LINEABILITY AND UNIFORMLY DOMINATED SETS OF SUMMING NONLINEAR OPERATORS

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Abstract. In this note we prove an abstract version of a result from 2002 due to Delgado and Piñero on absolutely summing operators. Several applications are presented; some of them in the multilinear framework and some in a completely nonlinear setting. In a final section we investigate the size of the set of non uniformly dominated sets of linear operators under the point of view of lineability.

1. Introduction

Let \( X,Y \) be Banach spaces over a fixed scalar field \( K = \mathbb{R} \) or \( \mathbb{C} \). By \( B_X = \{ x \in X : \| x \| \leq 1 \} \) we shall denote its closed unit ball and \( X^* \) shall be the topological dual of \( X \). If \( 1 \leq p < \infty \), a linear operator \( T : X \to Y \) is said to be absolutely \( p \)-summing if there exists a constant \( C_T \geq 0 \) such that

\[
\left( \sum_{i=1}^{n} \| T(x_i) \|^p \right)^{1/p} \leq C_T \sup_{\varphi \in B_{X^*}} \left( \sum_{i=1}^{n} |\varphi(x_i)|^p \right)^{1/p}
\]

for every finite set \( \{ x_1, ..., x_n \} \subset X \). For details, we refer to the classical book by Diestel, Jarchow, Tonge [13].

The class of absolutely \( p \)-summing linear operators from \( X \) to \( Y \) will be represented, as it is usual, by \( \Pi^p(X,Y) \) and the infimum of all \( C_T \) that satisfy the above inequalities defines a norm on \( \Pi^p(X,Y) \), denoted by \( \pi^p(T) \).

The Pietsch Domination Theorem is one of the most important results of the theory of summing operators and makes a surprising bridge linking Measure Theory and summing operators. It asserts that a continuous linear operator \( T : X \to Y \) between Banach spaces is absolutely \( p \)-summing if and only if there is a constant \( C_T \geq 0 \) and a Borel probability measure \( \mu \) on the closed unit ball of the dual of \( X \), \( (B_{X^*}, \sigma(X^*,X)) \), such that

\[
\| T(x) \|^p \leq C_T \cdot \left( \int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi) \right)^{\frac{1}{p}}
\]

for every \( x \in X \).

Let us recall that a subset \( M \) of \( \Pi^p(X,Y) \) is called uniformly dominated if there exists a positive Radon measure \( \mu \) defined on the compact space \( (B_{X^*}, \sigma(X^*,X)) \) such that

\[
\| T(x) \|^p \leq \int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi)
\]

for all \( x \in X \) and all \( T \in M \). The main result of [12] is a characterization of uniformly dominated sets when \( Y \) is a Banach space that has no finite cotype.

**Theorem 1.1.** (Delgado and Piñero [12]) Let \( 1 \leq p < \infty \), let \( X \) be a Banach space, \( Y \) be a Banach space that has no finite cotype, and \( M \subset \Pi^p(X,Y) \). The following statements are equivalent:

(a) \( M \) is uniformly dominated.

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(b) There is a constant $C > 0$ such that, for every $\{x_1, \ldots, x_n\} \subset X$ and $\{T_1, \ldots, T_n\} \subset M$, there exists an operator $T \in \Pi_p (X, Y)$ satisfying $\pi_p(T) \leq C$ and

$$
\|T_i(x_i)\| \leq \|T(x_i)\|, \ i = 1, \ldots, n.
$$

In this note we prove an abstract version of Theorem 1.1 and, as particular cases, we obtain versions of Theorem 1.1 in quite different contexts. Some of our arguments are essentially an abstraction of the results of [12]. We also estimate the size of the non uniformly dominated sets in the sense of the theory of lineability. For details on lineability we refer to [3, 4] and the references therein.

2. Preliminaries

In the recent years a series of works ([5, 6, 16, 17, 18]) related to the Pietsch Domination Theorem have shown that this cornerstone of the theory of summing operators in fact needs almost no linear structure to be valid (see Theorem 2.1 below); this new panorama of the subject has been proved to be useful in different frameworks (see, for instance, [1, 2, 8, 10, 11, 19]).

Let $X$, $Y$ and $E$ be (arbitrary) non-void sets, $\mathcal{H}(X; Y)$ be a non-void family of mappings from $X$ to $Y$, $G$ be a Banach space and $K$ be a compact Hausdorff topological space. Let $R: K \times E \times G \to [0, \infty)$ and $S: \mathcal{H}(X; Y) \times E \times G \to [0, \infty)$ be arbitrary mappings and $1 \leq p < \infty$. According to [6, 16] a mapping $f \in \mathcal{H}(X; Y)$ is $RS$-abstract $p$-summing if there is a constant $C_f \geq 0$ such that

$$
\left( \sum_{i=1}^{m} S(f, x_i, b_i)^p \right)^{\frac{1}{p}} \leq C_f \sup_{\varphi \in K} \left( \sum_{i=1}^{m} R(\varphi, x_i, b_i)^p \right)^{\frac{1}{p}},
$$

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$. We define

$$
\mathcal{H}_{RS,p}(X; Y) = \{ f \in \mathcal{H}(X; Y) : f \text{ is } RS\text{-abstract } p\text{-summing} \}
$$

and the infimum of the $C_f$’s satisfying (2.1) is denoted by $\pi_{RS,p}(f)$.

Suppose that $R$ is such that the mapping

$$
R_{x, b}: K \to [0, \infty) \text{ defined by } R_{x, b}(\varphi) = R(\varphi, x, b)
$$

is continuous for every $x \in E$ and $b \in G$. The formulation of the Pietsch Domination Theorem from [16] reads as follows:

**Theorem 2.1** (Abstract Pietsch Domination Theorem ([6, 16])). Suppose that $S$ is arbitrary, $R$ satisfies (2.2) and let $1 \leq p < \infty$. A map $f \in \mathcal{H}(X; Y)$ is $RS$-abstract $p$-summing if and only if there is a constant $C_f \geq 0$ and a Borel probability measure $\mu$ on $K$ such that

$$
S(f, x, b) \leq C_f \left( \int_{K} R(\varphi, x, b)^p d\mu \right)^{\frac{1}{p}}
$$

for all $x \in E$ and $b \in G$.

In the next section, we present the main result of this note, the general version for Theorem 1.1 which, as the Abstract Pietsch Domination Theorem, does not need much of the linear framework.

**Remark 2.2.** In the definition of $R$ we can have $R: K \times E \times G \to \mathbb{K}$ (with $\mathbb{K}$ instead of $[0, \infty)$) and when needed we replace $R$ by $|R|$. In this case we say $|R|$-abstract $p$-summing.

**Remark 2.3.** We stress that the constants from (2.7) and (2.5) can be chosen to be the same.
3. Main Result

The next definition is an abstract disguise of the notion of uniformly dominated operators presented in the introduction.

**Definition 3.1.** A subset $\mathcal{M}$ of $\mathcal{H}_{RS,p}(X;Y)$ is uniformly dominated if there exists a positive Radon measure $\mu$ defined on the compact space $K$ such that

\[
S(f,x,b)^p \leq \int_K R(\varphi,x,b)^p d\mu(\varphi)
\]

for all $f \in \mathcal{M}$, $x \in X$ and $b \in G$.

The following lemma is somewhat expected and simple, but useful.

**Lemma 3.2.** A subset $\mathcal{M}$ of $\mathcal{H}_{RS,p}(X;Y)$ is uniformly dominated if and only if there is a constant $C \geq 0$ such that

\[
\left( \sum_{i=1}^m S(f_i,x_i,b_i)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{i=1}^m R(\varphi,x_i,b_i)^p \right)^{\frac{1}{p}},
\]

for all $\{f_1,\ldots,f_m\} \subset \mathcal{M}$, $\{x_1,\ldots,x_m\} \subset E$ and $\{b_1,\ldots,b_m\} \subset G$.

**Proof.** Suppose that $\mathcal{M}$ is a uniformly dominated set of $\mathcal{H}_{RS,p}(X;Y)$. Given $\{f_1,\ldots,f_m\} \subset \mathcal{M}$, $\{x_1,\ldots,x_m\} \subset E$ and $\{b_1,\ldots,b_m\} \subset G$ there is a positive Radon measure $\mu$ such that

\[
\sum_{i=1}^m S(f_i,x_i,b_i)^p \leq \sum_{i=1}^m \int_K R(\varphi,x_i,b_i)^p d\mu(\varphi)
\]

\[= \int_K \sum_{i=1}^m R(\varphi,x_i,b_i)^p d\mu(\varphi)
\]

\[\leq \mu(K) \cdot \sup_{\varphi \in K} \sum_{i=1}^m R(\varphi,x_i,b_i)^p.
\]

Conversely, given $f \in \mathcal{M}$, $\{x_1,\ldots,x_m\} \subset E$ and $\{b_1,\ldots,b_m\} \subset G$ there is, by hypothesis, a constant $C \geq 0$ (not depending on $f$) so that

\[
\left( \sum_{i=1}^m S(f,x_i,b_i)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{i=1}^m R(\varphi,x_i,b_i)^p \right)^{\frac{1}{p}}.
\]

Then $f$ is $RS$-abstract $p$-summing and by Theorem 2.1 there is a Borel probability measure $\mu$ on $K$ such that

\[
S(f,x,b) \leq C \left( \int_K R(\varphi,x,b)^p d\mu \right)^{\frac{1}{p}}
\]

for all $x \in E$ and $b \in G$. Now, considering the positive Radon measure $\mu = C^p \mu$ we complete the proof. \qed

Henceforth $\frac{1}{p} + \frac{1}{p'} = 1$ and we suppose that $Y$ is a Banach space with no finite cotype (for details on cotype we refer to \cite[Theorem 14.1]{13}). For this reason, we know that $Y$ contains $\ell_\infty^n$’s uniformly (see \cite{13}). By \cite{13}, for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an isomorphism $J_n$ from $\ell_\infty^n$ onto a subspace of $Y$ satisfying $\|J_n^{-1}\| = 1$ and $\|J_n\| \leq (1 + \varepsilon)$ for all $n \in \mathbb{N}$. We define $y_j := J_n e_j$, where $e_j$ are the canonical vectors of $\ell_\infty^n$. From now on, for every $g_j \in L_{p'}(K,\mu)$, with $\|g_j\|_{p'} = 1$, $j = 1,\ldots,n$ we define $f : X \rightarrow Y$ by

\[
f(x) = \sum_{j=1}^n \left( \int_K R(\varphi,x,0) g_j(\varphi)d\mu(\varphi) \right) y_j.
\]
Lemma 3.3. \( f \) belongs to \( \mathcal{H}_{RS,p}(X;Y) \), with \( \pi_{RS,p}(f) \leq \mu(K)^{1/p} \).

Proof. Given \( x \in X \), by one of the versions of the Hahn–Banach Theorem, we have

\[
\|f(x)\| = \sup_{y^* \in B_{Y^*}} |\langle y^*, f(x) \rangle| \\
= \sup_{y^* \in B_{Y^*}} \left| \langle y^*, \sum_{j=1}^{n} \left( \int_{K} R(\varphi, x, 0) g_j(\varphi) d\mu(\varphi) \right) y_j \rangle \right| \\
\leq \sup_{y^* \in B_{Y^*}} \sum_{j=1}^{n} \left( \int_{K} R(\varphi, x, 0) |g_j(\varphi)| d\mu(\varphi) \right)^{1/p} \left( \int_{K} |g_j(\varphi)|^{p'} d\mu(\varphi) \right)^{1/p'} |\langle y^*, y_j \rangle| \\
\leq \left( \int_{K} R(\varphi, x, 0)^p d\mu(\varphi) \right)^{1/p} \sup_{y^* \in B_{Y^*}} \sum_{j=1}^{n} |\langle y^*, y_j \rangle| \\
\leq \left( \int_{K} R(\varphi, x, 0)^p d\mu(\varphi) \right)^{1/p} \|J_n^*\| \\
= \left( \int_{K} R(\varphi, x, 0)^p d\mu(\varphi) \right)^{1/p} \|J_n\| \\
\leq \left( \int_{K} R(\varphi, x, 0)^p d\mu(\varphi) \right)^{1/p} (1 + \varepsilon).
\]

Considering the probability measure \( \overline{\mu} = \mu/\mu(K) \), we get

\[
S(f, x, b) \leq (1 + \varepsilon) \mu(K)^{1/p} \left( \int_{K} R(\varphi, x, 0)^p d\overline{\mu}(\varphi) \right)^{1/p}.
\]

Hence \( f \in \mathcal{H}_{RS,p}(X;Y) \). Since \( \varepsilon > 0 \) is arbitrary we obtain \( \pi_{RS,p}(f) \leq \mu(K)^{1/p} \).

Now we state and prove our main result.

Theorem 3.4. Let \( Y \) be a Banach space with no finite cotype, \( \mathcal{M} \subset \mathcal{H}_{RS,p}(X;Y) \) and \( 1 \leq p < \infty \). The following statements are equivalent:

(a) \( \mathcal{M} \) is uniformly dominated.

(b) There is a constant \( C > 0 \) such that, for every \( \{x_1, \ldots, x_n\} \subset X \) and \( \{f_1, \ldots, f_n\} \subset \mathcal{M} \), there exists an operator \( f \in \mathcal{H}_{RS,p}(X;Y) \) satisfying \( \pi_{RS,p}(f) \leq C \) and

\[
\|f_i(x_i)\| \leq \|f(x_i)\|, \quad i = 1, \ldots, n.
\]

Proof. (a) \( \Rightarrow \) (b) By hypothesis, there exists a positive Radon measure \( \mu \) such that (recall that we are supposing that \( R \) is constant in \( G \)),

\[
\|u(x)\| \leq \left( \int_{K} R(\varphi, x, 0)^p d\mu(\varphi) \right)^{1/p}
\]

for all \( u \in \mathcal{M}, \ x \in X \).
Given \( \{f_1, \ldots, f_n\} \subset \mathcal{M}, \{x_1, \ldots, x_n\} \subset E \), by (3.6) we have

\[
\|f_i(x_i)\| \leq \left( \int_K R(\varphi, x_i, 0)^p \, d\mu(\varphi) \right)^{1/p}, \quad i = 1, \ldots, n.
\]

For every \( i = 1, \ldots, n \), take \( g_i \in L_{p'}(\mu) \) such that \( \|g_i\|_{p'} = 1 \) and

\[
\left( \int_K R(\varphi, x_i, 0)^p \, d\mu(\varphi) \right)^{1/p} = \int_K R(\varphi, x_i, 0) g_i(\varphi) \, d\mu(\varphi).
\]

From (3.7) and (3.8), we obtain

\[
\|f_i(x_i)\| \leq \int_K R(\varphi, x_i, 0) g_i(\varphi) \, d\mu(\varphi), \quad i = 1, \ldots, n.
\]

We now define \( y_i^* = e_i^* \circ J_n^{-1} \), where \( e_i^* \) are the canonical vectors of \((\ell^n_\infty)^* \cong \ell^n\). Notice that \( \|y_i^*\| \leq 1 \) for \( i = 1, \ldots, n \). We also denote by \( y_i^* \) a Hahn-Banach extension of \( e_i^* \circ J_n^{-1} \) to \( Y \). Thus, using \( f \) defined in (3.4) and Lemma 3.9, we obtain

\[
\|f(x_i)\| = \sup_{y_i^* \in B_Y^*} |\langle y_i^*, f(x_i) \rangle| \\
\geq |\langle y_i^*, f(x_i) \rangle| \\
= \left| \langle y_i^*, \sum_{j=1}^n \left( \int_K R(\varphi, x_i, 0) g_j(\varphi) \, d\mu(\varphi) \right) y_j \rangle \right| \\
= \left| \sum_{j=1}^n \left( \int_K R(\varphi, x_i, 0) g_j(\varphi) \, d\mu(\varphi) \right) \langle y_i^*, y_j \rangle \right| \\
= \left| \sum_{j=1}^n \left( \int_K R(\varphi, x_i, 0) g_j(\varphi) \, d\mu(\varphi) \right) \langle e_i^* \circ J_n^{-1}, J_n e_j \rangle \right| \\
= \left| \sum_{j=1}^n \left( \int_K R(\varphi, x_i, 0) g_j(\varphi) \, d\mu(\varphi) \right) \langle e_i^*, e_j \rangle \right| \\
= \int_K R(\varphi, x_i, 0) g_i(\varphi) \, d\mu(\varphi) \\
\geq \left( \int_K R(\varphi, x_i, 0)^p \, d\mu(\varphi) \right)^{1/p}.
\]

(b) \( \Rightarrow \) (a) It follows from Lemma 3.2. In fact, by hypotheses, there is a constant \( C > 0 \) and an operator \( f \in \mathcal{H}_{RS,p}(X;Y) \) satisfying \( \pi_{RS,p}(f) \leq C \) and

\[
\|f_i(x_i)\| \leq \|f(x_i)\|, \quad i = 1, \ldots, n
\]

for all \( \{x_1, \ldots, x_n\} \subset X \) and \( \{f_1, \ldots, f_n\} \subset \mathcal{M} \). Thus

\[
\left( \sum_{i=1}^n \|f_i(x_i)\|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n \|f(x_i)\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left( \sum_{i=1}^n R(\varphi, x_i, b_i)^p \right)^{\frac{1}{p}},
\]

for all \( \{f_1, \ldots, f_n\} \subset \mathcal{M}, \{x_1, \ldots, x_n\} \subset X \) and \( \{b_1, \ldots, b_n\} \subset G \). By Lemma 3.2, \( \mathcal{M} \) is uniformly dominated.

\[\square\]

4. Consequences of Theorem 3.4

In this section we apply Theorem 3.4 to characterize uniformly dominated sets in several usual classes of RS-abstract summing mappings when \( Y \) is a Banach space that has no finite cotype.
4.1. Absolutely $p$-summing linear operators. Observe that a continuous linear operator $T : X \to Y$ is absolutely $p$-summing if and only if it is $|R|S$-abstract $p$-summing with

$$E = X \text{ and } G = \mathbb{K}$$

and $K = B_{X^*},$ with the weak star topology, $\mathcal{H}(X;Y) = \mathcal{L}(X;Y)$ and $R$ and $S$ are defined by:

$$R : B_{X^*} \times X \times \mathbb{K} \to \mathbb{K}, \quad R(\varphi, x, b) = \varphi(x)$$

$$S : \mathcal{L}(X;Y) \times X \times \mathbb{K} \to [0, \infty), \quad S(T, x, b) = \|T(x)\|.$$ 

We can see that the hypotheses under $R$ and $S$ are straightforwardly satisfied, so Theorem 3.3 recovers Theorem 3.4.

4.2. Strongly $p$-summing multilinear mappings. Strongly $p$-summing multilinear mappings were introduced by Dimant [14]. Let $X_1, ..., X_n$ be Banach spaces. A continuous $n$-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is strongly $p$-summing if there is a constant $C > 0$ such that

$$\left( \sum_{i=1}^{m} \left\| T(x_1^i, ..., x_n^i) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{\mathcal{L}(X_1, ..., X_n)}} \left( \sum_{i=1}^{m} \left| \varphi(x_1^i, ..., x_n^i) \right|^p \right)^{\frac{1}{p}},$$

for every $m \in \mathbb{N}$ and $x_i^l \in X_i, \ l = 1, ..., n,$ where $\mathcal{L}(X_1, ..., X_n)$ is the space of all continuous $n$-linear forms on $X_1 \times \cdots \times X_n.$

Note that by choosing the parameters $E = X_1 \times \cdots \times X_n, \ K = B_{(X_1 \otimes \cdots \otimes X_n)^*}, \ G = \mathbb{K}, \ \mathcal{H} = \mathcal{L}(X_1, ..., X_n;Y)$ and $R$ and $S$ are defined by:

$$R : B(X_1 \otimes \cdots \otimes X_n)^* \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to \mathbb{K}, \ R(\varphi_1, ..., \varphi_n, b) = \varphi_1(x^1) \otimes \cdots \otimes \varphi_n(x^n)$$

$$S : \mathcal{L}(X_1, ..., X_n;Y) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0, \infty), \ S(T, x^1, ..., x^n) = \|T(x^1, ..., x^n)\|,$$

we can easily conclude that $T : X_1 \times \cdots \times X_n \to Y$ is strongly $p$-summing if and only if $T$ is $|R|S$-abstract $p$-summing.

It is immediate to verify that the hypotheses needed for $R$ and $S$ are also straightforwardly satisfied in this case. Thus, Theorem 3.4 provides a characterization of uniformly dominated set in classes of strongly $p$-summing multilinear mappings.

4.3. $p$-semi-integral multilinear mappings. The class $p$-semi-integral multilinear mappings was introduced in [13], inspired by previous work of R. Alencar and M.C. Matos. Let $X_1, ..., X_n$ be Banach spaces. A continuous $n$-linear mapping $T : X_1 \times \cdots \times X_n \to Y$ is $p$-semi-integral if there is a constant $C > 0$ such that

$$\left( \sum_{i=1}^{m} \left\| T(x_1^i, ..., x_n^i) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X_1^*}, j=1,...,n} \left( \sum_{i=1}^{m} \left| \varphi_1(x_1^i) \cdots \varphi_n(x_n^i) \right|^p \right)^{\frac{1}{p}},$$

for every $m \in \mathbb{N}$ and $x_i^l \in X_i, \ i = 1, ..., m, \ l = 1, ..., n.$

Choosing the parameters $E = X_1 \times \cdots \times X_n, \ K = B_{X_1^*} \times \cdots \times B_{X_n^*}, \ G = \mathbb{K}, \ \mathcal{H} = \mathcal{L}(X_1, ..., X_n;Y)$ and $R$ and $S$ are defined by:

$$R : (B_{X_1^*} \times \cdots \times B_{X_n^*}) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to \mathbb{K}$$

such that

$$R((\varphi_1, ..., \varphi_n), x^1, ..., x^n, b) = \varphi_1(x^1) \cdots \varphi_n(x^n)$$

and

$$S : \mathcal{L}(X_1, ..., X_n;Y) \times (X_1 \times \cdots \times X_n) \times \mathbb{K} \to [0, \infty), \ S(T, x^1, ..., x^n) = \|T(x^1, ..., x^n)\|,$$

we observe that $T : X_1 \times \cdots \times X_n \to Y$ is $p$-semi-integral if and only if $T$ is $|R|S$-abstract $p$-summing.

In this case, Theorem 3.4 provides a characterization of uniformly dominated sets in this context.
4.4. **Homogeneous mappings.** Let $X$ be a Banach space. In a continuous mapping $u : X \to Y$ such that
\begin{equation}
\|u(\lambda x)\| \geq |\lambda| \|u(x)\|
\end{equation}
for all $(x, \lambda) \in X \times K$ is called 1-subhomogeneous.

From ([7] Theorem 2.3 (b)) a 1-subhomogeneous map $u$ is absolutely $p$-summing if and only if there is a $C > 0$ such that
\begin{equation}
\left(\sum_{j=1}^{m} \|u(x_j)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X^*}} \left(\sum_{j=1}^{m} |\varphi(x_j)|^p\right)^{\frac{1}{p}}
\end{equation}
for every positive integer $m$. It is simple to note that a 1-subhomogeneous map $u$ is absolutely $p$-summing if and only if it is $RS$-abstract $p$-summing with $E = X$ and $G = \mathbb{R}$ and $K = B_{X^*}$, with the weak star topology, $\mathcal{H}(X; Y)$ is the space of continuous homogeneous mappings from $X$ to $Y$ and $R$ and $S$ are defined by
\begin{align*}
R : B_{X^*} \times X \times \mathbb{R} &\to [0, \infty) \subset \mathbb{R} , \quad R(\varphi, x, b) = |\varphi(x)| \\
S : \mathcal{H}(X; Y) \times X \times \mathbb{R} &\to [0, \infty) , \quad S(T, x, b) = \|T(x)\|.
\end{align*}

As expected, Theorem 5.1 also characterizes uniformly dominated sets in this classes of operators.

5. **Final Application: Absolutely summing arbitrary mappings**

Let $X$ be a Banach space. Following [6] Definition 4.1, an arbitrary mapping $u : X \to Y$ is absolutely $p$-summing at $a \in X$ if there is a $C > 0$ so that
\begin{equation}
\left(\sum_{j=1}^{m} \|u(a + x_j) - u(a)\|^p\right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X^*}} \left(\sum_{j=1}^{m} |\varphi(x_j)|^p\right)^{\frac{1}{p}}
\end{equation}
for every natural number $m$ and every $x_1, \ldots, x_m \in X$.

Choosing $E = X$, $G = \mathbb{R}$, $K = B_{X^*}$, with the weak star topology, $\mathcal{H}(X; Y)$ being the set of maps from $X$ to $Y$ and $R$, $S$ being defined by
\begin{align*}
R : B_{X^*} \times X \times \mathbb{R} &\to [0, \infty) \subset \mathbb{R} , \quad R(\varphi, x, b) = |\varphi(x)| \\
S : \mathcal{H}(X; Y) \times X \times \mathbb{R} &\to [0, \infty) , \quad S(h, x, b) = \|h(x) - h(0)\|
\end{align*}
we conclude that an arbitrary mapping $h : X \to Y$ is absolutely $p$-summing at 0 if and only if $h$ is $RS$-abstract $p$-summing.

A simple reformulation of the hypotheses that appear just below (3.4) allows us to prove that Theorem 3.4 is valid in this more arbitrary context, with $a = 0$ (we just need to ask that $S(g, x, b) = \|g(x)\|$ just for $g = f$ defined in 3.4 and not for all $g$). Besides, since
\begin{equation}
f(0) = \sum_{j=1}^{n} \left(\int_{K} R(\varphi, 0, 0) g_j(\varphi) d\mu(\varphi)\right) y_j = \sum_{j=1}^{n} \left(\int_{K} |\varphi(0)| g_j(\varphi) d\mu(\varphi)\right) y_j = 0,
\end{equation}
we in fact have $S(f, x, b) = \|f(x)\|$; so the re-formulation of Theorem 3.4 also characterizes uniformly dominated sets of absolutely summing arbitrary maps. To summarize:

**Theorem 5.1.** Let $1 \leq p < \infty$. Let $Y$ is a Banach space with no finite cotype and $\mathcal{M}$ be a subset of the set of all arbitrary mappings $u : X \to Y$ which are absolutely $p$-summing at 0 (denoted by $\mathcal{H}_{RS,p}(X; Y)$). The following statements are equivalent:

(a) $\mathcal{M}$ is uniformly dominated.
(b) There is a constant $C > 0$ such that, for every $\{x_1, \ldots, x_n\} \subset X$ and $\{f_1, \ldots, f_n\} \subset M$, there exists an operator $f \in H_{RS,p}(X;Y)$ satisfying $\pi_{RS,p}(f) \leq C$ and 
\[ \|f(x_i)\| \leq \|f(x_i)\|, \ i = 1, \ldots, n. \]

**Remark 5.2.** With some more technical effort it is also possible to prove the above result for $a \neq 0$ but we omit the details.

6. Lineability of $\Pi_p(X, \ell_\infty) \setminus M$

We say that the subset $M$ of a vector space $E$ is **lineable** if $M \cup \{0\}$ contains an infinite dimensional linear space (see [3, 4, 20] and the references therein).

In this section we investigate the size of the set $\Pi_p(X, \ell_\infty) \setminus M$ from the point of view of the theory of lineability. In fact, we show that, up to the null vector, $\Pi_p(X, \ell_\infty) \setminus M$ contains a subspace of dimension $c$ (cardinality of the continuum). Let us suppose that $M \subset \Pi_p(X, \ell_\infty)$ is uniformly dominated and $\Pi_p(X, \ell_\infty) \setminus M$ is non void. If $T \in \Pi_p(X, \ell_\infty) \setminus M$ and $\mu$ is a positive Radon measure defined on the compact space $(B_{X^*}, \sigma(X^*, X))$, then 
\[ \|T(x)\|^p > \int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi) \]
for some $x \in X$.

Now we separate the set of positive integers $\mathbb{N}$ into countably many infinite pairwise disjoint subsets $(A_k)_{k=1}^\infty$. For each positive integer $k$, let us denote 
\[ A_k = \{a^{(k)}_1 < a^{(k)}_2 < \cdots \} \]
and consider 
\[ \ell^{(k)} = \{x \in \ell_\infty : x_j = 0 \text{ if } j \notin A_k\}. \]

Now, for each fixed positive integer $k$, we define 
\[ T_k : X \to \ell^{(k)} \]
given by 
\[ (T_k(z))_{a^{(k)}_j} = (T(z))_j \]
for all positive integer $j$. Thus, for any fixed $k$, let $v_k : X \to \ell_\infty$ be given by 
\[ v_k = i_k \circ T_k, \]
where $i_k : \ell^{(k)} \to \ell_\infty$ is the canonical inclusion. It is plain that 
\[ \|v_k(z)\| = \|T_k(z)\| = \|T(z)\| \]
for every positive integer $k$ and $z \in X$. Thus, each $v_k$ satisfies (6.1). Note also that the operators $v_k$ have disjoint supports and thus the set $\{v_1, v_2, \ldots\}$ is linearly independent. Consider the operator 
\[ S : \ell_1 \to \Pi_p(X, \ell_\infty) \]
given by 
\[ S((a_k)_{k=1}^\infty) = \sum_{k=1}^\infty a_k v_k. \]

Since 
\[ \sum_{k=1}^\infty \|a_k v_k\| = \sum_{k=1}^\infty |a_k| \|v_k\| = \sum_{k=1}^\infty |a_k| \|T\| = \|T\| \sum_{k=1}^\infty |a_k| < \infty \]
we conclude that $S$ is well-defined and, moreover, $S$ is linear and injective. Since the supports of the operators $v_k$ are pairwise disjoint, we conclude that $S(\ell_1)$ satisfies [16]. In fact, suppose that $b_1,\ldots,b_m$ are all nonzero scalars and $a_1,\ldots,a_m$ are also scalars (and there is no loss of generality in supposing $a_1$ non null). For any positive Radon measure $\mu$ we have (recall the definition of the $v_k$ and that this norm is in $\ell_\infty$)

$$\left\|b_1^\infty \sum_{k=1}^\infty a_k v_k(x) + \cdots + b_m^\infty \sum_{k=1}^\infty a_k v_m(x)\right\|^p \geq \|b_1 a_1 v_1(x)\|^p = |b_1 a_1|^p \|v_1(x)\|^p \geq |b_1 a_1|^p \int_{B_{X^*}} |\varphi(x)|^p \, d\mu(\varphi).$$

Now replacing $\mu$ by the positive Radon measure $|b_1 a_1|^p \mu$, denoted by $\psi$, we have

$$\left\|b_1^\infty \sum_{k=1}^\infty a_k v_k(x) + \cdots + b_m^\infty \sum_{k=1}^\infty a_k v_m(x)\right\|^p \geq \int_{B_{X^*}} |\varphi(x)|^p \, d\psi(\varphi).$$

Thus, since the map $\mu \leftrightarrow |b_1 a_1|^p \mu$ is a bijection in the set of positive Radon measures we have

$$S(\ell_1) \subset (\Pi_p (X, \ell_\infty) \setminus \mathcal{M}) \cup \{0\}$$

and the proof is done, because $\dim (\ell_1) = c$.

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