A ONE DIMENSIONAL PROOF OF THE H"OLDER INEQUALITY USING THE HEAT EQUATION

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Abstract. This paper gives a proof of the Hölder Inequality by using supersolutions of the Heat Equation. The proof is based on a monotonicity formula for the heat equation presented in Tobias Colding’s lectures at MIT.

1. Introduction

In mathematics, the heat equation is a partial differential equation that models how heat diffuses through a region. A function $u$, solves the heat equation if

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t).$$

The fundamental solution of the heat equation is called the heat kernel. It is defined by

$$H(x, y, t) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x-y|^2}{4t}}.$$

We can prove that this is a solution by substituting in both sides of the equation and proving equality.

Any solution $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ of the heat equation (1.1) is determined by it’s initial values $u(\cdot, 0)$ by

$$u(x, t) = \int_{\mathbb{R}^n} H(x, y, t)u(y, 0)dy,$$

where $H$ is the heat kernel (1.2). We define $f$ and $g$ to be supersolutions of the heat equation when

$$\partial_t f \geq \Delta f, \partial_t g \geq \Delta g$$

In general, the heat equation has many applications in physics and mathematics. Brownian Motion, Black-Scholes equation, and Schrödinger’s equation are heat type equations. More recently, the heat equation has also been the basis for methods in machine learning. Graph Laplacian methods for example are based on the theory behind the heat equation.

The Hölder inequality is an inequality involving integrals named after the German mathematician Otto Hölder. It is a tool used in the study of Lebesgue spaces and can produce powerful results. Most notably it can be used to prove the triangle inequality. The standard
proof for the Hölder Inequality involves Young’s Inequality for Products which states that if \( p, q \in (1, \infty) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) then

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}
\]

for non-negative values of \( a \) and \( b \). Furthermore, equality is achieved when \( a^p = b^q \). Therefore, if we have positive functions \( f(x) \) and \( g(x) \) then

\[
|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}
\]

Now let \( X = \left( \int_S |f|^p d\mu \right)^\frac{1}{p} \) and let \( Y = \left( \int_S |g|^q d\mu \right)^\frac{1}{q} \). We now set \( a = \frac{f(x)}{X} \) and \( b = \frac{g(x)}{Y} \), therefore we have,

\[
\frac{|f(x)g(x)|}{XY} \leq \frac{|f(x)|^p}{pX^p} + \frac{|g(x)|^q}{qY^q}
\]

Integrating on both sides gives us,

\[
\frac{1}{XY} \int |f(x)g(x)| d\mu \leq \frac{1}{pX^p} \int |f|^p d\mu + \frac{1}{qY^q} \int |g|^q d\mu
\]

\( X^p \) and \( Y^q \) will be equal to \( \int f^p d\mu \) and \( \int g^q d\mu \) respectively. Therefore this inequality reduces to

\[
\int |f(x)g(x)| \leq \left( \int |f(x)|^p \right)^\frac{1}{p} \left( \int |g(x)|^q \right)^\frac{1}{q}
\]

This is the Hölder Inequality.

The main motivation behind this paper is to show an alternate proof of the Hölder Inequality involving the Heat Equation, and by extension will show us the relationship between these two seemingly unrelated topics. We will do this by using the supersolutions of the heat equation to prove the Hölder inequality. To do this we will start by proving a lemma; we will then proceed by defining the Central Limit Theorem, and will use both the results of the lemma and the Central Limit Theorem to prove the Hölder inequality.

### 2. Proof of the Hölder Inequality

**Lemma 2.1.** Let \( f \) and \( g \) be positive supersolutions of the heat equation. In other words, \((\partial_t - \Delta)f \geq 0\) and \((\partial_t - \Delta)g \geq 0\). If \( p, q \in (1, \infty) \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) then \((\partial_t - \Delta)(f^{\frac{1}{p}}g^{\frac{1}{q}}) \geq 0\), and equality implies \( f = cg \) for some \( c \in \mathbb{R} \).

**Proof.** Let’s define \( u \) to be a functions such that \( u = \log(f^{\frac{1}{p}}g^{\frac{1}{q}}) \). We take the exponential of both sides, thus giving us \( e^u = (f^{\frac{1}{p}}g^{\frac{1}{q}}) \). Therefore \((\partial_t - \Delta)(f^{\frac{1}{p}}g^{\frac{1}{q}}) = (\partial_t - \Delta)e^u \). Let’s now define a function \( e^{u(x,t)} \). If we take the partial of this function with respect to \( t \), we get \( \partial_t e^{u(x,t)} = e^{u(x,t)} \partial_t u(x,t) \). Now the Laplacian of this function will be given as, \( \Delta e^{u(x,t)} = \frac{\partial^2 e^{u(x,t)}}{\partial x^2} \). We can rewrite this using Chain Rule as \( \frac{\partial}{\partial x} \left( e^{u(x,t)} \frac{\partial u(x,t)}{\partial x} \right) \). We now apply product rule
giving us \( \frac{\partial}{\partial x} e^{u(x,t)} \frac{\partial u(x,t)}{\partial x} + e^{u(x,t)} \frac{\partial^2 u(x,t)}{\partial x^2} \). This can be simplified to \( e^{u(x,t)} \left( \frac{\partial u(x,t)}{\partial x} \right)^2 + e^{u(x,t)} \frac{\partial^2 u(x,t)}{\partial x^2} \).

We now can factor out the term \( e^{u(x,t)} \) thus giving us that

\[
\Delta e^{u(x,t)} = e^{u(x,t)} (\Delta u(x,t) + \left| \frac{\partial u(x,t)}{\partial x} \right|^2).
\]

Applying this method to our original expression, \((\partial_t - \Delta) e^u\), where \( u = \log(f^{\frac{1}{p}}g^{\frac{1}{q}}) \), we get

\[
(\partial_t - \Delta) \left( f^{\frac{1}{p}}g^{\frac{1}{q}} \right) = (\partial_t - \Delta) e^u = e^u((\partial_t - \Delta) u - \left| \frac{\partial u}{\partial x} \right|^2).
\]

We can expand this as

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \log(f^{\frac{1}{p}}g^{\frac{1}{q}}) = \frac{\partial}{\partial x} \left( \frac{1}{p} \log f + \frac{1}{q} \log g \right) = \frac{1}{p} \frac{\partial f}{\partial x} + \frac{1}{q} \frac{\partial g}{\partial x}.
\]

From that we know that \( \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} \log(f^{\frac{1}{p}}g^{\frac{1}{q}}) \). From here, we get that \( \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{1}{p} \frac{\partial f}{\partial x} + \frac{1}{q} \frac{\partial g}{\partial x} \right) \).

This is equal to \( \frac{1}{p} \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) + \frac{1}{q} \frac{\partial}{\partial y} \left( \frac{\partial g}{\partial y} \right) = \frac{1}{p} \left( \frac{\partial^2 f}{\partial x^2} - \frac{\partial f^2}{\partial f} \right) + \frac{1}{q} \left( \frac{\partial^2 g}{\partial y^2} - \frac{\partial g^2}{\partial g} \right) \). Now we compute

\[
\frac{\partial}{\partial x} (\log f^{\frac{1}{p}}g^{\frac{1}{q}}) = \frac{\partial}{\partial x} \left( \frac{1}{p} \log f + \frac{1}{q} \log g \right) = \frac{1}{p} \frac{\partial f}{\partial x} \log(f) + \frac{1}{q} \frac{\partial g}{\partial x} \log(g) = \frac{1}{p} \frac{\partial f}{\partial x} \left[ \log(f) \right] + \frac{1}{q} \frac{\partial g}{\partial x} \left[ \log(g) \right].
\]

Therefore we have,

\[
\Delta u = \frac{1}{p} \left( \frac{\Delta}{f} - \frac{|\nabla f|^2}{f^2} \right) + \frac{1}{q} \left( \frac{\Delta}{g} - \frac{|\nabla g|^2}{g^2} \right)
\]

and

\[
\partial_t u = \frac{\partial}{\partial t} \left[ \frac{f}{p} \right] + \frac{\partial}{\partial t} \left[ \frac{g}{q} \right]
\]

Now that we have \( \partial_t u \) and \( \Delta u \) by computation of terms we get

\[
(\partial_t - \Delta) u = \frac{1}{p} \left( \frac{(\partial_t - \Delta) f}{f} + \frac{|\nabla f|^2}{f^2} \right) + \frac{1}{q} \left( \frac{(\partial_t - \Delta) g}{g} + \frac{|\nabla g|^2}{g^2} \right)
\]

Therefore, since we know that \( f \) and \( g \) are supersolutions of the heat equation, we know that \( (\partial_t - \Delta) f = (\partial_t - \Delta) g \). Now from 2.3 we have \( (\partial_t - \Delta) \left( f^{\frac{1}{p}}g^{\frac{1}{q}} \right) = (\partial_t - \Delta) e^u = e^u((\partial_t - \Delta) u - \left| \frac{\partial u}{\partial x} \right|^2) \)

\[
\geq e^u(\partial_t - \Delta) u = \frac{1}{p} \left( \frac{(\partial_t - \Delta) f}{f} + \frac{|\nabla f|^2}{f^2} \right) + \frac{1}{q} \left( \frac{(\partial_t - \Delta) g}{g} + \frac{|\nabla g|^2}{g^2} \right) \geq 1 \frac{|\nabla f|^2}{f^2} + 1 \frac{|\nabla g|^2}{g^2} - \frac{1}{p} \frac{\nabla f}{f} + \frac{1}{q} \frac{\nabla g}{g}.
\]

Now by (2.3), we have \( (\partial_t - \Delta) e^u = e^u((\partial_t - \Delta) u - \left| \frac{\partial u}{\partial x} \right|^2) \geq e^u \frac{1}{p} \left( \frac{(\partial_t - \Delta) f}{f} + \frac{|\nabla f|^2}{f^2} \right) + \frac{1}{q} \left( \frac{(\partial_t - \Delta) g}{g} + \frac{|\nabla g|^2}{g^2} \right) \geq \frac{1}{p} \frac{|\nabla f|^2}{f^2} + \frac{1}{q} \frac{|\nabla g|^2}{g^2} - \frac{1}{p} \frac{\nabla f}{f} + \frac{1}{q} \frac{\nabla g}{g} \right) \). Thus we have, \( e^{-u}(\partial_t - \Delta) e^u = \frac{1}{pq} \left( \frac{|\nabla f|^2}{f^2} + \frac{|\nabla g|^2}{g^2} - \frac{2}{fg} (\nabla f, \nabla g) \right) \)

Thus, this expression reduces down to \( \frac{1}{pq} f^{\frac{1}{p}}g^{\frac{1}{q}} |\nabla \log(f^g)|^2 \). Since \( f \) and \( g \) are positive supersolutions of the heat equation, this expression will be \( \geq 0 \) and when equality does occur that implies that \( |\nabla \log(f^g)| = 0 \). Therefore, \( \log(f^g) = c \) so by taking the exponential on both sides we arrive to the expression \( f = e^c g \) when equality holds. \( \square \)
3. The Central Limit Theorem

The Central Limit Theorem tells us that if we have a function $u(x, t)$ which solves the heat equation and we define the function

$$v(x, t) = t^n u(\sqrt{t}x, t),$$

then $v$ converges to the Gaussian function

$$v(x, t) \to (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}} \int u_0(y) dy$$
as $t \to \infty$. We will use this property and Lemma 2.1 to prove the Hölder Inequality.

4. The Hölder Inequality

**Theorem 4.1.** Let $h_1$ and $h_2$ be functions with compact support. Let $\frac{1}{p} + \frac{1}{q} = 1$, where $(p, q) \in (1, \infty)$. The Hölder Inequality for integrals claims that

$$\int_R |h_1 h_2| \leq \left( \int_R |h_1|^p \right)^{\frac{1}{p}} \left( \int_R |h_2|^q \right)^{\frac{1}{q}}$$

and equality implies that $h_1 = ch_2$ for some $c \in \mathbb{R}$. Notice that the Hölder Inequality is a more general form of the Cauchy-Schwartz Inequality.

**Claim 4.3.** If $f$ and $g$ decay rapidly at infinity, then

$$\frac{d}{dt} \int_R f^\frac{1}{p} g^\frac{1}{q} \geq 0.$$ 

**Proof of Claim.** Since $f$ and $g$ decay at infinity, the divergence theorem and differentiation under the integral sign imply

$$\frac{d}{dt} \int_R f^\frac{1}{p} g^\frac{1}{q} = \int_R (\partial_t - \Delta)(f^\frac{1}{p} g^\frac{1}{q}).$$

We know that $(\partial_t - \Delta)(f^\frac{1}{p} g^\frac{1}{q}) = \frac{1}{pq} f^\frac{1}{p} g^\frac{1}{q} |\nabla \log(f_g)|^2$, thus by Lemma 2.1 we know that

$$\frac{1}{pq} f^\frac{1}{p} g^\frac{1}{q} |\nabla \log(f_g)|^2 \geq 0,$$

therefore proving that

$$\frac{d}{dt} \int_R f^\frac{1}{p} g^\frac{1}{q} \geq 0.$$ 

**Proof of Hölder inequality.** Let $f$ and $g$ be the solutions of the heat equation starting at $|h_1|^p$ and $|h_2|^q$, respectively. Then, let $u(x, t) = t^n f(\sqrt{t}x, t)$ and let $v(x, t) = t^n g(\sqrt{t}x, t)$. By applying the Central Limit Theorem, we have that $u(x, t)$ converges to

$$(4\pi)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4}} \int |h_1|^p$$
Similarly, \( v(x, t) \) converges to
\[
(4.8) \quad (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \int |h_2|^q.
\]
Since \( \int f_{\frac{1}{r}} g_{\frac{1}{q}} = \int u_{\frac{1}{p}} v_{\frac{1}{q}} \). Then from (4.6) and (4.7) we have as \( t \to \infty \)
\[
\int f_{\frac{1}{r}} g_{\frac{1}{q}} = \int u_{\frac{1}{p}} v_{\frac{1}{q}} \to \left( \int |h_1|^p \right)^{\frac{1}{p}} \left( \int |h_2|^q \right)^{\frac{1}{q}} \left( \int (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \right)^{\frac{1}{p} + \frac{1}{q}}
\]
We know that \( \frac{1}{p} + \frac{1}{q} = 1 \), so this simplifies into
\[
(4.10) \quad \int f_{\frac{1}{r}} g_{\frac{1}{q}} \to \left( \int |h_1|^p \right)^{\frac{1}{p}} \left( \int |h_2|^q \right)^{\frac{1}{q}}.
\]
From Claim 4.3 we know that \( \frac{d}{dt} \int_R f_{\frac{1}{r}} g_{\frac{1}{q}} \geq 0 \), therefore we have
\[
(4.11) \quad \int |h_1 h_2| dx \leq \left( \int |h_1|^p \right)^{\frac{1}{p}} \left( \int |h_2|^q \right)^{\frac{1}{q}},
\]
thus proving the H"older Inequality. \( \Box \)

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References

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