FACTORIZATION IN $SL_n(R)$ WITH ELEMENTARY MATRICES WHEN $R$ IS THE DISK ALGEBRA AND THE WIENER ALGEBRA

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Abstract. Let $R$ be the polydisc algebra or the Wiener algebra. It is shown that the group $SL_n(R)$ is generated by the subgroup of elementary matrices with all diagonal entries 1 and at most one nonzero off-diagonal entry. The result an easy consequence of the deep result due to Ivarsson and Kutzschebauch [4].

1. Introduction

Let $R$ be a commutative unital ring. Let $I_n$ denote the $n \times n$ identity matrix, that is the square matrix with all diagonal entries equal to $1 \in R$ and off-diagonal entries equal to $0 \in R$. Recall that an elementary matrix $E_{ij}(\alpha)$ over $R$ is a matrix of the form $I_n + \alpha e_{ij}$, where $i \neq j$, $\alpha \in R$, and $e_{ij}$ is the $n \times n$ matrix whose entry in the $i$th row and $j$th column is 1 and all other entries are zeros. Let $SL_n(R)$ be the group of all $n \times n$ matrices whose entries are elements of $R$ and whose determinant is 1. Let $E_n(R)$ be the subgroup of $SL_n(R)$ generated by the elementary matrices.

A classical question in commutative algebra is the following:

Question 1.1. Is $SL_n(R)$ equal to $E_n(R)$?

The answer to this question depends on the ring $R$, and here is a list of a few known results.

(1) If $R = \mathbb{C}$, then the answer is “Yes”, and this is standard exercise in linear algebra; see for example [1, Exercise 18.(c), page 71].

(2) Let $R$ be the polynomial ring $\mathbb{C}[z_1, \cdots, z_n]$ in the indeterminates $z_1, \cdots, z_n$ with complex coefficients.

If $n = 1$, then the answer is “Yes”, and this follows from the Euclidean Division Algorithm in $\mathbb{C}[z]$.

If $n = 2$, then the answer is “No”, and [2] gave the following counterexample:

$$
\begin{bmatrix}
1 + z_1 z_2 & z_1^2 \\
- \overline{z_2} & 1 - z_1 \overline{z_2}
\end{bmatrix} \in SL_2(\mathbb{C}[z_1, z_2]) \setminus E_2(\mathbb{C}[z_1, z_2]).
$$

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For $n \geq 3$, the answer is “Yes”, and this is the $K_1$-analogue of Serre’s Conjecture, which is the Suslin Stability Theorem [5].

(3) The case of $R$ being a ring of continuous functions was considered in [6]. Let $C(X; \mathbb{C})$ be the ring of continuous complex-valued functions on the finite-dimensional normal topological space $X$ with pointwise operations. $C_b(X; \mathbb{C})$ denotes the subring of $C(X; \mathbb{C})$ consisting of bounded functions. It was shown in [6] that for $R = C(X; \mathbb{C})$ or $C_b(X; \mathbb{C})$, the answer is “Yes” if there is no homotopy obstruction. Indeed, if $E$ is an elementary matrix, then $x \mapsto E(x) \in \text{SL}_n(\mathbb{C})$ is null-homotopic (to the constant map $x \mapsto I_n : X \to \text{SL}_n(\mathbb{C})$). So it follows that if $\pi(F)$ denotes the homotopy class of the map $x \mapsto F(x) : X \to \text{SL}_n(\mathbb{C})$ corresponding to $F \in \text{E}_n(C(X; \mathbb{C}))$ is that $\pi(F) = 0$. It turns out that this condition is also sufficient, and this is the content of [6, Theorem 4].

(4) Based on the above result, it is natural to consider the question also for the ring $\mathcal{O}(X)$ of holomorphic functions on Stein spaces in $\mathbb{C}^n$. This was posed as an explicit open problem by Gromov in [3], and was recently solved by Ivarsson and Kutzschebauch [4]. The main result in [4] is the following:

**Theorem 1.2** ([4]). If $X$ is a finite-dimensional reduced Stein space and $F : X \to \text{SL}_n(\mathbb{C})$ is a holomorphic mapping that is null-homotopic, then there exists a natural number $K$ and holomorphic mappings $G_1, \ldots, G_K : X \to \mathbb{C}^{m(m-1)/2}$ such that $F$ can be written as a product of upper and lower diagonal unipotent matrices

$$F(x) = M_1(G_1(x)) \cdots M_K(G_K(x)), \quad x \in X,$$

where the matrices $M_j(G_j(x))$ are defined by

$$M_j(G_j(x)) := \begin{bmatrix}
1 & 0 \\
& \ddots \\
G_j(x) & 1
\end{bmatrix} \quad \text{if } j \text{ is odd},$$

while

$$M_j(G_j(x)) := \begin{bmatrix}
1 & G_j(x) \\
& \ddots \\
0 & 1
\end{bmatrix} \quad \text{if } j \text{ is even}.$$

In particular, the assumption of null-homotopy is always satisfied if $X$ is contractible.

We wish to consider Question [11] for commutative, semisimple, unital complex Banach algebras $R$. A special case is when $R = C_b(X; \mathbb{R})$, where $X$ is a compact Hausdorff topological space, and item (3) above describes the answer in this special case. Motivated by this, we formulate the following question/conjecture, but first we introduce some convenient notation.
Let $R$ be a commutative, semisimple, unital complex Banach algebra with maximal ideal space denoted by $X_R$, equipped with the weak-* topology induced from the dual space $R^* := \mathcal{L}(R; \mathbb{C})$ of $R$.

Let $\hat{\varphi} : R \rightarrow C(X_R; \mathbb{C})$ denote the Gelfand transform. For $F \in SL_n(R)$, let $\hat{F}$ be the matrix with elements in $C(X_R; \mathbb{C})$ obtained by taking the Gelfand transform of the entries of $F$, and $\pi(\hat{F})$ denotes the homotopy class of $\varphi \mapsto \hat{F}(\varphi) : X_R \rightarrow SL_n(\mathbb{C})$.

**Conjecture 1.3.** Let $R$ be a commutative, semisimple, unital complex Banach algebra. $F \in SL_n(R)$ belongs to $E_n(R)$ if and only if $\pi(\hat{F}) = 0$.

We consider Question 1.1 for two important Banach algebras of holomorphic functions: the polydisc algebra $A(\mathbb{D}^d)$ and the Wiener algebra $W^+(\mathbb{D}^n)$.

Let $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ and $\overline{\mathbb{D}} := \{ z \in \mathbb{C} : |z| \leq 1 \}$. Let $d \in \mathbb{N}$. The Wiener algebra $W^+(\mathbb{D}^n)$ is the Banach algebra defined by

$$W^+(\mathbb{D}^d) = \left\{ \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} a_{(k_1, \ldots, k_d)} z_1^{k_1} \cdots z_d^{k_d} : \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} |a_{(k_1, \ldots, k_d)}| < \infty \right\},$$

with pointwise addition and multiplication, and the $\| \cdot \|_1$-norm given by

$$\|f\|_1 = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} |a_{(k_1, \ldots, k_d)}|, \quad f = \sum_{k_1=0}^{\infty} \cdots \sum_{k_d=0}^{\infty} a_{(k_1, \ldots, k_d)} z_1^{k_1} \cdots z_d^{k_d}.$$

The polydisc algebra $A(\mathbb{D}^d)$ is the Banach algebra of all continuous functions $f : \mathbb{D}^d \rightarrow \mathbb{C}$ which are holomorphic in $\mathbb{D}^d$, with pointwise addition and multiplication, and the supremum norm $\| \cdot \|_\infty$ given by

$$\|f\|_\infty := \sup_{(z_1, \ldots, z_d) \in \mathbb{D}^d} |f(z_1, \ldots, z_d)|, \quad f \in A(\mathbb{D}^d).$$

The ball algebra $A(\overline{\mathbb{D}}_d)$ is defined similarly, with the polydisc $\mathbb{D}^d$ replaced by the ball

$$\overline{\mathbb{D}}_d := \{ (z_1, \ldots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 \leq 1 \}.$$

For a $n \times n$ matrix $F$ with entries in $A(\mathbb{D}^d)$, $A(\overline{\mathbb{D}}_d)$ or $W^+(\mathbb{D}^d)$, we define

$$\|F\| := \sum_{i,j=1}^n \|F_{ij}\|_\infty,$$

where $F_{ij}$ denotes the entry in the $i$th row and $j$th column of $F$. Then $\|FG\| \leq \|F\| \|G\|$, for $n \times n$ matrices $F, G$ with entries from any of the Banach algebras $A(\mathbb{D}^d)$, $A(\overline{\mathbb{D}}_d)$ or $W^+(\mathbb{D}^d)$.

Our main result is the following.

**Theorem 1.4.** If $R = A(\mathbb{D}^d)$, $A(\overline{\mathbb{D}}_d)$ or $W^+(\mathbb{D}^d)$, then $SL_n(R) = E_n(R)$.
If \( R = A(\mathbb{D}^d) \) or \( W^+(\mathbb{D}^d) \), then in both cases, the maximal ideal space \( X_R \) can be identified with \( \overline{\mathbb{D}}^d \) as a topological space. Similarly, \( X_{A(\overline{\mathbb{D}}^d)} = \overline{\mathbb{D}}^d \). If Conjecture 1.3 is true, then Theorem 1.4 follows from the observation that \( \overline{\mathbb{D}}^d, \overline{\mathbb{B}}^d \) are contractible (since then \( \pi(\widehat{F}) \) is always trivial).

We will derive our main result as a consequence of the result from [4] quoted above, and [6, Lemma 9] reproduced below.

**Lemma 1.5** ([6]). Let \( R \) be a commutative topological unital ring such that the set of invertible elements of \( R \) is open in \( R \). If \( F \in SL_n(R) \) is sufficiently close to \( I_n \), then \( F \) belongs to \( E_n(R) \).

2. **Proof of Theorem 1.4**

**Proof.** We will simply prove the result in the case of the disc algebra \( A(\mathbb{D}^d) \); the proofs in the cases of the ball algebra \( A(\overline{\mathbb{B}}^d) \) and the Wiener algebra being analogous.

Let \( F \in SL_n(A(\mathbb{D}^d)) \). Let \( r \in (0, 1) \) (to be determined later). Define

\[
F_r(z_1, \ldots, z_d) := F(rz_1, \ldots, rz_d), \quad (z_1, \ldots, z_d) \in \mathbb{D}^d.
\]

As \( F_r \in \mathcal{O}(\frac{1}{r}\mathbb{D}^d) \), and \( \det F_r \equiv 1 \), it follows from Theorem 1.2 (since \( \frac{1}{r}\mathbb{D}^d \) is a contractible Stein domain) that there are elementary matrices \( G_1, \ldots, G_K \) belonging to \( E_n(\mathcal{O}(\frac{1}{r}\mathbb{D}^d)) \), such that

\[
F_r = E_1 \cdots E_K \in E_n(\mathcal{O}(\frac{1}{r}\mathbb{D}^d)) \subset E_n(A(\mathbb{D}^d)).
\]

Thus \( F(I_n + F^{-1}(F_r - F)) = F_r \in E_n(A(\mathbb{D}^d)) \). As \( \det F = \det F_r = 1 \), it follows that also \( \det(I_n + F^{-1}(F_r - F)) = 1 \). We will be done if we manage to show that \( I_n + F^{-1}(F_r - F) \in E_n(A(\mathbb{D}^d)) \) too. But this is clear by Lemma 1.5 since

\[
\left\| \left( I_n + F^{-1}(F_r - F) \right) - I_n \right\| = \|F^{-1}(F_r - F)\| \leq \|F^{-1}\||F_r - F|,
\]

and we can make \( \|F_r - F\| \) as small as we like by choosing \( r \) close enough to 1. \(\square\)

**Remark 2.1.** The above proof also works for some other Banach algebras of smooth functions contained in the polydisc algebra, for example, if \( N \in \mathbb{N} \), the Banach algebra \( \partial^{-N}A(\mathbb{D}^d) \) of all functions \( f \in A(\mathbb{D}^d) \) whose complex partial derivatives of all orders up to \( N \) belong to \( A(\mathbb{D}^d) \), with the norm

\[
\|f\|_{\partial^{-N}A(\mathbb{D}^d)} := \sum_{\alpha_1 + \cdots + \alpha_d \leq N} \frac{1}{\alpha_1! \cdots \alpha_d!} \sup_{(z_1, \ldots, z_d) \in \mathbb{D}^d} \left| \frac{\partial^{\alpha_1 + \cdots + \alpha_d} f}{\partial z_1^{\alpha_1} \cdots \partial z_d^{\alpha_d}} (z_1, \ldots, z_d) \right|.
\]

In light of Theorem 1.4, it is natural to ask the analogous question also for the Hardy algebra. Recall that if \( U \) is an open set in \( \mathbb{C}^d \), then the Hardy algebra \( H^\infty(U) \) is the Banach algebra of all complex-valued functions on \( U \) that are bounded and holomorphic in \( U \).
Conjecture 2.2. $SL_n(H^\infty(U)) = E_n(H^\infty(U))$ if $U$ is the polydisc $\mathbb{D}^d$ or open unit ball $U = \mathbb{B}_d := \{(z_1, \cdots, z_d) \in \mathbb{C}^d : |z_1|^2 + \cdots + |z_d|^2 < 1\}$.

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