Non-Hermitian quantum mechanics of bosonic operators

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Abstract

The spectral analysis of a non-Hermitian unbounded operator appearing in quantum physics is our main concern. The properties of such an operator are essentially different from those of Hermitian Hamiltonians, namely due to spectral instabilities. We demonstrate that the considered operator and its adjoint can be diagonalized when expressed in terms of certain conveniently constructed operators. We show that their eigenfunctions constitute complete systems, but do not form Riesz bases. Attempts to overcome this difficulty in the quantum mechanical set up are pointed out.

1 Introduction

The main motivation for this article is the following. In conventional formulations of non-relativistic quantum mechanics, the Hamiltonian operator is Hermitian (synonymously, self-adjoint), and so has real eigenvalues and an orthonormal set of eigenfunctions. These fundamental issues are in the heart of von Neumann quantum paradigm for physical observability and dynamical evolution. Certain relativistic extensions of quantum mechanics lead to non-Hermitian Hamiltonian operators, \( H \neq H^* \), for \( H^* \) the adjoint of \( H \) (e.g. see [11, Chapter VIII]). Extending the set of allowed operators by including \( \mathcal{PT} \)-symmetric ones, \( \mathcal{P} \) being the reflection operator, \( \mathcal{P} f(x) = f(-x) \) and \( \mathcal{T} \) the time reflection operator, \( \mathcal{T} f(x) = f(x) \), yields spectra with reflection symmetry with respect to the real
axis. In this context, non-Hermitian Hamiltonians with a purely discrete spectrum have been the object of intense research activity [1, 4, 6, 12, 15, 17, 18, 21], attempting to build quantum mechanical theories with physical observables described by these operators. The subtleties of non self-adjoint Hamiltonians deserved the attention of physicists and mathematicians, as they may originate new and unexpected phenomena, and the non self-adjoint theory has revealed difficulties and challenging problems.

We focus on the spectral analysis of non-Hermitian operators, a field with applications in many areas of physics, most remarkably in quantum mechanics. We summarise theoretical classical and recent developments on this topic, and the theory is illustrated with a concrete example. We explicitly determine the eigenfunctions and eigenvalues of a non-Hermitian operator describing two interacting bosons, concretely, a non self-adjoint 2D harmonic oscillator.

Let $H$ be an infinite dimensional separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$, and corresponding norm $\| \cdot \|$. For $H$ a non-Hermitian operator in $\mathcal{H}$ with a purely discrete spectrum, the strategy of finding a Hermitian operator $H_0$ and an invertible operator $P$ such that

$$H = PH_0P^{-1},$$

has been exploited. This idea refers that such a non-Hermitian $H$ can be viewed essentially as an alternative representation of a Hermitian operator with the same eigenvalues and multiplicities, whenever $P$, $P^{-1}$ are both bounded (see Ref. [13]).

The relation (1) is closely related to the quasi-Hermicity $QH = H^*Q$, (2)

where $Q = (P^{-1})^*P^{-1}$ is a positive definite operator called a metric operator. (The relation (2) means that $QH$ and $H^*Q$ have the same domain and act in the same manner). In fact, $H$ with property (1), is formally Hermitian with respect to the modified inner product $\langle Q \cdot, \cdot \rangle$,

$$\langle QHx, y \rangle = \langle x, H^*Qy \rangle = \langle x, QHy \rangle, \quad x, y \in \mathcal{H}.$$ 

The metric is said to be non singular if it is bounded, invertible and boundedly invertible, otherwise it is singular. The pathologies of non-Hermitian operators with singular metric may be serious, since the metric can transform a basis into a set without any reasonable basicity properties. Moreover, the spectral stability with respect to small perturbations is not ensured and complex eigenvalues can appear distant to the original ones [19].

The rest of this note is organized as follows. In Section 2 we recall some issues used subsequently. In Section 3 we investigate spectral properties of an unbounded non-Hermitian operator acting in $\mathcal{H}$, quadratic in a pair of bosonic operators, and of their adjoints. It is shown that the obtained eigenfunctions
are complete systems but do not form Riesz bases for $\mathcal{H}$, and so the validity of useful properties of Hermitian operators is not guaranteed. In fact, non-Hermitian operators behave in an essentially different way, due in particular to spectral instabilities and to the lack of basicity of the Hamiltonian eigenfunctions. In Section 4 we discuss incidences of non-Hermiticity in quantum physics.

2 Preliminaries

In the sequel, we shall be concerned with $\mathcal{H} = L^2(\mathbb{R}^2)$, the Hilbert space of square-integrable complex valued functions in two real variables $x, y$, equipped with the inner product

$$\langle \Phi, \Psi \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(x, y)\overline{\Psi(x, y)} \, dx \, dy,$$

and corresponding norm $\| \cdot \|$. Many important operators in quantum physics are unbounded, which restricts their domains of definition to adequate subsets of the Hilbert space where they live. For instance, these domains are considered to be dense.

The set of eigenfunctions of $H$, $\{\Psi_n\}_{n=1}^\infty$, forms a complete family in $\mathcal{H}$ if the span of $\Psi_n$ is dense in $L^2(\mathbb{R}^2)$, or, equivalently, the orthogonal complement of this linear span in the Hilbert space reduces to the zero function.

Eigenfunctions of non-Hermitian operators are in general not orthogonal or even do not form a complete family. It should be stressed that the completeness of an orthogonal set $\{\Psi_n\}_{n=1}^\infty$, adequately normalized, does not imply, by its own, basicity. An orthonormal set $\{\Psi_n\}_{n=1}^\infty$ does not guarantee that any $\psi \in \mathcal{H}$ admits a unique expansion of the form

$$\psi = \sum_{n=1}^\infty c_n \Psi_n,$$

where, in this event, it should be $c_n = \langle \psi, \Psi_n \rangle$, $n \geq 1$.

The concept of eigenbasis is very important in quantum mechanics. In contrast to the case of eigenfunctions of Hermitian operators, which are basis, for non-Hermitian operators this property fails in general, and so the notion of Riesz basis may be of interest. We say that $\{\Psi_n\}_{n=1}^\infty$ is a Riesz basis if for any $\psi \in \mathcal{H}$, there exists a positive constant $C$ independent of $\psi$ such that

$$C^{-1}\|\psi\|^2 \leq \sum_{n=1}^\infty |\langle \psi, \Psi_n \rangle|^2 \leq C\|\psi\|^2.$$ 

The eigenfunctions of an operator $H$ with purely discrete spectrum constitute a Riesz basis if and only if $H$ is quasi-Hermitian via a nonsingular bounded metric
Q (see [3]). The eigenfunctions of a non-Hermitian $H$, despite possibly being a complete family, may not form a Riesz basis, as occurs frequently with several models [13]. It should be noticed that Riesz basicity, like the spectrum, is not preserved by unbounded metrics.

We introduce, as a first example of familiar operators in quantum mechanics, the multiplication operators $x$ and $y$

$$f(x, y) \to xf(x, y), \quad f(x, y) \to yf(x, y),$$

defined in their maximal domains $\mathcal{D}(x) = \{ \psi \in L^2(\mathbb{R}^2) : x\psi \in L^2(\mathbb{R}^2) \}$ and $\mathcal{D}(y) = \{ \psi \in L^2(\mathbb{R}^2) : y\psi \in L^2(\mathbb{R}^2) \}$.

We consider the partial differential operators $\partial/\partial x$ and $\partial/\partial y$ defined by

$$f(x, y) \to \frac{\partial f(x, y)}{\partial x}, \quad f(x, y) \to \frac{\partial f(x, y)}{\partial y},$$

with domains $\mathcal{D}(\partial/\partial x) = W^{1,2}(\mathbb{R}^2)$ and $\mathcal{D}(\partial/\partial y) = W^{1,2}(\mathbb{R}^2)$, where $W^{1,2}(\mathbb{R}^2)$ stands for the usual Lebesgue space of functions in $L^2(\mathbb{R}^2)$, whose first partial derivatives belong to $L^2(\mathbb{R}^2)$. These operators are known as momentum operators. The partial differential operators $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ are defined by

$$f(x, y) \to \frac{\partial^2 f(x, y)}{\partial x^2}, \quad f(x, y) \to \frac{\partial^2 f(x, y)}{\partial y^2},$$

with domains $\mathcal{D}(\partial^2/\partial x^2) = W^{2,2}(\mathbb{R}^2)$ and $\mathcal{D}(\partial^2/\partial y^2) = W^{2,2}(\mathbb{R}^2)$, the Lebesgue space of functions in $L^2(\mathbb{R}^2)$ whose first and second derivatives belong to $L^2(\mathbb{R}^2)$.

The (standard) bosonic operators

$$a =: x + \frac{1}{2} \frac{\partial}{\partial x}, \quad b =: y + \frac{1}{2} \frac{\partial}{\partial y},$$

with domains

$$\mathcal{D}(a) = \mathcal{D}(b) = \{ \Psi \in W^{2,2}(\mathbb{R}^2) : x\Psi, y\Psi \in L^2(\mathbb{R}^2) \}$$

are useful in our discussion. The operators $a$ and $b$ are known to be densely defined, closed, and their adjoints read

$$a^* = x - \frac{1}{2} \frac{\partial}{\partial x}, \quad b^* = y - \frac{1}{2} \frac{\partial}{\partial y},$$

We recall that, conventionally, $a, b$ are said to be annihilation operators, while $a^*, b^*$ are creation operators. It is worth noticing that these operators are unbounded, and they satisfy the commutation rules (CR’s),

$$[a, a^*] = [b, b^*] = 1,$$  

(4)
where 1 is the identity operator on $L^2(\mathbb{R}^2)$. (This means that $aa^* f - a^* af = bb^* f - b^* bf = f$ for any $f$ in $D(a)$ and $D(b)$) Furthermore,

$$[a, b^*] = [b, a^*] = [a^*, b^*] = [a, b] = 0.$$  \hfill (5)

As it is well-known, the canonical commutation relations (1) and (2) characterize an algebra of Weil-Heisenberg (W-H). Moreover, the following holds,

$$a\Phi_0 = b\Phi_0 = 0,$$

for $\Phi_0 = e^{-(x^2+y^2)}$ in $D(a)$ and $D(b)$, a so-called vacuum state. The set of functions

$$\{\Phi_{m,n} = a^{*m}b^{*n}\Phi_0 : m, n \geq 0\},$$  \hfill (6)

constitutes a basis of $\mathcal{H}$, that is, every vector in $L^2(\mathbb{R}^2)$ can be uniquely expressed in terms of this system, which is complete, since 0 is the only vector orthogonal to all its elements.

### 3 Non self-adjoint Hamiltonian $H$ describing two interacting bosonic operators

#### 3.1 Model

We consider the operator in $L^2(\mathbb{R}^2)$,

$$H := -\frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \gamma \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + (x^2 + y^2), \quad \gamma \in \mathbb{R},$$  \hfill (7)

which is obviously non self-adjoint for $\gamma \neq 0$. The closedness of the operator is the essential starting point for the study of its spectrum. In the perspective of a convenient domain of $H$, aiming at defining $\mathcal{D}(H)$, we regard the cofactor of $\gamma$ in the non-self-adjoint term,

$$V = \left(y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}\right),$$

as a perturbation of the 2D harmonic oscillator,

$$\Re(H) = -\frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (x^2 + y^2).$$

It is easily seen that this term is relatively bounded with the relative bound less than one with respect to $\Re(H)$, provided that $\gamma$ is less than one. Under this condition, $H$ is a closed operator on the domain of the harmonic oscillator:

$$W^{2,2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (x^4 + y^4)dx\,dy).$$
Here, we have used the standard perturbation result that states the following. If \( \Re(H) \) is closed and \( V \) is relatively bounded with respect to \( \Re(H) \), with the relative bound smaller than 1, then \( \Re(H) + \lambda V \), with \( \lambda < 1 \), is closed (\cite[Theorem 3.3]{14}).

We notice that \( \mathcal{D}(H) \) contains the subspace \( \mathcal{S}(\mathbb{R}^2) \) constituted by functions \( f(x,y) \) such that \( e^{-2\gamma xy}f(x,y) \in \mathcal{S}(\mathbb{R}^2) \) and, in turn, this one contains \( C^\infty_0(\mathbb{R}^2) \).

In terms of creation and annihilation operators, \( H \) is equivalently defined as

\[
H = a^*a + bb^* + \gamma(a^*b^* - ab), \quad \gamma \in \mathbb{R},
\]

and so \( H \) is quadratic in the bosonic operators \( a, b \) and in their adjoints.

### 3.2 Spectrum

As it is well known, the resolvent set of \( H \), denoted by \( \rho(H) \), is constituted by all the complex numbers for which the resolvent operator \( \lambda \in \rho(H) \rightarrow (H - \lambda)^{-1} \) exists as a bounded operator on \( \mathcal{H} \). The complement

\[
\sigma(H) = \mathbb{C}\setminus\rho(H)
\]

is called the spectrum of \( H \). The set of all eigenvalues of \( H \) is called the point spectrum, denoted by \( \sigma_p(H) \), and formed by complex numbers \( \lambda \) for which \( H - \lambda : \mathcal{D}(H) \rightarrow \mathcal{H} \) is not injective. The spectrum of an operator on a finite dimensional Hilbert space is exhausted by the eigenvalues, but, in the infinite dimensional setting, there are additional parts to be considered. Those \( \lambda \) which are not eigenvalues but \( H - \lambda \) is not bijective constitute the continuous or residual spectrum, depending on the range \( \text{Ran}(H - \lambda) \) being, respectively, dense or not. The spectrum \( \sigma(H) \) is the union of these three disjoint spectra. The spectrum of self-adjoint operators is nonempty, real, and the residual spectrum is empty, while the spectrum of non self-adjoint operators can be empty or coincide with the whole complex plane (see, e.g., refs. \cite{19, 21}).

### 4 Accretivity

It may be advantageous to estimate the spectrum in terms of the numerical range:

\[
W(H) := \{ \langle H\psi, \psi \rangle : \psi \in \mathcal{D}(H), \|\psi\| = 1 \}.
\]

In general \( W(H) \) is neither open nor closed, even when \( H \) is a closed operator. However, it is always convex and for \( H \) bounded,

\[
\sigma(H) \subset \overline{W(H)}.
\]
Proposition 4.1  The numerical range of $H$ is bounded by the hyperbola

$$y^2 + \gamma^2(1-x^2) = 0, \quad x \geq 1.$$ 

Proof. The numerical range of $H$ is determined as follows. Recalling that $W(H)$ is convex, let us consider the supporting line of $W(H)$ perpendicular to the direction $\theta$. Recall that the distance of this line to the origin is the lowest eigenvalue of

$$\Re(e^{-i\theta}H) = (a^*a + b^*b)\cos \theta - i\gamma(a^*b - ab)\sin \theta,$$

provided this operator is bounded from below, which occurs for $-\pi/2 \leq \theta < \pi/2$. The eigenvalues of $\Re(e^{-i\theta}H)$ are readily determined by the EMM [9], and they are readily found to be

$$E_\theta(1 + n + m), \quad n, m \geq 0,$$

where

$$E_\theta = \sqrt{1 - \gamma^2 + \cos(2\theta) + \gamma^2\cos(2\theta)}.$$ 

Thus, the supporting line under consideration is given by

$$x \cos \theta + y \sin \theta = E_\theta.$$  \hspace{1cm} (9)$$

As it is well known, the equation of the boundary of $W(H)$ is found eliminating $\theta$ between (9) and

$$-x \sin \theta + y \cos \theta = E'_\theta,$$

being readily obtained as the branch of hyperbola

$$y = \pm \gamma \sqrt{x^2 - 1}, \quad x \geq 1.$$ 

\[\blacksquare\]

Sectorial operators are defined by the property that their numerical range is the subset of a sector

$$S_{w,\theta} = \{z \in \mathbb{C} : |\arg(z-w)| \leq \theta\}$$ 

with $w \in \mathbb{R}$ and $0 \leq \theta < \pi/2$, called, respectively, the vertex and semi-angle of $H$. An operator is said to be accretive if the vertex can be chosen at the origin, i.e., $W(H) \subset S_{0,\pi/2}$. An operator $H$ is $m$-accretive if its numerical range is contained in the right closed half-plane and the resolvent bound for any $\lambda$ with $\Re \lambda < 0$, holds:

$$\forall \lambda \in \mathbb{C}, \quad \Re \lambda < 0, \quad \|(H - \lambda)^{-1}\| \leq 1/|\Re \lambda|.$$

Any $m$-accretive operator is closed and densely defined ([3],p.251)
Proposition 4.2 The operator $H$ is $m$-accretive.

Proof. We show that for any $z \in \mathbb{C}$, with $\Re z < 0$, the resolvent bound holds. We have

$$\text{dist}(z, W(H)) \leq |\langle H\psi, \psi \rangle - z| = |\langle (H-z)\psi, \psi \rangle| \leq \| (H-z)\psi \|.$$

As $\text{dist}(z, W(H)) \geq |\Re z|$, the result follows, having in mind Proposition 4.1.

Any $m$-accretive operator is closed and densely defined. In fact, by Proposition 5.2.1 in [3] p. 246 if $H$ is not closed, then $\sigma(H) = \mathbb{C}$. A closed operator $H$ in $\mathcal{H}$ has a compact resolvent if $\rho(H) \neq \emptyset$ and $(H-\lambda)^{-1}$, for some $\lambda \in \rho(H)$, is a compact operator.

It is known that [14, Theorem IX, 2.3], if $H$ has a compact resolvent, then $\sigma(H) = \sigma_p(H)$. The operator $H$ has a compact resolvent if $\rho(H)$ lies on the positive real axis with compact resolvent and, moreover, $V$ is relatively bounded with respect to $\Re(H)$ with relative bound smaller than 1. Then, $\Re(H) + \lambda V$ has a compact resolvent [14, Theorem 5.4.1].

4.1 Eigenvectors and eigenvalues of $H$

To obtain the eigenvalues of $H$, we firstly consider the selfadjoint operator in $L^2(\mathbb{R}^2)$,

$$H_0 = -\frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (1 + \gamma^2)(x^2 + y^2), \quad \gamma < 1. \quad (10)$$

We know a priori that the eigenfunctions of $H_0$ (after normalization) form an orthonormal family in $L^2(\mathbb{R}^2)$ and the corresponding eigenvalues are real. The eigenvalues of $H_0$ are easily determined as

$$E_{mn} = (1 + m + n)\sqrt{1 + \gamma^2}, \quad m, n \geq 0,$$

and the associated eigenfunctions $\Phi_{mn}$ are

$$\Phi_{m,n} = KH_m((1 + \gamma^2)^{1/4}x)e^{-\sqrt{1+\gamma^2}x^2} \times H_n((1 + \gamma^2)^{1/4}y)e^{-\sqrt{1+\gamma^2}y^2}, \quad (11)$$

where $K = (2(1 + \gamma^2)/\pi)^{1/2}$ and $H_n(t)$ is the $n$th Hermite polynomial in $t$. That is, $\Phi_{mn}$ is factorized as follows

$$\Phi_{mn}(x, y) = \Phi_m(x)\Phi_n(y),$$

where $\Phi_n(t)$ are the usual eigenfunctions of the famous harmonic oscillator [3]. For the sake of completeness, we show how to obtain $\Phi_{mn}$. Indeed, let us consider the differential operators

$$g = \frac{1}{2(1 + \gamma^2)^{1/4}} \frac{\partial}{\partial x} + (1 + \gamma^2)^{1/4}x, \quad h = \frac{1}{2(1 + \gamma^2)^{1/4}} \frac{\partial}{\partial y} + (1 + \gamma^2)^{1/4}y,$$
\[ g^* = -\frac{1}{2(1 + \gamma^2)^{1/4}} \frac{\partial}{\partial x} + (1 + \gamma^2)^{1/4} x, \quad h^* = -\frac{1}{2(1 + \gamma^2)^{1/4}} \frac{\partial}{\partial y} + (1 + \gamma^2)^{1/4} y, \]

which satisfy the Weil-Heisenberg commutation rules,

\[ [g, g^*] = [h, h^*] = 1, [g, h] = [g^*, h^*] = [g^*, h] = [g, h^*] = 0. \]

We easily find that

\[ H_0 = \sqrt{1 + \gamma^2} (g^* g + h^* h + 1). \]

The groundstate eigenfunction of \( H_0 \), which is the vacuum of the operators \( g, h \), is

\[ \Phi_0(x, y) = \kappa_0 e^{-\sqrt{1 + \gamma^2}(x^2 + y^2)}, \]

being the remaining eigenfunctions,

\[ \Phi_{mn}(x, y) = \kappa_{mn} g^m h^n \Phi_0(x, y), \]

where \( \kappa_{m,n} \) are normalization factors.

The eigenfunctions \( \Phi_{mn}(x, y) \) constitute a complete set in \( L^2(\mathbb{R}^2) \), as can be easily verified by adapting the standard proof of completeness of Hermite functions. The functions are orthogonal, and the system forms a basis of \( L^2(\mathbb{R}^2) \).

Details can be found, for instance, in [9, 10].

To find the eigenvalues of \( H \) we notice that \( H \) is formally similar to \( H_0 \)

\[ H_0 = e^{-2\gamma xy} H e^{2\gamma xy}. \]  

(12)

The word “formally” refers to the fact that the operator \( P = e^{2\gamma xy} \) is unbounded. Nevertheless, the similarity relation is well defined on the eigenfunctions of \( H_0 \).

For \( \Phi(x, y) \) an arbitrary differentiable function in the domain of \( H_0 \), we have

\[
-\frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \gamma \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + (x^2 + y^2) \right) \Phi(x, y) \]

\[ = e^{2\gamma xy} \left( -\frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (1 + \gamma^2)(x^2 + y^2) \right) \Phi(x, y), \]

that is

\[ H e^{2\gamma xy} \Phi = e^{2\gamma xy} H_0 \Phi, \]

and the operator equality

\[ H e^{2\gamma xy} = e^{2\gamma xy} H_0 \]

holds.

Since

\[ H_0 \Phi_{mn} = E_{mn} \Phi_{mn}, \]

we easily get

\[ H e^{2\gamma xy} \Phi_{mn} = E_{mn} e^{2\gamma xy} \Phi_{mn}. \]
The eigenfunctions of $H$ are expressed as
\[ \Psi_{mn} = e^{2\gamma xy} \Phi_{mn}, \]
and the eigenvalues of $H$ are those of $H_0$. Thus, the eigenvalues of $H$ are real positive and $\Psi_{mn} \in \mathcal{D}(H)$, as it should be.

4.2 Eigenfunctions and eigenvalues of $H^*$

Following the procedure in Subsection 4.1, it can be shown that, formally, we may write,
\[ H_0 = e^{2\gamma xy} H^* e^{-2\gamma xy}, \]
implying that eigenfunctions and associated eigenvalues of $H^*$ are given respectively by
\[ \tilde{\Psi}_{m,n} = e^{-2\gamma xy} \Phi_{m,n} \in \mathcal{D}(H^*), \quad E_{m,n} = (m + n + 1)\sqrt{1 + \gamma^2}, \quad m, n \geq 0. \]

4.3 Biorthogonality of the eigenfunctions

It is straightforward to see that the set \{\(\Psi_{m,n} : m, n \geq 0\}\) is not constituted by orthogonal functions. However, the vector system \{\(\tilde{\Psi}_{m,n} : m, n \geq 0\}\), formed by the eigenfunctions of $H^*$, is biorthogonal to the eigensystem of $H$, \{\(\Psi_{m,n} : m, n \geq 0\}\}. Indeed, we have
\[ \langle \Psi_{m,n}, \tilde{\Psi}_{p,q} \rangle = \langle e^{2\gamma xy} \Phi_{mn}, e^{-2\gamma xy} \Phi_{pq} \rangle = \langle \Phi_{mn}, \Phi_{pq} \rangle = m! n! \delta_{mp} \delta_{nq}, \]
where $\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$, represents the Kronecker symbol.

4.4 Completeness of the eigenfunctions

The completeness of both eigensystems of $H$ and $H^*$ can be shown to hold. The operator is $m$-accretive as its numerical range lies a hyperbolical region limited by the branch of hyperbola $y = \sqrt{x^2 - 1}$, $x \geq 1$. Moreover, it can be easily seen that the imaginary part of the resolvent of $-iH$ at $\delta < 0$, is non-negative Hermitian
\[ \frac{1}{2i} \left( (-iH - \delta)^{-1} - (iH^* - \delta)^{-1} \right) \geq 0. \]
As the resolvent is a trace class function, by the completeness theorem [11, Theorem VII.8.1], we may conclude that the eigenfunctions of $H$ form a complete system. Analogous arguments are valid for the eigensystem of $H^*$. We observe that completeness does not imply that any \(\psi \in L^2(\mathbb{R}^2)\) has a unique expansion
\[ \psi = \sum_{m,n=0}^{\infty} c_{mn} \Psi_{mn}, \]
a fundamental issue in quantum mechanics.
4.5 Asymptotic behavior of the eigenfunctions

We wish to discuss the asymptotic behavior of the Hamiltonian eigenfunctions. To this end, it is convenient to introduce the Planck constant $\hbar$, explicitly. We also change the notation slightly. We replace the notation $x, y$, used for the particle coordinates, by $x_1, x_2$, and we denote the respective momenta by

$$p_1 = -i\hbar \frac{\partial}{\partial x_1}, \quad p_2 = -i\hbar \frac{\partial}{\partial x_2}.$$ 

For simplicity, we also consider $\gamma = 1$, so that (8) becomes

$$H = \frac{1}{4}(p_1^2 + p_2^2) + x_1^2 + x_2^2 + i(x_1 p_2 + x_2 p_1) - 1.$$ 

Next, we introduce the change of variables, which constitutes a canonical transformation,

$$X = \frac{1}{2}(x_1 + x_2), \quad x = x_1 - x_2, \quad P = p_1 + p_2, \quad p = \frac{1}{2}(p_1 - p_2).$$

In terms of the new variables, the Hamiltonian is the sum of two summands, each one involving only one type of variable, and becomes

$$H = \frac{1}{8}P^2 + 2X^2 + iXP + \frac{1}{2}p^2 + \frac{1}{2}x^2 + ixp - 1.$$ 

We observe that it is equivalent to consider $m, n \to \infty$ or $\hbar \to 0$. We firstly concentrate on the summand

$$H_X = \frac{1}{8}P^2 + 2X^2 + iXP.$$ 

In order to characterize the asymptotic behavior of the eigenfunctions we use the well known Wentzel-Kramers-Brillouin [WKB] approximation [9, Chapter III] in leading order. Indeed, we express the eigenfunction of the energy operator as

$$\Psi(X) = e^{iS(X)/\hbar}.$$ 

The function $S(X)$, in leading order, is determined by the Jacobi equation

$$\frac{1}{8} \left( \frac{dS}{dX} \right)^2 + 2X^2 + iX \frac{dS}{dX} = E,$$

where $E$ denotes the associated eigenvalue of the energy operator. Thus, we readily obtain,

$$\frac{dS}{dX} = 2\sqrt{2E - 8X^2} - 4iX,$$
and
\[ S = X\sqrt{2E - 8X^2} + \frac{E}{\sqrt{2}} \arctan \frac{2X}{\sqrt{E - 4X^2}} - 2iX^2. \]
Similarly, the eigenfunction of \( H^* \) is
\[ \tilde{\Psi}(X) = e^{i\tilde{S}(X)/\hbar}, \]
with
\[ \tilde{S} = X\sqrt{2E - 8X^2} + \frac{E}{\sqrt{2}} \arctan \frac{2X}{\sqrt{E - 4X^2}} + 2iX^2. \]
Therefore, the integral
\[ \int_{-\sqrt{E/2}}^{\sqrt{E/2}} \overline{\Psi} \Psi \, dX \]
approaches \(+\infty\) as \( \hbar \to 0 \), while the integral
\[ \int_{-\sqrt{E/2}}^{\sqrt{E/2}} \overline{\tilde{\Psi}} \tilde{\Psi} \, dX \]
approaches 0 as \( \hbar \to 0 \).

On the other hand, the integral
\[ \int_{-\sqrt{E/2}}^{\sqrt{E/2}} \overline{\Psi} \tilde{\Psi} \, dX \]
remains finite as \( \hbar \to 0 \).

The summand
\[ H_x = \frac{1}{2} p^2 + \frac{1}{2} x^2 + ixp \]
may be similarly treated.

As a consequence, we get
\[ \lim_{m,n \to \infty} \| \Psi_{mn} \| = \infty. \]
Thus, the system of eigenvectors of \( \{ \Psi_{mn} \} \) does not form a Riesz basis for \( L^2(\mathbb{R}^2) \).
Analogous conclusion holds for \( \{ \tilde{\Psi}_{mn} \} \).

### 4.6 Metric operator

The existence of a positive definite \( Q \) satisfying (2) is equivalent to the fact that the resolvent of \( H \) satisfies
\[ Q(H - z_0)^{-1} = (H^* - z_0)^{-1}Q \]
for \( z_0 \in \rho(H) \cap \rho(H^*) \cap \mathbb{R}^2 \).

As a consequence of the reality of the discrete spectrum of \( H \) and of the completeness of the corresponding eigenfunctions, it can be shown, using the procedure in [19], that there exists a bounded metric for \( H \). The proof relies on the fact that the existence of bounded metric for an unbounded \( H \) can be transferred to the same problem for its bounded resolvent.

Nevertheless, there does not exist a non-singular metric operator ensuring quasi-Hermiticity, because the eigenfunctions of \( H \) do not form a Riesz basis. As a consequence, in the similarity of \( H \) to a self-adjoint operator, the basicity properties of these operators may be very different (see [1] and [3]). However, despite these negative features, in the subspace \( S = \text{span}\{\Psi_{m,n}\} \), a new inner product may be meaningfully defined with the help of a metric operator (non-singular in \( S \)), with respect to which the restriction of \( H \) to \( S \), say \( h \), should represent \( H \).

### 4.7 Physical Hilbert space

In order to specify the physical Hilbert space associated with the Hamiltonian (7), we chose its domain to be the function space

\[
\text{Dom}(H) = \{\Psi : e^{-2\gamma xy}\Psi \in W^{2,2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (x^4 + y^4)dxdy)\},
\]

endowed with the inner product,

\[
\langle \Xi_1, \Xi_2 \rangle = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-4\gamma xy} \Xi_1(x, y) \Xi_2(x, y), \quad \Xi_1, \Xi_2 \in \text{Dom}(H),
\]

which involves the weight function \( e^{-4\gamma xy} \). Since \( H_0 \) in (10) is a closed operator on the domain of the harmonic oscillator

\[
\text{Dom}(H_0) = W^{2,2}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, (x^4 + y^4)dxdy),
\]

and

\[
He^{2\gamma xy}\Phi = e^{2\gamma xy}H_0\Phi, \quad \forall \Phi \in \text{Dom}(H_0),
\]

while

\[
e^{-2\gamma xy}H\Psi = H_0e^{-2\gamma xy}\Psi, \quad \forall \Psi \in \text{Dom}(H),
\]

it follows that \( H \) is a closed operator on \( \text{Dom}(H) \).

Let

\[
S = \text{span}\{\Psi_{m,n}\}; \quad \tilde{S} = \text{span}\{\tilde{\Psi}_{m,n}\};
\]

and \( D = \text{span}\{\Phi_{m,n}\} \).

In terms of the linear operators \( e^{2\gamma xy} : D \to S \) and \( e^{-2\gamma xy} : D \to \tilde{S} \) we may write

\[
\Psi_{m,n} = e^{2\gamma xy}\Phi_{m,n}, \quad \tilde{\Psi}_{m,n} = e^{-2\gamma xy}\Phi_{m,n}.
\]
From (13) it follows that

\[ \mathcal{S} = e^S(\mathcal{D}), \quad \tilde{\mathcal{S}} = e^{-S}(\mathcal{D}). \]

Since

\[ \langle e^{-S}\Psi_{mn}, e^{-S}\Psi_{pq} \rangle = \delta_{mp}\delta_{nq}, \quad S = 2\gamma xy, \]

it follows that the eigenfunctions \( \Psi_{mn} \) are orthogonal for this inner product,

\[ \langle \Psi_{mn}, \Psi_{pq} \rangle = \delta_{mp}\delta_{nq}. \]

Next we show that the linear space \( \mathcal{S} \), equipped with the inner product \( \langle \cdot, \cdot \rangle \), allows the probabilistic interpretation of quantum mechanics. The symmetry is obviously satisfied. It may be pointed out that, from the point of view of physics, only the action of the metric operator \( \exp(-2S) \) on \( \mathcal{S} \) is significant and, in this event, it is unambiguously defined. We verify that \( \exp(-2S) \) is a metric.

For \( \phi, \psi \in \mathcal{S} \), \( \langle \exp(-2S)\phi, \psi \rangle \) is finite, and so the operator \( \exp(-2S) \) is bounded in \( \mathcal{S} \). (Similarly, \( \exp(2S) \) is bounded in \( \tilde{\mathcal{S}} \).) Moreover,

\[ \langle \psi, \psi \rangle, \quad \psi \in \mathcal{S}, \]

is nonnegative. (We notice that \( H \) leaves \( \mathcal{S} \) invariant.)

The vectors \( \Psi_{mn} \) are orthogonal with respect to the new inner product and any \( \Psi \in \mathcal{S} \) may be expanded as a unique finite linear combination of \( \Psi_{nm} \),

\[ \Psi = \sum_{nm} c_{nm} \Psi_{nm}, \]

with

\[ \langle \Psi, \Psi_{nm} \rangle = c_{nm}. \]

In order to ensure the probabilistic interpretation of quantum mechanics, we impose the normalization \( \langle \Psi, \Psi \rangle = 1 \), so that

\[ \langle \Psi, \Psi \rangle = \sum_{nm} |c_{nm}|^2 = 1. \]

The probability amplitude, is given by \( c_{nm} \), and satisfies \( \sum_{nm} |c_{nm}|^2 = 1 \).

5 Conclusions

We have initially considered a non self-adjoint Hamiltonian \( H \) whose eigenvalues and corresponding eigenfunctions have been explicitly determined. The investigated Hamiltonian and (its adjoint) has real eigenvalues and systems of biorthogonal eigenvectors. They have infinite diagonal matrix representations in
the respective eigensystems, which are complete. Nevertheless, they do not form Riesz bases.

Viewing $H$ as the Hamiltonian of a physical model, problems arise from non Hermiticity. The original inner product defined in $\mathcal{H}$ is not adequate for the physical interpretation of the model. A new $Q$-metric, which is appropriate for that purpose, may be introduced (see Sub-Section 4.7). Following Mostafazadeh [13], one can define a subspace of the Hilbert space, and the restriction of the Hamiltonian operator to that subspace, so that it has the same spectrum and eigenfunctions as the original one. The referred subspace remains invariant under the action of $H$. Remarkably, stating that this Hermitian operator represents in a reasonable sense the non-Hermitian operator may be controversial, since relevant information on the Hamiltonian may not be captured in the mentioned subspace.

Non-Hermitian operators have typically non-trivial pseudospectra. It is known that the relation (1) holds via a bounded and boundedly invertible positive transformation if and only if (2) holds with a positive bounded and boundedly invertible metric [13]. Further, if (2) holds with a positive bounded and boundedly invertible metric, then the pseudospectrum of $H$ is trivial. The concept of pseudospectrum is of great relevance for the description of non-Hermitian operators in the context of quantum mechanics. A non trivial pseudospectrum ensures the non existence of a bounded metric.

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