Bilayer Quantum Hall System
As a Macroscopic Qubit

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In the bilayer quantum Hall system, a spontaneously charge imbalance state appears at the ground energy level. Gap in the collective excitation energy makes it stable against decoherence in macroscopic level. This state behaves as a spin $\frac{1}{2}$ representation of $SU(2)$ and can be controlled by applying the interlayer voltage. We suggest this system can be regarded as a macroscopic realization of a qubit for a quantum computer.

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1. Introduction

In recent years, a number of realization of a qubit for a quantum computer have been proposed. Many of them use a microscopic object, like an electron or a photon or a nucleus, as a carrier of quantum information. For example, in the NMR liquid quantum computer which is the first realization of quantum computing on the Earth, an ensemble of nuclear spins is manipulated. Though it has huge number of nuclei, quantum phase information is preserved only within a molecule in which a set of nuclei is embedded and computing is executed in every molecules. Then result is read as an average in a canonical ensemble of them. A kind of difficulty will exist with this microscopic qubits when number of qubits becomes large. It must manipulate an extremely large molecules with many nuclear spins. But when the size of a molecule becomes large, it becomes sensitive to external fluctuation and quantum phase information will be lost. In addition, there is the limit in resolution of NMR and it is difficult to read many spin degrees of freedom at once.

In this paper we investigate a possibility to realize a qubit with a collection of electrons which are strongly correlated and preserve phase information of collective modes. Now a day, many phenomena are known in which the quantum effect is observed in macroscopic level. An example is the superconducting circuit with a Josephson junction (SQUID) in which magnetic flux caused by movement of electrons in a circuit is constrained by an effective potential due to the Aharonov-Bohm effect. An application of this system to quantum computer was discussed in .

The quantum Hall effect is another novel example of the macroscopic quantum effect . The root cause of appearance of quantum effect in macroscopic level is excitation gaps in kinetic energy of state of electron. At low temperature limit, higher excitations become negligible and states in the lowest level are isolated from others. This suppresses the kinetic degrees of individual electrons and modulation in electron density (sound wave) and spin polarization (spin wave) are left as degree of freedom of the system. It is well known that there is no gapless density wave (phonon) excitation in a system with translation symmetries. On other hand, spin wave is a gapless excitation due to $SU(2)$ symmetry breaking of a vacuum. In the case of a real spin, an energy gap to first excitation is caused by the Zeeman interaction with external magnetic field and a vacuum becomes stable. At the ground energy level, we have two states with homogeneous electron density and fully spin polarized. This spontaneous spin polarization is called quantum Hall ferromagnet
caused by the exchange energy of Coulomb interacting fermions. If two layers of quantum
Hall liquid with a filling factor \( \nu = \frac{1}{2} \) are located near by each other, an electron in one
layer and a hole in another layer couple and develop coherence. In this system called
bilayer quantum Hall system, a pair of electron and hole behaves as a single particle with
pseudo spin \( \frac{1}{2} \). Here (pseudo) ferromagnet is observed again and charges of two layers are
imbalance spontaneously. Pseudo spin wave excitation is gapless too and we can make a
gap another way than Zeeman term. By turn on a magnetic field parallel to a quantum Hall
surface, long wave excitations are suppressed and discrete (quantized) excitation levels are
remained. We have two states fully pseudo spin polarized as the ground states and due to
gapes in excitation energy, they are stable against decoherence. Degeneracy of these two
are resolved by applying interlayer voltage, and the bilayer quantum Hall system can be
operated as a macroscopic qubit of quantum computer. In section 2 we quickly review the
quantum physics of the quantum Hall systems. In section 3 we argues its application to a
quantum computer. Note that another application of the quantum Hall effect to quantum
computer was discussed in [12].

2. Quantum Hall Systems

2.1. An electron moving in a magnetic field

Quantum Hall liquid is a system of electrons confined in a 2 dimensional surface and
suffered under the strong constant magnetic field which is orthogonal to the surface. This
magnetic field forces movement of an electron in a circle and its quantized excitation energy
becomes gapped. The Lagrangian for an electron is

\[
\mathcal{L}_{\text{electron}} = \frac{m}{2} \dot{x}_i^2 + eB\epsilon_{ij}\dot{x}_i x_j,
\]

where \( i, j = 1, 2 \) are index for coordinates of 2 dimensional surface in which electrons are
confined. This problem is solved as a 1 dimensional harmonic oscillator and its excitation
energy known as Landau level is

\[
\epsilon_n = (n + \frac{1}{2}) \frac{eB}{m}.
\]

This gap in the energy levels causes a quantum phenomenon named quantum Hall effect.
In a typical experiment environment, its energy scale is \( \sim 100K \). Now we consider the
limit in which the temperature is low enough and the constant magnetic field \( B \) is strong
enough. Where the energy gap becomes wide and each Landau level can be treated in
separate. This makes it possible to ignore higher kinetic excitations and explore physics
in the lowest Landau level with \( n = 0 \). The Lagrangian becomes

\[
\mathcal{L}_{\text{LLL}} = eB \epsilon_{ij} \dot{x}_i x_j. \tag{2.2}
\]

Here, the conjugate momentum for \( x_i \) is \( p_i = eB \epsilon_{ij} x_j \) and quantization of this system
make the space of coordinates noncommutative \([x_i, x_j] = \frac{i}{eB} \epsilon_{ij}\). Physical origin of this
noncommutative nature is the Aharonov-Bohm effect on electrons and truncation of the
system to the LLL is compensated by this noncommutativity. From this, an electron
occupies area \( \frac{1}{eB} \), and in other word, \( eB \) of electrons can degenerate in a unit area. A
rate of this density and actual electron density is called filling factor \( \nu \) and characterizes a
quantum Hall system. To understand collective behavior of electrons in the LLL, we move
to the effective field theory description of the system in next subsection.

2.2. Collective excitation modes

To move from the first quantized single electron picture to the second quantized field
theory picture [13], introduce fluid fields \( x_i(y) \) where \( y_i \) are the 2 dimensional coordinates
in which the density of electrons becomes a constant \( \rho_0 \) over the surface. Because of the
Coulomb interaction between electrons, a solution \( x_i(y) = y_i \) is a stable vacuum solution
and the quantum fluctuation is described by \( a_i \) as \( x_i = y_i + a_i \). With this variable, density
of electrons is \( \rho = \rho_0 \det | \delta_{ij} + \partial_i a_j | \). We get the field theory Lagrangian from (2.2) as
\( \mathcal{L} = eB \rho_0 \int dy^2 \epsilon_{ij} a_i a_j \). This is the 2+1 dimensional abelian Chern-Simons gauge theory
[14] in the temporal gauge. The gauge invariant form of the Lagrangian is

\[
\mathcal{L}_{\text{CS}} = \nu \int dy^2 \{ \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \}
\]

where \( \nu = eB \rho_0 \) is filling factor of quantum Hall system. Naively, the Hamiltonian for this
theory equals to zero and this theory dose not describe any dynamics. This is expected
because the kinetic degrees of freedom of original electrons are removed from an effective
theory for the LLL. Still we may have finite degrees of freedom which come from global
structure of a device. For accurate analysis, we need to take into account a boundary
condition of fields. If we consider a system on a compact Riemann surface, there remain
finite number of degrees of freedom depending on the topology of the surface. Gauge
symmetry of the theory is the area preserving diffeomorphism and gauge transformation
acts on $a$ as $a_i \rightarrow a_i + \partial_i \phi$. From the Cohomological argument, number of physical degree
doing freedom is given by $\dim H^1(\Sigma; R) = g$, where $\Sigma$ is the compact Rieman surface on which
the theory is defined and $g$ is number of genus on it. Each of degrees has $\nu$ of degenerated
states and vacuum degeneracy of quantum Hall system on a compact manifold is $\nu^0$. On
a sphere $g = 0$ there is no dynamics and the state stays on the ground. Physically, this
means there can not exist any sound wave excitation in the fluid of electrons propagating
on a sphere or a disk, if the boundary effect (edge excitation) can be ignored. Absence
of phonon like excitation in the LLL can be understand as the following. A wave of
momentum $k$ requires movement of electrons spread over area $\frac{1}{k^2}$. This area contains
$\frac{\nu eB}{k^2}$ of electrons and each of them has kinetic energy $\frac{k^2}{2m}$. This indicates whole energy of
excitation is independent of $k$ and more accurate calculation shows its value coincides with
Landau energy gap $\epsilon = \frac{\nu eB}{m}$. This means phonon excitation pushes one electron to next
Landau level.

We see, as far as only position of electrons are in consideration, there is no gapless
collective excitation in the fluid of electrons on a sphere. But once spin of electron is taken
into account, system develops collective modes with macroscopic coherence.

2.3. quantum Hall ferromagnet and bilayer quantum Hall system

With spin degree of freedom of electron, another novel phenomenon, called quantum
Hall ferromagnet [15,11], is observed where all spins are polarized in a direction at the
ground state. This is caused by the exchange energy of electrons. The electron is a fermion
and, if all spins are aligned in a same direction, the wave function of multiple electrons must
be anti-symmetric for permutation of any two of electrons in the system. This requires
vanishing of the wave function when two electrons close to each other. So the Coulomb
interaction energy among electrons is lower than other configuration where spins are not
polarized. To estimate excitation energy of the spin wave, it requires knowledge of the
LLL wave function.

Here, we shortly review the derivation of this excitation energy according to [11].
General form of the wave function for a single electron in the LLL is

$$\psi = f(z)e^{-\frac{1}{4}|z|^2} \quad (2.3)$$

where $z = \sqrt{eB}(x_1 + ix_2)$ and $f(z)$ is a polynomial of $z$. The innerproduct of two states
in the LLL is given by

$$\langle \psi_1 | \psi_2 \rangle = \int dz^2 f_1(z) f_2(z) e^{-\frac{1}{2}|z|^2}. \quad (2.4)$$
To represent a plane wave with momentum \( k \) in the LLL, we should use an operator
\[
e^{2ik\partial_z}e^{ikz} = e^{-\frac{1}{4}|k|^2}e^{2ik\partial_z+i\bar{k}z} \equiv \tau_k
\]
instead of usual c-function \( e^{ikz+i\bar{k}\bar{z}} \). The spin wave field with a momentum \( q \) can be written as

\[
S_\alpha(q) = \sum_i s_{\alpha,i}\tau_{q,i}
\]

with the spin raising operator \( s_\alpha \) where \( \alpha = 1, 2, 3 \) and \( i \) indexes electrons in the system.

The contribution to the Hamiltonian by the Coulomb interaction between electrons are written as

\[
H_{Coulomb} = \int dk^2 v(k)\rho_{-k}\rho_k
\]

where \( v(k) = e^2 \int dz^2 \frac{1}{|z|}e^{ikz} \) is the Coulomb interaction represented in the momentum space and the density operator is \( \rho_k = \sum_i \tau_{k,i} \). Now we find that \( S_\alpha(q) \) and \( H_{Coulomb} \) are not commute. This results raising of energy by the spin wave excitation of momentum \( q \) as

\[
[H, S_\alpha(q)] = \int dk^2 e^{-\frac{1}{2}|k|^2}v(k)\sin^2\left(\frac{1}{2}(q \times k)\right) \equiv \epsilon_q.
\]

This value becomes zero at \( q \to 0 \) that means this is gapless excitation. Small energy fluctuation brings the system unexpected states and this is very undesired for quantum computing. In the case of a real spin with magnetic moment, Zeeman interaction with external magnetic field perpendicular to quantum Hall surface causes additional gap in excitation energy and ground states becomes stable.

Aside from this spin wave excitation, there exists spin texture mode called Skyrmion whose excitation energy equals \( \frac{1}{4}\epsilon_\infty \). So any spin fluctuation costs finite energy and, at low temperature, state of this system is bound to one of two fully polarized ground states. This is the quantum Hall ferromagnet.

Because a Skyrmion excitation carries electric charge \( \nu e \) and it is the lowest cost way to inject a charge into quantum Hall system, Skyrmions appears in quantum Hall system in which filling factor \( \nu \) is slightly differed from 1 and spin polarization is rapidly broken at \( \nu \neq 1 \).

Same phenomenon as ferromagnet is observed in the bilayer quantum Hall system [16,17,18,11] where two layers of quantum Hall liquid are located in parallel and very closed to each other to turn on interlayer tunneling. An electron in one layer and a hole in another layer are in interlayer coherence and behaves as a particle with the pseudo spin. Charge and a filling factor in each layer is not definite because of tunneling and the
difference of charges between two layers is a value of a spin in the $z$ direction. The system with two of filling factor $\nu = \frac{1}{2}$ quantum Hall layers behaves as the quantum Hall liquid with filling factor $\nu = 1$ with pseudo spin $\frac{1}{2}$. States of this system are bound to two grounds again and vacuum transition takes place by applying the interlayer voltage to $z$ direction which appears in a Hamiltonian as the Zeeman term. Direct experimental evidence for this phenomenon was reported in [19]. Here we hope to use Zeeman interaction to control a vacuum transition and we can not fix it to make a gap in pseudo spin wave excitation energy. Fortunately, it is possible to make gaps in excitation energy by applying parallel magnetic field to bilayer surfaces [20]. A spin wave excited state with momentum $q$ is described as,

$$\phi_q(x) = \sum_i e^{i2\pi qx_i} |i\rangle$$

where $|i\rangle$ denotes a states with $i$th pseudo spin is inverted. Now move this inverted pseudo spin for a distance $l$. This operation causes movement of an electron in the lower layer and movement of a hole in the upper layer. This can be regarded as circle movement of an electron across both layer. If parallel magnetic field is applied, a quantum state picks phase factor $e^{ieB||d} \frac{d}{dl}$ during this operation, where $d$ is separation of two layers and $B||$ is a parallel magnetic field. Consistency requires

$$\phi_q(x) = \phi_q(x + \frac{2\pi}{eB||d}).$$

This constrains $q = eB||dm$ where $m = 1, 2, 3, ..$ and wave excitation lower than $eB||d$ is inhibited.

3. Macroscopic Qubit

3.1. Vacuum transition

Now consider a system of bilayer Quantum Hall liquids with filling factor $\nu = 1$ on a sphere at low temperature limit. In this theory, all kinetic excitations are frozen away and we have only two ground states in which all of electrons are in one of an upper or a lower layer. To calculate tunneling amplitude from one vacuums to another, semi-classical approximation can be used as

$$\Gamma \approx e^{-S_{cl}}$$
where $S_{cl}$ is the Euclidean Action at a bounce solution which is a path connecting two vacuums with minimizing the Action. We can construct a bounce solution from a Skyrmion solution with winding number equals 1. Skyrmion is a topological solution. Let’s remind that our theory is defined on a $S^2$ and the spin filed is a mapping $S^2 \to SU(2)$. This mapping are classified by counting how many times $S^2$ covers $SU(2)$. If this number equals to 1, a spin is up on a pole and down on a pole at opposite side and at intermediate region near $r = \lambda$, a spin is rotated 90 degrees. Explicit form of this solution in the polar coordinates $(r, \theta)$ is

\[
\begin{align*}
  s_x &= \frac{2\lambda r \cos(\theta)}{\lambda^2 + r^2} \\
  s_y &= \frac{2\lambda r \sin(\theta)}{\lambda^2 + r^2} \\
  s_z &= \frac{r^2 - \lambda^2}{\lambda^2 - r^2}.
\end{align*}
\]

As mentioned before, its excitation energy is $\frac{1}{4} \epsilon_\infty$.

Now the bounce solution is start $\lambda = 0$ and go to $\lambda = \infty$. During this process energy dose not depend on the value of $\lambda$ and $S_{cl}$ is proportional to $\delta t$ which is time required for process. Thus transition amplitude from spin up to down state within time $T$ is

\[
\Gamma(T) = \frac{\int_0^T dt e^{-\frac{1}{4} \epsilon_\infty t}}{\int_0^\infty dt e^{-\frac{1}{4} \epsilon_\infty t}} = 1 - e^{-\frac{1}{4} \epsilon_\infty T}.
\] (3.1)

3.2. Qubit operation

Let denote the global spin alignment operator $M_z$ whose eignestates are $M_z|+\rangle = |+\rangle$ and $M_z|−\rangle = −|−\rangle$ where $|+\rangle$ denotes all spins up and $|−\rangle$ denotes all spins down. $M_z$ is a good observable in our theory. It is measured by difference of electric charge between two layers and controlled by electric field $E$ caused by interlayer voltage which couples to $M_z$ by the Zeeman term $\mathcal{H} = eNEM_z$ where $N$ denotes number of electrons in the system. Initial state can be prepared by applying $E$ for enough interval. Then change the sign of $E$. After a certain period which is determined from (3.1) , we will get the state $|\Phi\rangle = cos(\theta)|+\rangle + sin(\theta)|−\rangle$. But this is not full $SU(2)$ rotation and the state dose not represent a universal qubit of [21]. To realize a qubit from this quantum mechanism, the method of [22] can be used in which a qubit is constructed from three spins governed by the Heisenberg interactions between every pair of them. To produce Heisenberg interaction, it requires a tunneling term between two qubits $M_{1,+} M_{2,-} + M_{1,-} M_{2,+}$ aside of $M_{1,z} M_{2,z}$. It will need tactical allocation of devices and a subject of further study. After accomplishing
this, a product of three spin states rotates under $SU(2)$. Now we obtain a qubit. In our system, a qubit is represented by the difference of charges between two layers which consists of many electrons and this affects other qubits via the electric field produced by this charge imbalance.

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