Anisotropic scaling limits of long-range dependent linear random fields on $\mathbb{Z}^3$

Donatas Surgailis *

Vilnius University, Faculty of Mathematics and Informatics, Naugarduko 24, 03225 Vilnius, Lithuania

May 10, 2018

Abstract

We provide a complete description of anisotropic scaling limits of stationary linear random field on $\mathbb{Z}^3$ with long-range dependence and moving average coefficients decaying as $O(|t|^\gamma_i)$ in the $i$th direction, $i = 1, 2, 3$. The scaling limits are taken over rectangles in $\mathbb{Z}^3$ whose sides increase as $O(\lambda^{\gamma_i})$, $i = 1, 2, 3$ when $\lambda \to \infty$, for any fixed $\gamma_i > 0$, $i = 1, 2, 3$. We prove that all these limits are Gaussian RFs whose covariance structure essentially is determined by the fulfillment or violation of the balance conditions $\gamma_i q_i = \gamma_j q_j$, $1 \leq i < j \leq 3$. The paper extends recent results in [26], [27], [23], [24] on anisotropic scaling of long-range dependent random fields from dimension 2 to dimension 3.

Keywords: scaling transition; long-range dependence; linear random field; operator self-similar random field; fractional Brownian sheet

1 Introduction

Let $X = \{X(t); t \in \mathbb{Z}^d\}$ be a stationary random field (RF) on $\mathbb{Z}^d$, $d \geq 1$, $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{R}_+^d$ be a collection of positive numbers (exponents), and

$$K_{\lambda, \gamma}(x) := \prod_{i=1}^d [1, [\lambda^{\gamma_i} x_i]], \quad x = (x_1, \cdots, x_d) \in \mathbb{R}_+^d, \quad (1.1)$$

be a family of $d$-dimensional ‘rectangles’ indexed by $\lambda > 0$, whose sides grow at possibly different rate $O(\lambda^{\gamma_i})$, $i = 1, \cdots, d$ as $\lambda \to \infty$. Consider the partial sums RF:

$$S_{\lambda, \gamma}^X(x) := \sum_{t \in K_{\lambda, \gamma}(x)} X(t), \quad x \in \mathbb{R}_+^d, \quad (1.2)$$

See the end of this section for all unexplained notation. We are interested in the limit distribution of normalized partial sums $\{1.2\}$:

$$A_{\lambda, \gamma}^{-1} S_{\lambda, \gamma}^X(x) \xrightarrow{fdd} V_{\gamma}^X(x), \quad x \in \mathbb{R}_+^d \quad (1.3)$$

*E-mail: donatas.surgailis@mii.vu.lt
as \( \lambda \to \infty \), where \( A_\lambda \gamma \to \infty \) is a normalization. Following \[23\], the family \( \{V_\gamma^X; \gamma \in \mathbb{R}_+^d\} \) of all scaling limits in (1.3) will be called the scaling diagram of RF \( X \).

The above problem is classical for RFs except that most previous work dealt with case \( \gamma_1 = \cdots = \gamma_d = 1 \) only. See \[1\], \[7\], \[8\], \[17\], \[18\], \[28\], \[9\], \[15\], \[12\] and the references therein. In the latter case, (1.3) is naturally referred to as isotropic scaling while that with \( \gamma \neq (1, \cdots, 1) \) as anisotropic scaling. For weakly dependent RFs anisotropic scaling is not very interesting since in such case, summation domains may have very general shape and the scaling diagram usually consists of a single point (white noise), or is empty. See e.g. \[5\]. Particularly, we note that \( d \)-dimensional rectangles in (1.1) satisfy van Hove’s condition for any \( \gamma \in \mathbb{R}_+^d \).

The situation is very different for long-range dependent (LRD) RFs. Although there is no single satisfactory definition of LRD, usually it refers to stationary RF \( X \) with nonsummable covariance function, or unbounded spectral density, see \[8\], \[17\], \[10\], \[15\], \[12\]. \[27\] observed that for a large class of LRD RFs \( X \) on \( \mathbb{Z}^2 \), nontrivial limits in (1.3) exist for any \( \gamma = (\gamma_1, \gamma_2) \in \mathbb{R}_+^2 \); moreover, there exists \( \gamma^0 > 0 \) such that \( V_\gamma^X \equiv V_\pm^X \) do not depend on \( \gamma = (\gamma_1, \gamma_2) \) for \( \gamma_2/\gamma_1 > \gamma^0 \) and \( \gamma_2/\gamma_1 < \gamma^0 \), respectively, and the RFs and \( V_+^X, V_-^X \) are different in the sense that \( V_\gamma^X \neq aV_0^X (\forall a > 0) \). \[27\] called the above phenomenon the scaling transition.

The existence of scaling transition was established in \[26\], \[27\], \[24\] for a wide class of Gaussian, linear and related nonlinear RFs on \( \mathbb{Z}^2 \). It turned out that for above classes RFs, the scaling limits \( V_\gamma^X, V_-^X \) have a very different dependence structure from \( V_0^X \), the value \( \gamma^0 \) being related to the intrinsic scale ratio (the ratio of Hurst exponents) of \( X \) along the vertical and horizontal axes. Since \( V_0^X, V_\pm^X \) arise in accordance or in violation of the ‘balance condition’ \( \gamma_2 = \gamma_0 \gamma_1 \), \[27\] termed \( V_0^X \) the well-balanced and \( V_\pm^X \) the unbalanced scaling limits of \( X \), respectively. A different kind of scaling transition was established for RFs arising by aggregation of network traffic and random-coefficient AR(1) time series models in telecommunications and economics, see \[11\], \[20\], \[13\], \[21\], \[19\], \[22\], \[16\], also Remark 2.3 in \[27\]. On the other hand, for some RFs in dimension 2 the scaling diagram may have more than three elements, see \[23\], and there are classes of LRD RFs which do not exhibit scaling transition (the scaling diagram consists of a single element), see \[20\], \[6\].

Since almost all of the above-mentioned work dealt with planar RF models, a challenging open problem raised in \[26\], \[24\] is anisotropic scaling and identification of the scaling diagrams of LRD RFs in dimensions \( d > 2 \). The present paper solves this problem for linear, or moving-average RFs in dimension \( d = 3 \):

\[
X(t) = \sum_{s \in \mathbb{Z}^3} a(t - s) \varepsilon(s), \quad t = (t_1, t_2, t_3) \in \mathbb{Z}^3, \tag{1.4}
\]

where \( \{\varepsilon(s); s \in \mathbb{Z}^3\} \) is an i.i.d. sequence with zero mean and unit variance, and \( \{a(t); t \in \mathbb{Z}^3\} \) are deterministic coefficients having the form

\[
a(t) = \frac{g(t)}{\left(\sum_{j=1}^3 c_j |t_j|^\nu_j/n_n\right)\nu}, \quad t = (t_1, t_2, t_3) \in \mathbb{Z}^3, \tag{1.5}
\]

where \( |t|_+ := |t| \wedge 1, t \in \mathbb{Z} \), \( g(t), t \in \mathbb{Z}^3 \) are bounded with \( \lim_{|t| \to \infty} g(t) =: g_\infty \in (0, \infty) \) and \( \nu > 0, q_j > 0 \).
0, \( c_j > 0, j = 1, 2, 3 \) are parameters satisfying the following inequalities:

\[
1 < Q := \sum_{i=1}^{3} \frac{1}{q_i} < 2. \tag{1.6}
\]

Condition (1.6) guarantees that

\[
\sum_{t \in \mathbb{Z}^3} |a(t)|^2 < \infty \quad \text{and} \quad \sum_{t \in \mathbb{Z}^3} |a(t)| = \infty. \tag{1.7}
\]

In particular, \( X \) in (1.4) is a well-defined stationary RF with zero mean, finite variance and covariance

\[
EX(t)X(0) = \sum_{s \in \mathbb{Z}^3} a(t - s)a(s), \quad t \in \mathbb{Z}^3. \tag{1.8}
\]

Notice that \( a(t) = O(|t_i|^{-q_i}) \) as \( |t_i| \to \infty \) meaning that when the \( q_i \) are different, the moving-average coefficients decay at different rate in different directions of \( \mathbb{Z}^3 \), in which case the RF \( X \) exhibits strong anisotropy. On the other hand, when \( q_i \equiv q, i = 1, 2, 3 \), the RF \( X \) is ‘nearly isotropic’ and conditions (1.6) reduce to \( 3/2 < q < 3 \). The parameter \( q_i \) representing ‘typical scale’ of \( X \) in the \( i \)th direction, \( i = 1, 2, 3 \), we may consider \( \gamma_{ij}^0 = q_i/q_j, i, j = 1, 2, 3, i > j \) as ‘intrinsic scale ratios’ leading to three balance conditions

\[
\gamma_3/\gamma_1 = \gamma_{31}^0, \quad \gamma_3/\gamma_2 = \gamma_{32}^0 \tag{1.9}
\]

among which only two are independent since any two of (1.9) imply the third one. Depending on which of the balance conditions in (1.9) are fulfilled or violated, we may expect different scaling limits \( V^X_\gamma \) of the partial sums in (1.2) of the linear RF \( X \).

The results of this paper confirm the above intuition. We prove that for linear RFs in (1.4) the limits \( V^X_\gamma \) in (1.3) exist for any \( \gamma \in \mathbb{R}_+^3 \); moreover depending on \( \gamma \) these limits can be divided into three groups: the well-balanced limit arising when \( \gamma \) satisfies all balance conditions in (1.9) (group 1); ‘partially unbalanced’ limits arising when \( \gamma \) satisfies only one of the balance conditions in (1.9) (group 2), and ‘completely unbalanced’ limits arising when \( \gamma \) satisfies none of the balance conditions in (1.9) (group 3). Furthermore, the limit RFs in group 3 agree with FBS (fractional Brownian sheet) \( B_{H_1,H_2,H_3} \) on \( \mathbb{R}_+^3 \) with at least two among three indices \( H_i \in (0,1], i = 1, 2, 3 \) equal to 1 or 1/2, while RFs in group 2 are not FBS but have some ‘partial FBS property’ in one direction.

Let us describe the contents of this work. Sec. 2.1 provides a formal definition of the partition of the set \( \mathbb{R}_+^3 = \{ \gamma \} \) of scaling exponents into 13 sets \( \Gamma_{000}, \ldots, \Gamma_{-111} \) induced by balance conditions (1.9). Sec. 2.2 identifies 3 regions (Regions I, II, and III) in the parameter space \( \{ q = (q_1,q_2,q_3) \in \mathbb{R}_+^3 : 1 < Q < 2 \} \) determined by (1.6) providing a classification of the linear RF \( X \) in (1.4) according to the convergence/divergence of the covariance function on coordinate axes/coordinate planes in \( \mathbb{Z}^3 \). In Sec. 3 we define limit Gaussian RFs as stochastic integrals w.r.t. white noise in \( \mathbb{R}^3 \) with kernels taking a different form in Regions I, II, and III, and discuss their self-similarity properties. We also relate some these limit RFs to FBS with two Hurst parameters equal to 1 or 1/2. Sec. 4 contains the main result (Theorem 4.1 and Corollary 4.2), by identifying all scaling limits in (1.3). Proofs of the main results are given in Sec. 5.
The following comments are in order. We expect that our results can be extended for linear RFs in dimension $d > 3$ with coefficients $a(t), t \in \mathbb{Z}^d$ having a similar form as in (1.5) (with (1.6) replaced by $1 < Q = \sum_{i=1}^{d} \frac{1}{q_i} < 2$); however, the description of the scaling limits when $d > 3$ seems cumbersome and we restrict ourselves to dimension $d = 3$ for relative transparency of exposition. Although the results of this paper can be interpreted as a scaling transition occurring at the boundaries of the balance partition, see Fig. 1 below, we do not attempt to provide a formal definition of the latter concept for RFs in dimensions $d = 3$ or higher. On the other hand, at present there are many open problems about anisotropic scaling even for linear RFs in dimension $d = 2$. Particularly, we mention the case (linear) RFs with infinite variance and/or negatively dependent RFs with coefficients as (1.5) but with $\sum_{t \in \mathbb{Z}^d} |a(t)| < \infty$ (or $Q < 1$) and satisfying $\sum_{t \in \mathbb{Z}^d} a(t) = 0$. See also [13] on isotropic scaling of negatively dependent linear RFs.

**Notation.** In what follows, $C, C_1, C_2$ denote generic positive constants which may be different at different locations. We write $\Rightarrow, \Leftrightarrow, \Leftrightarrow$ for the weak convergence, equality and inequality of finite-dimensional distributions, respectively. $\mathbb{R}_+^d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i > 0, i = 1, \ldots, d\}, \mathbb{R}_+^d := \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_i \geq 0, i = 1, \ldots, d\}, |x| := \max_{1 \leq i \leq d} |x_i|, \mathbb{R}_+ := \mathbb{R}_+^1, \mathbb{R}_+ := \mathbb{R}_+^2, \mathbb{R}_0^2 := \mathbb{R}_0^2 \setminus \{(0,0)\}$. $1(A)$ stands for the indicator function of a set $A$.

## 2 Preliminaries

The description of anisotropic limits in (1.3), or the limiting Gaussian RFs $V^X_q$, in the case $d = 3$ is considerably more complicated as in the case $d = 2$ in [20], [24]. The limit RFs take a different form in different regions depending both on $\gamma$ and $q$. These regions are specified in the following subsections.

### 2.1 The balance partition

For $(\gamma_{21}, \gamma_{31}, \gamma_{32}) \in \mathbb{R}_+^3, \gamma_{32} = \gamma_{31}/\gamma_{21}$ consider the partition

$$\mathbb{R}_+^3 = \bigcup_{i \in \varphi} \Gamma_i$$

(2.1)

of the set $\mathbb{R}_+^3$ of scaling exponents into 13 sets $\Gamma_i, i \in \varphi$ defined as

$$\Gamma_{000} := \{\gamma \in \mathbb{R}_+^3 : \gamma_2/\gamma_1 = \gamma_{21}, \gamma_3/\gamma_1 = \gamma_{31}/\gamma_{21}, \gamma_3/\gamma_2 = \gamma_{32}\},$$

$$\Gamma_{011} := \{\gamma \in \mathbb{R}_+^3 : \gamma_2/\gamma_1 = \gamma_{21}, \gamma_3/\gamma_1 > \gamma_{31}/\gamma_{21}, \gamma_3/\gamma_2 > \gamma_{32}\},$$

$$\Gamma_{-111} := \{\gamma \in \mathbb{R}_+^3 : \gamma_2/\gamma_1 < \gamma_{21}, \gamma_3/\gamma_1 > \gamma_{31}/\gamma_{21}, \gamma_3/\gamma_2 > \gamma_{32}\}.$$

That is, the index in $\Gamma_i$ is $i = k_{21}k_{31}k_{32}, k_{ij} \in \{1,0,-1\}$ means that $\gamma_i/\gamma_j > \gamma_{ij}$ if $k_{ij} = 1$, $\gamma_i/\gamma_j = \gamma_{ij}$ if $k_{ij} = 0$ and $\gamma_i/\gamma_j < \gamma_{ij}$ if $k_{ij} = -1$, for any $3 \geq i > j \geq 1$. Thus, the set $\varphi = \{i\}$ consists of 13 triples:

$$\varphi = \{000, 011, 110, 10-1, 0-1-1, -1-10, -101, 111, 11-1, 1-1-1, -1-1-1, -1-11, -111\}.$$  (2.2)
The corresponding partition of $\mathbb{R}^3_+ = \{(\gamma_2/\gamma_1, \gamma_3/\gamma_1)\}$ is shown in Figure 1 below. There, the line $\Gamma_{000} \subset \mathbb{R}^3_+$ satisfying all three balance conditions in (1.9) (the ‘well-balanced’ set) reduces to the single point $(\gamma_{111}^0, \gamma_{31}^0) = (q_1/q_2, q_1/q_3) \in \mathbb{R}^2_+$, the two-dimensional sets $\Gamma_{011}, \Gamma_{110}, \Gamma_{101}, \Gamma_{0-1-1}, \Gamma_{1-11}, \Gamma_{1-10}, \Gamma_{-111}, \Gamma_{-1-1-1}$ satisfying only one of the balance conditions in (1.9) (the ‘partly balanced’ sets) become line segments, and the three-dimensional sets $\Gamma_{111}, \Gamma_{11-1}, \Gamma_{1-11}, \Gamma_{1-1-1}, \Gamma_{-111}$ which violate two (or all) balance conditions in (1.9) (the ‘completely unbalanced’ sets) are projected as sets of dimension 2.

![Figure 1. Partition of the quotient space $\mathbb{R}^3_+ = \{(\gamma_2/\gamma_1, \gamma_3/\gamma_1)\}$ induced by balance partition (2.1). The shaded region corresponds to the subset in (2.3).](image)

**2.2 Covariance structure of linear LRD RF on $\mathbb{Z}^3$**

As noted above, the scaling limits $V^\chi_f$ depend on parameters $q_j, c_j, j = 1, 2, 3$ in (1.5). The dependence on the parameters is generally different in different regions $\Gamma_i, i \in \wp$. Essentially, it suffices to consider the region

$$\{\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3_+ : \gamma_1 q_1 \leq \gamma_2 q_2 \leq \gamma_3 q_3\} = \bigcup_{r=000,011,111,110} \Gamma_r$$

(2.3)

(the shaded region in Figure 1) only. Indeed, as shown in Corollary 4.2 below, for other $\gamma$’s, $V^\chi_f$ can be defined via a ‘permutation’ of $q_j, c_j, j = 1, 2, 3$. For $\gamma$ in (2.3), there are three parameter regions of $q = (q_1, q_2, q_3)$ defined as follows:

- **Region I:**
  \[ \frac{1}{2q_1} + \sum_{j=2}^3 \frac{1}{q_j} < 1 < \sum_{j=1}^3 \frac{1}{q_j}, \]  
  (2.4)

- **Region II:**
  \[ \sum_{j=1}^2 \frac{1}{2q_j} + \frac{1}{q_3} < 1 < \frac{1}{2q_1} + \sum_{j=2}^3 \frac{1}{q_j}, \]  
  (2.5)

- **Region III:**
  \[ \sum_{j=1}^3 \frac{1}{2q_j} < 1 < \sum_{j=1}^2 \frac{1}{2q_j} + \frac{1}{q_3}. \]  
  (2.6)

In the ‘isotropic’ case $q_1 = q_2 = q_3 =: q$, (2.4)-(2.6) reduce to $1.5 < q < 2$ (Region I), $2 < q < 2.5$ (Region II) and $2.5 < q < 3$ (Region III), respectively.

Since the dependence in RF $X$ generally decreases as the $q_j$’s increase, we may say that the dependence in $X$ increases from Region I to Region III. A more precise probabilistic meaning of the inequalities (2.4)-(2.6)
in terms of summability of the covariance function \( r_X(t) := \text{EX}(0)X(t) \) on coordinate axes and coordinate planes in \( \mathbb{Z}^3 \) is provided in the following proposition.

**Proposition 2.1** Let \( X = \{ X(t); t \in \mathbb{Z}^3 \} \) be a linear RF in \((1.4)-(1.5)\) satisfying \((1.6)\). Then the covariance \( r_X(t) = \text{EX}(0)X(t) \), \( t \in \mathbb{Z}^3 \) satisfies the following properties in respective parameter Regions I-III:

- **Region I:** \[ \sum_{(t_1,t_2,t_3) \in \mathbb{Z}^3} |r_X(t_1,t_2,t_3)| = \infty, \quad \sum_{(t_2,t_3) \in \mathbb{Z}^2} |r_X(0,t_2,t_3)| < \infty; \] (2.7)

- **Region II:** \[ \sum_{(t_2,t_3) \in \mathbb{Z}^2} |r_X(0,t_2,t_3)| = \infty, \quad \sum_{t_3 \in \mathbb{Z}} |r_X(0,0,t_3)| < \infty; \] (2.8)

- **Region III:** \[ \sum_{t_3 \in \mathbb{Z}} |r_X(0,0,t_3)| = \infty. \] (2.9)

**Remark 2.1** The divergence of the series in (2.9) can be interpreted as the LRD property of the sectional process \( \{ X(0,0,t_3); t_3 \in \mathbb{Z} \} \) on the coordinate axis \( t_3 \) in the parameter Region I. On the other hand, (2.8) say that, in the parameter Region II the last process is short-range dependent (SRD) but the sectional RF \( \{ X(0,t_2,t_3); (t_2,t_3) \in \mathbb{Z}^2 \} \) is LRD. Finally (2.9) say that in the parameter Region III the last sectional RF is SRD but the RF \( X \) on \( \mathbb{Z}^3 \) is LRD. Conditions (2.4)-(2.6) are not symmetric w.r.t. permutation of \( q_j, j = 1,2,3 \) and therefore the axes \( t_j, j = 1,2,3 \) generally cannot be exchanged in (2.7)-(2.9) except for the ‘isotropic’ case \( q_1 = q_2 = q_3 \).

**Remark 2.2** We expect that, under some additional conditions on \( g(s) \) in \((1.7)\), the linear RF \( X \) in Proposition 2.1 has a spectral density of the form

\[
\hat{f}_X(u) = \frac{\tilde{g}(u)}{(\sum_{i=1}^{3} \hat{c}_i |u_i|^{\alpha_i}/\bar{\nu})^{\bar{\nu}}}, \quad u \in \Pi^3 := [-\pi,\pi]^3,
\] (2.10)

where \( \bar{\nu} > 0, \hat{c}_i > 0, \alpha_i > 0, i = 1, 2, 3 \) are parameters, \( \tilde{g}(u) \geq 0, u \in \Pi^3 \) is a bounded function continuous at the origin with \( \tilde{g}(0) > 0 \), and the \( \alpha_i \)'s are related to the \( q_i \)'s as

\[
\alpha_i = 2q_i \left( \sum_{j=1}^{3} \frac{1}{q_j} - 1 \right), \quad q_i = \alpha_i \left( \sum_{j=1}^{3} \frac{1}{\alpha_j} - \frac{1}{2} \right). \] (2.11)

Under (2.11), the balance conditions in \((1.9)\) can be rewritten in spectral terms as \( \gamma_i/\gamma_j = \alpha_j/\alpha_i, 1 \leq i < j \leq 3 \). See also (27, p.2259). Particularly, \((1.6)\) is equivalent to \( \sum_{i=1}^{3} \frac{1}{\alpha_i} > 1 \) and \( \alpha_i > 0, i = 1, 2, 3 \). In terms of ‘spectral parameters’ \( \alpha_j, j = 1, 2, 3 \) in (2.11), Regions I-III in (2.4)-(2.6) correspond to \( \alpha_1 < 1 \) (Region I), \( \frac{1}{\alpha_1} < 1 < \sum_{j=1}^{2} \frac{1}{\alpha_j} \) (Region II), and \( \sum_{j=1}^{2} \frac{1}{\alpha_j} < 1 < \sum_{j=1}^{3} \frac{1}{\alpha_j} \) (Region III). The above conjecture agrees with Proposition 2.1. Indeed, the spectral density of the sectional RF \( \{ X(0,t_2,t_3); (t_2,t_3) \in \mathbb{Z}^2 \} \) is \( f_{23}(u_2,u_3) = \int_{\Pi} f_X(u_1,u_2,u_3)du_1 \) which satisfies \( C_1 f_{23}(u_2,u_3) \leq f_{23}(u_2,u_3) \leq C_2 f_{23}(u_2,u_3) \), \( \tilde{f}_{23}(u_2,u_3) := \int_{\Pi} \left( \sum_{j=1}^{3} |u_j|^{\alpha_j} \right)^{-1}du_1 \), see (2.12) below. Clearly, if \( \alpha_1 < 1 \) then \( \tilde{f}_{23}(u_2,u_3) \leq \int_{\Pi} |u_1|^{-\alpha_1}du_1 < C \) is bounded and hence \( f_{23}(u_2,u_3) \) is a bounded function on \( \Pi^2 \). The same fact follows from the summability of the covariance function \( r_X(0,t_2,t_3) \) in Region I. Similarly, \( 1 < \sum_{j=1}^{2} \frac{1}{\alpha_j} \) implies that the spectral density \( f_3(u_3) = \int_{\Pi^2} f_X(u_1,u_2,u_3)du_1du_2 \) of the sectional process \( \{ X(0,0,t_3); t_3 \in \mathbb{Z} \} \) is bounded, which agrees with the summability of the covariance function \( r_X(0,0,t_3) \) in Region II.
Proof of Proposition 2.1. We shall use the following elementary inequality. For any given \( q_j > 0, c_j > 0, \nu > 0, j = 1, 2, 3 \) there exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 \sum_{j=1}^{3} t_j^{q_j} \leq (\sum_{j=1}^{3} c_j t_j^{q_j/\nu})^\nu \leq C_2 \sum_{j=1}^{3} t_j^{q_j}, \quad \forall \, t = (t_1, t_2, t_3) \in \mathbb{R}_+^3.
\] (2.12)

Indeed, since \( c_j t_j^{q_j/\nu} \leq (\max_{1 \leq j \leq 3} c_j)(\sum_{j=1}^{3} t_j^{q_j/\nu})^\nu, j = 1, 2, 3 \) the second inequality in (2.12) holds with \( C_2 = (3 \max_{1 \leq j \leq 3} c_j)^\nu \) and the first inequality is similar. Denote

\[
\rho(t) := \sum_{j=1}^{3} |t_j|^{q_j}, \quad t \in \mathbb{R}_+^3,
\] (2.13)

then (2.12) and (1.5) imply

\[
C_1 \rho(t)^{-1} \leq |a(t)| \leq C_2 \rho(t)^{-1}, \quad t \in \mathbb{Z}_3.
\] (2.14)

We claim that (2.14) imply a similar inequality for the covariance \( r_X(t) = \sum_{s \in \mathbb{Z}_3} a(s)a(t + s) \), viz.,

\[
C_1 \rho(t)^{-2Q}(1 + o(1)) \leq |r_X(t)| \leq C_2 \rho(t)^{-2Q}(1 + o(1)), \quad |t| \to \infty,
\] (2.15)

where \( Q \in (1, 2) \) is as in (1.6). To check (2.15) consider the convolution

\[
(\rho^{-1} \ast \rho^{-1})(t) = \int_{\mathbb{R}_+^3} d_{s_1} d_{s_2} d_{s_3} \rho(t) \rho(s + t)^{-1}, \quad t \in \mathbb{R}_+^3.
\]

By change of variables \( s_j \to \rho^{1/q_j} s_j, j = 1, 2, 3 \) we obtain

\[
(\rho^{-1} \ast \rho^{-1})(t) = L(t) \rho(t)^{-2Q},
\] (2.16)

where

\[
L(t) := \int_{\mathbb{R}_+^3} d_{s_1} d_{s_2} d_{s_3} \sum_{j=1}^{3} |s_j|^{q_j} \sum_{k=1}^{3} |s_k + t_k/\rho(t)^{1/q_k}|^{q_k} = \int_{\mathbb{R}_+^3} d_s \rho(s) \rho(s + t)^{-1}, \quad \tilde{t} := (t_1/\rho(t)^{1/q_1}, t_2/\rho(t)^{1/q_2}, t_3/\rho(t)^{1/q_3}).
\] (2.17)

Let us show that

\[
0 < C_1 \leq L(t) \leq C_2 < \infty, \quad t \in \mathbb{R}_+^3.
\] (2.18)

Let \( B_\delta(t) := \{ s \in \mathbb{R}_+^3 : |t - s| \leq \delta \}, B_\delta^c(t) := \mathbb{R}_+^3 \setminus B_\delta(t) \). Let us prove first that for any \( h > 0, \delta > 0 \)

\[
\int_{B_\delta(0)} \rho(t)^{-h} dt < \infty \iff Q > h,
\] (2.19)

\[
\int_{B_\delta^c(0)} \rho(t)^{-h} dt < \infty \iff Q < h.
\] (2.20)

By (2.12), it suffices to prove (2.19), (2.20) for \( h = \delta = 1 \). We shall often use the elementary inequalities:

\[
\int_0^1 \frac{du}{(v + u)^h} \leq C \begin{cases} 1, & 0 < q < 1/h, 0 < v < 1, \\
|\log v|, & q = 1/h, 0 < v < 1, \\
v^{-h}, & q > 1/h, 0 < v < 1, \\
v^{(1/q) - h}, & q > 1/h, 0 < v < 1, \\
v^{-h}, & q > 0, v \geq 1,
\end{cases}
\] (2.21)
Let us prove the converse implication in (2.19), or \( I := \int_{[0,1]^3} \rho(t)^{-1} dt < \infty \) if \( Q > 1 \). Using (2.21), we get \( I < \infty \) when \( q_3 < 1 \) and \( I \leq C \int_0^1 dt_1 \int_0^1 (t_1^{q_1} + t_2^{q_2})^{-q_3} dt_2 \) when \( q_3 > 1 \). Using (2.21) again we get \( I < \infty \) if \( q_2(1 - (1/q_3)) < 1 \) and \( I \leq C \int_0^1 (t_1^{q_1/q_2} + (t_2^{q_2})^{-q_3}) dt_1 < \infty \) if \( q_2(1 - (1/q_3)) > 1 \) where the last integral converges since \( Q > 1 \). The case when \( q_3 = 1 \) and/or \( q_2(1 - (1/q_3)) = 1 \) follow similarly. The remaining implications in (2.19)-(2.20) follow in a similar fashion and we omit the details.

Next, we prove (2.18). Note \( \rho(t) = 1 \) and therefore \( |t| > \delta \) \( \forall t \in \mathbb{R}^3 \) for some \( \delta > 0 \). Split \( L(t) = L_1(t) + L_1(t) + L_{12}(t) \), where \( L_1(t) := \int_{B_{\delta}(0)} \rho(s)^{-1} \rho(s + t)^{-1} ds \), \( L_2(t) := \int_{B_{\delta}(-t)} \rho(s)^{-1} \rho(s + t)^{-1} ds \), \( L_{12}(t) := \int_{\mathbb{R}^3 \setminus (B_{\delta}(0) \cup B_{\delta}(-t))} \rho(s)^{-1} \rho(s + t)^{-1} ds \). Since \( \rho(s + t)^{-1} \) is bounded on \( B_{\delta}(0) \) for \( \delta > 0 \) small enough it follows that \( L_1(t) \leq C \int_{B_{\delta}(0)} \rho(s)^{-1} ds \leq C \) in view of (2.19) and \( Q > 1 \). Similarly, \( L_2(t) \leq \int_{B_{\delta}(-t)} \rho(s + t)^{-1} ds \leq C \). Finally, \( |L_{12}(t)|^2 \leq \left( \int_{B_{\delta}^c(0)} \rho(s)^{-2} ds \right)^{1/2} \left( \int_{B_{\delta}^c(-t)} \rho(s + t)^{-2} ds \right)^{1/2} \leq C \) according to (2.19) and \( Q < 2 \). This proves the upper bound in (2.18). The lower bound in (2.18) follows from the uniform boundedness from below of the integrand of (2.17) in a vicinity of the origin, viz., \( \inf_{t \in \mathbb{R}^3} \inf_{s \in B_{\delta}(0)} \rho(s)^{-1} \rho(s + t)^{-1} > C > 0 \) for any \( \delta > 0 \) small enough.

Let us prove (2.15). We use the following inequality: for all \( K > 0 \) large enough

\[
\rho(s_1)/2 < \rho(s_2) < 2\rho(s_1), \quad \forall |s| > K, i = 1, 2, |s_1 - s_2| \leq 1,
\]

which follows by Taylor expansion of \( \rho(t) \) in (2.13). For a large \( K > 0 \) we have \( r_X(t) = \sum_{|s| < K} a(s)a(t + s) + \sum_{|t| + |s| < K} a(s)a(t + s) + \sum_{|s| \geq K, |t| + |s| \geq K} a(s)a(t + s) =: \sum_{i=1}^3 T_i(t) \). Using (2.14) and \( Q > 1 \) we obtain that for any \( K > 0 \) fixed \( |T_i(t)| \leq CK^3 \rho(t)^{-1} = o(\rho(t)^{-Q}) \), \( i = 1, 2 \) as \( |t| \to \infty \). On the other hand, since \( \lim_{|t| \to \infty} \rho(t)a(t) \geq C \lim_{|t| \to \infty} g(t) \geq C > 0 \), see (2.12), (1.5) so for \( K > 0 \) large enough using (2.22) we infer that

\[
T_i(t) \geq C \sum_{|s| \geq K, |t| + |s| \geq K} \frac{1}{\rho(s)\rho(t + s)} \geq C \int_{|s| \geq K, |t| + |s| \geq K} \frac{ds}{\rho(s)\rho(t + s)} \leq C \int_{\mathbb{R}^3} \frac{ds}{\rho(s)\rho(t + s)} + O(\rho(t)^{-1}) = C(\rho^{-1} \ast \rho^{-1})(t) + O(\rho(t)^{-1}),
\]

proving the lower bound in (2.15) by (2.16) and (2.18). The proof of the upper bound in (2.15) follows similarly. This proves (2.15).

Let us prove (2.7). Using (2.15), (2.22) and (2.20) we have \( \sum_{t \in \mathbb{Z}^3} |r_X(t)| \geq C_2 \int_{|t| > K} \rho(t)^{-Q} dt = \infty \) since \( 2 - Q < 1 \). Next, using (2.15)

\[
\sum_{(t_2, t_3) \in \mathbb{Z}^2} |r_X(0, t_2, t_3)| \leq C + C \int_{|t_2| \leq 1} \left( t_2^{q_2} + t_3^{q_3} \right)^{-Q} dt_2 dt_3 \leq C + C \int_0^\infty t_2^{-q_2(2-Q-1/q_3)} dt_2 \int_0^\infty (1 + t_3^{q_3})^{-Q} dt_3 < \infty
\]

follows since \( q_3(2 - Q) > 1 \) and \( q_2(2 - Q - 1/q_3) > 1 \) is equivalent to \( \frac{1}{q_2} + \sum_{j=2}^3 \frac{1}{q_j} < 1 \). The proof of (2.8) and (2.9) follows similarly. Proposition 2.1 is proved. \( \square \)
3 Limiting Gaussian random fields

In this subsec. we define scaling limits $V^X_\gamma$ for $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ satisfying (2.3) (the shaded region in Fig. 1). The above mentioned limits are generally different in Regions I - III of parameters $q_i, j = 1, 2, 3$ determined by inequalities (2.4)-(2.6). In some cases these limits are particularly simple and agree with a Fractional Brownian Sheet (FBS) with special values of Hurst parameters.

Recall that a FBS $B_{H_1,H_2,H_3}$ with parameters $0 < H_i \leq 1, i = 1, 2, 3$ is a Gaussian process on $\mathbb{R}_+^3$ with zero mean and covariance

$$
E[B_{H_1,H_2,H_3}(x)B_{H_1,H_2,H_3}(y)] = \frac{1}{8}\prod_{j=1}^3(x_j^{2H_j} + y_j^{2H_j} - |x_j - y_j|^{2H_j}),
$$

(3.1)

$x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}_+^3$ which is a product of the covariances of a standard FBM with one-dimensional time parameter. Properties of FBS are discussed in [3].

Let us introduce some terminology extending the terminology in [26], [27]. If $\ell \subset \mathbb{R}^3$ is a line and $p \subset \mathbb{R}^3$ is a plane which are orthogonal to each other, we write $\ell \perp p$. Also write $y = (y_1, y_2, y_3) < x = (x_1, x_2, x_3)$ if $y_i < x_i, i = 1, 2, 3$. A rectangle is a set $K(y, x) = \prod_{i=1}^3(y_i, x_i) \subset \mathbb{R}^3, y < x, x, y \in \mathbb{R}^3$. We say that two rectangles $K = K(y, x), K' = K(y', x'), y < x, y' < x'$ are separated by plane $p \subset \mathbb{R}^3$ if $K$ and $K'$ lie on different sides of $p$. By rectangular increment of RF $V = \{V(x), x \in \mathbb{R}^3_+\}$ on rectangle $K(y, x)$ we mean the (triple) difference

$$
V(K(y, x)) := V(x_1, x_2, x_3) - V(x_1, x_2, y_3) - V(x_1, y_2, x_3) - V(y_1, x_2, x_3)
$$

$$
+ V(x_1, y_2, y_3) + V(y_1, x_2, y_3) + V(y_1, y_2, x_3) - V(y_1, y_2, y_3).
$$

We say that RF $V = \{V(x), x \in \mathbb{R}^3_+\}$ has stationary rectangular increments if for any $y \in \mathbb{R}^3_+, \{V(K(y, x)), y < x\} \overset{\text{fdd}}{=} \{V(K(0, x - y)), y < x\}$.

**Definition 3.1** Let $V = \{V(x), x \in \mathbb{R}^3_+\}$ be a RF with stationary rectangular increments and $\ell \subset \mathbb{R}^3$ be a line intersecting $\mathbb{R}^3_+$. We say that RF $V$ has:

(i) **independent rectangular increments in direction $\ell$** if for any orthogonal plane $p \perp \ell$ and any two rectangles $K, K' \subset \mathbb{R}^3_+$ separated by $p$, the increments $V(K)$ and $V(K')$ are independent;

(ii) **invariant rectangular increments in direction $\ell$** if $V(K) = V(K')$ for any two rectangles $K, K' \subset \mathbb{R}^3_+$ such that $K' = x + K$ for some $x \in \ell$.

It follows from Gaussianity and the covariance of FBS that for $H_1 = 1/2$, $B_{1/2, H_2, H_3}(x)$ has independent rectangular increments in the direction of the coordinate axis $x_1$ and, for $H_1 = 1$, $B_{1, H_2, H_3}(x_1, x_2, x_3) = x_1B_{H_2, H_3}(x_2, x_3)$ is a random line in $x_1$ having invariant rectangular increments in the same direction. The case when $H_i, i = 2, 3$ equal $1/2$ or $1$ is analogous. Particularly, $B_{1/2, 1, H_3}$ has independent increments in direction $x_1$ and invariant increments in direction $x_2$. Except for FBS, there are other Gaussian RFs which also enjoy the properties of increments in Definition 3.1. These RFs appear in the scaling limits of the linear RF $X$ in [1,4] and are defined below.
Let \( W(du) \) be real-valued Gaussian white noise on \( \mathbb{R}^3 \), that it, a random process defined on Borel sets \( A \subset \mathbb{R}^3 \) of finite Lebesgue measure \( \text{leb}(A) = \int_A du < \infty \) such that \( W(A) \) has a Gaussian distribution with mean zero and variance \( \text{leb}(A) \) and \( \mathbb{E}[W(A)W(B)] = \text{leb}(A \cap B) \) for any Borel sets \( A, B \subset \mathbb{R}^3 \) of finite Lebesgue measure. The stochastic integral \( I(g) = \int_{\mathbb{R}^3} g(u)W(du) \) is well-defined for any \( g \in L^2(\mathbb{R}^3) \) and has a Gaussian distribution \( I(g) \sim N(0, \|g\|^2) \), where \( \|g\|^2 = \int_{\mathbb{R}^3} g(u)^2 du \). Consider the following RFs defined as stochastic integrals w.r.t. \( W \):

\[
\mathcal{Y}_1(x) := \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^3} 1(0 < u_j < x_j, j = 2, 3, 0 < t_1 < x_1) dt, \quad \text{(Region I)} \tag{3.2}
\]

\[
\mathcal{Y}_2(x) := x_1 \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^2} 1(0 < u_3 < x_3, 0 < t_2 < x_2) dt_2 dt_3, \quad \text{(Region II)} \tag{3.3}
\]

\[
\mathcal{Y}_3(x) := x_1 x_2 \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^2} 1(0 < t < x_3) dt, \quad \text{(Region III)} \tag{3.4}
\]

\[
\mathcal{Y}_{12}(x) := \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^3} 1(0 < t_j < x_j, j = 1, 2, 0 < u_3 < x_3) dt, \quad \text{(Regions I&II)} \tag{3.5}
\]

\[
\mathcal{Y}_{23}(x) := x_1 \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^2} 1(0 < t_j < x_j, j = 2, 3) dt_2 dt_3, \quad \text{(Regions II&III)} \tag{3.6}
\]

\[
\mathcal{Y}_0(x) := \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^3} 1(0 < t_j < x_j, j = 1, 2, 3) dt, \quad \text{(Regions I&II&III)} \tag{3.7}
\]

We also write \( \mathcal{Y}_1(x) \equiv \mathcal{Y}_1(x; q, c), \ldots, \mathcal{Y}_0(x) \equiv \mathcal{Y}_0(x; q, c) \) to emphasize the dependence of these RFs on vector parameters \( q = (q_1, q_2, q_3) \in \mathbb{R}_+^3 \) and \( c = (c_1, c_2, c_3) \in \mathbb{R}_+^3 \).

**Theorem 3.1**

(i) The Gaussian RFs in \( \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \) depending on vector parameters \( q = (q_1, q_2, q_3) \in \mathbb{R}_+^3 \) and \( c = (c_1, c_2, c_3) \in \mathbb{R}_+^3 \) are well-defined in the indicated parameter regions given in (2.4)-(2.6). More precisely,

(i1) \( \mathcal{Y}_1 \) in (3.2) is well-defined for \( \frac{1}{2q_1} + \sum_{i=2}^3 \frac{1}{2q_i} < 1 < \sum_{i=1}^3 \frac{1}{2q_i} \).

(i2) \( \mathcal{Y}_2 \) in (3.3) is well-defined for \( \sum_{i=1}^2 \frac{1}{2q_i} + \frac{1}{q_3} < 1 < \frac{1}{2q_1} + \sum_{i=2}^3 \frac{1}{2q_i} \).

(i3) \( \mathcal{Y}_3 \) in (3.4) is well-defined for \( \sum_{i=1}^3 \frac{1}{2q_i} < 1 < \sum_{i=1}^2 \frac{1}{2q_i} + \frac{1}{q_3} \).

(i4) \( \mathcal{Y}_{12} \) in (3.5) is well-defined for \( \sum_{i=1}^2 \frac{1}{2q_i} + \frac{1}{q_3} < 1 < \sum_{i=1}^3 \frac{1}{2q_i} \).

(i5) \( \mathcal{Y}_{23} \) in (3.6) is well-defined for \( \sum_{i=1}^3 \frac{1}{2q_i} < 1 < \frac{1}{2q_1} + \sum_{i=2}^3 \frac{1}{2q_i} \).

(i6) \( \mathcal{Y}_0 \) in (3.7) is well-defined for \( \sum_{i=1}^3 \frac{1}{2q_i} < 1 < \sum_{i=1}^3 \frac{1}{2q_i} \).
(ii) RFs in [3.2)–[3.7) have stationary rectangular increments and satisfy the self-similarity properties:

\[ Y_1(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) \overset{fdd}{=} \lambda_1^{\lambda_1^{-1/2}} \lambda_2^{\lambda_2^{-1/2}} \lambda_3^{\lambda_3^{-1/2}} Y_1(x_1, x_2, x_3), \quad \forall \lambda_i > 0, \ i = 1, 2, 3, \]

\[ Y_2(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) \overset{fdd}{=} \lambda_1^{\lambda_2^{-1/2}} \lambda_2^{\lambda_3^{-1/2}} Y_2(x_1, x_2, x_3), \quad \forall \lambda_i > 0, \ i = 1, 2, 3, \]

\[ Y_3(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) \overset{fdd}{=} \lambda_1^{\lambda_2^{-1/2}} \lambda_3^{\lambda_2^{-1/2}} Y_3(x_1, x_2, x_3), \quad \forall \lambda_i > 0, \ i = 1, 2, 3, \]

\[ Y_{12}(\lambda^{1/q_1} x_1, \lambda^{1/q_2} x_2, \mu x_3) \overset{fdd}{=} \lambda^{\mu^{1/2}} Y_{12}(x_1, x_2, x_3), \quad \forall \lambda > 0, \ \mu > 0, \]

\[ Y_{23}(\lambda x_1, \mu^{1/q_2} x_2, \mu^{1/q_3} x_3) \overset{fdd}{=} \mu^{\lambda^{1/2}} Y_{23}(x_1, x_2, x_3), \quad \forall \lambda > 0, \ \mu > 0, \]

\[ Y_0(\lambda^{1/q_i} x_i, i = 1, 2, 3) \overset{fdd}{=} \lambda^{H_0} Y_0(x_1, x_2, x_3), \quad \forall \lambda > 0, \]

where

\[ H_1 := \frac{3}{2} - q_1(1 - \frac{1}{q_2} - \frac{1}{q_3}), \quad H_2 := \frac{3}{2} - q_2(1 - \frac{1}{q_1} - \frac{1}{q_3}), \quad H_3 := \frac{3}{2} - q_3(1 - \frac{1}{q_1} - \frac{1}{q_2}), \quad (3.8) \]

\[ H_{12} := \frac{3}{2} q_1 + \frac{3}{2} q_2 + \frac{1}{q_3} - 1, \quad H_{23} := \frac{1}{q_1} + \frac{3}{2} q_2 + \frac{3}{2} q_3 - 1, \quad H_0 := \sum_{i=1}^{3} \frac{3}{2} q_i - 1. \]

(iii) RFs \( Y_i, i = 1, 2, 3 \) agree, up to multiplicative constants \( \sigma_i := E^{1/2}[Y_i^2(1, 1, 1; q, c)] \), with FBS having two its parameters equal to either 1/2 or 1. Namely,

\[ Y_1 \overset{fdd}{=} \sigma_1 B_{H_1, 1.2, 1.2}, \quad Y_2 \overset{fdd}{=} \sigma_2 B_{1.2, H_2, 1.2}, \quad Y_3 \overset{fdd}{=} \sigma_3 B_{1.1, H_3}, \quad (3.9) \]

where \( H_i, i = 1, 2, 3 \) are defined in (3.8).

Proof. (i) In view of inequalities (2.12) and the form of the integrands, it suffices prove the existence of the stochastic integrals for \( c_1 = c_2 = c_3 = \nu = 1 \) and \( x_1 = x_2 = x_3 = 1 \).

(i1) It suffices to prove

\[ I := \int_{\mathbb{R}} du \left( \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{dt_1 dt_2 dt_3}{|t_1-u|^{q_1} + |t_2|^{q_2} + |t_3|^{q_3}} \right)^2 < \infty. \]

Split \( I = I_1 + I_2 \), where \( I_1 := \int_{|u|<2} \cdots, \ I_2 := \int_{|u|>2} \cdots \). Then \( I_1 \leq C \left( \int_0^1 \int_0^x \left( t_1^{q_1} + t_2^{q_2} + t_3^{q_3} \right)^{-1} dt_1 dt_2 dt_3 \right)^2 \).

Using twice the second inequality in (2.21), we obtain

\[ J(t) := \int_0^x \int_0^x \left( t_1^{q_1} + t_2^{q_2} + t_3^{q_3} \right)^{-1} dt_1 dt_2 dt_3 \leq C \int_0^x \int_0^x \left( t_1^{q_1} + t_2^{q_2} + t_3^{q_3} \right)^{1-q_1} dt_1 dt_2 \leq C^{q_3/q_1 + (q_2/q_3 - q_1)}, \]

and hence \( I_1 < \infty \) since \( q_1(1 - \frac{1}{q_2} - \frac{1}{q_3}) < 1 \). Similarly, \( I_2 \leq C \int_0^x \int_0^x \left( u^{q_1} + t_2^{q_2} + t_3^{q_3} \right)^{-1} dt_1 dt_2 dt_3 \) = \( C \int_0^x J^2(u) \) du < \( \infty \) since \( 2q_1(1 - \frac{1}{q_2} - \frac{1}{q_3}) > 1 \). This proves that \( Y_1 \) in (3.2) is well-defined.

(i2) It suffices to prove \( I := \int_{\mathbb{R}^2} du_1 du_2 \left( \int_0^1 I(u_1, t_2 - u_2) dt_2 \right)^2 < \infty \), where

\[ I(u, v) := \int_{\mathbb{R}} \left( |u|^{q_1} + |v|^{q_2} + |t|^{q_3} \right)^{-1} dt \leq C \left( |u|^{q_1} + |v|^{q_2} \right)^{\frac{1}{q_3}-1}, \quad (3.10) \]

see (2.21). Therefore, \( I \leq C \int_{\mathbb{R}^2} du_1 du_2 \left( \int_0^1 \left( |u_1|^{q_1} + |t_2 - u_2|^{q_2} \right)^{\frac{1}{q_3}-1} dt_2 \right)^2 = C \left( \int_{\mathbb{R}} du_1 \int_{|u_2|<2} du_2 (\cdots)^2 + \int_{\mathbb{R}} du_1 \int_{|u_2|>2} du_2 (\cdots) \right) =: C(I_1 + I_2) \), where \( I_1 \leq C \int_0^\infty F(u_1)^2 du_1 \) and

\[ F(u_1) := \int_0^1 \left( u_1^{\frac{1}{q_1}} + t_2^{\frac{1}{q_2}} \right)^{\frac{1}{q_3}-1} dt_2 \leq C \left\{ \begin{array}{ll} \frac{1}{q_1} (1 - \frac{1}{q_2} - \frac{1}{q_3}) & 1 > \frac{1}{q_2} - \frac{1}{q_3}, \\ 1 & 1 < \frac{1}{q_2} - \frac{1}{q_3}, \\ \frac{1}{q_2} - \frac{1}{q_3} & 1 = \frac{1}{q_2} - \frac{1}{q_3} \end{array} \right. \]
according to (2.21), implying \( I_2 = \int_1^\infty \frac{u_1 u_2}{(u_1^2 + u_2^2)^{2(\frac{1}{2} - \frac{1}{3})}} \, du_1 \leq \int_1^\infty u_2^{2 - 2q_1(1 - \frac{1}{2q_1}) - \frac{1}{3}} \, du_2 < \infty \) since \( 2q_2 (1 - \frac{1}{q_1}) - \frac{2q_2}{q_1} > 1 \). This proves that \( \mathcal{Y}_2 \) in (3.3) is well-defined.

(iii) It suffices to prove 

\[
I := \int_{\mathbb{R}^3} u_1 \, du_1 \, du_2 \left( \int_0^{t_1} u_1 \, dt_2 \right) \text{ in the case } 2q_2 (1 - \frac{1}{q_1}) > 1 \text{ we conclude that } \int_{u_1^2 + u_2^2 \leq 1} G(u_1, u_2)^2 \, du_1 \, du_2 \leq \int_0^{t_1} u_1 \, dt_2 \text{ follows trivially from (2.21).} \]

This proves (3.4) is well-defined.

(iiv) It suffices to show 

\[
I := \int_{\mathbb{R}^2} \, du_1 \, du_2 \left( \int_0^{t_1} u_1 \, dt_2 \right) \, du_2 < \infty \quad \text{since } 1 \leq \frac{1}{2q_2} < 1 \text{ the convergence of } \int_{u_1^2 + u_2^2 \leq 1} G(u_1, u_2)^2 \, du_1 \, du_2 < \infty \text{ in (iiv), in the case } 2q_2 (1 - \frac{1}{q_1}) > 1 \text{ we obtain that } \int_1^\infty u_1 \, dt_2 \text{ follows even easier. Next, by (2.21) } \int_{u_1^2 + u_2^2 \leq 1} G(u_1, u_2)^2 \, du_1 \, du_2 < \infty \quad \text{since } 2q_2 (1 - \frac{1}{q_1}) > 1 \text{ similarly, } \int_1^\infty u_1 \, dt_2 \text{ follows analogously.} \]

This proves that \( \mathcal{Y}_{23} \) in (3.6) is well-defined.

(ivi) It suffices to prove 

\[
I := \int_{\mathbb{R}^3} \, du_1 \, du_2 \left( \int_0^{t_1} u_1 \, dt_2 \right) \text{ in the case } 2q_2 (1 - \frac{1}{q_1}) > 1 \text{ we conclude that } \int_{u_1^2 + u_2^2 \leq 1} G(u_1, u_2)^2 \, du_1 \, du_2 \leq \int_0^{t_1} u_1 \, dt_2 \text{ follows trivially from (2.21).} \]

This proves that \( \mathcal{Y}_{23} \) in (3.6) is well-defined.
1 > \frac{1}{q_2} + \frac{1}{q_3}$ using (2.21) we obtain

\[ I_1 \leq C \int_{[0,1]^3} \left( \sum_{i=1}^3 \int_{t_i^{(q_i)}} dt_i \right)^{-1} dt_1 dt_2 dt_3 \leq C \int_0^1 \int_0^1 \left( \sum_{i=1}^2 \int_{t_i^{(q_i)}} dt_i \right)^{-\frac{1}{2}} dt_1 dt_2 \leq C \int_0^1 I^{q_1(1 - \frac{1}{q_2} - \frac{1}{q_3})} dt < \infty; \]

for $1 > \frac{1}{q_2} + \frac{1}{q_3}$ relation $I_1 < \infty$ follows easily. The remaining integrals can be easily evaluated, e.g.,

\[ I_2 \leq C \int_1^\infty du_1 \left( \int_0^1 \frac{dt_3}{u_1^{1/3} + \sum_{i=2}^3 t_i^{(q_i)}} \right)^2 \leq C \int_1^\infty u_1^{-2q_1} du_1 < \infty, \]

\[ I_3 \leq C \int_1^\infty \int_1^\infty du_1 du_2 \left( \int_0^1 \frac{dt_3}{\sum_{i=1}^2 u_i^{(q_i)}} \right)^2 \leq C \int_1^\infty \int_1^\infty \frac{du_1 du_2}{(\sum_{i=1}^2 u_i^{q_i})^2} \leq \int_1^\infty u_1^{-q_1(2 - \frac{1}{q_2})} du_1 < \infty, \]

\[ I_4 \leq C \int_1^\infty \int_1^\infty \frac{du_1 du_2}{(\sum_{i=1}^2 u_i^{q_i})^2} \leq C \int_1^\infty \int_1^\infty \frac{du_1 du_2}{(\sum_{i=1}^2 u_i^{q_i})^2} \leq \int_1^\infty u_1^{-q_1(2 - \frac{1}{q_2} - \frac{1}{q_3})} du_1 < \infty. \]

This proves that $\mathcal{Y}_1$ in (3.7) is well-defined, thereby completing the proof of part (i).

(ii) The self-similarity properties follow from scaling properties $\{W(\lambda_{1} u_1, \lambda_{2} u_2, \lambda_{3} u_3)\}$ $\overset{\text{fdd}}{=} \{(\lambda_{1} \lambda_{2} \lambda_{3})^{1/2} W(u_1, u_2, u_3)\}$ (for all $\lambda_i > 0$, $i = 1, 2, 3$) of the white noise and the integrands in (3.2)-(3.7). For example,

\[ \mathcal{Y}_{12}(\lambda_{1/3}^{1/3} x_1, \lambda_{1/2}^{1/2} x_2, \mu x_3; q, c) \overset{\text{fdd}}{=} \mathcal{Y}_{12}(x_1, x_2, x_3; q, c). \]

(iii) By Gaussianity, it suffices to show the agreement of the corresponding covariance functions. Using the definition in (3.2) we have that $E[\mathcal{Y}_{1}(x; q, c)]\mathcal{Y}_{1}(y; q, c)] = \int_{x_1}^x \int_{y_1}^y \theta(t-s)dt ds \prod_{i=2}^3 \delta(x_i - y_i)$, where

\[ \theta(t) := \int_{\mathbb{R}^3} \frac{du dt ds}{(c_1^3 t^{\frac{21}{6}} + \sum_{i=2}^3 c_i^3 s_i^{\frac{21}{6}})^\nu (c_1^3 t^{\frac{21}{6}} + \sum_{i=2}^3 c_i^3 s_i^{\frac{21}{6}})^\nu} = \theta(1)|t|^{1+2q_1(\frac{1}{q_2} + \frac{1}{q_3} - 1)}. \]

Hence using $3 + 2q_1(\frac{1}{q_2} + \frac{1}{q_3} - 1) = 2H_1$ we obtain

\[ \int_{x}^{y} \int_{0}^{x} \theta(t-s) dt ds = (C_1/2)(x^{2H_1} + y^{2H_1} - |x-y|^{2H_1}), \quad x, y \geq 0, \]

proving $E[\mathcal{Y}_{1}(x; q, c)]\mathcal{Y}_{1}(y; q, c)] = C_1 E[B_{H_1,1/2,1/2}(x)B_{H_1,1/2,1/2}(y)]$, $x, y \in \mathbb{R}^3$, for some constant $C_1 > 0$. Particularly, $C_1 E[B_{H_1,1/2,1/2}(1,1,1)] = C_1 = E[\mathcal{Y}_{1}^{2}(1,1,1)]$, or $C_1 = \sigma_{1}^{2}$. This proves the first relation in (3.9) and the other two relations (3.9) follow analogously. Theorem 3.1 is proved. \hfill $\square$

REMARK 3.1 The self-similarity properties in (ii) imply the following operator scaling properties of the corresponding RFs. For $\lambda > 0, \gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3_+$ denote the diagonal $3 \times 3$ matrix $\lambda^\gamma = \text{diag}(\lambda^{\gamma_i}, i = 1, 2, 3)$. Then for any $\lambda > 0$

\[ \mathcal{Y}_{1}^{i}(\lambda^{\gamma} x; q, a) \overset{\text{fdd}}{=} \lambda^{H_{1}(\gamma,q)} \mathcal{Y}_{1}^{i}(x; q, a), \quad i = 1, 2, 3, \quad (3.11) \]

\[ \mathcal{Y}_{1}^{ij}(\lambda^{\gamma} x; q, a) \overset{\text{fdd}}{=} \lambda^{H_{ij}(\gamma,q)} \mathcal{Y}_{1}^{ij}(x; q, a), \quad \gamma_{ij} = \gamma_{ji}, \quad 1 \leq i < j \leq 3, \quad j = i + 1, \quad (3.12) \]

\[ \mathcal{Y}_{1}^{0}(\lambda^{\gamma} x; q, a) \overset{\text{fdd}}{=} \lambda^{H_{0}(\gamma,q)} \mathcal{Y}_{1}^{0}(x; q, a), \quad \gamma_{12} = \gamma_{23} = \gamma_{31}, \quad (3.13) \]
where

\[
H_1(\gamma, q) := \gamma_1 H_1 + \frac{2 + \gamma_3}{2} = \frac{3\gamma_1 + 2 + \gamma_3}{2} + \gamma_1 q_1 \left( \frac{1}{q_1} + \frac{1}{q_2} - 1 \right),
\]

\[
H_2(\gamma, q) := \gamma_1 + \gamma_2 H_2 + \frac{\gamma_3}{2} = \gamma_1 + \frac{3\gamma_2 + \gamma_3}{2} + \gamma_2 q_2 \left( \frac{1}{q_1} + \frac{1}{q_2} - 1 \right),
\]

\[
H_3(\gamma, q) := \gamma_1 + \gamma_2 + \gamma_3 H_3 = \gamma_1 + \gamma_2 + \frac{3\gamma_3}{2} + \gamma_3 q_3 \left( \frac{1}{2 q_1} + \frac{1}{2 q_2} - 1 \right),
\]

\[
H_{12}(\gamma, q) := \gamma_1 q_1 H_{12} + \frac{\gamma_3}{2} = \frac{3(\gamma_1 + \gamma_2) + \gamma_3}{2} + \gamma_1 q_1 \left( \frac{1}{q_2} - 1 \right),
\]

\[
H_{23}(\gamma, q) := \gamma_1 + \gamma_2 q_2 H_{23} = \gamma_1 + \frac{3\gamma_2 + 3\gamma_3}{2} + \gamma_2 q_2 \left( \frac{1}{2 q_1} - 1 \right),
\]

\[
H_0(\gamma, q) := \gamma_1 q_1 H_0 = \gamma_1 q_1 \left( \sum_{i=1}^{3} \frac{1}{2 q_i} - 1 \right).
\]

See \[4\] for the definition and general properties of operator scaling RFs. Note that while (3.11) hold for any $\gamma \in \mathbb{R}_+^3$, the self-similarity properties in (3.12) and (3.13) hold for $\gamma \in \mathbb{R}_+^3$ satisfying one and two (all) balance conditions in (1.9), respectively. Also note that $H_1(\gamma, q) = H_2(\gamma, q) = H_{12}(\gamma, q)$ for $\gamma_1 = \gamma_2 q_2$, $H_2(\gamma, q) = H_3(\gamma, q) = H_{23}(\gamma, q)$ for $\gamma_2 q_2 = \gamma_3 q_3$, and that all scaling exponents in (3.14) coincide for $\gamma_1 = \gamma_2 q_2 = \gamma_3 q_3$.

**Remark 3.2** It follows from (3.9) that RFs $Y_i, i = 1, 2, 3$ in (3.2)-(3.4) have the (rectangular) increment properties of Definition 3.1 in two directions in $\mathbb{R}^3$. For instance, $Y_1$ has independent increments in $x_2$ and $x_3$, while $Y_3$ has invariant increments in $x_1$ and $x_2$. The RFs $Y_{12}$ and $Y_{23}$ have these properties in one direction, namely, $Y_{12}$ has independent increments in $x_3$ and $Y_{23}$ has invariant increments in $x_3$. These facts follow from the representations in (3.5) and (3.6) and the independent increment property of the white noise $W(\text{du})$. They are closely related to the number of balance conditions satisfied by $\gamma$’s as shown in the following sec.

### 4 The main result

In this sec. we formulate our main result about partial sums limits in (1.2) of the linear RF $X$ in (1.4). Theorem 4.1 specifies these limits for scaling exponents $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ satisfying (2.3). The general case $\gamma \in \mathbb{R}_+^3$ is treated in Corollary 4.1.

**Theorem 4.1** Let $X$ be a linear RF in (1.4) with standardized i.i.d. innovations $\{\varepsilon, \varepsilon(s); s \in \mathbb{Z}^3\}$, $E\varepsilon = 0, E\varepsilon^2 = 1$ and moving-average coefficients $a(t)$ in (1.5), where $\nu > 0, q_i > 0, c_i > 0, i = 1, 2, 3$ and $q = (q_1, q_2, q_3)$ satisfy (1.6). Moreover, we assume $\lim_{|t| \to \infty} g(t) = 1 \ w.l.g.$

(i) Let $\frac{1}{2 q_1} + \sum_{i=2}^{3} \frac{1}{q_i} < 1$ and $\gamma_1 q_1 < \gamma_2 q_2 \leq \gamma_3 q_3$. Then

\[
\lambda^{-H_1(\gamma, q)} S_{\lambda, \gamma}^X(x) \xrightarrow{f.d.d.} Y_1(x).
\]

(ii) Let $\sum_{i=1}^{2} \frac{1}{2 q_i} + \frac{1}{q_3} < 1 < \frac{1}{2 q_1} + \sum_{i=2}^{3} \frac{1}{q_i}$ and $\gamma_1 q_1 < \gamma_2 q_2 < \gamma_3 q_3$. Then

\[
\lambda^{-H_2(\gamma, q)} S_{\lambda, \gamma}^X(x) \xrightarrow{f.d.d.} Y_2(x).
\]
(iii) Let \( 1 < \sum_{i=1}^{2} \frac{1}{2q_i} + \frac{1}{q_1} \) and \( \gamma_1 q_1 \leq \gamma_2 q_2 < \gamma_3 q_3 \). Then
\[
\lambda^{-H_3(\gamma, q)} S_{\lambda, \gamma}^X(x) fdd \rightarrow \mathcal{Y}_3(x). \tag{4.3}
\]
(iv) Let \( \sum_{i=1}^{2} \frac{1}{2q_i} + \frac{1}{q_1} < 1 \) and \( \gamma_1 q_1 = \gamma_2 q_2 < \gamma_3 q_3 \). Then
\[
\lambda^{-H_2(\gamma, q)} S_{\lambda, \gamma}^X(x) fdd \rightarrow \mathcal{Y}_2(x). \tag{4.4}
\]
(v) Let \( 1 < \frac{1}{2q_1} + \sum_{i=2}^{3} \frac{1}{2q_i} \) and \( \gamma_1 q_1 < \gamma_2 q_2 = \gamma_3 q_3 \). Then
\[
\lambda^{-H_3(\gamma, q)} S_{\lambda, \gamma}^X(x) fdd \rightarrow \mathcal{Y}_3(x). \tag{4.5}
\]
(vi) Let \( \gamma_1 q_1 = \gamma_2 q_2 = \gamma_3 q_3 \). Then
\[
\lambda^{-H_0(\gamma, q)} S_{\lambda, \gamma}^X(x) fdd \rightarrow \mathcal{Y}_0(x). \tag{4.6}
\]

The limit RFs and the normalizing exponents in \((4.1)-(4.6)\) are defined in \((3.2)-(3.7)\) and \((3.14)\), respectively.

To describe the scaling limits in \((1.2)\) for general \( \gamma \in \mathbb{R}_+^3 \), we need some notation. Let \( \mathcal{P}_3 \) denote the set of all permutations \( \pi = (\pi(1), \pi(2), \pi(3)) \) of \( \{1, 2, 3\} \). Given a RF \( \mathcal{Y}(\cdot; q, c) = \{\mathcal{Y}(x; q, c); x \in \mathbb{R}_+^3\} \) depending on vector parameters \( c = (c_1, c_2, c_3), q = (q_1, q_2, q_3) \in \mathbb{R}^3 \), and a permutation \( \pi = (\pi(1), \pi(2), \pi(3)) \in \mathcal{P}_3 \), define a new RF \( \mathcal{Y}^\pi(\cdot; q, c) = \{\mathcal{Y}^\pi(x; q, c); x \in \mathbb{R}_+^3\} \) by
\[
\mathcal{Y}^\pi(x; q, c) := \mathcal{Y}(\pi x; \pi q, \pi c)
\]
where \( \pi y := (y_{\pi(1)}, y_{\pi(2)}, y_{\pi(3)}), y = (y_1, y_2, y_3) \in \mathbb{R}^3 \). The above definition requires some care since \( \mathcal{Y} \) and \( \mathcal{Y}^\pi \) need not exist simultaneously. For example, the existence of RFs \( \mathcal{Y}_1(x; q, c) \) in \((3.2)\) and
\[
\mathcal{Y}^\pi_1(x; q, c) = \int_{\mathbb{R}^3} W(du) \int_{\mathbb{R}^3} \frac{1}{t} (0 < u_i < x_i, i = \pi(2), \pi(3), 0 < t_{\pi(1)} < x_{\pi(1)}) dt
\]
require \( \frac{1}{2q_1} + \frac{1}{q_2} + \frac{1}{q_3} < 1 \) and \( \frac{1}{2q_{\pi(1)}} + \frac{1}{q_{\pi(2)}} + \frac{1}{q_{\pi(3)}} < 1 \), respectively, and the two conditions are generally different.

From the definition of the partition \((2.1)\) it is clear that any \( \gamma \in \mathbb{R}_+^3 \) can be ‘transformed’ into the region \((2.3)\) by a simultaneous permutation of indices of \( \gamma_i, q_i \), i.e., for any \( \gamma \in \mathbb{R}_+^3 \) there exists a \( \pi \in \mathcal{P}_3 \) such that
\[
\gamma_{\pi(1)} q_{\pi(1)} \leq \gamma_{\pi(2)} q_{\pi(2)} \leq \gamma_{\pi(3)} q_{\pi(3)} \tag{4.7}
\]
In general, the above \( \pi \) is not unique, e.g., the ‘well-balanced’ points \( \gamma \in \Gamma_{000} \) satisfy \((4.7)\) for any \( \pi \in \mathcal{P}_3 \).

For example, the region \( \gamma_2 q_2 \leq \gamma_3 q_3 \leq \gamma_1 q_1 = \Gamma_{-111} \cup \Gamma_{011} \cup \Gamma_{000} \cup \Gamma_{101} \) corresponds to \((4.7)\) and \( \pi(1) = 2, \pi(2) = 3, \pi(1) = 2 \).

**Corollary 4.2** Let RF \( X \) satisfy the conditions of Theorem 4.1. Let \( \pi \in \mathcal{P}_3 \) and \( \gamma \in \mathbb{R}_+^3 \) satisfy condition \((4.7)\).
(i) Let $\frac{1}{2q_{\pi(1)}} + \sum_{i=2}^{3} \frac{1}{q_{\pi(i)}} < 1$ and $\gamma_{\pi(1)}q_{\pi(1)} < \gamma_{\pi(2)}q_{\pi(2)} \leq \gamma_{\pi(3)}q_{\pi(3)}$. Then

$$\lambda^{-H_1}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} Y_1^\pi(x; q, c).$$

(4.8)

(ii) Let $\sum_{i=1}^{2} \frac{1}{2q_{\pi(i)}} + \frac{1}{q_{\pi(3)}} < 1 < \frac{1}{2q_{\pi(i)}} + \sum_{i=2}^{3} \frac{1}{q_{\pi(i)}}$ and $\gamma_{\pi(1)}q_{\pi(1)} < \gamma_{\pi(2)}q_{\pi(2)} < \gamma_{\pi(3)}q_{\pi(3)}$. Then

$$\lambda^{-H_2}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} Y_2^\pi(x; q, c).$$

(4.9)

(iii) Let $1 < \sum_{i=1}^{2} \frac{1}{2q_{\pi(i)}} + \frac{1}{q_{\pi(3)}}$ and $\gamma_{\pi(2)}q_{\pi(2)} < \gamma_{\pi(3)}q_{\pi(3)}$. Then

$$\lambda^{-H_3}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} Y_3^\pi(x; q, c).$$

(4.10)

(iv) Let $\sum_{i=1}^{2} \frac{1}{2q_{\pi(i)}} + \frac{1}{q_{\pi(3)}} < 1$ and $\gamma_{\pi(1)}q_{\pi(1)} = \gamma_{\pi(2)}q_{\pi(2)} < \gamma_{\pi(3)}q_{\pi(3)}$. Then

$$\lambda^{-H_4}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} Y_4^\pi(x; q, c).$$

(4.11)

(v) Let $1 < \frac{1}{2q_{\pi(1)}} + \sum_{i=2}^{3} \frac{1}{2q_{\pi(i)}}$ and $\gamma_{\pi(1)}q_{\pi(1)} < \gamma_{\pi(2)}q_{\pi(2)} = \gamma_{\pi(3)}q_{\pi(3)}$. Then

$$\lambda^{-H_5}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} Y_5^\pi(x; q, c).$$

(4.12)

The last corollary specifies the scaling limits in the ‘isotropic’ case $q_1 = q_2 = q_3$.

**Corollary 4.3** Let $X$ satisfy the conditions in Theorem 4.1 and $q_1 = q_2 = q_3 = : q$.

(I) Let $5/2 < q < 3$ (Region I). Then

$$\lambda^{-H_1}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \sigma_1 B_{\mathcal{H}_1,1/2,1/2}(x), \quad \gamma_1 < \gamma_2 \leq \gamma_3;$$

$$\lambda^{-H_2}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_{12}(x), \quad \gamma_1 = \gamma_2 < \gamma_3;$$

$$\lambda^{-H_0}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_0(x), \quad \gamma_1 = \gamma_2 = \gamma_3.$$ (I)

(II) Let $2 < q < 5/2$ (Region II). Then

$$\lambda^{-H_2}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \sigma_2 B_{1,\mathcal{H}_2,1/2}(x), \quad \gamma_1 < \gamma_2 < \gamma_3;$$

$$\lambda^{-H_2}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_{12}(x), \quad \gamma_1 = \gamma_2 < \gamma_3;$$

$$\lambda^{-H_0}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_2(x), \quad \gamma_1 < \gamma_2 = \gamma_3;$$

$$\lambda^{-H_0}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_0(x), \quad \gamma_1 = \gamma_2 = \gamma_3.$$ (II)

(III) Let $3/2 < q < 2$ (Region III). Then

$$\lambda^{-H_3}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \sigma_3 B_{1,\mathcal{H}_3}(x), \quad \gamma_1 \leq \gamma_2 < \gamma_3;$$

$$\lambda^{-H_3}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_{23}(x), \quad \gamma_1 < \gamma_2 = \gamma_3;$$

$$\lambda^{-H_0}(\gamma,q)S_{\lambda,\gamma}^X(x) \xrightarrow{\text{fdd}} \mathcal{Y}_0(x), \quad \gamma_1 = \gamma_2 = \gamma_3.$$ (III)

Here, the normalizations and the limit RFs are given as in Theorem 4.1. $\mathcal{H}_1 = \frac{7}{2} - q$, $\mathcal{H}_2 = 3 - q$, $\mathcal{H}_3 = \frac{5}{2} - q$. 

16
Remark 4.1 We expect that the results of Theorem 4.1 can be extended to the boundary situations

\[ \frac{1}{2q_1} + \sum_{j=2}^{3} \frac{1}{q_j} = 1 \]  

(4.13)

(the boundary between Regions I and II), and

\[ \sum_{j=1}^{2} \frac{1}{2q_j} + \frac{1}{q_2} = 1 \]  

(4.14)

(the boundary between Regions II and III), possibly under additional logarithmic normalization. See also [24], Remark 3.2. Note the exponents in (3.8) trivialize in the above cases: \( \mathcal{H}_1 = 1, \mathcal{H}_2 = 1/2 \) when (4.13) holds, and \( \mathcal{H}_2 = 1, \mathcal{H}_3 = 1/2 \) when (4.14) holds. If the above conjecture is true, we can expect in the limit (4.1) and (4.1) a (multiple of) FBS \( B_{1,1/2,1/2} \) under (4.13), and a (multiple of) FBS \( B_{1,1,1/2} \) under (4.14), in other words, an FBS with all its Hurst indices equal to 1 and/or 1/2.

5 Proof of Theorem 4.1

The proof of Theorem 4.1 reduces to the central limit theorem for linear forms in i.i.d.r.v.s \( \{\varepsilon(s), s \in \mathbb{Z}^3\} \). Moreover, the limits are written as stochastic integrals w.r.t. white noise \( W(du) \) on \( \mathbb{R}^3 \). The proof of such limit theorems is facilitated by the following criterion generalizing ([12], Prop.14.3.2) to linear forms \( S(h) := \sum_{s \in \mathbb{Z}^3} h(s)\varepsilon(s) \) with real coefficients \( \sum_{s \in \mathbb{Z}^3} h(s)^2 < \infty \).

Proposition 5.1 Let \( S(h_\lambda), \lambda > 0 \) be as in (5.1). Suppose \( h_\lambda(u) \) are such that for a real-valued function \( f \in L^2(\mathbb{R}^3) \) and some integers \( m_i = m_i(\lambda) \rightarrow \infty, \lambda \rightarrow \infty, i = 1, 2, 3 \) the functions

\[ \tilde{h}_\lambda(u) := (m_1m_2m_3)^{1/2}h_\lambda([m_1u_1], [m_2u_2], [m_3u_3]), \quad u = (u_1, u_2, u_3) \in \mathbb{R}^3 \]  

(5.2)

tend to \( f \) in \( L^2(\mathbb{R}^3) \), viz.,

\[ \|\tilde{h}_\lambda - f\|^2 = \int_{\mathbb{R}^3} |\tilde{h}_\lambda(u) - f(u)|^2du \rightarrow 0, \quad \lambda \rightarrow \infty. \]  

(5.3)

Then

\[ S(h_\lambda) \overset{d}{\rightarrow} I(f) := \int_{\mathbb{R}^3} f(u)W(du), \quad \lambda \rightarrow \infty. \]  

(5.4)

By Cramér-Wold device, the proof of finite-dimensional convergence in (1.3) reduces to the convergence of (scalar) linear combinations \( A_{\lambda,\nu}^1 = \sum_{k=1}^{p} \theta_k S_{\lambda,\nu, k}(x_k) \), for any \( p \geq 1, x_k \in \mathbb{R}_+^3, \theta_k \in \mathbb{R}, k = 1, \ldots, p \) which can be written as linear forms as in (5.1) with a suitable \( h \). For notational convenience, we restrict the proof of the last fact to to the case \( p = 1 = \theta_1, x_1 = x \), or to the one-dimensional convergence in (4.1)-(4.6) since the proof of finite-dimensional convergence is analogous. Moreover, for the same reason we will assume that \( \nu = 1, g(t) \equiv 1 \) in (1.5). We also use the notation \( V_\lambda(x) \) for normalized sums on the l.h.s. of (4.1)-(4.6), and drop \( \gamma, q \) in the notation of the exponents \( H_1(\gamma, q), \ldots, H_0(\gamma, q) \) in (3.14).
Proof of [4.1]. Using Proposition 5.1, let
\[ m_i := [\lambda^{q_i}], \quad \tilde{m}_i := \lambda^{q_i} q_i / q_i, \quad i = 1, 2, 3, \quad \kappa_\lambda := \frac{(m_1 m_2 m_3)^{1/2} \tilde{m}_1 \tilde{m}_2 \tilde{m}_3}{\lambda^{H_i} m_1^{q_i}} \rightarrow 1. \]
Then \( V_\lambda(x) = S(h_\lambda) \), where \( H_1 = \frac{3q_i + q_i + q_i}{2} + \gamma_1 q_1 \left( \frac{1}{q_2} + \frac{1}{q_3} - 1 \right) \), see (3.14),
\[
\begin{align*}
    h_\lambda(s) := \lambda^{-H_1} \sum_{1 \leq t_i \leq [\lambda^{t_i} x_i], i = 1, 2, 3} a(t - s) \\
    &= \lambda^{-H_1} \int_0^{[\lambda^{t_1} x_1]} \int_0^{[\lambda^{t_2} x_2]} \int_0^{[\lambda^{t_3} x_3]} \frac{dt_1 dt_2 dt_3}{c_1 [t_1] - s_1 |q_1} + \sum_{i=2}^3 c_i [t_i] - s_i |q_i} 
\end{align*}
\]
and
\[
\begin{align*}
    \tilde{h}_\lambda(u) &= (m_1 m_2 m_3)^{1/2} h_\lambda([m_1 u_1], [m_2 u_2], [m_3 u_3]) \\
    &= \frac{(m_1 m_2 m_3)^{1/2}}{\lambda^{H_1}} \int_0^{[\lambda^{t_1} x_1]} \int_0^{[\lambda^{t_2} x_2]} \int_0^{[\lambda^{t_3} x_3]} \frac{dt_1 dt_2 dt_3}{c_1 [t_1] - [m_1 u_1] |q_1} + \sum_{i=2}^3 c_i [t_i] - [m_i u_i] |q_i} \\
    &= \kappa_\lambda \int_0^{[\lambda^{t_1} x_1]/\lambda^{q_1}} \int_0^{[\lambda^{t_2} x_2]/\lambda^{q_2}} \int_0^{[\lambda^{t_3} x_3]/\lambda^{q_3}} \frac{dt_1 dt_2 dt_3}{c_1 ([\tilde{m}_1 t_1] - [m_1 u_1] |q_1} + \sum_{i=2}^3 c_i ([\tilde{m}_i t_i] - [m_i u_i] |q_i} \\
    &= \int_0^{[\lambda^{t_1} x_1]/\lambda^{q_1}} \int_\mathbb{R} \int_\mathbb{R} G_\lambda(t; u) dt_1 dt_2 dt_3, 
\end{align*}
\]
where
\[
G_\lambda(t; u) := \kappa_\lambda \mathbf{1}( -[m_1 u_1] m_1 < t_i < [\lambda^{t_i} x_i] - [m_i u_i], i = 2, 3) \\
&\quad \frac{1}{c_1 [t_1] - u_1 |q_1} + \sum_{i=2}^3 c_i [t_i] - u_i |q_i}. 
\]
Since \( \gamma_i / m_i \rightarrow 1 \) and \( m_i / \tilde{m}_i \rightarrow \infty \) due to \( \gamma_i q_i / \gamma_1 q_1 > 1, i = 2, 3 \), we see that
\[
G_\lambda(t; u) \rightarrow G_1(t; u) := \frac{1(0 < u_i < x_i, i = 2, 3)}{c_1 |t_1 - u_1|^{q_1} + \sum_{i=2}^3 c_i |t_i|^{q_i}} 
\]
point-wise for any fixed \( u = (u_1, u_2, u_3) \in \mathbb{R}^3, t = (t_1, t_2, t_3) \in \mathbb{R}^3, t_1 \neq y_1, u_i \neq 0, y_i \neq x_i, i = 2, 3 \). We claim that
\[
\begin{align*}
    \tilde{h}_\lambda(u) \rightarrow \int_{[0, x_1] \times \mathbb{R}^2} G_1(t; u) dt \\
    &= 1(0 < u_i < x_i, i = 2, 3) \int_{[0, x_1] \times \mathbb{R}^2} \frac{dt_1 dt_2 dt_3}{c_1 |t_1 - u_1|^{q_1} + \sum_{i=2}^3 c_i |t_i|^{q_i}} =: f_1(u) 
\end{align*}
\]
point-wise and in \( L^2(\mathbb{R}^3) \). Since \( Y_1(x) = \int_{\mathbb{R}^3} f_1(u) W(du), \) the one-dimensional convergence in [4.1] follows from [5.8] and Proposition 5.1.

To justify [5.8], note that for all \( \lambda \geq 1 \)
\[
[\tilde{m}_1 t_1] - [m_1 u_1] \geq \frac{|t_1 - u_1|}{2}, \quad [\tilde{m}_i t_i] \geq \frac{|t_i|}{2}, \quad i = 2, 3, \\
1( -[m_1 u_1] m_1 < t_i < [\lambda^{t_i} x_i] - [m_i u_i]) \leq 1(-2 < u_i < x_i + 2) + 1(u_i \leq -2, \frac{m_i u_i}{m_i} < t_i < \frac{m_i (u_i + x_i)}{m_i}) \\
+ 1(u_i \geq x_i + 2, \frac{m_i u_i - x_i}{m_i} < t_i < \frac{m_i u_i}{m_i}), \quad i = 2, 3. 
\]
Split $\tilde{h}_\lambda(u) = \sum_{j=0}^{1} \tilde{h}_{\lambda,j}(u)$, where $\tilde{h}_{\lambda,j}(u) := \int_0^{\lambda^{j+1}x_1/\lambda^{j+1}} \int_\mathbb{R} G_{\lambda,j}(t;u) dt$ and

$$
G_{\lambda,0}(t;u) := G_\lambda(t;u)1(-2 < u_i < x_i + 2, i = 2, 3),
G_{\lambda,1}(t;u) := G_\lambda(t;u)1(u_i \notin (-2, x_i + 2) \ (\exists i = 2, 3)).
$$

Relation (5.8) follows from

$$
\|\tilde{h}_{\lambda,0} - f_1\|^2 \to 0 \quad \text{and} \quad \|\tilde{h}_{\lambda,1}\|^2 \to 0. \quad (5.11)
$$

The first relation in (5.11) follows from (5.14) and the dominated convergence theorem since (5.9)-(5.10) imply the dominating bound

$$
0 \leq \tilde{h}_{\lambda,0}(u) \leq C1(-2 < u_i < x_i + 2, i = 2, 3) \int_{(0,x_1] \times \mathbb{R}} \frac{\text{d}t}{|t_1 - u_1|^{q_1} + \sum_{i=2}^{3} |t_i|^{q_i}} =: \bar{h}(u)
$$

with $\bar{h} \in L^2(\mathbb{R}^3)$; see the proof of Theorem 3.1 (i). With $\rho_i := \gamma_i - \gamma_1 q_1 / q_i > 0, i = 2, 3$, the second relation in (5.11) follows from

$$
\|\tilde{h}_{\lambda,1}\|^2 \leq C \int_{\mathbb{R}^3} \text{d}u \left\{ \int_{(0,x_1] \times \mathbb{R}^2} G_1(t;u) \sum_{i=2}^{3} (|t_i| > |u_i|^{\rho_i}) \text{d}t \right\}^2 = o(1)
$$

since $\int_{(0,1] \times \mathbb{R}^2} G_1(t;\cdot) \text{d}t \in L^2(\mathbb{R}^3)$ and $\sum_{i=2}^{3} (|t_i| > |u_i|^{\rho_i}) \to 0$ for any $t, u \in \mathbb{R}^3$ fixed. This proves (5.11) and completes the proof of (4.1). 

Proof of (4.2). We use Proposition 5.1 with

$$
m_1 := [\lambda^{2q_2}/q_1], \ m_2 := [\lambda^{q_2}], \ m_3 := [\lambda^{q_3}], \ m_1 := [\lambda^{\gamma_1}], \ m_2 := [\lambda^{\gamma_2}], \ m_3 := [\lambda^{\gamma_3}], \ m_1 := [\lambda^{\gamma_1}], \ m_2 := [\lambda^{\gamma_2}], \ m_3 := [\lambda^{\gamma_3}]. \quad (5.12)
$$

Then $V_\gamma(x) = S(h_\lambda)$, where $h_\lambda$ is defined as in (5.5) with $H_1$ replaced by $H_2$, and for $\tilde{h}_\lambda(u)$ defined in (5.2) we have the integral representation $\tilde{h}_\lambda(u) = \int_{\prod_{i=1}^{3} (0, [\lambda^{x_i}] / m_i)} G_\lambda(t;u) \text{d}t$, where

$$
G_\lambda(t;u) := \frac{\kappa_\lambda 1(-m_3 u_3 |m_3| < t_3 < \lambda^{\gamma_3} x_3 |m_3|)}{c_1([m_1 t_1 - m_1 u_1]|m_1|) t_1 + c_2([m_2 t_2 - m_2 u_2]|m_2|) t_2 + c_3([m_3 t_3]|m_3|) t_3}. \quad (5.13)
$$

Using $\gamma_2 q_2 / \gamma_1 q_1 > 1, \gamma_3 > \gamma_2 q_2 / q_3$ we see that $m_3 / m_3 \to \infty, m_3 / \lambda^{\gamma_3} \to 1, \lambda^{\gamma_1} / m_1 q_2 / q_1 \to 0, m_1 / m_2 q_2 / q_1 \to 1$ and hence

$$
G_\lambda(t;u) \to G_2(t;u) := \frac{1(0 < u_3 < x_3)}{c_1|u_1|^{q_1} + c_2|t_2 - u_2|^{q_2} + c_3|t_3|^{q_3}}. \quad (5.14)
$$

point-wise for any fixed $u = (u_1, u_2, u_3) \in \mathbb{R}^3, t = (t_1, t_2, t_3) \in \mathbb{R}^3, u_1 \neq 0, u_2 \neq t_2, t_3 \neq 0, u_3 \neq 0, u_3 \neq x_3$. Note $G_2(t;y)$ in (5.14) does not depend on $t_1$. Then

$$
\tilde{h}_\lambda(u) \to \int_{(0,x_1] \times (0,x_2] \times \mathbb{R}} G_2(t;u) \text{d}t \quad \text{and} \quad x_1 1(0 < u_3 < x_3) \int_{(0,x_2] \times \mathbb{R}} (c_1|u_1|^{q_1} + c_2|t_2 - u_2|^{q_2} + c_3|t_3|^{q_3})^{-1} \text{d}t_2 \text{d}t_3 =: f_2(u). \quad (5.15)
$$
point-wise and in $L^2(\mathbb{R}^3)$. Since $\mathcal{Y}_2(x) = \int_{\mathbb{R}^3} f_2(u) W(du)$, the one-dimensional convergence in (4.2) follows from (5.15) and Proposition 5.1. The proof of (??) uses the dominated convergence and a similar argument as in (??) and we omit the details.

**Proof of (4.3).** Let

$$m_i := \left[\lambda^{\gamma_3 q_i}/q_i\right], \quad \tilde{m}_i := \lambda^{\gamma_i}, \quad i = 1, 2, 3, \quad \kappa_\lambda := (m_1 m_2 m_3)^{1/2} \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \rightarrow 1.$$  \hfill (5.16)

Then $V_\lambda(x) = S(h_\lambda)$ and $\tilde{h}_\lambda(u) = (m_1 m_2 m_3)^{1/2} h_\lambda([m_i u_i], i = 1, 2, 3) = \int_{\Pi_{i=1}^3 [0, [u_i x_i]/\tilde{m}_i]} G_\lambda(t; u) dt$, where

$$G_\lambda(t; u) := \frac{1}{c_1 |u_1|^q + c_2 |u_2|^q + c_3 |t_3 - u_3|^q} =: G_3(t; u)$$ \hfill (5.17)

defined in (5.17) does not depend on $t_i, i = 1, 2$. Also note $\mathcal{Y}_3(x) = \int_{\mathbb{R}^3} f_3(u) W(du)$, where $f_3(u) := \int_{[0, x_1] \times [0, x_2] \times [0, x_3]} G_3(t; u) dt$. We omit the details of the proof of the convergence $\tilde{h}_\lambda \rightarrow f_3$ in $L^2(\mathbb{R}^3)$, which are similar to those in the proof of (4.1) and (4.2).

**Proof of (4.4).** Let $m_i, \tilde{m}_i, i = 1, 2, 3, \kappa_\lambda$ be defined as in (5.15). Note $H_2 = H_{12} = 3(\gamma_1 + \gamma_2)/2 + \gamma_1 q_1/3 - \gamma_1 q_1$ for $\gamma_1 q_1 = \gamma_2 q_2$. Then for $G_\lambda(t; u)$ defined in (5.15) we have the point-wise convergence

$$G_\lambda(t; u) \rightarrow G_{12}(t; u) := \frac{1(0 < u_3 < x_3)}{\sum_{i=1}^2 c_i |t_i - u_3|^{q_i} + c_3 |t_3|^{q_3}}$$
c.f. (5.14), for any fixed $u = (u_1, u_2, u_3) \in \mathbb{R}^3, t = (t_1, t_2, t_3) \in \mathbb{R}^3, u_1 \neq t_1, t_1, t_i, i = 1, 2, t_3 \neq 0, u_3 \neq 0, x_3$. Moreover, $\mathcal{Y}_2(x) = \int_{\mathbb{R}^3} f_2(u) W(du)$, where $f_2(u) := \int_{[0, x_1] \times [0, x_2] \times [0, x_3]} G_{12}(t; u) dt$. The details of the convergence $\tilde{h}_\lambda(u) := \int_{\Pi_{i=1}^3 [0, [u_i x_i]/\tilde{m}_i]} G_\lambda(t; u) dt \rightarrow f_2(u)$ in $L^2(\mathbb{R}^3)$ are similar as above.

**Proof of (4.5).** Let $m_i, \tilde{m}_i, i = 1, 2, 3, \kappa_\lambda$ be defined as in (5.16). Note $H_3 = H_{23} = \gamma_1 + 3(\gamma_2 + \gamma_3)/2 + \gamma_2 q_2/3 - \gamma_2 q_2$ when $\gamma_2 q_2 = \gamma_3 q_3$. Then for $G_\lambda(t; u)$ defined in (5.17) we have the point-wise convergence

$$G_\lambda(t; u) \rightarrow G_{23}(t; u) := \frac{1}{c_1 |u_1|^q + \sum_{i=2}^3 c_i |t_i - u_3|^q} =: G_{23}(t; u)$$
point-wise for any fixed $u = (u_1, u_2, u_3) \in \mathbb{R}^3, t = (t_1, t_2, t_3) \in \mathbb{R}^3, u_1 \neq 0, u_3 \neq t_i, i = 2, 3$. Moreover, $\mathcal{Y}_3(x) = \int_{\mathbb{R}^3} f_3(u) W(du)$, where $f_3(u) := \int_{[0, x_1] \times [0, x_2] \times [0, x_3]} G_3(t; u) dt$. The details of the convergence $\tilde{h}_\lambda(u) := \int_{\Pi_{i=1}^3 [0, [u_i x_i]/\tilde{m}_i]} G_\lambda(t; u) dt \rightarrow f_3(u)$ in $L^2(\mathbb{R}^3)$ are similar and omitted.

**Proof of (4.6).** Let $m_i := [\lambda^{\gamma_i}], \tilde{m}_i := \lambda^{\gamma_i}, i = 1, 2, 3, \kappa_\lambda := \prod_{i=1}^3 m_i^{1/2} \tilde{m}_i / \lambda H_0 m_1 q_1 \rightarrow 1$. As noted above, in this case $H_0 = \gamma_1 q_1 ((3/2) \sum_{i=1}^3 1/q_i - 1)$ agrees with any of $H_1, \cdots, H_{23}$ in the above proof. We also see that $G_\lambda(t; u)$ defined in (5.8)-(5.17) tends to $G_0(t; u)$, viz.,

$$G_\lambda(t; u) \rightarrow \frac{1}{\sum_{i=1}^3 c_i |t_i - u_3|^q} =: G_0(t; u)$$
point-wise for any fixed \( \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3, \mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3, u_i \neq t_i, i = 1, 2, 3 \), and \( \mathcal{Y}_0(x) = \int_{\mathbb{R}^3} f_0(\mathbf{u}) W(d\mathbf{u}) \), where \( f_0(\mathbf{u}) := \int_{[0,x_1] \times [0,x_2] \times [0,x_3]} G_0(t; \mathbf{u}) dt \). We omit the rest of the proof since it is similar as in the previous cases. Theorem 4.1 is proved. □

References

[1] Albeverio, S., Molchanov, S.A. and Surgailis, D. (1994) Stratified structure of the Universe and Burgers’ equation - a probabilistic approach. Probab. Th. Rel. Fields 100, 457–484.

[2] Anh, V.V., Leonenko, N.N. and Ruiz-Medina, M.D. (2013) Macroscale limit theorems for filtered spatiotemporal random fields. Stochastic Anal. Appl. 31, 460–508.

[3] Ayache, A., Leger, S. and Pontier, M. (2002) Drap Brownien fractionnaire. Potential Anal. 17, 31–43.

[4] Bierné, H., Meerschaert, M.M. and Scheffler, H.P. (2007) Operator scaling stable random fields. Stoch. Process. Appl. 117, 312-332.

[5] Bulinski, A., Spodarev, E. and Timmermann, F. (2012) Central limit theorems for the excursion sets volumes of weakly dependent random fields. Bernoulli 18, 100–118.

[6] Damarackas, J. and Paulauskas, V. (2017) Spectral covariance and limit theorems for random fields with infinite variance. J. Multiv. Anal. 153, 156-175.

[7] Dobrushin, R.L. (1979) Gaussian and their subordinated self-similar random generalized fields. Ann. Probab. 7, 1–28.

[8] Dobrushin, R.L. and Major, P. (1979) Non-central limit theorems for non-linear functionals of Gaussian fields. Probab. Th. Rel. Fields 50, 27–52.

[9] Doukhan, P., Lang, G. and Surgailis, D. (2002) Asymptotics of weighted empirical processes of linear random fields with long range dependence. Annales d’Institute de H. Poincaré 38, 879–896.

[10] Doukhan, P. (2003) Models, inequalities, and limit theorems for stationary sequences. In: P. Doukhan, G. Oppenheim and M.S. Taqqu (Eds.) Theory and Applications of Long-Range Dependence, pp. 43–100. Birkhäuser, Boston.

[11] Gaigalas, R. and Kaj, I. (2003) Convergence of scaled renewal processes and a packet arrival model. Bernoulli 9, 671–703.

[12] Giraitis, L., Koul, H.L. and Surgailis, D. (2012) Large Sample Inference for Long Memory Processes. Imperial College Press, London.

[13] Kaj, I. and Taqqu, M.S. (2008) Convergence to fractional Brownian motion and to the Telecom process: the integral representation approach. In: Vares, M.E. and Sidoravicius, V. (eds.) An Out of Equilibrium 2. Progress in Probability, vol. 60, pp. 383–427. Birkhäuser, Basel.

[14] Lahiri, S.N. and Robinson, P.M. (2016) Central limit theorems for long range dependent spatial linear processes. Bernoulli 22, 345–375.

[15] Lavancier, F. (2007) Invariance principles for non-isotropic long memory random fields. Statist. Inference Stoch. Process. 10, 255–282.

[16] Leipus, R., Philippe, A., Pilipauskaitė, V. and Surgailis, D. (2018) Sample autocovariances of random-coefficient AR(1) panel model. Preprint.
[17] Leonenko, N.N. (1999) *Random Fields with Singular Spectrum*. Kluwer, Dordrecht.

[18] Leonenko, N.N., Ruiz-Medina, M.D. and Taqqu, M.S. (2011) Fractional elliptic, hyperbolic and parabolic random fields. Electronic J. Probab. 16, 1134–1172.

[19] Lifshits, M. (2014) *Random Processes by Example*. World Scientific, New Jersey etc.

[20] Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? Ann. Appl. Probab. 12, 23–68.

[21] Pilipauskaitė, V. and Surgailis, D. (2014) Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. Stochastic Process. Appl. 124, 1011–1035.

[22] Pilipauskaitė, V. and Surgailis, D. (2015) Joint aggregation of random-coefficient AR(1) processes with common innovations. Statist. Probab. Lett. 101, 73–82.

[23] Pilipauskaitė, V. and Surgailis, D. (2016) Anisotropic scaling of random grain model with application to network traffic. J. Appl. Probab. 53, 857–879.

[24] Pilipauskaitė, V. and Surgailis, D. (2017) Scaling transition for nonlinear random fields with long-range dependence. Stochastic Process. Appl. 127, 2751–2779.

[25] Puplinskaitė, D. and Surgailis, D. (2010) Aggregation of random coefficient AR(1) process with infinite variance and idiosyncratic innovations. Adv. Appl. Probab. 42, 509–527.

[26] Puplinskaitė, D. and Surgailis, D. (2015) Scaling transition for long-range dependent Gaussian random fields. Stoch. Process. Appl. 125, 2256–2271.

[27] Puplinskaitė, D. and Surgailis, D. (2016) Aggregation of autoregressive random fields and anisotropic long-range dependence. Bernoulli 22, 2401–2441.

[28] Surgailis, D. (1982) Zones of attraction of self-similar multiple integrals. Lithuanian Math. J. 22, 185–201.