Lipschitz Bandit Optimization with Improved Efficiency

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Abstract

We consider the Lipschitz bandit optimization problem with an emphasis on practical efficiency. Although there is rich literature on regret analysis of this type of problem, e.g., \cite{1,2,20}, their proposed algorithms suffer from serious practical problems including extreme time complexity and dependence on oracle implementations. With this motivation, we propose a novel algorithm with an Upper Confidence Bound (UCB) exploration, namely Tree UCB-Hoeffding, using adaptive partitions. Our partitioning scheme is easy to implement and does not require any oracle settings. With a tree-based search strategy, the total computational cost can be improved to $O(T \log T)$ for the first $T$ iterations. In addition, our algorithm achieves the regret lower bound up to a logarithmic factor.

1 Introduction

In the classical stochastic bandit problem a gambler tries to maximize his profit by iteratively playing one of a finite number of slot machines which are associated with unknown payoff distributions. As the gambler pulls the arms of the machines one by one in a sequential manner, he learns about the machines’ payoff-distributions and gradually adapts his strategy of playing towards maximizing future profits. Specifically, the gambler chooses the next arm by taking into consideration both the urgency of gaining rewards and acquiring new information. The quality of the gambler’s strategy can be evaluated by the asymptotic rate of growth of his expected regret, which is defined as the difference between the total cumulative payoff and the cumulative best payoff obtained by always pulling the arm with the highest expected reward.

Although the original motivation of bandit problems comes from clinical trials \cite{23}, modern technologies have created many opportunities for new applications, including experimental design \cite{9,21,17}, recommender systems \cite{15} and data-efficient system control \cite{8,16,18}. A variety of methods have been proposed for these applications. Among these, the e-greedy algorithm \cite{22}, upper confidence bound (UCB) based algorithms \cite{14,3} and Thompson sampling \cite{4} are particularly popular. Research on bandit problems is also applicable in other areas. For example, the tree search algorithm used by AlphaGo \cite{19}, called UCT \cite{13}, applies UCB exploration on trees to guide Monte-Carlo planning and has excellent efficiency in practice.

Although these papers consider bandit optimization with finitely many arms, there is also a rich literature allowing infinitely many arms. This is relevant when the arms are identified by
continuous-valued parameters, resulting in online optimization problems over a metric space (usually assumed to be Euclidean space). Examples include pricing a new product with uncertain demand in order to maximize revenue, controlling the transmission power of a wireless communication system in a noisy channel to maximize the number of bits transmitted per unit of power, and tuning the hyperparameters (learning rate, regularization coefficient) of neural networks to realize better performance on certain learning tasks. Some works along this line model the expected payoff as a linear function of the arms \[6\] \[10\] \[15\] \[1\] \[5\] \[2\]; some algorithms model the expected payoff as Gaussian processes over the arms \[21\] \[11\]; some algorithms assume that the expected payoff is a Lipschitz function of the arms \[20\] \[12\] \[7\].

In this paper, we focus on stochastic Lipschitz bandit optimization, which has stronger theoretical support \[20\] \[12\] \[7\]. The space is assumed to be equipped with a metric and the expected payoff is Lipschitz continuous with respect to the metric. We consider the setting of both non-contextual and contextual bandits. Our methods adaptively partition the space such that the partition gets finer in the high pay-off region and we adopt UCB to explore the arm space. We demonstrate the contribution of our work with theoretical analysis on both regret and time complexity.

2 Related Work

Our paper is inspired by the Zooming bandit \[12\], the contextual Zooming bandit \[20\], and \(\chi\)-armed bandit \[7\]. Both the Zooming bandit and \(\chi\)-armed bandit have strong theoretical support, but suffer from practical inefficiency. Zooming bandit approaches construct a covering of the arm space. However, their algorithm adopts an oracle implementation for the covering check, which can be impractical to implement. \(\chi\)-armed bandit does not require any oracle implementation, but it is non-contextual. In addition, both of these algorithms require at least \(O(T^2)\) computational cost for \(T\) iterations.

We make the following contributions:
(i) Our algorithms deal with both non-contextual and contextual bandit problems;
(ii) We reduce the time complexity to \(O(T \log T)\) for non-contextual bandit and \(O(T^{\frac{2d}{d+2}})\), where \(d\) is the dimension of the arm space, for contextual bandit for the first \(T\) iterations;
(iii) Our algorithms are easy to implement and do not involve any oracle call.

| Algorithm                  | computational cost | oracle     | total regret          |
|---------------------------|--------------------|------------|-----------------------|
|                           | non-contextual     | contextual |                       |
| \(\chi\)-armed \[7\]      | \(O(T^2)\)         | n/a        | without oracle        |
| Zooming \[12\]            | \(O(T^{\frac{2d}{d+2}} \log Tf(T))\) | n/a        | with oracle \(O(T^{\frac{2d}{d+2}} + 1)\) |
| Contextual Zooming \[20\] | n/a                | \(O(T^2f(T))\) | with oracle           |
| Tree UCB-Hoeffding        | \(O(T \log T)\)    | \(O(T^{\frac{2d}{d+2}})\) | without oracle        |

Table 1: Comparisons of Lipschitz bandit algorithms.
3 Preliminaries

A stochastic bandit problem is a pair $(\mathcal{X}, M)$, where $\mathcal{X}$ is a measurable space and $M$ determines the distribution of rewards associated with each $x \in \mathcal{X}$. For the non-contextual bandit, $\mathcal{X}$ is the arm space $\mathcal{A}$, while for contextual bandit, $\mathcal{X}$ is the context-arm joint space $\mathcal{Z} \times \mathcal{A}$. $M$ is a mapping from $\mathcal{X}$ to the space of probability measures over the real line. We denote the distribution assigned to $x$ by $M_x$. We require that for each $x$, the distribution $M_x$ is integrable and the mean reward function

$$f(x) = \int y \, dM_x(y)$$

is measurable. In this paper, while we focus on stochastic Lipschitz bandit optimization [12][20][7], our algorithms can adapt to the problems without Lipschitz condition. The actual reward considered in our paper for any $x \in \mathcal{X}$ follows the distribution $\mathcal{N}(f(x), \sigma^2)$, where $f(\cdot) \in [0, 1]$ is the bounded mean-payoff satisfying the Lipschitz condition

$$|f(x) - f(x')| \leq L \|x - x'\|_\infty$$

and $\sigma^2$ is a variance. Furthermore, the metric space considered in this paper is a hyper-rectangle within $\mathbb{R}^d$, where $d = d_A$ or $d_Z + d_A$ depending on whether the context is involved. Our theory generalizes to arbitrary compact spaces by embedding such spaces within a hyper-rectangle.

The goal of Lipschitz bandit optimization is to locate the global maximum of the reward function with minimum queries. At each round $t$, based on past observations $(x_1, y_1, \ldots, x_{t-1}, y_{t-1})$ (and the context $z_t$ if it is given), where $x_i = a_i$ (for contextual bandit, $x_i = (z_i, a_i)$), the agent plays the arm $a_t$ under the proposed policy and observes reward $y_t$. The procedure is then repeated in this fashion. To measure the performance of the agent’s strategy, we define the regret at round $t$ as

$$r_t = f(x^*) - f(x_t),$$

where $a^*$ is the global maximum of $f(\cdot)$. Hence, the cumulative regret up to $T$ can be given by

$$R_T = \sum_{t=1}^{T} r_t = \sum_{t=1}^{T} [f(a^*) - f(a_t)].$$

Our algorithm adaptively partitions the entire space into finer parts as we observe more feedback and maintain an upper bound estimate of $f$ for each partition element separately. Every time we select a certain partition element with an UCB exploration and pull the arm randomly within the selected block. The following definition will be used to bound the difference of $f(\cdot)$ evaluated on two separate arms in the same partition:

**Definition 1. (Diameters)** Given the metric $d(\cdot, \cdot): \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ over the space $\mathcal{X}$, the diameter of $\mathcal{X}$ is defined by

$$D(\mathcal{X}) = \sup_{x, x' \in \mathcal{X}} d(x, x').$$

In our setting, the metric $d(\cdot, \cdot)$ is taken as $\|x - x'\|_\infty$ to adapt to the Lipschitz continuity.

The rest of the paper is organized as follows. In section 4, we describe our proposed algorithms and present theorems to bound the cumulative regret $R_T$. In section 5, we give improved upper bounds on the time complexity of our algorithms.
4 Proposed Method

Our proposed algorithms (see Algorithm 1 and 2) incrementally update a partition of the space, building an estimate of the mean pay-off function $f$ over the partition. The partition is initialized as the entire space (partition size is 1). It becomes finer as we gradually sub-divide its elements. This process of adaptive partitioning can be represented with a growing tree $T$, with which we can construct a one-to-one map from the collection of all leaf nodes to the latest partition. Specifically, the tree is initialized as a single root, whose associated partition is the entire space. At every iteration, we select a target leaf node of the tree using UCB exploration based upon the estimated reward function. An arm within the partition element associated with the target node will then be pulled arbitrarily. If the condition for sub-division is satisfied, we split the target partition element and add as many descendants as the number of resulting sub-elements to the target node, such that each descendant will be associated with one of the sub-elements. The estimation for $f$ will be iteratively updated based on the feedback of every pulled arm. The core idea is to perform finer partitioning towards the maxima of $f$ so that the expected average regret will converge with a desired rate.

In order to define the estimated reward, we need several statistics on all the leaf nodes. The statistics are maintained as a tuple $(P, n, v, l)$, where $P$ is the associated partition element, $v$ is the actual mean reward, $n$ is the number of arms over which the mean reward $v$ is evaluated and $l$ is the depth of the node of $T$. The Upper Confidence Bound (UCB) for $f$, denoted by $Q$ is given by

$$Q = v + \frac{2LD(X)}{\sqrt{n}} + \sqrt{4\sigma^2(4 - \log p + 2|\log t|)}$$

where $p$ is an arbitrary probability constant such that the regret bound holds with probability $1 - p$. $Q$ can be interpreted as the sum of the actual mean reward $v$ and the upper confidence gap. The term added to $v$ accounts for the uncertainty arising from the variation of the mean-payoff function over the region $P$ plus the random noise carried by the actual rewards. For non-leaf nodes, their $Q$ values are recursively defined as the maximum of their descendants.

4.1 Non-contextual Bandit

For non-contextual bandits, our algorithm (see Algorithm 1) performs greedy search from the root downwards and terminates at one of the leaves with the greatest $Q$. We call the selected node the target node. This yields the main improvement of our algorithm over $X$-armed bandit. In their method, the target node is split at each iteration independent of the statistics, hence the depth of the resulting tree may grow linearly with respect to the number of iterations. Consequently, the total time complexity for $T$ iterations can be $O(T^2)$. Our algorithm, however, gradually slows down the pace of splitting as the target node becomes deeper. Let us denote the target node at iteration $t$ as $u_t$ with the associated statistics $(A_t, n_t, v_t, l_t)$. After the arm is pulled, we sub-divide $A_t$ if and only if $n_t + 1 = 4^l$. This modification is critical to the analysis on both the regret and the time complexity. It can be shown that the depth of the tree can grow at most in logarithmic rate of $t$. Our splitting on $A_t$ contains binary splits for each of the $d$ dimensions resulting in $2^d$ descendants of $u_t$. The statistics defined on the parent node $u_t$ will be copied to each of its descendants. The whole procedure is then repeated until termination.

Under the stated assumptions, we can prove regret bounds of our Tree UCB-Hoeffding algorithm:
Algorithm 1: Tree UCB-Hoeffding

**Input:** \( v_0, l_0, n_0 = 0; \) probability \( p; \) Lipschitz constant \( L; \) action space \( \mathcal{A} \) and its dimensionality \( d; \) the initial search tree \( \mathcal{T} = \{ u_0 \} \) with statistics \( (\mathcal{A}, n_0, v_0, l_0) \) on node \( u_0; \) the estimated upper bound \( Q: \mathcal{T} \rightarrow \mathbb{R} \) initialized with \( Q(u_0) = 1; \)

1 for \( t = 1, 2, \ldots \) do
   2 Do greedy search on \( \mathcal{T} \) from the root to one of the leaves under priority \( Q \) and denote the target node \( u_t \) and its statistics \( (\mathcal{A}_t, n_t, v_t, l_t) \);
   3 Play the arm \( a_t \in \mathcal{A}_t \) (ties are broken randomly) and receive the reward \( y_t \);
   4 \( n' \leftarrow n_t + 1; \)
   5 \( v' \leftarrow \frac{1}{n} y_t + \frac{n'-1}{n} v_t; \)
   6 if \( n' \geq 4^l_t \) then
      7 \( l' \leftarrow l_t + 1; \)
      8 Split \( \mathcal{A}_t \) into \( 2^d \) sub-partitions \( \{ \mathcal{A}_{t,i} \}_{i=1}^{2^d} \) along the middle of each dimension;
      9 Add the descendants \( \{ u_{t,i} \}_{i=1}^{2^d} \) to \( u_t \) and assign statistics \( (\mathcal{A}_{t,i}, n', v', l') \) to \( u_{t,i} \);
   10 else
      11 Update the statistics of \( u_t \) to \( (\mathcal{A}_t, v', n', l_t) \);
   12 end
   13 For each leaf node \( u \in \mathcal{T} \) with its latest statistics denoted as \( (P, n, v, l) \), let
      14 \( Q(u) \leftarrow v + \frac{2LD(A)+\sqrt{4\sigma^2(1-\log p+2[\log t])}}{\sqrt{n}}; \)
   15 Update \( Q \) for each non-leaf node to be the maximum of its descendants recursively.
end

**Theorem 1.** For any constant \( p > 0 \), with probability at least \( 1 - p \), the cumulative regret \( R_T \) of the tree UCB algorithm is bounded by \( \tilde{O}(T^{\frac{d+1}{2}}) \).

To prove Theorem 1, we need several lemmas. Since the ties are broken randomly, the variation of the mean pay-off evaluated at \( a_t \) is associated with the diameter of the target partition \( \mathcal{A}_t \). Hence, it is essential to bound \( D(\mathcal{A}_t) \).

**Lemma 1.** It holds that

\[
D(\mathcal{A}_t) \leq \frac{D(A)}{\sqrt{n_t} + 1},
\]

in which \( (\mathcal{A}_t, n_t, v_t, l_t) \) denotes the statistics of the target node at time step \( t \).

**Proof.** Since the splitting is executed along the middle of each dimension, the diameter of the partition element of the descendant node is equal to half of its parent node’s. It follows that

\[
D(\mathcal{A}_t) = \frac{D(A)}{2^l_t}.
\]

Since we split the target node when \( n_t + 1 = 4^l_t \), it holds that

\[
n_t + 1 \leq 4^l_t.
\]

\( \tilde{O}(\cdot) \) is asymptotically equivalent to \( O(\cdot) \) up to a logarithmic factor
Therefore,
\[
D(A_t) = \frac{D(A)}{2^{n_t}} \leq \frac{D(A)}{\sqrt{n_t + 1}}.
\]

Lemma 1 relates the diameter of the target partition to the statistic \(n_t\). We can now move on to bound the variation \(|f(a) - v_t|\) for any \(a \in A_t\). The difference takes into consideration both the Lipschitz error and the randomness in the realized reward. It also explains our construction for \(Q\) as an upper bound of \(f\).

**Lemma 2.** For \(t \geq 2\), any \(0 < p < 1\) and any action \(a \in A_t\), it holds with probability at least \(1 - \frac{p}{2t^2}\) that
\[
|f(a) - v_t| \leq \frac{2LD(A)}{\sqrt{n_t}} \sqrt{\frac{1}{n_t}} \leq \sqrt{\frac{4\sigma^2(4 - \log p + 2\log t)}{n_t}}.
\]

**Proof.** Denote all the iteration indices \(i_1, i_2, \ldots, i_{n_t}\) before iteration \(t\) with \(i_1 \leq i_2 \leq \ldots \leq i_{n_t}\) satisfying \(a \in A_{i_j}\). It is clear that \(j = n_{i_j} + 1\). We have
\[
\left| f(a) - \frac{1}{n_t} \sum_{j=1}^{n_t} f(a_{i_j}) \right| \leq \frac{1}{n_t} \sum_{j=1}^{n_t} LD(A) \frac{1}{\sqrt{n_{i_j} + 1}} = \frac{1}{n_t} \sum_{j=1}^{n_t} L(D(A)) \sqrt{n_{i_j} + 1}
\]
according to Lemma 1. Since it holds that \(\sum_{j=1}^{n_t} \frac{1}{\sqrt{j}} \leq 2\sqrt{n_t}\), we thus have
\[
\left| f(a) - \frac{1}{n_t} \sum_{j=1}^{n_t} f(a_{i_j}) \right| \leq \frac{2LD(A)}{\sqrt{n_t}}.
\] (1)

By Hoeffding’s inequality,
\[
P\left( \left| \sum_{j=1}^{n_t} y_{i_j} - \sum_{j=1}^{n_t} f(a_{i_j}) \right| \geq \delta \right) \leq 2 \exp\left( -\frac{\delta^2}{2n_t\sigma^2} \right)
\]
for any \(\delta \geq 0\). With the change of variables, it holds with probability \(1 - \frac{p}{2t^2}\) that
\[
\left| v - \frac{1}{n_t} \sum_{j=1}^{n_t} f(a_{i_j}) \right| \leq \sqrt{\frac{4\sigma^2 \log \frac{4t^2}{p}}{n_t}} \leq \sqrt{\frac{4\sigma^2(4 - \log p + 2\log t)}{n_t}}.
\] (2)

Combining Equation 1 and 2, Lemma 2 is easily seen.

Equation 2 uses the simple fact \(\log \frac{4t^2}{p} \leq 4 - \log p + 2[\log t]\). This change reduces the computational consumption for updating \(Q\) due to the variation of \(t\), which will be explained in the next section. Lemma 3 gives the upper bound for the partition size at iteration \(T\), which is identical to the number of leaf nodes of \(\mathcal{T}\).
Lemma 3. The number of leaf nodes of $T$ at iteration $T$ is bounded by $O(T^{d/2})$.

Proof. Every time Line 8 of Algorithm 1 is executed, the number of leaves will be increased by $2^d - 1$. Without loss of generality, we denote by $\{t_i\}_{i=1}^{m_T}$ these iteration indices rearranged in the order such that $l_{t_1} \leq l_{t_2} \ldots \leq l_{t_{m_T}}$, where $m_T$ is the quantity we want to bound. It follows that

$$\sum_{i=1}^{m_T} (4^{l_{t_i}} - 4^{l_{t_i} - 1}) \leq T.$$ 

The above formula relates $m_T$ with $l_{t_i}$ and $T$. If we can further obtain a lower bound for $l_{t_i}$, we will thereby get rid of $l_{t_i}$ and bound $m_T$ as a function of $T$. Notice that the number of indices $i$ is at most $2^{dl}$, i.e., the maximum number of nodes with depth $l$. It follows that

$$l_{t_i} \geq \left\lceil \frac{\log_2(i2^d - i + 1)}{d} \right\rceil - 1.$$ 

Therefore,

$$T = \Omega \left( \sum_{i=1}^{m_T} 4^{l_{t_i}} \right) = \Omega \left( \sum_{i=1}^{m_T} 4^{\log_2(i2^d)/d} \right) = \Omega \left( \sum_{i=1}^{m_T} \frac{i^2}{t^d} \right) = \Omega \left( \frac{d+2}{m_T^{d/2}} \right).$$

Rewriting it as $m_T = O(T^{d/2})$ completes the proof.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Denote the node whose associated partition element contains $a^*$ at iteration $t$ as $u_t^*$, and its statistics as $(A_t^*, v_t^*, n_t^*, l_t^*)$. According to our algorithm, the $Q$ value estimated at the target node $u_t$ is no less than any other node including $u_t^*$ since we are doing greedy search with $Q$. Hence, it holds for $t \geq 2$ that

$$v_t + \frac{2LD(A_t) + \sqrt{4\sigma^2(4 - \log p + 2|\log t|)}}{\sqrt{n_t}} \geq v_t^* + \frac{2LD(A) + \sqrt{4\sigma^2(4 - \log p + 2|\log t|)}}{\sqrt{n_t^*}} \geq f(a^*)$$

with the last inequality holding with probability at least $1 - \frac{P}{2\sigma^2}$ as proved in Lemma 2. It follows that

$$r_t = f(a^*) - f(a_t) \leq f(a^*) - v_t + v_t^* - f(a_t) \leq \frac{4LD(A) + 2\sqrt{4\sigma^2(4 - \log p + 2|\log t|)}}{\sqrt{n_t}}.$$ 

Now it is sufficient to bound the term $\sum_{t=2}^{T} \frac{1}{\sqrt{n_t}}$. The strategy is that we first group the arms by the latest partition elements they belong to and then combine the summands of these groups together. We denote the partition elements associated with each leaf node at iteration $T$ as $P_1, P_2, P_{w_T}$ where $w_T$ is the number of leaf nodes. The order of this listing is arbitrary. It follows that
\[ \sum_{t=2}^{T} \frac{1}{\sqrt{n_t}} = \sum_{i=1}^{w_T} \sum_{\substack{2 \leq t \leq T : a_t \in P_i}} \frac{1}{\sqrt{n_t}} \]

Within the same partition element, \( n_t \) is strictly increasing, hence \( \sum_{2 \leq t \leq T : a_t \in P_i} \frac{1}{\sqrt{n_t}} \leq \sum_{j=1}^{p_i} \frac{1}{\sqrt{n_j}} \leq 2\sqrt{p_i} \), where \( p_i \) is the number of pulling arms contained in element \( P_i \). Finally,

\[ \sum_{t=2}^{T} \frac{1}{\sqrt{n_t}} \leq 2 \sum_{i=1}^{w_T} \sqrt{p_i} \leq 2 \sqrt{w_T} \sum_{i=1}^{w_T} p_i = 2\sqrt{w_T} \sum_{i=1}^{w_T} p_i \leq O(T^{\frac{d+1}{2}}) \]

with the second inequality derived from Cauchy’s inequality and the last inequality holding due to Lemma 3. Therefore, for any \( 0 < p < 1 \), with probability at least \( 1 - \sum_{t=2}^{\infty} \frac{p}{t} \leq 1 - p \), we have

\[ R_T \leq O(T^{\frac{d+1}{2}} \log T) = \tilde{O}(T^{\frac{d+1}{2}}) \]

since the regret at the first iteration is bounded by 1.

4.2 Contextual Algorithm

In contextual bandit problem, we observe the context information \( z \in \mathcal{Z} \) before we pull the arm, hence we are able to make decisions based on the past experience as well as the context information. The definitions of the statistics are identical with the non-contextual case. However, instead of running greedy search on the tree with respect to \( Q \), we must first select out all the leaf nodes whose associated partition elements contain the given context \( z \). We then pick the one with the greatest \( Q \) among the filtered leaves. This causes extra computation cost but is still an improvement over the state-of-the-art method [20], which takes more than \( O(T^2) \). It maintains the same regret bound as stated in Theorem 2.

**Theorem 2.** For any constant \( p > 0 \), with probability at least \( 1 - p \), the cumulative regret \( R_T \) of the Contextual Tree UCB algorithm (Algorithm 2) is bounded by \( \tilde{O}(T^{\frac{d+1}{2}}) \).

The proof of Theorem 2 follows directly from the same argument in Theorem 1.

5 Time Complexity

In this section, we analyze the time complexity of Algorithm 1 and 2. For the non-contextual bandit problem, we apply the greedy search on the partition tree. Since the number of descendants of any node is either \( 2^d \) or 0 depending on whether it is a leaf, the time complexity for the tree search process is bounded by the depth of the tree. Apart from the search, we also need to update the upper bound estimate \( Q \) after each arm is pulled. It consists of two parts. First, when \( \lfloor \log(t - 1) \rfloor = \lfloor \log t \rfloor \) for \( t \geq 2 \), the term involving \( t \) contained in the concentration error remains unchanged. Hence, we only update the \( Q \) value for the unique path from the root to the target node (including its new descendants if they exist) and other nodes remain unchanged. Clearly, the update is also bounded by the current depth of the tree per iteration. On the other hand, when \( \lfloor \log(t - 1) \rfloor < \lfloor \log t \rfloor \), we need to update \( Q \) for every node in \( T \). However, the frequency of this
Algorithm 2: Contextual Tree UCB-Hoeffding

**Input:** \( v_0, l_0, n_0 = 0; \) probability \( p; \) Lipschitz coefficient \( L; \) the context-action joint space \( \mathcal{X} = \mathcal{Z} \times \mathcal{A} \) and its dimensionality \( d = d_A + d_Z; \) the initial search Tree \( \mathcal{T} = \{ u_0 \}; \) with statistics \( \mathcal{(X, v_0, n_0, l_0)} \) on node \( u_0; \) the estimated upper bound \( Q: \mathcal{T} \rightarrow \mathbb{R} \) initialized with \( Q(u_0) = 1; \)

1. for \( t = 1, 2, \ldots \) do
2. Receive context \( z_t; \)
3. Find all leave nodes such that their associated partitions contain \( z_t; \)
4. Select the leaf \( u_t \) with the greatest \( Q \) value and denote its statistics \( \mathcal{(X_t \times A_t, v_t, n_t, l_t)}; \)
5. Play the arm \( a_t \in \mathcal{A}_t \) (Ties are broken randomly) and receive the reward \( y_t; \)
6. \( n'_t \leftarrow n_t + 1; \)
7. \( v'_t \leftarrow \frac{1}{n'} y_t + \frac{n'-1}{n'} v_t; \)
8. if \( n'_t \geq 4^t \) then
9. \( l'_t \leftarrow l_t + 1; \)
10. Split \( \mathcal{X_t \times A_t} \) into \( 2^d \) sub-partitions \( \mathcal{\{X_{t,i} \times A_{t,i}\}}^{2^d}_{i=1} \) along the middle of each dimension;
11. Add the descendants \( \mathcal{\{u_{t,i}\}}^{2^d}_{i=1} \) to \( u_t \) and assign statistics \( \mathcal{(X_{t,i} \times A_{t,i}, n', v', l')} \) to \( u_{t,i}; \)
12. else
13. Update the statistics of \( u_t \) to \( \mathcal{(X_t \times A_t, v', n', l_t)}; \)
14. end
15. For each leaf node \( u \in \mathcal{T} \) with its latest statistics denoted as \( \mathcal{(P, n, v, l)} \), let \( Q(u) \leftarrow v + \frac{2LD(\mathcal{X})+\sqrt{4\sigma^2(1-log p+2|log t|)}\sqrt{n}}{\sqrt{n}}; \)
16. Update \( Q \) for each non-leaf node to be the maximum of its descendants recursively.
17. end

update is logarithmic with respect to \( t. \) Therefore, in order to show that the total complexity is \( \mathcal{O}(T \log T) \) for the first \( T \) iterations, it is sufficient to prove that the depth of our constructed tree in \( \mathcal{O}(\log t) \) at iteration \( t. \)

**Lemma 4.** The depth of Tree \( \mathcal{T} \) by any iteration \( t \) is \( \mathcal{O}(\log t) \)

**Proof.** By the splitting condition (Line 8 of Algorithm 1), we have

\[ l_t \leq \log_4(n_t + 1) + 1 \leq \mathcal{O}(\log t). \]

It is clear that the depth of \( \mathcal{T} \) is at most \( \max_{t' \leq t} l_{t'} + 1 \) at time \( t \) because any leaf node must be explored before its descendants are defined.

**Corollary 1.** The total computational cost of Algorithm 1 for the first \( T \) iterations is \( \mathcal{O}(T \log T) \).

For the contextual bandit problem, the consumption for updating \( Q \) is unchanged. In each iteration, given the context \( z_t, \) we first collect the leaf nodes whose partition element’s contextual part contain \( z_t. \) In other words, our algorithm will traverse every node \( u \) (and its descendants) with statistics \( \mathcal{(Z' \times A', n', v', l')} \) satisfying \( z_t \in \mathcal{Z'}. \) Lemma 5 gives a tight bound on the cardinality of the collected nodes.
Lemma 5. At any iteration \( t \leq T \), the number of nodes \( u \) with statistics \( (Z', A', n', v', l') \) satisfying \( z_t \in Z' \) is bounded by \( \mathcal{O}(T^{d_A + 2}) \), where \( d_A \) is the dimension of the arm space.

Proof. We follow the argument of Lemma 3. Fixing any time step \( t \) and the given context \( z_t \), denote \( t_1, \ldots, t_m \) to be the iteration indices before \( t \), when the contextual part of the target partition \( Z_{t_i} \) contains \( z_t \) and such that the target node \( u_{t_i} \) is split after the arm \( a_{t_i} \) is pulled. Each split will increase the number of the nodes having \( z_t \in Z' \) by \( 2^{d_A} \). Hence it is sufficient to bound \( m_t \). Since, \( t \leq T \), we have

\[
 m_t \sum_{i=1}^{m_t} (4^{l_{t_i}} - 4^{l_{t_i} - 1}) \leq T
\]

Different from Lemma 3, the number of indices \( i \) satisfying \( l_{t_i} = l \) is at most \( 2^{d_A l} \) because the context \( z_t \) is fixed. Although the number of nodes with depth \( l \) can be at most \( 2^l \), those satisfying the context condition are no more than \( 2^{d_A l} \). It follows that

\[
 l_{t_i} \geq \lceil \frac{\log_2(i2^{d_A} - i + 1)}{d_A} \rceil - 1.
\]

Therefore,

\[
 T = \Omega\left( \sum_{i=1}^{m_t} 4^{l_{t_i}} \right) = \Omega\left( \sum_{i=1}^{m_t} 4^{\log_2(i2^{d_A} - i + 1)/d_A} \right) = \Omega\left( \sum_{i=1}^{m_t} 2^{2d_A} \right) = \Omega\left( m_t^{\frac{2d_A+2}{d_A}} \right),
\]

which can be rewritten as \( m_t = \mathcal{O}(T^{\frac{d_A}{d_A + 2}}) \).

Corollary 2. The total computational cost of Algorithm 2 for the first \( T \) iterations is \( \tilde{\mathcal{O}}(T^{\frac{2d_A+2}{d_A + 2}}) \).

Proof. The per iteration cost for \( t \leq T \) is \( \mathcal{O}(T^{\frac{d_A}{d_A + 2}}) + \mathcal{O}(\log T) \), derived from the previous argument. Therefore, the total time complexity for the first \( T \) iterations is at most \( \tilde{\mathcal{O}}(T^{\frac{2d_A+2}{d_A + 2}}) \).

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