Self-interference of a single Bose–Einstein condensate due to boundary effects

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Abstract
A simple model wavefunction, consisting of a linear combination of two free-particle Gaussians, describes many of the observed features seen in the interactions of two isolated Bose–Einstein condensates (BECs) as they expand, overlap and interfere. It is shown that a simple extension of this idea can be used to predict the qualitative time-development of a single expanding BEC produced near an infinite wall boundary, giving similar interference phenomena. Other possible time-dependent behaviours of single BECs in restricted geometries, such as wave packet revivals are also briefly discussed.

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1. Introduction

It can be argued that much of the early success of quantum theory can be traced to the fact that many exactly soluble quantum models are surprisingly coincident with naturally occurring physical systems, such as the hydrogen atom and the rotational/vibrational states of molecules. Many other exemplary quantum mechanical models, which have historically been considered as only textbook idealizations, have also recently found experimental realizations. Advances in areas such as materials science or laser trapping and cooling of atoms have allowed the production of approximations to a number of systems which had typically been relegated to lists of pedagogical examples. While some such examples have been found use as devices, many others have been applied to the study of fundamental quantum behaviour.

For example, one-dimensional (1D) quantum wells (infinite and finite) are a staple of textbooks and have found use in modelling quantum dots and other structures [1], including the use of asymmetric wells [2]. 2D quantum mechanical 'standing waves' have been observed in a wide variety of geometries [3], while evidence for bound quantum states of the neutron in the Earth’s gravitational field [4] (a problem which is often described as the ‘quantum bouncer’ in the pedagogical literature) has recently been presented. The ‘generation of nonclassical motional states’, such as coherent and squeezed states [5] of harmonically trapped ions, has also been demonstrated and studied in detail as an example of time-dependent designer wavefunctions in the most familiar of all model potentials.

The experimental realization of Bose–Einstein condensates (BECs) [6] has allowed for an even wider variety of fundamental tests, including the ‘Observation of interference between two Bose condensates’ [7]. In the original experiment [7], two samples of sodium atoms were evaporatively cooled ‘well below the transition temperature to obtain condensates’ such that they were initially well-separated in a double-well potential. The two condensates were then allowed to freely expand and interference effects were observed in the resulting overlap region, while no similar effects were seen for a single expanding condensate. Similar effects have been seen in other experimental realizations [8] and more recently have been observed with up to 30 uncorrelated condensates [9] produced in an optical trap.

As it is briefly reviewed in section 2, a simple model wavefunction consisting of a linear combination of two [10] (or more [9]) Gaussian terms, $\psi_G(x, t; x_0)$, (one for each condensate) captures many of the salient features observed experimentally. A (single particle) wavefunction of the form

$$\psi(x, t) = N[\psi_G(x, t; x_A) + e^{i\phi}\psi_G(x, t; x_B)]$$

(1)

can be used, with $x_0 = x_A, x_B$ describing the locations of the two isolated condensates and with fixed relative phase $\phi$ (and a normalization constant $N$) and the predictions of this simple model will also be discussed in the next section.

Such linear combination solutions (especially of Gaussians) have been frequently used in the pedagogical literature to describe the time development of wave packet solutions of the 1D Schrödinger equation (SE) describing an otherwise
free-particle impinging on an infinite wall or barrier (or ‘free
c Particle on the half-line’). ‘Mirror’ or ‘image’ solutions of the
form
$$\tilde{\psi}(x, t) = \begin{bmatrix} \tilde{N} \psi(x, t) - \psi(-x, t) \\ 0 \end{bmatrix} \frac{e^{-idx/2\beta_t^2} + e^{i(x+d)/2\beta_t^2} \sin t}{\beta_t} \cos \left( \phi + \frac{tdx}{\beta_t^2} \right),$$
(2)

(for a particle restricted to $x \leq 0$) satisfy the 1D free-
particle SE (if $\psi(x, t)$ does) and also automatically satisfy the
appropriate boundary condition at the infinite wall (assumed to be
at $x = 0$) for all times. Such solutions have been used in
a variety of pedagogical applications [12–14], but also in a
research context to discuss the deflection of ultracold quantum
particles (wave packets) from impenetrable boundaries or
mirrors [15]. Such analyses naturally explain the spatially
oscillatory behaviour of the position-space probability density
as the wave packet ‘hits’ the wall (as observed in numerical
calculations) as the interference between two overlapping
terms, much like the observed BEC effect.

In this paper, it is noted that a single BEC, produced
near an infinite wall boundary and allowed to expand freely,
will likely exhibit interference effects describable by such
‘mirror’ or ‘image’ solutions as in equation (2), and this
will be discussed in section 3. We also extend such ideas to
a localized BEC produced between two infinite boundaries
(as in an infinite well potential) to very briefly discuss other
possible effects, such as wave packet revivals.

2. Simple model of interfering BECs

A description of the initial (single particle) 1D wavefunction
for two separated BECs using isolated Gaussian forms
has been made by the authors of [10], who then
analyse the resulting time development of the BECs
using a Wigner distribution approach. (A generalization to
multiple BECs is given in [9].) As a review of such an
approach and its successes in qualitatively modelling more
sophisticated analyses (and the experimental data), consider
two BECs, separated by a distance $d$ and initially centred
at $x_0 = \pm d/2$, described by the time-dependent free-particle
wavefunction

$$\psi_{2\text{BEC}}(x, t) = \frac{N}{\sqrt{\pi} \beta \sqrt{(1 + it/\hbar)}},$$
$$\times \left[ e^{-(x-d)^2/2\beta^2(1+it/\hbar)} + e^{i\phi} e^{-(x+d)^2/2\beta^2(1+it/\hbar)} \right].$$

(3)

where the normalization factor is given by

$$N = \sqrt{\frac{1}{\sqrt{\pi}} \cos(\phi)} e^{-d^2/4\beta^2} \right)^{-1/2}.$$

(4)

The time-dependent spatial width for a single such Gaussian
term is given by

$$\Delta x_t = \frac{\beta_t}{\sqrt{2}} \equiv \frac{\beta_t}{\sqrt{2}} \sqrt{(1 + (t/\hbar)^2)} = \frac{d\beta_t^2}{\hbar},$$

(5)

and the corresponding spread in momentum-space (again,
for a single Gaussian) is $\Delta \beta_t = \Delta \beta_t = \hbar / (\beta_t \sqrt{2})$. The time-
dependent probability density for the two condensate states is

then given by

$$P_{2\text{BEC}}(x, t) = \frac{N^2}{\sqrt{\pi} \beta_t},$$
$$\times \left[ e^{-(x-d)^2/2\beta_t^2} + e^{i\phi} e^{-(x+d)^2/2\beta_t^2} + 2 e^{i(\phi_0 + \phi)} \cos \left( \phi + \frac{tdx}{\beta_t^2} \right) \right].$$

(6)

where the cross-term describes the interference effect.

For each Gaussian contribution, there are momentum
components of order $p \sim \Delta p / \beta \sqrt{2}$ so that the time
($T_0$) it takes such components to drift from one condensate
and overlap with the other is of order $T_0 \sim d / (p/\hbar \sim \sqrt{2} \hbar \beta \sqrt{2})$. For condensates which are initially
highly localized and well-separated, the time-dependent position-
spread is then dominated by the $(t/\hbar)^2$ term since $(T_0/T_0)^2 \sim 2(\beta / \hbar)^2 \gg 1$. In that limit, the oscillatory term is then
approximately given by

$$\cos \left( \phi + \frac{p \beta}{\hbar} \right) \rightarrow \cos \left( \phi + \frac{p \beta}{\hbar} \right),$$

(7)

so that the local wavelength variations seen in the interference
pattern are time-dependent and scale like

$$\frac{2\pi}{\lambda_x} = \frac{k_x}{\beta^2} \text{ or } \lambda = \frac{\hbar d}{m \lambda},$$

(8)

just as in equation (1) of [7]. The time-dependent real
and imaginary parts of the individual components of this
simple wavefunction are nicely consistent with more detailed
calculations [8] where the BEC is ‘characterized by a
phase that varies quadratically … across the condensate’;
for example, compare the pedagogical illustration of an
expanding $p = 0$ Gaussian wave packet in figure 2 of [16]
with figure 1 of [8]. Finally, the corresponding momentum-
space probability density is given by

$$|\psi_{2\text{BEC}}(p, t)|^2 = \frac{4N^2 \alpha}{\sqrt{\pi}} \cos^2 \left( \frac{pd}{2\hbar} \right) e^{-\alpha^2 p^2},$$

(9)

where $\alpha = \beta / \hbar$ so that there is indeed structure in momentum
space at integral multiples of $p = \hbar / d$, as discussed in more
detail in [17]. Thus, in many important ways, the simple
wavefunction in equation (3) encodes much of the physics
observed in the interference of two expanding BECs.

3. Single BEC near a infinite wall boundary
and related effects

Motivated then by earlier pedagogical papers on ‘mirror’ or
‘image’ solutions [12–14], it can be imagined that a single
BEC is produced close to an infinite barrier. If the barrier
is located at $x = 0$ and the single condensate is produced
at $x = -d/2$, the resulting ‘mirror’ solution in equation (2),
for $x \leq 0$, can be described by the form in equation (3) with
$\phi = \pi$ (so that $\cos(\phi) = -1$) and with normalizations simply
related by $N = \sqrt{2}N$. The resulting time-dependent solution
will then exhibit the same type of interference patterns.
observed for two isolated condensates. One possibility for such an infinite barrier might be an atomic mirror [18] of the type successfully used in a number of atomic physics applications [19].

The addition of a second infinite wall barrier (say at \( x = -d \)) to such a case might then be modelled by the standard infinite well problem of textbooks. The time development of wave packet propagation in this system can then be described in terms of an infinite number of image solutions [20] and discussions related to this approach go back to at least Einstein and Born [21]. For a single BEC in this restricted geometry, modelled as a \( p = 0 \) Gaussian, in addition to the spreading/coherence time (\( \tau_0 \)) and the time to overlap the other real or image condensate (\( T_0 \)), the only other relevant timescale is the quantum revival time, \( T_{\text{rev}} \); for a recent review of quantum wave packet revivals, see [22]. For the quantized energy eigenvalues in an infinite well of width \( d \) (as imagined here), the revival time for an arbitrary localized wave packet is \( T_{\text{rev}} = 4md^2/\hbar \pi^2 \). (This pedagogical paper contains one of the first references to what have become known as quantum revivals, examined in the context of the infinite well.) The ratio of revival time to overlap time is then \( T_{\text{rev}}/T_0 \sim 2d/\pi \Delta x_0 \gg 1 \). In the original two BEC experiments [7], the two condensates are allowed to fall freely as they expand, so a more detailed analysis of a specific experiment realization would be required to determine if the revival time is too long to be observed. One should note, however, that for the special case of a \( p = 0 \) Gaussian waveform produced precisely in the centre of such an infinite well, because only even energy eigenstates are excited, the effective revival time is actually \( T_{\text{rev}}/8 \) [22] for this very special geometry, which gives almost an order-of-magnitude shorter fall time to achieve a revival.

Other arrangements of two infinite plane barriers can also be imagined to give rise to interesting BEC interference effects, which can be modelled using ‘mirror’ or ‘image’ methods. For example, two such infinite walls could be placed at right angles, to form a ‘corner (90°)’ reflector’, defined by the potential

\[
V(x,y) = \begin{cases} 0 & \text{for } x > 0 \text{ and } y > 0, \\ +\infty & \text{otherwise.} \end{cases}
\]

A solution of the form

\[
\psi_{\text{corner}}(x, y; t) = N \psi(x, y; t) - \psi(-x, y; t) - \psi(x, -y; t) + \psi(-x, -y; t),
\]

making use of three auxiliary ‘image’ components, solves the SE in the allowed region (if \( \psi(x, y; t) \) is a free-particle solution) as well as satisfying the boundary conditions at the two walls. The normalization factor can be obtained explicitly in the case of an Gaussian solution, as above. The expansion of a single BEC near such a ‘corner’ would then result in a 2D interference pattern. The same construction can also be employed for other angles, \( \Theta \), between the two walls using familiar examples from optics, such as for the cases of \( \Theta = 45° \) and 60°.

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