ON DESCRIPTION OF BISTOCHASTIC KADISON-SCHWARZ OPERATORS ON $M_2(\mathbb{C})$

FARRUKH MUKHAMEDOV AND ABDUAZIZ ABDUGANIEV

Abstract. In this paper we describe bistochastic Kadison-Schwarz operators on $M_2(\mathbb{C})$. Such a description allows us to find positive, but not Kadison-Schwarz operators. Moreover, by means of that characterization we construct Kadison-Schwarz operators, which are not completely positive.

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1. Introduction

It is known that the theory of quantum dynamical systems provides a convenient mathematical description of irreversible dynamics of an open quantum system (see [2]) investigation of various properties of such dynamical systems have had a considerable growth. In a quantum setting, the matter is more complicated than in the classical case. Some differences between classical and quantum situations are pointed out in [10]. This motivates an interest to study dynamics of quantum systems (see [10]). One of the main objects of this theory is mapping (or channel) defined on matrix algebras. One of the main constraints to such a mapping is positivity and complete positivity. There are many papers devoted to this problem (see for example [3, 7, 14, 15]). In the literature the most tractable maps, the completely positive ones, have proved to be of great importance in the structure theory of $C^*$-algebras. However, general positive (order-preserving) linear maps are very intractable [7, 8, 9]. It is therefore of interest to study conditions stronger than positivity, but weaker than complete positivity. Such a condition is called Kadison-Schwarz (KS) property, i.e. a map $\phi$ satisfies the KS property if $\phi(a)^*\phi(a) \leq \phi(a^*a)$ holds for every $a$. Note that every unital completely positive map satisfies this inequality, and a famous result of Kadison states that any positive unital map satisfies the inequality for self-adjoint elements $a$. But KS-operators no need to be completely positive. In [13] relations between $n$-positivity of a map $\phi$ and the KS property of certain map is established. Some nice properties of the Kadison-Schwarz maps were investigated in [6, 12].

In this paper we are going to describe KS-operators which are unital, trace preserving linear mappings (i.e. bistochastic operators) defined on the algebra of 2 by 2 matrices $M_2(\mathbb{C})$. In Section 2 we show that the set of KS-operators forms a convex set. In section 3, we characterize bistochastic KS-operators on $M_2(\mathbb{C})$. Such a description allows us to find positive, but not Kadison-Schwarz operators. Moreover, by means of that characterization one can construct KS-operators, which are not completely positive. Note that trace-preserving maps arise naturally in quantum information theory [4, 5, 10, 11] and other situations in which one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system with which it interacts.

2. Preliminaries

Let $A$ and $B$ be unital $C^*$-algebras with identity $\mathbf{1}$. Recall that a linear mapping $\Phi : A \to B$ is called
Hence, from (2.2) - (2.3) one gets

\[ \tau = \text{identity matrix. By } \text{Tr} \text{ we mean trace on } \]

this completes the proof.

and all self-adjoint elements \( x \). A famous result of Kadison states that any positive unital map satisfies the inequality (2.1) for all self-adjoint elements \( x \in A \).

By \( \mathcal{KS}(A, B) \) we denote the set of all KS-operators mapping from \( A \) to \( B \).

**Theorem 2.1.** The following assertions hold true:

(i) Let \( \Phi, \Psi \in \mathcal{KS}(A, B) \), then for any \( \lambda \in [0, 1] \) the mapping \( \Gamma = \lambda \Phi + (1 - \lambda) \Psi \) belongs to \( \mathcal{KS}(A_1, A_2) \). This means \( \mathcal{KS}(A, B) \) is convex;

(ii) Let \( U, V \) be unitaries in \( A \) and \( B \), respectively, then for any \( \Phi \in \mathcal{KS}(A, B) \) the mapping \( \Psi_{U, V}(x) = U \Phi(V x V^*) U^* \) belongs to \( \mathcal{KS}(A, B) \).

**Proof.** (i). Let us show that \( \Gamma_\lambda \) satisfies (2.1). Let \( x \in A \), then one can see that

\[
\Gamma_\lambda(x^* x) = \lambda \Phi(x^* x) + (1 - \lambda) \Psi(x^* x)
\]

(2.2)

\[
\geq \lambda \Phi(x^* x) \Phi(x) + (1 - \lambda) \Psi(x^* x) \Psi(x)
\]

and

\[
\Gamma_\lambda(x) \Gamma_\lambda(x) = \lambda^2 \Phi(x)^* \Phi(x) + (1 - \lambda) \Phi(x)^* \Psi(x)
\]

\[
+ \lambda (1 - \lambda) \Psi(x)^* \Phi(x) + (1 - \lambda)^2 \Psi(x)^* \Psi(x)
\]

(2.3)

Hence, from (2.2) - (2.3) one gets

\[
\Gamma_\lambda(x^* x) - \Gamma_\lambda(x) \Gamma_\lambda(x) \geq \lambda (1 - \lambda) \big( \Phi(x) - \Psi(x) \big)^* \big( \Phi(x) - \Psi(x) \big) \geq 0,
\]

which proves the assertion.

(ii) For any \( x \in A \) one has

\[
\Psi_{U, V}(x^* x) = U \Phi((V x V^*)^* V x V^*) U^*
\]

\[
\geq U \Phi(V x V^*)^* U^* U \Phi(V x V^*) U^*
\]

\[
= \Psi_{U, V}(x)^* \Psi_{U, V}(x),
\]

this completes the proof. \( \square \)

Let us consider the set of 2 by 2 matrices \( M_2(\mathbb{C}) \) over \( \mathbb{C} \). In the sequel by \( \mathbf{1} \) we mean an identity matrix. By \( \text{Tr} \) we mean trace on \( M_2(\mathbb{C}) \). In what follows by \( \tau \) we denote a normalized trace, i.e. \( \tau = \frac{1}{2} \text{Tr} \).

A linear mapping \( \Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \) is called bistochastic if it is positive, unital and trace preserving, i.e. \( \tau(\Phi(x)) = \tau(x) \) for all \( x \in M_2(\mathbb{C}) \). Note that this terminology for maps that are both unital and stochastic was introduced in [1].

In the paper we are going to consider bistochastic KS-operators on \( M_2(\mathbb{C}) \). Therefore, by \( \mathcal{KS}(M_2(\mathbb{C})) \) we denote the set of all bistochastic KS-operators defined on \( M_2(\mathbb{C}) \). According to Theorem 2.1 the set \( \mathcal{KS}(M_2(\mathbb{C})) \) is convex.
3. Kadison-Schwarz operators on $M_2(\mathbb{C})$

It is known (see [2, 4]) that the identity and the Pauli matrices $\{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $M_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Every matrix $a \in M_2(\mathbb{C})$ can be written in this basis as $a = w_0 \mathbf{1} + w \cdot \sigma$ with $w_0 \in \mathbb{C}, w = (w_1, w_2, w_3) \in \mathbb{C}^3$, here by $w \cdot \sigma$ we mean the following

$$w \cdot \sigma = w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3.$$ 

The following facts holds (see [11]):

(a) a matrix $a \in M_2(\mathbb{C})$ is self-adjoint if and only if $w_0$ and $w$ are real;

(b) a matrix $a \in M_2(\mathbb{C})$ is positive if and only if $\|w\| \leq w_0$, where

$$\|w\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2};$$

(c) a matrix $a \in M_2(\mathbb{C})$ is normal if and only if $[w, \overline{w}] = [\overline{w}, w]$ for every $w \in \mathbb{C}^3$, where $[., .]$ stands for the cross product of vectors in $\mathbb{C}^3$.

Every $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ linear mapping can also be represented in this basis by a unique $4 \times 4$ matrix $\mathbf{F}$. It is trace preserving if and only if $\mathbf{F} = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}$ where $T$ is a $3 \times 3$ matrix and $0$ and $t$ are row and column vectors respectively so that

$$\Phi(w_0 \mathbf{1} + w \cdot \sigma) = w_0 \mathbf{1} + (w_0 t + T w) \cdot \sigma. \quad (3.1)$$

When $\Phi$ is also positive then it maps the subspace of self-adjoint matrices of $M_2(\mathbb{C})$ into itself, which implies that $T$ is real. A linear mapping $\Phi$ is unital if and only if $t = 0$. So, in this case we have

$$\Phi(w_0 \mathbf{1} + w \cdot \sigma) = w_0 \mathbf{1} + (T w) \cdot \sigma. \quad (3.2)$$

Hence, any bistochastic mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ has a form (3.2). Now we are going to give a characterization bistochastic KS-maps.

**Theorem 3.1.** Any bistochastic mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is KS-operator if and only if one has

$$\|T w\| \leq \|w\|, \quad T w = \overline{T \overline{w}} \quad (3.3)$$

$$\|T [w, \overline{w}] - [T w, \overline{T \overline{w}}]\| \leq \|w\|^2 - \|T w\|^2 \quad (3.4)$$

for all $w \in \mathbb{C}^3$.

**Proof.** 'if' part. Let $x \in M_2(\mathbb{C})$ be an arbitrary element, i.e. $x = w_0 \mathbf{1} + w \cdot \sigma$. Then $x^* = \overline{w_0} \mathbf{1} + \overline{w} \cdot \sigma$. Therefore

$$x^* x = (|w_0|^2 + \|w\|^2) \mathbf{1} + (w_0 \overline{w} + \overline{w_0} w - i [w, \overline{w}]) \cdot \sigma.$$ 

Consequently, we have

$$\Phi(x) = w_0 \mathbf{1} + (T w) \cdot \sigma, \quad \Phi(x^*) = \overline{w_0} \mathbf{1} + (T \overline{w}) \cdot \sigma \quad (3.5)$$

$$\Phi(x^* x) = (|w_0|^2 + \|w\|^2) \mathbf{1} + (w_0 \overline{T w} + \overline{w_0} T w - i [T w, \overline{w}]) \cdot \sigma \quad (3.6)$$

$$\Phi(x) \Phi(x) = (|w_0|^2 + \|T w\|^2) \mathbf{1} + (w_0 T \overline{w} + \overline{w_0} T w - i [T w, \overline{T w}]) \cdot \sigma \quad (3.7)$$
From (3.6)-(3.7) one gets
\[
\Phi(x^*x) - \Phi(x)^*\Phi(x) = \left(|w_0|^2 - \|T\mathbf{w}\|^2\right)\mathbf{I} + \left(w_0(T\mathbf{w} - \overline{T\mathbf{w}}) - i(T[\mathbf{w}, \overline{\mathbf{w}}] - [T\mathbf{w}, \overline{T\mathbf{w}}])\right) \cdot \sigma \geq 0
\]
Hence, due to (b) we conclude that \(\Phi\) should be positive, which means \(T\) is real, therefore one gets \(T\mathbf{w} = \overline{T\mathbf{w}}\). Consequently, the last inequality yields
\[
(3.8) \quad \left(|w_0|^2 - \|T\mathbf{w}\|^2\right)\mathbf{I} - i(T[\mathbf{w}, \overline{\mathbf{w}}] - [T\mathbf{w}, \overline{T\mathbf{w}}]) \cdot \sigma \geq 0
\]
which again with (b) implies the assertion.

'only if' part. Let (3.3)-(3.4) be satisfied. Then we have (3.8), which with (3.4) and (3.6)-(3.7) yields (2.1). This completes the proof. \(\square\)

Let \(\Phi\) be a bistochastic KS-operator on \(M_2(\mathbb{C})\), then it can be represented by (3.2). Following [4] let us decompose the matrix \(T\) as follows \(T = RS\), where \(R\) is a rotation and \(S\) is a self-adjoint matrix (see [4]). Define a mapping \(\Phi_S\) as follows
\[
(3.9) \quad \Phi_S(w_0\mathbf{I} + \mathbf{w} \cdot \sigma) = w_0\mathbf{I} + (S\mathbf{w}) \cdot \sigma.
\]
Every rotation is implemented by a unitary matrix in \(M_2(\mathbb{C})\), therefore there is a unitary \(U \in M_2(\mathbb{C})\) such that
\[
(3.10) \quad \Phi(x) = U\Phi_S(x)U^*, \quad x \in M_2(\mathbb{C}).
\]

On the other hand, every self-adjoint operator \(S\) can be diagonalized by some unitary operator, i.e. there is a unitary \(V \in M_2(\mathbb{C})\) such that \(S = V\sigma\lambda_1,\lambda_2,\lambda_3V^*\), where
\[
(3.11) \quad \sigma\lambda_1,\lambda_2,\lambda_3 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},
\]
where \(\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}\).

Consequently, the mapping \(\Phi\) can be represented by
\[
(3.12) \quad \Phi(x) = \tilde{U}\Phi_{\sigma\lambda_1,\lambda_2,\lambda_3}(x)\tilde{U}^*, \quad x \in M_2(\mathbb{C})
\]
for some unitary \(\tilde{U}\). Due to Theorem 2.1 the mapping \(\Phi_{\sigma\lambda_1,\lambda_2,\lambda_3}\) is also KS-operator. Hence, all bistochastic KS-operators can be characterized by \(\Phi_{\sigma\lambda_1,\lambda_2,\lambda_3}\) and unitaries. In what follows, for the sake of shortness by \(\Phi_{\lambda_1,\lambda_2,\lambda_3}\) we denote the mapping \(\Phi_{\sigma\lambda_1,\lambda_2,\lambda_3}\). It is clear to observe from (3.3) that \(|\lambda_k| \leq 1, k = 1,2,3\). It is easy to see that the mapping \(\Phi_D \mapsto U\Phi_D U^*\) is affine, therefore, if \(\Phi_D\) is an extreme point of \(KS(M_2(\mathbb{C}))\) then \(U\Phi_D U^*\) is an extreme point of \(KS(M_2(\mathbb{C}))\) as well. Denote
\[
(3.13) \quad \Delta = \{ (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R}^3 : \Phi_{\lambda_1,\lambda_2,\lambda_3} \in KS(M_2(\mathbb{C})) \}.
\]
According to Theorem 2.1 the set \(\Delta\) is convex. Now taking into account that the mapping \(\Phi_{\lambda_1,\lambda_2,\lambda_3} \mapsto \Phi_{\lambda_1,\lambda_2,\lambda_3}\) is affine, we infer that if \((\lambda_1,\lambda_2,\lambda_3)\) is an extreme point of \(\Delta\), then \(\Phi_{\lambda_1,\lambda_2,\lambda_3}\) is also extreme point of \(KS(M_2(\mathbb{C}))\).

**Example 1.** Let us consider a famous example of non completely positive operator defined by transposition, i.e. \(\Phi(x) = x^T\), where for \(x \in M_2(\mathbb{C})\) by \(x^T\) we denote its transposition. This mapping can be written in terms of \(\Phi_{\lambda_1,\lambda_2,\lambda_3}\) as follows \(\Phi = \Phi_{(1,-1,1)}\). First observe that by taking \(\mathbf{w} = (1,1,1)\) in (3.1) one finds
\[
2\sqrt{(\lambda_1 - \lambda_2\lambda_3)^2 + (\lambda_2 - \lambda_1\lambda_3)^2} \leq 2 - \lambda_1^2 - \lambda_2^2 + 1 - \lambda_3^2.
\]
Putting \(\lambda_3 = 1\), then the last one can be written as follows
\[
(3.14) \quad 2\sqrt{2|\lambda_1 - \lambda_2|} \leq 2 - \lambda_1^2 - \lambda_2^2.
\]
It is clear that at $\lambda_1 = 1$, $\lambda_2 = -1$, the inequality (3.14) is not satisfied, hence (3.4) as well. This means that $\Phi_{(1,-1,1)}$ is positive, but not KS-map.

In [11] it has been given a characterization of completely positivity of $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$. Namely, the following result holds.

**Theorem 3.2.** A map $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ is complete positive if and only if the following inequalities are satisfied

(3.15) \[(\lambda_1 + \lambda_2)^2 \leq (1 + \lambda_3)^2\]

(3.16) \[(\lambda_1 - \lambda_2)^2 \leq (1 - \lambda_3)^2\]

(3.17) \[1 - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \geq 4(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 - 2\lambda_1\lambda_2\lambda_3)\]

Let us characterize KS operators of the form $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$.

Using simple calculation from (3.1) with $T = D_{\lambda_1,\lambda_2,\lambda_3}$ we obtain the following

\[A|w_2\overline{w}_3 - w_3\overline{w}_2|^2 + B|w_1\overline{w}_3 - w_3\overline{w}_1|^2 + C|w_1\overline{w}_2 - w_2\overline{w}_1|^2 \leq (\alpha|w_1|^2 + \beta|w_2|^2 + \gamma|w_3|^2)^2,\]

where $w = (w_1, w_2, w_3) \in \mathbb{C}^3$ and

(3.19) $\alpha = |1 - \lambda_1^2|$, $\beta = |1 - \lambda_2^2|$, $\gamma = |1 - \lambda_3^2|$

(3.20) $A = |\lambda_1 - \lambda_2\lambda_3|^2$, $B = |\lambda_2 - \lambda_1\lambda_3|^2$, $C = |\lambda_3 - \lambda_1\lambda_2|^2$.

Due to the inequality $|2\Re(wv)| \leq |w|^2 + |v|^2$, we have

\[|w_i\overline{w}_j - w_j\overline{w}_i|^2 = |2\Re(w_iw_j)|^2 \leq |w_i|^4 + 2|w_i|^2|w_j|^2 + |w_j|^4 \quad (i \neq j)\]

hence, we estimate LHS of (3.18) by

\[A(|w_2|^4 + 2|w_2|^2|w_3|^2 + |w_3|^4) + B(|w_1|^4 + 2|w_1|^2|w_3|^2 + |w_3|^4) - C(|w_1|^4 + 2|w_1|^2|w_2|^2 + |w_2|^4)\]

Consequently, from (3.18) we derive the following one

(3.21) $\quad + 2|w_1|^2|w_2|^2(\alpha\beta - C) + 2|w_1|^2|w_3|^2(\alpha\gamma - A) + 2|w_2|^2|w_3|^2(\beta\gamma - A) \geq 0$

It is easy to see that (3.21) is satisfied if one has

\[\alpha^2 \geq B + C, \quad \beta^2 \geq A + C, \quad \gamma^2 \geq A + B, \quad \alpha\beta \geq C, \quad \alpha\gamma \geq B, \quad \beta\gamma \geq A.\]

Substituting above denotations (3.19), (3.20) to the last inequalities, and doing simple calculation one derives

(3.22) $\quad (1 + \lambda_1^2)(3 + \lambda_2^2 + \lambda_3^2 - \lambda_1^2) \leq 4(1 + \lambda_1\lambda_2\lambda_3);$

(3.23) $\quad (1 + \lambda_2^2)(3 + \lambda_1^2 + \lambda_3^2 - \lambda_2^2) \leq 4(1 + \lambda_1\lambda_2\lambda_3);$

(3.24) $\quad (1 + \lambda_3^2)(3 + \lambda_1^2 + \lambda_2^2 - \lambda_3^2) \leq 4(1 + \lambda_1\lambda_2\lambda_3);$

(3.25) $\quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1 + 2\lambda_1\lambda_2\lambda_3.$

Hence, we have the following

**Theorem 3.3.** If (3.22), (3.23), (3.24) and (3.25) are satisfied, then a map $\Phi_{(\lambda_1,\lambda_2,\lambda_3)}$ is a KS-operator.
The last theorem allows us to construct lots of KS-operators, which are not completely positive (see Example 2).

Let us consider mappings $\Phi_{(\lambda,\lambda,\mu)}$, and for the check the conditions of Theorems 3.2 and 3.3. Calculating (3.15), (3.16), (3.17), (3.22), (3.23), (3.24) and (3.25) we obtain the following
\[ \lambda^2 \leq \frac{(1 + \mu)^2}{4}; \quad \lambda^2 \leq \frac{1 + \mu}{3 - \mu}; \quad \lambda^2 \leq \frac{(1 + \mu)^2}{2}; \quad \lambda^2 \leq \frac{1 + \mu}{2}. \]

The graphics of above inequalities are the following

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Blue color indicates KS operators, which are not CP. Red color indicated CP maps.}
\end{figure}

From the graphic we see that class of KS-operators are much larger than the class of completely positive ones.

**Example 2.** Now we are going to construct KS-operators, which is not complete positive. Consider mappings of the form $\Phi_{(\lambda,\lambda,\lambda)}$. Let us first check conditions of Theorem 3.2, here as above $|\lambda| \leq 1$. From (3.15) we obtain the following inequality
\[ 4\lambda^2 \leq (1 + \lambda)^2. \]

Solving the last inequality one has
\[ (\lambda - 1)(\lambda + \frac{1}{3}) \leq 0. \]

If we take $\lambda$ such that $-1 \leq \lambda < -\frac{1}{3}$, then (3.15) is not satisfied. This means that $\Phi_{(\lambda,\lambda,\lambda)}$ is not complete positive.

Next we are going to check conditions of Theorem 3.3. From (3.22), (3.23), (3.24) and (3.25) one finds
\[ (1 + \lambda^2)(3 + \lambda^2) \leq 4(1 + \lambda^3). \]

Calculating the last we obtain
\[ (\lambda - 1)^2(\lambda - 1 - \sqrt{2})(\lambda - 1 + \sqrt{2}) \leq 0. \]

If $1 - \sqrt{2} \leq \lambda \leq 1$ then $\Phi_{(\lambda,\lambda,\lambda)}$ is KS-operator.

So, taking into account above we conclude that if $1 - \sqrt{2} \leq \lambda < -\frac{1}{3}$, then $\Phi_{(\lambda,\lambda,\lambda)}$ is KS-operator, but not complete positive one.
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FARRUKH MUKHAMEDOV, DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCE, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. BOX, 141, 25710, KUANTAN, PAHANG, MALAYSIA
E-mail address: far75m@yandex.ru, farrukh_m@iiu.edu.my

ABDUZZIZ ABDUGANIEV, DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCE, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. BOX, 141, 25710, KUANTAN, PAHANG, MALAYSIA
E-mail address: azizi85@yandex.ru