A CHARACTERIZATION OF THE $n$-ARY
MANY-SORTED CLOSURE OPERATORS AND A
MANY-SORTED TARSKI IRREDUNDANT BASIS
THEOREM

J. CLIMENT VIDAL AND E. COSME LLÓPEZ

Abstract. A theorem of single-sorted algebra states that, for a
closure space $(A, J)$ and a natural number $n$, the closure operator
$J$ on the set $A$ is $n$-ary if, and only if, there exists a single-sorted
signature $\Sigma$ and a $\Sigma$-algebra $A$ such that every operation of $A$
is of an arity $\leq n$ and $J = Sg_A$, where $Sg_A$ is the subalgebra
generating operator on $A$ determined by $A$. On the other hand,
a theorem of Tarski asserts that if $J$ is an $n$-ary closure operator
on a set $A$ with $n \geq 2$, and if $i < j$ with $i, j \in \text{IrB}(A, J)$, where
$\text{IrB}(A, J)$ is the set of all natural numbers $n$ such that $(A, J)$ has
an irredundant basis ($\equiv$ minimal generating set) of $n$ elements,
such that $\{i + 1, \ldots, j - 1\} \cap \text{IrB}(A, J) = \emptyset$, then $j - i \leq n - 1$.
In this article we state and prove the many-sorted counterparts of
the above theorems. But, we remark, regarding the first one under
an additional condition: the uniformity of the many-sorted closure
operator.

1. Introduction.

A well-known theorem of single-sorted algebra states that, for a
closure space $(A, J)$ and a natural number $n \in \mathbb{N} = \omega$, the closure operator
$J$ on the set $A$ is $n$-ary if, and only if, there exists a single-sorted sig-
nature $\Sigma$ and a $\Sigma$-algebra $A$ such that every operation of $A$
is of an arity $\leq n$ and $J = Sg_A$, where $Sg_A$ is the subalgebra generating operator on $A$ determined by $A$. On the other hand, in [3], it was stated
that, for an algebraic many-sorted closure operator $J$ on an $S$-sorted
set $A$, $J = Sg_A$ for some many-sorted signature $\Sigma$ and some $\Sigma$-algebra $A$ if, and only if, $J$ is uniform. And, by using, among others, the just
mentioned result, our first main result is the following characterization
of the $n$-ary many-sorted closure operators: Let $S$ be a set of sorts, $A$
an $S$-sorted set, $J$ a many-sorted closure operator on $A$, and $n \in \mathbb{N}$.
Then $J$ is $n$-ary and uniform if, and only if, there exists an $S$-sorted

signature $\Sigma$ and a $\Sigma$-algebra $A$ such that $J = \text{Sg}_A$ and every operation of $A$ is of an arity $\leq n$.

We turn next to Tarski’s irredundant basis theorem for single-sorted closure spaces. But before doing that let us begin by recalling the terminology relevant to the case. Given an $n \in \mathbb{N}$, a set $A$, and a closure operator $J$ on $A$, the closure operator $J$ is said to be an $n$-ary closure operator on $A$ if $J = J_{\leq n}^n$, where $J_{\leq n}^n$ is the supremum of the family $(J_{\leq n}^m)_{m \in \omega}$ of operators on $A$ defined by recursion as follows: for $m = 0$, $J_{\leq n}^0 = \text{Id}_{\text{Sub}(A)}$; for $m = k + 1$, with $k \geq 0$, $J_{\leq n}^{k+1}(X) = J_{\leq n} \circ J_{\leq n}^k$, where $J_{\leq n}$ is the operator on $A$ defined, for every $X \subseteq A$, as follows:

$$J_{\leq n}(X) = \bigcup \{J(Y) \mid Y \in \text{Sub}_{\leq n}(X)\},$$

where $\text{Sub}_{\leq n}(X)$ is $\{Y \subseteq X \mid \text{card}(Y) \leq n\}$.

Alfred Tarski in [3], on pp. 190–191, proved, as reformulated by S. Burris and H. P. Sankappanavar in [2], on pp. 33–34, the following theorem. Given a set $A$ and an $n$-ary closure operator $J$ on $A$ with $n \geq 2$, for every $i, j \in \text{IrB}(A, J)$, where $\text{IrB}(A, J)$ is the set of all natural numbers $n$ such that $(A, J)$ has an irredundant basis ($\equiv$ minimal generating set) of $n$ elements, if $i < j$ and $\{i+1, \ldots, j-1\} \cap \text{IrB}(A, J) = \emptyset$, then $j - i \leq n - 1$. Thus, as stated by Burris and Sankappanavar in [2], on p. 33, the length of the finite gaps in $\text{IrB}(A, J)$ is bounded by $n - 2$ if $J$ is an $n$-ary closure operator. And our second main result is the proof of Tarski’s irredundant basis theorem for many-sorted closure spaces.

2. MANY-SORTED SETS, MANY-SORTED CLOSURE OPERATORS, AND MANY-SORTED ALGEBRAS.

In this section, for a set of sorts $S$ in a fixed Grothendieck universe $\mathcal{U}$, we begin by recalling some basic notions of the theory of $S$-sorted sets, e.g., those of subset of an $S$-sorted set, of proper subset of an $S$-sorted set, of delta of Kronecker, of cardinal of an $S$-sorted set, and of support of an $S$-sorted set; and by defining, for an $S$-sorted set $A$, the concepts of many-sorted closure operator on $A$ and of many-sorted closure space. Moreover, for a many-sorted closure operator $J$ on $A$, we define the notions of irredundant or independent part of $A$ with respect to $J$, of basis or generator of $A$ with respect to $J$, of irredundant basis of $A$ with respect to $J$, and of minimal basis of $A$ with respect to $J$. In addition, we state that the notion of irredundant basis of $A$ with respect to $J$ is equivalent to the notion of minimal basis of $A$ with respect to $J$ and, afterwards, for a many-sorted closure space $(A, J)$, we define the subset $\text{IrB}(A, J)$ of $\mathbb{N}$ as being formed by choosing those natural numbers which are the cardinal of an irredundant basis of $A$ with respect to $J$. On the other hand, for a natural number $n$, we define the concept of $n$-ary many-sorted closure operator on $A$ and provide a
characterization of the \(n\)-ary many-sorted closure operators \(J\) on \(A\), in terms of the fixed points of \(J\). Besides, for a set of sorts \(S\), we define the concept of \(S\)-sorted signature, and, for an \(S\)-sorted signature \(\Sigma\), the notion of \(\Sigma\)-algebra and, for a \(\Sigma\)-algebra \(A\), the concept of subalgebra of \(A\) and the subalgebra generating many-sorted operator \(\text{Sg}_A\) on \(A\) determined by \(A\). Subsequently, once defined the notion of finitely generated \(\Sigma\)-algebra, we state that, for a finitely generated \(\Sigma\)-algebra \(A\), \(\text{IrB}(A, \text{Sg}_A) \neq \emptyset\).

**Definition 2.1.** An \(S\)-sorted set is a function \(A = (A_s)_{s \in S}\) from \(S\) to \(U\).

**Definition 2.2.** Let \(S\) be a set of sorts. If \(A\) and \(B\) are \(S\)-sorted sets, then we will say that \(A\) is a subset of \(B\), denoted by \(A \subseteq B\), if, for every \(s \in S\), \(A_s \subseteq B_s\), and that \(A\) is a proper subset of \(B\), denoted by \(A \subset B\), if \(A \subseteq B\) and, for some \(s \in S\), \(B_s - A_s \neq \emptyset\). We denote by \(\text{Sub}(A)\) the set of all \(S\)-sorted sets \(X\) such that \(X \subseteq A\).

**Definition 2.3.** Given a sort \(t \in S\) and a set \(X\) we call **delta of Kronecker** for \((t, X)\) the \(S\)-sorted set \(\delta^{t,x}\) defined, for every \(s \in S\), as follows:

\[
\delta^{t,x}_s = \begin{cases} X, & \text{if } s = t; \\
\emptyset, & \text{otherwise.} \end{cases}
\]

For a final set \(\{x\}\), to abbreviate, we will write \(\delta^{t,x}\) instead of the more accurate \(\delta^{t,\{x\}}\).

We next define, for a set of sorts \(S\), the concept of cardinal of an \(S\)-sorted set, for an \(S\)-sorted set \(A\), the notion of support of \(A\), and characterize the finite \(S\)-sorted sets in terms of its supports.

**Definition 2.4.** Let \(A\) be an \(S\)-sorted set. Then the **cardinal** of \(A\), denoted by \(\text{card}(A)\), is the cardinal of \(\coprod A\), where \(\coprod A\), the coproduct of \(A = (A_s)_{s \in S}\), is \(\bigcup_{s \in S}(A_s \times \{s\})\). Moreover, \(\text{Sub}_{\text{fin}}(A)\) denotes the set of all finite subsets of \(A\), i.e., the set \(\{X \subseteq A \mid \text{card}(X) < \aleph_0\}\), and, for a natural number \(n\), \(\text{Sub}_{\leq n}(A)\) denotes the set of all subsets of \(A\) with at most \(n\) elements, i.e., the set \(\{X \subseteq A \mid \text{card}(X) \leq n\}\). Sometimes, for simplicity of notation, we write \(X \subseteq_{\text{fin}} A\) instead of \(X \in \text{Sub}_{\text{fin}}(A)\).

**Definition 2.5.** Let \(S\) be a set of sorts. Then the **support** of \(A\), denoted by \(\text{supp}_S(A)\), is the set \(\{s \in S \mid A_s \neq \emptyset\}\).

**Proposition 2.6.** An \(S\)-sorted set \(A\) is finite if, and only if, \(\text{supp}_S(A)\) is finite and, for every \(s \in \text{supp}_S(A)\), \(\text{card}(A_s) < \aleph_0\).

**Definition 2.7.** Let \(S\) be a set of sorts and \(A\) an \(S\)-sorted set. A **many-sorted closure operator** on \(A\) is a mapping \(J\) from \(\text{Sub}(A)\) to \(\text{Sub}(A)\), which assigns to every \(X \subseteq A\) its \(J\)-closure \(J(X)\), such that, for every \(X, Y \subseteq A\), satisfies the following conditions:

1. \(X \subseteq J(X)\), i.e., \(J\) is extensive.
(2) If \( X \subseteq Y \), then \( J(X) \subseteq J(Y) \), i.e., \( J \) is isotone.

(3) \( J(J(X)) = J(X) \), i.e., \( J \) is idempotent.

Given two many-sorted closure operators \( J \) and \( K \) on \( A \), \( J \) is called \textit{smaller than} \( K \), denoted by \( J \leq K \), if, for every \( X \subseteq A \), \( J(A) \subseteq K(A) \).

A \textit{many-sorted closure space} is an ordered pair \((A, J)\) where \( A \) is an \( S \)-sorted set and \( J \) a many-sorted closure operator on \( A \). Moreover, if \( X \subseteq A \), then \( X \) is \textit{irredundant} (or \textit{independent}) with respect to \( J \) if, for every \( s \in S \) and every \( x \in X_s \), \( x \notin J(X - \delta^s_x) \), \( X \) is a \textit{basis} (or a \textit{generator}) with respect to \( J \) if \( J(X) = A \), \( X \) is an \textit{irredundant basis} with respect to \( J \) if \( X \) irredundant and a basis with respect to \( J \), and \( X \) is a \textit{minimal basis} with respect to \( J \) if \( J(X) = A \) and, for every \( Y \subset X \), \( J(Y) \neq A \).

We next state that the notion of irredundant basis of \( A \) with respect to \( J \) is equivalent to the notion of minimal basis of \( A \) with respect to \( J \). Moreover, for a many-sorted closure space \((A, J)\), we define \( \text{IrB}(A, J) \) as the intersection of the set of all natural numbers and the set of the cardinals of the irredundant basis of \( A \) with respect to \( J \).

**Proposition 2.8.** Let \((A, J)\) be a many-sorted closure space and \( X \subseteq A \). Then \( X \) is an irredundant basis with respect to \( J \) if, and only if, it is a minimal basis with respect to \( J \).

**Definition 2.9.** Let \( S \) be a set of sorts and \((A, J)\) a many-sorted closure space. Then we denote by \( \text{IrB}(A, J) \) the subset of \( \mathbb{N} \) defined as follows:

\[
\text{IrB}(A, J) = \mathbb{N} \cap \left\{ \text{card}(X) \left| \begin{array}{l}
X \text{ is an irredundant basis} \\
\text{of } A \text{ with respect to } J
\end{array} \right. \right\}.
\]

Later, in this section, after having defined, for a set of sorts \( S \) and an \( S \)-sorted signature \( \Sigma \), the concept of \( \Sigma \)-algebra, for a \( \Sigma \)-algebra \( A = (A, F) \), the uniform algebraic many-sorted closure operator \( S_gA \) on \( A \), called the subalgebra generating many-sorted operator on \( A \) determined by \( A \), and the notion of finitely generated \( \Sigma \)-algebra, we will state that, for a finitely generated \( \Sigma \)-algebra \( A \), \( \text{IrB}(A, S_gA) \neq \emptyset \).

**Definition 2.10.** Let \( A \) be an \( S \)-sorted set, \( J \) a many-sorted closure operator on \( A \), and \( n \) a natural number.

(1) We denote by \( J_{\leq n} \) the many-sorted operator on \( A \) defined, for every \( X \subseteq A \), as follows:

\[
J_{\leq n}(X) = \bigcup \{J(Y) \mid Y \in \text{Sub}_{\leq n}(X)\}.
\]

(2) We define the family \((J_{\leq n}^m)_{m \in \mathbb{N}}\) of many-sorted operator on \( A \), recursively, as follows:

\[
J_{\leq n}^m = \begin{cases} 
\text{Id}_{\text{Sub}(A)}, & \text{if } m = 0; \\
J_{\leq n} \circ J_{\leq n}^k, & \text{if } m = k + 1, \text{ with } k \geq 0.
\end{cases}
\]
(3) We denote by $J_{\leq n}^S$ the many-sorted operator on $A$ that assigns to an $S$-sorted subset $X$ of $A$, $J_{\leq n}^S(X) = \bigcup_{m \in \mathbb{N}} J_{\leq n}^m(X)$.

(4) We say that $J$ is $n$-ary if $J = J_{\leq n}^S$.

**Remark.** Let $J$ be a many-sorted closure operator on $A$. Then $J$ is 0-ary, i.e., $J = J_{\leq 0}^S$, if and only if, for every $X \subseteq A$, we have that

$$J(X) = X \cup J(\emptyset^S),$$

where $\emptyset^S$ is the $S$-sorted set whose $s$th coordinate, for every $s \in S$, is $\emptyset$.

We next prove that $J$ is 1-ary, i.e., that $J = J_{\leq 1}^S$, if and only if, for every $X \subseteq A$, we have that

$$J(X) = J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}).$$

Let us suppose that, for every $X \subseteq A$, $J(X) = J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x})$.

Then it is obvious that, for every $X \subseteq A$, $J(X) \subseteq J_{\leq 1}(X)$. Let us verify that, for every $X \subseteq A$, $J_{\leq 1}(X) = \bigcup \{J(Y) \mid Y \in \text{Sub}_{\leq 1}(X)\} \subseteq J(X)$. Let $Y$ be an element of Sub$_{\leq 1}(X)$. Then $Y = \emptyset^S$ or $Y = \delta^{t,a}$, for some $t \in S$ and some $a \in X_t$. If $Y = \emptyset^S$, then

$$J(\emptyset^S) \subseteq J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) = J(X).$$

If $Y = \delta^{t,a}$, then $J(\delta^{t,a}) \subseteq \bigcup_{s \in S, x \in X_s} J(\delta^{s,x})$, hence

$$J(\delta^{t,a}) \subseteq J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) = J(X).$$

Thus $J_{\leq 1}(X) \subseteq J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) = J(X)$. Therefore $J = J_{\leq 1}$.

Hence, for every $m \geq 1$, $J = J_{\leq m}^S$. Consequently $J$ is 1-ary.

Reciprocally, let us suppose that $J$ is 1-ary, i.e., that, for every $X \subseteq A$, $J(X) = \bigcup_{m \in \mathbb{N}} J_{\leq 1}^m(X)$. Then, obviously, we have that

$$J(X) \supseteq J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}).$$

Let us verify that, for every $m \in \mathbb{N}$, $J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) \supseteq J_{\leq m}^1(X)$. Evidently $J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) \supseteq J_{\leq 1}^0(X) \cup J_{\leq 1}^1(X)$. Let $k$ be $\geq 1$ and let us suppose that $J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) \supseteq J_{\leq k}^1(X)$.

We will show that $J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}) \supseteq J_{\leq k+1}^1(X)$. By definition we have that

$$J_{\leq k+1}^1(X) = J_{\leq 1}(J_{\leq k+1}^1(X)) = \bigcup \{J(Z) \mid Z \in \text{Sub}_{\leq 1}(J_{\leq k+1}^1(X))\}.$$

Let $Z$ be an element of Sub$_{\leq 1}(J_{\leq k+1}^1(X))$. Then $Z \subseteq J_{\leq k+1}^1(X)$. But we have that $J_{\leq 1}^k(X) = \bigcup \{J(Y) \mid Y \in \text{Sub}_{\leq 1}(J_{\leq k+1}^1(X))\}$. Therefore, for some $Y \in \text{Sub}_{\leq 1}(J_{\leq k+1}^1(X))$, $Z \subseteq J(Y)$. Thus $J(Z) \subseteq J(J(Y)) = J(Y)$. But $J(Y) \subseteq J_{\leq k+1}^1(X)$. Consequently $J(Z) \subseteq J_{\leq k+1}^1(X)$. Whence, by the induction hypothesis, $J(Z) \subseteq J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x})$. From this, since $Z$ was an arbitrary element of Sub$_{\leq 1}(J_{\leq k+1}^1(X))$, we infer that

$$J_{\leq k+1}^1(X) = \bigcup \{J(Z) \mid Z \in \text{Sub}_{\leq 1}(J_{\leq k+1}^1(X))\} \subseteq J(\emptyset^S) \cup \bigcup_{s \in S, x \in X_s} J(\delta^{s,x}).$$
Thus, for every $X \subseteq A$, we have that
\[ J(X) = J(\emptyset^S) \cup \bigcup_{s,x \in X} J(\delta^s_x). \]

**Remark.** Let $n$ be $\geq 1$, $A$ an $S$-sorted set, $X \subseteq A$, and $J$ a many-sorted closure operator on $A$. Then, for every $k \geq 0$ and every $Y \subseteq A$, if $Y \in \text{Sub}_{\leq_n}(J^{k}_{\leq_n}(X))$, then $Y \in \text{Sub}_{\leq_n}(J^{k+1}_{\leq_n}(X))$.

We next state, for a natural number $n \geq 1$ and a many-sorted closure operator $J$ on an $S$-sorted set $A$, that the family of many-sorted operators $(J^{m}_{\leq_n})_{m \in \mathbb{N}}$ on $A$ is an ascending chain and that $J^{\omega}_{\leq_n}$, which is the supremum of the above family, is the greatest $n$-ary many-sorted closure operator on $A$ which is smaller than $J$.

**Proposition 2.11.** For a natural number $n \geq 1$, an $S$-sorted set $A$, and a many-sorted closure operator $J$ on $A$, the family of many-sorted operators $(J^{m}_{\leq_n})_{m \in \mathbb{N}}$ on $A$ is an ascending chain, i.e., for every $m \in \mathbb{N}$, $J^{m}_{\leq_n} \leq J^{m+1}_{\leq_n}$. Moreover, $J^{\omega}_{\leq_n}$ is the greatest $n$-ary many-sorted closure operator on $A$ such that $J^{\omega}_{\leq_n} \leq J$.

We next provide a characterization of the $n$-ary many-sorted closure operators $J$ on an $S$-sorted set $A$ in terms of the fixed points $X$ of $J$ and of its relationships with the $J$-closures of the subsets of $X$ with at most $n$ elements.

**Proposition 2.12.** Let $A$ be an $S$-sorted set, $J$ a many-sorted closure operator on $A$, and $n$ a natural number. Then $J$ is $n$-ary if, and only if, for every $X \subseteq A$, if, for every $Z \in \text{Sub}_{\leq_n}(X)$, $J(Z) \subseteq X$, then $J(X) = X$ (i.e., if, and only if, for every $X \subseteq A$, $X$ is a fixed point of $J$ if $X$ contains the $J$-closure of each of its subsets with at most $n$ elements).

**Proof.** If $n = 0$, then the result is obvious. So let us consider the case when $n \geq 1$. Let us suppose that $J$ is $n$-ary and let $X$ be a subset of $A$ such that, for every $Z \in \text{Sub}_{\leq_n}(X)$, $J(Z) \subseteq X$. We want to show that $J(X) = X$. Because $J$ is extensive, $X \subseteq J(X)$. Therefore it only remains to show that $J(X) \subseteq X$. Since, by hypothesis, $J(X) = \bigcup_{m \in \mathbb{N}} J^{m}_{\leq_n}(X)$, to show that $J(X) \subseteq X$ it suffices to prove that, for every $m \in \mathbb{N}$, $J^{m}_{\leq_n}(X) \subseteq X$.

For $m = 0$ we have that $J^{0}_{\leq_n}(X) = X \subseteq X$.

Let us suppose that, for $k \geq 0$, $J^{k}_{\leq_n}(X) \subseteq X$. Then we want to show that $J^{k+1}_{\leq_n}(X) \subseteq X$. But, by definition, we have that
\[ J^{k+1}_{\leq_n}(X) = J_{\leq_n}(J^{k}_{\leq_n}(X)) = \bigcup \{ J(Y) \mid Y \in \text{Sub}_{\leq_n}(J^{k}_{\leq_n}(X)) \}. \]

Hence what we have to prove is that, for every $Y \in \text{Sub}_{\leq_n}(J^{k}_{\leq_n}(X))$, $J(Y) \subseteq X$. Let $Y$ be a subset of $J^{k}_{\leq_n}(X)$ such that $\text{card}(Y) \leq n$. Since $J^{k}_{\leq_n}(X) \subseteq X$, we have that $Y \subseteq X$ and $\text{card}(Y) \leq n$, therefore $J(Y) \subseteq X$. Consequently, for every $X \subseteq A$, if, for every $Z \in \text{Sub}_{\leq_n}(X)$, $J(Z) \subseteq X$, then $J(X) = X$. 


Reciprocally, let us suppose that, for every $X \subseteq A$, if for every $Z \in \text{Sub}_{\leq n}(X)$, $J(Z) \subseteq X$, then $J(X) = X$. We want to show that $J$ is $n$-ary, i.e., that $J = J_{\leq n}$. Let $X$ a subset of $A$. Then it is obvious that $J_{\leq n}(X) = \bigcup_{m \in \mathbb{N}} J^m_{\leq n}(X) \subseteq J(X)$. We now proceed to prove that $J(X) \subseteq J_{\leq n}(X)$. Since $J$ is isotone and, by the definition of $J_{\leq n}$, $X \subseteq J_{\leq n}(X)$, we have that $J(X) \subseteq J(J_{\leq n}(X))$. Therefore to prove that $J(X) \subseteq J_{\leq n}(X)$ it suffices to prove that $J(J_{\leq n}(X)) = J_{\leq n}(X)$. But the just stated equation follows from the supposition because, as we will prove next, for every $Z \in \text{Sub}_{\leq n}(J_{\leq n}(X))$, we have that $J(Z) \subseteq J_{\leq n}(X)$. Let $Z$ be a subset of $J_{\leq n}(X)$ such that $\text{card}(Z) \leq n$. Then, for some $\ell \in \mathbb{N}$, $\text{supp}_S(Z) = \{s_0, \ldots, s_{\ell-1}\}$ and, for every $\alpha \in \ell$, there exists an $n_\alpha \in \mathbb{N} - 1$ such that $Z_{s_\alpha} = \{z_{\alpha,0}, \ldots, z_{\alpha,n_\alpha-1}\}$. Therefore, for every $\alpha \in \ell$ and for every $\beta \in n_\alpha$ there exists an $m_{\alpha,\beta} \in \mathbb{N}$ such that $z_{\alpha,\beta} \in J_{\leq n}(X)_{s_\alpha}$. Since it may be helpful for the sake of understanding, let us represent the situation just described by the following figure:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
\cdots & \cdots & \cdots \\
z_{0,0} & z_{0,1} & \cdots & z_{0,n_0-1} \\
\vdots & \vdots & \vdots & \vdots \\
z_{\ell-1,0} & z_{\ell-1,1} & \cdots & z_{\ell-1,n_{\ell-1}-1} \\
\end{array}
\]

Hence, for every $\alpha \in \ell$ there exists a $\beta_\alpha \in n_\alpha$ such that $Z_{s_\alpha} \subseteq J_{\leq n}(X)_{s_\alpha}$. On the other hand, since the family of many-sorted operators $(J^m_{\leq n} \mid m \in \mathbb{N})$ on $A$ is an ascending chain, there exists an $m$ in the set $\{m_{\alpha,\beta} \mid \alpha \in \ell\}$ such that, for every $\alpha \in \ell$, $J_{\leq n}(X)_{s_\alpha} \subseteq J^m_{\leq n}(X)$. Thus $Z \subseteq J^m_{\leq n}(X)$. Therefore, since, in addition, $\text{card}(Z) \leq n$, we have that $Z \in \text{Sub}_{\leq n}(J^m_{\leq n}(X))$. Thus

\[
J(Z) \subseteq J^m_{\leq n}(X) = J_{\leq n}(J^m_{\leq n}(X)) = \bigcup \{J(K) \mid K \in \text{Sub}_{\leq n}(J^m_{\leq n}(X))\}
\]

Consequently $J(Z) \subseteq J^\omega_{\leq n}(X)$. Hence $J(X) \subseteq J^\omega_{\leq n}(X)$. Whence $J = J_{\leq n}^\omega$, which completes the proof.

We next recall the notion of free monoid on a set and, for a set of sorts $S$, we define, by using the the just mentioned notion, the concept of $S$-sorted signature and, for an $S$-sorted signature $\Sigma$, the concept of $\Sigma$-algebra.

**Definition 2.13.** Let $S$ be a set of sorts. The free monoid on $S$, denoted by $S^*$, is $(S^*, \lambda, \lambda)$, where $S^*$, the set of all words on $S$, is $\bigcup_{n \in \mathbb{N}} \text{Hom}(n, S)$, the set of all mappings $w: n \longrightarrow S$ from some $n \in \mathbb{N}$ to $S$, $\lambda$, the concatenation of words on $S$, is the binary operation on $S^*$ which sends a pair of words $(w, v)$ on $S$ to the mapping $w \lambda v$ from $|w| + |v|$ to $S$, where $|w|$ and $|v|$ are the lengths (\equiv domains) of the
and an $S$-sorted closure operator on $A$ to the set $\Sigma$ of the formal operations of arity $w$, sort (or coarity) $s$, and rank (or biarity) $(w, s)$.

**Definition 2.14.** Let $S$ be a set of sorts. Then an $S$-sorted signature is a function $\Sigma$ from $S^\ast \times S$ to $U$ which sends a pair $(w, s) \in S^\ast \times S$ to the set $\Sigma_{w, s}$ of the formal operations of arity $w$, sort (or coarity) $s$, and rank (or biarity) $(w, s)$.

**Definition 2.15.** Let $\Sigma$ be an $S$-sorted signature and $A$ an $S$-sorted set. The $S^\ast \times S$-sorted set of the finitary operations on $A$ is the family $(\text{Hom}(A_w, A_s))_{(w, s) \in S^\ast \times S}$, where, for every $w \in S^\ast$, $A_w = \prod_{i \in |w|} A_{w_i}$. A structure of $\Sigma$-algebra on $A$ is an $S^\ast \times S$-mapping $F = (F_{w, s})_{(w, s) \in S^\ast \times S}$ from $\Sigma$ to $(\text{Hom}(A_w, A_s))_{(w, s) \in S^\ast \times S}$. For a pair $(w, s) \in S^\ast \times S$ and a formal operation $\sigma \in \Sigma_{w, s}$, in order to simplify the notation, the operation from $A_w$ to $A_s$ corresponding to $\sigma$ under $F_{w, s}$ will be written as $F_\sigma$ instead of $F_{w, s}(\sigma)$. A $\Sigma$-algebra is a pair $(A, F)$, abbreviated to $A$, where $A$ is an $S$-sorted set and $F$ a structure of $\Sigma$-algebra on $A$.

Since it will be used afterwards, we next define, for a set of sorts $S$ and an $S$-sorted set $A$, the notions of algebraic and of uniform many-sorted closure operator on $A$.

**Definition 2.16.** A many-sorted closure operator $J$ on an $S$-sorted set $A$ is algebraic if, for every $X \subseteq A$, $J(X) = \bigcup_{K \subseteq \text{fin}} X J(K)$, and is uniform if, for every $X, Y \subseteq A$, if $\text{supp}_S(X) = \text{supp}_S(Y)$, then $\text{supp}_S(J(X)) = \text{supp}_S(J(Y))$.

We next prove that, for a many-sorted closure operator, the property of being $n$-ary is stronger than that of being algebraic.

**Proposition 2.17.** Let $n$ be a natural number. If a many-sorted closure operator $J$ on an $S$-sorted set $A$ is $n$-ary, then $J$ is an algebraic many-sorted closure operator on $A$.

**Proof.** Let $J$ be an $n$-ary many-sorted closure operator on an $S$-sorted set $A$ and let $X$ be a subset of $A$. Then, obviously, $\bigcup_{K \subseteq \text{fin}} X J(K) \subseteq J(X)$. Since $J(X) = J_{\leq n}(X) = \bigcup_{m \in \mathbb{N}} J^m_{\leq n}(X)$, to prove that $J(X) \subseteq \bigcup_{K \subseteq \text{fin}} X J(K)$ it suffices to prove that, for every $m \in \mathbb{N}$, $J^m_{\leq n}(X) \subseteq \bigcup_{K \subseteq \text{fin}} X J(K)$.

For $m = 0$, since $J^0_{\leq n}(X) = X$, we have that $J^0_{\leq n}(X) \subseteq \bigcup_{K \subseteq \text{fin}} X J(K)$.

Let $m$ be $k + 1$ with $k \geq 0$ and let us suppose that $J^k_{\leq n}(X) \subseteq \bigcup_{K \subseteq \text{fin}} X J(K)$. We want to prove that $J^{k+1}_{\leq n}(X) \subseteq \bigcup_{K \subseteq \text{fin}} X J(K)$.

However, by definition, $J^{k+1}_{\leq n}(X) = \bigcup\{J(Z) \mid Z \in \text{Sub}_{\leq n}(J^k_{\leq n}(X))\}$. Thus it suffices to prove that, for every $Z \in \text{Sub}_{\leq n}(J^k_{\leq n}(X))$, $J(Z) \subseteq \bigcup_{K \subseteq \text{fin}} X J(K)$.
Let $Z$ be a subset of $J_{\leq n}(X)$ such that $\text{card}(Z) \leq n$. Then, since, by the induction hypothesis, $J_{\leq n}(X) \subseteq \bigcup_{K \subseteq \text{fin}X} J(K)$, we have that $Z \subseteq \bigcup_{K \subseteq \text{fin}X} J(K)$ and, in addition, that $\text{card}(Z) \leq n$. Hence, for some $\ell \in \mathbb{N}$, $\text{supp}(Z) = \{s_0, \ldots, s_{\ell-1}\}$ and, for every $\alpha \in \ell$, there exists an $n_\alpha \in \mathbb{N} - 1$ such that $Z_{s_\alpha} = \{z_{\alpha,0}, \ldots, z_{\alpha,n_\alpha-1}\}$. Therefore, for every $\alpha \in \ell$ and every $\beta \in n_\alpha$ there exists a $K^{\alpha,\beta} \subseteq \text{fin}X$ such that $z_{\alpha,\beta} \in J(K^{\alpha,\beta})_{s_\alpha}$. Since it may be helpful for the sake of understanding, let us represent the situation just described by the following figure:

\[
\begin{array}{cccc}
  z_{0,0} \in J(K^{0,0})_{s_0} & \cdots & z_{0,n_0-1} \in J(K^{0,n_0-1})_{s_0} \\
  \vdots & \ddots & \vdots \\
  z_{\ell-1,0} \in J(K^{\ell-1,0})_{s_{\ell-1}} & \cdots & z_{\ell-1,n_{\ell-1}-1} \in J(K^{\ell-1,n_{\ell-1}-1})_{s_{\ell-1}}
\end{array}
\]

Then, for every $\alpha \in \ell$, $Z_{s_\alpha} \subseteq J(\bigcup_\beta K^{\alpha,\beta})_{s_\alpha}$, where $\bigcup_\beta K^{\alpha,\beta} \subseteq \text{fin}X$. So, for $L = \bigcup_{\alpha,\beta} K^{\alpha,\beta}$, we have that $L \subseteq \text{fin}X$ and $Z \subseteq J(L)$. Therefore $J(Z) \subseteq J(J(L)) = J(L) \subseteq \bigcup_{K \subseteq \text{fin}X} J(K)$.

We next define when a subset $X$ of the underlying $S$-sorted set $A$ of a $\Sigma$-algebra $A$ is closed under an operation $F_\sigma$ of $A$, as well as when $X$ is a subalgebra of $A$.

**Definition 2.18.** Let $A$ be a $\Sigma$-algebra and $X \subseteq A$. Let $\sigma$ be a formal operation in $\Sigma_{w,s}$. We say that $X$ is closed under the operation $F_\sigma: A_w \longrightarrow A_s$ if, for every $a \in X_w$, $F_\sigma(a) \in X_s$. We say that $X$ is a subalgebra of $A$ if $X$ is closed under the operations of $A$. We denote by $\text{Sub}(A)$ the set of all subalgebras of $A$ (which is an algebraic closure system on $A$).

**Definition 2.19.** Let $A$ be a $\Sigma$-algebra. Then we denote by $\text{Sg}_A$, the many-sorted closure operator on $A$ defined as follows:

\[
\text{Sg}_A \begin{cases} 
  \text{Sub}(A) \longrightarrow \text{Sub}(A) \\
  X \longmapsto \bigcap \{C \in \text{Sub}(A) \mid X \subseteq C\}. 
\end{cases}
\]

We call $\text{Sg}_A$ the subalgebra generating many-sorted operator on $A$ determined by $A$. For every $X \subseteq A$, we call $\text{Sg}_A(X)$ the subalgebra of $A$ generated by $X$. Moreover, if $X \subseteq A$ is such that $\text{Sg}_A(X) = A$, then we say that $X$ is an $S$-sorted set of generators of $A$, or that $X$ generates $A$. Besides, we say that $A$ is finitely generated if there exists an $S$-sorted subset $X$ of $A$ such that $X$ generates $A$ and card($X$) $< \aleph_0$.

**Proposition 2.20.** Let $A$ be a $\Sigma$-algebra. Then the many-sorted closure operator $\text{Sg}_A$ on $A$ is algebraic, i.e., for every $S$-sorted subset $X$ of $A$, $\text{Sg}_A(X) = \bigcup_{K \subseteq \text{fin}X} \text{Sg}_A(K)$.

For a $\Sigma$-algebra $A$ we next provide another, more constructive, description of the algebraic many-sorted closure operator $\text{Sg}_A$, which, in addition, will allow us to state a crucial property of $\text{Sg}_A$. Specifically, that $\text{Sg}_A$ is uniform.
Definition 2.21. Let $\Sigma$ be an $S$-sorted signature and $A$ a $\Sigma$-algebra.

1. We denote by $E_A$ the many-sorted operator on $A$ that assigns to an $S$-sorted subset $X$ of $A$, $E_A(X) = X \cup \left( \bigcup_{\sigma \in \Sigma \cdot s, s \in S} F_{\sigma}[X_{\text{ar}(\sigma)}] \right)_{s \in S}$, where, for $s \in S$, $\Sigma \cdot s$ is the set of all many-sorted formal operations $\sigma$ such that the coarity of $\sigma$ is $s$ and for $\text{ar}(\sigma) = w \in S^*$, the arity of $\sigma$, $X_{\text{ar}(\sigma)} = \prod_{i \in |w|} X_{w_i}$.

2. If $X \subseteq A$, then we define the family $(E_n^A(X))_{n \in \mathbb{N}}$ in Sub$(A)$, recursively, as follows:

\[
E_0^A(X) = X, \\
E_n^A(X) = E_A(E_n^A(X)), \quad n \geq 0.
\]

3. We denote by $E^\omega_A$ the many-sorted operator on $A$ that assigns to an $S$-sorted subset $X$ of $A$, $E^\omega_A(X) = \bigcup_{n \in \mathbb{N}} E_n^A(X)$.

Proposition 2.22. Let $A$ be a $\Sigma$-algebra and $X \subseteq A$, then $Sg_A(X) = E^\omega_A(X)$.

In [3], on pp. 82, we stated the following proposition (there called Proposition 2.7).

Proposition 2.23. Let $A$ be a $\Sigma$-algebra and $X, Y \subseteq A$. Then we have that

1. If $\text{supp}_S(X) = \text{supp}_S(Y)$, then, for every $n \in \mathbb{N}$, $\text{supp}_S(E_n^A(X)) = \text{supp}_S(E_n^A(Y))$.
2. $\text{supp}_S(Sg_A(X)) = \bigcup_{n \in \mathbb{N}} \text{supp}_S(E_n^A(X))$.
3. If $\text{supp}_S(X) = \text{supp}_S(Y)$, then $\text{supp}_S(Sg_A(X)) = \text{supp}_S(Sg_A(Y))$.

Therefore the algebraic many-sorted closure operator $Sg_A$ is uniform.

Proposition 2.24. If $A$ is a finitely generated $\Sigma$-algebra, then every $S$-sorted set of generators of $A$ contains a finite $S$-sorted subset which also generates $A$.

Corollary 2.25. If $A$ is a finitely generated $\Sigma$-algebra, then we have that $\text{IrB}(A, Sg_A)$ is not empty.

3. A characterization of the $n$-ary many-sorted closure operators.

A theorem of Birkhoff-Frink (see [1]) asserts that every algebraic closure operator on an ordinary set arises, from some algebraic structure on the set, as the corresponding generated subalgebra operator. However, for many-sorted sets such a theorem is not longer true without qualification. In [3], on pp. 83–84, Theorem 3.1 and Corollary 3.2, we characterized the corresponding many-sorted closure operators as precisely the uniform algebraic operators. We next recall the just mentioned characterization since it will be applied afterwards to provide
a characterization of the \( n \)-ary many-sorted closure operators on an \( S \)-sorted set.

Let us notice that in what follows, for a word \( w : |w| \to S \) on \( S \), with \(|w|\) the length of \( w \), and an \( s \in S \), we denote by \( w^{-1}[s] \) the set \( \{ i \in |w| \mid w(i) = s \} \), and by \text{Im}(w) \) the set \( \{ w(i) \mid i \in |w| \} \).

**Theorem 3.1.** Let \( J \) be an algebraic many-sorted closure operator on an \( S \)-sorted set \( A \). If \( J \) is uniform, then \( J = Sg_{\mathbf{A}} \) for some \( S \)-sorted signature \( \Sigma \) and some \( \Sigma \)-algebra \( \mathbf{A} \).

**Proof.** Let \( \Sigma = (\Sigma_{w,s})_{(w,s) \in S^* \times S} \) be the \( S \)-sorted signature defined, for every \((w,s) \in S^* \times S\), as follows:

\[
\Sigma_{w,s} = \{ (X, b) \in \bigcup_{X \in \text{Sub}(A)} \{ X \} \times J(X)_s \mid \forall t \in S (\text{card}(X_t) = |w|_t) \},
\]

where for a sort \( s \in S \) and a word \( w : |w| \to S \) on \( S \), with \(|w|\) the length of \( w \), the number of occurrences of \( s \) in \( w \), denoted by \(|w|_s\), is \( \text{card}(w^{-1}[s]) \).

Before proceeding any further, let us remark that, for \((w,s) \in S^* \times S\) and \((X, b) \in \bigcup_{X \in \text{Sub}(A)} \{ X \} \times J(X)_s\), the following conditions are equivalent:

1. \((X, b) \in \Sigma_{w,s}, \text{ i.e., for every } t \in S, \text{card}(X_t) = |w|_t.\)
2. \(\text{supp}_S(X) = \text{Im}(w)\) and, for every \( t \in \text{supp}_S(X)\), \( \text{card}(X_t) = |w|_t.\)

On the other hand, for the index set \( \Lambda = \bigcap_{Y \in \text{Sub}(A)} \{ Y \times \text{supp}_S(Y) \} \) and the \( \Lambda \)-indexed family \( \{ (Y_s) \mid s \in \Lambda \} \) whose \( (Y, s) \)-th coordinate is \( Y_s \), precisely the \( s \)-th coordinate of the \( S \)-sorted set \( Y \) of the index \( (Y, s) \in \Lambda \), let \( f \) be a choice function for \( (Y_s)_{(Y, s) \in \Lambda} \), i.e., an element of \( \prod_{(Y, s) \in \Lambda} Y_s \).

Moreover, for every \( w \in S^* \) and \( a \in \prod_{i \in |w|} A_{w(i)} \), let \( M_{w,a} = (M_{w,a}^s)_{s \in S} \) be the finite \( S \)-sorted subset of \( A \) defined as \( M_{w,a}^s = \{ a_i \mid i \in w^{-1}[s] \} \), for every \( s \in S \).

Now, for \((w,s) \in S^* \times S\) and \((X, b) \in \Sigma_{w,s}\), let \( F_{X,b} \) be the many-sorted operation from \( \prod_{i \in |w|} A_{w(i)} \) into \( A \) that assigns \( b \) to \( a \in \prod_{i \in |w|} A_{w(i)} \) such that \( M_{w,a} = (M_{w,a}^s)_{s \in S} = \text{supp}_S(X_1) \), for all \( s \in S \).

We will prove that the \( \Sigma \)-algebra \( \mathbf{A} = (A, F) \) is such that \( J = Sg_{\mathbf{A}} \). But before doing that it is necessary to verify that the definition of the many-sorted operations is sound, i.e., that for every \((w,s) \in S^* \times S\), \((X, b) \in \Sigma_{w,s}\) and \(a \in \prod_{i \in |w|} A_{w(i)}\), it happens that \( s \in \text{supp}_S(J(M_{w,a}))\), and for this it suffices to prove that \( \text{supp}_S(M_{w,a}) = \text{supp}_S(X)\), because, by hypothesis, \( J \) is uniform and, by definition, \( b \in J(X)_s\).

If \( t \in \text{supp}_S(M_{w,a})\), then \( M_{t}^{w,a} \) is nonempty, i.e., there exists an \( i \in |w| \) such that \( w(i) = t \). Therefore, because \((X, b) \in \Sigma_{w,s}\), we have that \( 0 < |w|_t = \text{card}(X_t) \), hence \( t \in \text{supp}_S(X)\).

Reciprocally, if \( t \in \text{supp}_S(X)\), \(|w|_t > 0\), and there is an \( i \in |w| \) such that \( w(i) = t \), hence \( a_i \in A_t\), and from this we conclude that \( M_{t}^{w,a} \neq \emptyset\),
i.e., that \( t \in \supp_S(M^{w,a}) \). Therefore, \( \supp_S(M^{w,a}) = \supp_S(X) \) and, by the uniformity of \( J \), \( \supp_S(J(M^{w,a})) = \supp_S(J(X)) \). But, by definition, \( b \in J(X)_s \), so \( s \in \supp_S(J(M^{w,a})) \) and the definition is sound.

Now we prove that, for every \( X \subseteq A \), \( J(X) \subseteq \text{Sub}_{\text{fin}}(A) \). Let \( X \) be an \( S \)-sorted subset of \( A \), \( s \in S \) and \( b \in J(X)_s \). Then, because \( J \) is algebraic, \( b \in J(Y)_s \), for some finite \( S \)-sorted subset \( Y \) of \( X \). From such \( Y \) we will define a word \( w_Y \) in \( S \) and an element \( a_Y \) of \( \prod_{i \in |w_Y|} A_{w_Y(i)} \)

\[
\begin{align*}
(1) \quad &Y = M^{w_Y,ay}, \\
(2) \quad & (Y, b) \in \sum_{w_Y,s}, \text{i.e., } b \in J(Y)_s \text{ and, for all } t \in S, \text{ card}(Y_t) = |w_Y|, \text{ and} \\
(3) \quad & a_Y \in \prod_{i \in |w_Y|} X_{w_Y(i)},
\end{align*}
\]

then, because \( F_{Y,b}(a_Y) = b \), we will be entitled to assert that \( b \in \text{Sub}_{\text{fin}}(A)_s \).

But given that \( Y \) is finite if, and only if, \( \supp_S(Y) \) is finite and, for every \( t \in \supp_S(Y) \), \( Y_t \) is finite, let \( \{ s_\alpha \mid \alpha \in m \} \) be an enumeration of \( \supp_S(Y) \) and, for every \( \alpha \in m \), let \( \{ y_{\alpha,i} \mid i \in p_\alpha \} \) be an enumeration of the nonempty \( s_\alpha \)-th coordinate, \( Y_{s_\alpha} \), of \( Y \). Then we define, on the one hand, the word \( w_Y \) as the mapping from \( |w_Y| = \sum_{\alpha \in m} p_\alpha \) into \( S \) such that, for every \( i \in |w_Y| \) and \( \alpha \in m \), \( w_Y(i) = s_\alpha \) if, and only if, \( \sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1 \) and, on the other hand, the element \( a_Y \) of \( \prod_{i \in |w_Y|} A_{w_Y(i)} \) as the mapping from \( |w_Y| \) into \( \bigcup_{i \in |w_Y|} A_{w_Y(i)} \) such that, for every \( i \in |w_Y| \) and \( \alpha \in m \), \( a_Y(i) = y_{\alpha,i} - \sum_{\beta \in \alpha} p_\beta \) if, and only if, \( \sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1 \). From these definitions follow (1), (2) and (3) above. Let us observe that (1) is a particular case of the fact that the mapping \( M \) from \( \bigcup_{w \in S^*} (\{w\} \times \prod_{i \in |w|} A_{w(i)}) \) into \( \text{Sub}_{\text{fin}}(A) \) that to a pair \((w, a)\) assigns \( M^{w,a} \) is surjective.

From the above and the definition of \( F_{Y,b} \) we can affirm that \( F_{Y,b}(a_Y) = b \), hence \( b \in \text{Sub}_{\text{fin}}(A)_s \). Therefore \( J(X) \subseteq \text{Sub}_{\text{fin}}(A) \).

Finally, we prove that, for every \( X \subseteq A \), \( \text{Sub}_{\text{fin}}(A)_s \subseteq J(X) \). But for this, by Proposition \[\text{Proposition 2.22}\], it is enough to prove that, for every subset \( X \) of \( A \), we have that \( E_A(X) \subseteq J(X) \). Let \( s \in S \) be and \( c \in E_A(X)_s \). If \( c \in X_s \), then \( c \in J(X)_s \), because \( J \) is extensive. If \( c \not\in X_s \), then, by the definition of \( E_A(X) \), there exists a word \( w \in S^* \), a many-sorted formal operation \( (Y, b) \in \sum_{w, s} \) and an \( a \in \prod_{i \in |w|} X_{w(i)} \) such that \( F_{Y,b}(a) = c \). If \( M^{w,a} = Y \), then \( c = b \), hence \( c \in J(Y)_s \), therefore, because \( M^{w,a} \subseteq X \), \( c \in J(X)_s \). If \( M^{w,a} \neq Y \), then \( F_{Y,b}(a) \in J(M^{w,a})_s \), but, because \( M^{w,a} \subseteq X \) and \( J \) is isotone, \( J(M^{w,a}) \) is a subset of \( J(X) \), hence \( F_{Y,b}(a) \in J(X)_s \). Therefore \( E_A(X) \subseteq J(X) \).

The just stated theorem together with Proposition \[\text{Proposition 2.23}\] entails the following corollary.
Corollary 3.2. Let $J$ be an algebraic many-sorted closure operator on an $S$-sorted set $A$. Then $J = Sg_A$ for some $S$-sorted signature $\Sigma$ and some $\Sigma$-algebra $A$ if, and only if, $J$ is uniform.

We next prove that for a natural number $n$, an $S$-sorted signature $\Sigma$, and a $\Sigma$-algebra $A$, under a suitable condition on $\Sigma$ related to $n$, the uniform algebraic many-sorted closure operator $Sg_A$ is an $n$-ary many-sorted closure operator on $A$.

Proposition 3.3. Let $\Sigma$ be an $S$-sorted signature, $A$ a $\Sigma$-algebra, and $n \in \mathbb{N}$. If $\Sigma$ is such that, for every $(w, s) \in S^* \times S$, $\Sigma_{w,s} = \emptyset$ if $|w| > n$—in which case we will say that every operation of $A$ is of an arity $\leq n$—, then the uniform algebraic many-sorted closure operator $Sg_A$ is an $n$-ary many-sorted closure operator on $A$, i.e., $Sg_A = (Sg_A)^{\leq n}$.

Proof. It follows from $Sg_A(X) = E_A(X)$ and from the fact that, for every $X \subseteq A$, $E_A(X) \subseteq (Sg_A)^{\leq n}(X) \subseteq Sg_A(X)$. The details are left to the reader. However, we notice that it is advisable to split the proof into two cases, one for $n = 0$ and another one for $n \geq 1$. □

Proposition 3.4. Let $A$ be an $S$-sorted set, $J$ a many-sorted closure operator on $A$, and $n \in \mathbb{N}$. If $J$ is $n$-ary (hence, by Proposition 2.17, algebraic) and uniform, then there exists an $S$-sorted signature $\Sigma'$ and a $\Sigma'$-algebra $A'$ such that $J = Sg_{A'}$ and every operation of $A'$ is of an arity $\leq n$.

Proof. If we denote by $A = (A, F)$ the $\Sigma$-algebra associated to $J$ constructed in the proof of Theorem 3.31 then taking as $\Sigma'$ the $S$-sorted signature defined, for every $(w, s) \in S^* \times S$, as: $\Sigma'_{w,s} = \Sigma_{w,s}$, if $|w| \leq n$; and $\Sigma'_{w,s} = \emptyset$, if $|w| > n$, and as $A' = (A', F')$ the $\Sigma'$-algebra defined as: $A' = A$, and $F' = F \circ \text{inc}^{\Sigma', \Sigma}$, where $\text{inc}^{\Sigma', \Sigma} = (\text{inc}^{\Sigma', \Sigma}_{w,s})_{(w, s) \in S^* \times S}$ is the canonical inclusion of $\Sigma'$ into $\Sigma$, then one can show that $J = Sg_{A'}$. □

From the just stated proposition together with Proposition 3.3 it follows immediately the following corollary, which is an algebraic characterization of the $n$-ary and uniform many-sorted closure operators.

Corollary 3.5. Let $J$ be a many-sorted closure operator on an $S$-sorted set $A$ and $n \in \mathbb{N}$. Then $J$ is $n$-ary and uniform if, and only if, there exists an $S$-sorted signature $\Sigma$ and a $\Sigma$-algebra $A$ such that $J = Sg_A$ and every operation of $A$ is of an arity $\leq n$.

4. The irredundant basis theorem for many-sorted closure spaces.

We next show Tarski’s irredundant basis theorem for many-sorted closure spaces.

Theorem 4.1 (Tarski’s irredundant basis theorem for many-sorted closure spaces). Let $(A, J)$ be a many-sorted closure space. If $J$ is an
n-ary many-sorted operator on the \( S \)-sorted set \( A \), with \( n \geq 2 \), and if \( i < j \) with \( i, j \in \text{IrB}_J(A) \) such that
\[
\{i + 1, \ldots, j - 1\} \cap \text{IrB}_J(A) = \emptyset,
\]
then \( j - i \leq n - 1 \). In particular, if \( n = 2 \), then \( \text{IrB}_J(A) \) is a convex subset of \( N \).

**Proof.** Let \( Z \subseteq A \) be an irredundant basis with respect to \( J \) such that \( \text{card}(Z) = j \) and \( \mathcal{K} = \{ X \in \text{IrB}_J(A) \mid \text{card}(X) \leq i \} \). Since \( J \) is \( n \)-ary, we can assert that \( J(Z) = A = \bigcup_{m \in \mathbb{N}} J^m(Z) \), so, for every \( s \in S \), \( J(Z)_s = A_s = \bigcup_{m \in \mathbb{N}} J^m(Z)_s \). Let \( X \) be an element of \( \mathcal{K} \). Then there exists a \( k \in \mathbb{N} - 1 \) such that \( X \subseteq J^k(Z) \). The natural number \( k \) should be strictly greater than 0, because if \( k = 0 \), \( X \subseteq J^0(Z) = Z \), but \( \text{card}(X) = i < j = \text{card}(Z) \), so \( Z \) would not be an irredundant basis. So that, for every \( X \in \mathcal{K} \), \( \{ k \in \mathbb{N} - 1 \mid X \subseteq J^k(Z) \} \neq \emptyset \).

Therefore, for every \( X \in \mathcal{K} \), we can choose the least element of such a set, denoted by \( d_Z(X) \), and there is fulfilled that \( d_Z(X) \) is greater than or equal to 1. For \( d_Z(X) - 1 \) we have that \( X \not\subseteq J^{d_Z(X)-1}(Z) \).

So we conclude that there exists a mapping \( d_Z : \mathcal{K} \rightarrow \mathbb{N} - 1 \) that to an \( X \in \mathcal{K} \) assigns \( d_Z(X) \). The image of the mapping \( d_Z \), which is a nonempty part of \( \mathbb{N} - 1 \), is well-ordered, hence it has a least element, which is, necessarily, non zero, \( t + 1 \), therefore, since \( \mathcal{K}/\text{Ker}(d_Z) \) is isomorphic to \( \text{Im}(d_Z) \), by transport of structure, it will also be well-ordered, then we can always choose an \( X \in \mathcal{K} \) such that, for every \( Y \in \mathcal{K} \), \( d_Z(X) \leq d_Z(Y) \), e.g., an \( X \) such that its equivalence class corresponds to the minimum \( t + 1 \) of \( \text{Im}(d_Z) \). Moreover, among the \( X \) which have the just mentioned property, we choose an \( X^0 \) such that, for every \( Y \in \mathcal{K} \) with \( Y \subseteq J^{d_Z(X)+1}(Z) \), it happens that
\[
\text{card}(X^0 \cap (J^{d_Z(X)+1}(Z) - J^t(Z))) \leq \text{card}(Y \cap (J^{d_Z(X)+1}(Z) - J^t(Z))).
\]

By the method of election we have that \( X^0 \subseteq J^{d_Z(X)+1}(Z) \) but \( X^0 \not\subseteq J^t(Z) \). Of the latter we conclude that there exists an \( s_0 \in S \) such that \( X^0 \not\subseteq J^t_s(Z) \), therefore
\[
(J^{d_Z(X)+1}(Z)_{s_0} - J^t(Z)_{s_0}) \cap X^0_{s_0} \neq \emptyset.
\]

Let \( a_0 \in (J^{d_Z(X)+1}(Z)_{s_0} - J^t(Z)_{s_0}) \cap X^0_{s_0} \). Then \( a_0 \in X^0_{s_0} \), \( a_0 \in J^{d_Z(X)+1}(Z)_{s_0} \) but \( a_0 \not\in J^t(Z)_{s_0} \). However, \( J^{d_Z(X)+1}(Z) = J_{\leq n}(Z)_{s_0} \), by definition, hence there exists a part \( F \) of \( J_{\leq n}(Z) \) such that \( \text{card}(F) \leq n \) and \( a_0 \in J(F)_{s_0} \). Let \( X^1 \) be the part of \( A \) defined as follows:
\[
X^1_s = \begin{cases} 
X^0_s \cup F_s, & \text{if } s \neq s_0; \\
(X^0_{s_0} - \{a_0\}) \cup F_{s_0}, & \text{if } s = s_0.
\end{cases}
\]

It holds that \( X^0 \subseteq J(X^1) \). Therefore \( J(X^0) \subseteq J(X^1) \), but \( J(X^0) = A \), hence \( J(X^1) = A \), i.e., \( X^1 \) is a finite generator with respect to \( J \), thus \( X^1 \) will contain a minimal generator \( X^2 \) with respect to \( J \). It
holds that \( \text{card}(X^2) \leq \text{card}(X^1) < \text{card}(X^0) + n \). It cannot happen that \( \text{card}(X^0) + n \leq j \). Because if \( \text{card}(X^0) + n \leq j \), then \( \text{card}(X^2) < j \), hence, since
\[
\{i+1, \ldots, j-1\} \cap \text{IrB}(A, J) = \emptyset,
\]
\( X^2 \in \mathcal{K} \), but \( X^2 \subseteq J_{\leq n}^{i+1}(Z) \) and, moreover, it happens that
\[
\text{card}(X^2 \cap (J_{\leq n}^{i+1}(Z) - J_{\leq n}^{i}(Z))) < \text{card}(X^0 \cap (J_{\leq n}^{i+1}(Z) - J_{\leq n}^{i}(Z))),
\]
because \( a_0 \not\in X^2_{a_0} \) but \( a_0 \in X^0_{a_0} \), which contradicts the choice of \( X^0 \).
Hence \( \text{card}(X^0) + n > j \). But \( \text{card}(X^0) \leq i \), therefore \( j - i < n \), i.e., \( j - i \leq n - 1 \).

\[\Box\]

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