Limitations on using the operator product expansion at small values of $x^*$

A.H. Mueller
Department of Physics, Columbia University
New York, New York 10027, USA

Abstract

Limits on the regions of $Q^2$ and $x$ where the operator product expansion can be safely used, at small values of $x$ are given. For a fixed large $Q^2$ there is an $x_0(Q^2)$ such that for Bjorken $x$-values below $x_0$ the operator product expansion breaks down with significant nonperturbative corrections occurring in the leading twist coefficient and anomalous dimension functions due to diffusion of gluons to small values of transverse momentum.

1 Introduction

For hard scattering processes involving two transverse momentum scales, one of which is the hard scattering scale, factorization and the Dokshitzer, Gribov, Lipatov, Altarelli, Parisi (DGLAP) equation[1-3] furnish the basis for a systematic description of the dependence of cross sections on the hard scale. For example, in deep inelastic lepton proton scattering the two scales are the inverse size of the proton, proportional to the QCD $\Lambda$-parameter, and the virtuality, $Q$, of the

*This research is supported in part by the Department of Energy under GRANT DE-FG02-94ER40819.
photon exchanged between the lepton and the proton. Factorization is given by the operator product expansion which separates the hard and soft scales while the DGLAP equation governs the $Q^2$-dependence.

For processes having only one (hard) transverse momentum scale the Balitsky, Fadin, Kuraev, Lipatov (BFKL) equation [4,5] determines the center of mass energy dependence when the hard transverse momentum scale is held fixed. BFKL effects also contribute to two-scale processes like deep inelastic scattering, and these effects are accounted for through a resummation of certain higher order terms in the anomalous dimension and coefficient functions which occur in the operator product expansion [6-12].

However, even for processes having only one transverse momentum scale the BFKL equation ultimately breaks down, at very high energies, because gluons diffuse away from the hard scale and reach transverse momenta of size $\wedge$ where perturbation theory no longer applies [5,13]. It can be expected that a similar phenomenon will occur when BFKL effects are included in the DGLAP equation through resummation. For deep inelastic scattering at moderate values of $x$ the leading twist terms in the operator product expansion represent the structure functions so long as $Q/\wedge >> 1$. However, if $Q/\wedge$ is fixed at some large value the operator expansion breaks down for sufficiently small values of $x$ because of diffusion, in both the anomalous dimension and coefficient functions, into regions of transverse momentum on the order of $\wedge$. This breakdown of the operator product expansion is not due to higher twist terms, but rather to an inability to properly separate hard and soft scales at very small values of $x$. Of course within a region of $x$, say $x_0 < x < 1$, one can always choose a $Q(x_0)$ such that if $Q > Q(x_0)$ then the operator product expansion applies.

As we shall see a little later on (See (40)) this region is determined by $\ell n Q(x_0)/\wedge \geq [7N_c\zeta(3)/\pi b]\ell n \bar{x}/x]^{1/3}$, with $b = [11N_c - 2N_f]/12\pi$, when a resummation of leading logarithmic terms is taken into account. $\bar{x}$ is a fixed parameter, $\bar{x} \approx 1/10$ perhaps, not determined in our analysis.

Our model of diffusion into the infrared is the fixed-coupling BFKL equation. Because the fixed-coupling BFKL equation underestimates diffusion into the infrared, and overestimates diffusion into the ultraviolet, our result on the region of applicability of the operator product expansion, (40), should be taken with some care. This uncertainty is implicit in (40) in that we are unable to specify the scale of the running coupling in D. In the limit for $\ell n Q(x_0)/\wedge$ given just above we have
taken \( \alpha = \alpha(Q(x_0)) \) in (40).

## 2 BFKL evolution and its limitations

The BFKL equation naturally applies to high energy cross sections where there is only a single transverse momentum scale, \( Q \), if that scale is hard enough so that the running coupling, \( \alpha(Q) \), is small. While there are a number of processes where BFKL evolution can be directly measured, for our purposes the heavy onium-heavy onium total cross section is the simplest to consider. The scale \( Q \) is determined by the inverse heavy onium radius. The cross section is given\[14\], in leading logarithmic approximation, by

\[
\sigma(Y) = \int d^2x \int_0^{1} dz \int d^2x' \int_0^{1} dz' \Phi(x', z') f(x', Y, x) \Phi(x, z) \tag{1}
\]

where \( \Phi(x, z) \) is the square of the light-cone wave function of the onium with \( x \) the transverse coordinate separation of the heavy quark and heavy antiquark while \( z \) is the longitudinal momentum fraction of the heavy quark. \( Y = \ell \ln s/M^2 \) with \( s \) the center of mass energy squared and \( M \) the mass of the onium. The BFKL impact parameter amplitude, \( f(x', Y, x) \), is given in terms of the momentum space amplitude by

\[
f(x', Y, x) = \int \frac{d^2k'}{(k')^2} \frac{d^2k}{k^2} e^{ik \cdot x - i(k') \cdot x'} f(k', Y, k). \tag{2}
\]

Asymptotically,

\[
f(x', Y, x) \sim 4 \pi x x' \alpha^2 \frac{e^{(\alpha_P - 1)Y}}{\sqrt{2} \alpha N_c \zeta(3) Y} e^{-\frac{\ell n^2 x'/x}{4DY}} \tag{3}
\]

and

\[
f(k', Y, k) \sim \frac{\alpha^2}{\pi k k'} \frac{e^{(\alpha_P - 1)Y}}{\sqrt{2} \alpha N_c \zeta(3) Y} e^{-\frac{\ell n^2 k'/k}{4DY}} \tag{4}
\]

where \( \alpha_P - 1 = \frac{4\alpha N_c}{\pi} \ell n \) and \( D = \frac{7\alpha N_c \zeta(3)}{2\pi} \). Using (3) in (1) gives
\[ \sigma \sim 16\pi R^2 \alpha^2 \frac{e^{(\alpha_P - 1)Y}}{\sqrt{2} \alpha N_c \zeta(3) Y} \] (5)

with R the radius of the heavy onium.

Eq.(5) breaks down at large Y for two separate reasons. (i) Even though the heavy onium state has a small radius, R, the gluons responsible for the growth of (3) and (4) with increasing Y diffuse to larger distances, determined by the final exponential factors in (3) and (4). When the diffusion reaches momenta as small as the QCD \( \Lambda \)-parameter the whole perturbative approach breaks down. (ii) When Y becomes large the cross section clearly grows faster than unitarity allows so that new corrections must become important which modify simple BFKL behavior.

Consider first the question of diffusion. Eqs.3 and 4 have been derived neglecting the running of the QCD coupling. Let \( k_0 = 1/R \). Then the minimum momentum to which gluons diffuse in an onium-onium collision in \( k_m \) given by

\[ \ell n^2 k_o/k_m = D Y \] (6)

as determined from (4) taking \( [exp\{-\frac{\ell n^2 k_0/k_m}{4D Y/2}\}]^2 = 1/e \) because diffusion is maximum at rapidities midway between the two onia. (We shall give a more complete derivation of (6) later on.) Then the condition for the running of the coupling to be negligible over the region where gluon momenta diffuse is

\[ \frac{\alpha(k_m) - \alpha(k_0)}{\alpha(k_m)} \ll 1 \] (7)

which leads to

\[ Y \ll \frac{\pi}{14N_c \zeta(3)b^2} \frac{1}{\alpha^3(k_0)} \] (8)

with \( b = \frac{11N_c - 2N_f}{12\pi} \) and where we have used \( \alpha(k) = \frac{1}{b k^2/\lambda^2} \).

If one is willing to use a running coupling in the BFKL equation, rapidity values

\[ Y \leq \frac{\pi}{14N_c \zeta(3)b^2} \frac{1}{\alpha^3(k_0)} \] (9)
can be reached. One cannot go beyond the limit set in (9) without having $k_m$-values less than $\wedge$.

The limit (8), or (9), is not too bad because unitarity constraints are expected to be reached, from (5), when rapidities become of size

$$Y \leq \frac{2}{\alpha P - 1} \ell n \frac{1}{\alpha(k_0)},$$

(10)

For $\alpha(k_0)$ very small unitarity constraints become important long before diffusion constraints. Thus questions of unitarity can in principle be studied completely within the fixed coupling approximation[15] for sufficiently heavy onia. However, when $\alpha(k_0)$ is of moderate size the diffusion limit (9) may be reached before unitarity corrections are important. In our leading logarithm discussion the limits (9) and (10) are comparable at $\alpha(k_0)$ is of size $1/3$ to $1/2$ although numerical studies suggest that one in fact must go significantly beyond (10) to see strong unitarity corrections[15]. Thus in most practical circumstances diffusion constraints will be reached before unitarity constraints.

3 Incorporating BFKL evolution in a DGLAP formalism

The asymptotic behavior (4) follows directly from the representation[4,5,16]

$$T(Y, Q/\mu) = \int \frac{d\lambda}{2\pi i} exp\{\frac{2\alpha N_c}{\pi} \chi(\lambda) Y + \lambda \ell n \frac{Q^2}{\mu^2}\}$$

(11)

where

$$T(Y, Q/\mu) = \frac{\pi Q^2}{4\alpha^2} f(Q, Y, \mu),$$

(12)

with

$$\chi(\lambda) = \psi(1) - \frac{1}{2} \psi(\lambda) - \frac{1}{2} \psi(1 - \lambda)$$

(13)

and where the path of integration in (11) goes from $\lambda = \lambda_0 - i \infty$ to $\lambda_0 + i \infty$ with $0 < \lambda_0 < 1$. In order to cast this into a form corresponding to the operator product expansion it is necessary to pick out the leading twist part of (11). If we take $Q/\mu > 1$ then higher
powers of \((\mu/Q)^2\) can discarded simply by changing the \(\lambda\)-integration in (11) to a circle, of radius less than 1, about the origin. Call this contour \(C\). Then the leading twist part of \(T\), denoted by \(L\), is

\[
L(Y,Q/\mu) = \int_C \frac{d\lambda}{2\pi i} \exp\left\{ \frac{2\alpha N_c}{\pi} \chi(\lambda)Y + \lambda \ln Q^2/\mu^2 \right\}
\]

\[
= \int \frac{dn}{2\pi i} C_n \exp\{\gamma_n \ln Q^2/\mu^2 + (n-1)Y\}
\]  

(14)

and determines the anomalous dimension and coefficient functions\[17\]. The \(n\)-integration runs parallel to the imaginary axis and to the right of the point \(n=1\). (The anomalous dimension, \(\gamma_n\), is unique in the leading logarithmic BFKL approximation. The coefficient function, \(C_n\), is not unique but the choice above, where matrix elements are normalized to 1 at \(\mu\), is natural and convenient for our purposes.) We note that when \(Y\) is large subleading twist terms are also subleading in \(Y\) since \(\lambda = 1/2\) is the largest of all the saddle points of \(\chi(\lambda)\).

Using (14) it is straightforward to find\[16\]

\[
\gamma_n = \chi^{-1}\left(\frac{\pi(n-1)}{2\alpha N_c}\right)
\]  

(15)

and

\[
C_n = \frac{d\gamma_n}{dn}
\]  

(16)

where \(\chi^{-1}\) is the function inverse to \(\chi\). If one writes

\[
\gamma_n = \sum_{N=1}^{\infty} \gamma^{(N)}\left(\frac{\alpha N_c}{\pi(n-1)}\right)^N
\]  

(17)

and

\[
C_n = \sum_{N=0}^{\infty} C^{(N)}\left(\frac{\alpha N_c}{\pi(n-1)}\right)^N \frac{1}{n-1},
\]  

(18)

then (16) gives \(C^{(N)} = N\gamma^{(N)}\). Using (14) one finds

\[
C^{(N)} = N! \sum_{k=0}^{N-1} \frac{1}{k!(k+1)!} \frac{\alpha N_c}{(N-1-k)!(N-1-k)!} \left\{\frac{d^k}{d\lambda^k}[\rho(\lambda)]^{N-1-k}\right\}_{\lambda=0}
\]  

(19)
with \( C^{(0)} = 1 \) and
\[
\rho(\lambda) = 2\chi(\lambda) - \frac{1}{\lambda}. \tag{20}
\]
(Eq.19 is most easily found by setting \( Q = \mu \) in (14), and then taking the term \( Y^N \) on each side of that equation.)

The coefficient function and anomalous dimension have a singularity at \( n = \alpha_P \) corresponding to the saddle point of \( \chi(\lambda) \) at \( \lambda = 1/2 \). One can determine the behavior of \( C_n \) and \( \gamma_n \) near \( n = \alpha_P \) either from (15) and (16) using
\[
\chi(\lambda) = 2\ell n 2 + 7\zeta(3)(\lambda - \frac{1}{2})^2 + \cdots \tag{21}
\]
or from (17) and (18) using
\[
C^{(N)} \underset{N \to \infty}{\sim} \frac{(4\ell n 2)^N}{\sqrt{N}} \frac{\ell n 2}{\sqrt{14\pi\zeta(3)}}. \tag{22}
\]
Near \( n = \alpha_P \) one finds[4,5,6-8,17]
\[
C_N \approx \frac{1}{4\sqrt{D(n - \alpha_P)}}, \tag{23}
\]
and
\[
\gamma_n - \frac{1}{2} \approx -\frac{1}{2} \sqrt{\frac{D}{D(n - \alpha_P)}}. \tag{24}
\]

We are now in a position to see more clearly how diffusion contributes to \( L \). It is convenient to separate the non-diffusion parts of \( L \) according to
\[
L = \frac{1}{2} \tilde{L}(Q/\mu) \ e^{(\alpha_P - 1)Y} \tag{25}
\]
where we have normalized \( \tilde{L} \) to be a probability distribution. Then,
\[
\tilde{L}(Y, Q/\mu) = \frac{1}{2} \int \frac{dn}{2\pi i} \ C_n \exp\{ (\gamma_n - \frac{1}{2})\ell n Q^2/\mu^2 + (n - \alpha_P)Y \} \tag{26}
\]
obeys, at large \( Y \), the diffusion equation[4,5,13]
\[
\left( \frac{\partial}{\partial Y} - D \frac{\partial^2}{\partial \ln^2 Q/\mu} \right) \tilde{L} = 0,
\]

(27)
as is easily seen using (24) in (26). The solution is

\[
\tilde{L}(Y, Q/\mu) = \exp\left\{ -\frac{\ln^2 Q/\mu}{4DY} \right\}
\]

(28)

which, using (25) and (12), gives (4),

4 Limitations on the use of DGLAP evolution at small x

Our normal picture of DGLAP evolution is not one of diffusion but of a monotonic increase of \(\ln Q/\mu\) with \(Y\). In order to see where the diffusion is hidden write \(L\), given in (14), as

\[
L(Y, Q/\mu) = \int_0^Y dy A(y, Q/\mu) C(Y - y)
\]

(29)

where asymptotically,

\[
C(Y) = \int \frac{dn}{2\pi i} C_n e^{(n-1)Y} \sim \frac{e^{(\alpha_P-1)Y}}{\sqrt{16\pi DY}}
\]

and

\[
A(Y, Q/\mu) = \int \frac{dn}{2\pi i} e^{\gamma_n \ln Q^2/\mu^2 + (n-1)Y}
\]

\[
\sim \frac{Q}{\mu Y} \frac{e^{(\alpha_P-1)Y}}{\sqrt{16\pi DY}} \exp\left\{ -\frac{\ln^2 Q/\mu}{4DY} \right\}
\]

(30)

A straightforward calculation shows that the values of \(y\) in (29) which dominate the integral are

\[
y \propto Y \left( 1 + \frac{\ln^2 Q/\mu}{4DY} \right).
\]

Thus, for \(Y \gg \frac{\ln^2 Q/\mu}{4DY}\) the coefficient function covers most of the rapidity region in \(\tilde{L}\), as given in (29), so that for very large values of \(Y\) most of the diffusion is hidden in \(C\).
However, this is not the whole story. There is also significant diffusion in $A$ despite the fact that the DGLAP equation

$$\frac{\partial A(Y, Q/\mu)}{\partial \ln Q/\mu} = \int_0^Y dy \gamma(y)A(Y - y, Q/\mu).$$

(31)

seems to suggest increasing $Q$-values and increasing $Y$-values go together. To see where the diffusion is hidden it is convenient to write

$$A = \frac{Q}{\mu}e^{(\alpha - 1)Y} \tilde{A}(Y, Q/\mu)$$

(32)

So that $\tilde{A}$ obeys

$$\frac{\partial \tilde{A}}{\partial \ln Q/\mu} = \int_0^Y dy(\gamma - 1/2)(y)\tilde{A}(Y - y, Q/\mu)$$

(33)

with

$$(\gamma - \frac{1}{2})(Y) = \int \frac{dn}{2\pi i}(\gamma_n - \frac{1}{2})e^{(n - \alpha P)Y}.$$  

(34)

It is now straightforward to see that the $y$-values dominating (33) are given by

$$y \propto \frac{4DY}{\ell n^2 Q/\mu + 4DY} Y.$$  

(35)

Thus when $Y \geq \frac{\ell n^2 Q/\mu}{4DY}$ the action of the anomalous dimension is very nonlocal in the DGLAP equation (33). This allows the diffusion, which according to (29) is certainly contained in $A$, to be hidden in the anomalous dimension $\gamma$.

To see the general limitations on what values of $Y$ allow a consistent operator product expansion, and a DGLAP formalism, we note that

$$L(Y, Q/\mu) = \int_0^\infty \frac{dk^2}{k^2}L(Y - y, Q/k)L(y, k/\mu).$$

(36)

From (36) one determines , using (25) and (28), that the smallest important values of $k$ in (36) are reached when one takes

$$y = y_m = \frac{Y}{2}(1 - \frac{\ell n Q/\mu}{\sqrt{4DY}})$$

(37)
with the effective minimum value, $k_m$, given by

$$\ell n^2 \frac{Q\mu}{k_m^2} = 4DY. \quad (38)$$

Taking $k_m = \wedge$ as the minimum allowed value for $k_m$ before the whole formalism breaks down we get the restriction

$$Y \leq \frac{1}{4D} (\ell n^2 \frac{Q\mu}{\wedge^2} = \frac{1}{4D} (\ell n Q/\mu + \ell n \mu^2/\wedge^2)^2. \quad (39)$$

If $\ell n \mu^2/\wedge^2 << \ell n Q/\mu$ one finds $Y \leq \frac{1}{4D} \ell n^2 Q/\mu$ and the breakdown occurs because of diffusion in the high order corrections to the anomalous dimension. If $\ell n Q/\mu << \ell n \mu^2/\wedge^2$ one finds $Y \leq \frac{1}{4D} \ell n^2 \mu^2/\wedge^2$ and the breakdown occurs in the coefficient function or, in more general schemes, in a combination of the coefficient function and the operator matrix element. In applying (39) to deep inelastic scattering one should identify $\mu$ with the transverse momentum scale where perturbation starts to apply while $Y = \ell n \bar{x}/x$ where $\bar{x}$ as the x-value where soft gluon emission starts to become important. As a rough estimate we take $\mu = \wedge$ and we suppose $\bar{x} \approx 1/10$. Then

$$\ell n \bar{x}/x \leq \frac{1}{4D} \ell n^2 Q/\wedge \quad (40)$$

gives the region of $x$ where the operator product expansion applies. (At our level of discussion we are unable to fix the scale at which $\alpha$ should be evaluated in D. $\alpha = \alpha(Q)$ would certainly give an upper bound in (40).)

If one wishes to use the DGLAP equation to evolve from $Q_0$ to $Q$ one must satisfy (39) with $\mu = Q_0$. However, the stronger constraint is that the operator product expansion be valid at $Q_0$, that is one needs $\ell n \bar{x}/x \leq \frac{1}{4D} \ell n^2 Q_0/\wedge$ for x-values which are important at $Q_0$ in determining the parton distribution at the desired small value of x. In case one chooses $Q_0 \geq Q$ then the strongest constraint is (40), the requirement that the operator product expansion apply at the x and $Q^2$ value one is interested in.

At the level we have calculated no breakdown of the operator product expansion is yet visible[10,11]. However, once momenta as low as $\wedge$ occur, and that will happen when (39) is not satisfied, higher order running coupling corrections will not be small. This signifies the fact that the anomalous dimension and/or the coefficient function cannot
be calculated perturbatively. Since this breakdown of perturbative oc-
curs at the leading twist level it corresponds to the breakdown of the
operator product expansion itself. Also, since our discussion concerns
leading twist effects this phenomena is different from the appearance
of renomalons. The breakdown of the perturbation series when higher
order running coupling effects are included will here not be associ-
ated with an n! growth of the perturbation series, but simply with
the fact that higher order running coupling corrections are not small.
It is likely that diffusion puts significant restrictions on the use of a
DGLAP analysis of present small-x and small-$Q^2$ deep inelastic scat-
tering data. At the same time we are presented with the interesting
challenge of how to analyze moderate $Q^2$ and very low x phenomena.

5 References

[1] Yu.L. Dokshitzer, Sov. Phys. JETP 73, (1977) 1216.
[2] V.N. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 78.
[3] G. Altarelli and G. Parisi, Nucl. Phys. B126 (1977) 298.
[4] E.A. Kuraev, L.N. Lipatov and V.S. Fadin, Sov. Phys. JETP 45
(1978) 199.
[5] Ya.Ya. Balitsky and L.N. Lipatov, Sov. J. Nucl. Phys. 28 (1978) 22.
[6] S. Catani, M. Ciafaloni and F. Hautmann, Nucl. Phys. B366 (1991) 135.
[7] S. Catani and F. Hautmann, Nucl. Phys. B27 (1994) 475.
[8] J.C. Collins and R.K. Ellis, Nucl. Phys. B360 (1991) 3.
[9] R.K. Ellis, F. Hautmann and B.R. Webber, Phys. Lett. B348 (1995) 582.
[10] R.D. Ball and S. Forte, Phys. Lett. B351 (1995) 313; Phys. Lett. B358 (1995) 365.
[11] J.R. Forshaw, R.G. Roberts and R.S. Thorne, Phys. Lett. B356 (1995) 79.
[12] R.S. Thorne, hep-ph/9610334 (1996).
[13] J. Bartels and H. Lotter, Phys. Lett. B309 (1993) 400.
[14] A.H. Mueller, Nucl. Phys. B437 (1995) 107.
[15] G. Salam, Nucl. Phys. B461 (1996) 512.

[16] A. Bassetto, M. Ciafaloni and G. Marchesini, Phys. Reports 100 (1983) 201.

[17] T. Jaroszewicz, Phys. Lett. B116 (1982) 291.