Entanglement of multiphoton polarization Fock states and their superpositions.

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Density matrices of pure multiphoton Fock polarization states and of arising from them reduced density matrices of mixed states are expressed in similar ways in terms of matrices of correlators defined as averaged products of equal numbers of creation and annihilation operators. Degree of entanglement of considered states is evaluated for various combinations of parameters of states and character of their reduction.

1. INTRODUCTION

There is a growing interest in the science of quantum information to multiparticle entanglement. It finds applications in quantum computing and quantum error correction [1, 2], as well as in quantum networks [3]. The latter includes, in particular, communication among many parties that is enhanced by shared multiparticle entanglement. The most promising recourse for establishing this type of entanglement are, of course, multi-phonon systems. Thus, there is a natural interest in studying entanglement properties of states of many photons. Recent experiments also shown that entanglement of up to ten systems. Thus, there is a natural interest in studying entanglement properties of states of many photons. Recent experiments also shown that entanglement of up to ten photons can be observed in a lab [4].

In this work we consider pure multiphoton polarization Fock states and their superpositions. We give a general definition of density matrices of such states as well as density matrices of mixed states arising from pure Fock states after their partial reduction over a series of photon variables. Elements of such density matrices are expressed in terms of correlators defined as averaged products of equal numbers of creation and annihilation operators. Parameters characterizing the degree of entanglement in such states are calculated, and their dependence is investigated on features of the original pure states and on the ways of their reduction.

2. DENSITY MATRICES

Let us consider an arbitrary pure state $|\Psi^{(n)}⟩$ of $n$ photons having identical frequencies and identical given propagation directions but distributed arbitrarily between two polarization modes, horizontal and vertical ones, $H$ and $V$. Two-mode polarization basis Fock states are states with given numbers of horizontally and vertically polarized photons $n_H$ and $n_V$ such that $n_H + n_V = n$

$$|\Psi_{n_H, n_V}⟩ = |n_H, n_V⟩ = \frac{a_H^{n_H} a_V^{n_V}}{\sqrt{n_H! n_V!}} |0⟩. \quad (1)$$

More general $n$-photon polarization states to be considered are superpositions of basis Fock states

$$|\Psi^{(n)}⟩ = \sum_{n_H=0}^{n} C_{n_H} |n_H, n_V⟩ |n_V = n - n_H⟩ \quad (2)$$

with $\sum_{n_H=0}^{n} |C_{n_H}|^2 = 1$. The wave functions of all states $|\Psi^{(n)}⟩$ depend on $n$ single-phonon variables $\sigma_i$, $\Psi^{(n)}(\{\sigma_i\}) = \langle \{\sigma_i\} |\Psi^{(n)}⟩$ and, explicitly, they are given by symmetrized products of $n$ single-phonon wave functions [3]. In the case of polarization modes the single-phonon wave functions in these products are $\psi_H(\sigma_i) = \delta_{\sigma_i, H}$ and $\psi_V(\sigma_j) = \delta_{\sigma_j, V}$. In the matrix representation $\psi_H(\sigma_i) = \left( \begin{array} {c} 1 \\ 0 \end{array} \right)$ and $\psi_V(\sigma_j) = \left( \begin{array} {c} 0 \\ 1 \end{array} \right)$. Direct products of such two-line columns $n$ form a basis of columns with $2^n$ elements (“rows”) and with different locations of a single unit in one of these “rows”. Written down in this basis explicitly the multiphoton wave function can be used for constructing the density matrix $\rho^{(n)}(\{\sigma_i\}, \{\sigma'_j\}) = \langle \{\sigma_i\} |\Psi^{(n)}⟩\Psi^{(n)}(\{\sigma'_j\})$. However at high values of the photon numbers $n$ this procedure is rather cumbersome to be reproduced explicitly. Fortunately, there is a much more compact representation of the multiphoton density matrices described below and used in this work for a further analysis. Though at any given $n$ correctness of the representations to be shown can be checked and confirmed directly by derivations via the multiphoton wave function described above.

Thus, for any two-mode multiphoton state $|\Psi^{(n)}⟩$ its $2^n \times 2^n$ density matrix can be presented symbolically in the following form

$$\rho^{(n)} = \frac{1}{n!} \left\{ \langle (a_H^{†} a_V^{†})^{n-k_2} (a_H^{†} a_V^{†})^{k_2} a_H^{n-k_1} a_V^{k_1} \rangle \right\} \quad (3)$$

with averaging understood as $\langle \cdots \rangle = \langle \Psi^{(n)}|\cdots|\Psi^{(n)}⟩$. Such mean products of the creation and annihilation operators can be referred to as correlators. The integers $k_1$ and $k_2$ (both $\geq 0$ and $\leq n$) in Eq. (3) numerate, correspondingly, columns and rows in the matrix. At any given values of $k_1$ and $k_2$ columns and rows repeat
themselves $C_{n}^{k_{1}2}$ times, where $C_{n}^{k} = n!/k!(n-k)!$ are the binomial coefficients.

The simplest examples are the density matrices of pure one-photon and two-photon polarization states

$$\rho^{(1)} = \begin{pmatrix} \langle a_H^1 a_H^1 \rangle & \langle a_H^1 a_H^2 \rangle \\ \langle a_H^2 a_H^1 \rangle & \langle a_H^2 a_H^2 \rangle \end{pmatrix}$$

and

$$\rho^{(2)} = \frac{1}{2} \begin{pmatrix} \langle a_H^2 a_H^2 \rangle & \langle a_H^2 a_H a_H a_H \rangle & \langle a_H^2 a_H a_H a_H \rangle \\ \langle a_H a_H^2 a_H \rangle & \langle a_H a_H^2 a_H a_H \rangle & \langle a_H a_H^2 a_H a_H \rangle \\ \langle a_H a_H^2 a_H \rangle & \langle a_H a_H^2 a_H a_H \rangle & \langle a_H a_H^2 a_H a_H \rangle \end{pmatrix},$$

and so on. The biphoton density matrix $\rho^{(2)}$ was written down by Klyshko [7] and used in Refs. [7–9]. Note however that the next step used for working with the density matrix [8] consisted in simple crossing out one of two coinciding rows and one of two coinciding columns. This reduces the 4th-order matrix to the 3-dimensional one, but it changes significantly features of the arising matrix. In particular, its trace becomes different from one in contrast to the density matrix [9]. Also it does not provide a correct transition to the so called coherence matrix of biphoton qutrits [9]. Indeed, the most general polarization biphoton state is qutrit, the state vector of which is

$$|\Psi^{(2)}\rangle = C_1 |2_H\rangle + C_2 |1_H, 1_V\rangle + C_3 |2_V\rangle$$

with $C_i$ being arbitrary complex constants obeying the normalization condition $\sum |C_i|^2 = 1$. The natural coherence matrix of this state is

$$\rho_{coh}^{(2)} = \begin{pmatrix} |C_1|^2 & C_1 C_2 & C_1 C_3 \\ C_2 C_1 & |C_2|^2 & C_2 C_3 \\ C_3 C_1 & C_3 C_2 & |C_3|^2 \end{pmatrix} = \begin{pmatrix} \langle a_H^2 a_H^2 \rangle & \frac{\langle a_H^2 a_H a_H a_H \rangle}{\sqrt{2}} & \frac{\langle a_H^2 a_H a_H a_H \rangle}{\sqrt{2}} \\ \frac{\langle a_H a_H^2 a_H \rangle}{\sqrt{2}} & \langle a_H a_H^2 a_H a_H \rangle & \frac{\langle a_H a_H^2 a_H a_H \rangle}{\sqrt{2}} \\ \frac{\langle a_H a_H^2 a_H \rangle}{\sqrt{2}} & \frac{\langle a_H a_H^2 a_H a_H \rangle}{\sqrt{2}} & \langle a_H a_H^2 a_H a_H \rangle \end{pmatrix}. (7)$$

Evidently, the last expression [10] does not coincide with that of Eq. [5] with, e.g., deleted the third column and third row. So, the procedure of crossing out repeated columns and rows can not be considered as mathematically correct. To make it correct, one has to make first the unitary transformation of the matrix [10], after which all elements in one of rows and one of columns in the $4 \times 4$ matrix turn zero. For the matrix [5] the required unitary transformation has the form

$$\rho^{(2)} \rightarrow \tilde{\rho}^{(2)} = U \rho^{(2)} U^\dagger$$

with

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

Only after this transformation the arising line and column with zero elements can be safely removed without changing general features of the original density matrix and providing the correct expression for the coherence matrix [4] [8]. In principle, similar transformations can be found also for density matrices of higher-order states, $n > 2$. But in the following discussion we will not use such transformations by keeping all the full $2^n$ dimensionality of the density matrices unchanged.

3. REDUCED DENSITY MATRICES

As known the degree of entanglement of pure quantum states is related directly to the degree of mixing of reduced state. The concept of reduced states arises when one represents a complicated pure states as if consisting of two parts. And then reduction is averaging over one of these two parts giving rise to possibly mixed state of the other part. In the simplest cases of $n = 2$ and $n = 3$ definitions of two parts are evident: these parts consist of two single-photon states in the case of biphotons, and they consist of a single-photon and two-photon states in the case of a pure three-photon original states. In the cases of states with large numbers of photons, $n \geq 4$, there are more than one ways of imagining how the original $n$-photon state can be divided for two parts. E.g. for $n = 4$ there are are two ways of the gedanken splitting this state for two parts: $4 = 2 + 2$ and $4 = 3 + 1$ [10]. Thus, in these cases one can speak about different degrees of entanglement corresponding to different ways of splitting the original state for two parts.

Mathematically, a standard way of reducing density matrices of pure states consists in using their wave-function representation $\rho^{(n)}(\{\sigma_i\}, \{\sigma'_i\}) = \Psi^{(n)}(\{\sigma_i\})\Psi^{(n)}(\{\sigma'_i\})$, equalizing one or several variables $\sigma_i = \sigma'_i$ and summing the product $\Psi^{(n)}(\Psi^{(n)}$ over the variable(s) $\sigma_m$. But the procedure is rather cumbersome for states with many photons and with all symmetry requirements to the multi-boson wave functions completely taken into account. Fortunately, the result of such calculations can be presented in a relatively simple form with elements of the reduced density matrices expressed in terms of correlators similar to those arising in the described above density matrices of pure states. By assuming that for an $n$-photon state we reduce the density matrix $\rho^{(n)}$ with respect to $n - m$ variables, we can write the following general expression for the resulting reduced $2^m$-order density matrix

$$\rho_{r}^{(m;n)} = \frac{(n - m)!}{n!} \left\{ \langle (a_H^{m-k_2})^2 (a_H^{k_2})^2 a_{H}^{m-k_1} a_{V}^{k_1} \rangle \right\}.$$  

(8)
with the previous definition of averaging in correlators \( \langle \ldots \rangle = |\Psi^{(n)}\rangle \ldots |\Psi^{(n)}\rangle \) and with the previous meaning of the integers \( k_1 \) and \( k_2 \) \((m \geq k_1,2 \geq 0)\) numerating columns and rows, which have to be repeated \( C_{m,k_1}^{k_2} \) times. Below are some examples of the reduced matrices.

The single-photon reduced density matrices of arbitrary pure \( n \)-photon states \( |\Psi^{(n)}\rangle \) arising at \( m = 1 \) have the form

\[
\rho^{(1; n)}_r = \frac{1}{n} \begin{pmatrix} \langle a_H^1 a_H \rangle & \langle a_H^1 a_V \rangle \\ \langle a_V^1 a_H \rangle & \langle a_V^1 a_V \rangle \end{pmatrix}.
\]

For basis Fock states \( |\Psi_{n_H,n_V}\rangle \) \(\text{(1)}\) these matrices are very simple

\[
\rho^{(1; n)}_r = \begin{pmatrix} n_H & 0 \\ 0 & n_V \end{pmatrix},
\]

and they correspond to the Schmidt entanglement parameter

\[
K(n, n_H) = \frac{1}{Tr[(\rho^{(1; n)}_r)^2]} = \frac{(n_H + n_V)^2}{n_H^2 + n_V^2} = \frac{n^2}{n^2 - 2n_Hn_V},
\]

(11)

where \( n = n_H + n_V \) and \( n \geq n_H \geq 0 \). In the case of even total number of photon \( n \), as a function of \( n_H \), the Schmidt parameter \( K \) achieves maximum at \( n_H = n_V = n/2 \) and \( K_{\text{max}} = 2 \). At other relations between of \( n_H \) and \( n_V \) the Schmidt parameter \( K \) is smaller than \( K_{\text{max}} \). In the cases of odd numbers of photons \( n \) the maximal values of the Schmidt parameter are achieved at \( n_H = \lfloor n/2 \rfloor \) and \( n_H = \lfloor n/2 \rfloor + 1 \), where the symbol \( \lfloor x \rfloor \) denotes in this case the integer closest to but smaller than \( x \). Maximal values of the Schmidt parameter in these cases are somewhat smaller than 2. The simplest example of the basis Fock state with odd \( n \) is that of three-photon states \( |1_H, 2_V\rangle \) and \( |2_H, 1_V\rangle \). In both cases Equation \(\text{(11)}\) gives \( K = 9/5 \) in agreement with the results of the work \( [12] \). The main conclusion from this brief analysis concerns achievable entanglement of \( n \)-photon basis Fock states with respect to division for subsystems of a single-photon and an \((n - 1)\)-photon states: entanglement of such states with respect to such division for subsystems does not exceed that occurring in the case of biphoton states, and the maximal entanglement with \( K = 2 \) or close to 2 is achieved in the states with maximally close numbers of horizontally and vertically polarized photons, \( n_H \) and \( n_V \).

The two-photon reduced density matrices of arbitrary pure \( n \)-photon states \( |\Psi^{(n)}\rangle \) arise in the cases of \( m = 2 \) and their general form is given by

\[
\rho^{(2; n)}_r = \frac{1}{n(n-1)} \times \begin{pmatrix}
\langle a_H^1 a_H^2 \rangle & \langle a_H^1 a_H a_V \rangle & \langle a_H^1 a_H a_{2V} \rangle & \langle a_H^1 a_{2H} a_V \rangle \\
\langle a_V^1 a_H^2 \rangle & \langle a_V^1 a_H a_V \rangle & \langle a_V^1 a_H a_{2V} \rangle & \langle a_V^1 a_{2H} a_V \rangle \\
\langle a_H^1 a_{2H} a_V \rangle & \langle a_H^1 a_{2H} a_{2V} \rangle & \langle a_{2H} a_{2H} a_V \rangle & \langle a_{2H} a_{2H} a_{2V} \rangle \\
\langle a_V^1 a_{2H} a_V \rangle & \langle a_V^1 a_{2H} a_{2V} \rangle & \langle a_{2H} a_{2H} a_V \rangle & \langle a_{2H} a_{2H} a_{2V} \rangle
d\end{pmatrix},
\]

(13)

etc.

Note at last that all the described matrices, both of pure and mixed states \( [3, 4, 6] \) and \( [3, 4, 12] \) obey the same important feature: their trace equals one.

For evaluating the degree of entanglement of multiphoton states \( |\Psi^{(n)}\rangle \) their reduced density matrices have to be diagonalized numerically after which the found eigenvalues \( \lambda_i^{(m; n)} \) can be used for finding the Schmidt entanglement parameter or the entropy of the reduced density matrices

\[
K = \frac{1}{\sum_i \lambda_i^2} \quad \text{and} \quad S_r = -\sum_i \lambda_i \log_2 \lambda_i.
\]

(14)

Before presenting specific results of calculations, it’s worth making a note concerning features of the described above density matrices and differences between their features in the cases of basis Fock states \( [1] \) and their superpositions \( [2] \). In the case of single basis Fock states their pure-state and reduced density matrices have many zeros. In fact, averaging over basis Fock states zeros all correlators containing products of creation and annihilation operators in one of two modes in different powers, e.g., such as \( \langle a_H^p a_{2H}^q \rangle \) with \( p \neq q \) and the same for the vertical-polarization mode. Owing to this, the density matrices of single Fock states turn out having a diagonal-block structure. The following Equation represents an example of such a diagonal-block second-order reduced density matrix \( [12] \) for the state \( |2_H, 2_V\rangle \) reduced with respect to two variables \( m = 2 \):

\[
\rho^{(2, 4)}_r = \begin{pmatrix}
1/6 & 0 & 0 & 0 \\
0 & 1/3 & 1/3 & 0 \\
0 & 1/3 & 1/3 & 0 \\
0 & 0 & 0 & 1/6
\end{pmatrix}
\]

(15)

In this matrix the diagonal blocks are located at the crossing of the first line and first column, at the crossing of the 2nd and 3rd lines with the 2nd and 3rd columns and at the crossing of the 4th line and 4th column. Each block gives only one non-zero eigenvalue, and they are equal to, correspondingly, \( 1/6, 2/3, \) and \( 1/6 \), which gives \( K = 2 \) in accordance with the result shown in Figure \( [1] \).

In a general case square blocks of the reduced density matrices \( [12] \) arise at crossings of the lines and columns with equal numbers of integers numerating lines and columns in \( [12] \), \( k_1 = k_2 \equiv k \), and the dimensionality of each such blocks is \( k \). All elements inside each block are equal to each other. Eigenvalues of the reduced density matrix can be calculated as non-zero eigenvalues of blocks. Owing to equality of elements inside each block, each block gives rise to only one nonzero eigenvalue of the reduced density matrix. All this simplifies significantly diagonalization of reduced density matrices in the case of single basis multiphoton Fock states \( [1] \). The situation appears to be absolutely different in the case of superpositions of basis of basis states \( [2] \). In this case the diagonal-block structure of matrices does not exist anymore and the reduced density matrices have to be diagonalized without any helping simplifications.
4. RESULTS

The results of calculations are presented in a series of pictures of Figures 1-6. The first of these pictures (Figure 1) corresponds to multiphoton states $|\Psi^{(n)}\rangle$ with the total number of photons $n$, where $n$ is taken even, and with equal numbers of photons with horizontal and vertical polarizations, $n_H = n_V = n/2$. The state is assumed to be imagined consisting of two parts with the same numbers of photons in each, $n/2$. The reduced density matrix of such subsystem is $\rho^{(2)}_{\frac{n}{2},\frac{n}{2}}$ ($m = n - m = n/2$ in notations of Equation 9). Its eigenvalues are $\lambda_i$ and the Schmidt entanglement parameter is determined by the first expression in Equation 9. In Figure 1 the Schmidt parameter is shown in its dependence on the total number of photons in the state $|\Psi^{(n)}\rangle$. As seen from the picture of Figure 1, in the considered case the Schmidt entanglement parameter and, hence, the degree of entanglement are monotonically growing function of the number of photons. In other words, multiphoton Fock states can have much higher resource of entanglement than usually considered biphoton states.

The same conclusion can be deduced from calculations of the entropy of reduced state $S_r$ defined by the second expression in Equations 1-3. For the same state as in the previous calculations the function $S_r(n)$ plotted in Figure 2 is seen to be monotonically growing and being very similar to the curve of Figure 1. This confirms the conclusion about growing degree of entanglement with the growing number of photons and confirms compatibility of the entropy and Schmidt parameter for characterization of the degree of entanglement.

The picture of Figure 3 describes dependencies of the Schmidt entanglement parameter $K$ on the relation between horizontally and vertically polarized photons in the Fock states with given total numbers of photons $n$: if the number of vertically polarized photons is $n_V = k \leq n$, the number of horizontally polarized photons is $n_H = n - k$, and the number $k$ varies along the horizontal axis in the picture of Figure 3. In this series of calculations the degree of reduction is taken to be as high as possible, $m = 1$, i.e., the reduced state is a single-photon one and its reduced density matrix is $\rho^{(1)}_{1,n}$ of Equation 3. As seen well from the pictures at all values of $n$ the Schmidt number $K$ and the degree of entanglement are maximal when the numbers of vertically and horizontally polarized photons in the state $|\Psi^{(n)}\rangle$ are equal ($k = n/2$) or maximally close to each other (in the case of odd $n$).

The picture of Figure 4 shows the dependence of the Schmidt entanglement parameter of the Fock state $|\Psi^{(n)}\rangle$ on $m/n$, i.e., on the ratio of number $m$ of variables remaining in the state after its reduction to the total number of photons (or their variables) $n$ in the original pure state. The picture shows clearly that entanglement of the state $|\Psi^{(n)}\rangle$ is maximal when it is considered as split for two parts with equal number of photons in each parts ($m/n = 0.5$).

The picture of Figure 5 shows the dependence of eigenvalues $\lambda_k$ on their numbers $k$ for the reduced density ma-

![Figure 1](image)

Figure 1: The calculated Schmidt entanglement parameter $K(n)$ for states $|\Psi^{(n)}\rangle = |n_H,n_V\rangle$ with even $n$, equal numbers of horizontally and vertically polarized photons, $n_H = n_V = n/2$, and with the gedanken splitting of the states for two multiphoton states with equal numbers of photons, $m = (n - m) = n/2$. The picture shows clearly that entanglement of photon states is maximal when they are considered as split for two parts with equal number of photons in each parts.

![Figure 2](image)

Figure 2: The same as in Figure 1 but for the entropy of the reduced state rather than for the Schmidt entanglement parameter.

![Figure 3](image)

Figure 3: The Schmidt number $K$ as a function of the number $k$ of vertically polarized photons in the state $|\Psi^{(n)}\rangle$, $n \geq k \geq 0$; values of $n$ are indicated by colored lines in the right column.
Let us consider now an example of states more complicated than a single basis Fock state. Let the state under consideration be given by

\[ |\Psi\rangle = \sum_{m=1}^{n} C_m |(n-m)_H, m_V\rangle. \quad (16)\]

Figure 5: Located in the diminishing order, eigenvalues \( \lambda_k \) of the reduced density matrices of the state with 120 photons and \( n_H = n_V = 60 \) for a series of different degrees of reduction determined by the imagined splittings of the state \( |\Psi_{n_H,n_V}\rangle \) \( \Box \) for pairs of states indicated in the right column.

The results shown in Figure 5 show that in spite of a growing degree of entanglement in strongly multiphoton states, eigenvalues of all reduced density matrices remain concentrated in a restricted region of not too high values. This means that the effective dimensionality of the corresponding Hilbert spaces remains not too high. This conclusion is important for approximate numerical calculations because it opens a possibility of performing these calculations in smaller- dimensionality matrices forming the main cores for finding relatively large eigenvalues \( \lambda_k \).

Let us consider now an example of states more complicated than a single basis Fock state. Let the state under consideration be given by

\[ |\Psi\rangle = \sum_{m=1}^{n} C_m |(n-m)_H, m_V\rangle. \quad (16)\]

Let us take the coefficients \( C_m \) in the Gaussian form

\[ C_m = N \exp \left(-\frac{(m-m_0)^2}{2\sigma^2}\right). \quad (17)\]

with the normalization factor \( N \) given by

\[ N = \left[ \sum_{m=0}^{n} \exp \left(-\frac{(m-m_0)^2}{\sigma^2}\right) \right]^{-1/2} \quad (18)\]

and \( m_0 \) is that value of \( m \) at which the squared coefficients \( |C_m|^2 \) are maximal. As mentioned above in this case diagonalization of the reduced density matrix is more complicated because this matrix does not have anymore a diagonal-block structure, and it has to be diagonalized as a whole, without any simplifications. Nevertheless, the results of such calculations are presented in Figure 6 for three different values of the parameter \( m_0 \) in the Gaussian distribution of Equation (17).

Figure 6: The Schmidt entanglement parameter \( K \) for the state \( \Box \) with \( n = 6 \) and \( m_0 = 3, 2, 1, 0 \) (from top to down at small values of \( \sigma \)).

One of the most interesting features of the curves in Figure 6 concerns disappearance of entanglement (\( K = 1 \)) at some definite point \( \sigma_0 \). In principle, this does not contradict, e.g., to the known features of the simplest superposition of Fock states - biphoton polarization qutrit \( \Box \) characterized by three constants \( C_1, C_2, C_3 \). As known \( \Box \), its degree of entanglement can be characterized either by the Schmidt entanglement parameter \( K \) or by the so called concurrence \( C = |2C_1C_3 - C_2^2| \) \( \Box \), which are related to each other by a simple formula

\[ C = \sqrt{2(1-K^{-1})}. \]

It’s known also that entanglement of qutrit disappears when \( C = 0 \) or \( 2C_1C_3 = C_2^2 \). This effect of disappearing entanglement at some specific relation between the qutrit’s parameter seems to be analogous to the effect of missing entanglement of the state \( \Box \) at \( \sigma = \sigma_0 \).

5. CONCLUSION

In this paper we described and exploited for calculations the method of constructing density matrices of both
pure and mixed (reduced) multiphoton quantum states. In this method, elements of all density matrices are expressed in terms of mean values of products of photon creation and annihilation operators. The method is applied to both single basis Fock states and superpositions of basis Fock states with the same given total numbers of photons. The degree of entanglement is evaluated and the conditions are identified when the resource if entanglement accumulated in multiphoton states is maximal. As the next step, it would be nice learning to extend the used method for superpositions of quantum states with different total numbers of photons. It’s early yet making any prediction of the possibility of such extension. But we hope to return to this problem elsewhere.

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