ON A MODEL OF A POPULATION WITH VARIABLE MOTILITY

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ABSTRACT. We study a reaction-diffusion equation with a nonlocal reaction term that models a population with variable motility. We establish a global supremum bound for solutions of the equation. We investigate the asymptotic (long-time and long-range) behavior of the population. We perform a certain rescaling and prove that solutions of the rescaled problem converge locally uniformly to zero in a certain region and stay positive (in some sense) in another region. These regions are determined by two viscosity solutions of a related Hamilton-Jacobi equation.

1. SETTING AND MAIN RESULTS

We study a reaction-diffusion equation with a nonlocal reaction term:

\begin{align*}
\frac{\partial n}{\partial t} &= \theta \frac{\partial^2}{\partial x^2} n + \alpha \frac{\partial}{\partial \theta} \rho n + r n (1 - \rho) \quad \text{for} \quad (x, \theta, t) \in \mathbb{R} \times \Theta \times (0, \infty), \\
\rho(x, t) &= \int_\Theta n(x, \theta, t) \, d\theta \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty), \\
\frac{\partial n}{\partial \theta} |_{x = \theta_m, t} &= \frac{\partial n}{\partial \theta} |_{x = \theta_M, t} = 0 \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty), \\
n(x, \theta, 0) &= n_0(x, \theta) \quad \text{for} \quad (x, \theta) \in \mathbb{R} \times \Theta.
\end{align*}

This problem, introduced by Bénichou, Calvez, Meunier, and Voituriez [6], models a population structured by a space variable $x$ and a motility trait $\theta \in \Theta$. Our analysis is focused on the case where the trait space $\Theta$ is a bounded subset of $\mathbb{R}^+$. The parameters $\alpha$ and $r$ are positive and represent, respectively, the rate of mutation and the strength of the reaction term. The problem $(E)$ is of Fisher-KPP type. The classical Fisher-KPP equation,

\begin{equation}
\frac{\partial m(x, t)}{\partial t} = \beta \frac{\partial^2}{\partial x^2} m(x, t) + r m(x, t)(1 - m(x, t)) \quad \text{for} \quad (x, t) \in \mathbb{R} \times (0, \infty),
\end{equation}

describes the growth and spread of a population structured by a space variable $x$. The diffusion coefficient $\beta$ is a positive constant. The behavior of solutions to $(F-KPP)$ has been widely studied, starting from its introduction in [23, 28]. The most important difference between $(E)$ and the Fisher-KPP equation is that the reaction term in $(E)$ is nonlocal in the trait variable. This is because competition for resources, which is represented by the reaction term, occurs between individuals of all traits that are present in a certain location. Another key feature of $(E)$ is that the trait $\theta$ affects how fast an individual moves – this is why the coefficient of the spatial diffusion in $(E)$ depends on $\theta$. In addition, the trait is subject to mutation, which is modeled by the diffusion term in $\theta$. Thus, $(E)$ describes the interaction between dispersion of a population and the evolution of the motility trait.

Main results. Throughout our paper, we assume that the trait space $\Theta$ is a bounded interval

$$\Theta = (\theta_m, \theta_M),$$

where $\theta_M$ and $\theta_m$ are positive constants. We study classical solutions of $(E)$ with initial condition that is non-negative and “regular enough.” We state these assumptions precisely as $(A1)$, $(A2)$ and $(A3)$ in Section 1.1.1.

A significant challenge for us is that $(E)$ does not enjoy the maximum principle. Nevertheless, we are able to establish a global upper bound for solutions of $(E)$. We prove:
Theorem 1.1. Suppose $n$ is nonnegative, twice differentiable on $\mathbb{R} \times \Theta \times (0, \infty)$ and satisfies (E) in the classical sense, with initial condition $n_0$ that satisfies (A2). There exists a constant $C$ such that
\[
\sup_{\mathbb{R} \times \Theta \times (0, \infty)} n \leq C.
\]

Theorem 1.1 is a key element in the proof of our second main result, which concerns the long-time and long-range behavior of solutions of (E). We perform the rescaling $(x, t) \mapsto (\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$, which leads us to consider solutions $n^\varepsilon$ of the $\varepsilon$-dependent problem,
\[
(E^\varepsilon) \begin{cases}
\varepsilon \partial_t n^\varepsilon = \varepsilon^2 \partial_x^2 n^\varepsilon + \alpha \partial_\theta^2 n^\varepsilon + r n^\varepsilon (1 - \rho^\varepsilon) & \text{for } (x, \theta, t) \in \mathbb{R} \times \Theta \times (0, \infty), \\
\rho^\varepsilon(x, t) = \int_\Theta n^\varepsilon(x, \theta, t) \, d\theta & \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \\
\partial_\theta n^\varepsilon(x, \theta, t) = 0 & \text{for } (x, \theta) \in \mathbb{R} \times \Theta.
\end{cases}
\]

We study the limit of the $n^\varepsilon$ as $\varepsilon \to 0$. We find that there exists a set on which the $n^\varepsilon$ converge locally uniformly to zero, and another set on which a certain limit of $\int n^\varepsilon \, d\theta$ stays strictly positive. These sets are determined by two viscosity solutions of the Hamilton-Jacobi equation,
\[
(HJ) \quad \max\{u, \partial_t u + \partial_x u c(\partial_x u)\} = 0.
\]

The function $c : \mathbb{R} \to \mathbb{R}$ arises from the eigenvalue problem (1.10) and is determined by $\theta_m$, $\theta_M$, $r$, and $\alpha$ (see Proposition 1.3). One may view the function $c$ as encoding the effect of the motility trait on the limiting behavior of the $n^\varepsilon$.

Viscosity solutions of (HJ) with infinite initial data play a key role in our analysis and we provide a short appendix where we discuss the relevant known results. For the purposes of the introduction, we state the following lemma:

Lemma 1.1. For any $\Omega \subset \mathbb{R}$, there exists a unique continuous function $u_\Omega$ that is a viscosity solution of (HJ) in $\mathbb{R} \times (0, \infty)$ and satisfies the initial condition
\[
(1.1) \quad u_\Omega(x, 0) = \begin{cases}
0 & \text{for } x \in \Omega \\
-\infty & \text{for } x \in \mathbb{R} \setminus \Omega.
\end{cases}
\]

In addition, we have $u_\Omega(x, t) \leq 0$ for all $x \in \mathbb{R}$ and $t \in (0, \infty)$.

We are interested in $u_\Omega$ for two sets $\Omega$ determined by the initial data $n_0$. We define these two sets, $J$ and $K$, by
\[
J = \{x \in \mathbb{R} : \text{ there exists } \theta \in \Theta \text{ such that } n_0(x, \theta) > 0\}
\]
and
\[
K = \{x \in \mathbb{R} : n_0(x, \theta) > 0 \text{ for all } \theta \in \Theta\}.
\]

We see that $x$ belongs to $J$ if initially there is at least some individual living at $J$, and $x$ belongs to $K$ individuals with all traits are present at $x$.

Our main result says that the limiting behavior of the $n^\varepsilon$ is determined by $u_J$ and $u_K$:

Theorem 1.2. Assume (A1), (A2) and (A3). Let $u_J$ and $u_K$ be the functions given by Lemma 1.1. Then,
\[
\lim_{\varepsilon \to 0} n^\varepsilon = 0 \text{ uniformly, at an exponential rate, on compact subsets of } \{u_J < 0\} \times \Theta.
\]

and
\[
\limsup_{\varepsilon \to 0} \rho^\varepsilon(x, t) = \limsup_{\varepsilon \to 0} \rho^\varepsilon(y, s) : \varepsilon' \leq \varepsilon, |y - x|, |t - s| \leq \varepsilon
\]
\[
\geq 1 \text{ on the interior of } \{u_K = 0\}.
\]

Let us remark on a special case of Theorem 1.2. Suppose the initial data $n_0$ is such that the two sets $J$ and $K$ are equal (this occurs if, for example, $n_0$ is independent of $\theta$). In this case, $u_J = u_K \leq 0$ and so $\mathbb{R} = \{u_J < 0\} \cup \{u_K = 0\}$, which means that Theorem 1.2 gives information about the limiting behavior of $n^\varepsilon$ almost everywhere on $\mathbb{R}$, for all times $t$.

We now explain the biological interpretation of Theorem 1.2. The question is, as time goes on, which territory will be occupied by the species and which will be left empty? To answer this question, it is enough
to determine where the functions $u_J$ and $u_K$ are zero. In fact, we have the following corollary, which gives information about the limit of the $n^\varepsilon$ simply in terms of the sets $J$, $K$ and the constants $r$, $\theta_m$ and $\theta_M$. For simplicity we consider only the case that $J$ and $K$ are intervals.

**Corollary 1.1.** Assume (A1), (A2) and (A3). Let us also suppose that each of $J$ and $K$ is a (possibly infinite) interval.

- If $\text{dist}(x, J) > 2t\sqrt{\frac{\theta_M}{r}}$, then $\lim_{\varepsilon \to 0} n^\varepsilon(x, \theta, t) = 0$ for all $\theta \in \Theta$.
- If $\text{dist}(x, K) < 2t\sqrt{\frac{\theta_m}{r}}$, then $\limsup_{\varepsilon \to 0} \rho^\varepsilon(x, t) \geq 1$.

Thus, we see that at time $t$, if we stand a point that is “far” from $J$, then there are no individuals at $x$. On the other hand, if we stand at a point $x$ that is “pretty close” to $K$, then there are some individuals living at $x$.

We remind the reader of the fact, due to Aronson and Weinberger [4], that $2\sqrt{\frac{\beta}{r}}$ is the asymptotic speed of propagation of fronts for (F-KPP). Thus, we may interpret Corollary 1.1 as saying that, in the limit, the population we’re considering spreads slower than one with constant motility $\theta_M$ and faster than one with constant motility $\theta_m$.

We further discuss the biological interpretation of our results, and give the proof of Corollary 1.1 in Section 1.2.

**Biological motivation.** Biologists are interested in the interplay between traits present in a species and how the species interacts with its environment – in other words, between evolution and ecology [37, 36, 35, 27]. It has been observed, for example in butterflies in Britain [39], that an expansion of the territory that a species occupies may coincide with changes in a certain trait – the butterflies that spread to new territory were able to lay eggs on a larger variety of plants than the butterflies in previous generations.

Some biologists have focused specifically on the interaction between ecology and traits that affect motility. Phillips et al [34] recently discovered a species of cane toads whose territory has, over the past 70 years, spread with a speed that increases in time. This is very interesting because this is contrary to what is predicted by the Fisher-KPP equation [23, 28, 4] and has previously been observed empirically [38]. Spacial sorting was also observed – the toads that arrive first in the new areas have longer legs than those in the areas that have been occupied for a long time. In addition, it was discovered that toads with longer legs are able to travel further than toads with shorter legs. It is hypothesised that the presence of this trait – length of legs – is responsible for both the front acceleration and the spacial sorting. Similar phenomena were observed in crickets in Britain over a shorter time period [39]. In that case, the motility trait was wingspan.

The cases we describe demonstrate the need to understand the influence of a trait – in particular, a motility trait – on the dynamics of a population.

**Literature review.** The Fisher-KPP equation has been extensively studied, and we refer the reader to [23, 28, 3, 22, 33] for an introduction. Hamilton-Jacobi equations similar to (HJ) are known to arise in the analysis of the long-time and long-range behavior of (F-KPP), other reaction-diffusion PDE, and systems of such equations – see, for example, Evans and Souganidis [20] and Barles, Evans and Souganidis [5] and Fleming and Souganidis [24]. The methods of [20, 5, 24] are a key part of our analysis of (E).

As we previously mentioned, (F-KPP) describes populations structured by space alone, while there is a need to study the interaction of dispersion and phenotypical traits (in particular, motility traits). Most models of populations structured by space and trait either consider a trait that does not affect motility or do not consider the effect of mutations. Champagnat and Méléard [14] start with an individual-based model of the such a population and derive a PDE that describes its dynamics. In the case that the trait affects only the growth rate and not the motility, Alfaro, Coville and Raoul [1] study this PDE, which is a reaction-diffusion equation with constant diffusion coefficient:

$$
(1.3) \quad n_t - \Delta_{x,\theta}n = \left( r(\theta - x) - \int_{\mathbb{R}} K(\theta - x, \theta' - x)n(x, \theta', t) d\theta' \right)n.
$$

The population modeled by (1.3) has a preferred trait that varies in space. Berestycki, Jin and Silvestre [8] analyze an equation similar to (1.3), but with a different kernel $K$ and growth term $r$ that represent the existence of a trait that is favorable for all individuals. The aims and methods of [1, 8] are quite different.
from those in this paper. The main result of [11] is the existence of traveling wave solutions of (1.3) for speeds above a critical threshold. In [8], the authors establish the existence and uniqueness of travelling wave solutions and prove an asymptotic speed of propagation result for the equation that they consider. We also mention that a local version of (1.3) was investigated by Berestycki and Chapuisat [7]. A model in which the dispersal rate does depend on the trait, but the trait is not subject to mutation, was studied by Desvillettes, Ferriere and Prevost [17] and Arnold, Desvillettes and Prevost [2].

There has also been analysis of traveling waves and steady states for equations of the form

\[
v_t = \Delta v + v(1 - v \ast \phi) \text{ in } \mathbb{R} \times (0, \infty),
\]

where the reaction term \(v \ast \phi\) is the convolution of \(v\) with some kernel \(\phi\). We refer the reader to Berestycki, Nadin, Perthame and Ryzhik [9], Hamel and Ryzhik [25] and the references therein. An important difference between (1.4) and (E) is that the reaction term of (E) is local in the space variable and nonlocal in the trait variable, while the reaction term in (1.4) is fully nonlocal. The long time behavior of solutions to (1.4) is studied in [25]. In addition, [25, Theorem 1.2] establishes a supremum bound for solutions of (1.4).

Bouin and Mirrahimi [12] analyze the reaction-diffusion equation

\[
v_t = D\Delta v + av_{xy} + rv(x, \theta, t) \left(a(x, \theta) - \int_\Theta v(x, \theta, t) \, d\theta\right) \text{ in } \mathbb{R}^d \times \Theta \times (0, \infty),
\]

with Neumann conditions on \(v\) on the boundary of \(\Theta\). The main difference between (1.5) and (E) is that the coefficient of spacial diffusion in (1.5) is constant, which means that (1.5) models a population where the trait does not affect motility. The methods used in [12] to analyze (1.5) are similar to those of [20, 5, 24] and to some of those in our paper. However, in general it is easier to obtain certain bounds for solutions of (1.5) than for solutions of (E). For example, because the coefficient of \(\Delta v\) in (1.5) is constant, integrating (1.5) in \(\theta\) implies that \(\int_\Theta v(x, \theta, t) \, d\theta\) satisfies a local equation in \(x\) and \(t\) that enjoys the maximum principle. This immediately implies that \(\int_\Theta v(x, \theta, t) \, d\theta\) is globally bounded [12, Lemma 2]. This strategy does not work for (E). Indeed, a serious challenge in studying (E), as opposed to (F-KPP) or nonlocal reaction diffusion equations with constant diffusion coefficient such as (1.5), is obtaining a global supremum bound for solutions of (E).

Let us discuss the literature that directly concerns (E) and (E). The problem (E) was introduced in [6]. The rescaling leading to (E) was suggested by Bouin, Calvez, Meunier, Mirrahimi, Perthame, Raoul, and Voituriez in [11]. In addition, formal results about the asymptotic behavior of solutions to (E) were obtained in [11]. In particular, Part (I) of Proposition 1.2 of our paper was predicted in [11, Section 2]. Bouin and Calvez [10] also study (E). They prove that there exist traveling wave solutions to (E) but do not analyze whether solutions converge to a traveling wave. In fact, to our knowledge, there are no previous rigorous results about the asymptotic behavior of solutions of (E) or the limiting behavior of solutions to (E). The main difficulty is the lack of comparison principle for (E).

Contribution of our work.

- To the best of our knowledge, Theorem 1.1 is the first global supremum bound for a Fisher-KPP type equation with a nonlocal reaction term and non-constant diffusion.
- Theorem 1.2 completes the program that was proposed in [11] for analyzing the asymptotic behavior of the model (E) in the case where the trait space \(\Theta\) is bounded.
- We view our main result, Theorem 1.2, as evidence that the presence of a motility trait does affect the limiting behavior of populations. We discuss this in subsection 1.2.
- We hope that our work is a step towards analyzing (E) in the case where the trait space \(\Theta\) is unbounded. It is in this case that the phenomena of accelerating fronts is predicted to occur [11].

Elements of the proofs of the main results. The proof of Theorem 1.1 is quite involved. The difficulty comes from the combination of the nonlocal reaction term and non-constant diffusion. Our proof of Theorem 1.1 uses regularity estimates for solutions of elliptic PDE, a heat kernel estimate, and an averaging technique similar to that of [25, Theorem 1.2]. We believe this combination of methods is new and may be useful in other contexts. We include a detailed outline in Subsection 2.1.

To analyze the limit of the \(n^\varepsilon\), we preform the transformation

\[
w^\varepsilon(x, \theta, t) = \varepsilon \ln(n^\varepsilon(x, t, \theta)).
\]

Such a transformation is used in [20, 5, 11, 12]. We prove locally uniform estimates on \(w^\varepsilon\):
Proposition 1.1. Assume (A1), (A2) and (A3) and let \( u^\varepsilon \) be given by (1.6). Suppose \( Q \) is a compact subset of \( \mathbb{R} \times (0, \infty) \). There exists a constant \( C \) that depends on \( Q, \alpha, r, \theta_m, \theta_M \) such that for all \( 0 < \varepsilon < 1 \) and for all \( (x, \theta, t) \) such that \( (x, t) \in Q \) and \( \theta \in \Theta \), we have

\[
u^\varepsilon(x, \theta, t) \geq -C
\]

and

\[
|u^\varepsilon_0(x, \theta, t)| \leq \varepsilon^{1/2}C.
\]

We define the half-relaxed limits \( \bar{u} \) and \( u \) of \( u^\varepsilon \) by,

\[
\bar{u}(x, t) = \limsup_{\varepsilon \to 0} \{ u^\varepsilon(y, \theta, s) : \varepsilon' \leq \varepsilon, |y - x|, |t - s| \leq \varepsilon, \theta \in \Theta \}
\]

and

\[
u(x, t) = \liminf_{\varepsilon \to 0} \{ u^\varepsilon(y, \theta, s) : \varepsilon' \leq \varepsilon, |y - x|, |t - s| \leq \varepsilon, \theta \in \Theta \}.
\]

The estimates of Proposition 1.1 imply that \( u \) and \( \bar{u} \) are finite everywhere on \( \mathbb{R} \times (0, \infty) \).

We use a perturbed test function argument (Evans [19]) and techniques similar to the proofs of [5, Propositions 3.1 and 3.2] to establish:

Proposition 1.2. Assume (A1), (A2) and (A3) and let \( \bar{u} \) and \( u \) be given by (1.7). Then:

(I): \( \bar{u} \) is a viscosity subsolution and \( u \) is a viscosity supersolution of \( (HJ) \) in \( \mathbb{R} \times (0, \infty) \); and

(II): we have

\[
\bar{u}(x, 0) = \begin{cases} 0 & \text{for } x \in J \\ -\infty & \text{for } x \in \mathbb{R} \setminus J \end{cases}
\]

and

\[
u(x, 0) = \begin{cases} 0 & \text{for } x \in K \\ -\infty & \text{for } x \in \mathbb{R} \setminus K \end{cases}
\]

Part (I) of Proposition 1.2 was predicted via formal arguments in [11, Section 2]. Theorem 1.2 follows easily from Proposition 1.2 by arguments similar to those in the proofs of [20, Theorem 1.1] and [5, Theorem 1].

Remark 1.3. An interesting question is whether Theorem 1.2 can be refined to obtain better information about the limit of the \( \rho^\varepsilon \) in the interior of the set \( \{u_K = 0\} \). For instance, is \( \liminf_{\varepsilon \to 0} \rho^\varepsilon \) bounded from below in this set?

Structure of our paper. In the next subsection, we state our assumptions, give notation, and state a certain known result about the spectral problem that defines the function \( c \). In subsection 1.2, we discuss the interpretation of Theorem 1.2 and give the proof of Corollary 1.1. The rather lengthy Section 2 is devoted to the proof of Theorem 1.1. This section is self-contained. In Section 3 we prove Proposition 1.1. The proof of Proposition 1.2 is in Section 4. The proof of Theorem 1.2 is in Section 5. We also provide an appendix with a discussion of results on comparison (Proposition A.1) and existence and uniqueness (Lemma 1.1) for Hamilton-Jacobi equations with infinite initial data.

We have organized our paper so that a reader who is interested mainly in our proof of Theorem 1.1 may only read Section 2. On the other hand, a reader who is interested in our results about the limit of the \( n^\varepsilon \), and not in the proofs of the supremum bound on \( n^\varepsilon \) and the estimates on \( u^\varepsilon \), may skip ahead to Sections 4 and 5 after finishing the introduction.

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Appendix A
Acknowledgements
References

1.1. Ingredients.

1.1.1. Assumptions. Our results hold under the following assumptions:

(A1) \( n^\varepsilon \) is a non-negative classical solution of \( \frac{\partial}{\partial t} n^\varepsilon \).

(A2) \( n_0(x, a(\theta)) \in C^{2,\eta}(\mathbb{R} \times \mathbb{R}) \) for some \( \eta \in (0,1) \), where \( a(\theta) : \mathbb{R} \to [\theta_m, \theta_M] \) is defined by (2.2).

(A3) \( \Theta = (\theta_m, \theta_M) \), where \( 0 < \theta_m < \theta_M \).

The assumption (A2) implies that \( n_0(x, \theta) \) is contained in \( C^{2,\eta}(\mathbb{R} \times \Theta) \) and satisfies \( \partial_\theta n_0(x, \theta_m) = \partial_\theta n_0(x, \theta_M) = 0 \) for all \( x \in \mathbb{R} \). We use condition (A2) in our proof of Theorem 1.1.

1.1.2. Notation. We will slightly abuse notation in the following way. If \( Q \) is a subset of \( \mathbb{R} \times (0, \infty) \), then we will use \( Q \times \Theta \) to denote the set of \((x, \theta, t)\) such that \((x, t) \in Q\) and \( \theta \in \Theta \). We record this as:

\[
Q \times \Theta = \{(x, \theta, t) : (x, t) \in Q \text{ and } \theta \in \Theta\}.
\]

1.1.3. A spectral problem. Next we state the relevant properties of the spectral problem (1.10) of [10, Proposition 5]. We remark that there is a difference in sign – what we call \( c(\lambda) \) is actually \( -\tilde{c}(\lambda) \), where \( \tilde{c} \) denotes the function in [10, Proposition 5].

**Proposition 1.3.** For all \( \lambda > 0 \), there exits a unique solution \((c(\lambda), Q_\lambda(\theta)) = (c(\lambda), Q(\theta, \lambda))\) of the spectral problem

\[
\begin{aligned}
(\lambda c(\lambda) + \theta \lambda^2 + r)Q(\theta, \lambda) + \alpha \partial_{\theta \theta}^2 Q(\theta, \lambda) = 0 & \quad \text{for } \theta \in \Theta \\
\partial_\theta Q(\theta_m, \lambda) = \partial_\theta Q(\theta_M, \lambda) = 0, \\
Q(\theta, \lambda) \geq 0 & \quad \text{for all } \theta, \lambda, \\
\int_\Theta Q(\theta, \lambda) d\theta = 1.
\end{aligned}
\]

Moreover, we have that the map \( \lambda \to c(\lambda) \) is continuous, and satisfies, for all \( \lambda \),

\[-\lambda^2 \theta_M - r \leq \lambda c(\lambda) \leq -\lambda^2 \theta_m - r.
\]

1.2. Interpretation of Theorem 1.2

**Informal discussion.** According to [20, Theorem 1.1], the behavior of solutions to (F-KPP) is characterized by a solution \( v \) of the Hamilton-Jacobi equation

\[
\max\{v_t - \beta \cdot (v_x)^2 - r, v\} = 0.
\]

(We remark that this is the negative of the equation that appears in [20; we write it this way to be consistent with the signs employed in the rest of this paper.) Let us compare (1.11) and (1.12). We see that both are of the form \( \max\{v_t - H(v_x), v\} \), but for different Hamiltonians \( H \). It is this difference that captures the effect of the trait. Indeed, according to Proposition 1.3, the term \( u_x c(u_x) \) satisfies

\[
-\theta_M (u_x)^2 - r \leq u_x c(u_x) \leq -\theta_m (u_x)^2 - r.
\]

Thus, we see that the term \( u_x c(u_x) \) in (1.12) is “like a quadratic”, but of different size than the quadratic in (1.11). In addition, in [10, Proposition 5] the function \(-\lambda c(\lambda)\) is characterized as

\[-\lambda c(\lambda) = -\lambda^2 \theta_M + \gamma(\lambda) - 1,
\]
where $\gamma(\lambda)$ is positive for all $\lambda$. Hence the Hamiltonian in (H3) is never exactly a quadratic, and hence must be different from that of (1.11).

Since (H3) and (1.11) characterize the behavior of populations with and without a motility trait, respectively, we interpret our results as evidence that the presence of the trait does affect the asymptotic behavior of a population. We make these ideas more precise in the proof of Corollary 1.1.

**Proof of Corollary 1.1.** Corollary 1.1 follows from Theorem 1.2 and the following lemma.

**Lemma 1.2.** The unique solution to (1.11) with initial data (1.1) is $v(x, t) = (rt - \frac{\text{dist}(x, \Omega)^2}{4t\beta}) \land 0$. And, $v(x, t) = 0$ if and only if $\text{dist}(x, \Omega) \leq 2t\sqrt{\frac{2}{r}}$.

**Proof of Lemma 1.2.** The proof of the first statement a direct modification of [32, Lemma 1.5], where this result is established in the case $\Omega$ is a single point. The expression for the zero set of $v$ follows easily. We omit the details.

**Proof of Corollary 1.1.** Let us denote by $\bar{v}$ the solution to (1.11) with $\beta = \theta_M$ and initial data (1.1) with $\Omega = J$. And, let us denote by $v$ the solution to (1.11) with $\beta = \theta_m$ and initial data (1.1) with $\Omega = K$. Finally, we take $u_J$ and $u_K$ to be as in Theorem 1.2.

At time $t = 0$ we have $u_J = \bar{v}$ and $u_K = v$. The estimates (1.12) on $-p c(p)$ imply that $u_J$ is a subsolution of the equation for $\bar{v}$ and $u_K$ is a supersolution of the equation for $v$. Since (1.11) satisfies the comparison principle (see [5, Section 4] or our discussion in the Appendix), we obtain, for all $(x, t) \in \mathbb{R} \times (0, \infty)$,

$$u_J(x, t) \leq \bar{v}(x, t) \quad \text{and} \quad v(x, t) \leq u_K(x, t).$$

We have that $u_J$ and $u_K$ are non-positive. Therefore, if $\bar{v}(x, t) < 0$ then $u_J(x, t) < 0$. And, if $v(x, t) = 0$ then $u_K(x, t) = 0$. In other words,

$$\{(x, t) : \bar{v}(x, t) < 0\} \subset \{(x, t) : u_J(x, t) < 0\} \quad \text{and} \quad \{(x, t) : v(x, t) = 0\} \subset \{(x, t) : u_K(x, t) = 0\}.$$

We now combine this with the conclusions of Theorem 1.2.

- If $\bar{v}(x, t) < 0$, then $\lim_{t \to 0} n(x, \theta, t) = 0$ for all $\theta \in \Theta$.
- If $(x, t)$ is in the interior of $\{\bar{v} = 0\}$, then $\lim_{t \to 0} \sup_{\rho} \rho(x, t) \geq 1$.

Using the exact expressions for the zero sets of $\bar{v}$ and $v$ given by Lemma 1.2 concludes the proof.

### 2. Supremum Bound

This section is devoted to the proof of:

**Theorem 2.1.** Suppose $n$ is nonnegative, twice differentiable on $\mathbb{R} \times \Theta \times (0, \infty)$ and satisfies (E) in the classical sense. Assume that $\Theta$ satisfies (A3) and the initial condition $n_0$ satisfies (A2). There exists a constant $C$ that depends only on $\theta_m, \theta_M, \alpha, r$ and $\eta$ such that

$$\sup_{\mathbb{R} \times \Theta \times (0, \infty)} n \leq C \max \left\{1, \sup_{\mathbb{R} \times \Theta} n_0, \left(|n_0(x, a(\theta))|_{L^\infty(\mathbb{R} \times \Theta)}^{1+2+2\gamma} \right) \right\}.$$  

(2.1)

We briefly explain notation for norms and seminorms, which we will be using only in this section. For $U \subset \mathbb{R}^{d+1}$ we denote:

$$[u]_{\eta, U} = \sup_{(x, t), (y, s) \in U} \frac{|u(x, t) - u(y, s)|}{|x - y| + |s - t|^{1/2}} \eta, \quad |u|_{\eta, U} = ||u||_{L^\infty(U)} + [u]_{\eta, U},$$

and

$$|u|_{2+\eta, U} = ||u||_{L^\infty(U)} + \sum_{i=1}^{d} |u_{x_i}|_{L^\infty(U)} + ||u_t||_{L^\infty(U)} + \sum_{i,j=1}^{d} |u_{x_i x_j}|_{\eta, U} + |u_t|_{\eta, U}.$$

We present the supremum bound for solutions of (E2):
Figure 1. On the left is a cartoon graph of the extension \( \theta \mapsto a(\theta) \). On the right is a cartoon graph of the extension \( \theta \mapsto \tilde{n}(x, \theta, t) \). The parts of the graphs that are in black and very thick represent the original functions \( \theta \mapsto \theta \) and \( \theta \mapsto n(x, \theta, t) \) on the domain \( \Theta \).

Corollary 2.1. Assume (A[1]), (A[2]) and (A[3]). There exists a constant \( C \) that depends only on \( \theta_m, \theta_M, \alpha, r, \eta, \) and \( |n_0(x, a(\theta))|_{2+\eta,R\times R} \) such that for all \( \varepsilon > 0 \),

\[
\sup_{R \times \Theta \times (0,\infty)} n^\varepsilon \leq C.
\]

Let us explain how Corollary 2.1 follows from Theorem 2.1.

Proof of Theorem 2.1. Let us fix some \( \varepsilon > 0 \) and suppose \( n^\varepsilon \) satisfies (E) in the classical sense and \( n_0 \) satisfies (A[2]). Let us define \( n(x, \theta, t) = n^\varepsilon(x, \theta, \varepsilon t) \). Then \( n \) satisfies (E) with initial data \( \tilde{n}_0(x, \theta) := n_0(x, \theta) \). We have that \( \tilde{n}_0 \) satisfies (A[2]). Therefore, according to Theorem 2.1 we have that the estimate (2.1) holds, with \( \tilde{n}_0 \) instead of \( n_0 \) on the right-hand side. Since we have \( \sup_{R \times \Theta} \tilde{n}_0 = \sup_{R \times \Theta} n_0 \) and \( |\tilde{n}_0(x, a(\theta))|_{2+\eta,R\times R} \leq |n_0(x, a(\theta))|_{2+\eta,R\times R} \), we obtain the conclusion of Corollary 2.1.

2.1. Outline. We introduce the following piecewise function \( a(\theta) : \mathbb{R} \to [\theta_m, \theta_M] \). For any \( \theta \in \mathbb{R} \) let \( k \in \mathbb{Z} \) be such that \( \theta - \theta_m \in [k(\theta_M - \theta_m), (k + 1)(\theta_M - \theta_m)) \). We define \( a(\theta) \) by

\[
a(\theta) = \begin{cases} 
\theta - k(\theta_M - \theta_m) & \text{if } k \text{ is even}, \\
(k + 1)(\theta_M - \theta_m) - \theta & \text{if } k \text{ is odd}.
\end{cases}
\]

Please see Figure 1.

We use \( a \) to extend \( \theta \mapsto n(x, \theta, t) \) to a function \( \tilde{n}(x, \theta, t) = n(x, a(\theta), t) \) defined for all \( \theta \in \mathbb{R} \). This is equivalent to extending \( \theta \mapsto n(x, \theta, t) \) by even reflection across \( \theta = \theta_M \) (since \( n \) satisfies Neumann boundary conditions, this gives a periodic function on \( (\theta_m, \theta_m + 2(\theta_M - \theta_m)) \)) with derivative zero on the boundary) and then by periodicity to the rest of \( \mathbb{R} \). Please see Figure 1.

If we use \( a \) to extend an arbitrary twice differentiable function on \( \Theta \) with derivative zero at the boundary to all of \( \mathbb{R} \), the result would still have continuous first derivative but would not necessarily be twice differentiable in \( \theta \). However, it turns out that since \( n \) solves the equation (E), the extension \( \tilde{n} \) is in \( C^{2,\eta} \). Moreover, the Hölder norm of the second derivatives of \( \tilde{n} \) is bounded in terms of the supremum of \( n \). We state this precisely in the following Proposition.

Proposition 2.1. Suppose \( n \) is nonnegative, twice differentiable on \( \mathbb{R} \times \Theta \times (0,\infty) \) and satisfies (E) in the classical sense. Assume that \( \Theta \) satisfies (A[3]) and the initial condition \( n_0 \) satisfies (A[2]). We define the extension \( \tilde{n} : \mathbb{R} \times \mathbb{R} \times [0,\infty) \to \mathbb{R} \) by

\[
\tilde{n}(x, \theta, t) = n(x, a(\theta), t).
\]

(1) We have that \( \tilde{n} \) is twice differentiable on \( \mathbb{R} \times \mathbb{R} \times (0,\infty) \) and satisfies

\[
\begin{aligned}
\tilde{n}_t &= \theta \tilde{n}_{xx} + \alpha \tilde{n}_{\theta \theta} + r \tilde{n}(1 - \rho) & \text{on } \mathbb{R} \times \mathbb{R} \times (0,\infty), \\
\tilde{n}(x, \theta, 0) &= n_0(x, \theta) & \text{for all } (x, \theta) \in \mathbb{R} \times \mathbb{R}.
\end{aligned}
\]

(2) Let \( T > 0 \) and assume \( \sup_{\mathbb{R} \times \Theta \times [0,T]} n = M \geq 1 \). There exists a positive constant \( \bar{C} \) that depends on \( \eta, \alpha, r, \theta_m \) and \( \theta_M \) so that

\[
|\tilde{n}|_{2+\eta,\mathbb{R}\times\mathbb{R}\times[0,T]} \leq \bar{C}(M^{2+\eta/2} + |n_0(x, a(\theta))|_{2+\eta,\mathbb{R}\times\mathbb{R}}).
\]

Proposition 2.1 follows from standard estimates and interpolation inequalities for seminorms.
Instead of working directly with $\bar{n}$, we work with its averages in $\theta$ on small intervals: we define the function $v$ by

$$v(x, \zeta, t) = \int_{\zeta-\sigma/2}^{\zeta+\sigma/2} \bar{n}(x, \theta, t) \, d\theta.$$  

Our idea to use these averages came from reading the proof of [25, Theorem 1.2]. We have that $v$ is twice differentiable on $\mathbb{R}^2 \times (0, \infty)$ and satisfies

$$v_t(x, \zeta, t) = \int_{\zeta-\sigma/2}^{\zeta+\sigma/2} a(\theta) \bar{n}_{xx}(x, \theta, t) \, d\theta + \alpha v_{\zeta\zeta}(x, \zeta, t) + rv(x, \zeta, t)(1 - \rho(x, t))$$

for all $x \in \mathbb{R}$, $\zeta \in \mathbb{R}$, and $t \in (0, \infty)$.

The first step in our proof of the supremum bound on $n$ is to obtain a bound on $\sup_{x,\zeta,t} v(x, \zeta, t)$ for some $\sigma > 0$. We then use the following proposition to extend the bound on $v$ to the supremum bound on $n$ itself.

**Proposition 2.2.** Suppose $\bar{u}$ is a non-negative classical solution of

$$\begin{cases}
\bar{u}_t = a(\theta)\bar{u}_{xx} + \alpha \bar{u}_{\theta\theta} & \text{on } \mathbb{R} \times \mathbb{R} \times (0, \infty),
\bar{u}(x, \theta, 0) = \bar{u}_0(x, \theta) & \text{on } \mathbb{R} \times \mathbb{R},
\end{cases}$$

where $\bar{u}_0(x, \theta) \in C(\mathbb{R} \times \mathbb{R})$ satisfies, for some $0 < \sigma \leq 1$ and for all $x \in \mathbb{R}$ and $\zeta \in \mathbb{R}$,

$$\int_{\zeta-\sigma/2}^{\zeta+\sigma/2} \bar{u}_0(x, \theta) \, d\theta \leq C_1.$$

There exists a constant $C_0$ that depends only on $\alpha$, $\theta_m$ and $\theta_M$ such that, for all $x$ and $\theta$,

$$\bar{u}(x, \theta, 1) \leq \frac{C_0 C_1}{\sigma}.$$

We will use the observation that $n(x, \theta, t)e^{-t}$ is a subsolution of the heat equation and apply this Proposition with initial data $n(x, \theta, t^*)$ at a certain fixed time $t^*$.

Proposition 2.2 follows from certain estimates for the heat kernel associated to the operator $\partial_t - a(\theta)\partial_x^2 - \partial_{\theta\theta}$. These estimates say that this kernel is like the kernel for the heat equation and were established by Aronson [3].

We postpone the proofs of the propositions and use them to present:

**2.2. Proof of the supremum bound.**

**Proof of Theorem 2.1.** Let $C_0$ be the constant from Proposition 2.2 and let $C$ be the constant from Proposition 2.1. Define the constants $\bar{C}$ and $M_0$ by

$$\bar{C} = \bar{C}(1 + |n_0(x, a(\theta))|_{2+\eta, \mathbb{R} \times \mathbb{R}})$$

and

$$M_0 = \max \left\{ (3eC_0)^{4+2\eta} \bar{C}^{4+2\eta} r^{-\eta} e \sup_{\bar{R} \times \theta} n_0, \frac{3eC_0}{\theta_M - \theta_m} \right\}.$$

We claim

$$\sup_{\mathbb{R} \times \theta \times (0, \infty)} n \leq M_0$$

for all numbers $M > M_0$.

If (2.7) holds, then we will have

$$\sup_{\mathbb{R} \times \theta \times (0, \infty)} n \leq M_0,$$

which will complete the proof. We proceed by contradiction: let us assume $M$ is a number with $M > M_0$ and

$$\sup_{\mathbb{R} \times \theta \times (0, \infty)} n > M.$$

Throughout the rest of the proof of this proposition, $C$ denotes a positive constant that may change from line to line and depends only on $\theta_m$, $\theta_M$, $r$, $\alpha$ and $\eta$ (in particular, $C$ does not depend on $M$).
Let us consider the map $S(t)$ that takes $t$ to the supremum of $n$ at time $t$, in other words:

$$S(t) = \sup_{x \in \mathbb{R}, \theta \in \Theta} n(x, \theta, t).$$

The map $S$ is continuous. In addition, since $n$ satisfies $[E]$, and $n$ and $\rho$ are non-negative, we have that $n(x, \theta, t)e^{-t}$ is a subsolution of

\begin{align}
&\begin{cases}
    u_t = \theta u_{xx} + \alpha u_{\theta \theta} & \text{on } \mathbb{R} \times \Theta \times (0, \infty), \\
    u(x, \theta, t) = u(x, \theta_m, t) = 0 & \text{for all } x \in \mathbb{R}, t \in (0, \infty), \\
    u(x, \theta, 0) = n_0(x, \theta) & \text{on } \mathbb{R} \times \Theta.
\end{cases} \\
&\text{(2.9)}
\end{align}

The equation \((2.9)\) satisfies the comparison principle. Therefore, we have the following bound on $n(x, \theta, t)e^{-t}$ from above: for all $x \in \mathbb{R}$ and $\theta \in \Theta$,

$$n(x, \theta, t)e^{-t} \leq u(x, \theta, t) \leq \sup_{x \in \mathbb{R}, \theta \in \Theta} n_0(x, \theta).$$

Taking supremum in $x$ and $\theta$ and multiplying by $e^t$ gives the bound

$$S(t) \leq e^t \sup_{x \in \mathbb{R}, \theta \in \Theta} n_0(x, \theta)$$

for all $t > 0$. In particular, taking supremum over $t \in (0, 1]$, we have

$$\sup_{t \in (0, 1]} S(t) \leq e \sup_{x \in \mathbb{R}, \theta \in \Theta} n_0(x, \theta) \leq M_0 < M,$$

where the second inequality follows from the definition of $M_0$. Line \((2.8)\) implies $\sup_t S(t) > M$. Since $S$ is continuous and $S(t) < M$ for $t \leq 1$, there exists a first time $T > 1$ for which $S(T) = M$. So, we have

$$\sup_{x \in \mathbb{R}, \theta \in \Theta} n = M \text{ and } \sup_{x \in \mathbb{R}, \theta \in \Theta} n(\cdot, \cdot, T) = M.$$

We will now work with the extension $\tilde{n}(x, \theta, t)$ defined in Proposition \ref{proposition:extension}. By the previous line, we have

$$\sup_{x \in \mathbb{R}, \theta \in \Theta} \tilde{n} = M \text{ and } \sup_{x \in \mathbb{R}, \theta \in \Theta} \tilde{n}(\cdot, \cdot, T) = M.$$

We apply Proposition \ref{proposition:extension} to $\tilde{n}$. Part \ref{part:extension1} implies that $\tilde{n}$ satisfies equation \((2.3)\). Part \ref{part:extension2} gives us the estimate

$$|\tilde{n}|_{2+\eta, \mathbb{R} \times \Theta \times (0, T]} \leq \tilde{C}(M^{2+\eta/2} + |n_0(x, a(\theta))|_{2+\eta, \mathbb{R} \times \mathbb{R}}) \leq \tilde{C}(M^{2+\eta/2} + n_0(x, a(\theta))|_{2+\eta, \mathbb{R} \times \mathbb{R}} M^{2+\eta/2}),$$

where the second inequality follows since $M \geq 1$. We use our choice of $\tilde{C}$ in \((2.5)\) to bound the right-hand side from the previous line from above and obtain,

$$|\tilde{n}|_{2+\eta, \mathbb{R} \times \Theta \times (0, T]} \leq \tilde{C} M^{2+\eta/2}.$$

Let us take

$$\sigma = \min\{1, \theta_M - \theta_m, \tilde{C} M^{\frac{4+\eta}{4+\eta}} M^{\frac{2+\eta}{4+\eta}} \}$$

and define $v(x, \zeta, t) : \mathbb{R} \times \Theta \times (0, \infty) \rightarrow \mathbb{R}$ by

$$v(x, \zeta, t) = \int_{\zeta - \sigma/2}^{\zeta + \sigma/2} \tilde{n}(x, \theta, t) d\theta.$$

**First Step:** We will prove

$$\sup_{x \in \mathbb{R}, \theta \in \Theta \times (0, T]} v \leq 3.$$

Since $n$ satisfies $[E]$, we have that $v$ satisfies

$$v_k(x, \zeta, t) = \int_{\zeta - \sigma/2}^{\zeta + \sigma/2} \rho(x) \tilde{n}_{xx}(x, \theta, t) d\theta + \alpha \rho(x) \tilde{n}_{\theta \theta}(x, \theta, t) + v(x, \zeta, t)(1 - \rho(x, t)).$$

Let us explain why we may assume, without loss of generality, that the supremum of $v$ on $\mathbb{R} \times \Theta \times (0, T]$ is achieved at some $(x_0, \zeta_0, t_0)$. Since $\zeta \mapsto v(x, \zeta - \theta_m, t)$ is periodic of period $2(\theta_M - \theta_m)$, and we are considering times $t$ in the bounded interval $(0, T]$, we know that there exist $\zeta_0 \in \Theta$, $t_0 \in [0, T]$ and a sequence $\{x_k\}_{k=1}^\infty$ with $v(x_k, \zeta_0, t_0) \rightarrow \sup_{x \in \mathbb{R}, \theta \in \Theta \times (0, T]} v$ as $k \rightarrow \infty$. For each $k$, we define the translated functions

$$\tilde{n}_{xx}^k(x, t) = \tilde{n}_{xx}(x + x_k, \theta, t),$$

where $\tilde{n}_{xx}^k(x, t)$ satisfies the comparison principle. Therefore, we have the following bound on $n(x, \theta, t)e^{-t}$ from above: for all $x \in \mathbb{R}$ and $\theta \in \Theta$,
\[ \rho^k(x,t) = \rho(x + x_k,t), \]

and

\[ v^k(x,t) = v(x + x_k, \theta,t). \]

We similarly define the translates of the first and second derivatives of \( v \). We have \( \bar{n}_{xx} \in C^n \), and according to (2.11),

\[ |\bar{n}_{xx}|_{0,R \times R \times [0,T]} \leq \bar{C}M^{2+n/2}. \]

Therefore, \( \bar{n}_{xx}^k, \rho^k, v^k \), and the translates of the first and second derivatives of \( v \) are uniformly bounded and uniformly equicontinuous on \( R \times R \times [0,T] \). Hence, there exists a subsequence (still denoted by \( k \)) and functions \( \rho^\infty, \bar{n}_{xx}^\infty \), and \( v^\infty \) such that \( \rho^k, \bar{n}_{xx}^k \) and \( v^k \) converge locally uniformly to \( \rho^\infty, \bar{n}_{xx}^\infty \) and \( v^\infty \), respectively; the derivatives of \( v^k \) converge locally uniformly to those of \( v^\infty \); and \( v^\infty(x,\zeta,t) = \int_{\zeta-\sigma/2}^{\zeta+\sigma/2} \bar{n}^\infty(x,\theta,t)\,d\theta \). Moreover, \( v^\infty \) satisfies

\[
\partial_t v^\infty(x,\zeta,t) = \int_{\zeta-\sigma/2}^{\zeta+\sigma/2} a(\theta)\bar{n}_{xx}^\infty(x,\theta,t)\,d\theta + \alpha v^\infty v^\infty(x,\zeta,t) + rv^\infty(x,\zeta,t)(1 - \rho^\infty(x,t))
\]

on \( R \times R \times (0,T) \) and we have, for all \( x \in R \), all \( \zeta \in \Theta \), and all \( t \leq T \),

\[ v^\infty(x,\zeta,t) \leq v^\infty(0,\zeta,t) = \sup_{R \times \Theta \times (0,T)} v; \]

in other words, \( v^\infty \) achieves its supremum on \( R \times R \times (0,T) \). We now drop the superscript \( \infty \).

At the point \((x_0,\zeta_0,t_0)\) where \( v \) achieves its supremum, we have

\[ v_t(x_0,\zeta_0,t_0) \geq 0, \quad v_{\zeta\zeta}(x_0,\zeta_0,t_0) \leq 0, \]

and

\[ 0 \geq v_{xx}(x_0,\zeta_0,t_0) = \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} \bar{n}_{xx}(x_0,\theta,t_0)\,d\theta. \]

We point out that \( v_{xx} \) does not appear in (2.12), the equation that \( v \) satisfies. We will bound from above the corresponding term that does appear in (2.12). This term is:

\[ \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} a(\theta)\bar{n}_{xx}(x_0,\theta,t_0)\,d\theta. \]

The inequality (2.14) implies that there exists \( \theta^* \in (\zeta_0 - \sigma/2, \zeta_0 + \sigma/2) \) with \( \bar{n}_{xx}(x_0,\theta^*,t_0) \leq 0 \). In addition, let \( \tilde{\theta} \in [\zeta_0 - \sigma/2, \zeta_0 + \sigma/2] \) be so that

\[ \min_{\theta \in [\zeta_0 - \sigma/2, \zeta_0 + \sigma/2]} a(\theta) = a(\tilde{\theta}). \]

Since \( a \) is positive, we multiply (2.14) by \( -a(\tilde{\theta}) \) and find

\[ 0 \leq \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} -a(\tilde{\theta})\bar{n}_{xx}(x_0,\theta,t_0)\,d\theta. \]

Adding the term (2.15) that we’re interested in to both sides of this inequality, we find

\[ \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} a(\theta)\bar{n}_{xx}(x_0,\theta,t_0)\,d\theta \leq \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} (a(\theta) - a(\tilde{\theta}))\bar{n}_{xx}(x_0,\theta,t_0)\,d\theta. \]

Let us recall that \( a(\theta) - a(\tilde{\theta}) \) is non-negative on \([\zeta_0 - \sigma/2, \zeta_0 + \sigma/2]\). We thus use the estimate (2.11) on the seminorm of \( \bar{n}_{xx} \) and the fact that \( \bar{n}_{xx}(x_0,\theta^*,t_0) \leq 0 \) to estimate the right-hand side of the previous line from above and obtain

\[ \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} a(\theta)\bar{n}_{xx}(x_0,\theta,t_0)\,d\theta \leq \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} (a(\theta) - a(\tilde{\theta}))\bar{n}_{xx}(x_0,\theta^*,t_0) + |\theta - \theta^*|^\alpha \bar{C}M^{2+n/2} \,d\theta \]

\[ \leq \int_{\zeta_0-\sigma/2}^{\zeta_0+\sigma/2} (a(\theta) - a(\tilde{\theta}))|\theta - \theta^*|^\alpha \bar{C}M^{2+n/2} \,d\theta. \]
Since \( \theta^* \in (\zeta_0 - \sigma/2, \zeta_0 + \sigma/2) \), we have \( |\theta - \theta^*| \leq \sigma \). In addition, \( a \) is Lipschitz with Lipschitz constant 1, so we have \( (a(\theta) - a(\bar{\theta})) \leq |\theta - \bar{\theta}| \). We use these two inequalities to bound the right-hand side of the previous line from above and find

\[
\int_{\zeta_0 - \sigma/2}^{\zeta_0 + \sigma/2} a(\theta) \bar{n}_{xx}(x_0, \theta, t_0) \, d\theta \leq \bar{C} \sigma^2 M^{2+\eta/2} \int_{\zeta_0 - \sigma/2}^{\zeta_0 + \sigma/2} |\theta - \bar{\theta}| \, d\theta \leq \frac{1}{2} \bar{C} \sigma^{2+\eta} M^{2+\eta/2}.
\]

The last inequality follows by an elementary calculus computation that relies on the fact that \( \bar{\theta} \) is contained in \( [\zeta_0 - \sigma/2, \zeta_0 + \sigma/2] \).

Using this estimate together with the information \( \bar{C} \) about the other derivatives of \( v \) at \( (x_0, \zeta_0, t_0) \) in the equation \( \bar{C} \) that \( v \) satisfies, we obtain

\[
0 \leq \frac{C}{2} \sigma^{2+\eta} M^{2+\eta/2} + rv(x_0, \zeta_0, t_0)(1 - \rho(x_0, t_0)).
\]

Since \( \sigma \leq \theta_M - \theta_m \) we may bound \( \rho(x_0, t_0) \) from below by \( \frac{v(x_0, \zeta_0, t_0)}{2} \):

\[
v(x_0, \zeta_0, t_0) = \int_{\zeta_0 - \sigma/2}^{\zeta_0 + \sigma/2} \bar{n}(x_0, \theta, t_0) \, d\theta \leq 2 \int_{\theta_m}^{\theta_M} n(x_0, \theta, t_0) \, d\theta = 2 \rho(x_0, t_0).
\]

We use the previous estimate to bound the right-hand side of \( \bar{C} \) from above and obtain

\[
0 \leq \frac{C}{2} \sigma^{2+\eta} M^{2+\eta/2} + rv(x_0, \zeta_0, t_0)(1 - \frac{1}{2} v(x_0, \zeta_0, t_0)).
\]

Upon rearranging, we find,

\[
\frac{r}{2} \bar{n}^2(x_0, \zeta_0, t_0) \leq \frac{C}{2} \sigma^{2+\eta} M^{2+\eta/2} + rv(x_0, \zeta_0, t_0).
\]

By our choice of \( \sigma \), we have \( \sigma \leq \bar{C} \sigma^{2+\eta} M^{2+\eta/2} = \bar{C} \sigma^{2+\eta} M^{2+\eta/2} \). We use this to bound the right-hand side of the previous line and find

\[
\frac{r}{2} \bar{n}^2(x_0, \zeta_0, t_0) \leq \frac{r}{2} + rv(x_0, \zeta_0, t_0),
\]

so

\[
v(x_0, \zeta_0, t_0) \leq \frac{1}{2} (1 + \sqrt{3}) \leq 3.
\]

Since \( v \) achieved its supremum on \( \mathbb{R} \times \mathbb{R} \times (0, T) \) at \( (x_0, \zeta_0, t_0) \), we conclude

\[
\sup_{\mathbb{R} \times \mathbb{R} \times (0, T)} v \leq 3.
\]

**Second Step:** We will now deduce a supremum bound on \( \bar{n} \) from the bound on \( v \). Let us fix any \( t^* \in (1, T) \). Let \( u \) be the solution of

\[
\begin{cases}
  u_t = \theta u_{xx} + u_{\theta\theta} & \text{on } \mathbb{R} \times \mathbb{R} \times (0, \infty), \\
  u(x, \theta, 0) = \bar{n}(x, \theta, t^* - 1) & \text{on } \mathbb{R} \times \mathbb{R}.
\end{cases}
\]

We have that \( \bar{n}(x, \theta, t + t^* - 1)e^{-t} \) is a subsolution of the equation for \( u \) for \( t \geq 0 \). Since they are equal at \( t = 0 \), the comparison principle for the equation for \( u \) implies the bound

\[
\bar{n}(x, \theta, t + t^* - 1)e^{-t} \leq u(x, \theta, t)
\]

for all \( x, \theta, \) and \( t \geq 0 \). In particular, we evaluate the above at \( t = 1 \) and take supremum in \( x \) and \( \theta \) to find

\[
\sup_{x \in \mathbb{R}, \theta \in \mathbb{R}} \bar{n}(x, \theta, t^*) \leq e \sup_{x \in \mathbb{R}, \theta \in \mathbb{R}} u(x, \theta, 1).
\]

We will now apply Proposition 2.2 to \( u \). The supremum bound \( 2.17 \) on \( v \) says exactly that assumption \( 2.4 \) is satisfied, with \( C_1 = 3 \). Therefore, Proposition 2.2 implies

\[
\sup_{x \in \mathbb{R}, \theta \in \mathbb{R}} u(x, \theta, 1) \leq \frac{3C_0}{\sigma}.
\]
Therefore, we may bound $u$ by $\frac{3C_0}{\sigma}$ on the right-hand side of (2.18) and find,
\[
\sup_{x \in \mathbb{R}, \theta \in \mathbb{R}} \tilde{n}(x, \theta, t^*) \leq \frac{3eC_0}{\sigma}.
\]
This holds for any $t^* \leq T$, so in particular at $t^* = T$. According to line (2.10), we have $\sup_{x, \theta} \tilde{n}(x, \theta, T) = M$, so we obtain
\[
M \leq \frac{3eC_0}{\sigma}.
\]
Recall that we chose $\sigma = \min\{\theta_M - \theta_m, 1, C \frac{1}{\frac{1}{\gamma} + M - \frac{1}{\gamma - r_{\frac{1}{\gamma}}}}\}$. If $\sigma = \theta_M - \theta_m$, then we find $M \leq \frac{3eC_0}{\theta_M - \theta_m}$; if $\sigma = 1$, then $M \leq 3eC_0$; and if $\sigma = C \frac{1}{\frac{1}{\gamma} + M - \frac{1}{\gamma - r_{\frac{1}{\gamma}}}}$ we obtain
\[
M \leq 3eC_0 C \frac{1}{\frac{1}{\gamma} + M - \frac{1}{\gamma - r_{\frac{1}{\gamma}}}} \leq 3eC_0 C \frac{1}{\frac{1}{\gamma} + M - \frac{1}{\gamma - r_{\frac{1}{\gamma}}}}.
\]
which implies, since $1 - \frac{4+\eta}{4+2\eta} = \frac{n}{4+2\eta}$, the following bound for $M$:
\[
M \leq (3eC_0) \frac{(4+2\eta)}{\eta} C \frac{4+2\eta}{\eta} r_{\frac{1}{\gamma}} \leq 3eC_0.
\]
Therefore,
\[
M \leq \max\left\{ \frac{3eC_0}{\theta_M - \theta_m}, (3eC_0) \frac{(4+2\eta)}{\eta} C \frac{4+2\eta}{\eta} r_{\frac{1}{\gamma}} \leq 3eC_0 \right\}.
\]
But we had taken $M > M_0$. Recalling our choice of $M_0$ in line (2.6) yields the desired contradiction and hence the proof is complete. □

2.3. Proofs of auxiliary results for the supremum bound.

Proof of Proposition [2.2] The proof of Proposition 2.2 relies on a certain $L^\infty$ estimates on the heat kernel, which we formulate as the following lemma.

Lemma 2.1. There exists a kernel $K$ such that if $\bar{u}_0$ is non-negative and $\bar{u}$ is a weak solution of
\[
\begin{align*}
\bar{u}_t &= \alpha(\theta)\bar{u}_{xx} + \alpha \bar{u}_{\theta \theta}, \quad \text{on } \mathbb{R} \times \mathbb{R} \times (0, \infty),
\bar{u}(x, \theta, 0) &= \bar{u}_0(x, \theta), \quad \text{on } \mathbb{R} \times \mathbb{R},
\end{align*}
\]
then
\[
\bar{u}(x, \theta, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(t, x, y, \theta, r)\bar{u}_0(y, r) \, dr \, dy.
\]
Moreover, there exist constants $c_1$ and $c_2$ that depend only on $\alpha$, $\theta_m$ and $\theta_M$ such that
\[
c_1 t^{-1} e^{-\frac{c_1}{(x-y)^2+(\theta-\theta)^2}} \leq K(t, x, y, \theta, r) \leq c_2 t^{-1} e^{-\frac{c_2}{(x-y)^2+(\theta-\theta)^2}}
\]
for all $x, y, \theta$ in $\mathbb{R}$, and for all $t > 0$.

Proof. We apply two theorems of Aronson [3]. That we have $\bar{u} = K * \bar{u}_0$ is the content of [3, Theorem 11]. The bound on $K$ is exactly [3, item (ii), Theorem 10]. Aronson’s results apply to any parabolic equation $u_t + \text{tr}(A(x, \theta)D^2u) = G$ in $\mathbb{R}^n \times (0, \infty)$ with bounded coefficients and with $G$ regular enough. In our case, $n = 2$ and $G \equiv 0$ (so, in particular, it satisfies Aronson’s hypotheses) and $A(x, \theta)$ is the diagonal matrix with entries $a(\theta)$ and $\alpha$. In the general case, the constants $c_1$ and $c_2$ depend on the ellipticity constant of $A$ and the $L^\infty$ norm of $A$. Since these depend only on $\alpha$, $\theta_m$ and $\theta_M$, we have that $c_1$ and $c_2$ depend only on $\alpha$, $\theta_m$ and $\theta_M$ as well. □

For the proof of Proposition 2.2 we also need the following lemma. Its proof is elementary enough to give as an exercise in a calculus class.

Lemma 2.2. For $a > 0$ we have $\sum_{i=1}^{\infty} e^{-\alpha} \leq \frac{\pi}{2a}$.

Proof. We have a bound on the series we’re interested in by the corresponding integral of $e^{-\alpha} x^2$:
\[
\sum_{i=1}^{\infty} e^{-\alpha} \leq \int_{0}^{\infty} e^{-\alpha} x^2 \, dx.
\]
By change of variables, we obtain
\[
\int_0^\infty e^{-a^2x^2} \, dx = \frac{1}{a} \int_0^\infty e^{-x^2} \, dx = \frac{\pi}{2a}.
\]

We proceed with:

Proof of Proposition 2.2. According to Lemma 2.1, we have, for all \( x \in \mathbb{R}, \theta \in \mathbb{R}, \) and all \( t > 0, \)
\[
\bar{u}(x, \theta, t) = \int_\infty^{-\infty} \int_\infty^{-\infty} K(t, x, y, \theta, r) \bar{u}_0(y, r) \, dr \, dy
\leq \int_\infty^{-\infty} \int_\infty^{-\infty} ct^{-1} e^{-c((x-y)^2 + (\theta-r)^2)} \bar{u}_0(y, r) \, dr \, dy.
\]

Let us take \( t = 1 \) to obtain the following bound for \( \bar{u}(x, \theta, 1): \)
\[
\bar{u}(x, \theta, 1) \leq c \int_\infty^{-\infty} \int_\infty^{-\infty} e^{-c((x-y)^2 + (\theta-r)^2)} \bar{u}_0(y, r) \, dr \, dy
= c \int_\infty^{-\infty} e^{-c(x-y)^2} \int_\infty^{-\infty} e^{-cr^2} \bar{u}_0(y, r - \theta) \, dr \, dy.
\]

We will now split up the integral in \( r \) into a sum of integrals over intervals of size \( \sigma. \) We have
\[
\int_\infty^{-\infty} e^{-cr^2} \bar{u}_0(y, r - \theta) \, dr = \sum_{i=-\infty}^\infty \int_{i\sigma}^{(i+1)\sigma} e^{-cr^2} \bar{u}_0(y, r - \theta) \, dr.
\]
For \( r \in (i\sigma, (i+1)\sigma) \) we have
\[
e^{-cr^2} \leq e^{-ci^2\sigma^2}.
\]

We use this to bound each of the integrals in \( r \) and find,
\[
\int_\infty^{-\infty} e^{-cr^2} \bar{u}_0(y, r - \theta) \, dr = \sum_{i=-\infty}^\infty e^{-ci^2\sigma^2} \int_{i\sigma}^{(i+1)\sigma} \bar{u}_0(y, r - \theta) \, dr.
\]

For each \( i, \) we can take \( \zeta = i - \sigma/2 \) in assumption (2.4) of this proposition to obtain
\[
\int_{i\sigma}^{(i+1)\sigma} \bar{u}_0(y, r - \theta) \, dr \leq C_1.
\]
Therefore,
\[
\int_\infty^{-\infty} e^{-cr^2} \bar{u}_0(y, r - \theta) \, dr \leq \sum_{i=-\infty}^\infty e^{-ci^2\sigma^2} C_1 = C_1 \left( 1 + 2 \sum_{i=1}^\infty e^{-ci^2\sigma^2} \right) \leq C_1 \left( 1 + \frac{\pi}{2\sigma} \right),
\]
where the last inequality follows from the elementary bound on the sum of the above convergent series of Lemma 2.2. Since \( \sigma \leq 1 \) we find
\[
\int_\infty^{-\infty} e^{-cr^2} \bar{u}_0(y, r - \theta) \, dr \leq \frac{CC_1}{\sigma}.
\]

We now use this bound on the integral in \( r \) in the estimate (2.19) for \( \bar{u}(x, \theta, 1) \) and obtain
\[
\bar{u}(x, \theta, 1) \leq \frac{CC_1}{\sigma} \int_\infty^{-\infty} e^{-c(x-y)^2} \, dy = \frac{CC_1}{\sigma}.
\]
This holds for all \( x \) and \( \theta, \) so the proof is complete.
Proof of Proposition 2.1. We use the following result on the solvability in $C^{2+\eta}$ of parabolic equations:

**Proposition 2.3.** Let us take $\eta \in (0,1)$ and suppose the coefficients $a^{ij}$ are uniformly elliptic, $a^{ij} \in C^{\eta}(\mathbb{R}^2)$, $f \in C^{\eta}(\mathbb{R}^2 \times (0,T))$, and $u_0 \in C^{2+\eta}(\mathbb{R}^2)$. Then the initial value problem

$$
\begin{align*}
  u_t - a^{ij}u_{x_i x_j} &= f \text{ on } \mathbb{R}^2 \times (0,T) \\
  u &= u_0 \text{ on } \mathbb{R}^2 \times \{t = 0\}
\end{align*}
$$

has a unique solution $u \in C^{2+\eta}(\mathbb{R}^2 \times (0,T))$. Moreover, there exists a constant $C$ that depends on $\eta$, the ellipticity constant of $a^{ij}$, and $|a^{ij}|_{\eta, \mathbb{R}^2}$ such that

$$
|u|_{2+\eta, \mathbb{R}^2 \times [0,T]} \leq C(|f|_{\eta, \mathbb{R}^2 \times [0,T]} + |u_0|_{\eta, \mathbb{R}^2} + |u_0|_{2+\eta, \mathbb{R}^2}).
$$

**Proof.** That there exists a unique solution $u \in C^{2+\eta}$ is part of the statement of Ladyzenskaja et al [3]. Chapter IV Theorem 5.1. The estimate (2.20) follows from Krylov [20] Theorem 9.2.2, which asserts that $u$ obeys the estimate,

$$
|u|_{2+\eta, \mathbb{R}^2 \times [0,T]} \leq C_1(|a^{ij}u_{x_i x_j} - u_t - u|_{\eta, \mathbb{R}^2 \times [0,T]} + |u_0|_{2+\eta, \mathbb{R}^2}),
$$

where the constant $C_1$ depends only on $\eta$, the ellipticity constant of $a^{ij}$ and $|a^{ij}|_{\eta, \mathbb{R}^2}$. We use the equation that $u$ satisfies to bound from above the first term on the right-hand side of (2.21) and find,

$$
|a^{ij}u_{x_i x_j} - u_t - u|_{\eta, \mathbb{R}^2 \times [0,T]} = |f - u|_{\eta, \mathbb{R}^2 \times [0,T]} \leq |f|_{\eta, \mathbb{R}^2 \times [0,T]} + |u|_{\eta, \mathbb{R}^2 \times [0,T]}.
$$

Using the previous line to bound from above the first term on the right-hand of (2.21) yields the estimate (2.20). \qed

In addition, we need the following lemma:

**Lemma 2.3.** Assume $\Theta$ satisfies (A3). Given $T > 0$ and continuous and non-negative functions $\rho(x,t)$ and $u_0(x,\theta)$, the solution $u$ of

$$
\begin{align*}
  u_t &= \theta u_{xx} + \alpha u_{\theta\theta} + u(1-\rho) \text{ on } \mathbb{R} \times \Theta \times (0,T), \\
  u_\theta(x,\theta,m,t) &= u_\theta(x,\theta_M,m,t) = 0 \text{ for all } x \in \mathbb{R}, t \in (0,T), \\
  u(x,\theta,0) &= u_0(x,\theta) \text{ on } \mathbb{R} \times \Theta
\end{align*}
$$

is unique.

We point out that here $\rho$ is a given function and the equation (2.22) for $u$ is local. Because of this, the proof of the lemma is standard and we omit it.

**Proof of Proposition 2.1.** We apply Proposition 2.3 with right-hand side $f = \bar{n}(1-\rho)$, initial condition $u_0 = \bar{n}_0$, and the matrix of diffusion coefficients being the diagonal matrix with entries $a(\theta)$ and $\alpha$. The assumption (A2) on $u_0$ says exactly that the assumption of Proposition 2.3 on the initial condition is satisfied. By Proposition 2.3, there exists a unique solution $\bar{u}$ of

$$
\begin{align*}
  \bar{u}_t - a(\theta)\bar{u}_{xx} - \alpha \bar{u}_{\theta\theta} &= \bar{n}(1-\rho) \text{ on } \mathbb{R}^2 \times (0,T) \\
  \bar{u} &= \bar{n}_0 \text{ on } \mathbb{R}^2
\end{align*}
$$

and we have the estimate

$$
|\bar{u}|_{2+\eta, \mathbb{R}^2 \times [0,T]} \leq C(|\bar{n}(1-\rho)|_{\eta, \mathbb{R}^2 \times [0,T]} + |\bar{u}_0|_{\eta, \mathbb{R}^2 \times [0,T]} + |\bar{u}_0|_{2+\eta, \mathbb{R}^2}),
$$

where $C$ depends on $\theta_m, \theta_M$ and $\alpha$. Let us remark that the maps $\theta \mapsto \bar{n}(x,\theta,t)(1-\rho(x,t))$ and $\theta \mapsto \bar{n}_0(x,\theta,t)$ satisfy

$$
v(k(\theta_M - \theta_m) - \theta) = v(k(\theta_M - \theta_m) + \theta) = v(\theta) \text{ for all } \theta \in \mathbb{R}, k \in \mathbb{Z}.
$$

Together with the fact that the solution $\bar{u}$ is unique, this implies $\bar{u}$ has this symmetry as well. Thus, $\bar{u}_\theta(x,\theta_m,t) = \bar{u}_\theta(x,\theta_M,t) = 0$ for all $x$ and for all $t > 0$. Therefore, $\bar{u}(x,\theta,t)$ with $\theta \in \Theta$ satisfies

$$
\begin{align*}
  \bar{u}_t &= \theta \bar{u}_{xx} + \alpha \bar{u}_{\theta\theta} + \bar{n}(1-\rho) \text{ on } \mathbb{R} \times \Theta \times (0,T), \\
  \bar{u}_\theta(x,\theta,m,t) &= \bar{u}_\theta(x,\theta_M,m,t) = 0 \text{ for all } x \in \mathbb{R}, t \in (0,T), \\
  \bar{u}(x,\theta,0) &= \bar{n}_0(x,\theta) \text{ on } \mathbb{R} \times \Theta.
\end{align*}
$$
Since \( n \) also satisfies this equation, Lemma 2.3 implies \( \bar{u}(x, \theta, t) = n(x, \theta, t) \) for all \( x \in \mathbb{R} \), all \( \theta \in \Theta \), and all \( t \in (0, T) \). Therefore, \( \bar{n} \equiv \bar{u} \). In particular, we have that item \( 2 \) of the proposition holds. In addition, the estimate \((2.23)\) holds for \( \bar{n} \) and reads,

\[
|\bar{n}|_{2+\eta, \mathbb{R} \times \Theta \times [0, T]} \leq C(|\bar{n}|(1-\rho)|_{\eta, \mathbb{R} \times \Theta \times [0, T]} + |\bar{n}|_{\eta, \mathbb{R} \times [0, T]} + |\bar{n}_0(x, a(\theta))|_{2+\eta, \mathbb{R} \times \mathbb{R}}),
\]

where the constant \( C \) depends on \( \eta, \alpha, r, \theta_m \) and \( \theta_M \). We will now establish item \( 2 \) of the proposition.

Let us recall that we assume

\[
\sup_{\mathbb{R} \times \Theta \times [0, T]} n = M
\]

and \( M \geq 1 \). We have \( \sup_{\mathbb{R} \times \Theta \times [0, T]} n = \sup_{\mathbb{R} \times \mathbb{R} \times [0, T]} \bar{n} \). For the remainder of the proof of this proposition, we drop writing the domains in the semi-norms and norms (it is always \( \mathbb{R} \times \mathbb{R} \times [0, T] \)). We will now bound the right-hand side of \((2.24)\) from above. To this end, we first recall the definition of \( |\cdot| \):

\[
(2.25) \quad |\bar{n}|(1-\rho)|_{\eta} = |\bar{n}|(1-\rho)|_{\eta} + ||\bar{n}|(1-\rho)||_\infty.
\]

For functions \( f \) and \( g \) we have the elementary estimate

\[
|fg|_{\eta} \leq ||f||_\infty |g|_{\eta} + ||g||_\infty |f|_{\eta}.
\]

We apply this with \( f = \bar{n} \) and \( g = (1-\rho) \) to obtain a bound from above on the first term of the right-hand side of \((2.25)\):

\[
|\bar{n}|(1-\rho)|_{\eta} \leq |\bar{n}|_{\eta}||(1-\rho)||_\infty + |1-\rho|_{\eta}||\bar{n}||_\infty + ||\bar{n}|(1-\rho)||_\infty.
\]

In addition, the definition of \( \rho \) implies \([(1-\rho)|_{\eta} \leq |\bar{n}|_{\eta} \) and \((||(1-\rho)||_\infty \leq M \). We use this to estimate the right-hand side of the previous line and find

\[
(2.26) \quad |\bar{n}|(1-\rho)|_{\eta} \leq 2M|\bar{n}|_{\eta} + M + M^2.
\]

Similarly we estimate the second term on the right-hand side of \((2.24)\):

\[
|\bar{n}|_{\eta} = |\bar{n}|_{\eta} + ||\bar{n}||_\infty \leq |\bar{n}|_{\eta} + M.
\]

We use \((2.26)\) and the previous line to bound from above the first and second terms, respectively, on the right-hand side of \((2.24)\) and obtain,

\[
|\bar{n}|_{2+\eta} \leq C((2M|\bar{n}|_{\eta} + M + M^2) + |\bar{n}|_{\eta} + M + |\bar{n}_0(x, a(\theta))|_{2+\eta})
\]

We now use \( M \geq 1 \) and obtain,

\[
(2.27) \quad |\bar{n}|_{2+\eta} \leq C(M|\bar{n}|_{\eta} + M^2 + |\bar{n}_0(x, a(\theta))|_{2+\eta}).
\]

By interpolation estimates for the seminorms \([\cdot]\) (for example, [29] Theorem 8.8.1), we have, for any \( \varepsilon > 0 \),

\[
(2.28) \quad |\bar{n}|_{\eta} \leq C(\varepsilon|\bar{n}|_{2+\eta} + \varepsilon^{-\eta/2}M).
\]

We use the estimate \((2.28)\) in the right-hand side of the bound \((2.27)\) for \(|\bar{n}|_{2+\eta}\) and obtain

\[
|\bar{n}|_{2+\eta} \leq C(M(\varepsilon|\bar{n}|_{2+\eta} + \varepsilon^{-\eta/2}M) + M^2 + |\bar{n}_0(x, a(\theta))|_{2+\eta})
\]

\[
= C_1M\varepsilon|\bar{n}|_{2+\eta} + CM^2\varepsilon^{-\eta/2} + CM^2 + C|\bar{n}_0(x, a(\theta))|_{2+\eta}.
\]

Choosing \( \varepsilon = \frac{1}{2C_1M} \), we obtain \( C_1M\varepsilon = \frac{1}{2} \) and \( M^2\varepsilon^{-\eta/2} = M^{2+\eta/2} \). Rearranging the above we thus obtain,

\[
|\bar{n}|_{2+\eta} \leq CM^2^{2+\eta/2} + CM^2 + C|\bar{n}_0(x, a(\theta))|_{2+\eta} \leq CM^2^{2+\eta/2} + C|\bar{n}_0(x, a(\theta))|_{2+\eta},
\]

where the last inequality follows since \( M \geq 1 \).
3. Estimates on $u^\varepsilon$

In this section we give the proof of Proposition 1.1. We believe that the arguments in this section are the most technical ones of the paper.

Let us remark that $u^\varepsilon(x, \theta, t) = \varepsilon \ln n^\varepsilon(x, \theta, t)$ satisfies

$$
(U_\varepsilon) \begin{cases} 
\partial_t u^\varepsilon = \varepsilon \partial_{xx} u^\varepsilon + \frac{\alpha}{\varepsilon} \partial_{\theta \theta} u^\varepsilon + \theta (\partial_x u^\varepsilon)^2 + \alpha \left( \frac{\partial_{xx} u^\varepsilon}{\varepsilon} \right)^2 + r(1 - \rho^\varepsilon) & \text{on } \mathbb{R} \times \Theta \times (0, \infty), \\
\partial_{\theta \theta} u^\varepsilon(x, \theta_m, t) = \partial_{\theta} u^\varepsilon(x, \theta_M, t) = 0 & \text{for all } (x, t) \in \mathbb{R} \times (0, \infty), \\
u^\varepsilon(x, \theta, 0) = \varepsilon \ln(n_0(x, \theta)).
\end{cases}
$$

In subsection 3.1 we establish the lower bound on $u^\varepsilon$ by a barrier argument that is quite similar to [20, Lemma 2.1] and [5, Lemma 1.2 item (i)]. We skip a few details in the proof because it is quite similar to the cited ones. In subsection 3.2 we prove the gradient estimate by an application of the so-called “Bernstein method” (see [20, Lemma 2.1] and [5, Lemma 2.2]). This argument is quite lengthy and has a some rather technical parts. We provide all the details.

3.1. Proof of lower bound of Proposition 1.1

In order to construct the barrier necessary for the proof of the lower bound, we need a lemma. The proof of the lemma is an elementary computation which we include for the sake of completeness.

Lemma 3.1. Given $R < 1/2$ we define the function $\xi(z)$ by,

$$
\xi(z) = \frac{1}{2z - R^2}.
$$

There exists a constant $C$ that depends only on $R$ such that for all $z \in (-R, R)$ and for all $0 < \varepsilon < 1$,

$$
-\varepsilon \xi_2(z) - (\xi_2(z))^2 \leq C \quad \text{and} \quad -\varepsilon \xi_2(z) - (\xi_2(z))^2 \leq C.
$$

Proof. Computing the derivatives of $\xi$, substituting them into $-\varepsilon \xi_2(z) - (\xi_2(z))^2$ and rearranging yields, for $z \in (-R, R)$,

$$
-\varepsilon \xi_2(z) - (\xi_2(z))^2 = \frac{2}{(z^2 - R^2)^2} \left( 1 + \frac{2z^2}{R^2 - z^2} \left( 2 - \frac{1}{R^2 - z^2} \right) \right).
$$

We have $R^2 - z^2 \leq R^2 \leq 1/4$, so that the term in the inner-most parentheses on the right-hand side of the previous line is bounded from above by $-2$. Consequently we find, for $z \in (-R, R)$,

$$
-\varepsilon \xi_2(z) - (\xi_2(z))^2 \leq \frac{2}{(z^2 - R^2)^2} \left( 1 - 2 \frac{2z^2}{R^2 - z^2} \right).
$$

We now consider $z$ such that $R \sqrt{2} \leq |z| \leq R$, in which case we have $0 \leq R^2 - z^2 \leq R^2 - R^2/2 = R^2/2$. Therefore,

$$
\frac{z^2}{R^2 - z^2} \geq \frac{2z^2}{R^2} \geq 1,
$$

where we have again used that $|z| \geq \frac{R}{\sqrt{2}}$. Hence the term in the parenthesis on the right-hand side of (3.1) is bounded from above by $-3$. Therefore, the right-hand side of (3.1) is negative for $\frac{R}{\sqrt{2}} \leq |z| \leq R$. Thus it is left to consider $z$ such that $|z| \leq \frac{R}{\sqrt{2}}$. But in this case, we see $R^2 - z^2 \geq R^2/2$. Using this, together with the fact that the term in the parenthesis on the right-hand side of (3.1) is bounded from above by 1, and find,

$$
-\varepsilon \xi_2(z) - (\xi_2(z))^2 \leq \frac{2}{(z^2 - R^2)^2} \leq \frac{8}{R^4}.
$$

Since $\xi_2(z)$ is non-negative for $z \in (-R, R)$, we have, for $0 < \varepsilon < 1$ and $z \in (-R, R)$,

$$
-\varepsilon \xi_2(z) \leq -\xi_2(z),
$$

so that,

$$
-\varepsilon \xi_2(z) - (\xi_2(z))^2 \leq -\xi_2(z) - (\xi_2(z))^2 \leq \frac{8}{R^2}.
$$

This completes the proof of the lemma.

$\square$
Proof of the lower bound of Proposition 1.1. Due to Corollary 2.1, there exists a positive constant $M$ that depends only on $\theta_m$, $\theta_M$, $\alpha$ and $r$ such that $u^\varepsilon$ is a supersolution of

$$
\partial_t u^\varepsilon = \varepsilon \theta \partial_{xx}^2 u^\varepsilon + \frac{\alpha}{\varepsilon} \partial_{\theta \theta} u^\varepsilon + \theta (\partial_x u^\varepsilon)^2 + \alpha \left( \frac{\partial_y u^\varepsilon}{\varepsilon} \right)^2 - M \text{ on } \mathbb{R} \times \Theta \times (0, \infty)
$$

Let us fix some cube $K_R = (x_0 - R, x_0 + R) \times (\theta_0 - R, \theta_0 + R)$ that satisfies $K_R \subset J$. Without loss of generality we may take $R < 1/2$. Since $n_0$ is continuous and positive in $K_R$, there exists a constant $\beta \leq 1$ such that $\inf_{K_R} n_0 \geq \beta$. Therefore, if $(x, \theta)$ is contained in $K_R$, we have

$$
u^\varepsilon(x, \theta, 0) = \varepsilon \ln u^0(x, \theta) \geq \ln(\beta).
$$

Let $C$ be the constant given by Lemma 3.1 and define $a = M + 2\theta_M C$. We define the function $\phi$ by,

$$
\phi(x, \theta, t) = \xi(x - x_0) + \varepsilon (\theta - \theta_0) - at + \ln(\beta) = \frac{1}{(x - x_0)^2 - R^2} + \frac{\varepsilon}{(\theta - \theta_0)^2 - R^2} - at + \ln(\beta).
$$

Using Lemma 3.1, it is easy to check that $\phi$ is a subsolution of (3.2) in $K_R \times (0, \infty)$. Moreover, $\phi$ lies below $u^\varepsilon$ on the parabolic boundary of $K_R \times (0, \infty)$. Indeed, for points $(x, \theta) \in K_R$, we have $\phi(x, \theta, 0) \leq \ln(\beta) \leq u^\varepsilon(x, \theta, 0)$. And, if $(x, \theta)$ is contained in $\partial K_R$, then $\phi(x, \theta, t) = -\infty \leq u^\varepsilon(x, \theta, t)$ for all $t > 0$. Therefore, the comparison principle for equation (3.2) implies

$$
u^\varepsilon \geq \phi \text{ on } K_R \times (0, \infty).
$$

Let us fix some $T > 0$. By (3.3) and the definition of $\phi$, we have,

$$
u^\varepsilon(x, \theta, t) \geq \frac{1}{4R^2} + \frac{\varepsilon}{4R^2} - aT + \ln(\beta) =: \tau, \text{ for all } (x, \theta, t) \in \bar{K}_{R/2} \times [0, T].
$$

We will now use this lower bound to build a barrier for $u^\varepsilon$ on $((\mathbb{R} \times \Theta) \setminus K_{R/2}) \times (0, T)$. We define the function $\psi$ by,

$$
\psi(x, \theta, t) = -\frac{b(x - x_0)^2}{t} - ct + \tau.
$$

We see that for $b$ and $c$ properly chosen and depending only on $T$, $R$, $\theta_m$, $\theta_M$, $\alpha$ and $r$, we have that $\psi$ is a subsolution of (3.2) on $((\mathbb{R} \times \Theta) \setminus K_{R/4}) \times (0, T)$. Moreover, we have $\psi(x, \theta, t) \leq \tau$ for all $(x, \theta, t)$. In particular, if $(x, \theta) \in \partial K_{R/2}$ and $t \in [0, T]$, then, according to (3.4), we have

$$
u^\varepsilon(x, \theta, t) \geq \tau \geq \psi(x, \theta, t).
$$

Since $\psi_\theta \equiv 0$ and $\psi(x, \theta, 0) = -\infty$ for all $(x, \theta)$, a comparison principle argument similar to that of Lemma 2.1 implies

$$
u^\varepsilon \geq \psi \text{ on } ((\mathbb{R} \times \Theta) \setminus K_{R/2}) \times (0, T).
$$

Together with (3.3), the previous estimate implies that if $Q \subset \subset \mathbb{R} \times (0, \infty)$, then there exists a constant $C$ that depends on $Q$, $\alpha$, $r$, $\theta_m$ and $\theta_M$ such that for all $0 < \varepsilon < 1$ and for all $(x, t) \in Q$ and all $\theta \in \Theta$, we have

$$
u^\varepsilon(x, \theta, t) \geq -C.
$$

3.2. Proof of the gradient bound of Proposition 1.1 Most of this subsection is devoted to the proof of:

**Proposition 3.1.** We denote $(x, \theta)$ by $y$. Suppose $u \in C^3(\mathbb{R} \times \Theta \times (0, \infty))$ satisfies

$$
\begin{align*}
\partial_t u &= \text{tr}(A(y)D^2 u) + H(Du, y) + f(y) \text{ on } \mathbb{R} \times \Theta \times (0, \infty), \\
\partial_\theta u(x, \theta_m, t) &= \partial_\theta u(x, \theta_M, t) = 0 \text{ for all } (x, t) \in \mathbb{R} \times (0, \infty),
\end{align*}
$$

where the coefficients $A$ and $H$ are given by

$$
A(y) = \begin{pmatrix} \varepsilon y_2 & 0 \\ 0 & \frac{\alpha}{\varepsilon} \end{pmatrix}, \quad H(p, y) = y_2 p_1^2 + \frac{\alpha}{\varepsilon} p_2^2
$$

and $f$ is independent of $y_2$ and satisfies

$$
-M \leq f \leq r
$$

\]
and
\begin{equation}
\|f_{y_1}\|_{L^\infty(\mathbb{R} \times \Theta \times (0, \infty))} \leq \varepsilon^{-1} M.
\end{equation}
Given \( Q \subset \subset Q' \subset \subset \mathbb{R} \times (0, T) \), there exists a constant \( C \) that depends only on \( Q, Q', \|u\|_{L^\infty(Q' \times \Theta)} \), \( \theta_m \), \( \theta_M \), \( \alpha \), \( r \) and \( M \) such that
\begin{equation}
\sup_{Q \times \Theta} u^2 \leq \varepsilon C.
\end{equation}

Proposition 3.1 is essentially the same as Part (II) of Proposition 1.1, but we believe the notation in Proposition 3.1 is more suited to the presentation of the proof. Let us now describe how Part (II) of Proposition 1.1 follows from Proposition 3.1.

**Proof of Part (II) of Proposition 1.1.** We apply Proposition 3.1 to \( u^\varepsilon \) with \( f = r(1 - \rho^\varepsilon) \). According to Part (I) of Proposition 1.1 which we have just established, we have that \( u^\varepsilon \) is bounded on \( Q' \times \Theta \), uniformly in \( \varepsilon \).

According to Corollary 2.1, we have that \( n^\varepsilon \), and hence \( \rho^\varepsilon \), is uniformly bounded from above. In addition, \( \rho^\varepsilon \) is non-negative. Hence the hypothesis (3.6) is satisfied.

Now let us demonstrate that (3.6) holds. Let us define \( n(x, \theta, t) = n^\varepsilon(\varepsilon x, \theta, \varepsilon t) \). Since \( n^\varepsilon \) satisfies (\( E_{\varepsilon} \)) and \( n_0 \) satisfies (\( A_2 \)), we have that \( n \) satisfies (\( E \)) with initial data that satisfies (\( A_2 \)). Thus, according to Proposition 2.1 and Theorem 2.1, \( n \) is bounded in \( C^2 \). Therefore, there exists a constant \( C \) so that, for all \( (x, \theta, t) \),
\[ |n^\varepsilon(x, \theta, t)| = \frac{1}{\varepsilon} |n_x(x, \theta, t)| \leq \frac{C}{\varepsilon}. \]
Integrating in \( \theta \) yields the desired estimate for \( \rho^\varepsilon \).

For the proof of Proposition 3.1 we will need the following elementary lemma.

**Lemma 3.2.** For any diagonal \( n \times n \) positive definite matrix \( M \), any \( n \times n \) matrices \( X, Y \), and any \( \beta > 0 \), we have
\begin{equation}
|\operatorname{tr}(MXY)| \leq \beta |\operatorname{tr}(MXX^T) + \frac{1}{4\beta} \operatorname{tr}(MYY^T)|.
\end{equation}
In addition, if \( X \) is symmetric then \( \operatorname{tr}(MXMX) = \operatorname{tr}(MMXX) \).

**Proof.** Let us fix \( M \) and define \( \langle \cdot, \cdot \rangle \) by \( \langle X, Y \rangle = \operatorname{tr}(MXY^T) \). We have that \( \langle \cdot, \cdot \rangle \) is an inner product on the space of \( n \times n \) matrices. Hence, \( \langle \cdot, \cdot \rangle \) satisfies the Cauchy Schwartz inequality. In other words, we have \( \langle X, Y \rangle^2 \leq \langle X, X \rangle \langle Y, Y \rangle \) for all \( n \times n \) matrices \( X \) and \( Y \). Taking square root and using Cauchy Schwartz for real numbers gives,
\[ |\langle X, Y \rangle| \leq \sqrt{\langle X, X \rangle} \sqrt{\langle Y, Y \rangle} \leq \beta \langle X, X \rangle + \frac{\langle Y, Y \rangle}{4\beta}. \]
Using the definition of \( \langle \cdot, \cdot \rangle \) gives that (3.9) holds.

For any three symmetric matrices \( A, B, C \), we have \( \operatorname{tr}(ABC) = \operatorname{tr}(BAC) \). We apply this with \( A = MX \), \( B = M \) and \( C = X \) and obtain the second claim of the lemma. \( \square \)

**Proof of Proposition 3.1.** Let \( \psi(y_1, t) \) be a cutoff function supported on \( Q' \) and identically 1 inside \( Q \). Let us define \( \zeta(y, t) = \psi^2(y_1, t) \) and
\[ \tilde{H}(p, y, t) = \zeta(y, t)H(p, y). \]
We record for future use the following facts:
\begin{equation}
H_p(Du, y) \cdot Du = 2H(Du, y)
\end{equation}
\begin{equation}
\tilde{H}_{pp}(Du, y) = \frac{2\zeta}{\varepsilon} A(y), \quad \text{and} \quad \tilde{H}_{yy}(Du, y) = \frac{2\psi^2}{\varepsilon} A(y) \cdot Du \otimes Du + \psi W + 2\zeta \psi W,
\end{equation}
where we denote \( W = \begin{pmatrix} 0 & u_{y_1} \\ 0 & 0 \end{pmatrix} \). Throughout the remainder of the argument we will use \( C \) and \( C_i \) with \( i = 1, 2, ... \) to denote constants that may depend on \( \alpha, r, \|u\|_{L^\infty(Q')} \), \( \theta_m, \theta_M, Q \) and \( M \). In addition, \( C \) may change from line to line. We also define the constant \( \lambda \) by
\[ \lambda = \frac{1}{16} \max \left\{ ||\zeta||_{L^\infty(Q')}, C_1, \frac{C_2}{\varepsilon}, \frac{C_4}{\varepsilon}, C_5 \sup_{Q'} (\zeta u_{y_1}), \frac{C_6}{\varepsilon} \right\} = \left\{ \frac{C}{\varepsilon}, C \sup_{Q'} (\zeta u_{y_1}) \right\}. \]
where the constants $C_i$ are specified later in the argument. We define the function
\begin{equation}
(3.12) \quad z(y, t) = \tilde{H}(Du, y, t) + \lambda u(y, t)
\end{equation}
and we claim
\begin{equation}
(3.13) \quad \text{if } (y_0, t_0) \text{ is the maximum of } z \text{ on } Q' \times \Theta \text{ and } (y_0, t_0) \notin (\partial Q') \times \Theta, \text{ then } H(Du(y_0, t_0), y_0) \leq C.
\end{equation}

**Why (3.13) implies the estimate (3.8).** Let us assume (3.13) holds, and we will establish (3.8).

**First step:** We will first prove the following upper bound on $\tilde{H}$ holds:
\begin{equation}
(3.14) \quad \tilde{H}(Du, y, t) \leq C + \lambda C \text{ for all } (y, t) \in Q' \times \Theta.
\end{equation}
To this end, we use (3.12) and the fact that $(y_0, t_0)$ is the maximum of $z$ to obtain,
\begin{equation}
(3.15) \quad \tilde{H}(Du, y, t) = z(y, t) - \lambda u(y, t) \leq z(y_0, t_0) - \lambda u(y_0, t_0) \text{ for all } (y, t) \in Q' \times \Theta.
\end{equation}
Next we consider two cases: the first is $(y_0, t_0) \in (\partial Q') \times \Theta$, and the second is $(y_0, t_0) \notin (\partial Q') \times \Theta$. In the first case, we have $\zeta(y_0, t_0) = 0$, so we find $z(y_0, t_0) = \lambda u(y_0, t_0)$. Therefore, the estimate (3.15) says,
\begin{equation}
\tilde{H}(Du, y, t) \leq \lambda u(y_0, t_0) - \lambda u(y, t) \text{ for all } (y, t) \in Q' \times \Theta.
\end{equation}
Since we have $||u||_{L^\infty(Q')} \leq C$, this establishes the estimate (3.14) in the case $(y_0, t_0) \in (\partial Q') \times \Theta$.

Now let us suppose $(y_0, t_0) \notin (\partial Q') \times \Theta$, so that, according to (3.13), we have
\begin{equation}
(3.16) \quad H(Du(y_0, t_0), y_0) \leq C.
\end{equation}
We now use (3.12) to express the right-hand side of (3.15) in terms of $\tilde{H}$ and $u$:
\begin{equation}
\tilde{H}(Du, y, t) \leq \tilde{H}(Du, y_0, t_0) + \lambda u(y_0, t_0) - \lambda u(y, t).
\end{equation}
We use that $\tilde{H} = \zeta H \leq H$ and the estimate (3.16) to bound the first term on the right-hand side of the previous line from above by $C$, and that $||u||_{L^\infty(Q' \times \Theta)} \leq C$ to bound the last two terms. This yields (3.14).

**Second step:** Next we will use (3.14) to verify the following upper bound on $u^2_{y_2}$ in terms of $C$ and $\lambda$:
\begin{equation}
(3.17) \quad \frac{\alpha}{\epsilon^2} u^2_{y_2}(y, t) \leq C + \lambda C \text{ for all } (y, t) \in Q \times \Theta.
\end{equation}
To this end, we use the definition of $\tilde{H}$ and the fact that $\zeta$ is identically 1 inside $Q$ to obtain,
\begin{equation}
\frac{\alpha}{\epsilon^2} u^2_{y_2}(y, t) = \tilde{H}(Du, y, t) - y_2 u^2_{y_1} \text{ for all } (y, t) \in Q \times \Theta.
\end{equation}
We use the estimate (3.14) to bound the first term on the right-hand side of the previous line from above by $C + \lambda C$. Since the second term is negative, we obtain the estimate (3.17).

**Third step:** There are two possible values for $\lambda$, and we will show that, together with the estimate (3.17) that we just established, either one implies the desired estimate (3.8). Let us first suppose $\lambda = C \epsilon^{-1}$. Upon substituting this on the right-hand side of (3.17) we find,
\begin{equation}
\frac{\alpha}{\epsilon^2} u^2_{y_2}(y, t) \leq C + \frac{C}{\epsilon},
\end{equation}
which yields the desired estimate (3.8) by multiplying both sides by $\epsilon^2$.

Now let us suppose that $\lambda$ takes on the other possible value, so that $\lambda = C \left( \sup_{Q'} |\zeta y_{y_1}| \right)$. We use that $\theta_m \leq y_2$, the definition of $\tilde{H}$, the line (3.14) and the value of $\lambda$ to obtain, for any $(y, t) \in Q' \times \Theta$,
\begin{equation}
\theta_m \zeta(y, t) u^2_{y_1}(y, t) \leq \zeta(y, t) y_2 u^2_{y_1}(y, t) \leq \tilde{H}(Du, y, t) \leq C + \frac{C}{\lambda} = C + C \left( \sup_{Q'} |\zeta y_{y_1}| \right).
\end{equation}
From this we deduce $\sup_{Q'} |\zeta y_{y_1}| \leq C$ and hence $\lambda \leq C$. Using this on the right-hand side of (3.17) implies that (3.8) holds.

**The proof of (3.13).** Let us suppose $(y_0, t_0)$ is the maximum of $z$ on $Q' \times \Theta$ and $(y_0, t_0) \notin (\partial Q') \times \Theta$, so that we have
\begin{equation}
(3.18) \quad 0 \leq z_t(y_0, t_0) - tr(AD^2_{y_2} z(y_0, t_0)), \text{ and } D_y z(y_0, t_0) = 0.
\end{equation}
(If $(y_0, t_0)$ is in the interior of $Q' \times \Theta$, then of course (3.18) holds. Otherwise, we have $(y_0, t_0) \in Q' \times \partial \Theta$, and the Neumann conditions for $u$ on $\partial \Theta$ imply that (3.18) holds.) We seek to establish
\begin{equation}
(3.19) \quad H(Du(y_0, t_0), y_0) \leq C.
\end{equation}
As we have just shown, once we establish (3.19) the proof of the proposition will be complete. To this end, we compute,

$$z_t - \text{tr}(AD^2u) = \lambda(u_t - \text{tr}(AD^2u)) + \zeta H_p(Du, y) \cdot (Du_t - \text{tr}(AD^3u)) + I$$

where $I$ is the sum of the left-over terms from $z_t$ and $-\text{tr}(AD^2z)$:

$$I = \tilde{H}_t(Du,y,t) - \text{tr} \left[ A \left( \tilde{H}_{pp}(Du, y, t)D^2uD^2u + 2\tilde{H}_{p}(Du, y, t)D^2u + \tilde{H}_{yy}(Du, y, t) \right) \right].$$

(Throughout, we use $D$ to denote the derivative in $y$.) We use that $u$ satisfies (3.5) to write the first term on the right-hand side of (3.20) as

$$\lambda(u_t - \text{tr}(AD^2u)) = \lambda(H(Du, y) + f(y)) \leq \lambda(H(Du, y) + r),$$

where we have used (3.6) to obtain the inequality. Now let us look at the second term on the right-hand side of (3.20). We recognize that $Du_t - \text{tr}(AD^3u)$ is “almost” the derivative of $u_t - \text{tr}(AD^2u)$, up to a term that involves a derivative of $A$. In addition, rearranging the equation that $u$ solves implies $u_t - \text{tr}(AD^2u) = H(Du, y) + f(y)$. We find:

$$Du_t - \text{tr}(AD^3u) = D(u_t - \text{tr}(AD^2u)) + \text{tr}(DAD^2u) = D(H(Du, y)) + Df(y) + \text{tr}(DAD^2u).$$

Multiplying by $\zeta$ we obtain,

$$\zeta(Du_t - \text{tr}(AD^3u)) = \zeta D(H(Du, y)) + \zeta Df(y) + \zeta \text{tr}(DAD^2u).$$

So far, our computations hold on all of $Q' \times \Theta$. Now we will specialize to $(y_0, t_0)$ and obtain an alternate expression for the first term on the right-hand side of the previous line. We recall the derivative of $z$ is zero at $(y_0, t_0)$, so that,

$$0 = Dz(y_0, t_0) = D(\tilde{H}(D(Du, y_0))) + \lambda Du = H(Du, y_0)D\zeta + \zeta D(H(Du, y_0)) + \lambda Du,$$

which, upon rearranging becomes,

$$\zeta D(H(Du, y_0)) = -\lambda Du - H(Du, y_0)D\zeta.$$

We substitute the right-hand side of the previous line for the first term of the right-hand side of (3.22) to obtain, at $(y_0, t_0)$,

$$\zeta(Du_t - \text{tr}(AD^3u)) = -\lambda Du - H(Du, y_0)D\zeta + \zeta Df(y_0) + \zeta \text{tr}(DAD^2u).$$

We take dot product with $H_p(y_0, t_0)$ and use the previous line to find:

$$\zeta H_p(Du, y_0) \cdot (Du_t - \text{tr}(AD^3u)) = -\lambda H_p(Du, y_0) \cdot Du + \Pi$$

where $\Pi$ is the sum of the left-over terms:

$$\Pi = H_p(Du, y_0) \cdot (-H(Du, y_0)D\zeta + \zeta Df(y_0) + \zeta \text{tr}(DAD^2u)).$$

Next, according to (3.10), we have $H_p(Du, y) \cdot Du = 2H(Du, y)$. We use this on the right-hand side of (3.23) and find,

$$\zeta H_p(Du, y_0) \cdot (Du_t - \text{tr}(AD^3u)) = -2\lambda H(Du, y_0) + \Pi.$$

Let us now consider (3.20) evaluated at $(y_0, t_0)$. According to (3.18), the left-hand side of (3.20) is non-negative. We use (3.21) to estimate the first term on the right-hand side of (3.20), and we use (3.24) for the second term, and find,

$$0 \leq \lambda(H(Du, y_0) + r - 2H(Du, y_0)) + \Pi + I = -\lambda H(Du, y_0) + r\lambda + \Pi + I.$$

We now claim that the sum of the leftover terms $I$ and $\Pi$ is bounded:

$$\Pi + I \leq C\lambda + \frac{\lambda}{2}H(Du, y).$$

We point out that once the bound (3.26) is established, the proof of (3.19), and hence of the proposition, will be complete. Indeed, using (3.20) to estimate the right-hand side of (3.25) yields

$$0 \leq -\frac{\lambda}{2}H(Du, y_0, t_0, y_0) + C\lambda,$$

which, upon rearranging and dividing by $\lambda > 0$ yields (3.19).
Proof of bound (3.26) on error terms. Let us start with I. We will prove,

\[ I \leq -\frac{\zeta}{\epsilon} \text{tr} [A^2 \cdot D^2 u \cdot D^2 u] + \frac{\lambda}{4} H(Du, y). \]

We use the expressions (3.11) for $\tilde{H}_{pp}$ and $\tilde{H}_{py}$ to rewrite I as

\[ I = \tilde{H}_I(Du, y) - \frac{2\psi}{\epsilon} \text{tr} [A^2 D^2 u D^2 u] - \frac{4\psi^2}{\epsilon} \text{tr} [A^2 \cdot Du \otimes D\psi D^2 u] - 4\zeta \text{tr} [A W D^2 u] - \text{tr} [A \tilde{H}_{yy}(Du, y)]. \]

Let us bound from above the third term in I by applying the inequality of Lemma 3.2 with $M = A^2$, $X = Du \otimes D\psi$, $Y = \psi D^2 u$ and $\beta = 2$. We obtain:

\[ -\frac{4\psi}{\epsilon} \text{tr} [A^2 D^2 u D^2 u] \leq \frac{8}{\epsilon} \text{tr}(A^2(Du \otimes D\psi)(Du \otimes D\psi)^T) + \frac{\psi^2}{2\epsilon} \text{tr}(A^2 D^2 u D^2 u). \]

Let us apply Lemma 3.2 again, this time with $M = Id$, $X = W$, $Y = AD^2 u$ and $\beta = 2$. We obtain the following upper bound for the fourth term in I:

\[ -\frac{\zeta}{\epsilon} \text{tr} [A W D^2 u] \leq 8\zeta \text{tr}(WW^T) + \frac{\zeta}{2} \text{tr}(A^2 D^2 u D^2 u) = 8\zeta u_1^2 + \frac{\zeta}{2} \text{tr}(A^2 D^2 u D^2 u), \]

where we have also used the second statement in Lemma 3.2. We use (3.29) and the previous line to bound from above the third and fourth terms, respectively, in on the right-hand side of (3.28). Notice that the terms involving $A^2 D^2 u D^2 u$ in the previous line and in (3.29) will be “absorbed” by the second term in I (we are using that $\psi^2 = \zeta$). We obtain:

\[ I \leq \tilde{H}_I(Du, y) - \frac{\zeta}{\epsilon} \text{tr} [A^2 D^2 u D^2 u] + \frac{8}{\epsilon} \text{tr}(A^2(Du \otimes D\psi)(Du \otimes D\psi)^T) + 8\zeta u_1^2 - \text{tr} [A \tilde{H}_{yy}(Du, y)]. \]

The first term on the right-hand side of (3.30) is simply $\zeta \lambda H(Du, y)$, which is less than $\frac{\lambda}{16} H(Du, y)$. In addition, we have $8\zeta u_1^2 \leq C_1 H(Du, y) \leq \frac{1}{16} H(Du, y)$. We use this to bound the right-hand side of (3.30) from above and find,

\[ I \leq \frac{\lambda}{8} H(Du, y) - \zeta \frac{\lambda}{\epsilon} \text{tr} [A^2 D^2 u D^2 u] + \frac{8}{\epsilon} \text{tr}(A^2(Du \otimes D\psi)^2) - \text{tr} [A \tilde{H}_{yy}(Du, y)]. \]

We will show now show that each of the last two terms on the right-hand side of the previous line is less than $\frac{\lambda}{16} H(Du, y)$. Once we show this, the estimate (3.27) will be established. To this end, we use that $\psi$ is independent of $y_2$, to compute

\[ \text{tr}(A^2(Du \otimes D\psi)(Du \otimes D\psi)^T) = y_2^2 \zeta u_{y_1}^2 \psi_{y_1}^2 + \frac{\lambda^2}{2} u_{y_2}^2 \psi_{y_2}^2. \]

Multiplying by $8/\epsilon$ and using the definitions of $H$ and $\lambda$ we obtain,

\[ \frac{8}{\epsilon} \text{tr}(A^2(Du \otimes D\psi)(Du \otimes D\psi)^T) \leq C \zeta u_{y_1}^2 + \frac{C_2}{\epsilon} H(Du, y) \leq \frac{\lambda}{16} H(Du, y). \]

Let us estimate the last term of (3.31). We compute $\tilde{H}_{yy}$ in terms of the derivatives of $H$ and $\zeta$ and find,

\[ \tilde{H}_{yy} = H_y \otimes D\zeta + D\zeta \otimes H_y + HD^2 \zeta = \begin{pmatrix} 0 & \frac{u_{y_1}^2 \zeta_{y_1}}{\epsilon} & u_{y_1} \zeta_{y_1} \\ u_{y_1} \zeta_{y_1} & 0 & \zeta_{y_1} \\ 0 & \zeta_{y_2} \end{pmatrix} + H \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

We multiply both sides of the previous line by $-A(y)$ and take trace. The first term gives zero. The second is simply $-H(Du, y)\zeta_{y_2} u_{y_2}^2$. We summarize this as,

\[ -\text{tr}(\tilde{A} \tilde{H}_{yy}(Du, y)) \leq C_4 \zeta u_{y_1} H(Du, y) \leq \frac{\lambda}{16} H(Du, y). \]

Thus we have proved (3.27). Now for II. We have

\[ II = -H_p(Du, y) \cdot Df(H(Du, y) + \zeta_H_p(Du, y) \cdot Df) + \zeta_H_p(Du, y) \cdot \text{tr}(DAD^2 u)). \]

Since $D\zeta = (\zeta_{y_1}, 0)$, the first term in II is simply

\[ -2y_2 \zeta u_{y_1} H(Du, y) \leq C_5 \left( \sup_{Q'} (\zeta u_{y_1}) \right) H(Du, y) \leq \frac{\lambda}{16} H(Du, y). \]
For the second term in $II$, we use that $Df = (f_{y_1}, 0)$, the Cauchy-Schwartz inequality, the assumption \[3.7\], and the definitions of $H$ and $\lambda$ to find,

$$
\zeta H_p(Du, y) \cdot Df(y) = \zeta y_2 u_{y_2}, f_{y_1} \leq 2y_2 \zeta (u_{y_2}^2 + 1) ||f_{y_1}||_{L^\infty(Q')} \leq \frac{C_6}{\varepsilon}(u_{y_2}^2 + 1) \leq \frac{\lambda}{16} H(Du, y) + \frac{\lambda}{16}.
$$

Finally we will bound the last term in $II$. We have $D_{y_1} A = 0$, so the last term is simply

$$
\zeta H_{p_2}(Du, y) \mathrm{tr}(D_{y_2} AD^2 u) = \zeta \frac{u_{y_2}}{\varepsilon^2} y_2 \varepsilon u_{y_1, y_1} \leq \frac{\zeta}{\varepsilon} \left( \frac{C_7}{\varepsilon^2} u_{y_2}^2 + \frac{\varepsilon^2}{2\theta_m} u_{y_1 y_1}^2 \right)
$$

where the first inequality follows by applying Cauchy Schwartz and the second from the definitions of $A$ and $H$. We therefore find,

$$II \leq \frac{3\lambda}{16} H(Du, y) + \frac{\lambda}{16} + \frac{\zeta}{2\varepsilon} \mathrm{tr}(A^2 D^2 u D^2 u).$$

Adding our upper bound \[5.27\] on $I$ to the previous line we obtain,

$$I + II \leq \frac{\lambda}{2} H(Du, y) + \frac{\lambda}{16} - \frac{\zeta}{2\varepsilon} \mathrm{tr}(A^2 D^2 u D^2 u) \leq \frac{\lambda}{2} H(Du, y) + \frac{\lambda}{16},$$

as desired. Thus the proof of Proposition \[3.1\] is complete.

\[\square\]

4. Limits of the $u^\varepsilon$ as $\varepsilon$ approaches zero

This section is devoted to studying the half-relaxed limits $\bar{u}$ and $\underline{u}$ of the $u^\varepsilon$ that we mentioned in the introduction. For the convenience of the reader, we recall their definitions here:

$$
\bar{u}(x, t) = \limsup_{\varepsilon \to 0} \{u^\varepsilon(y, \theta, s) : \varepsilon' \leq \varepsilon, |y - x|, |t - s| \leq \varepsilon, \theta \in \Theta \}
$$

and

$$
\underline{u}(x, t) = \liminf_{\varepsilon \to 0} \{u^\varepsilon(y, \theta, s) : \varepsilon' \leq \varepsilon, |y - x|, |t - s| \leq \varepsilon, \theta \in \Theta \}.
$$

We also summarize the results of the previous sections. We have established Corollary \[2.1\] which says that there exists a constant $C$ so that $||u^\varepsilon||_{L^\infty(\mathbb{R} \times [0, \infty])} \leq C$. We have also proved Proposition \[1.1\] which says that if $Q \subset \mathbb{R} \times (0, \infty)$, then there exists a constant $C$ that depends on $Q$ such that for $(x, t) \in Q$ and $\theta \in \Theta$, we have

$$u^\varepsilon(x, \theta, t) \geq -C$$

and

$$|u^\varepsilon_0(x, \theta, t)| \leq \varepsilon^{1/2} C.$$

In particular, these results imply that $\bar{u}$ and $\underline{u}$ are finite everywhere on $\mathbb{R} \times (0, \infty)$ (although they may be infinite at time $t = 0$, as we will demonstrate in the proof of Proposition \[1.2\]). As a consequence, we obtain that $\bar{u}$ and $\underline{u}$ are non-negative: indeed, since $||u^\varepsilon||_{L^\infty(\mathbb{R} \times \Theta \times [0, \infty))} \leq C$, we have $u^\varepsilon(x, \theta, t) \leq \varepsilon \ln C$ for any $(x, \theta, t) \in \mathbb{R} \times [0, \infty) \times \Theta$. Taking lim inf or lim sup as $\varepsilon \to 0$ implies,

$$\bar{u}(x, t) \leq 0 \text{ and } \underline{u}(x, t) \leq 0 \text{ for all } (x, t) \in \mathbb{R} \times [0, \infty).$$

In subsection \[4.1\] we prove that $\bar{u}$ and $\underline{u}$ are, respectively, a sub- and super- viscosity solution of the Hamilton-Jacobi equation \[11\]. This is part Part (I) of Proposition \[1.2\]. We use a perturbed test function argument \[19\] and some techniques similar to the proofs of \[5\] Propositions \[3.1\] and \[3.2\].

In subsection \[4.2\] we study the behavior of $\bar{u}$ and $\underline{u}$ at $t = 0$ and establish Part (II) of Proposition \[1.2\].

We follow the strategy of \[5\] Propositions \[3.1\], \[3.2\].

4.1. Proof of Part (I) of Proposition \[1.2\] Throughout this section we employ the notational convention we mentioned in Section \[1.1.2\] if $Q \subset \mathbb{R} \times (0, \infty)$, then we will use $Q \times \Theta$ to denote,

$$Q \times \Theta = \{(x, \theta, t) : (x, t) \in Q \text{ and } \theta \in \Theta \}.$$
4.1.1. *An auxiliary lemma.* We formulate the following lemma, which is similar to [15, Lemma 6.1].

**Lemma 4.1.** Let \( v(x, t) \) and \( Q(\theta) \) be smooth functions. Suppose \( u - v \) (resp. \( \bar{u} - v \)) has a local minimum (resp. maximum) at \((x_0, t_0)\). Define \( v^\varepsilon(x, \theta, t) = v(x, t) + \varepsilon Q(\theta) \). Then there exists a subsequence \( \{\varepsilon_n\}_{n=1}^\infty \) such that

(1) \( (x_{\varepsilon_n}, t_{\varepsilon_n}) \to (x_0, t_0) \),

(2) \( u^{\varepsilon_n} - v^{\varepsilon_n} \) has a local minimum (resp. maximum) at \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\) for some \( \theta_{\varepsilon_n} \), and

(3) \( \lim_{n \to \infty} u^{\varepsilon_n}(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) = u(x_0, t_0) \) (resp. \( = \bar{u}(x_0, t_0) \)).

We postpone the proof of the lemma until the end of subsection 4.1.

4.1.2. *Proof that \( u \) is a supersolution of \( HJ \).* We place one computation into a separate lemma.

**Lemma 4.2.** Suppose \( v(x, t) \) is a smooth function that satisfies, for some \((x_0, t_0)\),

\[
(4.2) \quad v_1(x_0, t_0) + \partial_x v(x_0, t_0) c(\partial_x v(x_0, t_0)) = -a.
\]

We set \( \lambda = \partial_x v(x_0, t_0) \) and take \( Q(\theta) \) to be the solution of the spectral problem (1.10) corresponding to this \( \lambda \). We define

\[
(4.3) \quad v^\varepsilon(x, \theta, t) = v(x, t) + \varepsilon \ln Q(\theta).
\]

There exist positive constants \( s, \varepsilon_1 \) such that for all \((x, \theta) \in B_s(x_0, t_0) \), all \( \theta \in \Theta \) and all \( 0 < \varepsilon \leq \varepsilon_1 \) we have

\[
(4.4) \quad v_1^\varepsilon(x, \theta, t) - \varepsilon \theta v_{\varepsilon x}(x, \theta, t) - \theta (v_{\varepsilon x}^\varepsilon)^2(x, \theta, t) - \frac{\alpha}{\varepsilon} v_{\theta \theta}(x, \theta, t) - \frac{\alpha}{\varepsilon^2} (v_{\varepsilon x}(x, \theta, t))^2 - r \leq -a/2.
\]

We postpone the proof of the lemma and proceed with:

*Proof that \( u \) is a supersolution of \( HJ \).* Let us fix a point \((x_0, t_0) \in \mathbb{R} \times (0, \infty) \) such that \( u(x_0, t_0) < 0 \). By Proposition 1.1, \( u(x_0, t_0) \) is finite, so we denote \( -\delta = u(x_0, t_0) \). We aim to prove that \( u \) satisfies \( HJ \) in the viscosity sense, so let us suppose \( u - v \) has a local minimum at \((x_0, t_0)\) for some smooth function \( v \). We shall show

\[
(4.5) \quad v_1(x_0, t_0) + \partial_x v(x_0, t_0) c(\partial_x v(x_0, t_0)) \geq 0.
\]

We proceed by contradiction and assume that (4.5) does not hold. Therefore, there exists \( a > 0 \) such that \( v \) satisfies (4.2). We define the perturbed test function \( \nu^\varepsilon(x, \theta, t) \) by (4.3). According to Lemma 4.2, there exist \( s > 0 \) and \( \varepsilon_1 > 0 \) such that \( \nu^\varepsilon \) satisfies (1.4) on \( B_s(x_0, t_0) \times \Theta \) for all \( \varepsilon \leq \varepsilon_1 \).

By Proposition 1.1, there exists some positive constants \( c, C, \varepsilon_0 \) such that for all \( \varepsilon < \varepsilon_0 \) and for all \((x, t, \theta) \in B_{c/2}(x_0, t_0) \times \Theta \),

\[
(4.6) \quad |\partial_\theta \nu^\varepsilon(x, \theta, t)| \leq \varepsilon^{1/2} C.
\]

Let \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\) be the sequence of points given by Lemma 4.1. According to item (3) of Lemma 4.1, we have

\[
\lim_{n \to \infty} u^{\varepsilon_n}(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) = u(x_0, t_0) = -\delta.
\]

Therefore, for all \( n \) large enough, we find,

\[
\nu^{\varepsilon_n}(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \leq -\delta/2.
\]

Together with the estimate (4.6) on \( \partial_\theta \nu^\varepsilon \), this implies that there exists \( N_1 > 0 \) such that for \( n > N_1 \) and for all \( \theta \in \Theta \),

\[
u^{\varepsilon_n}(x_{\varepsilon_n}, \theta, t_{\varepsilon_n}) \leq -\delta/4.
\]

The previous estimate implies that \( \rho^{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) \) is bounded from above, uniformly in \( n \). Indeed, we use the definition of \( \rho^\varepsilon \), the relationship \( n^\varepsilon(x, \theta, t) = e^{u^\varepsilon(x, \theta, t)} \) and the previous line to obtain

\[
\rho^{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) = \int e^{\frac{u^\varepsilon(x_{\varepsilon_n}, \theta, t_{\varepsilon_n})}{r}} d\theta \leq \int e^{\frac{-\delta}{4r}} d\theta.
\]

Thus there exists \( N_2 \) such that for all \( n > N_2 \), we have

\[
(4.7) \quad \rho^{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) \leq \frac{a}{4r}.
\]
Since \( u^\varepsilon_n - v^\varepsilon_n \) has a local minimum at \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\) and \( u^\varepsilon \) satisfies (1.3), we have, at \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\),

\[
v_t^\varepsilon_n - \varepsilon_n \theta v_x^\varepsilon_n - \theta (v^\varepsilon_n)^2 - \frac{\alpha}{\varepsilon_n} v_{\theta \theta}^\varepsilon_n - \frac{\alpha}{\varepsilon_n} \left( \frac{v_{\theta}^\varepsilon_n}{\varepsilon_n} \right)^2 - r(1 - \rho^\varepsilon_n) \geq 0.
\]

(4.8)

According to Lemma 4.2, \( v \) satisfies (4.5) in \( B_s(x_0, t_0) \times \Theta \). Since \((x_{\varepsilon_n}, t_{\varepsilon_n}) \rightarrow (x_0, t_0)\), we have \((x_{\varepsilon_n}, t_{\varepsilon_n}) \in B_s(x_0, t_0)\) for \( n \) large enough. Hence, at the point \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\), we have,

\[
v_t^\varepsilon_n - \varepsilon_n \theta v_x^\varepsilon_n - \theta (v^\varepsilon_n)^2 - \frac{\alpha}{\varepsilon_n} v_{\theta \theta}^\varepsilon_n - \frac{\alpha}{\varepsilon_n} \left( \frac{v_{\theta}^\varepsilon_n}{\varepsilon_n} \right)^2 - r \leq -a/2.
\]

Subtracting the previous line from (4.8) yields,

\[
r\rho^\varepsilon_n(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \geq a/2.
\]

But this contradicts (4.7). We have reached the desired contradiction and conclude that \( u \) is indeed a supersolution of (H).

The proof of Lemma 4.2 is a simple computation. We include it for the sake of completeness.

**Proof of Lemma 4.2.** We observe

\[
v_0^\varepsilon = \varepsilon \frac{Q_\theta}{Q}; \quad v_{\theta \theta}^\varepsilon = \varepsilon \frac{Q Q_{\theta \theta} - Q_\theta^2}{Q^2},
\]

so that we have,

\[
v_t^\varepsilon - \varepsilon \theta v_x^\varepsilon - \theta (v^\varepsilon)^2 - \frac{\alpha}{\varepsilon} v_{\theta \theta}^\varepsilon - \frac{\alpha}{\varepsilon^2} (v_\theta^\varepsilon)^2 - r(1 - \rho^\varepsilon) =
\]

\[
v_t - \varepsilon \theta v_x - \theta (v_x)^2 - \alpha \frac{Q Q_{\theta \theta} - Q_\theta^2}{Q^2} - \alpha \frac{Q_\theta^2}{Q^2} - r(1 - \rho^\varepsilon) =
\]

\[
v_t - \varepsilon \theta v_x - \theta (v_x)^2 - \alpha \frac{Q_{\theta \theta} Q}{Q} - r(1 - \rho^\varepsilon).
\]

Thus we have, for some \( s \) small, and for all \((x, \theta, t) \in B_s(x_0, t_0) \times \Theta \) and all \( \varepsilon \) small enough,

\[
v_t^\varepsilon(x, \theta, t) - \varepsilon \theta v_x^\varepsilon(x, \theta, t) - \theta (v^\varepsilon)^2(x, \theta, t) - \alpha \varepsilon v_{\theta \theta}^\varepsilon(x, \theta, t) - \alpha \varepsilon^2 (v_\theta^\varepsilon(x, \theta, t))^2 - r(1 - \rho^\varepsilon(x, t)) =
\]

\[
\leq v_t(x_0, t_0) - \theta (v_x)^2(x_0, t_0) - \alpha \frac{Q_{\theta \theta} Q}{Q} - r(1 - \rho^\varepsilon(x, t)) + a/2
\]

\[
= -a/2 - \partial_x v(x_0, t_0) c(\partial_x v(x_0, t_0)) - \theta (v_x)^2(x_0, t_0) - \alpha \frac{Q_{\theta \theta} Q}{Q} - r(1 - \rho^\varepsilon(x, t))
\]

\[
= -a/2 - \lambda c(\lambda) - \theta \lambda^2 - \alpha \frac{Q_{\theta \theta} Q}{Q} - r(1 - \rho^\varepsilon(x, t))
\]

\[
= -a/2 + r \rho^\varepsilon(x, t),
\]

where the last equality follows since \( Q \) satisfies (1.10).

4.1.3. **Proof that \( \bar{u} \) is a subsolution of (H)**.

**Proof that \( \bar{u} \) is a subsolution of (H).** This proof less involved than that for \( u \). According to (4.1) we have \( \bar{u} \leq 0 \).

Next, let us suppose that \( \bar{u} - v \) has a local maximum at \((x_0, t_0)\). We proceed as in the proof for \( u \): for contradiction, we assume

\[
v_t(x_0, t_0) + \partial_x v(x_0, t_0) c(\partial_x v(x_0, t_0)) = a,
\]

for some \( a > 0 \). We set \( \lambda = \partial_x v(x_0, t_0) \) and take \( Q(\lambda) \) to be the solution of the spectral problem (1.10) corresponding to this \( \lambda \). As in the previous proof, we define \( v^\varepsilon \) by (4.3) and find that for some \( s \) small, for all \((x, t) \in B_s(x_0, t_0) \times \Theta \), and for all \( \varepsilon \) small enough,

\[
v_t^\varepsilon(x, \theta, t) - \varepsilon \theta v_x^\varepsilon(x, \theta, t) - \theta (v_x^\varepsilon)^2(x, \theta, t) - \frac{\alpha}{\varepsilon} v_{\theta \theta}^\varepsilon(x, \theta, t) - \frac{\alpha}{\varepsilon^2} (v_\theta^\varepsilon(x, \theta, t))^2 - r \geq a/2.
\]

(4.9)
According to Lemma 4.1, there exists a subsequence of \( \varepsilon \to 0 \), also denoted \( \varepsilon \), and points \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\) such that \( u^\varepsilon - v^\varepsilon \) has a local maximum at \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\), and \((x_\varepsilon, t_\varepsilon) \to (x_0, t_0)\). Since \( u^\varepsilon \) satisfies (4.9), we obtain, at \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\),

\[
(4.10) \quad v^\varepsilon_t - \varepsilon \theta v^\varepsilon_x - \theta (v^\varepsilon_x)^2 - \frac{\alpha}{\varepsilon} \varepsilon^2_\theta - \frac{\alpha}{\varepsilon} (v^\varepsilon_x)^2 - r(1 - \rho^\varepsilon) \leq 0
\]

For \( \varepsilon \) small enough, \((x_\varepsilon, t_\varepsilon) \in B_s(x_0, t_0)\), so both (4.9) and (4.10) hold at \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\). Subtracting (4.9) from (4.10) yields

\[
r\rho^\varepsilon (x_\varepsilon, t_\varepsilon) \leq -a/2,
\]

which is impossible since \( \rho^\varepsilon \geq 0 \).

\[\square\]

4.1.4. Proof of the auxiliary lemma. We now present the proof of Lemma 4.1.

Proof of Lemma 4.1. We will give the proof of the case that \( u - v \) has a local minimum at \((x_0, t_0)\). The proof of the other case is similar.

Without loss of generality we assume that \( u - v \) has a strict local minimum at \((x_0, t_0)\) in \( B_r(x_0, t_0) \) for some \( r > 0 \). Let \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\) be a local minimum of \( u^\varepsilon - v^\varepsilon \) in \( B_{r/2}(x_0, t_0) \times \Theta \).

By the definition of \((u(x_0, t_0), \Theta)\), there exists a sequence \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\) with \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon) \to (x_0, t_0)\) and such that \( u^\varepsilon(x_\varepsilon, \theta_\varepsilon, t_\varepsilon) \to u(x_0, t_0) \).

We proceed with the proof of item (1) of the lemma. Let \( \{x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}\}_{n=0}^{\infty} \) be any subsequence of \((x_\varepsilon, t_\varepsilon)\) with \((x_{\varepsilon_n}, t_{\varepsilon_n}) \to (y, s) \in B_{r/2}(x_0, t_0)\). For \( n \) large enough we have \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \in B_{r/2}(x_0, t_0)\). Since \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\) is a local minimum of \( u^\varepsilon - v^\varepsilon \) on \( B_{r/2}(x_0, t_0) \times \Theta \), we obtain,

\[
(4.11) \quad u^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) - v^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \leq u^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) - v^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}).
\]

We take \( \limsup_{n \to \infty} \) of both sides of (4.11). The definition of \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\) implies that the first term on the right-hand side converges to \( u(x_0, t_0) \). In addition, since \( v \) is continuous and \((x_{\varepsilon_n}, t_{\varepsilon_n}) \) and \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \) converge to \((y, s)\) and \((x_0, t_0)\), respectively, we find,

\[
(4.12) \quad \limsup_{n \to \infty} (u^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})) - v(y, s) \leq u(x_0, t_0) - v(x_0, t_0).
\]

Since \((x_{\varepsilon_n}, t_{\varepsilon_n}) \to (y, s)\), the definition of \( u(y, s) \) implies,

\[
\limsup_{n \to \infty} (u^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})) \geq \liminf_{n \to \infty} (u^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})) \geq u(y, s).
\]

We use the previous line to estimate the left-hand side of (4.12) from below and obtain \( u(y, s) - v(y, s) \leq u(x_0, t_0) - v(x_0, t_0) \). Since \((x_0, t_0)\) is a strict local maximum, we see \((y, s) = (x_0, t_0)\). This completes the proof of items (1) and (2).

Next let us take \( \liminf_{n \to \infty} \) of both sides of (4.11). Since we now know \((x_{\varepsilon_n}, t_{\varepsilon_n}) \to (x_0, t_0)\) and \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \to (x_0, t_0)\), the terms with \( v \) are equal. In addition, the definition of \((x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})\) implies that the first term on the right-hand side converges to \( u(x_0, t_0) \). Thus we find

\[
\liminf_{n \to \infty} (u^\varepsilon(x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n})) \leq u(x_0, t_0).
\]

Since \((x_{\varepsilon_n}, t_{\varepsilon_n}) \to (x_0, t_0)\), the definition of \( u(x_0, t_0) \) implies that equality holds in the above. This completes the proof of the lemma.

4.2. Proof of Part (II) of Proposition 1.2. In the previous subsection we showed that \( \bar{u} \) and \( u \) are a supersolution and subsolution, respectively, of (4.10) on \( \mathbb{R} \times (0, \infty) \). In this section we study the behavior of \( \bar{u} \) and \( u \) at time \( t = 0 \).

Proof that \( \bar{u} \) satisfies (4.8). First we show that if \( x \in J \) then \( \bar{u}(x, 0) = 0 \). Indeed, if \( x \in J \) then there exists a point \( \theta^* \in \Theta \) such that \( n_0(x, \theta^*) = a > 0 \). Hence

\[
\bar{u}(x, 0) \geq \lim_{\varepsilon \to 0} \varepsilon \ln n^\varepsilon(x, \theta^*) = \lim_{\varepsilon \to 0} \varepsilon \ln a = 0.
\]

And, according to our observation (4.1) we have \( \bar{u}(x, 0) \leq 0 \), so we obtain \( \bar{u}(x, 0) = 0 \).

In the remainder of the proof we will analyze the behavior of \( \bar{u}(x, 0) \) for \( x \notin J \). To this end, we observe that, by the definition of \( J \),

\[
\mathbb{R} \setminus J = \{ x : \quad n_0(x, \theta) = 0 \text{ for all } \theta \in \Theta. \}
\]
We fix a constant $\nu > 0$ and a cutoff function $\zeta \in C^\infty(\mathbb{R})$ that satisfies
\begin{equation}
\begin{aligned}
\zeta &= 0 \text{ on } \bar{J}, \quad \zeta > 0 \text{ on } \mathbb{R} \setminus \bar{J} \\
0 &\leq \zeta \leq 1.
\end{aligned}
\end{equation}

**First step:** We claim that $\bar{u}$ is a viscosity subsolution of
\begin{equation}
\min\{\bar{u}_t + \bar{u}_x c(\bar{u}_x), \bar{u} + \mu \zeta\} \leq 0 \text{ on } \mathbb{R} \times \{0\},
\end{equation}
by which we mean if $\bar{u} - v$ has a local maximum at $(x_0, 0)$ for some test function $v$, then either
\begin{equation}
\bar{u}(x_0, 0) \leq -\mu \zeta(x_0)
\end{equation}
or
\begin{equation}
v_x + v_x c(v_x) \leq 0 \text{ at } (x_0, 0).
\end{equation}
For $x \in J$, we have $\bar{u}(x_0, 0) = 0$ and $\mu \zeta(x_0) = 0$, so we find (4.15) holds. Now let us suppose $x_0 \in \mathbb{R} \setminus \bar{J}$, (4.15) doesn’t hold, and $\bar{u} - v$ has a local maximum at $(x_0, 0)$ for some smooth $v$. We proceed as in the proof of Part (I) of Proposition 1.2 by which we mean if $\bar{u} - v$ has a local maximum at $(x_0, 0)$, then either $u^\varepsilon$ or $\bar{u}^\varepsilon$ has a local maximum at $(x_0, t_0)$ as $n \to \infty$, $u^\varepsilon - v^\varepsilon$ has a local maximum at $(x_0, t_0)$, and $\bar{u}(x_0, 0) = \lim u^\varepsilon(x_0, t_0)$ or $\bar{u}^\varepsilon(x_0, t_0)$.

Thus we find
\begin{equation}
\bar{u}(x_0, 0) = \lim_{n \to \infty} u^\varepsilon(x_0, \theta, t_0) = -\infty.
\end{equation}
But we had assumed (4.15) doesn’t hold, so that $\bar{u}(x_0, 0) > -\mu \zeta(x_0) > -\infty$. We have reached the desired contradiction, hence (4.17) must hold, and so we conclude that $\bar{u}$ is a viscosity subsolution of (4.14).

**Second step:** We will now use that $\bar{u}$ is a viscosity subsolution of (4.14) to prove $\bar{u}(x, 0) = -\infty$ on $\mathbb{R} \setminus \bar{J}$.
To this end, let us fix any $x_0 \in \mathbb{R} \setminus J$ and assume for contradiction
\begin{equation}
\bar{u}(x_0, 0) = -M > -\infty.
\end{equation}
For $\delta > 0$, let us define the test functions
\begin{equation}
v^\delta(x, t) = \frac{|x - x_0|^2}{\delta} + \nu(\delta)t,
\end{equation}
where we use $\nu(\delta)$ to denote
\begin{equation}
\nu(\delta) = \frac{8 \theta M}{\delta} + r.
\end{equation}
Since $\bar{u}$ is upper-semicontinuous, there exist $(x^\delta, t^\delta)$ such that $\bar{u} - v^\delta$ has a maximum at $(x^\delta, t^\delta)$ in $\mathbb{R} \times [0, \infty)$. In particular, we have
\begin{equation}
(\bar{u} - v^\delta)(x^\delta, t^\delta) \geq (\bar{u} - v^\delta)(x_0, 0) = \bar{u}(x_0, 0) = -M,
\end{equation}
where the equalities follow since $v^\delta(x_0, 0) = 0$ and from (4.18). We now use that $\bar{u} \leq 0$ and the definition of $v^\delta$ to estimate the left-hand side of the previous line from above by $\frac{|x^\delta - x_0|^2}{\delta}$. Thus we obtain an estimate on the distance between $x^\delta$ and $x_0$:
\begin{equation}
\frac{|x^\delta - x_0|^2}{\delta} \leq M \text{ for all } \delta.
\end{equation}

We will now establish the inequality
\begin{equation}
\nu(\delta) + \frac{2(x^\delta - x_0)}{\delta} c\left(\frac{2(x^\delta - x_0)}{\delta}\right) > 0 \text{ for all } \delta.
\end{equation}
Indeed, according to Proposition 1.3, we have \( \lambda c(\lambda) \geq \lambda^2 \theta_M - r \) for all \( \lambda \). We apply this with \( \lambda = \frac{2(x_\delta - x_0)}{\delta} \) and obtain,

\[
\frac{2(x_\delta - x_0)}{\delta} c \left( \frac{2(x_\delta - x_0)}{\delta} \right) \geq - \frac{4(x_\delta - x_0)^2}{\delta^2} \theta_M - r \geq - \frac{4\theta_M M}{\delta} + r,
\]

where the second inequality follows from (4.20). Let us add \( \nu(\delta) \) to both sides of the previous line. The left-hand side becomes exactly the left-hand side of (4.21). The right-hand side becomes \( \frac{4\theta_M M}{\delta} \), due to our choice of \( \nu \) in (4.19). Thus we find (4.21) holds.

We now recall that \( \bar{u} - v^\delta \) has a maximum at \((x_\delta, t_\delta)\) in \( \mathbb{R} \times [0, \infty) \). Let us suppose that \( t_\delta > 0 \) for some \( \delta \). According to Part (I) of Proposition 1.2, we define \( u \) in order for the previous line to hold, the second term inside the min must be non-positive. Thus we have,

\[
\nu(\delta) + 2\frac{(x_\delta - x_0)}{\delta} c \left( \frac{2(x_\delta - x_0)}{\delta} \right) \leq 0.
\]

But this is impossible, as we have just established (4.21). Therefore, we must have \( t_\delta = 0 \) for all \( \delta \). But we also know that \( \bar{u} \) is a subsolution of (4.14) on \( \mathbb{R} \times \{0\} \). Therefore, we have

\[
\min \left\{ \nu(\delta) + 2\frac{(x_\delta - x_0)}{\delta} c \left( \frac{2(x_\delta - x_0)}{\delta} \right), \bar{u}(x_\delta, 0) + \mu \zeta(x_\delta) \right\} \leq 0.
\]

But, again according to (4.21), we have that the first term inside the min must be strictly positive. Therefore, in order for the previous line to hold, the second term inside the min must be non-positive. Thus we have,

\[
\bar{u}(x_\delta, 0) \leq -\mu \zeta(x_\delta).
\]

Since \((x_\delta, 0)\) is a local maximum of \( \bar{u} - v^\delta \), we have

\[
\bar{u}(x_0, 0) - v^\delta(x_0, 0) \leq (\bar{u} - v^\delta)(x_\delta, 0).
\]

Because \( -M = \bar{u}(x_0, 0) \) and \( v^\delta(x_0, 0) = 0 \), we have that the left-hand side of the previous line is exactly \( -M \). In addition, we use that \( v^\delta \) is non-negative and the estimate (4.23) to bound the right-hand side of the previous line from above. We find,

\[
-M \leq -\mu \zeta(x_\delta).
\]

Since \( x_\delta \to x_0 \) as \( \delta \to 0 \) and \( \zeta \) is continuous, we obtain \( -M \leq -\mu \zeta(x_0) \), which is impossible since \( \zeta(x_0) > 0 \) and \( \mu \) is arbitrary. We have obtained the desired contradiction and thus (4.18) cannot hold. We conclude \( \bar{u}(x_0, 0) = -\infty \), and hence the proof is complete.

\[\square\]

**Proof that \( \bar{u} \) satisfies (1.9).** This proof is similar to the one for \( \bar{u} \). First we show that if \( x_0 \in \mathbb{R} \setminus K \), then \( \bar{u}(x_0, 0) = -\infty \). Indeed, since \( x_0 \in \mathbb{R} \setminus K \), there exists \( \theta_0 \) with \( n_0(x_0, \theta_0) = 0 \). Therefore,

\[
\bar{u}(x_0, 0) \leq \liminf_{\varepsilon \to 0} \bar{u}^\varepsilon(x_0, \theta_0, 0) = \liminf_{\varepsilon \to 0} \varepsilon \ln(n_0(x_0, \theta_0)) = -\infty.
\]

We will now prove that \( \bar{u}(x_0, 0) = 0 \) for \( x_0 \in K \).

**First step:** We prove that \( \bar{u} \) is a supersolution of

\[
\max\{\bar{u}, \bar{u}_1 + \bar{u}_2 c(\bar{u}_2)\} \geq 0 \text{ on } K \times \{0\}.
\]

To this end, let us suppose \( x_0 \in K \),

\[
\bar{u}(x_0, 0) < 0,
\]

and \( \bar{u} - v \) has a local minimum at \((x_0, 0)\) for some test function \( v \). We proceed as in the proof of Part (I) of Proposition 1.2 we define \( v^\varepsilon \) by (4.3), and find that there exist points \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\) such that \((x_\varepsilon, t_\varepsilon) \to (x_0, t_0)\) as \( n \to \infty \), \( u^\varepsilon - v^\varepsilon \) has a local minimum at \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\), and \( \bar{u}(x_0, 0) = \lim_{n \to \infty} u^\varepsilon(x_\varepsilon, \theta_\varepsilon, t_\varepsilon) \). We claim

\[
t_\varepsilon > 0 \text{ for all } n \text{ large enough.}
\]

If (4.17) holds, then the argument in the proof of Part (I) of Proposition 1.2 applies in this situation as well. Thus, once we show (4.26), we will find that \( v \) satisfies \( v_1 + v_2 c(v_2) \geq 0 \) at \((x_0, 0)\) and so we will have established that (4.21) in the viscosity sense.

We proceed by contradiction and assume that (4.26) does not hold. Thus, there exists a subsequence, also denoted \( \varepsilon_n \), with \( t_\varepsilon = 0 \). Since \( x_0 \in K \), there exists \( s > 0 \) such that \( B_s(x_0) \subset K \). Since \( x_n \to x_0 \), we
have that for all \( n \) large enough, \( x_{\varepsilon_n} \in B_s(x_0) \). Therefore, there exists \( m > 0 \) so that \( n_0(x_{\varepsilon_n}, \theta_{\varepsilon_n}) \geq m \) for all \( n \) large enough. Hence we find,

\[
\bar{u}(x_0,0) = \lim_{n \to \infty} \varepsilon_n(x_{\varepsilon_n}, \theta_{\varepsilon_n}, 0) = \lim_{n \to \infty} \varepsilon_n \ln(n_0(x_{\varepsilon_n}, \theta_{\varepsilon_n})) \geq \lim_{n \to \infty} \varepsilon_n \ln(m) = 0,
\]

where the first equality follows from the definition of the sequence \( (x_{\varepsilon_n}, \theta_{\varepsilon_n}, t_{\varepsilon_n}) \). But the previous line contradicts our assumption \((4.25)\). Therefore, \((4.26)\) must hold, and we conclude that \( \bar{u} \) is a viscosity supersolution of \((4.24)\).

**Second step:** Let us fix \( x_0 \in K \). We will prove \( \bar{u}(x_0,0) \geq 0 \), which, together with \((4.1)\), implies \( \bar{u}(x_0,0) = 0 \). Let us suppose for contradiction

\[
\bar{u}(x_0,0) < 0.
\]

We point out that \( \bar{u}(x_0,0) \) is finite. Indeed, if \( \bar{u}(x_0,0) = -\infty \) then \( \bar{u} - v \) has a minimum at \( (x_0,0) \) for all \( v \), and since we know \( \bar{u} \) is a supersolution of \((4.24)\), we find \( v_t + v_x c(v_x) \geq 0 \) at \((x_0,0)\) for all \( v \), which is of course impossible.

Since \( \bar{u} \) is lower-semicontinuous and finite at \((x_0,0)\), there exists a neighborhood \( Q \) of \((x_0,0)\) and some finite \( M > 0 \) such that if \((x,t) \in Q \), then \( \bar{u}(x,t) > -M \).

We define, for \( \delta > 0 \), the test functions

\[
u^\delta(x,t) = -\frac{|x - x_0|^2}{\delta} - \nu(\delta)t,\]

where we define \( \nu \) by \((4.19)\) as in the previous proof. There exists a sequence \( (x_\delta, t_\delta) \to (x_0,0) \) such that \( \bar{u} - v^\delta \) has a local minimum at \((x_\delta, t_\delta) \in Q \). We find, as in the previous proof,

\[-M + \frac{|x_\delta - x_0|^2}{\delta} \leq (\bar{u} - v^\delta)(x_\delta, t_\delta) \leq (\bar{u} - v^\delta)(x_0,0) \leq 0,
\]

which implies that the upper bound \((4.20)\) on \(|x_\delta - x_0|^2/\delta\) holds here as well. We use \((4.20)\), the definition of \( \nu(\delta) \) and the properties of \( c \) given in Proposition \(1.3\) to obtain,

\[
-\nu(\delta) - \frac{2(x_\delta - x_0)}{\delta} c \left( -\frac{2(x_\delta - x_0)}{\delta} \right) < 0\quad \text{for all } \delta.
\]

Let us suppose \( t_\delta > 0 \). Since \( \bar{u} - v^\delta \) has a minimum at \((x_\delta, t_\delta)\) and, according to Part (I) of Proposition \(1.2\), \( \bar{u} \) is a supersolution of \(\mathcal{H}J\) in \(\mathbb{R} \times (0, \infty)\), we find

\[
-\nu(\delta) - \frac{2(x_\delta - x_0)}{\delta} c \left( -\frac{2(x_\delta - x_0)}{\delta} \right) \geq 0,
\]

which is impossible, as we have established \((4.28)\). Therefore, we find \( t_\delta = 0 \) for all \( \delta \). According to \((4.24)\), we have that \( \bar{u} \) is a supersolution of \((4.24)\) on \( K \times \{0\} \), so we find

\[
\max \left\{ \bar{u}(x_\delta,0), -\mu - \frac{2(x_\delta - x_0)}{\delta} c \left( -\frac{2(x_\delta - x_0)}{\delta} \right) \right\} \geq 0.
\]

According to \((4.28)\) we see that the second term in the max is strictly negative. Hence we obtain \( \bar{u}(x_\delta,0) \geq 0 \) for all \( \delta \). Together with \( v^\delta(x_0,0) = 0 \), the fact that \((x_\delta,0)\) is a minimum of \( \bar{u} - v^\delta \), and \( v^\delta \leq 0 \), this implies,

\[
\bar{u}(x_0,0) = \bar{u}(x_0,0) - v^\delta(x_0,0) \geq \bar{u}(x_\delta,0) - v^\delta(x_\delta,0) \geq 0.
\]

But this contradicts our assumption \((4.27)\). Thus we find \( \bar{u}(x_0,t) \geq 0 \) and the proof is complete.

\[\Box\]

5. LIMITS OF THE \( n^\varepsilon \) AS \( \varepsilon \) APPROACHES ZERO

In this short section we give use the results that we’ve established in the rest of the paper to give the proof of our main result, Theorem \(1.2\). The arguments are similar to those in the proofs of \(20\) Theorem \(1.1\) and \(3\) Theorem \(1\).

**Proof of Theorem \(1.2\):** Since \(\mathcal{H}J\) satisfies the comparison principle (see Proposition \(A.1\), Proposition \(1.2\) and the definitions of \( u_J \) and \( u_K \) imply \( \bar{u} \leq u_J \) and \( \bar{u} \geq u_K \) on \( \mathbb{R} \times (0, \infty) \).

Let us fix some \( B_s(x_0,t_0) \subset \{ u_J < 0 \} \). Since \( u_J \) is continuous, there exists \( a > 0 \) such that \( u_J < -a \) for all \((x,t) \in B_s(x_0,t_0) \). Since \( \bar{u} \leq u_J \), we have \( \bar{u}(x,t) < -a \) for all \((x,t) \in B_s(x_0,t_0) \) as well. Therefore,

...
\[ u^\varepsilon(x, \theta, t) \leq -\frac{\varepsilon}{2} \text{ for all } (x, t) \in \bar{B}_s(x_0, t_0), \text{ for all } \theta \in \Theta \text{ and for all } \varepsilon \text{ small enough.} \]

Thus, for all \((x, t) \in \bar{B}_s(x_0, t_0)\) and for all \(\theta\), we have

\[ 0 \leq n^\varepsilon(x, \theta, t) = e^{u^\varepsilon(x, \theta, t)} - e^\frac{\varepsilon}{2}, \]

so we find that \(n^\varepsilon(x, \theta, t) \to 0\) uniformly at an exponential rate on \(\bar{B}_s(x_0, t_0) \times \Theta\).

Now let us suppose \((x_0, t_0)\) is a point in the interior of \(\{u_K = 0\}\), so that \(B_s(x_0, t_0) \subset \{u_K = 0\}\) for some \(s > 0\). According to (4.1), we have \(u \leq 0\) on \(\mathbb{R} \times (0, \infty)\). Since \(u \geq u_K\), we have \(u \geq 0\) on \(B_s(x_0, t_0)\), so we see \(u \equiv 0\) on \(B_s(x_0, t_0)\). We define the test function

\[ \phi(x, t) = -(x - x_0)^2 - (t - t_0)^2. \]

Since \(u \equiv 0\) on \(B_s(x_0, t_0)\), we find that \(u - \phi\) has a local minimum at \((x_0, t_0)\). Therefore there exists a sequence \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\) with \((x_\varepsilon, t_\varepsilon) \to (x_0, t_0)\) such that \(u^\varepsilon - \phi\) has a local minimum at \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\). Since \(u^\varepsilon\) satisfies (A.1) in the viscosity sense, we find, at \((x_\varepsilon, \theta_\varepsilon, t_\varepsilon)\),

\[ \phi_t \geq \varepsilon \theta_\varepsilon \phi_{xx} + \theta (\phi_x)^2 + r(1 - \rho^2). \]

Using the definition of \(\phi\) and rearranging yields,

\[ r \rho^2(x_\varepsilon, t_\varepsilon) \geq r - 2 \varepsilon \theta_\varepsilon + 2 \theta (x_\varepsilon - x_0)^2 + 2(t_\varepsilon - t_0). \]

Taking the limit \(\varepsilon \to 0\) of the previous line yields,

\[ \lim_{\varepsilon \to 0} \rho^2(x_\varepsilon, t_\varepsilon) \geq 1. \]

Recalling the definition of \(\limsup^*\) completes the proof. \(\Box\)

**APPENDIX A.**

In this appendix, we state a comparison principle for (HJ) with infinite initial data (Proposition A.1). This result was established in [5 Section 4] but not explicitly stated there, and we could not locate another reference in the literature. Because of this, we carefully explain how to obtain it from [5 Section 4]. Similarly, because we could not locate in the literature the proof of the exact existence result we need, Lemma 1.1 we sketch its proof here.

In addition to [20, 4], we also refer the reader to Crandall, Lions and Souganidis [16] for more about Hamilton-Jacobi equations with infinite initial data.

**Proposition A.1.** Suppose \(w\) is a viscosity solution of (HJ) on \(\mathbb{R} \times (0, \infty)\) with initial data (1.1). Suppose \(w\) and \(\bar{w}\) are, respectively, a viscosity subsolution and a viscosity supersolution of (HJ) on \(\mathbb{R} \times (0, \infty)\), and both with the initial data (1.1). Then we have

(A.1)

\[ w \leq \bar{w} \leq \bar{w}. \]

**Proof.** We explain why Proposition A.1 is exactly what was proven in [5 Section 4]. First, we point out that the Hamiltonian \(H\) in [5] is assumed to be convex, but this is not used in the arguments of [5 Section 4]. Second, there is a difference in sign between our paper and [5 Section 4], which we now address. We have that \(-w, \bar{w}\) and \(-\bar{w}\) are, respectively, a viscosity solution, supersolution, and subsolution of

\[ \min \{u, u_t + c(-u_x)u_x\} = 0 \]

with initial data

\[ \begin{cases} 0 & \text{for } x \in \Omega \\ \infty & \text{for } x \in \mathbb{R} \setminus \bar{\Omega}. \end{cases} \]

Thus we see that, if we take \(H(p) = c(-p)p\) and \(G_0 = \Omega\), then we are exactly in the situation of [5 Section 4]. We may take \(I = -w, v_0 = -\bar{w}, v = \bar{w}\) and \(v^* = -\bar{w}\). The conclusion of the comparison of the arguments in [5 Section 4] (specifically, lines (4.2) and (4.5)) is

\[ v^* \leq I \leq v_*. \]

Translating back to our notation, we see A.1 holds. \(\Box\)

Proposition A.1 implies that there exists at most one viscosity solution to (HJ) with infinite initial data (1.1). We briefly explain why there exists a solution to (HJ) with infinite initial data (1.1), as we could not find this exact result in the literature (in [20, 4] the convex structure of the Hamiltonian is used to establish existence). Hence we provide:
Sketch of proof of Lemma 1.7. Let $\zeta$ be a cutoff function satisfying (4.13) but with $\Omega$ instead of $J$. Then (HJ) with initial data $-\mu\zeta(x)$ has a unique solution $u_\mu$, according to Ishii’s [26, Theorem 1] (we use this particular result because it applies to equations set in the entire space $\mathbb{R} \times (0, \infty)$).

We define $u_*$ and $u^*$ as, respectively, the half relaxed limits $\liminf_* u_\mu$ and $\limsup_* u_\mu$ as $\mu \to \infty$ of the $u_\mu$. By definition, we have $u_* \leq u^*$. According to standard stability results for viscosity solutions (see, for example [15, Section 6]), we have that $u_*$ and $u^*$ are, respectively, a viscosity super- and sub- solution of (HJ) with initial data (1.1). So, by Proposition A.1, we obtain with $u_* \geq u^*$. But $u_*$ and $u^*$ also satisfy the opposite inequality, hence they agree and equal the unique viscosity solution of (HJ) with initial data (1.1). □

Acknowledgements

The author thanks her thesis advisor, Takis Souganidis, for his guidance and encouragement. The author is grateful to Vincent Calvez and Sepideh Mirrahimi for reading earlier drafts of this paper very thoroughly. Their remarks were invaluable.

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