String Tension Scaling in High-Temperature Confined SU(N) Gauge Theories

Peter N. Meisinger and Michael C. Ogilvie

Department of Physics, Washington University, St. Louis, MO 63130, USA

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SU(N) gauge theories, extended with adjoint fermions having periodic boundary conditions, are confining at high temperature for sufficiently light fermion mass \( m \). In the high temperature confining region, the one-loop effective potential for Polyakov loops has a \( Z(N) \)-symmetric confining minimum. String tensions associated with Polyakov loops are calculable in perturbation theory, and display a novel scaling behavior in which higher representations have smaller string tensions than the fundamental representation. In the magnetic sector, the Polyakov loop plays a role similar to a Higgs field, leading to an apparent breaking of SU(N) to \( U(1)^{N-1} \). This is turn yields a dual effective theory where magnetic monopoles give rise to string tensions for spatial Wilson loops. The spatial string tensions are calculable semiclassically from kink solutions of the dual system. We show that the spatial string tensions \( \sigma_{k}^{(s)} \) associated with each \( N \)-ality \( k \) obey a variant of Casimir scaling \( \sigma_{k}^{(s)}/\sigma_{1}^{(s)} \leq \sqrt{k(N-k)/(N-1)} \). Although lattice simulations indicate that the high temperature confining region is smoothly connected to the confining region of low-temperature pure SU(N) gauge theory, the electric and magnetic string tension scaling laws are different and readily distinguishable.

I. INTRODUCTION

One of the long-standing problems of modern strong-interaction physics is the origin of quark confinement \[1,2,3\]. Recently, results from theory and from lattice simulations \[4,5\] have indicated the existence of a class of models based on SU(N) gauge theories in which confinement can be understood using semiclassical arguments. The essential feature of this class of models is the deformation of the underlying gauge theory in such a way that the effective potential for the Polyakov loop is minimized in a confining phase in which the Polyakov loop apparently breaks SU(N) to \( U(1)^{N-1} \). The form of the Polyakov loop in this phase in turn leads to the existence of monopole solutions, and an explicit connection between monopoles and confinement along the lines developed by Polyakov \[5\].

In this class of models, at least one Euclidean direction is taken to be compact, of length \( L \). As \( L \) is decreased from infinity, the pure gauge theory undergoes a deconfining phase transition at a critical value of \( L \). Confinement is restored for small \( L \) by a deformation of the pure gauge action. The simplest deformation which does this is an addition to the gauge action of terms non-local in the compact direction \[5\]. Lattice simulations of this model are consistent with a smooth connection of the high-temperature confining phase to the confining phase of the pure gauge theory. A local action with similar properties is obtained by adding Dirac fermions in the adjoint representation to the gauge action, and it is this model we will consider here. The novel feature of this model is that the adjoint fermions are given periodic boundary conditions in the finite direction, rather than the standard antiperiodic boundary conditions. Recent lattice simulations of SU(3) with periodic adjoint fermions \[7\] observe many of the same features seen in \[3\]. We will refer to the compact direction as the timelike direction, writing \( L \) as \( \beta \), suggesting a system at finite temperature \( T = 1/\beta \). This is slightly misleading, because the transfer matrix for evolution in the compact direction is not positive-definite. Periodic boundary conditions in the timelike direction imply that the generating function of the ensemble, i.e., the partition function, is given by

\[
Z = \text{Tr} \left[ (-1)^F e^{-\beta H} \right]
\]  

(1)

where \( F \) is the fermion number. This graded ensemble, familiar from supersymmetry, can be obtained from an ensemble \( \text{Tr} \left[ \exp (\beta \mu F - \beta H) \right] \) with chemical potential \( \mu \) by the replacement \( \beta \mu \rightarrow i\pi \). This system can also be viewed as a gauge theory with periodic boundary conditions in one compact spatial direction of length \( L = \beta \). From this point of view, the transfer matrix is positive-definite, and the partition function is dominated by the ground state energy. We will use the language of finite temperature gauge theory because of the well-developed relation between \( Z(N) \) symmetry breaking and the confinement-deconfinement transition.
Finite temperature gauge theories are advantageous in many respects for the study of confinement. The Polyakov loop operator \( P \), given by

\[
P(\vec{x}) = P \exp \left[ i \int_0^\beta dt A_4(\vec{x}, t) \right]
\]

represents the insertion of a static quark into a thermal system of gauge fields. It is the order parameter for the deconfinement phase transition in pure SU(\( N \)) gauge theories, with \( \langle \text{Tr}_P P \rangle = 0 \) in the confined phase, and \( \langle \text{Tr}_P P \rangle \neq 0 \) in the deconfined phase. The deconfinement phase transition is associated with the spontaneous breaking of a global \( Z(N) \) symmetry \( P \rightarrow zP \) where \( z = \exp(2\pi i/N) \) is the generator of \( Z(N) \). The unbroken \( Z(N) \) symmetry of the confined phase leads to a rich set of conditions that expectation values must satisfy. The character \( \text{Tr}_R P \) of each irreducible representation of \( SU(N) \) transforms as \( \text{Tr}_R P \rightarrow z^k \text{Tr}_R P \) under \( P \rightarrow zP \) for some \( k \in \{0, \ldots, N-1\} \). In the confined phase, the expectation value \( \langle \text{Tr}_R P \rangle \) is 0 for all representations that transform non-trivially under \( Z(N) \), i.e., have \( k \neq 0 \). For each representation \( R \) with \( k \neq 0 \), there is a timelike string tension \( \sigma_R^{(t)} \) associated with the large-distance behavior of the correlation function

\[
\langle \text{Tr}_R P(\vec{x}) \text{Tr}_R P^*(\vec{y}) \rangle \simeq \exp \left[ -\frac{\sigma_R^{(t)}}{T} |\vec{x} - \vec{y}| \right].
\]

On physical grounds, it is generally believed that all representations with the same non-zero value of \( k \) have the same string tension \( \sigma_k^{(t)} \). This timelike string tension \( \sigma_k^{(t)} \) can be measured from the behavior of the correlation function

\[
\langle \text{Tr}_P P^k(\vec{x}) \text{Tr}_P P^{+k}(\vec{y}) \rangle \simeq \exp \left[ -\frac{\sigma_k^{(t)}}{T} |\vec{x} - \vec{y}| \right]
\]

at sufficiently large distances, where \( \sigma_k^{(t)} |\vec{x} - \vec{y}| \) represents the potential energy between widely separated groups of \( k \) quarks and \( k \) antiquarks. Thus the study of confinement at finite temperature using the Polyakov loop largely reduces to the study of \( Z(N) \) symmetry and the existence of a mass gap in Polyakov loop two-point functions, as opposed to the area law for Wilson loops appearing in confinement at zero temperature.

The addition of adjoint representation fermions to \( SU(N) \) gauge theories preserves the global \( Z(N) \) symmetry of the action. With normal antiperiodic boundary conditions for the fermions, the perturbative effective action for the Polyakov loop shows that the deconfined phase is favored at high temperature. As the adjoint fermion mass decreases from infinity, the critical temperature of the deconfinement transition decreases. With periodic boundary conditions for the fermions, however, this class of field theories can avoid the transition to the deconfined phase found in the pure gauge theory for sufficiently light fermion mass and high temperatures. If the number of adjoint fermion flavors \( N_f \) is less than 11/2, these systems are asymptotically free at high temperature, and therefore the effective potential for \( P \) is calculable using perturbation theory. As shown below, the system will lie in the confining phase at high temperature if the adjoint fermion mass \( m \) is sufficiently light and \( 1/2 < N_f < 11/2 \). Evidence from lattice simulations indicates that the high-temperature confining region is smoothly connected to the low-temperature confined phase of the pure gauge theory \([5]\), indicating a continuous change in string tensions from one region to the other. In the high-temperature confining region, electric string tensions can be calculated perturbatively from the effective potential, and magnetic string tensions arise semiclassically from non-Abelian magnetic monopoles. Thus the high-temperature confining region provides a realization of one of the oldest ideas about the origin of confinement.

The following section describes the calculation of the effective potential, and demonstrates the realization of confinement at high temperature. Section III derives the temporal string tensions via a perturbative expansion around the confining minimum of the effective potential. In section IV, we discuss spatial string tensions as measured by spatial Wilson loops. As in the classic treatment of the three-dimensional adjoint Higgs model by Polyakov \([4]\), monopoles are responsible for confinement. Following the recent work of Unsal and Yaffe \([4]\), we are able to calculate semiclassically the spatial string tensions, which exhibit a new string tension scaling law. A final section discusses our results and their possible confirmation by lattice simulation. An appendix details a mathematical identity useful for calculating the effective potential in the confined region.
II. HIGH TEMPERATURE CONFINEMENT

Consider a boson in a representation \( R \) of the gauge group with spin degeneracy \( s \) moving in a constant Polyakov loop background \( P \). The one-loop effective potential at non-zero temperature and density is given by [8][9]

\[
V_b = s T \int \frac{d^4k}{(2\pi)^4} Tr_R \left[ \ln \left( 1 - P e^{\beta \mu - \beta \omega_k} \right) + \ln \left( 1 - P^+ e^{-\beta \mu - \beta \omega_k} \right) \right]
\]

(5)

where periodic boundary conditions are assumed. With standard boundary conditions (periodic for bosons, antiperiodic for fermions), 1-loop effects always favor the deconfined phase. For the case of pure gauge theories in four dimensions, the one-loop effective potential can be written in the form

\[
V_{\text{gauge}} (P, \beta) = -\frac{2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^4}
\]

(6)

where the trace is over the adjoint representation. This series is minimized, term by term if \( P \in Z(N) \), so \( Z(N) \) symmetry is spontaneously broken at high temperature. The same minima are obtained for any bosonic field with periodic boundary conditions or for fermions with antiperiodic boundary conditions.

The use of periodic boundary conditions for the adjoint fermions dramatically changes their contribution to the Polyakov loop effective potential. In perturbation theory, the replacement \( \beta \mu \rightarrow i\pi \) shifts the Matsubara frequencies from \( \beta \omega_n = (2n + 1) \pi \) to \( \beta \omega_n = 2n \pi \). The one loop effective potential is now that of a bosonic field, but with an overall negative sign due to fermi statistics [10]. The sum of the effective potential for the fermions plus that of the gauge bosons gives

\[
V_{\text{1-loop}} (P, \beta, m, N_f) = \frac{1}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^2} \left[ 2N_f \beta^2 m^2 K_2 (n\beta m) - \frac{2}{n^2} \right].
\]

(7)

Note that the first term in brackets, due to the fermions, is positive for every value of \( n \), while the second term, due to the gauge bosons, is negative.

The largest contribution to the effective potential at high temperatures is typically from the \( n = 1 \) term, which can be written simply as

\[
\frac{1}{\pi^2 \beta^4} \left[ 2N_f \beta^2 m^2 K_2 (\beta m) - 2 \right] \left[ \left( Tr_F P \right)^2 - 1 \right]
\]

(8)

where the overall sign depends only on \( N_f \) and \( \beta m \). If \( N_f \geq 1 \) and \( \beta m \) is sufficiently small, this term will favor \( Tr_F P = 0 \). On the other hand, if \( \beta m \) is sufficiently large, a value of \( P \) from the center, \( Z(N) \), is preferred. Note that an \( N = 1 \) super Yang-Mills theory would correspond to \( N_f = 1/2 \) and \( m = 0 \), giving a vanishing perturbative contribution for all \( n \) [11][12]. In that case, non-perturbative effects lead to a confining effective potential for all values of \( \beta \). In the case of \( N_f \geq 1 \), each term in the effective potential will change sign in succession as \( m \) is lowered towards zero. This suggests that it should be possible to obtain a \( Z(N) \) symmetric, confining phase at high temperatures using adjoint fermions with periodic boundary conditions or some equivalent deformation of the theory.

The existence of a \( Z(N) \) symmetric, confining phase at high temperatures has been confirmed in \( SU(3) \), where both lattice simulations and perturbative calculations have been used to show that a gauge theory action with an extra term of the form \( \int d^4 x \, a_1 Tr_A P \) is confining for sufficiently large \( a_1 \) at arbitrarily high temperatures [5]. This simple, one-term deformation is sufficient for \( SU(2) \) and \( SU(3) \). However, in the general case, a deformation with at least \( \left[ \frac{N}{2} \right] \) terms is needed to assure confinement for representations of all possible non-zero \( k \)-alities. Thus the minimal necessary deformation is of the form

\[
\sum_{k=1}^{\left[ \frac{N}{2} \right]} a_k Tr_A P^k
\]

(9)

which is analyzed in detail in [13]. If all the coefficients \( a_k \) are sufficiently large and positive, the one-loop effective potential

\[
V_{\text{1-loop}} (P, \beta) = -\frac{2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^4} + \sum_{k=1}^{\left[ \frac{N}{2} \right]} a_k Tr_A P^k
\]

(10)
will be minimized by a unique set of Polyakov loop eigenvalues corresponding to exact $Z(N)$ symmetry. The unique set of $SU(N)$ Polyakov eigenvalues invariant under $Z(N)$ is \{w, wz, wz^2, ..., wz^{N-1}\}, where $z = e^{2\pi i/N}$ is the generator of $Z(N)$, and $w$ is a phase necessary to ensure unitarity. We will order these eigenvalues in a matrix $P_0$ as

$$P_0 = \begin{bmatrix} w & wz & wz^2 & \cdots & wz^{N-1} \end{bmatrix}.$$ \hspace{0.5cm} (11)

We can write $P_0$ in the form

$$(P_0)_{jk} = \delta_{jk} \exp (i\phi_{0j})$$ \hspace{0.5cm} (12)

where $\phi_{0j} = \frac{\pi}{N} (2j - N - 1)$ which represents uniform spacing of the eigenvalues around the unit circle. The matrix $P_0$ is gauge-equivalent to itself after a $Z(N)$ symmetry operation:

$$zP_0 = gP_0 g^+.$$ \hspace{0.5cm} (13)

This guarantees that $\text{Tr}_F [P_0^k] = 0$ for any value of $k$ not divisible by $N$, indicating confinement for all representations transforming non-trivially under $Z(N)$ [13].

To prove that $P_0$ is a global minimum of the effective potential, we use the high-temperature expansion for the one-loop effective potential of a particle in an arbitrary background Polyakov loop gauge equivalent to the matrix $P_{jk} = \delta_{jk} e^{i\phi}$. The one-loop effective potential can be written as

$$V_{1\text{-loop}} = \sum_{j,k=1}^N \left( 1 - \frac{1}{N} \delta_{jk} \right) V_B (\phi_j - \phi_k, 0) - 2N_f V_B (\phi_j - \phi_k, m)$$ \hspace{0.5cm} (14)

where $V_B (\theta, m)$ is given by [10]

$$V_B (\theta, m) = - \frac{m^2 T^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2 \left( n^2 \theta \right) \cos (\theta) = - \frac{2 T^4}{\pi^2} \left[ \frac{\pi^4}{90} - \frac{1}{48 \pi^2} \theta_+^2 (2\pi - \theta_+)^2 \right] + \frac{m^2 T^2}{2 \pi^2} \left[ \frac{\pi^2}{6} - \frac{1}{4} \theta_+ (2\pi - \theta_+) \right] - \frac{T^4}{2 \pi} \sum_l \left\{ \frac{1}{3} \left[ (\beta m)^2 + (\theta - 2\pi l)^2 \right]^{3/2} - \frac{1}{3} |\theta - 2\pi l|^3 - \frac{1}{2} |\theta - 2\pi l| \beta^2 m^2 - \frac{\beta^4 m^4}{16 \pi |l|} \right\} - \frac{m^4}{16 \pi^2} \ln \left( \frac{\beta m}{4\pi} \right) + \gamma - \frac{3}{4}$$ \hspace{0.5cm} (15)

and $\theta_+$ is $\theta$ shifted as necessary to lie between $0$ and $2\pi$. The prime on the summation over $l$ indicates that the term singular at $l = 0$ is to be omitted. The $T^4$ term dominates for $\beta m \ll 1$, giving

$$V_{1\text{-loop}} \approx \sum_{j,k=1}^N \left( 1 - \frac{1}{N} \delta_{jk} \right) \frac{2 (N_f - 1) T^4}{\pi^2} \left[ \frac{\pi^4}{90} - \frac{1}{48 \pi^2} \theta_+^2 (\phi_j - \phi_k)^2 (\phi_j - \phi_k - 2\pi)^2 \right]$$ \hspace{0.5cm} (16)

which has $P_0$ as a minimum provided $N_f > 1/2$. For any fixed value of $N$, a sufficiently small value of $\beta m$ will make the confined phase stable. Even if the adjoint fermion mass is enhanced by chiral symmetry breaking, as would be expected in a confining phase, it should be of order $gT$ or less, and the higher terms in the expansion of $V_{1\text{-loop}}$ can be neglected at sufficiently high temperature.

It is interesting to study the stability of the high-temperature confined region as a function of $N$. Myers and Ogilvie have determined the phase diagram as a function of $\beta m$ and $N$ up to $N = 20$ by numerically minimizing $V_{1\text{-loop}}$ as a function of all the Polyakov loop eigenvalues [13, 15]. These models display a rich phase structure, in which there may be many phase transitions. When $N$ is not prime, there are typically phases where $Z(N)$ center symmetry is spontaneously down to $Z(p)$ where $p$ is a factor of $N$. In the case of $N$ even, the confined phase gives way to a phase with unbroken $Z(N/2)$ symmetry as $\beta m$ increases. The phase with unbroken $Z(N/2)$ is the “least deconfined” of the partially confining phases. In the Appendix, we show that for $N$ even, the minima of the effective potential corresponding to the confined phase, which has $Z(N)$ symmetry, and to a partially confined phase which has $Z(N/2)$ symmetry are equal when

$$(2^{d+1} - 1) V_B (\beta, 0, 0) = 2N_f \left[ 2^{d+1} V_B (\beta, 0, Nm/2) - V_B (\beta, 0, Nm) \right].$$ \hspace{0.5cm} (17)
A solution exists only for $N_f > 1/2$ and $(\beta m)_c$ generally decreases as $N_f$ increases. For the case $N_f = 1$ and $d = 4$ we find

$$(\beta m)_c \simeq \frac{4.00398}{N}$$

which is consistent with the numerical results of Myers and Ogilvie. Thus, we see that for any given $T$ and $N$, there is a range of values of $m$ for which the system is in the confined phase.

For $\beta m \gg 1$, the one-loop potential favors the deconfined phase, so there must be at least one phase transition as $\beta m$ is varied. In the case of $SU(2)$, there is indeed a single phase transition separating the confined and deconfined phases. Somewhat surprisingly, the transition is first-order between the high-temperature confined region and the deconfined phase. This behavior is different from the Ising-like, second-order transition in the pure gauge theory \[16, 17, 18\], but both behaviors are possible in spin systems with $Z(2)$ symmetry. Typically, a negative quartic term in a scalar potential stabilized by higher-order terms leads to a tricritical point, where a line of second-order transitions intersects a first-order line. It seems very likely that both transitions are part of a critical line in the $m - T$ plane, and are separated by a tricritical point. For $N \geq 3$, the one-loop potential predicts that one or more phases separate the deconfined phase from the confined phase. In the case of $SU(3)$, a single new phase is predicted, and has been observed in lattice simulations. For higher values of $N$, there is a rich set of possible phases, including some where $Z(N)$ breaks down to a proper subgroup $Z(p)$. In such phases, particles in the fundamental representation are confined, but bound states of $N/p$ such particles are not \[13, 19\].

### III. TEMPORAL STRING TENSIONS

The timelike string tension $\sigma^{(t)}_k$ between $k$ quarks and $k$ antiquarks can be measured from the behavior of the correlation function

$$\langle Tr_F P^k(x) Tr_F P^{+k}(y) \rangle \simeq \exp \left[ - \frac{\sigma^{(t)}_k}{T} |x - y| \right]$$

at sufficiently large distances. Two widely-considered scaling behaviors for string tensions are Casimir scaling, characterized by

$$\sigma_k = \sigma_1 \frac{k (N - k)}{N - 1},$$

and sine-law scaling, given by

$$\sigma_k = \sigma_1 \frac{\sin \left[ \frac{\pi k}{N} \right]}{\sin \left[ \frac{\pi}{N} \right]}.$$  

For a more detailed discussion of string tension scaling laws, see \[3\].

Timelike string tensions are calculable perturbatively in the high-temperature confining region from small fluctuations about the confining minimum of the effective potential \[20\]. The one-loop three-dimensional effective Lagrangian is

$$\frac{T}{g^2} \sum_{j=1}^{N} (\nabla \phi_j)^2 + \frac{T^3}{\pi^2} \sum_{n=1}^{\infty} \frac{|Tr_F P^n|^2 - 1}{n^2} \left[ 2N_f \beta^2 m^2 K_2(n\beta m) - \frac{2}{n^2} \right]$$

For small fluctuations, we write

$$P = P_0 e^{i\delta \phi}$$

where $\delta \phi$ lies in the Cartan algebra of $SU(N)$, and therefore has $N - 1$ independent components. The powers of the Wilson loop are

$$Tr_F P^k = w^k \sum_{j=1}^{N} e^{i k \delta \phi_j}$$
If $k$ is not divisible by $N$, we have approximately

$$Tr_F P^k \simeq ikw^k \sum_{j=1}^{N} z^{jk} \delta \phi_j = ikw^k \tilde{\delta} \phi_k$$

(25)

where $\tilde{\delta} \phi$ is the discrete Fourier transform of $\delta \phi$, related by

$$\delta \phi_k = \sum_{j=1}^{N} z^{jk} \delta \phi_j$$

(26)

$$\delta \phi_j = \frac{1}{N} \sum_{k=1}^{N} z^{-jk} \tilde{\delta} \phi_k.$$  

(27)

Note that the reality of $\delta \phi$ implies that $\tilde{\delta} \phi^*_k = \tilde{\delta} \phi_{N-k}$; the last Fourier component, $\tilde{\delta} \phi_N$, is identically zero, due the tracelessness of $\delta \phi$. If $k$ is divisible by $N$, we have instead

$$Tr_F P^k \simeq N - \frac{1}{2} k^2 \sum_{j=1}^{N} (\delta \phi_j)^2 = N - \frac{1}{2N} k^2 \sum_{m=1}^{N} (\delta \phi_m \delta \phi_{N-m}).$$

(28)

These formulae allow us to write the three-dimensional effective action in terms of the $\tilde{\delta} \phi_k$ to quadratic order. For each value of $k$, the terms in the potential which contribute have $n \equiv k \text{mod } N$, $n \equiv N - k \text{mod } N$, or $n \equiv 0 \text{mod } N$. We obtain a different “mass” $\sigma^{(t)}_k/T$ for each Fourier component $\delta \phi_k$. The string tensions are of order $g$:

$$\left(\frac{\sigma^{(t)}_k}{T}\right)^2 = g^2 N \frac{2Nf m^2}{2\pi^2} \sum_{j=0}^{\infty} \left[K_2 ((k + jN)\beta m) + K_2 ((N-k+jN)\beta m) - 2K_2 ((j+1)N\beta m)\right] - g^2 N \frac{T^2}{3N^2} \left[3 \csc^2 \left(\frac{\pi k}{N}\right) - 1\right]$$

(29)

where the gluon contribution has been summed in the last term. Note that the symmetry $\sigma^{(t)}_k = \sigma^{(t)}_{N-k}$ is manifest in this formula. The string tensions are continuous functions of $\beta m$. The $m = 0$ limit has the simple form

$$\left(\frac{\sigma^{(t)}_k}{T}\right)^2 = \frac{(2Nf - 1) \beta T^2}{3N} \left[3 \csc^2 \left(\frac{\pi k}{N}\right) - 1\right]$$

(30)

and is a good approximation for $\beta m \ll 1$. This scaling law is not at all like either Casimir or sine-law scaling, because the usual hierarchy $\sigma^{(t)}_{k+1} \geq \sigma^{(t)}_k$ is here reversed. Because we expect on the basis of $SU(3)$ simulations that the high-temperature confining region is continuously connected to the conventional low-temperature region, there must be an inversion of the string tension hierarchy between the two regions for all $N \geq 4$. For the case $N_f = 1/2$, corresponding to a single multiplet of adjoint Majorana fermions, the perturbative string tension vanishes. As discussed in [11, 12], it is the non-perturbative contribution to the effective potential induced by monopoles that gives rise to the string tension in this case. The large-$N$ limit of eqn. (30) is smooth. For fixed $k$ as $N \to \infty$, we have

$$\left(\frac{\sigma^{(t)}_k}{T}\right)^2 \sim \frac{(2Nf - 1) \lambda T^2}{\pi^2 k^2}$$

(31)

where $\lambda$ is the ’t Hooft coupling $g^2 N$.

**IV. SPATIAL STRING TENSIONS**

The confining minimum $P_0$ of the effective potential breaks $SU(N)$ to $U(1)^{N-1}$. This remaining unbroken Abelian gauge group naively seems to preclude spatial confinement, in the sense of area law behavior for spatial Wilson loops.
However, as first discussed by Polyakov in the case of an $SU(2)$ Higgs model in $2 + 1$ dimensional gauge systems, instantons can lead to nonperturbative confinement \[3\]. In the high-temperature confining region, the dynamics of the magnetic sector are effectively three-dimensional due to dimensional reduction. The Polyakov loop plays a role similar to an adjoint Higgs field, with the important difference that $P$ lies in the gauge group, while a Higgs field would lie in the gauge algebra. The standard topological analysis \[21\] is therefore slightly altered, and there are $N$ fundamental monopoles in the finite temperature gauge theory \[22, 23, 24, 25, 26\] with charges proportional to the affine roots of $SU(N)$, given by $2\pi\alpha_j/g$ where $\alpha_j = \hat{e}_j - \hat{e}_{j+1}$ for $j = 1$ to $N - 1$ and $\alpha_N = \hat{e}_N - \hat{e}_1$. Monopole effects will be suppressed by powers of the Boltzmann factor $\exp[-E_j/T]$ where $E_j$ is the energy of a monopole associated with $\alpha_j$.

In the high-temperature confining region, monopoles interact with each other through both their long-ranged magnetic fields, and also via a three-dimensional scalar interaction, mediated by $A_4$. The scalar interaction is short-ranged, falling off with a mass of order $gT$. The long-range properties of the magnetic sector may be represented in a simple form by a generalized sine-Gordon model which generates the grand canonical ensemble for the monopole/antimonopole gas \[4\]. The action for this model represents the Abelian dual form of the magnetic sector of the $U(1)^{N-1}$ gauge theory. It is given by

$$S_{mag} = \int d^3x \left[ \frac{T}{2} (\partial \rho)^2 - 2\xi \sum_{j=1}^{N} \cos \left( \frac{2\pi}{g} \alpha_j \cdot \rho \right) \right]$$

where $\rho$ is the scalar field dual to the $U(1)^{N-1}$ magnetic field. The monopole fugacity $\xi$ is given by $\exp[-E_j/T]$ times functional determinantal factors \[27\].

This Lagrangian is a generalization of the one considered by Polyakov for $SU(2)$, and the analysis of magnetic confinement follows along the same lines \[3\]. The Lagrangian has $N$ degenerate inequivalent minima $\rho_{\mu k} = g\mu_k$ where the $\mu_k$'s are the simple fundamental weights, satisfying $\alpha_j \cdot \mu_k = \delta_{jk}$. In the basis we use, $\mu_k$ is an $N$-dimensional vector of the form $1/N \cdot \{k, k, ..., k - N, k - N\}$ with $(N - k)$ $k$'s and $k$ $(k - N)$'s. Note that $e^{2\pi i\mu_k} = z_k^{1/N}$ and the vector $\rho_{0N}$ is the zero vector. A spatial Wilson loop in the $x-y$ plane

$$W[C] = \mathcal{P} \exp \left[ i \oint_C dx_j \cdot A_j \right]$$

introduces a discontinuity in the $z$ direction in the field dual to the magnetic field tensor. We can move this discontinuity out to spatial infinity; then the string tension of the spatial Wilson loop is the interfacial energy of a kink $\rho(z)$ interpolating between the different vacua $\rho_{0k}$ as $z \rightarrow \pm \infty$ \[4\].

In the case of $SU(2)$, the Cartan algebra is one-dimensional, and the kink solution is the sine-Gordon soliton. A BPS-type inequality \[28\] gives the string tension as

$$2\sqrt{8\xi T^g \pi}$$

For higher $N$, the general kink solutions are not known analytically. It is likely that the $k = 1$ kink is given by a straight-line path along the $\mu_1$ direction. It is easy to check that motion of the Polyakov loop $TrFP(z) = \sum_{j=1}^{N} \exp \left( \frac{2\pi i}{g} \hat{e}_j \cdot \rho \right)$ is an arc in the complex plane from $N \cdot z_N$ along the boundary of allowed values of $TrFP$. In the case of $N = 3$, we can prove this is the global minimum. In this case the sum over affine roots can be written as

$$-2\xi \sum_{j=1}^{3} \cos \left( \frac{2\pi}{g} \alpha_j \cdot \rho \right) = \xi \left[ 3 - \sum_{j=1}^{3} \exp \left( \frac{2\pi i}{g} \hat{e}_j \cdot \rho \right)^2 \right]$$

and an argument similar to the one given in \[29\] shows that this solution is a global minimum. There is a simple ansatz that generalizes this $k = 1$ kink solution to higher values of $k$. The ansatz is a straight line through the Lie algebra, parametrized as $\rho(z) = g\mu_k q(z)$ \[30\]. Figure 1 shows the $k = 1, 2, 3$ paths in the $TrFP$ complex plane for $SU(6)$. For the general case, this straight line ansatz gives an action

$$S_{mag} = \int d^3x \left[ \frac{T}{2} g^2 \frac{k(N - k)}{N} (\partial q)^2 - 2\xi \cos (2\pi q) \right]$$

yielding an upper bound of the form

$$\sigma_k^{(s)} \leq \frac{8}{\pi} \left[ \frac{g^2 T\xi}{N} k (N - k) \right]^{1/2}$$

(37)
Figure 1: $k = 1, 2, 3$ paths in the $Tr_F P$ complex plane for $SU(6)$. The $k = 1$ path is along the boundary of allowed values of $Tr_F P$.

| $N$ | $k$ | Casimir | sine-Law | $\frac{\sigma_k^{(t)}}{\sigma_1^{(t)}}$ | $\frac{\sigma_k^{(s)}}{\sigma_1^{(s)}}$ |
|-----|-----|---------|----------|-------------------------------------|-------------------------------------|
| 4   | 2   | 1.33    | 1.41     | 0.40                                | 1.15                                |
| 5   | 2   | 1.50    | 1.62     | 0.30                                | 1.22                                |
| 6   | 2   | 1.60    | 1.73     | 0.27                                | 1.26                                |
| 6   | 3   | 1.80    | 2.00     | 0.18                                | 1.34                                |
| 7   | 2   | 1.67    | 1.80     | 0.26                                | 1.29                                |
| 7   | 3   | 2.00    | 2.25     | 0.14                                | 1.41                                |
| 8   | 2   | 1.71    | 1.85     | 0.26                                | 1.31                                |
| 8   | 3   | 2.14    | 2.41     | 0.13                                | 1.46                                |
| 8   | 4   | 2.29    | 2.61     | 0.10                                | 1.51                                |

Table I: Comparison of Casimir and sine-Law scaling with temporal and spatial string tension scaling in the high-temperature confining region.

This bound is exact for $N = 2$ or $3$ where there is only one independent string tension. If, as seems likely, this result is exact for $k = 1$ for all $N$, this would give a useful bound on string tension ratios of the form

$$\frac{\sigma_k^{(s)}}{\sigma_1^{(s)}} \leq \sqrt{\frac{k(N - k)}{N - 1}}.$$  \hspace{1cm} (38)

This square-root-Casimir behavior differs significantly from both Casimir and sine-law scaling, and should be easily distinguishable in lattice simulations. Table 1 compares the string tension scaling laws we have found in the high temperature confined region to Casimir and sine-Law scaling.

V. CONCLUSIONS

We have been able to predict analytically a number of properties of the high temperature confined region, which lattice simulations should be able to confirm. For all values of $N$, there exists a high-temperature confining region when the fermion mass is sufficiently small. The phase structure and thermodynamics of these models are particularly rich for $N \geq 4$. However, even in the case of $SU(2)$, there is an interesting prediction of a first-order transition between the
deconfined phase and the high-temperature confined region as the adjoint fermion mass is varied. It follows that there must be a tricritical point in the $\beta - m$ plane somewhere on the critical line separating the confined and deconfined phases, whose location could be determined with lattice simulations.

We have shown that there is a perturbative prediction for an inverted hierarchy of timelike string tensions, beginning at $N = 4$. This behavior is very different from the string tension scaling of pure $SU(N)$ gauge theory at zero temperature. It is very likely that the crossover between the two behaviors is dramatic. We have also developed a semiclassical bound for spacelike string tensions, which also indicates a deviation from the string tension scaling of pure $SU(N)$. However, the behavior of spacelike string tensions in the high temperature confined region is much closer to that expected in the pure gauge theory, and the crossover between the two regions might be smooth. It is intriguing that spacelike confinement is associated with monopole condensation, the oldest scenario for quark confinement. The predictions for both spacelike and timelike string tensions are potentially testable in lattice simulations.

The large-$N$ behavior of the high-temperature confining region is of obvious interest. Unsal and Yaffe have argued [4] that the description of this region as an $U(1)^{N-1}$ effective theory is valid only when the inequality $T > NA$ is satisfied, where $\Lambda$ is the usual renormalization-group invariant scale for an $SU(N)$ gauge theory. If this inequality holds, Abelian monopoles control are responsible for area-law behavior of spatial Wilson loops. On the other hand, we have shown that temporal string tensions and confinement, as measured by temporal Polyakov loops, require $T \gtrsim Nm$. In fact, these two inequalities may be compatible, because of dynamical generation of a fermion mass.

These predictions for the high-temperature region lead naturally to additional questions that lattice simulations can address, but semiclassical methods most likely cannot. Lattice simulations can explore the crossover from conventional, low-temperature confining behavior to the behavior predicted in the high-temperature confining region. Some features can be studied in simulations where a simple deformation of the action is used, as in [5]. In addition to the string tensions, these features include monopole and instanton densities, and the topological susceptibility. Other aspects will require the inclusion of adjoint dynamical fermions in lattice simulations. Chiral symmetry breaking is of particular interest. Unsal [31] [32] has proposed a detailed picture of chiral symmetry breaking which can be independently checked by simulation. The accessibility of lattice field configurations as well as conventional observables makes the high-temperature confined region a natural place to explore the overlap of theory and simulation.

Appendix

The determination of the critical point between the confined phase, with $Z(N)$ symmetry, and a phase with $Z(N/2)$ symmetry, is made possible by a generalization of a Bernoulli polynomial summation formula [33]. Let $f(\theta, M)$ have the form

$$f_p(\theta, M) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^p} g(nM) \left[ e^{in\theta} + e^{-in\theta} \right]$$

where $g$ is sufficiently well-behaved that the series converges. Consider the finite sum

$$\sum_{k=0}^{m-1} f_p\left(\theta + \frac{2\pi k}{m}, M\right) = \sum_{n=1}^{\infty} \sum_{k=0}^{m-1} \frac{1}{n^p} g(nM) \left[ e^{i\theta + 2\pi nk/m} + e^{-i\theta - 2\pi nk/m} \right].$$

(A.2)

The sum over $k$ gives a non-zero result only when $n$ is a multiple of $m$, giving

$$\sum_{k=0}^{m-1} f_p\left(\theta + \frac{2\pi k}{m}, M\right) = m^{1-p} \sum_{n=1}^{\infty} \frac{1}{n^p} g(nM) \left[ e^{in\theta m} + e^{-in\theta m} \right] = m^{1-p} f_p(m\theta, mM).$$

(A.3)

This formula can be directly applied to the basic quantity for constructing all one-loop effective potentials in $d$ spatial dimensions, the effective potential of a massive charged boson in a constant $U(1)$ Polyakov loop background. The $U(1)$ Polyakov loop is parametrized as $\exp \left( i \int_0^\beta dt A_\theta \right) = \exp \left( i\theta \right)$, and the standard periodic boundary conditions for bosons are used. The finite-temperature component of the effective potential is

$$V_B(\beta, \theta, M) = \frac{1}{\beta} \int \frac{d^4 k}{(2\pi)^d} \ln \left[ 1 - e^{-\beta \omega_k + i\theta} \right] + \frac{1}{\beta} \int \frac{d^4 k}{(2\pi)^d} \ln \left[ 1 - e^{-\beta \omega_k - i\theta} \right]$$

(A.4)

where $\omega_k$ is the energy $\sqrt{k^2 + M^2}$ [10]. This can be expanded as

$$V_B(\beta, \theta, M) = -2 \left( \frac{M}{2\pi\beta} \right)^{(d+1)/2} \sum_{n=1}^{\infty} \frac{1}{n^{(d+1)/2}} K_{(d+1)/2}(n\beta M) \left[ e^{i\theta n} + e^{-i\theta n} \right].$$

(A.5)
Applying our summation formula to \( V_B \), we have

\[
\sum_{k=0}^{m-1} V_B \left( \beta, \theta + \frac{2\pi k}{m}, M \right) = m^{-d}V_B (\beta, m\theta, mM) \tag{A.6}
\]

from which a series of useful formulas for evaluating the effective potential in phases with various \( Z(N) \) symmetries can be derived.

The effective potential of an adjoint \( SU(N) \) massive boson in the confining phase is given by

\[
\sum_{j,k=1}^{N} V_B \left( \beta, \frac{2\pi (j-k)}{N}, M \right) - V_B (\beta, 0, M) \tag{A.7}
\]

where the last term subtracts out the singlet term that would occur for \( U(N) \). Applying the summation formula, this is simply

\[
N^{1-d}V_B (\beta, 0, NM) - V_B (\beta, 0, M). \tag{A.8}
\]

In the case where \( N \) is even and \( Z(N) \) is spontaneously broken to \( Z(N/2) \), the effective potential is

\[
4 \sum_{j,k=1}^{N/2} V_B \left( \beta, \frac{2\pi (j-k)}{N/2}, M \right) - V_B (\beta, 0, M) \tag{A.9}
\]

which reduces to

\[
4 (N/2)^{1-d} V_B (\beta, 0, NM/2) - V_B (\beta, 0, M). \tag{A.10}
\]

Now consider massless gauge bosons with periodic boundary conditions combined with \( N_f \) adjoint Dirac fermions, also with periodic boundary conditions. The effective potential of the confined phase is

\[
\left[ N^{1-d}V_B (\beta, 0, 0) - V_B (\beta, 0, 0) \right] - 2N_f \left[ N^{1-d}V_B (\beta, 0, NM) - V_B (\beta, 0, M) \right] \tag{A.11}
\]

and the effective potential of the phase where \( Z(N) \) is spontaneously broken to \( Z(N/2) \) is

\[
\left[ 4 (N/2)^{1-d} V_B (\beta, 0, 0) - V_B (\beta, 0, 0) \right] - 2N_f \left[ 4 (N/2)^{1-d} V_B (\beta, 0, NM/2) - V_B (\beta, 0, M) \right]. \tag{A.12}
\]

The two minima of the effective potential are equal when

\[
(2^{d+1} - 1) V_B (\beta, 0, 0) = 2N_f \left[ 2^{d+1} V_B (\beta, 0, NM/2) - V_B (\beta, 0, NM) \right]. \tag{A.13}
\]

From this we find that \((\beta M)_c\) decreases as \(1/N\); in particular,

\[
(\beta M)_c \approx \frac{4.00398}{N} \tag{A.14}
\]

for \( d = 4 \) and \( N_f = 1 \), consistent with [13]. Note that a solution exists only for \( N_f > 1/2 \) and \((\beta m)_c\) generally decreases as \( N_f \) increases. This analysis can easily be extended to evaluate the effective potential when \( Z(N) \) is spontaneously broken to \( Z(p) \), where \( p \) is an arbitrary factor of \( N \). Such expressions are useful for cases such as \( N = 6 \), where all the phases have a \( Z(p) \) symmetry, but do not apply for general \( N \), due to the occurrence of phases without \( Z(p) \) invariance [15].

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