Two-loop renormalization of the effective field theory of a static quark

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Abstract We give a recurrence relation for two-loop integrals encountered in the effective field theory of an infinitely heavy quark, $Q$, interacting with gluons and $N_L$ massless quarks, $q$, from which we obtain exact two-loop results, in any dimension and covariant gauge, for the propagator of $Q$ and the vertex function of the heavy-light current $J \equiv Q \Gamma q$, at zero $q$-momentum. The anomalous dimension of the $Q$-field agrees with the recent result of Broadhurst, Gray and Schilcher. The anomalous dimension of the current is

$$\tilde{\gamma}_J \equiv \frac{d \log \tilde{Z}_J}{d \log \mu} = -\frac{\alpha_s}{\pi} \left\{ 1 + \frac{127 + 56\zeta(2) - 10N_L}{72} \frac{\alpha_s}{\pi} + O(\alpha_s^2) \right\}$$

which gives the new two-loop correction to the result of Voloshin and Shifman.

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1 Introduction

Recently an interesting new approach to QCD problems involving a heavy quark was proposed, namely the effective field theory (EFT) of a static quark [1-5]. (See also the review [6].) To leading order in the $1/m$ expansion, the EFT lagrangian is

$$L = Q_0^\dagger D_0 Q_0 + \bar{q}_0 i \gamma_\mu D_\mu q_0 - \frac{1}{4} G^a_{\mu\nu} G^a_{0\mu\nu} - \left( \partial_\mu A^a_{0\mu} \right)^2 / 2 a_0 + \text{(ghost term)}$$

where $q_0$ and $A^a_{0\mu}$ are the bare light-quark and gluon fields of conventional QCD and $Q_0$ is the bare static-quark field, which is a two-component spinor. (When used as a 4-component spinor, its lower components are assumed to vanish.) The free-field propagator of $Q_0$ is $1/\omega$ (equivalently, $(1 + \gamma_0)/2\omega$, in 4-component form) where $\omega \equiv p_0 - m$ is the energy by which it is off-shell. Its coupling to gluons is given by the vertex $i g_0 v_\mu t^a$, where $v_\mu = (1, 0)$ is its velocity. The lagrangian (1) possesses an SU(2) static-quark spin symmetry [4].

In this paper we consider the two-loop renormalization of EFT, using dimensional regularization in $D \equiv 4 - 2\epsilon$ dimensions. In the MS scheme the bare quantities in (1) are related to the corresponding renormalized quantities by

$$Q_0 = \bar{\mu}^{-\epsilon} \tilde{Z}_2^{1/2} Q, \quad q_0 = \bar{\mu}^{-\epsilon} \tilde{Z}_2^{1/2} q, \quad A^a_{0\mu} = \bar{\mu}^{-\epsilon} \tilde{Z}_3^{1/2} A^a_{\mu}, \quad g_0 = \bar{\mu}^{-\epsilon} \tilde{Z}_3^{1/2} g, \quad a_0 = \tilde{Z}_3 a$$

where $Z_2$, $Z_3$ and $Z_0$ are the same as in QCD [7], with $N_L$ light-quark flavours, and $\bar{\mu}^2 = \mu^2 e^\gamma/4\pi$. Here we calculate, to two loops, the static-quark wave-function renormalization constant $\tilde{Z}_2$ and the renormalization constant $\tilde{Z}_J$ of the heavy-light current $J_0 = \bar{Q}_0 \Gamma q_0 = \tilde{Z}_J J$, where $\Gamma$ is some (irrelevant) gamma matrix. For $\tilde{Z}_2$ we obtain the same result as Broadhurst, Gray and Schilcher [8], who have independently extracted it from the gauge-invariant singularities of on-shell wave-function renormalization of a finite-mass quark [9]. From $\tilde{Z}_J$ we obtain the two-loop correction to the leading-order result [10, 11] for the anomalous dimension $\tilde{\gamma}_J \equiv d \log \tilde{Z}_J / d \log \mu$. This is needed for the renormalization-group analysis of the extrapolations of lattice computations of heavy-meson decay constants [12].

Renormalization of EFT corresponds closely to renormalization of the Wilson line [13-17], which coincides with the static-quark propagator in coordinate space. In particular the lagrangian (1) is essentially the same as that proposed in [14], for investigating Wilson lines, and used in the two-loop calculations of [16, 17]. In [16] Aoyama calculated $\tilde{Z}_2$ in the Feynman gauge, at $N_L = 0$; we disagree with his result. In [17] Korchemsky and Radyushkin calculated the two-loop anomalous dimension of a heavy-heavy current, which depends on the relative velocity of initial and final quarks; the corresponding one-loop result [13] has recently been rediscovered in EFT [18].

2 One- and two-loop integrals

The static-quark bare self-energy term $-i \Sigma(\omega)$ is given, to two loops, by the diagrams of Fig 1, where the blob in Fig 1a includes the light-quark, gluon and ghost loops in the gluon propagator. We take the light quarks as massless and easily evaluate the master one-loop EFT integral

$$\int d^D k \left( \frac{-1}{k^2} \right)^\alpha \left( \frac{\omega}{\omega + k_0} \right)^p \equiv i \pi^{D/2} (-2\omega)^{(D-2\alpha)} I(\alpha, p)$$
using the parameterization
\[
\frac{1}{A^\alpha P^\rho} = \frac{\Gamma(\alpha + p)}{\Gamma(p)} \int_0^\infty \frac{y^{p-1} dy}{(A + yP)^{\alpha+p}}
\]
where we adopt the convention that massless lines have greek indices and static lines have roman ones. We find that
\[
I(\alpha, p) = \frac{\Gamma(2\alpha + p - D)\Gamma(D/2 - \alpha)}{\Gamma(\alpha)\Gamma(p)} \tag{2}
\]
which enables us to evaluate diagrams of the form of Fig 1a.

The master two-loop EFT integral, for diagrams of the form of Fig 1b, is
\[
\int d^D k \int d^D l \left( -\frac{1}{k^2} \right)^\alpha \left( -\frac{1}{l^2} \right)^\beta \left( -\frac{1}{(k-l)^2} \right)^\gamma \left( \frac{\omega}{\omega + k_0} \right)^p \left( \frac{\omega}{\omega + l_0} \right)^q
\]
\[
\equiv -\pi^D (-2\omega)^{2(D-\alpha-\beta-\gamma)} I(\alpha, \beta, \gamma, p, q)
\]
In degenerate cases one uses \([9]\) and the corresponding result \([8]\) for the master one-loop diagram of massless QCD
\[
G(\alpha, \beta) = \frac{\Gamma(\alpha + \beta - D/2)\Gamma(D/2 - \alpha)\Gamma(D/2 - \beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(D - \alpha - \beta)}
\]
to obtain
\[
I(\alpha, \beta, 0, p, q) = I(\alpha, p)I(\beta, q) \tag{3}
\]
\[
I(\alpha, 0, \gamma, p, q) = I(\alpha, 2\gamma + p + q - D)I(\gamma, q) \tag{4}
\]
\[
I(\alpha, \beta, \gamma, p, 0) = I(\alpha + \beta + \gamma - D/2, p)G(\beta, \gamma) \tag{5}
\]
which suffice for diagrams of the form of Fig 1c and Fig 1d, since the integrand of Fig 1c has only 4 factors in the denominator and the integrand of Fig 1d can be expressed as the sum of three such terms after multiplying by
\[
1 = \frac{1}{\omega} \{(\omega + k_0) + (\omega + l_0) - (\omega + k_0 + l_0)\}
\]
This is an enormous simplification in comparison with on-shell massive QCD calculations \([9, 10]\), where this last diagram was the most difficult.

In general we use a recurrence relation to reduce the non-degenerate case of the master integral to a sum of known degenerate cases. Using the method of integration by parts \([9, 10]\), we have found that
\[
(D - \alpha - 2\gamma - p - q + 1) I = \{\alpha A^+ (\Gamma^- - B^-) + (2D - \alpha - \beta - \gamma - p - q + 1)Q^- \} I \tag{6}
\]
where, for example, \(A^+\) raises \(\alpha\) and \(Q^-\) lowers \(q\). Again this is simpler than the triangle relation \([8]\) of massless QCD, because no more than three terms appear and no static-line index is increased.

Now it is a matter of programming to reduce every one-loop integral to a rational function of \(D \equiv 4 - 2\epsilon\) times
\[
\Gamma_1 \equiv (-2\omega)^{-2\epsilon} \Gamma(-\epsilon) \Gamma(1 + 2\epsilon)/(4\pi)^{D/2} \tag{7}
\]
and every two-loop integral from Fig 1 to a linear combination of $\Gamma_2^2$ and

$$\Gamma_2 \equiv (-2\omega)^{-4\epsilon}\Gamma^2(-\epsilon)\Gamma(1+4\epsilon)/(4\pi)^D$$

$$= (1 + 4\zeta(2)\epsilon^2 - 16\zeta(3)\epsilon^3)\Gamma_1^2 + O(\epsilon^2)$$

with coefficients which are rational functions of $D$. We used REDUCE [19] on a Vax and checked results with Mathematica [20] on a Macintosh. The programming effort and run times were far less than in the related QCD calculations [8, 9].

3 Anomalous dimension of the static-quark field

Using these methods, it is straightforward to compute the two-loop corrections to the bare propagator $S_0(\omega)$. We find that

$$\omega S_0(\omega) = \frac{\omega}{\omega - \Sigma(\omega)} = 1 + Z_1 \Gamma_1 C_F g_0^2 + \left\{ Z_{A1} C_A \Gamma_1^2 + (Z_{A2} C_A + Z_{F2} C_F + Z_{L2} T_F N_L) \Gamma_2 \right\} C_F g_0^4 + O(g_0^6)$$

with coefficients

$$Z_1 = a_0 - \frac{D - 1}{D - 3}$$

$$Z_{A1} = -\frac{Z_1}{D - 3}$$

$$Z_{A2} = \frac{(D - 2)(D - 5)}{2(2D - 7)(D - 3)^3(D - 6)} + \frac{(D - 4D + 5)Z_1}{2(D - 3)^2(D - 6)} - \frac{(D^2 - 9D + 16)Z_1^2}{8(D - 3)(D - 6)}$$

$$Z_{F2} = \frac{Z_1^2}{2}$$

$$Z_{L2} = \frac{2(D - 2)}{(2D - 7)(D - 3)(D - 6)}$$

and colour factors $C_A = N$, $C_F = (N^2 - 1)/2N$, $T_F = \frac{1}{2}$, for an SU($N$) gauge group, or $C_A = 0$, $C_F = T_F = 1$ for a U($1$) gauge group.

We now renormalize the bare static-quark field $Q_0 = \tilde{\mu}^{-\epsilon} \tilde{Z}_2^{1/2} Q$ using the minimal wave-function renormalization constant $\tilde{Z}_2$ which makes the renormalized propagator $S(\omega) = S_0(\omega)/\tilde{Z}_2$ finite. (Tildes are used here to distinguish EFT renormalization constants from those of QCD.) We work in the $\overline{\text{MS}}$ scheme, in which the bare coupling and gauge parameter are renormalized by

$$\frac{g_0^2}{g^2 \tilde{\mu}^{2\epsilon}} = Z_\alpha = 1 - \frac{\alpha_s}{4\pi\epsilon} \left( \frac{11}{3} C_A - \frac{4}{3} T_F N_L \right) + O(\alpha_s^2)$$

$$\frac{a_0}{a} = Z_3 = 1 + \frac{\alpha_s}{4\pi\epsilon} \left( \frac{13 - 3a}{6} C_A - \frac{4}{3} T_F N_L \right) + O(\alpha_s^2)$$

where $\alpha_s \equiv g^2/4\pi$. We posit a minimal form $\tilde{Z}_2 = 1 + (C_{11}(a)/\epsilon)\alpha_s/\pi + (C_{22}(a)/\epsilon^2 + C_{21}(a)/\epsilon)\alpha_s^2/\pi^2 + O(\alpha_s^3)$ and determine its coefficients from [10]. The renormalizability of the theory requires that this constant make $S_0(\omega)/\tilde{Z}_2$ finite for all $\omega$, which provides a highly non-trivial check on our result. Then, using the $\tilde{\mu}$ dependence of $\alpha_s$ and $a$, implied by [11,12], we obtain the anomalous dimension $\gamma_F \equiv d \log \tilde{Z}_2 / d \log \tilde{\mu}$ of the static $Q$ field. As a check on the whole procedure we mirror it, step by step,
by that for a massless quark, to obtain the corresponding QCD quantity $\gamma_F$ for the massless $q$ field from the recurrence relations of [13]. Our result is

$$
\tilde{\gamma}_F = \frac{(a - 3)C_F \alpha_s}{2\pi} + \left\{ \frac{a^2}{32} + \frac{a}{4} - \frac{179}{96} \right\} C_A + \frac{2}{3} T_F N_L \right\} \frac{C_F \alpha_s^2}{\pi^2} + O(\alpha_s^3)
$$

$$
= \gamma_F - \frac{3C_F \alpha_s}{2\pi} - \left\{ \frac{127}{48} C_A - \frac{3}{16} C_F - \frac{11}{12} T_F N_L \right\} \frac{C_F \alpha_s^2}{\pi^2} + O(\alpha_s^3)
$$

(13)

where the latter form was independently obtained in [15] from the gauge-invariant singularities of on-shell wave-function renormalization of a finite-mass quark. Note that (13) vanishes in QED in the renormalized Yennie gauge [21], $a = 3$, for which there is no ‘infrared catastrophe’ [22]. From (13) we find that $\gamma_F$ agrees with the QCD result of Egorian and Tarasov [23] (misquoted in [7], which contains sign errors).

Setting $a = 1$ and $N_L = 0$ in (13), we discover an error in [16], where the renormalization of the Wilson line was calculated (neglecting light-quark loops) in the Feynman gauge (where Fig 1b gives no contribution). The $\alpha_s^2/\epsilon^2$ terms in [16] are inconsistent with the renormalization group and the $\alpha_s^2/\epsilon$ terms include the contributions of Figs 1c,d with the wrong sign. This incorrect result was used in [24], where the authors also omitted a factor of 2 from the two-loop term in the anomalous dimension. In view of these approximations and errors the two-loop analysis of [24] needs reconsideration.

4 Anomalous dimension of the heavy-light current

The simplest method for finding the anomalous dimension $\tilde{\gamma}_J$ of the current $J_0 = \overline{q}_0\Gamma q_0$ is to consider the bare vertex function with zero light-quark momentum. The diagrams of Fig 2 then differ from those of Fig 1 only in their numerator structure and the indices of their denominators. The effect is to multiply the gamma matrix $\Gamma$ by a factor $\Gamma_0(\omega)$, which is gauge-dependent and more complicated than (10). But it becomes much simpler when one multiplies by the external static-quark bare propagator $S_0(\omega)$. We find that

$$
V_0(\omega) \equiv \omega S_0(\omega) \Gamma_0(\omega) = 1 + \frac{1}{\pi} \frac{D - 1}{D - 3} C_F \alpha_s^2 + \left\{ \frac{V_{A1} C_A + V_{F1} C_F}{\pi} \Gamma_1^2 \right\}
$$

$$+ \left\{ \frac{V_{A2} C_A + V_{F2} C_F + V_{L2} T_F N_L}{\pi} \Gamma_2 \right\} C_F \alpha_s^4 + O(\alpha_s^6)
$$

is gauge-invariant. The coefficients are

$$
V_1 = -\frac{D - 1}{D - 3}
$$

$$
V_{A1} = \frac{D^2 - 4D - 1}{2(D - 3)^2(D - 4)}
$$

$$
V_{F1} = \frac{D - 2}{(D - 3)(D - 4)}
$$

$$
V_{A2} = \frac{2D^4 - 22D^3 + 89D^2 - 151D + 78}{2(2D - 7)(D - 3)^2(D - 4)(D - 6)}
$$

$$
V_{F2} = \frac{(D^2 - 5D + 2)D}{2(2D - 7)(D - 3)(D - 4)(D - 6)}
$$

$$
V_{L2} = \frac{2(D - 2)}{(2D - 7)(D - 3)(D - 6)}
$$

$$
(14)
$$
independently of the spin structure of the current $J$.

The renormalization constant $\bar{Z}_J = \bar{Z}_V \bar{Z}_2^{-1/2} \bar{Z}_2^{1/2}$ may be calculated from the requirement that the renormalized vertex function $\Gamma(\omega) = \Gamma_0(\omega)/\bar{Z}_V$ be finite. Equivalently, but more conveniently, we have $\bar{Z}_J = \bar{Z}_V \bar{Z}_2^{-1/2} \bar{Z}_2^{1/2}$, where $\bar{Z}_V = \bar{Z}_T \bar{Z}_2$ makes $V(\omega) = V_0(\omega)/\bar{Z}_V$ finite. This gives two highly non-trivial checks on our results. First, the $\alpha_s^2/\epsilon^2$ terms in (14) must be such that division by a constant makes $V(\omega)$ finite for all $\omega$. Second, the anomalous dimension $\bar{\gamma}_V = d \log \bar{Z}_V/d \log \mu$ must be gauge-invariant, to ensure that $\bar{\gamma}_J = \bar{\gamma}_V - \frac{1}{2} \bar{\gamma}_F + \frac{1}{2} \bar{\gamma}_F$ is also gauge-invariant. As a third check, we mirrored this EFT calculation, step by step, by the QCD calculation of $\gamma_{qq}$, obtaining agreement with Tarrach \[25\]. These checks give us considerable confidence in our final result

$$\bar{\gamma}_J = \frac{-3 \alpha_s C_F}{4\pi} - \left\{ \frac{49}{96} C_A - \frac{5}{32} C_F - \frac{5}{24} T_F N_L - \left( \frac{1}{4} C_A - C_F \right) \zeta(2) \right\} \frac{C_F \alpha_s^2}{\pi^2} + O(\alpha_s^3)$$

$$= \frac{1}{2} \gamma_{qq} + \left\{ \frac{1}{2} C_A + \frac{1}{4} C_F + \left( \frac{1}{4} C_A - C_F \right) \zeta(2) \right\} \frac{C_F \alpha_s^2}{\pi^2} + O(\alpha_s^3) \quad (15)$$

whose second form we have also verified from the quark-condensate contributions to the correlator of $J$ \[24\].

5 Summary and conclusions

We have given a method for calculating one-scale one- and two-loop EFT integrals. The master two-loop integral can be reduced by the recurrence relation (6) to the degenerate cases (3,4,5), which yield the structures (7,8). We have used this method to calculate the static-quark propagator and a particular case of the heavy-light vertex function, obtaining (10,14) in an arbitrary spacetime dimension and in any covariant gauge. From these follow the anomalous dimensions of the static-quark propagator (13) and the heavy-light current (15). The first agrees with \[8\] and contradicts an old result for the Wilson line \[16\]. The second is also confirmed in \[26\]. It gives the radiative correction to the one-loop result of \[10,11\] and is necessary for matching QCD and EFT currents, with account of the radiative corrections of \[1\]. The lack of it in \[12\] limited the accuracy of extrapolation of lattice computation.

Novel features of EFT calculations include: the trivial simplification of diagrams such as Figs 1d and 2i, which are the most difficult in on-shell massive QCD calculations \[8,9\]; an algorithm more efficient than the triangle relation \[18\] of massless QCD; the appearance of $\zeta(2) = \pi^2/6$ in counterterms, via the relation (9) between the basic integral structures, in contrast to the situation in QCD, where $\zeta(3)$ is the first term to appear in the expansion.

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Figure 1: Heavy-quark self-energy diagrams, to two loops

Figure 2: Heavy-light proper vertex diagrams, to two loops