Semiclassical geometry of integrable systems

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Abstract
The main result of this paper is a formula for the scalar product of semiclassical eigenvectors of two integrable systems on the same symplectic manifold. An important application of this formula is the Ponzano–Regge type of asymptotic of Racah–Wigner coefficients.

Keywords: integrability, semiclassical, Racah–Wigner

(Some figures may appear in colour only in the online journal)

1. Introduction

One of the main motivations for this paper is the study of semiclassical asymptotics of 6j-symbols and their quantum analogs. Recall that 6j symbols, which are also known as Racah–Wigner coefficients, first appeared in the theory of angular momentum. They describe the transition matrix between two natural bases in the decomposition of the tensor product of three irreducible representations of SU(2) into irreducibles. More generally, they describe the associators in braided monoidal categories which naturally appear in representation theory of simple Lie algebras or corresponding quantum groups.

The semiclassical asymptotic in representation theory is the limit when all components of the highest weights go to infinity at the same rate. In this limit many features of representation theory can be expressed in terms of geometry of coadjoint orbits. For SU(2) the semiclassical asymptotic of 6j symbols was computed by Ponzano and Regge in [16]. A geometric interpretation of Ponzano–Regge asymptotic, involving symplectic geometry was made by Roberts in [17], where he observed that this is essentially the computation of the semiclassical
asymptotic of the scalar product of eigenfunctions of Hamiltonians of two integrable systems. A similar observation was made by Taylor and Woodward in [20, 21] where the authors computed the semiclassical asymptotic for $q - 6j$ symbols [11]. For references to earlier works see [16, 17]. See also [6] where the approach of [17] was extended to the tensor product of $N$ representations.

In this paper we give the general formula (16) and (20) for leading terms of the semiclassical asymptotic of the scalar product of eigenfunctions of two integrable systems on the same phase space. It is closely related to the quantization of Lagrangian submanifolds, see for example [4, 9, 10]. In the setting of the geometric quantization, eigenfunctions should be understood as half-densities, so it is better to say that we study scalar products of eigen half-densities. For the spectral analysis of quantum integrable systems see [15] and references therein.

In particular, the formula (20) for the scalar product gives the semiclassical asymptotic for $6j$ symbols (and $q - 6j$ symbols) for all simple Lie algebras in multiplicity free cases. We will not focus on this particular application here, but will address it in a separate publication.

Another natural place where a similar formula appears is the semiclassical asymptotic of the propagator in quantum mechanics when the initial quantum space of states and the target space of states are described in terms of geometric quantization when they correspond to different, transversal real polarizations of the phase space. This is discussed in the conclusion.

The plan of the paper is as follows. The first section is focused on the one dimensional case. In the second section the formula is derived for two integrable systems on a cotangent bundle. In the conclusion the semiclassical formula is stated for general symplectic manifolds, the relation to $6j$ symbols is discussed and the relation to topological quantum mechanics is outlined.

2. The one dimensional example

2.1. Hamiltonians which are quadratic in momentum

Here we recall some basic facts about the semiclassical, Wentzel–Kramers–Brillouin (WKB) asymptotic of eigenfunctions for the one dimensional Schrödinger operator.

Let $H(p, q) = \frac{p^2}{2} + V(q)$ be the Hamiltonian of a classical system describing a one dimensional particle with mass 1 in an external potential $V(q)$ which we assume to be a smooth function. This Hamiltonian is a function on the phase space $\mathbb{R}^2 = T^* \mathbb{R}$ which is equipped with the standard symplectic form $\omega = dp \wedge dq$, where $p$ is the coordinate along the cotangent fibers and $q$ is a coordinate on the configuration space $\mathbb{R}$. Generic level curves of $H$ are Lagrangian submanifolds in $\mathbb{R}^2$. For simplicity we will focus on the case when level curves are connected and compact $^1$.

With these assumptions the level curve $H(p, q) = b$

$$\mathcal{L}_b = \{(p, q) | \frac{p^2}{2} + V(q) = b\}$$  \hspace{1cm} (1)

is a double cover of the segment $q_1 < q < q_2$ where $q_i$ are turning points $V(q_i) = b$ (see figure 1).

We use $b$ for the energy because we consider it as a point of the Lagrangian fibration $H : \mathbb{R}^2 \to \mathbb{R}$ given by level curves of $H(p, q)$.

$^1$ In the case when level curves are not connected we will have to consider tunneling between different components. When they are noncompact there is no Bohr–Sommerfeld quantization condition.
Denote by $p_{\pm}(q, b)$ two branches of this level curve $p_{\pm}(q, b) = \pm p(q, b)$ where $p(q, b) = \sqrt{2(b - V(q))}$ and by $S_{\pm}(q, b)$ corresponding branches of the Hamilton–Jacobi action function:

$$\frac{\partial S_{\pm}(q, b)}{\partial q} = \pm p(q, b).$$

Here $q$ are two points on $\mathbb{L}_b$ over $q \in \mathbb{R}$.

Quantized of a classical Hamiltonian $H(p, q)$ is a differential operator, with $H(p, q)$ being its principal symbol. The Schrödinger operator

$$\hat{H} = -\frac{\hbar^2}{2} \frac{d^2}{dq^2} + V(q)$$

is the physically important quantization.

The semiclassical, i.e. $\hbar \to 0$, asymptotic of eigenfunctions of the quantized Hamiltonian

$$\hat{H}\psi_\hbar(q) = \hat{b}\psi_\hbar(q) \quad (2)$$

is well known as the WKB asymptotic:

$$\psi_\hbar(q) = \left(C_+(q) \exp\left(\frac{i}{\hbar} S_+(q, b) + \frac{\pi i}{4}\right) + C_-(q) \exp\left(\frac{i}{\hbar} S_-(q, b) - \frac{\pi i}{4}\right)\right) \sqrt{\frac{\partial p(q, b)}{\partial b}}. \quad (3)$$

Here $C_{\pm}(q) = C^{(0)}_{\pm}(1 + \sum_{n \geq 1} a_n^{(n)}(q) \hbar^n)$ are asymptotic (formal) power series. The coefficients $a_n^{(n)}(q)$ are determined recursively by the equation (2), uniquely up to constants that can be absorbed into $C^3$. Here we assume that as $\hbar \to 0$ the sequence of eigenvalues $\hat{b}$ converges to $b$.

The corresponding eigenvalues $\hat{b}$ should satisfy the Bohr–Sommerfeld (BS) condition

$$\int_{\mathbb{L}_b} \alpha = 2\pi \hbar (n + \frac{1}{2})(1 + O(\hbar)).$$

2 Here $\hbar$ is the Plank constant in appropriate units.
3 In the one dimensional case we have the identity $\frac{\partial \psi_\hbar(q)}{\partial b} = \frac{1}{p(q, b)}$ which allows one to express the WKB asymptotic in terms of $p(q, b)$. 

Figure 1. Level curve of a quadratic Hamiltonian with turning points.
Here α = pdq, n → ∞ as h → 0 such that hn is finite and \( \hat{b} \rightarrow b \). We summarized the derivation of the BS quantization condition through the analysis of solutions near turning points \( q_1, q_2 \) in the appendix.

The Hamilton–Jacobi function \( S(q_{\pm}, b) \) can be written as

\[
S_{\pm}(q, b) = \int_{\gamma_{\pm}(q, x_0)} \alpha_b.
\]

Here we made a choice of a reference point \( x_0 = (p_0, q_0) \in L_b \). Contours \( \gamma_{\pm} \) connect the point \( x_0 \) with points \((\pm p(q, b), q)\), respectively, on \( L_b \). The Bohr–Sommerfeld condition ensures that exponents in the semiclassical formulae do not depend on the choice of \( \gamma \) (mod \( 2\pi h \)). The change of the reference point \( x_0 \) changes the overall constant.

Natural objects in the semiclassical analysis are not functions but half-densities, see for example [4, 9, 10, 24], and references therein. Half-densities are particularly natural to consider when no metric or volume form on the configuration space is specified. In this case the space of square integrable functions is not defined but the space of square integrable half-densities is still naturally defined.

Thus, instead of (3) we shall consider the eigen half-density

\[
\psi_b(q) \sqrt{|dq|} = C(\exp(\frac{i}{h}S(q_+, b) + \frac{\pi}{4}) \left(1 + \sum_{n>0} h^n a_+^{(n)}(q, b)\right) + \exp(\frac{i}{h}S(q_-, b) - \frac{\pi}{4}) \left(1 + \sum_{n>0} h^n a_-^{(n)}(q, b)\right)) \sqrt{|dq|}
\]

where coefficients \( a_\pm^{(n)} \) are as in (3).

These semiclassical eigenfunctions for real values of \( h \) are exponentially decaying away from \( D_b = \{q|H(p, q) = b\} \), and away from this region behave as \( \exp(-c(q)h) \) as \( h \rightarrow 0 \), where \( c(q) \) is a positive function given by the Hamilton–Jacobi action. Note that to consider \( \psi_b(q) \) being a half-density in both \( b \) and \( q \) is even more natural:
This expression should be considered as the asymptotic of the kernel of the integral operator acting from the space of half-densities in $q$ to space of half-densities in $b$.

2.2. Hamiltonians which are polynomial in momentum

Now assume that the classical Hamiltonian $H(p, q)$ is a polynomial of order $n$ in momentum $p$ with coefficients which may depend analytically on $q$. Its quantization is an $n$th order self-adjoint differential operator $\hat{H}$ whose principal symbol is the classical Hamiltonian $H(p, q)$.

We want to describe its semiclassical eigenvectors, i.e. asymptotic behavior as $\hbar \to 0$ of square-integrable solutions to

$$\hat{H}(-i\hbar \frac{\partial}{\partial q}, q)\psi_b(q) = \tilde{b}\psi_b(q).$$

As in the previous section let us assume that the level curve

$$L_b = \{(p, q) | H(p, q) = b\}$$

of the classical Hamiltonian is connected and compact (for example, see figure 2). Otherwise we will have tunneling effect between components of WKB eigenfunctions supported on connected components of $L_b$, which deserves a separate discussion. From now on we will not indicate the difference between $\tilde{b}$ and $b$, assuming that this is clear.

Normalized $L_2$ eigen half-densities in the limit $\hbar \to 0$ have the following semiclassical asymptotic

$$\psi_b(q)\sqrt{|dqdb|} = C \sum_a \left|\frac{\partial p(a, b)}{\partial b}\right|^{\frac{1}{2}} \exp\left(\frac{i}{\hbar} S_{\gamma}(a, b) + \frac{\pi}{4} \mu_{\gamma}(a, \alpha_0)\right) (1 + O(\hbar)) \sqrt{|dqdb|}.$$  

Here $|C| = 1$ and the sum is taken over preimages of $q$ on branches of $L_b$ over $q$. The coefficients $\epsilon^n(a)$ are uniquely (up to a constant) determined by the equation (4) and can be computed recursively. We fixed a reference point $\alpha_0$ and a path $\gamma \subset L_b$ connecting each critical point with the reference point. The coefficient $\mu_{\gamma}(a, \alpha_0)$ is the difference between the number of negative and positive passings through turning points along $\gamma$ in the direction from $\alpha_0$ to $a$ (see the appendix). This number is also known as the Maslov index of $\gamma \subset L_b$.

The corresponding eigenvalue should satisfy the Bohr–Sommerfeld quantization condition:

$$\int_{L_b} \alpha = 2\pi \hbar (n + \frac{m}{2})(1 + O(\hbar)).$$
The integral is taken in the counter-clockwise direction with respect to the orientation on \( \mathbb{R}^2 \) given by the symplectic form \( dp \wedge dq \). Here \( m \) is the Maslov index of \( \mathcal{L}_b \) and \( m = 1/2 \) because all level curves are contractible in \( \mathbb{R}^2 \).

The Hamilton–Jacobi action function \( S_\gamma(a, b) \) is determined by the property \( \frac{\partial S_\gamma(a, b)}{\partial q} = p_a(q, b) \), where \( p_a(q, b) \) is the branch of the level curve over a neighborhood of \( q \) in the vicinity of the intersection point \( a \). It is given by the integral

\[
S_\gamma(a, b) = \int_{\gamma_{a,x_0}} \alpha_b.
\]

Here the form \( \alpha_b \) is the restriction of \( \alpha = pdq \) to \( \mathcal{L}_b \) and the path \( \gamma_{a,x_0} \) connects the reference point \( x_0 \) with \( a \) on \( \mathcal{L}_b \). In other words \( S_\gamma(a, b) \) is a generating function for \( \alpha \) restricted to \( \mathcal{L}_b \).

Bohr–Sommerfeld conditions ensure that the combination \( \frac{i}{\hbar} S_\gamma(a, b) + i \frac{\pi}{2} \mu_\gamma(a, x_0) \) does not depend on the choice of \( \gamma \) modulo \( 2\pi \mathbb{Z} \). Changing the reference point changes an overall constant \( C \).

When the Hamiltonian is not polynomial but level, the curve \( \mathcal{L}_b \) has finitely many fibers over each \( q \), the WKB eigenfunctions for eigenvalues in a neighborhood of \( b \) are given by the same expressions.

Observing that \( \frac{\partial p_a}{\partial b} = \frac{\partial^2 S_\gamma}{\partial q \partial b} \) we can write (5) as a section of the bundle of half-forms (see [9] for details) on \( B \times \mathbb{R} \):

\[
\psi_b(q) \sqrt{db \wedge dq} = \frac{C}{\sqrt{2\pi \hbar}} \sum_a \exp\left( \frac{\hbar}{i} S_\gamma(a, b) \right) (1 + O(h)) \sqrt{\frac{\partial^2 S_\gamma}{\partial q \partial b}} \, db \wedge dq.
\]

Here \( B = \{ b \in \mathbb{R} | H(p, q) = b, (p, q) \in T^* \mathbb{R} \} \) and the branch of the square root is determined by the Maslov index \( \mu_\gamma(a, x_0) \). Note that \( \frac{\partial^2 S_\gamma}{\partial q \partial b} \, db \wedge dq = dp \wedge dq \).

### 2.3. Two Hamiltonian systems on \( T^* \mathbb{R} \)

Let \( H_1 \) and \( H_2 \) be two Hamiltonian functions on \( T^* \mathbb{R} \approx \mathbb{R}^2 \). As above we will use coordinates \( p, q \). The standard symplectic form \( \omega = dp \wedge dq \) fixes an orientation on \( \mathbb{R}^2 \). Let \( H_1 \) and \( H_2 \) be self-adjoint differential operators quantizing \( H_1 \) and \( H_2 \), respectively.

The goal of this section is to describe the scalar product of normalized WKB eigen half-densities:

\[
\psi^{(k)}_{b_k}(q) \sqrt{|db_k dq|} = \frac{C_k}{(2\pi \hbar)^{1/2}} \sum_{a \in \mathcal{L}_k \cap \mathbb{R}^2} \exp \left( \frac{i}{\hbar} S_{\gamma(a,x_0^{(k)})}(a, b_k) + i \frac{\pi}{2} \mu_{\gamma}(a, x_0^{(k)}) \right) \sqrt{\frac{|\partial p^{(k)}_{a}(q, b_k)|}{\partial b_k}} (1 + O(h)) \sqrt{|db_k dq|}.
\]

Here \( k = 1, 2 \), \( b_k \in \mathbb{R}^k \) where \( B^k \) is the set of all possible values of \( H_k(p,q) \). Denote by \( \mathcal{L}_k = \{ (p,q) \in \mathbb{R}^2 | H_k(p,q) = b_k \} \) corresponding level curves of Hamiltonians. The sum in (6) is taken over fibers of \( \mathcal{L}_k \) over \( q \) and the rest of the notations is explained in the previous section. We made choices of reference points \( x_0^{(k)} \in \mathcal{L}_k \).

The scalar product of half-densities \( f = f(b_1, q) \sqrt{|db_1 dq|} \) and \( g = g(b_2, q) \sqrt{|db_2 dq|} \) on \( B_1 \times \mathbb{R} \) and \( B_2 \times \mathbb{R} \), respectively is the half-density on \( B_1 \times B_2 \) given by the integral over \( q \):
Here we integrate the density $\tilde{f} g = f(b_1, q)g(b_2, q)\,dq$ over $\mathbb{R}$.

**Proposition 1.** The scalar product of generic eigen half-densities of $\tilde{H}_1$ and $\tilde{H}_2$ has the following semiclassical asymptotic

$$
\langle \psi^{(2)}_{b_2}, \psi^{(1)}_{b_1} \rangle = C \sum_c \frac{1}{(2\pi \hbar)^{1/2}} \exp(\frac{i}{\hbar} S_{1,\gamma_1}(c, b_1, b_2) + \frac{\pi i}{2} \mu_{1,\gamma_1}(c, c_0)) \left| \frac{\partial^2 S_{1,\gamma_1}(c, b_1, b_2)}{\partial b_1 \partial b_2} \right|^{1/2} (1 + O(h)) \sqrt{|\mathrm{det} b_1 b_2|}.
$$

Here $|C| = 1$, the sum is taken over intersection points of $\mathcal{L}_1$ and $\mathcal{L}_2$, we made a choice of reference intersection point $c_0$ and $\mu_{1,\gamma_1}(c, x_0)$ is the number of points along $\gamma_1$ where $\gamma_1$ is parallel to level curves of $H^2$ and $S_{1,\gamma_1}(c, b_1, b_2) = S_{1,\gamma_1}(c, b_1) - S_{1,\gamma_1}(c, b_2)$. The exponent does not depend on the choice of $\gamma_1$ and $\gamma_2$ if the Bohr–Sommerfeld quantization conditions for $b_1$ and $b_2$ hold. The change of the reference point only changes the constant $C$.

Note that the difference $S_{1,\gamma_1}(c_1, b_1, b_2) - S_{1,\gamma_2}(c_2, b_1, b_2) = \int_{\gamma_1} \omega$ is the symplectic area of a disc which is bounded by curves $\gamma_1$ with its own orientation and by $\gamma_2$ taken with the opposite orientation between two intersection points $c_1$ and $c_2$. Because (7) is defined up to an arbitrary overall constant $C$, only such differences are important.

Here is an outline of the proof. We should evaluate the asymptotic of the integral

$$
\langle \psi^{(2)}_{b_2}, \psi^{(1)}_{b_1} \rangle = \frac{C \sqrt{\pi}}{2\pi \hbar} \sum_{a, c} \int_{q \in \mathbb{R}} \exp \left( \frac{1}{\hbar} S_{\gamma_1}(a, b_1) - S_{\gamma_1}(c, b_2) + \frac{i}{\hbar} \mu_{\gamma_1}(a, x_0^{(1)}) - \frac{i}{\hbar} \mu_{\gamma_1}(c, x_0^{(2)}) \right) \left| \frac{\partial p_1(a, b_1)}{\partial b_1} \frac{\partial p_2(c, b_2)}{\partial b_2} \right|^{1/2} (1 + O(h)) \sqrt{|\mathrm{det} b_1 b_2|}
$$

by the stationary phase method. Here $a \in T^*_q \mathbb{R} \cap \mathcal{L}^{(1)}_{b_1}$ and $c \in T^*_q \mathbb{R} \cap \mathcal{L}^{(2)}_{b_2}$.

Critical points of the exponent are solutions to the equation

$$
\frac{\partial S_{\gamma_1}(a, b_1)}{\partial q} = \frac{\partial S_{\gamma_1}(c, b_2)}{\partial q}
$$

which is equivalent to

$$
p_1(a, b_1) = p_2(c, b_2)
$$

where $(p^{(1)}(a, b_1), a)$ and $(p^{(2)}(c, b_2), c)$ are fibers over $q$ in $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Therefore, critical points of $S_{\gamma_1}(a, b_1) - S_{\gamma_1}(c, b_2)$ occur only when $a = c$ and they are intersection points of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Differentiating the equation $p_1(c, b_1) = p_2(c, b_2)$ we have

$$
\frac{\partial c}{\partial b_2} \frac{\partial p_2(c, b_2)}{\partial b_2} (c, b_2) + \frac{\partial p_2(c, b_2)}{\partial b_2} (c, b_2) = \frac{\partial c}{\partial b_2} \frac{\partial p_1(c, b_1)}{\partial b_1} (c, b_1).
$$

\[\text{Up to a constant, it is also equal to minus the number of points between } x_0 \text{ along } \gamma_2 \text{ where the level curves of } H_L \text{ are parallel to } \gamma_2. \text{ It is also known as Maslov index [13].}\]

\[\text{As it is shown in remark 4 the exponent is the generating function for local symplectomorphism near the intersection of point of } \mathcal{L}_1 \text{ and } \mathcal{L}_2 \text{ which bring one Lagrangian fibration to the other.}\]
For the second derivative of $S_{\gamma_1, \gamma_2}$ we have
\[
\frac{\partial^2 S_{\gamma_1, \gamma_2}(c, b_1, b_2)}{\partial b_1 \partial b_2} = \frac{\partial}{\partial b_2} \left( \int_{\gamma_1} \frac{\partial p_1}{\partial q}(c, b_1) \right) = \frac{\partial c}{\partial b_2} \frac{\partial p_1}{\partial b_1}(c, b_1).
\]

Combining these formulae we have the identity
\[
\frac{\partial^2 S_{\gamma_1, \gamma_2}(c, b_1, b_2)}{\partial b_1 \partial b_2} = \left( \frac{\partial p_1}{\partial q}(c, b_1) - \frac{\partial p_2}{\partial q}(c, b_2) \right)^{-1} \frac{\partial p_1}{\partial b_1}(c, b_1) \frac{\partial p_2}{\partial b_1}(c, b_2).
\]

Now we can compute the asymptotic of the integral by the stationary phase method:
\[
\langle \psi_{b_2}^{(2)}, \psi_{b_1}^{(1)} \rangle = \sum_{c} \frac{C_c C_{\gamma}}{(2\pi h)^{\gamma/2}} \exp \left( \frac{\pi i}{4} \text{sign} \frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) + \frac{i\pi}{2} \mu_{\gamma_1}(a, x_0^{(1)}) - \frac{i\pi}{2} \mu_{\gamma_2}(c, x_0^{(2)}) \right) \exp \left( \frac{i}{\hbar} (S_{\gamma_1}(c, b_1) - S_{\gamma_2}(c, b_2)) \right) \left| \frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) \right|^{1/2} (1 + O(h)) \sqrt{|\partial b_1 \partial b_2|}.
\]

One can show $\mu_{\gamma_1}(c, x_0^{(1)}) - \mu_{\gamma_2}(c, x_0^{(2)}) + \frac{i}{2} \text{sign} \frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) = \mu_{\gamma_1}(c, v_0) + A(c_0, x_0^{(1)}, x_0^{(2)})$. The constant $A(c_0, x_0^{(1)}, x_0^{(2)})$ does not depend on $c$ and is absorbed into an overall constant. Taking into account that $\frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) = \frac{\partial p_1}{\partial q}(c, b_1) - \frac{\partial p_2}{\partial q}(c, b_2)$ this gives the formula (7).

Note that (7) can also be written as section of the bundle of half-forms [9] over $B_1 \times B_2$
\[
\langle \psi_{b_2}^{(2)}, \psi_{b_1}^{(1)} \rangle = \sum_{c} \frac{C_c C_{\gamma}}{(2\pi h)^{\gamma/2}} \exp \left( \frac{i}{\hbar} S_{\gamma_1, \gamma_2}(c, b_1, b_2) \right) (1 + O(h)) \left| \frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) \right|^{1/2} \frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) \partial b_1 \wedge \partial b_2.
\]

Here the branch of the square root is determined by $\mu_{\gamma_1, \gamma_2}(c, v_0)$. Let $H_1^* \times H_2^* : \Omega^*(B_1 \times B_2) \to \Omega^*(\mathbb{R}^2)$ be the pullback of the projection to the space of level curves of $H_1$ and $H_2$. Then it is easy to check that at a smooth point
\[
H_1^* \times H_2^* \left( \frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial \tau^2}(c, b_1, b_2) \partial b_1 \wedge \partial b_2 \right) = \omega(c).
\]

**Remark 1.** It is also easy to see that
\[
\frac{\partial^2 S_{\gamma_1, \gamma_2}}{\partial b_1 \partial b_2}(c, b_1, b_2) = \{H_1, H_2\}(c)^{-1}.
\]

**Remark 2.** Scalar products $\langle \psi_{b_2}^{(2)}, \psi_{b_1}^{(1)} \rangle$ can be regarded as eigenfunctions of $\hat{H}_1$ when this operator is written in the basis of eigenvectors of $\hat{H}_2$ (or vice versa). If level curves of the second Hamiltonian are not compact, there is no Bohr–Sommerfeld quantization condition. When $H_2 = p$ this formula reduces to WKB eigenfunctions (5).
3. Semiclassical eigenfunctions for an integrable system on a cotangent bundle

3.1. An integrable system on a cotangent bundle

Recall that the cotangent bundle $T^*Q_n$ to a smooth manifold $Q_n$ has the natural symplectic structure which in local coordinates$^8$ is

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$  

This symplectic form is exact $\omega = d\alpha$ where $\alpha$ is a globally defined 1-form which in local coordinates is $\alpha = \sum_{i=1}^{n} p_i dq_i$.

The bundle projection $\pi: T^*Q_n \to Q_n$ is a Lagrangian fibration. The mapping $q \mapsto (0, q)$ is a section of this projection known as the zero-section. It gives an embedding of $Q_n \subset T^*Q_n$ as a Lagrangian submanifold.

Geometrically, an integrable system on a symplectic manifold $M_{2n}$ is a Lagrangian fibration. Algebraically this is a choice of a maximal Poisson commutative subalgebra in the algebra of functions on $T^*Q$. For example, it can be generated by $N$ Poisson commuting functions $f_1, \ldots, f_N$ of which only $n$ are linearly independent. The base of the fibration in this case is a variety $B$ defined by equations $P_1(f_1, \ldots, f_N) = P_2(f_1, f_N) = \cdots = 0$, where $P_1, \ldots, P_{N-n}$ are polynomials. The Lagrangian fibers in this case are level surfaces of $f_1, \ldots, f_N$.

For simplicity assume that we have $n$ independent functions $H^1, \ldots, H^n$ on $M_{2n}$ which Poisson commute. This defines Lagrangian fibration $H: M_{2n} \to B_N \subset \mathbb{R}^N, \ x \mapsto (H^1(x), \ldots, H^n(x))$ given by level surfaces of Hamiltonians $\{H\}$. Denote by $L_b$ the level surface

$$L_b = \{x \in M_{2n} | H^i(x) = b^i \}$$

i.e. the fiber of $H$ over $b$.

For generic $b^i$ the fiber $L_b$ is a Lagrangian submanifold. By the Liouville theorem such level surfaces are isomorphic to $T^k \times \mathbb{R}^{n-k}$ for some $k$, where $T^k$ is a $k$-dimensional torus.

Let $(p, q)$ be local Darboux coordinates on $M_{2n} = T^*Q_n$, such that $q^i$ are local coordinates on $Q_n$ and $p_i$ are coordinates on fibers. Let $c = (p_i(q, b), q)$ be the point on $L_b$ which projects on $q$. We will use abbreviated notation $p_i(c, b)$ instead of $p_i(q, b)$. Then on a tubular neighborhood of $L_b$ the symplectic form can be written as$^{10}$

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i = \sum_{i,j=1}^{n} \frac{\partial p_i(c, b)}{\partial b^j} db^j \wedge dq_i.$$

The pull-back of the one form

$^8$ Here $q^i$ are local coordinates on $Q_n$ and $p_i$ are corresponding coordinates on the fiber.

$^9$ For special $b$ it may degenerate and there is an interesting analysis of monodromies related to singularities, see for example [14, 3] and references therein, but we will ignore this here.

$^{10}$ The angle variables $\varphi_i$ are affine coordinates on level surfaces $L_b$ defined by the Hamiltonian flows generated by Hamiltonians $H_i$. They form a complementary set of coordinates to $b^i$ in a Darboux coordinate chart covering a neighborhood of $b$'s branch of $L_b$ and:

$$d\varphi_i = \sum_{j=1}^{m} \frac{\partial p_i(c, b)}{\partial b^j} db^j.$$  

So that $\omega = \sum_{i=1}^{m} db^i \wedge d\varphi_i$.
\[ \alpha = \sum_{i=1}^{n} p_i dq^i \]

to \(\mathcal{L}_b\) is closed because \(\mathcal{L}_b\) is Lagrangian and \(d\alpha = \omega\). Therefore, for a closed curve \(C \subset \mathcal{L}_b\) the integral \(\int_C \alpha\) does not depend on continuous deformations of \(C\) and is the pairing of the homology class of \(C\) with the de Rham cohomology class of \(\alpha\).

### 3.2. Quantization of the cotangent bundle

The algebra of differential operators on a smooth \(n\)-dimensional manifold \(Q\) forms a natural deformation quantization of the Poisson algebra \(\mathcal{O}(T^*Q)\) of functions which are smooth on \(Q\) and polynomial in cotangent directions.

Denote by \(D_h(Q)\) the algebra of differential operators on \(Q\) which locally have the form:

\[ D = \sum_{k_1, \ldots, k_n \geq 0} h^{k_1 + \cdots + k_n} v_{k_1, \ldots, k_n}(q) \partial_1^{k_1} \cdots \partial_n^{k_n} \]

where \(\partial_i = \frac{\partial}{\partial q^i}\) and \(v_{k_1, \ldots, k_n}(q)\) are some smooth functions. The space \(D_h(Q)\) is filtered by the degree of differential operators. Its associate graded space is naturally isomorphic to \(\mathcal{O}(T^*Q)\) with the grading being the degree of the polynomial in the cotangent direction. The multiplication of differential operators becomes the pointwise multiplication in \(\mathcal{O}(T^*Q)\) and the commutator becomes a Poisson bracket. This can be also written as

\[ D_1 D_2 = [D_1][D_2] + O(h), \quad D_1 D_2 - D_2 D_1 = h[[D_1], [D_2]] + O(h^2) \]

where \([D] \in \mathcal{O}(T^*Q)\) is the symbol of the differential operator \(D\).

Let \(f_1, \ldots, f_N\) be Poisson commuting functions on \(\mathcal{O}(T^*Q)\) defining an integrable system (with only \(n\) of them independent). A quantization of this integrable system is a collection of \(N\) Hermitian commuting differential operators \(\hat{H}_1, \ldots, \hat{H}_n\) of which only \(n\) are independent. For simplicity assume that \(Q\) is compact and that we have \(n\) independent commuting differential operators \(\hat{H}_1, \ldots, \hat{H}_n\).

### 3.3. Semiclassical eigenfunction for integrable systems on \(T^*Q_n\)

Assume we have an integrable system on \(T^*Q_n\), i.e. a Lagrangian fibration given by level curves of \(n\) Poisson commuting independent functions (as above).

From now on to avoid the discussion of tunneling effects we will assume that level surfaces \(\mathcal{L}_b\) are connected.

The projection \(\pi : \mathcal{L}_b \to Q_n\) is a branch cover over a sufficiently small neighborhood for generic \(q\). Choose a reference point \(x_0 = (p_0, q_0) \in \mathcal{L}_b\). Let \(a \in \mathcal{L}_b\) be a point in the fiber over \(q\). Define the action function \(S_\gamma(a, b)\) as a generating function for the form \(\alpha\) restricted to \(\mathcal{L}_b\), i.e. \(dS_\gamma(a, b) = \alpha\). It can be chosen as

\[ S_\gamma(a, b) = \int_{\gamma_{a,0}} \alpha. \quad (12) \]

Here the integration is taken along a path \(\gamma_{a,0}\) in \(\mathcal{L}_b\), connecting the point \(a\) and the reference point \(x_0 \in \mathcal{L}_b\).
The semiclassical asymptotic of normalized joint eigenfunctions (eigen-half-densities) of commuting quantum Hamiltonians $\hat{H}^i, \ i=1,\ldots,n$

$$\hat{H}^i \psi_b(q) = b^i \psi_b(q)$$

(13)
is well-known and is

$$\psi_b(q) = \frac{C}{(2\pi \hbar)^{\frac{1}{2}}} \sum_c \exp\left(\frac{i}{\hbar} S_c(c,b) + \frac{i}{2} \mu_\gamma(c,x_0)\right) \left|\det \left(\frac{\partial p_c(c,b)}{\partial b_j}\right)\right|^{\frac{1}{2}} \left(1 + \sum_{n>0} \hbar^n a_n(c,b)\right) \sqrt{|dbdq|}.\quad (14)$$

Here $|C| = 1$, the sum is taken over the fibers of the projection $\pi : L_b \to Q, \pi(p,q) = q$ over $q$, coefficients $a_n(c,b)$ can be computed recursively from (13) and $\sqrt{|dbdq|}$ is the Euclidean 1/2-density in local coordinates on $B \times Q_n$. The 1/2-density

$$\left|\det \left(\frac{\partial p_c(c,b)}{\partial b_j}\right)\right|^{\frac{1}{2}} \sqrt{|dbdq|}$$
is defined globally on $B \times Q$ and in particular, does not depend on the choice of local coordinates. In (14) we choose a base point $x_0 \in L_b$ and the path $\gamma$ as in the discussion above. The number $\mu_\gamma(c,x_0)$ (the appropriate Maslov index) is the oriented number of points on path $\gamma$ at which the tangent line (in $T(Q)$) lies in a tangent space to a fiber of $T^*Q \to Q$, i.e. to $T_{\pi(q)}(T^*Q)$. Changing $x_0$ will change the overall constant $C$.

The exponent does not depend on the choice of $\gamma$ if Bohr–Sommerfeld quantization conditions on $b$

$$\int_\beta \alpha_b = 2\pi \hbar (n_\beta + \frac{m(\beta)}{2})(1 + O(\hbar))$$

for each non contractible cycle $\beta \subseteq L_b$ with $n_\beta \in \mathbb{Z}$. Here $m(\beta)$ is the Maslov index of $\beta$.

Note that because

$$\frac{\partial p_c(c,b)}{\partial b_j} = \frac{\partial^2 S_c(c,b)}{\partial q^i \partial b^j}$$

the asymptotical eigen half-density (14) defines the following half-form on $B \times Q$:

$$\psi_b(q) = \frac{C}{(2\pi \hbar)^{\frac{1}{2}}} \sum_c \exp\left(\frac{i}{\hbar} S_c(c,b)\right) \left(1 + \sum_{n>0} \hbar^n a_n(c,b)\right) \sqrt{\left(\frac{\partial^2 S_c(c,b)}{\partial q^i \partial b^j} db^j \wedge dq^i\right)}.$$
3.4. Two integrable systems

Let $H_1$ and $H_2$ be corresponding Lagrangian fibrations by level surfaces of integrals of two integrable systems on $T^*Q$

$$H_1 : T^*Q \to B_1, \quad H_2 : T^*Q \to B_2.$$  \hfill (15)

Denote their fibers by $\mathcal{L}_{b_1}^{(1)}$ and $\mathcal{L}_{b_2}^{(2)}$, respectively. We assume that $H_1$ and $H_2$ are transversal Lagrangian fibrations, i.e. their generic fibers intersect transversally and, in particular, over finitely many points.

The formula for the semiclassical asymptotic of the scalar product of two WKB eigen half-densities corresponding $\mathcal{L}_{b_1}^{(1)}$ and $\mathcal{L}_{b_2}^{(2)}$ can be derived similarly to the one dimensional case discussed in the previous section.

$$\langle \psi_{b_1}^{(2)}, \psi_{b_1}^{(1)} \rangle = \frac{C}{(2\pi\hbar)^{n/2}} \sum_c \exp \left( \frac{i}{\hbar} S_{\gamma_1,\gamma_2}(c, b_1, b_2) + \frac{\pi i}{2} \mu_{\gamma_1,\gamma_2}(c, x_0) \right) \left| \det \left( \frac{\partial^2 S_{\gamma_1,\gamma_2}(c, b_1, b_2)}{\partial b_1 \partial b_2} \right) \right|^{1/2} \sqrt{|db_1 db_2| (1 + O(\hbar))}. \quad \hfill (16)$$

Here the sum is taken over intersection points of $\mathcal{L}_{b_1}^{(1)}$ and $\mathcal{L}_{b_2}^{(2)}$ and $\sqrt{|db_1 db_2|}$ is the Euclidean half-density on a coordinate neighborhood in $B_1 \times B_2$. We choose reference points $x_0^{(i)} \in \mathcal{L}_{b_i}^{(i)}$ and two paths, $\gamma_1$ connecting $x_0^{(1)}$ and $c$ in $\mathcal{L}^{(1)}$ and $\gamma_2$, connecting $x_0^{(2)}$ and $c$ in $\mathcal{L}^{(2)}$. The function $S_{\gamma_1,\gamma_2}(c, b_1, b_2)$ is defined as

$$S_{\gamma_1,\gamma_2}(c, b_1, b_2) = S_{\gamma_1}^{(1)}(c, b_1) - S_{\gamma_2}^{(2)}(c, b_2)$$

where $S_{\gamma_i}(c, b_i)$ are as in (14). Note that if $\gamma_1$ and $\gamma_2$ pass through two intersection points $a$ and $b$ we have

$$S_{\gamma_1,\gamma_2}(c_1, b_1, b_2) - S_{\gamma_1,\gamma_2}(c_2, b_1, b_2) = \int_{D_{\gamma_1,\gamma_2}} \omega$$

where $D_{\gamma_1,\gamma_2}$ is a disc with the boundary formed by segments of $\gamma_1$ and $\gamma_2$ between two intersection points $c_1$ and $c_2$. If Bohr–Sommerfeld quantization conditions hold, then, modulo $2\pi\hbar\mathbb{Z}$, this difference does not depend on paths. The integer $\mu_{\gamma_1,\gamma_2}(c, x_0)$ is the number of times when the tangent line to $\gamma_1$ inside the tangent plane to a level surface of $H^{(2)}$.

The exponent in (16) does not depend on choices of $\gamma_1$ and $\gamma_2$ if Bohr–Sommerfeld conditions hold for $\mathcal{L}^{(1)}$ and for $\mathcal{L}^{(2)}$.

Finally, the half-density (16) is globally defined. It is easy to see that it does not depend on local coordinates on $B_1 \times B_2$. Moreover

$$H_1^* \times H_2^* \left( \frac{\partial^2 S_{\gamma_1,\gamma_2}(c, b_1, b_2)}{\partial b_1 \partial b_2} \right) db_1 \wedge db_2 = \omega(c)$$

where $H_1^* \times H_2^* : T^*Q \to B_1 \times B_2$ is the product of the Lagrangian projections (15).

The proof of the formula (16) is similar to the one dimensional case. We should evaluate the asymptotic of the integral.

\footnote{As in the one dimensional case, this number, the Maslov index, is also equal to minus the number of times when the tangent plane to $\gamma_2$ is inside the tangent plane of a level surface of $H^{(1)}$.}
Applying the stationary phase method to these critical points we have

\[
\psi(2)_{b_j} \approx C_j \mathcal{C}_j \left( \frac{1}{2\pi h} \right)^n \sum \exp \left( \frac{i}{\hbar} (S_{b_j}(a, x_0) - S_{b_j}(c, x_0)) + \frac{i\pi}{2} h_{b_j}^{(1)}(a, x_0) - \frac{i\pi}{2} h_{b_j}^{(2)}(c, x_0) \right) \\
\left| \det \frac{\partial p^{(1)}(a, b_1)}{\partial b_1} \right| \left| \det \frac{\partial p^{(2)}(c, b_2)}{\partial b_2} \right|^{\frac{1}{2}} (1 + O(h)) \sqrt{|\det(\partial b_1 \partial b_2)|}
\]

as \( h \to 0 \) by the stationary phase method. Critical points of the exponent are solutions to the equation

\[
d_b S_{\gamma_1}^{(1)}(a, b_1) = d_b S_{\gamma_2}^{(2)}(c, b_2)
\]

which is exactly the equation \( \alpha_b(a) = \alpha_b(c) \). Its solutions are intersection points of two Lagrangian submanifolds \( L_{b_1}^{(1)} \cap L_{b_2}^{(2)} \). Applying the stationary phase method to these critical points we have

\[
\psi(2)_{b_j} \approx \frac{C_j \mathcal{C}_j}{(2\pi h)^n} \sum \exp \left( \frac{i}{\hbar} \left( S_{\gamma_1}^{(1)}(c, x_0) - S_{\gamma_2}^{(2)}(c, x_0) \right) + \frac{i\pi}{2} h_{\gamma_1}^{(1)}(a, x_0) - \frac{i\pi}{2} h_{\gamma_2}^{(2)}(c, x_0) \right) \\
+ \frac{i\pi}{4} \text{sign}(B(c, b_1, b_2)) \left| \frac{\det(\partial^2 S_{\gamma_1}^{(1)}/\partial q^2 \partial q^j)}{\det(B(c, b_1, b_2))} \right|^{\frac{1}{2}} (1 + O(h)) \sqrt{|\det(\partial b_1 \partial b_2)|}
\]

where

\[
B(c, b_1, b_2) = \partial^2 S_{\gamma_1}^{(1)}/\partial q^2 \partial q^j - \partial^2 S_{\gamma_2}^{(2)}/\partial q^2 \partial q^j = \partial p^{(1)}(c, b_1) - \partial p^{(2)}(c, b_2).
\]

For intersection points of \( L_{b_1}^{(1)} \) and \( L_{b_2}^{(2)} \) we have \( p^{(1)}(c, b_1) = p^{(2)}(c, b_2) \). Differentiating this identity in \( b_2 \) we obtain:

\[
\sum_k \frac{\partial c^k}{\partial b_2^j} \frac{\partial p^{(1)}(c, b_1)}{\partial c^k} = \sum_k \frac{\partial c^k}{\partial b_2^j} \frac{\partial p^{(2)}(c, b_2)}{\partial c^k} + \frac{\partial p^{(2)}(c, b_2)}{\partial b_2^j}.
\]

On the other hand,

\[
\frac{\partial^2 S_{\gamma_1, \gamma_2}^{(1)}(c, x_0)}{\partial b_1^j \partial b_2^k} = \frac{\partial}{\partial b_2^j} \int \sum_k \frac{\partial p^{(1)}(q, b_1)}{\partial q^k} dq^k = \sum_k \frac{\partial c^k}{\partial b_2^j} \frac{\partial p^{(1)}(c, b_1)}{\partial b_2^k}.
\] (17)

The first identity implies

\[
\frac{\partial c}{\partial b_2^j} = \left( \frac{\partial p^{(1)}(c, b_1)}{\partial c} - \frac{\partial p^{(2)}(c, b_2)}{\partial c} \right)^{-1} \frac{\partial p^{(2)}(c, b_2)}{\partial b_2^j}.
\]

Substituting this into (17) we obtain

\[
\frac{\partial^2 S_{\gamma_1, \gamma_2}^{(1)}(c, x_0)}{\partial b_1 \partial b_2} = \frac{\partial p^{(1)}(c, b_1)}{\partial b_1} \left( \frac{\partial p^{(1)}(c, b_1)}{\partial c} - \frac{\partial p^{(2)}(c, b_2)}{\partial c} \right)^{-1} \frac{\partial p^{(2)}(c, b_2)}{\partial b_2}.
\] (18)

The complete proof, with analytical details, and with details on the Maslov indices will be given elsewhere.
Remark 3. Let us show that
\[
\frac{\partial^2 S_{\gamma_1,\gamma_2}}{\partial b_1 \partial b_2 ^2}(c, x_0) = (\{H_1, H_2 \})^{-1}
\]  
(19)
where \((\{H_1, H_2 \})\) is the matrix of Poisson brackets.

Indeed, denote \(A_{ki} = \frac{\partial \phi_i}{\partial q_k}\) and \(B_{ki} = \frac{\partial \phi_i}{\partial p_k}\). We have:
\[
\{H_1', H_2' \}(c) = \frac{\partial H_1'}{\partial p_k}(c) \frac{\partial H_2'}{\partial q^k}(c) - \frac{\partial H_1'}{\partial q^k}(c) \frac{\partial H_2'}{\partial p_k}(c)
\]
\[
= (A^{-1})^k \frac{\partial H_1'}{\partial q^k}(c) - (B^{-1})^k \frac{\partial H_2'}{\partial q^k}(c)
\]
\[
= (A^{-1})^k (B^{-1})^k \left( \frac{\partial p_1^{(2)}}{\partial b_2} \frac{\partial H_2^n}{\partial q^k} - \frac{\partial p_1^{(2)}}{\partial q^k} \frac{\partial H_2^n}{\partial b_2} - \frac{\partial p_2^{(1)}}{\partial c} \frac{\partial H_1^n}{\partial q^k} \right)
\]
\[
= (A^{-1})^k (B^{-1})^k \left( \frac{\partial p_1^{(2)}}{\partial q^k} \frac{\partial H_2^n}{\partial b_2} - \frac{\partial p_2^{(1)}}{\partial c} \frac{\partial H_1^n}{\partial q^k} \right).
\]

Here \((A^{-1})^k = \frac{\partial H_1'}{\partial p_k}\) and \((B^{-1})^k = \frac{\partial H_2'}{\partial q_k}\). Now, take into account that the form \(\alpha = \sum \phi_i(c, b) dq^k\) is closed (since \(L_1\) is a Lagrangian submanifold). Therefore \(\frac{\partial \phi_i}{\partial q^k} = \frac{\partial \phi_i}{\partial c}\). Together with the formula above this gives
\[
\{H_1', H_2' \}(c) = (A^{-1})^k (B^{-1})^k \left( \frac{\partial p_1^{(2)}}{\partial q^k} \frac{\partial H_2^n}{\partial b_2} - \frac{\partial p_2^{(1)}}{\partial c} \frac{\partial H_1^n}{\partial q^k} \right).
\]

Combining this formula with (18) we obtain (19).

Remark 4. Let \(c \in L^{(1)}_{b_1} \cap L^{(2)}_{b_2}\) and \(U_1 \subset B_1, U_2 \subset B_2\) be open neighborhoods of \(\pi_1(c)\) and \(\pi_2(c)\), respectively. Then we have natural symplectomorphisms \(\pi_1^{-1}(U_1) \simeq W_1 \subset T^* U_1\) and \(\pi_2^{-1}(U_2) \simeq W_2 \subset T^* U_2\). Let \(\phi^{(1)}\) are affine coordinates (the angle variables) on fibers of \(M \rightarrow B_1\) generated by coordinates \(b_1\) and \(\phi^{(2)}\) are the angle variable corresponding to coordinates \(b_2\) on \(B_2\). In these coordinates
\[
\omega = db_1 \wedge d\phi^{(1)} = db_2 \wedge d\phi^{(2)}.
\]
Let \(\varphi : W_1 \rightarrow W_2\) be the natural symplectomorphism mapping \((b_1, \phi^{(1)}) \mapsto (b_2, \phi^{(2)}).\)

Theorem 1. The function \(S_{\gamma_1,\gamma_2}(c, b_1, b_2)\) is the generating function of this symplectomorphism.

We need to prove that
\[
\phi_i^{(1)} = \frac{\partial S_{\gamma_1,\gamma_2}(c, b_1, b_2)}{\partial b_i}, \quad \phi_i^{(2)} = -\frac{\partial S_{\gamma_1,\gamma_2}(c, b_1, b_2)}{\partial b_i^2}.
\]
Assume \(M = T^* Q\) (if not, choose appropriate local Darboux coordinates) and \(\alpha = pdq\). Let \((p, q)\) be coordinates of \(c \in L^{(1)}_{b_1} \cap L^{(2)}_{b_2}\). Then
\[ \frac{\partial S_{\gamma, \gamma}}{\partial b_1} = - \int q_j \frac{\partial p_j^{(1)}}{\partial b_1}, \quad \frac{\partial S_{\gamma, \gamma}}{\partial b_2} = \int p_j \frac{\partial p_j^{(2)}}{\partial b_2}. \]

Here \( \{ \phi_j^{(1)} \} \) are angle coordinates on \( \mathcal{L}^{(1)} \), corresponding to coordinates \( \{ b_i \} \) on \( B_1 \). Here we used the fact that \( c \in \mathcal{L}^{(1)} \cap \mathcal{L}^{(2)} \) and therefore \( p_i^{(1)}(b_1, q_c) = p_i^{(2)}(b_2, q_c) \). Similarly

\[ \frac{\partial S_{\gamma, \gamma}}{\partial b_1} = \int p_j \frac{\partial p_j^{(2)}}{\partial b_1} = \phi_j^{(2)}. \]

This proves the theorem.

4. Conclusion: some open problems and conjectures

4.1. General symplectic manifolds

Details of proofs and analytical aspects of the semiclassical asymptotic of eigen half-densities will be given in a separate publication. In this concluding section we will give some related conjectures and observations.

Let \( (M, \omega) \) be a symplectic manifold. Fix geometric quantization data which consist of the following:

- A line bundle \( L \) (a prequantization line bundle) with Hermitian structure on fibers and a Hermitian connection \( \alpha \) on \( L \) such that the symplectic form \( \omega \) is the curvature of \( \alpha \), i.e. \( d\alpha = \omega \) and the Hermitian product is covariantly constant.
- A real polarization \( P \subseteq TM \) which is an integrable tangent distribution on \( M \) such that each generic leaf is a Lagrangian submanifold in \( M \). We assume that the space of leaves \( B = M/P \) is almost always smooth, i.e. that the polarization is a Lagrangian fibration \( \pi : M \rightarrow B \) (generic fibers are Lagrangian).

The space of geometric quantization \( H_p^{(1/2)} \) is the space of half-densities on \( M \) which are covariantly constant (with respect to the connection \( \alpha \)) along \( P \). Locally it can be identified with functions on \( M/P \).

Let \( C_b(M) \) be the quantized algebra of functions on \( M \). Assume it acts on the space \( H_p^{(1/2)} \). See [22] for the microlocal setting where \( h \) is a formal variable.

A classical integrable system on \( M \) is a Lagrangian fibration\(^{13}\) \( \pi : M \rightarrow B \) which defines Poisson commuting subalgebra \( C(M, B) \subset C(M) \) in the algebra of functions on \( M \), \( C(M, B) = \pi^*(C(B)) \).

Assume that the subalgebra \( C(M, B) \) is quantized, i.e. deformed into a maximal commutative subalgebra \( \mathcal{C}_b(M, B) \subset \mathcal{C}_b(M) \). For 2\( n \)-dimensional \( M \) such subalgebra has rank \( n \).

Now, assume that we have a real polarization \( P \) with the corresponding Lagrangian fibration \( \pi_1 : M \rightarrow B_1 = M/P \) and an integrable system corresponding to the Lagrangian fibration \( \pi_2 : M \rightarrow B_2 \). Assume that fibers of both projections are generically transverse. Assume that the algebra \( \mathcal{C}_b(M) \) acts on the space \( \mathcal{H}_p^{(1/2)} \).

\(^{13}\)The generic fibers are Lagrangian submanifolds.
We will say a vector \( \psi_\chi \in H_p^{(1/2)} \) is an eigenvector of \( C_b(M, B_2) \) corresponding to the character \( \chi : C_b(M, B) \to \mathbb{R} \) if \( a \psi_\chi = \chi(a) \psi_\chi \) for any \( a \in C_b(M, B_2) \). Semiclassically, as \( h \to 0 \) the set of characters can be identified with \( B_2 \).

**Conjecture 1.** Semiclassical asymptotic of an eigen half-density of \( C_b(M, B_2) \) in the space \( H_p^{(1/2)} \) have the following structure:

\[
\psi(b_1)_{b_2} = \frac{C}{(2\pi h)^{n/2}} \sum_{c \in L_b^{(1)} \cap L_b^{(2)}} e^{i S_{\gamma_1 \gamma_2} (c; b_1, b_2) + i \sum_{\gamma_1 \gamma_2} c \mu_{\gamma_1 \gamma_2} (c)} \det \left( \frac{\partial^2 S_{\gamma_1 \gamma_2} (c; b_1, b_2)}{\partial b_1^i \partial b_2^j} \right)^{1/2} \sqrt{|db_1 db_2|}.
\]

(20)

Here as in (16) we made a choice of reference points \( x_0^{(i)} \in L_b^{(i)}, s_{\gamma_1 \gamma_2} = S^{(1)}_{\gamma_1} - S^{(2)}_{\gamma_2} \) where \( S^{(i)}_{\gamma} = \int_{c \in L_b^{(i)}} \alpha \) and \( \alpha \) is the prequantization connection, \( \gamma_1 \) and \( \gamma_2 \) are paths in \( L_b^{(1)} \) and \( L_b^{(2)} \), respectively, connecting corresponding reference points and \( c \). Lagrangian submanifolds \( L_{b_1}^{(1)} \) and \( L_{b_2}^{(2)} \) are fibers over \( b_1 \in B_1 \) and \( b_2 \in B_2 \), respectively. We assume that \( L_{b_1}^{(1)} \) and \( L_{b_2}^{(2)} \) are transverse. The number \( \mu_{\gamma_1 \gamma_2} (c) \) is the corresponding Maslov index. It is equal to the weighted number of points along \( \gamma_2 \) between \( x_0 \) and \( c \) where the tangent space to \( L_{b_2}^{(2)} \) intersect a fiber of \( \pi_1 \) over a line. The point counts with plus if it is crossed in the positive direction and with the minus if it is crossed in the negative direction. When quantization conditions for \( b_1 \) and \( b_2 \) hold, the exponent does not depend on the choice of \( \gamma_1 \) and \( \gamma_2 \).

Note that if \( M \) is the tangent bundle, this formula is equivalent to the formula for the scalar product of eigenfuctions of two integrable systems considered in previous sections.

### 4.2. Blattner–Kostant–Steinberg kernels and topological quantum mechanics

Let \( P_1 \) and \( P_2 \) be two real polarizations and \( H_p^{(1/2)} \) and \( H_p^{(1/2)} \) be two corresponding quantization spaces of half-densities with the algebra \( C_b(M) \) acting on them. We can naturally associate a classical integrable system on \( M \) with each polarization. Assume they both have quantizations \( C_b(M, B_2) \) where \( B_i = M/P_i \).

Let \( P \) be a third polarization. This polarization gives the representation space \( H_p^{(1/2)} \). It is clear that all three spaces should be isomorphic as representations of \( C_b(M) \)\(^{14} \)

Scalar products of eigen half-densities for integrable systems corresponding to \( P_1 \) and \( P_2 \) define the unitary linear map \( U_{P_1, P_2} : H_p^{(1/2)} \to H_p^{(1/2)} \). The formula (20) describes the semiclassical asymptotic of the integral kernel of this linear operator. Such integral kernels have been studied in the context of geometric quantization as well, and are known as Blattner–Kostant–Steinberg kernels. In terms of half-forms such kernel asymptotically can be written as

\[
U_{P_1, P_2}(b_1, b_2) = \frac{C}{(2\pi h)^{n/2}} \sum_{c \in L_b^{(1)} \cap L_b^{(2)}} e^{i \int_{b_1, b_2} \omega \mu \sqrt{|db_1 db_2|}} (1 + O(h)).
\]

(21)

The sign of the square is determined by the Malsov index \( \mu \) described above. This formula can be regarded as the quantization of the symplectomorphisms \( \varphi \) from the remark 4. The exponent is exactly the generating function of the mapping \( \varphi \).

\(^{14}\) Strictly speaking in such a general setting we should conjecture this.
The composition law of these integral operators satisfies the semigroup law and involves the Maslov index. Composing the semiclassical kernels (21) involves formal integration over the base of intermediate fibration and is a formal Gaussian computation. The details will be given in a separate publication.

One can argue that this formula corresponds to topological quantum mechanics and can be written as the path integral

$$U_{P_1,P_2}(b_1,b_2) = \int_{\gamma(0) \in \mathcal{L}_{b_1}(1)} e^{i f_{b_1}(\alpha) + f_{b_2}(1) - f_{b_2}(1)} D\gamma$$

where $f_{b_1}$ and $f_{b_2}$ are boundary contributions, defined, up to a constant, by the property $df_{b_\alpha} = \omega_\alpha$ where $\omega_\alpha : \mathcal{L}_{b_\alpha} \rightarrow \mathcal{M}$ are natural inclusions. The semiclassical expansion in all orders can be described in terms of the Poisson sigma model. This is work in progress [5].

### 4.3. Geometric asymptotic of 6j-symbols

An example of the formula (20) describes the Ponzano–Regge asymptotic of the Racah–Wigner coefficients, also known as 6j-symbols. For details see [17, 20].

Fix a root decomposition for the Lie algebra $\mathfrak{su}(2)$. Let $\mathcal{O}_s \subset \mathfrak{su}(2)^*$ be the coadjoint orbit for $SU(2)$ passing through the element $s$ of the dual space $\mathfrak{h} \subset \mathfrak{su}(2)^*$ to the Cartan subalgebra. Clearly $\mathcal{O}_s = \mathcal{O}_{-s}$. Define the moduli space

$$\mathcal{M}_{s_1,s_2,s_3,s_4} = \{ (x_1,x_2,x_3,x_4) | x_1 \in \mathcal{O}_s, x_1 + \cdots + x_4 = 0 \} / SU(2).$$

Here the quotient is taken with respect to the diagonal action of $SU(2)$ on the product of orbits. When $s_1, \ldots, s_4$ satisfy triangle inequalities this space is not empty and $\text{dim}(\mathcal{M}_{s_1,s_2,s_3,s_4}) = 2$. In this case it is compact, and almost always smooth. It is clear that the space $\mathcal{M}_{s_1,s_2,s_3,s_4}$ depends only on the equivalence classes $s_i \rightarrow -s_i$ and therefore only on lengths $l_i = |s_i|$ with respect to the metric induced by the Killing form $(-,-)$ on $\mathfrak{su}(2)^*$.

Level curves of functions

$$H_{12} = (x_1,x_2), \quad H_{23} = (x_2,x_3)$$

define Lagrangian fibrations with fibers

$$\mathcal{L}_{l_2}^{(1)} = \{ (x_1,x_2,x_3,x_4) | x_1 \in \mathcal{O}_s, x_1 + \cdots + x_4 = 0, x_1 + x_2 \in \mathcal{O}_{s_2} \} / SU(2)$$

$$\mathcal{L}_{l_3}^{(2)} = \{ (x_1,x_2,x_3,x_4) | x_1 \in \mathcal{O}_s, x_1 + \cdots + x_4 = 0, x_2 + x_3 \in \mathcal{O}_{s_3} \} / SU(2)$$

here $l_{12} = |s_{12}|$ is the length of $s_{12}$ and $l_{23} = |s_{23}|$ is the length of $s_{23}$ in the metric defined by the Killing form. Generic level curves of $H_{12}$ and $H_{23}$ intersect at two points.

The space $V_j$ of an irreducible representation of $\mathfrak{su}(2)$ can be regarded as the space of geometric quantization of $\mathcal{O}_s$. In the semiclassical limit $h \rightarrow 0, |s| = \lim(hj) \rightarrow \infty$. Similarly, the space of $SU(2)$-invariant vectors $V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4}^{SU(2)} \subset V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4}$ can be regarded as the space of geometric quantization of $\mathcal{M}_{s_1,s_2,s_3,s_4}$ [17, 20]. In the semiclassical limit $hj \rightarrow |s|$, while $h \rightarrow 0$. Casimir operators acting in $V_{j_1} \otimes V_{j_2}$, and in $V_{j_2} \otimes V_{j_3}$ are quantizations of $H_{12}$ and $H_{23}$, respectively. Denote their Eigen half-densities as $\psi_{l_2}$ and $\psi_{l_3}$, respectively.

The Racah–Wigner coefficients, also known as 6j-symbols, are scalar products $(\psi_{l_2}, \psi_{l_3})$ with respect to the natural scalar product in $V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4}$. The semiclassical asymptotic
of these scalar products was computed using geometric methods in [17] and is an example of (20):

\[
\langle \psi^j_{12}, \psi^j_{23} \rangle \sim \frac{C}{\sqrt{2\pi \hbar}} \left| \frac{\partial^2 S(a, b)}{\partial l_{12} \partial l_{23}} \right|^{1/2} \cos \left( \frac{1}{\hbar} \int_{D(a, b)} \omega + \frac{\pi}{4} \right) \left( 1 + O(\hbar) \right) \sqrt{|d_{12} d_{23}|}.
\]

Here \( C \) is an arbitrary constant and \( D(a, b) \) is the disc bounded by arcs of \( L_{12}^{(12)} \) and \( L_{23}^{(23)} \) confined between intersection points \( \{a, b\} = L_{12}^{(12)} \cap L_{23}^{(23)} \). For details see [17]. All quantities in this formula can be computed explicitly in terms of the geometry of tetrahedra in three-dimensional Euclidean space [16]. For more details about the semiclassical asymptotic of Racah–Wigner symbols see [12].

### 4.4. Other simple Lie algebras

For simple Lie algebras other than \( sl_2 \) the decomposition of the tensor product of two irreducible representations typically has multiplicities (for a discussion of \( 6j \)-symbols with multiplicities see for example [23]). But there are special cases of multiplicity free \( 6j \)-symbols. One such example is the tensor product of a generic finite dimensional irreducible \( j \)-module and an irreducible representation with the highest weight \( m \omega_1 \) (orth symmetric power of the vector representation of \( sl_n \)). The semiclassical limit of such multiplicity-free \( 6j \)-symbols is described by the formula (20) and by the geometry of corresponding moduli space.

Let \( g = sl_n \) be a compact real form of complex Lie algebra \( sl_n \). Consider four coadjoint orbits \( O_1, \ldots, O_4 \subset g^* \) with \( O_1 \) and \( O_3 \) being of rank 1 (orbits corresponding to irreducible \( su(n) \) modules with highest weight \( m \omega_1 \)). The symplectic manifold

\[
\mathcal{M}(O_1, \ldots, O_4) = \{ x_i \in O_i | x_1 + x_2 + x_3 + x_4 = 0 \} / Ad_G
\]

is the symplectic reduction of the product of symplectic manifold \( O_1 \times \cdots \times O_4 \) with respect to the diagonal action of \( G \).

It is easy to check that the dimension of \( \mathcal{M}(O_1, \ldots, O_4) \) is \( 2n - 2 \). There are two natural systems of Poisson commuting Hamiltonians \( H_{12}^{(k)} = tr((x_1 + x_2)^k) \) and \( H_{23}^{(k)} = tr((x_2 + x_3)^k) \) where \( k = 1, \ldots, n - 1 \). Thus, we have two integrable systems. One can show that generic fibers are transversal and intersect at \( n! \) points.

The semiclassical asymptotic of \( 6j \) symbols in this case is given by the formula (20). This particular multiplicity free case of \( q - 6j \) symbols is important for the computation of the HOMFLY polynomial in the semiclassical limit. The combination of the semiclassical limit and the limit \( n \to \infty \) was discussed in [1].

### 4.5. \( q - 6j \) symbols

The associativity of the tensor product of representations of \( U_q(g) \) in the basis of irreducible components is given by \( q - 6j \) symbols (see for example [23]). For \( sl_2 \) this asymptotic was computed in [20] using the difference equation which generalizes the computation by Ponzano and Regge.

One should note that this computation is correct only when the signs of the coefficients in this difference equation are suitably stable as \( \hbar \to 0 \). In this case solutions to the difference equation converge, in the appropriate analytical sense, to solutions of the corresponding differential equation. For \( q - 6j \) symbols this means that \( q = \exp(\frac{2\pi i}{n}) \) and \( n \to \infty \). The
number of irreducible representations of $U_q(sl_2)$ in this case is $n$. This case corresponds to the Chern–Simons theory and to the Wess–Zumino unitary conformal field theory.

When $q$ is another root of unity of degree $n$ the analysis based on the difference equation does not work and, as it was shown in [7, 8], the asymptotic is not of an oscillatory type. This agrees with the fact that for such roots of unity the corresponding conformal field theory is not unitary and does not correspond to Chern–Simons theory based on a compact simple Lie group.

When $q = \exp\left(\frac{2\pi i}{n}\right)$ the asymptotic of $q^{-6j}$ symbols can be expressed in terms of the geometry of conjugation orbits for $SU(2)$ and further in terms of the geometry of spherical tetrahedra [20]. For other Lie groups such asymptotic can be naturally computed in terms of cluster variables for conjugation orbits [18]. The semiclassical asymptotic of $q^{-6j}$ symbols will be analyzed in greater detail in a separate paper.

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Appendix. Turning points

For an integrable system on $T^*Q$ let $L_b = H^{-1}(b)$ be the energy surface of a complete set of Hamiltonians. A point $(p_0, q_0) \in L_b$ is simple critical if the intersection of the tangent plane to $L_b$ at this point with the tangent space to $T^*_{q_0}Q$ is a line. In this case $q_0 \in \mathbb{N}$ is called a simple turning point.

We assume that simple critical points form a submanifold of dimension $n - 1$ in $L_b$ which is generically smooth (i.e. that $L_b$ is sufficiently generic).

A.1. One dimensional case

Here we will recall the basic textbook derivation of Maslov indices $\mu$ in the semiclassical formula for eigen half-densities.

Let $\hat{H} = H_0(-i\hbar \frac{\partial}{\partial q}, q)$ be the differential operator with the principal symbol $H(p, q)$ and $(p_0, q_0)$ be a simple turning point on $L_b = \{(p, q) \in T^*_{q_0}Q | H(p, q) = b\}$. Near this point we have two branches of the level curve $p(q, b) = p_0 \pm \sqrt{|\alpha||q - q_0| + \ldots}$ where

$$\alpha = -\frac{\partial H}{\partial q}(p_0, q_0).$$

Possible types of simple turning points are shown on figure A1.

An eigenfunction of $\hat{H}$ with the eigenvalue $\hat{b}$ has the following asymptotical behaviour when $q \to q_0$ and $h \to 0$:

$$\psi_b(q) = e^{i\mu(\hat{b})_q} \phi(q - q_0) \left(1 + o(1)\right)$$

where $\phi(x)$ is an Airy function, i.e. it is a solution to the differential equation

$$\frac{d^2}{dx^2} \phi(x) - x \phi(x) = 0.$$
Here $\alpha$ is as above. The proof of this fact can be found in various textbooks.

**A.1.1. Airy functions.** These functions are solutions to the differential equation

\[ (-\frac{1}{2} \frac{d^2}{dx^2} + \alpha x)\psi(x) = 0. \]

They are given by contour integrals

\[ \psi_C(x) = \int_C e^{i(x \xi + \frac{\xi^3}{3})} d\xi. \]

Naturally, the integral depends only on the class of continuous deformations of $C$. 

---

**Figure A1.** Generic turning points $a;b;c;d$ corresponding to $\frac{\partial H}{\partial q} < 0, \frac{\partial^2 H}{\partial p^2} > 0$; $\frac{\partial H}{\partial q} < 0, \frac{\partial^2 H}{\partial p^2} < 0$; $\frac{\partial H}{\partial q} > 0, \frac{\partial^2 H}{\partial p^2} < 0$; $\frac{\partial H}{\partial q} > 0, \frac{\partial^2 H}{\partial p^2} > 0$, respectively.

**Figure A2.** The integration contour $C_a$ for $\alpha > 0$. The angle between the asymptote and the x-axis is $\frac{\pi}{3}$. 
1. $\alpha > 0$. The integral is convergent if the integration contour is approaching to infinity along the rays

$$ k = -it, \; k = -i\omega, \; k = -i\omega^{-1} $$

as $t \to +\infty$.

Its asymptotic when $|x| \to \infty$ is determined by critical points which for $\alpha > 0$ are:

$$ k_{1,2} = \pm i\sqrt{\alpha x}, \; x \to \infty,$$

$$ k_{1,2} = \pm \sqrt{\alpha x}, \; x \to -\infty.$$

The solution which is oscillatory asymptotic when $x \to -\infty$ and exponentially decaying when $x \to \infty$ correspond to the contour $C_+$, shown in figure A2.

The steepest descent method gives the following asymptotic of this function when $x \to \infty$ as the contribution from the critical point $k_1$:

$$ \psi_{C_+}(x) = \sqrt{2\pi} \left( \frac{\alpha}{2x} \right)^{\frac{1}{4}} e^{-\frac{i}{2} \sqrt{2\alpha x}^2} (1 + O(\frac{1}{x})). $$

The critical point $k_2$ does not contribute. When $x \to -\infty$ both critical points contribute and the solution $\psi_{C_-}$ has an oscillatory asymptotic:

$$ \psi_{C_-}(x) = \sqrt{2\pi} \left( \frac{\alpha}{2|x|} \right)^{\frac{1}{4}} (e^{-i\frac{2}{3} \sqrt{2\alpha x}^2 + i\frac{\pi}{4}} + e^{i\frac{2}{3} \sqrt{2\alpha x}^2 - i\frac{\pi}{4}})(1 + O(\frac{1}{|x|})). $$

2. Similarly for $\alpha < 0$, the contour of integration $C_-$, figure A3 gives the solution which exponentially decays when $x \to -\infty$ and has an oscillatory asymptotic when $x \to \infty$:

$$ \psi_{C_+}(x) = \sqrt{2\pi} \left( \frac{|\alpha|}{2x} \right)^{\frac{1}{4}} (e^{-i\frac{2}{3} \sqrt{2|\alpha|}^2 + i\frac{\pi}{4}} + e^{i\frac{2}{3} \sqrt{2|\alpha|}^2 - i\frac{\pi}{4}})(1 + O(\frac{1}{x})). $$

as $x \to \infty$ and

\[ Figure A3. \] The integration contour $C_-$ for $\alpha < 0$. The angle between the asymptote and the x-axis is $\xi$. 

\[ Figure A2. \] The integration contour $C_+$ for $\alpha > 0$. The angle between the asymptote and the x-axis is $\xi$. 

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\[
\psi_{c_+}(x) = \sqrt{2\pi} \left( \left| |x| \right| \right)^{\frac{1}{2}} e^{-\frac{i}{2} \sqrt{2|\alpha||x|}} (1 + O\left(\frac{1}{|x|}\right))
\]
as \(x \to -\infty\).

**A.1.2. Phase change at turning points.** From the asymptotic analysis as \(h \to 0\) of solutions to the eigenvalue problem

\[
\hat{H}\psi_b(q) = \hat{b}\psi_b(q),
\]
it is easy to derive that if \(\hat{b} \to b \in B\) we have

\[
\psi_b(q) = \sum_{a \in L_0 \cap T^*_q Q} C_{\gamma(a,a_0)} \left| \frac{\partial p(a,b)}{\partial q} \right| e^{iS_{\gamma(a,a_0)}(q,b)} (1 + O(h)). \tag{A.1}
\]

Here \(x_0\) is a reference point on \(L_0\). Since \(\phi_b(q)\) is defined up to a multiplication by an arbitrary constant the choice of \(x_0\) is not important and only the ratios \(C_{\gamma(a,a_0)} / C_{\gamma(c,c_0)}\) are important.

Now let us compute the ratios \(C_{\gamma(a,a_0)} / C_{\gamma(c,c_0)}\) by comparing the WKB asymptotic and the Airy asymptotic. Let \((p_0, q_0)\) be a simple turning point. Assume that \(q \to q_0\). We have two branches \(p_+(q, b) > p_-(q, b)\) with \(p_\pm(q_0, b) = p_0\) of the level curve \(H(p, q) = b\) over \(q\). For small \(q\), \(p(q, b) = p_0 \pm \sqrt{|\alpha||q - q_0| + \ldots}\). In the neighborhood of this turning point the exponential function in (5) corresponding to these branches behaves as

\[
S_\pm(q, b) = S(q_0) + p_0(q - q_0) \mp \frac{2}{3} \sqrt{2|\alpha||q - q_0|^{3/2}} + \ldots \tag{A.2}
\]
for \(\alpha > 0\) and \(q < q_0\) and

\[
S_\pm(q, b) = S(q_0) + p_0(q - q_0) \pm \frac{2}{3} \sqrt{2|\alpha||q - q_0|^{3/2}} + \ldots \tag{A.3}
\]
for \(\alpha < 0\), \(q > q_0\).

For the prefactors in (5) when \(q \to q_0\) we have

\[
\left| \frac{\partial p_\pm}{\partial q} \right| = \text{const} |q - q_0|^{1/4}(1 + o(1)).
\]

Let us isolate contributions from the neighborhood of the turning point \(q_0\) to the WKB asymptotic:

\[
\psi_b(q) = \left| \frac{\partial p_+(q, b)}{\partial b} \right|^\frac{1}{2} e^{iS_{\gamma(a, a_0)}(q, b)} C_+ \sqrt{|dbdq|} + \left| \frac{\partial p_-(q, b)}{\partial b} \right|^\frac{1}{2} e^{iS_{\gamma(c, c_0)}(q, b)} C_- \sqrt{|dbdq|} + \ldots.
\]

Here ‘…’ stands for other intersection points of \(T^*_q Q\) and \(L_q\) and we do not assume that \(\psi_b(q)\) is the WKB asymptotic of the eigen half-density of norm 1. Taking into account (A.3) and (A.2) we obtain the following asymptotic when \(\alpha > 0\) and \(q - q_0 = h^2/3 x \to 0\)

\[
\psi_b(q) = A + \frac{1}{|x|^{1/4}} e^{i\theta(p(q, q_0))} e^{i\sqrt{2|\alpha||x|}^{3/2}} + A - \frac{1}{|x|^{1/4}} e^{i\theta(p(q, q_0))} e^{-i\sqrt{2|\alpha||x|}^{3/2}} + \ldots.
\]

Here ‘…’ stand for higher order contributions and for the contributions from other intersection points in \(L_q \cap T^*_q Q\). Similarly, when \(\alpha < 0\) and \(q - q_0 = h^2/3 x > 0\) we have
\[ \psi_b(q) = A_+ \frac{1}{\chi^{1/4}} e^{ibn(q-q_0)} e^{-i \sqrt{2|\alpha|}\psi/2} + A_- \frac{1}{\chi^{1/4}} e^{ibn(q-q_0)} e^{i \sqrt{2|\alpha|}\psi/2} + \ldots. \]

Here in both cases \( A_+/A_- = C_+/C_- \).

Comparing these asymptotics with the asymptotic of Airy functions described before we conclude that
\[
\frac{A_+}{A_-} = \frac{C_+}{C_-} = e^{i \pi/2}
\]
for \( \alpha > 0 \) and
\[
\frac{A_+}{A_-} = \frac{C_+}{C_-} = e^{-i \pi/2}
\]
for \( \alpha < 0 \).

From here we conclude that
\[
\frac{C_\gamma(a,q_0)}{C_\gamma(c,q_0)} = e^{i \mu \gamma(a,c)}
\]
where \( \gamma(a,c) \) is the composition of \( \gamma(a,q_0) \) and \( \gamma(q_0,c) \). This gives the index \( \mu \) in (5).

The Bohr–Sommerfeld quantization condition is the consistency condition for (A.1) when coefficients \( C_\gamma(a,q_0) \) are given by the formula above.

A.2. Turning points for the cotangent bundle \( T^*Q \)

In this case the analysis is completely parallel. When \( q \) is close to a simple turning point, the Airy analysis in a transversal direction is completely parallel. We still have \( p_+(q,b) \) and \( p_-(q,b) \), one is ‘above’ the other [2] and the derivation of \( \mu \) is literally the same.

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