ON THE QUIVER PRESENTATION OF THE DESCENT ALGEBRA OF THE HYPEROCTAHEDRAL GROUP

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Abstract. In a recent article, we introduced a mechanism for producing a presentation of the descent algebra of the symmetric group as a quiver with relations arising from a new construction of the descent algebra as a homomorphic image of an algebra of binary forests. Here we extend the method to construct a similar presentation of the descent algebra of the hyperoctahedral group, providing a simple proof of the known formula for the quiver of this algebra and a straightforward method for calculating the relations.

1. Introduction

Let \((W, S)\) be a finite Coxeter system and let \(k\) be a field of characteristic zero. For all \(J \subseteq S\) we denote the parabolic subgroup \(\langle J \rangle\) of \(W\) by \(W_J\) and the set of minimal length left coset representatives of \(W_J\) in \(W\) by \(X_J\). In 1976 Solomon proved that the elements \(x_J = \sum_{x \in X_J} x\) of the group algebra \(kW\) for all \(J \subseteq S\) satisfy

\[ x_Jx_K = \sum_{L \subseteq S} c_{JKL} x_L \]

for certain integers \(c_{JKL}\) with \(J, K, L \subseteq S\) [8]. This implies that the linear span \(\langle x_J \mid J \subseteq S \rangle\) is a subalgebra of \(kW\). This algebra is called the descent algebra of \(W\) and is denoted by \(\Sigma(W)\). Thanks to the calculation of a complete set of primitive orthogonal idempotents of \(\Sigma(W)\) by Bergeron, Bergeron, Howlett, and Taylor [2] the complete representation theory of \(\Sigma(W)\) is known. The simple \(\Sigma(W)\)-modules are indexed by conjugacy classes of subsets of the generating set \(S\). Furthermore \(\Sigma(W)\) is a basic algebra and thereby admits a presentation as a quiver with relations.

The aim of this paper is to calculate and study the quiver presentation of \(\Sigma(W)\) when \(W\) is the Coxeter group of type \(B_n\), also known as the hyperoctahedral group. This is the symmetry group of a hypercube in \(\mathbb{R}^n\). We denote the hyperoctahedral group by \(W_n\) in this article. The quiver of \(\Sigma(W_n)\) was calculated by Saliola in 2008 using hyperplane arrangements [7]. In contrast, our approach is formulated in terms of binary forests. This has the advantage of providing a straightforward proof of the quiver in Theorem 19, the main result of this article. More importantly, it also allows us to deduce and compactly express the relations of the presentation of \(\Sigma(W_n)\) for a particular \(n \in \mathbb{N}\). Such calculations for small values of \(n\) resulted in

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the formulation of Conjecture 20, which proposes a generating set of the ideal of relations when \( n \) is arbitrary.

Similarly, the quiver of the Mantaci-Reutenauer algebra \( \Sigma'(W_n) \) was calculated in 2011 by Margolis and Steinberg [5]. The algebra \( \Sigma'(W_n) \) is a generalization of the descent algebra which contains both \( \Sigma(W_n) \) and \( \Sigma(S_n) \) as subalgebras, where \( S_n \) denotes the symmetric group on \( n \) letters.

For any set \( \Omega \) we denote the vector space of \( k \)-linear combination of elements of \( \Omega \) by \( k\Omega \). When \( \Omega \) is a monoid, the space \( k\Omega \) becomes an algebra, its product induced from the product in \( \Omega \). The group algebra \( kW \) mentioned above is an example of this construction. When \( \Omega \) is a category rather than a monoid, that is, when the products of certain pairs of elements are undefined, the space \( k\Omega \) becomes an algebra by taking the product of two elements of \( \Omega \) to be zero in \( k\Omega \) whenever that product is undefined in \( \Omega \). Viewing a quiver \( Q \) as the set of paths in \( Q \) the path algebra \( kQ \) is an example of this construction.

The main tool used in this article is the construction from [6] that we now briefly review. Let \((W,S)\) be any finite Coxeter system. Let \( A \) be the set of chains of subsets of \( S \). These are tuples \((J_0,J_1,\ldots,J_l)\) where \( S \supseteq J_0 \supseteq J_1 \supseteq \cdots \supseteq J_l \) and \( |J_i \setminus J_{i+1}| = 1 \) for all \( 0 \leq i \leq l-1 \). Since the concatenation of two chains might fail to be another chain, concatenation yields only a partial product making \( A \) into a category rather than a monoid.

The free monoid \( S^* \) acts on \( A \) by

\[
(J_0,J_1,\ldots,J_l) \cdot s = (J_0 \upharpoonright s, J_1 \upharpoonright s, \ldots, J_l \upharpoonright s)
\]

for \( s \in S \) where \( \omega = w_{J_0 \cup [s]} w_{J_1 \cup [s]} \ldots w_{J_l \cup [s]} \) and where \( w_K \) denotes the longest element in the parabolic subgroup \( W_K = \langle k \rangle \) of \( W \) for \( K \subseteq S \). Here we denote the conjugate \( x^{-1}yx \) by \( y^x \).

We define a difference operator \( \delta \) on the algebra \( kA \) as follows. If \( a = (J_0,J_1,\ldots,J_l) \) then we put \( \delta(a) = a \) if \( l = 0 \) or \( \delta(a) = b - b.s \) if \( l > 0 \), where \( b = (J_1,\ldots,J_l) \) and \( s \in J_0 \setminus J_1 \). Repeating \( \delta \) as many times as possible determines another difference operator \( \Delta \) defined by \( \Delta(a) = \delta^\infty(a) \) for \( a \) as above. So \( \Delta \) is a linear map \( kA \rightarrow k2^S \) where \( 2^S \) denotes the power set of \( S \).

Like a group action, it can be shown that the monoid action (1) partitions \( A \) into orbits. Identifying an \( S^* \)-orbit with the sum of its elements in \( kA \) it can also be shown that if \( X \subseteq kA \) denotes the set of orbits in \( A \) then \( kX \) is a subalgebra of \( kA \). The main tool used in this article is the following theorem extracted from [6] which constructs a presentation of the descent algebra of \( W \).

**Theorem 1** (Pfeiffer). There exist subsets \( \Lambda, \mathcal{E} \subseteq X \) such that

- \( \Lambda \) is a complete set of pairwise orthogonal primitive idempotents of \( kX \),
- \( \lambda (kX) \mu \cap X \) is a basis of the subspace \( \lambda (kX) \mu \) for all \( \lambda, \mu \in \Lambda \), and
- The pair \( (Q, \ker \Delta) \) is a quiver presentation of \( \Sigma(W)^{op} \) where \( Q \) is the quiver with vertices \( \Lambda \) and edges \( \mathcal{E} \).

Although [6] provides no uniform description of the set \( \mathcal{E} \), it can be calculated through an algorithm and consists of orbits of chains \((J_0,J_1,\ldots,J_l)\) with \( l \geq 1 \) which are irreducible in \( X \) and linearly independent in \( kX/\ker \Delta \). In contrast, the set \( \Lambda \) is given explicitly as the set of orbits of chains of the form \((J_0)\) for all \( J_0 \subseteq S \). If the orbit of \((J_0,J_1,\ldots,J_l)\) is in \( \mathcal{E} \) then we interpret it as an edge of \( Q \) whose source is the orbit of \((J_1)\) and whose destination is the orbit of \((J_0)\).
2. Trees and forests

Let \( \mathcal{T} \) be the minimal set containing the natural numbers and also containing the diagram \( \hat{UV} \) whenever \( U, V \in \mathcal{T} \). The elements of \( \mathcal{T} \) are called trees. Trees which are natural numbers are called leaves while trees of the form \( \hat{UV} \) are called (inner) nodes. The positions of the nodes in a tree can be specified by elements of the free monoid \( \{1, 2\}^* \) by labeling the tree itself by the empty word \( \emptyset \) and labeling the left and right children of the node labeled by \( w \in \{1, 2\}^* \) by \( w_1 \) and \( w_2 \) respectively. We designate the node of a tree \( U \) in position \( w \in \{1, 2\}^* \) by \( U_w \).

An unlabeled forest is a sequence of trees. Independent of the labeling convention described above, a labeled forest is a sequence of trees whose nodes are labeled by natural numbers in such a way that the label of every node is greater than that of its parent if it has one, and each number \( 1, 2, \ldots, l \) is the label of exactly one node, where \( l \) is the number of nodes in the sequence. For example, the nodes in positions \( \emptyset, 2, 21 \) of the first tree of the labeled forest

are labeled by \( 1, 4, 5 \) while the nodes in positions \( \emptyset, 1 \) of the second tree are labeled by \( 2, 3 \).

Next we introduce some invariants of a forest \( X \), regardless of whether \( X \) is labeled or unlabeled. The number of nodes in \( X \) is called its length and is denoted by \( \ell(X) \). The sequence of leaves in \( X \) is called its foliage and is denoted by \( X \). The sum of the leaves of a tree is called its value. The sequence of values of the trees in \( X \) is called its squash and is denoted by \( X \). Finally, the sum of the values of the trees of a forest is called its value. For example, if \( X \) is the forest shown above, then \( X = 97 \) and \( X = 1125124 \) while \( \ell(X) = 5 \) and the value of \( X \) is sixteen.

The sets of labeled and unlabeled forests of value \( n \in \mathbb{N} \) are denoted by \( L_n \) and \( M_n \) respectively. Both sets become categories through the partial product \( \bullet \) defined as follows. Whenever two forests \( X \) and \( Y \) satisfy \( X = Y \) we define \( X \bullet Y \) to be the forest obtained from \( X \) by replacing its leaves with the trees of \( Y \) in the same order. Note that this operation replaces each leaf of \( X \) with a tree from \( Y \) of exactly the same squash and increases the length of \( X \) by \( \ell(Y) \). In case \( X \) and \( Y \) are labeled forests, we also increment the node labels of \( Y \) by \( \ell(X) \) to ensure that the product \( X \bullet Y \) will also be a labeled forest.

Taking \( X \bullet Y \) to be zero whenever \( X \neq Y \) makes \( kL_n \) and \( kM_n \) into \( k \)-algebras. Naturally \( L_n \) is related to \( M_n \) through the functor \( E : L_n \to M_n \) which removes all the node labels of a forest. The functor \( E \) induces a homomorphism of algebras \( kL_n \to kM_n \).

The category \( \mathcal{A} \) associated to a Coxeter group \( W \) of rank \( n \) is equivalent to the category \( L_{n+1} \) of labeled forests of value \( n+1 \) through the functor \( \varphi \) that we now describe. We identify the Coxeter generating set \( S \) of \( W \) with the set \( \{1, 2, \ldots, n\} \). If \( J \subseteq S \) with \( |J| = n+1-j \) then we write \( S \setminus J = \{t_1, t_2, \ldots, t_{j-1}\} \) where \( t_1 < t_2 < \cdots < t_{j-1} \). We put \( t_0 = 0 \) and \( t_{j+1} = n+1 \) and let \( \varphi(J) \) be the composition \( q_1 q_2 \cdots q_{j} \) where \( q_i = t_i - t_{i-1} \) for all \( 1 \leq i \leq j \). Then \( \varphi \) is a bijection between the subsets of \( S \) and the compositions of \( n+1 \).
We extend $\varphi$ to a map $A \rightarrow L_{n+1}$ as follows. If $a = (J_0, J_1, \ldots, J_l) \in A$ is a chain of subsets of $S$ then $\varphi(J_0), \varphi(J_1), \ldots, \varphi(J_l)$ is a sequence of compositions of $n+1$. For each $1 \leq i \leq l$ the composition $\varphi(J_i)$ is a refinement of $\varphi(J_{i-1})$ obtained by replacing some part $x$ with two contiguous parts whose sum is $x$. There is a unique labeled forest of length one with squash $\varphi(J_{i-1})$ and foliage $\varphi(J_i)$ for each $1 \leq i \leq l$. We define $\varphi(a) = X_1 \cdot X_2 \cdots \cdot X_l$. The map $\varphi : A \rightarrow L_{n+1}$ is a bijection. We will therefore identify chains of subsets of $S$ with labeled forests in the remainder of this article.

We can now use Theorem 1 to construct a presentation of the descent algebra of the hyperoctahedral group $W_n$ in the setting of labeled forests rather than chains of subsets of $S$. This has the advantage that the combinatorial structure of labeled forests is more transparent than that of chains of subsets. For this purpose we briefly review the results from [3] of the program for the symmetric group. Let $S$ be the Coxeter generating set of $S_n$.

Let $X = X_1X_2 \cdots X_j \in L_n$ where $X_1, X_2, \ldots, X_j$ are labeled trees and suppose that $a \in A$ is such that $\varphi(a) = X$. Then the action of $S^*$ on $A$ in (1) is such that the orbit of a corresponds under $\varphi$ with the orbit of $X$ under the action of $S_j$ permuting the trees $X_1, X_2, \ldots, X_j$. In order to distinguish the $S^*$-actions in types $A$ and $B$, we call the $S_j$-orbit of a labeled forest $X$ with $j$ trees an $A$-orbit and we denote it by $[X]_A$.

Of course, it also makes sense to permute the trees of an unlabeled forest, even if this action does not correspond with an $S^*$-action. If $X = X_1X_2 \cdots X_j \in M_n$ is an unlabeled forest where $X_1, X_2, \ldots, X_j$ are trees, then we also call the $S_j$-orbit of $X$ an $A$-orbit and we denote it by $[X]_A$. Since the trees in a labeled forest have unique node labels, the $A$-orbit of a labeled forest $X$ typically has more elements than the $A$-orbit of $E(X)$. In fact $E[X]_A = \alpha_X |E(X)|_A$ where $\alpha_X$ is the index of the stabilizer of $X$ in $S_j$ in the stabilizer of $E(X)$ in $S_j$. We denote the sets of $A$-orbits in $kL_n$ and $kM_n$ by $L_n$ and $M_n$ respectively. Then $kL_n$ and $kM_n$ come out to be subalgebras of $kL_n$ and $kM_n$ and the map $E : kL_n \rightarrow kM_n$ is a homomorphism of algebras by the remark above.

Finally, since the elements of $M_n$ are forests with leaves in $N$ there is a natural map $\pi : kM_n \rightarrow kN^*$ given by recursively replacing each node of a tree with the Lie bracket in $kN^*$. That is, for unlabeled trees $U$ we put

$$\pi(U) = \begin{cases} \pi(U_1) \pi(U_2) - \pi(U_2) \pi(U_1) & \text{if } \ell(U) > 0 \\ U & \text{if } \ell(U) = 0 \end{cases}$$

and define $\pi(X_1X_2 \cdots X_j) = \pi(X_1) \pi(X_2) \cdots \pi(X_j)$ where $X_1, X_2, \ldots, X_j$ are unlabeled trees. Then under the equivalence $\varphi$ the map $\Delta : kA \rightarrow k2^S$ corresponds with a map $kL_n \rightarrow kN^*$ which comes out to be the composition $\pi \circ E$. However, to improve the legibility of this article, we observe that by extending the definition of $\pi$ to labeled forests by simply ignoring node labels, we can drop the factor $E$ and assert that $\Delta$ corresponds with $\pi$ rather than $\pi \circ E$ in type $A$.

The calculation above allows us to view the descent algebra $\Sigma(S_n)$ as the quotient of $kL_n$ by $\ker \pi$. It remains to find a quiver whose path algebra is $kL_n$ and to describe $\ker \Delta$ in terms of this path algebra. These results are not needed in this article.

We are now in a position to develop a similar program for the descent algebra of the hyperoctahedral group $W_n$. 

3. Preliminaries for type B

In addition to the definitions in §2 common to the Coxeter groups of types A and B we introduce the following constructions specific to type B. If $U$ is a labeled tree, then the tree defined by

$$\overline{U} = \begin{cases} \overline{U_2} & \text{if } U = \overline{U_1} \overline{U_2} \\ U & \text{if } \ell(U) = 0 \end{cases}$$

is called the mirror image of $U$. We write $\overline{\overline{U}} = U$. If $U$ is an unlabeled tree, then we modify the definitions of $\overline{U}$ and $\overline{U}$ by removing the node labels.

**Lemma 2.** If $U$ is an unlabeled tree then $\pi\left(\overline{U}\right) = (-1)^{\ell(U)} \pi(U)$ so that

$$\pi\left(\overline{U}\right) = \begin{cases} 2\pi(U) & \text{if } \ell(U) \text{ is odd} \\ 0 & \text{if } \ell(U) \text{ is even}. \end{cases}$$

**Proof.** Put $l = \ell(U)$. If $l = 0$ then $\overline{U} = U$ so that $\pi\left(\overline{U}\right) = \pi(U) = (-1)^{l} \pi(U)$. Suppose that $l > 0$ and put $l_1 = \ell(U_1)$ and $l_2 = \ell(U_2)$ so that $l = l_1 + l_2 + 1$. Then

$$\pi\left(\overline{U_2} \overline{U_1}\right) = \pi\left(\overline{U_2} \overline{U_1} - \overline{U_1} \overline{U_2}\right)$$

$$= \pi\left(\overline{U_2}\right) \pi\left(\overline{U_1}\right) - \pi\left(\overline{U_1}\right) \pi\left(\overline{U_2}\right)$$

$$= (-1)^{l_2} \pi(U_2) (-1)^{l_1} \pi(U_1) - (-1)^{l_1} \pi(U_1) (-1)^{l_2} \pi(U_2)$$

$$= (-1)^{l-1} \pi(U_2 U_1 - U_1 U_2)$$

$$= (-1)^{l} \pi(U_1 U_2 - U_2 U_1)$$

$$= (-1)^{l} \pi\left(\overline{U_1} \overline{U_2}\right)$$

by induction. \hfill \Box

**Corollary 3.** Suppose that $U$ is an unlabeled tree satisfying $\overline{\overline{U}} = U$. Then the following statements are equivalent.

1. $\pi(U) \neq 0$
2. $\ell(U)$ is even
3. $\ell(U) = 0$

**Proof.** Suppose that $\ell(U)$ is even. If $\ell(U) > 0$ then $\ell(U) = \ell(U_1) + \ell(U_2) + 1$ so one of $\ell(U_1)$ or $\ell(U_2)$ is even and the other is odd. But this is impossible because $\ell(U_1) = \ell\left(\overline{U_2}\right) = \ell(U_2)$. Therefore $\ell(U) = 0$. This proves the equivalence of (2) and (3). It also proves that $\pi(U) = 0$ implies that $\ell(U)$ is odd, since $\pi(V) = V \neq 0$ for any tree $V$ of length zero. Conversely, if $\ell(U)$ is odd, then $\pi(U) = \pi\left(\overline{U}\right) = -\pi(U)$ by **Lemma 2** so that $\pi(U) = 0$. This proves the equivalence of (1) and (2). \hfill \Box
4. Delta and the Monoid Action

In this section we translate the various elements of Theorem 1 to the context of labeled forests. If \((W, S)\) is a Coxeter system then for each \(s \in S\) the conjugate \(s^w_0\) is also an element of \(S\), where \(w_0\) denotes the longest element of \(W\). It follows that conjugation by \(w_0\) induces a permutation of \(S\) which can be extended to a permutation of the free monoid \(S^*\) by conjugating all the letters of a word by \(w_0\).

It is easy to see that conjugation by the longest element of the Coxeter group of type \(A_i\) reverses the word \(12\cdots i\) while conjugation by the longest element of the Coxeter group of type \(B_i\) fixes the word \(12\cdots i\). We can use this information to calculate the action (1) of \(S^*\) on \(A\) since that action is defined in terms of conjugation by longest elements.

Let \((W, S)\) be a Coxeter system of type \(B_n\). Suppose \(K \subseteq S\) and \(S \setminus K = \{t_0, t_1, \ldots , t_{j-1}\}\) where \(t_0 < t_1 < \cdots < t_{j-1}\). Let \(y_0, y_1, \ldots , y_j \in \mathbb{N}^*\) be such that \(12\cdots n = y_0t_1y_1t_1\cdots t_{j-1}y_j\). For each \(0 \leq i \leq j\) let \(K_i \subseteq S\) be the set of letters in the word \(y_i\) so that \(K = \bigcup_{i=0}^{j} K_i\). Then \(W_K\) is the direct product of the subgroups \(W_{K_i}\) and the longest element \(w_K\) of \(W_K\) is the product \(w_{K_0}w_{K_1}\cdots w_{K_j}\). Note that \(W_{K_0}\) is of type \(B\) while \(W_{K_i}\) is of type \(A\) for all \(1 \leq i \leq j\). Then by the comments above, conjugation by \(w_K\) is the permutation such that \((y_0y_1y_2\cdots y_j)^{w_{K}} = y_0y_1y_2\cdots y_j\) where here we denote the reverse \(s_p, s_{p-1}, \ldots , s_1\) of a word \(w = s_1s_2\cdots s_p \in S^*\) by \(\overline{w}\).

We can now compute the permutation of \(K\) induced by conjugation by \(\omega = w_Kw_{K \cup \{s\}}\) for \(s \in S\). If \(s \in K\) then \(\omega\) is the identity element of \(W\) and thereby induces the trivial permutation of \(K\). Otherwise \(s = t_i\) for some \(0 \leq i \leq j-1\). Continuing the calculation above, conjugation by \(w_Kw_{K \cup \{t_i\}}\) sends \(y_0y_1\cdots y_j\) to

\[
y_0\overline{y_1}y_2\cdots \overline{y_j} = y_0y_1y_2\cdots y_j
\]

while conjugation by \(w_Kw_{K \cup \{t_j\}}\) sends \(y_0y_1\cdots y_j\) to

\[
y_0\overline{y_1}\cdots \overline{y_{i+1}}y_{i+1}\cdots \overline{y_j} = y_0y_1\cdots y_{i+1}y_{i+1}\cdots y_j
\]

for \(1 \leq i \leq j-1\). This proves the following proposition.

**Proposition 4.** Let \((W, S)\) be a Coxeter system of type \(B_n\). Let \(X = UV_1 \cdots V_j \in L_{n+1}\) be a labeled forest where \(U, V_1, V_2, \ldots , V_j\) are trees and suppose that \(X\) corresponds with \((J_0, J_1, \ldots , J_l)\) \(\in A\) under \(\varphi\). Then

\[
X \cdot t_i = \begin{cases} 
UV_1 \cdots V_j & \text{if } i = 0 \\
UV_1 \cdots V_{i+1}V_i \cdots V_j & \text{if } 1 \leq i \leq j-1 
\end{cases}
\]

where \(S \setminus J_0 = \{t_0, t_1, \cdots , t_{j-1}\}\) and \(t_0 < t_1 < \cdots < t_{j-1}\). Thus \(\delta (X)\) is obtained from

\[
(2) \begin{cases} 
U_1U_2V_1 \cdots V_j & \text{if } U \text{ is the node of } X \text{ labeled } 1 \\
UV_1 \cdots V_{l-1} (V_1V_2 - V_2V_1) \cdot V_{i+1} \cdots V_j & \text{if } V_i \text{ is the node of } X \text{ labeled } 1
\end{cases}
\]
by subtracting 1 from all the node labels. Let \( m \) be the greatest integer for which \( U_1^m \) is defined. Then iterating (2) gives

\[
\Delta(X) = \pi \left( U_1^m \overbrace{U_1 \cdots U_1^m}^{\text{\# \( m \)}} V_1 \cdots V_j \right) \\
= \begin{cases} 
2^m \pi(U_1 = U_1 \cdots U_2 V_1 \cdots V_j) & \text{if } \ell(U_{1\cdots 2}) \text{ is odd for all } 0 \leq i \leq m-1 \\
0 & \text{otherwise}
\end{cases}
\]

by Proposition 4.

If \( X \) is as in Proposition 4 then it follows that the \( S^* \)-orbit of \( X \) corresponds with the orbit of \( X \) under the action of \( \mathcal{S}_2 \wr \mathcal{S}_j \) where \( \mathcal{S}_2 \) acts on the trees \( V_1, V_2, \ldots, V_j \) by \( \overleftarrow{\tau} \) and \( \mathcal{S}_j \) acts by permuting the trees \( V_1, V_2, \ldots, V_j \). Now if \( X = UV_1V_2 \cdots V_j \) is any labeled or unlabeled forest where \( U, V_1, V_2, \ldots, V_j \) are trees, then we call the \( \mathcal{S}_2 \wr \mathcal{S}_j \)-orbit of \( X \) a \( B \)-orbit and we denote it by \( [X]_B \). The greatest integer \( m \) for which \( U_1^m \) is defined is called the depth of the forest \( X \).

**Proposition 5.** Let \( (W, S) \) be a Coxeter system of type \( B_n \) and let \( X = UV_1V_2 \cdots V_j \in L_{n+1} \) be a labeled forest where \( U, V_1, V_2, \ldots, V_j \) are trees. If \( m \) is the depth of \( X \) and \( r \) is the number of \( V_i \) of positive length, then

\[
\Delta [X]_B = 2^{r+m} \pi \left( U_1^m U_1 \cdots U_2 [V_1 \cdots V_j]_A \right)
\]

if \( \ell(U_{1\cdots 2}) \) is odd for all \( 0 \leq i \leq m-1 \) and \( \ell(V_1) \) is even for all \( 1 \leq i \leq j \). Otherwise \( \Delta [X]_B = 0 \). In particular, the \( B \)-orbits of forests of odd length are in \( \ker \Delta \).

**Proof.** For any \( \sigma \in \mathcal{S}_j \) we observe that

\[
\pi \left( (V_{1,\sigma} + \hat{V}_{1,\sigma}) \cdots (V_{j,\sigma} + \hat{V}_{j,\sigma}) \right) = \begin{cases} 
2^\pi(V_{1,\sigma} \cdots V_{j,\sigma}) & \text{if } \ell(V_i) \text{ is even for all } i \\
0 & \text{otherwise}
\end{cases}
\]

by Lemma 2. Next we observe that expanding the product

\[
(3) \quad U \left( V_{1,\sigma} + \hat{V}_{1,\sigma} \right) \cdots \left( V_{j,\sigma} + \hat{V}_{j,\sigma} \right)
\]

results in the sum over all \( J \subseteq \{1, 2, \ldots, j\} \) of terms \( UV_{1,\sigma} \cdots V_{j,\sigma} \) having \( \hat{V}_{1,\sigma} \) in place of \( V_{1,\sigma} \) for all \( i \in J \). It follows that summing (3) as \( \sigma \) ranges over a set of representatives of the cosets of the stabilizer \( H \) of \( V_1V_2 \cdots V_j \) in \( \mathcal{S}_1 \) results in a multiple of \([X]_B \). Assuming that \( \ell(V_i) \) is even for all \( i \) we observe that each term of (3) will be duplicated for every \( i \) such that \( \hat{V}_i = V_i \). By Corollary 3 we have \( \hat{V}_i = V_i \) exactly when \( \ell(V_i) = 0 \). Therefore (3) has \( 2^r \) distinct terms each appearing \( 2^{j-r} \) times. Then by our description of the \( S^* \)-action in Proposition 4 we have

\[
(4) \quad 2^{j-r} [X]_B = \sum_{\sigma \in \mathcal{S}_j/H} U \left( V_{1,\sigma} + \hat{V}_{1,\sigma} \right) \cdots \left( V_{j,\sigma} + \hat{V}_{j,\sigma} \right)
\]
so that
\[
\Delta [X]_B = 2^{r-1} \sum_{\sigma} \Delta \left( U \left( V_{1,\sigma} + \hat{V}_{1,\sigma} \right) \cdots \left( V_{j,\sigma} + \hat{V}_{j,\sigma} \right) \right) \\
= 2^{r+m} \sum_{\sigma} \pi \left( U_1 U_{1m-1} \cdots U_2 \left( V_{1,\sigma} + \hat{V}_{1,\sigma} \right) \cdots \left( V_{j,\sigma} + \hat{V}_{j,\sigma} \right) \right) \\
= 2^{r+m} \sum_{\sigma} \pi \left( U_1 U_{1m-1} \cdots U_2 V_{1,\sigma} \cdots V_{j,\sigma} \right) \\
= 2^{r+m} \pi \left( U_1 U_{1m-1} \cdots U_2 \left\{ V_1 \cdots V_1 A \right\} \right)
\]
by Proposition 4.
\( \square \)

5. The Quiver

We define a quiver \( Q_n \) in this section and prove that \( Q_n \) is the quiver of \( \Sigma (W_n) \) in §11. According to Theorem 1 the vertices of \( Q_n \) correspond with the B-orbits of compositions of \( n+1 \). By Proposition 5 the B-orbits of forests of odd length are in \( \ker \Delta \). However, we will show in Theorem 19 that the B-orbits of forests of length greater than two are in \( \text{Rad}^2 (k \mathcal{L}_{n+1}/\ker \Delta) \). We will therefore select a set of B-orbits of labeled forests of length two, linearly independent modulo \( \ker \Delta \), to be the edges of \( Q_n \). Note that the source and destination of such an edge are the B-orbits of the foliage and squash of any forest in the orbit by the comments following Theorem 1.

Let \( X = UV_1 \cdots V_j \in \mathcal{L}_{n+1} \) have length two, where \( U, V_1, \ldots, V_j \) are labeled trees. Then assuming that \( [X]_B \notin \ker \Delta \) it follows from Proposition 5 that the length of \( V_i \) must be even for all \( i \) while the length of \( U_2 \) must be one if \( U_2 \) exists. Then by taking a different representative \( X \) of \([X]_B \) if necessary, we can assume that either

\[
X = a \xrightarrow{2} b \xrightarrow{c} q_1 q_2 \cdots q_j \quad \text{or} \quad X = q_0 \xrightarrow{a} b \xrightarrow{c} q_1 q_2 \cdots q_j \tag{5}
\]

for some \( a, b, c, q_0, q_1, \ldots, q_j \in \mathbb{N} \). In the first case

\[
\Delta [X]_B = 2^r \left( a \xrightarrow{b} c \left[ q_1 q_2 \cdots q_j \right]_A \right) = 2 (bc - cb) \left[ q_1 q_2 \cdots q_j \right]_A \tag{6}
\]

by Proposition 5. Then \( [X]_B \notin \ker \Delta \) if and only if \( b \neq c \). If \( b < c \) then we take \([X]_B \) to be an edge of \( Q_n \). We remark that if \( Y = a \xrightarrow{2} b \xrightarrow{c} q_1 q_2 \cdots q_j \) then \([Y]_B \) is another element of \( \mathcal{L}_{n+1} \) having the same foliage and squash as \([X]_B \). However, (6) shows that \([Y]_B \equiv -[X]_B \pmod{\ker \Delta} \). Therefore, we need not introduce an edge in \( Q_n \) corresponding with \([Y]_B \).

Next we consider the second case in (5). Observe that

\[
\begin{bmatrix}
q_0 \\
\xrightarrow{a} \\
\xrightarrow{b} \\
\xrightarrow{c} \q_1 q_2 \cdots q_j 
\end{bmatrix}_B + \begin{bmatrix}
q_0 \\
\xrightarrow{c} \\
\xrightarrow{a} \\
\xrightarrow{b} \q_1 q_2 \cdots q_j 
\end{bmatrix}_B \\
+ \begin{bmatrix}
q_0 \\
\xrightarrow{b} \\
\xrightarrow{c} \\
\xrightarrow{a} \q_1 q_2 \cdots q_j 
\end{bmatrix}_B \in \ker \Delta 
\]

by Proposition 5. If \( a, b, c \) are distinct, then we take only two of the terms in (7) to be edges of \( Q_n \). If \(|\{a, b, c\}| = 2 \) then it is easy to check that one of the terms of (7) is in \( \ker \Delta \) while the other two are negatives of one another modulo \( \ker \Delta \).
Therefore we take only one of the terms in (7) to be an edge of \( Q_n \) in this case. Finally, all three terms of (7) are in \( \ker \Delta \) when \( a = b = c \) so we introduce no edges in this case. As in the remark above, exchanging the children of the node labeled 2 of any of the terms of (7) results in the negative of that term, so we need not introduce edges for any of those terms.

**Remark 6.** By Proposition 4 two compositions \( q_0 q_1 \cdots q_i \) and \( r_0 r_1 \cdots r_j \) are in the same B-orbit if and only if \( q_1 \cdots q_i \) is a rearrangement of \( r_1 \cdots r_j \). Therefore each vertex of \( Q_n \) can be represented by a pair \( (q_0, q) \) where \( 1 \leq q_0 \leq n + 1 \) and \( q \) is a partition of \( n + 1 - q_0 \). Since we can recover \( q_0 \) from \( q \) we can drop \( q_0 \) from the notation and identify the vertices of \( Q_n \) with the partitions of the numbers \( 0, 1, \ldots, n \).

For concreteness we now specify the edges of \( Q_n \) in terms of their sources and destinations, which we identify with the partitions of \( 0, 1, \ldots, n \) as explained in Remark 6. Let \( p \) and \( q \) be vertices of \( Q_n \). Regarding a partition as an equivalence class of a composition under rearrangement, we will represent a partition by any convenient representative, which need not be non-decreasing.

(Q1) \( Q_n \) has an edge from \( p \) to \( q = q_1 q_2 \cdots q_i \) if \( p \) has parts \( b < c \) such that \( q \) can be obtained from \( p \) by deleting the parts \( b \) and \( c \). We take this edge to be \([q_0 \atop b \choose c]q_1 q_2 \cdots q_i \) where \( a = n + 1 - \sum_{i=1}^j q_i - b - c \).

(Q2) \( Q_n \) has two edges from \( p \) to \( q \) if \( p = abcq_1 q_2 \cdots q_i \) for some \( a < b < c \) such that \( q \) can be obtained from \( p \) by replacing \( a, b, c \) with \( a + b + c \). We take these edges to be \([q_0 \atop a \choose b \choose c]q_1 q_2 \cdots q_i \) and \([q_0 \atop b \choose a \choose c]q_1 q_2 \cdots q_i \).

(Q3) \( Q_n \) has one edge from \( p \) to \( q \) if \( p = aabq_1 q_2 \cdots q_i \) for some \( a \neq b \) such that \( q \) can be obtained from \( p \) by replacing \( a, a, b \) with \( 2a + b \). We take this edge to be \([q_0 \atop a \choose b \choose a]q_1 q_2 \cdots q_i \) if \( a < b \) or \([q_0 \atop a \choose b \choose a]q_1 q_2 \cdots q_i \) if \( a > b \) where \( q_0 = n + 1 - \sum_{i=1}^j q_i - 2a - b \).

Note that each edge of \( Q_n \) goes from a vertex with \( m \) parts to a vertex with \( m - 2 \) parts for some \( m \geq 2 \). Disregarding vertices not incident with any edges, it follows that \( Q_n \) has at least two connected components. For example, omitting the vertices \( 1^4, 1^5, 1^6, 2^2, 2^3, 3^2 \) the quiver \( Q_6 \) has two connected components, which are shown in Figure 1 and Figure 2. The full quiver \( Q_6 \) has 30 vertices corresponding with the 30 partitions of the numbers \( 0, 1, \ldots, 6 \).

**Lemma 7.** If \( p \) and \( q \) are vertices of \( Q_n \) then the images in \( kL_{n+1}/\ker \Delta \) of the edges from \( p \) to \( q \) are linearly independent.

**Proof.** It is easy to check that if \( Q_n \) has any edges from \( p \) to \( q \) then they must all be of the same type (Q1), (Q2), or (Q3). To prove the assertion it suffices to verify that the image of each edge of \( Q_n \) is nonzero, and in case (Q2) that the images of two edges with the same source and destination are linearly independent.

Let \( e \) be an edge of \( Q_n \). If \( e \) is of type (Q1) then \( \Delta(e) \neq 0 \) by (6). Otherwise suppose that \( e \) is of type (Q2) or (Q3). Then \( e = [q_0 \atop a \choose b \choose c]q_1 q_2 \cdots q_i \) for...
some $a, b, c, q_0, q_1, \ldots, q_j \in \mathbb{N}$ and

$$\Delta(e) = 2q_0\pi\left[\qquad\begin{array}{c}a \\ b \\ c \end{array}\right] q_1 q_2 \cdots q_j]_A = 2q_0(Y - Z)$$

by Proposition 5 where $Y$ is the sum of all rearrangements of $abcq_1 q_2 \cdots q_j$ in which $abc$ or $cba$ appear contiguously and $Z$ is the sum of all rearrangements of $abcq_1 q_2 \cdots q_j$ in which $acb$ or $bca$ appear contiguously. Assuming that $q_1 \leq q_2 \leq \cdots \leq q_j$ let $0 \leq i \leq j$ be such that

$$q_1 \leq \cdots \leq q_i \leq b < q_{i+1} \leq \cdots \leq q_j.$$ 

If $a \leq b$ then the letters of the term $q_1 \cdots q_i abcq_{i+1} \cdots q_k$ of $Y$ are non-decreasing from left to right except possibly at the segments $q_i a$ and $cq_{i+1}$. But all the terms of $Z$ contain deceasing segments $cb$ or $ca$, neither of which equals $q_i a$ or $cq_{i+1}$. This shows that $q_1 \cdots q_i abcq_{i+1} \cdots q_k$ cannot be canceled by a term of $Z$ so that $\Delta(e) \neq 0$.

In case $b < a$ we argue similarly that the term $q_1 \cdots q_i bcaq_{i+1} \cdots q_k$ of $Z$ has possible decreasing segments $ca$ and $aq_{i+1}$ and can thereby not be canceled by any term of $Y$, all of which have deceasing segments $ab$ or $cb$.

Finally, if $e$ is of type (Q2) with $a < b < c$ then let $f = \left[\begin{array}{c}q_0 \\ b \\ a \end{array}\right] q_1 q_2 \cdots q_j]_B$. Then the classes of $e$ and $f$ are linearly independent since every term of $\Delta(e)$ contains the segment $abc$ while at least one term of $\Delta(f)$ does not contain this segment. □
6. Admissibility and Right Alignment

A forest is called \textit{aligned} if each of its nodes $Z$ satisfies $\overline{Z}_1 < \overline{Z}_2$. Aligned forests play an important role in the quiver presentation of $\Sigma(\mathfrak{S}_n)$. Namely, the image of the anti-homomorphism $\iota$ consists precisely of the A-orbits of aligned forests. The property in type B corresponding with alignment is \textit{right alignment}, which we define for individual trees below. We will define right alignment for forests in §7.

The \textit{parity} of a tree $V$ of positive length is the pair

$$(\ell(V_1) \bmod 2, \ell(V_2) \bmod 2)$$

and is denoted by $p(V)$. We say that a tree $V$ is \textit{admissible} if no node of $V$ has parity $(1, 1)$. We say that $V$ is \textit{right aligned} if

1. no node of $V$ has parity $(1, 1)$ or $(1, 0)$,
2. each node $Z$ of $V$ of parity $(0, 0)$ satisfies $\overline{Z}_1 < \overline{Z}_2$, and
3. each node $Z$ of $V$ of parity $(0, 1)$ satisfies $\overline{Z}_1 < \overline{Z}_2$.

Since (3) implies that $\overline{Z}_1 \leq \overline{Z}_2 < \overline{Z}_2 + \overline{Z}_2 = \overline{Z}_2$, it follows that a right aligned tree is admissible and aligned. However, the converse need not hold, since an aligned admissible tree $V$ need not satisfy (3).

\textbf{Lemma 8.} Let $V$ be an unlabeled tree of odd length. Then there exist unlabeled trees $U_1, U_2, \ldots, U_p$ and integers $\alpha_1, \alpha_2, \ldots, \alpha_p$ such that $V \equiv \sum_{i=1}^p \alpha_i U_i \pmod{\ker \pi}$ where $U_{i1}, U_{i2}$ have even length and satisfy $\overline{U}_{i1} < \overline{U}_{i2}$ for all $1 \leq i \leq p$.

\textit{Proof.} We can assume that no node $Z$ of $V$ satisfies $Z_1 = Z_2 \in Z$ since otherwise $V \in \ker \pi$ and the assertion holds trivially. By replacing $V$ with $-\overrightarrow{V_2} \overrightarrow{V_1}$ if necessary, we can also assume that $V$ satisfies $\overrightarrow{V_1} \leq \overrightarrow{V_2}$. Now if $\ell(V) = 1$ then $V_1, V_2$ are leaves so that $V_1 < V_2$ by the assumptions above. Otherwise suppose that $\ell(V) \geq 3$ and that the assertion holds for trees of odd length less than $\ell(V)$. Observe that $\ell(V) = 1 + \ell(V_1) + \ell(V_2)$ so that $\ell(V_1), \ell(V_2)$ are both odd or both even. We consider these cases separately.

If $\ell(V_1), \ell(V_2)$ are both odd, then by applying induction to $V_1$ we have integers $\alpha_i$ and unlabeled trees $X_i, Y_i$ of even length such that $X_i < Y_i$ for all $i$ and $V_1 \equiv \sum_i \alpha_i \overrightarrow{X_i} \overrightarrow{Y_i}$. Then

$$V \equiv \sum_i -\alpha_i \left( \overrightarrow{V_2} \overrightarrow{X_i} \overrightarrow{Y_i} + \overrightarrow{Y_i} \overrightarrow{V_2} \overrightarrow{X_i} \right) = \sum_i \alpha_i \left( -\overrightarrow{Y_i} \overrightarrow{X_i} \overrightarrow{V_2} + \overrightarrow{X_i} \overrightarrow{Y_i} \overrightarrow{V_2} \right)$$

expresses $V$ as a linear combination of trees $U$ with $\ell(U_1), \ell(U_2)$ even and $\overrightarrow{U_1} < \overrightarrow{U_2}$.

Otherwise suppose that $\ell(V_1), \ell(V_2)$ are even. If $\overrightarrow{V_1} < \overrightarrow{V_2}$ then we have nothing to do. We assume therefore that $\overrightarrow{V_1} = \overrightarrow{V_2}$. By replacing $V$ with $-\overrightarrow{V_2} \overrightarrow{V_1}$ if necessary, we can also assume that $\ell(V_2) \geq 2$. Since $\ell(V_2)$ is even, one of its children has odd length, and again by replacing $V_2$ with $-\overrightarrow{V_2} \overrightarrow{V_1}$ if necessary, we can assume that $\ell(V_2)$ is odd. Applying induction to $V_2$, we can assume that $V_{221}$ and $V_{222}$ have even length. We denote $V_1, V_{21}, V_{221}, V_{222}$ by $A, B, C, D$ so that $V = \overrightarrow{D} \overrightarrow{C} \overrightarrow{B} \overrightarrow{A}$, where $A, B, C, D$ have even length and $\overrightarrow{A} = \overrightarrow{B} + \overrightarrow{C} + \overrightarrow{D}$. Then
by applying the Jacobi identity twice we find that

\[
\begin{array}{c}
\begin{array}{c}
A \\
B \\
C \\
D
\end{array}
\equiv - \begin{array}{c}
C \\
D \\
A \\
B
\end{array}
- B \\
\equiv - D \\
C \\
B \\
A
\end{array}
+ \begin{array}{c}
D \\
C \\
B \\
A
\end{array}
- B \\
\equiv - D \\
C \\
B \\
A
\end{array}
\]

expresses \( V \) as a linear combination of trees \( U \) with \( \ell(U_1), \ell(U_2) \) even and \( U_1 < U_2 \).

**Lemma 9.** Any unlabeled tree of even length is congruent modulo \( \ker \pi \) to a linear combination of right aligned trees.

**Proof.** Let \( V \) be an unlabeled tree of even length. If \( \ell(V) = 0 \) then we have nothing to do. Otherwise suppose that \( \ell(V) \geq 2 \). By replacing \( V \) with \( -V_2 \) if necessary, we can assume that \( \ell(V_1) \) is even and \( \ell(V_2) \) is odd. By applying Lemma 8 to \( V_2 \) we can assume that \( V_2 = A \overline{B} \) where \( \ell(A), \ell(B) \) are even and \( A < B \). Observe that

\[
V_1 \equiv B + A - B \overset{(8)}{=} - A \overset{(mod \ \ker \pi)}{\equiv} - B 
\]

so we can replace \( V \) with the right hand side of (8) in case \( V_1 > B \). Then applying induction to \( V_1, A, B \) expresses \( V \) as a linear combination of right aligned trees.

Finally, we will need a slightly stronger notion of right alignment in certain situations. A right aligned tree is called **strongly right aligned** if its nodes \( Z \) of parity \((0, 1)\) satisfy

1. \( Z_1 \neq Z_{21} \) unless \( Z_1 \) and \( Z_{21} \) are both leaves, and
2. \( Z_1 \neq Z_{22} \) unless \( Z_1 \) and \( Z_{22} \) are both leaves.

**Lemma 10.** Any unlabeled tree of even length is congruent modulo \( \ker \pi \) to a linear combination of strongly right aligned trees.

**Proof.** Let \( V \) be an unlabeled tree of even length. We can assume by Lemma 9 that \( V \) is right aligned. We consider separately the cases that \( V_1 = V_{21} \) and \( V_1 = V_{22} \).

Suppose that \( V_1 = V_{21} \). If \( V_1 \) and \( V_{21} \) are both leaves, then applying induction to \( V_{22} \) results in a linear combination of strongly right aligned trees. Otherwise one of \( V_1 \) or \( V_{21} \) has positive length. We consider these cases separately. In each case we will show that \( V \) can be expressed modulo \( \ker \pi \) as a linear combination of trees of the form \( \overrightarrow{RST} \) where \( R, S, T \) are distinct. The assertion then follows by applying induction to \( R, S, T \). To simplify the notation we will denote appropriate subtrees of \( V \) by \( A, B, C, D \) and write \((a, b, c, d) = (A, B, C, D)\).

1. Suppose that \( \ell(V_{21}) > 0 \). Then

\[
V = A \overrightarrow{BCD} \equiv - A \overrightarrow{CBD} + A \overrightarrow{BCD} \overset{(mod \ \ker \pi)}{=} - B \overrightarrow{ACD} 
\]
expresses $V$ in the required form since $c < a < b + d$ and $b < a < c + d$.

(2) Suppose that $\ell (V_1) > 0$. Then

$$V = \begin{array}{ccccc} & & & & \\
& A & B & C & D
\end{array} \equiv - \begin{array}{ccccc} & & & & \\
& A & C & D & B
\end{array} + \begin{array}{ccccc} & & & & \\
& B & C & D & A
\end{array} \pmod{\ker \pi}$$

expresses $V$ in the required form since $a < b < c + d$.

Suppose now that $V_1 = V_2$. If $V_1$ and $V_2$ are both leaves, then applying induction to $V_2$ results in a linear combination of strongly right aligned trees. Otherwise one of $V_1$ or $V_2$ has positive length. We consider these cases separately. In each case we will again show that $V$ can be expressed modulo $\ker \pi$ as a linear combination of trees of the form

$$\begin{array}{ccccc} & & & & \\
& R & S & T
\end{array}$$

where $R, S, T$ are distinct.

(3) If $\ell (V_1) > 0$ then again (2) expresses $V$ in the required form.

(4) Suppose that $\ell (V_2) > 0$. Then

$$V = \begin{array}{ccccc} & & & & \\
& A & B & C & D
\end{array} \equiv \begin{array}{ccccc} & & & & \\
& C & D & B & A
\end{array} + \begin{array}{ccccc} & & & & \\
& A & C & D & B
\end{array} \pmod{\ker \pi}$$

expresses $V$ in the required form for the following reasons. The first term can be expressed in the required form by (3). The second term satisfies $b < a + c$ and $d < a + c$. In case $b = d$ the second term can be expressed in the required form by (1) or (2). Similarly the third term satisfies $b < a + d$ and $c < a + d$. In case $b = c$ the third term can be expressed in the required form by (1) or (2).

\[\square\]

7. Right aligned forests

Next we define right alignment for forests.

**Definition 11.** Let $X = U V_1 \cdots V_j$ be an unlabeled forest where $U, V_1, \ldots, V_j$ are trees.

(1) We call $X$ even if each of the trees $V_1, \ldots, V_j$ has even length and each of the trees $U_2, U_1, \ldots, U_{j-1}$ has odd length, where $m$ is the depth of $X$.

(2) The nodes in positions $1, 1^2, \ldots, 1^{m-1}$ of $U$ are called the leftmost nodes of $X$.

(3) $X$ is called admissible if $X$ is even and no node of $X$ has parity $(1, 1)$.

(4) $X$ is called right aligned if $X$ is even and

(a) no node of $X$ has parity $(1, 1)$ or $(1, 0)$,

(b) each node $Z$ of $X$ of parity $(0, 0)$ satisfies $Z_1 < Z_2$, and

(c) each non-leftmost node $Z$ of $X$ of parity $(0, 1)$ satisfies $Z_1 \leq Z_2$.

(5) $X$ is called strongly right aligned if $X$ is right aligned and its non-leftmost nodes $Z$ of parity $(0, 1)$ satisfy

(a) $Z_1 \neq Z_2$ unless $Z_1$ and $Z_2$ are both leaves, and

(b) $Z_1 \neq Z_2$ unless $Z_1$ and $Z_2$ are both leaves.
Definition 12. A labeled forest $X$ is called admissible, right aligned, or strongly right aligned if $E(X)$ is admissible, right aligned, or strongly right aligned and in addition, the node label of each node of parity $(0,0)$ is one greater than the node label of its parent.

Proposition 13. Any labeled forest is equivalent modulo $\ker \Delta$ to a linear combination of strongly right aligned forests.

Proof. Let $X \in L_{n+1}$. We can assume that $X$ is even, since otherwise $X \in \ker \Delta$ by Proposition 5. Suppose that $E(X) = U V_1 \cdots V_j$ where $U, V_1, \ldots, V_j$ are unlabeled trees. Applying Lemma 10 to the nodes $V_1, V_2, \ldots, V_j$ and $U_{1,121}, U_{1,122}$ for all $i \geq 0$ after applying Lemma 8 to the nodes $U_{1,21}$ for all $i \geq 0$ expresses $E(X)$ modulo $\ker \pi$ as a linear combination of right aligned forests. Then any preimage under $E$ of this linear combination satisfying the labeling condition in Definition 12 is congruent to $X$ modulo $\ker \Delta$. For example, the map $F : M_{n+1} \to L_{n+1}$ defined in §10 provides such a preimage. \qed

8. The primary factorization

The purpose of this section is to introduce a mechanism for factorizing right aligned forests. We use it in inductive arguments in the following sections. Suppose that $X = X_0 X_1 \cdots X_j$ is a right aligned labeled forest where $X_0, X_1, \ldots, X_j$ are trees and let $0 \leq i \leq j$ be such that $X_i$ is the tree in $X$ with node label 1. We put

$$X' = \frac{x_0 x_1 \cdots x_{i-1}}{x_i} \frac{x_i \cdots x_j}{\underbrace{x_{i1} x_{i1} x_{i2} x_{i2}}}_1 \frac{x_{i1} \cdots x_j}{2} \frac{x_{i+1} \cdots x_j}{\underbrace{x_{i1} x_{i1} x_{i2} x_{i2}}}_2$$

where $x_0 x_1 \cdots x_{i-1} x_i x_{i1} x_{i1} x_{i2} x_{i2} x_{i+1} \cdots x_j = \overline{X''}$ where $X''$ is obtained from

$$X_0 X_1 \cdots X_{i-1} x_i x_{i1} x_{i1} x_{i2} x_{i2} x_{i+1} \cdots x_j$$

by reducing all the node labels by two. Then the factorization $X = X' \cdot X''$ is called the primary factorization of $X$.

Now if $X = X_0 X_1 \cdots X_j$ is a right aligned unlabeled forest, then the construction above can be carried out for every $0 \leq i \leq j$ for which $\ell(X_i) > 0$. We call the resulting factorization the primary factorization of $X$ with respect to $i$.

Iterating the primary factorization of a right aligned forest yields a factorization of the forest into a product of right aligned forests of length two. If the forest is unlabeled, then the factorization is not unique in general, since it depends on the choice of $i$ at every stage.

9. The product of edges of $Q_n$

Our aim in this section is to study the subalgebra of $kL_{n+1}$ generated by the edges of $Q_n$. This results in the formula in Lemma 14 for the product of various edges of $Q_n$. For an admissible labeled forest $X$ let $T_X$ be the set of non-leftmost subtrees of $X$ of even length. Consider the following transformations of $X$.

(P1) replacing $U \in T_X$ with its mirror image $\overline{U}$

(P2) exchanging $U$ and $V$, where $U, V \in T_X$ satisfy $\overline{U} = \overline{V}$ and the node labels of the parents of $U$ and $V$, if they exist, are smaller than the node labels of $U$ and $V$, if they exist.
(P3) Exchanging two non-leftmost trees of $X$, that is, exchanging two subtrees $U, V \in \mathcal{T}_X$ which are both in position $\emptyset$ of their trees.

We define an equivalence relation $\sim$ on admissible labeled forests by $X \sim Y$ if $Y$ can be obtained from $X$ by applying a sequence of moves (P1)–(P3). The condition on the node labels in move (P2) is meant to ensure that the resulting forest will also be a labeled forest. Moves (P1) and (P3) ensure that $\sim$ induces an equivalence relation on the $B$-orbits of admissible labeled forests. We also denote the induced relation on $\mathcal{L}_{n+1}$ by $\sim$. For example, the forests

\[
\begin{array}{c}
\begin{array}{c}
1 \ 2 \ 4 \ 5 \ 4 \ 1 \\
1 \ 2 \ 4 \ 5 \ 4 \ 1 \\
2 \ 1 \ 4 \ 5 \ 4 \ 1 \\
1 \ 2 \ 4 \ 5 \ 4 \ 1 \\
1 \ 2 \ 4 \ 5 \ 4 \ 1 \\
1 \ 2 \ 4 \ 5 \ 4 \ 1 \\
\end{array}
\end{array}
\]

are related by $\sim$. In fact, the forests shown in (9) represent all the distinct $B$-orbits of forests related by $\sim$ to any of the forests shown in (9).

**Lemma 14.** $e_1 \circ e_2 \circ \cdots \circ e_t = \sum_{[Y]_B - [X]_B} [Y]_B$ for any edges $e_1, e_2, \ldots, e_t$ of $Q_n$ where $X$ is any term of $e_1 \circ e_2 \circ \cdots \circ e_t$.

**Proof.** Observe that if $e_1 \circ e_2 \circ \cdots \circ e_t$ is nonzero, then it has a right aligned term, since $e_1, e_2, \ldots, e_t$ are represented by right aligned forests. We can therefore assume that $X$ is right aligned. The formula holds when $t = 1$ since $e_1 = [X]_B$ in this case. Otherwise suppose that $X' \circ X''$ is the primary factorization of $X$. Note that $X'$ is a term of $e_1$ and $X''$ is a term of $e_2 \circ \cdots \circ e_t$. Assuming by induction that $e_2 \circ \cdots \circ e_t = \sum_{[Z]_B - [X''']_B} [Z]_B$ we have

\[
(10) \quad e_1 \circ e_2 \circ \cdots \circ e_t = [X']_B \circ \sum_{[Z]_B - [X''']_B} [Z]_B.
\]

Note that all the terms $[Y]_B$ of (10) satisfy $[Y]_B \sim [X]_B$. Conversely, suppose that $[Y]_B$ is such that $[Y]_B \sim [X]_B$. We can assume that $Y$ can be obtained from $X$ by applying a single move (P1)–(P3) since $\sim$ is the reflexive and transitive closure of the set of all such pairs of forests. If the move exchanges the node labeled 1 with some other node, then both nodes must be in position $\emptyset$ of their trees, since the node labeled 1 can never be a proper subtree. It follows that $[X]_B = [Y]_B$. Similarly, if the move applies $\lleft\rright$ to the node labeled 1 then again $[X]_B = [Y]_B$. Otherwise the move involves nodes with labels greater than 2. It follows that $Y = X' \circ Z$ for some forest $Z$ satisfying $Z \sim X''$. This shows that $[Y]_B$ is a term of $e_1 \circ e_2 \circ \cdots \circ e_t$. \hfill $\square$

We can associate a path $P(X)$ to a right aligned labeled forest $X \in \mathcal{L}_{n+1}$ as follows. If $t(X) = 0$ then we define $P(X)$ to be the vertex $[X]_B$ of $Q_n$. Otherwise suppose that $X = X_0 X_1 \cdots X_j$ where $X_0, X_1, \ldots, X_j$ are trees and that $X_i$ is the tree with node label 1. We put

\[
P(X) = [X']_B \circ P(X'')
\]

where $X' \circ X''$ is the primary factorization of $X$. Note that $[X']_B$ is an edge of $Q_n$ by Definition 11. Here we denote the product in $kQ_n$ by $\circ$ in order to distinguish it from the product $\bullet$ in $k\mathcal{L}_{n+1}$. There is a natural anti-homomorphism $\iota: kQ_n \to k\mathcal{L}_{n+1}$ given by replacing $\circ$ with $\bullet$. Following is a reformulation of Lemma 14.
voiced in terms of \( t \) and \( P \). It follows with the observation that \( X \) is a term of \( t(P(X)) \).

**Corollary 15.** If \( X \) is a right aligned labeled forest, then \( t(P(X)) = \sum_{|Y|_B=|X|_A} [Y]_B \).

10. A TOTAL ORDER ON UNLABELED FORESTS

The purpose of this section is to develop an important component of the proof of the quiver in §11. This consists of defining a preferred preimage \( F(X) \in L_{n+1} \) under \( E \) of an unlabeled forest \( X \in M_{n+1} \) and a total order on unlabeled forests for which the forests in relation \( \sim \) with \( F(X) \) are smaller than \( X \) after applying \( E \).

The first step is to define a total order \(<\) on the set of admissible unlabeled trees of even length. Observe that if \( U \) is an admissible tree of even length, then one of its children \( U_1 \) or \( U_2 \) has even length and the other has odd length. We denote these trees by \( U_E \) and \( U_O \) respectively. Let \( U \) and \( V \) be admissible unlabeled trees of even length. We write \( U < V \) if one of the following conditions holds.

1. \( U < V \)
2. \( U = V \) and \( \ell(U) > \ell(V) \)
3. \( U = V \) and \( \ell(U) = \ell(V) \) and \( p(U) = (0, 1) \) and \( p(V) = (1, 0) \)
4. \( U = V \) and \( \ell(U) = \ell(V) \) and \( p(U) = p(V) \) and \( U_E < V_E \)
5. \( U = V \) and \( \ell(U) = \ell(V) \) and \( p(U) = p(V) \) and \( U_E = V_E \) and \( U_{O1} < V_{O1} \) and \( U_{O2} < V_{O2} \)

Note that in situations (4)-(6) the trees \( U_E, U_{O1}, U_{O2}, V_E, V_{O1}, V_{O2} \) have even length less than \( \ell(U) = \ell(V) \) and can therefore be compared by induction. The relation \(<\) is a total order on admissible unlabeled trees of even length.

The next step is to define the map \( F : M_{n+1} \rightarrow L_{n+1} \). Let \( X \in M_{n+1} \) be a right aligned unlabeled forest. If \( X \) has length zero, then \( X \) is also a labeled forest and we define \( F(X) = X \). If \( X \) has positive length, then suppose that \( X = X_0 X_1 \cdots X_j \) where \( X_0, X_1, \ldots, X_j \) are trees and let \( 0 \leq i \leq j \) be such that \( X_i \) is minimal with respect to \( < \) among the trees \( X_0, X_1, \ldots, X_j \) of positive length. We define

\[
F(X) = \begin{pmatrix}
X_0X_1\cdots X_{i-1} & & \\
& X_i & \\
& & X_{i+1}\cdots X_j
\end{pmatrix} \cdot F(X''')
\]

where

\[
X' = \begin{pmatrix}
X_0X_1\cdots X_{i-1} & & \\
& X_i & \\
& & X_{i+1}\cdots X_j
\end{pmatrix}
\]

and \( X'' \) is the primary factorization of \( X \) with respect to \( i \)

\[
x_0x_1\cdots x_{i-1}x_{i+1}x_{i+2}\cdots x_j = \overline{X^{(i)}}.
\]

Note that \( E(F(X)) = X \) by induction.

Now let \( X = X_0X_1 \cdots X_j \in M_{n+1} \) be an admissible unlabeled forest of depth \( m \) where \( X_0, X_1, \ldots, X_j \) are trees. Let \( T_{jm} \) be the set containing the positions of the trees \( X_1, X_2, \ldots, X_j \) in \( X \) and also containing the positions \( \{1121, 1122 | 0 \leq i \leq m - 1\} \) of the tree \( X_0 \). Then any admissible forest \( Y \) with \( j + 1 \) trees and depth \( m \) has a subtree in each of the positions in \( T_{jm} \). Suppose that \( \prec_X \) is a total order on \( T_{jm} \) compatible with the order \(<\) defined above in the sense that if \( U \) and \( V \) are subtrees
Figure 3. Example of the total order \(\prec_X\)

of \(X\) for which the position of \(U\) is smaller with respect to \(\prec_X\) than the position of \(V\) then \(U \prec V\). Then \(\prec_X\) induces a lexicographic order on the set of admissible unlabeled forests with \(j + 1\) trees and depth \(m\). Namely, one compares two such forests by comparing the subtrees in the positions in \(\mathcal{F}_{jm}\) with respect to \(<\) in the order specified by \(\prec_X\). This order is also denoted by \(\prec_X\).

**Example 16.** After applying the map \(E\) the forests shown in Figure 3 appear in increasing order according to \(\prec_X\) where \(X\) is the first forest shown. Note that there is only one possible lexicographic order \(\prec_X\) for the forest \(E(X)\) in this case. Here the node label \(0\) denotes ten. The sixteen forests shown in Figure 3 represent all the distinct \(B\)-orbits of forests related to \(X\) by \(\sim\). However, note that if we replace \(X\) with

\[
F(E(X)) = \begin{array}{c}
1 & 4 & 5 & 1 & 8 & 7 & 1 & 2 & 1 & 2 \\
1 & 4 & 5 & 1 & 8 & 7 & 1 & 2 & 1 & 2 \\
1 & 4 & 5 & 1 & 8 & 7 & 1 & 2 & 1 & 2 \\
1 & 4 & 5 & 1 & 8 & 7 & 1 & 2 & 1 & 2 \\
\end{array}
\]

then the only \(B\)-orbits related to \([X]_B\) are obtained by applying (P1) to the nodes labeled 7 or 9. Indeed, the purpose of the map \(F\) is to reduce the number of forests related to a given forest by \(\sim\).
Lemma 17. If \( X \in M_{n+1} \) is strongly right aligned and \( Z \in L_{n+1} \) is such that \( Z \sim F(X) \) but \( [Z]_B \neq [F(X)]_B \), then \( E(Z) \sim_X X \).

Proof. Let \( x_0x_1\cdots x_j = X \). Suppose that \( 2l = \ell(X) \) and let \( \{a_i, b_i, c_i \mid 1 \leq i \leq l\} \) be such that \( \begin{array}{c} a_i \overline{b_i} \overline{c_i} \\ \end{array} \) is the tree of positive length in some representative of the edge \( e_i \) for all \( 1 \leq i \leq l \) where \( e_1 \circ e_2 \circ \cdots \circ e_l = P(F(X)) \). Then any \( Z \in L_{n+1} \) such that \( Z \sim F(X) \) can be assembled from the trees

\[
\begin{array}{c}
x_0, x_1, \ldots, x_j, \begin{array}{c} a_1 \overline{b_1} \overline{c_1} \\ a_2 \overline{b_2} \overline{c_2} \\ \vdots \\ a_{2l-1} \overline{b_{2l-1}} \overline{c_{2l-1}} \end{array}
\end{array}
\]

by identifying a leaf equal to \( a_i + b_i + c_i \) in (11) with \( \begin{array}{c} a_i \overline{b_i} \overline{c_i} \\ a_i \overline{b_i} \overline{c_i} \\ \vdots \\ a_i \overline{b_i} \overline{c_i} \end{array} \) or its mirror image for all \( 1 \leq i \leq l \). This sequence of replacements defines an injective function \( \{1, 2, \ldots, l\} \to \{1, -1\} \times \{1, 2, \ldots, j + 3l\} \). Viewing \( F(X) \) and \( Z \) as functions in this way, the sequence of moves (P1)–(P3) transforming \( F(X) \) into \( Z \) is equivalent to an element of \( \mathcal{S}_2 \wr \mathcal{S}_{3l+1} \) which we view as a signed permutation of \( \{1, 2, \ldots, j + 3l\} \). Decomposing this permutation into a product of disjoint cycles, we find that each cycle permutes a set of subtrees of equal squash. Note that the set of trees permuted by such a cycle contains at most one leaf, since cycles containing more than one leaf can be further decomposed.

Since these cycles act on disjoint sets of subtrees, we can assume that the sequence of moves (P1)–(P3) transforming \( F(X) \) into \( Z \) is a single cycle permuting subtrees of the same squash, at most one of which being a leaf. Suppose that the cycle moves a subtree \( U \) of positive length to the position of a subtree \( V \). Note that \( U \) and \( V \) cannot both be in position \( \emptyset \) of their trees by the assumption that \( [Z]_B \neq [F(X)]_B \). Therefore, if \( V \) has no parent, then the squash of the tree containing \( U \) is strictly greater than \( V = \overline{U} \) so that the position of \( V \) is strictly smaller with respect to \( \prec_X \) than the position of the tree containing \( U \). If \( V \) has a parent, then the parent has a node label smaller than the node label of \( U \), so either \( V \) lies in a tree whose position is smaller with respect to \( \prec_X \) than the position of the tree containing \( U \), or else \( U \) and \( V \) are subtrees of the same tree and the parent of \( V \) has a smaller node label than the parent of \( U \). Here we use the fact that \( X \) is strongly right aligned, since otherwise it would be possible for \( U \) to be in position 1 and \( V \) in position 21 or 22 of some subtree, in which case the parent of \( V \) would have a larger node label than the parent of \( U \). In summary, the cycle moves each subtree of positive length to a tree whose position is smaller with respect to \( \prec_X \), or to a position in the same tree whose parent has a smaller node label than the parent of the node. We conclude that the cycle must contain a leaf that it replaces with a tree of positive length, resulting in a forest less than \( X \) with respect to \( \prec_X \) after applying \( E \). \( \square \)

11. Proof of the Quiver

Proposition 18. The map \( kQ_n \to k\mathcal{L}_{n+1}/\ker \Delta \) induced from \( \iota \) is surjective.

Proof. By Proposition 13 it suffices to show that [X]_B \equiv \iota(P) (mod \ker \Delta) for some \( P \in kQ_n \) where \( X \in L_{n+1} \) is any strongly right aligned labeled forest. First we
replace $X$ with $F(\mathbb{E}(X))$. Since this operation only relabels the nodes of $X$ it results in a forest equivalent to $X$ modulo $\ker\Delta$. Putting $P = P(X)$ we have $\iota(P) = \sum_{(Y)|B \sim (X)|B} \mathbb{E}(Y)$ by Corollary 15. Let $\mathbb{Y} = \iota(P) - \mathbb{X}_B$. We observe that under $\mathbb{E}$ all the terms of $\mathbb{Y}$ are smaller than $|\mathbb{E}(X)|_B$ with respect to $\prec_X$ by Lemma 17. We can repeat the argument for each term $\mathbb{Y}_B$ of $\mathbb{Y}$, taking the order $\prec_Y$ to be $\prec_X$ in each case. Then subtracting these results from $P$ results in an element of $kQ_n$ mapping to $[X]_B + \ker\Delta$ under $\iota$ by induction.

Theorem 19. $Q_n$ is the ordinary quiver of $\Sigma(W_n)$.

Proof. Let $I = \iota^{-1}(\ker\Delta)$ so that $kQ_n/I \cong \iota(kQ_n)/\ker\Delta$. But $\iota(kQ_n)/\ker\Delta = k\mathcal{C}_{n+1}/\ker\Delta$ by Proposition 18 and $k\mathcal{C}_{n+1}/\ker\Delta \cong \Sigma(W_n)^{\text{op}}$ by Theorem 1. Let $R$ be the Jacobson radical of $kQ_n$. Then $R$ is generated by all paths in $Q_n$ of positive length. Since $Q_n$ is the ordinary quiver of any quotient of $kQ_n$ by an ideal contained in $R^2$ by [1, Lemma 3.6] it suffices to show that $I \subseteq R^2$.

Let $P$ be any element of $I$. By multiplying $P$ on the left and on the right by various vertices of $Q_n$ we can split $P$ into a sum of elements of $I$ all of whose terms have the same source and destination. We can therefore assume that all the terms of $P$ have the same source and destination and hence the same length. If this length were zero or one, then $P$ would be a vertex or a linear combination of edges. But $\Delta(p) = p \neq 0$ for all vertices $p$ of $Q_n$ while no linear combination of edges can be in $\ker\Delta$ by Lemma 7. Therefore $P \in R^2$.

12. Examples

The quiver $Q_6$ is shown in Figure 1 and Figure 2. We observe that $Q_6$ has 30 vertices corresponding with the 30 partitions of the numbers 0, 1, ..., 6. Note that the vertices $1^4, 1^3, 1^3, 2^2, 2^1, 3^2$ are not shown, not being incident with any edges of $Q_6$. We count 28 paths of length one and 7 paths of length two so that $\dim(kQ_6) = 30 + 28 + 7 = 65$. Since $\Sigma(W_6)$ has dimension $2^6 = 64$ the presentation must have a single relation. We know that the paths in the relation must have the same source and destination since $\Delta$ is an anti-homomorphism. Then since the edges of $Q_6$ are linearly independent in $kQ_6/\iota^{-1}(\ker\Delta)$ by Lemma 7 the relation must be among paths of length two. The only possibility is that the relation is among paths going from 1122 to the empty partition. We calculate this relation directly as follows.

From the definition of $Q_n$ we write down the following forests.

$$
c = \begin{bmatrix} 1 & 1 & 5 \\
1 & 1 & 5 \\
\end{bmatrix}_B \hspace{1cm} d = \begin{bmatrix} 1 & 2 & 4 \\
1 & 2 & 4 \\
\end{bmatrix}_B
$$

$$
a = \begin{bmatrix} 1 & 1 & 2 \\
2 & 1 & 2 \\
\end{bmatrix}_B \hspace{1cm} b = \begin{bmatrix} 1 & 1 & 2 \\
1 & 1 & 2 \\
\end{bmatrix}_B
$$

Multiplying we have

$$
ac = \begin{bmatrix} 1 & 2 & 3 & 2 \\
1 & 2 & 3 & 2 \\
\end{bmatrix}_B \hspace{1cm} + \begin{bmatrix} 1 & 2 & 3 & 2 \\
1 & 2 & 3 & 2 \\
\end{bmatrix}_B
$$

$$
bd = \begin{bmatrix} 1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
\end{bmatrix}_B \hspace{1cm} + \begin{bmatrix} 1 & 2 & 3 & 1 \\
1 & 2 & 3 & 1 \\
\end{bmatrix}_B
$$
so that

\[
\Delta (ac) = 2\pi \left[ \begin{array}{ccc}
1 & 2 & 1 \\
2 & 2 & 1
\end{array} \right]_B + 2\pi \left[ \begin{array}{ccc}
1 & 2 & 1 \\
2 & 2 & 1
\end{array} \right]_B = 4\pi \left[ \begin{array}{ccc}
1 & 2 & 1 \\
2 & 2 & 1
\end{array} \right]_B
\]

\[
\Delta (bd) = 2\pi \left[ \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 1
\end{array} \right]_B + 2\pi \left[ \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 1
\end{array} \right]_B = 4\pi \left[ \begin{array}{ccc}
2 & 1 & 1 \\
1 & 1 & 1
\end{array} \right]_B
\]

by Proposition 5. We find that \( ac - bd \in \ker \Delta \) either by direct calculation, or by observing that

\[
0 = \pi \left( \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1
\end{array} + \begin{array}{ccc}
1 & 2 & 1 \\
2 & 2 & 1
\end{array} + 2 \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1
\end{array} \right) = \pi \left( \begin{array}{ccc}
1 & 2 & 1 \\
2 & 2 & 1
\end{array} - \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1
\end{array} \right)
\]

by the Jacobi relation. We conclude that \( \Sigma (W_6) \cong kQ_6/(ac - bd) \).

We can also calculate the presentation of \( \Sigma (W_7) \) by hand. The parts of \( Q_7 \) relevant to the following discussion are shown in Figure 4 and Figure 5. Namely, only those edges which contribute to paths of length at least two are shown. We have also eliminated any path sharing its source and destination with no other path. There conveniently remain exactly 26 edges which we can label \( a, b, \ldots, z \).
Calculating exactly as in the calculation for $\Sigma(W_6)$ we find the following relations.

$$2mt + px - nv - 2ry, \quad 2lt - 2mt - nv + px + 2ow,$$

be - cf, \quad ag - ch, \quad bi - cj, \quad qu - sz

13. The relations

In this section we state our conjecture about the generating set of the ideal of relations for the presentation of $\Sigma(W_n)$. This conjecture resulted from computer experiments with the rank of $W_n$ as large as computationally realizable.

In order to state the conjecture, we introduce the following monoid. Let $B$ be the set of symbols $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ c \\ d \end{pmatrix}$ for all $a, b, c, d \in \mathbb{N}$ with $b \leq d$ and $c < d$. We call the free monoid $B^*$ the branch monoid. The two different types of symbols in $B$ correspond with the two different types of edges of $Q_n$. Namely, the symbols $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ correspond with edges of type (Q1) while the symbols $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ correspond with edges of types (Q2) or (Q3) insofar as the possible entries $a, b, c$ of a symbol coincide with the possible leaves of the tree in the corresponding edge.

We can define an action of the branch monoid on $kQ_n$ as follows. If $P$ is a path in $Q_n$ with source $p = p_1p_2\cdots p_j$ where $p_1, p_2, \ldots, p_j \in \mathbb{N}$ then let $P.\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the path obtained from $P$ by appending the edge $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ on the left if $Q_n$ has such an edge, or put $P.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset$ otherwise. If $p$ has a part equal to $a + b + c$, say $p_1$, then let $P.\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the path obtained from $P$ by appending the edge $\begin{pmatrix} p_1 \\ a \\ b \\ c \end{pmatrix}$ on the left if $Q_n$ has such an edge, or put $P.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \emptyset$ otherwise. The algebra $kB^*$ acts on $kQ_n$ by extending the definitions above by linearity.

The branch monoid provides a convenient language for specifying paths in $Q_n$. Namely, we can uniquely specify any path $P$ as $pPB$ where $p$ is the destination of $P$ and $B$ is an element of $B^*$. There are also natural actions of $B^*$ on the various forest algebras. Although we have no use for these actions in this article, we remark that the notation and the name branch monoid are meant to reflect the fact that the action of $B^*$ on the forest algebras can be used to build forests, in the same way that its action on $kQ_n$ can be used to build paths.

Omitting the inner delimiters in the product of two or more elements of $B$ to simplify notation, we define the following classes of elements of $kQ_n$.

**B1** $\begin{pmatrix} a \\ b \\ f \end{pmatrix} - \begin{pmatrix} a \\ d \\ f \end{pmatrix}$ where $d + e + f \not\in \{b, c\}$

**B2** $\begin{pmatrix} a \\ b \\ f \end{pmatrix} - \begin{pmatrix} d \\ b \\ f \end{pmatrix}$ where $a + b + c \not\in \{d, e, f\}$ and $d + e + f \not\in \{a, b, c\}$

**B3** $\begin{pmatrix} a \\ d \\ g \\ h \\ i \end{pmatrix} + \begin{pmatrix} a \\ g \\ d \\ h \\ i \end{pmatrix} - \begin{pmatrix} a \\ g \\ d \\ e \\ h \\ i \end{pmatrix} - \begin{pmatrix} d \\ g \\ a \\ c \\ h \\ i \end{pmatrix}$

where $d + e + f = g + h + i \in \{b, c\}$ or $g + h + i \in \{b, c\} \cap \{d, e, f\}$
\( \langle a \quad d \quad g \rangle \langle b \quad c \quad h \rangle + \langle g \quad a \quad d \rangle \langle h \quad b \quad c \rangle - \langle a \quad g \quad d \rangle \langle h \quad b \quad e \rangle - \langle d \quad g \quad a \rangle \langle f \quad h \quad c \rangle \)

where \( a + b + c = d + e + f \in \{g, h, i\} \) or \( g + h + i \in \{a, b, c\} \cap \{d, e, f\} \).

It is easy to check that \( p.B \in \iota^{-1}(\ker(\Delta)) \) for all vertices \( p \) of \( kQ_n \) and all elements \( B \in k\mathbb{B}^* \) of the form (B1), (B2), (B3), or (B4).

Recall that if \( X \) is any labeled forest, then there exists a linear combination of strongly right aligned labeled forests congruent to \( X \) modulo \( \ker(\Delta) \) by Proposition 13. Such a linear combination is called a right aligned rendering of \( X \). Similarly, it follows from the proof of Proposition 13 that if \( X \) is any unlabeled forest, then the same procedure can be applied to \( X \) resulting in a linear combination of strongly right aligned unlabeled forests, any preimage under \( E \) of which is congruent modulo \( \ker(\Delta) \) to any preimage under \( E \) of \( X \). Such a linear combination is also called a right aligned rendering of \( X \). A right aligned rendering of a forest is not uniquely determined in general.

For aligned unlabeled trees \( X, Y, Z \) we denote \( X \bigotimes Y \bigotimes Z + Z \bigotimes X \bigotimes Y + Y \bigotimes Z \bigotimes X \) by \( j(X, Y, Z) \) to simplify the following definition. We define the following classes of elements of \( k\mathbb{M}_{n+1} \) where \( a, b, c, d, e, f, g, z, q_0, q_1, \ldots, q_j \) are any natural numbers.

\[
\begin{align*}
(\text{J1}) \quad & \left[ \begin{array}{c} z \bigotimes X \bigotimes Y \bigotimes \cdots \bigotimes q_j \\ a \bigotimes b \\
\end{array} \right]_B - \left[ \begin{array}{c} z \bigotimes Y \bigotimes X \bigotimes \cdots \bigotimes q_j \\ c \bigotimes d \\
\end{array} \right]_B - 2 \left[ \begin{array}{c} z \bigotimes X \bigotimes Y \bigotimes \cdots \bigotimes q_j \\ d \bigotimes e \bigotimes f \\
\end{array} \right]_B \\
\text{where } X = & \bigotimes c \quad d \\
\text{and } Y = & \bigotimes c \quad d \quad e \quad f \\
\text{or } \bigotimes a, b, c, d, e, f \end{align*}
\]

or \( \bigotimes a, b, c, d, e, f \).

\[
\begin{align*}
(\text{J2}) \quad & \left[ \begin{array}{c} z \bigotimes \cdots \bigotimes q_j \\ a \bigotimes b \bigotimes c \bigotimes d \bigotimes e \bigotimes f \bigotimes g \\
\end{array} \right]_B \\
\text{or } \bigotimes a, b, c, d, e, f \bigotimes g \end{align*}
\]

Note that any element of the form (J1), (J2), or (J3) is homogeneous of length two or three and that any preimage under \( E \) of such an element lies in \( \ker(\Delta) \).

**Conjecture 20.** The elements of the form \( p.B \) for all vertices \( p \) of \( kQ_n \) and all \( B \in k\mathbb{B}^* \) of the form (B1), (B2), (B3), or (B4) together with elements of \( k\mathbb{Q}_n \) mapping under \( E \) to right aligned renderings of all elements of the form (J1), (J2), or (J3) generate \( \iota^{-1}(\ker(\Delta)) \) as an ideal.

We have verified Conjecture 20 for all \( n \leq 16 \) through computer calculation using the GAP system [4].

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