Strong and weak convergence rates of finite element method for stochastic partial differential equation with non-globally Lipschitz coefficients

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Abstract: Strong and weak approximation errors of a spatial finite element method are analyzed for the stochastic partial differential equations (SPDEs) with non-globally Lipschitz coefficients, including the stochastic Allen–Cahn equation, driven by additive noise. In order to give the strong convergence rate of the finite element method, we present an appropriate decomposition and some a priori estimates of the discrete stochastic convolution. To investigate the weak error, we first regularize the original equation by the splitting technique and obtain the regularity of the corresponding regularized Kolmogorov equation. Meanwhile, we present the refined estimates and the regularity in Malliavin sense of the finite element methods. Combining with the regularity of regularized Kolmogorov equation and Malliavin integration by parts, we give the weak convergence rate of the finite element method.

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1. Introduction

Both strong and weak convergence rates of numerical approximations for SPDEs with globally Lipschitz continuous and regular nonlinearities have been studied over past decades (see [JK11, Kru14] and references therein). In contrast to the Lipschitz case, the strong and weak convergence rates of numerical approximations for SPDEs with non-globally Lipschitz continuous nonlinearity, especially the stochastic Allen–Cahn equation, become more involved recently (see, e.g., [BGJK17, BJ16, BCH18, BG18a, KLL18, LQ18, MP17, QW18, Wan18]) and are far from well-understand. We refer to [BGJK17, BJ16, BCH18, LQ18, MP17, QW18, Wan18] and references therein for the strong convergence rate results of many different temporal and spatial approximations, and to [BG18b, CH18] for the weak convergence rate result of temporal splitting type schemes. Up to now, there have been no essentially sharp weak convergence rates of spatial approximations for parabolic SPDEs with non-globally Lipschitz coefficients. The present work makes further contributions on the strong and weak convergence rates of spatial approximations for SPDEs with non-globally Lipschitz continuous nonlinearity but monotone nonlinearity driven by additive noise.

Let $H = L^2(O)$ be the real separable Hilbert space endowed with usual inner product and $O = [0, L]$. In this article, we mainly focus on the following semi-linear parabolic SPDE,

\begin{align}
\frac{dX(t) + AX(t)dt}{dt} = F(X(t))dt + dW(t), \quad t \in [0, T] \\
X(0) = X_0,
\end{align}

where $0 < T < \infty$, $A : D(A) \subset H \to H$ is a linear, densely defined, positive self-adjoint operator with compact inverse. $F$ is a Nemytskii operator defined by $F(X)(\xi) := f(X(\xi)), \xi \in O$, where $f$ is a real-valued non-linear function and satisfies Assumption 2.3. In particular, Eq. (1) is the stochastic Allen–Cahn equation if $F(X) = X - X^3$. The stochastic process $\{W(t)\}_{t \geq 0}$ is a generalized Q-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$. Under further assumptions on $X_0$, $Q$ and $f$ and $\|A^{-\frac{1}{2}}\|_{L^2} < \infty$, $\beta \in (0, 1]$, similar arguments in [BCH18, QW18] yield that there is a unique mild solution $X$ of Eq. (1), which possesses the optimal spatial regularity $\mathbb{E}\left[\|X(t)\|_{H^\beta}^p\right] \leq C(T, Q, X_0, p), p \geq 1$.

One main contribution of this article is applying the variational approach, combined with an appropriate error decomposition, to deduce the strong con-
vergence rate of the finite element method for Eq. (1) with non-globally Lipschitz coefficient under very mild assumption on $X_0$. The corresponding finite element approximation $X^h$ satisfies

$$dX^h(t) + A_hX^h(t)dt = P^hF(X^h(t))dt + P^h dW(t),$$

(2)

$$X(0) = X_0^h,$$

where $P^h$ is the Galerkin finite element projection and $A_h$ is the discretization of $A$. As the considered noise in Eq. (1) may be rougher than trace-class noise, a priori estimate of approximated noise is needed. We make use of the properties of $S^h$ and $P^h$ to get the non-uniform estimate of the approximated stochastic convolution $Z^h$, and we obtain the sharp strong convergence rate, for $X_0 \in C(O), T > 0, \ p \geq 1,$

$$\mathbb{E}[\|X(T) - X^h(T)\|_p^p] \leq C(X_0; T, p, \gamma)(1 + T^{-\frac{\gamma}{2}})^{p\beta\gamma},$$

where $\gamma \leq \beta$, if $\beta \in (\frac{1}{2}, 1]$ and $\gamma < \beta$, if $\beta \in (0, \frac{1}{2}]$.

Another main contribution is about the weak convergence rate of the finite element method for Eq. (1) with non-globally Lipschitz coefficient. In recent years, there already exist many different strategies on the weak error analysis for many different numerical schemes approximating parabolic SPDEs with Lipschitz coefficients. We refer to e.g. [AL16, BD17, Deb11] for the error analysis based on the associated Kolmogorov equation, to e.g. [CJK14, HJK16] for applying the mild Itô formula approach and to e.g. [AKL16, WG13] for other techniques. To the best of our knowledge, there exist no essentially sharp weak convergence rates of spatial approximations for parabolic SPDEs with non-globally Lipschitz coefficients. There are three key points to deduce the weak convergence rate of numerical approximations for Eq. (1) with non-globally Lipschitz coefficient: to give a priori estimate of the corresponding Kolmogorov equation, to deduce the uniform estimate of the spatial approximations and to get rid of the irregular terms of the weak error. Inspired by [BG18b] which shows the weak convergence order $\frac{1}{2}$ of the two temporal splitting type schemes approximating stochastic Allen–Cahn equation driven by space-time white noise, we first apply the splitting strategy to regularize the original equation. Then we utilize the properties of $S^h, P^h$ and $A_h$, as well as the non-uniform estimate of the approximated stochastic convolution $Z^h$ to get a priori estimate of the finite element approximations. At last, we use the Malliavin integration by parts, together with the regularity of the
regularized Kolmogorov equation and a priori estimate of the finite element approximation, to derive the optimal weak convergence rate result of \( X^h \), for \( \phi \in C^2_b(H), \ X_0 \in C(O), \ T > 0, \ \gamma < \beta \),

\[
\mathbb{E} \left[ \phi(X(T)) - \phi(X^h(T)) \right] \leq C(X_0, T, \gamma)(1 + T^{-\gamma})h^{2\gamma}.
\]

The outline of this paper is as follows. In the next section, some preliminaries are listed. Section 3 is devoted to giving the regularity and a priori estimate of Eq. (1), the strong convergence rate of the finite element method, as well as the a priori estimates of the finite element method and semi-discretized stochastic convolution. In Section 4, we propose a new regularizing procedure and give an approach to study the weak convergence rate of the finite element method by Malliavin calculus.

2. Preliminaries

In this section, we give assumptions on \( A, F \) and \( W(t) \), the abstract functional analytical framework of the considered equation and finite element method, and a brief introduction to Malliavin calculus.

Given two separable Hilbert spaces \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \), denote \( L(\mathcal{H}, \tilde{\mathcal{H}}) \) and \( L^1(\mathcal{H}, \tilde{\mathcal{H}}) \) by the Banach spaces of all linear bounded operators and the nuclear operators from \( \mathcal{H} \) to \( \tilde{\mathcal{H}} \), respectively. The trace of an operator \( T \in L^1(\mathcal{H}) \) is \( \text{tr}[T] = \sum_{k \in \mathbb{N}^+} (Tf_k, f_k)_{\mathcal{H}} \), where \( \{f_k\}_{k \in \mathbb{N}^+} (\mathbb{N}^+ = \{1, 2, \ldots\}) \) is any orthonormal basis of \( \mathcal{H} \). In particular, if \( T \geq 0 \), \( \text{tr}[T] = \|T\|_{L^1} \). Denote by \( L_2(\mathcal{H}, \tilde{\mathcal{H}}) \) the space of Hilbert–Schmidt operators from \( \mathcal{H} \) into \( \tilde{\mathcal{H}} \), equipped with the usual norm given by \( \|T\|_{L^2} = \left( \sum_{k \in \mathbb{N}^+} \|e_k\|_{\tilde{\mathcal{H}}}^2 \right)^{\frac{1}{2}} \). The following useful property and inequality hold

\[
\langle T, S \rangle_{L_2(\mathcal{H}, \tilde{\mathcal{H}})} = \text{tr}[T^*S] = \text{tr}[ST^*], \quad T, S \in L_2(\mathcal{H}, \tilde{\mathcal{H}}), \quad (3)
\]

\[
|\text{tr}[ST^*]| = \|TS\|_{L^1} \leq \|S\| \|T\|_{L^1}, \quad S \in L(\mathcal{H}, \tilde{\mathcal{H}}), \quad T \in L_1(\mathcal{H}, \tilde{\mathcal{H}}),
\]

where \( T^* \) is the adjoint operator of \( T \).

Given a Banach space \( \mathcal{E} \), we denote by \( R(\tilde{\mathcal{H}}, \mathcal{E}) \) the space of \( \gamma \)-radonifying operators endowed with the norm defined by \( \|T\|_{\gamma(\tilde{\mathcal{H}}, \mathcal{E})} = (\mathbb{E} \left\| \sum_{k \in \mathbb{N}^+} \gamma_k T e_k \right\|_{\mathcal{E}}^2)^{\frac{1}{2}} \), where \( \{\gamma_k\}_{k \in \mathbb{N}^+} \) is a sequence of independent \( \mathcal{N}(0, 1) \)-random variables on a probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \). For convenience, the notations \( \mathbb{H} = L^2(O), L^q = L^q(O), \ 1 \leq q < \infty \) and \( E = C(O) \) is frequently used. We also need
the following Burkholder inequality in martingale-type 2 Banach spaces $L^q$, $q \in [2, \infty)$, for some $C_{p,q} \in (0, \infty)$,

$$
\left\| \sup_{t \in [0,T]} \left\| \int_0^t \phi(r) d\tilde{W}(r) \right\|_{L^q(\Omega)} \right\|_{L^p(\Omega)} \leq C_{p,q} \|\phi\|_{L^p(\Omega; L^2[0,T]; \gamma(\mathbb{H}; L^q)))}
$$

\[(4)\]

$$
\leq C_{p,q} \left( \mathbb{E} \left( \int_0^T \left\| \sum_{k \in \mathbb{N}^+} (\phi e_k)^2 \right\|_{L^q^r}^2 \, dt \right)^{\frac{1}{r}} \right)^{\frac{1}{r}},
$$

where $\tilde{W}$ is the $\mathbb{H}$-valued cylindrical Wiener process (see, e.g., [vNVW08]).

### 2.1. Main assumptions

In this subsection, we introduce some useful notations and our main assumptions on $A$, $F$ and $W(t)$. Throughout this article, initial datum $X_0$ is assumed to be a deterministic function and belongs to $E$ for convenience. We use $C$ to denote a generic constant, independent of $h$, which differs from one place to another.

**Assumption 2.1.** Let $\mathcal{O} = (0, \mathcal{L})$, $\mathcal{L} > 0$ and $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ be the Laplacian with the homogenous Dirichlet boundary condition, i.e., $Au = -\Delta u$, $u \in D(A)$.

Such assumption implies that $A$ generates an analytic and contraction $C_0$-semigroup $S(t)$, $t \geq 0$ in $\mathbb{H}$ and $L^q$. It is also well-known that the assumption on $\mathcal{O}$ implies that the existence of the eigensystem $\{\lambda_k, e_k\}_{k \in \mathbb{N}^+}$ of $\mathbb{H}$, such that $\lambda_k > 0$, $Ae_k = \lambda_k e_k$ and $\lim_{k \to \infty} \lambda_k = \infty$. Let $\mathbb{H}^r$ be the Banach space equipped with the norm $\| \cdot \|_{\mathbb{H}^r} := \| A^{\frac{r}{2}} \cdot \|_{\mathbb{H}}$ for the fractional power $A^{\frac{r}{2}}$, $r \geq 0$.

**Assumption 2.2.** Let $W(t)$ be a Wiener process with covariance operator $Q$, where $Q$ is a bounded, linear, self-adjoint and positive definite operator on $\mathbb{H}$ and satisfies $\| A^{\frac{2-\beta}{2}} \|_{L_2^0} < \infty$ with $0 < \beta \leq 1$, where $L_2^0 = L_2(U_0, \mathbb{H})$, $U_0 = Q^{\frac{1}{2}}(\mathbb{H})$. In the case of $\beta \leq \frac{1}{2}$, we in addition assume that $Q$ commutes with $A$.

**Assumption 2.3.** Let $K \in \mathbb{N}^+$ and $L_f > 0$. Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$
|f(\xi)| \leq L_f (1 + |\xi|^K), \quad f'(\xi) \leq L_f, \quad |f'(\xi)| \leq L_f (1 + |\xi|^{K-1}).
$$

Let $F : L^{2K} \to \mathbb{H}$ be Nemytskii operator defined by $F(X)(\xi) = f(X(\xi))$. 


The above assumption ensures that $F : L^{2K} \to \mathbb{H}$ satisfies for $c_1 > 0$ and $K \geq 1$,
\[
\langle F(u) - F(v), u - v \rangle \leq c_1 \| u - v \|^2,
\]
\[
\| F(u) - F(v) \| \leq c_1 (1 + \| u \|^{K-1}_E + \| v \|^{K-1}_E) \| u - v \|,
\]
where $\| \cdot \|_E$ is the supremum norm. We remark that in the analysis of strong convergence rates, this assumption about the upper bound of the derivative of $f$ could be weakened to the one-sided Lipschitz condition or monotone condition. We also point out that when studying the weak convergence rates, more restricted condition on $F$ is needed. The typical example of $f$ is a cubic polynomial
\[
f(\xi) = a_3 \xi^3 + a_2 \xi^2 + a_1 \xi + a_0, \quad a_3 < 0, \quad a_2, a_1, a_0 \in \mathbb{R}.
\]
In this case, Eq. (1) is the stochastic Allen–Cahn equation.

2.2. Finite element method

Let $(V_h)_{h \in (0,1)}$ be a family of spaces of continuous piecewise linear function corresponding to a quasi-uniform family $(T_h)_{h \in (0,1)}$ of a triangulations of $\mathcal{O}$, and $N_h$ be the dimension of $V_h$. The parameter $h$ is the mesh size of $T_h$. Denote $P^h : \mathbb{H} \to V_h$ the orthogonal projection and $A_h : V_h \to V_h$ the discrete Laplacian satisfying $\langle A_h u, v \rangle = \langle \nabla u, \nabla v \rangle, u, v \in V_h$. It is well-known that the semi-discretization $-A_h u_h = P^h f$ is finite element approximation of $-Au = f$ and that $\| u - u_h \| = \| A_h^{-1} P^h f - A^{-1} f \| \leq C h^2 \| f \|$. The operator $-A_h$ generates an analytic semigroup $(S^h(t))_{t \geq 0}$. In particular, there is an orthonormal eigenbasis $(e^h_i)_{i=1}^{N_h}$ in $V^h$ equipped with the $\mathbb{H}$ norm, with eigenvalues $0 < \lambda^h_1 \leq \lambda^h_2 \leq \cdots \leq \lambda^h_{N_h}$ such that
\[
S^h(t) v_h = \sum_{i=1}^{N_h} e^{-\lambda^h_i t} \langle v_h, e^h_i \rangle e^h_i, \quad v_h \in V_h, \quad t \geq 0.
\]
We will often use the equivalence of the following two norms for $v_h \in V_h$, $\gamma \in [-\frac{1}{2}, \frac{1}{2}]$,
\[
c \| A^\gamma_h v_h \| \leq \| A^\gamma v_h \| \leq C \| A^\gamma_h v_h \|,
\]
the interpolation space \((\mathbb{H}^2_h)_{\beta \in (-1,1)}\) and the properties of the Ritz projection \(R^h : \mathbb{H}^1 \rightarrow V_h\) and \(P^h\),

\[
\|A^\frac{s}{2}(I - R^h)A^{-\frac{s}{2}}\|_{\mathcal{L}(\mathbb{H})} \leq C h^{r - s}, \quad 0 \leq s \leq 1, r \leq 2, \quad (6)
\]

\[
\|A^\frac{s}{2}(I - P^h)A^{-\frac{s}{2}}\|_{\mathcal{L}(\mathbb{H})} \leq C h^{r - s}, \quad 0 \leq s \leq 1, 0 \leq s \leq r \leq 2.
\]

In the setting of strong convergence rate result, we will need the error of the semigroups \(G^h(t) := S^h(t)P^h - S(t), t \geq 0\), (see, e.g., \([\text{Tho06, Yan05}]\)) for \(h \in (0, 1],\)

\[
\|G^h(t)x\| \leq C h^u t^{-\frac{u}{2-r}} \|x\|_{\mathbb{H}^v}, \quad x \in \mathbb{H}^v, \quad t > 0, \quad 0 \leq v \leq u \leq 2,
\]

\[
\|G^h(t)x\| \leq C t^\frac{s}{2} \|x\|_{\mathbb{H}^{-\rho}}, \quad x \in \mathbb{H}^{-\rho}, t > 0,
\]

\[
\|G^h(t)x\| \leq C t^{-1} h^{2-\rho} \|x\|_{\mathbb{H}^{-\rho}}, \quad x \in \mathbb{H}^{-\rho}, t > 0,
\]

\[
\left(\int_0^t \|G^h(s)x\|^2 ds\right)^{\frac{1}{2}} \leq C h^{1+\rho} \|x\|_{\mathbb{H}^\rho}, \quad x \in \mathbb{H}^\rho, t > 0.
\]

Besides the above properties of finite element methods, the other important parts for our analysis are that the smooth effect of \(S^h\) (see, e.g., \([\text{Yan05}]\))

\[
\|A^\gamma_h S^h(t)P_h\|_{\mathcal{L}(\mathbb{H})} \leq C \gamma t^{-\gamma}, \quad \gamma \geq 0, \quad t > 0, \quad (8)
\]

\[
\int_0^t \|A^\frac{s}{2}_h S^h P_h x\|^2 ds \leq C \|x\|^2, \quad x \in \mathbb{H},
\]

and the boundedness of \(P^h\) (see, e.g., \([\text{CLT94}]\))

\[
\|P^h\|_{L^p} \leq C, \quad 1 \leq p < \infty, \quad \|P^h\|_E \leq C.
\]

### 2.3. Malliavin calculus

In order to get the weak convergence rate, we recall the Malliavin calculus duality formula in Hilbert space, (see, e.g., \([\text{AL16, Nua06}]\)) which will be used to deal with the singular term appeared in the weak error. Since \(Q\) is a bounded, linear, self-adjoint and positive definite operator on \(\mathbb{H}\), the corresponding Cameron-Martin space is \(U_0 = Q^{\frac{1}{2}}(\mathbb{H})\). Let \(\mathcal{I} : L^2([0, T]; U_0) \rightarrow \mathbb{H}\).
\( L^2(\Omega) \) be an isonormal process such that for any \( \psi \in L^2([0,T]; U_0) \), \( \mathcal{I}(\psi) \) is the centered Gaussian variable and \( \mathbb{E}[\mathcal{I}(\psi_1)\mathcal{I}(\psi_2)] = \langle \psi_1, \psi_2 \rangle_{L^2([0,T]; U_0)} \), \( \psi_1, \psi_2 \in L^2([0,T]; U_0) \). We denote the smooth cylindrical random variables

\[
\mathcal{S} = \left\{ \mathcal{X} = g(\mathcal{I}(\psi_1), \ldots, \mathcal{I}(\psi_N)) : g \in C^\infty_c(\mathbb{R}^N), \psi_j \in L^2([0,T]; U_0), j = 1, \ldots, N \right\},
\]

and the smooth cylindrical \( \mathbb{H} \)-valued random variables

\[
\mathcal{S}(\mathbb{H}) = \left\{ G = \sum_{i=1}^{M} \mathcal{X}_i \otimes h_i : \mathcal{X}_i \in \mathcal{S}, h_i \in \mathbb{H}, M \geq 1 \right\}.
\]

Then the Malliavin derivative of \( G = \sum_{i=1}^{M} g_i(\psi_1, \ldots, \psi_N) \otimes h_i \) is defined by

\[
\mathcal{D}_s G = \sum_{i=1}^{M} \sum_{j=1}^{N} g_i(\psi_1, \ldots, \psi_N) \otimes (h_i \otimes \psi_j(s)).
\]

Since the derivative operator \( D \) is closable (see, e.g., \([\text{Nua06}]\)), we denote \( \mathbb{D}^{1,2} \) the closure of \( \mathcal{S}(\mathbb{H}) \) with respect to Malliavin derivative equipped with the norm

\[
\|G\|_{\mathbb{D}^{1,2}} = \left( \mathbb{E}[\|G\|^2] + \mathbb{E}\left[\int_0^T \|\mathcal{D}_s G\|^2 ds\right] \right)^{\frac{1}{2}},
\]

where \( \mathcal{D}_s G \) is the Malliavin derivative of \( G \). The key in the analysis of weak convergence rate is the following integration by parts (see, e.g., \([\text{AL16, Nua06}]\)). For any random variable \( G \in \mathbb{D}^{1,2}(\mathbb{H}) \) and any predictable process \( \Theta \in L^2(\Omega; \mathcal{L}^0_\mathbb{H}) \), we have

\[
\mathbb{E}\left[\left\langle \int_0^T \Theta(t) dW(t), G \right\rangle\right] = \mathbb{E}\left[\int_0^T \left\langle \Theta(t), \mathcal{D}_s G \right\rangle_{\mathcal{L}^0_\mathbb{H}} dt\right]. \tag{9}
\]

Moreover, we also need chain rule of the Malliavin derivative. Let \( \mathcal{V} \) be another separable Hilbert space, and \( \sigma \in \mathcal{C}_b^1(\mathbb{H}, \mathcal{V}) \). Then we have \( \sigma(G) \in \mathbb{D}^{1,2}(\mathcal{V}) \),

\[
\begin{align*}
\mathcal{D}_t^y(\sigma(G)) &= \mathcal{D} \sigma(G) \cdot \mathcal{D}_t^y G, \quad y \in U_0, \quad G \in \mathbb{D}^{1,2}(\mathbb{H}), \\
\mathcal{D}_t(\sigma(G)) &= \mathcal{D} \sigma(G) \mathcal{D}_t G, \quad G \in \mathbb{D}^{1,2}(\mathbb{H}).
\end{align*}
\]
3. A priori estimate and strong convergence rate

In this section, we present the a priori estimate and the strong convergence rates of the discrete stochastic convolution and the finite element method.

3.1. A priori estimate

Using the equivalence of Eq. (1) and the following random PDE
\[ dY = (AY + F(Y + Z))dt, \quad Y(0) = X_0, \]
\[ dZ = AZdt + dW(t), \quad Z(0) = 0 \]
and the similar arguments to [BCH18], we get the following a priori estimate on the exact solution of Eq. (1).

Lemma 3.1. Under the Assumptions 2.1-2.3, there exists a unique mild solution \( X \) of Eq. (1). In addition, \( X \) satisfies for \( t > 0, p \geq 1, \)
\[ \sup_{s \in [0,t]} E[\|X(s)\|_E^p] \leq C(T,p)(1 + \|X_0\|_E^p), \]
\[ E[\|X(t)\|_{H^\beta}^p] \leq (1 + t^{-\beta\alpha/p})C(T,p)(1 + \|X_0\|^p). \]

Now, we are in the position to derive a priori estimate for the semi-discretization Eq. (2). At first, we prove the smoothing property of \( S^h(t) \), \( t \geq 0. \)

Lemma 3.2. For \( 2 \leq p \leq \infty, \) we have
\[ \|S^h(t)P^h f\|_{L^p} \leq Ct^{-\frac{\beta}{2} + \frac{1}{p}}\|f\|_{L^2}, \quad f \in L^2, \; t > 0. \]

Proof. Since \( S^h(t)P^h f \in V^h, \) we have
\[ S^h(t)P^h f = \sum_{i=1}^{N_h} e^{-\lambda^h_i t} \langle f, e^h_i \rangle e^h_i. \]

Then the uniformly boundness of \( e^h_i \) and \( ci^2 \leq \lambda^h_i \leq C\bar{c}^2 \) in [Wal05] yield that
\[ \|S^h(t)P^h f\|_E \leq \|\sum_{i=1}^{N_h} e^{-\lambda^h_i t} \langle f, e^h_i \rangle e^h_i\|_E \leq (\sum_{i=1}^{N_h} e^{-2\lambda^h_i t})^{1/2} \|f\|_{L^2} \leq Ct^{-\frac{\beta}{2}} \|f\|_{L^2}. \]
and
\[ \| S^h(t) P^h f \|_{L^2} \leq \sum_{i=1}^{N^h} e^{-\lambda_i^h t} \langle f, e_i^h \rangle_{L^2} \leq C \| f \|_{L^2}. \]

The Riesz-Thorin interpolation theorem leads to the desired result.

The other tool to get the a priori estimate is using the weak discrete maximum principle in e.g. [CLT94, Tho06].

**Lemma 3.3.** Under the assumptions on \( T^h \) and \( V^h \), there exists a positive constant \( C \) such that, for any \( v^h \in V^h \),
\[ \| S^h(t)v^h \|_{L^\infty} \leq C \| v^h \|_{L^\infty}, \quad t > 0. \] (10)

We remark in general case the similar boundedness results of finite element methods still hold [CLT94, Tho06]. Next, we give the a priori estimate of the semi-discretized stochastic convolution \( Z^h \), which satisfies
\[ dZ^h(t) + A^h Z^h(t) = P^h dW(t), \quad Z^h(0) = 0. \]

**Lemma 3.4.** Let \( \mathcal{V} = E \) or \( L^{2q} \) \((q \geq 1)\). Under the Assumptions 2.1-2.2, the discretized stochastic convolution \( Z^h \) satisfies, for \( t > 0 \) and \( p \geq 1 \),
\[ \mathbb{E}\left[ \| Z^h(t) \|_{\mathcal{V}}^p \right] \leq C(T,p), \quad \text{if } \beta > \frac{1}{2}, \]
\[ \mathbb{E}\left[ \| Z^h(t) \|_{\mathcal{V}}^p \right] \leq C(T,p)(1 + \log\left(\frac{1}{h}\right))^\frac{p}{2}, \quad \text{if } 0 < \beta \leq \frac{1}{2}. \]

In particular, if \( Q = I \), \( \beta \in [0, \frac{1}{2}) \), then
\[ \mathbb{E}\left[ \| Z^h(t) \|_{\mathcal{V}}^p \right] \leq C(T,p). \]

**Proof.** The case of \( \beta > \frac{1}{2} \) is directly proven by using the Sobolev embedding theorem for \( d = 1 \), the Burkholder inequality, and the smoothy property of \( A^h \) (8). Now we focus on the case \( \beta \leq \frac{1}{2} \) and take \( \mathcal{V} = L^{2q}, \quad q \geq 1 \) as example. Similar arguments yield the case \( \mathcal{V} = E \). Notice that \( Z^h(t,\xi) = \sum_{k \in \mathbb{N}^+} f_0 \sum_{i=1}^{N^h} e^{-\lambda_i^h s} (\sqrt{q_k} e_k, e_i^h) e_i^h(\xi) d\beta_k(s) \), where \( \{e_k, q_k\}_{k \in \mathbb{N}^+} \) is
the eigensystem of $Q$. The Burkholder inequality and uniform boundedness of $e_i^h$ yield that

$$
\mathbb{E}\left[ \left\| Z^h(t) \right\|_{L^2}^p \right] \leq C \mathbb{E}\left[ \left( \int_0^t \left\| \sum_{i \in \mathbb{N}^+} e^{-\lambda_i^h s} (\sqrt{\gamma_k e_k, e_i^h})^2 \right\|_{L^2} ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq C \mathbb{E}\left[ \left( \int_0^t \sum_{i,j=1}^{N^h} e^{-(\lambda_i^h + \lambda_j^h) s} (Q^2 e_i^h, Q^2 e_j^h) e_i^h e_j^h \left\| e_i^h \right\|_{L^2} ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq C \mathbb{E}\left[ \left( \int_0^t \sum_{i,j=1}^{N^h} e^{-(\lambda_i^h + \lambda_j^h) s} \left\| e_i^h \right\|_{L^2} \left\| e_j^h \right\|_{L^2} ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq C \mathbb{E}\left[ \left( \int_0^t \sum_{i,j=1}^{N^h} e^{-(\lambda_i^h + \lambda_j^h) s} ds \right)^{\frac{p}{2}} \right].
$$

By the Weyl law and $N^h \leq O\left( \frac{1}{h} \right)$, we have

$$
\mathbb{E}\left[ \left\| Z^h(t) \right\|_{L^2}^p \right] \leq C \mathbb{E}\left[ \left( \int_0^t \sum_{i,j=1}^{N^h} e^{-(\lambda_i^h + \lambda_j^h) s} ds \right)^{\frac{p}{2}} \right] \leq C \mathbb{E}\left[ \left( \int_0^t \sum_{i=1}^{(N^h)^2} e^{-i c s} ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq C \left[ \left( \int_0^{h^l} \frac{ds}{h^2} \right)^{\frac{p}{2}} \right] + C \left[ \left( \int_0^t \int_1^\infty e^{-c s} d\xi ds \right)^{\frac{p}{2}} \right]
$$

$$
\leq C (h^{l-2} + C \log(1 + t) + l \log(1 + t))^{\frac{p}{2}}
$$

$$
\leq C (1 + (\log(1 + t))^{\frac{p}{2}} + (\log(\frac{1}{h}))^{\frac{p}{2}}),
$$

for a large $l \in \mathbb{N}^+$. In particular, if $Q = I$, then the logarithm factor can be
eliminated as
\[
\mathbb{E}\left[\|Z^h(t)\|_{L^q}^p\right] \leq C\mathbb{E}\left[\left(\int_0^t \left\|\sum_{k \in \mathbb{N}^+} \left(\sum_{i=1}^{N^h} e^{-\lambda_i^h s}(e_k, e_i^h) e_i^h\right)^2\right\|_{L^q} ds\right)^{\frac{p}{2}}\right]
\]
\[
\leq C\mathbb{E}\left[\left(\int_0^t \left\|\sum_{k \in \mathbb{N}^+} \sum_{i,j=1}^{N^h} e^{-2\lambda_i^h s}(e_i^h, e_j^h) e_i^h e_j^h\right\|_{L^q} ds\right)^{\frac{p}{2}}\right]
\]
\[
\leq C\mathbb{E}\left[\left(\int_0^t \left\|\sum_{i=1}^{N^h} e^{-2\lambda_i^h s}(e_i^h)^2\right\|_{L^q} ds\right)^{\frac{p}{2}}\right] \leq Ct^{\frac{p}{2}}.
\]

Summing up all the estimates, we finish the proof.

Remark 3.1. Using Sobolev embedding \(\mathbb{H}^1 \hookrightarrow L^6\) for \(d = 2, 3\), one can obtain the a priori estimate of \(Z^h\) in \(L^6\) norm, which is an essential part to derive the strong convergence rates of numerical schemes for additive trace class noise, i.e. \(\beta \geq 1\). We refer to \([QW18]\) for more details.

The following a priori estimate is very useful for deducing the weak convergence rate in Section 4 and has its own interest.

Proposition 3.1. Let \(\mathcal{V} = E\) or \(L^{2q}\) (\(q \geq 1\)). Under the Assumptions 2.1-2.3 with \(K < 5\), there exists a unique mild solution \(X^h\) of Eq. (2) which satisfies
\[
\sup_{t \in [0,T]} \mathbb{E}\left[\|X^h(t)\|_\mathcal{V}^p\right] \leq C(p, T, X_0)(1 + (\log(\frac{1}{h})))^{\frac{K^2p}{2}}, \text{ for } p \geq 1.
\]

Proof. For convenience, we only prove the case \(\mathcal{V} = E\). By the equivalence of the stochastic PDE
\[
dX^h = (-A_h X^h + P^h F(X^h)) dt + P^h dW(t), \quad X^h(0) = P^h X_0,
\]
and the random PDE
\[
dY^h = (-A_h Y^h + P^h F(Y^h + Z^h)) dt, \quad X^h(0) = P^h X_0,
\]
\[
dZ^h = -A_h Z^h dt + P^h dW(t),
\]
and Lemma 3.4, it suffices to bound $E\left[\|Y^h(t)\|_E^p\right]$. The higher regularity of $Y^h$ and the dissipativity of $F$ imply that
\[
\|Y^h(t)\|^2 + \int_0^t \langle \nabla Y^h, \nabla Y^h \rangle ds = \|X^h(0)\|^2 + \int_0^t \langle F(Y^h + Z^h), Y^h \rangle ds \\
\leq \|X^h(0)\|^2 + \int_0^t \langle F(Y^h + Z^h) - F(Z^h), Y^h \rangle ds + \int_0^t \langle F(Z^h), Y^h \rangle ds \\
\leq \|X^h(0)\|^2 + C \int_0^t \|Y^h\|^2 ds + C \int_0^t (1 + \|Z^h\|_{L^2}^2) ds.
\]
The $p$-moment boundedness of $\|Z^h\|_E$ and the Gronwall’s inequality yield that for $p \geq 1$,
\[
\sup_{t \in [0,T]} \|Y^h(t)\|^{2p} + \left( \int_0^T \|\nabla Y^h\|^2 ds \right)^p \\
\leq C(p, T)(1 + \|X^h(0)\|^{2p} + \int_0^T \|Z^h\|_{L^2}^{2Kp} ds).
\]
Next, based on the above estimate and Lemma 3.4, we are in position to prove the desired result. The mild form of $Y^h$, Lemmas 3.2 and 3.3, together with the Gagliardo–Nirenberg–Sobolev inequality $\|f\|_{L^{2K}} \leq C \|\nabla f\|^{\frac{K-1}{2K}} \|f\|^{\frac{K+1}{2K}}$, lead to
\[
\|Y^h(t)\|_E \leq \|S^h(t) P^h X_0\|_E + \int_0^t \|S^h(t) P^h F(Y^h + Z^h)\|_E ds \\
\leq C \|X_0\|_E + C \int_0^t (t - s)^{-\frac{1}{2}} \|F(Y^h + Z^h)\| ds \\
\leq C \|X_0\|_E + C \int_0^t (t - s)^{-\frac{1}{2}} (1 + \|Y^h\|_{L^{2K}}^K + \|Z^h\|_{L^{2K}}^K) ds \\
\leq C \|X_0\|_E + C \int_0^t (t - s)^{-\frac{1}{2}} (1 + \|Z^h\|_{L^{2K}}^K) ds \\
+ C \int_0^t (t - s)^{-\frac{1}{2}} \|\nabla Y^h\|^{\frac{K-1}{2}} \|Y^h\|^{\frac{K+1}{2}} ds.
\]
Taking $p$-moments on both sides, together with the Hölder and Young inequalities and the boundedness of $\int_0^T \|\nabla Y^h\|^2 ds$ and $Z^h$, yields that for $p \geq 1$
and $K < 5$,
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \| Y^h(t) \|_E^p \right] \\
\leq C \| X_0 \|_E^p + C(T, p)(1 + \log(\frac{1}{h}))^{\frac{Kp}{2}} \\
+ C \mathbb{E}\left[ \left( \int_0^t (t-s)^{-\frac{1}{2}} \| \nabla Y^h \|_{L^2}^{\frac{K-1}{2}} ds \right)^{2p} \sup_{s \in [0,T]} \| Y^h(s) \|_E^{(K+1)p} \right] \\
\leq C \sqrt{\mathbb{E}\left[ \left( \sup_{t \in [0,T]} \int_0^t (t-s)^{-\frac{1}{2}} \| \nabla Y^h \|_{L^2}^{\frac{K-1}{2}} ds \right)^{2p} \right]} \sqrt{\mathbb{E}\left[ \sup_{s \in [0,T]} \| Y^h(s) \|_E^{(K+1)p} \right]} \\
+ C(X_0, T, p)(1 + \log(\frac{1}{h}))^{\frac{Kp}{2}} \\
\leq C(X_0, T, p)(1 + \log(\frac{1}{h}))^{\frac{(K+1)Kp}{2}} \\
+ C(X_0, T, p)(1 + \log(\frac{1}{h}))^{\frac{Kp}{2}} \\
\leq C(X_0, T, p)(1 + \log(\frac{1}{h}))^{\frac{K^2p}{2}}.
\]

The triangle inequality
\[
\mathbb{E}\left[ \sup_{t \in [0,T]} \| X^h(t) \|_E^p \right] \leq C_p \mathbb{E}\left[ \sup_{t \in [0,T]} \| Y^h \|_E^p \right] + C_p \mathbb{E}\left[ \sup_{t \in [0,T]} \| Z^h(t) \|_E^p \right],
\]

together with the a priori estimate on $Y^h$ and $Z^h$, completes the proof. \qed

**Remark 3.2.** Under the assumptions of Proposition 3.1, if in addition $\| A_{\beta - 1}^{\frac{1}{2}} Q \|_{\mathcal{L}_2^0} < \infty$, $\beta > \frac{1}{2}$ or $Q = I$ and $\beta < \frac{1}{2}$, then we have the following optimal estimate
\[
\sup_{t \in [0,T]} \mathbb{E}\left[ \| X^h(t) \|_Y^p \right] \leq C(p, T, X_0).
\]

### 3.2. Strong convergence rate

In this subsection, we aim to give the strong convergence result under very mild assumptions on the initial data. We also remark that this approach to get strong convergence rates does not require the additional a priori estimate of the spatial approximation $X^h$ and that if $\| A_{\beta - 1}^{\frac{1}{2}} Q \|_{\mathcal{L}_2^0} < \infty$, $\beta > \frac{1}{2}$ or $Q = I$, $\beta < \frac{1}{2}$, then the logarithm term can be eliminated.
Theorem 3.1. Under the Assumptions 2.1-2.3, the finite element approximation $X^h(t)$ is strongly convergent to $X(t)$, $t > 0$ and satisfies, for $p \geq 1$,

$$
\mathbb{E}\left[ \|X(t) - X^h(t)\|_H^p \right] \leq C(X_0, T, p)(1 + t^{-\frac{\alpha}{2}})p h^{\beta p}, \quad \text{for } \beta > \frac{1}{2}
$$

$$
\mathbb{E}\left[ \|X(t) - X^h(t)\|_H^p \right] \leq C(X_0, T, p)(1 + t^{-\frac{\alpha}{2}} + (\log(\frac{1}{h}))^{\frac{p-1}{2}})p h^{\beta p}, \quad \text{for } \beta \leq \frac{1}{2}.
$$

Proof. Since $A$ does not commute with $P^h$, we could not use the usual strategy which divides the strong error $X(t) - X^h(t)$ into $(I - P^h)X(t)$ and $P^hX(t) - X^h(t)$. Thus, we introduce a new auxiliary process $\tilde{Y}^h$ which satisfies

$$
d\tilde{Y}^h = -Ah\tilde{Y}^h dt + P^h F(Y + Z)dt, \quad \tilde{Y}^h(0) = X^h(0).
$$

Now, we split strong error as

$$
X(t) - X^h(t) = Y(t) - Y^h(t) + Z(t) - Z^h(t)
$$

$$
= (Y(t) - \tilde{Y}^h(t)) + (\tilde{Y}^h(t) - Y^h(t)) + (Z(t) - Z^h(t)),
$$

and estimate the three terms, respectively. The treatment of $Z(t) - Z^h(t)$ is standard. Using the estimates (7) of $C^h(t) := S^h(t)P^h - S(t)$, $t \geq 0$ and Burkholder inequality, we get

$$
\mathbb{E}\left[\|Z(t) - Z^h(t)\|^p\right]
$$

$$
\leq C_p \mathbb{E}\left[\left( \int_0^t \|C^h(t - s)\|_{L_2}^2 ds \right)^\frac{p}{2}\right]
$$

$$
\leq C_p \mathbb{E}\left[\left( \int_0^t \|(R^h - I)S(t - s)\|_{L_2}^2 ds \right)^\frac{p}{2}\right]
$$

$$
\leq C_p \mathbb{E}\left[\left( \int_0^t \|(R^h - I)A^{-\beta}\|L_2^{\frac{\alpha}{2}} A^\frac{\beta}{2} S(t - s)A^{-\frac{\beta}{2}}\|_{L_2}^2 ds \right)^\frac{p}{2}\right]
$$

$$
\leq C(T, p)\|A^{\frac{\alpha - 1}{2}}\|L_2^{\frac{p}{2}} h^{\beta p}.
$$

The mild forms of $Y$ and $\tilde{Y}^h$, together with the a priori estimates of $Y$ and
and the properties (7) of finite element method, yield that for $0 \leq u < 2$,

$$
\mathbb{E}\left[\|Y(t) - \tilde{Y}^h(t)\|^p\right] \\
\leq C_p \mathbb{E}\left[\|G^h(t)X_0\|^p\right] + C_p \mathbb{E}\left[\left\| \int_0^t G^h(t-s)F(Y + Z)ds \right\|^p\right] \\
\leq C_p h^{up} t^{-\frac{np}{2}} \|X_0\|^p + C_p \mathbb{E}\left[\left\| \int_0^t G^h(t-s)F(Y + Z)ds \right\|^p\right] \\
\leq C_p h^{up} t^{-\frac{np}{2}} \|X_0\|^p + C(p,T) h^{up} \mathbb{E}\left[ \sup_{s \in [0,T]} \left( 1 + \|Y(s)\|_{L^2(K)} + \|Z(s)\|_{L^2(K)} \right)^p \right] \\
\leq C(X_0,T,p) h^{up}(1 + t^{-\frac{np}{2}}).
$$

Notice that the similar arguments as in the proof of [BCH18, Lemma 3.1] yield that $\mathbb{E}\left[\|\tilde{Y}^h(t)\|_E^p\right] \leq C(p,T,X_0)$. Next we deal with the term $\tilde{Y}^h(t) - Y^h(t)$. The random PDE forms of $\tilde{Y}^h(t)$ and $Y^h(t)$ lead that

$$
\|\tilde{Y}^h(t) - Y^h(t)\|^2 \\
\leq -2 \int_0^t \|\nabla (\tilde{Y}^h(s) - Y^h(s))\|^2 ds \\
+ \int_0^t 2 \langle F(Y(s) + Z(s)) - F(Y^h(s) + Z^h(s)), \tilde{Y}^h(s) - Y^h(s) \rangle ds \\
\leq \int_0^t 2 \langle F(Y(s) + Z(s)) - F(\tilde{Y}^h(s) + Z^h(s)), \tilde{Y}^h(s) - Y^h(s) \rangle ds \\
+ \int_0^t 2 \langle F(\tilde{Y}^h(s) + Z^h(s)) - F(Y^h(s) + Z^h(s)), \tilde{Y}^h(s) - Y^h(s) \rangle ds.
$$

For $\beta \leq \frac{1}{2}$, by the monotonicity of $F$ and Lemma 3.4 and taking expectation,
we have

\[
\mathbb{E}
\left[
\|\tilde{Y}^h(t) - Y^h(t)\|^2_p
\right]
\leq C_p \int_0^t \mathbb{E}
\left[
\|\tilde{Y}^h(s) - Y^h(s)\|^2_p
\right] ds
+ C_p \mathbb{E}
\left[
\left(
\int_0^t \|Y(s) - \tilde{Y}^h(s)\| + \|Z(s) - Z^h(s)\|
\right)
(1 + \|Y(s)\|^K_E - 1 + \|\tilde{Y}^h(s)\|^K_E - 1 + \|Z(s)\|^K_E - 1 + \|Z^h(s)\|^K_E - 1) ds
\right)^{2p}
\]
\leq C_p \int_0^t \mathbb{E}
\left[
\|\tilde{Y}^h(s) - Y^h(s)\|^2_p
\right] ds
+ C_p (X_0, T, p) \left(1 + \log\left(\frac{1}{h}\right)\right)^{(K-1)p} h^{2\beta_p}.
\]

Then the Gronwall’s inequality leads to

\[
\mathbb{E}
\left[
\|\tilde{Y}^h(t) - Y^h(t)\|^2_p
\right]
\leq C(X_0, T, p) (1 + \log\left(\frac{1}{h}\right))^2p h^{2\beta_p}.
\]

For \(\beta > \frac{1}{2}\), similar arguments, together with Lemma 3.4 and the boundedness of \(\|\tilde{Y}^h(t)\|_E\) imply that

\[
\mathbb{E}
\left[
\|\tilde{Y}^h(t) - Y^h(t)\|^2_p
\right]
\leq C(X_0, T, p) h^{2\beta_p}.
\]

Combing the strong error estimates of \(Y(t) - \tilde{Y}^h(t), \tilde{Y}^h(t) - Y^h(t)\) and \(Z(t) - Z^h(t)\) together, we finish the proof.

Remark 3.3. Under the assumptions of Theorem 3.1, if in addition \(X_0 \in H^\beta\), then

\[
\mathbb{E}
\left[
\sup_{t \in [0, T]} \|X(t) - X^h(t)\|^p_{H^\beta}
\right]
\leq C(X_0, T, p) h^{\beta_p}, \quad \beta > \frac{1}{2}.
\]

\[
\mathbb{E}
\left[
\sup_{t \in [0, T]} \|X(t) - X^h(t)\|^p_{H^\beta}
\right]
\leq C(X_0, T, p) (1 + \log\left(\frac{1}{h}\right)\left(k-1\right)p h^{2\beta_p}, \quad \beta \leq \frac{1}{2}.
\]

In particular, if \(Q = I, \beta < \frac{1}{2}\), then the logarithm factor can be eliminated. We also remark that this approach to deduce strong convergence rates of numerical schemes is available for the cases of the higher dimension and more regular noise.
4. Regularity of Kolmogorov equation and weak convergence rate

4.1. Regularity of Kolmogorov equation

Let \( C^2_b(H) \) be the space of not necessarily bounded functions from \( H \) to \( \mathbb{R} \) having the continuous and bounded Fréchet derivative of orders 1 and 2, and \( \phi \in C^2_b(H) \). Set \( U(t, x) = \mathbb{E}[\phi(X(t, x))] \), then formally, \( U \) is the solution of the Kolmogorov equation associated with Eq. (1):

\[
\frac{\partial U(t, x)}{\partial t} = \langle -Ax + F(x), DU(t, x) \rangle + \text{tr}[Q^{\frac{1}{2}}D^2U(t, x)Q^{\frac{1}{2}}].
\]

To give rigorous meaning of the Kolmogorov equation, we follow the approach of [BG18b]. We first apply the splitting strategy in [BCH18, BG18a] to regularize the original equation and get a regularized Kolmogorov equation. Then making use of the regularity of the regularized Kolmogorov equation and integration by parts in Malliavin sense, we obtain the weak convergence rate of the finite element method.

Now, we are in the position to give the rigorous meaning of the regularized Kolmogorov equation and uniform estimates on \( DU \) and \( D^2U \). The following lemma is useful in constructing the regularized PDE and its corresponding Kolmogorov equation. For a function \( f \) on \( \mathbb{R} \), we denote the first derivative and second derivative by \( f' \) and \( f'' \).

**Lemma 4.1.** Let \( L_f > 0, K \in \mathbb{N}^+ \) and \( f : \mathbb{R} \to \mathbb{R} \) satisfy

\[
|f(\xi)| \leq L_f(1 + |\xi|^K), \quad |f'(\xi)| \leq L_f(1 + |\xi|^{K-1}),
\]

\[
f''(\xi) \leq L_f, \quad |f''(\xi)| \leq L_f(1 + |\xi|^{(K-2)\vee 0}).
\]

Then the phase flow \( \Phi_t \) of the differential equation

\[
dx(t) = f(x(t))dt, \quad x(0) = \xi \in \mathbb{R}, \tag{11}
\]

satisfies for all \( \xi \in \mathbb{R} \)

\[
|\Phi_t(\xi)| \leq C(f, t)(1 + |\xi|), \quad \Phi'_t(\xi) \leq C(f, t), \quad |\Phi''_t(\xi)| \leq C(f, t)(1 + |\xi|^{K-2}).
\]

**Proof.** From the properties of \( f \) and the Young inequality, it follows that

\[
|x(t)|^2 = |\xi|^2 + \int_0^t 2(f(x(s)) - f(0)x(s))ds + \int_0^t 2f(0)x(s)ds
\]

\[
\leq |\xi|^2 + L_f^2 + \int_0^t (1 + L_f)|x(s)|^2ds.
\]
Then the Gronwall’s inequality implies that $|\Phi_t(\xi)| = |x(t)| \leq C(f, t)(1 + |\xi|)$. Similarly, using the differentiable dependence on initial data, we obtain

$$\Phi_t'(\xi) = 1 + \int_0^t f'(\Phi_s(\xi))\Phi_s'(\xi) ds,$$

which, together with the Gronwall’s inequality, yields that $0 \leq \Phi_t'(\xi) \leq e^{L f t}$. Similar arguments lead that

$$|\Phi''_t(\xi)|^2 \leq 2 \int_0^t f''(\Phi_s(\xi))(\Phi_s''(\xi))^2 ds + 2 \int_0^t f''(\Phi_s(\xi))(\Phi_s'(\xi))^2 ds,$$

which indicates that $|\Phi''_t(\xi)| \leq C(f, t)(1 + |\xi|^{K-2})$.

With the help of Lemma 4.1, we introduce our regularizing procedures. Based on the strategy of splitting method in [BG18a], we split the Eq. (1) into two sub-systems

$$dX_1 = F(X_1)dt, \quad dX_2 = -AX_2 dt + dW(t).$$

Then given a fixed time step size $\delta t > 0$, the splitting method is defined as

$$\tilde{X}_{n+1} := S(\delta t)\Phi_{\delta t}(\tilde{X}_n) + \int_{t_n}^{t_{n+1}} S(t_{n+1} - s)dW(s)$$

$$= S_{\delta t}\tilde{X}_n + \delta t S_{\delta t}\Psi_{\delta t}(\tilde{X}_n) + \int_{t_n}^{t_{n+1}} S(t_{n+1} - s)dW(s),$$

for $0 \leq n \leq N-1, N\delta t = T$, where $\Psi_t(x) := \frac{\Phi_t(x) - x}{t}$, $t > 0$ and $\Phi_0(x) = F(x)$. Notice that the splitting method can be used to approximate SPDE with non-monotone coefficients in strong and weak convergence senses (see, e.g., [CH18, CHLZ17]). Based on the idea that $\{\tilde{X}_n\}_{n=1, \ldots, N}$ is the exponential Euler method applied to stochastic PDE in [BCH18, BG18a], we introduce the auxiliary problem as

$$dX^{\delta t} = AX^{\delta t} + \Psi_{\delta t}(X^{\delta t}) dt + dW(t), \quad X^{\delta t} = X_0. \quad (12)$$

The differentiability of $\Psi_{\delta t}$ is listed on the following lemma, which is similar to [BG18a, Lemma 2.1].
Lemma 4.2. For $\delta t_0 \in (0,1]$, there exists $C(\delta t_0, f) > 0$ such that for all $\delta t \in [0, \delta t_0]$ and $\xi \in \mathbb{R}$,

$$\Psi_{\delta t}'(\xi) \leq e^{C\delta t_0}, \quad |\Psi_{\delta t}'(\xi)| \leq C(\delta t_0)(1 + |\xi|^{K-1}),$$

$$|\Psi_{\delta t}''(\xi)| \leq C(\delta t_0)(1 + |\xi|^{2K-3}), \quad |\Psi_{\delta t}(\xi) - \Psi_{0}(\xi)| \leq C(\delta t_0)\delta t(1 + |\xi|^{2K-1}).$$

Proof. By the definition of $\Psi_{\delta t}$ and the properties of $\Phi_{\delta t}$ in Lemma 4.1, we have

$$\Psi_{\delta t}'(\xi) = \frac{\Phi_{\delta t}'(\xi) - 1}{\delta t} = \frac{\int_{0}^{\delta t} f'(\Phi_{s}(\xi))\Phi_{s}'(\xi)ds}{\delta t} \leq C(f, \delta t_0),$$

$$|\Psi_{\delta t}'(\xi)| \leq \frac{|\int_{0}^{\delta t} f'(\Phi_{s}(\xi))\Phi_{s}'(\xi)ds|}{\delta t} \leq C(f, \delta t_0) \sup_{s \in [0,\delta t]} (1 + |\Phi_{s}(\xi)|^{K-1})$$

$$\leq C(f, \delta t_0)(1 + |\xi|^{K-1}),$$

$$|\Psi_{\delta t}''(\xi)| \leq \frac{|\int_{0}^{\delta t} f''(\Phi_{s}(\xi))(\Phi_{s}'(\xi))^2 + f'(\Phi_{s}(\xi))\Phi_{s}''(\xi)ds|}{\delta t}$$

$$\leq C(f, \delta t_0)(1 + |\xi|^{2K-3})$$

and

$$|\Psi_{\delta t}(\xi) - \Psi_{0}(\xi)| \leq \frac{|\int_{0}^{\delta t} \int_{0}^{1} f'(\theta \Phi_{s}(\xi) + (1 - \theta)\xi)(\Phi_{s}(\xi) - \xi)d\theta ds|}{\delta t}$$

$$\leq \sup_{s \in [0,\delta t]} \int_{0}^{1} |f'(\theta \Phi_{s}(\xi) + (1 - \theta)\xi)|d\theta \sup_{s \in [0,\delta t]} |(\Phi_{s}(\xi) - \xi)|$$

$$\leq C(f, \delta t_0)\delta t(1 + |\xi|^{2K-1}).$$

Based on Lemma 4.2, the coefficient $\Psi_{\delta t}(\cdot)$ of Eq. (12) is globally Lipschitz due to the fact that $\Psi_{\delta t}(\xi) = \frac{\Phi_{\delta t}(\xi) - \xi}{\delta t}$. However, the Lipschitz coefficients of $\Psi_{\delta t}, t \geq 0$ are not uniformly bounded with respect to $t$ (see, e.g., [BG18b]). Indeed, we can get the following convergence result of Eq. (12) approximating Eq. (1), whose proof is similar to [BG18a, Proposition 4.8].

Lemma 4.3. Let Assumptions 2.1-2.3 hold. Then the solution $X^{\delta t}$ of Eq. (12) is strongly convergent to the solution $X$ of Eq. (1) and satisfies, for any
$p \geq 1,$
\[
\mathbb{E}\left[\|X^{\delta t}(t)\|_E^p\right] \leq C(T, p)(1 + \|X_0\|_E^p),
\]
\[
\sup_{t \in [0, T]} \|X^{\delta t}(t) - X(t)\|_{L^p(\Omega)} \leq C(X_0, T, p)\delta t.
\]

The idea of deducing the sharp weak convergence rate lies on the decomposition of $\mathbb{E}\left[\phi(X(t)) - \phi(X^h(t))\right]$ into $\mathbb{E}\left[\phi(X(t)) - \phi(X^{\delta t}(t))\right]$ and $\mathbb{E}\left[\phi(X^{\delta t}(t)) - \phi(X^h(t))\right]$. The first term is estimated by Lemma 4.3 and possesses the positive convergence rate with respect to the parameter $\delta t$. The second error is dealt by utilizing the regularity of Kolmogorov equation with respect to Eq. (12) and integration by parts in the sense of Malliavin calculus. Similar to [BG18b, CJK14], to get the rigorous regularity result of the Kolmogorov equation, the noise term $dW(t)$ is regularized as $e^{\delta A}dW(t)$, $\delta > 0$. For convenience, we omit the procedure of regularizing the noise since the following proposition allows us to take the limit $\delta \to 0$.

Next, we give the regularity of Kolmogorov equation with respect to Eq. (12)
\[
\frac{\partial U^{\delta t}(t, x)}{\partial t} = \mathcal{L}U^{\delta t}(t, x) = \langle -Ax + \Psi^{\delta t}(x), DU^{\delta t}(t, x) \rangle
\]
\[
+ \frac{1}{2}tr[Q^\frac{1}{2}D^2U^{\delta t}(t, x)Q^\frac{1}{2}].
\]

Proposition 4.1. For every $\alpha, \beta, \gamma \in [0, 1)$, $\beta + \gamma < 1$, there exist $C(\delta t_0, \alpha, T)$ and $C(T, \delta t_0, \beta, \gamma)$ such that for $\delta t \in [0, \delta t_0]$, $x \in E, y, z \in H$ and $t \in (0, T]$,
\[
|DU^{\delta t}(t, x).y| \leq \frac{C(\delta t_0, \alpha, T)(1 + |x|_E^{\alpha-1})|\phi|_{C^1_b}}{t^\alpha}\|A^{-\alpha}y\|,
\]
\[
|D^2U^{\delta t}(t, x).y, z| \leq \frac{C(T, \delta t_0, \beta, \gamma)(1 + |x|_E^{\beta+\gamma})|\phi|_{C^2_b}}{t^{\beta+\gamma}}\|A^{-\beta}y\|\|A^{-\gamma}z\|.
\]

Proof. Similar arguments in [BG18b, Theorem 4.1] prove (14) and that for $0 \leq \alpha < 1$,
\[
\|
\eta^{\beta}(t, x)\| \leq \frac{C(T, \delta t_0, \alpha)}{t^\alpha}\sup_{s \in [0, T]}\|\Psi^{\delta t}(X^{\delta t}(t, x))\|_E\|(-A)^{-\alpha}y\|,
\]
where \( \eta^y \) satisfies
\[
d\eta^y(t, x) = (-A + \Psi_{\delta t}(X^{\delta t}(t, x)))\eta^y(t, x)dt, \ \eta^y(t, x) = y.
\]

Here we give a short proof for (15) which is different from the dual argument in [BG18b]. Notice that
\[
D^2U^{\delta t}(t, x).(y, z) = \mathbb{E}[D\phi(X^{\delta t}(t, x)).\zeta^{y, z}(t, x)] + \mathbb{E}[D^2\phi(X^{\delta t}(t, x)).(\eta^y(t, x), \eta^z(t, x))],
\]
where \( \zeta^{y, z} \) satisfies
\[
d\zeta^{y, z}(t, x) = (-A + \Psi_{\delta t}(X^{\delta t}(t, x)))\zeta^{y, z}(t, x)dt + \Psi''_{\delta t}(X^{\delta t}(t, x))\eta^y(t, x)\eta^z(t, x).
\]

Thus it suffices to prove the regularity of \( \zeta^{y, z} \) thanks to (16). Due to the fact that
\[
\zeta^{y, z}(t, x) = \int_0^t V(t, s)\left(\Psi''_{\delta t}(X^{\delta t}(s, x))\eta^y(s, x)\eta^z(s, x)\right)ds,
\]
where
\[
dV(t, s)z = (-A + \Psi'_{\delta t}(X^{\delta t}(t, x)))V(t, s)z, \quad V(s, s)z = z,
\]
we need to deduce more refined estimate of \( V(t, s)z \), \( 0 \leq s < t \leq T \). The property of \( \Psi'_{\delta t} \) in Lemma 4.2, combined with a energy estimate, yields that
\[
\|V(t, s)z\|^2 \leq C(T, \delta t_0)\|z\|^2. \quad \text{Moreover, we claim that for } 0 \leq s < t \leq T, 0 \leq \alpha < 1,
\]
\[
\|V(t, s)y\| \leq \frac{C(T, \delta t_0, \alpha)}{(t - s)^\alpha} \sup_{r \in [0, T]} \|\Psi'_{\delta t}(X^{\delta t}(t, x))\|_E \|A^{-\alpha}y\|. \quad (17)
\]

Indeed, let \( \tilde{V}(t, s)y = V(t, s)y - e^{(t-s)A}y, 0 \leq s < t \leq T \). Then we have for \( t > s \),
\[
d\tilde{V}(t, s)y = (-A + \Psi'_{\delta t}(X^{\delta t}(t, x)))\tilde{V}(t, s)y + \Psi'_{\delta t}(X^{\delta t}(t, x))e^{(t-s)A}y, \quad \tilde{V}(s, s)y = 0,
\]
and
\[
\|\tilde{V}(t, s)y\| \leq \int_s^t \|V(t, r)\left(\Psi'_{\delta t}(X^{\delta t}(r, x))e^{(r-s)A}y\right)\|dr
\]
\[
\leq C(T, \delta t_0) \sup_{r \in [0, T]} \|\Psi'_{\delta t}(X^{\delta t}(r, x))\|_E \int_s^t \|e^{(r-s)A}y\|dr
\]
\[
\leq C(T, \delta t_0, \alpha) \sup_{r \in [0, T]} \|\Psi'_{\delta t}(X^{\delta t}(r, x))\|_E (t - s)^{1-\alpha} \|A^{-\alpha}y\|,
\]
which implies that the estimate (17) holds. Now, we are in the position to prove (15). Based on (17) and Sobolev embedding theorem, we have for \( \alpha > \frac{1}{4}, \)

\[
\| \zeta^{y,z}(t, x) \| = \int_0^t \| V(t, s) \Psi''_{\delta t}(X^{\delta t}(s, x)) \eta^y(s, x) \eta^z(s, x) \| ds \\
\leq C(T, \delta t_0, \alpha) \int_0^t \sup_{r \in [0, T]} \| \Psi'_{\delta t}(X^{\delta t}(r, x)) \|_E (t - s)^{-\alpha} \\
\left\| (-A)^{-\alpha} \left( \Psi''_{\delta t}(X^{\delta t}(s, x)) \eta^y(s, x) \eta^z(s, x) \right) \right\| ds \\
\leq C(T, \delta t_0, \alpha) \int_0^t \sup_{r \in [0, T]} \| \Psi'_{\delta t}(X^{\delta t}(r, x)) \|_E (t - s)^{-\alpha} \| \Psi''_{\delta t}(X^{\delta t}(s, x)) \|_E \\
\| \eta^y(s, x) \| \| \eta^z(s, x) \| ds \\
\leq C(T, \delta t_0, \alpha) \sup_{r \in [0, T]} \left( \| \Psi'_{\delta t}(X^{\delta t}(r, x)) \|_E \| \Psi''_{\delta t}(X^{\delta t}(s, x)) \|_E \right) \\
\int_0^t (t - s)^{-\alpha} \| \eta^y(s, x) \| \| \eta^z(s, x) \| ds.
\]

Now using the estimation (16), the growth of \( \Psi_{\delta t}, \) and stability of \( X^{\delta t}, \) we obtain

\[
\mathbb{E}[\| \zeta^{y,z}(t, x) \|] \leq C(T, \delta t_0, \alpha, \beta, \gamma) \int_0^t (t - s)^{-\alpha} s^{-\beta-\gamma} ds \| A^{-\beta} y \| \| A^{-\gamma} y \| \\
\mathbb{E}\left[ \sup_{r \in [0, T]} \| \Psi'_{\delta t}(X^{\delta t}(r, x)) \|_E^2 \| \Psi''_{\delta t}(X^{\delta t}(r, x)) \|_E^2 \right] \\
\leq C(T, \delta t_0, \alpha, \beta, \gamma) t^{-\beta-\gamma} (1 + \| x \|_E^{5K-6}) \| A^{-\beta} y \| \| A^{-\gamma} y \|,
\]

which completes the proof. \( \square \)

**Remark 4.1.** Sobolev embedding theorem \( \| y \|_{L^\infty} \leq C \| y \|_{H^{d+\epsilon}}, \) \( y \in H^{d+\epsilon}, \) \( \epsilon > 0, \) \( d = 1, 2, 3, \) yields that \( \| A^{-\frac{d}{2}+\epsilon} y \| \leq C \| y \|_{L^1}. \) Thus the regularity result of Kolmogorov equation in Proposition 4.1 can be generalized to the higher dimension case \( (d=2,3) \) and more regular noise.

### 4.2. Weak convergence rate

Before studying the weak convergence rate, we show that the numerical solution \( X^h \) is differentiable in Malliavin sense.
Proposition 4.2. Let Assumptions 2.1-2.3 hold. Then the Malliavin derivative of \( X^h \) satisfies, for some constant \( C \),

\[
\mathbb{E} \left[ \left\| A_h^{-\frac{1}{2}} D_s X^h(t) \right\|_{L^2}^2 \right] \leq C(T, \beta, X_0, K) \left( 1 + \log \left( \frac{1}{h} \right) \right)^{2(K-1)K^2}, \quad 0 \leq s \leq t \leq T.
\]

Proof. Indeed, since for \( 0 \leq s \leq t \leq T, y \in U_0, \)

\[
D_s^y X^h(t) = S^h(t-s) P^h y + \int_s^t S^h(t-r) P^h D F(X^h(r)) \cdot D_s^y X^h(r) dr
\]
satisfies

\[
d D_s^y X^h(t) = -A_h D_s^y X^h(t) dt + P^h D F(X^h(t)) \cdot D_s^y X^h(t) dt, \quad (18)
\]

\[
D_s^y X^h(s) = P^h y.
\]

In order to get the estimate of \( \left\| A_h^{-\gamma} D_s X^h(t) y \right\| \), \( 0 \leq \gamma \leq \frac{1}{2} \), we define \( \tilde{\eta}_s(t, y) = D_s X^h(t) y - S^h(t-s) P^h y \). Then \( \tilde{\eta}_s(t, y) \) satisfies the following equation

\[
d \tilde{\eta}_s(t, y) = -A_h \tilde{\eta}_s(t, y) dt + P^h (DF(X^h(t)) \cdot \tilde{\eta}_s(t, y)) dt
\]

\[
+ P^h (DF(X^h(t)) \cdot S^h(t-s) P^h y) dt,
\]

\[
\tilde{\eta}_s(s, y) = 0.
\]

and

\[
\tilde{\eta}_s(t, y) = \int_s^t \hat{V}(t, r) P^h (DF(X^h(r)) \cdot S^h(r-s) P^h y) dr,
\]

where \( \hat{V}(t, r) z \) solves for \( z \in V^h, \)

\[
d \hat{V}(t, r) z = -A_h \hat{V}(t, r) z + P^h (DF(X^h(t)) \hat{V}(t, r) z), \quad \hat{V}(r, r) z = z.
\]

The energy estimate, combined with the Gronwall’s inequality, yields that for \( s \leq r \leq t, \)

\[
\left\| \hat{V}(t, r) z \right\|^2 \leq C(T) \left\| z \right\|^2.
\]

This implies that

\[
\left\| \tilde{\eta}_s(t, y) \right\| \leq C(T) \int_s^t \left\| P^h (DF(X^h(r)) \cdot S^h(r-s) P^h y) \right\| dr
\]

\[
\leq C(T, \gamma) \sup_{r \in [0, T]} \left[ 1 + \left\| X^h(r) \right\|_{E}^{K-1} \right] \int_s^t (r-s)^{-\gamma} dr \left\| A^{-\gamma} y \right\|.
\]
Combining with the fact that for \(0 \leq \gamma \leq \frac{1}{2}\), \(\|S^h(t-s)P^h y\| \leq C(T, \gamma)(t-s)^{-\gamma}\|A^{-\gamma}y\|\), we get

\[
\|D_s^y X^h(t)\| \leq C(T, \gamma) \sup_{r \in [0,T]} \left[ 1 + \|X^h(r)\|_{\mathbb{E}^{K-1}}^2 \right] (t-s)^{-\gamma}\|A^{-\gamma}y\|.
\]

Thus by the mild form of \(D_s X^h(t)y\) and the equivalence of norms in (5), we obtain for \(0 \leq \gamma \leq \frac{1}{2}\),

\[
\|A^{-\gamma}D_s^y X^h(t)\| \leq \|A^{-\gamma}S^h(t-s)P^h y\|
\]

\[
+ \int_s^t \|S^h(t-r)P^h D^s F(X^h(r)) : D_s^y X^h(r)\| dr
\]

\[
\leq C(\gamma) \|S^h(t-s)P^h y\|
\]

\[
+ C(T, \gamma) \sup_{r \in [0,T]} \left[ 1 + \|X^h(r)\|_{\mathbb{E}^{K-1}}^2 \right] \int_s^t \|D_s^y X^h(r)\| dr
\]

\[
\leq C(T, \gamma) \|(A)^{-\gamma}y\| \left( 1 + \sup_{r \in [0,T]} \|X^h(r)\|_{\mathbb{E}^{2K-2}}^2 \right) \|A^{-\gamma}y\|.
\]

Now, taking \(-\gamma = \frac{\beta - 1}{2}\), \(0 < \beta \leq 1\), \(y = Q^\frac{1}{2} e_i, i \in \mathbb{N}^+\), together with the stability result of \(X^h\) in Proposition 3.1, yields that

\[
\mathbb{E}\left[ \left\| A^{\frac{\beta - 1}{2}} D_s X^h(t) \right\|_{\mathbb{E}^{K-1}}^2 \right] \leq C(T, \beta) \sum_{i \in \mathbb{N}^+} \mathbb{E}\left[ \left( 1 + \sup_{r \in [0,T]} \|X^h(r)\|_{\mathbb{E}^{4(K-1)}}^2 \right) \|A^{\frac{\beta - 1}{2}} Q^\frac{1}{2} e_i\|^2 \right]
\]

\[
\leq C(T, \beta) \left( 1 + (\log\left(\frac{1}{h}\right))^2(2^{K-1})K^2 \|X_0\|_{\mathbb{E}^{4(K-1)}}\right).
\]

Now, we turn to estimate the weak error of \(\mathbb{E}\left[ \phi(X^{h\delta}(t)) - \phi(X^h(t)) \right]\).

**Theorem 4.1.** Let Assumptions 2.1-2.3 hold. Assume in addition that \(|f''(\xi)| \leq L_f(1 + |\xi|^{K-2})\), \(2 \leq K < 5\), then for every test functions \(\phi \in C^2_0(\mathbb{H})\), \(h \in (0,1], T > 0, \beta \in (0,1]\) and \(\gamma < \beta\),

\[
\left| \mathbb{E}\left[ \phi(X^{h\delta}(T)) - \phi(X^h(T)) \right] \right| \leq C(X_0, T, \beta) \left( h^{2\gamma} + \delta t (\log\left(\frac{1}{h}\right))^{2K^2} \right).
\]
Proof. Based on the property \( \mathbb{E}\left[\phi(X^h(T))\right] = \mathbb{E}\left[U^{\delta t}(0, X^h(T))\right] \), we split the weak error as

\[
\left| \mathbb{E}\left[\phi(X^{\delta t}(T)) - \phi(X^h(T))\right] \right| = \left| \mathbb{E}\left[\phi(X^{\delta t}(T))\right] - \mathbb{E}\left[U^{\delta t}(T, X^h(0))\right] \right|
+ \left| \mathbb{E}\left[U^{\delta t}(T, X^h(0))\right] - \mathbb{E}\left[\phi(X^h(T))\right] \right|.
\]

By the regularity of \( DU^{\delta t}(t, x) \) (14) in Proposition 4.1, we bound the first error as for \( 0 \leq \alpha < 1, \)

\[
\left| \mathbb{E}\left[\phi(X^{\delta t}(T))\right] - \mathbb{E}\left[U^{\delta t}(T, X^h(0))\right] \right|
= \left| U^{\delta t}(T, X(0)) - U^{\delta t}(T, X^h(0)) \right|
= C(T, \delta t_0) \left| \int_0^1 DU^{\delta t}(T, \theta X(0) + (1 - \theta)X^h(0))d\theta \cdot ((I - P^h)X(0)) \right|
\leq C(T, \delta t_0, \alpha, \phi)T^{-\alpha}\mathbb{E}\left[\left(1 + \|X_0\|_E^2 + \|X^h(0)\|_E^2\right)\|(A)^{-\alpha}(I - P^h)X(0)\| \right]
\leq C(T, \delta t_0, \alpha, \phi, X_0)T^{-\alpha}h^{2\alpha},
\]

where we use the fact \( \|(A)^{-\alpha}(I - P^h)\|_\mathcal{L} = \|(A)^{-\alpha}(I - P^h)^*\|_\mathcal{L} = \|I - P^h(A)^{-\alpha}\| \) and the estimation (6).

Next, we aim to estimate the left error \( \left| \mathbb{E}\left[U^{\delta t}(T, X^h(0))\right] - \mathbb{E}\left[\phi(X^h(T))\right] \right|. \)

We recall the Markov generator \( \mathcal{L}^h \) of \( X^h, \)

\[
(\mathcal{L}^h U)(x) = \langle -A^h x + P^h f(x), DU(x) \rangle + \frac{1}{2}tr[P^h Q P^h D^2 U(x)],
\]

\( U \in C^2(\mathbb{H}, \mathbb{R}), \ x \in V_h. \)

Then Itô formula and corresponding Kolmogorov equation (12) yield that

\[
\mathbb{E}\left[U^{\delta t}(T, X^h(0))\right] - \mathbb{E}\left[\phi(X^h(T))\right]
= \mathbb{E}\left[U^{\delta t}(T, X^h(0)) - U^{\delta t}(0, X^h(T))\right]
= -\mathbb{E}\left[\int_0^T \left(U^{\delta t}(T - t, X^h(t)) + \mathcal{L}^h U^{\delta t}(T - t, X^h(t))\right)dt\right]
= \mathbb{E}\left[\int_0^T \left((\mathcal{L}^h - \mathcal{L}^{\delta t}) U^{\delta t}(T - t, X^h(t))\right)dt\right].
\]
Based on the expressions of $\mathcal{L}^{\delta t}$ and $\mathcal{L}^h$, we obtain

$$
\left| \mathbb{E}\left[U^{\delta t}(T, X^h(0))\right] - \mathbb{E}\left[\phi(X^h(T))\right] \right| \\
\leq \left| \mathbb{E}\left[ \int_0^T \left\langle (A - A_h)X^h(t), DU^{\delta t}(T - t, X^h(t)) \right\rangle dt \right] \right| \\
+ \left| \mathbb{E}\left[ \int_0^T \left\langle \Psi_{\delta t}(X^h(t)) - P^h F(X^h(t)), DU^{\delta t}(T - t, X^h(t)) \right\rangle dt \right] \right| \\
+ \frac{1}{2} \left| \mathbb{E}\left[ \int_0^T \mathrm{tr}\left\{ QD^2 U^{\delta t}(T - t, X^h(t)) - P^h QP^h D^2 U^{\delta t}(T - t, X^h(t)) \right\} dt \right] \right| \\
:= e^1(T) + e^2(T) + e^3(T).
$$

Similar to [AL16], the relation $P^h = (A_h)^{-1} P^h A$ implies that

$$
\langle (A - A_h)X^h(t), DU^{\delta t}(T - t, X^h(t)) \rangle \\
= \langle (AP^h - P^h A_h)X^h(t), DU^{\delta t}(T - t, X^h(t)) \rangle \\
= \langle X^h, (P^h A - A_h P^h)DU^{\delta t}(T - t, X^h(t)) \rangle \\
= \langle X^h, A_h P^h ((A_h)^{-1}P^h A - I)DU^{\delta t}(T - t, X^h(t)) \rangle \\
= \langle X^h, A_h P^h (R^h - I)DU^{\delta t}(T - t, X^h(t)) \rangle.
$$

The above equality and the mild form of $X^h$ lead that

$$
ee^1(T) \leq \left| \mathbb{E}\left[ \int_0^T \left\langle S^h(t)X^h(0), A_h P^h (R^h - I)DU^{\delta t}(T - t, X^h(t)) \right\rangle dt \right] \right| \\
+ \left| \mathbb{E}\left[ \int_0^T \left\langle \int_0^t S^h(t - s)P^h F(X^h(s))ds, A_h P^h (R^h - I)DU^{\delta t}(T - t, X^h(t)) \right\rangle dt \right] \right| \\
+ \left| \mathbb{E}\left[ \int_0^T \left\langle \int_0^t S^h(t - s)dW(s), A_h P^h (R^h - I)DU^{\delta t}(T - t, X^h(t)) \right\rangle dt \right] \right| \\
:= e^{1,1}(T) + e^{1,2}(T) + e^{1,3}(T)\]$$

Applying the regularity of $DU^{\delta t}$ (14), the smoothy property of $S^h$ (8) and
the stability of $X^h$ in Proposition (3.1), if follows that for small $\epsilon > 0$,

$$e^{1,1}(T) = \left| \mathbb{E}\left[ \int_0^T \left\langle A_{h}^{-\epsilon} S^h(t) X^h(0), A_{h}^{-\epsilon} P^h(R^h - I)(A)^{-\alpha} A^{-\alpha} DU^\delta(t - t, X^h(t)) \right\rangle dt \right] \right|$$

$$\leq \mathbb{E}\left[ \int_0^T \left\| A_{h}^{-\epsilon} S^h(t) X^h(0) \right\| \left\| A_{h}^{-\epsilon} P^h(R^h - I)(A)^{-\alpha} A^{-\alpha} DU^\delta(t - t, X^h(t)) \right\| dt \right]$$

$$\leq C(T, \epsilon) h^{2\alpha - 2\epsilon} \int_0^T t^{-1+\epsilon} \| X_0 \| \mathbb{E}\left[ \left\| A^{-\alpha} DU^\delta(t - t, X^h(t)) \right\| \right] dt$$

$$\leq C(T, \epsilon, \alpha, \phi) h^{2\alpha - 2\epsilon} \int_0^T t^{-1+\epsilon} (T - t)^{-\alpha} \| X_0 \| \sup_{t \in [0, T]} \mathbb{E}\left[ 1 + \left\| X^h(t) \right\|_E^{K-1} \right] dt.$$

Similar arguments yield that

$$e^{1,2}(T) = \left| \mathbb{E}\left[ \int_0^T \left\langle \int_0^t A_{h}^{-\epsilon} S^h(t - s) P^h F(X^h(s)) ds, A_{h}^{-\epsilon} P^h(R^h - I)(A)^{-\alpha} A^{-\alpha} DU^\delta(t - t, X^h(t)) \right\rangle dt \right] \right|$$

$$\leq \mathbb{E}\left[ \int_0^T \left\| A_{h}^{-\epsilon} S^h(t - s) P^h F(X^h(s)) \right\| ds \left\| A_{h}^{-\epsilon} P^h(R^h - I)(A)^{-\alpha} \right\|_{\mathcal{L}(\mathbb{H})} \left\| A^{-\alpha} DU^\delta(t - t, X^h(t)) \right\| dt \right]$$

$$\leq C(T, \epsilon, \alpha) h^{2\alpha - 2\epsilon} \sup_{t \in [0, T]} \sqrt{\mathbb{E}\left[ \| F(X^h(t)) \|^2 \right]}$$

$$\int_0^T \int_0^t \sqrt{\mathbb{E}\left[ \| A^{-\alpha} DU^\delta(t - t, X^h(t)) \|^2 \right](t - s)^{-1+\epsilon} ds dt}$$

$$\leq C(T, \epsilon, \alpha, \phi) h^{2\alpha - 2\epsilon} \sup_{t \in [0, T]} \sqrt{\mathbb{E}\left[ 1 + \| X^h(t) \|^2_k \right]} \sup_{t \in [0, T]} \sqrt{\mathbb{E}\left[ 1 + \| X^h(t) \|^2_k \right]}$$

$$\int_0^T \int_0^t (T - t)^{-\alpha} (t - s)^{-1+\epsilon} ds dt.$$

To deal with $e^{1,3}(T)$, we make use of the integration by parts in Malliavin
sense (9) and chain rule to get

\[ e^{1.3}(T) = \left| \mathbb{E} \left[ \int_0^T \left\langle \int_0^t S^h(t-s)P^h \right. \right. \left. dW(s), A_h P^h (R^h - I) D U_{\delta t}^\alpha (T-t, X^h(t)) \right\rangle dt \right| \]

\[ = \left| \mathbb{E} \left[ \int_0^T \int_0^t \left\langle S^h(t-s), \right. \right. \left. A_h P^h (R^h - I) D^2 U_{\delta t}^\beta (T-t, X^h(t)) D_s X^h(t) \right\rangle \right| ds dt \right|. \]

Then by the property of Hilbert–Schmidt operator and Cauchy–Schwarz inequality, we have

\[ e^{1.3}(T) = \left| \mathbb{E} \left[ \int_0^T \int_0^t \left( A_{h1}^{1-\epsilon} S^h(t-s) P^h, \right. \right. \left. A_h P^h (R^h - I) A^{-\alpha} A^\alpha D^2 U_{\delta t}^\beta (T-t, X^h(t)) D_s X^h(t) \right) \right| dt \left. \right| ds dt \right|. \]

\[ \leq \left| \mathbb{E} \left[ \int_0^T \int_0^t \left\| A_{h1}^{1-\epsilon} S^h(t-s) P^h \right\|_{L_2} \left\| A_h P^h (R^h - I) (A)^{-\beta+1+t} \right\| \left\| A_{h1}^{1-\epsilon} D^2 U_{\delta t}^\beta (T-t, X^h(t)) A_{h1}^{1-\epsilon} \right\|_{L_2} \left\| A_{h1}^{1-\epsilon} D_s X^h(t) \right\| \right| \right| ds dt \right|.

Combining the regularity result of \( DU_{\delta t}^\beta \) (14) and \( D^2 U_{\delta t}^\beta \) (15), the smoothy property of \( S^h \) and the stability of \( X^h \) together, we obtain

\[ e^{1.3}(T) \leq C(T, \epsilon) h^{\beta+1-4\epsilon} \int_0^T \int_0^t (t-s)^{-1+\epsilon} \sqrt{\mathbb{E} \left[ \left\| A_{h1}^{\beta+1-\epsilon} D_s X^h(t) \right\|_{L_2}^2 \right]} ds dt \]

\[ \leq C(T, \epsilon) h^{\beta+1-4\epsilon} \int_0^T \int_0^t (t-s)^{-1+\epsilon} (T-t)^{-1+\epsilon} \sqrt{\mathbb{E} \left[ \left\| A_{h1}^{\beta+1-\epsilon} D_s X^h(t) \right\|_{L_2}^2 \right]} ds dt \]

\[ \leq C(T, \epsilon) h^{\beta+1-4\epsilon} \sup_{t \in [0,T]} \sqrt{\mathbb{E} \left[ 1 + \left\| X^h(t) \right\|_{L_2}^{10K-12} \right]} ds dt. \]

Combining with Proposition 4.2, it produces that

\[ e^{1.3}(T) \leq C(X_0, T_0, \epsilon, \beta) h^{\beta+1-4\epsilon} \left( 1 + \log \left( \frac{1}{h} \right) \right)^{\frac{10K-12}{2}}. \]
Thus we conclude that $e_1(T) \leq C(X_0, T, \epsilon, \beta)(1 + T^{-\beta} + \log(\frac{1}{h}))^\frac{(7K-8)K^2}{2})h^{2\beta - 2\epsilon}$.

Next, we turn to focus on $e_2(T)$. From $\Psi_0 = F$, it follows that

$$e_2(T) \leq \left| \mathbb{E}\left[ \int_0^T \left< \Psi_\delta t(X^h(t)) - \Psi_0(X^h(t)), DU^{\delta t}(T - t, X^h(t)) \right> dt \right] \right| + \left| \mathbb{E}\left[ \int_0^T \left< (I - P^h)F(X^h(t)), DU^{\delta t}(T - t, X^h(t)) \right> dt \right] \right|

:= e_1^{2.1}(T) + e_2^{2.2}(T).$$

By the continuity of $\Psi_t$ with respect to $t$ in Lemma 4.2 and the stability of $X^h$ in Proposition 3.1, it leads to

$$e_1^{2.1}(T) \leq \mathbb{E}\left[ \int_0^T \left| D\psi_\delta t(T - t, X^h(t)) \right| \left| \Psi_\delta t(X^h(t)) - \Psi_0(X^h(t)) \right| dt \right] \leq C(T, \delta t_0)\delta t \left( 1 + \sup_{t \in [0, T]} \|X^h(t)\|^{3K-2} \right).$$

The regularity of $DU^{\delta t}$ (14), the estimate (6), the growth of $F$ and the stability of $X^h$ yield that

$$e_2^{2.2}(T) \leq \mathbb{E}\left[ \int_0^T \left| (I - P^h)(A)^{-1+t} \right|_{L(H)} \left| (A)^{1-t} DU^{\delta t}(T - t, X^h(t)) \right|_{L(H)} \right] \leq C(T, \epsilon)h^{2\beta - 2\epsilon}\left[ \int_0^T (T - t)^{-1+\epsilon} \mathbb{E} \left[ \|F(X^h(t))\|^2 \right] \sqrt{\mathbb{E} \left[ 1 + \|X^h(t)\|^{2K-2} \right]} dt \right] \leq C(T, \epsilon)h^{2\beta - 2\epsilon} \sup_{t \in [0, T]} \mathbb{E} \left[ 1 + \|X^h(t)\|^{2K-1} \right].$$

Summing up the estimations of $e_1^{2.1}$ and $e_2^{2.2}$, we deduce that

$$e_2(T) \leq C(X_0, T, \epsilon) \left( h^{2\beta - 2\epsilon}(1 + (\log(\frac{1}{h}))^{\frac{(2K-1)K^2}{2}}) + \delta t(1 + (\log(\frac{1}{h}))^{\frac{(3K-2)K^2}{2}}) \right).$$
For the last term $e^3(T)$, we have

$$e^3(T) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \text{tr}\left\{ (I Q (I - P^h) + (I - P^h) Q P^h) D^2 U^\delta t (T - t, X^h(t)) \right\} dt \right]$$

$$\leq \frac{1}{2} \mathbb{E} \left[ \int_0^T \text{tr}\left\{ I Q (I - P^h) D^2 U^\delta t (T - t, X^h(t)) \right\} dt \right]$$

$$+ \frac{1}{2} \mathbb{E} \left[ \int_0^T \text{tr}\left\{ (I - P^h) Q P^h D^2 U^\delta t (T - t, X^h(t)) \right\} dt \right]$$

$$:= e^{3,1}(T) + e^{3,2}(T).$$

The properties of trace and Hilbert–Schmidt operator lead that

$$e^{3,1}(T) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \text{tr}\left\{ I Q (I - P^h) D^2 U^\delta t (T - t, X^h(t)) A^{1+\beta \over 2} A^{-1+\beta \over 2} \right\} dt \right]$$

$$= \frac{1}{2} \mathbb{E} \left[ \int_0^T \text{tr}\left\{ A^{1+\beta \over 2} Q (I - P^h) D^2 U^\delta t (T - t, X^h(t)) A^{-1+\beta \over 2} \right\} dt \right]$$

$$= \frac{1}{2} \mathbb{E} \left[ \int_0^T \text{tr}\left\{ A^{1+\beta \over 2} QA^{-1+\beta \over 2} (I - P^h) A^{-1+\beta \over 2} \right\} dt \right]$$

$$\leq \frac{1}{2} \int_0^T \mathbb{E}\left[ \left\| A^{1+\beta \over 2} \right\|_{L^2}^2 \right] \left\| A^{-1+\beta \over 2} (I - P^h) A^{-1+\beta \over 2} \right\|_{L(H)}$$

$$\left\| A^{1+\beta \over 2} D^2 U^\delta t (T - t, X^h(t)) A^{-1+\beta \over 2} \right\|_{L(H)} dt.$$
Similarly, we have

$$e^{3.2}(T) = \frac{1}{2} \left| \int_0^T \text{tr}\left\{ (A)^{-\frac{\beta+1}{2}+\epsilon} (I - P^h) A^\frac{1-\beta}{2} A^\frac{\beta-1}{2} Q A^\frac{\beta-1}{2} \right\} dt \right|$$

where we use the property ...

$$A^\frac{\beta+1}{2} P^h A^\frac{\beta-1}{2} \leq C \sup_{t \in [0, T]} \left| 1 + \|X(t)\|_{5K-6} \right| dt$$

where we use the property \( \|A^\frac{\beta+1}{2} P^h A^\frac{\beta-1}{2}\| \leq C \) proven by the equivalence of norms (5),

\[
\left| A^\frac{\beta+1}{2} P^h A^\frac{\beta-1}{2} \right|_{\mathcal{L}(H)} \leq C \left| A^\frac{\beta+1}{2} P^h A^\frac{\beta-1}{2} \right|_{\mathcal{L}(H)} \leq C \left| A^\frac{\beta+1}{2} A^\frac{\beta-1}{2} \right|_{\mathcal{L}(H)} \leq C.
\]

The estimations of \( e^{3.1}(T) \) and \( e^{3.2}(T) \) indicate

$$e^{3}(T) \leq C(T, \beta, \epsilon) h^{2\beta-2\epsilon} \left( 1 + \left( \log \left( \frac{1}{h} \right) \right)^{\frac{5K-6K^2}{2}} \right) \left( 1 + \|X_0\|_{5K-6} \right).$$

Summing up all the estimations of \( \left| \mathbb{E}\left[ \phi(X^{\delta t}(T)) - \mathbb{E}\left[ U^{\delta t}(T, X^h(0)) \right] \right] \right|, e^1(T), e^2(T) \) and \( e^3(T) \), we obtain that for any small \( \epsilon_1 > \epsilon \),

$$\left| \mathbb{E}\left[ \phi(X^{\delta t}(T)) - \phi(X^h(T)) \right] \right|$$

$$\leq C(X_0, T, \epsilon, \beta) h^{2\beta-4\epsilon} (1 + T^{-\beta} + \log \left( \frac{1}{h} \right)^{\frac{(5K-6)K^2}{2}})$$

$$+ C(X_0, T, \epsilon) \delta t \left( 1 + \left( \log \left( \frac{1}{h} \right) \right)^{\frac{(3K-2)K^2}{2}} \right)$$

$$\leq C(X_0, T, \beta, \epsilon) \left( h^{2\beta-4\epsilon_1} + \delta t \left( \log \left( \frac{1}{h} \right)^{\frac{(3K-2)K^2}{2}} \right) \right),$$

which, combined with a standard argument, finishes the proof.
Combining Lemma 4.3 and Theorem 4.1, we deduce the sharp weak convergence rate of the finite element method approximating the original equation. We remark that even though the logarithm factor can be eliminated if $\beta > \frac{1}{2}$ or $Q = I$, $\beta < \frac{1}{2}$, the weak convergence rate can not be improved.

**Theorem 4.2.** Let $T > 0$. Under the assumptions of Theorem 4.1, $X^h(T)$ is weakly convergent to $X(T)$ and satisfies

$$
\left| \mathbb{E} \left[ \phi(X(T)) - \phi(X^h(T)) \right] \right| \leq C(X_0, T, \beta) h^{2 \gamma},
$$

for $\phi \in C^2_b(\mathbb{H})$ and $\beta \in [0, 1)$, $\gamma < \beta$.

**Proof.** By triangle inequality, Lemma 4.3 and Theorem 4.1 and taking $\delta t = O(h^{2\beta})$, we have

$$
\left| \mathbb{E} \left[ \phi(X(T)) - \phi(X^h(T)) \right] \right| \leq \left| \mathbb{E} \left[ \phi(X(T)) - \phi(X^{\delta t}(T)) \right] \right| + \left| \mathbb{E} \left[ \phi(X^{\delta t}(T)) - \phi(X^h(T)) \right] \right|
$$

$$
\leq C(X_0, T, p) \delta t
$$

$$
+ C(X_0, T, \beta) \left( h^{2\gamma} + \delta t (\log \left( \frac{1}{h} \right) )^{\frac{3K - 2}{2}} K^2 \right)
$$

$$
\leq C(X_0, T, p) h^{2\gamma},
$$

which completes the proof.

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