Microwave background polarization in cosmological models

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Abstract

We introduce a new multipole formalism for polarized radiative transfer in general spacetime geometries. The polarization tensor is expanded in terms of coordinate-independent, projected symmetric trace-free (PSTF) tensor-valued multipoles. The PSTF representation allows us to discuss easily the observer dependence of the multipoles of the polarization, and to formulate the exact dynamics of the radiation in convenient 1+3 covariant form. For the case of an almost-Friedmann-Robertson-Walker (FRW) cosmological model we recast the Boltzmann equation for the polarization in to a hierarchy of multipole equations. This allows us to give a rigorous treatment of the generation and propagation of the polarization of the cosmic microwave background in almost-FRW models (with open, closed or flat geometries) without recourse to any harmonic decomposition of the perturbations. We also show how expanding the intensity and polarization multipoles in derivatives of harmonic functions gives a streamlined derivation of the mode-expanded multipole hierarchies. Integral solutions to these hierarchies are provided, and the relation of our formalism to others in the literature is discussed.

I. INTRODUCTION

With the growing body of data relating to the anisotropies of the cosmic microwave background radiation (CMB), and the potential impact on cosmology of conclusions drawn from the analysis of CMB data, there is a strong case for developing a flexible, but physically transparent, formalism for describing the propagation of the CMB in general cosmological models.

In a series of papers [1–10], a 1+3 covariant and gauge-invariant formalism has been developed with a view to giving a model-independent framework in which to study the physics of the CMB. The approach is based on the projected symmetric trace-free (PSTF) representation of relativistic kinetic theory due to Ellis, Treciokas, and Matravers [11], and Thorne [12], and builds on the covariant and gauge-invariant approach to perturbations in cosmology (e.g. Ref. [13]). Some of the benefits of the covariant approach, which were

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emphasised in Refs. [7,9], include: (i) clarity in the definition of the variables employed; (ii) covariant and gauge-invariant perturbation theory around a variety of background models; (iii) provision of a sound basis for studying non-linear effects; (iv) freedom to employ any coordinate system or tetrad. In its current form, the covariant formalism does not allow for the inclusion of polarization. It is this omission that we address here, by introducing a new multipole formalism which allows one to analyse CMB polarization in arbitrary cosmological models in the 1+3 covariant approach.

We set up the multipole decomposition of the polarization and the exact equations of radiative transfer (the Boltzmann equation) in convenient 1+3 covariant form. The polarization multipoles are covariantly-defined PSTF tensors which leads to significant simplifications in the algebraic structure of the scattering terms in the Boltzmann equation, and allows us to discuss easily how the multipoles transform under changes of reference frame. With the Boltzmann equation in 1+3 covariant form, the physical processes responsible for the production and evolution of anisotropies and polarization are particularly transparent. The PSTF multipole decomposition of the polarization suggests recasting the Boltzmann equation as a multipole hierarchy, in a similar manner to the treatment of the intensity in Refs. [11,12]. This turns out to be quite involved, and we shall only present the results for the special case of an almost-FRW model here. (The exact, non-linear multipole equations will be given in a subsequent paper.) These multipole equations allow us to give a rigorous treatment of the generation and propagation of polarization in the CMB in almost-FRW models for all spatial geometries.

Although the calculation of CMB anisotropies in an almost-FRW universe involves only linear perturbation theory, the complexity of the subject is increased when one allows for non-flat spatial geometries, vector and tensor perturbations, and the inclusion of polarization [14]. To our knowledge, the only published formalism for handling the most general almost-FRW models is the total angular momentum method of Hu et al. [14]. In their approach, considerable simplifications result from the introduction of a normal mode expansion for the radiation where the local angular and spatial distribution is explicit. In the 1+3 covariant approach the local angular distribution of the radiation is analysed in terms of the PSTF tensor-valued multipoles, which can be done without recourse to a decomposition of the spatial distribution in harmonic functions. The multipole equations can then be used to analyse the evolution of anisotropies and polarization for a quite general perturbation. If required, the spatial dependencies can be handled by expanding the radiation multipoles in derivatives of harmonic functions, which gives a streamlined derivation of the mode-expanded multipole hierarchies. Combining the angular and spatial expansions we find a normal mode representation of the radiation which is equivalent to that of Wilson [15] for scalar temperature perturbations, while for vector and tensor modes we obtain the obvious generalisation of Wilson’s method.

The paper is arranged as follows. In Sec. I we introduce the PSTF multipole decomposition of a polarized radiation field, relative to some timelike velocity field $u^a$, obtaining the 1+3 covariant forms of the electric and magnetic parts of the linear polarization tensor. We also give an exact discussion of the transformation properties of the polarization tensor under changes of the velocity field $u^a$, and give non-linear expressions to first-order in relative velocities for the transformations of the polarization multipoles. In Sec. III we discuss the Boltzmann equation for polarized radiative transport. We give new, exact expressions for...
the scattering term in the Thomson limit and the integral solution for the linear polarization, valid in general cosmological models. Specialising to almost-FRW models, we present the multipole form of the Boltzmann equation which shows explicitly that the evolution of the electric and magnetic parts of the linear polarization tensor are coupled through curl terms, much like the coupling of electric and magnetic fields in Maxwell’s equations. In Sec. IV we decompose the multipole equations into harmonic modes for scalar perturbations and provide the integral solution which is central to the line of sight algorithm for efficient numerical evaluation of the CMB power spectrum [16]. The analysis is repeated for tensor perturbations in Sec. V, where a very direct derivation of the integral solutions is also given. In Sec. VI we discuss the relation of our approach to others in the literature. Finally we close with our conclusions in Sec. VII. An appendix summarises the PSTF representation of the scalar and tensor harmonics.

We employ a (+ − − −) signature for the spacetime metric $g_{ab}$. Early lower case Roman indices $a, b$ etc. refer to a general basis, with $i$ and $j$ referring to the 1, 2 components in an orthonormal tetrad, $(e_i)^a$, $i = 1, 2$. Round brackets denote symmetrisation on the enclosed indices, square brackets antisymmetrisation, and angle brackets the PSTF part: $S_{(ab)} \equiv (h^c_{(a} h^d_{b)} - \frac{1}{3} h^e_{ab} h^{cd}) S_{cd}$, where the projection operator $h^{ab} \equiv g^{ab} - u^a u^b$ with $u^a$ the fundamental velocity of the 1+3 covariant approach. The index notation $A_l$ denotes the index string $a_1 \ldots a_l$, and the notation $e_{A_l}$ denotes the tensor product $e_{a_1} \ldots e_{a_l}$. We use units with $c = G = 1$ throughout.

**II. MULTIPOL DECOMPOSITION OF THE RADIATION FIELD**

We describe observations from the viewpoint of an observer comoving with the fundamental velocity field $u^a$. In a general cosmological model there is some freedom in the choice of $u^a$, although we must ensure that $u^a$ is defined in a physical manner (such as by the time-like eigenvector of the matter stress-energy tensor, or the 4-velocity of some particle species) so that in the FRW limit, $u^a$ correctly reduces to the fundamental velocity of the FRW model. This restriction on $u^a$ is necessary to ensure gauge-invariance of the 1+3 covariant perturbation theory. A photon with 4-momentum $p^a$ has an energy $E$ and propagation direction $e^a$ relative to $u^a$, where

$$p^a = E (u^a + e^a). \quad (2.1)$$

For a given propagation direction $e^a$, the observer can introduce a pair of orthogonal polarization vectors $(e_1)^a$ and $(e_2)^a$ which are perpendicular to $u^a$ and $e^a$:

$$H^a_1 (e_1)^b = (e_1)^a, \quad (2.2)$$

with a similar result for $(e_2)^a$. Here $H_{ab}$ is the screen projection tensor that projects perpendicular to both $u^a$ and $e^a$:

$$H_{ab} = h_{ab} + e_a e_b. \quad (2.3)$$

We shall refer to tensors like $(e_1)^a$, which are perpendicular to $e^a$ and $u^a$, as being transverse. The set of vectors $\{u^a, (e_1)^a, (e_2)^a, e^a\}$ form a right-handed orthonormal tetrad at
the observation point. Using the polarization basis vectors, the observer can decompose an arbitrary radiation field into Stokes parameters (e.g. Ref. [17]) $I(E, e^a), Q(E, e^a), U(E, e^a)$ and $V(E, e^a)$ along the direction $e^a$ at photon energy (or equivalently frequency) $E$. The transformation laws of the Stokes parameters under rotations of $(e_1)^a$ and $(e_2)^a$ lead one to introduce a second-rank transverse polarization tensor $P_{ab}(E, e^c)$. The only non-vanishing tetrad components of $P_{ab}(E, e^c)$ are

$$P_{ab}(e_i)^a(e_j)^b = \frac{1}{2} \begin{pmatrix} I + Q & U + V \\ U - V & I - Q \end{pmatrix},$$  

(2.4)

for $i$ and $j = 1, 2$, and we have left the arguments $E$ and $e^a$ implicit. Introducing the projected alternating tensor $\epsilon_{abc} \equiv \eta_{abcd} u^d$, where $\eta_{abcd}$ is the spacetime alternating tensor, we can write $P_{ab}(E, e^c)$ in the covariant irreducible form

$$P_{ab}(E, e^c) = -\frac{1}{2} I(E, e^d) \mathcal{H}_{ab} + \mathcal{P}_{ab}(E, e^d) + \frac{1}{2} V(E, e^d) \epsilon_{abc} e^c,$$

(2.5)

which defines the linear polarization tensor $\mathcal{P}_{ab}(E, e^c)$, where

$$\mathcal{P}_{ab}(e_i)^a(e_j)^b = \frac{1}{2} \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix},$$

(2.6)

which is transverse and trace-free. For a general second-rank tensor $S_{ab}$, we follow Thorne [18] by denoting the transverse trace-free (TT) part by $[S_{ab}]^{TT}$, so that

$$[S_{ab}]^{TT} = \mathcal{H}_a^c \mathcal{H}_b^d S_{cd} - \frac{1}{2} \mathcal{H}_{ab} \mathcal{H}^{cd} S_{cd}.$$  

(2.7)

The magnitude (squared) of the linear polarization is $Q^2 + U^2$ which can be written in the manifestly basis-independent form $2P_{ab}P^{ab}$.

Since $I(E, e^c)$ and $V(E, e^c)$ are scalar functions on the sphere at a point in spacetime, their local angular dependence can be handled by an expansion in PSTF tensor-valued multipoles [11,12]:

$$I(E, e^c) = \sum_{l=0}^{\infty} I_{A_l}(E) e^{A_l},$$  

(2.8)

$$V(E, e^c) = \sum_{l=0}^{\infty} V_{A_l}(E) e^{A_l}.$$  

(2.9)

The expansion in PSTF multipoles is the coordinate-free version of the familiar spherical harmonic expansion. Eqs. (2.8) and (2.9) can be inverted using the orthogonality of the $e^{(A_l)}$, for example,

$$I_{A_l}(E) = \Delta_l^{-1} \int d\Omega I(E, e^c) e^{(A_l)} \quad \text{where} \quad \Delta_l = \frac{4\pi(-2)^l (l)!^2}{(2l + 1)!.}$$  

(2.10)

For the TT tensor $\mathcal{P}_{ab}(E, e^c)$, we use the TT elements of the PSTF representation of the pure-spin tensor spherical harmonics (see Ref. [18] for a thorough review of the various representations of the tensor spherical harmonics), so that

\[ \]
\[ \mathcal{P}_{ab}(E, e^c) = \sum_{l=2}^{\infty} \left[ \mathcal{E}_{abC_{l-2}}(E)e^{C_{l-2}} \right]^{TT} + \sum_{l=2}^{\infty} \left[ e_{d_1}e^{d_2}(a_0B_0)d_2C_{l-2}(E)e^{C_{l-2}} \right]^{TT}. \] (2.11)

The first summation in Eq. (2.11) defines the electric part of the linear polarization, while the second summation defines the magnetic part. (The electric and magnetic parts of the polarization were denoted by \( G(\text{radient}) \) and \( C(\text{url}) \) in Ref. [19].) The \( l \)-th term in the multipole expansion of the electric part has parity \((-1)^l\), while the \( l \)-th term in the magnetic part has parity \((-1)^{l+1}\). Eq. (2.11) can be inverted to give

\[ \mathcal{E}_{A_1}(E) = M_l^2 \Delta l^{-1} \int d\Omega e_{\langle \Lambda_{l-2} \rangle \mathcal{P}_{a_{l-1}a_l}}(E, e^c), \] (2.12)

\[ -\mathcal{B}_{A_1}(E) = M_l^2 \Delta l^{-1} \int d\Omega e_{b^d(\langle \Lambda_{l-2} \rangle \mathcal{P}_{a_{l-1}d}d}(E, e^c), \] (2.13)

where \( M_l \equiv \sqrt{2l(l+1)/(l+2)} \). The multipole expansion in Eq. (2.11) is the coordinate-free version of the tensor spherical harmonic expansion introduced to the analysis of CMB polarization in Ref. [19], and the expansion in the Newman-Penrose spin-weight 2 spherical harmonics employed in Ref. [20].

A. Power spectra

To define the bolometric power spectrum it is convenient to define energy-integrated multipoles, for example

\[ I_{A_i} = \Delta l \int_0^\infty dE I_{A_i}(E), \] (2.14)

with equivalent definitions for \( \mathcal{E}_{A_i}, \mathcal{B}_{A_i} \) and \( V_{A_i} \). The factor \( \Delta l \) is included in Eq. (2.14) so that the three lowest intensity multipoles give the radiation energy density, energy flux and anisotropic stress respectively:

\[ I = \rho^{(\gamma)}, \quad I_a = q_a^{(\gamma)}, \quad I_{ab} = \pi_{ab}^{(\gamma)}. \] (2.15)

In the almost-FRW case, with an ensemble that is statistically isotropic, the power spectrum of the (bolometric) temperature anisotropies, \( C_{lII}^{TT} \), can be defined in terms of the covariance of the intensity multipoles [10]:

\[ \left( \frac{\pi}{7} \right)^2 \langle I_{A_i}I_{B_i}^{B'} \rangle = \Delta l C_{lII}^{TT} \delta_l^{l'} h_{(A_i)}^{(B_i)}; \] (2.16)

where \( h_{(A_i)}^{(B_i)} \equiv h_{(a_1 \ldots a_l)}^{(b_1 \ldots b_l)} \). For small anisotropies, the gauge-invariant fractional temperature difference from the all-sky mean is related to the \( I_{A_i} \) by

1\( I_{A_i} \) was denoted \( \Delta l \Pi_{A_i} \) in Ref. [7], and \( J^{(i)}_{A_i} \) in Ref. [8].
\[ \delta T(e^c) = \frac{\pi}{T} \sum_{l=1}^{\infty} \Delta_l I_{A_l} e^{A_l}, \quad (2.17) \]

so that the temperature correlation function evaluates to

\[ \langle \delta T(e^c) \delta T(e'^{c}) \rangle = \sum_{l=1}^{\infty} \frac{(2l + 1)}{4\pi} C^I_l P_l(X), \quad (2.18) \]

where \( X \equiv -e^a e'_a \), and \( P_l(X) \) is a Legendre polynomial. In deriving Eq. (2.18) we used the result \( e^{(A_l)} e'_{(A_l)} = (2l + 1) \Delta_l P_l(X)/(4\pi) \).

The power spectra for the polarization are defined by analogy with Eq. (2.16). We choose our conventions for the power spectra to conform with Ref. [20], so that it is necessary to include an additional factor of \( M_l/\sqrt{2} \) for each factor of the polarization. For example, for the electric polarization

\[ \langle \mathcal{E}_{A_l} \mathcal{E}^{B_l'} \rangle = \frac{l(l - 1)}{(l + 1)(l + 2)} \Delta_l C^E_l P_l(X), \quad (2.19) \]

Unfortunately, the definitions of the polarization power spectra given in Ref. [20] differ from those in Ref. [19] by factors of \( \sqrt{2} \); see Ref. [21] and Sec. VI for details. For a parity symmetric ensemble the magnetic polarization is uncorrelated with the electric polarization and the temperature anisotropies. In this case, we find that the generalisation of Eq. (2.18) to linear polarization is

\[ 2 \left( \frac{\pi}{T} \right)^2 \langle \mathcal{P}_{ab}(e^c) \mathcal{P}_{ab}(e'^{c}) \rangle = \sum_{l=2}^{\infty} \frac{(2l + 1)}{4\pi} [C^E_l P_{l-2}(X) + C^B_l P_{l-1}(X)], \quad (2.20) \]

where \( \mathcal{P}_{ab}(e^c) = \int_0^\infty dE \mathcal{P}_{ab}(E, e^c) \). When the directions \( e^c \) and \( e'^c \) coincide, Eq. (2.20) reduces to the ensemble average of the square of the degree of linear polarization (expressed as a dimensionless bolometric temperature). Note that the electric polarization quadrupole gives an isotropic contribution to the linear polarization correlation function. Integrating Eq. (2.20) over all directions \( e^c \), with \( e^c \) fixed, the right-hand side reduces to \( 5C^E_2 \). The presence of \( P_{l-2}(X) \) and \( P_{l-1}(X) \) in Eq. (2.20) reflects the opposite parities of the electric and magnetic contributions to \( \mathcal{P}_{ab}(e^c) \) for a given multipole \( l \).

**B. Transformation laws under changes of frame**

The polarization tensor \( \mathcal{P}_{ab}(E, e^c) \) has been expressed in 1+3 covariant form, so it is not invariant under changes of frame. If we consider a new velocity field \( \tilde{u}^a = \gamma(u^a + v^a) \), where \( v^a \) is the projected relative velocity in the \( u^a \) frame and \( \gamma \) is the associated Lorentz factor, for a given photon with 4-momentum \( p^a \) the energy and propagation directions in the \( \tilde{u}^a \) frame are given by the Doppler and aberration formulae:

\[ \tilde{E} = \gamma E(1 + e^a v_a), \quad (2.21) \]
\[ \tilde{e}^a = [\gamma(1 + e^b v_b)]^{-1}(u^a + e^a) - \gamma(u^a + v^a). \quad (2.22) \]
Note that \( \tilde{e}^a \) is a projected vector relative to \( \tilde{u}^a \). The screen projection tensor for a given null direction transforms to

\[
\tilde{H}_{ab} = H_{ab} - \frac{2\gamma}{E} p_{(a} H_{b)c} v^c + \frac{\gamma^2}{E^2} p_a p_b H_{cd} v^c v^d,
\]

(2.23)

while the polarization tensor transforms according to

\[
\tilde{E}^{-3} \tilde{P}_{ab}(\tilde{E}, \tilde{e}^c) = E^{-3} \tilde{H}^{d_1}_{ab} \tilde{H}^{d_2}_{b} P_{d_1 d_2}(E, e^c).
\]

(2.24)

Under this transformation law the intensity, circular polarization and linear polarization do not mix. The irreducible components of Eq. (2.24) give the frame invariance of \( I(E, e^c) / E^3 \) and \( V(E, e^c) / E^3 \), and show that the transformation law for \( P_{ab}(E, e^c) \) follows that for \( P_{ab}(e^c) \). The degree of linear polarization \( \bar{P}_{ab}(E, e^c) P_{ab}(E, e^c) \) is invariant under changes of frame. The transformation law for \( P_{ab}(E, e^c) \) ensures that the tetrad components of \( P_{ab}(E, e^c) / E^3 \) are invariant if the polarization basis vectors are transformed as

\[
(\tilde{e}_i)^a = \tilde{H}^a_b (e_i)^b,
\]

(2.25)

for \( i = 1, 2 \).

Under changes of frame, multipoles with different \( l \) mix because of Doppler beaming effects. To first-order in the relative velocity \( v^a \), the energy-integrated multipoles transform as

\[
\tilde{I}_{Al} = h^{(B)}_{(A)} I_{Bl} - (l - 2) v^b I_{bAl} - \frac{l(l + 3)}{2l + 1} v_{(a_l} I_{A_{l-1})},
\]

(2.26)

with an equivalent result for \( V_{Al} \). Although the linear polarization transforms irreducibly, the electric and magnetic parts do mix. To first-order in \( v^a \), we find

\[
\tilde{E}_{Al} = h^{(B)}_{(A)} E_{Bl} - \frac{l(l + 3)}{2l + 1} v_{(a_l} E_{A_{l-1})} - (l - 2)(l - 1)(l + 3) v^b E_{bAl}
\]

\[
- \frac{6}{l + 1} v^b \epsilon^{bc}_{(A_l} \tilde{B}_{A_{l-1}) c},
\]

(2.27)

\[
\tilde{B}_{Al} = h^{(B)}_{(A)} B_{Bl} - \frac{l(l + 3)}{2l + 1} v_{(a_l} B_{A_{l-1})} - (l - 2)(l - 1)(l + 3) v^b B_{bAl}
\]

\[
+ \frac{6}{l + 1} v^b \epsilon^{bc}_{(A_l} \tilde{E}_{A_{l-1}) c}.
\]

(2.28)

In an almost-FRW model the polarization is a first-order quantity, as are physically defined relative velocities. It follows that the electric and magnetic multipoles are frame-invariant in linear theory.

## III. BOLTZMANN EQUATION

The dynamics of the radiation field is described by the (exact) collisional Boltzmann equation
\[
\mathcal{L}[E^{-3}P_{ab}(E, e^c)] = K_{ab}(E, e^c),
\]
where the Liouville operator \( \mathcal{L} \) acts on transverse tensors \( A_{ab} = [A_{ab}]^{TT} \) along the photon path \( x^a(\lambda), p^a(\lambda) \) in phase space, with \( p^a = \text{d}a^a/\text{d}\lambda \), according to
\[
\mathcal{L}[A_{ab}(E, e^c)] = \mathcal{H}^{d_1}_{a} \mathcal{H}^{d_2}_{b} p^c \nabla_c A_{d_1 d_2}(E, e^c),
\]
where \( \nabla_a \) is the spacetime covariant derivative. The transverse scattering tensor \( K_{ab}(E, e^c) \) describes interactions of the radiation with matter. The Liouville operator \( \mathcal{L} \) preserves the irreducible decomposition of the polarization tensor \([\text{Eq. (2.5)}]\), so that
\[
\mathcal{L}[E^{-3}P_{ab}(E, e^c)] = -\frac{1}{2} \frac{\text{d}}{\text{d}\lambda} [E^{-3}I(E, e^c)] \mathcal{H}_{ab} + \mathcal{L}[E^{-3}P_{ab}(E, e^c)]
\]
\[
+ \frac{1}{2} \frac{\text{d}}{\text{d}\lambda} [E^{-3}V(E, e^c)] \epsilon_{abcd} e^d.
\]
In the absence of scattering, \( K_{ab}(E, e^c) = 0 \), the tetrad components of \( P_{ab}(E, e^c)/E^3 \) are constant along the photon path provided that the polarization basis vectors are transported along the null geodesic according to
\[
\mathcal{H}_{ab} p^c \nabla_c (e_i)^b = 0,
\]
and the constraint \( (e_i)^a = \mathcal{H}_{ab}^a (e_i)^b \).

It follows from Eqs. (2.24) and (3.2) that under changes of frame \( u^a \mapsto \tilde{u}^a \), the action of the Liouville operator transforms as
\[
\tilde{\mathcal{L}}[\tilde{E}^{-3}\tilde{P}_{ab}(\tilde{E}, \tilde{e}^c)] = \tilde{\mathcal{H}}_{ab}^{d_1} \tilde{\mathcal{H}}_{bd_2} \mathcal{L}[E^{-3}P_{d_1 d_2}(E, e^c)].
\]
With this result, we deduce from Eq. (3.1) that the scattering tensor \( K_{ab}(E, e^c) \) must transform as
\[
\tilde{K}_{ab}(\tilde{E}, \tilde{e}^c) = \tilde{\mathcal{H}}_{ab}^{d_1} \tilde{\mathcal{H}}_{bd_2} K_{d_1 d_2}(E, e^c).
\]
This result is useful since the scattering tensor is often simplest to evaluate in some preferred frame, picked out by the physics of the scattering process. The scattering tensor in a general frame can then be computed using Eq. (3.4).

Over the epochs relevant to the formation of anisotropies and polarization in the CMB the dominant coupling between the radiation and the matter is Compton scattering. To an excellent approximation we can ignore the effects of Pauli blocking, induced scattering, and electron recoil in the rest frame of the scattering electron, so that the scattering may be approximated by classical Thomson scattering in the electron rest frame. Taking the electron 4-velocity to be \( \tilde{u}^a \), and the (proper) number density of free electrons to be \( \tilde{n}_e \), the exact scattering tensor in the \( \tilde{u}^a \) frame in the classical Thomson limit is
\[
\tilde{E}^2 \tilde{K}_{ab}(\tilde{E}, \tilde{e}^c) = \tilde{n}_e \sigma_T \left\{ -\frac{1}{2} \tilde{\mathcal{H}}_{ab} \left[ -\tilde{I}(\tilde{E}, \tilde{e}^c) + \tilde{I}(\tilde{E}) + \frac{1}{10} \tilde{I}_{d_1 d_2}(\tilde{E}) \tilde{e}^{d_1} \tilde{e}^{d_2} + \frac{3}{5} \tilde{\mathcal{E}}_{d_1 d_2}(\tilde{E}) \tilde{e}^{d_1} \tilde{e}^{d_2} \right] \right.
\cdot \left[ -\tilde{P}_{ab}(\tilde{E}, \tilde{e}^c) + \frac{1}{10} \left[ \tilde{I}_{ab}(\tilde{E}) \right]^{TT} + \frac{3}{5} \left[ \tilde{\mathcal{E}}_{ab}(\tilde{E}) \right]^{TT} \right] \right.
\cdot \left[ -\tilde{V}(\tilde{E}, \tilde{e}^c) + \frac{1}{2} \tilde{V}_{d_2}(\tilde{E}) \tilde{e}^{d_2} \right] \right\},
\]
(3.7)
where $\sigma_T$ is the Thomson cross section. This expression for the scattering tensor follows from inserting the multipole decomposition of the polarization tensor into the Kernel for Thomson in-scattering (e.g. Ref. [17]), and integrating over scattering directions.

We have written Eq. (3.7) in irreducible form. The first set of terms in square brackets affects the evolution of the intensity in the electron rest frame, the second set of terms affect the linear polarization, and the third set affect the circular polarization. Concentrating first on the terms that affect the intensity, the lack of a monopole component is due to there being no energy transfer from Thomson scattering in the rest frame of the electron, while the presence of the intensity and electric polarization quadrupoles is due to the anisotropy and polarization dependence of Thomson scattering. Turning to the terms in $\tilde{K}_{ab}(\tilde{E}, e^c)$ which couple to the evolution of the linear polarization, the presence of the quadrupoles of the intensity and the electric polarization arises from Thomson in-scattering. These terms directly affect the evolution of the electric quadrupole alone in the electron rest frame, but this is not true in a general frame. Finally, we see from the terms in the final square bracket in Eq. (3.7) that the evolution of the circular polarization decouples from the intensity and the linear polarization. Since the circular polarization transforms irreducibly under changes of frame, circular polarization is not sourced by Thomson scattering for arbitrary frame choices.

### A. Integral solution for $\mathcal{P}_{ab}(e^c)$

The transformation of $\tilde{K}_{ab}(\tilde{E}, e^c)$ to a general frame gives a non-local expression in the energy $E$, since there is energy transfer in Thomson scattering from a moving electron. (See Ref. [22] for applications of this transformation in cluster physics.) It is therefore convenient to integrate over energy, in which case we find the following exact expressions for the evolution along the line of sight:

$$
\int_0^\infty dE E^2 \frac{d}{d\lambda}[E^{-3}I(E, e^c)] = -\tilde{n}_e\sigma_T \gamma (1 + v_d e^d) I(e^c) + \frac{1}{4\pi} \tilde{n}_e\sigma_T [\gamma (1 + v_d e^d)]^{-3} (\tilde{I} + \tilde{\zeta}_{ab} \tilde{e}^a \tilde{e}^b), \quad (3.8)
$$

$$
\int_0^\infty dE E^2 \mathcal{L}[E^{-3}\mathcal{P}_{ab}(E, e^c)] = -\tilde{n}_e\sigma_T \gamma (1 + v_d e^d) \mathcal{P}_{ab}(e^c) + \frac{1}{4\pi} \tilde{n}_e\sigma_T [\gamma (1 + v_d e^d)]^{-3} [\tilde{\mathcal{H}}_a^{\mathcal{C}1} \tilde{\mathcal{H}}_b^{\mathcal{C}2} \tilde{\zeta}_{c1c2}]^{\mathcal{T}T}, \quad (3.9)
$$

$$
\int_0^\infty dE E^2 \frac{d}{d\lambda}[E^{-3}V(E, e^c)] = -\tilde{n}_e\sigma_T \gamma (1 + v_d e^d) V(e^c) - \frac{3}{8\pi} \tilde{n}_e\sigma_T [\gamma (1 + v_d e^d)]^{-3} \tilde{V}_a \tilde{e}^a. \quad (3.10)
$$

The tildes denote that the quantity is evaluated in the rest frame of the electrons, which has four-velocity $\tilde{u}^a = \gamma (u^a + v^a)$, and we have defined e.g. $I(e^c) \equiv \int_0^\infty dE I(E, e^c)$. The quantity $\zeta_{ab}$, where

$$
\zeta_{ab} \equiv \frac{3}{4} I_{ab} + \frac{9}{2} E_{ab}, \quad (3.11)
$$
appears in Eqs. (3.8) and (3.9), evaluated in the \( \tilde{u}^a \) frame, because of the anisotropy and polarization dependence of Thomson scattering.

Formal solutions to Eqs. (3.8–3.10) can be obtained by integrating along the null cone. For the linear polarization we must work with the tetrad components of \( P_{ab}(e^c) \), introduced in Sec. II, since we can only meaningfully integrate scalar equations. Making use of

\[
(e_i)^a(e_j)^b \int_0^\infty dE \, E^2 \mathcal{L}[E^{-3} P_{ab}(E, e^c)] = (e^a + \tilde{u}^a) \nabla_a [P_{ab}(e^c)(e_i)^a(e_j)^b] + (e_i)^a(e_j)^b \int_0^\infty dE \, E^2 \frac{dE}{d\lambda} \partial \partial E^{-3}[E^{-3} P_{ab}(E, e^c)],
\]

and integrating the second term on the right by parts, we find the following exact solution for the energy-integrated linear polarization tensor at some point \( R \) in a general cosmological model:

\[
P_{ab}(e^c)(e_i)^a(e_j)^b|_R = \frac{1}{4\pi} \int_0^R \tilde{t} \tilde{n}_e \sigma T e^{-\tilde{\tau}} (1 + z)^{-4} [\gamma(1 + v de^d)]^{-3} [\tilde{H}^c_a \tilde{H}^c_b \tilde{\zeta}_e c_2]^T (e_i)^a(e_j)^b.
\]

Here, the integral is along the photon geodesic and the measure is \( dt \equiv dx^a u_a \). The redshift back from \( R \) along the line of sight is \( z \), and \( \tilde{\tau} \) is the optical depth along the line of sight, measured in the electron rest frame:

\[
\tilde{\tau}|_A \equiv \int_A^R \tilde{t} \tilde{n}_e \sigma T,
\]

where \( \tilde{t} \equiv \tilde{u}_a dx^a \), and \( A \) is a point on the null geodesic through \( R \). The integral solution, Eq. (3.13), shows clearly how the intensity and electric quadrupoles generate polarization on scattering, and the effect of subsequent gravitationally-induced rotation of the basis vectors \( (e_i)^a \) as the radiation free streams to us. The need to calculate the intensity and electric quadrupoles, the redshift, and the rotation of the basis vectors along the line of sight suggests that in a general (non-linear) model it will be more convenient to work directly with the polarization multipoles. However, in Sec. V we show that the linearised version of Eq. (3.13) does provide a very direct route through to the integral solutions for the mode-expanded multipoles in the almost-FRW case.

**B. Multipole decomposition of the Boltzmann equation**

The Boltzmann equation (I.1) can be written in multipole form by expressing \( P_{ab}(E, e^c) \) as a multipole expansion using Eqs. (2.8), (2.9), and (2.11), and decomposing the resulting equation into multipoles. This leads to four sets of multipole hierarchies for \( I_{A_l}(E) \), \( E_{A_l}(E) \), \( B_{A_l}(E) \), and \( V_{A_l}(E) \). In the absence of scattering the hierarchies for \( I_{A_l}(E) \) and \( V_{A_l}(E) \) decouple, but if Thomson scattering is included the hierarchy for \( I_{A_l}(E) \) includes the multipoles of the linear polarization as source terms. The exact multipole equations for \( I_{A_l}(E) \) in the absence of scattering were given in Refs. [7,11,12]; the same equations apply to the multipoles \( V_{A_l} \) of the circular polarization when there is no scattering. (The leading non-linear Thomson source terms in the hierarchy for the energy-integrated multipoles \( I_{A_l} \) were also
given in Ref. [7], but these did not include polarization effects.) The exact multipole equations for $E_A(l)$ and $B_A(l)$ are significantly more involved than those for the intensity or circular polarization. We defer the derivation of the exact polarization multipole equations to a subsequent paper. Here, we shall only consider the case of almost-FRW universes, in which case the polarization and $l > 0$ intensity multipoles are $O(\epsilon)$ in a smallness parameter, $\epsilon$, which characterises the departure of the cosmological model from exact FRW symmetry. For the rest of this paper we shall restrict attention to almost-FRW models, and work only to first-order in $\epsilon$ (linear perturbation theory).

To first-order in $\epsilon$, we find for the electric polarization

$$
\dot{E}_A(E) - \frac{1}{3} \Theta E^4 \frac{\partial}{\partial E} [E^{-3} E_A(E)] + D_{(a} E_{A_{l-1})} (E) - \frac{(l + 3)(l - 1)}{(l + 1)(2l + 3)} D^b E_{bA_l} (E)
$$

$$- \frac{2}{(l + 1)} \text{curl} \, B_A(l) = - n_e \sigma_T E_A(E) + \frac{1}{10} n_e \sigma_T [I_{a_1 a_2} (E) + 6 \varepsilon_{a_1 a_2} (E)] \delta_l^2,
$$

and for the magnetic polarization

$$
\dot{B}_A(E) - \frac{1}{3} \Theta E^4 \frac{\partial}{\partial E} [E^{-3} B_A(E)] + D_{(a} B_{A_{l-1})} (E) - \frac{(l + 3)(l - 1)}{(l + 1)(2l + 3)} D^b B_{bA_l} (E)
$$

$$+ \frac{2}{(l + 1)} \text{curl} \, E_A(E) = - n_e \sigma_T B_A(E),
$$

In these equations, an overdot denotes the action of $u^a \nabla_a$, $D_a$ is the totally projected derivative:

$$
D_a S_{b...c} \equiv h_a^d h_b^e \ldots h_c^f \nabla_d S_{e...f},
$$

for some arbitrary tensor $S_{b...c}$, and the volume expansion $\Theta \equiv \nabla_a u^a$. We have replaced $\tilde{n}_e$ by the electron number density in the $u^a$ frame, which is correct to the required order. The electric and magnetic multipoles are coupled through curl terms, where we have introduced the curl of a rank-1 PSTF tensor $S_{A_l}$,

$$
\text{curl} \, S_{A_l} \equiv \epsilon_{bc(a_l} D^b S_{A_{l-1})c}.
$$

The coupling of electric and magnetic multipoles through curl terms is reminiscent of Maxwell’s equations, as is the fact that only the electric multipoles have inhomogeneous source terms on the right. It is this curl coupling and the structure of the source terms that lead to the well-known result that scalar perturbations do not generate magnetic polarization (see Sec. [V]). This is critical for the detection of a gravitational wave component to the anisotropy [23].

It is often more convenient to work with the energy-integrated multipoles $E_A$ and $B_A$, since it follows from Eqs. (3.15) and (3.16) that the polarization has the same energy spectrum as the intensity anisotropies. Integrating over energies and then by parts, we find

2We adopt the convention that $E_A$ and $B_A$ vanish for $l < 2$, and $I_A$ and $V_A$ vanish for $l < 0$. 

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\[ \dot{E}_l + \frac{4}{3} \Theta E_l + \frac{(l+3)(l-1)}{(l+1)^2} D^b E_{b,l} - \frac{l}{(2l+1)} D_{(a_l E_{a_l-1})} \]

\[- \frac{2}{(l+1)} \text{curl} B_{A_l} = -n_e \sigma_T \left( E_{A_l} - \frac{2}{15} \zeta_{a_1 a_2} \delta_l^2 \right), \quad (3.19) \]

\[ \dot{B}_l + \frac{4}{3} \Theta B_l + \frac{(l+3)(l-1)}{(l+1)^2} D^b B_{b,l} - \frac{l}{(2l+1)} D_{(a_l B_{a_l-1})} \]

\[ + \frac{2}{(l+1)} \text{curl} E_{A_l} = -n_e \sigma_T B_{A_l}, \quad (3.20) \]

For the circular polarization, we find to \(O(\epsilon)\)

\[ \dot{V}_l + \frac{4}{3} \Theta V_l - \frac{l}{(2l+1)} D_{(a_l V_{a_l-1})} + D^b V_{b,l} = -n_e \sigma_T \left( V_{A_l} - \frac{1}{2} V_{a_1} \delta_l^1 \right), \quad (3.21) \]

and for the intensity

\[ \dot{I}_l + \frac{4}{3} \Theta I_l + D^b I_{b,l} - \frac{l}{(2l+1)} D_{(a_l I_{a_l-1})} + \frac{4}{3} I A_{a_1} \delta_l^1 - \frac{8}{15} I \sigma_{a_1 a_2} \delta_l^2 \]

\[ = -n_e \sigma_T \left( I_{A_l} - I \delta_l^0 - \frac{4}{3} I v_{a_1} \delta_l^1 - \frac{2}{15} \zeta_{a_1 a_2} \delta_l^2 \right), \quad (3.22) \]

which extends the result in Refs. [5, 7, 9] to include the polarization dependence of Thomson scattering. In Eq. (3.22), \(A_a \equiv u_a\) is the acceleration of \(u^a\) and \(\sigma_{ab} \equiv D_{(a} u_{b)}\) is the shear. Eqs. (3.19–3.22) provide a complete description of radiative transfer in almost-FRW models. They describe the evolution of the energy-integrated intensity and polarization multipoles along the integral curves of \(u^a\). The equations are valid for any type of perturbation (we have not decomposed the variables into their scalar, vector or tensor parts), and for any (physical) choice of \(u^a\). To close the equations it is necessary to supplement them with the 1+3 covariant hydrodynamic equations, e.g. Ref. [13], to determine the \(O(\epsilon)\) kinematic variables \(A_a, v_a,\) and \(\sigma_{ab}\).

**IV. SCALAR PERTURBATIONS**

Up to this point our discussion has been quite general. Although we specialised to almost-FRW models when discussing the multipole propagation equations, we did not split the perturbations into their constituent modes (scalar, vector, tensor etc.). However, for detailed calculation with the linearised, almost-FRW equations, it is convenient to exploit the linearity to break the problem into smaller pieces. Our strategy follows Refs. [5, 7, 9]: we expand all \(O(\epsilon)\) variables in PSTF tensors derived from appropriate irreducible eigenfunctions of the comoving Laplacian \(S^2 D^a D_a\), where \(S\) is the covariantly-defined scale factor with \(\dot{S}/S = \Theta/3\) and \(D_a S = O(\epsilon)\). Since the sources of anisotropy and polarization are at most rank-2 objects, there are non-vanishing contributions to the source terms only from the scalar, vector and rank-2 tensor eigenfunctions. The different perturbation types decouple at linear order, with each giving rise to a set of coupled, first-order ordinary differential equations. We consider the scalar modes in this section; tensor modes are treated in Sec. V.
Since vorticity dies away in expanding models in the absence of significant momentum densities and anisotropic stresses \cite{23}, we do not consider vector modes. If required, e.g. in defect models \cite{23}, vector modes can be easily included in the formalism. The mode-expanded 1+3 covariant equations describing the perturbations in the matter components other than the radiation, and the geometry, can be found in Refs. \cite{3,4,10}, so we need only consider the radiation here. Furthermore, we do not consider circular polarization any further since it is not generated by Thomson scattering.

For scalar perturbations we expand in the scalar eigenfunctions $Q^{(k)}$, where

$$S^2 D^a D_a Q^{(k)} = k^2 Q^{(k)}$$

(4.1)

at zero order. The $Q^{(k)}$ are constructed so that $\dot{Q}^{(k)} = O(\epsilon)$. The allowed eigenvalues $k^2$ depend on the spatial geometry of the background model. Defining $\nu^2 = (k^2 + K)/|K|$, where $6K/S^2$ is the curvature scalar of the spatial sections in the background model, the regular, normalisable eigenfunctions have $\nu \geq 0$ for open and flat models ($K \leq 0$). In closed models $\nu$ is restricted to integer values $\geq 1$ \cite{20,27}. The mode with $\nu = 1$ cannot be used to construct perturbations (its projected gradient vanishes globally), while the mode with $\nu = 2$ (which can only represent isocurvature perturbations \cite{28}) only contributes to the CMB dipole. An explicit PSTF representation of the scalar harmonics $Q^{(k)}$ is given in the appendix.

For the $l$-th multipoles of the radiation anisotropy and polarization we expand in rank-$l$ PSTF tensors, $Q^{(k)}_{A_l}$, derived from the scalar harmonics via

$$Q^{(k)}_{A_l} = \left(\frac{S}{k}\right)^l D_{(a_1} \ldots D_{a_l)l} Q^{(k)}.$$  

(4.2)

The recursion relation for the $Q^{(k)}_{A_l}$,

$$Q^{(k)}_{A_l} = \frac{k}{S} D_{(a_l} Q^{(k)}_{A_{l-1})},$$  

(4.3)

follows directly from Eq. (4.2). The factor of $(S/k)^l$ in the definition of the $Q^{(k)}_{A_l}$ ensures that $\dot{Q}^{(k)}_{A_l} = 0$ at zero-order. For $I_{A_l}$ and $E_{A_l}$ we write

$$I_{A_l} = I \sum_k \alpha_l^{-1} I^{(l)}_k Q^{(k)}_{A_l}, \quad l \geq 1,$$  

(4.4)

$$E_{A_l} = E \sum_k \alpha_l^{-1} E^{(l)}_k Q^{(k)}_{A_l},$$  

(4.5)

where we have defined $\alpha_l \equiv \prod_{n=1}^{l} \kappa_n$, with

$$\kappa_l \equiv [1 - (l^2 - 1)K/k^2]^{1/2}, \quad l \geq 1,$$  

(4.6)

$^3$The $I^{(l)}_k$ are related to the $J^{(l)}_k$ of Ref. 3 by $I^{(l)}_k = \alpha_l J^{(l)}_k$.  

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and $\alpha_0 = 1$. The symbolic summation in Eqs. (4.4) and (4.5) denotes a sum over the harmonics (see Eq. [A7]), and the mode coefficients, such as $I_{k}^{(l)}$, are $O(\epsilon)$ scalars with $O(\epsilon^2)$ projected gradients. We need not consider the magnetic polarization since it is not generated by scalar modes [19,20]. To see this, we need the first-order identity

$$\text{curl} \, D_{(a_{l+1}} S_{A_l)} = \frac{l}{(l+1)} D_{(a_{l+1}} \text{curl} \, S_{A_l)},$$

where $S_{A_l}$ is an $O(\epsilon)$, rank-$l$ PSTF tensor. Applying this identity repeatedly to the right-hand side of Eq. (4.3) we see that the curl of $Q_{(k)}^{(l)}$ vanishes, as we would expect for a scalar perturbation. It follows that in linear theory $\text{curl} \, \mathcal{E}_{A_l} = 0$ for scalar perturbations, so the inhomogeneous source terms in the propagation equation for $\mathcal{B}_{A_l}$ (Eq. [3.20]) vanish. Since primordial polarization is erased by Thomson scattering during tight coupling, scalar perturbations do not support magnetic polarization.

The decomposition of $I^{(e^2)}$ into angular multipoles $I_{A_l}$, and the subsequent expansion in the $Q_{A_l}^{(k)}$, combine to give a normal mode expansion which is equivalent to the Legendre tensor approach, first introduced by Wilson [15]. The advantage of handling the angular and scalar harmonic decompositions separately is that the former can be applied quite generally for an arbitrary cosmological model. Furthermore, extending the normal mode expansions to the polarization and tensor modes is then trivial once the angular decomposition has been performed.

Substituting Eq. (4.4) into Eq. (3.22), and using the zero-order identity [5,9]

$$D_{a_l} Q_{A_l}^{(k)} = \frac{k}{S(2l-1)} \left[ 1 - (l^2 - 1) \frac{K}{k^2} \right] Q_{A_{l-1}},$$

(4.8)

gives the Boltzmann hierarchy for the $I_{k}^{(l)}$:

$$\dot{I}_{k}^{(l)} + \frac{k}{S} \left[ \frac{l+1}{2l+1} \kappa_{l+1} I_{k}^{(l+1)} - \frac{l}{2l+1} \kappa_{l} I_{k}^{(l-1)} \right] + \frac{4}{3} \left( \frac{k}{S} Z_{k} - \Theta A_{k} \right) \delta_{l}^{0}$$

$$+ \frac{4 k}{3 S} A_{k} \delta_{l}^{1} - \frac{8}{15} S \kappa_{2} \sigma_{k} \delta_{l}^{2} = -n_{e} \sigma_{T} \left( I_{k}^{(l)} - \delta_{l}^{0} I_{k}^{(0)} - \frac{4}{3} \delta_{l}^{(1)} \nu_{k} - \frac{2}{15} \zeta_{k} \delta_{l}^{2} \right).$$

(4.9)

It should be noted that $I_{k}^{(0)}$ refers to the projected gradient of intensity monopole $I$,

$$\frac{D_{a} I}{I} = \sum_{k} \frac{k}{S} I_{k}^{(0)} Q_{a}^{(k)},$$

(4.10)

which is gauge-invariant. To obtain Eq. (4.4) for $l = 0$, take the projected gradient of Eq. (3.22) for $l = 0$ and commute the derivatives. This gives rise to the source term $Z_{k}$, where

$$D_{a} \Theta = \sum_{k} \left( \frac{k}{S} \right)^{2} Z_{k} Q_{a}^{(k)}.$$

(4.11)

The other kinematic source terms come from the acceleration $A_{a}$, the the baryon relative velocity $\nu_{a}$, and the shear $\sigma_{ab}$, with
\[ A_a = \sum_k \frac{k}{S} A_k Q_a^{(k)}, \quad (4.12) \]
\[ v_a = \sum_k v_k Q_a^{(k)}, \quad (4.13) \]
\[ \sigma_{ab} = \sum_k \frac{k}{S} \sigma_k Q_{ab}^{(k)}. \quad (4.14) \]

The term describing the dependence of Thomson scattering on the anisotropy and polarization is \( \zeta_k = 3 I_k^{(2)}/4 + 9 \mathcal{E}_k^{(2)}/2 \). Eq. (4.13) extends the 1+3 covariant results of Refs. [3,4] by including this polarization source term.

For the electric polarization we use Eq. (4.13) in Eq. (3.19) to find
\[ \mathcal{E}_k^{(i)} + \frac{k}{S} \left[ \frac{(l+3)(l-1)}{(2l+1)(l+1)} \kappa_{l+1} \mathcal{E}_k^{(l+1)} - \frac{l}{(2l+1)} \kappa_l \mathcal{E}_k^{(l-1)} \right] = -n_e \sigma_T \left( \mathcal{E}_k^{(i)} - \frac{2}{15} \zeta_k \delta^2 \right), \quad (4.15) \]
which is the 1+3 covariant analogue of equivalent results in Refs. [14,29]. Eqs. (4.9) and (4.13) can be integrated with 1+3 covariant hydrodynamic equations [5,9] to determine the anisotropy and polarization from scalar modes.

To obtain formal integral solutions to the Boltzmann hierarchies we note that the homogeneous form of Eq. (4.9), obtained by setting \( n_e = 0 \) and removing the kinematic source terms, is solved by \( \Phi_t(x) \), where \( x = \sqrt{K} (\eta_R - \eta) \), with \( \eta_R \) the conformal time at our current position \( R \). Here \( \Phi_t(x) \) are the ultra-spherical Bessel functions [27]. The full solution of Eq. (4.9) follows from Green’s method:
\[ I_k^{(i)} = 4 \int_{t_R}^{t} dt e^{-\tau} \left\{ \left( \frac{k}{S} \sigma_k + \frac{1}{4} n_e \sigma_T K_2^{-1} \zeta_k \right) \left[ \frac{1}{3} \Phi_t(x) + \frac{1}{(\nu^2 + 1)} \frac{d^2}{dx^2} \Phi_t(x) \right] \ight. \\
- \left( \frac{k}{S} A_k - n_e \sigma_T v_k \right) \frac{1}{\sqrt{\nu^2 + 1}} \frac{d}{dx} \Phi_t(x) - \left[ \frac{1}{3} \left( \frac{k}{S} Z_k - \Theta A_k \right) - \frac{1}{4} n_e \sigma_T I_k^{(0)} \right] \Phi_t(x) \right\}, \quad (4.16) \]
where \( \tau \) is the zero-order optical depth back to \( x \) (see Eq. 3.14). The geometric factors \( \Phi_t/3 + (\nu^2 + 1)^{-1/2} d^2 \Phi_t/dx^2 \) and \( (\nu^2 + 1)^{-1/2} d^2 \Phi_t/dx^2 \) arise from the projections of \( Q_a^{(k)} e^a \) and \( Q_a^{(k)} e^a \) respectively, at \( x \) back along the line of sight.

For the polarization we note that the homogeneous part of Eq. (4.13) is solved by \( l(l-1) \Phi_t(x)/(\sinh^2 x) \). The full solution is
\[ \mathcal{E}_k^{(i)} = \frac{l(l-1)}{(\nu^2 + 1)} \int_{t_R}^{t} dt n_e \sigma_T e^{-\tau} K_2^{-1} \zeta_k \Phi_t(x)/(\sinh^2 x), \quad (4.17) \]
Eqs. (4.16) and (4.17) are valid in an open universe. For closed models one should replace the hyperbolic functions by their trigonometric counterparts, and \( \nu^2 + n \) by \( \nu^2 - n \) where \( n \) is an integer.

To characterise the initial amplitude of the mode we introduce a set of random variables \( \phi_k \) with the covariance matrix
\[ \langle \phi_k \phi_{k'} \rangle = \frac{1}{|K|^{3/2} \nu(\nu^2 + 1)} \delta_{kk'}, \quad (4.18) \]
appropriate to a statistically isotropic and homogeneous ensemble. The $\delta_{kk'}$ is defined by $\sum_k \Delta_k \delta_{kk'} = \Delta_k$ for any mode functions $\Delta_k$ (see appendix). We follow the conventions of Lyth & Woszczyna [30], so that the scalar field $\sum_k \phi_k Q^{(k)}$ has a scale-invariant spectrum for $P_\phi(\nu) = \text{constant}$. We write the radiation mode coefficients in terms of the $\phi_k$ and transfer functions $T_\ell^{(l)}(\nu)$ and $T_\ell^{E(l)}(\nu)$:

$$I_k^{(l)} = T_\ell^{(l)}(\nu) \phi_k, \quad \mathcal{E}_k^{(l)} = \frac{M_l}{\sqrt{2}} T_\ell^{E(l)}(\nu) \phi_k,$$

where the factor $M_l/\sqrt{2}$ is for later convenience. To compute the anisotropy and polarization power spectra we use Eqs. (4.4) and (4.5) in Eqs. (2.16) and (2.19) to find

$$C_{lXY} = \frac{1}{16} \int_0^\infty \frac{\nu d\nu}{(\nu^2 + 1)} T_\ell^{X(l)}(\nu) T_\ell^{Y(l)}(\nu) P_\phi(\nu).$$

(4.20)

Here, $X$ and $Y$ represent either of $I$ or $E$. In a closed universe the integral over $\nu$ in Eq. (4.20) should be replaced by a sum over integral $\nu \geq l + 1$. To derive Eq. (4.20) we have made use of the result

$$\sum_k f(\nu) Q^{(k)}(A_l) Q^{(k)B_l} = \frac{1}{(4\pi)^2} \Delta_l h^{(B_l)}(A_l) \delta_k^{(l)} \int_0^{\infty} d\nu |K|^{3/2} \nu^2 f(\nu) \alpha_l^2,$$

(4.21)

for any scalar function $f(\nu)$, which follows from Eqs. (A6) and (A9) in the appendix.

**V. TENSOR PERTURBATIONS**

For tensor perturbations we follow the procedure in Ref. [10] and expand in rank-2 PSTF tensor eigenfunctions $Q^{(k)}_{ab}$ of the comoving Laplacian:

$$S^2 D^c D_a Q^{(k)}_{ab} = k^2 Q^{(k)}_{ab}.$$

(5.1)

The tensor harmonics are transverse, $D^a Q^{(k)}_{ab} = 0$, and constant along the integral curves of $u^a$; $\hat{Q}^{(k)}_{ab} = 0$, at zero-order. For tensor modes we define $\nu^2 = (k^2 + 3K)/|K|$. The regular, normalisable eigenmodes have $\nu \geq 0$ for flat and open models, while for closed models $\nu$ is an integer $\geq 3$. Explicit forms for the tensor harmonics in the PSTF representation are given in the appendix; see also Ref. [10]. The tensor harmonics can be classified as having electric or magnetic parity. We continue to denote the electric parity harmonics by $Q^{(k)}_{ab}$, but we use an overbar to distinguish the magnetic parity harmonics: $\bar{Q}^{(k)}_{ab}$. The electric and magnetic parity harmonics are related through the curl operation (see Eqs. [A13] and [A20]).

Following our treatment of scalar perturbations, we form rank-$l$ PSTF tensors $Q^{(k)}_{A_l}$ and $\bar{Q}^{(k)}_{A_l}$ from the electric and magnetic parity tensor harmonics respectively. We define

$$Q^{(k)}_{A_l} = \left( \frac{S}{k} \right)^{l-2} D_{(a_1} \ldots D_{a_{l-2}} Q^{(k)}_{a_{l-1} a_l)},$$

(5.2)
with an equivalent definition for the magnetic parity harmonics. The rank-\(l\) tensors satisfy \(Q^{(k)}_{A_l} = 0\) at zero-order. We expand the \(l\)-th multipoles of the intensity and linear polarization as

\[
I_{A_l} = I \sum_k \beta_l^{-1}(I^{(l)}_k Q^{(k)}_{A_l} + \tilde{I}^{(l)}_k \tilde{Q}^{(k)}_{A_l}), \quad l \geq 2, 
\]

\[
\mathcal{E}_{A_l} = I \sum_k \beta_l^{-1}(\mathcal{E}^{(l)}_k Q^{(k)}_{A_l} + \tilde{\mathcal{E}}^{(l)}_k \tilde{Q}^{(k)}_{A_l}), \quad l \geq 2, 
\]

\[
B_{A_l} = I \sum_k \beta_l^{-1}(B^{(l)}_k Q^{(k)}_{A_l} + \tilde{B}^{(l)}_k \tilde{Q}^{(k)}_{A_l}). \quad l \geq 2, 
\]

For tensor modes we have defined \(\beta_l \equiv \prod_{n=2}^l \kappa_n\), and

\[
\kappa_l \equiv [1 - (l^2 - 3)K/k^2]^{1/2}, \quad l \geq 2. 
\]

It is necessary to include the magnetic polarization for tensor modes since the source to the magnetic hierarchy does not vanish: \(\text{curl} \mathcal{E}_{A_l} \neq 0\).

The \(Q^{(k)}_{A_l}\) constructed from the tensor harmonics satisfy the same recursion relation, Eq. (4.3), as their scalar counterparts. However, their projected divergences evaluate to [10]

\[
\begin{align*}
D^a Q^{(k)}_{A_l} &= \frac{k}{l} \left(\frac{l^2 - 4}{2(l - 1)} \left[1 - (l^2 - 3)\frac{K}{k^2}\right] Q^{(k)}_{A_{l-1}}. 
\end{align*} 
\]

An equivalent relation holds for the \(\bar{Q}^{(k)}_{A_l}\). Using these results in Eq. (3.22) we find

\[
\begin{align*}
\dot{I}^{(l)}_k + \frac{k}{S} \left[\frac{(l + 3)(l - 1)}{(l + 1)(2l + 1)} \kappa_{l+1} I^{(l+1)}_k - \frac{l}{(2l + 1)} \kappa_l I^{(l-1)}_k\right] - \frac{8}{15} \frac{k}{S} \kappa_2 \sigma_k \delta_k^2 \\
= -n_e \sigma_T \left(\dot{I}^{(l)}_k - \frac{2}{15} \zeta_k \delta_k^2\right), 
\end{align*} 
\]

where \(\zeta_k = 3I_k^{(2)}/4 + 9\mathcal{E}_k^{(2)}/2\), and \(\sigma_k\) is the tensor mode coefficient for the shear,

\[
\sigma_{ab} = \sum_k \frac{k}{S} (\sigma_k Q^{(k)}_{ab} + \bar{\sigma}_k \bar{Q}^{(k)}_{ab}). 
\]

Eq. (5.8) extends the 1+3 covariant result in Ref. [10] to include the polarization source term. For the polarization hierarchies we need the result

\[
\text{curl} Q^{(k)}_{A_l} = \frac{2k}{7} \sqrt{1 + \frac{3K}{k^2}} \bar{Q}^{(k)}_{A_l}, \quad l \geq 2 
\]

which follows from repeated application of Eq. (4.4), and finally Eq. (A19) to express \(\text{curl} \bar{Q}^{(k)}_{A_{l+1}a_l}\) in terms of \(\bar{Q}^{(k)}_{A_{l+1}a_l}\). For the electric polarization, Eq. (3.19) gives

\[
\begin{align*}
\dot{\mathcal{E}}^{(l)}_k + \frac{k}{S} \left[\frac{(l + 3)^2(l - 1)^2}{(2l + 1)(l + 1)^3} \kappa_{l+1} \mathcal{E}^{(l+1)}_k - \frac{l}{(2l + 1)} \kappa_l \mathcal{E}^{(l-1)}_k\right] - \frac{4}{l(l + 1)} \frac{k}{S} \frac{3K}{k^2} \bar{B}^{(l)}_k \\
= -n_e \sigma_T \left(\mathcal{E}^{(l)}_k - \frac{2}{15} \zeta_k \delta_k^2\right). 
\end{align*} 
\]
In linearised form, we have
\[ \text{in an open universe. The geometric factor } l \ \text{Substituting the tensor mode expansion of } \] 
\[ \text{ics with magnetic parity. Using the representation of the tensor harmonics given in the } \] 
\[ \text{ics gives the corresponding equation for the magnetic polarization: } \]
\[ \dot{B}^{(l)}_k + \frac{k}{l} \left[ \left( \frac{l+3}{2l+1} \right)^2 \kappa_{l+1} B^{(l+1)}_k - \frac{l}{2l+1} \kappa_l B^{(l-1)}_k \right] + \frac{4}{l(l+1)} \frac{k}{S} \sqrt{1 + \frac{3K}{k^2}} \dot{\xi}^{(l)}_k \]
\[ = -n_e \sigma_T B^{(l)}_k. \]  
(5.12)

Equivalent equations hold for the barred variables \( \bar{\xi}^{(l)}_k \) and \( \bar{B}^{(l)}_k \). Note how power is transferred between \( \bar{B}^{(l)}_k \) and \( \bar{\xi}^{(l)}_k \) due to the curl coupling between the electric and magnetic multipoles in Eqs. (3.19) and (3.20).

The integral solution for the tensor contribution to the intensity anisotropy in a general almost-FRW model was given in Ref. \[14\]. Including polarization modifies the scattering source term, so that the integral solution becomes
\[ I^{(l)}_k = \frac{4l(l-1)}{([\nu^2+1][\nu^2+3])^{1/2}} \int t^R dt e^{-\tau} \left( \frac{k}{S} \sigma_k + \frac{1}{4} n_e \sigma_T \kappa_2^{-1} \xi_k \right) \Phi_R^x(x) \sinh^2 x, \]  
(5.13)
in an open universe. The geometric factor \( l(l-1) \Phi_R(x)/\sinh^2 x \) follows from the projection of \( Q_{ab}^{(k)} e^a e^b \) at \( x \) back along the line of sight.

To solve the coupled equations (5.11) and (5.12) for the linear polarization it is simplest to return to the integral solution for the tetrad components of \( \mathcal{P}_{ab}(e^c \), given as Eq. (3.13). In linearised form, we have
\[ I^{-1} \mathcal{P}_{ab}(e^c)(e_i)^a(e_j)^b \big|_R = \frac{1}{4\pi} \int t^R dt n_e \sigma_T e^{-\tau} I^{-1}[\zeta_{ab}]^{TT}(e_i)^a(e_j)^b. \]  
(5.14)

Substituting the tensor mode expansion of \( \zeta_{ab} \) gives terms in the integral like \( \left[ Q_{ab}^{(k)} \right]^{TT}(e_i)^a(e_j)^b \) from the electric parity harmonics, and similar terms from the harmonics with magnetic parity. Using the representation of the tensor harmonics given in the appendix, we find that
\[ \left[ Q_{ab}^{(k)} \right]^{TT}(e_i)^a(e_j)^b = T_3(x) \left[ Q_{abcL-2}^{(LM)} e^{C_{L-2}} \right]^{TT}(e_i)^a(e_j)^b, \]  
(5.15)

where \( T_3(x) \) is a \( \nu \) and \( L \)-dependent function given by Eq. (A13). The superscript \( (k) \) on the tensor harmonics represents the collection \( \nu, L, \) and \( M \), where \( L \) and \( M \) describe the orbital angular momentum of the harmonic; see the appendix for details. The \( Q^{(LM)}_{AL} \) are rank-\( L \) PSTF tensor fields, which are introduced in the appendix. They encode the \( 2L+1 \) degrees of freedom in the tensor harmonics with \( L \) units of orbital angular momentum. Using the transport properties of \( Q^{(LM)}_{AL} \), Eq. (A3), and of the tetrad vectors \( (e_i)^a \), Eq. (3.4), it is straightforward to show that to zero-order
\[ (u^a + e^a) \nabla_a \left[ \left[ Q_{abcL-2}^{(LM)} e^{C_{L-2}} \right]^{TT}(e_i)^a(e_j)^b \right] = 0, \]  
(5.16)
so that \( \left[ Q_{abcL-2}^{(LM)} e^{C_{L-2}} \right]^{TT}(e_i)^a(e_j)^b \) is constant along the line of sight. For the magnetic parity harmonics, the representation in the appendix gives
\[ \left[ Q_{ab}^{(k)} \right]^{TT}(e_i)^a(e_j)^b = \bar{T}_1(x) \left[ e_{cL} e^{C_{L-1}} \right]_{aL-1}^{(LM)} e^{C_{L-2}} \left[ Q_{b}^{(LM)} \right]^{TT}(e_i)^a(e_j)^b, \]  
(5.17)
where \( T_1(x) \) is given by Eq. (5.18). Working to zero-order, we find that
\[
(u^a + e^a) \nabla_a \{ [e_{cl} e^{cl-1}(a Q_{bL}) e^{CL-2}]^{TT} (e_i)^a (e_j)^b \} = 0,
\]
so that \( [e_{cl} e^{cl-1}(a Q_{bL}) e^{CL-2}]^{TT} (e_i)^a (e_j)^b \) is also constant along the line of sight at zero-order. With these results, Eq. (5.14) reduces to
\[
I^{-1} P_{ab}(e^c)|_R = \frac{1}{4\pi} \sum_k [Q_{abCL-2}(L^{LM})]^{TT} \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k T_3(x)
+ \frac{1}{4\pi} \sum_k [e_{cl} e^{cl-1}(a Q_{bL}) e^{CL-2}]^{TT} \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k \bar{T}_1(x).
\]
Evaluating the functions \( T_3(x) \) and \( \bar{T}_1(x) \), we find the integral solutions in the form
\[
\mathcal{E}_{AI}|_R = \frac{1}{4\pi} \sum_k \delta_{IL} \Delta_l \mathcal{Q}_{AL}^{(LM)}|_R \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k T_3(x),
\]
\[
\mathcal{B}_{AI}|_R = \frac{1}{4\pi} \sum_k \delta_{IL} \Delta_l \mathcal{Q}_{AL}^{(LM)}|_R \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k \bar{T}_1(x).
\]
To determine the harmonic coefficients, such as \( \mathcal{E}_{k}^{(l)} \), at \( t_R \) we note that \( \bar{Q}_{AI}^{(k)} = 0 \) at \( R \), so that only the electric parity harmonics contribute there. We can use Eq. (A18) to substitute for \( \delta_{IL} Q_{AI}^{(LM)} \) at each \( \nu \) in terms of \( Q_{AI}^{(k)}|_R \) in Eqs. (7.20) and (7.24). Comparing with Eqs. (5.4) and (5.5), we can read off the integral solutions
\[
\mathcal{E}_{k}^{(l)} = M_l \sqrt{\frac{\nu}{\nu^2 + 3}} \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k \Phi_l^\nu(x),
\]
\[
\mathcal{B}_{k}^{(l)} = M_l \sqrt{\frac{\nu}{\nu^2 + 3}} \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k \bar{\psi}_l^\nu(x).
\]
Evaluating the functions \( T_3(x) \) and \( \bar{T}_1(x) \), we can write the integral solutions in the form
\[
\mathcal{E}_{k}^{(l)} = \frac{M_l^2}{2([\nu^2 + 1](\nu^2 + 3))^{1/2}} \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k \Phi_l^\nu(x),
\]
\[
\mathcal{B}_{k}^{(l)} = \frac{M_l^2}{2([\nu^2 + 1](\nu^2 + 3))^{1/2}} \int_{t_R}^{t_R} dt n_e \sigma_T e^{-\tau} \zeta_k \bar{\psi}_l^\nu(x),
\]
where the geometric terms \( \Phi_l^\nu(x) \) and \( \bar{\psi}_l^\nu(x) \) are
\[
\Phi_l^\nu(x) = \frac{d^2}{dx^2} \Phi_l^\nu(x) + 4 \coth x \frac{d}{dx} \Phi_l^\nu(x) - (\nu^2 - 1 - 2 \coth^2 x) \Phi_l^\nu(x),
\]
\[
\bar{\psi}_l^\nu(x) = -2\nu \left[ \frac{d}{dx} \Phi_l^\nu(x) + 2 \coth x \Phi_l^\nu(x) \right]
\]
in an open universe. In closed models we replace \( \nu^2 + n \) by \( \nu^2 - n \), and the hyperbolic functions by their trigonometric counterparts (as for scalar perturbations). It is straightforward to verify that the integral solutions, Eqs. (5.24) and (5.25), satisfy the multipole
formalisms, it is worthwhile outlining their relation to the PSTF representation adopted first used for describing polarization in Ref. [20]. Given the widespread use of these two CMB research in Ref. [19], or the Newman-Penrose spin-weight 2 harmonics (e.g. Ref. [32]), based on either a coordinate representation of the tensor spherical harmonics, introduced to facts that $\bar{\zeta}$ modes from Eqs. (2.16) and (2.19). The result is

We can now compute the anisotropy and polarization power spectra for tensor modes from Eqs. (2.16) and (2.19). The result is

$$C_{l}^{XY} = \frac{1}{16} \frac{(l + 2)(l + 1)}{2l(l - 1)} \int_{0}^{\infty} \frac{\nu d\nu}{(\nu^2 + 1)} \frac{(\nu^2 + 3)}{\nu^2} T_{x}^{(l)}(\nu) T_{y}^{(l)}(\nu) \mathcal{P}(\nu),$$

(5.30)

where $XY$ is equal to $II, EE, BB$, or $IE$. In closed models the integral over $\nu$ is replaced by a discrete sum over integral $\nu \geq l + 1$. In deriving Eq. (5.30) we used the following result:

$$\sum_{k} f(\nu)(Q_{A_{i}}^{(k)}Q_{B_{r}}^{(k)}B_{r} + \bar{Q}_{A_{i}}^{(k)}\bar{Q}_{B_{r}}^{(k)}B_{r}) = \frac{1}{(4\pi)^2} M_{l}^{-2} \Delta h_{a_{l}}^{(B_{l})} \delta^l_{a_{l}} \int_{0}^{\infty} d\nu |K|^{3/2}(\nu^2 + 3) f(\nu) \beta_{l}^{2},$$

(5.31)

for any scalar function $f(\nu)$. This result is easily verified at $R$ by using Eq. (A15) and the fact that $\bar{Q}_{A_{i}}^{(k)}|_{R} = 0$.

VI. DISCUSSION

Most of the modern literature on the polarization of the CMB employs representations based on either a coordinate representation of the tensor spherical harmonics, introduced to CMB research in Ref. [19], or the Newman-Penrose spin-weight 2 harmonics (e.g. Ref. [32]), first used for describing polarization in Ref. [20]. Given the widespread use of these two formalisms, it is worthwhile outlining their relation to the PSTF representation adopted
here. We begin by introducing an orthonormal triad of projected vectors \( \{ (\gamma_1)^a, (\gamma_2)^a, (\gamma_3)^a \} \) at the observer’s location. At this point, the triad is used to define angular coordinates \( \{ \theta, \phi \} \) which cover the two-dimensional manifold of unit projected vectors \( \{ n^a \} \). The scalar spherical harmonics \( Y_{(lm)}(\theta, \phi) = Y_{(lm)}(n^a) \) can be represented by complex PSTF tensors \( Y_{A_l}^{(lm)} \), where

\[
Y_{A_l}^{(lm)} = \Delta_l^{-1} \int d\Omega Y_{(lm)}(n^c) n_{(Ai)},
\]

so that \( Y_{(lm)}(n^c) = Y_{A_l}^{(lm)} n^A_l \). Using the spherical harmonic addition theorem, one can show that the \( Y_{A_l}^{(lm)} \) satisfy the orthogonality relations

\[
Y_{A_l}^{(lm)} Y_{A_l'}^{(lm')} = \Delta_l^{-1} \delta_{mm'},
\]

\[
\sum_{m=-l}^{l} Y_{A_l}^{(lm)} Y_{A_l}^{(lm')} = \Delta_l^{-1} h_{(Ai)}^{(B_l)},
\]

which are useful for the discussion below.

If we compare the PSTF expansion of the temperature anisotropy, Eq. (2.17), with Eq. (2) of Ref. [19] we can read off the relation between the PSTF multipoles \( I_{A_l} \) and the complex scalar multipole coefficients \( a_T^{(lm)} \) of the anisotropy:

\[
\sum_{m=-l}^{l} a_T^{(lm)} Y_{A_l}^{(lm)} = \frac{\pi}{I\Delta_l} (-1)^l I_{A_l}.
\]

The factor of \((-1)^l\) in this equation arises because in Ref. [19] the unit projected vector \( n^a \) represents a direction on the sky, rather than the photon propagation direction, so that \( e^a = -n^a \). Using Eqs. (6.3) and (6.4) in the expression for the temperature power spectrum, Eq. (2.16), we find that

\[
\langle a_T^{(lm)} a_T^{(ln')} \rangle = C_l^{TT} \delta_{ll'} \delta_{mm'}.
\]

It follows that the \( C_l^{TT} \) of Ref. [19] coincides with the temperature power spectrum defined here.

For the linear polarization we need to relate the PSTF representation of the TT tensor spherical harmonics to the coordinate representation derived by covariant differentiation of the \( Y^{(lm)}(n^a) \). Taking covariant derivatives on the sphere of the PSTF representation of the \( Y^{(lm)}(n^a) \), and using the conventions for the \( G(\text{rad}) \) and \( C(\text{url}) \) tensor spherical harmonics of Ref. [19], we find that

\[
Y_{(lm)}^{G\,ab}(n^c) = M_l [\mathcal{Y}_{abc}^{(lm)}]^{TT},
\]

\[
Y_{(lm)}^{C\,ab}(n^c) = M_l [n_{d1} \epsilon^{d1d2} (a \mathcal{Y}_{b}^{(lm)} C_{d2}^{C_1-2} n^c)]^{TT}.
\]

Integrating Eq. (2.11) over energy and dividing by \( \pi/I \) expresses the linear polarization in dimensionless temperature units. Comparing with Eq. (2) of Ref. [19], we find the following
relations between the coordinate-dependent, scalar-valued multipole coefficients \( a^G_{(lm)} \) and \( a^C_{(lm)} \) defined there, and the PSTF multipoles \( \mathcal{E}_{A_i} \) and \( \mathcal{B}_{A_i} \) employed here:

\[
\frac{\pi}{I\Delta_i} (-1)^l \mathcal{E}_{A_i} = M_l \sum_{m=-l}^{l} a^G_{(lm)} \mathcal{Y}^{(lm)}_{A_i}, \tag{6.8}
\]

\[
\frac{\pi}{I\Delta_i} (-1)^{l-1} \mathcal{B}_{A_i} = M_l \sum_{m=-l}^{l} a^C_{(lm)} \mathcal{Y}^{(lm)}_{A_i}. \tag{6.9}
\]

Using these results in the definitions of the polarization power spectra, e.g. Eq. (2.19), we find the non-vanishing covariance structure

\[
2 \langle a^G_{(lm)} a^G_{(l'm')}^* \rangle = C_{lE}^{\delta \delta} \delta_{ll'} \delta_{mm'}, \tag{6.10}
\]

\[
2 \langle a^C_{(lm)} a^C_{(l'm')}^* \rangle = C_{lB}^{\delta \delta} \delta_{ll'} \delta_{mm'}, \tag{6.11}
\]

\[
\sqrt{2} \langle a^T_{(lm)} a^G_{(l'm')}^* \rangle = C_{lIE}^{\delta \delta} \delta_{ll'} \delta_{mm'}. \tag{6.12}
\]

in a parity-symmetric ensemble. Comparing with Eq. (8) of Ref. [19], we find that \( 2C_{lE}^{G} = C_{lE}^{\delta \delta} \) and \( 2C_{lC}^{C} = C_{lB}^{\delta \delta} \), while \( \sqrt{2} C_{lT}^{TG} = C_{lIE}^{\delta \delta} \).

To compare the PSTF representation of the linear polarization with the spin-weight 2 representation of Ref. [20], it is simplest to express the coordinate representation of the tensor spherical harmonics of Ref. [19] in terms of the spin-weight 2 spherical harmonics. Noting that the Stokes parameters in Ref. [20] are expressed on a right-handed basis with the 3-axis opposite to the direction of propagation (which flips the sign of \( U \)), we find that for the \( a_{E,lm} \) and \( a_{B,lm} \) multipole coefficients of Ref. [20], \( a_{E,lm} = -\sqrt{2} a^G_{(lm)} \) and \( a_{B,lm} = -\sqrt{2} a^C_{(lm)} \), whereas the temperature multipoles are equal. It follows that the polarization power spectra of Ref. [20] are related to those defined here by \( C_{EI}^{C} = C_{lE}^{\delta \delta} \), \( C_{BI}^{C} = C_{lB}^{\delta \delta} \), and \( C_{CI}^{C} = -C_{lIE}^{\delta \delta} \).

VII. CONCLUSION

We have introduced a new multipole formalism for describing polarized radiation on the sky. In this approach the polarization tensor is expanded in the PSTF representation of the transverse-traceless tensor spherical harmonics [18]. The PSTF representation is particularly convenient since the radiation multipoles are then coordinate-independent. This allows the equations of radiative transfer in a general spacetime to be recast as multipole equations, which is essential for the transfer problem in optically thin media. We applied the formalism to give a rigorous discussion of the generation and propagation of CMB polarization in cosmological models. We gave new results for the non-linear transformation properties of the polarization multipoles under changes of reference frame, the exact source term for Thomson scattering, and the multipole propagation equations in linearised form in an almost-FRW model. It was not necessary to split the perturbations into scalar, vector and tensor modes to obtain these results, and so they provide a solid foundation on which to build a complete, second-order analysis of CMB polarization. By expanding the linearised multipole equations in scalar and tensor harmonics we derived the mode-expanded multipole equations in almost-FRW models with general geometries, and their integral solutions. These results confirm those obtained earlier by Hu et al. [14] with the total angular momentum method.
The linearised results of this paper have been included in a publically available Fortran 90 code \cite{33}, based on CMBFAST \cite{16}, for calculating the CMB temperature anisotropy and polarization in general FRW models, including those with closed geometries, within the 1+3 covariant and gauge-invariant approach.

In a subsequent paper we will complete the development of a multipole transfer formalism in general spacetimes by providing the exact polarization multipole equations for an arbitrary geometry. This should simplify the modelling of a number of important astrophysical situations where relativistic flows or strong gravity effects are important, including non-linear effects in the CMB.

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APPENDIX: SCALAR AND TENSOR HARMONICS

In the text we make use of a specific PSTF representation of the scalar and tensor harmonics. This representation was described in detail in Ref. \cite{10}, so will only be summarised here. The starting point is to “coordinatise” the solution in terms of a projected vector field $e^a$, and a scalar field $\chi$. The restriction of these fields to the past lightcone through some point $R$ (which can conveniently be taken to be our point of observation) are the propagation direction of a free-streaming photon which will subsequently pass through $R$, and the conformal look-back time along the photon path respectively. The projected fields $e^a$ and $\chi$ are generated from their restriction to the past lightcone by Fermi transporting along the integral curves of $u^a$:

$$
\dot{e}^{(a)} = 0, \quad \dot{\chi} = 0.
$$

(A1)

It is straightforward to show that the field $e^a$ satisfies $e^a D_a e^b = 0$ at zero-order in an almost-FRW universe.

1. Scalar harmonics

In the PSTF representation the regular, normalisable scalar harmonics can be written as

$$Q^{(k)} = \Phi'_{L}(x)Q^{(LM)}_{AL} e^{AL}, \quad L \geq 0,
$$

(A2)

where the $Q^{(LM)}_{AL}$ with $M = -L \ldots L$ are rank-$L$ PSTF tensor fields satisfying the zero-order equations

$$e^b D_b Q^{(LM)}_{AL} = 0, \quad \dot{Q}^{(LM)}_{(AL)} = 0,
$$

(A3)

which determine the fields from their initial values at $R$. The $\Phi'_{L}(x)$ are ultra-spherical Bessel functions (see, e.g. Ref. \cite{27}) with $\nu^2 = (k^2 + K)/|K|$ and $x = \sqrt{|K|}\chi$. In closed
models $\nu$ is restricted to integer values with $\nu > L$. For convenience we label the harmonics with a lumped superscript $(k)$ which represents $\nu$, $L$ and $M$.

The scalar harmonics are normalised so that

$$\int_0^{\infty} d\Omega e^{a|xR|} \sinh^2 x Q^{(k)}(k') = \frac{\pi}{2} \delta_{LL'} \Delta_L Q^{(LM)}_{AL} Q^{(LM')}_{AL} \nu^{-2} \delta(\nu - \nu'), \quad (A4)$$

where $d\Omega e^{a|xR|}$ denotes an integral over solid angles at $R$. Eq. (A4) refers to an open universe. In closed models the hyperbolic functions replace their trigonometric counterparts, $\delta(\nu - \nu')$ replaces $\delta_{\nu\nu'}$, and the upper limit on the normalisation integral becomes $\pi$. It is very convenient to choose the $Q^{(LM)}_{AL}$ at $R$ so that

$$Q^{(LM)}_{AL} Q^{(LM')}_{AL} = \Delta_L^{-1} \delta_{MM'}, \quad (A5)$$

which implies

$$\sum_{M=-L}^{L} Q^{(LM)}_{AL} Q^{(LM')}_{AL} B_L = \Delta_L^{-1} h^{(B_L)}_{(AL)}, \quad (A6)$$

It follows from Eq. (A3) that Eqs. (A5) and (A6) hold at zero-order at all points. A scalar-valued, statistically homogeneous random field, say $\Delta(x^a)$, can be constructed from a superposition of the scalar harmonics:

$$\Delta(x^a) = \int_0^{\infty} |K|^{3/2} \nu^2 d\nu \sum_{L=0}^{\infty} \sum_{M=-L}^{L} \Delta_{\nu LM} Q^{(k)}, \quad (A7)$$

in an open universe. In the closed case the integral is replaced by a discrete sum over integer $\nu$, and the sum over $L$ is restricted to $L < \nu$. In the text we denote the sum over scalar harmonic modes in the symbolic form $\Delta(x^a) = \sum_k \Delta_k Q^{(k)}$. The covariance structure

$$\langle \Delta_k \Delta_{k'} \rangle = \Delta^2(\nu) \delta_{kk'} \quad (A8)$$

is sufficient to ensure statistical homogeneity and isotropy of the ensemble. The symbolic $\delta_{kk'}$ represents $\delta_{LL'} \delta_{MM'} |K|^{-3/2} \nu^{-2} \delta(\nu - \nu')$ in open models. For closed models $\delta(\nu - \nu')$ should be replaced by $\delta_{\nu\nu'}$.

From the $Q^{(k)}$ we can derive rank-$l$ PSTF tensors $Q^{(k)}_{Al}$ as in Eq. (4.2). For the specific representation given here we find that at the point $R$,

$$Q^{(k)}_{Al} |_{R} = \frac{1}{4\pi} \Delta_{lAL} Q^{(LM)}_{AL} \delta_{lL}, \quad (A9)$$

so that the only modes to contribute to the $l$-th multipole of the radiation anisotropy and polarization at $R$ have $l$ units of orbital angular momentum. In a closed universe, the $Q^{(k)}_{Al}$ vanish globally for $l \geq \nu$. 

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2. Tensor harmonics

a. Electric parity

The regular, normalisable tensor harmonics with electric parity are given by

$$Q^{(k)}_{ab} = T_1(x) \left( e_a e_b Q^{(LM)}_{CL} e^{CL} + \frac{1}{2} \mathcal{H}_{ab} Q^{(LM)}_{CL} e^{CL} \right)$$

$$+ T_2(x) e_a \mathcal{H}_{b}^{CL} Q^{(LM)}_{CL} e^{CL-1} + T_3(x) [Q^{(LM)}_{abCL-2} e^{CL-2}]^T_{TT}, \quad L \geq 2. \quad (A10)$$

Here, the (screen) projection tensor $\mathcal{H}_{ab} = h_{ab} + e^a e^b$, and the subscript $TT$ denotes the transverse (to $e^a$), trace-free part. In an open universe, $T_1(x)$ is given by

$$T_1(x) = \frac{1}{\nu \sqrt{2(\nu^2 + 1)}} \left[ \frac{(L + 2)!}{(L - 2)!} \right]^{1/2} \frac{\Phi_{L}^{\nu}(x)}{\sinh^2 x}, \quad (A11)$$

and $T_2(x)$ and $T_3(x)$ are determined by

$$T_2(x) = \frac{-2}{(L + 1) \sinh^2 x} \frac{d}{dx} [\sinh^3 x T_1(x)], \quad (A12)$$

$$T_3(x) = \frac{-L}{(L + 2)} T_1(x) - \frac{1}{(L + 2) \sinh^2 x} \frac{d}{dx} [\sinh^3 x T_2(x)]. \quad (A13)$$

The electric parity tensor harmonics are normalised so that

$$\int_0^\infty d\Omega e^a |_R dx \sinh^2 x Q^{(k)}_{ab} Q^{(k')}_{ab} = \frac{\pi}{2} \delta_{LL'} \Delta_L Q^{(LM)}_{A_L} Q^{(LM')}_{A_L} \nu^{-2} \delta(\nu - \nu'). \quad (A14)$$

For closed universes, the hyperbolic functions should be replaced by their trigonometric counterparts, $\nu^2 + n$ should be replaced by $\nu^2 - n$ with $n$ an integer, and $\delta(\nu - \nu')$ should be replaced by $\delta_{\nu\nu'}$. For closed models, the regular, normalisable modes have $\nu$ an integer $\geq 3$, restricted to $\nu > L$.

We can form rank-$l$ ($l \geq 2$) PSTF tensors $Q^{(k)}_{A_l}$ from the electric parity tensor harmonics as in Eq. (5.2). At the point $R$ we find that

$$Q^{(k)}_{A_l} |_R = \frac{1}{4\pi} \frac{\sqrt{\nu^2 + 3}}{\nu} M_l^{-1} \Delta_l \beta_l Q^{(LM)}_{A_L} \delta_{ll}, \quad (A15)$$

so that, as with scalar modes, the only contribution to the $l$-th multipole of the anisotropy and polarization at $R$ comes from those modes with $l$ units of orbital angular momentum. In closed models the $Q^{(k)}_{A_l}$ vanish for $l \geq \nu$.

b. Magnetic parity

For the magnetic parity tensor harmonics, which we denote with an overbar, the regular normalisable solutions are
\[ \check{Q}_{ab}^{(k)} = \check{T}_1(x)[e_{cl} e_{cLcL-1} (a \check{Q}_{b}^{(LM)} e^{cL-2})^{TT} + \check{T}_2(x)e_{d(a} e_{b)} d_{cL} Q_{cL}^{(LM)} e^{cL-1}], \]  

(A16)

with \( L \geq 2 \). In an open universe, \( \check{T}_2(x) \) is given by

\[
\check{T}_2(x) = \frac{1}{(L + 1)} \left[ \frac{2}{(\nu^2 + 1)} \right]^{1/2} \left[ \frac{(L + 2)!}{(L - 2)!} \right]^{1/2} \frac{\Phi_\nu^r(x)}{\sinh x}, \tag{A17}
\]

and \( \check{T}_1(x) \) is determined by

\[
\check{T}_1(x) = \frac{-1}{(L + 2) \sinh^2 x} \frac{d}{dx} \left[ \sinh^3 x \check{T}_2(x) \right]. \tag{A18}
\]

In a closed universe we make the same replacements as for the electric parity harmonics. The magnetic parity harmonics satisfy the same normalisation condition, Eq. (A14), as the electric parity harmonics, to which they are orthogonal. We can also form rank-\( l \) PSTF tensors \( \check{Q}_{A_i}^{(k)} \) from the magnetic parity harmonics, as in Eq. (3.2). However, we now find that the \( \check{Q}_{A_i}^{(k)} \) vanish at \( R \), so that the magnetic parity harmonics do not contribute to the anisotropy and polarization there. (However, magnetic parity modes do contribute at points not on the integral curve of \( u^a \) that passes through \( R \).) In closed models the \( \check{Q}_{A_i}^{(k)} \) vanish globally for \( l \geq \nu \).

The electric and magnetic parity tensor harmonics are related through the curl operation (which is parity reversing). For the conventions adopted here, we find

\[
\text{curl} \check{Q}_{ab}^{(k)} = \frac{k}{S} \sqrt{1 + \frac{3K}{k^2}} \check{Q}_{ab}^{(k)}, \tag{A19}
\]

\[
\text{curl} \check{Q}_{ab}^{(k)} = \frac{k}{S} \sqrt{1 + \frac{3K}{k^2}} Q_{ab}^{(k)}. \tag{A20}
\]

c. Statistically homogeneous tensor-valued random fields

We can construct a statistically homogeneous and isotropic tensor-valued random field by superposing the electric and magnetic parity tensor harmonics. For example, for the shear tensor \( \sigma_{ab} \), we write

\[
\sigma_{ab} = \int_0^\infty |K|^{3/2} \lambda^2 d\nu \sum_{L=2}^\infty \sum_{M=-L}^L \frac{k}{S} \left( \sigma_{\nu LM} Q_{ab}^{(k)} + \check{\sigma}_{\nu LM} \check{Q}_{ab}^{(k)} \right), \tag{A21}
\]

in an open universe. (Equation [A21] can easily be generalised to include supercurvature modes, if these are present in the initial conditions.) In a closed universe the integral over \( \nu \) is replaced by a sum over integer \( \nu \geq 3 \), and the sum over \( L > 2 \) is further restricted to \( L < \nu \). In the text we use the symbolic summation \( \sum_k \) to represent the sum over modes on the right-hand side of Eq. (A21). The requirements of statistical homogeneity and isotropy restrict the covariance structure of the \( \sigma_k \) and \( \check{\sigma}_k \) to the form
\begin{align*}
\langle \sigma_k \sigma_{k'} \rangle &= \sigma^2(\nu) \delta_{kk'} \\
\langle \bar{\sigma}_k \bar{\sigma}_{k'} \rangle &= \sigma^2(\nu) \delta_{kk'} \\
\langle \sigma_k \bar{\sigma}_{k'} \rangle &= 0,
\end{align*}
\tag{A22}

where the symbolic \( \delta_{kk'} \) is the same as for scalar modes. The shorthand \( \sigma_k \) is used to represent \( \sigma_{\nu L M} \), and similarly for the barred variables, in Eq. (A22) and also in the text.
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