ON $\tau$-TILTING SUBCATEGORIES

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Abstract. The main theme of this paper is to study $\tau$-tilting subcategories in an abelian category $\mathcal{A}$ with enough projective objects. We introduce the notion of $\tau$-cotorsion torsion triples and show a bijection between the collection of $\tau$-cotorsion torsion triples in $\mathcal{A}$ and the collection of $\tau$-tilting subcategories of $\mathcal{A}$, generalizing the bijection by Bauer, Botnan, Oppermann and Steen between the collection of cotorsion torsion triples and the collection of tilting subcategories of $\mathcal{A}$. General definitions and results are exemplified using persistent modules. If $\mathcal{A} = \text{Mod-}R$, where $R$ is an unitary associative ring, we characterize all support $\tau$-tilting, resp. all support $\tau^-$-tilting, subcategories of $\text{Mod-}R$ in term of finendo quasitilting, resp. quasicotilting, modules. As a result, it will be shown that every silting module, respectively every cosilting module, induces a support $\tau$-tilting, respectively support $\tau^-$-tilting, subcategory of $\text{Mod-}R$. We also study the theory in $\text{Rep}(Q, \mathcal{A})$, where $Q$ is a finite and acyclic quiver. In particular, we give an algorithm to construct support $\tau$-tilting subcategories in $\text{Rep}(Q, \mathcal{A})$ from certain support $\tau$-tilting subcategories of $\mathcal{A}$ and present a systematic way to construct $(n + 1)$-tilting subcategories in $\text{Rep}(Q, \mathcal{A})$ from $n$-tilting subcategories in $\mathcal{A}$.

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1. Introduction

Tilting theory is one of the most prominent tools in representation theory of artin algebras. The classical tilting modules were introduced by Brenner and Butler [BB], and Happel and Ringel [HR] as an axiomatisation of the reflection functors of Bernstein, Gelfand and Ponomarev [BGP] and Auslander, Platzeck and Reiten [APR]. It has been shown by Bongartz [Bo] that every partial tilting module can be completed to a tilting module and by Happel and Unger [HU] that every almost complete tilting module can be completed in at most two ways. However, there are examples of almost complete tilting modules that have exactly one complement.

Several years later, and inspired by the cluster algebras defined by Fomin and Zelevinskyy in [FZ], Adachi, Iyama and Reiten introduced $\tau$-tilting theory [AIR]. This is a generalization of classical tilting theory in which every almost complete support $\tau$-tilting module has exactly two complements, allowing to introduce a notion of mutation among these objects. The success of $\tau$-tilting theory was immediate, offering an explanation for several phenomena in the module category of artin algebras and offering new connections between representation theory and other areas of mathematics (see [T]).

Due to the effectiveness of $\tau$-tilting theory for the study of the categories of finitely presented modules, many mathematicians have introduced theories generalizing $\tau$-tilting theory, and its dual, to other contexts. For instance, there are the works of Angeleri-Hügel, Marks and Vitoria [AMV] and Breaz and Pop [BP] for the module category of rings; Iyama, Jørgensen and Yang [IJY] for functor categories; or Liu and Zhou for Hom-finite abelian categories with enough projective objects [LZh].

In this paper we are interested in studying $\tau$-tilting theory in arbitrary abelian categories with enough projective objects. Since in general there is no notion of Auslander-Reiten translation $\tau$ in such general categories, the definition of support $\tau$-tilting subcategories needs to be made with no mention to it. We follow [IJY,LZh] and define support $\tau$-tilting subcategories as follows.

**Definition 1.1.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{T}$ be an additive contravariantly finite full subcategory of $\mathcal{A}$. Then $\mathcal{T}$ is called a support $\tau$-tilting subcategory if

1. $\text{Ext}_A^1(T_1, \text{Fac}(T_2)) = 0$, for all $T_1, T_2 \in \mathcal{T}$.
2. For any projective $P$ in $\mathcal{A}$, there exists an exact sequence

$$P \xrightarrow{f} T^0 \rightarrow T^1 \rightarrow 0$$

such that $T^0$ and $T^1$ are in $\mathcal{T}$ and $f$ is a left $\mathcal{T}$-approximation of $P$.

The idea of generalizing a well-behaved theory from the category of finitely presented modules of an artin algebras to more general abelian categories is not exclusive to $\tau$-tilting theory. Indeed, Beligiannis introduced tilting theory for arbitrary abelian categories in [Be1] at the beginning of the millenium, see also [BR]. In recent years, the work of Bauer, Botnan, Oppermann and Steen [BBOS] has found a striking application of this theory in persistence theory and topological data analysis (TDA). One of the main results in [BBOS] is a bijection between tilting subcategories of an abelian category with enough projective objects and the collection of cotorsion torsion triples in the same category. The motivation that started this collaboration is to find a generalization of [BBOS, Theorem 2.29] to support $\tau$-tilting subcategories, but for that we first need an adequate notion of triple. Inspired by [BR, Lemma V.3.3] we give the definition of $\tau$-cotorsion torsion triple as follows.
Definition 1.2. Let \( \mathcal{A} \) be an abelian category with enough projective objects. A triple of full subcategories \((\mathcal{C}, \mathcal{D}, \mathcal{F})\) of \( \mathcal{A} \) is called a \( \tau \)-cotorsion torsion triple if

1. \( \mathcal{C} = \perp_1 \mathcal{D} \).
2. For every projective object \( P \in \mathcal{A} \), there exists an exact sequence
   \[ P \xrightarrow{f} D \rightarrow C \rightarrow 0, \]
   where \( D \in \mathcal{C} \cap \mathcal{D} \), \( C \in \mathcal{C} \) and \( f \) is a left \( \mathcal{D} \)-approximation.
3. \( \mathcal{C} \cap \mathcal{D} \) is a contravariantly finite subcategory of \( \mathcal{A} \).
4. \((\mathcal{D}, \mathcal{F})\) is a torsion pair in \( \mathcal{A} \).

Having the notion of \( \tau \)-cotorsion torsion triple we are able to show the desired bijection in an abelian category with enough projective objects.

Theorem 1.3 \((\text{Theorem 5.7})\). Let \( \mathcal{A} \) be an abelian category with enough projective objects. Then there are bijections

\[
\Phi : \{ \text{support } \tau \text{-tilting subcategories} \} \rightarrow \{ \tau \text{-cotorsion torsion triples} \}
\]

\[
\mathcal{T} \mapsto (\perp_1 (\text{Fac}(\mathcal{T})), \text{Fac}(\mathcal{T}), \mathcal{T}^\perp_0)
\]

\[
\Psi : \{ \tau \text{-cotorsion torsion triples} \} \rightarrow \{ \text{support } \tau \text{-tilting subcategories} \}
\]

\[
(\mathcal{C}, \mathcal{D}, \mathcal{F}) \mapsto \mathcal{C} \cap \mathcal{D}
\]

which are mutually inverse. Moreover these bijections restrict to bijections between the collection of tilting subcategories of \( \mathcal{A} \) and the collection of cotorsion torsion triples in \( \mathcal{A} \).

We note that Buan and Zhou \([\text{BZ}]\) introduced the notion of left weak cotorsion torsion triple in order to provide a version of \([\text{BBOS}, \text{Theorem 2.29}]\) for support \( \tau \)-tilting modules in mod-\( \Lambda \), where \( \Lambda \) is an artin algebra. In Theorem 4.10 we show that a triple \((\mathcal{C}, \mathcal{D}, \mathcal{F})\) of subcategories of mod-\( \Lambda \) is a left weak cotorsion torsion triple if and only if it is a \( \tau \)-cotorsion torsion triple.

One of the main properties of support \( \tau \)-tilting modules is that they generate functorially finite torsion classes in module categories. As a consequence of our results, we obtain the following corollary, see Theorem 5.7 and Corollary 4.8.

Corollary 1.4. Let \( \mathcal{A} \) be an abelian category with enough projective objects and let \( \mathcal{T} \) be a support \( \tau \)-tilting subcategory of \( \mathcal{A} \). Then \( \text{Fac}(\mathcal{T}) \) is a functorially finite torsion class in \( \mathcal{A} \).

For mod-\( \Lambda \), using results of \([\text{AIR}]\), it is shown that a module \( T \) in mod-\( \Lambda \) is a support \( \tau \)-tilting module if and only if \( \text{add}(T) \) is a support \( \tau \)-tilting subcategory of mod-\( \Lambda \). On the other hand, \( T \) is a support \( \tau^- \)-tilting module if and only if \( \text{add}(T) \) is a support \( \tau^- \)-tilting subcategory of mod-\( \Lambda \), see Propositions 5.9 and 6.11.

Suppose that \( \mathcal{A} \) is an abelian category with enough injective objects. Then in this category all the dual definitions and results can be stated and proved, see Section 6. Using this symmetry, when \( \mathcal{A} \) is an abelian category with enough projective and enough injective objects, we introduce the notion of a \( \tau^- \)-quadruple and use it to create a map from support \( \tau \)-tilting to support \( \tau^- \)-tilting subcategories of \( \mathcal{A} \). We show in Proposition 7.4 that this map is exactly the dual of the dagger map defined in \([\text{AIR}, \text{Theorem 2.15}]\) when \( \mathcal{A} = \text{mod-}\Lambda \).

Besides mod-\( \Lambda \), another prominent example of an abelian category with enough projective objects is Mod-\( R \), the category of modules over an unitary associative ring \( R \).
The theory of support $\tau$-tilting modules of $[AIR]$ is generalized to Mod-$R$ by Angeleri Hügel, Marks and Vitoria [AMV], where they introduced silting modules showing that finitely presented silting $\Lambda$-modules coincide with the support $\tau$-tilting modules of $[AIR]$. As the categorical dual of silting modules, Breaz and Pop introduced the notion of cosilting modules in Mod-$R$ and showed that finitely copresented cosilting $\Lambda$-modules coincide with the support $\tau^-$-tilting modules.

We study the connection between silting, resp. cosilting, modules in Mod-$R$ with the $\tau$-tilting, resp. $\tau^-$-tilting subcategories of Mod-$R$. Furthermore, we characterize all support $\tau$-tilting subcategories of Mod-$R$. In particular, we provide a bijection between the equivalence classes of all support $\tau$-tilting subcategories of Mod-$R$ and the collection of all equivalent classes of certain $R$-modules, the so-called finendo quasitilting $R$-modules, see Theorem 8.1.10. It is known that all silting modules are finendo quasitilting. Dually, we also characterize all support $\tau^-$-tilting subcategories of Mod-$R$ by providing a bijection between the equivalence classes of all support $\tau^-$-tilting subcategories of Mod-$R$ and the collection of all equivalent classes of the quasicotilting $R$-modules.

Towards the end of the paper, we provide applications of this theory to the category of the representations of a finite and acyclic quiver in an abelian category $A$ with enough projective objects. Let $Q$ be such a quiver. It is known that the category Rep($Q, \mathcal{A}$) of representations of $Q$ over $\mathcal{A}$ is again an abelian category with enough projective objects. See Section 9 for more details. We give a recipe to construct support $\tau$-tilting subcategories in Rep($Q, \mathcal{A}$) from certain support $\tau$-tilting subcategories of $\mathcal{A}$. More explicitly, we have the following result.

**Theorem 1.5 (Theorem 9.2.2).** Let $\mathcal{A}$ be an abelian category with enough projective objects and $Q$ be a finite acyclic quiver. Let $\mathcal{T}$ be a support $\tau$-tilting subcategory of $\mathcal{A}$ such that Fac($\mathcal{T}$) is closed with respect to the kernels of epimorphisms. Then

$$T = \operatorname{add}\{e_i^\rho(T) \mid i \in Q_0, T \in \mathcal{T}\}$$

is a support $\tau$-tilting subcategory of Rep($Q, \mathcal{A}$).

Let Prj($\mathcal{A}$) denote the category of all projective objects in $\mathcal{A}$. In [BBOS, Proposition 3.9] the authors have shown that

$$T = \operatorname{add}\{e_i^\rho(P) \mid i \in Q_0, P \in \operatorname{Prj}(\mathcal{A})\}$$

is a tilting subcategory of $\mathcal{A}$. Then the previous result can be seen as a generalization of this result, since $\mathcal{A} = \operatorname{Fac}(\operatorname{Prj}(\mathcal{A}))$ is clearly closed under kernels of epimorphisms.

There are other generalisations of tilting theory that we have not mentioned yet. One of these generalizations consists on defining $n$-tilting subcategories, where 1-tilting subcategories correspond to classical tilting subcategories, see Definition 2.1. By convention we consider Prj($\mathcal{A}$) as the 0-tilting subcategory of $\mathcal{A}$. Using similar methods as in the proof of Theorem 9.2.2 we were able to show another generalization of [BBOS, Proposition 3.9], which we believe might have interesting implications in the study of higher homological algebra, introduced by Iyama in [11, 12].

**Theorem 1.6 (Theorem 9.4.1).** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $n$ be a non-negative integer and let $Q$ be a finite and acyclic quiver. For an $n$-tilting subcategory $\mathcal{T}$ of $\mathcal{A}$ put

$$\mathcal{T}' = \operatorname{add}\{e_i^\rho(T) \mid i \in Q_0, T \in \mathcal{T}\}.$$

Then $\mathcal{T}'$ is an $(n + 1)$-tilting subcategory of Rep($Q, \mathcal{A}$).
Throughout the paper, several examples are provided using the theory of persistent modules, which are central objects of study in topological data analysis (TDA). We refer the reader to the survey [Ca] by Carlsson, or the book [O] by Oudot, for an introduction to topological data analysis and connections to quiver representations.

The paper is structured as follows. In Section 2 we fix notation and give the necessary background for the rest of the paper. In Section 3 we define support $\tau$-tilting subcategories in abelian categories with enough projective objects and we give some of its basic properties. Then, in Section 4 we introduce and study the notion of $\tau$-cotorsion torsion triples. We also compare $\tau$-cotorsion torsion triples with the left weak cotorsion torsion triples of [BZ]. In Section 5 we show the bijection between support $\tau$-tilting subcategories and $\tau$-cotorsion torsion triples. Section 6 is a compilation of dual definitions and results. Later, in Section 7 we relate support $\tau$-tilting subcategories with support $\tau^-$-tilting subcategories via $\tau^-\tau^-$-quadruples. In Section 8 we study support $\tau^-$ and $\tau^-$-tilting subcategories in $\text{Mod-}R$, where $R$ is a unitary associative ring, and show that how they are connected to the silting and co-silting theory. We end the paper in Section 9 where we study support $\tau$-tilting subcategories and $n$-tilting subcategories in the category $\text{Rep}(Q, \mathcal{A})$ of quiver representations over $\mathcal{A}$.

2. Preliminaries

Let $\mathcal{A}$ be an abelian category. In this paper by subcategory we always mean a full subcategory. Let $\mathcal{X}$ be a subcategory of $\mathcal{A}$. A morphism $\varphi : X \rightarrow A$, where $A$ is an object of $\mathcal{A}$, is called a right $\mathcal{X}$-approximation of $A$ if, $X \in \mathcal{X}$ and for every $X' \in \mathcal{X}$, the induced morphism $A(X', X) \rightarrow A(X', A) \rightarrow 0$ of abelian groups is exact. We say that $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{A}$ if every object $A$ of $\mathcal{A}$ admits a right $\mathcal{X}$-approximation. Dually, the notions of left $\mathcal{X}$-approximations and covariantly finite subcategories are defined. Moreover we say that $\mathcal{X}$ is a functorially finite subcategory of $\mathcal{A}$ if it is both a contravariantly finite and a covariantly finite subcategory of $\mathcal{A}$. If $\mathcal{X}$ is closed under taking finite direct sums and direct summands we say that it is additively closed.

Let $n$ be a non-negative integer. We define

$$\mathcal{X}^{\perp n} := \{ A \in \mathcal{A} \mid \text{Ext}^n_{\mathcal{A}}(\mathcal{X}, A) = 0 \} ,$$

$$^{\perp n}\mathcal{X} := \{ A \in \mathcal{A} \mid \text{Ext}^n_{\mathcal{A}}(A, \mathcal{X}) = 0 \} .$$

Note that $\text{Ext}^0$ is just the usual Hom-functor.

Moreover, we set

$$\mathcal{X}^{\perp} := \{ A \in \mathcal{A} \mid \text{Ext}^1_{\mathcal{A}}(\mathcal{X}, A) = 0, \forall i \geq 1 \} ,$$

$$^{\perp}\mathcal{X} := \{ A \in \mathcal{A} \mid \text{Ext}^1_{\mathcal{A}}(A, \mathcal{X}) = 0, \forall i \geq 1 \} .$$

Definition 2.1. Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $n$ be a non-negative integer. An additively closed subcategory $\mathcal{F}$ of $\mathcal{A}$ is called an $n$-tilting subcategory if

(i) $\mathcal{F}$ is a contravariantly finite subcategory of $\mathcal{A}$.
(ii) $\text{Ext}^n_{\mathcal{A}}(T_1, T_2) = 0$, for all $T_1, T_2 \in \mathcal{F}$ and all $i \geq 1$.
(iii) Every object $T \in \mathcal{F}$ has projective dimension at most $n$.
(iv) For every projective object $P$ in $\mathcal{A}$, there exists a short exact sequence

$$0 \rightarrow P \overset{f}{\rightarrow} T^0 \rightarrow T^1 \rightarrow \cdots \rightarrow T^n \rightarrow 0$$

with $T^i \in \mathcal{F}$.
If $\mathcal{T}$ only satisfies the conditions $(ii) - (iv)$, it is called a weak $n$-tilting subcategory of $\mathcal{A}$.

**Remark 2.2.** It follows, by breaking out the exact sequence of $(iv)$ to short exact sequences and using the vanishing of Ext-groups in $(ii)$ of the previous definition, that the map $f : P \to T^0$ is a left $\mathcal{T}$-approximation of $P$. Moreover, it is easy to see that the category $\text{Prj}(\mathcal{A})$ of all projective objects in $\mathcal{A}$ is the only 0-tilting subcategory of $\mathcal{A}$. In this paper, when we say tilting subcategory we mean a 1-tilting subcategory.

### 2.3. Cotorsion Torsion Triple

Here we recall the notion of torsion pairs, cotorsion pairs and cotorsion torsion triples. Let us begin by the following classical definition of Dickson [D].

**Definition 2.4.** Let $\mathcal{A}$ be an abelian category. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{A}$ is called a torsion pair if $\text{Hom}_\mathcal{A}(\mathcal{T}, \mathcal{F}) = 0$ and for every $A \in \mathcal{A}$ there is a short exact sequence

$$0 \to tA \to A \to fA \to 0$$

such that $tA \in \mathcal{T}$ and $fA \in \mathcal{F}$.

It is well known that $\mathcal{T}$ and $\mathcal{F}$ determine each others and the short exact sequence is functorial, see for instance [BR, §I.1].

**Definition 2.5.** (see [Sa]) Let $\mathcal{A}$ be an abelian category with enough projective objects. A pair $(\mathcal{C}, \mathcal{D})$ of full subcategories of $\mathcal{A}$ is called a cotorsion pair if $\mathcal{C} = \perp^1 \mathcal{D}$, $\mathcal{D} = \mathcal{C}^\perp$ and for every object $A \in \mathcal{A}$ there are short exact sequences

$$0 \to D \to C \to A \to 0$$

$$0 \to A \to D' \to C' \to 0$$

where $C$ and $C'$ are in $\mathcal{C}$ and $D$ and $D'$ are in $\mathcal{D}$.

This notion sometimes called complete cotorsion pair in the literature. See [BBOS, Remark 2.7] for more details.

**Remark 2.6.** It is immediate from the definition that in a cotorsion pair $(\mathcal{C}, \mathcal{D})$ both $\mathcal{C}$ and $\mathcal{D}$ are closed under extensions.

Let $\mathcal{A}$ be an abelian category with enough projective and enough injective objects. In view of [BR, Lemma V.3.3], a pair $(\mathcal{C}, \mathcal{D})$ of full subcategories of $\mathcal{A}$ is a cotorsion pair if and only if the following conditions are satisfied:

1. $\mathcal{C} = \perp^1 \mathcal{D}$,
2. for every object $A \in \mathcal{A}$, there exists a short exact sequence

$$0 \to A \xrightarrow{f} D \to C \to 0,$$

where $D \in \mathcal{D}$ and $C \in \mathcal{C}$.

Following proposition shows that when $\mathcal{D}$ is closed under factors it is enough, in the above definition, to check the Condition 2 only for projective objects.

**Proposition 2.7.** Let $\mathcal{A}$ be an abelian category with enough projective and enough injective objects. Let $(\mathcal{C}, \mathcal{D})$ be a pair of full subcategories of $\mathcal{A}$ such that $\mathcal{D}$ is closed under factors. Then $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair if and only if

1. $\mathcal{C} = \perp^1 \mathcal{D}$,
2. For every projective object $P \in \mathcal{A}$, there exists a short exact sequence
\[ 0 \to P \xrightarrow{f} D \to C \to 0, \]
where $D \in \mathcal{D} \cap \mathcal{C}$ and $C \in \mathcal{C}$.

Proof. We start by showing the necessity. Let $A$ be an arbitrary object of $\mathcal{A}$. Since $\mathcal{A}$ has enough projective objects, we have an epimorphism $P \to A \to 0$. By assumption $P$ fits into the short exact sequence
\[ 0 \to P \xrightarrow{f} D \to C \to 0, \]
where $D \in \mathcal{D} \cap \mathcal{C}$ and $C \in \mathcal{C}$.

The pushout diagram
\[
\begin{array}{cccccc}
0 & \to & P & \xrightarrow{f} & D & \to & C & \to & 0 \\
& & \downarrow{f} & & \downarrow{g} & & \\
0 & \to & A & \to & U & \to & C & \to & 0,
\end{array}
\]
induces the short exact sequence $0 \to A \to U \to C \to 0$, where $U \in \text{Fac}(\mathcal{D})$. Since $\mathcal{D}$ is closed under factors, we deduce that $U \in \mathcal{D}$ and hence this is the desired short exact sequence for $A$.

For the sufficiency, let $P$ be a projective object in $\mathcal{A}$. Then there exists a short exact sequence
\[ 0 \to P \xrightarrow{f} D \to C \to 0 \]
where $D \in \mathcal{D}$ and $C \in \mathcal{C}$. Since $P$ is projective, then $P \in \perp \mathcal{D} = \mathcal{C}$. Hence $D \in \mathcal{C}$ because $\mathcal{C}$ is closed under extensions and the result follows.

Following lemma shows that if the subcategory $\mathcal{C}$ of a cotorsion pair $(\mathcal{C}, \mathcal{D})$ of $\mathcal{A}$ is closed with respect to factors, then $\mathcal{C} \cap \mathcal{D}$ is contravariantly finite in $\mathcal{A}$.

Lemma 2.8. Let $(\mathcal{C}, \mathcal{D})$ be a cotorsion pair in $\mathcal{A}$ such that $\mathcal{D}$ is closed under factors. Then $\mathcal{C} \cap \mathcal{D}$ is a contravariantly finite subcategory of $\mathcal{A}$.

Proof. Since $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair, it follows easily that $\mathcal{D}$ is closed under extensions and products. So $\mathcal{D}$ is indeed a torsion class. Now, since $(\mathcal{C}, \mathcal{D})$ is a cotorsion pair, there exists a short exact sequence
\[ 0 \to D \to C \xrightarrow{i} tA \to 0 \]
such that $D \in \mathcal{D}$ and $C \in \mathcal{C}$. Here, since $D, tA \in \mathcal{F}$ and $\mathcal{F}$ is closed under extensions, we get $C \in \mathcal{C} \cap \mathcal{D}$. We claim that the composition $C \xrightarrow{i} tA \xrightarrow{j} A$ is a right $\mathcal{C} \cap \mathcal{D}$-approximation. To show the claim, let $X \xrightarrow{i} A$ be a morphism with $X \in \mathcal{C} \cap \mathcal{D}$. Since $X \in \mathcal{D}$ and every map from $\mathcal{D}$ to $A$, factors through $tA$, the morphism $\ell$ factors through $j$. Now since $X \in \mathcal{C}$ and $\text{Ext}^1_{\mathcal{A}}(X, D) = 0$, the morphism $\ell$ factors through $g$. Hence the claim is proved.

Definition 2.9. Let $\mathcal{A}$ be an abelian category with enough projective objects. A triple $(\mathcal{C}, \mathcal{F}, \mathcal{F})$ of full subcategories in $\mathcal{A}$ is called a cotorsion torsion triple, if $(\mathcal{C}, \mathcal{F})$ is a cotorsion pair and $(\mathcal{F}, \mathcal{F})$ is a torsion pair.

It has been recently shown in [BBOS] that cotorsion torsion triples and tilting subcategories are closely related. The explicit relation between these two concepts is as follows.
Theorem 2.10. ([BBOS, Theorem 2.29] Let $\mathcal{A}$ be an abelian category with enough projective objects. Then there is a bijection

\[
\{\text{tilting subcategories}\} \longleftrightarrow \{\text{cotorsion torsion triples}\}
\]

\[\mathcal{T} \longrightarrow (\mathcal{T}^{-1}(\text{Fac}(\mathcal{T})), \text{Fac}(\mathcal{T}), \mathcal{T}^{\perp})\]

\[\mathcal{C} \cap \mathcal{T} \longleftrightarrow (\mathcal{C}, \mathcal{T}, \mathcal{F})\]

where \text{Fac}(\mathcal{T}) is the full subcategory of $\mathcal{A}$ consisting of factors of objects in $\mathcal{T}$.

2.11. POINTWISE FINITE DIMENSIONAL REPRESENTATIONS. Let $k$ be a field and $(\mathcal{X}, \leq)$ be a poset. Consider $\mathcal{X}$ as a category. A persistence module is a (covariant and additive) functor $V : \mathcal{X} \to \text{Mod}-k$ from $\mathcal{X}$ to $\text{Mod}-k$, the category of $k$-vector spaces. A persistence module $V$ is called pointwise finite dimensional representation of $\mathcal{X}$ if it is a functor from $\mathcal{X}$ to $\text{mod}-k$, the category of finite dimensional vector spaces.

The abelian category of pointwise finite dimensional representations will be denoted by $\text{Rep}_{pfd}^k\mathcal{X}$. A subset $\mathcal{C}$ of $\mathcal{X}$ is called a convex subset if for every $x \leq y \in \mathcal{C}$, $x \leq z \leq y$ implies $z \in \mathcal{C}$. Let $\mathcal{C}$ be a convex subset of $\mathcal{X}$. The constant representation $k_{\mathcal{C}}$ is defined by $k_{\mathcal{C}}(x) = k$, for $x \in \mathcal{C}$, $k_{\mathcal{C}}(x) = 0$, for $x \notin \mathcal{C}$ and $k_{\mathcal{C}}(x \leq y) = \text{id}_k$. Set $P_x = k_{\{y \in \mathcal{X} \mid y \geq x\}}$. As application of Yoneda’s Lemma one can see that, for every $x \in \mathcal{X}$, $P_x$ is a projective object of $\text{Rep}_{pfd}^k\mathcal{X}$ and vice versa all projective modules are of this form, see e.g. [BBOS, §2]. A representation $V \in \text{Rep}_{pfd}^k\mathcal{X}$ is called finitely generated if there exists an epimorphism of functors $\bigoplus_{i \in J} P_{x_i} \longrightarrow V$, where $J$ is a finite indexing set. Moreover, if the kernel of this epimorphism is again finitely generated, it is called finitely presented. We denote the full subcategory of $\text{Rep}_{pfd}^k\mathcal{X}$ consisting of finitely presented representations by $\text{Rep}_{fd}^k\mathcal{X}$.

Of particular importance is the case $\mathcal{X} = \mathbb{R}_{\geq 0}$, the poset of non-negative real numbers and to concentrate on the abelian subcategory $\text{Rep}_{pfd}^{\mathbb{R}_{\geq 0}}$ of finitely presented representations. It is proved in [C, Theorem 1.1] that the indecomposable objects in this category are classified by the constant representation $k_{([x,y])}$, where $x < y \leq \infty$. Moreover, for each $x \geq 0$, the constant representations $k_{([x,\infty])}$ and $k_{([0,x])}$ are projective and injective objects, respectively.

3. $\tau$-TILTING SUBCATEGORIES

In this section, we recall the definition of $\tau$-tilting subcategories and study some of their properties. Following definition is motivated by [IJY, Definition 1.5] and [LZh, Definition 2.1].

**Definition 3.1.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{T}$ be an additive full subcategory of $\mathcal{A}$. Then $\mathcal{T}$ is called a weak support $\tau$-tilting subcategory of $\mathcal{A}$ if

(i) $\text{Ext}_{\mathcal{A}}^1(T_1, \text{Fac}(T_2)) = 0$, for all $T_1, T_2 \in \mathcal{T}$.

(ii) For any projective $P$ in $\mathcal{A}$, there exists an exact sequence

$$P \xrightarrow{f} T^0 \longrightarrow T^1 \longrightarrow 0$$

such that $T^0$ and $T^1$ are in $\mathcal{T}$ and $f$ is a left $\mathcal{T}$-approximation of $P$.

If furthermore $\mathcal{T}$ is a contravariantly finite subcategory of $\mathcal{A}$, it is called a support $\tau$-tilting subcategory of $\mathcal{A}$. A support $\tau$-tilting subcategory is simply called $\tau$-tilting if the approximation $f : P \longrightarrow T^0$ is non-zero for every projective object $P$.

**Remark 3.2.** Note that a key difference between tilting and $\tau$-tilting subcategories is that the approximations $f : P \longrightarrow T^0$ are always a monomorphism when $\mathcal{T}$ is a tilting subcategory.
It is immediate from the definition that every (weak) tilting subcategory is a (weak) $\tau$-tilting subcategory.

In the following example we present a support $\tau$-tilting subcategory which is not a tilting subcategory. Note that for $X \in \mathcal{A}$, we let $\text{add}(X)$ be the category of all direct summands of finite direct sums of copies of $X$.

**Example 3.3.** Let $\mathcal{A} = \text{Rep}_{k}^{fp} R \geq 0$, and set
\[
\mathcal{T} = \text{add}\{k_{[x,\infty]} \mid x \geq 1\} \cup \{k_{(0,x)} \mid x \leq 1\}.
\]
Then $\mathcal{T}$ is a support $\tau$-tilting subcategory. To see this, first note that,
\[
\text{Fac}(\mathcal{T}) = \text{add}\{k_{[x,y]} \mid 1 \leq x < y \leq \infty\} \cup \{k_{[0,x]} \mid x \leq 1\},
\]
and obviously $\text{Ext}^1_{\mathcal{T}}(\mathcal{T}, \text{Fac}(\mathcal{T})) = 0$.

Now let $k_{[a,\infty)}$ be an indecomposable projective in $\mathcal{A}$. If $a \geq 1$, the exact sequence
\[
0 \rightarrow k_{[a,\infty)} \rightarrow k_{[a,\infty)} \rightarrow 0,
\]
and if $0 \leq a < 1$, the exact sequence
\[
k_{[a,\infty)} \rightarrow k_{[0,1)} \rightarrow k_{[1,\infty)} \rightarrow 0,
\]
are the desired exact sequences.

Moreover, $\mathcal{T}$ is a contravariantly finite subcategory of $\mathcal{A}$. Indeed, depending on $a$ and $b$, for an indecomposable representation $k_{[a,b)}$ we have the following right $\mathcal{T}$-approximations
\[
\begin{cases}
  k_{[0,b)} \rightarrow k_{[0,b)}, & 0 = a < b \leq 1; \\
  k_{[1,\infty)} \rightarrow k_{[a,b)}, & 0 \leq a < b \leq \infty; \\
  0 \rightarrow k_{[a,b)}, & 0 < a < b \leq 1; \\
  k_{[a,\infty)} \rightarrow k_{[a,b)}, & 1 \leq a < b \leq \infty.
\end{cases}
\]
It is obviously not a tilting subcategory of $\mathcal{A}$, because $k_{[0,\infty)}$ is a projective object, for which we do not have a short exact sequence like Condition (iv) of Definition [BBOS].

One of the main reasons behind the success of $\tau$-tilting theory [AIR] is the formal inclusion of the notion of support from the start, see Definition 5.8. Note that in this definition we are defining objects in mod-$\Lambda$ by using a property of the object in a different category, namely mod-$\Lambda/e)$. So one needs to verify that the good properties of a support $\tau$-tilting object $M$ in mod-$\Lambda/e)$ can be transported to mod-$\Lambda$. This was done in [AIR, Lemma 2.1]. Note that mod-$\Lambda/e)$ is a functorially finite wide subcategory of mod-$\Lambda$ which is at the same time a torsion and a torsion-free class. Recall that a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is called a wide subcategory if it is closed under kernels, cokernels and extensions. In particular, this implies that $\mathcal{X}$ itself is an abelian category. It is known that a subcategory $\mathcal{X}$ of an abelian category $\mathcal{A}$ is a torsion class if it is a contravariantly finite subcategory of $\mathcal{A}$, closed with respect to quotients and extensions. Moreover, $\mathcal{X}$ is a torsion free class if it is a covariantly finite subcategory of $\mathcal{A}$ which is furthermore closed under subobjects and extensions.

**Lemma 3.4.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{X}$ be a wide and functorially finite full subcategory of $\mathcal{A}$. Then $\mathcal{X}$ is a torsion class of $\mathcal{A}$ if and only if $\mathcal{X}$ is a torsion free class of $\mathcal{A}$.
Proof. Let \( \mathcal{X} \) be a torsion class of \( \mathcal{A} \). In order to show that it is a torsion free class, we need to show that it is covariantly finite, closed under subobject and closed under extension. By the assumption \( \mathcal{X} \) is a covariantly finite subcategory. It is closed under extension, since \( \mathcal{X} \) is a torsion class. Moreover, it is closed under subobject, since it is a wide subcategory. The other implication follows similarly.

Our next result is inspired by [AIR, Lemma 2.1] and it justifies the name of support \( \tau \)-tilting subcategories.

**Proposition 3.5.** Let \( \mathcal{A} \) be an abelian category with enough projective objects. Let \( \mathcal{X} \) be a wide and functorially finite torsion class of \( \mathcal{A} \). Then every support \( \tau \)-tilting subcategory of \( \mathcal{X} \) is a support \( \tau \)-tilting subcategory of \( \mathcal{A} \).

**Proof.** Let \( \mathcal{T} \) be a support \( \tau \)-tilting subcategory of \( \mathcal{X} \). First we note that, since \( \mathcal{X} \) is a wide subcategory, then \( \text{Fac}(\mathcal{T}) \subseteq \text{Fac}(\mathcal{X}) = \mathcal{X} \). Therefore, since \( \mathcal{T} \) is a support \( \tau \)-tilting subcategory of \( \mathcal{X} \), we have \( \text{Ext}^1_{\mathcal{A}}(\mathcal{T}, \text{Fac}(\mathcal{T})) = 0 \). Hence \( \text{Ext}^1_{\mathcal{A}}(\mathcal{T}, \text{Fac}(\mathcal{T})) = 0 \).

Now let \( P \) be a projective object in \( \mathcal{A} \). By Lemma 3.4, there is a subcategory \( \mathcal{Y} \) of \( \mathcal{A} \) such that \((\mathcal{Y}, \mathcal{X})\) is a torsion pair of \( \mathcal{A} \). Consider the canonical short exact sequence

\[
0 \longrightarrow Y \longrightarrow P \overset{f}{\longrightarrow} X \longrightarrow 0
\]

of \( P \) with respect to this torsion pair. So \( Y \in \mathcal{Y} \) and \( X \in \mathcal{X} \). Let \( X' \) be an arbitrary object of \( \mathcal{X} \). By applying the functor \( \text{Hom}_{\mathcal{A}}(-, X') \) to the above short exact sequence, we get \( \text{Ext}^1_{\mathcal{A}}(\mathcal{T}, \text{Fac}(\mathcal{T})) = 0 \) and therefore \( \text{Ext}^1_{\mathcal{A}}(\mathcal{T}, X') = 0 \). Hence \( X \) is an Ext-projective object in \( \mathcal{X} \).

Since \( \mathcal{T} \) is a support \( \tau \)-tilting subcategory of \( \mathcal{X} \), there exists an exact sequence

\[
X \overset{g}{\longrightarrow} T^0 \longrightarrow T^1 \longrightarrow 0
\]

where \( T^0, T^1 \in \mathcal{T} \) and \( g \) is a left \( \mathcal{T} \)-approximation. Now the exact sequence

\[
P \overset{\varphi}{\longrightarrow} T^0 \longrightarrow T^1 \longrightarrow 0
\]

is the desired one. In fact, it is easy to see that \( gf \) is a left \( \mathcal{T} \)-approximation of \( P \).

Finally we show that \( \mathcal{T} \) is a contravariantly finite subcategory of \( \mathcal{A} \). Let \( A \in \mathcal{A} \). Since \( \mathcal{X} \) is a functorially finite subcategory of \( \mathcal{A} \), there exists a right \( \mathcal{X} \)-approximation \( X \overset{f}{\longrightarrow} A \) of \( A \).

Consider a right \( \mathcal{T} \)-approximation \( T \overset{g}{\longrightarrow} X \) of \( X \), which exists because \( \mathcal{T} \) is a support \( \tau \)-tilting subcategory of \( \mathcal{A} \). Now it is easy to see that \( T \overset{gf}{\longrightarrow} A \) is a right \( \mathcal{T} \)-approximation of \( A \). \( \square \)

We illustrate our previous result with the following example.

**Example 3.6.** Let \( \mathcal{A} = \text{Rep}_k^{\text{fp}} \mathbb{R}_{\geq 0} \) and \( \mathcal{B} = \text{Rep}_k^{\text{fp}} \mathbb{R}_{\geq 1} \). On one hand, since \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{\geq 1} \) are isomorphic as posets, it is clear that \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent as categories. On the other hand, the natural inclusion of \( \mathbb{R}_{\geq 1} \) into \( \mathbb{R}_{\geq 0} \) induces an embedding of \( \mathcal{B} \) into \( \mathcal{A} \). In fact, it is easy to see that \( \mathcal{B} \) is a functorially finite wide subcategory of \( \mathcal{A} \) which is both a torsion and a torsion free class. Set

\[
\mathcal{T} = \text{add}(\{k_{[1,y]} \mid 1 < y \leq \infty\}).
\]

It is easy to see that \( \mathcal{T} \) is a tilting subcategory of \( \mathcal{B} \) and hence a support \( \tau \)-tilting subcategory of \( \mathcal{B} \). So, Proposition 3.5 implies that \( \mathcal{T} \) is a support \( \tau \)-tilting subcategory of \( \mathcal{A} \). Note that \( \mathcal{T} \) is not a tilting subcategory of \( \mathcal{B} \) since the \( \mathcal{T} \)-approximation \( f : k_{[0,\infty)} \rightarrow k_{[1,\infty)} \) is not a monomorphism, in fact it is a zero morphism.
By [R, Proposition 3.42], if $\mathcal{A}$ is an abelian category with enough projective objects such that $\text{Prj}(\mathcal{A}) = \text{add}(P)$, then for every tilting subcategory $\mathcal{T}$ of $\mathcal{A}$ there exists an object $T \in \mathcal{T}$ such that $\mathcal{T} = \text{add}(T)$. In fact $T$ is a tilting object. Note that an object $T \in \mathcal{A}$ is called a tilting object if

(i) $\text{Ext}^1_{\mathcal{A}}(T, T) = 0$.
(ii) Projective dimension of $T$ is at most one.
(iii) For every projective object $P \in \mathcal{A}$, there exists a short exact sequence

$$0 \rightarrow P \rightarrow T^0 \rightarrow T^1 \rightarrow 0$$

such that $T^0, T^1 \in \text{add}(T)$.

We prove a version of this fact for $\tau$-tilting subcategories. To do this we provide the following definition of a $\tau$-tilting object in an abelian category.

**Definition 3.7.** Let $\mathcal{A}$ be an abelian category with enough projective objects. An object $T \in \mathcal{A}$ is called a support $\tau$-tilting object if

(i) $\text{Ext}^1_{\mathcal{A}}(T, \text{Fac}(T)) = 0$.
(ii) For every projective object $P \in \mathcal{A}$, there exists an exact sequence

$$P \rightarrow T^0 \rightarrow T^1 \rightarrow 0$$

such that $T^0, T^1 \in \text{add}(T)$ and $f$ is a left $\text{add}(T)$-approximation.

Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $T$ be a support $\tau$-tilting object in $\mathcal{A}$. It follows directly from the definition that $\text{add}(T)$ is a weak support $\tau$-tilting subcategory of $\mathcal{A}$. Next proposition provides a partial converse to this fact.

**Proposition 3.9.** Let $\mathcal{A}$ be an abelian category with enough projective objects such that $\text{Prj}(\mathcal{A}) = \text{add}(P)$, for some object $P \in \mathcal{A}$. If $\mathcal{T}$ is a support $\tau$-tilting subcategory of $\mathcal{A}$, then there exists a support $\tau$-tilting object $T \in \mathcal{T}$ such that $\text{Fac}(\mathcal{T}) = \text{Fac}(T)$.

**Proof.** Consider the exact sequence

$$P \rightarrow T^0 \rightarrow T^1 \rightarrow 0$$

which exists, because $\mathcal{T}$ is a support $\tau$-tilting subcategory. We claim that $\text{Fac}(T^0 \oplus T^1) = \text{Fac}(\mathcal{T})$. First we prove that $T^0 \oplus T^1$ is a support $\tau$-tilting object. To this end, we just need to show that every projective object $P' \in \text{Prj}(\mathcal{A}) = \text{add}(P)$ admits an exact sequence

$$P' \rightarrow T^0_{P'} \rightarrow T^1_{P'} \rightarrow 0$$

where $f'$ is a left $\text{add}(T^0 \oplus T^1)$-approximation and $T^0_{P'}, T^1_{P'} \in \text{add}(T^0 \oplus T^1)$. Since $\text{Prj}(\mathcal{A}) = \text{add}(P)$, there exists a non-negative integer $n$ and projective object $Q \in \text{add}(P)$ such that $P' \oplus Q = P^n$. So we get the exact sequence

$$P' \oplus Q \rightarrow (T^0)^n \rightarrow (T^1)^n \rightarrow 0.$$
Based on this sequence, we can construct the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & (T^0)^n & \rightarrow & (T^0)^n & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & 0 \\
0 & \rightarrow & P' \oplus Q & \rightarrow & (T^0)^n \oplus (T^0)^n & \rightarrow & \text{Coker } g \oplus \text{Coker } h & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & 0 \\
0 & \rightarrow & P' \oplus Q & \rightarrow & (T^0)^n & \rightarrow & (T^1)^n & \rightarrow & 0 \\
\end{array}
\]

Since the second vertical short exact sequence splits, both Coker \( g \) and Coker \( h \) are in \( \text{add}(T_0 \oplus T_1) \).
Therefore the exact sequence

\[ P' \rightarrow (T^0)^n \rightarrow \text{Coker } g \rightarrow 0 \]

is the desired one.

Now we show that \( \text{Fac}(\mathcal{F}) = \text{Fac}(T^0 \oplus T^1) \). It is clear that \( \text{Fac}(T^0 \oplus T^1) \subseteq \text{Fac}(\mathcal{F}) \). So it is enough to show the reverse inclusion. To do this, we show that every object \( \bar{T} \in \mathcal{F} \) lies in \( \text{Fac}(T^0 \oplus T^1) \). Let

\[ P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} \bar{T} \rightarrow 0 \]

be a projective presentation of \( \bar{T} \). Since \( T^0 \oplus T^1 \) is a support \( \tau \)-tilting object, for \( i = 0, 1 \), there are exact sequences

\[ P_i \xrightarrow{f_i} T_i^0 \xrightarrow{g_i} T_i^1 \rightarrow 0 \]

with \( T_i^0, T_i^1 \in \text{add}(T_0 \oplus T_1) \). Therefore, since \( f_i \) is a left \( \text{add}(T^0 \oplus T^1) \)-approximation of \( P_i \), we have the commutative diagram

\[
\begin{array}{ccccccc}
P_1 & \xrightarrow{h_1} & P_0 & \xrightarrow{h_0} & \bar{T} & \rightarrow & 0 \\
\downarrow f_1 & & \downarrow f_0 & & \downarrow & \downarrow & 0 \\
T_0^1 & \xrightarrow{k_1} & T_0^0 & \xrightarrow{k_0} & \bar{T} & \rightarrow & 0 \\
\downarrow g_1 & & \downarrow g_0 & & \downarrow & \downarrow & 0 \\
T_1^1 & \rightarrow & T_0^1 & \rightarrow & \bar{T} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

where the first row and the first column are exact. Since \( g_0k_1f_1 = 0 \) and \( k_0k_1f_1 = 0 \), by the cokernel property we have morphisms \( l_1 : T_1^1 \rightarrow T_0^1 \) and \( l_0 : T_1^1 \rightarrow \bar{T} \) such that \( g_0k_1 = l_1g_1 \).
and \( l_0 g_1 = k_0 k_1 \). To finish the proof we show that the sequence

\[
\begin{array}{c}
T_0^0 \oplus T_1^1 \\
\downarrow \begin{bmatrix} g_0 & l_0 \\ k_0 & l_1 \end{bmatrix} \\
T_0^1 \oplus \bar{T} \\
\downarrow 0
\end{array}
\]

is exact. Indeed, let \( x \oplus y \in T_0^0 \oplus T_1^1 \). Since \( g_0 \) is an epimorphism, there exists \( t_0' \in T_0^0 \) such that \( g_0(t_0') = x \). Now since \( h_0 \) is an epimorphism, there exists \( p_0 \in P_0 \) such that \( h_0(p_0) = k_0(t_0') - y \). Consider \( (t_0' - f_0(p_0)) \oplus 0 \in T_0^0 \oplus T_1^1 \). Therefore, it is easy to see that

\[
\begin{bmatrix} g_0 & l_0 \\ k_0 & l_1 \end{bmatrix} \begin{bmatrix} t_0' - f_0(p_0) \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Hence \( \bar{T} \in \text{Fac}(T_0^0 \oplus T_1^1) \subseteq \text{Fac}(T_0 \oplus T_1) \). \( \square \)

When \( \mathcal{S} \) is a weak tilting subcategory of \( \mathcal{A} \), then \( \text{Fac}(\mathcal{S}) = \mathcal{S} \perp_1 \), see [BBOS, Proposition 2.22]. The following example shows that this equality does not hold in general for \( \tau \)-tilting subcategories.

**Example 3.10.** Let \( A \) be the path algebra of the quiver

\[
\begin{array}{c}
2 \\
\downarrow \\
1 \\
\downarrow \\
3
\end{array}
\]

modulo the ideal generated by all paths of length 2. The Auslander-Reiten quiver of \( A \) is

\[
\begin{array}{c}
2 \\
\downarrow \\
3 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\downarrow \\
2 \\
\downarrow \\
3 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\downarrow \\
2
\end{array}
\]

Let \( \mathcal{S} = \text{add}(^2 \oplus 2 \oplus 1^2) \). Then \( \mathcal{S} \) is a \( \tau \)-tilting subcategory of \( \text{mod-}A \). It is easy to see that

\[
\text{Fac}(\mathcal{S}) = \text{add}(^2 \oplus 2 \oplus 1^2) \neq \text{add}(^3 \oplus 2 \oplus 1^2) = \mathcal{S} \perp_1.
\]

**4. \( \tau \)-COTORSION TORSION TRIPLES**

In this section we introduce a triple of full additive subcategories of \( \mathcal{A} \), that will be called a \( \tau \)-cotorsion torsion triple, or simply a \( \tau \)-triple, and show that there is a bijection between the collection of all \( \tau \)-tilting subcategories of \( \mathcal{A} \) and the collection of all \( \tau \)-triples. The Proposition 2.7 motivates the following definition.

**Definition 4.1.** Let \( \mathcal{A} \) be an abelian category with enough projective objects. A pair of full subcategories \((\mathcal{C}, \mathcal{D})\) of \( \mathcal{A} \) is called a \( \tau \)-cotorsion pair if

1. \( \mathcal{C} = \perp_1 \mathcal{D} \).
2. For every projective object $P \in \mathcal{A}$, there exists an exact sequence

$$P \xrightarrow{f} D \longrightarrow C \longrightarrow 0,$$

where $D \in \mathcal{C} \cap \mathcal{D}$, $C \in \mathcal{C}$ and $f$ is a left $\mathcal{D}$-approximation.

3. $\mathcal{C} \cap \mathcal{D}$ is a contravariantly finite subcategory of $\mathcal{A}$.

Example 4.2. Let $\mathcal{A} = \text{Rep}_k^{fp} \mathbb{R}_{\geq 0}$ and set

$\mathcal{C} = \text{add}(\{k_{[x, \infty)} | 0 \leq x < \infty\} \cup \{k_{[x,y)} | 0 \leq x < y < 1\})$

$\mathcal{D} = \text{add}(\{k_{[x,y)} | 1 \leq x < y \leq \infty\}).$

Then $(\mathcal{C}, \mathcal{D})$ is a $\tau$-cotorsion pair. Note that it is not a cotorsion pair, because $\mathcal{D}$ does not contain the injective representations. To see it is a $\tau$-cotorsion pair, first note that

$\mathcal{C} \cap \mathcal{D} = \text{add}(\{k_{[x, \infty)} | x \geq 1\})$

is a contravariantly finite subcategory of $\mathcal{A}$. Indeed, every indecomposable representation $k_{[a,b)}$ admits a right $\mathcal{C} \cap \mathcal{D}$-approximation as follows

$$\begin{cases} k_{[a,\infty)} \longrightarrow k_{[a,b)}, & 1 \leq a < b \leq \infty; \\
 k_{[1,\infty)} \longrightarrow k_{[a,b)}, & 0 < a < 1 < b \leq \infty; \\
 0 \longrightarrow k_{[a,b)}, & 0 \leq a < b \leq 1. 
\end{cases}$$

Now let $k_{[a,\infty)}$ be an indecomposable projective in $\mathcal{A}$. If $a \geq 1$, the exact sequence

$$0 \longrightarrow k_{[a,\infty)} \longrightarrow k_{[a,\infty)} \longrightarrow 0,$$

and if $a < 1$, the exact sequence

$$k_{[a,\infty)} \xrightarrow{0} k_{[b,\infty)} \longrightarrow k_{[b,\infty)} \longrightarrow 0$$

with $b \geq 1$, are the desired exact sequences. Finally, it is straightforward to see that $\mathcal{C} = \perp^{\perp}_1 \mathcal{D}$.

It is shown in [BBOS, Lemma 2.12] that if $\mathcal{C}$ and $\mathcal{D}$ are two subcategories of an abelian category $\mathcal{A}$ such that

1. $\text{Ext}^1_{\mathcal{A}}(\mathcal{C}, \mathcal{D}) = 0$;
2. $\mathcal{D}$ is closed under factor objects;
3. For every object $A \in \mathcal{A}$, there exists a short exact sequence $0 \longrightarrow A \xrightarrow{\varphi} D \longrightarrow C \longrightarrow 0$

where $D \in \mathcal{D}$ and $C \in \mathcal{C}$,

then every object in $\mathcal{C}$ is of projective dimension at most one. In particular, a cotorsion pair $(\mathcal{C}, \mathcal{F})$ satisfies all the above conditions when it is embedded in a cotorsion torsion triple $(\mathcal{C}, \mathcal{F}, \mathcal{F})$. So every object in $\mathcal{C}$ has projective dimension at most one.

The following example shows that the injectivity of the morphism $\varphi$ in Condition (3) is essential. We note that this is implicit in the proof of [BBOS, Lemma 2.12]. Also note that it follows automatically from the previous definition that the map $\varphi$ is a left $\mathcal{D}$-approximation.

Example 4.3. Let $A$ be the path algebra of the quiver

$$1 \xrightarrow{1} 2 \longrightarrow 3$$

modulo the ideal generated by the composition of the two arrows. The Auslander-Reiten quiver of $A$ is the following.
Let $\mathcal{T} = \text{add}(1 \oplus 1 \oplus 3)$. It is easy to see that $\mathcal{T}$ is a $\tau$-tilting subcategory of mod-$A$. Then we have

$$\mathcal{D} = \text{Fac}(\mathcal{T}) = \text{add}(1 \oplus 1 \oplus 3),$$
$$\mathcal{C} = \perp_1 \text{Fac}(\mathcal{T}) = \text{add}(1 \oplus 1 \oplus 3 \oplus 2).$$

It is clear that $\text{Fac}(\mathcal{T})$ is closed under factor modules. However the object $1 \in \mathcal{C}$ has projective dimension 2.

**Definition 4.4.** A triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ of full subcategories in $\mathcal{A}$ is called a $\tau$-cotorsion torsion triple, or simply a $\tau$-triple, if $(\mathcal{C}, \mathcal{T})$ is a $\tau$-cotorsion pair and $(\mathcal{T}, \mathcal{F})$ is a torsion pair.

**Example 4.5.** Let $\mathcal{A} = \text{Repl}_{k}^{fp} R_{\geq 0}$ and $(\mathcal{C}, \mathcal{D})$ be the $\tau$-cotorsion pair as in the Example 4.2. Since $\mathcal{D}$ is a contravariantly finite subcategory of $\mathcal{A}$ which is closed under factors and extensions, it is a torsion class. Hence $(\mathcal{C}, \mathcal{D}, \mathcal{D}^\perp)$ is a $\tau$-triple.

**Proposition 4.6.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Then every cotorsion torsion triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ is a $\tau$-cotorsion torsion triple.

**Proof.** Since $(\mathcal{F}, \mathcal{T})$ is a torsion pair, it is enough to show that $(\mathcal{C}, \mathcal{T})$ is a $\tau$-cotorsion pair. The Condition 1 of Definition 4.1, holds trivially. Now let $P$ be a projective object in $\mathcal{A}$ and consider the short exact sequence $0 \rightarrow P \rightarrow T \rightarrow C \rightarrow 0$, in which $T \in \mathcal{T}, C \in \mathcal{C}$ and $f$ is a left $\mathcal{T}$-approximation. It is exists by definition of a cotorsion pair. To show the validity of Condition 2 of Definition 4.1, we just need to show that $T \in \mathcal{C} \cap \mathcal{T}$. This follows using the facts that $P, C \in \mathcal{C}$ and $\mathcal{C}$ is closed under extensions. The contravariantly finiteness of $\mathcal{C} \cap \mathcal{T}$ follows from Lemma 2.8.

**Proposition 4.7.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be a $\tau$-triple in $\mathcal{A}$. Then for every object $A \in \mathcal{A}$ there exists an exact sequence

$$A \xrightarrow{g} T \rightarrow C \rightarrow 0,$$

where $T \in \mathcal{T}, C \in \mathcal{C}$ and $g$ is a left $\mathcal{T}$-approximation.

**Proof.** Let $A$ be an arbitrary object of $\mathcal{A}$. Let $P \rightarrow A \rightarrow 0$ be an epimorphism from a projective object in $\mathcal{A}$. By (3) of Definition 4.1, we can construct the following pushout diagram,
where \( C, \tilde{C} \in C \cap \mathcal{T} \) and \( f \) is a left \( \mathcal{T} \)-approximation

\[
\begin{array}{c}
P \xrightarrow{f} C \xrightarrow{g} T \\
\downarrow \pi \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
A \xrightarrow{g} T \xrightarrow{\tilde{C}} \tilde{C} \xrightarrow{t} 0
\end{array}
\]

We show that the second row in the above diagram is the desired exact sequence. First, we note that \( T \in \mathcal{T} \), because \( T \in \text{Fac}(\mathcal{T}) \) and \( \mathcal{T} \) is closed under quotients. In order to complete the proof, it remains to show that \( g : A \rightarrow T \) is a left \( \mathcal{T} \)-approximation. Let \( h : A \rightarrow T' \) be a morphism in \( \mathcal{A} \) with \( T' \in \mathcal{T} \). Since \( f \) is a left \( \mathcal{T} \)-approximation, there exists a morphism \( l : C \rightarrow T' \) such that \( lf = h \pi \). Thus the pushout property implies the existence of the morphism \( t : T \rightarrow T' \) such that the diagram

\[
\begin{array}{c}
P \xrightarrow{f} C \\
\downarrow \pi \\
A \xrightarrow{g} T \xrightarrow{t} T'
\end{array}
\]

is commutative. In other words we have that \( tg\pi = h\pi \). Since \( \pi : P \rightarrow A \) is an epimorphism, we obtain that \( tg = h \). Hence \( g \) is a left \( \mathcal{T} \)-approximation.

**Corollary 4.8.** Let \( \mathcal{A} \) be an abelian category with enough projective objects. Let \( (C, \mathcal{T}, F) \) be a \( \tau \)-triple in \( \mathcal{A} \). Then \( \mathcal{T} \) is a functorially finite torsion class of \( \mathcal{A} \).

**Proof.** The contravariantly finiteness of \( \mathcal{T} \) follows from the fact that it is a torsion class and its covariantly finiteness follows from the previous proposition. \( \square \)

**Remark 4.9.** In case \( \mathcal{A} = \text{mod-}\Lambda \), where \( \Lambda \) is an artin algebra, a generalization of the notion of a cotorsion pair, called a left weak cotorsion pair, is introduced and studied in [BZ]. Based on Definition 0.2 of [BZ] a pair \( (\mathcal{C}, \mathcal{D}) \) of subcategories of \( \text{mod-}\Lambda \) is a left weak cotorsion pair if

1. \( \text{Ext}^1_{\Lambda}(\mathcal{C}, \mathcal{D}) = 0 \).
2. For every \( M \in \text{mod-}\Lambda \), there are exact sequences

\[
\begin{array}{c}
M \xrightarrow{f} D \rightarrow C \\
0 \rightarrow D' \rightarrow C' \xrightarrow{g} M \rightarrow 0
\end{array}
\]

such that \( C, C' \in \mathcal{C}, D, D' \in \mathcal{D}, f \) is a left \( \mathcal{D} \)-approximation of \( M \) and \( g \) is a right \( \mathcal{C} \)-approximation of \( M \).

A triple \( (\mathcal{C}, \mathcal{T}, \mathcal{F}) \) of full subcategories in \( \text{mod-}\Lambda \) is called a left weak cotorsion torsion triple if \( (\mathcal{C}, \mathcal{T}) \) is a left weak cotorsion pair and \( (\mathcal{T}, \mathcal{F}) \) is a torsion pair.

In the following theorem we show that in the module category of an artin algebra \( \tau \)-triples are exactly left weak cotorsion torsion triples.
Theorem 4.10. Let $\mathcal{A} = \text{mod-}\Lambda$, where $\Lambda$ is an artin algebra. Then the triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ of full subcategories in $\mathcal{A}$ is a $\tau$-triple if and only if it is a left weak cotorsion torsion triple.

Proof. First, let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be a $\tau$-triple. Since $(\mathcal{T}, \mathcal{F})$ is already a torsion pair, we just need to show that $(\mathcal{C}, \mathcal{T})$ is a left weak cotorsion pair. The Condition (1) of the definition of a left weak cotorsion pair follows by the first condition of the definition of $\tau$-cotorsion pair. Also for $M \in \text{mod-}\Lambda$, by Proposition 4.7, there is an exact sequence

$$M \xrightarrow{f} T \rightarrow C \rightarrow 0$$

where $T \in \mathcal{T}$, $C \in \mathcal{C}$ and $f$ is a left $\mathcal{T}$-approximation. Now for $M \in \text{mod-}\Lambda$, we construct a short exact sequence as in the Condition (2) of the definition of left weak cotorsion pairs. Let $P_1 \xrightarrow{\beta} P_0 \rightarrow M \rightarrow 0$ be a projective presentation of $M$. By Condition (2) of the definition of a $\tau$-cotorsion pair, for projective module $P_1$, there exists an exact sequence

$$P_1 \xrightarrow{\alpha} T \rightarrow C \rightarrow 0.$$

Consider the pushout diagram

$$\begin{array}{ccc}
P_1 & \xrightarrow{\alpha} & T & \xrightarrow{\beta} & P_0 & \rightarrow & C' & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_0 & \xrightarrow{\gamma} & C' & \rightarrow & C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{M} & M & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

In view of the construction of pushouts in $\text{mod-}\Lambda$, we have the exact sequence

$$\phi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},$$

and therefore the short exact sequence

$$0 \rightarrow \text{Im} \phi \xrightarrow{\phi} T \oplus P_0 \rightarrow C' \rightarrow 0.$$

Now let $\pi : P_1 \rightarrow \text{Im} \phi \rightarrow 0$ and let $\psi : \text{Im} \phi \rightarrow T'$ be a morphism with $T' \in \mathcal{T}$. We show that $\psi$ factors through $\phi$. Since $\alpha$ is a left $\mathcal{T}$-approximation, then there exists a morphism $\gamma : T \rightarrow T'$ such that $\psi \pi = \gamma \alpha = \gamma \tau \pi$. Thus $\psi = \gamma \tau$. By applying the functor $\text{Hom}_\Lambda(-, \mathcal{T})$ on the above short exact sequence and using the fact that $T \oplus P_0 \in \mathcal{C}$, we get $C' \in \mathcal{C}$. Now since $\mathcal{T}$ is a torsion class, it is closed under quotients, and so we have $\text{Ker} g \in \mathcal{T}$. Therefore the short exact sequence

$$0 \rightarrow \text{Ker} g \rightarrow C' \xrightarrow{g} M \rightarrow 0$$

is the desired one.

Now we show the converse. Let $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ be a left weak cotorsion torsion triple. Again since $(\mathcal{T}, \mathcal{F})$ is a torsion pair, we just need to show that $(\mathcal{C}, \mathcal{T})$ is a $\tau$-cotorsion pair. By [BZ, Lemma 4.1], we have the Condition (1) of the definition of $\tau$-cotorsion pair. To conclude the Condition
(2) of the definition of $\tau$-cotorsion pair, we note that $\mathcal{T}$ is a functorially finite torsion class and by [AS] there exists an exact sequence

$$P \xrightarrow{f} T \rightarrow C \rightarrow 0,$$

where $f$ is a left $\mathcal{T}$-approximation, $T \in \mathcal{T} \cap \mathcal{C}$, $C \in \mathcal{C}$. Finally, the Condition (3) of the definition of $\tau$-cotorsion pairs follows by [BZ, Theorem 0.4]. □

5. $\tau$-tilting subcategories and $\tau$-triples

In this section, we show that there is a bijection between the collection of all support $\tau$-tilting subcategories of $\mathcal{A}$ and the collection of all $\tau$-cotorsion torsion triples in $\mathcal{A}$. In case we start with a tilting subcategory, this bijection specializes to the one introduced in [BBOS, Theorem 2.29]. We prepare the ground with some preliminary results.

**Lemma 5.1.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{T}$ be a support $\tau$-tilting subcategory of $\mathcal{A}$. Then $\perp 1 \perp \text{Fac}(\mathcal{T}) \cap \text{Fac}(\mathcal{T}) = \mathcal{T}$.

**Proof.** Since $\mathcal{A}$ is an abelian category with enough projectives $\text{Prj}(\mathcal{A})$, we have $\mathcal{A} \simeq \text{mod-Prj}(\mathcal{A})$, see [Be2, Corollaries 3.9 and 3.10]. Now the result follows by the part (ii) of the proof of [IJY, Proposition 5.3]. □

**Proposition 5.2.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{T}$ be a support $\tau$-tilting subcategory of $\mathcal{A}$. Then $(\perp 1 \perp \text{Fac}(\mathcal{T}), \text{Fac}(\mathcal{T}))$ is a $\tau$-cotorsion pair.

**Proof.** The first condition of Definition 4.1 holds trivially. For the second condition, consider the exact sequence

$$P \xrightarrow{f} T^0 \rightarrow T^1 \rightarrow 0$$

where $f$ is a left $\mathcal{T}$-approximation of $P$ and $T^0, T^1 \in \mathcal{T}$, which exists for every projective object $P$. Now by Lemma 5.1, we observe that $T^0, T^1 \in \perp 1 \perp \text{Fac}(\mathcal{T}) \cap \text{Fac}(\mathcal{T})$. So it remains to show that $f$ is a left $\text{Fac}(\mathcal{T})$-approximation of $P$. To show this, let $X \in \text{Fac}(\mathcal{T})$ and let $g : P \rightarrow X$ be a morphism. Consider an epimorphism $\pi : T \rightarrow X$ with $T \in \mathcal{T}$. Since $P$ is a projective object, there is a morphism $h : P \rightarrow T$ such that $\pi h = g$. Now because $f$ is a left $\mathcal{T}$-approximation, there is a morphism $t : T^0 \rightarrow T$ such that $tf = h$. Therefore a morphism $\pi t : T^0 \rightarrow X$ exists such that $\pi tf = \pi h = g$. Finally Lemma 5.1, implies that $\perp 1 \perp \text{Fac}(\mathcal{T}) \cap \text{Fac}(\mathcal{T}) = \mathcal{T}$ is a contravariantly finite subcategory of $\mathcal{A}$.

□

By [BBOS, Proposition 2.22] if $\mathcal{T}$ is a tilting subcategory of $\mathcal{A}$, then the pair $(\text{Fac}(\mathcal{T}), \mathcal{T}^{\perp 0})$ is a torsion pair. We use the same technique to prove the validity of the same result for $\tau$-tilting subcategories.

**Proposition 5.3.** Let $\mathcal{A}$ be an abelian category with enough projective objects. If $\mathcal{T} \subseteq \mathcal{A}$ is a $\tau$-tilting subcategory, then $(\text{Fac}(\mathcal{T}), \mathcal{T}^{\perp 0})$ is a torsion pair.

**Proof.** Let $X \in \text{Fac}(\mathcal{T})$. Then there exists an epimorphism $\mathcal{T} \rightarrow X \rightarrow 0$ with $T \in \mathcal{T}$. The exact sequence $0 \rightarrow \text{Hom}_\mathcal{A}(X, Y) \rightarrow \text{Hom}_\mathcal{A}(T, Y)$ shows that $\text{Hom}_\mathcal{A}(X, Y) = 0$ whenever $Y \in \mathcal{T}^{\perp 0}$.
Now let $A \in \mathcal{A}$ be an arbitrary object. Since $\mathcal{F}$ is contravariantly finite subcategory, there exists a right $\mathcal{F}$-approximation $\phi : T \rightarrow A$. Consider the short exact sequence

$$0 \rightarrow \text{Im} \phi \xrightarrow{f} A \rightarrow \text{Coker} \phi \rightarrow 0,$$

where $\text{Im} \phi \in \text{Fac}(\mathcal{F})$. By applying the functor $\text{Hom}_\mathcal{A}(\mathcal{F}, -)$, we have a long exact exact sequence

$$0 \rightarrow \text{Hom}_\mathcal{A}(-, \text{Im} \phi)|_\mathcal{F} \xrightarrow{f} \text{Hom}_\mathcal{A}(-, A)|_\mathcal{F} \rightarrow \text{Hom}_\mathcal{A}(-, \text{Coker} \phi)|_\mathcal{F} \rightarrow \text{Ext}^1_\mathcal{A}(-, \text{Im} \phi)|_\mathcal{F}.$$

Now since $f$ is a right $\mathcal{F}$-approximation, $f_*$ is an epimorphism. Also since $\mathcal{F}$ is $\tau$-tilting and $\text{Im} \phi \in \text{Fac}(\mathcal{F})$, $\text{Ext}^1_\mathcal{A}(-, \text{Im} \phi)|_\mathcal{F} = 0$. So $\text{Hom}_\mathcal{A}(-, \text{Coker} \phi)|_\mathcal{F} = 0$ and hence $\text{Coker} \phi \in \mathcal{F}^{\perp_0}$.

As a consequence of this proposition we can record the following corollary, which is a generalization of [AIR, Theorem 2.7], which proves the same result for the case when $\mathcal{A} = \text{mod-} \Lambda$, the category of finitely generated modules over an artin algebra $\Lambda$.

**Corollary 5.4.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $\mathcal{F}$ be a support $\tau$-tilting subcategory of $\mathcal{A}$. Then $\text{Fac}(\mathcal{F})$ is a functorially finite torsion class of $\mathcal{A}$.

**Proof.** By Propositions 5.2 and 5.3, $(\perp \text{Fac}(\mathcal{F}), \text{Fac}(\mathcal{F}), \mathcal{F}^{\perp_0})$ is a $\tau$-cotorsion torsion triple. So the result follows from Corollary 4.8.

**Lemma 5.5.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $(\mathcal{C}, \mathcal{F}, \mathcal{F})$ be a $\tau$-cotorsion torsion triple in $\mathcal{A}$. Then $\mathcal{C} \cap \mathcal{F}$ is a support $\tau$-tilting subcategory of $\mathcal{A}$.

**Proof.** First we observe that, by the definition of $\tau$-cotorsion torsion triples, $\mathcal{C} \cap \mathcal{F}$ is a contravarily finite subcategory of $\mathcal{A}$ and

$$\text{Ext}^1_\mathcal{A}(\mathcal{C} \cap \mathcal{F}, \text{Fac}(\mathcal{C} \cap \mathcal{F})) \subseteq \text{Ext}^1_\mathcal{A}(\mathcal{C}, \text{Fac}(\mathcal{F})) = \text{Ext}^1_\mathcal{A}(\mathcal{C}, \mathcal{F}) = 0.$$

Moreover, for every projective object $P$, there is an exact sequence

$$P \xrightarrow{f} T \rightarrow T' \rightarrow 0$$

where $T, T' \in \mathcal{C} \cap \mathcal{F}$ and $f$ is a left $\mathcal{F}$-approximation. To verify the last condition of support $\mathcal{F}$-tilting subcategories, it is enough to note that the left $\mathcal{F}$-approximation $f$ is also a left $\mathcal{C} \cap \mathcal{F}$-approximation.

**Lemma 5.6.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $(\mathcal{C}, \mathcal{F}, \mathcal{F})$ be a $\tau$-cotorsion torsion triple in $\mathcal{A}$. Then $\mathcal{F} = \text{Fac}(\mathcal{C} \cap \mathcal{F})$.

**Proof.** Since $\mathcal{F}$ is closed under factors, we observe that $\text{Fac}(\mathcal{C} \cap \mathcal{F}) \subseteq \mathcal{F}$. Now let $T \in \mathcal{F}$ and $P \xrightarrow{\pi} T \rightarrow 0$ be an epimorphism with projective object $P$. By Condition (3) of Definition 4.1, there is a left $\mathcal{F}$-approximation $f : P \rightarrow T'$, where $T' \in \mathcal{C} \cap \mathcal{F}$. So there exists an epimorphism $g : T' \rightarrow T$ such that $gf = \pi$. Hence $T \in \text{Fac}(\mathcal{C} \cap \mathcal{F})$.

**Theorem 5.7.** Let $\mathcal{A}$ be an abelian category with enough projective objects. Then there exists a bijection between the collection of all support $\tau$-tilting subcategories of $\mathcal{A}$ and the collections of all $\tau$-cotorsion torsion triples in $\mathcal{A}$. This bijection induces by the following maps

$$\{ \text{support } \tau\text{-tilting subcategories} \} \leftrightarrow \{ \text{\tau\text{-cotorsion torsion triples} \}} \quad \quad \mathcal{F} \xrightarrow{\Phi} (\perp \text{Fac}(\mathcal{F})), \text{Fac}(\mathcal{F}), \mathcal{F}^{\perp_0}) \quad \quad \mathcal{C} \cap \mathcal{F} \xrightarrow{\Psi} (\mathcal{C}, \mathcal{F}, \mathcal{F})$$
that are well defined and mutually inverse. Moreover, this bijection restricts to the bijection between the collection of all tilting subcategories of \( \mathcal{A} \) and the collections of all cotorsion torsion triples in \( \mathcal{A} \).

**Proof.** It follows from Propositions 5.2 and 5.3 and Lemma 5.5 that \( \Phi \) and \( \Psi \) are well defined. We show that they are mutually inverse. Let \( \mathcal{T} \) be a support \( \tau \)-tilting subcategory. We have that \( \mathcal{T} = \mathcal{T}^\perp (\text{Fac}(\mathcal{T})) \cap \text{Fac}(\mathcal{T}) \) by Lemma 5.1. Thus \( \Phi \Phi \cong 1 \).

Now let \( (\mathcal{C}, \mathcal{T}, \mathcal{F}) \) be a \( \tau \)-cotorsion torsion triple. First we note that by Lemma 5.6, \( \mathcal{C} = \text{Fac}(\mathcal{C} \cap \mathcal{T}) \) and so \( \mathcal{C} = \mathcal{T}^\perp (\text{Fac}(\mathcal{C} \cap \mathcal{T})) \). Also we have \( \mathcal{F} = (\mathcal{C} \cap \mathcal{T})^\perp 0 \). Hence \( \Phi \Psi \cong 1 \) and so the first statement holds.

We now prove that this bijections restrict to the bijections between cotorsion torsion triples in \( \mathcal{A} \) and tilting subcategories in \( \mathcal{A} \). Let \( \mathcal{T} \) be a tilting subcategory. We show that the \( \tau \)-triple \( (\mathcal{T}^\perp (\text{Fac}(\mathcal{T})), \text{Fac}(\mathcal{T}), \mathcal{T}^\perp 0) \)

is a cotorsion torsion triple. To see this, it is enough to show that \( (\mathcal{T}^\perp (\text{Fac}(\mathcal{T})), \text{Fac}(\mathcal{T})) \) is a cotorsion pair. First we note that, since \( \mathcal{T} \) is a tilting subcategory of \( \mathcal{A} \), for every object \( A \in \mathcal{A} \) there is a short exact sequence

\[
0 \to A \to X_A \to Y_A \to 0
\]

where \( X_A \in \text{Fac}(\mathcal{T}) \) and \( Y_A \in \mathcal{T}^\perp (\text{Fac}(\mathcal{T})) \). Next we construct the second short exact sequence in the definition of cotorsion pair. Let \( A \in \mathcal{A} \) and let \( 0 \to K \to P \to A \to 0 \) be a short exact sequence in \( \mathcal{A} \), where \( P \) is projective. Let \( 0 \to K \to X_K \to Y_K \to 0 \) be a short exact sequence such that \( X_K \in \text{Fac}(\mathcal{T}) \) and \( Y_K \in \mathcal{T}^\perp (\text{Fac}(\mathcal{T})) \). Consider the following pushout diagram

\[
\begin{array}{cccccc}
0 & & 0 \\
0 & & \downarrow & \downarrow & & A & \to 0 \\
\downarrow & & & \downarrow & & & \\
K & \to P & \to A & \to 0 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & \\
X_K & \to U & \to A & \to 0 \\
\downarrow & & & \downarrow & & & \\
Y_K & = & Y_K \\
\downarrow & & & \downarrow & & & \\
0 & & \downarrow & \downarrow & & 0 & \\
\end{array}
\]

Since \( P, Y_K \in \mathcal{T}^\perp (\text{Fac}(\mathcal{T})) \) and \( \mathcal{T}^\perp (\text{Fac}(\mathcal{T})) \) is closed under extensions, then \( U \in \mathcal{T}^\perp (\text{Fac}(\mathcal{T})) \). Hence the second row in the diagram is the desired short exact sequence. Finally, it is clear that \( \text{Ext}_{\mathcal{A}}^1 (\mathcal{T}^\perp (\text{Fac}(\mathcal{T})), \text{Fac}(\mathcal{T})) = 0 \). So we have verified all of the conditions of a cotorsion pair. \( \square \)

As a corollary of the above theorem, we can recover one of the main results of [BZ]. For this, we need some preparations. Let us begin by recalling the original definition of a support \( \tau \)-tilting module [AIR]. This definition is based on \( \tau : \text{mod-} \Lambda \to \text{mod-} \Lambda \), the Auslander-Reiten translation in \( \text{mod-} \Lambda \), see [ASS, Chapter IV].
Definition 5.8. (see [AIR, Definition 0.1]) Let $\Lambda$ be an artin algebra. A module $T$ in mod-$\Lambda$ is called $\tau$-rigid if $\text{Hom}_{\Lambda}(T, \tau T) = 0$. It is called $\tau$-tilting if it is $\tau$-rigid and $|T| = |\Lambda|$. A support $\tau$-tilting module $T$ in mod-$\Lambda$ is a module $T$ that is a $\tau$-tilting module in mod-$\langle \Lambda/\langle e \rangle \rangle$, where $\langle e \rangle$ is the ideal generated by some idempotent $e \in \Lambda$.

As it is mentioned in Remark 3.8, by [J, Proposition 2.14] $T$ is a support $\tau$-tilting module in mod-$\Lambda$ if and only if $\text{add}(T)$ is a support $\tau$-tilting subcategory of mod-$\Lambda$. We now show that every support $\tau$-tilting subcategory in mod-$\Lambda$ is of the form $\text{add}(T)$ for some $\tau$-tilting module $T$ in mod-$\Lambda$.

Proposition 5.9. Let $\Lambda$ be an artin algebra. Then every support $\tau$-tilting subcategory $\mathcal{F}$ of mod-$\Lambda$ is of the form $\text{add}(T)$, where $T$ is a support $\tau$-tilting module in mod-$\Lambda$.

Proof. Let $\mathcal{F}$ be a support $\tau$-tilting subcategory of mod-$\Lambda$. By Proposition 3.9, there exists a support $\tau$-tilting module $T$ such that $\text{Fac}(T) = \text{Fac}(\mathcal{F})$. We show that $\mathcal{F} = \text{add}(T)$. First we note that by the construction of $T$ in Proposition 3.9, we have $T \in \mathcal{F}$. Therefore $\text{add}(T) \subseteq \mathcal{F}$, since $T$ is additively closed subcategory of mod-$\Lambda$. Next, let $X \in \mathcal{F}$, then $X \in \text{Fac}(T)$. By Proposition 2.5 of [Z], there exists a short exact sequence $0 \rightarrow K' \rightarrow T' \rightarrow X \rightarrow 0$

where $T' \in \text{add}(T)$ and $K' \in \text{Fac}(T)$. Since $T$ is a $\tau$-tilting module, the above short exact sequence splits and hence $X \in \text{add}(T)$. Thus $\mathcal{F} = \text{add}(T)$.

Corollary 5.10. (see [BZ, Theorem 4.6]) Let $\mathcal{A} = \text{mod-}\Lambda$, where $\Lambda$ is an artin algebra. Then there is a bijection between the collection of all support $\tau$-tilting modules and the collection of all left weak cotorsion torsion triples.

Proof. This is a direct consequence of Theorem 4.10, Theorem 5.7 and Proposition 5.9.

6. Summary of dual results

In this section we collect the dual of our results in the previous sections. The proofs are similar, so we just list the results without their proofs. Throughout this section we assume that $\mathcal{A}$ is an abelian category with enough injective objects. For a subcategory $\mathcal{U}$ of $\mathcal{A}$, let $\text{Sub}\mathcal{U}$ be the full subcategory of $\mathcal{A}$ consisting of all subobjects of finite direct sums of objects in $\mathcal{U}$.

We start by the definition of a $\tau^-$-tilting subcategory of $\mathcal{A}$. Recall [AIR, §2.2] that a $\Lambda$-module $M$, where $\Lambda$ is an artin algebra, is called $\tau^-$-tilting if it is $\tau^-$-rigid, i.e. $\text{Hom}_{\Lambda}(\tau^- M, M) = 0$, and $|M| = |\Lambda|$. It follows from [AS, Proposition 5.6] that $M$ is $\tau^-$-rigid if and only if $\text{Ext}^1_{\mathcal{A}}(\text{Sub}M, M) = 0$. Here $\text{Sub}M$ means the subcategory of mod-$\Lambda$ consisting of all subobjects of add($M$). This motivates the following definition.

Definition 6.1. Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $\mathcal{U}$ be an additive full subcategory of $\mathcal{A}$. Then $\mathcal{U}$ is called a weak support $\tau^-$-tilting subcategory if

1. $\text{Ext}^1_{\mathcal{A}}(\text{Sub}\mathcal{U}, \mathcal{U}) = 0$.
2. For every injective object $I$ in $\mathcal{A}$, there exists an exact sequence

$$0 \rightarrow U^0 \rightarrow U^1 \overset{g}{\rightarrow} I$$

such that $U^0$ and $U^1$ are in $\mathcal{U}$ and $g$ is a right $\mathcal{U}$-approximation of $I$.

If furthermore $\mathcal{U}$ is a covariantly finite subcategory of $\mathcal{A}$, it is called a support $\tau^-$-tilting subcategory of $\mathcal{A}$. A support $\tau^-$-tilting subcategory $\mathcal{U}$ of $\mathcal{A}$ is called a $\tau^-$-tilting subcategory if the approximation $g : U^1 \rightarrow I$ is non-zero for every injective object $I$. 
Definition 6.2. Let $\mathcal{A}$ be an abelian category with enough injective objects. A pair of full subcategories $(\mathcal{C}, \mathcal{D})$ of $\mathcal{A}$ is called a $\tau^-$-cotorsion pair if

1. $\mathcal{D} = \mathcal{C}^\perp$,
2. For every injective object $I \in \mathcal{A}$, there is an exact sequence
   $$0 \rightarrow D \rightarrow C \xrightarrow{g} I,$$
   where $C \in \mathcal{C} \cap \mathcal{D}$, $D \in \mathcal{D}$ and $g$ is a right $\mathcal{C}$-approximation.
3. $\mathcal{C} \cap \mathcal{D}$ is a covariantly finite subcategory of $\mathcal{A}$.

Definition 6.3. A triple $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ of full subcategories in $\mathcal{A}$ is called a $\tau^-$-torsion cotorsion triple, or simply a $\tau^-$-triple, if $(\mathcal{T}, \mathcal{F})$ is a torsion pair and $(\mathcal{F}, \mathcal{D})$ is a $\tau^-$-cotorsion pair.

A triple $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ of subcategories of $\mathcal{A}$ is called a torsion cotorsion triple if $(\mathcal{T}, \mathcal{F})$ is a torsion pair and $(\mathcal{F}, \mathcal{D})$ is a cotorsion pair.

Proposition 6.4. (Dual of Proposition 4.6) Let $\mathcal{A}$ be an abelian category with enough injective objects. Then every torsion cotorsion pair $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ in $\mathcal{A}$ is a $\tau^-$-triple.

Proposition 6.5. (Dual of Proposition 4.7) Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ be a $\tau^-$-triple in $\mathcal{A}$. Then for every object $A \in \mathcal{A}$, there exists an exact sequence

$$0 \rightarrow D \rightarrow F \xrightarrow{g} A,$$

where $D \in \mathcal{D}$, $F \in \mathcal{F}$ and $g$ is a right $\mathcal{F}$-approximation.

Lemma 6.6. (Dual of Lemma 5.6) Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ be a $\tau^-$-triple in $\mathcal{A}$. Then $F = \text{Sub}(\mathcal{F} \cap \mathcal{D})$.

Proposition 6.7. (Dual of Propositions 5.2, 5.3 and Lemma 5.1) Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $\mathcal{U}$ be a support $\tau^-$-tilting subcategory of $\mathcal{A}$. Then

$$(\text{Sub}\mathcal{U}, (\text{Sub}\mathcal{U})^{\perp_1})$$

is a $\tau^-$-cotorsion pair and

$$(\text{Sub}\mathcal{U}, \text{Sub}\mathcal{U})$$

is a torsion pair in $\mathcal{A}$. Moreover,

$$(\text{Sub}\mathcal{U})^{\perp_1} \cap \text{Sub}\mathcal{U} = \mathcal{U}.$$

Corollary 6.8. (Dual of Corollary 5.4) Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{U}$ be a support $\tau^-$-tilting subcategory of $\mathcal{A}$. Then Sub$\mathcal{U}$ is a functorially finite torsion free class of $\mathcal{A}$.

Lemma 6.9. (Dual of Lemma 5.5) Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ be a $\tau^-$-triple in $\mathcal{A}$. Then $\mathcal{F} \cap \mathcal{D}$ is a support $\tau^-$-tilting subcategory of $\mathcal{A}$.

Let $\mathcal{U}$ be an additively closed full subcategory of $\mathcal{A}$. By [BBOS, Subsection 2.3], $\mathcal{U}$ is called a cotilting subcategory if it satisfies the following conditions.

(i) $\mathcal{U}$ is a covariantly finite subcategory of $\mathcal{A}$.
(ii) Ext$^1_{\mathcal{A}}(U_1, U_2) = 0$, for all $U_1, U_2 \in \mathcal{U}$.
(iii) Every object $U \in \mathcal{U}$ has injective dimension at most 1.
(iv) For every injective object $I$ in $\mathcal{A}$, there exists a short exact sequence

$$0 \rightarrow U_1 \rightarrow U_0 \rightarrow I \rightarrow 0,$$

with $U^i \in \mathcal{U}$.
If $\mathcal{U}$ only satisfies the conditions $(ii)-(iv)$, it is called a weak cotilting subcategory of $A$.

**Theorem 6.10.** (Dual of Theorem 5.7) Let $\mathcal{A}$ be an abelian category with enough injective objects. Then there is a bijection

$$
\{ \text{support } \tau^- \text{-tilting subcategories} \} \leftrightarrow \{ \tau^- \text{-torsion cotorsion triples} \}
$$

$$\mathcal{U} \rightarrow (\perp_0 \mathcal{U}, \text{Sub} \mathcal{U}, (\text{Sub} \mathcal{U})^\perp)
$$

$$\mathcal{F} \cap \mathcal{D} \leftarrow (\mathcal{F}, \mathcal{D}).$$

This bijection restricts to a bijection between the collection of all cotilting subcategories of $A$ and the collections of all torsion cotorsion triples in $A$.

**Proposition 6.11.** (Dual of Proposition 5.9) Let $\Lambda$ be an artin algebra. Then every support $\tau^- \text{-tilting subcategory } \mathcal{T}$ of $\text{mod-} \Lambda$ is of the form $\text{add}(T)$, where $T$ is a support $\tau^- \text{-tilting module}$ in $\text{mod-} \Lambda$.

### 7. $\tau^- \tau^-$-Quadruples

In this section, by combining the notions of $\tau$-triples and $\tau^-$-triples we are able to relate certain support $\tau^- \text{-tilting subcategories}$ to certain support $\tau^- \text{-tilting subcategories}$ of an abelian category with enough projective and enough injective objects. We show that this relation specializes to the dual of the dagger map introduced in [AIR, Theorem 2.15], when we restrict $\mathcal{A}$ to be $\text{mod-} \Lambda$, the category of finitely presented modules over an artin algebra $\Lambda$.

**Definition 7.1.** Let $\mathcal{A}$ be an abelian category with enough projective and enough injective objects. A quadruple $(\mathcal{C}, \mathcal{T}, \mathcal{F}, \mathcal{D})$ of full additively closed subcategories of $\mathcal{A}$ is called a $\tau^- \tau^- \text{-quadruple}$ if $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ is a $\tau^- \text{-triple}$ and $(\mathcal{T}, \mathcal{F}, \mathcal{D})$ is a $\tau^- \text{-triple}$.

**Remark 7.2.** Let $(\mathcal{C}, \mathcal{T}, \mathcal{F}, \mathcal{D})$ be a $\tau^- \tau^- \text{-quadruple}$. Then by Lemmas 5.5 and 6.9, the map given by

$$\dagger : \mathcal{C} \cap \mathcal{T} \rightarrow \mathcal{F} \cap \mathcal{D},$$

associates a $\tau$-tilting subcategory to a $\tau^- \text{-tilting subcategory}$.

Let $\mathcal{A} = \text{mod-} \Lambda$, where $\Lambda$ is an artin algebra. In [AIR, Theorem 2.15], using the dual of a special map, which is called the dagger map and is denoted by $(\cdot)^\dagger$, the authors constructed a bijection between the set of isomorphism classes of all support $\tau^- \text{-tilting } \Lambda$-modules, denoted by $\tau^- \text{-tilt} \Lambda$ and the set of all isomorphism classes of support $\tau^- \text{-tilting } \Lambda$-modules, denoted by $s\tau^- \text{-tilt} \Lambda$. This bijection is given by

$$(M, P) \mapsto (\tau M \oplus \nu P, \nu M_{\text{pr}}),$$

where $\nu$ is the Nakayama functor and $M_{\text{pr}}$ denotes the projective summand of $M$. The next proposition shows that the map $\dagger$ defined in Remark 7.2 can be considered as a generalization of this map. To this end we need the following easy lemma.

**Lemma 7.3.** Let $(\mathcal{C}, \mathcal{T}, \mathcal{F}, \mathcal{D})$ be a $\tau^- \tau^- \text{-quadruple}$. Then both $\mathcal{T}$ and $\mathcal{F}$ are functorially finite subcategories of $\mathcal{A}$. In this case, we say that $(\mathcal{T}, \mathcal{F})$ is a functorially finite torsion pair of $\mathcal{A}$.

**Proof.** Since $\mathcal{T}$ is a torsion class, it is always contravariantly finite. Moreover, Proposition 4.7 implies that $\mathcal{F}$ is a covariantly finite. Hence it is functorially finite. By similar argument, $\mathcal{F}$ is functorially finite. Thus $(\mathcal{T}, \mathcal{F})$ is a functorially finite torsion pair.

Recall that two modules $M$ and $N$ are additively equivalent if $\text{add}(M) = \text{add}(N)$.
Proposition 7.4. Let $\mathcal{A} = \text{mod} \Lambda$. Then the map associating $\mathcal{C} \cap \mathcal{T}$ to $\mathcal{F} \cap \mathcal{D}$ in every quadruple $(\mathcal{C}, \mathcal{T}, \mathcal{F}, \mathcal{D})$, is exactly the dual of dagger map defined in [AIR, Theorem 2.15], up to additive equivalences.

Proof. We show that there exists a bijection between the collection of all support $\tau$-tilting $\Lambda$-modules and the collection of all $\tau$-$\tau^-$-quadruples in mod-$\Lambda$. Let $M \in \text{mod} \Lambda$ be a support $\tau$-tilting module, that is, $(M, P)$ is a $\tau$-tilting pair, for some projective module $P$. Then $\mathcal{T} = \text{add}(M)$ is a $\tau$-tilting subcategory of mod-$\Lambda$ and so by Theorem 5.7,

$$(\perp \text{Fac}(M), \text{Fac}(M), M^{\perp_0})$$

is a $\tau$-triple. On the other hand, in view of Theorem 2.15 of [AIR], $\tau M \oplus \nu P$ is a support $\tau^-$-tilting module. More precisely, $(\tau M \oplus \nu P, \nu M_p)$ is a $\tau^-$-tilting pair, where $M_p$ denotes the projective summand of $M$. Hence $\mathcal{U} = \text{add}(\tau M \oplus \nu P)$ is a $\tau^-$-tilting subcategory and so by Theorem 6.10,

$$(\perp_0(\tau M \oplus \nu P), \text{Sub}(\tau M \oplus \nu P), (\text{Sub}(\tau M \oplus \nu P))^{\perp_1})$$

is a $\tau^-$-torsion cotorsion triple. Now $\perp_0(\tau M \oplus \nu P) = \text{Fac}(M)$ and $\text{Sub}(\tau M \oplus \nu P) = M^{\perp_0}$ by [AIR, Proposition 2.16.b]. In fact we get the following $\tau$-$\tau^-$-quadruple

$$(\perp_1 \text{Fac}(M), \text{Fac}(M), \text{Sub}(\tau M \oplus \nu P), (\text{Sub}(\tau M \oplus \nu P))^{\perp_1}).$$

Conversely, assume that $(\mathcal{C}, \mathcal{T}, \mathcal{F}, \mathcal{D})$ is a $\tau$-$\tau^-$-quadruple. Since $\mathcal{T} = \text{Fac}(T)$ for some support $\tau$-tilting module $T$, Proposition 5.9 implies that $\mathcal{C} \cap \mathcal{T} = \text{add}(T)$ and $\mathcal{F} \cap \mathcal{D} = \text{add}(\tau M \oplus \nu P)$. Hence the result follows.

Let $\mathcal{A} = \text{mod} \Lambda$. Then [Sm, Theorem 1] states that if $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ then $\mathcal{T}$ is functorially finite if and only if $\mathcal{F}$ is functorially finite. As we saw in the proof of the above theorem, we have that every $\tau$-triple $(\mathcal{C}, \mathcal{T}, \mathcal{F})$ can be completed to a $\tau$-$\tau^-$-quadruple. In particular, this implies that the map $\hat{\mathcal{T}}$ can be defined as a map from the set of all $\tau$-triples to the set of all $\tau^-$-triples. This is not true in an arbitrary abelian category with enough projective objects, as is shown by the following example.

Example 7.5. Let $\mathcal{A} = \text{Rep}_k^{fp}\mathbb{R}_{\geq 0}$. Set

$$\mathcal{T} = \text{add}(\{k_{[0,y]} \mid 0 < y \leq \infty\} \cup \{k_{[x,y]} \mid 1 \leq x < y \leq \infty\}),$$

$$\mathcal{F} = \text{add}(\{k_{[x,y]} \mid 0 < x < y \leq 1\}).$$

Then by [BBOS, Example 2.5], $(\mathcal{T}, \mathcal{F})$ is a torsion pair. Obviously $\mathcal{T}$ is a functorially finite subcategory of $\mathcal{A}$. We show that $\mathcal{F}$ is not a contravariantly finite subcategory of $\mathcal{A}$. To see this, note that objects such as $K_{[0,b)}$, where $b < 1$, have not a right $\mathcal{F}$-approximation. In fact, a right $\mathcal{F}$-approximation of $K_{[0,b)}$ should be of the form $\theta : k_{[a,b)} \rightarrow k_{[0,b)}$, for some $0 < a < b < 1$. But non-zero morphisms such as $\eta : k_{[x,y)} \rightarrow k_{[0,b)}$, with $0 \leq x < a$ and $b \leq y$, can not factor through $\theta$.

8. Connection to silting and cosilting modules

This section is divided to two subsections and is devoted to the study the connections between support $\tau$- and support $\tau^-$-tilting subcategories and silting and cosilting theories in Mod-$R$, where $R$ is an associative unitary ring. Based on these theories we are able to characterize all support $\tau$- and support $\tau^-$-tilting subcategories of Mod-$R$. 

8.1. Silting modules and support $\tau$-tilting subcategories. Our aim in this subsection is to characterize all support $\tau$-tilting subcategories of $\text{Mod-}R$. We do this by providing a bijection between the equivalence classes of all support $\tau$-tilting subcategories of $\text{Mod-}R$ and the collection of all equivalent classes of the certain $R$-modules, the so-called finendo quasitilting $R$-modules.

It is known that all silting modules are finendo quasitilting. As a result, it will be shown that $\text{Add}(S)$ is a $\tau$-tilting subcategory of $\text{Mod-}R$, where $S$ is a silting $R$-module.

For a module $M$ in $\text{Mod-}R$, let $\text{Add}(M)$ denote the class of all modules isomorphic to a direct summand of an arbitrary direct sum of copies of $M$. We also let $\text{Gen}(M)$ to be the subcategory of $\text{Mod-}R$ consisting of all $M$-generated modules, i.e. all modules isomorphic to an epimorphic images of modules in $\text{Add}(M)$. and $\text{Pres}(M)$ to be the subcategory of $\text{Mod-}R$ consisting of all $M$-presented modules, i.e. all modules that admit an $\text{Add}(M)$-presentation. Recall that an $\text{Add}(M)$-presentation of an $R$-module $X$ is an exact sequence

$$0 \rightarrow \text{Ker} \pi \rightarrow T(J) \xrightarrow{\pi} X \rightarrow 0,$$

with $M_1$ and $M_0$ in $\text{Add}(M)$.

Let $\sigma$ be a morphism in $\text{Prj}(R)$. Let $\mathcal{D}_\sigma$ be the class of all modules $M$ in $\text{Mod-}R$ such that the induced homomorphism $\text{Hom}_R(\sigma, M)$ is surjective.

**Definition 8.1.1.** (see [AMV, Definition 3.7]) An $R$-module $S$ is called a partial silting module if there exists a projective presentation $\sigma$ of $S$ such that $\mathcal{D}_\sigma$ contains $S$ and is a torsion class in $\text{Mod-}R$. $S$ is called a silting module if there is a projective presentation $\sigma$ of $S$ such that $\text{Gen}(S) = \mathcal{D}_\sigma$.

**Remark 8.1.2.** By [AMV, Remark 3.8], every silting module is a partial silting module, hence $\text{Gen}(S)$ is a torsion class. Support $\tau$-tilting modules over a finite dimensional $k$-algebra are examples of silting modules [AMV, Proposition 3.15].

**Definition 8.1.3.** (see [AMV, Lemdef 3.1]) An $R$-module $T$ is called quasitilting if $\text{Pres}(T) = \text{Gen}(T)$ and $T$ is Ext-projective in $\text{Gen}(T)$.

Following proposition collects some of the basic properties of the quasitilting modules. Their proofs can be found in Lemdef 3.1, Lemma 3.3 and Proposition 3.2 of [AMV]. Recall that an $R$-modules $T$ is called finendo if it is finitely generated over its endomorphism ring.

**Proposition 8.1.4.** Let $T$ be a quasitilting $R$-module. Then the following statements hold true.

1. If $X \in \text{Gen}(T)$ then, there exist a set $J$ and a short exact sequence

$$0 \rightarrow \text{Ker} \pi \rightarrow T(J) \xrightarrow{\pi} X \rightarrow 0,$$

such that $\text{Ker} \pi \in \text{Gen}(T)$ and $T(J)$ is the coproduct of copies of $T$ indexed by $J$. That is, $\text{Gen}(T)$ is closed with respect to the kernels of epimorphisms.

2. $\text{Add}(T)$ is the class of Ext-projective modules in $\text{Gen}(T)$.

3. The following are equivalent.

   (i) $T$ is a finendo quasitilting $R$-module.

   (ii) $\text{Gen}(T)$ is a torsion class and $T$ is a tilting $R/\text{ann}(R)$-module.

   (iii) $T$ is an Ext-projective module in $\text{Gen}(T)$ and there exists an exact sequence

$$R \xrightarrow{f} T_0 \rightarrow T_1 \rightarrow 0$$

such that $T_0, T_1 \in \text{Add}(T)$ and $f$ is a left $\text{Gen}(T)$-approximation.
Proposition 8.1.5. Let $T$ be a finendo quasitilting module in $\text{Mod-}R$. Then $\text{Add}(T)$ is a support $\tau$-tilting subcategory of $\text{Mod-}R$.

Proof. We show the validity of conditions of Definition 3.1. Note that $\text{Fac}(\text{Add}(T)) = \text{Gen}(T)$. Then $\text{Ext}^1_R(\text{Add}(T), \text{Fac}(\text{Add}(T))) = 0$ follows from the statement 2 of Proposition 8.1.4.

By Statement 3 of Proposition 8.1.4, there exists an exact sequence

$$R \xrightarrow{f} T_0 \rightarrow T_1 \rightarrow 0$$

where $T_0, T_1 \in \text{Add}(T)$ and $f$ is a left $\text{Gen}(T)$-approximation. Since $T_0 \in \text{Add}(T)$, $f$ is also a left $\text{Add}(T)$-approximation. Now let $P$ be a projective $R$-module. There exists a set $J$ and an epimorphism $R^{(J)} \rightarrow P \rightarrow 0$ which is split. Consider the pushout diagram

$$
\begin{array}{c}
R^{(J)} \xrightarrow{f^{(J)}} T_0^{(J)} \xrightarrow{h} T_1^{(J)} \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
P \xrightarrow{g} \tilde{T} \rightarrow T_1^{(J)} \rightarrow 0
\end{array}
$$

Since $h$ is a split epimorphism, we get $\tilde{T} \in \text{Add}(T)$. Also it follows easily from the universal property of the pushout diagrams that $g$ is a left $\text{Add}(T)$-approximation of $P$.

In order to complete the proof, we just need to show that $\text{Add}(T)$ is a contravariantly finite subcategory of $\text{Mod-}R$. Let $M \in \text{Mod-}R$. Since by Proposition 8.1.4.3, $\text{Gen}(T)$ is a torsion class, it is a contravariantly finite subcategory of $\text{Mod-}R$. Therefore, there is a monomorphism $0 \rightarrow X \rightarrow M$ such that $X \in \text{Gen}(T)$. By Proposition 8.1.4.1, there exist a set $J$ and a short exact sequence

$$0 \rightarrow \text{Ker} \pi \rightarrow T^{(J)} \xrightarrow{\pi} X \rightarrow 0$$

such that $\text{Ker} \pi \in \text{Gen}(T)$. We show that $T^{(J)} \xrightarrow{\pi} M$ is a right $\text{Add}(T)$-approximation. To do this, let $T' \in \text{Add}(T)$ and $T' \xrightarrow{\ell} M$ be a morphism. Since $\ell$ is a right $\text{Gen}(T)$-approximation, there is a morphism $T' \xrightarrow{\iota} X$ such that $\iota \ell = \ell$. By applying $\text{Hom}_R(T', -)$ on the above short exact sequence and using the fact that $\text{Ker} \pi \in \text{Gen}(T)$, we conclude that $\iota$ factors through $\pi$. Hence $\ell$ factors through $\pi \ell$ and the result follows.

As a direct consequence of the above proposition, we have the following.

Corollary 8.1.6. Let $S$ be a silting module in $\text{Mod-}R$. Then $\text{Add}(S)$ is a support $\tau$-tilting subcategory of $\text{Mod-}R$.

Proof. By [AMV, Proposition 3.10], every silting $R$-module is a finendo quasitilting $R$-module. Now the result follows by the above proposition.

We also have a kind of converse to the previous proposition.

Proposition 8.1.7. Let $\mathcal{F}$ be a support $\tau$-tilting subcategory of $\text{Mod-}R$ such that $\mathcal{F} = \text{Add}(T)$, for some $R$-module $T$. Then $T$ is a finendo quasitilting $R$-module.

Proof. Since $\mathcal{F}$ is a support $\tau$-tilting subcategory, there exists a short exact sequence

$$R \xrightarrow{f} T^0 \rightarrow T^1 \rightarrow 0$$

where $T^0, T^1 \in \mathcal{F}$ and $f$ is a left $\mathcal{F}$-approximation. We note that $f$ also is a left $\text{Fac}(\mathcal{F})$-approximation, see for instance the proof of Proposition 5.2 for the proof of this fact. On the
other hand, since Fac(\(\mathcal{T}\)) = Fac(Add(T)) = Gen(T) and \(\mathcal{T}\) is a support \(\tau\)-tilting subcategory, 
\(T\) is Ext-projective in Gen(T). Now Proposition 8.1.4.3 implies that \(T\) is a finendo quasitilting module. 

The following result is a Mod-version of Proposition 3.9.

**Proposition 8.1.8.** Let \(\mathcal{T}\) be a support \(\tau\)-tilting subcategory of Mod-\(R\). Then there exists a
finendo quasitilting module \(T\) such that Fac(\(\mathcal{T}\)) = Gen(\(T\)).

**Proof.** By Theorem 5.7, \((\perp_1 \text{Fac(\(\mathcal{T}\))}, \text{Fac(\(\mathcal{T}\))}, \perp_0)\) is a \(\tau\)-triple. Hence by Proposition 4.7, for 
every \(M \in \text{Mod-}R\), there exists an exact sequence 
\[M \xrightarrow{\phi} B \rightarrow C \rightarrow 0\]
such that \(\phi\) is a left Fac(\(\mathcal{T}\))-approximation and \(C \in \perp_1 \text{Fac(\(\mathcal{T}\))}\), that is, \(C\) is an Ext-projective in Fac(\(\mathcal{T}\)). Hence by [AMV, Theorem 3.4], we deduce that there exists a finendo quasitilting 
\(R\)-module \(T\) such that Fac(\(\mathcal{T}\)) = Gen(\(T\)).

**Definition 8.1.9.** Let \(\mathcal{A}\) be an abelian category with enough projective objects. Let \(\mathcal{T}\) and \(\mathcal{T}'\) be two support \(\tau\)-tilting subcategories of \(\mathcal{A}\). We say that \(\mathcal{T}\) and \(\mathcal{T}'\) are equivalent if 
Fac(\(\mathcal{T}\)) = Fac(\(\mathcal{T}'\)).

We now have enough ingredients for the proof of our main theorem. Recall that, by [AMV, Page 12], two quasitilting modules \(T_1\) and \(T_2\) are equivalent if Add(\(T_1\)) = Add(\(T_2\)).

**Theorem 8.1.10.** There is a bijection between equivalence classes of support \(\tau\)-tilting subcategories of Mod-\(R\) and equivalence classes of finendo quasitilting \(R\)-modules.

**Proof.** The result follows by Propositions 8.1.8 and 8.1.5.

**8.2. Cosilting modules and support \(\tau^-\)-tilting subcategories.** Our aim in this subsection is to characterize all support \(\tau^-\)-tilting subcategories of Mod-\(R\) in term of quasicotilting \(R\)-modules. It is known that all cosilting modules are quasicotilting. As a result, we show that Prod(\(C\)) is a \(\tau^-\)-tilting subcategory of Mod-\(R\), where \(C\) is a cosilting \(R\)-module, where Prod(\(M\)) denote the class of all modules isomorphic to an arbitrary direct product of copies of \(M\).

For a module \(M\) in Mod-\(R\), let Cogen(\(M\)) be the subcategory of Mod-\(R\) consisting of all \(M\)-cogenerated modules, i.e. all modules isomorphic to a submodule of modules in Prod(\(M\)) and let Copres(\(M\)) be the subcategory of Mod-\(R\) consisting of all \(M\)-copresented modules, i.e. all modules that admit a Prod(\(M\))-copresentation. Recall that a Prod(\(M\))-copresentation of an 
\(R\)-module \(X\) is an exact sequence 
\[0 \rightarrow X \rightarrow M_0 \rightarrow M_1,\]
with \(M_0\) and \(M_1\) in Prod(\(M\)).

Let \(\zeta\) be a morphism in \(\text{Inj}(R)\). Let \(\mathcal{B}_\zeta\) denote the class of all modules \(M\) in Mod-\(R\) such that the induced homomorphism \(\text{Hom}_R(M, \zeta)\) is surjective.

**Definition 8.2.1.** (see [BP, Definition 3.1]) An \(R\)-module \(C\) is called a partial cosilting module if there is an injective copresentation \(\zeta\) of \(C\) such that \(\mathcal{B}_\zeta\) contains \(C\) and the class \(\mathcal{B}_\zeta\) is closed under direct products. Moreover, \(C\) is called a cosilting module if there is an injective copresentation \(\zeta\) of \(C\) such that Cogen(\(C\)) = \(\mathcal{B}_\zeta\).

By [BP, Remark 3.2], every cosilting module is a partial cosilting module. In particular, for every cosilting module \(C\), Cogen(\(C\)) is a torsion-free class.
**Definition 8.2.2.** (see [ZW, Definition 2.1]) An $R$-module $T$ is called a quasicotilting module if $\text{Cogen}(T) = \text{Copres}(T)$, $\text{Hom}_R(-, T)$ preserves exactness of any short exact sequence in $\text{Cogen}(T)$ and $T$ is Ext-injective in $\text{Cogen}(T)$.

By [ZW, Proposition 2.4], if $T$ is a quasicotilting $R$-module, then $\text{Cogen}(T)$ is a torsion-free class. Also [ZW, Proposition 2.11] implies that all quasicotilting $R$-modules are cofinendo. Recall that $R$-module $T$ is cofinendo if and only if there exists a right $\text{Prod}(T)$-approximation of an injective cogenerator $\text{Mod}_R$, see [ATT, Proposition 1.6].

The following proposition collects some of the basic properties of the quasicotilting modules. The proofs can be found in Lemma 3.1 and Theorem 3.2 of [ZW].

**Proposition 8.2.3.** Let $E$ be an injective cogenerator of $\text{Mod}_R$. Let $T$ be a quasicotilting $R$-module. Then the following statements hold true.

1. If $X \in \text{Cogen}(T)$ then there exist a set $J$ and a short exact sequence
   \[ 0 \to X \rightarrowtail T^J \twoheadrightarrow \text{Coker} \rightarrowtail 0 \]
   such that $\text{Coker} \rightarrowtail \in \text{Cogen}(T)$ and $T^J$ is the product of copies of $T$ indexed by $J$. That is, $\text{Cogen}(T)$ is closed with respect to the cokernels of monomorphisms.
2. $\text{Prod}(T)$ is the class of Ext-injective module in $\text{Cogen}(T)$.
3. $T$ is quasicotilting if and only if $T$ is an Ext-injective in $\text{Cogen}(T)$ and there exists an exact sequence
   \[ 0 \to T_0 \twoheadrightarrow T_1 \rightarrowtail f \to E \]
   such that $T_0, T_1 \in \text{Prod}(T)$ and $f$ is a right $\text{Cogen}(T)$-approximation.

In the following we show that every quasicotilting module induces a support $\tau^-$-tilting subcategory of $\text{Mod}_R$. The proof is essentially dual to that of Proposition 8.1.5, but we include it for the sake of completeness.

**Proposition 8.2.4.** Let $E$ be an injective cogenerator of $\text{Mod}_R$. Let $T$ be a quasicotilting $R$-module. Then $\text{Prod}(T)$ is a support $\tau^-$-tilting subcategory of $\text{Mod}_R$.

**Proof.** We show the validity of the conditions of Definition 6.1. For the first condition, note that $\text{SubProd}(T) = \text{Cogen}(T)$. The fact that
\[ \text{Ext}^1_R(\text{SubProd}(T), \text{Prod}(T)) = 0 \]
follows from Proposition 8.2.3.2. For the second condition, consider the exact sequence
\[ 0 \to T_0 \to T_1 \to E \]
of the statement 3 of Proposition 8.2.3, in which $T_0, T_1 \in \text{Prod}(T)$ and $f$ is a right $\text{Cogen}(T)$-approximation of $E$. Now let $I$ be an injective $R$-module. There exists a set $J$ and a monomorphism $0 \to I \to E^J$ which is a split morphism. Consider the pullback diagram
\[
\begin{array}{ccc}
0 & 
\xrightarrow{0} & T_0^J \\
\downarrow & & \\
0 & 
\xrightarrow{0} & T_0^J
\end{array}
\quad
\begin{array}{ccc}
\tilde{T} & 
\xrightarrow{g} & I \\
\downarrow & & \\
T_1^J & 
\xrightarrow{f^J} & E^J
\end{array}
\]
Since $h$ is a split monomorphism, we get $\tilde{T} \in \text{Prod}(T)$. Moreover, it follows easily from the universal property of the pullback diagrams that $g$ is a right $\text{Prod}(T)$-approximation of $I$. 
To complete the proof it remains to show that $\text{Prod}(T)$ is a covariantly finite subcategory of $\text{Mod-R}$. Let $M \in \text{Mod-R}$. Since $\text{Cogen}(T)$ is a torsion-free class, it is a covariantly finite subcategory of $\text{Mod-R}$. Therefore, there is an epimorphism $M \xrightarrow{\pi} X \xrightarrow{i} 0$ such that $X \in \text{Cogen}(T)$. By Proposition 8.2.3.1, there exist a set $J$ and a short exact sequence

$$0 \rightarrow X \xrightarrow{i} T^J \xrightarrow{\pi} \text{Coker} i \xrightarrow{0}$$

such that $\text{Coker} i \in \text{Cogen}(T)$. We claim that $M \xrightarrow{i \pi} T^J$ is a left $\text{Prod}(T)$-approximation. Indeed, let $f : M \rightarrow \tilde{T}$ be a map with $\tilde{T} \in \text{Prod}(T)$. Then $f$ factors through $X$ because $\tilde{T} \in \text{Cogen}(T)$. Moreover, $\text{Cogen}(T)$ is closed under cokernels of monomorphisms by Proposition 8.2.3.1. Hence we can lift the factorisation $f$ through $X$ using $\pi$ and conclude that $f$ factors through $i \pi$. □

As a direct consequence of the above proposition, we have the following.

**Corollary 8.2.5.** Let $C$ be a cosilting module in $\text{Mod-R}$ with respect to an injective copresentation $\zeta$. Then $\text{Prod}(C)$ is a support $\tau$-tilting subcategory of $\text{Mod-R}$.

**Proof.** By [BP, Lemma 3.4], $C$ is Ext-injective in $\text{Cogen}(T)$. Moreover, by [BP, Corollary 3.5], we have $\text{Cogen}(C) = \text{Copres}(C)$. Now [BP, Proposition 2.4] implies that every cosilting $R$-module $C$ is quasicotilting. Hence the result follows by the above proposition. □

The following is a kind of converse to the previous proposition which is also the dual of Proposition 8.1.7.

**Proposition 8.2.6.** Let $E$ be an injective cogenerator of $\text{Mod-R}$. Let $\mathcal{U}$ be a support $\tau$-tilting subcategory of $\text{Mod-R}$ such that $\mathcal{U} = \text{Prod}(T)$, for some $R$-module $T$. Then $T$ is a quasicotilting $R$-module.

**Proof.** Since $\mathcal{U}$ is a support $\tau$-tilting subcategory, for $E$ there exists an exact sequence

$$0 \rightarrow U^0 \rightarrow U^1 \xrightarrow{f} E$$

where $U^0, U^1 \in \mathcal{T}$ and $f$ is a right $\mathcal{U}$-approximation. We note that $f$ also is a right $\text{Sub}\mathcal{U}$-approximation. On the other hand, since $\text{Sub}\mathcal{U} = \text{SubProd}(T) = \text{Cogen}(T)$ and $\mathcal{U}$ is a support $\tau$-tilting subcategory, $T$ is Ext-injective in $\text{Cogen}(T)$. Now Proposition 8.2.3.3 implies that $T$ is a quasicotilting module. □

The following result is a duall of Proposition 8.1.8.

**Proposition 8.2.7.** Let $\mathcal{U}$ be a support $\tau$-tilting subcategory of $\text{Mod-R}$. Then there exists a quasicotilting module $T$ such that $\text{Sub}\mathcal{U} = \text{Cogen}(T)$.

**Proof.** By Theorem 6.10, $(\perp \mathcal{U}, \text{Sub}\mathcal{U}, (\text{Sub}\mathcal{U})^{\perp})$ is a $\tau$-triple. Hence by Proposition 6.5, for every $M \in \text{Mod-R}$, there exists an exact sequence

$$0 \rightarrow B \rightarrow C \xrightarrow{\phi} M$$

such that $\phi$ is a right $\text{Sub}\mathcal{U}$-approximation and $B \in (\text{Sub}\mathcal{U})^{\perp}$, that is, $B$ is an Ext-injective module in $\text{Sub}\mathcal{U}$. Hence by [ZW, Theorem 3.5], we deduce that there exists a quasicotilting $R$-module $T$ such that $\text{Sub}\mathcal{U} = \text{Cogen}(T)$. □

**Definition 8.2.8.** Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $\mathcal{U}$ and $\mathcal{U}'$ be two support $\tau$-tilting subcategories of $\mathcal{A}$. We say that $\mathcal{U}$ and $\mathcal{U}'$ are equivalent if $\text{Sub}\mathcal{U} = \text{Sub}\mathcal{U}'$. 
Now we can state the main theorem of this subsection which is the dual of Theorem 8.1.10. Recall that, by [ZW, Page 12], two quasicotilting modules $T_1$ and $T_2$ are equivalent if $\text{Prod}(T_1) = \text{Prod}(T_2)$.

**Theorem 8.2.9.** There is a bijection between equivalence classes of support $\tau^-$-tilting subcategories of $\text{Mod-R}$ and equivalence classes of quasicotilting $R$-modules.

**Proof.** The result follows by Propositions 8.2.7 and 8.2.4. □

9. Applications to quiver representations

This section is devoted to produce support $\tau^-$-tilting and $n$-tilting subcategories of category of representations of quivers, where $n$ is a non-negative integer. We divide the section in four subsections. In the first subsection we recall some known definitions and properties of the category $\text{Rep}(Q, \mathcal{A})$ of representations of a finite acyclic quiver $Q$ over an abelian category $\mathcal{A}$ with enough projective objects. In the second subsection we produce support $\tau^-$-tilting subcategories in $\text{Rep}(Q, \mathcal{A})$ from certain $\tau^-$-tilting subcategories of $\mathcal{A}$. In the third we construct (co)silting modules in $\text{Mod-RQ}$ from (co)silting modules in $\text{Mod-R}$. Finally, in the last subsection, we use similar techniques to produce $(n + 1)$-tilting subcategories of $\text{Rep}(Q, \mathcal{A})$ from $n$-tilting subcategories of $\mathcal{A}$.

9.1. Notions on quiver representations. Let $\mathcal{A}$ be an abelian category with enough projective objects $\text{Prj}(\mathcal{A})$ and $Q = (Q_0, Q_1)$ be a finite acyclic quiver with vertex set $Q_0$ and arrow set $Q_1$. An arrow $\alpha \in Q_1$ of source $i = s(\alpha)$ and target $j = t(\alpha)$ is usually denoted by $\alpha : i \rightarrow j$.

We denote by $\text{Rep}(Q, \mathcal{A})$ the category of representations of $Q$ in $\mathcal{A}$ over an abelian category $\mathcal{A}$ with enough projective objects. An object $X$ in $\text{Rep}(Q, \mathcal{A})$ is defined by the following data:

1. To each vertex $i \in Q_0$ is associated an object $X_i$ in $\mathcal{A}$.
2. To each arrow $\alpha : i \rightarrow j$ in $Q_1$ is associated a morphism $X_\alpha : X_i \rightarrow X_j$.

A morphism $\varphi : X \rightarrow Y$ in $\text{Rep}(Q, \mathcal{A})$ is a family $\{\varphi_i : X_i \rightarrow Y_i\}_{i \in Q_0}$ of morphisms in $\mathcal{A}$ such that for each arrow $\alpha : i \rightarrow j$ in $Q_1$, the diagram

$$
\begin{array}{ccc}
X_i & \xrightarrow{\varphi_i} & Y_i \\
\downarrow^{X_\alpha} & & \downarrow^{Y_\alpha} \\
X_j & \xrightarrow{\varphi_j} & Y_j \\
\end{array}
$$

is commutative.

The category $\text{Rep}(Q, \mathcal{A})$ is an abelian category. Kernels, cokernels, and images in $\text{Rep}(Q, \mathcal{A})$ are computed vertex-wise in $\mathcal{A}$. In fact, a sequence $X \rightarrow Y \rightarrow Z$ in $\text{Rep}(Q, \mathcal{A})$ is exact if and only if for every vertex $i \in Q_0$, the sequence $X_i \rightarrow Y_i \rightarrow Z_i$ is exact in $\mathcal{A}$.

For each vertex $i \in Q_0$, there exists the evaluation functor $e_i : \text{Rep}(Q, \mathcal{A}) \rightarrow \mathcal{A}$ which sends each representation $X \in \text{Rep}(Q, \mathcal{A})$ to the object $X_i \in \mathcal{A}$ at vertex $i$. It is clear that the evaluation functor $e_i$ is exact and moreover it has an exact left and also an exact right adjoint, which will be denoted by $e_i^!$ and $e_i^\circ$, respectively. Let us recall the constructions of $e_i^!$, $e_i^\circ : \mathcal{A} \rightarrow \text{Rep}(Q, \mathcal{A})$ more explicitly, cf. [HJ].
Let $A \in \mathcal{A}$. Then $e_i^1(A)_j = \bigoplus_{Q(i,j)} A$, where $Q(i,j)$ denotes the set of paths starting in $i$ and ending in $j$. The morphisms are natural inclusions, that is, for any arrow $\alpha : j \to k$, we set $e_i^1(A)_\alpha : \bigoplus_{Q(i,j)} A \to \bigoplus_{Q(i,k)} A$.

The right adjoint $e_i^0$ is defined dually. Let $A \in \mathcal{A}$. Then $e_i^0(A)_j = \bigoplus_{Q(j,i)} A$. The morphisms are natural projections. Moreover, the functor $e_i^0$ has a right adjoint, which will be denoted by $Re_i^0$.

It is proved that the sets

$$\{e_i^1(P) : i \in Q_0, P \in \text{Prj}(\mathcal{A})\} \quad \text{and} \quad \{e_i^0(I) : i \in Q_0, I \in \text{Inj}(\mathcal{A})\}$$

are sets of projective generators and injective cogenerators for the category $\text{Rep}(Q, \mathcal{A})$, respectively. For details of the proofs see e.g. [EE] and [EER].

9.2. Constructing $\tau$-tilting subcategories of $\text{Rep}(Q, \mathcal{A})$. Our aim in this subsection is to provide a systematic technique to construct, starting from a certain $\tau$-tilting subcategory $\mathcal{I}$ of an abelian category $\mathcal{A}$, a new $\tau$-tilting subcategory in the category of representation of a finite and acyclic quiver in $\mathcal{A}$. For the proof of the main result of this subsection, we need the following lemma. Although it seems that it is known to the experts, we could not find a reference. So we provide a proof for the sake of completeness.

**Lemma 9.2.1.** Let $\mathcal{A}$ be an abelian category with enough projective objects and $Q = (Q_0, Q_1)$ be a finite and acyclic quiver. Then for every $X, Y \in \text{Rep}(Q, \mathcal{A})$ there exists the long exact sequence

$$0 \to \text{Hom}_{\mathcal{A}}(X, Y) \to \bigoplus_{r \in Q_0} \text{Hom}_{\mathcal{A}}(X_r, Y_r) \xrightarrow{\cdot \lambda} \bigoplus_{\alpha : r \to l} \text{Hom}_{\mathcal{A}}(X_r, Y_l)$$

$$\to \text{Ext}^1_{\mathcal{A}}(X, Y) \to \bigoplus_{r \in Q_0} \text{Ext}^1_{\mathcal{A}}(X_r, Y_r) \to \bigoplus_{\alpha : r \to l} \text{Ext}^1_{\mathcal{A}}(X_r, Y_l)$$

$$\to \text{Ext}^2_{\mathcal{A}}(X, Y) \to \cdots,$$

where here and throughout we set $\mathcal{A} := \text{Rep}(Q, \mathcal{A})$.

**Proof.** Let $X \in \mathcal{A} := \text{Rep}(Q, \mathcal{A})$. By [BBOS, Lemma 3.5], there exists a short exact sequence

$$0 \to \bigoplus\limits_{\alpha : r \to l} e_i^1(X_r) \to \bigoplus\limits_{r \in Q_0} e_i^1(X_r) \to X \to 0$$

which is natural in $X$. By applying the functor $\text{Hom}_{\mathcal{A}}(-, Y)$ to this sequence, we get the following long exact sequence

$$0 \to \text{Hom}_{\mathcal{A}}(X, Y) \to \bigoplus\limits_{r \in Q_0} \text{Hom}_{\mathcal{A}}(e_i^1(X_r), Y) \xrightarrow{\cdot \lambda} \bigoplus\limits_{\alpha : r \to l} \text{Hom}_{\mathcal{A}}(e_i^1(X_r), Y)$$

$$\to \text{Ext}^1_{\mathcal{A}}(X, Y) \to \bigoplus\limits_{r \in Q_0} \text{Ext}^1_{\mathcal{A}}(e_i^1(X_r), Y) \to \bigoplus\limits_{\alpha : r \to l} \text{Ext}^1_{\mathcal{A}}(e_i^1(X_r), Y)$$

$$\to \text{Ext}^2_{\mathcal{A}}(X, Y) \to \cdots.$$

Now the result follows in view of the adjoint pair $(e_i^1, e_i^0)$ and using the fact that the adjunction between $e_i^1$ and $e_i^0$ extends to $\text{Ext}^t$, for all $t \geq i$, see [HJ, Proposition 5.2].

Now we can state and prove the main result of this part.
Theorem 9.2.2. Let $\mathcal{A}$ be an abelian category with enough projective objects and $Q = (Q_0, Q_1)$ be a finite and acyclic quiver. Let $\mathcal{T}$ be a support $\tau$-tilting subcategory of $\mathcal{A}$ such that $\text{Fac}(\mathcal{T})$ is closed with respect to the kernels of epimorphisms. Then

$$\mathcal{T} = \text{add}\{e_i^\rho(T)\mid i \in Q_0, T \in \mathcal{T}\}$$

is a support $\tau$-tilting subcategory of $\text{Rep}(Q, \mathcal{A})$.

Proof. We show the validity of conditions of Definition 3.1. For the first condition, let $i, j \in Q_0$ and $T \in \mathcal{T}$. We show that

$$\text{Ext}^1_{\mathcal{A}}(e_i^\rho(T), \text{Fac}(e_j^\rho(T))) = 0.$$ 

Set $X := e_i^\rho(T)$ and pick $Y \in \text{Fac}(e_j^\rho(T))$. Then for every $r \in Q_0$, $X_r$ is the sum of some finite copies of $T$, maybe zero, and $Y_r$ is in $\text{Fac}(T)$. Moreover, for every $\alpha: r \to l \in Q_1$, $Y_\alpha : Y_r \to Y_l$ is an epimorphism. Hence since by assumption $\text{Fac}(\mathcal{T})$ is closed with respect to the kernels of epimorphisms, we deduce that $\text{Ker} Y_\alpha \in \text{Fac}(T)$. This in particular implies that for every $T \in \mathcal{T}$, the induced morphism

$$\text{Hom}_{\mathcal{A}}(T, Y_r) \to \text{Hom}_{\mathcal{A}}(T, Y_l)$$

is an epimorphism.

Hence, in the exact sequence

$$\bigoplus_{r \in Q_0} \text{Hom}_{\mathcal{A}}(X_r, Y_r) \xrightarrow{\varphi} \bigoplus_{r \in Q_0} \text{Hom}_{\mathcal{A}}(X_r, Y_l) \to \text{Ext}^1_{\mathcal{A}}(X, Y) \to \bigoplus_{r \in Q_0} \text{Ext}^1_{\mathcal{A}}(X_r, Y_r),$$

of the above lemma, we deduce that $\varphi$ is an epimorphism. So to show the result, it is enough to show that

$$\bigoplus_{r \in Q_0} \text{Ext}^1_{\mathcal{A}}(X_r, Y_r) = 0.$$ 

This follows from the fact $\text{Ext}^1_{\mathcal{A}}(\mathcal{T}, \text{Fac}(\mathcal{T}))$, because $\mathcal{T}$ is a support $\tau$-tilting subcategory of $\mathcal{A}$ and the fact that

$$\bigoplus_{r \in Q_0} \text{Ext}^1_{\mathcal{A}}(X_r, Y_r) \subseteq \text{Ext}^1_{\mathcal{A}}(\mathcal{T}, \text{Fac}(\mathcal{T})).$$

Now we show the validity of the second condition. It is enough to show it only for the projective generators of $\text{Rep}(Q, \mathcal{A})$, i.e. for representations of the form $e_i^\lambda(P)$, where $P$ is a projective object in $\mathcal{A}$. Let $\{\rho_1, \cdots, \rho_k\}$ be the set of all longest paths in $Q$ starting from $i$. Since $\mathcal{T}$ is a support $\tau$-tilting subcategory of $\mathcal{A}$, for $P$ there exists an exact sequence $P \xrightarrow{f} T^0 \xrightarrow{g} T^1 \to 0$, such that $T^0, T^1 \in \mathcal{T}$ and $f$ is a left $\mathcal{T}$-approximation of $P$. Since $\text{Fac}(\mathcal{T})$ is closed with respect to the kernels of epimorphisms, $\text{Ker}g \in \text{Fac}(\mathcal{T})$ and so the induced short exact sequence

$$0 \to \text{Ker}g \to T^0 \to T^1 \to 0$$

splits and so $\text{Ker}g \in \mathcal{T}$. Take the exact sequence

$$e_i^\lambda(P) \xrightarrow{\psi} \bigoplus_{q=1}^k \left(\bigoplus_{\alpha \in \mathcal{I}} e^\rho_{\lambda(q)}(T^0) \oplus e^\rho_{\lambda(q)}(T^1) \oplus e^\rho_{\lambda(q)}(\text{Ker}g)\right) \to 0$$

where $\mathcal{I} \subseteq Q_1$ is the set of arrows $\alpha$ of $Q$ such that $\alpha$ is not part of any path in the set $\{\rho_1, \cdots, \rho_k\}$ but there is a path in that set passing through $\lambda(q)$. An easy verification shows that this is the desired sequence for $e_i^\lambda(P)$. In particular, $\psi$ is a left $\mathcal{T}$-approximation of $e_i^\lambda(P)$.

In order to complete the proof, we have to show that $\mathcal{T}$ is a contravariantly finite subcategory of $\text{Rep}(Q, \mathcal{A})$. Let $X \in \text{Rep}(Q, \mathcal{A})$. For each $i$, consider a right $\mathcal{T}$-approximation $\pi^i : T^i \to
Let $\Lambda$ be an artin algebra and $e_j^\Lambda$ is the right adjoint of $e_i^\Lambda$. Following the same argument as in [BBOS, Proposition 3.9] one can show that

\[ \bigoplus_{i \in Q_0} e_i^\Lambda(T^i) \longrightarrow X \]

is a right $T$-approximation of $X$. \hfill \square

Following examples provide situations where a support $\tau$-tilting subcategory of $\mathcal{A}$ has the property that $\text{Fac}(\mathcal{F})$ is closed with respect to the kernels of epimorphisms.

**Example 9.2.3.** Let $\Lambda$ be an artin algebra and $S$ be a simple injective object in $\text{mod-}\Lambda$. Then $\text{add}(S)$ is a support $\tau$-tilting subcategory such that $\text{Fac}(S) = \text{add}(S)$ is closed under kernels of epimorphisms.

**Example 9.2.4.** Let $A$ be a finite dimensional algebra, $e$ be an idempotent of $A$ and $B = A/AeA$. Let $Q = (Q_0, Q_1)$ be a finite and acyclic quiver. By [BBOS, Proposition 3.9], we have $\mathcal{F} = \text{add}(\{e_i^\Lambda(B) \mid i \in Q_0\})$ is a tilting subcategory in $\text{Rep}(Q, \text{mod-B})$. On the other hand, it is obvious that $\text{add}(B)$ is a support $\tau$-tilting subcategory of $\text{mod-A}$ and $\text{Fac}(\text{add}(B)) = \text{mod-B}$ is closed under kernels of epimorphisms. So by Theorem 9.2.2, $\mathcal{F}$ is a support $\tau$-tilting of $\text{Rep}(Q, \text{mod-A})$.

We end this subsection by the following example which is also an application of the Theorem 3.5.

**Example 9.2.5.** Let $Q$ be a finite and acyclic quiver and $Q'$ be a full subquiver of $Q$. Then it is immediate that $\text{Rep}(Q', \mathcal{A})$ is a wide and functorially finite torsion class of $\text{Rep}(Q, A)$. By [BBOS, Proposition 3.9], $\mathcal{F} = \text{add}(\{e_i^\Lambda(P) \mid i \in Q'_0, P \in \text{Prj}(\mathcal{A})\})$ is a tilting subcategory of $\text{Rep}(Q', \mathcal{A})$. So by Theorem 3.5, $\mathcal{F}$ is a support $\tau$-tilting subcategory of $\text{Rep}(Q, \mathcal{A})$.

### 9.3. (Co)silting objects in $\text{Mod-RQ}$. Let $R$ be an associative ring with unity and $Q$ be a finite and acyclic quiver. In this subsection we construct silting, resp. cosilting, objects in the category of representations of $Q$ in $\text{Mod-R}$, $\text{Rep}(Q, \text{Mod-R})$, from silting, resp. cosilting, modules in $\text{Mod-R}$. Note that $\text{Rep}(Q, \text{Mod-R})$ is equivalent to the $\text{Mod-RQ}$, where $RQ$ denotes the path algebra of $Q$ over $R$. So by a silting, resp. cosilting, object in $\text{Rep}(Q, \text{Mod-R})$ we mean a silting, resp. cosilting, module in $\text{Mod-RQ}$.

**Theorem 9.3.1.** Let $Q = (Q_0, Q_1)$ be a finite and acyclic quiver.

(i) Let $S$ be a silting module in $\text{Mod-R}$. Let $i \in Q_0$ be an arbitrary vertex of $Q$. Then $e_i^\Lambda(S)$ is a silting object in $\text{Rep}(Q, \text{Mod-R})$.

(ii) Let $C$ be a cosilting module in $\text{Mod-R}$. Let $i \in Q_0$ be an arbitrary vertex of $Q$. Then $e_i^\Lambda(C)$ is a cosilting object in $\text{Rep}(Q, \text{Mod-R})$.

**Proof.** (i) Let $S$ be a silting $R$-module. By definition, there exists a projective presentation

\[ P_1 \xrightarrow{\sigma} P_0 \longrightarrow S \longrightarrow 0 \]

of $S$ such that $\mathcal{D}_\sigma = \text{Gen}(S)$. By applying the exact functor $e_i^\Lambda$ on the projective presentation of $S$ and using the fact that $e_i^\Lambda$ is an exact functor that preserves projectives, we get the projective presentation

\[ e_i^\Lambda(P_1) \xrightarrow{e_i^\Lambda(\sigma)} e_i^\Lambda(P_0) \longrightarrow e_i^\Lambda(S) \longrightarrow 0 \]

of $e_i^\Lambda(S)$. To complete the proof, we show that $\mathcal{D}_{e_i^\Lambda(\sigma)} = \text{Gen}(e_i^\Lambda(S))$. 


Let $X \in \mathcal{D}_{\lambda_i}(\sigma)$. So there exists an epimorphism

$$\text{Hom}_{\mathcal{A}}(e_1^i(P_0), X) \rightarrow \text{Hom}_{\mathcal{A}}(e_1^i(P_1), X) \rightarrow 0,$$

where $\mathcal{A}$ means $\text{Rep}(Q, \text{Mod-}R)$. The adjoint pair $(e_1^i, e_i)$ induces the epimorphism

$$\text{Hom}_R(P_0, X_i) \rightarrow \text{Hom}_R(P_1, X_i) \rightarrow 0.$$ 

This, in turn, implies that $X_i \in \mathcal{D} = \text{Gen}(S)$. Therefore, $e_1^i(X_i) \in e_1^i(\text{Gen}(S))$. On the other hand, by [BBOS, Lemma 3.5], there exists an epimorphism $\bigoplus_{i \in Q_0} e_1^i(X_i) \rightarrow X \rightarrow 0$ which shows that $X \in e_1^i(\text{Gen}(S))$. But it follows directly from the definition of $e_1^i$ that $e_1^i(\text{Gen}(S)) = \text{Gen}(e_1^i(S))$. Thus $\mathcal{D}_{\lambda_i}(\sigma) \subseteq \text{Gen}(e_1^i(S))$.

To see the reverse inclusion, let $X \in e_1^i(\text{Gen}(S))$. So $X = e_1^i(U)$ such that $U \in \text{Gen}(S)$. Since $\text{Gen}(S) = \mathcal{D}$, we have $U \in \mathcal{D}$. Therefore, there exists an epimorphism

$$\text{Hom}_R(P_0, U) \rightarrow \text{Hom}_R(P_1, U) \rightarrow 0.$$ 

By the using of adjoint properties of adjoint pair $(e_1^i, e_i)$, we have an epimorphism

$$\text{Hom}_{\mathcal{A}}(e_1^i(P_0), X) \rightarrow \text{Hom}_{\mathcal{A}}(e_1^i(P_1), X) \rightarrow 0$$

which shows that $X \in \mathcal{D}_{\lambda_i}(\sigma)$. So we show that $e_1^i(\text{Gen}(S)) = \text{Gen}(e_1^i(S)) \subseteq \mathcal{D}_{\lambda_i}(\sigma)$. Hence the proof is complete.

(ii) The proof is just dual of the proof of part (i), so we skip the proof. \qed

The following result provides a partial converse to the above theorem. Recall that a vertex $i \in Q_0$ is called a source, resp. a sink, of $Q$ if there is no arrows $\alpha \in Q_1$ such that $t(\alpha) = i$, resp. $s(\alpha) = i$.

**Theorem 9.3.2.** Let $Q = (Q_0, Q_1)$ be a finite and acyclic quiver.

(i) Let $X$ be a silting object in $\text{Rep}(Q, \text{Mod-}R)$. Then $e_1(X)$ is a silting module in $\text{Mod-}R$, provided $i \in Q_0$ is a source of $Q$.

(ii) Let $Y$ be a cosilting object in $\text{Rep}(Q, \text{Mod-}R)$. Then $e_i(Y)$ is a cosilting module in $\text{Mod-}R$, provided $i \in Q_0$ is a sink of $Q$.

**Proof.** (i) Since $X$ is a silting object in $\text{Rep}(Q, \text{Mod-}R)$, there exists a projective presentation

$$P^1 \rightarrow P^0 \rightarrow X \rightarrow 0$$

of $X$ such that $\mathcal{D} = \text{Gen}(X)$. By applying the exact functor $e_1$ on this exact sequence we get the exact sequence

$$P_1^i \rightarrow P_0^i \rightarrow X_i \rightarrow 0,$$

which is a projective presentation of $e_1(X) = X_i$.

In order to complete the proof, we have to show that $\mathcal{D}_{\lambda_i} = \text{Gen}(X_i)$. First let $M \in \mathcal{D}_{\lambda_i}$. Then there exists an epimorphism

$$\text{Hom}_R(P_0^i, M) \rightarrow \text{Hom}_R(P_1^i, M) \rightarrow 0.$$ 

By using the adjoint pair $(e_i, e_i^\rho)$, we get the epimorphism

$$\text{Hom}_{\mathcal{A}}(P_0^i, e_i^\rho(M)) \rightarrow \text{Hom}_{\mathcal{A}}(P_1^i, e_i^\rho(M)) \rightarrow 0$$

where $\mathcal{A}$ means $\text{Rep}(Q, \text{Mod-}R)$. So $e_i^\rho(M) \in \mathcal{D} = \text{Gen}(X)$. Therefore $M = e_i e_i^\rho(M) \in e_i(\text{Gen}(X))$. But $e_i(\text{Gen}(X)) = \text{Gen}(e_i(X)) = \text{Gen}(X_i)$. Hence $\mathcal{D}_{\lambda_i} \subseteq \text{Gen}(X_i)$.

For the reverse inclusion, let $Y_i \in \text{Gen}(X_i)$. We define $Y \in \text{Rep}(Q, \text{Mod-}R)$ by setting $Y_i$ at source vertex $i$, and 0 elsewhere. Since $i$ is a source, it follows easily that $Y \in \text{Gen}(X)$. Therefore
Y ∈ D which implies that Y_i ∈ D_i. Hence we have the equality D_i = Gen(X_i) and the proof is complete.

(ii) The proof is just dual of the proof of part (i). So we skip the proof. □

9.4. **Higher tilting subcategories of** Rep(Q, A).

Let **A** be an abelian category with enough projective objects. Let Prj( **A** ) denote the subcategory of **A** consisting of all projective objects. In [BBOS, Proposition 3.9] it is shown that

\[ \mathcal{J} = \text{add}(\{e_i(P)\mid i \in Q_0, P \in \text{Prj}(\mathcal{A})\}) \]

is a (1-)tilting subcategory of Rep(Q, A), where Q is a finite and acyclic quiver.

Now if we interpret Prj( **A** ) as a 0-tilting subcategory of **A**, then by the above result, starting from a 0-tilting subcategory of A we get a 1-tilting subcategory of Rep(Q, A). In our next and last result we provide a higher version of this result by constructing an (n+1)-tilting subcategory in Rep(Q, A) starting from an n-tilting subcategory of A.

**Theorem 9.4.1.** Let **A** be an abelian category with enough projective objects. Let n be a non-negative integer. Let Q be a finite and acyclic quiver. For an n-tilting subcategory **J** of **A** put

\[ \mathcal{J}' = \text{add}(\{e_i(T)\mid i \in Q_0, T \in \mathcal{J}\}). \]

Then **J**' is an (n+1)-tilting subcategory of Rep(Q, A).

**Proof.** We show the validity of the conditions of Definition 2.1. The validity of Condition (i) follows by the similar argument in the proof of Theorem 3.5. For the validity of Condition (ii), we have to show that for all t ≥ 1 and T_1, T_2 ∈ **J**,

\[ \text{Ext}^t_{\text{Rep}(Q,A)}(e_i(T_1), e_j(T_2)) = 0. \]

But Proposition 5.2 of [HJ] implies that

\[ \text{Ext}^t_{\text{Rep}(Q,A)}(e_i(T_1), e_j(T_2)) \cong \text{Ext}^t_{\mathcal{J}}(e_i(T_1), T_2). \]

Now e_i(T_1) is either zero or a sum of copies of T_1 and **J** is an n-tilting subcategory, hence

\[ \text{Ext}^t_{\mathcal{J}}(e_i(T_1), T_2) = 0. \]

Thus it follows that,

\[ \text{Ext}^t_{\text{Rep}(Q,A)}(e_i(T_1), e_j(T_2)) = 0. \]

For the Condition (iii), we have to show that every object in **J**' has projective dimension at most n+1. To this end, it is enough to show this fact for an additive generator e_i(T) of **J**', for some i ∈ Q_0 and some T in **J**. By [BBOS, Lemma 3.5], for every such generator, there exists a short exact sequence

\[ 0 \rightarrow \bigoplus_{\alpha \rightarrow \tau} e_{\alpha}(e_i(T)_\tau) \rightarrow \bigoplus_{r \in Q_0} e_r(e_i(T)_r) \rightarrow e_i(T) \rightarrow 0. \]

Since for every r, i ∈ Q_0, e_r(T)_r is zero or a sum of copies of T and projective dimension of T is at most n, projective dimension of e_r(T)_r is at most n. Now since for every t ∈ Q_0, e_i(T) preserves projective dimensions, projective dimension of e_i(T)_r is at most n. Hence the above short exact sequence shows that projective dimension of e_i(T) is at most n+1.
For the Condition (iv), we construct the desired exact sequence for projectives of the form \( e_i^\lambda(P) \), where \( P \) is a projective object in \( \mathcal{A} \). By the dual of Lemma 3.5 of [BBOS], there exists a short exact sequence

\[
0 \to e_i^\lambda(P) \to \bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(P)_j) \to \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(P)_j) \to 0.
\]

First note that, since \( \mathcal{T} \) is an \( n \)-tilting subcategory of \( \mathcal{A} \), there exists an exact sequence

\[
(9.1) 0 \to P \to T^0 \to T^1 \to \cdots \to T^n \to 0
\]

where \( T^\ell \in \mathcal{T}, \ell \in \{0, \cdots, n\} \).

Since functors \( e_i^\lambda \), \( e_i^\rho \) are all exact, the exact sequence (9.1) induces the following two exact sequences

\[
0 \to \bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(T^\ell)_j) \to \bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(T^0)_j) \to \cdots \to \bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(T^n)_j) \to 0,
\]

and

\[
0 \to \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(P)_j) \to \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(T^0)_j) \to \cdots \to \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(T^n)_j) \to 0.
\]

So we get the following diagram

\[
\begin{array}{ccc}
0 & \to & e_i^\lambda(P) \\
\downarrow & & \downarrow \\
\bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(P)_j) & \to & \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(P)_j) \\
\downarrow & & \downarrow \\
\bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(T^\ell)_j) & \to & \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(T^\ell)_j) \\
\vdots & \cdots & \vdots \\
\bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(T^n)_j) & \to & \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(T^n)_j) \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

Since for every \( r \geq 1, \) \( \text{Ext}_{\mathcal{T}}^r(\mathcal{T}, \mathcal{T}) = 0 \), and adjoint properties of the adjoint pairs \((e_i^\lambda, e_i^\rho)\) and \((e_i^\rho, e_i^\lambda)\) extends to \( \text{Ext}^1 \), for every \( \ell \in \{0, \cdots, n\} \), \( \bigoplus_{\alpha \to j} e_j^\rho(e_i^\lambda(T^\ell)_j) \) is a relative injective object with respects to \( \bigoplus_{j \in Q_0} e_j^\rho(e_i^\lambda(T^\ell)_j) \). Thus, we can construct the dotted maps starting from \( \theta \).
Now by considering the mapping cone of the above diagram and applying a simple diagram chasing, we get the long exact sequence

\[
0 \to e_\lambda^i(P) \to \bigoplus_{j \in Q_0} e_j^\rho(e_\lambda^i(T^0)_j) \to \bigoplus_{j \in Q_0} e_j^\rho(e_\lambda^i(T^1)_j) \oplus \bigoplus_{\alpha:t \to j} e_j^\rho(e_\lambda^i(T^0)_j) \to \cdots \to \bigoplus_{j \in Q_0} e_j^\rho(e_\lambda^i(T^n)_j) \oplus \bigoplus_{\alpha:t \to j} e_j^\rho(e_\lambda^i(T^n-1)_j) \to \bigoplus_{\alpha:t \to j} e_j^\rho(e_\lambda^i(T^n)_j) \to 0,
\]

such that all terms except \(e_\lambda^i(P)\) are in \(\mathcal{T}'\). This is the desired exact sequence. Hence the proof is complete. □

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