Turing Kernelization for Finding Long Paths and Cycles in Restricted Graph Classes*

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Abstract. We analyze the potential for provably effective preprocessing for the problems of finding paths and cycles with at least $k$ edges. Several years ago, the question was raised whether the existing superpolynomial kernelization lower bounds for $k$-Path and $k$-Cycle can be circumvented by relaxing the requirement that the preprocessing algorithm outputs a single instance. To this date, very few examples are known where the relaxation to Turing kernelization is fruitful. We provide a novel example by giving polynomial-size Turing kernels for $k$-Path and $k$-Cycle on planar graphs, graphs of maximum degree $t$, claw-free graphs, and $K_{3,t}$-minor-free graphs, for each constant $t \geq 3$. Concretely, we present algorithms for $k$-Path ($k$-Cycle) on these restricted graph families that run in polynomial time when they are allowed to query an external oracle for the answers to $k$-Path ($k$-Cycle) instances of size and parameter bounded polynomially in $k$. Our kernelization schemes are based on a new methodology called Decompose-Query-Reduce.

1 Introduction

Motivation. Kernelization is a formalization of efficient and provably effective data reduction originating from parameterized complexity theory. In this setting, each instance $x \in \Sigma^*$ of a decision problem is associated with a parameter $k \in \mathbb{N}$ that measures some aspect of its complexity. Work on kernelization over the last few years has resulted in deep insights into the possibility of reducing an instance $(x,k)$ of a parameterized problem to an equivalent instance $(x',k')$ of size polynomial in $k$, in polynomial time. By now, many results are known concerning problems that admit such polynomial kernelization algorithms, versus problems for which the existence of a polynomial kernel is unlikely because it implies the complexity-theoretic collapse $\text{NP} \subseteq \text{coNP}/\text{poly}$. (See Section 2 for formal definitions of parameterized complexity.)

In this work we study the possibility of effectively preprocessing instances of the problems of finding long paths or cycles in a graph. In the model of (Karp) kernelization described above, in which the output of the preprocessing algorithm is a single, small instance, we cannot guarantee effective polynomial-time preprocessing for these problems. Indeed, the $k$-Path and $k$-Cycle problems are or-compositional [6] since the disjoint union of graphs $G_1, \ldots, G_t$ contains a path (cycle) of length $k$ if and only if there is at least one input graph with such a structure. Using the framework of Bodlaender et al. [6] this proves that the problems do not admit Karp kernelizations of polynomial size unless $\text{NP} \subseteq \text{coNP}/\text{poly}$ and the polynomial hierarchy collapses to its third level [25].

More than five years ago [5], the question was raised how fragile this bad news is: what happens if we relax the requirement that the preprocessing algorithm outputs a single instance? Does a polynomial-time preprocessing algorithm exist that, given an instance $(G,k)$ of $k$-Path, builds a list of instances $(x_1,k_1), \ldots, (x_t,k_t)$, each of size polynomial in $k$, such that $G$ has a length-$k$ path if and only if there is at least one YES-instance on the output list? Such a cheating kernelization is possible for the $k$-Leaf Out-Tree problem [3] while it does not admit a polynomial Karp kernelization unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. Hence it is natural to ask whether this can be done for $k$-Path or $k$-Cycle.

A robust definition of such relaxed forms of preprocessing was given by Lokshtanov [20] under the name Turing kernelization. It is phrased in terms of algorithms that can query an oracle for the answer to small instances of a specific decision problem in a single computation step.1 Observe that the existence of an $f(k)$-size kernel for a parameterized problem $\mathcal{Q}$ shows that $\mathcal{Q}$ can be solved in polynomial time if we allow the algorithm to make a single size-$f(k)$ query to an oracle for $\mathcal{Q}$: apply the kernelization to

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1 Formally, such algorithms are oracle Turing machines (cf. [15, Appendix A.1]).
input \((x, k)\) to obtain an equivalent instance \((x', k')\) of size \(f(k)\), query the \(Q\)-oracle for this instance and output its answer. A natural relaxation, which encompasses the cheating kernelization mentioned above, is to allow the polynomial-time algorithm to query the oracle more than once for the answers to \(f(k)\)-size instances. This motivates the definition of Turing kernelization.

**Definition 1.** Let \(Q\) be a parameterized problem and let \(f : \mathbb{N} \rightarrow \mathbb{N}\) be a computable function. A Turing kernelization for \(Q\) of size \(f\) is an algorithm that decides whether a given instance \((x, k) \in \Sigma^+ \times \mathbb{N}\) is contained in \(Q\) in time polynomial in \(|x| + k\), when given access to an oracle that decides membership in \(Q\) for any instance \((x', k')\) with \(|x'|, k' \leq f(k)\) in a single step.

For practical purposes the role of oracle is fulfilled by an external computing cluster that computes the answers to the queries. A Turing kernelization gives the means of efficiently splitting the work on a large input into manageable chunks, which may be solvable in parallel depending on the nature of the Turing kernelization. Moreover, Turing kernelization is a natural relaxation of Karp kernelization that facilitates a theoretical analysis of the nature of preprocessing.

At first glance, it seems significantly easier to develop a Turing kernelization than a Karp kernelization. However, to this date there are only a handful of parameterized problems known for which polynomial-size Turing kernelization is possible but polynomial-size Karp kernelization is unlikely [1,23,22,8]. Last year, the first adaptive\(^2\) Turing kernelization was given by Thomassé et al. [23] for the \(k\)-INDEPENDENT SET problem restricted to bull-free graphs. Although this forms an interesting step forwards in harnessing the power of Turing kernelization, the existence of polynomial-size Turing kernels for \(k\)-PATH and related subgraph-containment problems remains wide open [5,3,16]. Since many graph problems that are intractable in general admit polynomial-size (Karp) kernels when restricted to planar graphs, it is natural to consider whether such a restriction makes it easier to obtain polynomial Turing kernels for \(k\)-PATH; this was presented as an open problem by several authors [20,21]. Observe that, as the disjoint-union argument remains valid even for planar graphs, we do not expect polynomial-size Karp kernels for \(k\)-PATH in this setting.

**Our results.** In this paper we introduce the Decompose-Query-Reduce framework for obtaining adaptive polynomial-size Turing kernelizations for the \(k\)-PATH and \(k\)-CYCLE problems on various restricted graph families, including planar graphs and bounded-degree graphs. The three steps of the framework consist of (i) decomposing the input \((G, k)\) into parts of size \(k^{O(1)}\) with constant-size interfaces between the various parts; (ii) querying the oracle to determine how a solution can intersect such bounded-size parts, and (iii) reducing to an equivalent but smaller instance using this information. In our case, we use a classic result by Tutte [24] concerning the decomposition of a graph into its triconnected components, made algorithmic by Hopcroft and Tarjan [17], to find a tree decomposition of adhesion two of the input graph \(G\) such that all torsos of the decomposition are triconnected minors of \(G\). We complement this with various known graph-theoretic lower bounds on the circumference of triconnected graphs belonging to restricted graph families to deduce that if this Tutte decomposition has a bag of size \(\Omega(k^{O(1)})\), then there must be a cycle (and therefore path) of length at least \(k\) in \(G\). If we have not already found the answer to the problem we may therefore assume that all bags of the decomposition have polynomial size. Consequently we may query the oracle for solutions involving only \(k^{O(1)}\) parts of the decomposition. We use structural insights into the behavior of paths and cycles with respect to bounded-size separators to reduce the number of bags that are relevant to a query to \(k^{O(1)}\). Here we use ideas from earlier work on kernel bounds for structural parameterizations of path problems [7]. Together, these steps allow us to invoke the oracle to instances of size \(k^{O(1)}\) to obtain the information that is needed to safely discard some pieces of the input, thereby shrinking it. Iterating this procedure, we arrive at a final equivalent instance of size \(k^{O(1)}\), whose answer is queried from the oracle and given as the output of the Turing kernelization. In this way we obtain polynomial Turing kernels for \(k\)-PATH and the related \(k\)-CYCLE problem (is there a cycle of length at least \(k\)) in planar graphs, graphs that exclude \(K_{3, t}\) as a minor for some \(t \geq 3\), graphs of maximum degree bounded by \(t \geq 3\), and claw-free graphs. We remark that the \(k\)-PATH and \(k\)-CYCLE problems remain NP-complete in all these cases [19]. Our techniques can be adapted to construct a path or cycle of length at least \(k\), if one exists; for each of the mentioned graph classes \(\mathcal{G}\), there is an algorithm that, given a pair \((G \in \mathcal{G}, k)\), either outputs a path (respectively cycle) of...
length at least \( k \) in \( G \), or reports that no such object exists. The algorithm runs in polynomial time when given constant-time access to an oracle that decides \(-\text{PATH}\) (respectively \(-\text{CYCLE}\)) on \( G \) for instances of size and parameter bounded by some polynomial in \( k \) that depends on \( G \).

Our results raise a number of interesting challenges and shed some light on the possibility of polynomial Turing kernelization for the unrestricted \(-\text{PATH}\) problem. A completeness program for classifying Turing kernelization complexity was recently introduced by Hermelin et al. \[16\]. They proved that a colored variant of the \(-\text{PATH}\) problem is complete for a class called \( \text{WK}[1]\) and conjectured that \( \text{WK}[1] \)-hard problems do not admit polynomial Turing kernels. We give evidence that the classification of the colored variant may have little to do with the kernelization complexity of the base problem: MULTICOLORED \(-\text{PATH}\) remains \( \text{WK}[1] \)-hard on bounded-degree graphs, while our framework yields a polynomial Turing kernel for \(-\text{PATH}\) in this case.

**Related work.** Non-adaptive Turing kernels of polynomial size are known for \( k \)-\text{LEAF OUT-TREE} \[3\], \( k \)-\text{COLORFUL MOTIF} on comb graphs \[1\], and \( s \)-\text{CLUB} \[22\]. Trotignon et al. \[23\] gave an adaptive Turing kernel of polynomial size for \( k \)-\text{INDEPENDENT SET} on bull-free graphs \[23\].

**Organization.** In Section 2 we give preliminaries on parameterized complexity and graph theory. In Section 3 we present Turing kernels for the \(-\text{CYCLE}\) problem. These are technically somewhat less involved than the analogues for \(-\text{PATH}\) that are described in Section 4. While the Turing kernels are phrased in terms of decision problems, we describe how to construct solutions in Section 5. In Section 6 we briefly consider MULTICOLORED \(-\text{PATH}\).

## 2 Preliminaries

**Parameterized complexity and kernels.** The set \( \{1,2,\ldots,n\} \) is abbreviated as \([n]\). For a set \( X \) and non-negative integer \( n \) we use \( \mathbin{\binom{X}{n}} \) to denote the collection of size-\( n \) subsets of \( X \). A parameterized problem \( Q \) is a subset of \( \Sigma^* \times \mathbb{N} \), the second component of a tuple \( (x,k) \in \Sigma^* \times \mathbb{N} \) is called the parameter. A parameterized problem is (strongly uniformly) **fixed-parameter tractable** if there exists an algorithm to decide whether \( (x,k) \in Q \) in time \( f(k)|x|^{\mathcal{O}(1)} \) where \( f \) is a computable function. A **Karp kernelization algorithm** (or **Karp kernel**) of size \( f: \mathbb{N} \rightarrow \mathbb{N} \) for a parameterized problem \( Q \subseteq \Sigma^* \times \mathbb{N} \) is an algorithm that, on input \( (x,k) \in \Sigma^* \times \mathbb{N} \), runs in time polynomial in \( |x| + k \) and outputs an instance \( (x', k') \) with \( |x'|, k' \leq f(k) \) such that \( (x,k) \in Q \iff (x', k') \in Q \). It is a polynomial kernel if \( f \) is a polynomial (cf. \[4\]). We refer to a textbook \[15\] for more background on parameterized complexity.

**Graphs and tree decompositions.** All graphs we consider are finite, simple, and undirected. An undirected graph \( G \) consists of a vertex set \( V(G) \) and an edge set \( E(G) \subseteq \binom{V(G)}{2} \). We write \( G \subseteq H \) if \( G \) is a subgraph of \( H \). The subgraph of \( G \) induced by a set \( X \subseteq V(G) \) is denoted \( G[X] \). We use \( G - X \) as a shorthand for \( G[V(G) \setminus X] \). When deleting a single vertex \( v \), we write \( G - v \) rather than \( G - \{v\} \). The open neighborhood of a vertex \( v \) in \( G \) is \( N_G(v) \). The open neighborhood of a set \( X \subseteq V(G) \) is \( \bigcup_{v \in X} N_G(v) \setminus X \). The degree of vertex \( v \) in \( G \) is \( \deg_G(v) := |N_G(v)| \). A **minor model** of a graph \( H \) in a graph \( G \) is a mapping \( \phi \) from \( V(H) \) to subsets of \( V(G) \) (called **branch sets**) which satisfies the following conditions: (a) \( \phi(u) \cap \phi(v) = \emptyset \) for distinct \( u, v \in V(H) \), (b) \( G[\phi(v)] \) is connected for \( v \in V(H) \), and (c) there is an edge between a vertex in \( \phi(u) \) and a vertex in \( \phi(v) \) for all \( \{u, v\} \in E(H) \). Graph \( H \) is a **minor** of \( G \) if \( G \) has a minor model of \( H \). A vertex of degree at most one is a **leaf**. A cut vertex in a connected graph \( G \) is a vertex \( v \) such that \( G - v \) is disconnected. A pair of distinct vertices \( u, v \) is a **separation pair** in a connected graph \( G \) if \( G - \{u,v\} \) is disconnected. A vertex (pair of vertices) is a cutvertex (separation pair) in a disconnected graph if it forms such a structure for a connected component. A graph \( G \) is **biconnected** if it is connected and contains no cut vertices. The biconnected components of \( G \) partition the edges of \( G \) into biconnected subgraphs of \( G \). A graph \( G \) is triconnected if removing at most three vertices from \( G \) cannot result in a disconnected graph.\(^3\) A **separation** of a graph \( G \) is a pair \( (A,B) \) of subsets of \( V(G) \) such that \( A \cup B = V(G) \) and \( G \) has no edges between \( A \setminus B \) and \( B \setminus A \). The latter implies that \( A \cap B \) separates the vertices \( A \setminus B \) from the vertices \( B \setminus A \). The order of the separation is \( |A \cap B| \). A **minimal separator** in a connected graph \( G \) is a vertex set \( S \subseteq V(G) \) such that \( G - S \) is disconnected and \( G - S' \) is connected for all \( S' \subseteq S \). A vertex set of a disconnected graph is a minimal separator if it is a minimal separator for all the connected components.

\(^3\) Some authors require a triconnected graph to contain more than three vertices; the present definition allows us to omit some case distinctions.
A walk in $G$ is a sequence of vertices $v_1, \ldots, v_k$ such that \( \{v_i, v_{i+1}\} \in E(G) \) for \( i \in [k-1] \). An xy-walk is a walk with $v_1 = x$ and $v_k = y$. A path is a walk in which all vertices are distinct. Similarly, an xy-path is an xy-walk consisting of distinct vertices. The vertices $x$ and $y$ are the endpoints of an xy-path. An $x$-path is a path that has vertex $x$ as an endpoint. The length of a path $v_1, \ldots, v_k$ is the number of edges on it: $k - 1$. The vertices $v_2, \ldots, v_{k-1}$ are the interior vertices of the path. A cycle is a sequence of vertices $v_1, \ldots, v_k$ that forms a $v_1v_k$-path such that, additionally, the edge $\{v_1, v_k\}$ is contained in $G$. The length of a cycle is the number of edges on it: $k$. For an integer $k$, a $k$-cycle in a graph is a cycle with at least $k$ edges; similarly a $k$-path is a path with at least $k$ edges. The claw is the complete bipartite graph $K_{1,3}$ with partite sets of size one and three. A graph is claw-free if it does not contain the claw as an induced subgraph.

**Proposition 1.** If the graph $G$ contains a cycle (path) of length at least $k$ as a minor, then it contains a cycle (path) of length at least $k$ as a subgraph.

**Proof.** We first prove the statement for the case of a cycle. Suppose that $G$ contains the cycle $C$ as a minor with $\ell \geq k$. Let $\phi: V(C) \to 2^{|V(G)|}$ be a minor model of $C$ in $G$. Let $v_1, \ldots, v_\ell$ be the vertices on $C$ in the order along the cycle. We show how to construct a $k$-cycle in $G$. For each $i \in [\ell]$, let $e_i$ be an edge in $G$ that connects a vertex $l_i$ in $\phi(v_i)$ to a vertex $r_i$ in $\phi(v_{i+1 \mod \ell})$; such edges exist by the definition of a minor model. Build a cycle starting with the edge $\{l_1, r_1\} = e_1$ connecting $\phi(v_1)$ to $\phi(v_2)$. Since $\phi(v_i)$ induces a connected subgraph of $G$ for each choice of $i$, there is a path through $\phi(v_2)$ from $r_1$ to the left endpoint $l_2$ of $e_2$, from which we can follow the edge $e_2$ to its right endpoint $r_2$ in the set $\phi(v_2)$. By appending a path from $r_3$ to $l_4$ through $\phi(v_4)$ and repeating this procedure, we obtain a path from $l_1$ to $r_4$ whose interior is contained in \( \bigcup_{2 \leq i \leq 3} \phi(v_i) \). Since $r_4$ is contained in $\phi(v_1)$, we can close the path into a cycle by adding a path between $r_2$ and $l_4$ through the set $\phi(v_3)$. The obtained cycle has length at least $\ell$ since it contains all the edges $e_i$ with $i \in [\ell]$. 

For the case of a path, the argumentation is similar except that we do not add the final path through $\phi(v_1)$ to connect the path into a cycle. \( \square \)

**Definition 2.** A tree decomposition of a graph $G$ is a pair $(T, X)$, where $T$ is a tree and $X: V(T) \to 2^{|V(G)|}$ assigns to every node of $T$ a subset of $V(G)$ called a bag, such that:

(a) $\bigcup_{i \in V(T)} X(i) = V(G)$.

(b) For each edge $\{u, v\} \in E(G)$ there is a node $i \in V(T)$ with $\{u, v\} \subseteq X(i)$.

(c) For each $v \in V(G)$ the nodes $\{i \mid v \in X(i)\}$ induce a connected subtree of $T$.

The width of the tree decomposition is $\max_{i \in V(T)} |X(i)| - 1$. The adhesion of a tree decomposition is $\max_{i, j \in V(T)} |X(i) \cap X(j)|$. If $T$ has no edges, we define the adhesion to be zero. For an edge $e = \{i, j\} \in E(T)$ we will sometimes refer to the set $X(i) \cap X(j)$ as the adhesion of edge $e$. If $(T, X)$ is a tree decomposition of a graph $G$, then the torso of a bag $X(i)$ for $i \in V(T)$ is the graph $\text{TORSO}(G, X(i))$ obtained from $G[X(i)]$ by adding an edge between each pair of vertices in $X(i)$ that are connected by a path in $G$ whose internal vertices do not belong to $X(i)$.

**Tutte decomposition.** The following theorem is originally due to Tutte, but has been reformulated in the language of tree decompositions. In Appendix A we give a proof of the current formulation, for completeness.

**Theorem 1** ([24], see [12, Exercise 12.20]). For every graph $G$ there is a tree decomposition $(T, X)$ of adhesion at most two, called a Tutte decomposition, such that (i) for each node $i \in V(T)$, the graph $\text{TORSO}(G, X(i))$ is a triconnected minor of $G$, and (ii) for each edge $\{i, j\}$ of $T$ the set $X(i) \cap X(j)$ is a minimal separator in $G$ or the empty set.

An algorithm due to Hopcroft and Tarjan [17] can be used to compute a Tutte decomposition in linear time by depth-first search.\(^4\)

**Observation 1** If $(T, X)$ is a tree decomposition of adhesion at most two of a graph $G$, then for every $i \in V(T)$ the graph $\text{TORSO}(G, X(i))$ is a minor of $G$.

\(^4\) We remark that the Hopcroft-Tarjan algorithm formally computes triconnected components of a graph, rather than a Tutte decomposition; this corresponds to a variant of Tutte decomposition where each torso is either a triconnected graph or a cycle. A decomposition matching our definition easily follows from their result.
Observation 2. If \((T, \mathcal{X})\) is a tree decomposition of adhesion at most two of a graph \(G\), then for every \(i \in V(T)\), we can assign to every edge \(\{u, v\} \in E(\text{TORSO}(G, \mathcal{X}(i))) \setminus E(G)\) a distinct connected component \(C_{(u, v)}\) of \(G - \mathcal{X}(i)\) such that there exists a \(uv\)-path in \(G\) whose interior vertices belong to \(C_{(u, v)}\).

Observation 3. If \(\{x, y\}\) is a minimal separator in a planar graph \(G\), then all graphs obtained from \(G\) by adding an \(xy\)-path consisting of a single edge \(\{x, y\}\), or consisting of new interior vertices of degree two, are planar.

Proposition 2. Let \((T, \mathcal{X})\) be a Tutte decomposition of a graph \(G\). If \(\{x, y\}\) is a minimal separator of \(G\) then for every bag \(\mathcal{X}(i)\) containing \(x\) and \(y\), the edge \(\{x, y\}\) is contained in \(\text{TORSO}(G, \mathcal{X}(i))\).

Proof. If \(\{x, y\}\) \(\in E(G)\) then the proposition is trivial, so assume that this is not the case. Since \(\{x, y\}\) is a minimal separator there are at least two connected components \(C_1, C_2\) of \(G - \{x, y\}\) that are both adjacent to \(x\) and \(y\). Consequently, there is an \(xy\)-path \(\mathcal{P}_1\) with interior vertices in \(C_1\), and an \(xy\)-path \(\mathcal{P}_2\) with interior vertices in \(C_2\). Using the fact that \(\{x, y\}\) separates \(C_1\) from \(C_2\) it is easy to verify that \(\mathcal{X}(i)\) cannot contain vertices from both \(C_1\) and \(C_2\), as \(\text{TORSO}(G, \mathcal{X}(i))\) is triconnected. Hence at least one of the paths \(\mathcal{P}_1\) or \(\mathcal{P}_2\) is an \(xy\)-path with interior vertices not in \(\mathcal{X}(i)\), which shows by the definition of torso that \(\text{TORSO}(G, \mathcal{X}(i))\) contains edge \(\{x, y\}\). \(\Box\)

Circumference of restricted classes of triconnected graphs. The circumference of a graph is the length of a longest cycle. Several results are known that give a lower bound on the circumference of a triconnected graph in terms of its order. We will use these lower bounds to deduce that if a Tutte decomposition of a graph has large width, then the graph contains a long cycle (and therefore also a long path).

Theorem 2. Let \(G\) be a triconnected graph on \(n \geq 3\) vertices and let \(\ell\) be its circumference.

(a) If \(G\) is planar, then \(\ell \geq n^{\log_3 2}\). [10]
(b) If \(G\) is \(K_{3,1}\)-minor free, then \(\ell \geq (1/2)^{(t-1)n^{\log_{12} 2}}\). [11]
(c) If \(G\) is claw-free, then \(\ell \geq (n/12)^{0.753} + 2\). [2]
(d) If \(G\) has maximum degree at most \(\Delta \geq 4\), then \(\ell \geq n^{\log_r 2} / 2 + 3\), where \(r := \max(64, 4\Delta + 1)\). [9].

Running times and kernel sizes. When analyzing the running time of graph algorithms, we use \(n\) to denote the number of vertices and \(m\) to denote the number of edges. Our algorithms need information about paths and cycles through substructures of the input graph to safely reduce its size without affecting the existence of a solution. Within the framework of Turing kernelization, which is defined with respect to decision oracles, we therefore need self-reduction techniques to transform decision algorithms into construction algorithms. The repeated calls to the oracle in the self-reduction contribute significantly to the running time. In practice, it may well be possible to run a direct algorithm to compute the required information (such as the length of a longest \(xy\)-path, for given \(x\) and \(y\)) directly, thereby avoiding the repetition inherent in a self-reduction, to give a better running time. To stay within the formal framework of Turing kernelization we will avoid making assumptions about the existence of such direct algorithms, however, and rely on self-reduction. Since the running time estimates obtained in this way are higher than what would be reasonable in an implementation using a direct algorithm, running time bounds using self-reduction are not very informative beyond the fact that they are polynomial. For this reason we will content ourselves with obtaining polynomial running time bounds in this paper, without analyzing the degree of the polynomial in detail.

Similar issues exist concerning the size of the kernel, i.e., the size of the instances for which the oracle is queried. For \(k\)-Cycle on planar graphs, we give explicit size bounds (Theorem 3). For \(k\)-Cycle on other graph families, and for \(k\)-Path, we use an NP-completeness transformation to allow the path-or cycle oracle to compute structures such as longest \(xy\)-paths. These transformations blow up the size of the query instance by a polynomial factor. However, in practice one might be able to use a direct algorithm to compute this information, thereby avoiding the NP-completeness transformation and the associated blowup of kernel size. For this reason it is not very interesting to compute the degree of the polynomial in the kernel size for the cases that NP-completeness transformations are involved. For this reason we only give an explicit size bound for planar \(k\)-Cycle, where these issues are avoided.
3 Turing kernelization for finding cycles

In this section we show how to obtain polynomial Turing kernels for \textit{k-Cycle} on various restricted graph families. After discussing some properties of cycles in Section 3.1, we start with the planar case in Section 3.2. In Section 3.3 we show how to adapt the strategy for \(K_{3,t}\)-minor-free, claw-free, and bounded-degree graphs.

3.1 Properties of cycles

We present several properties of cycles that will be used in the Turing kernelization. Recall that a \(k\)-cycle is a cycle with at least \(k\) edges. The following lemma shows that, after testing one side of an order-two separation for having a \(k\)-cycle, we may safely remove vertices from that side as long as we preserve a maximum-length path connecting the two vertices in the separator.

**Lemma 1.** Let \(A, B \subseteq V(G)\) be a separation of order two of a graph \(G\) with \(A \cap B = \{x, y\}\). Let \(V(P_A)\) be the vertices on a maximum-length \(xy\)-path \(P_A\) in \(G[A]\), or \(\emptyset\) if no such path exists. If \(G\) has a \(k\)-cycle, then \(G[A]\) has a \(k\)-cycle or \(G[V(P_A) \cup B]\) has a \(k\)-cycle.

**Proof.** Assume that \(G\) has a \(k\)-cycle \(C\) with edge set \(E(C)\) and vertex set \(V(C)\). If \(V(C) \subseteq A\) then \(G[A]\) contains the \(k\)-cycle \(C\) and we are done. Similarly, if \(V(C) \subseteq V(P_A) \cup B\) then the graph \(G[V(P_A) \cup B]\) contains the \(k\)-cycle \(C\) and we are done. We may therefore assume that \(C\) contains at least one vertex \(a \in A \setminus B\) and one vertex \(b \in B \setminus A\). Since a cycle provides two internally vertex-disjoint paths between any pair of vertices on it, \(C\) contains two internally vertex-disjoint paths between \(a\) and \(b\). Since \(\{x, y\} = A \setminus B\) separates vertices \(a\) and \(b\) by the definition of a separation, each of these two vertex-disjoint paths contains exactly one vertex of \(\{x, y\}\). Hence \(E(C) \cap E(G[A])\) is the concatenation of an \(xa\) and an \(ya\) path in \(G[A]\), and therefore forms an \(xy\)-path in \(G[A]\). Since \(P_A\) is a maximum-length \(xy\)-path in \(G[A]\), the number of edges on \(P_A\) is at least \(|E(C) \cap E(G[A])|\). Replacing the \(xy\)-subpath \(E(C) \cap E(G[A])\) of \(C\) by the \(xy\)-path \(P_A\) we obtain a new cycle, since all edges of \(G[A]\) that were used on \(C\) are replaced by edges of \(P_A\). As \(P_A\) has maximum length, this replacement does not decrease the length of the cycle. Hence the resulting cycle is a \(k\)-cycle on a vertex subset of \(G[V(P_A) \cup B]\), which concludes the proof. \(\square\)

We show how to use an oracle for the decision version of \textit{k-Cycle} to construct longest \(xy\)-paths, by self-reduction (cf. [14]). These paths can be used with the previous lemma to find vertices that can be removed from the graph while preserving a \(k\)-cycle, if one exists.

**Lemma 2.** There is an algorithm that, given a connected graph \(G\) with distinct vertices \(x\) and \(y\), and an integer \(k\), either (i) determines that \(G\) contains a \(k\)-cycle, (ii) determines that \(G\) contains an \(xy\)-path of length at least \(k - 1\), or (iii) outputs the (unordered) vertex set of a maximum-length \(xy\)-path in \(G\) (or \(\emptyset\) if no such path exists). The algorithm runs in \(O((n + k)(m + n + k))\) time when given access to an oracle that decides the \(k\)-Cycle problem. The oracle is queried for instances \((G', k)\) with \(|V(G')| \leq n + k\), where \(G'\) is obtained from an induced subgraph of \(G\) by adding an \(xy\)-path that is either a single edge or consists of new vertices of degree two.

**Proof.** Given an input \((G, k, x, y)\) we proceed as follows. The algorithm first invokes the \textit{k-Cycle} oracle with the instance \((G, k)\) to query whether \(G\) has a \(k\)-cycle. If this is the case, the algorithm reports this and halts. If \(x\) and \(y\) belong to different connected components, the algorithm returns the empty set (no \(xy\)-path exists) and halts. In the remainder we therefore assume that \(G\) contains an \(xy\)-path.

The algorithm adds the edge \(\{x, y\}\) to the graph (if it was not present already) to obtain \(G_0\) and queries whether \((G_0, k)\) has a \(k\)-cycle. If this is the case, then \(G_0\) contains an \(xy\)-path of length at least \(k - 1\): since \((G_0, k)\) contains a \(k\)-cycle but \((G, k)\) does not, the edge \(\{x, y\}\) must be used in any \(k\)-cycle in \((G_0, k)\). Removing the edge \(\{x, y\}\) from a \(k\)-cycle leaves an \(xy\)-path of length at least \(k - 1\). Hence in this case we may report that \(G\) contains an \(xy\)-path of length at least \(k - 1\), according to case (ii). If \((G_0, k)\) does not have a \(k\)-cycle then it is easy to see that the maximum length of an \(xy\)-path in \(G\) is less than \(k\). The goal of the algorithm now is to identify a maximum-length \(xy\)-path. The remainder of the procedure consists of two phases: determining the maximum length and finding the path.

\textit{Determining the length.} To determine the maximum length, we proceed as follows. We iteratively create a sequence of graphs \(G_1, \ldots, G_{k-2}\) where \(G_\ell\) is obtained from \(G\) by adding \(\ell\) new vertices \(v_1, \ldots, v_\ell\).
to $G$, along with the edges $\{v_i, v_{i+1}\}$ for $i \in [\ell-1]$ and the edges $\{x, v_1\}$ and $\{y, v_1\}$. The inserted vertices, together with $x$ and $y$, form an $xy$-path of length $\ell + 1$. For each graph $G_\ell$ we invoke the oracle for $(G_\ell, k)$ to determine whether $G_\ell$ has a $k$-cycle. Let $\ell^*$ be the smallest index for which the oracle reports the existence of a $k$-cycle. This is well defined since $(G_{k-2}, k)$ must contain a $k$-cycle, for the following reason. By assumption, $G$ contains an $xy$-path of length at least one. Combining this $xy$-path with the $xy$-path of length $k-1$ through the new vertices $v_1, \ldots, v_{k-2}$ we obtain a cycle of length at least $k$. Hence $\ell^* \leq k - 2$ is well defined. Since the circumference of $G_{i+1}$ is at most the circumference of $G_i$ plus one, it follows that the circumference of $G_{\ell^*}$ is exactly $k$ and thus that any $k$-cycle in $G_{\ell^*}$ has length exactly $k$.

For every $\ell$, an $xy$-path in $G$ of length $k'$ can be combined with the newly inserted $xy$-path of length $\ell + 1$ in $G_\ell$ to create a cycle of length $k' + \ell + 1$. If $\ell^*$ is the smallest index such that $G_{\ell^*}$ has a $k$-cycle, this implies that the length of a longest $xy$-path in $G$ is $k^* := k - (\ell^* + 1) \geq 1$. Hence by querying the $k$-cycle oracle for the instances $(G_1, k)\ldots,(G_{k-2}, k)$ the algorithm determines the value of $\ell^*$ and, simultaneously, the maximum length $k^*$ of an $xy$-path in $G$.

Finding the path. Using the value of $\ell^*$ the algorithm identifies a maximum-length $xy$-path as follows. Set $H_0 := G_{\ell^*}$. We order the vertices of $H_0$ from one to $n + \ell^*$ as $u_1, \ldots, u_{n+\ell^*}$ and perform the following steps for $i \in [n+\ell^*]$. Query the oracle for $(H_{i-1} - u_i, k)$ to determine if $H_{i-1}$ has a $k$-cycle that does not use $u_i$. If the oracle answers yes, let $H_i := H_{i-1} - u_i$, otherwise let $H_i := H_{i-1}$. Since $H_0$ contains a $k$-cycle and the algorithm maintains this property, the final graph $H_{n+\ell^*}$ has a $k$-cycle. Since vertex $u_i$ is removed from $H_{i-1}$ if $H_{i-1}$ contains a $k$-cycle avoiding $u_i$, we know that for each vertex in $H_{n+\ell^*}$ there is no $k$-cycle in $H_{n+\ell^*}$ without that vertex. As $H_{n+\ell^*}$ contains a cycle of length exactly $k$, it follows that $H_{n+\ell^*}$ consists of the vertex set of a cycle of length exactly $k$; it is a Hamiltonian graph on $k$ vertices. As $G_{\ell^*-1}$ does not have a $k$-cycle, all $k$-cycles in $G_{\ell^*} = H_0$ contain the vertices $v_1, \ldots, v_{k-2}$ on the inserted $xy$-path, and therefore the graph $H_{n+\ell^*}$ contains all these vertices. Removing these $\ell^*$ vertices from $H_{n+\ell^*}$ yields the vertex set of an $xy$-path in $G$ of length $k - (\ell^* + 1) = k^*$, which is a maximum-length $xy$-path in $G$ as observed above. The vertex set is given as the output for case (iii).

Let us verify that the oracle queries made by the algorithm are of the required form. The first oracle queries are made for $G$ and for $G$ with the edge $\{x, y\}$ inserted. During the length-determining phase, all query graphs consist of $G$ with an extra $xy$-path of length at least one (on at most $k-2$ vertices) inserted. In the second phase, the query graphs consist of induced subgraphs of $H_0 = G_{\ell^*}$. Since the latter is $G$ with an extra $xy$-path, the queries indeed take the stated form. To justify the running time, note that during the first phase the algorithm queries graphs of order at most $n + k$ containing at most $m + k + 1$ edges. It stops querying when the oracle reports that $(G_\ell, k)$ has a $k$-cycle, which is after less than $k$ queries. Since a query of a graph on $n'$ vertices and $m'$ edges takes $O(n' + m')$ time (to communicate the instance to the oracle), the first phase takes $O(k(n + m + k))$ time. The second phase consists of less than $n + k$ queries, one for each vertex in $H_0$. Hence this phase takes $O((n + k)(n + m + k))$ time, which dominates the overall running time.

When the self-reduction algorithm detects a long $xy$-path for a minimal separator $\{x, y\}$, the following proposition proves that there is in fact a long cycle.

**Proposition 3.** If $\{x, y\}$ is a minimal separator of a graph $G$ and $G$ contains an $xy$-path of length $k \geq 2$, then $G$ contains a $k + 1$-cycle.

**Proof.** Assume the stated conditions hold and let $\mathcal{P}$ be an $xy$-path of length $k \geq 2$ in $G$, which implies it is not a single edge. If $\{x, y\}$ is an edge of $G$ then this edge completes $\mathcal{P}$ into a cycle of length at least $k + 1$ and we are done. Assume therefore that $\{x, y\}$ is not an edge of $G$. The interior of the $xy$-path $\mathcal{P}$, which consists of at least one vertex as $\mathcal{P}$ has length at least two, is contained entirely within one connected component $C_{G}$ of $G - \{x, y\}$. Since removal of $\{x, y\}$ increases the number of connected components (by the definition of minimal separator), there is at least one other connected component $C'$ of $G - \{x, y\}$ that is adjacent to vertex $x$ or vertex $y$. If $C'$ is adjacent to only one of $\{x, y\}$, then that vertex would be a cut vertex, contradicting minimality of the separator $\{x, y\}$. Component $C'$ is therefore adjacent to both $x$ and $y$ and therefore contains an $xy$-path $\mathcal{P}'$. Since the interior vertices on this path lie in $C' \neq C_{G}$ it follows that the concatenation of $\mathcal{P}$ and $\mathcal{P}'$ is a cycle through $x$ and $y$ of length greater than $k$, which completes the proof. \qed
3.2 $k$-Cycle in planar graphs

**Theorem 3.** The planar $k$-CYCLE problem has a polynomial Turing kernel: it can be solved in polynomial time using an oracle that decides planar $k$-CYCLE instances with at most $(3k+1)k^{\log_3 3} + k$ vertices and parameter value $k$.

**Proof.** We present the Turing kernel for $k$-CYCLE on planar graphs following the three steps of the kernelization framework.

**Decompose.** Consider an input $(G, k)$ of planar $k$-CYCLE. First observe that a cycle in $G$ is contained within a single biconnected component of $G$. We may therefore compute the biconnected components of $G$ in linear time using the algorithm by Hopcroft and Tarjan [17] and work on each biconnected component separately. In the remainder we therefore assume that the input graph $G$ is biconnected. By another algorithm of Hopcroft and Tarjan [17] we can compute a Tutte decomposition $(T, \mathcal{X})$ of $G$ in linear time. For each edge $\{i, j\} \in E(T)$ of the decomposition tree, the definition of a Tutte decomposition ensures that $\mathcal{X}(i) \cap \mathcal{X}(j)$ is a minimal separator in $G$. Since $T$ has adhesion at most two, these minimal separators have size at most two. Using the biconnectedness of $G$ it follows that the intersection of the bags of adjacent nodes in $T$ has size exactly two.

**Claim 1** If there is a node $i \in V(T)$ of the Tutte decomposition such that $|\mathcal{X}(i)| \geq k^{\log_3 3}$, then $G$ has a $k$-cycle.

**Proof.** By the definition of a Tutte decomposition, the graph $\text{TORSO}(G, \mathcal{X}(i))$ is a triconnected minor of $G$. Since planarity is closed under minors, the torso is planar. Hence the torso is a triconnected planar graph on at least $k^{\log_3 3}$ vertices, which implies by Theorem 2 that its circumference is at least $(k^{\log_3 3})^{3/2} = k$. Consequently, there is a minor of $G$ that contains a $k$-cycle. By Proposition 1 this implies that $G$ has a $k$-cycle.\hfill \Box

The claim shows that we may safely output Yes if the width of $(T, \mathcal{X})$ exceeds $k^{\log_3 3}$. For the remainder of the kernelization we may therefore assume that $(T, \mathcal{X})$ has width at most $k^{\log_3 3}$. To prepare for the reduction phase we make a copy $G'$ of $G$ and a copy $(T', \mathcal{X}')$ of the decomposition. During the reduction phase we will repeatedly remove vertices from the graph $G'$ to reduce its size. Removing these vertices from the bags of the decomposition $(T', \mathcal{X}')$, we may violate the property of a Tutte decomposition that all torsos of bags are triconnected. However, we will maintain the fact that $(T', \mathcal{X}')$ is a tree decomposition of adhesion at most two and width at most $k^{\log_3 3}$ of $G'$. We root the decomposition tree $T'$ at an arbitrary vertex to complete the decomposition phase. We use the following terminology. For $i \in V(T')$ we write $T'[i]$ for the subtree of $T'$ rooted at $i$. For a subtree $T'' \subseteq T'$ we write $\mathcal{X}'(T'')$ for the union $\bigcup_{i \in V(T'')} \mathcal{X}'(i)$ of the bags of the nodes in $T''$. For a node $i$ in the rooted tree $T'$ we write $N_{T'}^+(i)$ to denote the children (the out-neighbors) of node $i$.

**Query and reduce.** We recursively query the $k$-CYCLE oracle to reduce the size of $G'$ while preserving a $k$-cycle, if one exists. At any point in the process we may find a $k$-cycle and halt. The procedure is given as Algorithm 1. It is initially called for the root node $r$ of $T'$. Intuitively, Algorithm 1 processes the decomposition tree $T'$ bottom-up, applying Lemma 1 to justify two types of size reductions for a node $i \in V(T')$. During the first for each loop the sizes $|\mathcal{X}'(T'[j])|$ of the subtrees $T'[j]$ rooted at children $j$ of $i$ are reduced by removing vertices that are avoided by some maximum-length $xy$-path. The second for each loop applies the same lemma to reduce the number of children of $i$ by effectively deleting connected components $C$ of $G'[A] - \{x, y\}$ when a maximum-length $xy$-path can be obtained through a different component $C' \neq C$.

After the procedure terminates, we make a final call to the planar $k$-CYCLE oracle for the remaining graph $G'$ and parameter $k$. The output of the oracle is given as the output of the procedure. By the postcondition of the procedure, each modification step preserves the existence of a $k$-cycle. The oracle answer to the final reduced graph $G'$ is therefore the correct answer to the original input instance $(G, k)$. It is easy to see that the algorithm runs in polynomial time using constant-time access to the oracle: note that each iteration of the while-loop removes at least one child subtree of node $i$ from the decomposition. Assuming that the algorithm acts according to its specification, it is also easy to see that the overall approach is correct. To establish Theorem 3 it therefore remains to prove that the algorithm adheres to its specifications and that it only queries the oracle for small instances of the $k$-PATH problem on planar graphs.
Algorithm 1 QueryReduceCycle($G', (T', \mathcal{X'})$, $i, k$)

**Precondition:** $G'$ is an induced subgraph of $G$ with a tree decomposition $(T', \mathcal{X'})$ of adhesion at most two. A node $i$ of $T'$ is specified.

**Postcondition:** The existence of a $k$-cycle in $G$ is reported, or the graph $G'$ and decomposition $(T', \mathcal{X}')$ are updated by removing vertices of $\mathcal{X'}(T'[i]) \setminus \mathcal{X'}(i)$, resulting in $|\mathcal{X'}(T'[i])| \leq k \cdot |E(\text{TORSO}(G, \mathcal{X}(i)))| + |\mathcal{X}(i)|$.

1: for each child $j$ of $i$ in $T'$ do
2: QueryReduceCycle($G', (T', \mathcal{X'})$, $j, k$)
3: Let $(x, y) := \mathcal{X'}(i) \cap \mathcal{X'}(j)$, let $A' := \mathcal{X'}(T'[j])$, and let $B := (V(G') \setminus A) \cup \{x, y\}$
4: Invoke the $k$-Cycle oracle on $(G'[A], k)$ and apply Lemma 2 to $(G'[A], k, x, y)$
5: if the oracle answers YES or Lemma 2 reports an $xy$-path of length $\geq k - 1$ then
6: Report the existence of a $k$-cycle and halt

7: else
8: Let $S$ be the vertex set of the $xy$-path computed by Lemma 2
9: Remove the vertices $\mathcal{X'}(T'[j]) \setminus S$ from $G'$ and $T'$
10: for each pair $(x, y) \in \{\mathcal{X'}(i)\}_{i \neq j}$ do
11: while $\exists j', j'' \in \mathcal{N}^+_G(i)$ with $j \neq j''$ and $\mathcal{X'}(i) \cap \mathcal{X'}(j') = \mathcal{X'}(i) \cap \mathcal{X'}(j'') = \{x, y\}$ do
12: Let $A := \mathcal{X'}(T'[j']) \cup \mathcal{X'}(T'[j''])$, let $B := (V(G') \setminus A) \cup \{x, y\}$
13: Invoke the $k$-Cycle oracle on $(G'[A], k)$ and apply Lemma 2 to $(G'[A], k, x, y)$
14: if the oracle answers YES or Lemma 2 reports an $xy$-path of length $\geq k - 1$ then
15: Report the existence of a $k$-cycle and halt
16: else
17: Let $S$ be the vertex set of the $xy$-path computed by Lemma 2
18: Choose $j' \in \{j, j''\}$ such that $\mathcal{X'}(T'[j']) \setminus \{x, y\}$ contains no vertex of $S$
19: Remove $T'[j']$ from $(T', \mathcal{X'})$ and remove $\mathcal{X'}(T'[j']) \setminus \{x, y\}$ from $G'$

Claim 2 If Algorithm 1 reports a $k$-cycle, then $G$ has a $k$-cycle.

**Proof.** Clearly, if the oracle decides that $G'[A]$ has a $k$-cycle in one of the two if-statements, then since $G'$ is an induced subgraph of $G$ it follows that $G$ has a $k$-cycle. Suppose that the algorithm of Lemma 2 reports that $G'[A]$ contains an $xy$-path of length at least $k - 1$ in one of the two if-statements. Since $\{x, y\}$ is the intersection of a pair of bags in $(T', \mathcal{X'})$ and the procedure does not add vertices to any bags of the decomposition, the set $xy \subseteq \{x, y\}$ is also the intersection of the bags of adjacent nodes in the original decomposition $(T, \mathcal{X})$. It therefore forms a minimal separator in $G$ by Definition 1. Hence by Proposition 3 the graph $G$ has a $k$-cycle. $\checkmark$

Claim 3 If the input to Algorithm 1 satisfies the precondition, then the output satisfies the postcondition.

**Proof.** The postcondition makes several claims that we must verify. The correctness of reporting a cycle is already established. It is easy to see that the algorithm only removes vertices contained in the bags of subtrees rooted at children of $i$. It does not remove any vertices from bag $\mathcal{X'}(i)$ itself, since the deleted vertices explicitly exclude $\{x, y\}$ or $S \supseteq \{x, y\}$, which are the only vertices common to the bag of $i$ and the bags of subtrees.

Let us show that, upon completion, the number of vertices in bags of $T'[i]$ has indeed been reduced to $k \cdot |E(\text{TORSO}(G, \mathcal{X}(i)))| + |\mathcal{X}(i)|$. The first for each loop removes, for each child $j$ of $i$, all but $|S|$ vertices from the bags of the subtree rooted at $j$. Since $S$ is the vertex set of an $xy$-path of length less than $k$ we have $|S| \leq k$; hence each child subtree represents at most $k$ vertices of $G'$. The second for each loop repeats while there are at least two children whose bags intersect the bag of $i$ in the same set of size two. Observe that, since $G$ was initially biconnected and a recursive call to a child $j$ does not remove vertices in the intersection of $j$ to its parent, each bag of a child of $i$ must have an intersection of size exactly two with the bag of $i$; this intersection is a minimal separator in $G$ by Theorem 1. Hence upon termination, for each remaining child of $i$ there is a unique minimal separator $\{x, y\}$ contained in $\mathcal{X'}(i) = \mathcal{X}(i)$. By Proposition 2, each such minimal separator yields an edge in $\text{TORSO}(G, \mathcal{X}(i))$. Hence the number of children of $i$ is reduced to $|E(\text{TORSO}(G, \mathcal{X}(i)))|$. Since each child represents at most $k$ vertices of $G'$, while the bag $\mathcal{X'}(i) = \mathcal{X}(i)$ adds another $|\mathcal{X}(i)|$ vertices to $\mathcal{X'}(T'[i])$, it follows that $|\mathcal{X'}(T'[i])| \leq k \cdot |E(\text{TORSO}(G, \mathcal{X}(i)))| + |\mathcal{X}(i)|$. 9
Finally we show that the algorithm preserves the existence of a \( k \)-cycle. There are two points in the procedure when vertices are removed. Consider line 9, which removes the vertices of \( X'(T'[j]) \setminus S \), where \( S \) is the vertex set of a maximum-length \( xy \)-path in \( G'[A] \). At that point of the execution, we are considering the order-two separation \( A, B \) of \( G' \). Through the oracle call we have established that \( G'[A] \) does not have a \( k \)-cycle. By Lemma 1 we find that if \( G' \) has a \( k \)-cycle, then either \( G'[A] \) has a \( k \)-cycle or \( G'[S \cup B] \) has a \( k \)-cycle. Since we excluded the first option, any potential \( k \)-cycle in \( G' \) must be contained in \( G'[S \cup B] \). Removing \( X'(T'[j]) \setminus S \) therefore preserves any potential \( k \)-cycle.

Now consider the second point where vertices are removed, in line 19. The argumentation here is similar, although the separation \( A, B \) is now defined based on two child subtrees. However, since the \( xy \)-path \( S \) must be contained fully in one of the two child subtrees, removing the vertices represented by the other child subtree preserves the graph \( G'[S \cup B] \) and therefore any potential \( k \)-cycle.

\[ \diamond \]

**Claim 4** Algorithm 1 only queries the \( k \)-Cycle oracle with parameter \( k \) on planar graphs of order at most \((3k + 1)k^{\log_3 k} + k\).

**Proof.** Since we are building a Turing kernel for planar \( k \)-Cycle, the oracle can only decide instances of planar \( k \)-Cycle. The self-reduction algorithm of Lemma 2 only queries instances of the \( k \)-Cycle problem, but we must still verify that all queried instances are planar and of bounded size. Lemma 2 guarantees that all queried instances can be obtained from an induced subgraph of the input \( G'[A] \) by inserting the edge \( \{x, y\} \) or an \( xy \)-path consisting of degree two vertices. Since \( \{x, y\} \) is the intersection of two adjacent bags in \((T', X')\) and therefore also in \((T, X)\), by Theorem 1 the set \( \{x, y\} \) is a minimal separator in \( G \). Hence, by Observation 3, any graph obtained from \( G \) by adding an \( xy \)-path in the form of a single edge or new degree-two vertices, is planar. Since \( G'[A] \) is a subgraph of \( G \), all such extensions of \( G'[A] \) are subgraphs of the planar extension of \( G \) and are therefore planar. Consequently, all the graphs for which the algorithm of Lemma 2 queries the oracle, are planar. All graphs for which the oracle is queried directly from Algorithm 1 are planar induced subgraphs of the planar graph \( G' \subseteq G \).

Finally, let us bound the order of the graphs that are queried to the oracle during the execution for some node \( i \in V(T') \). Recall that the width of \((T, X')\) is at most \( k^{\log_3 k} \) and therefore that \( |X(i)| \leq k^{\log_3 k} \). Since any minor of a planar graph is planar, the graph \( \text{TORSO}(G, X(i)) \) is a planar graph on at most \( k^{\log_3 k} \) vertices. Since an \( n \)-vertex planar graph has at most \( 3n \) edges, it follows that \( |E(\text{TORSO}(G, X(i)))| \leq 3k^{\log_3 k} \). Therefore the postcondition of the algorithm guarantees that upon termination for node \( i \), the number of vertices represented by the subtree rooted at \( i \) is at most \((3k + 1)k^{\log_3 k} \). This shows that when the oracle is queried for the graph \( G'[A] \) in line 4 during the first for each loop, the order of the query graph is at most \((3k + 1)k^{\log_3 k} \). (The same bound applies when the oracle is applied to the final graph \( G' \) after the reduction procedure has finished.) When the oracle is queried for the graph \( G'[A] \) in line 13, each child subtree has already been reduced to at most \( k \) vertices by the first for each loop and consequently the queried graph has order at most \( 2k \). It remains to analyze the size of the graphs that are queried in the invocation of Lemma 2. The lemma guarantees that the number of vertices in the graphs it queries, exceeds the order of its input graph by at most \( k \). Hence the maximum order of a query graph is dominated by the invocation of Lemma 2 during the first for each loop: \((3k + 1)k^{\log_3 k} + k \). \[ \diamond \]

This concludes the proof of Theorem 3.

\[ \square \]

### 3.3 \( k \)-Cycle in other graph families

There are two obstacles when generalizing the Turing kernel for \( k \)-Cycle on planar graphs to the other restricted graph families. In the decompose step we have to ensure that each torso of the Tutte decomposition still belongs to the restricted graph family, so that Theorem 2 may be used to deduce the existence of a \( k \)-cycle if the width of the Tutte decomposition is sufficiently large. Lemma 3 is used for this purpose. In the query step we have to deal with the fact that the alterations made to the graph by the self-reduction procedure may violate the defining property of the graph class, which can be handled by using an NP-completeness transformation before querying the oracle. Besides these issues, the kernelization is the same as in the planar case.

**Lemma 3.** Let \((T, X)\) be a tree decomposition of adhesion at most two of a graph \( G \), let \( i \in V(T) \), and let \( H \) be a graph.

1. If \( G \) is claw-free, then \( \text{TORSO}(G, X(i)) \) is claw-free.
Hence the degree of $k_a$ may answer of a Tutte decomposition is not bounded by a suitable polynomial in $G$. The torso does not exceed the maximum degree $v$ since the torso does not contain any vertex of the components $C$. Each edge $E$ contains $H$ as a minor. Consequently, let $C_j$ be a component containing the interior vertices of a $uvj$-path for each $\{v, u_j\}$ in $E^*$. We argue that the components $C_j$ are all distinct. Suppose that some component $C^*$ contains the interior vertices of both a $vu_i$-path and a $vu''_i$-path for distinct $\{v, u_i\}, \{v, u''_i\} \in E^*$. Then the connected component $C^*$ of $G - X_i$ is adjacent to the three vertices $\{v, u_i, u''_i\}$. Using the connectivity property of tree decompositions it is easy to verify that this implies $T, X$ has adhesion at least three: a contradiction to the assumption that $(T, X)$ is a Tutte decomposition. Hence we may assume that the components $C_j$ for $\{v, u_j\} \in E^*$ are all distinct. For each $\{v, u_j\} \in E^*$ let $w_j$ be the successor of $v$ in a $vu_j$-path through $C_j$. Since the components $C_j$ are all distinct, the chosen vertices $w_j$ are all distinct. Since the vertices $w_j$ belong to different connected components of $G - X_i$, they are mutually non-adjacent. Since the adhesion of $(T, X)$ is at most two, no vertex of $X_i \setminus \{v, u_j\}$ is adjacent to $w_j$. Hence we may replace each edge $\{v, u_j\} \in E^*$ in the claw by $\{v, w_j\}$ to obtain a claw in $G$; a contradiction to the assumption that $G$ is claw-free.

(2). Consider an arbitrary node $i \in V(T)$ and the corresponding graph $\text{TORSO}(G, X(i))$. For an arbitrary vertex $v \in X(i)$ we will show that its degree in the torso does not exceed its degree in $G$. The edges incident on $v$ in $\text{TORSO}(G, X(i))$ are of two types: there are edges that are also present in $G[X(i)]$, and there are edges that have been added because their endpoints can be connected by a path in $G$ whose interior vertices avoid $\text{TORSO}(G, X(i))$. Since edges of the first type contribute equally to the degree of $v$ in the torso and in $G$, it suffices to prove that for each edge of the second type incident on $v$ in the torso, there is an edge incident on $v$ in $G$ that is not included in the torso.

By Observation 2, for each edge $\{v, u_j\}$ of the second type incident on $v$ in the torso, there is a distinct connected component $C_{\{v, u_j\}}$ of $G - X(i)$ such that there is a $vu_j$-path in $G$ whose interior vertices belong to $C_{\{v, u_j\}}$. Since $\{v, u_j\} \not\in E(G)$ by the definition of type, such a path contains at least one vertex from $C_{\{v, u_j\}}$. This shows that for each edge $\{v, u_j\}$ of the second type incident on $v$ in the torso, vertex $v$ has an edge to a vertex in $C_{\{v, u_j\}}$. Observe that these edges are not included in the torso, since the torso does not contain any vertex of the components $C_{\{v, u_j\}}$. Hence for each edge of the second type incident on $v$ in the torso, there is an edge incident on $v$ in $G$ that is not included in the torso. Hence the degree of $v$ in the torso is at most its degree in $G$. Consequently, the maximum degree of the torso does not exceed the maximum degree $\Delta$ of $G$.

(3). We prove the contrapositive. If $\text{TORSO}(G, X(i))$ contains $H$ as a minor, then since $G$ contains $\text{TORSO}(G, X(i))$ as a minor by Observation 1, by transitivity of the minor relation we find that $G$ contains $H$ as a minor. 

**Theorem 4.** The $k$-Cycle problem has a polynomial Turing kernel when restricted to graphs of maximum degree $t$, claw-free graphs, or $K_{3, t}$-minor-free graphs, for each constant $t \geq 3$.

**Proof.** The main procedure is similar to that of Theorem 3. Since a biconnected component of a graph $G$ is an induced subgraph of $G$ it follows that if $G$ satisfies one of the mentioned structural restrictions such as being claw-free, then all its biconnected components are also claw-free. Consequently, we may again apply the Turing kernelization algorithm to all biconnected components individually. By Lemma 3 it follows that for each mentioned restricted graph family $G$ the torso of a node $i$ of a Tutte decomposition of $G \in G$ belongs to the same family $G$. By using the subresult of Theorem 2 corresponding to the particular choice of graph class we therefore establish the required analogue of Claim 1: if the width of a Tutte decomposition is not bounded by a suitable polynomial in $k$, then $G$ has a $k$-cycle and we may answer YES. We can apply Algorithm 1 without modifications to recursively reduce the instance. The argumentation of Claim 2 still goes through to show that the algorithm is correct when it reports a $k$-cycle. Since Claim 3 did not rely on any properties of planarity for its proof, it is also valid in this case. However, Claim 4 is obviously no longer valid when the input graph is not planar. There are two
issues: the size of the queried instances, and the type of instances that are queried; these may not belong
to the restricted graph class, and therefore the oracle may not be able to answer them.

It is easy to show that the size of the queried instances is still polynomial in \( k \). The postcondition
of the algorithm ensures that the preprocessing algorithm for node \( i \) decreases the number of vertices
represented in the subtree \( T'[i] \) to \( k \cdot |E(\text{TORSO}(G,X(i))| + |X(i)| \). Since the width of the decomposition
is polynomial in \( k \) and the number of edges of a graph is quadratic in its order, the sizes of subtrees are
reduced to a polynomial in \( k \) that depends on the graph class and its parameters.

It remains to consider the type of instances that are queried to the oracle during the procedure. The
direct invocations described in Algorithm 1 concern induced subgraphs of \( G \). Since all mentioned graph
families are hereditary, these induced subgraphs obviously fall within the same graph class. The oracle
queries that are made by Lemma 2 are a potential problem, since the queried graphs are not induced
subgraphs of \( G \), but result from an induced subgraph of \( G \) by adding a new \( xy \)-path. One can show that
the degree of the queried graphs does not exceed the degree of \( G \) and that no \( K_{3,3} \)-minors (for \( t \geq 3 \)) are
created in the query graphs that were not present in \( G \). However, it may be that a query graph contains
a claw whereas \( G \) was claw-free; in particular, this can happen if the vertices \( \{x,y\} \) are connected by
an edge in \( G \). While the self-reduction of Lemma 2 can be modified to ensure that the query graphs are
claw-free if the input is claw-free, for the sake of simplicity and generality we use a different approach to
deal with this issue.

Recall that the classical version of the \( k \)-CYCLE problem is NP-complete for all restricted graph
classes mentioned in the theorem. The algorithm of Lemma 2 needs to query the oracle for the answers
to instances \((H,k)\) of the \( k \)-CYCLE problem (on an unrestricted graph). Since \( k \)-CYCLE is in NP, the
NP-completeness [19] transformation from general \( k \)-CYCLE to the \( k \)-CYCLE problem restricted to the
relevant graph class can be used to transform instance \((H,k)\) in polynomial time into an equivalent
instance \((H',k')\). Since a polynomial-time transformation cannot blow up the instance size superpoly-
nomially, the order of \( H' \) is polynomial in the order of \( H \), which is polynomial in \( k \) in all applications
of Lemma 2. We can therefore modify the algorithm as follows: whenever the algorithm tries to query
the \( k \)-CYCLE oracle, we first use the NP-completeness transformation to obtain an equivalent \( k \)-CYCLE
instance on the appropriate restricted graph class, convert it to a parameterized instance, and query
that instead. By this adaptation the oracle is only queried for instances that it can answer. The size and
parameter of the queried instances remains polynomial in \( k \). As this resolves the last issue, this completes
the proof of Theorem 4.

\[ \square \]

4 Turing kernelization for finding paths

Now we turn our attention to the \( k \)-PATH problem. While the main ideas are the same as in the \( k \)-CYCLE
case, the details are a bit more technical, for two reasons. First of all, since a path may cross several
biconnected components, we can no longer restrict ourselves to biconnected graphs and therefore the
minimal separators formed by the intersections of adjacent bags of the Tutte decomposition may now
have size one or two. Additionally, there are several structurally different ways in which a path may
cross a separation of order two and we have to account for all possible options. To query for the relevant
information, we need a more robust self-reduction algorithm. We first develop the structural claims and
self-reduction tools in Section 4.1. In Section 4.2 we present the Turing kernels.

4.1 Properties of paths

The following two statements describe how longest paths intersect separations of order one and two.

**Observation 4** Let \( A,B \subseteq V(G) \) be a separation of order one of a graph \( G \) with \( A \cap B = \{x\} \). Let \( V(P_A) \)
be the vertices on a maximum-length path \( P_A \) in \( G[A] \) that ends in \( x \). If \( G \) has a \( k \)-path, then \( G[A] \) has
a \( k \)-path or \( G[V(P_A) \cup B] \) has a \( k \)-path.

Recall that for a vertex \( x \), an \( x \)-path is a path that has \( x \) as an endpoint.

**Lemma 4.** Let \( A,B \subseteq V(G) \) be a separation of order two of a graph \( G \) with \( A \cap B = \{x,y\} \). Let \( P_1,\ldots,P_6 \)
be subgraphs of \( G[A] \) such that:

1. \( P_1 \) is a maximum-length \( x \)-path in \( G[A] - \{y\} \).
2. $P_2$ is a maximum-length $y$-path in $G[A] - \{x\}$.
3. $P_3$ is a maximum-length $x$-path in $G[A]$.
4. $P_4$ is a maximum-length $y$-path in $G[A]$.
5. $P_5$ is a maximum-length $xy$-path in $G[A]$.
6. $P_6$ consists of two vertex-disjoint paths in $G[A]$, one $x$-path and one $y$-path, such that the combined length of these paths is maximized.

If $G$ has a $k$-path, then $G[A]$ has a $k$-path or $G[\bigcup_{i=1}^6 V(P_i)] \cup G[B]$ has a $k$-path.

Proof. Consider a $k$-path $P$ in $G$. Let $P_A$ be the subgraph of $P$ consisting of the edges $E(P) \cap E(G[A])$. Similarly, let $P_B$ be the subgraph of $P$ consisting of the edges $E(P) \cap (E(G[B]) \setminus E(G[A]))$ such that every edge on $P$ is contained in exactly one of $P_A$ and $P_B$. Observe that the lemma is trivial if $P$ is contained within $G[A]$ or within $G[B]$. In the remainder we may therefore assume that $P$ contains a vertex $a \in A \setminus B$ and a vertex $b \in B \setminus A$. Since $\{x, y\}$ is the separator corresponding to the separation $(A, B)$, path $P$ must traverse at least one vertex of $\{x, y\}$ to connect $a$ and $b$. This implies that there is a vertex $z \in \{x, y\}$ such that $\deg_{P_A}(z) = \deg_{P_B}(z) = 1$; in particular, we can choose $z$ by starting at vertex $a$ and traversing the path until the first time it is about to visit a vertex in $B \setminus A$; observe that this even holds if $\{x, y\}$ is an edge of $G$ that is contained in $P$. Since the set of subgraphs $P_1, \ldots, P_6$ is symmetric with respect to $x$ and $y$, we may assume without loss of generality that $\deg_{P_A}(x) = \deg_{P_B}(x) = 1$. We proceed by a case distinction on the values $\deg_{P_A}(y)$ and $\deg_{P_B}(y)$. Observe that $\deg_{P_A}(y) + \deg_{P_B}(y) \leq 2$ since the subgraphs $P_A$ and $P_B$ partition $P$ and a vertex on a path has at most two incident edges on that path.

1. If $\deg_{P_A}(y) = \deg_{P_B}(y) = 1$:
   (a) If $P_A$ is a connected subgraph of $P$, then since the vertices $x$ and $y$ have degree one in this subgraph (their other incident edges on the path $P$ are contained in subgraph $P_B$) the subgraph $P_A$ forms an $xy$-path in $G[A]$. Replacing this $xy$-subpath of $P$ by the maximum-length $xy$-path $P_5$ in $G[A]$ we therefore obtain a path that is at least as long, proving the existence of a $k$-path in $G[\bigcup_{i=1}^6 V(P_i)] \cup G[B]$.
   (b) Now assume that $P_A$ is a disconnected subgraph of $P$. Each connected component of $P_A$ contains one of the vertices $\{x, y\}$, since $P$ is connected and these are the only vertices in $G[A]$ for which some of their incident edges in $G$ are not contained in $G[A]$. Since $\deg_{P_A}(x) = \deg_{P_A}(y) = 1$, there are exactly two connected components in $P_A$ and each component contains one of $x$ and $y$ as a degree-one vertex. (Since $P_A$ would be connected if $\{x, y\}$ would be an edge on $P$, we know that $\{x, y\} \not\in P$.) Hence one of the components of $P_A$ is an $x$-path and the other one is a $y$-path. Since the combined length of these two paths is at most the combined length of the $x$-path and $y$-path in $P_6$, we can replace $P_A$ by $P_6$ to obtain a $k$-path in $G[\bigcup_{i=1}^6 V(P_i)] \cup G[B]$.

2. If $\deg_{P_A}(y) \geq 1$, then by the case above we have $\deg_{P_B}(y) = 0$. Since $P$ can only cross the separator $\{x, y\}$ at vertex $x$ (as $\deg_{P_A}(y) = 0$), the restriction of $P$ to $G[A]$ consists of a single connected component which forms an $x$-path in $G[A]$. Since $\deg_{P_A}(y) = 0$, vertex $y$ is not used on $P_B$. Therefore we can replace the $x$-path $P_A$ in $P$ by the $x$-path $P_3$ to obtain a new path; by the maximality of $P_3$, this path is at least as long as $P$ which proves that $G[\bigcup_{i=1}^6 V(P_i)] \cup G[B]$ contains a $k$-path.
3. Otherwise we have $\deg_{P_A}(y) = 0$, which implies that $P_A$ is an $x$-path in $G[A]$ that does not contain the vertex $y$. Replacing $P_A$ by $P_1$ we therefore obtain a path that is at least as long and which is contained in $G[\bigcup_{i=1}^6 V(P_i)] \cup G[B]$. Hence the latter graph contains a $k$-path.

As the case distinction is exhaustive, this concludes the proof of Lemma 4. □

Now we turn to the self-reduction that is needed for $k$-PATH. The self-reduction procedure of Lemma 2 suffices to obtain a Turing kernel for the $k$-CYCLE case. The Turing kernelization for $k$-CYCLE queries the oracle to compute longest $xy$-paths. In the case of $k$-PATH, Lemma 4 shows that we will need other information besides just a maximum $xy$-path. To avoid having to construct ad-hoc self-reductions for the various pieces of information needed in the lemma, we give a general theorem that shows how queries to an oracle for an arbitrary NP-complete language may be used to find maximum-size subgraphs specifying certain properties. We will need the following terminology.

**Definition 3.** A 2-terminal graph is a triple $(G, x, y)$ where $G$ is a graph and $x, y$ are distinguished terminal vertices in $G$ that are not necessarily distinct. A stable 2-terminal edge property is a function $\Pi$ which takes as parameters a 2-terminal graph $(G, x, y)$ and an edge subset $Y \subseteq E(G)$ and outputs TRUE or FALSE, such that the following holds: if $\Pi((G, x, y), Y) = \text{true}$ then for any subgraph $G'$ of $G$ that contains $x, y$, and the edge set $Y$, we have $\Pi((G', x, y), Y) = \text{true}$. 

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For example, observe that the properties “the edge set Y forms a path between x and y” and “the edge set Y consists of two vertex-disjoint paths, one ending in x and one ending in y” are stable 2-terminal edge properties. The following lemma shows how to find maximum-size edge sets satisfying a stable 2-terminal edge property by self-reduction.

**Lemma 5.** Let \( \mathcal{Q} \subseteq \Sigma^* \times \mathbb{N} \) be a parameterized problem such that the classical language \( \tilde{\mathcal{Q}} := \{ x\#1^k \mid (x,k) \in \mathcal{Q} \} \) is NP-complete; here \# is a new character that is added to the alphabet. Let \( \Pi \) be a polynomial-time-decidable stable 2-terminal edge property. There is an algorithm that, given a 2-terminal graph \((G,x,y)\), computes a maximum-cardinality set \( Y \subseteq E(G) \) that satisfies \( \Pi \), or determines that no nonempty edge set satisfies \( \Pi \). The algorithm runs in polynomial time when given access to an oracle that decides instances of \( \mathcal{Q} \) with size and parameter polynomial in \(|V(G)|\) in constant time.

**Proof.** The overall proof strategy is similar to that of Lemma 2 in that we first determine the maximum cardinality of a set that has the property and then use self-reduction to find it. The difference is that we have to use the NP-completeness transformation to \( \mathcal{Q} \) to make our queries to an oracle for \( \mathcal{Q} \) rather than an oracle that decides \( k \)-CYCLE.

**Determining the maximum size.** Consider following decision problem \( L_H \): given a 2-terminal graph \((G,x,y)\) and an integer \( k \), is there an edge set \( Y \) of size at least \( k \) such that \( \Pi((G,x,y),Y) = \text{true} \)? This decision problem is contained in NP since a non-deterministic algorithm can decide an instance in polynomial time by guessing an edge set \( Y \) of size at least \( k \) and then checking whether it satisfies \( \Pi \); the latter can be done in polynomial time by our assumption on \( \Pi \). Since \( L_{H} \) is contained in NP and \( \tilde{\mathcal{Q}} \) is NP-complete by assumption, there is a polynomial-time computable function \( f \) such that for any string \( s \in \Sigma^* \) we have \( s \in L_H \) if and only if \( f(s) \in \tilde{\mathcal{Q}} \). As the transformation is polynomial-time, it cannot output a string of length superpolynomial in the input size and therefore \(|f(s)|\) is polynomial in \(|s|\). Observe that we can easily split a well-formed instance \( f(s) = z\#1^k \) of \( \tilde{\mathcal{Q}} \) into an equivalent parameterized instance \((z,k)\). Since the value of \( k \) is encoded in unary in instances of \( \mathcal{Q} \), the size \(|z|\) of the parameterized instance and its parameter \( k \) are both bounded by a polynomial in \(|s|\). Using these transformations together with the oracle for \( \mathcal{Q} \), we can obtain the answer to any instance \( s \) of \( L_H \) by querying the \( \mathcal{Q} \)-oracle for a parameterized instance of size and parameter bounded polynomially in \(|s|\).

These observations allow us to determine the maximum cardinality of an edge set satisfying \( \Pi \), as follows. Let \( m \) the number of edges in \( G \). For \( i \in [m] \) we consider the instance \( s_i = ((G,x,y),i) \) of problem \( L_H \), which asks whether \((G,x,y)\) has an edge set of size at least \( i \) satisfying \( \Pi \). We transform each instance \( s_i \) of \( L_H \) to an instance of \( \mathcal{Q} \) and query the \( \mathcal{Q} \)-oracle for the corresponding instance. Observe that the queried instances have size polynomial in \( n := |V(G)| \). If the oracle only answers \( \text{no} \) then there is no nonempty edge set satisfying \( \Pi \) and we output this. Otherwise we let \( k^* \) be the largest index for which \( f(s_i) \) is a \( \text{yes} \)-instance of \( \mathcal{Q} \), which is clearly the maximum cardinality of a subset satisfying \( \Pi \).

**Finding a maximum-cardinality set.** Using the value of \( k^* \) we use self-reduction to find a corresponding satisfying edge set of size \( k^* \). Number the edges in \( G \) as \( e_1, \ldots, e_m \). Let \( H_0 := G \). For \( i \in [m] \), perform the following steps. Query the \( \mathcal{Q} \)-oracle for the parameterized instance corresponding to \( f(((H_{i-1} - \{e_i\},x,y),k^*)) \). The oracle then determines whether the graph obtained from \( H_{i-1} \) by removing edge \( e_i \) has a set of size \( k^* \) satisfying \( \Pi \). If the oracle answers \( \text{yes} \), then define \( H_i := H_{i-1} - \{e_i\} \); otherwise let \( H_i := H_{i-1} \). Since \( H_0 = G \) has a satisfying set of \( k^* \), the procedure maintains the invariant that \( H_i \) contains a satisfying edge set of size \( k^* \). By the definition of a stable edge property, a satisfying set will remain a satisfying set even when removing an edge that is not in the set from the graph. From this it easily follows that graph \( H_m \) contains exactly \( k^* \) edges, which form an edge set satisfying \( \Pi \). The edges that remain in graph \( H_m \) are therefore given as the output of the algorithm. It is easy to see that the algorithm takes polynomial-time given constant-time access to the oracle for \( \mathcal{Q} \). It only queries instances of size and parameter polynomial in \( n \).

**4.2 k-Path in restricted graph families**

**Theorem 5.** The \( k \)-PATH problem has a polynomial-size Turing kernel when restricted to planar graphs, graphs of maximum degree \( t \), claw-free graphs, or \( K_{3,t} \)-minor-free graphs, for each constant \( t \geq 3 \).

**Proof.** We will prove that, for each choice of restricted graph class \( \mathcal{G} \), an instance \((G \in \mathcal{G}, k)\) of \( k \)-PATH can be solved in polynomial time when given access to a constant-time oracle that decides \( k \)-PATH for instances \((H \in \mathcal{G}, k')\) in which \(|V(H)|\) and \( k' \) are bounded polynomially in \( k \). Since the \( k \)-PATH problem
is NP-complete for all graph classes in the theorem statement (cf. [19]), the classical language (in which the parameter is encoded in unary) underlying the parameterized $k$-PATH problem restricted to $G$ is NP-complete in all cases. We may therefore safely use Lemma 5 during the reduction algorithm.

Since a path is contained entirely within a single connected component, by running the algorithm separately on each connected component of the input graph we may assume that the input instance $(G, k)$ is connected.

**Decompose.** We compute a Tutte decomposition $(T, X)$ of $G$ [17]. Observe that if the circumference of a graph is $k + 1$, then it contains a $k$-path. Using the same argumentation as in Claim 1 (but a different polynomial bound, given by Theorem 2) we may therefore output *yes* if the width of $(T, X)$ exceeds some fixed polynomial in $k$. We make a copy $G'$ of $G$, a copy $(T', X')$ of the decomposition, and root $T'$ at an arbitrary vertex.

**Query and reduce.** The procedure that reduces the $k$-PATH instance based on information obtained by oracle queries is given as Algorithm 2 on page 16. We use it in the same way as for $k$-CYCLE: we apply the reduction algorithm to the root node of the decomposition $(T', X')$ of $G'$ with integer $k$. If the procedure reports the existence of a $k$-path then the Turing kernelization answers *yes*. If the procedure finishes without reporting a $k$-path, we query the $k$-PATH oracle for the final reduced graph $G'$ with parameter value $k$. By the postcondition of the reduction algorithm, after it completes on the root node the number of vertices that remain in the graph is $O(k \cdot (|E()()()()()()) + |X(i)|)$. Since $|X(i)|$ is bounded by a fixed polynomial in $k$ (that depends on the graph class) in the decomposition phase and the size of a graph is obviously at most quadratic in its order, the queried instance $(G', k)$ has size polynomial in $k$.

The answer of the oracle is given as the output of the Turing kernelization algorithm. Before proving the correctness of the algorithm we discuss the procedure informally to build some intuition.

The reduction algorithm is recursive. When invoked on a node $i$ of the tree decomposition $(T', X')$ of $G'$, it starts by recursively calling itself on each child $j$ of $i$ in $T'$. When the recursive call completes, we further reduce the number of vertices in the subtree rooted at $j$ (which is guaranteed to be polynomial in $k$ by the postcondition) to $O(k)$, as follows. We consider the separation $A, B$ of $G'$ defined by the edge $(i, j)$ of the tree decomposition. Since the recursive call has shrunk the subgraph $G'[A]$ of $G'$ represented by the subtree rooted at $j$ to size polynomial in $k$, we may query the oracle for $G'[A]$. If the oracle reports that no $k$-path exists, then depending on whether the separation $A, B$ has order one or two, we reduce the graph $G'[A]$ in different ways. If the separation has order one then Observation 4 shows that all that is relevant in $G'[A]$ is a longest path that ends in the cutvertex; if it has order two then the six structures described in Lemma 4 (which have size $O(k)$ each if no $k$-path exists in $G'[A]$) are relevant. The other vertices of $G'[A]$ are discarded from $G'$ and the decomposition is updated accordingly.

The procedure above shrinks the number of vertices represented by each subtree rooted at a child of $i$ to $O(k)$. To reduce the overall number of vertices in bags of the subtree rooted at $i$, it remains to reduce number of children of $i$. The second *for each* loop considers children that have an adhesion of size one to node $i$. If there is a pair of such children with the same adhesion of size one, then to preserve the existence of a $k$-path it suffices to keep only a child that realizes a maximum-length path to the adhesion vertex; the other child is discarded. The second *for each* loop considers children that have an adhesion of size two to node $i$. If there are 13 distinct children with the same separation pair as their adhesion to $i$, then we construct the corresponding separation of $G'$ and query the oracle to find the six types of relevant substructures for paths described by Lemma 4. Observe that the vertices of each substructure are contained in the union of the bags of at most two children of $i$. This follows from the fact that all substructures contain at least one of vertex of the separating pair $(x, y)$ that forms the common adhesion, while a path that ends in, e.g., vertex $x$ can visit at most two different connected components of $G' - \{x, y\}$ since it has to traverse $y$ to move to a different component. Hence the six substructures are contained in the bags of at most twelve children of $i$, implying that at least one of the 13 children of $i$ is not used by the six crucial substructures; by Lemma 4 the algorithm may safely delete it. In this way the algorithm reduces the number of children of $i$, which results in a polynomial bound on the number of vertices in the bags of the subtree rooted at $i$.

Let us now formally establish the correctness of the procedure. Using the observations above it follows that each *while* loop discards a child of $i$ in $T'$ and therefore terminates after a polynomial number of steps. Since Lemma 5 forms a polynomial-time subroutine, the entire algorithm runs in polynomial time when given access to the $k$-PATH oracle. The following three claims establish the overall correctness of the Turing kernelization.
Algorithm 2 QueryReducePath($G', (T', \mathcal{X}')$, i, k)

Precondition: $G'$ is an induced subgraph of $G$ with a tree decomposition $(T', \mathcal{X}')$ of adhesion at most two. A node $i$ of $T'$ is specified.

Postcondition: The existence of a $k$-path in $G$ is reported, or the graph $G'$ and decomposition $(T', \mathcal{X}')$ are updated by removing vertices of $\mathcal{X}'(T'[i]) \setminus \mathcal{X}'(i)$, resulting in $|\mathcal{X}'(T'[i])| \in O(k \cdot (|E(\text{torso}(G, \mathcal{X}(i)))| + |\mathcal{X}(i)|))$. If $G'$ initially contained a $k$-path, then the deletions preserve this property.

1: for each child $j$ of $i$ in $T'$ do  
2: QueryReducePath($G', (T', \mathcal{X}')$, $j$, $k$)  
3: Let $Z := \mathcal{X}'(i) \cap \mathcal{X}'(j)$, let $A' := \mathcal{X}'(T'[j])$, and let $B := (V(G') \setminus A) \cup Z$  
4: if the $k$-PATH oracle applied to $(G'[A], k)$ answers YES then  
5: Report the existence of a $k$-path and halt  
6: else if $Z = \{x, y\}$ has cardinality two then  
7: Apply Lemma 5 to $(G'[A], x, y)$ to find $P_1, \ldots, P_6 \subseteq G'[A]$ as in Lemma 4  
8: Remove the vertices $\mathcal{X}'(T'[j]) \setminus (\bigcup_{n=1}^{6} V(P_n))$ from $G'$ and $T'$  
9: else if $Z = \{x\}$ has cardinality one (observe $|Z| \in \{1, 2\}$) then  
10: Apply Lemma 5 to $(G'[A], x, x)$ to find a longest $x$-path $P_A$ in $G'[A]$  
11: Remove the vertices $\mathcal{X}'(T'[j]) \setminus V(P_A)$ from $G'$ and $T'$  
12: for each vertex $x \in \mathcal{X}'(i)$ do  
13: while $\exists j \neq j'$ such that $\mathcal{X}'(i) \cap \mathcal{X}'(j) = \{x\}$ do  
14: Let $A := \mathcal{X}'(T'[j]) \cup \mathcal{X}'(T'[j'])$, let $B := (V(G') \setminus A) \cup \{x\}$  
15: if the $k$-PATH oracle answers YES on instance $(G'[A], k)$ then  
16: Report the existence of a $k$-path and halt  
17: else  
18: Apply Lemma 5 to $(G'[A], x, x)$ to find a maximum-length $x$-path in $G'[A]$  
19: Let $j^* \in \{j, j'\}$ such that $\mathcal{X}'(T'[j^*]) \setminus \{x\}$ contains no vertex of the $x$-path  
20: Remove $\mathcal{X}'(T'[j^*]) \setminus \{x\}$ from $G'$ and remove $T'[j^*]$ from $(T', \mathcal{X}')$  
21: for each pair $(x, y) \in (\mathcal{X}'(i))$ do  
22: while $\exists$ distinct $j_1, \ldots, j_3 \in N_G(i)$ with $\forall s: \mathcal{X}'(i) \cap \mathcal{X}'(j_s) = \{x, y\}$ do  
23: Let $A := \bigcup_{s=1}^{3} \mathcal{X}'(T'[j_s])$ and let $B := (V(G') \setminus A) \cup \{x\}$  
24: if the $k$-PATH oracle answers YES on instance $(G'[A], k)$ then  
25: Report the existence of a $k$-path and halt  
26: else  
27: Apply Lemma 5 to $(G'[A], x, y)$ to find $P_1, \ldots, P_6 \subseteq G'[A]$ as in Lemma 4  
28: Let $j^* \in \{j_1, \ldots, j_3\}$ such that $\mathcal{X}'(T'[j^*]) \setminus \{x, y\}$ contains no vertex of $\bigcup_{n=1}^{6} V(P_n)$  
29: Remove $\mathcal{X}'(T'[j^*]) \setminus \{x, y\}$ from $G'$ and remove $T'[j^*]$ from $(T', \mathcal{X}')$

Claim 5 If Algorithm 2 reports a $k$-path, then $G$ has a $k$-path.

Proof. The algorithm reports a $k$-path in various cases. However, in all these cases the $k$-PATH oracle has decided that an induced subgraph $G'[A]$ of $G'$, which itself is an induced subgraph of $G$, contains a $k$-PATH. Hence the algorithm is correct when it reports a $k$-PATH.

Claim 6 If the input to Algorithm 2 satisfies the precondition, then the output satisfies the postcondition.

Proof. There are three main items to verify: (i) that the algorithm only removes vertices of $\mathcal{X}'(T'[i]) \setminus \mathcal{X}'(i)$, (ii) that the number of vertices represented by the subtree rooted at $i$ is sufficiently small upon completion, and (iii) that the existence of a $k$-path in $G'$ is preserved (unless the algorithm reports a $k$-PATH outright). The correctness of reporting a $k$-PATH has already been established.

(i) It is easy to see that the algorithm only removes vertices contained in the bags of subtrees rooted at children of $i$. To see that it does not remove any vertices from $\mathcal{X}'(i)$, observe that the only vertices of $G$ represented in a child subtree $T'[j]$ that are also contained in bag $\mathcal{X}'(i)$ are the adhesion vertices $\mathcal{X}'(i) \cap \mathcal{X}'(j)$, of which there are at most two by the precondition. When vertices are removed in the for each loop over single vertices in $\mathcal{X}'(i)$ or in the for each loop over pairs of vertices in $\mathcal{X}'(i)$, the adhesion $x$ or $\{x, y\}$ is explicitly not removed from the graph. When vertices are removed during the first for each loop, over the children $j$ of $i$, the vertices in the subgraphs $P_1, \ldots, P_6$ (in the case of adhesion size two) or of the path $P_A$ (in the case of adhesion size one) are explicitly not removed. Since these subgraphs must contain all vertices in $\mathcal{X}'(i) \cap \mathcal{X}'(j)$, these vertices are not removed from $G'$ and therefore no vertex of $\mathcal{X}'(i)$ is removed when the procedure is executed for node $i$.  

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(ii) We now prove that, upon completion, the number of vertices in bags of $T'[i]$ has indeed been reduced to $O(k \cdot (|E(\text{TORSO}(G, X(i)))| + |X(i)|))$. In the first for each loop we remove, for each child $j$ of $i$, all vertices in the bags of the subtree $T'[j]$ except for $\bigcup_{h=1}^{b} V(P_h)$ (in the first case) or $V(P_A)$ (in the second case). Since the algorithm halts when a $k$-path is found in the graph $G'[A]$, no subgraph $P_h$ of $G'[A]$ can contain a path of length $k$ or more (in the first case), nor can $P_A$ have length more than $k$ (in the second case). But since each subgraph $P_h$ is the disjoint union of at most two paths, which each have length less than $k$, each subgraph $P_h$ has at most $2k$ vertices. Similarly, the path $P_A$ consists of at most $k$ vertices. Therefore the first for each loop reduces the number of vertices in the bags of each child subtree $T'[j]$ to $O(k)$.

Since the initial graph $G$ is connected and a recursive call to a child $j$ does not remove vertices from bags $X'(i)$, the bag of child $j$ intersects the bag of $i$ in at least one vertex; since $(X', T')$ is a decomposition of adhesion at most two, the intersection has size at most two. Therefore all children are considered during one of the last two for each loops. The for each loop over single vertices $x$ removes a child subtree whenever there are two children whose adhesion to $i$ is $\{x\}$; this reduces the number of children with a size-one adhesion to $|X'(i)| = |X(i)|$. The for each loop over pairs $\{x, y\}$ removes a child subtree whenever there are $13$ children whose adhesion to $i$ is $\{x, y\}$. By the same argumentation as in Claim 3, for each size-two adhesion there is an edge in the graph $\text{TORSO}(G, X'(i))$. Hence the number of children with a size-two adhesion is reduced to $12|E(\text{TORSO}(G, X'(i)))| = 12|E(\text{TORSO}(G, X(i)))|$. As each child subtree represents $O(k)$ vertices of $G'$, while the bag $X'(i)$ equals the bag $X(i)$, the algorithm has reduced $|X'(T'[i])|$ to $O(k \cdot (|E(\text{TORSO}(G, X'(i)))| + |X(i)|))$ upon completion.

(iii) Finally we show that the algorithm preserves the existence of a $k$-path, if it does not report one outright. There are four points in the procedure when vertices are removed. However, in all cases we are considering a separation $A, B$ of $G'$ of order at most two and are using the oracle to make sure that $G'[A]$ does not contain a $k$-Path. We then delete vertices in $A$ that are not contained in one of the six maximum-size substructures mentioned of Lemma 1, or in a maximum-length $x$-path as in Observation 4, depending on the adhesion size. The lemma and the observation show that the graph reduction preserves the existence of a potential $k$-path in $G'$.

\textbf{Claim 7} Algorithm 2 only queries the $k$-Path oracle for instances of size and parameter bounded polynomially in $k$.

\textbf{Proof.} The oracle is queried directly during the algorithm and indirectly using Lemma 5. Since the lemma guarantees that the size and parameter of its queries is polynomial in the size of its input $(G, x, y)$ it suffices to verify that the graphs to which the oracle is directly applied, and the graphs to which Lemma 5 is applied, have size polynomial in $k$. Let us first consider the first for each loop. When the oracle is invoked to $G'[A] = G'[X'(T'[j])]$ there, the algorithm has already been recursively applied to node $j$. By the postcondition, this has reduced the number of vertices in bags of $T'[j]$ to $O(k \cdot (|E(\text{TORSO}(G, X'(j)))| + |X(j)|))$. By the decomposition step, the size of each bag is bounded by a fixed polynomial in $k$, hence the size of $G'[A]$ is polynomial in $k$. This also shows that when Lemma 5 is invoked in the first for each loop, the argument has size polynomial in $k$.

In the last two for each loops, the graph $G'[A]$ which is queried to the oracle and given to Lemma 2 is the union of vertices in a constant number of child subtrees of $i$. By then, each of these has already been reduced to size $O(k)$ as described in the proof of Claim 6. Hence these queries also involve graphs of size polynomial in $k$.

This concludes the proof of Theorem 5. \hfill $\Box$

5 Constructing solutions

Motivated by the definition of Turing kernelization, we have presented our results in terms of decision problems where the goal is to answer YES or NO; this also simplified the presentation. In practice, one might want to construct long paths or cycles rather than merely report their existence. Our techniques can be adapted to construct a path or cycle of length at least $k$, if one exists.

\textbf{Corollary 1.} For each graph class $\mathcal{G}$ as described in Theorem 5, there is an algorithm that given a pair $(G \in \mathcal{G}, k)$ either outputs a $k$-cycle (respectively $k$-path) in $G$, or reports that no such object exists.
The algorithm runs in polynomial time when given constant-time access to an oracle that decides \( k \)-Cycle (respectively \( k \)-Path) on \( G \) for instances of size and parameter bounded by some polynomial in \( k \) (whose degree depends on \( G \)).

Proof. Constructing paths. Let us first consider the \( k \)-Path case. If the Turing kernelization outputs YES because a \( k \)-Path oracle gives a YES answer on an instance \((G'[A], k)\) of size polynomial in \( k \), then a straight-forward self-reduction on this small instance using Lemma 5 can be used to construct a solution (the lemma guarantees that the oracle is only queried for instances of size polynomial in \(|V(G'[A])|\)). However, the situation is more complicated when the Turing kernelization answers YES based on Theorem 2 because there is a large bag in the Tutte decomposition: applying Lemma 5 to the torso of the bag would violate the size bound on the queried instances, since the torso can be arbitrarily large. For triconnected claw-free graphs \([2, \S 5.3]\) and bounded-degree graphs \([9, \S 6]\), polynomial-time algorithms are known that construct a path of length \( n^\alpha \) for some positive \( \alpha \), which can be used to construct a \( k \)-path if the width of the Tutte decomposition exceeds \( k^{1/\alpha} \). No such algorithmic results are known for planar or \( K_{3,1} \)-minor-free graphs. However, for these graph classes we can construct long paths by exploiting the fact that they are closed under edge deletions, through a self-reduction that calls the Turing kernelization algorithm, as follows.

Let \((G, k)\) be a planar or \( K_{3,1} \)-minor-free instance that contains a \( k \)-path. Order the edges of \( G \) as \( e_1, \ldots, e_m \) and set \( G_0 := G \). For \( i \in [m] \) we apply the Turing kernelization to the instance \((G_{i-1} - e_i, k)\). If it outputs YES then we set \( G_i := G_{i-1} - e_i \), otherwise we set \( G_i := G_{i-1} \). After \( m \) calls to the Turing kernelization the resulting graph \( G_m \) contains exactly the edges of a \( k \)-path. The fact that planar and \( K_{3,1} \)-minor-free graphs are closed under edge deletions ensures that we may safely apply the Turing kernelization to all graphs \( G_i \). Since the Turing kernelization only queries the oracle for instances of size polynomial in \( k \), we obtain an algorithm that constructs a \( k \)-path in polynomial time when given access to a \( k \)-Path oracle for the restricted graph class. The oracle is only invoked for instances of size and parameter polynomial in \( k \).

Constructing cycles. We move on to the \( k \)-Cycle case. If the Turing kernelization outputs YES because the oracle gives this answer on a small instance \( G'[A] \), or because the Tutte decomposition has a bag of large width, then we may proceed similarly as in the \( k \)-Path case to construct a solution. However, there is an extra complication for \( k \)-Cycle since the Turing kernelization may output YES because its call to Lemma 2 reports the existence of a long \( xy \)-path for some minimal separator \( \{x, y\} \). Note that Lemma 2 is only applied to instances \((G'[A], x, y)\) of size polynomial in \( k \). If the existence of a long \( xy \)-path is reported, we can therefore use the self-reduction of Lemma 5 to construct the edge set of a maximum-length \( xy \)-path \( P \) by using oracle queries of size polynomial in \(|V(G'[A])|\) (which is polynomial in \( k \)). The proof of Proposition 3 easily yields a polynomial-time algorithm to construct \( P \) into a \( k \)-cycle, which handles this last case and completes the proof.

6 Multicolored Paths in Bounded-Degree Graphs

An input for the Multicolored \( k \)-Path problem consists of a graph \( G \), an integer \( k \) and a (generally not proper) coloring \( f : V(G) \rightarrow [k+1] \) of its vertices. The question is whether there is a path of length \( k \) (which must contain \( k + 1 \) vertices) that contains exactly one vertex of each color. Hermelin et al. \([16]\) showed that Multicolored \( k \)-Path is WK[1]-complete under polynomial-parameter transformations. They conjectured that WK[1]-hard problems do not have polynomial-size Turing kernels.

Theorem 6. The Multicolored \( k \)-Path problem on graphs of maximum degree at most three is WK[1]-hard.

Proof. We prove the theorem by giving a polynomial-parameter transformation \([16, \text{Definition 3}]\) from the WK[1]-hard \([16, \text{Theorem 5}]\) \( n \)-Exact Set Cover problem to Multicolored \( k \)-Path on bounded degree graphs.

Consider an input \((\mathcal{F}, U)\) of \( n \)-Exact Set Cover with consists of a size-\( m \) set family \( \mathcal{F} \) over a finite universe \( U \) of size \( n \); the question is whether there is a subfamily \( \mathcal{F}' \subseteq \mathcal{F} \) such that each element of \( U \) is contained in exactly one set of \( \mathcal{F}' \). By duplicating some sets in \( \mathcal{F} \), which does not increase the instance size by more than two, we may assume that \(|\mathcal{F}| = 2^r - 1\) for an integer \( r \).

If \( m \geq 2^r \) then the straight-forward dynamic program for \( n \)-Exact Set Cover over subsets of the universe, which runs in \( O(2^r m^{O(1)}) \) \( \subseteq O(m \cdot m^{O(1)}) \) time, takes time polynomial in \( m \) and therefore
in the input size; we may apply it and output a constant-size instance with the same answer. In the remainder we may therefore assume that \( \log m \leq n \) which implies that we may afford to increase the parameter by a polynomial in \( \log m \).

The instance of Multicolored \( k \)-Path that we construct consists of \( 2(n-1) \) complete binary trees \( O_1, I_2, O_2, I_3, \ldots, O_{n-1}, I_n \) with \( 2^r \) leaves each. We color the vertices such that all vertices that belong to a common level of a common tree have the same color, while vertices of different trees or on different levels have different colors. Since a complete binary tree with \( 2^r \) leaves consists of \( r+1 \) levels, this requires \( 2(n-1)(r+1) \) different colors. We also create a unique color \( c(u_1), \ldots, c(u_n) \) for each element of \( U \). We connect the root of tree \( I_i \) to the root of tree \( O_i \) for all \( 2 \leq i \leq n-1 \). We encode the sets of the instance as follows. For each \( j \in [2^r-1] \), for each \( i \in [n-1] \), we do the following. Let \( F_j = \{u_{i_1}, \ldots, u_{i_\ell}\} \) be the \( j \)th set in \( F \). Create a path on \( \ell \) vertices and give the \( \ell \)th vertex on this path color \( c(u_{i_\ell}) \). We make the first vertex of the path adjacent to the \( j \)th leaf of \( O_i \) and the \( j \)th leaf of \( I_{i+1} \). Additionally, we make the \( 2^r \)th leaf of \( O_i \) adjacent to the \( 2^r \)th leaf of \( I_i \). After doing this for each choice of \( i \) and \( j \) we output the resulting colored graph with the parameter \( k' := n + (2(n-1)(r+1) - 1) \in \mathcal{O}(n \cdot r) \in \mathcal{O}(n \log m) \in \mathcal{O}(n^2) \). It is easy to see that the construction can be performed in polynomial time and that the parameter \( k' \) is suitably bounded for a polynomial-parameter transformation. The maximum degree of the resulting instance is three since it is obtained by gluing paths to the leaves of binary trees.

It remains to prove that \((F, U)\) has an exact set cover if and only if \( G \) has a multicolored \( k \)-path. In one direction, suppose that there is an exact set cover with \( \ell \) sets \( F_1, \ldots, F_\ell \); observe that \( \ell \leq n \) since each set contains at least one element and no element is allowed to be covered twice. Construct a multicolored path starting from the root of \( O_1 \), moving down the tree to the leaf corresponding to set \( F_1 \), traverse the path to the corresponding leaf of \( I_2 \), move up the tree to the root of \( I_2 \), traverse the edge to the root of \( O_2 \), move down the tree to the leaf that corresponds to \( F_2 \), and so on. After the \( \ell \) sets have all been used, traverse through the remaining trees to the root of \( I_n \) by using the direct connection between the \( 2^r \)th leaves of the relevant trees. If the sets cover \( U \) exactly then, since the elements on the used subpaths correspond to the colors of the universe elements, while one vertex of each level of each binary tree is used, the resulting \( k' \)-path is multicolored.

The other direction can be proven similarly. Since each color can be used only once by a path, the linear structure of the instance forces a multicolored \( k' \)-path to start at the root of \( O_1 \), traverse down to a leaf, and use a connection that either corresponds to a set of \( F \) or to skipping a set. To use all the available colors once, the path has to traverse subpaths corresponding to sets that cover the universe exactly. This concludes the proof of Theorem 5.

The theorem shows that the Multicolored \( k \)-Path problem remains WK[1]-hard on bounded-degree graphs. However, Theorem 5 shows that the uncolored \( k \)-Path problem admits a polynomial Turing kernel on bounded-degree graphs. This indicates that the colored problem may be significantly harder to preprocess than the uncolored version.

7 Conclusion

We presented polynomial-size Turing kernels for \( k \)-Path and \( k \)-Cycle on restricted graph families using the Decompose-Query-Reduce framework, thereby answering an open problem posed by Lokshakov [20] and Misra et al. [21]. Our results form the second [23] example of adaptive Turing kernelization of polynomial size. (We remark that our results were obtained independently without knowing of the work of Thomassé et al [23].)

The question remains whether \( k \)-Path admits a polynomial-size Turing kernel in general graphs. Theorem 6 indicates that the WK[1]-hardness of Multicolored \( k \)-Path [16, Theorem 6] may not be relevant for the \( k \)-Path problem, suggesting the possibility of a positive answer. Significant new ideas will be needed to solve this case in the positive. The Tutte decomposition employed here is of little use in general graphs, since the elementary building blocks of the decomposition (triconnected graphs) do not yield anything useful. While triconnected planar graphs have circumference \( \mathcal{O}(n^a) \) for a positive constant \( a \), the circumference of a general triconnected graph may be as low as \( \mathcal{O}(\log n) \), which is achieved by considering the join of a triangle with a complete binary tree. Different decomposition methods may be used in general graphs; for example, in linear time one can either find a \( k \)-path or establish that the treedepth (and therefore treewidth) is at most \( k \), which gives a decomposition of the graph as an embedding into the closure of a rooted tree of height \( k \) (cf. [13, Theorem 8.2]). However, since the adhesion
of the corresponding tree decomposition can be linear in \( k \), this does not seem as useful for identifying irrelevant parts of the input. Analyzing \( k\text{-PATH} \) on chordal graphs may be an intermediate step: the example above shows that even for triconnected chordal graphs the circumference may be \( \Omega(\log n) \).

Our results also prompt the investigation of other subgraph and minor testing problems. For example, does the problem of testing whether \( H \) is isomorphic to a subgraph of a planar graph \( G \) admit a polynomial Turing kernel, parameterized by \( |H| \)? The simplest unresolved case of this problem seems to be the \textsc{Exact} \( k\)-\textsc{Cycle} problem of finding a cycle of length exactly, rather than at least, \( k \). The present approach fails on this problem since it is already unclear how to deal with triconnected planar graphs. Similar questions can be asked for the problem of finding a graph \( H \) as a minor in a planar graph \( G \), parameterized by \( |H| \). To further understand the nature of Turing kernelization, one might also investigate whether the adaptive Turing kernel given here can be transformed into a non-adaptive Turing kernel, whose queries only depend on the input but not on the answers to earlier queries. Since the queries in a non-adaptive Turing kernel can be executed in parallel, this might offer practical advantages.

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\begin{thebibliography}{10}

1. A. M. Ambalath, R. Balasundaram, C. R. H., V. Koppula, N. Misra, G. Philip, and M. S. Ramanujan. On the kernelization complexity of colorful motifs. In \textit{Proc. 5th IPEC}, pages 14–25, 2010. \texttt{doi:10.1007/978-3-642-17493-3_4}.

2. M. Blininski, B. Jackson, J. Ma, and X. Yu. Circumference of 3-connected claw-free graphs and large Eulerian subgraphs of 3-edge-connected graphs. \textit{J. Comb. Theory, Ser. B}, 101(4):214–236, 2011. \texttt{doi:10.1016/j.jctb.2011.02.009}.

3. D. Binkele-Raible, H. Fernau, F. V. Fomin, D. Lokshtanov, S. Saurabh, and Y. Villanger. Kernel(s) for problems with no kernel: On out-trees with many leaves. \textit{ACM Trans. Algorithms}, 8(4):38, 2012. \texttt{doi:10.1145/2344422.2344428}.

4. H. L. Bodlaender. Kernelization: New upper and lower bound techniques. In \textit{Proc. 4th IWPEC}, pages 17–37, 2009. \texttt{doi:10.1007/978-3-642-11269-0_2}.

5. H. L. Bodlaender, E. D. Demaine, M. R. Fellows, J. Guo, D. Hermelin, D. Lokshtanov, M. Müller, V. Raman, J. v. Rooij, and F. A. Rosamond. Open problems in parameterized and exact computation - IWPEC 2008. Technical Report UU-CS-2008-017, Utrecht University, 2008.

6. H. L. Bodlaender, R. G. Downey, M. R. Fellows, and D. Hermelin. On problems without polynomial kernels. \textit{J. Comput. Syst. Sci.}, 75(8):423–434, 2009. \texttt{doi:10.1016/j.jcss.2009.04.001}.

7. H. L. Bodlaender, B. M. P. Jansen, and S. Kratsch. Kernel bounds for path and cycle problems. In \textit{Proc. 6th IPEC}, pages 145–158, 2011. \texttt{doi:10.1007/978-3-642-28050-4_12}.

8. H. L. Bodlaender, B. M. P. Jansen, and S. Kratsch. Kernelization lower bounds by cross-composition. \textit{SIAM J. Discrete Math.}, 2013. To appear. \texttt{arXiv:1206.5941}.

9. G. Chen, Z. Gao, X. Yu, and W. Zang. Approximating longest cycles in graphs with bounded degrees. \textit{SIAM J. Comput.}, 36(3):635–650, 2006. \texttt{doi:10.1137/050633263}.

10. G. Chen and X. Yu. Long cycles in 3-connected graphs. \textit{J. Comb. Theory, Ser. B}, 86(1):80–99, 2002. \texttt{doi:10.1006/jctb.2002.2113}.

11. G. Chen, X. Yu, and W. Zang. The circumference of a graph with no \( K_4 \)-minor, II. \textit{J. Comb. Theory, Ser. B}, 102(6):1211–1240, 2012. \texttt{doi:10.1016/j.jctb.2012.07.003}.

12. R. Diestel. \textit{Graph Theory}. Springer-Verlag, Heidelberg, 4th edition, 2010.

13. R. Downey and M. R. Fellows. \textit{Parameterized Complexity}. Monographs in Computer Science. Springer, New York, 1999.

14. M. R. Fellows and M. A. Langston. Fast self-reduction algorithms for combinatorical problems of VLSI-design. In \textit{Proc. 3rd AWOC}, pages 278–287, 1988. \texttt{doi:10.1007/978-3-642-44039-5}.

15. J. Flum and M. Grohe. \textit{Parameterized Complexity Theory}. Springer-Verlag New York, Inc., 2006.

16. D. Hermelin, S. Kratsch, K. Soltys, M. Wahlström, and X. Wu. A completeness theory for polynomial (Turing) kernelization. In \textit{Proc. 8th IPEC}, pages 202–215, 2013. \texttt{doi:10.1007/978-3-642-39369-8_18}.

17. J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. \textit{SIAM J. Comput.}, 2(3):135–158, 1973. \texttt{doi:10.1137/0203012}.

18. J. E. Hopcroft and R. E. Tarjan. Efficient algorithms for graph manipulation [H] (algorithm 447). \textit{Commun. ACM}, 16(6):372–378, 1973. \texttt{doi:10.1145/362248.362272}.

19. M.-C. Li, D. G. Corneil, and E. Mendelsohn. Pancyclicity and NP-completeness in planar graphs. \textit{Discrete Appl. Math.}, 98(3):219 – 225, 2000. \texttt{doi:10.1016/S0166-218X(99)00163-8}.

20.

20. D. Lokshtanov. *New Methods in Parameterized Algorithms and Complexity*. PhD thesis, University of Bergen, Norway, 2009.

21. N. Misra, V. Raman, and S. Saurabh. Lower bounds on kernelization. *Discrete Optim.*, 8(1):110–128, 2011. doi:10.1016/j.disopt.2010.10.001.

22. A. Schäfer, C. Komusiewicz, H. Moser, and R. Niedermeier. Parameterized computational complexity of finding small-diameter subgraphs. *Optim. Lett.*, 6(5):883–891, 2012. doi:10.1007/s11590-011-0311-5.

23. S. Thomassé, N. Trotignon, and K. Vuskovic. Parameterized algorithm for weighted independent set problem in bull-free graphs. arXiv 1310.6205, 2013. arXiv:1310.6205.

24. W. T. Tutte. *Connectivity in graphs*. Mathematical expositions. University of Toronto Press, 1966.

25. C.-K. Yap. Some consequences of non-uniform conditions on uniform classes. *Theor. Comput. Sci.*, 26:287–300, 1983. doi:10.1016/0304-3975(83)90020-8.
A Proof of Theorem 1

Proof. The proof uses induction on the order of $G$.

Triconnected. The base case of the induction is when $G$ is a triconnected graph; note that, by our definition, the single-vertex graph is triconnected. The trivial tree decomposition $(T, X)$ where $T$ consists of a single node $i$ and $X(i) = V(G)$ is a Tutte decomposition in this case; the adhesion is zero since $T$ has no edges, while the torso $\text{TORSO}(G, X(i))$ coincides with $G$. It is triconnected by assumption and a minor of $G$ by definition.

For the induction step, we assume that the statement is true for all graphs of order less than $|V(G)|$ and proceed by a case distinction.

Disconnected. If $G$ is disconnected, then let $C_1, \ldots, C_t$ be its connected components. By induction there are Tutte decompositions $(T_1, X_1), \ldots, (T_t, X_t)$ of each connected component. Obtain a Tutte decomposition $(T, X)$ of $G$ as follows. The tree $T$ is obtained from the forest $T_1 \cup \ldots \cup T_t$ by adding arbitrary edges to make the forest connected. Each node $i$ of $T$ belongs to a unique tree $T_j$; the associated bag $X(i)$ is simply $X_j(i)$. Since, by induction, each torso $\text{TORSO}(C_j, X_j(i))$ is a triconnected minor of the connected component $C_j$ of $G$, it is also a triconnected minor of $G$. Using $X_j(i) = X(i)$ we show that $\text{TORSO}(C_j, X_j(i)) = \text{TORSO}(G, X(i))$. This follows from the fact that $C_j[X_j(i)] = G[X(i)]$, that all paths in $C_j$ connecting vertices $u, v \in X_j(i)$ with interior vertices that avoid $X_j(i)$ also exist in $G$ (since $C_j$ is a subgraph of $G$), and that no such paths exist in $G$ that do not exist in $C_j$, since no vertex of $G - C_j$ can be reached from a vertex in $C_j$ since it belongs to a different connected component. Hence the torso of each bag of $(T, X)$ is a triconnected minor of $G$. It is easy to verify that $T$ is a tree decomposition of adhesion at most two. Each nonempty intersection of the bags of adjacent nodes in the resulting decomposition was also an intersection in one of the Tutte decompositions for the connected components; hence the intersection forms a minimal separator in one of the connected components by induction.

Cut vertex. Now assume that $G$ is connected but contains a cut vertex $v$. Let $C_1, \ldots, C_t$ be the connected components of $G - v$, and for each $i \in [t]$ let $C'_i := G[V(C_i)] \cup \{v\}$. Each edge of $G$ is contained in exactly one graph $C'_i$. By induction there are Tutte decompositions $(T_1, X_1), \ldots, (T_t, X_t)$ of the graphs $C'_1, \ldots, C'_t$. Since each graph $C'_i$ contains vertex $v$, each tree decomposition $T_i$ has a node $n_i$ such that $v \in X_i(n_i)$. The tree $T$ of the Tutte decomposition $(T, X)$ is obtained from the forest $T_1 \cup \ldots \cup T_t$ by adding an edge between $n_i$ and $n_j$ for each $i \geq 2$; the bags of $T$ correspond to the bags of the individual decompositions $T_i$ as before. To see that $\text{TORSO}(G, X(i))$ is a triconnected minor of $G$ for each $i \in V(T)$, we prove that $\text{TORSO}(C'_j, X'_j(i)) = \text{TORSO}(G, X(i))$ for each choice of $j \in [t]$ and $i \in V(T_j)$. This suffices to prove the claim, since each $\text{TORSO}(C'_j, X'_j(i))$ is a triconnected minor of $C'_j$ by induction, which is a minor of $G$ since $C'_j$ is a minor of $G$. To see that $\text{TORSO}(C'_j, X'_j(i)) = \text{TORSO}(G, X(i))$ note that (i) $C'_j[X'_j(i)] = G[X(i)]$, (ii) that each path in $C'_j$ whose interior vertices avoid $X'_j(i)$ and that connects two vertices in $X'_j(i)$, also exists in $G$ (since $C'_j$ is a subgraph of $G$), and finally that (iii) all paths in $G$ connecting two vertices in $X'_j(i)$ whose interior vertices avoid $X'_j(i)$ lie entirely within $C'_j$. The latter follows from the fact that $v$ is a cut vertex, which implies that the vertices of $G - C'_j$ are not adjacent to any vertex of $C'_j$ except $v$. Hence each torso of $(T, X)$ is a triconnected minor of $G$. Again it is easy to verify that the resulting structure is a tree decomposition of adhesion at most two. The only new edges introduced into the decomposition tree are those to connect the various trees together; the intersection of such bags is the cut vertex $v$ which is a minimal separator.

Separation pair. Finally, assume that $G$ is connected and contains no cut vertices, but contains a separation pair $u, v$. Let $C_1, \ldots, C_t$ be the connected components of $G - \{u, v\}$. As $G$ contains no cut vertices, each component $C_i$ is adjacent to both $u$ and $v$. For each $i \in [t]$ let $C'_i$ be the graph obtained from $G[V(C_i)] \cup \{u, v\}$ by adding the edge $\{u, v\}$ if it did not exist already. Let $(T_1, X_1), \ldots, (T_t, X_t)$ be Tutte decompositions of $C'_1, \ldots, C'_t$, which exist by induction. Since each graph $C'_i$ contains the edge $\{u, v\}$, by property (b) of Definition 2 each decomposition $(T_i, X_i)$ has a node $n_i$ whose bag $X_i(n_i)$ contains both $u$ and $v$. As in the previous case, the tree $T$ of the Tutte decomposition $(T, X)$ of $G$ is obtained from $T_1 \cup \ldots \cup T_t$ by adding the edges $\{n_1, n_t\}$ for all $i \geq 2$. These edges do not increase the adhesion of the decomposition beyond two since the bags corresponding to their endpoints have an intersection of size exactly two consisting of $u$ and $v$, since these are the only vertices occurring in more than one graph $C'_i$.

Observe that each graph $C'_i$ is a minor of $G$: if the edge $\{u, v\}$ did not exist already, it can be created in a minor model of $G$ by repeatedly contracting vertices of a connected component other
than \(C_j\) into vertex \(u\), as each such component is adjacent to both \(u\) and \(v\). Using this property, to prove that each torso of \((T, \mathcal{X})\) is a triconnected minor of \(G\), just as in the previous case it suffices to prove that \(\text{TORSO}(C'_j, \mathcal{X}(i)) = \text{TORSO}(G, \mathcal{X}(i))\) for each \(j \in [t]\) and \(i \in V(T_j)\).

To see that \(\text{TORSO}(G, \mathcal{X}(i))\) is a subgraph of \(\text{TORSO}(C'_j, \mathcal{X}(i))\), note that \(\mathcal{X}(i) = \mathcal{X}_j(i)\) and that for each pair of vertices \(p, q\) that are connected through a nontrivial path in \(C'_j\) whose interior avoids \(\mathcal{X}_j(i)\), such a path also exists in \(G\) since \(C'_j\) minus the edge \(\{u, v\}\) (which connects two vertices in \(\mathcal{X}(i)\) and therefore cannot be used on a nontrivial path whose interior avoids \(\mathcal{X}_j(i)\)) is a subgraph of \(G\). The edge \(\{u, v\}\) itself is present in \(\text{TORSO}(G, \mathcal{X}(i))\) since a \(uv\)-path whose interior avoids \(\mathcal{X}(i)\) can be realized through any component \(C_{j'}\) with \(j' \neq j\), as \(C_{j'}\) is adjacent to both \(u\) and \(v\). Hence \(\text{TORSO}(G, \mathcal{X}(i))\) is indeed a subgraph of \(\text{TORSO}(C'_j, \mathcal{X}(i))\). To prove equality of the two torsos it remains to prove the converse.

We now prove that \(\text{TORSO}(C'_j, \mathcal{X}(i))\) is a subgraph of \(\text{TORSO}(G, \mathcal{X}(i))\). Each edge of \(G[\mathcal{X}(i)]\) is also present in \(C'_j[\mathcal{X}(i)]\). For each pair of vertices \(p, q \neq \{u, v\}\) that are connected by a path in \(G\) whose interior avoids \(\mathcal{X}(i)\), this path is contained entirely within \(C'_j\): vertices of \(G - C'_j\) are not adjacent to any vertices of \(C'_j\) except \(u\) and \(v\), since \(u, v\) is a separation pair. Hence the edge that such a path adds to torso \(\text{TORSO}(G, \mathcal{X}(i))\), is also added to \(\text{TORSO}(C'_j, \mathcal{X}(i))\). For the pair \(\{u, v\}\) itself, the connecting edge is added explicitly to \(C'_j\) by its definition, and therefore also exists in the torso. This shows that \(\text{TORSO}(C'_j, \mathcal{X}(i))\) is a subgraph of \(\text{TORSO}(G, \mathcal{X}(i))\) and therefore that the torsos are identical. By arguments above this shows that each torso of \((T, \mathcal{X})\) is a triconnected minor of \(G\). Again it is easy to show that the resulting structure is a tree decomposition of adhesion at most two.

Finally let us prove that the intersections of the bags of adjacent nodes in \(T\) form minimal separators in \(G\). For each edge \(\{a, b\}\) of \(T\) that originates from one of the trees \(T_i\) that were obtained by induction, the set \(\mathcal{X}(a) \cap \mathcal{X}(b)\) is a minimal separator in \(C'_j\). The fact that it is a minimal separator in \(G\) can be proven using the fact that the edge \(\{a, b\}\) is present in \(C'_j\), showing that \(a\) and \(b\) belong to the same connected component with respect to any separator. For edges \(\{a, b\}\) of \(T\) that were added to connect the different decomposition trees together, note that \(\{u, v\} = \mathcal{X}(a) \cap \mathcal{X}(b)\) is the separation pair that defines this case. It is a minimal separator by the assumption that \(G\) does not have a cut vertex. This concludes the case that \(G\) has a separation pair.

Since a connected graph without cut vertices or separation pairs is triconnected, any graph that is not covered by one of these cases is triconnected. It is therefore covered by the base case, which concludes the proof of Theorem 1. \(\square\)