Higher Shell Configuration Mixing for Magnetic Moments

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Abstract

In many works on nuclear magnetic moments, shell model calculations must ignore certain configurations since the result in a model space is too large. Motivated by this we here construct a simple model which will enable us to evaluate the effects of high lying configurations. We start with 2 neutrons in the \(g_{7/2}\) single shell and then admix systematically other configurations: \(h_{11/2}h_{11/2}\), \(h_{11/2}h_{9/2}\) and \(h_{9/2}h_{9/2}\) and examine how configurations affect the magnetic moments \((g\) factors\) of \(J = 2^+, 4^+\) and \(6^+\) states.

We have a simple formula which explains the qualitative behaviour of the effects of higher shells.

Introduction

The nuclear magnetic moment of a free neutron in units of \(\mu_N\) is \(-1.913\) (+2.793 for a proton). The magnetic moment operator is \(g_sS + g_lL\). Allowing for quenching we have for a neutron \(g_s = -3.826x\) where for \(x = 1\) if there is no quenching. Also for a neutron we have \(g_l = 0\), while for a proton \(g_l = 1\). However often the values \(g_l = -0.2Z/A\) is used for the neutron and \(1 + 0.2N/A\) for the proton.

The Schmidt values for a single \(j\) shell:

\[
g = \begin{cases} 
\frac{1}{j}(g_l + \frac{g_s}{2}) & \text{if } j = l + \frac{1}{2} \\
\frac{1}{j+1}(g_l(l+1) - \frac{g_s}{2}) & \text{if } j = l - \frac{1}{2}
\end{cases}
\]  

For 2 nucleons in different shells \(j_1\) and \(j_2\) the \(g\) factors depend of the total angular momentum \(J = j_1 + j_2\).

\[
g = \frac{1}{2}(g_1 + g_2) + (g_1 - g_2)\frac{j_1(j_1+1) - j_2(j_2+1)}{2J(J+1)}
\]  

Where \(g_1\) and \(g_2\) are \(g\) factors of nucleon 1 and nucleon 2. We will use equation (1) and (2) to calculate the \(g\) factors of single nucleon state and equation (3) for \(g\) factors of two mixed nucleon state.

Often when large scale shell model calculations of nuclear magnetic moments are performed, one is frustrated by the fact that one cannot include configurations that might be important, because the model space becomes too large to handle. One example in which some of the current authors were involved is the work of Kumartzki et al\(^1\).

In this work we will consider a very simple system of 2 neutrons in various shells. We start with the configuration \([g_{7/2}g_{7/2}]^J\) with \(J = 2^+, 4^+\) and \(6^+\). The \(g\) factors for all 3 \(J\)s are the same and are equal to the single particle \(g\) factor in the \(g_{7/2}\) shell. The free value is 0.42511. We will see how this number changes when we use a more elaborate wave function:

\[
\Psi^J = a[g_{7/2}g_{7/2}]^J + b[h_{11/2}h_{11/2}]^J + c\sqrt{2}([h_{11/2}(1)h_{9/2}(2)]^J - [h_{11/2}(2)h_{9/2}(1)]^J) + d[h_{9/2}h_{9/2}]^J
\]
The four terms in the wave function associated with $a, b, c, d$ are chosen to be the basis states and $|a|^2$ is the probability finding the system in the state of $[g_{7/2} g_{7/2}]^J$, $|b|^2$ is the probability finding the system in the state of $[h_{11/2} h_{11/2}]^J$ etc. (Note, since the states on interest have positive parity, there are no $[g_{7/2}, h_j]^J$ terms). The $g$ factors of the basis states are respectively $+0.42511$ (from equation (2)), $-0.34782$ (from equation (1)), $-0.63767/J(J+1)$ (from equation (3)) and $+0.34782$ (from equation (2)). From equation (1) and (2) ($g_l = 0$ for neutron), we see that the $g$ factor in single state $h_{9/2}$ is equal but opposite to that of $h_{11/2}$. Equation (3) implies that $g$ factor for the mixed state of $h_{11/2}$ and $h_{9/2}$ is $J$ dependant.

Calculations

We will obtain the wave functions and $g$ factors using a surface delta interaction. This was used previously by us for $g$ factor calculations in $^{86}$Kr and $^{112}$Sn [2, 3].

The $T = 1$ matrix element of the SDI interaction can be written as following [4, 5, 6]:

$$< j_1 j_2 | SDI | j_3 j_4 > = C_0 f(j_1, j_2) f(j_3, j_4)$$

(5)

where $f(j_1, j_2) = (-1)^{j_2+1/2} \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)} \langle j_1 j_2 | \frac{1}{2} | J_0 \rangle$.

Note that the expression is separable. In previous works [2, 3] we chose $C_0$ to be $-0.200 MeV$. We begin in Table I by having the single particle splitting for $g_{7/2} - h_{11/2}$ and $g_{7/2} - h_{9/2}$ be identical and show results for a splitting $E$ with values of $E = +0.5 MeV$ and $E = +1.0 MeV$. It is really the ratio $E/C_0$ that is the relevant parameter.

| Configuration | $J$ | $g(E = 0.5)$ | $g(E = 1)$ |
|---------------|----|--------------|-------------|
| Case 1: $c = d = 0$ | 2  | 0.2870       | 0.3805      |
|               | 4  | 0.3767       | 0.4128      |
|               | 6  | 0.4085       | 0.4212      |
| Case 2: $c = 0$  | 2  | 0.2497       | 0.3599      |
|               | 4  | 0.3536       | 0.4085      |
|               | 6  | 0.4017       | 0.4202      |
| Case 3: $a, b, c, d \neq 0$ | 2  | 0.2015       | 0.3394      |
|               | 4  | 0.3100       | 0.3988      |
|               | 6  | 0.3656       | 0.4146      |
| Initial Case: $[g_{7/2} g_{7/2}]$ | All $J$ | 0.4251 | 0.4251 |

In case 1, we only include $[h_{11/2} h_{11/2}]^J$ and $[g_{7/2} g_{7/2}]^J$, since $c$ and $d$ are set to be 0. For Case 2, we will also include $[h_{9/2} h_{9/2}]^J$, that is, only $c$ is set to zero. Lastly in Case 3, we include all 4 configurations, that is $a, b, c, d$ are not 0. Let us first focus on the $g$ factors of $J = 2^+$ state. The $g$ factor of Initial Case is 0.4251, and it reduces to 0.2870 when we include $[h_{11/2} h_{11/2}]^J, (c = d = 0)$ i.e. a 32.5% reduction. This is due to the fact that the $[h_{11/2} h_{11/2}]^J$ state has a $g$ factor that is the opposite sign of that of $[g_{7/2} g_{7/2}]^J$. In general a $g$ factor of neutron in $j = l + 1/2$ state has the opposite sign from that of $j = l - 1/2$ state.

We then set $c = 0$. Here is where the counterintuitive behavior appears. The $g$ factor of the $h_{9/2}$ neutron is equal and opposite to that of a $h_{11/2}$ neutron. This would suggest that $h_{9/2}$ would undo the damage done by $h_{11/2}$ leading to a $g$ factor closer to that of a pure $[g_{7/2} g_{7/2}]$ configuration. But the opposite is true. The value in this case is 0.2497 as compared with 0.2870.
To understand what is happening, we show the detailed wave functions in Table II for the case $E = +0.5$. The important point is that the amplitude $a$ is much smaller in Case 2. Introducing the configuration $[h_9/2h_9/2]^J$ causes the $[g_7/2g_7/2]^J$ probability to be depleted. This depletion causes the overall $g$ factor to be come smaller. However the $[h_{11/2}h_{11/2}]^J$ component does not get depleted. In fact it is slightly enhanced. Clearly these effects are beyond first order perturbation theory.

We note that the $g$ factors, which were $J$ independent for the configuration $[h_9/2h_9/2]^J$ are now all different, with $g(J = 2)$ is the smallest, $g(J = 4)$ is in the middle, and $g(J = 6)$ is the largest.

Table II: Wave function coefficients of Eq (4) with $E = +0.5$ for Case 1, Case 2, and Case 3.

|       | Case 1 | Case 2 | Case 3 |
|-------|--------|--------|--------|
| $a$   | -0.9063| -0.7857| -0.7711|
| $b$   | 0.4227 | 0.4570 | 0.4574 |
| $c$   | 0      | 0      | -0.1418|
| $d$   | 0      | 0.4159 | 0.4174 |

In Table III we show the values of $g(J = 2)$ for various values of $E$. The behavior is a bit complex. Up to $E = +0.2$ the values in Case 1 ($[h_{11/2}h_{11/2}]$ included) are smaller than those in Case 2 (both $[h_{11/2}h_{11/2}]$ and $[h_9/2h_9/2]$ included). But small $E$ corresponds to strong coupling. For $E = +0.3$ and beyond we get a reversal with Case 2 yielding $g$ factors smaller than that of Case 1. But higher $E$ means weaker coupling, so one might have expected the opposite behavior. In Table II there is a considerable depletion of the $[g_7/2g_7/2]$ configuration when one goes from Case 1 ($|a|^2 = 82\%$) to Case 2 (only 62\%).

In Table IV we show the results with all the configurations present. We do it for Case 3 as before, where the $h_{11/2}$ and $h_{9/2}$ are degenerate at and energy $E$ above $g_7/2$ and a new Case 4 where $h_{11/2}$ is at an energy $E$ above $g_7/2$ and $h_{9/2}$ is raised above $h_{11/2}$ and at an energy $E + 0.25$.

We would expect Case 4 to give smaller values than Case 3, because we have raised the energy of the $h_{9/2}$ energy relative to $h_{11/2}$. This is indeed true, up to $E = 0.5$ but for $E = 1$ and $E = 2$ there is a reversal.
But for $E = 4$ and $E = 5$ we are back to the ordering for $E$ equal or less than 0.5. It is not surprising that as $E$ getting very large, $g(2^+)$ approaches to the single particle value of $g_{9/2}$, which equals $0.42511$.

**Explanation**

To attempt an explanation of the results we compare the results for Case 1 (We will denote $G$ as $g$ factor to avoid confusion between $g$ of $g$ factor and $g$ of $g_{7/2}$ state.

In Case 1 the wave function is

$$\Psi_1 = a_1[g_{7/2}g_{7/2}]^j + b_1[h_{11/2}h_{11/2}]^j$$  \hspace{1cm} (6)

The $g$ factor in Case 1 is

$$G_1(J) = a_1^2 G(g_{7/2}) + b_1^2 G(h_{11/2})$$  \hspace{1cm} (7)

We then use the fact that $a_1^2 + b_1^2 = 1$ to rewrite equation (7) as

$$G_1(J) = G(g_{7/2}) + b_1^2 (G(h_{11/2}) - G(g_{7/2}))$$  \hspace{1cm} (8)

The wave function for Case 2 is

$$\Psi_2 = a_2[g_{7/2}g_{7/2}] + b_2[h_{11/2}h_{11/2}] + d_2[h_{9/2}h_{9/2}]$$  \hspace{1cm} (9)

The $g$ factor is therefore given by following

$$G_2(J) = a_2^2 G(g_{7/2}) + b_2^2 G(h_{11/2}) + d_2^2 G(h_{9/2})$$  \hspace{1cm} (10)

Since $a_2^2 + b_2^2 + d_2^2 = 1$, and $G(h_{11/2}) = -G(h_{9/2})$ then we rewrite equation (10) as

$$G_2(J) = (1 - b_2^2 - d_2^2) G(g_{7/2}) + (b_2^2 - d_2^2) G(h_{11/2})$$  \hspace{1cm} (11)

Then using equation (8) to subtract equation (11), we have

$$G_1(J) - G_2(J) = (b_2^2 + d_2^2 - b_1^2) G(g_{7/2}) + (b_1^2 - b_2^2 + d_2^2) G(h_{11/2})$$  \hspace{1cm} (12)

In perturbation theory we have $b_1 = b_2$. Using this we find

$$G_1(J) - G_2(J) = d_2^2 (G(g_{7/2}) + G(h_{11/2}))$$  \hspace{1cm} (13)

Since $G(g_{9/2}) = 1.913/4.5$ and $G(g_{11/2}) = -1.913/5.5$, then $G_1(J) - G_2(J) = +0.07729 d_2^2$

We see that Case 1 gives a larger value (closer to the pure $g_{7/2}$ case) than Case 2 for all $d_2^2$ in this approximation. For the weak coupling or strong coupling, we get the same qualitative behavior i.e. introducing $h_{9/2}$ to the existing $h_{11/2}$ admixture does not cancel out the effect of the $h_{11/2}$ admixture but rather enhances it. In particular this is due to the fact that the $g$ factor for $h_{9/2}$, although positive, is smaller than the $g$ factor for $g_{7/2}$.

We can simulate perturbation theory by simply setting $a = 1$. Using $E = 0.5$ as an example. In the "exact" calculation, the values of $G(J=2)$ in the 3 cases were respectively 0.2870, 0.2497, 0.2015, now with $a = 1$ we obtain 0.3629, 0.4123, 0.3738. Whereas in the case where $a$ is less than one we go from large to smaller to smallest, in the perturbation calculation ($a = 1$) we go from small to large to smaller. There is a qualitative difference.

We repeat that it is of value to study simple systems. In this case we examine the 2 neutron system, because it enables us to include configurations that are at present not possible for more complicated systems. We can get some idea of what the missing elements of these more complex system do.
Appendix: Expressions for the $G$ factors

Again, we use the symbol $G$ for the $g$ factors so as to distinguish them form the $g$ in $g_{7/2}$. Note that in a single $j$ shell of particles of one kind, all $g$ factors are the same. In particular for this case the $g$ factor of two $g_{7/2}$ is the same as that of one. The same is true for $h_{9/2}$ and $h_{11/2}$.

The expression for the $g$ factors is:

$$G(J) = a^2G(g_{7/2}) + b^2G(h_{11/2}) + c^2G(h_{11/2}h_{9/2}) + d^2G(h_{9/2}) + CrosstermA + CrosstermB$$  (14)

We then use equation (3) to expand $G(h_{11/2}h_{9/2})$

$$G(h_{11/2}h_{9/2}) = \frac{1}{2}(G(h_{11/2}) + G(h_{9/2})) + (G(h_{11/2}) - G(h_{9/2})) \frac{11/2 \cdot 13/2 - 9/2 \cdot 11/2}{2J(J+1)}$$  (15)

$$CrossA = 4bc \cdot U(1,4.5,J;5.5,5.5,J) \frac{\langle h_{11/2} || \mu || h_{9/2} \rangle}{\sqrt{2(J+1)}}$$  (16)

$$CrossB = -4cd \cdot U(1,5.5,J,4.5;4.5,J) \frac{\langle h_{9/2} || \mu || h_{11/2} \rangle}{\sqrt{2J(J+1)}}$$  (17)

where $U$ is a unitary Racah coefficient and at the end we have reduced matrix elements of the magnetic moment operator $\mu = g_L L + g_S \sigma/2$.

The numerical values for bare operators which can be obtained from equation (1), (2), and (3) are: $G(g_{7/2}) = 0.4251$, $G(h_{11/2}) = -0.3478$, $G(h_{9/2}) = +0.34782$. $G(h_{11/2}h_{9/2}) = -3.8260/(J(J+1))$. The respective values for $J = 2, 4, 6$ are $-0.6377, -0.1913$ and $-0.0911$.

The bare values are $g_L = 0$ and $g_S = -3.826$. Our reduced matrix element is one used by Lawson [7]:

$$\langle \psi^J_{M_B} O^J_{M_A} \psi^J_{M_A} > = \langle J_A \lambda M_A \mu | J_B M_B > \langle J_B || O^J || J_A >$$  (18)

The reduced matrix elements are

$$\langle h_{11/2} || \sigma || h_{9/2} \rangle = -\sqrt{20/11}$$

$$\langle h_{9/2} || \sigma || h_{11/2} \rangle = \sqrt{24/11}$$

$$\langle h_{11/2} || L || h_{9/2} \rangle = \sqrt{5/11}$$

$$\langle h_{9/2} || L || h_{11/2} \rangle = -\sqrt{6/11}$$

The one used by Bohr and Mottelson is $\langle \psi^J_{M_B} O^J_{M_A} \psi^J_{M_A} > = \langle J_A \lambda M_A \mu | J_B M_B > \frac{\langle J_B || O^J || J_A >}{\sqrt{2J_B+1}}$ [8].

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References

[1] G. Kumbartzki et al., Phys. Rev C89, 064305 (2014)
[2] Larry Zamick, Brian Kleszyk, Yitzhak Sharon and Shadow Robinson, Phys. Rev. C90, 027305 (2014)
[3] Xiaofei Yu and Larry Zamick, Nuclear Physics, A949 (2016)
[4] I.M. Green and S.A. Moszkowski, Phys. Rev. 139B, 790 (1965)
[5] R. Arvieu and S.A. Moszkowski, Phys. Rev. 145, 830 (1966)
[6] I. Talmi, Simple Models of Complex Nuclei, Harwood Academic Publishers, Switzerland (1993)
[7] R.D. Lawson, Theory of the Nuclear Shell Model, Oxford Press (1980)
[8] A. Bohr and B.R. Mottelson, Nuclear Structure, Vol I, W.A. Benjamin Inc. New York (1969)