Shape space figure-8 solution of three body problem with two equal masses

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Abstract
In a preprint by Montgomery (https://people.ucsc.edu/~rmont/Nbdy.html), the author attempted to prove the existence of a shape space figure-8 solution of the Newtonian three body problem with two equal masses (it looks like a figure 8 in the shape space, which is different from the famous figure-8 solution with three equal masses (Chenciner and Montgomery 2000 Ann. Math. 152 881–901)). Unfortunately there is an error in the proof and the problem is still open.
Consider the \( \alpha \)-homogeneous Newton-type potential, \( 1/r^\alpha \), using action minimization method, we prove the existence of this solution, for \( \alpha \in (1,2) \); for \( \alpha = 1 \) (the Newtonian potential), an extra condition is required, which unfortunately seems hard to verify at this moment.

Keywords: three body problem, variational method, syzygy sequence

(Some figures may appear in colour only in the online journal)

1. Introduction
The motion of three point masses \( m_j > 0 \), \( j = 1, 2, 3 \), in a plane under \( \alpha \)-homogeneous Newtonian-like potential is described by the following equation

\[
m_j \ddot{x}_j = \frac{\partial}{\partial x_j} U(x), \quad j = 1, 2, 3,
\]

(1)
where \( x = (x_1, x_2, x_3) \in \mathbb{C}^3 \) with \( x_j \in \mathbb{C} \) representing the position of \( m_j \), and \( U(x) \) is the (negative) potential energy.

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\[ U(x) = \sum_{1 \leq j < k \leq 3} \frac{m_i m_k}{\alpha |x_j - x_k|^\alpha}. \]  

(2)

In the literature, the potential is usually referred as a strong force when \( \alpha \geq 2 \) and a weak force when \( 0 < \alpha < 2 \). The Newtonian potential corresponds to \( \alpha = 1 \). In this paper we only consider \( 1 \leq \alpha < 2 \) and to emphasize the specialty of the Newtonian potential, the term weak force will only refer to cases \( 1 < \alpha < 2 \).

Equation (1) is the Euler–Lagrange equation of the action functional

\[ A_L(x, [T_1, T_2]) = \int_{T_1}^{T_2} L(x, \dot{x}) \, dt, \quad x \in H^1([T_1, T_2], \mathbb{C}^3), \]  

(3)

where \( L(x, \dot{x}) = K(\dot{x}) + U(x) \) is the Lagrange and \( K(\dot{x}) = \frac{1}{2} \sum_{j=1}^{3} m_j |\dot{x}_j|^2 \) is the kinetic energy. For simplicity, set \( A_L(x, T) = A_L(x, [0, T]), \) for any \( T > 0 \).

As the three body problem is invariant under linear translation, the center of mass can be fixed at the origin, and we set

\[ \mathcal{X} = \{ x \in \mathbb{C}^3 : \sum_{j=1}^{3} m_j x_j = 0 \}. \]

The subset of collision-free configurations will be denoted by \( \hat{\mathcal{X}} = \{ x \in \mathcal{X} : x_j \neq x_k, \forall 1 \leq j < k \leq 3 \} \).

In the celebrated paper by Chenciner and Montgomery [6], the famous figure-8 solution (three equal masses chase each other on a fixed loop in a plane with the shape of figure 8) was proved. It belongs to a special class of solutions now known as simple choreographies, see [5, 23].

An interesting story was told in the appendix of [6] regarding the origin of the paper: A preprint titled ‘Figure 8s with Three Bodies’ by Montgomery [12], was submitted to Nonlinearity and Chenciner was asked to be the referee. In the preprint, Montgomery attempted to prove the existence of two zero angular momentum reduced periodic solutions (periodic after modulo a proper rotation) of the Newtonian three body problem. One under the condition that all three masses are equal and the other under the condition that two masses are equal to each other. While the first solution eventually leads to the proof of the figure-8 solution in [6], the proof of the second solution was found to be an error.

What is interesting is that the ‘figure 8s’ in the title of the preprint was referring to the second solution, because once we established its existence, it should have the following features: if one puts the system in a moving frame such that the line between the two equal masses is fixed and further apply a time-dependent homothety to the motion so that they are fixed points on the lines all the time, then the third mass moves along a curve having the topological type of a figure 8 with each circle of the 8 surrounding one of the two equal masses. This means after projecting the solution to the shape space, it is a figure 8 surrounding two of the binary collision rays, see section 2. Because of this, this solution will be called the shape space figure-8 solution throughout the paper.

The approach proposed by Montgomery in [12] was to find the desired solution as a minimizer of the action functional under certain symmetric and topological constraints. The main difficulty is to show the minimizer is collision-free. In the last 15 years, lots of progress has been made to overcome this. We briefly summarize in the following.

- **Local deformation**: based on asymptotic analysis near an isolated collision and the ‘blow-up’ technique, one tries to show after a local deformation of the collision path near
the isolated collision, there is a new path with strictly smaller action value. For the details see [4, 7, 14] and [21].

- **Level estimate**: one gives a sharp lower bound estimate of the action functional among all the collision paths in the set of admissible paths and then try to find a collision-free test path within the set of admissible paths, whose action value is strictly smaller than the previous lower bound estimate. See [1–3, 6].

Despite of all the progress, a proof of the shape space figure-8 solution is still missing for any $1 \leq \alpha < 2$.

In this paper, based on some new local deformation result near an isolated binary collision (see section 4), we show the approach laid out by Montgomery in [12] can be carried out and a shape space figure-8 solution exists, for weak force potentials ($1 < \alpha < 2$); for the Newtonian potential ($\alpha = 1$), an extra condition (4) is required. To be precise, we have the following results.

**Theorem 1.1.** Assume $m_1 = m_2$. When $1 < \alpha < 2$, for any $\bar{T} > 0$, there is a zero angular momentum periodic solution $x \in C^2(\mathbb{R}/\bar{T} \mathbb{Z}, \mathbb{C}^3)$ of equation (1) satisfying:

(a) when $t \in \{0, \bar{T}/2\}$, $x(t) \in \mathbb{R}^3 \subset \mathbb{C}^3$ and

\[
\begin{align*}
x_1(0) & < x_2(0) < x_3(0), \\
x_3(\bar{T}/2) & < x_1(\bar{T}/2) < x_2(\bar{T}/2);
\end{align*}
\]

(b) when $t \in \{\bar{T}/4, 3\bar{T}/4\}$, $x(t)$ is an Euler configuration with $x_3(t) = 0$ and $x_1(t) = -x_2(t)$;

(c) when $t \in [0, \bar{T}/2]$, $x(t)$ experiences exactly four collinear configurations at the instants $t = 0, \bar{T}/4, \bar{T}/2, \text{and} 3\bar{T}/4$;

(d) for any $t \in \mathbb{R}/\bar{T} \mathbb{Z}$,

\[
\begin{align*}
(x_1, x_2, x_3)(t) & = (-x_2, -x_1, -x_3)(\bar{T}/2 - t); \\
(x_1, x_2, x_3)(t) & = (\bar{x}_1, \bar{x}_2, \bar{x}_3)(-t),
\end{align*}
\]

where $\bar{x}_j$ is the complex conjugate of $x_j$.

When $\alpha = 1$, the above results hold if the following condition is satisfied

\[
A_L(x^*, \bar{T}/4) > \inf\{A_L(y, \bar{T}/4) : y \in \Omega\},
\]

(4)

where $x^*$ is a Schubart solution (it is a collinear collision solution defined in definition 2.2) and $\Omega$ is the set of admissible paths defined as in definition 2.1.

Only **local deformation** are used throughout the paper to rule out collisions and in some sense theorem 1.1 is the best result, one could expect based on this method, see remark 4.1. It will be really interesting, if someone can verify or disprove condition (4) for certain choices of masses. At this moment, this seems hard to do. In general, **level estimate** used in [2] and [3] can give a good lower bound estimate of the action value of a collision path. However it is not the case when the collision path is collinear (all three masses stay on a single line all the time), which is exactly the case for a Schubart solution.

2. Geometry of the shape space

A proof of theorem 1.1 will be given in this section. For this, we briefly recall the basics of the shape space and the syzygy sequences of the planar three body problem. The details can be found in a series of papers by Montgomery: [12–14, 16], as well as the beautiful expository article [17] by him.
Since the center of mass is fixed at the origin, the configuration space $\mathcal{X}$ has two complex dimension and we define the mass-dependent Jacobi map $\mathcal{J} : \mathcal{X} \to \mathbb{C}^2$ as following:

$$(z_1, z_2) = \mathcal{J}(x_1, x_2, x_3) := \left( \sqrt{\mu_1(x_2 - x_1)}, \sqrt{\mu_2(x_3 - \frac{m_1x_1 + m_2x_2}{m_1 + m_2})} \right)$$

where $\mu_1 = \frac{m_3}{m_1 + m_2}$ and $\mu_2 = \frac{m_1m_2}{m_1 + m_2}$. The action of $SO(2)$ on $\mathbb{C}^2$ corresponds to multiplication by $e^{i\theta}$; $e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$, $\theta \in \mathbb{R}$. The quotient space $\mathcal{C} = \mathbb{C}^2/\text{SO}(2)$ can be given by the Hopf map $\mathcal{H} : \mathbb{C}^2 \to \mathcal{C}$

$$w = (w_1, w_2, w_3) = \mathcal{H}(z_1, z_2),$$

where

$$w_1 := \frac{1}{2}(\vert z_1 \vert^2 - \vert z_2 \vert^2), \ w_2 + iw_3 := z_1\overline{z}_2.$$

In the following, we set

$$\pi : \mathcal{X} \to \mathcal{C} \cong \mathbb{R}^3; \ \pi(x) = \mathcal{H}(\mathcal{J}(x)).$$

We remark that $w_3$ represents the signed area of the corresponding triangle in the configuration space multiplied by a constant depending on the masses, so $\pi$ maps the collinear configurations (all three masses lie on a single straight line) to the plane $\{w_3 = 0\}$. In astronomy, a collinear configuration is also called a syzygy, see [11, 16]. Depending on which mass is in the middle, we call it a type-$j$ syzygy. The plane $\{w_3 = 0\} \subset \mathcal{C}$ will be called the syzygy plane.

Let $I(x) = \sum_{j=1}^{3} m_j \vert x_j \vert^2$ be the moment of inertia. When $w = \pi(x)$, by a direct computation, $I(x) = 4\vert w \vert$. The two dimensional sphere $S = \{w \in \mathcal{C} : \vert w \vert = 1/2\}$ represents all oriented similarity classes of triangles and will be referred as the shape sphere. We introduce a spherical coordinates $(R, \phi, \theta)$ in $\mathcal{C}$ according to $R = \sqrt{I(x)} = \sqrt{2}\vert w \vert$ and

$$(w_1, w_2, w_3)/\vert w \vert := (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi).$$

Although topologically the shape space $\mathcal{C}$ is homeomorphic to $\mathbb{R}^3$, as a metric space it is not isometric to $\mathbb{R}^3$, but to the cone $c(S)$, where its metric $ds_{\text{shape}}^2$ is given by

$$ds_{\text{shape}}^2 = dR^2 + \frac{R^2}{4}(\cos^2 \phi d\theta^2 + d\phi^2).$$

In general, the cone $c(M)$ over a Riemannian manifold $M$ with metric $ds^2$ is a metric space with one higher dimension. Topologically it is homeomorphic to $M \times [0, +\infty)/M \times \{0\}$ with $M \times \{0\}$ collapse to a single point. Given a $w \in S$ (or $w \in \mathcal{C}$), we use $c(w)$ (or $c(W)$) to represent the cone over $w$ (or $W$).

There are five different normalized central configurations (modulo rotations): two of them are the equilateral Lagrangian configurations $L^\pm$ with different orientations; the other three are the collinear Euler configurations $E_j, j = 1, 2, 3$, with $m_j$ in the middle. In the shape sphere, $E_j$s are located on the equator $S \cap \{w_3 = 0\}$, $L^+$ in the upper semi-sphere and $L^-$ in the lower semi-sphere (when all the masses are equal, $L^\pm$ are at the north and south poles correspondingly). See the left picture in figure 1.

The three $E_j$s divide the equator into three disjoint arcs. Each arc contains a binary collision $h_{jk}, 1 \leq j < k \leq 3$ (the sub-index means the collision is between $m_j$ and $m_k$). $c(h_{jk})$ represents the three binary collision rays emanating from the origin. After deleting them from
the shape space, \( C^* = C \setminus \{c(\beta_k) : 1 \leq j < k \leq 3 \} \) has nontrivial fundamental group. As \( C^* \) is homotopic to \( S \setminus \{b_{jk} : 1 \leq j < k \leq 3 \} \), its fundamental group is isomorphic to the projective colored braids group, see [13].

When the center of mass is fixed at the origin, through Saari’s decomposition of Kinetic energy

\[
K(\dot{x}) = K_{\text{shape}}(\dot{x}) + \frac{|J(x)|}{2I(x)} \quad \text{where} \quad 2K_{\text{shape}}(\dot{x}) = \ddot{R}^2 + \frac{R^2}{4} (\cos^2(\phi)\dot{\theta}^2 + \dot{\phi}^2).
\]

Here \( J(x) \) is the angular momentum.

Given any two points in the shape space, they represent two oriented congruence classes of triangles in \( X \) and are invariant under the \( SO(2) \) action. If \( x(t) \) is a minimizer of the action functional \( A_L \) among all paths connecting those two oriented congruence classes, it must have zero angular momentum. Because otherwise, we can find a proper path \( g(t) \in SO(2) \) with \( g(t)x(t) \) having zero angular momentum, and by (5), its action value will be strictly smaller than \( x(t) \), which is absurd.

As a result, the above minimization problem is equivalent to the minimization problem in the shape space with the corresponding Lagrange:

\[
L_{\text{shape}} = K_{\text{shape}} + U.
\]

Inspired by the basic theorem that every nontrivial free homotopy class of a compact Riemannian manifold can be realized by a minimal closed geodesic, Montgomery asked the following questions in [14] (similar questions were also asked by Wu–Yi Hsiang):

**Question 1.** Can each free homotopy class of \( C^* \) be realized by a zero angular momentum reduced periodic solution of the Newtonian three body problem?

**Question 2.** The same question as above but without the condition of angular momentum being zero.
A nice way to represent the free homotopy classes is using the syzygy sequences. Given a free homotopy class of $C^*$, a generic loop contained in this class has a discrete set of syzygies. Write down the syzygy types, namely 1, 2 or 3, experienced by the loop following their temporal order in a single period, we got a corresponding syzygy sequence of the loop. Such a sequence may have two or more of the same letters from $\{1, 2, 3\}$ appearing consecutively in a row. Such a phenomena will be called stutter. After canceling all the stutters, the remaining sequence will be called the reduced syzygy sequence of the loop.

If two generic loops are in two different free homotopy classes, they have different reduced syzygy sequences; if they are in the same free homotopy class, then they have the same reduced syzygy sequence (one may need to reverse the time on one of the loop). As a result, there is a 2-to-1 map from the periodic reduced syzygy sequences to the free homotopy classes of $C^*$, except the empty sequence, which is the only pre-image of the trivial free homotopy class.

Now questions 1 and 2 can be rephrased with free homotopy class replaced by reduced syzygy sequence.

In a recent paper [10] by Moeckel and Montgomery, question 2 was answered affirmatively using non-variational method. However their proof requires the angular momentum to be small but non-zero, which leaves question 1 still open. Meanwhile zero-angular momentum solutions can be found as action minimizations of the action functional defined by $L_C$. If we replace the Newtonian potential by a strong force potential, then question 1 was answered completely by Montgomery in [14] using variational approach, as in this case a minimizer can not have any collision. However when the potential is Newtonian or a weak force, for most free homotopy classes, an action minimizer is likely to contain collision, see [15].

Despite of this, for some free homotopy classes or reduced syzygy sequences, the variational approach may still work, if we can impose additional constraints on the paths. As it is well-known now, this can be done when some of the masses are equal. For example the figure-8 solution of three equal masses is an action minimizer under certain symmetric constraints, and it is a zero angular momentum solution realizing the periodic reduced syzygy sequence 123 123.

Besides the above example, when $m_1 = m_2$, 2313 may also be realizable by a zero angular momentum reduced periodic solution as suggested by Montgomery in [15]. In the following, the readers will see the zero angular momentum periodic solution obtained in theorem 1.1 does indeed realize this syzygy sequence. Obviously corresponding results hold for 3212 (or 2131), when $m_1 = m_3$ (or $m_2 = m_3$).

Following the approach given in [12], we first formulate the corresponding action minimization problem. Let $T_0 = T/4$ for the rest of the paper.

**Definition 2.1.** We define $\Omega$ as the subspace of $H^1([0, T_0], \mathbb{X})$ consisting of all paths, starting at a type-2 syzygy with all the three masses lying on the real axis in the order: $x_1 < x_2 < x_3$ of arbitrary size and ending at the Euler configuration $E_3$ of arbitrary size and arbitrary angle. The weak closure of $\Omega$ in $H^1([0, T_0], \mathbb{X})$ will be denoted by $\Omega$.

After projecting to the shape space $\pi(\Omega) = \{\pi(x) : x \in \Omega\}$ contains all the paths starting at the closed sector in the syzygy plane between $c(b_{12})$ and $c(b_{23})$, and ending at $c(E_3)$.

**Proposition 2.1.** Assume $m_1 = m_2$. When $1 < \alpha < 2$, the infimum of the action functional $A_L$ in $\Omega$ is a minimum and any action minimizer $x \in \Omega$ must be collision-free.
In the following, let $H^1([0, T_0], \mathbb{R}^3)$ be the space of collinear $H^1$-paths.

**Proposition 2.2.** Assume $m_1 = m_2$. When $\alpha = 1$, the infimum of the action functional $A_L$ in $\Omega$ is a minimum and any action minimizer $x \in \Omega$ must satisfies one of the followings:

1. $x$ is collision-free;
2. $x \in \Omega \cap H^1([0, T_0], \mathbb{R}^3)$ contains a single binary collision at $t = 0(x_1(0) < x_2(0) = x_3(0))$, and it is a quarter of a Schubart solution.

**Definition 2.2.** When $m_1 = m_2$, a collision solution $x^* : \mathbb{R}/T^2 \to \mathcal{X}$ of (1) is called a Schubart solution, if $x(t) \in \mathbb{R}^3$, for any $t$, and the following conditions are satisfied:

1. $x^*$ satisfies condition (d) in theorem 1.1;
2. $x_1^*(0) < 0 < x_2^*(0) = x_3^*(0)$ and $\dot{x}_i^*(0) = 0$;
3. for $t \in (0, T_0)$, $x_3^*(t)$ is strictly decreasing, $x_j^*(t)$, $j = 1, 2$, is strictly increasing and $x_1(t) < x_3(t) < x_2(t)$;
4. $x_1^*(T_0) = 0$ and $x_2^*(T_0) = -x_3^*(T_0)$.

This solution was discovered numerically by Schubart [18] in the case of Newtonian potential and the following was proved by Venturelli [22]

**Proposition 2.3.** When $\alpha = 1$ and $m_1 = m_2$, the infimum of the action functional $A_L$ in $\Omega \cap H^1([0, T_0], \mathbb{R}^3)$ is a minimum and an action minimizer is a quarter of a Schubart solution.

By this result, when $m_1 = m_2$, if a minimizer $x \in \Omega$ obtained in proposition 2.2 is not collision-free, it must be a quarter of a Schubart solution.

The proofs of propositions 2.1 and 2.2 will be given in sections 5 and 6. Right now we give a proof of theorem 1.1 based on them.

**Proof of Theorem 1.1.** First let us assume $1 < \alpha < 2$. By proposition 2.1, there is an $x \in C^3([0, T_0], \mathcal{X})$, which is a collision-free minimizer of $A_L$ in $\Omega$. Therefore $x$ is a solution of (1) with zero angular momentum.

Recall that the corresponding path $\pi(x(t)), t \in [0, T_0]$, in the shape space starts and ends on the syzygy plane. We claim it does not intersect the syzygy plane for all $t \in (0, T_0)$. First, if the path insect the syzygy plane tangentially, then it must stay on the syzygy plane for all $t \in [0, T_0]$, and for topological reason this will lead to a collision, which is a contradiction to proposition 2.1.

If $x$ intersects transversally the syzygy plane at a moment $t_0 \in (0, T_0)$, then the path $\tilde{x}$ defined as following

$$\tilde{x}(t) = \begin{cases} (x_1, x_2, x_3)(t), & \text{if } t \in [0, t_0], \\ (\bar{x}_1, \bar{x}_2, \bar{x}_3)(t), & \text{if } t \in [t_0, T_0], \end{cases}$$

belongs to $\Omega$ and $A_L(\tilde{x}) = A_L(x)$, but $\tilde{x}$ is not smooth at the moment $t = t_0$. This gives again a contradiction.

We can continue the solution $x|_{[0, T_0]}$ to the time interval $[T_0, 2T_0]$ by twisting it through the Euler configuration $E_3$ and reversing the time:

$$x(T_0 + t) = H_3(x(T_0 - t)), \quad \forall t \in [0, T_0],$$

where the twist $H_3(x_1, x_2, x_3) = -(x_2, x_1, x_3)$ is the composition of reflection about the syzygy plane with reflection about the plane $c(M_3) = \{ \pi(x) : |x_3 - x_1| = |x_3 - x_2| \} \cap \tilde{S}$ is a
meridian on the space sphere, see the left picture in figure 1). It is a symmetry of the action when \( m_1 = m_2 \). In the inertial plane, it is realized by the following equation
\[
(x_1, x_2, x_3)(T_0 + t) = (-x_2, -x_1, -x_3)(T_0 - t), \quad \forall t \in [0, T_0].
\]

Given a \( f = (f_1, f_2, f_3) \in H^1([0, T_0], \mathcal{X}) \), the first variation of \( A_L \) at \( x \) along \( f \) can be computed as following:
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} A_L(x + \varepsilon f, T_0) = \lim_{\varepsilon \to 0} \frac{A_L(x + \varepsilon f, T_0) - A_L(x, T_0)}{\varepsilon} \\
= \int_0^{T_0} \sum_{j=1}^3 m_j \langle \dot{x}_j, f_j \rangle + \sum_{j=1}^3 \langle \partial_x U(x), f_j \rangle \, dt \\
= \left[ \sum_{j=1}^3 m_j \langle \dot{x}_j(T_0), f_j(T_0) \rangle \right]_0^{T_0} + \int_0^{T_0} \sum_{j=1}^3 \langle -m_j \ddot{x}_j + \partial_x U(x), f_j \rangle \, dt \\
= \sum_{j=1}^3 m_j \langle \dot{x}_j(T_0), f_j(T_0) \rangle - \sum_{j=1}^3 m_j \langle \dot{x}_j(0), f_j(0) \rangle.
\]

If \( x + \varepsilon f \in \Omega \), for \( |\varepsilon| > 0 \) small enough, since \( x \) is a minimizer of \( A_L \) in \( \Omega \), the above first variation must be zero
\[
\sum_{j=1}^3 m_j \langle \dot{x}_j(T_0), f_j(T_0) \rangle - \sum_{j=1}^3 m_j \langle \dot{x}_j(0), f_j(0) \rangle = 0.
\]

By the boundary constraint the three masses must form an Euler configuration \( E_3 \) at \( t = T_0 \), then (7) implies \( \dot{x}_1(T_0) = \dot{x}_2(T_0) \). As a result, \( x|[0,2T_0] \) is a smooth path and a solution of (1).

Similarly the boundary constraint implies at \( t = 0 \), \( x \) must satisfy
\[
x_j(0) \in \mathbb{R}, \quad \forall j \in \{1, 2, 3\}, \quad \text{and} \quad x_1(0) < x_2(0) < x_3(0).
\]

Then (7) implies \( \text{Re} \ (\dot{x}_j(0)) = 0 \), for any \( j \in \{1, 2, 3\} \). Combining with (6), this shows \( \text{Re} \ (\dot{x}_j(2T_0)) = 0 \), for any \( j \in \{1, 2, 3\} \) and
\[
x_j(2T_0) \in \mathbb{R}, \quad \forall j \in \{1, 2, 3\}, \quad \text{and} \quad x_3(0) < x_1(0) < x_2(0).
\]

Defining
\[
x(t) = (x_1, x_2, x_3)(t) = (\bar{x}_1, \bar{x}_2, \bar{x}_3)(-t), \quad \forall t \in [-2T_0, 0],
\]
and extending \( x(t) \) by \( 4T_0 \)-periodicity, we get a \( 4T_0 \)-periodic solution satisfying all the required properties.

Furthermore such a periodic solution realize the syzygy sequence 2313. Notice that in the shape space the above identity represents reflection about the syzygy plane and reversing on time.

When \( \alpha = 1 \), by proposition 2.2, there is an \( x \in \Omega \), which is a minimizer of the action functional \( A_L \) in \( \Omega \). If condition (4) holds, then by propositions 2.2 and 2.3, \( x \) must be collision-free. The rest follows from the same argument given as above. □
3. Local deformation

We prove several local deformation results in this section that will be needed. Results given in this section hold for the \(N\)-body problem in \(\mathbb{R}^d\), for any \(N, d \geq 2\),

\[
m_j \ddot{x}_j = \frac{\partial}{\partial x_j} U(x), \quad j = 1, \ldots, N, \tag{8}
\]

with the corresponding Lagrangian \(L(x, \dot{x}) = K(\dot{x}) + U(x)\),

\[
K(\dot{x}) = \frac{1}{2} \sum_{j=1}^{N} m_j |\dot{x}_j|^2, \quad U(x) = \sum_{1 \leq j < k \leq N} \frac{m_j m_k}{\alpha |x_j - x_k|^{\alpha}}.
\]

There are two different approaches of local deformation: one of them gets a contradiction by showing the average action value of locally deformed paths along a large enough set of directions is smaller than the action value of the collision minimizer (this was first introduce by Marchal [4], then generalized by Ferrario and Terracini [7]); the other one gets a contradiction by showing directly the action value of some locally deformed path along some properly chosen direction is smaller than the action value of the collision minimizer (to our knowledge, this method first appeared in [12], and then [21]).

In general the first approach is more powerful. In particular when there is only symmetric constraints, it only requires the rotating circle property to be satisfied, see [7], although it does not work when topological constraints are also imposed. This makes it not applicable to the shape space figure-8 solution, as the corresponding symmetric constraints do not satisfy the rotating circle property and topological constraints are involved in some sense (although it makes boundary constraints after symmetries). While the second approach is not as powerful as the first one, it can be applied to problem with certain symmetric and topological constraints.

In the following, we prove some local deformation results based on the second approach. The results are essentially the same as those given in [12] and [21]. We include them here for two reasons: first in the previous references the results are given only for the Newtonian potential; second this gives us the opportunity to introduce some notations and technical results that will be needed in the next section.

We will follow the notations introduced in [7]. Let \(n := \{1, \ldots, N\}\), for any subset of indices \(k \subset n\), we set

\[
K_k(\dot{x}) = \frac{1}{2} \sum_{j \in k} m_j |\dot{x}_j|^2, \quad U_k(x) = \sum_{j \in k} \sum_{k \neq l \in k} \frac{m_j m_k}{\alpha |x_j - x_k|^{\alpha}}. \tag{9}
\]

Then the corresponding Lagrange \(L_k\) and energy \(E_k\) of the \(k\)-cluster is defined by

\[
L_k(x, \dot{x}) = K_k(\dot{x}) + U_k(x), \quad E_k(x, \dot{x}) = K_k(\dot{x}) - U_k(x). \tag{10}
\]

Any \(y = (y_j)_{j \in k}\) with \(y_j \in \mathbb{R}^d\) will be called a \(k\)-configuration, and a centered \(k\)-configuration, if \(\sum_{j \in k} m_j y_j = 0\). The set of all centered \(k\)-configurations will be denoted by

\[
X^k := \{y = (y_j)_{j \in k} : \sum_{j \in k} m_j y_j = 0\}.
\]

In this section, by a configuration we will only mean an \(n\)-configuration. Given a configuration \(x\) and a \(k\)-configuration \(y\), \(z = x + y\) is defined as

\[
z_j = x_j + y_j, \quad j \in k; \quad z_j = x_j, \quad j \in n \setminus k.
\]
Definition 3.1. Given an path \( x \in H^1([0, \delta], \mathcal{X}) \) with \( \delta > 0 \), we say \( x(0) \) is an isolated \( k \)-cluster collision, if \( x(t), t \in (0, \delta] \) is collision-free and \( x(0) \) has a \( k \)-cluster collision, i.e.

\[
x_k(0) = x_0(0), \quad \forall k \in \mathbf{k}; \quad x_j(0) \neq x_0(0), \quad \forall j \in \mathbf{n} \setminus \mathbf{k},
\]

where \( x_0(t) \) is the center of mass of the \( \mathbf{k} \)-body,

\[
x_0(t) = \frac{\sum_{k \in \mathbf{k}} m_k x_k(t)}{m_0}, \quad m_0 := \sum_{k \in \mathbf{k}} m_k. \tag{11}
\]

Furthermore if \( x(t), t \in (0, \delta) \) is solution of (8), then it will be called an isolated \( k \)-cluster collision solution.

In the rest of this section, \( x \in H^1([0, \delta], \mathcal{X}) \) will always represent an isolated \( k \)-cluster collision solution. Let \( I_k \) be the moment of inertia of the \( \mathbf{k} \)-cluster with respect to their center of mass \( x_0 \), and \( q \) the relative positions of \( m_k, k \in I_k \), with respect to \( x_0 \),

\[
I_k = I_k(x) = \sum_{k \in \mathbf{k}} m_k |x_k - x_0|^2, \tag{12}
\]

\[
q = (q_k)_{k \in \mathbf{k}} = (x_k - x_0)_{k \in \mathbf{k}}. \tag{13}
\]

While \( I_k \) represents the size of the \( k \) cluster, its shape can be described by the normalized centered \( k \)-configuration

\[
s = (s_k)_{k \in \mathbf{k}} = (I_k^{-\frac{1}{2}} (x_k - x_0))_{k \in \mathbf{k}} = (I_k^{-\frac{1}{2}} q_k)_{k \in \mathbf{k}}. \tag{14}
\]

Obviously \( I_k(s) = 1 \).

Choose a sequence of times \( \{t_n \} \) such that \( \lim_{n \to \infty} t_n = 0 \) and \( \{s(t_n)\} \) converges to a normalized centered \( \mathbf{k} \)-configuration \( s = (s_k)_{k \in \mathbf{k}} \) (such a sequence always exists as the set of normalized centered \( k \)-configurations is compact), then the following is a well-known result (for a proof see [7]).

Proposition 3.1. For an isolated \( k \)-cluster collision solution \( x \) and normalized centered \( k \)-configuration \( s \) defined as above, there is a sequence of instants \( \{t_n\} \) converging to 0 satisfying \( s = \lim_{n \to +\infty} x(t_n) \). Furthermore \( s \) must be a central configuration of the \( \mathbf{k} \)-body problem.

Let \( \sigma = (\sigma_k)_{k \in \mathbf{k}} \) be a normalized centered \( \mathbf{k} \)-configuration, i.e.

\[
I_k(\sigma) = \sum_{k \in \mathbf{k}} m_k |\sigma_k|^2 = 1; \quad \sum_{k \in \mathbf{k}} m_k \sigma_k = 0. \tag{15}
\]

The following proposition is the first local deformation result we obtain.

Proposition 3.2. Let \( \sigma_k = \sigma_j - \sigma_k \) and \( \delta_k = \delta_j - \delta_k \), if the following condition holds

\[
\sigma_j \neq \sigma_k \quad \text{and} \quad \langle \sigma_k, \delta_k \rangle \geq 0, \quad \forall \{j \neq k\} \subset \mathbf{k}
\]

then for \( \delta_0, \varepsilon_0 > 0 \) small enough, there is a new path \( y \in H^1([0, \delta], \mathcal{X}) \) satisfying

\[
A_L(y, \delta, \varepsilon_0) < A_L(x, \delta),
\]

(a) \( y(t) = x(t), \quad \forall t \in [\delta_0, \delta] \);
(b) \( y(t) \) is collision-free, for any \( t \in (0, \delta] \);
(c) \( y|_{[0,\delta]} \) is a small deformation of \( x|_{[0,\delta]} \) with \( y(0) = x(0) + \varepsilon_0 \sigma \), in particular \( y_k(0) \neq y_k(0) \), for any \( \{k_1 \neq k_2\} \subset \mathbf{k} \) and \( y_k(0) \neq y_k(0) \), for any \( k \in \mathbf{k} \) and \( j \notin \mathbf{k} \).
To get the above result, first we will prove a similar result for the homothetic-parabolic solution \( \bar{q}(t) \) related to \( s \) (its energy is zero), which is defined as following

\[
\bar{q}(t) = (\kappa t) \frac{\bar{s}}{\bar{s}^2}, \text{ for any } t \in [0, +\infty),
\]

where \( \kappa \) is a constant only depending on \( \alpha, \bar{s} \) and the masses.

**Lemma 3.1.** For any \( T > 0 \), if condition (16) in proposition 3.2 holds, then for \( \varepsilon > 0 \) small enough, there is a collision-free \( \bar{q}^\varepsilon \in H^1([0, T], \mathbb{R}^k) \) with \( A_{t_k}(\bar{q}^\varepsilon, T) < A_{t_k}(\bar{q}, T) \), where \( \bar{q}^\varepsilon(t) = \bar{q}(t) + \varepsilon f(t)\sigma \), for any \( t \in [0, T] \) and

\[
f(t) = \begin{cases} 
1, & \text{if } t \in [0, \varepsilon^{\frac{1}{\alpha_n}}], \\
1 + \frac{1}{\varepsilon} (\varepsilon^{\frac{1}{\alpha_n}} - t), & \text{if } t \in [\varepsilon^{\frac{1}{\alpha_n}}, \varepsilon^{\frac{1}{\alpha_n}} + \varepsilon], \\
0, & \text{if } t \in [\varepsilon^{\frac{1}{\alpha_n}} + \varepsilon, T].
\end{cases}
\]

**Proof.** By the definition of \( \bar{q}^\varepsilon \) and \( f \),

\[
A_{t_k}(\bar{q}^\varepsilon, T) - A_{t_k}(\bar{q}, T) = \int_0^T L_k(\bar{q}^\varepsilon, \dot{\bar{q}}^\varepsilon) - L_k(\bar{q}, \dot{\bar{q}})
\]

\[
= \int_0^{\varepsilon^{\frac{1}{\alpha_n}}} U_k(\bar{q}^\varepsilon) - U_k(\bar{q}) + \int_{\varepsilon^{\frac{1}{\alpha_n}}}^{\varepsilon^{\frac{1}{\alpha_n}} + \varepsilon} U_k(\bar{q}^\varepsilon) - U_k(\bar{q}) + \int_{\varepsilon^{\frac{1}{\alpha_n}} + \varepsilon}^{\frac{2}{\alpha_n} + \varepsilon} K_k(\bar{q}^\varepsilon) - K_k(\bar{q})
\]

\[
:= A_1 + A_2 + A_3.
\]

We will estimate each \( A_j \) in the following. For any \( \{ j < k \} \subset k \), let

\[
\bar{q}_{jk} = \bar{q}^\varepsilon - \bar{q}^\varepsilon, \quad \bar{q}_{jk} = \bar{q}_j - \bar{q}_k, \quad a_{jk} = \kappa \varepsilon^{\frac{1}{\alpha_n}} |\bar{q}_{jk}|, \quad c_{jk} = \kappa \varepsilon^{\frac{1}{\alpha_n}} (\bar{q}_{jk}, \sigma_{jk}).
\]

Notice that \( c_{jk} \geq 0 \) by condition (16).

Since \( f(t) \geq 0, \forall t \in [0, T] \), the following holds for any \( \{ j < k \} \subset k \)

\[
|\bar{q}_{jk}(t) + \varepsilon f(t)\sigma_{jk}|^\alpha = (|\bar{q}_{jk}(t)|^2 + 2\varepsilon f(t)|\varepsilon^{\frac{1}{\alpha_n}} c_{jk} + \varepsilon^2 f^2(t)|\sigma_{jk}|^2)^\frac{\alpha}{2} \geq |\bar{q}_{jk}(t)|^\alpha.
\]

This immediately tells us

\[
A_2 = \sum_{j, k \in k, j < k} \frac{m_j m_k}{\alpha} \int_{\varepsilon^{\frac{1}{\alpha_n}}}^{\varepsilon^{\frac{1}{\alpha_n}} + \varepsilon} |\bar{q}_{jk}(t) + \varepsilon f(t)\sigma_{jk}|^{-\alpha} - |\bar{q}_{jk}(t)|^{-\alpha} \, dt \leq 0. \tag{18}
\]

For \( A_1 \), we notice that when \( 0 \leq t \leq \varepsilon^{\frac{1}{\alpha_n}} \),

\[
U_k(\bar{q}^\varepsilon(t)) - U_k(\bar{q}(t)) = \sum_{j, k \in k, j < k} \frac{m_j m_k}{\alpha |\bar{q}_{jk}(t)|^\alpha} - \frac{m_j m_k}{\alpha |\bar{q}_{jk}(t)|^\alpha}
\]

\[
= \sum_{j, k \in k, j < k} \frac{m_j m_k}{\alpha} (|\bar{q}_{jk}(t) + \varepsilon \sigma_{jk}|^{-\alpha} - |\bar{q}_{jk}(t)|^{-\alpha})
\]

\[
= \sum_{j, k \in k, j < k} \frac{m_j m_k}{\alpha} [a_{jk}^2 t^\frac{1}{\alpha_n} + 2\varepsilon c_{jk} t^\frac{2}{\alpha_n} + \varepsilon^2 |\sigma_{jk}|^2]^{-\frac{\alpha}{2}} - (a_{jk} t^\frac{2}{\alpha_n})^{-\alpha}.
\]
Therefore
\[
A_1 = \sum_{j, k \in k, j < k} \frac{m_j m_k}{\alpha} \int_0^{\frac{2\alpha}{\epsilon}} [a_{jk}^2 t^{(2-\alpha)} + 2\varepsilon c_{jk} t^{(2-\alpha)} + \varepsilon^2 |\sigma_j|^2 t^{\alpha-2} - (a_{jk} t^{2-\alpha})^{-\alpha}] dt.
\]

After a time reparameterization \(\tau = t^{(2-\alpha)}/\varepsilon\),
\[
\frac{2\alpha}{2 + \alpha} A_1 = \sum_{j, k \in k, j < k} \frac{m_j m_k}{\alpha} \int_0^1 \frac{\varepsilon^{\frac{2\alpha}{\epsilon}} \tau^{\alpha}}{(a_{jk}^2 \tau^2 + 2\varepsilon c_{jk} \tau + \varepsilon^2 |\sigma_k|^2)^{\frac{\alpha}{2}}} \frac{\tau^{\alpha}}{(a_{jk} \tau)^{\alpha}} d\tau - \frac{\tau^{\alpha}}{(a_{jk} \tau)^{\alpha}} d\tau \\
\leq \varepsilon^{\frac{2\alpha}{\epsilon}} \sum_{j, k \in k, j < k} m_j m_k \int_0^1 [(a_{jk}^2 \tau^2 + |\sigma_j|^2)^{\frac{\alpha}{2}} - (a_{jk}^2 \tau^2)^{\frac{\alpha}{2}}] d\tau.
\]

The last inequality follows from \(c_{jk} \geq 0\). By (16), \(|\sigma_j| > 0\), for any \(\{j \neq k\} \subset k\). As a result, there is a constant \(C_1 > 0\), such that
\[
A_1 \leq -C_1 \varepsilon^{\frac{2\alpha}{\epsilon}}. \tag{19}
\]

In the rest of the paper, \(C\) and \(C_j, j \in \mathbb{Z}\), always represent positive constants. Meanwhile by a straightforward computation,
\[
A_3 = \int_0^{\frac{2\alpha}{\epsilon} + \varepsilon} K_k(\dot{\bar{q}}) - K_k(\bar{q}) dt = \sum_{k \in k} \frac{m_k}{2} \int_0^{\frac{2\alpha}{\epsilon} + \varepsilon} |\dot{\bar{q}}(t) - \sigma_k|^2 - |\bar{q}(t)|^2 dt \\
= \sum_{k \in k} \frac{m_k}{2} \int_0^{\frac{2\alpha}{\epsilon} + \varepsilon} |\sigma_k|^2 = \frac{4}{2 + \alpha} \kappa t^{\alpha-\frac{\alpha}{2}} (\sigma_k, \bar{s}_k) dr \\
= \frac{\varepsilon}{2} \sum_{k \in k} m_k |\sigma_k|^2 - \frac{2}{2 + \alpha} \kappa t^{\alpha-\frac{\alpha}{2}} (\sigma, \bar{s})_m \int_0^{\frac{2\alpha}{\epsilon} + \varepsilon} t^{-\frac{\alpha}{2}} dr.
\]

Notice that condition (16) implies \((\sigma, \bar{s})_m \geq 0\). Then by \(\sum_{k \in k} m_k |\sigma_k|^2 = 1\), we get
\[
A_3 \leq \varepsilon/2. \tag{20}
\]

Following the above estimates
\[
A_1 + A_2 + A_3 \leq \varepsilon/2 - C_1 \varepsilon^{\frac{2\alpha}{\epsilon}} < 0,
\]
for \(\varepsilon > 0\) small enough, as \(0 < \frac{2\alpha - \alpha}{2} < 1\) for any \(\alpha \in [1, 2]\). This finishes our proof.

To get a proof of proposition 3.2 using the above result, we need the blow-up technique introduced by Terracini, see [7].

**Definition 3.2.** Given arbitrary paths \(\bar{x} \in H^1([0, \delta], \mathcal{X})\) and \(\bar{q} \in H^1([0, \delta], \mathcal{X}^k)\). For any \(\lambda > 0\), we say \(\bar{x}^\lambda : [0, \delta/\lambda] \to \mathcal{X}\) is a \(\lambda\)-**blow up** of \(x\), if
and \(\tilde{q}^\lambda : [0, \delta/\lambda] \to \mathcal{X}^k\) is a \(\lambda\)-blow up of \(\tilde{q}\), if

\[
\tilde{q}^\lambda(t) = (\tilde{q}_k^\lambda(t))_{k \in \mathbb{N}}, \quad \forall t \in [0, \delta/\lambda].
\]

Let \(x\) be the isolated \(k\)-cluster collision solution given before and \(q\) defined by (13). Given a sequence \(\{\lambda_n > 0\}\) with \(\lim_{n \to \infty} \lambda_n = 0\), let \(x^{\lambda_n}\) and \(q^{\lambda_n}\) be the corresponding \(\lambda_n\)-blow up of \(x\) and \(q\). The following result was proved in \([7, 7.4]\).

**Proposition 3.3.** If \(x(\lambda_n) \in \mathcal{X}^k\) converges to \(\tilde{s}\), then for any \(T > 0\), the sequences \(\{q^{\lambda_n}\}\) and \(\{\lambda_n q^{\lambda_n}\}\) converge to the homothetic-parabolic solution \(q\) and its derivative \(\dot{q}\) respectively, uniformly on \([0, T]\) and on compact subsets of \([0, T]\) correspondingly.

Let \(\{\Psi_n \in H^1([0, T], \mathcal{X}^k)\}_{n=1}^\infty\) be a sequence of functions, defined as following

\[
\Psi_n(t) = \begin{cases} q(t) - q^{\lambda_n}(t), & \text{if } t \in [0, T - \frac{1}{S_n}]; \\ S_n(T - t)(q(t) - q^{\lambda_n}(t)), & \text{if } t \in [T - \frac{1}{S_n}, T], \end{cases}
\]

where the sequence \(\{S_n > 0\}\) satisfying \(\lim_{n \to \infty} S_n = +\infty\). Notice that \(\Psi_n(0) = \Psi_n(T) = 0\) and by proposition 3.3, \(\Psi_n\) converges uniformly to 0 on \([0,T]\). Furthermore define a function \(\Phi = (\Phi_j)_{j \in \mathcal{X}^k} : [0, T] \to \mathcal{X}^k\) as following

\[
\Phi(t) = \tilde{q}(t) - \dot{q}(t), \quad \forall t \in [0, T],
\]

where \(\tilde{q}\) is obtained through lemma 3.1. Since \(\Phi(t)\) is \(C^1\) in a neighborhood of \(T\) and \(\sum_{j \in \mathcal{X}^k} m_j \Phi_j(t) = 0\) for any \(t \in [0, T]\), the following was proved in \([7, 7.9]\).

**Proposition 3.4.** For any \(T \in (0, \delta)\), there is a sequence of integers \(\{S_n\} \not\to +\infty\), such that for \(\Psi_n\) and \(\Phi\) defined as above,

\[
\lim_{n \to +\infty} A_L(x^{\lambda_n} + \Phi + \Psi_n, T) - A_L(x^{\lambda_n}, T) = A_{L_k}(\tilde{q} + \Phi, T) - A_{L_k}(\tilde{Q}, T).
\]

The above two results give us a connection between the isolated \(k\)-cluster collision solution \(x\) and the homothetic-parabolic solution \(\tilde{q}\). Now we can prove proposition 3.2.

**Proof of proposition 3.2.** Choose a \(T \in (0, \delta)\), for each \(\lambda_n\), let

\[
y^{\lambda_n}(t) = \begin{cases} x^{\lambda_n}(t) + \Phi(t) + \Psi_n(t), & \text{if } t \in [0, T]; \\ x^{\lambda_n}(t), & \text{if } t \in [T, \frac{1}{S_n}]. \end{cases}
\]

They are well-defined as \(\Phi(T) = \Psi_n(T) = 0\). By proposition 3.4 and lemma 3.1,

\[
\lim_{n \to +\infty} A_L(y^{\lambda_n}, \delta/\lambda_n) - A_L(x^{\lambda_n}, \delta/\lambda_n) = \lim_{n \to +\infty} A_L(y^{\lambda_n}, T) - A_L(x^{\lambda_n}, T) = A_{L_k}(\tilde{q} + \Phi, T) - A_{L_k}(\tilde{Q}, T) - A_{L_k}(\tilde{q}, T) < 0.
\]

Hence for \(n\) large enough,

\[
A_L(x^{\lambda_n}, \delta/\lambda_n) < A_L(x^{\lambda_n}, \delta/\lambda_n).
\]

For each \(n\), we define a new path: \(y_n(t) = \frac{\lambda_n}{\lambda_{n+1}} y^{\lambda_n}(t/\lambda_n)\), for any \(t \in [0, \delta]\). Then
\[ y_n(t) - x(t) = \begin{cases} \lambda_n^{\frac{1}{\lambda_n}}[\Phi(t/\lambda_n) + \Psi(t/\lambda_n)], & \text{if } t \in [0, \lambda_n T]; \\ 0, & \text{if } t \in [\lambda_n T, \delta]. \end{cases} \]

This shows, for \( n \) large enough, \( y_n \) is just a small deformation of \( x \) in a small neighborhood of \( 0 \) and
\[ y_n(0) = x(0) + \lambda_n^{\frac{1}{\lambda_n}} \Phi(0) = x(0) + \lambda_n^{\frac{1}{\lambda_n}} \varepsilon \sigma, \]
so \( y_n(t) \) is collision-free, for any \( t \in (0, \delta] \) and \( y(0) \) satisfies statement (c) in proposition 3.2. At the same time, a straightforward computation shows
\[ A_L(y_n, \delta) - A_L(x, \delta) = \lambda_n^{\frac{1}{\lambda_n}} [A_L(y^{\frac{1}{\lambda_n}}, \frac{\delta}{\lambda_n}) - A_L(x^{\frac{1}{\lambda_n}}, \frac{\delta}{\lambda_n})] < 0. \]

Although we only talked about isolated collision happening at \( t = 0 \) so far, by simply reversing time, we can get similar results when an isolated collision occurs at \( t = T_0 \). At the same time, an isolated collision can also occur at \( t \in (0, T_0) \). To deal with this, we extend the definition of isolated k-cluster collision solution given in definition 3.1 to \( x(t), t \in [-\delta, \delta] \) in the obvious way with \( x(t), t \neq 0 \) being collision-free and \( x(0) \) an isolated k-cluster collision.

Let \( x(t) \) be the normalized centered k-configuration defined as in (14). If two sequences \( \{t_n^+ > 0\} \) and \( \{t_n^- < 0\} \) satisfies \( \lim_{n \to \infty} t_n^+ = 0 \) and \( \lim_{n \to +\infty} s(t_n^+) = s^+ \), by proposition 3.1, \( s^+ \) are two central configurations of the k-body problem. Let \( \sigma = (\sigma_j)_{j \in k} \) be a normalized centered k-configuration satisfying (15).

**Proposition 3.5.** If \( \sigma_j \neq \sigma_k \) and \( (\sigma_j, s^+_j) \geq 0 \), for any \( \{j < k\} \in k \), where \( \sigma_k = \sigma_j - \sigma_k \) and \( s^+_j = s^+_j - s^+_k \), then for \( \delta_0, \varepsilon_0 > 0 \) small enough, there is a new path \( y \in H^1([-\delta, \delta], X) \)

satisfying \( A_L(y, [-\delta, \delta]) < A_L(x, [-\delta, \delta]) \), and

(a) \( y(t) = x(t), \forall t \in [-\delta, -\delta_0] \cup [\delta_0, \delta]; \)
(b) \( y(t) \) is collision-free, for any \( t \in [-\delta, \delta] \setminus \{0\}; \)
(c) \( y|_{[\delta_0, \delta_0]} \) is a small deformation of \( x|_{[\delta_0, \delta_0]} \) with \( y(0) = x(0) + \varepsilon_0 \sigma \), in particular \( y_k(0) \neq y_k(0), \forall \{k_1 \neq k_2\} \in k \) and \( y_k(0) \neq y_j(0), \forall k \in k \) and \( j \notin k \).

**Proof.** By applying proposition 3.2 to \( x|_{[-\delta, 0]} \) and \( x|_{[0, \delta]} \) correspondingly (for \( x|_{[-\delta, 0]} \), one needs to first reverse the time and then shift it by \( \delta \)), we get two collision-free paths \( y^- \in H^1([-\delta, 0], X) \) and \( y^+ \in H^1([0, \delta], X) \) with \( y^-(0) = y^+(0) = x(0) + \varepsilon_0 \sigma \). Simply piece them together at \( t = 0 \), we get a path \( y \) with the desired properties.

### 4. Binary collision

The local deformation results obtained in section 3 imposed strong constraints on the possible directions of deformation, which limits their application. In this section, we show such constraints can be substantially relaxed for isolated binary collisions. This result was not available in [12] and is the key property in our proof of the main result. We believe it could be useful in other action minimizing problems as well, for example see [23, 25].

Like before, throughout this section, we always assume \( x \in H^1([0, \delta], X) \) is an isolated k-cluster collision solution with an isolated k-cluster collision at \( t = 0 \). However in this section we only consider the planar N-body problem (\( N \geq 2 \) and \( d = 2 \)) and the set \( k \) will only include two different indices. Without loss of generality, we set \( k = \{2, 3\} \) for the rest of this section.
We need a couple of results regarding the asymptotic behavior of the bodies as they approach to the collision. In the rest of the paper, for \( t > 0 \) small enough, by \( f(t) \sim Ct^\beta + o(t^\beta) \), we mean \( f(t) = Ct^\beta + o(t^\beta) \).

**Proposition 4.1.** Let \( x \) be an isolated \( k \)-cluster collision solution given as above and \( I_k(t) = I_k(x(t)) \) defined as in (12), then for \( t > 0 \) small enough

\[
I_k(t) \sim (\kappa t)^{-\alpha}, \quad I_k(t) \sim \frac{4}{2 + \alpha} \kappa(\kappa t)^{\frac{\alpha}{2 + \alpha}}, \quad I_k(t) \sim 4 \frac{2 - \alpha}{(2 + \alpha)^2} \kappa^2(\kappa t)^{\frac{-\alpha}{2 + \alpha}}.
\]

The constant \( \kappa \) is the same as the one given in (17).

This is the well-known Sundman’s estimates. It holds for any \( k \)-cluster collision not just binary collision (a proof can be found in [7, 6.25]).

Next we need to know the asymptotic directions of the masses as they approach to the collision. In this, it is better to use polar coordinates:

\[
q(t) = (q_2(t), q_3(t)) := (\rho_2(t)e^{\theta_2(t)}, \rho_3(t)e^{\theta_3(t)}).
\]

Here \( q(t) \) is defined as in (13). Then \( m_2q_2(t) + m_3q_3(t) = 0 \) implies

\[
\theta_2(t) = \theta_3(t) + \pi, \quad \rho_2(t) = \frac{m_3}{m_2}\rho_3(t) = \sqrt{\frac{m_2}{m_3(m_2 + m_3)}} I_k^{\frac{1}{2}}(t).
\]

The following result can be seen as a generalization of proposition 3.1 in the case of binary collision.

**Proposition 4.2.** There exist \( \theta_2^+ \) and \( \theta_2^+ = \theta_3^+ + \pi \), such that the following limits hold

1. \( \lim_{t \to 0^+} \theta_2(t) = \theta_2^+ \), \( \lim_{t \to 0^+} \theta_3(t) = \theta_3^+ \);
2. \( \lim_{t \to 0^-} \theta_2(t) = \lim_{t \to 0^-} \theta_3(t) = 0 \).

**Proof.** Let \( E_k(x, \dot{x}) \) be the energy of \( k \)-cluster defined in (10). A straight forward computation shows

\[
E_k(x, \dot{x}) = \frac{1}{2} \sum_{k \in \mathbf{k}} m_k |\dot{x}_j - \dot{x}_0|^2 + \frac{1}{2} (m_2 + m_3) |\dot{x}_0|^2 - U_k(x)
\]

\[
= \frac{m_2m_3}{2(m_2 + m_3)} |\dot{x}_2 - \dot{x}_3|^2 + \frac{1}{2} (m_2 + m_3) |\dot{x}_0|^2 - U_k(x),
\]

where \( x_0 \) is the center of mass of the \( k \)-cluster.

Because \( x(0) \) is an isolated \( k \)-cluster collision, there exists a \( \delta_0 > 0 \) small enough such that

\[
x_j(t) \neq x_k(t), \quad \text{for any} \ t \in [0, \delta_0], j \in \mathbf{n} \setminus k \text{ and } k \in \mathbf{k}.
\]

As a result, \( x_0(t) \) is continuous on \([0, \delta_0]\). Meanwhile \( E_k(x(t), \dot{x}(t)) \) is bounded (for a proof see [7, 6.24]). Combining these with (26), we get

\[
|\dot{x}_2(t) - \dot{x}_3(t)|^2 \leq C_1 + C_2 U_k(x(t)), \quad \forall t \in (0, \delta_0].
\]

Now consider the relative angular momentum between \( m_3 \) and \( m_2 \)

\[
J_k(t) = [x_3(t) - x_2(t)] \wedge [\dot{x}_3(t) - \dot{x}_2(t)].
\]
By (28),
\[ |J_k(t)|^2 \leq |x_3(t) - x_2(t)|^2 |\dot{x}_3(t) - \dot{x}_2(t)|^2 \leq |x_3(t) - x_2(t)|^2 (C_1 + C_2 U_k(x(t))) \]
\[ \leq C_1 |x_3(t) - x_2(t)|^2 + C_2 \frac{m_2 m_3}{\alpha} |x_3(t) - x_2(t)|^{2-\alpha}. \]

Since \(2 - \alpha > 0\), this implies
\[ \lim_{t \to 0^+} J_k(t) = 0. \tag{29} \]

Notice that \( \dot{J}_k = (x_3 - x_2) \wedge (\dot{x}_3 - \dot{x}_2) \), and by (8),
\[ \ddot{x}_3 - \ddot{x}_2 = -(m_2 + m_3) \frac{x_3 - x_2}{|x_3 - x_2|^{2+\alpha}} = \sum_{j \in n \setminus k} m_j \left( \frac{x_3 - x_j}{|x_3 - x_j|^{2+\alpha}} - \frac{x_2 - x_j}{|x_2 - x_j|^{2+\alpha}} \right). \]

As a result,
\[ \dot{J}_k = -(x_3 - x_2) \wedge \sum_{j \in n \setminus k} m_j \left( \frac{x_3 - x_j}{|x_3 - x_j|^{2+\alpha}} - \frac{x_2 - x_j}{|x_2 - x_j|^{2+\alpha}} \right). \]

Combining this with (27) and proposition 4.1, we get
\[ |J_k(t)| \leq C_3 |x_3(t) - x_2(t)| \leq C_4 t^{\frac{4}{3}} \quad \forall t \in (0, \delta_0]. \]

Along with (29), this shows
\[ |J_k(t)| \leq \int_0^t |J_k(\tau)| d\tau \leq C_4 t^{\frac{4}{3}}. \]

Meanwhile in polar coordinates,
\[ J_k(t) = |q_3(t) - q_2(t)| \wedge [\dot{q}_3(t) - \dot{q}_2(t)] = \frac{(m_2 + m_3)^2}{m_2^2} \rho_2^2(t) \theta_3(t). \tag{30} \]

This implies
\[ |\dot{\theta}_3(t)| \leq C_4 t^{\frac{4}{3}}. \]

As a result \( \lim_{t \to 0^+} \dot{\theta}_3(t) = 0 \), and there must be a \( \theta^+ \) with \( \lim_{t \to 0^+} \theta_3(t) = \theta^+ \). The rest of the proposition follows from \( \theta_2(t) = \theta_3(t) + \pi \) given in (25).

Let \( s^+ = (s_2^+, s_3^+) = (\dot{\rho}_2 e^{i \Phi_2}, \dot{\rho}_3 e^{i \Phi_3}) \), where
\[ \dot{\rho}_2 = \sqrt{\frac{m_3}{m_2 (m_2 + m_3)}}, \quad \dot{\rho}_3 = \sqrt{\frac{m_2}{m_3 (m_2 + m_3)}}. \tag{31} \]

Then \( s^+ \) is a normalized central configuration of the \( k \)-body problem. We point out that in the 2-body problem, every normalized centered configuration is a central configuration and it is unique up to rotation. By proposition 4.2, \( \lim_{t \to 0} s(t) = s^+ \), where \( s(t) = I_k^{-1} (t) q(t) \).

Given a normalized centered \( k \)-configuration \( \sigma = (\sigma_2, \sigma_3) \). Using polar coordinates, we can write \( \sigma = (\phi_2, \phi_3) = (\dot{\rho}_2 e^{i \Phi_2}, \dot{\rho}_3 e^{i \Phi_3}) \) with \( \phi_2 = \phi_3 + \pi \).
Proposition 4.3. If the following condition holds
\[
\begin{cases}
|\phi_3 - \theta^+| \leq \pi, & \text{when } \alpha \in (1, 2); \\
|\phi_3 - \theta^+| < \pi, & \text{when } \alpha = 1,
\end{cases}
\]
then for \( \delta_0, \varepsilon_0 > 0 \) small enough, there is a new path \( y \in H^1([0, \delta], \mathcal{X}) \) satisfying \( A_L(y, \delta) < A_L(x, \delta) \) and
\begin{enumerate}[(a)]
\item \( y(t) = x(t), \quad \forall t \in [\delta_0, \delta] \);
\item \( y(t) \) is collision-free, for any \( t \in (0, \delta) \);
\item \( y_{[0,\delta]} \) is a small deformation of \( x_{[0,\delta]} \) with \( y(0) = x(0) + \varepsilon_0 \sigma \), in particular \( y_2(0) \neq y_3(0) \) and \( y_k(0) \neq y_3(0) \), for any \( k \in \mathbb{k} \) and \( j \notin \mathbb{k} \).
\end{enumerate}

Remark 4.1. For Newtonian potential, \( \alpha = 1 \), by Gordon’s classical result on Kepler problem [9], the result we obtained in proposition 4.3 is optimal, i.e. the corresponding result does not hold for \( |\phi_3 - \theta^+| = \pi \), when \( \alpha = 1 \).

We notice that condition (16) in proposition 3.2 holds, when \( \phi_3 \in [\theta^+ - \frac{\pi}{2}, \theta^+ + \frac{\pi}{2}] \) and fails, when \( \phi_3 \in [\theta^+ - \pi, \theta^+ + \pi] \setminus [\theta^+ - \frac{\pi}{2}, \theta^+ + \frac{\pi}{2}] \). Therefore proposition 4.3 is a substantial improvement of proposition 3.2 in the case of binary collision.

The proof of the above result follows the same idea used in the proof of proposition 3.2. Let \( \bar{q} : [0, +\infty) \to \mathcal{X}^k \) be the homothetic-parabolic solution defined in (17) with \( \bar{s} \) replaced by \( s^+ = (s^+_2, s^+_3) = (\bar{\rho}_2 e^{\theta^+}; \bar{\rho}_3 e^{\theta^+}) \). The next result generalizes lemma 3.1 in the case of binary collision.

Lemma 4.1. For any \( T > 0 \), if condition (32) in proposition 4.3 holds, then there is a collision-free path \( z \in H^1([0, T], \mathcal{X}^k) \) satisfying \( A_{L_k}(z, T) < A_{L_k}(\bar{q}, T) \) and
\begin{enumerate}[(a)]
\item \( z(0) = \varepsilon \sigma = \varepsilon (\bar{\rho}_2 e^{i\phi_2}; \bar{\rho}_3 e^{i\phi_3}), \text{ for some } \varepsilon > 0 \), and \( z(T) = \bar{q}(T) \);
\item \( \arg z_k(0) = \phi_k \), and \( \arg (z_k(T)) = \theta^+ k \), for \( k = 2, 3 \).
\end{enumerate}

A proof of lemma 4.1 will be given in the appendix B. Now we will use it to prove proposition 4.3.

Proof of Proposition 4.3. Choose a \( T \in (0, \delta) \). By lemma 4.1 there is a \( z \in H^1([0, T], \mathcal{X}^k) \) satisfying all the properties listed there. Define \( \Phi \in H^1([0, T], \mathcal{X}^k) \) by \( \Phi(t) = z(t) - \bar{q}(t) \), just as in (22) with \( \bar{q}^* \) replaced by \( z \), and \( \Psi_n \in H^1([0, T], \mathcal{X}^k) \) as in (21). The rest of the proof is the same as proposition 3.2 and we will not repeat it here.

Like section 3, although we only talked about an isolated binary collision happening at \( t = 0 \). By reversing the time, similar results as above hold when an isolated binary occurs at \( t = T_0 \). Similarly a stronger result than proposition 3.5 can also be obtained in the case of an isolated binary collision. However this will not be needed in this paper, as proposition 3.5 will be enough for us.

5. Proof of proposition 2.1

The existence of such an action minimizer \( x \in \Omega \) follows from the coercivity and lower semi-continuity of \( A_L \), for the details see [12]. What is left is to prove \( \alpha(t) \) is collision-free, for any \( t \in [0, T_0] \). While Marchal’s approach can be used to show this when \( t \in (0, T_0) \) (although it does not work at \( t = 0 \) or \( T_0 \)), we show the alternative approach proposed by Montgomery in [12] will also work.
The path $\pi(x(t)), t \in [0, T_0]$ starts and ends on the syzygy plane. Without loss of generality, we may assume it always lies on or above the syzygy plane (i.e. never crosses it). Because if parts of $\pi(x(t))$ are below the syzygy plane, we can always replace them by their symmetries with respect to the syzygy plane, and due to the symmetry of the Lagrangian $L_C$ with respect to the syzygy plane the new path will have the same action value as the old one. Similarly we can also assume $\pi(x(t))$ never crosses $c(M_3)$, as $L_C$ is symmetric with respect to the isosceles plane $c(M_1)$ when $m_1 = m_2$. As a result, there is no loss of generality to assume
\[
  w_3(x(t)) \geq 0, \quad |x_3(t) - x_1(t)| \geq |x_3(t) - x_2(t)|, \quad \forall t \in [0, T_0].
\]  
(33)

Following the ideas from [12] and [4], first we will show $x(t)$ can only have isolated binary collisions.

**Lemma 5.1.** Let $k = \{1, 2\}, \{1, 3\}$ or $\{2, 3\}$, the set of $k$-collision instants of $x(t), t \in [0, T_0]$, is isolated in the set of collision instants.

**Proof.** Without loss of generality, let us assume $k = \{2, 3\}$. Since the action value of $x|_{[0,T_0]}$ is finite, the set of collision times is closed and has zero measure in $[0, T_0]$. Suppose $x(t^*)$ is $k$-cluster collision, for some $t^* \in [0, T_0]$. By the continuity of $x(t)$, for $t$ in a small neighborhood of $t^*$, we have
\[
x_1(t) \neq x_j(t), \quad \forall j \in k = \{2, 3\}.
\]  
(34)

Now let us assume $x(t^*)$ is not an isolated $k$-cluster collision. Then there is a sequence of intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$, such that $a_n$ and $b_n$ converge to $t^*$, as $n$ goes to infinity. Furthermore $x(t)$ is collision-free, for any $t \in (a_n, b_n)$, and
\[
  I_k(a_n) = I_k(b_n) = 0, \quad \forall n,
\]
where $I_k(t) = I_k(x(t))$ is the moment of inertia of the $k$-cluster with respect to its center of mass as defined in (12).

For any $n$, as $x(t)$ is collision-free in $t \in (a_n, b_n)$, $I_k(t)$ is $C^2$ in $(a_n, b_n)$. Then there is a $t_n \in (a_n, b_n)$, with $I_k(t_n) \leq 0$. Recall that the energy of the $k$-cluster $E_k(x(t), \dot{x}(t))$ defined as in (10) is bounded (for a proof see [7, 6.24]). Meanwhile a direct computation shows
\[
  \frac{1}{2} \ddot{I}_k(t) = 2E_k(t) + (2 - \alpha)U_k(x(t)) + R(t),
\]
where $R(t)$ is a continuous function of $t$ in a small neighborhood of $t^*$ (see [7, corollary 5.11]). As $2 - \alpha > 0$ and $U_k(x(t))$ goes to $+\infty$ as $t$ goes to $t^*$, we must have $\ddot{I}_k(t_n)$ goes to $+\infty$, as $n$ goes to infinity, which is absurd. This finishes our proof.  

Now suppose $x(t_0)$ is a $k$-cluster collision with $k = \{1, 2\}, \{2, 3\}$ or $\{1, 3\}$. By the above result it must be isolated. Let $s(t)$ be a normalized centered $k$-configuration defined as in (14). By proposition 4.2, $s(t) = I_k^{-1/2}(t)q(t)$ converges to $s^\pm$, a normalized central configuration of the $k$-body problem, as $t$ converges to $t_0^\pm$.

**Lemma 5.2.** If $t_0 \in (0, T_0)$, then $x(t_0)$ cannot be a binary collision.

**Proof.** We will give details for the case: $k = \{2, 3\}$, while the others are similar. Choose a $\delta > 0$ small enough such that $x(t_0)$ is the unique collision in $[t_0 - \delta, t_0 + \delta]$. In polar coordinates, set

\( q(t) = (q_2(t), q_3(t)) = (\rho_2 e^{i\theta_2(t)}, \rho_3 e^{i\theta_3(t)}) \).

By proposition 4.2, there are \( \theta_3^\pm \in [0, 2\pi) \) and \( \theta_j^\pm = \theta_3^\pm + \pi \), such that

\[
\lim_{t \to t_0^\pm} \theta_j(t) = \theta_j^\pm, \quad \forall j \in \{2, 3\}.
\]

As a result \( s_j^\pm = (s_2^j, s_3^j) = (\bar{\rho}_2 e^{i\phi_2}, \bar{\rho}_3 e^{i\phi_3}) \), where \( \bar{\rho}_2, \bar{\rho}_3 \) are defined in (31).

Since \( x(t_0) \) is a collinear configuration, let us assume all the three masses lying on the real axis at the moment \( t_0 \) with \( m_1 \) on the negative direction. We claim

\[
\theta_3^\pm \in [0, \pi], \quad \theta_j^\pm = \theta_3^\pm + \pi \in [\pi, 2\pi].
\]

Because if \( \theta_3^+ \in (\pi, 2\pi) \), then for \( t - t_0 > 0 \) small enough, \( x(t) \) is not collinear and the triangle formulated by the masses will have the same orientation as \( L^- \). This means \( \pi(x(t)) \) is below the syzygy plane, which violates (33). The argument for \( \theta_3^- \) is the same.

Let \( \sigma = (\sigma_2, \sigma_3) = (\bar{\rho}_2 e^{i\phi_2}, \bar{\rho}_3 e^{i\phi_3}) \) be a normalized centered \( k \)-configuration with

\[
\phi_2 = \frac{1}{2}(\theta_2^+ + \theta_2^-) ; \quad \phi_3 = \frac{1}{2}(\theta_3^+ + \theta_3^-).
\]

Because \( |\theta_j^+ - \theta_j^-| \leq \pi \) for any \( j \in \{2, 3\} \), we have \( (s_2^+, \sigma_2^+) ) \geq 0 \). Then by proposition 3.5, we can get a new path in \( \Omega \) with action value strictly smaller than \( x' \)’s and get a contradiction (see picture \((a)\) in figure 2).

**Lemma 5.3.** If \( t_0 \in \{0, T_0\} \), \( x(t_0) \) cannot be a binary collision.

**Proof.** When \( t_0 = T_0 \), by the definition of \( \Omega \), the only possible collision is a triple collision. The result holds automatically.

Assume \( x(0) \) is a binary collision, by the definition of \( \Omega \), it is either \( b_{12} \) or \( b_{23} \).

First, let us say \( x(0) \) is a \( b_{23} \) binary collision, so it is an isolated \( k \)-cluster collision with \( k = \{2, 3\} \). Following the same notations and argument used in the proof of the previous lemma, we may assume all the three masses are on the real axis at \( t = 0 \) with \( x_1(0) < 0 < x_2(0) = x_3(0) \) and
\[
\theta_3^+ \in [0, \pi], \quad \theta_2^+ = \theta_2^+ + \pi \in [\pi, 2\pi].
\] (35)

Let \( \sigma = (\sigma_2, \sigma_3) = (\rho_2 e^{i\phi_2}, \rho_3 e^{i\phi_3}) \) with \( \phi_3 = 0, \phi_2 = \pi. \) Then \( 0 \leq \theta_3^+ - \phi_3 \leq \pi. \)

As a result, we can use proposition 4.3 to get a new path \( y \in \Omega \) with \( A_L(y, T_0) < A_L(x, T_0) \) (see picture (b) in figure 2) and this is a contradiction. This shows \( x(0) \) cannot be a \( b_{12} \) binary collision.

Now let us assume \( x(0) \) is a \( b_{12} \) binary collision, so \( x(0) \) is an isolated \( k \)-cluster collision with \( k = \{1, 2\} \). Like before we assume all the three masses are in the real axis at \( t = 0 \) with \( x_1(0) = x_2(0) < 0 < x_3(0). \)

Now we let \( q_i(t), j = 1, 2, \) represents relative position of \( m_j \) with respect to the center of mass of \( m_1 \) and \( m_2 \) and in polar coordinates: \( q_j(t) = \rho_j(t)e^{i\phi_j(t)}. \) By proposition 4.2, there exist \( \theta_j^+ \in [0, 2\pi] \) and \( \theta_j^- = \theta_j^+ + \pi \) satisfying

\[
\lim_{t \to 0^+} \theta_j(t) = \theta_j^+, \quad \forall j \in \{1, 2\}.
\]

By our assumption, \( \pi(x(t)) \) is never below the syzygy plane. Similar argument as before shows \( \theta_j^+ \in [0, \pi]. \) Meanwhile the assumption that \( \pi(x(t)) \) never crosses \( c(M_3) \) (the second inequality in (33)) further implies \( \theta_1^+ \in [\pi/2, \pi]. \) Then \( \theta_2^+ \in [3\pi/2, 2\pi]. \)

Let \( \tilde{\sigma} = (\tilde{\sigma}_1, \tilde{\sigma}_2) = (\tilde{\rho}_1 e^{i\tilde{\phi}_1}, \tilde{\rho}_2 e^{i\tilde{\phi}_2}) \) be a normalized centered \( k \)-configuration with \( \tilde{\phi}_2 = \pi \) and \( \tilde{\phi}_2 = 2\pi. \) Then by proposition 3.2, we can get a new path \( x' \in \Omega \) with \( A_L(x', T_0) < A_L(x, T_0) \) (see picture (c) in figure 2), which is absurd. This shows \( x(0) \) can not be a \( b_{12} \) collision either.

So far we have proved \( x(t) \) is free of binary collision. In the rest of the section, we will try to rule out triple collision.

**Lemma 5.4.** The set of triple collision instants of \( x(t), t \in [0, T_0], \) is isolated in the set of collision instants.

**Proof.** Suppose \( x(t) \) is \( k \)-cluster collision with \( k = \{1, 2, 3\} \), for some \( t^* \in [0, T_0]. \) By a contradiction argument, let us assume it is not isolated. Since we have proved \( x(t) \) is free of binary collision, this means there exists a sequence of intervals \( \{[a_n, b_n]\} \), such that \( a_n \) and \( b_n \) converge to \( t^* \) as \( n \) goes to infinity. Furthermore for any \( n, x(t) \) is collision-free, for any \( t \in (a_n, b_n), \) and \( I_k(a_n) = I_k(b_n) = 0. \) A contradiction now can be reached by a similar argument used in the proof of lemma 5.1.

**Lemma 5.5.** If \( t_0 \in (0, T_0), x(t_0) \) cannot be a triple collision.

**Proof.** Assume \( x(t_0) = 0 \) is a triple collision. By lemma 5.4, \( t_0 \) is isolated in the set of collision instants. Let \( s(t) = I(t)^{-1}x(t). \) By proposition 3.1, there exist a sequence of instants \( \{t^+_n\} \) with each \( t^+_n > t_0 \) and a sequence of instants \( \{t^-_n\} \) with each \( t^-_n < t_0, \) such that

\[
\lim_{n \to +\infty} t^+_n = t_0, \quad \lim_{n \to +\infty} s(t^+_n) = s^+, \quad \lim_{n \to +\infty} s(t^-_n) = s^-,
\]

where \( s^+ \) are normalized central configurations of the three body problem.

By (33), \( \pi(x(t)) \) never goes below the syzygy plane. Hence \( s^- \) can not be \( L^- \). This left us with the following three cases.
Case 1: both \( s^+ \) and \( s^- \) are the Lagrange configuration \( L^+ \). Then after rotating \( x|_{t_0} \) by be a proper angle in the inertial plane with respect to the origin, we can have \( s^+ = s^- \).

Case 2: \( s^- \) is the Lagrange configuration \( L^+ \) and \( s^+ \) is an Euler configuration (the alternative case is similar). There are three different Euler configurations and by rotating \( x|_{t_0} \) with a proper angle, we may put \( s^- \) and \( s^+ \) in relative positions given by the pictures in figure 3, where \( j^\pm \) indicates the position of \( m_j \) in \( s^\pm \). The straight line containing the Euler configuration is perpendicular to one of the sides of the equilateral triangle as indicated in the pictures.

Case 3: both \( s^- \) and \( s^+ \) are Euler configurations. If \( s^- \) and \( s^+ \) are the same type, again we may assume \( s^+ = s^- \); otherwise like in Case 2, we may assume the relative positions of \( s^+ \) and \( s^- \) are given by the pictures in figure 4, where the straight lines containing the Euler configurations are perpendicular to each other.

For all the relative positions of \( s^\pm \) shown in figures 3 and 4, we can see \( s^j \neq s^k \) and \( \langle s^j, s^k \rangle > 0 \), for any \( 1 \leq j < k \leq 3 \). This is automatically true when \( s^- = s^+ \).

This allows us to use proposition 3.5 (for example let \( \sigma = s^+ \)) to make a local deformation of \( x \) near \( t = t_0 \) and get a new path \( x^\varepsilon \in \Omega \) with \( A_L(x^\varepsilon, T_0) < A_L(x, T_0) \), which is absurd.

**Lemma 5.6.** If \( t_0 \in \{0, T_0\} \), \( x(t_0) \) cannot be a triple collision.

**Proof.** First assume \( x(T_0) = 0 \) is a triple collision. Let \( s(t) = I(t)^{-\frac{1}{2}} x(t) \). By proposition 3.1, there exist a sequence of instants \( \{t_n^-\} \) with each \( t_n^- < t_0 \), such that

\[
\lim_{n \to +\infty} t_n^- = t_0, \quad \text{and} \quad \lim_{n \to +\infty} s(t_n^-) = s^-.
\]
where \( s^- \) is a normalized central configuration of the three body problem. By (33), \( s^- \) cannot be \( L^- \).

Due to the boundary constraints at \( t = T_0 \), after local deformation the resulting configuration needs to be an Euler configuration \( E_3 \). Because of this, we let \( \sigma \) be a normalized Euler configuration \( E_3 \).

If \( s^- = L^+ \), we put \( \sigma \) and \( s^- \) in relative positions as indicated in the third picture of figure 3. If \( s^- \) is Euler, we put \( \sigma \) orthogonally to \( s^- \) as indicated in the pictures of figure 4.

In any case, we always have \( \sigma_j \neq \sigma_k \) and \( \langle \sigma_j, s^-_k \rangle \geq 0 \), for any \( 1 \leq j < k \leq 3 \). By proposition 3.2, there is path \( x^\varepsilon \in \Omega \) (a local deformation of \( x \) near \( t_0 \)) satisfying \( AL(x^\varepsilon, T_0) < AL(x, T_0) \), which is absurd. Notice that for the \( \sigma \) we chosen, \( \pi(x^\varepsilon(T_0)) \in c(E_3) \).

Now assume \( x(0) = 0 \) is a triple collision. Then we can find a sequence of instants \( \{t_n^+\} \) with each \( t_n^+ > t_0 \) such that

\[
\lim_{n \to +\infty} t_n^+ = t_0, \quad \text{and} \quad \lim_{n \to +\infty} s(t_n^+) = s^+,
\]

where \( s^+ \) is a normalized central configuration of the three body problem. Again by (33), \( s^- \) can not be \( L^- \). In this case, after local deformation the resulting configuration needs to be an Euler configuration \( E_2 \), so we let \( \sigma \) be a normalized Euler configuration \( E_2 \), the rest of the argument is the same as above.

We have proved as a minimizer \( x \) must be collision-free and this finishes our proof of proposition 2.1.

6. Proof of proposition 2.2

The existence of an action minimizer in \( \Omega \) can be proved just like before. Let \( x \in \Omega \) be such a minimizer. Like in section 5, we assume it satisfies (33).

Recall that proposition 4.3 was used only once in section 5 to prove \( x(0) \) is free of \( b_{23} \) binary collision. In the other cases we used propositions 3.2 and 3.5, whose results hold for any \( \alpha \in [1, 2) \), so those results will still hold for \( \alpha = 1 \). Therefore the only possible collision is a \( b_{23} \) binary collision at \( t = 0 \) and for the rest we assume \( k = \{2, 3\} \).

Proposition 2.2 will follow directly from the following lemma.

**Lemma 6.1.** If \( x(0) \) has a \( b_{23} \) binary collision, then \( x(t) \in \mathbb{R}^3, \forall t \in [0, T_0] \) and it is a quarter of a Schubart solution.

**Proof.** We can still use the same notations and assumptions set up in the first half of the proof of lemma 5.3. By condition (32) in proposition 4.3, \( x(0) \) can have a \( b_{23} \) binary collision, only when \( \theta^+ \| = \pi \). As otherwise, proposition 4.3 can still be used like in the proof of lemma 5.3 to reach a contradiction, even when \( \alpha = 1 \). Let \( x(t) \) be the center of mass of \( m_2 \) and \( m_3 \). Set

\[
x_j(t) = u_j(t) + iv_j(t), \quad \text{for} \ j \in \{0, 1, 2, 3\} \quad \text{with} \ u_j(t), v_j(t) \in \mathbb{R}.
\]

We claim

\[
\dot{v}_1(0) = \lim_{t \to 0^+} \dot{v}_2(t) = \lim_{t \to 0^+} \dot{v}_3(t) = 0.
\]  

Let \( m_0 = m_2 + m_3 \), then
As an action minimizer, the angular momentum of \( x \) vanishes:

\[
J(x(t)) = 3 \sum_{j=1}^{3} m_j x_j \wedge \dot{x}_j(t) = 0, \quad \forall t \in (0, T_0).
\]

Rewrite the angular moment as following

\[
J(x) = \sum_{j=0}^{1} m_j x_j \wedge \dot{x}_j + \sum_{k=2}^{3} m_k (x_k(t) - x_0(t)) \wedge (\dot{x}_k(t) - \dot{x}_0) = 0.
\]

In the proof of proposition 4.2, we showed \( \lim_{t \to 0^+} J_k(t) = 0 \), where

\[
J_k(t) = (x_3(t) - x_2(t)) \wedge (\dot{x}_3(t) - \dot{x}_2(t)).
\]

A simple computation shows

\[
\frac{m_2 m_3}{m_0} J_k(t) = \sum_{k=2}^{3} m_k (x_k(t) - x_0(t)) \wedge (\dot{x}_k(t) - \dot{x}_0(t)).
\]

Combining the above results, we get

\[
\lim_{t \to 0^+} m_0 x_0(t) \wedge \dot{x}_0(t) + m_1 x_1(t) \wedge \dot{x}_1(t) = 0.
\]

Together with (37), they imply

\[
\lim_{t \to 0^+} x_1(t) \wedge \dot{x}_1(t) = \lim_{t \to 0^+} [u_1(t) \dot{v}_1(t) - v_1(t) \dot{u}_1(t)] = 0.
\]

Because \( m_1 \) is on the negative axis at \( t = 0 \) without involving in the collision,

\[
u_1(0) < 0, \quad v_1(0) = 0, \quad |\dot{u}_1(0)| < +\infty.
\]

This means the following must hold

\[
\dot{v}_1(0) = \lim_{t \to 0^+} \dot{v}_1(t) = 0.
\]

(37) also implies \( m_0 \dot{v}_0(t) + m_1 \dot{v}_1(t) = 0 \). Therefore

\[
\dot{v}_0(0) = \lim_{t \to 0^+} \dot{v}_0(t) = \lim_{t \to 0^+} m_1 \dot{v}_1(t)/m_0 = 0.
\]

To estimate \( \dot{v}_2(t) \) and \( \dot{v}_3(t) \), it is better to use polar coordinates

\[
x_j - x_0 = (u_j - u_0) + i(v_j - v_0) = \rho_j e^{i\theta_j}, \quad \forall j \in \{2, 3\}.
\]

Then

\[
\dot{v}_3 = \dot{v}_0 + \dot{\rho}_3 \sin \theta_3 + \rho_3 \dot{\theta}_3 \cos \theta_3.
\]

By proposition 4.1
\[
\dot{\rho}_3(t) \sim C_1 t^{-\frac{3}{2}}, \quad \rho_3(t) \sim C_2 t^{-\frac{3}{4}}.
\]

Meanwhile proposition 4.2 implies \(\lim_{t \to 0^+} \dot{\theta}_3(t) = 0\). Therefore
\[
\lim_{t \to 0^+} \rho_3(t) \dot{\rho}_3(t) \cos(\theta_3(t)) = 0.
\]

Recall that \(\theta_3^+ = \lim_{t \to 0^+} \theta_3(t) = \pi\), by Taylor expansion,
\[
|\dot{\rho}_3(t) \sin(\theta_3(t))| \leq C_3 t^{\frac{3}{2}}, \quad \text{for } t > 0 \text{ small enough}.
\]

As a result, \(\lim_{t \to 0^+} \dot{v}_3(t) = 0\). Similarly we can show \(\lim_{t \to 0^+} \dot{v}_2(t) = 0\). This establishes (36).

**Lemma 6.2.** Let \(x(t) \in C^3, t \in [0, T_0] \) be a zero angular momentum solution of the three body problem with an isolated binary collision at \(t = 0\), i.e.
\[
x_1(0) < 0, \quad x_2(0) = x_3(0) > 0, \quad x_j(0) \in \mathbb{R}, \forall j \in \{1, 2, 3\}.
\]

If condition (36) holds, then \(x_j(t) \in \mathbb{R}\) for any \(t \in [0, T_0]\) and for any \(j = 1, 2, 3\).

A proof of the above lemma will be given in appendix A using Levi-Civita regularization. The rest follows from proposition 2.3 and the fact that \(x\) is an action minimizer in \(\Omega\). □

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**Appendix A. Proof of lemma 6.2**

First define the *Jacobi coordinates*
\[
\xi_1 = x_3 - x_2, \quad \xi_2 = x_1 - \frac{m_2 x_2 + m_3 x_3}{m_2 + m_3}.
\]

Notice that this is different from the *Jacobi map* given in section 2, as the singularity here is a binary collision between \(m_2\) and \(m_3\).

Recall that \(m_1 = m_2\), for simplicity we assume \(m_1 = m_2 = 1\) and \(m_3 = m\). Further define
\[
\eta_1 = M_1 \xi_1, \quad \eta_2 = M_2 \xi_2,
\]

where \(M_1 = \frac{1}{1+m} \) and \(M_2 = \frac{1+m}{2(1+m)}\). Then equation (1) becomes
\[
\begin{align*}
\dot{\xi}_1 &= \frac{m_1}{M_1} \\
\dot{\xi}_2 &= \frac{m_2}{M_2} \\
\dot{\eta}_1 &= -\frac{m \xi_1}{|\xi_1|} - \frac{m \beta_3 (\beta_3 + \xi_3)}{|\beta_3 + \xi_3|^3} - \frac{\beta_3 (\beta_3 \xi_1 - \xi_2)}{|\beta_3 \xi_1 - \xi_2|^3} \\
\dot{\eta}_2 &= -\frac{m \xi_2 + m \beta_3 \xi_3}{|\xi_2 + m \beta_3 \xi_3|} - \frac{\beta_3 \xi_2}{|\xi_2 + m \beta_3 \xi_3|^3}.
\end{align*}
\]
where $\beta_0 = \frac{1}{1 + m}$ and $\beta_1 = \frac{m}{1 + m}$. In these coordinates the energy reads

$$E = \frac{|\eta_1|^2}{2M_1} + \frac{|\eta_2|^2}{2M_2} - \frac{m}{|\xi_1|} - \frac{m}{|\beta_0 \xi_1 + \xi_2|} - \frac{1}{|\beta_1 \xi_1 + \xi_2|}.$$

Now we perform the Levi-Civita regularization by introducing the new complex variables $z, w$ through

$$\xi_1 = -\frac{z^2}{2}, \quad \eta_1 = -\frac{w}{z}, \quad \xi_2 = \frac{z}{2}, \quad \eta_2 = \frac{w}{2z} \tag{A.2}$$

and the new time parameter $\tau$ by $d\tau = |z|^2 d\tau$.

First notice that the formula for energy now becomes

$$E = \frac{2m}{|\xi_1|^2} - \frac{2m}{|\xi_2|^2} - \frac{2m}{|2\xi_2 - \beta_0 z^2|} - \frac{2}{|2\xi_2 + \beta_1 z^2|} \tag{A.3}$$

Since energy is conserved along any solution of (A.1), for a fixed energy value $E$, in the new coordinates $(z, \xi_2, w, \eta_2)$ and time parameter $\tau$, equation (A.1) becomes

$$\begin{align*}
\frac{d\xi_2}{d\tau} &= \frac{w}{M_1}, \\
\frac{d\eta_2}{d\tau} &= \frac{|z|^2}{M_1} \eta_2 \\
\frac{dw}{d\tau} &= z \left( 2E - \frac{|\eta_1|^2}{M_2} + \frac{4m}{|2\xi_2 - \beta_0 z^2|} + \frac{4}{|2\xi_2 + \beta_1 z^2|} \right) \\
&\quad + |z|^2 z \left( \frac{4m/\beta_0 (2\xi_2 - \beta_0 z^2)}{|2\xi_2 - \beta_0 z^2|^2} + \frac{4/\beta_1 (2\xi_2 + \beta_1 z^2)}{|2\xi_2 + \beta_1 z^2|^2} \right) \\
\frac{dw}{d\tau} &= -|z|^2 \left( \frac{4m/\beta_0 (2\xi_2 - \beta_0 z^2)}{|2\xi_2 - \beta_0 z^2|^2} + \frac{4/\beta_1 (2\xi_2 + \beta_1 z^2)}{|2\xi_2 + \beta_1 z^2|^2} \right). \tag{A.3}
\end{align*}$$

We remark that a binary collision between $m_2$ and $m_3$, which corresponding to $z = 0$ and $\xi_2 \neq 0$, is no longer a singularity in equation (A.3).

Under the coordinate change given above, $x_3(0) = x_3(0), (x_1, x_2, x_3)(0) \in \mathbb{R}^3$ and (36) imply $z(0) = 0$ and both $\xi_2(0), \eta_2(0)$ are real numbers. For $w$, notice that

$$w = -\bar{z} \eta_1 = -M_1 \bar{z} \xi_1. \tag{A.4}$$

Put $\xi_1 = x_3 - x_2$ in polar coordinates: $\xi_1 = \rho e^{i\theta}$. By proposition 4.1

$$\rho(t) \sim C_1 t^\frac{1}{4}, \quad \rho(t) \sim C_2 t^{-1.5}. \tag{A.5}$$

Meanwhile recall that $\lim_{t \to 0^+} \theta_k(t) = 0, k = 2, 3$ and

$$\lim_{t \to 0^+} \theta_1(t) = \pi, \quad \lim_{t \to 0^+} \theta_2(t) = 2\pi.$$

This means

$$\lim_{t \to 0^+} \dot{\theta}(t) = 0, \quad \lim_{t \to 0^+} \theta(t) = -\pi. \tag{A.6}$$

Since $z^2 = -2\xi_1$, we have

$$z = \sqrt{2\rho(t)^{\frac{1}{4}}} \text{ or } \sqrt{2\rho(t)^{\frac{1}{4}}}. \tag{A.7}$$

Plug (A.5)–(A.7) into (A.4), a direction computation shows

$$w(0) = \lim_{t \to 0^+} w(t) = -\sqrt{2} M_1 C_3 \text{ or } \sqrt{2} M_1 C_3,$$

where $C_3$ is a positive constant, so $w(0)$ is a non-zero real number.
As a result, we have shown \((z, \xi_2, w, \eta_2)(0) \in \mathbb{R}^4\) with \(w(0) \neq 0\). By checking the coefficients in equation (A.3), we can see such an initial condition determines a unique non-trivial solution of (A.3) with \((z, \xi_2, w, \eta_2)(t) \in \mathbb{R}^4\), for any \(t\). This then implies \((\xi_1, \xi_2)(t) \in \mathbb{R}^2\) and \((x_1, x_2, x_3)(t) \in \mathbb{R}^3\), for any \(t \in [0, T_0]\), which finishes our proof.

**Appendix B. Proof of lemma 4.1**

Recall that \(k = \{2, 3\}\), so after blow up, we are dealing with the planar 2-body problem. It is equivalent to the *Kepler-type problem* or the 1-center problem, which describes the motion of a massless particle in a plane under the attraction of mass \(M\) fixed at the origin. The position function of the massless particle satisfies the following equation

\[
\dot{\gamma}(t) = \nabla V(\gamma(t)), \quad V(\gamma(t)) := \frac{M}{\alpha|\gamma(t)|^\alpha}.
\]

(B.1)

It is the Euler–Lagrange equation of the following action functional

\[
A_L(\gamma, \{T_0, T_2\}) = \int_{T_0}^{T_2} \dot{L}(\gamma, \dot{\gamma}) \, dt; \quad \dot{L}(\gamma, \dot{\gamma}) = \frac{1}{2}|\dot{\gamma}|^2 + V(\gamma).
\]

For simplicity, let \(A_L(\gamma, T) = A_L(\gamma, [0, T])\), for any \(T > 0\).

Given an arbitrary pair of angles \(\psi^\pm\), there is a corresponding parabolic collision-ejection solution \(\gamma : \mathbb{R} \to \mathbb{C}\) defined as following:

\[
\dot{\gamma}(t) = \begin{cases} \frac{2 + \alpha}{\sqrt{2\alpha}} \sqrt{M(t)} \frac{\pi}{M} e^{i\psi^+} & \text{if } t \leq 0, \\ \frac{2 + \alpha}{\sqrt{2\alpha}} \sqrt{M(t)} \frac{\pi}{M} e^{i\psi^-} & \text{if } t > 0. \end{cases}
\]

A straight forward computation shows, \(\dot{\gamma}(t)\) satisfies (B.1) with zero energy, for any \(t \neq 0\).

**Proposition B.1.** If the following condition holds

\[
\begin{cases} |\psi^+ - \psi^-| \leq 2\pi, & \text{when } \alpha \in (1, 2); \\ |\psi^+ - \psi^-| < 2\pi, & \text{when } \alpha = 1, \end{cases}
\]

then there is a solution of equation (B.1), \(\gamma \in C^2([-T, T], \mathbb{C} \setminus \{0\})\), satisfying

(a) \(\gamma(\pm T) = \gamma(\pm T)\);
(b) \(\text{Arg } \gamma(\pm T) = \psi^\pm\) and \(\text{Arg } (\gamma(T) - \text{Arg } (\gamma(-T))) = |\psi^+ - \psi^-|\);
(c) \(A_L(\gamma, [-T, T]) < A_L(\gamma, [-T, T]).\)

A proof of this result can be found in [24] by considering a minimization problem of fixed ends with obstacle constraints (the paths are required to have a positive minimum distance to the origin and then pass the minimum distance to zero). The ideas were first given in [20] and [19]. When \(\alpha = 1\), this result was attributed to C Marchal. Proofs can be found in [21, proposition 4.3.10] and the appendix of [8].

**Corollary B.1.** Given an angle \(\psi\), if the following condition holds

\[
\begin{cases} |\psi - \psi^+| \leq \pi, & \text{when } \alpha \in (1, 2); \\ |\psi - \psi^+| < \pi, & \text{when } \alpha = 1, \end{cases}
\]

then there is a collision-free path \(\xi \in H^1([0, T], \mathbb{C} \setminus \{0\})\) satisfying
(a) $\xi(T) = \bar{\gamma}(T)$;
(b) $\text{Arg} \left( \xi(0) \right) = \psi$, $\text{Arg} \left( \xi(T) \right) = \psi^+$ and $|\text{Arg} \left( \xi(T) \right) - \text{Arg} \left( \xi(0) \right)| = |\psi^+ - \psi|$;
(c) $A_L(\xi([0, T])) < A_L(\bar{\gamma}, [0, T])$.

**Proof.** Without loss of generality, let us assume $\psi^+ > 0$ and $\psi = 0$. We further set $\psi^- = -\psi^+$ and let $\gamma \in C^1([T, T], C^2 \setminus \{0\})$ be a solution of (B.1) satisfying the properties given in proposition B.1. Then there exist a $T_0 \in (-T, T)$, such that $\text{Arg} \left( \gamma(T_0) \right) = 0$. Obviously
\[
\min \{A_L(\gamma, [-T, T_0]), A_L(\bar{\gamma}, [T_0, T])\} < A_L(\bar{\gamma}, [-T, 0]) = A_L(\bar{\gamma}, [0, T])
\]
Without loss of generality, let us assume
\[
A_L(\gamma, [T_0, T]) < A_L(\bar{\gamma}, [-T, 0]) = A_L(\bar{\gamma}, [0, T])
\]
Notice that if $T_0 = 0$, then $\gamma(t), t \in [0, T]$ is just what we are looking for. However we don’t know if this is always the case. In the following we show that using $\gamma$ and $\bar{\gamma}$, we can always construct a path with the required properties.

Given a $T_1 > 0$ large enough, we define a new path $\eta \in H^1([0, T_1], \mathbb{C} \setminus \{0\})$ by
\[
\eta(t) = \begin{cases} \gamma(t + T_0), & \text{if } t \in [0, T - T_0]; \\ \gamma(t + T_1), & \text{if } t \in [T - T_0, T], \end{cases}
\]
where $a = \frac{T_1 - T}{T_1 - (T - T_0)}$ and $b = \frac{T_0}{T_1 - (T - T_0)}$. Notice that
\[
a(T - T_0) + b = T, \quad \text{and} \quad aT_1 + b = T_1.
\]
Recall that
\[
\varepsilon = A_L(\bar{\gamma}, [0, T]) - A_L(\eta, [0, T - T_0]) = A_L(\bar{\gamma}, [0, T]) - A_L(\gamma, [0, T]) > 0.
\]
Set $f(T_1) := A_L(\bar{\gamma}, [T, T_1]) - A_L(\eta, [T - T_0, T_1])$, then
\[
A_L(\gamma, [0, T_1]) - A_L(\eta, [0, T_1]) = f(T_1) + \varepsilon.
\]
By a straight forward computation,
\[
|f(T_1)| = |(1 - a) \int_T^{T_1} \frac{1}{2} |\dot{\gamma}(t)|^2 dt + (1 - a) \int_T^{T_1} \frac{M}{a|\gamma(t)|^\alpha} dt| 
\leq (C_1 \frac{|T_0|}{T_1 - (T - T_0)} + C_2 \frac{|T_0|}{T_1 - T}(T_1^{\frac{3+\alpha}{2}} - T^{\frac{3+\alpha}{2}}),
\]
As $\frac{3+\alpha}{2} \in (0, 1)$ for $\alpha \in [1, 2)$, we have
\[
|f(T_1)| \to 0, \quad \text{as} \quad T_1 \to \infty.
\]
Therefore for $T_1$ large enough,
\[
A_L(\gamma, [0, T_1]) - A_L(\eta, [0, T_1]) = f(T_1) + \varepsilon > 0.
\]
Let $\lambda = T_1 / T$ and define the following paths
\[
\bar{\gamma}^\lambda(t) = \lambda^{-\frac{3+\alpha}{2}} \bar{\gamma}(\lambda t), \quad \eta^\lambda(t) = \lambda^{-\frac{3+\alpha}{2}} \eta(\lambda t), \quad \forall t \in [0, T].
\]
By a direct computation,
\[ A_L(\gamma^\lambda, [0, T]) - A_L(\eta^\lambda, [0, T]) = \lambda^{-\frac{1}{2}} \left( A_L(\bar{\gamma}, [0, T_1]) - A_L(\bar{\eta}, [0, T_1]) \right) > 0. \]

Since \( \gamma^\lambda(t) = \gamma(t) \), for any \( t \), the proof is finished once we set \( T_1 > 0 \) large enough and define \( \xi(t) = \eta^\lambda(t) \), for any \( t \in [0, T] \).

Using the above result, we can give a proof of lemma 4.1.

**Proof of lemma 4.1.** Recall that the homothetic-parabolic solution \( \bar{q}|_{[0, \infty)} \) is defined as following:
\[ \bar{q}(t) = (\bar{q}_2(t), \bar{q}_3(t)) = (\kappa t)^{\frac{1}{2\alpha}} \left( \rho_2 e^{i \theta_j^0}, \rho_3 e^{i \theta_j^1} \right), \]
where \( s^+ = (\rho_2 e^{i \theta_j^0}, \rho_3 e^{i \theta_j^1}) \) is a normalized central configuration of the \( k \)-body problem \((k = \{2, 3\})\) with \( \theta_j^0 = \theta_j^1 + \pi \) and \( \rho_j, j = 2, 3, \) defined in (31). Then
\[
A_{L_k}(q, T) = \int_0^T \frac{1}{2} m_3 \dot{q}_2(t)^2 + \frac{1}{2} m_3 \dot{q}_3(t)^2 + \frac{m_2 m_3}{\alpha |\dot{q}_2(t) - \dot{q}_3(t)|^\alpha} dt
= \frac{m_0 m_3}{m_2} \int_0^T \frac{1}{2} \dot{q}_2(t)^2 + m_2 \left( \frac{m_2}{m_0} \right)^{1+\alpha} \frac{1}{\alpha |\dot{q}_3(t)|^\alpha} dt, \tag{B.2}
\]
where \( m_0 = m_2 + m_3 \), Let \( M = m_2(m_2/m_0)^{1+\alpha} \) be the mass of the Kepler-type problem, then
\[
A_{L_k}(q, T) = \frac{m_0 m_3}{m_2} A_L(\bar{q}_3, T). \tag{B.3}
\]

Since the energy of \( \bar{q}(t) \) is zero, we find
\[
\kappa = \frac{2 + \alpha}{\sqrt{2\alpha}} m_0^{-\frac{1}{2}} (m_2 m_3)^{\frac{1+\alpha}{2}}. \tag{B.4}
\]

Because \( \dot{q}_3(t) = (\rho_3)^{\frac{1+\alpha}{2}} \kappa t^{\frac{1+\alpha}{2}} e^{i \theta_j^1} \) and
\[
(\dot{\rho}_3)^{\frac{1+\alpha}{2}} \kappa = \frac{2 + \alpha}{\sqrt{2\alpha}} \sqrt{m_2(m_2/m_0)^{1+\alpha}} = \frac{2}{\sqrt{2\alpha}} \sqrt{M},
\]
\( \bar{q}_3|_{[0, \infty)} \) is half of a parabolic collision-ejection solution of (B.1).

When \( \alpha \in (1, 2) \) (or \( \alpha = 1 \)), for any \( |\phi_3 - \theta_j^1| \leq \pi \) (or \( |\phi_3 - \theta_j^1| < \pi \)), by corollary B.1, there is a \( C^2 \) function \( z_3(t) = |z_3(t)| e^{i \phi_3(t)}, t \in [0, T] \), satisfying
\[ z_3(T) = \bar{q}_3(T), \quad \phi_3(0) = \phi_3, \quad \theta_3(T) = \theta_j^3; \tag{B.5} \]
\[ A_L(z_3, T) < A_L(\bar{q}_3, T). \tag{B.6} \]

Let \( z_3|_{[0, T]} \) be the path of \( m_3 \), and \( z_2(t) = \frac{m_0}{m_2} |z_3(t)| e^{i \phi_3(t) + \pi}, t \in [0, T] \), be the path of \( m_2 \). Then \( z(t) := (z_2(t), z_3(t)), t \in [0, T], \) satisfies property (a) and (b) in lemma 4.1, and a similar computation as in (B.2) shows
\[
A_{L_k}(z, T) = \frac{m_0 m_3}{m_2} A_{L_k}(z_3, T). \]
Combining this with (B.3) and (B.6), we get

\[ A_{I_k}(z, T) < A_{I_k}(q, T). \]

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