On the bosonic behavior of mean-field fluctuations in atomic ensembles

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The mean field fluctuations of large atomic ensembles can behave like bosonic modes, i.e. they induce a state on an appropriate system of bosonic modes. The most prominent example is that, if the atomic ensemble is in a homogenous product state, then the mean-field fluctuations are inducing a Gaussian state on a system of bosonic modes. In the present paper we show that for atomic ensemble states with exponentially decaying correlations (e.g. with respect to the distance of atoms) the mean-field fluctuations are inducing (possibly non-Gaussian) states on a system of bosonic modes. This result is true for a general lattice of atomic systems that is equipped with a reasonable distance function.

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I. Introduction

In many body quantum systems, the mean-field theory is focusing on the average behavior of single constituents (see e.g. [4],[5],[10]). Large atomic ensembles are systems of this kind. Typical global states of these systems have the property that the state restricted to an single atom is independent of the individual atom. If we average the state restricted to a single atom over all atoms of the ensemble, we obtain the same state as restricted to each individual atom. For asymptotically large systems, this mean-field average corresponds to an effective classical systems. Since one is only concerned with expectation values of observable of single atoms, correlations between different atoms are irrelevant for the mean-field limit.

A kind of “first-order correction” to the classical mean-field limit are “mean-field fluctuations”. Observables which for testing mean-field fluctuations are build as follows: One looks at the deviation of a single atom observable from its mean-field expectation value. Whereas the mean-field expectation value is the same for each single atom, the expectation value of the deviation may depend on the individual atom. A “mean-field fluctuation observable” (fluctuation operator) is an appropriate average of the individual mean-field deviations over all atoms.

It is well known, that if a large atomic ensemble is prepared in a homogenous product state, i.e. each single atom is individually prepared in the same state, the mean-field fluctuations effectively behave like a systems of non-interacting bosonic modes. In other words, a homogenous product state of a large atomic ensemble, induce (via mean-field fluctuations) a “Gaussian state” on a system of bosonic modes. This statement has to be interpreted in the limit of infinitely large systems. This is also related to the well known “Holstein-Primakoff transformation” [3] which relates large spin systems to bosonic systems.

The “bosonic nature” of mean-field fluctuations for a large atomic ensemble can also be interpreted as “simulating” bosonic systems by large atomic ensembles.

It can be observed in experiments that mean-field fluctuations can have a “bosonic behavior” by building interfaces between atomic ensembles and light [2]. Here, a laser is appropriately interacting with a gas of atoms confined to a glass box at room temperature. The state of the laser field can be stored into the atomic ensemble by using the effective degrees of freedom of the mean-field fluctuations. Conversely, one can also perform an inverse process, by transferring the state of the mean-field fluctuations of an atomic ensemble to bosonic modes of a light field.

One can also imagine to use a similar technique in order to store the state of a laser field into an ensemble of atoms that are trapped by the periodic potential of an optical lattice. The interesting point is here, that the implementation of quantum cellular automata for optical lattice systems is a natural task. In an optical lattice, each atoms occupies a lattice site. A quantum cellular automaton is a global (mostly reversible) quantum operation whose local action on a single site subsystem only affects the neighboring sites [9].

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A very interesting issue combines the simulation of bosonic modes by atomic ensembles, on one hand, with the implementation of quantum cellular automata which acts on atomic ensembles, on the other hand: First, store the state of a laser field into an ensemble of atoms (arrange within a optical lattice), then implement an quantum cellular automaton acting on the atomic ensemble, and finally “release the light” from the atomic ensemble. By the overall process, we obtain an incoming light field and a outgoing scattered light field. The question is now:

What kind of effective operation is describing this “scattering process”? We expect to obtain operations that go beyond the “Gaussian world” which can be used as “non-Gaussian addons”. Since Gaussian operations are limited in their ability to perform quantum information tasks, this may open a door to perform new tasks.

Whether the mean-field fluctuations of a large atomic ensemble behave like bosonic modes depends on the state of the atomic ensemble. If we want to find an answer to the question given above, we will answer the following question first:

For which states of large atomic ensembles do the mean-field fluctuations behave like bosonic modes and is there a set of states with bosonic mean field fluctuations which is invariant under application of quantum cellular automata?

We shall see that Theorem 7 provides an answer to this question. For states of large atomic ensembles whose correlations are exponentially decaying with the distance of the single atoms (exponential clustering), the mean-field fluctuations behave like bosonic modes. In particular, states with exponential clustering are invariant under quantum cellular automata since the action of a quantum cellular automaton on a single site system only affects a finite set of neighbors.

As a consequence, the following process is possible: A laser field is interacting with a large atomic ensemble such that the Gaussian state of the laser field is encoded into a homogenous product state of the atoms. A quantum cellular automaton acting on the atomic ensemble is implemented. The resulting state of the atomic ensemble possesses again bosonic mean-field fluctuations. Finally, we can use again the interaction between the laser field and the atomic ensemble to transfer the state of the atomic ensemble (almost faithfully) to the bosonic modes of the laser.

The total process induces an operation on bosonic modes which maps an initial Gaussian state to some bosonic state, which can be non-Gaussian. With help of Theorem 5 the correlation functions of the resulting state can be written as a perturbation of a Gaussian state. This may be helpful in oder to decide which states of atomic ensembles correspond to Gaussian states.

Outline of the paper

In Section II we provide the appropriate mathematical tools for describing mean-field fluctuation. Fixing a given atomic ensemble, it depends on the state which kind of system of bosonic modes (if there is one) is corresponding to the mean-field fluctuations. This requires to compare different bosonic systems even if they differ by its canonical commutator relations. The tensor algebra provides a universal description that covers all different bosonic systems at once. Which bosonic system is realized is part of the state of this “universal continuous variable system”.

How to describe and analyze mean-field fluctuation by using the framework of universal continuous variable systems (tensor algebras) is discussed in Section III. Here, we also present the main results (Theorem 5, Theorem 7). Technical supplements in order to give self-contained proofs are postponed to the sections in the appendix.

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II. Universal description of continuous variable systems

Our goal is to relate systems of atomic ensembles with bosonic systems. In this section, we explain this relation in mathematical detail and generality, by using the algebraic approach to quantum mechanics. Here
systems are given in terms of their observable algebras, which, in our case, are \( C^* \)-algebras or more general \(^*\)-algebras when unbounded operators are included.

A. The tensor algebra a universal playground

We now introduce a formalism for describing general continuous variable systems in a uniform manner, which is not so frequently used, but which has the advantage that the Holstein-Primakov transformation can be implemented easily and naturally.

Let \( V \) be a complex vector space with a complex conjugation \( J \). The tensor algebra over \((V, J)\) is the unital associative \(^*\)-algebra that is given by the complex vector space

\[
\mathcal{F}(V, J) = \bigoplus_{n \in \mathbb{N}} V^\otimes n.
\]

The product, which is just given by the tensor product, is determined by

\[
(v_1 \otimes v_2 \otimes \cdots \otimes v_n)(w_1 \otimes w_2 \otimes \cdots \otimes w_m) = v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes w_1 \otimes w_2 \otimes \cdots \otimes w_m
\]

and the adjoint is determined by

\[
(v_1 \otimes v_2 \otimes \cdots \otimes v_n)^* = Jv_n \otimes Jv_{n-1} \otimes \cdots \otimes JV_1,
\]

where \( v_1, \ldots, v_n, w_1, \ldots, w_m \in V \). Note that \( V^\otimes 0 \cong \mathbb{C} \) corresponds to the multiples of the unit operator \( 1 \). Obviously, there is a linear embedding \( \Phi \) of \( V \) into the tensor algebra \( \mathcal{F}(V, J) \) such that \( \Phi(v)^* = \Phi(Jv) \).

In the following, we call the operators \( \Phi(v) \) “generalized field operators”.

The tensor algebra \( \mathcal{F}(V, J) \) represents the observable algebra for a wider class of continuous variable systems. The detailed type of the system, e.g. fermionic or bosonic, is encoded in the states under consideration. A state \( \omega \) is described by normalized positive linear functional on the tensor algebra, i.e. \( \omega(A^* A) \geq 0 \) and \( \omega(1) = 1 \). Each state is determined by the \( n \)-point correlation functions

\[
\omega_n(v_1, \ldots, v_n) = \omega(v_1 \otimes \cdots \otimes v_n)
\]

where \( \omega_n \) is a \( n \)-multi-linear functional on \( V \).

How is all this related to the ordinary Hilbert space formalism of quantum mechanics? Well, with help of the so called GNS representation we obtain a Hilbert space \( \mathcal{H}_\omega \), a vector \( \Omega_\omega \) and a \(^*\)-representation \( \pi_\omega \) by linear (but not necessarily bounded) operators on \( \mathcal{H}_\omega \) with \( \omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle \). This is just a consequence of the positivity of the functional \( \omega \).

B. Realizing bosonic systems

As a first example, let us have a look at the bosonic systems. For this purpose we construct a “quasi-free bosonic state” from a bilinear form, called the covariance \( W \), on \( V \). In order to obtain a positive functional, the positivity condition \( W(Jv, v) \geq 0 \) has to be fulfilled. The corresponding quasi-free state is determined according to the following conditions: For \( n > 0 \) we put

\[
\omega_W(v_1 \otimes \cdots \otimes v_n) := \sum_{P \in \Pi_2(n)} \prod_{(i, j) \in P} W(v_i, v_j)
\]

and \( \omega_w(1) = 1 \), where \( \Pi_2(n) \) is the set of ordered partitions of \( \{1, \ldots, n\} \) into two-elementary subsets. Note that the sum is empty for odd \( n \).

We associate to \( W \) the hermitian form \( \gamma \) which is given by \( \gamma(v_1, v_2) = W(Jv_1, v_2) - W(v_2, Jv_1) \). Using the Araki’s self-dual formalism, the self-dual CCR algebra \( CCR(V, J, \gamma) \) is the \(^*\)-algebra that is constructed as follows: Let \( \mathcal{J}_\gamma \) be the two sided ideal in \( \mathcal{F}(V, J) \) that is generated by the operators \( \Phi(v)^* \Phi(v') - \Phi(v')\Phi(v)^* - \gamma(v, v')1 \). Then the corresponding self-dual CCR algebra is given by the quotient \(^*\)-algebra

\[
CCR(V, J, \gamma) := \mathcal{F}(V, J)/\mathcal{J}_\gamma.
\]

As we will briefly sketch below, the state \( \omega_W \) annihilates the ideal \( \mathcal{J}_\gamma \), which implies that \( \omega_W \) induces a unique state on \( CCR(V, J, \gamma) \). Two quasi-free states \( \omega_W \) and \( \omega_{W'} \) on the tensor algebra belong to the same Bosonic system if \( W(Jv, v') - W(v', Jv) = W'(Jv, v') - W'(v', Jv) = \gamma(v, v') \). In this case, both states annihilates the ideal \( \mathcal{J}_\gamma \) and can be lifted to the same CCR algebra.
Remark 1 The quasi-free states $\omega_W$ have the special property to be “even”, i.e. the expectation value of a single generalized field operator is vanishing $\omega_W(\Phi(v)) = 0$. To obtain all quasi-free states, we take advantage of the following fact: Let $V^*_R$ be the real vector space of real continuous linear functionals on $V$. Note that a functional $u \in V^*$ is real if it fulfills the condition $u(Jv) = u(v)$ for all $v \in V$. If we regard $V^*_R$ with its addition as an Abelian group, then $V^*_R$ is acting by $*$-automorphisms on the tensor algebra $\mathcal{T}(V,J)$. For each $u \in V^*_R$, we define the $*$-automorphism $\alpha_u \in \text{Aut}(\mathcal{T}(V,J))$ according to

$$\alpha_u(\Phi(v)) := \Phi(v) + u(v)1.$$  

By construction, the group law is fulfilled, i.e. $\alpha_{u_1}\alpha_{u_2} = \alpha_{u_1+u_2}$ is valid for all $u_1,u_2 \in V^*_R$. To obtain a quasi-free state with a non-vanishing one-point function, we just “shift” an even quasi-free state $\omega_W$ by an appropriate automorphism $\alpha_u$ yielding the quasi-free state $\omega_{W,u} = \omega_W \circ \alpha_u$ which has the one-point function $\omega_{W,u}(\Phi(v)) = \omega_W(\Phi(v)) + u(v) = u(v)$.

C. Ideals to specify more detailed systems

The discussion of the previous subsection shows that the tensor algebra can be indeed used to describe various systems by one unified object. To specify a more particular sub-class of systems additional algebraic relations has to be respected. This corresponds to a proper two-sided ideal $\mathcal{I} \subset \mathcal{T}(V,J)$. By inclusion, the set of two-sided ideals is partially ordered. As larger the ideal, as more specific is the systems class under consideration. For instance, if the hermitian form $\gamma$ is non-degenerate, then the ideal $\mathcal{I}^{\gamma}$ which describes the corresponding CCR-relations is maximal: This can be interpreted as the most specific description of a system, here for a set of bosonic modes. Each state $\omega$ is accompanied with a natural system that is given by the quotient algebra $\omega_\omega := \mathcal{T}(V,J)/\mathcal{I}_\omega$, where $\mathcal{I}_\omega$ is the two-sided ideal $\mathcal{I}_\omega := \{\alpha \mid \forall B,C : \omega(B^*AC) = 0\}$. Note that, by construction, $\mathcal{I}_\omega$ does not contain the identity operator and is therefore a proper ideal.

D. Comparison of states

But what does it mean, that two states on the tensor algebra are close to each other? To give a precise answer to this question, we need to compare states quantitatively. For this purpose, we assume that $V$ is a Banach space. The dual space of the tensor algebra $\mathcal{T}(V,J)$ is denoted by $\mathcal{T}(V,J)^*$. It consists of all linear functionals $F : \mathcal{T}(V,J) \to \mathbb{C}$ for which for all $n \in \mathbb{N}$ the semi-norms

$$\nu_n(F) := \sup_{(v_1, \cdots, v_n) \in V^n_1} |F(v_1 \otimes \cdots \otimes v_n)| < \infty$$  

are bounded, where $V_1 = \{v \in V \parallel v \parallel = 1\}$ is the unit sphere. Now, $\mathcal{T}(V,J)^*$ is closed in the following topologies:

- The strong topology is the locally convex topology that is induced by the family of semi-norms $\nu_n$, $n \in \mathbb{N}$.
- The weak topology is the locally convex topology that is induced by the family of semi-norms $\nu_{(v_1, \cdots, v_n)}(F) := |F(v_1 \otimes \cdots \otimes v_n)|$ with $(v_1, \cdots, v_n) \in \bigcup_k V^k$.

From an experimental perspective, the strong topology is related to the comparison of two states $\omega$ and $\omega'$. Suppose we estimate for a finite family of vectors $v_1, \cdots, v_n \in V$ the correlation functions $\omega(v_1 \otimes \cdots \otimes v_n)$ and $\omega'(v_1 \otimes \cdots \otimes v_n)$. Then the modulus of the difference of the correlation functions can be used as a measure how “close” $\omega$ and $\omega'$ are to each other.

To give an example, we consider the correlation functions of two quasi-free states $\omega_W$ and $\omega_{W'}$, where the covariances $W,W'$ have a norm difference that is given by $\|W - W'\| = \sup_{v_1,v_2 \in V_1} |W(v_1,v_2) - W'(v_1,v_2)|$.

Proposition 2 Let $\omega_W$ and $\omega_{W'}$ be quasi-free states with covariances $W$ and $W'$ respectively, then for each $n \in \mathbb{N}$ the semi-norm difference of the quasi-free states satisfies the bound

$$\nu_n(\omega_W - \omega_{W'}) \leq \|W - W'\| \Pi_2(n) \sum_{k=1}^{n/2} \|W\|^{k-1} \|W'\|^{n/2-k}.$$  

(9)
A direct consequence of the proposition (which we prove in the appendix) is that, if \( W \to W' \) are converging in norm, then \( \omega_W \to \omega_W' \) converges in the strong topology. In other words the mapping \( W \to \omega_W \) is continuous in the respective topologies.

### III. Mean-field fluctuations

Many systems under consideration possessing a large number of independent degrees of freedom such that they can be idealized by infinite systems in the thermodynamic limit. Here we model this situation by an infinite (countable) lattice \( \Lambda \) that possesses a distance function \( d: \Lambda^2 \to \mathbb{R}_+ \). The observable algebra of the global system is the so called quasi-local algebra that is constructed by the infinite tensor product

\[
\mathfrak{A}(\Lambda) = \bigotimes_{x \in \Lambda} \mathfrak{A}(x)
\]

of single cell C*-algebras \( \mathfrak{A} \cong \mathfrak{A}(x) \). For a given lattice point \( x \in X \), the natural embedding of the single site algebra which identifies \( \mathfrak{A} \) with \( \mathfrak{A}(x) \subset \mathfrak{A}(\Lambda) \) is denoted by \( \iota_x \). We are going to use this mapping later on quite often.

However, in view of non-equilibrium thermodynamics, the nature of global states of an infinite systems can be very different from equilibrium states and the calculation of expectation values for such states may be a hard computational task. The analysis of global states, one looks at the asymptotic behavior of certain properties within the mesoscopic range. For this purpose, one restricts the global state to the local observable algebras that correspond to finite subsets sets \( X \subset \Lambda \) which is given by the finite tensor product

\[
\mathfrak{A}(X) = \bigotimes_{x \in X} \mathfrak{A}(x).
\]

Note that for an inclusion \( X \subset Y \subset \Lambda \), it follows immediately that \( \mathfrak{A}(X) \subset \mathfrak{A}(Y) \).

Taking a global state \( \omega_{\Lambda} \), we obtain for each finite subset \( X \subset \Lambda \) a restricted state \( \omega_X := \omega_{\Lambda}|_{\mathfrak{A}(X)} \) which lives on finitely many degrees of freedom. This yields a net of states \( (\omega_X)_{X \subset \Lambda} \) that is indexed by the partially ordered set of finite subsets of the lattice \( \Lambda \). Roughly speaking, the basic idea behind the Holstein-Primakoff transformation is to analyze the behavior of each of the states \( \omega_X \) concerning their “bosonic nature”, i.e. to what extend they “simulate” continuous variable systems. We shall see, that each restriction \( \omega_X \) induces a state \( \hat{\omega}_X \) on the tensor algebra \( \mathcal{F}(\Lambda, \ast) \), where \( \mathfrak{A} \) is the observable algebra of a single cell system. To be of “bosonic nature”, the induced state \( \hat{\omega}_X \) has to fulfill “almost” the canonical commutation relations. This means that there is an antisymmetric hermitian form \( \gamma \) such that the induced state \( \hat{\omega}_X \) is “almost” annihilating the ideal \( \mathcal{F}_\gamma \): A typical behavior is \( \hat{\omega}_X(A) = O(|X|^{-1/2}) \) for each operator \( A \) that belongs to the ideal \( \mathcal{F}_\gamma \).

#### A. Inducing states and \( \sqrt{\pi} \)-fluctuations

Let \( \omega_{\Lambda} \) be a state of the global system. Then we obtain the net of restricted states \( (\omega_X)_{X \subset \Lambda} \) that are indexed by the partially ordered set of finite subsets \( X \subset \Lambda \). The induction of states works by using “fluctuation operators” associated with the restricted state \( \omega_X \) and an operator \( a \in \mathfrak{A} \):

\[
\Phi_{\omega_X}(a) := \frac{1}{|X|^{1/2}} \sum_{x \in X} [\iota_x a - \omega_X(\iota_x a) 1].
\]

This yields a representation \( \Phi(a) \to \Phi_{\omega_X}(a) \) of the tensor algebra and the induced state \( \hat{\omega}_X \) is determined by its \( n \)-point functions according to

\[
\hat{\omega}_X(a_1 \otimes \cdots \otimes a_n) := \omega_X(\Phi_{\omega_X}(a_1) \cdots \Phi_{\omega_X}(a_n)).
\]

The main goal is now to study the asymptotic limit of large systems. For this purpose, let \( \Lambda \) be a countable lattice. Let \( (\omega_X)_{X \subset \Lambda} \) be a net of states that is indexed by finite subsets of \( \Lambda \), where \( \omega_X \) is a state on \( \mathfrak{A}(X) \). The asymptotic properties in the limit \( X \to \Lambda \) can be investigated by looking at the net of induced states \( (\hat{\omega}_X)_{X \subset \Lambda} \) according to the classification:
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- The state $\omega_\Lambda$ has $\sqrt{n}$-fluctuations if the induced net $⟨\hat{\omega}_X⟩_{X⊆\Lambda}$ converges $w - \lim_{X→\Lambda} \hat{\omega}_X = \hat{\omega}_\Lambda$ in the weak topology on $\mathcal{F}(\mathfrak{A}, \ast)$.

- The state $\omega_\Lambda$ has strongly $\sqrt{n}$-fluctuations if the induced net $⟨\hat{\omega}_X⟩_{X⊆\Lambda}$ converges $s - \lim_{X→\Lambda} \hat{\omega}_X = \hat{\omega}_\Lambda$ in the strong topology on $\mathcal{F}(\mathfrak{A}, \ast)$.

- The state $\omega_\Lambda$ has weakly $\sqrt{n}$-fluctuations if for the induced net $⟨\hat{\omega}_X⟩_{X⊆\Lambda}$ the each semi-norm $ν_n$, $n ∈ \mathbb{N}$, is uniformly bounded: $\sup_{X⊆\Lambda} ν_n(⟨\hat{\omega}_X⟩) < ∞$.

Obviously, strongly $\sqrt{n}$-fluctuations implies $\sqrt{n}$-fluctuations implies weakly $\sqrt{n}$-fluctuations. We are now considering states that are “single site homogenous”. These states are defined by the property that their restrictions to a single site is independent of the lattice point.

### B. Induced states for asymptotically large systems

Asymptotically large atomic ensembles can be described by an infinite lattice system which is in some state $\omega_\Lambda$. Suppose we assume that the corresponding induced net of states $⟨\hat{\omega}_X⟩_{X⊆\Lambda}$ has weakly $\sqrt{n}$-fluctuations. What conclusions can we draw from this property? What do we know about the asymptotic behavior of the correlation functions of the induced states $\hat{\omega}_X$?

Since each semi-norm $ν_n(⟨\hat{\omega}_X⟩)$ is uniformly bounded in the size of the subset $X$, we know that there are weak limit points. In order to analyze the properties of these limit points more systematically, we will give here an “operational” description of what limit points are.

Within a concrete experimental realization, the atomic ensemble under consideration will be always finite. If the setup is scalable, then, at least in principle, the same experiment can be performed for various sizes of the system, i.e. the subset $X$ can be regarded as a “classical configuration”. Here one can also think of a situation, where atoms occupy only finitely many sites of a lattice randomly. Thus we are dealing with a preparation device that prepares for each finite atomic ensemble $X ⊆ \Lambda$ a state $\omega_X$ with a certain probability $μ(X)$. The probability distribution $μ : X → μ(X)$ is nothing else but a classical state on the system of finite subsets $X ⊆ \Lambda$. The corresponding observable algebra consists of all bounded complex valued functions $f : \Lambda → f(X)$. The expectation value of an observable $f$ for the state $μ$ is then given by $μ(f) = \sum_{X⊆\Lambda} μ(X)f(X)$.

A general classical state on the system of finite subsets $X ⊆ \Lambda$ is a complex valued linear functional on the algebra of bounded complex valued functions $f : X → f(X)$ such that the following holds:

- Positivity: $η(f) ≥ 0$ for each $f ≥ 0$.
- Normalization: $η(1) = 1$.

A preparation device that produces asymptotically large atomic ensembles has the property that, in the limit $X → Λ$, the probability that only a finite number of lattice sites are occupied is vanishing. This corresponds to classical states $η$ with the following property:

- $η(f) = 0$ if $\lim_{X→\Lambda} f(X) = 0$.

A state $η$ with this property is called a “limit point”. To justify this notion, suppose that limit $\lim_{X} f(X) = c$ exists. In this case $\lim_{X} (f(X) - c) = 0$, and $η(f - c1) = η(f) - c = 0$ follows, which means that there expectation value $η(f) = c$ coincides for all limit points $η$.

What can we say about the limit points of the induced net $⟨\hat{\omega}_X⟩_{X⊆\Lambda}$ for a state $\omega_\Lambda$ that have weakly $\sqrt{n}$-fluctuations? To each operator $A ∈ \mathcal{F}(\mathfrak{A}, \ast)$ of the tensor algebra, we assign a bounded function which is given by $X → \hat{\omega}_X(A)$. Here we use the suggestive notation $η(X → f(X)) := η(f)$ to represent an expectation value.

The states $\hat{\omega}_\Lambda$ describe the mean-field fluctuations of asymptotically large atomic ensembles. The next propositions states that these mean-field fluctuations behave like bosonic modes. Consider a state $\omega_\Lambda$ that is single site homogenous with single site restriction $ω = ω_\Lambda ∘ τ_X$. Then there is a natural antisymmetric hermitian form $γ(a, b) := ω([a^*, b])$ on the observable algebra $\mathfrak{A}$ of the single site system. The ideal $\mathcal{F}_γ$, which represents the canonical commutation relations, is generated by the operators $I_γ(a, b) := [Φ(a), Φ(b)] - γ(a^*, b)1$. 


Proposition 3 Let \( \omega_\Lambda \) be a single site homogenous state having weakly \( \sqrt{\eta} \)-fluctuations. Then for each limit point \( \eta \), the state \( \hat{\omega}_\eta \) annihilates the ideal \( \mathcal{J}_\gamma \) and can uniquely be lifted to a state on the corresponding CCR algebra.

Proof: By Lemma 2 of the appendix, we conclude that \( \lim_X \hat{\omega}_X(A) = 0 \) for each operator in the ideal \( \mathcal{J}_\gamma \). This implies \( \hat{\omega}_\eta(A) = \eta(X \mapsto \hat{\omega}_X(A)) = 0 \) which implies that \( \hat{\omega}_\eta \) annihilates \( \mathcal{J}_\gamma \).

Remark 4 Proposition 3 can be interpreted, at least to a certain extend, by saying that a state \( \omega_\Lambda \) of a large atomic ensemble with weakly \( \sqrt{\eta} \)-fluctuations possess bosonic mean-field fluctuations. This is justified by the fact that each limit point \( \hat{\omega}_\eta \) is a state on the CCR algebra CCR(S, \( \ast \), \( \gamma \)) which describes a bosonic system. On the other hand, the CCR algebra is an algebra of unbounded operators and it might happen that the GNS representation associated to a limit state \( \hat{\omega}_\eta \) has “exotic” properties. Recall, that the GNS representation is given by a Hilbert space \( \mathcal{H} \) an algebra homomorphism \( \pi \) that assigns to each operator \( A \) in the CCR algebra a linear (unbounded) operator on \( \mathcal{H} \) as well as a normalized vector \( \Omega \in \mathcal{H} \) such that \( \omega_\eta(A) = \langle \Omega, \pi(A)\Omega \rangle \). The question that arises here is whether it is possible to build the exponential \( \exp(i\pi(\Phi(a))) \) of a field operator \( \pi(\Phi(a)) \) in the representation \( \pi \), where \( a = a^\ast \) is selfadjoint. If we can do this, then we obtain a representation of the Weyl algebra by bounded operators. If it is not possible to build the exponential we are dealing with an “exotic” case (see e.g. [6]). In order to exclude this kind of pathologies, we need to consider more specific examples of states.

C. States with exponential clustering

A state \( \omega_\Lambda \) on \( \mathcal{A}(\Lambda) \) has exponential clustering (with respect to \( d \)) if for local operators \( A \in \mathcal{A}(X) \) and \( B \in \mathcal{A}(Y) \) the identity

\[
\omega_\Lambda(AB) = \omega_\Lambda(A)\omega_\Lambda(B) + G_{X,Y}(A, B)e^{-d(X,Y)}
\]

(14)
is valid for a bounded bilinear function \( G_{X,Y} : \mathcal{A}(X) \times \mathcal{A}(Y) \to \mathbb{C} \) such that \( |G_{X,Y}(A, B)| \leq G_0||A||||B|| \) for all \( A, B \), for all finite regions \( X, Y \). Here \( G_0 \) is a constant that is independent of the localization regions. The bilinear forms \( G_{X,Y} \) express locally the deviations from the state to a product state, being scaled with the exponential of the distance. Therefore \( G_{X,Y} \) indicates the presence of correlations that are exponentially decreasing with the distance. To give a name, we call the family of bilinear maps \( G = (G_{X,Y})_{X,Y \subset \Lambda} \) the correlators. Note that, equivalently, exponential clustering is given by the condition

\[
|\omega_\Lambda(AB) - \omega_\Lambda(A)\omega_\Lambda(B)| \leq G_0 e^{-d(X,Y)}
\]

(15)
for all \( A \in \mathcal{A}(X), B \in \mathcal{A}(Y) \). Here \( d(X,Y) = \min_{x \in X, y \in Y} d(x, y) \) is the distance between the finite subsets \( X, Y \subset \Lambda \). We always require here, that the distance \( d \) is regular, i.e. the maximal number \( N(r) \) of lattice sites within a ball of radius \( r \) is bounded by a polynomial.

The exponential clustering property can be used to derive a useful cluster expansion in terms of expectation values of the single site restriction \( \omega \) and the correlators \( G \). In order to write down this expansion, we introduce the following objects:

- For each finite subset \( Y \subset \Lambda \) we introduce the “spread” \( \Delta(Y) := \max_{y \in Y} d(y, Y \setminus y) \) which measures the maximal distance of a point to its relative complement in \( Y \).

- An \( k \)-elementary subset \( \{y_1, \cdots, y_k\} \subset X \) is called “spread optimally enumerated” if the enumeration fulfills the condition \( d(y_l, \{y_{l+1}, \cdots, y_k\}) = \Delta(y_l, \cdots, y_k) \) for all \( l = 1, \cdots, k - 1 \). Note that each subset can be spread optimally enumerated.

- Given a tuple \( x \in X^n \), we choose a spread optimal enumeration of the range \( \text{Ran}(x) = \{y_1, \cdots, y_{|\text{Ran}(x)|}\} \) and we consider the correlators \( G_k := G_{\{y_k, \{y_{k+1}, \cdots, y_{|\text{Ran}(x)|}\}} \) which test the correlations for splitting the site \( y_k \) from the remaining points \( \{y_{k+1}, \cdots, y_{|\text{Ran}(x)|}\} \), where \( k = 1, \cdots, |\text{Ran}(x)| - 1 \).
For a family of operators $a_1, \ldots, a_n \in \mathfrak{A}$ and a tuple $x \in X^n$ whose range $\{y_1, \ldots, y_{|\text{Ran}(x)|}\}$ is spread optimally enumerated, we introduce the single site “cluster operators” $a^x_k \in \mathfrak{A}$ which are given by the ordered product $a^x_k := \prod_{j \in x^{-1}(y_k)} a_j$, where the ordering is according to the value of the index in $x^{-1}(y_k) = \{j = 1, \ldots, n | x_j = y_k\}$.

The following theorem, whose proof is given in the appendix, states that correlation functions of the induced states $\hat{\omega}_X$ admit a cluster expansion in terms of the single site restriction $\omega$, the correlators $G$ and cluster operators $a^x_k$:

**Theorem 5 (Cluster expansion)** Let $\omega_{A}$ be a single site homogenous state with single site restriction $\omega$ and exponential clustering with respect to $d$. For each $a_1, \ldots, a_n \in \ker(\omega)$ and for each finite subset $X \subset \Lambda$ the $n$-point correlation function of the induced state $\hat{\omega}_X$ can be written as

$$\hat{\omega}_X(a_1 \otimes \cdots \otimes a_n) = \hat{\omega}^{\otimes X}(a_1 \otimes \cdots \otimes a_n) + F_X(a_1 \otimes \cdots \otimes a_n),$$

where the correlation function of the induces homogenous product state $\hat{\omega}^{\otimes X}$ and the functional $F_X$ are given by

$$\hat{\omega}^{\otimes X}(a_1 \otimes \cdots \otimes a_n) = |X|^{-\frac{n}{2}} \sum_{x \in X^n} \omega(a^x_1) \cdots \omega(a^x_{|\text{Ran}(x)|})$$

$$F_X(a_1 \otimes \cdots \otimes a_n) = |X|^{-\frac{n}{2}} \sum_{x \in X^n} \sum_{k=1}^{\text{Ran}(x)-1} \omega(a^x_1) \cdots \omega(a^x_k-1)$$

$$\times G^x_k(a^x_k, a^x_{k+1} \cdots a^x_{|\text{Ran}(x)|}) e^{-\Delta(y_k, y_{k+1}, \ldots, y_{|\text{Ran}(x)|})},$$

where for each $x \in X^n$ the range $\text{Ran}(x)$ is spread optimally enumerated.

As the cluster expansion is stated above, it holds for all correlation functions for which the operators $a_1, \ldots, a_n$ are chosen in the kernel $\ker(\omega)$ of the single site restriction. If this is not the case, we can express the correlation function in terms of $a_t = a'_t + \omega(a_t)1$ where $a'_t \in \ker(\omega)$. The tensor product $a_1 \otimes \cdots \otimes a_n$ can be expanded in terms of the operators $a'_t \in \ker(\omega)$ according to

$$a_1 \otimes \cdots \otimes a_n = \sum_{J \subset \{1, \ldots, n\}} \bigotimes_{i \in J} a'_i \prod_{j \in J} \omega(a_j)$$

where the sum runs over all ordered subsets. To get the general cluster expansion for the full tensor algebra, one only has to apply Theorem 5 to (18) for each summand.

It is known, that the induced net $(\hat{\omega}^{\otimes X})_{X \subset \Lambda}$ converges weakly to a quasi-free state. We show here a slightly stronger result:

**Proposition 6** A homogenous product state $\omega^{\otimes A}$ has strongly $\sqrt{n}$-fluctuations. In particular, the induced net $(\hat{\omega}^{\otimes X})_{X \subset \Lambda}$ converges strongly to the quasi-free state $\omega_{qf}$ whose covariance is given by the truncated two-point function $W(a, b) = \omega(ab) - \omega(a)\omega(b)$.

Homogenous product states are the simplest among states that have exponential clustering. For the general case, the following is true:

**Theorem 7** Each single site homogenous state with exponential clustering has weakly $\sqrt{n}$-fluctuations.

The proof of the theorem is quite technical and therefore postponed to the appendix. However, it takes advantage of the cluster expansion of $F_X$ into single site expectation values and correlators. The basic idea to get a uniform bound for the semi norms $\nu_n(F_X)$ is to count the number of terms that are contributing to the cluster expansion. In total, we sum over all tuples in $X^n$ which gives $|X|^n$ terms. Since we normalize by multiplying $|X|^{-n/2}$, a naive counting would give the non-uniform bound $\nu_n(F_X) \leq O(|X|^{-n/2})$.

By a more careful analysis, it turns out that effectively only $|X|^{n/2}$ terms are contributing. By choosing $a_1, \ldots, a_n \in \ker(\omega)$, the single site expectation value of a cluster operator $\omega(a^x_k)$ is vanishing if $x^{-1}(y_k) = \{j\}$ contains only a single element. Note that in this case we just have $\omega(a^x_k) = \omega(a_j) = 0$. This reduces directly the number of terms which in the cluster expansion (17). A large number contributions are also coming from tuples $x$ with range $\{y_1, \ldots, y_{|\text{Ran}(x)|}\}$ for which the spreads $\Delta(y_k, \ldots, y_{|\text{Ran}(x)|})$ are large. These contributions are also of order $|X|^{n/2}$, since they are suppressed the exponential damping $\exp(-\Delta(y_k, \ldots, y_{|\text{Ran}(x)|}))$. 


Remark 8 We can derive from the cluster expansion that for asymptotically large atomic ensembles there is a quasi-free part from the product state contribution and a perturbation which comes from the correlators. Namely, for each weak limit point \( \eta \) the state \( \hat{\omega}_\eta \) can be written as

\[
\hat{\omega}_\eta = \hat{\omega}_{\text{qf}} + F_\eta
\]

with \( F_\eta(A) = \eta(X \mapsto F_X(A)) \). The functional \( F_\eta \) is a perturbation of the quasi-free limit state \( \hat{\omega}_{\text{qf}} \) which may depend on the limit functional \( \eta \). Note that Theorem 7 guarantees the existence of weak limit points \( F_\eta \), since \( \sup_{X \subset \Lambda} \nu_\eta(F_X) < \infty \).

IV. Conclusion

We have shown that states of large atomic ensembles whose correlations are exponentially decaying with the distance between atoms (exponential clustering) possess bosonic mean-field fluctuations. In addition to that, these states are invariant under applications of quantum cellular.

This enables the implementation of the following type of process: The bosonic modes of a light field are coupled to a large atomic ensemble such that the Gaussian state of the laser field is transferred almost perfectly (where the precision is here of order \( O(\sqrt{\text{number of single atom systems}}) \)) to a homogenous product state of the atoms. A quantum cellular automaton acting on the atomic ensemble is implemented. The resulting state of the atomic ensemble possesses again bosonic mean-field fluctuations and the resulting state of the atomic ensemble can be transferred back almost perfectly to the bosonic modes of the light field.

The total process induces an operation on bosonic modes which maps an initial Gaussian state to some bosonic state, which can be non-Gaussian. With help of Theorem 5 the correlation functions of the resulting state can be written as a perturbation of a Gaussian state. This may be helpful in order to decide which states of atomic ensembles correspond to Gaussian states.

It is still an open problem to decide in general from the state of the atomic ensemble whether the resulting induced state is Gaussian or not. Concerning states with exponential clustering, the cluster expansion (Theorem 5) appears to be a reasonable technique in order to address this problem. Here the correlation functions of the fluctuation operators can be expanded into the correlation functions of the homogenous product state \( \omega^{\otimes X} \) (here \( \omega \) is the restriction of the global state to a single atom) and some correction \( F_X \). For large atomic ensembles, the correlation functions of the homogenous product state \( \omega^{\otimes X} \) correspond to a Gaussian state, whereas \( F_X \) can be regarded as a “perturbation”.

Furthermore, it would be desirable to construct new examples of atomic ensemble states (in particular beyond homogenous product states) whose induced net has strongly \( \sqrt{n} \)-fluctuations or \( \sqrt{n} \)-fluctuations.

In order to archive more concrete results in this direction, one has to consider here more concrete examples. One suggestion is to consider ensembles of two-level atoms arranged in a one-dimensional lattice. A natural class of states for which mean-field fluctuations can be investigated are stabilizer states which are invariant under the action of so called Clifford quantum cellular automata (see [1][7][8] and references given therein).

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This function is indeed bounded which can be verified as follows: The operator $A$ can be written as a finite direct sum $\bigoplus_{k=0}^{\infty} A_k$ with $A_k \in \mathfrak{H}^k$. Since $\omega_\Lambda$ has weakly $\sqrt{n}$-fluctuations we obtain that $|\hat{\omega}_X(A)| \leq \sum_{k=0}^{\infty} |\hat{\omega}_X(A_n)| \leq \sum_{k=0}^{\infty} C_n \|A_n\| < \infty$ with $C_n = \sup_{X \subset \Lambda} \nu_n(\hat{\omega}_X)$. 
On the bosonic behavior of mean-field fluctuations

A. On quasi-free states (proof of Proposition 2)

Proof of Proposition 2 Let $W, W'$ be two bounded covariances (positive bounded bilinear form) on $V$. Recall that a bilinear form $F$ on $V$ is bounded if $\|F\| := \sup_{v_1, v_2 \in V} |F(v_1, v_2)| < \infty$. Moreover, recall that a valid covariance $W$ has to be positive in the sense that $W(v, v) \geq 0$ for all $v \in V$. We fix vectors $v_1, \ldots, v_n$ and use the expression (5) for the correlation functions of quasi-free states to calculate the difference of the $n$-point functions for the quasi-free states $\omega_W$ and $\omega_{W'}$. For each ordered partition $P \in \Pi_2(n)$ we choose an enumeration $P = ((i_1, j_1), \ldots, (i_{n/2}, j_{n/2}))$ and we introduce for $l = 1, \ldots, n/2$ the quantities $W_{P,l} := W(v_{i_l}, v_{j_l})$ and $W'_{P,l} := W'(v_{i_l}, v_{j_l})$. This yields for the difference of the corresponding correlation functions:

$$
\omega_W(v_1 \otimes \cdots \otimes v_n) - \omega_{W'}(v_1 \otimes \cdots \otimes v_n) = \sum_{P \in \Pi_2(n)} \sum_{l=1}^{n/2} W_{P,1} \cdots W_{P,l-1}(W_{P,l} - W'_{P,l})W'_{P,l+1} \cdots W'_{P,n/2} \cdot \tag{A1}
$$

Here we have used that the difference of products can be written as a sum in the following way:

$$
W_{P,1} \cdots W_{P,n/2} - W'_{P,1} \cdots W'_{P,n/2} = \sum_{l=1}^{n/2} W_{P,1} \cdots W_{P,l-1}(W_{P,l} - W'_{P,l})W'_{P,l+1} \cdots W'_{P,n/2} \cdot \tag{A2}
$$

By using the fact that $W$ and $W'$ are bounded bilinear forms, the modulus of $W_{P,l}$ (and similarly for $W'$ and $W - W'$) can be bounded by $|W_{P,l}| \leq \|v_i\| \|v_j\| |W|$ and we obtain the desired bound

$$
|\omega_W(v_1 \otimes \cdots \otimes v_n) - \omega_{W'}(v_1 \otimes \cdots \otimes v_n)| \leq \|v_1\| \cdots \|v_n\| \|W - W'\| \sum_{l=1}^{n/2} |W||W'|^{l-1}|W' n/l - 1| \cdot \tag{A3}
$$

B. Proving the bosonic behavior of mean-field fluctuations for large atomic ensembles

Roughly, the statement of Proposition 2 is that mean-field fluctuations for states having weakly $\sqrt{n}$-fluctuations behave like a bosonic system for large atomic ensembles. The following lemma provides the bounds which are used to prove this statement.

Lemma 9 Let $\omega_X$ be a single site homogenous state on the quasi-local $\mathcal{A}(\Lambda)$ such that the induced net $(\omega_X)_{X \subset \Lambda}$ has weakly $\sqrt{n}$-fluctuations. Then for each family of operators $a_1, \ldots, a_n$ the bound

$$
|\omega_X(\Phi(a_1) \cdots \Phi(a_i-1)I_{\gamma(a_i, a_{i+1})} \Phi(a_{i+2}) \cdots \Phi(a_n))| \leq 2|X|^{-1/2} C_{n-1} \prod_{i=1}^n |a_i| \tag{B1}
$$

holds with $C_n := \sup_{X \subset \Lambda} \nu_n(\omega_X)$.

Proof: The fluctuation operators define a representation $\pi_{\omega_X}(\Phi(a)) := \Phi_{\omega_X}(a)$ of the tensor algebra $\mathcal{F}(\mathcal{A}, \epsilon)$ by operators in $\mathcal{A}(X)$. The induced state $\omega_X = \omega_X \circ \pi_{\omega_X}$ is just the pullback of the state $\omega_X$ by the representation $\pi_{\omega_X}$. According to the definition of fluctuation operators, the identity

$$
\pi_{\omega_X}(I_{\gamma}(a, b)) = [\Phi_{\omega_X}(a), \Phi_{\omega_X}(b)] - \gamma(a^*, b) \mathbb{I} = |X|^{-1/2} \Phi_{\omega_X}([a, b]) \tag{B2}
$$
is valid for all single site operators $a, b$. Inserting this identity within the correlation function for $a_1, \cdots, a_n$ implies
\begin{align}
|\hat{\omega}_X(\Phi(a_1) \cdots \Phi(a_{i-1}) J_{\gamma}(a_i, a_{i+1}) \Phi(a_{i+2}) \cdots \Phi(a_n))| \\
= |X|^{-1/2} |\hat{\omega}_X(\Phi(a_1) \cdots \Phi(a_{i-1}) \Phi([a_i, a_{i+1}]) \Phi(a_{i+2}) \cdots \Phi(a_n))| \\
\leq 2 |X|^{-1/2} C_{n-1} \prod_{i=1}^{n} |a_i| .
\end{align}
which proves the proposition. Recall that we have used that all the semi-norms $\nu_n$ are uniformly bounded. \[\square\]

C. Cluster expansion for correlation functions (proof of Theorem)\[\square\]

This subsection provides the proof of the expansion (Theorem) of correlations functions for states $\omega_{\Lambda}$ with exponential clustering. We also assume here that $\omega_{\Lambda}$ is single site homogenous with single site restriction $\omega$.

We first derive here an simpler expansion for correlation functions of the form $\omega_{\Lambda}(A_1 \cdots A_k) = \omega_Y(A_1 \cdots A_k)$, where $Y = \{y_1, y_2, \cdots, y_k\} \subset \Lambda$ is spread optimally enumerated and the operator $A_i \in \mathfrak{A}(y_i)$ is localized at site $y_i$. The idea of the expansion is to expresses the expectation value $\omega_Y(A_1 \cdots A_k)$ in terms of single site expectation values $\omega(A_j)$ as well as terms $G_j(A_j, A_{j+1} \cdots A_k) := G_j(y_i, \{y_{i+1}, \cdots, y_k\}; \{A_j, A_{j+1} \cdots A_k\})$ that are given by the bilinear forms $G_j(X, Y)$. Recall, that for two operators $A \in \mathfrak{A}(X)$ and $B \in \mathfrak{A}(Y)$ the exponential clustering can be expressed by the identity
\begin{equation}
\omega_{\Lambda}(AB) = \omega_X(A)\omega_Y(B) + G_{(X,Y)}(A, B)e^{-d(X,Y)} .
\end{equation}

We use the following lemma in order to prove Proposition\[\square\]

**Lemma 10** Let $Y = \{y_1, \cdots, y_k\}$ be spread optimally enumerated and let $A_i \in \mathfrak{A}(y_i)$, $i = 1, \cdots, k$, be single site operators. Then the expectation value $\omega_Y(A_1 \cdots A_k)$ can be expressed as
\begin{equation}
\omega_Y(A_1 \cdots A_k) = \omega(A_1) \cdots \omega(A_k) + \sum_{l=1}^{k-1} \omega(A_1) \cdots \omega(A_{l-1}) G_l(A_l, A_{l+1} \cdots A_k) e^{-\Delta(y_l, \cdots, y_k)}
\end{equation}
where the bilinear form $G_l$ is defined as given above.

**Proof:** Let $Y = \{y_1, \cdots, y_k\}$ be spread optimally enumerated. Then $\{y_2, \cdots, y_k\}$ is also spread optimally enumerated. Suppose now the expansion is valid for spread optimally enumerated sets with $k - 1$ elements.
\begin{equation}
\omega_{\Lambda \setminus y_1}(A_2 \cdots A_k) = \omega(A_2) \cdots \omega(A_k) + \sum_{l=2}^{k-1} \omega(A_2) \cdots \omega(A_{l-1}) G_l(A_l, A_{l+1} \cdots A_k) e^{-\Delta(y_l, \cdots, y_k)}
\end{equation}
where the product $\omega(A_2) \cdots \omega(A_{l-1}) = 1$ is declared to be empty for $l = 2$. Then we can use the expansion
\begin{equation}
\omega_Y(A_1 A_2 \cdots A_k) = \omega(A_1) \omega_{\Lambda \setminus y_1}(A_2 \cdots A_k) + G_1(A_1, A_2 \cdots A_k) e^{-\Delta(y_1, \cdots, y_k)}
\end{equation}
By inserting the expansion for $\omega_{\Lambda \setminus y_1}(A_2 \cdots A_k)$ gives
\begin{equation}
\omega_Y(A_2 \cdots A_k) = \omega(A_1) \omega(A_2) \cdots \omega(A_k) + \sum_{l=2}^{k-1} \omega(A_1) \omega(A_2) \cdots \omega(A_{l-1}) G_l(A_l, A_{l+1} \cdots A_k) e^{-\Delta(y_l, \cdots, y_k)}
\end{equation}
\begin{equation}
+ G_1(A_1, A_2 \cdots A_k) e^{-\Delta(y_1, \cdots, y_k)}
\end{equation}
\begin{equation}
= \omega(A_1) \omega(A_2) \cdots \omega(A_k) + \sum_{l=1}^{k-1} \omega(A_1) \omega(A_2) \cdots \omega(A_{l-1}) G_l(A_l, A_{l+1} \cdots A_k) e^{-\Delta(y_l, \cdots, y_k)}
\end{equation}
Note that for a one-elementary set the statement is trivial and for a two elementary set \( \{ y_1, y_2 \} \) the expansion is also valid since we have \( \omega_{\Lambda}(A_1A_2) = \omega(A_1)\omega(A_2) + G_{(y_1,y_2)}(A_1,A_2)e^{-d(y_1,y_2)} \) and for any two elementary set the distance coincides with the spread.

**Proof of Theorem**\(^5\) Let \( \omega_{\Lambda} \) be a single site homogenous state with strong exponential clustering. Then for operators \( a_1, a_2, \cdots, a_n \in \ker(\omega) \) the correlation function for the induced state \( \tilde{\omega}_X \) can be written as

\[
\tilde{\omega}_X(\cdots) = \left| X \right|^{-n/2} \sum_{x \in X^n} \omega_X \left( \cdots \right) .
\]

where \( a_k^x = \prod_{j \in x^{-1}(y_k)} a_j \) are the cluster operators. If we enumerate for each \( x \in X^n \) the range \( \text{Ran}(x) \) spread optimally, we can apply Lemma 10 to the expectation values

\[
\omega_X \left( a_1^x a_2^x \cdots a_n^x \right) = \omega(a_1^x) \cdots \omega(a_n^x) + \sum_{k=1}^{\left| \text{Ran}(x) \right|-1} \omega(a_1^{x_k}) \cdots \omega(a_k^{x_{k-1}})
\]

\[
\times G_{\tilde{k}}(a_k^x, a_{k+1}^x \cdots a_{\left| \text{Ran}(x) \right|}^x) e^{-\Delta(y_k,y_{k+1}^r,\cdots,y_{\left| \text{Ran}(x) \right|})}
\]

where \( G_{\tilde{k}} = G((y_k,\{y_{k+1},\cdots,y_{\left| \text{Ran}(x) \right|}\})) \). The statement of Proposition 5 follows directly by summing over the elements in \( X^n \) and normalizing by \( \left| X \right|^{-n/2} \).

**D. Remarks on the strong topology of the tensor algebra**

Let \( \omega \) be a state of a C*-algebra \( \mathfrak{A} \). Besides the strong topology, we introduce here the “\( \omega \)-strong topology” on the space of continuous linear functionals \( \mathcal{T}(\mathfrak{A},*) \). It is defined to be induced by the family of semi-norms \( \nu_n^\omega \), \( n \in \mathbb{N} \), where \( \nu_n^\omega \) assign to each functional \( F \in \mathcal{T}(\mathfrak{A},*) \) the value

\[
\nu_n^\omega(F) := \sup_{a_1,\cdots,a_n \in \ker(\omega)} \left| a_1 \cdots a_n \right|^{-1} \left| F(a_1 \cdots a_n) \right| .
\]

The reason for introducing the semi-norms \( \nu_n^\omega \) is that for the functionals we are dealing with (correlation functions of fluctuation operators) the semi-norms \( \nu_n^\omega \) are easier to estimate. In view of this, the following lemma is helpful:

**Lemma 11** For a given state \( \omega \) on \( \mathfrak{A} \), the \( \omega \)-strong topology is equivalent to the strong topology on \( \mathcal{T}(\mathfrak{A},*) \). In particular the bounds

\[
\nu_n^\omega \leq \nu_n \leq \sum_{k=0}^{n} \binom{n}{k} 2^k \nu_k^\omega
\]

are valid for all \( n \in \mathbb{N} \).

**Proof:** Each operator \( a \in \mathfrak{A} \) can be written as \( a' + c1 \) with \( a' \in \ker(\omega) \) and \( c = \omega(a) \). This yields

\[
a_1 \cdots a_n = (a'_1 + c_1 1) \cdots (a'_n + c_n 1) = \sum_{I \subset \{1,\cdots,n\}} \prod_{j \in I} c_j \otimes \otimes a_i^j .
\]

Applying the linear functional on both sides and taking the modulus, we obtain

\[
\left| F(a_1 \cdots a_n) \right| \leq \sum_{I \subset \{1,\cdots,n\}} \prod_{j \in I} \left| a_j \right| \prod_{i \in I} \left| a_i^j \right| \nu_i^\omega(F) \]

\[
\leq \left| a_1 \cdots a_n \right| \sum_{I \subset \{1,\cdots,n\}} 2^{|I|} \nu_i^\omega(F) .
\]
Here we have used that \(\|a_i'\| \leq 2\|a_i\|\). This implies the desired bound
\[
\nu_n^\omega \leq \nu_n \leq \sum_{I \in \{1,2,\ldots,n\}} 2^{|I|} \nu_\omega^{|I|}.
\]
(Note that the semi-norm \(\nu_\omega^{|I|}\) is optimizing the modulus of the value of a functional over tuples \((a_1,\ldots,a_n)\) in the kernel \(\ker(\omega)\). This immediately implies that \(\nu_n \geq \nu_n^\omega\).

**E. States with exponential clustering have weakly \(\sqrt{n}\)-fluctuations (proof of Theorem 7)**

1. **Homogenous product states (proof of Proposition 6)**

**Proof of Proposition 6.** Let \(\omega_A = \omega \otimes \Lambda\) be a homogenous product state. For operators \(a_1,\ldots,a_n \in \ker(\omega)\) the correlation function of the induced state \(\hat{\omega}_X\) is given by
\[
\hat{\omega} \otimes X(a_1 \otimes \cdots \otimes a_n) = |X|^{-\frac{3}{2}} \sum_{x \in X^n} \omega(a_1^x) \cdots \omega(a_n^x) = |X|^{-\frac{3}{2}} \sum_{k=1}^n |E_k(X)| \sum_{(I_1,\ldots,I_k) \in \Pi(k,n)} \omega(a_{I_1}) \cdots \omega(a_{I_k}),
\]
where \(a_i^x\) is the cluster operator \(a_i^x = \prod_{j \in x^{-1}(y_k)} a_j\). Let \(E_k(X)\) the set of all enumerated \(k\)-elementary subsets in \(X\) and let \(\Pi(k,n)\) the set of all ordered partitions of \(\{1,\ldots,n\}\) into \(k\) non-empty subsets. Then (E2) can be written as
\[
\hat{\omega} \otimes X(a_1 \otimes \cdots \otimes a_n) = |X|^{-\frac{3}{2}} \sum_{k=1}^n |E_k(X)| \sum_{(I_1,\ldots,I_k) \in \Pi(k,n)} \omega(a_{I_1}) \cdots \omega(a_{I_k}),
\]
with cluster operators \(a_{I_j} := \prod_{i \in I_j} a_i\). Here we have used the fact that for each tuple \(x \in X^n\) there is a unique enumerated \(k\)-elementary subset \(\{y_1,\ldots,y_k\}\) (\(k \leq n\)) and a partition \((I_1,\ldots,I_k) \in \Pi(k,n)\) such that \(x_i = y_j\) for \(i \in I_j\). Suppose that for a partition \((I_1,\ldots,I_k)\) one of the sets \(I_j\) contains only one element \(I_j = \{q\}\), then this partition is not contributing to the sum since \(\omega(a_{I_j}) = \omega(a_{\{q\}}) = 0\). Hence, only those partitions with \(|I_j| \geq 2\) are contributing. For the \(n\)-point correlation function (restricted to \(\ker(\omega) \otimes \Lambda\)) we can write
\[
\hat{\omega} \otimes X(a_1 \otimes \cdots \otimes a_n) = \prod_{l=0}^{n/2-1} \left(1 - \frac{l}{|X|}\right) \sum_{(I_1,\ldots,I_{n/2}) \in \Pi_2(n)} \omega(a_{I_1}) \cdots \omega(a_{I_{n/2}})
\]
\[
+ \sum_{k=1, k \neq n/2}^n |X|^{-n/2 + k} \prod_{l=0}^{k-1} \left(1 - \frac{l}{|X|}\right) \sum_{(I_1,\ldots,I_k) \in \Pi_{>2}(k,n)} \omega(a_{I_1}) \cdots \omega(a_{I_k}),
\]
where \(\Pi_2(n)\) is the set of all ordered partitions into two elementary subsets and \(\Pi_{>2}(k,n)\) is the set of all ordered partitions into \(k\) subsets containing more than one element \((I_1,\ldots,I_k)\), where at least one subset \(I_j\) contains more than two elements. Note that the first term in (E3) is vanishing if \(n\) is odd. \(E_k(X)\) is the set of enumerated \(k\)-elementary subsets whose cardinality is \(|E_k(X)| = \prod_{l=0}^{k-1} (|X| - l)|. Restricted to the sub-algebra \(\mathcal{T}(\ker(\omega), *)\), the functional \(\hat{\omega} \otimes X\) can be written as
\[
\hat{\omega} \otimes X = \prod_{l=0}^{n/2-1} \left(1 - \frac{l}{|X|}\right) \hat{\omega}_{\text{def}} + D_X
\]
with a continuous functional \(D_X \in \mathcal{T}(\mathfrak{A}, *)\) that is given by
\[
D_X(a_1 \otimes \cdots \otimes a_n) := \sum_{k=1}^n |X|^{-n/2 + k} \prod_{l=0}^{k-1} \left(1 - \frac{l}{|X|}\right) \sum_{(I_1,\ldots,I_k) \in \Pi_{>2}(k,n)} \omega(a_{I_1}) \cdots \omega(a_{I_k}).
\]
for $a_1, \cdots, a_n \in \ker(\omega)$. To bound the semi-norm $\nu_n^\omega(D_X)$, we observe the bound

$$|D_X(a_1 \otimes \cdots \otimes a_n)| \leq \|a_1\| \cdots \|a_n\| \sum_{k=1}^{n} |X|^{-n/2+k} \prod_{l=0}^{k-1} \left(1 - \frac{l}{|X|}\right) |\Pi_{>2}(k,n)|. \quad (E6)$$

If we consider a partition $(I_1, \cdots, I_k)$ in $\Pi_{>2}(k,n)$ then the constraint $2k+1 \leq n$ has to be fulfilled. This implies $|X|^{-n/2+k} \leq |X|^{-1/2}$ and we obtain:

$$|D_X(a_1 \otimes \cdots \otimes a_n)| \leq \|a_1\| \cdots \|a_n\| |X|^{-1/2} \sum_{k=1}^{n} \prod_{l=0}^{k-1} \left(1 - \frac{l}{|X|}\right) S(k,n) \delta(2k+1 \leq n), \quad (E7)$$

where $S(k,n) = |\Pi(k,n)|$ is the number of all partitions of the set $\{1, \cdots, n\}$ into $k$ non-empty subsets (Stirling number of second kind). For a logical statement $S$, the $\delta$-function is defined by $\delta(S) = 1$, if $S$ is true, and $\delta(S) = 0$ if $S$ is false. By Lemma 11 it is sufficient to show convergence for the semi-norms $\nu_n^\omega$. From the inequality (E7) we obtain for $|X| > n$ the bound

$$\nu_n^\omega(D_X) \leq |X|^{-1/2} \sum_{k=1}^{n} S(k,n) \delta(2k+1 \leq n) \quad (E8)$$

which implies that $s - \lim_{X \subset A} D_X = 0$. Since $\lim_{X \subset A} \prod_{l=0}^{n/2-1} \left(1 - \frac{l}{|X|}\right) = 1$, the proposition follows finally from (E4):

$$s - \lim_{X \subset A} \hat{w}^{\otimes X} = s - \lim_{X \subset A} \prod_{l=0}^{n/2-1} \left(1 - \frac{l}{|X|}\right) \hat{w}_{\text{qf}} = \left(\lim_{X \subset A} \prod_{l=0}^{n/2-1} \left(1 - \frac{l}{|X|}\right) \right) \hat{w}_{\text{qf}} = \hat{w}_{\text{qf}}. \quad (E9)$$

2. Estimating the number of subsets for a given spread

To formulate our next lemma, we introduce the function $\Delta$ that assigns to each finite subset $Y \subset A$ the maximal distance of a point in $Y$ to its complement: $\Delta(Y) := \max_{y \in Y} d(y, Y \setminus y)$. To prove bound on correlation functions, one partial task is to count for a finite subset $X \subset A$ the number $N(X, k, r)$ of all $k$-elementary ($k \geq 2$) subsets $Y$ in $X$ such that $\Delta(Y) \leq r$. Whereas the spread $\Delta$ measures the spreading of one point sets, we can also look at the spreading of sets that contain more elements. We introduce the $k$-spread of a set $Y$ as $\Delta_k(Y) = \max_{J \subset P_k(Y)} d(J, Y \setminus J)$ which measures the largest distance of a $k$-elementary subset $J \subset P_k(Y)$ to its relative complement in $Y$ (here $P_k(Y)$ denotes the set of all $k$-elementary subsets in $Y$). According to this definition, the one-spread $\Delta_1 = \Delta$ is just the spread.

**Lemma 12** Let $Y$ be a subset of $X$ with $\Delta(Y) \leq r$ and $\Delta_2(Y) > r$. Then $Y = Z \cup \{x, y\}$ is the disjoint union of a set $Z$ and a two elementary subset $\{x, y\}$ such that $\Delta(Z) \leq r$ and $d(x, y) \leq r$.

**Proof:** Since the 2-spread of $Y$ is larger than $r$, there exists a pair of points $\{x, y\}$ such that $d(\{x, y\}, Y \setminus \{x, y\}) > r$. Let $Z = Y \setminus \{x, y\}$ be the relative complement of $\{x, y\}$. Then the spread of $Y$ can be expressed as

$$\Delta(Y) = \max_{z \in Z} \{\min\{d(z, Z \setminus z), d(z, x), d(z, y)\}, \min\{d(x, Z), d(x, y)\}, \min\{d(y, Z), d(x, y)\}\}. \quad (E10)$$

Since $\Delta(Y) \leq r$ the bound $\min\{d(y, Z), d(x, y)\} \leq r$ has to be fulfilled. This implies that $d(x, y) \leq r$ because $d(y, Z) > r$ holds by our assumption. Moreover, the inequality

$$\Delta(Y) \geq \max_{z \in Z} \{\min\{d(z, Z \setminus z), d(z, x), d(z, y)\}\} \quad (E11)$$

...
is obviously fulfilled. For each \( z \in Z \) we have \( r \geq \min\{d(z, Z \setminus z), d(z, x), d(z, y)\} = d(z, Z \setminus z) \) since \( d(z, x) > r \) and \( d(z, y) > r \) is valid according to our assumption that each point in \( Z \) has a distance \( > r \) to \( x \) and \( y \). This implies the inequality

\[
gr \geq \Delta(Y) \geq \max_{z \in Z} d(z, Z \setminus z) = \Delta(Z) . \tag{E12}
\]

Hence, \( Y \) can be decomposed into a disjoint union of a set \( Z \) and two points \( \{x, y\} \) such that \( \Delta(Z) \leq r \) and \( d(x, y) \leq r \).

**Lemma 13** Let \( Y \) be a subset such that \( \Delta(Y) \leq r \) and \( \Delta_2(Y) \leq r \). Then \( Y = Z \cup \{x\} \) is the disjoint union of a set \( Z \) and a single point \( x \) such that \( \Delta(Z) \leq r \).

**Proof:** Let \( \Delta_2(Y) \leq r \), i.e. for each pair \( \{x, y\} \), the distance to the relative complement \( d(\{x, y\}, Y \setminus \{x, y\}) \leq r \). In order to discuss this case, we introduce for each point \( y \in Y \) the number \( I(y, Y, r) = |\{z \in Y \setminus y| d(y, z) \leq r\}| \) of points in the relative complement of \( y \) whose distance to \( y \) is smaller than \( r \). Now we have to perform a further case distinction:

- There exists a point \( x \in Y \) such that \( I(x, Y, r) = 1 \). In this case the spread of \( Y \) can be expressed in terms of the set \( Z = Y \setminus x \) and the single point \( x \) as

  \[
  \Delta(Y) = \max_{z \in Z} \{\min\{d(z, Z \setminus z), d(z, x)\}, d(x, Z)\} . \tag{E13}
  \]

  Suppose now that \( d(x, z) \leq r \) holds. Then this is true for a unique point \( z = z^* \in Z \). Since the 2-spread fulfills the bound \( \Delta_2(Y) \leq r \), for all \( y \in Z \setminus z^* \) the bound \( d(\{x, z^*\}, y) \leq r \) follows. This implies that \( d(z^*, y) \leq r \) since \( d(x, y) > r \) holds for all points \( y \in Z \setminus z^* \). Hence \( d(z^*, Z \setminus z^*) \leq r \).

  In all other cases, we have \( d(x, z) > r \) which implies \( r \geq \min\{d(z, Z \setminus z), d(z, x)\} = d(z, Z \setminus z) \). Putting these things together implies finally

  \[
  \Delta(Z) = \max_{z \in Z} d(z, Z \setminus z) \leq r . \tag{E14}
  \]

- For all points \( y \in Y \) we have \( I(y, Y, r) \geq 2 \). In this case we can take any point \( x \) and express the spread of \( Y \) as

  \[
  \Delta(Y) = \max_{z \in Z} \{\min\{d(z, Z \setminus z), d(z, x)\}, d(x, Z)\} . \tag{E15}
  \]

  with \( Z = Y \setminus x \). Suppose now that \( d(z, x) \leq r \) for some \( z \in Z \). Since \( I(x, Y, r) \geq 2 \) there exists at least one \( y \in Z \) such that \( d(z, y) \leq r \) which implies \( d(z, Z \setminus z) \leq r \). On the other hand, if \( d(z, x) > r \), then \( d(z, Z \setminus z) \leq r \) since \( \min\{d(z, Z \setminus z), d(z, x)\} \leq r \). This implies again

  \[
  \Delta(Z) = \max_{z \in Z} d(z, Z \setminus z) \leq r . \tag{E16}
  \]

**Lemma 14** For each finite subset \( X \), for each \( r > 0 \), and for each \( n \geq 2 \) the bounds

\[
N(X, n, r) \leq q[n]|X|^{n/2}N(r)^{n/2}
\tag{E17}
\]

are valid, where the numbers \( q[n] \) are recursively determined by \( q[n+2] = (n+1)q[n+1] + q[n] \) with initial conditions \( q[2] = 1 \) and \( q[3] = 2 \).

**Proof:** According to Lemma 12 and Lemma 13 we have the following two cases:

1. Each subset \( Y \subset X \) with \( \Delta(Y) \leq r \) is a disjoint union \( Y = Z \cup \{x, y\} \) with \( \Delta(Z) \leq r \) and \( d(x, y) \leq r \).

2. \( Y \) is a disjoint union \( Y = Z \cup \{x\} \) with \( \Delta(Z) \leq r \).
Putting both cases together, we obtain a recursion formula for bounding \( N(X, n + 2, r) \). For the first case, we can choose for each \( n \)-elementary subset \( Z \) with \( \Delta(Z) \leq r \) a pair \( \{x, y\} \) with \( d(x, y) \leq r \). This gives less than \( N(X, n, r)\) \( |X|N(r) \) possibilities. We have to add the second case, where for each \( n + 1 \)-elementary subset \( Z \) with \( \Delta(Z) \leq r \) we can just add a point \( x \) such that \( \Delta(Z \cup \{x\}) \leq r \). Since there must be a point \( z \in Z \) with \( d(x, z) \leq r \), we have \( (n + 1)N(r) \) possibilities to choose \( x \) which gives less than \( (n + 1)N(r)N(X, n + 1, r) \) possibilities. In total, we obtain the recursive bound

\[
N(X, n + 2, r) \leq (n + 1)N(r)N(X, n + 1, r) + |X|N(r)N(X, n, r) .
\]

(E18)

To get a convenient explicit solution of this recursion, we allow to over-count a bit by using the inequality \( N(r) \leq N(r)^{1/2}|X|^{1/2} \):

\[
N(X, n + 2, r) \leq (n + 1)N(r)^{1/2}|X|^{1/2}N(X, n + 1, r) + |X|N(r)N(X, n, r) .
\]

(E19)

Now, we insert the ansatz \( N(X, n, r) \leq q[n]|X|^{n/2}N(r)^{n/2} \) into the recursion bound which gives the following consistency relation:

\[
q[n+2]|X|^{(n+2)/2}N(r)^{(n+2)/2} \leq (n + 1)N(r)^{1/2}|X|^{1/2}q[n+1]|X|^{(n+1)/2}N(r)^{(n+1)/2} + |X|N(r)q[n]|X|^{n/2}N(r)^{n/2} \leq [(n + 1)q[n+1] + q[n]] |X|^{(n+2)/2}N(r)^{(n+2)/2} .
\]

(E20)

Thus, if the sequence \( q[n] \) fulfills the recursion relation \( q[n+2] = (n + 1)q[n+1] + q[n] \), then we obtain a consistent upper bound. For getting the correct initial conditions with respect to \( n \), we first look at the case \( k = 2 \). The first point of the requested set can be chosen freely in \( X \), which gives \( |X| \) possibilities. The second point must have distance \( r \) to the one firstly chosen. Hence we obtain the bound \( N(X, 2, r) \leq |X|N(r) \). For \( k = 3 \) two points within the requested set must have distance \( r \). This gives at most \( |X|N(r) \) possibilities. The third point must have a distance \( \leq r \) to one of the chosen two points. This yields \( N(X, 3, r) \leq 2|X|N(r)^2 \leq 2|X|^{3/2}N(r)^{3/2} \).

3. Towards bounding the semi-norms \( \nu_n \)

Recall that \( \Lambda \) is a countably infinite lattice with a regular distance \( d \). Here regular means that the maximal number of lattice sites within a ball of radius \( r \) is bounded by a polynomial \( P(r) \), i.e. \( N(r) = \sup_{x \in \Lambda} |\{y \in \Lambda|d(x, y) \leq r\}| \leq P(r) \) for all \( r \in \mathbb{R}_+ \).

If we require that the distance \( d \) is bounded, than \( N(r) \) is bounded by a polynomial which implies that \( \hat{N}_k = \sum_{r=1}^{\infty} N(r)^k e^{-r} \) is finite and monotonous increasing in \( k \). For each finite subset \( X \), the semi-norms \( \nu_n \) of the induced state \( \hat{\omega}_X = \hat{\omega}^\otimes X + F_X \) can be bounded by bounding the corresponding semi-norms of \( \hat{\omega}^\otimes X \) and \( F_X \) separately. We know already from Proposition 6 that the semi-norms of \( \hat{\omega}^\otimes X \) are uniformly bounded in \( X \). Thus it is sufficient to bound the semi-norms of \( F_X \). For this purpose, we introduce the quantities

\[
B_n(X) := \sum_{x \in X^n} \sum_{k=1}^{\lfloor \frac{\text{Ran}(x)-1}{k-1} \rfloor} \prod_{l=1}^{k-1} \delta(|x^{-1}(y_l)| > 1) e^{-\Delta(y_k, y_{k+1}, \ldots, y_{\lfloor \text{Ran}(x) \rfloor})} .
\]

(E21)

Lemma 15 For each finite subset \( X \subset \Lambda \), \( B_n(X) \) fulfill the bound

\[
B_n(X) \leq \hat{B}_n |X|^{n/2}
\]

(E22)

for each finite subset \( X \subset \Lambda \), where \( \hat{B}_n \) are finite positive numbers that are given by

\[
\hat{B}_n = n! \sum_{k=1}^{n} S(k, n) \sum_{l=1}^{k-1} q[k-l+1] \hat{N}_{k-l+1} .
\]

(E23)

Here \( S(k, n) \) are the Stirling numbers of the second kind and the numbers \( q[n] \) are recursively determined by \( q[n+2] = (n + 1)q[n+1] + q[n] \) with initial conditions \( q[2] = 1 \) and \( q[3] = 2 \).
Proof: To bound \( B_n(n)\), we take advantage of the following fact: Let \( P(n, k) \) be the set of partitions of \( n \) into \( k \) non-vanishing summands. For each tuple \( x \in X \) there exists a \( k \)-elementary subset \( Y = \{y_1, \cdots, y_k\} \subset X \) with \( k \leq n \) and a partition \( (n_1, \cdots, n_k) \in P(k, n) \) as well as a permutation \( \sigma \in S_n \) such that \( x = \sigma(y_1^{n_1}, \cdots, y_k^{n_k}) \) where \( y^{x} \) is the \( l \)-tuple with constant entry \( y \). Note that the permutation \( \sigma \) is not unique, since permuting one of the sub-tuples \( y_i^{x} \) leaves \( x \) invariant. This implies the following bound for \( B_n(X) \):

\[
B_n(X) \leq n! \sum_{l=1}^{n} \sum_{y} \delta(n_1 > 1) \cdots \delta(n_{l-1} > 1) e^{-\Delta(y_1, \cdots, y_k)}. \tag{E24}
\]

For fixed \( l < k \), each partition \( (n_1, \cdots, n_k) \) which contributes to the sum has to fulfill the condition \( l + k - 1 \leq n \). The constrained given by the “delta-functions” \( \delta(n_1 > 1) \cdots \delta(n_{l-1} > 1) \) can be replaced by the weaker constraint \( \delta(l + k - 1 \leq n) \) which yields an upper bound of the right hand side:

\[
B_n(X) \leq n! \sum_{l=1}^{n} \sum_{y} \delta(l + k - 1 \leq n) e^{-\Delta(y_1, \cdots, y_k)}. \tag{E25}
\]

For each \( l < k \) the sum over \( k \)-elementary subsets can be estimated as follows:

\[
\sum_{\{y_1, \cdots, y_k\} \subset X} e^{-\Delta(y_1, \cdots, y_k)} \leq |X|^{l-1} \sum_{\{y_1, \cdots, y_k\} \subset X} e^{-\Delta(y_1, \cdots, y_k)} \leq |X|^{l-1} \sum_{r=1}^{\infty} N(X, k-l+1, r) e^{-r} \leq q[k-l+1]|X|^{(k+l-1)/2} \tilde{N}_{k-l+1}
\]

where we have used Lemma[14]. This yields the desired bound

\[
B_n(X) \leq n! \sum_{k=1}^{n} S(k, n) \sum_{l=1}^{k-1} q[k-l+1] \tilde{N}_{k-l+1} |X|^{n/2} \tag{E27}
\]

which completes the proof of the lemma.  

Proof of Theorem[7]: By taking advantage of the expansion of Proposition[8] we can write the state \( \hat{\omega}_X = \hat{\omega}^{\otimes X} + F_X \). Let us assume first, that the operators \( a_1, \cdots, a_n \) belong to the kernel of the single site restriction \( \omega \). We first bound the correlation function of the functional \( F_X \).

\[
|F_X(a_1 \otimes \cdots \otimes a_n)| \leq |X|^{-n/2} \sum_{x \in X^n} \prod_{k=1}^{[\text{Ran}(x)]} e^{-\Delta(y_k, y_{k+1}, \cdots, y_{[\text{Ran}(x)]})}, \tag{E28}
\]

where we have used the fact that \( \omega(a_k^{x}) = 0 \) whenever the pre-image \( |x^{-1}(y)| = 1 \) contains only one element. By inserting the definition of the quantities \( B_n(X) \) we find from Lemma[15]

\[
\nu_n^\omega(F_X) \leq |X|^{-n/2} G_0 B_n(X). \tag{E29}
\]

By Proposition[6] the semi-norms \( \nu_n^\omega(\hat{\omega}^{\otimes X}) \) are uniformly bounded, i.e. for each \( n \in \mathbb{N} \), the constant \( \hat{A}_n := \sup_{X \in A} \nu_n^\omega(\hat{\omega}^{\otimes X}) < \infty \) is finite. Therefore we obtain for the induced state \( \hat{\omega}_X \) the semi-norm bound:

\[
\nu_n^\omega(\hat{\omega}_X) \leq \hat{A}_n + G_0 \hat{B}_n \tag{E30}
\]
Hence the semi-norm $\nu_n$ can be bounded by $\nu_n^\omega$ according to Lemma [11]:

$$\nu_n(F) \leq \sum_{k=0}^{n} \binom{n}{k} 2^k \nu_k^\omega(F) .$$  \hspace{1cm} (E31)

By applying this bound to $\hat{\omega}_X$ we obtain

$$\nu_n(\hat{\omega}_X) \leq \sum_{k=0}^{n} 2^k \binom{n}{k} (\hat{A}_k + \hat{G}_0 \hat{B}_k) .$$  \hspace{1cm} (E32)

This implies that $\sup_{X \subseteq \Lambda} \nu_n(\hat{\omega}_X) < \infty$ and the net $(\hat{\omega}_X)_{X \subseteq \Lambda}$ has weakly $\sqrt{n}$-fluctuations.