Order parameter fluctuations at a critical point
- an exact result about percolation -

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Abstract. The order parameter of the system in the critical state, is expected to undergo large non-Gaussian fluctuations. However, almost nothing is known about the mathematical forms of the possible probability distributions of the order parameter. A remarkable exception is the site-percolation on the Bethe lattice, for which the complete order-parameter distribution has been recently derived at the critical point. Surprisingly, it appears to be the Kolmogorov-Smirnov distribution, well known in very different areas of mathematical statistics. In the present paper, we explain first how this special distribution could appear naturally in the context of the critical systems, under the assumption (still virtually unstudied) of the exponential distribution of the number of domains of a given size. In a second part, we present for the first time the complete derivation of the order-parameter distribution for the critical percolation model on the Bethe lattice, thus completing a recent publication [1] announcing this result.

1. Introduction

Phase transitions take a central place in statistical physics, because it is of prime importance in applications, and also because of the serious mathematical challenge. Such critical behavior happens when collective coherence spans suddenly the entire system for a definite value of the control parameter. The resulting correlations are infinitely-ranged and make the theoretical problem of the utmost difficulty. Only a very few exact results are known [2].

The order parameter is the key physical parameter to characterize the symmetry of the system state in any phase. This is a global macroscopic quantity, and its value may fluctuate because of the large number of different configurations that the system can access using energy fluctuations. However, the local order parameter, say: \( \eta \), must be defined as soon as we have to deal with spatial fluctuations.

Then, the global order parameter is simply the sum of its local values, and one defines more generally the moments of the local order parameter, \( \langle \eta^k \rangle \), where the notation \( \langle \cdot \rangle \) stems for the average value over the statistical fluctuations of the order parameter, for a given value of the control parameter. The usual task consists in analyzing the singular behavior of some moments, when the value of the control parameter crosses the critical point. Instead, one should analyze...
the order-parameter distribution (OPD), \( f(\eta) \), since all the moments are related to this function, namely:

\[
\langle \eta^k \rangle = \int \eta^k f(\eta) d\eta
\]

but this is much more complicated to deal with a function than with its moments, and it is a reason why this analysis is usually not carried out. A remarkable exception is the seminal analysis done by Landau [3], when he considered the density of free energy, \( F(\eta)/N \), of a system of \( N \) components. The thermodynamic quantity \( F(\eta)/N \) is simply related to the OPD:

\[
f(\eta) = \frac{1}{Z(\beta)} e^{-\beta F(\eta)/N},
\]

where \( Z(\beta) = \int \exp(-\beta F(\eta)/N) d\eta \) is the partition function of the system at the temperature \( T = 1/k_B \beta \). This equation links clearly the OPD to the thermodynamics of the system.

Another exceptional model where this approach has been done is the Potts model, for which the density of free energy is known [4].

The present paper is essentially divided into two parts. In the first part, we discuss a relation between the fluctuations of the order parameter and the distribution of domains. This way of investigation is not well known though it could provide a different point of view of the problem, and could be tackled with computer simulations. Particular families of possible critical OPD appear naturally in this framework. The second part is about a recent result about the exact OPD at the critical point of the percolation problem on the Bethe lattice. Amazingly, this distribution is known in various other scientific areas not connected to criticality. This result shows in particular that the critical OPD is not necessarily a very strange and complicated distribution.

2. Critical OPD for a system divided in domains.–

In a number of cases, the critical system is divided in areas of about the same value of the local order parameter. These areas are called ‘domains’. Detailed knowledge of the distribution of the number of domains of a given size, allows in principle, to determine the OPD of the system. This way of study has not been investigated in details up to now, and we describe hereafter the corresponding framework.

2.1. Relation between the OPD and the distributions of the number of domains of given size.

Let us consider a macroscopic system of size \( N \), embedded in the \( d \)-dimensional space, which experiences a critical behavior for a definite value of the control parameter. We suppose that the system can, at any given time, be split into well-defined homogeneous domains, each of them characterized by a value of the local order parameter.

It is convenient to consider a generic example to illustrate the idea: the global magnetization per spin, \( M_N/N \), of the \( N \)-spins Ising system is the sum of the magnetizations of the ‘up’-spins domains minus the sum of the magnetizations of the ‘down’-spins domains. If the spin ‘up’ corresponds to the value \( \eta = +1 \) of the local order parameter, and the spin ‘down’ to the value \( -1 \), one has:

\[
m_N \equiv M_N/N = \frac{2}{N} \sum_{s=1}^{N} s n_s^{(+)} - 1,
\]
where \( n_s^{(+)} \) is the number of domains of ‘up’-spins, of size \( s \). That way, the variable \( m_N \) is a number between -1 and 1. Clearly, the distribution of the magnetization is then directly related to the distribution, \( n_s^{(+)} \), of the number of the domains of size \( s \).

Let us be more precise. Let \( \hat{f}_{M_N}(H) \) denote the Laplace transform of the probability distribution of \( M_N \), i.e.:

\[
\hat{f}_{m_N}(\lambda) = \int_0^\infty f_{m_N}(m) e^{-\lambda m} dm .
\]

This function corresponds, in the context of the statistical physics, to the ratio between the partition function \( Z(\beta, H) \), in the field: \( H = -\lambda/\beta \), and the partition function \( Z(\beta, 0) \equiv Z(\beta, 0) \) in the null field.

The variables \( \{n_s^{(+)}\}_{s=1}^N \) are weakly correlated, because the only constraint acting on these quantities is the conservation of the mass:

\[
\sum_s n_s^{(+)} + \sum_s n_s^{(-)} = N .
\]

Then, the function \( \hat{f}_{M_N}(\lambda) \) is the product of the Laplace transforms of the \( n_s^{(+)} \)-distributions:

\[
\hat{f}_{m_N}(\lambda) = e^{-\lambda N} \prod_{s=1}^N \int_0^\infty f_{n_s^{(+)}}(n) e^{-2\lambda s n / N} dn ,
\]

with \( f_{n_s^{(+)}} \) the probability density of the random variable \( n_s^{(+)} \). This relation allows to compute the OPD if we know the distributions of the numbers of the domains of a given size \( s \).

2.2. The case of the exponentially-distributed domain-number distribution

At the critical point, the average distribution of the \( n_s^{(+)} \) is known to be an inverse power-law [5]:

\[
\langle n_s^{(+)} \rangle \simeq \alpha N s^{-\tau} \text{ with the exponent } \tau \geq 2, \text{ and } \alpha \text{ a positive constant which can be found using the condition: } \lim_{N \to \infty} \langle m_N \rangle = 0 \text{ at the critical point.}
\]

If we knew the probability distribution of the random variable \( n_s^{(+)} \), one could deduce from (2) the OPD of the Ising model. Unfortunately, the probability distribution of the \( n_s^{(+)} \) is generally unknown at the critical temperature [6], and the average values are only available [7]. However, we can make simple guess about the unknown distribution of the number of domains, to see which kind of OPD one could generally expect. The Gamma-distributions or the Poisson distribution can be studied this way.

Here, we consider the exponential distribution [8] as the simplest distribution for the number of domains of size \( s \), that is:

\[
f_{n_s^{(+)}}(n) = \frac{1}{\langle n_s^{(+)} \rangle} e^{-n / \langle n_s^{(+)} \rangle} ,
\]

as the density of probability to find \( n \) spins ‘up’ domains of size \( s \). The limit distribution for \( M_N \) can then be derived exactly for the integer values of \( \tau \), as explained below.

Considering the exponential-distributions (3) and using the Fisher result \( s \langle n_s^{(+)} \rangle / N \simeq \alpha s^{1-\tau} \), one obtains:

\[
\hat{f}_{m_N}(\lambda) \simeq e^{-\lambda \left[ \prod_{s=1}^N (1 + 2\alpha \lambda s^{1-\tau}) \right]^{-1} .
\]
When the value of \( \tau \) is an integer number, the asymptotic limit when \( N \to \infty \) of the product above, is written more simply as the finite product:

\[
\lim_{N \to \infty} \hat{f}_{m_N}(\lambda) = e^{-\lambda} \prod_{j=1}^{\tau-1} \Gamma \left( 1 - (2\alpha\lambda)^{1/(\tau-1)} e^{i\pi(2j+1)/(\tau-1)} \right) \tag{5}
\]

with \( \Gamma \) the Gamma function in the complex plane. The Equation (5) defines a one-parameter family of critical OPD, parametrized with the actual value of \( \tau \).

Two particular cases are important to discuss: \( \tau = 2 \) and \( \tau = 3 \).

- The case \( \tau = 2 \).
  Using the relation:
  \[
e^{\gamma x} \Gamma(1 + x) = \lim_{N \to \infty} \prod_{s=1}^{N} \frac{e^{x/s}}{1 + x/s},\]
  with \( \gamma \) the Euler constant, the moment generating function of \( m_N \) writes:
  \[
  \lim_{N \to \infty} \hat{f}_{m_N}(\lambda) \propto \Gamma(1 + 2\alpha\lambda),
  \]
  after proper rescaling. This generating function corresponds to the Gumbel distribution [9], that is (written in its simplest rescaled form):
  \[
  f_{Gumbel}(y) = e^{y} - e^{-y}.
  \]
  This result might be related to a recent series of articles by Bramwell et al [10] in which it is argued that Gumbel-like distributions could be typical of the OPD of the critical 2d XY-model [11]. The analysis was both from numerical simulations and approximated theoretical arguments. Bertin [12] gave a different argument for the 2d XY-model at the Kosterlitz-Thouless transition [10], using an argument close to the one presented here. Actually, the result could be explained with the Fisher’s result [5]: \( \tau = 2 + 1/\delta \), with \( \delta \) the critical exponent, known to be \( \delta = 15 \) in this model. Because the value of \( \delta \) is that large, the critical OPD should not differ too much from the Gumbel distribution, but it could only be an approximation. Indeed, it has been proved recently that the Gumbel distribution could not be the correct OPD for the 2d XY-model [13], though it is indeed a good approximation for all values of the magnetization, but the extreme ones.

- The case \( \tau = 3 \).
  When \( \tau \) is a general odd integer number, one can express the finite product (5) in a yet simpler form as:
  \[
  \lim_{N \to \infty} \hat{f}_{m_N}(\lambda) = \prod_{j=1}^{\tau-1} \left[ \frac{\pi(\alpha\lambda)^{1/(\tau-1)}}{\sin(\pi(\alpha\lambda)^{1/(\tau-1)} e^{i\pi(2j+1)/(\tau-1)})} \right]^{1/2}.
  \]
  For the particular case \( \tau = 3 \), one finds the limit Laplace transform:
  \[
  \lim_{N \to \infty} \hat{f}_{m_N}(\lambda) = e^{-\lambda} \frac{\sqrt{2\alpha\pi^2\lambda}}{\sinh\sqrt{2\alpha\pi^2\lambda}},
  \]
  which corresponds to the Kolmogorov-Smirnov distribution [14] (written here in a reduced form):
  \[
  f_{KS}(y) = \frac{d}{dy} \left[ \sum_{n=-\infty}^{\infty} (-1)^n e^{-n^2 y} \right].
  \]
This result has been shown previously in various contexts. Probably Watson was the first one to mention it [15], in connection with the distribution of the excursions in the Brownian bridge problem [16]. One can also read Doob’s paper [17].

In the physics of the critical systems, this distribution has been discovered in the OPD of the percolation-cluster size on the Bethe lattice, at the critical threshold [1]. A detailed derivation of this result is outlined in the next section.

3. Critical OPD for the site-percolation on the Bethe lattice.–

We shall derive now an exact critical OPD, which belongs to the family (4). The critical system is a mean-field percolation model. Note that, despite a number of available results [18], the distribution of the numbers of domains of given sizes is not known in this case, and we cannot conclude about a definite relation of this (possible) coincidence to the results developed in the preceding Section.

3.1. Scaling relations for the OPD

A fundamental relation resulting from self-similarity of the system at the critical point, is the so-called first scaling law [19]. For the normalized OPD function $P^{(N)}(\eta)$ of the finite system of size $N$, the first-scaling law writes:

$$\langle \eta \rangle N P^{(N)}(\eta) \sim F_c \left( \frac{\eta}{\langle \eta \rangle N} \right) .$$

with the scaling function $F_c$ independent of the size of the lattice.

In terms of the Laplace transform of the OPD:

$$\hat{P}^{(N)}(h) \equiv \int P^{(N)}(\eta) e^{-h\eta/\langle \eta \rangle N} d\eta ,$$

written here with the reduced variable $z_1 \equiv \eta/\langle \eta \rangle N$, the first-scaling law is equivalent to the assertion that $\hat{P}^{(N)}(h)$ tends to the Laplace transform of the scaling function, namely:

$$\hat{P}^{(N)}(h) \to \int_0^{\infty} F_c(z_1) e^{-hz_1} dz_1 ,$$

when $N \to \infty$.

Note that the definition (6) leads to the small-$h$ expansion of the Laplace transform of the OPD under the form:

$$\hat{P}^{(N)}(h) = 1 - h + O(h^2) .$$

3.2. The Bethe lattice

A finite Bethe lattice $B_n$ of branching parameter $z$, is built from an origin site, $O$. At the first generation ($n = 1$), $z$ sites are added, each linked to $O$ with one bond. The $z$ sites are peripheral, while $O$ becomes a bulk site. At the second generation ($n = 2$), $z - 1$ sites are added to every last-generation site, and so on. As $z(z - 1)^{n-1}$ sites are added at the generation $n$, the total number of sites in $B_n$ is:

$$N_{B_n} = 1 + z \frac{(z - 1)^n - 1}{z - 2} ,$$
and this corresponds to $N_{\mathcal{B}_n} - 1$ bonds.

In this work, we consider cases where the branching parameter $z$ has a finite value larger than 3. Figure 1 shows the second-generation Bethe lattice $\mathcal{B}_2$, with the branching parameter $z = 3$.

![Figure 1. Sketch of the finite Bethe lattice of the second generation. It contains 10 sites, of which 6 are peripheral.](image)

### 3.3. The site-percolation problem on the Bethe lattice

We suppose that every site of the Bethe lattice has the probability $p$ to be occupied [20]. We denote $q \equiv 1 - p$, the probability that a site be vacant. A cluster is defined as an ensemble of occupied sites connected together by neighboring. The site-percolation problem on the infinite Bethe lattice is a second-order critical phenomenon [21]: transition occurs suddenly when a giant cluster (the percolation cluster) spans the lattice. This dramatic event occurs for the definite value $p = p_c(z) = 1/(z - 1)$ of the occupation probability.

In this context, the order parameter for the transition, is the probability for a given site to belong to the percolation cluster. Clearly, it is 0 when the percolation cluster does not exist, and has a positive value when such a cluster appears in the system. In other words, the order parameter is the volume fraction of the percolation cluster.

We plan to analyze here the fluctuations of the order parameter. To complete this task, we have first to define properly the percolating cluster on the finite Bethe lattice. Then, we shall look precisely how the result for the finite lattice can be extended to the infinite size.

In the finite Bethe lattice, the percolation cluster is the cluster which links the lattice origin to at least one surface site. Within this definition, the percolation cluster is necessarily unique on the Bethe lattice since, by definition, it must include the origin, $O$, of the lattice.

### 3.4. Probability distribution of the percolating-cluster size
The starting point of the derivation is the recurrence relation concerning the number, \( g^{(n)}_{s, l} \), of possible different clusters of \( \mathcal{B}_n \), containing \( s \) sites, \( \mathcal{O} \) being one of these sites, and \( l \) sites of the cluster being peripheral.

Since any \( l \) peripheral site of generation \( n - 1 \) gives birth to \((z - 1)l\) sites at generation \( n \) (for any \( n \geq 2 \)), one has the relations:

\[
g^{(k)}_{s, l} = \sum_{l'} \binom{z - 1}{l'}^{l'} g^{(k-1)}_{s, l}, \quad \text{for } k \geq 2 ,
\]

\[
g^{(1)}_{s, l} = \binom{z}{s - 1} \delta_{s, l - 1},
\]

with \( \delta_{i,j} \) the Kronecker symbol. In Equ.(8), the sum runs over all values of \( l' \) such that:

\( l \leq (z - 1)l' \leq (z - 1)(s - l - 1) \).

The rest of the derivation consists in transforming step by step these recurrence equations in the exact OPD for the percolation-cluster size.

- **The generating function for the statistical weights \( g^{(k)}_{s, l} \).**

On the finite Bethe lattice \( \mathcal{B}_n \) and for a given value of the occupation probability \( p \), the probability that the cluster passing through \( \mathcal{O} \) be of size \( s \) with \( l \) peripheral sites, is:

\[
P^{(n)}_{s, l} = g^{(n)}_{s, l} p^{s-1} q^{(z-2)s+2-(z-1)l},
\]

with the normalization \( \sum_{s, l} P^{(n)}_{s, l} = 1 \). It is convenient to consider the following generating function:

\[
\Phi^{(n)}(x, y) = \sum_{s, l} P^{(n)}_{s, l} x^{s-1} y^l = q^z \sum_{s, l} g^{(n)}_{s, l} (pq^{z-2}x)^{s-1} (q^{1-z}y)^l.
\]

In particular, the value of \( \Phi^{(1)} \) is explicitly:

\[
\Phi^{(1)}(x, y) = (q + pxy)^z,
\]

and the normalization condition: \( \Phi^{(n)}(1, 1) = 1 \), holds for any value of \( p \) and of \( n \).

- **Definition of the probability distribution of the percolating-cluster size.**

The percolating cluster, when it exists, is the cluster passing through \( \mathcal{O} \), and connecting \( \mathcal{O} \) to the periphery of \( \mathcal{B}_n \). The probability of occurrence of a percolating cluster of size \( s \) is then the probability that the cluster has size \( s \) and any number \( l \) of peripheral sites, minus the probability that the cluster has size \( s \) but does not reach the periphery \((l = 0)\), namely:

\[
P^{(n)}_{\text{perco}}(s) = \sum_{l \neq 0} P^{(n)}_{s, l} = \sum_{l} P^{(n)}_{s, l} - P^{(n)}_{s, 0}.
\]

- **The Laplace transform of the distribution \( P^{(n)}_{\text{perco}}(s) \).**
From the previous relation, and using (9), we can write down the discrete Laplace transform of the probability distribution \( \mathcal{P}_{\text{perco}}^{(n)}(s) \) as:

\[
\hat{\mathcal{P}}_{\text{perco}}^{(n)}(\lambda) \equiv \sum_{s} \mathcal{P}_{\text{perco}}^{(n)}(s) e^{-\lambda s} / \sum_{s} \mathcal{P}_{\text{perco}}^{(n)}(s) ,
\]

\[
= e^{-\lambda / (s_{\text{perco}} n)} \frac{\Phi^{(n)}(\exp(-\lambda / (s_{\text{perco}} n)), 1) - \Phi^{(n)}(\exp(-\lambda / s_{\text{perco}} n)), 0)}{\Phi^{(n)}(1, 1) - \Phi^{(n)}(1, 0)},
\]

with the average size of the percolating cluster: \( \langle s_{\text{perco}} \rangle_n = \sum_s s \mathcal{P}_{\text{perco}}^{(n)}(s) / \sum_s \mathcal{P}_{\text{perco}}^{(n)}(s) \).

The running variable \( \lambda \) is a positive real number in order to insure the uniform convergence of the series in (11).

- The average percolating-cluster size.–

The definition (11) was written such that the expansion of \( \hat{\mathcal{P}}_{\text{perco}}^{(n)}(\lambda) \) in powers of \( \lambda \) begins as: \( \mathcal{P}_{\text{perco}}^{(n)}(\lambda) = 1 - \lambda + \mathcal{O}(\lambda^2) \), in accordance to (7). This constraint gives the value of the average size \( \langle s_{\text{perco}} \rangle_n \) of the percolation cluster through the relation:

\[
\langle s_{\text{perco}} \rangle_n = 1 + \frac{\partial \Phi^{(n)}(1, 1) - \partial \Phi^{(n)}(1, 0)}{\Phi^{(n)}(1, 1) - \Phi^{(n)}(1, 0)} .
\]

So far, we obtained the Laplace transform of the probability distribution of the size of the percolation cluster for the site-percolation problem on the Bethe lattice at a given value (not necessary the critical value) of the occupation probability \( p \). This Laplace transform is given by (12) and (13), in function of the generating functions \( \Phi^{(n)}(x, y) \) defined in (9).

3.5. Asymptotic behavior of the generating function \( \Phi^{(n)}(x_n, y) \)

In this Section, we shall use the recurrence equations (8) to study the asymptotic behavior of the \( \Phi^{(n)}(x, y) \) at the critical point \( p = p_c(z) \), when the generation number of the lattice, \( n \to \infty \).

- Recurrence relation for the generating functions \( \Phi^{(n)} \).

Because of the relations (8), one has the recurrence equations:

\[
\Phi^{(n)}(x, y) = \Phi^{(n-1)}(x, (q + p xy)^{z-1}) .
\]

Noting from (10) that the second argument in the right-hand term of (14) is \( (\Phi^{(1)}(x, y))^{(z-1)}/z \), one deduces the general relation: \( \Phi^{(n)}(x, y) = \Phi^{(n-j)}(x, (\Phi^{(j)}(x, y))^{(z-1)/z}) \) for any value \( j = 1, \cdots, n - 1 \). In particular, for \( j = n - 1 \), the recurrence relation writes:

\[
\Phi^{(n)}(x, y) = \left( q + px \left( \Phi^{(n-1)}(x, y) \right)^{(z-1)/z} \right)^z ,
\]

with Equ.(10) as the initial condition.

- Asymptotic behavior of the generating functions \( \Phi^{(n)} \) for \( p = p_c(z) \).

From now, we consider the system at the critical value of the occupation probability \( p = p_c(z) = 1/(z - 1) \), and we wish to derive the asymptotic (i.e. \( n \to \infty \)) behavior of \( \hat{\mathcal{P}}_{\text{perco}}^{(n)}(\lambda) \).
We study here the finite Bethe lattices of increasing size \( k \), until a maximum size \( k = n \), while all the other parameters are constant. Therefore, the \( \Phi^{(n)}(x, y) \) are derived recursively from Equ.(15), written as:

\[
\Phi^{(k)}(x_n, y) = (q_c + p_c x_n (\Phi^{(k-1)}(x_n, y))^{(z-1)/z})^{z},
\tag{16}
\]

\[
\Phi^{(0)}(x_n, y) = y^{z/(z-1)},
\]

where the initial condition was expressed conveniently for the index \( k = 0 \). In Equ.(16), the index \( k \) runs from 1 to \( n \), while the parameter \( x_n = e^{-\lambda/(s_{perco})} \) is a constant (i.e. independent of \( k \)).

In order to simplify Equ.(16), we introduce the auxiliary variable \( \varphi_n(k) \) defined by:

\[
\Phi^{(k)}(x_n, y) \equiv K_n^z e^{-\varphi_n(k)/n},
\]

where \( K_n \) is the coefficient:

\[
K_n \equiv \left( \frac{1}{(z-2) x_n p_c} \right)^{1/(z-1)}.
\]

Equ.(16) writes then:

\[
\varphi_n(k) = \frac{z - 2}{n} \kappa_n^2 - z n \ln \left( \frac{z - 2}{z - 1} + \frac{1}{z - 1} e^{-\frac{z - 1}{z} \varphi_n(k-1)} \right),
\tag{17}
\]

\[
\varphi_n(0) = \begin{cases} 
zn \ln K_n & \text{if } y = 1, \\
\infty & \text{if } y = 0,
\end{cases}
\tag{18}
\]

with

\[
\kappa_n^2 = 2 \frac{z - 2}{z - 1} n^2 \frac{\lambda}{s_{perco}}.
\tag{19}
\]

In the relation (19), \( x_n \) was replaced by its value: \( e^{-\lambda/(s_{perco})} \). The advantage of these auxiliary variables is that the parameter \( x_n \) appears only in the constant term in Equ.(17) and the initial condition (18).

- General solution for the auxiliary sequences \( \varphi_n(k) \).

We suppose that one can develop the right-hand term of Equ.(17) in successive powers of \( 1/n \). This assumption - which will be checked later on the solution - needs the condition:

\[
\lim_{n \to \infty} \varphi_n(n)/n = 0.
\tag{20}
\]

When (20) is fulfilled, Equ.(17) writes, at the second order in \( 1/n \), as the difference equation:

\[
\varphi_n(k) - \varphi_n(k-1) = \frac{z - 2}{2zn} \left[ \frac{z^2 \kappa_n^2}{(z-2)^2} - \varphi_n^2(k-1) \right].
\tag{21}
\]

Therefore, (21) is solved after considering the differential analog:

\[
f'(u) = \frac{z - 2}{2zn} \left[ \left( \frac{z \kappa_n}{z - 2} \right)^2 - f^2(u) \right],
\]

\[9\]
with the initial conditions: \( f(0) = zn \ln K_n \) for \( y = 1 \), and \( f(0) = \infty \) for \( y = 0 \). This leads to the form:

\[
\varphi_n(k) = \frac{z\kappa_n}{z-2} \frac{\rho \exp (\kappa_n k/n) - 1}{\rho \exp (\kappa_n k/n) + 1},
\]

with \( \rho \) a coefficient given by the initial conditions. Then, after using the identity:

\[
\frac{\rho e^x - 1}{\rho e^x + 1} = \tanh\left(\frac{x}{2} + \frac{1}{2} \ln \rho\right),
\]

the sequence \( \varphi_n(k) \) can be expressed in terms of hyperbolic functions:

\[
\varphi_n(k) = \begin{cases} 
\frac{z}{z-2} \kappa_n \tanh \left( \frac{\kappa_n k}{2} - a_0 \right) & \text{if } y = 1, \\
\frac{z}{z-2} \kappa_n \coth \left( \frac{\kappa_n k}{2} \right) & \text{if } y = 0,
\end{cases}
\]

with \( a_0 \) the coefficient defined by:

\[
a_0 = \frac{1}{2} \ln \frac{1 - \kappa_n/2n}{1 + \kappa_n/2n}. \tag{22}
\]

Note that \( a_0 \) tends to 0 as \( \sim \sqrt{\frac{\lambda}{\langle s_{\text{perco}} \rangle_n}} \). Moreover, the assumption (20) can be rewritten in terms of \( \kappa_n \) as \( \lim_{n \to \infty} \kappa_n/n = 0 \), which is well realized here.

At this stage, we obtained the asymptotic behavior of the generating function \( \Phi^{(n)}(x, y) \) under the form:

\[
\Phi^{(n)}(x, y) = K_n^x e^{-\varphi^{(n)}(n)/n},
\]

with the function \( \varphi^{(n)}(n) \) given by:

\[
\varphi^{(n)}(n) = \begin{cases} 
\frac{z}{z-2} \kappa_n \tanh \left( \frac{\kappa_n k}{2} - a_0 \right) & \text{if } y = 1, \\
\frac{z}{z-2} \kappa_n \coth \left( \frac{\kappa_n k}{2} \right) & \text{if } y = 0,
\end{cases} \tag{23}
\]

with \( \kappa_n \) as in (19) and \( a_0 \) as in (22).

### 3.6. Exact expression for the limit distribution \( \hat{\mathcal{P}}^{(\infty)}_{\text{perco}}(\lambda) \)

The explicit solutions (23) can be used in (12) to write the discrete Laplace transform of the probability distribution of \( \mathcal{P}^{(n)}_{\text{perco}}(s) \) under the asymptotic form:

\[
\hat{\mathcal{P}}^{(n)}_{\text{perco}}(\lambda) \simeq C \kappa_n \left( \coth\left( \frac{\kappa_n}{2} \right) - \tanh\left( \frac{\kappa_n}{2} - a_0 \right) \right). \tag{24}
\]

Then, the condition (7): \( \hat{\mathcal{P}}^{(n)}_{\text{perco}}(\lambda) = 1 - \lambda + \mathcal{O}(\lambda^2) \), defines the two missing parameters: \( C \) and \( \langle s_{\text{perco}} \rangle_n \) at the critical case point \( p = p_c(z) \).

The equations (24), (25) and (26) characterize completely the limit scaled distribution \( \hat{\mathcal{P}}^{(n)}_{\text{perco}} \).
Considering the asymptotic dependences $\kappa_n \to \sqrt{6\lambda}$ and $a_0 \to 0$, when $n \to \infty$, one deduces that the Laplace transform of the probability distribution of the percolation-cluster size tends to the non-trivial distribution:

$$\hat{P}_{perco}^{(n)}(\lambda) \to \frac{\sqrt{6\lambda}}{\sinh \sqrt{6\lambda}},$$

which is a universal function in the sense that it is independent on the value of branching parameter $z$.

### 3.7. Limit distribution

As explained in details in [22], the function (27) is the Laplace transform of the Kolmogorov-Smirnov distribution [14]:

$$F(z_1) = \frac{d}{dz_1} \left[ \sum_{n=-\infty}^{\infty} (-1)^n e^{-n^2\pi^2 z_1/6} \right],$$

(28)

$$= \frac{d}{dz_1} \left[ \frac{\sqrt{6}}{\pi \sqrt{z_1}} \sum_{n=-\infty}^{\infty} e^{-3(2n+1)^2/(2z_1)} \right].$$

(29)

Both expressions are equivalent because of the Poisson summation formula [23].

The expression (29) gives the singular small-$z_1$ tail:

$$F(z_1) \sim e^{-3/(2z_1)} \frac{e^{-3/(2z_1)}}{z_1^{3/2}} \text{ for } z_1 \to 0,$$

(30)

while (28) gives the exponential large-$z_1$ tail as:

$$F(z_1) \sim e^{-\pi^2 z_1/6} \text{ for } z_1 \to \infty.$$  

(31)

The Figure 2 shows results of numerical simulations of the site-percolation on the $z = 3$-Bethe lattice at the critical point. We can see the convergence of the scaled OPD to the limit distribution.

In particular, all the moments of the limit distribution exist since both tails (30) and (31) go to zero faster than any power of $z_1$. The moments of the Kolmogorov-Smirnov distribution can be obtained readily in a closed form from the development of (27) as a series of Bernoulli numbers. Translated into moments of the order parameter, they write:

$$\langle \eta^k \rangle = 2k! \left( \frac{1}{(2k)!} \right) \left( 2^{2k-1} - 1 \right) B_{2k},$$

for any non-negative value of the integer exponent $k$.

### 4. Conclusion

The order parameter of the non-critical $N$-body system, is the sum of uncorrelated local variables, thus the Central Limit Theorem must apply and lead essentially to Gaussian fluctuations. On the other hand, the problem is not so simple at the critical point because of the long-ranged correlations which embrace the entire system. No general theory is available in the critical case, while the order-parameter fluctuations are expected to be non-Gaussian.
Figure 2. Numerical simulations of the OPD for the site-percolation model on the $z = 3$-Bethe lattice of generations $n = 2$ to 24. All the data are shown as dots.

We have seen in this work that the probability distribution of the order parameter is tightly linked to the statistics of the domain sizes. However, lack of knowledge about this statistics forbids presently any definite conclusion. Then, any exact result is of prime importance to understand which kind of fluctuations the order parameter can experience at the criticality.

The percolation on the Bethe lattice provides such unique information. For the first time, the complete derivation was presented in details in the present paper. It requires only standard tools of asymptotic analysis, and it can easily be extended to system states out of the criticality. As a matter of fact, the derivation (not presented here) for given values of the occupation probability different from $p_c(z)$ leads to the results expected from the Central Limit Theorem, as it should be. For this special model at the critical point, the OPD appears to be a known distribution (the Kolmogorov-Smirnov distribution). It does not depend on the value of the coordination number $z$ which defines the lattice. In this sense, this critical OPD is expected to be universal and characteristics of the mean-field percolation on loopless lattices. This is a first result concerning the possible universality of the critical OPD. Such a question, for the other systems at the critical point, is still pending.

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