Intrinsic viscosity of a suspension of weakly Brownian ellipsoids in shear

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We analyze the angular dynamics of small triaxial ellipsoids in a viscous shear flow subject to weak thermal noise. By numerically integrating an overdamped angular Langevin equation, we find the steady angular probability distribution for a range of triaxial particle shapes. From this distribution we compute the intrinsic viscosity of a dilute suspension of triaxial particles. We determine how the viscosity depends on particle shape in the limit of weak thermal noise. While the deterministic angular dynamics depends very sensitively on particle shape, we find that the shape dependence of the intrinsic viscosity is weaker, in general, and that suspensions of rod-like particles are the most sensitive to breaking of axisymmetry. The intrinsic viscosity of triaxial particles is smaller than that of the corresponding axisymmetric particles.
I. INTRODUCTION

Einstein [1, 2] calculated the shear viscosity $\mu^*$ of a dilute suspension of small, non-interacting spheres in a viscous fluid. He found $\mu^* = \mu(1 + \eta \phi)$, where $\mu$ is the viscosity of the suspending fluid, $\eta = 5/2$ is the intrinsic viscosity, and $\phi$ is the concentration by volume of suspended spheres. The suspension viscosity is larger than that of the suspending fluid because the particle cannot deform as the suspension is sheared. There is extra stress in the particle to resist the surface traction from the flow, and therefore there is a contribution proportional to the volume fraction of particles $\phi$.

For a non-spherical particle, this additional stress depends on the orientation of the particle relative to the shear flow, and it also depends upon the particle shape. Jeffery [3] calculated the angular motion and dissipation for a small ellipsoidal particle in order to determine the intrinsic viscosity $\eta$ for a dilute suspension of ellipsoids. He found that the angular motion, and consequently the intrinsic viscosity, depends indefinitely on the initial orientation of the ellipsoid. This indeterminacy is physically unsatisfactory because the macroscopic suspension viscosity $\mu^*$ should not depend on the detailed microscopic initial conditions of the suspended particles after a long time.

For larger particles, the effects of inertia may break this indeterminacy [4–10]. But the long-time dynamics still depends on the initial condition for sufficiently flat disk-shaped particles [9], which could lead to hysteresis in the rheological functions of an inertial suspension.

For small particles, thermal fluctuations render the particle trajectories stochastic, and eventually independent of their initial conditions. In this case the intrinsic viscosity $\eta$ is a function of particle shape and noise strength, when averaged over an ensemble of stochastic realizations [11, 12]. For spheroidal particles subject to sufficiently weak noise, the stationary angular distribution is independent of noise strength [12, 13]. This is because the angular dynamics is well described by the deterministic Jeffery trajectories in this limit, but with occasional jumps to a nearby trajectory. After many such jumps, a stationary probability distribution over the deterministic trajectories is established, however, the time to reach equilibrium is longer for weaker noise strength. The dilute, weak-noise rheology is given by averaging over this stationary distribution. The intrinsic viscosity of a suspension of spheroids is larger than that of a suspension of spheres, and the shape dependence is stronger for prolate spheroids than oblate spheroids [12].

How do these results generalize to triaxial ellipsoids? Much less is known concerning particles that do not possess axisymmetry. In absence of noise, the angular trajectory of a triaxial ellipsoid in shear flow is doubly periodic or chaotic, but nevertheless depends indefinitely upon initial condition [14–17]. Similarly to the case of axisymmetric ellipsoids, thermal fluctuations eventually establish a stationary distribution over these trajectories, and this angular distribution determines the suspension rheology. For strong noise Rallison [15] and Haber and Brenner [16] determined the first deviations from the uniformly distributed equilibrium state. But the angular distributions and the resulting intrinsic viscosity in the weak noise regime remain unknown. It is hard to make analytical progress, because the deterministic dynamics is chaotic.

In this paper we numerically compute the angular distribution and resulting intrinsic viscosity for a range of triaxial ellipsoids in shear flow, subject to weak thermal noise. We derive the appropriate Langevin equation and solve it numerically for the stationary probability distribution. We show how the angular distribution reflects the underlying deterministic trajectories. We compute the resulting intrinsic viscosity for a dilute suspension and show that it is maximal for axisymmetric particle shapes. In general the shape dependence of the intrinsic viscosity is weaker than that of the deterministic angular dynamics, which depends very sensitively on particle shape.

The remainder of this paper is organized as follows. In Section II we present our notation, derive the Langevin equation, and give the relation between the angular distribution and the dilute suspension viscosity. Section III contains the numerical results from our Langevin simulations. We discuss the results in Section IV and conclude in Section V.

II. THEORY

A. Notation

Where possible we use vector notation without indices, but in some instances we find index notation necessary for clarity, and then we use the Einstein summation convention. The double dot product denotes contraction with two adjacent indices, for example $(\mathbf{A} : \mathbf{B})_{ij} = A_{ijk} B_{jkl}$.

We represent the shape and orientation of an ellipsoid by the lengths $(a_1, a_2, a_3)$ and directions $(n^1, n^2, n^3)$ of its principal semi-axes. Without loss of generality we take $a_1 \leq a_2 \leq a_3$. The two aspect ratios are $\lambda = a_3/a_1$ and $\kappa = a_2/a_1$. We denote the coordinate axes fixed with respect to the undisturbed fluid flow by $(e^1, e^2, e^3)$. The undisturbed flow takes the form $\mathbf{u}^\infty = \Omega^\infty \times \mathbf{r} + E^\infty \mathbf{r}$ where $\mathbf{r}$ is the spatial coordinate vector, $\Omega^\infty$ is half the
FIG. 1. Coordinate system. Flow direction $e^1$, shear direction $e^2$. The flow vorticity points along $-e^3$. The angle between the principal axis $n^3$ and the $e^3$-axis is $\theta$, and $\varphi$ is the angle between the $-e^2$-axis and the projection of $n^3$ onto the flow-shear plane. See appendix [11] for details concerning the definition of the angles.

Fluid vorticity, and $E^\infty$ is the strain-rate matrix of the flow. We take the undisturbed flow to be a simple shear, $u^\infty = (sr_2, 0, 0)$, as shown in Fig. 1.

We also use the convention that components of a tensor in the particle coordinate frame have Greek indices, while components in the fixed frame have Latin indices, for example

$$X = \sum_{i=1}^{3} X_i e^i = \sum_{\alpha=1}^{3} X_\alpha n^\alpha.$$  \hspace{1cm} (1)

The two sets of components are related by the rotation matrix $R$, such that

$$X_\alpha = R_{\alpha i} X_i, \quad X_i = R^{T}_{i\alpha} X_\alpha .$$  \hspace{1cm} (2)

In particular, the components of the particle orientation are

$$n^\beta_i = R^{T}_{i\alpha} n^\beta_\alpha = R_{\beta i} .$$  \hspace{1cm} (3)

In the remainder of this paper we employ dimensionless variables. We scale length by $V_p^{1/3}$ and time by $1/s$, where $V_p$ is the volume of the particle, and $s$ is the magnitude of the undisturbed shear rate. Stress is scaled by $\mu s$, where $\mu$ is the viscosity of the suspending fluid.

B. Orientational dynamics

Disregarding thermal noise, the hydrodynamic angular velocity of an ellipsoidal particle in a linear Stokes flow $u^\infty = \Omega^\infty \times r + E^\infty r$ is given by

$$\omega^H = \Omega^\infty + K^{-1} H : E^\infty .$$  \hspace{1cm} (4)

The resistance tensors $K$ and $H$ are given by Haber and Brenner [19] (see Table I for translation of notation). The components of the resistance tensors are constant when expressed in the body frame, but the components of the flow gradients ($\Omega^\infty$ and $E^\infty$) are constant when expressed in the fixed frame. Therefore the components of the angular velocity, either in the body or the fixed frame, depend on the orientation of the particle.

Thermal fluctuations randomize the particle orientation. The resulting angular probability distribution $P(\mathcal{R}, t)$ is governed by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} + \nabla \cdot (\omega^H P) = P e^{-1} \nabla \cdot (D \nabla P) .$$  \hspace{1cm} (5)

Here $Pe = \mu V_p s/(k_B T)$ is the Peclét number which is a dimensionless measure of the noise strength. The diffusion tensor is given by $D = K^{-1}$, where $K$ is the viscous resistance against a steady rotation of the particle. This fluctuation-dissipation relation was first deduced by Einstein [1, 2]. Einstein’s argument was adapted to the rotation of triaxial...
TABLE I. The dimensionless elements of the resistance tensors in relation to expressions in Haber and Brenner [19].

| Notation in this paper | Notation in Ref. [19] | Eqs. in Ref. [19] |
|------------------------|------------------------|-------------------|
| K                      | 6°K                    | Eqs. [3.1], [A1]  |
| H                      | 6τ                     | Eqs. [3.1], [A2]  |
| Z                      | 5Q                     | Eqs. [3.1], [A3], [A4] |

ellipsoids by Perrin [20]. Our notation is closest to that of Rallison [18] who studied this equation in the limit of strong noise.

The Fokker-Planck equation (5) is equivalent to the following single-particle Langevin equation [21]:

\[
\mathbb{R}(t + \delta t) = \mathbb{R}(t) + \delta \mathbb{R}(t),
\]

\[
\delta \mathbb{R}_{ai} = -\varepsilon_{\alpha\beta\gamma}\omega_{\beta}H(t)R_{\gamma i}(t)\delta t + Pe^{-1}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\gamma\rho\sigma}K_{\beta\rho}^{-1}R_{\sigma i}(t)\delta t,
\]

\[
\delta \mathbb{R}_{ai}\delta \mathbb{R}_{\mu p} = 2Pe^{-1}\varepsilon_{\alpha\beta\gamma}\varepsilon_{\mu\rho\sigma}R_{\gamma i}(t)R_{\sigma p}(t)K_{\beta\rho}^{-1}\delta t.
\]

These equations describe the random angular displacements \(\delta \mathbb{R}\) as during a time \(\delta t \gg St\) (but \(\delta t \ll 1\)). The Stokes number \(St = \rho_p s V^2/3 \mu\) is a dimensionless measure of the particle inertia, and \(\rho_p\) and \(\rho_f\) are the particle and fluid densities. In Eq. (6b) and (6c) the over-bar denotes average over the distribution of angular displacements \(\delta \mathbb{R}\), distinct from the thermal average \(\langle \cdots \rangle\).

To simulate Eq. (6) in practice, we represent the orientation by a unit quaternion instead of a rotation matrix [22]. The unit quaternion is better than the rotation matrix for numerical computation because it has four scalar components and a unit constraint \(|q| = 1\), whereas the rotation matrix has nine scalar components and an orthogonality constraint \(R^T R = 1\). The Langevin equation in quaternion coordinates is described in Appendix A.

C. Dilute suspension rheology

The macroscopic description of a particulate suspension is based on a statistical model of the microscopic fluid mechanics of all the suspended particles [23]. For a sufficiently homogeneous suspension a macroscopic observable, such as the stress tensor \(\sigma\), may be represented by an average of the microscopic configurations. In general this averaging is a very complicated task [3]. But for a dilute suspension it is sufficient to consider the stress contribution from an isolated particle and sum the independent contributions from all particles, because particle interactions are negligible. This gives the correct rheology to first order in the volume fraction of particles in the suspension [3, 23].

Batchelor [3] showed that the stress contribution from a single torque-free particle in steady Stokes flow is determined by the symmetric force dipole on the particle, the so-called stresslet. In terms of resistance tensors, the stresslet for a torque-free particle is

\[
\mathcal{S} = \mathcal{C} : \mathbb{E}^{\infty},
\]

where the components of \(\mathcal{C}\) are

\[
C_{ijkl} = R_{ai}R_{bj}R_{ck}R_{dl} \left( Z_{\alpha\beta\gamma\delta} - H_{\alpha\beta\gamma\delta}K_{\mu\nu}^{-1}H_{\mu\nu} \right).
\]

Eq. (7) was derived by Batchelor [3], and it is valid in the limit of weak thermal noise. The stress due to the presence of particles in a dilute suspension of volume \(V\) is therefore [3]

\[
\Sigma^{(p)} = \frac{V_p}{V} \sum_m \mathcal{S}_m,
\]

where \(\mathcal{S}_m\) denotes the stresslet from the \(m\):th particle and the sum is over all particles. The stresslet (7) depends on the the shape and orientation of the particle. If there are many identical particles in \(V\), the sum over particles may be replaced by an angular average over the distribution \(P(\mathbb{R})\):

\[
\Sigma^{(p)} = \phi \int d\mathbb{R} \mathcal{S}(\mathbb{R}) P(\mathbb{R}).
\]
Here $\phi = NV_p/V$ is the volume concentration of particles, with $N$ the number of particles in the volume $V$. The volume concentration is assumed to be small, $\phi \ll 1$. The intrinsic viscosity $\eta$ is determined by the shear stress $\Sigma_{12}^{(p)}$ due to the particles [18, 24]:
FIG. 3. Rows: four representative surfaces-of-sections for $\varphi = \pi/2 + n\pi/4$, for $n = 0, \ldots, 3$. These sections correspond to the directions parallel with the flow, of extending strain, perpendicular to the flow, and of compressing strain (Fig. 2). Columns: (1) the surface-of-section of deterministic trajectories; (2) the stationary angular distributions; (3) the stresslet element corresponding to the intrinsic viscosity of a dilute suspension; (4) the contribution to intrinsic viscosity, given by the product of the angular distribution and the stresslet element. Parameters: $\lambda = 10$, $\kappa = 5$, and $Pe = 200$.

The deterministic trajectories go around, see Fig. 2. Each transversal slice of constant $\varphi$ of this torus is a Poincaré surface-of-section [26], schematically shown in Fig. 2. To illustrate the angular distributions, we choose the four representative surfaces-of-sections for $\varphi = \pi/2 + n\pi/4$, for $n = 0, \ldots, 3$, corresponding to the directions parallel with the flow, of extending strain, perpendicular to the flow, and of compressing strain (Fig. 2). The surfaces of section in Refs. [15–17] are for $\varphi = n\pi$ ($n^3$ perpendicular to the flow).

We show the distributions for two example particle shapes: Fig. 3 is for a strongly triaxial ellipsoid with aspect ratios $\lambda = 10$ and $\kappa = 5$, while Fig. 4 is for a moderately triaxial particle with $\lambda = 10$ and $\kappa = 2$. Both particles are subject to weak noise ($Pe = 200$). In Figs. 3 and 4 we show, in four columns, (1) the surface-of-section of deterministic trajectories; (2) the stationary angular distributions; (3) the stresslet element corresponding to the intrinsic viscosity of a dilute suspension; (4) the contribution to intrinsic viscosity, given by the product of the angular distribution and the stresslet element.

At low thermal noise the distribution is dominated by the deterministic dynamics, and the only effect of the noise is to establish a distribution over the deterministic trajectories. For strong thermal noise, by contrast, the distribution $P(\mathbb{R})$ is nearly isotropic (not shown).
From Eq. (11) we compute the intrinsic viscosity in the limit of weak thermal noise, at large Péclet numbers. We choose Pe large enough so that the intrinsic viscosity converges to a Pe-independent plateau, as in the axisymmetric case [12, 13]. Fig. 5 shows the results for spheroidal particles, for λ = 5 and 10 as a function of κ. The error bars correspond to one standard deviation. The parameter κ ranges from κ = 1 to κ = λ. The limiting cases correspond to rotationally symmetric, ellipsoidal particles. In these special cases our numerical results agree with those of previous work. The values for λ = 10 and κ = 1, 10 are determined from Eq. (27) in [13], together with the angular averages from Table 1 in this paper. The angular averages for λ = 5 and κ = 1, 5 are taken from Table 3 in [12]. For the λ = 10-particle slightly higher Pe-values are needed to obtain this convergence, than for the λ = 5-particle. For the special case of spheres (λ = κ = 1) our numerical results are consistent with Einstein’s result, η = 5/2.
FIG. 5. Intrinsic viscosity of a dilute suspension of triaxial ellipsoids as a function of \( \kappa \). Left: \( \lambda = 5 \), Pe = 500 (○), Pe = 1000 (□), Pe = 2000 (△). Right: \( \lambda = 10 \), Pe = 2000 (○), Pe = 3000 (□). On the x-axis, the shapes represented by the \( \kappa \)-values range from rods (\( \kappa = 1 \)) to disks (\( \kappa = \lambda \)). Known limiting values for axisymmetric particles (•) are taken from Refs. [12, 13].

IV. DISCUSSION

A. Orientational distributions

The deterministic angular trajectories depend very sensitively on the shape of the particle. While axisymmetric ellipsoids tumble on periodic Jeffery orbits, a slight breaking of this symmetry can lead to doubly periodic, and even chaotic angular dynamics [15–17], as the surfaces-of-section in Figs. 3 and 4 show. The closed concentric lines near \( \cos \theta = 0 \) correspond to doubly-periodic tumbling, the black regions correspond to chaotic angular motion, while the almost horizontal lines near \( |\cos \theta| = 1 \) correspond to slightly perturbed Jeffery orbits (log rolling). We note that the surfaces-of-section look very similar to those of Hamiltonian dynamics [26]. This may appear surprising, because our dynamics is dissipative, not Hamiltonian. But it is no coincidence that the surfaces-of-section look so similar. While our system does not conserve energy, it is time-reversal invariant and exhibits discrete reflection symmetry [17] that constrains the angular dynamics in a way analogous to the symplectic structure of Hamiltonian dynamics [26].

For weak noise, the particle orientation tends to follow deterministic trajectories, but occasionally jumps to a neighboring trajectory. This process establishes an equilibrium distribution of the particle orientation over the deterministic trajectories after some time. Which orientations are most probable, and how does the distribution reflect the nature of the deterministic angular dynamics?

Figs. 3 and 4 show that the probability is highest in the flow-shear plane, when \( \mathbf{n}^3 \) aligns with the flow direction (first rows of Figs. 3 and 4). This is simply a consequence of the time-scale separation in the deterministic dynamics for \( \lambda \gg 1 \): rod-like particles spend most of their time aligned with the flow where the angular dynamics is slow. This orientation corresponds to a local minimum of shear stress (third panel in first row of Figs. 3 and 4).

The other three surfaces of section capture how the angular dynamics when the projection of \( \mathbf{n}^3 \) is not aligned with the flow direction. The probability is not uniformly distributed over the surfaces of section. Also in this case peaks in \( P(\mathbb{R}) \) are explained by slow angular dynamics. Consider the second row of Figs. 3 and 4 corresponding to \( \varphi = 3\pi/4 \). The probability is peaked at \( (\cos \theta, \psi) \approx (1, -\pi/4) \) and the symmetric point \((-1, \pi/4)\). The condition \( \cos \theta = \pm 1 \) corresponds to the log-rolling orbit, and when \( \cos \theta = 1 \) then \( \psi = -\pi/4 \) ensures that \( \mathbf{n}^3 \) aligns with the shear direction, so that the torque is minimal and the dynamics is slowest. The same argument holds for \( \cos \theta = -1 \) then \( \psi = \pi/4 \).

In rows 3 and 4 of Figs. 3 and 4 the situation is analogous: the probability \( P(\mathbb{R}) \) is largest for orientations where the torque is smallest. Comparing Figs. 3 and 4 we see that the maximal values of \( P(\mathbb{R}) \) are similar (first rows). This is expected because the parameter \( \lambda \) is the same. The probability in rows 2, 3, and 4 is larger in Fig. 3 (\( \kappa = 5 \)) compare with Fig. 4 (\( \kappa = 2 \)). A larger value of \( \kappa \) gives a smaller torque, corresponding to slower dynamics and thus higher probability.

In summary, the probability \( P(\mathbb{R}) \) of orientations in the weak-noise limit is strongly peaked where the deterministic dynamics is slowest, regardless of whether it is periodic, doubly periodic or possibly chaotic.
B. Intrinsic viscosity

The orientation-dependent contribution to intrinsic viscosity $S_{12}(\mathbb{R})$, however, has a local minimum where the probability density is concentrated (Figs. 3 and 4). Nevertheless, this direction dominates the contribution to the intrinsic viscosity at weak noise. With the major axis along the flow direction, the maximally dissipating configuration is when the particle is tilted 45°. Although those particle orientations are relatively unlikely, they contribute to the integral of $P(\mathbb{R})S_{12}(\mathbb{R})$ because of their relatively high shear stress.

Fig. 5 shows the resulting intrinsic viscosity. We see that $\text{Pe}$ is large enough so that the intrinsic viscosity is approximately independent of $\text{Pe}$, to within numerical accuracy. For the $\lambda=10$-particle slightly higher $\text{Pe}$-values are needed to obtain this convergence, than for the $\lambda=5$-particle.

For all particle shapes shown the intrinsic viscosity is larger than that of spheres ($\eta=5/2$). This is consistent with the observation that the intrinsic viscosity of a dilute suspensions of axisymmetric particles increases with larger particle aspect ratio $\kappa$, most strongly so for suspensions of prolate particles.

The effect of making the particles triaxial, however, is to decrease the resulting intrinsic viscosity, as can be seen in Fig. 5. The Figure shows, in fact, that intrinsic viscosity depends only weakly on $\kappa$, except for rod-like axisymmetric particles. We conclude that the intrinsic viscosity does not depend as sensitively on particle shape as the deterministic angular dynamics, even at low thermal noise where the angular dynamics follows deterministic trajectories for long times. Figs. 6 and 7 show that this is the consequence of two effects. First, the angular dynamics is most sensitive to particle shape near orientation where the particle spends least time. Second, the additional stress caused by the particle is comparatively small at these orientations.

A further consequence of our findings is that it is important to ensure that rod-like particles are axisymmetric to high precision when trying to achieve a maximal increase in suspension viscosity by adding rod-like particles to a suspension.

V. CONCLUSIONS

We analyzed the angular dynamics of triaxial ellipsoids in a shear flow subject to weak thermal noise (large Péclet numbers). By numerically integrating the corresponding angular Langevin equation, we found the stationary probability distribution for a range of asymmetric particle shapes at weak thermal noise. We showed that the probability is largest when the deterministic angular dynamics is slow, regardless of whether it is strictly periodic, doubly periodic, or chaotic.

We also compared how the angular distribution correlates with the orientation-dependent contribution to the intrinsic viscosity of a dilute suspension. We found that the angular probability is concentrated in a local minimum of the shear stress. In general though the shear stress is much less localized than the angular probability.

Finally, we computed the intrinsic viscosity of a dilute suspension of triaxial ellipsoids at weak noise, and found that the intrinsic viscosity decreases as particles deviate from axisymmetric shape, and most strongly so for rod-shaped particles. For example, at $\lambda=10$ changing $\kappa$ from 1 to 2 gives a 10%-reduction in intrinsic viscosity. In general, however, we found that the dependence of the intrinsic viscosity on particle shape is much less sensitive than the nature of the deterministic angular dynamics, because the angular probability is localized where the torque is small, regardless of the nature of the classical dynamics.

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Appendix A: Quaternion formulation

Our quaternion description follows that of Graf [22]. Here we give the practical details relevant for simulation of the Langevin equation (6). We represent the unit quaternion $q$ as a four-component unit vector $q = (W, X, Y, Z)$, $|q| = 1$. Its relation to the rotation matrix $R$ is

$$R = EQ^T,$$  \hspace{1cm} (A1)

where

$$E = \begin{pmatrix} -X & W & -Z & Y \\ -Y & Z & W & -X \\ -Z & -Y & X & W \end{pmatrix},$$  \hspace{1cm} (A2)

$$G = \begin{pmatrix} -X & W & Z & -Y \\ -Y & -Z & W & X \\ -Z & -Y & X & W \end{pmatrix}.$$  \hspace{1cm} (A3)
The equation of motion of $q$ is

$$\dot{q}_i = \frac{1}{2} G_{\alpha i} \omega_\alpha,$$  \hspace{1cm} (A4)

where $\omega$ is the angular velocity of the particle in body coordinates.

Einstein’s fluctuation-dissipation argument (see main text) gives the asymptotic form of the integral of the angular-velocity autocorrelation function:

$$\int_0^{\delta t} ds \langle \omega(s) \rangle \sim \omega^H \delta t, \quad 1 \gg \delta t \gg St,$$  \hspace{1cm} (A5)

$$\int_0^{\delta t} ds_1 \int_0^{\delta t} ds_2 (\omega(s_1) - \omega^H) (\omega(s_2) - \omega^H)^T \sim 2P_e^{-1}K^{-1}\delta t, \quad 1 \gg \delta t \gg St.$$  \hspace{1cm} (A6)

Here $\omega^H$ is the hydrodynamic angular velocity \[.\) With the angular velocity autocorrelation given in Eq. (A6) we derive

$$\langle \delta q_i \rangle = \frac{1}{2} G_{\alpha i} \omega^H \delta t - \frac{1}{4} P_e^{-1} K^{-1}_{\alpha \alpha} q_i \delta t + O(\delta t^2),$$  \hspace{1cm} (A7)

$$\langle \delta q_i \delta q_j \rangle = \frac{1}{2} P_e^{-1} K^{-1}_{\alpha \beta} G_{\beta j} \delta t + O(\delta t^2).$$  \hspace{1cm} (A8)

The numerical simulation starts with an initial orientation $q_0$, and updates according to $q_{n+1} = q_n + \delta q$, where $\delta q$ is Gaussian random with mean and variance given by Eq. (A7) and Eq. (A8).

**Appendix B: Euler angles**

We use Euler angle coordinates in the Goldstein $z$-$x'$-$z''$ convention \[27\]: starting from $n_i = e_i$, first rotate the $n_i$ by $\varphi$ around $n_3$, then by $\theta$ around the resulting $n_1$ and finally by $\psi$ around the resulting $n_3$. With the shorthand $cx = \cos x$ and $sx = \sin x$ the elements of the rotation matrix are

$$R = \begin{pmatrix}
c\varphi c\psi - c\theta s\varphi s\psi & c\psi s\varphi + c\theta c\varphi s\psi & s\theta s\psi \\
-c\theta c\varphi s\psi & c\varphi c\psi - s\varphi s\psi & c\psi s\theta \\
s\theta s\varphi & -c\varphi s\theta & c\theta
\end{pmatrix}.$$  \hspace{1cm} (B1)

This convention is the same as that adopted in Ref. \[15\], and Fig. 1 in their paper corresponds to our Fig. 1. Our axis $n^3$ corresponds to their $z'$-axis.