On orthogonal transformations of the Christoffel equations

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Abstract
We prove the equivalence—under rotations of distinct terms—of different forms of a determinantal equation that appears in the studies of wave propagation in Hookean solids, in the context of the Christoffel equations. To do so, we prove a general proposition that is not limited to $\mathbb{R}^3$, nor is it limited to the elasticity tensor with its index symmetries. Furthermore, the proposition is valid for orthogonal transformations, not only for rotations. The sought equivalence is a corollary of that proposition.

Keywords Orthogonal transformation · Christoffel equation · Tensor algebra · Algebraic formulation

Mathematics Subject Classification 86A15 Geophysics, seismology · 74J05 Mechanics of deformable solids, linear waves

1 Introduction

In this paper, we prove analytically a conjecture used in Ivanov and Stovas (2016, 2017, 2019), which is based on numerical considerations for rotations of slowness.
surfaces in anisotropic media. The conjecture is essential for their proposed technique of obtaining a set of mapping operators that establish point-to-point correspondences for travel times and relative-geometric-spreading surfaces between those calculated in nonrotated and rotated media. Thus, this proof results in mathematical rigor to substantiate relevant techniques of seismological modelling.

The existence and properties of three waves that propagate in a Hookean solid are a consequence of the Christoffel equations (e.g., Slawinski 2015, Chapter 9), whose solubility condition is

\[
\det \left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ij\ell} p_j p_\ell - \delta_{ik} \right] = 0, \quad i, k = 1, 2, 3, \quad (1)
\]

which is a cubic polynomial equation in \( p^2 \), whose roots are the eikonal equations (e.g., Slawinski 2015, Section 7.3). The matrix in condition (1),

\[
\left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ij\ell} p_j p_\ell \right] \in \mathbb{R}^{3 \times 3},
\]

is commonly referred to as the Christoffel matrix, where \( c_{ij\ell} \) is a density-normalized elasticity tensor and \( p \) is the wavefront-slowness vector. For a discussion of the Christoffel matrix and its use in seismic wave propagation in inhomogeneous anisotropic media, we refer readers to Červený (2001, Section 2.2), Chapman (2004, Section 5.3), Carcione (2015, Section 1.3), or Slawinski (2015, Chapter 9). For the symmetries of the elasticity tensor, we refer readers to Slawinski (2018, Chapter 3).

Studies of Hookean solids by Ivanov and Stovas (2016, equations (7)–(12)), Ivanov and Stovas (2017, equations (10), (11)) and Ivanov and Stovas (2019, equations (A.3), (A.5), (A.6)) invoke a property that we state as Corollary 1, which is a consequence of Proposition 1. Ivanov and Stovas (2016, 2017, 2019) verify the equivalence of the equations given in Corollary 1, without a general proof, hence, this paper.

The purpose of this paper is to prove Proposition 1 and, hence, Corollary 1. In doing so, we gain an insight into a tensor-algebra property that results in this corollary. The equivalence of the aforementioned equations is a consequence of two orthogonal transformations of \( c_{ij\ell} \) and \( p_i \) that result in two matrices that are similar to one another.

### 2 Proposition and its corollary

**Proposition 1** Let \( c_{ij\ell} \) and \( p_i \) be a tensor and, respectively a vector, in \( \mathbb{R}^d \). Then, the matrices

\[
\sum_{j=1}^{d} \sum_{\ell=1}^{d} c_{ij\ell} \hat{p}_j \hat{p}_\ell \quad (2)
\]

\( \in \mathbb{R}^{d \times d} \)
and

\[
\begin{bmatrix}
\sum_{j=1}^{d} \sum_{\ell=1}^{d} \hat{c}_{ijk \ell} P_j P_{\ell} \\
\end{bmatrix}_{1 \leq i, k \leq d} \in \mathbb{R}^{d \times d},
\]  

(3)

are similar and, consequently, have the same spectrum. Herein,

\[
\hat{p}_i := \sum_{j=1}^{d} A_{ij} P_j \quad \text{and} \quad \hat{c}_{ijk \ell} := \sum_{m=1}^{d} \sum_{n=1}^{d} \sum_{o=1}^{d} \sum_{p=1}^{d} A_{mi} A_{nj} A_{ok} A_{p \ell} c_{ijk \ell},
\]

\[
A \in \mathbb{R}^{d \times d} \text{ denotes an orthogonal transformation, } A^t A = AA^t = I, \quad \text{and }^t \text{ denote a transformation by } A \text{ and } A^t \text{, respectively, and }^t \text{ denotes the transpose.}
\]

**Proof** The fourth-rank tensor, \( c_{ijk \ell} \), in \( \mathbb{R}^d \) can be viewed as a \( d \times d \) matrix, whose entries are \( d \times d \) matrices,

\[
C = [C_{ik}]_{1 \leq i, k \leq d} \in \left( \mathbb{R}^{d \times d} \right)^{d \times d},
\]

with \( C_{ik} \in \mathbb{R}^{d \times d} \) and \( (C_{ik})_{j \ell} := c_{ijk \ell} \). Thus, matrix (2) can be written as

\[
[ \hat{p}_i C_{ik} \hat{p} ]_{1 \leq i, k \leq d} = [ (A^t p)^t C_{ik} (A p) ]_{1 \leq i, k \leq d} = [ p^t (A^t C_{ik} A) p ]_{1 \leq i, k \leq d} \in \mathbb{R}^{d \times d}.
\]

We claim that matrix (3) can be written as

\[
A^t [ p^t (A^t C_{ik} A) p ]_{1 \leq i, k \leq d} A \in \mathbb{R}^{d \times d}. \quad (4)
\]

To see this, we let matrix (4) be

\[
M = A^t X A,
\]

where \( X := [ p^t (A^t C_{ik} A) p ]_{1 \leq i, k \leq d} \), to write

\[
M_{ik} = \sum_{m=1}^{d} \sum_{o=1}^{d} (A^t)_{im} X_{mo} A_{ok} = \sum_{m=1}^{d} \sum_{o=1}^{d} A_{mi} X_{mo} A_{ok}.
\]

Defining \( Y := A^t C_{mo} A \), we have

\[
X_{mo} = \sum_{j=1}^{d} \sum_{\ell=1}^{d} Y_{j \ell} P_j P_{\ell},
\]
where
\[
Y_{j\ell} = \sum_{n=1}^{d} \sum_{q=1}^{d} (A^{\dagger})_{jn} (C_{mo})_{nq} A_{q\ell} = \sum_{n=1}^{d} \sum_{q=1}^{d} A_{nj} A_{q\ell} c_{mnoq}.
\]

Hence,
\[
X_{mo} = \sum_{j=1}^{d} \sum_{\ell=1}^{d} \left( \sum_{n=1}^{d} \sum_{q=1}^{d} A_{nj} A_{q\ell} c_{mnoq} \right) p_{j} p_{\ell}
\]
and, in turn,
\[
M_{ik} = \sum_{j=1}^{d} \sum_{\ell=1}^{d} \left( \sum_{m=1}^{d} \sum_{n=1}^{d} \sum_{o=1}^{d} \sum_{q=1}^{d} A_{mi} A_{nj} A_{ok} A_{q\ell} c_{mnoq} \right) p_{j} p_{\ell} = \sum_{j=1}^{d} \sum_{\ell=1}^{d} \tilde{c}_{ij\ell k} p_{j} p_{\ell},
\]
\[i, k = 1, \ldots, d,
\]
which is matrix (3), as required.

\[\square\]

**Corollary 1** From Proposition 1—and the aforementioned fact that similar matrices share the same spectrum, as well as the fact that the similarity of matrices is not affected by subtracting from them the identity matrix—it follows that
\[
\det \left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ij\ell k} \hat{p}_{j} \hat{p}_{\ell} - \delta_{ik} \right] = \det \left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} \tilde{c}_{ij\ell k} p_{j} p_{\ell} - \delta_{ik} \right], \quad i, k = 1, 2, 3,
\]
and, hence, equations
\[
\det \left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ij\ell k} \hat{p}_{j} \hat{p}_{\ell} - \delta_{ik} \right] = 0, \quad i, k = 1, 2, 3,
\]
\[\text{(5)}\]
and
\[
\det \left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} \tilde{c}_{ij\ell k} p_{j} p_{\ell} - \delta_{ik} \right] = 0, \quad i, k = 1, 2, 3,
\]
\[\text{(6)}\]
are equivalent to one another.

Corollary 1 is valid even without requiring the index symmetries of Hookean solids. Also, \(A \in O(3)\), not only \(A \in SO(3)\), which is more general than the property invoked by Ivanov and Stovas (2016, 2017).

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3 Numerical example

Consider an orthotropic tensor (Ivanov and Stovas 2016, Table 2), whose components are

\[ c_{1111} = 6.3, \quad c_{2222} = 6.9, \quad c_{3333} = 5.4, \]
\[ c_{1122} = c_{2211} = 2.7, \quad c_{1133} = c_{3311} = 2.2, \quad c_{2233} = c_{3322} = 2.4, \]
\[ c_{1212} = c_{2112} = c_{2121} = c_{1221} = 1.5, \quad c_{1313} = c_{3113} = c_{3131} = c_{1331} = 0.8, \]
\[ c_{2323} = c_{3223} = c_{3232} = c_{2332} = 1.0. \]

Also, consider vector \( p = [0, 0, \sqrt{\frac{1}{c_{3333}}}] \). Rotating this vector by

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}, \quad (7) \]

with an arbitrary angle of \( \theta = \pi / 5 \), and the tensor by \( A^t \), we obtain

\[ \hat{F} := \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ijk\ell} \hat{p}_j \hat{p}_\ell = \begin{bmatrix} 0.192934 & 0 & 0 \\ 0 & 0.331749 & -0.018424 \\ 0 & -0.018424 & 0.949406 \end{bmatrix} \]

and

\[ \tilde{F} := \sum_{j=1}^{3} \sum_{\ell=1}^{3} \tilde{c}_{ijk\ell} p_j p_\ell = \begin{bmatrix} 0.192934 & 0 & 0 \\ 0 & 0.562667 & -0.299407 \\ 0 & -0.299407 & 0.718488 \end{bmatrix}, \]

respectively. The eigenvalues of these matrices are the same, \( \lambda_1 = 0.949955 \), \( \lambda_2 = 0.33120 \) and \( \lambda_3 = 0.192934 \), as required for similar matrices. Their corresponding eigenvectors are related by transformation (7).

Herein, \( \det[\hat{F} - I] = -0.027013 = \det[\tilde{F} - I] \). In general, the two determinants are equal to one another. Hence, if \( \det[\hat{F} - I] = 0 \), so does \( \det[\tilde{F} - I] \), and vice versa.

The equivalence of Eqs. (5) and (6) does not imply their equivalence to

\[ \det \left[ \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ijk\ell} p_j p_\ell - \delta_{ik} \right] = 0, \quad i, k = 1, 2, 3. \]

The eigenvalues of

\[ F := \sum_{j=1}^{3} \sum_{\ell=1}^{3} c_{ijk\ell} p_j p_\ell = \begin{bmatrix} 0.148148 & 0 & 0 \\ 0 & 0.185185 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
are $\lambda_1 = 0.148148$, $\lambda_2 = 0.185185$ and $\lambda_3 = 1$, which are distinct from the eigenvalues of $\hat{\Gamma}$ and $\tilde{\Gamma}$. Herein—in view of $p$ and $c_{ijk\ell}$ representing, respectively, the slowness vector along the $x_3$-axis and its corresponding elasticity tensor—$\det[\Gamma - I] = 0$, which results in the eikonal equations. We emphasize, however, that Proposition 1 and Corollary 1 are valid for arbitrary vectors and fourth-rank tensors, even though, in this example, they are related by the Christoffel equations.

4 Conclusion

The corollary proven in this article is necessary to address problems of quantitative seismology (Ivanov and Stovas 2016, 2017, 2019). As shown in the proof of the proposition, it relies on certain intricacies of tensor algebra, which, intrinsically, do not exhibit any physical meaning but are pertinent to mathematical operations in examining geophysical concepts. In summary, the proof contributes the bridging of geoscience and mathematics through an examination of anisotropy, within the context of elasticity theory, for techniques in seismological modelling.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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