Estimates for the Green’s function of the discrete bilaplacian in dimensions 2 and 3

Stefan Müller∗ Florian Schweiger†

December 8, 2017

Dedicated to the memory of Eberhard Zeidler who has been an inspiration in so many ways. Who ever had the good fortune to meet him will never forget him.

Abstract

We prove estimates for the Green’s function of the discrete bilaplacian in squares and cubes in two and three dimensions which are optimal except possibly near the corners of the square and the edges and corners of the cube. The main idea is to transfer estimates for the continuous bilaplacian using a new discrete compactness argument and a discrete version of the Cacciopoli (or reverse Poincaré) inequality. One application that we have in mind is the study of entropic repulsion for the membrane model from statistical physics.

Keywords: discrete bilaplacian, finite differences, discrete Campanato spaces, membrane model, entropic repulsion, Gaussian field

Mathematics Subject Classification (2010): 65N06, 31B30, 39A14, 60K35, 82B41

1 Introduction

Let $V = [-1, 1]^n$ and $V_N = NV \cap \mathbb{Z}^n$ with $n \in \mathbb{N}^+$ and $N \in \mathbb{N}^+$. Consider the Hamiltonian $H_N(\psi) = \frac{1}{2} \sum_{x \in V_N} |\Delta_1 \psi_x|^2$, where $\Delta_1$ is the discrete Laplacian and $\psi \in \mathbb{R}^{V_N}$ is a function on $V_N$, extended by 0 to all of $\mathbb{Z}^n$. The associated Gibbs measure

$$P_N(\text{d}\psi) = \frac{1}{Z_N} \exp(-H_N(\psi)) \prod_{x \in V_N} \text{d}\psi_x \prod_{x \in \mathbb{Z}^n \setminus V_N} \delta_0(\text{d}\psi_x)$$
is then the distribution of a Gaussian random field on $\mathbb{Z}^n$ with 0 boundary data, the so-called membrane model. Its covariance matrix is given by the Green’s function $G_N$ of the discrete biharmonic on $V_N$ with zero boundary data outside $V_N$. We prove estimates for the Green’s function of the discrete biharmonic for $n = 2$ and $n = 3$ which are optimal except possibly near the corners of the square and the edges and corners of the cube.

One motivation for our work is to understand entropic repulsion, i.e. the probability of the event $\Omega_{V_N,+} = \{ \psi: \psi_x \geq 0 \ \forall x \in V_N \}$, and the behaviour of $P_N$ conditioned on $\Omega_{V_N,+}$. For this analysis a good understanding of the Green’s function $G_N$ is crucial. We focus here on the subcritical dimensions $n = 2$ and $n = 3$, since entropic repulsion for the membrane model has already been studied by Kurt in the supercritical case $n \geq 5$ [Kur07] and in the critical case $n = 4$ [Kur09]. For earlier results in the supercritical case see also Sakagawa [Sak03]. In the case $n = 1$ Hamiltonians of the form $\sum_{x \in V_N} V(\Delta_1 \psi_x)$ for convex $V$ have been studied by Caravenna and Deuschel [CD08, CD09] using renewal methods. Entropic repulsion in the gradient model corresponding to the Hamiltonian $\frac{1}{2} \sum_{x \in V_N} |\nabla_1 \psi_x|^2$ with 0 boundary condition was analysed by Deuschel [Deu96], see also Bolthausen-Deuschel-Giacomin [BDG01] and the survey by Velenik [Vel06].

More recently the membrane model with periodic boundary conditions has also been discussed as a scaling limit of the divisible sandpile model, see Levine et al. [LMPU16] for the expression of the odometer function as a shifted discrete biharmonic field and Cipriani, Subhra Hazra and Ruszel [CSR16b, CSR16a] for the convergence of the rescaled odometer to a continuum biharmonic field on the unit torus and further properties of the odometer function.

The analysis of the discrete Green’s function is very closely related to stability estimates for the inverse of the corresponding fourth order finite difference operator. In numerical analysis such stability estimates and related convergence results estimates go back to the seminal work of Courant, Friedrich and Lewy [CFL28], who followed a variational approach for second and fourth order equations and showed in particular apriori estimates for the $L^2$ norm of the discrete derivatives and convergence of discretely biharmonic functions to a continuous biharmonic function, and Gerschgorin [Ger30] who proved an error estimate of order $h^2$ for the Poisson equation on a grid of size $h$ if the continuous solution is in $C^4$. There has been a large amount of subsequent work in particular for second order equations, including estimates under weak regularity assumptions. See, e.g., the recent monographs by Jovanović and Suli [JS14] and Hackbusch [Hac17] for the state of the art and further references. For special domains such as a rectangle or an orthant explicit formulae for the Green’s function of the Laplace operator are available [MW40, CCI16]. The biharmonic case has been less studied. Early references include [DS58, Man67, Sim67] and error estimates for low regularity solutions have been obtained in [Hac81, Laz81, GMP83, GLMP83, HS86, JS14].
The main result of this paper is

**Theorem 1.1.** Let \( n = 2 \) or \( n = 3 \), let \( G_N \) be the Green’s function of the discrete bilaplacian with zero boundary data outside \( V_N \), and let \( d(z) = \text{dist}(z, \mathbb{Z}^n \setminus V_N) \). Then there exist \( c, C > 0 \) independent of \( N \) such that \( G_N \) and its discrete derivatives satisfy the following estimates.

i) For any \( x, y \in \mathbb{Z}^n \)

\[
|G_N(x, y)| \leq C \min \left( \frac{d(x)^2 + d(y)^2}{|x - y| + 1}, \frac{d(x)^2 d(y)^2}{|x - y| + 1} \right) \quad (1.1)
\]

\[
|\nabla_x G_N(x, y)| \leq C \min \left( \frac{d(y)^{3-n}}{|x - y| + 1}, \frac{(d(x) + 1)d(y)^2}{|x - y| + 1} \right) \quad (1.2)
\]

\[
|\nabla_x^2 G_N(x, y)| \leq \begin{cases} 
C \log \left( 1 + \frac{d(y)^2}{(|x-y|+1)^2} \right) & n = 2 \\
C \min \left( \frac{1}{|x-y|+1}, \frac{d(y)^2}{(|x-y|+1)^3} \right) & n = 3 
\end{cases} \quad (1.3)
\]

\[
|\nabla_x \nabla_y G_N(x, y)| \leq \begin{cases} 
C \log \left( 1 + \frac{d(x)+1)(d(y)+1)}{(|x-y|+1)^2} \right) & n = 2 \\
C \min \left( \frac{1}{|x-y|+1}, \frac{(d(x)+1)(d(y)+1)}{(|x-y|+1)^3} \right) & n = 3 
\end{cases} \quad (1.4)
\]

ii) For any \( x \in \mathbb{Z}^n \)

\[
G_N(x, x) \geq c d(x)^{4-n} \quad (1.5)
\]

\( G_N \) is symmetric in \( x \) and \( y \), so we also have the analogous estimates for \( |\nabla_y G_N(x, y)| \) and \( |\nabla_y^2 G_N(x, y)| \). For the optimality of these estimates, see the discussion after Theorem 1.3.

The estimates (1.1) and (1.5) immediately provide estimates for the variance and covariance of \( \psi \) under \( P_N \). From the estimates (1.1) and (1.5) one can also deduce that \( P_N(\Omega_{V_N,+}) \leq e^{-c^{N-n-1}} \) for \( n \in \{2, 3\} \) and some \( c > 0 \) [Kur, Sch16]. In addition Theorem 1.4 implies the following continuity estimates.

**Corollary 1.2.** Under \( P_N \), the random field \( \psi \) satisfies

\[
E_N(|\psi_x - \psi_y|^2) \leq \begin{cases} 
C|x - y|^2 \log \left( 2 + \frac{N}{|x-y|} \right) & n = 2, \\
C|x - y| & n = 3
\end{cases} \quad (1.6)
\]

To show (1.6) for \( n = 2 \) one uses the identity

\[
E_N(|\psi_x - \psi_y|^2) = G_N(x, x) - G_N(x, y) - G_N(y, x) + G_N(y, y) \quad (1.7)
\]

as well as a discrete counterpart of the identity

\[
H(x, x) - H(x, y) - H(y, x) + H(y, y)
\]

3
\[
= \int_0^1 \int_0^1 \partial_s \partial_t H(x + s(y - x), x + t(y - x)) \, ds \, dt,
\]
valid for every smooth function \(H\), and \((\mathbb{1.4})\). For \(n = 3\) one uses \((\mathbb{1.7})\) and the estimates for \(G(x, x) - G(x, y)\) and \(G(y, y) - G(y, x)\) provided by \((\mathbb{1.2})\) and its analogue for the \(y\)-derivative. Since \(\psi\) is a Gaussian field the estimate \((\mathbb{1.6})\) and the Kolmogorov continuity criterion imply that the rescaled fields \(\psi'_{x'} = N^{-2 + n/2} \psi'_{N,y'}\) are uniformly Hölder continuous with exponents \(\alpha < \alpha_n\) where \(\alpha_2 = 1\) and \(\alpha_3 = \frac{1}{2}\). More precisely
\[
P\left( \left\{ \psi' : \sup_{x' \neq y'} \frac{|\psi'_{x'} - \psi'_{y'}|}{|x' - y'|^\alpha} \leq K \right\} \right) \geq 1 - \varepsilon_\alpha(K)
\]
with \(\lim_{K \to \infty} \varepsilon_\alpha(K) = 0\).

In order to prove Theorem \((\mathbb{1.1})\) we need regularity improving estimates for discrete biharmonic functions and optimal decay estimates for various norms in annuli around the singularity. The corresponding estimates for continuous biharmonic functions can be proved using well-established techniques. One insight of this paper is that these estimates can be transferred to the discrete realm using two ingredients: a new compactness argument and the discrete version of the Cacciopoli (or reverse Poincaré) inequality. It should also be possible to transfer continuous estimate to discrete estimates by using error estimates in numerical analysis, see the discussion below Corollary \((\mathbb{1.4})\).

In order to derive the estimates in detail and to highlight the similarities between the continuous and discrete setting, it is convenient to change notation. In particular, we rescale our lattice to have width \(h\), while the domain is fixed. We also shift the boundary by \(h\) inwards.

Consider the lattice \((h\mathbb{Z})^n\), where we assume \(\frac{1}{h} \in \mathbb{N}\). Let \(\Lambda^h_n = [0, 1]^n \cap (h\mathbb{Z})^n\), \(\operatorname{int} \Lambda^h_n = \left[\frac{1}{h}, 1 - \frac{1}{h}\right]^n \cap (h\mathbb{Z})^n\) and let \(\Delta^2_h\) be the discrete Laplacian on \((h\mathbb{Z})^n\). Let \(G^h(x, y)\) be the Green’s function for \(\Delta^2_h\) on \(\operatorname{int} \Lambda^h_n\) with zero boundary values on \((h\mathbb{Z})^n \setminus \operatorname{int} \Lambda^h_n\). In this setting, Theorem \((\mathbb{1.1})\) becomes

**Theorem 1.3.** Let \(n = 2\) or \(n = 3\), and let \(d(z)\) denote the distance of \(z \in \operatorname{int} \Lambda^h_n\) to \((h\mathbb{Z})^n \setminus \operatorname{int} \Lambda^h_n\). Then there exist \(c, C > 0\) independent of \(h\) such that
\[
i) \text{ for any } x, y \in (h\mathbb{Z})^n
\]
\[
|G^h(x, y)| \leq C \min\left(\frac{d(x)^2 - \frac{n}{2}d(y)^2 - \frac{2}{2}}{(|x - y| + h)^n}\right),
\]
\[
(1.8)
\]
\[
|\nabla_{x,y} G^h(x, y)| \leq C \min\left(\frac{d(y)^3 - n}{(|x - y| + h)^n}\right),
\]
\[
(1.9)
\]
\[|\nabla^2_{h,x}G_h(x, y)| \leq \begin{cases} \frac{C \log \left(1 + \frac{d(y)^2}{(|x-y|+h)^2}\right)}{1} & n = 2 \\ \frac{C \min \left(\frac{1}{|x-y|+h}, \frac{d(y)^2}{(|x-y|+h)^2}\right)}{1} & n = 3 \end{cases} \]

(1.10)

\[|\nabla_{h,x}\nabla_{h,y}G_h(x, y)| \leq \begin{cases} \frac{C \log \left(1 + \frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^2}\right)}{1} & n = 2 \\ \frac{C \min \left(\frac{1}{|x-y|+h}, \frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^2}\right)}{1} & n = 3 \end{cases} \]

(1.11)

ii) for any \(x \in (h\mathbb{Z})^n\)

\[G_h(x, x) \geq cd(x)^{1-n}. \]

(1.12)

Theorem 1.1 can be easily derived from Theorem 1.3 if one chooses \(h = \frac{1}{2N+2}\), rescales by a factor of \(2N+2\) and observes that the estimates are scale-invariant. One can also obtain estimates for higher discrete derivatives, see Remark 8.4 below.

Comparison with the Green’s function of the continuous biharmonic in the ball (see [Bog05, eqn. (48)] or [GGS10, eqn. (2.65) and Thm. 4.7]), a general bounded smooth set [DS04, Thm. 3 and Thm. 12] or a half-space [GGS10, eqn. (2.66)] shows that the estimates in Theorem 1.3 are optimal in the interior and near the regular boundary points (edges for \(n = 2\) and faces for \(n = 3\)).

Near the singular boundary points (corners for \(n = 2\) and edges and corners for \(n = 3\)) the continuous regularity theory gives a more rapid decay of biharmonic functions (and their derivatives) and hence a more rapid decay for the Green’s function with a decay exponent \(\gamma\). Our compactness argument can be used to establish a similar decay estimate for all exponents \(\gamma' < \gamma\). Since the general continuum theory provides an open interval of admissible exponents \(\gamma\) (due to possible logarithmic terms) there is no loss in passing to the discrete estimates.

The general statement is rather tedious, so let us look instead at an illustrative example, the corner point 0 of the square \((0,1)^2\). In this case the distance of a point \(x\) from the corner point is given by \(|x|\). If \(|x| < \frac{1}{4}|y|\) then \(|x - y| \geq \frac{1}{2}|y| \geq \frac{1}{2}d(y)\) and the continuous theory implies that

\[|G(x, y)| \leq C \left(\frac{|x|}{|y|}\right)^{2+\theta/2} d^2(y). \]

(1.13)

where \(0 < \theta < \theta_0\) and \(\theta_0 \approx 3.47918\). To see this use Lemma 5.13 and note that

\[\|\nabla^2 G(\cdot, y)\|_{L^2(Q_{|y|/2} \cap (0,1)^2)} \leq C |y|^{-1} d^2(y)\]

(this follows from the continuous counterparts of (8.2) and Lemma 5.2). Moreover we have

\[\sup_{Q_r \cap (0,1)^2} G(\cdot, y) \leq 8 \|\nabla^2 G(\cdot, y)\|_{L^2(Q_r \cap (0,1)^2)}\]
by the Sobolev-Poincaré inequality and scaling.

The estimate (1.13) is better than the estimate
\[
G(x, y) \leq \frac{d^2(x) d^2(y)}{|x - y|^2} \sim C \frac{d(x)^2}{|y|^2} d^2(y)
\]
if
\[
\frac{d(x)}{|y|} \gg \left(\frac{|x|}{|y|}\right)^{1+\theta/4}.
\]

Note that this condition holds in particular if $|x|$ and $d(x)$ are comparable and $|x| \ll |y|$. The compactness argument shows that the discrete Green’s function $G_h$ satisfies a counterpart of (1.13) if we replace $\theta$ by any smaller exponent $\theta'$ and $C$ by $C_{\theta'}$.

It is also easy to show that the discrete Green’s function converges to the continuous Green’s function.

**Corollary 1.4.** Let $G(\cdot, y) \in W^{2,2}_0((0, 1)^n)$ denote the continuous Green’s function, i.e., the unique weak solution of $\Delta^2 G(\cdot, y) = \delta_y$. Extend $G_h(x, y)$ to $y \in (0, 1)^n$ by piecewise constant interpolation in the second variable. Then for each $y \in (0, 1)^n$ the following assertions hold.

i) We have
\[
I_{pc}^h G_h(\cdot, y) \to G(\cdot, y) \quad \text{uniformly},
\]
where $I_{pc}^h$ denotes the piecewise constant interpolation in the first variable.

ii) If $n = 2$ then $I_{pc}^h \nabla_h G_h(\cdot, y)$ converges uniformly to $\nabla G(\cdot, y)$ and $I_{pc}^h \nabla^2 G_h(\cdot, y)$ converges to $\nabla^2 G(\cdot, y)$ in $L^p((0, 1)^2)$ for all $p < \infty$.

iii) If $n = 3$ then $I_{pc}^h \nabla_h G_h(\cdot, y)$ is uniformly bounded and converges to $\nabla G(\cdot, y)$ in $L^p((0, 1)^3)$ for all $p < \infty$ and locally uniformly in $[0, 1)^3 \setminus \{y\}$. Moreover $I_{pc}^h \nabla^2_h G(h, \cdot, y)$ converges to $\nabla^2 G(\cdot, y)$ in $L^p$ for all $p < 3$.

A slight variant of the argument given below shows that the convergence in i) is also uniform in $y$, i.e., that we have uniform convergence of the piecewise constant interpolation of $G_h$ to $G$ in $(0, 1)^n \times (0, 1)^n$. The proof of assertion i) in Corollary 1.4 uses essentially only the elementary discrete $W^{2,2}$ estimate in Lemma 8.1 and the compact embedding from $W^{2,2}$ to $C^0$. The other two assertions follow from Theorem 1.3 and the local compactness argument in Section 5. See Section 8 for the details.

For $n = 2$ quantitative estimates for the discrete $W^{2,2}$ norm of difference between the solutions of the discretised and the continuous biharmonic equation under weak assumptions on the regularity of the continuous solution have been obtained by Lazarov [Laz81], Gavrilyuk, Makarov and Pironazarov [GMP83], Gavrilyuk et al. [GLMP83] and Ivanović, Jovanović and
Süli [ISS86], see also Chapter 2.7 in [JS14] which includes estimates for more general fourth order equations in divergence form with variable coefficients. More precisely, let \( u \in (W^{2,2}_0 \cap W^{s,2})((0,1)^2) \) and let \( \hat{u}_h \) be the solution of

\[
\Delta^2 \hat{u}_h = K_h \ast \Delta^2 u \quad \text{in } \text{int } \Lambda_h^2
\]

subject to the discrete boundary conditions

\[
\hat{u}_h(x) = 0 \quad \text{and } \quad \hat{u}_h(x+he_i) - \hat{u}_h(x-he_i) = 0 \quad \forall x \in \Lambda_h^2 \setminus \text{int } \Lambda_h^2 \quad \forall i \in \{1,2\}.
\]

Here \( K_h(x) = h^{-2}K(\frac{x}{h}) \) and \( K(z) = (1 - |z_1|_+) (1 - |z_2|)_+ \). The boundary condition (1.14) has the advantage that it leads to a higher order of consistency compared to our boundary condition \( u_h = 0 \) on \( (hZ)^2 \setminus \text{int } \Lambda_h \) (this latter condition is arguably more natural from the point of view of probability and statistical mechanics). For the discrete \( W^{2,2} \) norm the optimal error estimates

\[
\|u - \hat{u}_h\|_{W^{2,2}(\Lambda_h)} \leq C|h|^{s-2}\|u\|_{W^{s,2}((0,1)^2)}
\]

were established in [GMP83] for \( s = 3 \) and in [JS14, Thm. 2.69] for \( 5/2 < s < 7/2 \). In [GMP83] the estimate (1.15) is also proved for \( s = 4 \), but under the additional condition that that the symmetric extension \( \tilde{u} \) of \( u \) outside \( (0,1)^2 \) still belongs to \( W^{4,2} \). This holds only if the third normal derivatives of \( u \) (which exist in the sense of trace) vanish.

Because \( K_h \ast \delta = \delta_h \) these estimates can be used to compare the continuous Green’s function \( G_y \in W^{2,2} \) and the discrete Green’s function \( \hat{G}_{h,y} \) (defined using the boundary conditions (1.14) rather than \( G_{h,y} = 0 \) on \( (hZ)^2 \setminus \text{int } \Lambda_h \)) and one obtains \( \|G_y - \hat{G}_{h,y}\|_{W^{2,2}(\Lambda_h)} \leq C_s h^{s-2}d^{3-s}(y) \) for \( s \in (5/2,3) \). More precise estimates can be obtained if one applies the error estimates to \( u = G_y - \eta \hat{G}_{h,y} \) where \( \hat{G}_{h,y} \) is a suitable Green’s function in \( \mathbb{R}^2 \) and \( \eta \) is a suitable cut-off function (see below).

One can also use Theorem 1.3 to obtain quantitative error estimates for \( G_h - G \) and its discrete derivatives and we plan to pursue this elsewhere.

Let us briefly discuss some other approaches to prove Theorem 1.3. For \( n = 2 \) the estimates (1.8) and (1.12) as well as a discrete BMO estimate for the mixed derivative were proved in the second author’s MSc thesis [Sch10]. There a different approach was used to obtain the estimates near the corners. One starts from a discrete biharmonic function, defines a careful interpolation to get a continuous functions which is biharmonic up to a small error and uses the continuous theory to get good estimates for that interpolation which can then be transferred back to the original discrete function. This approach can in principle be extended to \( n = 3 \), but we found the compactness argument more flexible and more convenient to use.

Hackbusch [Hac83, Thm. 2.1] has developed a very general approach to derive discrete stability estimates on a scale of Banach spaces from the corresponding continuous estimates. One advantage of the compactness method
is that it avoids the construction of suitable discrete norms and restriction and prolongation operators which is a bit delicate near the singular boundary points.

Alternatively, for \( n = 2 \) and the symmetric boundary condition (1.14) one can use the optimal error estimates (1.15) in connection with the asymptotic expansion of the discrete Green’s function \( \hat{G}_{h,y} \) on \( (h\mathbb{Z})^2 \) in \( \frac{\sqrt{T}}{2} \) (see also Section 2). One applies the estimate (1.15) with \( s = 3 \) to \( u = G_y - \eta \hat{G}_y \) where \( \hat{G}_y \) is a suitable Green’s function in \( \mathbb{R}^2 \). It is not difficult to estimate the additional error term \( w_h = G_h - \eta \hat{G}_h - \hat{u}_h \) in the discrete \( W^{2,2} \) norm by computing \( \Delta^2_{h} w_h \) and testing with \( w_h \). This yields the estimate \( \| \hat{G}_{h,y} - G_y \|_{W^{2,2}(\Lambda^2)} \leq C h \) and the discrete inverse estimate implies that \( \| \hat{G}_{h,y} - G_y \|_{W^{2,\infty}(\Lambda^2)} \leq C \). Together with the known estimates for \( \nabla^2 G_y \) one concludes in particular that

\[
|\nabla^2_{h} \hat{G}_{h,y}| \leq C d^2(y)/(|x - y| + h)^2 \quad \text{for} \quad |x - y| \leq C d(y). \tag{1.16}
\]

To get the optimal estimate for \( |x - y| \gg d(y) \) one may proceed as follows. From the estimate for \( |x - y| \leq C d(y) \) one can obtain the crucial discrete \( L^\infty - L^2 \) estimate (1.1) for the second discrete derivatives for cubes of length \( 2r \) that touch the boundary by using the identity \( u(x) = \sum_{y \in \text{int} \Lambda_h} \hat{G}_h(x,y) \Delta^2_{h}(\eta u)(y) h^2 \) for an arbitrary lattice function \( u \) and a suitable cut-off function \( \eta \) with \( |\nabla^k_h \eta| \leq C_k r^{-k} \). For cubes which do not touch the boundary one can apply the identity \( v(x) = \sum_{y \in \text{int} \Lambda_h} \hat{G}_h(x,y) \Delta^2_{h}(\eta v)(y) h^2 \) to \( v(x) = u(x) - a - b \cdot x \) where \( a \) is the average of \( u \) over the cube and \( b \) is the average of \( \nabla_h u \). Together with the duality argument in Lemma 6.2 and Theorem 6.3 and similar estimates for the discrete \( y \)-derivatives of \( G_y - \hat{G}_{h,y} \) this yields the estimates in Theorem 1.3 for \( n = 2 \) for the Green’s function \( \hat{G}_{h,y} \) which satisfies the modified boundary conditions (1.14). The same argument applies to \( G_h \).

These estimates initially hold for \( \hat{G}_{h,y} \) and not for the function \( G_{h,y} \) in Theorem 1.3. Note, however, that \( \Delta^2_{h}(G_{h,y} - \hat{G}_{h,y}) = 0 \) in int \( \Lambda_h \). Using this fact as well as careful comparison of the different boundary conditions for \( \hat{G}_h \) and \( G_h \) one can show that \( \| G_{h,y} - \hat{G}_{h,y} \|_{W^{2,2}(\Lambda_h)} \leq C h \). This shows that the estimate (1.16) also holds for \( G_h \). For the estimates for \( |x - y| \gg d(y) \) one can then argue as for \( G_{h,y} \).

The remainder of this paper is organised as follows. In Section 2 we introduce some notation in the discrete setting and recall discrete counterparts of the product rule as well as Sobolev and Poincaré estimates. In Section 3 we give the weak and strong formulation of the discrete bilaplace equation and prove the Cacciopoli inequality (or reverse Poincaré inequality). The proof is very similar to the argument in the continuous case based on testing the equation with a cut-off function times the solution, but due to the discrete product rule some additional terms appear. In Section 4 we associate to
each discrete function a continuous function by discrete convolution with a B-spline and prove basic estimates of the interpolation.

Sections 5 and 6 contain the key estimates. The first key ingredient is an $L^\infty - L^2$ estimate for the discrete second derivative of discrete biharmonic functions in cubes which may intersect the boundary (see Theorem 6.1). This estimate is deduced from decay estimates for the second derivative of continuous biharmonic functions using a discrete version of the Kolmogorov-Riesz-Fréchet compactness criterion and the Cacciopoli inequality. The transition from continuous to discrete decay estimates is carried out in Section 5 separately for interior cubes, cubes near regular boundary points and cubes near singular boundary points.

The second key estimate is an $L^\infty$ decay estimate for discretely biharmonic functions in the complement of a cube (see Lemma 6.2 and Theorem 6.3). This follows by duality from the $L^\infty - L^2$ estimate in Theorem 6.1. The estimates in the interior and near regular boundary points can alternatively be derived by using discrete scaled $L^2$ estimates, i.e., by translating the continuous Campanato regularity theory to the discrete setting (see Dolzmann [Dol93, Dol99]). For the behaviour near the singular boundary points there seems to be no argument, however, which is only based on scaled $L^2$-norms and testing. For ease of exposition we use the compactness approach in all three regimes: interior points, regular boundary points and singular boundary points.

In Section 7 we recall Mangad’s [Man67] asymptotic expansion of a Green’s function $\tilde{G}_h$ of the discrete biharmonic operator in $(h\mathbb{Z})^n$. Finally in Section 8 we prove Theorem 1.3 and Corollary 1.4. An $L^2$ estimate for the second discrete derivatives of $G_h$ is easily obtained by testing with $G_h$ and Poincaré’s inequality. We then choose a suitable cut-off function $\eta_h$ and use the fact that $G_h(\cdot, y) - \eta_h(x)\tilde{G}_h(x - y)$ is biharmonic near $x = y$ to prove estimates for the mixed third discrete derivative $\nabla^2_{h,x} \nabla_{h,y} G_h$. The estimates for the lower derivatives now follow essentially by discrete integration over suitable paths (the relevant path are the discrete counterparts of the paths used in [DS04]). For the estimate for the first discrete derivatives for $n = 3$ we directly use the discrete Sobolev embedding since integration of the second derivative would generate an unnecessary additional logarithmic term.

2 Preliminaries

2.1 Notation

In the following $C$ denotes a constant that may change from line to line but is independent of $h$, unless stated otherwise.

Let $n \in \mathbb{N}^+$ (most of the time $n = 2$ or $n = 3$) be the dimension. We use standard notation for continuous quantities: We consider $\mathbb{R}^n$ with the
standard basis $e_1, \ldots, e_n$ and the usual Euclidean norm $|\cdot|$ and the $l^\infty$-norm $|\cdot|_\infty$. The differential of a map $f: \mathbb{R}^n \to \mathbb{R}^m$ is $Df = (Df_j)_{ij}$. For $\alpha \in \mathbb{N}^n$ we let $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. We also use the gradient $\nabla$, the Hessian $\nabla^2$, the Laplacian $\Delta$ and the divergence $\text{div}$.

By $B_r(x)$ we denote the open ball of radius $r$ around $x \in \mathbb{R}^n$, and by $Q_r(x) = x + (-r, r)^n$ the open cube with half-side length $r$ around $x$. If $x = 0$, we omit $x$.

Given $a \in \mathbb{R}^n$, we define $\tau_a f = f(\cdot + a)$ for any $f$. This corresponds to shifting $f$ by $-a$.

For a function $f$ we denote by $[f]_\Omega = \frac{1}{|\Omega|} \int_\Omega f \, dx$ its average over the bounded open set $\Omega$.

We use the standard $L^p$-norms $\| \cdot \|_{L^p}$, Sobolev-norms $\| \cdot \|_{W^{k,p}}$ and Hölder norms $\| \cdot \|_{C^{k,\alpha}}$ and Hölder seminorms $[\cdot]_{C^{0,\alpha}}$.

For discrete quantities we choose notation in such a way that it resembles the continuous notation. Let $h > 0$ be the (typically small) lattice width. We consider the lattice $(h\mathbb{Z})^n \subset \mathbb{R}^n$.

For $r \in \mathbb{R}$ we define $\lfloor r \rfloor_h := h \lfloor \frac{r}{h} \rfloor$, the largest element of $h\mathbb{Z}$ less than or equal to $r$.

For $x \in (h\mathbb{Z})^n$, $r \in \mathbb{R}$, $r \geq 0$ we define $Q_r^h(x) = \{ y \in (h\mathbb{Z})^n : |y - x|_\infty \leq r \} = Q_r(x) \cap (h\mathbb{Z})^n$. Then $Q_r^h(x)$ is a cube of sidelength $2\lfloor r \rfloor_h$ and center $x$. If $x = 0$, we omit $x$.

Given $A_h \subset (h\mathbb{Z})^n$, we define a corresponding subset $(A_h)_pc \subset \mathbb{R}^n$ as

$$(A_h)_pc = \text{int} \left( A + \left[ \frac{h}{2}, \frac{h}{2} \right]^n \right).$$

For example, for $x \in (h\mathbb{Z})^n$, $r \in h\mathbb{N}$, $(Q_r^h(x))_pc = Q_{r+\frac{h}{2}}^h(x)$. For a function $u_h: A_h \to \mathbb{R}$, we define its piecewise constant interpolation $P_h^{pc} u_h: A_{pc} \to \mathbb{R}$ by $P_h^{pc} u_h(y) = u_h(x)$ on each square $x + \left[ \frac{h}{2}, \frac{h}{2} \right]^n$, where $x \in A$.

Given $u_h: (h\mathbb{Z})^n \to \mathbb{R}$ and $x \in (h\mathbb{Z})^n$, define the forward derivative $D^h u_h(x) = \frac{1}{h} (u_h(x+h) - u_h(x))$ and the backward derivative $D_{-h} u_h(x) = \frac{1}{h} (u_h(x) - u_h(x-h))$. Furthermore, $\nabla_h u_h(x) = (D^h_{1} u_h(x), \ldots, D^h_{n} u_h(x))$ are the forward and backward gradient, $\text{div}_h u_h(x) = \sum_{i=1}^n D^h_{i} u_h(x)$ the forward and backward divergence, $\Delta_h u_h(x) = \sum_{i=1}^n D^2_{i, i} u_h(x)$ the Laplacian, and $\nabla^2_{h} u_h(x) = (D^2_{i, j} u_h(x))_{i,j}$ the Hessian matrix. Note that the Hessian matrix is in general not symmetric.

For a multi-index $\alpha \in \mathbb{N}^n$ we also define

$$(D^\alpha_{\pm h} u_h(x)) = (D^h_{\pm 1} \cdots D^h_{\pm n})^{\alpha} u_h(x),$$

and for $a \in \mathbb{N}$, $a > 2$ we set

$$(\nabla^a_{h} u_h(x) = (D^{\alpha}_{-i_1} D^{\alpha}_{i_2} \cdots D^{\alpha}_{i_a} u_h(x))_{i_1,i_2,\ldots,i_a}.$$ 

If $a \in (h\mathbb{Z})^n$ then $\tau_a$ also defines the shift $\tau_a x = a + x$ of $(h\mathbb{Z})^n$. We denote $\tau_{\pm h e_i}$ by $\tau_{\pm h i}$. Thus $\tau_{h i} f_h(x) = f_h(x \pm h e_i)$. 

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The discrete product rule then takes the form

\[ D_i^h (f_h g_h) = (D_i^h f_h) g_h + \tau_i^h f_h D_i^h g_h. \]

When dealing with functions of several variables we use a sub- or superscript to indicate the variable with respect to which a derivative is taken. So for example in \( \nabla_h x \nabla_h y G_h(x, y) \) we take one gradient in each variable.

As mentioned in the introduction, we set \( \Lambda^n_h = [0, 1]^n \cap (h\mathbb{Z})^n \) and \( \text{int } \Lambda^n_h = [\frac{1}{h}, 1 - \frac{1}{h}]^n \cap (h\mathbb{Z})^n \). We also set \( \partial \Lambda^n_h = \Lambda^n_h \setminus \text{int } \Lambda^n_h \).

### 2.2 Function spaces and inequalities

Let \( u_h, v_h : (h\mathbb{Z})^n \to \mathbb{R} \). For \( \Omega \subset \mathbb{R}^n \) measurable, \( p \in [1, \infty], k \in \mathbb{N}, \alpha \in [0, 1] \) we define (slightly abusing notation)

\[ \|u_h\|_{L^p(\Omega)} := \|I_{pc}^h u_h\|_{L^p(\Omega)}, \]

\[ (u_h, v_h)_{L^2(\Omega)} := (I_{pc}^h u_h, I_{pc}^h v_h)_{L^2(\Omega)}, \]

\[ \|u_h\|_{W^{k,p}(\Omega)} := \left( \sum_{|\alpha| \leq k} \|I_{pc}^h D^h_{\alpha} u_h\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}}, \]

\[ [u_h]_{C^{0,\alpha}_h(\Omega)} = \sup_{x, y \in \Omega, x \neq y} \frac{|I_{pc}^h u_h(x) - I_{pc}^h u_h(y)|}{|x - y|^\alpha}. \]

For \([\cdot]_{C^{0,\alpha}_h(\Omega)}\) we add the index \( h \) to emphasize the fact that we only take the supremum over \( x, y \) with \( |x - y| \geq h \).

For \( A_h \subset (h\mathbb{Z})^n \) these definitions take a familiar form. For example, if \( p < \infty \)

\[ \|u_h\|_{L^p(A_h)} = \left( \sum_{x \in A_h} h^n |u_h(x)|^p \right)^{\frac{1}{p}}, \]

\[ [u_h]_{C^{0,\alpha}_h(A_h)} = \sup_{x, y \in A_h, x \neq y} \frac{|u_h(x) - u_h(y)|}{|x - y|^\alpha}. \]

We extend these definitions to vector-valued functions by taking the Euclidean norm of the norms of the components.

We also set \([u_h]_{\Omega} = [I_{pc}^h u_h]_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} I_{pc}^h u_h \).

We then have the discrete analogues of Poincaré and Sobolev inequalities. All of them can be proved easily by applying their continuous counterpart to the piecewise multilinear interpolation of the function. We state the results that we will need.
Lemma 2.1 (Poincaré inequality on cubes with 0 boundary values). Let \( p \in [1, \infty) \), let \( u_h : (h \mathbb{Z})^n \to \mathbb{R}, x \in (h \mathbb{Z})^n, r \in h \mathbb{N} + \frac{h}{2} \), and suppose that \( u_h = 0 \) on at least one of the faces of \( Q_h^r(x) \). Then

\[
\|u_h\|_{L^p(Q_r(x))} \leq Cr\|\nabla_h u_h\|_{L^p(Q_r(x))}
\]

where \( C \) is independent of \( h \) and \( r \).

Lemma 2.2 (Poincaré inequality on annuli with 0 boundary values). Let \( p \in [1, \infty) \), let \( u_h : (h \mathbb{Z})^n \to \mathbb{R}, x \in (h \mathbb{Z})^n, r, s \in h \mathbb{N} + \frac{h}{2}, s < r \) and suppose that \( u_h = 0 \) on at least one of the faces of \( Q_r^s(x) \). Then

\[
\|u_h\|_{L^p(Q_r(x) \setminus Q_s(x))} \leq Cr\|\nabla_h u_h\|_{L^p(Q_r(x) \setminus Q_s(x))}
\]

where \( C \) only depends on \( \frac{s}{r}, p \) and \( n \).

Lemma 2.3 (Sobolev-Poincaré inequality on cubes with 0 boundary values). Let \( p \in [1, \infty) \), let \( u_h : (h \mathbb{Z})^n \to \mathbb{R}, x \in (h \mathbb{Z})^n, r \in h \mathbb{N} + \frac{h}{2} \), and suppose that \( u_h = 0 \) on at least one of the faces of \( Q_h^r(x) \).

If \( q \in [1, \infty) \) is such that \( \frac{n}{q} + 1 \geq \frac{n}{p} \) and \((p, q) \neq (n, \infty)\), then

\[
\|u_h\|_{L^q(Q_r(x))} \leq Cr^{1 + \frac{n}{q} - \frac{n}{p}}\|\nabla_h u_h\|_{L^p(Q_r(x))}
\]

and if \( \alpha \in (0, 1] \) is such that \( \alpha + \frac{n}{p} \leq 1 \), then

\[
[u_h]_{C^{0, \alpha}_h(Q_r(x))} \leq Cr^{1 - \frac{n}{p} - \alpha}\|\nabla_h u_h\|_{L^p(Q_r(x))}.
\]

3 The discrete bilaplacian equation

3.1 Definitions and basic properties

We consider the space of functions

\[
\Phi_h = \{ u_h : (h \mathbb{Z})^n \to \mathbb{R} : u_h(x) = 0 \ \forall x \in (h \mathbb{Z})^n \setminus \text{int } \Lambda_h^n \}.
\]

The discrete bilaplacian equation on \( \Lambda_h^n \) with 0 boundary data is the equation

\[
\Delta_h^2 u_h = f_h \ \text{in } \text{int } \Lambda_h^n
\]

(3.1)

where \( f_h : (h \mathbb{Z})^n \to \mathbb{R} \) is given and we are looking for a solution \( u_h \in \Phi_h \).

This equation is the discrete analogue of the bilaplace equation with clamped boundary conditions,

\[
\Delta^2 u = f \ \text{in } [0, 1]^n
\]

\[
\begin{align*}
0 &= u \ \text{on } \partial [0, 1]^n \\
D_n u &= 0 \ \text{on } \partial [0, 1]^n
\end{align*}
\]
If we multiply (3.1) with a test function \( \varphi_h \in \Phi_h \) and use summation by parts, we obtain the weak form of the bi Laplace equation

\[
(\nabla^2_h u_h, \nabla^2_h \varphi_h)_{L^2(\mathbb{R}^n)} = (f_h, \varphi_h)_{L^2(\mathbb{R}^n)} \quad \forall \varphi_h \in \Phi_h.
\]

It is easy to check that (3.1) and (3.2) are equivalent.

Written as a sum over lattice points, (3.2) becomes

\[
h^n \sum_{x \in \Lambda_h^n} \nabla^2_h u_h(x) : \nabla^2_h \varphi_h(x) = h^n \sum_{x \in \text{int} \Lambda_h^n} f_h(x) \varphi_h(x).
\]

Observe that the sum on the left-hand side has nonzero terms for \( x \in \Lambda_h^n \), whereas the right-hand side has nonzero terms only for \( x \in \text{int} \Lambda_h^n \).

If we choose \( \varphi_h = u_h \) in (3.2), we obtain

\[
(\Delta^2_h u_h, u_h)_{L^2(\mathbb{R}^n)} = (\nabla^2_h u_h, \nabla^2_h u_h)_{L^2(\mathbb{R}^n)} = \|\nabla^2_h u_h\|_{L^2(\mathbb{R}^n)}^2.
\]

Hence \( \Delta^2_h \), seen as a linear operator on \( \Phi_h \), is positive definite and hence invertible, and so (3.1) has a unique solution for any right-hand side \( f_h \).

The discrete Green’s function \( G_h \) is now defined as the inverse of \( \Delta^2_h \) (considered as a matrix operating on \( \mathbb{R}^{\text{int} \Lambda_h^n} \) with the scalar product \( \langle u_h, v_h \rangle = (u_h, v_h)_{L^2(\mathbb{R}^n)} \)).

Let us also give an alternative description of \( G_h \): The discrete delta function is given as

\[
\delta_{h,x}(y) = \begin{cases} \frac{1}{h^n} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}
\]

The discrete Green’s function \( G_h \) of \( \Lambda_h^n \) is then the function \((h\mathbb{Z})^n \times (h\mathbb{Z})^n \rightarrow \mathbb{R} \) such \( G_h(x, y) = 0 \) when \( y \notin \text{int} \Lambda_h^n \) and such that \( G_h(\cdot, y) \) is the unique solution in \( \Phi_h \) of

\[
\Delta^2_h u_h = \delta_{h,y} \text{ in } \text{int} \Lambda_h^n
\]

when \( y \in \text{int} \Lambda_h^n \).

As in the continuous case one can easily show that \( G_h \) is symmetric in \( x \) and \( y \). We will frequently denote \( G_h(x, \cdot) \) and \( G_h(\cdot, y) \) by \( G_{h,x} \) and \( G_{h,y} \), respectively.

Let us return our attention to (3.2) for a moment. If \( f_h \) is given in divergence form as \( \text{div}_h \text{div}_\cdot g_h \), this equation takes the form

\[
(\nabla^2_h u_h, \nabla^2_h \varphi_h)_{L^2(\mathbb{R}^n)} = (g_h, \nabla^2_h \varphi_h)_{L^2(\mathbb{R}^n)}
\]

and if we choose \( \varphi_h = u_h \), we obtain the energy estimate

\[
\|\nabla^2_h u_h\|_{L^2(\mathbb{R}^n)} \leq \|g_h\|_{L^2(\mathbb{R}^n)}.
\]
3.2 Caccioppoli inequalities

We will need a discrete counterpart of the Cacciopoli (or reverse Poincaré) estimate for biharmonic functions (see e.g. [Cam80] Cap. II, Lemma 1.II). It can be derived by testing $\Delta^2_h u_h = 0$ with $\eta_h u_h$ for a suitable cut-off function $\eta_h$ and some manipulations of the error terms.

Lemma 3.1. Let $u_h \in \Phi_h$, $x \in (h\mathbb{Z})^n$, $r > 0$ and assume that $\Delta^2_h u_h(y) = 0$ for all $y \in Q^h_{r-h}(x) \cap \text{int} \Lambda_h^n$. Then for any $0 < s \leq r - 4h$ we have

$$\|\nabla^2_h u_h\|_{L^2(Q_r(x))}^2 \leq \frac{C}{(r-s)^4} \|u_h\|_{L^2(Q_s(x))}^2 + \frac{C}{(r-s)^2} \|\nabla_h u_h\|_{L^2(Q_r(x))}^2.$$  

The proof is similar to the continuous case. However, the fact that the discrete chain rule only holds up to translations generates additional error terms. Therefore we will give the somewhat lengthy proof in full detail. The proof is adapted from that of Lemma 2.9 in [Dol93].

Proof. By replacing $r$ by $|r-h|_h + \frac{h}{2}$ and $s$ by $|s-h|_h + \frac{3h}{2}$, we can assume that $r, s \in h\mathbb{Z} + \frac{h}{2}$ and $s \leq r - 3h$.

Choose a discrete cut-off function $\eta_h$ with support in $Q_{r-2h}(x)$ that is 1 on $Q_{s+h}(x)$ and such that $|\nabla^2_h \eta| \leq \frac{C}{(r-s)^2}$ for $\kappa \leq 2$. Note that $\eta^4_h u_h \in \Phi_h$ and $\eta^4_h u_h = 0$ whenever $\Delta^2_h u_h \neq 0$. Thus the weak form of (3.2) with $\varphi_h = \eta^4_h u_h$ is

$$0 = (\Delta^2_h u_h, \eta^4_h u_h)_{L^2(\mathbb{R}^n)} = (\nabla^2_h u_h, \nabla^2_h (\eta^4_h u_h))_{L^2(\mathbb{R}^n)}.$$  

We can expand the right-hand side and obtain

$$0 = (\nabla^2_h u_h, \nabla^2_h (\eta^4_h u_h))_{L^2(\mathbb{R}^n)} = \sum_{i,j} \left( D^4_{i,j} D^4_{-i,j} u_h, \eta^4_h D^4_{i,j} D^4_{-i,j} u_h \right)_{L^2(\mathbb{R}^n)}$$  

$$+ \sum_{i,j} \left( D^4_{i,j} (\eta^4_h)^r_{i,j} D^4_{i,j} u_h + D^4_{i,j} (\eta^4_h)^r_{i,j} D^4_{i,j} u_h \right)_{L^2(\mathbb{R}^n)} + \sum_{i,j} \left( D^4_{i,j} (\eta^4_h)^r_{i,j} D^4_{i,j} u_h \right)_{L^2(\mathbb{R}^n)}.$$  

We can rewrite this as

$$\|\eta^4_h \nabla^2_h u_h\|_{L^2(\mathbb{R}^n)}^2 = \sum_{i,j} \left( \eta^2_h D^4_{i,j} D^4_{-i,j} u_h \right)_{L^2(\mathbb{R}^n)}^2$$  

$$\leq \sum_{i,j} \left( D^4_{i,j} D^4_{-i,j} u_h, D^4_{i,j} (\eta^4_h)^r_{i,j} D^4_{-i,j} u_h \right)_{L^2(\mathbb{R}^n)}.$$
\[
\begin{align*}
&\quad + \sum_{i,j} \left( D^h_{i} D^h_j u_h, D^h_{i} (\eta^h_i) \tau^h_i D^h_j u_h \right)_{L^2(\mathbb{R}^n)} \\
&\quad + \sum_{i,j} \left( D^h_{i} D^h_j u_h, D^h_{i} D^h_j (\eta^h_i) \tau^h_i \tau^h_j u_h \right)_{L^2(\mathbb{R}^n)}.
\end{align*}
\]

We will estimate the terms on the right-hand side separately.

Using \(\frac{a^3 - b^3}{a-b} = a^2 + ab^2 + b^3\) for \(a = \eta^h \circ \tau^h_j\) and \(b = \eta^h\) we can rewrite the summands of the first term as

\[
\begin{align*}
&\quad \left( D^h_{i} D^h_j u_h, D^h_{i} (\eta^h_i) \tau^h_i D^h_j u_h \right)_{L^2(\mathbb{R}^n)} \\
&= \left( D^h_{i} D^h_j u_h, \left( \eta^h_i \eta^h_j \eta^h_i + \eta^h_i \tau^h_i \eta^h_j + \tau^h_i \eta^h_j \right) D^h_{i} \eta^h_j \tau^h_i D^h_j u_h \right)_{L^2(\mathbb{R}^n)} \\
&= \left( D^h_{i} D^h_j u_h, 4 \eta^h_i \eta^h_j \eta^h_i \tau^h_i \tau^h_j D^h_i u_h \right)_{L^2(\mathbb{R}^n)} \\
&\quad + \left( D^h_{i} D^h_j u_h, \left( \eta^h_i \tau^h_i \eta^h_j - \eta^h_j \right) + \eta^h_i \left( \tau^h_j \eta^h_i - \eta^h_i \right) + \left( \tau^h_i \eta^h_j - \eta^h_i \right) \right) \left( D^h_{i} \eta^h_j \tau^h_i D^h_j u_h \right)_{L^2(\mathbb{R}^n)}.
\end{align*}
\]

The second term here is problematic\(^1\), because it does not contain a factor \(\eta^h_i \tau^h_i \eta^h_j\). We will control it by moving a factor \(\frac{1}{4}\) from the left-hand side to the right-hand side, so that we are no longer taking second derivatives of \(u_h\).

We obtain

\[
\begin{align*}
&\quad \left( D^h_{i} D^h_j u_h, D^h_{i} (\eta^h_i) \tau^h_i D^h_j u_h \right)_{L^2(\mathbb{R}^n)} \\
&= \left( \eta^h_i \eta^h_i \eta^h_i \eta^h_j \eta^h_i \tau^h_i \tau^h_j D^h_i u_h \right)_{L^2(\mathbb{R}^n)} \\
&\quad + \left( D^h_{i} \tau^h_i u_h - D^h_{i} u_h, \left( \eta^h_i \eta^h_j \eta^h_i \eta^h_j \eta^h_i \tau^h_i \tau^h_j D^h_i u_h \right) \right)_{L^2(\mathbb{R}^n)}.
\end{align*}
\]

Therefore, using the Cauchy-Schwarz inequality, \(ab \leq \delta a^2 + \frac{1}{\delta} b^2\) and the pointwise bounds on \(\eta_h\) and its derivatives we get

\[
\begin{align*}
&\quad \sum_{i,j} \left( D^h_{i} D^h_j u_h, D^h_{i} (\eta^h_i) \tau^h_i D^h_j u_h \right)_{L^2(\mathbb{R}^n)} \\
&= \sum_{i,j} \left( \eta^h_i \eta^h_i \eta^h_i \eta^h_j \eta^h_i \tau^h_i \tau^h_j D^h_i u_h \right)_{L^2(\mathbb{R}^n)} \\
&\quad + \sum_{i,j} \left( D^h_{i} \tau^h_i u_h - D^h_{i} u_h, \left( \eta^h_i \eta^h_j \eta^h_i \eta^h_j \eta^h_i \tau^h_i \tau^h_j D^h_i u_h \right) \right)_{L^2(\mathbb{R}^n)}.
\end{align*}
\]

\(^1\)Note that in a continuous setting this term would not occur at all.
and hence the right-hand side of (3.3). Then we obtain

Analogously we can find the same upper bound for the other two terms on the right-hand side of (3.3). Then we obtain

\[
\leq \frac{1}{4} \| \eta_h^2 \nabla_h^2 u_h \|_{L^2(\mathbb{R}^n)}^2 + \sum_{i,j} \left( \left\| \eta_h^2 D_j \eta_h \nabla_h D_{-i}^h u_h \right\|_{L^2(Q_{r-h}(x))}^2 \right)
\]

\[
+ \frac{1}{2} \left( \frac{r-s}{2} \right)^2 \sum_{i,j} \left( \left\| D_{-i}^h D_j^h u_h - D_{-i}^{h} u_h \right\|_{L^2(Q_{r-h}(x))}^2 \right)
\]

\[
+ \frac{(r-s)^2}{2} \sum_{i,j} \left( \left\| \eta_h^2 D_j \eta_h + \eta_h D_j^h (\eta_h^2) + D_j^h (\eta_h^3) \right\|_{L^2(Q_{r-h}(x))}^2 \right)
\]

\[
\leq \frac{1}{4} \| \eta_h^2 \nabla_h^2 u_h \|_{L^2(\mathbb{R}^n)}^2 + \frac{(r-s)}{\tau} \| u_h \|_{L^2(Q_r(x))}^2 + \frac{(r-s)^2}{\tau^2} \| \nabla_h u_h \|_{L^2(Q_r(x))}^2 .
\]

Analogously we can find the same upper bound for the other two terms on the right-hand side of (3.3). Then we obtain

\[
\| \eta_h^2 \nabla_h^2 u_h \|_{L^2(\mathbb{R}^n)}^2
\]

\[
\leq \frac{3}{4} \| \eta_h^2 \nabla_h^2 u_h \|_{L^2(\mathbb{R}^n)}^2 + \frac{C}{(r-s)^2} \| u_h \|_{L^2(Q_r(x))}^2 + \frac{C}{(r-s)^2} \| \nabla_h u_h \|_{L^2(Q_r(x))}^2 .
\]

and hence

\[
\| \eta_h^2 \nabla_h^2 u_h \|_{L^2(\mathbb{R}^n)}^2 \leq \frac{C}{(r-s)^2} \| u_h \|_{L^2(Q_r(x))}^2 + \frac{C}{(r-s)^2} \| \nabla_h u_h \|_{L^2(Q_r(x))}^2 .
\]

This implies the claim, once one notes that

\[
\| \nabla_h^2 u_h \|_{L^2(Q_r(x))} \leq \| \eta_h^2 \nabla_h^2 u_h \|_{L^2(\mathbb{R}^n)} .
\]

\[
\blacksquare
\]

4 Interpolation

We want to deduce discrete estimates from their continuous counterparts using compactness arguments. To do so, we need an interpolation operator that turns discrete functions into continuous functions having similar features. The most important property of this interpolation operator that we require is that the continuous derivatives of the output are comparable to the discrete derivatives of the input.

To construct such an operator we use B-splines (cf., e.g., [Sch81, §4.4]): For \( m \geq 1 \), \( x \in \mathbb{R} \) the \( m \)-th normalized B-spline is given by

\[
N^m(x) = m \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i} \text{max}(x - i, 0)^{m-1}}{m!} .
\]

The function \( N^m \) is piecewise a polynomial of degree \( m - 1 \), has support in \([0, m]\) and satisfies \( \sum_{z \in \mathbb{Z}} N^m(x - z) = 1 \) for all \( x \in \mathbb{R} \). Furthermore its discrete and continuous derivatives are closely related. Indeed we have

\[
\partial_x N^m(x) = N^{m-1}(x) - N^{m-1}(x-1) = D_{-1}^1 N^{m-1}(x) \]

\[(4.1)\]
Proposition 4.2. Let $u$ have just defined, and let $J$ so many properties that we still call $J$ to the lattice $N$.

Definition 4.1. Let $h > 0$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ with $\mu_i \geq 1$ let

$$N_h^{\mu}(x_1, \ldots, x_n) = N_{\mu_1}(\frac{x_1}{h}) \cdots N_{\mu_n}(\frac{x_n}{h}) .$$

It follows easily from (4.1) that for any $\alpha \in \mathbb{N}^n$ with $\alpha_i < \mu_i$ for all $i$ we have

$$D^\alpha N_h^{\mu} = D_{\alpha}^h N_h^{\mu-\alpha} .$$

Using this, we can define our interpolation operator:

Definition 4.1. Let $h > 0$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ with $\mu_i \geq 1$ for all $i$. Define $J_h^{\mu} : \mathbb{R}^{(h\mathbb{Z})^n} \to L_{loc}^1(\mathbb{R}^n)$ by

$$(J_h^{\mu} u_h)(x) = \sum_{z \in (h\mathbb{Z})^n} u_h(z) N_h^{\mu}(x - z)$$

and extend $J_h^{\mu}$ to vector-valued functions component-wise.

Note that $N_h^{\mu}$ has compact support so that the above sum has only finitely many nonzero terms.

$J_h^{\mu}$ does not interpolate the values of $u_h$ (i.e. in general we will not have $J_h^{\mu} u_h(x) = u_h(x)$ for all $x \in (h\mathbb{Z})^n$). The maps $J_h^{\mu} u_h$ and $u_h$, however, share so many properties that we still call $J_h^{\mu}$ an interpolation operator.

Let us collect some properties of $J_h^{\mu}$.

Proposition 4.2. Let $J_h^{\mu}$ be the family of interpolation operators that we have just defined, and let $u_h : (h\mathbb{Z})^n \to \mathbb{R}$.

i) $J_h^{\mu}$ is linear.

ii) $J_h^{\mu} u_h$ is piecewise a polynomial and is in the Sobolev space $W_{loc}^{(\min, \mu) - 1,2}.$

iii) $J_h^{\mu}$ is local in the sense that $(J_h^{\mu} u_h)(x)$ only depends on the values of $u_h$ in $Q_{(\max, \mu_i)}(x)$.

iv) $J_h^{\mu}$ preserves constant functions, i.e. $(J_h^{\mu} c)(x) = c$ for any $c \in \mathbb{R}$ and any $x \in \mathbb{R}^n$.

v) For every $\alpha$ with $\alpha_i < \mu_i$ we have $(D^\alpha J_h^{\mu} u_h)(x) = (J_h^{\mu-\alpha} (D_h^\alpha u_h))(x)$.

vi) For every $\alpha$ with $\alpha_i < \mu_i$ and any $p \in [1, \infty]$ there is a constant $C = C(\mu, \alpha, n, p)$ such that for any $x \in \mathbb{R}^n$ and any $r \geq s + (1 + \max_i \mu_i)h$ we have

$$\|D^\alpha J_h^{\mu} u_h\|_{L_p(Q_s(x))} \leq C \|D_h^\alpha u_h\|_{L_p(Q_r(x))}$$

and

$$\|D_h^\alpha u_h\|_{L_p(Q_s(x))} \leq C \|D^\alpha J_h^{\mu} u_h\|_{L_p(Q_r(x))} .$$
Proof. Properties i), ii) and iii) are obvious. Property iv) easily follows from \(\sum_{z \in \mathbb{Z}} N^{\mu}(x - z) = 1\) for all \(x \in \mathbb{R}\), so it remains to prove v) and vi).

For v), note that we can assume that \(u_h\) is zero far away from \(x\) by iii). This means that all sums in the following calculations have only finitely many nonzero terms. Now, using (4.2), we can calculate that

\[
(D^\alpha J^\mu_h u_h)(x) = D^\alpha \left( \sum_{z \in (h\mathbb{Z})^n} u_h(z) N^\mu_h(x - z) \right) \\
= \sum_{z \in (h\mathbb{Z})^n} u_h(z) D^\alpha N^\mu_h(x - z) \\
= \sum_{z \in (h\mathbb{Z})^n} u_h(z) D^\alpha N^{-\alpha}_h(x - z) \\
= \sum_{z \in (h\mathbb{Z})^n} D^\mu_h u_h(z) N^{-\alpha}_h(x - z) = (J^{\mu - \alpha}_h (D^\mu_h u_h))(x).
\]

Finally we prove vi). In view of v) it is sufficient to consider the case \(\alpha = 0\) here. We can also assume that \(x \in (h\mathbb{Z})^n\) and \(r, s \in h\mathbb{N} + \frac{h}{2}\), \(r \geq s + (\max_1 \mu_i)h\) (otherwise move \(x\) to the nearest lattice point, and replace \(r\) and \(s\) by \(|r - \frac{h}{2}|_h + \frac{h}{2}\) and \(|s - \frac{h}{2}|_h + \frac{3h}{2}\) respectively).

Let \(y \in Q^h_{\mu}(x)\). The definition of \(J^\mu_h\) immediately implies

\[
\|J^\mu_h u_h\|_{L^\infty(Q_{\mu/2}(y))} \leq C \sup_{z \in (h\mathbb{Z})^n} \|u_h(z)\|_{|z-y| \leq (\max_1 \mu_i)h}
\]

and thus

\[
\|J^\mu_h u_h\|^P_{L^P(Q_{\mu/2}(y))} \leq C \sum_{z \in (h\mathbb{Z})^n} \|u_h(z)\|^P_{|z-y| \leq (\max_1 \mu_i)h} \leq C\|u_h\|^P_{L^P(Q_{(\max_1 \mu_i+1/2)}h)(y)}.
\]

If we sum this over all \(y \in Q^h_{\mu}(x)\), we easily obtain (4.3).

For (4.4), by a similar argument it suffices to show

\[
|u_h(y)| \leq C\|J^\mu_h u_h\|_{L^P(Q_{\mu/2}(y))} \tag{4.5}
\]

for all \(y \in Q^h_{\mu}(x)\).

One can see this as follows: \(N^\mu_h\) has support \([0, \mu_1] \times \cdots \times [0, \mu_n]\). This means that the values of \(J^\mu_h u_h\) in \(Q_{\mu/2}(y)\) depend on the finitely many values \(\{u_h(z)\}_{z \in I_y}\), where \(I_y := [y_1 - \mu_1] \times \cdots \times [y_n - \mu_n] \cap (h\mathbb{Z})^n\) and no others. Furthermore by linear independence of the B-splines (see [Sch81] Theorem 4.18 for the one-dimensional case; the \(n\)-dimensional case is analogous) \(J^\mu_h u_h\) is identically 0 in \(Q_{\mu/2}(y)\) only if all \(\{u_h(z)\}_{z \in I_y}\) are 0. This means that
∥J^\mu_h u_h∥_{L^p(Q_h/2(y))} is not only a seminorm on \(\mathbb{R}^I\) but actually a norm. Now all norms on a finite-dimensional vector space are equivalent, so in particular

\[
\|u_h\|_{P(L)} = \left(\sum_{z \in I_y} |u_h(z)|^2 \right)^{\frac{1}{2}} \leq C \|J^\mu_h u_h\|_{L^p(Q_h/2(y))}
\]

for a constant \(C\) that is independent of \(y\). This immediately implies (4.5).

Using these interpolation operators \(J^\mu_h\) we define the two operators that we will actually use most often: One is \(J_h := J_h^{(3,3,\ldots,3)}\) and the other is the matrix interpolation operator \(\tilde{J}_h\) given by \((\tilde{J}_h)_{ij} = J_h^{(3,3,\ldots,3)} - e_i - e_j \circ \tau_h^I\) (for example \((\tilde{J}_h)_{11} = J_h^{(1,3,\ldots,3)} \circ \tau_h^I\)).

One easily checks using parts ii) and v) of Proposition 4.2 that for any \(f_h : (h\mathbb{Z})^n \rightarrow \mathbb{R}\) we have \(J_h f_h \in W^{2,2}_{loc}(\mathbb{R}^n)\) and

\[
\nabla^2 J_h f_h = \tilde{J}_h \nabla^2 f_h .
\]

(4.6)

5 Inner decay estimates for discrete biharmonic functions: special cases

Our goal is to prove an \(L^\infty-L^2\) estimate for discrete biharmonic functions (see Theorem 6.1): If \(u_h \in \Phi, x \in \Lambda_h^n, r > 0\) and \(\Delta_2 u_h(y) = 0\) for all \(y \in Q_{r-h}(x) \cap \text{int} \Lambda_h^n\), then, for all \(z \in Q_{r/2}(x) \cap \Lambda_h^n\),

\[
|\nabla^2 u_h(z)| \leq \frac{C}{r^2} \|\nabla^2 u_h\|_{L^2(Q_r(x))} .
\]

To prove this estimate it will be necessary to distinguish where \(x\) lies in relation to \(\partial \Lambda_h^n\): \(x\) can be far inside \(\Lambda_h^n\), near a face, near an edge or near a vertex. In the following subsections we will study these cases separately and prove some decay estimates that we will then assemble to prove the aforementioned estimate.

5.1 Full space

Lemma 5.1. Let \(u_h : (h\mathbb{Z})^n \rightarrow \mathbb{R}\), let \(x \in (h\mathbb{Z})^n, r > 0\). Suppose \(\Delta_2 u_h(y) = 0\) for all \(y \in Q_{r-h}(x)\). Then

\[
|\nabla^2 u_h(x)| \leq \frac{C}{r^2} \|\nabla^2 u_h\|_{L^2(Q_r(x))} .
\]

The main tool to prove this statement will be the following estimate:
Lemma 5.2. There exist constants $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ with the following property: Let $u_h : (h\mathbb{Z})^n \to \mathbb{R}$, $r > 0$, such that $\Delta_h^2 u_h(y) = 0$ for all $y \in Q_r - h$. Assume that $\rho r \geq M h$. Then we have that

$$
\left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q_{pr}} \right\|^2_{L^2(Q_{pr})} \leq \rho^{n+1} \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q_r} \right\|^2_{L^2(Q_r)}.
$$

We will prove this lemma by contradiction using a compactness argument and the following decay estimate for continuous biharmonic functions:

Lemma 5.3. Let $0 < s \leq \frac{\alpha}{2}$, $u \in W^{2,2}(Q_r)$ such that $\Delta^2 u = 0$ weakly in $Q_r$. Then we have

$$
\left\| \nabla^2 u - [\nabla^2 u]_{Q_s} \right\|^2_{L^2(Q_s)} \leq C \left( \frac{s}{r} \right)^{n+\frac{\alpha}{2}} \left\| \nabla^2 u - [\nabla^2 u]_{Q_r} \right\|^2_{L^2(Q_r)}.
$$

Proof. The estimate (5.1) expresses the fact that the second derivatives of biharmonic functions are in the Campanato space $L^{2, n+\frac{\alpha}{2}} \simeq C^{0, \frac{\alpha}{2}}$. The easiest way to show it is to use Schauder estimates for higher order elliptic equations as follows.

By scaling we can assume $r = 1$. By replacing $u$ with $u - \frac{1}{2} [\nabla^2 u]_{Q_1} : x \otimes x$ we can assume that $[\nabla^2 u]_{Q_1} = 0$. Now by Schauder estimates (see e.g. [Mor66, Theorem 6.4.8] or [Cam80, Cap. II, Teorema 6.1]) we have that any $C^{0, \alpha}$-Hölder seminorm of $\nabla^2 u$ in $Q_{1/2}$ is bounded by the $L^2$-norm of $\nabla^2 u$ in $Q_1$. In particular, we have

$$
[\nabla^2 u]_{C^{0, \frac{\alpha}{2}}(Q_{1/2})} \leq C \left\| \nabla^2 u \right\|_{L^2(Q_1)}.
$$

On the other hand, Jensen’s inequality easily yields that

$$
\left\| \nabla^2 u - [\nabla^2 u]_{Q_s} \right\|^2_{L^2(Q_s)} \leq \frac{1}{|Q_s|} \int_{Q_s} \int_{Q_s} |\nabla^2 u(y) - \nabla^2 u(y')|^2 dy dy' \\
\leq C s^{n+\frac{\alpha}{2}} \left[ \nabla^2 u \right]^{2}_{C^{0, \frac{\alpha}{2}}(Q_{1/2})}.
$$

Together with the previous estimate this yields the result. \hfill \square

We will also need a local version of the well-known Kolmogorov-Riesz-Fréchet compactness theorem.

Lemma 5.4. Let $p \in [1, \infty)$, let $U, V, W \subset \mathbb{R}^n$ be open with $U$ compactly contained in $V$, and $V$ compactly contained in $W$. Let $A$ be a subset of $L^p(W)$.

i) If $A$ is bounded in $L^p(W)$ and

$$
\lim_{\delta \to 0} \sup_{f \in A} \left\| \tau_\delta f - f \right\|_{L^p(V)} = 0
$$

then $A$ (or rather the restriction of the elements of $A$ to $U$) is precompact in $L^p(U)$. 


ii) If $A$ is precompact in $L^p(W)$ then
\[ \limsup_{\delta \to 0} \| \tau_\delta f - f \|_{L^p(V)} = 0. \]

Proof. Part i) follows by applying the usual Kolmogorov-Riesz-Fréchet compactness theorem (see e.g. [Bre11, Corollary 4.27 and Exercise 4.34]) to the family $\{\eta f : f \in A\}$, where $\eta$ is a smooth cut-off function that is 1 on $U$ and 0 outside of $V$.

For part ii) let $\tilde{V}$ be open such that $V$ is compactly contained in $\tilde{V}$ and $\tilde{V}$ is compactly contained in $W$, and let $\zeta$ be a cut-off that is 1 on $\tilde{V}$ and 0 outside of $W$. Then the family $\{\zeta f : f \in A\}$ is precompact in $L^p(\mathbb{R}^n)$ and the statement is obtained by applying the converse of the Kolmogorov-Riesz-Fréchet compactness theorem to that family.

After these preparations we can return to the proofs of Lemma \ref{lem:5.1} and Lemma \ref{lem:5.2}

Proof of Lemma \ref{lem:5.2}

Step 1: Set-up of the compactness argument

Let the constant $\rho \leq \frac{1}{2}$ be fixed later, and suppose that the statement for that fixed $\rho$ is wrong. Then for any $k \in \mathbb{N}$ there exist $M_k \geq k$, $h_k > 0$, $u_{h_k} : (h_k^\mathbb{Z})^n \to \mathbb{R}$, $r_k > 0$ such that
\[
\left\| \nabla^2_{h_k} u_{h_k} - \left[ \nabla^2_{h_k} u_{h_k} \right]_{Q_{rk}} \right\|_{L^2(Q_{rk})}^2 > \rho^{n+1} \left\| \nabla^2_{h_k} u_{h_k} - \left[ \nabla^2_{h_k} u_{h_k} \right]_{Q_{rk}} \right\|_{L^2(Q_{rk})}^2. \tag{5.2}
\]

By rescaling the lattice by a factor of $r_k$, we can assume that all the $r_k$ are equal to 1. Because $h_k \leq \frac{1}{1000}$, we have that $h_k \to 0$. Omitting finitely many $k$, we can assume that all $h_k$ are small (less than $\frac{1}{1000}$, say).

By replacing $u_{h_k}$ with $u_{h_k} - \frac{1}{h_k^2} \left[ \nabla^2_{h_k} u_{h_k} \right]_{Q_1} : x \otimes x$ we can assume that $\left[ \nabla^2_{h_k} u_{h_k} \right]_{Q_1} = 0$, and by scaling we can assume that $\left\| \nabla^2_{h_k} u_{h_k} \right\|_{L^2(Q_1)} = 1$ (note that $\nabla^2_{h_k} u_{h_k}$ cannot be identically 0, as then $u_{h_k}$ would be affine, and so both sides of (5.2) would be 0). Then (5.2) implies that
\[
\left\| \nabla^2_{h_k} u_{h_k} - \left[ \nabla^2_{h_k} u_{h_k} \right]_{Q_{rk}} \right\|_{L^2(Q_{rk})}^2 > \rho^{n+1}. \tag{5.3}
\]

Finally, we replace $u_{h_k}$ by $u_{h_k} - a_k - b_k \cdot x$, where $a_k \in \mathbb{R}$, $b_k \in \mathbb{R}^n$ are constants that will be chosen below (such that equation (5.4) is satisfied). This leaves $\nabla^2_{h_k} u_{h_k}$ unaffected, so all the above statements about $\nabla^2_{h_k} u_{h_k}$ remain true.

We let $v_k = J_{h_k} u_{h_k}$, where $J_{h_k} = J_{h_k}^{(3,...,3)}$ is the interpolation operator introduced in Section 4. From $\left\| \nabla^2_{h_k} u_{h_k} \right\|_{L^2(Q_1)} = 1$ and Proposition \ref{prop:4.2 vii} we immediately conclude that $\left\| \nabla^2 v_k \right\|_{L^3(Q_{1/3,1/4})} \leq C$. 21
Now we choose $a_k$ and $b_k$ in such a way that
\[ [v_k]_{Q_{13/14}} = 0, \quad [\nabla v_k]_{Q_{13/14}} = 0. \]  

(5.4)

The Poincaré inequality on $Q_{13/14}$ implies that
\[ \|v_k\|_{W^{2,2}(Q_{13/14})} \leq C\|\nabla^2 v_k\|_{L^2(Q_{13/14})} \leq C. \]

Therefore the $v_k$ are bounded in $W^{2,2}(Q_{13/14})$ and hence have a subsequence (not relabeled) that converges weakly to some $v \in W^{2,2}(Q_{13/14})$.

**Step 2: $\Delta^2 v = 0$**

We claim that $\Delta^2 v = 0$ weakly in $Q_{13/14}$. To prove this, let $\varphi \in C_c^\infty(Q_{13/14})$ be arbitrary and let $\varphi_{h_k}$ be its restriction to $(h_kZ)^n$. We need to prove that
\[ \int_{Q_{13/14}} \nabla^2 v : \nabla^2 \varphi \, dx = 0. \]

We have by (4.3) that
\[
\int_{Q_{13/14}} \nabla^2 v_k : \nabla^2 \varphi \, dx = \int_{Q_{13/14}} \nabla^2 j_{h_k} u_{h_k} : \nabla^2 \varphi \, dx
\]
\[
= \int_{Q_{13/14}} j_{h_k} \nabla^2 v_k : \nabla^2 \varphi \, dx
\]
\[
= \sum_{i,j=1}^n \int_{Q_{13/14}} \sum_{z \in (h_kZ)^n} N_{h_k}^{(3,3,...,3) - \varepsilon_i - \varepsilon_j} (x-z) D_i^{h_k} D_j^{h_k} u_{h_k}(z) D_i D_j \varphi(x) \, dx
\]
\[
= \sum_{i,j=1}^n \sum_{z \in (h_kZ)^n} D_i^{h_k} D_j^{h_k} u_{h_k}(z) \int_{Q_{13/14}} N_{h_k}^{(3,3,...,3) - \varepsilon_i - \varepsilon_j} (x-z) D_i D_j \varphi(x) \, dx.
\]

Now Taylor expansion and the fact that $\int_{Q_{13/14}} N_{h_k}^{(3,3,...,3) - \delta_i - \delta_j} = 1$ imply that
\[
\int_{Q_{13/14}} N_{h_k}^{(3,3,...,3) - \varepsilon_i - \varepsilon_j} (x-z) D_i D_j \varphi(x) \, dx = D_i D_j \varphi(z) + O(h_k)
\]
\[
= D_i^{h_k} D_j^{h_k} \varphi_{h_k}(z) + O(h_k)
\]

In addition, from $\Delta^2 v_{h_k} u_{h_k} = 0$ in $Q_{13/14}$ we conclude that
\[
\sum_{i,j=1}^n \sum_{z \in (h_kZ)^n} D_i^{h_k} D_j^{h_k} u_{h_k}(z) D_i^{h_k} D_j^{h_k} \varphi_{h_k}(z)
\]
\[
= \sum_{i,j=1}^n \sum_{z \in (h_kZ)^n} D_i^{h_k} D_j^{h_k} u_{h_k}(z) D_i^{h_k} D_j^{h_k} \varphi_{h_k}(z)
\]
= (\nabla^2_{h_k} u_{h_k}, \nabla^2_{h_k} \varphi_{h_k})_{L^2(\mathbb{R}^n)} = 0
and so we obtain
\[
\left| \int_{Q_{13/14}} \nabla^2 v_k : \nabla^2 \varphi \, dx \right| \leq C \| \nabla^2_{h_k} u_{h_k} \|_{L^2(Q_1)} h_k = C h_k.
\]
Using weak convergence of \nabla^2 v_k we can pass to the limit here and get
\[
\int_{Q_{13/14}} \nabla^2 v : \nabla^2 \varphi \, dx = 0.
\]

**Step 3: Strong convergence of v_k**

Let \( w_k = I_{h_k}^{\tau_k} \nabla^2_{h_k} u_{h_k} \). We claim that both \( \nabla^2 v_k \) and \( w_k \) converge strongly in \( L^2(Q_{1/2}) \) to \( \nabla^2 v \).

**Step 3.1: Precompactness of w_k**

We first prove that \((w_k)_{k \in \mathbb{N}}\) is precompact in \( L^2(Q_{1/2}) \).

Because \((\nabla^2_{h_k} u_{h_k})\) is bounded in \( L^2(Q_1) \), \( w_k \) is bounded in \( L^2(Q_1) \). So, according to Lemma 5.4 i), it suffices to verify that
\[
\lim_{|a| \to 0} \sup_{a \in \mathbb{Z}^n, k \in \mathbb{N}} \| \tau_a w_k - w_k \|_{L^2(Q_{5/7})} = 0. \tag{5.5}
\]

Let \( a \in (h\mathbb{Z})^n \) such that \(|a| \leq \frac{1}{7} \). Then \( \Delta^2_{h_k}(\tau_a u_{h_k} - u_{h_k}) = 0 \) in \( Q_{11/14} \), so by the Cacciopoli inequality we obtain
\[
\| \nabla^2_{h_k} (\tau_a u_{h_k} - u_{h_k}) \|_{L^2(Q_{5/7}(x))}^2 \leq C \| \tau_a u_{h_k} - u_{h_k} \|_{L^2(Q_{11/14}(x))}^2 \\
+ C \| \nabla_{h_k} (\tau_a u_{h_k} - u_{h_k}) \|_{L^2(Q_{11/14}(x))}^2.
\]

Here the left-hand side is equal to \( \| \tau_a w_k - w_k \|_{L^2(Q_{5/7})}^2 \), while we can use Proposition 1.2 vi) to bound the right-hand side. We obtain
\[
\| \tau_a w_k - w_k \|_{L^2(Q_{5/7})}^2 \leq C \| \tau_a v_k - v_k \|_{L^2(Q_{6/7}(x))}^2 + C \| \tau_a \nabla v_k - \nabla v_k \|_{L^2(Q_{6/7}(x))}^2.
\]

Recall that \((v_k)\) is bounded in \( W^{2,2}(Q_{13/14}) \). Hence by the compact Sobolev embedding, \((v_k)\) and \((\nabla v_k)\) are precompact in \( L^2(Q_{13/14}) \). Thus by Lemma 5.4 ii),
\[
\lim_{|a| \to 0} \sup_{k \in \mathbb{N}} \left( \| \tau_a v_k - v_k \|_{L^2(Q_{6/7}(x))}^2 + \| \tau_a \nabla v_k - \nabla v_k \|_{L^2(Q_{6/7}(x))}^2 \right) = 0
\]
(note that this expression is defined for all \( a > 0 \), not just those in \((h\mathbb{Z})^n\)).

In particular,
\[
\lim_{\delta \to 0} \sup_{k \in \mathbb{N}} \sup_{a \in (h_k \mathbb{Z})^n} \left( \| \tau_a v_k - v_k \|_{L^2(Q_{6/7}(x))}^2 + \| \tau_a \nabla v_k - \nabla v_k \|_{L^2(Q_{6/7}(x))}^2 \right) = 0
\]

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and therefore
\[ \limsup_{\delta \to 0} \sup_{k \in \mathbb{N}} \sup_{a \in (h_k \mathbb{Z})^n} \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} = 0. \]

It remains to consider shifts \( \tau_a \) where \( a \notin (h_k \mathbb{Z})^n \). This is possible because \( w_k \) is piecewise constant on cubes of sidelength \( h_k \). This easily implies that for any \( a \in \mathbb{R}^n \) we have
\[ \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} \leq C \sup_{b \in (h_k \mathbb{Z})^n} \| \tau_b w_k - w_k \|_{L^2(Q_{5/7}(x))}. \]
Combining this with the previous estimate we find that
\[ \limsup_{\delta \to 0} \sup_{k \in \mathbb{N}} \sup_{a \in \mathbb{R}^n} \sup_{|a| \leq \delta + h_k} \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} = 0. \]

Because \( h_k \to 0 \), this implies
\[ \limsup_{a \in \mathbb{R}^n} \limsup_{k \to \infty} \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} = 0. \tag{5.6} \]

We finally show that (5.6) already implies (5.5). It follows from (5.6) that for every fixed \( \varepsilon > 0 \) there are \( \delta > 0 \), \( K \in \mathbb{N} \) such that \( \sup_{k \geq K} \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} \leq \varepsilon \) for all \( a \) with \( |a| \leq \delta \). For the finitely many \( k < K \), we use that \( \limsup_{a \in \mathbb{R}^n} \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} = 0 \) to see that for a potentially smaller \( \delta' \) we have \( \sup_{k \in \mathbb{N}} \| \tau_a w_k - w_k \|_{L^2(Q_{9/14}(x))} \leq \varepsilon \) for all \( a \) with \( |a| \leq \delta' \).

Therefore the sequence \( (w_k) \) is precompact in \( L^2(Q_{4/7}(x)) \). Choose a subsequence (not relabeled) converging strongly to some \( w \in L^2(Q_{4/7}(x)) \).

**Step 3.2: Strong convergence of \( (\nabla^2 v_k) \) and \( w = \nabla^2 v \)**

We split \( w \) into a smooth part and a part with small \( L^2 \)-norm. Let \( \varepsilon > 0 \) be arbitrary, and choose a \( w^{(\varepsilon)} \) in \( C^\infty_c(Q_{4/7}) \) such that \( \| w - w^{(\varepsilon)} \|_{L^2(Q_{4/7})} \leq \varepsilon \).

We denote the restriction of \( w^{(\varepsilon)} \) to \( (h_k \mathbb{Z})^n \) by \( w^{(\varepsilon)}_{h_k} \). Using Taylor expansion, one immediately verifies that then \( \nabla^2 w^{(\varepsilon)}_{h_k} \) and \( \nabla^2 w^{(\varepsilon)}_{h_k} \) converge to \( w^{(\varepsilon)} \) in \( L^2(Q_{4/7}) \) and \( L^2(Q_{1/2}) \), respectively.

This means in particular that
\[ \lim_{k \to \infty} \| w^{(\varepsilon)}_{h_k} - \nabla^2 w^{(\varepsilon)}_{h_k} \|_{L^2(Q_{4/7})} = \| w^{(\varepsilon)} - w \|_{L^2(Q_{4/7})} \leq \varepsilon. \]

Using Proposition 4.2 vi), we conclude that
\[ \limsup_{k \to \infty} \| \nabla^2 w^{(\varepsilon)}_{h_k} \|_{L^2(Q_{1/2})} \leq C \varepsilon. \]
The left-hand side here equals \( \lim \sup_{k \to \infty} \| w - \nabla^2 v_k \|_{L^2(Q_{1/2})} \), and so we obtain

\[
\lim \sup_{k \to \infty} \| w - \nabla^2 v_k \|_{L^2(Q_{1/2})} \leq C \varepsilon .
\]

Since \( \varepsilon \) was arbitrary, we conclude that \( \nabla^2 v_k \) converges strongly in \( L^2(Q_{1/2}) \) to \( w \). But we already know that \( \nabla^2 v_k \) converges weakly in \( L^2(Q_{13/14}) \) to \( \nabla^2 v \), so we obtain that \( \nabla^2 v = w \) in \( Q_{1/2} \).

**Step 4: Conclusion of the argument**

We proved that \( w_k = I_{h_k}^\infty \nabla^2 u_{h_k} \) converges strongly in \( L^2(Q_{1/2}) \) to \( \nabla^2 v \). Because \( \rho \leq \frac{1}{2} \) then also \( \nabla^2 u_{h_k} - \left[ \nabla^2 u_{h_k} \right]_{Q_\rho} \) converges strongly in \( L^2(Q_{1/2}) \) to \( \nabla^2 v - \left[ \nabla^2 v \right]_{Q_\rho} \), and so from (5.3) we conclude that

\[
\left\| \nabla^2 v - \left[ \nabla^2 v \right]_{Q_\rho} \right\|_{L^2(Q_\rho)}^2 \geq \rho^{n+1}.
\]

In addition we know that \( \| \nabla^2 v_k \|_{L^2(Q_{13/14})} \leq C \) and that \( \nabla^2 v_k \) converges weakly in \( L^2(Q_{13/14}) \) to \( \nabla^2 v \). This implies

\[
\left\| \nabla^2 v - \left[ \nabla^2 v \right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2 \leq \left\| \nabla^2 v \right\|_{L^2(Q_{13/14})}^2 \leq \liminf_{k \to \infty} \left\| \nabla^2 v_k \right\|_{L^2(Q_{13/14})}^2 \leq C.
\]

In summary, we have proved that there is a constant \( C_1 \) independent of \( \rho \) such that

\[
\left\| \nabla^2 v - \left[ \nabla^2 v \right]_{Q_\rho} \right\|_{L^2(Q_\rho)}^2 \geq \frac{\rho^{n+1}}{C_1} \left\| \nabla^2 v - \left[ \nabla^2 v \right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2 .
\] (5.7)

On the other hand, \( \Delta^2 v = 0 \) in \( Q_{13/14} \), and thus Lemma [5.3] implies that

\[
\left\| \nabla^2 v - \left[ \nabla^2 v \right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2 \leq C_2 \left( \frac{\rho}{14} \right)^{n+\frac{5}{2}} \left\| \nabla^2 v - \left[ \nabla^2 v \right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2
\]

for a constant \( C_2 \) independent of \( \rho \).

This is a contradiction to (5.7) provided that we choose \( \rho \) small enough, namely \( \rho < \frac{1}{C_1 C_2} \left( \frac{125}{13} \right)^{2n+5} \). So we finally fix a \( \rho \) satisfying this condition, and proved that falsity of the claim leads to a contradiction. \( \square \)

Now we can return to Lemma 5.1

**Proof of Lemma 5.1.** We can assume w.l.o.g. that \( x = 0 \).

We claim that for any \( 0 < s' \leq s \leq r \) we have

\[
\left\| \nabla^2 u_h - \left[ \nabla^2 u_h \right]_{Q_{s'}} \right\|_{L^2(Q_{s'})}^2 \leq C \left( \frac{s'}{s} \right)^{n+1} \left\| \nabla^2 u_h - \left[ \nabla^2 u_h \right]_{Q_{s}} \right\|_{L^2(Q_{s})}^2 .
\] (5.8)
To prove this estimate, observe first that we can assume \( s' \geq \frac{h}{2} \), as otherwise the left-hand side is 0. We can also assume \( \frac{s}{\rho} \geq \frac{2M}{\kappa} \) (where \( M \) is the constant from Lemma 5.2), as otherwise we can trivially estimate

\[
\left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,s'} \right\|_{L^2(Q,s')}^2 \leq C \left( \frac{2M}{\kappa} \right)^{n+1} \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,s} \right\|_{L^2(Q,s')}^2,
\]

which holds for \( C \geq \left( \frac{2M}{\kappa} \right)^{n+1} \).

So we assume \( s' \geq \frac{h}{2} \) and \( \frac{s}{\rho} \geq \frac{2M}{\kappa} \). Then in particular \( s \geq Mh \). Consider the \( \rho \) from Lemma 5.2 and let \( \kappa \) be the largest integer such that \( \rho^\kappa s \geq \max(s',Mh) \). We can then apply Lemma 5.2 repeatedly with radii \( s, \rho s, \ldots, \rho^\kappa s \) to find

\[
\left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,\rho^\kappa s} \right\|_{L^2(Q,\rho^\kappa s)}^2 \leq \rho^\kappa(n+1) \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,s} \right\|_{L^2(Q,s)}^2.
\]

Because \( s' \leq \rho^\kappa s \), we also have

\[
\left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,s'} \right\|_{L^2(Q,s')}^2 \leq \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,\rho^\kappa s} \right\|_{L^2(Q,\rho^\kappa s)}^2.
\]

Here we have used the fact that \( \| f - [f]_{\Omega} \|_{L^2(\Omega)} \) is monotone in \( \Omega \). If we combine the last two estimates and observe that \( \rho^{\kappa+1} s < \max(s',Mh) \leq 2Ms' \), i.e. \( \rho^\kappa \leq \frac{2M}{\rho} s' \), we indeed obtain (5.8) with \( C = \left( \frac{2M}{\rho} \right)^{n+1} \).

Now using (5.8) to prove the lemma is a standard iteration argument as e.g. in [Gia93 Theorem 3.1]. For the sake of completeness we sketch the proof.

If we apply (5.8) with \( s = r \) and \( s' = \frac{r}{2} \) or \( s' = \frac{r}{2^{\lambda+1}} \), we can estimate

\[
\left\| [\nabla_h^2 u_h]_{Q,r/2^{\lambda+1}} - [\nabla_h^2 u_h]_{Q,r/2^{\lambda}} \right\|_{L^2(Q,r/2^{\lambda+1})}^2 \leq 2 \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,r/2^{\lambda}} \right\|_{L^2(Q,r/2^{\lambda})}^2 + 2 \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,r/2^{\lambda+1}} \right\|_{L^2(Q,r/2^{\lambda+1})}^2 \leq \frac{C}{2^{\lambda(n+1)}} \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,r} \right\|_{L^2(Q,r)}^2
\]

and hence

\[
\left\| [\nabla_h^2 u_h]_{Q,r/2^{\lambda+1}} - [\nabla_h^2 u_h]_{Q,r/2^{\lambda}} \right\|_{L^2(Q,r)} \leq \frac{C}{r^{n+1}} \left\| \nabla_h^2 u_h - [\nabla_h^2 u_h]_{Q,r} \right\|_{L^2(Q,r)}^2.
\]
If we sum this for $\lambda = 0, 1, \ldots$ and observe that for $\lambda$ small enough $[\nabla^2_h u_h]_{Q_{r/2^\lambda}} = \nabla^2_h u_h(0)$ we obtain

$$\left| \nabla^2_h u_h(0) - [\nabla^2_h u_h]_{Q_r} \right| \leq \frac{C}{r^\alpha} \left\| \nabla^2_h u_h - [\nabla^2_h u_h]_{Q_r} \right\|_{L^2(Q_r)}.$$ 

Now we can estimate

$$\left| \nabla^2_h u_h(0) \right|^2 \leq 2 \left| \nabla^2_h u_h(0) - [\nabla^2_h u_h]_{Q_r} \right|^2 + 2 \left( \left\| \nabla^2_h u_h \right\|_{L^2(Q_r)} \right)^2 \leq \frac{C}{r^\alpha} \left( \left\| \nabla^2_h u_h - [\nabla^2_h u_h]_{Q_r} \right\|_{L^2(Q_r)} + \left\| [\nabla^2_h u_h]_{Q_r} \right\|_{L^2(Q_r)} \right) = \frac{C}{r^\alpha} \left\| \nabla^2_h u_h \right\|_{L^2(Q_r)}^2,$$

which proves the claim. 

\[\square\]

### 5.2 Half-space

In the half-space we want to prove the following statement, which is a slightly weaker analogue of Lemma [5.1](#).

**Lemma 5.5.** Let $u_h : (h\mathbb{Z})^n \to \mathbb{R}$, let $x \in (h\mathbb{Z})^n$, $r > 0$, $\nu \in \{e_1, -e_1, \ldots, e_n, -e_n\}$. Suppose that $u_h(y) = 0$ for all $y \in Q_r^h(x)$ such that $(y - x) \cdot \nu \leq 0$, and $\Delta^2_h u_h(y) = 0$ for all $y \in Q_{r-h}^h(x)$ such that $(y - x) \cdot \nu > 0$. Then, for any $s \leq r$,

$$\left\| \nabla^2_h u_h \right\|_{L^2(Q_s(x))} \leq C \left( \frac{s}{r} \right)^{\frac{n}{2}} \left\| \nabla^2_h u_h \right\|_{L^2(Q_r(x))}.$$

The proof is mostly similar to that, Lemma 5.1, so we only give details where a new idea is required.

For $r > 0$ let $Q_{r,+} = Q_r \cap \{x_1 > 0\}$. The main step in the proof of Lemma 5.3 will be to prove the following estimate.

**Lemma 5.6.** There exist constants $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ with the following property: Let $u_h : (h\mathbb{Z})^n \to \mathbb{R}$, $r > 0$ be such that $u_h(y) = 0$ whenever $y \in Q_r^h$ and $y_1 \leq 0$, and $\Delta^2_h u_h(y) = 0$ for all $y \in Q_{r-h}^h$ such that $y_1 > 0$. Assume that $\rho r \geq M h$. Then we have

$$\left\| \nabla^2_h u_h - \left[ D_{-1}^- D_1^h u_h \right]_{Q_{\rho r,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{\rho r,+})} \leq \rho^{n+1} \left\| \nabla^2_h u_h - \left[ D_{-1}^- D_1^h u_h \right]_{Q_{r,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{r,+})}^2.$$

Using a compactness argument, we will deduce this estimate from the following continuous estimate.

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Lemma 5.7. Let $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q_{r,+})$. Assume that $\Delta^2 u = 0$ weakly in $Q_{r,+}$ and that $u = 0$, $D_1 u = 0$ on $\partial Q_{r,+} \cap \{x_1 = 0\}$ in the sense of traces. Then we have

$$\left\| \nabla^2 u - [D_1^2 u]_{Q_{r,+}} e_1 \otimes e_1 \right\|^2_{L^2(Q_{r,+})} \leq C \left( \frac{s}{n} \right)^{n+\frac{2}{n}} \left\| \nabla^2 u - [D_1^2 u]_{Q_{r,+}} e_1 \otimes e_1 \right\|^2_{L^2(Q_{r,+})}.$$

**Proof.** This follows like Lemma 5.3 from Schauder estimates up to the boundary (cf. [Mor66, Theorem 6.4.8]).

**Proof of Lemma 5.6.**

**Step 1: Preparations**

We follow the same strategy as in the proof of Lemma 5.2. That is, we assume that the claim is wrong for some fixed $\rho$, and consider a sequence of counterexamples $u_{h_k}$ and their interpolations $v_k = I_{h_k} u_{h_k}$. We can assume that $r_k = 1$.

Next observe that for $\omega_h(x) := \begin{cases} \frac{x_1(x_1+h)}{2} & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases}$ we have $\omega_h(x) = 0$ if $x_1 \leq 0$ and $D_h^1 D_h^2 \omega_h(x) = \begin{cases} 1 & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases}$. So by replacing $u_k$ with $u_k - [D_h^1 D_h^2 u_k]_{Q_{1,+}} \omega_h$ we can also assume $[D_h^1 D_h^2 u_k]_{Q_{1,+}} = 0$. Having normalized $u_k$ on $Q_{1,+}$ in this way, we now consider $Q_1$ again. We can assume

$$\left\| \nabla^2 u_{h_k} \right\|^2_{L^2(Q_1)} = 1.$$ (5.9)

Note that

$$\left\| \nabla^2 u_{h_k} \right\|^2_{L^2(Q_1)} = \left\| \nabla^2 u_{h_k} \right\|^2_{L^2(Q_{1,+})} + \left\| \nabla^2 u_{h_k} \right\|^2_{L^2((-h/2,0) \times (-1,1)^{n-1})}$$

and

$$\left\| \nabla^2 u_{h_k} \right\|^2_{L^2((-h/2,0) \times (-1,1)^{n-1})} = \left\| \nabla^2 u_{h_k} \right\|^2_{L^2((0,h/2) \times (-1,1)^{n-1})}.$$

Now (5.9) implies that $\left\| \nabla^2 u_{h_k} \right\|^2_{L^2(Q_{1,+})} \geq \frac{1}{2}$, so that

$$\left\| \nabla^2 u_{h_k} - [D_h^1 D_h^2 u_{h_k}]_{Q_{1}} e_1 \otimes e_1 \right\|^2_{L^2(Q_1)} > \frac{\rho^{n+1}}{2}. (5.10)$$

By (5.9), Proposition 4.2 and the Poincaré inequality with 0 boundary values $(v_k)$ is bounded in $W^{2,2}(Q_{3/4})$, and so a non-relabeled subsequence converges weakly to some $v$ in $W^{2,2}(Q_{3/4})$.

As in step 2 of the proof of Lemma 5.2 we can show that $\Delta^2 v = 0$ weakly in $Q_{3/4,+}$. We have $u_{h_k} = 0$ in $Q_1 \cap \{x_1 < 0\}$ and hence $v_k = 0$ in
Furthermore, we claim that for any \( y \in Q_{1/2} \cap \{ x_1 = 0 \} \) we have

\[
\lim_{\tilde{r} \to 0} \lim_{k \to \infty} \sup \| \nabla^2 h_k u_{h_k} \|_{L^2(Q_{\tilde{r}}(y))} = 0. \tag{5.11}
\]

To see this, let \( \tilde{r} > 0 \). For \( h_k \) small enough Lemma 3.1 and Proposition 1.2 imply that

\[
\| \nabla^2 h_k u_{h_k} \|_{L^2(Q_{\tilde{r}}(y))}^2 \leq \frac{C}{\tilde{r}^2} \| \nabla h_k u_{h_k} \|_{L^2(Q_{2\tilde{r}}(y))}^2 + \frac{C}{\tilde{r}^4} \| u_{h_k} \|_{L^2(Q_{2\tilde{r}}(y))}^2
\]

\[
\leq \frac{C}{\tilde{r}^2} \| \nabla v_k \|_{L^2(Q_{4\tilde{r}}(y))}^2 + \frac{C}{\tilde{r}^4} \| v_k \|_{L^2(Q_{4\tilde{r}}(y))}^2.
\]

Now \( v_k \) converges to \( v \) weakly in \( W^{2,2}(Q_{3/4}) \), so \( v_k \) and \( \nabla v_k \) converge strongly in \( L^2(Q_{3/4}) \). Hence we can pass to the limit in the above inequality and find

\[
\limsup_{k \to \infty} \| \nabla^2 h_k u_{h_k} \|_{L^2(Q_{\tilde{r}}(y))}^2 \leq \frac{C}{\tilde{r}^2} \| \nabla v \|_{L^2(Q_{4\tilde{r}}(y))}^2 + \frac{C}{\tilde{r}^4} \| v \|_{L^2(Q_{4\tilde{r}}(y))}^2.
\]

Furthermore \( v \) is 0 in \( Q_{4\tilde{r}}(y) \cap \{ x_1 < 0 \} \), so we can apply the Poincaré inequality to conclude

\[
\limsup_{k \to \infty} \| \nabla^2 h_k u_{h_k} \|_{L^2(Q_{\tilde{r}}(y))}^2 \leq C \| \nabla^2 v \|_{L^2(Q_{4\tilde{r}}(y))}^2.
\]

Now \( \nabla^2 v \) is a fixed \( L^2 \)-function, so if we pass to the limit \( \tilde{r} \to 0 \) here, we indeed obtain (5.11).

It is easy to see that (5.11) together with the fact that \( w_k = I_{h_k}^{pc} \nabla^2 h_k u_{h_k} \) converges to \( \nabla^2 v \) strongly in \( L^2_{loc}(Q_{5/8} \setminus \{ x_1 = 0 \}) \) imply that \( w_k \) actually converges to \( \nabla^2 v \) strongly in \( L^2(Q_{1/2}) \).

We have

\[
\limsup_{k \to \infty} \| \nabla^2 v_k \|_{L^2(Q_{\tilde{r}}(y))} \leq C \limsup_{k \to \infty} \| \nabla^2 h_k u_{h_k} \|_{L^2(Q_{2\tilde{r}}(y))}.
\]
and so from (5.11) we also conclude
\[
\lim_{r \to 0} \limsup_{k \to \infty} \|\nabla^2 v_k\|_{L^2(Q_r(y))} = 0.
\]

This in turn implies that also \(\nabla^2 v_k\) converges to \(\nabla^2 v\) strongly in \(L^2(Q_{1/2})\).

**Step 3: Conclusion of the argument**

We can now continue as in Step 4 of the proof of Lemma 5.2. The strong convergence of \(w_k\) to \(\nabla^2 v\) allows us to conclude from (5.10) that
\[
\left\| \nabla^2 v - \left[ D_{1}^2 v \right]_{Q_{s/3,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{s/3,+})}^2 \leq \rho^{n+1} \]

On the other hand, we have
\[
\left\| \nabla^2 v - \left[ D_{1}^2 v \right]_{Q_{s/4,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{s/4,+})}^2 \leq C
\]

and it is easy to check that we arrive at a contradiction to Lemma 5.7 once we choose \(\rho\) small enough.

**Proof of Lemma 5.6** The proof is similar to the first half of the proof of Lemma 5.1. One can assume that \(x = 0, \nu = e_1\). Then one first proves that, for any \(0 < s' \leq s \leq r\),
\[
\left\| \nabla^2 {\tilde{u}}_{uh} - \left[ D_{1}^h {\tilde{u}}_{uh} \right]_{Q_{s'/3,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{s'/3,+})}^2 \leq C \left( \frac{s'}{s} \right)^{n+1} \left\| \nabla^2 {\tilde{u}}_{uh} - \left[ D_{1}^h {\tilde{u}}_{uh} \right]_{Q_{s,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{s,+})}^2,
\]

which already looks similar to the claimed estimate. We can again use this with \(s = r\) and \(s' = \frac{\sqrt{r}}{2^k}\) or \(s' = \frac{\sqrt{r}}{2^{k+1}}\) to conclude
\[
\left| \left[ D_{-1}^h {D}_{1}^h u_{uh} \right]_{Q_{r/2^{k+1},+}} - \left[ D_{-1}^h {D}_{1}^h u_{uh} \right]_{Q_{r/2^{k},+}} \right| \leq C \left( \frac{r}{2^k} \right)^2 \left\| \nabla^2 {\tilde{u}}_{uh} - \left[ D_{1}^h {\tilde{u}}_{uh} \right]_{Q_{r,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{r,+})}.
\]

Let \(\lambda_0\) be the largest integer such that \(\frac{r}{2^k} \geq s\). We can apply this estimate with radii \(r, \frac{r}{2}, \ldots, \frac{r}{2^\lambda_0}\) and sum to conclude
\[
\left| \left[ D_{-1}^h {D}_{1}^h u_{uh} \right]_{Q_{r/2^{\lambda_0},+}} - \left[ D_{-1}^h {D}_{1}^h u_{uh} \right]_{Q_{r,+}} \right| \leq C \left( \frac{r}{2^k} \right)^2 \left\| \nabla^2 {\tilde{u}}_{uh} - \left[ D_{1}^h {\tilde{u}}_{uh} \right]_{Q_{r,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{r,+})}.
\]

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Using all this, we can estimate

$$\|\nabla_h^2 u_h\|_{L^2(Q_{r,+})}^2 \leq \|\nabla_h^2 u_h\|_{L^2(Q_{r/2\ell_0,+})}^2$$

\[
\leq 2 \left\| \nabla_h^2 u_h - \left[ D_{-1}^h D_1^h u_h \right]_{Q_{r/2\ell_0,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{r/2\ell_0,+})}^2 \\
+ 2 \left\| \left[ D_{-1}^h D_1^h u_h \right]_{Q_{r,+}} \right\|_{L^2(Q_{r,+})}^2 \\
\leq \left( \frac{C}{2\lambda_0(n+1)} + \frac{C}{2\lambda_0} \right) \left\| \nabla_h^2 u_h - \left[ D_{-1}^h D_1^h u_h \right]_{Q_{r,+}} e_1 \otimes e_1 \right\|_{L^2(Q_{r,+})}^2 \\
+ \frac{C}{2\lambda_0} \left\| \left[ D_{-1}^h D_1^h u_h \right]_{Q_{r,+}} \right\|_{L^2(Q_{r,+})}^2 \\
\leq \frac{C}{2\lambda_0} \left\| \nabla_h^2 u_h \right\|_{L^2(Q_{r,+})}^2,
\]

which implies

$$\|\nabla_h^2 u_h\|_{L^2(Q_{r,+})}^2 \leq C \left( \frac{r}{s} \right)^n \|\nabla_h^2 u_h\|_{L^2(Q_{r,+})}^2 \\
\leq C \left( \frac{r}{s} \right)^n \|\nabla_h^2 u_h\|_{L^2(Q_r)}^2 . \tag{5.12}$$

Now by the same argument as in Step 1 of the proof of Lemma 5.6 we have

$$\|\nabla_h^2 u_h\|_{L^2(Q_r)}^2 \leq 2 \|\nabla_h^2 u_h\|_{L^2(Q_{r,+})}^2 .$$

Combining this with (5.12) yields the result. \qed

5.3 Edges and vertices

It remains to prove the analogue of Lemma 5.5 near edges (in 3D) and vertices (in 2D and 3D). The actual compactness argument requires no new idea, so we will only give a very brief sketch of the proofs. However, this time the continuous estimate require a bit more work, so we will go into detail there. Let us first state the two results:

**Lemma 5.8.** Let \( u_h : (h\mathbb{Z})^3 \to \mathbb{R} \), let \( x \in (h\mathbb{Z})^3, r > 0, \nu_1, \nu_2 \in \{ e_1, -e_1, \ldots, e_n, -e_n \} \) such that \( \nu_1 \neq \pm \nu_2 \). Suppose that \( u_h(y) = 0 \) for all \( y \in Q^h_r(x) \) such that \( (y-x) \cdot \nu_1 \leq 0 \) or \( (y-x) \cdot \nu_2 \leq 0 \), and \( \Delta^2 u_h(y) = 0 \) for all \( y \in Q^h_{r-h}(x) \) such that \( (y-x) \cdot \nu_1 > 0 \) and \( (y-x) \cdot \nu_2 > 0 \). Then, for any \( s \leq r \),

$$\|\nabla_h^2 u_h\|_{L^2(Q_r(x))} \leq C \left( \frac{s}{r} \right)^{\frac{3}{2}} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))} .$$
Lemma 5.9. Let $n = 2$ or $n = 3$, $u_h : (h\mathbb{Z})^n \to \mathbb{R}$, let $x \in (h\mathbb{Z})^n$, $r > 0$, $\nu_i \in \{e_i, -e_i\}$ for $i \in \{1, \ldots, n\}$. Suppose that $u_h(y) = 0$ for all $y \in Q^h_r(x)$ such that $(y - x) \cdot \nu_i \leq 0$ for at least one $i$, and $\Delta^2 u_h(y) = 0$ for all $y \in Q^h_{r-n}(x)$ such that $(y - x) \cdot \nu_i > 0$ for all $i$. Then, for any $s \leq r$,

$$\|\nabla^2 u_h\|_{L^2(Q^s_r(x))} \leq C \left( \frac{s}{r} \right)^{\frac{3}{2}} \|\nabla^2 u_h\|_{L^2(Q^s_r(x))}.$$  

Proof of Lemma 5.8 and Lemma 5.9. This follows easily from the following two lemmata.  

Lemma 5.10. There are constants $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ with the following property: let $u_h : (h\mathbb{Z})^3 \to \mathbb{R}$, $r > 0$, such that $u_h(y) = 0$ for all $y \in Q^h_r$ such that $y_1 \leq 0$ or $y_2 \leq 0$, and $\Delta^2 u_h(y) = 0$ for all $y \in Q^h_{r-n}(x)$ such that $y_1 > 0$ and $y_2 > 0$. Then we have that

$$\|\nabla^2 u_h\|_{L^2(Q^s_r)} \leq \rho^{n} \|\nabla^2 u_h\|_{L^2(Q^s_r)}.$$  

Lemma 5.11. There are constants $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ with the following property: let $n = 2$ or $n = 3$, $u_h : (h\mathbb{Z})^n \to \mathbb{R}$, $r > 0$, such that $u_h(y) = 0$ for all $y \in Q^h_r$ such that $y_i \leq 0$ for at least one $i \in \{1, \ldots, n\}$, and $\Delta^2 u_h(y) = 0$ for all $y \in Q^h_{r-n}(x)$ such that $y_i > 0$ for all $i$. Assume that $\rho r \geq Mh$. Then we have that

$$\|\nabla^2 u_h\|_{L^2(Q^s_r)} \leq \rho^n \|\nabla^2 u_h\|_{L^2(Q^s_r)}.$$  

We will deduce these two lemmata from the following continuous estimates. $D_{\nu}$ denotes the derivative in normal direction.

Lemma 5.12. There is a constant $\theta > 0$ with the following property: let $n = 3$, $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q^s_{r,+})$, where $Q^s_r = Q_r \cap \{x_1 > 0, x_2 > 0\}$. Assume that $\Delta^2 u = 0$ weakly in $Q^s_{r,+}$ and that $u = 0$, $D_{\nu} u = 0$ on $\partial Q^s_{r,+} \cap \{x_1 = 0 \vee x_2 = 0\}$ in the sense of traces. Assume that $\rho r \geq Mh$. Then we have

$$\|\nabla^2 u\|_{L^2(Q^s_{r,+})} \leq C \left( \frac{s}{r} \right)^{3+\theta} \|\nabla^2 u\|_{L^2(Q^s_{r,+})}.$$  

Lemma 5.13. There is a constant $\theta > 0$ with the following property: let $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q^s_{r,n+})$, where $Q^s_{r,n+} = Q^s_{r,+} = Q^s_r \cap \{x_1 > 0, x_2 > 0\}$ if $n = 2$, and $Q^s_{r,n+} = Q^s_{r,+} = Q^s_r \cap \{x_1 > 0, x_2 > 0, x_3 > 0\}$ if $n = 3$. Assume that $\Delta^2 u = 0$ weakly in $Q^s_{r,n+}$ and that $u = 0$, $D_{\nu} u = 0$ on $\partial Q^s_{r,n+} \cap \{x_i = 0\}$ for some $i$ in the sense of traces. Then we have

$$\|\nabla^2 u\|_{L^2(Q^s_{r,n+})} \leq C \left( \frac{s}{r} \right)^{n+\theta} \|\nabla^2 u\|_{L^2(Q^s_{r,n+})}.$$
The proof of Lemma 5.12 and Lemma 5.13 relies heavily on the theory of elliptic equations in domains with singularities. We use results from [KMR97] and [MR10] and refer the reader to these monographs for more background information.

**Proof of Lemma 5.12.** Let \( \mathbb{R}^3_{++} = \mathbb{R}^3 \cap \{x_1 > 0, x_2 > 0\} \). For \( x \in \mathbb{R}^3_{++} \) write \( x = (x', x_3) \).

The statement is trivial if \( s \geq \frac{3}{2} \), so assume \( s < \frac{3}{2} \). Let \( \eta \in C_c^{\infty}(Q_r) \) be a cut-off function that is 1 on \( Q_{r/2,++} \), and such that \( |\nabla \eta| \leq \frac{C}{r^2} \). Then \( \eta \Delta^2 u = 0 \) in \( \partial \mathbb{R}^3_{++} \), and we can calculate (as an identity in the sense of distributions) that

\[
\Delta^2(\eta u) = (\Delta^2 \eta) u + 4 \nabla \Delta \eta \cdot \nabla u + 2 \Delta \eta \Delta u + 4 \nabla^2 \eta \cdot \nabla^2 u + 4 \nabla \eta \cdot \nabla \Delta u.
\]

In order to avoid terms with too many derivatives of \( u \) we rewrite the last term as

\[
\nabla \eta \cdot \nabla \Delta u = \text{div}(\nabla \eta \Delta u) - \Delta \eta \Delta u
\]

to obtain

\[
\Delta^2(\eta u) = \Delta^2 \eta u + 4 \nabla \Delta \eta \cdot \nabla u - 2 \Delta \eta \Delta u + 4 \nabla^2 \eta \cdot \nabla^2 u + 4 \text{div}(\nabla \eta \Delta u) =: f.
\]

Because \( u \in W^{2,2}(Q_{r,++}) \) with zero boundary values on \( \partial \mathbb{R}^3_{++} \), the right-hand side \( f \) is an element of \( W^{-2,2}(\mathbb{R}^3_{++}) \), while \( \eta u \) is in \( W^{2,2}_0(\mathbb{R}^3_{++}) \). Hence (cf. [MR10], Theorem 2.5.1) we can represent \( \eta u \) via the Green’s function of \( \mathbb{R}^3_{++} \), as

\[
(\eta u)(x) = \int_{\mathbb{R}^3_{++}} G(x, \xi) f(\xi) \, d\xi.
\]

For \( x \in Q_{s,++} \subset Q_{r/4,++} \) this implies

\[
\nabla^2 u(x) = \int_{\mathbb{R}^3_{++}} \nabla^2 G(x, \xi) f(\xi) \, d\xi.
\]

Now \( f \) is supported in \( Q_{r,++} \setminus Q_{r/2,++} \), whereas \( x \in Q_{s,++} \subset Q_{r/4,++} \). So a decay estimate for \( G \) will directly lead to a pointwise estimate for \( \nabla^2 u \).

In fact, Theorem 2.5.4 in [MR10] states that if \( |x - \xi| \geq \min(|x'|, |\xi'|) \) we have, for every \( \varepsilon > 0 \),

\[
|D^\alpha_{x'} D^j_{x_3} D^\beta_\xi D^k_\xi G(x, \xi)| \leq C_\varepsilon \frac{|x'|^{1+\delta_+ - |\alpha| - \varepsilon} |\xi'|^{1+\delta_- - |\beta| - \varepsilon}}{|x - \xi|^{1+\delta_+ + \delta_- + j + k - 2\varepsilon}}. \tag{5.13}
\]

Here \( \delta_+ \) and \( \delta_- \) are certain real parameters defined in terms of eigenvalue problems related to the bilaplacian (see [MR10] Section 2.4 for the precise definition). According to [MR10] Section 4.3 we have that \( \delta_+ = \delta_- \approx 2.73959 \). In particular, \( \delta_+ > 1 \), so we can choose \( \theta > 0 \) such that \( 1 + \frac{\theta}{2} < \delta_+ \). Then let \( \varepsilon = \delta_+ - 1 - \frac{\theta}{2} > 0 \).

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We are interested in the case where \( x \in Q_{s,++}, \xi \in Q_{r,++} \setminus Q_{r/2,++} \). In that case the inequality \(|x - \xi| \geq \min(|x'|, |\xi'|)\) certainly holds, and we can estimate \(|x'| \leq s, |\xi'| \leq r, |x - \xi| \geq r\), so that (5.13) turns into

\[
|D_x^\alpha D_{x_3}^j D_\xi^\beta D_{\xi_3}^k G(x, \xi)| \leq C \frac{s^{1 + \delta_+ - |\alpha| - \varepsilon} r^{2 - |\beta|} - j - k}{r^{1 + \frac{\theta}{2} + |\beta| + j + k}}.
\]

This estimate is sharp enough to allow us to estimate the terms of \( f \). For example we can calculate using the Poincaré and Hölder inequality that

\[
\left| \int_{\mathbb{R}^3_{++}} \nabla^2 G(x, \xi) \Delta^2 \eta u(\xi) \, d\xi \right| \leq C \int_{Q_{r,++}} \frac{s^{\frac{\theta}{2} - 2}}{r^{1 + \frac{\theta}{2} + 0 + 0}} \frac{1}{r^1} |u(\xi)| \, d\xi
\]

\[
= C \frac{s^{\frac{\theta}{2}}}{r^{1 + \frac{\theta}{2}}} \int_{Q_{r,++}} |u| \, d\xi
\]

\[
\leq C \frac{s^{\frac{\theta}{2}}}{r^{1 + \frac{\theta}{2}}} r^2 \left( \int_{Q_{r,++}} |\nabla^2 u|^2 \, d\xi \right)^{\frac{1}{2}}
\]

\[
\leq C \frac{s^{\frac{\theta}{2}}}{r^{1 + \frac{\theta}{2}}} \left( \int_{\mathbb{R}^3_{++}} |\nabla^2 u|^2 \, d\xi \right)^{\frac{1}{2}}
\]

and that

\[
\left| \int_{\mathbb{R}^3_{++}} \nabla^2 G(x, \xi) \text{div}(\nabla \eta \Delta u)(\xi) \, d\xi \right| = \left| \int_{\mathbb{R}^3_{++}} \nabla^2 G(x, \xi) \cdot \nabla \eta(\xi) \Delta u(\xi) \, d\xi \right|
\]

\[
\leq C \int_{Q_{r,++}} \frac{s^{\frac{\theta}{2}}}{r^{1 + \frac{\theta}{2}}} \frac{1}{r} |\Delta u(\xi)| \, d\xi
\]

\[
\leq C \frac{s^{\frac{\theta}{2}}}{r^{1 + \frac{\theta}{2}}} \left( \int_{\mathbb{R}^3_{++}} |\nabla^2 u|^2 \, d\xi \right)^{\frac{1}{2}}.
\]

We can estimate the other terms on \( f \) analogously. If we integrate the sum of the squares of all these inequalities with respect to \( x \) we immediately obtain the conclusion.

**Proof of Lemma 5.13** The proof in the case of a vertex is very similar. One can again deduce the representation

\[
\nabla^2 u(x) = \int_{\mathbb{R}^n} \nabla^2 G(x, \xi) f(\xi) \, d\xi \quad (5.14)
\]

for \( x \in Q_{r/4,n++} \), so that one only needs sharp estimates for the Green’s function to complete the argument.
If \( n = 2 \), we can use for this purpose Theorem 8.4.8 in combination with Theorem 6.1.2 in [KMR97]. Theorem 8.4.8 gives a Green’s function for right-hand sides in \( L^2 \). However, according to Theorem 6.1.2, the solution operator has a continuous extension to right-hand sides in \( W^{-2,2} \), so that (5.14) holds for this Green’s function. Now Theorem 8.4.8 also gives asymptotics for \( G \) in terms of the eigenvalues of a certain eigenvalue problem. If we stay in the eigenvalue-free strip, this estimate reads

\[
|D^α_x D^β_ξ G(x, ξ)| \leq C_ε |x|^{1 + δ_+ - |α| - ε} |ξ|^{1 - δ_+ - |β| + ε}
\]

where \( 2|α| \leq |ξ| \) and \( ε > 0 \) is arbitrary. Using this estimate we can continue as in the proof of Lemma 5.12.

The case \( n = 3 \) is slightly more complicated. We can use [MR10] Theorem 3.4.5], which states that if \( 2|α| \leq |ξ| \), then for any \( ε > 0 \)

\[
|D^α_x D^β_ξ G(x, ξ)|
\]

\[
\leq C_ε |x|^{3 - |α| - ε} |ξ|^{1 - Λ - |β| + ε} \prod_{j=1}^3 \left( \frac{r_j(x)}{|ξ|} \right)^{1 + δ + |α| - ε} \prod_{k=1}^3 \left( \frac{r_k(ξ)}{|ξ|} \right)^{1 + δ - |β| - ε}
\]

where \( δ_+ \) are as before, \( Λ_+ \) is another constant defined in terms of a certain eigenvalue problem (see [MR10] Section 3.4.5) for the precise definition) and \( r_j(x) \) denotes the distance of \( x \) to the line \( \{x_j = 0\} \). If we choose \( ε \leq δ_+ - 1 = δ_- - 1 \), then the exponents of the terms \( \frac{r_j(x)}{|ξ|} \) and \( \frac{r_k(ξ)}{|ξ|} \) are non-negative whenever \( |α| \leq 2 \) and \( |β| \leq 2 \). So we obtain under these assumptions

\[
|D^α_x D^β_ξ G(x, ξ)| \leq C_ε |x|^{3 - |α| - ε} |ξ|^{1 - Λ - |β| + ε}.
\]

In [MR10] Section 4.3 it is proved that \( Λ_+ \geq 3 \). This allows us to take \( θ > 0 \) such that \( 2 + \frac{θ}{2} \leq Λ_+ \) and \( 1 + \frac{θ}{2} < Λ_- \). By choosing \( ε = \min (Λ_+ - 2 - \frac{θ}{2}, δ_- - 1) \) we conclude

\[
|D^α_x D^β_ξ G(x, ξ)| \leq C_ε |x|^{3 - |α| - ε} |ξ|^{1 - Λ_+ - |β| + ε}.
\]

for \( |α| \leq 2 \) and \( |β| \leq 2 \). Now we can continue as in the proof of Lemma 5.12 (observe that in that proof we only needed estimates for \( D^α_x D^β_ξ G(x, ξ) \) with \( |α| \leq 2 \) and \( |β| \leq 1 \)).

\[\square\]

**Proof of Lemma 5.10 and Lemma 5.17.** We follow the proofs of Lemma 5.2 and Lemma 5.6. The proof is slightly easier than the proof of Lemma 5.6 because we no longer need to worry about the subtraction of the averages of \( u_κ \). We assume that the claim is wrong for some fixed \( ρ \), and consider a sequence of counterexamples \( u_κ \) and their interpolations \( v_κ = I_κ u_κ \). We can assume that \( r_κ = 1 \), and \( \|∇^2_κ u_κ\|_{L^2(Q)} = 1 \) and conclude that \( (v_κ) \)
is bounded in $W^{2,2}(Q_{3/4})$, and so a non-relabeled subsequence converges to some $v$ in $W^{2,2}(Q_{3/4})$.

As before we see that $\Delta^2 v = 0$ in $Q_{3/4,+++}$ and $Q_{3/4,n+}$ respectively and that $v$ has 0 boundary values. Also we obtain strong convergence of $\nabla^2 u_k$ and $w_k := \int_{\partial}^h \nabla^2 u_{hk}$ in $L^2_{loc}(Q_{5/8} \setminus \partial Q_{3/4,+++})$ and $L^2_{loc}(Q_{5/8} \setminus \partial Q_{3/4,n+})$, respectively. Now, as in Step 2 of the proof of Lemma 5.6 we find that $\nabla^2 u_{hk}$ does not concentrate at the boundary, so that $\nabla^2 v_k$ and $w_k$ actually converge strongly in $L^2(Q_{1/2})$.

This convergence allows us to pass to the limit in
\[ \| \nabla^2 u_k \|_{L^2(Q_{\rho})}^2 > \rho^n \]
so that we easily arrive at a contradiction to Lemma 5.12 or Lemma 5.13 once we choose $\rho$ small enough.

6 Inner and outer decay estimates for discrete biharmonic functions

6.1 Inner estimates

We can now combine the results from the previous section in one general decay estimate for biharmonic functions:

**Theorem 6.1.** Let $u_h \in \Phi_h$. Let $x \in \Lambda^n_h$, $r > 0$ and suppose that $\Delta^2 u_h(y) = 0$ for all $y \in Q_{r-h}(x) \cap \text{int} \Lambda^n_h$. Then, for all $z \in Q_{r/2}(x) \cap \Lambda^n_h$,
\[ |\nabla^2 u_h(z)| \leq \frac{C}{r^2} \| \nabla^2 u_h \|_{L^2(Q_r)} \cdot \]  \hspace{1cm} (6.1)

Observe that $\nabla^2 u_h = 0$ is zero in $(h\mathbb{Z})^n \setminus \Lambda^n_h$. Therefore we could equivalently only integrate over $Q_r(x) \cap (\Lambda^n_h)_{pc}$ on the right-hand side.

**Proof.** The proofs for the cases $n = 2$ and $n = 3$ are similar, but the latter is somewhat more tedious. Therefore we give the proof for $n = 2$ in detail and then describe how to adapt it to the case $n = 3$. So let $n = 2$.

We first prove the statement in the special case $z = x$. By rotating and reflecting $\Lambda^2_h$ we may assume $x_2 \leq x_1 \leq \frac{1}{2}$. We may also assume $r \geq \frac{h}{2}$, as otherwise we can replace $r$ by $\frac{h}{2}$ without changing (6.1).

Let $x^* = (x_1, 0)$ be a point on $\partial \Lambda^2_h$ closest to $x$. We consider the three cases $r \leq x_2$, $x_2 < r \leq x_1$ and $r > x_1$.

**Case 1:** $r \leq x_2$

In this case the interior estimate Lemma 5.1 applied to $Q_r(x)$ directly implies
\[ |\nabla^2 u_h(x)| \leq \frac{C}{r^2} \| \nabla^2 u_h \|_{L^2(Q_r)} \cdot \]
Case 2: $x_2 < r \leq x_1$

Apply first Lemma 5.1 to $Q_{x_2+h/2}(x)$ to find
\[
|\nabla^2 h u_h(x)| \leq \frac{C}{x_2 + \frac{h}{2}} \|\nabla^2_h u_h\|_{L^2(Q_{x_2+h/2}(x))}.
\]

If $r < 3x_2$ then this already implies (6.1) once we increase $C$ by a factor of 3. If $r \geq 3x_2$ we have $Q_{x_2+h/2}(x) \subset Q_{2x_2+h/2}(x^*) \subset Q_r(x^*) \subset Q_r(x)$ and so, by Lemma 5.4,
\[
|\nabla^2 h u_h(x)| \leq \frac{2x_2 + \frac{h}{r}}{r} \|\nabla^2_h u_h\|_{L^2(Q_r(0))}.
\]

This together with the previous equation implies (6.1).

Case 3: $x_1 < r$

As in the previous case we obtain
\[
|\nabla^2 h u_h(x)| \leq \frac{C}{x_1 + \frac{h}{2}} \|\nabla^2_h u_h\|_{L^2(Q_{x_1+h/2}(x^*))}.
\]

Now either $r < 3x_1$ and we are done, or we can continue with Lemma 5.9 to find
\[
\|\nabla^2_h u_h\|_{L^2(Q_{x_1+h/2}(x^*))} \leq \|\nabla^2_h u_h\|_{L^2(Q_{2x_1+h/2}(0))} \leq C \frac{2x_1 + \frac{h}{r}}{r} \|\nabla^2_h u_h\|_{L^2(Q_r(0))},
\]

which in combination with (6.2) implies (6.1).

This proves (6.1) in the case $z = x$. For general $z$, it suffices to observe that $Q_{r/2}(z) \subset Q_r(x)$ and apply the statement we have just proved to $Q_{r/2}(z)$.

The proof for $n = 3$ is analogous. However there is one more case and hence we need one more intermediate step, where we deal with the case of an edge. So one applies Lemmata 5.1, 5.5, 5.8, 5.9 in order until one reaches a radius of order $r$. We omit the details.

6.2 Outer estimates via duality

Theorem 6.1 states that if a discrete function is biharmonic in a subcube $Q_r(x)$ of $L^h_n$, then we have pointwise control over its second derivatives in a smaller subcube $Q_{r/2}(x)$. Remarkably, a dual statement is also true: If a discrete function is biharmonic outside a subcube $Q_r(x)$ of $L^h_n$, then we have control over its second derivatives outside of a larger subcube $Q_{2r}(x)$. The following lemma does not claim pointwise control, but only control in $L^2$.

However we will combine it with Theorem 6.1 into Theorem 6.3 where we actually obtain pointwise control.
Lemma 6.2. Let \( u_h \in \Phi_h \). Let \( x \in \Lambda^n_h \), \( r \geq d(x) \) and suppose that \( \Delta^2_h u_h(x) = 0 \) for all \( x \in \text{int} \Lambda^n_h \setminus Q_r(x) \). Then, for all \( s \geq r \),

\[
\| \nabla^2_h u_h \|_{L^2(\mathbb{R}^n \setminus Q_s(x))} \leq C \left( \frac{r}{s} \right)^{\frac{n}{2}} \| \nabla^2_h u_h \|_{L^2(\mathbb{R}^n \setminus Q_r(x))} .
\]  

(6.3)

Proof. Consider first the case \( r < h \). Then \( d(x) = 0 \), i.e. \( x \in \partial \Lambda^n_h \), and the assumptions imply \( \Delta^2_h u_h = 0 \) in \( \text{int} \Lambda^n_h \), i.e. \( u_h = 0 \) in \( \text{int} \Lambda^n_h \) by the uniqueness of the bilaplacian equation. So both sides of (6.3) are zero and the inequality holds.

So we can assume \( r \geq h \). The statement is trivial in the case that \( s < 23r \), so we can also assume \( s \geq 23r \). We can then replace \( r \) and \( s \) by \( \tilde{r} = |r - \frac{h}{2}|_h + \frac{2h}{3} \) and \( \tilde{s} = |s - \frac{h}{2}|_h + \frac{h}{2} \), respectively. It is easy to see that \( \tilde{r} \geq r \), \( \tilde{s} \leq s \) and \( \tilde{s} \geq 11\tilde{r} \), and it suffices to prove the theorem for \( \tilde{r}, \tilde{s} \).

Let \( f_h = \nabla^2_h u_h \chi_{\Lambda^n_h \setminus Q_s(x)} \), where \( \chi_{\Lambda} \) is the indicator function of a set \( \Lambda \). Let \( v_h \in \Phi_h \) be the unique solution of \( \Delta^2 v_h = \text{div}_h \text{div}_h f_h \). Then, for any \( \varphi_h \in \Phi_h \),

\[
\nabla^2_h v_h, \nabla^2 \varphi_h)_{L^2(\mathbb{R}^n)} = (f_h, \nabla^2 \varphi_h)_{L^2(\mathbb{R}^n)} .
\]  

(6.4)

Also let \( \zeta_h \) and \( \eta_h \) be discrete cut-off functions such that \( \zeta_h \) is 1 on \( \Lambda^n_h \setminus Q_{5r}(x) \), 0 on \( Q_{3r}(x) \cap \Lambda^n_h \), \( \eta_h \) is 1 on \( Q_{7r}(x) \cap \Lambda^n_h \), 0 on \( \Lambda^n_h \setminus Q_{9r}(x) \) and such that \( |\nabla^2 \zeta_h| \leq \frac{r}{\kappa^2} \) and \( |\nabla^2 \eta_h| \leq \frac{r}{\kappa} \) for \( \kappa \leq 2 \).

These choices ensure that

\[
\nabla^2_h (\zeta_h u_h) = \nabla^2_h u_h \text{ on the support of } f_h
\]  

(6.5)

and that

\[
\eta_h = 1 \text{ on the support of } \Delta^2_h (\zeta_h u_h) .
\]  

(6.6)

Indeed, for example the support of \( \Delta^2_h (\zeta_h u_h) \) is contained in \( Q_{5r+2h}(x) \setminus Q_{3r-2h}(x) \subset Q_{7r}(x) \).

This implies

\[
\| \nabla^2_h u_h \|_{L^2(\mathbb{R}^n \setminus Q_s(x))}^2 = (f_h, \nabla^2_h u_h)_{L^2(\mathbb{R}^n)}
\]

6.5

\[
= (f_h, \nabla^2_h (\zeta_h u_h))_{L^2(\mathbb{R}^n)}
\]

6.6

\[
= (\nabla^2_h v_h, \nabla^2_h (\zeta_h u_h))_{L^2(\mathbb{R}^n)}
\]

6.4

\[
= (v_h, \Delta^2_h (\zeta_h u_h))_{L^2(\mathbb{R}^n)}
\]

6.6

\[
= (\eta_h v_h, \Delta^2_h (\zeta_h u_h))_{L^2(\mathbb{R}^n)}
\]

6.6

\[
= (\nabla^2_h (\eta_h v_h), \nabla^2_h (\zeta_h u_h))_{L^2(\mathbb{R}^n)}
\]

6.6

\[
\leq \| \nabla^2_h (\eta_h v_h) \|_{L^2(\mathbb{R}^n)} \| \nabla^2_h (\zeta_h u_h) \|_{L^2(\mathbb{R}^n)} .
\]  

(6.7)

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Now by the product rule
\[
\nabla^2_h(\eta_i v_h) = \sum_{i,j=1}^n D^2_{i,j} \eta_i v_h + \tau^h_{i,j} D^2_{i,j} \eta_i v_h + \tau^h_{-i} D^2_{-i} \eta_i v_h + \tau^h_{-j} D^2_{-j} \eta_i v_h
\]
and so, using the Poincaré inequality\footnote{Here we have used the assumption $r \geq d(x)$ (or rather $7r \geq d(x)$): It ensures that we have zero boundary data somewhere on $Q_{9r}(x) \setminus Q_r(x)$ so that we can indeed use the Poincaré inequality.} on $Q_{9r}(x)$,
\[
\|
abla^2_h(\eta_i v_h)\|_{L^2(\mathbb{R}^n)} \leq C \frac{r}{\tau} \|
abla h v_h\|_{L^2(Q_{9r}(x))} + C \|
abla^2_h v_h\|_{L^2(Q_{9r}(x))} \leq C \|
abla^2_h v_h\|_{L^2(Q_{9r}(x))}.
\]
(6.8)

Similarly, by the Poincaré inequality on the annulus $Q_{7r}(x) \setminus Q_r(x)$,
\[
\|
abla^2_h(\zeta_i u_h)\|_{L^2(\mathbb{R}^n)} \leq C \frac{r}{\tau} \|u_h\|_{L^2(Q_{7r}(x) \setminus Q_r(x))} + C \|
abla_h u_h\|_{L^2(Q_{7r}(x) \setminus Q_r(x))} + C \|
abla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_r(x))} \leq C \|
abla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_r(x))}.
\]

If we plug the last two estimates into (6.7) and then use Theorem 6.1 for $v_h$ we obtain
\[
\|
abla^2_h u_h\|_{L^2(\Lambda^*_h \setminus Q_r(x))} \leq C \|
abla^2_h(\eta_i v_h)\|_{L^2(Q_{9r}(x))} \|
abla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_r(x))} \leq C \left(\frac{9r}{s}\right)^\frac{2}{3} \|
abla^2_h(v_h)\|_{L^2(\mathbb{R}^n)} \|
abla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_r(x))}.
\]
\[\text{This is equivalent to (6.3) once we use the energy estimate}\]
\[
\|
abla^2_h v_h\|_{L^2(\mathbb{R}^n)} \leq \|f_h\|_{L^2(\mathbb{R}^n)} = \|
abla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_r(x))}.
\]

Now we can combine this lemma with Theorem 6.1 to obtain a pointwise outer estimate.

**Theorem 6.3.** Let $u_h \in \Phi_h$. Let $x \in \Lambda^*_h$, $r > 0$ and suppose that $\Delta^2_h u_h(x) = 0$ for all $x \in \text{int} \Lambda^*_h \setminus Q_r(x)$.

Then, for all $y \in \Lambda^*_h \setminus Q_{2r}(x)$,
\[
|\nabla^2_h u_h(y)| \leq C \left(\frac{\max(d(x), r)}{|x - y|^n}\right)^\frac{2}{3} \|
abla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_r(x))}.
\]
Proof. As in the proof of Lemma 6.2 we see that \( d(x) = 0 \) implies \( u = 0 \) everywhere and (6.9) holds. So assume \( d(x) \geq h \).

Let \( y \in \Lambda^n \setminus Q_{2r}(x) \). If \( y \in Q_{2d(x)}(x) \) we use Theorem 6.1 on \( Q_{d(x)}(y) \subset \mathbb{R}^n \setminus Q_{2r}(x) \) to obtain

\[
|\nabla^2_h u_h(y)| \leq \frac{C}{d(x)^2} \|\nabla^2_h u_h\|_{L^2(Q_{d(x)}(y))} \leq \frac{C}{d(x)^2} \|\nabla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_{2r}(x))},
\]

which implies (6.9) because \( |x - y| \leq \sqrt{n} |x - y|_\infty \leq 2 \sqrt{n} d(x) \) and hence

\[
\frac{1}{d(x)} \leq 4 \sqrt{n} \frac{d(x)}{|x - y|}. \]

If, on the other hand, \( y \in \Lambda^n \setminus Q_{2d(x)}(x) \) then we use Theorem 6.1 on \( Q_{|x - y|_\infty / 2}(y) \) and then Lemma 6.2 as follows:

\[
|\nabla^2_h u_h(y)| \leq \frac{C}{|x - y|_\infty^2} \|\nabla^2_h u_h\|_{L^2(Q_{|x - y|_\infty / 2}(y))} \leq \frac{C}{|x - y|_\infty^2} \|\nabla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_{d(x)}(x))} \leq C \|\nabla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_{d(x)}(x))} \leq C \|\nabla^2_h u_h\|_{L^2(\mathbb{R}^n \setminus Q_{d(x)}(x))},
\]

which implies (6.9). \( \square \)

7 The discrete full-space Green’s function

In order to obtain estimates for \( G_h \), we will compare \( G_h \) with a Green’s function of \( (h\mathbb{Z})^n \). In the absence of boundary conditions such a Green’s function is not uniquely defined. We will choose a normalization that is best suited for our application. The necessary asymptotics for the Green’s function of \( (h\mathbb{Z})^n \), have been derived by Mangad [Man67] using Fourier-theoretic methods.

By \( \mathcal{F} \) we denote the Fourier transform of tempered distributions (where we use the convention \( (\mathcal{F} f)(x) = \int_{\mathbb{R}^n} f(\xi) e^{-2\pi i x \cdot \xi} d\xi \)).

Theorem 7.1 ([Man67], Section 4). Let \( n \in \mathbb{N}^+ \). Define \( F: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{R} \) by

\[
F(x, y) = \mathcal{F} \left( \frac{V(\xi)}{\left( 4 \sum_{j=1}^n \sin^2(\pi \xi_j) \right)^{n/2}} \right)(x - y)
\]
where $V \in C_0^\infty([-1, 1]^n)$ is chosen such that $V = 1$ near 0 and $\sum_{z \in \mathbb{Z}} V(x + z) = 1$ for all $x$ and $\frac{V(\xi)}{(4 \sum_{j=1}^{n} \sin^2(\pi \xi_j))^2}$ denotes the tempered distribution given by its finite part in the sense of Hadamard (see Sch66 Chapitre II, §2 and §3).

Then $F$ is a Green’s function for $\Delta^2_1$ in the sense that $\Delta^2_1 F(\cdot, y) = \delta_y$. It satisfies the following asymptotic expansion: If $n = 2$ and $z = x - y$,

$$F(x, y) = \frac{|z|^2 \log |z|}{8\pi} + \frac{(\gamma - 1 + \log \pi)|z|^2}{8\pi} - \frac{\log |z|}{16\pi} + \frac{4(z_1^4 + z_2^4)}{|z|^4}$$
$$- 12 \log \pi - 12\gamma - 3 + O \left( \frac{1}{|z|^2} \right)$$

where $\gamma$ is the Euler-Mascheroni constant, and if $n = 3$ and $z = x - y$,

$$F(x, y) = \frac{|z|}{8\pi} + \frac{z_1^2 + z_2^4 + z_3^4}{64\pi|z|^5} + \frac{1}{64\pi |z|} + O \left( \frac{1}{|z|^3} \right).$$

Let us briefly sketch how to prove this theorem: Observe that $\sigma(\xi) := \left(4 \sum_{j=1}^{n} \sin^2(\pi \xi_j)\right)^2$ is the symbol of $\Delta^2_1$, so that $\Delta^2_1 F(x, y) = F(V)(x - y)$. On the other hand one easily checks that $\sum_{z \in \mathbb{Z}} V(x + z) = 1$ implies that $F(V)(m) = \delta_0(m)$ for any $m \in \mathbb{Z}^n$. This proves that $F$ is a Green’s function. To derive the asymptotic expansion, one develops a Laurent series

$$\frac{1}{\sigma(\xi)} = \frac{1}{16\pi^2 |\xi|^2} + \frac{f_2(\xi)}{|\xi|^2} + f_0(\xi) + \cdots + o(|\xi|^N).$$

Then one can check using the explicit formulas for the Fourier transforms of $|\xi|^m$ (see Sch66) and the Riemann-Lebesgue lemma that

$$\mathcal{F} \left( \frac{V(\xi)}{\sigma(\xi)} \right) \frac{1}{16\pi^2 |\xi|^2} + \frac{f_2(\xi)}{|\xi|^2} + f_0(\xi) + \cdots = o(|x|^{-n-N}).$$

so it suffices to compute the Fourier transform of $\frac{1}{16\pi^2 |\xi|^2} + \frac{f_2(\xi)}{|\xi|^2} + f_0(\xi) + \cdots$. This one can again do explicitly and thereby obtain an asymptotic expansion for $F$ up to $O(|x|^{-N})$. For details we refer to Man67.

By scaling the lattice we can deduce from this estimates for Green’s functions on $(h\mathbb{Z})^n$. We state the estimates that we will need.

**Lemma 7.2.** Let $n = 2$ or $n = 3$, $h > 0$, $r \geq 4h$. There exists a function $\tilde{G}_h: (h\mathbb{Z})^n \times (h\mathbb{Z})^n \to \mathbb{R}$ such that $\Delta_h^2 \tilde{G}_h(\cdot, y) = \delta_h(y)$ and such that the following estimates are satisfied:

$$|\nabla_{h,y} \tilde{G}_h(x, y)| \leq Cr^{3-n} \quad \text{if } |x - y|_{\infty} \leq \frac{r}{2}, \quad (7.1)$$
$$|\nabla_{h,x}^2 \nabla_{h,y} \tilde{G}_h(x, y)| \leq \frac{C}{(|x - y| + h)^{n-1}} \quad \text{if } |x - y|_{\infty} \leq \frac{r}{2}, \quad (7.2)$$

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\[ |\nabla_{h,x}^2 \nabla_{h,y}^2 \tilde{G}_h(x, y)| \leq \frac{C}{(|x - y| + h)^n} \quad \text{if } |x - y|_\infty \leq \frac{r}{2} \]  
\text{(7.3)}

and
\[ |D_{h,x}^\alpha D_{h,y}^\beta \tilde{G}_h(x, y)| \leq Cr^{4-n-|\alpha|-|\beta|} \quad \text{if } \frac{r}{2} \leq |x - y|_\infty \leq r, |\alpha| + |\beta| \leq 4. \]  
\text{(7.4)}

For \( n = 2 \) the function \( \tilde{G}_h \) depends on \( r \), but we will suppress this dependence for ease of notation.

\textbf{Proof.} We begin with the slightly easier case \( n = 3 \). The asymptotic expansion in Theorem 7.1 easily implies that
\[ |D_{1,x}^\alpha D_{1,y}^\beta F(x, y)| \leq C|x - y|^{1-|\alpha|-|\beta|} \]
for \( |\alpha| + |\beta| \leq 4 \) and any \( x, y \) with \( |x - y| \geq 10 \), say (observe that \( g = O(|x|^{-3}) \) implies \( D_{1,x}^\alpha g(x) = O(|x|^{-3}) \), so we do not need to care about the error term).

On the other hand \( F \) is finite everywhere, so that
\[ |D_{1,x}^\alpha D_{1,y}^\beta F(x, y)| \leq C \]
for \( |\alpha| + |\beta| \leq 4 \) and any \( x, y \) with \( |x - y| < 10 \). If we combine these two estimates we conclude that we have
\[ |D_{1,x}^\alpha D_{1,y}^\beta \tilde{G}_h(x, y)| \leq C(|x - y| + 1)^{1-|\alpha|-|\beta|}. \]
Now if we set \( \tilde{G}_h(x, y) = hF \left( \frac{x}{h}, \frac{y}{h} \right) \) then \( \tilde{G}_h \) satisfies
\[ |D_{h,x}^\alpha D_{h,y}^\beta \tilde{G}_h(x, y)| \leq C(|x - y| + h)^{1-|\alpha|-|\beta|}, \]
which immediately implies the claimed estimates.

If \( n = 2 \) we need to take care of the logarithmic terms. So we set
\[ \tilde{F}(x, y) = F(x, y) + \frac{|x - y|^2 \log \left( \frac{h}{r} \right)}{8\pi}. \]
Then \( \tilde{F} \) has the asymptotic expansion
\[ \tilde{F}(z) = \frac{z^2 \log |z|}{8\pi} + \frac{(\log \left( \frac{h}{r} \right) + \gamma - 1 + \log \pi)z^2}{8\pi} - \frac{\log |z|}{16\pi} z^2 - \frac{4(z_1^4 + z_2^4)}{|z|^4} - 12 \log \pi - 12 \gamma - 3 + O \left( \frac{1}{|z|^4} \right) \]
and this implies
\[ |D_{1,x}^\alpha D_{1,y}^\beta \tilde{F}(x, y)| \leq C|x - y|^{2-|\alpha|-|\beta|} \left( \log |x - y| + \log \left( \frac{h}{r} \right) + 1 \right). \]
for $|\alpha| + |\beta| \leq 2$ and any $x, y$ with $|x - y| \geq 10$. Because $D_{1,x}^\alpha D_{1,y}^\beta \tilde{F}(x, y)$ is bounded by $C \left(1 + \log \left(\frac{|x - y|}{r}\right)\right)$ for $|x - y| < 10$, we conclude

$$|D_{1,x}^\alpha D_{1,y}^\beta \tilde{F}(x, y)| \leq C(|x - y| + 1)^{2-|\alpha|-|\beta|} \log \left(\frac{h(|x - y| + 1)}{r}\right).$$

We now set $\tilde{G}_h(x, y) = h^2 \tilde{F}(\frac{x}{h}, \frac{y}{h})$ and obtain

$$|D_{h,x}^\alpha D_{h,y}^\beta \tilde{G}_h(x, y)| \leq C(|x - y| + h)^{2-|\alpha|-|\beta|} \log \left(\frac{|x - y| + h}{r}\right).$$

It is easy to check that this implies (7.1) and (7.4) for $|\alpha| + |\beta| \leq 2$. If $|\alpha| + |\beta| \geq 3$ we need to be slightly more careful: Observe that third discrete derivatives of $|x - y|^2$ vanish, so that we actually have

$$|\nabla_{1,x}^\alpha \nabla_{1,y}^\beta \tilde{F}(x - y)| \leq \frac{C}{|x - y|^{|\alpha|+|\beta|-2}}$$

if $|x - y| \geq 10$ from which we conclude

$$|\nabla_{1,x}^\alpha \nabla_{1,y}^\beta \tilde{F}(x - y)| \leq \frac{C}{(|x - y| + 1)^{|\alpha|+|\beta|-2}}$$

for any $x, y$. Recalling that $\tilde{G}_h(x, y) = h^2 \tilde{F}(\frac{x}{h}, \frac{y}{h})$ we immediately obtain (7.2), (7.3) and (7.4) for $|\alpha| + |\beta| \geq 3$. \hfill $\square$

8 Proof of the main theorem

We are now able to prove Theorem 1.3. We first give the straightforward proof of part ii) and then continue with part i).

8.1 Lower bounds for $G_h(x, x)$

The proof is rather short and based on the choice of an appropriate test function.

**Proof of Theorem 1.3 ii).**

We can assume $d(x) \geq h$, as otherwise $d(x) = 0$ and hence $G_h(x, x) = 0$. If we test the equation $\Delta_h^2 G_{h,x} = \delta_{h,x}$ with $G_{h,x}$, we find

$$\|\nabla_h^2 G_{h,x}\|_{L^2(\mathbb{R}^n)}^2 = (\Delta_h^2 G_{h,x}, G_{h,x})_{L^2(\mathbb{R}^n)} = (\delta_{h,x}, G_{h,x})_{L^2(\mathbb{R}^n)} = G_h(x, x).$$

(8.1)

Now let $\varphi_h \in \Phi_h$. Then testing the equation $\Delta_h^2 G_{h,x} = \delta_{h,x}$ with $\varphi_h$ and using the Cauchy-Schwarz inequality we find

$$\varphi_h(x) = (\nabla_h^2 G_{h,x}, \nabla_h^2 \varphi_h)_{L^2(\mathbb{R}^n)}$$
\[
\begin{align*}
&\leq \|\nabla^2 G_h, x\|_{L^2(\mathbb{R}^n)} \|\nabla^2 \varphi_h\|_{L^2(\mathbb{R}^n)} \\
&= \sqrt{G_h(x, x)} \|\nabla^2 \varphi_h\|_{L^2(\mathbb{R}^n)}.
\end{align*}
\]

If \( \varphi_h \) is not identically zero this implies
\[
G_h(x, x) \geq \frac{(\varphi_h(x))^2}{\|\nabla^2 \varphi_h\|_{L^2(\mathbb{R}^n)}^2} \geq Cd(x)^{2-\frac{2}{n}}.
\]

and so it remains to find a \( \varphi_h(x) \) such that \( \frac{\varphi_h(x)}{\|\nabla^2 \varphi_h\|_{L^2(\mathbb{R}^n)}} \geq Cd(x)^{2-\frac{2}{n}} \). But this is easy:

Take \( \varphi_{h,x} \in \Phi_h \) supported in \( Q_{d(x)}(x) \) such that \( \varphi_{h,x}(x) = 1 \) and such that \( |\nabla^2 \varphi_{h,x}| \leq \frac{C}{d(x)} \) and extend it by 0 to all of \( \Lambda^u_h \).

8.2 Upper bounds for \( G_h(x, y) \)

In this section we prove part i) of Theorem 1.3.

We begin with a rather weak estimate for \( G_h(x, y) \).

Lemma 8.1. Let \( n = 2 \) or \( n = 3 \) and \( G_h \) be the Green’s function of \( \Lambda^u_h \). Then we have
\[
0 \leq G_h(x, x) = \|\nabla^2 G_h, x\|_{L^2(\mathbb{R}^n)}^2 \leq Cd(x)^{1-n} \tag{8.2}
\]
for any \( x \in \Lambda^u_h \) and
\[
|G_h(x, y)| \leq Cd(x)^{2-\frac{2}{n}} d(y)^{2-\frac{2}{n}} \tag{8.3}
\]
for any \( x, y \in \Lambda^u_h \).

Proof. We first prove (8.2). By (8.1) we have
\[
\|\nabla^2 G_h, x\|_{L^2(\mathbb{R}^n)}^2 = G_h(x, x). \tag{8.4}
\]

If \( x \in \partial \Lambda^u_h \) then \( G_h(x, x) = 0 \) and (8.2) holds. So assume \( x \in \text{int} \Lambda^u_h \), i.e. \( d(x) \geq h \). The Sobolev-Poincaré inequality implies that
\[
G_h(x, x) \leq \|G_h, x\|_{L^\infty(Q_{d(x)+h/2}(x))} \\
\leq C \left(d(x) + \frac{h}{2}\right)^{2-\frac{2}{n}} \|\nabla^2 G_h, x\|_{L^2(Q_{d(x)+h/2}(x))} \\
\leq Cd(x)^{2-\frac{2}{n}} \|\nabla^2 G_h, x\|_{L^2(Q_{2d(x)+h/2}(x))}.
\]

If we combine this estimate with (8.3) we find that
\[
\|\nabla^2 G_h, x\|_{L^2(\mathbb{R}^n)}^2 = G_h(x, x) \leq Cd(x)^{2-\frac{2}{n}} \|\nabla^2 G_h, x\|_{L^2(Q_{2d(x)+h/2}(x))} \\
\leq Cd(x)^{2-\frac{2}{n}} \|\nabla^2 G_h, x\|_{L^2(\mathbb{R}^n)}.
\]

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and hence
\[ 0 \leq G(x,x) = \| \nabla^2_{h} G_{h,x} \|^2_{L^2(\mathbb{R}^n)} \leq C d(x)^{4-n} . \]
This proves (8.2). For (8.3), we test \( \Delta^2_{h} G_{h,x} = \delta_{h,x} \) with \( G_{h,y} \) and use the Cauchy-Schwarz inequality to obtain
\[ |G(x,y)| = \left| \langle \delta_{h,x}, G_{h,y} \rangle_{L^2(\mathbb{R}^n)} \right| \]
\[ = \left| \langle \nabla^2_{h} G_{h,x}, \nabla^2_{h} G_{h,y} \rangle_{L^2(\mathbb{R}^n)} \right| \]
\[ \leq \| \nabla^2_{h} G_{h,x} \|_{L^2(\mathbb{R}^n)} \| \nabla^2_{h} G_{h,y} \|_{L^2(\mathbb{R}^n)} \]
\[ \leq C d(x)^{2-\frac{2-n}{2}} d(y)^{2-\frac{2-n}{2}} . \]

The next lemma gives estimates for \( G_h \) and its derivatives that are sharp when \( x \) and \( y \) are far apart. We first prove a pointwise estimate for \( \nabla^2_{h} G_h \) by applying Theorem 6.3 to a cut-off version of \( \nabla_{h,y} G_{h,y} \). Afterwards we integrate it along suitable paths to deduce the estimates in the lemma.

**Lemma 8.2.** Let \( n = 2 \) or \( n = 3 \) and \( G_h \) be the Green’s function of \( \Lambda^n_h \). If \( x,y \in \Lambda^n_h \) and \( |x-y|_\infty > \frac{d(y)}{8} \) then
\[ |G_h(x,y)| \leq C \left( \frac{(d(x) + h)^2(d(y) + h)^2}{|x-y|^n} \right), \quad (8.5) \]
\[ |\nabla_{h,x} G_h(x,y)| \leq C \left( \frac{(d(x) + h)(d(y) + h)}{|x-y|^n} \right), \quad (8.6) \]
\[ |\nabla^2_{h,x} G_h(x,y)| \leq C \left( \frac{(d(y) + h)}{|x-y|^n} \right), \quad (8.7) \]
\[ |\nabla_{h,x} \nabla_{h,y} G_h(x,y)| \leq C \left( \frac{(d(x) + h)(d(y) + h)}{|x-y|^n} \right). \quad (8.8) \]

**Proof.**

**Step 1: Pointwise estimate for \( \nabla^2_{h,x} \nabla_{h,y} G_h(x,y) \)**

We claim that if \( x,y \in \Lambda^n_h \) and \( |x-y|_\infty > \frac{d(y)}{8} \) then
\[ |\nabla^2_{h,x} \nabla_{h,y} G_h(x,y)| \leq C \frac{d(y) + h}{|x-y|^n}. \quad (8.9) \]

In the following all derivatives will be with respect to \( x \) unless we mark them with a sub- or superscript \( y \).

If \( d(y) < 160h \) we can use a trivial estimate: From Lemma 8.1 we know
\[ \| \nabla^2_{h} G_{h,y} \|_{L^2(\mathbb{R}^n)} \leq C d(y)^{2-\frac{n}{2}} \leq C h^{2-\frac{n}{2}} \]
if \(|y' - y|_\infty \leq h\). If we now use

\[
|D^hf_h(y)|^2 = \left(\frac{1}{h}(f_h(y + e_i) - f_h(y))\right)^2 \leq \frac{2}{h^2}(f_h(y + e_i)^2 + f_h(y)^2)
\]

with \(f = \nabla^2 G_h\) we get that

\[
\|\nabla^2 D^h_y G_h\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{2}{h^2} \left(\|\nabla^2 \tilde{r}^h G_h\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla^2 G_h\|_{L^2(\mathbb{R}^n)}^2\right) \leq C h^{2-n},
\]

i.e.

\[
\|\nabla^2 D^h_y G_h\|_{L^2(\mathbb{R}^n)} \leq C h^{1-\frac{n}{2}}.
\]

Then Theorem 6.3 with \(r = h\) implies

\[
|\nabla^2 D^h_y G_h(x, y)| \leq C \max(d(y, h)\frac{\eta}{|x - y|}, \|\nabla^2 D^h_y G_h\|_{L^2(\mathbb{R}^n)})
\]

\[
\leq C \frac{h^{\frac{n}{2}}}{|x - y|^n} h^{1-\frac{n}{2}} = C \frac{h}{|x - y|^n},
\]

which implies \((8.9)\) if we choose \(C\) there large enough.

So assume \(d(y) \geq 160h\). Let \(\eta_h\) be a discrete cut-off function that is 1 on \(Q_{d(y)/32+2h}, 0\) on \((hZ)^n \setminus Q_{d(y)/16-2h}(x)\), and such that \(|\nabla^\kappa \eta_h| \leq \frac{C}{d(y)^\kappa}\) for \(\kappa \leq 2\). Let \(H_h(x, y) = G_h(x, y) - \eta_h(x) \tilde{G}_h(x, y)\), where \(\tilde{G}_h\) is the function from Lemma \(7.2\) with \(r = \frac{d(y)}{16}\). We write \(H_{h,y}\) for \(H_h(\cdot, y)\).

Then, for \(i \in \{1, \ldots, n\}\), \(D^h_y H_{h,y} \in \Phi_h\). Also, the singularities near \(y\) cancel out, so that \(\Delta^2 D^h_y H_{h,y} = 0\) in \(Q_{d(y)/32}(y)\) and in \(\text{int} A_h^\kappa \setminus Q_{d(y)/16}(y)\). Next, we want to bound \(\|\nabla^2 D^h_y H_{h,y}\|_{L^2(\mathbb{R}^n)}\). To do so, we introduce another cut-off function \(\zeta_h\) that is 1 on \(\text{int} A_h^\kappa \setminus Q_{d(y)/32}(y)\), 0 on \(Q_{d(y)/64}(y)\) and such that \(|\nabla^\kappa \zeta_h| \leq \frac{C}{d(y)^\kappa}\) for \(\kappa \leq 2\). Then we have that

\[
\Delta^2 D^h_y H_{h,y} = \zeta_h \Delta^2 D^h_y \tilde{G}_h, y = -\zeta_h \Delta^2 D^h_y \left(\eta_h \tilde{G}_h, y\right) = -\zeta_h \Delta^2 \left(\eta_h D^h_y \tilde{G}_h, y\right)
\]

where we have used that \(\eta_h\) does not depend on \(y\). Thus

\[
\|\nabla^2 D^h_y H_{h,y}\|_{L^2(\mathbb{R}^n)}^2 = (\Delta^2 D^h_y H_{h,y}, D^h_y H_{h,y})_{L^2(\mathbb{R}^n)}
\]

\[
= - (\zeta_h \Delta^2 (\eta_h D^h_y \tilde{G}_h, y), D^h_y H_{h,y})_{L^2(\mathbb{R}^n)}
\]

\[
= - (\Delta^2 (\eta_h D^h_y \tilde{G}_h, y), \zeta_h D^h_y H_{h,y})_{L^2(\mathbb{R}^n)}
\]

\[
= - (\nabla^2 (\eta_h D^h_y \tilde{G}_h, y), \nabla^2 (\zeta_h D^h_y H_{h,y}))_{L^2(\mathbb{R}^n)}
\]

\[
\leq \|\nabla^2 (\eta_h D^h_y \tilde{G}_h, y)\|_{L^2(\mathbb{R}^n)} \|\nabla^2 (\zeta_h D^h_y H_{h,y})\|_{L^2(\mathbb{R}^n)}.
\]

(8.10)
If we use the pointwise estimates for $\tilde{G}_{h,y}$ from Lemma 7.2 we conclude
\[ |\nabla^2_h(\eta_h D_i^{h,y} \tilde{G}_{h,y})| \leq C d(y)^{1-n} \]
and hence
\[ \|\nabla^2_h(\eta_h D_i^{h,y} \tilde{G}_{h,y})\|_{L^2(\mathbb{R}^n)} \leq C d(y)^{1-\frac{n}{2}}. \]

Furthermore, as in (8.8), the Poincaré inequality on $Q_{d(y)+h/2}(y)$ and the pointwise estimates for $\zeta_h$ imply that
\[
\|\nabla^2_h(\zeta_h D_i^{h,y} H_{h,y})\|_{L^2(\mathbb{R}^n)} \\
\leq \frac{C}{d(y)^2} \|D_i^{h,y} H_{h,y}\|_{L^2(Q_{d(y)+h/2}(y))} + \frac{C}{d(y)} \|\nabla_h D_i^{h,y} H_{h,y}\|_{L^2(Q_{d(y)+h/2}(y))} \\
+ \|\nabla^2_h D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^n)} \\
\leq C \|\nabla^2_h D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^n)}.
\]

If we combine the last two estimates with (8.10) we conclude that
\[ \|\nabla^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^n)} \leq C d(y)^{1-\frac{n}{2}}. \]

We recall that $\Delta_h H_h = 0$ in $\text{int} \Lambda_h \setminus Q_{d(y)/16}(y)$ and use Theorem 6.3 to find that, for $x \in \Lambda_h \setminus Q_{d(y)/8}(y)$,
\[
|\nabla^2_h D_i^{h,y} H_h(x)| \leq C \frac{d(y)^\frac{n}{2}}{|x-y|^n} \|\nabla^2 D_i^{h,y} H_h\|_{L^2(\mathbb{R}^n)} \\
\leq C \frac{d(y)^\frac{n}{2}}{|x-y|^n} d(y)^{1-\frac{n}{2}} = C \frac{d(y)}{|x-y|^n}.
\]

This implies (8.8) because $D_i^{h,y} H_{h,y}$ is equal to $D_i^{h,y} G_{h,y}$ in $\Lambda_h \setminus Q_{d(y)/16}(y)$ and therefore $\nabla^2_h D_i^{h,y} H_{h,y}$ is equal to $\nabla^2_h D_i^{h,y} G_{h,y}$ in $\Lambda_h \setminus Q_{d(y)/8}(y)$.

Step 2: Proof of (8.8)
We can obtain (8.8) by integrating (8.9) along a well-chosen path in $x$. Let $(x^{(k)})_{k=0}^L$ be a path of length $Lh$ from $x^{(0)} = x$ to $x^{(L)} \in (h\mathbb{Z})^n \setminus \Lambda_h$ such that $|x^{(k+1)} - x^{(k)}|_{\infty} = h$, $|x^{(k)} - y| \geq |x - y|_{\infty}$ for all $k$, and $L \leq 2(d(x) + h)$.

To construct such a path begin with the straight path from $x$ to a closest point $x_* \in (h\mathbb{Z})^n \setminus \Lambda_h$ (which will have length $d(x) + h$). If this path does not intersect $Q_{d(x) + h}(x)$, we are done. Else we modify the path by taking a (shortest-possible) detour around $Q_{|x-y|_{\infty} - h}(y)$. This detour lengthens the path by at most $|x - y|_{\infty}$, and it is easy to check that if it is necessary then $y \in Q_{d(x)}(x)$, so that $|x - y|_{\infty} \leq d(x)$, and our path has length at most $d(x) + h + |x - y|_{\infty} \leq 2(d(x) + h)$.

Now, by (8.9),
\[
|\nabla^2_h \nabla_h G_h(x^{(k)}, y)| \leq C \frac{d(y) + h}{(|x^{(k)} - y| + h)^n}.
\]
\[ \frac{d(y) + h}{(|x| - y) \cdot h + h} \leq C \frac{d(y) + h}{(|x| - y) \cdot h + h} \leq C \frac{d(y) + h}{(|x| - y) \cdot h + h} . \]

Now we can perform discrete integration along \((x^{(k)})_{k=0}^L\): Observe that \(\nabla_{h,x} \nabla_{h,y} G_h(x^{(L)}, y) = 0\) and so

\[
|\nabla_{h,x} \nabla_{h,y} G_h(x, y)| \leq \sum_{k=0}^{L-1} |\nabla_{h,x} \nabla_{h,y} G_h(x^{(k+1)}, y) - \nabla_{h,x} \nabla_{h,y} G_h(x^{(k)}, y)|
\leq \sum_{k=0}^{L-1} h |\nabla_{h,x} \nabla_{h,y} G_h(x^{(k)}, y)|
\leq L \frac{d(y) + h}{(|x| - y) \cdot h + h},
\]

which implies (8.8).

**Step 3: Proof of (8.7)**

We proceed as in the previous step with the only difference that this time we integrate in \(y\) along a path that avoids \(x\). Let \((y^{(k)})_{k=0}^L\) be a path of length \(Lh\) from \(y^{(0)} = y\) to \(y^{(L)} \in (h\mathbb{Z})^n \setminus \Lambda_h^n\) such that \(|y^{(k+1)} - y^{(k)}| = h, |y^{(k)} - x| \geq |y - x|\) for all \(k\), and \(L \leq 2(d(y) + h)\). If we construct this path as in the previous step, we can in addition ensure that \(d(y^{(k)}) \leq d(y)\) for all \(k\) (then in particular \(|y^{(k)} - x| \geq \frac{d(y^{(k)})}{8}\), so that (8.9) is applicable for all \(y^{(k)}\)).

Now by (8.10)

\[
|\nabla_{h,x} \nabla_{h,y} G_h(x, y^{(k)})| \leq C \frac{d(y^{(k)}) + h}{(|x| - y^{(k)}) \cdot h + h}
\leq C \frac{d(y^{(k)}) + h}{(|x| - y^{(k)}) \cdot h + h} \leq C \frac{d(y) + h}{(|x| - y) \cdot h + h},
\]

and if we integrate this along \((y^{(k)})_{k=0}^L\), we obtain (8.7).

**Step 4: Proof of (8.6) and (8.9)**

We proceed as in the previous two steps. If we integrate (8.7) along a path \((x^{(k)})_{k=0}^L\) that avoids \(y\) once, we obtain (8.6), and if we integrate once more, we obtain (8.5).

Now we complement this lemma with an estimate when \(x\) and \(y\) are close:

**Lemma 8.3.** Let \(n = 2\) or \(n = 3\) and \(G_h\) be the Green’s function of \(\Lambda_h^n\). If \(x, y \in \Lambda_h^n\) and \(|x - y| \leq \frac{d(y)}{8}\) then

\[
|G_h(x, y)| \leq C(d(x) + h)^{2-\frac{n}{2}}(d(y) + h)^{2-\frac{n}{2}},
\]

\[
|\nabla_{h,x} G_h(x, y)| \leq C(d(y) + h)^{3-n},
\]

(8.11)
\[ |\nabla^2_{h,x}G_h(x,y)| \leq \begin{cases} C \log \left( \frac{d(y) + h}{|x - y| + h} \right) & n = 2, \\ \frac{C}{|x - y| + h} & n = 3, \end{cases} \quad (8.13) \]

\[ |\nabla_{h,x}\nabla_{h,y}G_h(x,y)| \leq \begin{cases} C \log \left( \frac{(d(x) + h)(d(y) + h)}{|x - y| + h^2} \right) & n = 2, \\ \frac{C}{|x - y| + h} & n = 3. \end{cases} \quad (8.14) \]

**Proof.**

**Step 1: Pointwise estimate for \( \nabla^2_{h,x}\nabla_{h,y}G_h(x,y) \)**

We claim that if \( x, y \in \Lambda^h_i \) and \( |x - y|_\infty \leq \frac{d(y)}{4} \) then

\[ |\nabla^2_{h,x}\nabla_{h,y}G_h(x,y)| \leq \frac{C}{(|x - y| + h^{n-1})}. \quad (8.15) \]

The fact that we prove this for \( |x - y|_\infty \leq \frac{d(y)}{4} \) will give us a bit of space to wiggle around in the following steps where we integrate \( (8.15) \). The proof of \( (8.15) \) is similar to the proof of \( (8.9) \). The main difference is that this time we choose the cut-off function further away from the singularity.

If \( d(y) < 10h \) we can again use a trivial estimate: By Lemma \( 8.1 \) \( G_h(x', y') \) is bounded by \( Cd(x')^{2 - \frac{n}{2}}d(y')^{2 - \frac{n}{2}} \leq Ch^{4-n} \) if \( |x' - x| \leq h \) and \( |y' - y|_\infty \leq h \), so that

\[ |\nabla^2_{h,x}\nabla_{h,y}G_h(x,y)| \leq C \frac{1}{h^4}h^{4-n} = Ch^{4-n}. \]

Therefore \( (8.15) \) holds if we choose \( C \) sufficiently large.

So assume that \( d(y) \geq 10h \). Let \( \eta_h \) be a discrete cut-off function that is 1 on \( Q_{d(y)/2 + 2h}(y) \) and 0 on \( (h\mathbb{Z})^n \setminus Q_{d(y) - 2h}(y) \) and such that \( |\nabla^k \eta_h| \leq \frac{C}{d(x)^\kappa} \) for \( \kappa \leq 2 \) and let \( H_h(x, y) = G_h(x, y) - \eta_h(x)\tilde{G}_h(x, y) \), where \( \tilde{G}_h \) is the function from Lemma \( 7.2 \) with \( r = d(y) \).

Then, for \( i \in \{1, \ldots, n\} \), \( D_{i}^{h,y}H_{h,y} \in \Phi_h \) and \( \Delta^2_{h}D_{i}^{h,y}H_{h,y} = 0 \) in \( Q_{d(y)/2}(y) \) and in \( \text{int} \Lambda^h_i \setminus Q_{d(y)}(y) \). We can estimate \( \|\nabla^2_{h}D_{i}^{h,y}H_{h,y}\|_{L^2(\mathbb{R}^n)} \) just as in Step 1 of the proof of Lemma \( 8.2 \) and obtain that

\[ \|\nabla^2_{h}D_{i}^{h,y}H_{h,y}\|_{L^2(\mathbb{R}^n)} \leq Cd(y)^{1 - \frac{n}{2}}. \quad (8.16) \]

Now recall that \( H_h \) is biharmonic in \( Q_{d(y)/2}(y) \). So Theorem \( 6.1 \) implies for \( x \in Q_{d(y)/4}(y) \)

\[ |\nabla^2_{h}D_{i}^{h,y}H_{h,y}(x)| \leq \frac{C}{d(y)^\frac{n}{2}}\|\nabla^2_{h}D_{i}^{h,y}H_{h,y}\|_{L^2(\mathbb{R}^n)} \leq Cd(y)^{1-n}. \]

Because \( \nabla^2_{h}D_{i}^{h,y}H_{h,y} = \nabla^2_{h}D_{i}^{h,y}G_{h,y} - \nabla^2_{h}D_{i}^{h,y}\tilde{G}_{h,y} \) in \( Q_{d(y)/2}(y) \) we can use \( (8.16) \) and obtain

\[ |\nabla^2_{h}D_{i}^{h,y}G_{h,y}(x)| \leq |\nabla^2_{h}D_{i}^{h,y}H_{h,y}(x)| + |\nabla^2_{h}D_{i}^{h,y}\tilde{G}_{h,y}(x)| \]

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\[
\leq C \left( \frac{1}{d(y)^{n-1}} + \frac{1}{(|x-y| + h)^{n-1}} \right).
\]

This implies (8.15) if we use that \(|x-y|_{\infty} \leq \frac{d(y)}{4}\) and \(d(y) \geq 10h\) so that \(|x-y| + h \leq C d(y)\).

**Step 2: Proof of (8.14)**

If \(d(y) < 4h\) we can repeat the trivial estimate from the previous step, so assume \(d(y) \geq 4h\).

We want to integrate (8.15) along a suitable path. So let \((x^{(k)})_{k=0}^{L}\) be a straight path from \(x^{(0)} = x\) to a closest point \(x^{(L)} \in Q_{d(y)/4}(y) \setminus Q_{d(y)/4-h}(y)\). This path will have length \(Lh = \left\lfloor \frac{d(y)}{4} \right\rfloor h - |x-y|_{\infty}\). By Lemma 8.2 we have

\[
|\nabla_{h,x} \nabla_{h,y} G_h(x^{(L)}, y)| \leq C \frac{(d(x^{(L)}) + h)(d(y) + h)}{|x^{(L)} - y|^{n}} \leq C \frac{d(y) + h}{|d(y) + h|^{n}} \leq C \frac{1}{|d(y) + h|^{n-2}}. \tag{8.17}
\]

Furthermore (8.15) implies that

\[
|\nabla_{h,x} \nabla_{h,y} G_h(x^{(k)}, y)| \leq \frac{C}{(|x^{(k)} - y| + h)^{n-1}} \leq \frac{C}{(|x - y| + (k+1)h)^{n-1}}. \tag{8.18}
\]

Now we can integrate (8.15) along \((x^{(k)})_{k=0}^{L}\) and use (8.17), and after a short calculation we arrive at (8.14).

**Step 3: Proof of (8.13)**

If \(d(y) < 77h\) we can again use the trivial estimate from Step 1, so assume \(d(y) \geq 77h\).

This is similar to the previous step: We choose a shortest-possible path \((y^{(k)})_{k=0}^{L}\) from \(y^{(0)} = y\) to a point \(y^{(L)} \in Q_{d(x)/6}(x) \setminus Q_{d(x)/6-h}(y)\). Then \(|y^{(k)} - x|_{\infty} \leq \frac{d(x)}{6}\), so that \(\frac{5}{6}d(x) \leq d(y^{(k)}) \leq \frac{7}{6}d(x)\) and hence

\[
|y^{(k)} - x|_{\infty} \leq \frac{d(x)}{6} \leq \frac{d(y^{(k)})}{5}.
\]

Therefore we can apply (8.15) at the point \((x, y^{(k)})\) for each \(k\) and conclude

\[
|\nabla_{h,x} \nabla_{h,y} G_h(x, y^{(k)})| \leq \frac{C}{(|x - y^{(k)}| + h)^{n-1}} \leq \frac{C}{(|x - y| + (k+1)h)^{n-1}}. \tag{8.19}
\]

On the other hand,

\[
d(y^{(L)}) \geq \frac{5}{6}d(x) \geq \frac{5}{6} \cdot \frac{7}{6}d(y) \geq 56h
\]

so that

\[
|y^{(L)} - x|_{\infty} \geq \frac{d(x)}{6} - h \geq \frac{d(y^{(L)})}{7} - h > \frac{d(y^{(L)})}{8}.
\]
This means that we can apply (8.8) at the point \((x, y^{(L)})\) and conclude

\[
|\nabla_{h,x}^2 G_h(x, y^{(L)})| \leq C \frac{(d(y^{(L)}) + h)^2}{|x - y^{(L)}|^n} \leq C \frac{(d(y) + h)^2}{|d(y) + h|^n} \leq C, \quad (8.20)
\]

Now we can integrate (8.19) along the path \((y^{(k)})_{k=0}^L\) and use the estimate (8.20) for the one endpoint to obtain (8.13).

**Step 4: Proof of (8.12)**

We could try to prove this by integrating (8.13) along a path. However, this turns out to be not sharp enough at least if \(n = 3\) (we would get a logarithmic term instead of a constant term). Instead we will use the Sobolev inequality on the function \(H_{h,y}\) from Step 1. Thereby we get a bound for \(\nabla_{h,y} G_h(x, y)\) if \(x, y\) are close. By the symmetry of \(G_h\) we can turn this into a bound for \(\nabla_{h,x} G_h(x, y)\).

If \(d(y) < 10h\) we can again use the trivial estimate from Step 1, so assume \(d(y) \geq 10h\). Recall the function \(H_{h,y}\) from Step 1. If we use the Sobolev and Poincaré inequality on \(Q_{d(y)+h/2}(y)\) and the estimate (8.16) we obtain

\[
\|D^{h,y}_{i} H_{h,y}\|_{L^\infty(Q_{d(y)+h/2}(y))} \leq C(d(y) + h/2)^{2-\frac{n}{2}} \|\nabla_{h}^2 D^{h,y}_{i} H_{h,y}\|_{L^2(Q_{d(y)+h/2}(y))} \leq C(d(y))^{2-\frac{n}{2}} \|\nabla_{h} D^{h,y}_{i} H_{h,y}\|_{L^2(\mathbb{R}^n)} \leq C(d(y))^{3-n}
\]

and therefore

\[
|\nabla_{h,y} H_{h,y}(x)| \leq C(d(y))^{3-n}
\]

for any \(x \in Q_{d(y)}(y)\). Now we can use (8.12) and the fact that \(D^{h,y}_{i} H_{h,y} = D^{h,y}_{i} G_{h,y} - D^{h,y}_{i} \tilde{G}_{h,y}\) in \(Q_{d(y)/2}(y)\) and obtain

\[
|D^{h,y}_{i} G_{h,y}(x)| \leq |D^{h,y}_{i} H_{h,y}(x)| + |D^{h,y}_{i} \tilde{G}_{h,y}(x)| \leq C(d(y))^{3-n}
\]

for any \(x \in Q_{d(y)/2}(y)\) and any \(i \in \{1, \ldots, n\}\). By the symmetry of \(G_h\) in \(x\) and \(y\) we conclude that also

\[
|D^{h,x}_{i} G_{h,x}(y)| \leq C(d(x))^{3-n} \quad (8.21)
\]

for any \(y \in Q_{d(x)/2}(x)\).

Now in the setting of (8.12) we are given \(x, y\) with \(|y - x|_\infty \leq \frac{d(y)}{4}\). These satisfy \(\frac{3}{4}d(y) \leq d(x) \leq \frac{5}{4}d(y)\), so that \(|y - x|_\infty \leq \frac{1}{2}d(x)\) and in particular \(y \in Q_{d(x)/2}(x)\). Thus we can apply (8.21) and obtain

\[
|D^{h,x}_{i} G_{h,x}(y)| \leq C d(x)^{3-n} \leq C d(y)^{3-n},
\]

which implies (8.12).

**Step 5: Proof of (8.11)**

This follows immediately from (8.3). \(\square\)
Proof of Theorem 1.3 i). Now that we have proved Lemma 8.3 and Lemma 8.2 the proof is straightforward. First observe that it suffices to consider \( x, y \in \Lambda^n \) as otherwise \( G_h \) and its relevant derivatives are trivially 0.

We claim that we can combine (8.8) and (8.14) to obtain (1.11). Indeed, if \( |x - y| \leq \frac{d(y)}{8} \) we have \( d(y) \leq \frac{8}{3}d(x) \) and \( |x - y| + h \leq \sqrt{n}|x - y| + h < d(y) + h \) which implies

\[
1 \leq \frac{(d(x) + h)(d(y) + h)}{|x - y|^n}
\]

and we are done by (8.14).

If however \( |x - y| \geq \frac{d(y)}{8} \), then we have in particular \( |x - y| \geq h \), so that \( |x - y| + h \leq 2|x - y| \). We also have \( d(y) \leq 8|x - y| \) and \( d(x) \leq 9|x - y| \) and we easily see that

\[
\frac{(d(x) + h)(d(y) + h)}{|x - y|^n} \leq \frac{C}{(|x - y| + h)^{n-2}}
\]

so we are done by (8.8).

Similarly, we can combine (8.7) and (8.13) into the estimate

\[
|\nabla^2_{h, x} G_h(x, y)| \leq \begin{cases} C \log \left( 1 + \frac{(d(y) + h)^2}{|x - y| + h} \right) & n = 2 \\ C \min \left( 1 + \frac{(d(y) + h)^2}{|x - y| + h}, \frac{(d(y) + h)^2}{|x - y| + h} \right) & n = 3 \end{cases}
\]

This is not quite (1.10), but it implies (1.10) unless \( d(y) = 0 \). On the other hand, if \( d(y) = 0 \) then \( y \in \partial \Lambda^h \). Therefore \( G_{h,y} \) is identically 0, so that \( \nabla_{h,x} G_h(x, y) = 0 \) and (1.10) holds as well.

Similarly we can combine (8.9) and (8.12), and (8.8) and (8.14) into

\[
|\nabla_{h, x} G_h(x, y)| \leq C \min \left( (d(x) + h)^{3-n}, \frac{(d(x) + h) d(y) + h}{|x - y| + h} \right),
\]

\[
|G_h(x, y)| \leq C \min \left( (d(x) + h)^{2-n} (d(y) + h)^2 \frac{2}{n}, \frac{(d(x) + h)^2 (d(y) + h)^2}{(|x - y| + h)^n} \right)
\]

respectively. These estimates imply (1.10) and (8.11), except in the cases \( d(x) = 0 \) or \( d(y) = 0 \), which are again trivial.

Remark 8.4. As a byproduct of the proofs of Lemma 8.3 and Lemma 8.2 we proved the estimates (8.9) and (8.15) which can easily be combined into the estimate

\[
|\nabla^2_{h, x} \nabla_{h, y} G_h(x, y)| \leq C \min \left( \frac{1}{(|x - y| + h)^{n-1}}, \frac{d(y) + h}{(|x - y| + h)^n} \right)
\]

(8.22)

for any \( x, y \in (h\mathbb{Z})^n \).
With the same method of proof it is possible to prove an estimate for \( \nabla^2_{h,x} \nabla^2_{h,y} G_h \) as well. One again considers \( H_{h,y} = G_{h,y} - \eta_h \tilde{G}_{h,y} \) in Lemma 8.3 and Lemma 8.2 and derives estimates for \( \| \nabla^2_{h,x} \nabla^2_{h,y} H_{h,y} \|_{L^2(\mathbb{R}^n)} \). In combination with the pointwise estimates for \( \tilde{G}_h \) (in particular (7.3)) these again yield estimates for \( \nabla^2_{h,x} \nabla^2_{h,y} G_h \) in the two regimes where \( x \) and \( y \) are far away and close together, respectively. The final result is

\[
|\nabla^2_{h,x} \nabla^2_{h,y} G_h(x,y)| \leq \frac{C}{(|x-y| + h)^n} 
\]

for any \( x, y \in (h\mathbb{Z})^n \).

Actually it is even possible to derive estimates for higher derivatives \( \nabla^a_{h,x} \nabla^b_{h,y} G_h \), at least when \( a \leq 2 \) or \( b \leq 2 \). However we cannot expect these estimates to be optimal any more, because high derivatives are increasingly divergent near the singular boundary points, and our approach does not really capture this behaviour.

### 8.3 Convergence of Green’s functions

**Proof of Corollary 1.4**. We begin with the proof of assertion i). We can assume that \( h \leq \frac{1}{3} \). There exists a unique \( y_h \in \Lambda^n_h \) such that \( y \in y_h + [-\frac{h}{2}, \frac{h}{2})^n \). Set \( u_h(x) = G_h(x,y_h) \). We extend \( u_h \) by zero to \( (h\mathbb{Z})^n \setminus \text{int} \Lambda^n_h \).

To prove (i) we have to show that \( u_h \) converges uniformly to \( G(\cdot, y) \).

Testing the equation for \( \Delta^2_h u_h \) with \( u_h \) we get (see Lemma 8.1)

\[
\| \nabla^2_h u_h \|_{L^2(\mathbb{R}^n)} \leq Cd^{2-\frac{n}{2}}(y_h) \leq C.
\]

The discrete Sobolev-Poincaré inequality implies in particular that the \( u_h \) are uniformly Hölder continuous

\[
[u_h]_{C^{0,\frac{1}{4}}_h(\mathbb{R}^n)} \leq C. 
\]

(8.24)

Denote by \( J_h \) the interpolation operator introduced in Section 4. From Proposition 4.2 vi) and the Poincaré inequality we deduce that the sequence \( J_h u_h \) is bounded in \( W^{2,2}(\mathbb{R}^n) \) and \( J_h u_h = 0 \) in \( \mathbb{R}^n \setminus (-3h,1+3h)^n \). It follows that for a subsequence

\[
J_h u_h \rightharpoonup u \quad \text{in} \quad W^{2,2}(\mathbb{R}^n), \quad u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus (0,1)^n.
\]

From the uniform Hölder continuity (8.24) and Proposition 4.2 iii), iv) and vi) we deduce that, for any \( x \in (-3h,1+3h)^n \),

\[
|J_h u_h(x) - I^{pc}_{h_k} u_h(x)| = |J_h(u_h(\cdot) - u_h(x))(x)| \leq C\|u_h - u_h(x)\|_{L^\infty(Q_{3h_k}(x))} \leq Ch_k^\frac{1}{4}
\]

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and therefore
\[ \sup_{x \in (-1,2)^n} |J_{h_k} u_{h_k}(x) - I^{pc}_{h_k} u_{h_k}(x)| \leq C h_k^{\frac{3}{2}}. \]

In connection with the compact embedding from \( W^{2,2}_0((-1,2)^n) \) to \( C^0((-1,2)^n) \) we conclude that
\[ I^{pc}_{h_k} u_{h_k} \to u \quad \text{uniformly}. \quad (8.25) \]

If we can show that \( u(x) = G(x,y) \) then by uniqueness of the limit it follows that the convergences above do not only hold along a particular subsequence \( h_k \to 0 \) but for every subsequence \( h_k \to 0 \) and we are done.

To show that \( u(x) = G(x,y) \) we use that by definition of \( G_h(\cdot,y_h) \) we have for each \( \varphi \in C^\infty_c((0,1)^n) \)
\[
\varphi(y_h) = \sum_{x \in \text{int} \Lambda_h} \Delta^2_{h_k} u_{h_k}(x) \varphi(x) h_k^6 = \sum_{x \in \text{int} \Lambda_h} u_{h_k}(x) \Delta^2_{h_k} \varphi(x) h_k^6
\]
\[ = \int_{(0,1)^n} I^{pc}_{h_k} u_{h_k} I^{pc}_{h_k} \Delta^2_{h_k} \varphi(x) \, dx. \]

Now by Taylor expansion \( |I^{pc}_{h_k} \Delta^2_{h_k} \varphi - \Delta^2 \varphi| \leq C h_k \). Together with (8.25) we get
\[ \varphi(y) = \lim_{k \to \infty} \varphi(y_h) = \lim_{k \to \infty} \int_{(0,1)^n} I^{pc}_{h_k} u_{h_k} I^{pc}_{h_k} \Delta^2_{h_k} \varphi \, dx = \int_{(0,1)^n} u \Delta^2 \varphi \, dx. \]

Thus \( \Delta^2 u = \delta_y \) in the sense of distributions. Since we also know that \( u \in W^{2,2}_0((0,1)^n) \) we conclude that \( u(x) = G(x,y) \) as desired.

To prove ii) note that the estimates in Theorem 1.3 show that the second discrete derivatives are bounded in \( L^p \) for all \( p < \infty \). Hence by the discrete Sobolev embedding theorem the discrete first derivatives are bounded in \( C^{0,\alpha} \) for all \( \alpha < 1 \). This implies that
\[ |I^{pc}_{h} \nabla u - \nabla J_{h_k} u| \leq C h^{\alpha}. \quad (8.26) \]

Moreover the \( L^p \) bound on the discrete second derivatives and (1.3) give a bound of \( J_{h_k} u_{h_k} \) in \( W^{2,p} \). Hence a subsequence of \( J_{h_k} u_{h_k} \) converges in \( C^{1,\alpha} \) to \( G(\cdot,y) \). Since the limit is unique, the whole sequence converges in \( C^{1,\alpha} \) to \( G \). Together with (8.26) this yields uniform convergence of the discrete first derivatives.

The local compactness argument in Section \( \Box \) (and a diagonalisation argument) shows that a subsequence of \( I^{pc}_{h} \nabla^2 u_{h_k} \) converges in \( L^2_{\text{loc}}((0,1)^2 \setminus \{y\}) \) to a function \( v \). Since \( I^{pc}_{h} \nabla^2 u_{h_k} \) is also bounded in \( L^q \) for some \( q > 2 \) we get strong convergence in \( L^2((0,1)^2) \). Using again the \( L^q \) bound we get strong convergence in all \( L^p \) with \( p < q \). Since we have \( L^q \) bounds for all \( q < \infty \) we get strong convergence for all \( p < \infty \). It remains to show that \( v = \nabla^2 G(\cdot,y) \).
To obtain this identity we can use discrete integration by parts and pass to the limit on both sides, as in the proof that $\Delta^2 u = \delta_y$.

The proof of (iii) is similar. Uniform boundedness of the discrete derivatives follows directly from Theorem 1.3. This theorem also shows that the second discrete derivatives are uniformly bounded on the complement of any cube $Q_r(y)$. It follows that the functions $u_h$ are uniformly Lipschitz on the complement of any cube $Q_r(y)$ and we obtain locally uniform convergence of $I_h^\kappa \nabla_h u_h$ in the complement of those cubes as in the proof of (ii). Combined with the uniform boundedness we immediately conclude convergence of $I_h^\kappa \nabla_h u_h$ in $L^p$ for all $p < \infty$.

The proof of $L^p$ convergence of $I_h^\kappa \nabla_h^2 u_h$ for $p < 3$ is again analogous to the argument for $n = 2$.

Acknowledgements

The authors are very grateful to G. Dolzmann for many inspiring conversations and suggestions. Indeed, G. Dolzmann and the first author derived in unpublished work estimates in the interior and near the regular boundary points using discrete Campanato estimates in the spirit of [Dol93, Dol99] and developed a version of the duality argument in Section 6.2. The authors are also very grateful to E. Süli for his insightful comments on a preliminary version of this manuscript.

This work has its roots in very interesting discussions with J.D. Deuschel, N. Kurt and E. Bolthausen in the framework of the Berlin-Leipzig DFG Research Unit 718 ‘Analysis and stochastics in complex physical systems’. The authors would like to thank A. Cipriani for pointing out connections to sandpile models.

SM was supported by the DFG through the Hausdorff Center for Mathematics (EXC59) and the CRC 1060 ‘The mathematics of emergent effects’, project A04. FS was supported by the Hausdorff Center for Mathematics through the Bonn International Graduate School for Mathematics (BIGS), and by the German National Academic Foundation. SM also strongly benefitted from the Trimester Programme ‘Mathematical challenges of materials science and condensed matter physics’ at the Hausdorff Research Institute for Mathematics (HIM).

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