Metrological Nonlinear Squeezing Parameter

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The well known metrological linear squeezing parameters (such as quadrature or spin squeezing) efficiently quantify the sensitivity of Gaussian states. Yet, these parameters are no longer sufficient to characterize the much wider class of highly sensitive non-Gaussian states. We introduce a class of metrological nonlinear squeezing parameters obtained by analytical optimization of measurement observables among a given set of accessible (possibly nonlinear) operators. This allows for the metrological characterization of non-Gaussian quantum states of discrete and continuous variables. Our results lead to optimized and experimentally-feasible recipes for high-precision moment-based estimation of a phase parameter and can be used to systematically construct multipartite entanglement and non-classicality witnesses for complex quantum states.

Introduction.—A central quest in quantum metrology is to relate the reduced variance of an observable to the possible enhancement of sensitivity in parameter estimation [1–4]. For instance, quadrature squeezing can enhance the sensitivity of homodyne interferometers beyond the shot-noise limit [1], as experimentally demonstrated with squeezed vacuum states of light [5, 6] and atoms [7], and envisaged for third-generation gravitational wave detectors [8, 9]. Moreover, multi-mode squeezing can reveal mode entanglement [10–15] and Einstein-Podolski-Rosen correlations [16–19]. Squeezing of a collective spin [2] currently represents the leading strategy to obtain quantum-enhanced sensitivities in Ramsey interferometers [3], with direct applications to atomic clocks [20], magnetometers [21], and matter-wave interferometers [22]. Spin squeezing is also a witness of metrologically-useful multiparticle-entanglement [23–26] and Bell correlations [27–29]. Squeezing of linear observables of discrete [23–26, 30–37] or continuous variables [38–42], e.g., collective spins or quadratures, has proven to be a successful concept to characterize the class of Gaussian quantum states with phase-estimation sensitivities beyond the classical limit [3], and is hereinafter indicated as metrological linear squeezing.

Yet, some highly sensitive continuous-variable states are non-Gaussian [43–46] and Gaussian spin states form a small and non-optimal class of useful states for metrology [47–49]. Non-Gaussian states further hold the promise of opening up classically intractable pathways for quantum information processing [50–52]. These perspectives have led to a growing interest in the generation of non-Gaussian quantum states in both discrete- and continuous-variable systems [53] using nonlinear processes [54, 55], photon-addition or -subtraction [56–58] or measurement techniques [44, 45, 59]. More refined tools are required to characterize highly sensitive non-Gaussian states, as the linear squeezing coefficient becomes too coarse to capture non-Gaussian features [54]. It would be highly desirable to reveal the metrological sensitivity of non-Gaussian states using only the mean value and variance of some accessible nonlinear observables, beyond the limitations of linear squeezing. However, this possibility has been demonstrated only for specific cases—e.g., using squared spin operators for twin-Fock states [60, 61], or the parity operator for GHZ states [62–64]—and it is not known how to systematically identify optimal nonlinear observables for arbitrary states. While in principle, we may theoretically determine an optimal projective measurement that will fully reveal the metrological potential of any quantum state [65], such a measurement is experimentally unfeasible in most cases.

In this manuscript, we extend the concept of metrological linear squeezing to arbitrary (nonlinear) observables and provide a systematic way to optimize it. Specifically, we analytically identify the optimal measurement observable out of any given family of accessible operators for arbitrary quantum states. Measurement of this observable will lead to the highest achievable metrological sensitivity in a quantum phase estimation experiment within this family of accessible operators. If this family includes only linear observables, we recover the well known linear squeezing parameters. If also nonlinear operators are accessible, we obtain metrological nonlinear squeezing parameters that are suitable to characterize the sensitivity of a wider class of non-Gaussian quantum states, as we illustrate with a series of examples. When all possible measurement operators are accessible, the ensemble of states detected by the nonlinear squeezing parameters equals the full ensemble of metrologically-useful quantum states detected by the quantum Fisher information. Our results provide scalable tools for the development of feasible quantum phase estimation strategies beyond Gaussian states and the identification of multiparticle entanglement in increasingly complex many-body quantum systems.

Metrological nonlinear squeezing parameter.—One possibility to estimate an unknown parameter $\theta$ encoded in a quantum state $\hat{\rho}(\theta)$ is given by the method of moments (see [49] for a review). Within this approach $\theta$ is estimated as the parameter $\theta_{\text{est}}$ that yields equality of the sample mean $\bar{x}_\mu = \sum_{i=1}^N x_i/\mu$ of a sequence of independent measurements and the expectation value $\langle \hat{X} \rangle_{\hat{\rho}(\theta)} = \text{Tr}(\hat{X} \hat{\rho}(\theta))$ that is determined in a calibration experiment beforehand, i.e., $\bar{x}_\mu = \langle \hat{X} \rangle_{\hat{\rho}_{\text{est}}}$. In the central limit, for $\mu$ sufficiently large, the random variable $\bar{x}_\mu$ is normally distributed around $\langle \hat{X} \rangle_{\hat{\rho}(\theta)}$ with variance $(\Delta \bar{x}_\mu)^2/\mu$. Then, the phase uncertainty is given by $(\Delta \theta_{\text{est}})^2 = x^2(\hat{p}(\theta)/\mu, \hat{X}^2)$, where $x^2(\hat{p}(\theta), \hat{X}) = (\Delta \hat{X})^2(\partial \hat{X}/\partial \theta)^2$ is the squeezing parameter of $\hat{p}$ associated with the measurement of the observable $\hat{X}$. It can be obtained by error propaga-
tion and quantifies the squeezing of the measurement variance \((\Delta X)^2_{\rho,\theta}\) with respect to the variation of the expectation value \(\langle X \rangle_{\rho,\theta}\) of \(X\). The squeezing parameter fulfills the chain of inequalities \(\chi^2(\rho,\theta) \leq F(\rho(\theta),\tilde{X}) \leq F_O[\rho(\theta),\tilde{X}]\) that is saturable by an optimal \(\tilde{X}\) [49, 65–67]. Here, \(F(\rho(\theta),\tilde{X}) = \sum_x p(x|\theta) \log p(x|\theta)^2\) is the Fisher information, where \(p(x|\theta) = \text{Tr}([\Gamma(\rho(\theta))_{x}])\) describes the full counting statistics for the observable \(\tilde{X}\) with spectral decomposition \(\tilde{X} = \sum_x x_1 x\). Finally, the quantum Fisher information \(F_Q[\rho(\theta),\tilde{X}] = \max_{\tilde{X}} F(\rho(\theta),\tilde{X})\) defines the state’s full metrological potential by providing the quantum Cramér–Rao sensitivity limit \((\Delta \theta)_{\text{stat}}^2 \geq (\mu F_Q[\rho(\theta),\tilde{X}])^{-1}\) for arbitrary unbiased estimation strategies [65, 68]. For unitary evolutions \(\rho(\theta) = e^{-i\theta \rho} e^{i\theta \rho}\) generated by \(\tilde{H}\), the parameter-encoding Hamiltonian \(\tilde{H}\) and the observable \(\tilde{X}\).

The achievable sensitivity (1) depends on the choice of the observable \(\tilde{X}\). This motivates the introduction of an optimal metrological squeezing parameter for a family of accessible operators \(\tilde{H} = (\tilde{H},\ldots,\tilde{H}_k)\) as

\[
\chi^2_{\rho,\tilde{H},\tilde{X}} = \frac{\langle \hat{H}_m, \hat{H}_m \rangle_{\rho}^{\frac{1}{2}}}{\langle \hat{H}_m, \hat{H}_m \rangle_{\rho}^{\frac{1}{2}}} = n^T M[\hat{\rho}, \hat{\tilde{H}}] n,
\]

for all \(n\) and \(m\), where we introduced the moment matrix

\[
M[\hat{\rho}, \hat{\tilde{H}}] = C[\hat{\rho}, \hat{\tilde{H}}]^T \Gamma[\hat{\rho}, \hat{\tilde{H}}]^{-1} C[\hat{\rho}, \hat{\tilde{H}}],
\]

Here, \(\Gamma[\hat{\rho}, \hat{\tilde{H}}]\) is the covariance matrix with elements \((\Gamma[\hat{\rho}, \hat{\tilde{H}}])_{ij} = \text{Cov}(\hat{H}_i, \hat{H}_j)\), which is symmetric \(\Gamma[\hat{\rho}, \hat{\tilde{H}}] = \Gamma[\hat{\rho}, \hat{\tilde{H}}]^T\) and positive semidefinite for all \(\tilde{H}\). We further assume \(\Gamma[\hat{\rho}, \hat{\tilde{H}}]\) to be positive definite and hence invertible, which excludes the situation where \(\tilde{H}\) has zero variance for some \(\tilde{H}\). The real-valued, skew-symmetric commutator matrix \(C[\hat{\rho}, \hat{\tilde{H}}] = -C[\hat{\rho}, \hat{\tilde{H}}]^T\) has elements \((C[\hat{\rho}, \hat{\tilde{H}}])_{ij} = -i(\langle \hat{H}_i, \hat{H}_j \rangle)\). The maximum in (3) is reached for

\[
m = \alpha \Gamma[\hat{\rho}, \hat{\tilde{H}}]^{-1} C[\hat{\rho}, \hat{\tilde{H}}] n,
\]

where \(\alpha \in \mathbb{R}\) is a normalization constant. To prove Eq. (3), we write \((\Delta \hat{H}_m)^2_{\rho,\theta} = m^T \Gamma[\hat{\rho}, \hat{\tilde{H}}] m\) and \((\langle \hat{H}_m, \hat{H}_m \rangle_{\rho}) = \sum_n m_n \delta_n (\langle \hat{H}_m, \hat{H}_m \rangle) = m^T C[\hat{\rho}, \hat{\tilde{H}}] m\). The Cauchy-Schwarz inequality \((u^T v)^2 \leq (v^T v)(u^T u)\) holds for arbitrary vectors \(u\) and \(v\) and is saturated if and only if \(u = \alpha v\) with some normalization constant \(\alpha\). Inserting \(u = \Gamma[\hat{\rho}, \hat{\tilde{H}}] m\) and \(v = \Gamma[\hat{\rho}, \hat{\tilde{H}}]^{-1} C[\hat{\rho}, \hat{\tilde{H}}] m\) yields the statement.

The optimized squeezing coefficient (2) depends on the available set \(\tilde{H}\) of accessible observables. If \(\tilde{H}\) contains all linear observables, Eq. (2) provides an analytically-optimized linear squeezing parameter. By adding nonlinear observables, we can introduce metrological nonlinear squeezing coefficients for non-Gaussian states, as we will discuss below. For all \(\tilde{X}, \tilde{H}\), and \(\tilde{H}\), the coefficients \(\chi^2_{\rho, \tilde{X}}, \tilde{X}\) and \(\chi^2_{\rho, \tilde{X}}, \tilde{X}\) are convex functions of \(\rho\) [70], which ensures that they are maximized by pure states. Finally, maximizing over all possible observables \(\tilde{X}\), we have that \(\max_{\tilde{X}} \chi^2_{\rho, \tilde{X}}, \tilde{X}\) is the Fisher information [71].

Besides determining the optimal measurement observable \(\tilde{H}_m\), we may use Eq. (3) to find the evolution Hamiltonian \(\tilde{H}_n\) that leads to the highest possible phase estimation sensitivity. This Hamiltonian is identified as \(\tilde{H}_\text{max}\), where \(\tilde{m}_\text{max}\) is the maximum eigenvector of \(\tilde{M}[\hat{\rho}, \hat{\tilde{H}}]\) and the obtained sensitivity is the corresponding eigenvalue \(\lambda_{\text{max}}(\tilde{M}[\hat{\rho}, \hat{\tilde{H}}])\). If \(\tilde{C}[\hat{\rho}, \hat{\tilde{H}}]\) is a maximal eigenvector of \(\Gamma[\hat{\rho}, \hat{\tilde{H}}]^{-1}\), the optimal measurement \(\tilde{m}\) is achieved by \(\tilde{m}_{\text{max}} = \alpha^{\dagger} C[\hat{\rho}, \hat{\tilde{H}}] \tilde{C}[\hat{\rho}, \hat{\tilde{H}}] n_{\text{max}} = \alpha^{\dagger} C[\hat{\rho}, \hat{\tilde{H}}] n_{\text{max}}\), where we used the eigenvalue property and \(\alpha, \alpha'\) are normalization constants.

Finally, to quantify the achievable metrological sensitivity enhancement, we introduce

\[
\xi_{\text{opt}}^2[\rho, \tilde{H}, \tilde{H}] := \frac{F_{\text{SN}}[\tilde{H}]}{\chi^2_{\text{opt}}[\rho, \tilde{H}, \tilde{H}]},
\]

where \(F_{\text{SN}}[\tilde{H}] = \max_{\tilde{H}} F_Q[\rho, \tilde{H}]\) indicates the shot-noise (SN) limit, i.e., the maximal quantum Fisher information for classical states namely particle-separable states in a many-spin system [47] or coherent states in the continuous-variable regime [72, 73].

Nonlinear spin squeezing coefficients.—Let us consider the case of an \(N\)-qubit system described by collective spin operators \(\tilde{J} = (\tilde{J}_x, \tilde{J}_y, \tilde{J}_z)\) where \(\tilde{J}_n = \sum_{k=1}^{N} \delta_{n k}^{(2)} / 2\) and the \(\delta_{n k}^{(2)}\) are the Pauli matrices for \(\alpha = x, y, z\). The operators \(\tilde{J}\) are linear in a sense that they do not involve spin-spin interactions. Particle-separable states \(\tilde{\rho}_{\text{p-sep}} = \sum_y p_y \tilde{\rho}_y^{(N)}\) can at most have a sensitivity of \(F_{\text{SN}}[\tilde{J}_n] = \max_{p_y} F_Q[\tilde{\rho}_{\text{p-sep}}, \tilde{J}_n] = N [47]\), where \(p_y\) describes a probability distribution and the \(\tilde{\rho}_y^{(N)}\) are local states of the \(k\)th qubit. As the achievable quantum-enhancement increases with the number of entangled particles, we obtain with Ref. [74] that \(\xi_{\text{opt}}^2[\rho, \tilde{J}_n, \tilde{H}] > k\) reveals multiparticle entanglement of at least \(k\) qubits, where \(\tilde{J}_n = n \cdot \tilde{J}\) is an arbitrary collective spin operator with \(n \in \mathbb{R}^3\) and \(|n|^2 = 1\) and the elements of \(\tilde{H}\) can be nonlinear [75].

Using Eq. (6), we now introduce a fully optimized (linear) spin squeezing coefficient as \(\xi_{\text{opt}}^2[\rho, \tilde{J}_n, \tilde{H}] = \min_n \xi_{\text{opt}}^2[\rho, \tilde{J}_n, \tilde{H}]\). This is equivalent to the spin squeezing coefficient first introduced by Wineland et al. [2]. Both directions for the measurement and the parameter-encoding evolution are optimized for sensitivity. The implied optimization problem can be involved (see, e.g., Ref. [37]). Using Eq. (3) and
max \( n \), \( \chi_{\text{opt}}^2(\hat{\rho}, \hat{J}_n, \hat{\mathbf{J}}) = \lambda_{\text{max}}(M[\hat{\rho}, \hat{\mathbf{J}}]) \), we obtain

\[
\xi_{(1)}^2(\hat{\rho}) = \min_{m, n} \frac{N(\Delta J^2_m)^2}{\{[J^2_m, J_n]\}_\hat{\rho}} = \frac{N}{\lambda_{\text{max}}(M[\hat{\rho}, \hat{\mathbf{J}}])},
\]

(7)

which solves this problem analytically for arbitrary states \( \hat{\rho} \) without constraints. A state \( \hat{\rho} \) is spin squeezed if \( \xi_{(1)}^2(\hat{\rho}) < 1 \).

To assess the sensitivity of non-Gaussian spin states we define optimized nonlinear spin-squeezing coefficients of order \( K \) as \( \xi_{(K)}^2(\hat{\rho}) = \min_n \chi_{\text{opt}}^2(\hat{\rho}, J_n, \hat{\mathbf{J}}^K) \). The measurement observable is optimized for sensitivity over families of operators \( \hat{\mathbf{J}}^K \) that include, beyond the linear operators \( \hat{\mathbf{J}} \), also symmetric products of up to \( K \) linear operators. For example, the second-order spin-squeezing coefficient is obtained from Eq. (3) as

\[
\xi_{(2)}^2(\hat{\rho}) = \min_{m, n} \frac{N(\Delta J^2_m)^2}{\{[J^2_m, J_n]\}_\hat{\rho}} = \frac{N}{\lambda_{\text{max}}(M[\hat{\rho}, \hat{\mathbf{J}}^2])},
\]

(8)

where \( \hat{\mathbf{J}}^2 = (\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}_r, \hat{J}_f, \hat{J}_y, \frac{1}{2}(\hat{J}_x, \hat{J}_y), \frac{1}{2}(\hat{J}_x, \hat{J}_y), \frac{1}{2}(\hat{J}_x, \hat{J}_y)) \) contains all linear and symmetric quadratic collective spin operators and \( \hat{J}_m^2 = m \cdot \hat{\mathbf{J}}^2 \) [76]. Note that we always assume a linear generator \( \hat{\Pi} = J_m \) for the phase imprinting evolution. The optimization problem for \( \hat{\mathbf{m}} \) is solved using Eq. (3), where \( M[\hat{\rho}, \hat{\mathbf{J}}^2] = \Pi M[\hat{\rho}, \hat{\mathbf{J}}^2] \Pi \) is the moment matrix in the linear subspace, obtained via the projector \( \Pi = (1, 1, 1, 0, 0, 0, 0, 0, 0) \). The higher-order coefficients \( \xi_{(K)}^2(\hat{\rho}) \) are obtained analogously. The procedure is experimentally challenging for large values of \( K \) but less costly than full quantum tomography [77]. This defines a hierarchy \( \xi_{(1)}^2(\hat{\rho}) \leq \xi_{(2)}^2(\hat{\rho}) \leq \xi_{(3)}^2(\hat{\rho}) \leq ... \) that generalizes and improves the linear spin squeezing coefficient, capturing a larger and larger set of metrologically useful states.

Collective spin systems: Nonlinear evolution—Benchmark examples of non-Gaussian spin states are obtained by the nonlinear one-axis-twisting (OAT) [30] evolution \( |\Psi_{\text{OAT}}(\tau)\rangle = e^{-i\tau \hat{J}_z} |\Psi(0)\rangle \) starting from a coherent spin state pointing in the \( z \) direction, i.e., \( |\Psi(0)\rangle = |N/2, N/2\rangle \) is a simultaneous eigenstate of \( \hat{J}_x \) and \( \hat{J}_z \). For short times, the evolution generates spin-squeezed states, as captured by the linear spin-squeezing coefficient [3, 30]. For \( \tau \approx 1/ \sqrt{N} \) spin squeezing is lost as the state wraps around the Bloch sphere and becomes non-Gaussian, generating a GHZ state at \( \tau = \pi/2 \) and a full revival of the initial spin coherent state at \( \tau = \pi \). This dynamical has been realized on relatively short time scales [3] with large ensembles of Bose-Einstein condensates [31], ultracold atoms in a cavity system [34], or trapped ions [33], and on longer times in experiments with smaller numbers of trapped ions [64] as well as, recently, for the electronic spin \( J = 8 \) of dysprosium atoms [78]. Lower bounds on the Fisher information for the states \( |\Psi_{\text{OAT}}(\tau)\rangle \) have been studied numerically in Ref. [67]. In Fig. 1 we show the analytically optimized linear and nonlinear spin-squeezing coefficients \( \xi_{(K)}^2(\Psi_{\text{OAT}}(\tau)) \) for \( K = 1, \ldots, 5 \) as a function of the evolution time, compared to the maximal quantum Fisher density \( f_{\text{max}}(\Psi_{\text{OAT}}(\tau)) = \max_n F_{\Pi}[\Psi(\tau), J_n]/N \). For \( N = 16 \) the metrological sensitivity over almost the entire evolution period is captured by \( \xi_{(K)}^2(\hat{\rho}) \) with \( K \leq 5 \) [Fig. 1 a)]. For \( N = 100 \), sensitivities up to \( N/2 \) are still revealed [Fig. 1 b)] and are achieved by highly non-Gaussian states [Fig. 1 c)]. For long evolution times, the characterisation is complemented by the spin parity squeezing coefficient, \( \xi_{(K)}^2(\hat{\rho}) = \xi_{(2)}^2(\hat{\rho}, \hat{J}_z, \hat{P}) \), where \( \hat{P} = e^{i\tau \hat{J}_z} \hat{P}_{\Pi} e^{-i\tau \hat{J}_z} \). The coefficient \( \xi_{(2)}^2(\hat{\rho}) \) is particularly suitable in the vicinity of the GHZ state [62] at \( \tau = \pi/2 \), as is shown by the green dashed line in Fig. 1 a).

A faster generation of entanglement is possible by the so-called twist-and-turn evolution \( |\Psi_{\text{TAT}}(\tau)\rangle = e^{-i\tau \hat{J}_x - \frac{1}{2} \hat{J}_y} |\Psi(0)\rangle \) [54, 79, 80]. Also in this case, whereas the onset of non-Gaussianity produces a rapid decay of the linear squeezing coefficient \( (K = 1) \), nonlinear squeezing coefficients of moderate order are sufficient to capture large sensitivities and reveal sig-
significant amounts of particle entanglement at experimentally relevant short times [Fig. 1 d) and e)]. The Wigner functions [Fig. 1 c) and f)] reflect the non-Gaussian nature of the generated states revealed by these methods.

**Nonlinear continuous-variable squeezing coefficients and Fock-state sensing.**—In the continuous variable case, the nonlinear squeezing coefficient is defined from higher-order combinations of $\hat{x} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and $\hat{p} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$, i.e., phase space quadrature operators for a bosonic single-mode field with annihilation operator $\hat{a}$. A particularly important application is the sensing of displacement amplitudes, which can be used to estimate small forces and fields [46]. Displacements are generated by linear combinations of $\hat{x}$ and $\hat{p}$, i.e., $\hat{D}(\theta) = \exp(\theta \hat{a}^\dagger - \theta^* \hat{a}) = \exp(i\theta \hat{a}^\dagger \hat{a})$. Here the amplitude of $\theta = \theta e^{-i\phi}/\sqrt{2}$ characterizes the phase parameter $\theta$ and its phase $\phi$ determines the “direction” of the displacement via the quadrature $\hat{q}_n = n_1 \hat{x} + n_2 \hat{p}$ with $n_1 = \sin(\phi)$ and $n_2 = -\cos(\phi)$. Shot-noise sensitivity is attained by coherent states $|\theta\rangle = \hat{D}(\theta)|0\rangle$, generated from the vacuum $|0\rangle$. The displacement sensitivity of $|\theta\rangle$ is independent of $\theta$ and we find $F_{SN} = \max_{|\theta\rangle=|\theta\rangle=|0\rangle} F_Q(|\theta\rangle, \hat{q}_n) = F_Q(|0\rangle, \hat{q}_n) = 2$.

Fock states $|n\rangle = (\hat{a}^\dagger)^n/\sqrt{n!}|0\rangle$, have particularly appealing properties for displacement amplitude sensing. They can be generated in a variety of quantum systems, e.g., by manipulating the motion of trapped ions [82], by strong atom-cavity interactions [45], or by optical nonlinear [55] or superradiant processes [83]. Due to their isotropic concentric fringes in phase space, Fock states yield sub-shot-noise sensitivity for any displacement generated by $\hat{q}_n$. Fock states lead to $F_Q(|n\rangle, \hat{q}_n) = 4n + 2$, which indicates a quantum enhancement for all $n > 0$, independently of $n$.

As Fock states are non-Gaussian, their characteristics cannot be sufficiently uncovered by measuring linear observables. In fact, even second-order observables are insufficient. Let us therefore extend the family of accessible observables. In fact, even second-order observables are insufficiently uncovered by measuring linear observables.

Nonlinear squeezing relates the metrological quantum enhancement of non-Gaussian states to the squeezed variance of an optimal nonlinear observable, which is identified from the accessible set. The required order of non-linearity provides an indication of the states’ degree of non-Gaussianity. This paves the way for implementable strategies to characterize metrological sensitivity and entanglement of non-Gaussian states, and to harness their increased potential in quantum metrology experiments. Our methods can be readily applied to non-Gaussian states of quantum light, as well as oversqueezed spin states of cold atoms or trapped ions.

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For unitary evolutions, we consider the case \( \theta = 0 \) for ease of notation and replace the argument \( \hat{\rho}(\theta) \) by the initial state \( \hat{\rho} \) and the Hamiltonian \( \hat{H} \).

For \( \hat{\rho} = \sum p_\gamma \hat{\rho}_\gamma \), concavity of the variance implies that
\[
\chi^2[\hat{\rho}, \hat{H}, \hat{X}_\gamma] \leq \sum p_\gamma \chi^2[\hat{\rho}_\gamma, \hat{H}, \hat{X}_\gamma],
\]
from the Cauchy-Schwarz inequality with \( u_\gamma = \sqrt{\rho_\gamma} / \chi^2[\hat{\rho}_\gamma, \hat{H}, \hat{X}_\gamma] \) follows that
\[
\chi^2[\hat{\rho}, \hat{H}, \hat{X}] \leq \sum p_\gamma \chi^2[\hat{\rho}_\gamma, \hat{H}, \hat{X}] / \chi^2[\hat{\rho}_\gamma, \hat{H}, \hat{X}_\gamma].
\]
Convexity of \( \chi^2[\hat{\rho}, \hat{H}, \hat{X}] \) then follows from
\[
\chi^2[\hat{\rho}, \hat{H}, \hat{X}_\gamma] = \chi^2[\hat{\rho}, \hat{H}, \hat{X}_{\gamma opt}] \leq \sum p_\gamma \chi^2[\hat{\rho}_\gamma, \hat{H}, \hat{X}_{\gamma opt}] \leq \sum p_\gamma \max_{\xi_{\gamma opt}} \chi^2[\hat{\rho}_\gamma, \hat{H}, \hat{X}].
\]
Given an orthonormal set of projectors \( \hat{P} = (\hat{P}_I) \), equivalence of the squeezing parameter (1) and the Fisher information, \( \chi^2[\hat{\rho}(\theta), \hat{X}] = f[\hat{\rho}(\theta), \hat{X}] \), is established, e.g., for \( \hat{X} = \sum (\hat{X}_\gamma + \sum \log p(\theta)) \hat{P}_I \). The squeezing parameter thus equals the quantum Fisher information when choosing \( \hat{P} \) as the set of projectors that yields equality between the classical and quantum Fisher information [65].

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In fact a sharper bound for \( N \)-qubit states with up to \( k \) entangled qubits [74] is given as \( \epsilon^2[\hat{\rho}, \hat{H}, \hat{X}] \leq \epsilon^2[\hat{\rho}] + \epsilon^2[\hat{X}] \) where \( \epsilon = \sqrt{N/k} \) and \( r = N - k \).

For instance \( \langle J_x, J_x \rangle \) can be accessed by measuring \( \langle J_x + J_x \rangle^2 / 2 \) and \( \langle J_y - J_y \rangle^2 / 2 \).

The cost of full state tomography grows exponentially with \( N \). In general, the number of elements in the set \( J^K \) scales exponentially with \( K \), i.e., all unique products of up to \( K \) elements of \( J \) (rendered Hermitian by adding the same product in reverse order and dividing by two). For the states considered in this manuscript, restricting to fully symmetric products is sufficient. That is, instead of treating, e.g., \( (J_x, J_x, J_x + J_y) / 2 \) and \( (J_x, J_x, J_x) / 2 \), as two independent observables, we only consider fully symmetric products of the kind \( (J_x, J_x, J_x + J_y, J_y) / 3 \), etc., which reduces the effective number of elements to \( (K + 1)(K + 2) / 2 \) and a more favorable quadratic scaling with \( K \).

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