HARNACK INEQUALITY FOR A CLASS OF DEGENERATE
ELLiptic OPERATORS

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Abstract. We prove a Harnack inequality for a class of two-weight degenerate
elliptic operators. The metric distance is induced by continuous Grushin-type
vector fields. It is not known whether there exist cutoffs fitting the metric balls.
This obstacle is bypassed by means of a covering argument that allows the use
of rectangles in the Moser iteration.

1. Introduction

Perhaps inspired by David and Semmes’ work [5], Franchi, Gutierrez and Whee-
den proved in [10] a very deep generalization of the classical Sobolev-Poincaré in-
equality, unifying several other previous results. The importance of Sobolev-Poin-
caré-type inequalities to the study of elliptic equations has been well known for
decades [18]. In particular, the so-called Moser iteration technique [22] still
is the basis upon which are built more recent proofs of Harnack-type inequalities
for non-negative solutions of degenerate elliptic equations [1, 3, 6, 7, 13, 14, 15].

The main result in [10] thus paved the way for the proof of a more ge-
neral Harnack inequality. Indeed, in [11], Theorem II, the same authors stated a result
which has as particular cases the Harnack inequalities proven in [3] and [7]. As
they pointed out, that new version would apply to solutions of the equation

\[
\frac{\partial}{\partial x} \left( (|x|^\sigma + |y|) \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( (|x|^\sigma + |y|) \frac{\partial f}{\partial y} \right) = 0
\]

in an open set \( \Omega \subset \mathbb{R}^2 \) containing the origin, with \( \kappa \) and \( \sigma \) arbitrary positive
numbers. None of the other available results includes this example.

The proof of Theorem II in [11], however, is not complete. It depends on the (not
proven) existence of certain cut-off functions fitting the metric balls defined by the
operator. It is easy to construct (see our Proposition 14 below) cut-offs which are
identical to one or nonzero not on metric balls, but on certain “rectangles” which are
products of Euclidean balls with variable ratio of the radii. If one insists in using
balls contained or containing those rectangles, there remains a gap between the
two balls which provokes an explosion of the constants that appear in the iteration
process.

In this paper, we prove Theorem II of [11] without using cut-offs adapted to balls,
applying instead a covering technique, based on a theorem in [8], already used in
the study of degenerate parabolic equations by the first author [8]. The building
block of the Moser iteration used here turns out to be not exactly a Sobolev-
Poincaré inequality, but rather its consequence stated in Theorem 2, which is a
Sobolev-Poincaré inequality for rectangles, with the one on the right \( \epsilon \) times larger
than the one on the left and with a negative power of \( \epsilon \) on the right. The main
point of Section 4 is to show that a sequence \( \varepsilon_k \) can be chosen in such a way that the iteration converges. We show that the Moser-type iteration designed by Chanillo and Wheeden in [3] also works in this context. Propositions which are straightforward adaptations of results in [3] are stated here without proof.

We will assume as a hypothesis that the Sobolev-Poincaré inequality we need is true, without explicitly stating Franchi, Gutierrez and Wheeden’s Theorem I of [10], which is nonetheless our main motivation (since it provides the main example). One important aspect of that theorem is that it allows the presence of two (possibly non-comparable and non-Muckenhoupt) weights in the ellipticity condition.

The existence of cutoffs suitable to the study of regularity properties of weak solutions of degenerate elliptic equations has been independently proven by Franchi, Serapioni and Serra Cassano [14], and by Garofalo and Nhieu [17]. Their results would apply in our context, however, only if we required that the function \( \lambda \), defined in our Section 2, be Lipschitz continuous (for the operator in (1), the natural choice of \( \lambda \) would be \( \lambda(x) = |x|^{\sigma}, \sigma > 0 \)). Under this additional assumption, Theorem 1.3 in [17], or Proposition 2.9 in [14] (together with, for example, the composition argument in the proof of Theorem 1.5 in [17]), would imply the existence of the test functions needed for the proof of Theorem II in [11] to work.

A different approach was taken by Biroli and Mosco [1]. Within a very general framework, they proved the existence of cutoffs which satisfy, in stead of a pointwise estimate (as in [17], Theorem 1.5, for example), a weaker requirement, in integral form ([1], Proposition 3.3). That also suffices for the proof of Harnack-type inequalities (Theorem 1.1 in [1]; Theorem 1 in [15]). Working directly with the bilinear form defined by the elliptic operator, they did not have to to deal with the regularity of the vector fields usually used to define the metric.

2. Preliminaries and statement of the main result

The operators considered in this paper are of type

\[
Lf = \sum_{i,j=1}^{N} \frac{\partial}{\partial z_i} \left( a_{ij}(z) \frac{\partial f}{\partial z_j} \right),
\]

where \( z = (z_1, \ldots, z_N) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^N = \mathbb{R}_n \times \mathbb{R}_m \), the matrix \( A = (a_{ij}) \) is symmetric and the functions \( a_{ij} \) are real, measurable and satisfy the (degenerate) ellipticity condition

\[
v(z)(|\xi|^2 + \lambda(x)^2|\eta|^2) \leq \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \leq u(z)(|\xi|^2 + \lambda(x)^2|\eta|^2),
\]

for all \( \zeta = (\xi, \eta) \in \mathbb{R}_n \times \mathbb{R}_m \), with the functions \( \lambda, u \) and \( v \) non-negative and satisfying several hypotheses which are specified in what follows.

Throughout this paper, \( aB \) will denote, for \( a > 0 \) and \( B \) a ball in some metric space, another ball with the same center and \( a \)-times the radius as \( B \).

We require that the function \( \lambda \), defined on \( \mathbb{R}_n \), satisfy:

**H1:** It is non-negative, continuous, and vanishes possibly only on a set of isolated points.
**H2:** It is doubling with respect to the Euclidean metric and the Lebesgue measure, with doubling constant $C_1$; i.e.,
\[ \int_{2B_r} \lambda(x) dx \leq C_1 \int_{B_r} \lambda(x) dx, \]
for every Euclidean ball $B_r \subset \mathbb{R}^n$.

**H3:** There exists a constant $C_2$ such that
\[ \sup_{x \in B_r} \lambda(x) \leq C_2 \frac{1}{|B_r|} \int_{B_r} \lambda(x) dx, \]
for every Euclidean ball $B_r \subset \mathbb{R}^n$, with $|\cdot|$ denoting the Lebesgue measure.

**Definition 1.** Given $z_0 = (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^N$ and $r > 0$, we define
\[ \Lambda(z_0, r) = \sup_{\{x:|x-x_0|<r\}} \lambda(x) \]
and denote
\[ Q(z_0, r) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m; |x-x_0|<r, |y-y_0|<r\Lambda(z_0, r)\}. \]
If $Q = Q(z_0, r)$ and $t > 0$, $tQ$ will denote $Q(z_0, tr)$.

**Remark 2.** If follows from (H2) and (H3) that
\[ \Lambda(z_0, 2r) \leq C_1 C_2 \frac{1}{2^n} \Lambda(z_0, r) \]
for all $z_0 \in \mathbb{R}^N$ and all $r > 0$; and, hence, $C_1 C_2 \geq 2^n$ must hold.

**Lemma 3.** If $z \in Q(z_0, r)$ and $w \in Q(z, s)$, then $w \in Q(z_0, r+s)$.

**Definition 4.** An absolutely continuous curve in $\mathbb{R}^N$ is subunit if, for every $\zeta = (\xi, \eta) \in \mathbb{R}^N$ and for almost every $t$ in its domain, we have
\[ \langle \gamma'(t), \zeta \rangle^2 \leq |\xi|^2 + \lambda(\gamma(t))^2|\eta|^2, \]
with $\langle \cdot, \cdot \rangle$ denoting the usual inner-product of $\mathbb{R}^N$. Given $z$ and $w$ in $\mathbb{R}^N$, let $\rho(z, w)$ denote the infimum of all $T \geq 0$ such that there is a subunit curve joining the two points with domain $[0, T]$.

The function $\rho$ corresponds to the metric on $\mathbb{R}^N$ associated to the Grushin-type vector fields $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \lambda(x) \frac{\partial}{\partial y_1}, \cdots, \lambda(x) \frac{\partial}{\partial y_m}$ in a way which has by now become standard [12]. If $\lambda$ is smooth and does not vanish, one can see that $\rho$ is equal to the geodesic distance associated to the Riemannian metric $ds^2 = \sum_{i=1}^n dx_i^2 + \lambda(x)^{-2} \sum_{j=1}^m dy_j^2$.

An elementary proof of the following proposition can be given. For a somewhat different but closely related result, we refer to [9].

**Proposition 5.** The function $\rho$ above defines a metric on $\mathbb{R}^N$ and there exists a constant $b$, depending only on $n$ and $m$, such that the double inclusion
\[ Q(z_0, r/b) \subseteq B(z_0, r) \subseteq Q(z_0, br) \]
holds for every $z_0 \in \mathbb{R}^N$ and $r > 0$, where $B(z_0, r)$ denotes the ball with respect to this new metric with center $z_0$ and radius $r$.

**Remark 6.** Only (H1) is required for the proof of Proposition 5. As shown in Proposition 2.1.1 of [13], one may take $b = \max\{3, \sqrt{m}, \sqrt{n}\}$. Proposition 5 and (H1) imply that the metric $\rho$ induces in $\mathbb{R}^N$ its usual topology.
We require that $u$ and $v$ be weights on $\mathbb{R}^N$, (non-negative non-trivial locally integrable functions), which are doubling with respect to the $\rho$-metric and the Lebesgue measure, i.e., such that there are positive constants $C_3$ and $C_4$, with

$$\int_{2B} u(z) dz \leq C_3 \int_B u(z) dz \quad \text{and} \quad \int_{2B} v(z) dz \leq C_4 \int_B v(z) dz,$$

holding for all $\rho$-balls $B$. For every measurable $E \subseteq \mathbb{R}^N$, we will denote by $u(E)$ and $v(E)$ the integrals over $E$ of $u$ and $v$, respectively. Notice that \cite{2} and Proposition \cite{5} imply that $u(E)$ and $v(E)$ are positive if $E$ has non-empty interior.

For every locally integrable function $g$, we will denote by $m_E(g)$ the $u$-average $u(E)^{-1} \int_E gu$.

Last we state the strongest hypothesis we impose on $u$, $v$ and $\lambda$: that the following Sobolev-Poincaré inequality holds. For sufficient conditions for its validity see, for example, the papers \cite{2} \cite{7} \cite{9} \cite{10} \cite{12} \cite{16} \cite{21} \cite{25} and their references.

**SP:** There exist $q > 2$ and $C_5 > 0$, constants depending only on $u$, $v$, $\lambda$, $n$ and $m$, such that the inequality

$$\left[ \frac{1}{u(B)} \int_B |g(z) - m_B(g)|^q u(z) dz \right]^{\frac{1}{q}} \leq C_5 \left[ \frac{1}{v(B)} \int_B |\nabla g(z)|^2 v(z) dz \right]^{\frac{1}{2}},$$

holds for every Lipschitz continuous function $g$ and every ball $B$ with respect to the metric $\rho$ induced by $\lambda$, with $r$ denoting the radius of $B$, and $\nabla g$ denoting the vector field

$$\nabla g(z) = \left( \frac{\partial q}{\partial x_1}(z), \ldots, \frac{\partial q}{\partial x_n}(z), \lambda(x) \frac{\partial q}{\partial y_1}(z), \ldots, \lambda(x) \frac{\partial q}{\partial y_m}(z) \right).$$

Weak solutions of $Lf = 0$ in a bounded open set $\Omega \subset \mathbb{R}^N$ are defined (as in \cite{11}) in $H(\Omega)$, the completion of the space $\text{Lip}_p(\overline{\Omega})$ of the Lipschitz continuous functions on $\overline{\Omega}$, the closure of $\Omega$, with respect to the norm

$$||f||_H^2 = \sum_{i,j=1}^N \int_\Omega a_{ij}(z) \frac{\partial f}{\partial z_i}(z) \frac{\partial f}{\partial z_j}(z) dz + \int_\Omega f(z)^2 u(z) dz.$$

Using \cite{8} and \cite{6}, one can show, similarly as in \cite{3}, that the equation above indeed defines a norm. Moreover, if we denote by $H_\omega(\Omega)$ the closure in $H(\Omega)$ of the space $\text{Lip}_p(\Omega)$ of the Lipschitz continuous functions of compact support in $\Omega$, it can be proven, and for that (SP) is required, that the bilinear form $a_\omega$ on $\text{Lip}_p(\Omega)$

$$a_\omega(f,g) = \sum_{i,j=1}^N \int_\Omega a_{ij}(z) \frac{\partial f}{\partial z_i}(z) \frac{\partial g}{\partial z_j}(z) dz,$$

induces on $H_\omega(\Omega)$ an inner-product whose corresponding norm is equivalent to $|| \cdot ||_H$.

**Definition 7.** An element $f \in H(\Omega)$ is a weak solution of $Lf = 0$ if $a_\omega(f,\theta) = 0$ for all $\theta \in H_\omega(\Omega)$.

Applying Lax-Milgram’s Theorem, existence and uniqueness of a suitably defined weak version of the Dirichlet problem on $\Omega$ can be proven, in exactly the same way as in \cite{3}.

We still need two more definitions. The inequality $\int_\Omega f(x)^2 u(x) dx \leq ||f||_H^2$ follows from \cite{9} and the definition of $|| \cdot ||_H$. A natural mapping $H(\Omega) \rightarrow L^2(\Omega, u(z) dz)$,
Let $f \mapsto \tilde{f}$, is then defined. We stress we are not claiming that this is an injection, even though that could be proven under additional hypotheses. Finally, we will call an $f \in H(\Omega)$ non-negative, and denote this by $f \geq 0$, if there is a sequence of non-negative functions $f_k \in \text{Lip}(\Omega)$ converging to $f$ in $H(\Omega)$.

**Remark 8.** If $U \subset \Omega$ is open and $f \in H(\Omega)$ is a weak solution of $Lf = 0$ in $\Omega$, the restriction $f|_U \in H(U)$ is then a weak solution of $Lf = 0$ in $U$. We also have $\tilde{f}|_U = f|_U$.

We are ready to state our main result.

**Theorem 1.** Suppose that $\lambda$ satisfies (H1), (H2) and (H3), $u$ and $v$ are doubling weights and also that (SP) holds. Then there is a constant $K$, depending only on $C_1, C_2, C_3, C_4, C_5, q, m$ and $n$, such that, if $\Omega$ is a bounded open subset of $\mathbb{R}^N$ and $f \in H(\Omega)$ is a non-negative weak solution of $Lf = 0$, with $L$ satisfying $\text{a}$ and $\text{b}$, then

$$\text{ess sup}_B \tilde{f} \leq e^{K\mu} \text{ess inf}_B \tilde{f},$$

for every $\rho$-ball $B$ such that $2b^4B \subset \Omega$, where $\mu = u(B)^\frac{1}{2}v(B)^{-\frac{1}{2}}$.

3. **Application of a Covering Technique**

All hypotheses of Theorem 1 are assumed to be true for the rest of the paper, even if not explicitly. By a “constant” we will always mean a positive number which may depend only on the constants that arise in the hypotheses of Theorem 1. $C_1, C_2, C_3, C_4, C_5, q, m$ and $n$. We start with a Sobolev-Poincaré inequality for the rectangles $Q$ of Definition 1.

**Proposition 9.** There exists a constant $C_6$ such that

$$\frac{1}{u(Q)} \int_Q |g(z)|^p u(z)dz \leq C_6 r \left[ \frac{1}{v(Q)} \int_{\lambda Q} |\nabla g(z)|^2 u(z)dz \right]^\frac{1}{2}$$

$$+ \left[ \frac{1}{u(Q)} \int_{\lambda Q} g(z)^2 u(z)dz \right]^\frac{1}{2},$$

holds for every Lipschitz function $g$ and every $Q = Q(z, r)$, where $q > 2$ is the constant provided by (SP).

**Proof:** Using (5), we see that

$$\left[ \frac{1}{u(Q)} \int_Q |g(z) - m_{bB}(g)|^p u(z)dz \right]^\frac{1}{p}$$

is bounded by

$$\left[ \frac{u(bB)}{u(\frac{1}{b} B)} \frac{1}{u(bB)} \int_{bB} |g(z) - m_{bB}(g)|^p u(z)dz \right]^\frac{1}{p}.$$

Using that $u$ is doubling and the inequality (SP) for the ball $bB$, we get:

$$\left[ \frac{1}{u(Q)} \int_Q |g - m_{bB}(g)|^p u \right]^\frac{1}{p} \leq C_6 r \left[ \frac{1}{v(Q)} \int_{\lambda Q} |\nabla g|^2 v \right]^\frac{1}{2},$$

with $C_6 = bC_3^\frac{1}{2} C_5$, where $l$ is an integer such that $b^2 < 2^l$. 


To prove (12), we start by applying to $g = \|g - m_B(g)\| + m_B(g)$ the triangle inequality in $L^p(Q, u(z)dz)$, followed by (13), then by the Cauchy-Schwarz inequality for $L^2(Q, u(z)dz)$ and finally (14).

We will call a metric space homogeneous if it can be equipped with a Borel measure $\nu$ such that $\nu(2B) \leq D \nu(B)$ for every ball $B$, for some doubling-factor $D$. The following proposition is a particular case of Theorem 1.2 of [1].

**Proposition 10.** If $\{B(x, r)\}$ is a family of balls of constant radius covering a subset $E$ of a homogeneous metric space $X$, then there is a finite sub-family $\{B(x_i, r); i = 1, \cdots, m\}$ of disjoint balls such that $\{B(x_i, 4r); i = 1, \cdots, m\}$ still covers $E$.

**Proposition 11.** The metric space $(\mathbb{R}^N, \rho)$ is homogeneous.

Proof: Let $z_o = (x_o, y_o) \in \mathbb{R}^n \times \mathbb{R}^m$ and $r > 0$ be given. By (4), we have
\[
\Lambda(z_o, t) \leq C^l_7 \Lambda(z_o, t^{\frac{1}{2}})
\]
for every non-negative integer $l$ and every $t > 0$, with $C_7 = 2^{-n}C_1C_2$. Using Proposition 6 we then get
\[
|B(z_o, 2r)| \leq \omega_n \omega_m(2b)^N \Lambda(z_o, 2br)^m \leq C^m_7(2b^2)^N|Q(z_o, r/b)|,
\]
if $l$ is chosen so that $2b^2 \leq 4$, with $\omega_k$ denoting the volume of the unit ball in $\mathbb{R}^r$. Since $Q(z_o, r/b) \subseteq B(z_o, r)$, this shows that the Lebesgue measure is doubling with doubling-factor $C^m_7(2b^2)^N$.

**Proposition 12.** Given $z \in \mathbb{R}^N$ and $0 < r < s$, there exist $z_1, \cdots, z_p$ in $Q(z, s)$, such that the family $\{Q(z_1, r), \cdots, Q(z_p, r)\}$ covers $Q(z, s)$, with $Q(z_j, \frac{r}{4b^2})$ and $Q(z_k, \frac{r}{4b^2})$ disjoint when $j \neq k$. Moreover, there are constants $\beta$ and $C_8$ such that
\[
p \leq C_8 \left( \frac{s}{r} \right)^\beta.
\]

Proof: The first statement of this proposition follows straightforwardly from Proposition 5, Proposition 10 (with $\frac{r}{4b^2}$ replacing $r$) and Proposition 11. In order to prove (13), let us first remark that there is a constant $\beta$ such that the inequality
\[
|Q(w, \theta t)| \geq C^{-m} t^{-\beta} |Q(w, t)|
\]
holds for all $0 < \theta < 1$, $t > 0$ and $w \in \mathbb{R}^N$. Indeed, let $\beta$ be defined by $\beta = N + m \log C_7/\log 2$. Using $|Q(w, t)| = \omega_n \omega_m t^N A(w, t)^m$, we get (12) by applying (11) to the integer $l$ such that $\theta^2 < 2^{-l} \leq \theta$. It follows from Remark 2 that $C_7 \geq 1$ and thus $\beta$ is positive.

By Lemma 2 and since $s + \frac{r}{4b^2} < (b^2 + 1)s$, each $Q_j = Q(z_j, \frac{r}{4b^2})$ is contained in $Q(z, (b^2 + 1)s)$. Since the $Q_j$'s are mutually disjoint, we have:
\[
|Q(z, (b^2 + 1)s)| \geq \sum_{j=1}^p |Q(z_j, \frac{r}{4b^2})|.
\]
Now let us apply (13) to $w = z_j$, $t = (2b^2 + 1)s$ and $\theta = r/(8b^4 + 4b^2)s$. We get:
\[
|Q(z_j, \frac{r}{4b^2})| \geq \left( \frac{r}{8s} \right)^\beta \frac{|Q(z_j, (2b^2 + 1)s)|}{C^m_7(8b^4 + 4b^2)^\beta}.
\]
By (5b), \(z\) is in \(Q(z_j, b^2 s)\). Now Lemma 3 implies: \(Q(z_j, (2b^2 + 1)s) \supseteq Q(z, (b^2 + 1)s)\).

This, (14) and (15) together imply:

\[
|Q(z, (b^2 + 1)s)| \geq p \left( \frac{r}{s} \right)^\beta |Q(z, (b^2 + 1)s)| C_7^\beta (8b^4 + 4b^2)^\beta.
\]

This proves (12) with \(C_8 = C_7^{\beta} (8b^4 + 4b^2)^\beta\). □

**Lemma 13.** There are constants \(C_0\), and \(\gamma\) such that

\[
(16) \quad \frac{u(sQ)}{u(rQ)} \leq C_0 \left( \frac{s}{r} \right)^\gamma \quad \text{and} \quad \frac{v(sQ)}{v(rQ)} \leq C_0 \left( \frac{s}{r} \right)^\gamma
\]

for every “rectangle” \(Q\) and for every \(0 < r < s\).

**Proof:** It follows from (16) and (10) that, if \(l\) is an integer such that \(b^2 < 2^l\), then \(u(2Q) \leq C_3^{l+1} u(Q)\) and \(v(2Q) \leq C_3^{l+1} v(Q)\) for all \(Q\). Arguing similarly as for the proof of (13), we can get (16) with \(C_0 = \max\{C_3^{l+1}, C_4^{l+1}\}\), and \(\gamma = \log C_0 / \log 2\). □

The following theorem plays here the role of Theorem D in [5]. The explicit form of the constants in (17), valid for arbitrarily small \(\epsilon\), is needed for an efficient control of the constants that show up in the iteration process.

**Theorem 2.** Under the hypotheses of Theorem 1 there are constants \(\alpha\) and \(C_{10}\) such that the estimate

\[
(17) \quad \frac{\alpha}{C_{10}} \left[ \frac{1}{u(Q)} \int_Q |g(z)|^q u(z) dz \right] \leq 
\]

\[
\left[ \left( \frac{s^2}{v(Q)} \right)^{1/(1+\epsilon)Q} \int_{(1+\epsilon)Q} |\nabla \chi g(z)|^2 v(z) dz \right]^{\frac{1}{q}} + \left( \frac{1}{u(Q)} \int_{(1+\epsilon)Q} g(z)^2 u(z) dz \right)^{\frac{1}{q}}\]

holds for every \(Q = Q(z, s)\), for every \(0 < \epsilon < 1\), and for every Lipschitz continuous function \(g\), where \(q > 2\) is the constant provided by (SP).

**Proof:** Let us apply Proposition 12 with \(r = \epsilon s / b^2\) and let the \(Q_j\)’s then obtained be denoted by \(Q_j = Q(z_j, r)\), \(j = 1, \ldots, m\). By (9) we get:

\[
\int_Q |g(z)|^q u(z) dz \leq \sum_{j=1}^p u(Q_j) \left[ C_0 r \left( \frac{1}{v(Q_j)} \int_{Q_j} |\nabla \chi g(z)|^2 v(z) dz \right)^{\frac{1}{q}} + \left( \frac{1}{u(Q_j)} \int_{Q_j} g(z)^2 u(z) dz \right)^{\frac{1}{q}} \right]^q.
\]

By Lemma 3 we have \(b^2 Q_j \subseteq Q(z, s + b^2 r)\), and hence the integrals on \(b^2 Q_j\) inside the brackets in (18) may be replaced by integrals on \((1+\epsilon)Q\). We then estimate \(u(Q_j) / u(Q_j)\) and \(v(Q_j) / v(Q_j)\) using (10) and \(Q(z, s) \subseteq Q(z_j, (b^2 + 1)s)\) (which follows from Lemma 3 and Proposition 5). This way we see that the expression between brackets in (18) is bounded by the expression between brackets in (17) times \(C_0^\alpha\max\{C_0, 1\}^{(|b^2 + 1)s/r|}C_7^\beta\). Next we use that \(Q_j \subseteq 2Q_j\) (which follows from Lemma 3), to get \(u(Q_j) \leq C_9 u(Q_j)\) (by the proof of Lemma 13). After using (12), we finally get (17) with

\[
C_{10} = C_0 C_9^{\frac{2+\delta}{\delta}} \max\{C_0, 1\}^{\frac{q}{2}} (b^4 + b^2)^\frac{2+\delta}{\delta} b^2\beta
\]
and \(\alpha = \beta + q\gamma/2\).

4. Moser iteration and Harnack inequality

We start this section with the construction of the test functions adapted to rectangles mentioned in the Introduction.

**Proposition 14.** Given any \(z_0 \in \mathbb{R}^N\) and any \(0 < r_1 < r_2\), there is a smooth function \(\eta\) equal to one everywhere on \(Q(z_0, r_1)\), with support contained in \(Q(z_0, r_2)\), and such that \(0 \leq \eta(z) \leq 1\) and \(|\nabla \eta(z)| \leq C_{11}/(r_2 - r_1)\) for all \(z \in \mathbb{R}^N\), with \(C_{11}\) denoting the constant \(2\sqrt{N}\).

**Proof:** Choose \(\psi\) a smooth function on \(\mathbb{R}\) identical to one on \((-\infty, 0]\), with support contained in \((-\infty, 1)\), and such that \(0 \leq \psi(t) \leq 1\) and \(|\psi'(t)| \leq 2\) for all \(t \in \mathbb{R}\). Given \(z_0 = (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m\) and \(0 < r_1 < r_2\), define

\[
\eta(x, y) = \varphi \left( \frac{|x - x_0|}{r_2} \right) \varphi \left( \frac{|y - y_0|}{r_2 \lambda(z_0, r_2)} \right), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m,
\]

where \(\varphi(t) = \psi \left( \frac{r_2 - r_1}{r_2 - r_1} t \right)\). It is straightforward to check that this \(\eta\) does it. \(\square\)

**Definition 15.** An element \(f \in H(\Omega)\) is a weak subsolution of \(Lf = 0\) if \(a_\omega(f, \theta) \leq 0\) for all non-negative \(\theta \in H_0(\Omega)\).

**Definition 16.** Given \(M > 0\) and \(d \geq 1\), let the function \(H_{M,d}\) (continuously differentiable with bounded derivative) be defined by \(H_{M,d}(t) = t^d \) if \(t \in [0, M]\), and \(H_{M,d}(t) = M^d + dM^{d-1}(t - M)\) if \(t > M\).

**Proposition 17.** Let \(f \in H(Q), Q = Q(z_0, h),\) be a non-negative subsolution of \(Lf = 0\) and let \(f_k\) be a sequence of non-negative Lipschitz continuous functions on \(\overline{Q}\) converging to \(f\) in \(H(Q)\). Given \(\frac{1}{2} \leq s < t \leq 1\), \(M > 0\) and \(\beta \geq 1\), there are a subsequence \(f_{k_j}\) of \(f_k\) and a sequence \(\delta_j \geq 0\), \(\delta_j \to 0\), such that for all \(j\) we have:

\[
\int_{\Omega} |\nabla H_{M,d} \circ f_{k_j}|^2 v \leq \delta_j + \frac{4C_{11}^2}{(t-s)^2} \int_{\Omega} |f_{k_j} \cdot (H_{M,d}' \circ f_{k_j})|^2 u.
\]

**Proposition 18.** If \(f \in H(Q), Q = Q(z_0, h),\) is a non-negative subsolution of \(Lf = 0\), then the estimate

\[
\left( \text{ess sup}_{aQ} \tilde{f} \right)^p \leq \frac{C_{12}}{(1-a)\delta (\mu(Q))^{\frac{2}{p} - 1}} \frac{1}{u(Q)} \int_Q \tilde{f}^p u
\]

holds for every \(a \in \left(\frac{1}{2}, 1\right)\) and every \(p \geq 2\), with \(\delta\) and \(C_{12}\) denoting constants explicitly defined below (at the end of the proof), and \(u(Q) = u(Q)^{\frac{2}{p} - 1} v(Q)^{-\frac{1}{p}}\). (We recall that \(q\) arises in (SP).)
Proof: Given $\frac{1}{2} \leq s < t \leq 1$ and $d \geq 1$, let us first use Proposition 17 to extract a subsequence $f_{k_j}$ of a sequence $f_k$ of non-negative Lipschitz continuous functions on $\overline{Q}$ converging to $f$ in $H(Q)$ for which (14) is true. Then, let us apply (17) to the rectangle $sQ$, for some $\epsilon$ satisfying $(1 + \epsilon)s < t$ and for $g = H_{M,d} \circ f_{k_j}$. Then let us apply (19) with $(1 + \epsilon)s$ replacing $s$. Next, we use (16) with $t$ and $s$ replacing $s$ and $r$, respectively, and taking advantage of the fact that $1 < t/s \leq 2$. Finally, after using that $H_{M,d}(\varphi) \leq \varphi H'_{M,d}(\varphi)$ for all $\varphi \in \mathbb{R}$, we get

\begin{equation}
\left[ \frac{1}{u(sQ)} \int_{sQ} |H_{M,d} \circ f_{k_j}|^q u \right]^\frac{1}{q} \leq \frac{C_{10}^q - \frac{\mu}{s} \frac{sh}{\delta_j} + 2^q C_9^q C_{10}^q \epsilon - \frac{\mu}{s}}{v(sQ)}.
\end{equation}

Now we want to let $j$ first, and then $M$, go to infinity. We may suppose, passing to another subsequence if necessary, that $f_{k_j}$ converges to $\tilde{f}$ pointwise, almost everywhere with respect to the measure $u(z)dz$. Using Fatou’s Lemma on the left-hand side and Lebesgue’s convergence theorem on the right (again, this is the same argument as Chanillo and Wheeden’s, on page 1120 of [3]), one can see that it is legitimate to replace $f_{k_j}$ by $\tilde{f}$ in (21), and then $H_{M,d} \circ \tilde{f}$ by $\tilde{f}^d$ and $H'_{M,d} \circ \tilde{f}$ by $\tilde{f}^{d-1}$.

Since $\frac{1}{2} \leq s < (1 + \epsilon)s < t \leq 1$, then $s/t - (1 + \epsilon)s$ is greater than one. By [3], it follows that $\mu(sQ) \geq 1$. Hence, the “+1” inside the first pair of brackets at the right-hand side of the inequality (21) may be absorbed by the constant at its left, which will then be multiplied by two. Next we raise to the $\frac{1}{q}$-th power both sides of the inequality and change notation, writing $r = 2d$ and $q = 2\sigma$. After all that is taken into account, we will have deduced from (21) the estimate

\begin{equation}
\left[ \frac{1}{u(sQ)} \int_{sQ} \tilde{f}^r u \right]^\frac{1}{r} \leq \frac{C_{13}^q - A^q \mu(sQ) rs}{t - (1 + \epsilon)s} \left[ \frac{1}{u(tQ)} \int_{tQ} \tilde{f}^r u \right]^\frac{1}{r},
\end{equation}

for all $r \geq 2$, with $C_{13} = 2^{\frac{2q}{q-1}} C_9^q C_{10}^q C_{11}$ and $A = \frac{\mu}{s} q$.

Let $a \in [\frac{1}{2}, 1)$ and $p \geq 2$ be given and define $a_j = a + (1 - a)/(j + 1)$. For each non-negative integer $j$, let us apply (22) with $t = a_j$, $s = a_{j+1}$, $\epsilon = \epsilon_j = (a_{j+1} - a_{j+2})/a_{j+1}$ and $r = \sigma^j p$. Let us apply (22) again to the right-hand side of the inequality thus obtained, but with $t = a_{j-1}$, $s = a_j$, $\epsilon = \epsilon_{j-1}$ and $r = \sigma^{j-1} p$. By repeating this procedure, after $j + 1$ steps we will get:

\begin{equation}
\left[ \frac{1}{u(a_{j+1}Q)} \int_{a_{j+1}Q} \tilde{f}^{p\sigma^{j+1}} u \right]^\frac{1}{p\sigma^{j+1}} \leq \left\{ \prod_{k=0}^{j} \frac{C_{13}^q - A^q a_{k+1} Q p \sigma^k u_{a_k} u_{a_k}}{a_k - (1 + \epsilon_k) a_{k+1}} \right\}^\frac{1}{p\sigma^j} \left[ \frac{1}{u(Q)} \int_Q \tilde{f}^{p u} \right]^\frac{1}{p u}.
\end{equation}

Since $a < a_{j+1} < 2a$ for all $j$, it follows from Lemma 13 that the left-hand side of (23) is greater than or equal to

\begin{equation}
\frac{2^q C_9}{u(aQ)} \int_a Q \tilde{f}^{p\sigma^{j+1}} u \right]^\frac{1}{p\sigma^{j+1}}.
\end{equation}
which converges to $\text{ess sup}_{aQ} \tilde{f}$ as $j$ tends to infinity. On the right-hand side of (24) we may replace $\mu(a_{j+1}, Q)$ by $\sqrt{2/\sqrt{aQ}}$, due to Lemma 12. Hence, all we need is to find a precise estimate for the product

\[
(24) \quad \prod_{k=0}^{\infty} \left[ \frac{C_{13}\sqrt{2/\sqrt{aQ}} \mu(Q)}{\mu(Q)} \frac{a_{k+1}}{a_k}\right]^{\frac{2}{p-a}} \prod_{k=0}^{\infty} \left[ \frac{\epsilon_k a_{k+1}}{a_k - (1 + \epsilon_k) a_{k+1}} \right]^{\frac{2}{s+a}}.
\]

The first of these products equals $[C_{13}\sqrt{2/\sqrt{aQ}} \mu(Q)]^{\frac{2}{p-a}}$. The second expression between brackets is equal to the left side of (25).

Proposition 19. Let $f \in H(Q)$, $Q = Q(z, h)$, be a strictly positive $(f \geq \epsilon_0 > 0)$ solution of $Lf = 0$ and let $f_k, f_k \geq \epsilon_0$, be a sequence in $\text{Lip}(Q)$ converging to $f$ in $H(Q)$. Given $\frac{1}{2} \leq s < t \leq 1$ and $\beta \leq 1$, with $-1 \neq \beta \neq 0$, there is a subsequence $f_{k_j}$ of $f_k$ and a sequence of non-negative reals $\delta_j \to 0$ such that for all $j$ we have:

\[
(25) \quad \int_{sQ} |\nabla f_{k_j}^{1/2}|^2 v \leq \delta_j + \frac{(\beta + 1)^2}{\beta^2} \frac{C_{11}^2}{(t-s)^2 k^2} \int_{tQ} f_{k_j}^{\beta+1} u.
\]

A proof for Proposition 19 can be given following exactly the same steps as in the first half of the proof of Lemma (3.11) in [3], pages 1121 and 1122, making the adaptations already described after the statement of Proposition 11.

The proof of the following proposition follows the steps of Lemma (3.11) of [3], for $p < 0$ or $p \geq 2$. For $0 < p < 2$, we use a technique of Hardy and Littlewood, as in Lemma (3.17) of [20].

Proposition 20. If $f \in H(Q)$, $Q = Q(z, h)$, is a non-negative solution of $Lf = 0$, then the estimate

\[
(26) \quad \left(\text{ess sup}_{aQ} \tilde{f}\right)^p \leq \frac{C_{14}}{(1-a)^{\beta}} \left[1 + |p| \mu(Q)\right]^{1-p} \frac{1}{u(Q)} \int_{Q} \tilde{f}^p u
\]

holds for every $a \in (\frac{1}{2}, 1)$ and every $0 \neq p \in \mathbb{R}$, with $\delta$ and $\mu(Q)$ as defined in Proposition 16 and $C_{14}$ denoting the constant explicitly defined below, at the end of the proof.

Proof. We may suppose that $f \geq \epsilon_0 > 0$ and later let $\epsilon_0$ tend to zero, as long as we make sure that none of the constants depends on $\epsilon_0$. 

Given $\beta \leq 1$, $-1 \neq \beta \neq 0$, $\frac{1}{2} \leq s < t \leq 1$ and $\epsilon > 0$ such that $(1 + \epsilon)s < t$, we may combine (17) for $g = \int_{k_i}^{k_i+1}$ and (24), and then let $j$ go to infinity. Similarly as just before (22), with $r = \beta + 1$ and $\sigma = q/2$, we get:

\[
\left( \frac{C_{13}\epsilon^{-A}}{2} \right)^{\frac{1}{2}} \left[ \left\| \frac{|r|}{|r-1|} \cdot \frac{s\mu(s)}{t - (1 + \epsilon)s} + 1 \right\| \left[ \frac{1}{u(tQ)} \int_{tQ} \hat{f}^ru \right]^{\frac{1}{2}r} \right.
\]

for all $r \leq 2$, $0 \neq r \neq 1$.

Now let $a \in \left[ \frac{1}{2}, 1 \right)$ and $p < 0$ be given and let $a_j$ and $\epsilon_j$ be defined as in the proof of Proposition 18. For each integer $j$, let us then apply (27) with $r = \sigma^kp$, $t = a_k$, $s = a_{k+1}$ and $\epsilon = \epsilon_k$, for $k = 0, 1, \ldots, j$. Iterating the $j + 1$ inequalities just obtained and letting $j$ tend to infinity, similarly as before, we get:

\[
\text{ess sup}_aQ \hat{f}^{-1} \leq K_0 \left[ \frac{1}{u(Q)} \int_Q \hat{f}^ru \right]^{\frac{1}{|p|\sigma^k}},
\]

with

\[
K_0 = \prod_{k=0}^{\infty} \left[ \frac{C_{13}\epsilon^{-A}}{2} \left( \frac{\sigma^k|p|\mu(a_{k+1}Q)a_{k+1}}{\sigma^k|p| - 1 \cdot a_k - (1 + \epsilon_k)a_{k+1}} + 1 \right) \right]^{\frac{2}{|p|\sigma^k}}.
\]

Since at this point we are assuming $p < 0$, we have $|\sigma^k - 1| \geq 1$ for all $k \geq 0$. Taking into account also that $1 \leq \sigma^k$, that $1 \leq a_{k+1}/a_k - (1 + \epsilon_k)a_{k+1}$, that $\mu(a_{k+1}Q) \leq \sqrt{2\pi}C_{10} \mu(Q)$ and that $1 \leq \sqrt{2\pi}C_{9}$, the infinite product above is seen to be bounded by:

\[
K_0 \leq \prod_{k=0}^{\infty} \left[ \frac{C_{13}\sqrt{2\pi}C_{9}}{2} \left( 1 + |p|\mu(Q) \right) \frac{\sigma^k|p|\mu(a_{k+1}Q)a_{k+1}}{a_k - (1 + \epsilon_k)a_{k+1}} \right]^{\frac{2}{|p|\sigma^k}}.
\]

We may here use the estimates obtained at the end of the proof of Proposition 18 to conclude that (20) holds, if there we replace $C_{14}$ by $C_{12}$. It follows from Proposition 13 that the same is true for $p \geq 2$.

In case $0 < p < 2$, we have $\sigma^p$ tending to infinity, but smaller than two for some values of $j$. Let us first suppose that $\sigma^p \neq 1$, for every integer $k \geq 0$. Let then $l$ be the integer such that $\sigma^lp < 2 \leq \sigma^{-1}p$. We may iterate as before, but using (27) at the first $l + 1$ steps of the iteration and (22) after that. We get:

\[
\text{ess sup}_aQ \hat{f} \leq K_1 \left[ \frac{1}{u(Q)} \int_Q \hat{f}^ru \right]^{\frac{1}{p}},
\]

with

\[
K_1 = \prod_{k=0}^{l} \left[ \frac{C_{13}\epsilon^{-A}}{2} \left( \frac{\sigma^kp\mu(a_{k+1}Q)a_{k+1}}{\sigma^k|p| - 1 \cdot a_k - (1 + \epsilon_k)a_{k+1}} + 1 \right) \right]^{\frac{2}{|p|\sigma^k}} \prod_{k=l+1}^{\infty} \left[ \frac{C_{13}\epsilon^{-A}\sigma^kp\mu(a_{k+1}Q)a_{k+1}}{a_k - (1 + \epsilon_k)a_{k+1}} \right]^{\frac{2}{|p|\sigma^k}}.
\]

In order to get a good estimate for $K_1$, let us further suppose that $p = \sigma^j(\sigma+1)/2$, for some $j \in \mathbb{Z}$. Then it will hold that $|\sigma^p - 1| \geq (\sigma - 1)/(2\sigma)$, for every integer
We may proceed as we did for the other infinite products, using in addition that $1 < 2\sigma/(\sigma - 1)$, and prove that $I_\theta$ holds for these values of $p$, with $C_{14}$ replaced by $C_{15} \cdot (2\gamma C_9)^{\frac{\theta}{\theta - 1}}$.  

By Remark 25, we may apply the result we have just obtained with $\alpha Q$ replacing $Q$, for any $\alpha \in (0, 1)$. Given $\frac{1}{2} \leq \alpha' \leq \alpha \leq 1$ and $p$ belonging to $X = \{\sigma^j(\sigma + 1)/2; j \in \mathbb{Z}\}$, we get:

\begin{equation}
\tag{29}
(\text{ess sup}_{\alpha'Q} \hat{f})^p \leq \frac{C_{15} \cdot (2\gamma C_9)^{\frac{\theta}{\theta - 1}}}{(\alpha - \alpha')^\theta} [1 + p\mu(Q)]^\frac{\theta}{\theta - 1} \int_{\alpha Q} \hat{f}^p u, 
\end{equation}

where we have used Lemma 15 and 1 $\leq C_9$ in order to replace $\mu(\alpha Q)$ by $\mu(Q)$ inside the brackets.

Let us define

\[ I_p = \frac{[1 + p\mu(Q)]^{\frac{\theta}{\theta - 1}}}{u(Q)} \int_{\alpha Q} \hat{f}^p u, \quad p \in (0, 2), \]

and $E(\alpha) = \text{ess sup}_{\alpha Q} \hat{f}$. Given any $p \in (0, 2) \setminus X$, let $\overline{p} \in X$ be such that $\frac{1}{2} < p < \overline{p}$. By Lemma 26 and (29), we get

\begin{equation}
\tag{30}
E(\alpha')^{\overline{p}} \leq \frac{C_{16}}{(\alpha - \alpha')^\theta} E(\alpha)^{\overline{p} - p} I_p,
\end{equation}

with $C_{16} = C_{15} \cdot (2\gamma C_9)^{\frac{\theta^2 - 1}{\theta - 1}} \cdot \sigma^{\frac{\theta^2}{\theta - 1}}$.

Given $a \in [\frac{1}{2}, 1)$, let $\alpha_k$ be a strictly increasing sequence such that $\alpha_0 = a$, and $\lim \alpha_k < 1$. Let us take the logarithm of (30) and iterate, with $\alpha' = \alpha_k$ and $\alpha = \alpha_{k+1}$, $k = 0, 1, \cdots$. With $\theta = (\overline{p} - p)/\overline{p}$, we get

\[ \log E(a) \leq \frac{1}{\overline{p}} \sum_{k=0}^{\infty} \theta^k \log \frac{C_{16}}{(\alpha_{k+1} - \alpha_k)^\theta} \]

\begin{equation}
\tag{31}
+ \limsup_{k \to \infty} \theta^{k+1} \log E(\alpha_{k+1}) + \frac{1}{\overline{p}(1 - \theta)} \log I_p;
\end{equation}

noting that, since $C_{16} > 1$ and $\alpha_{k+1} - \alpha_k < 1/2$, the terms of the series in the above inequality are positive.

It follows from Proposition 18 for $p = 2$ that $E(\lim \alpha_k)$ is finite. Since $\theta < 1$, we then get $\limsup_{k \to \infty} \theta^{k+1} \log E(\alpha_{k+1}) = 0$. To estimate the sum in (31), we need to make a precise choice of $\alpha_k$. If we let

\[ \alpha_k = a + (1 - a) \sum_{j=1}^{k} \frac{j^2}{2} \sum_{j=1}^{k} j - 2, \quad k \geq 1, \]

we get $\alpha_{k+1} - \alpha_k \geq (1 - a)/[4(k+1)^2]$. Since $\overline{p}(1 - \theta) = p$, we get:

\[ p \log E(a) \leq \log \frac{C_{16}^{4\delta}}{(1 - a)^\theta} \sum_{k=0}^{\infty} \theta^k \log (k + 1)^{2\delta} + \log I_p. \]

Exponentiating both sides of this inequality and defining

\[ C_{14} = \max \left\{ C_{12}, C_{15}, 4^4 C_{16} \exp \left[ \sum_{k=0}^{\infty} \theta^k \log (k + 1)^{2\delta} \right] \right\} \]

finishes the proof. \qed
Proposition 21. Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, and let $f \in H(\Omega)$ be a positive weak solution of $Lf = 0$, bounded below by a positive number. Let $z_0 \in \Omega$ and $h > 0$ be such that $bB \subseteq \Omega$, where $B = B(z_0, h)$. For each $\alpha \in [\frac{1}{2}, 1)$, define $k(\alpha, f)$ by $\log k(\alpha, f) = m_{abB}(\log \hat{f})$. (See page 4.) Then there is a constant $C_{17}$ such that, if $z_0$ and $h$ are such that $b^2Q \subseteq \Omega$, where $Q = Q(z_0, h)$, then the inequality

$$u(\{x \in \alpha Q; |\log \frac{\hat{f}(x)}{k(\alpha, f)}| > t\}) \leq \frac{C_{17} \mu(Q) u(\alpha Q)}{(1 - \alpha)t}$$

holds for every $t > 0$ and every $\alpha \in [\frac{1}{2}, 1)$.

Proof: This proposition can be given a proof very similar to that of Lemma (3.13) of [3]. We are going to highlight a few points, referring to Chanillo and Wheeden’s article for more details.

Let $f_k$ denote a sequence of positive Lipschitz continuous functions, uniformly bounded away from zero, converging to $f$ in $H(\Omega)$. With the aid of the test function $\eta$ (built in Proposition 13—here we take $r_1 = \alpha h$ and $r_2 = h$), we can extract from $f_k$ a subsequence, which we will still denote by $f_k$, such that

$$\int_{\alpha Q} |\nabla \lambda(\log f_k)|^2 u \leq \frac{4C_{17}^2 \mu(Q)}{(1 - \alpha)^2} + \delta_k,$$

for some $\delta_k \to 0$.

With $q = \log f_k$, let us apply (14) with $q$ replaced by 2 (this is allowed by Hölder’s inequality) and $Q$ replaced by $\alpha Q$. Next, let us apply (33) with $Q$ replaced by $b^2Q$. Using also Lemma 22 we get:

$$\int_{\alpha Q} |\log (f_k) - m_{abB}(\log f_k)|^2 u \leq \frac{C_{17}^2}{(1 - \alpha)^2} \mu(Q) u(\alpha Q) + \delta'_k,$$

with $C_{17} = 2C_\epsilon C_\delta C_\eta b^{-2}$ and $\delta'_k \to 0$. Using that $f_k$ is uniformly bounded away from zero, one can see that the lim$_k$ of the left-hand side of (34) is equal to $\int_{\alpha Q} |\log \hat{f} - \log k(\alpha, f)|^2 u$. The Proposition now follows from Chebyshev’s and Cauchy-Schwarz’s inequalities.

The following lemma for $w \equiv 1$ is essentially Lemma 3 of [24], whose proof also works for the case of an arbitrary weight $w$.

Lemma 22. (Bombieri-Moser) Let $w$ be a (non-negative) weight on $\mathbb{R}^N$, and let $f$ be a bounded non-negative measurable function defined on a bounded measurable set $E$. Suppose there is a family $E_t$, $t \in (0, 1]$, of measurable sets with $w(E_t) > 0$ for all $t$, $E_1 = E$ and $E_s \subseteq E_t$ if $s < t$. Assume there are $\mu, c, d > 0$, such that

$$\text{ess sup}_{E_s} f^p \leq \frac{c}{(t-s)^d} \frac{1}{w(E_1)} \int_{E_t} f^p w,$$

for all $p$, $s$, and $t$ such that $0 < p < \mu^{-1}$ and $\frac{1}{2} \leq s \leq t \leq 1$; and

$$w(\{x \in E_1; \log f(x) > \tau\}) \leq \frac{C_\mu}{\tau} w(E_1),$$

for all $\tau > 0$. Then there exists $C > 0$, depending only on $c$, such that

$$\text{ess sup}_{E_\alpha} f \leq \exp \left( \frac{C_\mu}{(1 - \alpha)^{2d}} \right),$$

for all $\alpha \in [\frac{1}{2}, 1)$.
Proof of Theorem 1. We may suppose that \( \tilde{f} \) is bounded away from zero, otherwise we could add an \( \epsilon > 0 \) and later let \( \epsilon \to 0 \).

Let \( B = B(z_0, h) \) be such that \( 2b^3B \subseteq \Omega \) and let \( Q = Q(z_0, h) \). With \( w = u \) and \( E_t = \frac{d}{t}Q \), we are going to apply Lemma 22 to the functions \( \tilde{f}/k \) and \( k/\tilde{f} \), where \( k = \exp[m_{2B}^1(\log f)] \). Notice that \( \tilde{f} \) is bounded on \( E_1 \), since the closure of \( E_1 \) is contained in \( \Omega \), and we may then apply Proposition 18 with \( p = 2 \) for a rectangle slightly larger than \( E_1 \). Choosing, for example,

\[
c = \max\{4C_17\sqrt{2^7C_9}, 2^7C_9C_14(2^7+1)C_9 \sqrt{d} \},
\]

we can check that (35) and (36) with \( \epsilon > 0 \) wise we could add an \( \epsilon \) and see that (38) holds. We remark that \( 2^d \) is doubling, those two quantities are comparable.

Now let \( B = B(z_0, h) \) be such that \( 2b^4 \subseteq \Omega \). We may apply (38) for the rectangle \( bQ \). By (5) we thus have

\[
\text{ess sup}_{bQ} f \leq \exp(2C3^d \mu) \text{ess inf}_{bQ} \tilde{f}.
\]

This proves (3) with \( K = 2C3^d \) but with \( \mu = \mu(bQ) \) instead of \( \mu = u(B) \frac{1}{2} v(B)^{-\frac{1}{2}} \). Since \( u \) and \( v \) are doubling, those two quantities are comparable. \( \square \)

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