The heavy-quark pole masses in the Hamiltonian approach

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Abstract

From the fact that the nonperturbative self-energy contribution $C_{SE}$ to the heavy meson mass is small: $C_{SE}(b\bar{b}) = 0$; $C_{SE}(c\bar{c}) \equiv -40$ MeV, strong restrictions on the pole masses $m_b$ and $m_c$ are obtained. The analysis of the $b\bar{b}$ and the $c\bar{c}$ spectra with the use of relativistic (string) Hamiltonian gives $m_b(2\text{-loop}) = 4.78 \pm 0.05$ GeV and $m_c(2\text{-loop}) = 1.39 \pm 0.06$ GeV which correspond to the $\overline{\text{MS}}$ running mass $\overline{m_b}(\overline{m_c}) = 4.19 \pm 0.04$ GeV and $\overline{m_c}(\overline{m_c}) = 1.10 \pm 0.05$ GeV. The masses $\omega_c$ and $\omega_b$, which define the heavy quarkonia spin structure, are shown to be by $\sim 200$ MeV larger than the pole ones.

1 Introduction

The spectrum of heavy quarkonia (HQ) is very rich and provides a unique opportunity to study the static interaction in the infrared (IR) region, and hyperfine and fine structure effects. To use that opportunity one needs to know, besides such fundamental parameters as the string tension and the strong coupling, also the heavy-quark mass, which cannot directly be measured since a quark is not observed as a physical particle. Therefore the quark mass $m_Q$ has to be determined indirectly, e.g. from the study of hadronic properties like $e^+e^- \rightarrow b\bar{b}$, hadronic decays, and the $Q\bar{Q}$ spectra.

In the QCD Lagrangian the mass parameter depends on the renormalization scheme and by convention this current mass is taken in the $\overline{\text{MS}}$ scheme. In perturbation theory it is convenient to introduce the pole quark mass, i.e. the pole of the quark propagator, and at present the $\overline{\text{MS}}$ pole mass is known to three-loops \cite{2,3}:

$$m_Q = \overline{m_Q}(\overline{m_Q}) \left\{ 1 + \frac{4}{3} \frac{\alpha_s(\overline{m_Q})}{\pi} + \xi_2 \left( \frac{\alpha_s}{\pi} \right)^2 + \xi_3 \left( \frac{\alpha_s}{\pi} \right)^3 \right\},$$

(1)
where the Lagrangian current masses,

\[ \bar{m}_b(\bar{m}_b) = (4.25 \pm 0.25) \text{ GeV}, \quad \bar{m}_c(\bar{m}_c) = (1.20 \pm 0.20) \text{ GeV}, \quad (2) \]

are known now with an accuracy of 17% (6%) respectively for the $c$ quark ($b$ quark). Most calculations of the pole masses $m_b$ and $m_c$ have been done in the QCD sum rules approach [4], lattice QCD [5], and different perturbative approaches [2, 3].

For three decades many properties of HQ like the spectra, electromagnetic transitions, hadronic and semileptonic decays, were successfully studied in different potential models (PM) [6]-[13], however, the heavy quark masses used in PM are considered “to make sense in the limited context of a particular quark model” [2] i.e. as fitting parameters.

However, in the last decade the situation has changed and in Ref. [11] the relativistic (string) Hamiltonian was derived directly from the QCD Lagrangian, starting with the gauge-invariant meson Green’s function in Fock-Feynman-Schwinger (FFS) representation. In Refs. [11]-[13] it was established that for the orbital angular momentum $L \leq 5$ and not too large string corrections, as in HQ, the string Hamiltonian reduces to the well-known Hamiltonian $H_0$ used in the relativized potential model (RPM) for many years [7, 8]:

\[ H_0 = 2\sqrt{\vec{p}^2 + m_q^2} + V_{\text{static}}(r) \quad (3) \]

It follows from the derivation of $H_0$ in Ref. [11] that the mass $m_q$ in (3) coincides with the MS running mass $\bar{m}_q(\bar{m}_q)$, if the perturbative interaction is neglected, or with the pole mass $m_Q$ [4] if the perturbative self-energy corrections are taken into account. Therefore this Hamiltonian can be used to extract the pole mass $m_Q$ from the analysis of the HQ spectrum.

Nevertheless, if one looks at the heavy-quark masses used in PM, a large variety of $m_b$ and $m_c$ values can be found in different analyses: $m_c$ in the range 1.30 GeV $\div$ 1.84 GeV and $m_b$ in the range 4.20 $\div$ 5.17 GeV [7]-[10]. The main reason behind this wide spread in the values of $m_b$ and $m_c$ (even for the same Hamiltonian $H_0$) is the presence of a negative arbitrary constant $C_0$ in the mass formula (or in the chosen static potential). We give three examples: in Ref. [7] $m_c = 1.327$ GeV and $C_0 = 0$ is used by the Wisconsin Group; in Ref. [8] $m_c = 1.628$ GeV and $C_0 = -253$ MeV (in both cases the Hamiltonian Eq. [8] was used); in Ref. [6] $m_c = 1.84$ GeV and $C_0 \cong -800$ MeV are taken, i.e. the magnitude of $C_0$ is always larger for larger heavy-quark mass.

The meaning of the constant $C_0$ was understood recently and in Ref. [1] it was shown that the negative contribution to the meson mass comes from the nonperturbative (NP) interaction of the color-magnetic moment of a quark (antiquark) with the background (vacuum) field. Moreover, this self-energy NP term $C_{\text{SE}}$ was analytically calculated with 10% accuracy [1] (see Eq. (54)):

\[ C_{\text{SE}}(nL) = \frac{4\sigma}{\pi\omega_q(nL)} \eta(m_q) \quad (4) \]
for a quark and an antiquark with equal masses. In the expression Eq. (4) $m_q$ is the pole mass which determines the factor $\eta(m_q)$ Eq. (57) while

$$\omega_q(nL) = \sqrt{p^2 + m_q^2} \langle nL \rangle$$

is the dynamical quark mass. For low-lying states in charmonium and bottomonium $\omega_Q$ turns out to be $\sim 200$ MeV larger $m_Q$.

The essential fact (for light and heavy-light mesons) is that $C_{SE}(nL)$ depends on the quantum numbers and just due to this the correct intercept of the Regge-trajectory has been obtained in Ref. [13]. In HQ the situation appears to be much more simple. The factor $\eta(m_q)$ in Eq. (4) depends on the flavor through the pole mass $m_q$ and from the analytical expression (57) one obtains a small value: $\eta_c \approx 0.35 \div 0.27$ for $m_c$ in the range $1.37 \div 1.70$ GeV and $\eta_b \approx 0.07$ for $m_b \approx 4.7 \div 5.0$ GeV. As a result $C_{SE}(b\bar{b}) \approx -3$ MeV, i.e. is compatible with zero, and $C_{SE}(c\bar{c})$ is also small:

$$C_{SE}(b\bar{b}) = 0; \quad C_{SE}(c\bar{c}) = (-40 \pm 10) \text{ MeV}$$

Thus the self-energy contributions to HQ states are well defined and therefore there is no opportunity anymore to vary the pole mass by introducing a fitting constant. We shall show in this paper that the condition (6) puts strong restrictions on the values of $m_b(m_c)$ needed to describe the $b\bar{b}(c\bar{c})$ spectrum. The extracted pole masses $m_b$ and $m_c$ in our analysis will be determined with an accuracy better than 60 MeV and the main uncertainty in their values comes not from the method used (for fixed string tension and the strong coupling ($\Lambda_{QCD}$) the uncertainty is $\pm 10$ MeV) but from the uncertainty in our knowledge of the strong coupling in the IR region. We shall show that HQ spectra, in particular high excitations and the recently discovered 1D state in bottomonium, can give very important information about the strong coupling in the IR region.

Our analysis of HQ spectra shows that in bottomonium $m_b(2\text{-loop}) < 4.70$ GeV and values $m_b > 4.84$ GeV turn out to be incompatible with the condition $C_{SE} = 0$. In charmonium the admissible $m_c$ values, $m_c = 1.39 \pm 0.06$ GeV, appear to be rather small and agree with the one obtained by Narison with the use of the QCD sum rules for the (pseudo)-scalar current [4]. Our calculations of the HQ spectra are done with the use of only three fundamental quantities: the string tension, the QCD constant $\Lambda(n_f)$ and the pole mass $m_Q$. The main emphasis in our fit lies on the excited (not ground) states.

This paper is organized as follows. In Section 2 the mass formula, following from the relativistic Hamiltonian, as well as the approximations to that, are presented and the notion of dynamical mass is introduced. In Section 3 the static potential and the strong coupling in the IR region which is defined in background perturbation theory (BPT), are discussed. In Section 4 from the analysis of the $b\bar{b}$ spectrum (with special accent on high excitations) the pole mass $m_b(2\text{-loop})$ is obtained. In Section 5 the pole mass $m_c$ is extracted from the $c\bar{c}$ spectrum. In Section 6 our Conclusions are presented and in the Appendix the method and NP self-energy term are discussed.
2 The mass formula

The string corrections are small in HQ and therefore the simplified form of the Hamiltonian $H_R$ may be used (see Appendix):

$$H_R = \frac{\vec{p}^2}{\omega} + \omega + \frac{m_q^2}{\omega} + V_{\text{static}}.$$  

(7)

To derive this Hamiltonian in the FFS representation one needs to go over from the proper time $\tau$ in the meson Green’s function to the actual time $t$ and at this point a new variable $\omega(t)$ must be introduced:

$$\omega(t) = \frac{1}{2} \frac{dt}{d\tau}.$$  

(8)

This variable is the canonical coordinate and since $H_R$ does not depend on its derivative, the requirement that the canonical momentum $\pi_\omega = 0$ is preserved in time, corresponds to the extremum condition

$$\dot{\pi}_\omega = \{\pi, H_R\} = \frac{\partial H}{\partial \omega} = 0.$$  

(9)

From this extremum condition the operator $\omega$,

$$\omega = \sqrt{\vec{p}^2 + m_q^2},$$  

(10)

is defined as the kinetic energy operator. Substituting the definition into the $H_R$ one arrives at the Hamiltonian $H_0$ Eq. (3):

$$H_0 = 2\sqrt{\vec{p}^2 + m_q^2} + V_{\text{static}}(r),$$  

(11)

which does explicitly not depend on the variable $\omega$. However, to calculate different corrections to the meson mass, like spin and string corrections, or the self-energy corrections which can be considered as a perturbation and by derivation depend on the $\omega$, we shall use the approximation when for a given state the operator $\omega$ will be substituted by its average:

$$\omega(nL) = \langle \sqrt{\vec{p}^2 + m_q^2} \rangle_{nL}.$$  

(12)

This mass $\omega(nL)$ can be called the dynamical mass since its difference with respect to the current (pole) mass $m_q$ is fully determined by the dynamics. Note that for vanishing pole mass the value of $\omega(nL)$ is finite and determines the constituent mass of a light quark.

It is also important that perturbative corrections to the current mass, which are essential at small quark-antiquark separations, $r \lesssim 0.1$ fm, are included in the pole mass $m_Q$ occurring in $H_0$. On the other hand the static potential $V_{\text{static}}(r)$ is well defined at $Q\bar{Q}$ separations $r \gtrsim T_g \approx 0.2$ fm where $T_g$ is the gluonic correlation length. The eigenvalues of $H_0$, denoted as $M_0(nL)$,

$$\left\{2\sqrt{\vec{p}^2 + m_Q^2} + V_{\text{static}}(r)\right\} \psi_{nL}(r) = M_0(nL)\psi_{nL}(r)$$  

(13)
together with the self-energy term Eq. (6) define the heavy-meson masses. As shown in the Appendix, in bottomonium $C_{SE} = 0$ and therefore the spin-averaged mass $M(nL)$ for a given $b\bar{b}$ state coincides with the eigenvalue $M_0(nL)$:

$$M(nL, b\bar{b}) = M_0(nL),$$  \hspace{1cm} (14)

while in charmonium from Eq. (57) $C_{SE} \approx -40$ MeV and

$$M(nL, c\bar{c}) = M_0(nL) + C_{SE}.$$  \hspace{1cm} (15)

There exist two approximations to the solution of the spinless Salpeter equation (SSE) (13) leading to two approximations to the meson mass $M(nL)$. First, the nonrelativistic (NR) approximation when the mass $M_{NR}(nL)$ is given by:

$$M_{NR}(nL) = 2m_Q + E_{NR}(nL) + C_{SE},$$  \hspace{1cm} (16)

where $E_{nL}^R(m_Q)$ is the eigenvalue of the Schrödinger equation with the reduced mass equal to $\frac{1}{2}m_Q$.

There is also another, so called “einbein” approximation (EA) to the solutions of SSE (13), where the meson mass given by

$$M_{EA}(nL) = \omega(nL) + \frac{m_Q^2}{\omega_{nL}} + E_{nL}(\omega_Q) + C_{SE}$$  \hspace{1cm} (17)

appears to be closer to the exact solution $M(nL)$ than in the NR approximation (13). In EA the binding energy $E_{nL}(\omega_Q)$ is defined as the solution of the Schrödinger equation with the reduced mass equal to $\frac{1}{2}\omega_Q(nL)$ (not the pole mass $\frac{1}{2}m_Q$) while $\omega_Q(nL)$ is to be defined from the selfconsistent equation:

$$\frac{\partial M_{EA}}{\partial \omega} = 0, \text{ or } \omega_{nL} = \frac{m_Q^2}{\omega_{nL}} - \omega_{nL} \frac{\partial E_{nL}(\omega)}{\partial \omega_{nL}}.$$  \hspace{1cm} (18)

Through $\omega_{nL}$ in EA the relativistic corrections are taken into account in the mass formula (17). Moreover, owing to the special form of the mass formula (17) the choice of $\omega(b\bar{b}) \approx 5.0$ GeV turns out to be compatible with the condition $C_{SE} = 0$, while in NR approximation the admissible values of $m_b$ are about 200 MeV smaller.

It is worthwhile to notice that in bottomonium where both $\omega_Q(nL)$ and $m_Q$ are large, around 5 GeV, the difference between NR, EA, and relativistic cases is small, $|\delta_R| = M(nL) - M^{NR}(nL)$ is about $10 \div 20$ MeV. In charmonium the difference depends on the quantum numbers and for high excitations can reach $\sim 100$ MeV (see the discussion in Section 5).

### 3 Static potential

The static potential contains perturbative and NP contributions where the NP linear potential can directly be derived from the meson Green’s function if the
qq separation is larger than the gluonic correlation length. From the analysis of the Regge trajectories of light and heavy-light mesons the value of the string tension, $\sigma = 0.185 \pm 0.005$ GeV$^2$ [13], is fixed while the perturbative interaction in coordinate space can be presentread in the form,

$$V_P(r) = -\frac{4}{3} \frac{\alpha_{\text{static}}(r)}{r},$$

where $\alpha_{\text{static}}(r)$ is well known only in the perturbative region, i.e. at very small distances, $r \lesssim 0.1$ fm [15, 16].

However, the r.m.s. radii in bottomonium and charmonium, span a very wide range:

$$R(\Upsilon(1S)) = 0.2 \text{ fm}, \quad R(\chi_b(1P)) = 0.4 \text{ fm}, \quad R(\chi_b(2P)) = 0.65 \text{ fm},$$

$$R(\Upsilon(4S)) = 0.9 \text{ fm}, \quad R(\Upsilon(6S)) \cong 1.3 \text{ fm},$$

and

$$R(J/\psi) \cong 0.4 \text{ fm}, \quad R(\chi_c(1P)) = 0.6 \text{ fm}, \quad R(\psi(1D)) = 0.8 \text{ fm}$$

$$R(\psi(3S)) = 1.1 \text{ fm}, \quad R(\psi(4S)) \cong 1.4 \text{ fm}.$$  \hspace{1cm} (21)

Apparently, with the exception of $\Upsilon(1S)$ the sizes of these states lie outside the perturbative region.

Therefore the problem arises how to define the strong coupling $\alpha_{\text{static}}(r)$ at all distances, in particular in the IR region. In PM it has always been assumed and later this fact has been supported by direct measurement of the static potential in lattice QCD, that the strong coupling freezes and reaches a critical (saturated) value at large $r$. Unfortunately, at present there is no consensus about the true value of $\alpha_{\text{crit}}$ and different values were used. In the phenomenological analysis of Ref. [8]) $\alpha_{\text{crit}} = 0.60$ was used, but in analytical perturbation theory [17] the large value $\alpha_{\text{crit}} = 4\pi/\beta_0 \cong 1.4$ appeared. In the background perturbation theory (BPT) which will be used here, $\alpha_{\text{crit}}$ is smaller and fully defined by $\Lambda_{QCD}$ [16, 18]. For the definition of $\alpha_{\text{static}}(r)$ it is better to start with the vector coupling in momentum space:

$$V_B(q) = -4\pi C_F \frac{\alpha_B(q)}{q^2}$$

This background coupling $\alpha_B(q)$ is defined in Euclidean momentum space at all $q^2$, including $q^2 = 0$, i.e. it has no Landau singularity,

$$\alpha_B(q, 2\text{-loop}) = \frac{4\pi}{\beta_0 t_B} \left\{ 1 - \frac{\beta_1}{\beta_0} \ln t_B \right\}.$$  \hspace{1cm} (23)

The logarithm

$$t_B = \ln \frac{q^2 + M_B^2}{\Lambda_B^2}$$

\hspace{1cm} (24)
contains the background mass $M_B$ which appears due to the interaction of a gluon with the background field at small $q^2$. This mass $M_B \approx 1$ GeV has the meaning of the lowest hybrid excitation: $M_B = M(Q\bar{Q}q) - M(Q\bar{Q})$ \[1\] and from the comparison with the static potential $M_B$ determined on the lattice, was found to be equal to 1 GeV \[16\]. The $t_B$ \[24\] coincides in form with the parametrization of $\alpha_s(q)$ suggested in Refs.\[20\] where instead of the background mass $M_B$ two gluonic masses ($2m_g$) enter. However, the physical gluon cannot have a mass while $M_B = M(Q\bar{Q}gg) - M(Q\bar{Q}g)$ is a well defined physical quantity ($M_Q$ is supposed to be large) and can be calculated in different theoretical approaches \[21\] and on the lattice.

By definition $\alpha_B(q)$ has the correct asymptotic freedom (AF) behavior at large $q^2$ and in this region the connection between the vector coupling $\alpha_B(q)$ and $\alpha_s(q)$ in the $\overline{\text{MS}}$ renormalization scheme is very simple, so that the QCD constant $\Lambda_V$ (in the vector-scheme) can be expressed through $\Lambda_{\overline{\text{MS}}}$ \[22\]:

$$\Lambda_B^{(n_f)} = \Lambda_{\overline{\text{MS}}}^{(n_f)} \exp \left( \frac{a_1}{2\beta_0} \right)$$ \[25\]

Here $a_1 = \frac{31}{3} - \frac{10}{3}n_f$. At present the value $\Lambda_{\overline{\text{MS}}}^{(5)}$ (2-loop) = 215 ± 15 MeV is established from high energy processes \[2\], while in quenched QCD the value $\Lambda_{\overline{\text{MS}}}^{(0)} = 240 \pm 20$ MeV was calculated on the lattice \[23\]. Then from the relation \[25\] it follows that in quenched approximation the QCD constant in the vector scheme \[25\] has the value

$$\Lambda_B^{(0)} = 385 \pm 30 \text{ MeV}$$ \[26\]

and for $n_f = 5$

$$\Lambda_B^{(5)} = 290 \pm 30 \text{ MeV}. \[27\]$$

Our choice of $\Lambda_B$ in this paper will be in accord with the numbers \[26\] and \[27\].

In coordinate space the background coupling $\alpha_B(r)$ is defined as the Fourier transform of $\alpha_B(q)$:

$$\alpha_B(r) = \frac{2}{\pi} \int_0^\infty dq \frac{\sin qr}{q} \alpha_B(q),$$ \[28\]

so that the perturbative part of the static potential is

$$V_B(r) = -\frac{4}{3} \frac{\alpha_B(r)}{r}$$ \[29\]

and the static potential is the sum of $V_B(r)$ and the NP linear potential

$$V_{\text{static}}(r) = \sigma r - \frac{4}{3} \frac{\alpha_B(r)}{r}. \[30\]$$

In phenomenology the approximation where the coupling $\alpha_B(r)$ is constant is often used. This approximation is valid, because at distances $r \gtrsim 0.4$ fm the
Table 1: The effective coupling $\alpha_{\text{eff}}(nL)$ for different $b\bar{b}$ and $c\bar{c}$ states for the static potential with $m_b = 4.78$ GeV, $m_c = 1.45$ GeV, $\alpha_{\text{crit}} = 0.547, \sigma = 0.185$ GeV$^2$, $\Lambda^{(4)} = 360$ GeV ($n_f = 4$).

| State | 1S  | 2S  | 3S  | 4S  | 5S  | 6S  |
|-------|-----|-----|-----|-----|-----|-----|
| $\alpha_{\text{eff}}(nL, b\bar{b})$ | 0.386 | 0.419 | 0.427 | 0.430 | 0.431 | 0.432 |
| $\alpha_{\text{eff}}(nL, c\bar{c})$ | 0.441 | 0.447 | 0.446 | 0.445 | –   | –   |

Note that the critical values of $\alpha_B(q)$ and $\alpha_B(r)$ coincide,

$$\alpha_B(q = 0) = \alpha_B(r \to \infty) = \alpha_{\text{crit}},$$

and their characteristic values in two-loop approximation are given below, \(\Lambda^{(n_f)} \equiv \Lambda^{(n_f)}_B\)

- $\alpha^{(0)}_{\text{crit}}(\Lambda^{(0)}) = 385$ MeV $= 0.428$
- $\alpha^{(3)}_{\text{crit}}(\Lambda^{(3)}) = 370$ MeV $= 0.510$
- $\alpha^{(4)}_{\text{crit}}(\Lambda^{(4)}) = 340$ Mev $= 0.515$.

Our calculations show that the bottomonium spectrum appears to be rather sensitive to the AF behavior while in charmonium the approximation $\alpha_B = \text{const}$ can be used with good accuracy. It is also instructive to look at the effective coupling $\alpha_{\text{eff}}$ for different $c\bar{c}$ and $b\bar{b}$ states, which can be defined as

$$\left\langle \frac{\alpha_B(r)}{r} \right\rangle_{nL} = \alpha_{\text{eff}}(nL) \left\langle r^{-1} \right\rangle_{nL}$$

and is dependent on the quantum numbers (see Table I) and about 20% is smaller than $\alpha_{\text{crit}}$.

From Table I one can see that in bottomonium due to the different character of the wave functions the effective coupling is smaller for the $nS$ states and rather large for orbital excitations like the $nD$ states.

## 4 Bottomonium

To extract the pole mass $m_b$ the $b\bar{b}$ spectrum will be studied here as a whole. We mostly ignore the ground state – the $\Upsilon(1S)$ mass, for which high perturba-


Table 2: The spin-averaged masses $M(nL)$ (in GeV) in bottomonium for the potential (30) with the parameters (35) (NR case).

| State | Set I  | Set II | Set III | Set 3 | experiment |
|-------|--------|--------|---------|-------|------------|
| 1S    | 9.460  | 9.406  | 9.379   | 9.460 | 9.460 ± 0.0003 |
| 2S    | 10.013 | 10.001 | 9.988   | 10.233| 10.233 ± 0.0003 |
| 3S    | 10.367 | 10.359 | 10.359  | 10.355| 10.355 ± 0.0005 |
| 1P    | 9.900  | 9.900  | 9.900   | 9.901 | 9.901 ± 0.0006 |
| 2P    | 10.267 | 10.267 | 10.270  | 10.260| 10.260 ± 0.0006 |
| 1D    | 10.150 | 10.156 | 10.161  | 10.161| 10.161 ± 0.0016 |

Above $B\bar{B}$ threshold, $M_{\text{th}} = 10.558$ GeV

| State | Set I  | Set II | Set III | Set 3 | experiment |
|-------|--------|--------|---------|-------|------------|
| 4S    | 10.659 | 10.647 | 10.649  | 10.5800| 10.5800 ± 0.0035 |
| 5S    | 10.917 | 10.900 | 10.902  | 10.865 | 10.865 ± 0.008 |
| 6S    | 11.146 | 11.131 | 11.132  | 10.019 | 10.019 ± 0.008 |

\(a\) The mass $M(1S)$ increases by an amount of $\sim 50 \div 80$ MeV if the AF correction is taken into account.

Corrections can be important \[3\], but rather concentrate on the following experimental splittings \[2, 14\]

\[
\begin{align*}
M_{\text{cog}}(1D) - M(1P) &\cong M(1^3D_2) - M(1P) = 261.8 \pm 1.8 \text{ MeV (exp)} \\
M(2P) - M(1P) &\cong 360.0 \pm 1.2 \text{ MeV (exp)}. \quad (34)
\end{align*}
\]

It is important that all $1P$, $2P$, and $1D$ states lie below the $B\bar{B}$ threshold and have no hadronic shifts. Also for these splittings the small relativistic corrections are partly cancelled and the calculations can be done either with the use of SSE or in NR approximation.

First, we consider the case with $\alpha_{\text{static}} = \text{const}$ and give the $b\bar{b}$ spectrum in Table 2 for three sets of parameters with different $m_b$:

- Set I: $m_b = 4.727$ GeV, $\sigma = 0.20$ GeV$^2$, $\alpha_{\text{static}} = 0.3345$,
- Set II: $m_b = 4.765$ GeV, $\sigma = 0.19$ GeV$^2$, $\alpha_{\text{static}} = 0.390$,
- Set III: $m_b = 4.778$ GeV, $\sigma = 0.188$ GeV$^2$, $\alpha_{\text{static}} = 0.415$. \quad (35)

In all cases $C_{\text{SE}} = 0$ is taken in the mass formula \[14\].

From the masses presented in Table 2 one can see that

1. For small $m_b = 4.727$ GeV (Set I) the mass $M(1D)$ appears to be about 10 MeV lower than the experimental value even for very large $\sigma = 0.20$ GeV$^2$.

2. For Set II and Set III almost identical fits are obtained with exception of the 1D state where good agreement with experiment can be reached only for a larger value of the coupling, as for Set III.
Table 3: The dynamical mass \( \omega_b(nL) \) (in GeV) and the difference between the dynamical mass and the current quark mass (in MeV) for SSE with Cornell potential \( (m_b = 4.78 \text{ GeV}, \sigma = 0.185 \text{ GeV}^2, \alpha_{\text{static}} = 0.4125) \).

| State | 1S | 2S | 3S | 4S | 5S | 6S | 1P | 2P | 1D |
|-------|----|----|----|----|----|----|----|----|----|
| \( \omega(nL) \) | 5.043 | 5.008 | 5.028 | 5.057 | 5.088 | 5.127 | 4.959 | 4.989 | 4.962 |
| \( \omega(nL) - m_b \) | 263 | 228 | 248 | 277 | 308 | 347 | 179 | 209 | 182 |

3. The spin-averaged \( 1D - 1P \) splitting,
\[
\Delta = M_{\text{cog}}(1D) - M_{\text{cog}}(1P),
\]

has remarkable properties – it is practically independent of the relativistic correction \( \delta_R \) and small variations of the string tension, and therefore \( \Delta \) can be considered the best and very stable criterium to determine the critical value of the strong coupling as well as the pole mass \( m_b \).

4. In NR approximation Eq. (16) and for SSE for the \( b \)-quark masses \( m_b \leq 4.70 \text{ GeV} \) or \( m_b \geq 4.8 \text{ GeV} \) the condition \( C_{\text{SE}} = 0 \) cannot be combined with a reasonably good fit to the \( b\bar{b} \) spectrum.

However, if one uses EA Eq. (17) instead of the NR mass formula Eq. (16) the values of the dynamical mass \( \omega_b(nL) \) are larger and the difference \( \omega_b(nL) - m_b \) varies in the range \( (180 \div 300 \text{ MeV}) \) (see Table 3), from which one can see that the dynamical mass \( \omega_b(nL) \) is slightly different for different \( nL \) states. The \( b\bar{b} \) spectrum calculated with \( \omega_b(nL) = 5.0 \text{ GeV} \) and \( m_b = 4.78 \text{ GeV} \) with the use of the mass formula (17) gives values coinciding within \( \pm 5 \text{ MeV} \) with those from Table 2.

Thus from our fits, when the coupling is taken constant, the extracted value of the pole mass
\[
m_b = 4.76 \pm 0.02 \text{ GeV} \quad (\alpha_{\text{static}} = \text{const})
\]
is obtained.

The picture does not change much if the AF behavior of \( \alpha_3(r) \) in two-loop approximation is taken into account.

However, in this case the admissible values of \( m_b \) appear to be larger by \( \sim 50 \text{ MeV} \). The \( b\bar{b} \) spectrum for \( m_b \cong 4.82 \text{ GeV} \) and \( m_b = 4.83 \text{ GeV} \) for the number of the flavors \( n_f = 4, 5 \) and also in quenched approximation is presented in Table 4.

As seen from Table 4 a good agreement for Sets A and B is obtained for the \( 1P \) and \( 1D \) states but for the \( 1S \) level the mass is \( \sim 10 \text{ MeV} \) higher than the \( M(\Upsilon(1S)) \) value. This fact can be connected with the contribution of the 3-loop perturbative correction which is neglected here.
Table 4: The $b\bar{b}$ spectrum defined by the mass formula (16) for the spinless Salpeter equation.

| State | Set A | Set B | Set C | experiment$^a$ |
|-------|-------|-------|-------|---------------|
|       | $m_b = 4.816$ | $m_b = 4.83$ GeV | $m_b = 4.817$ | $\Lambda^{(5)} = 360$ MeV | $\Lambda^{(4)} =$390 MeV | $\Lambda^{(0)} (1$-loop)$=365$ MeV | |
| 1S    | 9.471 | 9.478 | 9.470 | 9.460 |
| 2S    | 10.025 | 10.032 | 10.023 | 10.023 |
| 3S    | 10.376 | 10.386 | 10.375 | 10.355 |
| 1P    | 9.900 | 9.901 | 9.900 | 9.900 |
| 2P    | 10.271 | 10.278 | 10.266 | 10.260 |
| 1D    | 10.159 | 10.162 | 10.152 | 10.161 |

$^a$) The experimental errors of the masses are given in Tab. 2

Our conclusion is that for the vector constant $\Lambda^{(4)} = 390$ MeV, or $\Lambda^{(5)} \leq 365$ MeV, and $\sigma = 0.186 \pm 0.004$ GeV$^4$, the extracted pole mass lie in the narrow range

$$m_b (2 - \text{loop}) = 4.81 \pm 0.02 \text{ GeV}. \quad (38)$$

Thus, if the AF behavior of $\alpha_B(r)$ is taken into account the extracted pole mass is about 50 MeV larger than $m_b$ for $\alpha_{\text{static}} = \text{const}$.

Then combining Eqs. (37) and (36) for different choices of $\alpha_B$ one obtains that the extracted pole mass of the $b$ quark lies in the range

$$m_b = 4.78 \pm 0.05 \text{ GeV}. \quad (39)$$

Then by definition of the two-loop pole mass (11) ($n_f = 5$) where in the relation (11) the parameter $\xi_2$ is

$$\xi_2(n_f = 5) \approx -1.0414 \sum_k^{N_L} \left( 1 - \frac{4}{3} \bar{m}_{Q_k} \bar{m}_Q \right) + 13.4434,$$  

(40)
and the sum over $k$ extends over the $N_L$ flavors $Q_k$ which are lighter than $Q$, one finds $\xi_2(n_f = 5) \approx 9.6 \pm 9.7$, and for $\alpha_s(\bar{m}_b) = 0.217$ it follows from (11) that

$$\bar{m}_b(\bar{m}_b) = (4.19 \pm 0.04) \text{ GeV}, \quad (41)$$

This number for the MS mass (11) appears to be in good agreement with the conventional value from Ref. 2 but has smaller theoretical error, $\sim 50$ MeV, than the number quoted in Eq. 2.

5 Charmonium

The $c\bar{c}$ spectrum has several differences in comparison to bottomonium.
Table 5: The $c\bar{c}$ spin-averaged masses $M(nL)$ (in MeV) in R and NR cases with the same static potential ($\alpha_{\text{static}} = 0.42; \sigma = 0.18 \text{ GeV}^2$, $m_c = 1.41 \text{ GeV}$ and different $C_{SE}$ (MeV). The quantity $\delta_R$ is the relativistic shift $M(nL) - M^{NR}(nL)$.

| State | Experiment | R case | NR case | $\delta_R$ |
|-------|------------|--------|---------|------------|
|       |            | $C_{SE}^{R} = -35$ | $C_{SE}^{NR} = -57$ |            |
| 1S    | 3067±0.7   | 3067   | 3067 (fit) | 0          |
| 2S    | 3673±6     | 3661   | 3688     | -27        |
| 1P    | 3525±0.6   | 3528   | 3510     | 18         |
|       | Above $DD$ threshold |
| 1D    | 3770±2.5   | 3823   | 3812     | 12         |
|       | 3872±1.2   |        |          |            |
| 2P    | -          | 3965   | 3984     | -19        |
| 1F    | -          | 4067   | 4067     | 0          |
| 3S    | 4040±10    | 4082   | 4141     | -59        |
| 2D    | 4159±20    | 4200   | 4227     | -27        |
| 4S    | 4415±6     | 4433   | 4527     | -94        |

First, the self-energy contribution to the mass $M(nL)$ is nonzero, about $-40 \text{ MeV}$, being practically the same for different $nL$ states, and therefore can be taken constant for all states with the accuracy $1 \pm 3 \text{ MeV}$.

Secondly, relativistic corrections are not small in charmonium and $M(nL)$ is in the relativistic case (R) always smaller, so

$$\delta_R(nL) = M(nL) - M^{NR}(nL) \quad (42)$$

is negative. Note that the self-energy term must be the same in both cases. However, if the spin-averaged mass of the 1S state, $M(1S)=3067 \text{ MeV}$, is used for the fit, then $C_{SE}$ are different in R and NR cases and $\delta_R(nL)$ has irregular behavior (see Table 5). From Table 5, where $C_{SE} = -35 \text{ MeV}$ in the R case and $C_{SE}^{NR} = -57 \text{ MeV}$ in the NR case, one can see that $\delta_R(nL)$ is positive ($\sim 10 \div 20 \text{ MeV}$) for the 1P and 1D states; equal to zero for the 1F state, and negative for the nS states, 2P and 2D, and higher states. It is important that for the 4S (3S) state $|\delta(nL)|$ is large, $\sim 100 \text{ MeV}$ (60 MeV) and therefore the $c\bar{c}$ spectrum has to be calculated with a relativistic Hamiltonian.

The interesting observation is that while the ground state mass is fitted well, the relativistic corrections to $M_{NR}(nL)$ for the 1P and 1D states turn out to be positive, (since a negative value for $C_{SE}^{NR}$ in NR with larger magnitude was taken).

A third difference refers to the choice of $\alpha_B(r)$. Since the $c\bar{c}$ states have larger sizes than the $b\bar{b}$ ones the AF behavior of $\alpha_B(r)$ appears to be less important in charmonium and the approximation $\alpha_B(r) = \text{constant}$ is valid with good
Table 6: The spin-averaged masses in charmonium in $R$ case for the static potential with the parameters given in Eq. (43).

| State | Set A   | Set B   | Experiment          |
|-------|---------|---------|---------------------|
| $1S$  | 3067    | 3067    | $3067 \pm 0.7$      |
| $2S$  | 3660    | 3668    | $3673 \pm 8$       |
| $1P$  | 3528    | 3510    | $3525 \pm 0.6$     |
| Above $D\bar{D}$ threshold | | | |
| $1D$  | 3823    | 3805    | $3871.8 \pm 1.2$   |
|       |         |         | $3770 \pm 2.5$     |
| $2D$  | 4199    | 4198    | $4159 \pm 20$     |
| $3S$  | 4080    | 4109    | $4040 \pm 10$     |
| $4S$  | 4424    | 4459    | $4415 \pm 6$      |
| $2P$  | 3964    | 3954    | –                   |

accuracy. For example for two sets of parameters:

Set A $m_c = 1.42$ GeV, $\sigma = 0.18$ GeV, $\alpha_B = 0.42$; $C_{SE} = -35$ MeV
Set B $m_c = 1.42$ GeV, $\sigma = 0.185$ GeV, $\Lambda(4) = 360$ MeV, $C_{SE} = -30$ MeV

close values of $M(nL)$ in the relativistic case are obtained (see Table 6).

As seen from Table 6 the higher excitations, like the $3S$, $4S$, and $2D$ states, lie $\sim 40$ MeV higher than the experimental values. All these states have large r.m.s. radii: $R(3S) = 1.1$ fm, $R(4S) \equiv 1.4$ fm, and $R(2D) \equiv 1.4$ fm. At such distances the confining potential is flattening due to quark-antquark pair creation [24] and it results in a correlated shift of the radial excitations down as it takes place for light mesons [25]. This phenomenon can be illustrated taking instead of the linear potential $\sigma_0 r$ the modified confining potential $\sigma(r) r$ which was proposed in Ref. [26],

$$\sigma(r) = \sigma_0 (1 - \gamma_0 f(r)) \text{ with } f(r) = \frac{\exp(\sqrt{\sigma_0} (r - a))}{B + \exp(\sqrt{\sigma_0} (r - a))}$$  \hspace{1cm} (44)

with the parameters

$$\sigma = 0.185 \text{ GeV}^2, \ \gamma_0 = 0.40, \ a = 6.0 \text{ GeV}^{-1}, \ B = 20.$$  \hspace{1cm} (45)

For this set of parameters the $c\bar{c}$ spectrum (R case) is given in Table 7 together with the one for the standard linear potential $\sigma_0 r$ with $\sigma_0 = 0.185 \text{ GeV}^2$. The value $\alpha_{\text{static}} = 0.42$ is taken in both cases.

From Table 7 one can see that for the modified potential $\sigma(r) r$ the masses $M(4S)$ and $M(3S)$ of the radial excitations are shifted down by 50 MeV and 20 MeV respectively, and turn out to be closer to the experimental values. It is also worthwhile to look at the dynamical masses for the low-lying states which are larger than the pole mass by the constant amount

$$\omega_c(nL) - m_c \approx 220 \div 250 \text{ MeV},$$  \hspace{1cm} (46)
Table 7: The comparison of the spin-averaged masses in charmonium (\(R\) case) for confining \(\sigma_0 r\) potential and modified potential \(^{(14)}\) \((m_c = 1.42\ \text{GeV}, C_{SE} = -42\ \text{MeV}, \alpha_{\text{static}} = 0.42\ \text{in both cases})\).

| State | \(\sigma_0 = \text{const} = 0.185\ \text{GeV}^2\) | \(\sigma = \sigma(r)\) with parameters Eq. \(^{(15)}\) | experiment |
|-------|-----------------|-----------------|-----------|
| 1S    | 3068            | 3067            | 3067      |
| 2S    | 3670            | 3664            | 3672      |
| 3S    | 4097            | 4077            | 4040\pm10|
| 4S    | 4454            | 4403            | 4415\pm6  |
| 1P    | 3535            | 3530            | 3525\pm0.6|
| 2P    | 3979            | 3965            | –         |
| 1D    | 3835            | 3828            | 3779\pm25 |
| 2D    | 4217            | 4194            | 4159\pm20 |

Table 8: The dynamical masses \(\omega_c(nL)\) for the potential with the parameters Eq. \(^{(13)}\) and \(m_c = 1.42\ \text{GeV}\) (Set A).

| State \(\omega_c(nL)\) in GeV | 1S | 2S | 3S | 4S | 1P | 2P | 1D | 2D |
|-------------------------------|----|----|----|----|----|----|----|----|
| \(\omega_c(nL)\) in GeV     | 1.65 | 1.69 | 1.74 | 1.76 | 1.63 | 1.69 | 1.66 | 1.77 |

while for high excitations this difference can reach \(300 \div 340\ \text{MeV}\) (see Table \(^8\)). The observed difference between the dynamical and the pole mass can be essential for such physical characteristics as the hyperfine and fine-structure splittings, which are determined by the dynamical mass \(^{(20)}\) and it causes a decrease of the hyperfine splitting, e.g. for the \(2S\) state in charmonium \(^{(27)}\).

In our analysis the best fit to the \(c\bar{c}\) spectrum together with the correct choice of the self-energy contribution Eq. \(^{(6)}\) gives the pole mass \(m_c\) in the range

\[
m_c(2\text{-loop}) = 1.39 \pm 0.01\text{GeV (theory)} \pm 0.04(\alpha_B). \quad (47)
\]

Then the \(\overline{\text{MS}}\) running mass Eq. \(^{(2)}\) \((n_f = 4, \text{the coefficient } \xi_2 \approx 10.5)\) from Eq. \(^{(47)}\) is

\[
\bar{m}_c(\bar{m}_c) = (1.10 \pm 0.05)\text{GeV.} \quad (48)
\]

The value obtained turns out to be in good agreement with the conventional value for \(\bar{m}_c(\bar{m}_c)\) \(^{(2)}\), but has a smaller theoretical error.

6 Conclusion

Our study of the \(b\bar{b}\) and \(c\bar{c}\) spectra has been performed with the use of the relativistic Hamiltonian \(H_0\) and correct NP self-energy contribution to the meson mass.
By derivation the kinetic part of $H_0$ contains the pole quark mass $m_Q$ and it can directly be extracted from the analysis of the $QQ$ spectrum. In our study all meson masses are expressed through only two parameters: the string tension and the QCD constant $\Lambda^{(n_f)}$ (in the Vector scheme). In charmonium the strong coupling $\alpha_B(r)$ can be approximated by a constant with good accuracy.

The spin-averaged splittings like 1D-1P and 2P-1P in bottomonium and 2S-1P and 1P-1S in charmonium appear to be very sensitive to the chosen freezing (critical) value of the strong coupling. A good description of the HQ spectra was reached only if $\alpha_{\text{crit}}$ was taken rather large, $\alpha_{\text{crit}} \approx 0.55 \pm 0.02$ while the constant value for $\alpha_{\text{eff}}$ taken in the Coulomb potential is about 20% smaller.

From our analysis one can conclude that

(i) The dynamical quark mass $\omega_b(\omega_c)$ is about 200 MeV larger than the pole mass $m_b(m_c)$ for low-lying states. This difference should be taken into account when the spin structure in heavy quarkonia is studied, and it is especially important in charmonium.

(ii) The pole masses, $m_b(2\text{-loop}) = 4.78 \pm 0.05$ GeV and $m_c = 1.39 \pm 0.06$ GeV, were extracted from our fit to the $QQ$ spectra which correspond to the MS running masses: $\tilde{m}_b(\tilde{m}_c) = 4.19 \pm 0.04$ GeV and $\tilde{m}_c(\tilde{m}_c) = 1.10 \pm 0.05$ GeV. The numbers obtained are in good agreement with the conventional values but have smaller theoretical error. The error we found is small, because in our analysis only one parameter, $\Lambda$ (or $\alpha_{\text{crit}}$), is actually varied while a second parameter—the string tension—was taken the same as for light mesons.

A Relativistic Hamiltonian

Here we present the principal steps to derive the Hamiltonian $H_R$ Eq. (7) and the NP self-energy term Eq. (4) taken from Refs. [1, 11]. The starting point is the gauge-invariant meson Green’s function written in FFS representation [1, 26] with the use of the QCD action

$$G_M(x, y) = \langle Tr \Gamma_1 G_q(x, y) \Gamma_2 G_{\bar{q}}(x, y) \rangle_B \tag{49}$$

In Eq. (49) the averaging goes over the background field $B_\mu$ and $G_q(x, y)(G_{\bar{q}}(x, y))$ is the Euclidean quark (antiquark) Green’s function

$$G_q(x, y) = (\tilde{m}_q + \tilde{D})_{x,y}^{-1} = (\tilde{m}_q - \tilde{D})_x (\tilde{m}_q^2 - \tilde{D}^2)^{-1}_{x,y} \tag{50}$$

where the factors $R_a$, $R_B$, and $R_F$ are given by,

$$R_a = P_a \exp \left( i q \int_0^{\infty} \frac{x}{y} a_\mu dz_\mu \right) .$$
\[ R_B = P_B \exp \left( \frac{ig}{y} \int B_\mu dz_\mu \right), \]
\[ R_F = P_F \exp \left( \int_0^s g\sigma_{\mu\nu} F_{\mu\nu} d\tau \right) \]  
(51)

(The factors corresponding to the antiquarks are defined similarly.) Here \( P_a, P_B, \) and \( P_F \) are the ordering operators of the matrices \( a_\mu, B_\mu, \) and \( F_{\mu\nu} \) respectively, and
\[ \sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \frac{\partial \hat{H}}{\partial \hat{E}} & \frac{\partial \hat{E}}{\partial \hat{H}} \end{pmatrix}, \]  
(52)

\[ \sigma_{\mu\nu} = \frac{1}{4i}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \]  
represents the interaction of the quark (antiquark) color magnetic moment with the NP field strength \( F_{\mu\nu} \).

In Eq. (50) the kinetic energy term \( K \) is defined as the integral over the proper time \( \tau \):
\[ K = m_q^2 s + \frac{1}{4} \int_0^s (\dot{z}_\mu)^2 d\tau. \]  
(53)

The quark moving along the trajectory \( z_\mu(\tau) \) interacts with the field of the valence gluon \( a_\mu \) and by its color charge with the NP background field \( B_\mu \).

In Eq. (53) the quantity \( m_q \) is the Lagrangian current mass usually taken in the \( \overline{\text{MS}} \) renormalization scheme. The factor \( R_a \) is responsible for the standard perturbative corrections to the quark mass \( \overline{m}_q \) (as in Eq. (1)), i.e. for the appearance of the pole mass in the QCD Action (Hamiltonian) \[ 3 \]. Finally, the factors \( R_a \) and \( \overline{R}_a \) (from the quark and the antiquark) provide the perturbative static interaction \[ 20 \].

The other two factors \( R_B \) and \( \overline{R}_B \) (from the quark and the antiquark) in \( G_M(x, y) \) Eq. (50) are responsible for the full NP (string) dynamics and were considered in detail in Ref. [11], where after several steps the meson Green’s function was presented in following form
\[ G_M(r) = \int d\omega \ d\nu \ dr \ \exp(-A_R), \]  
(54)

where the action \( A_R \) in coordinate space is expressed through two auxiliary fields \( \omega \) and \( \nu \). Since this action (see Ref. [11]) does not depend on the derivatives \( \dot{\omega} \) and \( \dot{\nu} \), the integration over \( \omega, \nu \) in Eq. (54) is equivalent to the canonical quantization of the Hamiltonian \( H_R \) which corresponds to the action \( A_R \). It results in the following Hamiltonian,
\[ H_R = \frac{p^2}{\overline{m}_q^2} + \omega(t) + \frac{\beta^2}{\nu^2} \left[ \omega + 2 \int_0^1 d\beta \nu(\beta) \right]^{-1} \]
\[ + \frac{1}{2} \sigma^2 \frac{d}{d\omega(\beta)} + \frac{1}{2} \int_0^1 d\beta \nu(\beta), \]  
(55)

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where the field operator $\omega(t)$ is defined as $\omega(t) = \frac{1}{2} \frac{d^2}{dt^2}$ and $t$ is the actual time.

With the use of the extremal conditions ($\nu_0 = \sigma r$) and considering the string corrections as a perturbation (the procedure is described in Ref. [13]) one obtains the Hamiltonian $H_0$. The terms $R_F$ and $\bar{R}_F$ (from quark and antiquark) provide the NP self-energy contribution $C_{SE}$ (gauge-invariant) to the meson mass [1], where for the quark (antiquark) the self-energy correction $\Delta m^2_q$ to the pole mass $m_q$ appears to be expressed through the string tension $\sigma$ and the factor $\eta$:

$$\Delta m^2_q(m_q) = -\frac{4\sigma}{\pi} \eta(m_q)$$  \hspace{1cm} (56)

The factor $\eta(m_q)$ was calculated in analytical form in Ref. [1] and for $m_q > T_g$, where $T_g$ is the gluonic correlation length ($\delta = T_g^{-1}$), $\eta(m_q)$ is given by the expression

$$\eta(m_q) = -\frac{3m_q^2 \delta^4}{(m_q^2 - \delta^2)^{5/2}} \arctan \frac{\sqrt{m_q^2 - \delta^2}}{\delta} + \frac{\delta^2(2m_q^2 + \delta^2)}{(m_q^2 - \delta^2)^2}$$  \hspace{1cm} (57)

A straightforward calculation gives $\eta(m_B \approx 5.0) \approx 0.07$; $\eta(m_q = 1.70) = 0.24$ and $\eta(m_q \approx 1.40) \approx 0.30$. Then the unperturbed part of the string Hamiltonian Eq. (7) acquires the correction Eq. (56)

$$H'_0 \to \frac{\hat{p}^2 + m_q^2 + \Delta m^2_q}{\omega_q} + \omega_q + V_{static} = H_0 + C_{SE}$$  \hspace{1cm} (58)

with the self-energy correction

$$C_{SE} = \frac{\Delta m^2_q}{\omega_q} = -\frac{4\sigma}{\pi \omega_q} \eta(m_q).$$  \hspace{1cm} (59)

If this self-energy correction is considered as a perturbation, the operator $\hat{\omega}$ in Eq. (59) can be replaced by the average of this operator Eq. (5), i.e. by the dynamical mass

$$\hat{\omega}_q \to \omega_q = \langle \sqrt{\hat{p}^2 + m_q^2} \rangle_{nL}$$  \hspace{1cm} (60)

and through $\omega_q$ the NP self-energy term $C_{SE}$ appears to be dependent on the quantum numbers $nL$. However, in bottomonium

$$C_{SE}(b\bar{b}) \approx -3 \text{ MeV}$$  \hspace{1cm} (61)

is small and can be neglected in the mass formula. In charmonium however, for $m_c \approx 1.40 \text{ GeV}$ the factor $\eta_c = 0.29$ and the value of $C_{SE} \approx -40 \text{ MeV}$ is obtained which is practically the same for different $nL$ states because of the weak dependence of $\omega_c(nL)$ on the quantum numbers (see Table 8).

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