Stability and instability of Navier boundary layers

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Abstract

We study the inviscid limit problem for the incompressible Navier-Stokes equation on a half-plane with a Navier boundary condition depending on the viscosity. On one hand, we prove the $L^2$ convergence of Leray solutions to the solution of the Euler equation. On the other hand, we show the nonlinear instability of WKB expansions in the stronger $L^\infty$ and $H^s$ ($s > 1$) norms.

Key words: inviscid limit problem, nonlinear instability, Navier boundary condition and boundary layers, shear flows

1 Introduction

1.1 The inviscid limit problem

We consider the two-dimensional incompressible Navier-Stokes equation on the half-plane $\Omega = \mathbb{R} \times \mathbb{R}^+$ with viscosity $\epsilon > 0$:

\[
(\text{NS}_\epsilon) : \begin{cases}
\partial_t u^\epsilon + u^\epsilon \cdot \nabla u^\epsilon - \epsilon \Delta u^\epsilon + \nabla p^\epsilon = 0 \quad \text{in } \mathbb{R}^+ \times \Omega \\
div u^\epsilon = 0 \quad \text{in } \mathbb{R}^+ \times \Omega \\
u^\epsilon|_{t=0} = u^\epsilon_0 \quad \text{in } \Omega
\end{cases}
\]

$u^\epsilon = (u^\epsilon_1, u^\epsilon_2)$ is the two-dimensional fluid speed and $p^\epsilon$ the internal pressure. We add two boundary conditions: first, the standard non-penetration condition for the component of $u^\epsilon$ normal to the boundary

\[u^\epsilon \cdot \vec{n} = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega\]

where $\vec{n} = (0, -1)$, and a Navier condition that depends on the viscosity $\epsilon$ and that describes the tangential part of the fluid’s speed on the boundary:

\[(Su^\epsilon \cdot \vec{n} + a^\epsilon u^\epsilon)_{\text{tan}} = 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega\]

where $a^\epsilon \in \mathbb{R}$, and $Su^\epsilon = \frac{1}{2}(\nabla u^\epsilon + \nabla u^\epsilon^T)$ is called the strain tensor.

Unlike the homogeneous Dirichlet (no-slip) boundary condition, $u|_{\nu=0} = 0$, the Navier condition allows the fluid to slide along the boundary. Physically, the Navier condition is more accurate than the no-slip condition when one takes into account interaction at the boundary. The slip condition is therefore used...
to homogenise the no-slip condition on rough or porous walls (see \cite{16} and \cite{12}), and to model, for example, the flow of blood in capillary vessels that are a few microns wide \cite{25}. Mathematically, it is shown in \cite{22} that the Navier condition can be derived by taking the limit when the mean-free path goes to zero of renormalised solutions of the Boltzmann equation with Maxwell reflection boundary condition.

This paper will deal with the inviscid limit problem for $(\text{NS}_\varepsilon)$, i.e. we study the behaviour of a family of Leray solutions of $(\text{NS}_\varepsilon)$ with Navier boundary condition, $(u^\varepsilon)_{0 < \varepsilon \leq \varepsilon_0}$, relative to $v$, a solution of the incompressible Euler equation \begin{equation}
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla q &= 0 \\
\text{div } v &= 0 \\
v|_{t=0} &= v_0 \\
u_2|_{y=0} &= 0 
\end{aligned}
\end{equation}
when the viscosity $\varepsilon$ goes to 0.

The inviscid limit problem is well understood in the whole space or with periodic boundary conditions \cite{20}. However, in domains with boundaries, the lack of compactness due to the presence of boundary layers makes the problem considerably more difficult. In recent years, much progress has been made in the case of non-characteristic boundaries. For incompressible fluids, these appear in the case of injection or suction boundary conditions (see \cite{34}). For related results for compressible models and general hyperbolic-parabolic systems, we refer to \cite{23}. In this situation the boundary layer is of size $\varepsilon$ and has an amplitude $O(1)$, and sharp stability and instability conditions can be shown (see for example \cite{6} and \cite{29} for the study of the stability or instability of the Ekman layer in rotating fluids).

On the half-plane, the inviscid limit problem remains mainly unsolved in the case of Navier-Stokes equations with the homogeneous Dirichlet boundary condition. Formally, one expects the boundary layer to be of size $\sqrt{\varepsilon}$ and hence to write a solution $u^D$ as
\begin{equation}
u^D(t, x, y) = u^I(t, x, y) + u^h(t, x, \varepsilon^{-1/2} y)
\end{equation}
in which $u^I$ solves (E) and $u^h$ solves the Prandtl equation
\begin{equation}
\begin{aligned}
\partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - \partial^2_{yy} u_1 &= \left(\partial_t u^I + u^I \cdot \nabla u^I\right)_1|_{y=0} \text{ in } \Omega \\
\text{div } u &= 0 \text{ in } \Omega \\
u|_{y=0} &= 0 \\
\lim_{y \to +\infty} u &= u^I|_{y=0}
\end{aligned}
\end{equation}
A major difficulty resides in the existence of such an expansion, because, while the Prandtl system is well-posed with monotonous initial conditions in the strip $[0, L] \times \mathbb{R}^+$ (a fact known since the 60s, \cite{24}), it has recently been proved by D.Gérard-Varet and E.Dormy in \cite{11} that the Prandtl equation is ill-posed in Sobolev spaces on $\mathbb{T} \times \mathbb{R}^+$. This shows that the ansatz \cite{1} is only formal in general, and E.Grenier showed in \cite{13} that, even when such an ansatz can be justified, eg in analytic framework (see \cite{32}), it may not be valid in $H^1$, and instability occurs on the derivatives in $L^\infty$. The instability phenomenon is linked to the linear instability of shear flows for the Euler equation.
For the Navier-Stokes equation with Navier boundary condition, when $a_\varepsilon$ is a fixed number independent of $\varepsilon$, the inviscid limit problem is solved in $L^2$ framework, see [1], [4], [17], [14]. Moreover, asymptotic expansions of the form

$$u^N(t, x, y) = u^i(t, x, y) + \sqrt{\varepsilon}u^b(t, x, \varepsilon^{-1/2}y)$$

with the amplitude of the boundary layer much smaller than in (1), are rigorously justified in [15] and uniform conormal estimates in agreement with this formal behaviour are obtained in [21]. We also refer to [2], [35], and [3] for the study of special cases in 3D.

Here, we shall be interested in the case where the Navier condition is as above with $a(\varepsilon) = a\varepsilon^{-1/2}$, $a \neq 0$ constant. Denoting the two components of the fluid’s speed by $u^\varepsilon_1$ and $u^\varepsilon_2$, the boundary conditions translate into the following:

\[(NP) : \quad u^\varepsilon_2(t, x, 0) = 0\]
\[(NCa_\varepsilon) : \quad \frac{1}{2}\partial_y u^\varepsilon_1(t, x, 0) = \frac{a}{\sqrt{\varepsilon}}u^\varepsilon_2(t, x, 0)\]

Our motivation for this is to understand the transition between the unstable Dirichlet case and the stable Navier case studied in [15]. Note that for $\varepsilon$ small, our slip condition seems to approximate the no-slip Dirichlet condition, and we expect the formal asymptotic behaviour of the solution to replicate the ansatz of the Dirichlet case, ie

$$u^N(t, x, y) = u^i(t, x, y) + u^b(t, x, \varepsilon^{-1/2}y)$$

and hence we expect to observe some instability.

As of now, the term “solution of the Navier-Stokes equation (NSa_\varepsilon)” will designate a Leray solution of (NS_\varepsilon) satisfying the boundary conditions (NP) and (NCa_\varepsilon) (the full definition is given in section 2). We obtain two results: firstly, convergence in $L^2$ of every sequence of Leray solutions of (NSa_\varepsilon) to a smooth solution of (E) when $\varepsilon$ goes to zero, and secondly, we prove that there exist boundary layer profiles such that the WKB expansion, which is also of the form (1) in our case, is unstable in $L^\infty$.

### 1.2 The $L^2$ stability result

We show that $u^\varepsilon$ converges to $v$ in $L^2(\Omega)$, extending the result obtained by D.Iftimie and G.Planas in [14] for a positive constant Navier boundary condition, ie $(Su \cdot \vec{n} + au)_{\text{tan}} = 0$ with $a > 0$.

**Theorem 1.** Let $u^\varepsilon$ be a solution of the Navier-Stokes equation (NSa_\varepsilon), and $v$ be the solution of the Euler equation (E). We assume that the initial condition $v_0$ is in $H^s(\Omega)$ for some $s > 2$. If $u^\varepsilon_0$ converges to $v_0$ in $L^2(\Omega)$ as $\varepsilon$ goes to 0, then for every $T > 0$,

$$\sup_{t \in [0, T]} \|u^\varepsilon(t) - v(t)\|_{L^2(\Omega)} \xrightarrow{\varepsilon \to 0} 0$$

In the case of a positive boundary condition, ie (NCa_\varepsilon) with $a > 0$, we merely check that their calculations, based on energy estimates, allow us to conclude. The negative case ($a < 0$) will require little more work.
1.3 The nonlinear instability result

Proving our nonlinear instability result below will closely follow an argument by E. Grenier, with which he proved that linear instability implies nonlinear instability for the Euler equation in [13], and for some viscous boundary layers in [6] with B. Desjardins. We will therefore require a form of linear instability as a starting point: we will start with a shear flow that is linearly unstable for the Euler equation.

**Definition.** A smooth 2D vector field $u_s$ is a shear flow if it is written as $u_s = (u_s(y), 0)$ (the vector field and its first component are indifferently called $u_s$). Note that a shear flow is automatically a stationary solution of the Euler equation. The shear flow $u_s$ is linearly unstable if there exist $k \in \mathbb{R}$, $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, and a function $\Psi \in H^1_0(\mathbb{R}^+)$ such that

$$u(t, x, y) = (e^{\lambda t}e^{ikx}\Psi'(y), -ike^{\lambda t}e^{ikx}\Psi(y))$$

is a solution of the Euler equation linearised around $u_s$,

$$\text{(EL)}: \quad \partial_t u + u_s \cdot \nabla u + u \cdot \nabla u_s = 0 \quad \text{in } \mathbb{R}^+ \times \Omega$$

with (NP) condition on the boundary.

**Theorem 2.** Let $u_s$ be a linearly unstable shear flow for the Euler equation. Setting $Y = \varepsilon^{-1/2}y$, we generate a time-dependent boundary layer $\overline{u}_\varepsilon$ as solution of the heat equation

$$\begin{cases}
\partial_t \overline{u}_\varepsilon(t, Y) - \partial_{YY}^2 \overline{u}_\varepsilon(t, Y) &= 0 \\
\overline{u}_\varepsilon(0, Y) &= u_s(Y) \\
\frac{1}{\varepsilon} \partial_Y \overline{u}_\varepsilon(t, 0) &= ae^{-1/2}u_s(t, 0)
\end{cases}$$

the boundary condition being dictated by (NCaε): $\overline{u}_\varepsilon(t, y/\sqrt{\varepsilon})$ therefore solves the Navier-Stokes system. Then, for any large $n \in \mathbb{N}$, there exist $\delta_0$ and $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have $u^\varepsilon$ a Leray solution of (NSaε) such that

$$\|u^\varepsilon(0, x, y) - u_s(y/\sqrt{\varepsilon})\|_{H^s(\Omega)} \leq C\varepsilon^n$$

for some $s > 0$, and at a time $T^\varepsilon \sim n \ln(\varepsilon^{-1})\sqrt{\varepsilon}$,

$$\|u^\varepsilon(T^\varepsilon, x, y) - u_s(T^\varepsilon, y/\sqrt{\varepsilon})\|_{L^\infty(\Omega)} \geq \delta_0 \quad (2)$$

Moreover, for $s > 1$,

$$\|u^\varepsilon(T^\varepsilon, x, y) - u_s(T^\varepsilon, y/\sqrt{\varepsilon})\|_{H^s} \xrightarrow{\varepsilon \to 0} +\infty$$

This is a nonlinear instability result in $L^\infty$ in the usual sense for WKB expansions of the form (1). We show that when the boundary layer profile $u^\varepsilon$ is linearly unstable for the Euler equation, then the WKB expansion is unstable,
in the sense that arbitrarily small perturbations yield instantaneous amplification on $[0, T^\varepsilon]$ to reach the amplitude $O(1)$. Note that $T^\varepsilon$ converges to 0 as $\varepsilon$ goes to 0. An explicit example of unstable boundary layer which matches the assumptions of Theorem 2 is given at the end of the paper.

Our result differs from the result obtained by E.Grenier in [13] on the inviscid limit problem for Navier-Stokes equations with homogeneous Dirichlet boundary conditions: in (2), one obtains $\delta_0 \varepsilon^{1/4}$ instead of $\delta_0$, and therefore, the result with Dirichlet conditions is not a standard instability result. This is due to the Prandtl boundary layers, the amplitude of which is the same as that of the internal layer. Indeed, after rescaling $(t, x, y) \mapsto \varepsilon^{-1/2}(t, x, y)$ (these new coordinates are indifferently denoted $(t, x, y)$), a solution of $(\text{NS}^{\sqrt{\varepsilon}})$ with homogeneous Dirichlet boundary conditions can formally be written as (1). In our case of Navier condition depending on $\sqrt{\varepsilon}$, the rescaled boundary condition no longer depends on the viscosity, so, as shown by D.Iftimie and F.Sueur in [15], we can write

$$u^\varepsilon(t, x, y) \sim u^s(y) + u^i(t, x, y) + \varepsilon^{1/4} u^b(t, x, \frac{y}{\varepsilon^{1/4}})$$

and the factor $\varepsilon^{1/4}$ in front of the boundary layer in the ansatz neutralises the one that appears when differentiating with respect to $y$. In the Dirichlet case, this compensation does not take place, and considering times that are $O\left(\left(n - \frac{1}{4}\right) \ln(\varepsilon^{-1})\right)$ leads to the apparent loss of instability.

The proof of Theorem 2 will follow Grenier’s approach, which mainly relies on the construction of a WKB approximate solution to $(\text{NS}^{ae})$ starting with a linearly unstable solution to the Euler equation, and the use of a resolvent estimate on the linearised Euler equation (our Theorem 3). The strength of the method comes from the fact that the resolvent estimate only needs to be shown on a subspace - typically spaces of functions with one or a finite number of Fourier modes. This has allowed it to be used in other circumstances than the study of boundary layers: F.Rousset and N.Tzvetkov used it to prove transverse nonlinear instability of solitary waves for KdV and water-wave equations (see [31] and [30]).

Organisation of the paper. In the next section, we prove the $L^2$ convergence result, Theorem 1. In section 3, we first prove the resolvent estimate before proving Theorem 2 and finally we give an example of shear flow that fits the instability result.

2 Proof of the $L^2$ stability (Theorem 1)

We remind the reader of the notion of Leray solution to the Navier-Stokes system:

**Definition.** $u^\varepsilon : \mathbb{R}^+ \times \Omega \to \mathbb{R}^2$ is a Leray solution of $(\text{NS}^{ae})$ if:

1. $u^\varepsilon \in C_w(\mathbb{R}^+, L^2_\sigma) \cap L^2([0, T], H^1_\sigma)$ for every $T > 0$, where, if $E(\Omega)$ is a functional space on $\Omega$, $E_\sigma$ designates the space of divergence-free vector fields, tangent to the boundary and belonging to $E(\Omega)$. 

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2. \( u^\varepsilon \) is a weak solution to \((\text{NS}a\varepsilon)\) in the following sense: we have

\[
-\int_{\mathbb{R}^+} \int_{\Omega} u^\varepsilon \cdot \partial_t \varphi + 2a\sqrt{\varepsilon} \int_{\mathbb{R}^+} \int_{\partial\Omega} u^\varepsilon \cdot \varphi + 2\varepsilon \int_{\mathbb{R}^+} \int_{\Omega} Su^\varepsilon : S\varphi \\
- \int_{\mathbb{R}^+} \int_{\Omega} (u^\varepsilon \cdot \nabla \varphi) \cdot u^\varepsilon = \int_{\Omega} u^\varepsilon(0) \cdot \varphi(0) \tag{3}
\]

for every \( \varphi \in H^1(\mathbb{R}^+, H^1_{\sigma}) \), where \( A : B = \sum A_{i,j} B_{i,j} \) is the contracted product of matrices \( A \) and \( B \).

3. For every \( t \geq 0 \), \( u^\varepsilon \) satisfies the following energy estimate:

\[
\|u^\varepsilon(t)\|_{L^2(\Omega)}^2 + 4a\sqrt{\varepsilon} \int_0^t \int_{\partial\Omega} |u^\varepsilon|^2 + 4\varepsilon \int_0^t \int_{\Omega} |Su^\varepsilon|^2 \leq \|u^\varepsilon(0)\|_{L^2(\Omega)}^2 \tag{4}
\]

For \( a \) and \( \varepsilon \) fixed, such solutions are known to exist and are global in time. Moreover, in 2D, we have uniqueness (the proof uses the Galerkin method, see [15]). Likewise, the existence and uniqueness of a classical global-in-time solution \( v \) to the Euler equation \((E)\) when \( v_0 \in H^s(\Omega) \), \( s > 2 \), are well-known, and for every \( T > 0 \), \( v \in L^\infty([0,T], H^s(\Omega)) \) (see [20] and references therein).

Integration by parts leads to the following: let \( f, g \in H_2^2 \), with \( f \) satisfying \((NCa\varepsilon)\). Then

\[
-\int_{\Omega} \Delta f \cdot g = \frac{2a}{\sqrt{\varepsilon}} \int_{\partial\Omega} f \cdot g + 2 \int_{\Omega} Sf : Sg \tag{5}
\]

This allows us to confirm that the weak formulation (3) contains the Navier-Stokes equation, the initial condition and the Navier boundary condition ((NP) and the divergence-free conditions being given by the choice of the spaces in point 1 of the definition).

Let \( w^\varepsilon = u^\varepsilon - v \). \( w^\varepsilon \) solves the equation

\[
\partial_t w^\varepsilon + u^\varepsilon \cdot \nabla w^\varepsilon + w^\varepsilon \cdot \nabla u - \varepsilon \Delta w^\varepsilon + \nabla(p^\varepsilon - q) = 0 \tag{6}
\]

For \( t > 0 \), \( w^\varepsilon \) satisfies the inequality

\[
\frac{1}{2} \|w^\varepsilon(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_{\Omega} (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon + 2a\sqrt{\varepsilon} \int_0^t \int_{\partial\Omega} u^\varepsilon \cdot w^\varepsilon + 2\varepsilon \int_0^t \int_{\Omega} Su^\varepsilon : Sw^\varepsilon \\
\leq \frac{1}{2} \|w^\varepsilon(0)\|_{L^2(\Omega)}^2 \tag{7}
\]

which can be seen as (6) multiplied by \( w^\varepsilon \), integrated by parts in space and integrated in time. Note that this is possible only because we are working in 2D, which gives \( w^\varepsilon \) the right regularity to be used as a test function. In 3D or higher, (7) also holds; to prove it, one starts by writing the energy inequality (3), then get three energy equalities: (I) by multiplying the Euler equation by \( v \), (II) by again multiplying \((E)\) by \( u^\varepsilon \), and (III) by multiplying the Navier-Stokes equation by \( v \). One then gets (7) by doing (4)+(I)-(II)-(III) and using (5).
Let \( a > 0 \). For every \( x, y \in \mathbb{R}^2 \), let \( z = x - y \). We have
\[
2(x \cdot z) = 2 \left| z + \frac{y}{2} \right| - \frac{1}{2} |y|^2
\]
and the same goes for the contracted matrix product. So, from \( 8 \) we get
\[
\|w^\varepsilon(t)\|_{L^2}^2 + 4a \sqrt{\varepsilon} \int_0^t \int_{\partial \Omega} |w^\varepsilon + \frac{\varepsilon}{2}|^2 + 4\varepsilon \int_0^t \int_{\Omega} |S \left( w^\varepsilon + \frac{\varepsilon}{2} \right)|^2
\]
\[
\leq \|w^\varepsilon(0)\|_{L^2}^2 - 2 \int_0^t \int_{\partial \Omega} (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon + a \sqrt{\varepsilon} \int_0^t \int_{\partial \Omega} |v|^2 + \varepsilon \int_0^t \int_{\Omega} |Sv|^2
\]
The left side is greater than \( \|w^\varepsilon(t)\|_{L^2}^2 \), and we estimate each term on the right-hand side as follows:
- \( \int_{\partial \Omega} (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon \leq \|\nabla v\|_{L^\infty} \|w^\varepsilon\|_{L^2}^2 \) because \( v(\tau) \in H^s(\Omega) \) for every \( \tau \geq 0 \) and for a certain \( s > 2 \), so \( v(\tau) \in C^2(\Omega) \) and \( \nabla v(\tau) \in L^2(\Omega) \);
- \( \int_{\partial \Omega} |v|^2 \leq C_1 \|v\|_{H^1}^2 \) for a certain \( C_1 > 0 \);
- \( \|Sv\|_{L^2}^2 \leq \|v\|_{H^1}^2 \) given that \( Sv = \frac{1}{2}(\nabla v + \nabla v) \);
so \( 9 \) becomes
\[
\|w^\varepsilon(t)\|_{L^2}^2 \leq \|w^\varepsilon(0)\|_{L^2}^2 + \varepsilon \left( \frac{aC_0}{\sqrt{\varepsilon}} + 1 \right) \int_0^t \|v\|_{H^1}^2 \ d\tau + 2 \int_0^t \|\nabla v\|_{L^\infty} \|w^\varepsilon\|_{L^2}^2 \ d\tau
\]
to which we apply Grönwall’s lemma, and we obtain
\[
\|w^\varepsilon(t)\|_{L^2}^2 \leq \left[ \|w^\varepsilon(0)\|_{L^2}^2 + \varepsilon \left( \frac{aC_0}{\sqrt{\varepsilon}} + 1 \right) \int_0^t \|v\|_{H^1}^2 \ d\tau \right] \exp \left( 2 \int_0^t \|\nabla v\|_{L^\infty} \ d\tau \right)
\]
Fix \( T > 0 \), and, if \( (E(\Omega), \|\cdot\|_E) \) is a Banach space of functions on \( \Omega \), we denote \( \|f\|_{\infty,E} = \sup_{t \in [0,T]} \|f(t)\|_E \). Using the facts that \( v \in L^\infty([0,T],H^1(\Omega)) \) and that for every \( t \in [0,T] \), \( v(t) \in H^2(\Omega) \subset \text{Lip}(\Omega) \), we rewrite \( 10 \) as
\[
\|w^\varepsilon(t)\|_{L^2} \leq \left[ \|w^\varepsilon(0)\|_{L^2} + \sqrt{\varepsilon} (aC_0 + \sqrt{\varepsilon}) T \|v\|_{\infty,H^1}^2 \right] \exp \left( 2T \|v\|_{\text{Lip}} \right)
\]
The right-hand side of this last inequality is bounded uniformly in \( t \), and goes to 0 as \( \varepsilon \to 0 \), so we have \( \|w^\varepsilon\|_{\infty,H^2(\Omega)} \xrightarrow{\varepsilon \to 0} 0 \). □

Let \( a < 0 \). Following the method above leads to a situation in which the coefficient in front of the exponential term after applying Grönwall’s lemma no longer converges to zero. Instead of \( 8 \), we use \( x \cdot z = |x|^2 + y \cdot z \) in \( 7 \) to get
\[
\|w^\varepsilon(t)\|_{L^2}^2 + 2a \int_0^t \int_{\partial \Omega} |w^\varepsilon + \frac{\varepsilon}{2}|^2 + 2\varepsilon \int_0^t \int_{\Omega} |Sv| |w^\varepsilon|^2 - 2a \varepsilon \int_0^t \int_{\Omega} (w^\varepsilon \cdot \nabla v) \cdot w^\varepsilon
\]
\[
-2a \varepsilon \int_0^t \int_{\Omega} Sv : Sw^\varepsilon + 2|a\sqrt{\varepsilon} \int_0^t \int_{\partial \Omega} |w^\varepsilon|^2 + 2|a| \sqrt{\varepsilon} \int_0^t \int_{\partial \Omega} v \cdot w^\varepsilon
\]
Regrouping all but the last two terms on the right-hand side, we write
\[
\|w^\varepsilon(t)\|_{L^2}^2 + 2a \int_0^t \int_{\partial \Omega} |w^\varepsilon|^2 \leq T_1 + T_2 + T_3
\]

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• \(|T_1| \leq \frac{1}{2} \|w^\varepsilon(0)\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^\infty} \|w^\varepsilon\|_{L^2}^2 + \varepsilon \int_0^t (\|S(v)\|_{L^2}^2 + \|S(w^\varepsilon)\|_{L^2}^2);

• \(T_2 \leq 2|a|\sqrt{\varepsilon C_{\gamma}} \int_0^t \|w^\varepsilon\|_{H^{1/2}}^2\) by trace continuity. We then use, in that order, the Sobolev interpolation inequality, \(\|w^\varepsilon\|_{H^{1/2}}^2 \leq \|w^\varepsilon\|_{L^2} \|w^\varepsilon\|_{H^1}\), Young’s inequality with a positive parameter \(\eta > 0\) depending on \(\varepsilon\) to be chosen later, and Korn’s inequality to get

\[
T_2 \leq 2|a|\sqrt{\varepsilon C_{\gamma}} \int_0^t \|S w^\varepsilon\|_{L^2}^2 + (\eta + 2)C_{\gamma}|a|\sqrt{\varepsilon} \int_0^t \|w^\varepsilon\|_{L^2}^2 := T'
\]

We recall Korn’s inequality: if \(f\) is a \(H^1\) vector field on the half-space \(\Omega\), tangent to the boundary, then \(\|\nabla f\|_{L^2(\Omega)}^2 \leq 2 \|Sf\|_{L^2(\Omega)}^2\). If \(f \in H_0^1\), we in fact have an equality.

• and finally \(T_3 \leq |a|\sqrt{\varepsilon} \int_0^t \|\nabla v\|_{L^\infty} \|w^\varepsilon\|^2 \leq |a|\sqrt{\varepsilon} C_{\gamma} \int_0^t \|w^\varepsilon\|_{L^2}^2 + \frac{T'}{2}
\)

So (12) becomes

\[
\frac{1}{2} \|w^\varepsilon(t)\|_{L^2}^2 + 2\varepsilon \int_0^t \|S w^\varepsilon\|_{L^2}^2 \leq \frac{1}{2} \|w^\varepsilon(0)\|_{L^2}^2 + \sqrt{\varepsilon + |a|C_{\gamma}} \int_0^t \|w^\varepsilon\|_{H^1}^2 + \frac{T'}{2}
\]

\[
+ \int_0^t \left(\|\nabla v\|_{L^\infty} + \frac{3}{2} (\eta + 2)C_{\gamma}|a|\sqrt{\varepsilon}\right) \|w^\varepsilon\|_{L^2}^2 + \varepsilon \left(1 + \frac{3|a|C_{\gamma}}{\eta\sqrt{\varepsilon}}\right) \int_0^t \|S w^\varepsilon\|_{L^2}^2
\]

We fix \(\varepsilon\) and choose \(\eta = \frac{3|a|C_{\gamma}}{\sqrt{\varepsilon}}\) so that the terms with \(\|S w^\varepsilon\|_{L^2}\) cancel each other out. Then we take \(T > 0\), use Grönwall’s lemma and estimate the right-hand side independently of \(t \in [0,T]\) and get

\[
\|w^\varepsilon\|_{L^2}^2 \leq \left(\|w^\varepsilon(0)\|_{L^2}^2 + 2\sqrt{\varepsilon(\varepsilon + |a|C_{\gamma})T \|v\|_{H^1}^2}\right)
\]

\[
\times \exp \left[T \left(\|v\|_{\infty, \text{Lip}} + 3|a|\sqrt{\varepsilon} C_{\gamma} + \frac{9(aC_{\gamma})^2}{2}\right)\right]
\]

Again, we can conclude that \(\|w^\varepsilon\|_{\infty, L^2} \xrightarrow{\varepsilon \to 0} 0\). \(\square\)

**Remark:** one can study a more general Navier boundary condition depending on the viscosity: \(\partial_t u_1(t,x) = a \varepsilon^{-\beta} u_1(t,x,0)\).

\(\sqrt{\varepsilon}\) is then replaced by \(\varepsilon^{1-\beta}\) in (3), (11) and (12). When \(a \geq 0\), in the same way as we got (11), we obtain

\[
\|w^\varepsilon(t)\|_{L^2} \leq \|w^\varepsilon(0)\|_{L^2} + a A T_{\varepsilon} \varepsilon^{1-\beta} \exp(B_{T_{\varepsilon}}^+)
\]

When \(a < 0\), we use (3), the Sobolev, Young and Korn inequalities to get

\[
\|w^\varepsilon(t)\|_{L^2} \leq \|w^\varepsilon(0)\|_{L^2} + |a| A T_{\varepsilon} \varepsilon^{1-2\beta} \exp(B_{T_{\varepsilon}}^-)
\]

where \(A T_{\varepsilon}, A' T_{\varepsilon} \sim 1 + \varepsilon^{\theta}, \theta > 0\). The difference in powers of \(\varepsilon\) is due to the use of Young’s inequality when \(a < 0\): to cancel out the terms involving \(\|S w^\varepsilon\|_{L^2}\), one needs a coefficient \(\eta \sim \varepsilon^\beta\).

Thus, in both cases, we have ranges of \(\beta\) for which the \(L^2\) stability result holds: for \(a > 0\), we have convergence when \(\beta < 1\); for \(a = 0\) we have stability for any \(\beta\); and for \(a < 0\), when \(\beta \leq 1/2\) (convergence for the critical value \(\beta = 1/2\) having been proved above).
3 Proof of the nonlinear instability result

As mentioned in the Introduction, a resolvent estimate will be required, we prove it first. Then we use an asymptotic expansion of solutions of (NSε) around a linearly unstable shear profile $u_s$, as in [13]. Like for Theorem 1 there will be a slight difference between the positive case (with $a > 0$) and the negative case. An example of linearly unstable shear flow will be given in the final paragraph.

3.1 Preliminary results on the linearised Euler equation

3.1.1 Linear instability of the Euler equation

We set $\sigma(k)$ the highest real part of complex numbers $\lambda$ that are unstable eigenvalues of the linear Euler equation; for each wave number $k$, it is finite, and $\sigma$ is an even analytic function of $k$. In turn, the function $\sigma$ has a maximum $\sigma_0 > 0$ (see the study of the Rayleigh equation in [13], paragraph 4). We assume that this maximum is nondegenerate.

Fix a wave number $k$, and consider the space $V_k$ of divergence-free Fourier modes written as $u(t, x, y) = v(t, y) e^{ikx}$, with $v(t, \cdot) \in H^s$ for every $s > 0$. We write divergence-free fields tangent to the boundary by using stream functions:

$$u(t, x, y) = e^{ikx} (\partial_y \Psi(t, y), -ik \Psi(t, y)) = \nabla_\perp (e^{ikx} \Psi(t, y))$$

with $\Psi(t, 0) = 0$. Note that we have $\text{rot } u = -\Delta (e^{ikx} \Psi(t, y))$. For $u \in V_k$, we use the norm $\|u\|_{l} = \sqrt{\|u\|_{H^s}^2 + \|\text{rot } u\|_{H^s}^2}$. As $\text{rot } u = (k^2 - \partial_{yy}) \Psi$, standard elliptic regularity (see [18] for example) gives us

$$\|\nabla u(t)\|_{L^2(\Omega)} = \|\nabla^2 (e^{ikx} \Psi)\|_{L^2(\Omega)} \sim \|\text{rot } u(t)\|_{L^2(\Omega)} \quad (13)$$

Using this, and standard Sobolev inequalities, we get that if $u \in V_k$ and $u' \in V_{k'}$, then $u \cdot \nabla u' \in V_{k+k'}$ and

$$\|u \cdot \nabla u'\|_{l} \leq C \|u\|_{l_2} \|u'\|_{l_2} \quad (14)$$

**Theorem 3.** Let $\lambda' > \sigma_0$, and $w(t, x, y) \in V_k$ such that

$$\|w(t)\|_{l} \leq C_w \frac{e^{\lambda' t}}{(1 + t)^\alpha} \quad (15)$$

for every $l \geq 2$ and for some $\alpha \geq 0$. Consider the linearised Euler equation with source term

$$(\text{ELS}) : \quad \begin{cases} 
\partial_t u + u_s \cdot \nabla u + u \cdot \nabla u_s + \nabla p = w \\
\text{div } u = 0 \\
u|_{t=0} = u_0 \in V_k \\\nu|_{y=0} = u_1 \in V_k
\end{cases}$$

with $|u_1(t)| \leq C_w (1 + t)^{-\alpha} e^{\lambda' t}$, and likewise for $\partial_t u_1$. The initial condition $u_0$ must be compatible with the boundary condition $u_1$. Then the solution of this system satisfies the estimate

$$\|u(t)\|_{l-2} \leq C (1 + t)^{-\alpha} e^{\lambda' t}$$
for \( t > 0 \), with \( u(t) \in V_k \), and \( C \), which depends on \( w, u_0, u_1, l \) and \( k \), and is locally bounded in the parameter \( k \).

We show the result for functions with a single Fourier mode, but it extends to wave packets

\[
U(t, x, y) = \int_{\mathbb{R}} \varphi(k)u(k; t, x, y) \, dk
\]

with \( u(k) \in V_k \) and \( \varphi \) a smooth compact-supported function.

**Proof:** we adapt the arguments used to prove the resolvent estimate for the KdV equation in [31]. Set

\[
u(t, y) = e^{ikz} \begin{pmatrix} 0 \\ -ik \Psi(t, y) \end{pmatrix} \quad \text{and} \quad w(t, x, y) = e^{ikz} \bar{w}(t, y)
\]

and note that rot \( u = (k^2 - \partial_{yy})\Psi : = B_k \Psi \), where, for \( k \neq 0 \), the differential operator \( B_k : H^{l+2}(\mathbb{R}^+) \cap H^2_0(\mathbb{R}^+) \to H^l(\mathbb{R}^+) \) is invertible.

We first examine the equation

\[
\partial_t B_k \Psi + ik(u_x B_k \Psi + u''_x \Psi) = \text{rot } w
\]

We then set

\[
\Psi_0(t, y) = \Psi(t, y) - e^{-\mu t} \Psi(0, y) - e^{-\nu y} \Psi(t, 0) + e^{-\nu(t+y)} \Psi(0, 0)
\]

with arbitrary \( \mu > 0 \), thus \( \Psi_0(0, y) = 0 \) and \( \Psi_0(t, 0) = 0 \). \( \Psi_0 \) solves

\[
\partial_t B_k \Psi_0 + ik(u_x B_k \Psi_0 + u''_x \Psi_0) = F_0
\]

We will not give the detailed expression of \( F_0 \), but we point out that integrating the \( L^2 \) hermitian dot-product of \( F_0 \) by \( \Psi_0 \) by parts leads to

\[
|(F_0| \Psi_0) | \leq C \left( \| f_0 \|_{L^2}^2 + \| \nabla_k \Psi_0 \|_{L^2}^2 \right)
\]

where \( f_0 \) contains data terms \( w, u_0 \) and \( u_1 \), and \( \| \nabla_k \Psi_0 \|_{L^2} \) is the norm of the velocity:

\[
\nabla_k \Psi_0 = (k \Psi_0, \partial_y \Psi_0)
\]

Note that \( f_0 = \Phi_1 + \Phi_2 e^{-\mu t} \), where \( \Phi_1 \) contains rot \( w \) and terms with \( y = 0 \) (namely \( u_1 \) and \( \partial_t u_1 \)), while \( \Phi_2 \) contains terms with \( t = 0 \).

Using the Laplace transform in time, \( \mathcal{L}_t g(z, y) = \int_0^\infty e^{-iz} g(s, y) \, ds \), and writing \( \tilde{\Psi} = \mathcal{L}_t \Psi_0 \) and \( F = \mathcal{L}_t F_0 \), we turn this differential equation into the eigenvalue problem:

\[
z B_k \tilde{\Psi} + ik(u_x B_k \tilde{\Psi} + u''_x \tilde{\Psi}) = F
\]

We choose \( \gamma_0 \in ]\sigma_0, \lambda[ \), and set \( z = \gamma_0 + i\tau \). \( \gamma_0 \) being fixed, we abbreviate \( \tilde{\Psi}(z) = \tilde{\Psi}(\tau) \). When \( \tau \) evolves in \( \mathbb{R} \), we get the following estimates:

**Lemma 4.** Let \( l \geq 0 \), and \( \tilde{\Psi} \) solve \((17)\) with \( \tilde{\Psi}(\tau, 0) = 0 \), and \( F \) verifying

\[
|(F| \tilde{\Psi}) | \leq C \left( \| \nabla_k \tilde{\Psi} \|_{L^2}^2 + \| f \|_{L^2}^2 \right)
\]
where \( f = L_t f_0 \) depends on the data. There exists \( C \) depending on \( k, l \) and \( u_s \), locally bounded in the parameter \( k \), such that

\[
\left\| B_k \Psi(\tau) \right\|_{H^l}^2 \leq C \| f(\tau) \|_{H^{l+2}}^2 \quad \text{and} \quad \left\| \nabla_k \Psi \right\|_{H^l}^2 \leq C \| f \|_{H^{l+2}}^2
\]

Note that the lemma provides estimates on both the vorticity and the velocity. We prove this lemma in the next sub-paragraph.

By Parseval's equality, we have

\[
\int_{0}^{+\infty} e^{-2\gamma_0 t} \| B_k \Psi_0 \|_{H^l}^2 \, dt = \int_{-\infty}^{+\infty} \left\| B_k \Psi(\tau) \right\|_{H^l}^2 \, d\tau
\]

\[
\leq C \int_{-\infty}^{+\infty} \| f(\tau) \|_{H^{l+2}}^2 \, d\tau
\]

\[
\leq C \int_{0}^{+\infty} e^{-2\gamma_0 t} \| \Phi_1(t) \|_{H^{l+2}}^2 \, dt
\]

\[
+ \int_{0}^{+\infty} e^{-2(\gamma_0 + \rho)t} \| \Phi_2 \|_{H^{l+2}}^2 \, dt
\]

\[
\leq C \int_{0}^{+\infty} e^{-2\gamma_0 t}(\| \Phi_1(t) \|_{H^{l+2}}^2 + \| \Phi_2 \|_{H^{l+2}}^2) \, dt
\]

Replacing \( \Phi_1 \) by \( \Phi_1 1_{[0, T]} \times \mathbb{R}^+ \) for some \( T > 0 \) does not affect the solution on \([0, T] \times \mathbb{R}^+\), so \((15)\) gives us

\[
\int_{0}^{T} e^{-2\gamma_0 t} \| B_k \Psi_0 \|_{H^l}^2 \, dt \leq C \int_{0}^{T} \frac{e^{2(\lambda' - \gamma_0)t}}{(1 + t)^{\alpha}} \, dt
\]

where \( C \) depends on \( \| u(0) \|_{H^l}, \gamma_0, l \) and \( k \), and is a locally bounded function of \( k \). Noticing that

\[
\| \Psi_0(t) \|_{H^l} \geq \| \Psi(t) \|_{H^l} - \| u_0 \|_{H^l} - \| e^{-\mu} \|_{H^l} (|u_1(t)| + |\Psi(0, 0)|)
\]

we get

\[
\int_{0}^{T} e^{-2\gamma_0 t} \| B_k \Psi \|_{H^l}^2 \, dt \leq C \int_{0}^{T} \frac{e^{2(\lambda' - \gamma_0)t}}{(1 + t)^{\alpha}} \, dt \leq C \frac{e^{2(\lambda' - \gamma_0)T}}{(1 + T)^{\alpha}}
\]

(19)

(the last inequality is obtained by integrating by parts).

Note that, using the same procedure, estimate \((19)\) also holds for \( \| u \|_{H^l}^2 \):

\[
\int_{0}^{T} e^{-2\gamma_0 t}(\| k \|_{H^l}^2 + \| \partial_x \Psi \|_{H^l}^2) \, dt \leq C \frac{e^{2(\lambda' - \gamma_0)T}}{(1 + T)^{\alpha}}
\]

(20)

A quick energy estimate on \((16)\) gives us

\[
\frac{1}{2} \partial_t \| B_k \Psi(t) \|_{H^l}^2 \leq C(\| B_k \Psi(t) \|_{H^{l+2}}^2 + \| u(t) \|_{H^l}^2 + \| \text{rot} w(t) \|_{H^l}^2)
\]

Multiply this by \( e^{-2\gamma_0 t} \), and using and the hypothesis on \( w \), we get

\[
\partial_t (e^{-2\gamma_0 t} \| B_k \Psi \|_{H^l}^2) \leq C \left( e^{-2\gamma_0 t} \| B_k \Psi \|_{H^{l+2}}^2 + e^{-2\gamma_0 t} \| u \|_{H^l}^2 + \frac{e^{2(\lambda' - \gamma_0)T}}{(1 + t)^{\alpha}} \right)
\]
Thus, for \( |k| \) large from the case \( |k| \) close to zero. Take \( |k| \geq k_0 \) to be chosen later, and the imaginary part of the \( L^2 \) dot-product of equation (17) and \( \Theta = B_k \Psi \). The elliptic regularity estimate \( \|\Psi\|_{H^2} \leq \max(1, k_0^{-2}) \|B_k \Psi\|_{L^2} \) then gives us

\[
|\tau| \|\Theta\|_{L^2}^2 \leq \left[ C|k| + \frac{1}{2} \right] \|\Theta\|_{L^2}^2 + \frac{1}{2} \|F\|_{L^2}^2
\]

with \( C \) depending on \( u_s \) and \( k_0 \). Setting \( g(\tau) = |\tau| - |k| \|u_s\|_{L^\infty} - C|k| - \frac{1}{2} \) and \( M \) so that \( g(M) = 1 \), we have, for \( |\tau| \geq M \), \( \|\Theta\|_{L^2} \leq \|F\|_{L^2} \leq \|f\|_{H^1} \).

Now take \( |k| \leq k_0 \). The imaginary part of the dot-product of (17) and \( \Theta \) leads to

\[
|\tau| \|\Theta\|_{L^2}^2 \leq (C_s|k| + 1) \|\Theta\|_{L^2}^2 + \frac{1}{2} \|F\|_{L^2}^2 + C_s \left\| \nabla_k \bar{\Psi} \right\|_{L^2}^2
\]

using the Young inequality, and that \( \left\| k \bar{\Psi} \right\|_{L^2}^2 \leq \left\| \nabla_k \bar{\Psi} \right\|_{L^2}^2 \). Using the inequality (15), we consider the real part of the dot-product of (16) and \( \bar{\Psi} \): as \( (u_s' \bar{\Psi} |\bar{\Psi}) \) is real, we get

\[
\gamma_0 \left\| \nabla_k \bar{\Psi} \right\|_{L^2}^2 \leq \left[ C|k| + \frac{\gamma_0}{2} \right] \left\| \nabla_k \bar{\Psi} \right\|_{L^2}^2 + C \|f\|_{L^2}^2
\]

Thus, for \( |k| \leq k_0 \) such that \( Ck_0 + \frac{\gamma_0}{2} = \frac{3\gamma_0}{4} \), we have \( \left\| \nabla_k \bar{\Psi} \right\|_{L^2}^2 \leq C \|f\|_{L^2}^2 \), with \( C \) depending only on \( \gamma_0 \) and \( u_s \). This allows us to conclude the estimate (21):

\[
h(\tau) \|\Theta\|_{L^2}^2 = (|\tau| - C_s|k| - 1) \|\Theta\|_{L^2}^2 \leq C \|f\|_{L^2}^2
\]

and again, we choose \( |\tau| \geq M \) so that \( h(M) = 1 \). Note that \( M \) depends affinely on \( |k| \), so we have a uniform bound on \( \|\Theta(\tau)\|_{L^2} \) for \( |k| \leq \rho \) and \( |\tau| \geq M(\rho) \).
$H^l$ estimates are easily obtained by induction: examine the dot-product of $\partial^l \Theta$ by $\partial^l \Theta$, and use Young’s inequality and the induction hypothesis $\|\Theta\|_{H^{l-1}} \leq C\|F\|_{H^{l-1}}$ to get $\frac{\gamma_0}{2} \|\partial^l \Theta\|^2 \leq C\|F\|_{H^l}^2,$ with $C$ depending on $\gamma_0, |k|$ and $l$.

Step 2: we seek a solution to (17) as $\tilde{\Psi} = \Psi_1 + \Psi_2$, where

\begin{equation}
-zB_k \Psi_1 = F \tag{22}
\end{equation}

\begin{equation}
-(z +iku)_{\alpha}B_k \Psi_2 + iku^{\prime\prime}_{\alpha} \Psi_2 = -ik(u_{\beta} B_k \Psi_1 + u^{\prime\prime}_{\beta} \Psi_1) := kG \tag{23}
\end{equation}

Immediately, we get $\|B_k \Psi_1\|_{H^l} \leq \frac{\gamma_0}{2} \|F\|_{H^l}$, and multiplying (22) by $\Psi_1$, we get

$$\frac{\gamma_0}{2} \|\nabla_k \Psi_1\|^2 \leq \frac{1}{\gamma_0} (\|\nabla_k \Psi_1\|^2 \big|_{t=0} + \|F\|_{L^2}^2)$$

so $\|\Psi_2\|_{H^l} \leq C\|kG\|_{H^l}$ estimate suffices to prove the result, as

$$\|kG\|_{H^l} \leq c|k| \|B_k \Psi_1\|_{H^l} + c' \|\nabla_k \Psi_1\|_{H^l}$$

First rewrite (23) as an ordinary differential system:

$$\partial_y U(y) = A(k, z; y)U(y) + H(y) \tag{24}$$

with $U = (\Psi_2, \partial_y \Psi_2)$, $H = (0, kG)$ and

$$A(k, z; y) = \begin{pmatrix}
0 & 1 \\
-k^2 - \frac{iku^{\prime\prime}_{\alpha}(y)}{z + ik u_{\alpha}(y)} & 0
\end{pmatrix} = A_\infty(k) + B(k, z; y)$$

where $A_\infty(k) = \lim_{\eta \to +\infty} A(k, z; y)$, and $B(k, z; y) = O(e^{-\eta y})$ for some $\eta > 0$ since $u_{\alpha}^{\prime\prime}$ decays exponentially. The eigenvalues of $A_\infty(k)$ are real, so by the roughness of exponential dichotomy [3], the system $\partial_y U = A(k, z)U$ has an exponential dichotomy on $\mathbb{R}^+$; this means that if $T(k, z; y, y')$ is the fundamental solution of this last equation with $T(k, z; y, y) = I_2$, there exist a projection $P(k, z; y)P(k, z; y') = P(k, z; y)T(k, z; y, y')$, and positive constants $C$ and $\alpha$, with all of these depending smoothly on $(k, z)$, such that for any $\xi \in \mathbb{C}^2$,

$$[T(k, z; y, y')P(k, z; y')\xi] \leq C(k, z)e^{-\alpha(y-y')}|\xi| \quad \text{if } y \geq y' \geq 0$$

$$[T(k, z; y, y')(I - P(k, z; y'))\xi] \leq C(k, z)e^{\alpha(y-y')}|\xi| \quad \text{if } y' \geq y \geq 0$$

For any $0 < \rho < \rho'$, $C(k, z)_{\rho, \rho'}$ is uniformly bounded in the set

$$K_{\rho, \rho'} = \{(k, \gamma_0 + i\tau) \mid |k| \in [\rho, \rho'] \text{ and } |\tau| \leq M\}$$

but we would like $C(k, z)$ to be bounded in $K_{0, \rho'}$ for a certain $\rho'$.

To get that $C(k, z)$ is bounded near $k = 0$, we follow the proof of the persistence of ordinary dichotomies as done in [5]. $A_\infty(k)$ cannot be uniformly diagonalised near $k = 0$, because the basis of diagonalisation for $k \neq 0$ is

$$\begin{pmatrix}
1 & 1 \\
-k & |k|
\end{pmatrix}.$$ Instead, we change to the basis

$$(v_1(k), v_2(k)) = \begin{pmatrix}
1 & 0 \\
-k & 1
\end{pmatrix}$$

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Consider $E$ such that the first component vanishes at $y = 0$. Let $U$ be a solution of the equation $\partial U = A_\infty(k)U$. There exists a projection $\Pi$ on this space such that

$$|T_\infty(k; y, y')(I - \Pi(k; y'))[\xi]| \leq e^{-|k|(y-y')}|\xi| \quad \text{if } y \geq y' \geq 0$$

$$|T_\infty(k; y, y')\Pi(k; y')[\xi]| \leq e^{|k|(y-y')}|\xi| \quad \text{if } y' \geq y \geq 0$$  \hspace{1cm} (25)

Consider $E(k)$, the Banach space of functions $V$ such that

$$\|V\|_{E(k)} := \|V e^{[y]}\| < +\infty$$

and $S$ the linear mapping defined by

$$SV(y) = \int_0^y T_\infty(k; y, y')(I - \Pi(k; y'))B(k, z; y')V(y') \, dy'$$

$$- \int_y^{+\infty} T_\infty(k; y, y')\Pi(k; y')B(k, z; y')V(y') \, dy'$$

Note that $|B(k, z; y)| \leq b|k|e^{-ny'}$. Using (25), we get that if $V \in E(k)$, then $SV \in E(k)$ with the estimate

$$|SV(y)e^{[y]}| \leq \|V\|_{E(k)} \left(\int_0^{+\infty} b|k|e^{-ny'} \, dy'\right) \leq \frac{1}{2} \|V\|_{E(k)} (26)$$

when $|k| \leq \rho'$ small enough. So $S$ is a contracting endomorphism of $E(k)$.

Let $U_\infty(k) \in E(k)$ be a solution of $\partial U = A_\infty(k)U$. By the Duhamel formula, a bounded solution of the equation $\partial U = AU$ is a fixed point of the affine transform $\tilde{S} = S + U_\infty$; thanks to (26), Picard’s fixed point theorem allows us to conclude that such a solution $U$ exists in $E(k)$. Finally, we must uniformly bound $\|U\|_{E(k)}$. Choosing $U_\infty(k) = (e^{-|k|y}, -|k|e^{-|k|y})$, the decreasing eigenfunction of $U_\infty$, (26) gives us the wanted bound.

The end of the proof is the same as in [31]; we provide it for completeness. Note that for $k \neq 0$, $I - P(k, z)$ is the projection on the subspace of solutions that go to 0 when $y \to +\infty$; let us define $Q(k, z)$ the projection on the subspace of solutions such that the first component vanishes at $y = 0$. As the linearised Euler equation does not have an eigenvalue with real part $\gamma_0$, necessarily

$$\mathcal{R}(I - P(k, z; 0)) \cap \mathcal{R}(Q(k, z; 0)) = \{0\} \hspace{1cm} (27)$$

and we define a basis of solutions $(e_1(k, z), e_2(k, z))$, with $e_1 \in \mathcal{R}(I - P(k, z; 0))$ and $e_2 \in \mathcal{R}(Q(k, z; 0))$. Define a new projection

$$P'(k, z) = (e_1(k, z), e_2(k, z)) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (e_1(k, z), e_2(k, z))^{-1}$$

and $P'(k, z; y) = T(k, z, y, 0)P'(k, z)$, so that $\mathcal{R}(P'(k, z; y)) = \mathcal{R}(Q(k, z; y))$ and $\mathcal{R}(I - P'(k, z; y)) = \mathcal{R}(I - P(k, z; y))$. We also have the estimates

$$|T(k, z, y, y')P'(k, z, y')[\xi]| \leq C'(k, z)e^{-\alpha(y-y')}|\xi| \quad \text{if } y \geq y' \geq 0$$

$$|T(k, z, y, y')(I - P'(k, z; y'))[\xi]| \leq C'(k, z)e^{\alpha(y-y')}|\xi| \quad \text{if } y' \geq y \geq 0 \hspace{1cm} (28)$$
Again, we must check that \( C'(k, z) \) is bounded in \( K_{0,p} \). To do so, we point out that the projections \( Q(k, z) \) and \( I - P(k, z) \) can be continued up to \( k = 0 \): we have that \( Q(0, z) \) is the subspace of solutions to the equation \( z \Psi'' = 0 \) with \( \Psi(0) = 0 \), and \( I - P(0, z) \) is the subspace of bounded solutions to \( z \Psi'' = 0 \) (constants). As the only bounded solution of \( z \Psi'' = 0 \) with \( \Psi(0) = 0 \) is \( \Psi \equiv 0 \), \( \Psi'(0) \) is true up to \( k = 0 \), so \( C'(k, z) \) is bounded for \( k \) near \( 0 \).

Finally, by Duhamel’s formula, a bounded solution of (24) is, for fixed \((k, z)\),

\[
U(y) = \int_{0}^{y} T(y, y') P(y') \, dy' = \int_{0}^{+\infty} T(y, y')(I - P(y')) \, dy'
\]
since the only bounded solution to \( \partial_y U = A(k, z) U \) is zero. So, by using (28) and standard convolution estimates, we have \( \|U\|_{L^2} \leq C(k, z) \|H\|_{L^2} \). To get estimates on the \( L^2 \) norms of the derivatives, just differentiate the equation \( l \) times, notice that it is the same type of system and conclude by induction. \( \square \)

### 3.2 Proof of the instability result (Theorem 2)

Until the last sub-paragraph, we consider the positive case, which is that of the \((\text{NS})_\varepsilon\) equation with the \((\text{NC}a \varepsilon)\) Navier boundary condition with \( a > 0 \).

First and foremost, as the initial profile depends on \( \varepsilon^{-1/2}y \), we rescale the variables \((t, x, y) \rightarrow \varepsilon^{1/2}(t, x, y)\), and these new variables will be indifferently noted \((t, x, y)\). From now on, the faster variable will be \( Y = \varepsilon^{-1/2}y \). Indeed, the system we get after rescaling is

\[
\begin{align*}
\text{(NS):} & \quad \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon - \sqrt{\varepsilon} \Delta u^\varepsilon + \nabla p^\varepsilon = 0 \\
\text{div} u^\varepsilon &= 0 \\
\text{(NP):} & \quad u^\varepsilon(0, x, y) = u_0^\varepsilon(x, y) \\
\text{(NC + 1):} & \quad \frac{1}{\varepsilon} \partial_y u^\varepsilon(t, x, 0) = u_1(t, x, 0)
\end{align*}
\]

with the initial condition \( u_0^\varepsilon \) close to a linearly unstable shear flow \( u_s \). The Navier boundary condition no longer depends on the viscosity, and we have the following asymptotic expansion (from [13]):

\[
u^\varepsilon(t, x, y) \sim \pi(t, y) + u^i(t, x, y) + \varepsilon^{1/4} u^b(t, x, Y) + O(\varepsilon^{1/2}) \tag{29}
\]

where \( u^i \) solves the Euler equation, \( u^b \) is the boundary layer and \( O(\varepsilon^{1/2}) \) is the amplitude of the error in \( L^\infty([0, T], L^2(\Omega)) \) for any \( T > 0 \).

Starting with a linearly unstable solution to the Euler equation, we will construct an approximate solution to the Navier-Stokes system \((\text{NS'})\) above that will allow us to prove the instability inequality (2).

#### 3.2.1 Building an approximate solution

We write \( u^{0p} = \pi + \sum_{j=1}^{N} \varepsilon^{jn} U^j \) and \( p^{0p} = \sum_{j=1}^{N} \varepsilon^{jn} P^j \), where \( N \) is large and to be chosen later, as an approximate solution of the \((\text{NS'})\) system, where each \( U^j \) will be written as a wave packet. Letting \( k_0 \) be such that \( \sigma(k_0) = \sigma_0 \), and setting \( \varphi^j \) as compact-supported smooth functions of \( k \) with \( k_0 \subset \text{supp}(\varphi) \), we write

\[
U^j(t, x, y) = \int_{\mathbb{R}} \varphi^j(k)V^j(k; t, y)e^{ikx} \, dk = \int_{\mathbb{R}} \varphi^j(k)V^j(k; t, x, y) \, dk
\]
with $V^j(k)$, called terms of the main expansion, being in the previously-defined $V_k$ space. We shall write these wave packets more precisely in the next paragraph. The equation that the $V^j$ are supposed to solve are Navier-Stokes equations linearised around $\mathbf{\pi}_s$:

$$\partial_t V^j + \mathbf{u}_s \cdot \nabla V^j + V^j \cdot \nabla \mathbf{u}_s - \sqrt{\varepsilon} \Delta V^j + \nabla P^j + \sum_{j_1 + j_2 = j} V^{j_1} \cdot \nabla V^{j_2} = 0 \quad (30)$$

We will not necessarily have $\text{div } V^j = 0$, but in total we must have $\text{div } u^{ap} = 0$. As we can only solve (30) approximately, we write a sub-expansion

$$V^j = \sum_{m=0}^{8n-1} \epsilon^{m/8} v^{i,q,jn+m}(t,x,y) + \epsilon^{(m+2)/8} v^{b,q,jn+m}(t,x,Y) \quad (31)$$

following the expansion (29), and likewise for the pressure - there is a gap of $\varepsilon^{1/4}$ between corresponding internal and boundary layers. The first term, $V^1$, solves the linearised Navier-Stokes equation above with an error:

$$\partial_t V^1 + \mathbf{u}_s \cdot \nabla V^1 + V^1 \cdot \nabla \mathbf{u}_s - \sqrt{\varepsilon} \Delta V^1 + \nabla P^1 = \varepsilon^n \mathcal{E}^j$$

For $j > 1$, we ask $V^j$ to take the previous error into account, but the fact that we once again solve the equation for $V^j$ approximately creates a new error. So, $V^j$ solves

$$\partial_t V^j + \mathbf{u}_s \cdot \nabla V^j + V^j \cdot \nabla \mathbf{u}_s - \sqrt{\varepsilon} \Delta V^j = -\mathcal{E}^{j-1} + \varepsilon^n \mathcal{E}^j - \sum_{j_1 + j_2 = j} V^{j_1} \cdot \nabla V^{j_2}$$

Fixing a wave number $k$ and writing $q = 8jn + m$, we must understand the equations that $v^{b,q} = v^{b,q}(k)$ solve to get estimates on these functions.

We plug the ansatz (31) into each equation of the linearised Navier-Stokes system, and consider the equations obtained when asking terms of a same order $\varepsilon^{q/8}$ to cancel out. We get that the internal layers $v^{i,q}$, which solve only the $\mathbf{NS}$ Euler part of the $(\mathbf{NS} \sqrt{\varepsilon})$ equation, are solutions of linear Euler systems

$$(\mathbf{EL}(q)) : \begin{cases} (E_q) : & \partial_t v^{i,q} + u_s \cdot \nabla v^{i,q} + v^{j,q} \cdot \nabla u_s + r^{i,q} + \nabla p^{i,q} = 0 \\
 & \text{div } v^{i,q} = 0 \\
 & v^{i,q}(t,x,0) + v^{b,q}(t,x,0) = 0 \end{cases}$$

where we linearise around $u_s$ instead of $\mathbf{\pi}_s$, because we have a better understanding of the linear equation around time-independent profiles, and

$$r^{i,q} = \frac{u_s - u_s \cdot \nabla v^{i,q-1} + v^{j,q-1} \cdot \nabla}{\varepsilon^{1/8}} - \Delta v^{i,q-2} + \sum_{8n+q_1+q_2=q} v^{i,q_1} \cdot \nabla v^{i,q_2}$$

We also get that the boundary layers each solve a Stokes problem with Navier boundary conditions

$$(\mathbf{S}(q)) : \begin{cases} (S_q) : & \partial_t v^{b,q} + u^{b,q} \cdot \nabla v^{b,q} - \partial_{yy} v^{b,q} + \nabla p^{b,q} = 0 \\
 & \partial_y v^{b,q} + \partial_x v^{b,q-2} = 0 \\
 & \partial_y v^{b,q} + \partial_x v^{b,q-2} = 0 \end{cases} \begin{cases} (N_q) : & \left( \frac{1}{\varepsilon} \partial_y v^{b,q} + \frac{1}{\varepsilon} \partial_y v^{i,q} - v^{i,q} - v^{b,q-2} \right) (t,x,0) = 0 \end{cases}$$

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the second boundary condition being that \( v^{b,q} \) should vanish as \( Y \to +\infty \). We do not give any detail on \( v^{b,q} \) other than that depends on \( \varepsilon^{-1/8} \) and terms \( v^{i,m} \) and \( v^{b,m} \) with \( m < q \).

In both systems, initial conditions are chosen to be compatible with the boundary conditions, and rapidly decreasing functions of \( y \) (like \( v(0, x, 0)e^{-\eta y} \)), therefore \( V^{(i)}|_{t=0} \) is \( \mathcal{O}(\varepsilon^{n}) \). For low values of \( q \), any term with a non-positive index is of course ignored, and we must add to the approximate solution a corrective term \( \varepsilon^{(N+1)\epsilon_{8}^{1/8}}\omega(t, x, Y) \), to deal with the remainders of the divergence equation and the (NP) boundary condition:

\[
\text{div} \omega = -\partial_{x}v^{b,M-1}_{1} - \varepsilon^{1/8} \partial_{x}v^{b,M}_{1} \quad \text{and} \quad \omega_{2}(t, x, 0) = (-v^{b,M-1}_{2} - \varepsilon^{1/8} v^{b,M}_{2})(t, x, 0)
\]

where \( M = 8nN \). Thus the total, \( w^{\text{ap}} \), solves the Navier-Stokes equation with an error term and an initial condition \( \varepsilon^{n} \)-close to \( u_{s} \) in \( H^{\sigma} \), but is divergence-free and satisfies all the boundary conditions of the (NS) system, although each individual \( U^{j} \) does not.

Now we know the equations that each layer is supposed to solve, we can construct the approximate solution by induction. As announced, we start by choosing \( v^{i,0} \) as an unstable solution of the linear Euler equation (\( E_{i} \)) with homogeneous non-penetration condition. In fact, we choose the most unstable mode for each \( k \), in other words

\[
v^{i,0}(k; t, x, y) = e^{\lambda(k)t} \tilde{v}(k; y)e^{ikx}
\]

with \( \Re(\lambda(k)) = \sigma(k) \). Then, \( (D_{1}) \) and the limit condition allow us to choose \( v^{b,0}_{1} = 0 \), then \( p^{b,0} = 0 \) and \( v^{b,0} \) solves a heat equation.

Having built \( v^{i,m} \) and \( v^{b,m} \) for \( m < q \), \( v^{i,q} \) is the solution of the linear Euler system \((E(q)) \). Now we build \( v^{b,q} \) by taking the equations of the Stokes system \((S(q)) \) one after another. First, \( (D_{q}) \) is the equation relative to the divergence-free condition: integrating it between \( Y \) and \( +\infty \), we get an expression of the normal component of \( v^{b,q} \) that goes to 0 as \( Y \to +\infty \); the rapid-decrease property will follow from the induction below, and nothing guarantees that we have \( v^{b,q}_{2}(t, x, 0) = 0 \), so this is where the boundary condition \((P_{q}) \) in the Euler system comes from. Then, notice that thanks to the structure of \( \overline{u_{s}} \), the normal part of the Stokes equation \((S_{q}) \) depends only on \( v^{b,j}_{2} \) and \( p^{b,q} \) (and of course previous terms), so we get \( p^{b,q} \) by the same method. Finally, the first component of \((S_{q}) \) is now a heat equation, with a boundary condition given by the Navier condition \((N_{q}) \).

### 3.2.2 Estimates on the approximate solution

The important estimates are those on the first term \( U^{1} \), or more precisely, those on the wave packet of unstable modes of the Euler equation

\[
u^{i,0}(t, x, y) = \int_{\mathbb{R}} \varphi^{i}(k) \nu^{i,0}(k; t, x, y) \, dk
\]

with \( \varphi^{i} \) supported in \( I = [-k_{0} - \eta, -k_{0} + \eta] \), where \( \eta \) is small enough so that \( I \subset \{k \mid \sigma(k) > 0\} \), and such that \( k_{0} \) is the only critical point of \( \sigma \) in \( [k_{0} - \eta, k_{0} + \eta] \). By Parseval’s equality, we have

\[
\|u^{i,0}(t)\|_{H^{\sigma}}^{2} = \int_{\mathbb{R}} \varphi^{i}(k)^{2} \|\nu^{i,0}(k; t)\|_{H^{\sigma}}^{2} \, dk = \int_{\mathbb{R}} \varphi^{i}(k)^{2} \|\tilde{v}(k; y)\|_{H^{\sigma}}^{2} \varepsilon^{2\sigma(k)t} \, dk
\]

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so, as \( \pm k_0 \) is the only nondegenerate maximum of \( \sigma \) in each sub-interval of \( I \), using the Laplace method, we get \( \|u_{t0}(t)\|_{H^s} \sim t^{-1/2}e^{2\sigma t} \) when \( t \to +\infty \), so we have, given that \( \|u_{t0}(t)\|_{H^s} \) is bounded near \( t = 0 \),

\[
C' \frac{e^{\sigma t}}{(1 + t)^{1/4}} \leq \|u_{t0}(t)\|_{H^s} \leq C \frac{e^{\sigma t}}{(1 + t)^{1/4}}
\] (32)

This allows us to estimate the wave packets associated with the other terms of \( U^j \), which are the \( u^{i,q} \) and \( u^{b,q} \) with \( q < 8n \); this is done by induction. Fix \( j < 8n \), and suppose that the internal layers \( u^{i,q} \) with \( 0 \leq q < j \) satisfy

\[
\|u^{i,q}(t)\|_{H^s} \leq C_j \exp \left[ \sigma_0 \left( 1 + \frac{1}{32} \right) \frac{t}{(1 + t)^{1/4(1 + j/8n)}} \right]
\] (33)

As \( e^{\sigma_0 t}(1 + t)^{-\alpha} \xrightarrow{t \to +\infty} +\infty \), we have

\[
\|u^{i,q}(t)\|_{H^s} \leq C_j \exp \left[ \sigma_0 \left( 1 + \frac{1}{32} \right) \frac{t}{(1 + t)^{1/4(1 + j/8n)}} \right]
\] (34)

Note that we are studying the stability of \( \pi(\sqrt{t}) \), and that by energy estimates on the heat equation with the Robin boundary condition induced by the Navier condition,

\[
\left\| \pi(\sqrt{t}) - u_s \right\|_{H^s} \leq \frac{e^{C\sqrt{t}} - 1}{e^{1/8}}
\]

so \( e^{-1/8}(\pi(\sqrt{t}) - u_s) \) is bounded in \( H^s \) uniformly for \( t \leq e^{-1/32} \). Therefore, as the remainder \( r^{i,j} \) in the Euler equation \( (E_j) \) verifies (34), as does \( \partial_r r^{i,j} \), \( u^{i,j} \) also satisfies (33) by Theorem 3.

Now we consider the boundary layer \( u^{b,j} \). Suppose that \( u^{b,q} \), \( q < j \), are rapidly decreasing functions in the \( Y \)-variable and satisfy (33). By construction, \( u_2^{b,j} \) and \( p^{b,j} \) are rapidly decreasing functions in the \( Y \)-variable. Then, as \( e^{-1/8}(\pi(t,Y)) \) is also uniformly bounded in \( \varepsilon \) for \( t \leq e^{-1/32} \), \( Y \in \mathbb{R}^+ \) and \( \varepsilon < \varepsilon_0 \) small enough (however there is no guarantee that \( e^{-1/8}\sqrt{u_s} \) verifies such an estimate, so we have to leave it as a term in the equation (34)), we get, by using the Green function of the heat equation with Neumann boundary conditions (see 33), we get that \( u^{b,j} \) is a rapidly decreasing function of \( Y \) satisfying

\[
\|u^{b,j}\|_{H^s} \leq CV \frac{e^{\sigma_0 (1 + j/8n) t}}{(1 + t)^{1/4(1 + j/8n)}} \leq C_j \exp \left[ \sigma_0 \left( 1 + \frac{j}{8n} \right) \frac{t}{(1 + t)^{1/4(1 + j/8n)}} \right]
\]

Overall, we have

\[
\|\varepsilon^n U^j(t)\|_{L^2} \leq C_j \sum_{j=0}^{8n-1} \frac{\varepsilon^{n+j/8}}{(1 + t)^{1/4(1 + j/8n)}} \exp \left[ \sigma_0 \left( 1 + \frac{j}{8n} \right) \frac{t}{(1 + t)^{1/4(1 + j/8n)}} \right]
\] (35)

But we will be interested in times \( t = T_0^\varepsilon - \tau \), with \( T_0^\varepsilon \) such that

\[
\frac{\varepsilon^n e^{\sigma_0 T_0^\varepsilon}}{\sqrt{1 + T_0^\varepsilon}} = 1
\]

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The important manoeuvre here is to write \( t = T_0^\epsilon - \tau \), so that

\[
\frac{e^{\sigma_0 t}}{\sqrt{1 + t}} = e^{-\sigma_0 \tau} \sqrt{\frac{1 + T_0^\epsilon}{1 + T_0^\epsilon - \tau}} := K_0^\epsilon(\tau) e^{-\sigma_0 \tau}
\]

with \( K_0^\epsilon(\tau) \in [1, 2] \) for \( \epsilon \) small enough, and with \( 0 \leq \tau \leq \tau_0 \) where \( \tau_0 \) does not depend on \( \epsilon \). As \( T_0^\epsilon \sim c \ln(\epsilon^{-1}) \), we can choose \( \epsilon < \epsilon_0 \) so that \( T_0^\epsilon - \tau_0 > 0 \). \( \tau_0 \), and therefore \( \epsilon_0 \), will be chosen at the end of the proof in the next paragraph.

We show how this manoeuvre works in detail. Writing \( t = T_0^\epsilon - \tau \), (35) becomes

\[
\|e^{\sigma_0}(T_0^\epsilon - \tau)\|_{L^2} \leq (1 + T_0^\epsilon - \tau)^{1/4} C_1 K_0^\epsilon(\tau) e^{-\sigma_0 \tau} \frac{\sum_{j=0}^{8n-1} (K_0^\epsilon(\tau)e^{-\sigma_0 \tau})^j}{8n}
\]

Thus, by returning to the \( t \)-variable, for \( t \leq T_0^\epsilon \),

\[
\|U^1(t)\|_{L^2} \leq (1 + t)^{1/4} C_1 e^{\sigma_0 t} \frac{1}{\sqrt{1 + t}}
\]

(36)

Note that this estimate also means that \( \|U^1(t)\|_{L^2} \) behaves like \( \|u^{i,0}(t)\|_{L^2} \).

In Theorem 2 we want an \( L^\infty \) under-estimate, so we need to work more in order to get an under-estimate of a local \( L^2 \) norm. If \( \alpha = \Im(\lambda(k_0)) \) and \( \beta = -\sigma'(k_0) > 0 \), then

\[
|u^{i,0}(t, x, y)| \sim e^{\sigma_0 t} \left| \int \varphi^1(k) \tilde{u}(k, y) \exp(i(x + \alpha t)(k - k_0) - \beta t(k - k_0)^2) \, dk \right|
\]

But, writing \( u = k - k_0 \), we have (by factorising)

\[
\int_\mathbb{R} \exp(i(x + \alpha t)u - \beta t u^2) \, du = \frac{c}{\sqrt{t}} \exp\left(\frac{-t(x + \alpha t)^2}{4t}\right)
\]

with \( c \) complex, so there exists \( C \) such that

\[
|u^{i,0}(t, x, y)| \geq \frac{C e^{\sigma_0 t}}{\sqrt{1 + t}} \exp\left(\frac{-t(x + \alpha t)^2}{4t}\right)
\]

Integrating \( |u^{i,0}(t)|^2 \) on the bounded domain

\[
\Omega_A(t) = \{(x, y) \mid y \leq A \text{ and } |x + \alpha t| \leq A \sqrt{1 + t}\}
\]

for \( A \) large enough, and using the fact that \( u^{i,0} \) is the dominant term in \( U^1 \), we have

\[
\|U^1(t)\|_{L^2(\Omega_A(t))} \geq C_1^\prime (1 + t)^{1/4} \frac{e^{\sigma_0 t}}{\sqrt{1 + t}}
\]

(37)

As the measure of \( \Omega_A(t) \) is \( A^2 \sqrt{1 + t} \), this gives us an under-estimate of the norm \( \|U^1(t)\|_{L^\infty} \). In fact, \( \|U^1(t)\|_{L^\infty} \sim C(1 + t)^{-1/2} e^{\sigma_0 t} \).

Now we can get estimates on \( U^j \) for \( j > 1 \). Remember that \( V^1 \) behaves like \( \psi^{i,0} \), so induction using Theorem 3 gives us

\[
\|V^j(k; t)\|_{H^s} \leq c_\epsilon e^{\sigma_0 t}
\]

(38)
Indeed, for \( j > 1 \), the internal part of \( V^j \) solves a linearised Euler equation with a source whose predominant terms are \( V^{j_1} \cdot \nabla V^{j_2} \) with \( j_1 + j_2 = j \), which verify (38) by (14), and as above, the boundary layers are rapidly decreasing functions of \( Y \) satisfying (34), with \( q \geq 8n \). Now we specify the structure of \( U^j \): we write

\[
U^j(t, x, y) = \int_I \cdots \int_I V^j(k_1, \cdots, k_j; t, x, y) \, dk_1 \cdots dk_j
\]

with \( |V^j(k_1, \cdots, k_j; t, x, y)| \leq C \exp[(\sigma(k_1) + \cdots + \sigma(k_j))t] \), which gives us

\[
\|U^j(t)\|_{L^\infty} \leq C \int_I \cdots \int_I \exp[(\sigma(k_1) + \cdots + \sigma(k_j))t] \, dk_1 \cdots dk_j
\]

\[
\leq C \left( \int_R \exp(\sigma_0 t - \beta(k - k_0)^2 t) \, dk \right)^j \leq \frac{C e^{\sigma_0 t}}{(1 + t)^{3/2}}
\]

(39) thanks to the second-order Taylor inequality

\[
\sigma(k) \leq \sigma_0 - \beta(k - k_0)^2
\]

(40)

For an \( L^2 \) estimate, we rewrite \( U^j \) as

\[
U^j(t, x, y) = \int_{jk \in I} \int_{k_1 + \cdots + k_j = jk} V^j(k_1, \cdots, k_j; t, y) e^{ijkx}
\]

with \( jI = I + \cdots + I \). By Parseval’s equality, we have

\[
\|U^j(t)\|_{L^2}^2 = \int_{jk \in I} \left\| \int_{k_1 + \cdots + k_j = jk} V^j(k_1, \cdots, k_j; t) \right\|_{L^2}^2
\]

=: \int_{jk \in I} N(t)^2

Using (10), and noticing that

\[
\sum_{m=1}^j (k_m - k_0)^2 = j(k - k_0)^2 + \sum_{m=1}^j (k - k_m)^2
\]

we have

\[
N(t)^2 \leq Ce^{(j\sigma_0 - j\beta(k - k_0)^2)t} \int_{\sum_{m=1}^j k_m = jk} \exp(-\beta \sum_{m=1}^j (k_m - k)^2 t)
\]

As \( k_j \) is imposed by the other \( j - 1 \) \( k_m \)-variables, we integrate \( j - 1 \) gaussian functions, so

\[
\|U^j(t)\|_{L^2}^2 \leq \int_R \frac{Ce^{2j\sigma_0 t - 2j\beta(k - k_0)^2 t}}{t^{j-1}} \, dk
\]

Integrating this final gaussian function, and taking into account the boundedness of \( \|U^j(t)\|_{L^2} \) near \( t = 0 \), we have

\[
\|U^j(t)\|_{L^2} \leq C_j (1 + t)^{1/4} e^{j\sigma_0 t} \frac{1}{(1 + t)^{3/2}}
\]

(41)

This easily gives us a final estimate on the remainder of the equation that \( u^{ap} \) solves:

\[
\partial_t u^{ap} + u^{ap} \cdot \nabla u^{ap} - \sqrt{\varepsilon} \Delta u^{ap} + \nabla p^{ap} = R
\]
The remainder $R$ contains terms of the type $\delta^{j+l}U_j \cdot \nabla U^l$ with $j + l > N$, so by using (11), we have the estimate

$$
\|R(t)\|_{L^2} \leq C(1 + t)^{1/4} \delta^{N+1} \sum_{j=N+1}^{2N} \frac{e^{j\sigma_0 t}}{(1 + t)^{j/2}}
$$

Replacing $t$ by $T_0^\varepsilon - \tau$, we get

$$
\|R(T_0^\varepsilon - \tau)\|_{L^2} \leq C K^2 N (1 + T_0^\varepsilon - \tau)^{1/4} \sum_{j=N+1}^{2N} e^{-j\sigma_0 \tau} 
\leq C N 2^{N-1} (1 + T_0^\varepsilon - \tau)^{1/4} K_0^\varepsilon(\tau)^{N+1} e^{-j(N+1)\tau}
$$

and so, for $t \in [T_0^\varepsilon - \tau_0, T_0^\varepsilon]$,

$$
\|R(t)\|_{L^2} \leq \frac{C_R \delta^{N+1}}{(1 + t)^{(N+1)/2}} (1 + t)^{1/4} e^{(N+1)\sigma_0 t}
$$

(42)

$C_R$ can be adjusted so that the inequality is valid for $t \in [0, T_0^\varepsilon]$. 

3.2.3 Proof of the instability

For this paragraph, we denote $\delta = \varepsilon^n$, and we choose $N$ such that for $t \geq 0$, $2N\sigma_0 > 2 \|\nabla u_0\|_{L^\infty} + 1$. Let $u^\varepsilon$ be an exact Leray solution to the ($NS'$) system with initial condition $u_0^\varepsilon$. We have $\|u^\varepsilon(0) - u_0\|_{L^2} \leq \varepsilon^n$. The aim is to get a lower bound of $\|u^\varepsilon - \nabla u_0\|_{L^2}$, so we will look for a local $L^2$ lower bound.

Let us first estimate $\|u^{o\varepsilon} - \nabla u_0\|_{L^2(\Omega, t)}$; using (11) and (37), we have

$$
\|u^{o\varepsilon}(t) - \nabla u_0(t)\|_{L^2(\Omega, t)} \geq (1 + t)^{1/4} \left( \frac{C_0 \delta}{(1 + t)^{1/2}} e^{\sigma_0 t} - \sum_{j=2}^{N} \frac{C_j \delta^j}{(1 + t)^{j/2}} e^{j\sigma_0 t} \right)
$$

Writing $t = T_0^\varepsilon - \tau$, we get

$$
\|u^{o\varepsilon}(T_0^\varepsilon - \tau) - \nabla u_0(T_0^\varepsilon - \tau)\|_{L^2(\Omega, t)} \geq (1 + T_0^\varepsilon - \tau)^{1/4} K_0^\varepsilon(\tau) \left( C_0 e^{-\sigma_0 \tau} - C_0' e^{-2\sigma_0 \tau} \right)
$$

with $C_0' = N 2^N \max_{1 \leq j \leq N} C_j$. There exists $\tau_1 > 0$ such that

$$
C_0 e^{-\sigma_0 \tau} - C_0' e^{-2\sigma_0 \tau} \geq \frac{C_0}{2} e^{-\sigma_0 \tau}
$$

for $\tau \geq \tau_1$. So, choosing $\tau_0 > \tau_1$, we have, for $T_0^\varepsilon - \tau_0 \leq t \leq T_0^\varepsilon - \tau_1$,

$$
\|u^{o\varepsilon}(t) - \nabla u_0(t)\|_{L^2(\Omega, t)} \geq (1 + t)^{1/4} \frac{C_0 \delta}{2(1 + t)^{1/2}} e^{\sigma_0 t}
$$

(43)

Note that $\tau_1$ does not depend on $\delta$.

Now we estimate $\|w(t)\|_{L^2(\Omega)}$, where $w = u^\varepsilon - u^{o\varepsilon}$ and $q = p^\varepsilon - p^{o\varepsilon}$. The pair $(w, q)$ solves the equation

$$
\partial_t w + w \cdot \nabla w - \sqrt{\varepsilon} \Delta w + \nabla q = -u^{o\varepsilon} \cdot \nabla w - w \cdot \nabla u^{o\varepsilon} + R
$$

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We compute the $L^2$ dot-product of this equation and $w$, integrating by parts, using the divergence-free condition on the nonlinear term, the $(NC + 1)$ and $(NP)$ boundary conditions on the Laplacian term and Young’s inequality on the remainder, we get the energy estimate

$$\partial_t \|w\|^2_{L^2} + 2\sqrt{\varepsilon} \|\nabla w\|^2_{L^2} + 4\sqrt{\varepsilon} \int_{\partial\Omega} |w_1|^2 \leq (2 \|\nabla w^p\|_{L^\infty} + \rho) \|w\|^2_{L^2} + \frac{1}{\rho} \|R\|^2_{L^2}$$

with $0 < \rho < 1/3$. The $L^\infty$ estimate on $\nabla w^p$ is crucial here: as each $U^j$ can be written $U^j = U^{j+1}(t, x, y) + \varepsilon^{1/4}U^{b,j}(t, x, y)$, the first order derivatives of $U^j$ verify (39)-type estimates, since $\varepsilon^{1/4}\nabla(U^{b,j}(t, x, \varepsilon^{-1/4}y)) = (\nabla U^{b,j})(t, x, y)$, so

$$\|\nabla w^p(t)\|_{L^\infty} \leq \|w(t)\|_{L^\infty} + \sum_{j=1}^{N} \frac{C\delta^j\varepsilon^{\sigma_j t}}{(1+t)^{j/2}}$$

Using the $t = T_0^\varepsilon - \tau$ manoeuvre, the sum on the right is smaller than $\rho$ for $t \leq T_0^\varepsilon - \tau_2$, with $\tau_2 \geq \tau_1$ independent from $\delta$, and with (12) and the choice of $N$ we made at the beginning of the paragraph, the energy estimate becomes

$$\partial_t \|w(t)\|^2_{L^2} \leq 2N\sigma_0 \|w(t)\|^2_{L^2} + \frac{C\delta^{N+1} \varepsilon^{\sigma_N t}}{\rho(1 + t)^{(N+1)/2}} (1 + t)^{1/4} e^{(N+1)\sigma_N t}$$

Therefore,

$$\|w(t)\|_{L^2(\Omega)} \leq \frac{C\delta^{N+1} \varepsilon^{\sigma_N t}}{\rho(1 + t)^{(N+1)/2}} (1 + t)^{1/4} e^{(N+1)\sigma_N t} \quad (44)$$

by applying the following lemma, which is obtained using a variant of Grönwall’s lemma from (26) and integration by parts:

**Lemma 5.** Let $\varphi$ be a function such that

$$\partial_t \varphi(t) \leq \lambda \varphi(t) + C\frac{\varepsilon^{\sigma t}}{(1+t)^{\alpha}}$$

for every $t \geq 0$, and for parameters $\mu > \lambda \geq 0$ and $\alpha > 1$, then

$$\varphi(t) \leq \frac{C\varepsilon^{\sigma t}}{(1+t)^{\alpha}}$$

with $C'$ depending on $\varphi(0)$, $\lambda$, $\mu$ and $C$.

With (43) and (44), we can now conclude:

$$\|w^\varepsilon(t) - w^\varepsilon\|^2_{L^2(\Omega_A(t))} \geq \|w^p(t) - w^\varepsilon\|^2_{L^2(\Omega_A(t))} - \|w(t)\|^2_{L^2(\Omega)}$$

$$\geq (1 + t)^{1/4} \left[ \frac{C\delta^{\sigma t}}{2(1+t)^{1/2}} \right]$$

Writing $t = T_0^\varepsilon - \tau$ for the last time, we have

$$\frac{C\delta^{\sigma t}}{2(1+t)^{1/2}} \frac{C\delta^{N+1} \varepsilon^{(N+1)\sigma_N t}}{(1 + t)^{(N+1)/2}} \geq \frac{C\delta^{\sigma t}}{4(1+t)^{1/2}}$$

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for $t \leq T_0 - \tau_3$, with $\tau_3 \geq \tau_2$ independent from $\delta$. Setting $T^\varepsilon = T_0 - T$, with $\tau_3 < T < \tau_0$ fixed (here we can finally fix $\tau_0$ and $\varepsilon_0$), we have

$$
\|u^\varepsilon(T^\varepsilon) - \overline{u}_a(T^\varepsilon)\|_{L^2(\Omega_A(T^\varepsilon))} \geq (1 + T^\varepsilon)^{1/4} C_0 e^{-\sigma_0 T} := (1 + T^\varepsilon)^{1/4} \delta_0
$$

where $\delta_0$ does not depend on $\varepsilon$. Thus, $\|u^\varepsilon(T^\varepsilon) - \overline{u}_a(T^\varepsilon)\|_{L^\infty} \geq C \delta_0$.

Also, as $\dot{H}^s \hookrightarrow L^\infty$, we have $\|u^\varepsilon(T^\varepsilon) - \overline{u}_a(T^\varepsilon)\|_{\dot{H}^s} \geq C \delta_0$. Remember that we are still using the fast variables, so, returning to the original scale of time and space (again without changing notation), we get (2) and

$$
\left\|u^\varepsilon(T^\varepsilon, x, y) - \overline{u}_a(T^\varepsilon, \varepsilon^{-1/2} y)\right\|_{\dot{H}^s} \geq C \delta_0 \varepsilon^{(1-s)/2} \varepsilon^{-\sigma_0 T} \varepsilon^{-\tau_3} + \infty
$$

In the original scale of time, $T^\varepsilon = O(n \sqrt{\varepsilon \ln(\varepsilon^{-1})})$.

3.2.4 The negative case

The proof in the negative case is exactly the same up until the energy estimate of $w$. Here, the Laplacian term becomes

$$
- \int_\Omega \Delta w \cdot w = \|\nabla w\|^2_{L^2(\Omega)} - 2|a| \|w_1\|^2_{L^2(\partial \Omega)}
$$

The second term cannot be ignored, so we use the same idea as in paragraph 2: the trace theorem, Sobolev interpolation and the Young inequality allow us to get

$$
4|\alpha| \|w\|^2_{L^2(\partial \Omega)} \leq 2 \|\nabla w\|^2_{L^2(\Omega)} + 2 C_0 \sqrt{\varepsilon} \|w\|^2_{L^2(\Omega)}
$$

where $C_0$ does not depend on $\varepsilon$. Choosing $N$ so that $N \sigma_0 > \|\nabla u_a\|_{L^\infty} + C_0 + \frac{1}{2}$, everything works as above.

3.3 An example of unstable profile

Examples of piecewise-linear flows that are linearly unstable for the Euler equation are given in [7] and [13], the latter stating that close-enough regularisations of these are smooth unstable shear flows. We wish to provide an explicit example of smooth linearly unstable profile, which is the object of Proposition 9 below. From this example, we easily deduce the expression of a smooth unstable profile that fits Theorem 2.

**Proposition 6.** For every $\delta > 0$ and $\zeta \in \mathbb{R}$, the shear flow $(u_\delta, 0)$ with

$$
u_\delta(y) = \tanh(y - \delta) + \zeta$$

is a linearly unstable for the Euler equation.

**Proof:** writing $\lambda = -ikc$ for $\Re(\lambda) > 0$ and $k \neq 0$, and taking the curl of equation (EL), the Euler equation linearised around $u_\delta$, we obtain a linear second-order differential equation for $\Psi$: the Rayleigh equation

$$(\mathcal{R}(c, k)) : (u_\delta - c)(\partial^2_{yy} \mathcal{L} - k^2)\Psi - u_\delta'' \Psi = 0$$

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with the conditions $\Psi(0) = 0$ and $\lim_{y \to +\infty} \Psi(y) = 0$. Our problem is now finding $c$ in the complex upper-half-plane, and real numbers $k$ such that $(R(c, k))$ has a solution in $H^1_0(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$.

It is well known that if the Rayleigh equation has an unstable solution, then $u_*$ must have an inflection point in $\mathbb{R}^+$ (Rayleigh’s theorem, [27]), and $u_\delta$ has exactly one inflection point, $y_0 = \delta$, with inflection value $u_\delta(y_0) = \zeta$. Also, the function

$$K_\delta(y) = \frac{-u_*''(y)}{u_*(y) - u_0} = \frac{-(\tanh(y - \delta))^\prime}{\tanh(y - \delta)} = 2(1 - \tanh(y - \delta)^2)$$

has a limit when $y \to y_0$, and is a positive continuous function on $\mathbb{R}^+$, vanishing at infinity. Thus, choosing $c = u_0$, we can divide $(R(c, k))$ by $u_* - u_0$ (which is not usually possible when $c$ is in the range of $u_*$), and we have a Sturm-Liouville problem

$$-\Psi''(y) - K_\delta(y)\Psi(y) = -k^2\Psi(y)$$

in which the square of the wave number $k$ intervenes as an eigenvalue of the operator $S_\delta = -\partial^2 - K_\delta$. We shall therefore use the following result by Z. Lin (Theorem 1.5 in [19]):

**Theorem 7.** Let $U$ be a $C^2$ profile which has a limit $l$ as $y \to +\infty$, and an inflection point $y_0$ such that, writing $u_0 = U(y_0)$, the function $K(y) = \frac{-U''(y)}{U(y) - u_0}$ is a positive continuous function which goes to zero as $y \to +\infty$. $U(y)$ can take the value $l$ only a finite number of times. If the lowest eigenvalue of the operator $S = -\partial^2 - K$, defined on the Sobolev space $H^1_0(\mathbb{R}^+) \cap H^2(\mathbb{R}^+)$, is strictly negative, then, letting $-\mu^2$ be the lowest eigenvalue, for every $k \in (0, \mu)$, there exists $c(k)$ such that the Rayleigh equation $(R(c(k), k))$ has an unstable solution.

The result is first shown on a finite interval; one proves that eigenfunctions of $S$ with negative eigenvalues are limits of unstable solutions, and this is done by using the Picard fixed point theorem. A compactness argument is then used to get the result on the half-line. We detail the proof no further.

In the case of $u_\delta$, it remains to show that $S_\delta = -\partial^2 - K_\delta$ has a negative eigenvalue. We hope to use a standard Sturm-Liouville argument, and notice that $\varphi_\delta(y) = \tanh(y - \delta)$ is a solution to the equation

$$-\varphi''(y) - K_\delta(y)\varphi(y) = 0$$

(45)

that has exactly one zero in $]0, +\infty[. Since $K_\delta$ decreases exponentially to 0, the multiplication by $K_\delta$ is a compact perturbation of $-\partial^2$, so by Weyl’s theorem (see [26]), the essential spectrum of $S_\delta$ is $\mathbb{R}^+$. But that means trouble, because the 0 in (45) corresponds to the lowest point in the essential spectrum, and the one zero of $\varphi_\delta$ means that $S_\delta$ has between 2 and 3 negative eigenvalues, according to the classical Sturm-Liouville analysis shown in [8], chapter XIII.

So we must use a different tool to determine that $S_\delta$ has a strictly negative eigenvalue. Fix $\delta$ and let $Q_\delta$ be the quadratic form associated with $S_\delta$: for $u \in H^1_0(\mathbb{R}^+)$, we define

$$Q_\delta(u) = \int_0^{+\infty} |u'(y)|^2 - K(y)|u(y)|^2 \, dy$$

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Let \( v_\eta(y) = \tanh(y - \eta) \) for \( \eta > 0 \), and \( \chi \) be a \([0, 2]\)-supported smooth function, with \( \chi(y) = 1 \) for \( y \leq 1 \). For any \( n \in \mathbb{N}^* \), we define

\[
 w^n_\eta(y) = \begin{cases} 
 v_\eta(\delta) \chi \left( n \left( 1 - \frac{y}{\delta} \right) \right) & \text{if } y \leq \delta \\
 v_\eta(y) \chi \left( \frac{y}{\delta} \right) & \text{if } y \geq \delta
\end{cases}
\]

\( w^n_\eta \) is equal to \( v_\eta \) on \([\delta, n]\), continuous on \( \mathbb{R}^+ \), and is in \( H^1_0(\mathbb{R}^+) \), and since \( v'_\eta \) and \( \sqrt{K_\eta v_\eta} \) are square-integrable on \( \mathbb{R}^+ \), we have

\[
 \lim_{n \to +\infty} Q_\delta(w^n_\eta) = \int_{\delta}^{+\infty} |v'_\eta|^2 - K_\eta |v_\eta|^2 := Q(\eta)
\]

By integrating by parts, we have \( Q(\delta) = 0 \), and \( Q \) is a differentiable function of \( \eta \), so we look at \( \partial_\eta Q(\delta) \). We have

\[
 \partial_\eta Q(\eta) = \int_{\delta}^{+\infty} -2v''_\eta v'_\eta + 2K_\eta v_\eta v'_\eta = \int_{\delta}^{+\infty} 2v_\eta v'_\eta(K_\eta + K_\delta)
\]

Thus, since \( v_\eta(y) \), \( v'_\eta(y) \) and \( K_\delta(y) \) are positive for \( y > \delta \), \( \partial_\eta Q(\delta) > 0 \). So there exists \( \eta_0 \) such that, for \( \eta \in [\eta_0, \delta] \), \( Q(\eta) < 0 \), and therefore, for a given \( \eta \) in that interval, there exists \( n \) large enough such that \( w^n_\eta \) verifies \( Q_\delta(w^n_\eta) < 0 \).

We have proved that the lowest point of the spectrum of \( S_\delta \) is negative, and \( u_3 \) verifies all the hypotheses of \textbf{Theorem 2} \textbf{Proposition 1} is proved. \(\square\)

Remark: \( S_\delta \) has in fact exactly one negative eigenvalue. Remember that \( \varphi_\delta(y) = \tanh(y - \delta) \) is a bounded continuous solution of \(-\varphi''_\delta - K_\delta \varphi_\delta = 0 \) that vanishes at \( y_0 = \delta \), and that \( Q_\delta(u) = \int_0^{+\infty} |u'|^2 - K_\delta |u|^2 \) is the quadratic form on \( H^1_0(\mathbb{R}^+) \) associated with the operator \( S_\delta \). We shall show that \( Q_\delta \geq 0 \) on \( F_\delta = \{ u \in H^1_0(\mathbb{R}^+) \mid u(\delta) = 0 \} \). A function \( u \in F_\delta \) can be written as \( u = \varphi_\delta v \), with \( v \in H^1_0(\mathbb{R}^+) \). Let us take \( v \in C^\infty_0(\mathbb{R}^+) \), with \( \text{supp}(v) \subset [h, y_0 - h] \cup [y_0 + h, +\infty[ \) for a certain \( h > 0 \). Replacing \( u \) by \( \varphi_\delta v \) in \( Q(u) \), we get

\[
 -u'' - K_\delta u = -\varphi''_\delta v - 2\varphi'_\delta v' - \varphi_\delta v'' - K \varphi_\delta v = -2\varphi'_\delta v' - \varphi_\delta v''
\]

so, using \( Q_\delta(u) = \int_0^{+\infty} S_\delta u \cdot u \), we have \( Q_\delta(\varphi_\delta v) = \int_0^{+\infty} -2\varphi'_\delta v'v'' - \varphi_\delta v''v'' \).

Integrating by parts, we get the factorisation \( Q_\delta(\varphi_\delta v) = \int_0^{+\infty} \varphi_\delta^2 v'^2 \), ie

\[
 Q_\delta(u) = \int_0^{+\infty} \varphi_\delta(y)^2 \left( \frac{u(y)}{\varphi_\delta(y)} \right)'^2 dy \quad \text{for all } u \in F_\delta \quad (46)
\]

as \( C^\infty_0((0, \delta) \cup (\delta, +\infty[) \) is dense in \( F_\delta \). We now have \( Q_\delta \geq 0 \) on \( F_\delta \), which is a hyperplane of \( H^1_0(\mathbb{R}^+) \), thus \( Q_\delta \) is negative only on a subspace of dimension 1: \( S_\delta \) has only one negative eigenvalue.

We can now give an example of unstable shear flow for \textbf{Theorem 2} it is one of the linearly unstable profiles of the Euler equation, \( u_3(y) = \tanh(y - \delta) + \zeta \) with \( \zeta \) and \( \delta \) to be chosen such that it satisfies the rescaled Navier condition,

\[
 \frac{1}{2} \partial_y u_3(0) = a u_3(0)
\]

(47)
This example will cover all cases except \( a = 0 \), firstly because \( u'_{\delta} \) never vanishes, and secondly because when \( a = 0 \), instability does not occur (see (11)). (17) is an equation of the variable \( X = \tanh(\delta) \) with parameter \( \zeta \); precisely it is
\[
\frac{1}{4}X^2 - aX + a\zeta - \frac{1}{2} = 0;
\]
the discriminant is \( \Delta = a^2 - 2a\zeta + 1 \). We then have two cases:

- for \( a \notin \mathbb{Z} \), we choose \( \zeta \) so that \( \Delta = E[a]^2 \), therefore one of the solutions of the polynomial equation is \( X_0 = a - E[a] \in [0, 1] \):

- for \( a \in \mathbb{Z}^* \), we choose \( \zeta \) such that \( \Delta = \left( a - \frac{1}{2} \right)^2 \), so \( X_0 = \frac{1}{2} \).

Setting \( \tanh^{-1}(X_0) := \delta_0 \), \( u_{\delta_0} \) is provides the wanted example; explicitly

\[
u_s(y) = \begin{cases} 
\tanh \left( y - \tanh^{-1}(a - E[a]) \right) + \frac{2}{3} - \frac{E[a]^2 - 1}{2a} & \text{if } a \notin \mathbb{Z} \\
\tanh \left( y - \tanh^{-1}\left( \frac{1}{2} \right) \right) + \frac{1}{2} + \frac{3}{8a} & \text{if } a \in \mathbb{Z}^*
\end{cases}
\]

are flows that fit Theorem 2.

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**References**

[1] C. Bardos, Existence et unicité de la solution de l’équation d’Euler en dimension deux, 1972, *J. Math. Anal. Appl.* 40

[2] H. Beirão da Veiga, Vorticity and regularity for flows under the Navier boundary condition, 2006, *Commun. Pure Appl. Anal.* 5

[3] H. Beirão da Veiga, F. Crispo, Concerning the \( W^{k,p} \)-inviscid limit for 3-d flows under a slip boundary condition., 2009, *J. Math. Fluid Mech.* , online at SpringerLink

[4] T. Clopau, A. Mikelić, R. Robert, On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions, 1998, *Nonlinearity* 11

[5] W. A. Coppel, *Dichotomies in Stability Theory*, LNM 629, 1978, Springer

[6] B. Desjardins, E. Grenier, Linear instability implies nonlinear instability for various types of viscous boundary layers, 2003, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* 20

[7] P. G. Drazin, W. H. Reid, *Hydrodynamic Stability*, 1981, Camb. Univ. Press

[8] N. Dunford, J. T. Schwartz, *Linear Operators, Part II*, 1963, Wiley

[9] K. O. Friedrichs, On the Boundary-Value Problems of the Theory of Elasticity and Korn’s Inequality, 1947, *Annals of Math.* 48

[10] F. Gallaire, D. Gérard-Varet, F. Rousset, Three-dimensional instability of planar flows, 2007, *Arch. Ration. Mech. Anal.* 186

26
[11] D. Gérard-Varet, E. Dormy, On the ill-posedness of the Prandtl equation, 2010, *J. AMS* **23**

[12] D. Gérard-Varet, N. Masmoudi, Relevance of the Slip Condition for Fluid Flows Near an Irregular Boundary, 2010, *Comm. Math. Phys.* **295**

[13] E. Grenier, On the nonlinear instability of Euler and Prandtl equations, 2000, *Comm. Pure Appl. Math.* **53**

[14] D. Iftimie, G. Planas, Inviscid limits for the Navier-Stokes equations with Navier friction boundary conditions, 2006, *Nonlinearity* **19**

[15] D. Iftimie, F. Sueur, Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions, 2011, *Arch. Ration. Mech. Anal.* **199**

[16] W. Jäger, A. Mikelić, On the Roughness-Induced Effective Boundary Conditions for an Incompressible Viscous Flow, 2001, *J. Diff. Equvs.* **170**

[17] J. P. Kelliher, Navier-Stokes equations with Navier boundary conditions for a bounded domain in the plane, 2006, *SIAM J. Math. Anal.* **38** (electronic)

[18] N. V. Krylov, *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, GSM **96**, 2008, AMS

[19] Z. Lin, Instability of some ideal plane flows, 2003, *SIAM J. Math. Anal.* **35**

[20] A. J. Majda, A. L. Bertozzi, *Vorticity and Incompressible Flow*, 2002, Camb. Univ. Press

[21] N. Masmoudi, F. Rousset, Uniform regularity for the Navier-Stokes equations with Navier boundary condition, 2010, preprint, online at arxiv.org

[22] N. Masmoudi, L. Saint-Raymond, From the Boltzmann Equation to the Stokes-Fourier System in a Bounded Domain, 2003, *Comm. Pure Appl. Math.* **56**

[23] G. Métivier, K. Zumbrun, Large viscous boundary layers for noncharacteristic nonlinear hyperbolic problems, 2005, *Mem. Amer. Math. Soc.* **175**, n°826

[24] O. A. Oleinik, On the mathematical theory of boundary layer for an unsteady flow of incompressible fluid, 1966, *J. Appl. Math. Mech.* **30**

[25] D. Pal, N. Rudraiah, R. Devanathan, The effects of slip velocity at a membrane surface on blood flow in the microcirculation, 1988, *J. Math. Biol.* **26**

[26] L. C. Piccinini, G. Stampacchia, G. Vidossich, *Ordinary Differential Equations in $\mathbb{R}^n$, Problems and Methods*, 1984, Springer

[27] Lord Rayleigh (J.W.Strutt), On the stability, or instability, of certain fluid motions, 1880, *Proc. London Math. Soc.* **11**

[28] M. Reed, B. Simon, *Methods of Modern Mathematical Physics IV, Analysis of Operators*, 1978, Academic Press
[29] F. Rousset, Stability of large Ekman boundary layers in rotating fluids, 2004, *Arch. Ration. Mech. Anal.* **172**

[30] F. Rousset, N. Tzvetkov, Transverse instability of the line solitary water waves, 2010, *Invent. Math.*, online at SpringerLink

[31] F. Rousset, N. Tzvetkov, Transverse nonlinear instability for two-dimensional dispersive models, 2009, *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **26**

[32] M. Sammartino, R. E. Caflisch, Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space, I & II, 1998, *Comm. Math. Phys.* **192**

[33] W. A. Strauss, *Partial Differential Equations, an Introduction*, 1992, Wiley

[34] R. Temam, X. Wang, Boundary layers associated with the incompressible Navier-Stokes equations: the noncharacteristic boundary case, 2002, *J. Diff. Eqns.* **179**

[35] Y. Xiao, Z. Xin, On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, 2007, *Comm. Pure Appl. Math.* **60**

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