Two-point Green functions of free Dirac fermions in single-layer graphene ribbons with zigzag and armchair edges

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Abstract
Green function technique is a very efficient theoretical tool for the study of dynamical quantum processes in many-body systems. For the study of dynamical quantum processes in graphene ribbons it is necessary to know two-point Green functions of free Dirac fermions in these materials. The purpose of present work is to establish explicit expressions of two-point Green functions of free Dirac fermions in single-layer graphene ribbons with zigzag and armchair edges. By exactly solving the system of Dirac equations with appropriate boundary conditions on the edges of graphene ribbons we derive formulae determining wave functions of free Dirac fermions in these materials. Then the quantum fields of free Dirac fermions are introduced, and explicit expressions of two-point Green functions of free Dirac fermions in single-layer graphene ribbons with zigzag and armchair edges are established.

Keywords: graphene, ribbon, zigzag, armchair, green function

Classification numbers: 2.01, 3.00, 5.15

1. Introduction
The discovery of graphene by Geim, Novoselov et al [1–4] has stimulated the development of a new multidisciplinary area of science and technology of graphene-based nanomaterials [5, 6]. Recently a new approach to the theoretical study of these nanomaterials as well as to the electromagnetic interaction processes in single-layer graphene using mathematical tools of quantum field theory was proposed [7, 8]. In particular, a comprehensive study on two-point Green functions of Dirac fermions in Dirac fermion gas of an infinitely large graphene single layer was performed [7]. The purpose of present work is to study two-point Green functions of Dirac fermions in the Dirac fermion gas of graphene ribbons with zigzag and armchair edges. It was known that hexagonal crystalline lattice of graphene comprises two interpenetrating sublattices with triangular symmetry [4]. Throughout the work following notations and conventions will be used.

The distance between two nearest carbon atoms in the hexagonal graphene lattice is denoted a. Then the distance between two nearest vertices in each triangular sublattice is \( a_0 = \sqrt{3}a \). Denote \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \) the translation vectors of the triangular crystalline sublattice, and \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) those of its reciprocal sublattices

\[
\mathbf{a}_i \mathbf{b}_j = 2\pi \delta_{ij}. \tag{1}
\]

We chose the \( xOy \) Cartesian coordinate system as follows: \( Ox \) axis is parallel to the direction of the length of the ribbon, while \( Oy \) axis is parallel to that of its width. Then for the graphene ribbon with zigzag edges we have the crystalline...
lattice structure represented in figure 1(a) and the reciprocal lattice represented in figure 1(b), while for that with armchair edges the crystalline lattice structure is represented in figure 2(a) and the reciprocal lattice is represented in figure 2(b).

For the simplicity we shall limit our study to the case of a Dirac fermion gas with the Fermi energy level \( E_F = 0 \) and at the vanishing absolute temperature \( T = 0 \). The extension to other cases is straightforward.

In section 2 we study the quantum fields of Dirac fermions in graphene ribbon with zigzag edges, and the subject of section 3 is the study of quantum fields of Dirac fermions in graphene ribbon with armchair edges. Conclusion and discussion are presented in section 4.

2. Graphene ribbon with zigzag edges

2.1. Wave functions of free Dirac fermions

With the above-mentioned convention on the choice of Cartesian coordinate system \( xOy \) we have

\[
\begin{align*}
\mathbf{a}_1 &= a_0 (1, 0), \\
\mathbf{a}_2 &= a_0 \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \\
\mathbf{b}_1 &= \frac{4\pi}{\sqrt{3}a_0} \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right), \\
\mathbf{b}_2 &= \frac{4\pi}{\sqrt{3}a_0} (1, 0).
\end{align*}
\]

(2)

Each Brillouin zone has two inequivalent vertices \( \mathbf{K} \) and \( \mathbf{K}' \). In the first Brillouin zone we can choose

\[
\begin{align*}
\mathbf{K} &= \frac{4\pi}{3a_0} (1, 0), \\
\mathbf{K}' &= \frac{4\pi}{3a_0} (-1, 0)
\end{align*}
\]

(3)

(figure 1(b)).
Wave functions $\varphi^K(r)$ and $\varphi^{K'}(r)$ of Dirac fermions satisfy Dirac equations
\begin{equation}
-iv_F (\tau \nabla) \varphi^K(r) = E\varphi^K(r)
\end{equation}
and
\begin{equation}
-iv_F (\tau \nabla) \varphi^{K'}(r) = E\varphi^{K'}(r),
\end{equation}
where two components $\tau_1$ and $\tau_2$ of the $2 \times 2$ vector matrix $\tau$ are the Pauli matrices.

We set
\[ \varepsilon = \frac{E}{v_F} \]
and rewrite Dirac equations in the form
\begin{equation}
-i(\tau \nabla) \varphi^K(r) = \varepsilon \varphi^K(r)
\end{equation}
and
\begin{equation}
-i(\tau \nabla) \varphi^{K'}(r) = \varepsilon \varphi^{K'}(r).
\end{equation}

Dirac equations (7) and (8) must be invariant with respect to the translations along the Ox axis which do not change the graphene ribbon crystalline lattice as a whole. These translations form a group called the translational symmetry group of this crystalline lattice. According to the Bloch theorem [9] eigenfunctions of Dirac equations (7) and (8) have following general form
\begin{equation}
\varphi^{K,K'}(r) = e^{ikz}\begin{pmatrix} \alpha^{K,K'}_1(y) \\ \beta^{K,K'}_1(y) \end{pmatrix}
\end{equation}
with a real number $k$ playing the role of the wave vector of a wave propagating along the Ox axis. Let us choose the lower edge of the zigzag ribbon to have the ordinate $y = 0$ and the upper one to have the ordinate $y = L$. Then functions $\alpha^{K,K'}_k(y)$ and $\beta^{K,K'}_k(y)$ must satisfy following boundary conditions
\begin{equation}
\alpha^{K,K'}_k(L) = 0,
\end{equation}
and
\begin{equation}
\beta^{K,K'}_k(0) = 0.
\end{equation}

In [4] it was shown that by solving Dirac equations (7) and (8) one obtains eigenvalues $\varepsilon$ determined by relation
\begin{equation}
\varepsilon^2 = \omega(k, \lambda)^2, \quad \omega(k, \lambda) = \sqrt{k^2 - \lambda^2}
\end{equation}
with some real constants $\lambda$ and eigenfunctions (9), where $\alpha^{K,K'}_k(y)$ and $\beta^{K,K'}_k(y)$ have the expressions
\begin{equation}
\alpha^{K}_k(y) = \frac{1}{\varepsilon} [A^K(k - \lambda)e^{\lambda y} + B^K(k + \lambda)e^{-\lambda y}],
\end{equation}
\begin{equation}
\alpha^{K'}_k(y) = \frac{1}{\varepsilon} [A^{K'}(k + \lambda)e^{\lambda y} + B^{K'}(k - \lambda)e^{-\lambda y}],
\end{equation}
\begin{equation}
\beta^{K,K'}_k(y) = A^{K,K'}e^{\lambda y} + B^{K,K'}e^{-\lambda y}.
\end{equation}

Due to the boundary condition (11) between the constants $A^{K,K'}$ and $B^{K,K'}$ there exists following relation
\begin{equation}
B^{K,K'} = -A^{K,K'}.
\end{equation}

According to formula (12) Dirac equations (7) and (8) have two common eigenvalues
\begin{equation}
\varepsilon_1 = \omega(k, \lambda), \quad \varepsilon_2 = -\omega(k, \lambda).
\end{equation}
In the first case with $\varepsilon = \varepsilon_1(k, \lambda)$ formulae (13) and (14) become
\begin{equation}
\alpha^{K}_k(y) = A^K \left[ \frac{k - \lambda}{k + \lambda} e^{\lambda y} \right. - \left. \frac{k + \lambda}{k - \lambda} e^{-\lambda y} \right],
\end{equation}
\begin{equation}
\alpha^{K'}_k(y) = A^{K'} \left[ \frac{k + \lambda}{k - \lambda} e^{\lambda y} \right. - \left. \frac{k - \lambda}{k + \lambda} e^{-\lambda y} \right],
\end{equation}
while in the second case with $\varepsilon = \varepsilon_2(k, \lambda)$ formulae (13) and (14) become
\begin{equation}
\alpha^{K}_k(y) = -A^K \left[ \frac{k - \lambda}{k + \lambda} e^{\lambda y} - \frac{k + \lambda}{k - \lambda} e^{-\lambda y} \right],
\end{equation}
\begin{equation}
\alpha^{K'}_k(y) = -A^{K'} \left[ \frac{k + \lambda}{k - \lambda} e^{\lambda y} - \frac{k - \lambda}{k + \lambda} e^{-\lambda y} \right].
\end{equation}

In both case we have
\begin{equation}
\beta^{K,K'}_k(y) = A^{K,K'}e^{\lambda y} - e^{-\lambda y}.
\end{equation}

Let us now study the conditions determining the values of the parameter $\lambda$. From boundary condition (10) for function (18) we derive following algebraic equation
\begin{equation}
e^{-2\lambda L} = \frac{k - \lambda}{k + \lambda},
\end{equation}
while from the same boundary condition (10) for function (19) we obtain another one
\begin{equation}
e^{-2\lambda L} = \frac{k + \lambda}{k - \lambda}.
\end{equation}

In [4] it was noted that whenever $k$ is positive ($k > 0$), equation (23) for $\lambda$ has real solutions corresponding to surface waves propagating near the edges of the graphene ribbon. Similarly, whenever $k$ is negative ($k < 0$), equation (24) for $\lambda$ also has real solutions corresponding to surface waves propagating near the edges of the graphene ribbon.

Thus we have demonstrated that Dirac equations (7) and (8) have eigenvalues determined by equation (17). In the case of positive eigenenergies $\varepsilon_1(k, \lambda)$ the corresponding eigenfunctions are
\begin{equation}
\varphi^{K,K'}(r) = e^{ikz}\begin{pmatrix} \alpha^{K}_k(y) \\ \beta^{K}_k(y) \end{pmatrix},
\end{equation}
\begin{equation}
\varphi_{e,K,\lambda}^{K,K'}(r) = \begin{pmatrix} \alpha^{K}_k(y) \\ \beta^{K}_k(y) \end{pmatrix},
\end{equation}
\begin{equation}
u^{K,K'}_e(r) = \begin{pmatrix} \alpha^{K}_k(y) \\ \beta^{K}_k(y) \end{pmatrix},
\end{equation}
\begin{equation}
u_{e,K,\lambda}^{K,K'}(r) = \begin{pmatrix} \alpha^{K}_k(y) \\ \beta^{K}_k(y) \end{pmatrix}.
\end{equation}
with following components
\[ \alpha_{k,\lambda}^K(y) = A_{k,\lambda}^K \left[ \frac{k - \lambda}{k + \lambda} e^{\lambda y} - \frac{k + \lambda}{k - \lambda} e^{-\lambda y} \right], \]  
\[ \alpha_{k,\lambda}^{K'}(y) = A_{k,\lambda}^{K'} \left[ \frac{k + \lambda}{k - \lambda} e^{\lambda y} - \frac{k - \lambda}{k + \lambda} e^{-\lambda y} \right], \]  
\[ \beta_{k,\lambda}^{K,K'}(y) = A_{k,\lambda}^{K,K'} (e^{\lambda y} - e^{-\lambda y}), \]  
while in the case of negative eigenenergies \( \varepsilon_2(k, \lambda) \) the corresponding eigenfunctions are
\[ \varphi_{k,\lambda}^{K,K'}(r) = e^{ikr} v_{k,\lambda}^{K,K'}(y), \]  
\[ v_{k,\lambda}^{K,K'}(y) \text{ being two-component spinors} \]
\[ v_{k,\lambda}^{K,K'}(y) = \begin{pmatrix} -\alpha_{k,\lambda}^{K'}(y) \\ \beta_{k,\lambda}^{K,K'}(y) \end{pmatrix} \]  
The magnitudes of constants \( A_{k,\lambda}^{K,K'} \) are determined by the condition of normalization of wave functions:
\[ \int_0^L u_{k,\lambda}^{K,K'}(y) u_{k,\lambda}^{K,K'}(y) dy = \int_0^L v_{k,\lambda}^{K,K'}(y) v_{k,\lambda}^{K,K'}(y) dy = \int_0^L (|\alpha_{k,\lambda}^{K,K'}|^2 + |\beta_{k,\lambda}^{K,K'}|^2) dy = 1. \]  
It is easy to verify that
\[ \int_0^L u_{k,\lambda}^{K,K'}(y) v_{k,\lambda}^{K,K'}(y) dy = \int_0^L v_{k,\lambda}^{K,K'}(y) u_{k,\lambda}^{K,K'}(y) dy = 0. \]

### 2.2. Two-point Green functions of free Dirac fermions

Denote \( a_{k,\lambda}^{K,K'} \) and \( a_{k,\lambda}^{K,K'+} \), \( \nu = 1, 2 \), the destruction and creation operators of free Dirac fermions in the quantum states with energies \( \varepsilon_\nu(k, \lambda) \) and with corresponding wave functions \( \varphi_{k,\lambda}^{K,K'} \) determined by formulae (25) and (30).

These operators satisfy following canonical anticommutation relations
\[ \{ a_{k,\lambda}^K, a_{k',\lambda'}^{K'} \} = \{ a_{k,\lambda}^{K,K'+}, a_{k',\lambda'}^{K,K'+} \} = \delta_{\lambda,\lambda'} \delta_{k, k'}, \]  
\[ \{ a_{k,\lambda}^{K,K'}, a_{k,\lambda'}^{K,K'} \} = \{ a_{k,\lambda}^{K,K'+}, a_{k,\lambda'}^{K,K'+} \} = 0, \]  
\[ \{ a_{k,\lambda}^{K,K'}, a_{k',\lambda'}^{K,K'} \} = \{ a_{k,\lambda}^{K,K'+}, a_{k',\lambda'}^{K,K'+} \} = 0. \]

Quantum fields of free Dirac fermions are
\[ \psi^{K,K'}(r, t) = \psi^{K,K'}(x, y, t) = \frac{1}{\sqrt{2\pi}} \int dk \]
\[ \times \sum_\lambda \{ e^{ikx} e^{-i\varepsilon_\lambda t} \} a_{k,\lambda}^{K,K'}(y) a_{k,\lambda}^{K,K'} \]
\[ + e^{-ikx} e^{-i\varepsilon_\lambda t} \} v_{k,\lambda}^{K,K'}(y) a_{k,\lambda}^{K,K'} \}

By means of standard calculations [7] it is straightforward to derive following explicit expression of two-point Green functions of free Dirac fermions in a single-layer graphene ribbon with zigzag edges
\[ \Delta^{K,K'}(r, r'; t - t')_{\lambda,\lambda'} = \frac{1}{\sqrt{2\pi}} \int dk \]
\[ \times \frac{1}{\varepsilon - \omega(k, \lambda) + i\delta} \]
\[ + \frac{1}{\varepsilon - \omega(k, \lambda) - i\delta} \]  

### 3. Graphene ribbon with armchair edges

#### 3.1. Wave functions of free Dirac fermions

In the case of graphene ribbon with armchair edges the quantum states of Dirac fermions with wave vectors near both inequivalent Dirac points \( \textbf{K} \) and \( \textbf{K}' \) must be simultaneously taken into account, so that wave functions of stationary states are two orthogonal and normalized linear combinations
\[ \Phi_1(r) = \frac{1}{\sqrt{2}} \{ e^{ikr} \varphi^K(r) + e^{ik' r} \varphi^{K'}(r) \}, \]  
\[ \Phi_2(r) = \frac{1}{\sqrt{2}} \{ e^{ikr} \varphi^K(r) - e^{ik' r} \varphi^{K'}(r) \}, \]  
\( \varphi^K(r) \) and \( \varphi^{K'}(r) \) being the Bloch wave functions with the wave vectors near to \( \textbf{K} \) and \( \textbf{K}' \), respectively.

With above-mentioned convention on the choice of Cartesian coordinate system \( xOy \) we have now (figure 2(b))
\[ a_1 = a_0(0,1), \quad a_2 = a_0 \left( \frac{\sqrt{3} \ 1}{2} \right), \]
\[ b_1 = \frac{4\pi}{\sqrt{3} a_0} \left( -\frac{1}{2} \frac{\sqrt{3}}{2} \right), \quad b_2 = \frac{4\pi}{\sqrt{3} a_0} (1,0). \]  
In the first Brillouin zone we choose following two inequivalent vertices
\[ \textbf{K} = \frac{4\pi}{3a_0} (0,1), \quad \textbf{K}' = \frac{4\pi}{3a_0} (0, -1). \]
Due to the invariance of the Dirac equations with respect to the translations of the symmetry group of the graphene ribbon crystalline lattice, wave functions $\varphi^K(\mathbf{r})$ and $\varphi^{K'}(\mathbf{r})$ must be periodic with respect to the coordinate $x$:

$$\varphi^{K,K'}(\mathbf{r}) = e^{iKx}u^{K,K'}(y)$$

(42)

with two-component spinors

$$u^{K,K'}(y) = \begin{pmatrix} \alpha_k^{K,K'}(y) \\ \beta_k^{K,K'}(y) \end{pmatrix}.$$  

(43)

Let us consider separately two different cases with wave functions $\Phi_1(\mathbf{r})$ and $\Phi_2(\mathbf{r})$. Wave function $\Phi_1(\mathbf{r})$ must satisfy following boundary conditions

$$\Phi_1(\mathbf{r})|_{y=0} = 0$$

(44)

and

$$\Phi_1(\mathbf{r})|_{y=L} = 0$$

(45)

meaning that

$$u^K_k(0) + u^{K'}_k(0) = 0$$

(46)

and

$$e^{iKx}u^K_k(L) + e^{-iKx}u^{K'}(L) = 0.$$  

(47)

Thus, due to the boundary condition at the edge $y = 0$ and $y = L$ of the ribbon there takes place the mixing between the states with wave vectors in the neighbors of both equivalent Dirac points $K$ and $K'$. In [4] it was shown that functions $\beta^K_k(y)$ and $\beta^{K'}_k(y)$ have following general forms:

$$\beta^K_{k,\lambda_0^1}(y) = Ae^{i\lambda_0^1y} + Be^{-i\lambda_0^1y}$$

(48)

and

$$\beta^{K'}_{k,\lambda_0^1}(y) = Ce^{i\lambda_0^1y} + De^{-i\lambda_0^1y}.$$  

(49)

Corresponding wave functions $\alpha^K_{k,\lambda_0^1}(y)$ and $\alpha^{K'}_{k,\lambda_0^1}(y)$ are expressed in terms of wave functions $\beta^K_{k,\lambda_0^1}(y)$ and $\beta^{K'}_{k,\lambda_0^1}(y)$ through relations

$$\alpha^K_{k,\lambda_0^1}(y) = \frac{1}{\varepsilon}(k - \partial_y)\beta^K_{k,\lambda_0^1}(y)$$

(50)

and

$$\alpha^{K'}_{k,\lambda_0^1}(y) = \frac{1}{\varepsilon}(k + \partial_y)\beta^{K'}_{k,\lambda_0^1}(y).$$

(51)

Using formulae (50) and (51) of $\beta^K_{k,\lambda_0^1}(y)$ and $\beta^{K'}_{k,\lambda_0^1}(y)$, we obtain

$$\alpha^K_{k,\lambda_0^1}(y) = \frac{1}{\varepsilon}[A(k - i\lambda_0^1)e^{i\lambda_0^1y} + B(k + i\lambda_0^1)e^{-i\lambda_0^1y}]$$

(52)

and

$$\alpha^{K'}_{k,\lambda_0^1}(y) = \frac{1}{\varepsilon}[C(k + i\lambda_0^1)e^{i\lambda_0^1y} + D(k - i\lambda_0^1)e^{-i\lambda_0^1y}].$$

(53)

Now we study the consequences of the boundary conditions (46) and (47). Applying these conditions to the components $\beta^K_{k,\lambda_0^1}(y)$ and $\beta^{K'}_{k,\lambda_0^1}(y)$ determined by formulae (48) and (49), we obtain a system of two linear algebraic equations

$$A + B + C + D = 0$$

(54)

and

$$Ae^{i(K+\lambda_0^1)L} + De^{-i(K+\lambda_0^1)L} + Be^{i(K-\lambda_0^1)L} + Ce^{-i(K-\lambda_0^1)L} = 0.$$  

(55)

Equation (54) is satisfied if

$$A = -D, \quad B = C = 0.$$  

(56)

In this case from equation (55) we derive a condition for the parameters $\lambda_0^{(1)}$

$$\sin[(K + \lambda_0^{(1)})L] = 0.$$  

(57)

Thus parameters $\lambda_0^{(1)}$ can have following values

$$\lambda_0^{(1)} = \frac{n\pi}{L} - \frac{4\pi}{3a_0}$$

(58)

with all integers $n = 0, \pm 1, \pm 2 \cdots$

Similarly, equation (54) can be also satisfied if

$$A = D = 0, \quad B = -C.$$  

(59)

In this case from equation (55) we obtain another condition for the parameters $\lambda_0^{(1)}$

$$\sin[(K - \lambda_0^{(1)})L] = 0.$$  

(60)

Thus parameters $\lambda_0^{(1)}$ can have also other values

$$\lambda_0^{(1)} = \frac{n\pi}{L} + \frac{4\pi}{3a_0}.$$  

(61)

Consider now the case of wave function $\Phi_2(\mathbf{r})$. From boundary conditions

$$\Phi_2(\mathbf{r})|_{y=0} = 0$$

(62)

and

$$\Phi_2(\mathbf{r})|_{y=L} = 0,$$

(63)

it follows that

$$u^K_k(0) - u^{K'}_k(0) = 0$$

(64)

and

$$e^{iKx}u^K_k(L) - e^{-iKx}u^{K'}_k(L) = 0.$$  

(65)

Instead of expressions (48) and (49) now we have following expressions of functions $\beta^K_k(y)$ and $\beta^{K'}_k(y)$:

$$\beta^K_{k,\lambda_0^1}(y) = A'e^{i\lambda_0^1y} + B'e^{-i\lambda_0^1y}$$

(66)
and
\[ \beta_{k,l}^\nu(y) = C e^{i\lambda_{(2)}^\nu y} + D e^{-i\lambda_{(2)}^\nu y}. \]

Corresponding wave functions \( \alpha_{k,l}^\nu(y) \) and \( \alpha_{k,l}^{\nu'}(y) \) are expressed in terms of wave functions \( \beta_{k,l}^\nu(y) \) and \( \beta_{k,l}^{\nu'}(y) \) through relations
\[ \alpha_{k,l}^\nu(y) = \frac{1}{\epsilon} (k - \partial_y) \beta_{k,l}^\nu(y), \]
and
\[ \alpha_{k,l}^{\nu'}(y) = \frac{1}{\epsilon} (k + \partial_y) \beta_{k,l}^{\nu'}(y). \]

Using formulae (66) and (67) of \( \beta_{k,l}^\nu(y) \) and \( \beta_{k,l}^{\nu'}(y) \), we obtain
\[ \alpha_{k,l}^\nu(y) = \frac{1}{\epsilon} [A'(k - i\lambda_{(2)}^\nu) e^{i\lambda_{(2)}^\nu y} + B'(k + i\lambda_{(2)}^\nu) e^{-i\lambda_{(2)}^\nu y}] \]
and
\[ \alpha_{k,l}^{\nu'}(y) = \frac{1}{\epsilon} [A'(k + i\lambda_{(2)}^{\nu'}) e^{i\lambda_{(2)}^{\nu'} y} + B'(k - i\lambda_{(2)}^{\nu'}) e^{-i\lambda_{(2)}^{\nu'} y}]. \]

Now we consider the consequences of the boundary conditions (64) and (65). Applying these conditions to the components \( \beta_{k,l}^\nu(y) \) and \( \beta_{k,l}^{\nu'}(y) \), we derive a system of two linear algebraic equations
\[ A' + B' - C' - D' = 0 \]
and
\[ A' e^{i(K+\lambda_{(2)}^\nu)L} - D' e^{-i(K+\lambda_{(2)}^\nu)L} + B' e^{i(K-\lambda_{(2)}^\nu)L} - C' e^{-i(K-\lambda_{(2)}^\nu)L} = 0. \]

Equation (72) is satisfied if
\[ A' = D', \quad B' = C' = 0. \]

In this case from equation (73) we obtain a condition for the parameters \( \lambda_{(2)}^\nu \)
\[ \sin [(K + \lambda_{(2)}^\nu)L] = 0, \]
which is the same as that for the parameters \( \lambda_{(1)}^\nu \). Thus parameters \( \lambda_{(2)}^\nu \) can have also following values
\[ \lambda_{(2)}^\nu = \frac{n\pi}{L} + \frac{4\pi}{3d_0}. \]

Similarly, equation (72) can be also satisfied if
\[ B' = C', \quad A' = D' = 0. \]

In this case from equation (73) we obtain following condition for the parameters \( \lambda_{(2)}^\nu \)
\[ \sin [(K - \lambda_{(2)}^\nu)L] = 0, \]
which is again the same as that for the parameters \( \lambda_{(1)}^\nu \). Thus parameters \( \lambda_{(2)}^\nu \) can have also following values
\[ \lambda_{(2)}^\nu = \frac{n\pi}{L} + \frac{4\pi}{3d_0}. \]

For each state with a parameter \( \lambda_{(j)}^\nu \) the eigenvalue \( \varepsilon(k, \lambda_{(j)}^\nu) \) is determined by equation
\[ \varepsilon(k, \lambda_{(j)}^\nu)^2 = k^2 + \lambda_{(j)}^\nu. \]

Therefore with each given set of two parameters \( k \) and \( \lambda_{(j)}^\nu \) there exist two opposite values of \( \varepsilon(k, \lambda_{(j)}^\nu) \), namely
\[ \varepsilon_1(k, \lambda_{(j)}^\nu) = \omega(k, \lambda_{(j)}^\nu), \quad \varepsilon_2(k, \lambda_{(j)}^\nu) = -\omega(k, \lambda_{(j)}^\nu) \]
with
\[ \omega(k, \lambda_{(j)}^\nu) = \sqrt{k^2 + \lambda_{(j)}^\nu}. \]

For the convenience in writing formulae we set
\[ \varepsilon(k, \lambda_{(j)}^\nu, \nu) = \varepsilon_\nu(k, \lambda_{(j)}^\nu) \]
with \( \nu = 1, 2. \)

Wave functions \( \varphi_{K,K'}(r) \), two-component spinors \( u_{K,K'}(y) \) and their components \( \alpha_{k,l}^{K,K'}(y) \), \( \beta_{k,l}^{K,K'}(y) \) depend on the indices \( \nu, n \) as well as on the values \( i = 1, 2 \) and \( j = 1, 2 \) of the pair \( (i, j) \) of two indices labeling different cases of using different solutions of equations (57), (60), (75) and (78). Therefore the complete notations of the physical quantities in formulae (42) and (43) must be \( \varphi_{K,K',0,0}^{K,K'}(r), u_{K,K',0,0}^{K,K'}(y), \alpha_{K,K',0,0}^{K,K'}(y), \beta_{K,K',0,0}^{K,K'}(y) \). In order to shorten formulae we denote the set of 3 indices \( i, j, \lambda_{(j)}^\nu \) by a new notation \( \sigma \) and have following shortened notations \( \varphi_{K,K',\sigma}^{K,K'}(r), u_{K,K',\sigma}^{K,K'}(y), \alpha_{K,K',\sigma}^{K,K'}(y), \beta_{K,K',\sigma}^{K,K'}(y) \). Then formulae (42) and (43) become
\[ \varphi_{K,K',\sigma}^{K,K'}(r) = e^{i\delta_\sigma} u_{K,K',\sigma}^{K,K'}(y), \]
and
\[ u_{K,K',\sigma}^{K,K'}(y) = \begin{pmatrix} \alpha_{K,K',\sigma}^{K,K'}(y) \\ \beta_{K,K',\sigma}^{K,K'}(y) \end{pmatrix}. \]

### 3.2. Two-point Green functions of free Dirac fermions

Denote \( a_{K,K',\sigma,\nu}^{\nu',0,0} \) and \( a_{K,K',\sigma,\nu}^{\nu',0,0} \) the destruction and creation operators of free Dirac fermions in the quantum states with corresponding quantum numbers \( \sigma, \nu \) and wave vector \( k \) along the Ox axis near the Dirac points \( K \) and \( K' \). They satisfy following canonical anticommutation relations
\[ \begin{align*}
\{a_{K,K',\sigma,\nu}^{\nu',0,0}, a_{K',\nu',0,0}^{K,\sigma'}\} &= \{a_{K,\nu',0,0}^{K',\sigma}, a_{K',\nu',0,0}^{K,\sigma'}\} = \delta_{\sigma\sigma'} \delta_{\nu\nu'} \delta(k-k'), \\
\{a_{K,K',\sigma,\nu}^{K',\sigma',\nu'} + a_{K,\nu',0,0}^{K',\sigma}, a_{K',\nu',0,0}^{K,\sigma'}\} = 0, \\
\{a_{K,\nu',0,0}^{K',\sigma}, a_{K,\nu',0,0}^{K',\sigma'}\} &= \{a_{K,\nu',0,0}^{K',\sigma}, a_{K',\nu',0,0}^{K,\sigma'}\} = 0.
\end{align*} \]
Quantum fields of free Dirac fermions are spinor operators

\[ \psi^{K, K'}(r, t) = \psi^{K, K'}(x, y, t) = \frac{1}{\sqrt{2\pi}} \int dk \left\{ e^{i[kx - \omega(k, \sigma)t]} u^{K, K'}_{k, \sigma}(y) a^{K, K'}_{k, \sigma, 1} + e^{-i[kx - \omega(k, \sigma)t]} v^{K, K'}_{k, \sigma}(y) a^{K, K'}_{k, \sigma, 2} \right\} \]  

(87)

with the components

\[ \psi^{K, K'}(r, t)_\alpha = \psi^{K, K'}(x, y, t)_\alpha = \frac{1}{\sqrt{2\pi}} \int dk \left\{ e^{i[kx - \omega(k, \sigma)t]} u^{K, K'}_{k, \sigma}(y) a^{K, K'}_{k, \sigma, 1} + e^{-i[kx - \omega(k, \sigma)t]} v^{K, K'}_{k, \sigma}(y) a^{K, K'}_{k, \sigma, 2} \right\} \]  

(88)

By means of standard calculations it is straightforward to derive following explicit expression of two-point Green functions of free Dirac fermion in a single-layer graphene ribbon with armchair edges

\[ \Delta^{K, K'}(r, r'; t - t')_{\alpha \beta} = \frac{1}{\sqrt{2\pi}} \int d\xi e^{-i\omega (r - r')} \times \sum_{\sigma} \left\{ e^{i(\xi - \xi')} u^{K, K'}_{k, \sigma} (y) u^{K, K'}_{k, \sigma} (y')^+ + \frac{1}{\xi - \omega(k, \sigma) + i0} + e^{i(\xi - \xi')} v^{K, K'}_{k, \sigma} (y) v^{K, K'}_{k, \sigma} (y')^+ \times \frac{1}{\xi + \omega(k, \sigma) - i0} \right\}. \]  

(89)

4. Conclusion

For the future application in the study of dynamical quantum processes in single-layer graphene ribbons with zigzag and armchair edges, we have derived formulae determining the wave function of free Dirac fermions in these materials, taking into account appropriate boundary conditions at the edges of the ribbons. Then these wave functions were used for construction of quantum fields of free Dirac fermions, and explicit expressions of their two-point Green functions were established.

In the present work we have considered the simplest case of the free Dirac fermion gas with Fermi level \( E_F = 0 \) at vanishing absolute temperature \( T = 0 \). A lot of works should be done to extend obtained results to more general cases such as the free Dirac fermion gas with Fermi level \( E_F \neq 0 \) and at non-vanishing absolute temperature \( T \neq 0 \) as well as to the case of non-equilibrium free Dirac fermion gas. Moreover, the study of the interaction between the electromagnetic field and Dirac fermions in graphene ribbons would be also interesting scientific subject with practical applications.

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