A NOTE ON THE BRAUER GROUP AND THE BRAUER-MANIN SET OF A
PRODUCT

CHANG LV

Abstract. We generalize the results of Skorobogatov and Zarhin considering the commutativity of Brauer groups (and Brauer-Manin sets) with taking product of two varieties, by relaxing the condition that varieties are projective.

1. Introduction

Let $X$ be a variety over a number field $k$, $\text{Br} X = H^2_{\text{ét}}(X, \mathbb{G}_m)$ the cohomological Brauer-Grothendieck group \cite{colliot-thelene2000, skorobogatov2001, skorobogatov2002}, $X(\mathbb{A}_k)^{\text{Br} X}$ the Brauer-Manin set \cite{skorobogatov2002}. Consider the product of two varieties $X \times Y$, and it is natural to ask about the commutativity of Brauer groups (and Brauer-Manin sets) with taking product.

Skorobogatov and Zarhin \cite{skorobogatov2002} investigated the cokernel of the natural map $\text{Br} X \oplus \text{Br} Y \to \text{Br}(X \times Y)$ and the relation between $X(\mathbb{A}_k)^{\text{Br} X} \times Y(\mathbb{A}_k)^{\text{Br} Y}$ and $(X \times Y)(\mathbb{A}_k)^{\text{Br}(X \times Y)}$, namely, if $X$ and $Y$ are smooth, projective, geometrically integral, then $\text{coker}(\text{Br} X \oplus \text{Br} Y \to \text{Br}(X \times Y))$ is finite and $X(\mathbb{A}_k)^{\text{Br} X} \times Y(\mathbb{A}_k)^{\text{Br} Y} = (X \times Y)(\mathbb{A}_k)^{\text{Br}(X \times Y)}$.

The aim of this note is to relax the projectivity constraint on $X$ and $Y$. Let $k$ be a field finitely generated over $\mathbb{Q}$, with fixed separable closure $\overline{k}$ and $\mathfrak{g}_k = \text{Gal}(\overline{k}/k)$. Let $X$ and $Y$ be smooth quasi-projective geometrically integral varieties over $k$ whose base changes to $\overline{k}$ are denoted by $\overline{X}$ and $\overline{Y}$. Then in Proposition 2.2 we show that $\text{coker}((\text{Br} \overline{X})^{\mathfrak{g}_k} \oplus (\text{Br} \overline{Y})^{\mathfrak{g}_k} \to \text{Br}(\overline{X} \times \overline{Y}))$ is finite, which generalizes \cite[Thm. A]{skorobogatov2002}.

Assume that $(X \times Y)(k) \neq \emptyset$ or $H^3(k, \overline{k}^*) = 0$. Then in Theorem 2.13 we show that $\text{coker}(\text{Br} X \oplus \text{Br} Y \to \text{Br}(X \times Y))$ is finite, generalizing the first main result Thm. B of \cite{skorobogatov2002}.

If $k$ is a number field and $X$ and $Y$ are smooth geometrically integral varieties over $k$, then in Theorem 3.1 we show that $(X \times Y)(\mathbb{A}_k)^{\text{Br}(X \times Y)} = X(\mathbb{A}_k)^{\text{Br} X} \times Y(\mathbb{A}_k)^{\text{Br} Y}$, generalizing the second main result Thm. C of \cite{skorobogatov2002}. As mentioned in \cite{skorobogatov2002}, by using étale homotopy of Artin and Mazur, Harpaz and Schlank \cite[Cor. 1.3]{harpaz2014} proved a statement similar to Theorem 3.1 where the Brauer-Manin set is replaced by the étale Brauer-Manin set and varieties do not need to be proper.

For Theorem 2.13 the idea of proof is to compare the various cohomological groups of varieties with the ones of their smooth compactifications, which uses a result of by Colliot-Thélène and Skorobogatov \cite{colliot-thelene2000}, implying that $(\text{Br} \overline{X})^{\mathfrak{g}_k} \to (\text{Br} \overline{X})^{\mathfrak{g}_k}$ has finite cokernel, where $\overline{X}$ is a smooth compactification of $X$. Since the compactifications are projective, one can then make use of the original result of \cite{skorobogatov2002}. The proof of Theorem 3.1 relies on the original results and the generalizable construction \cite{skorobogatov2002} of a certain homomorphism using universal torsors of $n$-torsion, and a modified...
version (Lemma 3.7) of a lemma of Cao [11] considering the relation between \( H^2_{\text{ét}}(X, \mu_n) \oplus H^2_{\text{ét}}(Y, \mu_n) \) and \( H^2_{\text{ét}}(X \times Y, \mu_n) \).

Notations. Let \( k \) be a field of characteristic zero and fix a separable closure \( \overline{k} \). Let \( g_k = \text{Gal}(\overline{k}/k) \).

A \( k \)-variety \( X \) is a separated \( k \)-scheme of finite type. For any field \( K \) containing \( k \), denote by \( X_K \) the base change \( X \times_{\text{Spec} \, k} \text{Spec} \, K \). We also write \( \overline{X} \) for \( X_{\overline{k}} \). For two varieties \( X \) and \( Y \), we write \( X \times Y \) (resp. \( \overline{X} \times \overline{Y} \)) for \( X \times_{\text{Spec} \, k} \text{Spec} \, Y \) (resp. \( \overline{X} \times_{\text{Spec} \, \overline{k}} \overline{Y} \)). Denote by \( p_X \) and \( p_Y \) the two projections from \( X \times Y \) to \( X \) and \( Y \).

Since \( k \) is of characteristic zero, if \( X \) is smooth, by Hironaka’s theorem, the smooth compactification of \( X \) exists. Let \( \overline{X} \) be a smooth compactification of \( \overline{X} \). Then we have a dense open immersion \( X \hookrightarrow \overline{X} \). If \( X \) is also quasi-projective, \( \overline{X} \) can be made projective.

Denote by \( S(X) \), \( Ab \) and \( g_k \text{-Mod} \) for the categories of étale sheaves on \( X \), abelian groups and discrete \( g_k \)-modules. Let \( D^+(X) \) and \( D^+(Ab) \) be the corresponding derived categories of complexes bounded below. Since \( S(X) \) has enough injectives, the right derived functor \( RF : D^+(X) \rightarrow D^+(Ab) \) of any additive functor \( F : S(X) \rightarrow Ab \) exists. We write \( R^iF = H^iRF \) for the \( i \)-th hypercohomology functor. Let \( H(X, -) = R\Gamma(X_{\text{ét}}, -) \) and \( H^i(X, -) = R\Gamma(X_{\text{ét}}, -) \), where \( X_{\text{ét}} \) is the small étale site over \( X \). Let \( S(k) = S(\text{Spec} \, k) \cong g_k \text{-Mod} \). Define \( D^+(k) = D^+(\text{Spec} \, k) \).

If \( n \) is a positive number and \( A \) an abelian group, write \( A_n \) and \( A/n \) for the kernel and cokernel of the homomorphism \( A \overset{n} \rightarrow\ A \).

2. The finiteness of the cokernel of the Brauer groups

For a contravariant functor \( F \) from the category of \( k \)-varieties to \( Ab \), let \( X \) and \( Y \) be two \( k \)-varieties with compactifications \( \overline{X} \) and \( \overline{Y} \). Define

\[
\text{coker}(F) = \text{coker}(F(X) \oplus F(Y) \rightarrow F(X \times Y)),
\]

\[
\text{coker}(F) = \text{coker}(F(\overline{X}) \oplus F(\overline{Y}) \rightarrow F(\overline{X} \times \overline{Y})),
\]

\[
\text{coker}(F) = \text{coker}(F(\overline{X}) \oplus F(\overline{Y}) \rightarrow F(\overline{X} \times \overline{Y})),
\]

where the maps are induced by \( (p_X^*, p_Y^*) \). We use similar notation for ker.

Define \( \text{Br}_1 X \) and \( \text{Br}_2 X \) to be the kernel and cokernel of the natural map \( \text{Br} \, X \rightarrow (\text{Br} \, \overline{X})^{g_k} \).

We first consider \( \text{coker}(\text{Br}^{g_k}) \).

Lemma 2.1. Let \( X \) and \( Y \) be two varieties over \( k \) and \( G \) an étale sheaf defined by a commutative \( k \)-group scheme. Then for \( n \geq 1 \), we have \( \text{ker}(\text{coker}(H^n(-, \mathcal{G}))) = 0 \), that is, the natural map induced by \( (p_X^*, p_Y^*) \)

\[
H^n(\overline{X}, \mathcal{G}) \oplus H^n(\overline{Y}, \mathcal{G}) \rightarrow H^n(\overline{X} \times \overline{Y}, \mathcal{G})
\]

is an injective homomorphism of \( g_k \)-modules.

Proof. See [11] Prop. 1.5 (iv) where varieties are assumed to be projective. One checks that it holds for general varieties. \( \square \)

Proposition 2.2. Let \( k \) be a field finitely generated over \( \mathbb{Q} \). Let \( X \) and \( Y \) be smooth quasi-projective geometrically integral varieties over \( k \). Then the group \( \text{coker}(\text{Br}^{g_k}) \), which is the cokernel of

\[
(\text{Br} \, \overline{X})^{g_k} \oplus (\text{Br} \, \overline{Y})^{g_k} \rightarrow \text{Br}(\overline{X} \times \overline{Y})^{g_k},
\]

is finite.
Proof. Since in the projective case, \( \text{coker}(\text{Br}^g) \) is already finite [11 Thm. A], the result follows from Lemmas 2.3 [1] and 2.4 below, which imply that \( \text{coker}(\text{Br}^g) \) is finite if and only if \( \text{coker}(\text{Br}^g) \) is.

**Lemma 2.3.** Let \( k \) be a field finitely generated over \( \mathbb{Q} \) and \( X \) a smooth quasi-projective geometrically integral variety over \( k \). Then we have

(i) the natural homomorphism \( \alpha(X) : (\text{Br}^g)_{\eta^k} \to (\text{Br}^g)_{\bar{\eta}^k} \) induced by \( \eta \to \bar{\eta} \) is injective and has finite cokernel,

(ii) the natural homomorphism \( \text{Br} X \to (\text{Br}^g)_{\bar{\eta}^k} \) has finite cokernel, i.e., \( \text{Br}_2 X \) is finite.

**Proof.** The second statement of the lemma is [2 Thm. 6.2 (i)], whose proof, based on the proper base change theorem and Weil conjecture, contains the first statement of the lemma (p. 164, l. 4-5, loc. cit.). \( \square \)

**Lemma 2.4.** With notations as Proposition 2.2, let \( \beta : \text{coker}(\text{Br}^g) \to \text{coker}(\text{Br}^g) \) be the natural homomorphism induced by \( \eta \to \bar{\eta} \) and \( \bar{\eta} \to \bar{\eta} \). Then we have the exact sequence

\[
0 \to \ker \beta \to \text{coker}(\text{Br}^g) \oplus \text{coker}(\text{Br}^g) \to \text{coker}(\text{Br}^g) \to 0,
\]

where \( \alpha \) is defined in Lemma 2.3 [1].

**Proof.** By Lemma 2.3 [1], there exists a functorial short exact sequence

\[
(2.5) \quad 0 \to (\text{Br}^g)_{\eta^k} \to (\text{Br}^g)_{\bar{\eta}^k} \to \text{coker}(\text{Br}^g) \to 0.
\]

Next, by Lemma 2.1 we have the functorial short exact sequence

\[
(2.6) \quad 0 \to (\text{Br}^g)_{\eta^k} \oplus (\text{Br}^g)_{\bar{\eta}^k} \to (\text{Br}^g)_{\eta^k} \times (\text{Br}^g)_{\bar{\eta}^k} \to \text{coker}(\text{Br}^g) \to 0,
\]

which fits into the commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \to & (\text{Br}^g)_{\eta^k} & \oplus & (\text{Br}^g)_{\bar{\eta}^k} & \to & (\text{Br}^g)_{\eta^k} \times (\text{Br}^g)_{\bar{\eta}^k} & \to & \text{coker}(\text{Br}^g) & \to & 0 \\
& & \alpha(X) \oplus \alpha(Y) \downarrow & & \alpha(X \times Y) \downarrow & & \beta \downarrow & & \text{coker}(\text{Br}^g) \downarrow & & 0 \\
0 & \to & (\text{Br}^g)_{\eta^k} & \oplus & (\text{Br}^g)_{\bar{\eta}^k} & \to & (\text{Br}^g)_{\eta^k} \times (\text{Br}^g)_{\bar{\eta}^k} & \to & \text{coker}(\text{Br}^g) & \to & 0
\end{array}
\]

The desired exact sequence then follows from (2.5) and the snake lemma. \( \square \)

Next, we investigate \( \text{coker}(\text{Br}_1) \).

**Proposition 2.7.** Let \( k \) be a field finitely generated over \( \mathbb{Q} \). Let \( X \) and \( Y \) be smooth quasi-projective geometrically integral varieties over \( k \). Assume that \( (X \times Y)(k) \neq \emptyset \) or \( H^3(k, \bar{\eta}^X) = 0 \). Then \( \text{coker}(\text{Br}_1) \), the cokernel of the natural map

\[
\text{Br}_1 X \oplus \text{Br}_1 Y \to \text{Br}_1 (X \times Y),
\]

is finite.

For the proof, it is enough to use Lemmas 2.9, 2.10 and 2.12 below.

For an arbitrary \( k \)-variety \( X \), let \( p : X \to \text{Spec} k \) be the structure morphism. Following Harari and Skorobogatov [6], define \( \text{KD}(X) = (\tau_{\leq 1} \mathcal{R}p_{!*} \mathcal{G}_{m,k})[1] \) and

\[
\text{KD}'(X) = \text{cone}(\mathcal{G}_{m,k} \to \tau_{\leq 1} \mathcal{R}p_{!*} \mathcal{G}_{m,k})[1].
\]

Thus we have the distinguished triangle in \( D^+(k) \)

\[
(2.8) \quad \mathcal{G}_{m}[1] \to \text{KD}(X) \to \text{KD}'(X) \to .
\]
Lemma 2.9. Let $k$ be a field of characteristic zero. Let $X$ and $Y$ be two $k$-varieties. Assume that $(X \times Y)(k) \neq \emptyset$ or $H^3(k, \mathbb{R}^\times) = 0$. Then we have a natural isomorphism $\text{coker}(\text{Br}_1) \cong \text{coker}(\mathbb{H}^1(k, \text{KD}'))$.

Proof. By applying $\mathbb{H}(k, -)$ to the distinguished triangle (2.8) and taking cohomology, the exact sequence [6, (8.5)] extends to

$$\text{Br}k \rightarrow \text{Br}_1 X \rightarrow \mathbb{H}^1(k, \text{KD}'(X)) \rightarrow H^3(k, \mathbb{R}^\times) \cong H^2(k, \text{KD}(X)).$$

We have an obvious commutative diagram

$$\begin{array}{ccc}
G_m[1] & \rightarrow & \text{KD}(X) \\
\downarrow & & \downarrow \\
R\mathcal{F}_* G_m[1] & & \\
\end{array}$$

On applying $\mathbb{H}^2(k, -)$ and using $\mathbb{H}(k, R\mathcal{F}_* -) = \mathbb{H}(X, -)$ we obtain

$$H^3(k, \mathbb{R}^\times) \xrightarrow{e} \mathbb{H}^2(k, \text{KD}(X)) \xrightarrow{p^*} H^3(X, G_m)$$

where $p^*$ is an injection by the assumption that $(X \times Y)(k) \neq \emptyset$ or $H^3(k, \mathbb{R}^\times) = 0$ (see the comments after (21) in [11]), and so is the horizontal arrow $e$. Thus by functoriality we have the commutative diagram with exact rows

$$\begin{array}{ccc}
\text{Br}k \oplus \text{Br}k & \rightarrow & \text{Br}_1 X \oplus \text{Br}_1 Y \\
\downarrow & & \downarrow \\
\text{Br}k & \rightarrow & \text{Br}_1 (X \times Y) \\
\end{array}$$

Clearly the left vertical arrow is surjective. Thus the proof is complete by using the snake lemma. □

Lemma 2.10. Let $k$ be a field of characteristic zero. Let $X$ and $Y$ be geometrically integral $k$-varieties. Then we have a distinguished triangle in $D^+(k)$

$$\text{KD}'(X) \oplus \text{KD}'(Y) \rightarrow \text{KD}'(X \times Y) \rightarrow \text{coker}((\text{Pic})) \rightarrow .$$

In particular, we have a natural injection $\text{coker}(\mathbb{H}^1(k, \text{KD}')) \hookrightarrow H^1(k, \text{coker}((\text{Pic})))$.

Proof. Let $U(X) = \text{cone}(G_m \rightarrow p_* G_m)$. We have the obvious distinguished triangles

$$G_m[1] \rightarrow p_* G_m[1] \rightarrow U(X)[1] \rightarrow ,$$

$$p_* G_m[1] \rightarrow \text{KD}(X) \rightarrow (\tau_1) R\mathcal{F}_* G_m[1] \rightarrow ,$$

$$G_m[1] \rightarrow \text{KD}(X) \rightarrow \text{KD}'(X) \rightarrow .$$

Note $(\tau_1) R\mathcal{F}_* G_m[1] = R^1 p_* G_m = \text{Pic} X$. Then by the octahedron axiom of triangulated categories, we have the distinguished triangle

$$U(X)[1] \rightarrow \text{KD}'(X) \rightarrow \text{Pic} X \rightarrow .$$
Thus by functoriality we have the commutative diagram of distinguished triangles

\[
\begin{array}{ccc}
U(X)[1] \oplus U(Y)[1] & \to & KD'(X) \oplus KD'(Y) \to \text{Pic } X \oplus \text{Pic } Y \\
\downarrow & & \downarrow & & \downarrow \\
U(X \times Y)[1] & \to & KD'(X \times Y) \to \text{Pic } (X \times Y) \\
\downarrow & & \downarrow & & \downarrow \\
\text{cone}(g) & \to & \text{coker}(\text{Pic})
\end{array}
\]

Next, by Rosenlicht lemma (c.f. [8, Lem. 6.5 (ii)]), we obtain a natural isomorphism

(2.11) \( U(X \times Y) \cong U(X) \oplus U(Y) \).

Thus the desired distinguished triangle is obtain using the octahedron axiom.

Applying \( H(k, -) \) we have the long exact sequence

\[
\ldots \to \mathbb{H}^i(k, KD'(X)) \oplus \mathbb{H}^i(k, KD'(X)) \to \mathbb{H}^i(k, KD'(X \times Y)) \to \\
H^i(k, \text{coker(Pic)}) \to \mathbb{H}^{i+1}(k, KD'(X)) \oplus \mathbb{H}^{i+1}(k, KD'(X)) \to \ldots.
\]

Taking \( i = 1 \) we obtain the desired injection. The proof is complete. \( \square \)

**Lemma 2.12.** Let \( k \) be a field finitely generated over \( \mathbb{Q} \), and let \( X \) and \( Y \) be smooth quasi-projective geometrically integral varieties over \( k \). Then we have a natural isomorphism of \( g_k \)-modules

\[ \text{coker}(\text{Pic}) \cong \text{coker}(\text{Pic}) \]

induced by \( X \hookrightarrow X' \) and \( Y \hookrightarrow Y' \). In particular, as an abelian group, \( \text{coker}(\text{Pic}) \) is finitely generated and torsion-free, and \( H^1(k, \text{coker}(\text{Pic})) \) is finite.

**Proof.** Let \( \text{Div}_{X \setminus X} X' = \bigoplus_{x \in (X \setminus X)^{(1)}} \mathbb{Z} \) be the group of divisors of \( X' \) outside \( X \), where \( (X' \setminus X)^{(1)} \) is the set of codimension 1-point in \( X' \). Since \( X \) is smooth, we have a natural exact sequence (c.f. [10] (4.18), (4.19), pp. 72)

\[
0 \to U(X) \to \text{Div}_{X \setminus X} X' \to \text{Pic } X' \to \text{Pic } X \to 0
\]

which, by functoriality and Lemma 2.1 fits into the following exact commutative diagram of \( g_k \)-modules

\[
\begin{array}{ccc}
0 & 0 \\
0 & U(X) \oplus U(Y) & \to & \text{Div } X' \setminus X' \oplus \text{Div } Y' \setminus Y' \to \text{Pic } X' \oplus \text{Pic } Y' \to \text{Pic } X \oplus \text{Pic } Y \\
\downarrow & & \downarrow & & \downarrow \\
0 & U(X \times Y) & \to & \text{Div } X' \times Y' \setminus X' \times Y' \to \text{Pic } (X' \times Y') \to \text{Pic } (X \times Y) \\
\downarrow & & \downarrow & & \downarrow \\
\text{coker}(\text{Pic}) & \to & \text{coker}(\text{Pic}) \\
0 & 0
\end{array}
\]
where the first vertical arrow is an isomorphism by \([2.11]\) and so is the second one since by definition, 
\((\overline{X} \times Y)^{(1)} \cap (\overline{X} \times Y \setminus \overline{X} \times Y) = ((\overline{X})^{(1)} \cap (\overline{X} \setminus \overline{X})) \cup ((\overline{Y})^{(1)} \cap (\overline{Y} \setminus \overline{Y})). \) Then the desired isomorphism follows from the diagram and the snake lemma.

Next, denote by \(A\) the Picard variety of \(X^c\) and \(B^t\) the Albanese variety of \(Y^c\). Then the abelian group \(\text{coker}(\text{Pic}) = \text{Hom}(\text{Br}^t, A)\) is finitely generated and torsion-free \([11, \text{Prop. 1.7 and Rem. 1.9}]\), and so is \(\text{coker}(\text{Pic})\). It follows that \(H^1(k, \text{coker}(\text{Pic}))\) is finite. The proof is complete. \(\square\)

Now we obtain the desired finiteness of \(\text{coker}(\text{Br})\).

**Theorem 2.13.** Let \(k\) be a field finitely generated over \(\mathbb{Q}\). Let \(X\) and \(Y\) be smooth quasi-projective geometrically integral varieties over \(k\). Assume that \((X \times Y)(k) \neq \emptyset\) or \(H^3(k, \overline{X}) = 0\). Then the group \(\text{coker}(\text{Br})\), which is the cokernel of the natural map
\[
\text{Br} X \oplus \text{Br} Y \to \text{Br}(X \times Y),
\]
is finite.

**Proof.** Consider the following commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Br}_1 X & \oplus & \text{Br}_1 Y & \rightarrow & \text{Br} X & \oplus & \text{Br} Y & \rightarrow & (\text{Br} X)^{\text{G}_k} \oplus (\text{Br} Y)^{\text{G}_k} & \rightarrow & \text{Br}_2 X & \oplus & \text{Br}_2 Y & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & \text{Br}_1 (X \times Y) & \rightarrow & \text{Br}(X \times Y) & \rightarrow & \text{Br}(X \times Y)^{\text{G}_k} & \rightarrow & \text{Br}_2 (X \times Y) & \rightarrow & 0
\end{array}
\]
The idea is to break rows into short exact sequences and use the snake lemma. By Lemma \([2.2, 11]\), \(\ker(\text{Br}_2)\) is finite. Proposition \([2.2, 11]\) implies \(\ker(\text{Br}_2)^{\text{G}_k}\) is finite. Also by Proposition \([2.7, 11]\) \(\ker(\text{Br}_1)\) is finite. Finally we conclude that \(\ker(\text{Br})\) is finite. The proof is complete. \(\square\)

3. The Brauer-Manin set of a product

**Theorem 3.1.** Let \(k\) be a number field. Let \(X\) and \(Y\) be smooth geometrically integral varieties over \(k\). Then
\[
(X \times Y)(\mathbb{A}_k)^{\text{Br}(X \times Y)} = X(\mathbb{A}_k)^{\text{Br} X} \times Y(\mathbb{A}_k)^{\text{Br} Y}.
\]

**Proof.** Note that the Brauer group of a smooth variety is torsion \([3, \text{Prop. 1.4}]\), so we only need to show that \([11, \text{Lems. 5.2, 5.3}]\) are still available for non-projective varieties. Actually, the proof of Lemma \(5.3\) did not make any use of projective property. It suffices to prove Lemma \([8, 7]\) below, a variant of Lemma \(5.2, \text{loc. cit.}\), which does not need the projective assumption on varieties. \(\square\)

Let \(S_X\) be the \(k\)-group of multiplicative type dual to \(H^1(X, \mu_n)\), that is,
\[
S_X = \text{Hom}_{\text{-gps}}(S_X, G_m) = H^1(X, \mu_n)
\]
and the same for \(S_Y\). Let \(T_X\) be a universal torsor of \(n\)-torsion for \(X\), which is a torsor over \(X\) under \(S_X\). It can be defined for a general smooth geometrically integral variety over a field of characteristic zero \([11, \text{Def. 2.1}]\), and exists if \(X(\mathbb{A}_k)^{\text{Br} X} \neq \emptyset\) \([5, \text{Cor. 8.17}]\). Then there is a homomorphism
\[
epsilon : \text{Hom}_k(S_X, S_Y^e) \to H^2(X \times Y, \mu_n) : \phi \mapsto \phi_*[T_X] \cup [T_Y],
\]
which is originally defined by Skorobogatov and Zarhin \([11]\) and can be generalized for arbitrary smooth geometrically integral varieties. By \([11, \text{Prop. 2.2}]\), we have \([T_X] = [T_Y]\) in \(H^1(X, S_X)\) and there is a natural isomorphism
\[
\tau_X : S_X = \text{Hom}_{\text{-gps}}(S_X, \mu_n) \iso H^1(X, \mu_n) : \phi \mapsto \phi_*([T_X]).
\]
Twisting by $\mathbb{Z}/n(-1) = Hom(\mu_n, \mathbb{Z}/n)$, we obtain the isomorphism
\[
\tau_X(-1) : Hom_T(S_X, \mathbb{Z}/n) \cong H^1(X, \mathbb{Z}/n) : \psi \mapsto \psi_*([T_X]).
\]
We also obtain that pairing with $[T_X]$ induces the identity on $H^1(X, \mu_n)$ and $H^1(X, \mathbb{Z}/n)$. Similar properties hold if we replace $X$ by $Y$. Thus pairing with $[T_X] \cup [T_Y]$ induces the cup product
\[
H^1(X, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} H^1(Y, \mu_n) \xrightarrow{\cup} H^2(X \times Y, \mu_n)
\]
and we have the commutative diagram
\[
(\text{Hom}_T(S_X, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} S_Y^*)^\mathfrak{g}_k \xrightarrow{\epsilon} H^2(X \times Y, \mu_n)
\]
\[
\tau_X(-1) \cup \tau_Y : (H^1(X, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} H^1(Y, \mu_n))^\mathfrak{g}_k \xrightarrow{\cup} H^2(X \times Y, \mu_n)^{\mathfrak{g}_k}
\]
The above discussion can be found in Section 2, loc. cit.

Let us first recall

**Lemma 3.3** ([11 Lemma 5.3]). With notations and assumptions as Theorem 3.1 for any positive integer $n$, we have
\[
X(A_k)^{(Br, X)}_n \times Y(A_k)^{(Br, Y)}_n \subseteq (X \times Y)(A_k)_{Im\varepsilon}.
\]

**Lemma 3.4.** Assume that $k$ is a field of characteristic zero and $X$, $Y$ are smooth geometrically integral $k$-varieties. Then we have the following K"{u}nneth decompositions of $g_k$-modules
\[
(p^*_X, p^*_Y) : H^1(X, \mu_n) \oplus H^1(Y, \mu_n) \xrightarrow{\sim} H^1(X \times Y, \mu_n),
\]
(3.5)
\[
(p^*_X, p^*_Y) : H^2(X, \mu_n) \oplus H^2(Y, \mu_n) \oplus (H^1(X, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} H^1(Y, \mu_n)) \xrightarrow{\sim} H^2(X \times Y, \mu_n),
\]
(3.6)
where $\cup$ is the cup product.

**Proof.** The original proof assuming projectivity is in [11]. A proof without properness assumption can be found in [1] Prop. 2.6. \hfill \Box

**Lemma 3.7.** With notations as Theorem 3.1, suppose that $X(A_k)^{Br, X}$ and $Y(A_k)^{Br, Y}$ are both non-empty. Then we have a natural surjection
\[
H^2(X, \mu_n) \oplus H^2(Y, \mu_n) \oplus \text{Hom}_k(S_X, S_Y^*) \xrightarrow{(p^*_X, p^*_Y, \epsilon)} H^2(X \times Y, \mu_n).
\]

**Proof.** We have seen the assumption that $X(A_k)^{Br, X}$ and $Y(A_k)^{Br, Y}$ are both non-empty ensures the existence of $T_X$ and $T_Y$.

It should be noted that if we assume that $X(k) \neq \emptyset$, fix $u \in X(k)$, and define
\[
H^i_u(X, \mu_n) = \ker(H^i(X, \mu_n) \xrightarrow{u^*} H^i(k, \mu_n)).
\]
Then [1] Cor. 2.7 asserts that there is an isomorphism
\[
H^2_u(X, \mu_n) \oplus H^2(Y, \mu_n) \oplus \text{Hom}_k(S_X, S_Y^*) \xrightarrow{(p^*_X, p^*_Y, \epsilon)} H^2(X \times Y, \mu_n),
\]
where $X$, $Y$ are $U$, $V$ there. Thus clearly (3.8) is surjective.
Now it is not necessarily that \( X(k) \neq \emptyset \), but by modifying the proof of Cor. 2.7, loc. cit., we can also show that (3.8) is surjective. Consider the Hochschild-Serre spectral sequences

\[
E_2^{p,q}(X) = H^p(k, H^q(X, \mu_n)) \Rightarrow E^{p,q}(X) = H^{p+q}(X, \mu_n)
\]

for \( X \) and \( E_2^{p,q}(Y) \) (resp. \( E_2^{p,q}(X \times Y) \)) for \( Y \) (resp. \( X \times Y \)). Let

\[
\phi_2^{p,q} : E_2^{p,q}(X) \oplus E_2^{p,q}(Y) \to E_2^{p,q}(X \times Y)
\]

be the morphism of spectral sequences induced by \((p_X^*, p_Y^*)\). Let \( E_2^2(X) = \ker(E^2(X) \to E_2^{0,2}(X)) \) and we define \( E_2^2(Y) \) and \( E_2^2(X \times Y) \) in the same manner.

Let \( p : X \to \text{Spec} \, k \) be the structure morphism. Applying \( H(k, -) \) to the distinguished triangle

\[
\tau_0 \mathbb{R}p_*\mu_n \to \tau_{\leq 1} \mathbb{R}p_*\mu_n \to \tau_1 \mathbb{R}p_*\mu_n \to \tau_0 \mathbb{R}p_*\mu_n
\]

and taking cohomology, we obtain the exact sequence

\[
E_2^{3,0}(X) \to \mathbb{H}^3(k, \tau_{\leq 1} \mathbb{R}p_*\mu_n) \to E_2^{3,1}(X) \to E_2^{3,0}(X) \to \mathbb{H}_3(k, \tau_{\leq 1} \mathbb{R}p_*\mu_n)
\]

which extends the low term exact sequence associated to the spectral sequence \( E_2^{p,q}(X) \) since by a similar argument as in [9] pp. 413-414 we have \( \mathbb{H}^2(k, \tau_{\leq 1} \mathbb{R}p_*\mu_n) = E_2^1(X) \).

We have an obvious commutative diagram

\[
\begin{array}{ccc}
\tau_{[0]} \mathbb{R}p_*\mu_n & \to & \tau_{\leq 1} \mathbb{R}p_*\mu_n \\
\downarrow & & \downarrow \\
\mathbb{R}p_*\mu_n & \to & \mathbb{R}p_*\mu_n
\end{array}
\]

On applying \( \mathbb{H}^3(k, -) \), we obtain

\[
E_2^{3,0}(X) \xrightarrow{f} \mathbb{H}^3(k, \tau_{\leq 1} \mathbb{R}p_*\mu_n) \xrightarrow{p^*} E^3(X)
\]

By assumption \( X(\mathbb{A}_k) \neq \emptyset \), we have the injection \( p^* : H^3(k, \mu_n) \to H^3(X, \mu_n) \) by [11] p. 765, l. 6-14). Note that we have

\[
H^0(X, \mu_n) = \mu_n(k).
\]

Hence \( E_2^{3,0}(X) = H^3(k, \mu_n) \to E^3(X) = H^3(X, \mu_n) \) is injective. It follows that \( f \) is also an injection and (3.11) becomes

\[
E_2^{2,0}(X) \to E_2^{2}(X) \to E_2^{1,1}(X) \to 0
\]

which by functoriality fits into the commutative diagram with exact rows and vertical arrows induced by \( \phi_2^{p,q} \)

\[
\begin{array}{ccccccc}
E_2^{2,0}(X) \oplus E_2^{2,0}(Y) & \xrightarrow{\phi_2^{2,0}} & E_2^1(X) \oplus E_2^1(Y) & \xrightarrow{\phi_2^{1,1}} & E_2^{1,1}(X) \oplus E_2^{1,1}(Y) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
E_2^{2,0}(X \times Y) & \xrightarrow{\phi_2^{2,0}} & E_2^1(X \times Y) & \xrightarrow{\phi_2^{1,1}} & E_2^{1,1}(X \times Y) & \to & 0
\end{array}
\]
Since for any $p$, $\phi_2^{p,0}$ is clearly surjective by (3.12) and $\phi_2^{p,1}$ is an isomorphism by (3.5), the left vertical arrow is surjective and the right one is an isomorphism. It follows by the five lemma that the middle one is surjective.

Using the injectivity of $E_2^{2,0}(X) \to E_2^{3}(X)$ again, a standard computation of the spectral sequence (3.10) yields the exact sequence

$$0 \to E_1^{2}(X) \to E_2^{2}(X) \to E_2^{0,2}(X) \to E_2^{2,1}(X).$$

Thus we have the commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \to & E_1^{2}(X) \oplus E_1^{2}(Y) & \to E_2^{2}(X) \oplus E_2^{2}(Y) \to E_2^{0,2}(X) \oplus E_2^{0,2}(Y) \to E_2^{2,1}(X) \oplus E_2^{2,1}(Y) \\
0 & \to & E_1^{2}(X \times Y) & \to E_2^{2}(X \times Y) \to E_2^{0,2}(X \times Y) \to E_2^{2,1}(X \times Y) \\
0 & \to & E_1^{2}(X \times Y) & \to E_2^{2}(X \times Y) \to E_2^{0,2}(X \times Y) \to E_2^{2,1}(X \times Y) \\
\end{array}
$$

where the vertical arrows are induced by $\phi_2^{p,q}$, of which the first one is a surjection and the last one an isomorphism, by the previous discussion. A diagram chasing yields the exact sequence

(3.13) $E_2^{2}(X) \oplus E_2^{2}(Y) \xrightarrow{(p_X,p_Y)} E_2^{2}(X \times Y) \to \text{coker } \phi_2^{0,2}$.

Then by (3.6) and commutative diagram (3.2), one shows that the composition

$$\text{Hom}_k(S_X, S_Y^\ast) \xrightarrow{\alpha} H^2(X \times Y, \mu_n) \to \text{coker } \phi_2^{0,2}$$

is an isomorphism. Along with (3.13) we obtain the desired result that (3.8) is surjective. \hfill \Box

ACKNOWLEDGMENT

The author would like to thank the referees for valuable suggestions, and Dasheng Wei, Junchao Shentu and Jiangxue Fang for helpful discussions.

REFERENCES

[1] Yang Cao, Sous-groupe de Brauer invariant et obstruction de descente itérée, arXiv preprint arXiv:1704.05425v4 (2019).
[2] Jean-Louis Colliot-Thélène and Alexei N Skorobogatov, Descente galoisienne sur le groupe de brauer, Journal für die reine und angewandte Mathematik (Crelles Journal) 2013 (2013), no. 682, 141–165.
[3] Alexander Grothendieck, Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 46–66. MR 244269
[4] , Le groupe de Brauer. II. Théorie cohomologique, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 67–87. MR 244270
[5] , Le groupe de Brauer. III. Exemples et compléments, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188. MR 244271
[6] David Harari and Alexei N. Skorobogatov, Descent theory for open varieties, Torsors, étale homotopy and applications to rational points, London Math. Soc. Lecture Note Ser., vol. 405, Cambridge Univ. Press, Cambridge, 2013, pp. 250–279. MR 3077172
[7] Yonatan Harpaz and Tomer M. Schlank, Homotopy obstructions to rational points, Torsors, étale homotopy and applications to rational points, London Math. Soc. Lecture Note Ser., vol. 405, Cambridge Univ. Press, Cambridge, 2013, pp. 280–413. MR 3077173
[8] J-J Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres., Journal für die reine und angewandte Mathematik 327 (1981), 12–80.
[9] A. N. Skorobogatov, Beyond the Manin obstruction, Inventiones Mathematicae 135 (1999), no. 2, 399–424.
[10] , Torsors and rational points, vol. 144, Cambridge University Press, 2001.
[11] Alexei N. Skorobogatov and Yuri G. Zarhin, The Brauer group and the Brauer-Manin set of products of varieties, J. Eur. Math. Soc. (JEMS) 16 (2014), no. 4, 749–768. MR 3191975
State Key Laboratory of Information Security, Institute of Information Engineering, Chinese Academy of Sciences, Beijing 100093, P.R. China

E-mail address: lvchang@amss.ac.cn