Equilibrium Pricing in an Order Book Environment: Case Study for a Spin Model

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1. Introduction

In models for financial markets, the traders’ decision to buy or to sell is typically mapped to supply and demand, which are then balanced to determine the resulting price change, see [1–4]. This equilibrium pricing is a classical concept in the economics literature [5, 6]. On the other hand, a real stock exchange uses a double auction order book. It provides a trading platform to every registered participant in which all offers to buy or sell are listed, ensuring that all traders have the same information. A price is quoted whenever a buy or sell orders match. Thus, the real price formation is quite different from the concept of equilibrium pricing. Here, we confront a model based on equilibrium pricing with the dynamics of a double auction order

Abstract

When modelling stock market dynamics, the price formation is often based on an equilibrium mechanism. In real stock exchanges, however, the price formation is governed by the order book. It is thus interesting to check if the resulting stylized facts of a model with equilibrium pricing change, remain the same or, more generally, are compatible with the order book environment. We tackle this issue in the framework of a case study by embedding the Bornholdt–Kaizoji–Fujiwara spin model into the order book dynamics. To this end, we use a recently developed agent based model that realistically incorporates the order book. We find realistic stylized facts. We conclude for the studied case that equilibrium pricing is not needed and that the corresponding assumption of a “fundamental” price may be abandoned.

Keywords: decision making, agent-based modeling, order book, spin model
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Due to the rich variety of existing equilibrium pricing models, we have to restrict ourselves to case studies. In a previous work \cite{7}, we looked at a rather simple-minded decision making model that we analyzed with equilibrium pricing and, alternatively, in an order book environment. Here, we wish to address the more advanced Ising-type of spin models which are known to properly capture several aspects of financial markets. We choose the Bornholdt–Kaizoji–Fujiwara model \cite{8, 9} as a particularly interesting representative. We sketch its salient features in Sec. \ref{sec:2}. To apply order book dynamics to this model, we employ an agent-based model that fully accounts for the double auction order book as used in stock exchanges. It obviously makes sense to choose a “minimalistic” agent-based model free of additional features that might influence the resulting picture. Such an agent-based model was recently put forward and successfully tested in Ref. \cite{10}. In Sec. \ref{sec:3} we briefly present its setup and adjust it to the Bornholdt–Kaizoji–Fujiwara model. This amounts to applying the decision making part of the Bornholdt–Kaizoji–Fujiwara model, but to then let the order book work. In particular, this lifts the constraints due to equilibrium pricing. We statistically analyze selected resulting quantities in Sec. \ref{sec:4}. We summarize and conclude in Sec. \ref{sec:5}.

2. Bornholdt–Kaizoji–Fujiwara Model

Considering their paramount success for the study of phase transitions in statistical mechanics, it is not surprising that the application of spin models, in particular those of the Ising type, to financial markets has a long history, see \cite{8, 9, 11–16}. Bornholdt \cite{8} introduced an additional coupling constant to the ferromagnetic nearest neighbor interaction which then couples the individual spins to the total magnetization. This is motivated by two conflicting economic driving forces:

1. “Do what your neighbors do”, by aligning your spin to your neighbors.
2. “Do what the minority does”, by coupling to the magnetization.

As there is no quantifiable stock price in the original version of the model \cite{8}, Kaizoji, Bornholdt and Fujiwara \cite{9} extended it accordingly by setting up a stock market with two groups of traders. We refer to this version as Bornholdt–Kaizoji–Fujiwara model.

There are \( n \) interacting traders \( i \) whose investment attitude is represented by one spin variable \( S_i(t) = \pm 1, \ i = 1, \ldots, n \) each. The dynamics of the spins...
is governed by a heat bath that depends on a local Hamiltonian \( h_i(t) \), \( i = 1, \ldots, n \) and determines a probability \( q \) such that

\[
S_i(t+1) = +1 \quad \text{with} \quad q = \frac{1}{1 + \exp(-2\beta h_i(t))}
\]

(1)

\[
S_i(t+1) = -1 \quad \text{with} \quad 1 - q .
\]

(2)

Here, \( S_i(t) = 1 \) (\( S_i(t) = -1 \)) represents a positive (negative) investment attitude, meaning that trader \( i \) buys (sells) the stock. The traders’ perception of the market is driven by two kinds of information. Locally, he is only influenced by the nearest interacting traders, but globally, it will also affect him whether or not he belongs to the majority group. This is measured by the absolute value \(|M(t)|\) of the magnetization

\[
M(t) = \frac{1}{n} \sum_{i=1}^{n} S_i(t) .
\]

(3)

To accumulate wealth, the trader has to be in the majority group and the majority has to expand over the next trading period. However, if \(|M(t)|\) already has a large value, further increase is hampered. Traders in the majority group then tend to switch to the minority to avert a loss. On the other hand, a trader in the minority group tends to switch to the majority in the quest for profit. Altogether, the larger \(|M(t)|\), the larger is the tendency for any group member to switch sides. The local Hamiltonian \( h_i(t) \) entering Eq. (2) reads

\[
h_i(t) = \sum_{j=1}^{n} J_{ij} S_j(t) - \alpha S_j(t)|M(t)| ,
\]

(4)

with interaction \( J_{ij} = J \) for nearest neighbors \( i, j \) and with \( J_{ij} = 0 \) for all other pairs \( i, j \). The global coupling constant \( \alpha \) is positive, \( \alpha > 0 \).

How is the stock price \( p(t) \) determined in this model? — A number \( m \) of fundamentalist traders is introduced whose decisions are driven by supply and demand. They assume to have a reasonable knowledge of the fundamental value \( p^*(t) \) of the stock price. If the price \( p(t) \) falls below that threshold \( p^*(t) \), the fundamentalists buy the stock, otherwise they sells it. The fundamentalists’ excess demand is given by

\[
x^F(t) = am \left( \log p^*(t) - \log p(t) \right) = am \log \frac{p^*(t)}{p(t)} ,
\]

(5)

where \( a \) is a parameter characterizing how strongly the fundamentalists react to the price difference between fundamental value and current price. On the
other hand, the interacting traders’ excess demand is governed by the total magnetization,
\[ x^I(t) = bnM(t) \] (6)
with \( b \) being the corresponding strength parameter.

The crucial assumption in the Bornholdt–Kaizoji–Fujiwara model is now the balance of supply and demand such that
\[ 0 = x^F(t) + x^I(t) \]
\[ = am \log \frac{p^*(t)}{p(t)} + bnM(t) . \] (8)

From this the relative price change \( r_l(t) \) after the time step from \( t \) to \( t + \Delta t \) follows according to
\[ r_l(t) = \log \frac{p(t + \Delta t)}{p(t)} = \log p(t + \Delta t) - \log p(t) \]
\[ = \log \frac{p^*(t + \Delta t)}{p^*(t)} + \lambda \left( M(t + \Delta t) - M(t) \right) \] (10)

with the combination of constants as given by
\[ \lambda = \frac{bn}{am} . \] (11)

Here, we slightly differ from Ref. [9] where the price change is defined from the time step \( t - \Delta t \) to \( t \) with fixed \( \Delta t = 1 \). For the sake of simplicity, we set \( p^*(t) \) constant implying
\[ r_l(t) = \lambda \left( M(t + \Delta t) - M(t) \right) . \] (12)

As the above sketch shows, the price \( p(t) \) in the Bornholdt–Kaizoji–Fujiwara model results from a supply and demand mechanism. The authors establish a link between magnetization and trading volume and give an interpretation of the aperiodic switching between bull and bear markets. They also show in a detailed analysis that their model reproduces, in an impressive fashion, stylized facts of real financial data, see Ref. [17], such as clustered volatilities, positive cross–correlation between trading volume and volatility, powerlaw fat tails and certain similarities of the volatilities at different time scales.

Nevertheless, the concept of a fundamental price \( p^*(t) \), which may be interpreted in the spirit of the “fair price” in the efficient market model [5], raises questions. Why should such a fair or fundamental price exist at all?
— It is an artificial criterion applied from outside the market. Is this consistent with the reality in which “the market makes the price”? — The zero intelligence trading model \cite{18,19} works without such an external concept and still produces results which are equivalent to those of the efficient market model. Consequently, it is interesting to study the Bornholdt–Kaizoji–Fujiwara model in a realistic setting by dropping the concept of the fundamental price and applying the full order book dynamics instead. Put differently, we only use the decision making procedure of the Bornholdt–Kaizoji–Fujiwara model and leave the rest to the trading via the order book.

3. Agent–Based Model

After reviewing the model setup of Ref. \cite{10} in Sec. 3.1, we introduce in Sec. 3.2 the IsingTrader that is especially adjusted to the Bornholdt–Kaizoji–Fujiwara model.

3.1. Sketch of the Model

Agent–based models provide a useful microscopic framework for the understanding of stylized facts, even though their complexity often outrules a one–to–one assignment of input and output. There are numerous agent–based models for financial markets involving different approaches, see e.g. \cite{17,20–23}.

In Ref. \cite{10}, a “minimalistic” agent–based model was introduced that implements a double–auction order book by only resting upon absolutely essential features. It possesses a variety of traders that follow a given set of rules. The order book stores the limit orders ascending from the cheapest buy to the most expensive sell order. Prices and time are discretized to the tick–size and simulation steps, respectively. Orders are cleared, whenever they are marketable, i.e., whenever some buy and sell orders match. In each simulation step an arbitrary number of traders are active, the order of the trading actions is randomized. Each active trader can place one order and draws his next time of action or rather his waiting time $t_{\text{wt}}$ from an exponential distribution

$$f(t_{\text{wt}}) = \frac{1}{\mu_{\text{wt}}} \exp(-t_{\text{wt}}/\mu_{\text{wt}}) \quad \text{with} \quad \mu_{\text{wt}} = cN \quad (13)$$

at the end of his trading action. The mean value $\mu_{\text{wt}}$ is the product of the number $N$ of traders and a parameter $c$ calibrated to achieve approximately 5.4 trades per minute. This choice coincides with the average trade frequency of the top 75% stocks in the S&P 500 index traded in 2007 \cite{10}. (To avoid confusion, we mention that the distributions denoted $f$ in the present study
are denoted \( p \) in Ref. [10]. Limit orders are placed with a lifetime \( t_{lt} \) drawn from an exponential distribution

\[
f(t_{lt}) = \frac{1}{\mu_{lt}} \exp\left(-\frac{t_{lt}}{\mu_{lt}}\right)
\]

with mean value \( \mu_{lt} \). We notice that traders of different type can choose different lifetimes \( t_{lt} \) corresponding to their governing rules. The virtual trading is carried out by the RandomTrader whose buy or sell limit orders are random with normal distributed prices centered around the current best price. The order sizes \( \nu \) are exponentially distributed,

\[
f(\nu) = \frac{1}{\mu_{vol}} \exp\left(-\frac{\nu}{\mu_{vol}}\right)
\]

with mean value \( \mu_{vol} \). Short-selling is allowed and the traders have unlimited credit.

In Ref. [10], a gap structure in the order book is identified as the reason for fat tails. Two mechanisms yield such gaps: canceling of older limit orders and orders placed far away from the current midpoint. The more liquidity is provided by limit orders, the less likely are extreme price shifts. The finiteness of lifetimes \( t_{lt} \) of limit orders placed by the RandomTraders produces gaps, if the lifetime \( t_{lt} \) is in the range of the rate at which new orders are placed. Consistent with Farmer et al. [24], price gaps between limit orders are at low liquidity even relevant close to the current midpoint. It also happens that large order volumes yield fat tails because orders far away from the midpoint are very unlikely and lead to large gaps, implying that the liquidity deep in the order book is very low. These gaps can only be reached with very large volumes which explains the observations. For very small order lifetimes \( t_{lt} \), the number of limit orders becomes so small that the order book effectively plays a minor role. The price formation is then mainly driven by the specific behavior of the trader, in particular by the distribution used to determine the order price. To study the relevance of the order book, we should not focus on this regime. Following Ref. [10], we only consider order lifetimes which are sufficiently large, i.e., \( \mu_{lt} > 40 \) time steps.

### 3.2. IsingTrader

We now introduce the IsingTrader as a new trader type in the agent-based model of Ref. [10]. With the IsingTrader we implement the Bornholdt–Kaizoji–Fujiwara model apart from the concept of the fundamental price. The order book dynamics alone generates the price.

We identify every lattice site in the Bornholdt–Kaizoji–Fujiwara model as an independent IsingTrader. The spin \( S_i(t) \) of every IsingTrader determines,
when he is active, whether he will buy or sell in the particular time step. We do not modify the mechanism that activates the traders to avoid interference with the time evolution in the agent–based model, and neither do we change the drawing of the order volumes as compared with the RandomTrader, because the Bornholdt–Kaizoji–Fujiwara model does not provide a corresponding appropriate rule. We also have to decide which type of order the IsingTrader places. According to the Bornholdt–Kaizoji–Fujiwara model, we ought to derive an appropriate limit price and a lifetime to place limit orders. This, however, would be quite complicated and, importantly, not in line with our intention to abandon the concept of equilibrium pricing. The better choice is the market order, because the trader’s spin value in the Bornholdt–Kaizoji–Fujiwara model directly translates to a demand at the current point in time. This means in the framework of the agent–based model that the trader has to buy or sell the stock directly, depending on his spin value. This can only be achieved by a market order. Another question arises regarding the lattice dynamics. In the agent–based model, we wish to affect the lattice dynamics as little as possible. We have to account for the facts that, first, not every IsingTrader is active in every timestep and that, second, too many market orders at one time would wipe the whole order book clean. Hence, it is reasonable to couple the rate of sweeps over the spin grid to the trading dynamics in the agent–based model by a sweep probability $q_{\text{sweep}}$.

For comparative reasons, we also introduce the LiquidityTaker [25] who is a RandomTrader placing market orders. Comparing the results for IsingTrader and LiquidityTaker, we are able to trace statistical features back to the way how the IsingTrader makes his decisions.

4. Results

We performed simulations with $N_{\text{Random}} = 2160$ RandomTraders with limit-order lifetimes of $\mu_{lt} = 600$ time steps, because, as shown in Ref. [10], fat–tails or related stylized facts do not occur in this case. The RandomTraders provide a neutral background for the trading activities of the $N_{\text{Ising}} = 144$ IsingTraders. The IsingTraders’ parameters are set to $J = 1.0$, $\alpha = 4.0$ and $\beta = 1.45$ in accordance with Ref. [8]. The mean value $\mu_{\text{wt,Ising}}$ of the waiting time distribution (13) for the IsingTraders only scales with the number of IsingTraders,

$$\mu_{\text{wt,Ising}} = cN_{\text{Ising}} .$$

A good value for the sweep probability was found to be $q_{\text{sweep}} = 0.001$. To this end we looked at the price stability resembled in the ratio of limit order placing RandomTraders and market order placing IsingTraders. Importantly, the simulations only very weakly depend on the exact value of $q_{\text{sweep}}$. 7
A crucial quantity is the average ratio $Q$ of IsingTraders who can place their orders before the lattice is updated for the first time. It is easily seen to be

$$Q = \sum_{n=1}^{\infty} q_{\text{sweep}} (1 - q_{\text{sweep}})^{n-1} \left(1 - e^{(n-1)/\mu_{\text{Ising}}}\right)$$

$$\approx 0.78.$$  

Knowledge of $Q$ gives a grip on the interplay between lattice and trading dynamics. This is so, because only if the lattice is updated slowly enough the simulation “sees” something of the lattice dynamics. Altogether, these choices yield on average a trading frequency of 5.4 trades per time step. We ran 1000 simulations with $T = 5 \cdot 10^5$ time steps. We checked that all of them were stable far beyond the chosen time scale.

From the traded prices $p(t)$, we calculate the returns

$$r(t) = \frac{p(t + \Delta t) - p(t)}{p(t)}$$

with return interval $\Delta t$ as well as the standard deviation

$$\sigma = \sqrt{\langle r(t)^2\rangle_T - \langle r(t)\rangle_T^2},$$

where the angular brackets indicate the sample mean over the entire trading time $T$.

To study the behavior of the return distribution, we normalize the returns to zero mean and unit standard deviation,

$$g(t) = \frac{r(t) - \langle r(t)\rangle_T}{\sigma}.$$ 

In Fig. we compare the distributions of the normalized returns $g(t)$ for six different return intervals $\Delta t$ to a normal distribution. We observe heavy tails which are the lower the larger the return intervals $\Delta t$.

To also investigate the volatility distributions, we calculate time dependent volatilities by moving a window of 1000 time steps through the data. The resulting distributions for different return intervals are displayed in Fig. They agree well with a log–normal distribution, which is consistent with empirically found volatility distributions. 

What is the effect of the IsingTrader’s strategy? — In Fig. We compare his time autocorrelation

$$\text{acf}(\tau) = \langle g^2(t)g^2(t + \tau)\rangle_T$$
Figure 1: Distributions of the normalized returns $g$ on a logarithmic scale for six return interval $\Delta t = 10, 30, 60, 360, 540$ and 720 time steps. A normal distribution is shown as dashed line.
Figure 2: Volatility distributions for six different return intervals $\Delta t$. A log-normal distribution is shown as dashed line.
Figure 3: Autocorrelation function of the squared returns $g^2$ for six different return intervals $\Delta t$ versus time lag $\tau$, for LiquidityTakers as red and for IsingTraders as black lines.
for squared returns as function of a time lag \(\tau\) to that of the LiquidityTaker. Interestingly, the IsingTrader creates long–range autocorrelation independently of the chosen return interval. A similar behavior has indeed been observed in empirical data [17]. This behavior can be traced back to the autocorrelation of the squared magnetization changes as observed in Ref. [8]. However, in the case of small return intervals, we also observe a non zero autocorrelation for lags greater than the return interval for the LiquidityTaker. This is a result of the rate of trades compared to the return interval. If the trade frequency is very low compared to the return interval, the price does not change and we obtain many successive zero returns. This can be seen in Fig. 1 for \(\Delta t = 10\) and \(\Delta t = 30\). A persistent autocorrelation of the squared returns results. We therefore find similar behavior for the LiquidityTaker and the IsingTrader at relatively small lags. It is less pronounced and eventually disappears when the return interval increases.

5. Conclusion

Models for financial markets typically consist of two parts: decision making and price formation. Due to the \textit{a priori} limited information about the individual trader, statistical concepts have to be invoked to model the decision making. The challenge is then to capture salient features of the highly complex dynamics by stochastic ingredients. On the other hand, the price formation is, in real stock exchanges, a microscopically well defined and deterministic process. The reason why the price formation part of most models relies on equilibrium pricing and does not involve the order book is probably twofold: First, the concept of equilibrium pricing is deeply rooted in the economics literature. Second, even though the order book dynamics is microscopically well defined, it is not possible to directly map it on a simple schematic equation.

Hence, we find it worthwhile to critically examine the price formation part of stock market models by putting their specific decision making part in a realistic order book setting. In a previous study [7], we carried out this program for a model whose decision making part is much simpler than in the Bornholdt–Kaizoji–Fujiwara model that we investigated in the present study. The price formation part of the latter explicitly employs the concept of a “fundamental” price which is as questionable as the closely related concept of a “fair” price in the efficient market model. To model the dynamics of a system, \textit{i.e.}, the stock market in the present case, only quantities should enter which have some empirical justification. The “fundamental” price, however, is an external criterion that cannot, not even indirectly, be measured.
We implemented the decision making part of the Bornholdt–Kaizoji–Fujiwara model in a minimalistic agent–based model and performed numerical simulations. They yield realistic stylized facts, including non–trivial features such as long–range temporal autocorrelations. This implies that the equilibrium pricing mechanism is fully obsolete. We thus conclude that the decision making part of the Bornholdt–Kaizoji–Fujiwara model grasps essential features of the market dynamics in a reliable manner.

Furthermore, we may draw a second conclusion. In an indirect way, our study shows that the equilibrium pricing mechanism can be used to coarsely and effectively mimick the order book dynamics. Put differently, the highly complex order book dynamics generates upon average a schematic rule that seems to be largely equivalent to an equilibrium pricing mechanism. Contrary to the common assumption in economics, there is no need whatsoever to require that the individual trades result from such an equilibrium pricing mechanism. Admittedly, we can base these statements only on the present case study, but we are tempted to believe that they are more general. Nevertheless, it is not necessary to evoke the sometimes almost ideological reasoning behind the concept of equilibrium pricing. Abandoning such a strict and schematic view also helps to understand turbulent market situations in which the assumption of equilibrium pricing is even less plausible than during quiet times. Equilibrium pricing should at best only be seen as an averaged result of the true dynamics: Everything is in the order book.

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