Geodesics or autoparallels from a variational principle?

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Abstract

Recently it has been argued that autoparallels should be the correct description of free particle motion in spaces with torsion, and that such trajectories can be derived from variational principles if these are suitably adapted. The purpose of this letter is to call attention to the problems that such attempts raise, namely the requirement of a more elaborate structure in order to formulate the variational principle and the lack of a Hamiltonian description for the autoparallel motion. Here is also raised the problem of how to generalize this proposed new principle to quantum mechanics and to field theory. Since all applications known of such a principle are equally well described in terms of geodesics in non-holonomic frames we conclude that there is no reason to modify the conventional variational principle that leads to geodesics.

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1 Introduction

The motion of a free particle is a fundamental problem in physics. It is the ultimate manifestation of the principle of inertia, that has been formulated in different manners throughout the history of physics, and which must be expressed somehow in any good theory. In quantum theory, for example, where point particles are not fundamental objects, it must nevertheless be possible to address this problem in a suitable limit of the theory.

One of the great successes of General relativity was to replace the gravitational force acting in flat space by motion of a free particle in curved space along geodesics, incorporating in this way gravitation in the principle of inertia. In the discussion that follows we will use $x^\mu$ to describe either the evolution in time of a particle in space, $x^i(t)$, or the evolution in proper-time of a particle in space-time, $x^\mu(s)$, and $\dot{x}^\mu$ to represent the derivative with respect to its parameter. The two cases differ mathematically only by the signature of the metric, which is not relevant for the discussion. The geodesic equation is

$$\ddot{x}^\mu + g^{\mu\nu} \left( \partial_\alpha g_{\beta\nu} - \frac{1}{2} \partial_\nu g_{\alpha\beta} \right) \dot{x}^\alpha \dot{x}^\beta = 0,$$  

(1)

and it defines the lines of shortest length computed using the given metric.

This equation can also be given another geometric interpretation, as the autoparallels in a space with a rule for parallel transport, that is where a linear connection $\Gamma^\chi_{\mu\nu}$ is defined. The autoparallels are the lines along which the covariant derivative of the velocity vanishes,

$$\ddot{x}^\mu + \Gamma^\chi_{\alpha\beta\mu} \dot{x}^\alpha \dot{x}^\beta = 0.$$  

(2)

And this equation is equivalent to eq.(1) if one takes the connection to be the Riemannian one, which is given in a coordinate system by the Christoffel symbols,

$$\Gamma^\alpha_{\beta\mu} = \left\{ \begin{array}{c} \mu \\ \alpha \beta \end{array} \right\} = \frac{1}{2} g^{\mu\nu} \left( \partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right).$$  

(3)

The Riemannian connection is the only connection that can be constructed solely from the metric tensor. Actually since the totally antisymmetric Levi-Civita symbol $\epsilon_{\alpha_1...\alpha_n}$ is defined in any orientable manifold, for the particular case of three dimensions the tensor $\sqrt{|g|} \epsilon_{\alpha\beta\gamma} g^{\gamma\mu}$ could be added to the Riemannian connection in order to produce another connection. But such a term would nevertheless not affect the geodesic equation since it is antisymmetric in its first two indices. And if we content ourselves with a metric description of space (or space-time) the story would end here.

If a metric is not defined in space one can still have a definition of parallel transport if a connection is given. In such a case only two tensors can be constructed out of the connection (plus of course functions of these),

$$R_{\mu\nu\alpha}^\beta = \partial_\nu \Gamma^\beta_{\mu\alpha} - \partial_\mu \Gamma^\beta_{\nu\alpha} + \Gamma^\gamma_{\mu\alpha} \Gamma^\beta_{\gamma\nu} - \Gamma^\gamma_{\nu\alpha} \Gamma^\beta_{\gamma\mu},$$  

(4)
Torsion: \[ T_{\mu\nu}^\beta = \Gamma_{\mu\nu}^\beta - \Gamma_{\nu\mu}^\beta. \] (5)

We note in passing that two natural and covariant conditions that can be imposed on a space endowed with a connection are
\[ R_{\mu\nu\alpha}^\beta = 0 \] (6)
or
\[ T_{\mu\nu}^\beta = 0. \] (7)

Imposing both simultaneously would restrict the connection to be trivial, that is, it would lead to ordinary flat space.

Since the work of Cartan, interest has been raised in looking upon space-time as a space with parallel transport. But the metric is essential for the description of Nature, so we should consider spaces with a connection and a metric. Besides, also on physical grounds, one should impose the condition that the metric is covariantly conserved, which is usually called the metric condition. Doing so, the connection can be decomposed as
\[ \Gamma_{\alpha\beta}^\mu = \bar{\Gamma}_{\alpha\beta}^\mu + K_{\alpha\beta}^\mu, \] (8)
where the second term on the right hand side is a combination of torsion tensors called the contorsion tensor,
\[ K_{\alpha\beta\gamma} = \frac{1}{2} (T_{\alpha\beta\gamma} - T_{\alpha\gamma\beta} - T_{\beta\gamma\alpha}). \] (9)

This is the starting point for Einstein-Cartan theory where one considers besides the metric the presence of torsion \[ \bar{\Gamma}. \] As we can see it differs conceptually from Einstein’s General relativity in looking upon space-time as “a space with a connection and a metric that must be compatible among themselves” rather than as “a metric space with a natural rule for parallel transport derived from the metric”. Such theories gained particular relevance after the analogy with the remaining interactions of Nature (electroweak and strong) which are also formulated as theories of connections and not as metric ones.

We remind the readers of the well known result that in a space with a metric and a compatible connection, Riemannian geometry is recovered if the extra condition of vanishing torsion eq.(\ref{eq:R}) is imposed, since then the connection is expressible in terms of the metric by eq.\((\ref{eq:G})\). If we impose vanishing curvature eq.\((\ref{eq:R})\) but not vanishing torsion, we are lead to teleparallel space, which can be described with a set of tetrad fields by
\[ g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu \] (10)
\[ \Gamma_{\mu\nu}^\chi = g^{\chi\omega} \vec{e}_\omega \cdot \partial_\mu \vec{e}_\nu. \] (11)
The vector arrow refers to as many indices as the dimension of space and the internal product is to be performed in those indices, using the flat metric with
the appropriate signature. Here and in what follows we will use vector arrows and internal product dot notation in flat spaces, reserving the indices notation for spaces where the metric is not trivial. We note that both Riemannian and teleparallel spaces allow for arbitrary metrics. See for example [2] and references therein. See also Schrödinger [3].

In this context, of a space with parallel transport, the autoparallel equation, eq.(2), would seem more natural as a description of free particle motion. And it must be noted that in such a space autoparallels and geodesics do not coincide in general. But unlike in general relativity, since contorsion is a tensor, it can be added freely to the autoparallel equation in order to generate other covariant equations. In other words, it can be added or subtracted to the original connection, generating other connections, so that the connection is not unique anymore. In particular, the geodesic equation shall still be a candidate. Actually, among the infinitely many candidates, the most respectable ones are the autoparallels and the geodesics, for their particular geometrical meanings, as the “straightest” lines and the “shortest” lines respectively.

The debate on whether geodesics or autoparallels should describe free particle motion can however be carried on only as far as derivation of the motion from a Lagrangean is not concerned. It has been known since long that from a conventional variational principle only geodesics can be obtained [1].

But recently Kleinert and collaborators proposed in a series of interesting papers [4, 5, 6] a modification of the conventional variational principle in torsionful spaces that leads to autoparallels as the equations of motion for free particles.

2 Tensor equations in non-coordinate frames

Kleinert [4, 7] suggested that an inconsistency could be present in the conventional way of deriving equations of motion in torsionful spaces by an argument that goes as follows.

Starting from the Lagrangean for a free particle in flat space,

\[ L = \frac{1}{2} \dot{x} \cdot \dot{x} , \]

and performing a change of frame,

\[ \dot{x} = \dot{\epsilon}_\mu q^\mu , \]

where the tetrads depend on the \( q \)-variables, \( \dot{\epsilon}_\mu = \dot{\epsilon}_\mu(q) \), one gets

\[ L = \frac{1}{2} g_{\mu \nu} \dot{q}^\mu \dot{q}^\nu , \]

with

\[ g_{\mu \nu} = \epsilon_\mu \cdot \epsilon_\nu \]
the induced metric on $q$-space.

Variation of this Lagrangean with respect to $q$ will lead as is well known to the geodesic equation, eq.(1). But variation with respect to $\vec{x}$ leads to

$$\ddot{\vec{x}} = 0$$ \hspace{1cm} (16)

from which follows by the use of eq.(13)

$$\ddot{q}^\alpha + g^{\mu\nu} \vec{e}_\nu \cdot \partial_\alpha \vec{e}_\beta \dot{q}^\alpha \dot{q}^\beta = 0$$ \hspace{1cm} (17)

This is an autoparallel equation for the connection eq.(11), which is in general torsionful,

$$T_{\alpha\beta\mu} = g^{\mu\nu} \vec{e}_\nu \cdot (\partial_\alpha \vec{e}_\beta - \partial_\beta \vec{e}_\alpha)$$ \hspace{1cm} (18)

The two results eq.(1) and eq.(2) are then in contradiction if the tetrads do not obey $\partial_\alpha \vec{e}_\beta = \partial_\beta \vec{e}_\alpha$, that is, if the transformation is not holonomic (integrable).

We would like to stress that this inconsistency is only apparent. This disagreement between the two derivations is precisely what one should expect. Paths in $x$-space and $q$-space are related by

$$\vec{x}(t) = \int^t \dot{e}_\mu(q(s)) \dot{q}^\mu(s) ds$$ \hspace{1cm} (19)

If the transformation is non-holonomic a set of paths with fixed end points in $x$-space is not mapped into a set of paths with fixed end points in $q$-space. While the geodesic equation, eq.(1), arises from a variation with fixed end points in $q$-space, eq.(17) arises from a variation with fixed endpoints in $x$-space.

The conventional way of deriving equations of motion states that one should vary coordinates. So, if $q$-space is a coordinate one, then the correct equations of motion are the geodesics. Eq.(17) is then wrong, for the derivation of equations of motion in a non-coordinate frame ($x$-space in this case) must follow a different procedure [8].

On the other hand, if we assume $x$-space to be a coordinate one, then the correct equations of motion are of course eq.(1). The connection eq.(11) is not symmetric. However, since we are now assuming $q$-space to be a non-coordinate system, we have to account for the non-vanishing structure coefficients

$$c_{\alpha\beta\mu} = g^{\mu\nu} \vec{e}_\nu \cdot (\partial_\alpha \vec{e}_\beta - \partial_\beta \vec{e}_\alpha)$$ \hspace{1cm} (20)

We further remind the reader of the generalization of the definitions of some quantities to non-holonomic frames (cf. e.g. [9]):

$$R_{\mu\nu\alpha}^\beta = \partial_\nu \Gamma_{\mu\alpha}^\beta - \partial_\mu \Gamma_{\nu\alpha}^\beta + \Gamma_{\mu\alpha}^\gamma \Gamma_{\nu\gamma}^\beta - \Gamma_{\nu\beta}^\gamma \Gamma_{\mu\gamma}^\alpha + c_{\mu\nu}^\gamma \Gamma_{\gamma\alpha}^\beta$$ \hspace{1cm} (21)

$$T_{\mu\nu}^\beta = \Gamma_{\mu\nu}^\beta - \Gamma_{\nu\mu}^\beta - c_{\mu\nu}^\beta$$ \hspace{1cm} (22)

$$\bar{\Gamma}_{\mu\nu}^\chi = \left\{ \begin{array}{l} \chi \\ \mu\nu \end{array} \right\} + \frac{1}{2} c_{\mu\nu}^\chi \cdot \left[ g^{\alpha\chi}(g_{\beta\mu}c_{\nu\alpha}^\beta + g_{\beta\nu}c_{\mu\alpha}^\beta) \right]$$ \hspace{1cm} (23)
As can be seen from eq.(23), which involves besides the Christoffel symbols a non-symmetric term, the Riemannian connection is in general not symmetric in a non-coordinate frame, and it is indeed given in the present case by eq.(11). What we have called the torsion tensor, eq.(18), should then be interpreted as the set of structure coefficients eq.(20), and eq.(17) represents in fact the geodesic equation in this frame. We should also clarify that if torsion vanishes in \textit{x}-frame it will also vanish in \textit{q}-frame, according to eq.(22). After all torsion is a tensor, if it vanishes in one frame it must vanish in all frames.

The purpose of this section was to emphasize that there is no inconsistency in the conventional variational principle, and to remind that the key point is that variation with fixed endpoints in coordinate and non-coordinate systems are not equivalent, and that a choice is needed, the conventional one being the selection of coordinate systems, leading to the geodesic equation no matter what the value of torsion is. We shall see in the next section that Kleinert has proposed a different choice.

### 3 The proposed variational principle

Kleinert tried to obtain autoparallels from a variational principle by changing the convention of varying the action with fixed end points in “coordinate systems” to varying it in “a non-coordinate system where the connection vanishes”. This means that in the previous section, if \textit{q}-space is a coordinate system and the connection is given by eq.(11) in this system, then the correct equations of motion are the ones derived in \textit{x}-space, that is eq.(17), which are the autoparallels. We shall use in the remaining of this article this same meaning when we shall speak of \textit{x} and \textit{q} variables.

We shall not go into the details of the calculations, the interested reader may look in the references [4, 5]. But it can be realized directly from the description given above of the proposed variational principle that it has got a restricted application because “a frame where the connection vanishes” can be found if and only if the space is teleparallel, eq.(6).

In his derivation [3] Kleinert starts from a Lagrangean that may depend both on \textit{q}-variables and on \textit{x}-variables and arrives at a set of integro-differential equations that we do not reproduce here and that are non-causal. He reaches the conclusion that if and only if the Lagrangean can be written solely in terms of \textit{x}-variables,

\[
L = L(\vec{x}, \vec{x}) = L(\vec{e}_\mu(q) \dot{q}^\mu, \vec{x}),
\]

then the equations of motion are causal, and they take the form of a system,

\[
\vec{e}_\mu \cdot \frac{\partial L}{\partial \vec{x}} + \frac{\partial L}{\partial q^\mu} - \left(\frac{\dot{\partial L}}{\dot{\partial q^\mu}}\right) - T_{\mu\nu} \dot{q}^\nu \frac{\partial L}{\partial \dot{x}^\nu} = 0
\] (25)
\[ \vec{x} = \int_0^t \vec{e}_\mu(q(\tau)) \dot{q}^\mu(\tau) d\tau \quad . \]

We see thus that this derivation is even more restrictive than requiring teleparallelism of space because it rules out the possibility of introducing position dependent potentials. The remaining freedom of having a dependence on \( x \)-variables will also be suppressed if we want the dynamical equations to be differential and not integro-differential.

In a more recent paper \[6\] the previously described procedure was changed by allowing \( x \)-space to have a higher dimensionality than \( q \)-space. The vanishing curvature condition applies then to the whole of \( x \)-space, leaving freedom for induced curvature in \( q \)-space, in much the same way as it happens with the embedding of Riemannian manifolds in flat spaces by means of holonomic mappings \[10\]. We summarize this procedure by remembering that a torsionless \( n \)-dimensional metric manifold can be described in terms of its embedding into a \( N \)-dimensional flat manifold with \( N > n \), and that a curvatureless \( n \)-dimensional metric manifold can be described in terms of the embedding of its tangent space into a \( n \)-dimensional flat manifold. A general metric manifold with non-vanishing curvature and torsion tensors could then be described by a combination of the two methods, that is, the embedding of the tangent space into a \( N \)-dimensional vector space with \( N > n \).

This seems to be an interesting generalization of these methods and it also seems to be the correct formulation of Kleinert’s original claims, though the authors left still unproved that arbitrary curvature and torsion can be generated in this way, and only a lower bound for the dimensionality of the embedding space was computed. However it leaves unsolved the problem of the incorporation of position dependent potentials.

We shall see in the next section that a bigger drawback affects this proposed action principle even in its more recent form \[6\]: the lack of a consistent hamiltonian formulation.

## 4 Hamiltonian formulation and quantum mechanics

In their paper \[6\] the authors commented on the fact that a hamiltonian description was lacking. A Legendre transformation is always possible to perform,

\[ p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = g_{\mu\nu} \dot{q}^\nu \quad \]

\[ H = p_\mu \dot{q}^\mu - L = \frac{1}{2} g^{\mu\nu} p_\nu \quad , \]
and phase space with a symplectic form can be defined in the usual way, along with canonical transformations. Poisson brackets in canonical coordinates are

\[ \{ A, B \} = \frac{\partial A}{\partial q^\mu} \frac{\partial B}{\partial p_\mu} - \frac{\partial B}{\partial q^\mu} \frac{\partial A}{\partial p_\mu} \].  

(29)

The difference with the conventional formulation comes with the time evolution. Since in phase space we only need to specify the position variables endpoints the variation of the action should follow the same procedure as in the Lagrangean case, that is, varying in \( x \)-space and then coming back to \( q \)-space. The hamiltonian formulation in \( x \)-space is trivial and leads to

\[ H = \frac{1}{2} \pi \cdot \pi \]  

(30)

\[ \dot{x} = \pi \]  

(31)

\[ \dot{\pi} = 0 \]  

(32)

Restricting ourselves to orbits in \( x \)-space that are mapped from \( q \)-space, we have

\[ \pi = \tilde{e}_{\mu} g^{\mu\nu} p_\nu \].  

(33)

From this equation and eq.(13) one gets

\[ \dot{q}^\mu = g^{\mu\nu} p_\nu \]  

(34)

\[ \dot{p}_\mu = \Gamma^{\alpha\beta\gamma}_{\mu} g^{\alpha\gamma} p_\beta p_\gamma \]  

(35)

This last set of equations describe correctly the autoparallel motion, as can be checked by direct computation, and it can be derived from the hamiltonian eq.(28), the Poisson brackets eq.(29) and the time evolution equation

\[ \dot{F} = \{ F, H \} - T_{\mu\nu} \chi \frac{\partial F}{\partial p_\mu} \frac{\partial H}{\partial p_\nu} p_\chi \]  

(36)

which will also give the right time evolution for any quantity defined in phase space. However this time evolution is not generated by a hamiltonian vector field. Consequently time evolution is not described anymore by a canonical transformation and most of the classical results concerning hamiltonian systems are not valid anymore. As an example, Liouville’s theorem does not apply anymore and the time derivative of the volume in phase space of an ensemble of systems is given by

\[ \frac{dV}{dt} = \int_V T_{\mu\nu} \chi \frac{\partial H}{\partial p_\mu} p_\chi \]  

(37)

We observe that one could also make the attempt of starting from the Lagrangean eq.(12), subject it to the constraints eq.(13) and follow Dirac’s constrained systems analysis. The constraints eq.(13) can actually be written without reference to the embedded tangent space as

\[ \epsilon^{\mu_1 \ldots \mu_n} e_{\mu_1}^i(q) \ldots e_{\mu_n}^i(q) \epsilon_{i_1 \ldots i_N} = 0 \]  

(38)
being \( n \) and \( N \) the dimensions respectively of \( q \) and \( x \) spaces. But since one cannot integrate \( q \), Dirac’s method cannot be used.

The fact that the dynamics of this system is not hamiltonian is of course of major importance. Concerning quantization that means that one cannot describe time evolution by means of an unitary transformation and a prescription to write the Schrödinger equation is missing. Kleinert \([7]\) arrives to a Schrödinger equation that we analyse in what follows.

**Schrödinger equation**

Kleinert’s extension of the Schrödinger equation can be obtained by replacing in the equation

\[
i\partial_t \psi = -\frac{1}{2m} \nabla^2 \psi \tag{39}\]

ordinary derivatives by covariant ones and contracting them with the metric. We get

\[
i\partial_t \psi = -\frac{1}{2m} g^{ij} (\nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k) \psi \tag{40},\]

which is equivalent to

\[
i\partial_t \psi = -\frac{1}{2m} \nabla_i (\sqrt{g} g^{ij} \nabla_j \psi) + \frac{1}{2m} T^{ij} \nabla_i \psi \tag{41}.\]

For a description of quantum mechanics in curved space see \([11]\). In a coordinate representation by scalar wave functions, the inner product is given by

\[
<\phi|\psi> = \int \sqrt{g} \phi^* \psi dx \tag{42},\]

and the momenta conjugate to the coordinates are

\[
p_i = -i \left( \partial_i + \frac{1}{2} \Gamma_{ij}^k \right) \tag{43},\]

The hamiltonian corresponding to eq.\([11]\) can then be written as

\[
H = \frac{1}{2m} g^{-1/4} p_i g^{1/2} g^{ij} p_j g^{-1/4} + \frac{i}{2m} T^{ij} g^{1/4} p_i g^{-1/4} \tag{44}.\]

The first term in this formula is obviously hermitian, while the second, the one involving torsion, is not. Kleinert noticed this problem and comments on it in sec.11.5 of his book \([8]\). However he states that it can be solved for the particular application he has got in mind (the hydrogen atom) and that in Nature it will not show up, invoking Einstein-Cartan’s theory where the torsion tensor is totally antisymmetric \([9]\). But the point is that Einstein-Cartan’s theory is precisely the type of theory derived from a conventional variational principle that Kleinert is questioning.
We should also notice that this extension of the Schrödinger equation cannot ever lead to autoparallels in the classical limit for an arbitrary torsion, because autoparallels depend on the symmetric part of torsion which has got $n(n^2 - 1)/3$ independent components, and eq. (41) involves only the contracted torsion tensor that has got $n$ independent components. Thus eq. (41) cannot cover the general case of spaces with arbitrary metric and torsion tensors.

5 Final remarks

Field theory

All the considerations done so far in this article concerned “particles”, that is, the degrees of freedom were the coordinates themselves. But there is the belief that matter should be described by fields. In this line of reasoning the trajectories of free particles must be a suitable limit of a field theory.

But in a field theory this problem acquires a completely different form, for the space coordinates become “labels” for the degrees of freedom and not the degrees of freedom themselves. The proposed variational method that we have been discussing does not make sense any longer for now the conventional variational principle states that “the values of the fields at any space point must be fixed at the ends of the path”. Paths are not defined on space but rather on the space of field configurations. Besides, using a non-coordinate frame would affect the very counting of degrees of freedom, since the transformation back to coordinates would act on what are now labels for the degrees of freedom.

We are not imagining how to generalize the previous principle to fields. From the point of view of the equations of motion themselves it seems that the analogy would be to replace the Riemannian connection by the full connection in the equations. The Klein-Gordon equation, for example, would become:

$$g^\mu\nu D_\mu \partial_\nu \phi - m^2 \phi = 0 \quad \Leftrightarrow \quad (45)$$

$$\Leftrightarrow \quad g^\mu\nu \partial_\mu \partial_\nu \phi - \Gamma^\nu_{\mu\nu} \partial_\nu \phi - m^2 \phi = 0 \quad \Leftrightarrow \quad (46)$$

$$\Leftrightarrow \quad \partial_\mu (\sqrt{-g} g^{\nu\mu} \partial_\nu \phi) - \sqrt{-g} (\Gamma^\nu_{\mu\nu} \partial_\nu \phi + m^2 \phi) = 0 \quad , \quad (47)$$

which is not derivable from a variational principle in the conventional way, and which leads in the non-relativistic limit to Kleinert’s extension of the Schrödinger equation, eq. (41).

We would like to note that it is possible to derive from a variational principle a generalization of the Dirac equation to torsionful spaces involving torsion, namely in Einstein-Cartan theory [1]. But in such an equation only the totally antisymmetric part of torsion shows up. If one tries the rule of replacing the ordinary derivative by the covariant one in the Dirac equation, again non-hermitian terms will show up in the non-relativistic limit. And such an equation will equally well be non-derivable from an action. The symmetric part of torsion,
precisely the one involved in the autoparallel equation, is the one responsible for these undesirable features in field equations.

**Motivations for these ideas**

It has been stated several times that torsion arises naturally in a space with parallel transport. That is true. Of course there is no reason why a connection should be symmetric in a coordinate frame, as we saw in sec.1. But when dealing with a metric space we also saw in that section that there is a naturally defined connection, namely the Riemannian connection. And insisting that the metric space should allow for the most general connection compatible with it is by no means an obvious assumption.

Moreover vanishing of torsion provides the theory with the nice property that antisymmetrized covariant derivatives are equal to ordinary exterior derivatives on forms, essential ingredient in the construction of Lagrangians, since for an arbitrary form $A_\mu$,

$$D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + T_{\mu\nu}{}^\chi A_\chi.$$  (48)

Our final remarks concern the original motivations for these works. They seem to have originated from the description of defects in crystals by means of torsionful spaces [12] (more recent works and applications in this field can be found in [7, 13] and references therein). In this context autoparallels would describe the motion of an observer “who would care only about the local crystalline structure and follow his way as if he were in a defectless crystal”. This description should be understood as an effective description of a much more complicated system bearing no direct link to the fundamental principles of mechanics. Moreover the actual framework for this theory of defects in crystals seems to be best described in terms of teleparallel spaces rather than in terms of spaces with arbitrary metric and torsion fields.

Another important motivation were the works of Kleinert on the quantization of the hydrogen atom by the path integral method, using the Kustaanheimo-Stiefel transformation [14]. It turns out however that both in this case and in the case of the solid body equations of motion [14] the same results would have been obtained in the conventional way by calling the frames used non-holonomic ones and by calling torsion the structure coefficients. The equations would be the same but “autoparallels in a coordinate system in the presence of torsion” would be called “geodesics in a non-holonomic frame, no matter the value of torsion”. No example has been given in the quoted works of a system with torsion that is not derivable from a torsionless space by a non-holonomic transformation.

**6 Conclusions**

We conclude that the quoted works offer no decisive argument on why should one change such fundamental principles of physics like the variational principle
and the result that geodesics describe the trajectories of test particles in General relativity, and that on the contrary they raise new and difficult problems. The results of these works should probably be seen like useful tools in effective theories where torsion may play a role. We summarize the main disadvantages of the proposed variational principle:

- The variational principle proposed looks much more artificial than the simpler principle of varying coordinates.
- It does not allow for the introduction of position dependent potentials.
- Time evolution is not described by a hamiltonian vector field in phase space and consequently standard quantization breaks down.
- The Schrödinger equation proposed involves a non-hermitian hamiltonian and fails to lead to autoparallels in the classical limit for an arbitrary torsion field.
- The more fundamental problem of field theory falls out of this scheme.
- In a metric space the vanishing of torsion seems to be a natural and convenient choice.
- The previous works that suggest the use of autoparallels are equally well described by means of geodesics in non-holonomic frames.
- There is no experimental result suggesting the need for a modification of the conventional variational principle.

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