A $C^\infty$-REGULARITY THEOREM FOR NONDEGENERATE CR MAPPINGS

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Abstract. We prove the following regularity result: If $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ are smooth generic submanifolds and $M$ is minimal, then every $C^k$-CR-map from $M$ into $M'$ which is $k$-nondegenerate is smooth. As an application, every CR diffeomorphism of $k$-nondegenerate minimal submanifolds in $\mathbb{C}^N$ of class $C^k$ is smooth.

1. Introduction and statement of results

We first briefly describe the setting for the results which we want to discuss. Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be generic, real submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively. We shall denote by $dM$ the real codimension of $M$, and by $d'$ the real codimension of $M'$, and write $n = N - d$, $n' = N' - d'$. Recall that $M$ is generic if there is a smooth defining function $\rho = (\rho_1, \ldots, \rho_d)$ for $M$ such that the vectors $\rho_1, Z(p), \ldots, \rho_d, Z(p)$ are linearly independent for $p \in M$. Here for any smooth function $\phi$ we let $\phi_Z = (\frac{\partial \phi}{\partial z_1}, \ldots, \frac{\partial \phi}{\partial z_n})$ be its complex gradient.

We also fix points $p_0 \in M$ and $p'_0 \in M'$ (which we will assume to be equal to 0 for most of this paper). A $C^k$-mapping $H$ from $M$ into $M'$ is said to be CR if its differential $dH$ satisfies $dH(T^c_p M) \subset T^c_{H(p)} M'$ for $p \in M$, where $T^c_p M$ denotes the complex tangent space to $M$ at $p$, that is, the largest subspace of the real tangent space $T_p M$ invariant under the complex structure operator $J$ in $\mathbb{C}^N$. Equivalently, if $H = (H_1, \ldots, H_{N'})$ for any system of holomorphic coordinates in $\mathbb{C}^{N'}$, each $H_j$ is a CR-function on $M$. (For further reference on these definitions, the reader is referred to the book of Baouendi, Ebenfelt and Rothschild [1].)

The following definition is from [9]. We shall give it in a slightly modified form.

Definition 1. Let $M, M'$ be as above. Let $\rho' = (\rho'_1, \ldots, \rho'_{d'})$ be a defining function for $M'$ near $H(p_0)$, and choose a basis $L_1, \ldots, L_n$ of CR-vector fields tangent to $M$ near $p_0$. We shall write $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$ for any multiindex $\alpha$. Let $H : M \to M'$ be a CR-map of class $C^m$. For $0 \leq k \leq m$, define the increasing sequence of subspaces $E_k(p_0) \subset \mathbb{C}^{N'}$ by

\begin{equation}
E_k(p_0) = \text{span}_\mathbb{C} \{ L^\alpha \rho'_l Z(H(Z), \overline{H(Z)}) |_{Z=p_0} : 0 \leq |\alpha| \leq k, 1 \leq l \leq d' \}.
\end{equation}

We say that $H$ is $k_0$-nondegenerate at $p_0$ (with $0 \leq k_0 \leq m$) if $E_{k_0 - 1}(p_0) \neq E_{k_0}(p_0) = \mathbb{C}^{N'}$.

The invariance of this definition under the choices of the defining function, the basis of CR vector fields and the choices of holomorphic coordinates in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$ is easy to show; the reader can find proofs for this in [9] or [1].

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Recall that if $\Gamma \subset \mathbb{R}^d$ is an open convex cone, $p_0 \in M$, and $U \subset \mathbb{C}^N$ is an open neighbourhood of $p_0$, then a wedge $W$ with edge $M$ centered at $p_0$ is defined to be a set of the form $W = \{Z \in U : \rho(Z) \in \Gamma\}$, where $\rho$ is a local defining function for $M$. We can now state our main theorem.

**Theorem 2.** Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be smooth generic submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, $p_0 \in M$ and $p'_0 \in M'$, $H : M \to M'$ a $C^{k_0}$-CR-map which is $k_0$-nondegenerate at $p_0$ and extends continuously to a holomorphic function in a wedge $W$ with edge $M$. Then $H$ is smooth in some neighbourhood of $p_0$.

This theorem is the smooth version of the main result in [3]. Let us recall that $M$ is said to be minimal at $p_0$ if there does not exist any CR-submanifold through $p_0$ strictly contained in $M$ with the same CR dimension as $M$. By a theorem of Tumanov, if $M$ is minimal, every continuous CR-function $f$ on $M$ near $p$ extends continuously to a holomorphic function into a wedge $W$ with edge $M$. Hence we have the following corollary.

**Corollary 3.** Let $M \subset \mathbb{C}^N$, $M' \subset \mathbb{C}^{N'}$ be smooth generic submanifolds of $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, respectively, $p_0 \in M$ and $p'_0 \in M'$, $M$ minimal at $p_0$, $H : M \to M'$ a $C^{k_0}$ map which is $k_0$-nondegenerate at $p_0$. Then $H$ is smooth in some neighbourhood of $p_0$.

Note that by a regularity theorem of Rosay ([13], see also [1]), if the boundary value of a holomorphic function in a wedge $W$ with edge $M$ is $C^k$ on $M$, then the extension is also of class $C^k$ up to the edge. Hence, for the proof of Theorem 2 we will assume that $H$ extends in a $C^{k_0}$-fashion to a wedge $W$ centered at $p_0$.

We would like to mention one particular instance of this theorem. If $M$ is a manifold whose identity map is $k_0$-nondegenerate in the sense of Definition 1, then we say that $M$ is $k_0$-nondegenerate. This notion has been introduced for hypersurfaces by Baouendi, Huang and Rothschild in [2]; for a thorough introduction to this nondegeneracy condition for submanifolds and its connection with holomorphic nondegeneracy in the sense of Stanton ([13]), see [1], or the paper of Ebenfelt [3]. In particular, every CR-diffeomorphism of class $C^{k_0}$ of a $k_0$-nondegenerate submanifold is $k_0$-nondegenerate in the sense of Definition 1. Theorem 2 implies the following regularity result for $k_0$-nondegenerate smooth submanifolds.

**Corollary 4.** Assume that $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are $k_0$-nondegenerate smooth submanifolds of real codimension $d$, $M$ minimal at $p_0$, and $H : M \to M'$ is a CR-diffeomorphism of class $C^{k_0}$. Then $H$ is smooth.

If $d = 1$, we can drop the assumption of minimality, since in the hypersurface case, $k_0$-nondegeneracy implies minimality. In the case where $N = N' = 2$ and $d = 1$, Corollary 4 is basically contained in the thesis of Roberts [12]. The Levi-nondegenerate hypersurface case is well understood; the connection with the results proved in this paper is that Levi-nondegeneracy of hypersurfaces is equivalent to $1$-nondegeneracy. In fact, for Levi-nondegenerate hypersurfaces, Corollary 4 is due to Nirenberg, Webster and Yang [11], and of course we should not forget to mention Fefferman’s mapping theorem ([3]) (however, we shall not deal with the $C^1$-extension here). A proof for strictly pseudoconvex hypersurfaces of finite smoothness was given by Pinchuk and Khasanov [13]. More recently, Tumanov [3] has proved the corresponding theorem for Levi-nondegenerate targets of higher codimension. For results for pseudoconvex targets, we want to refer the reader to the historical
discussion in the paper by Coupet and Sukhov and the newer results for convex hypersurfaces by Coupet, Gaussier and Sukhov.

The paper is organized as follows. In section 3, 4 and 5 we present the technical foundations for the proof. Although these results are well known, they are not easy to find in the literature; so, in order to make this paper as self contained as possible, we have decided to include the proofs. Theorem 2 is then proved in section 5.

2. Boundary values of functions of slow growth

In this section, we will develop an integral representation for a partially bounded function of slow growth (in a wedge with straight edge). Let us first fix notation. Let \( U \subset \mathbb{C}^n \), \( V \subset \mathbb{R}^d \) be open subsets, and let \( \delta = (\delta_1, \ldots, \delta_d) \in \mathbb{R}^d \) with \( 0 < \delta_j \) for \( 0 \leq j \leq d \). We set \( \Omega_+ = \{(z, s, t) \in U \times V \times \mathbb{R}^d : 0 < t < \delta\} \), \( \Omega_- = \{(z, s, t) \in U \times V \times \mathbb{R}^d : 0 > t > -\delta\} \) and \( \Omega_0 = U \times V \times \{0\} \), and we will write \( z = (x, y) \) for the underlying real variables. Throughout the paper, \( d\mu \) will denote Lebesgue measure. Let \( \mathcal{B}(\Omega_+) \) be the space of all functions \( h \in C^1(\Omega_+) \) that extend smoothly to the set \( E = \{(z, s, t) \in \Omega_+ : t \neq 0\} \) which have the following property: For each compact set \( K \subset U \times V \), there exist positive constants \( C_1 \), \( \mu \) and \( C_2 \) (depending on \( K \) and \( h \)) such that

\[
\sup_{(z,s) \in K, 0 < t < \delta} |t|^{\mu} |h(x, y, s, t)| \leq C_1
\]

and

\[
\sup_{(z,s) \in K, 0 < t < \delta} |\partial_j h(x, y, s, t)| \leq C_2, \quad 1 \leq j \leq d.
\]

Here we write \( \partial_j = \frac{1}{2} \left( \frac{\partial}{\partial y_j} + i \frac{\partial}{\partial t_j} \right) \). We have the following (probably well known) result, which we state for \( \mathcal{B}(\Omega_+) \); however, we define \( \mathcal{B}(\Omega_-) \) in a similar manner, and all the results stated in this section hold equally well for \( \mathcal{B}(\Omega_-) \).

**Theorem 5.** Let \( h \in \mathcal{B}(\Omega_+) \). Then the limit

\[
\langle b_+ h, \phi \rangle = \lim_{\epsilon = (\epsilon_1, \ldots, \epsilon_d) \to 0} \int_{U \times V} h(x, y, s, \epsilon) \phi(x, y, s) \, d\mu
\]

exists for each \( \phi \in C_0^\infty(\Omega_0) \) and defines a distribution \( b_+ h \) called the boundary value of \( h \). Furthermore, for each compact set \( K \) there exists an integer \( v_0 \) such that for \( v \geq v_0 \), for each \( j = 1, \ldots, d \), \( 0 \leq \delta_j \leq \delta \), we have the following integral representation for \( \phi \in C_0^\infty(U \times V) \) with \( \text{supp } \phi \subset \bar{K} \):

\[
\langle b_+ h, \phi \rangle = \int_{U \times V} h(x, y, s, 0, \ldots, \delta', \ldots, 0) S_v \phi(x, y, s, 0, \ldots, \delta', \ldots, 0) \, d\mu
\]

\[
+ 2i \int_{U \times V} \int_0^{\delta'} \partial_1 h(x, y, s, 0, \ldots, t_j, \ldots, 0) S_v \phi(x, y, s, 0, \ldots, t_j, \ldots, 0) \, dt_j \, d\mu
\]

\[
+ 2i \int_{U \times V} \int_0^{\delta'} h(x, y, s, 0, \ldots, t_j, \ldots, 0) D_{s_j}^{v+1} \phi(x, y, s) t_j^v \, dt_j \, d\mu,
\]

where

\[
S_v \phi(x, y, s, t) = \sum_{|\alpha| \leq v} \frac{1}{\alpha!} D_\alpha \phi(x, y, s) t^\alpha.
\]
Proof. Let \( S_e \phi \) be defined by (1). We are going to prove the formula under the assumption that \( j = 1 \). Fix \((x, y), s_2, \ldots, s_d \) and \( 0 < \delta' < \delta_1 \), and assume \( 0 < \epsilon_1 < \delta_1 - \delta' \). First we are going to assume that \( K = \text{supp} \phi \) is contained in a product of the form \( U_1 \times [a, b] \times [a_2, b_2] \times \ldots \times [a_d, b_d] \) contained in a relatively compact open subset \( W \subset U \times V \). In this case, define

\[
u(s_1, t_1) = h(x, y, s_1, s_2, \ldots, s_d, \epsilon_1 + t_1, \epsilon_2, \ldots, \epsilon_d)S_e \phi(z, s, t_1, 0, \ldots, 0).
\]

Clearly, \( u \) is \( C^1 \) on the square \( \omega = [a, b] \times [0, \delta'] \) and \( u(s_1, t_1) = 0 \) if \( s_1 \geq b \) or \( s_1 \leq a \).

By Stokes formula,

\[
\int_{\partial \omega} u(s_1, t_1) \, dw = 2i \int_{\omega} \bar{\partial} u(s_1, t_1) \, dm,
\]

where we have set \( w = s_1 + it_1 \) and \( \bar{\partial} = \bar{\partial}_1 \). This formula translates into

\[
\int_a^b h(x, y, s, \epsilon) \phi(x, y, s) \, ds_1 = 
\int_a^b h(x, y, s, \epsilon_1 + \delta', \epsilon_2, \ldots, \epsilon_d)S_e \phi(x, y, s, \delta', 0, \ldots, 0) \, ds_1 
+ 2i \int_0^{\delta'} \bar{\partial}_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \ldots, \epsilon_d)S_e \phi(x, y, s, t_1, 0, \ldots, 0) \, ds_1 \, dt_1 
+ 2i \int_0^{\delta'} \bar{\partial}_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \ldots, \epsilon_d)D_s^{\epsilon_1+1} \phi(x, y, s) t_1^v \, ds_1 \, dt_1.
\]

We integrate this formula with respect to \((x, y, s_2, \ldots, s_d) \) to obtain

\[
\int_W h(x, y, s, \epsilon) \phi(x, y, s) \, dm = 
\int_W h(x, y, s, \epsilon_1 + \delta', \epsilon_2, \ldots, \epsilon_d)S_e \phi(x, y, s, \delta', 0, \ldots, 0) \, dm 
+ 2i \int_W \int_0^{\delta'} \bar{\partial}_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \ldots, \epsilon_d)S_e \phi(x, y, s, t_1, 0, \ldots, 0) \, dt_1 \, dm 
+ 2i \int_W \int_0^{\delta'} \bar{\partial}_1 h(x, y, s, \epsilon_1 + t_1, \epsilon_2, \ldots, \epsilon_d)D_s^{\epsilon_1+1} \phi(x, y, s) t_1^v \, dt_1 \, dm.
\]

For each of these integrals, we can use the bounded convergence theorem to take the limit as \( \epsilon \to 0 \), provided that we choose \( v \geq \mu_K \), where \( \mu_K \) denotes the least integer \( \mu \) for which (3) holds on \( K \) and to obtain an estimate of the form \( \| \partial_{+} h, \phi \| \leq C \| \phi \|_{l+1} \) (where \( \| \phi \|_k = \max_{x \in U \times V, |\alpha| \leq k} |\phi^{\alpha}(x)| \)).

Now we pass to the case of general \( K \) by covering with finitely many sets of the form considered above and using a partition of unity. The details are easy and left to the reader. \( \square \)

Consider now the class \( \mathfrak{A}(\Omega_+) \) of functions \( h \) which are smooth on \( E \) with the property that for all \( \alpha, \beta \) we have that \( D_{x,y}^{\alpha}D_s^{\beta} h \in \mathfrak{B}(\Omega_+) \). If \( h \in \mathfrak{A}(\Omega_+) \), for \( K \subset U \times V \) we let \( \mu_l(h, K) \) the smallest integer \( \mu \) such that

\[
\sup_{(x,y) \in K, 0 < t < \delta} |t|^{\mu} |D_{x,y}^{\alpha}D_s^{\beta} h(x, y, s, t)| \leq C_1, \quad |\alpha| + |\beta| \leq l
\]

for some constant \( C_1 \). Let us also introduce the space \( \mathfrak{A}_\infty(\Omega_+) \) of functions in \( \mathfrak{A}(\Omega_+) \) with the additional property that for any compact set \( K \subset U \times V \), for any
multiindices $\alpha$ and $\beta$, and for any nonnegative integer $k$ there exists a constant $C$ such that

\begin{equation}
\sup_{(x,y) \in K, 0 < t < \delta} |D_{x,y}^\alpha D_s^\beta \tilde{\partial}_j h(x,y,s,t)| \leq C |t|^k, \quad 1 \leq j \leq d.
\end{equation}

Of course, we define the spaces $\mathcal{A}(\Omega_-)$ and $\mathcal{A}_\infty(\Omega_-)$ analogously, and the results stated below for $\mathcal{A}(\Omega_+)$ and $\mathcal{A}_\infty(\Omega_+)$ also hold for $\mathcal{A}(\Omega_-)$ and $\mathcal{A}_\infty(\Omega_-)$. This can be seen most easily by noting the following useful fact: If $h(x,y,s,t) \in \mathcal{A}(\Omega_-)$ (or $\mathcal{A}_\infty(\Omega_-)$, respectively), $\tilde{h}(x,y,s,-t) \in \mathcal{A}(\Omega_+)$ (or $\mathcal{A}_\infty(\Omega_+)$, respectively).

We will also need the space of functions which are almost holomorphic on $U \times V$.

This is the space

\begin{equation}
\mathcal{A}(U \times V) = \{a \in C^\infty(U \times V \times \mathbb{R}^d) : D_{x,y}^\alpha D_s^\beta \tilde{\partial}_j a(x,y,s,0) = 0, 1 \leq j \leq d\}.
\end{equation}

**Lemma 6.** Let $h \in \mathcal{A}(\Omega_+)$, $a \in \mathcal{A}(U \times V)$, and set $a_0(x,y,s) = a(x,y,s,0)$. Then $ah \in \mathcal{A}(\Omega_+)$, and $b_+ ah = a_0 b_+ h$ in the sense of distributions. Furthermore, if $h \in \mathcal{A}_\infty(\Omega_+)$, so is $ah$.

**Proof.** By the Leibniz rule, $D_{x,y}^\alpha D_s^\beta ah$ is a sum of products of derivatives of $a$ and $h$. It is clear that such a sum fulfills (2). To see that it also fulfills (3), note that by (11) every derivative of $\tilde{\partial}_j a$ vanishes to infinite order on $t = 0$.

To see that $b_+ ah = a_0 b_+ h$ we use Taylor development to write $a(x,y,s,t) = \sum_{|\beta| \leq k} \frac{1}{\beta!} D_s^\beta a(x,y,s,0)(it)^\beta + O(|t|^{k+1})$ (uniformly on compact subsets of $U \times V$). Now choose $k \geq \mu_0(h, K)$ and substitute into (4) for $\phi$ with $\text{supp } \phi \subset K$. The claim follows now by taking the limit and using Theorem 3. 

Basically the same proof shows the following Lemma.

**Lemma 7.** Assume that $X$ is a vector field on $U \times V \times \mathbb{R}^d$ which is tangent to all subspaces of the form $t = c$, where $c \in \mathbb{R}^d$ is a constant vector, and such that all the coefficients of $X$ are in $\mathcal{A}(U \times V)$. Set $X_0 = X|_{t=0}$. If $h \in \mathcal{A}(\Omega_+)$, then $Xh \in \mathcal{A}(\Omega_+)$, and $b_+ Xh = X_0 b_+ h$ in the sense of distributions. Furthermore, if $h \in \mathcal{A}_\infty(\Omega_+)$, so is $Xh$.

### 3. An almost holomorphic edge-of-the-wedge theorem

The main result of this section is the following theorem. Our presentation follows closely [2], but we also want to refer the reader to [1]. We keep the notation from the proceeding section and since we shall use the Fourier transform we also introduce the following new variables: $\xi \in \mathbb{R}^n$, $\tau \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^d$. For a distribution $\phi$ on $U \times V$ we will write $\phi(\xi, \tau, \sigma) = \langle \phi, \exp(-i(x\xi + y\tau + s\sigma)) \rangle$ for its Fourier transform.

**Theorem 8.** Assume that $h_+ \in \mathcal{A}(\Omega_+)$, $h_- \in \mathcal{A}(\Omega_-)$, and that $b_+ h_+ = b_- h_- = h$. Then $h$ is smooth.

The proof follows from the next Lemma.

**Lemma 9.** Let $h \in \mathcal{A}(\Omega_+)$, and $\phi \in C_0^\infty(U \times V)$. Then for every $k \in \mathbb{N}$ there exists a constant $C_k$ such that if $\zeta = (\xi, \tau, \sigma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^d$ with $\sigma_j \leq 0$ for some $j$, $1 \leq j \leq d$, then

\begin{equation}
|\phi b_+ h(\zeta)| \leq \frac{C_k}{(1 + |\zeta|^2)^d}.
\end{equation}
Here, $C_k$ depends on $k$, $\phi$, and $h$. The same result holds with $A(\Omega_+)$ replaced by $A(\Omega_-)$ if $\sigma_j \geq 0$ for some $j$.

Proof. For the moment, fix $\zeta$; for simplicity, assume that $j = 1$, so that $\sigma_1 = 0$. We shall write $a(x, y, s, t) = \exp(-i(x\xi + y\tau + sa) + t\sigma)$. Then $a \in A\delta(U \times V)$—in fact, $\partial_i a = 0$, $1 \leq j \leq d$. We let $\Delta$ be the real Laplacian in the $2n + d$ variables $(x, y, s)$, that is,

$$
\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} + \sum_{j=1}^{d} \frac{\partial^2}{\partial s_j^2}.
$$

We then have that $(1 + \Delta)^k a(x, y, s, t) = (1 + |\zeta|^2)^k a(x, y, s, t)$. Recall that we write $a_0(x, y, s) = a(x, y, s, 0)$. By Lemma 3 we see that $\phi b_+ h(\zeta) = \langle \phi b_+ h, a_0 \rangle = \langle b_+, h, \phi a_0 \rangle = \langle b_+, ah, \phi \rangle$. We apply the integral formula 3 from Theorem 5 for $j = 1$, and some $\delta'$, which implies that

$$
\langle b_+ ah, \phi \rangle =
\int_{U \times V} h(x, y, s, \delta', 0)e^{-i(x\xi + y\tau + sa)} e^{\delta' \sigma_1} S_\phi(x, y, s, \delta', 0) \, dm + 2i \int_{U \times V} \int_{0}^{\delta'} (\partial_1 h(x, y, s, t, 0)) e^{-i(x\xi + y\tau + sa)} e^{t_1 \sigma_1} S_\phi(x, y, s, t, 0) \, dt_1 \, dm + 2i \int_{U \times V} \int_{0}^{\delta'} h(x, y, s, t, 0) e^{-i(x\xi + y\tau + sa)} e^{t_1 \sigma_1} D_{s_1}^{c+1} \phi(x, y, s, t_1) \, dt_1 \, dm = I_1 + I_2 + I_3.
$$

We now replace $e^{-i(x\xi + y\tau + sa)}$ by $\frac{1}{(1 + (\zeta^2))^k} (1 + \Delta)^k e^{-i(x\xi + y\tau + sa)}$ in all three integrals above. Then we integrate by parts and estimate, where we choose $\nu \geq \mu_k(h, K)$ (see 3 for the definition of this number) with $K = \text{supp} \phi$. Since all the estimates are easy, we do not write them out; the reader can easily check them.

Proof of Theorem 3. Let $p \in U \times V$. Choose a function $\phi \in C^\infty_c(U \times V)$ which is equal to 1 in some open neighbourhood of $p$. By Lemma 3, since $h_+ \in A(\Omega_+)$ and $h_- \in A(\Omega_-)$, we have that

$$
|\widehat{\phi h}(\zeta)| \leq \frac{C_k}{(1 + |\zeta|^2)^k}.
$$

for all $\zeta \in \mathbb{R}^{2n+d}$. Hence, $\phi h$ is smooth (see for Example 3), and so $h$ is smooth in some neighbourhood of $p$, since $\phi \equiv 1$ there. Since $p$ was arbitrary, the claim follows.

4. A VERSION OF THE IMPLICIT FUNCTION THEOREM

We will need the following, “almost holomorphic”, implicit function theorem.

**Theorem 10.** Let $U \subset \mathbb{C}^N$ be open, $0 \in U$, $A \in \mathbb{C}^p$, $F : U \times \mathbb{C}^p \to \mathbb{C}^N$ be smooth in the first $N$ variables and polynomial in the last $p$ variables, and assume that $F(0, A) = 0$ and $F_z(0, A)$ is invertible. Then there exists a neighbourhood $U' \times V'$ of $(0, A)$ and a smooth function $\phi : U' \times V' \to \mathbb{C}^N$ with $\phi(0, A) = 0$, such that if
$F(Z, ar{Z}, W) = 0$ for some $(Z, W) \in U' \times V'$, then $Z = \phi(Z, \bar{Z}, W)$. Furthermore, for every multiindex $\alpha$, and each $j$, $1 \leq j \leq N$,

(16) \[ D^\alpha \frac{\partial \phi}{\partial Z_k}(Z, \bar{Z}, W) = 0, \quad 1 \leq k \leq N, \]

if $Z = \phi(Z, \bar{Z}, W)$, and $\phi$ is holomorphic in $W$. Here, $D^\alpha$ denotes the derivative in all the real variables.

Proof. Let us write $F(Z, \bar{Z}, W) = F(x, y, W)$ where $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ are the underlying real coordinates in $\mathbb{C}^N$, as usual identified by $Z_j = x_j + iy_j$. Let us also choose a neighbourhood $U_0 \subset \mathbb{R}^N$ of 0 with the property that $U_0 \times U_0 \subset U$.

We extend $F$ in the first $2N$ variables almost holomorphically; that is, we have a function $\tilde{F} : U_0 \times \mathbb{R}^N \times U_0 \times \mathbb{R}^N \times \mathbb{C}^p \to \mathbb{C}^N$ with the property that

(17) \[ \tilde{F}(x, x', y, y', W) |_{x' = y' = 0} = F(x, y, W) \]

and, if we introduce complex coordinates $\xi_k = x_k + ix_k', \eta_k = y_k + iy_k'$, $1 \leq k \leq N$, then

(18) \[ D^\alpha \frac{\partial \tilde{F}_j}{\partial \xi_k} \bigg|_{x' = y' = 0} = D^\alpha \frac{\partial \tilde{F}_j}{\partial \eta_k} \bigg|_{x' = y' = 0} = 0, \quad 1 \leq j, k \leq N. \]

Also, $\tilde{F}$ is still polynomial in $W$. We introduce new coordinates $\chi = (\chi_1, \ldots, \chi_N) \in \mathbb{C}^N$ by

\[ \xi_k = \frac{z_k + \chi_k}{2}, \quad \eta_k = \frac{z_k - \chi_k}{2i}, \quad 1 \leq k \leq N, \]

and write $G(Z, \bar{Z}, \chi, \bar{\chi}, W) = F(\xi, \bar{\xi}, \eta, \bar{\eta}, W)$. $G$ is smooth in the first $2N$ complex variables in some neighbourhood of the origin, and polynomial in $W$. We will now compute the real Jacobian of $G$ with respect to $Z$ at $(0, A)$. At $(0, A)$, $\frac{\partial G}{\partial Z}(0, A) = \frac{\partial \tilde{F}}{\partial Z}(0, A)$ and $\frac{\partial G}{\partial \bar{Z}}(0, A) = 0$, so that we have

\[ \det \left( \begin{array}{cc} \frac{\partial G}{\partial \bar{Z}} & \frac{\partial G}{\partial Z} \\ \frac{\partial \bar{G}}{\partial \bar{Z}} & \frac{\partial \bar{G}}{\partial Z} \end{array} \right) (0, A) = \left| \det \frac{\partial F}{\partial Z}(0, A) \right|^2 \neq 0 \]

by assumption. Hence, by the implicit function theorem, there exists a smooth function $\psi$ defined in some neighbourhood of $(0, A)$, valued in $\mathbb{C}^N$, such that $Z = \psi(\chi, \bar{\chi}, W)$ solves the equation $G(Z, \bar{Z}, \chi, \bar{\chi}, W) = 0$ uniquely. Here we have already taken into account that $\psi$ depends holomorphically on $W$, a fact that the reader will easily check. Since $G(Z, \bar{Z}, Z, W) = F(Z, \bar{Z}, W)$, this implies that if $F(Z, \bar{Z}, W) = 0$, then $Z = \psi(\bar{Z}, Z, W)$.

We let $\phi(Z, \bar{Z}, W) = \psi(\bar{Z}, Z, W)$ and claim that $\phi$ satisfies (16). In fact, computation shows that $\phi_Z(Z, \bar{Z}, W) = \psi_\chi = -(G_Z - G_{\bar{Z}} G^{-1}_Z G_Z)^{-1}(G_{\bar{\chi}} + G_Z G^{-1}_Z G_Z G_{\bar{\chi}})$, where the right hand side is evaluated at $(\psi(Z, \bar{Z}, W), \psi(\bar{Z}, Z, W), \bar{Z}, Z, W)$. This formula shows that each $\phi_{Z, \bar{Z}}$ is a sum of products each of which contains a factor which is a derivative of $G$ with respect to $\bar{Z}$ or $\bar{\chi}$.

By the definition of $G$, we have that

\[ \frac{\partial G}{\partial Z} = \frac{1}{2} \frac{\partial \tilde{F}}{\partial \xi} + \frac{1}{2i} \frac{\partial \tilde{F}}{\partial \eta}, \quad \frac{\partial G}{\partial \bar{Z}} = \frac{1}{2} \frac{\partial \tilde{F}}{\partial \bar{\xi}} - \frac{1}{2i} \frac{\partial \tilde{F}}{\partial \bar{\eta}}. \]

By (18) every derivative of those vanishes if $x' = y' = 0$, which is in turn the case if $\text{Im} \frac{\partial (Z, \bar{Z})}{\partial \bar{Z}} + \frac{\bar{Z}}{2} = 0$ and $\text{Im} \frac{\partial (Z, \bar{Z})}{\partial Z} - \frac{Z}{2i} = 0$. But this is clearly fulfilled if $Z = \phi(Z, \bar{Z})$.
The proof is now finished by applying the Leibniz rule, the chain rule and the observations made above.

Note that it is clear from the usual implicit function theorem that we can solve for $N$ of the real variables $(x, y)$. What this theorem asserts is that we can do so in a special manner.

5. Proof of Theorem 2

Let us start by choosing coordinates. There is a neighbourhood $U$ of $p_0 = 0$ in $\mathbb{C}^N$ and a smooth function $\phi : \mathbb{C}^N \times \mathbb{R}^d \to \mathbb{R}^d$ defined in a neighbourhood $V$ of 0 such that $M \cap U = \{(z, s + i\phi(z, \bar{z}, s)): (z, s) \in V\}$ with the property that $\nabla \phi(0) = 0$. Since the conclusion of the theorem is local, we shall replace $M$ by $M \cap U$, and use this representation. For suitably chosen open sets $U \subset \mathbb{C}^N$ and $V \subset \mathbb{R}^d$, consider the diffeomorphism $\Psi : U \times V \to M$, $\Psi(z, \bar{z}, s) = (z, s + i\phi(z, \bar{z}, s))$. We extend this diffeomorphism almost holomorphically to a map, again denoted by $\Psi$, from $U \times V \times \mathbb{R}^d$ to $\mathbb{C}^N$. $\Psi$ is a diffeomorphism in an open neighbourhood of $U \times V \times \{0\}$, and it has the property that for every component $\Psi_i$ of $\Psi$, it has the property that for every component $\Psi_i$ of $\Psi$, and it has the property that for every component $\Psi_i$ of $\Psi$,

\[
D_x^\alpha D_y^\beta \Psi_i(z, s, 0) = 0, \quad (z, s) \in U \times V,
\]

where the derivative is in all the real variables. Equivalently,

\[
D_x^\alpha D_y^\beta \Psi_i(z, s, 0) = O(|t|^{\infty}), \quad (z, s) \in U \times V,
\]

uniformly on compact subsets of $U \times V$. That is, for each $\alpha, \beta, K \subset U \times V$ compact and every $l \in \mathbb{N}$ there exists a constant $C_l = C_l(\alpha, \beta, K)$ such that

\[
|D_x^\alpha D_y^\beta \Psi_i(z, s, t)| \leq C_l|t|^l, \quad (z, s) \in K.
\]

We assume that each component $H_j$ of $H$ extends continuously (and, consequently by a theorem of Rosay already alluded to above, in a $C^k$-fashion) to a holomorphic function into a wedge with edge $M$. Let us recall that this means that with an open convex cone $\Gamma$ in $\mathbb{R}^d$ each $H_j$ extends continuously to the set $W_\Gamma = \{Z \in U_0: \rho(Z, Z) \in \Gamma\}$, where $U_0$ is an open neighbourhood of 0 in $\mathbb{C}^N$. By choosing $\Gamma$ accordingly, and possibly shrinking $U_0$, we can in addition assume that each $H_j$ is continuous and bounded on the closure of $W_\Gamma$, and in fact smooth up to $bW_\Gamma \setminus M$.

There exists another open, convex cone $\Gamma'$, relatively closed in $\Gamma$, neighbourhoods $U' \subset U$ and $V' \subset V$ of 0 in $\mathbb{C}^N$ and 0 in $\mathbb{R}^d$, respectively, and $\delta = (\delta_1, \ldots, \delta_d) > 0$ such that the wedge $\tilde{W}_{\Gamma'} = \{(z, s, t) \in U' \times V' \times \Gamma': 0 < t < \delta\}$ with flat edge $U' \times V'$ satisfies $\tilde{W}_{\Gamma'} = \Psi(\tilde{W}_{\Gamma'}) \subset W_\Gamma$. Hence, $h_j = H_j \circ \Psi$ is well defined on $\tilde{W}_{\Gamma'}$ for $1 \leq j \leq d$, extends continuously to $\tilde{W}_{\Gamma'}$ and is smooth up to $b\tilde{W}_{\Gamma'} \setminus U' \times V'$. Since the conclusion of the theorem is local, we can replace $U$ by $U'$ and $V$ by $V'$. Furthermore, by shrinking the the neighbourhoods once more if necessary, we have that there exist positive constants $C_1$ and $C_2$ such that (here, $d(A, B)$ denotes the distance between a compact set $A$ and a closed set $B$)

\[
C_1 d((z, s, t), b\tilde{W}_{\Gamma'}) \leq d(\Psi(z, s, t), b\tilde{W}_{\Gamma'}) \leq C_2 d((z, s, t), b\tilde{W}_{\Gamma'}).
\]

Our next claim is that we can replace $\Gamma'$ by the standard cone $\mathbb{R}^d_+ = \{t \in \mathbb{R}^d: t > 0\}$. In fact, since $\Gamma'$ is open, we can find $d$ linearly independent vectors $v_1, \ldots, v_d$ in $\Gamma'$. The linear mapping $T$ which maps $v_j$ to the $j$-th standard basis vector $e_j$ is invertible, and $T^{-1}(\mathbb{R}^d_+) \subset \Gamma'$ by convexity. Then we can make a
complex linear change of coordinates by setting \((z', s', t') = (z, T^{-1}s, T^{-1}t)\). Since this coordinate change is linear and there exist positive constants \(C_1\) and \(C_2\) with \(C_1|t| \leq |t'| \leq C_2|t|\), (19), (20), and (21) also hold in the new coordinates. We need just one more coordinate change.

**Claim 1.** There exists a \(\delta > 0\), coordinates \((z, s, t)\) and positive constants \(C_1\) and \(C_2\) such that \(\Psi(z, s, t) \in \tilde{W}_T\) for \((z, s) \in U \times V, 0 < t < \delta\) and \(C_1|t| \leq d(b\tilde{W}_T, \Psi(z, s, t)) \leq C_2|t|\) for \((z, s) \in U \times V, 0 \leq t \leq \delta\).

**Proof.** Let \(e_j\) denote the \(j\)-th standard basis vector in \(\mathbb{R}^d\), \(1 \leq j \leq d\). If \(t = (t_1, \ldots, t_d) \in \mathbb{R}^d\), then clearly \(d(t, b\mathbb{R}^d_+) = \min_{j=1}^d t_j\). For \(\epsilon > 0\) consider the vectors \(v_j = e_j + \epsilon \sum_{l \neq j} e_l, 1 \leq j \leq d\). For \(\epsilon\) small enough, these are linearly independent. We now consider the linear change of coordinates given by \(z = z', t' = (t'_1, \ldots, t'_d) \mapsto \sum_{j=1}^d t'_j v_j, s' = (s'_1, \ldots, s'_d) \mapsto \sum_{j=1}^d s'_j v_j\). By (22) it is enough to show that there exist positive constants \(C_1\) and \(C_2\) such that \(C_1|t'| \leq d(t, b\mathbb{R}^d_+) \leq C_2|t'\). The existence of \(C_2\) is clear. But if \(\epsilon \leq 1\), then \(d(t, b\mathbb{R}^d_+) = \min_{j=1}^d t_j = \min_{j=1}^d (t'_j + \epsilon \sum_{l \neq j} t'_l) \geq \epsilon (t'_1 + \ldots t'_d) \geq \frac{\epsilon}{d} |t|\). An appropriate choice for \(\delta\) finishes the argument. \(\blacksquare\)

We are going to use the notation introduced in section 2, that is, we let \(\Omega_+ = U \times V \times \{t \in \mathbb{R}_d^+: 0 < t < \delta\}\). We let \(h_j = H_j \circ \Psi\) on \(\Omega_+\).

**Claim 2.** \(h_j \in \mathfrak{A}_\infty(\Omega_+)^{N'}\) for \(1 \leq j \leq N'\).

**Proof.** By all the choices above, \(h_j\) satisfies the smoothness assumptions. Let us first check that every derivative of \(h_j\) is of slow growth. Since \(H_j\) is holomorphic in \(\tilde{W}_T\) and continuous on its closure, the Cauchy estimates imply that we have an estimate of the form

\[
|\partial^\beta H_j(Z)| \leq C_{\beta}(d(Z, b\tilde{W}_T))^{-|\beta|}
\]

for each \(\beta\), where \(\partial^\beta\) denotes \(\frac{\partial^{|eta|}}{\partial z^{\beta}}\). By the chain rule, \(D_{x,y,s}^\alpha h_j(z, s, t)\) is a sum of products of derivatives of \(\Psi\) (which are bounded) and a derivative of \(H_j\) with respect to \(Z\), evaluated at \(\Psi(z, s, t)\), of order at most \(|\alpha|\). Hence, by (23) and claim 2 we conclude that there exists a positive constant \(C\) such that

\[
|D_{x,y,s}^\alpha h_j(z, s, t)| \leq C_{\alpha}|t|^{-|\alpha|}.
\]

We now have to estimate the derivatives of \(\tilde{\partial}_m h_j\) for \(1 \leq m \leq d\). But \(\tilde{\partial}_m h_j = \sum_{l=1}^{N'} \partial_{Z_l} h_j\tilde{\partial}_m \Psi_l\). Hence, if we take an arbitrary derivative of \(\tilde{\partial}_m h_j\), we get a sum of products of derivatives of components of \(\Psi\) and a derivative of \(H_j\) with respect to \(Z\) each of which contains a term of the form \(\tilde{\partial}_m \Psi_l\). By (23) and (24) we conclude that for each compact set \(K \subset U \times V\) and each \(k \in \mathbb{N}\) there exists a positive constant \(C_k\) with \(|D_{x,y,s}^\alpha \tilde{\partial}_m h_j(z, s, t)| \leq C_k|t|^k\). This proves claim 2. \(\blacksquare\)

We now equip \(U \times V\) with the CR-structure of \(M\); that is, a basis of the CR-vector fields near 0 is given by \(\Lambda_j = \Psi^* L_j\) for \(1 \leq j \leq n\). We almost holomorphically extend the coefficients of the \(\Lambda_j\) to get smooth vector fields on an open subset of \(\mathbb{C}^n \times \mathbb{R}^d \times \mathbb{R}^d\) containing 0.

**Claim 3.** For each \(j, 1 \leq j \leq N'\), there exists a smooth function \(\phi_j(Z', \tilde{Z}', W)\) defined in an open neighbourhood of \((0, (\Lambda^\alpha h(0))_{|\alpha| \leq k_0})\) in \(\mathbb{C}^N \times \mathbb{C}^{K(k_0)}\) \((K(k_0)

\[
\]
denoting \(N'([\alpha : |\alpha| \leq k_0])\) such that
\begin{equation}
(25) \quad h_j(z, s, 0) = \phi_j(h(z, s, 0), \bar{h}(z, s, 0), (\Lambda^\alpha h(z, s, 0))_{|\alpha| \leq k_0});
\end{equation}
here, we write \(h = (h_1, \ldots, h_{N'})\). Furthermore, after possibly shrinking \(U\) and \(V\), the right hand side of \((25)\) defines a function in \(\mathfrak{A}(\Omega_-)\).

This last claim of course establishes Theorem \(2\); since \(h_j \in \mathfrak{A}(\Omega_+)\) by Claim \(3\) and by Claim \(3\) \(h_j \in \mathfrak{A}(\Omega_-)\), we can apply Theorem \(3\) to see that \(h_j\) is smooth.

**Proof.** By the chain rule, we have smooth functions \(\Phi_{l,0}(Z, Z', W)\) for \(|\alpha| \leq k_0\), \(1 \leq l \leq d'\), defined in a neighbourhood of \(\{0\} \times \mathbb{C}^{K(k_0)}\) in \(\mathbb{C}^N \times \mathbb{C}^{K(k_0)}\), polynomial in the last \(K(k_0)\) variables, such that
\begin{equation}
(26) \quad \Lambda^\alpha \rho_l^j(h, \bar{h})(z, s, 0) = \Phi_{l,0}(h(z, s, 0), \bar{h}(z, s, 0), (\Lambda^\alpha \bar{h}(z, s, 0))_{|\alpha| \leq k_0}),
\end{equation}
and \(\Lambda^\alpha \rho_{l,Z}^j(h, \bar{h})[a] = \Phi_{l,a,Z}(0, 0, (\Lambda^\alpha \bar{h}(0, 0, 0))_{|\alpha| \leq k_0})\). By Definition \(2\) we can choose \(\alpha^1, \ldots, \alpha^{N'}\) and \(l^1, \ldots, l^{N'}\) such that if we set \(\Phi = (\Phi_{l^1,0}, \ldots, \Phi_{l^{N'},0})\), then \(\Phi_{l^j}(0)\) is invertible. Hence, we can apply Theorem \(10\); let us call the solution \(\phi\). Then \(\phi_j\) satisfies \((23)\), and we shrink \(U\) and \(V\) and choose \(\delta\) in such a way that \(g_j(z, s, t) = \phi_j(h(z, s, -t), \bar{h}(z, s, -t), (\Lambda^\alpha h(z, s, -t))_{|\alpha| \leq k_0})\) is well defined and continuous in a neighbourhood of \(\Omega_-\). It is easily checked that \(g_j\) is a function in \(\mathfrak{A}(\Omega_-)\) as a consequence of \((17)\) and the fact that each \(h_j \in \mathfrak{A}_\infty(\Omega_+)\). First note that this implies \(g_j(z, s, -t) \in \mathfrak{A}_\infty(\Omega_-)\), and by Lemma \(3\), \(\Lambda^\alpha g_j(z, s, -t) \in \mathfrak{A}_\infty(\Omega_-)\) for each \(\alpha\). Now, each derivative \(D^\beta\) of \(g_j\) is a sum of products of derivatives of \(\phi_j\) (which are uniformly bounded on \(\Omega_-\)) and derivatives of \(h, \bar{h}\), and \(\Lambda^\alpha \bar{h}\), all of which fulfill the analog of \((3)\) on \(\Omega_-\). So \(g_j\) fulfills the analog of \((3)\) on \(\Omega_-\). Next, we compute the derivative of \(g_j\) with respect to \(\bar{w}_k\). We have that
\[
\frac{\partial g_j}{\partial \bar{w}_k} = \sum_{l=1}^{N'} \frac{\partial \phi_j}{\partial \bar{Z}_l} \frac{\partial h_l}{\partial \bar{w}_k} + \sum_{l=1}^{N'} \frac{\partial \phi_j}{\partial \bar{Z}_l} \frac{\partial \bar{h}_l}{\partial \bar{w}_k} + \sum_{|\alpha| \leq k_0} \frac{\partial \phi}{\partial W_\alpha} \frac{\partial \Lambda^\alpha \bar{h}}{\partial \bar{w}_k}.
\]
Applying any derivative \(D^\beta\), we see that the first sum gives rise to products of derivatives of \(\partial \phi_j/\partial \bar{Z}_l\) and derivatives of \(h, \bar{h}\), and \(\Lambda^\alpha \bar{h}\). Now the derivatives of \(\phi_j\) fulfill \((16)\). Since on \(t = 0\), \(h = \phi(h, \bar{h}, (\Lambda^\alpha \bar{h})_{|\alpha| \leq k_0})\), we conclude that \(h - \phi(h, \bar{h}, (\Lambda^\alpha \bar{h})_{|\alpha| \leq k_0}) = O(|t|)\). But by \((16)\), any derivative of \(\partial \phi_j/\partial \bar{Z}_l\) evaluated at \((h, \bar{h}, (\Lambda^\alpha \bar{h})_{|\alpha| \leq k_0})\) is \(O(|Z - \phi(Z, \bar{Z}, W)|^\infty)\), so that derivatives of \(\partial \phi_j/\partial \bar{Z}_l\) are \(O(|t|^\infty)\). All the other terms in the product are \(O(|t|^{-s})\) for some \(s\), so that the terms coming from the first sum are actually \(O(|t|^\infty)\). For the second and third sum, a similar argument using that \(\bar{h}\) and \(\Lambda^\alpha \bar{h}\) are in \(\mathfrak{A}_\infty(\Omega_-)\) implies that all the terms arising from them are \(O(|t|^\infty)\). All in all, we conclude that \(g_j \in \mathfrak{A}_\infty(\Omega_-)\), which finishes the proof.

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