A smooth surface in $\mathbb{P}^4$ not of general type has degree at most 46

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This is a continuation of the papers of Braun and Fløystad [BF], Cook [C1] and Braun and Cook [BC] bounding the degree of smooth surfaces not of general type in $\mathbb{P}^4$ using Generic Initial Ideal Theory. We will use and expand on the results of [BC] using Liaison Theory. We will further restrict the configurations of the generic initial ideal of a generic hyperplane section of the surface by considering the geometric implications of having between one and five generators of the generic initial ideal in degree much higher than the other generators. We prove the following

**Theorem 1.**

Let $S$ be a smooth surface of degree $d$ in $\mathbb{P}^4$ not of general type. Then $d \leq 46$.

1 Definitions and Introduction

First let us recall some of the language of generic ideal theory. (For a full treatment refer to [B] and [G].) As we will mainly concern ourselves with a generic hyperplane section of a surface in $\mathbb{P}^4$ we will restrict our definitions to curves in $\mathbb{P}^3$.

Let $C[x_0, x_1, x_2, x_3]$ be the ring of polynomials of $\mathbb{P}^3$ with the reverse lexicographical ordering $\succ$. Let $C$ be a curve of degree $d$ in $\mathbb{P}^3$, with ideal $I_C$. After a generic change of basis, the monomial ideal of initial terms of elements of $I_C$ under $\succ$ is called the *generic initial ideal* of $C$ and denoted $\text{gin}(I_C)$.

Let $\Gamma$ be a generic hyperplane section of $C$. Then the generic initial ideal of $\Gamma$, $\text{gin}(I_\Gamma)$, is an ideal in $C[x_0, x_1, x_2]$ and can be defined by

$$\text{gin}(I_\Gamma) = (\text{gin}(I_C)|_{x_3=0})^{\text{sat}}$$
where the saturation is with respect to \( x_2 \).

As \( I_C \) and \( I_\Gamma \) are saturated ideals, \( \text{gin}(I_C) \) is generated by monomials of the form \( x_0^i x_1^j x_2^k \) and the generic initial ideal of \( \Gamma \) is of the form

\[
\text{gin}(I_\Gamma) = (x_0^{s}, x_0^{s-1} x_1^{\lambda_{s-1}}, \ldots, x_1^{\lambda_0}),
\]

where \( \sum \lambda_i = d \) and \( \lambda_i \geq \lambda_{i+1} + 1 \). Furthermore, as the points of \( \Gamma \) are in uniform position, the work of Gruson and Peskine [GP] tells us that \( \lambda_{i+1} + 2 \geq \lambda_i \). The \( \{\lambda_0, \lambda_1, \ldots, \lambda_{s-1}\} \) are called the \textit{connected invariants} of \( \Gamma \) or \( C \).

\textbf{Definition.} A monomial \( x_0^a x_1^b x_2^c \) is a \textit{sporadic zero} of \( C \) if \( x_0^a x_1^b x_2^c \notin \text{gin}(I_C) \), but \( x_0^a x_1^b \in \text{gin}(I_\Gamma) \). (i.e. there exists \( c' > c \) such that \( x_0^a x_1^b x_2^{c'} \in \text{gin}(I_C) \).)

Notice that every generator \( x_0^a x_1^b x_2^c \) of \( \text{gin}(I_C) \) with \( c' > 0 \) gives rise to sporadic zeros \( x_0^a x_1^b x_2^{c'} \) for all \( 0 \leq c < c' \).

In [BF], it was shown that if \( S \) is a smooth surface of degree \( d \) in \( \mathbb{P}^4 \) not of general type, whose generic hyperplane section, \( C \), has \( \alpha_t \) sporadic zeros in degree \( t \) and invariants \( \{\lambda_0, \lambda_1, \ldots, \lambda_{s-1}\} \), then the following equation must be satisfied

\[
0 \geq d^2 - 5d - 18 - 10 \sum_{i=0}^{s-1} \left( \frac{\lambda_i}{2} \right) + (i-1)\lambda_i + 12 \sum_{i=0}^{s-1} \left( \frac{\lambda_i + i - 1}{3} \right) - \left( \frac{i-1}{3} \right) - \sum_t \alpha_t (12t - 22). \tag{1}
\]

For each degree \( d \), there are only a few possibilities for \( \{\lambda_0, \lambda_1, \ldots, \lambda_{s-1}\} \) and for each \( d \) and \( \{\lambda_0, \lambda_1, \ldots, \lambda_{s-1}\} \) there is an upper bound on the number of sporadic zeros by the work of Ellingsrud and Peskine [EP]. Thus, in order to bound \( d \), we need to find the smallest possible upper bound for \( A = \sum_{t=0}^{\alpha_t} \alpha_t t \), for each \( d \) and \( \{\lambda_0, \lambda_1, \ldots, \lambda_{s-1}\} \).

We will bound \( A \) by bounding the degree of the generators of \( \text{gin}(I_C) \). We will show that if there are \( a \) generators in a high degree \( r > \frac{d}{2} \), and all the others are in degree \( \leq r - 2 \), then \( a \geq 6 \). Thus we can find an upper bound for \( A \) by assuming that either all generators of \( \text{gin}(I_C) \) are in degree \( \leq \frac{d}{4} \) or if there exists a generator in degree \( r > \frac{d}{2} \) then either there is a generator in degree \( r - 1 \) or there are six generators in degree \( \geq r \). Having done this
we find that \( d \leq 48 \). We will then use the connectedness of the invariants of \( C \) (see [C2]) to lower the bound to 46 by considering the last few examples on a case by case basis. We will also give an example of a connected Borel-fixed monomial ideal in degree 46 where all the conditions of this paper are satisfied. (However, whether or not this example corresponds to an actual curve is another matter.)

2 Restricting the generators in high degree

Let \( S \) be a smooth surface not of general type in \( \mathbb{P}^4 \) with generic hyperplane section \( C \). Let \( \{\lambda_0, \lambda_1, \ldots, \lambda_{s-1}\} \) be the invariants of \( C \). Assume that \( d > (s-1)^2 + 1 \). In [BC] we found that \( s \leq 7 \) and \( d \leq 66 \). Furthermore, if \( s \leq 3 \), \( d \leq 8 \) and if \( s = 6 \) or \( 7 \), \( d \leq 44 \). Thus we may restrict ourselves to the case \( s = 4 \) or 5.

Let \( \gamma \) be the number of sporadic zeros of \( C \). [EP] show that

\[
\text{for } s = 4 \quad \gamma \leq 1 + \sum_{i=0}^{s-1} \left( \frac{\lambda_i}{2} \right) + (i-1)\lambda_i - \frac{d^2}{8} + \frac{9d}{8},
\]

\[
\text{for } s = 5 \quad \gamma \leq 1 + \sum_{i=0}^{s-1} \left( \frac{\lambda_i}{2} \right) + (i-1)\lambda_i - \frac{d^2-5d+10}{10}.
\]

For equation (1) to hold for large degree, \( A \) will need to be large. Every generator of \( \text{gin}(I_C) \) of the form \( x_0^{a}x_1^{b}x_2^{c} \) with \( c > 0 \) gives a sporadic zero in each degree \( n \) for \( a + b \leq n \leq a + b + c - 1 \). So, one would like generators of \( \text{gin}(I_C) \) in as high a degree as possible.

We saw in [BC], that if there were only one generator in degree \( r > \frac{d}{2} \) and all others are in degree \( \leq r - 2 \), this would imply that there were an secant line of \( C \) of order \( r \) which leads to a contradiction if \( d > 50 \). In Appendix A we give a slight improvement on this Lemma, showing that one can obtain a contradiction for \( d > 42 \). (Although 42 is not optimal, in that we could lower this bound on \( d \) if we were more careful, it is quite sufficient for our needs.)

Thus, if there is one generator in degree \( r > \frac{d}{2} \), there must be another in degree \( \geq r - 1 \). We will continue with this argument and show that if there are \( a \) generators in degree \( \geq r > \frac{d}{2} \) and all the others are in degree \( \leq r - 2 \), then \( a \geq 6 \).
Consider the following situation. Let $C$ be a non-degenerate curve in $\mathbb{P}^3$ of degree $d$. Suppose $\text{gin}(I_C)$ has $a$ generators in degree $\geq r > \frac{d}{2}$, where $1 \leq a \leq 5$ and the rest in degree $\leq r - 2$. (Assume from this point, that a generic change of coordinates has been made so that $\text{gin}(I_C) = \text{in}(I_C)$, the initial ideal of $C$.)

Let $J$ be the ideal generated by elements of $I_C$ in degree $\leq r - 1$. Then the generators of $\text{gin}(J)$ are the generators of $\text{gin}(I_C)$ in degree $\leq r - 2$. By considering the Hilbert function associated to $J$, one finds that $\deg(V(J)) = \deg(C) + a$. Hence, $V(J) = C \cup X$ and $\deg(X) = a$. Thus $X$ contains a pure one-dimensional scheme $Y$ of degree $a$.

We will show that $Y$ must either be non-reduced or reducible. Then given that $a \leq 5$, $Y$ must contain (perhaps with a multiple structure) a line or a conic. Furthermore this line (respectively conic) must meet $C$ in $\geq \frac{d}{2}$ (respectively $> d$) points (up to multiplicity). We will then proceed as in [BC] to show in Proposition 4 that $Y$ cannot contain such a line or such a conic.

**Proposition 2**

If $I_C$ has two generators of degree $m$ and $n$ so that $m + n - 2 \leq \frac{d}{2}$, then $Y$ is either non-reduced or reducible.

(The condition of Proposition 2 is easily satisfied, if our curves arise as generic hyperplane sections of surfaces not of general type. Each of these curves lies on a surface of degree $s \leq 7$ (and in the cases we are interested in $s = 4$, or 5). By connectedness $\lambda_{s-1} \leq \frac{d}{s} - \frac{s-1}{2} < \frac{d}{s}$. Furthermore, as there are only a few chains of sporadic zeros, the curve will also lie on a surface of degree $\lambda_{s-1} + (s - 1) + \epsilon$ where $\epsilon$ is small and hence $s + \lambda_{s-1} + (s - 1) + \epsilon$ will be much less than $\frac{d}{2}$.)

**Proof.**

Suppose, for a contradiction, that $Y$ is reduced and irreducible.

**Step 1**

$H^1(\mathcal{I}_Y(d)) = 0$ for all $d \leq 0$.

Let $H$ be a generic hyperplane section of $\mathbb{P}^3$, then $Y \cap H$ is a set of points, $\Gamma'$, and we have the following short exact sequence

$$O \rightarrow \mathcal{I}_Y(d-1) \rightarrow \mathcal{I}_Y(d) \rightarrow \mathcal{I}_{\Gamma'}(d) \rightarrow O.$$
\( H^0(\mathcal{I}_Y(d)) = 0 \) for \( d \leq 0 \), therefore

\[ O \rightarrow H^1(\mathcal{I}_Y(d - 1)) \rightarrow H^1(\mathcal{I}_Y(d)) \rightarrow \]

for \( d \leq 0 \) and hence \( h^1(\mathcal{I}_Y(d - 1)) \leq h^1(\mathcal{I}_Y(d)) \) for \( d \leq 0 \).

Now, the restriction sequence

\[ O \rightarrow \mathcal{I}_Y \rightarrow \mathcal{O}_{pa} \rightarrow \mathcal{O}_Y \rightarrow O \]

gives the exact sequence of cohomology

\[ O \rightarrow H^0(\mathcal{I}_Y) \rightarrow H^0(\mathcal{O}_{pa}) \rightarrow H^0(\mathcal{O}_Y) \rightarrow H^1(\mathcal{I}_Y) \rightarrow O. \]

As \( Y \) is reduced and irreducible \( h^0(\mathcal{O}_Y) = 1 \). \( H^0(\mathcal{I}_Y) = 0 \) and hence \( H^0(\mathcal{O}_{pa}) \cong H^0(\mathcal{O}_Y) \). This implies that \( H^1(\mathcal{I}_Y) = 0 \) and hence \( H^1(\mathcal{I}_Y(d)) = 0 \) for all \( d \leq 0 \).

**Step 2**

All the generators of \( \text{gin}(I_{C \cup X}) = \text{gin}(J) \) are in degree \( \leq t_0 \leq r - 2 \). We will show that all the generators of \( \text{gin}(I_{C \cup Y}) \) are also in degree \( \leq t_0 \leq r - 2 \).

(Recall that the maximum degree of the minimal generators of \( \text{gin}(I_C) \) is the same as the regularity of \( C \). (See [B]))

Let \( H \) be a generic hyperplane. Then \( (C \cup X) \cap H = (C \cup Y) \cap H = \Gamma \) and we have the following commutative diagram

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
O & \mathcal{I}_{C \cup X}(t - 1) & \mathcal{I}_{C \cup X}(t) \\
\downarrow \alpha_1 & \downarrow \alpha_2 & \downarrow \alpha_3 \\
O & \mathcal{I}_{C \cup Y}(t - 1) & \mathcal{I}_{C \cup Y}(t) \\
\downarrow \beta & \downarrow \gamma & \downarrow \delta \\
H^0(\mathcal{I}_{C \cup X}(t + 1)) & H^0(\mathcal{I}_Y(t + 1)) & O \\
\downarrow \gamma & \downarrow \delta \\
H^0(\mathcal{I}_{C \cup Y}(t + 1)) & H^0(\mathcal{I}_Y(t + 1)) & H^1(\mathcal{I}_{C \cup Y}(t)) \\
\end{array}
\]

where \( \alpha_3 \) is an isomorphism and \( \alpha_1 \) and \( \alpha_2 \) are injections. All generators of \( \text{gin}(I_{C \cup X}) \) are in degree \( \leq t_0 \), hence \( \mathcal{I}_{C \cup X} \) is \( t_0 \)-regular and \( H^1(\mathcal{I}_{C \cup X}(t)) = 0 \) for \( t \geq t_0 - 1 \).

Taking cohomology of the diagram above we get for \( t \geq t_0 - 1 \)

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \\
H^0(\mathcal{I}_{C \cup X}(t + 1)) & H^0(\mathcal{I}_Y(t + 1)) & O \\
\downarrow \gamma & \downarrow \delta \\
H^0(\mathcal{I}_{C \cup Y}(t + 1)) & H^0(\mathcal{I}_Y(t + 1)) & H^1(\mathcal{I}_{C \cup Y}(t)) \\
\end{array}
\]
α is onto and δ is an isomorphism, therefore δ ○ α is onto. β ○ γ = δ ○ α, and so β is onto. Hence the coker(β) = 0 for all t ≥ t_₀ − 1. This means there are no sporadic zeros of C ∪ Y in degree t ≥ t_₀ and hence all generators of \( \text{gin}(I_{C ∪ Y}) \) are in degree ≤ t_₀.

**Step 3**

C ⊂ c.i.(n, m) a complete intersection of type (m, n), where m and n ≤ r − 2. Thus C is linked via this complete intersection to a curve containing the curve Y. Let’s call this curve C’ ∪ Y. C’ is also a curve (perhaps non-reduced or reducible) and it is linked via the same complete intersection to C ∪ Y.

Now
\[ O \to \mathcal{O}_{C’ ∪ Y} \to \mathcal{O}_{C’} \oplus \mathcal{O}_Y \to \mathcal{O}_{C’ \cap Y} \to O \]
and hence for all t
\[ H^0(\mathcal{O}_{C’ ∪ Y}(t)) \hookrightarrow H^0(\mathcal{O}_{C’}(t)) \oplus H^0(\mathcal{O}_Y(t)). \]

Also for an arbitrary scheme S ⊂ P^3
\[ O \to \mathcal{I}_S \to \mathcal{O}_{P^3} \to \mathcal{O}_S \to O \]
and hence for t < 0, \( H^0(\mathcal{O}_S(t)) \cong H^1(\mathcal{I}_S(t)) \)

By Step 1, \( H^1(\mathcal{I}_Y(t)) = 0 \) for t ≤ 0 and hence
\[ H^0(\mathcal{O}_{C’ ∪ Y}(t)) \hookrightarrow H^0(\mathcal{O}_{C’}(t)) \] for all t < 0
or equivalently
\[ H^1(\mathcal{I}_{C’ ∪ Y}(t)) \hookrightarrow H^1(\mathcal{I}_{C’}(t)) \] for all t < 0. (4)

Now \( \text{gin}(I_C) \) has a generator in degree r and hence C has a sporadic zero in degree r − 1 which means that \( H^1(\mathcal{I}_C(r − 2)) \neq 0 \). C is linked to C’ ∪ Y via a complete intersection of type (m, n) and so by a fundamental theorem from liaison theory ([PS])
\[ H^1(\mathcal{I}_C(t)) \cong H^1(\mathcal{I}_{C’ ∪ Y}(m + n − 4 − t)). \]

Hence \( H^1(\mathcal{I}_{C’ ∪ Y}(m + n − 4 − (r − 2))) \neq 0 \).
On the other hand C’ is linked to C ∪ Y via a complete intersection of type (m, n) and all the sporadic zeros of C ∪ Y are in degree ≤ t_₀ − 1 ≤ r − 3.
Hence $H^1(\mathcal{I}_{C \cup Y}(t)) = 0$ for all $t \geq t_0 - 1$. Again by the relationship between the deficiency modules of linked curves

$$H^1(\mathcal{I}_{C \cup Y}(t)) \cong H^1(\mathcal{I}_{C'}(m + n - 4 - t))$$

and so $H^1(\mathcal{I}_{C'}(m + n - 4 - t)) = 0$ for all $t \geq t_0 - 1$. Therefore $H^1(\mathcal{I}_{C'}(m + n - 4 - (r - 2))) = 0$. Furthermore, by the assumptions of the proposition $m + n - r - 2 < 0$. This contradicts the injection in equation (4).

Let $I_C = (J, f_1, \ldots, f_b)$ where $1 \leq b \leq a \leq 5$ and $\frac{d}{2} < r = \deg(f_1) \leq \ldots \leq \deg(f_b)$. As $Y$ is either non-reduced or reducible and the degree of $Y$ is at most 5, $Y$ must contain either a line, $l$, or a conic, $Q$ perhaps with a multiple structure.

**Lemma 3**

There exists $i, 1 \leq i \leq b$ such that $l \cap F_i$ (respectively $Q \cap F_i$) in $\deg(f_i) > \frac{d}{2}$ (respectively $2\deg(f_i) > d$) points (up to multiplicity). (Here $F_i = \{x \in \mathbb{P}^3 | f_i(x) = 0\}$.)

**Proof**

As the line $l$ or conic $Q$, with perhaps a multiple structure, is contained in $V(J)$, then the reduced scheme will certainly be contained in $V(J)$ as the non-reduced structure is really information about the embedding. Thus $l$ or $Q$ is contained in $V(J)$.

If $b = 1$, $I_C = (J, f_1)$ and $l$ (respectively $Q$) $\notin F_1$. By Bezout's theorem $l \cap F_1$ (respectively $Q \cap F_1$) in $\deg(f_1) > \frac{d}{2}$ (respectively $2\deg(f_i) > d$) points (up to multiplicity).

If $b > 1$, consider the ideals $I_j = (J, f_1, \ldots, \hat{f}_j, \ldots, f_b), j = 1, \ldots, b$. As $\text{in}(f_i)$ is of the form $x_0^{i_0}x_1^{i_1}x_2^{i_2}$ with $i_2 > 0$, $\deg V(I_j) > \deg(C)$. Hence $V(I_j)$ contains a pure one dimensional scheme $Y_j$. $C \subset C \cup Y_j \subset C \cup Y$ and hence $Y_j$ contains some component of $Y$.

If $Y_j$ contains a line or a conic (with possible multiple structure) we are done as in the case of $b = 1$, replacing $J$ with $I_j$.

The only other possibility is that $Y_j$ only contains the unique cubic or quartic (possibly) contained in $Y$. However this is only possible for at most one choice of $j$, (otherwise the cubic or quartic is contained in $C$.) Therefore there exists some $j$ such that $Y_j$ contains a line or a conic.
Proposition 4
Let $S$ be a smooth surface not of general type of degree $d$ in $\mathbb{P}^4$ with generic hyperplane section $C$. Suppose $\text{gin}(I_C)$ has at least four generators, at least two in degree $\leq r - 2$ and $a$ in degree $\geq r > \frac{d}{2}$. Then $d \leq 42$ or $a \geq 6$.

Proof
The proof is in two parts. The first uses the (slightly improved) result of [BC], which can be found at the end of this paper, which states that if $S$ is a smooth surface not of general type of degree $d$ in $\mathbb{P}^4$ with generic hyperplane section $C$ and $d > 42$, then $C$ cannot have a secant line of order $r > \frac{d}{2}$.

The second deals with the conic. Let $Q$ be the conic as above. Let $\deg(f_i) = t \geq r > \frac{d}{2}$ and $F = \{f_i = 0\}$. We may assume $Q$ is reduced and irreducible and hence (as we are in $\mathbb{P}^3$) a smooth conic. By Bezout’s Theorem $F \cap Q$ in $2t > d$ points (up to multiplicity) and all these points must lie on $C$. Let $F \cap Q = \sum m_i p_i$ where $p_i$ are the points of $C$.

Claim
$Q$ meets $C$ at $p_i$ with multiplicity $m_i$.

Given the claim, as $Q$ is a conic $Q \subset P$ a plane in $\mathbb{P}^3$ and so $C \cap P$ in at least $2t > d$ points. But this means that $C$ is contained in the plane which is not possible.

Proof of the Claim
The definition of intersection multiplicity given in [H] pg 427, is as follows. Let $X$ and $Y$ be varieties (in $\mathbb{P}^3$ for simplicity) meeting at a point $p$, then the intersection multiplicity

$$i(X, Y, p) = \sum_j (-1)^j \text{length}(\text{Tor}_j^A(A/\mathfrak{a}, A/\mathfrak{b})),$$

where $A = \mathcal{O}_{p, \mathbb{P}^3}$ is the local ring of $p$ and $\mathfrak{a} = I_X$ and $\mathfrak{b} = I_Y$ are the ideals of $X$ and $Y$ in $A$.

As $C$ is smooth, it is a local complete intersection, locally cut out at $p$ by coprime polynomials $g_1$ and $g_2$. We may assume that these polynomials correspond to polynomials in $I_C$ of degree $t$. Suppose the conic $Q$ meets $V(g_1)$ and $V(g_2)$ at $p$ with multiplicity $m$ and thus

$$m = \sum_j (-1)^j \text{length}(\text{Tor}_j^A(A/\mathfrak{a}, A/\mathfrak{b})).$$
Where \( a = I_Q \) and \( b_i = (g_i) \) for \( i = 1, 2 \).

We have the following short exact sequence

\[
0 \to \frac{b_2}{b_1} \to \frac{A}{b_1} \to \frac{A}{(g_1, g_2)} \to 0
\]

and hence via the long exact sequence of Tor,

\[
m = \sum_j (-1)^j \text{length}(\text{Tor}_j^A(\frac{A}{a}, \frac{b_2}{b_1})).
\]

Thus we need to show that

\[
\sum_j (-1)^j \text{length}(\text{Tor}_j^A(\frac{A}{a}, \frac{b_2}{b_1})) = 0.
\]

Now,

\[
O \to b_1 \cap b_2 \to b_2 \to \frac{b_2}{b_1} \to O
\]

Thus we need to show that

\[
\sum_j (-1)^j \text{length}(\text{Tor}_j^A(\frac{A}{a}, b_1 \cap b_2)) = \sum_j (-1)^j \text{length}(\text{Tor}_j^A(\frac{A}{a}, b_2)).
\]

\( b_1 \cap b_2 = (g_1, g_2) \) and \( b_1 = (g_1) \) are principal ideals and hence have minimal resolutions

\[
O \to A \to b_1 \cap b_2 \to O
\]

\[
O \to A \to b_1 \to O
\]

Therefore for any \( A \)-module \( B \), \( \text{Tor}_i^A(B, b_1 \cap b_2) = \text{Tor}_i^A(B, b_1) = 0 \) if \( i \geq 1 \). Moreover \( \text{Tor}_0^A(\frac{A}{a}, b_1 \cap b_2) = \frac{A}{a} \otimes b_1 \cap b_2 = \frac{b_1 \cap b_2}{a \cap b_1 \cap b_2} \) and \( \text{Tor}_0^A(\frac{A}{a}, b_1) = \frac{A}{a} \otimes b_1 = \frac{b_1}{a \cap b_1} \).

Let \( \Phi : \frac{b_1}{a \cap b_1} \to \frac{b_1 \cap b_2}{a \cap b_1 \cap b_2} \) be defined by \( \Phi(g_1a + b_1) = g_1g_2a + a \cap b_1 \cap b_2 \).

This is an isomorphism, hence we are done.

Thus if \( a \leq 5 \), \( Y \) either contains a secant line of \( C \) of order \( t > \frac{d}{2} \) which contradicts [BC] if \( d > 42 \) or \( Y \) contains a conic meeting \( C \) in \( > d \) points which again leads to a contradiction. Therefore either \( d \leq 42 \) or \( a \geq 6 \).  \( \Box \)
Note that if \( Y \) were of degree 6, there is the possibility that \( Y \) is a union of two cubics, or a double cubic, which cannot be eliminated by the above methods. However, given the bounds on the number of sporadic zeros it would be very difficult to have more than 6 generators in degree \( > \frac{d}{2} \). And as we will see in the last example this is not the obstruction to lowering the bound further.

### 3 Bounding A

Thus if the degree of \( S > 42 \), a generic hyperplane section, \( C \), of \( S \) cannot have an \( t \)-secant line with \( t > \frac{d}{2} \) nor a conic meeting the curve in \( 2t > d \) points. In terms of the generic initial ideal of \( C \), this means that either

1. all generators of \( \text{gin}(I_C) \) are in degree \( \leq \frac{d}{2} \)

or

2. if there exists a generator of \( \text{gin}(I_C) \) in degree \( r > \frac{d}{2} \) then either there exists a generator in degree \( r - 1 \) or there exist a five more generators in degree \( \geq r \).

The idea now is to build Borel-fixed monomial ideals with as many generators in degree \( \geq \lfloor \frac{d}{2} \rfloor \) as possible.

If there can be 6 generators in degree \( \geq \lfloor \frac{d}{2} \rfloor + 1 \), let the ideal have 5 generators in degree \( \lfloor \frac{d}{2} \rfloor + 1 \) and the sixth generator be in as high a degree as possible using all the remaining sporadic zeros. (This configuration is unlikely to arise from a curve. However, for the purposes of Bounding \( A \) it will give a good enough upper bound.) If \( s = 5 \) the bound on the number of sporadic zeros means that it is impossible to have 6 or more generators in degree \( \lfloor \frac{d}{2} \rfloor + 1 \), and it is \( s = 5 \) which is giving the upper bound on \( d \).

Otherwise let the ideal have one generator in degree \( \lfloor \frac{d}{2} \rfloor \), one in degree \( \lfloor \frac{d}{2} \rfloor + 1 \) and so on until all the sporadic zeros have been used up.

The rest of the calculations are done by computer. For each \( d \leq 66 \) (the bound we obtained in [BC]) we find all connected invariants \( \{\lambda_0, \ldots, \lambda_{s-1}\} \) with \( s = 4 \) or 5 using a program of Rich Liebling. We then use equations (2) and (3) to find the maximal number of sporadic zeros, \( z \). We then use Mathematica to find the maximal \( A \) using the criteria above and see which examples satisfy equation (1). The following is a list of the examples with
the highest degree the Mathematica program gives which satisfy the criteria above and equation (1).

**s = 4**

For $s = 4$ the only possibilities are in degree $\leq 46$. Those of highest degree being

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\# & \text{degree} & z & \{\lambda_i\} & A\text{-bound} & \text{neg} \\
\hline
1 & 46 & 50 & 14, 12, 11, 9 & 921 & -2 \\
2 & 45 & 49 & 14, 12, 10, 9 & 882 & -18 \\
3 & 45 & 48 & 13, 12, 11, 9 & 854 & -2 \\
\hline
\end{array}
\]

**s = 5**

For $s = 5$ the only possibilities are in degree $\leq 48$. Those of highest degree being

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\# & \text{degree} & z & \{\lambda_i\} & A\text{-bound} & \text{neg} \\
\hline
1 & 48 & 45 & 13, 11, 10, 8, 6 & 810 & -14 \\
2 & 47 & 44 & 13, 11, 9, 8, 6 & 770 & -20 \\
3 & 47 & 43 & 12, 11, 10, 8, 6 & 743 & -4 \\
4 & 46 & 44 & 13, 11, 9, 7, 6 & 770 & -56 \\
\hline
\end{array}
\]

**Notes:**
1. $z$ is the maximal number of sporadic zeros.
2. *A-bound* is the upper bound on $A$ found using the criteria above.
3. The column *neg* gives the value of equation (1) divided by 12 then rounded up. Hence for each of the examples above in degree $\geq 47$ we need to show that $A < A\text{- bound} - \text{neg}$ in order to eliminate that particular case.
4. The input list was not actually the full list of possible connected invariants for each degree. Certain configurations give rise to Arithmetically Cohen-Macaulay (ACM) curves, or equivalently, curves without sporadic zeros. If a curve has no sporadic zeros then $A = 0$, then in order to satisfy equation (1) we must have for $s = 4$, $d \leq 10$ and for $s = 5$, $d \leq 17$. Invariants which give ACM curves (for $s \geq 4$) are
(i) those whose consecutive invariants differ by two in which case \( C \) is a complete intersection of type \((s, \lambda_{s-1} + s - 1)\). For example the invariants \( \{6, 8, 10, 12, 14\} \) correspond to a complete intersection of type \((5, 10)\).

(ii) those whose consecutive invariants differ by two except \( \lambda_0 = \lambda_1 + 1 \) in which case \( C \) is linked to a line by a complete intersection of type \((s, \lambda_s - 1 + s - 1)\). Then as a line is ACM, \( C \) is ACM. (See Rao [R] or Migliore [M]) For example the invariants \( \{6, 8, 10, 12, 13\} \) correspond to a curve linked by a complete intersection of type \((5, 10)\) to a line.

5. We will show that example 4 for \( s = 5 \) gives rise to a Borel-fixed, connected monomial ideal which satisfies all the conditions of this paper.

**Eliminating (13, 11, 10, 8, 6)**

We will eliminate the degree 48 configuration using connectedness. The two configurations in degree 47 can also be eliminated in a similar way.

If \( x_0^4 x_1^6 \) is a generator of \( \text{gin}(I_C) \) then \( C \) is contained in a complete intersection of type \((5, 10)\) and

\[
\text{gin}(I_C) \supseteq (x_0^5, x_0^4 x_1, x_0^3 x_1, x_0^2 x_1^5, x_0 x_1^{12}, x_1^{14}).
\]

This means that the only monomials of the form \( x_0^a x_1^b x_2^c \) with \( c > 0 \) which can be generators of \( \text{gin}(I_C) \) are those with \((a, b) = (1, 11)\) or \((0, 13)\). Then the best one could hope for is that one is of degree 24, and the other is of degree 25 and

\[
A \leq \sum_{t=12}^{23} t + \sum_{t=13}^{24} t = 156 < 810 - 14.
\]

So this is not a possibility.

Therefore \( x_0^4 x_1^6 \) is a sporadic zero. As \( z = 45 \), there are at most three generators in degree \( \geq 24 \) and taking connectedness into account (See [C2]) there are only a few possibilities. The one giving the best bound on \( A \) would be if \( x_1^{13} x_2^{12}, x_0 x_1^{11} x_2^{12} \) and \( x_0^4 x_1^6 x_2^{16} \) are generators. This leaves 5 sporadic zeros which one can use up by making \( x_0^2 x_1^8 x_2^5 \) a generator. This gives an upper bound on \( A \) of 777 < 810 - 14, therefore this particular set of invariants is not possible.

**Example (13, 11, 9, 7, 6)**

The degree 46 configuration for \( s = 5 \) is actually possible. (i.e. one can create a Borel-fixed monomial ideal which is connected and still get an upper
bound on $A$ which is big enough.) Let $\text{gin}(I_C)$ be defined as follows

$$\text{gin}(I_C) = (x_0^5, x_0^4x_1^6x_2, x_0^4x_1^7, x_0^3x_1^9x_2^{13}, x_0^3x_1^8, x_0^2x_1^9x_2^{13}, x_0^2x_1^{10}, x_0x_1^{11}x_2^{13}, x_0x_1^{12}, x_1^{13}x_2^4, x_1^{14})$$

(Notice that $x_0^4x_1^6$ cannot be a generator of $\text{gin}(I_C)$ otherwise $C$ would be linked to a plane quartic and hence Arithmetically Cohen-Macaulay.) In this case we get $A = 731 > 770 - 56$.

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**Appendix A**

We saw in [BC], that if $S$ is a smooth surface not of general type in $\mathbb{P}^4$, whose hyperplane section $C_h$ has an secant line of order $r$, with $r > \frac{d}{2}$, then $S$ contains a plane curve of degree $r > \frac{d}{2}$. In [BC] we showed that this was impossible if $d > 50$, here we will improve the lemma slightly to show that this is impossible if $d > 42$.

**Lemma** If $S$ is a smooth surface not of general type in $\mathbb{P}^4$ of degree $d > 42$, $S$ cannot contain a plane curve of degree $r > \frac{4}{2}$.

**Proof**

We may assume that $d > (s - 1)^2 + 1$ then using the bounds

$$1 + \sum_{i=0}^{s-1} \left( \frac{\lambda_i}{2} \right) + (i - 1)\lambda_i \leq \frac{d^2}{2s} + (s - 4)\frac{d}{2} + 1 \quad (5)$$

$$\sum_{t=0}^{s-1} \left( \frac{\lambda_t + t - 1}{3} \right) - \left( \frac{t - 1}{3} \right) \geq s \left( \frac{\frac{d}{s} + \frac{s-3}{2}}{3} \right) + 1 - \left( \frac{s - 1}{4} \right) \quad (6)$$

found in [GP] and [BF] (respectively), equation (1) can be approximated by
\[ 0 \geq d^2 - 5d - 18 - 10\left(\frac{d^2}{2s} + (s - 4)\frac{d}{2}\right) + 12s \left(\frac{d}{s} + \frac{s-3}{2}\right) \]
\[ +12\left(1 - \left(\frac{s-1}{4}\right)\right) - \sum_{t=0}^{m} \alpha_t(12t - 22). \]  
(7)

Let us first find a lower bound for the number of sporadic zeros of a generic hyperplane section of \( S \).

For \( s = 4 \), suppose that the number of sporadic zeros is \( \leq \frac{3d}{4} \) then, naively, \( A \leq \sum_{0}^{\lambda_0 + \frac{4d}{4} - 1} t \). By connectedness \( \lambda_0 \leq \frac{d}{4} + 3 \) and hence \( A \leq \frac{5}{32}d^2 + \frac{13}{8}d - 3 \).

Substituting back into equation (7) we get
\[ 0 \geq \frac{d^3}{8} - \frac{23}{8}d^2 - \frac{17}{2}d + 33 \]
and hence \( d \leq 25 \). Therefore we may assume that the number of sporadic zeros is \( > \frac{3d}{4} \), then \( g(C_h) < \frac{d^2}{8} + 1 - \frac{3d}{4} \).

Similarly for \( s = 5 \), suppose that the number of sporadic zeros is \( \leq \frac{2d}{5} \) then \( A \leq \sum_{0}^{\lambda_0 + \frac{4d}{4} - 1} t \). By connectedness \( \lambda_0 \leq \frac{d}{5} + 4 \) and hence \( A \leq \frac{4}{25}d^2 + \frac{7}{5}d \).

Substituting back into equation (7) we get
\[ 0 \geq \frac{d^3}{25} - \frac{24}{25}d^2 - 10d - 9 \]
and hence \( d \leq 35 \). Therefore we may assume that the number of sporadic zeros is \( > \frac{2d}{5} \), then \( g(C_h) < \frac{d^2}{16} + \frac{d}{2} + 1 - \frac{2d}{5} \).

Let \( C \subset P \) be a plane curve of degree \( r > \frac{d}{2} \) contained in \( S \). Let \( H \) be a hyperplane containing \( C \).

Then \( S \cap H = C_h = C \cup C_{res} \).

We have
\[ 0 \rightarrow \mathcal{O}_{C \cup C_{res}} \rightarrow \mathcal{O}_C \oplus \mathcal{O}_{C_{res}} \rightarrow \mathcal{O}_{C \cap C_{res}} \rightarrow 0 \]

therefore
\[ h^1(\mathcal{O}_{C_h}) \geq h^1(\mathcal{O}_C) + h^1(\mathcal{O}_{C_{res}}) \]
and hence
\[ g(C_h) \geq g(C) + g(C_{res}) \geq g(C). \]
$C$ is a plane curve of degree $d_C \geq \frac{d}{2}$ and so
\[
g(C) = \frac{(d_C - 1)(d_C - 2)}{2} - \delta \geq \frac{(\frac{d}{2} - 1)(\frac{d}{2} - 2)}{2}
\]

Hence
\[
\text{for } s = 4 \quad \frac{d^2}{8} + 1 - \frac{3d}{4} > \frac{(\frac{d}{2} - 1)(\frac{d}{2} - 2)}{2} \\
\text{for } s = 5 \quad \frac{d^2}{10} + \frac{d}{2} + 1 - \frac{2d}{5} \geq \frac{(\frac{d}{2} - 1)(\frac{d}{2} - 2)}{2} \\
\text{for } s = 6, 7 \quad \frac{d^2}{2s} + (s - 4) \frac{d}{2} + 1 \geq \frac{(\frac{d}{2} - 1)(\frac{d}{2} - 2)}{2}
\]

This means for $s = 4$ we have a contradiction and hence $d \leq 25$, for $s = 5$, $d < 34$ and for $s = 6$ and 7, $d \leq 42$, $\square$

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