On Casimir Pistons

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Abstract

In this paper we study the Casimir force for a piston configuration in $R^3$ with one dimension being slightly curved and the other two infinite. We work for two different cases with this setup. In the first, the piston is "free to move" along a transverse dimension to the curved one and in the other case the piston "moves" along the curved one. We find that the Casimir force has opposite signs in the two cases. We also use a semi-analytic method to study the Casimir energy and force. In addition we discuss some topics for the aforementioned piston configuration in $R^3$ and for possible modifications from extra dimensional manifolds.

Introduction

More than 60 years have passed since H. Casimir's [1] originating paper, stating that there exist's an attractive force between two neutral parallel conducting plates. After 20 years, the scientific society had appreciated this work and extended this work in various areas, from solid state physics to quantum field theory and even cosmology [10, 8]. The Casimir effect is closely related with the existence of zero point quantum oscillations of the electromagnetic field in the case of parallel conducting plates. The boundaries polarize the vacuum and that results to a force acting on the boundary [8]. The Casimir force can be either repulsive or attractive. That depends on the nature of the background field in the vacuum, the geometry of the boundary, the dimension and the curvature of the spacetime. The regularization of the Casimir energy is of particular important in order physical results become clearer.

One very interesting configuration is the so called Casimir piston. This configuration was originally treated [6] as a single rectangular box with three parallel plates. The one in the

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middle is the piston. The dimensions of the piston are \((L - a) \times b\) and \(a \times b\), with the piston being located in \(a\). In [6] the Casimir energy and Casimir force for a scalar field was calculated. The boundary conditions on the 'plates' where Dirichlet. There exists a large literature on the subject [5, 11, 4, 3, 2, 7] calculating the Casimir force for various piston configurations and for various boundary conditions of the scalar field. Also most results were checked at finite temperature. The configurations used were extended to include extra dimensional spaces which form a product spacetime with the piston topology, that is \(M_{\text{Piston}} \times M^n\), with \(M_{\text{Piston}}\) and \(M^n\) the piston spacetime topology and the extra dimensional spacetime topology. In addition to the known Neumann and Dirichlet boundary conditions, in reference [4] Robin boundary conditions were considered for a Kaluza-Klein piston configuration. In most cases the scalar field was taken massless but there exists also literature for the massive case [12].

Motivated by a recent paper [2], we shall consider a piston configuration in \(R^3\) space without the extra dimensions. In this topology we consider a piston geometry living in three space. We shall study two configurations which can be seen in Figures 2 and 3. In the first case Fig. 2 the \(x_1\) and \(x_3\) dimensions are sent to infinity and we have "parallel plates" having distance \(L\) between them. The piston consists of two such three dimensional...
chambers with the piston plate having distance $\alpha$ and $L - \alpha$ from the boundary plates. In addition to this we shall assume that in one of the two infinite dimensions, there is curvature. As can be seen the infinite dimension $x_1$ seems to be a part of a circle. We shall work in the case where the radius of the circle goes to infinity, that is when $R \to \infty$. We shall give the solution for the Laplace equation for a scalar field (with Dirichlet boundary conditions at all boundaries, see below) corresponding to this configuration and find the eigenfrequencies. Next we calculate the Casimir energy and the Casimir force for each chamber and for the whole configuration using standard techniques [8, 10]. Additionally we study a similar to the above situation, described by Figure 3. Particularly in this case the piston is free to move along the curved dimension and $x_2$ and $x_3$ are infinite dimensions. These cases shall be presented in section 1. In section 2 we discuss the results we found in section 1. In section 3 we present a semi-analytic calculation of the Casimir energy and Casimir force. Finally the conclusions with a discussion follow.

1 The Piston Setup

The configuration of Figure 2 can be described as follows: The curvature of the infinite length curved dimension is $\frac{1}{R}$ and the width between plates is $\alpha$ and $L - \alpha$ for the two chambers. We shall treat only the $\alpha$ chamber first. The generalization to the other case is straightforward. In order to describe this slightly curved piston setup more efficiently, we choose the coordinates $s, t$ to describe the plane $x_1 - x_2$ and $x_3$ remains the same, as it appears in Figure 2. The local element in the plane $s - t$ is $dA = h(s, t)dsdt$, with $s$ the length along the infinite slightly curved dimension and $t$ the transverse dimension, with $h(s, t) = 1 - \frac{1}{R}t$. The Laplacian in terms of $s, t, x_3$ looks like (acting on $\psi$, with $\psi$ a scalar field),

$$\nabla^2 \psi = \frac{1}{h} \frac{\partial}{\partial t}(h \frac{\partial \psi}{\partial t}) + \frac{1}{h} \frac{\partial}{\partial s}(h \frac{\partial \psi}{\partial s}) + \frac{\partial^2 \psi}{\partial x^2_3}. \quad (1)$$

If in the above relation we expand $h(s, t)$ for $R \to \infty$, then the non-vanishing terms in first order approximation yield the usual Laplacian in Cartesian coordinates. An ansatz
solution of $\nabla^2 \psi = 0$ is:

$$
\psi(s, t, x_3) = \frac{u(s)}{\sqrt{h(s, t)}} \sin \left( \frac{n\pi t}{\alpha} \right) e^{ik_3 x_3},
$$

(2)

subject to the Dirichlet boundary conditions, $\psi(s, t, x_3) = \psi(s + \alpha, x_3) = 0$ for the transverse coordinate $t$. Relation (2) for infinite $R$ yields (keeping first order non vanishing terms):

$$
\psi(s, t, x_3) = u(s) \sin \left( \frac{n\pi t}{\alpha} \right) e^{ik_3 x_3},
$$

(3)

with $u(s)$ satisfying:

$$
d^2 u \frac{ds}{ds^2} + \left( k^2 - \frac{\pi^2}{\alpha^2} + \frac{1}{R^2} \right) u(s) = 0.
$$

(4)

Then the eigenfrequencies of the above configuration is,

$$
\omega^2 = k_1^2 + k_2^2 + \left( \frac{n\pi}{\alpha} \right)^2 - \frac{\pi^2}{\alpha^2} + \frac{1}{R^2}.
$$

(5)

Notice that in the limit $R \to \infty$, the term $-\frac{\pi^2}{\alpha^2} + \frac{1}{R^2}$ is negative. In the next section we shall comment on this.

Now the Casimir energy per unit area for the chamber with transverse length $\alpha$ is [8] (bearing in mind that $-\frac{\pi^2}{\alpha^2} + \frac{1}{R^2}$ is much less than 1):

$$
E_{\text{ren}}(\alpha) = \frac{3}{2\pi} \left( \sum_{n=1}^{\infty} \sqrt{k_1^2 + k_2^2 + \left( \frac{n\pi}{\alpha} \right)^2 - \frac{\pi^2}{\alpha^2} + \frac{1}{R^2}} \right.
$$

$$
- \frac{a}{\pi} \sqrt{\frac{-\pi^2}{\alpha^2} + \frac{1}{R^2}} \left( \sum_{n=1}^{\infty} \sqrt{k_1^2 + k_2^2 + \left( \frac{n\pi}{\alpha} \right)^2 - \frac{\pi^2}{\alpha^2} + \frac{1}{R^2}} \right).
$$

(6)

where the left term is the finite $\alpha$ part and the right term the infinite $\alpha$ part. The substraction of the continuum part from the finite part corrections is a very well known renormalization technique for the Casimir energy [8]. Now, performing the $k_1, k_2$ integration, one obtains:

$$
E_{\text{ren}}(\alpha) =
$$

$$
\frac{3}{2\pi} \left( \sum_{n=1}^{\infty} \left[ \left( \frac{n\pi}{\alpha} \right)^2 - \frac{\pi^2}{\alpha^2} + \frac{1}{R^2} \right]^{3/2} - \frac{a}{\pi} \sqrt{-\frac{\pi^2}{\alpha^2} + \frac{1}{R^2}} \left[ \left( \frac{n\pi}{\alpha} \right)^2 - \frac{\pi^2}{\alpha^2} + \frac{1}{R^2} \right]^{3/2} \right).
$$

(7)

The $k_1, k_2$ integration is done using,

$$
\int dk_1 dk_2 \sum_{n=1}^{\infty} \left[ k_1^2 + k_2^2 + \left( \frac{n\pi}{\alpha} \right)^2 - m^2 \right]^{-s},
$$

(8)
which after integrating over $k_1$ and $k_2$ we obtain,

$$\Gamma(s-1)\pi\sum_{n=1}^{\infty} \left[ \left( \frac{n\pi}{a} \right)^2 - m^2 \right]^{1-s}. \quad (9)$$

Analytically continuing the above to $s = -\frac{1}{2}$ we obtain relation (7). Upon using the Abel-Plana formula [8, 9],

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(n)dn = \frac{1}{2} f(0) + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} \left[ f(it) - f(-it) \right], \quad (10)$$

the Casimir energy (7) becomes,

$$E_{\text{ren}}(\alpha) = \frac{3}{2\alpha} \left( \frac{1}{2} (1 - b^2)^{3/2} \right)$$

$$- \int_0^{\infty} dt \frac{1}{e^{2\pi t} - 1} \left[ 2t \sqrt{\frac{\sqrt{(1 - b^2 - t^2)^2 + 4t^2} + 1 - b^2 - t^2}{2}} \right]$$

$$+ (1 - b^2 - t^2) \sqrt{\frac{\sqrt{(1 - b^2 - t^2)^2 + 4t^2} - 1 + b^2 + t^2}{2}}$$

$$+ \frac{1}{8} \left( (5b^2 - 2) \sqrt{-b^2 + 1 + 3b^4(\ln b - \ln(1 + \sqrt{1 - b^2}))} \right),$$

with

$$b = \frac{\alpha \sqrt{-1 + \frac{\pi^2}{a^2}}}{\pi}. \quad (12)$$

Following the same steps as above, with $a \to L - \alpha$, we obtain the renormalized Casimir energy for the chamber $L - \alpha$:

$$E_{\text{ren}}(L - \alpha) = \frac{3}{2(L - \alpha)} \left( \frac{1}{2} (1 - \lambda^2)^{3/2} \right)$$

$$- \int_0^{\infty} dt \frac{1}{e^{2\pi t} - 1} \left[ 2t \sqrt{\frac{\sqrt{(1 - \lambda^2 - t^2)^2 + 4t^2} + 1 - \lambda^2 - t^2}{2}} \right]$$

$$+ (1 - \lambda^2 - t^2) \sqrt{\frac{\sqrt{(1 - \lambda^2 - t^2)^2 + 4t^2} - 1 + \lambda^2 + t^2}{2}}$$

$$+ \frac{1}{8} \left( (5\lambda^2 - 2) \sqrt{-\lambda^2 + 1 + 3\lambda^4(\ln \lambda - \ln(1 + \sqrt{1 - \lambda^2}))} \right),$$

and for this case

$$\lambda = \frac{(L - \alpha) \sqrt{-1 + \frac{\pi^2}{(L - \alpha)^2}}}{\pi}. \quad (14)$$

Now the total Casimir energy for the piston is obtained by adding relations (11) and (13), namely,

$$E_{\text{Piston}} = E_{\text{ren}}(L - \alpha) + E_{\text{ren}}(\alpha). \quad (15)$$
In Figures 4, 5 and we plot the Casimir energy for the chamber $\alpha$, $L - \alpha$ respectively, for the numerical values $R = 10^{100}$, $L = 10^{11}$. Also in Figures 6, 7 and 8 we plot the Casimir force for the chambers $\alpha$, $L - \alpha$ and the total Casimir force, respectively.

Figure 4: The Casimir Energy for the chamber $\alpha$, for $R = 10^{100}$, $L = 10^{11}$ (Piston 1 Case)

Figure 5: The Casimir Energy for the chamber $L - \alpha$, for $R = 10^{100}$, $L = 10^{11}$ (Piston 1 Case)

1.1 Another Piston Configuration

Let us now study the case for the piston configuration appearing in Figure 3. Following the previous steps, the eigenvalue spectrum of the Laplacian is:

$$\omega^2 = k_1^2 + k_2^2 + \left(\frac{n\pi}{\alpha}\right)^2 - \frac{1}{R^2}. \quad (16)$$

Now the Casimir energy for the $\alpha$ piston is (after integrating on the infinite dimensions),

$$E_{ren}(\alpha) = \frac{3}{2\pi} \left( \sum_{n=1}^{\infty} \left[ \left(\frac{n\pi}{\alpha}\right)^2 - \frac{1}{R^2} \right]^{3/2} - \frac{a}{\pi} \int_{\frac{1}{R}}^{\infty} \left[ \left(\frac{n\pi}{\alpha}\right)^2 - \frac{1}{R^2} \right]^{3/2} \right). \quad (17)$$
Upon using again the Abel-Plana formula [8, 9], the Casimir energy (17) becomes,

$$E_{ren}(\alpha) = \frac{3}{2\alpha} \left( \frac{1}{2}(1 - b^2)^{3/2} \right)$$

$$- \int_0^\infty dt \frac{1}{e^{2\pi t} - 1} \left[ 2t \sqrt{\frac{(1 - b^2 - t^2)^2 + 4t^2 + 1 - b^2 - t^2}{2}} ight. \
+ (1 - b^2 - t^2) \left( \sqrt{(1 - b^2 - t^2)^2 + 4t^2 - 1 + b^2 + t^2} \right) \
+ \left. \frac{1}{8} \left( (5b^2 - 2) \sqrt{-b^2 + 1 + 3b^4 (\ln b - \ln(1 + \sqrt{1 - b^2}))} \right) \right],$$

with

$$b = \frac{\alpha}{R\pi}.$$
Following the same steps above, with $\alpha \rightarrow L - \alpha$, we obtain the renormalized Casimir energy for the chamber $L - \alpha$:

$$E_{\text{ren}}(L - \alpha) = \frac{3}{2(L - \alpha)} \left( \frac{1}{2}(1 - \lambda^2)^{1/2} \right. \\
- \int_0^\infty dt \frac{1}{e^{2\pi t} - 1} \left[ 2t \sqrt{\frac{(1 - \lambda^2 - t^2)^2 + 4t^2 - 1 - \lambda^2 - t^2}{2}} \\
+ (1 - \lambda^2 - t^2) \sqrt{(1 - \lambda^2 - t^2)^2 + 4t^2 - 1 + \lambda^2 + t^2} \\
\left. + \frac{1}{8}(5\lambda^2 - 2) \sqrt{-\lambda^2 + 1 + 3\lambda^4(\ln \lambda - \ln(1 + \sqrt{1 - \lambda^2}))} \right] \\
+ 18 \left( 5\lambda^2 - 2 \right) \sqrt{-\lambda^2 + 1 + 3\lambda^4(\ln \lambda - \ln(1 + \sqrt{1 - \lambda^2})))}, \right)$$

(20)

and for this case

$$\lambda = \frac{(L - \alpha)}{R\pi}. \quad (21)$$

Now the total Casimir energy for the piston is obtained by adding relations (18) and (20), namely,

$$E_{\text{Piston}} = E_{\text{ren}}(L - \alpha) + E_{\text{ren}}(\alpha). \quad (22)$$

In Figures 9 we plot the Casimir energy for the chamber $\alpha$ and for the numerical values $R = 10^{100}, L = 10^{11}$. Also in Figures 10, 11 and 13 we plot the Casimir force for the chamber $\alpha$ the chamber $L - \alpha$ and the total Casimir force.

2 Brief Discussion

Let us discuss the results of the previous section. We start with the piston 1 that appears in Figure 2. The Casimir force that stems out of this configuration is described by Figures 6, 7 and 8. As it can be seen, the total Casimir force is negative for small $\alpha$. Also for large $\alpha$, for values near $L$, the total Casimir energy is positive. In conclusion the Casimir force is attractive when the piston is near to the one end. In addition it is repulsive if the piston goes to the other end. This kind of behavior is a known result for pistons (see [2]).
Figure 9: The Casimir Energy for the chamber $\alpha$, for $R = 10^{100}$, $L = 10^{11}$ (Piston 2 Case)

Figure 10: The Casimir Force for the chamber $\alpha$, for $R = 10^{100}$, $L = 10^{11}$ (Piston 2 Case)

Remember this case corresponds to a piston that is "free to move" along one of the non curved dimensions.
In the case of Piston 2 of Figure 3 the results are different. Now the piston is "free to move" along the slightly curved dimension of total length $L$. This case is best described by Figures 10, 11 and 13. As we can see, the total Casimir force is positive for small $\alpha$ values and negative for large $\alpha$ values (but still smaller than $L$). This means that the Casimir force is repulsive to the one end and attractive to the other one. Note that this behavior is similar to the Piston 1 case with the difference that the force is attractive (repulsive) to different places. However the qualitative behavior that is described by repulsion to one end and attraction to the other, still holds. In the next section we shall verify this using a semi-analytic approximation.

3 A Semi-analytic Approach for the Piston Casimir Energy and Casimir Force

In this section we shall consider as in previous sections a three dimensional piston $R^3$ and for simplicity the piston of Figure 8. Our study will be focused on the semi-analytic
calculation of the Casimir energy and Casimir force. The piston configuration consists of two chambers with lengths $\alpha$ and $L - \alpha$, and with Dirichlet boundary conditions on the boundaries and on the moving piston. The energy eigenfrequency for this setup is:

$$\omega^2 = k_1^2 + k_2^2 + \left(\frac{n\pi}{\alpha}\right)^2 - y^2,$$

for the $\alpha$ chamber, and

$$\omega^2 = k_1^2 + k_2^2 + \left(\frac{n\pi}{L - \alpha}\right)^2 - y^2,$$

for the $L - \alpha$ chamber. In the above two relations the parameter $y$ stands for a positive number with physical significance analogous to the ones we described in the previous sections. The Casimir energy with no regularization for this system is given by:

$$E_{Piston} = E_P(L - \alpha) + E_P(\alpha),$$

where $E_P(\alpha)$ is (upon integrating the infinite dimensions):

$$E_P(\alpha) = \frac{3}{2\pi} \sum_{n=1}^{\infty} \left[ \left(\frac{n\pi}{\alpha}\right)^2 - y^2 \right]^{3/2}.$$
As we said, the above sum contains a singularity and needs regularization. In order to see how the singularity "behaves" we use the binomial expansion (or a Taylor expansion for small $y$, which is exactly the same as can be checked):

$$(a^2 - b^2)^s = \sum_{l=0}^{s} \frac{s!}{(s-l)!l!} (a^2)^l \left(-b^2\right)^{s-l},$$ \hspace{1cm} (27)$$

and rearranging the sum as:

$$E_P(\alpha) = \frac{3\pi^2}{2\alpha^3} \sum_{n=1}^{\infty} (n^2 - m^2)^{3/2},$$ \hspace{1cm} (28)$$

and $m = \frac{\alpha\pi}{2}$ we obtain:

$$E_P(\alpha) = \sum_{n=1}^{\infty} \frac{3}{2\pi} \frac{3}{\alpha^3} \left(\frac{n^3}{\alpha^3} - \frac{3y^2n}{2\alpha\pi^2} + \frac{3\alpha y^4}{8n\pi^4} + \frac{\alpha^3 y^6}{16n^3\pi^6} + \frac{3\alpha^5 y^6}{128n^5\pi^8} + \frac{3\alpha^7 y^{10}}{256n^7\pi^{10}} + \frac{7\alpha^9 y^{12}}{1024n^9\pi^{12}}...\right).$$ \hspace{1cm} (29)$$

Using the zeta regularization method [10] the above relation [29] becomes,

$$E_P(\alpha) = \frac{3}{2\pi} \left(\frac{\zeta(-3)}{\alpha^3} - \frac{3y^2\zeta(-1)}{2\alpha\pi^2} + \frac{3\alpha y^4\zeta(1)}{8\pi^4} + \frac{\alpha^3 y^6\zeta(3)}{16\pi^6} + \frac{3\alpha^5 y^6\zeta(5)}{128\pi^8} + \frac{3\alpha^7 y^{10}\zeta(7)}{256\pi^{10}} + \frac{7\alpha^9 y^{12}\zeta(9)}{1024\pi^{12}}...\right).$$ \hspace{1cm} (30)$$

Notice that the singularity is contained in $\zeta(1)$ and is a pole of first order. Now the other chamber of the piston contributes to the Casimir energy as:

$$E_P(L - \alpha) = \frac{3}{2\pi} \left(\frac{\zeta(-3)}{(L - \alpha)^3} - \frac{3y^2\zeta(-1)}{2(L - \alpha)^2\pi^2} + \frac{3(L - \alpha) y^4\zeta(1)}{8\pi^4} + \frac{(L - \alpha)^3 y^6\zeta(3)}{16\pi^6} + \frac{3(L - \alpha)^5 y^6\zeta(5)}{128\pi^8} + \frac{3(L - \alpha)^7 y^{10}\zeta(7)}{256\pi^{10}} + \frac{7(L - \alpha)^9 y^{12}\zeta(9)}{1024\pi^{12}}...\right).$$ \hspace{1cm} (31)$$

Figure 13: The total Piston Casimir Force, for $R = 10^{100}$, $L = 10^{11}$ and for a wide range of $\alpha$ (Piston 2 Case)
Before proceeding we must mention that we could reach the same results as above by Taylor expanding the initial sum of relation (26) for $y \to 0$.

Let us now discuss the above. It can be easily seen that the total Casimir force, 
\[
F_c = - \frac{\partial E_P(L - \alpha)}{\partial \alpha} - \frac{\partial E_P(\alpha)}{\partial \alpha},
\]  
(32)
is free or singularity. The reason is obvious and it is because the pole containing term is linear to the length of the piston chamber. Due to this linearity, the derivative of the energy cancels this dependence and the two singularities cancel each other. Thus we see how in our case also, a slightly curved piston configuration results to a singularity free Casimir force.

Finally, let us discuss something different. One quantity that is also free of singularity is, 
\[
N(\alpha) = \frac{E_P(\alpha)}{\alpha} - \frac{E_P(L - \alpha)}{L - \alpha}.
\]  
(33)
It is not accidental that this quantity has dimensions Energy per length which is actually force. The finiteness of the piston Casimir force and the force related quantity $N(\alpha)$ show once more, as is well known [6, 2], that the piston configuration has many attractive field theoretic features, even in the presence of very small curvature in one of the dimensions. The same arguments hold regarding the quantity (33), also when we consider a massive scalar field in an usual $R^3$ piston configuration (with no curved dimensions).

### 3.1 Semi-analytic Analysis for the Piston Casimir Force

According to the above, the Casimir energy can be approximated by relations (30) and (31). Thus when the radius of the slightly curved dimension is very large, the Casimir force can be approximated by,
\[
F_c(\alpha) = -\frac{9\pi \zeta(-3)}{\alpha^4} + \frac{9y^2\zeta(-1)}{2\alpha^2\pi} + \frac{9\alpha^2y^6\zeta(3)}{16\pi^5},
\]  
(34)
for the chamber $\alpha$, and
\[
F_c(L - \alpha) = -\frac{9\pi \zeta(-3)}{(L - \alpha)^4} + \frac{9y^2\zeta(-1)}{2\alpha^2\pi} + \frac{9(L - \alpha)^2y^6\zeta(3)}{16\pi^5},
\]  
(35)
for the $L - \alpha$ chamber. When $R$ is much larger than $\alpha$, (with $y = 1/R$) then the approximation we used above is adequate to describe the Casimir force. In Figures 14, 15 and 16 we plot the Casimir force for the chamber $\alpha$, $L - \alpha$ and the total force, for $R = 10^{20}$ and $L = 10^{11}$ As it can be seen from the Figures, the behavior of the Casimir force is the same as the one we found in section 1. Thus for the Piston 2 configuration, the Casimir force is positive for the $\alpha$ chamber and negative for the $L - \alpha$ chamber (when $\alpha$ approaches the size of $L$). Also the total Casimir force is positive for small $\alpha$ and negative for large $\alpha$.

A similar analysis can be carried for the Piston 1 configuration and similar results hold. Before closing this section we plot the quantity $N(\alpha)$ of relation (33). In Figure 17 we can see that the behavior of $N(\alpha)$ resembles that of the total Casimir force we studied previously, for the case of Piston 2, and for the $\alpha$ chamber.
Conclusions

In this paper we studied the Casimir force for two piston configurations. We used Dirichlet boundary conditions and the pistons had a slightly curved dimension. We found the eigenfunctions (in first order approximation with respect to \( R \)) and the \( R \)-dependent eigenvalues, for two piston configurations. These configurations appear in Figures 2 and 3. For the case of Piston 1, we found that the Casimir force is attractive when the piston is near to the one end and particularly in the end which is near to \( \alpha \to 0 \). In addition it is repulsive if the piston goes to the other end. Remember this case corresponds to a piston that is "free to move" along one of the non-curved dimensions.

In the case of Piston 2 (Figure 3) the results are different. Now the piston is free to move along the slightly curved dimension of total length \( L \). The total Casimir force is positive for small \( \alpha \) values and negative for large \( \alpha \) values (but still smaller than \( L \)). This means that the Casimir force is repulsive to the one end and attractive to the other one. Note that this behavior is similar to the Piston 1 case with the difference that the force is attractive (repulsive) to different places.

This said behavior, that is, the Casimir force on the piston being attractive in the one end and repulsive to the other, is a feature of Casimir pistons, see [2]. We verified this
behavior following a semi-analytic method. Also we found that the quantity,

$$N(\alpha) = \frac{E_P(\alpha)}{\alpha} - \frac{E_P(L - \alpha)}{L - \alpha}$$

which has dimensions of force, has the same behavior as the piston force for the same chamber $\alpha$. Notice that $E_P(\alpha)$ appears first in $N(\alpha)$ and $E_P(L - \alpha)$ is subtracted from it.

In conclusion we saw how a slightly curved dimension alters the Casimir force for a Casimir piston with Dirichlet boundary conditions. It would be interesting to add the contribution of an extra dimensional space. Particularly a three dimensional compact manifold, since in most cases the predictions of ADD models for large extra dimensions corrections of the Newton law, rule out manifolds [16] with dimensions less than 3 (of course with TeV compactification scale). Indeed in table 1 this is seen clearly.

| number of extra dimensions | $R$ (m)       |
|---------------------------|---------------|
| n=1                       | $\sim 10^{14}$|
| n=2                       | $\sim 10^{-3}$|
| n=3                       | $\sim 10^{-8}$|
In the setup we used in this paper, the incorporation of a Ricci flat manifold or a positive curved manifold could be done using standard techniques [2, 11, 3]. One interesting case would involve hyperbolic manifolds and especially with non Poisson spectrum of their eigenvalues. Also a single $R^3$ piston configuration with one curved dimension could not support Neumann boundary conditions for the scalar field. However such a configuration with an extra dimensional structure could hold if the extra dimensional space has non zero index or if it has an orbifold structure. We shall report on these issues soon.

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