A Singular Value Inequality for Heinz Means

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Abstract: We prove a matrix inequality for matrix monotone functions, and apply it to prove a singular value inequality for Heinz means recently conjectured by X. Zhan.

1 Introduction

Heinz means, introduced in [2], are means that interpolate in a certain way between the arithmetic and geometric mean. They are defined over \( \mathbb{R}^+ \) as

\[
H_\nu(a, b) = (a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu})/2,
\]

for \( 0 \leq \nu \leq 1 \). One can easily show that the Heinz means are “inbetween” the geometric mean and the arithmetic mean:

\[
\sqrt{ab} \leq H_\nu(a, b) \leq (a + b)/2.
\]

Bhatia and Davis [3] extended this to the matrix case, by showing that the inequalities remain true for positive semidefinite (PSD) matrices, in the following sense:

\[
|||A^{1/2}B^{1/2}||| \leq |||H_\nu(A, B)||| \leq |||(A + B)/2|||,
\]

where \(|||.|||\) is any unitarily invariant norm and the Heinz mean for matrices is defined identically as in (1).

In fact, Bhatia and Davis proved the stronger inequalities, involving a third, general matrix \( X \),

\[
|||A^{1/2}XB^{1/2}||| \leq |||(A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu})/2||| \leq |||(AX + XB)/2|||.
\]

X. Zhan [6, 7] conjectured that the second inequality in (3) also holds for singular values. Namely: for \( A, B \geq 0 \),

\[
\sigma_j(H_\nu(A, B)) \leq \sigma_j((A + B)/2),
\]

is conjectured to hold for all \( j \). These inequalities have been proven in a few special cases. The case \( \nu = 1/2 \) is known as the arithmetic-geometric mean inequality for singular values, and has been proven by Bhatia and Kittaneh [4]. The case \( \nu = 1/4 \) (and \( \nu = 3/4 \)) is due to Y. Tao [5]. In the present paper, we prove (5) for all \( 0 \leq \nu \leq 1 \). To do so, we first prove a general matrix inequality for matrix monotone functions (Section 3). The proof of the Conjecture is then a relatively straightforward application of this inequality (Section 4).

Remark: One might be tempted to generalise the first inequality in (3) to singular values as well:

\[
\sigma_j(A^{1/2}B^{1/2}) \leq \sigma_j(H_\nu(A, B)).
\]

These inequalities are false, however. Consider the following PSD matrices (both are rank 2):

\[
A = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix}.
\]

Then \( \sigma_2(A^{1/2}B^{1/2}) > \sigma_2(H_\nu(A, B)) \) for \( 0 < \nu < 0.13 \).
2 Preliminaries

We denote the eigenvalues and singular values of a matrix $A$ by $\lambda_j(A)$ and $\sigma_j(A)$, respectively. We adhere to the convention that singular values and eigenvalues (in case they are real) are sorted in non-increasing order.

We will use the positive semidefinite (PSD) ordering on Hermitian matrices throughout, denoted $A \geq B$, which means that $A - B \succeq 0$. This ordering is preserved under arbitrary conjugations: $A \geq B$ implies $XAX^* \succeq XBX^*$ for arbitrary $X$.

A matrix function $f$ is matrix monotone iff it preserves the PSD ordering, i.e. $A \geq B$ implies $f(A) \succeq f(B)$. If $A \geq B$ implies $f(A) \leq f(B)$, we say $f$ is inversely matrix monotone. A matrix function $f$ is matrix convex iff for all $0 \leq \lambda \leq 1$ and for all $A, B \succeq 0$,

$$f(\lambda A + (1 - \lambda) B) \leq \lambda f(A) + (1 - \lambda) f(B).$$

Matrix monotone functions are characterised by the integral representation $[1, 7]$ 

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{\lambda t}{t + \lambda} d\mu(\lambda),$$

where $d\mu(\lambda)$ is any positive measure on the interval $\lambda \in [0, \infty)$, $\alpha$ is a real scalar and $\beta$ is a non-negative scalar. When applied to matrices, this gives, for $A \succeq 0$,

$$f(A) = \alpha \mathbb{1} + \beta A + \int_0^\infty \lambda A (A + \lambda \mathbb{1})^{-1} d\mu(\lambda).$$

The primary matrix function $x \mapsto x^p$ is matrix convex for $1 \leq p \leq 2$, matrix monotone and matrix concave for $0 \leq p \leq 1$, and inversely matrix monotone and matrix convex for $-1 \leq p \leq 0$ $[1]$.

3 A matrix inequality for matrix monotone functions

In this Section, we present the matrix inequality that we will use in the next Section to prove Zhan’s Conjecture.

**Theorem 1** For $A, B \succeq 0$, and any matrix monotone function $f$:

$$Af(A) + Bf(B) \succeq \left(\frac{A + B}{2}\right)^{1/2} \left(f(A) + f(B)\right) \left(\frac{A + B}{2}\right)^{1/2}.$$ 

(9)

**Proof.** Let $A$ and $B$ be PSD. We start by noting the matrix convexity of the function $t \mapsto t^{-1}$. Thus

$$\frac{A^{-1} + B^{-1}}{2} \succeq \left(\frac{A + B}{2}\right)^{-1}.$$ 

(10)

Replacing $A$ by $A + \mathbb{1}$ and $B$ by $B + \mathbb{1}$,

$$(A + \mathbb{1})^{-1} + (B + \mathbb{1})^{-1} \succeq 2(\mathbb{1} + (A + B)/2)^{-1}.$$ 

(11)

Let us now define

$$C_k := \frac{A^k}{A + \mathbb{1}} + \frac{B^k}{B + \mathbb{1}},$$

and

$$M := (A + B)/2.$$ 

With these notations, (11) becomes

$$C_0 \succeq 2(\mathbb{1} + M)^{-1}.$$ 

(12)
This implies
\[ C_0 + \sqrt{MC_0 \sqrt{M}} \geq 2(\mathbb{I} + M)^{-1} + 2\sqrt{M}(\mathbb{I} + M)^{-1}\sqrt{M} = 2\mathbb{I}, \]  
where the last equality follows easily because all factors commute.

Now note: \( C_k + C_{k+1} = A^k + B^k \). In particular, \( C_0 + C_1 = 2\mathbb{I} \), and thus becomes
\[ \sqrt{M}(2\mathbb{I} - C_1)\sqrt{M} \geq C_1. \]  
Furthermore, as \( C_1 + C_2 = 2M \), this is equivalent with
\[ C_2 \geq \sqrt{MC_1 \sqrt{M}}, \]  
or, written out in full:
\[ \frac{A^2}{A} + \frac{B^2}{B} \geq \left( \frac{A+B}{2} \right)^{1/2} \left( \frac{A}{A + \mathbb{I}} + \frac{B}{B + \mathbb{I}} \right) \left( \frac{A+B}{2} \right)^{1/2}. \]  

We now replace \( A \) by \( \lambda^{-1}A \) and \( B \) by \( \lambda^{-1}B \), for \( \lambda \) a positive scalar. Then, after multiplying both sides with \( \lambda^2 \), we obtain that
\[ \frac{\lambda A^2}{A + \mathbb{I}} + \frac{\lambda B^2}{B + \mathbb{I}} \geq \left( \frac{A+B}{2} \right)^{1/2} \left( \frac{\lambda A}{A + \mathbb{I}} + \frac{\lambda B}{B + \mathbb{I}} \right) \left( \frac{A+B}{2} \right)^{1/2} \]  
holds for all \( \lambda \geq 0 \). We can therefore integrate this inequality over \( \lambda \in [0, \infty) \) using any positive measure \( d\mu(\lambda) \).

Finally, by matrix convexity of the square function, \( ((A + B)/2)^2 \leq (A^2 + B^2)/2 \) \[ \| \mathbb{I} \| \]  
, we have, for \( \beta \geq 0 \),
\[ A(\alpha \mathbb{I} + \beta A) + B(\alpha \mathbb{I} + \beta B) \geq \left( \frac{A+B}{2} \right)^{1/2} (2\alpha \mathbb{I} + \beta(A + B)) \left( \frac{A+B}{2} \right)^{1/2}. \]  
Summing this up with the integral expression just obtained, and recognising representation \( 8 \) in both sides finally gives us \( 9 \). \[ \square \]

Weyl monotonicity, together with the equality \( \lambda_j(AB) = \lambda_j(BA) \), immediately yields

**Corollary 1** For \( A, B \geq 0 \), and any matrix monotone function \( f \):
\[ \lambda_j(Af(A) + Bf(B)) \geq \lambda_j \left( \frac{A+B}{2} (f(A) + f(B)) \right). \]  

**4 Application: Proof of (5)**

As an application of Theorem 1 we now obtain the promised singular value inequality \( 10 \) for Heinz means, as conjectured by X. Zhan:

**Theorem 2** For \( A, B \in M_n(\mathbb{C}) \), \( A, B \geq 0 \), \( j = 1, \ldots, n \), and \( 0 \leq s \leq 1 \),
\[ \sigma_j(A^s B^{1-s} + A^{1-s} B^s) \leq \sigma_j(A + B). \]  

**Proof.** Corollary 1 applied to \( f(A) = A^r \), for \( 0 \leq r \leq 1 \), yields
\[ \lambda_j(A^{r+1} + B^{r+1}) \geq \frac{1}{2} \lambda_j ((A + B)(A^r + B^r)) \]
\[ = \frac{1}{2} \lambda_j \left( \begin{pmatrix} A^{r/2} & \sqrt{B} \end{pmatrix} (A + B) \begin{pmatrix} A^{r/2} & \sqrt{B} \end{pmatrix} \right) \]
\[ = \frac{1}{2} \lambda_j \left( \begin{pmatrix} A^{1/2} & \sqrt{B} \end{pmatrix} (A^r + B^r) \begin{pmatrix} A^{1/2} & \sqrt{B} \end{pmatrix} \right). \]
Tao’s Theorem \[5\] now says that for any $2 \times 2$ PSD block matrix $Z := \begin{pmatrix} M & K \\ K^* & N \end{pmatrix} \geq 0$ (with $M \in M_m$ and $N \in M_n$) the following relation holds between the singular values of the off-diagonal block $K$ and the eigenvalues of $Z$, for $j \leq m, n$:

$$\sigma_j(K) \leq \frac{1}{2} \lambda_j(Z).$$ (23)

The inequality (21) therefore yields

$$\lambda_j(A^{r+1} + B^{r+1}) \geq \sigma_j \left( A^{r/2} (A + B) B^{r/2} \right) = \sigma_j(A^{1+r/2} B^{r/2} + A^{r/2} B^{1+r/2}).$$ (24)

Replacing $A$ by $A^{1/(r+1)}$ and $B$ by $B^{1/(r+1)}$ then yields (20) for $s = (1 + r/2)/(1 + r)$, hence for $0 \leq s \leq 1/4$ and $3/4 \leq s \leq 1$.

If, instead, we start from (22) and proceed in an identical way as above, then we obtain (20) for $s = (r + 1/2)/(1 + r)$, which covers the remaining case $1/4 \leq s \leq 3/4$. \[\square\]

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