : S_p^n \to M_n\|_d \leq n^{-\frac{1}{p}}$. Now, according to the factorization theorem there exist positive elements $a, b \in M_n$ verifying $\|a\|_{2p} = \|b\|_{2p} = 1$ such that

$$\|(1_{M_d} \otimes T)(x)\|_{S_p^d[M_n]} \leq \|(1_{M_d} \otimes a)x(1_{M_d} \otimes b)\|_{S_p(\ell_2^d \otimes \ell_2^d)}$$

for every $x \in M_{dn}$.

Now, for every $x \in M_{dn}$ we have

$$\|(1_{M_d} \otimes T)(x)\|_{S_p^d[M_n]} = \|(1_{M_d} \otimes g)((1_{M_d} \otimes T)(x))(1_{M_d} \otimes g^{-1})\|_{S_p^d[M_n]}$$

$$= \|(1_{M_d} \otimes T)((1_{M_d} \otimes g)x(1_{M_d} \otimes g^{-1}))\|_{S_p^d[M_n]}$$

$$\leq \|(1_{M_d} \otimes ag)x(1_{M_d} \otimes g^{-1}b)\|_{S_p(\ell_2^d \otimes \ell_2^d)}$$

for every $g \in G$. Therefore, according to [25, Lemma 1.14] we obtain

$$\|(1_{M_d} \otimes T)(x)\|_{S_p^d[M_n]} \leq \left( \int_G \|(1_{M_d} \otimes ag)x(1_{M_d} \otimes g^{-1}b)\|_{S_p(\ell_2^d \otimes \ell_2^d)}^p \right)^{\frac{1}{p}}$$

$$\leq \left\| \left( 1 \otimes \left( \int_G (g^{-1}a g^{2p} dg) \right)^{\frac{1}{2p}} \right) x \left( 1 \otimes \left( \int_G (g^{-1}b g^{2p} dg) \right)^{\frac{1}{2p}} \right) \right\|_{S_p(\ell_2^d \otimes \ell_2^d)}$$

$$= \left\| \left( 1 \otimes \left( \int_G g^{-1}a^{2p} g dg \right)^{\frac{1}{2p}} \right) x \left( 1 \otimes \left( \int_G g^{-1}b^{2p} g dg \right)^{\frac{1}{2p}} \right) \right\|_{S_p(\ell_2^d \otimes \ell_2^d)}$$

$$= n^{-\frac{1}{p}} \|x\|_{S_p^d[S_p]}.$$

This concludes the proof. \qed

The previous lemma says that if we are dealing with a covariant channel $\mathcal{N}$ we can always take $a = n^{-\frac{1}{2p}}1_{M_n}$ in the factorization given by Theorem 2.1. Thus, in order to compute $C^d(\mathcal{N})$ for these kinds of channels we will have to differentiate the norm $\|\mathcal{N} : S_1^n \to S_p^n\|_d$ instead of the $\pi_{p,d}(\mathcal{N}^*)$-norm. Indeed, we have

**Corollary 4.2.** For any covariant quantum channel $\mathcal{N} : S_1^n \to S_1^n$ we have

$$C^d(\mathcal{N}) = \ln n + \frac{d}{dp} \|[\mathcal{N} : S_1^n \to S_p^n]\|_{p=1}$$

for every $1 \leq d \leq n$. 
Proof. First of all, note that $N^*$ also verifies condition 2. in Definition 4.1. Therefore, applying Lemma 4.1 we obtain

\[
C^d(N) = \lim_{q \to \infty} q \ln \pi_{p,d}(N^*) = \ln n + \lim_{q \to \infty} q \ln \|N^* : S_q^n \to M_n\|_d \\
= \ln n + \lim_{q \to \infty} q \ln \|N : S_1^n \to S_p^n\|_d \\
= \ln n + \frac{d}{dp}[\|N : S_1^n \to S_p^n\|_d]_{p=1}.
\]

\[\square\]

In particular, in this case we have an easy relation between the (unlimited) entanglement-assisted classical capacity of a quantum channel, $C_E(N)$, and the $cb$-min entropy of $N$ introduced in [8]:

\[
C_{CB,min}(N) := -\frac{d}{dp}[\|N : S_1^n \to S_p^n\|_{cb}]_{p=1}.
\]

We obtain that for every covariant quantum channel the equality

\[
C_E(N) = \ln n - C_{CB,min}(N)
\]

holds. As promised, we finish this section by proving Theorem 1.3.

Proof of Theorem 1.3. We will consider the following two channels. First,

\[
N_1 = \idd_{\ell_1^n} \circ P : S_1^n \to \ell_1^n \to \ell_1^n \hookrightarrow S_1^n.
\]

That is, the classical identity regarded as a quantum channel. Using Theorem 1.2 we easily deduce that

\[
C^d(N_1) = \ln n
\]

for every $1 \leq d \leq n$.

On the other hand, we will consider the depolarizing channel $N_2^\lambda : S_1^n \to S_1^n$ defined by

\[
N_2^\lambda = D_\lambda = \lambda \idd + (1 - \lambda)tr(\cdot)\frac{1}{n}M_n.
\]

We will use the estimate

\[
C^d(N_2^\lambda) \simeq \lambda \ln(nd),
\]

where we use $\simeq$ to denote equality up to universal additive constants independent of $n$ and $d$. This estimate is a particular case of the more general study in ([16]).
Let us take $\lambda = \frac{2}{3} \in (0, 1)$ and $1 < d = \sqrt{n} < n$ so that

$$C^d(N_2^\lambda) \simeq \ln n,$$

and let us remove the dependence of $\lambda$ from $N_2$.

Note that

$$C^d(N_1 \otimes N_2) \geq C^1(N_1) + C^d(N_2) \simeq \ln n + \lambda \ln(nd^2)$$

$$= \ln n + \lambda \ln d + \lambda \ln(nd) \simeq \ln n + C^d(N_1) + C^d(N_2).$$

Here, the first inequality follows from the fact that if we are using entanglement dimension $d^2$ the capacity is greater or equal than the capacity given by the specific protocol in which Alice and Bob use all the entanglement in the second channel and they use independently the first channels without using any entanglement. Replacing the values of $\lambda$ and $d$ we obtain

$$C^n(N_1 \otimes N_2) \geq C^1(N_1) + C^n(N_2) \simeq \frac{1}{3} \ln n + C^{\sqrt{n}}(N_1) + C^{\sqrt{n}}(N_2).$$

Since $N_2 : S_1^n \rightarrow S_1^n$ is a covariant quantum channel we can apply Lemma 4.1 to conclude that

$$\pi_{q,d}(N_2^*) = n^\frac{1}{q} \|N_2^* : S_q^n \rightarrow M_n\|_d$$

for every $q \geq 1$ and $d$. Therefore,

$$C^d(N_2) = \lim_{q \rightarrow \infty} q \ln \pi_{q,d}(N_2^*) = \ln n + \lim_{q \rightarrow \infty} q \ln \|N_2^* : S_q^n \rightarrow M_n\|_d$$

for every $d$. Since we know that $C^{\sqrt{n}}(N_2) \simeq \ln n$, we deduce that

$$\lim_{q \rightarrow \infty} q \ln \|N_2^* : S_q^n \rightarrow M_n\|_{\sqrt{n}} \simeq 0. \tag{4.1}$$

Let

$$N' : S_1^n \oplus_1 S_1^n \rightarrow S_1^n$$

be the quantum channel defined via $N'^* : M_n \rightarrow M_n \oplus_\infty M_n$ such that $N'^*(A) = N_1^*(A) \oplus N_2^*(A)$. Then, we know that for every $q$ and $d$,

$$\pi_{q,d}(N'^*) \leq n^\frac{1}{q} \|N'^* : S_q^n \rightarrow M_n \oplus_\infty M_n\|_d = n^\frac{1}{q} \max \left\{\|N_1^* : S_q^n \rightarrow M_n\|_d, \|N_2^* : S_q^n \rightarrow M_n\|_d\right\}.$$

Therefore,

$$C^{\sqrt{n}}(N') = \lim_{q \rightarrow \infty} q \ln \pi_{q,\sqrt{n}}(N'^*) \leq \lim_{q \rightarrow \infty} q \ln n^\frac{1}{q} \max \left\{\|N_1^* : S_q^n \rightarrow M_n\|_{\sqrt{n}}, \|N_2^* : S_q^n \rightarrow M_n\|_{\sqrt{n}}\right\}$$
\[ = \ln n + \lim_{q \to \infty} q \ln \max \{ 1, \| N_2^n : S_q^n \to M_n \|_{\sqrt{n}} \} \simeq \ln n, \]

where we have used Equation (4.1) in the last equality. Actually, we have

\[ C\sqrt{n}(\mathcal{N}) \simeq \ln n \]

since the converse inequality is very easy.

Finally, if we consider \( \mathcal{N} \otimes \mathcal{N} : (S_1^n \oplus 1 S_1^n) \otimes (S_1^n \oplus 1 S_1^n) \to S_1^n \otimes S_1^n \) we see that this channel extends \( \mathcal{N}_1 \otimes \mathcal{N}_2 \). Therefore, we have

\[ C^n(\mathcal{N} \otimes \mathcal{N}) \geq C^n(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq \frac{1}{3} \ln n + C\sqrt{n}(\mathcal{N}_1) + C\sqrt{n}(\mathcal{N}_2) \simeq \frac{1}{3} \ln n + 2C\sqrt{n}(\mathcal{N}). \]

This concludes the proof. \( \square \)

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5. Appendix: Physical interpretation of the restricted capacities of quantum channels

In the first part of this section we will explain the notion of classical capacity of a quantum channel in more detail. In particular, we will state Holevo-Schumacher-Westmoreland’s Theorem and Bennett, Shor, Smolin, Thapliyal’s Theorem. In the second part of the section we will prove Theorem 1.1 and we will discuss the connections between the $d$-restricted classical capacity of a quantum channel and the capacity considered in [29].

5.1. Physical interpretation of the restricted classical capacities of quantum channels. As we said in Section 1, given a quantum channel $\mathcal{N} : S_1^n \to S_1^n$, the $d$-restricted classical capacities of the channel can be defined within the following common ratio-expression:

\begin{equation}
\lim_{\epsilon \to 0} \limsup_{k \to \infty} \left\{ \frac{m}{k} : \exists A, \exists B \text{ such that } \|id_{\ell^2_{1}} - B \circ \mathcal{N}^{\otimes k} \circ A\| < \epsilon \right\}. \tag{5.1}
\end{equation}

Let us assume $d = 1$, so that Alice and Bob are not allowed to use entanglement in their protocol to encode-transmit-decode information. Then, $\mathcal{A} : \ell^2_{1} \to \otimes^k S_1^n$ will be a quantum channel representing Alice’s encoder from classical information to quantum information. On the other hand, Bob will decode the information he receives from Alice via the $k$ times uses of the channel, $\mathcal{N}^{\otimes k}$, by means of a quantum channel $B : \otimes^k S_1^n \to \ell^2_{1}$. The key point here is that they want this composition to be asymptotically close to the identity map. That is, they want to get $\|id - B \circ \mathcal{N}^{\otimes k} \circ \mathcal{A}\| < \epsilon$.

The case in which Alice and Bob are allowed to share an entangled state is a little bit more difficult to describe. Let us assume that they share a $d$-dimensional state $\rho \in S_1^d \otimes S_1^d (= S_1(H_A) \otimes S_1(H_B))$ in the protocol. Then, a general encoder for Alice will be described by the channel:

$\mathcal{A} = (\mathcal{M}_A \otimes 1_{M_d}) \circ i : \ell^2_{1} \to \ell^2_{1} \otimes (S_1^d \otimes S_1^d) \to \otimes^k S_1^n \otimes S_1^d$,  

where $i : \ell^2_{1} \to \ell^2_{1} \otimes (S_1^d \otimes S_1^d)$ is the map defined by $i(x) = x \otimes \rho$ for every $x \in \ell^2_{1}$ and $\mathcal{M}_A : \ell^2_{1} \otimes S_1^d \to \otimes^k S_1^n$ is a quantum channel. Since Alice has not access to Bob’s part of $\rho$, the state received by Bob will be of the form $((\mathcal{N}^{\otimes k} \otimes 1_{M_d}) \circ \mathcal{A})(x)$, where $x$ is the message that Alice wants to transmit. Finally, Bob’s decoder will be a quantum channel $B : \otimes^k S_1^n \otimes S_1^d \to \ell^2_{1}$.  

As in the previous case, the goal is to have \( \|id - B \circ (N \otimes k \otimes 1_{M_d}) \circ A\| < \epsilon \).

The following diagram represents the two previous situations:

Note that, since the definition of \( C_d(N) \) refers to dimension \( d \) per channel use, we should consider a state \( \rho \) of dimension \( d^k \) if we are using \( k \) times the channel in our protocol.

Motivated by the noisy channel coding theorem (1.1), one could try to obtain a similar result for the \( d \)-restricted capacities. However, the situation is more difficult in the quantum case. A first approach to the problem is based on restricting the kinds of protocols that Alice and Bob can perform. Holevo-Schumacher-Westmoreland’s theorem (HSW Theorem) gives a nice formula when Alice (the sender) is not allowed to distribute one entangled state among more than one channel use. That is, Alice encodes her messages into product states: \( \rho_1 \otimes \cdots \otimes \rho_k \in \otimes^k S^n_1 \) (see [14], [27]).

**Theorem 5.1** (HSW). Given a quantum channel \( N : S^n_1 \to S^n_1 \), we define

\[
C^1(N) := \sup \left\{ S\left( \frac{1}{N} \sum_{i=1}^{N} \lambda_i \rho_i \right) - \sum_{i=1}^{N} \lambda_i S(N(\rho_i)) \right\},
\]

where the sup is taken over all \( N \in \mathbb{N} \), all probability distributions \( (\lambda_i)_{i=1}^{N} \) and all states \( \rho_i \in S^n_1 \) for all \( i = 1, \cdots, N \). Then, \( C^1(N) \) is the classical capacity of the channel when the sender is not allowed to distribute one entangled state among more than one channel use.

When we compute a capacity imposing this restriction we usually talk about **product state capacity of** \( N \). Product state classical capacity, \( C^1 \), is also called **Holevo capacity** and denoted by \( \chi(N) \). It is very easy to see that the classical capacity of a quantum channel \( N \) is the regularization version of \( \chi(N) \):

\[
C_c(N) = \chi(N)_{\text{reg}} := \sup_k \frac{\chi(N \otimes k)}{k}.
\]
It follows by definition that $\chi(\mathcal{N}) \leq C_c(\mathcal{N})$ and it was a major question in QIT for a long time whether $C_c(\mathcal{N}) = \chi(\mathcal{N})$ for every quantum channel $\mathcal{N}$. Finally, Hastings solved the problem showing that both capacities are not the same for certain channels ([13]). We refer ([6], [11], [1]) for a more complete explanation of the problem and some open questions in the area. Hastings’ result says that we do need to consider the regularization (5.2) to compute the classical capacity. This fact encourages us to understand the different capacities of a channel as a regularization of the product state version of the capacity (rather than dealing directly with a general definition of the form (5.1)). Actually, one can check that the product state version of the classical capacity of a classical channel is given by the formula in (1.1). The reason to have the same expression for the general capacity $C_c$ is that such an expression is additive on classical channels: $C_c(\mathcal{N}_1 \otimes \mathcal{N}_2) = C_c(\mathcal{N}_1) + C_c(\mathcal{N}_2)$.

One could argue that the form of $C^1(\mathcal{N})$ given in Theorem 5.1 does not look so much like an analogue formula to (1.1). Recall that, if we denote by $H(X)$ the Shannon entropy of a random variable $X$, the mutual information of two random variables $X, Y$ is defined as $H(X : Y) = H(X) + X(Y) - H(X,Y)$. Since in the quantum setting Shannon entropy is replaced by the von Neumann entropy of a quantum channel $S(\rho) \equiv -\text{tr}(\rho \log_2 \rho)$, the quantum generalization of the mutual information for a bipartite mixed state $\rho^{AB} \in S_1^A \otimes S_1^B$, which reduces to the classical mutual information when $\rho^{AB}$ is diagonal in the product basis of the two subsystems, is

$$S(\rho^A) + S(\rho^B) - S(\rho^{AB}),$$

where $\rho^A = \text{tr}_B \rho^{AB}$ and $\rho^B = \text{tr}_A \rho^{AB}$. Thus, the expression

$$\max_{\rho \in S_1(H_A) \otimes S_1(H_B)} \left\{ S(\rho^B) + S(\mathcal{N}(\rho^A)) - S(\mathcal{N} \otimes 1_{B(H_B)})(\rho) \right\}$$

is a natural generalization of the classical channel’s maximal input:output mutual information (1.1) to the quantum case and it is equal to the classical capacity whenever $\mathcal{N}$ is a classical channel. However, it was shown in [5] that this amount exactly describes the entangled-assisted classical capacity $C_E(\mathcal{N})$.

**Theorem 5.2** (Bennett, Shor, Smolin, Thapliyal. [5]). For a noisy quantum channel $\mathcal{N}$ the (unlimited) entanglement-assisted classical capacity is given by the expression:

$$C_E(\mathcal{N}) = \max_{\rho \in S_1^A \otimes S_1^A} \left\{ S\left((\text{tr}_M \otimes 1_{M_n})(\rho)\right) + S\left((1_{M_n} \otimes \text{tr}_M)(\rho)\right) - S\left((\mathcal{N} \otimes 1_{M_n})(\rho)\right) \right\}.$$
There is a crucial difference between Theorem 5.1 and Theorem 5.2. While in the first case the authors described the product state classical capacity, the last theorem describes the general (unlimited) entanglement-assisted classical capacity. Actually, it can be seen that the expression $C_E(N)$ in Theorem 5.2 describes the product state version of the capacity but, furthermore, it is additive on quantum channels. So, no regularization is required in this case.

In [29] the author studied the classical capacity of a quantum channel with restricted assisted entanglement (which involves, in particular, the two previous capacities). However, the restriction studied there is as a function of the entropy of entanglement per channel use (see [29, Theorem 1]) while in our work the restriction is imposed in the dimension of the entanglement dimension per channel use. It can be seen that both restrictions give rise to the same capacity. However, such a capacity is defined in both works via the regularization of a product state version of the capacity and it is this last quantity which is characterized. As we will show later, our characterization of the product state version of the $d$-restricted classical capacity is analogous to the one given in [29], but we can not assure that they are equal.

5.2. **Proof of Theorem 1.1 and some additional comments.** To prove Theorem 1.1 we will essentially mimic the argument in [29]. However, we find interesting for the non familiar reader writing the steps of the protocol in detail. On the other hand, we will give just a brief sketch of the proof. The proof consists of two parts. In the first one, we show that the expression (1.3) is an upper bound for the capacity described by conditions a) and b) in Theorem 1.1.

**Proof of the upper bound in Theorem 1.1.** Let us assume that we use $k$ times the channel in our protocol. Then, by assumption a) Alice and Bob are allowed to share $k$ pure entangled states, each of dimension $d$: $\eta_1, \ldots, \eta_k \in S^d_1 \otimes S^d_1$. In other words, they start the protocol by sharing the state $\eta = \eta_1 \otimes \cdots \otimes \eta_k \in S^{kd}_1 \otimes S^{kd}_1$. The most general thing Alice can do is to apply a global channel $\phi_x : S^{dk}_1 \to S^{nk}_1$ on her half of $\eta \in S^{dk}_1 \otimes S^{dk}_1$ to obtain $(id_{M_{dk}} \otimes \phi_x)(\eta) \in S^{dk}_1 \otimes S^{nk}_1$, where $\phi_x$ depends on the classical data $x = 1, \cdots, D$ she wants to send. Then, she sends her half of $\phi_x(\eta)$ through the channel $N \otimes k$ formed by the tensor product of $k$ uses of the channel $N$. However, by assumption b) we can assume that $\phi_x = \phi^x_1 \otimes \cdots \otimes \phi^x_k$ for every $x = 1, \cdots, D$, where $\phi^x_i : S^d_1 \to S^n_1$ is a quantum channel for every $i = 1, \cdots, k$. Thus, after the action of Alice, Bob will receive the ensemble
\( \{(p_x^{D_{x=1}}, \rho_x^{D_{x=1}}) \} \), where \( \rho_x = (1_{M_d} \otimes \mathcal{N} \circ \phi_i^x)(\eta_1) \otimes \cdots \otimes (1_{M_d} \otimes \mathcal{N} \circ \phi_i^x)(\eta_k) \) for every \( x = 1, \cdots, D \). According to Holevo-Schumacher-Westmoreland’s Theorem the classical capacity obtainable by this ensemble is given by

\[
S\left( \sum_x p_x \rho_x \right) - \sum_x p_x S(\rho_x).
\]

As a consequence of the subadditivity of the von Neumann entropy, the previous expression is upper bounded by

\[
S\left( (1_{M_d} \otimes \text{tr}_{M_{n_k}}) \sum_x p_x \rho_x \right) + S\left( (\text{tr}_{M_{n_k}} \otimes 1_{M_{n_k}}) \sum_x p_x \rho_x \right) - \sum_x p_x S(\rho_x).
\]

Now, using that \( (1_{M_d} \otimes \text{tr}_{M_{n_k}})(\rho_x) \) does not depend on \( x \), using the concavity and subadditivity of the von Neumann entropy and noting that

\[
\sum_x p_x S(\rho_x) = \sum_x p_x \sum_{i=1}^k S\left( (1_{M_d} \otimes \mathcal{N} \circ \phi_i^x)(\eta_i) \right),
\]

we deduce that the capacity obtainable with this protocol, \( \frac{1}{k} \left( S\left( \sum_x p_x \rho_x \right) - \sum_x p_x S(\rho_x) \right) \), is upper bounded by \( C^d(\mathcal{N}) \), considering the ensemble \( \{(p_x^{1\leq k_{x=1}}, (1_{M_d} \otimes \mathcal{N} \circ \phi_i^x)(\eta_i^{1\leq k_{x=1}}) \} \).

The second part of the proof of Theorem 1.1 consists of showing that, given an ensemble \( \{(\lambda_i)_{i=1}^N, (\eta_i)_{i=1}^N, (\phi_i)_{i=1}^N \} \) optimizing \( C^d(\mathcal{N}) \), we can define a protocol verifying conditions a) and b) which asymptotically achieves the capacity \( C^d(\mathcal{N}) \). The protocol is the same as in [29] up to slight modifications to adapt it to our case. It consists of the use of \( k \) times the channel and it will use \( k \) states distributed according to \( ( (\lambda_i)_{i=1}^N, (\eta_i)_{i=1}^N ) \).

**Proof of the lower bound in Theorem 1.1.** Since \( \eta_k \in S_1^d \otimes S_1^d \) is a pure state for every \( i = 1, \cdots, N \), we can write \( \eta_k = |\alpha_i\rangle \langle \alpha_i| \) for certain unit elements \( |\alpha_i| \) in \( \ell_2^d \otimes \ell_2^d \). Moreover, according to the singular value decomposition we can write \( |\alpha_i\rangle = \sum_{j=1}^d \alpha_i^j |v_{i,j}\rangle \) for certain orthonormal basis \( \{|v_{i,j}\rangle\}_{j=1}^d \) of \( \ell_2^d \) and positive elements \( (\alpha_i^j)^2 \) verifying \( \sum_{j=1}^d (\alpha_i^j)^2 = 1 \). Now, given \( i = 1, \cdots, N \) and \( P \subseteq \{2, \cdots, d \} \) we will denote by

---

\(^7\)To simplify notation we are assuming that the \( |\alpha_i| \)’s are positive operators. The general case follows straightforward with slight modifications of the unitaries in the protocol.
$U^P_i : \ell^d_2 \to \ell^d_2$ the unitary operator defined by

$$U^P_i |v_{i,j}\rangle = \begin{cases} -|v_{i,j}\rangle & \text{if } j \in P \\ |v_{i,j}\rangle & \text{if } j \in \{2, \cdots, d\} \setminus P. \end{cases}$$

Note that $\{U^P_i\}_{P \subseteq \{2, \cdots, d\}}$ consists of $2^{d-1}$ different unitaries for every $i = 1, \cdots, N$. It is easy to check that

$$\sum_{P \subseteq \{2, \cdots, d\}} (1_{M_d} \otimes U^P_i) \eta_1 (1_{M_d} \otimes (U^P_i)^*) = \sum_{j=1}^d (\alpha_j^2)|v_{i,j}\rangle \langle v_{i,j}| \otimes |v_{i,j}\rangle \langle v_{i,j}|$$

for every $i = 1, \cdots, N$. That is, these unitaries “disentangle” Alice and Bob’s entangled states $\eta_i$.

The protocol starts with Alice and Bob sharing $k$ pure states in $S^d_1 \otimes S^d_1$ according to the distribution defined by $((\lambda_i)_{i=1}^N, (\eta_i)_{i=1}^N)$. That is, they share the state $\delta_1 \otimes \cdots \otimes \delta_k \in S^d_1 \otimes S^d_1$ such that the first $k_1$ $\delta_i$‘s are equal to $\eta_1$, the following $k_2$ $\delta_i$‘s are equal to $\eta_2$ and so on; where $k_i \approx k\lambda_i$ for every $i = 1, \cdots, N$. We just consider this order to simplify notation. We will see that the protocol considers all possible permutations so the only relevant thing is the numbers $k_i$. The protocol uses $k!2^{(d-1)k}$ signal states each of them with probability $\frac{1}{k!2^{(d-1)k}}$. More precisely, for any $P_1 \subseteq \{2, \cdots, d\}, \cdots, P_k \subseteq \{2, \cdots, d\}$ and any permutation $\pi : [k] \to [k]$ Alice applies the channel $\Psi_{P_1, \cdots, P_k, \pi} : S^{d^k}_1 \to S^{d^k}_1$ on her part of $\eta$ (with probability $\frac{1}{k!2^{(d-1)k}}$), where $\Psi_{P_1, \cdots, P_k, \pi}$ is defined as follows. First, consider the channel $A_{P_1, \cdots, P_k} : S^{d^k}_1 \to S^{d^k}_1$ defined by

$$A_{P_1, \cdots, P_k} = \vartheta_{U_1^{P_1}} \otimes \cdots \otimes \vartheta_{U_1^{P_{k_1}}} \otimes \cdots \otimes \vartheta_{U_N^{P_{k_{k-1}+1}}} \otimes \cdots \otimes \vartheta_{U_N^{P_{k}}}.$$ 

We denote $\vartheta_{U^P_j}$ the channel defined by the unitary $U^P_j$, which depends on $\eta_i$ (via the base $(|v_{i,j}\rangle)_{j=1}^d$ we consider). Next, consider the channel $\Theta : S^{d^k}_1 \to S^{d^k}_1$ given by

$$\Theta = \phi_1 \otimes \cdots \otimes \phi_1 \otimes \cdots \otimes \phi_N \otimes \cdots \otimes \phi_N,$$

where the $\phi_i$’s are the channels appearing in the ensemble $(\lambda_i)_{i=1}^N, (\eta_i)_{i=1}^N, (\phi_i)_{i=1}^N$ optimizing $C^d(N)$. Finally, we consider the channel

$$\Lambda_{\pi} : S^{d^k}_1 \to S^{d^k}_1.$$


defined on product states by \( \Lambda_\pi(\rho_1 \otimes \cdots \otimes \rho_k) = \rho_{\pi(1)} \otimes \cdots \otimes \rho_{\pi(k)} \). Then,

\[ \Psi_{P_1, \ldots, P_k, \pi} = \Lambda_\pi \circ \Theta \circ A_{P_1, \ldots, P_k}. \]

Once Alice applies the channel \( \Psi_{P_1, \ldots, P_k, \pi} \) with probability \( \frac{1}{k! 2^{(d-1)k}} \) to her part of \( \eta \), she sends the obtained state through the use of \( k \) times the channel \( \mathcal{N} \otimes k \). Therefore, Bob will obtain the ensemble

\[ \{ \left( \frac{1}{k! 2^{(d-1)k}} \right)^{-1} P_1, \ldots, P_k, \pi \} \left( (1_{S_1^d} \otimes \mathcal{N} \otimes k \circ \Psi_{P_1, \ldots, P_k, \pi})(\eta) \right)_{P_1, \ldots, P_k, \pi} \}. \]

As a consequence of HSW theorem, and following the same ideas as in [29] one can conclude that the considered protocol, which verifies conditions a) and b) in Theorem 1.1, achieves asymptotically the capacity \( C^d(\mathcal{N}) \). This concludes the proof. \( \square \)

Expression (1.3) is not completely analogous to the one in [29]. This is because (1.3) is expressed in the “tensor product form”. However, one can show the following result.

**Theorem 5.3.** Given a quantum channel \( \mathcal{N} : S_1^n \rightarrow S_1^n \), we have

\[
C^d(\mathcal{N}) = \sup \left\{ S \left( \sum_{i=1}^N \lambda_i (\mathcal{N} \circ \phi_i)(\delta_i) \right) + \sum_{i=1}^N \lambda_i \left[ S(\delta_i) - S \left( (1_{S_1^d} \otimes (\mathcal{N} \circ \phi_i))(\chi_{\delta_i}) \right) \right] \right\}.
\]

Here, the supremum runs over all \( N \in \mathbb{N} \), all probability distributions \((\lambda_i)_{i=1}^N\), all states \( \delta_i \in S_1^d \) and all quantum channels \( \phi_i : S_1^d \rightarrow S_1^n \) for every \( i = 1, \ldots, N \). \( \chi_{\delta_i} \) denotes any purification of \( \delta_i \) for every \( i = 1, \ldots, N \).

This Theorem gives us an expression of \( C^d(\mathcal{N}) \) completely analogous to the expression in [29] Theorem 1]. The expressions do not coincide since the restriction considered by Shor is as a function of the entropy of entanglement per channel use instead of as a function of the entanglement dimension. Note that if we call \( R^d(\mathcal{N}) \) the capacity described in [29] Theorem 1], we trivially have \( C^d(\mathcal{N}) \leq R^d(\mathcal{N}) \). Furthermore, following the spirit of [29] we may consider the same restricted capacity as a function of the number of singlets per channel use, \( E^d \), and we would trivially obtain \( E^d(\mathcal{N}) \leq C^d(\mathcal{N}) \). In order to obtain the general capacities (rather than the product state version) one has to consider the regularization. It is not difficult to see that in this case the regularization is given by

\[
C^d(\mathcal{N})_{\text{reg}} = \sup_k \frac{C^{\otimes k}(\mathcal{N})}{k}
\]
and analogously for $R^d$ and $E^d$. However, using that entanglement is an interconvertible resource (see [20]) one can conclude that

$$E^{\log_2 d}(\mathcal{N})_{\text{reg}} = C^d(\mathcal{N})_{\text{reg}} = R^{\log_2 d}(\mathcal{N})_{\text{reg}}.$$ 

Therefore, the three capacities $E^d$, $C^d$ and $R^d$ represent a product state version of the same capacity. Theorem 1.3 tells us that we do need to consider this regularization since $C^d(\mathcal{N})_{\text{reg}}$ can be very different from $C^d(\mathcal{N})$. Actually, Theorem 1.3 can be proved exactly in the same way for $R^{\log_2 d}(\mathcal{N})$ and $E^{\log_2 d}(\mathcal{N})$. The proof of Theorem 5.3 is based on the following proposition.

**Proposition 5.4.** Let $\xi \in S_1^m \otimes S_1^d$ be a state and let $\gamma \in S_1^k \otimes S_1^d$ be a purification of $\eta = (\text{tr}_{M_k} \otimes \text{id}_{M_d})(\xi)$. Then, there is a completely positive trace preserving map $\Phi : S_1^k \to S_1^m$ such that

$$(\Phi \otimes \text{id}_{M_d})(\gamma) = \xi.$$ 

We will not prove Proposition 5.4 since it can be obtained by using standard techniques. With this at hand, we will easily prove Theorem 5.3.

**Proof of Theorem 5.3.**

Inequality $\geq$ is very easy. Indeed, given the ensemble $\{((\lambda_i)^N_{i=1}, (\delta_i)^N_{i=1}, (\phi_i)^N_{i=1})\}$ optimizing the right hand side in Equation (5.4), it suffices to consider $\{((\lambda_i)^N_{i=1}, (\chi_i)^N_{i=1}, (\phi_i)^N_{i=1})\}$ in the expression (5.3) of $C^d$.

To see the converse inequality consider the element $\{((\lambda_i)^N_{i=1}, (\eta_i)^N_{i=1}, (\phi_i)^N_{i=1})\}$ which maximizes the expression (5.3) and let $\delta_i = (1_{M_d} \otimes \text{tr}_{M_d})(\eta_i) \in S_1^d$ for every $i = 1, \ldots, N$. According to Proposition 5.4 there exist quantum channels $\psi_i : S_1^d \to S_1^d$ such that $(1_{M_d} \otimes \psi_i)(\chi_{\delta_i}) = \eta_i$ for every $i = 1, \ldots, N$. Then, using the ensemble $\{((\lambda_i)^N_{i=1}, (\delta_i)^N_{i=1}, (\phi_i \circ \psi_i)^N_{i=1})\}$ we can deduce that the right hand side in (5.4) is greater or equal $C^d(\mathcal{N})$. Indeed, note that the first two terms can be written respectively as

$$S\left(\sum_{i=1}^N \lambda_i (\mathcal{N} \circ \phi_i \circ \psi_i)(\delta_i)\right) = S\left(\sum_{i=1}^N \lambda_i (\mathcal{N} \circ \phi_i)((\text{tr}_{M_d} \otimes 1_{M_d})(\eta_i))\right)$$ 

and

$$\sum_{i=1}^N \lambda_i S((1_{M_d} \otimes \mathcal{N} \circ \phi_i \circ \psi_i)(\chi_{\delta_i})) = \sum_{i=1}^N \lambda_i S((1_{M_d} \otimes \mathcal{N} \circ \phi_i)(\eta_i)).$$
while the last term is
\[
\sum_{i=1}^{N} \lambda_i S(\delta_i) = \sum_{i=1}^{N} \lambda_i S((1_{M_d} \otimes tr_{M_d})(\eta_i)).
\]
\[
\square
\]

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