Earth’s rotation in the framework of general relativity: rigid multipole moments

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ABSTRACT. A set of equations describing the rotational motion of the Earth relative to the GCRS is formulated in the approximation of rigidly rotating multipoles. The external bodies are supposed to be mass monopoles. The derived set of formulas is supposed to form the theoretical basis for a practical post-Newtonian theory of Earth precession and nutation.

1. Introduction

The relation between the International Celestial Reference System (ICRS) and the corresponding terrestrial one (ITRF) is a central problem of astrometry, geodesy and related disciplines. From a theoretical point of view this requires not only a precise determination of the ICRS and the ITRS, but also a detailed modelling of Earth’s rotation. Due to the high accuracy requirements it is obvious that all of these problems have to be formulated in Einstein’s theory of gravity, at least in its first post-Newtonian approximation. Until now most Newtonian treatments of Earth’s rotation are based upon some highly accurate rigid body theory such as SMART97 (Bretagnon et al., 1998) and add effects from elasticity, the atmosphere, the oceans, the core etc. in a perturbative manner. Actually, the concept of a rigid body is very powerful in Newton’s theory where the three fundamental axes, the total angular momentum or spin axis, the rotation axis and the figure axis, can be introduced without efforts. Unfortunately rigid bodies with an internal velocity field of the form \( \mathbf{v} = \omega \times \mathbf{x} \) in general do not exist in General Relativity. Nevertheless one might introduce a certain class of models, where the time behavior of potential coefficients, moments of inertia tensor, etc. is completely determined by some quantity \( \omega(T) \). We call such models “rigidly rotating multipole” models.

The aim of this paper is to summarize a set of formulas describing the rotational motion of the Earth with respect to the Geocentric Celestial Reference System (GCRS) in the approximation of rigidly rotating multipole moments. The GCRS is defined in the post-Newtonian approximation of General Relativity by the IAU Resolution B1.3 adopted at the 24th General Assembly of the IAU (Manchester, 2000) and published in the IAU Information Bulletin No. 88 (see also erratum in Bulletin No. 89). Full text of this Resolution can be also found at [http://danof.obspm.fr/IAU_resolutions/Resol-UAI.htm](http://danof.obspm.fr/IAU_resolutions/Resol-UAI.htm). The approximation of rigidly rotating multipoles used in this paper is a phenomenological model which allows one to simplify the mathematical description of rotational motion almost to the level of Newtonian theory. Likely, the model of rigidly rotating multipoles is not consistent with general relativity in the sense that there is no physical equation of state and local conditions of matter that will support
the model. However, a theory of motion within this model can be used as a first approximation to be refined later by methods of perturbation theory. Within this model the accuracy of the given formulas is not worse than 0.1 µas.

2. Post-Newtonian equations of rotational motion

The post-Newtonian equations of rotational motion of the Earth relative to the Geocentric Celestial Reference System (GCRS) can be derived from the metric tensor of the GCRS (Voinov, 1988; Damour, Soffel, Xu, 1993; Klioner, 1996) in the form

\[
\frac{d}{dT_{\mathrm{CGS}}} S^a = L^a + O(c^{-4}).
\]

(1)

where \( S^a \) is the post-Newtonian spin defined as an explicit integral over the body of the Earth

\[
S^a = \varepsilon_{abc} \int_V X^b p^c d^3X + O(c^{-4}).
\]

(2)

Here,

\[
p^a = \Sigma^a \left( 1 + \frac{4}{c^2} W \right) - \frac{1}{2c^2} G \Sigma \int_V \Sigma^b(T, X') \frac{7 \delta^{ab} + n^a n^b}{|X - X'|} d^3X' + O(c^{-4}),
\]

(3)

\[
n^a = \frac{X^a - X'^a}{|X - X'|},
\]

(4)

and \( \Sigma \) and \( \Sigma^a \) are defined by the components of the energy-momentum tensor in the GCRS, \( T^{\alpha\beta} \), as \( \Sigma = T^{\alpha\alpha} \) and \( \Sigma^a = T^{ba} \). \( W \) is the potential appearing in the metric tensor of the GCRS. This definition of spin was first derived by Fock (1955) and thoroughly discussed in, e.g., Damour, Soffel and Xu (1993). The right-hand side of (1) represents the post-Newtonian torque which can be represented as

\[
L^a = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \varepsilon_{abc} M_{bL} G_{cL} + \frac{1}{c^2} \frac{l + 1}{l + 2} \varepsilon_{abc} S_{bL} H_{cL} \right) + \frac{d}{dT_{\mathrm{CGS}}} \tilde{S}^a.
\]

(5)

Here \( M_L \) and \( S_L \) are the Blanchet-Damour mass and spin multipole moments characterizing the Earth, and \( G_L \) and \( H_L \) for \( l \geq 2 \) are the gravitoelectric and gravitomagnetic tidal moments of the external gravitational field experienced by the Earth. The moments \( M_L \) are equivalent to the set of coefficients \( C_{lm} \) and \( S_{lm} \) in the conventional expansion of the gravitational potential of the Earth in terms of spherical functions. Explicit formulas for \( G_L \) within the adopted model will be given below. The moment \( H_a \) describes the inertial forces induced by the rotation of the GCRS relative to the locally inertial reference system (these forces appear because the GCRS is defined to be kinematically non-rotating with respect to the BCRS):

\[
\Omega^a_{\text{iner}} = \frac{1}{2c^2} H_a = - \frac{3}{2c^2} \varepsilon_{aij} v^i_E \frac{\partial}{\partial x^j} w_{\text{ext}}(x_E) - \frac{2}{c^2} \varepsilon_{aij} \frac{\partial}{\partial x^j} w_{\text{ext}}^i(x_E) + \frac{1}{2c^2} \varepsilon_{aij} v^i_E G_b,
\]

(6)

where \( v^i_E \) is the barycentric velocity of the Earth, \( w_{\text{ext}}(x_E) \) and \( w_{\text{ext}}^i(x_E) \) are the external BCRS potential evaluated at the geocenter, and \( G_a \) is the acceleration of the geocenter with
respect to the geodetic motion (all these quantities are defined and briefly explained in the IAU Resolution B1.3). The quantity $\Omega_{\text{iner}}$ represent the angular velocity of precession of the GCRS with respect to the locally inertial axis. This relativistic precession consist of geodetic, Lense-Thirring and Thomas precessions (the three terms in (6), respectively) and amounts to $\sim 1.92''$ per century plus periodic terms with maximal amplitudes of 0.15 mas. It is easy to estimate that $|\Omega| < 3.1 \cdot 10^{-15} \text{s}^{-1}$.

The last term in (5) represent a total time derivative and can be in principle included in the definition of the post-Newtonian spin $S^a$ as suggested by Damour, Soffel and Xu (1993). Here we prefer to retain the definition of $S^a$ to be (2)–(4). An explicit formula for $S^a$ is given by Eqs. (2.14)–(2.16) of Damour, Soffel, Xu (1993). Numerical estimates of this term and the way to cope with it will be published elsewhere.

3. Post-Newtonian angular velocity and tensor of inertia

In order to be able to discuss the rotational motion of the Earth it is not sufficient to consider only the time dependence of the spin $S^a$ described by (1). In classical Newtonian (Eulerian) theory of a gyroscope the concepts of a figure axis, tensor of inertia and an angular velocity of the body play a central role and one expects the introduction of corresponding quantities in a relativistic framework to be very fruitful.

Different approaches leading to the same results are possible. Both restricted rigid body models (Thorne, Gürsel, 1983; Soffel, 1994) and a theory of post-Newtonian Tisserand axes for a deformable body (Klioner, 1996) allows one to derive the same definition of the post-Newtonian tensor of inertia and split the post-Newtonian spin $S^a$ defined by (2)–(4) in the Newtonian-looking way

$$S^a = C^{ab} \omega^b, \quad (7)$$

where $C^{ab}$ is the post-Newtonian tensor of inertia and $\omega^b$ is the angular velocity of rotation of the post-Newtonian Tisserand axes (Klioner, 1996).

The explicit formula for $C^{ab}$ as an integral over the volume of the Earth is given in Klioner (1996). Although the definition of $C^{ab}$ contains a number of explicit relativistic terms usually we do not compute $C^{ab}$ from the distribution of density, pressure, etc. within the Earth, but determine the values of $C^{ab}$ from observations (as numerical parameters of the models). Therefore, for practical purposes we can simply use the fact that the spin $S^a$ can be represented in the form given in (7).

4. Rigidly rotating multipole moments

Up to now the tensor of inertia $C^{ab}$ and the multipole moments of the Earth’s gravitational field $M_L$ and $S_L$ were considered as arbitrary functions of time. In Newtonian theory of Earth’s rotation a rigid Earth plays a very important role as a first order approximation. This rigid model i) crucially simplifies the mathematical description of the rotational motion and ii) is not too far from reality, so that the effect of non-rigidity can be than added to the model by means of perturbation theory. The reason why the rigid body model substantially simplifies the rotational equations of motion is that both the mass multipole moments $M_L^{\text{Newt}}$ and the tensor of inertia $C^{ab}_{\text{Newt}}$ rotate rigidly with the same angular velocity. In other words there exist a rigidly rotating reference system $Y^a = P^{ab}(T) X^a$, where $P^{ab}(T)$ is some time-dependent orthogonal matrix, where both $M_L^{\text{Newt}}$ and $C^{ab}_{\text{Newt}}$ are constant. Moreover, in Newtonian theory one can easily prove
that the time dependence of matrix \( P_{ab}(T) \) defines the same angular velocity which appears in the Newtonian analog of (7). All this can be proved starting from the fundamental assumption that the velocity \( \mathbf{v} \) of matter inside the body is described by a rigid rotation \( \mathbf{v} = \omega \times \mathbf{X} \).

It is well known that in general relativity it is impossible to define a rigid non-isolated body in a self-consistent way even in the first post-Newtonian approximation (see, e.g., Thorne, Gürsel, 1983). However, we can assume the same nice properties of the relativistic tensor of inertia \( C_{ab} \) and multipole moments \( M_L \) and \( S_L \) that we had in Newtonian theory. Thus, we define the model of rigidly rotating multipole moments by means of a series of assumptions:

\[
C_{ab} = P^{ac} P^{bd} \mathcal{C}^{cd}, \quad \mathcal{C}^{cd} = \text{const} \tag{8}
\]

\[
M_{a_1a_2...a_l} = P^{a_1b_1} P^{a_2b_2} \ldots P^{a_lb_l} M_{b_1b_2...b_l}, \quad M_{b_1b_2...b_l} = \text{const}, \quad l \geq 2, \tag{9}
\]

\[
S_L = C^{bL} \omega^b, \quad l \geq 2, \tag{10}
\]

\[
C^{b_1a_1...a_l} = P^{bd} P^{a_1c_1} P^{a_2c_2} \ldots P^{a_lc_l} \mathcal{C}_{c_1c_2...c_l}, \quad \mathcal{C}_{c_1c_2...c_l} = \text{const}, \quad l \geq 2, \tag{11}
\]

where \( P_{ab}(T) \) is the orthogonal matrix related to \( \omega^a \) by the kinematical Euler equations

\[
\omega^a(T) = \frac{1}{2} \varepsilon_{abs} P^{db}(T) \frac{d}{dT} P^{dc}(T). \tag{12}
\]

Note that we assume that \( \omega^a \) in (10) and (12) is identical with \( \omega^a \) from (7). We assume also that to bring \( C^{ab} \) into a diagonal form one more time-independent rotation is required. The matrix \( P_{ab}(T) \) can be parametrized by three Euler angles \( \psi, \theta, \varphi \), and the time-derivatives of these angles define the angular velocity of rotation according to (12). Relations (10)–(11) for the higher spin moments \( S_L, l \geq 2 \) and for \( C_{iL} \) are only necessary to Newtonian accuracy since they appear only in relativistic terms of (5).

Let us stress again that in Newtonian theory one one can derive (7) and (8)–(12) from the fundamental property of rigidity of the body \( \mathbf{v} = \omega \times \mathbf{X} \). On the contrary, in general relativity we define the model by assuming the properties of \( C^{ab}, M_L \) and \( S_L \) without further restrictions of the local flow of matter.

The experience of Newtonian models of Earth’s rotation shows that the phenomenological model (8)–(12) can be used as a first-order approximation for a description of the global rotational motion of the Earth. As in Newtonian theory, such a model serves as a basis for considering the effects of non-rigidity in the rotational motion of the Earth.

5. Further simplifying assumptions

A number of additional simplifying assumptions will be adopted here. Some of these assumptions are justified by numerical estimations of the corresponding terms in the equations of rotational motion of the Earth.

- It is natural to assume that the GCRS is defined in such a way that the mass dipole \( M_3 \) of the Earth vanishes (this assumption is actually supported by the IAU Resolution B1.4 in (IAU, 2001)), i.e. the origin of the GCRS is assumed to agree with the post-Newtonian center of mass of the Earth.
- In order to numerically estimate the influence of the terms in the right-hand side of (5) produced by \( S_L \) with \( l \geq 2 \) (and for that purpose only!) let us consider the following model for the matter of the Earth
\[ \Sigma^a = \Sigma \varepsilon_{abc} \omega^b X^c. \]  
(13)

Since we want to numerically estimate post-Newtonian terms it is sufficient to consider (13) as a Newtonian assumption of a rigidly rotating body. Substituting (13) into the definition of \( S_L \) for \( l \geq 2 \) one can prove that

\[ S_L = C_{aL} \omega^a, \]  
(14)

\[ C_{aL} = -M_{aL} + \frac{l+1}{2l+1} \delta_{a<b} N_{L-1}, \]  
(15)

where the moments \( M_L \) and \( N_L \) to Newtonian order read

\[ M_L \equiv \int_E \sum X^L d^3X, \]  
(16)

\[ N_L \equiv \int_E \sum X^2 \hat{X}^L d^3X. \]  
(17)

Here the angle brackets "\(<\ldots>\)" as well as the caret "\(^\hat{\ldots}\)" indicate symmetric and tracefree (STF) part of the corresponding expression. Eqs. (14)–(16) allow one to estimate the torque \( \delta \dot{S}_L \) due to \( S_L \), \( l \geq 2 \) as

\[ \left. \frac{\delta \dot{S}_a}{S^a} \right|_{S_L} \sim \max(|J_{l+1}^E|, |J_{l-1}^E|) \cdot \sum_A \frac{GM_A v_{EA}}{c^2 r_{EA}^3} \left( \frac{R_E}{r_{EA}} \right)^l < 10^{-20} \text{s}^{-1}. \]  
(18)

This estimate gives typical angular velocity of precession due to \( S_L \). This can be compared, e.g., with the relativistic precession due to \( \Omega_{\text{iner}} \). It is easy to see that the precession due to \( S_L \) for \( l \geq 2 \) is at least a factor \( 10^5 \) smaller than the relativistic precession due to \( \Omega_{\text{iner}} \) (i.e., than the geodetic precession). This implies that these terms for any \( l \geq 2 \) can be neglected at the accuracy level of 0.1 \( \mu \)as. This circumstance makes the assumption (10)–(11) for \( S_L \), \( l \geq 2 \) superfluous.

- All external bodies are supposed to be mass-monopoles, that is point masses characterized only by their masses \( M^A \) and BCRS positions \( x_A(t), t = \text{TCB} \). Since the multipole structure (e.g., oblateness) of external bodies is not taken into account in the modern Newtonian theories of nutation of the rigid Earth, such an assumption does not prevent us to achieve the required accuracy of 0.1 \( \mu \)as also in the relativistic framework. In the framework of this model one can derive explicit formulas for the external tidal moments \( G_L \) influencing the Earth \( (E) \). One has

\[ G_L = \sum_{A \neq E} GM_A g^A_L, \]  
(19)

where \( g^A_L \) are functions of i) the BCRS position \( x_E \), velocity \( v_E \) and acceleration \( a_E \) of the Earth, ii) the BCRS position \( x_A \), velocity \( v_A \) and acceleration \( a_A \) of other bodies, iii) the mass \( M_E \) of the Earth, iv) the masses \( M_A \) of other bodies, v) the higher-order multiple moments \( M_L, l \geq 2 \) of the Earth. Note that in Newtonian physics only the positions of
the Earth and body $A$ ($x_E$ and $x_A$) appear in $g_L^A$. In the post-Newtonian approximation one has

$$g_L^A = \frac{(-1)^l (2l - 1)!!}{r_{EA}^{l+1}} \left[ \hat{n}_{EA}^L \left\{ \frac{1}{c^2} \left( 2 v_{EA}^2 - \frac{1}{2} a_A r_{EA} - l \overline{\omega}_E(x_E) - \overline{\omega}_A(x_A) \right) - \frac{1}{2} (2l + 1) (v_A n_{EA})^2 \right\} \right. $$

$$- \frac{1}{c^2} \frac{(l - 1) (l - 8)}{2 (2l - 1)} v_{EA}^{(i_l)} v_{EA}^{(i_{l-1})} n_{EA}^{L-2} + \frac{1}{c^2} \frac{1}{2l - 1} r_{EA} a_{(i_l)} n_{EA}^{L-1} + \frac{1}{c^2} \frac{l}{2} (v_E n_{EA}) v_{EA}^{(i_l)} n_{EA}^{L-1} - \frac{1}{c^2} (l v_A n_{EA} + 4 v_A n_{EA}) v_{EA}^{(i_l)} n_{EA}^{L-1} \right],$$

where

$$a = (l^2 - l + 4) a_E + \frac{1}{2} (l - 8) a_A, \quad (21)$$

$$\overline{\omega}_E(x_E) = \sum_{B \neq E} \frac{G M_B}{r_{EB}}, \quad (22)$$

$$\overline{\omega}_A(x_A) = \sum_{B \neq A} \frac{G M_B}{r_{AB}} + G \sum_{l=2}^{\infty} \frac{(-1)^l (2l - 1)!!}{l! r_{EA}^{l+1}} M_L \hat{n}_{EA}^L, \quad (23)$$

and for any $A$ and $B$ one has $r_{AB} = x_A - x_B$, $v_{AB} = v_A - v_B$, $n_{AB}^L = r_{AB} a_{q_i} \cdots r_{AB} a_{q_i}$. 

6. Reduced equations of motion

Taking into account all the components of the model and the simplifying assumptions the equation of rotational motion of the Earth with respect to the GCRS can be written as

$$\frac{d}{d TCG} \left( C^{ab\omega,b} \right) = \sum_{l=0}^{\infty} \frac{1}{l!} \varepsilon_{abc} M_{bL} G_{cL} + \varepsilon_{abc} \Omega_{b \text{iner}}^b C^{cd\omega,d}. \quad (24)$$

The GCRS is kinematically non-rotating and this is the reason why the post-Newtonian Coriolis force proportional to $\Omega_{\text{iner}}$ appears in the right-hand side of (24). The equations of rotational motion of the Earth relative to a dynamically non-rotating local geocentric reference system does not contain this additional torque. However, the use of a dynamically non-rotating reference system does not seem to be advantageous since the slow precession of its spatial axes relative to those of the BCRS must be taken into account while computing the external tidal moments $G_L$ in the dynamically non-rotating coordinates, which is by no means simpler than using (24). Note also that the relative orientation of the GCRS and that local dynamically non-rotating reference system is well known (see, e.g., Brumberg, Bretagnon, Francou (1991)) and this can be used as a check of theories of precession and nutation constructed in these two reference systems. However, it does not mean that a theory of precession and nutation of the Earth in one of these two reference systems can be constructed in a purely Newtonian way, as it was assumed in all modern theories of Earth nutation, where a purely Newtonian theory was interpreted as a theory in dynamically non-rotating coordinates.
The quantities characterizing the Earth, $C^{ab}$, $\omega^a$ and $M_L$, are functions of TCG or TT while the quantities appearing in the external tidal moments $G_L$ (e.g., the BCRS positions of the bodies) are functions of TCB or TDB (since BCRS Solar system ephemerides should be used here to evaluate those quantities). To avoid the recomputing of $G_L$ from TCB (or TDB) to TCG (or TT) during the numerical integration one may want to use TDB as the independent variable in (24). This version of the equations of rotational motion reads

$$\frac{d}{dTDB} (C^{ab} \omega^b) = \left( \frac{d}{dTDB} (C^{ab} \omega^b) \right)_{\text{geocenter}} \sum_{l=1}^{\infty} \frac{1}{l!} \epsilon_{abc} M_{bL} G_{cL} + \epsilon_{abc} \Omega^b_{\text{mer}} C^{cd} \omega^d. \quad (25)$$

A solution of this equation gives the Euler angles parametrizing the orthogonal matrix $P_{ab}$ from (8)–(9) as functions of TDB which should be re-calculated as functions of TCG or TT afterwards.

The factor $\frac{dTCG}{TDB}$ gives a scaling of the torque as well as additional periodic signal in the torque. Its practical importance should be further investigated.

Another important issue is that the equations (24) and the formulas (20)–(23) for $G_L$ are valid only if both time and space coordinates are not scaled in both GCRS and BCRS. Since in practice one employs scaled time scales TT and TDB as well as associated scaled spatial coordinates, the corresponding scaling factors must be taken into account while computing $G_L$ and using (24) or (25).

In a further publication the equations of rotational motion (25) will be re-written in a form which can be directly used for practical construction of a post-Newtonian theory of Earth’s rotation along the lines of Bretagnon et al. (1997, 1998).

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