Description of mesoscale pattern formation in shallow convective cloud fields by using time-dependent Ginzburg-Landau and Swift-Hohenberg stochastic equations

Diana L. Monroy and Gerardo G. Naumis

Dept. de Sistemas Complejos, Instituto de Física,
Universidad Nacional Autónoma de México
Apdo. Postal 20-364, 01000,
México City, CDMX, MEXICO.

e-mail: naumis@fisica.unam.mx

The time-dependent Ginzburg-Landau equation and the Swift-Hohenberg equation, both added with a stochastic term, are proposed to describe cloud pattern formation and cloud regime phase transitions of shallow convective clouds organized in mesoscale systems. The starting point is the Stechmann-Hottovy linear spatio-temporal stochastic model for tropical precipitation, used to describe the dynamics of water vapor and tropical convection. By taking into account that shallow stratiform clouds are close to a self-organized criticality and that water vapor content is the order parameter, it is observed that sources must have non-linear terms in the equation to include the dynamical feedback due to precipitation and evaporation. The inclusion of this non-linearity leads to a kind of time-dependent Ginzburg-Landau stochastic equation, originally used to describe superconductivity phases. By performing numerical simulations, pattern formation is observed, specially for cellular convective phases. These patterns are much better compared with real satellite observations than the pure linear model. This is done by comparing the spatial Fourier transform of real and numerical cloud fields. Finally, by considering fluctuation terms for the turbulent eddy diffusion we arrive to a Swift-Hohenberg equation. The obtained patterns are much more organized that the patterns obtained from the Ginzburg-Landau equation in the case of closed cellular and roll convection.

I. INTRODUCTION

Convective clouds are well known to be crucial components of weather and climate, being a key process not only in the transport of heat, moisture, momentum, and dynamical quantities in the atmosphere but also by strongly affecting solar and longwave radiation budgets from local to global scales [1, 2]. Historically, most research involving convective clouds has focused on deep rather than shallow clouds. However, shallow convective clouds have significant impacts on the mesoscale as well as for large scale atmospheric dynamics [3].

The study of shallow clouds is worthy for at least two reasons: first, they cool our planet reflecting a significant portion of the incoming solar radiation back to space contributing only marginally to the greenhouse effect; and second, shallow clouds cover large fractions of our planets sub-tropical oceans [2, 4]. Even changes in the order of 1% in cloud cover or other properties may significantly affect the overall radiation balance [5]. As a consequence, cloud feedback influences significantly the response of the climate system to global warming [1, 6].

Shallow clouds exhibit spatial organization over a wide range of scales [2, 7]. Compared to spatially homogeneous low clouds, these modes of organization could be significant for the radiative effect of convective organization. They presumably affect the interaction of convection with atmospheric humidity and thus cloudiness plays a role in climate variability [8]. Cloud systems formed by shallow convection have horizontal dimensions ranging from several to 100 or 200 kilometers. They are often characterized as mesoscale patterns [9] and are largely ignored in actual climate models [1].

Therefore, mesoscale systems need to be considered in climate-model parameterizations of the physical processes that affect the shallow clouds radiative response to climate perturbations [10]. At the same time, this is one of the challenges in climate sciences as contemporary climate models cannot resolve the length scales where it occurs [2]. Even the driving mechanisms responsible for these patterns are not completely well understood [11].

Stratocumulus clouds (Sc) are relevant examples of mesoscale organization of shallow convection on stratiform cloudiness. They have been studied in recent years due to their impact on the amount of sunlight reflected back to space [1, 12]. Covering approximately one-fifth of Earth’s surface in the annual mean, Sc are the dominant cloud type by area covered. Thus, there are few regions of the planet where these clouds are not climatologically important [13]. Sc are characterized by honeycomb-like patterns of stratiform cloudiness, arranged in either open or closed cells controlled by processes from the micrometer to the kilometer scale which interact in and above the scale $O(10-100km)$ of large-scale models [14].

The organization of Sc into cellular or roll convection could be considered in a first approximation as a form of Rayleigh-Bénard convection in the atmospheric boundary layer [15]. However, this mechanism does not completely explain the multiscale turbulent character of the mesoscale cloud convection (MCC) seen in observations, whereby other theories have been proposed to explain the driving of these patterns [10]. For Sc, in addition to the temperature difference between the lower boundary
There have been proposed many theoretical and numerical models. Two of the most investigated mechanisms are (1) cloud-aerosol precipitation interactions \cite{20} and (2) advection over warmer surface temperature and air temperature; instead of that, convection when there is a large difference between sea and cloudy air at their borders, closed cells (Fig. 1a) are formed in presence of upward motion and cloudy air in their centers and descending air at their interfaces. Heating from below is the key responsible process in open-cell convection when there is a large difference between sea surface temperature and air temperature; instead of that, radiative cooling of cloud tops is the key responsible process for closed-cell convection \cite{13 14 19}.

The transition from closed to open cellular convection is interesting from the system dynamics as well as from the perspective of radiative forcing of the climate but is not clearly understood yet. There have been proposed many theoretical and numerical models. Two of the most investigated mechanisms are (1) cloud-aerosol precipitation interactions \cite{20} and (2) advection over warmer surface temperature and air temperature; instead of that, convection when there is a large difference between sea and cloudy air at their borders, closed cells (Fig. 1a) are formed in presence of upward motion and cloudy air in their centers and descending air at their interfaces. Heating from below is the key responsible process in open-cell convection when there is a large difference between sea surface temperature and air temperature; instead of that, radiative cooling of cloud tops is the key responsible process for closed-cell convection \cite{13 14 19}.

FIG. 1. The four distinctive phases of shallow cloud organization: closed-cell stratocumulus, pockets of open-cell stratocumulus, and shallow cumulus viewed from satellite in panels a) to d), generated by the Stechmann and Hottovy model (Eq. 3) with the parameters proposed in Ref \cite{17} in panels f) to j) and by the non-linear idealized model (Eq. 11 in panels j) to m). See Supplemental Material for the parameter values \cite{18}. The data of the real fields was taken from the Moderate Resolution Imaging Spectroradiometer (MODIS) data, and from the Geostationary Satellite Server (GOES) data from NOAA.

**II. THE STECHMANN AND HOTTOVY LINEAR STOCHASTIC MODEL FOR MESOSCALE SHALLOW PATTERNS**

In this section, we explain the basic details of the Stechmann and Hottovy model \cite{26}, based upon a idealization of water vapor dynamics as a stochastic diffusion process. In this model, several effects of the physical processes involved in cellular convection are included: evaporation, turbulent advection-diffusion of water vapor and precipitation.

The Stechmann and Hottovy Model \cite{26} was proposed as a model for the dynamics of the cloudy boundary layer following the idealized simplification of models of phase transitions in other contexts. The model starts by considering the evolution of the total moisture content \( q = q(r, t) \) (water vapor plus condensed water, i.e, liquid and ice) in each planetary boundary layer (PBL) column at a horizontal spatial location \((x, y)\), normalized and shifted so that \( q = 0 \) represents the saturation level \cite{17}. Spatio-temporal changes, given by the convective derivative of \( q \), must be equal to the contribution of all...
sources and sinks such as precipitation or evaporation,
\[ \frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = S \]  
(1)

where \( \mathbf{v} \) is the velocity. By decomposing \( q \) as \( q = \bar{q} + q' \), where \( \bar{q} \) is a large-scale average component and \( q' \) is a small fluctuation part. Using Eq. (1), we obtain an equation for the large component [17],
\[ \frac{\partial \bar{q}}{\partial t} = \bar{S} - \nabla \cdot (\bar{q} \mathbf{v}) - \nabla \cdot (q' \mathbf{v}') \]  
(2)

where it was used that \( \bar{q}' = 0 \) and \( \mathbf{v}'_x = \mathbf{v}'_y = 0 \). Next the small-scale \( ux \) convergence term \( \nabla \cdot (q' \mathbf{v}') \) is approximated by a laplacian \( b \nabla^2 q' \), used to represent eddy diffusion and mixing due to turbulence. The parameter \( b \) is an effective diffusion constant. The nonlinear turbulent effects contained in \( \nabla (q \bar{v}) \) are taken into account by additional turbulent damping \( -q/\tau_0 \) and stochastic forcing, \( D \dot{W} \) [30]. The term \( \bar{q}/\tau_0 \) represents a relaxation, where the parameter \( \tau_0 \) is obtained through a careful analysis of the column-integrated water and precipitation rate [17]. The term \( D \dot{W} \) represents a stochastic forcing, and is used as the simplest model for the turbulent fluctuations and others physical processes with a random component, such as the entrainment. Finally, the source term \( S \) represents the net water sources and sinks, including precipitation and evaporation of water from the ocean surface. It is considered to contribute with a constant and deterministic forcing \( F_0 \), and a partial stochastic contribution, taken already into account in the constant \( D \).

Finally, the temporal evolution is given by the following equation [17],
\[ \frac{\partial q}{\partial t} = b \nabla^2 q - \frac{1}{\tau_0} q + F_0 + D \dot{W} \]  
(3)

where here, and to avoid overburden the notation, \( q \) represents the average part \( \bar{q} \). In what follows, the same convention will be used.

It has been shown that this model can be translated into a spin-like Hamiltonian system which presents phase transitions [26] once \( q \) is discretized using a function that takes the values 0 or 1 depending on the sign of \( q \). Typical clouds fields obtained through numerical simulations using this equation are shown in Fig. 1. Therein, we include real images from satellite to provide a comparison.

Although the model is able to reproduce the overall aspect of the fields and the phase transitions between them, it is also clear that there is much more organization in real cloud patterns for closed phases. To account for this, we have calculated the spatial Fourier transform of real closed-cell patterns taken from satellite photographs as well as from the outcome of Stechmann and Hottovy model, as seen in Fig. 2 a) and b).

Notice that in the case of the satellite photographs, we adjust the contrast and exposure of the original image -showed in Fig. 2 a)- before converting the grayscale image into a binary image. This is done to define the cells with more details and precision.

![Fourier spectrum](image-url)

**FIG. 2.** Fourier transform of the closed-cell phase. Panels in the left column show the cellular pattern taken from a) satellite photograph, c) Stechmann and Hottovy model, e) Ginzburg-Landau stochastic model and h) Swift-Hohenberg model. In the right column, we present the respective Fourier spectrums of each pattern. We can identity in panels d) and h) a dominant frequency with radial symmetry indicated by red arrows, corresponding to a characteristic length of \( \approx 14km \). The maximal spatial frequencies in panels h) and f) are determined by the resolution of the grid used in the simulation given in the units of \( k_x \) (see text). See Supplemental Material for the parameter values [18]. The data of the real fields was taken from the Moderate Resolution Imaging Spectroradiometer (MODIS) data and from the Geostationary Satellite Server (GOES) data from NOAA.
In Fig. 2 panel b) we can identify one spatial frequency (wave-vector) that reveals the existence of a particular structure. Nevertheless, in Fig. 2 d) we see that the Fourier transform of the outcomes obtained from the Stechmann and Hottovy model does not show any characteristic dominant structure. This is expected as the Stechmann and Hottovy is a linear model which does not couple modes [26].

It is important to remark here that in Fig. 2 b) there is a lower cut-off of the spectrum when compared with 2 d). This is due to the resolution of the grid used. Although one can increase the cut-off frequency by growing the number of points in the simulation mesh, it turns out that the phases and parameters of the Stechmann and Hottovy model depend upon the mesh. On the other hand, decreasing the resolution of the real cloud fields leads to a lower-quality Fourier image. A trade-off is thus needed to keep the original parameters of the Stechmann and Hottovy model and the best resolution of the real cloud fields. To solve this conundrum, here we adopted the policy of using absolute units in reciprocal space. These units are determined by the length \((L = 500)\) in \(\text{Km}\) of the real space field and the resolution of the photograph \((N_{\text{pixels}} \times N_{\text{pixels}} = 500 \times 500)\), resulting in the cut-off frequency \(k_x = \pm \pi N_{\text{pixels}}/L = \pm \pi [\text{Km}^{-1}]\). For the simulation, the mesh has \(N \times N\) points resulting in a cut-off frequency \(k_x = \pm \pi N/L = \pm \pi (N/500)[\text{Km}^{-1}]\).

### III. NON-LINEAR MODEL: TIME-DEPENDENT GINZBURG-LANDAU STOCHASTIC EQUATION

One of the most important points in the work of Stechmann and Hottovy is the recognition of \(q\) as an order parameter [26]. In general, pattern formation is governed by order parameters whose spatio-temporal behavior is determined by nonlinear partial differential equations [31]. This suggests that the extra features seen in real cloud patterns are due to non-linear effects. Following this idea, here we consider the cellular convective pattern described by a state vector \(p(r,t)\) which in this case corresponds to the cloud cover. Its evolution equation takes the general form of a partial differential equation [31]:

\[
\frac{\partial p(r,t)}{\partial t} = N[\nabla, p(r,t)]
\]

(4)

where \(N\) denotes a nonlinear function. The behavior of the state vector \(q(r,t)\) of the pattern forming system can be represented as a functional of one or several order parameters, denoted by \(\Phi(r,t)\) that often can be directly related to a physical observable [31],

\[
p(r,t) = Q[\Phi(r,t)]
\]

where \(Q\) is a functional of \(\Phi(r,t)\). In order to recover the linear equation proposed by Stechmann and Hottovy, in our model we identify \(\Phi(r,t) = q(r,t)\), i.e., the CWV in each column of the lattice. Thus, instead of solving the determining equations for the state vector \(p(r,t)\), the spatio-temporal evolution is in general determined by an equation for the order parameter field [31]. The most simple case is the following,

\[
\frac{\partial q}{\partial t} = L(\Delta)q + N[q]
\]

(5)

Here \(L(\Delta)\) is a linear operator and \(N[q,t]\) the non-linear functional that is approximated by a polynomial expansion of \(q\) in its low order derivatives.

Therefore, by comparing with Eq. (5) we can identify the operator \(L(\Delta)\) with \(\tau_0^{-1} + b\nabla^2\), while \(D\) and \(F_0\) are the tunable parameters that determine the strength of the random and deterministic forcing generated by internal forcing due to small scale cloud processes and large-scale external forcing, respectively. The transition of cloud area fraction (CAF) from a regime of closed cellular convection to a regime of pockets of open cells is determined by both parameters [7].

It is important to remark that Eq. (5) is linear in \(\Delta\). However, it is also possible to have non-linearity in \(\Delta\). A well known example is the Swift-Hohenberg equation, which arises as an order parameter equation for the Rayleigh convection [32, 33]. In the next section this will be explained in detail.

For the moment, let us stick with the simple model given by Eq. (5) to indicate how non-linear terms arise. We start by pointing out that several observational data and numerical studies have documented how the relationship between precipitation and water vapor is crucial for precipitation prediction in the context of convective parametrizations. Peters and Neelin [22, 25] showed that there is a critical value \(q_c\) of the CWV where the mean precipitation \((P(q))\) increases rapidly as an approximate power law, i.e., \((P(q)) \sim (q - q_c)^{\beta}\) for \(q > q_c\). As \(\beta < 1\), the precipitation variance has a strong peak at the critical value \(q_c\) and then diminishes [33, 36].

It has been argued that the mechanism presents a tendency to self-maintain at criticality instead of being simply controlled by an external parameter [22, 25]. In fact, self-organized critically (SOC) has been proposed to describe macroscopic critical phenomena such as organized structures associated with atmospheric convection [37].

This organization mechanism is supported by observations which exhibit that, even when the system hardly exceeds \(q_c\), the CWV tends to decay more slowly than an exponential rate toward the higher values, reflecting the tendency towards SOC [22, 25]. The same studies show a scale invariance suggesting a scaling law (fractal) for atmospheric convection. Moreover, the invariance under spatial averaging suggests the applicability of the renormalization group (RNG), also supported by the SOC approach [14, 25].

In the original Stechmann and Hottovy model, the relaxation time \(\tau_0^{-1}\) and the forcing \(F_0\) was adjusted in such a way that different assumed models of the precipi-
tation ratio fitted the results of Peters and Nelling for the precipitation conditional probability. If \( r_{i,j} \) is the precipitation ratio for a cell with integer coordinates \((i,j)\) in a square mesh, there are two precipitation models, the first model is the BettsMiller-like rain rate model \[20\],

\[
 r_{i,j} = |F_0|\sigma_{i,j}
\]

(6)

the other was provided by Stechmann and Hottovy \[17\],

\[
 r_{i,j} = |F_0| + q_{i,j}/\tau_0 \sigma_{i,j}
\]

(7)

where \( \sigma_{i,j} = 1 \) if \( q > 0 \), and \( \sigma_{i,j} = 0 \) otherwise. Notice that \( \sigma_{i,j} \) is analogous to a spin variable. Its role is to signal whenever \( q \) is above the precipitation threshold \( q = 0 \). Then is possible to have rain.

While the conditional probability for precipitation can be obtained from the distribution function of \( q \), the linear model does not provide a feedback threshold due to precipitation in the source term \( S \). In other words, the precipitation can be calculated a posteriori once the model is solved, but in the linear model it does not enter into the calculation. We require that \( S \) depends on \( q \).

Therefore, to improve the model one needs to include the fact that once the threshold for precipitation is reached, indicated by the spin variable \( \sigma_{i,j} \), the source term will change. In fact, \( \sigma_{i,j} \) can be used to derive an equivalent Ising Hamiltonian for the cloud field \[17\]. Now comes the question, what is the most simple and natural choice for the feedback term? Following the Ising analogy, we can replace the spins \( \sigma_{i,j} \) by the known Ising mean field, \( \bar{\sigma} \approx (1 + \tanh(q/T))/2 \) with \( T \) a constant. Notice how the field is shifted to have \( \sigma_{i,j} \approx \bar{\sigma} = 0 \) for \( q \rightarrow -\infty \) and \( \bar{\sigma} = 1 \) for \( q \rightarrow +\infty \). This results on two possible average precipitation rates \( \bar{\tau} \) depending upon the used model,

\[
\bar{\tau} \approx \frac{1 + \tanh(q/T)}{2} |F_0|
\]

(8)

or,

\[
\bar{\tau} \approx \frac{1 + \tanh(q/T)}{2} \left[ |F_0| + \frac{q}{\tau_0} \right]
\]

(9)

As we are interested in the region around the threshold, i.e., near the linear model, we can expand the hyperbolic tangent to obtain, using Eq. (8), the simplest model,

\[
\bar{\tau} \approx \left( 1 + \frac{q}{T} - \frac{1}{3} \left( \frac{q}{T} \right)^3 + \frac{2}{15} \left( \frac{q}{T} \right)^5 + \ldots \right) |F_0|/2
\]

(10)

Thus, we generated a non-linear term able to model dynamically a precipitation threshold. Although in principle we can just modify the sources term in Eq. (3) by using \( \bar{S} \rightarrow \bar{S} - \bar{\tau} \), it will be unwise not to recognize that sources must also depend dynamically on \( q \), as for example, the conditional probability of having an increased \( q \) grows once precipitation occurs \[28, 39\]. Thus, we left open the possibility of having an interplay between sources and sinks by the replacement \( \bar{S} \rightarrow F_0 + DW - \bar{\tau} + \bar{s} \) where \( \bar{s} \) is an average dynamic source. The most simple model is to assume \( \bar{s} = f\bar{r} \) where \( f \) controls the relative weight between sources, like evaporation, and precipitation. The parameter \( f \) allows an interplay between two kinds of non-linear regimes, one dominated by sinks the other by sources.

Finally, we can now include, up to third order, the sources and sinks terms in Eq. (3) to obtain the following non-linear model built from Eq. (9) BettsMiller-like rain rate precipitation model,

\[
\frac{\partial q}{\partial t} = b\nabla^2 q + Eq - Kq^3 + DW + F
\]

(11)

where the constants are given by,

\[
E = \frac{1}{\tau_s} - \frac{1}{\tau_0}, \quad K = \frac{1}{3\tau_s^2 T^2}, \quad F = \left( \frac{f + 1}{2} \right) |F_0|
\]

(12)

with,

\[
\frac{1}{\tau_s} = \left( \frac{f - 1}{2} \right) |F_0|/T
\]

(13)

The model given by Eq. (11) take the same form of the celebrated time-dependent Ginzburg-Landau equation \[40, 41\], now added with stochastic noise. This coincides with the idea that most classical models for phase transitions are inherently nonlinear \[12\] and at the same time, satisfies one of the conditions of SOC: non-linear interaction, normally in the form of thresholds \[13\]. In Eq. (11), the threshold transition parameter \( T \) and the ratio \( f \) control the time parameter \( \tau_s \). This is a new characteristic time that competes with the damping time \( \tau_0 \).

Also, we can use the alternative Stechmann-Hottovy precipitation model given by Eq. (9). Up to terms of order \( q^3 \), we obtain a general model that contains the Ginzburg-Landau as a particular case,

\[
\frac{\partial q}{\partial t} = b\nabla^2 q + \frac{q}{\tau_s} + Gq^2 - Kq^3 + DW + F
\]

(14)

where \( G \) defined as,

\[
G = f - 1 \frac{2T}{\tau_0},
\]

(15)

The main difference between Eq. (11) and (14) is the quadratic term which vanishes in the Betts Miller-like rain rate model, resulting in the Ginzburg-Landau equation. As is well known, the quadratic term in the Ginzburg-Landau equation does not appear due to symmetry considerations. Here we will only study the Ginzburg-Landau equation, as the resulting pattern obtained from the second model were very different from real fields.

Fig. (1 i)-l) shows the outcomes of the first model found solving numerically Eq. (11). Further details of
FIG. 3. Phase diagram of shallow cloud regimes for the Ginzburg-Landau non-linear stochastic model given by Eq. (11). The plot shows the mean cloud area fraction (CAF) as a function of variability, D, and the net source/sink parameter F. The transition from open to close cells is clearly seen as a transition from high to low values of the CAF.

the simulations are explained in the Supplemental Material. Clearly, much more structure is observed in the non-linear model when compared with the pure linear one. This is especially visible for intermediate regimes where the POCs are well defined.

As was done previously with the linear model, we further compare the outcomes of our non-linear model with the original cloud formations using Fourier spectrum and the closed-cell convection as reference. Fig. 2f) reveals the presence of a dominant frequency as observed in real patterns (Fig. 2b). Thus, even the most simple Ginzburg-Landau model presented here show results closer to reality than the linear model.

A. Phase Transitions Diagrams

The model outputs in Fig. 1, panels e)-h) present the four phases of cloud organization shown in observational data from panels a)-d), respectively. It is possible to see the transition from closed-cells to pockets of open cells (POCs). These four cloud regimes correspond to four distinct parameter regimes of Eq. (11) where F and D are the tuning parameters which determine the transition of phase.

To clearly see the change between one and another phase, we use a phase diagram of cloud regimes using statistics moments as shown in Figures 3 and 4. In the first diagram, the mean cloud area fraction (CAF) is calculated as a function of D and F, i.e., \( \langle \sigma \rangle = \langle \sigma(F,D) \rangle = \sum_{i,j} \sigma_{i,j} \) in the stationary state and by fixing \( \tau_0 \) and b. Moreover, the plot in Fig. 4 provides the standard deviation, which is a measure of the statistical sensitivity. Its maximum is associated with the transition of regime in the phenomena of self-organized criticality [22, 25, 34].

In Fig. 3 is notorious the phase diagram regions belonging to each regime: the closed-cell regime corresponds to \( F > 0 \) and the open-cell regime corresponds to \( F < 0 \), as indicated by the mean CAF, since while the average value cloud area of open cells is 1, the mean of the closed ones is 0. On the other hand, the POCs could be seen in the middle of both regimes as their transition in the region around \( F = 0 \) with intermediate values of the mean CAF between 0 and 1. All these cellular regimes are associated with intermediate values of D.

Finally, to have a measure of the climate response or climate uncertainty, in Fig. 4 we present the standard deviation of the cloud area fraction (STDCAF). The open and closed cellular regimes are associated with low values of the STDCAF. The POCs and shallow phases are associated with high values of the STDCAF, indicating how small changes in F or D lead to very large changes in the CAF. It also shows how the STDCAF increases drastically out of the regions where it presents the closed or open cellular patterns.

IV. STOCHASTIC SWIFT-HOHENBERG MODEL

In spite that the non-linear models already show much more organization, figures a) and reveal that real
cloud fields still can be much more organized. The next natural step is to change \( L(\Delta) \) by a non-linear operator for which, it has resorted to the amplitude equations formalism used for many types of pattern forming systems \[14\, 15\]. The method to derive the non-linear operator has been described multiple times before; briefly, it consists of an expansion of the full equation solution by writing the leading term of this expansion as the product of a slowly varying amplitude and a critical solution of the linearized equations of motion that, in the Rayleigh-Bénard case, corresponds to a critical wavenumber \( k_c \). \[45, 46\].

Therefore, considering the expansion of \( N[q] \) in Eq. (14), we obtain a stochastic Swift-Hohenberg equation,

\[
\frac{\partial q}{\partial t} = \left[ \epsilon - (k_c^2 + \nabla^2)^2 \right] q + Gq^2 - Kq^3 + F + D\dot{W}
\]

The solutions of Eq. (16) are still in the process of being investigated \[17\]. This is the general form, and probably the most simple model in the development of the Ginzburg-Landau theory of amplitude equations \[18\]. In fact, the Ginzburg-Landau model could be recovered by rescaling the long spatial and time scales \[45, 49, 50\].

Eq. (16) can be solved numerically through implicit finite differences and a successive over-relaxation (SOR) method as proposed by S. Sanchez Prez-Moreno et al. \[51\]. In Fig. 2g) and Fig. 5c) we show the formation of two particular patterns that arise in the Rayleigh-Bénard convection, hexagons and rolls. Further details of the simulations are explained in the Supplemental Material \[18\]. Both patterns have been identified as ways of organization in Sc clouds and their formation depends on the parameter \( g \) that controls the strength of the quadratic nonlinearity. In Fig. 2 panels a), g) and in Fig. 5 panels a), c) we compare satellite photographs with simulations of hexagons and rolls, respectively; we can see clear similarities with the satellite patterns. To confirm the similarities, the Fourier spectrums of the real and simulated cloud formations were performed.

In Fig. 2 panels b), h), the hexagonal pattern spectrum reveals the presence of a dominant frequency for a cut along a certain direction. Although not shown here, the spectrum has nearly radial symmetry. In Fig. 2 we can identify a principal frequency and other harmonics of lower amplitude. This coincides with the spectrum of a cellular pattern with defects and not highly ordered as a result of the forcing added in Eq. (16), which generates different sizes of cells without a particular tessellation. On the other hand, in Fig. 5 panels b), c) we show the presence of a dominant frequency with axial symmetry that corresponds to a pattern formed by parallel rolls in real space. In both kinds of convection, the simulations recover the structures formed in real clouds fields.

FIG. 5. Fourier transform of the horizontal convective rolls. Panels in the left column show the horizontal convection pattern taken from a) satellite photograph and c) the Swift-Hohenberg model given by Eq. (16). In the right column are presented the respective Fourier spectra. We can identity in panels b) and d) a dominant frequency with axial symmetry indicated by red arrows. See Supplemental Material for the parameter values \[18\]. The data of the real fields was taken from the Moderate Resolution Imaging Spectroradiometer (MODIS) data, and from the Geostationary Satellite Server (GOES) data from NOAA.

V. CONCLUSIONS

Following the work of Stechmann and Hottovy, we proposed a non-linear differential equation for an order parameter field given by the column water vapor \( q(r,t) \) to describe the transitions of various pattern formations in mesoscale shallow clouds systems. The main modification introduced to the original linear model is the possibility of a feedback due to sources. In particular, we used two precipitation rate models, one leading to a time-dependent stochastic Ginzburg-Landau equation while the other adds a quadratic term to this equation.

On the other hand, following the theory of order parameter, we used the Swift-Hohenberg equation, proposed as a simple model for the Rayleigh-Bénard instability of roll and hexagonal waves, to describe both formations present in clouds fields. By adding a deterministic and stochastic damping, we found the closed-cellular and horizontal convection phases.

The success of both models can be appreciated by observing the real patterns in Fig. 1. Therein, we identified that the three patterns corresponding to MCC are not in a perfectly hexagonal arrangement (highly ordered) nor are they arranged in complete randomness (highly disor-
The distributions of cumulus, both in closed and open-cells, appear in some arrangement between these two extremes. Both proposed non-linear models are closer from this dominant structure that the linear one, while the non-linear operator $L(\Delta)$ present in the Swift-Hohenberg equation allows the formation of patterns with a clear organization for two characteristic convective regimes.

**ACKNOWLEDGMENTS**

In Fig.1, 2 and 5 the satellite images were taken from the Moderate Resolution Imaging Spectroradiometer (MODIS) data, available from NASA at https://earthobservatory.nasa.gov, and from the Geostationary Satellite Server (GOES) data from NOAA at https://www.nesdis.noaa.gov.

We thank UNAM DGAPA-PROJECT IN102620. D. L. Monroy thanks a scholarship from DGAPA-UNAM. We also thank Graciela B. de Raga and Michel Flores for sharing comments and clarifying certain points. We also acknowledge helpful advice from Gerardo Ruiz-Chavarria on how to properly perform the simulations of the Swift-Hohenberg equation.

[1] T. Schneider, J. Teixeira, C. S. Bretherton and F. Brient, “Climate goals and computing the future of clouds,” Nature Climate Change 7(1), 35 (2017).
[2] L. Nuijens and A. P. Siebesma, “Boundary layer clouds and convection over subtropical oceans in our current and in a warmer climate,” Current Climate Change Reports 5(2), 8094 (2019).
[3] A. Deng, N. L. Seaman, and J. S. Kain, “A shallow-convection parameterization for mesoscale models. part i: Submodel description and preliminary applications,” Journal of the Atmospheric Sciences 60, 3456 (2002).
[4] Stephan Rasp, Hauke Schulz, Sandrine Bony, and Bjorn Stevens, “Combining crowd-sourcing and deep learning to understand meso-scale organization of shallow convection,” (2019).
[5] G Dagan, I Koren, O Altaratz1, and G Feingold, “Feedback mechanisms of shallow convective clouds in a warmer climate as demonstrated by changes in buoyancy,” Environmental Research Letters 13(5) (2018).
[6] P. Ceppi, F. Brient, M. D. Zelinka, and D. L. Hartmann, “Cloud feedback mechanisms and their representation in global climate models,” Wiley Interdisciplinary Reviews: Climate Change 8(4) (2017).
[7] Boualem Khouider and Alexander Bihlo, “A new stochastic model for the boundary layer clouds and stratocumulus phase transition regimes: Open cells, closed cells, and convective rolls,” Journal of Geophysical Research: Atmospheres, 124, 367386 (2019).
[8] S. Bony, I. Tobin and and R. Roca, “Observational evidence for relationships between the degree of aggregation of deep convection, water vapor, surface fluxes, and radiation,” Journal of Climate 25, 6885904 (2012).
[9] Jr. Robert A. Houze, Cloud Dynamics (Oxford, UK, 1998).
[10] R. Vogel, L. Nuijens, and B. Stevens, “Influence of deepening and mesoscale organization of shallow convection on stratiform cloudiness in the downstream trades,” Quarterly Journal of the Royal Meteorological Society 146, 174185 (2019).
[11] C Robert Wood and Jonathan P. Taylor, “Liquid water path variability in unbroken marine stratocumulus cloud,” Quarterly Journal of the Royal Meteorological Society 127(578) (2006).
[12] T. Schneider, M. K. Colleen, and K. G. Pressel, “Possible climate transitions from breakup of stratocumulus decks under greenhouse warming,” Nature Geoscience 12(3), 164–168 (2019).
[13] Robert Wood, “Review: Stratocumulus clouds,” 140, 2373–2423 (2012).
[14] Saskia Noteboom, “Open cell convection and closed cell convection,” De Bilt: KNMI (2006).
[15] E. M. Agee, T. S. Chen, and K. E. Dowell, “A review of mesoscale cellular convection,” Bulletin of the American Meteorological Society 54, 10041012 (1973).
[16] C. S. Bretherton and P. N. Blossey, “Understanding mesoscale aggregation of shallow cumulus convection using large eddy simulation,” Nature Geoscience 9(8), 2798–2821 (2017).
[17] Scott Hottovy and Samuel N. Stechmann, “A spatiotemporal stochastic model for tropical precipitation and water vapor dynamics,” 72, 4721–4738 (2015).
[18] See Supplemental Material at [URL will be inserted by publisher] for more details on the numerical solutions of the models presented, specifically the parameters values used in each cloud regime.
[19] Isabel L. McCoy, Robert Wood, and Jennifer K. Fletcher, “Identifying meteorological controls on open and closed mesoscale cellular convection associated with marine cold air outbreaks,” Journal of Geophysical Research: Atmospheres 122(21) (2017).
[20] G. Feingold, I. Koren, T. Yamaguchi, and J. Kazil, “On the reversibility of transitions between closed and open cellular convection,” Atmospheric Chemistry Physics 15, 73517367 (2005).
[21] T. Yamaguchi and G. Feingold, “On the relationship between open cellular convective cloud patterns and the spatial distribution of precipitation.” Atmospheric Chemistry and Physics 15(3), 2371251 (2015).
[22] Ole Peters, J. David Neelin, and Stephen W. Nesbitt, “Mesoscale convective systems and critical clusters,” 66, 2913–2924 (2009).
Franziska Glassmeier and Graham Feingold, “Network approach to patterns in stratocumulus clouds,” PNAS (2017), 10.1073/pnas.1706495114.

Per Bak, Chao Tang, and Kurt Wiesenfeld, “Self-organized criticality,” Physical Review A 38(1), 364–365 (1988).

Ole Peters and J. David Neelin, “Critical phenomena in atmospheric precipitation,” Nature Physics 2, 393–396 (2006).

Scott Hottovy and Samuel N. Stechmann, “Cloud regimes as phase transitions,” 43, 65796587 (2016).

B. Stevens, G. Vali, R. Wood K. Comstock, M.C. van Zanten, P.H, C.S. Bretherton Austin, and D.H. Lenschow, “Pockets of open cells ans drizzle in marine stratocumulus,” Bulletin of the American Meteorological Society 86, 5158 (2005).

Scott Hottovy and Samuel N. Stechmann, “Convective and strati-form components of the precipitationmoisture relationship,” Geophysical Research Letters 42 (23) (2015).

A. K. Betts and M. J. Miller, “A new convective adjustment scheme. part ii: Single column tests using gate wave, bomex, atex and arctic air-mass data sets.” 112(473), 693709 (1986).

A. J. Majda and M. J. Grote, “Mathematical test models for superparametrization in anisotropic turbulence.” Proceedings of the National Academy of Sciences 106(14), 54705474 (2009).

Svetlana V. Gurevich, “Numerical methods for complex systems ii chapter 4: Swift-hohenberg equation,” Westflische Wilhelms-Universitt (2017).

M. C. Cross and P. C. Hohenberg, “Pattern formation outside of equilibrium,” Reviews of Modern Physics 65(3), 8511112 (1993).

J. Swift and P. C. Hohenberg, “Hydrodynamic fluctuations at the convective instability,” Physical Review A 15(1), 319328 (1977).

Samuel N. Stechmann and J. David Neelin, “A stochastic model for the transition to stong convection.” Journal of the Atmospheric Sciences 68(12), 2955–2970 (2011).

Christopher E. Holloway and J. David Neelin, “Moisture vertical structure, column water vapor, and tropical deep convection.” Journal of Atmospheric Sciences 66, 1665–1683 (2009).

Christopher S. Bretherton, Matthew E. Peters, and Larissa E. Back, “Relationships between water vapor path and precipitation over the tropical oceans,” Journal of Climate 17, 15171528 (2004).

Jun-Ichi Yano, Changhai Liu, and Mitchell W. Moncrief, “Self-organized criticality and homeostasis in atmospheric convective organization,” Journal of the Atmospheric Sciences 69, 3449–3462 (2012).

M. D. Lebsock, T. S. LEcuyer, and R. Pincus, “An observational view of relationships between moisture aggregation, cloud, and radiative heating profiles.” 38, 12371254 (2017).

C. E. Holloway and J. D. Neelin, “Temporal relations of column water vapor and tropical precipitation.” Journal of the Atmospheric Sciences 67, 10911105 (2010).

J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, “The theory of critical phenomena: An introduction to the renormalization group,” (1992) Chap. 7.

N. Komin, L. Lacasa, and R. Toral, “Critical behavior of a ginzburglandau model with additive quenched noise.” Journal of Statistical Mechanics: Theory and Ex-periment 12, P12008 (2010).

J. M. Yeomans, “Statistical mechanics of phase transitions,” (1992).

Nicholas W. Watkins, Gunnar Pruessner, Sandra C. Chapman, Norma B. Crosby, and Henrik J. Jensen, “25 years of self-organized criticality: Concepts and controverses,” Space Science Reviews 198, 3–44 (2016).

M. Cross, “Lecture 8 supplementary notes: Amplitude equations.” (2006).

M. van Hecke, P. C. Hohenberg, and W. van Saarloos, “Amplitude equations for pattern forming systems.” V8. 245278 (1994).

M. Cross and H. Greenside, Pattern formation and dynamics in nonequilibrium systems (Cambridge University Press, 2009).

Peng Gao, “The stochastic swift–hohenberg equation,” Nonlinearity 30, 3516–3559 (2017).

A. Doelman and G. Schneider, “Lecture notes in the complex ginzburg-landau equation,” (2001).

K. Klepel, D. Blmker, and W. Mohammed, “Amplitude equation for the generalized swift hohenberg equation with noise.” Zeitschrift fr angewandte Mathematik und Physik 65 (2013), 10.1007/s00033-013-0371-8.

W.V. Saarloos, “The complex ginzburglandau equation for beginners.”, 1931 (1994).

Sebastian Sanchez Perez-Moreno, Mara Sabina Ruiz Chavarria., and Gerardo Ruiz-Chavarra, “Numerical solution of the swifthohenberg equation,” in Experimental and Computational Fluid Mechanics. Environmental Science and Engineering, edited by J. Klapp and A. Medina (2014) pp. 409–416.
Supplemental Material

The domain and discretization, initial and boundary conditions, as well as the parameters values used in the numerical solutions of the models presented in the main text, are explain in detail here. For each cloud regime formed by the models, we specify the tuning parameters that were used.

A. The Stechmann and Hotovy linear Stochastic Model for mesoscale shallow patterns

In Fig. 1 panels a)-d), the outcomes of the Eq. (3) were numerically solved using implicit finite differences with the same parameter values proposed by Stechmann and Hotovy [17, 26]. A two-dimensional discrete spatial grid in a domain of $L$ by $L$, where $L = 500$ km divided in a $N$ by $N$ lattice with $N = 100$ and lattice spacing of $\Delta x = \Delta y = 5$ km; this was chosen to be roughly the smallest width of individual cells of tropical deep convection. The boundary and initial conditions were considered as periodic and random, respectively. It was defined $q_{i,j}(t)$ as the integrated CWV and $W_{i,j}(t)$ as the independent white noises, denoted formally as the derivative of a Wiener process [17, 26], in the $(i,j)th$ column of the atmosphere for $i,j = 1,...,N$.

The parameters $b$ and $\tau_0$ conserves the values $b = 25 \text{ mm}^2 \cdot \text{hr}^{-1}$ and $\tau_0 = 100 \text{ hr}$ proposed in [17, 26]. In each phase of Fig. 1, the parameter values used were a) $D = 1.55 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = 0.12 \text{ mm} \cdot \text{day}^{-1}$, b) $D = 1.94 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = 0.048 \text{ mm} \cdot \text{day}^{-1}$ c) $D = 1.55 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = -0.12 \text{ mm} \cdot \text{day}^{-1}$ and d) $D = 11.62 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = -0.72 \text{ mm} \cdot \text{day}^{-1}$.

B. Non-linear model: time-dependent Ginzburg-Landau stochastic equation

In Fig. 1 panels i)-j), the outcomes of the Eq. (12) used the same domain and discretization as well as initial and boundary conditions of the linear model simulations. The parameters $b$ and $\tau_0$ conserves the same value proposed by Stechmann and Hotovy [17, 26], while different values of $F$ and $D$, in the same range used by them ($F_0 \sim \pm 1 \text{ mm} \cdot \text{day}^{-1}$ and $D \sim 10 \text{ mm} \cdot \text{hr}^{-1/2}$), were explored to find the regimens observed in Fig. 1, panels i)-j). The dynamics of the non-linear terms in Eq.(12) was determined by the parameters $E$ and $K$ whose values, after an exploration of different orders of magnitude, were fixed in $E = 8.5 \text{ hr}^{-1}$ and $K = 6.5 \text{ mm}^2 \cdot \text{hr}^{-1}$. The increase of both parameters is associated with a major percolation in the boundaries around open or closed clusters to the same $F$ and $D$ values.

In particular, the parameter values used in Fig. 1 for Eq. (12) were i) $D = 8.5 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = 1 \text{ mm} \cdot \text{day}^{-1}$, j) $D = 9 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = 0.2 \text{ mm} \cdot \text{day}^{-1}$ k) $D = 8.55 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = -1 \text{ mm} \cdot \text{day}^{-1}$ and l) $D = 10.25 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = -0.4 \text{ mm} \cdot \text{day}^{-1}$.

C. Stochastic Swift-Hohenberg model

In Fig. 2 g) and Fig. 5 c) we show the formation of two particular patterns that arise in the Rayleigh-Bnard convection, hexagons and rolls. Eq. (16) was solved numerically through implicit finite differences and a successive over-relaxation (SOR) method as proposed by S. Sanchez Prez-Moreno et al. [51].

For the simulations showed, the numerical method used a two-dimensional discrete spatial grid in a domain of $L$ by $L$, where $L = 500$ km was divided in a $N$ by $N$ lattice with $N = 200$ and lattice spacing of $\Delta x = \Delta y = 2.5$ km. In this case, this discretization was chosen to approximate the cell diameter of the real ones. The boundary and initial conditions were considered again as periodic and random. In the SOR method, it was used as the iteration step $k = 15$ and as the relaxation factor $w = 1.3$.

To form each pattern, the parameters were fixed as follows: in Fig.2 g) $\epsilon = 0.1$, $k_c = 1.3 \text{ m}^{-1}$, $g = 1$, $D = 0.15 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = 0.1 \text{ mm} \cdot \text{day}^{-1}$, and in Fig.5 c) $\epsilon = 0.3$, $k_c = 1.2 \text{ m}^{-1}$, $g = 0$, $D = 0.3 \text{ mm} \cdot \text{hr}^{-1/2}$, $F = 0.25 \text{ mm} \cdot \text{day}^{-1}$.