The translational $\mu$-\(\tau\) reflection symmetry of Majorana neutrinos

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Abstract

The present neutrino oscillation data allow \(m_1 = 0\) (or \(m_3 = 0\)) for the neutrino mass spectrum and support \(\theta_{23} \simeq \pi/4\) and \(\delta \simeq -\pi/2\) as two good approximations for the PMNS lepton flavor mixing matrix \(U\). We show that these intriguing possibilities can be a very natural consequence of the translational \(\mu\)-\(\tau\) reflection symmetry — the effective Majorana neutrino mass term keeps invariant under the transformations

$$
\nu_{eL} \to (\nu_{eL})^c + U_{ei}^* z_\nu^c, \quad \nu_{\mu L} \to (\nu_{\tau L})^c + U_{\tau i}^* z_\nu^c \quad \text{and} \quad \nu_{\tau L} \to (\nu_{\mu L})^c + U_{\mu i}^* z_\nu^c \quad \text{(for} \ i = 1 \ \text{or} \ 3),
$$

where \(z_\nu^c\) is the charge conjugation of a constant spinor field \(z_\nu\). Extending such a working flavor symmetry to the canonical seesaw mechanism at a superhigh-energy scale, we calculate its soft breaking effects at the electroweak scale by using the one-loop renormalization-group equations.

Keywords: Majorana neutrinos, flavor mixing and CP violation, translational \(\mu\)-\(\tau\) reflection symmetry, minimal seesaw mechanism, renormalization-group equations

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1 Introduction

The facts that the three known (active) neutrinos have nonzero but tiny masses and lepton flavors are significantly mixed \[1\] tell us that the standard model (SM) of particle physics is incomplete at least in its flavor aspects, although this great theory of electromagnetic and weak interactions has proved to be a huge success in describing how the Universe works. A particular characteristic of massive neutrinos is the flavor oscillation — a pure quantum phenomenon, as their flavor eigenstates $\nu_\alpha$ (for $\alpha = e, \mu, \tau$) are the quantum superpositions of their mass eigenstates $\nu_i$ (for $i = 1, 2, 3$):

$$
\begin{pmatrix}

\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix}_{L} = 
\begin{pmatrix}

U_{e1} & U_{e2} & U_{e3} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} \\
U_{\tau1} & U_{\tau2} & U_{\tau3}
\end{pmatrix}
\begin{pmatrix}

\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix}_{L},
$$

(1)

where $U_{\alpha i}$ stand for the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton flavor mixing matrix elements $[2–4]$. A global analysis of current neutrino oscillation data in the three-flavor scheme leaves us with the following information on neutrino masses, lepton flavor mixing and CP violation $[5,6]$:  

- The mass ordering of three neutrinos can be either normal (namely, $m_1 < m_2 < m_3$) or inverted (namely, $m_3 < m_1 < m_2$), but the latter is slightly disfavored as compared with the former. Now that the absolute neutrino mass scale has been undetermined, the possibility of $m_1 = 0$ or $m_3 = 0$ is actually allowed as an extreme but intriguing case of the neutrino mass spectrum.

- The $3 \times 3$ PMNS matrix $U$ exhibits an approximate but suggestive $\mu$-$\tau$ interchange symmetry $|U_{\mu i}| \simeq |U_{\tau i}|$ (for $i = 1, 2, 3$), which is rather different from the observed pattern of the Cabibbo-Kobayashi-Maskawa (CKM) quark flavor mixing matrix $[7,8]$. This observation, together with the preliminary experimental evidence for significant CP violation in neutrino oscillations $[9,10]$, implies that $\theta_{23} = \pi/4$ and $\delta = -\pi/2$ may naturally hold in the standard parametrization of $U$ at a given energy scale.

It is well known that making use of some proper discrete symmetry groups (such as $A_4$ and $S_4$) to describe the flavors of charged leptons and massive neutrinos constitutes the most popular way to achieve $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ $[11–17]$. In particular, the minimal or residual flavor symmetry of this kind is expected to be the $\mu$-$\tau$ reflection symmetry in the neutrino sector $[18]$. A very natural way of obtaining $m_1 = 0$ or $m_3 = 0$ is to invoke the minimal type-I seesaw model in which two right-handed neutrino fields are introduced and lepton number violation is required $[19–21]$. The so-called Friedberg-Lee symmetry $[22–27]$, which demands the effective neutrino mass term to be invariant under the translational transformations $\nu_\alpha \rightarrow \nu_\alpha + \eta_\alpha z_\nu$ with $z_\nu$ being a constant spinor field and $\eta_\alpha$ denoting a flavor-dependent complex number (for $\alpha = e, \mu, \tau$), is also an interesting option because it
can not only predict one of the neutrinos to be massless but also help constrain the flavor texture of massive neutrinos if the values of $\eta_\alpha$ are explicitly assumed.

So far some attempts have been made to build a simple but viable neutrino mass model by combining the $\mu$-$\tau$ reflection symmetry with either the minimal seesaw scenario [28–40] or a generalized version of the Friedberg-Lee symmetry [41–47], in order to simultaneously obtain $m_1 = 0$ (or $m_3 = 0$), $\theta_{23} = \pi/4$ and $\delta = \pm \pi/2$. Along the latter line of thought, the present paper is intended to go further and show that such three intriguing predictions can naturally result from a translational $\mu$-$\tau$ reflection symmetry in the neutrino sector; namely, the effective Majorana neutrino mass term keeps invariant under the following transformations of three left-handed neutrino fields:

\[
\begin{align*}
\nu_{eL} &\to (\nu_{eL})^c + U^*_{ei} z^c_\nu, \\
\nu_{\mu L} &\to (\nu_{eL})^c + U^*_{\tau\nu} z^c_\nu, \\
\nu_{\tau L} &\to (\nu_{\mu L})^c + U^*_{\mu\nu} z^c_\nu,
\end{align*}
\]

where $U_{\alpha i}$ denote the elements of the 3 $\times$ 3 PMNS flavor mixing matrix $U$ in its first or third column (i.e., $i = 1$ or 3), and $z^c_\nu$ is the charge conjugation of a constant spinor field $z_\nu$ which anticommutes with the neutrino fields [47–49].

As first pointed out in Ref. [47], a massless Majorana neutrino field $\nu_i$ and its charge-conjugated counterpart $\nu^c_i = \nu_i$ satisfy the same Dirac equation $i\gamma^\mu \partial_\mu \nu_i = 0$ which is invariant under the translational transformation $\nu_i \to \nu_i + z_\nu$ or $\nu^c_i \to \nu^c_i + z^c_\nu$ in free space. Such a transformation of $\nu_i$ is equivalent to $\nu_{\alpha L} \to \nu_{\alpha L} + U_{\alpha i} z_\nu$ or $\nu_{\alpha L} \to (\nu_{\alpha L})^c + U^*_{\alpha i} z^c_\nu$ in the flavor space as a consequence of lepton flavor mixing. This novel observation clearly explains why the flavor-dependent coefficients $\eta_\alpha$ of $z_\nu$ in the original Friedberg-Lee transformation or its generalized form are not really arbitrary but can be uniquely identified as the PMNS matrix elements $U_{\alpha i}$, and thus it naturally motivates us to conjecture the translational $\mu$-$\tau$ reflection transformations of $\nu_{\alpha L}$ in Eq. (2). In this respect our conjecture is certainly new as compared with those made before (see, e.g., Refs. [50–52]) and may provide a new angle of view to understand the flavor issues of Majorana neutrinos. Moreover, we shall extend such a simple working flavor symmetry to the canonical seesaw framework [53–58] so as to naturally arrive at a constrained version of the minimal seesaw model.

The remaining parts of this paper are organized as follows. In section 2, we are going to show that $m_i = 0$ (for $i = 1$ or 3) will definitely hold if the effective Majorana neutrino mass term is invariant under the transformations made in Eq. (2). Section 3 is devoted to extending such a translational $\mu$-$\tau$ reflection symmetry to the right-handed neutrino sector in the well-known canonical seesaw mechanism at a superhigh-energy scale, and section 4 is intended to calculate its soft breaking effects on three neutrino masses, three flavor mixing angles and two CP-violating phases at the electroweak scale. A brief summary and some concluding remarks will be made in section 5.
2 The symmetry and its consequences

Regardless of how the tiny masses of three Majorana neutrinos are generated, here let us simply focus on their effective mass term $\mathcal{L}_{\text{mass}}$ at a given energy scale:

$$-\mathcal{L}_{\text{mass}} = \frac{1}{2} \overline{\nu}_L M_\nu (\nu_L)^c + \text{h.c.}, \quad (3)$$

where $\nu_L = (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^T$ denotes the column vector of three left-handed neutrino fields, $(\nu_{\alpha L})^c \equiv C \nu_{\alpha L}^T$ stands for the charge conjugate of $\nu_{\alpha L}$ (for $\alpha = e, \mu, \tau$) with $C$ being the charge conjugation operator which satisfies $C \gamma_\mu C^{-1} = -\gamma_\mu$, $C \gamma_5 C^{-1} = \gamma_5$ and $C^{-1} = C^\dagger = C^T = -C$, and $M_\nu$ is the $3 \times 3$ symmetric Majorana neutrino mass matrix. If the traditional CP transformations are made for the left-handed neutrino fields, namely $\nu_\alpha L \rightarrow (\nu_\alpha L)^c$ (for $\alpha = e, \mu, \tau$) as shown in Fig. 1, then the invariance of $\mathcal{L}_{\text{mass}}$ requires $M_\nu = M_\nu^*$, implying that this effective Majorana neutrino mass term must be CP-conserving.

Now let us introduce a real orthogonal $(\mu, \tau)$-associated permutation matrix, or equivalently one of the six elements of the non-Abelian $S_3$ group, of the form

$$\mathcal{P} = \mathcal{P}^T = \mathcal{P}^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (4)$$

With the help of this notation, we may rewrite the translational $\mu$-$\tau$ reflection transformations proposed in Eq. (2) as follows:

$$\nu_L \rightarrow \mathcal{P} [ (\nu_L)^c + \xi_i^* z_i^c ], \quad (5)$$

together with the charge-conjugated transformation $(\nu_L)^c \rightarrow \mathcal{P} (\nu_L + \xi_i z_i)$, where $\xi_i$ is a column vector of the PMNS lepton flavor mixing matrix $U$ defined as $\xi_i \equiv (U_{ei}, U_{\mu i}, U_{\tau i})^T$ corresponding to $\nu_i$ (for $i = 1, 2, 3$). Figure 1 provides us with a clear comparison between the traditional CP transformations and the $\mu$-$\tau$-interchanging CP transformations for three left-handed neutrino fields. Under the latter transformations the effective Majorana neutrino mass term in Eq. (3) changes to

$$-\mathcal{L}'_{\text{mass}} = \frac{1}{2} \overline{\nu}_L (\mathcal{P} M_\nu^* \mathcal{P}) (\nu_L)^c + \overline{\nu}_L (\mathcal{P} M_\nu^* \mathcal{P}) \xi_i^* z_i^c + \overline{\nu}_L \xi_i^\dagger (\mathcal{P} M_\nu^* \mathcal{P}) (\nu_L)^c$$

$$+ \overline{\nu}_L \xi_i^\dagger (\mathcal{P} M_\nu^* \mathcal{P}) \xi_i^* z_i^c + \text{h.c.}. \quad (6)$$

The necessary and sufficient conditions for $\mathcal{L}'_{\text{mass}} = \mathcal{L}_{\text{mass}}$ turn out to be the relation

$$M_\nu = \mathcal{P} M_\nu^* \mathcal{P}, \quad (7)$$

together with three dependent relations

$$M_\nu \xi_i^* = 0, \quad \xi_i^\dagger M_\nu = 0^T, \quad \xi_i^\dagger M_\nu \xi_i^* = 0, \quad (8)$$
Figure 1: A straightforward comparison between the traditional CP transformation and the \( \mu-\tau \)-interchanging CP transformation for three left-handed neutrino fields.

where \( \mathbf{0} \) denotes the zero column vector. The condition in Eq. (7) is a natural consequence of the \( \mu-\tau \) reflection symmetry and provides a quite strong constraint on the flavor texture of \( M_\nu \); and those in Eq. (8) are apparently required by the translational symmetry of \( \mathcal{L}_{\text{mass}} \) and should simply lead us to the result \( m_i = 0 \).

To prove the last point, let us denote the elements of \( M_\nu \) as \( \langle m \rangle_{\alpha\beta} \) (for \( \alpha, \beta = e, \mu, \tau \)) and express them in terms of the neutrino masses \( m_i \) and the PMNS matrix elements \( U_{\alpha i} \) and \( U_{\beta i} \) (for \( i = 1, 2, 3 \)) in the basis where the mass eigenstates of three charged leptons are identified with their flavor eigenstates. Namely, \( M_\nu = UD_\nu U^T \) with \( D_\nu \equiv \text{Diag} \{ m_1, m_2, m_3 \} \) holds in this flavor basis, and therefore

\[
(M_\nu)_{\alpha\beta} \equiv \langle m \rangle_{\alpha\beta} = \sum_{i=1}^{3} (m_i U_{\alpha i} U_{\beta i}) .
\]  

(9)

With the help of the unitarity of \( U \), we combine Eq. (8) with Eq. (9) and then obtain

\[
\sum_{\beta} \langle m \rangle_{\alpha\beta} U_{\beta i}^* = m_i U_{\alpha i} = 0 ,
\]

\[
\sum_{\alpha} \sum_{\beta} U_{\alpha i}^* \langle m \rangle_{\alpha\beta} U_{\beta i}^* = m_i = 0 .
\]  

(10)

So \( m_i = 0 \) is proved to be another consequence of the translational \( \mu-\tau \) reflection symmetry of \( \mathcal{L}_{\text{mass}} \). Now that \( m_2 > m_1 \) has been fixed \(^1\), we are left with the possibility of either \( m_1 = 0 \) (normal ordering) or \( m_3 = 0 \) (inverted ordering) for the neutrino mass spectrum. Combining \( m_1 = 0 \) or \( m_3 = 0 \) with the available data on the neutrino mass-squared differences \( \Delta m_{ij}^2 \equiv m_i^2 - m_j^2 \), one may immediately determine the three neutrino masses. To be more explicit, we have the special normal neutrino mass hierarchy

\[
m_1 = 0 ,
m_2 = \sqrt{\Delta m_{21}^2} = 8.61^{+0.12}_{-0.11} \times 10^{-3} \text{ eV} ,
m_3 = \sqrt{\Delta m_{31}^2} = 5.01^{+0.03}_{-0.03} \times 10^{-2} \text{ eV} ,
\]  

(11)

\(^1\)In Eq. (8) the second condition is actually a transpose of the first condition, and thus they are equivalent.
where $\Delta m^2_{21} = 7.42^{+0.21}_{-0.20} \times 10^{-5}$ eV$^2$ and $\Delta m^2_{31} = 2.510^{+0.027}_{-0.027} \times 10^{-3}$ eV$^2$ as the best-fit values plus the $\pm 1\sigma$ ranges have been input; or the special inverted neutrino mass hierarchy

$$m_1 = \sqrt{|\Delta m^2_{31}|} = 4.92^{+0.03}_{-0.03} \times 10^{-2} \text{ eV} ,$$
$$m_2 = \sqrt{|\Delta m^2_{32}|} = 4.99^{+0.03}_{-0.03} \times 10^{-2} \text{ eV} ,$$
$$m_3 = 0 ,$$

(12)

where $\Delta m^2_{21} = 7.42^{+0.21}_{-0.20} \times 10^{-5}$ eV$^2$ and $\Delta m^2_{32} = -2.490^{+0.026}_{-0.028} \times 10^{-3}$ eV$^2$ have been used for a numerical illustration. The next-generation neutrino oscillation experiments are going to determine which neutrino mass ordering is really true, and the next-generation precision measurements of the cosmic microwave background anisotropies and large scale structures may hopefully help to probe the absolute neutrino mass scale and examine whether $m_1$ or $m_3$ is essentially vanishing or vanishingly small.

No matter whether $m_1 = 0$ or $m_3 = 0$ holds, we are left with only two nontrivial CP-violating phases $\delta$ and $\sigma$ in the PMNS matrix $U$:

$$U = P_\nu \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta} & -c_{13}s_{23} \\
    s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta} & c_{13}c_{23}
\end{pmatrix},$$

(13)

where $P_\nu = \text{Diag} \{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\}$ has no physical effects, but $P_\nu = \text{Diag} \{1, e^{i\sigma}, 1\}$ with $\sigma$ being the Majorana CP phase is sensitive to lepton number violation. Substituting Eq. (13) into Eq. (9) and then taking account of

$$\langle m \rangle_{ee} = \langle m \rangle^*_{e\mu} , \quad \langle m \rangle_{\mu\tau} = \langle m \rangle^*_{\mu\tau} , \quad \langle m \rangle_{e\tau} = \langle m \rangle^*_{e\mu} , \quad \langle m \rangle_{\tau\tau} = \langle m \rangle^*_{\mu\mu},$$

(14)

derived from Eq. (7), we simply arrive at

$$\theta_{23} = \frac{\pi}{4} , \quad \delta = \pm \frac{\pi}{2} , \quad \sigma = 0 \text{ or } \frac{\pi}{2}$$

(15)

as a natural consequence of the $\mu$-$\tau$ reflection symmetry. No matter whether $m_1 = 0$ or $m_3 = 0$ holds, together with the conditions $\phi_e = 0$ or $\pm \pi$ and $\phi_\mu + \phi_\tau = 0$ or $\pi$. The preliminary T2K measurement favors the possibility of $\delta = -\pi/2$. At present the measurement of a neutrinoless double-beta ($0\nu2\beta$) decay mode, whose half-life is sensitive to the magnitude of $\langle m \rangle_{ee}$, is the only experimentally feasible way to probe the Majorana nature of massive neutrinos and constrain the corresponding phase parameters. In the above translational $\mu$-$\tau$ reflection symmetry limit, we have

$$|\langle m \rangle_{ee}|_{m_1=0} = m_2s^2_{12}c^2_{13} \mp m_3s^2_{13} \simeq \begin{cases} 
    1.45 \times 10^{-3} \text{ eV} & (\sigma = 0) , \\
    3.67 \times 10^{-3} \text{ eV} & (\sigma = \pi/2) ;
\end{cases}$$

$$|\langle m \rangle_{ee}|_{m_3=0} = c^2_{13} (m_1c^2_{12} \pm m_2s^2_{12}) \simeq \begin{cases} 
    4.83 \times 10^{-2} \text{ eV} & (\sigma = 0) , \\
    1.91 \times 10^{-2} \text{ eV} & (\sigma = \pi/2) ,
\end{cases}$$

(16)
where the best-fit values \( s_{12}^2 = 0.304 \) and \( s_{13}^2 = 0.022 \) \cite{5,6} together with the numerical results of \( m_i \) (for \( i = 1, 2, 3 \)) obtained in Eqs. (11) and (12) have been used as the typical inputs. Eq. (16) may give one a ball-park but promising feeling that the translational \( \mu-\tau \) reflection symmetry is so powerful that it can really lead us to a definite prediction for the effective electron-neutrino mass of the 0\( \nu \)2\( \beta \) decays. In this special case an experimental measurement of \( |\langle m \rangle_{ee}| \) will allow us not only to determine the neutrino mass ordering but also to pin down the Majorana CP-violating phase \( \sigma \).

As a by-product, the other three independent matrix elements of \( M_\nu \) in the \( \mu-\tau \) reflection symmetry limit can similarly be determined. We have the results as follows:

\[
|\langle m \rangle_{\mu\mu}|_{m_1=0} = \frac{1}{2} \sqrt{\left| m_2 \left( c_{12}^2 - s_{12}^2 s_{13}^2 \right) \pm m_3 c_{13}^2 \right|^2 + 4 m_2^2 c_{12}^2 s_{12}^2 s_{13}^2}
\]

\[
\simeq \begin{cases} 
2.75 \times 10^{-2} \text{ eV} & (\sigma = 0), \\
2.15 \times 10^{-2} \text{ eV} & (\sigma = \pi/2);
\end{cases}
\]

\[
|\langle m \rangle_{\mu\mu}|_{m_3=0} = \frac{1}{2} \sqrt{\left| m_1 \left( s_{12}^2 - c_{12} s_{13}^2 \right) \pm m_2 \left( c_{12}^2 - s_{12}^2 s_{13}^2 \right) \right|^2 + 4 (m_1 \mp m_2)^2 c_{12}^2 s_{12}^2 s_{13}^2}
\]

\[
\simeq \begin{cases} 
2.43 \times 10^{-2} \text{ eV} & (\sigma = 0), \\
1.22 \times 10^{-2} \text{ eV} & (\sigma = \pi/2);
\end{cases}
\]

and

\[
|\langle m \rangle_{e\mu}|_{m_1=0} = \frac{c_{13}}{\sqrt{2}} \sqrt{m_2^2 c_{12}^2 s_{12}^2 + (m_2 s_{12} \pm m_3)^2 s_{13}^2}
\]

\[
\simeq \begin{cases} 
6.13 \times 10^{-3} \text{ eV} & (\sigma = 0), \\
5.65 \times 10^{-3} \text{ eV} & (\sigma = \pi/2);
\end{cases}
\]

\[
|\langle m \rangle_{e\mu}|_{m_3=0} = \frac{c_{13}}{\sqrt{2}} \sqrt{(m_1 \mp m_2)^2 c_{12}^2 s_{12}^2 + (m_1 c_{12} \pm m_2 s_{12})^2 s_{13}^2}
\]

\[
\simeq \begin{cases} 
5.13 \times 10^{-3} \text{ eV} & (\sigma = 0), \\
3.19 \times 10^{-2} \text{ eV} & (\sigma = \pi/2);
\end{cases}
\]

as well as

\[
|\langle m \rangle_{\mu\tau}|_{m_1=0} = \frac{1}{2} \left[ m_3 c_{13} \pm m_2 \left( c_{12}^2 + s_{12}^2 s_{13}^2 \right) \right]
\]

\[
\simeq \begin{cases} 
2.15 \times 10^{-2} \text{ eV} & (\sigma = 0), \\
2.75 \times 10^{-2} \text{ eV} & (\sigma = \pi/2);
\end{cases}
\]

\[
|\langle m \rangle_{\mu\tau}|_{m_3=0} = \frac{1}{2} \left[ m_2 \left( c_{12}^2 + s_{12}^2 s_{13}^2 \right) \pm m_1 \left( s_{12}^2 + c_{12}^2 s_{13}^2 \right) \right]
\]

\[
\simeq \begin{cases} 
2.54 \times 10^{-2} \text{ eV} & (\sigma = 0), \\
9.68 \times 10^{-3} \text{ eV} & (\sigma = \pi/2).
\end{cases}
\]

Note that all the results in Eqs. (16)–(19) are insensitive to the two-fold uncertainties of \( \delta \) (i.e., \( \delta = \pm \pi/2 \)) in the translational \( \mu-\tau \) reflection symmetry limit. The smallness of these matrix elements makes it extremely difficult, if not impossible, to probe the relevant
lepton-number-violating processes mediated by $\nu_i$ (for $i = 1, 2, 3$), as such reactions are
directly associated with the magnitudes of $\langle m \rangle_{\alpha\beta}$ [59–61].

For the sake of illustration, here let us briefly comment on a typical example of this kind:
the lepton-number-violating decay modes $B^-_u \rightarrow \pi^+ \alpha^- \beta^-$ which involve all the six effective
Majorana neutrino masses $\langle m \rangle_{\alpha\beta}$ (for $\alpha, \beta = e, \mu, \tau$). The two tree-level Feynman diagrams
responsible for $B^-_u \rightarrow \pi^+ \alpha^- \beta^-$ are shown in Fig. 2. Quite similar to the more familiar
case of those $0\nu2\beta$ processes which are directly associated with $|\langle m \rangle_{ee}|^2$, the decay rates
$\Gamma(B^-_u \rightarrow \pi^+ \alpha^- \beta^-)$ are simply proportional to $|\langle m \rangle_{\alpha\beta}|^2$. So the $\mu$-$\tau$
reflection symmetry under discussion allows us to obtain $\Gamma(B^-_u \rightarrow \pi^+ e^- \tau^-) = \Gamma(B^-_u \rightarrow \pi^+ e^- \mu^-)$ and $\Gamma(B^-_u \rightarrow \pi^+ \tau^- \tau^-) = \Gamma(B^-_u \rightarrow \pi^+ \mu^- \mu^-)$. But there are only very preliminary upper bounds on
the branching ratios of $B^-_u \rightarrow \pi^+ e^- e^-$, $B^-_u \rightarrow \pi^+ e^- \mu^-$ and $B^-_u \rightarrow \pi^+ \mu^- \mu^-$ [1] which
cannot provide any meaningful constraints on $|\langle m \rangle_{ee}|$, $|\langle m \rangle_{e\mu}|$ and $|\langle m \rangle_{\mu\mu}|$ as compared
with the more realistic and feasible $0\nu2\beta$-decay experiments. This situation is expected to
be improved to some extent in the upcoming precision measurement era characterized by
the LHCb experiment at the high-luminosity LHC [62] and the Belle-II experiments at the
KEK super-B factory [63].

So far we have simply assumed the translational $\mu$-$\tau$ reflection symmetry of three active
Majorana neutrinos at low energies. It is certainly appropriate to conjecture such a working
flavor symmetry at a superhigh-energy scale where there may exist a well-defined mechanism
responsible for the origin of tiny neutrino masses and a suitable flavor symmetry to constrain
the neutrino mass texture. In this regard the popular canonical seesaw mechanism [53–58],
which attributes the smallness of three active neutrino masses to the existence of three
heavy Majorana neutrinos and the violation of lepton number conservation, is no doubt the
most natural theoretical framework at hand.
3 An extension in the seesaw framework

The canonical seesaw mechanism is a straightforward and natural extension of the SM with three right-handed neutrino fields $\nu_{\alpha R}$ (for $\alpha = e, \mu, \tau$) and lepton number violation\footnote{Here we only make the $\mu\tau$ reflection transformation instead of the translational $\mu\tau$ reflection transformation for the left-handed neutrino fields. Otherwise, the constraint on $M_D$ would be too strong to result in a viable texture for the effective Majorana neutrino mass matrix $M_\nu$. If the $\mu\tau$ reflection transformation $\nu_L \rightarrow \mathcal{P}(\nu_L)^c$ were not made, on the other hand, the constraint on $M_\nu$ would be too weak to be desirable for our purpose of enhancing the predictability of this model.}. In this theoretical framework the gauge- and Lorentz-invariant neutrino mass terms can be written as

$$-\mathcal{L}_\nu = \ell_L^\dagger Y_\nu \tilde{H} N_R + \frac{1}{2} (N_R)^c M_R N_R + \text{h.c.} ,$$  
(20)

where $\ell_L$ denotes the $\text{SU}(2)_L$ doublet of the left-handed lepton fields, $\tilde{H} \equiv i\sigma_2 H^*$ with $H$ being the Higgs doublet and $\sigma_2$ being the second Pauli matrix, $N_R = (N_{eR}, N_{\mu R}, N_{\tau R})^T$ is the column vector of three right-handed neutrino fields which are the SU(2)$_L$ singlets, $(N_R)^c \equiv C N_R^T$ is the charge conjugate of $N_{\alpha R}$, $Y_\nu$ represents an arbitrary $3 \times 3$ Yukawa coupling matrix, and $M_R$ stands for a symmetric $3 \times 3$ Majorana mass matrix. After spontaneous electroweak gauge symmetry breaking, Eq. (20) becomes

$$-\mathcal{L}_\text{mass} = \nu_L^\dagger M_D N_R + \frac{1}{2} (N_R)^c M_R N_R + \text{h.c.} ,$$  
(21)

where $\nu_L$ is the column vector of $\nu_{eL}, \nu_{\mu L}$ and $\nu_{\tau L}$ as already given in Eq. (3), and $M_D \equiv Y_\nu \langle H \rangle$ with $\langle H \rangle \simeq 174 \text{ GeV}$ being the vacuum expectation value of the Higgs field. Note that $M_D$ is in general neither Hermitian nor symmetric. In a way similar to Eq. (5)\footnote{Here we only make the $\mu\tau$ reflection transformation instead of the translational $\mu\tau$ reflection transformation for the left-handed neutrino fields. Otherwise, the constraint on $M_D$ would be too strong to result in a viable texture for the effective Majorana neutrino mass matrix $M_\nu$. If the $\mu\tau$ reflection transformation $\nu_L \rightarrow \mathcal{P}(\nu_L)^c$ were not made, on the other hand, the constraint on $M_\nu$ would be too weak to be desirable for our purpose of enhancing the predictability of this model.}, we may make the following transformations for the left- and right-handed neutrino fields

$$\nu_L \rightarrow \mathcal{P}(\nu_L)^c , \quad N_R \rightarrow \mathcal{P} [(N_R)^c + \zeta_i^* z_N^T] ,$$  
(22)

together with $(\nu_L)^c \rightarrow \mathcal{P} \nu_L$ and $(N_R)^c \rightarrow \mathcal{P} (N_R + \zeta_i z_N)$, where the $(\mu, \tau)$-associated permutation matrix $\mathcal{P}$ has been given in Eq. (4), $\zeta_i$ is a column vector of the $3 \times 3$ unitary flavor mixing matrix $U'$ used to diagonalize $M_R$ and defined as $\zeta_i \equiv (U'_{e\mu}, U'_{e\tau}, U'_{\mu\tau})^T$ corresponding to the heavy neutrino mass eigenstate $N_i$ with the mass $M_i$ (for $i = 1, 2, 3$), and $z_N$ is another constant spinor field like $z_\nu$ in Eq. (2). Then Eq. (21) becomes

$$-\mathcal{L}_\text{mass}' = \nu_L^\dagger (\mathcal{P} M_D^* \mathcal{P}) N_R + \frac{1}{2} [(N_R)^c (\mathcal{P} M_D^* \mathcal{P}) N_R + (N_R)^c (\mathcal{P} M_D^* \mathcal{P}) \zeta_i z_N^T] \mathcal{P} M_D^* \mathcal{P} \zeta_i z_N + \text{h.c.} .$$  
(23)

In this case the necessary and sufficient conditions for $\mathcal{L}_\text{mass}' \equiv \mathcal{L}_\text{mass}$ turn out to be

$$M_D = \mathcal{P} M_D^* \mathcal{P} , \quad M_R = \mathcal{P} M_R^* \mathcal{P} ,$$  
(24)
together with the dependent relations
\[ M_D \zeta_i = 0, \quad M_R \zeta_i = 0, \quad \zeta_i^T M_R = 0^T, \quad \zeta_i^T M_R \zeta_i = 0, \]
where 0 stands for the zero column vector. The flavor textures of \( M_D \) and \( M_R \) constrained by Eq. (24) can therefore be expressed as follows \[64,65\]:
\[
M_D = \begin{pmatrix}
 a & b & b^* \\
 e & c & d \\
 e^* & d^* & c^*
\end{pmatrix},
\]
\[
M_R = \begin{pmatrix}
 A & B & B^* \\
 B & C & D \\
 B^* & D & C^*
\end{pmatrix},
\]
(26)
where the matrix elements \( a, A \) and \( D \) are real. On the other hand, Eq. (25) implies that \( M_i = 0 \) must hold. To assure the seesaw mechanism to work, however, only one of the three mass eigenstates \( N_i \) (for \( i = 1, 2, 3 \)) is allowed to vanish.

Without loss of any generality, let us take \( M_1 = 0 \). Now that there is no way to measure \( U' \) in any realistic experiments, we simply choose the flavor basis defined by \( \zeta_1 \equiv (1, 0, 0)^T \). Then Eq. (25) immediately leads us to
\[
M_D = \begin{pmatrix}
 0 & b & b^* \\
 0 & c & d \\
 0 & d^* & c^*
\end{pmatrix},
\]
\[
M_R = \begin{pmatrix}
 0 & 0 & 0 \\
 0 & C & D \\
 0 & D & C^*
\end{pmatrix},
\]
(27)
As argued in Ref. [43], the flavor eigenstate of the massless neutrino \( N_1 \) simply decouples and has only gravitational interactions due to its kinetic energy term. We actually have no idea about whether such a massless sterile neutrino state could really exist in nature. Here our focus is on the point that the translational \( \mu-\tau \) reflection symmetry of \( \tilde{L}_{\text{mass}} \) can be regarded as a reasonable way to reduce the degrees of freedom associated with the canonical seesaw mechanism and constrain its flavor textures. Through this purely phenomenological operation, we only need to consider the remaining two right-handed neutrino fields and may simply rewrite \( \tilde{L}_{\text{mass}} \) in Eq. (21) as
\[
-\tilde{L}_{\text{mass}} = \nu^c_L \tilde{M}_D \nu'_R + \frac{1}{2} (\nu'_R)^c \tilde{M}_R \nu'_R + \text{h.c.},
\]
(28)
where \( N'_R \equiv (N_{\mu R}, N_{\tau R})^T \) is defined, and the effective neutrino mass matrices \( \tilde{M}_D \) and \( \tilde{M}_R \) are of the following textures:
\[
\tilde{M}_D = \begin{pmatrix}
 b & b^* \\
 c & d \\
 d^* & c^*
\end{pmatrix}, \quad \tilde{M}_R = \begin{pmatrix}
 C & D \\
 D & C^*
\end{pmatrix},
\]
(29)
which assure $\mathcal{L}_{\text{mass}}$ itself to satisfy the $\mu$-$\tau$ reflection symmetry. Such a scenario was first proposed in the minimal seesaw framework [32], and here it is reproduced from the canonical seesaw mechanism with the help of the translational $\mu$-$\tau$ reflection symmetry.

Given Eqs. (28) and (29), in which the heavy degrees of freedom can be integrated out to obtain the unique dimension-five Weinberg operator for three active Majorana neutrino fields [66], the resulting effective Majorana neutrino mass matrix

\[ \hat{M}_\nu \simeq -\hat{M}_D\hat{M}_R^{-1}\hat{M}_D^T \]  

must be of rank two and thus have a zero eigenvalue (i.e., either $m_1 = 0$ or $m_3 = 0$) \(^3\)

Substituting Eq. (29) into Eq. (30), one may easily show that $\hat{M}_\nu$ satisfies

\[ \hat{M}_\nu = \mathcal{P}\hat{M}_\nu^*\mathcal{P}, \]  

just as $M_\nu$ in Eq. (7). So the relevant matrix elements and flavor mixing parameters of $\hat{M}_\nu$ are constrained as in Eqs. (14) and (15).

Now that $\hat{M}_\nu$ is obtained at a superhigh-energy scale $\Lambda_{\mu\tau}$ where both the minimal seesaw mechanism and the translational $\mu$-$\tau$ reflection symmetry take effect, it will unavoidably receive some quantum corrections at the electroweak scale $\Lambda_{\text{EW}} \sim 10^2$ GeV through the renormalization-group-equation (RGE) running effects. At the one-loop level such RGE-induced quantum corrections provide a very simple and natural way of softly breaking the exact $\mu$-$\tau$ reflection symmetry but keeping one of the three active neutrinos massless. Let us proceed to elaborate on this point.

4 RGE-induced soft symmetry breaking

The RGE approach, which was originally invented and developed in the 1950s [68,70], is a very powerful theoretical tool. It has proved to be staggering success in the study of critical phenomena in condensed matter physics [71,72] and in the discovery of asymptotic freedom of strong interactions in quantum field theories and particle physics [73,74]. The key point of the RGE tool is that the physical parameters at one renormalization point or energy scale can be related to their counterparts at another renormalization point or energy scale with the help of the relevant differential equations, such that the theory keeps its form invariance or self similarity between the two points or scales. Regarding the flavor issues of charged fermions and massive neutrinos, one is especially interested in the RGE-induced connection between a superhigh energy scale characterized by a fundamental mechanism of neutrino mass generation and the electroweak scale where the physical flavor parameters can be experimentally measured [15,75].

\(^3\)Note that this observation is a natural consequence of the “seesaw fair play rule” [67] which is independent of the approximation made in obtaining the seesaw formula in Eq. (30).
The one-loop differential RGE for the effective Majorana neutrino mass matrix is well known to us \([76, 79]\), and its integral form can also be found in the literature (see, e.g., Refs. \([80–83]\)). Given \(M_\nu(\Lambda_\mu)\) at the \(\mu\tau\) reflection symmetry scale and in the basis where the mass and flavor eigenstates of three charged leptons are chosen to be identical, its counterpart \(M_\nu(\Lambda_{EW})\) at the electroweak scale is explicitly expressed as

\[
M_\nu(\Lambda_{EW}) = I_\nu(\Lambda_{EW}) \left[ T_I(\Lambda_{EW}) \cdot M_\nu(\Lambda_\mu) \cdot T_I^\dagger(\Lambda_{EW}) \right]
\]

in the safe \(\tau\)-flavor dominance approximation, where \(T_I \simeq \text{Diag}\{1,1,1+\Delta_\tau\}\) and the scale-dependent one-loop RGE evolution functions \(I_\nu\) and \(\Delta_\tau\) are defined as \([83]\)

\[
I_\nu(\mu) \equiv \exp \left[ -\frac{1}{16\pi^2} \int_{\ln(\mu/\Lambda_{EW})}^{\ln(\mu/\Lambda_{EW})} \alpha_\nu(t) \, dt \right],
\]

\[
\Delta_\tau(\mu) \equiv -\frac{C_\nu}{16\pi^2} \int_{\ln(\mu/\Lambda_{EW})}^{\ln(\mu/\Lambda_{EW})} y_\tau^2(t) \, dt.
\]

In Eq. \([33]\), \(t \equiv \ln(\mu/\Lambda_{EW})\) with \(\Lambda_{EW} \leq \mu \leq \Lambda_\mu\), \(C_\nu = -3/2\) and \(\alpha_\nu \simeq -3y_\tau^2 + 6y_t^2 + \lambda\) in the SM framework with \(g_2\) being the gauge coupling constant of weak interactions, \(y_t\) being the top-quark Yukawa coupling eigenvalue and \(\lambda\) being the Higgs self-coupling parameter; or \(C_\nu = 1\) and \(\alpha_\nu \simeq -6g_2^2/5 - 6g_2^2 + 6y_t^2\) in the minimal supersymmetric SM (MSSM) with \(g_1\) being the gauge coupling constant of electromagnetic interactions. Moreover, \(y_\tau = m_\tau/\langle H \rangle\) in the SM or \(y_\tau = m_\tau/\langle H \rangle\) in the MSSM, where \(\cos \beta\) originates from \(\tan \beta\) which is defined as the ratio of the two Higgs doublets’ vacuum expectation values. The smallness of \(y_\tau\) means that \(\Delta_\tau\) is expected to be small and become zero at \(\mu = \Lambda_\mu\), as numerically illustrated in Ref. \([83]\). Therefore, we have

\[
M_\nu(\Lambda_{EW}) \simeq M_\nu(\Lambda_\mu) + \begin{pmatrix} 0 & 0 & \langle m \rangle_{e\mu}^* \\ 0 & 0 & \langle m \rangle_{\mu\tau} \\ \langle m \rangle_{e\mu} & \langle m \rangle_{\mu\tau} & 2\langle m \rangle_{\tau\tau} \end{pmatrix} \Delta_\tau(\Lambda_{EW}),
\]

where the second term with \(\langle m \rangle_{\alpha\beta}\) or its complex conjugates at \(\Lambda_\mu\) characterizes some small effects of the \(\mu\tau\) reflection symmetry breaking. Note that \(M_\nu(\Lambda_{EW})\) is of rank two as \(M_\nu(\Lambda_\mu)\), and thus it contains a vanishing eigenvalue.

Adopting the standard parametrization of the \(3 \times 3\) PMNS matrix \(U\) in Eq. \([13]\), we have the form invariance of \(M_\nu = UD_\nu U^T\) with \(D_\nu \equiv \text{Diag}\{m_1,m_2,m_3\}\) at any energy scale between \(\Lambda_{EW}\) and \(\Lambda_\mu\). A combination of this decomposition and Eq. \([34]\) allows us to establish some intriguing relations between the physical parameters of three active Majorana neutrinos at these two scales. As for the neutrino masses, we explicitly have

\[
m_1(\Lambda_{EW}) \simeq 0,
\]

\[
m_2(\Lambda_{EW}) \simeq I_\nu(\Lambda_{EW}) \left[ 1 + \Delta_\tau(\Lambda_{EW}) \left(1 - s_{12}^2 c_{13}^2 \right) \right] m_2(\Lambda_\mu),
\]

\[
m_3(\Lambda_{EW}) \simeq I_\nu(\Lambda_{EW}) \left[ 1 + \Delta_\tau(\Lambda_{EW}) c_{13}^2 \right] m_3(\Lambda_\mu).
\]

(35)
in the \( m_1(\Lambda_{\mu\tau}) = 0 \) case; or

\[
\begin{align*}
    m_1(\Lambda_{\text{EW}}) & \simeq I_\nu(\Lambda_{\text{EW}}) \left[ 1 + \Delta_\tau(\Lambda_{\text{EW}}) \left( 1 - c_{12}^2 c_{13}^2 \right) \right] m_1(\Lambda_{\mu\tau}) , \\
    m_2(\Lambda_{\text{EW}}) & \simeq I_\nu(\Lambda_{\text{EW}}) \left[ 1 + \Delta_\tau(\Lambda_{\text{EW}}) \left( 1 - s_{12}^2 c_{13}^2 \right) \right] m_2(\Lambda_{\mu\tau}) , \\
    m_3(\Lambda_{\text{EW}}) & \simeq 0
\end{align*}
\]  

(36)

in the \( m_3(\Lambda_{\mu\tau}) = 0 \) case, where \( \theta_{23}(\Lambda_{\mu\tau}) = \pi/4, \delta(\Lambda_{\mu\tau}) = \pm\pi/2 \) and \( \sigma(\Lambda_{\mu\tau}) = 0 \) or \( \pi/2 \) have been input, and \( \theta_{12} \) and \( \theta_{13} \) take their values measured at \( \Lambda_{\text{EW}} \). As for the flavor mixing angles and CP-violating phases, let us define

\[
\begin{align*}
    \Delta \theta_{ij} & \equiv \theta_{ij}(\Lambda_{\text{EW}}) - \theta_{ij}(\Lambda_{\mu\tau}) , \\
    \Delta \delta & \equiv \delta(\Lambda_{\text{EW}}) - \delta(\Lambda_{\mu\tau}) , \\
    \Delta \sigma & \equiv \sigma(\Lambda_{\text{EW}}) - \sigma(\Lambda_{\mu\tau}) ,
\end{align*}
\]

(37)

to describe small deviations of their values at \( \Lambda_{\text{EW}} \) from those at \( \Lambda_{\mu\tau} \) where the translational \( \mu-\tau \) reflection flavor symmetry is exact. Following the generic analytical results obtained in Ref. \[83\], we simply arrive at

\[
\begin{align*}
    \Delta \theta_{12} & \simeq -\frac{1}{2} c_{12} s_{12} (1 - s_{13}^2 \zeta_{32}) \Delta_\tau(\Lambda_{\text{EW}}) , \\
    \Delta \theta_{13} & \simeq -\frac{1}{2} c_{13} s_{13} \left( c_{12}^2 + s_{12}^2 \zeta_{32} \right) \Delta_\tau(\Lambda_{\text{EW}}) , \\
    \Delta \theta_{23} & \simeq -\frac{1}{2} \left( s_{12}^2 + c_{12}^2 \zeta_{32} - \eta_{32} \right) \Delta_\tau(\Lambda_{\text{EW}})
\end{align*}
\]

(38)

in the \( m_1(\Lambda_{\mu\tau}) = 0 \) case; or

\[
\begin{align*}
    \Delta \theta_{12} & \simeq -\frac{1}{2} c_{12} s_{12} c_{13}^2 \zeta_{21} \eta_{32} \Delta_\tau(\Lambda_{\text{EW}}) , \\
    \Delta \theta_{13} & \simeq +\frac{1}{2} c_{13} s_{13} \Delta_\tau(\Lambda_{\text{EW}}) , \\
    \Delta \theta_{23} & \simeq +\frac{1}{2} \Delta_\tau(\Lambda_{\text{EW}}) 
\end{align*}
\]

(39)

in the \( m_3(\Lambda_{\mu\tau}) = 0 \) case, where \( \zeta_{ij} \equiv (m_i - m_j) / (m_i + m_j) \) is defined at \( \Lambda_{\text{EW}} \) and \( \eta_{\sigma} \equiv \cos 2\sigma = \pm 1 \) denotes one of the two possible options of \( \sigma(\Lambda_{\mu\tau}) \) in the \( \mu-\tau \) reflection symmetry limit \[84\]. Furthermore, we have

\[
\begin{align*}
    \Delta \delta & \simeq -\frac{c_{12} s_{12} \eta_{32}}{2 s_{13}} \left[ 1 - \zeta_{32} - \frac{s_{13}^2}{c_{12}^2 s_{12}^2} \left( 1 - s_{12}^4 + c_{12}^4 \zeta_{32} \right) \right] \Delta_\tau(\Lambda_{\text{EW}}) , \\
    \Delta \sigma & \simeq c_{12} s_{12} s_{13} \eta_{32} (1 - \zeta_{32} - \eta_{32}) \Delta_\tau(\Lambda_{\text{EW}}) 
\end{align*}
\]

(40)

in the \( m_1(\Lambda_{\mu\tau}) = 0 \) case; or

\[
\begin{align*}
    \Delta \delta & \simeq -\frac{s_{13} \eta_{32}}{2 c_{12} s_{12}} \left( 1 - 2 s_{12}^2 \zeta_{21} - \eta_{32} \right) \Delta_\tau(\Lambda_{\text{EW}}) , \\
    \Delta \sigma & \simeq \frac{s_{12} s_{13} \eta_{32}}{2 c_{12}} (1 - \zeta_{21}^2) \Delta_\tau(\Lambda_{\text{EW}}) 
\end{align*}
\]

(41)
in the $m_3(\Lambda_{\mu\tau}) = 0$ case, where $\eta_\delta \equiv \sin \delta = \pm 1$ stands for one of the two possible options of $\delta(\Lambda_{\mu\tau})$ in the $\mu$-$\tau$ reflection symmetry limit. Some immediate comments are in order.

- $m_i(\Lambda_{\mu\tau}) = 0$ will automatically lead to $m_i(\Lambda_{EW}) = 0$ (for $i = 1$ or 3) as a natural consequence of the one-loop RGE running. Note that a finite value of $m_i(\Lambda_{EW})$ may result from $m_i(\Lambda_{\mu\tau}) = 0$ at the two-loop level \[85\] \[87\], but it is extremely tiny (typically of $O(10^{-13})$ eV in the SM or $O(10^{-10})$ eV to $O(10^{-8})$ eV in the MSSM) and can thus be treated as zero in practice. Due to the smallness of $\Delta_\tau$, the two nonzero neutrino masses essentially have the same running effects characterized by $I_\nu$.

- Among the three flavor mixing angles, $\theta_{13}$ is most insensitive to the RGE-induced quantum correction because of its smallness. That is to say, $\Delta\theta_{13}$ is proportional to $s_{13}$ no matter whether $m_1(\Lambda_{\mu\tau}) = 0$ or $m_3(\Lambda_{\mu\tau}) = 0$. The smallness of $\Delta m_{21}^O$ implies that only $\Delta\theta_{12}$ in the $m_3(\Lambda_{\mu\tau}) = 0$ case with $\sigma(\Lambda_{\mu\tau}) = 0$ (i.e., $\eta_\sigma = 1$) may be significantly enhanced. The deviation of $\theta_{23}(\Lambda_{EW})$ from $\pi/4$ is expected to be quite mild unless $\tan \beta$ assumes a large value in the MSSM framework.

- One can see that the magnitude of $\Delta\sigma$ is suppressed by the smallness of $\theta_{13}$, no matter whether $m_1(\Lambda_{\mu\tau}) = 0$ or $m_3(\Lambda_{\mu\tau}) = 0$. In the latter case the magnitude of $\Delta\delta$ is also suppressed by $s_{13}$ but it can simultaneously be enhanced by $\zeta_{21}$ with $\eta_\sigma = -1$ (i.e., $\sigma(\Lambda_{\mu\tau}) = \pi/2$), so can the magnitude of $\Delta\sigma$. In comparison, the magnitude of $\Delta\delta$ is somewhat enhanced by $1/s_{13}$ in the $m_1(\Lambda_{\mu\tau}) = 0$ case.

It is worth mentioning that the signs of $\Delta\theta_{ij}$, $\Delta\delta$ and $\Delta\sigma$ in the SM and those in the MSSM are always opposite, because of $\Delta_\tau \propto C_\nu$ having the opposite signs in these two frameworks.

To numerically illustrate our results obtained above, let us only consider the possibility of $\delta = -\pi/2$ (i.e., $\eta_\delta = -1$) at $\Lambda_{\mu\tau} \approx 10^{14}$ GeV and input the best-fit values $s_{12} \approx 0.551$ and $s_{13} \approx 0.148$ at $\Lambda_{EW} \approx 10^2$ GeV. Using Eqs. (11) and (12), we obtain $\zeta_{32} \approx 0.707$ in the $m_1(\Lambda_{\mu\tau}) = 0$ case and $\zeta_{21} \approx 7.564 \times 10^{-3}$ in the $m_3(\Lambda_{\mu\tau}) = 0$ case at $\Lambda_{EW}$. As for the RGE-induced quantum corrections, we quote $I_\nu(\Lambda_{EW}) \approx 0.748$ and $\Delta_\nu(\Lambda_{EW}) \approx 2.822 \times 10^{-5}$ in the SM; $I_\nu(\Lambda_{EW}) \approx 0.879$ and $\Delta_\nu(\Lambda_{EW}) \approx -1.354 \times 10^{-3}$ in the MSSM with $\tan \beta = 10$; or $I_\nu(\Lambda_{EW}) \approx 0.871$ and $\Delta_\nu(\Lambda_{EW}) \approx -1.346 \times 10^{-2}$ in the MSSM with $\tan \beta = 30$ \[83\]. Then we arrive at the numerical result

\[
\frac{m_2(\Lambda_{EW})}{m_2(\Lambda_{\mu\tau})} \approx \frac{m_3(\Lambda_{EW})}{m_3(\Lambda_{\mu\tau})} \approx I_\nu(\Lambda_{EW}) \approx \begin{cases} 0.75 & \text{(SM)}, \\ 0.88 & \text{(MSSM with } \tan \beta = 10), \\ 0.87 & \text{(MSSM with } \tan \beta = 30), \end{cases} \quad (42)
\]

in the $m_1(\Lambda_{\mu\tau}) = 0$ case from Eq. (35); and the same approximate result for the ratios $m_1(\Lambda_{EW})/m_1(\Lambda_{\mu\tau})$ and $m_2(\Lambda_{EW})/m_2(\Lambda_{\mu\tau})$ in the $m_3(\Lambda_{\mu\tau}) = 0$ case can be obtained from Eq. (36). The numerical values of $\Delta\theta_{ij}$ (for $ij = 12, 13, 23$), $\Delta\delta$ and $\Delta\sigma$ are calculated by using Eqs. (38) — (41) for either $\sigma(\Lambda_{\mu\tau}) = 0$ or $\sigma(\Lambda_{\mu\tau}) = \pi/2$, and they are concisely listed...
Table 1: A numerical illustration of the RGE-induced quantum corrections to three flavor mixing angles and two CP-violating phases in the canonical seesaw model constrained by the translational $\mu$-$\tau$ reflection symmetry, where $\Lambda_{\mu\tau} \simeq 10^{14}$ GeV and $\delta(\Lambda_{\mu\tau}) = -\pi/2$ have been taken, and the results of $\mathcal{O}(10^{-5})$ degrees or smaller have been denoted as $\sim 0^\circ$.

| $m_3(\Lambda_{\mu\tau}) = 0$ | SM | MSSM (tan $\beta = 10$) | MSSM (tan $\beta = 30$) |
|--------------------------|-----------------|-----------------|-----------------|
| $\sigma(\Lambda_{\mu\tau})$ | $\sigma(\Lambda_{\mu\tau})$ | $\sigma(\Lambda_{\mu\tau})$ | $\sigma(\Lambda_{\mu\tau})$ |
| $\Delta \theta_{12}$ | $\sim 0^\circ$ | $\sim 0^\circ$ | $0.018^\circ$ | $0.017^\circ$ | $0.175^\circ$ | $0.172^\circ$ |
| $\Delta \theta_{13}$ | $\sim 0^\circ$ | $\sim 0^\circ$ | $0.005^\circ$ | $0.006^\circ$ | $0.051^\circ$ | $0.064^\circ$ |
| $\Delta \theta_{23}$ | $-0.001^\circ$ | $-0.0006^\circ$ | $0.050^\circ$ | $0.031^\circ$ | $0.497^\circ$ | $0.307^\circ$ |
| $\Delta \delta$ | $-0.0004^\circ$ | $0.001^\circ$ | $0.021^\circ$ | $-0.056^\circ$ | $0.207^\circ$ | $-0.552^\circ$ |
| $\Delta \sigma$ | $\sim 0^\circ$ | $\sim 0^\circ$ | $-0.002^\circ$ | $0.0015^\circ$ | $-0.022^\circ$ | $0.015^\circ$ |

in Table 1. These results are certainly compatible with our above observations based on the analytical approximations. Two particular remarks are in order.

- Now that the neutrino mass spectrum has been fixed by the translational flavor symmetry (i.e., either $m_1 = 0$ or $m_3 = 0$ at $\Lambda_{\mu\tau}$) and the observed neutrino mass-squared differences, there is no freedom to adjust the strength of the RGE-induced quantum corrections in the SM case. That is why the numerical results of $\Delta \theta_{ij}$, $\Delta \delta$ and $\Delta \sigma$ are all too small to be experimentally distinguishable. In the MSSM case taking relatively large values of $\tan \beta$ may help enhance the significance of the RGE-induced effects, simply because $\Delta \tau(\Lambda_{EW}) \propto (1 + \tan^2 \beta)$ holds.

- Table 1 shows that $\Delta \theta_{12}$ can be significantly enhanced for $m_3(\Lambda_{\mu\tau}) = 0$ and $\sigma(\Lambda_{\mu\tau}) = 0$ in the MSSM framework with a sufficiently large input of $\tan \beta$. The reason is simply that $\Delta \theta_{12}$ is proportional to both $\zeta_{21}^{-1} = (m_1 + m_2)^2 / \Delta m_{21}^2 \simeq 132.2$ and $(1 + \tan^2 \beta) \gtrsim 10^2$ for $\tan \beta \gtrsim 10$ and thus gets enhanced in this case. In comparison, $\Delta \delta$ and $\Delta \sigma$ can be remarkably enhanced for the same reasons if $m_3(\Lambda_{\mu\tau}) = 0$ and $\sigma(\Lambda_{\mu\tau}) = \pi/2$ are taken in the MSSM, although these two quantities are apparently suppressed by the smallness of $\theta_{13}$ as shown in Eq. (41).

In addition, the small magnitude of $\Delta \theta_{23}$ in the MSSM with $\tan \beta = 30$ implies that the
RGE-induced quantum correction to $\theta_{23}(\Lambda_{\mu}) = \pi/4$ at $\Lambda_{EW}$ is in general too mild to resolve the octant issue of $\theta_{23}$ in the minimal seesaw scenario.

If the values of $\theta_{23}$ and $\delta$ to be measured in the future long-baseline neutrino oscillation experiments (e.g., Hyper-Kamiokande [88] and DUNE [89]) turn out to be remarkably different from their values constrained by the translational $\mu$-\tau reflection symmetry under discussion, it will be difficult to naturally account for such a discrepancy with the help of the above RGE-induced soft symmetry breaking effects. In this case one may either invoke an explicit $\mu$-\tau reflection symmetry breaking scenario to fit the experimental data (see, e.g., a recent scenario proposed in Ref. [90]), which certainly seems contrived, or go beyond the $\mu$-\tau reflection symmetry by invoking a larger flavor symmetry group for model building. In either case it will be useful to explore the direct correlation between thermal leptogenesis and CP violation in neutrino oscillations (see, e.g., the pioneering attempts in Refs. [21,91] and a recent comprehensive review in Ref. [37]).

5 Summary

The strong hierarchies associated with the mass spectra of charged leptons, up-type quarks and down-type quarks, together with a clear hierarchy of the off-diagonal CKM matrix elements, strongly indicate that the flavors of charged fermions should have specific structures instead of a random nature. This expectation has been further strengthened in recent years with the very observation that the PMNS lepton flavor mixing matrix $U$ exhibits an approximate $\mu$-\tau interchange symmetry, although whether the three active neutrinos have a normal, inverted or nearly degenerate mass spectrum is still an open question. So far a lot of phenomenological efforts have been made towards deeper understanding of the issues of tiny neutrino masses and significant lepton flavor mixing and CP violation with the help of current neutrino oscillation data, but new ideas are always called for because we still have a long way to go before the true theory finally emerges.

Along this line of thought, we have coherently combined two simple but suggestive flavor symmetries — the $\mu$-\tau reflection symmetry and the translational flavor symmetry by proposing a translational $\mu$-\tau reflection symmetry for the effective Majorana neutrino mass term $\mathcal{L}_{\text{mass}}$ in this work. Namely, $\mathcal{L}_{\text{mass}}$ is required to keep invariant under the transformations $\nu_{eL} \rightarrow (\nu_{eL})^c + U_{e\alpha} z_\nu^c$, $\nu_{\mu L} \rightarrow (\nu_{\tau L})^c + U_{\tau i} z_\nu^c$ and $\nu_{\tau L} \rightarrow (\nu_{\mu L})^c + U_{\mu i} z_\nu^c$, where $z_\nu$ is a constant spinor field and $U_{\alpha i}$ are the PMNS matrix elements corresponding to $m_i = 0$ (for $\alpha = e, \mu, \tau$ and $i = 1$ or 3). Extending such a flavor symmetry to the canonical seesaw mechanism by requiring the relevant neutrino mass terms to be invariant under similar translational $\mu$-\tau reflection transformations for the right-handed neutrino fields, we show that a minimal seesaw scenario can naturally be reproduced and it automatically respects the $\mu$-\tau reflection symmetry. As a by-product, the soft breaking effects of this flavor symmetry have been studied with the help of the one-loop RGEs.
We admit that the translational $\mu$-$\tau$ reflection symmetry is not a real flavor symmetry for the whole Lagrangian of a Majorana neutrino mass model. Instead, it is just an effective or working flavor symmetry for the effective neutrino mass term, but it can greatly help constrain the corresponding flavor textures. Motivated by the principle of Occam’s razor, we find that such simple but viable and predictive neutrino mass models really deserve attention. First, their predictions can be more easily tested in the near future. Second, it is very likely that the empirical flavor symmetries of this kind are actually the residual symmetries of some larger flavor symmetry groups in an underlying fundamental and UV-complete flavor theory. In this sense, we may argue that the translational $\mu$-$\tau$ reflection symmetry under discussion, among many other existing flavor symmetries, might be located in the neutrino landscape instead of the neutrino swampland [92–94].

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