Mathematical aspects of optimal control of transference processes in spatial networks

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Abstract. The new kind of practical issues was born due to modern trend on globalization and transition to the concept of industry 4.0. These are the processes in commercial networks representing the movement of goods and finance, which have specific economic indicators; migration of populations and labor resources; transfer of one-parameter continuous media (diffusion effect). The issues with higher level of complexity occur during the research of multi-phase environments dynamics in network-like media and mathematical modeling of such processes. The authors divide these two problems and offer the formalized description for the laminar processes and dynamic convective processes with feedback. There was developed an approach to solve the problems of optimal control for these processes. The research is based on methods for analyzing systems of partial differential equations with distributed parameters on a graph or network-like domain.

1. Introduction
This article is devoted to mathematical description of two essential transference networks processes: 1) established process with pronounced laminar character; 2) process with inverse relationship, which dynamics has convective character. The first can be found in the study of economic laws in commercial networks in the analysis of the movement of goods, money, human migration, the transfer of one-parameter continuous media, etc. [1-3]. The second process occurs in the study of dynamics of multi-phase medium like hydro- and aero networks, radio and TV network [4-8].

At the same time, the approach to the solution of the actual problem of network transference processes is developed, that is the optimal control of such processes, and its subsequent analysis. The mathematical formalism of differential systems of partial differential equations with distributed parameters on a graph (network) or network-like domain is used, the state of the differential system is determined by its weak solution.

2. Processes with established laminar character
These include regularities or phenomena observed on one-parameter carriers, which include spatial oriented connected graphs Γ with edges γ parametrized by a single interval [0,1].
In the area $\Gamma_r = \Gamma \times (0, T)$ is considered the equation

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial y(x,t)}{\partial x} \right) + b(x) y(x,t) = f(x,t),$$

(1)

representing a system of differential equations with distributed parameters on each edge $\gamma \in \Gamma$. The state of system (1) in the area $\big[0, T\big]\times\Gamma$ is determined by a weak solution $(y(x,t))$ of equation (1) satisfying the compatibility conditions

$$\sum_{\gamma_j \in R(\xi)} a(1)_{\gamma_j} \frac{\partial y(1,t)}{\partial x} = \sum_{\gamma_j \in R(\xi)} a(0)_{\gamma_j} \frac{\partial y(0,t)}{\partial x},$$

(2)

($R(\xi)$ and $R(\xi)$ – sets of edges, oriented, respectively, "to the node $\xi"$ and "from the node $\xi,"$ $u(\cdot)_{\gamma}$ is the restriction of the function $u(\cdot)$ to the edge $\gamma$), initial and boundary conditions

$$y_{|_{t=0}} = \phi(x), \quad x \in \Gamma, \quad a(x) \frac{\partial y}{\partial x} \big|_{x \in \partial \Gamma} = v(x,t), \quad t \in (0, T).$$

The function $v(x,t)$ is the boundary control action on the system (1) (the boundary control system (1)); $f(x,t), \phi(x), a(x), b(x)$ – defined functions.

2.1. Definition 1.

A weak solution of the initial-boundary value problem (1) – (3) is a function $y(x,t) \in V^{1,0}(a, \Gamma_r)$ that satisfies an integral identity.

$$\int_{\Gamma} y(x,t) \eta(x,t) dx - \int_{\Gamma} y(x,t) \frac{\partial \eta(x,t)}{\partial t} \ dx dt + \ell_t(y, \eta) =$$

$$= \int_{\Gamma} \phi(x) \eta(x,0) dx + \int_{\Gamma} \phi(x,t) \eta(x,t) dx dt + \int_{\Gamma} f(x,t) \eta(x,t) dx dt$$

(3)

for any $t \in [0, T]$ and for any function $\eta(x,t) \in W(\Gamma_r)$; $V^{1,0}(a, \Gamma_r)$ and $W^1(\Gamma_r)$ – Sobolev spaces whose elements satisfy conditions (2) and have generalized first order derivatives, $\ell_t(y, \eta)$ is a bilinear form, defined by the relation

$$\ell_t(y, \eta) = \int_{\Gamma} \left( a(x) \frac{\partial y(x,t)}{\partial x} \frac{\partial \eta(x,t)}{\partial x} + b(x) y(x,t) \eta(x,t) \right) dx dt.$$

2.2. Theorem 1.

The initial-boundary value problem (1) – (3) has at least one weak solution in space $V^{1,0}(a, \Gamma_r)$.

Optimal boundary control with delay. As is known, any dynamic processes are accompanied by the effect of a time delay (see, for example, the works [3, 5] and bibliography there), which is a reason to consider equation (1) as a system of equations with a constant delay $h \in (0, T)$:

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial y(x,t)}{\partial x} \right) + b(x) y(x,t) + c(x) y(x,t-h) = f(x,t),$$

(4)

here $x,t \in \Gamma_{h,T} = \Gamma \times (h, T)$, the coefficient $c(x)$ is a bounded measurable on $\Gamma$ function. Each solution $y(x,t), \quad x,t \in \Gamma_h$ of system (4) is determined by the initial function $\theta(x,t)$:

$$y(x,t) = \theta(x,t), \quad x,t \in \Gamma_h,$$

(5)
and boundary condition

\[ a(x) \frac{\partial y}{\partial x} \frac{\partial}{h_{t,T}} = v(x,t). \]  

(6)

We obtain the initial boundary-value problem (4) - (6), whose weak solution \( y(v)(x,t) \) determines the state \( y(v)(x,t) \) of system (4) in space \( V^{1,0}(a, \Gamma_{h,T}) \), \( v(x,t) \) - the boundary effect on system (4).

We present equation (4) in a form more convenient for analysis [5]. Let \( Z : W^1(a, \Gamma_T) \rightarrow W^1(a, \Gamma_T) \) be a linear continuous operator (delay operator) defined by the relation

\[ Zy = \begin{cases} y(x,t - h), & x,t \in \Gamma_{h,T}, \\ 0, & x,t \in \Gamma_h. \end{cases} \]

We define a function \( \theta(x,t) \in V^{1,0}(a, \Gamma_h) \) that satisfies the boundary condition

\[ a(x) \frac{\partial \theta(x,t)}{\partial x} \frac{\partial}{h_{t,T}} = v(x,t) ; \]  

on \( \Gamma_T \) we introduce the function

\[ F(x,t) = \left( \frac{\partial \theta(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial \theta(x,t)}{\partial x} \right) + b(x) \theta(x,t), x,t \in \Gamma_h, \right. \]

(how to understand this expression on \( \Gamma_h \) will be clear below) and the function \( y_0(x) = \theta(x,0) \) (so \( y_0(x) \in L_2(\Gamma) \)). Then the problem (4) – (6) takes the form

\[ \frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial y(x,t)}{\partial x} \right) + b(x)y(x,t) + c(x)Zy(x,t) = F(x,t), x,t \in \Gamma_T, \]

(7)

\[ y \big|_{t=0} = y_0(x), \; x \in \Gamma, \; a(x) \frac{\partial y}{\partial x} \big|_{x \in \partial \Gamma_T} = v(x,t), \; t \in (0,T). \]

(8)

Consider the system (7), the state of which is defined as the solution \( y(v)(x,t) \) of the initial-boundary value problem (7), (8) in space \( V^{1,0}(a, \Gamma_T) \).

2.3. **Definition 2**

A weak solution to the initial-boundary value problem (7), (8) is a function \( y(v)(x,t) \in V^{1,0}(a, \Gamma_T) \) that satisfies an integral identity

\[
\int_I y(v)(x,t) \eta(x,t) dx - \int_I y(v)(x,t) \frac{\partial \eta(x,t)}{\partial t} dx dt + \int_I c(x)Zy(v)(x,t)\eta(x,t) dx dt = \\
= \int_0^T y_0(x) \eta(x,0) dx + \int_I v(x,t) \eta(x,t) dx dt + \int_I F(x,t) \eta(x,t) dx dt
\]

for any \( t \in [0,T] \) and for any function \( \eta(x,t) \in W^1(a, \Gamma_T) \).

Let the space \( U = L_2(\partial \Gamma_T) \) (here \( \partial \Gamma_T \) - the boundary of \( \Gamma_T \)) of the acceptable controls \( v(x,t) \) of the system (7) be given and let \( L_2(\partial \Gamma_T) \) be the observation space, which is the boundary and have an important for applications form \( C \big|_{\partial \Gamma_T} = M \big|_{\partial \Gamma_T} \) \( (M : L_2(\partial \Gamma_T) \rightarrow L_2(\partial \Gamma_T) - \) linear continuous operator, \( M^* \) – conjugate operator to it, \( C \) - observation operator), where \( y(v) \big|_{\Gamma_T} \) is the function trace on the surface \( \partial \Gamma_T \) (other types of observations are also possible, for example, the final). The
functional \( J(v) \) that requires minimization on \( U \) is taken as \( J(v) = PC_y(v) - z_0 P_{L_2(\Omega_T)} + (Nv,v)_U \), here 
\( z_0(x,t) \in L_2(\partial \Omega_T) \) the given observation, \( N : U \to U \) – the linear continuous Hermitian operator, 
\( (Nv,v)_U \geq \zeta P v P_U \) (\( \zeta > 0 \) is a fixed constant).

The presence of a term \( (Nv,v)_U \) in the representation of the functional \( J(v) \) guarantees the coercivity of the quadratic component of the functional \( J(v) \). The task of optimal boundary control of system (7) is to find \( \inf_{v \in U_0} J(v) \) on a convex closed subset \( U_0 \subset U \) [2, 3].

2.4. Theorem 2

The problem of optimal boundary control of system (7) has a unique solution \( v^* \in U_0 \), i.e.

\[ J(v^*) = \inf_{v \in U_0} J(v) ; v^* \in U_0 \] - optimal control of system (7).

Synthesis of optimal boundary control. For system (7), we define the conjugate state \( \omega(x,t) \), taking into account the representation of the conjugate operator

\[ Z^* : W^1(a,\Gamma_T) \to W^1(a,\Gamma_T) \] in the form

\[ Z^* p = \begin{cases} p(x,t+h), & x,t \in \Gamma_{T-h}, \\ 0, & x,t \in \Gamma_{T-a}, \end{cases} \]

as a weak solution \( \omega(x,t) \) of the initial-boundary value problem in space \( W^1(a,\Gamma_T) \):

\[ -\frac{\partial \omega(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial \omega(x,t)}{\partial x} \right) + b(x)\omega(x,t) + c(x)Z^*\omega(x,t) = 0, \]

\[ \omega(x,T) = 0, \quad x \in \Gamma, \quad a(x) \frac{\partial \omega}{\partial x} \bigg|_{x=\Gamma} = M^*(My(x,t) - z_0(x,t)). \]

2.5. Theorem 3

In order for an element \( u(x) \in U_0 \) to be the optimal control of system (7), and therefore (4), it is necessary and sufficient that the following relations are satisfied:

\[ \int_0^t \int y(u(x,t))\eta(x,t)dxdt - \int_0^t \int y(u(x,t))\frac{\partial \eta(x,t)}{\partial t}dxdt + \ell_1(y(u),\eta) + \int_0^t c(x)Zy(u(x,t))\eta(x,t)dxdt = 0, \]

\[ = \int_0^t y_0(x)\eta(x,h)dx + \int_0^t u(x,t)\eta(x,t)dxdt + \int F(x,t)\eta(x,t)dxdt \]

for any \( t \in [0,T] \) and for any function \( \eta(x,t) \in W^1(a,\Gamma_T) \);

\[ -\int_{ \Gamma_T } \frac{\partial \omega(u(x,t))}{\partial t} \xi(x,t)dxdt + \ell_1(\omega(u),\xi) + \int_{ \Gamma_T } c(x)Z^*\omega(u(x,t))\xi(x,t)dxdt = 0, \]

\[ = \int_{ \Gamma_T } M^*(My(v)(x,t) - z_0(x,t))\xi(x,t)dxdt \]

for any functions

\[ \xi(x,t) \in W^{1,0}(a,\Gamma_T) ; \int_{ \Gamma_T } (\omega(u)(x,t) + Nu(x,t))(v(x,t) - u(x,t))dxdt \geq 0 \]

for any \( v \in U_0 \). Here \( y(u) \in V^{1,0}(a,\Gamma_T) \), \( \omega(u) \in W^1(a,\Gamma_T) \) and \( \omega(u)(x,T) = 0, \quad x \in \Gamma \).
We consider the problem of synthesis of optimal boundary control for the case of the absence of restrictions on control: \( \mathbf{U} \) coincides with \( \mathbf{U} \). Then, in relation (11), we can put \( v = u \pm \nu \) and by virtue of arbitrariness \( v \in \mathbf{U} \) it is transformed into equality, and therefore, \( \omega(u(x,t)) + Nu(x,t) = 0 \).

Excluding with its help \( u \ (u = -N^{-1}\omega(u(x,t))) \), we come to the conclusion that the optimal control is determined from the solution of the system of two integral identities (variational relations) (9) and (10) for any \( t \in [0,T] \) and, thus, the optimal control has the form

\[
u(x,t) = -N^{-1}\omega(u(x,t)).
\]

Relation (12) synthesizes the optimal boundary control of the system (7) (and therefore (4)) with a delay: the optimal control is determined through the conjugate state \( \omega(u(x,t)) \) of this system.

3. Processes with convective character

Mathematical modeling of the processes transference liquid media and gases (as well as their mixtures) in a network-like system is in the formative stage and represented only by fragmentary results. One of these models (of course, extremely simplified) is presented below. For the vector function \( Y(x,t) = \{y_1(x,t), y_2(x,t), \ldots, y_n(x,t)\} \ (x = \{x_1, x_2, \ldots, x_n\}) \) defined in the domain \( \mathfrak{A} = \mathfrak{A} \times (0,T) \ (T < \infty) \), we consider the classical Navier-Stokes system

\[
\frac{\partial Y}{\partial t} - \nu \Delta Y + \sum_{i=1}^{n} \frac{\partial Y}{\partial x_i} + \nabla p = f, \quad \text{div} Y = 0
\]

(13)

relative to a pair of functions \( \{Y(x,t), p(x,t)\} \) (the Navier-Stokes system in the evolutionary case; \( Y = \{Y_1, Y_2, \ldots, Y_n\} \) – definable vector function, \( p \) – scalar function, \( x \in \mathbb{R}^n \) describes the dynamics of an incompressible viscous medium (\( \nu \) – viscosity) with the velocity vector \( Y \) of the hydraulic flow and the convective component determined by the term \( \sum_{i=1}^{n} \frac{\partial Y}{\partial x_i} \). Wherein, in each nodal place \( \xi \), are valid the relations:

\[
Y|_{S_i^-(\xi)} = Y|_{S_i^+(\xi)}, \quad \sum_{i=1}^{n} \frac{\partial Y}{\partial n_i^+}|_{S_i^+(\xi)} + \sum_{i=1}^{n} \frac{\partial Y}{\partial n_i^-}|_{S_i^-(\xi)} = 0,
\]

(14)

referred to in the literature as conjugation conditions [8 – 12]; \( S_i^-(\xi) \) and \( S_i^+(\xi) \) – one-sided surfaces for \( S_i(\xi) \), determined by the direction of the normal \( n_i^- \), \( n_i^+ \) to the surfaces \( S_i(\xi), S_i^-(\xi), S_i^+(\xi) \). Attaching to (13), (14) the initial \( Y(x,0) = Y_o(x), x \in \mathfrak{A} \) at the time moment \( t = 0 \) and the boundary conditions \( Y|_{\partial \mathfrak{A}} = 0 \) on the boundary of the volume of the continuous medium, we obtain the initial boundary value problem for finding the functions \( Y(x,t) \) and \( p(x,t) \) in the closed region \( \mathfrak{A} \) \((\mathfrak{A} = (\mathfrak{A} \cup \partial \mathfrak{A}) \times [0,T])\). In applied problems of hydrodynamics, the net-like region \( \mathfrak{A} \) is the hydraulic system, conditioned by the pressure \( p \ (\nabla p = \text{grad} p – \text{pressure gradient}) \) and distributing fluid flows (multi-phase medium), the function \( Y(x,t) \) describes the velocity vector of the hydraulic flow in the area \( \mathfrak{A} \), relations (13), (14) describe the dynamics of an incompressible fluid with viscosity \( \nu > 0 \) in the region \( \bigcup_{t} \mathfrak{A} \times (0,T) \), the balanced equalities (5), (6) determine the conditions for the flow of fluid in the nodal points of the hydraulic system \( \mathfrak{A} \), \( f(x,t) \) – the density of external forces; the transference process of a multiphase medium is isothermal.
4. Conclusions

The initial-boundary value problem for system (13), (14) is a natural continuation of research in the direction of the dimension increasing both for the spatial variable \( x \in \mathbb{R}^n \), \( n \geq 2 \), and functions describing the state of the system under study. In the simpler case of the absence of a convective effect, the following results were obtained: the unique solvability of the corresponding initial-boundary problem, the analysis of the problems of optimal control (distributed and starting) that often meet in practice. Sufficient attention is paid to the synthesis of optimal control. It should be noted the possibility of using the results of the work in the stabilization processes of physical phenomena [9-18] and related problems study [19-21].

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