Quantum phase transition in the Dzyaloshinskii–Moriya interaction with inhomogeneous magnetic field: Geometric approach

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Abstract In this paper, we generalize the results of Oh (Phys Lett A 373:644–647, 2009) to Dzyaloshinskii–Moriya model under non-uniform external magnetic field to investigate the relation between entanglement, geometric phase (or Berry phase) and quantum phase transition. We use quaternionic representation to relate the geometric phase to the quantum phase transition. For small values of DM parameter, the Berry phase is more appropriate than the concurrence measure, while for large values, the concurrence is a good indicator to show the phase transition. On the other hand, by increasing the DM interaction the phase transition occurs for large values of anisotropy parameter. In addition, for small values of magnetic field the concurrence measure is appropriate indicator for quantum phase transition, but for large values of magnetic field the Berry phase shows a sharp changes in the phase transition points. The results show that the Berry phase and concurrence form a complementary system from phase transition point of view.

1 Introduction

Phase transition is a non-analytic change in the ground-state energy as a function of system’s parameters is associated with level crossings or avoided crossings between the ground and exited energy levels [1], and these phase transition points are important in physics. In classical systems, there are formal rules to determine phase transition

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but in quantum systems this is an open problem. Quantum phase transition is a phase transition in the zero temperature of quantum systems \[1\]. So it is interesting to find quantum mechanical quantity that can determine the level crossing. In Ref. \[5\], the author showed that the geometric phase (Berry phase) and concurrence measure can detect the phase transition point corresponding to level crossing in anisotropy Heisenberg XY model with external magnetic field. In this paper, we will generalize the results of Ref. \[5\] to general Heisenberg XYZ Hamiltonian with the DM interaction and anisotropic magnetic field. We will show the influence of DM interaction and anisotropic magnetic field in phase transition by using concurrence measure and geometric phase. In addition, the quaternionic representation that has Bloch sphere demonstration will be used for two-qubit systems. Quaternionic representation gives a simple geometric interpretation of concurrence measure and geometric phase for two-qubit systems. The results show that the geometric phase is proportional to parallel transportation with Mannoury–Fubini–Study metric in quaternionic Hilbert space of two-qubit states. Also, the concurrence measure is a quaternionic part of stereographic projection in quaternionic Hopf fibration.

Single-qubit pure states can be identified by points on the surface of the Bloch sphere \(S^2\), and mixed states are characterized by points inside the Bloch sphere. The generalization of this concept in two- and three-qubit states are described by the Hopf fibration. The relation between Hopf fibration, single-qubit and two-qubit states, has been studied by Mosseri and Dandoloff \[3\] in quaternionic skew field and subsequently has been generalized to three-qubit state based on octonions by Bernevig and Chen \[4\]. However, there is also one more reason to look for Hopf fibration and stereographic projections. For two-qubit pure states, the concurrence measure appears explicitly in quaternionic stereographic projection which geometrically means that non-entangled states are mapped from \(S^7\) onto a two-dimensional planar subspace of the target space \(\mathbb{R}^4\). On the other hand, it has been shown that the quaternionic representation has a geometric description of geometric phase. The geometric phase is the magnetic flux due to magnetic monopoles located at the level crossing points \[6,7\]. In quaternionic representation of quantum state \[3,4,8,9\], Levay provided a elegant interpretation of the geometric phase as the parallel transformation of quaternionic spinors due to Mannoury–Fubini–Study metric in Hilbert space of two-qubit states \[10,11\]. The relation between geometric phases, phase transition and level crossings for the Heisenberg XY model with transverse magnetic field has been investigated by Oh et al \[12\].

The entanglement property is one of the most fascinating features of quantum mechanics, and this property provides a fundamental resource in quantum information theory \[13–16\]. The entanglement has been discussed at the early years of quantum mechanics as a specific quantum computation and quantum information \[17–20\]. In spin chain systems, the entangled subsystems of whole vector states cannot be separated into a product of the subsystem states. A measurement on one subsystem in quantum entangled system not only gives information about the other subsystem, but also provides possibility of manipulating it. Thus, these various aspects of entanglement make it as a main tool in quantum information processing, quantum phase transition, quantum optic, and etc. \[21\].
The single-qubit gates are local operators, and it is clear that the local operators unable to generate entanglement in an N-qubit system. To generate entanglement state in N-qubit system, we need an inter-qubit interaction such as a two-qubit gate. The simplest two-qubit interaction is described by the Ising interaction between spin half particles in the form of \( J_z \sigma^z_1 \sigma^z_2 \). More general interaction between two qubits is given by the Heisenberg model with magnetic field and Dzyaloshinskii–Moriya (DM) interactions. Recently, entanglement of two qubits and its dependence on external magnetic fields, anisotropy and temperature have been considered in several Heisenberg models [22–32].

This paper studies the behavior of quantum correlations and quantum phase transition in the anisotropic XYZ spin-half chain with uniform and non-uniform external magnetic field and DM interaction \((D, (\sigma_1 \times \sigma_2))\) [33–35]. The DM interaction arising from extension of the Anderson superexchange interaction theory by including the spin–orbit coupling, and it is important for the weak ferromagnetism and for the spin arrangement in antiferromagnetic of low symmetry. It also plays a significant role in performing universal quantum computation [36,37]. In this state, we find non-analytic dependence of concurrence measure \([38,39]\) and geometric phase on the DM interaction and establish their relations with the quantum phase transition. In addition, we will show in some regions that entanglement is not an appropriate indicator for the phase transition, the geometric phase is a good one and vice versa. In other words, geometric phase and the ground state entanglement are complementary systems that can exhibit quantum phase transition in spin chain systems.

2 Heisenberg XYZ model with Dzyaloshinskii–Moriya interaction

In this section, we study the quantum phase transition in a system of two-qubits Heisenberg XYZ model with Z-component DM coupling and non-uniform external magnetic field.

2.1 The model

The Hamiltonian of the system is read as

\[
H = -\frac{1 + \gamma}{2} \sigma^x_1 \sigma^x_2 - \frac{1 - \gamma}{2} \sigma^y_1 \sigma^y_2 - J_z \sigma^z_1 \sigma^z_2 - \frac{D_z}{2} (\sigma^x_1 \sigma^y_2 - \sigma^y_1 \sigma^x_2) - \frac{\lambda_b + b_z}{2} \sigma^z_1 - \frac{\lambda_b - b_z}{2} \sigma^z_2,
\]

(2.1)

where \( \gamma \) is an anisotropy factor, \( J_z \) is a real coupling coefficient, \( D_z \) is the Z-component Dzyaloshinskii–Moriya (DM) coupling parameter, \( \lambda \) and \( b_z \) are uniform and non-uniform external Z-component magnetic field parameters, respectively, \( \sigma^a_i \) are the Pauli matrices of the i’th qubit with \( a = x, y, z \). The coupling constant \( J_z > 0 \) corresponds to the ferromagnetic case, and \( J_z < 0 \) corresponds to the antiferromagnetic case. The Hamiltonian (2.1) is the general form of a Heisenberg Hamiltonian, which is exactly solvable and becomes a paradigmatic example in the study of quantum phase
transitions. The matrix form of Hamiltonian (2.1) can be written as:

\[
H = \begin{pmatrix}
-\lambda - J_z & 0 & 0 & -\gamma \\
0 & b_z + J_z & -1 - iD_z & 0 \\
0 & -1 + iD_z & -b_z + J_z & 0 \\
-\gamma & 0 & 0 & \lambda - J_z
\end{pmatrix} = H^{\text{even}} + H^{\text{odd}}.
\]  

(2.2)

One may define the Hamiltonian

\[
H^{\text{even}} = \begin{pmatrix}
-\lambda - J_z & -\gamma \\
-\gamma & \lambda - J_z
\end{pmatrix},
\]  

(2.3)

on the subspace spanned by \{\ket{00}, \ket{11}\}. It is easy to write down the eigenvalues and eigenvectors of \(H^{\text{even}}\) as

\[
E^e_\pm = -J_z \pm \sqrt{\lambda^2 + \gamma^2},
\]

\[
|E^e_+\rangle = \cos \left( \frac{\theta_1}{2} \right) |00\rangle + \sin \left( \frac{\theta_1}{2} \right) |11\rangle,
\]

\[
|E^e_-\rangle = \sin \left( \frac{\theta_1}{2} \right) |00\rangle - \cos \left( \frac{\theta_1}{2} \right) |11\rangle,
\]

(2.4)

where \(\tan(\theta_1) = -\frac{\lambda}{\gamma}\). On the other hand, the Hamiltonian \(H^{\text{odd}}\) on the subspace \{\ket{01}, \ket{10}\} is given by

\[
H^{\text{odd}} = \begin{pmatrix}
b_z + J_z & -1 - iD_z \\
-1 + iD_z & \lambda - J_z
\end{pmatrix}.
\]  

(2.5)

The spectrum of \(H^{\text{odd}}\) is easily obtained as

\[
E^o_\pm = J_z \pm \sqrt{b_z^2 + D_z^2 + 1},
\]

\[
|E^o_+\rangle = \cos \left( \frac{\theta_2}{2} \right) |01\rangle + e^{-i\varphi} \sin \left( \frac{\theta_2}{2} \right) |10\rangle,
\]

\[
|E^o_-\rangle = \sin \left( \frac{\theta_2}{2} \right) |01\rangle - e^{-i\varphi} \cos \left( \frac{\theta_2}{2} \right) |10\rangle,
\]

(2.6)

where \(\tan(\theta_2) = \frac{\sqrt{D_z^2 + 1}}{b_z}\) and \(\tan(\varphi) = D_z\). The even and odd eigenvectors of Hamiltonian (2.1) confine to the subspace of even and odd parity operator \(\sigma_1^z \otimes \sigma_2^z\), respectively, i.e., \(\sigma_1^z \otimes \sigma_2^z |E^e_\pm\rangle = |E^e_\pm\rangle\) and \(\sigma_1^z \otimes \sigma_2^z |E^o_\pm\rangle = -|E^o_\pm\rangle\). Consider the Hamiltonian (2.1), whose degrees of freedom reside on the sites of a lattice, and which varies as a function of a dimensionless coupling, \(\gamma, J_z, \lambda, b_z\) and \(D_z\). At zero temperature limit, the system occupies its ground state \(E^e_-\) or \(E^o_-\). For the case of a finite lattice, this ground-state energy will generically be a smooth and analytic function of Hamiltonian couplings. In \(E^e_- = E^o_-\), the level crossings occur between the
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Fig. 1 (Color online) a Ground energy as a function of $\gamma$ and $\lambda$ for Hamiltonian (2.1). The level crossing (white line) for parameters $J_z = 0.15, D_z = 0.2, b_z = 0.1$. b Ground energy as a function of $\gamma$ and $b_z$ for Hamiltonian (2.1). The level crossing (white line) for parameters $J_z = 0.1, D_z = 0.4, \lambda = 0.85$

ground and first exited states. An avoided level crossing between the ground and an excited state of Hamiltonian in a finite lattice could become progressively sharper as the lattice size increases, leading to a non-analyticity at $E^e_\lambda = E^o_\lambda$ in the infinite lattice limit. We shall identify any point of non-analyticity in the ground-state energy of the Hamiltonian system as a quantum phase transition: The non-analyticity could be either the limiting case of an avoided level crossing or an actual level crossing. Corresponding to $E^e_\lambda < E^o_\lambda$, $E^e_\lambda = E^o_\lambda$ and $E^e_\lambda > E^o_\lambda$, the system stay at paramagnetic (P), ordered ferromagnetic (F) and the oscillatory phase (O), respectively (see Fig. 1). In Fig. 1a, the ground-state energy is plotted with respect to $\gamma$ and $\lambda$, the ordered ferromagnetic line shows quantum phase transition points, which by increasing the external magnetic field the phase transition occurs for small value of anisotropy parameter $\gamma$. To show the importance of inhomogeneous magnetic field $b_z$ in quantum phase transition we plot Fig. 1b which implies that by increasing the inhomogeneous magnetic field the quantum phase transition occurs for large value of anisotropy parameter.

2.2 Quaternionic representation and Hopf fibration

Consider the $\mathcal{H}_A^C$ for four-dimensional complex Hilbert space which is the tensor product of the individual Hilbert spaces $\mathcal{H}_2^C \otimes \mathcal{H}_2^C$ with a direct product basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A general two-qubit pure state reads

$$|\psi\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle, \quad a_0, ..., a_3 \in \mathbb{C}. \quad (2.7)$$

This state is called separable, if it can be represented in the product form $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, where $|\psi\rangle_A \in \mathcal{H}_2^C$ and $|\psi\rangle_A \in \mathcal{H}_2^C$. This occurs if and only if there exists only one nonzero Schmidt coefficient, $\lambda_1 = 1$, i.e., the reduced state $\rho_A$ or $\rho_B$ is pure. In the opposite case, the state $|\psi\rangle$ is called entangled. The normalization condition $|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1$ identifies $\mathcal{H}_A^C$ to the seven-dimensional real sphere $S^7$, embedded in $\mathbb{R}^8$. In geometric point of view, the unit sphere $S^7$ can be parame-
terized in many different ways as a product of manifolds, but for understanding the geometry of two-qubit entanglement it is useful to fiber $S^7$ over the four-dimensional sphere $S^4$ with $S^3$ fibers by employing the second Hopf fibration. This idea can be illustrated by introducing a quaternionic representation for two-qubit state, i.e., using the quaternionic skew field $\mathbb{Q}$, we can equivalently restate every $|\psi\rangle \in \mathcal{H}_4$ by a quaterbit $|\psi\rangle_Q \in \mathcal{H}_2^Q$ as

$$|\psi\rangle_Q := q_0|0\rangle_Q + q_1|1\rangle_Q,$$

(2.8)

where $q_0 = a_0 + a_1j$ and $q_1 = a_2 + a_3j$ are two quaternion numbers. In quaternion Hilbert space $\mathcal{H}_2^Q$, the state (2.7) can be recast as (2.8) with the following representation

$$|00\rangle \rightarrow |0\rangle_Q, \quad |01\rangle \rightarrow j|0\rangle_Q, \quad |10\rangle \rightarrow |1\rangle_Q, \quad |11\rangle \rightarrow j|1\rangle_Q.$$

(2.9)

Quaternion is an associative and non-commutative algebra of rank 4 on real space $\mathbb{R}$ whose every element can be written as $q = q_0 + q_1i + q_2j + q_3k \in \mathbb{Q}$, where the quaternionic units $i, j$ and $k$ with squares equal to -1 satisfy the usual relations $ij = -ji = k$, and similar ones obtained by employing cyclic permutations of the symbols $ijk$. The quaternion can be equivalently defined in term of complex numbers $z_1 = q_0 + q_1i$ and $z_2 = q_2 + q_3i$ in the form $q = z_1 + z_2j$. The conjugate quaternion $\bar{q}$ is obtained by $\bar{q} = (q_0 - q_1i) - (q_2 + q_3i)j$. Note that in term of quaternion numbers the normalization condition of state (2.8) is given by $|q_0|^2 + |q_1|^2 = 1$. Now we define the second Hopf fibration by the map as the composition of a stereographic projection $\mathcal{P}$ from $S^7$ to $\mathbb{R}^4 + \{\infty\}$, followed by an inverse stereographic projection $\mathcal{S}$ from $\mathbb{R}^4 + \{\infty\}$ to $S^4$:

$$\mathcal{P} : \begin{cases} S^7 & \rightarrow \mathbb{R}^4 + \{\infty\} \\ (q_1, q_2) & \rightarrow Q = q_2\bar{q}_2 \end{cases}, \quad q_1, q_2 \in \mathbb{Q},$$

$$\mathcal{S} : \begin{cases} \mathbb{R}^4 + \{\infty\} & \rightarrow S^4 \\ Q & \rightarrow M(x_i) \end{cases}, \quad \sum_{i=0}^{4} x_i^2 = 1,$$

(2.10)

or explicitly the fibration $\mathcal{P}$ maps the state $|\psi\rangle_Q$ as

$$\mathcal{P}|\psi\rangle_Q := \frac{q_0\bar{q}_1j}{|q_1|^2} = \frac{S + Cj}{|q_1|^2},$$

(2.11)

where $S = a_0\bar{a}_2 + a_1\bar{a}_3$ and $C = a_0a_3 - a_1a_2$ denote, respectively, the Schmidt and concurrence terms in quantum information theory. In Hopf fibration (2.10), the $x_i$ are Cartesian coordinates for $S^4$ and define as follow:

$$x_0 = |q_0|^2 - |q_1|^2, \quad x_1 = 2Re(S),$$
\[ x_2 = 2 \text{Im}(S), \]
\[ x_3 = 2 \text{Re}(C), \]
\[ x_4 = 2 \text{Im}(C). \]

(2.12)

For \( S = 0 \), the two-qubit pure state (2.7) has Schmidt form \( \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle \). On the other hand, for two-qubit pure state \( C = 2|C| \) is concurrence measure and for \( C = 0 \) the two-qubit pure state (2.7) is reduced to a separable state, which implies that the base space is restricted to \( S^2 \) for non-entangled two-qubit state.

According to Eq. (2.9), the quaternionic form of ground state for \( E_e < E_o \) and \( E_e > E_o \) is given by

\[
\begin{align*}
|E_e^e\rangle \rightarrow |E_e^e\rangle_Q &= \sin \left( \frac{\theta_1}{2} \right) |0\rangle_Q - \cos \left( \frac{\theta_1}{2} \right) j|1\rangle_Q, \\
|E_o^o\rangle \rightarrow |E_o^o\rangle_Q &= \sin \left( \frac{\theta_2}{2} \right) j|0\rangle_Q - e^{-i\phi} \cos \left( \frac{\theta_2}{2} \right) |1\rangle_Q.
\end{align*}
\]

(2.13)

According to Hopf fibration (2.11), the concurrence of ground-state energy takes the form

\[ C(\text{ground state}) = \begin{cases} 
\sin(\theta_1) & E_e^e < E_o^o, \\
\sin(\theta_2) & E_e^e > E_o^o. 
\end{cases} \]

(2.14)

As shown in Fig. 2a, the entanglement changes abruptly in phase transition point for \( \lambda = 0 \) and \( 0 < \gamma < 1 \) then it seems that the entanglement works well as an indicator to quantum phase transitions in this region. But for \( \gamma = 0 \) and \( 0 < \lambda < 1 \), the concurrence has not tangible change and it is not a suitable indicator for detect the phase transition. There is similar behavior in Fig. 2b, i.e., for small value of inhomogeneous magnetic field \( b_z \) and \( 0 < \gamma < 1 \) the concurrence has intangible change, and for large value of \( b_z \) the concurrence changes abruptly in phase transition point. For \( b_z = J_z = 0 \) and \( D_z = 0 \), the concurrence (2.14) reduces to the results of Ref. [5].

(a)  
(b)  

Fig. 2 (Color online) a Concurrence of ground state as a function of \( \gamma \) and \( \lambda \) for Hamiltonian (2.1). The level crossing (white line) for parameters \( J_z = 0.15, D_z = 0.2, b_z = 0.1 \). b Concurrence of ground state as a function of \( \gamma \) and \( b_z \) for Hamiltonian (2.1). The level crossing (white line) for parameters \( J_z = 0.1, D_z = 0.4, \lambda = 0.85 \).
2.3 Geometric phase and its geometrical structure in quantum phase transition

2.3.1 Geometric phase on one-qubit Bloch sphere

Quantum states are represented as vectors in a complex vector space, and these vectors are only defined up to a global phase which is a unit modulus complex number. Look at the amplitude between the two states $|\psi_I\rangle$ and $|\psi_F\rangle$ in the polar decomposition:

$$\langle \psi_I | \psi_F \rangle = re^{i\xi_{IF}},$$

(2.15)

where the $\xi_{IF}$ is the relative phase between the two states. The states $e^{i\alpha_1} |\psi_I\rangle$ and $e^{i\alpha_2} |\psi_F\rangle$, which differ from the original states by an overall arbitrary phase, have a different relative phases by the amount of $\Delta \alpha = \alpha_1 - \alpha_2$. There are infinitely choices for $\Delta \alpha$, and they all look equally appropriate which formally says that this definition of phase is gauge dependent (phase dependent). Consider the path connecting the two states, $|\psi(t)\rangle$, such that when $t = 0$ we have $|\psi_I\rangle$ and when $t = 1$ we have $|\psi_F\rangle$. One can transport the states $|\psi(t)\rangle$ from the position $I$ to the position $F$ and see how different the final phase is to that of $|\psi_F\rangle$ through interference. If the states $|\psi_F\rangle$ and $|\psi_I\rangle$ transported to $|\psi_F\rangle$ interfere constructively, then they are in phase, and the degree of interference can define the phase difference (see Fig. 3) [40].

From employing the differential geometry, we know that the transport itself does not introduce any additional “twists and turns” in the phases so that we are actually comparing some different phases to the original ones. Suppose, we have a curved manifold and we have a vector at a point $I$ and another at a point $F$. The relative phase between the two vectors can be measured by transport one of them to the other one, which the angle between the two vectors is relative phase. The straightest possible path is known as a geodesic, and the corresponding evolution along this path is known as the parallel transport. To define a parallel transport, let’s look at the infinitesimal evolution, from $|\psi(t)\rangle$ to $|\psi(t + dt)\rangle$. If we do not want there to be any twists and turns in the phase, even infinitesimally, then the two states should be in phase. So, we require that $Arg[\langle \psi(t) | \psi(t + dt) \rangle] = 0$. This is the same as asking that $\langle \psi(t) | \psi(t + dt) \rangle$ be purely real, i.e., up to second-order $Im[\langle \psi(t) | \psi(t + dt) \rangle] = Im[\langle \psi(t) | d| \psi(t) \rangle] = 0$. But $\langle \psi(t) | d| \psi(t) \rangle$ is purely imaginary, hence the parallel transport condition becomes $\langle \psi(t) | d| \psi(t) \rangle = 0$. This definition of parallel transport
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is not automatically gauge invariant. By this we mean that if instead of the state $|\psi(t)\rangle$, we use the state

$$|\psi'(t)\rangle = e^{i\alpha t} |\psi(t)\rangle,$$

then the parallel transport condition changes by the amount

$$\langle \psi'(t) | d | \psi'(t) \rangle = \langle \psi(t) | d | \psi(t) \rangle + i \frac{d\alpha}{dt} dt,$$

as can easily be checked. In order to obtain something that is gauge invariant, we can integrate the expression $\langle \psi(t) | d | \psi(t) \rangle$ over a closed loop, giving us the expression for the geometric phase, and then exponentiate the result. So, the geometric phase resulting from the parallel transport is

$$B = \int_i^f \langle \psi(t) \rangle \frac{d}{dt} |\psi(t)\rangle dt,$$

and its exponential (over a closed loop) is gauge independent, but not path independent. It is also interesting that the underlying space is curved, and it is the curvature that is reflected in the phase difference; in fact, the curvature is the phase difference up to a constant factor. When a quantity vanishes infinitesimally, but its integral over a finite region does not, then this quantity is called non-integrable. Therefore, geometric phases are a manifestation of non-integrable phase factors in quantum mechanics. Let’s look at two level systems to illustrate this point. Suppose that we now evolve from the state $|0\rangle$ to the state $|S^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then to $|S^+_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i |1\rangle)$ and finally back to the state $|0\rangle$. On the Bloch sphere, we are going from the north pole to the equator, then we move on the equator by an angle of $\pi/2$ and finally we move back to the north pole. To see the corresponding geometric phase, we start with a tangential vector initially at the north pole pointing in some direction. If we now parallel transport this vector along the described path, then we end up with a vector pointing in a different direction to the original one (even though, infinitesimally, the phase vector has always stayed parallel to itself). The angle between the two is $\pi/2$, which is exactly equal to the area covered by the state vector during the transport, or the corresponding solid angle of the transport (see Fig. 4). It is interesting that for two-qubit pure states there are Bloch sphere representation in quaternionic Hilbert space and we can generalize the concept of parallel transform from complex Bloch sphere to quaternionic Bloch sphere.

2.3.2 Geometric phase in two-qubit states and quaternion representation

For discussion of geometric phases and criticality in spin chain systems, we are interested in the Hamiltonian that can be obtained by applying a rotation by angle $\eta$, around the $z$ direction, to each spin, i.e., $H' = H^{\text{even}} + H^{\text{odd}} = U^\dagger_z(\eta)H U_z(\eta)$
where \( U_z(\eta) = \exp[-i \frac{\eta}{2}(\sigma_z^1 + \sigma_z^2)] \). Then the rotational Hamiltonian is given by

\[
H_{\text{even}}^\prime = H_{\text{even}} = \begin{pmatrix}
-2b_z - j_z & 2i D_z + 1 \\
-2i D_z + 1 & 2b_z - j_z
\end{pmatrix},
\]

\[
H_{\text{odd}}^\prime = \begin{pmatrix}
2\lambda + j_z & \gamma e^{2i\eta} \\
\gamma e^{-2i\eta} & j_z - 2\lambda
\end{pmatrix},
\]

(2.19)

where the \( H_{\text{even}}^\prime \) is invariant under transformation, then the instantaneous ground state \(|\psi_0\rangle\) satisfying \( H'(r)|\psi_0\rangle = E_0|\psi_0\rangle\), and according to (2.9) the transformed ground state reads

\[
\begin{cases}
|E_{-}^\prime\rangle_Q = e^{-i\eta} \sin\left(\frac{\theta_1}{2}\right) |0\rangle_Q - e^{i\eta} \cos\left(\frac{\theta_1}{2}\right) j|1\rangle_Q & E_- < E_+, \\
|E_{+}^\prime\rangle_Q = \sin\left(\frac{\theta_2}{2}\right) j|0\rangle_Q - e^{-i\eta} \cos\left(\frac{\theta_2}{2}\right) |1\rangle_Q & E_+ > E_-.
\end{cases}
\]

(2.20)

For two quaternionic spinors \(|\psi\rangle_Q\) and \(|\phi\rangle_Q\), the scalar product is defined by

\[
\langle \phi | \psi \rangle_Q := \bar{\phi}^{\alpha} \psi_{\alpha} = \bar{\phi}_1 \psi_1 + \bar{\phi}_2 \psi_2, \quad \psi_1, \psi_2, \phi_1, \phi_2 \in \mathbb{Q}.
\]

(2.21)

Note that right multiplication of quaternionic spinors with the nonzero quaternion \(q\) yields the expression \( \langle \phi q | \psi q \rangle_Q = \bar{q} \langle \phi | \psi \rangle_Q q \). The vector space \( \mathcal{H}_2^Q \) of quaternionic spinors with this scalar product is a quaternionic Hilbert space. Let us consider the quaternionic spinor (2.8) in useful form

\[\hfill \]
\[ |\psi\rangle_Q = \left( \frac{q_1}{q_2} \right) = \frac{1}{\sqrt{1 + |x|^2}} \left( \begin{array}{c} 1 \\ x \end{array} \right) q, \tag{2.22} \]

where \( x = \frac{q_0\bar{q}}{|q_1|} \) come from stereographic projection \( P \) in Hopf fibration (2.10) and \( q \) is an unit quaternion \( (q\bar{q} = 1) \). The distance between two nonidentical, non-orthogonal spinor states \( |\psi\rangle \) and \( |\phi\rangle \) in base space \( S^4 \) is defined as

\[ \cos^2 \frac{\Delta \phi}{2} = \frac{ \langle \phi | \psi \rangle_Q^2 }{4} \quad 0 < \Delta < \pi. \tag{2.23} \]

In the representation Eq. (2.22), the Fubini–Study metric in quaternion spinor is defined by [11]

\[ dl^2 = g_{ij} dx^i \otimes dx^j = 4(1 - |\langle \psi + d\psi | \psi \rangle_Q|^2) = \frac{4d\bar{x}dx}{(1 + |x|^2)^2}, \tag{2.24} \]

and its corresponding connection is defined by

\[ \Gamma = 1 - \langle \psi + d\psi | \psi \rangle_Q = \bar{q} \left( \frac{Im(\bar{d}dx)}{1 + |x|^2} \right) q + \bar{q}dq, \tag{2.25} \]

where \( Im(x) \) is imaginary part of quaternion \( x \). The quantity \( A = Im(\bar{d}dx)_{1+|x|^2} \) is a non-Abelian gauge field (one-form) which equivalent to the standard \( SU(2) \) instanton with self-dual curvature and second Chern number \( C_2 = 1 \) and called Berry connection [11]. Note that according to Eq. (2.25) the \( \Gamma = 0 \) corresponding to parallel transformation of quaternionic phases. The differential equation of the parallel transformation is determined with a suitable boundary condition in Eq. (2.25). The standard path ordered solution for parallel transportation with initial and end points \( q(0) = 1 \) and \( q(\tau) \), respectively, is obtain by

\[ q(\tau) = P \exp \left( -\int_{\text{curve}} A \right). \tag{2.26} \]

For convenience in the next step, we consider the following state

\[ |u\rangle_Q = \left( \begin{array}{c} \cos \left( \frac{\theta}{2} \right) p \\ \sin \left( \frac{\theta}{2} \right) q \end{array} \right), \tag{2.27} \]

where \( q, p \) are quaternionic phases with \( |q|^2 = |p|^2 = 1 \) and the parametrization in term of \( \theta, p \) and \( q \) has the same form as the well-known parametrization of a complex spinor associated with the Bloch sphere. According to Eq. (2.25), the connection for state Eq. (2.27) is given by

\[ \Gamma = \frac{1 - \cos(\theta)}{2} \bar{q} (Im(\bar{p}dp)) q + \bar{q}dq, \tag{2.28} \]
Fig. 5  (Color online) a Geometric phase of ground state as a function $\gamma$ and $\lambda$ for Hamiltonian (2.1). The level crossing (white line) for parameters $J_z = 0.15, D_z = 0.2, b_z = 0.1$. b Geometric phase of ground state as a function $\gamma$ and $b_z$ for Hamiltonian (2.1). The level crossing (white line) for parameters $J_z = 0.1, D_z = 0.4, \lambda = 0.85$

and

$$A = \frac{1}{2}(1 - \cos(\theta)) Im(\bar{p}dp). \tag{2.29}$$

The ground states of $H'$ in Eq. (2.20) take the form of state (2.27), then the Berry connection of the ground states can be estimated as

$$A = \begin{cases} (1 - \cos(\theta))d\eta & E^e < E^o, \\ 0 & E^e > E^o. \end{cases} \tag{2.30}$$

The parallel transformation of ground states with respect to parameter $\eta$ gives the Berry phase $B = -\int A$ as follows

$$B = \begin{cases} -2\pi(1 - \cos(\theta)) & E^e < E^o, \\ 0 & E^e > E^o. \end{cases} \tag{2.31}$$

Figure 5 shows the behavior of Berry phase for three regime. It seems that the Berry phase works as well an indicator to quantum phase transitions in Heisenberg Hamiltonian. However, a comparison of the Berry phase and concurrence shows that in the regions that concurrence is not a appropriate indicator for the phase transition, the geometric phase is a appropriate indicator, and vice versa, in the sense that in regions where Berry phase does not show phase transition (i.e., the region $\lambda = 0$ and $0 < \gamma < 1$ in Fig. 5a and the region $b_z = 1$ and $0 < \gamma < 1$ in Fig. 5b) the concurrence indicates the quantum phase transition (see the corresponding region in Fig. 2a, b). In other words, geometric phase and the ground state entanglement are complementary systems that can exhibit quantum phase transition. For $b_z = J_z = 0$ and $D_z = 0$, the geometric phase (2.31) reduces to the results of Ref. [5]. Figure 6 displays the ground-state energy, concurrence and Berry phase as a function of anisotropic parameter for the nearest-neighbor spins in the Heisenberg model (2.1) with different values of the DM interaction. It is clear that for small value of $D_z$ the Berry phase has a significant changes in phase transition points, but the concurrence measure has a smooth change.
Fig. 6 (Color online) Energy, concurrence and geometric phase of ground state as a function of $\gamma$ for Hamiltonian (2.1) for parameters $J_z = 0.5$, $\lambda = 0.1$, $b_z = 0.1$ and different value of $D_z$. a $D_z = 0.4$, b $D_z = 0.6$, c $D_z = 0.8$ and d $D_z = 1.2$

Fig. 7 (Color online) Energy, concurrence and geometric phase of ground state as a function of $\gamma$ for Hamiltonian (2.1) for parameters $J_z = 0.1$, $D_z = 0.2$, $b_z = 0.2$ and different value of $\lambda$. a $\lambda = 0.1$, b $\lambda = 0.3$, c $\lambda = 0.6$ and d $\lambda = 0.9$
in this region. It is interesting that for a large value of $D_z$ concurrence changes are sharper than the Berry phase. Therefore, it is reasonable to conclude that in small value of $D_z$ the Berry phase is good indicator than concurrence measure, and for large value of $D_z$ the concurrence measure is good indicator than Berry phase. On the other hand, by increasing the DM interaction the phase transition occurs for large value of anisotropy parameter $\gamma$.

Figure 7 shows the ground-state energy, concurrence and Berry phase as a function of anisotropic parameter for the nearest-neighbor spins in the Heisenberg model (2.1) with different values of the magnetic field $\lambda$. It shows that for small value of magnetic field the concurrence measure is good indicator for quantum phase transition, but for large value of magnetic field the Berry phase shows a sharp changes in phase transition point.

3 Conclusion

In summary, we have considered the anisotropic XYZ Hamiltonian with uniform and non-uniform external magnetic field and DM interaction. It was shown that the geometric phase and concurrence measure are appropriate indicators for detecting the quantum phase transition in generalized Heisenberg model. In fact, the geometric phase and concurrence measure individually do not capture a level crossing completely, which happens in ground state, but they are complementary systems that can detect quantum phase transition. In other words, the Berry phase and concurrence measure can be used as indicators to detect the quantum phase transitions in Heisenberg Hamiltonian; however, a comparison of these indicators shows that, where concurrence is not a good indicator for the phase transition, the geometric phase is a appropriate one and vice versa. On the other hand, the quaternionic representation maps a usual two-qubit pure state to the one quaterbit which simply reminds us the ordinary Bloch sphere. In this representation, the geometric phase is proportional to parallel transportation with Mannoury–Fubini–Study metric in Hilbert space of one quaterbit states. Also, the quaternionic part of stereographic projection is related to the concurrence measure.

We have showed that, for small values of $D_z$ and $b_z$, but large magnetic field $\lambda$, the Berry phase has significant changes in phase transition points, while the concurrence measure has a smooth change in this region. On the other hand, for large values of $D_z$ and $b_z$, but small magnetic field $\lambda$, the concurrence has more significant change than Berry phase. Moreover, by increasing the DM interaction, the phase transition occurs for large value of anisotropy parameter $\gamma$.

References

1. Sachdev, S.: Quantum Phase Transitions, 2nd edn. Cambridge University Press, Cambridge (2011)
2. Goldenfeld, N.: Lectures on Phase Transitions and the Renormalization Group. University of Illinois, Urbana-Champaign (1992)
3. Mosseri, R., Dandoloff, R.: Geometry of entangled states, Bloch spheres and Hopf fibrations. J. Phys. A Math. Gen. 34, 10243 (2001)
4. Bernevig, B.A., Chen, H.D.: Geometry of the three-qubit state, entanglement and division algebras. J. Phys. A Math. Gen. 36, 8325 (2003)
5. Oh, S.: Geometric phases and entanglement of two qubits with XY type interaction. Phys. Lett. A 373, 644–647 (2009)
6. Berry, M.V.: Quantal phase factors accompanying adiabatic changes. Proc. R. Soc. Lond. Ser. A 392, 45 (1984)
7. Shapere, A., Wilczek, F.: Geometric Phases in Physics. World Scientific, Singapore (1989)
8. Najarbashi, G., Ahadpour, S., Fashti, M.A., Tavakoli, Y.: Geometry of a two-qubit state and intertwining quaternionic conformal mapping under local unitary transformations. J. Phys. A Math. Theor. 40, 6481–6487 (2007)
9. Najarbashi, G., Seifi, B., Mirzaei, S.: Two- and three-qubit geometry, quaternionic and octonionic conformal maps, and intertwining stereographic projection. Quantum Inf. Process. 15, 509528 (2016)
10. Najarbashi, G., Seifi, B.: Relation Between Stereographic Projection and Concurrence Measure in Bipartite Pure States. Int. J. Theor. Phys. (2016). doi:10.1007/s10773-016-3071-2
11. Lévy, P.: The geometry of entanglement: metrics, connections and the geometric phase. J. Phys. A Math. Gen. 37, 1821 (2004)
12. Oh, S., Huang, Z., Peskin, U., Kais, S.: Entanglement, Berry phases, and level crossings for the atomic Breit–Rabi Hamiltonian. Phys. Rev. A 78, 062106 (2008)
13. Bennett, C.H., Wiesner, S.J.: Communication via one-and two-particle operators on Einstein–Podolsky–Rosen states. Phys. Lett. A 108, 2881 (1992)
14. Bennett, C.H., Brassard, G., Crépeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. 70, 1895 (1993)
15. Ekert, A.K.: Quantum cryptography based on Bells theorem. Phys. Rev. Lett. 67, 661 (1991)
16. Murao, M., Jonathan, D., Plenio, M.B., Vedral, V.: Quantum teleporting and multipartite entanglement. Phys. Rev. A 59, 156 (1999)
17. Schrödinger, E.: Probability relations between separated systems. Proc. Camb. Phil. Soc. 31, 555 (1935)
18. Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935)
19. Bell, J.S.: On the Einstein–Podolsky–Rosen paradox. Physics 1, 195 (1964)
20. Maleki, Y., Khashami, F., Mousavi, Y.: Entanglement of three-spin states in the context of SU(2) coherent states. Int. J. Theor. Phys. 54, 210 (2015)
21. Angelakis, D.G., Christandl, M., Ekert, A., Kay, A., Kulik, S.: Quantum Information Processing: From Theory to Experiment, vol. 199. Computer and Systems Sciences, vol. 199. IOS Press, NATO Science Series, Amsterdam (2006)
22. Gunlycke, D., Kendon, V.M., Vedral, V., Bose, S.: Thermal concurrence mixing in a one-dimensional Ising model. Phys. Rev. A 64, 042302 (2001)
23. Yang, Z., Yang, L., Dai, J., Xiang, T.: Rigorous solution of the spin-1 quantum ising model with single-ion anisotropy. Phys. Rev. Lett. 100, 067203 (2008)
24. Kamta, G.L., Starace, A.F.: Anisotropy and magnetic field effects on the entanglement of a two qubit Heisenberg XY chain. Phys. Rev. Lett. 88, 107901 (2002)
25. Wang, X.: Thermal and ground-state entanglement in Heisenberg XX qubit rings. Phys. Rev. A 66, 034302 (2002)
26. Sun, Y., Chen, Y., Chen, H.: Thermal entanglement in the two-qubit Heisenberg XY model under a nonuniform external magnetic field. Phys. Rev. A 68, 044301 (2003)
27. Kao, Z.C., Ng, J., Yeo, Y.: Three-qubit thermal entanglement via entanglement swapping on two-qubit Heisenberg XY chains. Phys. Rev. A 72, 062302 (2005)
28. Zhu, S.L.: Scaling of geometric phases close to the quantum phase transition in the XY spin chain. Phys. Rev. Lett. 96, 077206 (2006)
29. Asoudeh, M., Karimipour, V.: Thermal entanglement of spins in an inhomogeneous magnetic field. Phys. Rev. A 71, 022308 (2005)
30. Zhang, G.F.: Thermal entanglement and teleportation in a two-qubit Heisenberg chain with Dzyaloshinskii–Moriya anisotropic antisymmetric interaction. Phys. Rev. A 75, 034304 (2007)
31. Zhang, G.F., Li, S.S.: Thermal entanglement in a two-qubit Heisenberg XXZ spin chain under an inhomogeneous magnetic field. Phys. Rev. A 72, 034302 (2005)
32. Kargarian, M., Jafari, R., Langari, A.: Renormalization of entanglement in the anisotropic Heisenberg (XXZ) model. Phys. Rev. A 77, 032346 (2008)
33. Dzyaloshinsky, I.: A thermodynamic theory of weak ferromagnetism of antiferromagnetics. J. Phys. Chem. Solids 4, 241 (1958)
34. Moriya, T.: New mechanism of anisotropic super exchange interaction. Phys. Rev. Lett. 4, 228 (1960)
35. Kheirandish, F., Akhtarshenas, S.J., Mohammadi, H.: Effect of spin–orbit interaction on entanglement of two-qubit Heisenberg XYZ systems in an inhomogeneous magnetic field. Phys. Rev. A 77, 042309 (2008)
36. Wu, L.A., Lidar, D.A.: Universal quantum logic from Zeeman and anisotropic exchange interactions. Phys. Rev. A 66, 062314 (2002)
37. Wu, L.A., Lidar, D.A.: Dressed qubits. Phys. Rev. Lett. 91, 097904 (2003)
38. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245 (1998)
39. Hill, S., Wootters, W.K.: Entanglement of a pair of quantum bits. Phys. Rev. Lett. 78, 5022 (1997)
40. Vedral, V.: Modern Foundations of Quantum Optics. University of Leeds, Imperial College Press, Leeds (2005)