Topological Color-Hall Insulators: SU(3) Fermions in Optical Lattices

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We discuss the emergence of topological color insulators in optical lattices as quantum phases of SU(3) ultra-cold neutral fermions. We construct the Chern matrix and classify all insulating phases in terms of three topological invariants: the charge-charge, the color-charge and the color-color Chern numbers. Our classification transcends that of SU(2) systems which require only the charge-charge (charge-Hall) and spin-charge (spin-Hall) Chern numbers. To illustrate the topological classification of the insulating phases of SU(3) fermions, we construct phase diagrams of chemical potential and color-orbit parameter versus color-flip fields for fixed magnetic flux ratio.

The identification of topological invariants for charged SU(2) fermions has been very important in the distinction between trivial and non-trivial insulators found in condensed matter physics. The integer quantum Hall effect is a typical example of the importance of topological invariants, where the quantum Hall conductance (charge-charge response) is proportional to an integer \[ \frac{e^2}{h} \] in two-dimensional lattices at high magnetic fields. This topological invariant, known as the TKNN \([1]\) integer, is also identified as the first Chern number of a \( U(1) \) principal fiber bundle on a torus \([2]\). The TKNN integer counts the number and chirality of edge states in two-dimensional SU(2) systems with open boundary conditions, as indicated by the bulk-edge correspondence \([3,4]\).

More recently, topological insulators, with zero quantum Hall and non-zero quantum spin-Hall conductances, were found to exist in graphenelike two-dimensional lattices without Zeeman or magnetic fields, but in the presence of spin-orbit coupling \([5]\). When spin is conserved, the quantization of the spin-charge (spin-Hall) response is the result of spin-dependent TKNN integers having opposite values, that is, \( c^\ell_{\text{TKNN}} = -c^\ell_{\text{TKNN}} \), in which case spin-orbit coupling plays a pivotal role. This state of matter is coined the quantum spin-Hall phase, and exhibits spin-filtered edge states carrying opposite currents for opposite spins, while possessing a bulk energy gap. Extensions of these findings for finite Zeeman \([6]\) and magnetic \([7]\) fields in SU(2) graphenhelike lattices have indicated that quantum spin-Hall phases can survive the breaking of time-reversal symmetry.

However, in spite of a deeper understanding of topological properties of SU(2) fermions in two-dimensional lattices found in condensed matter systems, very little is known about the topological properties of neutral SU(N ≥ 3) fermions, such as \(^{173}\)Yb, that have been loaded into optical lattices \([8,10]\). Many experiments involving cold atoms have focused on studying topological properties of neutral SU(2) systems due to their direct connections to their charged SU(2) cousins. A few experiments have attempted to explore neutral SU(2) systems in fictitious magnetic and spin-orbit fields with the goal of studying the analogues of the quantum charge-charge (charge-Hall) and spin-charge (spin-Hall) effects \([11,12]\).

In this paper, we show that neutral SU(3) fermions are qualitatively different from their neutral or charged SU(2) relatives. By labeling the internal states of the atoms as colors, we find that the topological insulating phases are characterized by a set of three topological invariants: charge-charge, color-charge, and color-color Chern numbers. This is in contrast with SU(2) systems, where only charge-charge (charge-Hall) and spin-charge (spin-Hall) Chern numbers are necessary to classify topological insulators \([6]\). We show examples of our classification for phase diagrams of neutral SU(3) fermions loaded into two-dimensional lattices and in the presence of fictitious magnetic, color-flip and color-orbit fields.

Hamiltonian: To describe topological phases of SU(3) fermions loaded into two-dimensional optical lattices, we start from the Hamiltonian

\[
\hat{H} = - \sum_{\mathbf{r},\mathbf{r}+\mathbf{\ell}} t_\mathbf{\ell} \psi_\mathbf{r}^{\dagger} e^{-i\mathbf{k}_\mathbf{\ell} \theta_\mathbf{r}} \psi_{\mathbf{r}+\mathbf{\ell}} - h_z \sum_{\mathbf{r}} \psi_\mathbf{r}^{\dagger}(\mathbf{r}) \mathbf{J}_{\mathbf{z}} \psi_{\mathbf{r}}(\mathbf{r}),
\]

where \( t_\mathbf{\ell} \) are hopping energies along the \( \mathbf{\ell} = \{x, y\} \) direction, and \( h_z \) plays the role of a color-flip field along the \( z \)-direction. The phase operators \( \theta_\mathbf{r} \) describe the effect of artificial color-orbit coupling \( \mathbf{\theta}_\mathbf{r} = k_T \eta_x \mathbf{J}_x + \mathbf{A}_y \eta_y \mathbf{1} \), with momentum transfer \( k_T \) and artificial gauge field \( \mathbf{\theta}_\mathbf{y} = \mathbf{A}_y \eta_y \mathbf{1} \), where \( \mathbf{A}_y = eH_z x / h_\mathbf{c} \) plays the role of the \( y \) component of an artificial vector potential with dimension of inverse length. Here, \( H_z \) is identified as a synthetic magnetic field along the \( z \)-axis. The vector potential \( \mathbf{A}_y \) may be generated by laser assisted tunneling \([14,15]\), while the color-dependent momentum transfer \( k_T \) and color-flip field \( h_z \) may be created via counter-propagating Raman beams \([10]\) or via radio-frequency chips \([16]\).

The vectors \( \eta_x = \eta_x \mathbf{\hat{x}} \), with \( \eta_x = \pm a_x \) and \( \eta_y = \pm a_y \), indicate the position \( \mathbf{r} + \eta_x \mathbf{\hat{x}} \) of nearest neighbors with respect \( \mathbf{r} = \{x, y\} \), where fermion creation operators are defined by three-component vectors \( \psi_\mathbf{r}^{\dagger}(\mathbf{r}) = \left[ \psi_{\mathbf{r}}^{\dagger}(\mathbf{r}), \psi_{\mathbf{r}}^{\dagger}(\mathbf{r}), \psi_{\mathbf{r}}^{\dagger}(\mathbf{r}) \right] \) with color \( c = \{R, G, B\} \) (Red, Green and Blue). The unit cell lengths are \( a_x (a_y) \) along the \( x \) (\( y \)) direction, \( \mathbf{\hat{\ell}} = \{\mathbf{\hat{x}}, \mathbf{\hat{y}}\} \) are the corresponding unit vectors, while the operators \( \mathbf{J}_x \) and \( \mathbf{J}_z \) are pseudospin-1 matrices with states \( \{|\uparrow, 0, \downarrow\} \) representing colors \( \{R, G, B\} \), respectively, and \( \mathbf{1} \) is the identity matrix.
Under the color-gauge transformation $\psi(r) = e^{ikT x J^z} \tilde{\psi}(r)$, the Hamiltonian of Eq. (1) becomes

$$\hat{H} = -\sum_{r, \eta} t_r \tilde{\psi}^\dagger(r) e^{-i \theta_r} \tilde{\psi}(r) - h_x \sum_r \tilde{\psi}^\dagger(r) \tilde{J} \tilde{\psi}(r),$$

(2)

with $\theta_r = 0$, $\theta_y = \theta_y$, and $\tilde{J} = J_x \cos(k_T x) + J_y \sin(k_T x)$. When $h_x = 0$ the color-orbit coupling can be gauged away (color-gauge symmetry), since the resulting Hamiltonian and its eigenvalues are independent of $k_T$.

We transform the second quantization Hamiltonian of Eq. (1) into the first quantization Hamiltonian matrix

$$\hat{H} = \begin{pmatrix} \varepsilon_R(k) & -h_x \sqrt{2} & 0 \\ -h_x \sqrt{2} & \varepsilon_G(k) & -h_x / \sqrt{2} \\ 0 & -h_x / \sqrt{2} & \varepsilon_B(k) \end{pmatrix}$$

(3)

that describes a color generalization of the original Harper’s Hamiltonian for SU(2) fermions [18]. The matrix elements are

$$\varepsilon_R(k) = -2t_x \cos((k_x - k_T) a_x) - 2t_y \cos((k_y - A_y) a_y)$$

(4)

corresponding to the kinetic energy of the $R$ state,

$$\varepsilon_G(k) = -2t_x \cos((k_x a_x) - 2t_y \cos((k_y - A_y) a_y)$$

(5)

corresponding to the kinetic energy of the $G$ state, and

$$\varepsilon_B(k) = -2t_x \cos((k_x + k_T) a_x) - 2t_y \cos((k_y - A_y) a_y)$$

(6)

corresponding to the kinetic energy of the $B$ state. Lastly, $h_x$ is a color-flip field along the $x$ direction, whose physical origin is a Rabi term that couples Red and Green, as well as, Green and Blue internal states of the atom. The Hamiltonian matrix in Eq. (3) acts on a three-color wavefunction $\Psi(r) = [\Psi_R (r), \Psi_G (r), \Psi_B (r)]^T$, where $T$ indicates transposition.

Rewriting Eq. (3) in terms of spin-1 matrices $J_\ell$, with $\ell = \{x, y, z\}$, leads to

$$\hat{H} = \varepsilon_G(k) \mathbf{1} - h_x J_x - h_z(k) J_z + b_z(k) J_z^2$$

(7)

where $h_z$ plays the role of a Zeeman field along the $x$ axis in spin-space, $h_z(k) = \frac{\varepsilon_B(k) - \varepsilon_R(k)}{2}$ represents momentum dependent Zeeman field along the $z$ axis in spin-space, and $b_z(k) = \frac{\varepsilon_B(k) + \varepsilon_R(k)}{2} - \varepsilon_G(k)$ describes a momentum dependent quadrupole field along the $z$ axis in spin-space. The explicit forms of the operators are $h_z(k) = 2t_x \sin(k_x a_x) \sin(k_y a_y)$ and $b_z(k) = 4t_x \sin^2(k_x a_x / 2) \cos(k_y a_y)$. The term $b_z(k) J_z^2$ describes a momentum dependent color-quadrupole (or pseudo-spin-quadrupole) coupling, reflecting the entanglement of momentum and tensorial degrees of freedom [19-21]. The presence of the color fields $h_x$, $h_z(k)$ and $b_z(k)$ breaks SU(3) symmetry [22], however the color-gauge transformation restores SU(3) symmetry when $h_z = 0$ for any value of $k_T$. The term $b_z(k) J_z^2$ is absent for SU(2) fermions in the presence of spin-orbit coupling, but, here, it plays a very important role in the determination and classification of topological insulating phases that emerge between degenerate SU(3) symmetric color insulators at $h_x = 0$ and fully polarized color insulators at $h_x \to \infty$.

**Eigenspectrum:** We choose first a cylindrical geometry with periodic boundary conditions along the $x$ direction, and a finite number $M_x$ of sites along the $x$ direction. In this case, $k_y$ is a good quantum number, but $k_x$ is not, leading to the color-dependent Harper’s matrix

$$\mathbf{H} = \begin{pmatrix} A_{m-2} & B & 0 & 0 \\ B^* & A_{m-1} & B & 0 \\ 0 & B^* & A_m & B \\ 0 & 0 & B^* & A_{m+1} \end{pmatrix},$$

(8)

which has a tridiagonal block structure coupling neighboring sites $(m-1, m, m+1)$ along the $x$ direction, with $x = ma_x$, and discrete translational symmetry along the $y$ axis. The matrices $A$, $B$ and the null matrix $0$ consist of $3 \times 3$ blocks with entries labeled by internal color states $\{R, G, B\}$ or pseudo-spin-1 states $\{\uparrow, \downarrow\}$. The size of the space labeled by the site index $m$ is $M_x$, thus the total dimension of the matrix $\mathbf{H}$ in Eq. (8) is $3M_x \times 3M_x$.

The matrix indexed by position $x = ma_x$ is

$$A_m = \begin{pmatrix} A_{mR} & -h_x / \sqrt{2} & 0 \\ -h_x / \sqrt{2} & A_{mG} & -h_x / \sqrt{2} \\ 0 & -h_x / \sqrt{2} & A_{mB} \end{pmatrix},$$

with $A_{mR} = A_{mG} = A_{mB} = -2t_y \cos(k_y a_y - 2\pi m\alpha)$. Here, $\alpha = \Phi / \Phi_0$ is the ratio of the magnetic flux through a lattice plaquette $\Phi = H z a_x a_y$ to the flux quantum $\Phi_0 = \hbar c / e$. The matrix containing the color-orbit coupling is

$$B = \begin{pmatrix} -t_x e^{-ikT a_x} & 0 & 0 \\ 0 & -t_x & 0 \\ 0 & 0 & -t_x e^{ikT a_x} \end{pmatrix},$$

where $k_T$ ($-k_T$) corresponds to the momentum transfer along the $x$ direction for state $R$ ($B$), while the momentum transfer for state $G$ is zero.

We consider $M_x = 50$ sites along the $x$ direction, with three states $\{R, G, B\}$ per site, but periodic boundary conditions along the $y$ direction. The eigenvalues $E_{n_k}(k_y)$ are labeled by a discrete band index $n_k$ and by momentum $k_y$, and are functions of the color-orbit coupling $k_T$, color-flip field $h_z$, and flux ratio $\alpha = \Phi / \Phi_0$. In Fig. 1(a) we show $E_{n_k}(k_y)$ for flux ratio $\alpha = 1/3$ in four cases. In Fig. (a) with $k_T a_x = 0$ and $h_z / t_y = 0$, there are three sets of degenerate bulk bands connected by color-degenerate edge states. In Fig. 1(b) with $k_T a_x = \pi / 8$ and $h_z / t_y = 0$, the plots are identical to case (a) because of the color-gauge symmetry allows gauging away the color-orbit coupling. Notice in (a) and (b) that the bulk band gaps are connected by edge states at filling
The boundary condition on the many-particle wavefunction \( \psi \) or \( \{ | \psi \rangle \} \) is the length vector along \( \ell \) = \((x, y)\) direction, and \( \phi_c \) is the phase twist \( \frac{2\pi}{m_c} \) along \( \ell \) for color \( c \).

Under the transformation \( \psi(\mathbf{r}_{1c}, \ldots, \mathbf{r}_{jc}, \ldots, \mathbf{r}_{N_c}) = e^{i\phi_c} \psi(\mathbf{r}_{1c}, \ldots, \mathbf{r}_{jc}, \ldots, \mathbf{r}_{N_c}) \) with \( 0 \leq \phi_c < 2\pi \), the wavefunction \( \tilde{\psi}(\mathbf{r}_{1c}, \ldots, \mathbf{r}_{jc}, \ldots, \mathbf{r}_{N_c}) \) is periodic in \( \phi_c \), and we can define the Chern matrix \( \mathcal{C}_{cc'} \)

\[
\mathcal{C}_{cc'} = \frac{i}{4\pi} \int d\phi_{xc} d\phi_{yc'} \mathcal{F}_{xy}(\phi_{xc}, \phi_{yc'}),
\]

where the purely imaginary curvature function

\[
\mathcal{F}_{xy}(\phi_{xc}, \phi_{yc'}) = \left\langle \frac{\partial \tilde{\psi}}{\partial \phi_{xc}} \frac{\partial \tilde{\psi}}{\partial \phi_{yc'}} \right\rangle - \left\langle \frac{\partial \tilde{\psi}}{\partial \phi_{yc'}} \frac{\partial \tilde{\psi}}{\partial \phi_{xc}} \right\rangle,
\]

is integrated over the torus \( T^2 \), that is, over the range of phase twists \( 0 \leq \phi_{xc} < 2\pi \) and \( 0 \leq \phi_{yc'} < 2\pi \). The dimension of the Chern number matrix is \( 3 \times 3 \), since there are three color states \( c = \{ R, G, B \} \) and three pseudo-spin 1 states \( \tau = \{ \uparrow, 0, \downarrow \} \).

In passing, we note that the SU(N) generalization for \( N \geq 3 \) leads to an \( N \times N \) Chern matrix.

The expression given in Eq. (9) is an integer just like in SU(2) systems \( \mathcal{C}_{ch}^\tau = \sum_{c'} \mathcal{C}_{cc'}^\tau \).

Three topological invariants can be obtained from the Chern matrix above. The first invariant is the charge-charge (charge-Hall) Chern number \( \mathcal{C}_{ch}^\tau = \sum_{c'} \mathcal{C}_{cc'}^\tau \), or charge-color Chern number \( \mathcal{C}_{ch}^{\sigma} = \sum_{c'} \mathcal{C}_{cc'}^{\sigma} \), since \( \mathcal{C}_{ch} = \mathcal{C}_{sp} \).

The third topological invariant is the color-color Chern number \( \mathcal{C}_{co} = \sum_{cc'} m_c m_{c'} \), where \( m_c \) is the color quantum number with \( m_R = +1 \), \( m_G = 0 \), and \( m_B = -1 \), as identified from the pseudo-spin 1 representation \( m_{\uparrow} = +1 \), \( m_0 = 0 \), and \( m_{\downarrow} = -1 \).

A simple way to connect these results to conventional SU(2) condensed matter physics of electrons and holes is to look at the current density \( J_{\lambda \tau} \), where \( \lambda \) refers to either charge or color, that is, \( \lambda = \{ \text{ch}, \text{co} \} \) and the conductivity tensor \( \sigma_{\lambda \tau}^\lambda \) through the generalized relation

\[
J_{\lambda \tau}^\lambda = \sigma_{\lambda \tau}^\lambda E_{\tau \lambda}^\tau, \quad E_{\tau \lambda}^\tau \text{ plays the role of a generalized electric field with } \tau = \{ \text{ch}, \text{co} \}.
\]

To simplify our notation we drop the \( xy \) labels, define the conductivity tensor \( \sigma_{\lambda \tau}^\lambda \equiv \sigma_{\lambda \tau}^\lambda \), and work finally with the conductance tensor \( \sigma_{\lambda \tau}^\sigma \equiv \sigma_{\lambda \tau}^\sigma \).

If we were dealing with fermions with charge \( e \) and conserved pseudo-spin 1 projection along a global quantization axis, then the charge-charge (charge-Hall) conductance would be \( \sigma_{ch}^{\sigma} = (e^2/h) \mathcal{C}_{ch}^{\sigma} \), the color-charge (color-Hall) conductance would be \( \sigma_{ch}^{\sigma} = (e^2/h)(e_c/e_{ch}) \mathcal{C}_{ch}^{\sigma} \), and the color-color conductance would be \( \sigma_{co}^{\sigma} = (e^2/2\pi)(h/e) \mathcal{C}_{co}^{\sigma} = (h/2\pi) \mathcal{C}_{co}^{\sigma} \).

However, our fermions are really neutral and their colors represent three internal states of the atoms, thus one can only hope to probe the charge-charge (charge-Hall) and color-charge (color-Hall) and color-color Chern numbers in analogy with measurement proposals \( \mathcal{C}_{ch}^{\sigma} \) or actual measurements \( \mathcal{C}_{co}^{\sigma} \) of Chern numbers for atomic systems with one and two internal states.
which has the same value as charge-charge (charge-Hall) Chern number $C_{ch}^0 = \sum_{x} C_{\sigma\sigma}$, since $m^2 = 1$. Since $C_{ch}^0$ does not add any additional information about the topological nature of insulating phases for SU(2) systems, it is sufficient to stop the topological classification at the spin-charge (spin-Hall) level, such as the $Z_2$ classification used in the case of quantum spin-Hall phases of graphene-like structures [6, 8]. However, in the SU(3) case, $C_{co}$ provides new topological information and can be used to refine the topological classification of the non-trivial insulating phases. We note that for SU(N) fermions with $N > 3$ flavors, the generalized flavor-flavor Chern number $C_{ff}^0$ will also provide additional topological information about the insulating states.

**Phase Diagrams:** In Fig. 2 we show the phase diagram of chemical potential $\mu/t_y$ versus Zeeman field $h_x/t_y$ for fixed value of the magnetic flux ratio $\alpha = 1/3$, hoppings $t_x = t_y$ and two values of the color-orbit parameter: (a) $k_T a_x = 0$, and (b) $k_T a_x = \pi/8$. The white regions indicate conducting phases, while the regions with other colors correspond to insulating phases. The color palette in Fig. 2 indicates the charge-charge (charge-Hall) Chern numbers $C_{ch}^0$ associated with the corresponding colored regions. However, $C_{ch}^0$ is not sufficient to classify the topological insulating phases of SU(3) fermions for arbitrary $k_T a_x$ and $h_x/t_y$, as we also need the color-charge (color-Hall) $C_{ch}^{co}$ and color-color $C_{co}^{co}$ Chern numbers.

In Figs. 2(a) and 2(b), we see our classification at work. The gray regions, occurring at filling factors $\nu = 0, 1, 2, 3$, are topologically trivial with either non-chiral or no edge states at all. They have charge-charge (charge-Hall) Chern number $C_{ch}^0 = 0$ and also zero color-charge (color-Hall) and color-color Chern numbers $C_{co}^0 = C_{co}^{co} = 0$. The magenta region at $\nu = 1$ has Chern numbers $C_{ch}^0 = +3, C_{ch}^{co} = 0$ and $C_{co}^{co} = +2$, while the cyan region at $\nu = 2$ has Chern numbers $C_{ch}^0 = -3, C_{ch}^{co} = 0$ and $C_{co}^{co} = -2$. These regions, occurring at low values of $h_x/t_y$, are the analogue of the traditional quantum Hall phases of SU(2) fermions. However, the yellow region at $\nu = 2/3$ with $C_{ch}^0 = +2, C_{ch}^{co} = +1$ and $C_{co}^{co} = +1$, and green region at $\nu = 7/3$ with $C_{ch}^0 = -2, C_{ch}^{co} = +1$ and $C_{co}^{co} = -1$, have no counterparts for SU(2) fermions.

The orange regions, occurring at filling factors $\nu = 1, 2, 3$ have $C_{ch}^0 = 0$, but, unlike the gray regions, they are topologically non-trivial. Each orange region has two chiral edge states and non-zero Chern numbers are listed.

In Fig. 2(b), there are additional insulating phases induced by color-orbit coupling, such as the red region at high values of $h_x/t_y$, represented by the brown and purple regions. The brown region at $\nu = 1/3$ has $C_{ch}^0 = +1, C_{ch}^{co} = +1$ and $C_{co}^{co} = +1$; the brown region at $\nu = 4/3$ has $C_{ch}^0 = +1, C_{ch}^{co} = 0$ and $C_{co}^{co} = 0$; and the brown region at $\nu = 7/3$ has $C_{ch}^0 = +1, C_{ch}^{co} = -1$ and $C_{co}^{co} = +1$. The purple region at $\nu = 2/3$ has $C_{ch}^0 = -1, C_{ch}^{co} = -1$ and $C_{co}^{co} = -1$; the purple region at $\nu = 5/3$ has $C_{ch}^0 = -1, C_{ch}^{co} = 0$ and $C_{co}^{co} = 0$; and the purple region at $\nu = 3/3$ has $C_{ch}^0 = -1, C_{ch}^{co} = -1$ and $C_{co}^{co} = -1$. In Fig. 2(b), there are additional insulating phases induced by color-orbit coupling, such as the red region at
\[ \nu = 4/3 \] with \( C^\text{ch}_{\nu 3} = +1 \), \( C^\text{ch}_{\nu 2} = -2 \) and \( C^\text{co}_{\nu 0} = 0 \); the blue region at \( \nu = 5/3 \) has \( C^\text{ch}_{\nu 3} = -1 \), \( C^\text{ch}_{\nu 2} = -2 \) and \( C^\text{co}_{\nu 0} = 0 \); the green region at \( \nu = 4/3 \) has \( C^\text{ch}_{\nu 3} = -2 \), \( C^\text{ch}_{\nu 2} = 0 \) and \( C^\text{co}_{\nu 0} = 0 \); and the yellow region at \( \nu = 5/3 \) has \( C^\text{ch}_{\nu 3} = +2 \), \( C^\text{ch}_{\nu 2} = 0 \) and \( C^\text{co}_{\nu 0} = 0 \).

We plot phase diagrams of color-orbit coupling parameter \( k_{\alpha 3} \) versus color-flip field \( h_x/t_y \) for \( \nu = 1 \) in Fig. 3(a) and for \( \nu = 4/3 \) in Fig. 3(b). Two phases not yet discussed arise in Fig. 3(a), a light green region with \( C^\text{ch}_{\nu 3} = +6 \), \( C^\text{ch}_{\nu 2} = -1 \) and \( C^\text{co}_{\nu 0} = +3 \), and a cyan region with \( c^\text{ch}_{\nu 3} = -3 \), \( C^\text{ch}_{\nu 2} = -1 \) and \( C^\text{co}_{\nu 0} = -1 \), while no new phases arise in Fig. 3(b). In order to relate SU(3) fermions to their SU(2) cousins, we show schematically edge states in Fig. 3(c) linked to the orange region in Fig. 3(a), and edge states in Fig. 3(d) linked to the red region Fig. 3(b).

Conclusions: For SU(3) fermions in optical lattices, we showed that the classification of topological color insulators requires three topological invariants: the charge, the color-charge and the color-color Chern numbers. We analyzed SU(3) fermions in the presence of artificial magnetic, color-flip and color-orbit fields, and indicated that our classification transcends that of SU(2) fermions, where only charge-charge (charge-Hall) and spin-charge (spin-Hall) Chern numbers are necessary to characterize topological insulating phases. Our findings open an avenue for the exploration of topological insulators of SU(N \( \geq 3 \)) fermions with and without interactions, and also suggest that such phases may be found in lattice quantum chromodynamics models.

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