Abstract

Let $F : T^n \times I \to T^n$ be a homotopy on a n-dimensional torus. The main purpose of this paper is to present a formula for the one-parameter Nielsen number $N(F)$ of $F$ in terms of its induced homomorphism. If $L(F)$ is the one-parameter Lefschetz class of $F$ then $L(F)$ is given by $L(F) = N(F)\alpha$, for some $\alpha \in H_1(\pi_1(T^n), \mathbb{Z})$.

1 Introduction

Let $F : X \times I \to X$ be a homotopy on a finite CW complex $X$ and $G = \pi_1(X, x_0)$. Here $I$ will denote the unit interval. We say that $(x, t) \in X \times I$ is a fixed point of $F$ if $F(x, t) = x$. We denote the fixed points set of $F$ by $\text{Fix}(F)$. R. Geoghegan and A. Nicas in [8] developed a one-parameter theory and defined the one-parameter trace $R(F)$ of $F$ to study the fixed points set of $F$. From trace $R(F)$ we define the one-parameter Nielsen number $N(F)$ of $F$ and the one-parameter Lefschetz class $L(F)$. These invariants are computable, depending only on the homotopy class of $F$ relative to $X \times \{0, 1\}$.

The study of the fixed points of a homotopy has been considered by many authors, see for example [12], [2] and [6]. Here is important to point that only the reference [2] uses the approach developed in [8]. Following [8] we have an important application of the trace $R(F)$. Given a smooth flow $\Psi : M \times \mathbb{R} \to M$ on a closed oriented manifold one may regard any finite portion of $\Psi$ as a homotopy. Write $F = \Psi| : M \times [a, b] \to M$. The traces $L(F)$ and $R(F)$ recognize dynamical meaning of $\Psi$. When $a > 0$, $L(F)$ detects the Fuller homology class, derived from Fuller’s index theory, see [4]. Thus it is possible to study periodic orbits of $\Psi$ using the one-parameter theory, see [9].

The result of this paper allows as to solve the important problem which is the calculation of periodic orbits of a flow on the n-torus. In fact, given a smooth flow $\Psi : T^n \times \mathbb{R} \to T^n$ on n-torus we write $F = \Psi| : T^n \times [a, b] \to T^n$ for a finite portion of $\Psi$. In the case $n = 2$, in [8, Example 5.10, pg 431], was presented an example of calculation of periodic orbits. In this paper we prove that the Lefschetz class $L(F)$ of $F$ is given by $L(F) = N(F)\alpha$, for some $\alpha \in H_1(\pi_1(T^n), \mathbb{Z})$, and we present a formula for $N(F)$.

Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus and $v = [(0, 0, \ldots, 0)]$. We denote

\[ \pi_1(T^n, v) = \langle u_1, u_2, \ldots, u_n | u_i u_j u_i^{-1} u_j^{-1} = 1, \text{ for all } i \neq j \rangle. \]
We say that a homotopy $H : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n$ is affine if there exist $H' : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$ such that $H \circ (p_{\mathbb{T}^n} \times Id) = p_{\mathbb{T}^n} \circ H'$, where $p_{\mathbb{T}^n} : \mathbb{R}^n \rightarrow \mathbb{T}^n$ is the natural projection and $H'$ is given by

$$H'(x_1, \cdots, x_n, t) = \left( \sum_{j=1}^{n} a_{ij} x_j + c_i t + \epsilon_1, \cdots, \sum_{j=1}^{n} a_{nj} x_j + c_n t + \epsilon_n \right),$$

for some $a_{ij}, c_i \in \mathbb{Z}$ and $0 \leq \epsilon_i < 1$.

Given $F : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n$ a homotopy we denote by $w = F(v, I)$. We will assume that the homotopy class of $F$ relative to $\mathbb{T}^n \times \{0, 1\}$ contains one affine homotopy where the $\epsilon_i$ are chosen such that $F$ has no fixed points in $\mathbb{T}^n \times \{0, 1\}$ when the classical Nielsen number $N(F|\mathbb{T}^n)$ is zero. From this hypothesis follows that $w$ is a loop in $\mathbb{T}^n$. We can write

$$[w] = u_1^{c_1} u_2^{c_2} \cdots u_n^{c_n}$$

for some integers $c_1, c_2, \ldots, c_n$. Let $\phi$ be the homomorphism given by the following composition:

$$\pi_1(\mathbb{T}^n \times I, (v, 0)) \xrightarrow{F_{\#}} \pi_1(\mathbb{T}^n, F(v, 0)) \xrightarrow{c_{[\tau]}^{-1}} \pi_1(\mathbb{T}^n, v),$$

where $\tau$ is the path in $\mathbb{T}^n$ from $v$ to $F(v, 0)$ and $c_{[\tau]}$ is the isomorphism that changes the base point. Suppose that the Nielsen number of $F$ restricted to $\mathbb{T}^n$, $N(F|\mathbb{T}^n) = |det([\phi] - I)|$, is zero. Let $w_1$ be an eigenvector of $[\phi]$ associated to 1. Complete $\{w_1, w_2, \ldots, w_n\}$ for a basis of $\mathbb{R}^n$. In this new basis the matrix of $\phi$ is given by:

$$[\phi] = \begin{pmatrix}
1 & b_{12} & \cdots & b_{1n} \\
0 & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & b_{n2} & \cdots & b_{nn}
\end{pmatrix}.$$

If $P : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n$ is the projection then $[\phi] - [P_{\#}] = \begin{pmatrix}
0 & b_{12} & \cdots & b_{1n} \\
0 & b_{22} - 1 & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & b_{n2} & \cdots & b_{nn} - 1
\end{pmatrix}$. We denote

$$A = \begin{pmatrix}
b_{12} & \cdots & b_{1n} & c_1 \\
b_{22} - 1 & \cdots & b_{2n} & c_2 \\
\vdots & \vdots & \ddots & \vdots \\
b_{n2} & \cdots & b_{nn} - 1 & c_n
\end{pmatrix}.$$

With the above hypothesis and notations we present the main result of this paper.

**Theorem 1.** Given a homotopy $F : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n$ then the one-parameter Lefschetz class of $F$ is given by:

$$L(F) = N(F) \alpha,$$

where $N(F)$ is the one-parameter Nielsen number of $F$ and $\alpha$ is a class in $H_1(\pi_1(\mathbb{T}^n), \mathbb{Z})$. The one-parameter Nielsen number of $F$ is given by:

$$N(F) = \begin{cases}
|det(A)| & \text{if } N(F|\mathbb{T}^n) = 0, \\
0 & \text{otherwise.}
\end{cases} \quad (1)$$
2 One-parameter Fixed Point Theory

To facilitate the reading of this paper we will do in this section a brief review of definition of the one-parameter trace for a homotopy \( F : X \times I \to X \) where \( X \) is a finite CW complex and \( F \) is cellular. For more details see [8].

Let \( R \) be a ring and \( M \) an \( R-R \) bimodule, that is, a left and right \( R \)-module satisfying \((r_1m)r_2 = r_1(mr_2)\) for all \( m \in M \), and \( r_1, r_2 \in R \). The Hochschild chain complex \( \{C_n(R, M), d\} \) is given by \( C_n(R, M) = R^{\otimes n} \otimes M \) where \( R^{\otimes n} \) is the tensor product of \( n \) copies of \( R \), taken over the integers, and

\[
d_n(r_1 \otimes \ldots \otimes r_n \otimes m) = r_2 \otimes \ldots \otimes r_n \otimes mr_1 \\
+ \sum_{i=1}^{n-1} (-1)^i r_1 \otimes \ldots \otimes r_i r_{i+1} \otimes \ldots \otimes r_n \otimes m \\
+ (-1)^n r_1 \otimes \ldots \otimes r_{n-1} \otimes r_n m.
\]

The \( n \)-th homology of this complex is the Hochschild homology of \( R \) with coefficient bimodule \( M \), it is denoted by \( HH_n(R, M) \). There are other ways of presenting the definition of Hochschild homology, for example see [11].

In the particular cases \( n = 1, 2 \) we have the formula \( d_2(r_1 \otimes r_2 \otimes m) = r_2 \otimes mr_1 - r_1 r_2 \otimes m + r_1 \otimes r_2 m \) and \( d_1(r \otimes m) = mr - rm \). Using the expression of \( d_2 \) and the 2-chain \( 1 \otimes 1 \otimes m \) we obtain;

**Lemma 2.** If \( 1 \in R \) is the unit element and \( m \in M \) then the 1-chain \( 1 \otimes 1 \otimes m \) is a boundary.

For the definition of \( R(F) \) we use the Hochschild homology in the following situation: Let \( G \) be a group and \( \phi : G \to G \) an endomorphism. Also denote by \( \phi \) the induced ring homomorphism \( \mathbb{Z}G \to \mathbb{Z}G \). Take the ring \( R = \mathbb{Z}G \) and \( M = (\mathbb{Z}G)^{\phi} \) the \( \mathbb{Z}G-\mathbb{Z}G \) bimodule whose underlying abelian group is \( \mathbb{Z}G \) and the bimodule structure is given by \( g.m = gm \) and \( m.g = m\phi(g) \).

We say that two elements \( g_1, g_2 \) in \( G \) are semiconjugate if there exists \( g' \in G \) such that \( g_1 = g'g_2\phi(g'^{-1}) \). We write \( C(g) \) for the semiconjugacy class containing \( g \) and \( G_{\phi} \) for the set of semiconjugacy classes. Thus, we can decompose \( G \) in the union of its semiconjugacy classes. This partition induces a direct sum decomposition of \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \).

Note that each generating chain \( \gamma = g_1 \otimes \ldots \otimes g_n \otimes m \) can be written in canonical form as \( g_1 \otimes \ldots \otimes g_n \otimes g_n^{-1} \ldots g_1^{-1} \) where \( g = g_1 \ldots g_n m \). We will say that \( g \) “marks” a semiconjugacy class. The decomposition \( (\mathbb{Z}G)^{\phi} \cong \bigoplus_{C \in G_{\phi}} \mathbb{Z}C \) as a direct sum of abelian groups determines a decomposition of chains complexes \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \cong \bigoplus_{C \in G_{\phi}} C_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C \) where \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C \) is the subgroup of \( C_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \) generated by those generating chains whose markers lie in \( C \). Therefore, we have the following isomorphism: \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi}) \cong \bigoplus_{C \in G_{\phi}} HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C \) where the summand \( HH_*(\mathbb{Z}G, (\mathbb{Z}G)^{\phi})_C \) corresponds to the homology classes marked by the elements of \( C \). This summand is called the \( C \)-component.
Let $Z(h) = \{ g \in G \mid h = gh\phi(g^{-1}) \}$ be the semicentralizer of $h \in G$. Choosing representatives $g_C \in C$, then we have the following proposition whose proof is in [8].

**Proposition 3.** Choosing representatives $g_C \in C$ then we have

$$HH_\ast(ZG, (ZG)^\phi) \cong \bigoplus_{C \in G_\phi} H_\ast(Z(g_C))_C$$

where $H_\ast(Z(g_C))_C$ corresponds to the summand $HH_\ast(ZG, (ZG)^\phi)_C$.

**Lemma 4.** If $G = \pi_1(X, v)$ is an abelian group then the cardinality of semiconjugacy classes in $G$ is the cardinality of $\text{coker}(\phi - P_\#)$ in $G$, where $P : X \times I \to X$ is the projection.

**Proof.** In fact, two elements $g_1$ and $g_2$ in $G$ belong to the same semiconjugacy class if and only if there exists $g \in G$ such that $g_1 = gg_2\phi(g^{-1})$. This is equivalent to $g_2 - g_1 = \phi(g) - P_\#(g)$, because $G$ is abelian. On the other hand, the last equation is equivalent to say that $g_1$ and $g_2$ belong the same class in $\text{coker}(\phi - P_\#)$ in $G$. 

\[ \square \]

### 2.1 One-parameter trace $R(F)$

Let $X$ be a finite connected CW complex and $F : X \times I \to X$ a cellular homotopy. We consider $I = [0,1]$ with the usual CW structure and orientation of cells, and $X \times I$ with the product CW structure, where its cells are given the product orientation. Pick a basepoint $(v,0) \in X \times I$, and a basepath $\tau$ in $X$ from $v$ to $F(v,0)$. We identify $\pi_1(X \times I, (v,0)) \cong G$ with $\pi_1(X, v)$ via the isomorphism induced by projection $p : X \times I \to X$. We write $\phi : G \to G$ for the homomorphism;

$$\pi_1(X \times I, (v,0)) \xrightarrow{F_\#} \pi_1(X, F(v,0)) \xleftarrow{\partial} \pi_1(X, v)$$

For each cell $E$ in $X$, we choose a lift $\tilde{E}$ in the universal cover $\tilde{X}$ and we orient $\tilde{E}$ compatibly with $E$. Let $\tilde{\tau}$ be the lift of the basepath $\tau$ which starts in the basepoint $\tilde{v} \in \tilde{X}$ and $\tilde{F} : \tilde{X} \times I \to \tilde{X}$ the unique lift of $F$ satisfying $\tilde{F}(\tilde{v},0) = \tilde{\tau}(1)$. We can regard $C_\ast(\tilde{X})$ as a right $ZG$ chain complex as follows: if $\omega$ is a loop at $\tilde{v}$ which lifts to a path $\tilde{\omega}$ starting at $\tilde{v}$ then $\tilde{E}[\omega]^{-1} = h_{[\omega]}(\tilde{E})$, where $h_{[\omega]}$ is the covering transformation sending $\tilde{v}$ to $\tilde{\omega}(1)$. The homotopy $\tilde{F}$ induces a chain homotopy $\tilde{D}_k : C_k(\tilde{X}) \to C_{k+1}(\tilde{X})$ given by

$$\tilde{D}_k(\tilde{E}) = (-1)^{k+1}\tilde{F}_k(\tilde{E} \times I) \in C_{k+1}(\tilde{X}),$$

for each cell $\tilde{E} \in \tilde{X}$. This chain homotopy satisfies; $\tilde{D}(\tilde{E}g) = \tilde{D}(\tilde{E})\phi(g)$ and the boundary operator $\partial_k : C_k(\tilde{X}) \to C_{k-1}(\tilde{X})$ satisfies; $\partial(\tilde{E}g) = \partial(\tilde{E})g$. Define endomorphism of $\oplus_k C_k(\tilde{X})$ by $\tilde{D}_\ast = \oplus_k(-1)^{k+1}D_k$, $\partial_\ast = \oplus_k \partial_k$, $\tilde{F}_0 = \oplus_k(-1)^k\tilde{F}_k$ and $\tilde{F}_1 = \oplus_k(-1)^k\tilde{F}_k$. We consider trace($\partial_\ast \otimes \tilde{D}_\ast$) in $HH_1(ZG, (ZG)^\phi)$. This is a Hochschild 1-chain whose boundary is; $\text{trace}(\partial_\ast \otimes \tilde{D}_\ast - \partial_\ast \tilde{D}_\ast)$. We denote by $G_\phi(\partial(F))$ the subset of $G_\phi$ consisting of semiconjugacy classes associated to fixed points of $F_0$ or $F_1$.

**Definition 5.** The one-parameter trace of homotopy $F$ is:

$$R(F) \equiv T_1(\partial_\ast \otimes \tilde{D}_\ast; G_\phi(\partial(F))) \in \bigoplus_{C \in G_\phi - G_\phi(\partial(F))} HH_1(ZG, (ZG)^\phi)_C$$

$$\cong \bigoplus_{C \in G_\phi - G_\phi(\partial(F))} H_1(Z(g_C)).$$
Let \((x, t)\) and \((y, s)\) be two points in \(\text{Fix}(F)\). We say that these points are in the same fixed point class if there exists a path \(\gamma : I \to X \times I\) with \(\gamma(0) = (x, t), \gamma(1) = (y, s)\) and \((P \circ \gamma)(F \circ \gamma)^{-1}\) is homotopically trivial. Here \(P : X \times I \to X\) is the projection. This defines an equivalence relation \(\sim\) on \(\text{Fix}(F)\). The function \(\Psi : \text{Fix}(F) / \sim \to G_\phi\) defined by \(\Psi([([x, t])]) = [(P \circ \nu)(F \circ \nu)^{-1} \circ \tau^{-1}]\) is injective, where \(\nu\) is any path from base point \((v, 0)\) to \((x, t)\).

Supposing that \(F\) is transverse the projection \(P : X \times I \to X\) then \(\text{Fix}(F)\) is composed by circles in \(X \times (0, 1)\) and arcs connecting \(X \times \{0, 1\}\), see [8]. By Definition 5 we are only interested in the circles in \(X \times (0, 1)\) because all semiconjugacy classes associated to fixed point classes that intersect \(X \times \{0, 1\}\) has no contribution to the expression of \(R(F)\). From [2] one can always reduce to the case in which only one circle occurs in each fixed point class.

**Definition 6.** Let \(K\) a fixed point class of \(F\) and \(\Psi(K) = C \in G_\phi\). The one-parameter fixed point index of \(K\) is the \(C\)-component of \(R(F)\), \(i(F, C)\), in \(HH_1(ZG, (ZG)^\phi)\). The one-parameter fixed point index \(i(F, C)\) is zero if \(i(F, C)\) is the trivial homology class.

**Definition 7.** Given a cellular homotopy \(F : X \times I \to X\) the one-parameter Nielsen number, \(N(F)\), of \(F\) is the number of components \(i(F, C)\) with nonzero fixed point index.

**Definition 8.** The one-parameter Lefschetz class, \(L(F)\), of \(F\) is defined by;

\[
L(F) = \sum_{C \in G_\phi - G_\phi(\partial F)} j_C(i(F, C))
\]

where \(j_C : H_1(Z(g_C)) \to H_1(G)\) is induced by the inclusion \(Z(g_C) \subset G\).

**Remark 9.** From [8, Theorem 1.9 item c], to compute the one-parameter trace \(R(F)\) of \(F : X \times I \to X\) is enough compute \(R(F')\) for \(F'\) where \(F'\) is a map homotopic to \(F\), relative to \(X \times \{0, 1\}\), which is cellular.

## 3 Semiconjugacy classes on n-torus

In this section we describe some results about the semiconjugacy classes on a n-torus related to a homotopy \(F : \mathbb{T}^n \times I \to \mathbb{T}^n\).

Let \(\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n\) be the n-torus and \(v = [(0, 0, ..., 0)]\). We denote

\[G = \pi_1(\mathbb{T}^n, v) = \langle u_1, u_2, ..., u_n | u_iu_ju_i^{-1}u_j^{-1} = 1, \text{ for all } i \neq j \rangle.\]

Given \(F : \mathbb{T}^n \times I \to \mathbb{T}^n\) a homotopy, where \(I\) is the unit interval, we denote by \(w = F(v, I)\) the path in \(\mathbb{T}^n\). Assume that \(w\) is a loop in \(\mathbb{T}^n\). Therefore we can write

\[w = u_1^{c_1}u_2^{c_2}...u_n^{c_n}\]

for some integers \(c_1, c_2, ..., c_n\). Let \(\phi\) be the homomorphism given by the following composition:

\[\pi_1(\mathbb{T}^n \times I, (v_0)) \xrightarrow{\pi_1(I, (v_0))} \pi_1(\mathbb{T}^n, v_0) \xrightarrow{\phi} \pi_1(\mathbb{T}^n, v),\]

where \(\tau\) is a base path from \(v_0\) to \(F(v_0)\).

Let us consider the isomorphism \(\Theta : G = \pi_1(\mathbb{T}^n, v) \to \mathbb{Z}^n\) defined by \(\Theta(u_1^{k_1}...u_n^{k_n}) = (k_1, \cdots, k_n)\).

By abuse of notation we will sometimes write \(\Theta(g) = g\).

Two elements \(g_1\) and \(g_2\) in \(G\) belong to the same semiconjugacy class if, and only if, there exists \(g \in G\) such that \(g_1 = gg_2\theta(g^{-1})\), in this case this is equivalent to saying;

\[(\phi - P_\#)(\Theta(g)) = \Theta(g_2) - \Theta(g_1),\]

where \(P : \mathbb{T}^n \times I \to \mathbb{T}^n\) is the projection. Thus we have:
Lemma 10. For each \( g \in G \) the semicentralizer \( Z(g) \) is isomorphic to the kernel of \( (\phi - P_\#) \).

Proof. It follows from the definition of \( Z(g) \) given on page 3. \( \square \)

Proposition 11. Let \( F : \mathbb{T}^n \times I \to \mathbb{T}^n \) be a homotopy. If the Nielsen number of \( F \) restricted to \( \mathbb{T}^n \) is nonzero then \( R(F) = 0 \), which implies \( L(F) = 0 \) and \( N(F) = 0 \).

Proof. If \( N(F|_{\mathbb{T}^n}) \neq 0 \) then by \([1]\) we have \( |\det((\phi) - I)| \neq 0 \). From Lemma 10 the semicentralizer \( Z(g) \) is trivial for all \( g \in G \). Thus \( H_1(Z(gC)) \) is trivial for each \( g_c \) which represents a semiconjugacy class \( C \). By decomposition presented in Section 2 we must have \( HH_1(ZG, (ZG)^\phi) = 0 \). Therefore, we obtain \( R(F) = 0 \), which implies \( L(F) = 0 \) and \( N(F) = 0 \). \( \square \)

Note that in the situation of Proposition 11 the cardinality of \( G_\phi \) is infinite.

From now on, we will assume that the classical Nielsen number of \( F : \mathbb{T}^n \times I \to \mathbb{T}^n \) restricted to \( \mathbb{T}^n \) is zero, that is, \( |\det((\phi) - I)| = 0 \). Let \( w_1 \) a eigenvector of \( [\phi] \) associated to 1. Complete \( \{w_1, w_2, ..., w_n\} \) for a basis of \( \mathbb{R}^n \). With respect to this new base the matrix of \( [\phi] \) has the following expression:

\[
[\phi] = \begin{pmatrix}
1 & b_{12} & \cdots & b_{1n} \\
0 & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & b_{nn} & \cdots & b_{nn}
\end{pmatrix}.
\]

We will assume from now on that \([\phi]\) has the above expression. Also we denote \( B = u_1^{k_1} \ldots u_n^{k_n} \) and \( D = u_1^{l_1} \ldots u_n^{l_n} \) elements in \( G \), where \( k_j, l_j \in \mathbb{Z} \), for all \( 1 \leq j \leq n \).

Lemma 12. The 1-chain \( B \otimes D \) is a cycle in \( HH_1(ZG, (ZG)^\phi) \) if, and only if, the element \((k_1, ..., k_n) \in \mathbb{Z}^n \) belongs to the kernel of \( ([\phi] - I) \). Therefore, if \( \text{rank}([\phi] - I) = n - 1 \) then \( B \otimes D \) is a cycle if, and only if, \( k_2 = ... = k_n = 0 \).

Proof. In fact, the 1-chain \( B \otimes D \) is a cycle if and only if \( d_1(B \otimes D) = 0 \), that is, if and only if \( 0 = D\phi(B) - BD \). Since \( G \) is abelian then this is equivalent \((\phi - I)(B) = 0 \). The last equation is equivalent to say that \((k_1, ..., k_n) \in \ker([\phi] - I) \). In other words \(([\phi] - I)(B) = 0 \) is equivalent to

\[
\begin{pmatrix}
0 & b_{12} & \cdots & b_{1n} \\
0 & b_{22} & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & b_{nn} & \cdots & b_{nn} - 1
\end{pmatrix}\begin{pmatrix}
k_1 \\
k_2 \\
\vdots \\
k_n
\end{pmatrix} = 0.
\]

Thus if \( \text{rank}([\phi] - I) = n - 1 \) then the system of equations above implies \( k_2 = ... = k_n = 0 \), and therefore the 1-cycle \( B \otimes D \) is written as \( u_1^{k_1} \otimes D \).

Let \( E = u_1^{d_1} \cdots u_n^{d_n} \). Given a 2-chain \( B \otimes D \otimes E \in C_2(ZG, (ZG)^\phi) \) then

\[
d_2(B \otimes D \otimes E) = D \otimes E\phi(B) - BD \otimes E + B \otimes DE.
\]

The above expression will be used in the proof of the next result.

Proposition 13. The 1-chain \( u_1^{k_1} \otimes D \in C_1(ZG, (ZG)^\phi) \) is homologous to the 1-chain \( k_1 u_1 \otimes u_1^{k_1 - 1} D \) for all \( k_1 \in \mathbb{Z} \).

Proof. For \( k_1 = 1 \) the proposition is clearly true. For \( k_1 = 0 \) the result is a consequence of Lemma 2. Let us assume that for some \( s > 0 \) the 1-chain \( u_1^s \otimes D \) is homologous to \( su_1 \otimes u_1^{s - 1} D \) for any \( D \) in \( G \). Taking the 2-chain, \( u_1^s \otimes u_1 \otimes D \in C_2(ZG, (ZG)^\phi) \), we obtain

\[
d_2(u_1^s \otimes u_1 \otimes D) = u_1 \otimes D u_1^s - u_1^{s+1} \otimes D + u_1^s \otimes u_1 D = (s + 1)u_1 \otimes u_1^s D - u_1^{s+1} \otimes D.
\]
Therefore \((s + 1)u_1 \otimes u_1^{(s+1)-1}D \sim u_1^{s+1} \otimes D\). By induction the result follows. The proof for case \(k_1 < 0\) is made in an analogous way.

**Proposition 14.** If \(\text{rank}([\phi] - I) = n - 1\) then the 1-cycle \(u_1^{-1} \otimes D\) is not homologous to zero, for any \(D \in G\).

**Proof.** We can write \(u_1^{-1} \otimes D\) as follows; \(u_1^{-1} \otimes u_1 g\), where \(g = u_1^{-1}D\). It follows from Lemma 10 that the semicentralizer \(Z(g)\) is isomorphic to \(\text{ker}([\phi] - I)\) for each \(g \in G\). Since \(\text{rank}([\phi] - I) = n - 1\) then \(Z(g) = \{u_1^s | s \in \mathbb{Z}\} \cong \mathbb{Z}\). Therefore \(H_1(Z(g)) \cong \mathbb{Z}\). From [8, pg 433] there is a sequence of natural isomorphisms;

\[
H_1(Z(g)) \rightarrow H_1(G, \mathbb{Z}(G/Z(g))) \rightarrow H_1(G, \mathbb{Z}(C(g))) \rightarrow HH_1(ZG, (ZG)^\phi)_{C(g)}.
\]

The class of element \(u_1^s\) is sent in the class of the 1-cycle \(u_1^s \otimes u_1^{-s}g\), which is homologous to a 1-cycle; \(-su_1^{-1} \otimes u_1 g = -s(u_1^{-1} \otimes u_1 g)\). Thus, if the 1-cycle was homologous to zero we would have \(H_1(Z(g)) \cong 0\) which is a contradiction. \(\square\)

Let us denote by \(B_i = u_1^{k_i} \cdots u_n^{k_n}\) and \(D_i = u_1^{l_i} \cdots u_n^{l_i}\) elements in \(G\), where \(k_i, l_i \in \mathbb{Z}\).

**Proposition 15.** If \(\text{rank}([\phi] - I) = n - 1\) then each 1-cycle \(\sum_{i=1}^{t} a_i B_i \otimes D_i \in C(ZG, (ZG)^\phi)\) is homologous to a 1-cycle with the following expression: \(\sum_{i=1}^{t} \bar{a}_i u_1 \otimes D'_i\).

**Proof.** Using Propositions 13 and 14, this is an easy generalization of [14, Proposition 4.18]. \(\square\)

**Corollary 16.** If the cycles \(u_1 \otimes D_i\) and \(u_1 \otimes D_j\) are in different semiconjugacy classes for all \(i \neq j\), \(i, j \in \{1, ..., t\}\), then \(\sum_{i=1}^{t} u_1 \otimes D_i\) is a nontrivial cycle. Furthermore, \(u_1 \otimes D_i\) projects to the same class \([u_1] \in H_1(G)\).

## 4 Proof of the main result

This section shall be devoted to proof Theorem 1.

**Proof.** Given \(H: \mathbb{T}^n \times I \rightarrow \mathbb{T}^n\) by hypothesis there exists an affine homotopy \(F\) homotopic to \(H\) relative to \(\mathbb{T}^n \times \{0, 1\}\). We have \(F \circ (p_{\mathbb{T}^n} \times Id) = p_{\mathbb{T}^n} \circ F'\) where

\[
F'(x_1, \cdots, x_n, t) = \left(\sum_{j=1}^{n} b_{ij} x_j + c_1 t + \epsilon_1, \cdots, \sum_{j=1}^{n} b_{ij} x_j + c_n t + \epsilon_n\right),
\]

for some \(b_{ij}, c_i \in \mathbb{Z}\) and \(0 \leq \epsilon_i < 1\).

By [8, Theorem 1.9 item a] we have \(R(H) = R(F)\), therefore is enough to check the proprieties of Theorem 1 for the homotopy \(F\). Based on that we will compute \(R(F)\). From [8, Theorem 1.9 item c] we can suppose \(H\) cellular. By Proposition 11 is enough to consider the case where \(N(F|_{\mathbb{T}^n}) = 0\), and therefore we can assume;

\[
[\phi] = \begin{pmatrix}
1 & b_{12} & \cdots & b_{1n} \\
0 & b_{22} & \cdots & b_{2n} \\
& \vdots & \ddots & \vdots \\
0 & b_{n2} & \cdots & b_{nn}
\end{pmatrix},
\]

(2)
Note that \( w = F(v, I) \) is a loop in \( \mathbb{T}^n \). Thus \([w] = u_1^c_1 u_2^c_2 \cdots u_n^c_n \), for some integers \( c_1, c_2, \ldots, c_n \). We denote by \( A \) the following matrix:

\[
A = \begin{pmatrix}
   b_{12} & \cdots & b_{1n} & c_1 \\
   b_{22} - 1 & \cdots & b_{2n} & c_2 \\
   \vdots & \ddots & \vdots & \vdots \\
   b_{n2} & \cdots & b_{nn} - 1 & c_n \\
\end{pmatrix}.
\]

(3)

Our proof breaks into two cases. The case \( \text{rank}(A) = n \) and \( \text{rank}(A) < n \). Firstly we assume \( \text{rank}(A) = n \). Note that this implies \( \text{rank}([\varphi] - I) = n - 1 \).

Since \( \mathbb{T}^n \) is a polyhedron, it has a structure of a regular CW-complex. We take an orientation for each k-cell \( E^k_j \) in \( \mathbb{T}^n \). From [8, Proposition 4.1] the trace \( R(F) \) is independent of the choice of orientation of cells on \( \mathbb{T}^n \). This independence is in terms of homology class.

On the universal covering space \( \mathbb{R}^n \) we choose a k-cell \( \tilde{E}^k_j \) which projects on \( E^k_j \). We orient \( \tilde{E}^k_j \) compatible with \( E^k_j \). We can suppose that \( \tilde{E}^k_j \) is contained in \( Y = [0, 1] \times \cdots \times [0, 1] \subset \mathbb{R}^n \). Considering \( C_*(\mathbb{R}^n) \) as a right \( \mathbb{Z}[\pi_1(\mathbb{T}^n)] \) chain complex as defined in Section 2 we have

\[
\partial_i(e^i_k) = \sum_j [e^i_k : e^{i-1}_j]e^{i-1}_j
\]

with \([E^i_k : E^{k-1}_j] = [e^i_k : e^{i-1}_j]\) where \([E^i_k : E^{k-1}_j]\) is the incidence of a k-cell \( E^i_k \) to a \((k-1)\)-cell. Since that \( \mathbb{T}^n \) is a regular CW complex then \([E^i_k : E^{k-1}_j]\) belongs to the set \( \{0, 1, -1\} \), see [16]. From definition of the right \( \mathbb{Z}G \) action on \( C_*(\mathbb{R}^n) \) and the fact that each k-cell is contained in \( Y \) then, for each \( j = 1, \ldots, n \), the entries of matrices of operators \( \partial_j \) will be composed by the following elements: \( 0, \pm 1, \pm u_i^{-1} \), where \( 1 \leq i \leq n \). By definition;

\[
R(F) = \text{tr} \begin{pmatrix}
-[[\partial_1] \otimes [\tilde{D}_0] & 0 & 0 & \cdots & 0 \\
0 & [[\partial_2] \otimes [\tilde{D}_1] & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & (-1)^{n+1}[[\partial_n] \otimes [\tilde{D}_{n-1}]]
\end{pmatrix},
\]

where the elements of matrices \([\partial_j]_{ik}\) belong to the set \( \{0, \pm 1, \pm u_i^{-1}\} \), \( 1 \leq i \leq n \). Thus the general expression of \( R(F) \) in \( C_1(\mathbb{Z}G, \mathbb{Z}G^\phi) \) would be;

\[
R(F) = -1 \otimes (\sum_{j=1}^m E_j) + 1 \otimes (\sum_{j=1}^m D_j) + \sum_{j=1}^n u_i^{-1} \otimes \sum_{j=1}^n A_j - \sum_{j=1}^n u_i^{-1} \otimes \sum_{j=1}^p B_j,
\]

where \( E_j, D_j, A_j, B_j \) are elements in \( G \).

If there exists \( \overline{F} : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n \) homotopic to \( F \), relative to \( \mathbb{T}^n \times \{0, 1\} \), such that \( \text{Fix}(\overline{F}) = \emptyset \) then \( R(F) \) is trivial and therefore \( L(F) = N(F) = 0 \). From now on, we assume that each homotopy \( \overline{F} : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n \) homotopic to \( F \), relative to \( \mathbb{T}^n \times \{0, 1\} \), contains isolated circles in \( \text{Fix}(\overline{F}) \). The number of these isolated circles for each \( \overline{F} \) is finite because \( \mathbb{T}^n \) is compact.

From Lemma 2 each 1-chain \( 1 \otimes E_j \) is a boundary. Therefore, the 1-chains \( 1 \otimes E_j \) and \(-1 \otimes D_j \) are homologous to zero in \( C_1(\mathbb{Z}G, \mathbb{Z}G^\phi) \). By Lemma 12 the 1-chain \( u_i^{-1} \otimes A_j \) is not a cycle for each \( 2 \leq i \leq n \). Therefore, the 1-chains \( u_i^{-1} \otimes A_j \) and \(-u_i^{-1} \otimes B_j \), for \( i \geq 2 \), can not appear in the expression of \( R(F) \) since \( R(F) \) is a cycle in \( HH_1(\mathbb{Z}G, \mathbb{Z}G^\phi) \).

Now let us calculate \( \text{Fix}(F) \). Note that \( F(x, 0) = F(x, 1) \). Therefore the homotopy \( F \) induces a map \( \overline{F} : \mathbb{T}^{n+1} = \mathbb{T}^n \times S^1 \rightarrow \mathbb{T}^n \) defined by \( \overline{F}(x, [t]) = F(x, t) \). It is not too difficult to see that the number of path components in \( \text{Fix}(F) \) and \( \text{Fix}(\overline{F}) \) are the same. Furthermore \( \overline{F}(x, t) = H(x, t) + (\epsilon_1, \cdots, \epsilon_n) \)
where $H : \mathbb{T}^{n+1} \rightarrow \mathbb{T}^n$ is linear and the $\epsilon_i$ are chosen such that $F$ has no fixed point in $\mathbb{T}^n \times \{0,1\}$.

The number of path components in $Fix(H)$ is the same as in $Fix(\bar{F})$.

Follows from [10, Theorem 3.3] that the number of path components in $Fix(F)$ is $D([H_#] - [P_#]) = D([\bar{F}_#] - [P_#]) = |det(A)|$, because the first column of $[\bar{F}_#] - [P_#]$ is null. More precisely $Fix(F)$ is composed by $|det(A)|$ disjoint circles. In fact, we have $Fix(F) = p_{\mathbb{T}^n}(F' - P)^{-1}(\mathbb{Z}^n)$). Follows from expression of $\bar{F}'$ that a point $z = (x_1, \ldots, x_n, t)$ belongs to the set $Fix(F')$ if and only if $z$ is a solution of the following system:

$$
\begin{pmatrix}
0 & b_{12} & \cdots & b_{1n} & c_1 \\
0 & b_{22} - 1 & \cdots & b_{2n} & c_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & b_{n2} & \cdots & b_{nn} - 1 & c_n
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
t
\end{pmatrix}
= 
\begin{pmatrix}
-\epsilon_1 + l_1 \\
-\epsilon_2 + l_2 \\
\vdots \\
-\epsilon_n - l_{n-1} + l_n \\
-\epsilon_n + l_n
\end{pmatrix}
$$

for some $l_1, \ldots, l_n \in \mathbb{Z}$. Note that in the System (5) we have $0 \leq x_1 \leq 1$, which implies that each path component in $Fix(F)$ is a isolated circle.

Note that two different circles belong to the different fixed point classes. In fact, let $C_1 = (x_1, x_1^1, \ldots, x_n^1, t^1)$ and $C_2 = (x_2, x_2^2, \ldots, x_n^2, t^2)$ two circles in $Fix(F)$. Each fixed point class has the form $p_{\mathbb{T}^n}(Fix(\bar{F}))$ where $\bar{F}$ is a lift of $F$. We have $C_1 = \bar{F}(C_1)$, $C_2 = \bar{F}(C_2)$. Note that each lift of $F$ has the form $F'(x, t) + (m_1, \ldots, m_n)$ where $m_j \in \mathbb{Z}$. Thus $C_1 - C_2 = \bar{F}(C_1 - C_2)$ which implies $(\bar{F} - P)(C_1 - C_2) = (0, \ldots, 0)$ where $P : \mathbb{R}^n \times \mathbb{I} \rightarrow \mathbb{R}^n$ is the projection. As $\text{rank}(A) = n$ then we must have $x_2^1 = x_2^2, \ldots, x_n^1 = x_n^2, t^1 = t^2$, and therefore $C_1 = C_2$.

The cardinality of the semiconjugacy classes $G_\phi$ is equal to $|det(A)|$, that is, is the same as the number of isolated circles in $Fix(F) \cap (0,1)$. In fact, by Lemma 4 the cardinality of the set $G_\phi$ is given by: $\#(\text{coker}(\phi - P_#))$. We have $[w] = u_1^{c_1}u_2^{c_2}\ldots u_n^{c_n}$ for some integers $c_1, c_2, \ldots, c_n$. Therefore the image of $(\phi - P_#)$ in $\pi_1(\mathbb{T}^n)$ is generated by columns of the following matrices:

$$
[\phi] - [P_#] = 
\begin{pmatrix}
0 & b_{12} & \cdots & b_{1n} \\
0 & b_{22} - 1 & \cdots & b_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & b_{n2} & \cdots & b_{nn} - 1
\end{pmatrix}
and
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{pmatrix},
$$

that is, the image of $(\phi - P_#)$ is generated by the columns of matrix $A$ where $A$ is given by:

$$
A = 
\begin{pmatrix}
b_{12} & \cdots & b_{1n} & c_1 \\
b_{22} - 1 & \cdots & b_{2n} & c_2 \\
\vdots & \vdots & \vdots & \vdots \\
b_{n2} & \cdots & b_{nn} - 1 & c_n
\end{pmatrix}.
$$

From hypothesis we have $\text{rank}(A) = n$. Therefore $\#(\text{coker}(\phi - P_#)) = \#(\pi_1(\mathbb{T}^n)/\text{im}(\phi - P_#)) = \#(\mathbb{Z}^n/A(\mathbb{Z}^n)) = |det(A)|$ since $A$ is non-singular.

Since each circle in $Fix(F')$ is parallel to the axis of $x_1$ then choosing an orientation for a circle all the others will have the same orientation according to the orientation of these circles defined in [3]. But the orientation of the circles in $Fix(F)$ is compatible with the orientation of the circles in $Fix(F')$. Thus, all circles in $Fix(F)$ have the same orientation and therefore all cycles in $R(F)$ will have the same signal. From these facts, the one-parameter trace of $F$ will have the following expression in $\text{HH}_1(\mathbb{Z}G, (\mathbb{Z}G)\phi)$:

$$
R(F) = u_1^{-1} \otimes \sum_{j=1}^m A_j
$$

(6)
or

\[ R(F) = -u_1^{-1} \otimes \sum_{j=1}^{m} B_j \tag{7} \]

where \( A_j, B_j \) are elements in \( G \). Let us consider the expression (6). The proof using the expression (7) is analogous.

From Proposition 14 each 1-cycle \( u_1^{-1} \otimes A_j \) is non trivial and thus represents a nonzero \( C \)-component. Furthermore, by definition gives in [8, Page 434] each circle in \( \text{Fix}(F) \) contributes to only one element in \( R(F) \). Thus, each semiconjugacy class contains only one cycle \( u_1^{-1} \otimes A_j \) in \( R(F) \), and therefore the one-parameter Nielsen number of \( F \) is:

\[ N(F) = m = |\det(A)|. \]

From Section 2, the one-parameter Lefschetz class is the image of \( R(F) \) in \( H_1(\pi_1(\mathbb{T}^n), \mathbb{Z}) \) by homomorphism induced by inclusion \( i : Z(gC) \to \pi_1(\mathbb{T}^n) \). By Proposition 14 each cycle \( u_1^{-1} \otimes A_j \) will be sent in the same class \(-[u_1]\). Therefore the image of \( R(F) \) in \( H_1(\pi_1(\mathbb{T}^n), \mathbb{Z}) \) is:

\[ L(F) = \sum_{i=1}^{m} -[u_1] = -m[u_1] = -N(F)[u_1]. \]

Thus \( L(F) = N(F)\alpha \), where \( \alpha = [u_1] \) or \(-[u_1]\).

Now we assume \( \text{rank}(A) < n \). In this case we have \( \text{im}(\phi - P_\#) \subseteq \mathbb{Z}^n \). Let \( w_0 \notin \text{im}(\phi - P_\#) \).

Define \( \overline{F} : \mathbb{T}^n \times I \to \mathbb{T}^n \) by \( \overline{F}(x, t) = F(x, t) + w_0\sin(2t\pi) \). The map \( Q : \mathbb{T}^n \times I \times I \to \mathbb{T}^n \) define by \( Q(x, t, s) = F(x, t) + s\omega_0\sin(2t\pi) \) is a homotopy between \( F \) and \( \overline{F} \) relative to \( \mathbb{T}^n \times \{0, 1\} \). Since \( w_0 \notin \text{im}(\phi - P_\#) \) then there are no circles in \( \text{Fix}(\overline{F}) \cap (\mathbb{T}^n \times (0, 1)) \). Therefore \( R(\overline{F}) = 0 \) which implies \( R(F) = 0, N(F) = 0 \) and \( L(F) = 0 \). \( \square \)

5 Applications

In this section we present some applications of Theorems 1 for compute the minimum number of path components in the fixed point set of some maps.

I. Let \( X \) be a finite CW complex and \( F : X \times I \to X \) be a homotopy such that \( F(x, 0) = F(x, 1) \). For example, when \( X = \mathbb{T}^n \), all linear homotopies satisfies \( F(x, 0) = F(x, 1) \), because \( F(x, 1) = F(x, 0) + (d_1, ..., d_n) \), where \( d_1, ..., d_n \) are integer numbers. Denote \( S^1 = \frac{I}{0 \sim 1} \). The homotopy \( F \) induces a map \( \overline{F} : X \times S^1 \to X \) defined by

\[ \overline{F}(x, [t]) = F(x, t). \]

It is not difficult to see that each homotopy \( H : X \times I \times I \to X \) from \( F \) to a map \( F' \) relative to \( X \times \{0, 1\} \) is equivalent to a homotopy \( \overline{H} : X \times S^1 \times I \to X \) from \( \overline{F} \) to \( \overline{F}' \) relative to \( (v, [0]) \). If \( F \) has no fixed points in \( X \times \{0, 1\} \) then we must have \( N(F|_X) = 0 \), and the minimum number of path components in \( \text{Fix}(F) \) and \( \text{Fix}(\overline{F}) \), as \( F \) runs over a homotopy class of maps \( X \times I \to X \) relative to \( X \times \{0, 1\} \), must coincide.

Let us consider \( X = \mathbb{T}^n \) and \( F \) a affine homotopy. Suppose that \( N(F|_{\mathbb{T}^n}) = 0 \). In this case the one-parameter Nielsen number of \( F \) given in Theorem 1 coincides with the invariant \( D([P_\#] - [\overline{P}_\#]) \) presented in [10, Theorem 3.3], where \( P \) is the projection and the matrix of \( F_\# \) is as in Theorem 1. In fact, from [10] \( D([P_\#] - [\overline{P}_\#]) \) is defined by

\[ D([P_\#] - [\overline{P}_\#]) = \gcd\{([F_\#] - [\overline{F}_\#])_{ai}, \quad 1 \leq i \leq n + 1\}, \]
guarantees that the one-parameter Nielsen
number.

In our case we have;

\[ D((\mathcal{F}_n) - (\mathcal{P}_n)) = |\text{det}(A)| = N(F), \]

where \( A \) is as in Theorem 1. In this case the affine homotopies “realize” the one-parameter Nielsen number.

In which case where \( N(F|_{\mathbb{T}^n}) \neq 0 \) the Proposition 11 guarantees that the one-parameter Nielsen number \( N(F) \) is zero. But in this case we have \( D((\mathcal{F}_n) - (\mathcal{P}_n)) \neq 0 \). This happens because arcs connecting \( \mathbb{T}^n \times \{0\} \) to \( \mathbb{T}^n \times \{1\} \) in \( \text{Fix}(F) \) will produce circles in \( \text{Fix}(\mathcal{F}) \).

II. Let \( M \) be a fiber bundle with base \( S^1 \) and fiber \( \mathbb{T}^n \). The total space \( M \) is given by

\[ M = M(h) = \frac{\mathbb{T}^n \times I}{(z, 0) \sim (h(z), 1)} \]

where \( h \) is a homeomorphism of \( \mathbb{T}^n \). For more details in the cases \( n = 1 \) or \( n = 2 \) see [5] and [7]. The projection map \( p : M(h) \rightarrow S^1 = I/0 \approx 1 \) is given by \( p(< z, t >) =< t >, \) where \( < z, t > \) denotes the class of \( (z, t) \) in \( M(h) \). Therefore each map \( f : M(h) \rightarrow M(h) \), over \( S^1 \), is given by

\[ f(< z, t >) =< F(z, t), t >, \]

where \( F : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n \) is a homotopy. The term over \( S^1 \) means \( p \circ f = p \). By the long exact sequence of the fibration \( p \) in homotopy we obtain;

\[ \pi_1(M(h), 0) \approx \pi_1(\mathbb{T}^n) \times \pi_1(S^1). \]

We denote the generators of \( \pi_1(M(h), 0) \) by \( u_1, \ldots, u_n, d \).

We are interested to use the Theorem 1 to compute the minimal path components of \( \text{Fix}(f) \). This phrases means, we want to find a map \( g \) fiberwise homotopic to \( f \) such that the path components in \( \text{Fix}(g) \) is minimal.

Given \( f : M(h) \rightarrow M(h) \) we will assume that \( N(f|_{\mathbb{T}^n}) = 0 \). Let us take \( G : \mathbb{T}^n \times I \rightarrow \mathbb{T}^n \) defined by

\[ G(x_1, \ldots, x_n, t) = \left( \sum_{j=1}^{n} b_{1j} x_j + c_1 t + \epsilon_1, \ldots, \sum_{j=1}^{n} b_{nj} x_j + c_n t + \epsilon_n, \right), \]

where \([\phi] = [(H|_{\mathbb{T}^n})_#] = (b_{ij}) = [(f|_{\mathbb{T}^n})_#] \) and \( c_i \) are given by \([f(< v, t >)] = u_1^{c_1} \cdots u_n^{c_n} \). Since \( N(f|_{\mathbb{T}^n}) = 0 \) we can suppose that \([\phi] \) has the same expression as in (2).

We consider \( H \) the set of all homeomorphism \( h \) of \( \mathbb{T}^n \) such that \( G(z, 0) = G(h(z), 1) \). In our application we consider \( M(h) \) with \( h \in H \). If \( h \in H \) then \( G \) induces a fiber map \( g : M(h) \rightarrow M(h) \) over \( S^1 \) defined by \( g(< z, t >) =< G(z, t), t >, \) for an example of this in case \( n = 2 \) see [15].

By construction we have \( g_# = f_# \). Since \( M(h) \) is \( K(\pi, 1) \) then \( g \) is homotopic to \( f \) by a homotopy over \( S^1 \). We can choose \( 0 \leq \epsilon_i < 1 \) such that \( G \) has no fixed points for \( t = 0, 1 \), which implies that \( g \) also has no fixed points for \( t = 0, 1 \). Therefore \( \text{Fix}(G) \approx \text{Fix}(F) \).

Note that \( G(z, 0) = G(z, 1) \) in \( \mathbb{T}^n \), where \( z = (x_1, \ldots, x_n) \). In this case each homotopy of \( G \) relative to \( \mathbb{T}^n \times \{0, 1\} \) is equivalent to a homotopy over \( S^1 \) of \( g \) relative to \( (v, [0]) \). Therefore to
minimize the path components of $Fix(g)$ by homotopies over $S^1$ relative to $(v, [0])$ is equivalent to minimize the path components of $Fix(G)$ by homotopies relative to $\mathbb{T}^n \times \{0, 1\}$.

The one-parameter Nielsen number $N(G)$ is a lower bound for the number of path components in $Fix(H)$ for each $H$ homotopic to $G$ relative to $\mathbb{T}^n \times \{0, 1\}$. Follows from Theorem 1 that the minimum number of path components, or the minimum number of circles, in $Fix(G)$ is given by $N(G) = |det(A)|$, where $A$ is as in (3), and therefore the minimum number of path components in $Fix(f)$ is $|det(A)|$.

In case $n = 2$ the number $|det(A)|$ appeared in the main theorem of [7] to say only when a fiber map $f : M(h) \to M(h)$ can be deformed or not to a fixed point free map over $S^1$. Here we give a complete description because we proved that $|det(A)|$ is the minimum number of path components in $Fix(f)$ up to deformations over $S^1$. Therefore $f$ can be deformed to a fixed point free map over $S^1$ if and only if $|det(A)| = 0$ in the case where $h \in H$.

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