TERMINAL FANO THREEFOLDS AND THEIR SMOOTHINGS

PRISKA JAHNKE AND IVO RADLOFF

INTRODUCTION

Let $X$ be some irreducible projective variety. By a smoothing of $X$ we mean a flat projective morphism

$$\pi : \mathcal{X} \to \Delta$$

onto the unit disc $\Delta$, such that $\mathcal{X}_0 \simeq X$ while $\mathcal{X}_t$ is smooth for $t \neq 0$. The total space $\mathcal{X}$ is a reduced complex space but not necessarily smooth. We are interested in smoothings in the case of singular (almost) Fano threefolds $X$. Recall that $X$ is called Fano if some multiple of $-K_X$ is free and ample (big and nef in the case almost Fano).

The existence of a smoothing is known here in the following cases:

1.) $X$ is a Gorenstein Fano threefold with at most terminal singularities ([N97]). Here $\mathcal{X}_t$ is Fano.

2.) $X$ is an almost Fano threefold with at most $\mathbb{Q}$–factorial terminal Gorenstein singularities ([Mi01]). Here $\mathcal{X}_t$ is almost Fano.

Associated to any (almost) Fano threefold we have several discrete quantities, among them the degree, by definition the top intersection number $(-K_X)^3 > 0$, and the Picard number

$$\rho(X) = \text{rk}_\mathbb{Z}\text{Pic}(X).$$

The degree is of course constant in a smoothing. The Picard number may jump.

We show that the Picard number does not jump in the above two cases and give some examples around smoothings of Gorenstein Fanos in the case of canonical singularities.

1. Terminal Gorenstein Singularities

Throughout this section, $\mathcal{X}$ is a smoothing of a terminal Gorenstein (almost) Fano threefold $X = \mathcal{X}_0$. Simplest examples are terminal degenerations of hypersurfaces of degree at most four in $\mathbb{P}_4$.

1.1. Proposition. $\mathcal{X}$ is normal having at most isolated terminal factorial Gorenstein singularities.

Proof. Terminal Gorenstein singularities in dimension 3 are isolated hypersurface singularities ([R87]). Through any point $P \in \mathcal{X}$ we have the reduced Cartier divisor $\mathcal{X}_t$ having at most singularities of this type. Sequence

$$N_{\mathcal{X}_t/X}^* \to \Omega_{\mathcal{X}_t/X}^1 \to \Omega_{\mathcal{X}_t}^1 \to 0$$
shows $e_P(\mathcal{X}) = 4$ or $5$ for the local embedding dimension. Either $P$ is a smooth point or an analytic isolated hypersurface singularity. We conclude that $\mathcal{X}$ is Cohen–Macaulay, normal and Gorenstein. By Inversion of Adjunction ([KM98], Theorem 5.50) $(X, 0)$ klt implies $(\mathcal{X}, X)$ plt. The singularities of $\mathcal{X}$ are hence terminal. Finally, Grothendieck’s proof of Samuel’s Conjecture ([G68], Corollaire 3.14) implies that all stalks are factorial, i.e., $Cl(\mathcal{O}_{\mathcal{X}, P}) = 0$ (compare [GPR94], p.85 for the analytic case). Then $\mathcal{X}$ is factorial. □

Recall the definition of a *simultaneous resolution*. It means a diagram

$$
\begin{array}{ccc}
\mathcal{X}' & \to & \mathcal{X} \\
\pi' \downarrow & & \downarrow \pi \\
\Delta & & \\
\end{array}
$$

where each $\mathcal{X}'_t$ is smooth and $\mathcal{X}'_t \to \mathcal{X}_t$ is birational for all $t$. The next Corollary follows from the fact that $\mathcal{X}$ is factorial:

**1.2. Corollary.** A *simultaneous resolution does not exist* (unless $X$ is smooth).

We come to the Picard groups. The Kawamata–Viehweg vanishing Theorem says $H^i(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$ for $i > 0$ and all $t$. By the Leray spectral sequence $H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = 0$ for $i > 0$. The exponential sequence gives

$$
\text{Pic}(X) \simeq H^2(X, \mathbb{Z}) \quad \text{and} \quad \text{Pic}(\mathcal{X}) \simeq H^2(\mathcal{X}, \mathbb{Z}).
$$

The inclusion $X \hookrightarrow \mathcal{X}$ is a deformation retract. Hence

$$(1.3) \quad \text{Pic}(X) \simeq H^2(X, \mathbb{Z}) \simeq H^2(\mathcal{X}, \mathbb{Z}) \simeq \text{Pic}(\mathcal{X})$$

by restriction and these groups are torion free. A priori the Picard number of $\mathcal{X}_t$ might be different from the Picard number of $X$.

**1.4. Theorem.** We have

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\text{Pic}(X) \simeq \text{Pic}(\mathcal{X}) \quad \text{and} \quad \text{Pic}(\mathcal{X}_t) \simeq \text{Pic}(\mathcal{X}).
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In the case $X$ smooth this just follows from (1.3) and the fact that the smoothing is differentiably trivial. Example 2.2 below illustrates some steps in the following proof.

**Proof.** Only the second isomorphism has to be proved. The vanishing $H^{2,0}(\mathcal{X}_t) = 0$ implies that the primitive part of $H^2(\mathcal{X}_t)$ defines a rational polarized variation of Hodge structure of pure type $(1, 1)$. From the fact that the isometries of an integral lattice form a finite group, we get that the corresponding monodromy is finite, i.e., the image of

$$
\rho : \pi_1(\Delta^*) \simeq \mathbb{Z} \longrightarrow H^2(\mathcal{X}_t, \mathbb{Z})
$$

is a finite cyclic group of some order $N$. After base change

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the monodromy becomes trivial. We have $\mathcal{X}'_0 \simeq X$, i.e., $\mathcal{X}'$ is another smoothing of $X$. By Proposition $\ref{prop:smoothness}$, $\mathcal{X}'$ is normal with at most factorial terminal Gorenstein singularities. Our aim is to show

\begin{equation}
\text{(1.5)} \quad \text{Pic}(\mathcal{X}') \otimes \mathbb{Q} \simeq H^2(\mathcal{X}', \mathbb{Q}) \simeq H^2(\mathcal{X}'_0, \mathbb{Q}) \simeq \text{Pic}(\mathcal{X}'_0) \otimes \mathbb{Q}.
\end{equation}

Then, as $H^2(\mathcal{X}, \mathbb{Q}) \simeq H^2(X, \mathbb{Q}) \simeq H^2(\mathcal{X}', \mathbb{Q})$, we find $H^2(\mathcal{X}, \mathbb{Q}) \simeq H^2(\mathcal{X}_0, \mathbb{Q})$. The image of $H^2(\mathcal{X}, \mathbb{Q}) \rightarrow H^2(\mathcal{X}_0, \mathbb{Q})$ consists of monodromy invariant classes. If this map is an isomorphism, the monodromy must be trivial. Then $\ref{prop:smoothness}$ shows the second identity for rational line bundles, i.e., $\text{Pic}_\mathbb{Q}(\mathcal{X}) \simeq \text{Pic}_\mathbb{Q}(\mathcal{X}_0)$.

For simplicity denote $\mathcal{X}'$ by $\mathcal{X}$. In order to prove $\ref{prop:smoothness}$ we have to show that any rational cohomology class $\xi_t \in H^2(\mathcal{X}_t, \mathbb{Q})$ comes by restriction from a class in $H^2(\mathcal{X}, \mathbb{Q})$. The monodromy being trivial, we may think of $\xi_t$ as being defined on $\mathcal{X}^* = \mathcal{X} \setminus \Delta$, varying holomorphically with $t$. Notice that even if some multiple of $\xi_t$ is represented by an effective divisor, we cannot just take the closure in $\mathcal{X}$, in the analytic setting this closure need not be a divisor. We will use the Local Invariant Cycle Theorem (LICT) instead to extend $\xi_t$ to $\mathcal{X}_{\text{reg}}$, and then make use of the fact that $\mathcal{X}$ is factorial to extend it to $\mathcal{X}$.

We first recall the statement of the LICT. Let $\pi : \mathcal{W} \rightarrow \Delta$ be a semistable degeneration of Kähler manifolds. As above, any rational monodromy invariant cohomology class on some $\mathcal{W}_t$, $t \neq 0$, extends to $\mathcal{W}^*$ by the degeneration of the Lerner spectral sequence. The LICT says that it even extends to the whole of $\mathcal{W}$, i.e., we have a surjection:

$$H^k(\mathcal{W}, \mathbb{Q}) \rightarrow H^k(\mathcal{W}_t, \mathbb{Q})_{\text{inv}} \rightarrow 0.$$ 

The LICT is due to Clemens in the Kähler case (\cite{C77}. See also \cite{D71}, \cite{V03}. Semistable degeneration might not be necessary here, compare \cite{C77}, p.230).

In order to apply the LICT let $\mathcal{Y}$ be a log resolution of $(\mathcal{X}, X)$ and $\mathcal{W}$ be a semistable reduction of $\mathcal{Y}$

\begin{equation}
\begin{array}{ccc}
\mathcal{W} & \rightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\Delta & \rightarrow & \Delta
\end{array}
\end{equation}

The pullback of $\xi_t$ to $\mathcal{W}^*$ comes by restriction from a class $\xi_W \in H^2(\mathcal{W}, \mathbb{Q})$. The rational map from $\mathcal{W}$ to $\mathcal{X}$ is a Galois covering over $\mathcal{X}_{\text{reg}}$. The Galois trace of $\xi_W$ induces a class $\xi \in H^2(\mathcal{X}_{\text{reg}}, \mathbb{Q})$ and extends $\xi_t$ to $\mathcal{X}_{\text{reg}}$.

As $\mathcal{X}$ is Cohen–Macaulay, $H^2(\mathcal{X}_{\text{reg}}, \mathcal{O}_{\mathcal{X}_{\text{reg}}}) = 0$. This follows from the long exact sequence of cohomology with support. The exponential sequence shows once again that a multiple of $\xi$ corresponds to a line bundle on $\mathcal{X}_{\text{reg}}$. As $\mathcal{X}$ is normal and factorial, this line bundle comes from a line bundle on $\mathcal{X}$. Its Chern class gives the desired extension of $\xi_t$ to $\mathcal{X}$ and shows $\ref{prop:smoothness}$.

Now $\mathcal{X}' = \mathcal{X}$ by $\ref{prop:smoothness}$ and $\text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(\mathcal{X}_0)$ is an isomorphism after tensorizing with $\mathbb{Q}$. We now prove surjectivity over $\mathbb{Z}$. Notice that the above argumentation involves a trace not defined over $\mathbb{Z}$.

Let $L_t \in \text{Pic}(\mathcal{X}_t)$. After tensorizing with $\mathcal{O}_\mathcal{X}(-mK_\mathcal{X})$ for some $m \gg 0$ we may assume that each $L_t$ is effective and consider a family $D_t \in |L_t|$. Again the closure of $D_t$ in $\mathcal{X}$ might not be a divisor. But take a pointwise limit $D_{tn}$ as $t_n \rightarrow 0$. The limit $D_0 \in \text{Cl}(\mathcal{X})$ depends on $(t_n)$. 

3
By (1.5), \( L^{\otimes r} \) comes by restriction of a line bundle \( B \) on \( \mathcal{X} \). Then \( rD_0 \) is Cartier. As \( X \) is Gorenstein with terminal singularities, any \( \mathbb{Q} \)-Cartier Weil divisor is Cartier (Kan. Lemma 5.1). By (1.6), \( D_0 \) induces a line bundle \( L \in \text{Pic}(\mathcal{X}) \). As \( L \otimes r = B \) we get \( L|_{\mathcal{X}_t} \simeq L_t \), i.e., \( L \) is the desired extension of \( L_t \) to \( \mathcal{X} \). □

2. Canonical Gorenstein Singularities

In this section we collect examples. From now on \( X \) is an (almost) Fano variety with at most canonical Gorenstein singularities. Here a smoothing need not exist (Example 2.1). If it exists, the Picard number may jump (Example 2.2). Different smoothings are possible (Example 2.3).

Recall the definition of a cDV (compound DuVal) point. If \( P \in X \) is a three-dimensional canonical singularity, then a general hyperplane section through \( P \) either has a DuVal or an elliptic singularity. In the first case, \( P \) is called cDV, in the latter non–cDV. In Example 2.1 and Example 2.3 we have non-cDV singularities.

2.1. Example. The system of cubics embeds \( \mathbb{P}_2 \) into \( \mathbb{P}_9 \). Let \( X \) be the cone in \( \mathbb{P}_{10} \). The vertex is a non–cDV canonical point. Its blowup yields the almost Fano threefold \( Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(3)) \) of degree \((-K_X)^3 = (-K_Y)^3 = 72 \) (see [JPR05], table A.3, no.1). A smooth Fano threefold of degree 72 does not exist, implying that \( X \) is not smoothable.

Examples of smoothings of canonical Fanos where the Picard number jumps are easy. A simple one in the case of surfaces is the degeneration of \( \mathbb{P}_1 \times \mathbb{P}_1 \) into the quadric cone in \( \mathbb{P}_3 \); for a threefold take the product with \( \mathbb{P}_1 \). This example and the simultaneous resolution quite nicely illustrates some steps in the proof of Theorem 1.4:

2.2. Example. Consider the following concrete degeneration of \( \mathbb{P}_1 \times \mathbb{P}_1 \) into the quadratic cone \( Q_0 \)

\[ \mathcal{X} = \{ tw^2 + x^2 + y^2 + z^2 = 0 \} \subset \mathbb{P}_3 \times \Delta. \]

The total space \( \mathcal{X} \) is smooth but the monodromy is non–trivial: choose a branch of the square root. For fixed \( t \neq 0 \), where this branch is defined, the two rulings \( f_1, f_2 \) of \( \mathcal{X}_t \simeq \mathbb{P}_1 \times \mathbb{P}_1 \) are given by the lines

\[ \pm \sqrt{t} \ w = \sqrt{-1} \ x \ \text{and} \ \ y = \sqrt{-1} \ z. \]

Going around the origin, \( \sqrt{t} \) becomes \(-\sqrt{t} \), i.e., the monodromy interchanges the rulings:

\[ T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{on} \quad H^2(\mathcal{X}_t, \mathbb{Z}) = \mathbb{Z}[f_1] \oplus \mathbb{Z}[f_2]. \]

In order to trivialize the monodromy we make a base change \( t \mapsto t^2 \). We get \( \mathcal{X}' \) given by

\[ t^2 w^2 + x^2 + y^2 + z^2 = 0 \quad \text{in} \quad \mathbb{P}_3 \times \Delta. \]

Now the monodromy is trivial. The central fiber is still \( \simeq Q_0 \). But \( \mathcal{X}' \) is no longer smooth, it has one terminal point in the central fiber at \([1:0:0:0]\), lying over the vertex of the cone \( Q_0 \). This singularity is not factorial, and we cannot continue as above.

The problem is the now globally on \( \mathcal{X}' \) defined (Weil–) divisor

\[ tw = \sqrt{-1} \ x \ \text{and} \ \ y = \sqrt{-1} \ z. \]
which is not $\mathbb{Q}$–Cartier. The blowup of this divisor gives a small resolution $\mathcal{Y} \to \mathcal{X}'$ (a simultaneous resolution of the canonical family $\mathcal{X}$).

The central fiber of $\mathcal{Y}$ is the second Hirzebruch surface $\Sigma_2$, i.e., $\mathcal{Y}$ comes from letting degenerate the non–split extension

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

to the splitting one, parametrised by $\mathbb{C} = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$. On $\mathcal{Y}$ we can argue as above, finding that the general and the central fiber have the same Picard number, which is clear.

2.3. Example. Let $X$ be the cone over the del Pezzo surface $S$ of degree six. This is a Gorenstein Fano threefold with canonical singularities and Picard number one. Here we have two smoothings, one with general fiber $\mathbb{P}(T_{\mathbb{P}^2})$ and one with fiber $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.

Indeed, we may think of $S$ as a hyperplane section in the half anticanonical system of $\mathbb{P}(T_{\mathbb{P}^2})$ or of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$. Any smooth projective manifold flatly degenerates to the cone over its general hyperplane section.

With this remark it is easy to give explicit equations: the anticanonical system of $S$ embeds $S$ into $\mathbb{P}_6$. Think of $S$ as the blowup $\text{Bl}_{p_1, p_2, p_3}(\mathbb{P}^2)$. After some projective transformation we may assume $p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0], p_3 = [0 : 0 : 1]$. The anticanonical embedding is then given by all cubics in $\mathbb{P}^2$ through these points:

$$x_0 = x^2 y, \quad x_1 = x^2 z, \quad x_2 = xyz, \quad x_3 = xy^2, \quad x_4 = y^2 z, \quad x_5 = xz^2, \quad x_6 = yz^2$$

and a homogeneous ideal of $S$ in $\mathbb{P}_6$ is given by all $2 \times 2$ minors of

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_2 & x_4 \\ x_2 & x_5 & x_6 \end{pmatrix}.$$  

The ideal of $\mathbb{P}_2 \times \mathbb{P}_2$ embedded into $\mathbb{P}_8$ by Segre is given by the rank one locus of any general $3 \times 3$ matrix of linear forms (if $s^p_1, s^p_2, s^p_3$ are homogeneous coordinates of the $p$–th factor, write $x_{i,j} = s^p_1 s^p_2 s^p_3$ for the homogeneous coordinates of $\mathbb{P}_8$). The rank one locus of such a general matrix over $\mathbb{P}_7$ is therefore isomorphic to a hyperplane section of $\mathbb{P}_2 \times \mathbb{P}_2$, i.e., to $\mathbb{P}(T_{\mathbb{P}^2})$. The idea is hence to degenerate the general matrix into $A$.

The smoothing with general fiber $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ is obtained similarly: the ideal of $S$ can as well be described as minors of the cube

$$A' = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_2 & x_4 \\ x_2 & x_5 & x_6 \end{pmatrix}.$$  

What one has to do here is to compute all minors along side faces and diagonals. As above we compare this with the general cube with linear entries over $\mathbb{P}_7$. The general cube cuts out $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ (if $s^p_1, s^p_2, s^p_3$ are homogeneous coordinates of the $p$–th factor, write $x_{i,j,k} = s^p_1 s^p_2 s^p_3$ for the homogeneous coordinates of $\mathbb{P}_7$).

Notice that the above description of the anticanonical embedding also shows that the general hyperplane section through the vertex of $X$ has an elliptic Gorenstein
point, i.e., the vertex is a non–cDV point (as is any isolated canonical non–terminal point).

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