Multiple scan data association by convex variational inference

Jason L. Williams and Roslyn A. Lau

Abstract—Data association, the reasoning over correspondence between targets and measurements, is a problem of fundamental importance in target tracking. Recently, belief propagation (BP) has emerged as a promising method for estimating the marginal probabilities of target/measurement association, providing fast, accurate estimates. The excellent performance of BP in the particular formulation used may be attributed to the convexity of the underlying free energy which it implicitly optimises. This paper studies multiple scan data association problems, i.e., problems that reason over correspondence between targets and several sets of measurements, which may correspond to different sensors or different time steps. We find that the multiple scan extension of the single scan BP formulation is non-convex and demonstrate the undesirable behaviour that can result. A convex free energy is constructed using the recently proposed fractional free energy, and optimised using a primal-dual coordinate ascent. Finally, based on a variational interpretation of joint probabilistic data association (JPDA), we develop a sequential variant of the algorithm that is similar to JPDA, but retains consistency constraints from prior scans. The performance of the proposed methods is demonstrated on a bearings only target localisation problem.

I. INTRODUCTION

Multiple target tracking is complicated by data association, the unknown correspondence between measurements and targets. The classical problem arises under the assumption that measurements are received in scans (i.e., a collection of measurements made at a single time), and that within each scan, each target corresponds to at most one measurement, and each measurement corresponds to at most one target.

Techniques for addressing data association may be classified as either single scan (considering a single scan of data at a time) or multiple scan (simultaneously considering multiple scans), and as either maximum a posterior (MAP) (finding the most likely correspondence), or marginal-based (calculating the full marginal distribution for each target). Common methods include:

1) Global nearest neighbour (GNN), e.g., [1], is a single scan MAP method, which finds the MAP correspondence in the latest scan, and proceeds to the next scan assuming that correspondence was correct.
2) Multiple hypothesis tracking (MHT) [2]–[4] is a multiple scan MAP method, which in each scan seeks to find the MAP correspondence over a recent history of scans.
3) Joint probabilistic data association (JPDA) [5] is a single scan marginal-based method, which calculates the marginal distribution of each track, and proceeds by approximating the joint distribution as the product of its marginals.

Classical JPDA additionally approximates the distribution of each target as a moment-matched Gaussian distribution; in this paper, we use the term JPDA more generally to refer to the approach that retains the full marginal distribution of each target in a manner similar to [6]–[10].

Compared to GNN, MHT and JPDA, multiple scan variants of JPDA, e.g., [11], have received less attention. One may posit that this is due to their formidable computational complexity: While there exist fast approximations to the multiple scan MAP problem such as Lagrangian relaxation [12,13], no such equivalents have existed for either single scan or multiple scan JPDA.

Variational inference (e.g., [14,15]) describes the collection of methods that use optimisation (or calculus of variations) to approximate difficult inference problems in probabilistic graphical models (PGMs). Methods within this framework include belief propagation (BP) [16,17], mean field [18], tree-reweighted sum product (TRSP) [19] and norm-product BP (NPBP) [20]. Excellent performance has been demonstrated in a variety of problems, typified by the recognition that turbo coding is an instance of BP [21].

Variational inference was first applied to data association in [22]–[24], addressing the problem of distributed tracking using wireless sensor networks. The problem was formulated with vertices corresponding to targets and sensors, where sensor nodes represent the joint association of all sensor measurements in the scan to targets. A related sensor network application was studied in [25].

In contrast to these methods, which hypothesise the joint association of all sensor measurements via a single variable, the approach in [26]–[30] formulates the single scan problem in terms of a bipartite graph, where vertices hypothesise the measurement associated with a particular target, or the target associated with a particular measurement. Empirically, it was found that BP converges reliably and produces excellent estimates of both the marginal association probabilities [28,30] and the partition function [29]. Convergence of BP in the two related formulations was proven in [31,32].

In [33] it was shown that, when correctly parameterised, the variational inference problem that underlies the bipartite graph...
formulation is convex. This may be understood to be the source of the empirically observed robustness of the approach when applied to the bipartite model. For example, it was shown in [34] that the approximations produced by BP tend to be either very good, or very bad. This may be understood through the intuition that BP converges to a “good” approximation if the objective function that it implicitly optimises is locally convex in the area between the starting point and the optimal solution, and a “bad” approximation if it is non-convex. Accordingly, if the problem is globally convex, a significant source of degenerate cases is eliminated.

PGM methods provide a straightforward extension of the bipartite model to multiple scan problems; this has been studied in works including [28][35]. An alternative PGM formulation of the same problem was utilised in [36] for the purpose of parameter identification. However, these approaches lose the convexity property of the single scan bipartite formulation, which is understood to be the source of the robustness of this special case.

A. Contributions

This paper addresses the multiple scan data association problem using a convexification of the multiple scan model utilised in [28][35]. The model is developed based on a variational interpretation of JPDA, which at each iteration calculates the marginal distribution of each target, and approximates the posterior as the product of its marginals. We develop a convex alternative for the variational problem, which uses as a building block for each scan the fractional free energy (FFE) introduced in [37]. In section IV the convex approach is demonstrated to provide high-quality, conservative estimates of marginal distributions through a bearings only localisation problem. This is in contrast to multiple scan BP (MSBP), which is shown to give vanishingly small likelihood to the true solution in a significant portion of cases. Improved performance is also demonstrated in comparison to JPDA.

The proposed method operates by iteratively solving single scan FFE problems. We prove convergence of a BP-like algorithm for optimising the single scan FFE of [37]. The proof of convergence is based on [33], but admits cases in which some elements of the matrix are zero.

II. BACKGROUND

A. Multiple scan data association

The problem we consider is that of data association across multiple scans, involving many targets, the state of which is to be estimated through point measurements. We assume that many targets are present, each target gives rise to at most one measurement (excluding so-called extended target problems, where targets may produce multiple measurements), and each measurement is related to at most one target (excluding so-called merged measurement problems, e.g., where multiple targets fall within a resolution cell). We assume that false alarms occur according to a Poisson point process (PPP). Denote the joint state of all targets at time $t$ by $X_t = (x_1^t, \ldots, x_n^t)$ in the vector case, and the set of measurements at time $t$ by $Z_t = \{z_1^t, \ldots, z_m^t\}$.

Assumptions, terminology and notation broadly follow [10]. For ease of understanding, the formulation is described in terms of the vector case (i.e., in which the number of targets present is known), but it is equally applicable to the random finite set (RFS) formulation in [10] which admits an unknown, time-varying number of targets.

The goal of the formulation is to avoid the need to explicitly enumerate or reason over global hypotheses, i.e., hypotheses in the joint state space. Instead, marginal distributions of each target are stored, and dependencies between targets are accounted for using variational methods.

Definition 1. A global association history hypothesis (or global hypothesis for short) is a partitioning of measurements received so far into subsets, where each subset is hypothesised to correspond to a particular target.

Definition 2. A single target association history hypothesis (or single target hypothesis for short) is a subset of measurements that are hypothesised to correspond to the same target.

A global hypothesis at time $t$ may be represented as $A_t = (A_1^t, \ldots, A_n^t)$, where $A_i^t = \{a_1^t, \ldots, a_l^t\}$ indexes the single target association history hypothesis utilised for the $i$-th target, and $a_j^t \in \{0, \ldots, m_t\}$ is the index of the measurement used by track $i$ in scan $\tau \leq t$ (or zero for a missed detection). Each hypothesis is equipped with a hypothesis weight $w_t^{i,A_i^t}$ (utilised in the calculation of the probability of the global hypotheses), and the target state probability density function (PDF) conditioned on the hypothesis $f_t^{i,A_i^t}(x_t)$.

Since we do not detail prediction steps in this work (which follow well-known forms), we simplify notation by using a single time index $t$ in PDFs such as $f_t^{i,A_i^t}(\cdot)$ and quantities such as $w_t^{i,A_i^t}$ to indicate both the quantity at time $t$ and conditioning on measurements up to and including time $t$, i.e., $Z_t = (Z_1^t, \ldots, Z_t)$.

Definition 3. A track is the collection of information available about a particular target, consisting of a collection of single-target hypotheses (and the accompanying information) representing different possibilities of measurement sequences corresponding to the target.

Note that this definition is more in line with the use of the term in JPDA and related methods than MHT.

We denote by $n$ the number of tracks, and by $A_i^t$ the set of single-target hypotheses in track $i \in \{1, \ldots, n\}$ conditioned on measurements up to and including time $t$. The set of all feasible global hypotheses (i.e., those in which no two tracks utilise the same measurement) can be written as:

$$A^t = \left\{ (A_1^t, \ldots, A_n^t) \mid A_i^t = (a_1^t, \ldots, a_l^t) \in A_i^t; a_j^t \neq a_k^t \forall \tau, i, j \text{ s.t. } i \neq j, a_k^t \neq 0 \right\}$$

The term “track” in [3] refers to the single-target hypothesis of definition 2 while the target tree, or the set of track hypotheses in [3] refers to the track of definition 3.
As shown in [10], the probability of a global hypothesis 
\(A_t = (A_1^t, \ldots, A_T^t) \in A_T\) can be written in the form:

\[
p_t(A_t) \propto \prod_{i=1}^{n} w_{t_i}^{i, A_i^t}
\]

(2)

In the vector case, the joint PDF of all targets can be represented through a total probability expansion over all global hypotheses:

\[
f_t(X_t) = \sum_{A_t \in A_T} p_t(A_t) \prod_{i=1}^{n} f_{i}^{i, A_i^t}(x_i^t)
\]

(3)

Similar expressions for the RFS case can be found in [10]. It is of interest to obtain the marginal distributions of global hypothesis probabilities:

\[
p_t^i(A_i^t) = \sum_{A_t \in A_T} p_t(A_t) f_{i}^{i, A_i^t}(x_i^t)
\]

(4)

From these marginal association distributions, we can find the marginal state PDF of each target:

\[
f_t^i(x_i^t) = \sum_{A_t \in A_T} p_t^i(A_i^t) f_t^{i, A_i^t}(x_i^t)
\]

(5)

Prediction steps follow the well-known expressions, e.g., [38]. The update step constructs new hypotheses for each old single-target hypothesis, first assuming that a missed detection occurred, and second updating the hypothesis with one of the measurements \(Z_t = \{z_1^t, \ldots, z_n^t\}\). The parameters for the hypothesis \(A_i^t = (A_{i-1}^t, 0)\), which considers the possibility that a missed detection occurred, can be calculated using the expression:

\[
w_{i}^{i, A_i^t} = \frac{w_{i-1}^{i, A_i^t-1}}{1 - P_d(x_i^t)} f_{i}^{i, A_i^t-1}(x_i^t) \int dx_i^t
\]

(6)

\[
f_t^{i, A_i^t}(x_i^t) \propto \left[1 - P_d(x_i^t)\right] f_t^{i, A_i^t-1}(x_i^t)
\]

(7)

where \(f_{i}^{i, A_i^t-1}(x_i^t)\) is the PDF predicted from the previous time step (generally speaking the weights \(w_{i}^{i, A_i^t-1}\) are not changed under prediction), and \(P_d(x)\) is the probability of detection of a target in state \(x\). The hypothesis \(A_i^t = (A_{i-1}^t, j)\), which updates old single-target hypothesis \(A_{i-1}^t\) with measurement \(z_j^t\), is calculated using the expressions:

\[
w_{i}^{i, A_i^t} = \frac{w_{i-1}^{i, A_i^t-1}}{f(z_j^t | x_i^t) P_d(x_i^t)} \int dx_i^t
\]

(8)

\[
f_t^{i, A_i^t}(x_i^t) \propto f(z_j^t | x_i^t) P_d(x_i^t) f_t^{i, A_i^t-1}(x_i^t)
\]

(9)

where \(f(z_j^t | x_i^t)\) is the intensity of the false alarm Poisson point process at the measurement location \(z\). The RFS-based extension of these prediction and update steps to accommodate an unknown, time-varying number of targets can be found in [10].

B. Probabilistic graphical models and variational inference

PGMs [14, 15, 39] aim to represent and manipulate the joint probability distributions of many variables efficiently by exploiting factorisation. The Kalman filter [40] and the hidden Markov model (HMM) [41] are two examples of algorithms that exploit sparsity of a particular kind (i.e., a Markov chain) to efficiently conduct inference on systems involving many random variables. Inference methods based on the PGM framework generalise these algorithms to a wider variety of state spaces and dependency structures.

PGMs have been developed for undirected graphical models (Markov random fields), directed graphical models (Bayes nets) and factor graphs. In this work we consider a subclass of pairwise undirected models, involving vertices (i.e., random variables) \(v \in V\), and edges (i.e., dependencies) \(e \in E \subset V \times V\), and where the joint distribution can be written as:

\[
p(x_V) \propto \prod_{v \in V} \psi_v(x_v) \prod_{(i,j) \in E} \psi_{i,j}(x_i, x_j)
\]

As an example, a Markov chain involving variables \((x_1, \ldots, x_T)\) may be formulated by setting \(\psi_1(x_1) = p(x_1)\) for the initial prior, \(\psi_{t-1,t}(x_{t-1}, x_t) = 1\) for \(t > 1\), and edges \(\psi_{t-1,t}(x_{t-1}, x_t) = p(x_t|x_{t-1})\), \(t \in \{2, \ldots, T\}\) representing the Markov transition kernels, although other formulations are possible.

Exact inference can be conducted on tree-structured graphs using belief propagation (BP), which operates by passing messages between neighbouring vertices. We denote by \(\mu_{i \rightarrow j}(x_j)\) the message sent from vertex \(i \in V\) to vertex \(j \in V\) where \((i, j) \in E\). The iterative update equations are then:

\[
\mu_{i \rightarrow j}(x_j) \propto \sum_{x_i} \psi_{i,j}(x_i, x_j) \psi_i(x_i) \prod_{(j', i) \in E \setminus E' \setminus (j, i)} \mu_{j' \rightarrow i}(x_i)
\]

(10)

For obvious reasons, this is also known as the sum-product algorithm. If the summations are replaced with maximisation operations, then we arrive at max-product BP, which generalises the Viterbi algorithm [42], providing the MAP joint state of the Markov model (HMM) [41] are two examples of algorithms that exploit sparsity of a particular kind (i.e., a Markov chain) to efficiently conduct inference on systems involving many random variables. Inference methods based on the PGM framework generalise these algorithms to a wider variety of state spaces and dependency structures.

Inference in cyclic graphs (graphs that have cycles, i.e., that are not tree-structured) is far more challenging. Conceptually, one can always convert an arbitrary cyclic graph to a tree by merging vertices (e.g., so-called junction tree representations) [15, 39], but in practical problems, the dimensionality of the agglomerated variables tends to be prohibitive. BP may be applied to cyclic graphs; practically, this simply involves repeated application of (10) until convergence occurs (i.e., until the maximum change between subsequent messages is
less than a pre-set threshold. Unfortunately, this is neither guaranteed to converge to the right answer, nor to converge at all. Nevertheless, and perhaps surprisingly, it has exhibited excellent empirical performance in many practical problems\cite{24}. For example, the popular iterative turbo decoding algorithm has been shown to be an instance of BP applied to a cyclic graph\cite{21}.

The current understanding of BP in cyclic graphs stems largely from\cite{17}. It has been shown (e.g.,\cite{13} Theorem 3.4) that one can recover exact marginal probabilities from an optimisation of a convex function known as the Gibbs free energy. In the single-vertex case (or if all variables are merged into a single vertex), this can be written as described in lemma 1.

**Lemma 1.** The Gibbs free energy variational problem for a single random variable $x$ can be written as:

$$\min_{q(x)} - H(x) - E[\log \psi(x)]$$

subject to $q(x) \geq 0$, $\sum_x q(x) = 1$  \hspace{1cm} (12)

where $E[\log \psi(x)] \triangleq \sum_x q(x) \log \psi(x)$, and $H(x) = -E[\log q(x)]$ is the entropy of the distribution $q(x)$ (all expectations and entropies are under the distribution $q$). The solution of the optimisation is

$$q(x) = \frac{\psi(x)}{\sum_{x'} \psi(x')} \propto \psi(x)$$

Similar expressions apply for continuous random variables, replacing sums with integrals. The objective in (12) can be recognised as the Kullback-Leibler (KL) divergence between $q(x)$ and the (unnormalised) distribution $\psi(x)$. If the graph is a tree, the entropy can be decomposed as:

$$H(x) = \sum_{v \in \mathcal{V}} H(x_v) - \sum_{(i,j) \in \mathcal{E}} I(x_i; x_j)$$

where $I(x_i; x_j) = H(x_i) + H(x_j) - H(x_i, x_j)$ is the mutual information between $x_i$ and $x_j$\cite{43}. Accordingly, the variational problem can be written as:

$$\min_{q(x_v), q(x_i, x_j)} - \sum_{v \in \mathcal{V}} \{ H(x_v) + E[\log \psi_v(x_v)] \}$$

$$- \sum_{(i,j) \in \mathcal{E}} \{ -I(x_i; x_j) + E[\log \psi_{i,j}(x_i, x_j)] \}$$

subject to $q(x_i, x_j) \geq 0 \forall (i,j) \in \mathcal{E}$, $\forall x_i, x_j$

$$\sum_{x_i} q(x_i, x_j) = 1 \forall v \in \mathcal{V}$$

$$\sum_{x_j} q(x_i, x_j) = q(x_i) \forall (i,j) \in \mathcal{E}$$

$$\sum_{x_j} q(x_i, x_j) = q(x_j) \forall (i,j) \in \mathcal{E}$$

(17)

For tree-structured graphs, BP can be shown to converge to the optimal value of this convex, variational optimisation problem. It has further been shown that the feasible set described by\cite{18}-\cite{21} is exact, i.e., any feasible solution can be obtained by a valid joint distribution, and any valid joint distribution maps to a feasible solution.

If a graph contains a leaf vertex $x_i$ that is connected only to vertex $x_j$ via factor $\psi_{i,j}(x_i, x_j)$\footnote{Without loss of generality, assume that neither vertex has a vertex factor, as this can be incorporated into the edge factor.}, then inference can be performed equivalently by eliminating vertex $x_i$, and replacing the vertex factor for $x_j$ with $[15]$ ch 9

$$\tilde{\psi}_j(x_j) = \sum_{x_i} \psi_{i,j}(x_i, x_j)$$

(22)

Given the resulting marginal distribution for $p(x_j)$ (or an approximation thereof), the pairwise joint distribution (or belief) of $(x_i, x_j)$ can be reconstructed as:

$$p(x_i, x_j) = p(x_j) \frac{\psi_{i,j}(x_i, x_j)}{\tilde{\psi}_j(x_j)}$$

(23)

**Lemma 2.** Let $J[q(x_j)]$ be the solution of the following optimisation problem:

$$J[q(x_j)] = \min_{q(x_i, x_j) \geq 0} - H(x_i | x_j) - E[\log \psi_{i,j}(x_i, x_j)]$$

subject to $\sum_{x_i} q(x_i, x_j) = q(x_j) \forall x_j$  \hspace{1cm} (27)

Then

$$J[q(x_j)] = - \sum_{x_j} q(x_j) \log \sum_{x_i} \psi_{i,j}(x_i, x_j)$$

(29)

and the minimum of the optimisation of (27) is attained at $q(x_i, x_j) = q(x_i)q(x_j | x_i)$, where

$$q(x_i | x_j) = \frac{\psi_{i,j}(x_i, x_j)}{\sum_{x_i'} \psi_{i,j}(x_i', x_j)}$$

(30)

Proof of this result can be found in\cite{20} App C. On tree-structured graphs, BP may be viewed as successive applications of variable elimination, followed by reconstruction.

If the graph has cycles (i.e., is not a tree), then the entropy does not decompose into the form in (15)-(17). Furthermore, the feasible set in\cite{18}-\cite{21} is an outer bound to the true feasible set, i.e., there are feasible combinations of marginal distributions that do not correspond to a valid joint distribution. Nevertheless,
as an approximation, one may solve the optimisation in (17)-(21). The objective in (17) is referred to as the Bethe free energy (BFE) after [43], a connection identified in [17]. For a cyclic graph, it differs from the Gibbs free energy, but is a commonly utilised approximation.

It was shown in [17] that, if it converges, the solution obtained by BP is a local minimum of the BFE. The BP message iterates in (10) can be viewed as a general iterative method for solving a series of fixed point equations derived from the optimality conditions of the BFE variational problem (see [14], 4.1.3]). The marginal probability estimates obtained using BP, denoted in this paper by the symbol $q$, are referred to as beliefs.

TRSP [14,19] provides a convex alternative to the BFE, by applying weights $\gamma_{i,j} \in [0,1]$ to the mutual information terms $I(x_i;x_j)$. If the weights correspond to a convex combination of embedded trees (i.e., a convex combination of weighted, tree-structured sub-graphs, where in a given graph $\gamma_{i,j} = 1$ if the edge is included, and $\gamma_{i,j} = 0$ otherwise), then the resulting free energy is convex. A rigorous method for minimising energy functions of this form was provided in [20].

C. BP data association

It was shown in [28,30] that, for a single scan, the data association problem in section II-A can be formulated via a graphical model with the structure shown in figure 1. The value of the random variable $a_i^t \in \{0, \ldots, m_t\}$ represents the index of the measurement with which target $i$ is associated (or zero for missed detection), and the value of the variable $b_i^t \in \{0, \ldots, n\}$ represents the index of the target with which measurement $j$ is associated (or zero if the measurement is a false alarm). The vertex factors $\psi^t_i(a_i^t)$ were shown to be of the form (6), (8) (where it is assumed that there is a single prior hypothesis), the edge factors were shown to be:

$$\psi^{i,j}(a_i^t, b_i^t) = \begin{cases} 0, & a_i^t = j, b_i^t \neq i \text{ or } b_i^t = i, a_i^t \neq j \\ 1, & \text{otherwise} \end{cases}$$  

(31)

This redundant representation implicitly ensures that each measurement corresponds to at most one target, and each target corresponds to at most one measurement.

The graphical model in figure 1 excludes the continuous states $x_i^t$ since it is assumed that they have been marginalised (e.g., as described in [22]). In this paper, we work with the graphical model of the form figure [2](a), which includes a hypothesis history variable $A_i^t$ in addition to $a_i^t$ (which represents the association in the current scan). This graph results from performing variable elimination on the underlying Markov chain of continuous states. The single-target association weight is incorporated via the vertex factor $\psi^t_i(A_i^t) = w_{ti}^{i,A_i^t}$ (a vertex factor for $a_i^t$ is not required), and correspondence of $A_i^t$ and $a_i^t$ is ensured through the edge factor:

$$\psi^t_i(A_i^t, a_i^t) = \begin{cases} 1, & A_i^t = (A_i^{t-1}, a_i^t) \\ 0, & A_i^t = (A_i^{t-1}, \tilde{a}), \tilde{a} \neq a_i^t \end{cases}$$  

(32)

A similar structure can be used to represent a particle filter, replacing the hypothesis variable $A_i^t$ with a particle index $p^t$, where the prior particle weight is $w_{ti}^{i,p^t}$ and the particle state is $x_i^{t,p^t}$. The vertex factor encodes the prior particle weight, $\psi^t_i(p^t) = w_{ti}^{i,p^t}$, while the edge factor $\psi^t_i(p^t, a_i^t)$ encodes the likelihood of the measurement given the particle:

$$\psi^t_i(p^t, a_i^t) = \begin{cases} 1 - P^t_i(x_i^{t,p^t}), & a_i^t = 0 \\ f(z, x_i^{t,p^t}, p^t_i(x_i^{t,p^t})) \lambda_{i}(x_i^{t,p^t}), & a_i^t = j > 0 \end{cases}$$  

(33)

This representation is used in our experiments in section IV.

Following similar lines to [33], we show in appendix A-A that the Bethe variational problem can be solved by minimising the objective:

$$F_B([q^t_i(A_i^t)], [q^t_i(A_j^t), q^t_{i,j}]) = - \sum_{i=1}^{n} \{H(A_i^t) + \mathbb{E}[\log \psi^t_i(A_i^t)] + H(a_i^t|A_i^t) + \mathbb{E}[\log \psi^t_i(A_i^t, a_i^t)] \} + \sum_{j=1}^{m} q^0_{i,j} \log q^0_{i,j} - \sum_{i=1}^{n} \sum_{j=1}^{m} (1 - q^i_{t,j}) \log (1 - q^i_{t,j})$$  

(34)

subject to the constraints:

$$q^t_i(A_i^t, a_i^t) \geq 0, \quad q^0_{i,j} \geq 0$$  

(35)

$$\sum_{a_i^t=0}^{m_i} q^t_i(A_i^t, a_i^t) = q^t_i(A_i^t)$$  

(36)

$$q^{i,j}_t = \sum_{A_i^t \in A_i^t} q^t_i(A_i^t, j)$$  

(37)

$$q^0_{i,j} = 1 - \sum_{i=1}^{n} q^{i,j}_t$$  

(38)

$$\sum_{A_i^t \in A_i^t} q^t_i(A_i^t) = 1$$  

(39)

As shown in appendix A-A, performing a partial minimisation over the pairwise joint $q^t_i(A_i^t, a_i^t)$, we arrive at the equivalent problem:

$$\text{minimise } \sum_{i=1}^{n} \sum_{j=0}^{m_i} q^{i,j}_t \log \frac{q^{i,j}_t}{w_{ti}^{i,A_i^t}} + \sum_{j=1}^{m_t} q^0_{i,j} \log q^0_{i,j}$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{m_t} (1 - q^{i,j}_t) \log (1 - q^{i,j}_t)$$  

(40)

subject to:

$$\sum_{j=0}^{m_t} q^{i,j}_t = 1 \forall i \in \{1, \ldots, n\}$$  

(41)

$$\sum_{i=0}^{n} q^{i,j}_t = 1 \forall j \in \{1, \ldots, m_t\}$$  

(42)

$$0 \leq q^{i,j}_t \leq 1$$  

(43)
where \( w_{i,j} = \sum_{i,j} A_i^j \psi^i(A_i^j) \psi^j(A_i^j) \), and \( q_i^{t,j} = q(a_i^j = j) \).

The constraints in (41) and (42) are referred to as consistency constraints, as they are a necessary condition for the solution to correspond to a valid joint association event distribution.

In [37] it was shown that, if the correct fractional coefficient \( \gamma \in [−1, 1] \) is incorporated on the final term in the objective (40), the value of the modified objective function at the optimum is the same as the exact free energy objective. In the formulation that incorporates false alarms and missed detections, the fractional free energy (FFE) objective is:

\[
F(t)=[(q_i^{t,j})]=\frac{n}{m} \sum_{i=1}^{n} \sum_{j=1}^{m} q_i^{t,j} \log q_i^{t,j} + \gamma \sum_{i=1}^{n} q_i^{0,j} \log q_i^{0,j} - \gamma \sum_{i=1}^{n} \sum_{j=1}^{m} (1 - q_i^{t,j}) \log(1 - q_i^{t,j}) \tag{44}
\]

Despite the fact that the “right” value of \( \gamma \) is not known for any particular problem, it was shown in [46] that fractional values \( \gamma \in [0, 1] \) can yield improved beliefs for high SNR problems. Practically, the value could be chosen a priori based on the problem parameters.

D. Conjugate duality and primal-dual coordinate ascent

The Fenchel-Legendre conjugate dual of a function \( f(q) \) is defined as: [47]

\[
f^*(\lambda) = \sup_q \lambda^T q - f(q) \tag{45}
\]

The dual \( f^*(\lambda) \) is convex regardless of convexity of \( f(q) \), since it is constructed as the supremum of a family of linear functions. The key outcome of conjugate duality is that, if \( f(q) \) is closed and convex, then the conjugate dual of \( f^*(\lambda) \) is the original function \( f(q) \), thus \( f(q) \) and \( f^*(\lambda) \) are alternate representations of the same object.

Dual functions are useful in constrained convex optimisation since the optimal value of the primal

\[
\min_q f(q) \quad \text{subject to } Aq = 0 \tag{46}
\]

is the same as the optimal value of the dual optimisation

\[
\max_{\lambda} -f^*(A^T \lambda) \tag{47}
\]

and if \( f(q) \) is strictly convex then, given the optimal solution \( \lambda^* \) of the dual, the optimal value \( q^* \) of the primal can be recovered as the solution of the unconstrained optimisation

\[
\min_q f(q) + (A^T \lambda^*)^T q \tag{48}
\]

The additional usefulness of conjugate duality over Lagrangian duality is its ability to tractably address objectives that decompose additively. In this work, we utilise the primal-dual framework developed in [20], which addresses problems of the form:

\[
\min_q f(q) + \sum_{i=1}^{n} h_i(q) \tag{49}
\]

where \( f(q) \) and \( h_i(q) \) are proper, closed, convex functions. Constraints are addressed by admitting extended real-valued functions. It is shown in [20,48] that the dual of (49) is

\[
\max_{\lambda_1, \ldots, \lambda_n} -f^* \left( -\sum_{i=1}^{n} \lambda_i \right) - \sum_{i=1}^{n} h_i^*(\lambda_i) \tag{50}
\]

Thus, assuming smoothness of \( f^* \) (or strict convexity of \( f \)), the dual optimisation can be performed via block coordinate ascent, iteratively performing the following steps for each \( i \):

\[
\mu := \sum_{j \neq i} \lambda_j \tag{51}
\]

\[
\lambda_i := \arg \max_{\lambda_i} -f^*(\lambda_i - \mu) - h_i^*(\lambda_i) \tag{52}
\]

The method in [20] shows that the block optimisations required in (52) can be performed via primal minimisations, i.e., the updated value \( \lambda_i \) can be obtained through the optimisation

\[
q^* := \arg \min_q f(q) + h_i(q) + \mu^T q \tag{53}
\]

\[
\lambda_i := -\mu - \nabla f(q^*) \tag{54}
\]

In [20], the authors develop the norm-product belief propagation algorithm for convexifications of general PGM inference problems. While these methods could be applied to the problem of interest, they do not exploit the unique problem structure and resulting convexity discussed in section II-C. Consequently, the necessary convexification procedure would produce an unnecessarily large change to the BFE. Thus we adopt the optimisation framework of [20], but the solution does not exactly fit the NPBP algorithm, and so it is necessary to develop it from the basic framework.

III. VARIATIONAL MULTIPLE SCAN DATA ASSOCIATION

The standard approach to tracking using JPDA and related methods is to calculate the marginal distribution of each track, and proceed to the next scan approximating the posterior as the product of the single-target marginal distributions. In many cases, this approach is surprisingly effective. The method proposed in this work is based on a variational interpretation of the JPDA approach, which gives rise to a family of formulations that includes the JPDA-like approach and MSBP.

In particular, we examine the single scan BP approach (e.g., [10]) which calculates association beliefs, and approximates the joint as the product of the beliefs (although the steps taken are not unique to the BP estimate of marginal probabilities). Consider a particular scan, \( (t-1) \), at which time the beliefs are calculated through the optimisation of (54), and denote

\[\gamma \text{Note that we reverse the sign of } \gamma \text{ in comparison to [37], so that } \gamma = 1 \text{ yields the regular BFE.}

\[\gamma \text{In high SNR cases (very high } \gamma \text{, very low } \lambda \text{), the BFE objective tends to yield solutions that are almost integral, i.e., are closer to MAP solutions rather than marginal probabilities. The inclusion of the } \gamma \text{ coefficient on the term involving } q_i^{t,j} \text{ retains the property of the BFE that the optimisation provides a near-exact result when targets are well-spaced.} \]
the solution as \( q_{t-1}^{i,j} (A_{t-1}^{i}, a_{t-1}^{i,j}) \) and \( q_{t-2}^{i,j} \). By approximating the joint by the product of marginals, JPDA effectively approximates the graph in figure 2(a) via the graph in figure 2(b), modifying the factor \( \psi_{t-1}^{i,j} (A_{t-1}^{i}, a_{t-1}^{i,j}) \) such that the modified formulation results in the same solution as the original problem in figure 2(a). Comparing the variational problems at time 2 in figures 2(b) (JPDA) and 2(c) (MSBP), we note the following differences:

1) The concave term \( -\sum_{i=1}^{n} \sum_{j=1}^{m} (1 - q_{t-1}^{i,j}) \log(1 - q_{t-1}^{i,j}) \) appears only for scan 2 in figure 2(b), but for both scans in figure 2(c);
2) The constraint \( \sum_{i=1}^{n} q_{t-1}^{i,j} = 1 \) appears only for scan 2 in figure 2(b), but for both scans in figure 2(c);
3) The factors \( \psi_{t-1}^{i,j} (A_{t-1}^{i}, a_{t-1}^{i,j}) \) have been modified in figure 2(b) but remain at their original values in figure 2(c);
4) The objective function in figure 2(b) is convex, whereas the objective in figure 2(c) is non-convex.

It can be shown that the modification of the factors \( \psi_{t-1}^{i,j} (A_{t-1}^{i}, a_{t-1}^{i,j}) \) effectively incorporates the Lagrange multipliers for the constraints that are being relaxed, and a linearisation of the concave term is reminiscent of a single iteration of the convex-concave procedure in [49].

The proposed solution commences from the variational problem in 2(c), and makes the following modifications:

1) As in the FFE objective in (44), we apply fractional weights \( \gamma_{t'} \) to the concave terms \( -\sum_{i=1}^{n} \sum_{j=1}^{m} (1 - q_{t'}^{i,j}) \log(1 - q_{t'}^{i,j}) \), such that if \( \sum_{t'=1}^{T} \gamma_{t'} < 1 \) then the objective function will be strictly convex.
2) We retain constraints \( \sum_{i=1}^{n} q_{t-1}^{i,j} = 1 \) for both scans.
3) If the coefficient \( \gamma_{t'} \) differs from the value used for that scan at the previous time step, we modify the factor \( \psi_{t'}^{i,j} (A_{t}^{i}, a_{t}^{i,j}) \) such that the solution remains unchanged (before the following scan is incorporated).

The reason for the latter step is that it was found to be desirable to use values \( \gamma_{t'} \) close to 1 in the current (most recent) scan, and reducing to zero in earlier scans. In our experiments, we use the selection \( \gamma_{t} \in \{0.55,0.75,0.95\} \) in the current scan, and \( \gamma_{t'} = 0 \) in past scans. In this case, the algorithm approximates the objective function in a similar manner to JPDA (using BP), but the consistency constraints from previous scans are retained. Thus, unlike JPDA, when later scans are processed, the constraints which ensure that the origin of past measurements is consistently explained remain enforced. This allows new information to cause significant adjustment to past association beliefs. In section IV, we will see that this can result in improved performance.

Because the remainder of the section concerns the solution of the association problem for the current scan \( t \), we drop the subscript \( t \) from the association history event \( A_{t}^{i} \).

A. Bethe free energy function

In appendix A-B we show that the Bethe free energy for the multiple scan formulation in section II-A (and figure 2(c))
can be written as:

\[
F_B([q^i(A^i)], [q^i_t(A^i, a^i_t)], [q^i_{t'}]) = \\
- \sum_{i=1}^{n} \{H(A^i) + \mathbb{E}[\log \psi^i_t(A^i)]\} \\
- \sum_{t' \in S} \sum_{i=1}^{n} \{H(a^i_t | A^i) + \mathbb{E}[\log \psi^i_{t'}(A^i, a^i_t)]\} \\
+ \sum_{t' \in S} \sum_{j=1}^{m_{t'}} q^i_{t'} \log q^i_{t'} - \sum_{t' \in S} \sum_{i=1}^{n} \sum_{j=1}^{m_{t'}} (1 - q^i_{t'}) \log(1 - q^i_{t'})
\]  \hspace{1cm} (55)

subject to the constraints:

\[
q^i_t(A^i, a^i_t) \geq 0, \quad q^i(A^i) \geq 0
\]  \hspace{1cm} (56)

\[
\sum_{A^i \in A^i_t} q^i_t(A^i, a^i_t) = 1, \quad \sum_{A^i \in A^i_t} q^i(A^i) = 1
\]  \hspace{1cm} (57)

\[
q^i_{t'} = \sum_{A^i \in A^i_t} q^i_t(A^i, j) \quad \forall \ t' \in S
\]  \hspace{1cm} (58)

\[
q^i_{t'} \geq 0 \quad \forall \ i, j, \forall \ t' \in S
\]  \hspace{1cm} (59)

\[
\sum_{i=0}^{n} q^i_{t'} = 1 \quad \forall \ t' \in S
\]  \hspace{1cm} (60)

\[
\sum_{j=0}^{m_{t'}} q^i_{t'} = 1 \quad \forall \ t' \in S
\]  \hspace{1cm} (61)

where \(q^i(A^i)\) is the belief (i.e., approximate probability) of single-target hypothesis \(A^i\) for track \(i\), \(q^i_{t'}\) is the belief that track \(i\) is associated with measurement \(z^i_{t'}\), \(q^i_{t'}\) is the belief that measurement \(z^i_{t'}\) is not associated with any track, and the set \(S\) indexes the measurement scans under consideration.

**B. Convexification of energy function**

In this work, we consider convex free energies of the form:

\[
F_B^{\gamma,\beta}([q^i(A^i)], [q^i_t(A^i, a^i_t)], [q^i_{t'}]) = \\
- \sum_{i=1}^{n} \{H(A^i) + \mathbb{E}[\log \psi^i_t(A^i)]\} \\
- \sum_{t' \in S} \sum_{i=1}^{n} \{H(a^i_t | A^i) + \mathbb{E}[\log \psi^i_{t'}(A^i, a^i_t)]\} \\
+ \sum_{t' \in S} \sum_{j=1}^{m_{t'}} \beta_{t'} q^i_{t'} \log q^i_{t'} \\
- \sum_{t' \in S} \sum_{i=1}^{n} \sum_{j=1}^{m_{t'}} (1 - q^i_{t'}) \log(1 - q^i_{t'})
\]  \hspace{1cm} (63)

Subsequently, we will show that (63) is convex if \(\eta \geq 0\) (strictly convex if \(\eta > 0\)), where

\[
\eta = 1 - \sum_{t' \in S} \gamma_{t'}
\]  \hspace{1cm} (64)

The use of re-weighting to obtain a convex free energy is closely related to the TRSP algorithm [19].

In the development which follows, we provide a decomposition of the convex free energy that permits application of primal-dual coordinate ascent (PDCA). First we give the basic form, and show that the components are convex. Then we state weights which ensure that the objective is the same as (63).

Then, in section [III-D], we provide algorithms to minimise each block that needs to be solved in PDCA.

In order to optimise (63), we consider decompositions of the expression of the form:

\[
F_B^{\gamma,\beta}([q^i(A^i)], [q^i_t(A^i, a^i_t)], [q^i_{t'}]) = f([q^i(A^i)], [q^i_t(A^i, a^i_t)]) \\
+ \sum_{t' \in S} \{h_{t',1}([q^i(A^i)], [q^i_t(A^i, a^i_t)]) + h_{t',2}([q^i(A^i)], [q^i_t(A^i, a^i_t)])\}
\]  \hspace{1cm} (65)

where

\[
f([q^i(A^i)], [q^i_t(A^i, a^i_t)]) = \\
- \sum_{i=1}^{n} \{\kappa_{f,A} H(A^i) + \mathbb{E}[\log \psi^i_t(A^i)]\} \\
- \sum_{t' \in S} \sum_{i=1}^{n} \{\kappa_{f,t'} H(A^i, a^i_t) + \mathbb{E}[\log \psi^i_{t'}(A^i, a^i_t)]\}
\]  \hspace{1cm} (66)

\[
h_{t',1}([q^i(A^i)], [q^i_t(A^i, a^i_t)]) = \\
- \sum_{i=1}^{n} \{\kappa_{t',1, A} H(A^i) + \kappa_{t',1, t'} H(A^i, a^i_t)\} \\
+ \beta_{t'} \sum_{j=1}^{m_{t'}} q^i_{t'} \log q^i_{t'} \\
- \gamma_{t'} \sum_{i=1}^{n} \sum_{j=1}^{m_{t'}} (1 - q^i_{t'}) \log(1 - q^i_{t'})
\]  \hspace{1cm} (67)

\[
h_{t',2}([q^i(A^i)], [q^i_t(A^i, a^i_t)]) = \\
- \sum_{i=1}^{n} \{\kappa_{t',2, A} H(A^i) + \kappa_{t',2, t'} H(A^i, a^i_t)\}
\]  \hspace{1cm} (68)

Note that we do not consider (65) or (67) to depend on \([q^i_{t'}]\) as these are uniquely determined from \([q^i_t(A^i, a^i_t)]\) by the constraints in (59)–(62) (which will be enforced whenever we consider the block \(h_{t',1}\), the only block that includes these variables).

Immediate statements that can be made regarding convexity of (66)–(68) include:

1) \(f\) is strictly convex if \(\kappa_{f,A} > 0\) and \(\kappa_{f,t'} > 0\); this is the consequence of the convexity of entropy
2) \(h_{t',2}\) is convex if the consistency constraints (58) are enforced for scan \(t'\), \(\kappa_{t',2, A} + \kappa_{t',2, t'} \geq 0\).

\[
\kappa_{t',2, A} + \kappa_{t',2, t'} \geq 0
\]  \hspace{1cm} (69)
Convexity of \( h_{t',1} \) is proven in the lemma below. Subsequently, the decomposition in terms of \( f, h_{t',1} \) and \( h_{t',2} \) (for each \( t' \in S \)) is utilised in the PDCA framework introduced in section II-D.

**Lemma 3.** If constraints (56)–(62) are enforced for scan \( t' \), \( \kappa_{t',1,t'} \geq 0 \), \( \beta_{t'} \geq 0 \) and \( \kappa_{t',1,t'} + \kappa_{t',1,A} \geq \gamma_{t'} \), then \( h_{t',1} \) is convex.

The proof of lemma [3] is in appendix [B]. The following lemma provides coefficients which fit (63) into the form (65), in order to permit solution using PDCA.

**Lemma 4.** Given the following, (65) is equivalent to (63):

\[
\begin{align*}
\kappa_{f,A} &= \kappa_{f,t'} = \frac{\eta}{S + 1} \\
\kappa_{t',1,t'} &= \frac{-\eta}{S + 1} \\
\kappa_{t',1,A} &= \gamma_{t'} + \frac{\eta}{S + 1} \\
\kappa_{t',2,A} &= \left(1 - \gamma_{t'} - \frac{2\eta}{S + 1}\right) \\
\kappa_{t',2,t'} &= 1 - \gamma_{t'} - \frac{2\eta}{S + 1}
\end{align*}
\]

where \( S = |S| \) is the number of scans in the problem.

Lemma 4 can be shown by substituting the coefficients into (65), and showing that the coefficients of \( H(A') \) sum to \(-S-1\), and the coefficients of \( H(A^i, a^i_t) \) sum to 1. Examining the values in lemma 4 we find (65) is convex (under the constraints (56)–(62)). This in turn shows that (63) is convex.

### C. Solution of convex energy

The convex free energy in (65) is of the form (49), so we propose a solution using PDCA. As discussed in section II-D, this is achieved by iterating (51), (53) and (54), commencing with \( \lambda_{t'} = 0 \) \( \forall t' \in S \). To begin, we note that if the gradient of a block \( h_i \) with respect to a subset of variables is zero, then those updated dual variables in (53) will be zero.

The algorithm proceeds by repeatedly cycling through blocks \( h_{t',1} \) and \( h_{t',2} \) for each \( t' \in S \). The following lemmas provide algorithms for solving each block in turn; the proofs can be found in appendix [B].

**Lemma 5.** Consider the problem to be solved in block \((t', 1)\):

\[
\text{minise} \ f([q^i(A^i)], [q^i_t(A^i, a^i_t)]) + h_{t',1}([q^i(A^i)], [q^i_t(A^i, a^i_t)]) \\
+ \sum_{i=1}^{n} \sum_{A^i \in A_i} \mu^i(A^i) q^i_t(A^i) \\
+ \sum_{t \in S} \sum_{A^i \in A_i} \sum_{a^i_t=0}^{m^i_t} \mu^i_t(A^i, a^i_t) q^i_t(A^i, a^i_t)
\]

under the constraints (56)–(62), where (58) is applied only for scan \( t' \), and \( \kappa_{f,A} + \kappa_{t',1,A} = 0 \). The solution of this problem is given by:

\[
q^i_t(A^i, a^i_t) = q^i_t(a^i_t) \times \frac{\exp \left\{ \phi^i(A^i) + \phi^i_t(A^i, a^i_t) \right\}}{\exp \phi^i_t(a^i_t)}
\]

where

\[
\begin{align*}
\phi^i(A^i) &= \log \psi^i(A^i) - \mu^i_t(A^i) \\
\phi^i_t(A^i, a^i_t) &= \log \psi^i_t(A^i, a^i_t) - \mu^i_t(A^i, a^i_t)
\end{align*}
\]

The single scan marginal \( q^i_t(A^i, a^i_t) \) is the solution of the following sub-problem:

\[
\text{minimise} \ \sum_{i=1}^{n} \sum_{j=0}^{m^i_t} q^i_t(A^i, a^i_t) \log q^i_t + \sum_{j=1}^{m^i_t} q^i_t \log q^i_t \\
- \beta_{t'} \sum_{i=1}^{n} \sum_{j=1}^{m^i_t} \left(1 - q^i_t(a^i_t)\right) \log \left(1 - q^i_t(a^i_t)\right)
\]

subject to (60)–(62), where \( \beta_{t'} = \frac{\beta_{t'}}{\kappa_{f,t'} + \kappa_{t',1,t'}} \), and \( \hat{\beta}_{t'} = \frac{\gamma_{t'} \beta_{t'}}{\kappa_{f,t'} + \kappa_{t',1,t'}} \). This sub-problem is studied in theorem [7]. The update to \( \lambda \) is

\[
\lambda_{t',1,t'}(A^i) \leq \phi^i_t(A^i) - \kappa_{f,A} \log q^i_t(A^i) \\
\lambda_{t',1,t'}(A^i, a^i_t) \leq \phi^i_t(A^i, a^i_t) - \kappa_{f,A} \log q^i_t(A^i, a^i_t) \\
q^i_t(A^i) = \sum_{a^i_t=0}^{m^i_t} q^i_t(A^i, a^i_t)
\]

where \( \leq \) denotes equality up to an additive constant. For other scans \( \tau \in S, \tau \neq t' \), \( \lambda_{t',1,\tau}(A^i, a^i_t) = 0 \).

**Lemma 6.** The solution of block \((t', 2)\):

\[
\begin{align*}
\text{minimise} f([q^i(A^i)], [q^i_t(A^i, a^i_t)]) \\
&+ h_{t',2}([q^i(A^i)], [q^i_t(A^i, a^i_t)]) \\
&+ \sum_{i=1}^{n} \sum_{A^i \in A_i} \mu^i(A^i) q^i_t(A^i) \\
&+ \sum_{t \in S} \sum_{A^i \in A_i} \sum_{a^i_t=0}^{m^i_t} \mu^i_t(A^i, a^i_t) q^i_t(A^i, a^i_t)
\end{align*}
\]

under the constraints (56)–(62) (including (58) only for scan \( t' \)) is:

\[
q^i_t(A^i, a^i_t) = q^i_t(A^i) \times \frac{\exp \left\{ \phi^i(A^i) + \hat{\phi}^i_t(A^i) \right\}}{\exp \phi^i_t(A^i)}
\]

As discussed in [20], constraints for each block can be incorporated into \( h_i \), so this condition also implies that no constraints are incorporated for the subset of variables in block \( h_i \).
Theorem 1. The iterative procedure in algorithm 1 converges to the minimum of the overall convex free energy, provided that weights are as given in lemma 4, \( \gamma \nu \geq 0 \), and \( \sum_{t' \in S} \gamma_{t'} < 1 \). This theorem is proven in appendix B.

Theorem 2. The iterative procedure in algorithm 2 converges to the minimum of the problem in (79), provided that \( \mu_{t'}(A^i) \) and \( \lambda_{t'}.1.A(A^i) \), \( \lambda_{t',2}.A(A^i), a^i_{t'} \), \( \gamma_{t'} \), and \( \beta_{t'} \) are as defined. This theorem is proven in appendix B.

Theorem 3. The iterative procedure in algorithm 2 converges to the minimum of the overall convex free energy, provided that weights are as given in lemma 4, \( \gamma \nu \geq 0 \), and \( \sum_{t' \in S} \gamma_{t'} < 1 \). This is a corollary of theorem 1.

Algorithm 1: Fractional BP algorithm for optimising single scan fractional free energy (79), where \( \alpha = 1 \) is the coefficient of \( q_{i}^{0} \log \frac{2}{q_{i}^{0}} \). Calculations marked \( \forall \) or \( \exists \) are over the range \( i \in \{1, \ldots, n\} \) (excluding \( i = 0 \), while those marked \( \forall j \) or \( \exists j \) are over the range \( j \in \{1, \ldots, m\} \) (excluding \( j = 0 \)).

\[
\phi^i(A^i) = \log \psi^i(A^i) - \mu^i(A^i)
\]

\[
\phi_{t'}(A^i, a_{t'}^i) = \log \psi^i(A^i, a_{t'}^i) - \mu_{t'}(A^i, a_{t'}^i)
\]

\[
\tilde{\phi}_{t'}(A^i) = \left( \kappa_{f,t'} + \kappa_{t',2,t'} \right) \log \left[ \sum_{a_{t'}^i=0}^{m_{t'}} \exp \left( \frac{\phi_{t'}(A^i, a_{t'}^i)}{\kappa_{f,t'} + \kappa_{t',2,t'}} \right) \right]
\]

The update to \( \lambda \) is

\[
\lambda_{t',2,.A}(A^i) \triangleq \phi^i(A^i) - \mu_{t'.A}(A^i)
\]

\[
\lambda_{t',2,.t}(A^i, a_{t'}^i) \triangleq \phi_{t'}(A^i, a_{t'}^i) - \mu_{t'.t}(A^i, a_{t'}^i) \tag{89}
\]

For other scans \( \tau \in S \), \( \tau \neq t' \), \( \lambda_{t',2,.t}(A^i, a_{t'}^i) = 0 \).

D. Modification of factors between scans

As discussed at the beginning of this section (and illustrated in figure 2), we propose solving a problem at time \( t \) involving a recent history of scans of measurements \( t' \in S \) with fractional weights configured to give high accuracy in the newest scan, e.g., \( \gamma_{t} = 0.95 \), and using lower values in earlier scans, e.g., \( \gamma_{t'} = 0, t' < t \). The main goal of retaining historical scans is to ensure that consistency constraints from past scans remain enforced.

Suppose that we solve the multiple scan problem at time \( t \). When we move to time \( t + 1 \), we will set \( \gamma_t = 0 \), this time using the larger value for \( \gamma_{t+1} \). Thus we seek to modify the problem parameters at time \( t \) to counteract the change of reducing \( \gamma_t \) to zero. This is similar to the approximation that JPDA makes, approximating the posterior as the product of the marginals.

More generally, suppose we have been using weights \( \gamma_{t'} \), \( t' \in S \), and at the next time, we will change these to \( \tilde{\gamma}_{t'} \). Similarly, suppose that the coefficients of the terms involving \( q_{t'}^{0} \) were \( \beta_{t'} \), and will be changed to \( \tilde{\beta}_{t'} \). The following theorem gives the change to the problem parameters necessary to ensure that the solution of the problem remains unchanged.

Algorithm 2: PDCA algorithm for minimising convex free energy based on decomposition described in lemma 4.
Theorem 3. Let \([q^t_i(A^i), q^t_{ij}(A^i, a^i_j)]\) and \([\hat{q}^t_i(A^i), \hat{q}^t_{ij}(A^i, a^i_j)]\) be the solution of the problem in (42) using fractional weights \(\gamma^t_i\) and \(\beta^t_i\), \(t' \in S\). Suppose that the weights are changed to \(\tilde{\gamma}^t_i = \gamma^t_i + \Delta \gamma^t_i\) and \(\tilde{\beta}^t_i = \beta^t_i + \Delta \beta^t_i\), and the problem parameters are changed as follows:

\[
\log \tilde{\psi}^t_i(A^i, a^i_j) = \log \psi^t_i(A^i, j) + \Delta \gamma^t_i \left[1 + \log(1 - q^t_{ij})\right] - \Delta \beta^t_i \left[1 + \log \hat{q}^t_{ij}\right] \quad (91)
\]

for \(j > 0\), \(\log \tilde{\psi}^t_i(A^i, 0) = \log \psi^t_i(A^i, 0)\) remains unchanged. Then the solution of the modified problem, denoted \([\tilde{q}^t_i(A^i)], [\tilde{q}^t_{ij}(A^i, a^i_j)]\), is unchanged.

The proof of the theorem is in appendix C. Algorithm 2 includes a step to incorporate these modifications.

IV. EXPERIMENTS

The proposed method is demonstrated through a simulation which seeks to estimate the marginal distribution of several targets using bearings only measurements. The region of interest is the square \([-100, 100]^2 \subset \mathbb{R}^2\). There are four sensors on the corners of the square. Tracks are initialised using a single accurate bearing measurement from one sensor, corrupted by Gaussian noise with 0.1° standard deviation (e.g., as may be provided if there is an accurate, low false alarm rate sensor providing bearing measurements from a single location); particle filter representations of each track are initialised by randomly sampling from the posterior calculated by combining these measurements with a uniform prior on the region of interest. The proposed algorithm is then utilised to refine the sensor positions using either one scan of measurements from each sensor (i.e., four scans in total) or three scans of measurements from each sensor (i.e., twelve scans in total). False alarms follow a Poisson distribution with one per scan on average, and targets are detected with probability 0.9. Target-originated bearing measurements are corrupted with 1° standard deviation Gaussian noise. False alarms are uniform over the 90° bearing range covering the region of interest. The number of targets is Poisson distributed with expected value of eight (but in each simulation the true number is known by the estimator; the method can be extended to accommodate an unknown number of targets using [10]).

The baseline method with which we compare is JPDA, adapted to maintain a particle representation of each target location, and to approximate data association probabilities using BP; we refer to this as JPDA-BP. The algorithm approaches multi-sensor data by sequentially processing individual sensors.

For the proposed method, we compare:

1) \(\gamma^t_i = \frac{1}{\sqrt{t+1}}\) for each scan, weights according to Lemma 4 and solving using algorithm 2, not utilising sequential modification (i.e., the final line); we refer to this as the convex variational (CV) algorithm

2) The method using sequential modification, introducing a new scan at each step with \(\gamma^t_i \in \{0.55, 0.75, 0.95\}\), and for past scans setting \(\gamma^t_i = 0\), again solving using algorithm 2, we refer to this as the CV-sequential (CVS) algorithm

The results in figures 3 and 5 show the cumulative distribution function (CDF) of the HPD value (top) and the entropy for the beliefs of each target over 200 Monte Carlo trials.

The goal of all methods is to correctly characterise the marginal distributions. The performance on this task is measured by evaluating the entropy of the beliefs produced by each method, and the high probability density (HPD) value in which the true location lies. The HPD value is defined as the total probability under a distribution that is more likely than a given point; e.g., given a point \(x^*\) and a distribution \(p\), the HPD value is

\[
\text{HPD}(p, x^*) = \int_{x \mid p(x) \geq p(x^*)} p(x) dx \quad (92)
\]

Thus if \(\text{HPD}(p, x^*) \approx 1\) then \(x^*\) is in the tails of \(p(x)\) and is assigned very low likelihood, while if \(\text{HPD}(p, x^*) = 0\) then \(x^*\) falls on the most likely value of \(p(x)\) (i.e., it is a MAP estimate). Under mild conditions, it can be shown that if \(x^* \sim p(x)\) (i.e., if \(p(x)\) correctly characterises the uncertainty in \(x^*\)) then \(\text{HPD}(p, x^*) \sim U\{0, 1\} [50]\) section 9.7.2]. If the distribution of HPD values is concentrated at the lower end, then the beliefs generated are conservative, i.e., they overestimate uncertainty in such a way that the true value rarely falls in the tails. If the distribution of HPD values is concentrated at the higher end, then the beliefs are non-conservative, i.e., they underestimate uncertainty, and the true value is often falling in the tails.

The results show that MSBP produces significantly non-conservative results. The entropies of the MSBP beliefs are significantly smaller than the other methods, while the true location is less likely than 99.99% of the belief in 12% of
cases. These results may be useful in particular applications (where smaller entropy is desirable) but when consistency and accuracy of the beliefs is essential, it is quite undesirable. 

JPDA-BP also produces non-conservative results, since the CDF of the HPD value consistently lies below the $x = y$ line (i.e., the CDF of a uniform distribution). The effect is more noticeable in the twelve scan case. In this case, 20% of targets are at a true location that is less likely than 89% of the belief (for a correct marginal distribution, the number should be 11%).

The convex variational (CV) method produces conservative beliefs, since the CDF of the HP value consistently lies above the $x = y$ line. The cost of this conservatism is beliefs with higher entropy; in certain applications this may be preferable. For the CV approach, 20% of targets are at a true location that is less than 72% of the belief; this value, combined with the larger entropies shown, indicate that the method is very conservative. For the CVS approach, the equivalent values are 84% (with $\gamma_t = 0.95$), 80% (with $\gamma_t = 0.75$) and 77% (with $\gamma_t = 0.55$). Thus, the method can be tuned to provide the desired trade-off between conservatism and entropy.

A large family of heuristic methods can be developed by employing algorithm with weights that do not ensure that each component is convex. We consider an instance of this, which sets

$$\kappa_{f,A} = \kappa_{f,t'} = 1$$

$$\kappa_{t',1,A} = -1$$

$$\kappa_{t',1,t'} = \kappa_{t',2,A} = \kappa_{t',2,t'} = 0$$

As long as $\kappa_{f,A} + \sum_{t' \in \mathcal{S}} [\kappa_{t',1,A} + \kappa_{t',2,A}] = -|\mathcal{S}| + 1$ and $\kappa_{f,t'} + \kappa_{t',1,t'} + \kappa_{t',2,t'} = 1$ for all $t' \in \mathcal{S}$, the original objective remains unchanged, and if the algorithm converges, the result is optimal (assuming convexity). Experimentally, convergence appears to be both reliable and rapid, though not guaranteed. In the rare case that convergence is not obtained, weights may be reverted (immediately or via a homotopy) to the form in lemma 4 for which convergence is guaranteed, but slower in practice.

With $\gamma_{t'} = 1$ for all $t' \in \mathcal{S}$, this can be seen to be equivalent to multiple scan BP (which is non-convex since $\sum_{t' \in \mathcal{S}} \gamma_{t'} > 1$).

Figure 5 shows the results of the heuristic approach. The algorithms with guaranteed convergence are marked with CV and CVS(0.95); the heuristic equivalents are CVH and CVSH(0.95) respectively. CVSH1(0.95) uses a single backward-forward sweep at each time step. The slight difference between the method with guaranteed performance and the heuristic method is caused by the different values of $\beta_{t'}$ used (since we must ensure that $\beta_{t'} > 0.5$; in each case we select $\beta_{t'}$ to set $\beta_{t'} = 1$). The results demonstrate that almost the same performance can be obtained with a single sweep.

V. CONCLUSION

This paper has shown how the BP data association method of [80] can be extended to multiple scans in a manner which preserves convexity. In doing so, it was demonstrated that the resulting beliefs are conservative, whereas the estimates provided by MSBP are significantly non-conservative. The method provides a variational interpretation of the standard JPDA approach which yields a continuum between this approach and MSBP, which can be tuned to provide conservative estimates of marginal distributions.

REFERENCES

[1] S. S. Blackman and R. Popoli, Design and Analysis of Modern Tracking Systems. Norwood, MA: Artech House, 1999.
where \( \lambda_{ij}^0(a_i^*) \) and \( \lambda_{ij}^0(b_i^*) \) are dual variables. Solving the dual function yields the solution:

\[
q^{ij}_t(a_i^*, b_i^*) = \frac{1}{c_{ij}^t} \exp(-\lambda_{ij}^0(a_i^*) - \lambda_{ij}^0(b_i^*))
\]

(100)

where \( c_{ij}^t \) is the normalisation constant.

Using the marginalisation constraints (98), the pairwise joint (100) and the definition of \( \psi^{ij}_t(a_i^*, b_i^*) \) in (31), we find that

\[
q^{ij}_t = q^{ij}_t(a_i^* = j, b_i^* = i),
\]

which is related to the dual variables by

\[
q^{ij}_t = \frac{1}{c_{ij}^t} \exp(-\lambda_{ij}^0(a_i^*) \exp(-\lambda_{ij}^0(b_i^*))
\]

(101)

Secondly, for \( i' \neq i \) and \( j' \neq j \), we find that \( q^{ij}_t(a_i^* = j') = q^{ij}_t(a_i^* = j, b_i^* = i') \), and \( q^{ij}_t(b_i^* = i') = q^{ij}_t \) are related to the dual variables by

\[
q^{ij}_t = \frac{1}{c_{ij}^t} \exp(-\lambda_{ij}^0(a_i^*) \exp(-\lambda_{ij}^0(j'))
\]

(102)

\[
q^{ij}_t = \frac{1}{c_{ij}^t} \exp(-\lambda_{ij}^0(b_i^*) \exp(-\lambda_{ij}^0(j'))
\]

(103)

The structure of \( \psi^{ij}_t(a_i^*, b_i^*) \) ensures that \( q^{ij}_t(a_i^*, b_i^*) = 0 \) if \( a_i^* = j, b_i^* \neq i \) or \( b_i^* = i, a_i^* \neq j \). As we have already seen, \( q^{ij}_t = q^{ij}_t(a_i^* = j, b_i^* = i) \). In the remaining case, \( a_i^* = j' \neq j, b_i^* = i' \neq i \),

\[
q^{ij}_t(a_i^*, b_i^*) = \frac{1}{c_{ij}^t} \exp(-\lambda_{ij}^0(a_i^*) \exp(-\lambda_{ij}^0(b_i^*))
\]

(104)

Substituting (101)-(104) into the pairwise normalisation constraint (99) yields:

\[
q^{ij}_t + \frac{1}{c_{ij}^t} \sum_{i' = 0}^n \sum_{i' \neq i}^n \exp(-\lambda_{ij}^0(b_i^*)) \sum_{j' = 0}^m \exp(-\lambda_{ij}^0(a_i^*)) = 1
\]

(105)

Subsequently, we find for \( i' \neq i, j' \neq j \),

\[
q^{ij}_t(a_i^* = j', b_i^* = i') = \frac{q^{ij}_t}{1 - q^{ij}_t}
\]

(106)

Substituting the marginals and the pairwise joint into the
entropies $H(a_i^t), H(b_i^t)$ and $H(a_i^t, b_i^t)$ yields:

\[-H(a_i^t) = \sum_{j=0}^{m_i} q_{i,j}^t \log q_{i,j}^t \tag{107}\]

\[-H(b_i^t) = \sum_{j=0}^{n} q_{i,j}^t \log q_{i,j}^t \tag{108}\]

\[-H(a_i^t, b_i^t) = q_{i,j}^t \log q_{i,j}^t + \sum_{j'=0}^{m_i} q_{i,j'}^t q_{i,j'}^t \log q_{i,j'}^t q_{i,j'}^t = q_{i,j}^t \log q_{i,j}^t - (1 - q_{i,j}^t) \log (1 - q_{i,j}^t) \tag{109}\]

so that the mutual information \[(16)\] is:

\[I(a_i^t; b_i^t) = -q_{i,j}^t \log q_{i,j}^t - (1 - q_{i,j}^t) \log (1 - q_{i,j}^t) \tag{110}\]

Substituting \[(107)-(110)\] into the single scan formulation \[(96)-(99)\], we arrive at the equivalent Bethe variational problem \[(40)-(43)\] where $w_{ij} = \psi_i(a_i^t = j)$.

The Bethe variational problem in section II-C involving random variables $A_i^t, a_i^t$ and $b_i^t$ (illustrated in figure 2(a)) can be solved by minimising:

\[F_B'[q_i^t(A_i^t), [q_i^t(A_i^t), a_i^t), [q_i^t(A_i^t), a_i^t, b_i^t), [q_i^t(b_i^t)] = \]

\[-\sum_{i=1}^{n} \{ H(A_i^t) + \mathbb{E}[\log \psi_i^t(A_i^t)] \} - \sum_{i=1}^{n} H(a_i^t) - \sum_{j=1}^{m_i} H(b_i^t) \]

\[-\sum_{i=1}^{n} \{ -I(A_i^t; a_i^t) + \mathbb{E}[\log \psi_i^t(A_i^t, a_i^t)) \} \]

\[-\sum_{i=1}^{n} \{ -I(a_i^t; b_i^t) + \mathbb{E}[\log \psi_i^t(a_i^t, b_i^t)] \} \tag{111}\]

subject to the constraints \[(97)-(99)\] and

\[q_i^t(A_i^t, a_i^t) \geq 0, \quad q_i^t(a_i^t) \geq 0 \tag{112}\]

\[\sum_{a_i^t=0}^{m_i} q_i^t(A_i^t, a_i^t) = q_i^t(A_i^t), \quad \sum_{a_i^t=0}^{m_i} q_i^t(A_i^t, a_i^t) = q_i^t(a_i^t) \tag{113}\]

\[\sum_{A_i^t \in A_i^t} \sum_{a_i^t=0}^{m_i} q_i^t(A_i^t, a_i^t) = 1, \quad \sum_{A_i^t \in A_i^t} q_i^t(A_i^t) = 1 \tag{114}\]

Partial minimisation over the pairwise joint $q_{ij}^{t,j}(a_i^t, b_i^t)$ arrives at the Bethe variational problem \[(34)-(39)\]. A rearrangement of the Bethe free energy \[(34)\] is:

\[F_B'[q_i^t(a_i^t), [q_i^t(A_i^t), a_i^t), [q_i^t(A_i^t), a_i^t, b_i^t), [q_i^t(b_i^t)] = \]

\[-\sum_{i=1}^{n} \{ H(A_i^t) + \mathbb{E}[\log \psi_i^t(A_i^t)] \} \]

\[+ \sum_{j=0}^{m_i} q_i^0,j q_{i,j}^t \log q_{i,j}^t - \sum_{j=0}^{m_i} \sum_{j'=0}^{m_i} (1 - q_{i,j}^t) \log (1 - q_{i,j}^t) \tag{115}\]

Let $q_i^t(a_i^t)$ and $q_{i,j}^t$ be fixed and feasible. Minimising the Bethe free energy \[(115)\] with respect to $q_i^t(A_i^t, a_i^t)$ subject to the constraints \[(35)-(39)\] yields the solution (using \[(23)\]):

\[q_i^t(A_i^t, a_i^t) = \frac{q_i^t(A_i^t)\psi_i^t(A_i^t)\psi_i^t(A_i^t, a_i^t)}{\sum_{A_i^t \in A_i^t} \psi_i^t(A_i^t)\psi_i^t(A_i^t, a_i^t)} \tag{116}\]

Substituting $q_i^t(A_i^t, a_i^t)$ into the Bethe free energy \[(115)\] results in the equivalent Bethe variational problem \[(40)-(43)\] where $w_{ij} = \psi_i^j(A_i^t, a_i^t)$, and $q(a_i^t = j) = q_{i,j}^t$.

**B. Multiple scans**

The Bethe variational problem in section II-A which is represented by figure 2(c), can be solved by minimising:

\[F_B'[q_i^t(A_i^t), [q_i^t(A_i^t), a_i^t), [q_i^t(A_i^t), a_i^t, b_i^t), [q_i^t(b_i^t)] = \]

\[\sum_{i=1}^{n} \{ H(A_i^t) + \mathbb{E}[\log \psi_i^t(A_i^t)] \} \]

\[-\sum_{i=1}^{n} \sum_{t' \in S} H(a_i^t) - \sum_{i=1}^{n} \sum_{t' \in S} H(b_i^t) \]

\[-\sum_{t' \in S} \{ -I(A_i^t; a_i^t) + \mathbb{E}[\log \psi_i^t(A_i^t, a_i^t)) \} \]

\[-\sum_{t' \in S} \{ -I(a_i^t; b_i^t) + \mathbb{E}[\log \psi_i^t(a_i^t, b_i^t)] \} \tag{117}\]

subject to the constraints \[(97)-(99)\] and \[(112)-(114)\] for $t' \in S$. Partial minimisation over $q_{ij}^{t,j}(a_i^t, b_i^t)$ and rearrangement produces the equivalent variational problem \[(55)-(57)\].

**APPENDIX B**

**PROOF OF ALGORITHMS FOR MINIMISING PDCA SUB-PROBLEMS**

In this section, we prove lemmas \[5\] and \[6\] i.e., we derive algorithms for minimising the blocks utilised in the PDCA algorithm. Before we begin, we prove the preliminary result in lemma \[3\] which shows that the block $h_{t',1}$ is convex (convexity of $f$ and $h_{t',2}$ is straight-forward).

**Proof of lemma \[5\]** If $\kappa_{t',1,A} \geq 0$, then convexity with respect to $q_i^t(A_i^t)$ is immediate, and we let $\tilde{\kappa} = \kappa_{t',1,A}$. Otherwise, $-\kappa_{t',1,A} > 0$; let $\tilde{\kappa} = \kappa_{t',1,A} + \kappa_{t',1,A} \geq 0$, and rewrite the first line of \[(67)\] as:

\[-\sum_{i=1}^{m_i} \{ -\kappa_{t',1,A} H(a_i^t|A_i^t) + \kappa H(A_i^t|a_i^t) + \kappa H(a_i^t) \} \]

The first two terms are convex by definition of conditional entropy, as is the second line in \[(67)\], so we focus on the remainder of the expression:

\[-\tilde{\kappa} \sum_{i=1}^{m_i} H(a_i^t) - \gamma_t \sum_{i=1}^{m_i} (1 - q_{ij}^{t,j}) \log (1 - q_{ij}^{t,j}) \tag{118}\]

where $-H(a_i^t) = \sum_{j=0}^{m_i} q_{ij}^{t,j} \log q_{ij}^{t,j}$. Recognising \[(118)\] as:

\[\gamma_t \left( \sum_{j=0}^{m_i} q_{ij}^{t,j} \log q_{ij}^{t,j} - \sum_{j=0}^{m_i} (1 - q_{ij}^{t,j}) \log (1 - q_{ij}^{t,j}) \right) \]

\[+ (\tilde{\kappa} - \gamma_t) \sum_{i=0}^{m_i} q_{ij}^{t,j} \log q_{ij}^{t,j} + \gamma_t \sum_{i=0}^{m_i} (1 - q_{ij}^{t,j}) \log (1 - q_{ij}^{t,j}) \]

\[(119)\]
we obtain the desired result; the first line is convex by theorem 20 in [33], which shows that the function $S(\xi) = \sum_j \xi_j \log \xi_j - \sum_j (1 - \xi_j) \log(1 - \xi_j)$ is convex on the domain $\xi_j \geq 0$, $\sum_j \xi_j = 1$; the second line is convex by convexity of $x \log x$.

Proof of lemma 3 Collecting terms, the objective to be minimised is:

$$F_{\mu_{t',2}}^\mu([q^t(A^i)], [q^t(A^i), a^i_{t'}]) = -\sum_{t'=1}^n \left\{ (\kappa_{f,t'} + \kappa_{f',t'} A^i) H(A^i) + \mathbb{E}[\phi_t^i(A^i)] \right\}$$

$$- \sum_{\tau \in S \setminus \{t'\}} \sum_{t'=1}^n \left\{ \kappa_{f,t'} H(A^i, a^i_{t'}) + \mathbb{E}[\phi_t^i(A^i, a^i_{t'})] \right\}$$

$$- \sum_{t'=1}^n \sum_{j=1}^{m_{t'}} \left\{ \kappa_{f,t'} \log q^t_{0,j} - \gamma t' \sum_{i=1}^n \sum_{j=1}^{m_{t'}} (1 - q^t_{i,j}) \log(1 - q^t_{i,j}) \right\}$$

which is the result in (80). Following identical steps for $q^t_{f'}(A^i, a^i_{t''})$ gives the result in (81).

Proof of lemma 6 Collecting terms, the objective to be minimised is:

$$F_{\mu_{t',2}}^\mu([q^t(A^i)], [q^t(A^i), a^i_{t'}]) = -\sum_{t'=1}^n \left\{ (\kappa_{f,t'} + \kappa_{f',t'} A^i) H(A^i) + \mathbb{E}[\phi_t^i(A^i)] \right\}$$

$$- \sum_{\tau \in S \setminus \{t'\}} \sum_{t'=1}^n \left\{ \kappa_{f,t'} H(A^i, a^i_{t'}) + \mathbb{E}[\phi_t^i(A^i, a^i_{t'})] \right\}$$

$$- \sum_{t'=1}^n \sum_{j=1}^{m_{t'}} \left\{ \kappa_{f,t'} \log q^t_{0,j} - \gamma t' \sum_{i=1}^n \sum_{j=1}^{m_{t'}} (1 - q^t_{i,j}) \log(1 - q^t_{i,j}) \right\}$$

For $\tau \in S \setminus \{t'\}$, (58) is not enforced, and $h_{t',2}$ is constant with respect to $q^t_{f'}(A^i, a^i_{t'})$, so $\lambda_{t',2}(A^i, a^i_{t'}) = 0$. Since the constraint (58) is enforced for time $t'$, (127) can be written equivalently as

$$F_{\mu_{t',2}}^\mu([q^t(A^i)], [q^t(A^i), a^i_{t'}]) = -\sum_{t'=1}^n \left\{ \tilde{\kappa} H(A^i) + \mathbb{E}[\phi_t^i(A^i)] \right\}$$

$$- \sum_{t'=1}^n \left\{ (\kappa_{f,t'} + \kappa_{f',t'} A^i) H(A^i) + \mathbb{E}[\phi_t^i(A^i, a^i_{t'})] \right\}$$

$$- \sum_{\tau \in S \setminus \{t'\}} \sum_{t'=1}^n \left\{ \kappa_{f,t'} H(A^i, a^i_{t'}) + \mathbb{E}[\phi_t^i(A^i, a^i_{t'})] \right\}$$

(128)

where $\tilde{\kappa} = \kappa_{f,t'} + \kappa_{f',t'}$. Using lemma 2 to perform a partial minimisation of (128) with respect to $q^t_{f'}(A^i, a^i_{t'})$, holding $q^t(A^i)$ fixed, we find the result in (85) and the remaining problem:

$$F_{\mu_{t',2}}^\mu([q^t(A^i)]) = -\sum_{t'=1}^n \left\{ \tilde{\kappa} H(A^i) + \mathbb{E}[\phi_t^i(A^i)] \right\}$$

(129)

Using lemma 1 we obtain the result in (84).

Following similar steps to (126) gives the updates for $\lambda_{t',2}(A^i)$ and $\lambda_{t',2}(A^i, a^i_{t'})$.

**APPENDIX C**

**PROOF OF SEQUENTIAL MODIFICATION**

This section proves theorem 3 i.e., that the solution of the problem in (63) is the same as the solution of the modified problem of the same form, changing $\gamma t'$ to $\gamma t' = \gamma t' + \Delta \gamma t'$, $\beta t'$ to $\beta t' = \beta t' + \Delta \beta t'$, and $\psi_t^i(A^i, a^i_{t'})$ as described in (91).

Proof of theorem 3 Let $F_B^{(\gamma),(\beta)}$ be the original problem (in (63)) and $F_B^{(\gamma),(\beta)}$ be the modified problem. Note that the modifying term in (91) depends only on $a^i_{t'}$, so we can equivalently implement the modification by retaining the unmodified $\psi_t^i(A^i, a^i_{t'})$ and incorporating an additive term

$$- \mathbb{E}[\phi_t^i(a^i_{t'})] = - \sum_{j=0}^{m_{t'}} q^t_{0,j} \phi_t^i(j)$$

(130)

where

$$\phi_t^i(j) = \Delta \gamma t' \lambda_{t'} \log(1 + \log(1 - q^t_{i,j})) - \Delta \beta t' \lambda_{t'} \log(1 + \log(1 - q^t_{i,j}))$$

(131)
Consider the Karush-Kuhn-Tucker (KKT) optimality conditions [51] relating to $[q_{ij}^0]^j_i$, since all modifications relate to these variables. The conditions in the original problem are:

$$\begin{align}
\nu_i^j + \rho_i^j + \sigma_i^j + \gamma^j \log(1 - q_{ij}^j) + \gamma^j = 0 \quad (132)\\
\nu_i^j + \sigma_i^j + \beta^j \log q_{ij}^j + \beta^j = 0 \quad (133)
\end{align}$$

where $\nu_i^j$ is the dual variable for the constraint in (59), $\rho_i^j$ is the dual variable for (62), and $\sigma_i^j$ is the dual variable for (61). For the modified problem, the two KKT conditions are:

$$\begin{align}
\nu_i^j + \rho_i^j + \sigma_i^j + \gamma^j \log(1 - \bar{q}_{ij}^j) + \gamma^j - \phi_i^j(j) = 0 \quad (134)\\
\nu_i^j + \sigma_i^j + \beta^j \log q_{ij}^j + \beta^j = 0 \quad (135)
\end{align}$$

Substituting in (131) and expanding $\gamma^j$ and $\bar{\beta}_i^j$, we find:

$$\begin{align}
\bar{\nu}_i^j + \rho_i^j + \sigma_i^j + [\gamma^j + \Delta \gamma^j] \log(1 - q_{ij}^j) + \gamma^j + \Delta \gamma^j - \Delta \gamma^j(1 + \log(1 - q_{ij}^j)) + \Delta \beta^j(1 + \log q_{ij}^j) = 0 \quad (136)
\end{align}$$

Subsequently, by setting $\bar{q}_{ij}^j = q_{ij}^j$, $\bar{q}_{ij}^j = q_{ij}^j$, $\bar{\nu}_i^j = \nu_i^j$, $\rho_i^j = \nu_i^j$ and

$$\sigma_i^j = \sigma_i^j - \Delta \beta^j(1 + \log q_{ij}^j) \quad (137)$$

we find a primal-dual solution with identical primal values $[q_{ij}^j]$ which satisfies the KKT conditions for the modified problem.

\section*{APPENDIX D}
\textbf{PROOF OF CONVERGENCE OF SINGLE SCAN ITERATION}

In this section, we prove convergence of an iterative algorithm for solution of the single scan block, the PGM for which was illustrated in figure 1

$$\begin{align}
\text{minimise} \quad &\sum_{i=1}^{n} \sum_{j=1}^{m} q_{ij} \log \frac{q_{ij}}{w_{ij}} \\
&+ \alpha \sum_{i=1}^{n} q_{ij} \log \frac{q_{ij}}{w_{ij}} + \beta \sum_{j=1}^{m} q_{ij} \log \frac{q_{ij}}{w_{ij}} \\
&- \gamma \sum_{j=1}^{m} (1 - q_{ij}) \log(1 - q_{ij}) \quad (138)\\
\text{subject to} \quad &\sum_{j=1}^{m} q_{ij} = 1 \forall i \in \{1, \ldots, n\} \\
&\sum_{i=1}^{n} q_{ij} = 1 \forall j \in \{1, \ldots, m\} \\
&0 \leq q_{ij} \leq 1 \quad (140)
\end{align}$$

where $\gamma \in [0, \alpha] \cap [0, \beta] \cap [0, 1], \alpha \in (0.5, \infty), \beta \in (0.5, \infty)$. 

In our analysis, we permit values $w_{ij} = 0$, maintaining a finite objective by fixing the corresponding $q_{ij} = 0$, and defining $q_{ij}/w_{ij} = 1$; since these take on fixed values, we do not consider them to be optimisation variables. While we state the algorithm more generally, we prove convergence for three cases:

1) $n = m$ and $w_{ij} = 0$ (i.e., no missed detection/false alarm events)

2) $w_{ij} > 0 \forall i, w_{ij} = 0 \forall j, \alpha = 1$ (i.e., missed detections but no false alarms)

3) $w_{ij} > 0 \forall i, w_{ij} > 0 \forall j, \alpha = 1$ (i.e., missed detections and false alarms)

In case 1 above, $\alpha$ and $\beta$ have no effect since $q_{ij} = 0$ and $q_{ij} = 0$. Similarly, in case 2, $\beta$ has no effect since $q_{ij} = 0$. Assumption 1 ensures that the problem has a relative interior (again, we exclude the $q_{ij}$ variables for which $w_{ij} = 0$, since they are fixed to zero).

\textbf{Assumption 1} There exists a feasible point in the relative interior, i.e., that there exists $q_{ij}$ satisfying the constraints (139)-(141) such that $0 < q_{ij} < 1 \forall (i, j)$ s.t. $w_{ij} > 0$.

\textbf{Assumption 2} The graph is connected, i.e., we can travel from any left-hand side vertex $a^i$, $i \in \{1, \ldots, n\}$ to any right-hand side vertex $b^j$, $j \in \{1, \ldots, m\}$ by following a path consisting of edges $(i', j')$ with $w_{i'j'} > 0$.

Assumption 1 can easily be shown to be satisfied if $w_{ij} > 0 \forall i$ and $w_{ij} > 0 \forall j$ (i.e., missed detection and false alarm likelihoods are non-zero). In problems without false alarm or missed detection, the condition excludes infeasible problems (e.g., where two measurements can only be associated with a single track), and problems with trivial components (e.g., where a measurement can only be associated with one track, so that measurement and track can be removed and the smaller problem solved via optimisation). Assumption 2 ensures that the problem is connected. Any problem in which the graph is not connected can be solved more efficiently by solving each connected component separately. Thus we utilise the property for the proof in case 1, although the proof can be adapted to lift the requirement.

\textbf{Lemma 7} The solution of (138)-(141) lies in the relative interior, i.e., $0 < q_{ij} < 1 \forall (i, j)$ s.t. $w_{ij} > 0$.

\textbf{Proof} Rewrite the objective in (138) in the form:

$$\begin{align}
&\sum_{i=1}^{n} \sum_{j=1}^{m} q_{ij} \log \frac{q_{ij}}{w_{ij}} \\
&+ \alpha \sum_{i=1}^{n} q_{ij} \log \frac{q_{ij}}{w_{ij}} + \beta \sum_{j=1}^{m} q_{ij} \log \frac{q_{ij}}{w_{ij}} \\
&- \gamma \sum_{j=1}^{m} (1 - q_{ij}) \log(1 - q_{ij}) \quad (138)\\
&+ (\alpha - \gamma) \sum_{i=1}^{n} q_{ij} \log \frac{q_{ij}}{w_{ij}} + \beta \sum_{j=1}^{m} q_{ij} \log \frac{q_{ij}}{w_{ij}} \\
&+ \gamma \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} q_{ij} \log \frac{q_{ij}}{w_{ij}} - \sum_{i=1}^{n} \sum_{j=1}^{m} (1 - q_{ij}) \log(1 - q_{ij}) \right] \quad (142)
\end{align}$$

Consider two feasible points $q^0$ and $q^1$, where $q^0$ is on the boundary and $q^1$ is in the relative interior (such a point exists by assumption 1). Let $q^1 = \lambda q^0 + (1 - \lambda)q^0$, and denote the objective evaluated at $q^1$ by

$$f(\lambda) = g(\lambda) + h(\lambda)$$

where $g(\lambda)$ is the first two lines of (142) evaluated at $q^1$, and $h(\lambda)$ is the final line. Lemma 3 shows that $h(\lambda)$ is convex, therefore its gradient is monotonically non-decreasing. Consequently it must be the case that:

$$\lim_{\lambda \to 0} \frac{d}{d\lambda} h(\lambda) = c < \infty \quad (143)$$
The derivative of $g(\lambda)$ is given by:

$$
g'(\lambda) = (1-\gamma) \sum_{i=1}^{n} \sum_{j=1}^{m} \left[ \log \lambda q_{ij}^1 + (1-\lambda)q_{ij}^0 \right] \frac{w_{ij}}{w_{ij}} + (\alpha - \gamma) \sum_{i=1}^{n} (q_{i0}^1 - q_{i0}^0) \left[ \log \lambda q_{i0}^1 + (1-\lambda)q_{i0}^0 \right] \frac{w_{i0}}{w_{i0}} + \beta \sum_{j=1}^{m} (q_{0j}^1 - q_{0j}^0) \left[ \log \lambda q_{0j}^1 + (1-\lambda)q_{0j}^0 \right] \frac{w_{0j}}{w_{0j}} + 1
$$

Since $q^0$ is on the boundary and $q^1$ is not, we must have:

$$
\lim_{\lambda \to 0} g'(\lambda) = -\infty \quad (144)
$$

By (143) and (144), we thus have that $f'(<) < 0 \forall \lambda \in (0, \epsilon)$ for some $\epsilon > 0$. Thus the optimum cannot lie on the boundary.

Lemma 8. The KKT optimality conditions for the problem in (138) are:

$$
\log \frac{q_{ij}}{w_{ij}} + \gamma \log(1-q_{ij}) + 1 + \gamma - \lambda_i - \mu_j = 0 \forall i,j > 0 \quad (145)
\alpha \log \frac{q_{i0}}{w_{i0}} + \alpha - \lambda_i = 0 \forall i > 0 \quad (146)
\beta \log \frac{q_{0j}}{w_{0j}} + \beta - \mu_j = 0 \forall j > 0 \quad (147)
$$

as well as the primal feasibility conditions (139)-(141). The conditions are necessary and sufficient for optimality. In case 1 (where $w_{ij} = 0 \forall i$ and $w_{i0} = 0 \forall j$) the solution is unique up to a constant $c$ being added to $\lambda_i \forall i$ and subtracted from $\mu_j \forall j$. In other cases, the solution is unique.

Proof: One complication is that the objective is not convex on $\mathbb{R}^{n+1 \times m+1}$ but rather only on the subspace in which either (139) or (140) is satisfied. We show that the regular KKT conditions are still necessary and sufficient in this case. Relaxing the non-negativity condition, the problem can be expressed as:

$$
\begin{align*}
\text{minimise} & \quad f(q) \\
\text{subject to} & \quad A_1 q = b_1, \quad A_2 q = b_2 \quad (148)
\end{align*}
$$

The KKT conditions for this problem are:

$$
\begin{align*}
\nabla f(q) - A_1^T \lambda - A_2^T \mu = 0 \\
A_1 q = b_1, \quad A_2 q = b_2
\end{align*}
$$

(149)

Given a solution $q_0$ which satisfies $A_1 q_0 = b_1$, we can express any feasible $q$ as $q_0 + P(q - q_0)$ where $P = I - A_1^T (A_1 A_1^T)^{-1} A_1$ is the matrix that projects onto the null-space of $A_1$. Thus we can equivalently solve

$$
\begin{align*}
\text{minimise} & \quad f(q_0 + P(q - q_0)) \\
\text{subject to} & \quad A_1 q = b_1, \quad A_2 q = b_2
\end{align*}
$$

Since the argument of $f$ lies in the feasible subspace for the first constraint, this problem is convex, and under Assumption [1] the Slater condition is satisfied, so the KKT conditions are necessary and sufficient. The KKT conditions for this modified problem are:

$$
\begin{align*}
P \nabla f(q) - A_1^T \lambda - A_2^T \mu = 0 \quad (150) \\
A_1 q = b_1, \quad A_2 q = b_2 \quad (151)
\end{align*}
$$

where, after taking the gradient of $f$ in (150), we substitute $q_0 + P(q - q_0) = q$ since the point must satisfy the constraints (151). The projection of the gradient is:

$$
P \nabla f(q) = \nabla f(q) - A_1^T (A_1 A_1^T)^{-1} A_1 \nabla f(q)
$$

Thus a point $(q^*, \lambda^*, \mu^*)$ satisfying the KKT conditions for the modified problem (150) and (151) corresponds to a point $(q^*, \lambda^*, \mu^*)$ in the KKT conditions for the original problem (148)-(149), where

$$
\hat{\lambda}^* = \lambda^* + (A_1 A_1^T)^{-1} A_1 \nabla f(q)
$$

Similarly, given a point satisfying the KKT conditions for the original problem, we can find a corresponding point satisfying the modified KKT conditions (150)-(151) by inverting (152).

The expressions in (145)-(147) are found by forming the Lagrangian and taking derivatives. Uniqueness of the solution comes from strict convexity of $f$. The freedom to choose a constant offset is the result of linear dependence of the constraints in case 1 (since each set of constraints implies that $\sum_{ij} q_{ij} = n$).

The optimisation methodology we adopt is to define an iterative method and prove that it converges to a point that satisfies the KKT conditions, motivated by analysis of the BP iteration in [30][33]. Defining $\hat{\lambda}_i = \lambda_i - \alpha, \hat{\mu}_i = \mu_i - \beta$ and $\kappa = -1 - \gamma + \alpha + \beta$, the KKT conditions in (145)-(147) can be rewritten as:

$$
\begin{align*}
q_{ij} &= \frac{w_{ij} \exp\left(\frac{\hat{\lambda}_i + \hat{\mu}_j + \kappa}{1 - q_{ij}}\right)}{(1 - q_{ij})^\gamma} \forall i,j > 0
q_{i0} &= \frac{w_{i0} \exp\left(\frac{\hat{\lambda}_i}{\alpha}\right)}{\beta} \forall i > 0
q_{0j} &= \frac{w_{0j} \exp\left(\frac{\hat{\mu}_j}{\beta}\right)}{\gamma} \forall j > 0
\end{align*}
$$

While these expressions do not permit us to immediately solve for $q_{ij}$, they permit application of an iterative method, in which we repeatedly calculate new LHS values of $q_{ij}$ by updating either $\hat{\lambda}_i$ or $\hat{\mu}_j$ via the equation:

$$
\exp \hat{\lambda}_i = \left[w_{i0} \exp\left(\frac{\frac{1}{\alpha} - 1}{1 - q_{ij}}\right) \right] + \sum_{j=1}^{m} \frac{w_{ij} \exp\left(\hat{\mu}_j + \kappa\right)}{(1 - q_{ij})^\gamma} \quad (153)
$$

or $\hat{\mu}_j$ via the equation:

$$
\exp \hat{\mu}_j = \left[w_{ij} \exp\left(\frac{\frac{1}{\beta} - 1}{1 - q_{ij}}\right) \right] + \sum_{i=1}^{n} \frac{w_{i0} \exp\left(\hat{\lambda}_i + \kappa\right)}{(1 - q_{ij})^\gamma} \quad (157)
$$

where the RHS values of $q_{ij}, \hat{\lambda}_i$ and $\hat{\mu}_j$ refer to the previous iterates. The updates in (156) and (157) are applied alternately. After each update, the values of $q_{ij}$ are recalculated using (153)-(155).

\footnotetext[10]{Alternatively, define $f(q) = \infty$ for points violating the constraint.}
The iteration may be written equivalently in terms of the parameterisation $x_{ij}$ and $y_{ij}$, where
\begin{align*}
x_{i0} &= \exp\{\frac{1}{\beta} - 1\bar{\lambda}_i \}, \quad x_{i0} = \exp\{\bar{\mu}_i \} \quad \text{(158)}
y_{j0} &= \exp\{\bar{\lambda}_j \}, \quad y_{j0} = \exp\{\frac{1}{\beta} - 1\bar{\mu}_j \} \quad \text{(159)}
\end{align*}
Algebraic manipulation yields equivalent iterations in terms of $x_{ij}$ and $y_{ij}$ as:
\begin{align*}
x_{ij}^{(k+1)} &= r_{ij}(y^{(k)}) \triangleq \left( w_{0j} y_{0j} + \sum_{i' \neq i} w_{ij} y_{ij}^{(k)} \right)^{-(1-\gamma)} \times \left( w_{0j} y_{0j} + \sum_{i' \neq i} w_{ij} y_{ij}^{(k)} \right)^{-\gamma} \times e^\kappa \quad \text{(160)}
y_{ij}^{(k+1)} &= s_{ij}(x^{(k+1)}) \triangleq \left( w_{0i} x_{0i} + \sum_{j' \neq j} w_{ij} x_{ij'}^{(k+1)} \right)^{-(1-\gamma)} \times \left( w_{0i} x_{0i} + \sum_{j' \neq j} w_{ij} x_{ij'}^{(k+1)} \right)^{-\gamma} \times e^\kappa \quad \text{(161)}
\end{align*}
and
\begin{align*}
x_{i0}^{(k+1)} &= r_{i0}(y^{(k)}) \triangleq (y_{i0}^{(k)})^{\frac{1}{\beta} - 1} \quad \text{(162)}
y_{j0}^{(k+1)} &= s_{j0}(x^{(k+1)}) \triangleq (x_{j0}^{(k+1)})^{\frac{1}{\beta} - 1} \quad \text{(163)}
\end{align*}
\begin{align*}
\left[ 1 - \frac{1}{q^*_{ij}} \right] &= 1 - \frac{1 - w_{ij} \exp\{\bar{\lambda}_i + \gamma \bar{\mu}_j + \kappa \}}{(1 - q^*_{ij})^\gamma} \quad \text{(164)}
\end{align*}
The shorthand $\sum_{i'}$ represents the sum over the set $i' \in \{1, \ldots, n\}$, while $\sum_{i' \neq i}$ represents the same summation, excluding the $i$-th element. Similarly, $\sum_{j'}$ represents the sum over the set $j' \in \{1, \ldots, n\}$, while $\sum_{j' \neq j}$ represents the same summation, excluding the $j$-th element. The structure of this iterative method is illustrated in figure 6. Note that if $\gamma = 1$, this reduces to the BP iteration of $x^{(k+1)} = r(y^{(k)})$ and $y^{(k+1)} = s(x^{(k)})$.

At this point, we have stated but not derived the iteration $x^{(k+1)} = r(y^{(k)})$. The validity of the expressions is established by proving that the KKT conditions are a fixed point of the iteration (in lemma 9), and then showing that repeated application of the expressions yields a contraction, which is guaranteed to converge to the unique fixed point.

**Definition 4.** An operation $g(x)$ is a contraction with respect to $d(\cdot, \cdot)$ if there exists $\alpha \in (0, 1)$ such that for all $x$, $\bar{x}$
\begin{align*}
d[g(x), g(\bar{x})] &\leq \alpha d(x, \bar{x}) \quad \text{(171)}
\end{align*}
If the expression is satisfied for $\alpha = 1$, then $g(x)$ is a non-expansion.

**Lemma 10.** Let $g(\cdot)$ be the operator taking the weighted combination with non-negative weights $w_{ij} \geq 0$:
\begin{align*}
g_{kl}(x) &= \sum_{i,j} w_{ij} x_{ij} \quad \text{for all } (i,j) \in \mathbb{R}
\end{align*}
$g(\cdot)$ is non-expansive with respect to $d$.

**Proof:** Let $L = \exp\{d(x, \bar{x})\} < \infty$ (otherwise there is nothing to prove), so that $\frac{1}{L} x_{ij} \leq \bar{x}_{ij} \leq L x_{ij}$. Then
\begin{align*}
g_{kl}(\bar{x}) &= \sum_{i,j} w_{ij} \bar{x}_{ij} \leq L \sum_{i,j} w_{ij} x_{ij} = L g_{kl}(x) \quad \text{(172)}
\end{align*}
and

$$g_k(x) = \sum_{i,j} w_{ijk} x_{ij} = \frac{1}{L} \sum_{i,j} w_{ijk} x_{ij} = \frac{1}{L} g_k(x) \quad (173)$$

Lemma 11. Let \( f(\cdot) \) be formed from two operators \( g(\cdot) \) and \( h(\cdot) \) as

$$f_{ij}(x) = g_{ij}(x)^{s_{k}} h_{ij}(x)^{\rho_{k}} \quad (174)$$

Suppose that \( g(\cdot) \) and \( h(\cdot) \) are contractions or non-expansions with coefficients \( \alpha_g \) and \( \alpha_h \). If \( \alpha_f = \alpha_g \rho_g + \alpha_h \rho_h \), then \( f(\cdot) \) is a contraction with respect to \( d \). If \( \alpha_f = 1 \) then \( f(\cdot) \) is a non-expansion.

Proof:

$$d[f(x), f(\bar{x})] = \max_{i,j} \log \left| \frac{f_{ij}(x)}{f_{ij}(\bar{x})} \right| \quad (175)$$

$$\leq \max_{i,j} \max \left\{ \rho_g, \rho_h \right\} \left[ \frac{g_{ij}(x)}{g_{ij}(\bar{x})} \right] + \max_{i,j} \max \left\{ \rho_g, \rho_h \right\} \left[ \frac{h_{ij}(x)}{h_{ij}(\bar{x})} \right] \quad (176)$$

$$\leq (\max \left\{ \rho_g, \rho_h \right\} \alpha_g + \max \left\{ \rho_g, \rho_h \right\} \alpha_h) d(x, \bar{x}) \quad (177)$$

The following results immediately from lemmas 10 and 11.

Corollary 1. The operators \( s(\cdot) \) and \( s(\cdot) \) defined in (160) are non-expansions.

Lemma 12. If \( d(x, \bar{x}) \leq \log L < \infty \), then the operator \( s(\cdot) \) is a contraction in cases 2 and 3 with a contraction factor dependent on \( L \) and \( u_{ij} \).

Proof: Lemma 2 in [30] shows the update:

$$y_{ij} = \frac{u_{ij}}{1 + \sum_{j' \neq j} w_{ij} x_{ij'}} \quad (178)$$

is a contraction. The proof of lemma 2 in [30] may be trivially modified to show that the updates:

$$c_1 \sum_{j' \neq j} w_{ij} x_{ij'} \quad (179)$$

are also contractions for any \( c_1 > 0 \), \( c_2 > 0 \). In cases 2 and 3, \( \alpha = 1 \), so \( x_{ij} = 1 \). Therefore, these results combined with lemma 11 show that (163) and (164) are contractions. Since \( \frac{1}{L} - 1 < 1 \), (165) is also a contraction.

This is adequate to prove convergence in cases 2 and 3: \( s(\cdot) \) is a contraction, and \( \theta(\cdot) \) is a non-expansion, so the composite operator is a contraction. Combined with lemma 8 this shows that the iteration converges to the unique solution of the KKT conditions.

The final step is to prove convergence in case 1. For this, we prove that \( n \) successive iterations of applying operators \( \theta(\cdot) \) and \( s(\cdot) \) collectively form a contraction. The proof is based on [33], but adapts it to address \( \gamma < 1 \), and to admit cases where some edges have \( w_{ij} = 0 \).

As discussed in lemma 8, the solution in case 1 is not changed by adding any constant \( c \) to \( \tilde{\lambda}_j \) \( \forall \; i \) and subtracting it from \( \tilde{\mu}_j \) \( \forall \; j \); this is clear from [153], and was termed message gauge invariance in [33] remark 30. Incorporating any such constant simply offsets all future iterations by the value, having no impact on the \( q_j \) iterates produced. Thus, for the purpose of proving convergence, when analysing \( \theta(\cdot) \), we scale \( y_{ij} \) such that \( \max_{i,j} y_{ij}^{(k)} = 1 \), and we denote \( \max_{i,j} y_{ij}^{(k-1)} / y_{ij}^{(k)} \) for some \( L \) with \( 1 < L < \infty \) (if \( L = 1 \) then convergence has occurred, and \( L = \infty \) will only occur if \( y_{ij}^{*} = 0 \), which contradicts lemma 7). The result is similar to that obtained using by changing the distance to Hilbert’s projective metric (e.g., [52]). We emphasise that this rescaling does not need to be performed in the online calculation; rather we are exploiting an equivalence to aid in proving convergence.

Lemmas 13 and 14 establish an induction which shows that after \( n \) steps, we are guaranteed to have reduced \( \max_{i,j} y_{ij}^{(k+n)} / y_{ij}^{*} \). The induction commences with a single edge with \( y_{ij}^{(k)} / y_{ij}^{*} = 1 \), setting \( T^{(k)} = \{ (i,j) \} \), and \( u^{(k)} = 1 \). As the induction proceeds, the set \( T^{(k)} \) (or, alternately, \( S^{(k)} \)) represents the edges for which improvement in the bound \( L \) is guaranteed, and \( u^{(k)} < L \) (or, alternately, \( u^{(k)} \)) represents the amount of improvement that is guaranteed. The induction proceeds by alternately visiting the left-hand vertices and right-hand vertices (e.g., in figure 1), at each stage adding to the set \( S^{(k)} \) edges (or, alternatively, \( T^{(k)} \)) for which \( w_{ij} > 0 \), and an edge that is incident on the vertex \( j \) is in \( T^{(k)} \) (or, alternatively, \( T^{(k)} \)) that could be traversed by starting form a vertex \( i \) represented by an edge in \( S^{(k)} \).

Lemma 13. At iteration \( (k-1) \), suppose that \( 1 \leq y_{ij}^{(k-1)} / y_{ij}^{*} \leq L < \infty \) \( \forall \; i, j \), and \( y_{ij}^{(k-1)} / y_{ij}^{*} \leq u^{(k-1)} \leq L \) \( \forall \; (i,j) \in T^{(k-1)} \). Then \( 1/L < x_{ij}^{(k)} / x_{ij}^{*} \leq 1 \) \( \forall \; (i,j) \), and \( 1/u^{(k)} \leq x_{ij}^{(k)} / x_{ij}^{*} \leq 1 \) \( \forall \; (i,j) \in S^{(k)} \), where

$$S^{(k)} = \{ (i,j) \in \{1,\ldots,n\}^2 | \exists j' \text{ s.t. } (i',j) \in T^{(k-1)}, w_{ij} > 0 \} \quad (181)$$

and

$$u^{(k)} = \max_{i,j} \left[ \theta^{(k-1)}_j y_{ij}^{(k-1)} + (1-\theta^{(k-1)}_j) L \right]^{1-\gamma} L^\gamma < L \quad (182)$$

where

$$\theta^{(k-1)}_j = \sum_{i \in (i,j) \in T^{(k-1)}} w_{ij} y_{ij}^{*} / \sum_i w_{ij} y_{ij}^{*} \quad (183)$$

Proof: Consider the sum in the first factor in (160) (remembering that \( w_{0j} = 0 \)):

$$\sum_i w_{ij} y_{ij}^{(k)} \quad (184)$$

The proof follows from the definition of the update. The final step is to prove convergence in case 1.
so that \( \sigma_j^{(k)} / \sigma_j^* \geq 1 \), and

\[
\sigma_j^{(k)} / \sigma_j^* \leq \frac{v^{(k-1)} \sum_{i \in T^{(k-1)}} w_{ij} y_{ij} + L \sum_{i,j \in T^{(k-1)}} w_{ij} y_{ij}}{\sum_{i \neq j} w_{ij} y_{ij}} = \theta_j^{(k-1)} v^{(k-1)} + (1 - \theta_j^{(k-1)}) L
\]

While a similar analysis could be applied to the second factor of the expression for \( r(y) \) (as in \[33\]), to prove convergence in the case with \( \gamma < 1 \), it is adequate to simply bound it by:

\[
1 \leq \frac{\sum_{i,j \in T^{(k-1)}} w_{ij} y_{ij}}{\sum_{i \neq j} w_{ij} y_{ij}} \leq L
\]

Substituting these bounds into \[160\] gives the desired result.

**Lemma 14.** At iteration \( k \), suppose that \( 0 < \frac{1}{L} \leq x_{ij}^{(k)} / x_{ij}^* \leq 1 \) \( \forall i, j \), and \( 1 / L < 1 / v^{(k)} \leq x_{ij}^{(k)} / x_{ij}^* \) \( \forall (i, j) \in S^{(k)} \). Then \( 1 \leq y_{ij}^{(k)} / y_{ij}^* \leq L \) \( \forall (i, j) \) and \( 1 \leq y_{ij}^{(k)} / y_{ij}^* \leq v^{(k)} \) \( \forall (i, j) \in T^{(k)} \), where

\[
T^{(k)} = \{(i, j) \in \{1, \ldots, n\}^2 \mid \exists j' \text{ s.t. } (i, j') \in S^{(k)}, w_{ij} > 0 \}
\]

and

\[
1 / v^{(k)} = \min_i \left[ \frac{\omega_i^{(k)}}{u^{(k)}} + \frac{1 - \omega_i^{(k)}}{L} \right] > \frac{1}{L} \geq \frac{1}{L}
\]

where

\[
\omega_i^{(k)} = \sum_{j \in \{i, j \} \in S^{(k)}} w_{ij} x_{ij}^* / \sum_j w_{ij} x_{ij}^*
\]

**Proof:** Following similar steps to the proof of lemma \[13\] we define:

\[
\tau_i^* = \sum_j w_{ij} x_{ij}^*, \quad \tau_i^{(k)} = \sum_j w_{ij} x_{ij}^{(k)}
\]

so that \( \tau_i^{(k)} / \tau_i^* \leq 1 \), and

\[
\frac{\tau_i^{(k)}}{\tau_i^*} \geq \frac{1}{v^{(k)}} - \frac{1}{L} \sum_{j \in \{i, j \} \in S^{(k)}} w_{ij} x_{ij}^* + \frac{1}{L} \sum_{j \in \{i, j \} \in S^{(k)}} w_{ij} y_{ij}^*
\]

\[
= \frac{\omega_i^{(k)}}{u^{(k-1)}} + (1 - \omega_i^{(k)}) \frac{1}{L}
\]

The second factor in \[163\] can be bounded by the expression

\[
1 / L \leq \sum_{j \not\in \{i, j \} \in S^{(k)}} w_{ij} x_{ij}^* / \sum_{j \not\in \{i, j \} \in S^{(k)}} w_{ij} x_{ij}^* \leq 1
\]

Substituting these bounds into \[163\] gives the desired result.

**Lemma 15.** \( \alpha(L) \) is continuous and non-decreasing in \( L \), i.e., its left and right derivatives everywhere satisfy

\[
\partial_- \alpha(L) \geq 0, \quad \partial_+ \alpha(L) \geq 0
\]

**Proof:** Taking either the left or right derivative of \( \alpha(L) \) in \[191\]:

\[
\partial_- \alpha(L) \geq \frac{\partial v(L) \cdot \log L}{v(L)} - \frac{1}{L} \log v(L)
\]

Thus it suffices to show that (omitting the iteration index superscript from \( v \))

\[
\partial_- v(L) \geq \frac{v(L) \log v(L)}{L \log L}, \quad \partial_+ v(L) \geq \frac{v(L) \log v(L)}{L \log L}
\]

We prove by induction, showing that the property in \[193\] is maintained through the recursion in \[182\] and \[187\]. The base case is established by noting that if \( v^{(k-1)} = 1 \), then \[193\] holds. Now suppose that the property \[193\] is held for some iteration \((k + l - 1)\) with upper bound \( v^{(k+l-1)}(L) \). Then let

\[
\bar{u}_j^{(k+l)}(L) = \theta_j^{(k+l-1)} v^{(k+l-1)}(L) + (1 - \theta_j^{(k+l-1)}) L
\]

By the second result in lemma \[16\], \( \bar{u}_j^{(k+l)} \) will satisfy the property \[193\] for each \( j \). The pointwise maximum in \[182\] will introduce a finite number of points where \( \bar{u}_j^{(k+l)}(L) \) is continuous but the derivative is discontinuous. However, the one-sided derivative at any of these points will satisfy \[193\] since each component in the pointwise maximum satisfied it. Finally, the result of \[182\] is:

\[
u^{(k+n)}(L) \geq \left[ \bar{u}_j^{(k+l)}(L) \right]^{1-\gamma} L^\gamma
\]

The first result in lemma \[16\] shows that this will satisfy \[193\].

Now we need to prove that the other half-iteration \[187\] maintains the property \[193\]. First, note that if \( \bar{L} = 1 / L \) and \( \bar{u}_j^{(k+l)} = 1 / u_j^{(k+l)}(L) \) then the final result in lemma \[16\] shows that if \( u_j^{(k+l)}(L) \) satisfies \[193\], then:

\[
\partial_- \bar{u}_j^{(k+l)}(\bar{L}) \geq \frac{\bar{u}_j^{(k+l)}(\bar{L}) \log \bar{u}_j^{(k+l)}(\bar{L})}{\bar{L} \log \bar{L}}
\]

Subsequently, if:

\[
\bar{u}_j^{(k+l)}(\bar{L}) \geq \frac{\bar{u}_j^{(k+l)}(\bar{L}) \log \bar{u}_j^{(k+l)}(\bar{L})}{\bar{L} \log \bar{L}}
\]

then the second result in lemma \[16\] establishes that \( \bar{u}_j^{(k+l)} \) will satisfy the property \[194\] for each \( i \). As with the pointwise
maximum in the the previous case, the pointwise minimum \[187\]:

\[\bar{v}^{(k+l)}(L) = \min_i \bar{v}_i^{(k+l)}(L)\]

will introduce a finite number of points where \(\bar{v}^{(k+l)}(L)\) is continuous but the derivative is discontinuous. However, the one-sided derivative at any of these points will satisfy \[194\].

The first result in lemma \[16\] shows that the composition:

\[\bar{v}^{(k+l)}(L) = \left[\bar{v}^{(k+l)}(L)\right]^{1-\gamma} L^{\gamma}\]

will also satisfy \[194\]. Finally, the result of \[187\] is:

\[v^{(k+l)}(L) = 1/\bar{v}^{(k+l)}(1/L)\]

which will satisfy \[194\] by the final result in lemma \[16\].

**Lemma 16.** Suppose that \(du(x)/dx \geq \frac{u(x) \log u(x)}{x \log x}\). Then if \(v(x)\) is given by any of the following:

1. \(v(x) = u(x)^{1-\gamma} x^{\gamma}\)
2. \(v(x) = \theta u(x) + (1-\theta) x\)

then \(dv(x)/dx \geq \frac{v(x) \log v(x)}{x \log x}\). Finally, if \(y = 1/x\) and \(v(y) = 1/u(x) = 1/u(1/y)\)

then \(dv(y)/dy \geq \frac{v(y) \log v(u)}{y \log y}\).

**Proof:** For the first case:

\[
\frac{dv(x)}{dx} = (1 - \gamma) \frac{du(x)}{dx} u(x)^{-\gamma} x^{\gamma} + \gamma u(x)^{1-\gamma} x^{\gamma-1}
\]

\[
\geq (1 - \gamma) \frac{u(x) \log u(x)}{x \log x} u(x)^{-\gamma} x^{\gamma} + \gamma u(x)^{1-\gamma} x^{\gamma-1}
\]

\[
= \frac{u(x)^{1-\gamma} x^{\gamma} [(1 - \gamma) \log u(x) + \gamma \log x]}{x \log x}
= \frac{v(x) \log v(x)}{x \log x}
\]

For the second case:

\[
\frac{dv(x)}{dx} = \theta \frac{du(x)}{dx} + (1 - \theta)
\]

\[
\geq \theta \frac{u(x) \log u(x)}{x \log x} + (1 - \theta)
\]

\[
= \frac{\theta u(x) \log u(x) + (1-\theta) x \log x}{x \log x}
\]

\[
\geq \frac{v(x) \log v(x)}{x \log x}
\]

where the final inequality is the result of convexity of \(x \log x\).

For the final result, let \(f(x) = 1/u(x)\) and \(g(y) = 1/y\) and apply the chain rule:

\[
\frac{dv(y)}{dy} = f'(g(y)) \times g'(y)
\]

\[
= \frac{u'(1/y)}{u(1/y)^2} \times \frac{1}{y^2}
\]

\[
\geq \frac{1}{u(1/y)^2 y^2} \times \frac{u(1/y) \log u(1/y)}{(1/y) \log (1/y)}
\]

\[
= \frac{[1/u(1/y)] \log [1/u(1/y)]}{y \log y} = \frac{v(y) \log v(y)}{y \log y}
\]