Magnetic Field Symmetry of Dynamic Capacitances

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While the static capacitance matrix is always symmetric and thus an even function of the magnetic field, the dynamic capacitance matrix of multi-terminal samples obeys in general only the weaker Onsager-Casimir symmetry relations. Our results are in accordance with recent experimental observations of asymmetric dynamic capacitances in quantized Hall systems. We predict quantization of the four-terminal resistances of an insulating Hall sample.

Symmetry relations play an important role in both theoretical and experimental physics. In mesoscopic conductors, microreversibility coupled with the (irreversible) thermodynamics of electron reservoirs leads to capacitances which obey the Onsager-Casimir reciprocity relations under magnetic field reversal. A manifestly symmetric formulation of the dc-transmission approach to electrical conductance proved to be close to experimental reality and has been applied to a wide range of problems including the quantized Hall effect. In this Letter we are concerned not with stationary dc-conductances, the static capacitance matrix can be obtained from thermodynamic potentials. As a consequence, the admittance of a two-terminal sample is determined by a single scalar admittance and is thus an even function of the magnetic field. An asymmetric conductance, on the other hand, requires at least a three-terminal sample. While asymmetric dc-conductances are well established, the investigation of an asymmetric ac-response is relatively new. Two years ago, Chen et al. reported on an asymmetric capacitance of a gate close to the edge of a quantum Hall bar. This effect was explained in the framework of the edge-state picture and with the help of a novel theory of the low-frequency admittance of mesoscopic conductors provided by Ref. . Importantly, this theory is based on the concept of partial densities of states which are sensitive to the breaking of time-reversal symmetry. In a succeeding work we have extended the theory to the general case of a multi-terminal Hall sample with an arbitrary arrangement of edge states. It is now understood that the capacitance between two contacts is in general not an even function of the magnetic field, if one of these contacts allows transmission of charge carriers to other contacts. If, however, both contacts are purely capacitively coupled to the rest of the sample and to each other, the static capacitance between them is always symmetric. It would be tempting to extend this statement to the dynamical capacitance, i.e. to finite frequencies. However, in a recent experiment, Sommerfeld et al. investigated the symmetry properties of gates which are capacitively coupled to a disc-shaped two-dimensional electron gas. While in the two-terminal case the expected symmetric capacitance was observed, in the three-terminal case the capacitance turned out to be asymmetric. In the present work we show that the above mentioned theory of low-frequency admittance predicts the observed result. In particular, we show that the dynamic capacitance between two conductors which are both capacitively coupled to the rest of the sample and to each other can be asymmetric, and we discuss an example similar to the experiment of Sommerfeld et al.

Consider a general multi-terminal sample consisting of conductors $j = 1, \ldots, M$ and which is connected via contacts to electron reservoirs $\alpha = 1, \ldots, N$. For low frequencies we expand the admittance in powers of frequency,

$$G_{\alpha\beta}(\omega) = G_{\alpha\beta}^{(0)} - i\omega E_{\alpha\beta} + \omega^2 K_{\alpha\beta} + \mathcal{O}(\omega^3) ,$$

(3)
where \( \hat{K}_{\alpha\beta} \) is the dc-conductance matrix, \( E_{\alpha\beta} \) is the emittance matrix \[12\], and \( K_{\alpha\beta} \) is related to a charge relaxation time \[2\]. If all conductors are purely capacitively coupled, the dc-conductance vanishes and the emittance is just the electrochemical capacitance which is a symmetric function of the magnetic field \[1\]. In the sequel, we show that the matrix \( K_{\alpha\beta} \) is in general asymmetric even for purely capacitive contacts. To do this, we assume the presence of a voltage probe at a contact \( N \). Hence, contact \( N \) is connected to at least one other contact such that \( \hat{G}^{(0)}_{N,N} \neq 0 \), and so the net current through this contact vanishes, \( \delta I_N = 0 \). After an elimination of \( \delta V_N \) from the Eqs. \[1\] one obtains the reduced admittance matrix \( \hat{G}_{\alpha\beta} \) for the remaining \( N-1 \) terminals. Now, consider two purely capacitive contacts denoted by \( \gamma \) and \( \delta \). (A specific example of such a structure is shown in row 3 of table \[1\].) Thus, an element of the dc-conductance matrix \( \hat{G}^{(0)}_{\alpha\beta} \) vanishes whenever one of the indices takes the value \( \gamma \) or \( \delta \). An expansion of the reduced admittance with respect to frequency yields then the reduced coefficients

\[
\hat{G}_{\gamma\delta} = 0 \ , \quad (4)
\]

\[
\hat{E}_{\gamma\delta} = E_{\gamma\delta} \ , \quad (5)
\]

\[
\hat{K}_{\gamma\delta} = K_{\gamma\delta} + \frac{E_{\gamma N} E_{\delta N}}{\hat{G}^{(0)}_{NN}} \ . \quad (6)
\]

Since contact \( N \) is not purely capacitive, the emittance elements \( E_{\gamma N} \) and \( E_{\delta N} \) are in general asymmetric \[1\]. Thus, \( \hat{K}_{\gamma\delta} \) is in general asymmetric, too, even for a symmetric \( K_{\gamma\delta} \).

Starting from a \( N \)-terminal sample with a symmetric dynamic capacitance, one can arrive at a sample with an asymmetric capacitance for an \( N-1 \)-terminal sample by using one terminal as a voltage probe. Obviously, this is only possible if \( N > 3 \) and provided that there are contacts with a non-vanishing dc-conductance. This last condition was not satisfied in the experiment by Sommerfeld et al. \[6\], where apparently only capacitive contacts were present. Their sample, however, was macroscopic in size such that electron motion is not phase coherent. But phase-breaking processes (dissipation) can be modeled with the help of fictitious voltage probes \[13\]. We expect thus an asymmetric dynamic capacitance of a three-terminal sample in the presence of dissipation.

Consider a disc of a two-dimensional electron gas in a perpendicular magnetic field as shown in Fig. \[1\]. Three equivalent gates (grey regions) connected to the contacts (1, 2, 3) are located at the sample boundary in a symmetric way and couple only capacitively. For simplicity, assume that the electron gas is at the first Hall plateau such that one edge channel runs along the sample boundary. Due to the fictitious voltage probes (4, 5, 6), this edge channel is divided into three pieces which couple each to the closest gate with a dynamic electrochemical capacitance of the form \( C_{\mu}(\omega) = C_{\mu} + i\omega(h/e^2)C_{\mu}^2 + O(\omega^2) \) \[1\], where \( C_{\mu} \) is the static electrochemical capacitance. Here, the dynamic correction term of the capacitance can be associated with charge relaxation. The conductance of these channels is given by \( e^2/h \) for a spin split Landau level. The dc-relationship between currents and voltages is obtained from the well-known transmission approach applied to quantized Hall systems \[7\]. We neglect all capacitances except those between gates and the closest piece of the edge channel. The ac-part of the admittance can then be obtained as follows \[11\]. A voltage oscillation in reservoir \( \beta \) induces a displacement current through contact \( \alpha \), if there is a capacitive coupling between the conductors in which contact \( \beta \) injects charge, and the conductors which emit charge into contact \( \alpha \). For example, \( \delta I_4 \propto -i\omega(-C_{\mu}(\omega)\delta V_1) \). As usual, off-diagonal capacitance elements occur with a minus sign. If a contact \( \beta \) transmits carriers into contact \( \alpha \), the associated capacitance element is obtained from current conservation, e.g., \( \delta I_4 \propto -i\omega C_{\mu}(\omega)\delta V_0 \). In the purely capacitive case this capacitance element corresponds to a diagonal element of the capacitance, e.g., \( \delta I_1 \propto -i\omega C_{\mu}(\omega)\delta V_1 \).

This receipt yields

\[
\begin{align*}
\delta I_1 &= -i\omega C_{\mu}(\omega)(\delta V_1 - \delta V_6) \\
\delta I_2 &= -i\omega C_{\mu}(\omega)(\delta V_2 - \delta V_4) \\
\delta I_3 &= -i\omega C_{\mu}(\omega)(\delta V_3 - \delta V_5) \\
\delta I_4 &= (e^2/h)(\delta V_4 - \delta V_6) - i\omega C_{\mu}(\omega)(\delta V_6 - \delta V_1) \\
\delta I_5 &= (e^2/h)(\delta V_5 - \delta V_4) - i\omega C_{\mu}(\omega)(\delta V_4 - \delta V_2) \\
\delta I_6 &= (e^2/h)(\delta V_6 - \delta V_5) - i\omega C_{\mu}(\omega)(\delta V_5 - \delta V_3) .
\end{align*}
\] (7)

Using the conditions for voltage probes, \( \delta I_4 = \delta I_5 = \delta I_6 = 0 \), the elimination of \( \delta V_4 \), \( \delta V_5 \), and \( \delta V_6 \) leads to the reduced admittance matrix \( \hat{G}_{\alpha\beta} \) for the contacts 1, 2, 3. We find

\[
\begin{align*}
\hat{G}_{12} &= \frac{1}{3} (i\omega C_{\mu} - (h/e^2)\omega^2 C_{\mu}^2) \quad (8) \\
\hat{G}_{21} &= \frac{1}{3} i\omega C_{\mu} . \quad (9)
\end{align*}
\]

The other elements of \( \hat{G}_{\alpha\beta} \) are determined by the above mentioned sum rules and by the specific symmetry of the sample (i.e. \( \hat{G}_{12} = \hat{G}_{23} = \hat{G}_{31} \) and \( \hat{G}_{21} = \hat{G}_{13} = \hat{G}_{32} \)). There is obviously an asymmetry \( (\hat{G}_{12})_B - (\hat{G}_{12})_{-B} = -(h/3e^2)\omega^2 C_{\mu}^2 \) in \( O(\omega^2) \). For the modulus it holds \( | \langle \hat{G}_{12} \rangle_B | > | \langle \hat{G}_{12} \rangle_{-B} | \), which is in accordance with the experiment \[6\]. An additional elimination of contact 3 leads to a (scalar and thus symmetric) two-terminal capacitance equal to 3/2 times the capacitance associated with \( \hat{G}_{12} \) of Eq. \[8\]. This discussion suggests that dissipation plays an essential role. It would be interesting to investigate the asymmetry in the capacitance elements as a function of temperature or for models in which the dissipation can be gradually switched off.

Instead of the just considered case where three contacts are capacitively coupled to three pieces of an edge channel, we mention the analogous case with four gates coupled to four pieces of an edge channel. For such a sample one can calculate the longitudinal resistance, \( R_L \), and the
quantum Hall resistance, \( R_H \). Interestingly, we find exact quantization, \( R_H = \frac{h}{e^2} \) and \( R_L = 0 \) independent of \( \omega \). Note that, due to the purely capacitive coupling of the reservoirs, this sample is insulating, i.e. the dc resistances diverge. Clearly, for the four-terminal analog of Fig. 1 such a behavior is to be expected: Four probe measurements are independent of the contact impedances. Furthermore, if the sample exhibits inelastic relaxation and equilibration the manner in which current is introduced into the sample is unimportant. We remark that the quantization of the four-terminal resistances is not a consequence of the symmetry of the model of Fig. 1 and is valid for arbitrary capacitances between the gates and the closest edge. However, the absence of capacitive coupling along the edge and across the sample is crucial. In general, there are displacement current contributions and relaxation contributions to both the Hall resistance and the longitudinal resistance \( \frac{1}{4}, \frac{1}{2} \). Finally, we mention that even so the purely capacitively coupled sample of Fig. 1 is an insulator in terms of a strict dc-measurement, it is not a model of a Hall insulating state \( \frac{1}{4} \) for which it is expected that the Hall resistance remains finite but the longitudinal resistance diverges. We conclude with the remark that measurements on such samples are made with ohmic contacts, but that these contacts provide increasingly capacitive coupling only when the sample is driven deep enough into the insulating phase.

To conclude, we summarize the main results on the symmetry properties of capacitances in table \( \frac{1}{3} \) for two- and three-terminal samples. In order to discuss the symmetry properties of the admittance under reversal of a magnetic field one must first distinguish the leading order response (\( E_{\alpha \beta} \), proportional to \( \omega \)) and the higher order response (\( K_{\alpha \beta} \), proportional to \( \omega^2 \), or even higher powers of frequency). Secondly, one must distinguish between different conducting topologies. The static capacitance is always symmetric since it is the second derivative of an appropriate thermodynamic potential. The dynamic capacitance of two-terminal samples (row 1 of table \( \frac{1}{3} \)) is also always symmetric. This is a direct consequence of the Onsager-Casimir relations, gauge invariance, and current conservation. Purely capacitive, phase-coherent multi-terminal samples (row 2 of table \( \frac{1}{3} \)) exhibit dynamic capacitance coefficients which are even functions of the magnetic field. In the case of a spatially dependent potential along a conductor (e.g. in the presence of a voltage probe), the emittance of capacitive contacts is symmetric but higher-order terms in frequency are not (row 3 of table \( \frac{1}{3} \)). In a measurement for which at least one of the conductors is connected to two or more contacts (and thus permits transmission from one of its contacts to another) one finds an emittance \( \frac{1}{4} \) which is not an even function of the magnetic field (row 4 of table \( \frac{1}{3} \)).

Table \( \frac{1}{3} \) demonstrates that dynamic capacitances have a far richer magnetic field symmetry behaviour than dc conductances.

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FIG. 1. Disc-shaped 2-dimensional electron gas at the first Landau level. Three gates (1, 2, 3) interact electrically with the edge states. The fictitious contacts (4, 5, 6) act as voltage probes and model dissipation.

TABLE I. Symmetry properties of the dynamic capacitance of two- and three-terminal samples; a circle (V) indicates a voltage probe.