Scaling of success probabilities for linear optics gates

Stefan Scheel and Koenraad M R Audenaert
Quantum Optics and Laser Science, Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2BW, UK
E-mail: s.scheel@imperial.ac.uk and k.audenaert@imperial.ac.uk

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Abstract. By using the abstract linear-optical network derived by Scheel and Lütkenhaus (2004 New J. Phys. 6 51) we show that for the lowest possible ancilla photon numbers the probability of success of realizing a (single-shot) generalized nonlinear sign-shift gate on an \((N + 1)\)-dimensional signal state scales as \(1/N^2\). We limit ourselves to single-shot gates without conditional feed-forward. We derive our results by using determinants of Vandermonde-type over a polynomial basis which is closely related to the well-known Jacobi polynomials.

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1. Introduction

One of the promising routes to implementing small-scale quantum information processing networks for tasks in quantum communication or cryptography is by using linear-optical networks. Already a number of experiments have been performed that accomplish the task of generating controlled-NOT gates as the basic non-trivial two-mode operation [1, 2]. The necessary nonlinearities on the single-photon level are produced via conditional measurements [3, 4].

Some two-mode gates such as the controlled-$\sigma_z$ gate can be generated by acting separately on both modes within a Mach–Zehnder interferometric setup. In that way, the complexity of a two-mode gate is reduced to a single-mode gate, the nonlinear sign-shift gate. A number of recent theoretical works [5]–[8] give a variety of networks capable of performing the nonlinear sign-shift gate with the following transformation rule:

$$c_0|0\rangle + c_1|1\rangle + c_2|2\rangle \mapsto c_0|0\rangle + c_1|1\rangle - c_2|2\rangle,$$

which can serve as a building block for the two-mode controlled-phase gate. The crucial question for the ability to concatenate linear-optical gates are their probabilities of success. Recently, it has been shown that the nonlinear sign-shift gate cannot be realized with a success probability of more than $1/4$ if one abstains from using conditional feed-forward [9, 10] which is stronger than the bound of $1/2$ obtained in [11].

These results were obtained by considering an abstract type of network that includes all possible networks by dividing it into an ‘active’ beam splitter ‘A’ that mixes signal and ancilla states, a preparation stage ‘P’ and a detection stage ‘D’ that are there to generate and disentangle a purpose-specific ancilla, respectively. The general scheme is reproduced in figure 1. In short, the abstraction consists in the observation that each U($N$) network can be decomposed into a triangle-shaped network of at most $N(N - 1)/2$ beam splitters (and some additional phase shifters) [12], which in fact, is nothing else but doing a Householder-type diagonalization of the unitary matrix associated with the network. Then, it is easy to see that the signal mode can be chosen to impinge only on a single beam splitter (here called ‘A’) with the rest of the network conveniently divided into the ancilla preparation and detection stages ‘P’ and ‘D’, respectively.

What we will try to do in this paper is to answer the question how the probability of success scales as the signal state becomes higher dimensional, thereby giving a hint on the possible scaling law for multimode quantum gates (by virtue of the above-mentioned fact that generalized nonlinear sign-shift gates can be combined in Mach–Zehnder interferometer-type setups to yield multimode controlled-phase gates). However, as we have seen previously, a general answer is hard to obtain. Fortunately, it turned out that the most important special case can be dealt with analytically. It has been argued already in [9] and confirmed in [10] that the success probability, taken as a function of the chosen ancilla photon numbers, reaches its maximum if the ancilla contains as few photons as possible. In addition, the restriction to the lowest possible photon number is justified by the fact that in this case the least quantum-state engineering is needed to generate the ancilla in the preparation stage ‘P’ and decoherence affects the ancilla only minimally.

This paper is organized as follows. In section 2, we provide the necessary framework that is suitable for our set task. Specializing to the lowest admissible photon numbers, we show in section 3 that there exist networks that realize the generalized nonlinear sign shift (to be defined...
Figure 1. Preparation stage ‘P’ of the ancilla state that is fed into the ‘active’ beam splitter ‘A’, and decomposition stage ‘D’ for detection.

in section 2) with a probability of success of 1/N^2 which constitutes the main result of our paper. Some details of the calculation are provided in the appendices. Conclusions are drawn in section 4.

2. \( (N+1) \)-dimensional signal state with \( N \)-dimensional ancilla

A possible building block for multimode quantum gates via some Mach–Zehnder interferometer-type setups [7] is the generalized nonlinear sign-shift gate. Let us therefore consider an \( (N+1) \)-dimensional signal state

\[
|\psi_{\text{in}}\rangle = \sum_{k=0}^{N} c_k |k\rangle.
\]

The generalized nonlinear sign-shift gate is defined by the following transformation rule:

\[
c_N \mapsto -c_N,
\]

leaving all other coefficients untouched. As thoroughly described in [9], not all ancilla states are admissible. In fact, the ancilla must consist of a superposition of states with fixed photon number. Moreover, the number of photons that are detected after the disentangling stage ‘D’ in figure 1 must be the same as the initial ancilla photon number. The reason for this requirement is essentially that the Hilbert spaces of incoming and outgoing signal states have to be isomorphic.

We will restrict ourselves to the case in which we use an \( N \)-dimensional ancilla for transforming an \( (N+1) \)-dimensional signal state. This choice is somewhat restrictive but, in view of minimizing experimental resources, the most accessible choice. Let us denote the \( N \) different photon numbers in the ancilla by \( n_l, l = 1, \ldots, N \). Then, the ancilla can be represented in the form

\[
\sum_{l=1}^{N} \gamma_l |n_l\rangle |A_{n-n_l}\rangle
\]

with \( \sum_l \gamma_l^2 = 1 \) (the expansion coefficients can be taken to be real) and \( n = \sum_l n_l \). The state \( |A_{n-n_l}\rangle \) is arbitrary as concerns the number of modes as long as it contains exactly \( n - n_l \) photons. The advantage of introducing the abstract network in figure 1 is apparent since the \( |A_{n-n_l}\rangle \) do not play any role in what follows.
Now we transform the signal and ancilla states at the ‘active’ beam splitter ‘A’ and use the fact that the matrix elements \( \langle k, n_i | \hat{U} | k, n_i \rangle \) can be written as
\[
\langle k, n_i | \hat{U} | k, n_i \rangle = (T^*)^{n_i-k} P_k^{(0,n_i-k)} (2|T|^2 - 1),
\]
where \( P_k^{(0,n_i-k)}(x) \) is a Jacobi polynomial defined by
\[
P_k^{(0,n_i-k)}(2|T|^2 - 1) = \sum_{m=0}^{k} \binom{k}{m} \binom{n_i+m}{m} (|T|^2 - 1)^m,
\]
and \( T \) (in general complex) the transmission coefficient of the beam splitter ‘A’. Then, the (unnormalized) states \( |\psi_k\rangle \) \( (k = 1, \ldots, N+1) \) after the transformation can be defined as
\[
|\psi_k\rangle = \sum_{l=1}^{N} \gamma_l (T^*)^{n_i-k+1} P_k^{(0,n_i-k+1)} (P) |n_i\rangle |A_{n-n_i}\rangle,
\]
where we have abbreviated the argument \( P = 2|T|^2 - 1 \) of the Jacobi polynomial to emphasize its dependence on the permanent of the beam splitter matrix. This is a reminder of the fact that matrix elements of unitary operators can always be written as permanents of a matrix associated with the symmetric tensor power of the beam splitter matrix [13].

The state we project upon is denoted by
\[
|\psi\rangle = \sum_{i=1}^{N} \alpha_i |n_i\rangle |A_{n-n_i}\rangle,
\]
where \( \sum_i |\alpha_i|^2 = 1 \). Then, in order to realize the generalized nonlinear sign shift, the following \( N \) equations need to be satisfied:
\[
\langle \psi | \psi_{N+1} \rangle = -\langle \psi | \psi_k \rangle, \quad k = 1, \ldots, N.
\]
That specifies all weights \( \alpha_l \) as well as the transmission coefficient \( T \) which can be used to compute the probability as
\[
p = |\langle \psi | \psi_k \rangle|^2 \quad \text{for any } k.
\]
In fact, equation (9) is a homogeneous matrix equation of the form
\[
\sum_{l=1}^{N} a_{kl} \gamma_l \alpha_l = 0,
\]
with matrix elements
\[
a_{kl} = a_{kl}^{(1)} + a_{kl}^{(2)} = (T^*)^{n_i-N} P_N^{(0,n_i-N)} (P) + (T^*)^{n_i-k+1} P_k^{(0,n_i-k+1)} (P).
\]
From linear algebra it is known that the homogeneous system of equation (11) has non-trivial solutions only if the determinant of the coefficient matrix vanishes. That gives a condition for the transmission coefficient $T$.

From the expansion theorem of determinants, we immediately have the solutions for $\alpha_l\gamma_l$ given in terms of the cofactors $A_{kl}$ of the matrix elements $a_{kl}$, i.e. $\alpha_l\gamma_l \propto A_{kl}$. Because the success probability (10) is nothing else but $p = \left| \sum_l \alpha_l\gamma_l a_{kl}^{(2)} \right|^2$, it is clearly maximized if the magnitudes of $\alpha_l$ and $\gamma_l$ are the same. From the normalization condition for $|\psi\rangle$, we then get the solution

$$p_{\text{max}} = \frac{\left| \sum_l A_{kl} a_{kl}^{(2)} \right|^2}{(\sum_l |A_{kl}|)^2} \quad \text{for some } k.$$  \hspace{1cm} (13)

We will shortly see that the choice of $k$ does not matter.

3. Maximal success probability

In most cases, one needs to refer to numerical methods to maximize the success probability (13) for a given set of ancilla photon numbers $n_l$. However, there is one case in which an analytical result can be obtained. For this purpose, let us consider the photon numbers $n_l = l - 1$. This is the ancilla with the lowest possible photon numbers and as such the experimentally most important situation. Then, we find that there exists at least one solution of the form

$$T = 1 - 2^{1/N}. \quad \hspace{1cm} (14)$$

For $N = 2$, i.e. the nonlinear sign-shift gate with up to two photons in the signal state, this is the solution $T = 1 - \sqrt{2}$ encountered in [5, 9]. For this particular choice, the maximal success probability reaches

$$p_{\text{max}} = \frac{1}{N^2}. \quad \hspace{1cm} (15)$$

This result is obviously true for $N = 1$, since the corresponding gate can indeed be obtained deterministically, and for $N = 2$ that is the result obtained in [9, 10]. Note the surprising fact that the success probability does not scale exponentially but rather quadratically with $N$. Equation (15) constitutes the main result of this paper. We will spend the following subsections deriving it.

3.1. Determinant of the coefficient matrix

In the calculation of the projection vector $|\psi\rangle$, we encountered the matrix consisting of the elements $a_{kl}$ (equation (12)). As one can see, the $a_{kl}^{(1)}$ are actually independent of the row index $k$ and therefore constant columns. In addition, each column consists of two subcolumns, and the determinant can thus be expanded as a sum of determinants, each of which contains one of the subcolumns [14]. Hence,

$$\det(a_{kl}) = \sum_{m_1,\ldots,m_N = 1}^2 \det(a_{kl}^{(m)}) \hspace{1cm} (16)$$
Together with the property that a determinant vanishes if it contains more than one constant column, this means that only $N + 1$ out of the total $2^N$ terms eventually have to be computed. The first term is just $\det(a_{kl}^{(2)})$ and the other $N$ terms are obtained by replacing the $m$th column of $(a_{kl}^{(2)})$ by the corresponding column of $(a_{kl}^{(1)})$. Thus,

$$\det(a_{kl}) = \det(a_{kl}^{(2)}) + \sum_{m=1}^{N} \det(a_{kl}^{(2)} : a_{km}^{(2)} \mapsto a_{km}^{(1)}).$$

(17)

All higher-order replacements vanish due to the fact that determinants of matrices with more than one constant column vanish identically. Let us first consider the term $\det(a_{kl}^{(2)})$. By inspection of the structure of the matrix elements $a_{kl}^{(2)}$, one realizes that this determinant can be written as

$$\det(a_{kl}^{(2)}) = \det[S_{k-1}^{(T)}(n_l)]$$

(18)

Due to the special choice of photon number $n_l = l - 1$, all prefactors containing the transmission coefficient cancel out. The next observation is crucial for the following discussion. The Jacobi polynomials $P_{k-1}^{(0,l-k)}(2|T|^2 - 1)$, being polynomials of order $k - 1$ in $2|T|^2 - 1$, are also polynomials of order $k - 1$ in the photon numbers $n_l = l - 1$. This can be seen as follows. Recall the definition of the Jacobi polynomials (equation (6), we briefly revert to the general notation $n_l$ for the photon numbers) and rewrite the binomial factor

$$\binom{n_l + m}{m}$$

as

$$\frac{1}{m!} \prod_{i=1}^{m} (n_l + i) = \frac{1}{m!} \sum_{p=0}^{m} n_l^p \sigma^{(m)}_{m-p},$$

(19)

which itself is a polynomial of degree $m$ in $n_l$. Hence, the Jacobi polynomial $P_{k-1}^{(0,n_l-k)}(2|T|^2 - 1)$ is also a polynomial of degree $k$ in the photon numbers $n_l$. The $\sigma^{(m)}_{m-p}$ are the elementary symmetric polynomials of degree $m - p$ of $m$ variables which in our case are just the integers $1, \ldots, m$.

We will now define a new class of polynomials $S_k^{(n)}(n_l)$ by setting

$$S_k^{(n)}(n_l) = \sum_{p=0}^{k} c_{k,p} n_l^p = \sum_{p=0}^{k} n_l^p \sum_{m=p}^{k} \binom{k}{m} (x^2 - 1)^m m! \sigma^{(m)}_{m-p}. $$

(20)

These polynomials fulfil the identity

$$S_k^{(n)}(n_l) = P_k^{(0,n_l-k)}(2|T|^2 - 1).$$

(21)

From their definition (equation (20)), it is easy to show that these polynomials fulfil a three-term recursion relation. Some properties of the $S$-polynomials are collected in appendix A.

With this definition, we can finally rewrite the determinant (18) with the help of the $S$-polynomials as

$$\det(a_{kl}^{(2)}) = \det[S_{k-1}^{(T)}(n_l)],$$

(22)

which is nothing but a Vandermonde determinant over the $S$-polynomial basis. For later use, let us denote this Vandermondian by $V_{S,N}(\{n_l\})$, where the subscripts $S$ and $N$ denote the polynomial
basis and the dimension, respectively. For such a determinant, we have the relation (see e.g. [14])

\[ V_{S,N}(\{n_l\}) = V_N(\{n_l\}) \prod_{k=0}^{N-1} c_{kk}, \]  

(23)

where \( V_N(\{n_l\}) \) is the usual Vandermonde determinant of the photon numbers \( n_l \). This implies that

\[ \det(a^{(2)}_{kl}) = \left( \prod_{l=1}^{N} \frac{1}{(l-1)!} \right) (|T|^2 - 1)^{\frac{N}{2} - (N-1)} V_N(\{n_l\}). \]  

(24)

Specializing again to \( n_l = l - 1 \), in which case \( V_N(\{n_l\}) = \prod_l (l - 1)! \), we are left with

\[ \det(a^{(2)}_{kl}) = (|T|^2 - 1)^{\frac{N}{2} - (N-1)}. \]  

(25)

Next, in order to compute the determinant in equation (17), we also need the terms in which the \( m \)th column of \( (a^{(2)}_{kl}) \) has been replaced by \( (a^{(1)}_{km}) \). Expanding the determinants

\[ \det(a^{(2)}_{kl}) \mapsto a^{(1)}_{km} \]

with respect to the column \( m \), we obtain

\[ \sum_{m=1}^{N} \det(a^{(2)}_{kl} : a^{(2)}_{km} \mapsto a^{(1)}_{km}) = \sum_{m,n=1}^{N} a^{(1)}_{nm} A^{(2)}_{nm}, \]  

(26)

where the \( A^{(2)}_{nm} \) are the cofactors of the matrix \( (a^{(2)}_{kl}) \). They are, disregarding some prefactors, nothing but generalized Vandermondisians over the \( S \)-polynomial basis with one gap, denoted by \( V_{S,N\setminus\{n_l\}} \), with the polynomials of order \( n - 1 \) and photon number \( n_m = m - 1 \) missing. With that, equation (26) becomes

\[ \sum_{m=1}^{N} \det(a^{(2)}_{kl} : a^{(2)}_{km} \mapsto a^{(1)}_{km}) = \sum_{n=1}^{N} (-1)^{N-n} (T^n)^{n-N-1} \sum_{m=1}^{N} (-1)^m S_N^{(T^n)}(m - 1) V_{S,N\setminus\{n_l\}}. \]  

(27)

The last summation in equation (27) is just the definition of \( V_{S,N+1\setminus\{n_l\}} \). Such a quantity can be computed in the following way. Analogous to the elementary symmetric polynomials \( \sigma^{(j)}_i \) defined in equation (19), we define the elementary symmetric polynomials \( s_i(x; N; N) \) with respect to the \( S \)-polynomial basis by

\[ \frac{(x^2 - 1)^N}{N!} \prod_{i=1}^{N} (n - n_i) = \sum_{j=0}^{N} (-1)^{N-j} s_{N-j}(x; N; N) S_j^{(x)}(n). \]  

(28)

These polynomials \( s_i(x; N; N) \) are \( i \)th-order symmetric polynomials over the \( N \) different photon numbers \( n_l, l = 1, \ldots, N \). Expanding \( V_{S,N+1} \) on one hand with respect to a Vandermondian of lower order and on the other hand with respect to its last column, namely

\[ V_{S,N+1} = \frac{(x^2 - 1)^N}{N!} \prod_{i=1}^{N} (n_{N+1} - n_i) V_{S,N} \]

\[ = \sum_{m=0}^{N} (-1)^{N-m} S_m^{(x)}(n_{N+1}) V_{S,N+1\setminus\{m\}}, \]  

(29)
and using the definition (28) for the symmetric polynomials, we obtain that
\[
V_{S,N+1}\{n-1\} = s_{N-n+1}(|T|; N; N) V_{S,N}
\]  
(30)
(see also [15]).

The symmetric polynomials \(s_i(x; N; N)\) can be computed by inserting equation (19) into equation (28). When specifying yet again to the choice of photon numbers \(n_l = l - 1\), the solution is simply
\[
s_k(|T|; N; N) = \binom{N}{k} |T|^{2k}, \tag{31}
\]  
which can be easily checked by inserting into equation (28) and using the definition (20) for the S-polynomials. Note that the symmetry with respect to the interchange of any two photon numbers \(n_k = k - 1\) and \(n_l = l - 1\) is hidden inside the binomial coefficient in equation (31). For a constructive proof of the statement (31) and generalizations of it, we refer the reader to appendix B.

Combining the knowledge about the symmetric polynomials and the Vandermondians defined earlier, we arrive at the expression
\[
\sum_{n,m=1}^{N} (-1)^{N-n+m} (T^*)^{n-N-1} s_n^{(T)}(m-1) V_{S,N\{n-1\}}(m-1) = \sum_{n=1}^{N} (-1)^{N-n} (T^*)^{n-N-1} V_{S,N+1\{n-1\}}
\]
\[
= \sum_{n=1}^{N} (-1)^{N-n} (T^*)^{n-N-1} \left( \begin{array}{c} N \\ n-1 \end{array} \right) |T|^{2(N-n+1)} V_{S,N}
\]
\[
= (|T|^2 - 1) \frac{N(N-1)}{2} [2 - (1 - T)^N]. \tag{32}
\]
Combined with the result for the determinant \(\det(a_{kl}^{(2)})\) (equation (25)), this yields the final result for the determinant of the coefficient matrix as
\[
\det(a_{kl}) = (|T|^2 - 1) \frac{N(N-1)}{2} [2 - (1 - T)^N]. \tag{33}
\]
A non-trivial condition for this determinant to vanish is thus
\[
T = 1 - 2^{1/N}, \tag{34}
\]
just as we set out to show. What we have seen is that calculating the value of the transmission coefficient \(T\) of the active beam splitter ‘A’ is still mathematically involved despite the reduction of the problem to the abstract network.

3.2. Success probability

In order to compute the success probability (13), we need to calculate the cofactors \(A_{kl}\) of the generalized Vandermondian \((a_{kl})\). It turns out that it is useful to choose \(k = N\) in this relation since the cofactors \(A_{Nl}\) are essentially generalized Vandermondians over the S-polynomial basis which can be computed in the same way as shown above. Although this particular choice for \(k\) is not unique, it represents the most easily tractable situation. Clearly, any other \(k\) could have been chosen. However, since the solution \(T = 1 - 2^{1/N}\) is unique and is a simple root of equation (11),
the result cannot depend on the particular choice of \( k \). Since now \( T \in \mathbb{R} \), we can drop asterisks to simplify the notation.

The numerator in equation (13) can be calculated in the following way. By using simple row manipulations, it is shown to be equal to

\[
\sum_{l=1}^{N} a_{Nl}^{(2)} A_{Nl} = \det(a_{kl}^{(2)}) + \sum_{n=1}^{N-1} \det(a_{n}^{(2)} : a_{nl}^{(2)} \mapsto a_{nl}^{(1)})
\]

\[
= - \det(a_{kl}^{(2)} : a_{Nl}^{(2)} \mapsto a_{Nl}^{(1)}),
\]

where we have made use of the fact that the determinant \( \det(a_{kl}) \) vanishes for \( T = 1 - 2^{1/N} \). All subsequent formulas have to be understood with the solution for \( T \) in mind.

Inserting the definitions for the matrix elements \( a_{kl}^{(1,2)} \), it turns out that this is a Vandermondian with one gap. More precisely,

\[
\det(a_{kl}^{(2)} : a_{Nl}^{(2)} \mapsto a_{Nl}^{(1)}) = -T^{-1} V_{S,N+1\setminus(N-1)}^{(N)}.
\]

Expanding this determinant with respect to the last row, we get

\[
V_{S,N+1\setminus(N-1)}^{(N)} = \sum_{k=0}^{N-1} (-1)^{N+k+1} S_{N}^{(T)}(k) V_{S,N-1}^{(k)}.
\]

The Vandermondian over the polynomial basis can be easily computed from the corresponding Vandermondian over the power basis as (see appendix C)

\[
V_{S,N-1}^{(k)} = (T^2 - 1)^{\frac{N-2}{2}} \binom{N-1}{k}.
\]

With that and the definition of the \( S \)-polynomials, one finds that

\[
\sum_{k=0}^{N-1} (-1)^{N+k+1} \binom{N-1}{k} S_{N}^{(T)}(k) = \sum_{k=0}^{N-1} \sum_{m=1}^{N} (-1)^{N+k+1} \binom{N-1}{k} \binom{k+m}{m} \binom{N}{m} (T^2 - 1)^m
\]

\[
= \sum_{m=1}^{N} \binom{N}{m} (T^2 - 1)^m \binom{m}{N-1} = NT^2(T^2 - 1)^{N-1}.
\]

where the last line follows from the fact that only \( m = N - 1, N \) give nonzero contributions to the sum. Combining all the results, we find that

\[
\sum_{l=1}^{N} a_{Nl}^{(2)} A_{Nl} = NT(|T|^2 - 1)^{\frac{N-2}{2}}.
\]

The remaining task is to compute the normalization factor \( (\sum_{l} |A_{Nl}|^2) \) for \( T = 1 - 2^{1/N} \). Recalling the expression (12) for the matrix elements \( a_{kl} \), inserting the expression (6) for the Jacobi polynomials and changing the indexing such that \( k \) and \( l \) now range from 0 to \( N - 1 \), we get

\[
da_{kl} = \sum_{m=0}^{N} \binom{l+m}{m} (T^2 - 1)^m T^l \left[ \binom{N}{m} T^{-N} + \binom{k}{m} T^{-k} \right].
\]
To find the cofactors \( A_{kl} \) of \( a = (a_{kl})_{0 \leq k, l \leq N-1} \), we make use of the fact that they are given by

\[
A_{kl} = \det(a)(a^{-1})_{kl}.
\]

We need \( A_{N-1,l} \) and this can thus be found by solving the equation \( ax = b \), where \( b_k = \delta_{k,N-1} \), and \( A_{N-1,l} = \det(a)x_l \). The equation \( ax = b \) gives rise to the following \( N \) equations in \( x \):

\[
\sum_{m=0}^{N} \binom{N}{m} T^{-N} + \binom{k}{m} T^{-k} y_m = \delta_{k,N-1},
\]

where we have defined the \( N + 1 \) quantities \( y_m \) as

\[
y_m = (T^2 - 1)^m \sum_{l=0}^{N-1} \binom{l+m}{m} T^l x_l.
\]

We will now proceed by inverting the two systems (42) and (43) in succession. The former system is of course underdetermined, which means that the solution for \( y_m \) will contain one free parameter. This parameter will be solved for when solving (43) for \( x_l \). Let us now choose \( \eta \) as a free parameter with

\[
\eta = \sum_{m=0}^{N} \binom{N}{m} T^{-N} y_m,
\]

then (42) turns into

\[
\sum_{m=0}^{N} \binom{k}{m} y_m = (\delta_{k,N-1} - \eta) T^k.
\]

Since \( k \) is at most \( N - 1 \), the term for \( m = N \) in the left-hand side vanishes and we can as well sum from 0 to \( N - 1 \). This system is now easily inverted using the sum

\[
\sum_{k=0}^{N-1} (-1)^{k+m} \binom{m}{k} \binom{k}{l} = \delta_{l,m},
\]

giving

\[
y_m = \sum_{k=0}^{N-1} (-1)^{k+m} \binom{m}{k} (\delta_{k,N-1} - \eta) T^k
\]

\[
= (-1)^{N-1+m} \binom{m}{N-1} T^{N-1} - \eta \sum_{k=0}^{N-1} (-1)^{k+m} \binom{m}{k} T^k,
\]

for \( 0 \leq m \leq N - 1 \). Noting that the first term in (46) is actually zero for all \( m < N - 1 \), and \( T^{N-1} \) for \( m = N - 1 \), and simplifying further, we find

\[
y_m = -\eta(T - 1)^m, \quad 0 \leq m \leq N - 2,
\]

\[
y_{N-1} = T^{N-1} - \eta(T - 1)^m.
\]
The value of $y_N$ follows from (44) as

$$y_N = T^N \eta - \sum_{m=0}^{N-1} \binom{N}{m} y_m = \eta \left[ T^N + \sum_{m=0}^{N-1} \binom{N}{m} (T - 1)^m \right] - NT^{N-1}$$

We can now solve (43) for $x_l$ and $\eta$. To perform the inversion, we use the sum

$$\sum_{m=0}^{N-1} (-1)^{k-m} \left[ \sum_{p=0}^{N-1} \binom{p}{k} \binom{p}{m} \right] \binom{l+m}{m} = \delta_{kl},$$

giving, for $0 \leq l \leq N - 1$,

$$T^l x_l = \sum_{m=0}^{N-1} (-1)^{l-m} \left[ \sum_{p=0}^{N-1} \binom{p}{l} \binom{p}{m} \right] \frac{y_m}{(T^2 - 1)^m}$$

$$= -\eta \sum_{m=0}^{N-1} (-1)^{l-m} \left[ \sum_{p=0}^{N-1} \binom{p}{l} \binom{p}{m} \right] \frac{1}{(T^2 - 1)^m} \sum_{k=0}^{N-1} (-1)^{k+m} \binom{m}{k} T^k$$

$$+ (-1)^{l-N+1} \binom{N-1}{l} \frac{T^{N-1}}{(T^2 - 1)^{N-1}}$$

$$= -\eta (-1)^l \sum_{m=0}^{N-1} \binom{p}{l} \left( \frac{T}{T+1} \right)^p + (-1)^{l-N+1} \binom{N-1}{l} \frac{T^{N-1}}{(T^2 - 1)^{N-1}}.$$  (50)

We can solve $\eta$ from (43) and (49), using (50)

$$\eta [2T^N - (T - 1)^N] - NT^{N-1} = (T^2 - 1)^N \sum_{l=0}^{N-1} \binom{l+N}{N} T^l x_l$$

$$= -\eta (T^2 - 1)^N \sum_{l=0}^{N-1} \binom{l+N}{N} (T^2 - 1) \sum_{p=0}^{N-1} \binom{p}{l} \left( \frac{T}{T+1} \right)^p$$

$$+ (T^2 - 1)^N \sum_{l=0}^{N-1} \binom{l+N}{N} (-1)^{l-N+1} \binom{N-1}{l} T^{N-1}.$$  

Using the sum

$$\sum_{l=0}^{N-1} (-1)^l \binom{l+N}{N} \binom{p}{l} = (-1)^p \binom{N}{p},$$  (51)

this simplifies to

$$\eta [2T^N - (T - 1)^N] - NT^{N-1} = -\eta (T^2 - 1)^N \left[ \left( 1 - \frac{T}{T+1} \right)^N - \left( -\frac{T}{T+1} \right)^N \right]$$

$$+ (T^2 - 1)^N \frac{T^{N-1}}{(T^2 - 1)^{N-1}}$$

$$= -\eta (T - 1)^N (1 - (-T)^N) + N(T^2 - 1)T^{N-1},$$

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giving, after solving for $\eta$ and simplifying,

$$\eta = \frac{NT}{[2 - (1 - T)^N]}.$$  \hfill (52)

Inserting this in (50) gives the final solution for $x_l$. However, we actually need $A_{N-1,l}$, which is given by $A_{N-1,l} = \text{det}(a) x_l$. With the value for $\text{det}(a)$ calculated as (33), this becomes

$$A_{N-1,l} = (T^2 - 1) \frac{N^{N-1}}{2} [2 - (1 - T)^N] T^{-l} \left[ -\frac{NT}{(2 - (1 - T)^N)} (-1)^{l} \sum_{p=0}^{N-1} \binom{p}{l} \left( \frac{T}{T+1} \right)^p \right]$$

$$+ (-1)^{l-N+1} \binom{N-1}{l} \frac{T^{N-1}}{(T^2 - 1)^{N-1}}$$

$$= N(T^2 - 1) \frac{N^{N-1}}{2} (-1)^{-l+1} T^{-l+1} \sum_{p=0}^{N-1} \binom{p}{l} \left( \frac{T}{T+1} \right)^p$$

$$+ (-1)^{N-l+1} (T^2 - 1) \frac{N^{N-1}}{2} [2 - (1 - T)^N] \binom{N-1}{l} T^{N-l-1}.  \hfill (53)$$

We now specialize to the case of interest, namely, where $T$ takes the optimal value $T = 1 - 2^{1/N}$. Then the second term of (53) vanishes, giving

$$A_{N-1,l} = N(T^2 - 1) \frac{N^{N-1}}{2} (-1)^{-l+1} T^{-l+1} \sum_{p=0}^{N-1} \binom{p}{l} \left( \frac{T}{T+1} \right)^p.  \hfill (54)$$

We have not substituted the optimal value for $T$ in every part of the expression, because no simplification occurs at this point.

We finally have to add up $|A_{N-1,l}|$ over all $l$. We first show that in the optimal $T$, the $A_{N-1,l}$ alternate in sign. The factor that could change sign in $(-1)^{N-l-1} A_{N-1,l}$ is

$$s_l = T^{-l} \sum_{p=0}^{N-1} \binom{p}{l} \left( \frac{T}{T+1} \right)^p.$$

The generating function of this sequence is

$$S(z) = \sum_{l=0}^{N-1} s_l z^l = \frac{(T + 1)^N - (T + z)^N}{(T + 1)^{N-1} (1 - z)},$$

whence it follows that $s_l$ can be rewritten as

$$s_l = \frac{1}{(T + 1)^{N-1}} \sum_{j=l+1}^{N} \binom{N}{j} T^{N-j} = \frac{1}{(T + 1)^{N-1}} \sum_{t=0}^{N-l-1} \binom{N}{t} T^t.$$
We consider now the functions $f_{j,N}(T) := \sum_{t=0}^{j} \binom{N}{t} (N-t)^t$ and show that they are non-negative for $T \geq -1/N$, which covers the case of the optimal $T = 1 - 2^{1/N}$. First put $T = z - 1/N$, which gives after some simplifications

$$f_{j,N}(z - 1/N) = \sum_{l=0}^{j} z^l \binom{N}{l} \sum_{t=0}^{j-l} \binom{N-l}{t} (-1/N)^t.$$  

This is non-negative if the coefficients $\sum_{t=0}^{j-l} \binom{N-l}{t} (-1/N)^t$ are non-negative. This in turn would follow from the statement $f_{j-l,N-l}(-1/(N-l)) \geq 0$. Now the terms in the sum

$$f_{j,N}(T) = \sum_{t=0}^{j} \binom{N}{t} (-1/N)^t$$

constitute an alternating sequence, with the terms for even $t$ positive and the terms for odd $t$ negative. Non-negativity of the sum then follows from the fact that the terms decrease in absolute value: indeed, the absolute value of the $t$-term divided by the $t-1$ term is $(N-t+1)/Nt$, which does not exceed 1 for $t > 0$. This finally shows non-negativity of the sequence $s_j$.

As a consequence, we can just add up all $(-1)^{N-l-1} A_{N-1,l}$ and take the absolute value afterwards. This gives

$$\sum_{l=0}^{N-1} (-1)^{N-l-1} A_{N-1,l} = N(T^2 - 1) \frac{N(N-1)}{2} (-1)^N T \sum_{l=0}^{N-1} \sum_{p=0}^{N-1} \binom{p}{l} T^{-l} \left( \frac{T}{T+1} \right)^p$$

$$= N(T^2 - 1) \frac{N(N-1)}{2} (-1)^N T \sum_{p=0}^{N-1} (1 + 1/T)^p \left( \frac{T}{T+1} \right)^p$$

$$= N^2(T^2 - 1) \frac{N(N-1)}{2} (-1)^N.$$

The final result is then

$$\left( \sum_{m=1}^{N} |A_{Nm}| \right)^2 = N^4 T^2 (T^2 - 1)^{N(N-1)}. \quad (55)$$

Combining the results from equations (40) and (55) and inserting into the expression (13) for the maximal success probability, we obtain

$$p_{\text{max}} = \frac{1}{N^2}, \quad (56)$$

just as we set out to show. This means that there exists a beam splitter network with an $N$-dimensional ancilla state containing photon numbers $n_l = 0, \ldots, N-1$ such that a generalized nonlinear sign shift can be performed on an $(N+1)$-dimensional signal state with a probability of $1/N^2$.

4. Discussion and conclusions

In this paper, we have shown how the abstract view on linear-optical networks can be used to derive scaling laws for success probabilities. Thus far, we have limited ourselves to single-shot gates without conditional feed-forward dynamics which could in principle be incorporated
by concatenating several of those abstract networks. We found that the maximal probability of success of conditionally generating a (single-shot) generalized nonlinear sign-shift gate on \((N+1)\)-dimensional signal states using \(N\)-dimensional ancillas with the lowest possible photon numbers \(n_1 = 0, \ldots, N - 1\) scales as \(1/N^2\). To our knowledge, this is the first time such scaling laws have been found. It also hints towards scaling of success probabilities of certain classes of \(N\)-qubit gates. This is due to the fact that multiqubit quantum gates acting on tensor-product states with constant photon number can be decomposed into a multimode Mach–Zehnder interferometer-type setup where single-mode conditional gates are inserted into the interferometer’s paths \([7]\). Note, however, that this is not necessarily the optimal way to implement such gates. For example, a controlled-\(\sigma_z\) gate would work in only 1/16 of all cases, whereas a more general network has been found in \([16]\) that works with a slightly higher probability of success of 2/27. In order to find proper upper bounds on such networks, the existing abstract network would have to be modified by replacing the single ‘active’ beam splitter by a U(2\(M\))-network, \(M\) being the number of modes to be acted upon.

We have shown that computing the success probability within the framework of the abstract network reduces to the calculation of various Vandermonde-type determinants. To do so, we defined a class of polynomials related to the Jacobi polynomials. These \(S\)-polynomials obey a three-term recursion relation which is given in appendix A, and the elementary symmetric polynomials associated with them are derived in appendix B.

We have restricted our attention to the experimentally most accessible case in which the ancilla state contains the lowest possible photon numbers \(n_1 = 0, \ldots, N - 1\). This choice represents the only analytically solvable case thus far but, on the other hand, is motivated by the fact that low photon numbers also means low decoherence which is desirable for possible applications in quantum information processing. However, the theory presented above is, in principle, valid for any admissible choice of ancilla states. It should be added that the proof technique for deriving upper bounds by considering dual convex optimization problems \([10]\) can similarly be applied to the situation considered in this paper.

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Appendix A. Some properties of the \(S\)-polynomials

In this appendix, we briefly summarize some properties of the \(S\)-polynomials. We recall again their definition \((20)\):

\[
S_j^{(s)}(l) = P_j^{(0,l-j)}(2x^2 - 1),
\]

in terms of the Jacobi polynomials

\[
P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{m=0}^{n} \binom{n + \alpha}{m} \binom{n + \beta}{n - m} (x - 1)^{n-m} (x + 1)^m.
\]

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From this definition one can derive a three-term recursion relation as
\[ kS_k^{(x)}(n) = [(x^2 - 1)(n + k) + 2k - 1]S_{k-1}^{(x)}(n) - (k - 1)x^2 S_{k-2}^{(x)}(n). \] (A.1)

Then, according to Favard’s theorem [17], which states that if \( P = (P_n)_{n \geq 0} \) is a polynomial sequence which satisfies

(i) \( P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x); \quad P_0(x) = 1, \quad P_{-1}(x) := 0 \) and

(ii) \( A_n A_n C_n > 0, \)

then there exists a positive measure \( \mu \) on the real line such that \( P \) is an orthogonal sequence with orthogonality measure \( \mu \), the polynomials \( S_k^{(x)}(n) \) do form an orthogonal sequence for all \( x \neq 0, \pm 1 \). This can be seen by noting that \( k^2 A_n A_n C_n = (x^2 - 1)x^2 \) in our case. As for the excluded values of the parameters \( x \), we have that \( S_k^{(\pm 1)}(n) = 1 \) and \( S_k^{(0)}(n) = (p)^0 \) both of which certainly do not form orthogonal sequences. Orthogonality is important if one needs to find an inversion relation between the powers of \( n \) and the polynomials \( S_k^{(x)}(n) \).

Appendix B. Symmetric polynomials over the S-polynomial basis

We now present a number of technical results about the expansion of binomial coefficients and related polynomials in terms of the \( S \)-polynomials.

For any positive integer \( N \), for any \( l \), and for an integer \( p \) satisfying \( p \leq N \),

\[ (x^2 - 1)^N \binom{l}{p} = \sum_{j=0}^{N} (-1)^{N-j} s_{N-j}(x; p; N) S_j^{(x)}(l), \] (B.1)

with coefficients

\[ s_{N-j}(x; p; N) = \binom{p}{j} x^{2(p-j)} (1 - x^2)^{N-p}, \] (B.2)

which are polynomials of degree \( N - j \) in \( x^2 \). Specifically, for \( p = N \) these formulas simplify to

\[ (x^2 - 1)^N \binom{l}{N} = \sum_{j=0}^{N} (-1)^{N-j} \binom{N}{j} x^{2(N-j)} S_j^{(x)}(l). \] (B.3)

To prove (B.1) and (B.2), we simplify the right-hand side of (B.1) with (B.2) and (20) plugged in:

\[ \sum_{j=0}^{N} (-1)^{N-j} \binom{p}{j} x^{2(p-j)} (1 - x^2)^{N-p} \sum_{r=0}^{j} \binom{j}{r} (l+r)(x^2 - 1)^r. \]

Since the binomial coefficients take on the value 0 for \( j < 0, j > p, r < 0 \) and \( r > j \), and since we have the condition \( p \leq N \), we can drop the bounds on the summation signs and subsequently
move them up front, giving:

\[ \sum_{j,r} (-1)^{p-j} \binom{p}{j} \binom{j}{r} \binom{l+r}{r} (x^2)^{p-j} (x^2 - 1)^{N-p+r}. \]

Now we notice \( \binom{j}{r} \binom{j}{r} = \binom{p}{r} \binom{p-r}{r} \), so that we can calculate the sum over \( j \) easily, using

\[ \sum_j \left( \binom{p-r}{j-r} (-x^2)^{p-j} = (1 - x^2)^{p-r}. \]

It remains to calculate

\[ \sum_r \binom{p}{r} \binom{l+r}{r} (1 - x^2)^{p-r} (x^2 - 1)^{N-p+r}, \]

which is nothing but

\[ (x^2 - 1)^N \sum_r \binom{p}{r} \binom{l+r}{r} (-1)^{p-r}. \]

It now remains to show that the sum in this expression reduces to \( \binom{1}{l} \). To do that, we note that \( \binom{1}{l} \) is a polynomial of degree \( p \) in \( l \), with zeros \( 0, 1, \ldots, p-1 \) and leading term \( l^p / p! \). We will confirm that the same holds for the sum \( \sum_r \binom{p}{r} \binom{l+r}{r} (-1)^{p-r} \). Consider first the specific values \( l = 0, 1, \ldots, p-1 \). The factor \( \binom{l+r}{r} \) is a polynomial of degree \( l \) in \( r \). It can, therefore, be written as a linear combination of the degree \( j \) polynomials \( \binom{j}{r} \). Thus, with \( c_j \) independent of \( r \),

\[ \sum_{r=0}^{p} \binom{p}{r} \binom{l+r}{r} (-1)^{p-r} = \sum_{j=0}^{l} c_j(l) \sum_{r=0}^{p} \binom{p}{r} \binom{r}{j} (-1)^{p-r} \]

\[ = \sum_{j=0}^{l} c_j(l) \binom{p}{j} \sum_{r=0}^{p} \binom{p-j}{r-j} (-1)^{p-r}. \]

The sum over \( r \) in the latter expression is clearly 0 for \( j < p \), hence the whole expression is indeed zero for \( l = 0, 1, \ldots, p-1 \). Now, concerning the sum’s leading term in \( l \), we note that it can only come from the term with summation index \( r = p \), \( \binom{l+p}{p} \), whose leading term in \( l \) is \( l^p / p! \), as required. This completes the proof of (B.2).

The binomial coefficient \( \binom{l}{p} \) is a polynomial in \( l \):

\[ \binom{l}{p} = \frac{1}{p!} l(l-1) \cdots (l-p+1) = \frac{1}{p!} \prod_{k=0}^{p-1} (l-k). \]

We now look at related polynomials where one or two of the factors \( (l-k) \) are missing.

Consider first the polynomial

\[ \frac{\binom{l}{j}}{l-q} = \frac{1}{p!} \prod_{k=q}^{p-1} (l-k), \]
where \( q \) is an integer in the range \( 0 \leq q < p \). This polynomial can be expanded in terms of the binomial coefficients \( \binom{l}{r} \), with \( r < p \):

\[
\frac{\binom{l}{q}}{l - q} = \frac{1}{p^{\binom{p-1}{q}}} \sum_{r=0}^{p-1} (-1)^{p-1-r} \binom{r}{q} \binom{l}{r}.
\]

The proof of this expansion goes as follows. Consider the numerator:

\[
\sum_{r=0}^{p-1} (-1)^{p-1-r} \binom{r}{q} \binom{l}{r}.
\]

This can again be rewritten using

\[
\sum_{r=0}^{p-1} (-1)^{p-1-r} (l - q) \binom{r}{q} = \sum_{s=0}^{p-q-1} (-1)^{p-q-1-s} \binom{l - q}{s} = \binom{l - q - 1}{p - q - 1}.
\]

Wrapping up everything yields the expression

\[
\frac{\binom{l}{q}}{p^{\binom{p-1}{q}}},
\]

which is very easily seen to be identical to \( \binom{l}{q}/(l - q) \).

Using this expansion and (B.3), one can easily show the following. For any positive integer \( N \), for any \( l \), and for an integer \( q \) satisfying \( 0 \leq q < N \),

\[
(x^2 - 1)^N \frac{\binom{l}{N+1}}{l - q} = \sum_{j=0}^{N} (-1)^{N-j} s^{(q)}_{N-j}(x; N) S^I_j(l), \quad (B.4)
\]

with coefficients

\[
s^{(q)}_{N-j}(x; N) = \frac{1}{(N + 1) \binom{N}{q}} \sum_{r=0}^{N} (-1)^{N-r} \binom{r}{q} s_{N-j}(x; r; N) = \frac{1}{(N + 1) \binom{N}{q}} \sum_{r=0}^{N} (-1)^{N-r} \binom{r}{q} \binom{r}{j} x^{2(r-j)} (1 - x^2)^{N-r}. \quad (B.5)
\]

**Appendix C. Simple Vandermondians with gaps in their arguments**

We have seen that we need to calculate simple Vandermondians of degree \( N - 1 \) with integer arguments \([0 \ldots N - 1] \setminus \{k\}\) (see equation (38)). Because of equation (24), it is enough to concentrate on Vandermondians over the power basis. Firstly, note that the Vandermonian
$V_N(x_1, \ldots, x_N)$ can be decomposed as

$$V_N(x_1, \ldots, x_N) = \prod_{i=1}^{N-1} (x_N - x_i) V_{N-1}(x_1, \ldots, x_{N-1}).$$

Hence, we can write

$$V_N^{(s-1)} = \frac{V_N}{\prod_{m \neq s} |s - m|}. \quad (C.1)$$

The product in the denominator can be computed as

$$\prod_{m=1, m \neq s}^{N} |s - m| = \prod_{m=1}^{s-1} (s - m) \prod_{m=s+1}^{N} (m - s)$$

$$= \prod_{q=1}^{s-1} q \prod_{p=1}^{N-s} p = (s - 1)! (N - s)!. \quad (C.2)$$

Noting that $V_N = \prod_{n=1}^{N} (n - 1)! = V_{N-1}(N - 1)!$, we obtain

$$V_N^{(s-1)} = V_{N-1} \left( \frac{N - 1}{s - 1} \right). \quad (C.3)$$

With $k = s - 1$ and the additional factor $|T|^2 - 1)^{(N-1)(N-2)/2}$, this is just equation (38).

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