Abstract

This paper is concerned with index pairs in the sense of Conley index theory for flows relative to pseudo-gradient vector fields for $C^1$-functions satisfying Palais-Smale condition. We prove a deformation theorem for such index pairs to obtain a Lusternik-Schnirelmann type result in Conley index theory.

Keywords: Critical point theory, index pair, Palais-Smale condition, pseudo-gradient vector field, relative Lusternik-Schnirelmann category.

Subject Classification: 37B30, 58E05

1 Introduction

Conley’s homotopy index has proved to be a useful tool in critical point theory. In most of the applications of critical point theory, we deal with a $C^1$-function on a complete Finsler manifold satisfying Palais-smale condition and we desire to estimate the number of critical points of such a function. There are two different viewpoints: the number of analytically distinct critical points which is discussed by Morse theory and the number of geometrically distinct critical points which is investigated by Lusternik-Schnirelmann theory [12]. The applications of Conley index in Morse theory has been studied in details [1, 2, 3, 7]. This paper is concerned with the applications in Lusternik-Schnirelmann theory. (See also [18, 19, 20, 21].) We consider the flow relative to a pseudo-gradient vector field for a $C^1$-function on a complete Finsler manifold satisfying Palais-Smale condition. Then we prove the deformation results for index pairs in the sense of Conley index theory.
for such a flow. Using a modification of the relative Lusternik-Schnirelmann category, we obtain a Lusternik-Schnirelmann type result in Conley index theory.

We need some basic results from Conley index theory which are presented in Section 2. Here we have extended the concept of regular index pair [25] for the noncompact case. Then in Section 3, we define the concept of relative Lusternik-Schnirelmann category with a little modification so that it can be applied to index pairs and yield sharper results in critical point theory. Finally in Section 4, we prove our result concerning the existence of critical points for a \( C^1 \)-function on a complete Finsler manifold.

2 Index Pair

Let \( \varphi^t \) be a continuous flow on a metric space \( X \). An isolated invariant set is a subset \( I \subset X \) which is the maximal invariant subset in a closed neighborhood of itself. Such a neighborhood is called an isolating neighborhood. In order to define the concept of index pair, we follow [1] and [22].

Given a closed pair \( (N, L) \) in \( X \), we define the induced semi-flow on \( N/L \) by

\[
\varphi^t I : N/L \to N/L, \quad \varphi^t I(x) = \begin{cases} 
\varphi^t(x) & \text{if } \varphi^{[0,t]}(x) \subset N - L \\
[L] & \text{otherwise.}
\end{cases}
\]

In [20, 22] it is proved that \( \varphi^t I : N/L \times \mathbb{R}^+ \to N/L \) is continuous if and only if

(i) \( L \) is positively invariant relative to \( N \), i.e.

\[
x \in L, t \geq 0, \varphi^{[0,t]}(x) \subset N \Rightarrow \varphi^{[0,t]}(x) \subset L.
\]

(ii) Every orbit which exits \( N \) goes through \( L \) first, i.e.

\[
x \in N, \varphi^{[0,\infty]}(x) \not\subset N \Rightarrow \exists t \geq 0 \text{ with } \varphi^{[0,t]}(x) \subset N, \varphi^t(x) \in L,
\]

or equivalently if \( x \in N - L \) then there is a \( t > 0 \) such that \( \varphi^{[0,t]}(x) \subset N \).

**Definition.** An index pair for an isolated invariant set \( I \subset X \) is a closed pair \( (N, L) \) in \( X \) such that \( N - L \) is an isolating neighborhood for \( I \) and the semi-flow \( \varphi^t I \) induced by \( \varphi^t \) is continuous.
For every subset $A \subset X$ and $T \in \mathbb{R}^+$, we define
$$G^T(A) = \{ x \in A | \varphi^{[-T,T]}(x) \subset A \},$$
$$\Gamma^T(A) = \{ x \in G^T(A) | \varphi^{[0,T]}(x) \cap \partial A \neq \emptyset \}.$$

Now suppose that $G^T(A) \subset \text{int}(A)$ for a closed subset $A \subset X$. Then $A$ is an isolating neighborhood for $I := \bigcap_{T > 0} G^T(A)$. In [1], Benci proved that $(G^T(A), \Gamma^T(A))$ is an index pair for $I$. This index pair has a special property, that is for every $x \in \Gamma^T(A)$, $\varphi^{[0,3T]}(x) \not\subset G^T(A)$. To see this, suppose that $\varphi^{[0,3T]}(x) \subset G^T(A)$, then $\varphi^{[T,2T]}(x) \subset G^{2T}(A) \subset \text{int}(G^T(A))$. So $\varphi^T(x) \not\in \Gamma^T(A)$ and by (i), $x \not\in \Gamma^T(A)$.

**Definition.** An index pair $(N, L)$ is called regular if the exit time map
$$\tau_+ : N \to [0, +\infty], \quad \tau_+(x) = \begin{cases} \sup \{ t | \varphi^{[0,t]}(x) \subset N - L \} & \text{if } x \in N - L, \\ 0 & \text{if } x \in L, \end{cases}$$
is continuous. For every regular index pair $(N, L)$, we define the induced semi-flow on $N$ by
$$\varphi^{t}_\flat : N \times \mathbb{R}^+ \to N, \quad \varphi^{t}_\flat(x) = \varphi^{\min\{t, \tau_+(x)\}}(x)$$

The reader is referred to [25] to see the details about regular index pairs and the proof of the following useful criterion.

**Proposition 2.1.** An index pair $(N, L)$ is regular provided that $\varphi^{[0,t]}(x) \not\subset N - L$ for every $x \in L$ and $t > 0$.

**Definition.** An index pair $(N, L)$ is called weakly regular if for every $x \in L$, there exists $t \in \mathbb{R}^+$ such that $\varphi^{t}(x) \not\in N - L$.

We showed that if $G^T(A) \subset \text{int}(A)$ for a closed subset $A \subset X$, then $(G^T(A), \Gamma^T(A))$ is a weakly regular index pair. Now let $(N, L)$ be an index pair and $V := N - L$. We define $I^+(V) = \{ x \in V | \varphi^{[0,\infty]}(x) \subset V \}$. It is not hard to see that $I^+(V)$ is closed subset of $V$. Notice that if $(N, L)$ is a weakly regular index pair, then $I^+(V) \cap L = \emptyset$. The following theorem asserts that there is a Lyapunov function for $\varphi^{t}_\flat$ on $N/L$ which separates $[L]$ and $I^+(V)$. If we consider the natural projection $\pi : N \to N/L$, then we obtain a Lyapunov function on $N$ which separates $L$ and $I^+(V)$. (See [23] for a similar result in the compact case.)
Theorem 2.2. Let \((N, L)\) be a weakly regular index pair and \(V = N - L\). Then there exists a continuous function \(g : N/L \to [0, 1]\) such that

(i) \(g^{-1}(0) = [L]\) and \(g^{-1}(1) = I^+(V)\).

(ii) If \(0 < g(x) < 1\) and \(t \in \mathbb{R}^+\), then \(g(\varphi_t^L(x)) < g(x)\).

Proof. Let \(\rho : N/L \to [0, 1]\) be a continuous function with \(\rho^{-1}(0) = [L]\) and \(\rho^{-1}(1) = I^+(V)\). We define \(f : N/L \to [0, 1]\) by \(f(x) = \sup_{t \geq 0} \rho(\varphi_t^L(x))\). Then \(f^{-1}(0) = [L]\), \(f^{-1}(1) = I^+(V)\) and \(f(\varphi_t^L(x)) \leq f(x)\) for every \(x \in N/L\). Moreover it is not hard to check that \(f\) is continuous in \(I^+(V)\). Now suppose that \(0 < f(x) < 1\) for some \(x \in N - L\). Then there is \(t_0 \in \mathbb{R}^+\) such that \(\varphi_{t_0}(x) \in L\). Since \((N, L)\) is a weakly regular index pair there is a \(t_1 \in \mathbb{R}^+\) such that \(\varphi_{t_0 + t_1} \not\in V\). Thus there is a neighborhood \(A\) of \(x\) such that \(\varphi_{t_0 + t_1}(A) \cap V = \emptyset\) which means that \(\varphi_{t_0 + t_1}(A) = [L]\). Therefore \(\sup_{t \geq 0} \rho(\varphi_t^L(y)) = \sup_{0 \leq t \leq t_0 + t_1} \rho(\varphi_t^L(y))\) for every \(x \in A\), hence \(f\) is continuous at \(x\). It remains to show the continuity of \(f\) at \([L]\). Let \(\varepsilon\) be a given positive number. For every \(x \in L\), there is a \(t \in \mathbb{R}^+\) such that \(\varphi^t(x) \not\in V\). Thus there exists a neighborhood \(U_x\) of \(x\) such that \(\rho(\varphi_{t, [0, t]}^L(U_x)) \not\in V\) and \(\varphi^t(U_x) \cap V = \emptyset\). If we set \(U = \bigcup_{x \in L} U_x\), then \(U\) is a neighborhood of \([L]\) in \(N/L\) with \(\rho(\varphi_{t, [0, +\infty]}^L(U)) \not\in V\), hence \(f|U < \varepsilon\). The above argument shows that \(f\) is continuous. Now it is not hard to check that \(g := \int_{0}^{+\infty} e^{-t} f(\varphi_t^L(x)) dt\) is the desired function \([25]\). \(\square\)

Corollary 2.3. Let \((N, L)\) be a weakly regular index pair for an isolated invariant set \(I\). Then there exists a subset \(L' \subset N\) such that \(L \subset L'\) and \((N, L')\) is a regular index pair for \(I\).

Proof. Let \(g : N/L \to [0, 1]\) be the Lyapunov function described above. Take \(L' := \pi^{-1}(\pi^{-1}(\varepsilon))\) where \(\pi : N \to N/L\) is the natural projection and \(\varepsilon \in (0, 1]\). Now by Proposition 2.1., \((N, L')\) is a regular index pair for \(I\). \(\square\)

3 Relative Category

The relative Lusternik-Schnirelmann category introduced by Fadell and Husseini \([1]\) has shown to give important information about the existence of critical points \([3, 10]\). For
every topological space $X$ and a closed subset $A \subset X$, the relative category $\text{cat} (X, A)$ is defined to be the minimum $n$ for which there exists an open cover $X = U_0 \cup \ldots \cup U_n$ such that $U_0$ can be retracted to $A$ and $U_i$ is contractible in $X$ for $1 \leq i \leq n$. Unfortunately this definition is not so efficient in Conley index theory. Indeed in this theory we deal with a pair in the form of $(N/L, [L])$ where $(N, L)$ is an index pair for a continuous flow. Now one can see that if $L$ is a large subset of $N$, then $\text{cat}(N/L, [L])$ is possibly a small number. Since the relative category is used to find a lower bound for the number of critical points, we can not obtain good estimates in critical point theory. (See [18, 19, 20].) For this reason, we use a little modification of the relative category which is more efficient in Conley index theory.

**Definition.** (Relative HLS category) Let $X$ be a topological space and $A \subset X$ be a closed subset. The relative Homotopy Lusternik-Schnirelmann category $\nu_H(X, A)$ is defined to be the minimum of $n$ for which there exists an open covering $X = U_0 \cup \ldots \cup U_n$ such that $A$ is a deformation retract of $U_0$ and $U_i$ is contractible in $X - A$ for $1 \leq i \leq n$.

It is easy to see that $\nu_H(X, A) \geq \text{cat}(X, A)$. Unfortunately the relative category is not easy to compute. It is more convenient to use cohomological category [14] to find lower bounds for it. (See [3, 13, 16, 23] and references therein for other tools.) Suppose that $H^*$ is a cohomology functor on $X$. A subset $S \subset X$ is called cohomologically trivial if the restriction map $H^k(X) \twoheadrightarrow H^k(S)$ is zero for every $k \in \mathbb{N}$. Similarly for $A \subset S \subset X$, we say that $(S, A)$ is cohomologically trivial in $(X, A)$ if the restriction map $H^k(X, A) \twoheadrightarrow H^k(S, A)$ is zero for every $k \in \mathbb{N}$.

**Definition.** (Relative CLS category) The relative Cohomology Lusternik-Schnirelmann category $\nu_C(X, A)$ is defined to be the minimum $n$ for which there exists an open covering $U_0, \ldots, U_n$ such that $A \subset U_0$ and $(U_0, A)$ is cohomologically trivial in $(X, A)$ and $U_i \subset X - A$ is cohomologically trivial in $X - A$ for $1 \leq i \leq n$. If such a covering does not exist, we set $\nu_C(X, A) = +\infty$. When $A = \emptyset$, then $\nu_C(X) := \nu_C(X, \emptyset)$ is called CLS category.

Recall that the cuplength of a topological space $X$ is defined to be the minimum integer $N > 0$ such that for any set of cohomology classes $\alpha_j \in H^{k_j}(M)$, $j = 1, \ldots, N$ of degree $k_j \geq 1$, the class $(\alpha_1 \cup \cdots \cup \alpha_N)|_A = j^*(\alpha_1 \cup \cdots \cup \alpha_N)$ is zero. It is well-known that $\nu_C(X) \geq \text{cuplength}(X)$. (cf. [14].) Now we introduce the concept of relative cuplength

5
which gives a lower bound for the relative CLS category.

**Definition.** (Relative Cuplength) For $A \subset X$, we define the relative cuplength $CL(X, A)$ to be the minimum integer $N > 0$ such that for any set of cohomology classes $\alpha_j \in H^{k_j}(X)$, $j = 1, \cdots, N$ and $\alpha_0 \in H^{k_0}(X, A)$ of degree $k_j > 0$, the class $(\alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_N)$ is zero in $H^*(X, A)$.

**Proposition 3.1.** $\nu_C(X, A) \geq CL(X, A)$.

**Proof.** Let $U$ and $V$ be open subsets of $X$ with $A \subset V$. Suppose that $\alpha \in H^*(X, A)$ and $\beta \in H^*(X)$ such that $\alpha|_V = 0$ in $H^*(V, A)$ and $\beta|_U = 0$ in $H^*(U)$. Since $U$ and $V$ are excisive subsets of $X$, the cup product map $H^*(X, V) \otimes H^*(X, U) \rightarrow H^*(X, V \cup U)$ is defined and the following diagram is commutative with exact columns:

\[
\begin{array}{ccc}
H^*(X, V) & \otimes & H^*(X, U) \\
\downarrow & & \downarrow \\
H^*(X, A) & \otimes & H^*(X) \\
\downarrow & & \downarrow \\
H^*(V, A) & \otimes & H^*(U) & H^*(V \cup U, A)
\end{array}
\]

A diagram chase shows that $(\alpha \cup \beta) = 0$ in $H^*(V \cup U, A)$. Now the proposition is obvious by induction. □

**Example 3.2.** Let $\pi : E \rightarrow B$ be an $m$-dimensional orientable vector bundle over a metric space $B$. Then $E$ admits a Euclidean metric and we may consider the disk bundle $D(E)$ and sphere bundle $S(E)$. Using the Thom isomorphism, we conclude that $CL(D(E)/S(E), [S(E)]) = CL(D(E), S(E)) = cuplength(M)$. (See [13, 14] for the details.)

4 Critical Point Theory

Let $X$ be a complete Finsler manifold and $f \in C^1(X, \mathbb{R})$. In [1, 17], it has been shown that $f$ admits a pseudo-gradient vector field i.e. a map $Y : X \rightarrow T(X)$ such that
(i) The equation $\dot{x} = Y(x)$ has a unique solution for every initial point $x_0 \in X$,

(ii) $Y.f := \langle Df(x), Y(x) \rangle \geq \alpha(\|Df(x)\|)$ where $\alpha$ is a strictly increasing continuous function with $\alpha(0) = 0$,

(iii) $\|Y\|$ is bounded.

Therefore we can consider the flow relative to a pseudo-gradient vector field for $f$. From now on, suppose that $\varphi^t$ is the flow relative to the pseudo-gradient vector field $Y$ for $f \in C^1(X, R)$ and $(N, L)$ is an index pair for $\varphi^t$ such that $f$ is bounded on $V := \overline{N - L}$. Moreover we assume that $f$ satisfies Palais-Smale condition in $V$. (See also [24].)

**Definition.** We say that $f$ satisfies Palais-Smale condition in $V$ if any sequence $\{x_n\} \subset V$ such that $f(x_n)$ is bounded and $\|Df(x_n)\| \to 0$ possesses a convergent subsequence.

The following lemma shows that under the above assumptions, $(N, L)$ is a weakly regular index pair. Therefore we can use Corollary 2.3. to show the existence of a regular index pair in this situation.

**Lemma 4.1.** There exists $T \in \mathbb{R}^+$ such that for every $x \in L$, $\varphi^{[0, T]}(x) \not\subset V$.

**Proof.** Suppose the contrary, then there is a sequence $x_i \in L$ and $t_i \in \mathbb{R}^+$ such that $t_i \to +\infty$ and $\varphi^{[0, t_i]}(x_i) \subset V$. Since $L$ is positively invariant relative to $N$, we get $\varphi^{[0, t_i]}(x_i) \subset L$. Now $L \cap V$ is a closed set which does not contain any critical point of $f$. Since $f$ satisfies Palais-Smale condition in $V$, there is $\delta > 0$ such that $Y.f(x) > \delta$ for every $x \in L \cap V$. Therefore $f(\varphi^{t_i}(x_i)) - f(x_i) > \varepsilon t_i$. Since $t_i \to +\infty$, it follows that $f(\varphi^{t_i}(x_i)) \to +\infty$ which contradicts the boundedness of $f$ on $V$. □

**Lemma 4.2.** For large values of $t \in \mathbb{R}^+$, $(\varphi^t_2)^{-1}([L])$ is a neighborhood of $[L]$ in $N/L$.

**Proof.** By the above lemma, there exists $T \in \mathbb{R}^+$ such that $\varphi^{[0, T]}(x) \not\subset V$ for every $x \in L$. Thus there is a neighborhood $U_x$ of $x$ such that $\varphi^{[0, T]}(y) \not\subset V$ for every $y \in U_x$. Now $\bigcup_{x \in L} U_x$ is a neighborhood of $L$, hence it defines a neighborhood $U$ of $[L]$ with $\varphi^T_2(U) = [L]$. Therefore $(\varphi^t_2)^{-1}([L])$ is a neighborhood of $[L]$ in $N/L$ for every $t \geq T$. □
For every \( a \in \mathbb{R} \), we define \((N/L)^a = [L] \cup \{x \in N-L|f(x) \geq a\}\). The following two lemmas are reformulation of Deformation Theorems for index pairs.

**Lemma 4.3.** (First Deformation Theorem) Suppose that \( f \) has no critical points in \((N - L) \cap f^{-1}[a,b]\) for some \( a < b \). Then there exists \( T \in \mathbb{R}^+ \) such that \((N/L)^a \subset (\varphi_T^{-1})^-((N/L)^b)\).

**Proof.** Since \( f \) has no critical points in the closed subset \( V \cap f^{-1}[a,b]\) and \( f \) satisfies Palais-Smale condition in \( V \), there is a \( \delta > 0 \) such that \( Y.f > \delta \) in \( V \cap f^{-1}[a,b]\). Now for every \( T \geq \frac{b-a}{\delta} \) and \( x \in V \cap f^{-1}[a,b]\), we have \( f(\varphi_T(x)) \geq a + \delta T \geq b \). \( \square \)

**Lemma 4.4.** (Second Deformation Theorem) Suppose that \( f \) has exactly \( m \) critical points \( y_1, \ldots, y_m \) in \((N - L) \cap f^{-1}[c - \varepsilon_0, c + \varepsilon_0]\) and \( f(y_i) = c \) for some \( c \in \mathbb{R} \) and \( \varepsilon_0 > 0 \). If \( U_i \) is a neighborhood of \( x_i \) in \( N - L \) and \( T \in \mathbb{R}^+ \), then there is an \( \varepsilon \in [0,\varepsilon_0] \) such that \((N/L)^{c-\varepsilon} \subset (\varphi_T^{m})^{-1}((N/L)^{c+\varepsilon}) \cup \bigcup_{i=1}^{m} U_i\).

**Proof.** Since \( y_i \) is a rest point for \( \varphi^t \), there is an open set \( V_i \subset U_i \) such that \( \varphi^{[0,T]}(x) \cap V_i = \emptyset \) for every \( x \notin U_i \). Since \( f \) has no critical point in \( V \cap f^{-1}[c - \varepsilon_0, c + \varepsilon_0] - \bigcup_{i=1}^{m} V_i \) which is a closed subset of \( V \) and \( f \) satisfies Palais-Smale condition in \( V \), there exists \( \delta > 0 \) such that \( Y.f > \delta \) in \( V \cap f^{-1}[c - \varepsilon_0, c + \varepsilon_0] - \bigcup_{i=1}^{m} V_i \). If we set \( \varepsilon = \frac{T\delta}{2} \), then for every \( x \in V \cap f^{-1}[c - \varepsilon, c + \varepsilon] - \bigcup_{i=1}^{m} U_i \), we have \( f(\varphi^{T}(x)) \geq f(x) + T\delta \geq c - \varepsilon + 2\varepsilon = c + \varepsilon \). \( \square \)

**Proposition 4.5.** Suppose that \( f \) has a finite number of critical points \( x_1, \ldots, x_n \) in \( V \) and \( U_i \) is a neighborhood of \( x_i \) in \( N - L \) for \( 1 \leq i \leq n \). Then there are \( t_0, t_1, \ldots, t_n \in \mathbb{R}^+ \) such that \( N/L = (\varphi_{t_0}^{-1})([L]) \cup \bigcup_{i=1}^{n} (\varphi_{t_i}^{-1})(U_i) \). In particular if \( (N,L) \) is a regular index pair, then \( N = (\varphi_{t_0}^{-1})(L) \cup \bigcup_{i=1}^{n} (\varphi_{t_i}^{-1})(U_i) \).

**Proof.** We use induction on the number of critical values of \( f|_V \). For \( k = 0 \), \( f \) has no critical points in \( V \). Since \( f \) is bounded on \( V \), there are \( a < b \) such that \((N/L)^b = [L]\) and \((N/L)^a = N/L\). Thus by Lemma 4.3., \((\varphi_T^{-1})([L]) = N/L\) for some \( T \in \mathbb{R}^+ \). Now suppose that \( f \) has \( k + 1 \) critical values \( c_0 < c_1 < c_2 \ldots < c_k \) and \( x_{\ell}, \ldots, x_n \) are critical points with \( f(x_i) = c_k \) for \( \ell \leq i \leq n \). If we use Lemma 4.4. for \( \varepsilon_0 = \frac{Tc_k-c_{k-1}}{2} \) and \( T = 1 \), we obtain an \( \varepsilon \in [0,\varepsilon_0] \) such that \((N/L)^{c-\varepsilon} \subset (\varphi_T^{-1})(N/L)^{c+\varepsilon}) \cup \bigcup_{i=1}^{n} U_i \). Moreover by Lemma 4.3., there is a \( T \in \mathbb{R}^+ \) such that \((N/L)^{c-\varepsilon} \subset (\varphi_T^{-1})([L]) \). Therefore we have \((N/L)^{c-\varepsilon} \subset (\varphi_T^{-1})([L]) \cup \bigcup_{i=1}^{n} U_i \). Now if we set \( L_1 = N \cap f^{-1}[c - \varepsilon, +\infty) \), then
$(N, L_1)$ is an index pair with $k$ critical values, hence there are $t'_0, t'_1, \ldots, t'_{\ell-1}$ such that $N/L_1 = (\varphi^{t_0}_t)^{-1}([L_1]) \cup \bigcup_{i=1}^{\ell-1} (\varphi^{t_i}_t)^{-1}(U_i)$. Now we set $t_0 = t'_0 + T + 1, t_i = t'_i$ for $1 \leq i \leq \ell - 1$ and $t_i = t'_0$ for $\ell \leq i \leq n$. Then it is not hard to check that

$$N/L = (\varphi^{t_0}_t)^{-1}([L]) \cup \bigcup_{i=1}^{n} (\varphi^{t_i}_t)^{-1}(U_i).$$

□

**Theorem 4.6.** $f$ has at least $\nu_H(N/L, [L])$ critical points in $V$. Moreover if $(N, L)$ is a regular index pair, then $f$ has at least $\nu_H(N, L)$ critical points in $V$.

**Proof.** We may assume that $f$ has a finite number of critical points $x_1, \ldots, x_n$ in $V$. Since $X$ is a Banach manifold, we may choose a contractible open set $U_i \subset N - L$ which contains $x_i$ for $1 \leq i \leq n$. Now by the above proposition, there are $t_0, t_1, \ldots, t_n \in \mathbb{R}^+$ such that $N/L = (\varphi^{t_0}_t)^{-1}([L]) \cup \bigcup_{i=1}^{n} (\varphi^{t_i}_t)^{-1}(U_i)$. It is easy to see that each $(\varphi^{t_i}_t)^{-1}(U_i)$ is contractible in $N - L$. The only problem is that $(\varphi^{t_0}_t)^{-1}([L])$ is not an open set. By Lemma 4.2, there is $T \in \mathbb{R}^+$ such that $int((\varphi^{T}_t)^{-1}([L])) \neq \emptyset$. Now we replace $(\varphi^{t_0}_t)^{-1}([L])$ by $int((\varphi^{t_0+T}_t)^{-1}([L]))$ which is obviously invariant under $\varphi^{t}_t$ and contractible to $[L]$. Therefore we obtain an open covering which follows that $\nu_H(N/L, [L]) \leq n$. A similar argument for $\varphi^{t}$ shows that if $(N, L)$ is a regular index pair, then $\nu_H(N, L) \leq n$. □

**Example 4.7.** Let $E$ be an orientable vector bundle over a Finsler manifold $X$ and $B$ be a closed subset of $X$. We denote the disk and sphere bundles of $E|_B$ by $D$ and $S$ respectively. Let $f : E \to \mathbb{R}$ be a $C^1$ function satisfying Palai-Smale condition in $B$ and $\varphi^t$ be the flow relative to a pseudo-gradient vector field for $f$ such that $(D, S)$ is an index pair for $\varphi^t$. Then by Example 2.3. and the above theorem, $f$ has at least $\nu_H(D, S) \geq \text{cuplength}(B)$ critical points in $D$. If we consider the case $X = M \times \mathbb{R}^n$, $E = X \times \mathbb{R}^m$ and $B = M \times D^n \subset M \times \mathbb{R}^n$, then we obtain a noncompact version of a well-known result of [3].

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