Equilibrium states for partially hyperbolic diffeomorphisms with hyperbolic linear part

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Abstract
We address the problem of existence and uniqueness (finiteness) of ergodic equilibrium states for a natural class of partially hyperbolic dynamics homotopic to Anosov. We propose to study the disintegration of equilibrium states along central foliation as a tool to develop the theory of equilibrium states for partially hyperbolic dynamics homotopic to Anosov on $T^3$.

Keywords: equilibrium state, partially hyperbolic, conditional measure
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1. Introduction
In this paper we address the problem of existence, uniqueness (or finiteness) of equilibrium states for a class of partially hyperbolic diffeomorphisms which are homotopic to Anosov (‘Derived from Anosov’). The novelty with respect to previous works in this topic is the use of disintegration of measures along central foliation of partially hyperbolic dynamics.

Definition 1.1. Let $M$ be a closed manifold. A diffeomorphism $f : M \rightarrow M$ is called partially hyperbolic if the tangent bundle $TM$ admits a $Df$-invariant decomposition $TM = E^s \oplus E^c \oplus E^u$ such that all unit vectors $v^\sigma \in E^\sigma_x$ ($\sigma = s, c, u$) for all $x \in M$ satisfy:

$$\| Df(x)v^s \| < \| Df(x)v^c \| < \| Df(x)v^u \|$$

and moreover $\| Df \|_{E^c} < 1$ and $\| Df^{-1} \|_{E^c} < 1$.

We call $f$ absolutely partially hyperbolic, if it is partially hyperbolic and for any $x, y, z \in M$

$$\| Df(x)v^s \| < \| Df(y)v^c \| < \| Df(z)v^u \|$$

where $v^s \in E^s_x$, $v^c \in E^c_x$ and $v^u \in E^u_x$. 


For partially hyperbolic diffeomorphisms, it is well known that there are foliations $\mathcal{F}_\sigma$ tangent to the sub-bundles $E_\sigma$ for $\sigma = s, u$. The leaf of $\mathcal{F}_\sigma$ containing $x$ will be called $\mathcal{F}_\sigma(x)$, for $\sigma = s, u$. In general, it is not true that there is a foliation tangent to the central sub-bundle $E^c$. However, by Brin, Burago and Ivanov [3] all absolutely partially hyperbolic diffeomorphism on $T^3$ admit center foliation tangent to $E^c$. Hertz, Hertz and Ures [23] gave examples of partially hyperbolic diffeomorphisms (non absolutely partially hyperbolic) on $T^3$ which do not admit center foliation. In their example the central bundle is not uniquely integrable and there is no foliation tangent to the center bundle.

**Definition 1.2.** Let $f: T^3 \to T^3$ be a partially hyperbolic diffeomorphism. $f$ is called Derived from Anosov (DA) diffeomorphism if is homotopic to linear Anosov automorphism $A: T^3 \to T^3$ with three invariant subbundle $T(T^3) = E^s \oplus E^c \oplus E^u$.

**Remark 1.3.** In this paper we assume that $f$ preserves the orientation on central leaves.

Let $f$ be a partially hyperbolic diffeomorphism homotopic to Anosov defined before as DA. By a well-known result of Franks [9] $f$ is semiconjugate to $A$. More specifically, there exists $H: T^3 \to T^3$ homotopic to the identity such that $H \circ f = A \circ H$. It is possible to show that $H$ is Hölder continuous. (see appendix.)

**Theorem 1.4 (Ures, [27]).** Let $f: T^3 \to T^3$ be a (absolutely partially hyperbolic) DA diffeomorphism. Then, $f$ has a unique measure of maximal entropy.

**Remark 1.5.** Ures in [27] showed that:

1. For all $x \in T^3$, $H^{-1}(x)$ is a compact connected interval (including the case of just a point) in a center manifold.
2. $C = \{x \in T^3 : \#H^{-1}(H(x)) > 1\}$ is $f$-invariant and $H^{-1}(H(C)) = C$. By the previous item we know that $C$ is union of compact intervals. We call these intervals as collapse intervals.
3. If $\mu$ is an $f$-invariant measure, then $h_\mu(f) = h_\mu(A)$ where $\nu = H_* (\mu)$.

Let us recall that as the semi-conjugacy fails to be injective only on uniformly bounded intervals, $h_{\text{top}}(f) = h_{\text{top}}(A)$. By corollary 2.8 in [11] we know that any derived from Anosov diffeomorphism is dynamically coherent and leaf conjugated to its linearization. Moreover, the above results of Ures hold in the more general setting of partially hyperbolic derived from Anosov diffeomorphisms and not just absolutely partially hyperbolic case (We thank Hammerlindl and Ures for explaining us this point).

**Definition 1.6.** Consider a continuous map $f: M \to M$ on a compact metric space $M$. We say that an $f$-invariant Borel probability measure $\mu$ is an equilibrium state for $f$ with respect to a potential $\phi \in C^0(M, \mathbb{R})$ if it satisfies

$$h_\mu(f) + \int \phi d\mu = \sup \{h_\eta(f) + \int \phi d\eta : \eta \in \mathcal{M}(f)\}$$

where $\mathcal{M}(f)$ denotes the set of $f$-invariant Borel probability measures.

If $\phi \equiv 0$, then any $\mu \in \mathcal{M}(f)$ such that $h_\mu(f) = \sup \{h_\eta(f) : \eta \in \mathcal{M}(f)\}$ is called measure of maximal entropy.

Ures [27] proved that DA diffeomorphisms have a unique measure of maximal entropy. Climenhaga, Fisher and Thompson [5] showed that robustly transitive diffeomorphisms introduced by Mañé and Bonatti–Viana have a unique equilibrium states with respect to Hölder
continuous potentials under a small variation condition. Here we study conditional measures of equilibrium states along center foliation. This enables us to find simpler technical conditions for the uniqueness (in some cases finiteness) of equilibrium states.

1.1. Preliminary and statement of results

The existence of equilibrium states for partially hyperbolic systems in \( T^3 \), associated to any continuous potential is guaranteed as a consequence of the work of Diaz, Fisher, Pacifico and Vieitez [8]. However, we can ask whether or not it is true that any Hölder potential admits a unique equilibrium state for a derived from Anosov diffeomorphism. The following theorems give a partial answer to this question. For a partially hyperbolic diffeomorphism with central foliation, we say an invariant measure is ‘virtually hyperbolic’ if there exists a full measurable invariant subset which intersects each center leaf in at most one point. See definition 2.9.

**Theorem A.** Let \( f : T^3 \to T^3 \) be a DA diffeomorphism (preserving orientation of center foliation) and \( \psi : T^3 \to \mathbb{R} \) a continuous potential such that \((A, \psi)\) has unique equilibrium state and the equilibrium state has full support. Define \( \phi := \psi \circ H \) and \( \mu \) be any ergodic equilibrium states of \( f \) with respect to \( \phi \):

1. If \( \mu(C) = 0 \), then \( \mu \) is the unique equilibrium state.
2. If \( \mu(C) = 1 \), then \( \mu \) is virtually hyperbolic and there exists necessarily another equilibrium state \( \eta \) for \((f, \phi)\).

Observe that clearly the first item of the above theorem implies that in the second case (if it occurs) any other equilibrium state gives total mass to the union of collapse intervals \( C \).

**Remark 1.7.** It is well known by work of Bowen [2] that when \( \psi : T^3 \to \mathbb{R} \) is Hölder continuous, then it satisfies the conditions of theorem A.

**Question 1.8.** Is there any \( \psi : T^3 \to \mathbb{R} \) Hölder continuous such that \( \phi := \psi \circ H \) verifies the case 2 of theorem A?

Currently we do not know \( \phi = \psi \circ H \) with \( \psi \) Hölder continuous that satisfies item 2 of the above theorem, although continuous examples exist which verifies the case 2 of the theorem A. Let \( f : T^3 \to T^3 \) be a DA diffeomorphism such that \( \text{vol}(C) = 1 \) which exists by the work of Ponce–Tahzibi (see [18]). In fact, they construct example of diffeomorphisms \( f \) preserving volume and ergodic such that the central Lyapunov exponent of \( f \) is negative whereas those of \( A \) (action of \( f \) on homotopy) is positive. Since vol is ergodic, then \( \nu = H_\ast \text{vol} \) is ergodic with respect to \( A \) and \( \text{supp}(\nu) = T^3 \). On the other hand, by a result of Jenkinson (see [14], theorem 5) there exists a continuous potential \( \psi : T^3 \to \mathbb{R} \) such that \( \nu \) is the unique equilibrium state for \( A \) associated to \( \psi \). Define \( \phi = \psi \circ H \) and observe that (using the fact that \( H \) preserves entropy of measures by remark 1.5) \( \text{vol} \) is an equilibrium states for \((f, \phi)\).

The proof of the above theorem enables us to conclude a result about finiteness of ergodic equilibrium states. Let \( f \) and \( \phi \) be as before:

**Theorem B.** Let \( f \) and \( \phi \) be as in theorem A. Then at least one of the following occurs:

- There is an ergodic non-hyperbolic equilibrium state.
- The number of ergodic equilibrium states is finite.
Recall that $f$ has a unique measure of maximal entropy. Indeed, we can show that under small variation hypothesis of the potential, the equilibrium state is unique.

**Theorem C.** Let $f$ and $\phi$ be as in theorem A. If the potential satisfies $\sup_{T} \phi - \inf_{T} \phi < \lambda_2$, then there exists a unique equilibrium state for $\phi$.

The ‘small’ variational condition in the theorem C is quite common in the literature of this topic to achieve uniqueness of equilibrium states and it has been considered by Oliveira and Viana [17] for non-uniformly expanding maps on compact manifolds, by Rios and Siqueira [22], Ramos and Siqueira [21] for partially hyperbolic horsehoes, by Hofbauer and Keller [12] for piecewise monotomic maps, by Bruin and Todd [4] for interval maps, and by Denker and Urbánski [7] for rational maps on the Riemann sphere.

Climenhaga, Fisher and Thompson [5, 6] proved uniqueness of equilibrium states for natural class of potentials in the setting of Mañé and Bonatti–Viana class of robustly transitive diffeomorphisms. We observe that in one hand their result is more general, as it treats non partially hyperbolic setting and much more general potentials. On the other hand, the class under consideration in their result is the special type of systems which are localized perturbations of uniformly hyperbolic dynamics. In fact their result ‘gives a quantitative criterion for existence and uniqueness of equilibrium state involving the topological pressure, the norm and variation of the potential, the tail entropy, and the $C^0$ size of the perturbation from the original Anosov map for the Mañé and Bonatti type examples (quote from [5])’. As they consider general Hölder potentials, a small variation (closedness to Anosov diffeomorphism) hypothesis is required for the dynamics.

We also mention that Spatzier and Visscher [25] proved uniqueness of equilibrium state for frame flows on closed, oriented, negatively curved $n$-manifold, $n$ odd and $(n \neq 7)$ and potentials induced by potentials defined on unit tangent bundles, i.e. constant on the fibers of the bundle $FM \to SM$ where $FM$ and $SM$ are respectively frame bundle and unit tangent bundle.

### 1.2. Higher dimensional case

We remark that the results of theorems (A), (B) and (C) are valid for absolutely partially hyperbolic $f : T^n \to T^n$ with one-dimensional (1D) center bundle and quasi-isometric stable and unstable foliations. We say $F$ is quasi-isometric if there are constants $C,D$ such that after lifting $F$ to the universal cover, for any two points $x, y$ in the same leaf, one has $d_F(x, y) \leq Cd(x, y) + D$ where $d_F$ and $d$ are respectively distance along the leaf and on the universal cover.

Let $f : T^n \to T^n$ be an absolutely partially hyperbolic diffeomorphism with 1D center bundle $(T(T^n)) = E^s \oplus E^c \oplus E^u)$ and quasi-isometric strong foliations.

Under the hypothesis of absolutely partial hyperbolicity and quasi-isometric strong foliations, remark 1.5 holds as it is pointed out in [27]. We write the statement and proof of higher dimensional version of theorem C for completeness (theorem D in section 5.1). The proofs of theorems A and B are exactly the same as in three dimensional setting.

### 2. Disintegration of measures

In this paper, in order to prove uniqueness (or finiteness) of equilibrium states, we propose to study the conditional measures of equilibrium states on the leaves of center foliation. In what follows we review some basic properties of disintegration of measures.
Let \((M, \mu, B)\) be a probability space, where \(M\) is a compact metric space, \(\mu\) a probability measure and \(B\) the borelian \(\sigma\)-algebra. Given a partition \(P\) of \(M\) by measurable sets, we associate the probability space \(\tilde{M} := M/P, \tilde{\mu}, \tilde{B}\) by the following way. Let \(\pi : M \rightarrow \tilde{M}\) be the canonical projection, that is, \(\pi\) associates a point \(x\) of \(M\) to the partition element \(P(x)\) that contains it. Then we define \(\tilde{\mu} := \pi_*\mu\) and \(\tilde{B} := \pi_*B\).

**Definition 2.1.** Given a partition \(P\). A family \(\{\mu_P\}_{P \in \mathcal{P}}\) is a system of conditional measures for \(\mu\) (with respect to \(P\)) if

(i) given \(\phi \in C^0(M)\), then \(P \mapsto \int \phi \mu_P\) is measurable;
(ii) \(\mu_P(P) = 1\) \(\tilde{\mu}\)-a.e.;
(iii) \(\mu = \int_M \mu_P \text{d}\tilde{\mu}\), i.e. if \(\phi \in C^0(M)\), then \(\int \phi \text{d}\mu = \int_M \int_P \phi \text{d}\mu_P \text{d}\tilde{\mu}\).

When it is clear which partition we are referring to, we say that the family \(\{\mu_P\}\) disintegrates the measure \(\mu\). There exists an equivalent form of writing the disintegration formula above:

\[
\mu = \int_M \mu_x \text{d}\mu
\]

by considering the conditional measures \(\mu_x, x \in M\) where \(\mu_y = \mu_x\) if \(y \in P(x)\). In this paper we use both formulation to simplify the notations whenever it is necessary.

**Proposition 2.2 ([10, 26]).** If \(\{\mu_P\}\) and \(\{\nu_P\}\) are conditional measures that disintegrate \(\mu\), then \(\mu_P = \nu_P\) \(\tilde{\mu}\)-a.e.

**Corollary 2.3.** If \(f : M \rightarrow M\) preserves a probability \(\mu\) and the partition \(P\), then \(f_*\mu_P = \mu_{f(P)}\) \(\tilde{\mu}\)-a.e. 

**Proof.** It follows from the fact that \(\{f_*\mu_P\}_{P \in \mathcal{P}}\) is also a disintegration of \(\mu\) and essential uniqueness of system of disintegration. \(\square\)

**Definition 2.4.** We say that a partition \(P\) is measurable (or countably generated) with respect to \(\mu\) if there exist a measurable family \(\{A_i\}_{i \in \mathbb{N}}\) and a measurable set \(F\) of full measure such that if \(B \in P\), then there exists a sequence \(\{B_i\}\), where \(B_i \in \{A_i, A_i^c\}\) such that \(B \cap F = \bigcap_i B_i \cap F\).

**Theorem 2.5 (Rokhlin’s disintegration [26]).** Let \(P\) be a measurable partition of a compact metric space \(M\) and \(\mu\) a borelian probability. Then there exists a disintegration by conditional measures for \(\mu\).

Let us state a simple but useful remark which comes from essential uniqueness of disintegration.

**Remark 2.6.** Let \((M, B, \mu)\) be a probability space, \(P\) a measurable partition of \(M\) and \(X \subset M\) a measurable subset of positive measure. Then \(X\) is called \(P\)-saturated if for any \(x \in X\) then \(P(x)\), the atom of partition containing \(x\), is contained in \(X\). Let \(\mu|_X\) be the normalized (probability) restriction of \(\mu\) on \(X\). For any \(P \in \mathcal{P}\) such that \(P \subset X\), the conditional measures of \(\mu\) and \(\mu|_X\) coincide, i.e. \(\mu_P = (\mu|_X)_P\).

More generally, if \(X \subset M\) is a measurable subset with positive measure then \(P\) induces a measurable partition on \(X\). Namely,
\[ \mathcal{P}_X := \{ P_X | P_X := P \cap X; P \in \mathcal{P} \} \]

is a measurable partition of \( X \). So, by Rokhlin theorem we consider the conditional measures \( \{ (\mu|_X)_{P_X} \} \) obtaining by disintegration of the probability \( \mu|_X \) on the atoms of partition \( \mathcal{P}_X \). We will use later in the paper the following fact which can be verified using the essential uniqueness of conditional measures: \( (\mu|_X)_{P_X} = (\mu_P)|_{P_X} \) and consequently \( \mu_P \leq (\mu|_X)_{P_X} \) on \( P_X \subseteq P \).

### 2.1. Atomic disintegration along foliations

In general the partition by leaves of a foliation may be non-measurable. It is for instance the case for the stable and unstable foliations of Anosov diffeomorphisms with respect to measures of non vanishing metric entropy. Therefore, by disintegration of a measure along the leaves of a foliation we mean the disintegration on compact foliated boxes. In principle, the conditional measures depend on the foliated boxes, however, two different foliated boxes induce proportional conditional measures. See [1] for a discussion on this issue.

**Definition 2.7.** We say that a foliation \( F \) has atomic disintegration with respect to a measure \( \mu \) if the conditional measures on any foliated box are sum of Dirac measures.

Equivalently we could define atomic disintegration as follows: there exist a full measurable subset \( Z \) such that \( Z \) intersects all leaves in at most a countable set.

Although the disintegration of a measure along a general foliation is defined in compact foliated boxes, it makes sense to say that a foliation \( F \) has a quantity \( k_0 \in \mathbb{N} \) of atoms per leaf. The meaning of ‘per leaf’ should always be understood as a generic leaf, i.e. almost every leaf. That means that there is a set \( A \) of \( \mu \)-full measure which intersects a generic leaf on exactly \( k_0 \) points.

Let us state a recent result of Viana and Yang\(^1\) [28].

**Theorem 2.8.** Let \( f \) be a DA diffeomorphism and \( \mu \) an invariant measure with \( h_{\mu}(f) > \lambda_1 \) then the disintegration of \( \mu \) along center foliation can not be atomic.

Let \( f \) be a derived from Anosov (or more generally any partially hyperbolic diffeomorphism) diffeomorphism.

**Definition 2.9.** An \( f \)-invariant measure \( \mu \) is called virtually hyperbolic if there exists a full measurable invariant subset \( Z \) such that \( Z \) intersects each center leaf in at most one point.

The above definition was given in [16] in the context of algebraic automorphisms. If \( \mu \) is virtually hyperbolic, then the center foliation is measurable with respect to \( \mu \) and conditional measures along center leaves are Dirac measure. Indeed the partition into central leaves is equivalent to the partition into points.

### 3. Proof of theorem A

**Proof of theorem A.** From remark 1.5 we have that \( H_* \mu \) is an equilibrium state for \( (A, \psi) \) and it is the unique one by hypothesis.

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\(^1\) We thank Yang for making us aware of the existence of this result when we were finishing this paper at the same time.
1. Let us prove the first part of the theorem. So, suppose that \( \mu(C) = 0 \). If \( \nu \) is another equilibrium state for \( (f, \psi \circ H) \), then \( H_* \mu = H_* \nu \). Let \( \varphi: \mathbb{T}^3 \to \mathbb{R} \) be any continuous map. Since \( H^{-1} H(C) = C \) we have \( \nu(C) = 0 \). Hence,

\[
\int \varphi \, d\mu = \int_{\mathbb{T} \setminus C} \varphi \, d\mu = \int_{\mathbb{T} \setminus C} \varphi \circ H^{-1} \circ H \, d\mu = \int_{\mathbb{T} \setminus C} \varphi \circ H^{-1} \, dH_* \mu \\
= \int_{\mathbb{T} \setminus C} \varphi \circ H^{-1} \, dH_* \nu = \int_{\mathbb{T} \setminus C} \varphi \, d\nu
\]

This implies that \( \mu = \nu \).

2. Now we prove the second item of the theorem. Let us begin to prove the virtually hyperbolic property of equilibrium state in the case where \( \mu(C) = 1 \).

Firstly we consider the partition into collapse intervals:

\[
\mathcal{N} := \{ H^{-1}(x) : H^{-1}(x) \text{ is a non trivial closed interval} \}.
\]

It is a measurable partition and we can speak about disintegration of \( \mu \) along collapse intervals. We denote by \( \mu_{\mathcal{N}(x)} \) the conditional measure supported on the collapse interval containing \( x \). Of course, if \( \mathcal{N}(x) = \mathcal{N}(y) \) then \( \mu_{\mathcal{N}(x)} = \mu_{\mathcal{N}(y)} \).

We prove in the lemma 3.1 that the disintegration is atomic and there exist at most one atom in each collapse interval.

**Lemma 3.1.** The disintegration is atomic and there exist one atom in each collapse interval.

**Proof.** Fix an orientation for the central leaves and for each collapse interval (any element of the partition \( \mathcal{N} \)) and consider the left extreme point of them. It can be proved that the left extreme point of collapse intervals form a measurable set (see [20] right after lemma 3.2). Observe also that by assumption \( f \) preserves orientation along center leaves. We call these sets as point zero, that is if \( x \in C \) then \( 0_x \) means the left extreme point associated to the segment \( \mathcal{N}(x) \) which contains \( x \). If \( y \in \mathcal{N}(x) \) then \([0_x,y]\) stands for the segment inside the center leaf which contains \( 0_x \) and \( y \) as its endpoints.

For \( 0 \leq \alpha \leq 1 \), we consider the set

\[
H_\alpha = \{ y : [0_x,y] \subset \mathcal{N}(x) \mid \mu_{\mathcal{N}(x)}([0_x,y]) \leq \alpha \}.
\]

We claim that \( H_\alpha \) is an invariant set. This comes from the fact that \( f(\mathcal{N}(x)) = \mathcal{N}(f(x)) \) and \( f_* \mu_{\mathcal{N}(x)} = \mu_{\mathcal{N}(f(x))} \). From the definition of disintegration and \( H_\alpha \), notice that \( \mu(H_\alpha) \leq \alpha \).

By ergodicity we have \( \mu(H_\alpha) = 0 \) for all \( \alpha < 1 \). This implies that the conditional measures have exactly one atom on each collapse interval.

As there are at most countably many collapse intervals in each center leaf, we get a full measurable subset \( M \subset \mathbb{T}^3 \) which intersects each center leaf in at most countably many points.

**Proposition 3.2.** If \( \mu \) is as above, then there exist finitely many atoms per (global) leaf.

**Proof.** There exists a similar result (although just for volume measure) in ([19], theorem B). For completeness we give the proof here in our context of equilibrium states. We aim to show
that there exists a full measurable subset which intersects almost all center leaves in a finite (uniformly bounded) number of points.

By contradiction, suppose that this is not the case. Define $\nu := H_*\mu$ which is an invariant measure by the linear hyperbolic automorphism.

Let $\{R_i\}$ be a Markov partition for $A$ and consider the partition $\mathcal{P} := \{F^R_i(x), x \in R_i \}$ for some $1$. Where $F^R_i(x)$ denotes the connected component of $F(x) \cap R_i$ containing $x$. The partition $\mathcal{P}$ is a measurable partition and by Rohlin theorem we can disintegrate $\nu$ along the elements of this partition. Here we are neglecting the boundary of elements of partition. As $\nu$ is an equilibrium state of $\phi$ for Anosov automorphism with full support, it gives zero mass to the boundary of Markov partition. Here is the unique place where we use the hypothesis of full support for the unique equilibrium state of $(A, \psi)$ in theorem $A$.

Let $\nu_x$ be the conditional measure supported on $F^R_i(x)$. Observe that, as $H(\mathcal{M})$ intersects typical leaves in a countable number of points, the conditional measures $\nu_x$ must be atomic.

**Lemma 3.3.** There is a natural number $\alpha_0 \in \mathbb{N}$ such that for $\nu$-almost every $x$, $\nu_x$ contains exactly $\alpha_0$ atoms.

**Proof.** Firstly we observe that:

**Sub-Lemma 3.4.** $A_*\nu_x \leq \nu_{A(x)}$ restricted to the subsets of $F^R_i(A(x))$.

**Proof.** Observe that $A_*\nu_x$ and $\nu_{A(x)}$ are probability measures defined respectively on $A(F^R_i(x))$ and $F^R_i(A(x))$. Fix an element of Markov partition $R_i$. By remark 2.6, $\nu_{x}, x \in R_i$ coincides with the disintegration of the normalized restriction of $\nu$ on $R_i$ which we denote by $\nu|_{R_i}$. As $\nu$ is invariant $A_*\nu_R|_{R_i} = \nu|_{A(R_i)}$, by essential uniqueness of disintegration, $A_*\nu_x$ coincides with the disintegration of $\nu|_{A(R_i)}$ along the partition $A(F^R_i(x)), x \in R_i$.

For any $j$ such that $A(R_i) \cap R_j \neq \emptyset$, by Markov property $A(R_i)$ crosses $R_j$ completely in the center-unstable direction and so for all $x \in R_i$,

$$F^R_i(A(x)) \subset A(F^R_i(x)).$$

Again by remark 2.6 we conclude that $A_*\nu_x \leq \nu_{A(x)}$ on $F^R_i(A(x))$.

Given any $\delta > 0$ consider the set $K_\delta = \{ x \in \mathbb{T}^3 | \nu_{A}(\{x\}) > \delta \}$, that is, the set of atoms with weight at least $\delta$. If $x \in K_\delta$ then

$$\delta < \nu_{A}(\{x\}) = A_*\nu_x(\{A(x)\}) \leq \nu_{A(x)}(\{A(x)\}).$$

Thus $A(K_\delta) \subset K_\delta$, and by the ergodicity of $A$ we have that $\nu(K_\delta)$ is zero or one, for each $\delta > 0$. Note that $\nu(K_0) = 1$ and $\nu(K_1) = 0$. Let $\delta_0$ be the critical point for which $\nu(K_\delta)$ changes value, i.e., $\delta_0 = \sup \{ \delta : \nu(K_\delta) = 1 \}$. This means that all the atoms have weight $\delta_0$. Due to the atomicity of disintegration, the value of $\delta_0$ has to be a strictly positive number. Since $\nu_x$ is a probability we have an $\alpha_0 := 1/\delta_0$ number of atoms as claimed in lemma.

In particular the above lemma shows that given a fixed length $L \in \mathbb{R}^+$ there exist $N \in \mathbb{N}$ such that the number of atoms in any typical center plaque of size $L$ is at most $N$. Recall that we had supposed that $H(\mathcal{M})$ intrinsically intersects center leaves in infinitely many points (or non uniformly finite). So, take a center plaque $D \subset F_i^R$ with more than $N$ atoms. By backward contraction along central leaves by $A$ we get a large $n > 0$ such that the length of $A^{-n}(D)$ is
less than \(L\). As \(\nu\) is invariant and disintegration is unique we get a center plaque with length less than \(L\) containing more than \(N\) atoms which is absurd. This ends the proof of proposition.

Up to now we have proved that the number of atoms is finite and constant by ergodicity on almost all center leaves. Take an orientation on the center foliation. As \(f\) preserves orientation we conclude that the number of atoms is one. Indeed, the set of left extremal atoms on each center leaf form an invariant subset with positive measure and hence full measure by ergodicity. If \(f\) does not preserve orientation the same argument yields two atoms on almost all leaves.

Now we complete the proof of the second item of theorem. Up to now, we have seen that if \(\mu(C) = 1\) then the center foliation is measure theoretically equivalent to the partition of \(T^3\) into points and consequently measurable. We denote by \((\tilde{M}, \tilde{\mu})\) the quotient space \(T^3/F^c\) equipped with the quotient measure. Observe that by virtual hyperbolicity proved above, any element \(\tilde{x} \in \tilde{M}\) can be considered as a unique collapse interval inside the center leaf \(F^c(\tilde{x})\). From now on we denote this collapse interval by \(N(\tilde{x})\). We denote by \(\tilde{f} : \tilde{M} \to \tilde{M}\) the induced map on the quotient space. Clearly as \(\mu\) is invariant by \(f\) then \(\tilde{\mu}\) obtained by natural quotient is invariant by \(\tilde{f}\).

We can write:

\[
\mu = \int \delta_{a(\tilde{x})} d\tilde{\mu}(\tilde{x}),
\]

where \(a(\tilde{x}) \in N(\tilde{x})\) and \(N(\tilde{x})\) is the collapse interval corresponding to \(\tilde{x}\). Choose \(b(\tilde{x}) \neq a(\tilde{x})\) the left (or right) extreme point of \(N(\tilde{x})\). We define

\[
\eta = \int \delta_{b(\tilde{x})} d\tilde{\mu}(\tilde{x})
\]

\(\eta\) is well defined because \(\{b(\tilde{x})\}\) is a measurable set. We claim that \(\eta\) is \(f\)-invariant, \(H_*\mu = H_*\eta\) and ergodicity of \(\mu\) implies ergodicity of \(\eta\). To show that \(\eta\) is invariant take any continuous \(\xi\) and observe that:

\[
\int \xi \circ f d\eta = \int \int \xi \circ f \delta_{b(\tilde{x})} d\tilde{\mu} = \int \xi(b(\tilde{x})))d\tilde{\mu} = \int \xi(b(\tilde{x}))d\tilde{\mu} = \int \xi d\eta,
\]

where the third equality comes from the invariance of collapse intervals and that \(f\) preserves orientation on center foliation. The fourth equality is consequence of invariance of \(\tilde{\mu}\) by \(\tilde{f}\).

To prove the ergodicity of \(\eta\), consider any invariant subset \(D\) with \(\eta(D) > 0\). Observe that \(\tilde{\mu}\) is ergodic as an invariant measure of \(\tilde{f}\). As \(f(b(\tilde{x})) = b(\tilde{f}(\tilde{x}))\) and \(D\) is invariant we have that \(\{\tilde{x} : \chi_D(b(\tilde{x})) = 1\}\) is an \(\tilde{f}\) invariant subset of \(\tilde{M}\). So, ergodicity of \(\tilde{\mu}\) implies that it has full measure. This implies that \(\eta(D) = 1\).

By essential uniqueness of disintegration \(\eta \neq \mu\).

On the other hand, as \(H(a(\tilde{x})) = H(b(\tilde{x}))\) and \(\phi = \psi \circ H\) we have:

\[
\int \phi d\eta = \int \phi(b(\tilde{x})) d\tilde{\mu} = \int \phi(a(\tilde{x})) d\tilde{\mu} = \int \phi d\mu.
\]

Finally, let us observe that \(h_\mu(f) = h_\eta(f)\). Indeed, \((f, \mu)\) and \((f, \eta)\) are measure theoretically isomorphic by the map that sends \(a(\tilde{x})\) to \(b(\tilde{x})\). This implies that \(\eta\) is an equilibrium state for \((f, \phi)\).
4. Proof of theorem B

Let \( \mu \) and \( \nu \) be two ergodic equilibrium states for \( (f, \phi) \) such that \( \mu(C) = \nu(C) = 1 \). By the proof of theorem A we have that

\[
\mu = \int \delta_{a(\tilde{x})} \, d\tilde{\mu}(\tilde{x}) \quad \text{and} \quad \nu = \int \delta_{b(\tilde{x})} \, d\tilde{\mu}(\tilde{x}).
\]

We mention that the atoms \( a(\tilde{x}), b(\tilde{x}) \) can not be in the same Pesin stable manifold if \( \mu \neq \nu \). More specifically we prove:

**Lemma 4.1.** Let \( \mu = \int \delta_{a(\tilde{x})} \, d\tilde{\mu}(\tilde{x}) \) and \( \nu = \int \delta_{b(\tilde{x})} \, d\tilde{\mu}(\tilde{x}) \) be \( f \)-invariant measures. If \( \lim_{n \to \infty} d(f^n(a(\tilde{x}))), f^n(b(\tilde{x})) = 0 \) for \( \tilde{\mu} \)-a.e. \( \tilde{x} \), then \( \mu = \nu \).

**Proof.** Since \( f_*\mu = \mu \) and \( f_*\nu = \nu \), we have \( \mu = f_*\mu = \int \delta_{f^n(a(\tilde{x}))} \, d\tilde{\mu} \) and \( \nu = f_*\nu = \int \delta_{f^n(b(\tilde{x}))} \, d\tilde{\mu} \).

Let \( \varphi : \mathbb{T}^3 \to \mathbb{R} \) be any Lipschitz map. Hence,

\[
\left| \int \varphi \, d\mu - \int \varphi \, d\nu \right| = \left| \int \varphi(f^n(a(\tilde{x}))) \, d\tilde{\mu} - \int \varphi(f^n(b(\tilde{x}))) \, d\tilde{\mu} \right|
\leq \int | \varphi(f^n(a(\tilde{x}))) - \varphi(f^n(b(\tilde{x}))) | \, d\tilde{\mu}
\leq \int kd(f^n(a(\tilde{x})), f^n(b(\tilde{x}))) \, d\tilde{\mu}.
\]

This implies that \( \int \varphi \, d\mu = \int \varphi \, d\nu \). Since Lipschitz maps are dense in \( C(\mathbb{T}^3) \), we have last equality is holds for \( \varphi \in C(\mathbb{T}^3) \).

**Proof of theorem B.** Suppose that there are infinitely many ergodic equilibrium states and all of them are hyperbolic measures. Observe that by theorem A, all of the ergodic equilibrium states disintegrate to atomic Dirac measure along collapse intervals. We claim that all of them have the same quotient measure. Recall that if \( \pi : \mathbb{T}^3 \to \mathbb{T}^3/F^c \) denotes the natural projection, then for a measure \( \mu \), the quotient measure is \( \tilde{\mu} = \pi_*\mu \). See section 2.

To proof of the claim comes from the fact that \( H_*(\mu_1) = H_*(\mu_2) \) for any two such measures (equal to the equilibrium measure for \( A \)) and \( H \) sends central leaves to central leaves. Indeed, define \( \hat{H} : \mathbb{T}^3/F^c \to \mathbb{T}^3/V^c \) induced by \( H \), where \( V^c \) is the center foliation of \( A \). As the semi conjugacy sends central leaves of \( f \) to central leaves of \( A \) we have

\[
\hat{H} \circ \pi = \pi \circ H.
\]

So we conclude that \( \hat{H}_*\pi_*(\mu_1) = \pi_*H_*(\mu_i) \) for \( i = 1, 2 \). But, \( \pi_*H_*(\mu_i) = \pi_*\nu \). This implies that

\[
\hat{H}_*\pi_*(\mu_1) = \hat{H}_*\pi_*(\mu_2)
\]

and as \( \hat{H} \) is bijective, we conclude that \( \pi_\mu_1 = \pi_\mu_2 \).

Take \( \mu_1 \) one of such measures and
\[ \mu_1 = \int \delta_{\hat{a}_1(x)} \, d\hat{\mu}(\tilde{x}) \]

where \( \hat{\mu} \) is independent of the choice of \( \mu_1 \).

Without loss of generality suppose that \( \hat{a}_1(\tilde{x}) \) is not an extreme point of the collapse interval \( \mathcal{N}(\tilde{x}) \) for almost every \( \tilde{x} \). Now we take into account the central Lyapunov exponent of \( \mu_1 \). We assume that \( \lambda_c(\mu_1) < 0 \). The process of the proof in the case of positive Lyapunov exponent will be similar. We can construct another ergodic equilibrium state, which we denote by twin measure of \( \mu_1 \):

\[ \eta_1 = \int \delta_{b_1(\tilde{x})} \, d\hat{\mu}, \]

where \( b_1(\tilde{x}) \) is one of the boundary points of the Pesin center manifold of \( \hat{a}_1(\tilde{x}) \). Without loss of generality we take always left boundary point. (See figure 1) By Pesin center manifold here we mean the arc which is the intersection of Pesin stable manifold with the center leaf through \( \hat{a}_1(\tilde{x}) \). Let us clarify the choice of \( b_1(\tilde{x}) \). We claim that at least one of the (two) boundary points of Pesin center manifold of \( \hat{a}_1(\tilde{x}) \) is not an extreme point of \( \mathcal{N}(\tilde{x}) \) for \( \hat{\mu} \)-a.e. \( \tilde{x} \). Indeed, if by contradiction there exists a positive \( \hat{\mu} \) set \( \hat{B} \subset \hat{M} \) of \( \tilde{x} \) such that both boundary points of \( \mathcal{N}(\tilde{x}) \) belongs to the closure of Pesin center manifold of \( \hat{a}(\tilde{x}) \), by invariance of both collapse intervals and Pesin center manifolds, ergodicity of \( \hat{\mu} \) implies that \( \hat{\mu}(\hat{B}) = 1 \). Now, this gives a contradiction with infinitely many hyperbolic ergodic equilibrium states. Indeed, by lemma 4.1 the atoms of conditional measures of any other hyperbolic ergodic equilibrium state can not belong to the interior of \( \mathcal{N}(\tilde{x}) \) for \( \hat{\mu} \)-a.e. \( \tilde{x} \).

So, let suppose that \( b_1(\tilde{x}) \) is the left boundary point of the Pesin center manifold of \( \hat{a}_1(\tilde{x}) \) and it is not extreme point of \( \mathcal{N}(\tilde{x}) \). As \( b_1(\tilde{x}) \) is a boundary point of stable manifold, then \( \lambda_c(b_1(\tilde{x})) \geq 0 \). In fact as we are assuming that all ergodic equilibrium states are hyperbolic it comes out that \( \lambda_c(b_1(\tilde{x})) > 0 \).

Continuing the similar procedure for \( \eta_1 \) we obtain a new ergodic equilibrium state \( \mu_2 \) such that

\[ \mu_2 = \int \delta_{\hat{a}_2(x)} \, d\hat{\mu} \quad \text{and} \quad \lambda_c(\mu_2) < 0. \]

Observe that \( \hat{a}_2(\tilde{x}) \) is the (left) boundary point of Pesin center manifold of \( b_1(\tilde{x}) \) understanding now, Pesin center manifold as the intersection of Pesin unstable manifold with center leaf.

If this process stops in finite steps, then we conclude that there are only finitely many ergodic equilibrium states which contradicts our assumption. So, consider the sequence of measures
\[ \mu_n = \int \delta_{a_n(\tilde{x})} d\tilde{\mu}, \eta_n = \int \delta_{b_n(\tilde{x})} d\tilde{\mu} \]

constructed as above where \( \lambda_c(\mu_n) < 0 \) and \( \lambda_c(\eta_n) > 0 \). Let
\[ c(\tilde{x}) = \inf a_n(\tilde{x}) = \inf b_n(\tilde{x}) \]

and define
\[ \zeta = \int \delta_{c(\tilde{x})} d\tilde{\mu}. \]

By construction \( f(a_n(\tilde{x})) = a_n(f(\tilde{x})) \) and consequently \( f(c(\tilde{x})) = c(f(\tilde{x})) \). So, \( \zeta \) is invariant measure and by a similar argument as appeared in the proof of second item of theorem A, we have that \( \zeta \) is an ergodic equilibrium state. We claim that \( \lambda_c(\zeta) = 0 \).

Indeed, if \( \lambda_c(\zeta) < 0 \) (similar argument applies for positive exponent) then there exist \( N \in \mathbb{N} \) and \( \tilde{C} \subset \tilde{M} \) such that \( \tilde{\mu}(\tilde{C}) > 0 \) and \( a_n(\tilde{x}) \) belong to the Pesin center manifold of \( c(\tilde{x}) \) for all \( n \geq N \). Again we use ergodicity argument to conclude that \( \tilde{\mu}(\tilde{C}) = 1 \). Now applying lemma 4.1 it comes out that \( \mu_n = \mu_m \) for \( m, n \geq N \) which is a contradiction.

This ends the proof of theorem. \( \square \)

5. Proof of theorem C

Theorem C is a consequence of theorems A and 2.8. However, we include a proof which is interesting by itself. Let \( \lambda_1 < 0 < \lambda_2 < \lambda_3 \) be Lyapunov exponents of \( A \). Let \( \mu \) be an ergodic equilibrium state for \( (f, \phi = \psi \circ H) \).

Proposition 5.1. If \( \mu(C) = 1 \), then \( h_\mu(f) \leq \lambda_1 \).

Firstly we state two lemmas relating the dimension of a measure and those of its conditional measures.

Lemma 5.2. Let \( m \) be a probability measure on \( \mathbb{R}^p \times \mathbb{R}^q \), \( \pi \) projection onto \( \mathbb{R}^p \), \( m_t \) conditional measures of \( m \) along the fibers of \( \pi \). Define
\[ \gamma(t) = \lim_{\epsilon \to 0} \inf \frac{\log m(\pi^{-1}B^p(t, \epsilon))}{\log \epsilon} \]

and let \( \delta \geq 0 \) be such that at \( m \)-a.e. \( (s, t) \)
\[ \delta \leq \lim_{\epsilon \to 0} \inf \frac{\log m_B(s, \epsilon)}{\log \epsilon}. \]

Then, at \( m \)-a.e. \( (s, t) \)
\[ \delta + \gamma(t) \leq \lim_{\epsilon \to 0} \inf \frac{\log m_{B^p+q}(s, t, \epsilon)}{\log \epsilon}. \]

Proof. (See [15], lemma 11.3.1). \( \square \)
By the proof of theorem A, we have that if \( \mu(C) = 1 \), then \( \mu \) is virtually hyperbolic. Let \( \nu = H, \mu \) and \( R \) be a Markov’s rectangle of \( A \). We normalize the restriction of \( \nu \) on \( A \). Let \( F^u \) be a typical unstable leaf of \( A \). Consider \( R^u = R \cap F^u \). Observe that \( R^u \) is foliated by strong unstable plaques and also by central (weak unstable) plaques. Denote by \( \nu^u \) the conditional measure of \( \nu \) (normalized and restricted on \( R \)) on \( R^u \).

Since disintegration of \( \nu \) along center foliation is mono-atomic, we have

\[
\nu^u = \int \delta_{a(t)} d\nu^{u\mu}(t)
\]

where \( a(t) \) is the unique atom on the central leaf of \( t \) and \( \nu^{u\mu} \) is the quotient measure on the quotient of \( R^u \) by central plaques. This quotient space can be identified by a strong unstable plaque.

**Lemma 5.3.** If \( \delta^u \) denotes the pointwise dimension of \( \nu^u \), then

\[
\delta^u := \liminf_{\epsilon \to 0} \frac{\log \nu^u(B^u(x, \epsilon))}{\log \epsilon} = \delta^u
\]

where \( B^u(x, \epsilon) \) is the open ball with center \( x \) and radius \( \epsilon \) on the strong unstable leaf of \( x \).

**Proof.** The inequality

\[
\delta^u \geq \liminf_{\epsilon \to 0} \frac{\log \nu^u(B^u(x, \epsilon))}{\log \epsilon}
\]

is immediate by lemma 5.2.

Now we prove the other inequality. We consider the set

\[
D = \{ x \in R^u : \exists \alpha > 0, \exists \epsilon_0 > 0 \text{ s.t. } \frac{\nu^u(B^u(x, \epsilon) \times B^c(x))}{\nu^u(B^u(x, \epsilon) \times F^c(x))} \geq \alpha, \forall \epsilon \leq \epsilon_0\}
\]

where \( B^c(x, \epsilon) \) denote the open ball with center \( x \) and radius \( \epsilon \) in the central leaf of \( x \).

We claim that \( \nu^u(D) = 1 \). In fact, we prove that all atoms \( a(x) \) are in \( D \). By definition of conditional measure

\[
1 = \delta_{a(x)}(B^c(a(x), \gamma)) = \lim_{\epsilon \to 0} \frac{\nu^u(B^u(a(x), \epsilon)) \times B^c(a(x), \gamma))}{\nu^u(B^u(a(x), \epsilon) \times F^c(x))}
\]

since \( a(x) \) is the unique atom on the central leaf of \( x \), we have that the last equality holds for all \( \gamma > 0 \).

Hence,

\[
1 = \lim_{\epsilon \to 0} \frac{\nu^u(B^u(a(x), \epsilon)) \times B^c(a(x), \epsilon))}{\nu^u(B^u(a(x), \epsilon) \times F^c(x))}
\]

We take a large enough \( n \) such that there exist \( \epsilon_0 > 0 \) satisfying the following:

\[
\frac{n-1}{n} \nu^u(B^u(a(x), \epsilon) \times F^c(x)) < \nu^u(B^u(a(x), \epsilon)) \times B^c(a(x), \epsilon), \forall \epsilon < \epsilon_0
\]
this proves the claim.

If \( x \in D \) and since \( h^c \nu^u = \nu^u \) (\( h^c \) is the central holonomy in \( R^c \)), then
\[
\nu^u(B^u(x, \epsilon) \times B^c(x, \epsilon)) \geq \alpha \nu^u(B^u(x, \epsilon) \times \mathcal{F}^c(x)) = \alpha \nu^u(B^u(x, \epsilon))
\]
so,
\[
\delta^u \leq \liminf_{\epsilon \to 0} \frac{\log \nu^u(B^u(x, \epsilon)) + \log \alpha}{\log \epsilon}.
\]

Therefore the lemma is proved. \( \square \)

**Proof of the proposition 5.1.** By Ledrappier and Young’s formula \([15]\) and \( h^u(A) = h_\mu(f) \), it comes out that
\[
h_\mu(f) = \lambda_1 \delta_1 + \lambda_2 (\delta^u - \delta_1)
\]
where \( \delta_1 \) is the pointwise dimension of the measure on the strong unstable leaf. By lemma 5.2, we have \( \delta_1 \leq \delta^u \) and by lemma 5.3, we have \( \delta_1 \leq \delta^u \).

So,
\[
\delta^u (\lambda_1 - \lambda_2) \geq \delta_1 (\lambda_1 - \lambda_2)
\]
by lemma 5.3,
\[
\delta^u \lambda_1 \geq \lambda_1 \delta_1 + \lambda_2 (\delta^u - \delta_1) = h_\mu(f)
\]
since that \( \delta^u \leq 1 \), we have \( h_\mu(f) \leq \lambda_1 \). \( \square \)

**Proof of theorem C.** We claim that if the potential \( \phi \) satisfies the low variational hypothesis of the theorem then the entropy of any equilibrium state of \( \phi \) is larger than \( \lambda_1 \). To see this it is enough to take \( \mu \) as any equilibrium state of \( \phi \) and \( \eta \) the measure of maximal entropy. Recall that \( h_{\top}(f) = h_{\top}(A) \) where \( A \) is the linearization of \( f \).

Recall that \( h_\mu(f) + \int \phi d\mu \geq h_\eta(f) + \int \phi d\eta = \lambda_1 + \lambda_2 + \int \phi d\eta. \)

So,
\[
h_\mu(f) \geq \lambda_1 + \lambda_2 + (\int \phi d\eta - \int \phi d\mu) \geq \lambda_1 + \lambda_2 - (\sup \phi - \inf \phi) > \lambda_1.
\]

By proposition 5.1, we have that all equilibrium state ergodic that satisfies the low variational hypothesis give zero mass to the union of collapse intervals \( C \). Hence, by item 1 of theorem A, if the potential \( \phi \) satisfies the low variational hypothesis, then \( (f, \phi) \) has a unique equilibrium state. \( \square \)

5.1. **Theorem C in higher dimension**

To prove a high dimensional version of theorem C, we need to generalize proposition 5.1 to higher dimensional case.
Proposition 5.4. Let \( f : \mathbb{T}^n \to \mathbb{T}^n \) be an absolutely partially hyperbolic diffeomorphism with 1D center bundle \((T(\mathbb{T}^n)) = E^s \oplus E^c \oplus E^u)\) and quasi-isometric stable and unstable foliations. Suppose \( \mu \) is an invariant measure with atomic disintegration along center foliation. Then \( h_\mu(f) \leq \sum_{j=1}^{k} \lambda_j d_j \) where \( \sum_{j=1}^{k} d_j \) is the dimension of \( E^u \) and \( \lambda_1 > \lambda_2 > \cdots > \lambda_k \) are the \( k \) first positive Lyapunov exponents (with multiplicity \( d_j, j = 1, \cdots, k \)) of \( A \), the linearization of \( f \).

Proof. We follow like in the proof of proposition 5.1. Here \( \alpha_1, \alpha_2, \cdots, \alpha_{k+1}, \alpha_j \leq d_j, j = 1, \cdots, k+1 \) are the fractal dimensions as appears in the Ledrappier–Young result to guarantee

\[
h_\nu(A) = \sum_{j=1}^{k+1} \lambda_j \alpha_j.
\]

We denote by \( \delta^{cu} \) the dimension of the conditional measure of \( \nu \) along unstable foliation of \( A \). (center-unstable, if \( A \) is considered as partially hyperbolic.)

A higher dimensional version of lemma 5.3 implies that

\[
\delta^{cu} = \delta^{wu} \leq \sum_{j=1}^{k} d_j
\]

where \( \delta^{wu} \) is the dimension of the quotient of \( \nu^{cu} \) by center foliation of \( A \).

\[
h_\mu(f) = h_\nu(A) = \sum_{j=1}^{k+1} \lambda_j \alpha_j
\]

\[
= \sum_{j=1}^{k} \lambda_j \alpha_j + \lambda_{k+1}(\delta^{cu} - \sum_{j=1}^{k} \alpha_j)
\]

\[
= \sum_{j=1}^{k} (\lambda_j - \lambda_{k+1}) \alpha_j + \lambda_{k+1} \delta^{wu}
\]

\[
\leq (1) \sum_{j=1}^{k} (\lambda_j - \lambda_{k+1}) \alpha_j + d_j \lambda_{k+1}
\]

\[
\leq \sum_{j=1}^{k} \lambda_j d_j.
\]

The above proposition and theorem A in higher dimension implies:

Theorem D. Let \( f : \mathbb{T}^n \to \mathbb{T}^n \) be a absolutely partially hyperbolic diffeomorphism with 1D center bundle and quasi isometric stable and unstable foliations. Let \( \lambda_{k+1} \) be the central Lyapunov exponent of \( A \), the linearization of \( f \), and \( \phi \) be a continuous potential satisfying hypothesis of theorem A with

\[
\sup_{x \in \mathbb{T}^n} \phi(x) - \inf_{x \in \mathbb{T}^n} \phi(x) \leq \lambda_{k+1}
\]

then \( \phi \) has a unique equilibrium state.
6. Center Lyapunov exponent and equilibrium state

**Theorem 6.1.** Let $\mu$ be an equilibrium state for $f : \mathbb{T}^3 \to \mathbb{T}^3$ (absolutely partially hyperbolic and derived from Anosov) w.r.t. a potential $\phi = \psi \circ H$. If $\lambda_c^c(\mu) > 0$, then

$$\lambda_2 \leq \lambda_c^c(\mu) + \sup \phi - \inf \phi.$$  

The proof of this result is similar to arguments of Ures ([27], theorem 5.1) and it is based on a Pesin-Ruelle-like inequality proved by Hua, Saghin and Xia in [13]. Here we assume the stronger definition of partial hyperbolicity to guarantee the quasi-isometric property of unstable foliation in the universal cover ([3]). However, if a pointwise partially hyperbolic $T^3$ diffeomorphism has no $cu$ or $cs$ tori, then there is unique foliations tangent to $E^c$, $E^s$ and $E^u$ and all $W^c$, $W^s$ and $W^u$ are quasi-isometric in the universal cover. (This comes from [11] and has been remarked to us by Hammerlindl.) In particular transitive partially hyperbolic diffeomorphisms have quasi-isometric invariant foliations and satisfy the above theorem.

Let $\mathcal{W}$ be a foliation. Let $W_r(x)$ be the ball of the leaf $W(x)$ with radius $r$ and centered at $x$. Let

$$\chi_{\mathcal{W}}(x,f) = \limsup_{n \to \infty} \frac{1}{n} \log(\text{vol}(f^n(W_r(x))))$$

$\chi_{\mathcal{W}}(x,f)$ is the volume growth rate of the foliation at $x$. Let

$$\chi_{\mathcal{W}}(f) = \sup_{x \in \mathcal{M}} \chi_{\mathcal{W}}(x,f).$$

Then, $\chi_{\mathcal{W}}(f)$ is the maximum volume growth rate of $\mathcal{W}$ under $f$. Let us denote $\chi_u(f) = \chi_{W^u}(f)$ when $f$ is a partially hyperbolic diffeomorphism.

**Theorem 6.2 ([13]).** Let $f$ be a $C^{1+\alpha}$ partially hyperbolic diffeomorphism. Let $\mu$ be an ergodic measure and $\lambda_c^c(\mu)$ the center Lyapunov exponent of $\mu$. Then,

$$h_{\mu}(f) \leq \chi_u(f) + \sum_{\lambda_{c} > 0} \lambda_{c}^c(\mu).$$

**Proof of theorem 6.1.** Firstly observe that $\chi_u(f) \leq \lambda_1$. Indeed, we know that $\chi_u(f) = \chi_u(\tilde{f})$ where $\tilde{f}$ is any lift of $f$ to universal cover. On the one hand, since $W^u$ is quasi-isometric, we have that

$$\frac{1}{n} \log(\text{vol}(\tilde{f}^n(W^u_r(\tilde{x})) \leq \frac{1}{n} \log(Q \text{diam}(\tilde{f}^n(W^u_r(\tilde{x}))$$

for some constant $Q$. On the other hand, $\tilde{H}(\tilde{f}^n(W^u_r(\tilde{x})) = \tilde{A}^n(\tilde{H}(W^u_r(\tilde{x}))).$ Let $C = \text{diam}(\tilde{H}(W^u_r(\tilde{x}))).$ Then, $\text{diam}(\tilde{A}^n(\tilde{H}(W^u_r(\tilde{x}))) = C \exp(n\lambda_1).$ Since $\tilde{H}$ is at bounded distance from the identity we have that there exists a constant $K$ such that $\text{diam}(\tilde{f}^n(W^u_r(\tilde{x})) \leq C \exp(n\lambda_1) + K$. Thus,

$$\frac{1}{n} \log(\text{vol}(\tilde{f}^n(W^u_r(\tilde{x}))) \leq \frac{1}{n}(Q(C \exp(n\lambda_1) + K)).$$
Then, $\chi_u(f) \leq \lambda_1$. To conclude the proof observe that
\[ \lambda_1 + \lambda_2 + \int \phi d\eta = h_\eta(f) + \int \phi d\eta \leq h_\mu(f) + \int \phi d\mu \]
where $\eta$ is such that $H_\star \eta = \text{vol}$. Hence,
\[ \lambda_1 + \lambda_2 + \int \phi d\eta \leq \chi_u(f) + \lambda_1^c(\mu) + \int \phi d\mu \]
and then,
\[ \lambda_2 \leq \lambda_1^c(\mu) + \int \phi d\mu - \int \phi d\eta \leq \lambda_1^c(\mu) + \sup \phi - \inf \phi. \]

Therefore the theorem is proved.

By the proof of the above theorem, we have the next corollary.

**Corollary 6.3.** If $\mu$ is the unique equilibrium state for $f$ w.r.t. a potential $\phi = \psi \circ H$ with $\sup \phi - \inf \phi < \lambda_2$. Then, the center Lyapunov exponent of $\mu$ is positive.

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**Appendix**

In this appendix we show a result whose proof may not be original, but we did not find it written anywhere.

Let $A : T^3 \to T^3$ be a linear Anosov diffeomorphism with $T^3 = E^u \oplus E^c \oplus E^s$ and $f : T^3 \to T^3$ a partially hyperbolic diffeomorphism which is dynamically coherent and semi-conjugacy to $A$ by a continuous function $h$, i.e. $A \circ h = h \circ f$. We show that $h$ is Hölder continuous.

So, there are three foliations $F^s$, $F^c$ and $F^u$ which are invariant by $f$ and we treat $A$ as a partially hyperbolic diffeomorphism and consider three corresponding foliations $W^c$, $W^s$ and $W^u$. Clearly both $W^c(x)$ and $W^u(x)$ are inside unstable manifold of $x$ which in partially hyperbolic notations is $W^u(x)$.

The idea is similar to the classical argument in the hyperbolic case and may be found in Katok–Hasselblatt book. In fact it is sufficient to prove that the restriction of $h$ to the three invariant foliations of $f$ are Hölder continuous. (See proposition 19.1.1 and theorem 19.1.2 of [24].)
Proposition A.1. \( h \) restricted to invariant foliations \( F^* \) is Hölder continuous, \(* \in \{ s, c, u \} \).

Proof. We recall that 
\[
    h(F^c(x)) = W^c(h(x)), \quad h(F^s(x)) = W^s(h(x)).
\]
Moreover,
\[
    h(F^u(x)) \in W^{cu}(h(x)).
\]
Let us show that \( h \) restricted to \( F^u \) is Hölder. Take \( c < 1 < C \) such that \( C \) is a Lipschitz constant for \( f \) and \( c \) is a Lipschitz constant for \( A^{-1} \) restricted to \( W^u \) and let \( \alpha > 0 \) be such that \( c C^\alpha < 1 \). Fix \( \epsilon_0 > 0 \). By compactness and continuity of \( h \) there exists \( \delta_0 > 0 \) such that 
\[
    d(x, y) < \delta_0 \implies d(h(x), h(y)) < \epsilon_0.
\]
Now if \( y \in F^u(x) \) and \( \delta := d(x, y) \) is sufficiently small, there exists \( n \in \mathbb{N} \) such that 
\[
    d(f^n(x), f^n(y)) \leq C^n \delta < \delta_0 \leq C^{n+1} \delta.
\]
Hence, 
\[
    d(h(f^n(x)), h(f^n(y))) \leq \epsilon_0. \quad \text{Now using } c C^\alpha < 1 \text{ we have:}
\]
\[
    d(h(x), h(y)) = d(A^{-n}hf^n(x), A^{-n}hf^n(y)) < c^n \epsilon_0 = c^n \delta_0^\alpha \frac{\epsilon_0}{\delta_0^\alpha} \leq C^n \frac{\epsilon_0}{\delta_0^\alpha} d(x, y) = C^n \frac{\epsilon_0}{\delta_0^\alpha} d(x, y)^\alpha.
\]
Observe that the distance \( d(x, y) \) above is on the manifold. However, for small distances it may be changed by the distance along unstable leaf. \( \square \)

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