A Strengthening of Erdős-Gallai Theorem and Proof of Woodall’s Conjecture

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• 1. Main result

• 2. Applications

• 3. Kelmans operation
Terminology

• **Degree**: the degree of a vertex $v$ in $G$ is the number of its neighbors.

• **Path**: $P=(v_0,e_1,v_1,e_2,\ldots,e_k,v_k)$ where $e_i=v_{i-1}v_i$ and $v_i\neq v_j$ for $i\neq j$. An $(x,y)$-path is one from $x$ to $y$.

• **Cycle**: $C=(v_0,e_1,v_1,e_2,\ldots,e_k,v_0)$

• **Connectivity**: $G$ is connected if for each two vertices $u$, $v$ of $G$, there is a $(u,v)$-path. $G$ is $k$-connected if $G-S$ is connected for any vertex set $S$ with $|S|<k$. 
Degree, path, cycle
Erdős-Gallai Theorem:

- Erdős-Gallai [1]:

If $G$ is 2-connected, $x, y \in V(G)$, and every vertex other than $x, y$ has degree at least $k$, then $G$ has an $(x, y)$-path of length at least $k$.

[1] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.
Generalizations of Erdős-Gallai Theorem:

Erdős-Gallai Theorem has some other generalization, e.g. For Ore’s degree sum condition, for Fan-type degree condition, for paths passing through given vertex set, for weighted graphs.

**Enomoto [2]:** If $G$ is 2-connected, $x, z \in V(G)$, and every vertex other than $x, z$ has degree at least $k$, then for any given vertex $y$ of $G$, $G$ has an $(x, y, z)$-path of length at least $k$.

[3] H. Enomoto, Long paths and large cycles in finite graphs, J. Graph Theory 8 (1984) 287-301.
Generalizations

**Fan [3]:** Let $G$ be a 2-connected graph and $x$ and $y$ be two distinct vertices of $G$. If the average degree of the vertices other than $x$ and $y$ is at least $k$, then $G$ contains an $(x, y)$-path of length at least $k$.

**Bondy-Fan [4]:** Let $G$ be a 2-connected weighted graph and $d$ a real number. Let $x$ and $z$ be two distinct vertices of $G$. If $d^w(v) \geq k$ for all $v \in V(G) \setminus \{x, y\}$, then $G$ contains an $(x, y)$-path of weight at least $k$.

[3] G. Fan, Long cycles and the codiameter of a graph, I. J. Combin. Theory Ser B 49, 151–180 (1990).

[4] J.A. Bondy, G. Fan, Optimal paths and cycles in weighted graphs, Ann. Discrete Math. 41 (1989) 53{69.
Bondy-Jackson Theorem:

Set \( n_k(x, y) = |\{z \in V(G) \setminus \{x, y\}: d(z) \geq k\}|. \)

- Erdős-Gallai Theorem:
  \[ n_k(x, y) \geq n - 2 \rightarrow \text{an} \ (x, y) \text{--path of length} \geq k. \]

- Bondy-Jackson Theorem [5]:
  - If \( G \) is 2-connected with \( |V(G)| \geq 4, \ x, y \in V(G), \) and every vertex in \( V(G) \setminus \{x, y\}, \) with possibly one exception, has degree at least \( k, \) then \( G \) has an \((x, y)\)-path of length at least \( k. \)

  \[ n_k(x, y) \geq n - 3 \rightarrow \text{an} \ (x, y) \text{--path of length} \geq k. \]

[5] J.A. Bondy, B. Jackson, Long paths between specified vertices of a block, Ann. Discrete Math. 27 (1985), 195–200.
Our Main Theorem

• It is natural to ask what is the best lower bound on $n_k(x, y)$ which still ensures the same conclusion.

Theorem 1 [6]:

• If $G$ is 2-connected of order $n$, $x, y \in V(G)$, and there are at least $\frac{n-1}{2}$ vertex in $V(G) \setminus \{x,y\}$ has degree at least $k$, then $G$ has an $(x, y)$-path of length at least $k$.

\[ n_k(x, y) \geq \frac{n-1}{2} \rightarrow \text{an } (x, y)-\text{path of length } \geq k. \]

[6] B. Li, B. Ning, A strengthening of Erdős-Gallai Theorem and proof of Woodall’s conjecture, J. Combin. Theory Ser B 146, 76-95 (2021).
Extremal graphs

• Extremal graphs:

• Let $k \geq 5$ be an odd integer, $H = K_{k-1} \vee K_{k-1}^2$. Let $G$ be from $t$ disjoint copies of $H$, by adding two new vertices $x$ and $y$ followed by all edges between $x$, $y$ and the $K_{k-1}^2$ subgraph of each copy of $H$.

• There are exactly $\frac{n-2}{2}$ vertices other than $\{x, y\}$ of degree at least $k$, and the length of a longest $(x, y)$-path $= k - 1$. 
Dirac’s Theorem:

• Dirac [8]: Let $G$ be a 2-connected graph on $n$ vertices. If every vertex of $G$ has degree at least $d$, then $G$ contains a cycle of length at least $\min\{2d, n\}$.

• Extremal Graphs: complete bipartite graph $K_{d-1, n-d+1}$.

[8] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3) 2 (1952), 69--81
Generalization of Dirac's theorem:

• **Pósa [9]:** Let $G$ be a 2-connected graph on $n$ vertices. If the degree sum of every two non-adjacent vertices of $G$ is at least $2d$, then $G$ contains a cycle of length at least $\min\{2d, n\}$.

• **Fan [10]:** Let $G$ be a 2-connected graph on $n$ vertices. If every pair of vertices $x, y$ with distance 2 satisfies that $\max\{d(x), d(y)\} \geq d$, then $G$ contains a cycle of length at least $\min\{2d, n\}$.

[9] L. Pósa, On the circuits of finite graphs. Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1963) 355–361.

[10] G. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory Ser. B 37 (1984) 221–227.
Generalization of Dirac's theorem:

- **Erdős, Gallai [1]:** Let $G$ be a graph on $n$ vertices and $m$ edges. If $m \geq n$, then $G$ contains a cycle of length at least $\frac{2m}{n-1}$.

- **Locke [11]:** Let $G$ be a 2-connected graph on $n$ vertices. If every vertex of $G$ has degree at least $k$, then for any two vertex $x, y$ of $G$, $G$ contains a cycle of length at least $\min\{2k, n\}$ passing through $x, y$.

[1] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.
[11] S.C. Locke, A generalization of Dirac's theorem, Combinatorics 5 (2) (1985) 149-159.
Woodall’s Conjecture

• Woodall’s Conjecture [12]:

Let $G$ be a 2-connected graph on $n$ vertices. If there are at least $\frac{n}{2} + k$ vertices of degree at least $k$, then $c(G) \geq 2k$.

[12] D. R. Woodall, Maximal circuits of graphs II, Studia Sci. Math. Hungar. 10 (1975), 103–109.
Woodall’s Conjecture

• Häggkvist-Jackson [13]: \( n \leq 3k - 2 \) (2k vertices of degree \( \geq k \)).
• Li-Li [14]: \( n \leq 4k - 6 \).
• Häggkvist-Li [15]: 3-connected, \( k \geq 25 \).
• Li [15]: Under the condition, \( c(G) \geq 2k - 13 \).
• Li: \( (k \geq 683) \).

[13] R. Häggkvist, B. Jackson, A note on maximal cycles in 2-connected graphs, Ann. Discrete Math. 27 (1985), 205–208.
[14] D. Li, H. Li, On longest cycles in graphs, Rapport de recherche no 1160, LRI, URA 410 du CNRS, Bat. 490, Univ de Paris sud, 91405-Orsay, France.
[15] H. Li, On a conjecture of Woodall, J. Combin. Theory Ser. B 86 (1) (2002) 172–185.
Woodall’s conjecture

• Theorem 2 [6]:
Let $G$ be a 2-connected graph on $n$ vertices. If there are at least $\frac{n}{2} + k$ vertices of degree at least $k$, then $c(G) \geq 2k$.

[6] B. Li, B. Ning, A strengthening of Erdős-Gallai Theorem and proof of Woodall’s conjecture, J. Combin. Theory Ser B 146, 76-95 (2021).
• 2. Applications
Fan lemma

Let $G$ be a graph, $C$ a cycle of $G$, and $H$ a component of $G−C$. An $(H,C)$-fan consists of paths $P_1, P_2, ..., P_t$, $t\geq 2$, such that: (1) all $P_i$ have the same origin $v \in V(H)$ and pairwise different termini $u_i \in V(C)$; (2) all internal vertices of $P_i$ are in $H$ and $P_i$’s are pairwise internally disjoint.

Theorem (Fujisawa, Yoshimoto, Zhang[16]). Let $G$ be a 2-connected graph, $C$ a cycle of $G$, and $H$ a component of $G − C$. If each vertex in $H$ has degree at least $k$ in $G$, then there is an $(H, C)$-fan with at least $k$ edges.

[16] J. Fujisawa, K. Yoshimoto, S. Zhang, Heavy cycles passing through some specified vertices in weighted graphs, J. Graph Theory 49 (2005) (2), 93–103.
Woodall-type fan lemma

Theorem 3, Woodall-type Fan Lemma [4]: Let $G$ be a 2-connected graph, $C$ a cycle of $G$, and $H$ a component of $G−C$. If there are at least $\frac{|H|+1}{2}$ vertices in $V(H)$ of degree at least $k$ in $G$, then there is an $(H,C)$-fan with at least $k$ edges.

Lemma. Let $G$ be a 2-connected non-hamiltonian graph, $C$ a longest cycle and $H$ a component of $G − C$. If there is an $(H,C)$-fan with at least $k$ edges, then $|C| \geq 2k$. 
Proof of Woodall’s conjecture

\[ \frac{n}{2} + k \text{ vertices of degree } \geq k \rightarrow c(G) \geq 2k \]

Let \( C \) be a longest cycle of \( G \). By condition \( n \geq 2k \). If \( G \) is Hamiltonian then \( |C| \geq 2k \). So let \( H_i, i = 1, \ldots, t, \) be the components of \( G - C \).

If \( H_i \) contains more than half vertices of degree \( \geq k \), then an \((H_i, C)\)-fan of edge number \( \geq k \), then \( |C| \geq 2k \).

So the number of vertices of degree \( \geq k \) is at most

\[ |C| + \sum_{i=1}^{t} \frac{|H_i|}{2} = \frac{n+|C|}{2}. \]

We have \( \frac{n+|C|}{2} \geq \frac{n}{2} + k \) and \( |C| \geq 2k \).
Bermond’s conjecture

Bermond’s Conjecture [17]. Let $G$ be a 2-connected graph with vertex set $V = \{x_i: 1 \leq i \leq n\}$ and $c \leq n$. If for any pair of vertices $(x_i, x_j)$, $i < j$, one of the following holds:

(i) $i + j < c$;  
(ii) $x_i x_j \in E(G)$;  
(iii) $d(x_i) > i$;  
(iv) $d(x_j) \geq j$;  
(v) $d(x_i) + d(x_j) \geq c$,

then $c(G) \geq c$.

Theorem 4 [6]. Bermond’s conjecture is true for $n \geq 3c - 1$.

[17] J.C. Bermond, On Hamiltonian walks, Congr. Numer. 15 (1976) 41–51.
Bazgan-Li-Woźniak’s theorem

Lobel, Komlos and Sós’ conjecture: Suppose that $G$ is an $n$-vertex graph with at least $n/2$ vertices of degree at least $k$. Then $G$ contains each tree of order $k+1$.

• The following is a special case of Loebl-Komlós-Sós’ conjecture

Theorem 7 (Bazgan, Li, Woźniak [18])
Let $G$ be a graph on $n$ vertices. If there are at least $\frac{n}{2}$ vertices of degree at least $k$, then $G$ contains a path of length at least $k$.

[18] C. Bazgan, H. Li, M. Woźniak, On the Loebl-Komlós-Sós conjecture, J. Graph Theory 34 (2000), no. 4, 269–276.
Some theorems on cycles

- **Theorem** (Dirac [8]). Let $G$ be a 2-connected graph on $n$ vertices. If the degree of every vertex is at least $k$, then $c(G) \geq \min\{n, 2k\}$.

- **Theorem** (Erdős, Gallai [1], Bondy [19]). Let $G$ be a 2-connected graph on $n$ vertices. If the degree of every vertex other than one is at least $k$, then $c(G) \geq \min\{n, 2k\}$.

[8] G.A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. 2 (1952) 69–81.

[19] J.A. Bondy, Large cycles in graphs, Discrete Math. 1 (1971/72), no. 2, 121–132.
Some theorems on cycles

• **Theorem 8 [6].** Let $G$ be a 2-connected graph on $n \geq 2ks + 3$ vertices and $x, y \in V(G)$. If $\max\{d(v) : v \in S\} \geq k$ for any independent set $S \subset V(G) \setminus \{x, y\}$ with $|S| = s + 1$, then $G$ has an $(x, y)$-path of length at least $k$.

• **Theorem 9 [6].** Let $G$ be a 2-connected graph on $n \geq 2k(s+1)$ vertices. If $\max\{d(v) : v \in S\} \geq k$ for any independent set $S \subset V(G)$ with $|S| = s + 1$, then $c(G) \geq 2k$. 
Bondy’s conjecture

- **Theorem** (Fournier, Fraisse [20]). Let $G$ be an $s$-connected graph on $n$ vertices where $s \geq 2$. If the degree sum of any independent set of size $s + 1$ is at least $m$, then $c(G) \geq \min \left\{ \frac{2m}{s+1}, n \right\}$.

[20] I. Fournier, P. Fraisse, On a conjecture of Bondy, J. Combin. Theory Ser. B 39 (1985), no. 1, 17–26.
Häggkvist-Jackson’s conjecture

Häggkvist-Jackson’s conjecture [21]

Let $G$ be a 2-connected graph on $n$ vertices. If $G$ contains at least $\max\left\{2k - 1, \frac{n+k}{2} + 1\right\}$ vertices of degree at least $k$, then $G$ has a cycle of length at least $\min\{n, 2k\}$.

[21] R. Häggkvist, B. Jackson, A note on maximal cycles in 2-connected graphs, Ann. Discrete Math. 27 (1985), 205–208.
Häggkvist-Jackson’s conjecture

- Let $G_1 = K_2 \lor (K_{2k-4} + \overline{K_t})$.
- Let $H_1 = K_{\frac{k-1}{2}} \lor \overline{K_{\frac{k-1}{2}}}$, $H_2 = K_{k+1}$ ($k \geq 3$ is odd). Let $G_2$ be the graph obtained from one copy of $H_2$ and $t$ disjoint copies of $H_1$ by joining each vertex in the $K_{\frac{k-1}{2}}$ subgraph of $H_1$ to two fixed vertices of $H_2$.
- One can see $G_1$ has $2k-2$ vertices of degree at least $k$, $G_2$ has $\frac{n+k+1}{2}$ vertices of degree at least $k$, and $c(G_1) = c(G_2) = 2k - 1$. 
Li’s conjecture

- Li [21] conjectured that for any 2-connected graph $G$ of order $n$, there is a cycle of length at least $2k$ if the number of vertices of degree at least $k$ is at least $\frac{n+k}{2}$. The constructions $G_1$ and $G_2$ mentioned above disprove Li’s conjecture.

[21] H. Li, Generalizations of Dirac’s theorem in Hamiltonian graph theory—a survey, Discrete Math. 313 (2013), no. 19, 2034–2053.
A conjecture

• We suggest the following conjecture, which is a generalization of Theorem 1.

• Conjecture 1 [6]. Let $G$ be a 2-connected graph on $n$ vertices and $x,y \in V(G)$. Let $0 < \alpha \leq 1/2$. If $V(G) \setminus \{x, y\}$ contains more than $\alpha(n-2)$ vertices of degree at least $k$, then $G$ contains an $(x,y)$-path of length at least $2\alpha k$. 
A conjecture

• When $\alpha$ is a rational number, we choose $k$ such that $\alpha(k-1)$ is an integer. Let $H = K_{\alpha(k-1)} \cup K_{(1-\alpha)(k-1)}$. Let $G$ be obtained from $t$ copies of $H$, by adding two new vertices $\{x, y\}$ and all possible edges between $\{x,y\}$ and the $K_{\alpha(k-1)}$ subgraph of each $H$. The number of vertices of degree at least $k$ in $G$ is $\alpha(|G|-2)$ and a longest $(x,y)$-path is of length $2\alpha(k-1)$.

• This example shows that Conjecture 1 is sharp for infinite values of integers $n$ and $k$. 
• 3. Kelmans operation
Kelmans operation

• Let $G$ be a graph and $u,v \in V(G)$. A new graph $G':=G[u\to v]$ is a Kelmans graph [22] of $G$ (from $u$ to $v$), if $V(G')=V(G)$ and $E(G') = (E(G)\setminus \{uw: w \in N[u]\setminus N[v]\}) \cup \{vw: w \in N[u]\setminus N[v]\}$.

[22] A.K. Kelmans, On graphs with randomly deleted edges, Acta Math. Acad. Sci. Hungar. 37 (1981)77–88.
Order of degree sequences

• Assume \( \tau(G) = (d_1, d_2, \ldots, d_n) \), \( \tau(G') = (d'_1, d'_2, \ldots, d'_n) \) are non-increasing degree sequences of \( G \) and \( G' \), respectively. If there exists an integer \( j \) such that \( d_i = d'_i \) for \( 1 \leq i \leq j-1 \) and \( d_j = d'_j \), then we say that \( \tau(G) \) is larger than \( \tau(G') \) and denote it by \( \tau(G) > \tau(G') \).
A lemma on Kelmans operation

• Lemma [6]. Let $G$ be a graph, $x$, $y$, $u$ be distinct vertices of $G$, and $v \in N(u)$ (possibly $v \in \{x, y\}$). Let $G' = G[u \rightarrow v]$.

(i) If neither $N[u] \subseteq N[v]$ nor $N[v] \subseteq N[u]$, then $\tau(G') > \tau(G)$.

(ii) If $G'$ has an $(x, y)$-path of length at least $k$, then so does $G$. 
Thanks!