Random walks on random horospheric products

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Abstract. By developing the entropy theory of random walks on equivalence relations and analyzing the asymptotic geometry of horospheric products we describe the Poisson boundary for random walks on random horospheric products of trees.

Introduction

Horospheric products of trees were first introduced in the work of Diestel and Leader [DL01] in an attempt to answer a question of Woess [Woe91] on existence of vertex-transitive graphs not quasi-isometric to Cayley graphs. Although the fact that the Diestel–Leader graphs indeed provide such an example was only recently proved by Eskin, Fisher and Whyte [EFW07], in the meantime the construction of Diestel and Leader attracted a lot of attention because of its numerous interesting features (see [Woe05, BNW08] and the references therein). For instance, as it was observed by Woess, the horospheric product of two homogeneous trees of the same degree $p+1$ is isomorphic to the Cayley graph of the lamplighter group (the wreath product $\mathbb{Z} \wr \mathbb{Z}_p$) with respect to an appropriate generating set. This observation was the starting point for Bartholdi and Woess [BW05a] who showed that, along with lamplighter groups, horospheric products of homogeneous trees (not necessarily of the same degree!) provide one of very few examples of infinite graphs, for which all spectral invariants can be exhibited in an absolutely explicit form (and, in addition, the spectrum happens to be pure point).

The construction of horospheric products is very natural from a geometrical viewpoint. Namely, by choosing a point $\gamma$ on the boundary $\partial T$ of an infinite tree $T$ one converts it into the genealogical tree generated by the “mythological progenitor” $\gamma$. The Busemann cocycle $\beta_\gamma$ on $T$ can be interpreted as the signed “generations gap” in the genealogical tree: its level sets are the “generations” in $T$ as seen from $\gamma$. Given another pointed at infinity tree $(T', \gamma')$, the horospheric product (or, rather, products) of $(T, \gamma)$ and $(T', \gamma')$ are then the level sets of the aggregate...
cocycle $\beta_\gamma + \beta_{\gamma'}$ on $T \times T'$, and the Busemann cocycles determine a natural “height cocycle” on individual horospheric products. It is important for what follows that each horospheric product is endowed with two boundaries (“lower” and “upper”) isomorphic to the punctured boundaries $\partial T \setminus \{\gamma\}$ and $\partial T' \setminus \{\gamma'\}$ of the trees $T$ and $T'$.

In the previous paper [KS10] we considered the problem of *stochastic homogenization* for horospheric products. The approach that we used there was an implementation of the ideas from [Kai03]: to consider random graphs as leafwise graphs of an appropriate graphed equivalence relation; stochastic homogenization means that there is a probability measure invariant with respect to this relation. Here we are continuing to apply the ideas from [Kai03] to random horospheric products by looking at random walks on them, or, in view of the aforementioned reduction, at random walks along classes of graphed equivalence relations, leafwise graphs of which are horospheric products.

The problem we address is that of the boundary behaviour of such leafwise random walks, more precisely, the problem of identification of their *Poisson boundaries*. In the case of isotropic random walks on horospheric products of homogeneous trees this problem (or, actually, even the more general problem of describing the Martin boundary) was solved by Brofferio and Woess [Woe05, BW05b, BW06]. However, their approach (as is almost always the case with the Martin boundary) heavily depends on explicit estimates of the Green kernel only possible for highly symmetrical Markov chains. Following [Kai03], instead of this we use the *entropy theory*. Originally developed for dealing with the Poisson boundary of random walks on groups (see [KV83, Kai00] and the references therein), it is actually applicable in all situations when there is an appropriate probability path space endowed with a measure preserving time shift, in particular for random walks on equivalence relations in the presence of a global stationary measure. In this setup the entropy theory was already outlined by the first author in [Kai98, Kai03]; here we give a more detailed exposition.

In the group case the entropy theory produces not only the entropy criterion of *boundary triviality*, but also very efficient geometrical conditions for identification of the Poisson boundary (“ray” and “strip” approximations). Both these conditions readily carry over to random walks along classes of graphed equivalence relations as well. In order to apply them to horospheric products we establish the necessary geometrical ingredients. Namely, we completely characterize geodesics in horospheric products and give necessary and sufficient conditions for a sequence of points to be regular, i.e., to follow a geodesic with a sublinear deviation.

As a consequence we establish our **main result** (Theorem 2.22), according to which the Poisson boundary of a random walk on an equivalence relation graphed by horospheric products in the presence of a global stationary probability measure is completely determined by the height drift $h$ (the expectation of the height cocycle): if $h = 0$, then a.e. leafwise Poisson boundary is trivial, whereas if $h \neq 0$ then a.e. leafwise Poisson boundary coincides with the corresponding (lower or upper, depending on the sign of $h$) boundary of the underlying horospheric product endowed with the corresponding limit (hitting) distribution. This description is in perfect keeping with the situation for horospheric products of homogeneous trees.
or for lamplighter groups \([\text{Kai91}]\) (whose Cayley graphs for an appropriate choice of generators are horospheric products of homogeneous trees of the same degree \([\text{Woe05}]\)).

The main result implies that for reversible random walks on random horospheric products the leafwise Poisson boundaries are almost surely trivial, because in this situation the height drift (being the expectation of an additive cocycle) vanishes. This is the case for simple random walks on stochastically homogeneous horospheric products considered in \([\text{KS10}]\), in particular, for horospheric products of augmented Galton–Watson trees with the same offspring expectation.

On the other hand, although already lamplighter groups and horospheric products of homogeneous trees readily provide examples of random walks on horospheric products with non-zero height drift, it would be interesting to have more “probabilistically natural” examples of this kind.

It is worth mentioning in this respect that the homesick simple random walk with an integer parameter \(d\) on a pointed at infinity tree \(T\) can be interpreted as the projection of the usual simple random walk on the horospheric product of \(T\) and the homogeneous tree of degree \(d + 1\) (usually “homesickness” is defined with respect to a reference point inside the graph, e.g., see \([\text{LPP96a}]\), but this definition in an obvious way adapts to pointed at infinity trees as well). For usual simple random walks on random Galton–Watson trees existence of a linear rate of escape was established in \([\text{LPP95}]\) by using an explicit stationary measure on the space of trees. In the homesick case, although a linear rate of escape still exists \([\text{LPP96b}]\), no such construction is known.

Yet another link between horospheric products and homesick random walks worth further investigation is provided by a rather unexpected behavior of the rate of escape of homesick random walks on the lamplighter group exhibited in \([\text{LPP96b}]\) (although homesickness in \([\text{LPP96b}]\) is defined with respect to the standard generating set rather than the one whose Cayley graph is a horospheric product).

Let us finally mention that our results (with rather straightforward modifications) carry over to horospheric products with more than two multipliers which were introduced in \([\text{KW02}]\) p. 356 and further studied in \([\text{BNW08}]\).

The paper has the following structure. In Section 1 we study the asymptotic geometry of individual horospheric products. After reminding the necessary definitions concerning trees (Section 1.A) and their horospheric products (Section 1.B), in Section 1.C we reprove Bertacchi’s formula \([\text{Ber01}]\) for the distance in a horospheric product (Proposition 1.5). Our argument is somewhat different and provides an explicit description of geodesic segments in horospheric products, on the base of which we further describe geodesic rays and bilateral geodesics (Proposition 1.8 and Proposition 1.9 respectively). In Section 1.D we give criteria for a sequence of points in a horospheric product to be regular (Theorem 1.11). Finally, in Section 1.E we discuss boundaries of horospheric products.

Section 2 contains the probabilistic part of our arguments. We begin by reminding the basic definitions concerning graphed equivalence relations and random graphs (Section 2.A). In Section 2.B we discuss Markov chains along classes of an equivalence relation endowed with a quasi-invariant measure; the exposition here is based on \([\text{Kai98}]\). We express the action of the corresponding Markov operator on measures in terms of the leafwise transition probabilities and the Radon–Nikodym
c cocycle of the equivalence relation (Proposition 2.2) and give a necessary and sufficient condition for stationarity of a measure on the state space (Corollary 2.10). In particular, an invariant measure of a graphed equivalence relation becomes stationary for the leafwise simple random walk after multiplication by the density equal to the vertex degree function (Corollary 2.11).

In Section 2.C we develop the entropy theory for random walks on equivalence relation. The exposition here follows the outlines given in [Kai98, Kai03] and is completely parallel to the entropy theory for random walks in random environment on groups [Kai90] (which, in turn, was inspired by the case of the usual random walks on groups [KV83]). First we prove that the leafwise tail and Poisson boundaries coincide \( P_x - \mod 0 \) for a.e. initial point \( x \) (Theorem 2.13), after which we define the asymptotic entropy \( h \) and prove that the leafwise tail (≡ Poisson) boundaries are a.e. trivial if and only if \( h = 0 \) (Theorem 2.17). By passing to an appropriate boundary extension of the original equivalence relation [Kai05], Theorem 2.17 is also applicable to the problem of description of non-trivial Poisson boundaries of leafwise Markov chains. Indeed, a quotient of the Poisson boundary is maximal (i.e., coincides with the whole Poisson boundary) if and only if for almost all conditional chains determined by the points of this quotient the Poisson boundary is trivial. Thus, the criterion from Theorem 2.17 allows one to carry over the ray and the strip criteria used for identification of the Poisson boundary in the group case [Kai00] to the setup of random walks along classes of graphed equivalence relations.

Finally, in Section 2.D we formulate and prove the main result of the present paper: the aforementioned description of Poisson boundaries of random walks along random horospheric products (Theorem 2.22).

1. Asymptotic geometry of horospheric products

1.A. Trees. We begin by recalling that a tree is a connected graph without cycles. Any two vertices \( x, y \) in a tree \( T \) can be joined with a unique segment \([x, y]\) which is geodesic with respect to the standard graph distance \( d \). Throughout the paper we will only be considering trees “without leaves”, i.e., such that the degree of any vertex is at least 2.

Any locally finite tree \( T \) has a natural compactification \( \overline{T} = T \sqcup \partial T \) obtained in the following way: a sequence of vertices \( x_n \), which goes to infinity in \( T \) converges in this compactification if and only if for a certain (≡ any) reference point \( o \in T \) the geodesic segments \([o, x_n]\) converge pointwise. Thus, for any reference point \( o \in T \) the boundary \( \partial T \) can be identified with the space of geodesic rays issued from \( o \) (and endowed with the topology of pointwise convergence). There are many other equivalent descriptions of the boundary \( \partial T \) (and of the compactification \( \overline{T} \)), in particular, as the space of ends of \( T \) and as the hyperbolic boundary of \( T \).

A tree \( T \) with a distinguished boundary point \( \gamma \in \partial T \) is called pointed at infinity (≡ remotely rooted; in the terminology of Cartier [Car72] the point \( \gamma \) is called a “mythological progenitor”). We shall use the notation \( \partial_0 T = \partial T \setminus \{\gamma\} \) for the punctured boundary of a pointed at infinity tree \( (T, \gamma) \). A triple \( T_\circ = (T, o, \gamma) \) with \( o \in T \) and \( \gamma \in \partial T \) is a rooted tree pointed at infinity.

Any two geodesic rays converging to the same boundary point eventually meet, so that any boundary point \( \gamma \in \partial T \) determines the associated additive \( \mathbb{Z} \)-valued
**Busemann cocycle** on $T$. It is defined as

$$\beta_\gamma(x, y) = d(y, z) - d(x, z),$$

where $z = x \Join_\gamma y$ is the *confluence* of the geodesic rays $[x, \gamma)$ and $[y, \gamma)$, see Figure 1.

![Figure 1.](image1)

Obviously,

$$|\beta_\gamma(x, y)| \leq d(x, y) \quad \forall x, y \in T, \gamma \in \partial T.$$

The Busemann cocycle can also be defined as

$$\beta_\gamma(x, y) = \lim_{z \to \gamma} [d(y, z) - d(x, z)],$$

so that it is a “regularization” of the formal expression $d(y, \gamma) - d(x, \gamma)$. In the presence of a reference point $o \in T$ one can also talk about the *Busemann function*

$$b_\gamma(x) = \beta_\gamma(o, x).$$

The level sets

$$H_k = \{x \in T : b_\gamma(x) = k\}$$

of the Busemann function (≡ of the Busemann cocycle) are called *horospheres* centered at the boundary point $\gamma$, see Figure 2.

![Figure 2.](image2)
1.B. Horospheric products.

Definition 1.2. Let $T = (T, o, \gamma)$ and $T' = (T', o', \gamma')$ be two rooted trees pointed at infinity, and let $b = \beta_\gamma(o, \cdot)$, $b' = \beta_{\gamma'}(o', \cdot)$ be the corresponding Busemann functions. The horospheric product $T \uparrow_{\gamma} T'$ is the graph with the vertex set 

$$\{(x, x') \in T \times T' : b(x) + b'(x') = 0\}$$

and the edge set

$$\{( (x, x'), (y, y') ) : (x, y) and (x', y') are edges in T, T', respectively \} .$$

Geometrically one can think about the horospheric products in the following way [KW02]. Draw the tree $T'$ upside down next to $T$ so that the respective horospheres $H_k(T)$ and $H_{-k}(T')$ are at the same level. Connect the two origins $o, o'$ with an elastic spring. It can move along each of the two trees, may expand or contract, but must always remain horizontal. The vertex set of $T \uparrow_{\gamma} T'$ consists then of all admissible positions of the spring. From a position $(x, x')$ with $b(x) + b'(x') = 0$ the spring may move downwards to one of the “sons” of $x$ and at the same time to the “father” of $x'$, or upwards in an analogous way. Such a move corresponds to going to a neighbour $(y, y')$ of $(x, x')$, see Figure 3.

See [Woe05, BNW08, KS10] and the references therein for the historical background and recent works on horospheric products of trees (aka Diestel–Leader graphs or horocyclic products).

We shall use capital letters for denoting points of the horospheric product $T \uparrow_{\gamma} T'$ (so that $X = (x, x')$ with $x \in T, x' \in T'$, etc.). In particular, we denote by $O = (o, o')$ the reference point in $T \uparrow_{\gamma} T'$. The graph $T \uparrow_{\gamma} T'$ is endowed with the height cocycle

$$\beta(X, Y) = \beta_\gamma(x, y) = -\beta_{\gamma'}(x', y') .$$

Figure 3.
For simplicity below we shall use the “height function” on $T \updownarrow T'$

$$\overline{x} = -B(O, X)$$

defined in accordance with Figure 3 (so that the “higher” is the level, the bigger is the value $\overline{x}$). In the same way, we put $x = -b(x)$ and $\overline{x'} = b'(x')$ for any $x \in X, x' \in X'$, so that

$$\overline{x} = x = \overline{x'} \quad \forall X = (x, x') \in T \updownarrow T'.$$

1.C. Geodesic segments and rays. Before establishing an explicit formula for the graph metric on the horospheric product $T \updownarrow T'$ let us first notice that the sheer existence of the natural projections of $T \updownarrow T'$ onto $T, T'$ and $Z$ (the latter by the height function) implies the obvious inequalities

(1.4) \hspace{1cm} d(x, y), d(x', y'), |B(X, Y)| \leq d(X, Y)

for all pairs of points $X = (x, x'), Y = (y, y') \in T \updownarrow T'$.

Formula (1.6) below for the graph metric in $T \updownarrow T'$ was first established by Bertacchi [Ber01] Proposition 3.1 (although Bertacchi considered horospheric products of homogeneous trees only, her arguments are actually valid in the general case as well). We shall give here a somewhat different argument, which, in particular, allows us to obtain an explicit description of all geodesics in $T \updownarrow T'$.

**Proposition 1.5.** The graph distance in the horospheric product $T \updownarrow T'$ is

(1.6) \hspace{1cm} d(X, Y) = d(x, y) + d(x', y') - |B(X, Y)| .

for all $X = (x, x'), Y = (y, y') \in T \updownarrow T'$.

**Proof.** Let $\Phi$ be a path joining the points $X$ and $Y$. Then its projection $\varphi$ to $T$ (resp., its projection $\varphi'$ to $T'$) joins $x$ and $y$ (resp., $x'$ and $y'$). Since $T$ and $T'$ are trees, $\varphi$ and $\varphi'$ should pass through all edges of the geodesics $[x, y]$ and $[x', y']$, respectively. Let $x, y = x, y$ and $x', y' = x', y'$ be the conflations of the geodesic rays $[x, \gamma)$, $[y, \gamma)$ and $[x', \gamma')$, $[y', \gamma')$, respectively. The geodesic $[x, y]$ in $T$ consists of the ascending part $[x, x, y]$ (along which the height increases) and the descending part $[x, y, y]$ (along which the height decreases). In the same way the geodesic $[x', y']$ in $T'$ consists of the descending part $[x', x', y']$ and the ascending part $[x', y', y']$.

Thus,

the projection $\varphi$ of $\Phi$ to $T$ (by the height function) joins the points $\overline{x}, \overline{y} \in Z$

(1.7) \hspace{1cm} and contains all the edges (with the appropriate orientation) from the oriented segments $[\overline{x}, \overline{x}, y]$, $[\overline{x}, \overline{x}, y']$, $[\overline{x}, \overline{x}, y']$, $[\overline{x}, \overline{x}, y']$.

These segments do not overlap (if their orientation is taken into account), except for the oriented segment $[\overline{x}, \overline{y}]$ which appears twice (see Figure 4, where $\overline{X} < \overline{Y}$).

Therefore, the length of $\Phi$ satisfies the inequality

$$|\Phi| \geq |\overline{x}, \overline{x}, y| + |\overline{x}, \overline{x}, y'| + |\overline{x}, \overline{x}, y'| = |\overline{x}, \overline{x}, y'| + |\overline{x}, \overline{x}, y'| - |\overline{x}, \overline{y}|,$$

the right-hand side of which being precisely the right-hand side of equation (1.6), so that we have proved the inequality

$$d(X, Y) \geq d(x, y) + d(x', y') - |B(X, Y)|.$$
Now we shall show that paths of length $d(x, y) + d(x', y') - |\mathcal{B}(X, Y)|$ joining $X$ and $Y$ do exist, and, moreover, we shall explicitly describe all of them. Let us consider three cases.

(i) $X < Y$. Then there exists a unique path $\phi$ in $Z$ of length $d(x, y) + d(x', y') - |\mathcal{B}(X, Y)|$ satisfying condition (1.7). Indeed, there is only one way to make a path joining $X$ and $Y$ by using (one time each) all the oriented edges contained in the segments from (1.7). This is the path $\phi = [X, x \lor y] [x', x \lor y'] [x \lor y, y]$.

In order to lift it to $T \uparrow T'$ one has to choose a point $z \in T$ with $\overrightarrow{z} = x \lor y$ and such that $z$ is a descendant of $x$ (i.e., $x$ lies on the geodesic ray $[z, \gamma]$), and a point $z' \in T'$ with $\overrightarrow{z'} = x \lor y'$ and such that $z'$ is a descendant of $y'$. Then the resulting path $\Phi = (\varphi, \varphi')$ with the projections $\varphi = [x, z] [x, x \lor y] [x \lor y, y]$ and $\varphi' = [x', x' \lor y'] [x' \lor y', z'] [z', y']$ is a geodesic joining $X$ and $Y$, and all geodesics between $X$ and $Y$ have this form, see Figure 5.
(ii) $\overline{X} > \overline{Y}$. Mutatis mutandis, the situation is precisely the same as in case (i), see Figure 6.

![Figure 6](image_url)

(iii) $\overline{X} = \overline{Y}$. In this case, due to absence of the $[\overline{X}, \overline{Y}]$ segment, there are two paths in $\mathcal{Z}$ satisfying condition (1.7):

$$
\phi_1 = [\overline{X}, x', y'] [x', \lambda y', x, \lambda y] [\overline{x}, \lambda y, \overline{Y}].
$$

and

$$
\phi_2 = [\overline{X}, x, \lambda y] [x, \lambda y, x', \lambda y'] [x', \lambda y', \overline{Y}].
$$

Correspondingly, there are two types of geodesics joining $X$ and $Y$, see Figure 7.

![Figure 7](image_url)

By letting the length of geodesics go to infinity in the classification obtained in the proof of Proposition 1.5, we obtain the following description of geodesic rays and bilateral geodesics in $T \uparrow T'$:

**Proposition 1.8.** Given a point $X = (x, x') \in T \uparrow T'$, any pair $(z, \omega') \in T \times \partial_{\omega} T'$ with $z = x' \lambda \omega'$ determines a geodesic ray $\Phi = (\varphi, \varphi')$ in $T \uparrow T'$ issued from $(x, x')$ with the projections

$$
\varphi = [x, z] [z, \gamma], \quad \varphi' = [x', \omega'];
$$
any pair \((\omega, z') \in \partial_T \times T'\) with \(\overrightarrow{z'} = x \lambda \omega\) determines a geodesic ray \(\Phi = (\varphi, \varphi')\) in \(T \uparrow T'\) issued from \((x, x')\) with the projections
\[
\varphi = [x, \gamma], \quad \varphi' = [x', z'] [z', \omega'],
\]
and all geodesic rays in \(T \uparrow T'\) are of this form, see Figure 8.

**Figure 8.**

**Proposition 1.9.** All bilateral geodesics \(\Phi\) in \(T \uparrow T'\) belong to one of the following 3 classes described in terms of their projections \(\phi\) to \(Z\) (by the height functions) and \(\varphi, \varphi'\) to \(T, T'\), respectively:

(i) \(\phi\) coincides with \(Z\) (run in either positive or negative direction), and \(\varphi, \varphi'\) are, respectively, the bilateral geodesics \((\omega, \gamma)\) and \((\gamma', \omega')\) (both run in either the positive or the negative direction) with \(\gamma \in \partial_T\) and \(\gamma' \in \partial_T'\);

(ii) There is \(h \in \mathbb{Z}\) such that \(\phi\) is the concatenation \(\langle -\infty, h \rangle\langle h, -\infty \rangle\) of two copies of the geodesic ray \([h, -\infty)\) run in the opposite directions, \(\varphi\) is the geodesic \((\omega_1, \omega_2)\) for certain \(\omega_1 \neq \omega_2 \in \partial_T\) with \(\overrightarrow{\omega_1} = \overrightarrow{\omega_2} = h\), and \(\varphi' = (\gamma', x') [x', \gamma']\) for a certain \(x' \in T'\) with \(\overrightarrow{x'} = h\);

(iii) The same as (ii) with \(T\) and \(T'\) exchanged: \(\phi = (\langle -\infty, h \rangle \langle h, \infty \rangle, \varphi = (\langle \gamma, x \rangle [x, \gamma], \varphi' = (\omega_1', \omega_2')\) for \(\omega_1' \neq \omega_2' \in \partial_T'\) with \(\overrightarrow{\omega_1'} = \overrightarrow{\omega_2'} = h\).

**1.D. Regular sequences.**

**Definition 1.10.** A sequence of points \((x_n)\) in a connected graph \(X\) is called regular if there exist a geodesic ray \(\Phi\) (with the natural parameterization) and a real number \(\ell \geq 0\) (the rate of escape) such that
\[
d(x_n, \Phi(\ell n)) = o(n).
\]
If \(\ell = 0\), then \((x_n)\) is called a trivial regular sequence.

This notion was introduced by Kaimanovich [Kai89] by analogy with the notion of Lyapunov regularity for sequences of matrices. Any non-trivial regular sequence in a tree \(T\) converges to a boundary point in the compactification \(\overline{T} = T \cup \partial T\) (e.g., see [CKW94]).
Theorem 1.11. For a sequence of points $X_n = (x_n, x'_n)$ in the horospheric product of trees $T \upharpoonright T'$ the following conditions are equivalent:

(i) The sequence $(X_n)$ is regular with the rate of escape $\ell \geq 0$;
(ii) $d(X_n, X_{n+1}) = o(n)$ and $\overline{x_n} = hn + o(n)$ for a constant (which we call height drift) $h$ with $|h| = \ell$;
(iii) The sequences $(x_n)$ and $(x'_n)$ are regular in the trees $T$ and $T'$, respectively, with the same rate of escape $\ell$.

Proof. (i) $\implies$ (ii). Obvious in view of inequalities (1.4) and the description of geodesic rays in $T \upharpoonright T'$ from Proposition 1.8.

(ii) $\implies$ (iii). By (1.4) condition (ii) for the sequence $(X_n)$ implies that the analogous condition is satisfied for its projections $(x_n)$ and $(x'_n)$ to the trees $T$ and $T'$, respectively, i.e.,

$$d(x_n, x_{n+1}) = o(n), \quad d(x'_n, x'_{n+1}) = o(n), \quad \overline{x_n} = \overline{x'_n} = hn + o(n).$$

Then by [CKW94] Proposition 1 the sequences $(x_n), (x'_n)$ are both regular with the rate of escape $|h|$.

(iii) $\implies$ (i). If $\ell = 0$, then $(X_n)$ is a trivial regular sequence by formula (1.0). If $\ell > 0$, then both $(x_n)$ and $(x'_n)$ are non-trivial regular sequences. Since $\overline{x_n} = \overline{x'_n}$, one of these sequences converges to the distinguished boundary point of the corresponding tree, whereas the other sequence converges to a “plain” boundary point. For instance, let $\lim x_n = \gamma$ and $\lim x'_n = \omega' \in \partial T'$ (which corresponds to positivity of the height drift $h$). Take the geodesic ray $\Phi$ in $T \upharpoonright T'$ with the projections

$$\varphi = [o, z] [z, \gamma], \quad \varphi' = [o', \omega'],$$

where $z = o' \overline{\omega'}$ (cf. Proposition 1.8), then $d(X_n, \Phi(\ell n)) = o(n)$.

\[
1.E. \text{Boundaries of horospheric products.} \text{ For the horospheric product } T \upharpoonright T' \text{ there is a natural compactification} \tag{1.12}
\]

obtained by embedding $T \upharpoonright T'$ into the product $T \times T'$ and further taking the closure in $\overline{T \times T'}$, where $\overline{T}$ and $\overline{T'}$ are the canonical compactifications of the trees $T$ and $T'$, respectively. One can easily check (see [Ber01] Proposition 3.2) for details) that the boundary of this compactification is

$$\partial(T \upharpoonright T') = (\{\gamma\} \times \overline{T}) \cup (\overline{T} \times \{\gamma'\}).$$

Let

$$\partial_1(T \upharpoonright T') = \{\gamma\} \times \partial_\circ T' \subset \partial(T \upharpoonright T')$$

and

$$\partial_1(T \upharpoonright T') = \partial_\circ T \times \{\gamma\} \subset \partial(T \upharpoonright T')$$

be, respectively, the upper and the lower boundaries of the horospheric product $T \upharpoonright T'$. Similar pairs of boundaries arise for the dyadic-rational affine group [Kai91] or for treebolic spaces [BSCSW11].

Proposition 1.8 and Theorem 1.11 imply

Proposition 1.13. A non-trivial regular sequence in $T \upharpoonright T'$ converges in the compactification (1.12) either to a point from $\partial_1(T \upharpoonright T')$ (if the height drift is positive) or to a point from $\partial_1(T \upharpoonright T')$ (if the height drift is negative).
Remark 1.14. It is not true (unlike in the tree case) that any boundary point is the limit of a certain non-trivial regular sequence. It might be an instructive exercise to look at the Busemann compactification of the horospheric product $T \uparrow T'$ (which should not be difficult in view of the explicit descriptions of geodesics in $T \uparrow T'$ obtained in Section 1.C).

Proposition 1.9 describes which pairs of boundary points can be joined with a bilateral geodesic in $T \uparrow T'$ (which is necessarily unique, as it follows from Proposition 1.9). In particular,

Corollary 1.15. For any pair of boundary points from $\partial T \uparrow T'$ there exists a unique bilateral geodesic in $T \uparrow T'$ joining these points.

2. Random horospheric products

2.A. Graphed equivalence relations and random graphs. In the present article we shall consider random graphs from the point of view of the theory of graphed measured equivalence relations. Let us remind the basic definitions (see [FM77, Ada90, Kai97]).

Let $(X, \mu)$ be a Lebesgue measure space (below all the properties related to measure spaces will be understood mod 0, i.e., up to measure 0 subsets). A partial transformation of $(X, \mu)$ is a measure class preserving bijection between two measurable subsets of $X$. An equivalence relation $R \subset X \times X$ is called discrete measured if it is generated by an at most countable family of partial transformations. Then there exists a multiplicative Radon–Nikodym cocycle $\Delta = \Delta \mu : R \to \mathbb{R}_+$ such that for any partial transformation $f : A \to B$ whose graph is contained in $R$

$$\Delta(x, y) = \frac{df^{-1}\mu}{d\mu}(x) = \frac{d\mu}{df\mu}(y).$$

Alternatively, the Radon–Nikodym cocycle can be defined as the Radon–Nikodym ratio of the left and the right counting measures on $R$:

$$\Delta(x, y) = \frac{d\tilde{M}}{dM}(x, y),$$

where the left counting measure $M$ on $R$ is the result of integration of the counting measures $\#_x$ on the classes of the equivalence relation (considered as the fibers of the left projection $\pi : (x, y) \to x$ from $R$ onto $X$) against the measure $\mu$ on the state space $X$:

$$dM(x, y) = d\mu(x)d\#_x(y) = d\mu(x),$$

and the right counting measure $\tilde{M}$ is the image of the left one under the involution $(x, y) \mapsto (y, x)$.

If the Radon–Nikodym cocycle $\Delta$ is identically 1, then the measure $\mu$ is called $R$-invariant (≡ the equivalence relation $R$ preserves the measure $\mu$).

A (non-oriented) graph structure on a discrete measured equivalence relation $(X, \mu, R)$ is determined by a measurable symmetric subset $K \subset R \setminus \text{diag}$. The result of the restriction of this graph structure to an equivalence class $[x]$ gives the leafwise graph denoted by $[x]^R$ (by analogy with the theory of foliations classes of a discrete equivalence relation are often called leaves). We shall call $(X, \mu, R, K)$ a graphed equivalence relation. We shall always deal with the graph structures which are locally finite, i.e., any vertex has only finitely many neighbours, and denote by $\text{deg}$
the integer valued function which assigns to any point \( x \in \mathcal{X} \) the degree (valency) of \( x \) in the graph \([x]^K\). We shall also always assume that the graph structure is leafwise connected, i.e., a.e. leafwise graph \([x]^K\) is connected. Let us denote by \([x]_\bullet = ([x]^K, x)\) the graph \([x]^K\) rooted at the point \( x \). Thus, we have the map \( x \mapsto [x]_\bullet \) from \( \mathcal{X} \) to the space of connected locally finite rooted graphs \( \mathcal{G} \) endowed with the usual ball-wise convergence topology. In particular, if \( \mu \) is a probability measure, then its image under the above map is a probability measure on the space of rooted graphs \( \mathcal{G} \), i.e., a random rooted graph.

2.B. Random walks on equivalence relations.

**Definition 2.1 ([Kai98]).** A random walk along equivalence classes of a discrete measured equivalence relation \((\mathcal{X}, \mu, R)\) is determined by a measurable family of leafwise transition probabilities \( \{\pi_x\}_{x \in \mathcal{X}} \), so that any \( \pi_x \) is concentrated on the equivalence class of \( x \), and

\( (x, y) \mapsto p(x, y) = \pi_x(y) \)

is a measurable function on \( R \subset \mathcal{X} \times \mathcal{X} \). By

\( p^n(x, y) = \pi_x^n(y), \quad n \geq 1, \)

we shall denote the corresponding \( n \)-step transition probabilities which are then also measurable as a function on \( R \).

Since the measure class of \( \mu \) is preserved by the equivalence relation \( R \), the associated Markov operator \( P \) on the space \( L^\infty(\mathcal{X}, \mu) \) is well-defined (cf. Proposition 2.2 below). The dual operator then acts on the space of measures \( \lambda \) absolutely continuous with respect to \( \mu \) (notation: \( \lambda \prec \mu \)). Following a probabilistic tradition, we shall denote this action by \( \lambda \mapsto \lambda P \). The density of the measure \( \lambda P \) with respect to \( \mu \) can be explicitly described in terms of the density of \( \lambda \) and of the Radon–Nikodym cocycle \( \Delta = \Delta_\mu \) of the measure \( \mu \).

**Proposition 2.2 ([Kai98]).** For any \( \sigma \)-finite measure \( \lambda \prec \mu \)

\[
\frac{d\lambda P}{d\mu}(y) = \sum_{x \in y} p(x, y) \Delta(y, x) \frac{d\lambda}{d\mu}(x).
\]

**Proof.** Let us run the Markov chain determined by the operator \( P \) from the initial (time 0) distribution \( \lambda \). Then the time 1 distribution is, by definition, the measure \( \lambda P \), and the joint distribution of the positions of the chain at times 0 and 1 is the measure

\[
d\Pi(x, y) = d\lambda(x) p(x, y),
\]

which is obviously absolutely continuous with respect to the counting measure \( M \). The corresponding Radon–Nikodym derivative is

\[
\frac{d\Pi}{dM}(x, y) = \frac{d\lambda}{d\mu}(x) p(x, y).
\]

Since the left and the right counting measures are equivalent,

\[
\frac{d\Pi}{dM}(x, y) = \frac{d\Pi}{dM}(x, y) \frac{dM}{dM}(x, y) = \frac{d\lambda}{d\mu}(x) p(x, y) \Delta(y, x).
\]
On the other hand, since $\lambda P$ is the result of the right projection of the measure $\Pi$ to $X$,

\begin{equation}
\frac{d\Pi}{d\mathcal{M}}(x, y) = \frac{d\lambda P}{d\mu}(y) \hat{p}(y, x) ,
\end{equation}

where $\hat{p}(\cdot, \cdot)$ are the corresponding cotransition probabilities (cf. formula (2.5)), whence summation of (2.6) over $x \in [y]$ yields the claim. □

**Remark 2.8.** If the measure $\lambda$ is infinite, then the density from formula (2.3) may well be infinite on a set of positive measure $\mu$; however, even in this case the measure $\lambda P$ is absolutely continuous with respect to $\mu$ in the sense that any null set of $\mu$ is also null with respect to $\lambda P$.

Comparison of (2.5) and (2.7) leads to the following useful formula relating transition and cotransition probabilities:

\begin{equation}
\frac{d\lambda}{d\mu}(x)p(x, y) = \frac{d\lambda P}{d\mu}(y) \hat{p}(y, x) \Delta(x, y) ,
\end{equation}

or, in a somewhat informal way,

\begin{equation}
d\lambda(x)p(x, y) = d\lambda P(y)\hat{p}(y, x) ,
\end{equation}

which is what one could expect.

Given a measure $\lambda \prec \mu$, we denote by $\{\lambda_x\}_{x \in X}$ the family of leafwise measures on the equivalence classes of $X$ defined as

\begin{equation}
\lambda_x(y) = \frac{d\lambda(y)}{d\mu(x)} = \frac{d\lambda}{d\mu}(y)\Delta(x, y) .
\end{equation}

The measures $\lambda_x$ corresponding to different equivalent points $x$ are obviously all proportional.

**Corollary 2.10.** A measure $\lambda \prec \mu$ is $P$-stationary (i.e., $\lambda = \lambda P$) if and only if the leafwise measures $\lambda_x$ are almost surely stationary with respect to the transition probabilities $p(\cdot, \cdot)$.

If the measure $\lambda$ is stationary, then, as it follows from a comparison of formulas (2.6) and (2.7), the cotransition probabilities

\begin{equation}
\hat{p}(y, x) = p(x, y)\Delta_y(x, x) ,
\end{equation}

where $\Delta_y$ is the Radon–Nikodym cocycle of the measure $\lambda$, determine another Markov chain along equivalence classes with the same stationary measure $\lambda$. It is called the *time reversal* of the original random walk. If it coincides with the original walk, then the latter is called *reversible*.

If the equivalence relation $(X, \mu, R)$ is endowed with a graph structure $K$, then the transition probabilities

\begin{equation}
p(x, y) = \begin{cases} 1/\deg x , & (x, y) \in K \\ 0 , & \text{otherwise} \end{cases}
\end{equation}

determine the *simple random walk* along the classes of the equivalence relation $R$.

**Corollary 2.11.** Let $(X, \mu, R, K)$ be a graphed equivalence relation. If the measure $\mu$ is $R$-invariant, then the measure $\lambda = \deg \cdot \mu$ is stationary with respect to the simple random walk along the classes of $R$ (i.e., $\lambda = \lambda P$, where $P$ is the Markov operator of the simple random walk).
Remark 2.12. Actually, \( R \)-invariance of the measure \( \mu \) is precisely equivalent to the combination of the above stationarity condition with reversibility of the leafwise simple random walk with respect to the measure \( \deg \cdot \mu \), see [Kai98 Proposition 2.4.1 and its Corollary].

2.C. Entropy and leafwise Poisson boundaries. Below we shall be interested in describing the Poisson boundary of Markov chains along individual classes of an equivalence relation. We remind, without going into details, that the Poisson boundary is responsible for describing the stochastically significant behaviour of a Markov chain at infinity. The main tools used for its identification are the general 0-2 laws and the entropy theory (see [Kai92] and the references therein). The latter one is especially expedient when dealing with random walks on groups [KV83, Kai00].

As it was on numerous occasions mentioned by the first author (e.g., see [Kai86, Kai88, Kai90, Kai98]), the entropy theory is also applicable in all situations when there is an appropriate probability path space endowed with a measure preserving time shift. The most general currently known setup is provided by random walks on groupoids [Kai05] with a finite stationary measure on the space of objects. A particular case of it consists of Markov chains along classes of an equivalence relation in the presence of a global stationary probability measure [Kai98], and, as it was pointed out in [Kai98], the entropy theory is perfectly applicable in this situation, providing criteria for triviality and identification of the Poisson boundary of leafwise random walks (also see [Kai03] where this theory was used for describing the Poisson boundary on the graphed equivalence relations associated with the fractal limit sets of iterated function systems). In other particular cases (most of which can actually be completely described in terms of random walks on equivalence relations) the entropy theory was along the same lines implemented in [KW02] (for Markov chains with a transitive group of symmetries), [ACFdC11] (for random walks along orbits of pseudogroups acting on a measure space), [Bow10] (for random walks on random Schreier graphs), [BC10] (for simple random walks on unimodular and stationary random graphs).

Since [Kai98] and [Kai03] contain only a brief outline of the entropy theory for random walks along equivalence relations, we shall give more details here (although these arguments are essentially the same as in the case of random walks in random environment [Kai90]). For the rest of this section we shall assume that \((\mathcal{X}, \mu, R)\) is a discrete measured equivalence relation endowed with a Markov operator \(P\) determined by a measurable family of transition probabilities \(\pi_x\), and that \(\lambda \prec \mu\) is a \(P\)-stationary probability measure. Denote by \(P_x\) the probability measure in the space of paths of the associated leafwise Markov chain issued from a point \(x \in \mathcal{X}\). The one-dimensional distributions of \(P_x\) are the \(n\)-step transition probabilities \(\pi^n_x\) from the point \(x\).

Theorem 2.13. For \(\lambda\)-a.e. point \(x \in \mathcal{X}\) the tail and the Poisson boundaries of the leafwise Markov chain coincide \(P_x \mod 0\).

Proof. Let

\[
\varphi_n(x) = ||\delta_x P^n - \delta_x P^{n+1}|| = ||\pi^n_x - \pi^{n+1}_x||, \quad x \in X.
\]
Then
\[ \varphi_{n+1}(x) = \| \delta_x P^{n+1} - \delta_x P^{n+2} \| = \| (\delta_x P^n - \delta_x P^{n+1})P \| \leq \| \delta_x P^n - \delta_x P^{n+1} \| = \varphi_n(x), \]
so that there exists a limit
\[ \varphi(x) = \lim_{n} \varphi_n(x). \]
Moreover,
\[ \varphi_{n+1}(x) = \| \delta_x P^{n+1} - \delta_x P^{n+2} \| = \| \delta_x P^n - P^{n+1} \| \leq \sum_y p(x, y) \| \delta_y P^n - \delta_y P^{n+1} \| = \sum_y p(x, y) \varphi_n(y), \]
whence \( \varphi \) is subharmonic:
\[ \varphi \leq P \varphi. \]
The function \( \varphi \) is clearly measurable. Then
\[ \langle \lambda, P \varphi \rangle = \langle \lambda P, \varphi \rangle = \langle \lambda, \varphi \rangle \]
by stationarity of the measure \( \lambda \), so that in fact \( \varphi \) is harmonic. Therefore, by a classical property of Markov chains with a finite stationary measure \( \varphi \) must be constant along a.e. sample path (e.g., see [Kai92]). By the corresponding 0–2 law (see again [Kai92]) in this situation \( \varphi \) can take values 0 and 2 only (obviously, in the ergodic case only one of these two options may occur). In the first case the Poisson and the tail boundary coincide for an arbitrary initial distribution, whereas in the second case for any \( x \in \mathcal{X} \) the one-dimensional distributions \( \pi^n_x \) are all pairwise singular, so that the Poisson and the tail boundaries coincide \( P_x \mod 0. \)

Let
\[ H_n(x) = H(\pi^n_x) \]
be the entropies of \( n \)-step transition probabilities, and let
\[ H_n = \int H_n(x) \, d\lambda(x) \]
be their averages over the space \((\mathcal{X}, \lambda)\). In terms of the shift-invariant measure
\[ P_\lambda = \int P_x \, d\lambda(x) \]
on the space of sample paths \( \mathbf{x} = (x_n) \in \mathcal{X}^{\mathbb{Z}^+} \) which corresponds to the stationary initial distribution \( \lambda \),
\[ H_n = - \int \log \pi^n_{x_0} (x_n) \, dP_\lambda(x). \]
In yet another language, that of measurable partitions and their (conditional) entropies (e.g., see [Roh67]),
\[ H_n = H_\lambda(\alpha_n | \alpha_0) = \int H_x(\alpha_n) \, d\lambda(x), \]
where \( \alpha_k \) denotes the \( k \)-th coordinate partition in the path space \( \mathcal{X}^{\mathbb{Z}^+} \), and \( H_\lambda \) (resp., \( H_x \)) denotes the (conditional) entropy with respect to the measure \( P_\lambda \) (resp., \( P_x \)).
Theorem 2.17. If $H_1 < \infty$, then all the average entropies $H_n$ are also finite, there exists a limit (the asymptotic entropy)

\begin{equation}
\mathfrak{h} = \mathfrak{h}(P, \lambda) = \lim_{n \to \infty} \frac{H_n}{n} < \infty,
\end{equation}

and $\mathfrak{h} = 0$ if and only if for $\lambda$-a.e. point $x \in X$ the Poisson boundary of the leafwise Markov chain is trivial with respect to the measure $P_x$.

Let us put

\begin{equation}
\alpha^n_k = \bigvee_{i=k}^n \alpha_i,
\end{equation}

where, as before, $\alpha_i$ are the coordinate partitions in the path space. For proving Theorem 2.17 we shall need the following

Lemma 2.19. For any $k \leq n$ the conditional entropy of the partition $\alpha^k_1$ with respect to the partition $\alpha_n^\infty$ in the path space $(X^{X^+}, P_\lambda)$ is

\begin{equation}
H_\lambda (\alpha^k_1 \mid \alpha_0 \vee \alpha_n^\infty) = H_\lambda (\alpha^k_1 \mid \alpha_0 \vee \alpha_n) = kH_1 + H_{n-k} - H_n.
\end{equation}

Proof. Formula (2.20) and its proof are completely analogous to the group case considered in [KV83]. Indeed, the leftmost identity in (2.20) immediately follows from the Markov property, whereas

\begin{equation}
H_x (\alpha^k_1 \mid \alpha_0 \vee \alpha_n) = - \int \log P_x (\alpha^k_1 (x) \mid \alpha_n (x)) dP_x (x),
\end{equation}

cf. formula (2.10). By the definition of conditional entropy, for any $x \in X$

\begin{equation}
H_x (\alpha^k_1 \mid \alpha_n) = - \int \log P_x (\alpha^k_1 (x) \mid \alpha_n (x)) dP_x (x),
\end{equation}

where $\xi(x)$ denotes the element of a partition $\xi$ which contains a sample path $x$. Now,

\begin{equation}
P_x (\alpha^k_1 (x) \mid \alpha_n (x)) = \frac{P_x (\alpha^k_1 (x) \cap \alpha_n (x))}{P_x (\alpha_n (x))}
= \frac{p(x_0, x_1)p(x_1, x_2) \cdots p(x_{k-1}, x_k)p^{n-k}(x_k, x_n)}{p^n(x_0, x_n)},
\end{equation}

which implies the claim in view of formula (2.15) and shift invariance of the measure $P_\lambda$.

Proof of Theorem 2.17. Formula (2.15) in combination with shift invariance of the measure $P_\lambda$ easily implies subadditivity of the sequence $H_n$ and existence of the limit (2.18). Moreover, the sequence of functions $\varphi_n(x) = - \log \pi^n_{x_0} (x_n)$ satisfies conditions of Kingman’s subadditive ergodic theorem, which implies existence of individual limits

\begin{equation}
\lim_{n \to \infty} \frac{-1}{n} \log \pi^n_{x_0} (x_n)
\end{equation}

for $P_\lambda$-a.e. sample path $x = (x_n)$ as well. If the shift $T$ is ergodic, then these individual limits almost surely coincide with $\mathfrak{h}$. Note that ergodicity of $T$ is equivalent to absence of non-trivial subsets of the state space $X$ invariant with respect to the operator $P$ (by aforementioned general property of Markov chains with a finite stationary measure), which, in the case when pairs of points $(x, y) \in R$ with $\pi_x(y) > 0$ generate the relation $R$, is equivalent to ergodicity of $R$. 


Actually, Lemma 2.19 provides a stronger form of existence of the limit (2.18). Namely, since the sequence of partitions $\alpha_0 \vee \alpha_n^\infty$ is decreasing on $n$, monotonicity properties of conditional entropy (e.g., see [Roh67]) imply that $H_\lambda (\alpha_1 | \alpha_0 \vee \alpha_n^\infty)$ increases on $n$. In view of formula (2.20), it means that not only the limit $h = \lim H_n/n$ exists, but also that $[H_{n+1} - H_n] \searrow h$.

As we have already seen on a similar occasion in the proof of Lemma 2.19, the left-hand side of formula (2.20) can be rewritten as

$$H_\lambda (\alpha_1^k | \alpha_0 \vee \alpha_n^\infty) = \int H_x (\alpha_1^k | \alpha_n^\infty) d\lambda(x).$$

By continuity of conditional entropy (see again [Roh67]), for any $x \in \mathcal{X}$

$$H_x (\alpha_1^k | \alpha_n^\infty) \nearrow H_x (\alpha_1^k | \alpha^\infty) \leq H_x (\alpha_1^k),$$

where $\alpha^\infty = \lim_n \alpha_n^\infty$ is the tail partition. The right-hand side in the above formula is integrable, and

$$\int H_x (\alpha_1^k) = kH_1,$$

cf. formula (2.21). Therefore, after passing in (2.20) to a limit as $n \to \infty$ we conclude that for any $k > 0$

$$\int H_x (\alpha_1^k | \alpha^\infty) d\lambda(x) = k(H_1 - h)$$

and

$$k h = \int \left[ H_x (\alpha_1^k) - H_x (\alpha_1^k | \alpha^\infty) \right] d\lambda(x).$$

It means that $h = 0$ if and only if for $\lambda$-a.e. $x \in \mathcal{X}$ the tail partition $\alpha^\infty$ is $P_x$-independent of all coordinate partitions $\alpha_1^k$, the latter condition being equivalent to triviality of the tail partition $P_x \mod 0$.

Finally, by Theorem 2.13, for $P_\lambda$-a.e. $x \in \mathcal{X}$ the tail and the Poisson boundaries coincide $P_x \mod 0$, which completes the proof.

By passing to an appropriate boundary extension of the original equivalence relation [Kai05], Theorem 2.17 is also applicable to the problem of description of non-trivial Poisson boundaries of leafwise Markov chains. Indeed, a quotient of the Poisson boundary is maximal (i.e., coincides with the whole Poisson boundary) if and only if for almost all conditional chains determined by the points of this quotient the Poisson boundary is trivial. Thus, the criterion from Theorem 2.17 allows one to carry over the ray and the strip criteria used for identification of the Poisson boundary in the group case [Kai00] to the setup of random walks along classes of graphed equivalence relations.

2.D. The Poisson boundary of random horospheric products. Below by a random horospheric product we shall mean a graphed equivalence relation $(\mathcal{X}, \mu, R, K)$ such that a.e. leafwise graph is a horospheric product. Moreover, we shall assume that the “orientations” (signs) of leafwise height cocycles (1.3) are chosen in a consistent way, i.e., that there exists a global $\mathbb{Z}$-valued measurable cocycle $B$ on $R$ such that its restriction to a.e. leaf is a height cocycle. Therefore, a.e. leafwise graph $[x]^K$, being a horospheric product, is endowed with its lower and upper boundaries $\partial^- [x]^K$ and $\partial^+ [x]^K$, respectively, and one can easily see that the corresponding boundary bundles over $(\mathcal{X}, \mu, R, K)$ are measurable (cf. [Kai04]).
Theorem 2.22. Let \((X, \mu, \mathcal{B})\) be a random horospheric product with uniformly bounded vertex degrees, and \(P\) — the Markov operator of a random walk along classes of the equivalence relation \(R\) determined by a measurable family of leafwise transition probabilities \(\{\pi_x\}_{x \in X}\). If \(\lambda \prec \mu\) is a \(P\)-stationary probability measure such that the transition probabilities \(\{\pi_x\}_{x \in X}\) have a finite first moment
\[
\int_R d(x, y) d\Pi(x, y),
\]
where \(d\) is the leafwise graph distance, and \(\Pi\) is the measure \((2.4)\), then the Poisson boundaries of leafwise random walks are determined by the global height drift
\[
h = \int_R \mathcal{B}(x, y) d\Pi(x, y).
\]
If \(h = 0\), then the Poisson boundary is a.s. trivial, whereas when \(h > 0\) (resp., \(h < 0\)) a.e. leafwise Poisson boundary coincides (mod 0) with the upper (resp., lower) leafwise boundary endowed with the corresponding limit distribution (which is well-defined by Proposition 1.13).

Proof. Theorem 1.11 in combination with the standard ergodic arguments (cf. [Kai00]) implies that a.e. sample path is regular with the height drift \(h\). If \(h = 0\), then regularity implies vanishing of the asymptotic entropy, and therefore triviality of leafwise Poisson boundaries. If \(h \neq 0\), then by Proposition 1.13 a.e. sample path converges to the corresponding boundary (the upper, if \(h > 0\), and the lower, if \(h < 0\)) of leafwise horospheric products. The fact that these boundaries are actually maximal (i.e., coincide with the leafwise Poisson boundaries) then follows from the ray criterion (or Corollary 1.15 in combination with the strip criterion) in precisely the same way as in the group case, cf. [Kai00]. □

Remark 2.24. Finiteness of the entropies \((2.14)\) (which is crucial for Theorem 2.17) follows, in the usual way, from finiteness of the first moment \((2.23)\) and uniform boundedness of vertex degrees, cf. [Der86, p. 259] or [Kai00, Lemma 5.2].

Obviously, if the operator \(P\) is reversible with respect to a stationary measure \(\lambda\) (see the discussion at the end of Section 2.13 for the definition), then the integral of any additive cocycle on \(R\) with respect to \(\Pi\) vanishes. In particular, in this case the global height drift \(h\) vanishes, whence

Corollary 2.25. Under conditions of Theorem 2.22, if the operator \(P\) is reversible with respect to the measure \(\lambda\), then the leafwise Poisson boundaries are a.s. trivial.

Corollary 2.26. Under conditions of Theorem 2.22, if \(\lambda = \text{deg} \cdot \mu\) is the stationary measure of the leafwise simple random walk corresponding to a finite \(R\)-invariant measure \(\mu\) (see Corollary 2.11), then the Poisson boundary of the leafwise simple random walks is a.s. trivial.

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