Uniform spanning forest on the integer lattice with drift in one coordinate

BY GUILLERMO MARTINEZ DIBENE*
Mathematics Department
The University of British Columbia

Abstract

In this article we investigate the Uniform Spanning Forest (USF) in the nearest-neighbour integer lattice \( Z^{d+1} = Z \times Z^d \) with an assignment of conductances that makes the underlying (Network) Random Walk (NRW) drifted towards the right of the first coordinate. This assignment of conductances has exponential growth and decay; in particular, the measure of balls can be made arbitrarily close to zero or arbitrarily large. We establish upper and lower bounds for its Green’s function. We show that in dimension \( d = 1, 2 \) the USF consists of a single tree while in \( d \geq 3 \), there are infinitely many trees. We then show, by an intricate study of multiple NRWs, that in every dimension the trees are one-ended; the technique for \( d = 2 \) is completely new, while the technique for \( d \geq 3 \) is a major makeover of the technique for the proof of the same result for the graph \( Z^d \). We finally establish the probability that two or more vertices are USF-connected and study the distance between different trees.

Keywords. Uniform Spanning Tree, Uniform Spanning Forest, Biased Random Walk, Drifted Random Walk, Green’s function, Harmonic Functions, Dirichlet Functions, Liouville Property, Second Moment Method.

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*e-mail: gmomtzcanada@gmail.com
1 Introduction

All graphs are assumed directed, connected, without multiple edges or self-loops and denumerable. Given an edge $e$, the direction of $e$ is given by a starting vertex $e^-$ and a ending vertex $e^+$. All conductances are assumed strictly positive.

This work is based on the PhD thesis of the author; should the reader need more details, consult [MD20]. A graph without cycles is called a tree; the tree is spanning if it contains all vertices of the graph. Given a finite connected graph $G$, the number of subgraphs is also finite and thus, there are only finitely many spanning trees in that graph. The uniform spanning tree is, by definition, the probability measure on the set of spanning trees of $G$ satisfying $P(t) = (\# \text{ of spanning trees})^{-1}$. This can readily be generalised to the network case, $P(t) = k \cdot \text{weight}(t)$, where $k$ is a normalising constant. Any of these measures (or the underlying random object) is called “uniform spanning tree.”

Suppose now that $(G, \mu)$ is an infinite network. The first paper to introduce UST on infinite graphs was Robin Pemantle’s [Pem91]. He considered $\mathbb{Z}^d$-balls corresponding to the norm $||x||_\infty = \max_{1 \leq i \leq d} |x_i|$. He showed that the uniform spanning tree measure $\mu_n$ on the trees of $B_n$ converges weakly to some measure $\mu$. This is the uniform spanning measure on $\mathbb{Z}^d$. We will employ the notation $\text{UST}$ to denote it.

Later, Itai Benjamini, Russel Lyons, Yuval Peres and Oded Schramm, in [BLPS01] and using tools unavailable to Pemantle, extended his result from $\mathbb{Z}^d$ to general networks. They started by considering an exhaustion of the graph created by considering increasing families of finite sets of vertices and then considering the networks induced on these sets. If $(G_n)$ is any such exhaustion, they showed that the uniform spanning forests measure of $G_n$ defines a sequence of measures that converge weakly to some measure $\mu^F$. They named $\mu^F$ as “free uniform spanning forest measure.” The second way to construct the uniform spanning forest measure is by considering constraints in the boundary of $G_n$. Indeed, the considered a sequence $(V_n)$ of families of finite sets of vertices and then constructed the network $G^W_n$, which is the network induced on $V_n$ with the boundary of $V_n$ wired into a vertex. Again, they showed that the uniform spanning forest measures of $G^W_n$ converge to a measure $\mu^W$ and they called this measure the “wired uniform spanning forest measure.” Several properties of $\mu^F$ and $\mu^W$ were studied, in particular, necessary and sufficient conditions were given as to when $\mu^F = \mu^W$. We will employ $\text{FSF}$ and $\text{WSF}$ instead of $\mu^F$ and $\mu^W$, respectively.

1.1 Basic definitions

Consider any network $(G, \mu)$. For a vertex $v$, we define the conductance of $v$ by $\mu(v) = \sum_{e^- = v} \mu(e)$. The network random walk (NRW) in $(G, \mu)$ is the Markov chain with state space $V$ (the vertex set of $G$) and transition probability $p(u, v) = \mu(u)^{-1} \left( \sum_{e^- = u, e^+ = v} \mu(e) \right)$. If $(S_n)_{n \in \mathbb{Z}^+_+}$ is the NRW of $(G, \mu)$, we will write $\tau_H = \inf \{ n \in \mathbb{Z}^+_+ \mid S_n \in H \}$ and $\tau^+_H = \inf \{ n \in \mathbb{N} \mid S_n \in H \}$ where $H$ is any set of vertices. For sake of simplicity, we write $\tau_a = \tau(a)$, $\tau^+_a = \tau^+_1(a)$, $\tau_H$ the hitting time of $H$ and $\tau_a$ the visit time of $a$; if $H = K^d$, we will say exit time of $K$.

Suppose $S = (S_n)_{n \in \mathbb{Z}^+_+}$ is the NRW on $G$. Often we will use $P^x (S_n = y) = P (S_n = y \mid S_0 = x)$ and $E^x (f(S_n)) = E (f(S_n) \mid S_0 = x)$. The Green’s function “of the network $G$” is, by definition $G(x, y) = \sum_{n \in \mathbb{Z}^+_+} P^x (S_n = y) = E^x \left( \sum_{n=0}^\infty \mathbb{1} (S_n = y) \right)$, for all $(x, y) \in V^2$. Similarly, define the Green’s function restricted to leaving $V_0$ by $G_{V_0} (x, y) = \sum_{m=0}^\infty P^x \left( S_m = y, m < \tau_{V_0}^0 \right)$. 


1.2 Wilson’s algorithm

David Wilson published his algorithm in [Wil96]. (This algorithm is for finite networks.)

(WA) Order the vertices and set \( t_0 \) to be the tree consisting solely of the first vertex. Having defined \( t_{k-1} \), start an independent \( \text{NRW} \) from the first vertex not in \( t_{k-1} \) and run this random walk until it hits \( t_{k-1} \), consider the \( \text{LE} \) of this path and call \( t_k \) the tree obtained as the union of this \( \text{LE} \)-path with \( t_{k-1} \).

When all vertices have been searched, return \( \mathcal{T} \) the final tree constructed.

It is known that \( \mathcal{T} \) follows the \( \text{UST} \) law of the given (finite) network. A typical way of using Wilson’s algorithm is via a more intricate structure; while Wilson introduced this structure, a more detailed exposition can be found in [Bar16]. We refer to this version of Wilson’s algorithm as “Wilson’s algorithm with stacks.” See also [LPT16], Ch. 4. With the definitions of these references, we have the following result.

\( \text{Proposition (1.1)} \) Assume \( (G, \mu) \) is a finite network. Fix a root \( r \in V \). With probability one, there are finitely many cycles that can ever be popped. Any two ordering of the searches of the vertices in \( V \setminus \{r\} \) will pop all these cycles and the popping procedure will leave the same visible graph for the two orderings. This visible graph sitting on top of the stacks is a spanning tree and the distribution of this random tree is that of the \( \text{UST} \) of the given network.

Observe that Wilson’s algorithm run with stacks on a finite graph produces always the same tree provided the root is fixed in advance.
1.3 Definition of $\Gamma_d(\lambda)$ and summary of this paper

We mention now the usual way to construct WSF on an infinite network $(G, \mu)$. Consider an exhaustion $V_q$ of the vertex set and denote by $(G_q, \mu)$ the network induced on $V_q$ with its boundary $\partial V_q$ wired into a vertex. Run Wilson’s algorithm with stacks taking $\partial V_q$ as root. This produces a random tree $\mathcal{T}_q$. Then, $\mathcal{T}_q \xrightarrow{q \to \infty} \mathcal{T}$, where $\mathcal{T}$ has distribution WSF, see [LP16], sect. 10.1. There exists an alternative more dynamic (it provides a.s. convergence as opposed to weakly) method to sample the WSF-distributed random object, which was first introduced in [BLPS01].

Suppose $G$ is any network. Denote by $V$ the set of vertices of $G$ and by $E$ that of edges. Suppose $\xi : \mathbb{N} \to V$ is a bijection (an “ordering” of the vertex set). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which is possible to define an independent family $(S^v)_{v \in V}$ of NRWs of this network, such that $S^v$ is a random walk started at $v$. Define inductively the following random spanning subgraphs of $G$:

**Start.** $L^\xi_1 = \mathfrak{S}^\xi_1 = \text{LE} \left( S^\xi_m \right)_{m \in \mathbb{Z}_+}$.

**Inductive step.** Having defined $L^\xi_k$ and $\mathfrak{S}^\xi_k$, define $L^\xi_{k+1} = \text{LE} \left( S^\xi_m \right)_{m=0,\ldots,\tau_k}$ and $\mathfrak{S}^\xi_{k+1} = \mathfrak{S}^\xi_k \cup L^\xi_{k+1}$, where $\tau_k = \tau^\xi_{S^\xi_m}$, that is, $\tau_k$ is the hitting time of the forest currently present $\mathfrak{S}^\xi_k$ by the random walk $S^\xi_{k+1}$.

Consider finally the random spanning subgraph of $G : \mathfrak{S}^\xi = \bigcup_{k \in \mathbb{N}} \mathfrak{S}^\xi_k$. What Wilson’s algorithm rooted at infinity [BLPS01] Theorem 5.1] states is that $\mathfrak{S}^\xi \sim \text{WSF}_G$, in other words, the distribution of $\mathfrak{S}^\xi$ is that of the wired spanning forest of the network $G$, and this happens independently of the choice of ordering $\xi$.

1.3 Definition of $\Gamma_d(\lambda)$ and summary of this paper

In this paper we will study $\Gamma_d(\lambda) = (\mathbb{Z}^{d+1}, \mu)$, where $\mu$ is given by

$$\mu((n,x),(n',x')) = e^{\lambda \max(n,n')} \text{ for all } (n,x) \sim (n',x').$$

It can easily be seen that this assignment of conductances makes the underlying NRW to have a uniform drift to the right of the first axis. We will then investigate the main basic properties of the forest. In the remainder of this section we provide an overview of the results of this paper.

In sect. 2 we will prove the following theorem

**Theorem.** (Theorem 2.22) There exist four constants $c_i (i = 1, \ldots, 4)$ such that for all vertices $z = (n,x) \neq 0$ of the network $\Gamma_d(\lambda)$,

$$G(0,z) \leq c_1 \begin{cases} e^{-c_2 ||z||} & \text{if } ||z|| > n, \\ e^{-c_2 \frac{||z||^2}{n^2}} ||z||^{-\frac{d}{2}} & \text{if } ||z|| \leq n, \end{cases}$$

and

$$G(0,z) \geq c_3 \begin{cases} e^{-c_4 ||z||} & \text{if } ||z|| > n, \\ e^{-c_4 \frac{||z||^2}{n^2}} ||z||^{-\frac{d}{2}} & \text{if } ||z|| \leq n, \end{cases}$$

To prove this bounds we will employ a decomposition of the Green’s function using the continuous-time NRW and then use several “well-known” estimates on the latter.

In sect. 3 we will establish the following.
1.3 Definition of $\Gamma_d(\lambda)$ and summary of this paper

Theorem. (Theorem (3.8)) On $\Gamma_d(\lambda)$, we have $\text{WSF} = \text{FSF}$.

This is done using the Liouville property and a coupling argument. This theorem then shows that USF is well defined on $\Gamma_d(\lambda)$. Next, using Fourier inversion theorem together with Plancherel’s theorem, we will prove the following result, which we call “bubble condition.”

Theorem. (Theorem (3.10)) The Green’s function $z \mapsto G(0,z)$ of $\Gamma_d(\lambda)$ belongs to $L^2$ if and only if $d \geq 3$.

Shortly after, as a corollary of the bubble condition, we will also show that USF consists of a single tree when $d = 1, 2$, and it consists of infinitely many infinite trees for $d \geq 3$.

Theorem. (Theorem (3.9)) In $d = 1, 2$, WSF of $\Gamma_d(\lambda)$ is a.s. one tree; in $d \geq 3$, there are a.s. infinitely many trees.

In sect. 4 we estimate the frequency at which two independent NRW of $\Gamma_d(\lambda)$ will cross each other. We will prove the following result

Theorem. (Theorems (4.1) and (4.2), and Corollary (4.15.2)) Let $S$ and $S'$ be two independent network random walks in $\Gamma_d(\lambda)$ started at $0$.

(a) If $d = 1$, then a.s. there exists infinitely many $n$ such that $S_n = S'_n = 0$.

(b) If $d = 2$, then a.s. there exist infinitely many pairs $(n, n')$ such that $S_n = S'_{n'}$.

(c) If $d \geq 3$, there exists a positive probability that for no pair $(n, n')$, $S_n = S'_{n'}$.

In fact, in $d = 2$, we will estimate the probability that they will cross is roughly $\log n$ if they walk $n$ steps (a precise formulation of this is Theorem (4.15)). The proofs of dimensions $d \neq 2$ are relatively easy. The proof of dimension $d = 2$ is rather involved. We explain the gist of the idea now. We will consider a paraboloid given by the vertices $z = (n, x) \in \mathbb{Z}_+ \times \mathbb{Z}^d$ such that $\|x\| \leq n^2$ and inside each of these, we will consider cylinders (4.7) of appropriate dimensions. We will show that for certain regions (4.10), the NRW started in the boundary of these regions will not go back to the testing cylinder (except with a small probability that can be controlled) and therefore, the Green’s function virtually does not change value when we restrict it to leave these regions. We then employ a second-moment method to estimate the aforementioned probability. The calculations are quite tight and require a careful set up.

In sect. 5 we prove

Theorem. (Theorems (5.1) and (5.15) and (5.16)) In $\Gamma_d(\lambda)$, a.s. every tree in USF has one end.

The proofs of dimensions $d = 1$, $d = 2$ and $d \geq 3$ need to be done separately. In $d = 1$ we use planar duality and the rather easy-to-establish fact that $\Gamma_d(\lambda)$ is essentially its own dual; the idea is a make over of the proof of the same result for the graph $\mathbb{Z}^2$. In $d \geq 3$, we followed closely the proof of one-endedness for the graphs $\mathbb{Z}^d$ given in [LP16]. We made adaptation according to our conveniences but the underlying idea is that should a tree have two ends, there will be strong conductivity in the boundary of boxes. The proof of case $d = 2$ is perhaps the most innovative contribution of this paper and was introduced from scratch. The first property we establish is that Wilson’s algorithm rooted at infinity has some stochastic-stability (the law remains unaltered) if we fix the first steps and reorder future vertices (a stronger result for finite-graphs is proposition (1.1)). This will show that probabilities of future events depending on the past do not actually depend of the past. The next step is to show that it suffices to prove that the component of zero is one-ended.
1.3 Definition of $\Gamma_d(\lambda)$ and summary of this paper

To this aim, we employ Wilson’s algorithm rooted at infinity and take the first step of the algorithm to be the origin. Call $L_0$ this part of the forest. If the component at zero had at least two ends, then it would cross the boundary of each cylinder (centred at the origin) in at least two vertices. The aforementioned invariance of the law from reordering future searches having fixed the past will allow us to fix $L_0$ and then prove that, in a given a large cylinder $C = \{|n| \leq r, \|x\| \leq r\}$, most vertices on $\partial C$ have infinitesimal probability to connect to zero, this will reduce the work to study a part of the left base of $C$, namely $C_{r,0} = \{n = -r, \|x\| \leq r^\frac{d}{2}\}$.

We will then take a sparse subset here ($C'_{r,0}$) and construct the branches from its vertices. The subset $C'_{r,0}$ needs to be sparse enough so that the probability that the NRW started from its vertices create a second end from the origin is small. If now $z$ is any vertex in $C_{r,0} \setminus C'_{r,0}$, then we will show that the NRW started at $z$ has an overwhelming probability to hit the branch created by some $z' \in C'_{r,0}$ and since this branch does not create a second end from the origin, $z$ will also not create a second end. The proof will conclude by gluing together all the estimates. Admittedly, the proof is elaborate and to ease the reading, we have provided well-detailed arguments.

In sect. 6 we will show

**THEOREM.** (Theorems (6.2) and (6.5)) There exists a metric $\eta(z) = \max\{|n|^\frac{2}{d}, \|x\|\}$ such

$$P\left( z \text{ and } z' \text{ are in the same USF-component} \right) \approx \eta(z - z')^{-(d-2)}.$$

To prove this, we follow similar techniques as those established in [BKPS04]. In a nutshell, we will need to investigate bounds of the convolution $\sum_{z'' \in \mathbb{Z}^{d+1}} G(z, z'')G(z', z'')$.

In sect. 7 we will establish the following

**THEOREM.** (Theorem (7.13)) Let $D(z, z')$ be the minimal number of $\mathbb{Z}^{d+1}$-edges outside the USF-forest that connects the tree at $z$ and that at $z'$. Almost surely, $\max_{z, z'} D(z, z') = \left\lfloor \frac{d - 2}{4} \right\rfloor$.

The proof of this is done by adaptation of most of the methods of [BKPS04] to the network $\Gamma_d(\lambda)$.

### 1.3.1 Biased random walk

We make special mention of an article appearing in the arXiv.org in 2018. In [SSS+18] Zhan Shi, Vladas Sidoravicius, He Song, Longmin Wang, Kainan Xiang investigated properties of USF for a random walk they called “biased random walk.” They studied the network $\mathbb{Z}^d$ with assignment of conductances $c_\lambda(e) = \lambda^{-|e|}$, where $|e|$ is the graph-distance from the origin to the edge $e$ and $\lambda \in (0, 1)$ is a fixed parameter. The network considered in this paper and $\Gamma_d(\lambda)$ both have conductances which are far from uniform. However, the two assignments of conductes give rise to very different random walks, and very different properties of the uniform spanning forest.

In [SSS+18] the NRW has a drift away from the origin, while in our case it has a uniform drift towards the right. Our assignments of conductances make the network transitive (indeed, translation-invariant), while theirs is not. They exploited this fact by noticing that the axes play a special rôle. This also leads to quite different properties of the uniform spanning forest. For example, the USF in [SSS+18] has $2^d$ trees if $d = 2, 3$, while in our case this number is one. Their methods and ours are also different: they used “spectral radius” and “speed” of the random walk, while we will not mention these. Another display of the difference in nature of the results in their paper when compared to this paper is when they count the number of intersections of random walk paths. They estimated these intersections by using a local limit theorem. Typically, local limit
theorems have error terms of a polynomial decay in \( ||z|| \). We will show that the Green’s function in \( \Gamma_d(\lambda) \) has exponential decay in some directions. Thus, a local limit theorem did not prove useful for us. Finally, another contrast in the two assignment of conductances. The fact that \( c_\lambda(e) \) is bounded from below allowed them to prove one-endedness using the isoperimetric condition \([\text{LP16} \text{ Theorem 10.43}] \). Our conductances decay exponentially, and so we cannot use the isoperimetric condition for one-endedness (or any other known technique so far); a new technique was derived from scratch to prove it.

2 Green’s function bounds

2.1 Basic statistics of \( \Gamma_d(\lambda) \)

We will often write \( z \in \mathbb{Z}^{d+1} = \mathbb{Z} \times \mathbb{Z}^d \) as \( z = (n, x_1, \ldots, x_d) = (n, x) \) and we will refer to \( n \) as the “drifted coordinate” or the “zeroth coordinate.” A unit vector in \( \mathbb{Z}^{d+1} \) has one coordinate equal to 1 or -1 and all the rest equal to zero; for simplicity, \( u \) and \( u_i \) will denote unit vectors in \( \mathbb{Z}^{d+1} \) and \( u_i \) is such that its \( i \)th coordinate is non zero \((0 \leq i \leq d) \). The functions \( \rho_t : \mathbb{Z}^{d+1} \to \mathbb{Z} \), for \( i = 0, 1, \ldots, d \), are the projections and are defined in the obvious way. We will call \( \mathbb{Z}u_i \) the “\( i \)th factor” as well as “\( i \)th axis” \((i = 0, \ldots, d) \). Let \( \lambda > 0 \), fixed throughout the rest of this document. Recall \( \Gamma_d(\lambda) \) is the graph \( \mathbb{Z}^{d+1} \) with conductances \([2.1] \).

From these conductances, one can get the transition kernel for network random walk. The probability that the next step should be \( u \in \mathbb{Z}^{d+1} \), a unit vector with our conventions, is given by

\[
(2.1) \quad p(u) = \begin{cases} 
(2d + 1 + e^\lambda)^{-1} & \text{if } u = (-1, 0) \text{ or } u = u_i, i = 1, \ldots, d, \\
e^\lambda(2d + 1 + e^\lambda)^{-1} & \text{if } u = (1, 0).
\end{cases}
\]

The \( \frac{1}{2} \)-lazy version of the above transition density is \( p^L(0) = \frac{1}{2} \), \( p^L(u) = \frac{1}{2} p(u) \) for unit \( u \).

Denote by \( \zeta \cdot \varepsilon_i \) the measure on \( \mathbb{Z}^d \) such that it has total mass equal to \( \zeta \) at the vertex \( t \). Notice that if \( Y = (Y^{(0)}, \ldots, Y^{(d)}) \) is a random element of \( \mathbb{Z}^{d+1} \) with distribution \((2.1)\), then \((i = 1, \ldots, d) \)

\[
(2.2) \quad Y^{(0)} \sim (2d + 1 + e^\lambda)^{-1} \left( \varepsilon_{-1} + 2d \cdot \varepsilon_0 + e^\lambda \cdot \varepsilon_1 \right), \quad Y^{(i)} \sim (2d + 1 + e^\lambda)^{-1} \left( \varepsilon_{-1} + (2d - 1 + e^\lambda) \cdot \varepsilon_0 + \varepsilon_1 \right),
\]

which immediately allows one to conclude that

\[
(2.3) \quad \mathbb{E}(Y) = (a, 0) = \left( \frac{\varepsilon_0 \cdot \varepsilon_1}{2d + 1 + e^\lambda}, 0 \right), \quad \text{Cov}(Y) = \begin{bmatrix} \sigma_0^2 & 0 \\ 0 & \sigma^2 I_d \end{bmatrix} = \begin{bmatrix} \frac{\varepsilon_0 \cdot \varepsilon_1}{2d + 1 + e^\lambda} & 0 \\ 0 & \sigma^2 I_d \end{bmatrix},
\]

where \( I_d \) is the identity on \( \mathbb{R}^d \). From the Central Limit Theorem, it follows immediately that if \( (Y_n) \) is a family of independent random elements following \((2.1)\),

\[
\frac{1}{\sqrt{n}} \sum_{j=1}^n (Y_j - (a, 0)) \text{ weakly }_{n \to \infty} \text{ Norm} \left( 0; \begin{bmatrix} \sigma_0^2 \\ 0 \end{bmatrix} \right).
\]

The Fourier transform of \( Y \) is

\[
(2.4) \quad \varphi_Y(h) = \mathbb{E} \left( e^{i(h \cdot Y)} \right) = e^{\frac{h_0^2}{2d + 1 + e^\lambda}} + \frac{2}{2d + 1 + e^\lambda} \sum_{j=1}^{d} \cos(h_j),
\]

where \( h = (h_0, \ldots, h_d) \in [-\pi, \pi]^{d+1} \). Observe also \( \varphi_Y(h) = 1 \) if and only if \( 1 = e^{ih_0} = \cos h_j \) for \( j = 0, \ldots, d \), and this is the same as saying that \( h_0 = \ldots = h_d = 0 \).
2.2 Continuous time network random walk

Recall the definition of the continuous time network random walk. In $\Gamma_d(\lambda)$, we can write

$$\bar{X}_t = \left(\bar{B}_t, \bar{X}_t\right), \quad (t \geq 0).$$

The following properties come from well-known facts regarding Poisson processes or are otherwise given a reference. Set $p = \mathbb{P}(Y = u) = (2d + 1 + e^{2\lambda})^{-1}$.

(a) $\bar{B}$ and $\bar{X}$ are independent continuous-time Markov processes.

(b) $\bar{X}_t = X_{M_t}$, where $X$ is a standard random walk in $\mathbb{Z}^d$ and $M = (M_t)_{t \geq 0}$ is a Poisson process in $\mathbb{Z}_+$ with constant intensity equal to $2dp = \frac{2d}{2d+1+e^{2\lambda}}$.

(c) $\bar{B}_t = B_{L_t}$, where $B$ is a biased random walk in $\mathbb{Z}$ with jumps to $\pm 1$, $L = (L_t)_{t \geq 0}$ is a Poisson process in $\mathbb{Z}_+$ with constant intensity equal to $\sigma^2 = (1 + e^{2\lambda})p = \frac{1+c^2}{2d+1+e^{2\lambda}}$, and the probability of $B$ having a positive jump is $\frac{e^{2\lambda}}{1+e^{2\lambda}}$.

(d) Let $q^X$ be the transition probability of $\bar{X}$, that is, $q^X(t,y) = q^X(t,y) = \mathbb{P}^0(\bar{X}_t = y)$. Then [Bar17, Theorem 5.17] provides upper bounds

$$q^X(t,y) \leq \begin{cases} \frac{2d}{e^t} & \text{if } \|y\| \leq 2dpt, \\ \frac{1}{2d} \exp\left(-2dpt - \|y\| \log\left(\frac{\|y\|}{2dpt}\right)\right) & \text{if } \|y\| \geq 2dpt; \end{cases}$$

and lower bounds, valid for $\|y\| \geq 2dpt > 0$,

$$q^X(t,y) \geq c \exp\left(-c' \|y\| \left[1 + \log\left(\frac{\|y\|}{2dpt}\right)\right]\right).$$

(e) By Theorem 6.19 bearing in mind Definition 3.28, Corollary 3.30 and Definition 5.19 of [Bar17] there exists four constants $c_i > 0$ ($i = 1, \ldots, 4$) such that for all $(t,y)$ for which $2dpt > \|y\|$,

$$q^X(t,y) \leq c_1 t^{-\frac{4}{d}} e^{-c_2 \|y\|^{\frac{2}{d}}},$$

$$q^X(t,y) \geq c_3 t^{-\frac{4}{d}} e^{-c_4 \|y\|^{\frac{2}{d}}}.\quad (2.8)$$

(f) There exists a pair of universal constants $c, c' > 0$ such that for all $r, T \geq 1$, $\mathbb{P}\left(\sup_{0 \leq s \leq T} \|\bar{X}_s\| \geq r\right) \leq ce^{-c' r^{\frac{4}{d}}}$. This follows from Theorem 4.33 of [Bar17] applied to the standard random walk (so $\beta = 2$), you will need to recall Definitions 4.14 and 4.18, and use Lemma 4.20, which holds with $\alpha = d$ and $\beta = 2$ for the standard random walk on $\mathbb{Z}^d$.

We conclude this section with the following easy remark.

**Remark (2.9)** Let $c > 0$.

(a) If $z = (n, x)$ satisfies $|n| \leq c \|x\|$, then $\|x\| \leq \|z\| \leq \sqrt{1 + c^2} \|x\|$.

(b) If $z = (n, x)$ satisfies $\|x\| \leq c|n|$, then $|n| \leq \|z\| \leq \sqrt{1 + c^2} |n|$.

(c) There exists $c' > 0$ such that for all $z \in \mathbb{Z}^{d+1}$, $e^{-c' \|z\|} \leq e^{c \|z\|^{\frac{2}{d}}}$.\quad (2.9)
2.3 Some estimates of sums

We collect together here some estimates of sums which will be used later. All proofs can be found in [MD20], sect. 2.3.

Proposition (2.10) Let L ∈ N and f : [L, ∞) → R\_+ be a monotone function such for a pair a, b > 0, we have af(j + 1) ≤ f(j) ≤ bf(j + 1) for integers j ≥ L. Let r > L + 1. Then, \[ \int f(|x|) dx \leq \sigma_d \int \frac{r}{L} f(t)^{d-1}, \] where \( \sigma_d \) is the “surface area” of the d-dimensional sphere and any implicit constant may depend solely on dimension and on the pair (a, b).

Proposition (2.11) Let a ≥ 0, b, γ > 0 and r > 2.

(a) If \( a > 0 \), \( \sum_{|x| \leq r} \|x\|^a e^{-\gamma \|x\|^b} \approx \min (\tau^b, \gamma^{-1}) \frac{\tau^a}{\gamma} \).

(b) If \( a = 0 \), \( \sum_{0 < |x| \leq r} e^{-\gamma \|x\|^b} \approx \min (\tau^b, \gamma^{-1}) \frac{\tau^a}{\gamma} \). Thus, if \( r^{-b} \leq \gamma \leq 1 \), \( \sum_{|x| \leq r} e^{-\gamma \|x\|^b} \approx \gamma^{-\frac{a}{b}} \).

(Any implicit constant may depend on a, b and dimension but not on \( \gamma \) or on \( r \).)

Proposition (2.12) Let a, b > 0. There exists a pair of constants \( c_1, c_2 > 0 \) such that for all \( t \in \mathbb{Z}^d \) and all \( m \in \mathbb{N} \),

(a) if \( ||t|| > bm \), \( e^{-a||t||} \leq \sum_{|x| \leq bm} e^{-a \left( \frac{|x|^2}{m} + ||x-t|| \right)} \leq c_1 e^{-c_2||t||} \);

(b) if \( ||t|| \leq bm \), \( e^{-a \frac{|t|^2}{m}} \leq \sum_{|x| \leq bm} e^{-a \left( \frac{|x|^2}{m} + ||x-t|| \right)} \leq c_1 e^{-c_2 \frac{|t|^2}{m}} \).

Proposition (2.13) For every \( \alpha > 1 \) and all integers \( m \geq 2 \), \( \frac{1}{\alpha^m} m^{-(\alpha-1)} \leq \sum_{k \leq m} k^{-\alpha} \leq \frac{c^{\alpha-1}}{\alpha^m} m^{-(\alpha-1)} \).

Proposition (2.14) For every \( a \in \mathbb{R} \) and \( b \geq 1 \), there exists a constant \( c = c(a, b) > 0 \) such that for all \( \gamma > 1 \), \( \sum_{1 \leq j \leq br^2} j^a e^{-\gamma j^b} \leq c \gamma^{a+1} \).

Proposition (2.15) There exists a universal constant \( c > 0 \) such that for all systems \( (a, p, q, r, t) \) with \( r > 2, a, p, q > 0 \) and \( t \in \mathbb{Z}^d \),

\[ \sum_{|x| \leq r} e^{-a \left( \frac{|x|^2}{p} + \frac{|x-t|^2}{q} \right)} \leq \begin{cases} c \min \left( \frac{r^2}{p}, \frac{|t|^2}{a} \right)^{\frac{a}{2}} e^{-\frac{a |t|^2}{q}} + \min \left( \frac{|t|^2}{d}, \frac{q}{a} \right)^{\frac{q}{2}} e^{-\frac{q |t|^2}{r}} & \text{if } ||t|| > 2r \\ c \left[ \min \left( \frac{r^2}{p}, \frac{|t|^2}{a} \right)^{\frac{a}{2}} e^{-\frac{a |t|^2}{q}} + \min \left( \frac{|t|^2}{d}, \frac{q}{a} \right)^{\frac{q}{2}} e^{-\frac{q |t|^2}{r}} \right] & \text{if } ||t|| \leq 2r. \end{cases} \]

2.4 Some properties of (network) random walk

These properties are well-known and are included for completeness of the exposition. We denote by \( S = (S_n) \) the network random walk of a network \( \Gamma \) and \( G \) will denote its Green’s function.
Proposition (2.16) Let $\Gamma$ be any network. For any two vertices $o$ and $v$ of $\Gamma$, $G(o, v) = \int_0^\infty dt \, q_t^S(o, v)$. If $\Gamma$ is transitive and $v$ is any vertex, $G(o, v) = G(o, o)P^o (\tau_v < \infty)$, where $\tau_v = \tau_v (\tilde{S}) = \inf \{ t \geq 0; \tilde{S}_t = v \}$.

Proposition (2.17) Consider a random walk $S$ on $\mathbb{Z}$ with step-lengths distributed as $q, \varepsilon_{-1} + r_\varepsilon_0 + p, \varepsilon_1$. Assume $1 > p + q > p > q > 0$. Consider the events $C_u = \{ \text{for exactly one } n, S_n = u \} = \bigcup_{n \in \mathbb{Z}_+} \bigcap_{m \neq n} \{ S_n = u, S_m \neq u \}$ and $E_u = \{ \text{for at least one } n, S_n = u \} = \bigcup_{n \in \mathbb{Z}_+} \{ S_n = u \}$. Then, for $u \geq 0$ and $v < 0$,

(a) $P^0 (C_u) = P^0 (\tau_0^+ = \infty) = p - q$,

(b) $P^0 \left( \bigcup_{u=0}^n C_u \right) \geq 1 - (1 - (p - q))^n - n \left( \frac{q}{p} \right)^n$,

(c) $P^0 (E_v) = \left( \frac{p}{q} \right)^v$.

Proof. Items (a) and (c) are well-known. For convenience of the reader, we provide details of item (b).

Consider $D_u$ the complement of $C_u$. It suffices to show $P^0 \left( \bigcup_{u=0}^n D_{un} \right) \leq (1 - (p - q))^n + n \left( \frac{q}{p} \right)^n$. With respect to $P^0$, the events $D_u (u \geq 0)$ are a.s. the events where the state $u$ is visited twice or more. Introduce $\theta_j$ defined by "the time of first return to $t'$": $\theta_j = \inf \{ k > \tau_1 | S_k = t \}$ ($t \in \mathbb{Z}_+$. It is clear that $(\tau_{un})_{u=1,...,n}$ is strictly increasing, that $\tau_t < \theta_t$ for all $t \in \mathbb{Z}_+$, and also that the complement of the event $\bigcup_{j=1}^n \{ \theta_{(j-1)n} < \tau_j \}$ is contained in the event $\bigcup_{j=1}^n \{ \tau_j < \theta_{(j-1)n} \}$. Therefore, $P^0 \left( \bigcup_{u=0}^n D_{un} \right) \leq P^0 (\theta_0 < \tau_n < ... < \tau_n < ... < \tau_{n^2} < \theta_{n^2} < \infty) + \sum_{j=1}^n P^0 (\tau_j < \theta_{(j-1)n} < \infty)$. By the strong Markov property, $P^0 (\tau_j < \theta_{(j-1)n} < \infty) = P^0 (E_{-u}) = \left( \frac{q}{p} \right)^n$. Set $\varepsilon_n = \mathcal{F}_{\tau_n} = \sigma(S_k; 0 \leq k \leq \tau_n)$ and

$\gamma_k \overset{\text{def}}{=} P^0 (\theta_0 < \tau_n < \theta_n < ... < \tau_{kn} < \theta_{kn} < \infty)$.

Observe that the strong Markov property implies

$\gamma_n = P^0 (\theta_0 < \tau_n < \theta_n < ... < \tau_{n^2} < \theta_{n^2} < \infty)
= E^0 \left( \mathbf{1}_{\{ \theta_0 < \tau_n < \infty \}} E^0 \left( \mathbf{1}_{\{ \theta_0 < ... < \tau_n < \infty \}} \big| \varepsilon_n \right) \right)
= P^0 (\theta_0 < \tau_n < \infty) \gamma_n - 1 \leq P^0 (\tau_0^+ < \infty) \gamma_{n-1}$,

which leads at once to $\gamma_n \leq (1 - (p - q))^n$ since $\gamma_1 \leq 1 - (p - q)$.

2.5 A hitting time for the continuous time network random walk

Again, we compile some properties of certain random times associated with the continuous time random walk (2.13). These properties are arguably well-known and are included for completeness of the exposition. In what follows, we use $p = (2d + 1 + e^\lambda)^{-1}$, $a = (e^\lambda - 1)p$ and $\sigma^2 = (e^\lambda + 1)p$. Introduce

$\tau_n^0 = \inf \{ t \geq 0; B_t = n \}$ ($n \in \mathbb{N}$).

Then, $\tau_n^0$ is an almost surely finite stopping time, it is independent of the process $\tilde{X}$ and by successive applications of the strong Markov property at the times $\tau_j$ ($1 \leq j \leq n - 1$), it is immediately seen that
the law of \( \tau_n^0 \) is the nth convolution power of the law of \( \tau_1^0 \). Let \( \zeta \) denote the Laplace transform of \( \tau_1^0 \), that is, \( \zeta(t) = \mathbb{E}^0( e^{t \tau_1^0} ) \). The following proposition follows from elementary results and its proof is therefore omitted, see [MD20], propositions (2.5.1) and (2.5.2).

**Proposition (2.18)**  
(a) We have \( \mathbb{E}^0 ( \tau_0^0 ) = \frac{1}{\alpha} \).
(b) Then \( \zeta(t) < \infty \) for \( t < (e^{\frac{\lambda}{\alpha}} - 1)^2 p \) and \( \zeta(t) = \infty \) for \( t > (e^{\frac{\lambda}{\alpha}} - 1)^2 p \). We have the following bounds
\[
\zeta(t) \leq \begin{cases} 
\frac{2 e^{\frac{\lambda}{\alpha} - \gamma}}{2 e^{\frac{\lambda}{\alpha} - \gamma} - \delta} & \text{for } -\infty < t \leq 0, \\
\frac{2 e^{\frac{\lambda}{\alpha} - \gamma}}{2 e^{\frac{\lambda}{\alpha} - \gamma} - \delta} & \text{for } 0 < t < (e^{\frac{\lambda}{\alpha}} - 1)^2 p.
\end{cases}
\]

**Proposition (2.19)**  
(a) For \( \alpha \leq \frac{n}{2 e^{\lambda} p} \), \( \mathbb{P}^0( \tau^0 \leq \alpha ) \leq \exp \left( -2 e^{\lambda} p + n \log \left( \frac{2 e^{\lambda + 1} p}{e^{\frac{\lambda}{\alpha}} - 1} \right) \right) \).
(b) For \( 0 < \beta < \frac{1}{2 e^{\lambda} p} \) there exists a constant \( u(\beta) > 0 \) such that for all \( n, \) \( \mathbb{P}^0( \tau^0 \leq \beta n ) \leq e^{-u(\beta)n} \).
(c) There exists two constants \( T_0, K_0 > 0 \) such that \( \mathbb{P}^0( \tau^0 \geq \alpha ) \leq e^{-T_0 (\alpha - \frac{n}{2 e^{\lambda} p})} \) valid for \( \alpha \geq (K_0 + \frac{1}{\alpha})n \).

**Proof.** The bound in (c) is a Corollary of Theorem 1 and Lemma 5, in Chapter III of [Pet75], and observe we can apply these results since the previous proposition shows the Laplace transform of \( \tau_1^0 \) to be absolutely convergent on an interval around zero. We now prove the bound for \( \mathbb{P}^0( \tau_0^0 \leq \alpha ) \). Markov’s inequality shows, for \( t < 0 \), \( \mathbb{P}^0( \tau^0 \leq \alpha ) = \mathbb{P}^0( e^{t \tau_0^0} \geq e^{t \alpha} ) \leq e^{-t \alpha} \zeta(t)^n \), where \( \zeta \) is the Laplace transform of \( \tau_1^0 \).

From the previous proposition, we know \( \zeta(t) \leq \frac{2 e^{\frac{\lambda}{\alpha} - \gamma}}{2 e^{\frac{\lambda}{\alpha} - \gamma} - \delta} \) for \( t \leq 0 \), and so \( \mathbb{P}^0( \tau^0 \leq \alpha ) \leq e^{-t \alpha + n \log \left( \frac{2 e^{\lambda + 1} p}{e^{\frac{\lambda}{\alpha}} - 1} \right)} \) for \( t < 0 \). Minimising gives the minimiser \( t = 2 e^{\frac{\lambda}{\alpha} - \gamma} \), which is valid only when \( \alpha < \frac{2 e^{\frac{\lambda}{\alpha} - \gamma}}{2 e^{\frac{\lambda}{\alpha} - \gamma} - \delta} \). When \( \alpha = \frac{n}{2 e^{\lambda} p} \), we reach the trivial bound \( \mathbb{P}^0( \tau^0 \leq \alpha ) \leq 1 \), and so it is also valid in this case. Finally, the substitution \( \alpha = \beta n \) implies \( \mathbb{P}^0( \tau^0 \leq \beta n ) \leq e^{-u(\beta)n} \), where \( u(\beta) = 2 e^{\lambda} p \beta - \log(2 e^{\lambda + 1} p \beta) \). A standard minimisation shows \( u(\beta) > u \left( \frac{2 e^{\lambda} p}{e^{\frac{\lambda}{\alpha}} - 1} \right) = 0 \) for \( 0 < \beta < \frac{1}{2 e^{\lambda} p} \).

**Proposition (2.20)**  
Recall from (2.16) the definition of \( \tau_z \) for \( z = (n, x) \in \mathbb{Z}^d \). There exists \( c_1, c_2 > 0 \) such that, for all \( x \in \mathbb{Z}^d \), \( \mathbb{P}^0( \tau_{(0,x)} < \infty ) \leq c_1 e^{-c_2 \|x\|} \).

**Proof.** Let \( \gamma = r_x \) be the smallest integer \( k \geq \|x\| \). Then \( \mathbb{P}^0( \tau_{(0,x)} < \infty ) \leq \mathbb{P}^0( \tau_{(0,x)} \leq \gamma ) + \mathbb{P}^0( \tau_{(0,x)} \leq \gamma, \tau_\gamma > c r_x ) + \mathbb{P}^0( \tau_\gamma > c r_x, \tau_{(0,x)} < \infty ) \), where \( c \geq K_0 + \frac{1}{\alpha} \) is chosen so that \( c r_x \geq (K_0 + \frac{1}{\alpha}) r_x \) for all \( x \in \mathbb{Z}^d \), \( x \neq 0 \) and \( K_0 > 0 \) from (2.19). Then, recalling \( \bar{S} = (\bar{B}, \bar{X}) \) and item 1 in (2.22)
\[
\mathbb{P}^0( \tau_{(0,x)} \leq \gamma, \tau_\gamma > c r_x ) \leq \mathbb{P}^0 \left( \sup_{0 \leq s \leq \gamma} \|\bar{X}_s\| \geq \|x\| \right) \leq c' e^{-c'' \frac{\|x\|^2}{\|x\|}} = c' e^{-c'' \|x\|},
\]
for two constants \( c', c'' > 0 \). From (2.13), \( \mathbb{P}^0( \tau_{r_x} < \tau_{(0,x)} < \infty ) \leq \mathbb{P}^0( \bar{S} \bar{r}^0 (\tau_{(0,x)} < \infty ) \leq \mathbb{P}^0( \bar{B} \text{ ever reaches } -r ) = \mathbb{P}^0( B \text{ ever reaches } -r ) \leq c' \|x\|, \)
for some \( c' > 0 \).
Proposition (2.21) Recall from (2.16) the definition of $\tau_z$ for $z \in \mathbb{Z}^{d+1}$. If $z = (n, x)$, we have $G(0, z) = G(0, 0) \sum_{y \in \mathbb{Z}^d} \mathbb{P}^0 (\tau_{0, x-y} < \infty) \int_0^\infty q^X_t(y) \mathbb{P} (\tau_n^0 \in dt)$.

Proof. By (2.16), we have $G(0, z) = G(0, 0) \mathbb{P}^0 (\tau_z < \infty)$. Apply the strong Markov property at the time $\tau_n^0$ and use that $\tau_n^0$ and $X$ are mutually independent,

$$
\mathbb{P}^0 (\tau_z < \infty) = \mathbb{E}^0 \left( \mathbb{P}^{\tilde{\tau}_n^0} (\tau_z < \infty) \right) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}^{(n, y)} (\tau_{(n, x)} < \infty) \mathbb{P}^0 \left( \tilde{S}_{\tau_n^0} = (n, y) \right)
= \sum_{y \in \mathbb{Z}^d} \mathbb{P}^0 (\tau_{0, x-y} < \infty) \int_0^\infty \mathbb{P}^0 \left( \tilde{X}_t = y \right) \mathbb{P} (\tau_n^0 \in dt).
$$

Since $q^X_t(y) = \mathbb{P}^0 \left( \tilde{X}_t = y \right)$, we reach the desired conclusion.

2.6 Statement and proof of the Theorem

Consider the Green’s function $G$ of $\Gamma_d(\lambda)$. By transience, the probability $p_0 = \mathbb{P}^0 (\tau_0 < \infty)$ that the network random walk started at zero returns to zero is (strictly) less than one. Thus, the number of visits to zero, started at zero, has finite expectation since this number follows a geometric law with probability of success $p_0$, in other words, $G(0, 0)$ is some finite number, depending only on $\lambda$ (as such, regarded as universal). The next theorem deals with the bounds $G(0, z)$ for $z \neq 0$.

Theorem (2.22) There exist four constants $c_i$ ($i = 1, \ldots, 4$) such that for all vertices $z = (n, x) \neq 0$ of the network $\Gamma_d(\lambda),$

$$
G(0, z) \leq c_1 \begin{cases} 
    e^{-c_2 \|x\|} & \text{if } \|x\| > n, \\
    e^{-c_2 \frac{\|x\|^2}{n} \|x\|^{-\frac{d}{2}}} & \text{if } \|x\| \leq n,
    \end{cases}
$$

and

$$
G(0, z) \geq c_3 \begin{cases} 
    e^{-c_4 \|x\|} & \text{if } \|x\| > n, \\
    e^{-c_4 \frac{\|x\|^2}{n} \|x\|^{-\frac{d}{2}}} & \text{if } \|x\| \leq n,
    \end{cases}
$$

Proof. The proof is rather lengthy and we divide it into several steps.

Proof of the upper bounds.

We begin with proving the upper bound when $z \in \mathbb{Z}_- \times \mathbb{Z}^d$. We start with the following simple observation: bearing in mind Remark (2.3), if $z = (n, x) \in \mathbb{Z}_- \times \mathbb{Z}^d$, the upper bound will be established if we prove $G(0, z) \leq ce^{-c' \max(|n|, \|x\|)}$ for a pair of constants $c, c' > 0$ independent of $z$.

We analyse separately the cases $\|x\| < |n|$ and $\|x\| \geq |n|$.

Assume first $z \in \mathbb{Z}_- \times \mathbb{Z}^d$ is such that $\|x\| < |n|$, notice $n \leq 0$. We claim there exists a pair of constants $c, c' > 0$ (independent of $z = (n, x)$) such that $G(0, z) \leq ce^{-c' |n|}$. To see this, recall (2.16) and (2.17), so that

$$
G(0, z) = G(0, 0) \mathbb{P}^0 (\tau_z < \infty) \leq G(0, 0) \mathbb{P}^0 (B \text{ ever reaches level } n) = G(0, 0) e^{c' n},
$$

where $\tau_z = \tau_z (\tilde{S})$ is the visit time of the point $z$ by the continuous time random walk $\tilde{S}$ (2.15), proving the claim.

Assume now $z \in \mathbb{Z}_- \times \mathbb{Z}^d$ satisfies $\|x\| \geq |n|$, notice $n \leq 0$. We have that $G(0, z) = \frac{\mu(z)}{\mu(0)} G(0, -z)$, and since $\mu(z) \asymp e^{\lambda n}$, we have the following preliminary result.
Lemma (2.22.1) If the Green’s function bounds hold on the region of \( z \in \mathbb{Z}_+ \times \mathbb{Z}^d \) such that \( 0 \leq n \leq \|x\| \), then they also hold on the region of points \( z \in \mathbb{Z}_- \times \mathbb{Z}^d \) such that \( |n| \leq \|x\| \).

Observe that (2.20) shows that the Green’s function bounds hold on the set \( \{0\} \times \mathbb{Z}^d \). Thus, with the previous lemma in mind, the case \( z \in \mathbb{Z}_- \times \mathbb{Z}^d \) has been established.

Let us focus now on the case \( z = (n, x) \in \mathbb{Z}_+ \times \mathbb{Z}^d \). By virtue of (2.20) and (2.21), we have

\[
(*) \quad G(0, z) \leq c \sum_{y \in \mathbb{Z}^d} e^{-c\|x-y\|} \int_0^\infty q_t^X(y) \mathbb{P} \left( \tau_0^X \in dt \right),
\]

where \( \tau_0^X \) is the visit time of \( n \) by the walk \( \hat{B} \) and \( q_t^X \) is the continuous time transition probability of \( \hat{X} \).

Lemma (2.22.1) There exist two constants \( c_1, c_2 > 0 \), independent of \( z \), such that

\[
\int_0^\infty q_t^X(y) \mathbb{P} \left( \tau_0^X \in dt \right) \leq \left\{ \begin{array}{ll}
 c_1 e^{-c_2 \|n\|} + n^{-\frac{d}{2}} e^{-c_2 \|y\|^2} & \text{if } \|y\| \leq \frac{d}{c_1 + 1} n \\
 c_1 e^{-c_2 \|y\|} & \text{if } \|y\| > \frac{d}{c_1 + 1} n.
\end{array} \right.
\]

Proof of lemma. Recall \( p = (2d + 1 + e^\lambda)^{-1} \). We have \( \int_0^\infty q_t^X(y) \mathbb{P} \left( \tau_0^X \in dt \right) = \int_0^\infty q_t^X(y) \mathbb{P} \left( 2dp \tau_0^X \in dt \right) \). We will tackle the two cases.

Case 1: \( \|y\| \leq \frac{d}{c_1 + 1} n \). We split the integral as follows

\[
\int_0^\infty q_t^X(y) \mathbb{P} \left( 2dp \tau_0^X \in dt \right) = \left[ \int_0^\infty \right] + \left[ \int_{\|y\|}^{2dp(K_0 + \frac{1}{d})n} \right] + \left[ \int_{2dp(K_0 + \frac{1}{d})n}^\infty \right] q_t^X(y) \mathbb{P} \left( 2dp \tau_0^X \in dt \right),
\]

where \( K_0 \) is as in (2.19). In what follows, we will use the bounds for \( \mathbb{P} \left( \tau_0^X \leq \alpha \right) \) and \( \mathbb{P} \left( \tau_0^X \leq \beta n \right) \) of this proposition without further notice. The first two integrals will be bounded using the bounds (2.6). In the region \( 0 \leq t \leq \|y\| \), we have \( q_t^X(y) \leq \frac{1}{2} e^{-\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} \) and the function \( t \mapsto -t - \|y\| \log \left( \frac{\|y\|}{c \|x\|} \right) \) is easily seen to be increasing on \((0, \|y\|)\), hence

\[
\int_0^{\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} q_t^X(y) \mathbb{P} \left( 2dp \tau_0^X \in dt \right) \leq \frac{1}{2} e^{-\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} \mathbb{P} \left( \tau_0^X \leq \|y\| \right) \leq \frac{1}{2} e^{-\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} \mathbb{P} \left( \tau_0^X \leq \|y\| \right) \leq e^{-\frac{\|y\|^2}{2c_1 \|x\|^2}} \leq e^{-\frac{\|y\|^2}{2d}}.
\]

Similarly, in the region \( \|y\| \leq \|x\| \leq t < \infty \), we have the bound \( q_t^X(y) \leq \frac{2}{d} e^{-\frac{\|y\|^2}{2c_1 \|x\|^2}} \), and the term on the right is an increasing function of \( t > 0 \). Hence,

\[
\int_{\|y\|}^{\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} q_t^X(y) \mathbb{P} \left( 2dp \tau_0^X \in dt \right) \leq \frac{2}{d} e^{-\left(\frac{\|y\|^2}{2c_1 \|x\|^2}\right)} \int_{\|y\|}^{\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} q_t^X(y) \mathbb{P} \left( \tau_0^X \leq \|y\| \right) \leq e^{-\left(\frac{\|y\|^2}{2d}\right)} \|y\|^2 \leq e^{-\|y\|^2} \|x\|^2 \leq e^{-cn}
\]

for some constant \( c = u \left( \frac{1}{2} e^{-\frac{\|y\|^2}{2c_1 \|x\|^2}} \right) > 0 \). We use now the bounds (2.8). Consider the interval \( \frac{2dp(K_0 + \frac{1}{d})}{c_1 + 1} n \leq t \leq 2dp(K_0 + \frac{1}{d}) n \), we have a Gaussian decay \( q_t^X(y) \leq c_1 t^{-\frac{d}{2}} e^{-c_2 \|y\|^2} \). On this interval, \( t \geq n \) and so, there is a pair of constants \( c, c' > 0 \) such that

\[
\int_0^{\|y\| \log \left( \frac{\|y\|}{c \|x\|} \right)} q_t^X(y) \mathbb{P} \left( 2dp \tau_0^X \in dt \right) \leq c_n t^{-\frac{d}{2}} e^{-c_2 \|y\|^2} \leq c_n t^{-\frac{d}{2}} e^{-c_2 \|y\|^2}.
\]

Next, for
2.6 Statement and proof of the Theorem

Recall the function \( t \) and the integrals decay exponentially in the lemma and (in which

\[
\ln 2.6. \text{Statement and proof of the Theorem}
\]

\[
t \geq 2dp \left( K_0 + \frac{1}{a} \right) n, \text{ we use the factor } t^{-\frac{d}{2}} \text{ of the Gaussian estimate for } q_{n, y}^X \text{ to conclude there exists } c > 0 \text{ such that } q_{n, y}^X \leq c_1 t^{-\frac{d}{2}} \text{ for all } \|x\| \leq cn^{-\frac{d}{2}}. \text{ Thus,}
\]

\[
\int_{2dp(K_0 + \frac{1}{a})^n} q_{n, y}^X(y) \mathbb{P} \left( 2dp \tau^0_n \in dt \right) = cn^{-\frac{d}{2}} \mathbb{P} \left( \tau^0_n \geq (K_0 + \frac{1}{a}) n \right) = cn^{-\frac{d}{2}} e^{-T_0 K_0 n} \leq e^{-T_0 K_0 n},
\]

again by (2.10). Putting all the bounds found, this concludes the case \( \|y\| \leq \frac{d}{c\alpha + 1} n. \)

Case 2: \( \|y\| > \frac{d}{c\alpha + 1} n. \) Here we divide

\[
\int_0^\infty q_{n, y}^X(y) \mathbb{P} \left( 2dp \tau^0_n \in dt \right) = \left[ \int_0^{\frac{1}{\alpha c}} + \int_{\frac{1}{\alpha c}}^{2\sigma_0^2(K_0 + \frac{1}{a})\|y\|} + \int_{2\sigma_0^2(K_0 + \frac{1}{a})\|y\|}^\infty \right] q_{n, y}^X(y) \mathbb{P} \left( 2dp \tau^0_n \in dt \right).
\]

Recall the function \( t \mapsto -t - \|y\| \ln \left( \frac{\|y\|}{ct} \right) \) is increasing on \((0, \|y\|). As in Case 1, we can bound

\[
q_{n, y}^X(y) \leq \begin{cases} 
  e^{-\frac{\|y\|}{ct}} & \text{if } t \leq \frac{\|y\|}{ct}, \\
  \frac{2}{ct} e^{-c\|y\|} & \text{if } \frac{\|y\|}{ct} \leq t \leq 2\sigma_0^2(K_0 + \frac{1}{a}) \|y\|, \\
  c_1 & \text{if } t \geq 2\sigma_0^2(K_0 + \frac{1}{a}) \|y\|,
\end{cases}
\]

where \( c_1 \) is the constant appearing on the Gaussian bound of \( q_{n, y}^X \) and \( c = (2\sigma_0^2 e^2(K_0 + \frac{1}{a}))^{-1}. \) We may assume, should the need arise, \( K_0 \) is so large so that \( 2\sigma_0^2(K_0 + \frac{1}{a}) \geq 1. \) It is obvious then that the first two integrals decay exponentially in \( \|y\|, \text{ as for the third one, we employ (2.10) to obtain}
\]

\[
\int_{2\sigma_0^2(K_0 + \frac{1}{a})\|y\|}^\infty q_{n, y}^X(y) \mathbb{P} \left( 2dp \tau^0_n \in dt \right) \leq c_1 \mathbb{P} \left( \tau^0_n \geq \frac{\sigma_0^2(K_0 + \frac{1}{a})\|y\|}{dp} \right),
\]

the condition \( \frac{d}{c\alpha + 1} n < \|y\| \) guarantees \( \mathbb{P} \left( \tau^0_n \geq \frac{\sigma_0^2(K_0 + \frac{1}{a})\|y\|}{dp} \right) \leq e^{-T_0 K_0 \frac{c\alpha + 1}{d}} \|y\|. \) This completes Case 2 and the lemma.

With the lemma proven, let us return to the proof of the upper bounds for the Green’s function. The reader is reminded that there only remains to prove the bounds for \( z = (n, x) \text{ in } \mathbb{Z}_n^+ \times \mathbb{Z}^d \). The previous lemma and (*) imply that there exist two constants \( c, c' > 0 \) such that

\[
G(0, z) \leq c \left( \sum_{\|y\| \leq bn} e^{-c' \|x-y\| + \|n\|} + \sum_{\|y\| \geq bn} n^{-\frac{d}{2}} e^{-c' \left( \|x-y\| + \frac{1}{\alpha c} \|n\| \right)} + \sum_{\|y\| > bn} e^{-c' \left( \|x-y\| + \|y\| \right)} \right),
\]

where \( b = \frac{d}{c\alpha + 1}. \)

Bearing in mind (2.10), we only need to prove there exists a pair of constants \( c, c' > 0 \) such that \( G(0, z) \leq cn^{-\frac{d}{2}} e^{-c\|x\|} \) for all \( \|x\| > bn \) and that there exists a pair of constants \( c, c' > 0 \) such that \( G(0, z) \leq cn^{-\frac{d}{2}} e^{-c\|x\|} \) for all \( \|x\| \leq bn. \) We consider first a sum of the form \( \sum_{\|y\| \leq bn} e^{-c(\|x-y\| + \|n\|)} \). Set \( c' = \min \left( c, \frac{\alpha c}{20} \right); \) then

\[
\sum_{\|y\| \leq bn} e^{-c(\|x-y\| + \|n\|)} \leq \sum_{\|y\| \leq bn} e^{-c\|x-y\| - \|n\|} \leq \sum_{\|y\| \leq bn} e^{-c'\|x-y\|} \leq e^{-c'\|x\|} Ln^d e^{-\frac{d}{2} n} \leq c'' e^{-c'\|x\| - \frac{d}{2} n},
\]

in which \( L \) is a constant, depending only in dimension, satisfying that \( \operatorname{card} (B_{\mathbb{Z}_n^d} (0; r)) \leq L r^d \) for all \( r \geq 1 \) and \( c'' = \sup_{n \in \mathbb{N}} L n^d e^{-\frac{d}{2} n} < \infty. \) Thus, we have proven the following lemma.
LEMMA (2.22.2) Let $\alpha, \beta, \gamma > 0$. There exist constants $c, c' > 0$ such that $\sum_{\|y\| \leq \beta n} e^{-\gamma (\|x\| + \|y\|)} \leq ce^{-c'\|n-x\|}$, for all $(n, x) \in \mathbb{Z} \times \mathbb{Z}^d$.

Next, we consider the sum $\sum_{\|y\| > \beta n} e^{-c'(\|x-y\|+\|y\|)}$. In the case that $\|x\| \leq bn$, then we may bound $e^{-c'(\|x-y\|+\|y\|)} \leq e^{-c'bn}e^{-c'|x-y|}$, thus $\sum_{\|y\| > \beta n} e^{-c'(\|x-y\|+\|y\|)} \leq e^{-c'bn} \sum_{y \in \mathbb{Z}^d} e^{-c'|y|}$. In the case that $\|x\| > bn$, we may bound $\|x-y\| \geq \|x\| - \|y\|$, thus $e^{-c'(\|x-y\|+\|y\|)} \leq e^{-c'(\|x\| - \|y\|)} \leq e^{-\frac{c'}{2}\|x\| - \frac{c}{2}|y|}$. Therefore $\sum_{\|y\| > \beta n} e^{-c'(\|x-y\|+\|y\|)} \leq e^{-\frac{c}{2}\|x\|} \sum_{y \in \mathbb{Z}^d} e^{-\frac{c}{2}|y|}$. We have established the following lemma.

LEMMA (2.22.3) Let $\alpha, \beta > 0$. There exists a pair of constants $c, c' > 0$ such that $\sum_{\|y\| > \beta n} e^{-\alpha (\|x-y\|+\|y\|)} \leq ce^{-c'\|n-x\|}$ for every $(n, x) \in \mathbb{Z} \times \mathbb{Z}^d$.

By virtue of the previous lemmas, it remains to show that

$$\sum_{\|y\| \leq \beta n} n^{-\frac{d}{2}} e^{-c'\left(\|x-y\| + \frac{\|y\|^2}{\beta n}\right)} \leq \begin{cases} ce^{-c'|x|} & \text{if } \|x\| > bn, \\ ce^{-c'\frac{\|x-y\|^2}{\beta n}}n^{-\frac{d}{4}} & \text{if } \|x\| \leq bn. \end{cases}$$

Assume first that $\|x\| > bn > 0$. The estimate [2.14] shows at once $\sum_{\|y\| \leq \beta n} e^{-c'(\|x-y\| + \frac{\|y\|^2}{\beta n})} \leq ce^{-c'|x|}$ for a pair of constants $c, c' > 0$. This concludes the case $\|x\| > bn > 0$. Assume now that $\|x\| \leq bn$. By (2.12), we also have $\sum_{\|y\| \leq \beta n} n^{-\frac{d}{2}} e^{-c'(\|x-y\| + \frac{\|y\|^2}{\beta n})} \leq cn^{-\frac{d}{2}} e^{-c'\frac{\|x-y\|^2}{\beta n}}$ for (possibly) another pair $c, c' > 0$. This concludes the case $\|x\| \leq bn$, and thus, the proof of the upper bounds by virtue of Lemma (2.22.1).
Proof of lower bounds.

These are much simpler. We have \( G(0, z) = \frac{\mu(z)}{\mu(0)} G(0, -z) \). If \( z = -(n, x) \) with \( n > 0 \), then \( \mu(z) \asymp e^{-\lambda n} \) and so \( G(0, z) \geq c e^{-\lambda n} G(0, -z) \geq c e^{-\lambda \|z\|} G(0, -z) \). Therefore, if we prove lower bounds for \( n > 0 \), we will reach the lower bounds for \( n < 0 \). For the case \( n = 0 \) observe the following: by (2.16),

\[
G(0, (0, x)) = G(0, 0)^0 (\tau_{0,x} < \infty) \geq G(0, 0)^0 (\tau_{0,x} < \infty, \tilde{S}_{\sigma_1} = (1, 0)),
\]

where \( \sigma_1 \) is the time of the first jump of the Poisson process \( N \) in the definition of \( \tilde{S} \). Now,

\[
P^0 (\tau_{0,x} < \infty, \tilde{S}_{N_\sigma_1} = (1, 0)) = e^{\lambda p} \int_0^\infty dt e^{-t} P^{(1,0)} (\tau_{0,x} < \infty) = e^{\lambda p} P^0 (\tau_{-1,x} < \infty).
\]

Thus, if the correct bounds hold for \( n < 0 \), then \( G(0, (0, x)) \geq e^{\lambda p} G(0, (-1, x)) \geq c e^{-\|1, x\|} \|1, x\|^{-\frac{d}{2}} \), and it is obvious that \( \|1, x\| \asymp \|(0, x)\| \) (2.19). Whence, the proof of the lower bounds reduces to the case \( n > 0 \).

When \( n > 0 \), we can apply (2.21). Now,

\[
\int_0^\infty \int_{\mathbb{R}^n} q^X (y) P (\tau_n^0 \in dt) \geq \int_{\mathbb{R}^n} q^X (y) P (2dp \tau_n^0 \in dt).
\]

Observe that the lower bounds of (2.7) and (2.8) are of the same type (with possibly different constants) whenever \( \|y\| \asymp t \). On the interval of integration, \( t \asymp n \), therefore

\[
q^X (y) \geq \begin{cases} cn^{-\frac{d}{2}} e^{-c' \|y\|^2} & \text{if } n \geq \|y\|, \\ c \|y\|^{-\frac{d}{2}} e^{-c' \|y\|} & \text{if } n < \|y\|. \end{cases}
\]

Also, (2.19) gives \( P (\frac{2d}{n} n \leq 2dp \tau_n^0 \leq 4(K_0 + \frac{1}{a}) dp n) \geq c \), for a constant \( c > 0 \) independent of \( n \). Therefore,

\[
\frac{G(0, z)}{G(0, 0)} \geq c \left( \sum_{n \geq \|y\|} P^0 (\tau_{0,x-y} < \infty) n^{-\frac{d}{2}} e^{-c' \|y\|^2} + \sum_{n < \|y\|} P^0 (\tau_{0,x-y} < \infty) \|y\|^{-\frac{d}{2}} e^{-c' \|y\|} \right).
\]

We discard all the terms except \( y = x \), and reach

\[
G(0, z) \geq \begin{cases} c G(0, 0) n^{-\frac{d}{2}} e^{-c' \|y\|^2} & \text{if } n \geq \|x\|, \\ c G(0, 0) \|x\|^{-\frac{d}{2}} e^{-c' \|x\|} & \text{if } n < \|x\|. \end{cases}
\]

This terminates the proof of Theorem (2.22).
3.1 Coupling of random walks

Consider two vertices $x$ and $y$ in the network $(G, \mu)$. We will say that the network random walk started at $x$ can be **classically coupled** with the network random walk started at $y$ if the following condition holds:

\[(\text{CC}) \text{ There exists a probability space } (\Omega, \mathcal{F}, \mathbb{P}), \text{ a filtration } (\mathcal{F}_n)_{n \in \mathbb{Z}_+} \text{ in this probability space, two Markov processes with respect to this filtration and defined on this probability space, denoted by } (S_n, \mathcal{F}_n) \text{ and } (S'_n, \mathcal{F}_n), \text{ both of them having for transition probability that defined by the conductances of the network, } S_0 = x, S'_0 = y \text{ and there exists a stopping time } \tau \text{ relative to the filtration } (\mathcal{F}_n)_{n \in \mathbb{Z}_+}, \text{ such that } \tau < \infty \mathbb{P}\text{-a.e. and for those } \omega \in \Omega \text{ for which } \tau(\omega) < \infty, S_{\tau(\omega)+n}(\omega) = S'_{\tau(\omega)+n}(\omega) \text{ for any } n \in \mathbb{N}.)\]

When this happens, we will call $\tau$ the **coupling time** and the condition $\tau < \infty \mathbb{P}\text{-a.e.}$ is expressed by saying that “$\tau$ is successful.” Obviously, for a classical coupling to exist, it is necessary that $x$ and $y$ should be connected in $(G, \mu)$, but this is far from sufficient.

**Remark (3.1)** Observe that by definition, a classical coupling is a coupling $T = (S, S')$ of the network random walk $S$ started at $x$ and the network random walk $S'$ started at $y$ satisfying strong conditions and it does not necessarily exist; for instance, the standard lattice $\mathbb{Z}$ and a random walk $S_3$.

**Proposition (3.2)** Denote by $u_i \ (1 \leq i \leq d)$ the canonical unit vectors of $\mathbb{Z}^d$. Let $\rho$ be a probability measure on $\mathbb{Z}^d$ such that

(a) the relation $\rho(x) > 0$ implies $x$ is an integer multiple of some $u_i$;

(b) $\rho(Z^i u_i) > 0$ for each $i = 1, \ldots, d$.

Introduce two probability measures, first $\phi(\cdot) = \rho(Z^i u_i) + \frac{\rho(0)}{d\phi(1)} \ (1 \leq i \leq d)$ and then $\nu_i(k) = \frac{\rho(ku_i)}{\phi(1)}$ for $k \in \mathbb{Z}$ and $\nu_i(0) = \frac{\rho(0)}{\phi(1)}$. Suppose $F_j \sim \phi$ and $X_j^{(i)} \sim \nu_i$ are independent random elements ($1 \leq i \leq d$ and $j \in \mathbb{Z}_+$). Set $Y_j = X_j^{(F_j)} = \sum_{i=1}^d X_j^{(i)} 1_{\{F_j = i\}}$. Then, for any $S_0$ independent of the families $X_j^{(i)}$ and $F_j$, $S_n = S_0 + Y_1 + \ldots + Y_n$ is a random walk on $\mathbb{Z}^d$ started at $S_0$ with transition probability $\rho : \mathbb{P}(Y_j = x) = \rho(x)$ ($x \in \mathbb{Z}^d$).

**Remark (3.3)** When $\rho$ of (3.2) is given by the $\frac{1}{2}$-lazy version of (2.1), the appropriate distributions are $\phi(0) = (1 + e^{\lambda})(2(2d+1+e^{\lambda}))^{-1} + (2(d+1))^{-1}$, and for $i = 1, \ldots, d$, $\phi(i) = (2d+1+e^{\lambda})^{-1} + (2(d+1))^{-1}$. While the $\nu_i$ are given as $\nu_0(-1) = (2(2d+1+e^{\lambda})\phi(0))^{-1}$, $\nu_0(0) = (2d+1)\phi(0))^{-1}$, $\nu_0(i) = e^{\lambda}(2(2d+1+e^{\lambda})\phi(0))^{-1}$ and, for $i = 1, \ldots, d$, $\nu_i(1) = \nu_i(-1) = (2(2d+1+e^{\lambda})\phi(i))^{-1}$, $\nu_i(0) = (2d+1)\phi(i))^{-1}$.

The following theorem is arguably well-known while a source where it is proven is elusive to find. The reader can find a complete proof in [MD20], theorem (3.2.3). The proof-strategy follows by classically coupling each coordinate at a time.

**Theorem (3.4)** Let $\rho$ be a probability measure on $\mathbb{Z}^d$ such that

(a) the relation $\rho(x) > 0$ implies $x = ku_i$ for some $k \in \mathbb{Z}$ and some canonical unitary vector $u_i$;

(b) $\rho(Z^i u_i) > 0$ for each $i = 1, \ldots, d$;
(c) for every $i = 1, \ldots, d$, the probability measures $\nu_i$, defined in (3.2), and $\bar{\nu}_i$ defined by $\bar{\nu}_i : k \mapsto \nu_i(-k)$ $(k \in \mathbb{Z})$ are such that their convolution $\nu_i \ast \bar{\nu}_i$ defines a transition probability kernel for a Markov chain on $\mathbb{Z}$ that is irreducible and recurrent.

Then, for any pair of vertices $x$ and $y$ in $\mathbb{Z}^d$, there exists a classical coupling of random walks started at $x$ and $y$, respectively.

### 3.2 The Liouville property

We will say that a network satisfies

- (LP) The **Liouville property** if every bounded harmonic function on the network is constant.
- (SLP) The **strong Liouville property** if every positive harmonic function on the network is constant.
- (DLP) The **Dirichlet Liouville property** if every Dirichlet harmonic function is constant.

**Remark (3.5)**

(a) For any network, we have the following implications from the above properties

$$ (\text{SLP}) \implies (\text{LP}) \implies (\text{DLP}) $$

The first of these implications is trivial for if $h$ is a bounded harmonic function, then $h + c$ is a positive harmonic function for large enough constant $c$, then condition (SLP) implies $h + c$ is constant, and so is $h$. The second of this implications is harder and it fully proved in [MD20], appendix D (the author followed closely the steps of Soardi [Soa94]).

(b) It is a well-known fact that the graph $\mathbb{Z}^d$ satisfies the strong Liouville property. One proof is given in [Bar17]. Here, M. Barlow followed an already established strategy: showing that $\mathbb{Z}^d$ satisfies the so-called **elliptic Harnack inequality** (EHI) and also showing that for any network (EHI) $\implies$ (SLP).

(c) The network $\Gamma_d(\lambda)$ does not satisfy (SLP). Consider the function $h(z) = e^{-\lambda n}$ (with $z = (n, x)$ as usual). Then, with $p = (2d + 1 + e^{\lambda})^{-1}$,

$$ \sum_{z' \sim z} p(z, z') h(z') = pe^{-\lambda(n+1)} + 2dpe^{-\lambda n} + e^{\lambda}pe^{-\lambda(n+1)} = e^{-\lambda n} = h(z), $$

so that $h$ is harmonic, and proving the claim.

(d) If on a given network and for any pair of vertices, two lazy network random walks, one started at each vertex, can be classically coupled, then this network satisfies the Liouville property. In other words,

$$ (\text{CC}) \implies (\text{LP}). $$

First notice that a function $h : V \to \mathbb{R}$ is harmonic relative to the NRW if and only if it is harmonic relative to the $\frac{1}{2}$-lazy NRW. Next, consider a bounded harmonic function $h$, any two vertices $x, y$ and the two $\frac{1}{2}$-lazy NRWs $S$ and $S'$, started at $x$ and $y$, respectively, with successful coupling time $\tau$. Since $h(S)$ and $h(S')$ are bounded martingales, we obtain $h(x) = E(h(S_\tau)) = E(h(S'_\tau)) = h(y)$.

**Theorem (3.6)** The network $\Gamma_d(\lambda)$ has the Liouville property.

**Proof.** By virtue of the previous proposition, it suffices to prove that any two lazy network random walks (started at any two vertices) of $\Gamma_d(\lambda)$ can be classically coupled. In this case, the hypothesis of theorem (3.4) are satisfied: the measure $\nu_i \ast \bar{\nu}_i$, where $\nu_i$ is as in (3.3) is clearly symmetric and bounded, it is aperiodic by lazyness, thus $\nu_i \ast \bar{\nu}_i$ is also recurrent and irreducible. \qed
3.3 Liouville property and the uniform spanning forest measure

The Liouville property is relevant to the context of uniform spanning forests due to the following theorem.

**Theorem (3.7)** Let $\Gamma$ be any network. A sufficient condition for the measures WSF and FSF, of the wired and free uniform spanning forest on $G$, to be the same, is that this network should satisfy the Liouville property.

For the proof, see [BLPS01, Theorem 7.3]. As a corollary of the foregoing.

**Theorem (3.8)** $\text{WSF} = \text{FSF}$ on $\Gamma_d(\lambda)$.

By virtue of the equality between WSF and FSF on $\Gamma_d(\lambda)$, we will denote this measure as USF from now onwards.

3.4 The number of components

To count the number of components, the reader should recall the definition of transitive network. We know $\Gamma_d(\lambda)$ is transitive by considering translations.

We need the following results:

[LP16], Theorem 10.24. In any transient transitive network $(G, \mu)$, with vertex set $V$ and Green’s function $G$, the number of intersections of any two independent network random walks is almost surely infinity if $\sum_{z \in V} G(0, z)^2 = \infty$ and is almost surely finite otherwise.

[BLPS01], Theorem 9.2. On any network, for the number of components in WSF to be one, it is necessary and sufficient that, two independent network random walk started at any two vertices, should intersect (a.s.).

[BLPS01], Theorem 9.4. If there exists a vertex $o$ such that, for almost every realisation, the number of intersections of two independent copies of the network random walk started at $o$ is finite, then the number of components in WSF is infinite.

**Theorem (3.9)** Consider the network $\Gamma_d(\lambda)$. The USF here consists of a single tree when $d = 1, 2$ and it has an infinite number of components when $d \geq 3$. From now onwards, we use UST in lieu of USF for $d = 1, 2$.

**Proof.** By the theorems stated above, it all reduces to show that $\sum_{z \in \mathbb{Z}^{d+1}} G(0, z)^2$ diverges for $d = 1, 2$ and that it converges otherwise. In the sake of clarity, we separate the calculations in the next proposition, with them the theorem is proved.

As mentioned in the proof above, the next theorem states when the Green’s function belongs to $L^2$.

**Theorem (3.10)** (The “bubble condition.”)

(a) If $d = 1, 2$, $\sum_{z \in \mathbb{Z}^{d+1}} G(0, z)^2 = \infty$.

(b) If $d \geq 3$, $\sum_{z \in \mathbb{Z}^{d+1}} G(0, z)^2 < \infty$. 

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Proof. Recall the Fourier transform of the step-lengths of the network random walk started at zero and \( \varphi_n \) is its Fourier transform of \( S_n \), then \( \varphi_n(h) = \varphi_Y(h)^n \). Hence, for all \( h \neq 0 \), \( \hat{G}(h) = \sum_{n=0}^{\infty} \varphi_n(h) = (1 - \varphi_Y(h))^{-1} \), where \( \hat{G} \) denotes the Fourier transform of the Green’s function \( G \). By Parseval’s relation,

\[
(3.11) \quad \sum_{z \in \mathbb{Z}^{d+1}} G(0, z)^2 = \frac{1}{(2\pi)^{d+1}} \int_{[-\pi, \pi]^{d+1}} dh \ |1 - \varphi_Y(h)|^{-2},
\]

where the two integrals will converge to the same value, or they will both diverge.

Using second order expansion in (2.4) and with \( a \) as in (2.3) and \( p = (2d + 1 + e^\lambda)^{-1} \), we reach

\[
\varphi_Y(h) = 1 + iah_0 - \frac{h_0^2}{2} - p \sum_{j=0}^{d} h_j^2 + o \left( h_0^2 \right) + O \left( ||h||^4 \right), \quad h \to 0.
\]

Denote \( \alpha_j = p \) for \( j = 1, \ldots, d \) and \( \alpha_0 = p + \frac{h_0^2}{2} \), \( \varphi_Y(h) = 1 + iah_0 - \sum_{j=0}^{d} \alpha_j h_j^2 + o \left( h_0^2 \right) + O \left( ||h||^4 \right) \). Then,

\[
|1 - \varphi_Y(h)|^2 = a^2 h_0^2 + iah_0 \sum_{j=0}^{d} \alpha_j h_j^2 + o \left( h_0^2 \right) + O \left( ||h||^4 \right)
- iah_0 \sum_{j=0}^{d} \alpha_j h_j^2 + \left( \sum_{j=0}^{d} \alpha_j h_j^2 \right)^2 + O \left( ||h||^6 \right)
= a^2 h_0^2 + \left( \sum_{j=0}^{d} \alpha_j h_j^2 \right)^2 + o \left( h_0^2 \right) + O \left( ||h||^4 \right) + o \left( ||h||^4 \right).
\]

Divide the right-hand side of (3.11) as the sum of \( I_\varepsilon + J_\varepsilon \), where \( I_\varepsilon \) has domain of integration \( K_\varepsilon = [-\varepsilon, \varepsilon] \times B_{\mathbb{R}^d} (0; \varepsilon) \) and \( J_\varepsilon \) has domain of integration \( [-\pi, \pi]^{d+1} \setminus K_\varepsilon \). It follows at once \( J_\varepsilon < \infty \). We will show the existence of a positive \( \varepsilon \) such that \( I_\varepsilon = \infty \) for \( d = 1, 2 \) and \( I_\varepsilon < \infty \) for \( d \geq 3 \).

Assume first \( d = 1, 2 \). Then by (\ast) for \( \varepsilon \) small enough, \( |1 - \varphi_Y(h)|^2 \leq c^{-1} (h_0^2 + ||h||^4) \) for some \( c > 0 \). Then,

\[
I_\varepsilon \geq c \int_{B_{\mathbb{R}^d}(0; \varepsilon)} dh' \int_{-\varepsilon}^\varepsilon \int_{B_{\mathbb{R}^d}(0; \varepsilon)} dh_0 \frac{1}{h_0 + ||h'||} = 2c \int_{B_{\mathbb{R}^d}(0; \varepsilon)} dh' \int_{0}^\varepsilon \int_{B_{\mathbb{R}^d}(0; \varepsilon)} dh_0 \frac{1}{h_0 + ||h'||}.
\]

Changing measures to surface measure on the spheres of \( \mathbb{R}^d \), there is a constant \( c > 0 \) such that

\[
I_\varepsilon \geq c \int_{0}^\varepsilon \int_{0}^\varepsilon \int_{0}^\varepsilon \frac{1}{h_0 + ||h'||} = c \int_{0}^\varepsilon \int_{0}^\varepsilon \frac{1}{h_0 + ||h'||} = c \int_{0}^\varepsilon \int_{0}^\varepsilon \frac{1}{h_0 + ||h'||} = \infty.
\]

Similarly from (\ast), we can assume \( \varepsilon > 0 \) is small enough so that for some constant \( c > 0 \), \( |1 - \varphi_Y(h)|^2 \geq c^{-1} (h_0^2 + ||h||^4) \). As before, we change to surface measure of spheres and use that \( \rho^{d-1} \leq \rho^2 \) since \( \varepsilon \) can be assumed smaller than unity and \( d \geq 3 \), then

\[
I_\varepsilon \leq c \int_{K_\varepsilon} \frac{1}{h_0 + ||h'||} dh_0, h' \leq c' \int_{0}^\varepsilon \int_{0}^\varepsilon \frac{1}{h_0 + ||h'||} = \infty.
\]

Therefore, we reach that the sum (3.11) is finite for all \( d \geq 3 \).
4 Crossings of NRW

4.1 Dimensions \(d = 1, 3, 4, \ldots\)

**Theorem (4.1)** Let \(d \geq 3\). Consider two independent network random walks \(S\) and \(S'\) on \(\Gamma_\lambda(\lambda)\). Suppose they have arbitrary (random) initial points \(S_0\) and \(S'_0\), respectively. Then, almost surely, the paths of \(S\) and \(S'\) will cross finitely many times.

**Proof.** By the bubble condition \([3.10]\), \(\sum_{z \in \mathbb{Z}^{d+1}} G(0, z)^2 < \infty\) in this range of dimensions. The theorem then follows from [LP10] Theorem 10.24.

The previous theorem is not true in dimension \(d = 1, 2\). However, there is an important distinction between the cases \(d = 1\) and \(d = 2\). The gist of the distinction lies in how frequently the two random paths cross. In this section we consider the case \(d = 1\).

**Theorem (4.2)** Let \(d = 1\). Consider two independent network random walks \(S\) and \(S'\) on \(\Gamma_1(\lambda)\) with arbitrary starting points, \(S_0\) and \(S'_0\). Then, \(S - S'\) is a recurrent random walk; in particular, for almost every realisation, there will be infinitely many \(n\) for which \(S_n - S'_n = S_0 - S'_0\). Thus, if both \(S\) and \(S'\) start at the same point, then \(S\) and \(S'\) will collide infinitely often.

**Proof.** By means of a translation by \(-(S_0 - S'_0)\), we can assume \(S_0 - S'_0 = 0\). The difference \(S - S'\) is a random walk, started at zero and has step-length distribution

\[
\mu = p^2 e^\lambda (\varepsilon_{(-2,0)} + \varepsilon_{(2,0)}) + p^2 (\varepsilon_{(0,2)} + \varepsilon_{(0,-2)})
+ p^2 (e^\lambda + 1)(\varepsilon_{(-1,-1)} + \varepsilon_{(1,1)} + \varepsilon_{(-1,1)} + \varepsilon_{(1,-1)}) + p^2 (e^{2\lambda} + 3)\varepsilon_{(0,0)},
\]

where \(\varepsilon_{(h,k)}\) denotes the measures of unitary total mass at \((h,k)\) in \(\mathbb{Z}^2\) and \(p = \frac{1}{3+e^\lambda}\). Observe that \(S - S'\) is a symmetric, aperiodic and irreducible random walk with bounded step lengths on the lattice generated by \(u_1 = (1,1)\) and \(u_2 = (-1,1)\). By [LL10] Theorem 4.1.1 \(S - S'\) is recurrent.

The remainder of this chapter is devoted into estimating how frequently there will be a crossing between two network random walk paths on \(\Gamma_2(\lambda)\). Whence, in the upcoming sections we will develop the necessary tools to tackle dimension \(d = 2\).

4.2 Two elementary results

For the sake of reference, we write the following. The “second moment inequality” states that for a non-negative random variable

\[
(4.3) \quad \mathbb{P}(Z > \frac{1}{2} \mathbb{E}(Z)) \geq \frac{\mathbb{E}(Z)^2}{4 \mathbb{E}(Z^2)}.
\]

Consider now an increasing sequence of \(\sigma\)-fields \((\mathcal{E}_n)_{n \in \mathbb{Z}^+}\), with \(\mathcal{E}_0\) the \(\sigma\)-field generated by \(\emptyset\), on some probability space \((\mathbb{E}, \mathcal{E}, \mathbb{P})\). Suppose that \(A_n\) is an event measurable up to time \(n\), that is \(A_n \in \mathcal{E}_n\). Set \(q_n = \mathbb{P}(A_n \mid \mathcal{E}_{n-1})\) (so that \(q_1 = \mathbb{P}(A_1)\)). We have the following result.

**Proposition (4.4)** The event \(N = \{A_n \ i.o.\} \Delta \left\{ \sum_{n=1}^{\infty} q_n = \infty \right\}\) is \(\mathbb{P}\)-null (“Levy’s extension of Borel-Cantelli lemma”).

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As before, we will write $z = (n, x)$; any affix, such as a sub or superscript, will be given to the three symbols: if we write $z'$ at some point, it will be assumed that $z' = (n', x')$, similarly, if we talk about $n_1$ and $x_1$, we will be assuming $z_1 = (n_1, x_1)$, etc.

Divide the space $\mathbb{Z} \times \mathbb{Z}^2$ into the two sets $S_1 = \{ z; n \leq \|x\| \}$ and $S_2 = \{ z; n > \|x\| \}$. Notice $0 \in S_1$ and $S_1 \cap S_2 = \emptyset$. To simplify notation, we define (for any $y \in \mathbb{Z}^r$, $r \geq 1$)

\[
\langle y \rangle = \max(\|y\|, 1),
\]

and notice $\langle y \rangle = \|y\|$ unless $y = 0$; also $\langle y \rangle = \|y\| + 1 \approx \sqrt{\|y\|^2 + 1}$.

Consider the following tessellation ($p, q \in \mathbb{N}$ and $i = 1, 2$) of the “positive half-space” $\mathbb{Z}_+ \times \mathbb{Z}^2$:

\[
\begin{align*}
R_1(p, q) &= \{ z; 9^p < n \leq 2 \cdot 9^p, 3^q - 1 < \|x\| \leq 3^q \}, \\
R_2(p, q) &= \{ z; 2 \cdot 9^p < n \leq 9^p + 1, 3^q - 1 < \|x\| \leq 3^q \} \\
R_1(p, 0) &= \{ z; 9^p < n \leq 2 \cdot 9^p, \|x\| \leq 1 \}, \\
R_2(p, 0) &= \{ z; 2 \cdot 9^p < n \leq 2 \cdot 9^p + 1, \|x\| \leq 1 \} \\
R_1(0, q) &= \{ z; 0 \leq n \leq 2, 3^q - 1 < \|x\| \leq 3^q \} \\
R_2(0, q) &= \{ z; 3 \leq n \leq 9, 3^{q-1} < \|x\| \leq 3^q \} \\
R_1(0, 0) &= \{ z; 0 \leq n \leq 2, \|x\| \leq 1 \} \\
R_2(0, 0) &= \{ z; 3 \leq n \leq 9, \|x\| \leq 1 \}
\end{align*}
\]

For $p \in \mathbb{N}$, we shall consider the following region

\[
(4.7) \quad D_p = \bigcup_{q=0}^p R_1(p, q) = \{ z; 9^p < n \leq 2 \cdot 9^p, \|x\| \leq 3^p \}.
\]

The following easy remark contains all the technical estimates that will be employed later in this section. Consider the Green’s function bounds and fix the constants $c_i$ ($1 \leq i \leq 4$) of (2.22).

**Remark** (4.8) For $i = 1, 2$ and $(p, q) \in \mathbb{Z}_+^2$, we have the following.

(a) $\text{card} (R_i(p, q)) \asymp 9^{p+q}$.

(b) $R_1(i_1, q_1)$ and $R_1(i_2, q_2)$ are disjoint unless $(i_1, p_1, q_1) = (i_2, p_2, q_2)$.

(c) $\text{card} (D_p) \asymp 9^p \sum_{q=0}^p 9^q \asymp 9^{2p}$.

(d) We have $R_1(p, q) \subset S_1$ if $2p + 3 \leq q$ and $R_1(p, q) \subset S_2$ if $q \leq 2p$.

(e) If $z \in \mathbb{Z}_+ \times \mathbb{Z}^2$ is in $S_1$ and $c$ is a positive constant, then $0 \leq n \leq \|x\|$, this implies $\|x\| \leq \|z\| \leq \sqrt{2} \|x\|$, and then $\frac{1}{\sqrt{2}} e^{-\sqrt{2}c\|x\|} \langle x \rangle^{-1} \leq e^{-c\|x\|} \langle x \rangle^{-1} \leq e^{-c\|x\|} \langle x \rangle^{-1}$. Therefore, if $3^{q-1} \leq \|x\| \leq 3^q$,

\[
\frac{1}{\sqrt{2}} e^{-\sqrt{2}c\|x\|} \langle x \rangle^{-1} \leq e^{-c\|x\|} \langle x \rangle^{-1} \leq 3e^{-\frac{\|x\|}{3^q}}.
\]
(f) If $z \in \mathbb{Z}_+ \times \mathbb{Z}^2$ is in $S_2$ and $c$ is a positive constant, then $\|x\| < n$, this implies $0 < n \leq \|z\| \leq \sqrt{2}n$, and then $\frac{1}{9\sqrt{2}} e^{-c\|x\|^2} n^{-1} \leq e^{-c\|z\|^2} n^{-1} \leq e^{-c\|x\|^2} n^{-1}$. Therefore, if $9^p \leq n \leq 9^{p+1}$ and $3^{q-1} \leq \|x\| \leq 3^n$ $\frac{1}{9\sqrt{2}} e^{-c9^p} 9^{-p} \leq e^{-c\|z\|^2} n^{-1} \leq e^{-c9^p} 9^{-p}$.

(g) Define the function $\varphi_U : \mathbb{Z} \times \mathbb{Z}^2 \to \mathbb{R}$ given by $\varphi_U(z) = c_1 e^{-c_2 \|z\|^2} \|z\|^{-1}$ if $z$ lies in region $S_1$ and $\varphi_U(z) = c_1 e^{-c_2 \|z\|^2} \|z\|^{-1}$ for $z$ within $S_2$. Then $G(0, z) \leq \varphi_U(z)$. If $z \in R_1(p, q) \cup R_2(p, q)$,

$$\varphi_U(z) \leq \begin{cases} 3c_1 e^{-\frac{c_2}{9} 3^q 3^{-q}} & \text{if } 2p + 3 \leq q, \\
 c_1 e^{-\frac{c_2}{9} 3^q} 9^{-p} & \text{if } q \leq 2p + 2. \end{cases}$$

(h) Define $\varphi_L(z) = c_3 e^{-c_4 \|z\|^2} \|z\|^{-1}$ if $z \in S_1$ and $\varphi_L(z) = c_3 e^{-c_4 \|z\|^2} \|z\|^{-1}$ if $z \in S_2$. Then, $G(0, z) \geq \varphi_L(z)$. If $z \in R_i(p, q)$ ($i = 1, 2$),

$$\varphi_L(z) \geq \begin{cases} 3c_1 e^{-\sqrt{2}c_4 3^q 3^{-q}} & \text{if } 2p + 3 \leq q, \\
 c_1 e^{-\frac{c_2}{9} 3^q} 9^{-p} & \text{if } q \leq 2p + 2. \end{cases}$$

(i) For $z = (n, x) \in \mathbb{Z} \times \mathbb{Z}^2$ define $\hat{z} = (-n, x) \in \mathbb{Z} \times \mathbb{Z}^2$ (the “reflection” of $z$ through the “plane” \{0\} \times \mathbb{Z}^2). Then, $z \mapsto \hat{z}$ is a bijection of $\mathbb{Z}_+ \times \mathbb{Z}^2$ onto $\mathbb{Z}_\pm \times \mathbb{Z}^2$, it is the identity when restricted to \{0\} \times \mathbb{Z}^2. For every $z \in \mathbb{Z}_+ \times \mathbb{Z}^2$, $\varphi_U(z) \leq \varphi_U(\hat{z})$.

(j) There are constants $d_1 = 3c_1$, $d_2 = \frac{c_1}{3^1}$, $d_3 = \frac{c_1}{18}$ and $d_4 = 2c_4$, such that the relation $z \in R_i(p, q)$ ($p, q \in \mathbb{Z}_+$, $i = 1, 2$) imply

$$G(0, z) \leq \varphi_U(z) \leq \begin{cases} d_1 e^{-d_2 3^q 3^{-q}} & \text{if } 2p + 1 \leq q, \\
 d_1 e^{-d_2 9^p} 9^{-p} & \text{if } q \leq 2p. \end{cases}$$

and

$$G(0, z) \geq \varphi_L(z) \geq \begin{cases} d_3 e^{-d_4 3^q 3^{-q}} & \text{if } 2p + 1 \leq q, \\
 d_3 e^{-d_4 9^p} 9^{-p} & \text{if } q \leq 2p. \end{cases}$$

(k) $G(0, z) \approx 9^{-p}$ with any constant being uniform for $z \in D_p$ and $p \in \mathbb{N}$.

### 4.3.1 Some preliminary sums of the Green’s function of $\Gamma_2(\lambda)$.

In what follows, the estimates in Remark 4.3 will be used, often without referencing to this remark.

**Proposition 4.9** Let $S$ and $S'$ be two independent network random walks started at 0 in $\Gamma_2(\lambda)$. Then, with any implicit constants being independent of $p \in \mathbb{N}$,

(a) $\sum_{z \in D_p} G(0, z)^2 \asymp 1$; and

(b) $\sum_{z \in D_p - D_p} G(0, z) G(0, -z) \asymp 1$.

(c) Also, there exists a constant $c > 0$ such that $\sum_{z \in D_p - D_p} G(0, z)^2 \leq c p$, for all $p \in \mathbb{N}$.
Proof. We know $G(0,z)^2 \leq 9^{-2p}$ and $\text{card}(D_p) \asymp 9^{2p}$ for $z \in D_p$ and the implicit constants independent of $p$; this proves the first assertion $\sum_{z \in D_p} G(0,z)^2 = 1$. There remains to show the other two estimates. First, it is clear that $\sum_{z \in D_p - D_p} G(0,z)G(0,-z) \geq G(0,0)^2$. Hence, there remains to prove the existence of constants $c > 0$ and $c' > 0$ such that $\sum_{z \in D_p - D_p} G(0,z)G(0,-z) \leq c$ and $\sum_{z \in D_p - D_p} G(0,z)^2 \leq c'p$, for all $p \in \mathbb{N}$. Set $Q(p) = D_p - D_p$ and $Q(p)^+ = Q(p) \cap (\mathbb{Z}_+ \times \mathbb{Z}_2)$. Consider the function $\varphi_U$ introduced in (4.8), we regard the constants $c_k$ and $d_k$ ($k = 1, \ldots, 4$) of (4.8) to be fixed during the argument. Then $\sum_{z \in Q(p)} G(0,z)G(0,-z) \leq 2 \sum_{z \in Q(p)} G(0,z)^2 \leq 2 \sum_{z \in Q(p)} \varphi_U(z)\varphi_U(-z)$ where the second inequality follows from considering the reflection $z \mapsto z$ (4.8)(i). To simplify notation, introduce $R(s,t)$ to be the union $R_1(s,t) \cup R_2(s,t)$, these sets being defined in (170). We know that there exists a constant $L > 0$ such that for all $s,t \in \mathbb{Z}_+$, $\text{card}(R(s,t)) \leq 9^{s+t}$. Then, one can write $Q(p)^+ \subset \bigcup_{(s,t)} R(s,t),$ where the indices run over all choices of $0 \leq s, t \leq p + 1$. Now, the sets $R(s,t)$ are pairwise disjoint and, therefore, we split $\sum_{z \in Q(p)} \varphi_U(z)\varphi_U(-z) \leq \sum_{(s,t) \in R(s,t)} \sum_{z \in Q(p)} \varphi_U(z)\varphi_U(-z) = P_1 + P_2$, and, in a similar fashion,

$\sum_{z \in Q(p)} \varphi_U(z)^2 \leq T_1 + T_2$, where $P_1$ and $T_1$ are the sums corresponding to $t < s$ and $P_2$ and $T_2$, the sums for $t \geq s$. We shall show that $P_1$ and $P_2$ are bounded by constants independent of $p$, and $T_1 \leq cp$ and $T_2 \leq c'p$ for a pair of constants $c, c' > 0$ independent of $p$; having this, we will conclude $\sum_{z \in D_p - D_p} G(0,z)G(0,-z) \leq c$ and $\sum_{z \in D_p - D_p} G(0,z)^2 \leq c'p$ for some constants $c > 0$ and $c' > 0$, as desired.

We now prove $P_1$ and $P_2$ are bounded by constants independent of $p$. Observe from (4.8)(g) that for all $z \in R(s,t)$, either $\varphi_U(z) \leq d_1 e^{-d_3' \ell} 3^{-t}$ or $\varphi_U(z) \leq d_1 e^{-d_3' \ell} 9^{-s}$. Also, for $z \in Q(p)$ we have $\varphi_U(-z) = c_1 e^{-c_2\|z\| \|z\|^{-1}}$. Therefore (use (4.8)(a)),

$$P_2 \leq \sum_{s=0}^{p+1} \sum_{t=s}^{p+1} \sum_{z \in R(s,t)} \varphi_U(z)\varphi_U(-z) \leq 3c_1 d_1 L \sum_{s=0}^{p+1} \sum_{t=s}^{p+1} (e^{-d_3' \ell} 9^s + e^{-d_3' \ell} 3^t) e^{-\frac{d_3'}{2} 3^t}$$

$$\leq 3c_1 d_1 L \sum_{t=0}^{\infty} e^{-\frac{d_3'}{2} 3^t} \left( \sum_{k=0}^{\infty} e^{-d_3 \ell} 9^k \right) 9^t + \sum_{k=0}^{\infty} e^{-d_3 \ell} 3^k 3^t < \infty.$$
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Therefore, with $C = d_1^2 L$,

$$
T_2 \leq C \sum_{s=0}^{p+1} \sum_{t=s}^{p+1} \left( e^{-2d_2 x^t} g_{s-t} + e^{-2d_2 y^t} g_{s-t} \right) g_{s+t} = C \sum_{s=0}^{p+1} \sum_{t=s}^{p+1} \left( e^{-2d_2 x^t} g_s + e^{-2d_2 y^t} g_t \right)
$$

\[ \leq C \sum_{k=0}^{\infty} e^{-d_2 x^k} g_k \sum_{k=0}^{\infty} e^{-d_2 x^k} + C(p+2) \sum_{k=0}^{\infty} e^{-2d_2 y^k} g_k \leq C' p, \]

where $C' = C \sum_{k=0}^{\infty} e^{-d_2 x^k} g_k \sum_{k=0}^{\infty} e^{-d_2 y^k} + 3C \sum_{k=0}^{\infty} e^{-2d_2 y^k} g_k < \infty$ is independent of $p$. Similarly, notice one can express $T_1 = \sum_{s=0}^{p+1} \sum_{t=s}^{p+1} \varphi_U(z)^2$, bearing this in mind and that $\varphi_U(z) \leq d_1 9^{-s}$ for $0 \leq t \leq s$ it follows immediately that, for some positive constants $c$ and $c'$, $T_1 \leq c \sum_{s=0}^{p+1} \sum_{t=0}^{s} 9^{s+t} 9^{-2s} \leq c' p$, as desired. $\square$

4.3.2 Estimates on the Green’s function on some regions of $\Gamma_2(\lambda)$.

Denote with $S$ and $S'$ two independent network random walks of $\Gamma_2(\lambda)$.

Consider the following region ($p \in \mathbb{N}$)

$$
(4.10) \quad U_p = \left\{ (n, x) \in \mathbb{Z}^3 : |n| \leq 4 \cdot 9^{p k_0}, \|x\| \leq 3(p+1) k_0 \right\},
$$

where $k_0$ is a positive integer to be determined later on. Define the “separating cylinders” (creating a barrier between $D_{p k_0}$ and $D_{(p+1) k_0}$ for $p \in \mathbb{N}$) to be the (vertex) boundary $F_p = \partial U_p$, and set $F_0 = \{0\}$. Having these sets, define ($p \in \mathbb{Z}_+$)

$$
(4.11) \quad T_p = \tau_{F_p}(S)
$$

the hitting time of $F_p$ by $S$. If $z \in \mathbb{Z}^3$, let

$$
(4.12) \quad p_z = \min\{p \mid z \in U_p\}.
$$

Having defined $p_z$, observe that when $S_0 = o$, then a.s. the sequence of random variables $(T_p)_{p \geq p_z}$ is increasing and each one of them is finite (by transience).

Next, by definition $F_p \cap U_p = \emptyset$. It will be useful to divide the set $F_p$ into several regions. Geometrically, the set $F_p$ can be thought as a cylinder and we need to consider the body and the two bases separately. Analytically, if $(n, x) \in F_p$ then $|n| > 4 \cdot 9^{p k_0}$ or else $\|x\| > 3(p+1) k_0$. We can specialise further for either $|n| = 4 \cdot 9^{p k_0} + 1$ and $\|x\| \leq 3(p+1) k_0$ or else, $|n| \leq 4 \cdot 9^{p k_0}$ and $3(p+1) k_0 \leq \|x\| \leq 3(p+1) k_0 + 1$. Thus, the exponential scaling being considered permit us to assume that $\|z\|$, for $z \in F_p$, satisfies:

(a) $|n| = 4 \cdot 9^{p k_0}$ and $\|x\| \leq 3(p+1) k_0$, or

(b) $|n| < 4 \cdot 9^{p k_0}$ and $\|x\| = 3(p+1) k_0$.

With this assumption, we introduce the following sets. The “left-hand base of the cylinder $F_p$” is the set of $z \in F_p$ such that $n = -4 \cdot 9^{p k_0}$; the “right-hand base of the cylinder $F_p$,” is the set of $z \in F_p$ such that $n = 4 \cdot 9^{p k_0}$. The points in $F_p$ that belong to neither the left nor the right-hand base of the cylinder will be referred to as the “body of the cylinder $F_p$,” in other words, the body of the cylinder is the set of points $z \in F_p$ such that $|n| < 4 \cdot 9^{p k_0}$ and $\|x\| = 3(p+1) k_0$. 

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Define \((T'_p)\) as above changing the process \(S\) for the process \(S'\). Observe that \((T_p)\) and \((T'_p)\) are independent families.

Having the times \((T_p)\) and \((T'_p)\) we now consider “sections” of the network random walks and the number of intersections in \(D_{pk_0}\) of two such sections. Thus, define \((p \in \mathbb{N})\)

\[
M_p = \sum_{z \in D_{pk_0}} \sum_{m=T_{p-1}}^{T_p} \sum_{m'=T'_{p-1}}^{T'_p} \mathbb{1}(S_m = z) \mathbb{1}(S'_{m'} = z).
\]

Observe that \(M_p\) counts the number of intersections of the two random walks inside the region \(D_{pk_0}\) starting in the separating cylinder \(F_{p-1}\). We remark that, if \(S_0 = o\), then \(S\) cannot be inside \(D_{pk_0}\) before the time \(T_{p-1}\) provided \(p - 1 \geq p_0\); similarly, \(S'\) cannot be inside \(D_{pk_0}\) before time \(T'_{p-1}\) provided \(S'_0 = o'\) and \(p - 1 \geq p_0'\).

We will study the Green’s function restricted to the region \(U_p : G_{U_p}(0, z) = E^0 \left( \sum_{m=0}^{T_p-1} \mathbb{1}(S_m = z) \right)\), for \(z \in \mathbb{Z}^3\). For consistency we denote \(U_0 = \{0\}\). For the sake of simplicity, we will write \(G_p = G_{U_p}\).

**Proposition (4.14)** For every \(p \in \mathbb{N}\), \(D_{pk_0} - U_{p-1} \subset U_p\). Moreover, for any \(\varepsilon > 0\) there exists an index \(k_0 = k_0(\varepsilon)\) such that for some \(p_0 \in \mathbb{N}\) the relations \(p \geq p_0\) and \(z \in D_{pk_0} - U_{p-1}\) imply \(G(0, z) \leq (1 + \varepsilon)G_p(0, z)\).

**Proof.** The proof contains several steps. We begin by noticing that for \(z_1 \in D_{pk_0}\) and \(z_2 \in U_{p-1}\) one has by (4.7) and (4.10) \(g^{pk_0} < n_1 \leq 2 \cdot g^{pk_0}\) and \(\|x_1\| \leq g^{pk_0}\), and \(-4 \cdot g^{(p-1)k_0} \leq n_2 \leq 4 \cdot g^{(p-1)k_0}\) and \(\|x_2\| \leq 3^{pk_0}\), which implies

\[\text{(4.14.1) Any} (n, x) \in D_{pk_0} - U_{p-1} \text{ satisfies} \quad \frac{1}{2} \cdot g^{pk_0} \leq n \leq \frac{5}{2} g^{pk_0}, \quad \|x\| \leq 2 \cdot 3^{pk_0}.\]

In particular, \(z_1 - z_2\) belongs to the region \(U_p\) so this establishes the first assertion in (4.14).

We now prove the second assertion of (4.14). First, \(G(0, z) = G_p(0, z) + E^0 (G(S_{T_p}, z))\). Observe that \(S_{T_p}\) is in \(F_p\) and, therefore, we need to estimate \(G(z_3, z)\) for \(z_3 \in F_p\). This will be done in the upcoming three lemmas.

We will pause momentarily the proof of the proposition in order to establish these lemmas and resume it once they have been proven.

**Lemma (4.14.2) (a) For every \(\varepsilon > 0\), there exists a \(p_0 = p_0(\varepsilon)\) such that for all \(p \geq p_0\), all \(k_0 \in \mathbb{N}\), and all \(z_1 \in D_{pk_0}, z_2 \in U_{p-1}\) and \(z_3 \in F_p\) for which \(n_1 - n_2 - n_3 \leq 2 \cdot 3^{(p+1)k_0}\) and \(z_3\) does not lie in the left-hand base of \(F_p\), we have \(G(z_3, z_1 - z_2) \leq \varepsilon G(0, z_1 - z_2)\).

(b) There exists a constant \(c > 0\) such that for all \(p \in \mathbb{N}\) and all \(z_1 \in D_{pk_0}, z_2 \in U_{p-1}\) and \(z_3 \in F_p\) for which \(n_1 - n_2 - n_3 \geq 2 \cdot 3^{(p+1)k_0}\) and \(z_3\) does not lie in the left-hand base of \(F_p\) then \(G(z_3, z_1 - z_2) \leq c 9^{-k_0} G(0, z_1 - z_2)\).

**Proof of lemma.** Suppose the constants \(c_k\) \((1 \leq k \leq 4)\) of (2.22) are given. By (4.14.1) and the estimates on the Green’s function, we have \(G(0, z_1 - z_2) \approx g^{-pk_0}\).

We prove part (a) first. Assume \(z_1, z_2, z_3\) are as stated. Consider first the case \(n_3 < 4 \cdot 3^{pk_0}\), that is to say, consider first when \(z_3\) is not in the right-hand base of \(F_p\). Then \(\|x_3\| = 3^{(p+1)k_0}\), which in turn yields \(\frac{1}{3} \cdot 3^{(p+1)k_0} \leq \|x_1 - x_2 - x_3\| \leq 2 \cdot 3^{(p+1)k_0}\). Suppose first that \(n_1 - n_2 - n_3 \leq \frac{1}{3} \cdot 3^{(p+1)k_0}\), then \(z_1 - z_2 - z_3\) is in \(S_1\), implying by (2.22)

\[
G(z_3, z_1 - z_2) \leq c_1 e^{-c_2 \|x_1 - x_2 - x_3\|} \|z_1 - z_2 - z_3\|^{-1} \leq 3 c_1 e^{-\frac{c_2}{3} 3^{(p+1)k_0}} 3^{-(p+1)k_0} = 3 c_1 e^{-c_3 3^{(p+1)k_0}} 9^{-pk_0} \leq 3 c_1 e^{-c_3 3^{pk_0}} 3^{pk_0} 9^{-pk_0}.
\]
Define $u_1(t) = 3c_1e^{-\frac{c}{3}n^3}3^t$, so that $G(z_3, z_1 - z_2) \leq u_1(pk_0)9^{-pk_0}$, for $z_1$, $z_2$ and $z_3$ as stated. Next, suppose that $\frac{1}{3} \cdot 3^{(p+1)k_0} \leq n_1 - n_2 - n_3 \leq 2 \cdot 3^{(p+1)k_0}$. Then,

$$G(z_3, z_1 - z_2) \leq c_1 \left(e^{-c_2\|z_1 - z_2 - z_3\|} + e^{-c_2\frac{\|z_1 - z_2 - z_3\|^2}{n_1 - n_2 - n_3}}\right)\|z_1 - z_2 - z_3\|^{-1}$$

$$\leq 3c_1 \left(e^{-\frac{c}{3}3^{(p+1)k_0}} + e^{-\frac{c}{3}3^{(p+1)k_0}}\right)3^{-(p+1)k_0}$$

$$= \left[3c_1 \left(e^{-\frac{c}{3}3^{(p+1)k_0}} + e^{-\frac{c}{3}3^{(p+1)k_0}}\right)3^{(p-1)k_0}\right]9^{-pk_0}$$

$$\leq \left[3c_1 \left(e^{-\frac{c}{3}3^{pk_0}} + e^{-\frac{c}{3}3^{pk_0}}\right)3^{pk_0}\right]9^{-pk_0} = u_2(pk_0)9^{-pk_0},$$

where $u_2(t)$ is defined to be $3c_1 \left(e^{-\frac{c}{3}3^t} + e^{-\frac{c}{3}3^t}\right)3^t$. Consider now the case where $z_3$ is in the right-hand base of the cylinder, that is to say, assume $n_3 = 4.9pk_0$ and $\|x_3\| \leq 3^{(p+1)k_0}$ but in this case, $-\frac{7}{2} \cdot 9^{pk_0} \leq n_1 - n_2 - n_3 \leq -\frac{3}{2} \cdot 9^{pk_0}$ and again, by (2.22), $G(z_3, z_1 - z_2) \leq c_1e^{-c_2\|n_1 - n_2 - n_3\|/n_1 - n_2 - n_3} \leq 2c_1e^{-\frac{c}{3}9^{pk_0}9^{-pk_0}}$. Here we set $u_3(t) = \frac{2c_1}{3}e^{-\frac{c}{3}3^t}$, so that in this case $G(z_3, z_1 - z_2) \leq u_3(pk_0)9^{-pk_0}$. Notice that in all three cases above, we bounded $G(z_3, z_1 - z_2) \leq u_i(pk_0)9^{-pk_0}$, and it is clear that $u_i(t) \to 0$ as $t \to \infty$ in each case. Then, there exists a $t_0$ such that sup max $u_i(t) \leq \varepsilon$. We can define $p_0$ to be the least integer greater than or equal to $t_0$; this $p_0$ works for all $k_0$ since $pk_0 \geq p$. This finishes part (a).

We now prove part (b). Suppose that $z_1$, $z_2$, $z_3$ are as stated in (b). This implies the relation $n_1 - n_2 - n_3 \geq 2 \cdot 3^{(p+1)k_0}$. By (14.1), we have $n_1 - n_2 - n_3 \leq 7 \cdot 9^{pk_0}$, and, therefore $n_1 - n_2 - n_3$ lies in an interval of the form $J_\alpha = [\alpha - 3, \alpha + 3] \cdot 3^{(p+1)k_0}$, where $\alpha = 3, \ldots, 7 \cdot 3^{(p-1)k_0}$. Regardless the value of $\alpha$, the relation $n_1 - n_2 - n_3 \in J_\alpha$ implies $z_1 - z_2 - z_3 \in S_2$ and these relations imply

$$G(z_3, z_1 - z_2) \leq c_1e^{-\frac{c_2\|z_1 - z_2 - z_3\|^2}{n_1 - n_2 - n_3}}(n_1 - n_2 - n_3)^{-1} \leq 2c_1e^{-\frac{c_2\|z_1 - z_2 - z_3\|^2}{n_1 - n_2 - n_3}} \cdot \alpha^{-3} \cdot 3^{(p+1)k_0}.$$

Observe that for any constant $c > 0$, the function $u_e(t) = t^{-1}e^{-ct^{-1}}$ for $t > 0$ has an absolute and global maximum at $t = c$. Its maximum equals $u_e(c) = e^{-1}c^{-1}$. We deduce $G(z_3, z_1 - z_2) \leq 2c_1e^{-\frac{c_2\|z_1 - z_2 - z_3\|^2}{n_1 - n_2 - n_3}} \cdot \alpha^{-3} \cdot 3^{(p+1)k_0} \leq c\cdot 9^{-k_0}9^{-pk_0}$, the desired conclusion since $G(0, z_1 - z_2) < 9^{-pk_0}$, with any implicit constant independent of $p$ and of $k_0$.

**Lemma (4.14.3) For any $\varepsilon > 0$, there exist indices $k_0 = k_0(\varepsilon) \in \mathbb{N}$ and $p_0 = p_0(\varepsilon) \in \mathbb{N}$ such that the relations $p \geq p_0$ and $z \in D_{pk_0} - U_{p-1}$ imply $\sup_{n_3 > -4 \cdot 9^{pk_0}} G(z_3, z) \leq \varepsilon G(0, z)$.**

**Proof of lemma.** By (4.14.2), given $\varepsilon > 0$ there is a $p_0 \in \mathbb{N}$ such that if $p \geq p_0$, $k_0 \in \mathbb{N}$ and for all $z_1 \in D_{pk_0}$, $z_2 \in U_{p-1}$ and $z_3 \in F_p$ with $n_3 > -4 \cdot 9^p$, we either have $G(z_3, z_1 - z_2) \leq \varepsilon G(0, z_1 - z_2)$ or $G(z_3, z_1 - z_2) \leq c\cdot 9^{-k_0}G(0, z_1 - z_2)$, where the constant $c$ is universal. Choose $k_0 \in \mathbb{N}$ such that $c\cdot 9^{-k_0} \leq \varepsilon$. Hence, $G(z_3, z_1 - z_2) \leq \varepsilon G(0, z_1 - z_2)$ uniformly in $z_3 \in F_p$ with $n_3 > -4 \cdot 9^p$ for all $p \geq p_0$, which is the conclusion to be reached.

**Lemma (4.14.4) For any $\varepsilon > 0$, there exist indices $k_0 = k_0(\varepsilon) \in \mathbb{N}$ and $p_0 = p_0(\varepsilon) \in \mathbb{N}$ such that the relation $p \geq p_0$ implies $\mathbb{F}^0 \left( S_{\alpha_p} \in \{ z \in F_p : n = -4 \cdot 9^p \} \right) \leq \varepsilon$. Also, there is a universal constant $c > 0$ such that for all $p \in \mathbb{N}$ and all $z \in D_{pk_0} - U_{p-1}$, $\sup_{n_3 > -4 \cdot 9^p} G(z_3, z) \leq \varepsilon G(0, z)$.**
Proof of lemma. Set $A_p = \{ z \in F_p : n = -4 \cdot 9^{p_k_0} \}$. Consider the network random walk $S$ in $\Gamma_2(\lambda)$ started at zero and define $B = \Pr(B)$. By \( \textbf{(4.17)} \), \( \Pr(S_{T_p} \in A_p) \leq \Pr(B) \leq 4 \cdot 9^{p_k_0} \). This proved the first claim. To prove the second claim, use \( \textbf{(4.14.1)} \), if $z_3 \in A_p$ and $z \in D_{p_k_0} - U_{p-1}$ then $G(z_3, z) \sim 9^{p_k_0}$ with any implicit constant being universal in $p, k_0, z, z_3$. Similarly, $G(0, z) \sim 9^{p_k_0}$ for $z \in D_{p_k_0} - U_{p-1}$ with any implicit constant universal. The second claim follows readily.

We are ready to continue with the proof of Proposition \( \textbf{(4.14)} \). Let us write $A_p = \{ z \in F_p : n_3 = -4 \cdot 9^p \}$ for the left-hand base of $F_p$ and $B_p = F_p \setminus A_p$ for the remainder of the separating cylinder. For the time being, consider $\varepsilon > 0$ to be any positive number. By Lemmas \( \textbf{(4.14.2)} \) and \( \textbf{(4.14.4)} \), there are indices $k_0$ and $p_0$, only depending on $\varepsilon$, and a universal constant $c > 0$ (not depending on any index) satisfying for all $p \geq p_0$, $z_1 \in D_{p_k_0}$, $z_2 \in U_{p-1}$ sup $G(z_3, z_1 - z_2) \leq c G(0, z_1 - z_2)$, $P^0(S_{T_p} \in A_p) \leq \varepsilon$ and for all $z \in D_{p_k_0} - U_{p-1}$, sup $G(z_3, z) \leq c G(0, z)$. Write now, for $z \in D_{p_k_0} - U_{p-1}$,

\[
G(0, z) = G_p(0, z) + \mathbb{E}^0(G(S_{T_p}, z) \mathbb{1}_{A_p}(S_{T_p}) + G(S_{T_p}, z) \mathbb{1}_{B_p}(S_{T_p})) \\
\leq G_p(0, z) + c G(0, z) P^0(S_{T_p} \in A_p) + \varepsilon G(0, z) P^0(S_{T_p} \in B_p) \\
\leq G_p(0, z) + (c + 1) \varepsilon G(0, z).
\]

Hence, for any $\varepsilon > 0$, there are indices $k_0$ and $p_0$, depending only on $\varepsilon$, so that the relations $p \geq p_0$ and $z \in D_{p_k_0} - U_{p-1}$ imply $G(0, z) \leq (1 + \varepsilon) G_p(0, z)$, which yields $G(0, z) \leq (1 + \varepsilon) G_p(0, z)$. \( \square \)

### 4.4 Dimension $d = 2$: “logarithmic scale of crossings” of $\Gamma_2(\lambda)$

We will establish a lower bound on the probability of how frequently two random walk paths in $\Gamma_2(\lambda)$ will cross each other.

Invoking Proposition \( \textbf{(4.14)} \), for $\varepsilon = 1$ there exists two indices $k_0$ and $p_0$, depending only on $\varepsilon$, such that the conclusions of this proposition hold. For the remainder of this chapter, we assume both $k_0$ and $p_0$ are fixed and given as stated.

**Theorem (4.15)** Consider two independent network random walks $S$ and $S'$ in $\Gamma_2(\lambda)$, with respective starting points $o$ and $o'$. Define the random variables $T_p$ and $T'_p$ in \( \textbf{(4.11)} \) using $S$ and $S'$, accordingly; now introduce $M_p$ as in \( \textbf{(4.13)} \). Consider now the $\sigma$-fields $E_p = F^S_{T_p} \vee F^S'_{T_p}$. Define the event $A_p = \{ M_p > 0 \}$ and the random variable $q_{p+1} = \Pr(A_{p+1} \mid E_p)$. Then, $A_p \subseteq E_p$ and there exists a positive integer $P = P(o, o')$ and a universal constant $c > 0$ (depending neither on $o$ nor on $o'$) such that, almost surely, $q_p > \frac{c}{p}$ for all $p \geq P$.

**Proof.** It is clear that $A_p \subseteq E_p$. Consider $p_1 = \max(p_0, p_{o'})$, that is, $p_1$ is the first index such that $o$ and $o'$ belong to $U_{p_1}$. Suppose $p$ is any integer $\geq \max(p_0, p_1)$. By \( \textbf{(4.3)} \), we have $q_{p+1} = \Pr(A_{p+1} \mid S_{T_p}, S'_{T_p})$. It is now straightforward to check that

\[
q_{p+1} = \sum_{(z_1, z_2) \in \mathbb{F}_p^2} \Pr(A_{p+1} \mid S_{T_p} = z_1, S'_{T_p} = z_2) \mathbb{1}_{\{S_{T_p} = z_1, S'_{T_p} = z_2\}}.
\]

We will use \( \textbf{(4.3)} \). By the strong Markov property for a pair of independent Markov processes \( \textbf{(4.3)} \) gives (inside the expectation on the right of the first equality we use abuse of notation and assume $(S_m)$ and $(S'_m)$
are independent network random walks started at zero)

\[
\mathbb{E} \left( M_{p+1} \mid S_{T_p} = z_1, S'_{T_p} = z_2 \right) = \sum_{z_3 \in \mathcal{D}_k} \mathbb{E} \left( \sum_{m_0=0}^{T_{p+1}} \sum_{m=0}^{T'_{p+1}} \mathbb{I} \{ S_m + z_1 = z_3 \} \mathbb{I} \{ S'_m + z_2 = z_3 \} \right)
\]

\[
\approx \sum_{z_3 \in \mathcal{D}_k} G_{p+1}(0, z_3 - z_1) G_{p+1}(0, z_3 - z_2)
\]

\[
\approx \sum_{z_3 \in \mathcal{D}_k} G(0, z_3 - z_1) G(0, z_3 - z_2), \quad \text{by (4.14)}
\]

We shall now estimate the second moment of \( M_p \) with respect to the conditional measure. We have

\[
M_{p+1}^2 = \sum_{(z_3, z_4) \in \mathcal{D}_k^2} \sum_{m_1, m_2 = T_p}^{T_{p+1}} \sum_{m'_1, m'_2 = T'_p}^{T'_{p+1}} \mathbb{I} \{ S_m = z_3, S_m = z_4 \} \mathbb{I} \{ S'_{m'} = z_3, S'_{m'} = z_4 \}.
\]

Now apply the conditional expectation, by the strong Markov property (5.3) and independence between \( S \) and \( S' \) (we again employ abuse of notation and assume \( S \) and \( S' \) are two independent network random walks started at zero after having used the strong Markov property for two independent Markov processes)

\[
\mathbb{E} \left( M_{p+1}^2 \mid S_{T_p} = z_1, S'_{T_p} = z_2 \right) = \sum_{(z_3, z_4) \in \mathcal{D}_k^2} \mathbb{E} \left( \sum_{m_1, m_2 = 0}^{T_{p+1}} \mathbb{I} \{ S_m = z_3 - z_1, S_m = z_4 - z_1 \} \right)
\]

\[
\times \mathbb{E} \left( \sum_{m'_1, m'_2 = 0}^{T'_{p+1}} \mathbb{I} \{ S'_{m'_1} = z_3 - z_2, S'_{m'_2} = z_4 - z_2 \} \right)
\]

\[
\leq B_1 + B_2 + B_3 + B_4,
\]

in which

\[
B_1 = \sum_{m_1 \leq m_2, m'_1 \leq m'_2} \mathbb{P}(S_{m_1} = z_3 - z_1, S_{m_2} = z_4 - z_1) \mathbb{P}(S'_{m'_1} = z_3 - z_2, S'_{m'_2} = z_4 - z_2),
\]

\[
B_2 = \sum_{m_1 \leq m_2, m_1 \geq m'_2} \mathbb{P}(S_{m_1} = z_3 - z_1, S_{m_2} = z_4 - z_1) \mathbb{P}(S'_{m'_1} = z_3 - z_2, S'_{m'_2} = z_4 - z_2),
\]

\[
B_3 = \sum_{m_1 \geq m_2, m_1 \leq m'_2} \mathbb{P}(S_{m_1} = z_3 - z_1, S_{m_2} = z_4 - z_1) \mathbb{P}(S'_{m'_1} = z_3 - z_2, S'_{m'_2} = z_4 - z_2),
\]

and

\[
B_4 = \sum_{m_1 \geq m_2, m_1 \geq m'_2} \mathbb{P}(S_{m_1} = z_3 - z_1, S_{m_2} = z_4 - z_1) \mathbb{P}(S'_{m'_1} = z_3 - z_2, S'_{m'_2} = z_4 - z_2).
\]

We handle each sum separately.
(a) By independent increments and writing $p^k(z, z') = P(S_k = z' \mid S_0 = z)$,

\[
B_1 = \sum_{m_1 \leq m_2} p^{m_1}(0, z_3 - z_1)p^{m_2 - m_1}(0, z_4 - z_3)p^m(0, z_3 - z_2)p^{m_2 - m_1}(0, z_4 - z_3) \leq \sum_{(z_3, z_4) \in \mathbb{D}^2} G(0, z_3 - z_1)G(0, z_4 - z_3)G(0, z_4 - z_2),
\]

the last equality being an immediate consequence of Lebesgue-Tonelli’s theorem and the definition of the Green’s function of $\Gamma_2(\lambda)$. By virtue of \(4.14.1\) and \(4.9\) we can proceed further,

\[
B_1 \leq \sum_{(z_3, z_4) \in \mathbb{D}^2} 9^{-2(p+1)k_0} G(0, z_4 - z_3)^2 \leq c k_0(p + 1) \sum_{z_3 \in \mathbb{D}^2} 9^{-2(p+1)k_0} \asymp p.
\]

(b) Here we get

\[
B_2 = \sum_{(z_3, z_4) \in \mathbb{D}^2} G(0, z_3 - z_1)G(0, z_4 - z_3)G(0, z_4 - z_2)G(0, z_3 - z_4)
\]

\[
\leq \sum_{z_3 \in \mathbb{D}^2} \sum_{z_4 \in \mathbb{D}^2} 9^{-2(p+1)k_0} G(0, z_4 - z_3)G(0, z_3 - z_4)
\]

\[
\leq \sum_{z_3 \in \mathbb{D}^2} 9^{-2(p+1)k_0} \asymp 1,
\]

where the second $\asymp$ is a consequence of \(4.9\).

(c) Similarly to $B_2$, we get $B_3 \asymp 1$.

(d) As with $B_1$, $B_4 \leq cp$, for some universal constant $c > 0$.

Therefore, we have shown that there exist two positive constants $c$ and $c'$, such that, whenever $p \geq \max(p_0, p_1)$ and $(z_1, z_2) \in \mathbb{F}^2_p$, $E\left(M_{p+1} \mid S_{T_p} = z_1, S'_{T_p} = z_2\right) \geq c$ and $E\left(M^2_{p+1} \mid S_{T_p} = z_1, S'_{T_p} = z_2\right) \leq c'p$. By \(4.3\), we finally reach that there exists a constant $c > 0$ and a $P = \max(p_0, p_1)$ such that, for all $p \geq P$ and all $(z_1, z_2) \in \mathbb{F}^2_p$, $P\left(A_{p+1} \mid S_{T_p} = z_1, S'_{T_p} = z_2\right) \geq \frac{c}{p}$, comparing with the expression for $q_{p+1}$ in \(\ast\), we have substantiated the claims.

**Corollary (4.15.1)** Let $S$ and $S'$ be two independent network random walks in $\Gamma_2(\lambda)$, with starting points $S_0 = o$ and $S'_0 = o'$. There is a universal constant $c > 0$ and an index $P = P(o, o')$, such that for all $p \geq P$, the probability that the two random walk paths $(S_m)_{m=T_{p-1}, \ldots, T_p}$ and $(S'_m)_{m'=T_{p-1}, \ldots, T'_p}$ cross each other inside $D_{p0}$ is at least $\frac{c}{p}$. \hfill \square

The following corollary follows using \(4.3\) and \(4.6\).

**Corollary (4.15.2)** Any two independent network random walks on $\Gamma_2(\lambda)$ will have infinitely many intersections.

**Remark (4.16)** There is a much simpler way to obtain Corollary \(4.15.2\). Denote $S = (B, X)$, that is, $S_n = (B_n, X_n)$ for $n \in \mathbb{Z}_+$; here $B$ is the biased random walk on $\mathbb{Z}$ and $X$ a random walk on $\mathbb{Z}^2$. Define $\zeta_k = \tau_{(k) \times \mathbb{Z}}(S)$ for $k \in \mathbb{Z}_+$, which is the time at which the random walk $B$ visits $k$. With these hitting times, consider $Y_n = X_{\zeta_n}$. Similarly, define $Y'_n$ using $S' = (B', X')$, which is an independent copy of $S$. In a similar manner as theorem \(4.2\), it is seen that $Y_n = Y'_n$ for infinitely many indices $n$. Having this, it is now clear that $S_n = S'_n$, for infinitely many pairs $(n, n')$. Of course, this result is much coarser than \(4.15.1\).
5 One endedness for \( \Gamma_d(\lambda) \)

Let \( G \) be a directed infinite tree. A \textit{ray} is an infinite path that does not repeat vertices; any two rays of \( G \) will be disjoint or will eventually merge, this creates equivalence classes on the set of rays of \( G \). Any of these classes is called an \textit{end}.

5.1 One endedness for \( \Gamma_1(\lambda) \)

We refer the reader to [LP16], see Section 6.5, Section 9.2 and Section 10.3 for the definitions and notations of planar graph and duality of graphs.

**Theorem (5.1)** UST in \( \Gamma_1(\lambda) \) \(3.7\) has, almost surely, one end.

**Proof.** It is easy to see that the dual network \( \Gamma_1(\lambda) \) of \( \Gamma_1(\lambda) \) is the network image of the latter under the reflection \( (n,x) \mapsto (-n,x) \). In particular, the wired and free uniform spanning forest of \( \Gamma_1(\lambda) \) coincide and the number of USF trees in \( \Gamma_1(\lambda) \) is the same as that of \( \Gamma_1(\lambda) \); that is to say, for almost every realisation, just one tree, by \(3.9\). Bearing this in mind and the duality between UST in \( \Gamma_1(\lambda) \) with UST in \( \Gamma_1(\lambda) \), we see at once that the tree in \( \Gamma_1(\lambda) \) is almost surely one ended; for if it were two ended, it would split the plane into two disconnected regions making it impossible for the dual tree to be a tree. \( \square \)

**Remark (5.2)** The core idea of the proof of previous theorem is implicit in \([LP16\), Theorem 10.36].

5.2 One endedness for \( \Gamma_2(\lambda) \)

We will prove the following theorem: UST in \( \Gamma_2(\lambda) \) has, almost surely, one end.

The proof is rather involved and to facilitate the reading of it we are going to develop it in several steps and draw the conclusion at the end.

5.3 Some preliminary results on Markov processes

Consider a discrete time Markov process \((S_n, \mathcal{F}_n)_{n \in \mathbb{Z}_+}\), with values on a metrisable separable space \( E \), endowed with Borel sets \( \mathcal{B} \) as \( \sigma \)-algebra. Let \( \eta \) be an a.s. finite stopping time for this Markov process. We may construct the \( \sigma \)-field \( \mathcal{F}_\eta = \mathcal{F}_n^\eta \) of all events \( G \in \mathcal{B} \) such that \( G \cap \{ \eta \leq n \} \in \mathcal{F}_n \) (“stopped filtration up to time \( \eta \)”). In a similar way, we may construct \( S_\eta : \omega \mapsto S_{\eta(\omega)}(\omega) \). If no filtration is specified, it is assumed the canonical filtration is used \( \mathcal{F}_n = \sigma(S_k; 0 \leq k \leq n) \). Consider now the product space \( E^{\mathbb{Z}_+} \), which is also a metrisable separable space, and its Borel \( \sigma \)-field coincides with \( \bigotimes_{n \in \mathbb{Z}_+} \mathcal{B} \). For any bounded measurable function \( \varphi : E^{\mathbb{Z}_+} \to \mathbb{R} \), there exists a version of the conditional expectation \( \mathbb{E}_x \left( \varphi(S_j)_{j \in \mathbb{Z}_+} \mid S_0 = x \right) \); denote this function as \( \mathbb{E}_x^\eta \left( \varphi(S_j)_{j \in \mathbb{Z}_+} \mid \mathcal{F}_n \right) = \mathbb{E}_n^\eta \left( \varphi(S_j)_{j \in \mathbb{Z}_+} \right) \). The strong Markov property states that \( \mathbb{E}_x \left( \varphi(S_j)_{j \in \mathbb{Z}_+} \mid \mathcal{F}_n \right) = \mathbb{E}_x \left( \varphi(S_j)_{j \in \mathbb{Z}_+} \right) \).

The following results follow from well-known techniques (monotone-class, Dynkin’s theorem, etc.). The reader may consult [MD20], propositions (1.12.3) and (1.12.4).

**Proposition (5.3)** Suppose \( S = (S_n)_{n \in \mathbb{Z}_+} \) and \( S' = (S'_n)_{n \in \mathbb{Z}_+} \) are two independent Markov processes, defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values on some metrisable, separable space \( E \), endowed with Borel \( \sigma \)-field \( \mathcal{B} \). Let \( \psi : E^{\mathbb{Z}_+} \times E^{\mathbb{Z}_+} \to \mathbb{R} \) be a bounded measurable function, relative to the Borel sets of both \( \mathbb{R} \) and \( E^{\mathbb{Z}_+} \times E^{\mathbb{Z}_+} \). Define \( v_\psi(x,x') = \mathbb{E}_x \left( \psi(S_j, S'_j)_{(j,j') \in \mathbb{Z}_+^2} \mid S_0 = x, S'_0 = x' \right) \), for \((x,x') \in E^2\), that is \( v_\psi \)
is a version of \( \psi \left( S_j, S'_j \right) \) for \( j \geq 0 \) given \((S_0, S'_0)\). Suppose \( \eta \) and \( \eta' \) are two stopping times, \( \eta \) relative to \( S \) and \( \eta' \), to \( S' \). With \( \psi \) as before, 

\[
E \left( \psi \left( S_j, S'_j \right) \right| \mathcal{F}_n^S \vee \mathcal{F}_{\eta'}^S) = v_\psi (S_n, S'_n). 
\]

**Proposition (5.4)** Let \( E = \mathbb{R}^d \) (or any separable, metrisable, Fréchet space). Suppose \((S_n)_{n \in \mathbb{Z}_+}\) and \((S'_n)_{n \in \mathbb{Z}_+}\) are two \( E \)-valued stochastic processes with stationary independent increments. Suppose that \( \eta \) and \( \eta' \) are stopping times relative to \( S \) and \( S' \), respectively. Then, \( \mathcal{F}_\eta^S \vee \mathcal{F}_{\eta'}^S \) is independent of \( \mathcal{F}_0^S \vee \mathcal{F}_{\eta'}^S \), with \( \mathcal{G}_\eta^S = \sigma(S_{n+j} - S_n, j \in \mathbb{Z}_+) \), and a corresponding definition for \( \mathcal{G}_{\eta'}^S \) with \( (S'_j, \eta') \) replacing \( (S, \eta) \). In particular, if \( A \) is an event depending on the paths \((S_j)_{0 \leq j \leq \eta}\) and \((S'_j)_{0 \leq j \leq \eta'}\) and \( B \), depending on the paths \((S_{n+j} - S_n)_{j \in \mathbb{Z}_+}\) and \((S'_{n+j} - S'_n)_{j \in \mathbb{Z}_+}\), then \( A \) and \( B \) are independent.

### 5.3.1 Invariance of the chronology of Wilson’s algorithm rooted at infinity.

We start with a theorem related to the study of the chronological construction of Wilson’s algorithm rooted at infinity on general networks. Denote by \( \mathcal{G} \) the (metrisable compact) topological space of spanning subgraphs of the graph \( G \) and recall the notations from the introduction. With these notations, we construct the following random object \( \left( \tilde{\mathcal{F}}^\xi, (L_k^\xi)_{k \in \mathbb{R}} \right) = \left( \tilde{\mathcal{F}}^\eta, L_1^\xi, \ldots, L_M^\xi \right) \). This random object is \( \mathcal{G} \times \mathcal{G}^\mathbb{Z}_+ \)-valued. Notice that the sequence \( (L_k^\xi)_{k \in \mathbb{R}} \) may be referred to as a “chronology of the WSF-forest.” Wilson’s algorithm rooted at infinity gives \( \tilde{\mathcal{F}}^\xi \sim \tilde{\mathcal{F}}^\eta \). Observe that if \( \xi \) and \( \eta \) are two orderings satisfying \( \xi(j) = \eta(j) \) for \( 1 \leq j \leq M \) then \( (L_1^\xi, \ldots, L_M^\xi) = (L_1^\eta, \ldots, L_M^\eta) \) a.s. Furthermore, in the finite case we may run Wilson’s algorithm with stacks \([1,1]\) to obtain that \( \tilde{\mathcal{F}}^\xi \sim \tilde{\mathcal{F}}^\eta \) a.s., thus in the finite case \( \left( \tilde{\mathcal{F}}^\xi, L_1^\xi, \ldots, L_M^\xi \right) = \left( \tilde{\mathcal{F}}^\eta, L_1^\eta, \ldots, L_M^\eta \right) \).

**Theorem (5.5)** If \( \xi \) and \( \eta \) are two orderings of the vertex set of \( G \) such that \( \xi(j) = \eta(j) \) for \( 1 \leq j \leq M \), then \( \left( \tilde{\mathcal{F}}^\xi, L_1^\xi, \ldots, L_M^\xi \right) \sim \left( \tilde{\mathcal{F}}^\eta, L_1^\eta, \ldots, L_M^\eta \right) \). (The two random objects have the same law.)

The proof of this theorem follows from finite approximation, see [MD20, Theorem (5.2.1)].

**Corollary (5.5.1)** The statistical properties of events depending on WSF and its first branches are not affected by reordering vertices not yet searched. In other words, if \( \varphi : \mathcal{G}^{M+1} \rightarrow \mathbb{R} \) is any bounded measurable function or any non-negative measurable function, and \( \xi, \eta : \mathbb{N} \rightarrow V \) are two orderings of the vertices such that \( \xi(j) = \eta(j) \) for \( 1 \leq j \leq M \), then 

\[
E \left( \varphi \left( \tilde{\mathcal{F}}^\xi, L_1^\xi, \ldots, L_M^\xi \right) \right) = E \left( \varphi \left( \tilde{\mathcal{F}}^\eta, L_1^\eta, \ldots, L_M^\eta \right) \right).
\]

The importance of the previous corollary is that one can see a partial forest, and to estimate probabilities of successive branches, we can choose the next vertices to depend on the partial tree. (This is no surprise, since Wilson’s algorithm with stacks allows for a stronger result in the finite case: the next vertices may depend on the partial tree and the final tree cannot change.)

### 5.3.2 Preliminary estimates in the proof of one-endedness for \( \Gamma_2(\lambda) \).

We denote, for \( r > 0 \),

\[
B(x; r) = \{ y \in \mathbb{Z}^d \mid \| x - y \| \leq r \}, \quad S(x; r) = \{ y \in \mathbb{Z}^d \mid r \leq \| x - y \| \leq r + 1 \}.
\]

We remark that there exists a constant \( L = L(d) > 0 \) such that for all \( x \in \mathbb{Z}^d \) and all \( r \geq 1 \),

\[
\max \left( \card (B(x; r)), \card (S(x; r)) \right) \leq L \cdot r^d.
\]
Also, $B(x; r) = x + B(0; r)$ and $S(x; r) = x + S(0; r)$. Further, we show that $\partial B(x; r) \subset S(x; r)$. To see this, suffices to show this when $x = 0$. In this case, the relation $y \in \partial B(0; r)$ signifies there exists a $y' \in B(0; r)$ such that $\|y - y'\| = 1$, and then $\|y\| \leq r + 1$. The inequality $\|y\| \geq r$ for $y \in \partial B(0; r)$ follows from definition of the vertex boundary of a set.

**Proposition (5.6)** For each $p \in \mathbb{N}$, define the sets $A_p = [−p, p] \times S(0; p)$, $B_p = \{p\} \times B(0; p)$ and $C_p = \{−p\} \times B(0; p)$. Then, there is a pair of constants $c, c' > 0$ such that for all $p \in \mathbb{N}$, $\sum_{z \in A_p \cup B_p} G(z, 0) \leq c e^{−c'p}$. Furthermore, there exists another pair of constants $c, c' > 0$ and a positive number $B$, obeying the following: for every $b \geq B$ we can split $C_p = C_{p, 0} \cup C_{p, 1}$, where $C_{p, 0} = C_{p, 0}(b) = \left(\{−p\} \times B(0; bp^2)\right) \cap C_p$ and $C_{p, 1} = C_{p, 1}(b) = C_p \setminus C_{p, 0}$, and with this division, $\sum_{p \in \mathbb{N}} \sum_{z \in C_{p, 1}} G(z, 0) \leq c e^{−c'b^2}$. In particular, for any $\varepsilon > 0$, there exists an index $p_1 = p_1(\varepsilon) \in \mathbb{N}$ and a positive number $b_1 = b_1(\varepsilon)$ such that, for any $p \geq p_1$ and $b \geq b_1$, we can divide $C_p = C_{p, 0} \cup C_{p, 1}$ as before, and with this division $\sum_{p \geq p_1, z \in C_{p, 1}} G(z, 0) \leq \varepsilon$, where $E_p = A_p \cup B_p \cup C_{p, 1}$.

Proof. If $z \in A_p \cup B_p$ then the bounds on the Green’s function (2.22) give at once $G(z, 0) = G(0, −z) \leq c_1 e^{−c_2\|z\|} \|z\|^{−d} \leq c_1 e^{−c_2p} \|z\|^{−d}$. Now, $\sum_{z \in A_p \cup B_p} G(z, 0) \leq c_3 e^{−c_2p} \|z\|^{−d} \leq c_4 e^{−c_2p}$. Now, for any $b > 0$, we may construct $C_{p, 0}$ and $C_{p, 1}$ as in the statement; if $b \geq p^2$, then $C_{p, 1} = \emptyset$, which is fine. Write $C_{p, 1} \subset \bigcup_{b \geq p^2} \{−p\} \times S(0; k)$, so that $\sum_{z \in C_{p, 1}} G(z, 0) \leq b \left(\sum_{z \in C_{p, 1}} \text{card} \{S(0; k)\} c_1 e^{−c_2k^2} p^{−d} \right) \leq b \left(\sum_{b \geq p^2} \left(\sum_{z \in C_{p, 1}} \text{card} \{S(0; k)\} e^{−c_2k^2} p^{−d} \right) \right)$, for a constant $c_5 > 0$. The function $t \mapsto e^{−c_2t^2} t^d$ has a maximum at $t = \left(\frac{d}{2c}\right)^{\frac{1}{2}} p^2\frac{1}{2}$ and it is decreasing for $t \geq \left(\frac{d}{2c}\right)^{\frac{1}{2}} p^2\frac{1}{2}$; any value $b \geq \left(\frac{d}{2c}\right)^{\frac{1}{2}}$ allows bounding $\sum_{z \in C_{p, 1}} G(z, 0) \leq c_5 \int_{b \geq p^2} dt e^{−c_2t^2} p^{−d} t^{d−1} = c_5 \int_{b \geq p^2} dt e^{−c_2t^2} t^{d−1} \leq c_6 e^{−c_2p^2}$. Hence, putting all together, we have established that, for any $p \in \mathbb{N}$ and $b \geq \left(\frac{d}{2c}\right)^{\frac{1}{2}}$, $\sum_{z \in A_p \cup B_p} G(z, 0) \leq c e^{−c'p}$ and $\sum_{z \in C_{p, 1}} G(z, 0) \leq C e^{−c'b^2}$, where all these constants $c, c', C, C' > 0$ are independent of both $p$ and $b$. The desired results are now clear.

**Proposition (5.7)** Let $\varepsilon > 0$ and we maintain the notations and conclusions of (5.6). Then, there exists a positive integer $p_2 = p_2(\varepsilon)$ and a family of sets $(C'_{p, 0})_{p \geq p_2}$, such that $C'_{p, 0} \subset C_{p, 0}$, $\text{card} \{C'_{p, 0}\} \leq \varepsilon \cdot \text{card} \{C_{p, 0}\}$ and $\sup_{p \geq p_2} \sum_{z \in C'_{p, 0}} \|z - C'_{p, 0}\| < \infty$.

The proof follows from elementary techniques and can be found in [MD20 Proposition (5.2.3)].

For the next proposition, the reader should recall the definition of $\text{LE}$.

**Proposition (5.8)** Let $S$ and $S'$ be two independent transient irreducible Markov chains on the same countable state space (having same transition density), with initial states $o$ and $o'$, respectively. Let $U$ be any subset of the states containing $o$ and $o'$. Denote by $T$ and $T'$ the exit times of $U$ by $S$ and $S'$, respectively. If the event $\mathcal{H}_U$ defined by “there exists $0 \leq m \leq T$ and $0 \leq m' \leq T'$ with $S_m = S'_m$,” has positive probability,
then $P \left( \text{LE}(S^0_m)_{m=0}^{T} \cap (S_m)_{m=0}^{T} \neq \emptyset \right) \geq 2^{-8}P(\mathcal{H}_U)$. (This inequality obviously also holds if $\mathcal{H}_U$ has probability zero.)

We only provide a reference for this proposition: see Remark 1.3 and Lemma 4.1 of [LPS03].

In [4.13] we proved a lower bound for the probability that two independent sections of the network random walk intersect. That was an involved theorem. The following proposition provides an upper bound for the probability that two network random walks intersect in a narrow strip.

**Proposition (5.9)** Let $d = 2$. Consider $k_0$ and $p_0$ as in [4.14] with $\varepsilon = 1$ and define the “strips”

$$J_p^+ = [4 \cdot 9^p_0 - 2 \cdot 3^p_0, 4 \cdot 9^p_0 + 2 \cdot 3^p_0]$$

and

$$J_p^- = [4 \cdot 9^p_0, 4 \cdot 9^p_0 + 2 \cdot 3^p_0]$$

With $S$ and $S'$ being two independent network random walks in $\Gamma_d(\lambda)$ started at $0$ and 0, respectively, denote by $I_p^+(o)$ the number of intersections of the paths $S$ and $S'$ inside the strip $J_p^+$. Then, there exists a constant $c > 0$ such that for all $\delta > 0$, there exists an index $p_3 = p_3(\delta)$ such that, for all $p \geq p_3$,

$$\max P(I_p^+(0) > 0) \leq c^{3^p_0}p_0.$$

**Proof.** Take $p_{3,1} = p_{3,1}(\delta)$ the first integer $s$ such that the ball $B(0; \delta + 1)$ is a subset of $U_{s-1}$ defined by (4.10). Assume $p \geq p_{3,1}$. Write $I_p^+$ instead of $I_p^+(o)$ for simplicity and we will show the bounds are universal, provided $p$ is large enough (depending on $\delta$); since the argument is very similar, suffices to consider the case of $I_p^+$. We have $I_p^+ = \sum_{z \in J_{p,0}^+} \sum_{m, m = 0}^{\infty} \mathbb{1}(S = z) \mathbb{1}(S' = z)$, and by independence, $P(I_p^+ > 0) \leq E(I_p^+) = \sum_{z \in J_{p,0}^+} G(o, z) G(0, z)$. Write now

$$J_{p,0}^+ = [4 \cdot 9^p_0, 4 \cdot 9^p_0 + 2 \cdot 3^p_0] \times \mathbb{B}(0; 3^p_0)$$

and

$$J_{p,1}^+ = [4 \cdot 9^p_0, 4 \cdot 9^p_0 + 2 \cdot 3^p_0] \times \mathbb{B}(0; 3^{(t+1)p_0}) \setminus \mathbb{B}(0; 3^{p_0})$$

so that $J_{p}^+$ is the union of the pairwise disjoint sets $J_{p}^{+}_{t}$ ($t \in \mathbb{N}$).

It easily follows that $G(o, z) \sim G(0, z)$ for $z \in J_{p}^+$ and all $p$ large enough with any implicit constant universal. Denote now $p_3 = \max(p_{3,1}, p_{3,2}, p_{3,3})$. There exists $c > 0$ such that $G(o, z) \leq cG(0, z)$. Substitute this into the equation above, $P(I_p^+ > 0) \leq \sum_{z \in J_{p}^+} G(o, z) G(0, z) \leq c \sum_{z \in J_{p}^+} G(0, z)^2 = c \sum_{t \in \mathbb{Z}_+} \sum_{z \in J_{p}^{+}_{t}} G(0, z)^2$. For $t = 0$, by (4.22),

$$\sum_{z \in J_{p,0}^+} G(0, z) \leq c_1^2 9^{-2p_0} \text{card}(J_{p,0}^+) \asymp 3^{-4p_0} \times 3^{p_0} \cdot 3^{2p_0} = 3^{-p_0}.$$

For $t = 1$, and with $\text{card}(B(x; r)) \leq Lr^2$ for $x \in \mathbb{Z}^2$ and $r \geq 1$, we have

$$\sum_{z \in J_{p,1}^+} G(0, z)^2 \leq c_1^2 \sum_{n=4 \cdot 9^p_0}^{\infty} \frac{r!}{r!} 3^{p_0} (r+1)^2 3^{(r+1)p_0} e^{-2c_1^2 L r^2 n^{-2}} \leq c_2^2 9^{-2p_0} \cdot 3^{2p_0} \sum_{r=1}^{\infty} L(r+1)^2 3^{(r+1)p_0} e^{-c_2^2 L r^2} \leq c_3 e^{-c_2^2 ||x||^2}.$$

For $t \geq 2$ and $z = (n_x, x_z) \in J_{p,t}^+$, the condition $3^{p_0} \leq ||x|| \leq 3^{(t+1)p_0}$ guarantees the bound $G(0, z) \leq c_1 e^{-c_2^2 ||x||^2}$, and we obtain bounds that are negligible compared with the two bounds already obtained. Indeed,

$$\sum_{t \geq 2} \sum_{z \in J_{p,0}^+} G(0, z)^2 \leq c_1^2 L \sum_{t \geq 2} \sum_{n=4 \cdot 9^p_0}^{\infty} e^{-c_2^2 3^{p_0} 3^{(t+1)p_0}} \leq c_1^2 L \sum_{t \geq 2} e^{-c_3^2 3^{p_0} 3^{(2t+3)p_0}} \leq c_4 e^{-c_3^2 3^{p_0}}.$$
Before continuing with the next proposition, we introduce the following terminology. A set $H$ of the form $H = \{h\} \times \mathbb{Z}^d$ will be called a splitting hyperplane (splitting plane if $d = 2$) for the path $(v_n)$ (either finite or infinite) if for one, and exactly one value of $m$, we have $p_m v_m = h$. In this case, we will call $h$ to be a splitting level for this path $(v_n)$.

**Proposition (5.10)** Let $S$ be the network random walk on $\Gamma_d(\lambda)$ started at $S_0$. Suppose that $\zeta \leq \eta$ are two stopping times for $S$, with $\zeta < \infty$ almost surely ($\eta = \infty$ with positive probability not excluded). Denote by $S_h(\zeta, \eta)$ the event that $h \in \mathbb{Z}$ is a splitting level for the path $S = (S_m)_{\zeta \leq m \leq \eta}$ (whenever $\eta = \infty$, replace $m \leq \eta$ by $m < \eta$). On $S_h(\zeta, \eta)$, denote by $m_0$ to be the unique positive integer $m$ for which $p_{m_0} S_m = h$. Then, on the event $S_h(\zeta, \eta)$, $\text{LE}(S) = \text{LE}(S_m)_{\zeta \leq m \leq m_0} \vee \text{LE}(S_m)_{m_0 \leq m \leq \eta}$.

The proof is very easy and therefore, omitted.

The next proposition shows that splitting levels do occur with a universal overwhelming probability if one let appropriate scales to be considered. In other words, if the random walk moves enough so that the initial and last state considered are $n$ levels apart, at the very least, one in roughly every $\sqrt{n}$ consecutive levels will be splitting. The gist of its proof lies in random walk in one dimension, see (2.17).

**Proposition (5.11)** Consider the network random walk $S$ on $\Gamma_d(\lambda)$ started at zero. For every positive integer $a$, every pair $\zeta_p < \eta_p$ of stopping times for $S$ such that, for almost every realisation, $\zeta_p \leq \tau(\sigma) \times \mathbb{Z}^2(S)$, $\tau(a + p^2) \times \mathbb{Z}^2(S) \leq \eta_p$, and for $h \in \mathbb{Z}_+$, consider the event $S_a(h, \eta_p)$ that $h$ is a splitting level for $(S_m)_{m = \zeta_p, \ldots, \eta_p}$. There exists a universal constant $c > 0$ and an index $p_4 \in \mathbb{N}$ such that, for all $p \geq p_4$, all systems $(a, \zeta_p, \eta_p)$ as above and all integers $b$ satisfying $a \leq b$ and $b + p \leq a + p^2$, we have $\mathbb{P}\left(\bigcup_{h=b}^{b+p} S_h(\zeta_p, \eta_p)\right) \geq 1 - e^{-c p}$.

**Proof.** Let $\mathcal{K}_p$ be the event “some $h = b, \ldots, b + p$ is a splitting level for the path $(S_m)_{m \in \mathbb{Z}_+}$.” We have $\mathcal{K}_p \subset \bigcup_{h=b}^{b+p} S_h(\zeta_p, \eta_p)$ and therefore $\mathbb{P}\left(\bigcup_{h=b}^{b+p} S_h(\zeta_p, \eta_p)\right) \geq \mathbb{P}(\mathcal{K}_p)$. The result follows at once from (2.17). □

**Proposition (5.12)** Let $d = 2$. Consider $S$ and $S'$ two independent network random walks in $\Gamma_2(\lambda)$ started at $0$ and $0$, respectively. Define the following stopping time $\sigma_p = \tau_{\text{H}_p}(S)$, where $\text{H}_p = \{4 \cdot 9^p\mathbb{Z}\} \times \mathbb{Z}^2$, in other words, $\sigma_p$ is the hitting time of the “plane” $\text{H}_p$ by $S$. Define $\sigma'_p$ similarly, using $S'$ in lieu of $S$. Denote $L' = \text{LE}(S_m')_{m = \sigma_p(\text{H}_p - 1) + 1, \ldots, \sigma_p}$, which is the loop-erasure of the section of $S'$ between the planes $\text{H}_{2p(2)}$ and $\text{H}_{2p}$; with this, define $\beta_p(o) = \sum_{m = \sigma_p(\text{H}_p - 1)}^{\sigma_p} \sum_{z \in L'} I_{(S_m = z)}$, which is the number of times the section $(S_m)_{m = \sigma_p(\text{H}_p - 1), \ldots, \sigma_p}$ crosses the path $L'$. Then, there exists a constant $c > 0$ such that for every $\delta > 0$, there exists a $p_5 = p_5(\delta) \in \mathbb{N}$ satisfying $\min_{||o|| \leq \delta} \mathbb{P}(\beta_p(o) > 0) \geq \frac{c}{p}$, for all $p \geq p_5$.

**Proof.** Define $M_p(o)$ as in (4.13) using $S$ and $S'$. Define $p_{5,1} = p_{5,1}(\delta) \in \mathbb{N}$ to be the first index $p$ such that $B(0; \delta + 1) \subset U_{p-1}$. Notice that for $p \geq p_{5,1}$, $T_p \leq \sigma_p$ and $T'_p \leq \sigma'_p$. Since $\sigma_{p-1}$ and $\sigma'_{p-1}$ are the hitting times of the plane $\text{H}_{p-1}$, we find that

$$M_p(o) = \sum_{z \in D_{p-k_0}} \sum_{m = \sigma_{p-1} - 1 \ldots \sigma_p} \sum_{m' = \sigma_p} \sum_{m'' = \sigma_p} I_{(S_m = z)} I_{(S'_m = z)} \leq \sum_{z \in D_{p-k_0}} \sum_{m = \sigma_{p-1}} \sum_{m'' = \sigma_p} \sum_{m'' = \sigma_p} I_{(S_m = z)} I_{(S'_m = z)} \overset{\text{def.}}{=} N_p(o).$$

Thus, $\mathbb{P}(M_p(o) > 0) \leq \mathbb{P}(N_p(o) > 0)$ and Theorem (4.15) gives $\mathbb{P}(M_p(o) > 0) \geq \frac{c}{p}$, provided $p \geq P(o, 0)$, the constant $c$ being universal. Define $p_{5,2} = p_{5,2}(\delta) = \max P(o, 0)$. Finally, (5.8) gives $\mathbb{P}(\beta_p(o) > 0) \geq$
2^{−8} \mathbb{P}(\hat{N}_p(o) > 0), where \( \hat{N}_p(o) \) is the number of intersection between the two sections \((S_m)_{m=\sigma_2(p-1), \ldots, \sigma_{2p}}\) and \((S'_m)_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}}\). Clearly \( \hat{N}_p(o) \geq N_p(o) \), and the result follows with \( p_5 = \max(p_5, 1, p_5, 2) \).

**Proposition (5.13)** Let \( d = 2 \). With the hypotheses and notations of (5.12). Define \( \alpha_p(o) \) to be the number of intersections of \( \mathcal{L}'(S_m')_{m'=0, \ldots, \sigma_{2p}} \) and \((S_m)_{m=0, \ldots, \sigma_{2p}}\). For every \( \delta > 0 \), \( \lim_{p \to \infty} \max_{\|o\| \leq \delta} \mathbb{P}(\alpha_p(o) = 0) = 0 \).

**Proof.** Let \( \varepsilon > 0 \). We can then find the indices \( p_1(\varepsilon) \) and \( p_2(\varepsilon) \), and a constant \( b_1(\varepsilon) \) satisfying the conclusions of (5.9) and (5.11), we take \( b = b_1(\varepsilon) \). The \( \delta \) in the hypothesis, together with (5.9) and (5.12) give two indices \( p_3(\delta) \) and \( p_5(\delta) \), satisfying the corresponding conclusions of these results. And (5.11) gives yet another index \( p_4 \) satisfying the outcome of that proposition. Observe that the choice of \( p_4(\delta) \) guarantees that the closed ball \( B(0; \delta + 1) \) is contained in \( U_{p_4(\delta + 1)} \). Let \( P = \max(p_1(\varepsilon), p_2(\varepsilon), p_3(\delta), p_4, p_5(\delta)) \).

Write \( \alpha_p \) in place of \( \alpha_p(o) \) for simplicity, all the bounds will be shown to be independent of \( \delta \). Assume \( p \geq P \). Define the events

\[
\mathcal{L}'_p = \left\{ \text{h is a splitting level for } (S_m')_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}} \right\}
\]

(A)

\[
\mathcal{R}'_p = \left\{ \text{h is a splitting level for } (S_m')_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}} \right\},
\]

that is, the events where one of the \( 2 \cdot 9^{(p-1)k_0} \) left-most levels between the planes \( H_{2(p-1)} \) and \( H_{2p} \) and one of the \( 2 \cdot 9^{k_0} \) right-most levels in the same region is a splitting level for the stated section of \( S' \). Denote by \( \mathcal{R}'_p \) and \( \mathcal{L}'_p \) their corresponding complements; we also define \( \beta_p = \beta_p(o) \) from (5.12). Then,

\[
\mathbb{P}(\alpha_p = 0) = \mathbb{P}\left( \{\alpha_p = 0\} \cap \mathcal{L}'_p \right) + \mathbb{P}\left( \{\alpha_p = 0\} \cap \mathcal{R}'_p \right)
\]

\[
\leq \mathbb{P}\left( \{\alpha_p = 0\} \cap \mathcal{L}'_p \cap \{\beta_p = 0\} \right) + \mathbb{P}\left( \{\alpha_p = 0\} \cap \mathcal{L}'_p \cap \{\beta_p > 0\} \right) + \mathbb{P}\left( \mathcal{L}'_p \right).
\]

We analyse now what happens on the intersection \( \{\alpha_p = 0\} \cap \mathcal{L}'_p \cap \{\beta_p > 0\} \). Assume that this event occurs. Then, there exists a splitting level \( h \in \{4 \cdot 9^{2(p-1)k_0}, \ldots, 4 \cdot 9^{2(p-1)k_0} + 2 \cdot 9^{(p-1)k_0}\} \) for the section \((S_m')_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}}\). Denote by \( h'_p \) the first of such splitting levels and by \( m'_p \) the unique index for which \( p_0 S_{m'_p} = h'_p \). According to (5.10), we have

(B)

\[
\mathcal{L}(S_m')_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}} = \mathcal{L}(S_m')_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}} \cup \mathcal{L}(S_m')_{m' = m'_p, \ldots, \sigma'_{2p}}.
\]

Since \( \sigma'_2(p-1) \) is the hitting time of the set \( H_{2(p-1)} \), it is then obvious that \( h'_p \) is also a splitting level for the section \((S_m')_{m' = 0, \ldots, \sigma_{2p}}\) and so, we also have

(C)

\[
\mathcal{L}(S_m')_{m' = 0, \ldots, \sigma_{2p}} = \mathcal{L}(S_m')_{m' = 0, \ldots, m'_p} \cup \mathcal{L}(S_m')_{m' = m'_p, \ldots, \sigma_{2p}}.
\]

That \( \alpha_p = 0 \) entails that the section \((S_m)_{m=0, \ldots, \sigma_{2p}}\) does not intersect the set in (C) and the relation \( \beta_p > 0 \) means that it does intersect the set in (B). Therefore, on the event under consideration, the intersection of \((S_m)_{m=0, \ldots, \sigma_{2p}}\) with \( \mathcal{L}(S_m')_{m' = \sigma'_2(p-1), \ldots, \sigma'_{2p}} \) occurs at some \( S'_m \) with \( m' \in \{\sigma'_2(p-1), \ldots, m'_p\} \); in particular, \( S \) and \( S' \) intersect inside the strip \( J_{2(p-1)}^+ \) (notation of (5.9)). In other words, with the notation \( I_p^+ \) of (5.9),

\[
\mathbb{P}(\{\alpha_p = 0\} \cap \mathcal{L}'_p \cap \{\beta_p > 0\}) \leq \mathbb{P}\left( I_{2(p-1)}^+ > 0 \right).\)

Whence, by the arguments of the previous two paragraphs,

(D)

\[
\mathbb{P}(\alpha_p = 0) \leq \mathbb{P}(\{\alpha_p = 0\} \cap \mathcal{L}'_p \cap \{\beta_p = 0\}) + \mathbb{P}\left( \mathcal{L}'_p \right) + \mathbb{P}\left( I_{2(p-1)}^+ > 0 \right).
\]
Now observe that

\[ \{ \alpha_p = 0 \} \cap \mathcal{L}'_p \cap \{ \beta_p = 0 \} = (\{ \alpha_p = 0, \alpha_{p-1} = 0 \} \cap \mathcal{L}'_p \cap \{ \beta_p = 0 \}) \]

\[ \cup \left( \{ \alpha_p = 0, \alpha_{p-1} > 0 \} \cap \mathcal{L}'_p \cap \{ \beta_p = 0 \} \right) \]

\[ \subset \left( \{ \alpha_{p-1} = 0 \} \cap \mathcal{L}'_p' \cap \{ \beta_p = 0 \} \right) \cup \{ \alpha_p = 0, \alpha_{p-1} > 0 \}, \]

and from this it follows that

\[ \mathbb{P} \left( \{ \alpha_p = 0 \} \cap \mathcal{L}'_p \cap \{ \beta_p = 0 \} \right) \leq \mathbb{P} \left( \{ \alpha_{p-1} = 0 \} \cap \mathcal{L}'_p \cap \{ \beta_p = 0 \} \right) + \mathbb{P} (\alpha_p = 0, \alpha_{p-1} > 0). \]

By (5.4), \( \{ \alpha_{p-1} = 0 \} \) and \( \mathcal{L}'_p \cap \{ \beta_p = 0 \} \) are independent events. Write now

\[ \{ \alpha_p = 0, \alpha_{p-1} > 0 \} = \left( \{ \alpha_p = 0, \alpha_{p-1} > 0 \} \cap \mathcal{R}_{p-1}' \right) \cup \left( \{ \alpha_p = 0, \alpha_{p-1} > 0 \} \cap \mathcal{R}_{p-1}^c \right); \]

as done before, on the set \( \{ \alpha_p = 0, \alpha_{p-1} > 0 \} \cap \mathcal{R}_{p-1}' \), the two random walks \( S \) and \( S' \) intersect in the strip \( J_{2(p-1)} \), yielding \( \mathbb{P} \left( \{ \alpha_p = 0, \alpha_{p-1} > 0 \} \cap \mathcal{R}_{p-1}' \right) \leq \mathbb{P} \left( I_{2(p-1)}^+ \right) \). With the aforementioned considerations and substituting into (14), we reach

\[ \mathbb{P} (\alpha_p = 0) \leq \mathbb{P} (\alpha_{p-1} = 0) \mathbb{P} (\beta_p = 0) \]

\[ + \mathbb{P} \left( I_{2(p-1)}^+ \right) \mathbb{P} \left( \mathcal{R}_{p-1}^c \right) + \mathbb{P} \left( \mathcal{L}_{p-1}^c \right) + \mathbb{P} \left( I_{2(p-1)}^+ \right). \]

By virtue of propositions (5.9) and (5.11), we have \( \mathbb{P} \left( I_{2(p-1)}^+ \right) \mathbb{P} \left( \mathcal{R}_{p-1}^c \right) + \mathbb{P} \left( \mathcal{L}_{p-1}^c \right) + \mathbb{P} \left( I_{2(p-1)}^+ \right) \leq ce^{-c'p} \) and by (5.12) \( \mathbb{P} (\beta_p = 0) \leq 1 - \frac{c''}{p} \), for \( p \geq P \), where the constants \( c, c' \) and \( c'' \) are all universal. We can conclude then that \( \mathbb{P} (\alpha_{p+p} = 0) \leq \prod_{q=0}^{p} \left( 1 - \frac{c''}{p+q} \right) + ce^{-c'p} \sum_{q=0}^{p} e^{c'q} \). The conclusion of the proposition is now clear given the last inequality and bearing in mind that all estimates are uniform in \( ||o|| \leq \delta \).

\textbf{Proposition (5.14)} Let \( d = 2 \). With the hypotheses of (5.13), let \( \tilde{\alpha}_p(o) \) be the number of intersections of \( \text{LE}(S_{m'}_{m=0,\ldots,2p}) \). Then, for every \( \delta > 0 \), \( \lim_{p \to \infty} \max_{||o|| \leq \delta} \mathbb{P} (\tilde{\alpha}_p(o) = 0) = 0. \)

\textbf{Proof.} We use the notation of (5.11). Simply write

\[ \mathbb{P} (\tilde{\alpha}_p(o) = 0) = \mathbb{P} (\tilde{\alpha}_p(o) = 0, \alpha_p(o) = 0) + \mathbb{P} (\tilde{\alpha}_p(o) = 0, \alpha_p(o) > 0) \]

\[ \leq \mathbb{P} (\alpha_p(o) = 0) + \mathbb{P} \left( \{ \tilde{\alpha}_p(o) = 0, \alpha_p(o) > 0 \} \cap \mathcal{R}_p \right) \]

\[ + \mathbb{P} \left( \{ \tilde{\alpha}_p(o) = 0, \alpha_p(o) > 0 \} \cap \mathcal{R}_p^c \right). \]

Exactly as in the proof of (5.13), the two relations \( \tilde{\alpha}_p(o) = 0 \) and \( \alpha_p(o) > 0 \) on the event \( \mathcal{R}_p \) imply that two independent random walks will intersect in the strip \( J_{2p} \), hence the result follows immediately upon invoking (5.9), (5.11) and (5.13).

\subsection{The proof of the main theorem.}

\textbf{Theorem (5.15)} For almost every realisation of \( \text{UST} \) in \( \Gamma_{2}(\lambda) \), the tree has one end.

\textbf{Proof.} Denote by \( \text{UST} \) the measure of uniform spanning tree on the network \( \Gamma_{2}(\lambda) \). When we define an event on spanning trees, we use \( \text{UST} \) to refer to the tree itself. We know that \( \text{UST} \) has at least one end, up to a
UST-null event. Consider \( \mathcal{N} = \{ \text{UST has at least two ends} \} \). We aim at showing \( \text{UST}(\mathcal{N}) = 0 \). For a vertex \( z \in \mathbb{Z}^3 \), consider \( \mathcal{N}_z = \{ \text{there are two disjoint rays starting at } z \text{ in UST} \} \). Then, \( \mathcal{N} = \bigcup_{z \in \mathbb{Z}^3} \mathcal{N}_z \). Further, the translation invariance of the probability kernel of the network random walk of \( \Gamma_2(\lambda) \) shows that \( \text{UST}(\mathcal{N}_z) \) is a value independent of \( z \in \mathbb{Z}^3 \). Hence, to show \( \mathcal{N} \) is UST-null, it suffices to show \( \mathcal{N}_0 \) is UST-null.

Consider now a set of vertices \( K \) of \( \mathbb{Z}^3 \) with the following property (a “cutset” between 0 and infinity): every ray from 0 must cross \( K \) at some vertex. Recall an “edge of \( K \)” is an edge of the subgraph induced on \( K \). Then, \( \mathcal{N}_0 \) is contained in the event

\[
\mathcal{C}_K = \{ \text{there are two disjoint paths in UST from } K \text{ to } 0 \text{ that use no edge of } K \}.
\]

In particular, we have the following: \( \text{UST}(\mathcal{N}_0) \leq \inf_{K \text{ is a cutset between } 0 \text{ and } \infty} \text{UST}(\mathcal{C}_K) \).

As in proposition 5.3, define the sets \( A_p = [-p, p] \times S(0; p) \), \( B_p = \{ p \} \times B(0; p) \) and \( C_p = \{ -p \} \times B(0; p) \). Their union \( K_p \) is a cutset between 0 and \( \infty \). In particular,

\[
\text{(A)} \quad \text{UST}(\mathcal{N}_0) \leq \inf_{p \in \mathbb{N}} \text{UST}(\mathcal{C}_{K_p}).
\]

Construct \( \mathcal{I} \) the UST of \( \mathbb{Z}^3 \) using Wilson’s algorithm rooted at infinity and following some starting at zero predefined ordering of the vertices of \( \mathbb{Z}^3 \), and assume it is defined on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). (For instance, we can follow the order in which we start with 0, and having searched all vertices such that \( \| z \|_1 = k \), we search the vertices satisfying \( \| z \|_1 = k + 1 \) arranging them lexicographically.) Thus, we have a family of independent network random walks \( \{ S^z \}_{z \in \mathbb{Z}^3} \) defined on this probability space with \( S^z \) started at \( z \). Since 0 is the first vertex searched, \( L_0 = LE(S^0_m)_{m \in \mathbb{Z}^+} \subset \mathcal{I} \) (both trees canonically identified with their sets of edges). Up to a \( \mathbb{P} \)-null event, \( L_0 \) is an infinite branch. Thus, \( \{ \mathcal{I} \in \mathcal{C}_{K_p} \} = \{ \text{some vertex of } K_p \text{ is connected to } 0 \text{ in } \mathcal{I} \setminus L_0 \} \). Denote \( \mathcal{C}(z) = \{ z \text{ is connected to } 0 \text{ in } \mathcal{I} \setminus L_0 \} \), so that \( \{ \mathcal{I} \in \mathcal{C}_{K_p} \} = \bigcup_{z \in K_p} \mathcal{C}(z) \). By virtue of (A.6) we reach the existence of an index \( p_1(\varepsilon) \) and a number \( b_1(\varepsilon) \) such that we may divide \( C_p = C_{p,0} \cup C_{p,1} \) as in this proposition (taking \( b = b_1(\varepsilon) \)) and for all \( p \geq p_1(\varepsilon) \),

\[
\sum_{z \in E_p} G(z, 0) \leq \varepsilon,
\]

with the notation \( E_p = A_p \cup B_p \cup C_{p,1} \) of the proposition (in particular, \( G \) is the Green’s function of \( \Gamma_2(\lambda) \)). Next,

\[
\text{UST}(\mathcal{C}_{K_p}) = \mathbb{P} \left( \mathcal{I} \in \mathcal{C}_{K_p} \right) \leq \sum_{z \in E_p} \mathbb{P} \left( \mathcal{C}(z) \right) + \mathbb{P} \left( \bigcup_{z \in C_{p,0}} \mathcal{C}(z) \right).
\]

By corollary 5.5.1, when calculating \( \mathbb{P} \left( \mathcal{C}(z) \right) \) we may assume that the order in which \( \mathcal{I} \) was constructed is \((0, z, \ldots)\). If the order in the construction of \( \mathcal{I} \) were \((0, z, \ldots)\), then \( \mathcal{C}(z) \) would be the event where \( S^z \) hits \( L_0 \) for the first time at zero, and then \( \mathcal{C}(z) \) would be contained in the event where \( \tau_0(S^z) < \infty \). Then, \( \mathbb{P} \left( \mathcal{C}(z) \right) \leq G(z, 0) \). Substituting this inequality above we obtain that, for all \( p \geq p_1(\varepsilon) \),

\[
\text{UST}(\mathcal{C}_{K_p}) \leq \varepsilon + \mathbb{P} \left( \bigcup_{z \in C_{p,0}} \mathcal{C}(z) \right).
\]

Using (5.7), there is an index \( p_2(\varepsilon) \) and a family of sets \( \{ C'_{p,0} \}_{p \geq p_2(\varepsilon)} \), for which the conclusions of this proposition hold. In particular, we may define \( \delta_0 < \infty \) as follows

\[
\delta_0 = \sup_{p \geq p_2(\varepsilon)} \max_{z \in C_{p,0}} \| z - C'_{p,0} \|.
\]

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For each \( z' \in C_{p,0}' \), consider the set \( C_{p,0}(z') = \{ -p \} \times B(z'; \delta_0) \). The definition of \( \delta_0 \) shows that at once \( C_{p,0} \subset \bigcup_{z' \in C_{p,0}'} C_{p,0}(z') \). Then, \( P \left( \bigcup_{z \in C_{p,0}} \mathcal{E}(z) \right) \leq \sum_{z \in C_{p,0}'} P \left( \bigcup_{z \in C_{p,0}(z')} \mathcal{E}(z) \right) \). By definition, \( \bigcup_{z \in C_{p,0}(z')} \mathcal{E}(z) \) is an event depending solely on \( \mathcal{F} \) and \( L_0 \). Thus, (5.5.5) shows that its probability is independent of the ordering of the vertices of \( \mathbb{Z}^3 \setminus \{0\} \), in particular, we may assume that \( \mathcal{F} \) was constructed using the ordering \((0, z', \ldots)\). Consider now the following event \( \mathcal{H}(z') = \big\{ S'_{\tau_{L_0}(z')} \neq 0 \big\} \). In other words, \( \mathcal{H}(z') \) is the event where \( S' \) hits \( L_0 \) for the first time anywhere except zero (recall \( S' \) will hit \( L_0 \) a.s.), which is the same as the event where \( z' \) is not connected to 0 in \( \mathcal{F} \setminus L_0 \) (with the assumed order \((0, z', \ldots)\) of \( \mathbb{Z}^3 \)). Then, \( 1 - P(\mathcal{H}(z')) = P \left( S'_{\tau_{L_0}(S')} = 0 \right) \leq P \left( \tau_0 (S') < \infty \right) \leq G(z',0) \). Since \( z' \in C_{p,0}' \subset C_{p,0} = \{ -p \} \times B \left( 0; b p^{1 \over 2} \right) \), we have by theorem (5.22), \( G(z',0) \leq c_1 \| z' \|^{-1} \leq c_1 p^{-1} \). Then, \( P \left( \bigcup_{z \in C_{p,0}(z')} \mathcal{E}(z) \right) \leq P \left( \bigcup_{z \in C_{p,0}(z')} \mathcal{E}(z) \cap \mathcal{H}(z') \right) + c_1 p^{-1} \). Summing over \( z' \in C_{p,0}' \), we have that for \( p \geq p_2(\varepsilon) \), \( c_1 \| z' \|^{-1} \leq \sum_{z' \in C_{p,0}'} P \left( \mathcal{E}(z) \right) \). By (5.7), we have that for \( p \geq p_2(\varepsilon) \), \( c_1 \| z' \|^{-1} \leq \sum_{z' \in C_{p,0}'} P \left( \mathcal{E}(z) \right) \). By virtue of (A) and (C), we reduce the proof of the theorem to showing following: there exists a function \( \varphi(p) \) such that

\[
\sum_{z' \in C_{p,0}'} \sum_{z \in C_{p,0}(z')} P \left( \mathcal{E}(z) \cap \mathcal{H}(z') \right) \leq \varphi(p)
\]

and \( \varphi(p) \to 0 \) via some subsequence.

We begin by estimating \( P(\mathcal{E}(z) \cap \mathcal{H}(z')) \). Again by (5.5.1), to calculate the probability of the event \( \mathcal{E}(z) \cap \mathcal{H}(z') \) we may assume that \( \mathcal{F} \) was constructed using the ordering \((0, z', \ldots)\) of \( \mathbb{Z}^3 \). Assuming this ordering of \( \mathbb{Z}^3 \), and on the event \( \mathcal{E}(z) \cap \mathcal{H}(z') \), the first three steps of the construction of \( \mathcal{F} \) proceed as follows:

1) The first branch at zero is created, we call it \( L_0 \).

2) The random walk \( S' \) runs, it hits the first branch at zero at some non zero vertex. Denote by \( L_{z'} \) this second branch; by definition, \( L_{z'} = LE \left( S'_{m \in \ldots \tau_{L_0}(S')} \right) \).

3) The random walk \( S' \) runs, and it hit \( L_0 \) at 0 before hitting \( L_{z'} \). Introduce the event \( N(z, z') = \{ \tau_0(S') < \tau_{L_0}(S') \} \), which is the event where \( S' \) hits \( L_0 \) at zero before it touches \( L_{z'} \) anywhere. The aforementioned first three steps show that

\[
\mathcal{E}(z) \cap \mathcal{H}(z') \subset N(z, z').
\]

In what follows, we will consider the planes \( H_p = \{ 4 \cdot 9^{2 pk_0} \} \times \mathbb{Z}^2 \), and for \( z, z' \in \mathbb{Z}^3 \), the hitting times \( \sigma_p(z, z') = \tau_{H_p + z'}(S^2) \). As such, we will make a “change of scales” and consider from now onwards indices of
the form $4 \cdot g^{2k_0}$. Assume then that $z' \in C'_{4, g^{2k_0}, 0}$ and $z \in C_{4, g^{2p_k}, 0}(z')$. Denote $a_p = -4 \cdot g^{2k_0} + 4 \cdot g^{2(p-1)k_0}$, so that $H_{p-1} + z' = \{a_p\} \times \mathbb{Z}^2$. Consider the events

$$E_p = \{S^0 \text{ ever reaches } -9^{p_k}\},$$

$$S_p(z') = \bigcup_{h=a_p-2 \cdot 9^{(p-1)k_0}} \{h \text{ is a splitting level for } (S'_m)_{m' \in \mathbb{Z}^+} \}.$$  

By (2.17) and (5.11) we know there exists a constant $c > 0$ and an index $p_4 \in \mathbb{N}$ such that for all $p \geq p_4$, $P(E_p) = e^{-\lambda g^{p_k}}$ and $P(S_p(z')) \leq 1 - e^{-\lambda g^{p_k}}$. Hence,

$$(F) \quad P(N(z, z')) \leq P(N(z, z') \cap S_p(z') \cap E_p^c) + e^{-\lambda g^{p_k}} + e^{-\lambda g^{p_k}}.$$  

We introduce the following “good event” $S_p(z') = S_p(z') \cap E_p^c$: on this event $S^0$ never goes left of the plane $\{-9^{p_k}\} \times \mathbb{Z}^2$ and $S'$ has a splitting level in the stated range. On the good event $S_p$, the random walk $S'$ must have crossed the plane $H_{p-1} + z'$ before hitting the first branch. In particular, $\sigma_{p-1}(z', z') < \tau_{L_0}(S'_{z'}) < \infty$ on $S_p(z')$. Also, on the good event $S_p$, there is a splitting plane for $(S'_m)_{m' \in \mathbb{Z}^+}$: introduce $h_p(z')$ and $m_p(z') \leq \sigma_{p-1}(z', z')$ to be the last splitting level (amongst the levels $h = a_p - 2 \cdot g^{(p-1)k_0}, \ldots, a_p$) and the only index $m'$ such that $pr_0(S'_{m'}) = h_p(z')$. By $5.10$, on the good event,

$$L_{z'} = \text{LE}((S^z_{m'})_{m'=0, \ldots, \tau_{L_0}(S')} \cup \text{LE}((S'_{m'})_{m'=0, \ldots, m_p(z')} \setminus \tau_{L_0}(S')$$

Consider now the “partial branch at $z'$ (until its hitting time of $H_{p-1} + z'$)” which is

$$L_{z'}^{p-1} = \text{LE}((S^z_{m'})_{m'=0, \ldots, \sigma_{p-1}(z', z')}$$

On $S_p(z')$,

$$L_{z'}^{p-1} = \text{LE}((S^z_{m'})_{m'=0, \ldots, \sigma_{p-1}(z', z')} = \text{LE}(S_{m'}^{z'})_{m'=0, \ldots, m_p(z')} \cup \text{LE}(S_{m'}^{z'})_{m'=m_p(z')}, \ldots, \tau_{L_0}(S')$$

Define $\alpha_{p-1}(z, z') = \sum_{m=0, \ldots, \sigma_{p-1}(z', z')} \sum_{z'' \in L_{z'}^{p-1}} \mathbb{1}_{\{S^z_{m'} = z''\}}$, which is the number of intersections between the partial branch at $z'$, and the section $(S^z_m)_{m=0, \ldots, \sigma_{p-1}(z', z')}$. Similarly, define $\beta_{p-1}(z, z')$ to be the number of intersections between $L_z$ and the same section of $S'$, that is, $\beta_{p-1}(z, z') = \sum_{m=0, \ldots, \sigma_{p-1}(z', z')} \sum_{z'' \in L_z} \mathbb{1}_{\{S^z_m = z''\}}$. We have the following limit \(\lim_{p \to \infty} \max_{\|z-z'\| \leq 5 \delta_0} \mathbb{P}(\alpha_{p-1}(z, z') = 0) = 0\), where the maximum runs over all pairs $(z, z') \in C_{4, g^{2k_0}, 0} \times C'_{4, g^{2k_0}, 0}$. This last limit is a corollary of proposition (5.13): the translation invariance of the probability kernel shows at once that the maximum inside the limit does not depend on the pair $(z, z')$ above but solely on the distance separating them.

**Lemma (5.15.1)** \(\lim_{p \to \infty} \max_{\|z-z'\| \leq 5 \delta_0} \mathbb{P}(S_p(z') \cap \{\beta_{p-1}(z, z') = 0\}) = 0\), where the maximum runs over all pairs $(z, z') \in C_{4, g^{2k_0}, 0} \times C'_{4, g^{2k_0}, 0}$.

**Proof of lemma.** Observe that

$$\mathbb{P}(S_p(z') \cap \{\beta_{p-1}(z, z') = 0\}) \leq \mathbb{P}( \{\beta_{p-1}(z, z') > 0, \beta_{p-1}(z, z') = 0\}) + \mathbb{P}(\alpha_{p-1}(z, z') = 0) \leq 40.$$
The second of these two terms tends uniformly to zero for the set of stated pairs \((z, z')\), as was shown before the lemma. On the good event \(\mathcal{G}_p\) we can apply the two decompositions above for \(L_{z'}\) and \(L_{z'}^\perp\), giving

\[ \mathcal{G}_p \cap \{ \alpha_{p-1}(z, z') > 0, \beta_{p-1}(z, z') = 0 \} \subset \{ S^z \text{ cross } S^{z'} \text{ inside } J_{p-1} \}, \]

where \(J_{p-1} = \{ a_p - 2 \cdot \eta^{(p-1)\eta_0}, \ldots, a_p \} \times \mathbb{Z}^2\). By (5.9), we have that there exists a universal constant \(c > 0\) and an index \(p_3\) such that for \(p \geq p_3\), \(P \{ S^z \text{ cross } S^{z'} \text{ inside } J_{p-1} \} \leq c3^{-pk_0}\), this bound being uniform for \(z' \in C_{4g^{2k_0}}\) and \(z \in C_{4g^{2k_0}}(z')\).

We continue now with the proof of the theorem. Recall we consider \(z' \in C_{4g^{2k_0}}\) and \(z \in C_{4g^{2k_0}}(z')\). Introduce the following random time \(\theta_{p-1}(z, z') = \min \{ \sigma_{p-1}(z, z'), \tau_{\mathcal{G}_p}(S^z) \}\) which is the hitting time of \((H_{p-1} + z') \cup L_{z'}\) by \(S^z\). Then, \(\theta_{p-1}(z, z')\) is a stopping time for the filtration

\[ \mathcal{F}_k^z(z) = \sigma \{ S^z_m \mid 0 \leq m \leq k \} \cup \mathcal{F}_\infty(\{0, z'\}) \]

where \(\mathcal{F}_\infty(T) = \sigma(S^t_m \mid t \in T, m \in \mathbb{Z}_+\). Observe that \(\mathcal{G}_p(z') \in \mathcal{F}_\infty(\{0, z'\})\). We apply the strong Markov property to the process \((\mathcal{F}_k^z(z))^z(z')\) \(k \in \mathbb{Z}_+\),

\(\text{(G)}\) \(E \left( I_{N(z, z') \cap \mathcal{G}_p(z')} \right) = E \left( E \left( I_{N(z, z')} \mathcal{G}_p(z') \mid \mathcal{F}_{\theta_{p-1}^{-1}(z, z')(z)} \right) \right) \). \(\text{The definition of the stopping time } \theta_{p-1}(z, z') \text{ shows that } P \left( N(z, z') \mid \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \right) = 0 \text{ for all } z' \in L_{z'}, \quad \text{lying to the left or on the plane } H_{p-1} + z'. \text{ Therefore, we have \(} \right) \)

\[ P \left( N(z, z') \mid \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \right) = P \left( N(z, z') \mid \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \right) I \{ \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \notin L_{z'} \}. \]

Clearly \(N(z, z') \subset \{ \tau_0(S^z) < \infty \}\), also, \(\theta_{p-1}(z, z') = \sigma_{p-1}(z, z')\), on the event \(\{ \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \notin L_{z'} \}\); this implies \(\mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \in H_{p-1} + z'\) on this event. Whence,

\[ P \left( N(z, z') \mid \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \right) \leq \sup_{z' \in H_{p-1} + z'} P \left( \tau_0(S^z) < \infty \mid S^z_0 = z'' \right) I \{ \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \notin L_{z'} \}. \]

We know from the bounds of the Green’s function that

\(\text{sup}_{z' \in H_{p-1} + z'} P \left( \tau_0(S^z) < \infty \mid S^z_0 = z'' \right) \leq c_1 |a_p|^{-1} \leq c9^{-2pk_0}, \)

where \(c > 0\) is a universal constant.

Use the bound of (I) in (H), and then apply this to (G). Thus, we have shown up to this point in the argument the existence of a universal constant \(c > 0\) such that

\[ P \left( N(z, z') \cap \mathcal{G}_p(z') \right) \leq c9^{-2pk_0} P \left( \mathcal{G}_p(z') \cap \{ \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \notin L_{z'} \} \right). \]

It is also obvious that \(\mathcal{G}_p(z') \cap \{ \mathcal{F}_{\theta_{p-1}^{-1}(z, z')} \notin L_{z'} \} \subset \mathcal{G}_p(z') \cap \{ \beta_{p-1}(z, z') = 0 \}\). Hence,

\[ P \left( N(z, z') \cap \mathcal{G}_p(z') \right) \leq c9^{-2pk_0} \Delta_p, \]

with \(\Delta_p = \max_{\|z-z'\| \leq \delta_0} P \left( \mathcal{G}_p(z') \cap \{ \beta_{p-1}(z, z') = 0 \} \right) \to 0 \) as \(p \to \infty\), the pairs \((z, z') \in C_{4g^{2k_0}} \times C_{4g^{2k_0}}\).
We are in conditions to conclude the proof of the theorem. By the foregoing,

\[
P(\mathcal{C}(z) \cap \mathcal{H}(z')) \leq P(N(z, z'))
\]

use (E)

\[
\leq P(N(z, z') \cap \mathcal{G}_p(z')) + e^{-c'g_{pk_0}} + e^{-\lambda g_{-2pk_0}},
\]

use (I)

\[
\leq c g_{-2pk_0} \Delta_p + e^{-c'g_{pk_0}} + e^{-\lambda g_{-2pk_0}},
\]

use (J)

with the two constants \(c, c' > 0\) being universal. We consider now \(z' \in C_{4, g_{2pk_0}, 0}\) and sum over \(z \in C_{4, g_{2pk_0}, 0}(z')\), this latter set has cardinality bounded above by \(c'' \delta_0^2\), with \(c'' > 0\) another universal constant. This shows, for every \(z' \in C_{4, g_{2pk_0}, 0}\),

\[
\sum_{z \in C_{4, g_{2pk_0}, 0}(z')} P(\mathcal{C}(z) \cap \mathcal{H}(z')) \leq c'' \delta_0^2 \left(c g_{-2pk_0} \Delta_p + e^{-c'g_{pk_0}} + e^{-\lambda g_{-2pk_0}}\right),
\]

this bound being uniform for \(z'\). We know also that \(\text{card}(C_{4, g_{2pk_0}, 0}) \leq \text{card}(C_{4, g_{2pk_0}, 0}) \leq c''', g_{2pk_0}\), for yet another universal constant \(c'''> 0\). Whence, if we sum over \(z' \in C_{4, g_{2pk_0}, 0}\), we reach a bound of the form

\[
\sum_{z' \in C_{4, g_{2pk_0}, 0}} \sum_{z \in C_{4, g_{2pk_0}, 0}(z')} P(\mathcal{C}(z) \cap \mathcal{H}(z')) \leq c \left(\Delta_p + g_{2pk_0} \left(e^{-c'g_{pk_0}} + e^{-\lambda g_{-2pk_0}}\right)\right),
\]

for a constant \(c > 0\), not depending on \(p\). The right hand side of this inequality tends to zero, and this completes the proof of Theorem (5.15) by (D).

\[\square\]

### 5.4 One endedness for \(\Gamma_d(\lambda), \text{with } d \geq 3\)

We finally deal with the case of “high dimensions.” Here, the proof is considerably simpler than \(d = 2\). We will make use of several general results of electrical networks, see [LP16] or [MD20, Appendix C].

**Theorem (5.16)** Let \(d \geq 3\). Then, for almost every realisation of the uniform spanning forest in \(\Gamma_d(\lambda)\), all the components in this realisation have one end.

**Proof.** For every \(p \in \mathbb{N}\), consider the sets \(A_p, B_p, C_p\), defined in proposition (5.6). Using the notation of this proposition, \(p^\sharp\) will be larger than \(B\) starting at some index \(p^*\). As such, we may divide \(C_p = C_{p,0} \cup C_{p,1}\), with \(C_{p,0} = \{-p\} \times B(0;p^\sharp)\) and \(C_{p,1} = C_p \setminus C_{p,0}\).

(5.6.1) For every \(p \geq p^*\), \(\sum_{z \in A_p \cup B_p} G(z, 0) \leq ce^{-c'p^\sharp}\) and \(\sum_{z \in C_{p,1}} G(z, 0) \leq ce^{-c'p^\sharp}\), where \(c, c'\) are positive constants.

Assume \(\mathfrak{F}\) is the random spanning forest of \(\Gamma_d(\lambda)\) as constructed by Wilson’s algorithm rooted at infinity. To be more precise, assume that on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) there is a family of independent network random walks of \(\Gamma_d(\lambda)\), say \((S^z)_{z \in \mathbb{Z}^{d+1}}\) with \(S^z\) started at \(z\), and then construct \(\mathfrak{F}\) using some predefined order starting at zero of \(\mathbb{Z}^{d+1}\) following Wilson’s algorithm. (For instance, start at zero and then search the \(\|\cdot\|_1\)-spheres by first increasing the radii one unit at a time and then lexicographically ordering each sphere.) We know then that \(\mathfrak{F}\) is a random spanning forest following law USF of \(\Gamma_d(\lambda)\); this signifies that \(\mathbb{P}(\mathfrak{F} \in \mathfrak{E}) = \text{USF}(\mathfrak{E})\), for all events \(\mathfrak{E}\) in the probability space of USF spanning forests of \(\Gamma_d(\lambda)\).

Because of the order chosen to construct \(\mathfrak{F}\), \(L_0 = \mathbf{L}(S^z)_{m \in \mathbb{Z}_+}\) is (a.s.) an infinite path, we will call it “first branch of the random forest \(\mathfrak{F}\).” Define the following set in \(\mathbb{Z}^{d+1}\) (consider \(z \in \mathbb{Z}^{d+1}\) and events in
5.4 One endedness for $\Gamma_d(\lambda)$, with $d \geq 3$

$\mathcal{F}$:

$$\mathcal{E}_p = A_p \cup B_p \cup C_{p,1},$$
$$\mathcal{E}(z) = \{z \text{ is connected to } 0 \text{ in } \mathcal{F} \setminus L_0\},$$
$$\mathcal{E}_p = \bigcup_{z \in \mathcal{E}_p} \mathcal{E}(z) = \{\text{some vertex in } \mathcal{E}_p \text{ is connected to } 0 \text{ in } \mathcal{F} \setminus L_0\}.$$

Then, $\mathbb{P}(\mathcal{E}_p) \leq \sum_{z \in \mathcal{E}_p} \mathbb{P}(\mathcal{E}(z))$. To calculate $\mathbb{P}(\mathcal{E}(z))$ we may apply corollary [5.5.1] and assume that $\mathcal{F}$ was constructed using the ordering $(0, z, \ldots)$ of $\mathbb{Z}^{d+1}$. With this ordering, the event $\mathcal{E}(z)$ is the event where $S^z$ hits $L_0$ for the first time at vertex $0$, whence $\mathbb{P}(\mathcal{E}_p) \leq \sum_{z \in \mathcal{E}_p} \mathbb{P}(\tau_0(S^z) < \infty) \leq \sum_{z \in \mathcal{E}_p} G(z, 0) < ce^{-c'p^{\frac{d}{2}}}$, where the last inequality follows from [5.16.1]. By virtue of Borel-Cantelli lemma, we reach the existence of a random index $P : \Omega \to \mathbb{Z}_+$ (measurable relative to $\mathcal{F}$) such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\omega \notin \bigcup_{p \geq P(\omega)} \mathcal{E}_p$.

Let $U$ denote the (random connected) component of $0$ in the forest $\mathcal{F} \setminus L_0$. Denote by $U_p$ the graph induced by $U$ in the cylinder $V_p = [-p, p] \times B(0; p)$ of $\mathbb{Z} \times \mathbb{Z}^d$.

**Lemma (5.16.2)** If $\mathbb{P}(U \text{ is finite}) = 1$, then $\mathbb{P}(\text{all the components of } \mathcal{F} \text{ have one end}) = 1$.

**Proof of lemma.** For every $z \in \mathbb{Z}^{d+1}$, denote by $\mathcal{T}(z)$ the event where the component of $z$ in $\mathcal{F}$ is not one ended. Then,

$$\mathcal{F}^d \{\text{there exists a component of } \mathcal{F} \text{ that is not one ended}\} = \bigcup_{z \in \mathbb{Z}^{d+1}} \mathcal{T}(z).$$

We want to prove $\mathbb{P}(\mathcal{F}) = 0$. By translation invariance, $\mathbb{P}(\mathcal{T}(z))$ is independent of $z$, and therefore, it suffices to prove that $\mathcal{T}(0)$ is a null event.

Now, since $0$ is a vertex of $L_0 \subset \mathcal{F}$ and this is an infinite branch (a.s.), the relation that the component of $0$ in $\mathcal{F}$ is not one ended signifies that $U$ does not possess a ray, and this is equivalent to $U$ being finite since it is a locally finite tree. We established that $\mathcal{T}(0)^d = \{U \text{ is finite}\}$. The conclusion of the lemma is now clear.

We know that for $p \geq P$, no edge in $U$ is adjacent to $E_p$, and the only way $U$ can have infinitely many edges is by using edges adjacent to $C_{p,0}$. Let $\mathfrak{R}_{p-1}$ be the (random) set of edges of $V_p = [-p, p] \times B(0; p)$ that are adjacent to $U_{p-1}$ and $C_{p,0}$. We need to study the size of the sets $\mathfrak{R}_p$, the following lemma will be needed.

**Lemma (5.16.3)** Suppose that $\mathfrak{A}$ is a random spanning subgraph of $G$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathfrak{A}$ satisfies the following: if $v$ and $v'$ are two vertices that are end-points of edges of $\mathfrak{A}$, then $v$ and $v'$ are in the same connected-component relative to $\mathfrak{A}$. (Think of $\mathfrak{A}$ as a connected subgraph of $G$ plus all the vertices of $G$.) Denote by $(V_n)$ an exhaustion of $G$ such that $\bigcup_{n=1}^\infty V_n$ for every $n \in \mathbb{N}$. Let $\mathcal{F}_n$ denote the $\sigma$-algebra generated by the events $\{v \in \mathfrak{A}\}$ as $v$ runs through the edges of the graph induced on $V_n$. Define $\mathfrak{A}_n$ to be the spanning subgraph of $G$ whose edge set consists of the edges of $\mathfrak{A}$ with both end-points belonging to $V_n$ (with $\mathfrak{A}_0 = \emptyset$). Consider the events $\{n \in \mathbb{N} \mid \mathfrak{A}_n \neq \mathfrak{A}_{n-1}\}$, so that $\mathfrak{S}_n$ is the event where there was “growth by edges” of $\mathfrak{A}$ in $V_n$. Set $Y_n = \mathbb{P}(\mathfrak{S}_n \mid \mathcal{F}_{n-1})$ ($n \in \mathbb{N}$). Then $(\mathfrak{S}_n)_{n \in \mathbb{N}}$ is a decreasing sequence, if $\mathfrak{S}_\infty = \bigcap_{n \in \mathbb{N}} \mathfrak{S}_n$, then $\mathfrak{S}_\infty$ is the event where $\mathfrak{A}$ has infinite edges; furthermore, on $\mathfrak{S}_\infty$, $Y_n \to 1$. 

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Proof of lemma. It is clear that \( G_n^c \subset G_{n+1}^c \), hence \( G_n^c \) is decreasing. Next, \( 1 \geq P \left( \bigcup_{n \in \mathbb{N}} G_n^c \cap G_{n-1} \right) = \sum_{n \in \mathbb{N}} \mathbb{E} \left( E \left( I_{G_n^c} I_{G_{n-1}} | F_{n-1} \right) \right) \). It is clear that \( G_{n-1} \in F_{n-1} \) and so, \( 1 \geq \mathbb{E} \left( \sum_{n \in \mathbb{N}} I_{G_{n-1}} (1 - Y_n) \right) \). Therefore, the positive random variable \( \sum_{n \in \mathbb{N}} I_{G_{n-1}} (1 - Y_n) \) is finite for \( \mathbb{P} \)-a.e. realisation \( \omega \in \Omega \). Thus, \( \sum_{n \in \mathbb{N}} (1 - Y_n) \) is finite \( \mathbb{P} \)-a.s. on the event \( \mathcal{G}_\infty \), which immediately implies the conclusion to be reached, namely, that \( Y_n \to 1 \) on \( \mathcal{G}_\infty \).

Let us return to the proof of Theorem 4.10. We apply the previous lemma to the random graph \( \mathcal{U} \), and the exhaustion \( V_p = [-p, p] \times B (0 ; p) \). Then, the events constructed in the lemma are \( \mathcal{S}_p = \{ \mathcal{U} \neq \mathcal{U}_{p-1} \} \) and the \( \sigma \)-algebras are \( \mathcal{F}_p = \sigma (\{ e \in \mathcal{U}_p \}; e \) is an edge of the graph induced on \( V_p \) by \( \mathbb{Z}^{d+1} \). By construction, \( \mathcal{K}_{p-1} \) is \( \mathcal{F}_{p-1} \)-measurable. Set \( Y_p = P ( \mathcal{S}_p | \mathcal{F}_{p-1} ) \), so that \( \mathbb{P} \)-a.s. \( Y_p \to 1 \) on the event where \( \mathcal{U} \) is infinite. Now,

\[ 1 - Y_p = P ( \mathcal{U} = \mathcal{U}_{p-1} | \mathcal{F}_{p-1} ) = P ( \mathcal{U} = \mathcal{U}_{p-1}, p \geq P | \mathcal{F}_{p-1} ) + P ( \mathcal{U} = \mathcal{U}_{p-1}, p < P | \mathcal{F}_{p-1} ) \]

\[ = P ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset, p \geq P | \mathcal{F}_{p-1} ) + P ( \mathcal{U} = \mathcal{U}_{p-1}, p < P | \mathcal{F}_{p-1} ) \]

Observe that \( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset \), since, by definition, \( \mathcal{K}_{p-1} \) are the edges adjacent to both \( \mathcal{C}_{p,0} \) and \( \mathcal{U}_{p-1} \) in \( V_p \). Thus, \( 1 - Y_p = P ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset | \mathcal{F}_{p-1} ) - P ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset, \mathcal{U} \neq \mathcal{U}_{p-1}, p < P | \mathcal{F}_{p-1} ) \).

We know that \( P ( \mathcal{E}_p ) \to 0 \), so the sequence \( ( P ( \mathcal{E}_p | \mathcal{F}_{p-1} ) ) \) converges to zero in \( L^1 \). We may assume, passing through a subsequence should the need arise, that \( ( P ( \mathcal{E}_p | \mathcal{F}_{p-1} ) ) \) converges to zero \( \mathbb{P} \)-a.s. Therefore, \( 1 - Y_p \geq P ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset | \mathcal{F}_{p-1} ) - P ( \mathcal{E}_p | \mathcal{F}_{p-1} ) \), thus, on the event where \( \mathcal{U} \) is an infinite component, \( P ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset | \mathcal{F}_{p-1} ) \to 0 \).

Denote by \( \mathcal{G}_p \) the random graph obtained from \( \mathbb{Z}^{d+1} \) where all edges \( e \in \mathcal{U}_p \) are shorted and the edges \( e \) in \( V_p \) that are not in \( \mathcal{U}_p \) are cut \( ( p \in \mathbb{N} ) \). Elementary properties of wired spanning forest measures show that \( P ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset | \mathcal{F}_{p-1} ) = \text{WSF}_{\mathcal{G}_{p-1}} ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset ) \), the right hand side meaning the WSF measure on the random graph \( \mathcal{G}_{p-1} \); in this graph however, \( \mathcal{K}_{p-1} \) is not random, while \( \mathcal{S} \) is its WSF-distributed random object. We can apply \cite{LP} Lemma 10.40] and conclude

\[ \text{WSF}_{\mathcal{G}_{p-1}} ( \mathcal{S} \cap \mathcal{K}_{p-1} = \emptyset ) \geq \prod_{e \in \mathcal{K}_{p-1}} \frac{1}{1 + \mu ( e ) \text{R}^W_{\mathcal{G}_{p-1} \setminus \mathcal{K}_{p-1}} ( e^- , e^+ )} \]

where \( \text{R}^W_{\mathcal{G}} ( e^- , e^+ ) \) stands for the wired effective resistance between \( e^- \) and \( e^+ \) in the network \( G \). We know that \( \mu ( e ) \propto e^{-\lambda p} \) for \( e \in \mathcal{K}_{p-1} \) and \( \text{R}^W_{\mathcal{G}_{p-1} \setminus \mathcal{K}_{p-1}} ( e^- , e^+ ) \leq \text{R}_{\mathcal{G}_{p-1} \setminus \mathcal{K}_{p-1}} ( e^- , \infty ) + \text{R}_{\mathcal{G}_{p-1} \setminus \mathcal{K}_{p-1}} ( e^+ , \infty ) \). The following lemma is crux in the proof and shows at once why \( d \geq 3 \) is needed (which has not been needed so far).

Lemma (5.16.4) Let \( d \geq 3 \). For \( e \in \mathcal{K}_{p-1} \), we have \( \text{R}_{\mathcal{G}_{p-1} \setminus \mathcal{K}_{p-1}} ( e^\pm , \infty ) \propto e^{\lambda p} \) (any implicit constant being universal).

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Proof of lemma. By Thompson’s principle $\mathcal{R}_G(v, \infty) = \inf_\theta \mathcal{E}(\theta)$, where $\theta$ runs on the sets of unit flows from $v$ to $\infty$. Since $d \geq 3$, we have that the standard random walk of $\mathbb{Z}^d$ is transient and T. Lyon’s criterion shows that the network $G$ of $\mathbb{Z}^d$ contains the hyperplane $H_p = \{ -p \} \times \mathbb{Z}^d$. Observe that $H_p$ as a subnetwork of $\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}$ has constant resistances equal to $e^{\lambda \rho}$. Translate $\theta_0$ to $H_p$ in the obvious way and denote the new flow by $\theta_p$. Clearly, $\mathcal{E}_{H_p}(\theta_p) = e^{\lambda \rho} \mathcal{E}_{Z^d}(\theta_0)$, where $\mathcal{E}_{\mathcal{G}}(\theta)$ is the energy of the flow $\theta$ on the network $G$. Extend $\theta_p$ to $\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}$ by sending zero flow through the edges outside $H_p$. The energy of the extension of $\theta_p$, denoted by $\mathcal{E}_{\mathcal{G}}(\theta_p)$, does not change, that is to say, $\mathcal{E}_{H_p}(\theta_p) = \mathcal{E}_{\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}}(\theta_p)$.

Using (5.5.1), we may calculate $P(2(\lambda + 1)e^{\lambda \rho} e^{-\lambda \rho})$. Thus, we proved that there exists a $c = e^{\lambda \rho} (\theta_0) > 0$ such that for all $p \in \mathbb{N}$ and all edges $e \in \mathcal{R}_{p-1}$, $\mathcal{E}_{\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}}(e^\pm, \infty) \leq c e^{\lambda \rho}$.

The reverse inequality is not needed but we prove it for completeness. Here we use that $\mathcal{E}(v, \infty) = \mathcal{R}(v, \infty)^{-1}$ and we bound from above the effective conductance. We apply Dirichlet’s principle. Consider a vertex $z = (-p, x)$ and define $\varphi = 1_z$. Then, $\mathcal{E}_{\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}}(z, \infty) \leq \mathcal{E}_{\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}}(\varphi) \leq \sum_{e \text{ and edge of } \Gamma_d(\lambda)} \mu(e) \leq 2(d+1)e^{\lambda \rho}$. Thus, for $e^{-1} = 2(d+1)e^{\lambda \rho} > 0$, we have $\mathcal{E}_{\mathcal{G}_{p-1} \setminus \mathcal{R}_{p-1}}(z, \infty) \geq c e^{\lambda \rho}$. □

We may continue with the proof of one endedness in $\Gamma_d(\lambda)$. By virtue of the previous lemma, there exists a universal $\delta > 0$ such that $P_a.s.$ $\mathcal{W}_G \mathcal{E}_{\mathcal{G}_{p-1}}(z, \infty) \geq \delta \text{card}(\mathcal{S}_{p-1})$. On the event $\mathcal{U}$ is infinite, we proved that $\mathcal{W}_G \mathcal{E}_{\mathcal{G}_{p-1}}(z, \infty) \geq \delta \text{card}(\mathcal{S}_{p-1})$. On the event $\mathcal{U}$ is infinite, we proved that $\mathcal{W}_G \mathcal{E}_{\mathcal{G}_{p-1}}(z, \infty) \geq \delta \text{card}(\mathcal{S}_{p-1})$. On the event $\mathcal{U}$ is infinite, we proved that $\mathcal{W}_G \mathcal{E}_{\mathcal{G}_{p-1}}(z, \infty) \geq \delta \text{card}(\mathcal{S}_{p-1})$.

Consider then the random variables $Z_{p-1} = \text{card}(\mathcal{S}_{p-1})$. Notice that $(-p, x) = C_{p,0}$ is adjacent to an edge of $\mathcal{R}_{p-1}$ if and only if $(-p+1, x)$ is adjacent to an edge in $\mathcal{U}_{p-1}$. Thus,

$$E(Z_{p-1}) = \sum_{z \in C_{p-1,0}} P(z \text{ is connected to } 0 \text{ in } \mathcal{U}_{p-1}) \leq \sum_{z \in C_{p-1,0}} P(z \text{ is connected to } 0 \text{ in } \mathcal{S} \setminus L_0).$$

Using (5.5.1), we may calculate $P(z \text{ is connected to } 0 \text{ in } \mathcal{S} \setminus L_0)$ assuming that the order in which the vertices of $\mathbb{Z}^{d+1}$ were searched was $(0, z, \ldots)$. Then, the event where $z$ is connected to $0$ in $\mathcal{S} \setminus L_0$ is the event where $S^z$ visits $0$ before hitting $L_0$ anywhere else. In particular, $E(Z_{p-1}) \leq \sum_{z \in C_{p-1,0}} P(\tau_0(S^z) < \infty) \leq \sum_{z \in C_{p-1,0}} G(z, 0) \leq c$ by (5.16.1). This implies $P(Z \to \infty) = 0$ (by Fatou’s lemma). Finally, we showed that $P(\text{card}(\mathcal{U}) = \infty) \leq P(Y_p \to 1) \leq P(Z \to \infty) = 0$. The proof of (5.16) is complete by Lemma (5.16.2). □

5.5 On the rays of $\Gamma_d(\lambda)$

We will say that a path $\gamma = (v_j)_{j \in \mathbb{Z}_+}$ is ultimately inside a region $R$ if there exists an almost surely finite random index $J$ such that $v_j \in R$ for all $j \geq J$.

Denote $P = \{ z = (n, x) \in \mathbb{Z} \times \mathbb{Z}^d \mid \| x \|^2 \leq n \}$ and $P_\varepsilon = \{ z \mid \| x \|^2 \leq n^{1+\varepsilon} \}$.

Theorem 5.17 Let $z$ be any vertex of $\Gamma_d(\lambda)$. Then, there exists one and only one ray in USF (UST in $d = 1, 2$) starting at $z$. This ray is a random object (is USF-measurable). Furthermore, for every $\varepsilon > 0$, this ray is ultimately inside $z + P_\varepsilon$, for every $\varepsilon > 0$.

Proof. By Theorems 5.1, 5.15 and 5.16 there exists one and only one ray, this proves at once such ray is USF-measurable. There remains to prove that said ray is ultimately inside $z + P_\varepsilon$ for every $\varepsilon > 0$. We may construct USF in $d = 1, 2$ or USF for $d \geq 3$ using Wilson’s algorithm rooted at infinity with the ordering of $\mathbb{Z}^{d+1}$ to be $(z, \ldots)$. Thus, suffices to show that if $S^z$ is the network random walk of $\Gamma_d(\lambda)$ started at $z$, then $S^z$ is ultimately inside $z + P_\varepsilon$; furthermore, by translation invariance, we may assume $z = 0$.
6 Probability of same component

Let $S$ denote the network random walk of $\Gamma_\lambda(\zeta)$ started at zero. To show that $S$ is ultimately inside $P$ is the same as showing that, for every $\varepsilon > 0$, $S$ will only be finitely many times outside $P^\varepsilon$. We may apply Borel-Cantelli lemma to show this and as such, it all reduces to show that $\sum_{n \in \mathbb{Z}_+} P(S_n \notin P^\varepsilon) < \infty$. The Green’s function estimates (2.22) give at once the probability that $0$ and $z$ are in the same tree of $\Gamma_\lambda$, thus no decay. We remark that for every $\varepsilon > 0$, $\eta \in \mathcal{G}(\zeta)$ has the following properties, all of which are obvious. (a) $\eta(z) \geq 0$, and $\eta(z) = 0$ is equivalent to $z = 0$. (b) $\eta(z) = \eta(-z)$; (c) $\eta(z + z') \leq \eta(z) + \eta(z')$.

Thus, the function $(z, z') \mapsto \eta(z - z')$ is a metric on $\mathbb{Z}^{d+1}$.

**Theorem (6.2)** Let $d \geq 3$. Then, $\text{USF}(0)$ is connected to $z \preceq \eta(z)^{-(d-2)}$, with any constant depending solely on dimension.

**Proof.** We remark that $\eta(z)^{-(d-2)} \asymp \max\left(n^{\frac{2}{d+1}}, |x|^{-(d-2)}\right)$, and thus it suffices to show the bounds for any of these expressions. We also remark that for every $c > 0$, there exists $c' > 0$ such that $e^{-c|z|} \leq c' \eta(z)^{-(d-2)}$; thus, if at some point, we show an upper exponential bound in an estimate, we are done with that particular estimate.

Consider the event $\mathcal{G}(0, z)$ that $0$ is connected to $z$ in $\text{USF}$. Since the measure $\text{USF}$ is translation invariant in $\Gamma_\lambda(\zeta)$, $\text{USF}(\mathcal{G}(0, z)) = \text{USF}(\mathcal{G}(z', z + z'))$ for any $z' \in \mathbb{Z}^{d+1}$. It is also clear that $\mathcal{G}(0, z) \preceq \mathcal{G}(z, 0)$. Thus, we may assume $z \in \mathbb{Z}_+ \times \mathbb{Z}^d$. Construct now $\text{USF}$ using Wilson’s algorithm rooted at infinity using the order $(0, z, \ldots)$ of $\mathbb{Z}^{d+1}$, denote by $\mathfrak{F}$ the random forest constructed in this way. Thus, we have on some probability space $(\Omega, \mathcal{F}, P)$ a family of independent network random walks $(S^u)_{u \in \mathbb{Z}^{d+1}}$, with $S^u$ started at $v$. We know $\mathfrak{F} \sim \text{USF}$, and by construction $\{\mathfrak{F} \in \mathcal{G}(0, z)\} = \{S^u \cap \text{LE}(S^0_m)_{m \in \mathbb{Z}_+} \neq \emptyset\}$. Thus, and with the aid of (5.5),

(A) \[ \text{USF}(\mathcal{G}(0, z)) = P \left( S^u \cap \text{LE}(S^0_m)_{m \in \mathbb{Z}_+) \neq \emptyset} \right) \approx P \left( S^u \cap S^0 \neq \emptyset \right). \]
Denote by $K_z$ the number of intersections between $S^z$ and $S^0$. In other words, \( K_z = \sum_{z' \in \mathbb{Z}^d+1} \sum_{(p,q) \in \mathbb{Z}_+^2} 1 \{ S_p^z = z', S_q^0 = z' \} \).

Clearly, $\mathbb{P}(K_z > 0) = \mathbb{P}(S^z \cap S^0 \neq \emptyset)$, Independence and the Lebesgue-Tonelli theorem show at once
\[
E(K_z) = \sum_{z' \in \mathbb{Z}^d+1} G(z, z') G(0, z').
\]

**Lemma (6.2.1)** We have $\mathbb{P}(K_z > 0) \approx E(K_z)$, where any implicit constant is universal (does not depend on $z$).

**Proof of Lemma.** Since $K_z$ is a $\mathbb{Z}_+$-valued random variable, we have $\mathbb{P}(K_z > 0) \leq E(K_z)$. There remains to prove a lower bound of the form $\mathbb{P}(K_z > 0) \geq c E(K_z)$, for a universal constant $c > 0$. Using the second moment inequality \( L13 \), suffices to establish that $E(K_z)^2 \leq c E(K_z)$, for some universal constant $c > 0$. By definition,
\[
K_z^2 = \sum_{(z', z'') \in \mathbb{Z}^d+1 \times \mathbb{Z}^d+1} \sum_{(p, q, a, b) \in \mathbb{Z}_+^4} 1 \{ S_p^z = z', S_q^0 = z'' \} 1 \{ S_p^0 = z', S_q^0 = z'' \}.
\]

By adding terms corresponding to $p = a$ or $q = b$, then expanding the sum as to whether $p \leq a$ or $p \geq a$ and $q \leq b$ or $q \geq b$, we reach a bound with four sums
\[
K_z^2 \leq L_1 + L_2 + L_3 + L_4,
\]
and
\[
E(L_1) = \sum_{(z', z'') \in \mathbb{Z}^d+1 \times \mathbb{Z}^d+1} G(z, z') G(z', z'')^2 G(0, z'),
\]
\[
E(L_2) = \sum_{(z', z'') \in \mathbb{Z}^d+1 \times \mathbb{Z}^d+1} G(z, z') G(z', z'') G(z'', z') G(0, z''),
\]
\[
E(L_3) = \sum_{(z', z'') \in \mathbb{Z}^d+1 \times \mathbb{Z}^d+1} G(z, z'') G(z'', z') G(0, z'),
\]
\[
E(L_4) = \sum_{(z', z'') \in \mathbb{Z}^d+1 \times \mathbb{Z}^d+1} G(z, z'') G(z'', z'')^2 G(0, z'').
\]

It is clear that $E(L_1) = E(L_4)$ and $E(L_2) = E(L_3)$, thus
\[
E(K_z^2) \leq 2 \left[ E(L_1) + E(L_2) \right]
\]

We first handle the expectation of $L_1$, we obtain
\[
E(L_1) = \sum_{z' \in \mathbb{Z}^d+1} G(z, z') G(0, z') \sum_{z'' \in \mathbb{Z}^d+1} G(z'', z'')^2 = \left[ \sum_{z'' \in \mathbb{Z}^d+1} G(0, z'')^2 \right] \sum_{z' \in \mathbb{Z}^d+1} G(z, z') G(0, z'),
\]
and $\sum_{z'' \in \mathbb{Z}^d+1} G(0, z'')^2$ is a universal constant by the bubble condition \( L11 \).

We now handle $L_2$. First, the translation invariance of the Green’s function shows at once that
\[
E(L_2) = \sum_{(z', z'') \in \mathbb{Z}^d+1} G(z, z') G(z'', z') G(0, z'') = \sum_{z''} G(z, z') \sum_{z''} G(0, z'') G(z'', 0) G(0, z' + z'').
\]

In the following calculations we are going to establish that the inner sum in the line above, when viewed as a function of $z'$, possesses the same type of upper bounds as the Green’s function’s upper bounds \( L22 \).

There are two cases to consider. Write $z' = (n', x')$ and similarly $z'' = (n'', x'')$. 

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Case 1. Here we assume $\|x'\| \leq n'$. We then divide the inner sum into two, the first consists of all $z''$ satisfying $|n''| \leq \frac{n'}{2}$ and $\|x''\| \leq \frac{\|x'\|}{2}$ and the second sum is over all other $z''$. In the first sum, we reach $\|x' + x''\| \approx \|x'\|$ and $|n' + n''| \approx n' \approx \|z'\|$, then $G(0, z' + z'') \leq c e^{-c \frac{\|z''\|^2}{n''}} \|z''\|^{-\frac{d}{2}}$ for some constants $c', c'' > 0$. This entails

$$\sum_{|n''| \leq \frac{n'}{2}, \|x''\| \leq \frac{\|x'\|}{2}} G(z'', 0)G(0, z'')G(0, z' + z'') \leq c e^{-c \frac{\|z''\|^2}{n''}} \|z''\|^{-\frac{d}{2}} \sum_{z'' \in \mathbb{Z}^{d+1}} G(z'', 0)G(0, z''),$$

for a pair of constants $c, c' > 0$. We may apply the inequality $ab \leq a^2 + b^2$, and obtain that $\sum_{z'' \in \mathbb{Z}^{d+1}} G(z'', 0)G(0, z'') < \infty$ by the bubble condition (3.10). This shows that

$$\sum_{|n''| \leq \frac{n'}{2}, \|x''\| \leq \frac{\|x'\|}{2}} G(z'', 0)G(0, z'')G(0, z' + z'') \leq c e^{-c \frac{\|z''\|^2}{n''}} \|z''\|^{-\frac{d}{2}},$$

for a pair of positive constants $c, c', c'' > 0$. Next, we are going to handle the second sum, which is over all $z''$ for which either $|n''| > \frac{n'}{2}$ or $\|x''\| > \frac{\|x'\|}{2}$. Here, we will use that one of the two factors $G(z'', 0)$, $G(0, z'')$ has an upper bound of the form $e^{-c \|z''\|}$. We will write $e^{-c \|z''\|} = e^{-\frac{c}{2} \|z''\|} e^{-\frac{c}{2} \|z''\|}$ and use the fact that the Green’s function is bounded by some constant $L > 0$. Since $\|z''\| \approx n'$, we may write

$$\sum_{|n''| > \frac{n'}{2}, x'' \in \mathbb{Z}^{d}} G(z'', 0)G(0, z'')G(0, z' + z'') \leq L^2 e^{-\frac{c}{2} \|z''\|} \sum_{z'' \in \mathbb{Z}^{d+1}} e^{-\frac{c}{2} \|z''\|} \leq c' e^{-c' \|z''\|},$$

for positive constants $c, c', c'' > 0$. It is clear that $c' e^{-c' \|z''\|} \leq C e^{-c' \frac{\|z''\|^2}{n''}} \|z''\|^{-\frac{d}{2}}$ for another pair of constants $C, C' > 0$. We finally deal with the sum corresponding to $|n''| \leq \frac{n'}{2}$ and $\|x''\| > \frac{\|x'\|}{2}$. Here we decompose again $e^{-c \|z''\|} = \left(e^{-\frac{c}{2} \|z''\|}\right)^2$. The assumption $\|x'\| \approx n'$ shows that $\|x''\| \approx \frac{\|z''\|^2}{n''}$, and thus

$$e^{-\frac{c}{2} \|z''\|} \leq e^{-\frac{c}{2} \|x''\|} \leq e^{-\frac{c}{2} \|z''\|} \leq e^{-\frac{c}{2} \frac{\|z''\|^2}{n''}}.$$

Also, it is clear that $\|G(0, z' + z'')\| \leq c' \|z''\|^{-\frac{d}{2}}$ for some positive constant $c' > 0$ since $\|z' + z''\| \approx \|z''\|$. Thus,

$$\sum_{|n''| \leq \frac{n'}{2}, \|x''\| > \frac{\|x'\|}{2}} G(z'', 0)G(0, z'')G(0, z' + z'') \leq c' e^{-\frac{c}{2} \frac{\|z''\|^2}{n''}} \|z''\|^{-\frac{d}{2}},$$

for a pair of constants.

Case 2. Here we assume $|n'| < \|x'\|$. Split $\sum_{z'' \in \mathbb{Z}^{d+1}} G(z'', 0)G(0, z'')G(0, z' + z'') = A + B$, where $A$ is the sum corresponding to all $z'' = (n'', x'')$ satisfying $|n''| \leq \frac{\|x''\|}{2}$ and $\|x''\| \leq \frac{\|x'\|}{2}$, $B$ is the sum over all other $z''$. For $z''$ in the range of $A$, we obtain $\|x' + x''\| \approx \|x'\| \approx \|z''\|$ and $|n' + n''| \leq c \|x'\| \leq c \|z''\|$, and this shows that, on this range, $G(0, z' + z'') \leq e^{-c \|z''\|}$ for a pair of constants $c, c' > 0$. Then, there exists two universal constants $c, c' > 0$ such that (recall from Case 1 that $G(z'', 0)G(0, z'')$ is a summable family for $z'' \in \mathbb{Z}^{d+1}$) $A \leq c e^{-c \|z''\|}$. In the range of $B$, we have that either $|n''| > \frac{\|x''\|}{2} \approx \|z''\|$ or $\|x''\| > \frac{\|x'\|}{2} \approx \|z''\|$, thus $\|z''\| \geq c \|z''\|$ for a universal constant $c > 0$. Next, one of the two factor
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\( G(z'', 0) \) or \( G(0, z'') \) is bounded from above by \( e^{-c\|z''\|} \) by (2.22). Let \( L > 0 \) be a bound for the Green’s function of \( \Gamma_d(a) \). Then \( B \leq L^2 e^{-c\|z''\|} \sum_{z'' \in \mathbb{Z}^{d+1}} e^{-\frac{\|z''\|}{2}} \|z''\| \leq Ce^{-c\|z''\|}. \)

By virtue of the two cases above, we have established that there exists two constants \( c, c' > 0 \) such that (write \( z' = (n', x') \))

\[
\sum_{z'' \in \mathbb{Z}^{d+1}} G(z'', 0)G(0, z'')G(z' + z'') \leq \begin{cases} 
    ce^{-c\|z''\|} & \text{if } \|x'\| > n', \\
    ce^{-c\|z''\|} & \text{if } \|x'\| \leq n'. 
\end{cases}
\]

The proof of (6.2.1) is now an immediate consequence of the next lemma. Indeed, the next lemma shows at once that \( \mathbb{E}(K_z) \approx \eta(z)^{-(d-2)} \) and that \( \mathbb{E}(L_2) \leq c\eta(z)^{-(d-2)} \). Therefore, \( \mathbb{E}(K_Z^2) \leq c\eta(z)^{-(d-2)} \leq c'\mathbb{E}(K_Z) \), as wanted.

**Lemma (6.2.2)** Suppose that \( \varphi \) and \( \psi \) are two bounded functions defined on \( \mathbb{Z}^{d+1} \) and with positive values. Furthermore, suppose that there exists two constants \( c_1, c_2 > 0 \) such that

\[
\max(\varphi(z), \psi(z)) \leq \begin{cases} 
    c_1 e^{-c_2 \|z\|} & \text{if } \|x\| > n, \\
    c_1 e^{-c_2 \|z\|} & \text{if } \|x\| \leq n. 
\end{cases}
\]

Then, there exists a universal constant \( c > 0 \) such that for any \( z \in \mathbb{Z}^{d+1} \), \( \sum_{z' \in \mathbb{Z}^{d+1}} \varphi(z' - z)\psi(z') \leq c\eta(z)^{-(d-2)}. \)

Likewise, if there exists two constants \( c_3, c_4 > 0 \) such that

\[
\min(\varphi(z), \psi(z)) \geq \begin{cases} 
    c_3 e^{-c_4 \|z\|} & \text{if } \|x\| > n, \\
    c_3 e^{-c_4 \|z\|} & \text{if } \|x\| \leq n. 
\end{cases}
\]

then there exists a universal constant \( c > 0 \) such that for any \( z \in \mathbb{Z}^{d+1} \), \( \sum_{z' \in \mathbb{Z}^{d+1}} \varphi(z' - z)\psi(z') \geq c\eta(z)^{-(d-2)}. \)

**Proof of Lemma.** We divide the proof of the upper and lower bounds. We write \( z = (n, x, z') \) and so on with other affixes (such as superscripts).

1. **Upper bound.** There is no loss of generality to assume \( \varphi = \psi \). Decompose \( \sum_{z' \in \mathbb{Z}^{d+1}} \varphi(z' - z)\varphi(z') = H + I + J \), where

\[
H = \sum_{n \leq 0} \sum_{x' \in \mathbb{Z}^d} \varphi(z' - z)\varphi(z'), \quad I = \sum_{1 \leq n \leq n-1} \sum_{x' \in \mathbb{Z}^d} \varphi(z' - z)\varphi(z'), \quad J = \sum_{n \geq n} \sum_{x' \in \mathbb{Z}^d} \varphi(z' - z)\varphi(z')
\]

We shall bound now each of these terms \( H, I, J \) and we will see, at the end, that the largest of these bounds is of the claimed form. We will use the bounds in the hypothesis and the estimates of (2.3) to handle the resulting sums.

We start by bounding \( H \). Since \( z \in \mathbb{Z}_+ \times \mathbb{Z}_+ \) and \( z' \in \mathbb{Z}_- \times \mathbb{Z}_- \), we have the bounds \( \varphi(z' - z)\varphi(z') \leq ce^{-c'((\|z-z'\|+\|z'\|)} \), for a pair of positive constants. Now,

\[
ce^{-c'((\|z-z'\|+\|z'\|)} \leq ce^{-\frac{c'}{2}(\|z\|-\|z'\|)} - c'\|z'\| \leq ce^{-\frac{c'}{2}\|z\|}e^{-\frac{c'}{2}\|z'\|},
\]

summing over \( z' \in \mathbb{Z}_- \times \mathbb{Z}_- \), we reach

\[
H \leq ce^{-c'\|z\|}, \quad (E)
\]
for a pair of constants $c, c' > 0$.

We now handle $I$ from (D). Divide into two parts as follows

$$I = I_1 + I_2 = \sum_{1 \leq n' \leq n-1} \varphi(z' - z) \varphi(z') + \sum_{1 \leq n' \leq n-1} \varphi(z' - z) \varphi(z').$$

Then, $I_1 \leq c \sum_{1 \leq n' \leq n-1} \exp \left( -c' \left( \frac{\|z - z'|| + \|z'\|^2}{n'} \right) \right) (n')^{-\frac{d}{2}}$. Notice $\|z - z'||^2 = (n - n')^2 + \|x - x'||^2 \geq \frac{(n - n')^2}{2} \left( \frac{\|x - x'||}{\sqrt{2}} + \frac{\|z'\|}{\sqrt{2}} \right)^2$, thus $\|z - z'|| \geq \frac{|n - n'| + \|x - x'||}{\sqrt{2}}$. Hence,

$$I_1 \leq c \sum_{n' = 1}^{n-1} \left( n' - \frac{d}{2} e^{-c'(n-n')} \sum_{\|x\| \leq n'} \exp \left( -c' \left( \frac{\|x - x'|| + \|z'\|^2}{n'} \right) \right) \right).$$

For convenience, we now divide into two cases. But first, by (2.12), we can bound

$$\sum_{\|x\| \leq n'} \exp \left( -c' \left( \frac{\|x - x'|| + \|z'\|^2}{n'} \right) \right) \leq \begin{cases} ce^{-c'|x|} & \text{if } \|x\| > n' \\ ce^{-c'\frac{1}{n' + 2}} & \text{if } \|x\| \leq n'. \end{cases}$$

We may now proceed with the two cases.

**Case 1.** Here we assume $n \leq \|x\|$. Then, there exists two constants $c, c' > 0$,

$$I_1 \leq ce^{-c'|x|} \sum_{n' = 1}^{n-1} (n')^{-\frac{d}{2}} e^{-c'(n-n')} \leq c \left( \sum_{k = 1}^{n} k^{-\frac{d}{2}} \right) e^{-c'|x|} \leq Ce^{-C'|x|}.$$

**Case 2.** Here we assume $\|x\| < n$. Then, we divide the sum in the right of (E) into two parts. The first part consists on all terms for which $1 \leq n' \leq \|x\|$ and the second, $\|x\| < n' \leq n - 1$. Then, for a pair of constants $C, C'' > 0$,

$$\sum_{1 \leq n' \leq \|x\|} (n')^{-\frac{d}{2}} e^{-\frac{c'}{2} (n-n')} \sum_{\|x\| \leq n'} e^{-\frac{c'}{2} \left( \frac{\|x - x'|| + \|z'\|^2}{n'} \right)} \leq C \sum_{1 \leq n' \leq \|x\|} (n')^{-\frac{d}{2}} e^{-\frac{c'}{2} (n-n')} e^{-C'|x|}.$$

By setting $c'' = \min \left( \frac{c'}{2}, C' \right)$, it is clear that $e^{-\frac{c'}{2} (n-n')} e^{-C'|x|} \leq e^{-c''(n-n') + \|x\|} \leq e^{-c''n'}$ for $1 \leq n' \leq \|x\|$. Then, the first part in the division of $I_1$ is bounded above by $ce^{-c'|x|}$ for a pair of constants $c, c' > 0$. As for the second part,

$$\sum_{\|x\| < n' \leq n-1} (n')^{-\frac{d}{2}} e^{-\frac{c'}{2} (n-n')} \sum_{\|x\| \leq n'} e^{-\frac{c'}{2} \left( \frac{\|x - x'|| + \|z'\|^2}{n'} \right)} \leq C \sum_{\|x\| < n' \leq n-1} (n')^{-\frac{d}{2}} e^{-\frac{c'}{2} (n-n')} e^{-C'\frac{\|x\|^2}{n'}},$$

where $C, C' > 0$ are two constants. We have shown up to this point that under the assumption $\|x\| < n$, there exists two constants $c, c' > 0$, such that

$$I_1 \leq ce^{-c'|x|} + c \sum_{\|x\| < n' \leq n-1} (n')^{-\frac{d}{2}} e^{-c'\left( n-n' + \frac{\|x\|^2}{n'} \right)}.$$
(a) Assume first $\|x\| \leq n^{\frac{1}{2}}$. Here, we will bound $e^{-c\frac{\|x\|^2}{n}} \leq 1$. Then,

\[
\sum_{\|x\|<n^\frac{1}{2}} (n')^{-\frac{3}{4}} e^{-c\left(n-n'+\frac{\|x\|^2}{n}\right)} = \sum_{\|x\|<n^\frac{1}{2}} (n')^{-\frac{3}{4}} e^{-c\left(n-n'\right)} + \sum_{n^\frac{1}{2}<n'<n} (n')^{-\frac{3}{4}} e^{-c\left(n-n'\right)}.
\]

In the first of these sums, $e^{-c\left(n-n'\right)} \leq e^{-c\left(n-n^\frac{1}{2}\right)} \leq Ce^{-C'n^\frac{1}{2}}$, for two constants $C, C' > 0$. We reach

\[
\sum_{\|x\|<n^\frac{1}{2}} (n')^{-\frac{3}{4}} e^{-c\left(n-n'\right)} \leq C \left[ \sum_{k=1}^{\infty} k^{-\frac{3}{4}} \right] e^{-c'n^\frac{1}{2}}.
\]

Observe now that if $n^\frac{1}{2} < n' \leq n$, there exists a unique integer $j$ between 1 and $n^\frac{1}{2}$ such that $jn^\frac{1}{2} < n' \leq (j+1)n^\frac{1}{2}$ (there are $n^\frac{1}{2}$ indices $n'$ in this range). Thus,

\[
\sum_{n^\frac{1}{2}<n'<n} (n')^{-\frac{3}{4}} e^{-c\left(n-n'\right)} \leq n^{-\frac{3}{4}} n^\frac{1}{2} \sum_{1 \leq j \leq n^\frac{1}{2}} j^{-\frac{3}{4}} e^{-c\left(n-n+\frac{1}{4}j\right)} = n^{-\frac{3}{4}} n^\frac{1}{2} \sum_{1 \leq j \leq n^\frac{1}{2}} j^{-\frac{3}{4}} e^{-c\left(n^\frac{1}{2}-\min\left(j+1,n^\frac{1}{2}\right)\right)}.
\]

In the last sum, consider first the range $1 \leq j \leq \frac{n^\frac{1}{2}}{4}$ to obtain $j+1 \leq 2j \leq \frac{n^\frac{1}{2}}{2}$, giving $e^{-c\left(n^\frac{1}{2}-\min\left(j+1,n^\frac{1}{2}\right)\right)} \leq e^{-c\left(n^\frac{1}{2}\right)}$. Next, consider the range $\frac{n^\frac{1}{2}}{4} < j \leq n^\frac{1}{2}$, so that $j \approx n^\frac{1}{2}$. Then,

\[
\sum_{n^\frac{1}{2}<n'<n} (n')^{-\frac{3}{4}} e^{-c\left(n-n'\right)} \leq Cn^{-\frac{3}{4}} n^\frac{1}{2} e^{-c\left(n^\frac{1}{2}\right)} + Cn^{-\frac{3}{4}} n^\frac{1}{2} n^{-\frac{3}{4}} n^\frac{1}{2} \leq Cn^{-\frac{3}{4}} + 1.
\]

The foregoing shows

\[
\sum_{\|x\|\leq n^\frac{1}{2}} (n')^{-\frac{3}{4}} e^{-c\left(n-n'+\frac{\|x\|^2}{n}\right)} \leq ce^{-c'n} + Cn^{-\frac{3}{4}} + 1 \leq Cn^{-\frac{3}{4}} + 1.
\]

(b) Now assume $n^\frac{1}{2} < \|x\|$. We bound the sum appearing in (G). We use (2.14),

\[
\sum_{\|x\|<n^\frac{1}{2}} (n')^{-\frac{3}{4}} e^{-c\left(n-n'+\frac{\|x\|^2}{n}\right)} \leq \sum_{n^\frac{1}{2} \leq \|x\|^2} (n')^{-\frac{3}{4}} e^{-c\left(x^2\right)} \leq e^{c\|x\|^{-(d-2)}}.
\]

Bearing in mind the previous two items, and (G), we have established that $I_1 \leq c\eta(z)^{-(d-2)}$, for a universal constant $c > 0$. This completes Case 2.

We now need to bound $I_2$. Again, by the bounds in the hypothesis,

\[
I_2 = \sum_{1 \leq n \leq n^\frac{1}{2}} \sum_{\|x\| \geq n^\frac{1}{2}} \varphi(z' - z) \varphi(z') \leq c^2 \sum_{1 \leq n \leq n^\frac{1}{2}} \sum_{\|x\| \geq n^\frac{1}{2}} e^{-c\|\|z'-z\|-\|z'\||} \leq e^{-c\|z\|},
\]

where we proceeded as in (E). Thus, summing the upper bounds of $I_1$ and $I_2$ we have shown that

\[
I \leq c\eta(z)^{-(d-2)},
\]

for a universal constant $c > 0$, as desired.
We now proceed to upper bound the sum $J$ of (16). First we rewrite, $J = \sum_{n'\geq 0} \sum_{x'=2^{n'}} \varphi(0, z') \varphi(0, z' + z) = J_1 + J_2$, where $J_1 = \sum_{n'\geq 0} \sum_{|x'|\leq n'} \varphi(0, z') \varphi(0, z' + z)$ and $J_2 = \sum_{n'\geq 0} \sum_{|x'|> n'} \varphi(0, z') \varphi(0, z' + z)$. We further divide $J_1$ as follows: $J_1 = J_{1,1} + J_{1,2}$, with $J_{1,1} = \sum_{n'\geq 0} \sum_{|x'| \leq n'} \varphi(0, z') \varphi(0, z' + z)$ and $J_{1,2} = \sum_{n'\geq 0} \sum_{|x'| > n'} \varphi(0, z') \varphi(0, z' + z)$. Let us handle $J_{1,1}$ first. In this case, we have the bound

$$J_{1,1} \leq c \sum_{n'\geq 0} (n')^{-\frac{d}{2}} (n + n')^{-\frac{d}{2}} e^{-c \|x\|^2} \sum_{|x'| \leq n'} \sum_{|x + x'| \leq n + n'} e^{-c \|x + x'\|^2} \leq c \sum_{n'\geq 0} (n + n')^{-\frac{d}{2}} \leq c' n^{-\frac{d}{2} + 1},$$

by (2.11) and (2.13) with all $c, c', c''$ positive constants. We may assume for the rest of the bounding of $J_{1,1}$ that $|x| > n^2$. By virtue of (2.15), in the range $0 \leq n' < \frac{|x|}{2}$, we may bound

$$e^{-c \|x\|^2} \sum_{|x'| \leq n'} \sum_{|x + x'| \leq n + n'} \leq c (n')^{-\frac{d}{2}} e^{-c \|x\|^2} \sum_{|x'| \leq n'} \sum_{|x + x'| \leq n + n'}$$

with $c$ a universal constant. Next,

$$\sum_{0 \leq n' \leq \frac{|x|}{2}} (n')^{-\frac{d}{2}} (n + n')^{-\frac{d}{2}} e^{-c \|x\|^2} \sum_{0 \leq n' \leq \frac{|x|}{2}} \leq c \sum_{0 \leq n' \leq \frac{|x|}{2}} (n + n')^{-\frac{d}{2}} e^{-c \|x\|^2} \sum_{n \leq m \leq n + \frac{|x|}{2}} m^{-\frac{d}{2}} e^{-c \|x\|^2}.$$

By the assumption $|x| > n^2$, we have

$$\sum_{n \leq m \leq n + \frac{|x|}{2}} m^{-\frac{d}{2}} e^{-c \|x\|^2} \leq \sum_{m \leq 2|\|x\|^2|} m^{-\frac{d}{2}} e^{-c \|x\|^2} \leq c \|x\|^{-(d-2)},$$

by (2.14). To sum up, we have established so far in the bounding of $J_{1,1}$ (with the assumption that $|x| > n^2$),

$$J_{1,1} \leq c\eta(z)^{-(d-2)} + c \sum_{n' \geq \frac{|x|}{2}} (n')^{-\frac{d}{2}} (n + n')^{-\frac{d}{2}} e^{-c \|x\|^2} \sum_{|x'| \leq n'} \sum_{|x + x'| \leq n + n'}$$

If $|x| \leq 2n'$, we obtain from (2.15)

$$\sum_{|x'| \leq n'} \sum_{|x + x'| \leq n + n'} e^{-c \|x\|^2} \sum_{|x'| \leq n'} \sum_{|x + x'| \leq n + n'} \leq c \left[ (n')^{-\frac{d}{2}} e^{-c \|x\|^2} + (n + n')^{-\frac{d}{2}} e^{-c \|x\|^2} \right].$$

Substituting this result above,

$$J_{1,1} \leq c\eta(z)^{-(d-2)} + c \sum_{n' \geq \frac{|x|}{2}} \left[ (n + n')^{-\frac{d}{2}} e^{-c \|x\|^2} + (n')^{-\frac{d}{2}} e^{-c \|x\|^2} \right]$$

$$\leq c\eta(z)^{-(d-2)} + 2c \sum_{n' > \frac{|x|}{2}} (n')^{-\frac{d}{2}} e^{-c \|x\|^2}.$$
Split the appearing sum in the line above into \( \|x\| < n' \leq \|x\|^2 \) and \( n' > \|x\|^2 \); bound the first resulting sum by (2.14) to obtain an upper bound of the form \( c \|x\|^{-(d-2)} \), and the second, by

\[
\sum_{n' > \|x\|^2} (n')^{-\frac{d}{2}} e^{-c' \frac{n'^2}{n^2}} \leq c_n \sum_{n' > \|x\|^2} (n')^{-\frac{d}{2}} e^{-c' \frac{n'^2}{n^2}},
\]

the last \( \approx \) was obtained from (2.13). We have established,

(I) \( J_{1,1} \leq c_\eta(z)^{-(d-2)} \),

c being a universal constant depending solely on dimension.

We now will handle \( J_{1,2} \). The hypotheses give

\[
J_{1,2} \leq c_1 \sum_{n' \geq 0} \sum_{\|x\|^2 \leq n'} (n')^{-\frac{d}{2}} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \leq c_1 \sum_{n' \geq 0} (n')^{-\frac{d}{2}} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \leq c \sum_{n' \geq 0} (n')^{-\frac{d}{2}} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \leq c (n')^{-\frac{d}{2}},
\]

It is now clear that we may bound \( J_{1,2} \leq ce^{-c'n} \), with a pair of constants \( c, c' > 0 \) depending solely on dimension. This establishes the case \( n^* \geq \|x\|^2 \) and thus, we may assume now \( \|x\| > n^* \) in what follows and bound \( e^{-c\|x\|^2} \leq 1 \). Consider an integer \( 0 \leq n' < \|x\| \), invoking (2.14) we reach

\[
\sum_{\|x\| \leq n'} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \leq ce^{-c'\|x\|},
\]

for universal constants \( c, c' > 0 \).

0. If \( n' \geq \|x\| \), bound \( \|x+x'\| \geq 0 \) and use (2.11): \( \sum_{\|x\| \leq n'} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \leq c(n')^{-\frac{d}{2}}, \) and then

\[
\sum_{n' \geq \|x\|} (n')^{-\frac{d}{2}} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \leq c \sum_{n' \geq \|x\|} e^{-c \left( \|x+z\|^2 + \frac{\|x\|^2}{n} \right)} \approx e^{-c\|x\|}.
\]

Thus, we may assert, there exists a constant \( c > 0 \) such that for all \( z = (n, x) \) satisfying \( \|x\| > n^* \), \( J_{1,2} \leq c_\eta(z)^{-(d-2)} \); the same type of bound for \( \|x\| \leq n^* \) was established already. Bearing in mind these bounds, (I), and since \( J_1 = J_{1,1} + J_{1,2} \), we have finally established

(J) \( J_1 \leq c_\eta(z)^{-(d-2)} \),

with \( c > 0 \) is a constant that does not depend on \( z \).

We will now prove the upper bound for \( J_2 \). Similarly as was done with \( J_1 \), we split \( J_2 = J_{2,1} + J_{2,2} \), being

\[
J_{2,1} = \sum_{n' \geq 0} \sum_{\|x\|^2 > n'} \varphi(z') \varphi(z' + z) \quad \text{and} \quad J_{2,2} = \sum_{n' \geq 0} \sum_{\|x\|^2 > n'} \varphi(z') \varphi(z' + z).
\]

Here, \( J_{2,2} \) is easy for we have for \( x = 0 \) the last equation of (2.14) and obtain

(K) \( J_{2,2} \leq c_1 \sum_{z \in \mathbb{Z}^{d+1}} e^{-c_2 \left( \|z'^\| + \|z^2 + z'^2\| \right)} \leq ce^{-c'\|z\|} \),

with \( c_1, c_2 \) the constants in the hypothesis and \( c, c' \) two positive universal constants. There remains to handle
6.1 Probability that two vertices are USF-connected

\[ J_{2, 1} \leq c_{1}^{2} \sum_{n' \geq 0} (n + n') - \frac{d}{4} e^{-\frac{c_{2} \lambda}{n + n'}} \sum_{n' \leq n + n'} e^{-\frac{d}{4} \sqrt{2} \frac{\lambda}{n + n'}} (n + n')^{-\frac{d}{4}} \]

\[ = c_{1}^{2} \sum_{n' \geq 0} (n + n') - \frac{d}{4} e^{-\frac{c_{2} \lambda}{n + n'}} \sum_{n' \leq n + n'} e^{-\frac{d}{4} \sqrt{2} \frac{\lambda}{n + n'}} \]

Suppose first that \( ||x|| \leq n^{\frac{d}{2}} \). The bounds of (2.2.12) give

\[ \sum_{n' \leq n + n'} e^{-\frac{d}{4} \sqrt{2} \frac{\lambda}{n + n'}} \leq ce^{-c_{e} \frac{\lambda^{2}}{n + n'}} \leq c, \] for a pair of constants \( c, c' > 0 \). Then, \( J_{2, 1} \leq c \sum_{n' \geq 0} (n + n') - \frac{d}{4} \leq c' n^{-\frac{d}{2} + 1} = c' \eta(z)^{-d-2} \), where the constants \( c, c' \) are positive and depending on nothing but, perhaps, dimension. This would finish the bound for \( J_{2, 1} \)

This yields \( c_{1}^{2} \sum_{n' \geq 0} \frac{(n + n') - \frac{d}{4} e^{-\frac{c_{2} \lambda}{n + n'}} \sum_{n' \leq n + n'} e^{-\frac{d}{4} \sqrt{2} \frac{\lambda}{n + n'}} \}

If \( ||x|| \geq n' + n \) then the inner sum is bounded by an expression of the form \( ce^{-c' ||x||} \), again by (2.12).

This yields \( c_{1}^{2} \sum_{n' \geq 0} \frac{(n + n') - \frac{d}{4} e^{-\frac{c_{2} \lambda}{n + n'}} \sum_{n' \leq n + n'} e^{-\frac{d}{4} \sqrt{2} \frac{\lambda}{n + n'}} \}

We may now use (2.13) and (2.14) as follows:

\[ \sum_{n' + n > ||x||} (n + n') - \frac{d}{4} e^{-\frac{c' (n' + n + \frac{1}{2})^{2}}{n + n'}} \leq c \sum_{n' + n > ||x||} (n + n') - \frac{d}{4} e^{-\frac{c' (n' + n + \frac{1}{2})^{2}}{n + n'}} + \sum_{n' + n > ||x||} (n + n') - \frac{d}{4} e^{-\frac{c' (n' + n + \frac{1}{2})^{2}}{n + n'}} \]

\[ \leq c ||x||^{-(d-2)} + c' ||x||^{-(d-2)} = (c + c') ||x||^{-(d-2)}. \]

Thus, we showed that there exists a constant \( c > 0 \) such that for all \( z \in \mathbb{Z}^{d+1} \),

\[ J_{2, 1} \leq c \eta(z)^{-(d-2)}. \]

Then, by (K) and (L), we reach \( J_{2} \leq c \eta(z)^{-(d-2)} \), with \( c \) not depending on \( z \). Furthermore, the previous bound on \( J_{2} \) together with (J) show that there exists a \( c > 0 \) such that for all vertices \( z \) in \( \Gamma_{d}(\lambda) \),

\[ J \leq c \eta(z)^{-(d-2)}. \]
Combining the results of (E), (H) and (M), we finally reach that \( \sum_{z' \in \mathbb{Z}^{d+1}} \varphi(z' - z) \psi(z') \leq c \eta(z)^{-d} + 1 \).

### II. LOWER BOUND
Assume the hypothesis for the lower bound. Then
\[
\sum_{z' \in \mathbb{Z}^{d+1}} \varphi(z' - z) \psi(z') \geq \sum_{n' > \max(n, \|x\|^2)} \sum_{\|x' - x\|^2 \leq n' - n} \varphi(z' - z) \psi(z').
\]

The two relations \( \|x\|^2 < n' \) and \( \|x' - x\|^2 \leq n' - n \) imply \( \|x'\|^2 \leq 2 \|x\|^2 + 2 \|x' - x\|^2 \leq 4n' \). Then, the bounds of the hypothesis give (the last \( \asymp \) follows from (2.13))
\[
\sum_{n' > \max(n, \|x\|^2)} \sum_{\|x' - x\|^2 \leq n' - n} \varphi(z' - z) \psi(z') 
\geq c \sum_{n' > \max(n, \|x\|^2)} (n')^{-\frac{d}{2}} (n' - n)^{-\frac{d}{4}} \sum_{\|x' - x\|^2 \leq n' - n} e^{-c' \|x'\|^2}
\geq c' \sum_{n' > \max(n, \|x\|^2)} (n')^{-\frac{d}{4}} \max(n, \|x\|^2)^{-\frac{d}{4} + 1}.
\]

The lower bounds have been substantiated keeping in mind (A). \( \square \)

With the two lemmas proved, we are in conditions to finish the theorem. By virtue of (A), (B), (C) and (6.2.1) we have that \( \text{USF}(\mathcal{E}(0, z)) = E(K_z) = \sum_{z' \in \mathbb{Z}^{d+1}} G(z, z') G(0, z') \). Applying (6.2.2) (bear in mind the Green’s function bounds (2.22)), we reach that \( \text{USF}(\mathcal{E}(0, z)) \asymp \eta(z)^{-(d-2)} \). \( \square \)

### 6.2 An upper bound on the probability that finitely many points are USF-connected

We are going to generalise the previous result to any subset of vertices. We will follow closely [BKPS04].

First, we introduce the definition of \textit{spread} of a set of vertices. For any vertex \( z \in \mathbb{Z}^{d+1} \) and any finite subset \( W \subset \mathbb{Z}^{d+1} \) (with two or more vertices) we define their spreads (relative to the metric \( \eta \)) as
\[
\langle z \rangle_{\eta} = \max(1, \eta(z))
\]
\[
\langle W \rangle_{\eta} = \min_{E} \prod_{\{z, z'\} \in E} (z - z')_{\eta},
\]
where the minimum runs over \textit{all} \( E \subset \mathcal{P}(W) \) making \( (W, E) \) an undirected spanning tree. If \( W \) consists of two (different) vertices \( \alpha \) and \( \beta \), then \( \langle W \rangle_{\eta} = \eta(\alpha - \beta) \); in this case we will write \( \langle \alpha \beta \rangle_{\eta} \) in lieu of \( \langle W \rangle_{\eta} \). Observe that (6.3) defines the spread of a point relative to the Euclidean metric. For simplicity, if \( W = \{\alpha, \beta, \ldots, \xi\} \), we will write \( \langle \alpha \beta \cdots \xi \rangle_{\eta} \), in particular, \( \langle 0z \rangle_{\eta} = \langle z \rangle_{\eta} \). Observe that [BKPS04] also defined the concept of “spread of subset of vertices of \( \mathbb{Z}^{d} \)” but they use the Euclidean metric (see their Definition 2.1). When \( W = \{\alpha, \beta, \gamma\} \), then
\[
\langle W \rangle_{\eta} = \min\{\langle \alpha \beta \rangle_{\eta}, \langle \alpha \gamma \rangle_{\eta}, \langle \alpha \beta \rangle_{\eta}, \langle \beta \gamma \rangle_{\eta}, \langle \alpha \gamma \rangle_{\eta}, \langle \beta \gamma \rangle_{\eta}\}.
\]

**Proposition (6.4)** [BKPS04, Lemma 2.6] For every \( L > 0 \) there exists a constant \( c > 0 \) such that for every \( z \in \mathbb{Z}^{d+1} \), and every \( W \subset \mathbb{Z}^{d+1} \) with card (\( W \)) \( \leq L \), we have \( \langle W \cup \{z\} \rangle_{\eta} \leq \langle W \rangle_{\eta} \min_{w \in W} \langle wz \rangle_{\eta} \leq c \langle W \cup \{z\} \rangle_{\eta} \).
Proof. The proof of this proposition proceeds word by word like that of Lemma 2.6 in [BKPS04].

We may now generalise theorem (6.2) to any finite subset of vertices. We will only prove an upper bound.

Theorem (6.5) Let \( d \geq 3 \). For any \( \mathcal{W} \subset \mathbb{Z}^{d+1} \), denote by \( \mathcal{C}_\mathcal{W} \) the event that all the points in \( \mathcal{W} \) are in the same USF component. For every integer \( \mathcal{L} > 0 \), there exists a constant \( c > 0 \), depending only on \( \mathcal{L} \) and dimension, such that for whatever subset \( \mathcal{W} \) with at most \( \mathcal{L} \) vertices, \( \text{USF}(\mathcal{C}_\mathcal{W}) \leq c \langle \mathcal{W} \rangle_\eta^{-(d-2)} \).

Proof. We apply the same reasoning as that used in the proof of Theorem 4.3 of [BKPS04]. We will apply induction on the cardinality of \( \mathcal{W} \). The case of two points is precisely theorem (6.2). Assume then the result holds for \( \text{card}(\mathcal{W}) - 1 \geq 2 \). Let \( z_1, z_2 \) be two different vertices of \( \mathcal{W} \) and denote \( \mathcal{C}_\mathcal{W}(z_1, z_2) \) the event which is the intersection of \( \mathcal{C}_\mathcal{W} \) with the event where the USF-path connecting \( z_1 \) and \( z_2 \) is edge-disjoint from the USF-rays starting at the vertices of \( \mathcal{V} = \mathcal{W} \setminus \{z_1, z_2\} \). Let \( \mathcal{C}_\mathcal{W}^*(z_1, z_2) \) denote the event which is the intersection of \( \mathcal{C}_\mathcal{W}(z_1, z_2) \) and the event where the ray at \( z_1 \) meets the ray at \( z_2 \) at the vertex \( z \). Then

\[
\text{USF}(\mathcal{C}_\mathcal{W}(z_1, z_2)) \leq \sum_{z \in \mathbb{Z}^{d+1}} \text{USF}(\mathcal{C}_\mathcal{W}^*(z_1, z_2)).
\]

To calculate the probability of \( \text{USF}(\mathcal{C}_\mathcal{W}^*(z_1, z_2)) \) we may assume that we construct the USF-forest \( \mathcal{F} \) following Wilson’s algorithm rooted at infinity by first ordering all the vertices in \( \mathcal{V}_z \equiv \mathcal{V} \cup \{z\} \), followed by the vertices \( z_1 \) and \( z_2 \), and then the rest of the vertices of \( \mathcal{V}^{d+1} \). We see that

\[
\{ \mathcal{F} \in \mathcal{C}_\mathcal{W}^*(z_1, z_2) \} \subset \{ \mathcal{F} \in \mathcal{C}_\mathcal{V}_z \cap \{ \tau_z(S^{z_1}) < \infty \} \cap \{ \tau_z(S^{z_2}) < \infty \} \},
\]

and \( \{ \mathcal{F} \in \mathcal{C}_\mathcal{V}_z \} \) is an event depending solely in the random walks \( S^v \) for \( v \in \mathcal{V}_z \). By independence and induction,

\[
\text{USF}(\mathcal{C}_\mathcal{W}^*(z_1, z_2)) = \mathbb{P}(\mathcal{F} \in \mathcal{C}_\mathcal{W}^*(z_1, z_2)) \\
\leq \mathbb{P}(\mathcal{F} \in \mathcal{C}_\mathcal{V}_z) \mathbb{P}(\tau_z(S^{z_1}) < \infty) \mathbb{P}(\tau_z(S^{z_2}) < \infty) \\
= \text{USF}(\mathcal{C}_\mathcal{V}_z) G(z_1, z) G(z_2, z) \\
\leq c \langle \mathcal{V} \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z) \\
\leq c' \langle \mathcal{V} \rangle_\eta^{-(d-2)} \sum_{v \in \mathcal{V}_z} \langle v_z \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z),
\]

the last inequality by virtue of (6.4). Thus,

\[
(*) \quad \text{USF}(\mathcal{C}_\mathcal{W}(z_1, z_2)) \leq \langle \mathcal{V} \rangle_\eta^{-(d-2)} \sum_{v \in \mathcal{V}_z} \sum_{z \in \mathbb{Z}^{d+1}} \langle v_z \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z).
\]

We isolate some calculations first.

(6.5.1) For any \( a \in \mathbb{R} \) and \( r > 2 \), we have \( \sum_{\langle z \rangle_\eta \leq r} \frac{1}{\langle z \rangle_\eta} \leq 1 + r^{d+2} \), the implicit constants depending only on \( a \) and dimension.

Proof of (6.5.1). Divide the cases \( a < 0 \) and \( a \geq 0 \), use the fact that there exists an integer \( k \geq 1 \) such that \( 2^{k-1} \leq r < 2^k \) and use that

\[
\bigcup_{j=1}^{k-1} \left\{ 2^{j-1} \leq \langle z \rangle_\eta < 2^j \right\} \subset \left\{ \langle z \rangle_\eta \leq r \right\} \subset \bigcup_{j=1}^{k} \left\{ 2^{j-1} \leq \langle z \rangle_\eta < 2^j \right\}.
\]

Keep in mind that \( \langle 0 \rangle_\eta = 1 \).
There exists a universal constant $c > 0$ such that for any two vertices $z_1, z_2 \in \mathbb{Z}^{d+1}$, we have
\[
\sum_{z \in \mathbb{Z}^{d+1}} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z)G(z_1, z) \leq c \langle 0z_1z_2 \rangle_{\eta}^{-(d-2)}.
\]

**Proof of (6.5.2).** By symmetry, we may assume $\langle z_1 \rangle_{\eta} \leq \langle z_2 \rangle_{\eta}$. Write $z_i = (n_i, x_i)$, for $i = 1, 2$. Then,
\[
\sum_{z \in \mathbb{Z}^{d+1}} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z)G(z_2, z) = \sum_{\langle z \rangle_{\eta} \geq \frac{1}{2} \langle z_1 \rangle_{\eta}} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z)G(z_2, z) + I,
\]
and we may bound easily
\[
\sum_{\langle z \rangle_{\eta} \geq \frac{1}{2} \langle z_1 \rangle_{\eta}} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z)G(z_2, z) \leq \langle z_1 \rangle_{\eta}^{-(d-2)} \sum_{z \in \mathbb{Z}^{d+1}} G(z_1, z)G(z_2, z)
\]
\[
\leq c \langle z_1 \rangle_{\eta}^{-(d-2)} \langle z_1 - z_2 \rangle_{\eta}^{-(d-2)} \leq c \langle 0z_1z_2 \rangle_{\eta}^{-(d-2)},
\]
where the penultimate inequality is obtained from (6.2.2) and the last inequality by definition of the spread. The rest of the calculation is to bound $I$; this will be done in several cases. By definition
\[
I = \sum_{\langle z \rangle_{\eta} \leq \frac{1}{2} \langle z_1 \rangle_{\eta}} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z)G(z_2, z).
\]

We will handle four cases, we will make use of the Green’s function estimates (2.22). Notice that $G(z', z'') \leq c \|z' - z''\|^{- \frac{d}{2}}$ in all cases and that $\langle x \rangle \leq \langle z \rangle_{\eta}$ for every $z \in \mathbb{Z}^{d+1}$ (the left hand side is the spread relative to the Euclidean metric $|.|$).

**Case 1.** Here we assume that $\langle z_1 \rangle_{\eta} = |n_1|^{ \frac{d}{2}}$ and $\langle z_2 \rangle_{\eta} = |n_2|^{ \frac{d}{2}}$. In this case, $\langle z \rangle_{\eta} \leq \frac{1}{2} \langle z_1 \rangle_{\eta}$ implies that $|n| \leq \frac{1}{3}|n_1| \leq \frac{1}{3}|n_2|$ and $\|x - x_1\| \leq \|x\| + \|x_1\| \leq \frac{3}{2}\|x\|$, and similarly $\|x - x_2\| \leq \frac{3}{2}\|n_2\|^{ \frac{d}{2}}$, obtaining $I \leq c|n_1|^{- \frac{d}{2}}|n_2|^{- \frac{d}{2}} \sum_{\langle z \rangle_{\eta} \leq \frac{1}{2} \langle z_1 \rangle_{\eta}} \langle z \rangle_{\eta}^{-(d-2)}$. We may apply (6.5.1) to reach $I \leq c \langle z_1 \rangle_{\eta}^{-d} \langle z_2 \rangle_{\eta}^{-d} \langle z_1 \rangle_{\eta}^{4} \leq c \langle z_1 \rangle_{\eta}^{-d} \langle z_2 \rangle_{\eta}^{-d} \langle z_1 \rangle_{\eta}^{-d} \leq c \langle 0z_1z_2 \rangle_{\eta}^{-d}$. This finishes **Case 1**.

**Case 2.** Here we assume $\langle z_1 \rangle_{\eta} = |n_1|^{ \frac{d}{2}}$ and $\langle z_2 \rangle_{\eta} = \|x_2\|$. Here we have $G(z_1, z) \leq c|n_1|^{- \frac{d}{2}}$. We may split $I$ into the sum over $n$ and over $x$. This gives $I \leq c|n_1|^{- \frac{d}{2}} \sum_{\|x\| \leq \frac{1}{3}|n_1|^{ \frac{d}{2}}} \langle x \rangle^{-(d-2)} \sum_{|n_1| \leq \frac{1}{2}|n_1|^{ \frac{d}{2}}} \sum_{|n-n_2| \leq \frac{1}{2}|x_2|} G(z_2, z)$. For every $\|x\| \leq \frac{1}{3}|n_1|^{ \frac{d}{2}}$, we have $\|x - x_2\| \leq \|x\| + \|x_2\| \leq \frac{4}{3}\|x_2\|$ and $\|x - x_2\| \geq \|x\|$. Next, break the inner sum into two parts, the first one being over all $n$ such that $|n - n_2| \leq \frac{1}{3}\|x_2\|$ and the second one being over all other $n$. In the first part, we have a bound of the form $G(z_2, z) \leq ce^{-c'}\|x_2\|$,
\[
\sum_{|n| \leq \frac{1}{2}|n_1|^{ \frac{d}{2}}} \sum_{|n-n_2| \leq \frac{1}{2}|x_2|} G(z_2, z) \leq c\|x_2\|^{ \frac{d}{2}} e^{-c'}\|x_2\|^{ \frac{d}{2}} \leq c''\|x_2\|^{-(d-2)}.\]

Focus now in the second part of this inner sum. Here, we have $\frac{1}{3}\|x_2\| \leq |n - n_2| \leq |n| + |n_2| \leq \frac{2}{3}\|x_2\|^{ \frac{d}{2}}$. Thus, we can bound $G(z_2, z) \leq ce^{-c'}\frac{|x_2|^2}{k} k^{- \frac{d}{2}}$ for some $1 \leq k \leq \frac{5}{4}\|x_2\|^{ \frac{d}{2}}$. Then, we may bound the second part of the inner sum of $I$ as follows
\[
\sum_{|n| \leq \frac{1}{2}|n_1|^{ \frac{d}{2}}} \sum_{|n-n_2| > \frac{1}{2}|x_2|} G(z_2, z) \leq c \sum_{1 \leq k \leq \frac{5}{4}\|x_2\|^{ \frac{d}{2}}} \sum_{|n-n_2| > \frac{1}{2}|x_2|} e^{-c'\frac{|x_2|^2}{k} k^{- \frac{d}{2}}} \leq c \|x_2\|^{2-(d-2)+1} = c \|x_2\|^{-(d-2)}.\]
Therefore, we may bound $I$ as follows
\[
I \leq c|n_1|^{- \frac{d}{2}} \sum_{\|x\| \leq \frac{1}{3}|n_1|^{ \frac{d}{2}}} \langle x \rangle^{-(d-2)} \|x_2\|^{-(d-2)}.
\]

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We have \( \sum_{\|x\| \leq \frac{1}{2}|n_1|^{\frac{1}{2}}} \langle x \rangle^{-(d-2)} \leq 1 + c \int_1^{\frac{1}{2}|n_1|^{\frac{1}{2}}} dt \ t^{-(d-2)} t^{d-1} = 1 + c \int_1^{\frac{1}{2}|n_1|^{\frac{1}{2}}} dt \ t \leq c'|n_1| \). Substituting above yields \( I \leq c|n_1|^{-\frac{d}{2}+1} \|x_2\|^{-(d-2)} = c \langle z_1 \rangle_{\eta}^{-(d-2)} \langle z_2 \rangle_{\eta}^{-(d-2)} \leq c \langle 0z_1z_2 \rangle_{\eta}^{-(d-2)} \). This completed Case 2.

**Case 3.** Here we assume \( \langle z_1 \rangle_{\eta} = |x_1| \) and \( \langle z_2 \rangle_{\eta} = |n_2|^{\frac{1}{2}} \). Then \( |n-n_2| \approx |n_2| \) since \( |n| \leq \frac{\|x_1\|}{2} \leq \frac{|n_2|}{2} \) and we may bound \( G(z, z_2) \leq c \|z - z_2\|^{-\frac{d}{2}} \leq c'|n_2|^{-\frac{d}{2}} \). This gives \( I \leq c|n_2|^{-\frac{d}{2}} \sum_{\|n\| \leq \frac{3}{2}|x_1|} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z) \).

We may now employ the same split of this sum as in **Case 2**, deducing

\[
I \leq c|n_2|^{-\frac{d}{2}} \sum_{\|x\| \leq \frac{3}{2}|x_1|} \langle x \rangle^{-(d-2)} \sum_{\|n\| \leq \frac{3}{2}|x_1|} G(z_1, z).
\]

We may break again the inner sum over those \( n \) for which \( |n - n_1| \leq \frac{1}{2} \|x_1\| \) and over all other \( n \), and we will find that

\[
\sum_{\|x\| \leq \frac{3}{2}|x_1|} \langle x \rangle^{-(d-2)} \leq c \|x_1\|(|x_1| \sum_{\|n\| \leq \frac{3}{2}|x_1|} e^{-c'|n_2| \|x_1\|^2} \sum_{k \leq \frac{3}{2}|x_1|} e^{-c'\|x_1\|^2} k^{-\frac{d}{2}}).
\]

And again, we apply (2.11) and (2.14), obtaining

\[
\sum_{\|x\| \leq \frac{3}{2}|x_1| \|x\|} \langle x \rangle^{-(d-2)} \leq c \|x_1\|^2 \sum_{k \leq \frac{3}{2}|x_1|} e^{-c'\|x_1\|^2} k^{-\frac{d}{2}} \leq c \|x_1\|^{2(-\frac{d}{2}+1)} = c \|x_1\|^{-(d-2)}.
\]

It is also clear that \( \|x_1\| e^{-c'|n_2| \|x_1\|} \leq c \|x_1\|^{-(d-2)} \). Substituting yields \( I \leq c|n_2|^{-\frac{d}{2}} \|x_1\|^{2} \|x_1\|^{-(d-2)} \).

Since we assume \( \langle z_1 \rangle_{\eta} \leq \langle z_2 \rangle_{\eta} \), we have \( \|x_1\|^2 \leq \|n_2\| \) so that \( |n_2|^{-\frac{d}{2}} \|x_1\|^2 \leq \|n_2\|^{-\frac{d}{2}} = \langle z_2 \rangle_{\eta}^{-(d-2)} \). The proof of **Case 3** is finished since \( \langle z_1 \rangle_{\eta} = \|x_1\| \) and the definition of spread shows at once \( \langle z_1 \rangle_{\eta}^{-(d-2)} \langle z_2 \rangle_{\eta}^{-(d-2)} \leq \langle 0z_1z_2 \rangle_{\eta}^{-(d-2)} \).

**Case 4.** Here we assume \( \langle z_1 \rangle_{\eta} = |x_1| \) and \( \langle z_2 \rangle_{\eta} = \|x_2\| \). Consider first the set of \( z \in \mathbb{Z}^{d+1} \) such that \( \langle z \rangle_{\eta} \leq \frac{1}{2} \langle z_1 \rangle_{\eta} \), and \( |n-n_1| \leq \|x_1\| \). For \( z \) in this set, \( G(z_1, z) \leq c e^{-c'|x_1|} \), this entails

\[
\sum_{\langle z \rangle_{\eta} \leq \frac{1}{2} \langle z_1 \rangle_{\eta} \|n-n_1\| \leq \|x_1\|} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z) G(z_2, z) \leq c e^{-c'|x_1|} \sum_{\|x\| \leq \frac{3}{2}|x_1|} \langle x \rangle^{-(d-2)} \sum_{\|n\| \leq \frac{3}{2}|x_1|} G(z_2, z)
\]

Recall we assume \( \|x_1\| \leq \|x_2\| \), and, as done in **Case 2** and **Case 3**

\[
\sum_{\|n\| \leq \frac{3}{2}|x_2|} G(z_2, z) \leq c \|x_2\|^{-(d-2)}
\]

Therefore,

\[
\sum_{\langle z \rangle_{\eta} \leq \frac{1}{2} \langle z_1 \rangle_{\eta} \|n-n_1\| \leq \|x_1\|} \langle z \rangle_{\eta}^{-(d-2)} G(z_1, z) G(z_2, z) \leq c \|x_1\|^2 e^{-c'|x_1|} \|x_2\|^{-(d-2)}
\]

\[
\leq c' \|x_1\|^{-(d-2)} \|x_2\|^{-(d-2)} \leq c'' \langle 0z_1z_2 \rangle_{\eta}.
\]
There remains to handle the terms defining $I$ for which $\langle z \rangle_\eta \leq \frac{1}{2} \|x_1\|$ and $|n - n_1| > \|x_1\|$. Let $J$ be the least integer for which $\frac{1}{2} \|x_1\| \leq 2^J \|x_1\| + \frac{1}{2}$. In other words, $J$ is the least integer for which $\|x_1\| \leq 4^{J+1}$. In particular, $4^J < \|x_1\| < 4^{J+1}$. Next, observe that the relation $\langle z \rangle_\eta \leq \frac{1}{2} \|x_1\|$ implies $|n - n_1| \leq |n| + |n_1| \leq \frac{5}{4} \|x_1\|^2 \leq 5 \cdot 4^J \|x_1\|$. Define

$$A_j = \left\{ z; \ 4^{j-1} \|x_1\| < |n - n_1| \leq 4^j \|x_1\|, \langle z \rangle_\eta \leq \frac{1}{2} \|x_1\| \right\}, \quad 1 \leq j \leq J - 1$$

$$A_J = \left\{ z; \ 4^{J-1} \|x_1\| < |n - n_1| \leq 5 \cdot 4^J \|x_1\|, \langle z \rangle_\eta \leq \frac{1}{2} \|x_1\| \right\}.$$

We can write $I \leq c \langle 0 z_1 z_2 \rangle_\eta^{-(d-2)} + \sum_{j=1}^J \sum_{z \in A_j} \langle z \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z)$. If $z \in A_j$, then $G(z_1, z) \leq c e^{-c' \|x_1\|-4^{-j+\frac{d}{4}}}$. Then,

$$\sum_{j=1}^J \sum_{z \in A_j} \langle z \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z) \leq c \|x_1\|^{-\frac{d}{4}} \sum_{j=1}^J e^{-c' \|x_1\|-4^{-j+\frac{d}{4}}} \sum_{z \in A_j} \langle z \rangle_\eta^{-(d-2)} G(z_2, z).$$

We may break the inner sum in terms of $n$ and $x$ (for details, see Case 2 and Case 3) and this will give

$$\sum_{z \in A_j} \langle z \rangle_\eta^{-(d-2)} G(z_2, z) \leq \sum_{\|x_2\|^2 \|x_1\|^2} \sum_{|n| \leq \frac{5}{4} \|x_1\|^2} G(z_2, z) \leq c \|x_1\|^2 \|x_2\|^{-(d-2)}.\quad \text{Whence,}$$

$$\sum_{j=1}^J \sum_{z \in A_j} \langle z \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z) \leq c \|x_1\|^{-\frac{d}{4}} \|x_2\|^{-(d-2)} \sum_{j=1}^J e^{-c' \|x_1\|-4^{-j+\frac{d}{4}}}.$$}

Finally, in the inner sum reverse the order of summation, $\sum_{j=1}^J \sum_{z \in A_j} \langle z \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z) \leq c \|x_1\|^{-\frac{d}{4}} \|x_2\|^{-(d-2)} \sum_{k=0}^{J-1} 4^{-k} e^{-c' \|x_1\| \cdot 4^{(k-J)+\frac{d}{4}}}$.

This completes Case 4.

With the four cases established, we have reached the conclusion of (6.5.2).

A corollary of (6.5.2) is that for any vertex $v \in \mathbb{Z}^{d+1}$,

$$\sum_{z \in \mathbb{Z}^{d+1}} \langle uz \rangle_\eta^{-(d-2)} G(z_1, z) G(z_2, z) \leq c \langle \{0, z_1 - v, z_2 - v\} \rangle_\eta^{-(d-2)} = c \langle v z_1 z_2 \rangle_\eta^{-(d-2)}.$$

Substituting this inequality in (*) allows deducing, with the aid of (6.4),

$$\text{USF}(\mathcal{G}_W(z_1, z_2)) \leq c \langle V \rangle_\eta^{-(d-2)} \sum_{v \in V} \langle vz_1 z_2 \rangle_\eta^{-(d-2)} \leq c' \langle W \rangle_\eta^{-(d-2)},$$

where the last inequality follows from the fact that the union of a tree on $\{v, z_1, z_2\}$ and a tree on $V$ gives a tree on $W = V \cup \{z_1, z_2\}$. We will show now that

$$\mathcal{G}_W = \bigcup_{z_1 \neq z_2} \mathcal{G}_W(z_1, z_2).$$
where the vertices \( z_1 \) and \( z_2 \) run on \( W \). For each vertex \( w \in W \), denote by \( R^w \) the USF-ray from \( w \). For every \( z_1 \neq z_2 \), two vertices of \( W \), their corresponding rays \( R^{z_1} \) and \( R^{z_2} \) will meet at a vertex \( v(z_1, z_2) \). Furthermore, on the event \( \mathcal{E}_W \), the set of vertices belonging to all \( R^w \), for \( w \in W \), is non-empty and thus, there exists a vertex \( z^* \) satisfying the following: \( z^* \) is the closest vertex in the graph-distance of the USF-tree, amongst all vertices belonging to all rays \( R^w \) \( (w \in W) \), to each one of the vertices \( w \in W \). In other words, \( z^* \) is the vertex where all the rays meet. Consider now a pair \( z_1 \neq z_2 \) such that the graph-distance of \( v(z_1, z_2) \) and \( z^* \) is maximal. Then, \( \mathcal{E}_W(z_1, z_2) \) occurs. The proof of theorem (6.3) is complete.

7 The separation between components: transitions in dimensions

\( d = 2, 6, 10, ... \)

Given a vertex \( z \) of \( \Gamma_d(\lambda) \), denote by \( \Sigma_z \) the component of \( z \) in the USF-forest, regarded as spanning subgraph. Construct now \( \Gamma_d(\lambda)/\text{USF} \) from \( \Gamma_d(\lambda) \) by shorting all edges of all \( \Sigma_z \) as \( z \) runs on the vertices of \( \Gamma_d(\lambda) \). Therefore, \( \Sigma_z \) is a vertex of the new network \( \Gamma_d(\lambda)/\text{USF} \). Define \( D : \mathbb{Z}^d \to \mathbb{Z}_+ \) by

\[
D(z, z') = d_{\Gamma_d(\lambda)/\text{USF}}(\Sigma_z, \Sigma_{z'}),
\]

the right hand side being the graph-distance of \( \Gamma_d(\lambda)/\text{USF} \). We call this the separation between the components of \( z \) and \( z' \) relative to the chemical distance of the graph \( \mathbb{Z}^{d+1} \). Another way of describing \( D(z, z') \) is as follow: it is the minimal number of edges of \( \Gamma_d(\lambda) \) that do not belong to any USF-tree in a path connecting \( z \) and \( z' \). In this section we are going to show that, for almost every realisation,

\[
\max_{z, z' \in \mathbb{Z}^{d+1}} D(z, z') = \left\lfloor \frac{d-2}{4} \right\rfloor.
\]

This section is substantially based on [BKPS03].

7.1 Random relations and stochastic dimension

We will be considering relations between vertices of \( \Gamma_d(\lambda) \). If \( R \) is one such relation, we will use the common notation \( z_1 R z_2 \) to mean \( (z_1, z_2) \in R \). Also, the composition of two relations \( L \) and \( R \) is, by definition, the set of pairs \( (z_1, z_2) \) such that there exists a \( z \) satisfying \( z_1 L z \) and \( z R z_2 \). We write \( LR \) to denote the composition of \( L \) and \( R \) (in this order). We will say that a random relation \( R \) on the vertices of \( \Gamma_d(\lambda) \) has stochastic dimension \( \alpha \in [0, d+1) \), relative to the metric \( \eta \) (6.1), and write \( \dim_{\eta}(R) = \alpha \) if there exists a constant \( c = c(R) > 0 \) such that for all vertices \( z_1, z_2, z_3, z_4 \) of \( \Gamma_d(\lambda) \)

\[
c\mathbb{P}(z_1 R z_2) \geq \langle z_1 z_2 \rangle_\eta^{-(d+1-\alpha)},
\]

and

\[
\mathbb{P}(z_1 R z_2, z_3 R z_4) \leq c \langle z_1 z_2 \rangle_\eta^{-(d+1-\alpha)} \langle z_3 z_4 \rangle_\eta^{-(d+1-\alpha)} + c \langle z_1 z_2 z_3 z_4 \rangle_\eta^{-(d+1-\alpha)}.
\]

Remark (7.1) That a random relation \( R \) has stochastic dimension \( d+1 \) is the same as saying that

\[
\inf_{z_1, z_2} \mathbb{P}(z_1 R z_2) > 0.
\]

Remark (7.2) Let \( d \geq 3 \). Denote by \( U_{\Gamma_d(\lambda)} \) the random relation: “\( z_1 \) and \( z_2 \) are in the same USF-component.” Then, \( U_{\Gamma_d(\lambda)} \) has stochastic dimension three, relative to \( \eta \), this is a consequence of the definition and of theorems (6.2) and (6.5). Indeed, write \( z \sim z' \) to mean \( z U_{\Gamma_d(\lambda)} z' \). Then, (6.2) shows that
Suppose that \( \text{dim} \ > \) the probability of this event is bounded above by a universal multiple \( F \cdot \). The event \( \{ z_1 z_2 \} \) is the event where all the four vertices are in the same USF-tree, this the probability of this event is bounded above by a universal multiple of \( \langle z_1 z_2 z_3 z_4 \rangle^{\eta -(d+1-3)} \) by (6.3). To estimate the probability of the event \( \{ z_1 z_2 z_3 z_4, z_1 \neq z_3 \} \) we may construct the USF-forest using Wilson’s algorithm rooted at infinity with ordering of vertices \( (z_1, z_2, z_3, z_4, \ldots) \). If \( S^z \) denotes the random walk in this construction of \( \mathcal{G} \), then
\[
\mathbb{P} \left( z_1 \sim z_2, z_3 \sim z_4, z_1 \neq z_3 \right) \leq \mathbb{P} \left( z_1 \sim z_2, z_3 \sim z_4, z_1 \sim z_4 \right) + \mathbb{P} \left( z_1 \sim z_2, z_3 \sim z_4, z_1 \neq z_3 \right).
\]
The last inequality by virtue of lemmas (6.2.1) and (6.2.2).

Remark (7.3) We are going to base the results of this section on those of the paper [BKPS04], making the necessary adaptations according to our conveniences. We will see that, in a sense, the stochastic dimension relative to \( \eta \) is the same the stochastic dimension they defined (that is, relative to the Euclidean distance) minus one.

We introduce the following annuli: for \( n < N \) two integers and \( z \in \mathbb{Z}^{d+1} \),
\[ A^N_n(z) = \{ z' \in \mathbb{Z}^{d+1} | 2^n \leq \langle z-z' \rangle_\eta < 2^{n+1} \}. \]
Notice \( \text{card} \left( A^N_n(z) \right) \sim 2^{N(d+2)} \).

**Proposition (7.4)** ([BKPS04], Lemma 2.8) Let \( L \) and \( R \) be two independent random relations of \( \Gamma_d(\lambda) \). Suppose that \( \text{dim}_\eta(L) = d+1-\alpha \) and \( \text{dim}_\eta(R) = d+1-\beta \), both exist. Denote \( \gamma = \alpha + \beta - d - 2 \). For \( z_1, z_2 \in \mathbb{Z}^{d+1} \) and \( 1 \leq n \leq N \), let \( S_{z_1, z_2} = S_{z_1, z_2}(n, N) \) def. = \( \sum_{z \in A^N_n(z_1)} 1_{\{ z \sim L z \}} 1_{\{ z \sim R z \}} \). If \( \langle z_1 z_2 \rangle_\eta < 2^{n-1} \) and \( N \geq n \), \( \mathbb{P} \left( S_{z_1, z_2} > 0 \right) \geq c \frac{\sum_{k=0}^{N} 2^{-k\gamma}}{\sum_{k=0}^{N} 2^{-k\gamma}} \), where \( c \) is a constant that may depend solely on \( L, R \) and dimension.

**Proof.** We may repeat word by word, with very minor notational modifications, the proof of Lemma 2.8 of [BKPS04] bearing in mind that the only substantial thing that changes is \( \text{card} \left( A^N_k(z_1) \right) \sim 2^{k(d+2)} \). Also keep in mind that their Lemma 2.6 is our Proposition (6.4). Thus, their formula (2.6) changes to
\[
\mathbb{E} \left( S_{z_1, z_2} \right) \sim \sum_{k=n}^{N} 2^{(d+2)} 2^{-k\alpha} 2^{-k\beta} = \sum_{k=n}^{N} 2^{-k\gamma},
\]
the rest of the proof goes on without major modification except that we need to replace \( d \) by \( d+2 \) in their formula (2.7).

**Proposition (7.5)** ([BKPS04], Corollary 2.9) Let \( L \) and \( R \) be two independent random relations of \( \Gamma_d(\lambda) \). Suppose that \( \text{dim}_\eta(L) = d+1-\alpha \) and \( \text{dim}_\eta(R) = d+1-\beta \), both exist. Denote \( \gamma = \max(0, \alpha + \beta - d - 2) \). There exists a constant \( c > 0 \) such that for all \( z_1, z_2 \in \mathbb{Z}^{d+1} \),
\[
\mathbb{P} \left( z_1 L R z_2 \right) \geq c \langle z_1 z_2 \rangle^{\eta - \gamma}.
\]
7.1 Random relations and stochastic dimension

Proof. The proof of Corollary 2.9 of [BKPS04] applies almost verbatim: in their second sentence replace “lemma” with “Proposition (6.4).” □

**Proposition (7.6) (BKPS04 Lemma 2.10)** Let \( \alpha, \beta \in [0,d+1) \) satisfy \( \alpha + \beta > d+2 \). Let \( \gamma = \alpha + \beta - d - 2 \). Then, for all vertices \( z_1, z_2 \) of \( \mathbb{Z}^{d+1} \),

\[
\sum_{\mathbb{Z}^{d+1}} (z_1 z)^{-\alpha} (z_2 z)^{-\beta} \approx (z_1 z_2)^{-\gamma}
\]

for all vertices \( z_1, z_2 \) of \( \mathbb{Z}^{d+1} \), and with any implicit constant being universal.

**Proof.** We may use the same proof as that of Lemma 2.11 of [BKPS04] keeping in mind that their Lemma 2.11 of [BKPS04] applies without changes except to notation. □

**Proposition (7.7) (BKPS04 Lemma 2.11)** Let \( M \) be a positive integer and let \( \alpha, \beta \in [0,d+1) \) satisfy \( \alpha + \beta > d+2 \). Denote \( \gamma = \alpha + \beta - d - 2 \). There exists a constant \( c > 0 \) such that for all subsets \( V \) and \( W \) of \( \mathbb{Z}^{d+1} \) with at most \( M \) points, we have

\[
\sum_{\mathbb{Z}^{d+1}} (V \cup \{z\})^\alpha (W \cup \{z\})^{-\beta} \leq c (V^\alpha (V \cup W)^{-\beta} \leq c (V \cup W)^{-\gamma}.
\]

**Proof.** We may use the same proof as that of Lemma 2.11 of [BKPS04] keeping in mind that their Lemma 2.6 is our Proposition (6.4) and, Lemma 2.10 is Proposition (7.6). □

**Proposition (7.8) (BKPS04 Lemma 2.12)** Let \( \alpha, \beta \in [0,d+1) \) satisfy \( \alpha + \beta > d+2 \). Set \( \gamma = \alpha + \beta - d - 2 \).

(a) There exists a constant \( c > 0 \) such that for all vertices \( z_1, z_2, z_3 \in \mathbb{Z}^{d+1} \),

\[
\sum_{\mathbb{Z}^{d+1}} (z_1 z)^{-\alpha} (z_2 z)^{-\beta} (z_3 z)^{-\gamma} \leq c (z_1 z_2 z_3)^{-\gamma}
\]

(b) There exists a constant \( c > 0 \) such that for all vertices \( z_1, z_2, z_3, z_4 \in \mathbb{Z}^{d+1} \),

\[
\sum_{\mathbb{Z}^{d+1}} (z_1 z_2 z)^{-\alpha} (z_2 z_3)^{-\beta} (z_4 z)^{-\gamma} \leq c (z_1 z_2 z_3 z_4)^{-\gamma}
\]

**Proof.** The proof is essentially the same as that of Proposition 2.12 of [BKPS04] and their formula (2.9), except we need to use (6.5.1) in some step. For convenience of the reader, we repeat the proof.

We begin by establishing the first item. By changing the indices should the need arise, we may assume that \( \langle z_1 z_3 \rangle \leq \langle z_2 z_3 \rangle \). Denote by \( A \) the set of \( z \in \mathbb{Z}^{d+1} \) satisfying \( \langle z_3 z \rangle \leq 2 \langle z_1 z_3 \rangle \). When \( z \in A \) we have, \( \langle z_1 z \rangle \geq \langle z_1 z_3 \rangle - \langle z_2 z_3 \rangle \geq \frac{3}{2} \langle z_1 z_3 \rangle \), and, similarly, since \( \langle z_1 z_3 \rangle \leq \langle z_2 z_3 \rangle \), we also have \( \langle z_2 z_3 \rangle \geq \frac{1}{2} \langle z_2 z_3 \rangle \) and, similarly, since \( \langle z_1 z_3 \rangle \leq \langle z_2 z_3 \rangle \), we also have \( \langle z_2 z_3 \rangle \geq \frac{1}{2} \langle z_2 z_3 \rangle \). Therefore,

\[
\sum_{z \in A} \langle z_2 z \rangle^{-\beta} \langle z_3 z \rangle^{-\gamma} \leq 2^\alpha \langle z_1 z_3 \rangle^{-\alpha} \langle z_2 z_3 \rangle^{-\beta} \sum_{z \in A} \langle z_3 z \rangle^{-\gamma}.
\]

Apply (6.5.1) to obtain the bound \( \sum_{z \in A} \langle z_3 z \rangle^{-\gamma} \leq c \langle z_1 z_3 \rangle^{d+2-\gamma} \). Substitute above,

\[
\sum_{z \in A} \langle z_1 z_3 \rangle^{-\alpha} \langle z_2 z_3 \rangle^{-\beta} \langle z_3 z \rangle^{-\gamma} \leq c \langle z_1 z_3 \rangle^{-\alpha} \langle z_2 z_3 \rangle^{-\beta} \langle z_1 z_3 \rangle^{d+2-\gamma}
\]

\[
= c \langle z_1 z_3 \rangle^\gamma \langle z_2 z_3 \rangle^{-\gamma} \langle z_1 z_3 \rangle^{d+2-\alpha} \langle z_2 z_3 \rangle^{d+2-\alpha} \leq c \langle z_1 z_2 z_3 \rangle^{-\gamma},
\]
7.2 Tail triviality of random relations

Recall that an event in any probability space is said to be trivial if its probability is either zero or one, equivalently, if its probability $p$ satisfies $p = p^2$. It is easy to see that the set of trivial events is a $\sigma$-field, called the trivial $\sigma$-field (relative to the given probability measure). A $\sigma$-algebra is said to be trivial if it is contained in the trivial $\sigma$-field. If $R$ is a random relation on the vertices of $\Gamma_d(\lambda)$, we denote by $\mathcal{R}_z^{R}$, where $A \subset \mathbb{Z}^{d+1} \times \mathbb{Z}^{d+1}$, the $\sigma$-algebra generated by the events $\{zRz'\}$ as $(z, z') \in A$. If $z$ is a vertex, we define the tail $\sigma$-field based at $z$ on the left of $R$, denoted by $\mathcal{L}^{R}_z$, to be the $\sigma$-field generated by the events $\{zRz'\}$ as $z' \to \infty$, in other words, $\mathcal{L}^{R}_z$ is the intersection of all $\mathcal{R}^{R}_{\{z\} \times K}$ as $K$ runs on all finite subsets of $\mathbb{Z}^{d+1}$. For simplicity, we will say “left tail at $z$” to refer to $\mathcal{L}^{R}_z$. In a similar manner we define the tail $\sigma$-field $\mathcal{L}^{R}_z$ based at $z$ on the right of $R$, to be the intersection of all $\mathcal{R}^{R}_{K \times \{z\}}$ as $K$ runs on all finite subsets of $\mathbb{Z}^{d+1}$, we will talk about “right tail at $z$” to refer to this $\sigma$-algebra. We also define the (proper) tail $\sigma$-algebra of $R$ to be the intersection of all $\mathcal{R}^{R}_{K_1 \times K_2}$ as $K_1$ and $K_2$ run over all finite subsets of $\mathbb{Z}^{d+1}$, we denote it as $\mathcal{R}$. Finally, we define the restricted composition (relative to $\eta$) $L \circ R$ of two relations $L$ and $R$ (in that order) to be the set of all pairs $(z_1, z_2)$ such that there exists a $z \in \mathbb{Z}^{d+1}$ satisfying the relations $z_1Lz$ and $zRz_2$ and $z$ satisfies

$$\langle z_1z_2 \rangle_\eta \leq \min(\langle z_1z \rangle_\eta, \langle z_2z \rangle_\eta).$$

Theorem (7.10) ([BKPS04, Theorem 3.3]) Let $L$ and $R$ be two independent random relations on the vertices of $\Gamma_d(\lambda)$. If all left tail $\sigma$-fields $\mathcal{L}^{L}_z$ of $L$ are trivial and $R$ has a trivial tail $\sigma$-field $\mathcal{R}$, then the restricted composition $L \circ R$ has trivial left tail $\sigma$-fields $\mathcal{L}^{L \circ R}_z$. 

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7.3 The WSF connectedness relation

Let $\Gamma$ be any network. We will consider the following relation $U_\Gamma$ on the vertices of $\Gamma$ : “$v$ and $v'$ belong to the same WSF-component.” We have the following very general result regarding tail triviality of $U_\Gamma$.

**Theorem (7.11)** Suppose $\Gamma$ is any network satisfying the usual assumptions. Assume further that

(a) The network random walk of $\Gamma$ is transient.

(b) The network possesses the Liouville property (LP) [LZ].

Then, all left and right tail $\sigma$-algebras of $U_\Gamma$ are trivial. If, additionally to the previous hypotheses,

(c) The WSF of $\Gamma$ is one ended, that is to say, for almost every realisation, every tree in the WSF has one end.

Then, the tail $\sigma$-field of $U_\Gamma$ is also trivial.

For the proof see Theorem 4.5 of [BKPS04], together with Remark 4.6.

We need one last ingredient. Consider two random subsets of vertices, $A$ and $Z$, of a network $\Gamma$. It is said that $Z$ stochastically dominates $A$ if there exists a coupling $\mu$ of $A$ and $Z$ such that $\mu(A \subset Z) = 1$.

**Theorem (7.12)** ([BKPS04], Theorem 4.1] Let $\Gamma$ be any network. Let $\mathcal{F}, \mathcal{F}_0, \ldots, \mathcal{F}_m$ be independent samples of the WSF of the network $\Gamma$. If $x$ is a vertex of $\Gamma$, denote by $V_x^\mathcal{F}$ the vertex set of the component of $x$ in $\mathcal{F}$. Fix any vertex $o$ of $\Gamma$ and set $V_0 = V_o^\mathcal{F}$. For $j \geq 1$, define inductively $V_j$ to be the union of all vertex components of $\mathcal{F}_j$ that are contained in, or adjacent to, $V_{j-1}$; in other words, $V_j$ is the union of $V_x^\mathcal{F}$ as $x$ runs on $\overline{V}_{j-1}$ (closure in $\Gamma$). Let $Q_0 = V_0^\mathcal{F}$. For $j \geq 1$, define inductively $Q_j$ to be the union of all vertex components of $\mathcal{F}_j$ that intersect $Q_{j-1}$; in other words, $Q_j$ is the union of all $V_x^\mathcal{F}$ such that $V_x^\mathcal{F} \cap Q_{j-1} \neq \emptyset$. Then, $V_m$ stochastically dominates $Q_m$.

**Proof.** It is a simple adaptation of Theorem 4.1 of [BKPS04], indeed in their first sentence change $B_R$ and replace it with $B_R = \{v;d(v,o) < R\}$, where $d$ is the graph-distance of $\Gamma$. The remainder of their proof proceeds verbatim. \qed
7.4 The main theorem

Theorem (7.13) With the notation at the beginning of the section, \[ \max_{z, z' \in \mathbb{Z}^{d+1}} D(z, z') = \left\lceil \frac{d-2}{2} \right\rceil, \quad \text{a.s.} \]

Proof. The proof follows the lines of that of Theorem 1.1 and Proposition 5.1 of [BKPS04]. For convenience of the reader, we provide full details.

When \( d = 1, 2 \), we know that USF of \( \Gamma_d(\lambda) \) is a tree and thus the separation between components is zero, this agrees with the formula given. Assume then that \( d \geq 3 \).

Let \( m = \left\lceil \frac{d-2}{2} \right\rceil \). Consider \( m + 1 \) independent copies of the USF relation. By (7.11), bearing in mind that USF is one ended (see sect. 5) this random relation possesses trivial tail \( \sigma \)-field as well as trivial left tail and right tail \( \sigma \)-algebras. We may apply item (d) of (7.10) to conclude that, if \( R \) is the composition of the \( m + 1 \) copies of \( \text{USF}_{\Gamma_d(\lambda)} \) then \[ \inf_{z_1, z_2 \in \mathbb{Z}^{d+1}} P(z_1 R z_2) = 1. \]

Finally, define \( S \) to be the relation \( D(z, z') \leq m \). By virtue of (7.12), we know that \( P(z_1 S z_1) \geq P(z_1 R z_2) \) and thus, \( P(D(z, z') \leq m, \forall z, z') = 1. \)

The proof of the cases \( 3 \leq d \leq 6 \) is complete, so it only remains to prove the lower bound for the cases \( d \geq 7 \). To prove the lower bounds we will establish first

\[ (*) \quad P(D(z, z') \leq k) \leq c \langle z z' \rangle_{\eta}^{4k - (d-2)} \]

for all \( z, z' \in \mathbb{Z}^{d+1} \) and \( k \in \mathbb{N} \), the constant \( c \) depending on \( k \) and \( d \), but not on \( z \) or \( z' \). When \( k \geq m \), the criterion holds since then \( 4k \geq d - 2 \geq 5 \) and \( \langle z z' \rangle_{\eta} \geq 1 \). Assume \( k < m \), so that \( 4k - (d-2) \leq -1 \).

Consider a finite sequence \( (z_j, z_j')_{j=0, \ldots, k} \) of pairs of vertices of \( \Gamma_d(\lambda) \) with \( z_0 = z, z_k' = z' \) and \( z_j' \sim z_{j+1} \) for \( j = 0, \ldots, k-1 \). Let \( A = A(z_j, z_j') \) be the event in which \( z_j' \) is a vertex of \( T_z \), for \( j = 0, \ldots, k \) and for each of these \( j, z_j' \) is not a vertex of \( T_z \), for \( i \neq j \). Then,

\[ \{D(z, z') = k\} \subset \bigcup_{(z_j, z_j')} A(z_j, z_j') \]

where the pairs \( (z_j, z_j') \) run over all sequences \( (z_j, z_j') \) as stated. To estimate the probability of a particular \( A(z_j, z_j') \) we may construct the USF-forest using Wilson’s root at infinity, beginning with the random walks started at \( z_0, z_0', \ldots, z_k, z_k' \). For the event \( A(z_j, z_j') \) to hold, it is necessary that for each \( j = 0, \ldots, k \), the random walk started at \( z_j \) must intersect the path of the random walk started at \( z_j' \). Hence, if \( S^z \) denotes the random walk started at \( z \) with the different random walks mutually independent, \[ P(A) \leq \prod_{j=0}^{k} P\left(S^{z_j} \cap S^{z_j'} \neq \emptyset\right) = \prod_{j=0}^{k} \sum_{z \in \mathbb{Z}^{d+1}} G(z_j, z) G(z_j', z) \approx \prod_{j=0}^{k} \langle z_j z_j' \rangle_{\eta}^{-(d-2)}, \]

where \( \approx \) follows from (6.22). Since \( z_j' \sim z_{j+1} \), we can bound

\[ P(D(z, z') = k) \leq c \sum_{(z_j)} \prod_{j=0}^{k} \langle z_j z_{j+1} \rangle_{\eta}^{-(d-2)}, \]

the sum extending over all finite paths \( (z_j) \) starting at \( z \) and ending at \( z' \). We may finally break up this sum, starting with the sum over all \( z_k \) and applying (7.13) with \( \alpha = \beta = d-2 \), so that \( \gamma = \alpha + \beta - d-2 = d-6 > 0 \), and then continue with the sum over \( z_{k-1} \), and so on until \( z_1 \), this gives the desired bound \( (*) \). Having establish \( (*) \), we can terminate Theorem (7.13). Indeed, consider \( z \in \mathbb{Z}^{d+1} \), the event \( \left\{ \max_{z' \in \mathbb{Z}^{d+1}} D(z, z') = k \right\} \)
belongs to the left-tail at $z$ relative to the relation $U_{\Gamma_{d}(\lambda)}$, as such it is a trivial event (7.11). It is clear from (* ) that this event has probability zero, therefore the event \( \left\{ \max_{z,z' \in \mathbb{Z}^{d+1}} D(z, z') = k \right\} \) also have probability zero. 

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