Delone sets that are not Lipschitz rectifiable

Rodolfo Viera

Abstract. We prove that there exist Delone sets in $\mathbb{R}^d$, $d \geq 2$, which cannot be mapped onto the standard lattice $\mathbb{Z}^d$ by Lipschitz bijections. The impossibility of the Lipschitz-rectifiability crucially uses ideas of Lipschitz regular maps recently introduced by M. Dymond, V. Kaluža and E. Kopecká.

Introduction

Motivated by problems in many branches of mathematics (e.g. metric embedding theory [13, 16], geometric group theory [12], information theory [9], mathematical physics of quasicrystals [1]), over the last years there has been a lot of activity on Lipschitz embeddings of discrete sets. In this work, we focus on a particular aspect of this wide theory, namely the Lipschitz embeddability of Delone subsets of the Euclidean space into the standard lattice. Recall that a Delone set $D$ of a metric space $X$ is a subset that is discrete and coarsely dense in a uniform way. This means that there exist positive constants $\sigma, \Sigma$ such that $d(x, y) \geq \sigma$ for all $x \neq y$ in $D$, and for each $z \in X$ there exists $x \in D$ for which $d(x, z) \leq \Sigma$.

Furstenberg and, independently, Gromov, asked whether for every Delone subset of $\mathbb{R}^d$, $d \geq 2$, there exists a bi-Lipschitz bijection onto $\mathbb{Z}^d$ (i.e. whether every Delone set in $\mathbb{R}^d$ is bi-Lipschitz rectifiable). Furstenberg was interested on dynamical aspects of this question (see [3] for a broader discussion), while Gromov was motivated by instances of geometric group theory [12]. Their question was answered in the negative by Burago and Kleiner [2] and, independently, by McMullen [14]. However, their results only yield existence of non bi-Lipschitz rectifiable Delone sets. Concrete examples were recently produced by Cortez and Navas in [4]. These examples can be constructed with supplementary properties of “dynamical type”. In particular, they can be built so that they are repetitive, which means that the translation action on the space of Delone sets (endowed with an appropriate Chabauty topology) is minimal. Equivalently, for each $r > 0$, there exists $R > 0$ such that every pattern that appears in a ball of radius $r$ actually appears in every ball of radius $R$. Besides the mathematical relevance of this property, it is worth mentioning that all known examples of real (physical) quasicrystals lead to repetitive Delone sets.

In order to answer in the negative the Furstenberg-Gromov’ question, Burago and Kleiner in [2] and McMullen in [14] show that the existence of a non-rectifiable Delone set is a consequence of the following result of analytical nature: there exists a bounded away from zero density function $\rho : [0, 1]^2 \to \mathbb{R}$, for which the prescribed Jacobian equation

$$
\text{Jac}(F) = \rho \quad \text{a.e.,}
$$

(0.1)

has no bi-Lipschitz solution $F : [0, 1]^2 \to \mathbb{R}^2$; such densities $\rho : [0, 1]^2 \to \mathbb{R}$ which cannot be
realizable as the Jacobian of a bi-Lipschitz map are called non-bi-Lipschitz-realizable. For more details on the prescribed Jacobian equation, we refer for instance to [5, 15, 17]. We point out that “almost all” positive functions $\rho \in L^\infty$ can be used to construct non-rectifiable Delone sets, as was shown in [19].

Motivated by a fundamental problem in discrete geometry and information theory (see [9] and the references therein), Dymond, Kaluža and Kopecká recently adapted the Burago-Kleiner / McMullen techniques to answer in the negative a question raised by Feige in [13]. More precisely, in [8], they proved the following remarkable fact:

**Main Theorem.** For each $d \geq 2$, there exist Delone subsets of $\mathbb{R}^d$ that admit no Lipschitz bijection with $\mathbb{Z}^d$.

Since non-rectifiable Delone sets do exist, it would be interesting to know explicit examples of Delone sets that are Lipschitz rectifiable without being bi-Lipschitz rectifiable.

**Some words on notation.** Throughout this work we will denote by $\| \cdot \|_\infty = \| \cdot \|$ the supremum-norm in $\mathbb{R}^d$, and we denote by $B(x, r)$ the open ball with center $x \in \mathbb{R}^d$ and radius $r > 0$ with this norm. Given $\varepsilon > 0$ and a bounded set $A \subset \mathbb{R}^d$, let $B(A, \varepsilon)$ be the $\varepsilon$-neighbourhood of $A$. The set $C(I^d, \mathbb{R})$ will denote the space of real-valued continuous functions defined on the unit square $I^d := [0, 1]^d$, with the supremum-norm $\|f\|_\infty := \max\{|f(x)| : x \in I^d\}$. In addition, let $C_0(I^d, \mathbb{R})$ be the set of functions in $C(I^d, \mathbb{R})$ with compact support.

Given two metric spaces $(X, d_1)$ and $(Y, d_2)$, a function $f : X \to Y$ is said to be Lipschitz, if there exists $L > 0$ such that for every $x, y \in X$ $d_2(f(x), f(y)) \leq Ld_1(x, y)$. Moreover, $f$ is called bi-Lipschitz if there is $L \geq 1$ such that for every $x, y \in X$ $\frac{1}{L}d_1(x, y) \leq d_2(f(x), f(y)) \leq Ld_1(x, y)$. 

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Given a differentiable map $f : U \subset \mathbb{R}^d \to \mathbb{R}^d$, the (determinant) Jacobian of $f$ is denoted by $\text{Jac}(f) := \det(Df)$. Recall that by a classical theorem due to Rademacher (see Theorem 3.1.6 in [10]), every Lipschitz map $f : U \subset \mathbb{R}^d \to \mathbb{R}^n$ is differentiable almost everywhere. Thus, the expression “$\text{Jac}(f)$ a.e” make sense for Lipschitz maps from $\mathbb{R}^d$ to itself.

Given an integrable function $\rho : I^d \to [0, +\infty)$, let $\rho \lambda$ be the measure defined by

$$\rho \lambda(A) := \int_A \rho \lambda, \quad \text{for every measurable set } A \subset I^d$$

For a measurable map $F : A \to \mathbb{R}^d$ and a measure $\mu$ in $A$, consider the pushforward measure $F_\# \mu$ given by

$$F_\# \mu(B) := \mu(F^{-1}(B)) \quad \text{for every measurable set } B \subset F(I^d).$$

We say that a Delone set $D \subset \mathbb{Z}^2$ satisfy the $2\mathbb{Z}^2$-property if $2\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times 2\mathbb{Z}$ are subsets of $D$. Finally, we say that a subset $R$ of an integer square $T = ([i, i+k] \times [j, j+k]) \cap \mathbb{Z}^2$, where $i, j, k \in \mathbb{Z}$, satisfies the $2\mathbb{Z}^2$-property, if $T \cap (2\mathbb{Z} \times \mathbb{Z})$ and $T \cap (\mathbb{Z} \times 2\mathbb{Z})$ are contained in $R$.

**Sketch of the proof.** Our proof crucially follows the construction proposed in [2], [8] and [7], with some mild though crucial changes along the way. Our strategy is described in the following five steps:

1.- Let $\rho : I^2 \to \mathbb{R}$ be a positive continuous function such that $8/9 \leq \min \rho < \max \rho \leq 1$. As in [2], consider a Delone subset $D_\rho$ of $\mathbb{Z}^2$ emulating the behaviour of $\rho$ at bigger and bigger scales (see Section 1 for a precise construction).

2.- If there is a Lipschitz bijection $f : D_\rho \to \mathbb{Z}^2$, then $f$ must be “regular”, which means that the preimage of a ball $B$ under this map cannot contain a $2 \cdot \text{radius}(B)$-separated set with more than a certain prescribed number of elements (see Definition 2.1 and Proposition 2.2 below).

3.- By renormalizing $D_\rho$ and by passing to the limit the above Lipschitz bijection, we obtain a Lipschitz limit map $F : I^2 \to \mathbb{R}^2$ defined on the unit square, which can be also showed to be regular.

4.- By one of the main results in [8], there is an open subset of $\mathbb{R}^2$ whose preimage under $F$ is made up of finitely many disjoint open sets restricted to which the limit map is bi-Lipschitz; moreover, the number of these sets is uniformly controlled.

5.- Finally, we show that there is a closed ball $Q \subset F(I^2)$ such that $F$ satisfies the equation

$$F_\#(\rho \lambda)|_Q = \lambda|_Q. \quad (0.3)$$

The proof of that $F$ satisfies equation (0.3) is made by mean of a control of the loss of mass in $Q$ under the renormalization. More precisely, we prove that the escape of mass in $Q$ occurs always close to its boundary.
Therefore, if we choose a density map \( \rho \) for which there is no a closed ball \( Q \) such that (0.3) has Lipschitz regular solutions, we obtain a Delone set \( D_\rho \) which cannot be Lipschitz rectifiable. This function \( \rho \) must exist as a consequence of Proposition 3.1 (see Theorem 4.1 in [8])

The preceding steps can be summarized as follows: every bijection \( f : D_\rho \to \mathbb{Z}^2 \) having certain regularity (e.g. Lipschitz, bi-Lipschitz) induces certain type of regularity over the continuous density \( \rho \). Thus, the desired bad-behaved Delone set \( D_\rho \) can be found by choosing \( \rho \) lying outside of this regularity class.

1 Constructing a Delone set from an anomalous density.

In this section we explain how to produce a Delone set from a bounded away from zero density, as in [2] and [8] (see also [7]). To simplify computations, notations and figures, we will restrict ourselves to the 2-dimensional case; the higher dimensional case follows analogously.

For our purposes, we will consider a continuous density \( \rho : I^2 \to \mathbb{R} \) such that \( 8/9 \leq \min \rho < \max \rho \leq 1 \). Let \( (l_n)_{n \in \mathbb{N}} \) and \( (m_n)_{n \in \mathbb{N}} \) be two sequence of positive integers, where \( l_n \) is a multiple of \( m_n \) and such that \( l_n, m_n \to +\infty \) and \( l_n/m_n \to +\infty \). Let \( (S_n)_{n \in \mathbb{N}} \) be a sequence of disjoint squares with sides parallel to the coordinate axis, vertices having integer coordinates and with side-length equal to \( l_n \). For each \( n \in \mathbb{N} \), let \( (T_{n,i})_{i=1}^{m_n^2} \) be a subdivision of \( S_n \) by \( m_n^2 \) squares with sides parallels to the coordinate axis and side-length equal to \( l_n/m_n \) (see Figure 1). Finally, let \( \phi_n : S_n \to I^2 \) be an affine linear map sending the square \( S_n \) onto the unit square \( I^2 \).

We build a Delone set \( D_\rho \subset \mathbb{Z}^2 \) which "emulates" the behaviour of \( \rho \) as follows: in each square \( T_{n,i} \) we put \( \left[ \int_{T_{n,i}} \rho \circ \phi_n \, d\lambda \right] \) points with integer coordinates in such a way that each \( T_{n,i} \) satisfies the \( 2\mathbb{Z}^2 \)-property (see Figure 2); notice that this is possible since \( 8/9 \leq \min \rho < \max \rho \leq 1 \). Outside of \( \bigcup_{n \in \mathbb{N}} S_n \), put one point in each integer coordinate. This construction provides a set \( D_\rho \subset \mathbb{Z}^2 \) which is actually a Delone set satisfying the \( 2\mathbb{Z}^2 \)-property since \( 8/9 \leq \min \rho < \max \rho \leq 1 \).
2 Rescaling up to the limit and Lipschitz regularity

In this section we show, after renormalization and passing to the limit, that a Lipschitz bijection $f : D_\rho \to \mathbb{Z}^2$ induces a Lipschitz regular map from the unit square. We start this section with some basic background and recent results on Lipschitz regular maps. For additional information about these mappings, we refer to the reader to [6].

**Definition 2.1.** Let $X$ and $Y$ be two metric spaces. We say that a Lipschitz map $f : X \to Y$ is Lipschitz regular if there is a constant $C \in \mathbb{N}$ such that for every ball $B \subset Y$ of radius $r > 0$, the set $f^{-1}(B)$ can be covered by at most $C$ balls of radii $Cr$. The smallest such $C$ (that works for every $r > 0$) is called the regularity constant of $f$, and is denoted by $\text{Reg}(f)$.

A useful equivalent interpretation of this definition is provided by the next lemma. (The proof is straightforward and is left to the reader.)

**Lemma 2.2.** A Lipschitz map $f : X \to Y$ is Lipschitz regular if and only if there is a constant $C \in \mathbb{N}$ such that for every ball $B \subset Y$ of radius $r > 0$, the set $f^{-1}(B)$ does not contain a $Cr$-separated set with more than $C$ elements. If this is the case, then $\text{Reg}(f) \leq C$. Conversely, if $f$ is Lipschitz regular, then $C$ can be taken as being equal to $2\text{Reg}(f)$.

One of the main results of [8] (namely, Theorem 2.10 therein) is that every Lipschitz regular map defined on a bounded region of $\mathbb{R}^d$ can be “densely decomposed” into bi-Lipschitz pieces, as stated below.

**Theorem 2.3.** Let $U \subset \mathbb{R}^d$ be a nonempty open set. If $f : \overline{U} \to \mathbb{R}^d$ is a Lipschitz regular map, then there exist pairwise disjoint open sets $(A_n)_{n \in \mathbb{N}}$ in $U$ such that $\bigcup_{n \in \mathbb{N}} A_n$ is dense in $\overline{U}$ and, for each $n \in \mathbb{N}$, the map $f|_{A_n}$ is bi-Lipschitz with lower bi-Lipschitz constant $b = b(\text{Reg}(f))$.

As a consequence of Theorem 2.3, it is showed in [8] that the image by a Lipschitz regular map contains an open set whose preimage is made of a controlled number of open subsets where the map is bi-Lipschitz, with lower bi-Lipschitz constant depending only on the regularity constant. This is stated below, and corresponds to Proposition 2.15 in [8].
Proposition 2.4. Let $U \subset \mathbb{R}^d$ be a nonempty open set. If $f : U \to \mathbb{R}^d$ is a Lipschitz regular map, then there exist a nonempty open set $T \subset f(U)$, an integer $N \in \{1, \ldots, \text{Reg}(f)\}$, and pairwise disjoint open sets $W_1, \ldots, W_N \subset U$, such that $f^{-1}(T) = \bigcup_{i=1}^N W_i$ and, for each $1 \leq i \leq N$, the map $f|_{W_i} : W_i \to T$ is a bi-Lipschitz homeomorphism, with lower bi-Lipschitz constant $b = b(\text{Reg}(f))$. Actually, one may take $b = \frac{1}{2\text{Reg}(f)^2}$.

The following result deals with maps defined on discrete sets of points. It asserts that every Lipschitz bijection defined from a Delone set that satisfies the $2\mathbb{Z}^2$-property onto the integer lattice must be Lipschitz regular. Actually, as we will see along the proof, the $2\mathbb{Z}^2$-property may be replaced by any other property ensuring that densities of points in large balls are everywhere bounded from below (away from zero). Given $x \in \mathbb{R}^2$, $r > 0$ and a subset $\mathcal{L}$ of $\mathbb{R}^2$, we will denote the set $B(x, r) \cap \mathcal{L}$ by $B_{\mathcal{L}}(x, r)$.

Proposition 2.5. Let $\mathcal{D} \subset \mathbb{Z}^2$ be a Delone set satisfying the $2\mathbb{Z}^2$-property. If $f : \mathcal{D} \to \mathbb{Z}^2$ is an $L$-Lipschitz bijection, then $f$ is Lipschitz regular, with $\text{Reg}(f) \leq C(L + 1)^2$ for some universal constant $C > 0$.

Proof. Let $y \in \mathbb{Z}^2$ and $r > 0$. Consider $\Gamma \subset f^{-1}(B(y, r))$ a maximal $2r$-separated set, and write $\Gamma = \{x_1, \ldots, x_{|\Gamma|}\}$. Then, by the $L$-Lipschitz condition we have that

$$f \left( \bigcup_{i=1}^{|\Gamma|} B_{\mathcal{D}}(x_i, r) \right) \subset B_{\mathbb{Z}^2}(y, r + rL).$$

Observe that, for $i = 1, \ldots, |\Gamma|$, the open balls $B(x_i, r)$ are pairwise disjoint. Since $f$ is a bijection we obtain that, for a certain constant $C_1 \geq 1$,

$$\sum_{i=1}^{|\Gamma|} |B_{\mathcal{D}}(x_i, r)| = \left| f \left( \bigcup_{i=1}^{|\Gamma|} B_{\mathcal{D}}(x_i, r) \right) \right| \leq |B_{\mathbb{Z}^2}(y, r + rL)| \leq C_1 r^2(L + 1)^2. \quad (2.1)$$

Now, by the $2\mathbb{Z}^2$-property, the cardinality $|B_{\mathcal{D}}(x_i, r)|$ is at least $C_2 r^2$ for another universal constant $C_2 > 0$. (The value of $C_2$ can be taken as $8/9 - \varepsilon$ provided $r$ is large enough.) Thus, by (2.1),

$$C_2 r^2 |\Gamma| \leq C_1 r^2(L + 1)^2.$$

We hence conclude that $|\Gamma| \leq C_1(L + 1)^2/C_2$. Therefore, by Lemma 2.2, $f$ is Lipschitz regular with regularity constant at most $C_1(L + 1)^2/C_2$. \hfill \Box

From now on, let $\mathcal{P}_n := \mathcal{D}_\rho \cap S_n$, where $\mathcal{D}_\rho$ and $(S_n)_{n \in \mathbb{N}}$ are the Delone set and the sequence of squares given in §1, respectively. Let $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ be the homothety that maps the square $S_n$ onto $I^2$. In addition, define $\mathcal{R}_n := \phi_n(\mathcal{P}_n)$.

Assume there is an $L$-Lipschitz bijection $f : \mathcal{D}_\rho \to \mathbb{Z}^2$. As in [2] and [8], we proceed to normalize $f$ to each square $\mathcal{P}_n$, that is, to consider the map $f_n : \mathcal{R}_n \to \frac{1}{n} \mathbb{Z}^2$ defined by
\[
f_n(x) := \frac{1}{n} \left( f \circ \phi_n^{-1}(x) - f \circ \phi_n^{-1}(\bar{x}_n) \right), \tag{2.2}
\]
where \( \bar{x}_n \in \mathcal{R}_n \) is some base point. Notice that \( \text{Lip}(f_n) \leq L \) for all \( n \geq 1 \). By Kirszbraun’s extension theorem\(^1\) (see, for instance, Theorem 2.10.43 in [10]), each function \( f_n \) can be extended to an \( L \)-Lipschitz map
\[
\hat{f}_n : I^2 \to \mathbb{R}^2. \tag{2.3}
\]
By the Arzelà-Ascoli’s theorem, there exists a subsequence \((\hat{f}_{n_k})_{k \in \mathbb{N}}\) of \((\hat{f}_n)_{n \geq 1}\), converging to an \( L \)-Lipschitz map \( F : I^2 \to \mathbb{R}^2 \); from now on, the subsequence \((\hat{f}_{n_k})_{k \geq 1}\) will be just denoted \((\hat{f}_n)_{n \geq 1}\). As we next show, the Lipschitz regularity is inherited from \( f \) to \( F \).

**Proposition 2.6.** The map \( F : I^2 \to \mathbb{R}^2 \) built above is Lipschitz regular, with \( \text{Reg}(F) \leq 42 \text{ Reg}(f) \).

**Proof.** Let \( y \in F(I^2) \) and \( r > 0 \). Consider a maximal \( r \)-separated set \( \Gamma = \{x_1, \ldots, x_{|\Gamma|}\} \) contained in \( F^{-1}(B(y, r)) \). Given

\[
0 < \varepsilon < \min \left\{ \frac{(\sqrt{2} + \sqrt{2} - 1) r}{\sqrt{2 + \sqrt{2}}}, \text{dist}(\Gamma, \partial F^{-1}(B(y, r))) \right\},
\]

by the convergence of \( \hat{f}_n \) to \( F \), there is a positive integer \( n_0 = n_0(\varepsilon) \) such that, for every \( n \geq n_0 \), there exist \( p_1, \ldots, p_{|\Gamma|} \in \mathcal{R}_n \) for which the following hold:

- for every \( i = 1, \ldots, |\Gamma| \), we have that \( ||p_i - x_i|| < \varepsilon/2 \),
- the set \( \Gamma_n := \{p_1, \ldots, p_{|\Gamma|}\} \) is contained in \( F^{-1}(B(y, r)) \) and,
- \( f_n(\Gamma_n) \subset B(y, r) \).

Observe that \( \Gamma_n \) is \((r - \varepsilon)\)-separated, since \( \Gamma \) is \( r \)-separated.

We will delete some points in \( \Gamma_n \) in an appropriate way in order to obtain a set \( \Gamma'_n \subset f_n^{-1}(B(y, r)) \) that is \( r \)-separated and such that \( |\Gamma'_n| \geq |\Gamma|/21 \). By Lemma 2.2, this will imply that

\[
|\Gamma| \leq 21 |\Gamma'_n| \leq 42 \text{ Reg}(f),
\]
hence \( F \) is Lipschitz-regular with \( \text{Reg}(F) \leq 42 \text{ Reg}(f) \).

To build the set \( \Gamma'_n \), we consider the angle

\[
\alpha = \arccos \left( \frac{(r)^2 + (r - \varepsilon)^2 - (r - \varepsilon)^2}{2r(r - \varepsilon)} \right) = \arccos \left( \frac{r}{2(r - \varepsilon)} \right) \geq \arccos \left( \frac{\sqrt{2 + \sqrt{2}}}{2} \right) = \frac{\pi}{8},
\]

\(^1\)Actually, we do not really need to keep the same Lipschitz constant \( L \) for the extension map, but just another (larger) constant that depends only on \( L \), and a weaker form of Kirszbraun’s theorem proving this is much easier to establish.
where the inequality follows from the condition
\[
\varepsilon < \frac{(\sqrt{2} + \sqrt{2} - 1) r}{\sqrt{2} + \sqrt{2}}.
\]
Moreover, if \( \varepsilon \) is sufficiently small, then \( \alpha < \pi/4 \); this is the angle that appears in the picture below. In the area depicted in black, no pair of points in \( \Gamma_n \) is at distance \( > r - \varepsilon \). The same happens in a similar region with angle \( \pi/8 \). Since 20 of these polygonal regions cover exactly the anular region between a square of radius \( r - \varepsilon \) and another of radius \( r \) (with the same center), we deduce -by the pigeonhole principle- that no more than 20 points in this anular region can be \( (r - \varepsilon) \)-separated.

![Figure 3: In the black region there is no a pair of points in \( \Gamma_n \) at distance \( > r - \varepsilon \).](image)

Now, for each \( i \in \{1, \ldots, |\Gamma|\} \), let \( \Gamma_n^i \) be the set of all points \( p \in \Gamma_n \) such that \( r - \varepsilon \leq ||p_i - p|| \leq r \). We have shown that this set contains at most 20 points. We erase those corresponding to \( p_1 \), then those corresponding to the \( p_i \) with minimal index that survive after the first deletion (\( i \geq 2 \)), and so on. At the end, we get the subset \( \Gamma'_n \) with the desired properties.

\[ \square \]

3 Proof of the Main Theorem

In this section we are dedicated in the proof of the Main Theorem. Exploiting the bi-Lipschitz decomposition of Lipschitz regular maps recently introduced by Dymond, Kaluža and Kopecká in [8], in this section we show that the Lipschitz-rectifiability of \( D_\rho \) induces certain regularity over the positive continuous density \( \rho : I^2 \to \mathbb{R} \). Thus, the existence of a non-Lipschitz-rectifiable Delone set \( D_\rho \) will be a consequence of the existence of a continuous function \( \rho \) which does not belong to this regularity class.

Let \( (A_n)_{n \in \mathbb{N}} \) be a basis for the topology of the unit square \( I^2 \). As in [8], let \( \mathcal{E}_{C,L,n} \) be the set of positive continuous functions \( \rho : I^2 \to \mathbb{R} \) for which the following holds: there are pairwise disjoint
open sets \( Y_1, \ldots, Y_N \subset I^2 \), \( Y_1 = A_n \), where \( 1 \leq N \leq C \), an open set \( V \subset \mathbb{R}^2 \), and a family of \((b(C), L)\)-bi-Lipschitz homeomorphisms \( F_i : Y_i \to V \) such that

\[
\rho(y) = |Jac(F_i)(y)| - \sum_{i=2}^{n} \rho(F_i^{-1} \circ F_1)(y)|Jac(F_i^{-1} \circ F_1)(y)| \quad \text{a.e in } Y_1. \tag{3.1}
\]

In [8] it is shown that “almost all” positive continuous functions have not a bi-Lipschitz decomposition as in (3.1). This corresponds to Theorem 4.1 in [8].

**Proposition 3.1** (Dymond, Kaluža and Kopecká, 2018). A generic positive function \( \rho : I^2 \to \mathbb{R} \) does not belong to \( \bigcup_{C,L,n \in \mathbb{N}} \mathcal{E}_{C,L,n} \).

From now on let \( \rho \) be a positive continuous function such that \( 8/9 \leq \min \rho < \max \rho \leq 1 \) as in Proposition 3.1 and let \( D_{\rho} \subset \mathbb{Z}^2 \) be the corresponding Delone set constructed as in section §1. Assume that there is an \( L\)-Lipschitz bijection \( f : D_{\rho} \to \mathbb{Z}^2 \) and let \( F : I^2 \to \mathbb{R}^2 \) be the (limit) Lipschitz regular map obtained as in §2. By Proposition 2.4 there exist a non-empty open set \( W \subset F(I^2) \) and open disjoint subsets \( V_1, \ldots, V_N \subset I^2 \), where \( N \leq Reg(F) \), such that \( \bigcup_{i=1}^{N} V_i = F^{-1}(W) \) and, for each \( i \leq i \leq N \), the map \( F|_{V_i} : V_i \to W \) is bi-Lipschitz, with lower bi-Lipschitz constant \( b = \frac{1}{2Reg(F)} \). Thereupon, the Main Theorem is a consequence of the next proposition.

**Proposition 3.2.** For \( \rho, F \) and \( W \) as in the previous paragraph, there exists a closed ball \( Q \subset W \) such that

\[
F_\#(\rho \lambda)|_Q = \lambda|_Q. \tag{3.2}
\]

**Proof of Main Theorem from Proposition 3.2.** For every \( i = 1, \ldots, N \) write \( F_i := F|_{V_i} \) and let \( n \) be a natural number such that \( A_n \subset V_1 \cap F^{-1}(Q) \); besides, for each \( i = 2, \ldots, N \) denote by \( A_{i,n} := F^{-1}(F(A_n)) \cap V_i \) and \( A_{1,n} := A_n \). From the equation (3.2) we have that

\[
\sum_{i=1}^{N} \int_{A_{i,n}} \rho d\lambda = \lambda(F(A_n)).
\]

By a change of variable and the Euclidean Area formula for bi-Lipschitz maps, the previous equation can be rewritten as

\[
\int_{A_n} \sum_{i=1}^{N} \rho(F_i^{-1} \circ F_1)|Jac(F_i^{-1} \circ F_1)|d\lambda = \int_{A_n} |Jac(F_1)|d\lambda,
\]

which is equivalent to the equation

\[
\rho(y) = |Jac(F_1)(y)| - \sum_{i=2}^{n} \rho(F_i^{-1} \circ F_1)(y)|Jac(F_i^{-1} \circ F_1)(y)| \quad \text{a.e in } A_n.
\]

Thus, we conclude that \( \rho \in \mathcal{E}_{C,L,n \in \mathbb{N}} \) for \( C = 1/2Reg(F)^2 \), which contradicts the choice of \( \rho \). Therefore, \( D_{\rho} \) cannot be mapped onto \( \mathbb{Z}^2 \) by Lipschitz bijections, as announced. \( \square \)

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3.1 Mass-loss control under renormalization

In what follows we are concerned to prove Proposition 3.2, where we actually show that equation (3.2) is satisfied locally in \( W \). Let \( Q \subset W \) be a closed ball such that \( d(Q, \partial W) > 0 \) and define \( \mathcal{H}_i := F^{-1}(Q) \cap V_i \) (recall that we consider “balls” for the \( || \cdot ||_\infty \)-norm in \( \mathbb{R}^2 \)) and \( \mathcal{H} := \bigcup_{i=1}^{N} \mathcal{H}_i \); in addition, choose \( \varepsilon > 0 \) such that the closure of the \( \varepsilon \)-neighbourhood \( B(Q, \varepsilon) \) of \( Q \) is contained in \( W \). We start by showing that from a large-enough \( n \in \mathbb{N} \) the points in \( \mathcal{R}_n \) lying to \( \mathcal{H} \) are mapped under \( f_n \) into \( B(Q, \varepsilon) \), and \( f_n^{-1}(Q \cap (1/l_n)\mathbb{Z}^2) \) is completely contained in the \( \varepsilon \)-neighbourhood of \( \mathcal{H} \) (with \( f_n \) defined as in (2.2)); nevertheless, it might exist some points in \( Q \cap (1/l_n)\mathbb{Z}^2 \) which have not a pre-image under \( f_n \), producing loss of mass under the renormalization (see Figure 4). This is established in the Lemma 3.4 below.

![Figure 4: The set \( \mathcal{H} \cap \mathcal{R}_n \) is mapped under \( f_n \) into \( B(Q, \varepsilon) \) and \( f_n^{-1}(Q \cap (1/l_n)\mathbb{Z}^2) \) is contained in the \( \varepsilon \)-neighbourhood of \( \mathcal{H} \).](image)

**Lemma 3.3.** Given \( 0 < \varepsilon < \min\{d(\mathcal{H}, \partial F^{-1}(W)); \min_{i<j} d(\mathcal{H}_i, \mathcal{H}_j); d(Q, \partial W)\} \), there is a positive integer \( n_0 = n_0(\varepsilon) \) such that for every \( n \geq n_0 \) the following holds:

i) \( f_n(\mathcal{H} \cap \mathcal{R}_n) \) is contained in \( B(Q, \varepsilon) \); 

ii) \( f_n^{-1}(Q \cap (1/l_n)\mathbb{Z}^2) \subset B(\mathcal{H}, \varepsilon) \).

**Proof.** Assume that i) does not hold. Then there must exist an increasing sequence of integers \( k_n \) and a sequence of points \( x_n \in \mathcal{H} \cap \mathcal{R}_{k_n} \) such that \( f_{k_n}(x_n) \) does not belong to \( B(Q, \varepsilon) \). From the compactness of \( \mathcal{H} \) and after passing to a subsequence, we may assume that \( (x_n)_{n \geq 1} \) converges to a point \( x \in \mathcal{H} \). By the uniform convergence of \( f_n \) to \( F \) (recall that \( f_n \) was defined in (2.2)), we have that \( f_{k_n}(x_n) \rightarrow F(x) \). Since \( B(Q, \varepsilon) \) is an open set, \( F(x) \) cannot belong to \( B(Q, \varepsilon) \). However, this contradicts the fact that \( \mathcal{H} \subset F^{-1}(B(Q, \varepsilon)) \).

To prove ii) we proceed also by contradiction. Suppose that there exist an increasing sequence of positive integers \( (i_n)_{n \in \mathbb{N}} \) and a sequence of points \( (u_{i_n})_{n \in \mathbb{N}} \) such that for every \( n \geq 1 \) we have that \( u_{i_n} \) belongs to \( f_{i_n}^{-1}(Q \cap (1/l_n)\mathbb{Z}^2) \setminus B(\mathcal{H}, \varepsilon) \). Observe that \( (u_{i_n})_{n \in \mathbb{N}} \) converges, up to a subsequence, to an element \( u \in I^2 \setminus B(\mathcal{H}, \varepsilon) \). On the other hand, by the compactness of \( Q \) and since \( f_n \) converges uniformly to \( F \), we have that \( f_{i_n}(u_{i_n}) \) converges to \( F(u) \in Q \). However this would implies that \( u \in \mathcal{H} \), which is impossible since \( u \in I^2 \setminus B(\mathcal{H}, \varepsilon) \). 

\( \square \)
Given a subset $\mathcal{L} \subset \mathbb{R}^2$ we write $\mathcal{L}^{(n)} := \phi^{-1}_n(\mathcal{L})$, where $\phi_n$ is the homothety sending $S_n$ to $I^2$. Moreover given $k \in \mathbb{N}$, for each $j \in \mathbb{N}$ let $Q_{k,j}$ be the set of points $y \in B(Q, 1/k)$ for which $d(y, \partial B(Q, 1/k)) > jL/l_n$; notice that $Q_{k,j}$ can be empty for a sufficiently large positive integer $j$. Finally, let $c_n$ be the center of the square $S_n$. The key result to prove Proposition 3.2 is given by Lemma 3.4 below, which asserts that for sufficiently large positive integers $k, n$, there is a closed ball $M_{k,n} \subset I^2$ centred at the origin such that $f^{-1}(Q \cap (1/l_n)\mathbb{Z}^2) \cap M_{k,n}$ is mapped bijectively under $f_n$ onto a closed ball contained in $B(Q, 1/k) \cap (1/l_n)\mathbb{Z}^2$ with the same center as $Q$. The proof of Lemma 3.4 was inspired in the proof of Lemma 6 in [4] and relies strongly on the geometry of $\mathcal{D}_\rho$, namely, on the $2\mathbb{Z}^2$-property. Concretely, the Lemma 3.4 is a consequence of that points in the pre-image under $f$ of a closed ball $B$ contained in $W^{(n)}$ which are distant to $c_n$, must be mapped by $f$ “close” to the boundary of $B$.

**Lemma 3.4.** Given $k \in \mathbb{N}$ such that $1 < 1/k < \min \left\{ \frac{d(H, \phi^{-1}_F(W))}{2\log(F^2 + 1)}, \min d(H_i, H_j), d(Q, \partial W) \right\}$, there exists $n_k \in \mathbb{N}$ for which the following holds: for every $n \geq n_k$ there are a closed ball $M_{k,n}^{(n)} \subset S_n$ centered at $c_n$, and a closed ball $M_{k,n}^{(n)} \subset B(Q, 1/k)^{(n)}$ with the same center as $Q^{(n)}$, such that $f$ maps $M_{k,n}^{(n)} \cap f^{-1}(B(Q, 1/k)^{(n)})$ onto $M_{k,n}^{(n)}$ bijectively.

**Proof.** Let $x_{k,0}^n$ be a point in $f^{-1}(B(Q, 1/k)^{(n)})$ with the property that $x_{k,0}^n$ is at maximal distance from $c_n$. We claim that the distance from $f(x_{k,0}^n)$ to $\partial B(Q, 1/k)^{(n)}$ must be less or equal than $L$; indeed, if $d(f(x_{k,0}^n), \partial B(Q, 1/k)^{(n)}) > L$, then $B(f(x_{k,0}^n), L)$ is strictly contained in $B(Q, 1/k)^{(n)}$. On the other hand, the $2\mathbb{Z}^2$-property implies that at least one of the points $x_{k,0}^n \pm e_1, x_{k,0}^n \pm e_2, x_{k,0}^n \pm e_1 \pm e_2$ (where $e_1 = (1, 0)$ and $e_2 = (0, 1)$) belongs to $\mathcal{D}_\rho$ and lies at distance from $c_n$ larger than that $x_{k,0}^n$; see Figure 5 for a picture of this situation when $x_{k,0}^n$ lies to the left-hand side of the square $B(c_n, ||x_{k,0}^n - c_n||)$.

![Figure 5: A possible configuration of $\mathcal{D}_\rho$ around $x_{k,0}^n$.](image)

Then, from the $L$-Lipschitz condition, this point is mapped by $f$ into $B(f(x_{k,0}^n), L) \subset B(Q, 1/k)^{(n)}$, contradicting the choice of $x_{k,0}^n$.

In general, let $x_{k,j}^n \in f^{-1}(Q_{k,j}^{(n)})$ which is at maximal distance from $c_n$. As before we must have that $d(f(x_{k,j}^n), \partial Q_{k,j}^{(n)}) \leq L$, as long as $Q_{k,j}^{(n)} \neq \emptyset$. In addition, from the maximality of $||x_{k,j}^n - c_n||$,
the set \( \overline{B}(c_n, ||x^n_{k,j} - c_n||) \cap f^{-1}(B(Q, 1/k)^{(n)}) \) is sent bijectively by \( f \) onto \( \overline{Q}_{k,j}^{(n)} \cap \mathbb{Z}^2 \).

Let \( i(n) := \max\{i \in \mathbb{N} : Q_{k,i} \neq \emptyset \} \), i.e., \( i(n) \) satisfies that the side-length of \( Q_{k,i}^{(n)} \) is less or equal than \( 2L \). We shall prove that \( x^n_{k,i(n)} \) belongs to \( S_n \) for a sufficiently large \( n \in \mathbb{N} \). In fact, if \( x^n_{k,i(n)} \notin S_n \) and since \( \overline{B}(c_n, ||x^n_{k,i(n)} - c_n||) \cap f^{-1}(B(Q, 1/k)^{(n)}) \) is mapped bijectively onto \( \overline{Q}_{k,i(n)}^{(n)} \cap \mathbb{Z}^2 \) under \( f \), then by part \( i \) of Lemma \( 3.3 \) there exists a large-enough positive integer \( n_k \geq n_0 \) (where \( n_0 \) is the integer obtained in Lemma \( 3.3 \) for \( \varepsilon = 1/k \)) such that for every \( n \geq n_k \), the set \( f(H^{(n)} \cap D_\rho) \) is contained in \( \overline{Q}_{k,i(n)}^{(n)} \cap \mathbb{Z}^2 \) and so that \( |H^{(n)} \cap D_\rho| > (2L + 1)^2 \). Then the side-length of \( Q_{k,i(n)}^{(n)} \) must be greater than \( 2L \), and thus \( Q_{k,i(n)}^{(n)} = \{ y \in Q_{k,i(n)}^{(n)} : d(y, \partial Q_{k,i(n)}^{(n)}) > L \} \) is non-empty, contradicting the definition of \( i(n) \). Therefore \( x^n_{k,i(n)} \in S_n \) for every \( n \geq n_k \), as we claimed.

Thus, we can define \( j(n) := \min\{j \in \mathbb{N} : x^n_{k,j} \in S_n \} \) and \( x^n_{k,j(n)} \in f^{-1}(Q_{k,j(n)}^{(n)}) \) as in the preceding construction. Observe that \( x^n_{k,j(n)} \) necessarily belongs to \( B(H^{(n)}, (2\text{Reg}(F)^2 + 1)/k_n/k) \), which is contained in \( S_n \), as a consequence of the choice of \( k \), the part \( ii \) of Lemma \( 3.3 \) and the lower bi-Lipschitz constant in the decomposition of \( F^{-1}(W) \). Therefore, the conclusions of Lemma \( 3.4 \) hold for \( M_{k,n}^{(n)} = \overline{B}(c_n, ||x^n_{k,j(n)} - c_n||) \) and \( \hat{M}_{k,n}^{(n)} = \overline{Q}_{k,j(n)}^{(n)} \) (see Figure 6).

![Figure 6: Mapping \( \overline{B}(c_n, ||x^n_{k,j(n)} - c_n||) \cap f^{-1}(B(Q, 1/k)^{(n)}) \) onto \( \overline{Q}_{k,j(n)}^{(n)} \).](image)

Henceforth, let \( x_{k,j} := \phi_n(x^n_{k,j}) \) where the \( x^n_{k,j} \)'s are the points obtained in the proof of Lemma \( 3.4 \). Notice that for every \( k \in \mathbb{N} \) satisfying the hypothesis of Lemma \( 3.4 \) the point \( x_{k,j(n_k)} \) belongs to \( B(H, (2\text{Reg}(F)^2 + 1)/k_n/k) \) (where \( n_k \) and \( j(n_k) \) are the positive integers given in the proof of Lemma \( 3.4 \)), and hence the sequence \( (x_{k,j(n_k)})_{k \geq 1} \) converges (under a subsequence) to an element \( x_{\text{lim}} \in \mathcal{H} \).

Let \( r_{k,j} \) be the radius of the ball \( Q_{k,j} = \phi_n(Q_{k,j}^{(n)}) \); since \( \overline{Q}_{k,j}^{(n)} \) is contained in \( B(Q, 1/k) \) for every \( n \geq n_k \), we have that the sequence of radii \( (r_{k,j(n_k)})_{k \in \mathbb{N}} \) converges (under a subsequence) to a radius \( r \) which is less or equal than the radius of \( Q \). If we denote by \( \mathcal{M} \) the closed ball of radius \( r \) having the same center than \( Q \), then it is direct that \( \mathcal{M} \subset Q \). Our goal is to prove, firstly that
\( \mathcal{M} = \mathcal{Q} \) and, secondly that Proposition 3.2 holds for the closed ball \( \mathcal{Q} \).

In the next lemma we claim that \( \mathcal{H} \) must be contained in \( B(0, ||x_{lim}||) \) and, as a consequence, the sequence of closed balls \( \mathcal{Q}_{k,j(n_k)} \) “converges” to \( \mathcal{Q} \) (see Figure 7).

**Lemma 3.5.** The set \( \mathcal{H} \) is contained in \( \overline{B}(0, ||x_{lim}||) \). In particular, \( \mathcal{M} = \mathcal{Q} \).

**Proof.** Suppose there is a point \( u \in \text{int}(\mathcal{H}_i) \setminus \overline{B}(0, ||x_{lim}||) \) for some \( 1 \leq i \leq N \). Thereupon there must exist a positive number \( \alpha \) such that \( \overline{B}(u, \alpha) \) is contained in \( \text{int}(\mathcal{H}_i) \setminus \overline{B}(0, ||x_{lim}||) \). Thus, from the proof of Lemma 3.4 and by the definition of \( j(n_k) \), there is a sufficiently large \( k_0 \in \mathbb{N} \) such that \( f_{n_k}(\overline{B}(u, \alpha) \cap \mathcal{R}_{n_k}) \) is contained in \( \mathcal{Q}_{k,j(n_k)-1} \setminus \mathcal{Q}_{k,j(n_k)} \) for every \( k \geq k_0 \) (we point out that this last step is not necessarily true, for instance, for \( x_{k,i(n_k)} \), with \( i(n_k) \) being as in the proof of Lemma 3.4), and hence, by passing to the limit, we obtain that \( F(\overline{B}(u, \alpha)) \subset \partial \mathcal{M} \), i.e, \( F \) maps a set with positive Lebesgue measure into a set with zero-Lebesgue measure, which is impossible since \( F|_{\mathcal{H}_i} \) is a bi-Lipschitz homeomorphism. Thus, \( \text{int}(\mathcal{H}) \subset \overline{B}(0, ||x_{lim}||) \) and since each connected component of \( \mathcal{H} \) is homeomorphic to a closed ball, we conclude that \( \mathcal{H} \subset \overline{B}(0, ||x_{lim}||) \).

On the other hand, to show that \( \mathcal{M} = \mathcal{Q} \) it suffices to prove that \( \mathcal{Q} \subset \mathcal{M} \). Let \( v \in \mathcal{Q} \) and \( u \in \mathcal{H} \) such that \( F(u) = v \). By lemma 3.4 and since \( \mathcal{H} \subset \overline{B}(0, ||x_{lim}||) \), there exists a sequence \( (u_{j(n_k)})_{k \geq 1} \) converging to \( u \) so that \( u_{j(n_k)} \in f_{n_k}^{-1}(\mathcal{Q}_{k,j(n_k)} \cap (1/l_{n_k})\mathbb{Z}^2) \) for every \( k \in \mathbb{N} \). From the uniform convergence it follows that \( f_{n_k}(u_{j(n_k)}) \to F(u) \), and hence \( v = F(u) \in \mathcal{M} \), as we claimed.

![Figure 7: The set \( \overline{B}(0, ||x_{lim}||) \) contains \( \mathcal{H} \).](image)

**3.2 Proof of Proposition 3.2**

In order to prove Proposition 3.2 we need to estimate the cardinality of the set of points in \( B \cap (1/l_n)\mathbb{Z}^2 \) which have no pre-image under \( f_n : \mathcal{R}_n \to (1/l_n)\mathbb{Z}^2 \). To treat these points, we require the following result which corresponds to a slightly different version of Lemma 3.1 in [7] and whose proof is analogous.
Lemma 3.6. Let $U \subset \mathbb{R}^d$ be a closed set which is the image of $I^d$ under a bi-Lipschitz map and let $g : U \to \mathbb{R}^d$ be a homeomorphism and $h : U \to \mathbb{R}^d$ be continuous. Then $h(U) \Delta g(U) \subset B(\partial g(U), ||g - h||_\infty)$, where $\Delta$ denotes the symmetric difference.

Before showing Proposition 3.2, we introduce some terminology. For every $n \in \mathbb{N}$ consider the normalized counting measures $\mu_n$ and $\nu_n$ defined by

$$
\mu_n(A) := \frac{|A \cap \mathcal{R}_n|}{l_n^2}, \quad \nu_n(C) := \frac{|C \cap \frac{1}{n}\mathbb{Z}^2|}{l_n^2}
$$

Notice that $(\nu_n |_{F(I^2)})_{n \in \mathbb{N}}$ converges weakly to the Lebesgue measure in $F(I^2)$. Moreover, it can be shown that $\mu_n$ converges weakly to the measure $\rho \lambda$ in $I^2$, facts whose proof follows the very same lines than Claims 5.3.1 and 5.3.2 in [8].

Lemma 3.7. The sequences of measures $(\mu_n)_{n \in \mathbb{N}}$ and $(\nu_n)_{n \in \mathbb{N}}$ converge weakly to $\rho \lambda$ and $\lambda$, respectively. In particular, $(\hat{f}_n \# (\mu_n))$ converges weakly to $F_\# (\rho \lambda)$.

Now we are in the position to prove Proposition 3.2. This is made of by showing that the loss of mass in $Q$ only happens close to its boundary.

Proof of Proposition 3.2. To verify that $F_\# (\rho \lambda)|_Q = \lambda|_Q$, by Lemma 3.7 it is sufficient to show that the sequence $(\hat{f}_n |_{F^{-1}(Q)})\# (\mu_n)$ converges weakly to $\lambda|_Q$. We proceed in a similar way as in proof of Lemma 3.4 in [7], but with some variations along our demonstration. By definition of weak convergence of measures it is sufficient to prove that for a given function $\varphi \in C_0(I^2, \mathbb{R})$, the expression

$$
\left| \int_Q \varphi d\nu_n - \int_{\hat{f}_n |_{F^{-1}(Q)}} \varphi d(\hat{f}_n |_{F^{-1}(Q)}) \# (\mu_n) \right| \quad (3.3)
$$

tends to 0 when $k$ goes to $+\infty$. Observe that by the triangle inequality, the expression (3.3) can be bounded from above by

$$
\left| \int_Q \varphi d\nu_n - \int_{\hat{f}_n |_{F^{-1}(Q)}} \varphi d\nu_n \right| + \left| \int_{\hat{f}_n |_{F^{-1}(Q)}} \varphi d\nu_n - \int_{\hat{f}_n |_{F^{-1}(Q)}} \varphi d(\hat{f}_n |_{F^{-1}(Q)}) \# (\mu_n) \right| \quad (3.4)
$$

we denote by $T_{1,n_k}$ and $T_{2,n_k}$ the first and the second term in (3.4), respectively.

Notice that $T_{1,n_k}$ is at most $||\varphi||_\infty \nu_n(Q \Delta \hat{f}_n |_{F^{-1}(Q)})$. This last expression can be shown to be convergent to 0. Indeed, we have that
\[ Q \Delta \hat{f}_{n_k}(F^{-1}(Q)) \subset \bigcup_{i=1}^{N} F(H_i) \Delta \hat{f}_{n_k}(H_i) \subset \bigcup_{i=1}^{N} B(\partial F(H_i), ||F - \hat{f}_{n_k}||_\infty), \]

where in the second step we use Lemma 3.6 for \( U = H_i \). Thus, by using the weakly convergence of \( \nu_{n_k} \) to \( \lambda \), it follows that

\[ \nu_{n_k}(Q \Delta \hat{f}_{n_k}(F^{-1}(Q))) \leq \nu_{n_k}(B(\partial Q, ||F - \hat{f}_{n_k}||_\infty)) \longrightarrow 0. \]

Therefore, the first term in (3.4) converges to 0.

To prove that \( T_{2,n_k} \) tends to 0 we firstly note that this expression can be bounded from above by

\[ \frac{||\varphi||_\infty |A_{n_k}|}{l_{n_k}^2}, \tag{3.5} \]

where \( A_n := \hat{f}_n(H) \cap \frac{1}{l_n} \mathbb{Z}^2 \setminus f_n(H \cap R_n) \). Observe that \( \hat{f}_{n_k}(H) \) is contained in \( B(Q, ||F - \hat{f}_{n_k}||_\infty) \) and thus

\[ A_{n_k} \subset \left( B(Q, ||F - \hat{f}_{n_k}||_\infty) \cap \frac{1}{l_{n_k}} \mathbb{Z}^2 \right) \setminus f_{n_k}(H \cap R_{n_k} \cap B(0, ||x_{k,j(n_k)}||)). \tag{3.6} \]

From Lemmas 3.4 and 3.5 there is a large-enough positive integer \( k_0 \) such that for every \( k \geq k_0 \) the set \( H \cap R_{n_k} \cap \overline{B}(0, ||x_{k,j(n_k)}||) \) is non-empty and is mapped by \( f_{n_k} \) into \( \overline{Q}_{k,j(n_k)} \cap (1/l_{n_k}) \mathbb{Z}^2 \). In addition, we have that

\[ f_{n_k}(H \cap R_{n_k} \cap \overline{B}(0, ||x_{k,j(n_k)}||)) = (\overline{Q}_{k,j(n_k)} \cap 1/l_{n_k} \mathbb{Z}^2) \setminus f_{n_k}(f_{n_k}^{-1}(B(Q, 1/k)) \cap R_{n_k} \setminus H); \tag{3.7} \]

(remind that from Lemma 3.3 the set \( f_{n_k}^{-1}(B(Q, 1/k)) \) is contained in \( B(\mathcal{H}, (2Reg(F)^2 + 1)/k) \). On the other hand, Lemma 3.3 and the L-Lipschitz condition imply that \( f_{n_k}(f_{n_k}^{-1}(B(Q, 1/k)) \cap R_{n_k} \setminus H) \) is a subset of \( B(\partial Q, ||F - \hat{f}_{n_k}||_\infty + L(2Reg(F)^2 + 1)/k) \). Hence from (3.6) and (3.7) we obtain that

\[ A_{n_k} \subset B(\partial Q, \delta_{n_k}) \cap \frac{1}{l_{n_k}} \mathbb{Z}^2, \]

where \( \delta_{n_k} := \max\{|r - r_{k,j(n_k)}|, ||F - \hat{f}_{n_k}||_\infty + L(2Reg(F)^2 + 1)/k\} \), and \( r \) and \( r_{k,j(n_k)} \) are the radii of \( Q \) and \( Q_{k,j(n_k)} \), respectively. Thus, we can bound from above the expression (3.5) as

\[ ||\varphi||_\infty \frac{|A_{n_k}|}{l_{n_k}^2} \leq ||\varphi||_\infty \lambda \left( B \left( \partial Q, \delta_{n_k} + \frac{1}{l_{n_k}} \right) \right). \tag{3.8} \]

Notice that the right-hand side of (3.8) (and therefore the second term in (3.4)) tends to 0, as desired. \( \square \)
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Rodolfo Viera
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Av. Vicuña Mackenna 4860, Macul, Santiago, Chile
Email: rodolfo.viera@mat.uc.cl