GLOBAL EXISTENCE AND STABILITY FOR A HYDRODYNAMIC SYSTEM IN THE NEMATIC LIQUID CRYSTAL FLOWS

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Abstract. In this paper we consider a coupled hydrodynamical system which involves the Navier-Stokes equations for the velocity field and kinematic transport equations for the molecular orientation field. By applying the Chemin-Lerner’s time-space estimates for the heat equation and the Fourier localization technique, we prove that when initial data belongs to the critical Besov spaces with negative-order, there exists a unique local solution, and this solution is global when initial data is small enough. As a corollary, we obtain existence of global self-similar solution. In order to figure out the relation between the solution obtained here and weak solutions of standard sense, we establish a stability result, which yields in a direct way that all global weak solutions associated with the same initial data must coincide with the solution obtained here, namely, weak-strong uniqueness holds.

1. Introduction. In this paper, we study the following hydrodynamical system modeling the flow of nematic liquid crystal in $\mathbb{R}^n$:

\begin{align}
\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla P &= -\lambda \text{div} (\nabla d \circ \nabla d), \\
\partial_t d + u \cdot \nabla d &= \gamma (\Delta d - f(d)), \\
\text{div} u &= 0, \\
(u, d)|_{t=0} &= (u_0, d_0),
\end{align}

where $u$ and $P$ denote the velocity field and the pressure of the flow, respectively, $d$ denotes the (averaged) macroscopic/continuum molecule orientation field, $\nu, \lambda, \gamma$ are positive constants, and $f(d)$ is a Ginzburg-Landau approximation function. The

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notation $\nabla d \odot \nabla d$ denotes the $n \times n$ matrix whose $(i,j)$-th entry is given by $\partial_i d \cdot \partial_j d$ ($1 \leq i, j \leq n$). As in [8] and [14], we assume $f(d) = 0$ for simplicity. Besides, since the sizes of the viscosity constants $\nu$, $\lambda$ and $\gamma$ do not play a special role in our discussion, we assume that they all equal to the unit.

The above system (1)–(4) describes the time evolution of nematic liquid crystal materials (cf. [12]). Equation (1) is the conservation of linear momentum (the force balance equation). Equation (2) is the conservation of angular momentum, in which the left hand side represents the kinematic transport by the flow field, while the right hand side represents the internal relaxation due to the elastic energy. Finally, equation (3) represents the incompressibility of the fluid. This system was first introduced by Lin [12] as a simplified version of the liquid crystal model proposed by Ericksen in [3] and Leslie in [10], and retained most of the interesting mathematical properties of the liquid crystal model. Some basic results concerning the mathematical theory of this system were obtained by Lin and Liu in [14] and [15]. More precisely, by using the modified Galerkin method combined with some compactness argument, they proved in [14] the global existence of weak solutions of (1)–(4) with $f(d) = \nabla F(d)$ for some smooth bounded function $F$. Moreover, when $f(d) = 0$, they established global existence of strong solutions if the initial data is sufficiently small (or if the viscosity $\nu$ is sufficiently large). In [15] they proved that the one-dimensional space-time Hausdorff measure of the singular set of “suitable” weak solutions is zero. Recently, when $f(d) = 0$, by using the maximal regularity of Stokes equations and the parabolic equations, Hu and Wang [8] proved global existence of strong solutions to the system (1)–(4) for small initial data belonging to Besov spaces of positive-order. They also proved that when the strong solution exists, all global weak solutions constructed by [14] must be equal to the unique strong solution. In [13] and [16], the authors studied the system (1)–(4) with $f(d) = |\nabla d|^2 d$ in two dimensions. They established the global existence, uniqueness and partial regularity of weak solutions and performed the blow-up analysis at each singular time. Hong [7] proved independently the global existence of weak solutions of the system (1)–(4) in two dimensions. In [19], Wang established global well-posedness of (1)–(4) with $f(d) = \nabla d \odot \nabla d$ for small initial data in $\text{BMO}^{-1} \times \text{BMO}$ (see [19] for definitions of these function spaces). Here we refer the reader to see [4], [6], [11], [18], [20], [21] and the references therein for more details of the physical background of this problem and some different models of similar equations.

As in the work of Hu and Wang [8], we let $F = \nabla d$. Then, taking the gradient of (2), noticing the facts that $F \circ F = F^T F$ ($F^T$ denotes the transpose of $F$) and $
abla u$ and applying the Leray-Hopf projector $P$ to eliminate the pressure $P$, we see that the system (1)–(4) can be reduced into the following system:

\[
\begin{align*}
\partial_t u - \Delta u &= -P u \cdot \nabla u - P \text{div} (F^T F), \\
\partial_t F - \Delta F &= -u \cdot \nabla F - F \nabla u, \\
(u, F)|_{t=0} &= (u_0, F_0),
\end{align*}
\]

where $F_0 = \nabla d_0$. Let us recall that $P = I + \nabla (-\Delta)^{-1} \text{div}$, i.e., $P$ is the $n \times n$ matrix pseudo-differential operator in $\mathbb{R}^n$ with the symbol $(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2})_{i,j=1}^n$, where
I represents the unit operator and δ_{ij} is the Kronecker symbol. Later on we shall consider the Cauchy problem (5)–(7).

The purpose of this paper is to prove global existence and stability of solutions to the problem (5)–(7) in the critical Besov space \( B_{p,q}^{-1+n/p}(\mathbb{R}^n) \) of negative-order. It is easy to verify that (5) and (6) have the same scaling property as the Navier-Stokes equations (which are equations obtained by putting \( d = 0 \) in (1)–(4)), namely, if \((u, F)\) is a solution of (5) and (6) with initial data \((u_0, F_0)\), then for any \( \delta > 0 \), by letting

\[
\begin{align*}
  u_\delta &= \delta u(\delta x, \delta^2 t), \\
  F_\delta &= \delta F(\delta x, \delta^2 t),
\end{align*}
\]

we see that \((u_\delta, F_\delta)\) is a solution of (5) and (6) with initial data \((\delta u_0(\delta x), \delta F_0(\delta x))\). The so-called critical space for the equations (5) and (6) is a function space of \((u, F)\) such that the norm in it is invariant under the scaling (8). The so-called self-similar solutions are solutions satisfying the scaling relation \( u(t, x) = u_\delta(t, x), F(t, x) = F_\delta(t, x) \) (for all \( \delta > 0, x \in \mathbb{R}^n \) and \( t \geq 0 \)). Obviously, if \((u, F)\) is a self-similar solution, then we must have

\[
\begin{align*}
  u_0(\lambda x) &= \lambda^{-1} u_0(x), \\
  F_0(\lambda x) &= \lambda^{-1} F_0(x).
\end{align*}
\]

Such initial data do not belong to any Lebesgue and Sobolev spaces due to their strong singularity at \( x = 0 \) as well as slow decay as \( |x| \to \infty \), however, they belong to some homogeneous Besov spaces with negative order. This is the reason why we study the problem (5)–(7) in the critical Besov space of negative-order. By making use of Chemin-Lerner’s time-space estimates of the heat equation and the Fourier localization technique, we shall prove that when initial data \((u_0, F_0)\) belongs to the critical Besov space \( B_{p,q}^{-1+n/p}(\mathbb{R}^n) \) for some suitable \( p \) and \( q \), there exists a unique local solution, and this solution is global when initial data is small enough. As a corollary, we get existence of self-similar solutions; see Section 3. In order to figure out the relation between the solution obtained here and the weak solution studied by Lin and Liu in [14], we shall prove a stability result, which yields in a direct way that all global weak solutions with the same initial data must coincide with solutions obtained here, which is called the weak-strong uniqueness.

Our main results are as follows (for notations we refer the reader to see Section 2):

**Theorem 1.1** (Existence). Let \( n \geq 2, 2 \leq p < 2n \) and \( 1 \leq r \leq \infty \). Suppose that \((u_0, F_0) \in B_{p,r}^{-1+n/p}(\mathbb{R}^n) \) and \( \|u_0\| = 0 \). Then there exists \( T > 0 \) and a unique solution \((u, F)\) of the Cauchy problem (5)–(7) such that

\[
(u, F) \in \bigcap_{1 < q \leq \infty} L^q(0, T; \dot{B}_{p,r}^{-1+n/p+2/q}(\mathbb{R}^n)).
\]

Moreover, there exists \( \varepsilon > 0 \) such that if \( \|u_0, F_0\|_B^{1+n/p} \leq \varepsilon \), then the above assertion holds for \( T = \infty \), i.e., the solution \((u, F)\) is global. Furthermore, if \((u, F)\) and \((\tilde{u}, \tilde{F})\) are two solutions of (5)–(7) with initial conditions \((u_0, F_0)\) and \((\tilde{u}_0, \tilde{F}_0)\), respectively, \( \|u_0, F_0\| = 0 \) and \( \|\tilde{u}_0, \tilde{F}_0\| = 0 \), then there exists a constant \( C > 0 \) such that

\[
\|u - \tilde{u}, F - \tilde{F}\|_{L^q(0, T; \dot{B}_{p,r}^{-1+n/p+2/q})} \leq C\|u_0 - \tilde{u}_0, F_0 - \tilde{F}_0\|_{\dot{B}_{p,r}^{1+n/p}}.
\]

**Remark 1.** (i) If \((u_0, F_0)\) belongs to the closure of \( S(\mathbb{R}^n) \) in \( B_{p,q}^{-1+n/p}(\mathbb{R}^n) \), we actually have \((u, F) \in C([0, T], \dot{B}_{p,q}^{-1+n/p}(\mathbb{R}^n))\).

(ii) Since the appearance of the term \( F \nabla u \) in (6), we need the condition \( 2 \leq p < 2n \). For the Navier-Stokes equations, similar results hold for all \( 2 \leq p < \infty \).
Let Corollary 1. 

deduce the existence of self-similar solutions of (5)–(7). 

However, when we consider the problem (5)–(7) in the critical Besov space $B^{-1+n/p}_{p,q}({\mathbb{R}}^n)$, these $L^\infty$ estimates cannot be obtained. We shall use the Chemin-Lerner’s time-space estimates of the heat equation and the Fourier localization technique to prove Theorem 1.1.

Now by the standard procedure, based on the uniqueness of solutions, we can deduce the existence of self-similar solutions of (5)–(7).

**Corollary 1.** Let $n \geq 2$, $n \leq p < 2n$. Suppose that $(u_0, F_0) \in \dot{B}^{-1+n/p}_{p,\infty}({\mathbb{R}}^n)$ and div $u_0 = 0$, and furthermore, $u_0$ and $F_0$ are homogeneous functions with degree $-1$, i.e., 

$$u_0(x) = \delta u_0(\delta x), \quad F_0(x) = \delta F_0(\delta x).$$

Then the global solution $(u, F)$ constructed in Theorem 1.1 is a self-similar solution.

**Theorem 1.2** (Blow-up criterion). Under the hypotheses of Theorem 1.1, we denote by $T^{*}$ the maximum existence time. If $T^{*} < \infty$, then for any $2 \leq p < 2n$ and $2 < q < \infty$ satisfying $n/p + 2/q > 3/2$, we have

$$\| (u, F) \|_{L^q(0,T^{*}; \dot{B}^{-1+n/p+2/q}_{p,q})} = \infty.$$

For given $(u_0, F_0) \in \dot{B}^{-1+n/p}_{p,q}({\mathbb{R}}^n)$, by Theorem 1.1, there exists $T > 0$ such that system (5)–(7) has a unique solution $(u, F) \in \cap_{1 < q \leq \infty} L^q(0, T; \dot{B}^{-1+n/p+2/q}_{p,q}({\mathbb{R}}^n)).$

If $(u_0, F_0)$ is additionally in the space $L^2({\mathbb{R}}^n)$, then it is not difficult to see that $(u, F)$ is also a weak solution (in the standard sense, see [14]). A natural question is the following: Do all weak solutions coincide with the one we obtained in Theorem 1.1? In order to answer this question, we establish the following stability theorem.

**Theorem 1.3** (Stability). Let $n \geq 2$, $2 \leq p < \infty$, $2 < q < \infty$ and $n/p + 2/q > 1$. Assume that $(u_0, F_0)$ and $(\tilde{u}_0, \tilde{F}_0)$ be two vector fields in $L^2({\mathbb{R}}^n)$ such that div $u_0 = 0$ and div $\tilde{u}_0 = 0$, and $(\tilde{u}, \tilde{F})$ and $(u, F)$ be two weak solutions associated with initial conditions $(\tilde{u}_0, \tilde{F}_0)$ and $(u_0, F_0)$, respectively. Let $w = u - \tilde{u}$, $E = F - \tilde{F}$. If we assume further that $(u, F) \in L^q(0, T; \dot{B}^{-1+n/p+2/q}_{p,q}({\mathbb{R}}^n))$, then for any $0 < t < T$,

$$\| (w, E) \|_{L^q_t L^2_x}^2 + 2 \int_0^t \| (\nabla w, \nabla E) \|_{L^2_x}^2 \, d\tau \leq \| (w_0, E_0) \|_{L^q_t L^2_x}^2 \times \exp \left( C \int_0^t \| (u, F) \|_{B^{-1+n/p+2/q}_{p,q}}^q \, d\tau \right),$$

where $w_0 = u_0 - \tilde{u}_0$ and $E_0 = F_0 - \tilde{F}_0$, and $C > 0$ is a constant.

It is clear that if $u_0 = \tilde{u}_0$ and $F_0 = \tilde{F}_0$, then Theorem 1.3 implies that $w = E = 0$, i.e., $u = \tilde{u}$ and $F = \tilde{F}$. Hence, we have the following weak-strong uniqueness result for the system (5)–(7).

**Corollary 2** (Weak-strong uniqueness). Let $n \geq 2$, $2 \leq p < \infty$, $2 < q < \infty$ and $n/p + 2/q > 1$. Assume that $(u_0, F_0) \in L^2({\mathbb{R}}^n)$ (div $u_0 = 0$), and $(u, F)$ be a weak solution of the problem (5)–(7) with initial data $(u_0, F_0)$. If we assume further that $(u, F) \in L^q(0, T; \dot{B}^{-1+n/p+2/q}_{p,q}({\mathbb{R}}^n))$, then all weak solutions associated with initial data $(u_0, F_0)$ must coincide with $(u, F)$ on the time interval $[0, T).$
Remark 2. For initial data \((u_0, F_0) \in \dot{B}^{-1+\frac{n}{p}}_{p,r}(\mathbb{R}^n)\), when \(2 \leq p < 2n, 1 \leq r < \infty\) such that \(\frac{n}{p} + \frac{2}{r} > 1\), by Theorem 1.1, there exists \(2 < q < \infty\) such that \(\frac{n}{p} + \frac{2}{q} > 1\), and the system (5)–(7) has a unique solution \((u, F) \in L^q(0, T; \dot{B}^{-1+\frac{n}{p}+\frac{2}{q}}_{p,q}(\mathbb{R}^n))\). Hence, in this case, by Corollary 2, weak-strong uniqueness holds for the system (5)–(7).

This paper is organized as follows. In Section 2, we recall some basic facts about the Littlewood-Paley decomposition and Besov spaces. In Section 3, we present the proof of Theorem 1.1, which yields existence of global self-similar solutions. In Section 4, we prove Theorem 1.2. Section 5 is devoted to the proof of Theorem 1.3.

2. Preliminaries. We first recall some basic notions and preliminary results used in the proof of our main results. Let \(S(\mathbb{R}^n)\) be the Schwartz space and \(S'(\mathbb{R}^n)\) be its dual. Given \(f \in S(\mathbb{R}^n)\), the Fourier transform of it, \(\hat{f}(\xi)\), is defined by

\[\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx.\]

Let \(D_1 = \{\xi \in \mathbb{R}^n, |\xi| < \frac{1}{4}\}\) and \(D_2 = \{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}\). Choose two non-negative functions \(\phi, \psi \in S(\mathbb{R}^n)\) supported, respectively, in \(D_1\) and \(D_2\) such that

\[\psi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^n,\]

\[\sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.\]

We denote \(\phi_j(\xi) = \phi(2^{-j} \xi), h = \mathcal{F}^{-1}\phi\) and \(\tilde{h} = \mathcal{F}^{-1}\psi\), where \(\mathcal{F}^{-1}\) is the inverse Fourier transform. Then the dyadic blocks \(\Delta_j\) and \(S_j\) can be defined as follows:

\[\Delta_j f = \phi(2^{-j} D) f = 2^{jn} \int_{\mathbb{R}^n} h(2^j y) f(x - y) dy,\]

\[S_j f = \psi(2^{-j} D) f = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y) f(x - y) dy.\]

Here \(D = (D_1, D_2, \ldots, D_n)\) and \(D_i = i^{-1} \partial_{x_i} (i^2 = -1)\). The set \(\{\Delta_j, S_j\}_{j \in \mathbb{Z}}\) is called the Littlewood-Paley decomposition. Formally, \(\Delta_j = S_j - S_{j-1}\) is a frequency projection to the annulus \(|\xi| \sim 2^j\), and \(S_j = \sum_{k \leq j-1} \Delta_k\) is a frequency projection to the ball \(|\xi| \leq 2^j\). For more details, please refer to [1] and [9]. Let \(Z(\mathbb{R}^n) = \{f \in S(\mathbb{R}^n) : \mathcal{F} f(0) = 0, \forall \alpha \in (\mathbb{N} \cup \{0\})^n\}\), and denote by \(Z'(\mathbb{R}^n)\) the dual of it.

Definition 2.1. Let \(s \in \mathbb{R}, (p, r) \in [1, \infty] \times [1, \infty]\), the homogeneous Besov space \(\dot{B}^s_{p,r}(\mathbb{R}^n)\) is defined by

\[\dot{B}^s_{p,r}(\mathbb{R}^n) = \left\{ f \in Z'(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,r}} < \infty \right\},\]

where

\[\|f\|_{\dot{B}^s_{p,r}} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\Delta_j f\|_{L^p} \right)^{1/r} \quad \text{for} \ 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\Delta_j f\|_{L^p} \quad \text{for} \ r = \infty. \end{cases}\]

Remark 3. The above definition does not depend on the choice of the couple \((\phi, \psi)\). Recall that if either \(s < \frac{n}{p}\) or \(s = \frac{n}{p}\) and \(q = 1\), then \((\dot{B}^s_{p,q}(\mathbb{R}^n), \|\cdot\|_{\dot{B}^s_{p,q}})\) is a Banach space.
Let us now state some basic properties for the homogeneous Besov spaces.

**Lemma 2.2** (Bernstein’s inequality [1]). Let $k \in \mathbb{Z}^+$. There exists a constant $C$ independent of $f$ and $j$ such that for all $1 \leq p \leq q \leq \infty$, the following estimate holds:

$$\text{supp } \hat{f} \subset \{|\xi| \leq 2^j\} \implies \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \leq C 2^{j(k+\min(1/p-1/q))}\|f\|_{L^p}. \quad (9)$$

**Lemma 2.3** ([1]). Let $k \in \mathbb{Z}^+$ and $|\alpha| = k$ for multi-index $\alpha$. There exists a constant $C_k$ depending only on $k$ such that for all $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the following estimate holds:

$$C_k^{-1}\|\partial^\alpha f\|_{\dot{B}^s_{p,r}} \leq \|f\|_{\dot{B}^s_{p,r}+k} \leq C_k\|\partial^\alpha f\|_{\dot{B}^s_{p,r}}. \quad (10)$$

We now recall the definition of the Chemin-Lerner space $\mathcal{L}^q(0,T;\dot{B}^s_{p,r}(\mathbb{R}^n))$:

**Definition 2.4.** Let $s \in \mathbb{R}$, $1 \leq p, q, r \leq \infty$, and let $T > 0$ be a fixed number, the space $\mathcal{L}^q(0,T;\dot{B}^s_{p,r}(\mathbb{R}^n))$ is defined by

$$\mathcal{L}^q(0,T;\dot{B}^s_{p,r}(\mathbb{R}^n)) := \left\{ f \in \mathcal{S}'((0,T),\mathcal{S}'(\mathbb{R}^n)) : \|f\|_{\mathcal{L}^q(0,T;\dot{B}^s_{p,r})} < \infty \right\},$$

where

$$\|f\|_{\mathcal{L}^q(0,T;\dot{B}^s_{p,r})} = \left( \sum_{j \in \mathbb{Z}} 2^{jsr}\|\Delta_j f\|_{L^q(0,T;L^r)} \right)^{1/r}. \quad (11)$$

**Remark 4.** (i) We define the usual space $L^q(0,T;\dot{B}^s_{p,r}(\mathbb{R}^n))$ equipped with the norm

$$\|f\|_{L^q(0,T;\dot{B}^s_{p,r})} = \left( \int_0^T \left( \sum_{j \in \mathbb{Z}} 2^{jsr}\|\Delta_j f\|_{L^r}^r \right)^{q/r} dt \right)^{1/q}. \quad (12)$$

(ii) By Minkowski’s inequality, it is readily to verify that

$$\|f\|_{\mathcal{L}^q(0,T;\dot{B}^s_{p,r})} \leq \|f\|_{L^q(0,T;\dot{B}^s_{p,r})} \quad \text{if} \quad q \leq r, \quad (13)$$

$$\|f\|_{L^q(0,T;\dot{B}^s_{p,r})} \leq \|f\|_{\mathcal{L}^q(0,T;\dot{B}^s_{p,r})} \quad \text{if} \quad r \leq q. \quad (14)$$

The following product between functions will enable us to estimate nonlinear terms appeared in (5) and (6).

**Lemma 2.5** ([2], [17]). Let $1 \leq p, q, r, q_1, q_2 \leq \infty$, $s_1, s_2 < \frac{n}{p}$, $s_1+s_2 > 0$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then there exists a positive constant $C$ depending only on $s_1, s_2, p, q, r, q_1, q_2$ and $n$ such that

$$\|fg\|_{\mathcal{L}^q(0,T;\dot{B}^{s_1}_{p,r}+s_2^{q/q_1}/r)} \leq C \|f\|_{\mathcal{L}^{q_1}(0,T;\dot{B}^{s_1}_{p,r})} \|g\|_{\mathcal{L}^{q_2}(0,T;\dot{B}^{s_2}_{p,r})}. \quad (15)$$

**Notations:** The product of Banach spaces $\mathcal{X} \times \mathcal{Y}$ will be equipped with the usual norm $\|(f,g)\|_{\mathcal{X} \times \mathcal{Y}} = \|f\|_{\mathcal{X}} + \|g\|_{\mathcal{Y}}$, and if $\mathcal{X} = \mathcal{Y}$, we use $\|(f,g)\|_{\mathcal{X}}$ to denote by $\|(f,g)\|_{\mathcal{X} \times \mathcal{X}}$. For two $n \times n$ matrices $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$, we denote $A : B = \sum_{i,j=1}^n a_{ij}b_{ij}$. Throughout the paper, $C$ stands for a generic constant, and its value may change from line to line.
3. Local and global existence of solution. In this section we prove Theorem 1.1. Let $p$ and $r$ be as in Theorem 1.1, i.e., $2 \leq p < 2n$ and $1 \leq r \leq \infty$, and let $1 < q \leq \infty$. We choose a number $2 < q_1 \leq 2q$ such that $\frac{2}{q_1} + \frac{n}{r} > \frac{3}{2}$. For a constant $T > 0$ to be chosen later, we denote $X_T = \mathcal{L}^q(0, T; \dot{B}^{1+n/p+2/q_1}_p(R^n))$. In order to establish the desired estimates in $X_T$, let us recall the solvability of the Cauchy problem of the heat equation:

\[
\begin{align*}
\partial_t u - \Delta u &= f(x,t), \quad x \in \mathbb{R}^n, \ t > 0, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n.
\end{align*}
\]  

Proposition 1 ([2]). Let $s \in \mathbb{R}$ and $1 \leq p, q < \infty$, and let $T > 0$ be a real number. Assume that $u_0 \in \dot{B}^{s}_{p,r}(\mathbb{R}^n)$ and $f \in \mathcal{L}^{q_1}(0, T; \dot{B}^{s+2/q_1-2}_{p,r}(\mathbb{R}^n))$. Then the Cauchy problem (14) has a unique solution

\[
u \in \bigcap_{q_1 \leq q \leq \infty} \mathcal{L}^q(0, T; \dot{B}^{s+2/q}(\mathbb{R}^n)).
\]

Moreover, there exists a constant $C > 0$ depending only on $n$ such for any $q_1 \leq q \leq \infty$,

\[
\|u\|_{\mathcal{L}^q(0,T;\dot{B}^{s+2/q}_p)} \leq C \left(\|u_0\|_{\dot{B}^{s}_{p,r}} + \|f\|_{\mathcal{L}^{q_1}(0,T;\dot{B}^{s+2/q_1-2}_p)}\right).
\]  

Furthermore, if $u_0$ belongs to the closure of $\mathcal{S}(\mathbb{R}^n)$ in $\dot{B}^{s}_{p,r}(\mathbb{R}^n)$, then we have $u \in C([0, T], \dot{B}^{s}_{p,r}(\mathbb{R}^n))$.

We also recall an existence and uniqueness result for an abstract operator equation in a generic Banach space. For the proof we refer the reader to see Lemarié-Rieusset [9].

Proposition 2 ([9]). Let $\mathcal{X}$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a bilinear bounded operator, $\|\cdot\|_{\mathcal{X}}$ being the $\mathcal{X}$-norm. Assume that for any $u_1, u_2 \in \mathcal{X}$, we have

\[
\|B(u_1, u_2)\|_{\mathcal{X}} \leq C_0 \|u_1\|_{\mathcal{X}} \|u_2\|_{\mathcal{X}}.
\]

Then for any $y \in \mathcal{X}$ such that $\|y\|_{\mathcal{X}} \leq \varepsilon < \frac{1}{4C_0}$, the equation $u = y + B(u, u)$ has a solution $u$ in $\mathcal{X}$. Moreover, this solution is the only one such that $\|u\|_{\mathcal{X}} \leq 2\varepsilon$ and depends continuously on $y$ in the following sense: if $\|y\|_{\mathcal{X}} \leq \varepsilon$, $\bar{u} = \bar{y} + B(\bar{u}, \bar{u})$ and $\|\bar{u}\|_{\mathcal{X}} \leq 2\varepsilon$, then

\[
\|u - \bar{u}\|_{\mathcal{X}} \leq \frac{1}{1 - 4\varepsilon C_0} \|y - \bar{y}\|_{\mathcal{X}}.
\]

Now for given $(u, F) \in \mathcal{X}_T$, we define by $G(u, F) = (\bar{u}, \bar{F})$, where $(\bar{u}, \bar{F})$ is a solution of the following equations:

\[
\begin{align*}
\partial_t \bar{u} - \Delta \bar{u} &= -F \cdot \nabla u - \operatorname{div}(F^T F), \\
\partial_t \bar{F} - \Delta \bar{F} &= -u \cdot \nabla F - F \nabla u, \\
(\bar{u}, \bar{F})|_{t=0} &= (u_0, F_0).
\end{align*}
\]  

Proposition 3. Let $(u, F) \in \mathcal{X}_T$. Then we have $(\bar{u}, \bar{F}) \in \mathcal{X}_T$. In addition, the following estimates hold:

\[
\begin{align*}
\|\bar{u}\|_{\mathcal{X}_T} &\leq \|e^{t\Delta} u_0\|_{\mathcal{X}_T} + C(\|u\|^2_{\mathcal{X}_T} + \|F\|^2_{\mathcal{X}_T}), \\
\|\bar{F}\|_{\mathcal{X}_T} &\leq \|e^{t\Delta} F_0\|_{\mathcal{X}_T} + C\|u\|_{\mathcal{X}_T}\|F\|_{\mathcal{X}_T}.
\end{align*}
\]
Lemma 2.5 by choosing $\mu$ technique to obtain existence of local solution of the problem (5)–(7). To this end, in this case we shall use the Fourier localization

**Case 2.**

We prove only the results for $\bar{u}$, it can be done analogous for $\bar{F}$. By the Duhamel principle, (16) can be transformed into the following equivalent integral equations:

$$
\bar{u}(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}P(u \cdot \nabla u)(\tau)d\tau - \int_0^t e^{(t-\tau)\Delta}\text{div } (FTF)(\tau)d\tau.
$$

Since we have assumed $2 \leq p < 2n, 2 < q_1 < \infty$ and $\frac{2}{p} + \frac{2}{q_1} > \frac{3}{2}$, we can apply Lemma 2.5 by choosing $s_1 = -1 + \frac{2}{p} + \frac{2}{q_1}$ and $s_2 = -2 + \frac{2}{p} + \frac{2}{q_1}$ and Lemma 2.3 to obtain that

$$
\|u \cdot \nabla u\|_{L^{q_1/2}(0,T;B^{s_1+n/p+4/q_1}_{p,r})} \leq C\|u\|_{L^{q_1}(0,T;B^{-2n/p+2/q_1}_{p,r})} \|\nabla u\|_{L^{q_1}(0,T;B^{-2n/p+2/q_1}_{p,r})} \leq C\|u\|^2_{L^{q_1}(0,T;B^{-1+n/p+2/q_1}_{p,r})}. \tag{21}
$$

Similarly, by Lemmas 2.3 and 2.5,

$$
\|\text{div}(FTF)\|_{L^{q_1/2}(0,T;B^{-3n/p+4/q_1}_{p,r})} \leq C\|FTF\|_{L^{q_1/2}(0,T;B^{-2n/p+4/q_1}_{p,r})} \leq C\|F\|^2_{L^{q_1}(0,T;B^{-1+n/p+2/q_1}_{p,r})}. \tag{22}
$$

Hence, from Proposition 1 we know $u \in X_T$. Moreover, using the boundedness of $P$ in the homogeneous Besov spaces, (21) and (22), we get

$$
\|\bar{u}\|_{X_T} \leq \|e^{t\Delta}u_0\|_{X_T} + C\|u \cdot \nabla u\|_{L^{q_1/2}(0,T;B^{-3n/p+4/q_1}_{p,r})} + C\|\text{div } (FTF)\|_{L^{q_1/2}(0,T;B^{-3n/p+4/q_1}_{p,r})} \leq \|e^{t\Delta}u_0\|_{X_T} + C(\|u\|^2_{X_T} + \|F\|^2_{X_T}). \tag{23}
$$

This proves Proposition 3.

\[ \square \]

The Proposition 3 implies that $\mathcal{G}$ is well-defined and maps $X_T$ into itself. Moreover, from (19) and (20), we know there exists a constant $C_0 > 0$ such that for all $(u, F) \in X_T$ and $(\bar{u}, \bar{F}) = \mathcal{G}(u, F)$, we have the following estimate:

$$
\|\bar{u}, \bar{F}\|_{X_T} \leq \|(e^{t\Delta}u_0, e^{t\Delta}F_0)\|_{X_T} + C_0\|u, F\|^2_{X_T}. \tag{24}
$$

**Case 1.** (The small initial data) Taking $T = \infty$ and denoting $X = X_{\infty}$. From Proposition 1, there exists a constant $C_1$ such that we can rewrite (24) as follows:

$$
\|\bar{u}, \bar{F}\|_{X_T} \leq C_1\|u_0, F_0\|_{B^{-1+n/p}_{p,r}} + C_0\|u, F\|^2_{X}. \tag{25}
$$

Now if we choose $\varepsilon > 0$ sufficiently small such that $C_1\|u_0, F_0\|_{B^{-1+n/p}_{p,r}} \leq \varepsilon < \frac{1}{4C_0}$, that is to say, $\|(u_0, F_0)\|_{B^{-1+n/p}_{p,r}} \leq \frac{\varepsilon}{C_1} < \frac{1}{4C_0}$, then by Proposition 2, the system (5)–(7) has a global solution.

**Case 2.** (The large initial data) In this case we shall use the Fourier localization technique to obtain existence of local solution of the problem (5)–(7). To this end, we split $u_0 = u_{01} + u_{02}$ such that $\tilde{u}_0(\xi) = \tilde{u}_{01}(\xi) + \tilde{u}_{02}(\xi)$, where $1_p$ represents the characteristic function on the domain $\mathcal{D}$. Similarly, we split $F_0 = F_{01} + F_{02}$. Since $2 \leq p < 2n$, by using the properties of the Besov spaces, there exists $N \in \mathbb{Z}^+$ such that $C_1\|(u_{01}, F_{01})\|_{B^{-1+n/p}_{p,r}} \leq \frac{\varepsilon}{2}$, we see that

$$
\|(e^{t\Delta}u_0, e^{t\Delta}F_0)\|_{X_T} \leq \frac{\varepsilon}{2} + \|(e^{t\Delta}u_{02}, e^{t\Delta}F_{02})\|_{X_T}. \tag{26}
$$
Applying Bernstein’s inequality, there exists a constant $C_2$ such that
\[
\| (e^{t\Delta}u_{02}, e^{t\Delta}F_{02}) \|_{\mathcal{X}_T} = \| (e^{t\Delta}u_{02}, e^{t\Delta}F_{02}) \|_{L^{q_1}(0,T;\dot{B}^{-1+n/p+2/q}_{p,r})} \\
\leq C_2^{2(2N)/q_1,1} \| (e^{t\Delta}u_{02}, e^{t\Delta}F_{02}) \|_{L^{q_1}(0,T;\dot{B}^{-1+n/p}_{p,r})} \\
\leq C_2^{2(2N)/q_1,1} T^{1/q_1} \| (u_0, F_0) \|_{\dot{B}^{-1+n/p}_{p,r}}.
\]
Hence, if we choose $T$ small enough so that $C_2^{2(2N)/q_1,1} T^{1/q_1} \| (u_0, F_0) \|_{\dot{B}^{-1+n/p}_{p,r}} \leq \frac{\varepsilon}{2}$, i.e.,
\[
T \leq \left( C_2^{2(2N)/q_1,1} \| (u_0, F_0) \|_{\dot{B}^{-1+n/p}_{p,r}} \right)^{q_1},
\]
then we have $\| (e^{t\Delta}u_{02}, e^{t\Delta}F_{02}) \|_{\mathcal{X}_T} \leq \frac{\varepsilon}{2}$. This result together with (26) yield the fact that for such $T$ defined by (27), we have $\| (e^{t\Delta}u_0, e^{t\Delta}F_0) \|_{\mathcal{X}_T} \leq \varepsilon$. By applying Proposition 2 again, there exists a local solution to the system (5)–(7).

If $(u, F) \in \mathcal{X}_T$ is a solution of the system (5)–(7), then one can proceed the same way as the proof of Proposition 3 to obtain that
\[
P u \cdot \nabla u, \quad \text{P} \text{div} (F^T F), \quad u \cdot \nabla F, \quad F \nabla u \in \mathcal{L}^{q_1/2}(0,T;\dot{B}^{-3+n/p+4/q_1}_{p,r}(\mathbb{R}^n)).
\]
Hence, for any $\frac{2}{q} \leq q \leq \infty$, we have
\[
(u, F) \in \mathcal{L}^{q}(0,T;\dot{B}^{-1+n/p}_{p,r}(\mathbb{R}^n)).
\]
Moreover, if $(u_0, F_0)$ belongs to the closure of $\mathcal{S}(\mathbb{R}^n)$ in $\dot{B}^{-1+n/p}_{p,r}(\mathbb{R}^n)$, then we have $(u, F) \in C([0,T],\dot{B}^{-1+n/p}_{p,r}(\mathbb{R}^n))$.

Finally, we consider the uniqueness of solution. Note that in Proposition 2 we obtained only a partial answer to the uniqueness problem of solution, i.e., in the closed ball $B_{2T}$, the solution of (5)–(7) is unique. Now we intend to get rid of this restrictive condition. Let $(u, F)$ and $(\tilde{u}, \tilde{F})$ be two solutions of (5)–(7) in $\mathcal{X}_T$ associated with initial data $(u_0, F_0)$ and $(\tilde{u}_0, \tilde{F}_0)$, respectively. Set $w = u - \tilde{u}$ and $E = F - \tilde{F}$. Then $(w, E)$ satisfies the following equations:
\[
\begin{cases}
\partial_t w - \Delta w + w \cdot \nabla u + \tilde{u} \cdot \nabla \tilde{w} + \nabla \cdot (E^T F) + \tilde{E} \cdot (\tilde{F}^T E) = 0, \\
\partial_t E - \Delta E + w \cdot \nabla F + \tilde{u} \cdot \nabla E + E \nabla u + \tilde{F} \nabla w = 0,
\end{cases}
\]
\[
w(x,0) = w_0(x) = u_0(x) - \tilde{u}_0(x), \quad E(x,0) = E_0(x) = F_0(x) - \tilde{F}_0(x).
\]
As the proof of Proposition 3, we can prove that
\[
\| w \cdot \nabla u + \tilde{u} \cdot \nabla \tilde{w} + \nabla \cdot (E^T F) + \nabla \cdot (\tilde{F}^T E) \|_{\mathcal{X}_T} \\
\leq C\left( \| u \|_{\mathcal{X}_T} + \| \tilde{u} \|_{\mathcal{X}_T} + \| F \|_{\mathcal{X}_T} + \| \tilde{F} \|_{\mathcal{X}_T} \right) \|(w, E)\|_{\mathcal{X}_T}
\]
and
\[
\| w \cdot \nabla F + \tilde{u} \cdot \nabla E + E \nabla u + \tilde{F} \nabla w \|_{\mathcal{X}_T} \\
\leq C\left( \| u \|_{\mathcal{X}_T} + \| \tilde{u} \|_{\mathcal{X}_T} + \| F \|_{\mathcal{X}_T} + \| \tilde{F} \|_{\mathcal{X}_T} \right) \|(w, E)\|_{\mathcal{X}_T}.
\]
Hence, by Proposition 1, we get
\[
\| (w, E) \|_{\mathcal{X}_T} \leq C_1 \| (w_0, E_0) \|_{\dot{B}^{-1+n/p}_{p,r}} + C_0 \left( \| u \|_{\mathcal{X}_T} + \| \tilde{u} \|_{\mathcal{X}_T} + \| F \|_{\mathcal{X}_T} + \| \tilde{F} \|_{\mathcal{X}_T} \right) \|(w, E)\|_{\mathcal{X}_T}.
\]
Denoting $M(T) := C_0 \left( \| u \|_{\mathcal{X}_T} + \| \tilde{u} \|_{\mathcal{X}_T} + \| F \|_{\mathcal{X}_T} + \| \tilde{F} \|_{\mathcal{X}_T} \right)$, by the Lebesgue dominated convergence theorem, we know that $M(T)$ is a continuous nondecreasing
Theorem 1.2. Let \( M(T_1) \leq \frac{1}{2} \), then
\[
\|(w, E)\|_{X_{T_1}} \leq 2C_1\|(w_0, E_0)\|_{B_{p,r}^{-1+n/p}}.
\]  
(28)

Repeating the above procedure to the interval \([0, T_1), [T_1, 2T_1), \ldots\) allows us to conclude that there exists a constant \( C \) such that
\[
\|(w, E)\|_{X_T} \leq C\|(w_0, E_0)\|_{B_{p,r}^{-1+n/p}}.
\]  
(29)

This implies the uniqueness result immediately.

4. The proof of Theorem 1.2. Let \( 2 \leq p < 2n, 1 \leq r \leq \infty, 2 < q < \infty \) such that \( \frac{n}{p} + \frac{2}{q} > \frac{3}{2} \). Assume that \( \| (u, F) \|_{L^\infty(0,T;B_{p,r}^{-1+n/p+2/q})} < \infty \). By the embedding relation (12) and the proof of Theorem 1.1 we see that
\[
\|(u, F)\|_{L^\infty(0,T;B_{p,r}^{-1+n/p})} \leq C_1\|(u_0, F_0)\|_{B_{p,r}^{-1+n/p}} + C_0\|(u, F)\|_{L^\infty(0,T;B_{p,r}^{-1+n/p+2/q})} = M < \infty.
\]

It suffices to prove that if \( \| (u, F)\|_{L^\infty(0,T;B_{p,r}^{-1+n/p+2/q})} < \infty \), then \( T^* > T \). In other words, if \( T^* < \infty \), then \( \| (u, F)\|_{L^\infty(0,T;B_{p,r}^{-1+n/p+2/q})} = \infty \). To this end, for any \( t \in [0, T) \), we take \( (u(x, t), F(x, t)) \) as an initial data of the problem (5)–(7), and split \( u(x, t) = u_1(x, t) + u_2(x, t) \) such that
\[
\hat{u}(\xi, t) = \hat{u}_1(\{\xi| > 2N\}(\xi, t) + \hat{u}_1(\{\xi| \leq 2N\}(\xi, t) := \hat{u}_1(\xi, t) + \hat{u}_2(\xi, t).
\]

Similarly, we split \( F(x, t) := F_1(x, t) + F_2(x, t) \) since \( 2 \leq p < 2n \), by using the properties of the Besov spaces, there exists a sufficiently large constant \( N \in \mathbb{N} \) such that
\[
C_1\|(u_1(x, t), F_1(x, t))\|_{B_{p,r}^{-1+n/p}} \leq \frac{\varepsilon}{2}.
\]  
(30)

On the other hand, if we choose \( \hat{T} > t \) such that
\[
\hat{T} - t \leq \left( \frac{\varepsilon}{C_22^{1+(2N)/q}M} \right)^q := T_\varepsilon,
\]

then we can obtain \( \|(e^{\varepsilon^2u_2}, e^{\varepsilon^2F_2})\|_{X_{T_\varepsilon}} \leq \frac{\varepsilon}{2} \). This result together with (30), by Proposition 2, yield that there exists a constant \( T_\varepsilon \) depending only on \( \varepsilon \) and \( M \) such that for any \( t \in (0, T) \), the problem (5)–(7) has a solution on the time interval \([t, t + T_\varepsilon]\). By the uniqueness we know that all solutions obtained in this way are equal in their common existence interval, so that the solution can be extended to the time interval \([0, T + T_\varepsilon]\). That is to say \( T^* > T \), we complete the proof of Theorem 1.2.

5. Stability and weak-strong uniqueness. The aim of this section is to prove Theorem 1.3. Let us recall the definition of weak solutions to the system (5)–(7).

**Definition 5.1.** The vector-valued function \((u, F)\) is called a weak solution of (5)–(7) on \( \mathbb{R}^n \times (0, T) \) if it satisfies the following conditions:

1. \((u, F) \in L^\infty(0,T;L^2(\mathbb{R}^n)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^n)) := (WS)\), where \( \dot{H}^1(\mathbb{R}^n) = B_{2,2}^1(\mathbb{R}^n) \) is the usual homogeneous Sobolev space.
(2) \((u, F)\) satisfies the system (5)–(7) in the distributional sense, i.e., \(\text{div } u = 0\) in the distributional sense and for all \(v \in C_0^\infty(\mathbb{R}^n \times (0, T))\) and \(G \in C_0^\infty(\mathbb{R}^n \times (0, T))\) with \(\text{div } v = 0\), we have

\[
\int_0^T \int_{\mathbb{R}^n} u \partial_t v dx dt - \int_0^T \int_{\mathbb{R}^n} \nabla u : \nabla v dx dt - \int_0^T \int_{\mathbb{R}^n} u \cdot \nabla u \cdot \nabla v dx dt = -\int_0^T \int_{\mathbb{R}^n} F^T F : \nabla v dx dt,
\]

and

\[
\int_0^T \int_{\mathbb{R}^n} F : \partial_t G dx dt - \int_0^T \int_{\mathbb{R}^n} \nabla F : \nabla G dx dt - \int_0^T \int_{\mathbb{R}^n} u \cdot \nabla F : G dx dt = \int_0^T \int_{\mathbb{R}^n} F \nabla u : G dx dt.
\]

(3) The following energy inequality holds:

\[
\int_{\mathbb{R}^n} (|u(t)|^2 + |F(t)|^2) dx + 2 \int_0^t \int_{\mathbb{R}^n} (|\nabla u|^2 + |\nabla F|^2) dx dt \leq \int_{\mathbb{R}^n} (|u_0|^2 + |F_0|^2) dx.
\]

Remark 5. Formally, taking \(v = u\) and \(G = F\), and adding them together, we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} (|u(t)|^2 + |F(t)|^2) dx + \int_{\mathbb{R}^n} (|\nabla u|^2 + |\nabla F|^2) dx = 0,
\]

which implies the above energy inequality. Here we have used the fact \(AB : C = A : CB^T = B : AT C\) for any three \(n \times n\) matrices \(A\), \(B\), and \(C\).

Let \((\tilde{u}_0, \tilde{F}_0) \in L^2(\mathbb{R}^n)\), \((u_0, F_0) \in L^2(\mathbb{R}^n)\), and we denote by \((\tilde{u}, \tilde{F})\) and \((u, F)\) two weak solutions in the space \(W_S\) associated with initial conditions \((\tilde{u}_0, \tilde{F}_0)\) and \((u_0, F_0)\), respectively. Assume that \((u, F) \in L^q(0, T; B_{p,q}^{-1+n/p+2/q}(\mathbb{R}^n))\), where \(2 \leq p < \infty\) and \(2 < q < \infty\) satisfying \(\frac{2}{q} + \frac{2}{q} > 1\). Obviously, the above energy inequality yields that

\[
\| (\tilde{u}(t), F(t)) \|^2_{L^2} + 2 \int_0^t \| (\tilde{\nabla} u(\tau), \tilde{\nabla} F(\tau)) \|^2_{L^2} d\tau \leq \| (\tilde{\nabla} u_0, \tilde{\nabla} F_0) \|^2_{L^2},
\]

and

\[
\| (u(t), F(t)) \|^2_{L^2} + 2 \int_0^t \| (\nabla u(\tau), \nabla F(\tau)) \|^2_{L^2} d\tau \leq \| (u_0, F_0) \|^2_{L^2}.\]

(31)

(32)

Let \(w = u - \tilde{u}, E = F - \tilde{F}, w_0 = u_0 - \tilde{u}_0\) and \(E_0 = F_0 - \tilde{F}_0\). To prove Theorem 1.3, it suffices to prove the following result:

Proposition 4. Under the hypotheses of Theorem 1.3, we have

\[
\| (w(t), E(t)) \|^2_{L^2} + 2 \int_0^t \| (\nabla w(\tau), \nabla E(\tau)) \|^2_{L^2} d\tau \leq \| (w_0, E_0) \|^2_{L^2} \times \exp \left( C \int_0^t \| (u(\tau), F(\tau)) \|^q_{B_{p,q}^{-1+n/p+2/q}(\mathbb{R}^n)} d\tau \right).
\]

(33)
Note that by (31) and (32),
\[
\|(w(t), E(t))\|_{L^2}^2 + 2 \int_0^t \| (\nabla w(\tau), \nabla E(\tau)) \|_{L^2}^2 d\tau = \|(u(t), F(t))\|_{L^2}^2 + \|(\tilde{u}(t), \tilde{F}(t))\|_{L^2}^2
\]
\[
+ 2 \int_0^t \| (\nabla u(\tau), \nabla F(\tau)) \|_{L^2}^2 d\tau + 2 \int_0^t \| (\nabla \tilde{u}(\tau), \nabla \tilde{F}(\tau)) \|_{L^2}^2 d\tau - 2(u(t)|\tilde{u}(t))
\]
\[
- 2(F(t)|\tilde{F}(t)) - 4 \int_0^t (\nabla u(\tau)|\nabla \tilde{u}(\tau)) d\tau - 4 \int_0^t (\nabla F(\tau)|\nabla \tilde{F}(\tau)) d\tau
\]
\[
\leq \|(u_0, F_0)\|_{L^2}^2 + \|(\tilde{u}_0, \tilde{F}_0)\|_{L^2}^2 - 2(u(t)|\tilde{u}(t)) - 2(F(t)|\tilde{F}(t))
\]
\[
- 4 \int_0^t (\nabla u(\tau)|\nabla \tilde{u}(\tau)) d\tau - 4 \int_0^t (\nabla F(\tau)|\nabla \tilde{F}(\tau)) d\tau.
\]

Here (·, ·) denotes the scalar product in $L^2(\mathbb{R}^n)$. In order to prove Proposition 4, we need to establish the following lemma.

**Lemma 5.2.** Under the hypothesis of Theorem 1.3, the following equality holds for all $t \leq T$,
\[
(u(t)|\tilde{u}(t)) + (F(t)|\tilde{F}(t)) + 2 \int_0^t (\nabla u(\tau)|\nabla \tilde{u}(\tau)) d\tau + 2 \int_0^t (\nabla F(\tau)|\nabla \tilde{F}(\tau)) d\tau
\]
\[
= (u_0|\tilde{u}_0) + (F_0|\tilde{F}_0) - \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla w \cdot u d\tau + \int_0^t \int_{\mathbb{R}^n} \tilde{F}^T \tilde{F} : \nabla u d\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^n} F^T \tilde{F} : \nabla u d\tau - \int_0^t \int_{\mathbb{R}^n} \nabla u : (\tilde{F}^T F + F^T \tilde{F}) d\tau
\]
\[
- \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla F : \tilde{F} d\tau + \int_0^t \int_{\mathbb{R}^n} F : \tilde{F} \nabla w d\tau.
\]

We shall use Lemma 1.1 in [5] to prove Lemma 5.2.

**Lemma 5.3** ([5]). Let $n \geq 2$, $2 \leq p < \infty$ and $2 < q < \infty$ such that $\frac{n}{p} + \frac{2}{q} > 1$. Then for every $T > 0$, the trilinear form
\[
(u, v, w) \in W S \times W S \times L^q(0, T; \dot{B}^{-1+n/p+2/q}_{p,q}(\mathbb{R}^n)) \mapsto \int_0^T \int_{\mathbb{R}^n} u \cdot \nabla v \cdot w dx dt
\]
is continuous. In particular, the following estimate holds:
\[
\left| \int_0^T \int_{\mathbb{R}^n} u \cdot \nabla v \cdot w dx dt \right|
\]
\[
\leq C \left\| u \right\|_{L^\infty(0, T; L^2)}^{2/q} \left\| \nabla u \right\|_{L^2(0, T; L^2)}^{1-2/q} \left\| \nabla v \right\|_{L^2(0, T; L^2)} \left\| w \right\|_{L^\infty(0, T; \dot{B}^{-1+n/p+2/q}_{p,q})}
\]
\[
+ \left\| u \right\|_{L^\infty(0, T; L^2)}^{2/q} \left\| \nabla u \right\|_{L^2(0, T; L^2)} \left\| \nabla v \right\|_{L^2(0, T; L^2)}^{1-2/q} \left\| w \right\|_{L^\infty(0, T; \dot{B}^{-1+n/p+2/q}_{p,q})}
\]
\[
+ \left\| u \right\|_{L^\infty(0, T; L^2)}^{1/q} \left\| \nabla u \right\|_{L^2(0, T; L^2)} \left\| v \right\|_{L^\infty(0, T; L^2)} \left\| \nabla v \right\|_{L^2(0, T; L^2)} \left\| w \right\|_{L^\infty(0, T; \dot{B}^{-1+n/p+2/q}_{p,q})}.
\]

**Remark 6.** Checking the proof of Lemma 1.1 in [5] in detail, we find that the special structure of the Navier-Stokes equations (div $u = 0$) was not used, and (35) holds in both scalar and vector cases.
The proof of Lemma 5.2. Let us consider two smooth sequences of \(\{\tilde{u}_n, \tilde{F}_n\}\) and \(\{u_n, \tilde{F}_n\}\) such that \(\text{div} \, \tilde{u}_n = 0\) and \(\text{div} \, u_n = 0\), and

\[
\begin{align*}
\lim_{n \to \infty} (\tilde{u}_n, \tilde{F}_n) &= (\tilde{u}, \tilde{F}) \quad \text{in} \quad L^2(0, T; \dot{H}^1(\mathbb{R}^n)), \\
\lim_{n \to \infty} (\tilde{u}_n, \tilde{F}_n) &= (\tilde{u}, \tilde{F}) \quad \text{weakly-star in} \quad L^\infty(0, T; L^2(\mathbb{R}^n))
\end{align*}
\]

and

\[
\begin{align*}
\lim_{n \to \infty} (u_n, F_n) &= (u, F) \quad \text{in} \quad L^2(0, T; \dot{H}^1(\mathbb{R}^n)) \cap L^q(0, T; \dot{B}_{p,q}^{-1+n/p+2/q}(\mathbb{R}^n)), \\
\lim_{n \to \infty} (u_n, F_n) &= (u, F) \quad \text{weakly-star in} \quad L^\infty(0, T; L^2(\mathbb{R}^n)).
\end{align*}
\]

We split the proof into the following two steps.

**Step 1.** Taking the scalar product with \(\tilde{u}_n\) and \(u_n\) of the equation (5) on \(u\) and \(\tilde{u}\) respectively, after integration in time and integration by parts in the space variables, we get

\[
\int_0^t \left( (\partial_\tau u|\tilde{u}_n) + (\nabla u|\nabla \tilde{u}_n) + (u \cdot \nabla u|\tilde{u}_n) + (\text{div} \,(F^T F)|\tilde{u}_n) \right) d\tau = 0 \quad (36)
\]

and

\[
\int_0^t \left( (\partial_\tau \tilde{u}|u_n) + (\nabla \tilde{u}|\nabla u_n) + (\tilde{u} \cdot \nabla \tilde{u}|u_n) + (\text{div} \,(F^T \tilde{F})|u_n) \right) d\tau = 0. \quad (37)
\]

Note that we have assumed that both \(\nabla u_n\) and \(\nabla \tilde{u}_n\) converge in \(L^2(0, T; L^2(\mathbb{R}^n))\) towards \(\nabla u\) and \(\nabla \tilde{u}\) respectively, it is obvious that

\[
\lim_{n \to \infty} \int_0^t (\nabla u|\nabla \tilde{u}_n) d\tau + \int_0^t (\nabla \tilde{u}|\nabla u_n) d\tau = 2 \int_0^t (\nabla u|\nabla \tilde{u}) d\tau. \quad (38)
\]

Since \(u_n\) converges to \(u\) in \(L^q(0, T; \dot{B}_{p,q}^{-1+n/p+2/q}(\mathbb{R}^n))\), by Lemma 5.3, one obtains that

\[
\lim_{n \to \infty} \int_0^t (\tilde{u} \cdot \nabla \tilde{u}_n|u_n) d\tau = \int_0^t (\tilde{u} \cdot \nabla \tilde{u}|u) d\tau. \quad (39)
\]

Due to the fact \(\text{div} \, u = 0\), this yields, by Lemma 5.3 again,

\[
\lim_{n \to \infty} \int_0^t (u \cdot \nabla u|\tilde{u}_n) d\tau = - \lim_{n \to \infty} \int_0^t (u \cdot \nabla \tilde{u}_n|u) d\tau = - \int_0^t (u \cdot \nabla \tilde{u}|u) d\tau. \quad (40)
\]

Applying (35), it is also obvious that the following two equalities hold:

\[
\lim_{n \to \infty} \int_0^t (\text{div} \, (F^T F)|\tilde{u}_n) d\tau = \lim_{n \to \infty} \int_0^t (\text{div} \, (F^T F)|\nabla \tilde{u}_n) d\tau
\]

\[
= - \int_0^t (\text{div} \, (F^T F)|\tilde{u}) d\tau = \int_0^t (\text{div} \, (F^T F)|\tilde{u}) d\tau. \quad (41)
\]

and

\[
\lim_{n \to \infty} \int_0^t (\text{div} \, (\tilde{F}^T \tilde{F})|u_n) d\tau = \lim_{n \to \infty} \int_0^t \left( \sum_{i=1}^n (\partial_{x_i} \tilde{F}^T \tilde{F} + \tilde{F}^T \partial_{x_i} \tilde{F})|u_n) d\tau
\]

\[
= \int_0^t \left( \sum_{i=1}^n (\partial_{x_i} \tilde{F}^T \tilde{F} + \tilde{F}^T \partial_{x_i} \tilde{F})|u) d\tau = \int_0^t (\text{div} \, (\tilde{F}^T \tilde{F})|u) d\tau. \quad (42)
\]
Since \( \partial_t \tilde{u} = \Delta \tilde{u} - F \cdot \nabla \tilde{u} - \text{div} (\tilde{F}^T \tilde{F}) \) holds in the sense of distribution, the estimates (38)–(42) imply in particular that
\[
\lim_{n \to \infty} \int_0^t (\partial_s \tilde{u}) u_n) d\tau = - \lim_{n \to \infty} \int_0^t \left( (\nabla \tilde{u} | \nabla u_n) + (\tilde{u} \cdot \nabla \tilde{u} | u_n) + (\text{div} (\tilde{F}^T \tilde{F}) | u_n) \right) d\tau \\
= - \int_0^t \left( (\nabla \tilde{u} | \nabla u) + (\tilde{u} \cdot \nabla \tilde{u} | u) + (\text{div} (\tilde{F}^T \tilde{F}) | u) \right) d\tau \\
= \int_0^t (\partial_s \tilde{u} | u) d\tau.
\]
Similarly, we have
\[
\lim_{n \to \infty} \int_0^t (\partial_s u | u_n) d\tau = \int_0^t (\partial_s u | u) d\tau.
\]
Putting these estimates together, and noticing that
\[
\int_0^t (\partial_s \tilde{u} | u) + (\partial_s u | \tilde{u}) d\tau = (u(t) | \tilde{u}(t)) - (u_0 | \tilde{u}_0)
\]
and
\[
\int_0^t \left( (\tilde{u} \cdot \nabla \tilde{u} | u) + (u \cdot \nabla u | u) \right) d\tau = \int_0^t (w \cdot \nabla w | u) d\tau,
\]
we find
\[
(u(t) | \tilde{u}(t)) + 2 \int_0^t (\nabla u | \nabla \tilde{u}) d\tau = (u_0 | \tilde{u}_0) - \int_0^t (w \cdot \nabla w | u) d\tau \\
+ \int_0^t \int_{\mathbb{R}^n} \tilde{F}^T \tilde{F} : \nabla u dxd\tau + \int_0^t \int_{\mathbb{R}^n} \tilde{F}^T F : \nabla u dxd\tau.
\]

**Step 2.** In this step we derive the estimate for \( F \) and \( \tilde{F} \). Proceeding in the same way as (36) and (37), we obtain that
\[
\int_0^t \left( (\partial_s F | \tilde{F}_n) + (\nabla F | \nabla \tilde{F}_n) + (u \cdot \nabla F | \tilde{F}_n) + (\tilde{F} \nabla u | \tilde{F}_n) \right) d\tau = 0
\]
and
\[
\int_0^t \left( (\partial_s \tilde{F} | F_n) + (\nabla \tilde{F} | \nabla F_n) + (\tilde{u} \cdot \nabla \tilde{F} | F_n) + (\tilde{F} \nabla \tilde{u} | F_n) \right) d\tau = 0.
\]

Since we have assumed that both \( \nabla F_n \) and \( \nabla \tilde{F}_n \) respectively converge to \( \nabla F \) and \( \nabla \tilde{F} \) in \( L^2(0, T; L^2(\mathbb{R}^n)) \) and \( F_n \) converge to \( F \) in \( L^q(0, T; B_{p,q}^{-1+n/p+2/q}(\mathbb{R}^n)) \), by using (35) and the fact \( \text{div} u = 0 \), it is clear that
\[
\lim_{n \to \infty} \left( \int_0^t (\nabla F | \nabla \tilde{F}_n) d\tau + \int_0^t (\nabla \tilde{F} | \nabla F_n) d\tau \right) = 2 \int_0^t (\nabla F | \nabla \tilde{F}) d\tau,
\]
\[
\lim_{n \to \infty} \int_0^t (\tilde{u} \cdot \nabla \tilde{F}_n) d\tau = \int_0^t (\tilde{u} \cdot \nabla \tilde{F} | F) d\tau,
\]
\[
\lim_{n \to \infty} \int_0^t (u \cdot \nabla F | \tilde{F}_n) d\tau = \lim_{n \to \infty} \int_0^t (u \cdot \nabla \tilde{F}_n | F) d\tau = - \int_0^t (u \cdot \nabla \tilde{F} | F) d\tau,
\]
\[
\lim_{n \to \infty} \int_0^t (\tilde{F} \nabla \tilde{u} | F_n) d\tau = \int_0^t (\tilde{F} \nabla \tilde{u} | F) d\tau,
\]
and
\[
\lim_{n \to \infty} \int_0^t (F \nabla u | \tilde{F}_n) dt = \int_0^t (F \nabla u | \tilde{F}) dt.
\] (52)

The last identity holds because $\nabla \tilde{F}_n$ converge to $\nabla \tilde{F}$ in $L^2(0, T; L^2(\mathbb{R}^n))$ and $\{ \tilde{F}_n \}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R}^n))$, which was ensured by the Banach-Steinhaus theorem due to $\tilde{F}_n$ weakly-star converge to $\tilde{F}$ in $L^\infty(0, T; L^2(\mathbb{R}^n))$. Hence, combining the estimates (48)–(52), as in the derivation of estimate (43) and (44), we have
\[
\lim_{n \to \infty} \int_0^t ((\partial_\tau F | \tilde{F}_n) dt = \int_0^t (\partial_\tau F | \tilde{F}) dt
\] (53)
and
\[
\lim_{n \to \infty} \int_0^t (\partial_\tau \tilde{F} | F_n) dt = \int_0^t (\partial_\tau \tilde{F} | F) dt.
\] (54)

It is obvious that
\[
\int_0^t (\partial_\tau F | \tilde{F}) + (\partial_\tau \tilde{F} | F) dt = (F(t) | \tilde{F}(t)) - (F_0 | \tilde{F}_0).
\]

Moreover, since $\text{div} \, \tilde{u} = 0$, we have
\[
\int_0^t \int_{\mathbb{R}^n} \left( \tilde{u} \cdot \nabla \tilde{F} : F + \tilde{u} \cdot \nabla F : \tilde{F} \right) dx \, dt = \int_0^t \int_{\mathbb{R}^n} \tilde{u} \cdot \nabla (\tilde{F} : F) dx \, dt = 0
\]
and
\[
\tilde{F} \nabla u : F + \tilde{F} : F \nabla u = \nabla u : (\tilde{F}^T F + F^T \tilde{F}).
\]

These two facts imply that
\[
\int_0^t \int_{\mathbb{R}^n} \left( u \cdot \nabla F : \tilde{F} + \tilde{u} \cdot \nabla \tilde{F} : F + F \nabla u : \tilde{F} + \tilde{F} \nabla \tilde{u} : F \right) dx \, dt
= \int_0^t \int_{\mathbb{R}^n} \left( \nabla u : (\tilde{F}^T F + F^T \tilde{F}) + (u - \tilde{u}) \cdot \nabla F : \tilde{F} - \tilde{F} \nabla (u - \tilde{u}) \right) dx \, dt.
\]

Finally, putting all above estimates together yield
\[
(F(t) | \tilde{F}(t)) + 2 \int_0^t (\nabla F | \nabla \tilde{F}) dt = (F_0 | \tilde{F}_0)
- \int_0^t \int_{\mathbb{R}^n} \left( \nabla u : (\tilde{F}^T F + F^T \tilde{F}) + w \cdot \nabla F : \tilde{F} - \tilde{F} \nabla w \right) dx \, dt.
\] (55)

It is clear that (34) follows from (45) and (55). This completes the proof of Lemma 5.2.
The proof of Proposition 4. Note that \( \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla F : F dx \, dt = 0 \) since \( \text{div} \, w = 0 \). Then by Lemma 5.2, we get

\[
\|(w(t), E(t))\|_{L^2_{\infty}}^2 + 2 \int_0^t \|\nabla (w(\tau), \nabla E(\tau))\|_{L^2_{\infty}}^2 \, d\tau \leq \|(u_0, F_0)\|_{L^2_{\infty}}^2 + \|(\tilde{u}_0, \tilde{F}_0)\|_{L^2_{\infty}}^2
\]

\[
- 2(u_0) \cdot \tilde{u}_0 - 2(F_0) \cdot \tilde{F}_0 + 2 \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla w \cdot udx \, dt - 2 \int_0^t \int_{\mathbb{R}^n} \tilde{F} \cdot \nabla udx \, dt
\]

\[
- 2 \int_0^t \int_{\mathbb{R}^n} FT \cdot \nabla \tilde{u} dx \, dt + 2 \int_0^t \int_{\mathbb{R}^n} \nabla u \cdot (\tilde{F}T + F \tilde{T}) dx \, dt
\]

\[
+ 2 \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla \cdot \tilde{F} dx \, dt - 2 \int_0^t \int_{\mathbb{R}^n} F \cdot \nabla w dx \, dt
\]

\[
\leq \|(w_0, E_0)\|_{L^2_{\infty}}^2 + 2 \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla w \cdot udx \, dt - 2 \int_0^t \int_{\mathbb{R}^n} T^E \cdot \nabla udx \, dt
\]

\[
- 2 \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla F : Edx dx \, dt + 2 \int_0^t \int_{\mathbb{R}^n} T^E F : \nabla w dx \, dt.
\]

Using the similar argument as the proof of Lemma 5.3 (see [5]), we obtain that

\[
\left| \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla w : udx \, dt \right| \leq C \int_0^t \|w\|_{L^2}^2 \|\nabla w\|_{L^2}^2 \|u\|_{B^{1+n/p+2/q}_{p,q}} d\tau
\]

\[
\leq \frac{1}{2} \int_0^t \|\nabla w\|_{L^2}^2 \, d\tau + C \int_0^t \|w\|_{L^2}^2 \|u\|_{B^{1+n/p+2/q}_{p,q}}^q \, d\tau;
\]

(57)

\[
\left| \int_0^t \int_{\mathbb{R}^n} T^E \cdot \nabla w dx \, dt \right| = \left| - \int_0^t \int_{\mathbb{R}^n} \text{div} (T^E) \cdot udx \, dt \right|
\]

\[
\leq C \int_0^t \|E\|_{L^2}^{2/q} \|\nabla E\|_{L^2}^{2-2/q} \|u\|_{B^{1+n/p+2/q}_{p,q}} \, d\tau
\]

\[
\leq \frac{1}{2} \int_0^t \|\nabla E\|_{L^2}^2 \, d\tau + C \int_0^t \|E\|_{L^2}^2 \|u\|_{B^{1+n/p+2/q}_{p,q}}^q \, d\tau;
\]

(58)

\[
\left| \int_0^t \int_{\mathbb{R}^n} w \cdot \nabla F : Edx dx \, dt \right| = \left| \int_0^t \int_{\mathbb{R}^n} (w \otimes F) \cdot \nabla E dx \, dt \right|
\]

\[
\leq \frac{1}{2} \int_0^t \|\nabla w, \nabla E\|_{L^2}^2 \, d\tau + C \int_0^t \|w, E\|_{L^2}^2 \|F\|_{B^{1+n/p+2/q}_{p,q}}^q \, d\tau;
\]

(59)

\[
\left| \int_0^t \int_{\mathbb{R}^n} T^E \cdot \nabla w dx \, dt \right| \leq \frac{1}{2} \int_0^t \|\nabla w, \nabla E\|_{L^2}^2 \, d\tau
\]

\[
+ C \int_0^t \|w, E\|_{L^2}^2 \|F\|_{B^{1+n/p+2/q}_{p,q}}^q \, d\tau,
\]

(60)

where \( \otimes \) denotes the tensor product. Returning back to the estimate (56) and putting (57)–(60) together yield that

\[
\|(w(t), E(t))\|_{L^2_{\infty}}^2 + 2 \int_0^t \|\nabla (w(\tau), \nabla E(\tau))\|_{L^2_{\infty}}^2 \, d\tau \leq \|(w_0, E_0)\|_{L^2_{\infty}}^2
\]

\[
+ C \int_0^t (\|w\|_{L^2}^2 + \|E\|_{L^2}^2) (\|u\|_{B^{1+n/p+2/q}_{p,q}}^q + \|F\|_{B^{1+n/p+2/q}_{p,q}}^q) \, d\tau.
\]

(61)
This estimate together with Gronwall’s inequality yield the desired estimate (33) immediately. We complete the proof of Proposition 4.

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