Edge Disjoint Spanning Trees in an Undirected Graph with \( E=2(V-1) \)

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Abstract

Given a connected undirected graph \( G = [V; E] \) where \(|E| = 2(|V| - 1)\), we present two algorithms to check if \( G \) can be decomposed into two edge disjoint spanning trees, and provide such a decomposition when it exists. Unlike previous algorithms for finding edge disjoint spanning trees in general undirected graphs, based on matroids and complex in description, our algorithms are based on simple graph reduction techniques and thus easy to describe and implement. Moreover, the running time for our solutions is asymptotically faster. Specifically, ours are the first algorithms to achieve a running time that is a polylog factor from linear, approaching the 1974 linear time algorithm of Robert E. Tarjan for directed graphs. A direct implication of our result is that minimally rigid graphs, also called Laman graphs, can be recognized in almost linear time, thus answering a long standing open problem.

1 Introduction

We consider the problem of decomposing a connected undirected graph \( G = [V; E] \), with \(|E| = 2(|V| - 1)\), into two edge disjoint spanning trees when this is possible and to indicate it is not possible when that is the case. This problem was considered by R.E. Tarjan for the directed case in which both directed trees shared a common root [8]. He presented a linear time algorithm for directed graphs with \(|E|\) edges and \(|V|\) vertices and left open the development of a matching algorithm for undirected graphs. The undirected version is a special case of the matroid partition problem considered by J. Edmonds [4] for which the best known result, due to H. N. Gabow and H.H. Westermann [5], has a worst case complexity of \( O(n \sqrt{m'}) \) time, where \( n = |V| \) and \( m' = O(|E| + n \log n) \). This complexity for our case, where \(|E| = 2(|V| - 1)\), comes to \( O(n \sqrt{n \log n}) \). In this paper we propose two algorithms for our special case that take \( O(n \log^2 n) \) and \( O(n \log^2 n / \log \log n) \) time, respectively. Unlike previous approaches, our algorithms are much simpler and different in nature. The main novelty is in the way we approach the solution. In particular, we allow the introduction of new edges while performing a graph reduction. We proceed by first reducing the graph using Henneberg like reduction steps [6] and then build the trees in reverse order of this reduction.

In [3], Daescu and Kurdia proved that one can answer if a given graph with \( n \) vertices is minimally rigid (also called Laman) in \( O(T_{2ST} + |V| \log |V|) \) time, where \( T_{2ST} \) is
the time to decide if an undirected graph \( G = [V; E] \), with \( |E| = 2(|V| - 1) \), admits two edge disjoint spanning trees. A direct consequence of our result is \( O(|V| \log^4 |V|) \) and \( O(|V| \log^2 |V|/\log \log |V|) \) time algorithms to recognize if a given graph is minimally rigid. It has been a long standing question whether this problem can be solved in close to linear time. (Note: Since the \( O(|V| \log |V|) \) part of those algorithms is simple as well, our result implies two significantly faster and, at the same time, simple algorithms that can be easily implemented and used as examples in both undergraduate and graduate algorithms courses.)

Our first algorithm relies on fully dynamic 2-edge connectivity in undirected graphs and we use the result in [7], that has \( O(\log^2 n) \) amortized time for updates (edge deletion and insertion) and \( O(\log^4 n) \) time for checking whether two given vertices are 2-edge connected. Our second algorithm relies on fully dynamic graph connectivity in undirected graphs and we use the result in [8], that supports updates (edge insertions/deletions) in \( O(\log^2 n/\log \log n) \) amortized time and connectivity queries in \( O(\log n/\log \log n) \) worst-case time. Any improvement in any of the algorithms from [7, 8] that we use here, including for special undirected graphs with \( |E| = 2(|V| - 1) \), would imply an improvement in the running time of our algorithms.

## 2 Preliminary Results

Since a spanning tree of an undirected graph \( G = [V; E] \) has \( |V| - 1 \) edges, a graph whose edges can be decomposed into two spanning trees must have exactly \( 2(|V| - 1) \) edges. Moreover, since each spanning tree has at least one edge crossing each cut of the graph, graphs whose edges can be decomposed into two spanning trees must have at least two edges crossing each cut. In particular, we cannot have any node with degree equal to 1 in the given graph if it is so decomposable. A stronger result due to J. Edmonds is the following:

► **Theorem 1 (J. Edmonds).** We can partition a subset \( S \subseteq E \) into two disjoint forests, if and only if

\[
|A| \leq 2r(A) \quad \forall A \subseteq S
\]

where \( r(A) = |A| - k(A) \) and \( k(A) \) is the number of components of the subgraph \( G_A = [V, A] \).

This theorem can be used to show that a given graph cannot be decomposed by exhibiting a subset that violates this condition.

► **Definition 2.** An edge of \( G = [V; E] \) is called a bridge if the deletion of this edge makes the graph disconnected.

If edges of \( G \) can be decomposed into two edge disjoint spanning trees of \( G \), then \( G \) cannot contain a bridge since this violates the requirement that there should be at least two edges crossing any cut. Suppose \( G \) is decomposable and for vertex \( u, \deg_G(u) = 2 \). Let the edges in \( G \) incident at \( u \) be \( (u, v) \) and \( (u, w) \). Then, these two edges must belong to different spanning trees, say \( (u, v) \in T_1 \) and \( (u, w) \in T_2 \). Thus, it follows that \( T_1 - \{(u, v)\} \) and \( T_2 - \{(u, w)\} \) are edge disjoint spanning trees of the graph \( H = [V - \{u\}; E - \{(u, v), (u, w)\}] \). Moreover, if we can decompose \( H \) into two edge disjoint spanning trees, then by arbitrarily adding edge \( (u, v) \) to one of the trees and edge \( (u, v) \) to the other one, we get two edge disjoint spanning trees of \( G \). Thus, we can dispense with a node whose degree is 2. This can be repeated until the graph has no such nodes, including degree 2 nodes that might appear while removing other degree 2 nodes, if the graph is indeed decomposable. Note that this process of removing vertices of degree two corresponds to what is known as a Henneberg Step I reduction when related to Laman graph construction.
Lemma 3. If graph $G = [V; E]$, with $|E| = 2(|V| - 1)$, admits two edge-disjoint spanning trees then by removing vertices of degree 2 the new graph $G'$ stays connected and decomposable. If $G'$ disconnects in process then $G$ does not admit two edge disjoint spanning trees.

Proof. Assume first that $G$ is decomposable into two edge-disjoint spanning trees, say $T_1$ and $T_2$. Any vertex $u$ of degree 2 in graph $G$ must have one edge in $T_1$ and another in $T_2$. Therefore $u$ must be a leaf node in both trees $T_1$ and $T_2$. Let $T'_1$ and $T'_2$ be new trees obtained by deleting $u$ from $T_1$ and $T_2$. Only a leaf node is removed from both $T_1$ and $T_2$, therefore the new trees $T'_1$ and $T'_2$ remain connected and the graph $G'$ obtained by deleting vertex $u$ from graph $G$ is also decomposable. This process can now be repeated inductively. An example is shown in Figure 1.

![Graph $G$](image)

![Graph $G'$](image)

![Graph $G''$](image)

![Diagram $T_1$, $T_2$, $T'_1$, $T'_2$, $T_1''$, $T_2''$](image)

Figure 1 | $u$ is a degree 2 vertex in $G$ and a leaf node in both $T_1$ and $T_2$. Graph $G'$ is obtained by removing vertex $u$ from graph $G$, which is connected and decomposable. Removing degree 2 vertices, $v$ and $q$, from graph $G'$ does not disconnect the resulting graph $G''$.

If removal of a degree 2 vertex $u$ disconnects $G'$ into two components, then at least one of
the edges incident to \( u \) is a bridge in the reduced graph \( G' \) and graph \( G \) is not decomposable. This is shown in Figure 2.

Figure 2 Vertices \( P \) and \( Y \) of \( G \) have degree 2. Their edges can be assigned either as shown in (i) or (ii) (except for swapping notations for \( T_1 \) and \( T_2 \)). After reducing \( P \) and \( Y \), edges incident to vertex \( Q \) are bridges in \( G' \) and hence \( G \) is not decomposable.

From now on we assume that we have a graph \( G = [V; E] \), with \( |E| = 2(|V| - 1) \), which has (i) no vertex with degree 1; and (ii) no bridge. Graphs that violate any of these conditions cannot be decomposed. [Note that each of these checks can be done in \( O(|V| + |E|) \) time.]

We also assume that there is no vertex of degree equal to 2, since we can remove these and add them back after finding a decomposition in the remaining graph – again in \( O(n) \) time.

Thus, our inputs consist of graphs \( G = [V; E] \) which have no bridge, each node has degree at least 3, and \( |E| = 2(|V| - 1) \). Since, for any graph \( G \), \( \sum_{i} \text{deg}(v_i) = 2|E| \), we have \( \sum_{i} \text{deg}(v_i) = 4(|V| - 1) \) for any decomposable graph. From \( \text{deg}(v_i) \geq 3 \), \( \forall v_i \in V \), it follows that a graph \( G \) that is decomposable has at least four nodes with degree equal to 3. For each vertex of \( G \) whose degree is 3, we must split the edges incident at such a vertex between the two trees and so one gets two edges and one gets one edge. We need to decide how this partition is done. To summarize all this, we have the following preprocessing steps of the algorithm:
Preprocessing:
1. Find the degree of each vertex in the input graph $G = [V; E]$
2. $V' \leftarrow V; E' \leftarrow E$ // $G' = [V', E']$ is the remaining graph
3. while there is a vertex $u \in V'$ such that $\deg_{G'}(u) = 2$ and $(u, v) \in E'$, $(u, w) \in E'$ do
4. $V' \leftarrow V' - \{u\}; E' \leftarrow E' - \{(u, v), (u, w)\}; \deg_{G'}(v) \leftarrow \deg_{G'}(v) - 1; \deg_{G'}(w) \leftarrow \deg_{G'}(w) - 1$
5. return $G' = [V', E']$

The output of this algorithm has no vertex with degree equal to 2.

If the resulting graph $G'$ has any vertex with degree equal to 1, or if it is not connected, or if it has a bridge, then the original graph is not decomposable. Each of these can be checked in $O(|V| + |E|)$ time. Moreover, $|E'| = 2(|V'| - 1)$ and $G'$ is decomposable if and only if $G$ is decomposable and such decomposition can be obtained for $G'$ from one for $G$ and conversely.

In the next section, we show how to partition the edges of $G'$ when this is possible and show when this is not possible as to why that is so.

3 First Algorithm

The input is a connected undirected graph $G = [V; E]$ such that the degree of each vertex is greater than or equal to 3, $G$ has no bridges, and $|E| = 2(|V| - 1)$ (see Figure 11). If graph $G$ is decomposable into two edge disjoint spanning trees, $T_1$ and $T_2$, then for any vertex $u$ of degree 3, two of its incident edges must be part of one tree, say $T_1$, and the remaining edge is in the other tree, $T_2$. Let vertices $x$, $y$, and $z$ be neighbors of vertex $u$ such that edges $(u, x)$, $(u, y) \in T_1$ and $(u, z) \in T_2$. We remove vertex $u$ and all its incident edges from graph $G$ and add an extra edge $(x, y)$; removing $u$ disconnects $T_1$ and adding edge $(x, y)$ reconnects it back. Note that it is possible $G$ already had an edge from $x$ to $y$ (part of $T_2$), so it is possible to have multiple $(x, y)$ edges in the resulting graph. Let $G' = [V'; E']$ be new graph formed by this transformation. Thus, we have the following lemma.

**Lemma 4.** If graph $G = [V; E]$ is decomposable into two edge disjoint spanning trees then by removing a vertex of degree 3 and adding a new edge as described above the resulting graph $G' = [V'; E']$ satisfies $|E'| = 2(|V'| - 1)$ and is decomposable into two edge disjoint spanning trees.

If graph $G$ is decomposable then subgraphs obtained by repeating the process in Lemma 4 are also decomposable. Assuming two edge disjoint spanning trees exist in $G$, if we can somehow get information about the new edge that we need to add after removing a vertex of degree 3, then we can repeat the process until we are left with two vertices with two parallel edges, where one edge belongs to one tree and another edge to the other tree.

A graph $G$ is 2-edge connected if and only if it is connected and contains no bridges. Stated differently, there exist 2 edge disjoint paths between any two vertices of $G$. At any step of the reduction of $G$, if two edge disjoint paths between any two neighbors of $u$ do not exist then graph $G$ is not decomposable.

Assume $G$ is decomposable, and thus 2-edge connected. Let $u$ be a degree 3 vertex of $G$ and let $x$, $y$, and $z$ be the vertices of $G$ adjacent to $u$. Remove vertex $u$ and all its incident edges.

**Lemma 5.** After removing $u$, only one of its adjacent vertices has two edge disjoint paths to the other two vertices.
Figure 3 Connected undirected graph $G$ with no bridges, where degree of each vertex is greater than or equal to 3 and $|E| = 2(|V| - 1)$

**Proof.** With our notation above ($(u, x), (u, y) \in T_1$ and $(u, z) \in T_2$), removal of $u$ disconnects $x$ and $y$ in $T_1$ and thus we are left only with the path from $x$ to $y$ in $T_2$. Thus, the pair $(x, y)$ no longer has two edge disjoint paths. Vertex $z$ remains connected to $x$ and $y$ in $T_2$ ($u$ is a leaf node in $T_2$ and thus the paths of $z$ are not affected), however it disconnects from either $x$ or $y$ in $T_1$, so either $z$ is no longer 2-edge connected with $x$ or $z$ is no longer 2-edge connected with $y$. If the path from $z$ to $x$ in $T_1$ does not include $y$ then $(y, z)$ is the pair no longer 2-edge connected, otherwise $(x, z)$ is the one. $\blacksquare$

By Lemma 5 given a degree 3 vertex $u$ as stated, we can identify the problem pair by checking if $(x, z)$ and $(y, z)$ are 2-edge connected or not. Assume $x$ is no longer 2-edge connected with $y$ and $z$. We then connect $x$ with either $y$ or $z$.

To summarize, let $G$ be decomposable into two edge disjoint spanning trees $T_1$ and $T_2$, and let $u$ be a degree 2 node in $T_1$. After removing $u$, $T_2$ remains connected as $u$ is a leaf node in $T_2$, while $T_1$ is disconnected into two subtrees $T_{11}$ and $T_{12}$, and the pair $(x, y)$ is no longer 2-edge connected. We decide which of $x$ and $y$ is no longer 2-edge connected to $z$. If vertex $x$ does not have 2 edge disjoint paths to $y$ and $z$ then $x$ belongs to one subtree, say $T_{11}$ and $y$ and $z$ are in $T_{12}$. To connect $T_{11}$ and $T_{12}$ we add an edge $(x, y)$ or $(x, z)$ to $G$ (edge $(x, y) \in T_1$). This is shown in Figure 3.

If $G$ is decomposable in two edge disjoint spanning trees then by repeating the process above we will be left with a graph $G$ with two vertices and two parallel edges. We then construct two edge disjoint spanning trees for $G$ following the reverse order of reductions. If a reduction was for a degree two vertex $u$, with $y$ and $z$ adjacent vertices, we arbitrarily add edge $(u, y)$ to $T_1$ and edge $(u, z)$ to $T_2$. Otherwise $u$ was of degree three, with adjacent vertices $x$, $y$, and $z$. This implies edge $(x, y)$ was added back at the time of reduction. Thus, at this time, edge $(x, y)$ is in $T_1$ or in $T_2$ (it can be in both). Assume it is in $T_1$. Then, we remove edge $(x, y)$ from $T_1$, add edges $(u, x)$ and $(u, y)$ to $T_1$, and add edge $(u, z)$ to $T_2$. 

![Figure 3](image-url)
Figure 4 $T_1$ and $T_2$ are two edge disjoint spanning trees of the graph in Figure 1. Vertex $T$ is a degree 3 node in graph $G$ while it is degree 2 node in $T_1$ and leaf node in $T_2$. Removing $T$ disconnects $T_1$. To reconnect $T_1$ we can either connect $(Y, B)$ or $(Y, S)$ and we again have two edge disjoint spanning trees.
The following algorithm checks if graph $G$ is decomposable and if so outputs two edge disjoints spanning trees. Steps 1-8 are the reduction of $G$ while Steps 9-11 construct the two spanning trees.

**Algorithm 1.**

1. Stack $S = \emptyset$ to record the order in which vertices are removed
2. Queue $Q_1 = \emptyset$ to store vertices of degree 1
3. Queue $Q_2 = \emptyset$ to store vertices of degree 2
4. Queue $Q_3 = \emptyset$ to store vertices of degree 3
5. $n \leftarrow$ number of vertices of $G$
6. For each vertex $u \in G$, create a list $V_u[]$ with three entries.
7. while($n > 2$)
   a. if $\exists$ a degree 1 vertex (check $Q_1$) then exit and output "Graph $G$ is not decomposable into two edge disjoint spanning trees"
   b. if $\exists$ a degree 2 vertex (check $Q_2$)
      i. let $v$ be a degree 2 vertex, adjacent to vertices $x$ and $y$
      ii. $S$.push($v$)
      iii. remove vertex $v$ (and its incident edges) from $G$ and update the degree counts for $x$ and $y$. If the count of any of $x$ and $y$ becomes 1 then add it to $Q_1$, if it becomes 2 then add it to $Q_2$, and if it becomes 3 add it to $Q_3$
      iv. $V[v] = \text{null, } x, y$
      v. $n = n - 1$;
   c. else if $\exists$ a degree 3 vertex
      i. Let $v$ be a degree 3 vertex whose neighbors are $x$, $y$, and $z$
      ii. $S$.push($v$)
      iii. remove vertex $v$ (and its incident edges) from $G$ and update the degree counts for $x$, $y$, and $z$.
      iv. Find which pair among $(x, y), (y, z), (z, x)$ returns true for 2-edge connectivity. If none of the above pairs are 2-edge connected, then graph $G$ is not decomposable; Exit and output "Graph $G$ is not decomposable into two edge disjoint spanning trees". Else, let $(y, z)$ be the pair that has two edge disjoint paths in $G$
      v. Add edge $(x, y)$ to $G$ and update counts for $x$ and $y$ (note we might need to remove them from various queues)
      vi. $V[v] = x, y, z$
      vii. $n = n - 1$
8. Graph $G$ has only two vertices and two edges. If any edge is a self loop then graph $G$ is not decomposable.
9. Let $T_1, T_2$ be two spanning trees where $T_1 = \emptyset, T_2 = \emptyset$.
10. Let $a$ and $b$ be the two vertices in $G$
   a. $T_1 = T_1 \cup (a, b)$
   b. $T_2 = T_2 \cup (a, b)$
11. while ($S \neq \emptyset$)
   a. $u = s$.pop
   b. $x, y, z = V_u[]$
   c. if $x$ is null
      i. $T_1 = T_1 \cup (u, y)$
      Adding node of degree 2
ii. \[ T_2 = T_2 \cup (u, z) \]

d. else

Adding node of degree 3

i. if edge \((x, y) \in T_1\)

A. \[ T_1 = T_1 - (x, y) \cup (u, x) \cup (u, y) \]

B. \[ T_2 = T_2 \cup (u, z) \]

ii. else

A. \[ T_1 = T_1 \cup (u, z) \]

B. \[ T_2 = T_2 - (x, y) \cup (u, x) \cup (u, y) \]

A fully-dynamic graph connectivity structure is a data structure that dynamically maintains information about the connected components of a graph, where insertion and deletion of vertices or edges is allowed. In a fully dynamic 2-edge connectivity problem, the edge updates may be interspersed with queries asking whether two given vertices are 2-edge connected. In [7], Holm, de Lichtenberg, and Thorup provide an algorithm that supports edge insertions and deletions in \(O(\log^2 n)\) amortized time per update, and 2-edge connectivity queries between two vertices in \(O(\log^4 n)\) worst case time. Using this algorithm, we can check if there exist two edge disjoint paths between two vertices of \(G\) or not. While reducing a degree 3 vertex, we remove one vertex and 3 edges, and add another edge. We also check for 2-edge connectivity which results in a total of 8 operations. We reduce a total of \(n-2\) vertices and thus the total number of operations we need to perform is \(O(n)\). Therefore all update and 2-edge connectivity query operations take a total of \(O(n \log^4 n)\) time. Updating various queues used in the algorithm takes \(O(1)\) time per update and \(O(n)\) overall, using standard techniques (maintain \(O(1)\) space information at each vertex).

Deciding if an edge \((x, y) \in T_i, i = 1, 2\), can be done in constant time. Thus, overall, our algorithm takes \(O(n \log^4 n)\) time.

Lemma 6. Given an undirected graph \(G = [V; E]\), with \(|E| = 2(|V| - 1)\), it can be decided in \(O(n \log^4 n)\) time if \(G\) is decomposable in two edge disjoint spanning trees. Moreover, if \(G\) is decomposable, the two trees can be reported within the same time bound.

4 An Improved Algorithm

The algorithm in the previous section is slowed down by 2-edge connectivity queries. In this section we will show how to replace those queries with simple connectivity queries by maintaining additional information during the graph reduction process. Specifically, we will always maintain a spanning tree of (the reduced) \(G\) that will help us quickly decide when removing a degree three vertex \(u\) which of the pairs \((x, y), (x, z), \text{ and } (y, z)\) are no longer 2-edge connected, where \(x, y, \text{ and } z\) are the vertices adjacent to \(u\).

We start by performing a depth-first search of graph \(G\). For each vertex \(u \in V\), let \(t_{s,u}\) and \(t_{f,u}\) be the start and finish time obtained while performing the depth-first search. Let the depth-first tree be denoted by \(T_{DFS} = G_1\). If an edge \((a, b) \in T_{DFS}\) then we mark it as a tree edge in graph \(G\), else we mark it as a non-tree edge. Let \(G_2 = [V, E']\), where \(|E'| = |V| - 1\) is the set of non-tree edges of \(G\). If graph \(G_2\) is connected then it forms a spanning tree, we have two edge-disjoint spanning trees \(G_1\) and \(G_2\) in \(G\), and thus we are done.

Assume \(G_2\) is not connected and consists of a few connected components, \(CC_1, CC_2, \ldots CC_k\), for some \(k < n\).

A vertex \(u\) of degree 3 in \(G\) has either one tree edge, or two tree edges, or three tree edges incident to it. Note that if vertex \(u\) has three tree edges then the connected component of
Figure 5 Graph $G_1 = T_{DFS}$ obtained by depth-first search of $G$ shown in Figure 11 with start time and finish time of each vertex.
u in $G_2$, $CC_u$, consist of only vertex $u$. While reducing a vertex $u$ of degree 3, we want to maintain a (possibly modified) spanning tree $T_{DFS}$ in the resulting graph.

Let $u$ be a degree 3 vertex in $G$, with adjacent vertices $x$, $y$, and $z$. Before reduction, $u$, $x$, $y$, and $z$ are pairwise 2-edge connected. After reduction of $u$ we want $x$, $y$, and $z$ to be pairwise 2-edge connected. If that is not possible then $G$ cannot be decomposed in two edge disjoint spanning trees (see previous section). We next show how to perform the reduction of $u$ depending upon the number of tree edges incident to it.

CASE 1: $u$ is incident to one tree edge and two non-tree edges.

In this case $u$ is a leaf node (possibly root) in $T_{DFS}$ as it has only one tree edge, while it is an interior node in $CC_u$. Removal of $u$ and its incident edges leaves $T_{DFS}$ connected but it might disconnect $CC_u$.

Let $(u, x)$ be the tree edge and let $(u, y)$ and $(u, z)$ be non-tree edges. Removal of $u$ does not affect the paths in $T_{DFS}$ between the pairs $(x, y)$, $(x, z)$, and $(y, z)$. Vertices $y$, $z$, and $u$ belong to the same connected component of $G_2$, that is $CC_y = CC_z = CC_u$, while $x$ might be part of a different connected component. Thus, the following two case can occur:

1. $CC_x = CC_y = CC_z = CC_u$

Consider the $x$-to-$y$ path in this connected component (no $T_{DFS}$ edge involved). If this path does not contain edges $(u, y)$ and $(u, z)$ before removal of $u$ then the path is preserved after removal of $u$ and $x$ and $y$ remain 2-edge connected in $G$. This implies that $z$ is no longer two edge connected with $x$ and $y$, and we can add either $(z, y)$ or $(z, x)$ as a new non-tree edge (see previous section). Otherwise, the path was broken along the sequence $z, u, y$ and thus the $x$-to-$z$ path in the connected component does not contain edges $(u, y)$ and $(u, z)$ before removal of $u$. Thus, $x$ and $z$ remain 2-edge connected in $G$ and we can add either $(y, x)$ or $(y, z)$ as a new non-tree edge. Looking at the two choices it then suffices to add a new edge $(y, z)$ to maintain pairwise 2-edge connectivity for $x$, $y$, and $z$.

To summarize, we:
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1. Add a new edge $(y, z)$ to graph $G$ (as part of $G_2$)
2. Mark the new edge as non-tree edge.

2. $CC_x \neq CC_y = CC_z = CC_u$

   a. If in $T_{DFS}$ vertex $z$ is on the path from $x$ to $y$ we add $(x, y)$ as non-tree edge.
   b. If in $T_{DFS}$ vertex $y$ is on the path from $x$ to $z$ we add $(x, z)$ as non-tree edge.
   c. If in $T_{DFS}$ vertex $x$ is on the path from $y$ to $z$ we add $(y, z)$ as non-tree edge.

Notice that for this case, to decide which edge to add, we only need to answer dynamic connectivity information for $T_{DFS}$ and $G_2$.

**Figure 7** Reduction of degree 3 vertex with one tree edge. Non tree edges are represented as thick dashed lines (green) while tree edges are thick solid lines (red). A path between two vertices in $T_{DFS}$ which consist of only tree edges is represented by thin solid line (magenta) while a path between two vertices in $G_2$ is represented by thin dashed lines (olive green). Dotted line is a non-tree edge added after removing vertex $u$.

### CASE 2: $u$ is incident to one non-tree edge and two tree edges.

In this case $u$ is a degree one node in $G_2$, as it has only one non tree edge, while it is an interior node in $T_{DFS}$. Removal of $u$ disconnects $T_{DFS}$ into two trees, $T^1$ and $T^2$. After removing vertex $u$ and its incident edges we need to add a new tree edge to reconnect $T_{DFS}$.

Let $(u, x)$ and $(u, y)$ be tree edges and let $(u, z)$ be the non-tree edge. Assume that $x$ is parent of $u$ and $y$ is child of $u$ in $T_{DFS}$. If we remove $u$ from $T_{DFS}$ then $x \in T^1$ and $y \in T^2$. To reconnect $T_{DFS}$ we need to add a new tree edge, $(x, y)$ or $(x, z)$ if $z \in T^2$, or $(y, x)$ or $(y, z)$ if $z \in T^1$.

Notice that vertices $z$ and $u$ belong to the same connected component, that is $CC_z = CC_u$. The following cases can occur:
1. \( CC_x = CC_y = CC_z = CC_u \)
   Removing vertex \( u \) does not affect any connection between \( x, y, \) and \( z \) in \( G_2 \), so they remain connected in \( G_2 \). If \( z \in T^2 \) we can add either \((x, y)\) or \((x, z)\) to reconnect \( T_{DFS} \). If \( z \in T^1 \) we can add either \((y, x)\) or \((y, z)\). Since \((x, y)\) is a common option to both cases, we add edge \((x, y)\). To summarize, we:
   a. Add a new edge \((x, y)\) to \( T_{DFS} \) (and \( G \)) so that \( T_{DFS} \) remains connected.
   b. Mark the new edge as tree edge in \( G \).

2. \( CC_z \neq CC_y = CC_z = CC_u \)
   In this case, we have a non-tree edge path \( \pi_{yz} \) from \( y \) to \( z \) in \( G_2 \). Moreover, for each of the pairs \( \{x, y\}, \{x, z\}, \) and \( \{y, z\} \), one of the paths between them in \( G \) is unaffected by the removal of \( u \). We argue it is enough to add a new tree edge \((x, y)\) to reconnect \( T_{DFS} \), and maintain 2-edge connectivity for \( x, y, \) and \( z \) (see Figure 8).

![Figure 8 Case 2-2 where non tree edges are represented as thick dashed lines (green) while tree edges are thick solid lines (red). A path between two vertices in \( T_{DFS} \) which consist of only tree edges is represented by thin solid line (magenta) while a path between two vertices in \( G_2 \) is represented by thin dashed lines (olive green).](image)

As stated earlier, we have two sub-cases, depending on whether \( z \) is an ancestor of \( x \) or a descendant of \( y \) in \( T_{DFS} \). Assume first that \( z \) is ancestor of \( x \). Then, we have a cycle in \( G \) formed by the \( T_{DFS} \) path from \( z \) to \( x \), the tree edges \((x, u)\) and \((u, y)\), and the path \( \pi_{yz} \) in \( G_2 \). Replacing the pair of edges \((x, u)\) and \((u, y)\) with a new tree edge \((x, y)\) preserves the cycle, so \( x, y, \) and \( z \) remain pairwise 2-edge connected.

Assume now that \( z \) is a descendant of \( y \). Then, \( y \) and \( z \) are 2-edge connected since we have a path from \( y \)-to-\( z \) in \( T_{DFS} \) and the non-tree path \( \pi_{yz} \). By results in previous section, if \( G \) is decomposable, then we can add either \((x, y)\) or \((x, z)\) as a new edge to maintain 2-edge connectivity for \( x, y, \) and \( z \), and we can mark it as tree edge, to reconnect \( T_{DFS} \).

Thus, after removal of \( u \), it suffices to add a new tree edge \((x, y)\).

3. \( CC_y \neq CC_x = CC_z = CC_u \)
   This case is symmetric to the previous case, so we add a new tree edge \((x, y)\) to reconnect \( T_{DFS} \).

4. \( CC_x = CC_y \neq CC_z = CC_u \)
   In this case, \( x \) and \( y \) are connected in \( G_2 \). If \( z \in T^1 \) then we add a new tree edge \((y, z)\), otherwise \((z \in T^2)\) we add a new tree edge \((x, z)\) to reconnect \( T_{DFS} \).
5. \(CC_x \neq CC_y \neq CC_z = CC_u\)

In this case we swap a \(T_{DFS}\) edge incident to \(u\) with a non-tree edge in \(G_2\) such that \(T_{DFS}\) remains connected (the tree edge becomes non-tree edge and the non-tree edge becomes tree edge). Assume that \(z\) is an ancestor of \(x\) in \(T_{DFS}\) and refer to Figure 9 (the case when \(z\) is a descendant of \(y\) in \(T_{DFS}\) is symmetric). Consider a pair of vertices adjacent to \(u\), say \(x\) and \(y\). If a path \(\pi_{xy}\) in \(G\), from \(x\) to \(y\), contains vertex \(u\) (uses two edges incident to \(u\)) then any other \(x\)-to-\(y\) path in \(G\) that is edge disjoint with \(\pi_{xy}\) cannot contain \(u\). Thus, there must exist a cycle in \(G\) formed by: (1) a path in \(T_{DFS}\) that starts at a vertex \(A\) that is an ancestor of \(x\) (possible \(x\) itself), goes to vertex \(x\) and then uses tree edges \((x, u)\) and \((u, y)\), and ends at a vertex \(B\) that is descendant of \(y\) in \(T_{DFS}\) (possible \(y\) itself) and (2) a "back edge" \((B, A)\) \(((B, A) \in G_2)\). This cycle makes \(x\) and \(y\) two edge connected. Assume \(z\) is ancestor of \(A\) in \(T_{DFS}\), as shown in Figure 9. The case where \(z\) is a descendant of \(A\) is similar. Vertices \(x\) and \(z\) are two edge connected along the cycle from \(z\) to \(x\) in \(T_{DFS}\), followed by tree edge \((x, u)\) and non-tree edge \((u, z)\). Moreover, \(z\) and \(y\) are two edge connected along the cycle the follows the path \(z\)-to-\(A\) in \(T_{DFS}\), then the non-tree edge \((A, B)\), the \(T_{DFS}\) path from \(B\) to \(y\), the tree edge \((y, u)\), and finally the non-tree edge \((u, z)\).

We will address later on how to efficiently identify the edge \((A, B)\). For now, we make \((A, B)\) a tree edge (add it to \(T_{DFS}\)) and make \((x, u)\) a non-tree edge (add it to \(G_2\)). With this swap \(T_{DFS}\) remains connected, the three cycles in \(G\) described above are preserved \((x, y, z, \text{and } u \text{ are 2-edge connected})\), and we are now in Case 1, addressed earlier.

![Figure 9](image_url) Case 2-5 where each neighbor of vertex \(u\) belongs to a different component.

CASE 3: \(u\) has three tree edges.

When all edges incident to vertex \(u\) are tree edges it is easy to notice that \(CC_u\) contain only vertex \(u\). We convert this case to case 1 by swapping two three edges incident to \(u\) with two non-tree edges, as we did in Case 2-5.

Consider Figure 10 and notice that there is no path from \(y\) to \(z\) in \(G\). Since \(x, y, z, \text{and } u\) are pairwise 2-edge connected, the cycles for both pairs \(\{x, y\}\) and \(\{x, z\}\) are like in Case 2-5. We can find the non-tree edges \((A, B)\) and \((A', B')\) and make them tree edges, and
make \((u, y)\) and \((u, z)\) non-tree edges. Thus, \(T_{DFS}\) remains connected and \(u\) is now a Case 1 vertex.

![Figure 10 Case 3 when all edges incident to \(u\) are tree edges.](image)

Notice that it is possible that after reduction of a degree three vertex we create a degree two vertex \(u\) that has both incident edges, \((u, x)\) and \((u, y)\) as tree edges. Reduction of this vertex would disconnect \(T_{DFS}\). Again, we can take care of such case by swapping one of the two edges incident to \(u\) with a non-tree edge. Since \(u\) is on a cycle, we identify the non-tree edge as in Case 2-5.

We now analyze the time needed to perform the reduction. We have (a) vertex removal, edge removal and insertion, and connectivity queries in both \(T_{DFS}\) and \(G_2\), which can be handled by using the dynamic graph connectivity data structure in \([9]\), and (b) edge swapping between \(T_{DFS}\) and \(G_2\). Notice that in this case we change the labeling of the two edges and perform two edge removals and two edge insertions, which are handled by the dynamic connectivity data structures on \(T_{DFS}\) and \(G_2\). It then remains to explain how to quickly find the non-tree edge used in the swap.

In what follows, we provide an efficient way to find the non-tree edge to swap with a tree edge incident to a degree three vertex \(u\), as in Case 2-5. We remark that other ways to find such edge are possible.

To perform edge swapping, we use top trees from Alstrup et al.\(^{[1]}\) in similar way as it is used in \([7, 9]\). For finding connectivity between two vertices in a graph and performing updates operation (insertion and deletion of edges) \([7]\) and \([9]\) maintain a spanning forest \(F\) of graph \(G\) where edges of \(F\) are referred as tree edges. Deleting a tree edge may split some tree in \(F\) into two subtrees. In \([7]\), they showed how to use top trees to find the replacement (non-tree) edge so as to reconnect the split tree in \(F\) in \(O(\log^2 n)\) time. Nilson \([9]\) improved the update time by an \(O(\log \log n)\) factor, providing an \(O(\log^2 n / \log \log n)\) update time. Thus, using same algorithm, we can find and swap a non-tree edge with a tree edge.

We summarize our findings in the following lemma.

\textbf{Lemma 7.} The reduction of a vertex \(u\) of degree three can be done in \(O(\log^2 n / \log \log n)\) time.

The following algorithm checks if a given graph \(G\) is decomposable and if so outputs two edge disjoints spanning trees. Steps 1-11 are the reduction of \(G\) while Steps 12-15 construct the two spanning trees.
Algorithm 2.

1. Stack $S = \phi$ to record the order in which vertices are removed
2. Tree $T_{DFS} =$ depth-first tree of graph $G$
3. Graph $G_2 = G \setminus T_{DFS}$
4. if $G_2$ is connected, return $T_{DFS}$ and $G_2$ as two edge disjoint spanning trees
5. Queue $Q_0 = \phi$ to store vertices of degree 0
6. Queue $Q_1 = \phi$ to store vertices of degree 1
7. Queue $Q_2 = \phi$ to store vertices of degree 2
8. Queue $Q_3 = \phi$ to store vertices of degree 3
9. $n \leftarrow$ number of vertices of $G$
10. For each vertex $u \in G$, create a list $V_u[]$ with three entries.
11. while $(n > 2)$
   
   I. if $\exists$ a degree 0 or degree 1 vertex (check $Q_0$ or $Q_1$ respectively) then exit and output
      "Graph $G$ is not decomposable into two edge disjoint spanning trees"
   II. else if $\exists$ a degree 2 vertex (check $Q_2$)
      A. let $v$ be a degree 2 vertex, adjacent to vertices $x$ and $y$
      B. if edges $(v, x)$ and $(v, y)$ are part of $T_{DFS}$, perform edge-swapping
      C. $S$.push($v$)
      D. remove vertex $v$ (and its incident edges) from $G$ and update the degree counts for $x$ and $y$. If the count of any of $x$ and $y$ becomes 1 then add it to $Q_1$, if it becomes 2 then add it to $Q_2$, and if it becomes 3 add it to $Q_3$
      E. $V[v] = null, x, y$
      F. $n = n - 1$
   III. else if $\exists$ a degree 3 vertex (check $Q_3$)
      A. Let $v$ be a degree 3 vertex whose neighbors are $x, y$ and $z$
      B. if vertex $v$ has two edges in $T_{DFS}$
         i. if none of the neighbors of $v$ are connected in $G_2$, perform edge-swapping and update graph $G_2$ and $T_{DFS}$
         ii. else
            a. $S$.push($v$)
            b. remove vertex $v$ (and its incident edges) from $G$ and update the degree counts for $x$, $y$, and $z$.
            c. Let $(x, y)$ be the pair that has to be added according to conditions described in case 2
            d. Add edge $(x, y)$ to $G$ and update counts for $x$ and $y$ (note we might need to remove them from various queues)
            e. $V[v] = x, y, z$
            f. $n = n - 1$
      C. if vertex $v$ has one edge in $T_{DFS}$
         i. Let $v$ be a degree 3 vertex whose neighbors are $x, y$ and $z$
         ii. $S$.push($v$)
         iii. remove vertex $v$ (and its incident edges) from $G$ and update the degree counts for $x$, $y$, and $z$.
         iv. Let $(x, y)$ be the pair that has to be added according to conditions described in case 2
         v. Add edge $(x, y)$ to $G$ and update counts for $x$ and $y$ (note we might need to remove them from various queues)
vi \( V[v] = x, y, z \)

\[ n = n - 1 \]

D if vertex \( v \) has three edges in \( T_{DFS} \)

i Let \( v \) be a degree 3 vertex whose neighbors are \( x, y \) and \( z \)

ii Perform edge-swapping as described in case 3 and update graph \( G_2 \) and \( T_{DFS} \)

12 Graph \( G \) has only two vertices and two edges. If any edge is a self loop then graph \( G \) is not decomposable. Exit and output 'Graph \( G \) is not decomposable into two edge disjoint spanning trees'.

13 Let \( T_1, T_2 \) be two spanning trees where \( T_1 = \phi, T_2 = \phi \).

14 Let \( a \) and \( b \) be the two vertices in \( G \)

\[ T_1 = T_1 \cup (a, b) \]

\[ T_2 = T_2 \cup (a, b) \]

15 while \( (S \neq \phi) \)

I \( u = s.\text{pop} \)

II \( x, y, z = V_u \)

III if \( x \) is null

Adding node of degree 2

A \( T_1 = T_1 \cup (u, y) \)

B \( T_2 = T_2 \cup (u, z) \)

IV else

Adding node of degree 3

A if edge \( (x, y) \in T_1 \)

i \( T_1 = T_1 - (x, y) \cup (u, x) \cup (u, y) \)

ii \( T_2 = T_2 \cup (u, z) \)

B else

i \( T_1 = T_1 \cup (u, z) \)

ii \( T_2 = T_2 - (x, y) \cup (u, x) \cup (u, y) \)

Time analysis. Using results from \[7, 9\] and Lemma \[7\], each iteration of Step 11 takes \( O(\log^2 n / \log \log n) \) time, and Step 11 executes \( O(n) \) times. Updating various queues used in the algorithm takes \( O(1) \) time per update and \( O(n) \) overall, using standard techniques (maintain \( O(1) \) space information at each vertex). Thus overall, our algorithm takes \( O(n \log^2 n / \log \log n) \) time.

Theorem 8. Given an undirected graph \( G = [V; E] \), with \(|E| = 2(|V| - 1)\), it can be decided in \( O(n \log^2 n / \log \log n) \) time if \( G \) is decomposable into two edge disjoint spanning trees. Moreover, if \( G \) is decomposable, the two trees can be reported within the same time bound.

We apply Theorem 8 to the result in \[3\] and obtain the following.

Lemma 9. Given an undirected graph \( G = [V; E] \), it can be verified in \( O(n \log^2 n / \log \log n) \) time if \( G \) is minimally rigid.

5 Conclusion

While a linear time algorithm for finding two edge disjoint spanning trees in a directed graph has been known for more than four decades \[8\], it has remained elusive to find a matching algorithm for undirected graphs. In this paper we presented algorithms that are a small polylog factor from linear, for a special case, where the graph \( G = [V; E] \) has \(|E| = \)}
Our algorithms are simple in nature, with the most complex operations being related to dynamic graph connectivity. Thus, improved solutions for dynamic connectivity in undirected graph would speed up our algorithms as well. We leave open extending our algorithms to general undirected graphs, where $|E| > 2(|V| - 1)$.

A direct implication of our result is an almost linear time solution for checking if an undirected graph is minimally rigid; the question of whether this is possible has been long standing.

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6 Appendix-1

In this appendix we show the step by step reduction of a graph $G$, which is decomposible into two edge disjoint spanning trees. For better understanding, we maintain four different queues for degree 3 vertices instead of just one queue, as described in Algorithm 2. Queue $Q^p_3$ stores degree 3 vertices, each with two parallel edges. Let $v$ be such a degree 3 vertex, whose neighbor vertices are $x$ and $y$. After removing vertex $v$, we do not need to check connectivity between vertex $x$ and $y$; we can directly add a new edge $(x, y)$. We define the queues for degree three vertices as follows:

1. Queue $Q^p_3 = \phi$ to store vertices of degree 3 with two parallel edges
2. Queue $Q^1_3 = \phi$ to store vertices of degree 3 with one edge in $T_{DFS}$
3. Queue $Q^2_3 = \phi$ to store vertices of degree 3 with two edges in $T_{DFS}$
4. Queue $Q^3_3 = \phi$ to store vertices of degree 3 with three edge in $T_{DFS}$

While performing reduction of a degree 3 vertices, we reduce vertices from various queues in the following order: $Q^p_3, Q^2_3, Q^1_3$ and $Q^3_3$.

![Figure 11](image-url) Connected undirected graph $G$ with no bridges, where degree of each vertex is greater than or equal to 3 and $|E| = 2(|V| - 1)$
Given graph $G$ shown in above figure, we perform depth-first search. Solid edges are tree edges while dashed edges are non-tree edges. Edges/Vertices of similar color are connected in graph $G_2$.

- $Q_1 = \{\emptyset\}$
- $Q_2 = \{\emptyset\}$
- $Q_3^p = \{\emptyset\}$
- $Q_1^3 = \{3, 15, 1\}$
- $Q_2^3 = \{2, 5, 8, 11, 19, 10\}$
- $Q_3^3 = \{9\}$

We reduce vertex 2 and add new tree edge $(3, 1)$. 
Figure 13 Adding edge (3, 1) we update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q_3^0 = \{ \phi \} \]
\[ Q_1^3 = \{ 3, 15, 4 \} \]
\[ Q_2^3 = \{ 5, 8, 11, 19, 10, 1 \} \]
\[ Q_3^3 = \{ 9 \} \]

We reduce vertex 5 and add new tree edge (7, 6).
Adding edge (7, 6) we update queues:

**Figure 14** Adding edge (7, 6) we update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ 8 \}$
- $Q_3' = \{ \phi \}$
- $Q_4' = \{ 3, 15, 4 \}$
- $Q_5' = \{ 11, 19, 10, 1 \}$
- $Q_6' = \{ 9 \}$

We reduce vertex 8.
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q_1^4 = \{ 3, 15, 4 \} \]
\[ Q_2^3 = \{ 11, 19, 10, 1 \} \]
\[ Q_3^3 = \{ 9 \} \]

We reduce vertex 11 and add new tree edge (24, 9).
We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ \phi \}$
- $Q_3' = \{ \phi \}$
- $Q_3 = \{ 3, 15, 4 \}$
- $Q_4 = \{ 19, 10, 1, 26 \}$
- $Q_5 = \{ 9 \}$

We reduce vertex 19 and add new tree edge (23, 18).
Figure 17 We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{ \phi \}$

$Q_3^\phi = \{ \phi \}$

$Q_3^1 = \{3, 15, 4\}$

$Q_3^2 = \{10, 1, 26\}$

$Q_3^3 = \{9\}$

We reduce vertex 10 and perform edge swapping between (9, 10) and (7,25).
Figure 18 We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q_3^* = \{ 3, 15, 4, 10 \} \]
\[ Q_3^* = \{ 1, 26, 9 \} \]
\[ Q_3^* = \{ \phi \} \]

We reduce vertex 1 and add new tree edge (3, 13)
Figure 19 We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{ \phi \}$

$Q_3^* = \{ \phi \}$

$Q_3^1 = \{3, 15, 4, 10\}$

$Q_3^2 = \{26, 9, 14\}$

$Q_3^3 = \{ \phi \}$

We reduce vertex 26 and add new tree edge (25, 27)
We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{ \phi \}$

$Q_3 = \{ \phi \}$

$Q_3' = \{ 3, 15, 4, 10 \}$

$Q_4' = \{ 9, 14, 12 \}$

$Q_3'' = \{ \phi \}$

We reduce vertex 9 and add new tree edge (24, 10)
We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ 12 \}$
- $Q_3^0 = \{ \phi \}$
- $Q_3^1 = \{ 3, 15, 4 \}$
- $Q_3^2 = \{ 14, 10 \}$
- $Q_3^3 = \{ \phi \}$

We reduce vertex 12.
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q_4^1 = \{ 3, 15, 4 \} \]
\[ Q_4^2 = \{ 14, 10, 25 \} \]
\[ Q_4^3 = \{ \phi \} \]

We reduce vertex 14 and add new tree edge (17, 13)
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ 15 \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q_{13} = \{ 3, 4 \} \]
\[ Q_{23} = \{ 10, 25 \} \]
\[ Q_{33} = \{ \phi \} \]

We reduce vertex 15
Figure 24 We update queues: 
\[
\begin{align*}
Q_1 &= \{ \phi \} \\
Q_2 &= \{ \phi \} \\
Q_3 &= \{ \phi \} \\
Q_1' &= \{ 3, 4 \} \\
Q_2' &= \{ 10, 25, 16, 6 \} \\
Q_3' &= \{ \phi \}
\end{align*}
\]
We reduce vertex 10 and perform edge swapping between edges (10, 16) and (24, 22)
We update queues:

$Q_1 = \{\phi\}$

$Q_2 = \{\phi\}$

$Q_p^3 = \{\phi\}$

$Q_{13} = \{3, 4, 10, 16\}$

$Q_{23} = \{25, 6\}$

$Q_{33} = \{\phi\}$

We reduce vertex 25 and add new tree edge (7, 27)
We update queues:

$$Q_1 = \{ \phi \}$$
$$Q_2 = \{ \phi \}$$
$$Q_3 = \{ \phi \}$$
$$Q_4 = \{ 3, 4, 10, 16 \}$$
$$Q_5 = \{ 6 \}$$
$$Q_6 = \{ \phi \}$$

We reduce vertex 6 and add new tree edge (13, 7)
We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{ \phi \}$

$Q_3 = \{ \phi \}$

$Q_3^1 = \{ 3, 4, 10, 16 \}$

$Q_3^2 = \{ 17 \}$

$Q_3^3 = \{ \phi \}$

We reduce vertex 17 and add tree edge (13, 16)
Figure 28 We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ 10 \}$
- $Q'_3 = \{ \phi \}$
- $Q_3 = \{ 3, 4, 16 \}$
- $Q'_4 = \{ \phi \}$
- $Q_4 = \{ \phi \}$

We reduce vertex 10.
Figure 29 We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{16\}$

$Q_3^n = \{ \phi \}$

$Q_3 = \{3, 4, 24\}$

$Q_3^n = \{ \phi \}$

$Q_3^w = \{ \phi \}$

We reduce vertex 16
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ 13 \} \]
\[ Q_3^u = \{ \phi \} \]
\[ Q_1^u = \{ 3, 4, 24 \} \]
\[ Q_3^d = \{ \phi \} \]
\[ Q_1^d = \{ \phi \} \]

We reduce vertex 13 and perform edge swapping.
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ 3 \} \]
\[ Q_p^0 = \{ \phi \} \]
\[ Q_1^0 = \{ 4, 24 \} \]
\[ Q_2^0 = \{ 7 \} \]
\[ Q_3^0 = \{ \phi \} \]

We reduce vertex 3.
We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{ \phi \}$

$Q_3 = \{ \phi \}$

$Q_4 = \{ 4, 24, 18 \}$

$Q_5 = \{ 7, 20 \}$

$Q_6 = \{ \phi \}$

We reduce vertex 7 and add tree edge (4, 27).
We update queues:

- $Q_1 = \{\phi\}$
- $Q_2 = \{\phi\}$
- $Q_3 = \{\phi\}$
- $Q_1' = \{4, 24, 18\}$
- $Q_2' = \{20\}$
- $Q_3' = \{\phi\}$

We reduce vertex 20 and add tree edge (21, 27).
We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ \phi \}$
- $Q_3^1 = \{ \phi \}$
- $Q_3^1 = \{ 4, 24, 18 \}$
- $Q_3^3 = \{ 22 \}$
- $Q_3^3 = \{ \phi \}$

We reduce vertex 22 and add tree edge (24, 23).
We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ \phi \}$
- $Q_3^1 = \{ \phi \}$
- $Q_3^1 = \{ 4, 24, 18 \}$
- $Q_3^1 = \{ 27 \}$
- $Q_3^1 = \{ \phi \}$

We reduce vertex 27 and add tree edge (24, 4s).
Figure 36 We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ \phi \}$
- $Q_1^3 = \{ \phi \}$
- $Q_2^3 = \{4, 18, 21\}$
- $Q_3^3 = \{24\}$
- $Q_3^3 = \{ \phi \}$

We reduce vertex 24 and add tree edge (4, 23).
Figure 37 We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q_3^1 = \{ 4, 18, 21 \} \]
\[ Q_3^2 = \{ \phi \} \]
\[ Q_3^3 = \{ 23 \} \]

We reduce vertex 4 and add new non-tree edge (21, 18).
We update queues:

- $Q_1 = \{\phi\}$
- $Q_2 = \{23\}$
- $Q_1' = \{\phi\}$
- $Q_3' = \{18, 21\}$
- $Q_2' = \{\phi\}$
- $Q_3'' = \{\phi\}$

We reduce vertex 23 and perform edge swapping.

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We update queues:

- $Q_1 = \{\phi\}$
- $Q_2 = \{18, 21\}$
- $Q_1' = \{\phi\}$
- $Q_3' = \{\phi\}$
- $Q_2'' = \{\phi\}$
- $Q_3'' = \{\phi\}$. 
Figure 40 Edge-disjoint trees of graph $G$ shown in Figure 11
We modify graph $G$ shown in Figure 11 in such a way that modified graph is not decomposable into two edge-disjoint spanning trees. We show step by step reduction of modified graph which is shown in Figure 41.

![Figure 41 Modified graph $G' = G \cup \{(1, 6), (13, 17), (4, 5)\} - \{(5, 6), (7, 25), (4, 21)\}$.](image)

It is easy to notice that the graph in Figure 41 is not decomposable. Consider the cut that divides vertices of graph $G'$ into two sets, where one of the set is $\{1, 6, 10, 13, 14, 15, 16, 17\}$. By using Theorem 1, the number of vertices in this cut $= r(A) = 7$, number of edges $= |A| = 15$. For this cut $|A| \geq 2r(A)$ which implies that graph is not decomposable. Below we show the step by step reduction of modified graph.
Figure 42 Given graph $G$ shown figure 41, we perform depth-first search. Solid edges are tree edges while dashed edges are non-tree edges. Edges/Vertices of similar color belongs to same connected component.

$Q_1 = \{\phi\}$
$Q_2 = \{\phi\}$
$Q_3 = \{\phi\}$
$Q_1' = \{3, 15\}$
$Q_2' = \{2, 7, 5, 8, 11, 25, 19, 10\}$
$Q_3' = \{9\}$
We reduce vertex 2 and add new tree edge $(3, 1)$. 
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q'_1 = \{ \phi \} \]
\[ Q''_3 = \{ 3, 15, 4 \} \]
\[ Q_{3}^2 = \{ 7, 5, 8, 11, 25, 19, 10 \} \]
\[ Q_{3}^3 = \{ 9 \} \]

We reduce vertex 7 and add new tree edge \((4, 5)\).
We update queues:

$Q_1 = \{\phi\}$

$Q_2 = \{\phi\}$

$Q_3^0 = \{\phi\}$

$Q_3^1 = \{3, 15, 4\}$

$Q_3^2 = \{5, 8, 11, 25, 19, 10, 21\}$

$Q_3^3 = \{9\}$

We reduce vertex 5 and add new tree edge (4, 8).
Figure 45 We update queues:
$Q_1 = \{\phi\}$
$Q_2 = \{4\}$
$Q'_2 = \{\phi\}$
$Q_3 = \{3, 15\}$
$Q_4 = \{8, 11, 25, 19, 10, 21\}$
$Q_5 = \{9\}$
We reduce vertex 4.
Figure 46 We update queues:

$Q_1 = \{ \phi \}$
$Q_2 = \{ 8 \}$
$Q_3 = \{ \phi \}$
$Q_3' = \{ 3, 15, 18 \}$
$Q_3'' = \{ 11, 25, 19, 10, 21 \}$
$Q_3''' = \{ 9 \}$

We reduce vertex 8.
Figure 47 We update queues:

- $Q_1 = \{\phi\}$
- $Q_2 = \{\phi\}$
- $Q_3 = \{\phi\}$
- $Q_3^1 = \{3, 15, 18\}$
- $Q_3^2 = \{11, 25, 19, 10, 21\}$
- $Q_3^3 = \{9\}$

We reduce vertex 11 and add new tree edge (24, 9).
We update queues:

$Q_1 = \{ \phi \}$

$Q_2 = \{ \phi \}$

$Q_3^5 = \{ \phi \}$

$Q_3^1 = \{3, 15, 18\}$

$Q_3^2 = \{25, 19, 10, 21, 26\}$

$Q_3^3 = \{9\}$

We reduce vertex 25 and add new tree edge $(12, 26)$.
We update queues:

- $Q_1 = \{ \emptyset \}$
- $Q_2 = \{ \emptyset \}$
- $Q_3 = \{ \emptyset \}$
- $Q_{1}^3 = \{3, 15, 18\}$
- $Q_{2}^3 = \{19, 10, 21, 26\}$
- $Q_{3}^3 = \{9\}$

We reduce vertex 19 and add new tree edge (18, 23).
We update queues:

$Q_1 = \{\phi\}$
$Q_2 = \{\phi\}$
$Q_2^c = \{\phi\}$
$Q_3 = \{3, 15, 18\}$
$Q_3^c = \{10, 21, 26\}$
$Q_3^c = \{9\}$

We reduce vertex 10 and add new tree edge (9, 16).
We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ \phi \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q_3^1 = \{ 3, 15, 18 \} \]
\[ Q_3^2 = \{ 21, 26 \} \]
\[ Q_3^3 = \{ 9 \} \]

We reduce vertex 21 and add new tree edge (20, 23).
Figure 52 We update queues:
\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ 18 \} \]
\[ Q_3^p = \{ \phi \} \]
\[ Q_3^1 = \{ 3, 15 \} \]
\[ Q_3^2 = \{ 26 \} \]
\[ Q_3^3 = \{ 9 \} \]
We reduce vertex 18.
We update queues:

$Q_1 = \{ \phi \}$
$Q_2 = \{ 3 \}$
$Q''_2 = \{ \phi \}$
$Q_3 = \{ 15 \}$
$Q_2^3 = \{ 26, 23 \}$
$Q''_3^3 = \{ 9 \}$

We reduce vertex 3.
Figure 54 We update queues:

$Q_1 = \{\phi\}$

$Q_2 = \{\phi\}$

$Q_3^n = \{\phi\}$

$Q_3^1 = \{15, 1\}$

$Q_3^2 = \{26, 23, 20\}$

$Q_3^3 = \{9\}$

We reduce vertex 26 and add tree edge (12, 27).
We reduce vertex 23 and add tree edge (12, 27).

Figure 55 We update queues:
- $Q_1 = \{\phi\}$
- $Q_2 = \{\phi\}$
- $Q_3 = \{\phi\}$
- $Q_4 = \{15, 1\}$
- $Q_5 = \{23, 20, 12\}$
- $Q_6 = \{9\}$

We reduce vertex 23 and add tree edge (12, 27).
Figure 56 We update queues:

- $Q_1 = \{\phi\}$
- $Q_2 = \{\phi\}$
- $Q_3^n = \{\phi\}$
- $Q_3 = \{15, 1, 24\}$
- $Q_3^2 = \{20, 12\}$
- $Q_3^3 = \{9\}$

We reduce vertex 20 and add tree edge (22, 27).
Figure 57 We update queues:

$Q_1 = \{\phi\}$

$Q_2 = \{\phi\}$

$Q_3 = \{5, 1, 24, 22\}$

We reduce vertex 12 and add tree edge (9, 27).
We update queues:

- $Q_1 = \{\phi\}$
- $Q_2 = \{\phi\}$
- $Q_3 = \{\phi\}$
- $Q_4 = \{15, 1, 24, 22\}$
- $Q_5 = \{\phi\}$
- $Q_6 = \{9\}$

We reduce vertex 15 and add non-tree edge (6,16).
We reduce vertex 14 and add tree edge (17, 13).
Figure 60 We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ 1 \} \]
\[ Q'_3 = \{ \phi \} \]
\[ Q^*_3 = \{ 24, 22 \} \]
\[ Q_3 = \{ \phi \} \]
\[ Q^3_3 = \{ 9 \} \]

We reduce vertex 1.
Figure 61 We update queues:

- $Q_1 = \{ \phi \}$
- $Q_2 = \{ \phi \}$
- $Q'_3 = \{ \phi \}$
- $Q_3 = \{ 24, 22, 6 \}$
- $Q'_4 = \{ \phi \}$
- $Q_4 = \{ 9 \}$

We reduce vertex 24 and add non-tree edge (9, 22).
Figure 62 We update queues:

\[ Q_1 = \{\phi\} \]
\[ Q_2 = \{\phi\} \]
\[ Q_3 = \{\phi\} \]
\[ Q_4 = \{22, 6\} \]
\[ Q_5 = \{27, 9\} \]
\[ Q_6 = \{\phi\} \]

We reduce vertex 27 and add non-tree edge (9, 22).
Figure 63: We update queues:

\[ Q_1 = \{ \phi \} \]
\[ Q_2 = \{ 22 \} \]
\[ Q'_2 = \{ \phi \} \]
\[ Q_3 = \{ 6 \} \]
\[ Q_2^3 = \{ 9 \} \]
\[ Q_3^3 = \{ \phi \} \]

We reduce vertex 22.
**Figure 64** We update queues:

- $Q_1 = \{9\}$
- $Q_2 = \{\phi\}$
- $Q_1^* = \{\phi\}$
- $Q_2^* = \{6\}$
- $Q_1^* = \{\phi\}$
- $Q_2^* = \{\phi\}$

We have a degree 1 vertex. Hence graph is not decomposable.