MULTISCALE DYNAMICS OF AN ADAPTIVE CATALYTIC NETWORK

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Abstract. We study the multiscale structure of the Jain–Krishna adaptive network model. This model describes the co-evolution of a set of continuous-time autocatalytic ordinary differential equations and its underlying discrete-time graph structure. The graph dynamics is governed by deletion of vertices with asymptotically weak concentrations of prevalence and then re-insertion of vertices with new random connections. In this work, we prove several results about convergence of the continuous-time dynamics to equilibrium points. Furthermore, we motivate via formal asymptotic calculations several conjectures regarding the discrete-time graph updates. In summary, our results clearly show that there are several time scales in the problem depending upon system parameters, and that analysis can be carried out in certain singular limits. This shows that for the Jain–Krishna model, and potentially many other adaptive network models, a mixture of deterministic and/or stochastic multiscale methods is a good approach to work towards a rigorous mathematical analysis.

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1. Introduction

Catalytic reactions have been studied in a wide variety of contexts, classically in chemistry [30], but more recently across mathematics, physics and the life sciences. A particular area, where catalytic reactions have been employed, is the origins of life problem, i.e., how to form biology out of pre-biotic systems. One proposed mechanism is that various molecules can catalyze each other to form a self-sustaining and self-organized reaction network, which is then able to form more complex structures. This paradigm is linked to the notions of autocatalytic sets [16, 25, 26] as well as to hypercycles [8, 9], which have both become standard ideas in complex systems [13].

A new twist has been added to this line of research by more recent models in network science. In particular, autocatalytic reaction mechanisms have been paired with the idea of adaptive (or co-evolutionary) networks [15], where the dynamics of the network is fully coupled to the dynamics on the network. For example, adaptive networks have been used as epidemic models [14, 17, 33], for evolutionary games on networks [29, 31], and for modelling self-organized criticality (SOC) [5, 27].

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In this work, we study an adaptive network model proposed by Jain and Krishna \cite{Jain2019}. The Jain–Krishna model builds upon two well-established mathematical ideas: ordinary differential equations (ODEs) for autocatalytic systems, and network structure formation models via random graph theory. Each vertex/node in the model is part of a graph/network on which ODE dynamics takes place. Each vertex has an associated concentration, which is the variable for its ODE, which is coupled to the ODE of other vertices via the graph structure. The idea is that once the concentrations in the ODEs have reached a suitable asymptotic limit set in forward time, we update the graph structure by deleting the vertex with the lowest asymptotic concentration and insert a new vertex with random connections to the rest of the graph; see also Section 2 for a precise definition of the model.

It seems natural to conjecture that multiscale analysis is going to play a key role for the Jain–Krishna model, and in fact almost all other adaptive network models. For example, one may ask whether there is a natural time scale separation between the process of the network relative to the dynamics on the network. In fact, there are even further time scales, such as the convergence rate to invariant sets for the ODEs, or the time scale to reach a structurally different graph via adaptation. If we can identify the different time scales in the problem, then this would open up the entire methodology of multiple time scale dynamical systems methods such as geometric singular perturbation theory \cite{Kuehn2011, Kuehn2014, Kuehn2015}; see also \cite{Kuehn2018} for a more detailed background on multiple time scale methods. However, even if we have identified time scales, the multiscale paradigm suggests to separate the scales first in suitable singular limits, i.e., fast dynamics for frozen slow variables or slow dynamics for constrained/averaged fast variables. The problem only simplifies if these singular limits are mathematically tractable. In this work, we make the following progress towards this multiscale program for the Jain–Krishna model:

(R1) We provide a proof that the autocatalytic ODEs have equilibrium points and solutions generically converge to an equilibrium exponentially fast. We also algebraically characterize the sets of equilibrium points.

(R2) In the context of the ODE proof, we uncover a relation to projective spaces. We also provide an intertwining lemma linking network topology to convergence structure.

(R3) For the network adaptation rule, we formulate four conjectures\footnote{Although it may be evident to almost all readers, the author would be very interested to see any proof of these conjectures, or counter-examples with improved conjectures/theorems in future works.} regarding the relevant time scales based upon network size $d$ and the edge probability parameter $p$. In particular, we study the formation time of the first cycle as well as the formation time of a single autocatalytic set (ACS) encompassing the entire graph.

(R4) Although we do not prove the four conjectures from (R3), we provide heuristic/formal asymptotic calculations to motivate the intrinsic scalings appearing in all four cases.

In summary, we have advanced the understanding of the Jain–Krishna model as our results clearly show a multiple time scale structure based upon the ODE convergence time as well as the graph structure formation time for various ranges of $p$ and $d$. Furthermore, (R1)–(R2) show that the singular limit for a frozen graph structure is tractable. The formal calculations in (R3)–(R4) indicate that the graph adaptation process can be analyzed as well. However, to provide a full singular perturbation analysis in the finite time-scale separation regime is beyond our approach here and remains a challenge for future work. Numerical simulations strongly indicate \cite{Jain2019} that it is possible to study the finite time-scale case. In fact, the numerical results show on a coarse-grained level typical fast-jump, slow-drift, relaxation-oscillation, or bursting structures commonly encountered for multiple time scale problems. These observations were actually a main motivation to understand the apparent visual link to multiscale dynamics in more detail.

The structure of the paper is as follows: in Section 2, we define the Jain–Krishna model; we also point out an ill-defined variant of it in Appendix A, which has been used in the literature. In Section 3, we study the continuous-time dynamics on the network given by a system of autocatalytic-reaction ODEs and prove results (R1)–(R2) regarding existence and stability of equilibrium points. In Section 4, we consider the discrete-time
dynamics of the network. We identify two different regimes based upon giant-component existence or non-existence. Then we split these two cases into (a) the first cycle formation problem, which is linked back to classical literature on random graph dynamics, and (b) the formation problem of large ACSs. Based upon these ideas we provide approximate and asymptotic calculations to motivate four conjectures on the discrete-time Jain–Krishna model graph update rule.

2. The Jain–Krishna model

The version of the Jain–Krishna model we follow can be found in [19]. Fix a dimension $d \in \mathbb{N}$, $d \geq 2$, representing the number of species in the reaction pool. Let $C = C[s] = (c_{ij}[s]) = (c_{ij})$ for $s \in \mathbb{N}_0$ be a $d \times d$ matrix of interaction coefficients, where $s$ represents a discrete time step, which we shall, as indicated above, sometimes drop from the notation. We can view $C$ as the transpose of the adjacency matrix of a directed graph $G = G(V, E)$ with vertex set $V = \{1, 2, \ldots, d\}$ and edge set $E = \{(i, j) : c_{ij} = 1\}$, i.e., if there is an edge from $j$ to $i$ then $c_{ij} = 1$ and $c_{ij} = 0$ otherwise; self-loops are not allowed so that $c_{jj} = 0$ for all $j \in \{1, 2, \ldots, d\}$.

Whenever we consider a subgraph of $G' = G'(V', E') \subset G$, we not only require $V' \subset V$ and $E' \subset E$ but also consider induced subgraphs, where start- and end-points of edges are contained in the subgraph. We denote the set of all possible allowed matrices $C$ as $M := \{M = (m_{ij}) \in \mathbb{R}^{d \times d} : m_{ij} \in \{0, 1\}, m_{jj} = 0 \forall i, j\}$.

Let $t \in \mathbb{R}$ be a continuous time and consider the following ODEs for the values $x_j = x_j(t)$ at each vertex

$$\frac{dx_j}{dt} = x'_j = (Cx)_j - x_j \sum_{k=1}^{d} (Cx)_k, \quad j \in \{1, 2, \ldots, d\}, \quad (2.1)$$

with initial condition $x(0)$ and we set $x = (x_1, x_2, \ldots, x_d)^T$; here $(\cdot)^T$ denotes the transpose so we work with column vectors. We impose the two constraints

$$\sum_{k=1}^{d} x_k = 1 \quad \text{and} \quad x_k \geq 0 \quad \forall k, \quad (2.2)$$

which we refer to as mass and non-negativity conservation. Let $X$ be the subset of $\mathbb{R}^d$ containing all $x \in \mathbb{R}^d$ for which (2.2) holds. We shall prove in Proposition 3.1 that if the initial condition $x(0)$ satisfies (2.2), then $x(t)$ satisfies (2.2) as well for any $t > 0$. Note that (2.1) can be written more compactly as

$$x' = Cx - |Cx|_1 x =: f(x), \quad |x|_1 := \sum_{j=1}^{d} x_j, \quad (2.3)$$

which shall not lead to any confusion with the usual 1-norm as all components of $x$ will always be non-negative. We shall prove in Theorems 3.13 and 3.17 that (2.1), for fixed $C$, always converges for sufficiently generic initial data to an equilibrium point $x_*$ as $t \to +\infty$.

As yet, the network $C$ is static. The model is turned into a fully adaptive (or co-evolutionary) network as follows. The first matrix $C[0]$ is sampled as an Erdős–Renyi-type random graph, i.e.,

$$\mathbb{P}(c_{ij} = 1) = p \quad \text{and} \quad \mathbb{P}(c_{ij} = 0) = 1 - p \quad (2.4)$$

2Classically, one would work with the transpose $C^\top$ as it is the classical adjacency matrix but to keep with the standard conventions for the Jain–Krishna model, we use $C$. 

for \( i \neq j \) and a fixed parameter \( p \in (0, 1) \). Assume that we converged to an equilibrium point \( x_* \) for (2.1) for \( C = C[0] \). Then we define the following set of indices

\[
J_* := \left\{ j \in \{1, 2, \ldots, d\} : x_j = \min_k x_k \right\},
\]

which are the species with minimum prevalence. Next, pick some \( j_* \) at random with equal probability from \( J_* \) and re-sample \( c_{ij_*} \) and \( c_{j_*i} \) for \( i \neq j \) according to (2.4) keeping all the other entries fixed. This step corresponds to eliminating one of the species that has performed worst in the autocatalytic reaction process (2.1). This yields a new matrix \( C = C[1] \). Now we can repeat the process, including a new equilibrium point \( x_* = x_*[1] \).

The adaptive network dynamical system can formally be written as a mapping

\[
\phi : \mathbb{N}_0 \times \mathcal{X} \times \mathcal{M} \to \mathcal{X} \times \mathcal{M}, \quad \phi(s, x_*[0], C[0]) = (x_*[s], C[s]).
\]

It is already an interesting question under which assumptions/modifications the process just described generates a random dynamical system in the sense of [1]. This question will be considered in future work. Here, we focus on the two basic components, the static ODE dynamics and the graph structure. These components should be viewed as singular limits of the fully adaptive network, which is usually simulated using a finite-time scale for the ODE dynamics at each vertex.

3. Results for the continuous-time vertex dynamics

In this section, we study the ODEs (2.1)-(2.2). Our main goal is to provide a detailed analysis of all possible cases, which can occur depending upon a fixed and given matrix \( C \).

3.1. Well-posedness & consistency

The next result shows that \( \mathcal{X} \) is positively invariant so that the two constraints are always satisfied.

**Proposition 3.1.** Suppose \( x(0) \in \mathcal{X} \), then \( x(t) \in \mathcal{X} \) for all \( t > 0 \).

*Proof.* Regarding mass conservation, we compute

\[
\frac{d}{dt} \left( \sum_{j=1}^d x_j \right) = \sum_{j=1}^d (Cx)_j - \sum_{j=1}^d x_j \sum_{k=1}^d (Cx)_k = \left(1 - \sum_{j=1}^d x_j\right) \sum_{j=1}^d (Cx)_j
\]

and mass is indeed conserved if \( \sum_{j=1}^d x_j(0) = 1 \). For non-negativity, we decompose \( \partial \mathcal{X} \) and define \( \partial \mathcal{X}_j := \partial \mathcal{X} \cap \{x \in \mathbb{R}^d : x_j = 0\} \). Suppose \( x \in \partial \mathcal{X}_j \), then the \( j \)th component of the ODEs (2.1) is given by

\[
x'_j = (Cx)_j \geq 0,
\]

where the last inequality follows as we start inside \( \mathcal{X} \) and since \( C \) has non-negative entries. Since \( j \) was arbitrary and vector field either vanishes or points inside on \( \partial \mathcal{X} \), the non-negativity conservation follows.

We remark that there are variants of the Jain–Krishna model in the literature, which are not well-posed as shown in Appendix A. Hence, one has to be very careful, which variant of the model is discussed in various sources.

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3Good physical intuition and numerical calculations exist already for the ODEs [19–22]; however, it seems useful to have a detailed analysis of the mathematical structures behind the ODE convergence.
3.2. Existence of equilibria

Let \( 0 := (0,0,\ldots,0)^\top \) and observe that although \( f(0) = 0 \), the origin is not an equilibrium point for \((2.1)-(2.2)\) as \( 0 \not\in \mathcal{X} \). We briefly recall that for a general matrix \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \) one calls \( A \) positive and writes \( A > 0 \) if \( a_{ij} > 0 \) for all \( i,j \). Similar component-wise definitions apply to non-negative matrices \( A \geq 0 \) and naturally specialize to the vector case \( n = 1 \). The existence of a non-negative and non-zero equilibrium point \( x_* \), such that

\[
f(x_*) = 0, \quad \sum_{j=1}^d (x_*)_j = 1, \quad x_* \neq 0, \quad x_* \geq 0,
\]

is already an interesting problem. Before analyzing it, recall that a matrix \( A \) is irreducible if and only if there exists a permutation matrix \( P \) such that \( P^\top A P \) is block-upper triangular; if \( A \) is not irreducible, it is called reducible. Furthermore, viewing \( A \) as the adjacency matrix of a directed graph \( \mathcal{H} \), then one can easily prove that \( A \) is irreducible if and only if \( \mathcal{H} \) is strongly connected, i.e., if in \( \mathcal{H} \) there is a path from any vertex to any other vertex. The next two examples illustrate one of the major obstacles to determine \( x_* \).

**Example 3.2.** Let \( d = 3 \) and consider the matrix \( C \) with \( c_{ij} = 1 \) for \((i,j) = (2,1),(1,2),(1,3)\) and \( c_{ij} = 0 \) otherwise. Then one checks that \( x_* = (\frac{1}{2},\frac{1}{2},0)^\top \) is an equilibrium point. Note that \( C \) is reducible but the subgraph formed by vertices 1 and 2 is irreducible.

The last example suggests that we might want to use the Perron–Frobenius theorem. Recall that there are several versions. The strong Perron–Frobenius states that for a matrix \( A > 0 \) there exists an eigenvalue \( \lambda = \lambda(A) > 0 \) of algebraic multiplicity 1 such that \( \rho(A) = \lambda \), where \( \rho(A) \) is the spectral radius of \( A \). Furthermore, there exists a unique eigenvector \( v \), called the Perron vector, such that

\[
Av = \lambda v, \quad |v|_1 = 1, \quad v > 0,
\]

and \( v_1 \) is the unique non-negative eigenvector up to positive multiples. Obviously, this version of the strong Perron–Frobenius theorem is not applicable to the Jain–Krishna model as \( C > 0 \) never holds. A more general version of the strong Perron–Frobenius theorem states that the same conclusions apply if \( A \geq 0 \) and \( A \) is irreducible.

**Example 3.3.** Hence, applying the strong Perron–Frobenius theorem in Example 3.2 to the sub-graph/matrix formed by the first two vertices, taking \((x_*)_1, (x_*)_2 = v_1^\top \) and augmenting the result by \( (x_*)_3 = 0 \) yields an equilibrium point. Yet, this naive algorithm is flawed. Let \( d = 3 \) and consider the matrix \( C \) with \( c_{ij} = 1 \) for \((i,j) = (2,1),(1,2),(3,1)\) and \( c_{ij} = 0 \) otherwise so that we just reserved the edge between vertex 1 and 3 in comparison to Example 3.2. Then one checks that \( x_* = (\frac{1}{3},\frac{1}{3},\frac{1}{3})^\top \) is an equilibrium point and \( x_* = (\frac{1}{2},\frac{1}{2},0)^\top \) is not an equilibrium point.

Hence, applying the strong Perron–Frobenius theorem naively to a reducible \( C \geq 0 \) or an irreducible sub-graph with appended zeros to obtain the existence of equilibrium points is not possible. The weak Perron–Frobenius theorem states that if \( A \geq 0 \), then \( \rho(A) \) is eigenvalue (not necessarily positive) and there exists an eigenvector \( v \geq 0, v \neq 0 \).

**Theorem 3.4.** Suppose \( G \) has at least one edge, then there exists \( x_* \) satisfying (3.1).

**Proof.** Applying the weak form of the Perron–Frobenius theorem to \( C \), we get a non-zero and non-negative eigenvector \( v \) with eigenvalue \( \lambda = \rho(C) \). We define

\[
x_* = \frac{1}{|v|_1} v,
\]
so that $x_* \neq 0$, $x_* \geq 0$, $|x_*|_1 = 1$. Then we compute

$$f(x_*) = Cx_* - x_* |Cx_*|_1 = \frac{1}{|v|_1} C v - \frac{1}{(|v|_1)^2} v|Cv|_1 = \frac{1}{|v|_1} (\lambda v - \lambda v \cdot 1) = 0,$$

which finishes the proof.

Yet, the last result is not very constructive as we would like to know, how many non-zero components equilibrium points really have. This information is not provided by the weak form of Perron–Frobenius; cf. Examples 3.2–3.3. Furthermore, the dimension of the set of equilibrium points is of interest as the next two examples show.

**Example 3.5.** Let $d = 4$ and consider the matrix $C$ with $c_{ij} = 1$ for $(i,j) = (2,1)$, $(1,2)$, $(4,3)$, $(3,4)$ and $c_{ij} = 0$ otherwise. Then $v_1 = (\frac{1}{7}, \frac{1}{7}, 0, 0)^\top$ and $v_2 = (0, 0, \frac{1}{7}, \frac{1}{7})^\top$ are Perron–Frobenius eigenvectors (PFEs) for the matrices with just $(i,j) = (2,1), (1,2)$, respectively $(i,j) = (4,3), (3,4)$ as non-zero indices splitting the associated graph into its two 2-cycles. One checks that $x_* = bv_1 - (1 - b)v_2$ is an equilibrium point for any $b \in [0,1]$.

Therefore, we have shown that equilibrium points can be non-unique. Just using a general convex combination of PFEs of irreducible subgraphs is also not enough.

**Example 3.6.** We use the same $C$ as in Example 3.5 with the added entry $c_{31} = 1$. Then one checks that $x_* = bv_1 - (1 - b)v_2$ is an equilibrium point only for $b = 0$.

**Definition 3.7.** An ACS is an (induced) subgraph $G'(V', E') \subset G$ such that for each $i \in V'$ there exists $j \neq i$ such that $(i,j) \in E'$.

For completeness we record the next simple and well-known result [20] together with a short proof.

**Lemma 3.8.** Let $C$ be the matrix associated to a directed graph $G$.

(L1) We have the following implications

$$G \text{ is a cycle } \Rightarrow \text{ G is irreducible } \Rightarrow \text{ G is an ACS.} \quad (3.2)$$

The converse implications are false.

(L2) If $G$ has no cycle, then $\lambda(C) = 0$. If $G$ has a cycle, then $\lambda(C) \geq 1$.

(L3) An ACS must contain a cycle. Furthermore, suppose $\lambda(C) \geq 1$ and let $v$ be the eigenvector associated to $\lambda(C)$. Then the subgraph with vertices $\{j : v_j > 0\}$ is an ACS.

**Proof.** The implications (3.2) in (L1) easily follow from the definitions, e.g., if $G$ is a cycle, then there exists a path from every vertex to every other by going along the cycle. The converse implications are obviously false, e.g., Example 3.3 is an ACS, which is not irreducible. Regarding (L2), if $G$ has no cycle, then $C$ is nilpotent, so $\lambda(C) = 0$. If $G$ has a cycle, then $C^n \neq 0$ for any $n \in \mathbb{N}$. Since the eigenvalues of $C^n$ are $n$th powers of those of $C$ and $c_{ij} \in \{0,1\}$, it follows that $\lambda(C) \geq 1$. The first part in (L3) is easy since no cycles mean there is a vertex without an edge pointing to it. If $\lambda(C) \geq 1$, then we can re-order the vertices such that $\{1,2,\ldots,j_*\} = \{j : v_j > 0\}$ and compute for $i \leq j_*$ that

$$0 < \lambda v_i = (Cv)_i = \sum_{j=1}^{j_*} c_{ij} v_j.$$

Hence, $c_{ij} = 1$ for at least one index $j$, which means that the graph induced by $\{c_{ij}\}_{j=1}^{j_*}$ is an ACS. \qed
The easiest parts of $\mathcal{G}$ to deal with are graphs with no cycles. Let

$$\mathcal{T} = \{j : \exists i \text{ s.t. } c_{ji} = 1, c_{ij} = 0 \forall i\}$$

be the set of terminal vertex indices. Re-order vertex labels such that $\mathcal{G}$ has as first indices those in $\mathcal{T}$ so that $\mathcal{T} = \{1, 2, \ldots, j^*\}$. Let $e_j$ denote the $j$th standard basis vector.

**Proposition 3.9.** Suppose $\mathcal{G}$ has no cycles. Then

$$X^* := \left\{ \sum_{j=1}^{j^*} b_j e_j : \sum_{j=1}^{j^*} b_j = 1 \right\}$$

are equilibrium points.

**Proof.** Let $x^* \in X^*$. We find $(Cx^*) = 0$ since $Cx^*$ only contains entries for the non-terminal vertices. Therefore, we have $f(x^*) = 0$. \hfill \Box

Although $X^*$ contains many equilibrium points, we shall see below that not all of them are stable. There is a well-defined subset of $X^*$, which is going to be stable up to measure-zero initial data.

**Definition 3.10.** Suppose $\mathcal{G}$ has no cycles and terminal vertex set $\mathcal{T}$. Suppose $j \in \mathcal{T}$ and Let $p(j) := |\{i : \exists \text{ a path from } i \text{ to } j\}|$. Then define the maximal input equilibrium points

$$X_* := \left\{ \sum_{j=1}^{j^*} b_j e_j : \sum_{j=1}^{j^*} b_j = 1, b_j \neq 0 \iff p(j) = \max_{k \in \mathcal{T}} p(k) \right\} .$$

Clearly, $X_*$ contains only equilibrium points by Proposition 3.9. The next step is to consider graphs with cycles. Any graph with at least one cycle has an ACS. The previous ideas can be generalized.

**Definition 3.11.** Suppose $\mathcal{G}$ has at least one cycle. Let $v_1, \ldots, v_{j^*}$ be the set of Perron–Frobenious eigenvectors with eigenvalue $\lambda = \rho(C)$ associated to $\mathcal{G}$. Then define

$$X_* := \left\{ \sum_{j=1}^{j^*} b_j v_j : \sum_{j=1}^{j^*} b_j = 1, b_j \neq 0 \text{ if } j \text{ is part of an ACS} \right\} .$$

One checks that $X_*$ contains only equilibrium points. Note that the last two definitions imply that $X_*$ is a well-defined for every graph $\mathcal{G}$. $X_*$ is non-empty and we shall show below that it is an attracting set of equilibrium points up to measure zero of initial conditions.

### 3.3. Stability

In this section, we discuss one possible proof for stability. Some parts are based upon formal calculations, mentioned in the work of Jain and Krishna [20, 22], which we make rigorous here adding a new geometric viewpoint. Consider the system of ODEs (2.1). Then consider the mapping

$$G^{-1} : \mathbb{R}^d \to \mathcal{X}, \quad y \to \frac{y}{|y|} .$$

The inverse mapping $G : \mathcal{X} \to \mathbb{R}^d$ is defined via the pre-image so that $G$ is a multi-valued mapping, i.e., one should view the transformation (3.3) not as a coordinate change but as an unfolding of the phase space $\mathcal{X}$ onto
a larger phase space consisting of the non-negative quadrant/cone

\[ \mathbb{R}^d_+ := \{ x \in \mathbb{R}^d : x_j \geq 0 \text{ for all } j \}. \]

The next proposition shows that we just have a differently scaled and restricted version of real projective space.

**Proposition 3.12.** The following results hold:

(P1) Fix any vector \( v \in \mathbb{R}^d_+ \), then all lines with direction \( v \) through the origin in \( \mathbb{R}^d_+ \) map to the same point under \( G^{-1} \);

(P2) On \( \mathbb{R}^d_+ \), the ODEs are given by

\[ y' = Cy - \phi y, \]  

(3.4)

valid for any choice of \( \phi \in \mathbb{R} \).

**Proof.** For (P1), consider the line \( y = l_v(a) = av \) for \( v \in [0, \infty)^d \), \( v \neq 0 \), \( v \) fixed and \( a \in (0, \infty) \), then we calculate

\[ G^{-1}(av) = \frac{av}{|av|_1} = \frac{v}{|v|_1} = x_v, \]

where \( x_v \) is fixed since \( v \) is fixed for the line so all points on the line map to the same point under \( G \). For (P2), note that we have on the cone \( \mathbb{R}^d_+ \) the simpler version of the 1-norm

\[ |y|_1 = \sum_{j=1}^d y_j \]

so that

\[ \frac{dx}{dt} = \frac{y' |y|_1 - y |y'|_1}{|y|^2} = \frac{Cy - \phi y}{|y|} - \frac{y \sum_{j=1}^d (Cy)_j - \phi y_j}{|y|_1} = Cx - |Cx|x. \]

The free parameter \( \phi \) essentially gives the additional degree-of-freedom due to the lack of mass conservation in (3.4).

Due to (P1), we can view the model (2.1)–(2.2) as posed on part of projective space \( \mathbb{R}P^{d-1} = \mathbb{R}^d / \langle x \sim bx \rangle \) for \( b \neq 0 \). Indeed, in our context we just start with the non-negative quadrant \( \mathbb{R}^d \) and then apply the equivalence relation \( x \sim bx \) for \( b \neq 0 \); note that we use the 1-norm instead of the usual 2-norm to identify a unique point in \( X \) but this makes no difference from a topological point of view. For stability, the ODEs (3.4) can be used. Let \( \mu_X \) denote the \((d - 1)\)-dimensional Lebesgue measure induced on \( X \).

**Theorem 3.13.** Suppose \( G \) has at least one cycle. For almost every \( x_0 \in X \) (wrt \( \mu_X \)), there exists \( x_* \in X_* \) such that

\[ \lim_{t \to +\infty} x(t) = x_* \quad x(t) \text{ solves } (3.4) \text{ with } x(0) = x_0. \]

**Proof.** As before, fix \( v \neq 0 \), \( v \in \mathbb{R}^d_+ \), and set \( l_v := \{ y \in \mathbb{R}^d_+ : \exists a \geq 0 \text{ s.t. } av = y \} \). Let \( \mu_Y \) denote the Lebesgue measure induced via \( G \) on \( \mathbb{R}^d_+ \), i.e.,

\[ \mu_Y(B) := \mu_X(\{ v \in X : l_v \cap B \neq \emptyset \}). \]

Since \( G \) has at least one cycle, any \( x_* \) is a weighted convex combination of vectors \( \{ v_j \}_{j=1}^J \) in the eigenspace associated to the leading Perron–Frobenius eigenvalues. Consider the system (3.4) and fix any \( y_0 = y(0) \). Select \( \phi < 0 \) such that all solutions \( y(t) \) diverge sufficiently fast, i.e., \( |y|_1 \geq p(t) e^{kt} \) for all \( t \geq t_0 > 0 \) and some
polynomial $p(t)$. The solution to (3.4) can be written as

$$y(t) = \sum_{j=1}^{d} p_j(t)e^{\tilde{\lambda}_j t}\tilde{v}_j,$$

(3.5)

where $(\tilde{\lambda}_j, \tilde{v}_j)$ are (generalized) eigenpairs for $C - \phi \text{Id}$, $\text{Re}(\tilde{\lambda}_j) > 0$, and $p_j(t)$ are polynomials in $t$. Note that up to a re-ordering of variables, we can assume that $\tilde{\lambda}_j$ for $j = 1, 2, \ldots, j_*$ are the leading eigenvalues associated to the Perron–Frobenius eigenvalues of $C$ as defined in Section 3.2 by a shift of $-\phi$. In particular, we have

$$\text{Re}(\tilde{\lambda}_{j_*}) > \text{Re}(\tilde{\lambda}_j) \quad \forall j \geq j_*.$$  

(3.6)

Given any $\varepsilon > 0$ and $y_0 \in \mathcal{H} := \{y : y \notin \text{span}(v_j : j > j_*)\}$, there exists $T$ such that

$$d_H(y(t), \text{span}(v_j : j \leq j_*)) < \varepsilon \quad \forall t > T,$$

(3.7)

where $d_H$ is the usual Hausdorff distance and we used that the strongest expanding directions in (3.5) eventually dominate any weaker expanding direction, i.e., we have

$$\lim_{t \to +\infty} \frac{p_j(t)e^{\tilde{\lambda}_j t}}{p_{j_*}(t)e^{\tilde{\lambda}_{j_*} t}} = 0$$

for any non-trivial polynomials $p_j(t)$ and $p_{j_*}(t)$ by (3.6) from which we can conclude (3.7). Note that we just excluded a set of measure $\mu_Y$ in the last argument as initial conditions satisfy $\mu_Y(\mathcal{H}) = 0$ since Lebesgue measure vanishes on subspaces of dimension strictly less than the space dimension. Next, we observe that any point in subspaces contained

$$\text{span}(v_j : j \leq j_*)$$

is associated to a line $l_v$ such that $l_v$ maps to a point $x_* \in X_*$ under $G$. Since $\varepsilon > 0$ was arbitrary, the result follows.

The intuition is that the PFEs define the subspace, which is fastest growing so all initial conditions except those in non-leading eigenspaces are attracted to the span of the PFEs as $t \to +\infty$. To cover the case of no cycles, we start with a definition.

**Definition 3.14.** The power-weighted and power-weighted average variables $R_n$ and $r_n$ are defined as follows

$$R_n := C^n x \quad \text{and} \quad r_n := \sum_{j=1}^{d} (C^n x)_j.$$  

**Lemma 3.15.** (Adaptive network dynamics intertwining (ANDI)) The variables $r_n = r_n(t)$ satisfy the infinite-dimensional ODE system

$$r_n' = r_{n+1} - r_n r_1 \quad n \in \mathbb{N}.$$  

(3.8)

The variables $R_n = R_n(t)$ satisfy the infinite-dimensional ODE system

$$R_n' = R_{n+1} - R_n |R_{11}| \quad n \in \mathbb{N}.$$  

(3.9)
Proof. We just compute
\[
\begin{align*}
 r'_n &= \sum_{j=1}^d (C^n x'_j) = \sum_{j=1}^d (C^{n+1} x - C^n x | C x)_j \\
 &= \sum_{j=1}^d (C^{n+1} x)_j - \sum_{j=1}^d (C^n x)_j | C x|_1 \\
 &= r_{n+1} - r_n r_1,
\end{align*}
\]
which shows (3.8). The computation for \( R_n \) is even easier
\[
R'_n = C^n x' = C^{n+1} x - |C x|_1 C^n x = R_{n+1} - |R_1|_1 R_n,
\]
which finishes the proof.

The ANDI Lemma 3.15 connects the topology and cycle structure of the graph with the vertex dynamics into a bigger dynamical system. Although the proof is very simple, the insight is still substantial as we have connected the dynamics on the network modelled by (2.1)–(2.2) with the dynamics of the network encoded in \( C \). Since this general approach to find a dynamical system encoding both components of an adaptive network is not restricted to the Jain–Krishna model, we conjecture there are ANDI lemma results for many other types of adaptive networks. Using Lemma 3.15 it is easy to check that certain components must decay to zero.

**Proposition 3.16.** Suppose \( G \) has no cycles and \( j \notin T \), then \( x_j(t) \to 0 \) as \( t \to +\infty \).

**Proof.** First, we can eliminate all vertices \( j \), which are not connected to any other vertices since they satisfy the equation
\[
x'_j = -x_j |C x|_1.
\]
Similarly, if \( G \) has multiple connected components, we can analyze each component separately so we restrict attention to a single connected (sub-)graph. Order the indices so that \( \{1, 2, \ldots, j^*\} = T \). Since there are no cycles, \( C \) is nilpotent. Therefore, \( R_n = 0 \) for all \( n > n^* \) for some \( n^* \geq 2 \) and so
\[
R'_1 = R_2 - R_1 |R_1|, \\
\vdots \quad \vdots, \\
R'_{n^*} = -R_{n^*} |R_1|.
\]
Therefore, \( R_{n^*} \to 0 \) as \( t \to +\infty \) and we eventually get by eliminating everything up to the first equation that \( R_1 \to 0 \) as well. However, \( R_1 \) only contains linear combinations of non-terminal vertex variables, which proves the result.

**Theorem 3.17.** Suppose \( G \) has no cycles and \( G \) has at least one edge. For almost every \( x_0 \in X \) (wrt \( \mu_X \)), there exists \( x_* \in X_* \) such that
\[
\lim_{t \to +\infty} x(t) = x_* \quad \text{\( x(t) \) solves (3.4) with \( x(0) = x_0 \).}
\]

**Proof.** By Proposition 3.16, we already know that all components associated to non-terminal vertices vanish as \( t \to +\infty \), so any accumulation point \( x_* \) of \( \{x(t) : t \in [0, \infty)\} \) must lie in \( X_* \). We still have to prove that \( x_* \in X_* \), i.e., only those terminal components with the maximum number of input paths can remain. Without
loss of generality, suppose we just have a graph $G$ with a single connected component. Since $C$ is the transpose of the classical adjacency matrix, it follows that

$$(C^n)_{ij} = |\{\text{paths from } j \text{ to } i \text{ of length } n\}|.$$ 

Hence, if $\{1, 2, \ldots, j^c\}$ with $j^c \leq j^*$ is the set of indices of terminal vertices with the maximum number of paths pointing into them, we have

$$\sum_{j=1}^{d} \sum_{n=0}^{\infty} (C^n)_{i_1 j} > \sum_{j=1}^{d} \sum_{n=0}^{\infty} (C^n)_{i_2 j} \quad \text{if } i_1 \leq j^c \text{ and } i_2 > j^c. \quad (3.10)$$

Of course, the summation over $n$ is actually finite since $C$ is nilpotent as there are no cycles. Switching again to the formulation of the ODEs (3.4) on projective space we have

$$(y(t))_i = \sum_{j=1}^{d} \sum_{n=0}^{\infty} \frac{t^n}{n!} ((C - \phi \text{Id})^n)_{ij} (y(0))_j.$$ 

Hence, using that the identity matrix commutes with every matrix, applying (3.10), and picking $\phi$ so that all components grow, similar to the idea in the proof of Theorem 3.13, we see that those components with $i \leq j^c$ grow fastest so after projection into $x$-variables, only those terminal vertices with maximal number of input paths will remain in the equilibrium point.

In summary, the result for the ODE dynamics of the Jain–Krishna model is relatively simple: generically, we obtain an equilibrium point, which either is non-trivial and governed by one (or more) ACS, or the graph has no cycles and we concentrate on the terminal vertices with maximal input. In the infinite time-scale separation limit, this essentially means that we can regard the ODE dynamics as a discrete map, which produces upon a given input $C$ generically a vector $x^*$.

4. Observations for the discrete-time edge dynamics

In this section, we are going to state several observations and conjectures regarding the discrete-time Jain–Krishna graph update rule (or “JK update”, for short) stated in Section 2. In contrast to Section 3, we do not attempt to provide full proofs of our observations in this section, which is a task left for future work.

We assume that the time-scale separation is infinite in the sense that the ODEs in Section 3 converge instantaneously to an equilibrium point $x^*$. Consider the index sets

$$\mathcal{I}_*: = \{i \in \{1, 2, \ldots, d\} : (x^*)_i > 0\} \quad \text{and} \quad \mathcal{K}_*: = \{i \in \{1, 2, \ldots, d\} : (x^*)_i = 0\}.$$ 

We may always assume that $\mathcal{I}_*$ is non-empty as equilibrium points for the ODEs treated in Section 3 satisfy $|x^*_i| = 1$. Furthermore, observe that if $\mathcal{K}_*$ is non-empty then $\mathcal{K}_* = \mathcal{J}_*$, which is a natural starting point. Recall that in this case, the JK update eliminates all edges to and from a randomly chosen vertex in $\mathcal{K}_*$, and then inserts new edges to and from all other vertices with a fixed probability $p$. We are interested in characterizing cycles, respectively, also ACS, as they determine, how many vertices are active, i.e., the number of elements $|\mathcal{I}_*|$ in the set $\mathcal{I}_*$. 
4.1. Cycle distribution for random graphs

Since cycles are a key component of an ACS, let us start with looking at the cycles for the first graph \( G[0] = G \), which is of Erdős–Rényi type. We start with the undirected case and denote the ensemble of Erdős–Rényi random graphs by \( \text{ER}_d(p) \), where \( p \) is the probability for each edge being present, or by \( \text{ER}_d(\theta/d) \) where \( \theta := pd \). The degree distribution is well-known \([2, 35]\)

\[
p_k := \mathbb{P}(\text{degree}(v) = k) = \binom{d-1}{k} p^k (1-p)^{d-1-k} = \frac{(d-1)!}{k!(d-1-k)!} p^k (1-p)^{d-1-k}.
\]

The binomial distribution \( \text{Bin}(d, \theta/d) \) converges to the Poisson distribution \( \text{Poi}(\theta) \)

\[
\lim_{d \to \infty} \mathbb{P} \left( \text{Bin}(d, \theta/d) = k \right) = e^{-\theta} \frac{\theta^k}{k!},
\]

for each \( k \in \mathbb{N}_0 \). So, \( p_k = e^{-\theta} \theta^k / k! \) is an approximation for the degree distribution of very large Erdős–Rényi graphs. Hence, one expects that the Poisson distribution should also appear in the context of the random variable \( C_k \) counting the number of cycles of length \( k \) in \( \text{ER}_d(\theta/d) \). Let \( k \geq 3 \), consider \( \theta > 0 \) and define \( \mu := \theta^k/(2k) \). Then one can obtain \([4]\) convergence in distribution

\[
C_k \overset{d}{\to} \text{Poi}(\mu) =: c_k.
\]

One nice proof uses Stein’s method as discussed in \([4]\). For directed graphs, we note that for a given undirected cycle of length \( k \), there are \( 2^k \) possible edge orientations possible and only two form a directed cycle, so the distribution is just given by

\[
\tilde{c}_k = \frac{c_k 2^{k-1}}{\sum_{k=3}^\infty c_k 2^{k-1}}, \quad \text{for } k \geq 3.
\]

We can also include directed two-cycles by noticing that we need two consecutive successful binomial trials to generate a two-cycle. Although it is nice to know, how likely cycles are for the first graph, and although it is evident that

\[
\mathbb{P}(\text{no cycles}) \to 0 \quad \text{as } d \to \infty,
\]

there is clearly often the situation that either no cycles, or very few cycles, or just a very small ACS, appear in a graph \( G[s] \) in the Jain–Krishna model. Indeed, the Poisson distribution decays very rapidly and because even very large ACS/cycle structures can be destroyed if all edges are part of an ACS/cycle, we have to consider the time scale it takes to generate a cycle by the Erdős–Rényi graph sampling construction viewed as a dynamical process in discrete time \( s \in \mathbb{N}_0 \) (recall we re-sample the transpose of the adjacency matrix at each time step \( s \) for a single vertex).

4.2. The first cycle

Fix a starting time, wlog \( s = 0 \) and recall that \( G = G[0] \) is fully characterized by the transpose of the adjacency matrix \( C[0] \). Suppose there is no undirected cycle at \( s = 0 \). In this section, we just consider the formation of a cycle in the undirected case but we expect that similar results/conjectures hold upon suitable modification also for the directed case, i.e., although the graph is directed, we are only interested in the formation time...
of the first undirected cycle for simplicity. Since the re-sampling at each discrete time step is according to an Erdős–Rényi-type rule, we have to examine cycle formation for this rule in further detail.

Recall that the mean degree of $\text{ER}_d(p)$ is $dp$. There is a phase transition [10] at $dp = 1$ leading to a giant component for $dp > 1$. Assume that the undirected version $\mathcal{H}$ of $\mathcal{G}$ has an Erdős–Rényi graph structure $\text{ER}_d(p)$ and that $\mathcal{H}$ has no cycle.

**Conjecture 1:** If $dp > 1$, then the formation of a cycle according to the Jain–Krishna update requires $O(1)$ (as $d \to \infty$ with $dp$ kept constant) time steps independent of any other parameters.

Roughly speaking, the conjecture states that cycle formation for $dp > 1$ is extremely fast. Let us provide a heuristic to motivate the conjecture. Consider deleting and then adding a single vertex to $\mathcal{H}$ with edge probability $p$. To form a cycle it is most likely that the cycle has at least one vertex part of the giant component. Then we have

$$
\mathbb{P}(\text{“formation of at least one new cycle”}) \approx 1 - \sum_{j=0}^{1} \binom{d-1}{j} p^j (1-p)^{d-1-j},
$$

where the second term arises as it is the probability that we only connected at most one edge to the giant component. This means in the usual Poisson approximation we get

$$
\mathbb{P}(\text{“formation of a at least one new cycle”}) \approx 1 - e^{-dp}(1 + dp).
$$

Hence, we get the expected result that for a giant component regime, we just need very few trials to generate a cycle. Since in the Jain–Krishna model, only directed cycles are considered, a few additional re-samplings might be required but we shall not discuss this issue here. It is more interesting to look at the case $dp < 1$, since in this case, the approximation of all components by the single giant component is clearly wrong.

**Conjecture 2:** If $dp < 1$, then the number of steps $s_*$ to generate the first cycle by the Jain–Krishna rule is $O(d/p)$.

The last conjecture says that it may take a long time until a first cycle is formed. Again we provide a heuristic argument. Suppose $\mathcal{H}$ has $d$ vertices. Since there is no giant component, forming small loops with existing edges is very unlikely. Therefore, the cycle formation problem can be approximated by the problem to form the first cycle in an empty graph $\mathcal{H}$, which is filled in each time step with edges according to the JK update.

Instead of the JK update, we break the problem into smaller steps just looking at adding single edges in each time step. Enumerate all possible pairs of edges $e_{ij}$ between vertices $i$ and $j$ with $1 \leq i \leq j \leq d$. Now we go through these pairs iteratively and decide at each step with probability $p$, to add the edge to the graph. This is the classical permutation model of the Erdős–Rényi graph $\text{ER}_d(p)$. Another possible and related construction is the uniform model, where $i$ and $j$ are drawn uniformly from $\{1, 2, \ldots, d\}$ and the edge $e_{ij}$ is added with probability $p$. Of course, this means self-loops and double edges can appear generating a random multigraph, which has actually very similar properties to the random graph generated via the permutation model.

It is now a classical question, how the first cycle appears in these constructions and what its expected length is. In fact, this line of research goes back directly to the original work by Erdős and Rényi [10] and has been studied by a variety of combinatorial [12] and probabilistic techniques [3, 23].

It is known that for the uniform and the permutation model, the expected length of this first cycle is $O(d^{1/6})$ as $d \to \infty$ with standard deviation $O(d^{1/4})$. For us, it is more interesting to estimate the time it takes to generate the first cycle. The expected number of edges when the first cycles appears is [12]

$$
\frac{1}{3}d + O(d^{5/6}) \quad \text{as } d \to \infty
$$

Interestingly, even the same idea of using catalytic feedback loops to describe pre-biotic evolution appears in the context of these classical works on first cycle generation [3] so this can be viewed as an early variant the Jain–Krishna model already.
for the uniform model, while it is
\[ \frac{1}{2} (1 - p_*) d + \mathcal{O}(d^{5/6}) \quad \text{as } d \to \infty \]  
(4.2)
for a constant \( p_* \approx 0.12 \) in the permutation model. Since we add one edge per step at probability \( p \), there is an average of \( ps_* \) edges after \( s_* \) steps for each model. Discarding the higher-order terms we get that the condition
\[ \frac{1}{3} d \frac{1}{3} ps_* \Rightarrow s_* = \frac{1}{3} d / \theta = \frac{1}{3} d^2 / \theta \]
is conjectured to give a good approximation to obtain a first cycle in \( s_* \) steps of the uniform model. A similar condition appears for the permutation model so we always have \( s_* = \mathcal{O}(d^2 / \theta) \), where obviously \( \theta = \theta(d) \) may depend upon the size of the graph in most applications.

Applying these results to the Jain–Krishna model is not immediately possible since (a) we have a directed graph in this context, (b) we do not allow for loops, (c) we add not one but potentially many edges at each time step, and (d) we do not generate the graph completely but only re-sample one vertex at each time step.

Scalings with leading-order term \( \mathcal{O}(d) \) as in (4.1)–(4.2) also hold for directed graphs [3], which means we can conjecture that (a) does not play a crucial role. The case of loops is also excluded in the directed context in [3] not affecting the scaling so (b) can also be disregarded. The problem (c) is essentially just a time re-scaling for large graphs as instead of counting one edge at a time, we add a certain average number of edges \( 2p(d-1) \) at once. However, the problem (d) could be substantial as we have a lot more edges available already as we do not start from an empty graph but this should be covered by having a very sparse graph upon using \( dp < 1 \).

4.3. Generating large ACS

Now suppose \( G[0] = G \) has at least one cycle \( Y \subseteq G \) and suppose \( K_* \) is non-empty. Obviously, the cycle only consists of edges in \( I_* \) so it will not be destroyed by the JK update as long as \( K_* \) is non-empty. We are now interested in, how long it is going to take to form one large ACS \( Z \). As before, we expect that the size of \( dp \) is crucial to distinguish several cases. Again, we restrict to the undirected case and ask the simpler question, how long it is going to take to form one large cycle.

**Conjecture 3:** If \( dp > 1 \) and we start from a typical cycle, then the formation of a single ACS according to the JK update requires at most \( \mathcal{O}(1) \) time steps independent of any other parameters.

The reasoning for this conjecture is that there will be relatively large cycles in the case of a giant component and then just the same reasoning as for Conjecture 1 applies, so we shall not discuss the reasoning in additional detail. As before, the interesting case seems to be \( dp < 1 \).

**Conjecture 4:** If \( dp < 1 \) is fixed and \( d \to +\infty \), the mean waiting time until an ACS has formed containing all vertices is \( \mathcal{O}(d) \).

Again we give some heuristics to motivate the conjecture. Suppose our current ACS \( Z[0] = Z \) in \( G[0] \) has \( k \) vertices. To attach a vertex \( v \) via the JK update the ACS generation requires the generation of at least one edge from \( Z \) to \( v \). The probability of this event is
\[
\sum_{i=1}^{k} \binom{k}{i} p^i (1 - p)^{k-i} = 1 - (1 - p)^k =: r(k, p) = r.
\]
Therefore, we have for the mean waiting time $\tau_k$ until the update rule has attached a vertex $v$ that is repeatedly selected

$$
E[\tau_k] = \sum_{j=1}^{\infty} jr(1-r)^{j-1} = r \sum_{j=1}^{\infty} j(1-r)^{j-1} = \frac{r}{(1-(1-r))^2} = \frac{1}{r},
$$

where the usual differentiation of the geometric series has been used. Now summing over all the waiting times gives the total waiting time

$$
\sum_{k=1}^{d} E[\tau_k] = \sum_{k=1}^{d} \frac{1}{1-(1-p)^k}.
$$

The largest term in this sum is actually the first one as one would intuitively expect as it is easier to attach to an ACS if the ACS is already large. However, there is also the effect of $d$, particularly for large graphs. We just calculate

$$
\sum_{k=1}^{d} \frac{1}{1-(1-p)^k} \approx \int_{1}^{d} \frac{1}{1-(1-p)^x} \, dx
$$

$$
= \int_{1}^{d} \frac{1}{1-e^{x \ln(1-p)}} \, dx = x - \frac{1-e^{x \ln(1-p)}}{\ln(1-p)} \bigg|_{x=1}^{d}
$$

$$
= d - 1 - \frac{(1-p)^d}{\ln(1-p)} + \frac{p}{\ln(1-p)}.
$$

So, if $p$ is small and $d$ is large while $dp < 1$ is fixed in the asymptotic limit $p \to 0$, we easily check using L'Hôpital's rule that the last expression diverges like $1/p$ as $p \to 0$. Since $\theta = pd < 1$ is fixed, we have now motivated our Conjecture 4 as the divergence is $O(d)$ as $d \to +\infty$. Therefore, it takes very long to really form a full ACS activating every vertex, even if we start with a cycle already in the first graph. Yet, it does not take as long as forming cycles in the first place since we essentially form an ACS in a more deterministic way by judiciously eliminating vertices not part of an ACS at each step.

5. Summary and outlook

In this work, we have analyzed the two singular limits of the Jain–Krishna model for adaptive catalytic networks. One limit consists in freezing the dynamics of the network working on a fixed graph. In this case, we have proven the existence of stationary solutions for the ODE vertex dynamics. We have characterized the equilibria via Perron–Frobenius vectors, including a distinction between different cycle structures and ACSs of the underlying graph. Then we have rigorously proven that the dynamics of the catalytic ODEs converges (up to initial conditions of measure zero) to a stationary solution. The proof uses the insight that working in a projective space is the correct mathematical setting and it uses an intertwining lemma, which yields an infinite-dimensional system of ODEs including the graph structure so that the dynamics can be treated hierarchically. Then we studied the second singular limit, where the vertex dynamics are assumed to be infinitely fast, yet the dynamics of the network yields a changing graph. In this context, we formulated four conjectures, which we justified with formal asymptotic calculations for the two parameters $d$ (number of vertices) and $p$ (connection probability between vertices). More precisely there seems to be a substantial distinction for the Jain–Krishna model between the cases, when the graph has a giant component ($dp > 1$) and when it does not ($dp < 1$). More precisely, in the giant component case, we conjectured that the formation of an initial cycle and the formation of an ACS are of orders $O(d^2)$ and $O(d)$ respectively as $d \to +\infty$. In the case without a giant component, these events are conjectured to occur typically at order $O(1)$. 


In summary, we can conclude that for the non-giant component case \( dp < 1 \), the dynamics is a true multiple time scale dynamical system: the ODE convergence of the vertex dynamics to equilibrium is generically exponentially fast, and the network is slowly driven by the adaptation of the graph at a very slow time scale of at least \( \mathcal{O}(d) \). Yet, if we are very close to, or even have \( dp > 1 \), we expect a mixing of the time scales.

The critical regime \( dp \approx 1 \) seems to be particularly interesting as there could be links to SOC in networks [5, 7, 32], where time-scale separation again plays a crucial role [27]. SOC exploits that the network is near a topologically critical point, e.g., being marginally connected, to enhance information processing. From the viewpoint of applications, one may hence infer that it could be beneficial to study the Jain–Krishna model also very close to the critical regime \( dp \approx 1 \), where it can change from a multiple time scale network to a more single-scale structure between ODE vertex dynamics and network adaption.

For the regime \( dp < 1 \), it seems to be a natural next step to exploit the scale separation to make predictions about the dynamics. For example, we know that an ACS can only form very slowly. The ACS can only be destroyed once it encompasses the entire graph, which means that we may be able to derive early warning signs for drastic sudden transitions in such networks. Indeed, this has been shown to be possible in several other adaptive network models already [6, 17, 18, 29].

From the standpoint of rigorous mathematical analysis, it seems plausible that it might be possible to give a fully rigorous analysis of certain patterns observed in time series of the Jain–Krishna model. At least for the case \( dp < 1 \) one may use the equilibrium point results we proved in Section 3 and try to establish Conjectures 2 and 4. Then another step employing (singular) perturbation theory should yield the existence of long phases of slow network adaptation toward an ACS followed by a sudden collapses, which is the main phenomenon exhibited by the Jain–Krishna model. The work presented here provides at least a first step in this, potentially very involved, path toward a complete mathematical analysis.

**Appendix A. An inconsistent model definition**

In this appendix, we prove that the definition of the model in the paper [21] is incorrect. We briefly repeat the definition from [21]. Given \( C \in \mathbb{R}^{d \times d} \), we start by sampling an interaction matrix \( C = (c_{ij}) \) so that

\[
\mathbb{P}(c_{ij} \neq 0) = p \quad \text{and} \quad \mathbb{P}(c_{ij} = 0) = 1 - p
\]

for some fixed \( p \in (0, 1) \). Then the entry \( c_{ij} \) is determined by sampling uniformly from \([-1, 1]\) if \( i \neq j \) and from \([-1, 0]\) if \( i = j \), which is the first main difference to Section 2. Then we set as above

\[
f_i := (Cx)_i - x_i \sum_{k=1}^{d} (Cx)_k \quad (A.1)
\]

but define the ODEs for \( x = x(t) \in \mathbb{R}^d \) as

\[
x_i' = \begin{cases} 
  f_i & \text{if } x_i > 0 \text{ or } f_i \geq 0, \\
  0 & \text{if } x_i = 0 \text{ and } f_i < 0.
\end{cases} \quad (A.2)
\]

Furthermore, we impose the constraints

\[
0 \leq x_i \leq 1, \quad \sum_{i=1}^{d} x_i = 1. \quad (A.3)
\]

defining the phase space \( \mathcal{X} \).

**Theorem A.1.** The model definition (A.1)–(A.3) is consistent in the interior \( \text{int}(\mathcal{X}) \), yet it is inconsistent, i.e., not solvable, once points on \( \partial \mathcal{X} \) are considered.
Proof. Let us start with the interior, there the proof is easy since we only have to check whether the conservation of mass

\[ \sum_{j=1}^{d} x_j = 1 \quad (A.4) \]

is consistent with (A.2). Indeed, this is easy since differentiating (A.4) we get

\[ 0 = \frac{d}{dt} \left( \sum_{j=1}^{d} x_j \right) = \sum_{j=1}^{d} f_j \]

as we are in the interior. So, we find

\[ 0 = \sum_{j=1}^{d} f_j = \sum_{j=1}^{d} (Cx)_j - \sum_{j=1}^{d} x_j \sum_{k=1}^{d} (Cx)_k = \sum_{j=1}^{d} (Cx)_j - \sum_{k=1}^{d} (Cx)_k. \]

Hence, the model is consistent in the sense that the physical conservation law is also enforced by the ODEs themselves. This means we actually do not really need the conservation law in int\(\mathcal{X}\). However, on \(\partial \mathcal{X}\) the same calculation fails. Consider a point with \(x_r = 0\) and \(x_j > 0\) for \(j \neq r\) with \(f_r < 0\); it is easy to see that we can reach such a point in finite time for an open set of matrices \(C\), i.e., this is not a special case. Now we compute

\[ 0 = \sum_{j=1}^{d} x_j' = \sum_{j=1,j \neq r}^{d} f_j \]

\[ = \sum_{j=1,j \neq r}^{d} (Cx)_j - \sum_{j=1,j \neq r}^{d} x_j \sum_{k=1}^{d} (Cx)_k = \sum_{j=1,j \neq r}^{d} (Cx)_j - \sum_{k=1}^{d} (Cx)_k \]

and the last expression on the right-hand side is in general not equal to zero for some open set of possible matrices \(C\).

\[ \square \]

In fact, one can easily see that there are cases, where trajectories leave \(\mathcal{X}\), i.e., the ODEs are inconsistent with the mass constraint. Hence, one should follow the model definitions in [20, 22] instead of [21].

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References

[1] L. Arnold, Random Dynamical Systems. Springer, Berlin, Heidelberg, Germany (2003).
[2] B. Bollobás, Random Graphs. CUP, Cambridge (2001).
[3] B. Bollobás and S. Rasmussen, First cycles in random directed graph processes. Discrete Math. 75 (1989) 55–68.
[4] C. Bordenave, Random Graphs and Probabilistic Combinatorial Optimization. Lecture Notes. Available at: https://www.math.univ-toulouse.fr/~bordenave/coursRG.pdf (2016).
