Resilience of realism-based nonlocality to local disturbance

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Employing a procedure called monitoring—via a completely positive trace-preserving map that is able to interpolate between weak and projective measurements—we investigate the resilience of the recently proposed realism-based nonlocality to local and bilocal weak measurements. This analysis indicates realism-based nonlocality as the most ubiquitous and persistent form of quantumness within a wide class of quantum-correlation quantifiers. In particular, we show that the set of states possessing this type of quantumness forms a strict superset of symmetrically discordant states and, therefore, of discordant, entangled, steerable, and Bell-nonlocal states. Moreover, we find that, under monitoring, realism-based nonlocality is not susceptible to sudden death.

I. INTRODUCTION

By now it is safe to say that the notion of Bell nonlocality, as referring to a violation of Bell’s hypothesis of local causality [1], is very well defined formally [2] and convincingly demonstrated experimentally [3–6], although some debate persists about the real meaning of such violations [7,8]. It is also fair to state that these developments have been trig-
ered by a firm notion of local realism, which was shared by many physicists of the beginning of the twentieth century and led Einstein, Podolsky, and Rosen (EPR) to suggest that quantum mechanics could not be the whole story about nature [9].

Recently, a notion of nonlocality has been put forward that profoundly differs from Bell nonlocality [10]. The concept was constructed by using, as primitive mechanism, a quantifier of the irreality degree [11]

\[ \mathcal{I}(\rho) := S(\Phi_A(\rho)) \leq S(\rho) = S(\rho \| \Phi_A(\rho)) \]  

of an observable \( A \) acting on \( \mathcal{H}_A \) given a preparation \( \rho \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \), where \( S(\rho) \) stands for the von Neumann entropy of \( \rho \) and \( S(\rho \| \sigma) \) is the quantum relative entropy of \( \rho \) and \( \sigma \). Roughly, the irreality (1) gives an entropic distance between the state \( \rho \) under scrutiny and a state

\[ \Phi_A(\rho) := \sum_a A_a \rho A_a = \sum_a p_a A_a \otimes \rho_{B_a} \]  

where \( p_a = \text{Tr}(A_a \otimes \mathbb{1}) \) and \( \rho_{B_a} = \text{Tr}_B(A_a \otimes \mathbb{1}) \rho_{A_a} \). The state \( \Phi_A(\rho) \) can be viewed as resulting from a projective measurement of the observable \( A = \sum_a a A_a \) (where \( A_a A_{a'} = \delta_{aa'} A_a \)) in a scenario where the outcome is omitted. Since \( A \) has been measured and thus became well defined, \( \Phi_A(\rho) \) is a state of reality for \( A \). Hence, \( \mathcal{I}(\rho) \) quantifies the violation of the hypothesis of realism, \( \Phi_A(\rho) = \rho \). Notice, in particular, that the reduced state \( \rho_{A} = \text{Tr}_B \Phi_A(\rho) = \sum_a p_a A_a \) has no quantum coherence \( A_a \)’s eigenbasis, that is, it is just a mixture of states \( A_a \) with elements of reality \( a \) for \( A \). In addition, \( \Phi_A \Phi_A(\rho) = \Phi_A(\rho) \), showing that further unrevealed measurements do not change a preestablished reality. Finally, since \( \Phi_A(\rho) \) is a completely positive trace-preserving map, it follows from the monotonicity of the von Neumann entropy that \( \mathcal{I}(\rho) \geq 0 \), with the equality holding for \( \rho = \Phi_A(\rho) \). The work [10] then focused on the difference

\[ \eta_{AB}(\rho) := \mathcal{I}(\rho) - \mathcal{I}(\Phi_B(\rho)) \]  

which measures alterations in the irreality of \( A \) induced by unrevealed projective measurements of \( B \) conducted in a remote site \( B \). This object has been introduced in Ref. [11] as a measure of the realism-based nonlocality of a context defined by \( [A, B, \rho] \), as it quantifies violations of the hypothesis of local realism, \( \mathcal{I}(\rho) = \mathcal{I}(\Phi_B(\rho)) \), for a particular context. At this point we can readily recognize the conceptual difference between Bell nonlocality and realism-based nonlocality: the former accuses violations of the local causality hypothesis, whereas the latter refers to a violation of local realism in the terms delineated above. Using the definition (1), we also note that

\[ \eta_{AB}(\rho) = S(\Phi_A(\rho)) + S(\Phi_B(\rho)) - S(\Phi_A(\Phi_B(\rho)) - S(\rho) \]  

which reveals the presence of the symmetry \( A \leftrightarrow B \). This means that \( \eta_{AB}(\rho) \) also quantifies changes in \( B \)’s irreality induced by unrevealed measurements of \( A \) in a remote site. Finally, the realism-based nonlocality \( N(\rho) \) of a preparation \( \rho \) was introduced as the maximum difference \( \eta_{AB}(\rho) \) over all possible pair of observables \( [A, B] \) measured in distant sites:

\[ N(\rho) := \max_{A,B} \eta_{AB}(\rho). \]  

As pointed out in Refs. [10,11], \( \eta_{AB} \) is a nonnegative quantity that vanishes only for fully uncorrelated states \( (\rho = \rho_A \otimes \rho_B) \) or for states \( \rho = \Phi_{AB}(\rho) \) with reality for \( A \) \( (B) \). In addition, \( N \) vanishes for \( \rho = \rho_A \otimes \rho_B \) and reduces to the entanglement entropy for pure states, thus being free of the anomaly that affects some Bell-nonlocality measures [12–15].

Unlike the Bell nonlocality framework, for which much progress has already been reached (see, e.g., Ref. [2] for formal characterization and related concepts, applications, experimental aspects, and multipartite extensions, Ref. [16] for a resource theory and quantifiers, and Ref. [17] for the connection with entanglement), the research on realism-based nonlocality is rather incipient. Among several open questions concerning this quantity, one may cite the search for a resource theory, operational meaning, extension to multipartite scenarios, and applications in information-theoretic or thermodynamic tasks. Also, it is not known yet how realism-based nonlocality behaves under local operations. Such a study is crucial to assess whether realism-based nonlocality can be viewed as a genuine measure of nonlocal quantum correlations. In particular, one may wonder whether realism-based nonlocality is susceptible to sudden death, a phenomenon that
II. THEORETICAL PRELIMINARIES

We start by indicating the model of weak measurements we shall employ throughout this work. In Ref. [19] a procedure was introduced—the so-called monitoring—which is mathematically described by the completely positive trace-preserving map

\[ M'_O(\rho) := (1 - \epsilon)\rho + \epsilon \Phi_O(\rho), \]

for a real parameter \( \epsilon \in (0, 1) \), \( \rho \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \), and a generic observable \( O \) acting on either \( \mathcal{H}_A \) or \( \mathcal{H}_B \). This map continuously interpolates between a regime of no measurement at all (\( M'_O^{|\text{no}} = \Phi_O \)) to the one of unrevealed projective measurements (\( M'_O^{|\text{r}} = \Phi_O \)). Between these extrema, that is, for \( 0 < \epsilon < 1 \), the map is then said to correspond to an unrevealed weak measurement. The adjective “unrevealed” emphasizes that the outcome of the measurement is not accessed by the experimentalist, so that no collapse or information gain occurs. In this sense, the map \( M'_O \) differs from the standard weak-measurement approach that is defined via conjugation of weak interactions with post selections. Despite its relevance, the term “unrevealed” will be henceforth omitted. As pointed out in Ref. [19], the monitoring of \( O \) can also be thought of as deriving from a von Neumann premeasurement through which this quantity gets correlated with an ancillary system that is posteriorly discarded. By direct application of the definitions (2) and (6) one shows that

\[ M'_O \Phi_O = \Phi_O M'_O = \Phi_O. \]

Successive applications of the same monitoring yields

\[ [M'_O]^n(\rho) = (1 - \epsilon)^n \rho + (1 - (1 - \epsilon)^n) \Phi_O(\rho), \]

which implies that \( [M'_O]^{|\text{no}} = \Phi_O \). This shows that a sequence of infinitely many monitorings is equivalent to a projective unread measurement. Most importantly, it can be shown [11, 19] that

\[ S(\Phi_O(\rho)) \geq S([M'_O]^n(\rho)), \]

for any finite integer \( n \), with the equality holding for \( \rho = \Phi_O(\rho) \), and

\[ \Delta R^{\epsilon}_{A}(\rho) := S(M'_A(\rho)) - S(\rho), \]

for the weak increase in the reality of a monitored observable \( A \) given the preparation \( \rho \). Clearly, there is no reality increase for \( \epsilon \rightarrow 0 \) and maximum increase of \( \Delta R(\rho) \) for \( \epsilon \rightarrow 1 \). Recently, using state tomography Mancino et al. measured \( \Delta R \) for the polarization of a photon as a function of the intensity \( \epsilon \) of the monitoring in a photonics experiment [23].

In connection with the context realism-based nonlocality [4], and for future convenience, we prove a useful result for the following quantity:

\[ \delta_{AB}^{\epsilon}(\rho) := S(M_A^\epsilon(\rho)) + S(M_B^\epsilon(\rho)) - S(M_A^\epsilon M_B^\epsilon(\rho)) - S(\rho), \]

where \( M_A^\epsilon(\rho) \) is a local monitoring of the observable \( A(B) \) on \( \mathcal{H}_A(\mathcal{H}_B) \) with intensity \( \epsilon, \epsilon' \in (0, 1) \). Notice that \( \delta_{AB}^{\epsilon}(\rho) = 0 \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \), with the initial state \( \Phi_0 = \rho \otimes |x\rangle \langle y| \), with \( |x\rangle \langle y| \) being any orthogonal states.

\[ \text{Proof.} \quad S(\sigma_{ab}) + S(\sigma_{ec}) \leq S(\sigma_{ace}) + S(\sigma_{a'b'}) . \]

and the analogous relation for \( \sigma_{acb} \), which proves the first part of the theorem. From the additivity of the entropy, the property (7), and the definition (11) it follows that

\[ S(\rho_{AB}) = S(\rho_A \otimes \rho_B) = S(\Phi_{AB}(\rho)) = \delta_{AB}^{\epsilon}(\rho) + \delta_{AB}^{\epsilon'}(\rho) - \delta_{AB}^{\epsilon \epsilon'}(\rho), \]

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\[ S(\rho_A \otimes \rho_B) = S(\Phi_{AB}(\rho)) = \delta_{AB}^{\epsilon}(\rho) + \delta_{AB}^{\epsilon'}(\rho) - \delta_{AB}^{\epsilon \epsilon'}(\rho), \]
These relations reveal an interesting point. One might be tempted at a first sight to interpret \(\delta^\epsilon \text{ rel} (\rho)\) as mathematical extension of the context realism-based nonlocality \(\delta^\epsilon\) to the regime of monitorings, after all, \(\delta^\epsilon_{\text{AB}} (\rho) = \eta_{\text{AB}} (\rho)\). However, one has that \(\delta^\epsilon \text{ rel} (\rho) = \delta^0_{\text{AB}} (\rho) = \eta_{\text{AB}} (\rho) = 0\) even when \(\eta_{\text{AB}} (\rho) > 0\); that is, this interpretation would cause a clear incompatibility among the predictions given by the quantifiers \(\eta_{\text{AB}}\) and \(\delta^\epsilon_{\text{AB}}\). It follows, therefore, that \(\delta^\epsilon \text{ rel} (\rho)\) is to be interpreted in a rather different way. Let us specialize the analysis of Eq. (13) to the quantity \(\delta^\epsilon_{\text{AB}} (\rho) = \eta_{\text{AB}} (\rho) - \eta_{\text{AB}} (M^\rho_B)\) for a while. Since \(\eta_{\text{AB}} (\rho)\) and \(\eta_{\text{AB}} (M^\rho_B)\) are quantifiers of the amount of realizability-based nonlocality for the contexts \((A, B, \rho)\) and \((A, B, M^\rho_B)\), respectively, then \(\delta^\epsilon_{\text{AB}} (\rho)\) can be interpreted as the amount of realizability-based nonlocality that is suppressed when \(\rho\) is replaced with its monitored counterpart \(M^\rho_B\). In other words, \(\delta^\epsilon_{\text{AB}} (\rho)\) turns out to be a measure of the amount of realizability-based nonlocality that is destroyed upon local monitoring of the observable \(B\) with intensity \(\epsilon\).

Accordingly, we see that no realizability-based nonlocality is destroyed for \(\epsilon \rightarrow 0\) whereas all realizability-based nonlocality is destroyed for \(\epsilon \rightarrow 1\). This can also be verified through the following formulation. First, using precedent formulas notice that \(\delta^\epsilon_{\text{AB}} (M^\rho_B) = \eta_{\text{AB}} (M^\rho_B) - \eta_{\text{AB}} (M^\rho_B)\) and, therefore, \(\delta^\epsilon \text{ rel} (\rho) = \delta^\epsilon \text{ rel} (M^\rho_B)\), which, via Eq. (8), can be identified to \(\delta^\epsilon_{\text{AB}} (\rho)\) with \(\epsilon' = 1 - (1 - \epsilon)^{k+1}\). By summing terms for higher-order monitorings we find

\[
\sum_{k=0}^{n} \delta^\epsilon_{\text{AB}} (M^\rho_B) = \delta^\epsilon_{\text{AB}} (\rho) \Rightarrow \eta_{\text{AB}} (\rho),
\]

where \(\epsilon_k = 1 - (1 - \epsilon)^{k+1}\). This shows that adding the amount \(\delta^\epsilon \text{ rel} (\rho)\) of realizability-based nonlocality that is suppressed upon \(\epsilon\)-intensity monitorings \(M^\rho_B\) of infinitely many orders \(k\) is equal to the total amount \(\eta_{\text{AB}} (\rho)\) associated with the state \(\rho\).

### III. RESILIENCE TO LOCAL WEAK MEASUREMENT

#### A. Realism-based nonlocality suppression via local monitoring

We now introduce and analyze a quantifier that will prove informative with respect to the resilience of realizability-based nonlocality under local monitoring. Inspired by the previous discussion, we take

\[
\Delta^\epsilon_{\text{AB}} (\rho) := \max_{A, B} \left[ \eta_{\text{AB}} (\rho) - \eta_{\text{AB}} (M^\rho_B) \right]
\]

as a quantifier of the optimized suppression implied to the context realizability-based nonlocality when the state \(\rho\) on \(\mathcal{H}_A \otimes \mathcal{H}_B\) is submitted to a local monitoring in the site \(B\). An analog quantity \(\Delta^\epsilon_{\text{AB}}\) can be constructed for monitorings in the site \(A\), and it should be clear, by construction, that it is not necessary that \(\Delta^0_{\text{AB}} (\rho) = \Delta^\epsilon_{\text{AB}}\) for generic states. Notice that \(\Delta^0_{\text{AB}} (\rho) = 0\), meaning that no realizability-based nonlocality is destroyed when the system suffers no monitoring. The quantity \(\Delta^\epsilon_{\text{AB}}\) can be simplified. With Eq. (3), we write

\[
\Delta^\epsilon_{\text{AB}} (\rho) = \max_{A, B} \left[ \mathcal{S} (A|\rho) - \mathcal{S} (A|\rho) - \chi \right],
\]

where \(\chi \equiv \mathcal{S} (A|\Phi_B (\rho)) - \mathcal{S} (A|\Phi_B M^\rho_B)\). The relation (9) then implies that \(\chi \geq 0\). Hence, the maximization in Eq. (17) with respect to \(B\) is attained for \(\chi = 0\). According to Eq. (7), this can be ensured for a generic \(\rho\) via the choice \(B' = B\). Using the relation \(\delta^\epsilon_{\text{AB}} (\rho) = \eta_{\text{AB}} (\rho) - \eta_{\text{AB}} (M^\rho_B)\) one obtains

\[
\Delta^\epsilon_{\text{AB}} (\rho) = \max_{A, B} \left[ \eta_{\text{AB}} (\rho) - \eta_{\text{AB}} (M^\rho_B) \right] = \max_{A, B} \delta^\epsilon_{\text{AB}} (\rho),
\]

which will be the figure of merit considered from now on. By virtue of Theorem 1 we have \(\Delta^\epsilon_{\text{AB}} (\rho) \geq 0\) for any \(\epsilon \in (0, 1)\). Because \(\eta_{\text{AB}}\) is nonnegative, it is also clear that \(\Delta^\epsilon_{\text{AB}} (\rho) \leq N (\rho)\). This has to be so, since the amount of realizability-based nonlocality destroyed by the monitoring can never be grater than the available realizability-based nonlocality. Summing up, we have shown that the bounds

\[
0 \leq \Delta^\epsilon_{\text{AB}} (\rho) \leq N (\rho)
\]

hold for \(\epsilon \in (0, 1)\) and all \(\rho\) on \(\mathcal{H}_A \otimes \mathcal{H}_B\), with the equalities simultaneously applying for a..专业人士...
On the other hand, by direct use of the relation (21) and adapting the inequality (23) to $S(M^B_A(\rho) - S(A_\rho(\rho)))$, we arrive at a lower bound $L B_2$. These results are written as
\begin{align}
\Delta^e_{\rho}(\rho) &\leq e \eta_{AB}(\rho) + (1 - e) \Xi(B|\rho) \equiv UB_2, \quad (24a) \\
\Delta^e_{\rho}(\rho) &\geq e \eta_{AB}(\rho) - (1 + e) \Xi(B|\rho) \equiv LB_2. \quad (24b)
\end{align}
Clearly, while $\Delta^e_{\rho}(\rho) = 0$ these bounds do not necessarily vanish with $\epsilon$, so they cannot always be tight. Yet, as will be numerically shown later, each one of the bounds derived above may reveal its particular superiority in specific instances.

The set of bounds defined by the relations (19), (22), and (24) constitute the fundamental result of this work. $L B_1$, in particular, guarantees that no matter how tiny the monitoring is there will always be some suppression of realism-based nonlocality, which is at least of the order $O(\epsilon)$. Therefore, it is certain that realism-based nonlocality will not increase under monitoring. On the other hand, the suppression will never equal the total realism-based nonlocality of the state and can be limited to the order $O(\epsilon)$, as indicated by $U B_1$. As a consequence, even when the amount of realism-based nonlocality is small for a given state, the local monitoring will never destroy all the available realism-based nonlocality, which then implies no sudden death whatsoever.

As for pure states $\zeta = |\psi\rangle\langle\psi|$ are concerned, one can show that the suppression (18) of realism-based nonlocality under local monitoring is bounded as
\[
e \epsilon E(\zeta) \leq \Delta^e_{\rho}(\zeta) \leq E(\zeta), \tag{25}\]
where $E(\zeta) = S(\text{Tr}_{\mathcal{R}(B)}\rho)$ is the entanglement entropy of $|\psi\rangle$. The proof goes as follows. Since $N(|\zeta\rangle = E(\zeta)$ for any pure state $\zeta = |\psi\rangle\langle\psi|$ (see Ref. [10]), the second inequality above trivially follows from the bounds (19). To prove the first inequality, we use Eq. (18), the definition of irreality (1), the fact that $S(\zeta) = 0$, and the monotonicity relation $S(\Phi_A\Phi_B(\zeta)) \geq S(\Phi_A\Phi_B(\zeta))$. All this allows us to write
\[
\Delta^e_{\rho}(\zeta) \geq \max_{A,B} \left\{ S\left(M^B_A(\rho) - S(\Phi_A\Phi_B(\zeta)) - S(\Phi_A(\rho)) \right) \right\}, \tag{26}\]
where the equality holds for $\Phi_B(\zeta) = M^B_A(\zeta)$. Since the term in square brackets is nonnegative, the maximization will occur for Schmidt operators $A_S = \sum a_i |i\rangle\langle i|$ and $B_S = \sum b_j |j\rangle\langle j|$ (the ones that define the basis $|\psi\rangle = \sum \sqrt{a_i} |i\rangle$), for which in this case it follows that $S(\Phi_A\Phi_B(\zeta)) = S(\Phi_A(\rho))$. Then, using the definition (6) and the concavity of entropy we find
\[
\Delta^e_{\rho}(\zeta) \geq S(\Phi_B(\rho)) \geq e S(\Phi_B(\zeta)). \tag{27}\]

The observation that $S(\Phi_B(\rho)) = S(\text{Tr}_{\mathcal{R}(B)}\rho) = E(\zeta)$ completes the proof. It is worth noticing that by employing the Schmidt operators we can arrive at the result (25) departing from the bounds (24), which certifies the consistency of the approach. The bounds (25) show that local monitorings can never destroy more realism-based nonlocality (entanglement, for pure states) than the amount available in the preparation, but will destroy at least an amount $e E(\zeta)$. Thus, just as in the case of mixed states, increase and sudden death of realism-based nonlocality are both prevented.

1. Example

Let us consider the two-parameter state of two qubits
\[
\rho^{\alpha\beta} = (1 - \beta)\frac{1}{2}\sigma_z + \beta|\psi_\alpha\rangle\langle\psi_\alpha|, \tag{28}\]
where $|\psi_\alpha\rangle = \sqrt{\alpha}(1 - \alpha)|0\rangle - \sqrt{\alpha}(1 - \alpha)|1\rangle$ and $\alpha, \beta \in [0, 1]$.

We first focus on the state $\rho^{\beta|\beta}$, which corresponds to a Werner state with $|\psi_\beta\rangle$ being the singlet state. To compute the realism-based nonlocality suppression through the formulas (18) and (11) we consider the generic observable $A = \sum_{a=\pm} \alpha_a A_a$ with projectors $A_a = |\pm\rangle\langle\pm|$ such that $|\pm\rangle = \cos(\frac{\theta}{2}|0\rangle + e^{i\varphi_\pm} \sin(\frac{\theta}{2})|1\rangle$ and $|-\rangle = -\sin(\frac{\theta}{2})|0\rangle + e^{i\varphi_-} \cos(\frac{\theta}{2})|1\rangle$, and a similar parametrization for $B$ in terms of $\{|\phi_\beta\rangle, \{0\}, \{1\}\} \in \mathcal{H}_B$. Analytical calculations show that the maximum of $\delta^e_{AB}$ in (18) is attained for $\theta_{ab} = 0$, which implies that $A = \sigma_z$ and $B = \sigma_z$. With that, we find
\[
\Delta^e_{\rho}(\rho^{\beta|\beta}) = \frac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{j} \lambda_{ij} \ln \lambda_{ij}, \tag{29}\]
where $\lambda_{ij} = 1 + \beta [4i - 1 + 2j(e(1 - 2i)]$. This function is plotted in Fig. 1(a), where we can see that it indeed has the expected behavior: it is always less than realism-based nonlocality, increases with $\epsilon$, and is correctly positioned in between the bounds (19), (22), and (24). For the state under scrutiny, we have numerically verified that $\Delta^e_{\rho} = \Delta^e_{\rho}$. We now consider the pure state $\rho^{\alpha|\alpha}$. Instead of pursuing the analytical maximization demanded by the formula (18), which is harder to perform in this case, we proceed by similarity. It has been shown in Ref. [11] that
\[
\min_{A,B} \delta^{11}_{AB}(\zeta) = \eta_{AB}^{11}(\zeta) = 0 \tag{30}\]
for any pure state $\zeta = |\psi\rangle\langle\psi|$, the Schmidt operator $A_S$, and the maximally incompatible operator $B_S$ (that is, the one for which the commutator $[B_S, B_S]$ is maximum). On the other hand, from Ref. [10] one has that
\[
\max_{A,B} \delta^{11}_{AB}(\zeta) = \eta_{AB}^{11}(\zeta) = E(\zeta). \tag{31}\]
which occurs for Schmidt operators $A = A_S$ and $B = B_S$. By taking $A = \sigma_x$ and $B = \sigma_z$ we have been able to analytically check that $\eta_{AB}^{11}(\rho^{\alpha|\alpha}) = 0$ for $\epsilon \in (0, 1)$, just as in Eq. (30). Following Ref. [10], we then take $A = \sigma_x$ and $B = \sigma_z$, in order to attain higher values for $\Delta^e_{\rho}$, and then conjecture, with basis in comparisons made with many other choices for $A$ and $B$, that $\Delta^e_{\rho}(\rho^{\alpha|\alpha}) = \eta_{AB}^{11}(\rho^{\alpha|\alpha})$. This yields
\[
\Delta^e_{\rho}(\rho^{\alpha|\alpha}) = -\ln \sqrt{\Lambda_x} - \ln \sqrt{1 - 4\Lambda_x} \ \text{arctanh} \sqrt{1 - 4\Lambda_x}. \tag{32}\]
where $\Lambda_x = \epsilon \alpha (2 - \epsilon) (1 - \alpha)$. Although a rigorous proof is lacking, we believe that this is the searched-for solution for the maximization problem. In particular, it gives $\Delta^e_{\rho}(\rho^{\alpha|\alpha}) = 0$ and $\Delta^e_{\rho}(\rho^{\alpha|\alpha}) = E(\rho^{\alpha|\alpha}) = -\alpha \ln \alpha - (1 - \alpha) \ln (1 - \alpha)$, as expected. Also, we found that $\Delta^e_{\rho}(\rho^{\alpha|\alpha}) = \Delta^e_{\rho}(\rho^{\alpha|\alpha})$. The behavior of the function (32) is illustrated in Fig. 1(b) for two values of the monitoring intensity.
Do not hallucinate.

The bounds given by the result (24) provide no more information than bound by (22), illustrating of the results (19), (22), and (24). The bounds given by the relations (19), plotted in thin gray lines, whereas the ones defined by (22), LB, and UB, appear in dashed red lines. For the state \( \rho^{AB} \) under concern, we found that LB, UB, UB, so that the bounds given by the result (22) provide no more information than the previous ones. (b) Illustration of the bounds (25), with the pure state \( \rho^{(\alpha)} = \psi_\alpha \psi_\alpha^\dagger \), where \( \psi_\alpha = \sqrt{\alpha} |01\rangle - \sqrt{1 - \alpha} |10\rangle \). In each graph, the upper (lower) dashed red line denotes the upper (lower) bound \( E(\rho^{(\alpha)}) \) [\( E(\rho^{(\alpha)}) \)] indicated in the result (25).

B. Realism-based nonlocality suppression via bilocal monitoring

A natural extension of the study conducted in the previous section can be proposed in terms of the quantity

\[
\Delta^{\epsilon\epsilon} (\rho) := \max_{\lambda, \beta} \delta_{\lambda\beta}^{\epsilon\epsilon}(\rho),
\]

with \( \delta_{\lambda\beta}^{\epsilon\epsilon}(\rho) \) given by Eq. (11) and \( (\epsilon, \epsilon') \in (0, 1) \). Notice that this quantity respects the symmetry \( A \rightleftarrows B \) for \( \epsilon = \epsilon' \), reduces to \( N(\rho) \) as \( \epsilon = \epsilon' \to 1 \), and to \( \Delta^{00}(\rho) \) as \( \epsilon \to 1 \). In addition, \( \Delta^{\epsilon\epsilon}(\rho) = \Delta^{\epsilon\epsilon}(\rho) = 0 \) for all \( \rho \). It is instructive to observe, via Eq. (10), that

\[
\Delta^{\epsilon\epsilon}(\rho) = \max_{\lambda B} \left[ \Delta^{\epsilon\epsilon}_{\lambda A}(\rho) - \Delta^{\epsilon\epsilon}_{\lambda A}(\rho) \right];
\]

that is, \( \Delta^{\epsilon\epsilon} \) can be viewed as a quantifier of the maximum variation in the reality change in site \( A \) upon remote monitoring in \( B \). Hence, it indeed captures aspects of realism-based nonlocality, although it is not identical to \( N(\rho) \). On the other hand, since \( \Delta^{\epsilon\epsilon} = \Delta^{\epsilon\epsilon} \), it is clear that, in this limit, it quantifies the realism-based nonlocality destroyed via local monitoring in this. It is, therefore, reasonable to interpret \( \Delta^{\epsilon\epsilon} \) as the amount of realism-based nonlocality that is suppressed due to monitoring conducted in each one of the distant sites.

Now, the identities (14), Theorem 1, and the result (19) immediately imply that

\[
0 \leq \Delta^{\epsilon\epsilon}(\rho) \leq \Delta^{\epsilon\epsilon}_{\beta A}(\rho) \leq N(\rho),
\]

for any \((\epsilon, \epsilon') \in (0, 1) \) and \( \rho \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \). Equations apply for \( \rho = \rho_A \otimes \rho_B \). Notice the nontrivial result \( \Delta^{\epsilon\epsilon}(\rho) = \Delta^{\epsilon\epsilon}_{\beta A}(\rho) \), which means that the realism-based nonlocality suppression is smaller for bilocal monitorings.

Hereafter, we restrict our analysis to the symmetrical case, \( \epsilon = \epsilon' \). With a procedure similar to the one employed to derive the bounds (22), we use the relations (20) and (21) to obtain

\[
\Delta^{\epsilon\epsilon}(\rho) \leq \Gamma^{\epsilon\epsilon}_{\beta A}(\rho) = \text{ub}_1,
\]

\[
\Delta^{\epsilon\epsilon}(\rho) \geq \epsilon \delta_{\lambda\beta}^{\epsilon\epsilon}(\rho) - \Gamma^{\epsilon\epsilon}_{\beta A}(\rho) = \text{lb}_1,
\]

where \( \{\tilde{A}, \tilde{B}\} \) are the observables that maximize \( \delta_{\lambda\beta}^{\epsilon\epsilon} \) in Eq. (33). Once again we can conclude, from the bounds (35) and (36), that while there will always be some suppression of realism-based nonlocality due to bilocal monitoring, thus precluding any increase of realism-based nonlocality under local disturbance, the total realism-based nonlocality available will never be fully destroyed, which implies no sudden death.

1. Example

We consider once again the state \( \rho^{AB} \), given by the formula (28), and assume that the bilocal suppression is given by

\[
\Delta^{\epsilon\epsilon}(\rho^{AB}) = \eta_{\lambda AB}(\rho^{AB}), \quad \text{with } r = s \in [x, y, z] \quad (37)
\]

and Pauli matrices \( \sigma_{r,s} \). This assumption is based on several analytical incursions through which we have comparatively verified, for example, that smaller values occur for \( r \neq s \). Also, the analytical result obtained from the conjecture above correctly leads to the limit \( \Delta^{\epsilon\epsilon}(\rho^{AB}) \to N(\rho^{AB}) \) as \( \epsilon \to 1 \) and to \( \Delta^{\epsilon\epsilon}(\rho^{AB}) \to 0 \) as \( \epsilon \to 0 \). These results can be appreciated in Fig. (2b), where the smoothness of \( \Delta^{\epsilon\epsilon}(\rho^{AB}) \) with respect to monitoring strength \( \epsilon \) is illustrated. In Fig. (2b), full agreement with the result (35) can be observed in the parametric plots of \( N(\rho^{AB}), \Delta^{\epsilon\epsilon}(\rho^{AB}), \) and \( \Delta^{\epsilon\epsilon}(\rho^{AB}) \), against \( N(\rho^{AB}) \), as a function of \( \beta \) and \( \epsilon \).

So far we have discussed the behavior of realism-based nonlocality, as measured by \( N(\rho) \), upon (bi)local monitoring. Our findings demonstrate that we can never destroy all the realism-based nonlocality encoded in a given preparation via this kind of weak disturbance. However, it is trivial to show that a revealed measurement of an observable \( A' \) on \( \mathcal{H}_A \) is sufficient to remove all realism-based nonlocality. This is so because upon the reduction \( \rho \mapsto A'_A \otimes \rho_{B\mu} \), one obtains a fully uncorrelated state, which gives \( \eta_{AB}(A'_A \otimes \rho_{B\mu}) = 0 \) and, therefore, \( N(A'_A \otimes \rho_{B\mu}) = 0 \).

IV. HIERARCHY

The demonstrated resilience of realism-based nonlocality to (bi)local monitoring raises the question of how this quantumness notion compares to other measures of quantum correlations, for instance, quantum discord [25, 26], whose robustness under local disturbance, in particular to noisy channels, is well known [18, 27]. In Ref. [10] we have already

FIG. 1. Amount \( \Delta^{\epsilon\epsilon} \) (thick black lines) of realism-based nonlocality that is removed from the state \( \rho^{AB} \) [Eq. (28)] for local monitorings of intensities \( \epsilon = 0.1 \) (lower panels) and \( \epsilon = 0.6 \) (upper panels). (a) Illustration of the results (19), (22), and (24). The bounds given by the relations (19) are plotted in thin gray lines, whereas the ones defined by (22), LB, and UB, appear in dashed red lines. For the state \( \rho^{AB} \) under concern, we found that LB, UB, UB, so that the bounds given by the result (22) provide no more information than the previous ones. (b) Illustration of the bounds (25), with the pure state \( \rho^{(\alpha)} = \psi_\alpha \psi_\alpha^\dagger \), where \( \psi_\alpha = \sqrt{\alpha} |01\rangle - \sqrt{1 - \alpha} |10\rangle \). In each graph, the upper (lower) dashed red line denotes the upper (lower) bound \( E(\rho^{(\alpha)}) \) [\( E(\rho^{(\alpha)}) \)] indicated in the result (25).
Let us add another restriction, namely, \( p(b|B,A) = \text{Tr}(B_0 \rho_0^A) \). In this scenario, where both marginal probability distributions are consistent with quantum theory, the hypothesis \( \text{RBN} \leq \text{B} \) reduces to \( p(a,b|A,B) = \text{Tr}(A_0 \otimes B_0 \rho_0), \) with \( \rho_0 = \sum_i p_i \rho_i^A \otimes \rho_i^B \) being a manifestly separable state. Clearly, the existence of such a solution falsifies entanglement \( (E = 0) \). Again, because we have a more restrictive local model to rule out quantumness, the resulting set of entangled states, \( \mathcal{S}_E \), is more inclusive than the previous ones. We then obtain the relation \( \mathcal{S}_{\text{B}} \subseteq \mathcal{S}_E \subseteq \mathcal{S}_D \subseteq \mathcal{S}_{\text{SD}} \), as has been established in Ref. \( [29] \).

We are now ready to position realism-based nonlocality within the above hierarchy of quantumness notions. Let us focus on the classical-classical state \( \rho_{cc} = \sum_i p_i \rho_i^A \otimes \rho_i^B \), for which none of the aforementioned quantumness exists. By direct application of the joint-entropy theorem we can show that

\[
\eta_{AB}(\rho_{cc}) = H(p_{A|B}) + \sum_{i,B} \left( \Phi_A(\rho_i^A) \otimes \Phi_B(\rho_i^B) \right),
\]

(39)

where \( H(p_{A|B}) \) is the Shannon entropy of the probability distribution \( p_{A|B} \). In order to prove that \( H(p_{A|B}) > 0 \), it is sufficient to exhibit at least one pair \( [A,B] \) for which \( \eta_{AB}(\rho_{cc}) \) does not vanish. This will be the case for observables \( A \) and \( B \) that are maximally incompatible with \( A' = \sum_i a_i' A_i^A \) and \( B' = \sum_i b_i' B_i^B \), respectively. In this case, \( \Phi_A(\rho_i^A) = 1/d_A\text{dim } \mathcal{H}_{AB} \) and \( \Phi_B(\rho_i^B) = 1/d_B\text{dim } \mathcal{H}_{AB} \). With these results, Eq. \( (39) \) readily reduces to \( \eta_{AB}(\rho_{cc}) = H(p_{A|B}) \), which is positive for a generic distribution \( p_{A|B} \). This completes the proof that realism-based nonlocality can exist even when all the other quantumness quantifiers vanish. On the other hand, it is worth noticing that, for pure states, realism-based nonlocality becomes identical to entanglement \( [10] \), just like discord does, which is a clear evidence that entanglement implies discord, which implies realism-based nonlocality. All this together indicates that the set \( \mathcal{S}_{\text{B}} \) of states with realism-based nonlocality \( (N > 0) \) is a strict superset of all the other sets discussed so far. We then find the following hierarchy of quantumness notions:

\[
\mathcal{S}_{\text{B}} \subseteq \mathcal{S}_E \subseteq \mathcal{S}_D \subseteq \mathcal{S}_{\text{SD}} \subseteq \mathcal{S}_{\text{B}}.
\]

(40)

This result attests that realism-based nonlocality is the most ubiquitous species of quantumness within the class here considered (see illustration in Fig. 5). Indeed, according to the
V. SUMMARY

In this work, we investigated further properties of the recently introduced realism-based nonlocality—a notion of nonlocality that emerges from the violation of the hypothesis that a remote disturbance in site \( B \) does not change the degree of irreality in site \( A \). Specifically, we studied the resilience of this quantumness notion to (bi)local weak measurements, which were implemented through the monitoring map. Since monitoring is a particular class of local operations, our results show that realism-based nonlocality does not increase under these local actions, which is a desirable feature of quantum-correlation quantifiers. In particular, we demonstrated that, apart from the regime of strictly projective measurements, realism-based nonlocality can never be fully destroyed. Therefore, unlike Bell nonlocality, EPR steering, and entanglement, realism-based nonlocality is not susceptible to sudden death. As follows from the hierarchy demonstrated in Sec. IV, this behavior is consistent with the fact that no sudden death has ever been observed for discord or symmetrical discord. The remarkable robustness of realism-based nonlocality naturally raises questions about eventual applications of such nonlocal aspects in physical tasks, such as those of information theory and quantum thermodynamics. These questions are, however, left open for future work.

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[1] J. S. Bell, On the Einstein-Podolsky-Rosen paradox, Physics 1, 195 (1964).
[2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. 86, 419 (2014); Publisher’s Note: Bell nonlocality [Rev. Mod. Phys. 86, 419 (2014)], 86, 839 (2014).
[3] B. Hensen et al., Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, Nature (London) 526, 682 (2015).
[4] M. Giustina et al., Significant loophole-free test of Bell’s theorem with entangled photons, Phys. Rev. Lett. 115, 250401 (2015).
[5] L. K. Shalm et al., Strong loophole-free test of local realism, Phys. Rev. Lett. 115, 250402 (2015).
[6] B. Hensen et al., Loophole-free Bell test using electron spins in diamond: Second experiment and additional analysis, Sci. Rep. 6, 30289 (2016).
[7] N. Gisin, Non-realism, Deep thought or a soft option?, Found. Phys. 42, 80 (2012).
[8] F. J. Tipler, Quantum nonlocality does not exist, Proc. Natl. Acad. Sci. U.S.A. 111, 11281 (2014).
[9] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935).
[10] V. S. Gomes and R. M. Angelo, Nonanomalous realism-based measure of nonlocality, Phys. Rev. A 97, 012123 (2018).
[11] A. L. O. Bilobran and R. M. Angelo, A measure of physical reality, Europhys. Lett. 112, 40005 (2015).
[12] A. Acín, T. Durt, N. Gisin, and J. I. Latorre, Quantum nonlocality in two-three-level systems, Phys. Rev. A 65, 052325 (2002).
[13] A. A. Méthot and V. Scarani, An anomaly of non-locality, Quantum Inf. Comput. 7, 157 (2007).
[14] T. Vidick and S. Wehner, More nonlocality with less entanglement, Phys. Rev. A 83, 052310 (2011).
[15] S. Camalet, Measure-independent anomaly of non-locality, Phys. Rev. A 96, 052332 (2017).
[16] J. I. de Vicente, On nonlocality as a resource theory and nonlocality measures, J. Phys. A: Math. Theor. 47, 424017 (2014).
[17] F. Buscemi, All entangled quantum states are nonlocal, Phys. Rev. Lett. 108, 200401 (2012).
[18] A. C. S. Costa, R. M. Angelo, and M. W. Beims, Generalized discord, entanglement, Einstein-Podolsky-Rosen steering, and Bell nonlocality in two-qubit systems under (non-)Markovian channels: Hierarchy of quantum resources and chronology of deaths and births, Physica A 461, 469 (2016).
[19] P. R. Dieguez and R. M. Angelo, Information-reality complementarity: The role of measurements and quantum reference frames, Phys. Rev. A 97, 022107 (2018).
[20] Y. Aharonov, D. Z. Albert, and L. Vaidman, How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100, Phys. Rev. Lett. 60, 1351 (1988).
[21] Y. Aharonov, E. Cohen, and A. C. Elitzur, Foundations and applications of weak quantum measurements, Phys. Rev. A 89,
052105 (2014).

[22] O. Oreshkov and T. A. Brun, Weak measurements are universal, Phys. Rev. Lett. 95, 110409 (2005).

[23] L. Macino et al., Information-reality complementarity in photonic weak measurements, Phys. Rev. A 97, 062108 (2018).

[24] M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).

[25] H. Ollivier and W. H. Zurek, Quantum discord: A measure of the quantumness of correlations, Phys. Rev. Lett. 88, 17901 (2001).

[26] L. Henderson and V. Vedral, Classical, quantum and total correlations, J. Phys. A: Math. Gen. 34, 6899 (2001).

[27] L. C. Cáleri, J. Maziero, and R. M. Serra, Theoretical and experimental aspects of quantum discord and related measures, Int. J. Quantum Inform. 9, 1837-1873 (2011).

[28] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, entanglement, nonlocality, and the Einstein-Podolsky-Rosen paradox, Phys. Rev. Lett. 98, 140402 (2007).

[29] C. C. Rulli and M. S. Sarandy, Global quantum discord in multipartite systems, Phys. Rev. A 84, 042109 (2011).