Dynamics of mobile interacting ferromagnetic films: theory and numerical implementation

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Abstract
By coating two nearby bodies with thin ferromagnetic films below the Curie temperature, one obtains two interacting sets of magnetic domains. The dynamical properties of the bodies in the presence of this domain interaction have never been investigated thus far. In this work I derive a set of equations to simultaneously describe both the domain evolution within the two films and the dynamics of the coated bodies. The shape, size and mobility of the domains can easily be controlled with an external magnetic field or by properly choosing the material properties, thus unravelling how the domain characteristics influence the system dynamics. This can thus be of great technological relevance, providing new means to control and actuate mechanical motion at the micro- and nano-scale.

Keywords: sliding friction, non-linear dynamics, ferromagnetic films, domain dynamics, driven disordered systems

The possibility of controlling friction and thus the sliding motion of two bodies has been extensively investigated both at the fundamental and applied levels, being closely tied to progress in transportation, manufacturing and energy conversion and thus impacting on innumerable aspects of our health and environment. Not all of the control techniques available at the macro-scale, such as the use of lubricants, surface patterning or the application of mechanical vibrations, are straightforwardly applicable to micro- and nano-mechanical systems because of the different scaling of physical laws with system size. At the micro- and nano-scale, however, new physical phenomena can be exploited for the control of motion, such as atomic lattice commensurability or the superlubric transition. The possibility of controlling sliding friction through the occurrence of a structural phase transition in one or both of the sliding bodies as been recently suggested [1]. This technique allows the control of the phase
transition and thus the sliding motion dynamically and reversibly by means of an external electric field or by applying pressure to the sliding bodies. Along the same lines, I believe that the sliding motion can be controlled by coating the two sliding bodies with thin ferromagnetic films (FFs), as depicted in figure 1. Below the Curie temperature, the presence of magnetic domains can give rise to very strong forces which are able to modify the sliding dynamics, moreover the domain shape and size can be controlled by an external field, thus enabling the dynamical and reversible control of motion. The aim of this work is to develop the necessary mathematical and computational tools to investigate the mutual influence of magnetic domain interaction and the sliding motion of the coated bodies, namely to set up a system of equations to simultaneously describe the domain dynamics within each FF and the sliding motion of the coated bodies.

Motivated by data storage technology needs, the physics of magnetic domains in thin FFs has been extensively investigated in past decades. This work focuses on FFs with perpendicular anisotropy, i.e. the easy axis of the magnetization is perpendicular to the film surface. This behavior is typical of Co/Pt and Fe/Ni multilayers, permalloy and garnet films, to name a few examples. In these FFs, the domains exhibit stable disordered maze-like patterns but, under the influence of an external magnetic field, the domains can be ordered into parallel stripes or bubble lattices [2, 3]. The characteristic domain size, ranging from tens of nm to tens of $\mu$m, can be controlled by the materials and the film thickness [4]. By changing the deposition rate one controls the homogeneity of the FFs, promoting the presence of defects and impurities that serve as pinning sites for the domains, thus controlling the domain mobility [5].

Experiments to test this new suggested control mechanism can be set up in several ways. As illustrated in figure 1, the two FFs can be grown on a substrate and on a colloidal probe tip with a large curvature radius so that their interaction can be probed by atomic force microscopy in the non-contact mode. The atomic force microscope can also be used to study the contact sliding between two large plates [6], a meso-scale friction tester [7] or a surface force apparatus [8] can be used as well. When the two coated bodies slide in contact mode, the two FFs can
be protected from wear by a capping layer and they can be maintained at a constant distance with sub-nanometric precision by a non-magnetic spacing layer. Mechanical friction can be reduced by the use of lubricants.

In section 1 I recall, generalize and comment on the existing theory for the description of domain evolution in a single isolated FF; in section 2 I extend the theory in order to describe the case of two interacting parallel films; in section 3 I introduce Newton equations for describing the FF dynamics, i.e. motion of the coated bodies; and finally in section 4 I discuss the algorithms for the numerical implementation of the new set of equations.

1. Single FF

The magnetic properties of a FF below its Curie temperature can be modeled in several ways. Traditional modeling encompasses statistical approaches, such as the Preisach one, as well as micromagnetics [2, 3]. While the former allows the description of the hysteresis loop of macroscopic samples without any suggestion of the real microscopic domain dynamics, the latter can be used to access the fine details of the domain structure and motion, although the higher computational cost limits the size of the treatable systems. An intermediate phase-field approach exists which, starting from the micromagnetics equations by means of suitable approximations, allows the investigation of the detailed domain dynamics in FFs over large length-scales such as the ones accessible to magnetic force microscopy (MFM), a few µm², or to optical techniques, hundreds of µm². The first numerical studies using this approach were performed by Jagla who investigated the possible stable and unstable domain patterns in thin perpendicular anisotropy FFs [9] and the role of the domain dynamics in determining the hysteresis loop shape [10]. In more recent works, the same kind of modeling has been adopted to investigate return point memory effects [5], Barkhausen avalanche distributions and critical exponents [11], and the role of defects in the domain reorientation under the influence of an oscillating external field [12]. Recently, it has been also demonstrated that this kind of modeling reproduces quantitatively both the domain dynamics at the micro-scale and the macroscopic hysteresis properties of exchange-bias Co/Pt multilayers [13].

1.1. Hamiltonian and domain equation of motion

The starting point for the study of domain dynamics is the Landau–Lifshitz–Gilbert equation (LLGE), ruling the precession motion of the magnetization vector \( \mathbf{M}(r,t) \) associated to the infinitesimal medium volume \( d^3r \) around a local field \( \mathbf{B}(r,t) \) due to the presence of the rest of the medium and to external sources:

\[
\frac{\partial \mathbf{M}(r,t)}{\partial t} = -\gamma \mathbf{M}(r,t) \times \mathbf{B}(r,t) - \gamma \frac{\partial \mathbf{M}(r,t)}{\partial t}.
\]  

where \( \gamma \) is the gyromagnetic ratio of the electron spin and \( \eta \) is the characteristic damping time of the material, representing the irreversible energy transfer to microscopic degrees of freedom such as magnons, phonons and eddy-currents. The magnetization can be written as \( \mathbf{M}(r,t) = M_s \mathbf{m}(r,t) \), separating its modulus, i.e. the saturation magnetization \( M_s \), a material parameter, from the dimensionless versor \( \mathbf{m} \). Defining the dimensionless constant \( \alpha = \gamma \eta M_s \), in the limit \( \alpha \ll 1 \), (1) can be rewritten as [14]:

\[
\frac{\partial \mathbf{M}(r,t)}{\partial t} = -\gamma \mathbf{M}(r,t) \times \mathbf{B}(r,t) - \frac{\gamma \alpha}{M_s} \mathbf{M}(r,t) \times \mathbf{M}(r,t) \times \mathbf{B}(r,t).
\]  

Theoretical calculations and experimental measurements have demonstrated that the assumption \( \alpha \ll 1 \) is fulfilled by most of the ferromagnetic materials in their bulk, multilayer
and thin film forms, although in the latter case $\alpha$ can be slightly dependent on film thickness and growing conditions [15–18]. The field $B$ depends on the material properties and on the sample shape and it is known once the system Hamiltonian $H$ is given:

$$B(r, t) = -\frac{1}{M_s} \frac{\delta H[m(r, t)]}{\delta m(r, t)} + Q(r, t).$$  \hspace{1cm} (3)

The first term is the functional derivative of the Hamiltonian while the second one is a Gaussian stochastic process representing the thermal fluctuations the system experiences when in contact with an heat-bath at temperature $T$ [19]. More precisely the stochastic process $Q$ can be characterized giving its average and correlation:

$$\langle Q(r, t) \rangle = 0 \hspace{1cm} \langle Q(r, t)Q(r', t') \rangle = 2K_B T \frac{\alpha}{\gamma M_s} \delta(t - t')\delta(r - r'),$$ \hspace{1cm} (4)

$K_B$ is the Boltzmann constant; from the two Dirac delta in the correlation function it is seen that the stochastic process is uncorrelated in time and space. At finite temperatures (2) a Langevin equation can be seen as, in which the balancing of the competing damping and stochastic terms allows the sampling of the precession trajectories from a canonical ensemble.

Micromagnetic simulations can be performed starting from (2) and evolving the magnetization in time on a three-dimensional mesh [20], the field $B$ felt by every magnetic dipole, associated to the infinitesimal medium volume, will be the sum of the field due to all the other dipoles. This non-locality, together with the full vectorial treatment of the problem, is responsible for the high computational cost of these kind of simulations, limiting the size of the simulated samples. However, to describe the physics of certain systems with a specific symmetry, one component of the magnetization might be more relevant than the others. This is the case for perpendicular anisotropy FF in which, except for the domain wall regions, the magnetization is mostly aligned perpendicular to the film plane as depicted in figure 2(a). In this simplified picture the magnetization is assumed to be uniform along the $z$-axis, in the
approximation of thin domain walls [10], only its \( z \) component plays a relevant role, thus the domain dynamics can be described solely by a scalar function \( m \) varying only on the film plane, i.e. \( \mathbf{M} = M_s m(x, y) \mathbf{\hat{z}} = M_s m(r_\parallel) \mathbf{\hat{z}} \). Notice that, by construction, \( m(x, y) \) must vary continuously in the interval \([-1, +1]\). To give the magnetization a preferential orientation (easy-axis) along the \( z \) direction, the Hamiltonian must contain a quadratic term in \( m \):

\[
\mathcal{H}_{\text{anisotropy}} = -\frac{K_u}{2} \int m(r_\parallel)^2 \, d^3 r,
\]

so that the energy is lowered the most when \( |m| \to 1 \) irrespective of the sign, i.e. irrespective of the upward or downward orientation of the magnetic dipole moments. \( K_u \) is the anisotropy constant of the material determining the strength of the energy gain with the dipole moment’s alignment.

Every dipole moment of the film feels a stray field (also referred to as a demagnetizing or dipolar field) due to the presence of the other dipole moments, as illustrated in figure 2(b). For the simple geometry of a plane FF of thickness \( t_b \), the stray field energy can be calculated exactly starting from the potential \( \phi \) due to a magnetization distribution:

\[
\phi(r, t) = \frac{M_s}{4\pi} \left( -\int \frac{\nabla \cdot \mathbf{m}(r', t)}{|r - r'|} \, d^3 r' + \int \frac{\mathbf{m}(r', t) \cdot \mathbf{n}}{|r - r'|} \, d\Sigma' \right),
\]

where \( \mathbf{n} \) is the versor normal to the surface, the first integral is on the film volume, the second one is on the film surface. With the choice for \( \mathbf{m}(r, t) \) only the surface integral survives and the potential reduces to:

\[
\phi(r, t) = \frac{M_s}{4\pi} \int \left( \frac{m(r_\parallel, t)}{\sqrt{(r_\parallel - r_\parallel')^2 + (z - t_b)^2}} - \frac{m(r_\parallel', t)}{\sqrt{(r_\parallel - r_\parallel')^2 + z^2}} \right) \, d^2 r_\parallel',
\]

now \( r_\parallel \) and \( r_\parallel' \) span the \( xy \) plane only, i.e. \((r_\parallel - r_\parallel')^2 = (x - x')^2 + (y - y')^2\). The first contribution to the integral comes from the upper surface \((z' = t_b)\), the second one comes from the lower surface of the film \((z' = 0)\), see figure 2(b). The self-energy of a magnetization distribution can be calculated as:

\[
\mathcal{H}_{\text{stray}} = \frac{\mu_0 M_s}{2} \int \nabla \phi(r, t) \cdot \mathbf{m}(r, t) \, d^3 r
\]

\[
= \frac{\mu_0 M_s}{2} \left( \left. \int \phi(r, t) \, d^2 r_\parallel \right|_{z=0} \right)^2,
\]

the factor \( 1/2 \) is used to avoid double counting in the sum of all the dipole–dipole contributions (i.e. the double integral on \( r \) and \( r' \)); \( \mu_0 \) is the vacuum permeability. The second step comes from a simple integration by parts, taking into account that the magnetization is directed along \( z \) only and does not vary along the film thickness, i.e. is not a function of \( z \). The energy can be explicitly obtained substituting (7) into (8):

\[
\mathcal{H}_{\text{stray}} = \frac{\mu_0 M_s^2}{4\pi} \int \left( \frac{m(r_\parallel, t)m(r_\parallel', t)}{|r_\parallel - r_\parallel'|} - \frac{m(r_\parallel', t)m(r_\parallel, t)}{\sqrt{(r_\parallel - r_\parallel')^2 + t_b^2}} \right) \, d^2 r_\parallel \, d^2 r_\parallel'.
\]

Keeping in mind the streamlines of the magnetic field generated by a single dipole moment, it is easy to understand that, in order to minimize the total energy, each dipole tries to align oppositely to the neighboring ones.

Due to electronic interactions the system gains energy leaving the neighboring dipoles aligned along the same direction. The Hamiltonian term accounting for this behavior must
contain a space derivative of \( m(r, t) \) in order to lose energy at every spatial variation of the magnetization:

\[
\mathcal{H}_{\text{exchange}} = \frac{A}{2} \int [\nabla m(r, t)]^2 \, d^3 r, \tag{10}
\]

where \( A \) is the exchange stiffness, representing the energy cost to misalign neighboring dipole moments, and the square is necessary to treat upward and downward spatial variations in the same way. This term is in competition with the stray field one and the characteristic domain size arises from the balancing of the two, see section 1.4. In order for the model to be able to describe the domain manipulation via external magnetic field \( H_{\text{ext}} \), the last ingredient to be included in the Hamiltonian is given by:

\[
\mathcal{H}_{\text{extern}} = -\mu_0 M_s \int H_{\text{ext}} \cdot m(r, t) \, d^3 r = -\mu_0 M_s \int H_{\text{ext}} m(r, t) \, d^3 r, \tag{11}
\]

the second step comes from the assumption that the external field is completely aligned along the \( z \)-axis. Notice the absence of the \( 1/2 \) factor with respect to (8), this is in fact the energy contribution due to an external field, not a self-energy. To summarize, the full Hamiltonian for a single FF reads:

\[
\mathcal{H} = \mathcal{H}_{\text{anisotropy}} + \mathcal{H}_{\text{exchange}} + \mathcal{H}_{\text{extern}} + \mathcal{H}_{\text{stray}}
\]

\[
= \int \left[ -K_u m(r) \frac{m(r)}{2} + \frac{A}{2} [\nabla m(r, t)]^2 - \mu_0 M_s m(r) H_{\text{ext}} \right. \\
+ \left. \frac{\mu_0 M_s^2}{4\pi t_F} \int d^2 r' \left( \frac{m(r) m(r')}{|r - r'|} - \frac{m(r) m(r')}{\sqrt{(r - r')^2 + t_F^2}} \right) \right] \, d^3 r, \tag{12}
\]

the \( 1/t_F \) in the stray field term arises from the need to restore a volume integral in (9), this is done by putting \( d^2 r = d^2 r' \, dz / t_F = d^3 r / t_F \). From the functional derivative (3) one can thus calculate the field \( B \) which is parallel to the \( z \)-axis (from now on I drop the subscript \( \parallel \) and \( r \) and \( r' \) are intended to run on the \( xy \) plane only):

\[
B = \left[ K_u \frac{m(r)}{M_s} + \mu_0 \frac{H_{\text{ext}}}{t_F} - \frac{\mu_0 M_s}{2\pi t_F} \int \left( \frac{m(r')}{|r - r'|} - \frac{m(r')}{\sqrt{(r - r')^2 + t_F^2}} \right) d^2 r' \right] \hat{z} \\
+ Q(r, t) \hat{z} + \frac{A}{M_s} \nabla^2 m(r, t), \tag{13}
\]

the gradient term has been treated with the ‘thin domain wall’ approximation as described in [10]. Substituting the previous expression into (2) one immediately see that the fist term on the rhs vanishes and one is left with:

\[
\frac{\partial m}{\partial t} = \gamma a \left\{ (1 - m^2) \left( K_u m + \mu_0 H_{\text{ext}} - \frac{\mu_0 M_s}{2\pi t_F} \int \left( \frac{m(r')}{|r - r'|} - \frac{m(r')}{\sqrt{(r - r')^2 + t_F^2}} \right) d^2 r' \right) + Q(r, t) \right\} + \frac{A}{M_s} \nabla^2 m, \tag{14}
\]

1.2. Small thickness approximation and useful limits

In the early works by Jagla the stray field term has been treated in the small thickness approximation \( t_F \to 0 \). Power expanding the second term in the rhs of (9) for small \( t_F \),
it is seen that the zero order contribution cancels out with the first term leaving only the contribution in \( t^2 \) (the first order contribution is zero for parity reasons):

\[
H_{\text{stray}} = \frac{\mu_0 M_s^2}{8\pi} \int \frac{m(r', t)m(r, t)}{|r - r'|^3} d^2r d^2r'.
\]  
(15)

From this simplified expression the tendency of the stray field to anti-align the dipole moments is immediately clear: if the spin at the point \( r \) is oriented in the same direction as the one at \( r' \), the product \( m(r')m(r) \) is positive and the total energy increases, to gain energy the two spins at \( r \) and \( r' \) must be oppositely oriented so that \( H_{\text{stray}} < 0 \).

It is also important to notice that, when completely saturated, i.e. \( m(x, y) \equiv \pm 1 \forall x, y \), the FF behaves like a uniformly charged plane capacitor. This means that the outer field is zero while the inner one is constant and it depends only on the material parameters. The field expression (13), with \( m(r', t) = 1 \), at zero temperature and in absence of any external field becomes:

\[
B(r) = \frac{K_u}{M_s} - \frac{\mu_0 M_s}{2\pi t_F} \int \frac{1}{|r'|} - \frac{1}{\sqrt{(r^2 + t_F^2)}} d^2r'.
\]  
(16)

the integrand depends only on \(|r - r'|\) thus, if the film is infinitely extended along \( x \) and \( y \), one as translational invariance, i.e. the integral over \( r' \) gives the same result for every \( r \). This symmetry can be exploited to solve the integral for the convenient choice \( r = 0 \):

\[
B = \frac{K_u}{M_s} - \frac{\mu_0 M_s}{2\pi t_F} \int \left( \frac{1}{|r'|} - \frac{1}{\sqrt{r'^2 + t_F^2}} \right) d^2r' = \frac{K_u}{M_s} - \frac{\mu_0 M_s}{t_F} \lim_{\ell \to \infty} \int_0^{2\pi} \int_0^{\ell} \left( \frac{1}{\sqrt{r^2 + t_F^2}} \right) d^2r = \frac{K_u}{M_s} - \frac{\mu_0 M_s}{t_F} \left( \ell + t_F - \sqrt{\ell^2 + t_F^2} \right) = \frac{K_u}{M_s} - \frac{\mu_0 M_s}{t_F},
\]  
(17)

the second step has been obtained by moving to polar coordinates. The same can be done for the energy density \( \mathcal{E} \), i.e. the integrand in the rhs of (12) which, for \( m(r)m(r') = 1 \) and calculated in the infinitesimal volume \( d^3r \) centered at \( r = 0 \), reads:

\[
\mathcal{E} = -\frac{K_u}{2} + \frac{\mu_0 M_s^2}{4\pi t_F} \int \left( \frac{1}{|r'|} - \frac{1}{\sqrt{r'^2 + t_F^2}} \right) d^2r' = -\frac{K_u}{2} + \frac{\mu_0 M_s^2}{2}.
\]  
(18)

again the last step holds for small \( t_F \).

1.3. Pinning disorder

Under the influence of an external magnetic field \( H_{ext} \), the magnetization of a ferromagnetic material can be manipulated, promoting nucleation, growth and coalescence of domains. However the magnetization does not vary smoothly with external field strength, its dynamics is characterized by sudden jumps due to the discontinuous motion of the domain walls pinned by crystalline defects and grain boundaries, these jumps are known as Barkhausen avalanches. The disorder and the inhomogeneities of the material serve also as nucleation points at the initial stage of the magnetization reversal process. The pinning effect due to the sample inhomogeneities can be included in the model by letting one ore more material properties fluctuate randomly on the \( xy \) plane. Contrary to the disorder introduced by thermal fluctuations
which changes in time, this new source of randomness is fixed once and for all at \( t = 0 \). This frozen disorder can be introduced in the anisotropy constant (random anisotropy model), in the exchange stiffness constant (random bond model) or simply by means of a magnetic field \( H_{\text{random}} \) (random field model):

\[
K_u(r) = K_u [1 - c_1 p(r)], \\
A(r) = A [1 - c_2 p(r)], \\
H_{\text{random}}(r) = c_3 p(r),
\]

(19)

where \( c_i \) are parameters determining the amplitude of the time independent spatial fluctuations, i.e. the strength of the pinning inhomogeneities, \( K_u \) and \( A \) are the macroscopic average material parameter, and \( p(r) \) is a Gaussian stochastic process with \( \langle p(r) \rangle = 0 \) and a given correlation \( \langle p(r)p(r') \rangle \).

As will be shown in section 4, the LLGE must be solved numerically on a discrete mesh of spacing \( \Delta \); if this quantity is bigger than the characteristic length scale of the FF inhomogeneities, the pinning disorder fluctuations will be uncorrelated (white noise), i.e. \( \langle p(r)p(r') \rangle = \delta(r - r') \). More generally, the disorder can be correlated on a characteristic length scale dictated by the micro-structure of the FF, for instance the average crystalline grain size. A correlated random field has recently been used to study how the domain dynamics is affected by the presence of uncompensated spins at the interface between ferromagnetic and anti-FFs in exchange-bias systems [13].

The physics of the three fluctuating noises is of course different: the fluctuations entering the Hamiltonian through the anisotropy or the exchange terms, which are quadratic in \( m \), are not sensitive to the magnetization sign, i.e. they serve as nucleation points for both upward and downward oriented domains, and the up-to-down and down-to-up hysteresis semi-loops are exactly identical (with the same domain patterns). To have different nucleation points in the two hysteresis branches a random field model is needed, its Hamiltonian term is linear in \( m \) and thus sensitive to the magnetization sign. Typically, in a real FF, all the three sources of randomness are simultaneously present.

1.4. Model parameters and domain behavior

The material parameters \( K_u, A \) and \( c \) determine the domain morphology and dynamics. Starting from a simplified version of (12), it is easy to demonstrate that the energy cost of a domain wall is proportional to \( \sqrt{K_uA} \) [2, 3, 21], this is also easy to understand qualitatively: in a domain wall the magnetic dipoles are misaligned with respect to the neighbors, thus an energy proportional to \( A \) must be paid, but they are also misaligned with respect to the anisotropy easy axis, thus an anisotropy energy proportional to \( K_u \) must also be paid. The characteristic average domain width is set by the competition of the stray field and the energy cost of a wall, while the former tends to create a large amount of small domains to demagnetize the film, the latter tries to minimize the number of domain walls and thus domains. As explicitly visible from (9), the stray field strength depends solely on the film thickness, thus at fixed \( t_F \), the domain size can be tuned by varying only the domain wall cost and it becomes \( \sqrt{K_uA} \). On the other hand, working with a given material, one can tune the domain width by properly choosing the film thickness; the thickness dependence is, however, non-trivial [4, 22]. Figure 3(a) shows the magnetization \( m(x, y) \) resulting from two different simulations with the same thickness but a different \( K_uA \) product. Again starting from (12) it is easy to show that the domain wall thickness becomes \( \sqrt{A/K_u} \), figure 3(b) shows how the simulated domain walls get narrower as one decreases the \( A/K_u \) ratio. For many ferromagnetic materials the \( K_u \) and \( A \) values are tabulated [2], for thin films and multilayers they are known to be dependent on the thickness.
and deposition conditions. However, $K_u$ and $A$ can also be estimated starting from a measured domain image; in fact, knowing both the real domain width and domain wall thickness, both the product $K_u A$ and the ratio $A/K_u$ can be fixed and unique values for $K_u$ and $A$ which satisfy both the conditions exists. Using this idea it is possible to extract the $K_u$ and $A$ values directly from the simulations once the simulated domain morphology resembles the measured one [13]. For Co/Pt multilayers a very good agreement is found between the calculated and experimentally estimated $K_u$ values whereas $A$ is always overestimated. This last result is easily explicable recalling that in (14) a term in $(\nabla m)^2$ is neglected for the sake of simplicity and computational feasibility assuming that its main effect is simply to renormalize the $A$ constant [10]. The inhomogeneity strength $c$ is also important in determining the domain shape, as shown in figure 3(c): increasing the inhomogeneity of the FF, the domain boundaries become more irregular. The $c$ value also determines the domain mobility under the influence of an external magnetic field, a large $c$ value results is strong domain pinning with a very irregular and sudden domain motion, smaller $c$ values lead to a more smooth and continuous motion. For this reason the inhomogeneity strength can be estimated from the size distribution of the measured Barkhausen avalanches [11]. The two remaining material parameters $M_s$ and $\alpha$ play no role in determining the domain characteristics, they simply define the absolute strength of the field and the absolute time scale for the domain motion, respectively. This is explicitly shown in section 4.1 where (14) is rescaled in order to be dimensionless, $M_s$ becomes the unit field while $\alpha$ enters the unit time and both of them disappear from the rescaled equation.

2. Two interacting FFs

The possible experimental set-ups illustrated in figure 1 can be modeled by the idealized geometry of two plane interacting FFs of thickness $t_U$ and $t_L$ kept at a given distance $d$, as illustrated in figure 4.

2.1. Hamiltonian and domain equation of motion

When $d \to \infty$ the two films are well described by two independent Hamiltonians such as (12) where one simply replaces $m$, $M_s$, $K_u$ and $A$ with $m_U$, $M_U$, $K_U$ and $A_U$ for the upper film and with $m_L$, $M_L$, $K_L$ and $A_L$ for the lower film. The full Hamiltonian is thus $\mathcal{H} = \mathcal{H}_U + \mathcal{H}_L$ with:

$$\begin{align*}
\mathcal{H}_U &= \int_U \left[ -\frac{K_U}{2} \frac{m_U(r)^2}{2} + \frac{A_U}{2} \left( \nabla m_U(r) \right)^2 - \mu_0 M_U m_U(r) H_{ext} + \frac{\mu_0 M_U^2}{4\pi t_U} \int \left( \frac{m_U(r)m_U(r')}{|r-r'|} - \frac{m_U(r)m_U(r')}{\sqrt{(r-r')^2 + t_U^2}} \right) d^3r \right]
\end{align*}$$

Figure 3. Simulated domain morphologies. (a) Two different stable domain configurations obtained by decreasing $\sqrt{K_u A}$. (b) Domain walls of decreasing thickness obtained by decreasing $\sqrt{A/K_u}$. (c) The same stable domain configuration obtained by decreasing the anisotropy fluctuation strength $c_1$. 

\[(a)\text{ domain width } \propto \sqrt{K_u A} \quad (b)\text{ domain wall size } \propto \sqrt{A/K_u} \quad (c)\text{ boundary roughness } \propto c_1\]
\[ H_{\text{int}} = \int_L \left[ -K_L \frac{m_L(r)^2}{2} + \frac{A_L}{2} \left| \nabla m_L(r) \right|^2 - \mu_0 M_{\text{L}} m_L(r) H_{\text{ext}} \right. \\
+ \left. \frac{\mu_0 M_{\text{U}}^2}{4\pi r_L} \int \left( \frac{m_U(r)m_L(r')}{|r-r'|} - \frac{m_L(r)m_U(r')}{\sqrt{(r-r')^2 + t_U^2}} \right) \right] d^3r. \] (21)

When the two films are brought closer, each one feels the field generated by the other and their magnetic domains start to interact. The new Hamiltonian term responsible for this interaction can be obtained by calculating the energy of the upper film in the presence of the field generated by the lower one. To this end one must start from (11) and transform the integral on the upper film volume into a surface integral exactly as was done in (8):

\[
H_{\text{int}} = \mu_0 M_{\text{U}} \int_U \nabla \phi_{\text{L}}(r, t) \cdot m(r, t) \, d^3r \\
= \mu_0 M_{\text{U}} \int_U \frac{\partial \phi_{\text{L}}(r, t)}{\partial z} m_U(r, t) \, d^3r \\
= \mu_0 M_{\text{U}} \left( \int \phi_{\text{L}}(r, t) m_U(r, t) \, d^2r \right) \bigg|_{z=t_U+d}^{-z_0},
\] (22)

note again the absence of the factor 1/2, this is not a self-energy term. Substituting the potential due to the lower film \( \phi_{\text{L}}(r, t) \), given by (7), one obtains:

\[
H_{\text{int}} = \frac{\mu_0 M_{\text{U}} M_{\text{L}}}{4\pi} \int \left( \frac{m_U(r, t)m_L(r', t)}{\sqrt{(r-r')^2 + d^2}} \right. \\
+ \left. \frac{m_U(r, t)m_L(r', t)}{\sqrt{(r-r')^2 + (d + t_U)^2}} + \frac{m_U(r, t)m_L(r', t)}{\sqrt{(r-r')^2 + (d + t_L)^2}} \right) d^2r \, d^2r',
\] (23)

a term which is symmetric with respect to the upper and lower films and depends on their material and geometric parameters. Note that the same expression can be derived by calculating the interaction energy of the lower film in the field due to the upper one using:

\[
H_{\text{int}} = \mu_0 M_{\text{L}} \left( \int \phi_{\text{U}}(r, t) m_{\text{L}}(r, t) \, d^2r \right) \bigg|_{z=0}^{z=t_L},
\] (24)
and calculating $\phi_U(r, t)$ analogously to (7). The two field expressions obtained by differentiating $\mathcal{H}_{\text{int}}$ with respect to $m_L$ or $m_U$ become:

$$B_U = -\frac{\mu_0 M_L}{4\pi t_U} \int \left( -\frac{m_L(r', t)}{(r-r')^2 + d'^2} - \frac{m_U(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} \right. $$
$$+ \left. \frac{m_L(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} + \frac{m_U(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} \right) d^2r',$n

$$B_L = -\frac{\mu_0 M_U}{4\pi t_L} \int \left( -\frac{m_L(r', t)}{(r-r')^2 + t_L^2} - \frac{m_U(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} \right. $$
$$+ \left. \frac{m_L(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} + \frac{m_U(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} \right) d^2r',$n

with the field acting on one film depending only on the magnetization of the other. Also, to perform the functional derivative of $\mathcal{H} = \mathcal{H}_U + \mathcal{H}_L + \mathcal{H}_{\text{int}}$, here one has to restore a volume integral in (23). Before differentiating with respect to $m_U$ it is thus necessary to put $d^2r = d^2r'/t_U$ and before differentiating with respect to $m_L$ one has to set $d^2r' = d^2r'/t_L$. The two magnetizations $m_L$ and $m_U$ evolve in time according to (2), and one ends up with the two coupled equations:

$$\frac{\partial m_U}{\partial t} = \gamma \alpha_U \left\{ (1 - m_U^2) \left[ K_U \frac{M_U}{M_U} m_U + \mu_0 H_{\text{ext}} - \frac{\mu_0 M_U}{2\pi t_U} \int \left( -\frac{m_L(r', t)}{|r-r'|} \right. \right. $$
$$- \left. \frac{m_L(r', t)}{(r-r')^2 + t_L^2} \right) d^2r' - \mu_0 M_U \int \left( -\frac{m_L(r', t)}{|r-r'|} \right. \right. $$
$$- \left. \frac{m_L(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} \right) d^2r' + Q_U(r, t) \left] + \frac{A_U}{M_U} \nabla^2 m_U \right\},$$n

$$\frac{\partial m_L}{\partial t} = \gamma \alpha_L \left\{ (1 - m_L^2) \left[ K_L \frac{M_L}{M_L} m_L + \mu_0 H_{\text{ext}} - \frac{\mu_0 M_L}{2\pi t_L} \int \left( -\frac{m_U(r', t)}{|r-r'|} \right. \right. $$
$$- \left. \frac{m_U(r', t)}{(r-r')^2 + t_L^2} \right) d^2r' - \mu_0 M_L \int \left( -\frac{m_U(r', t)}{|r-r'|} \right. \right. $$
$$- \left. \frac{m_U(r', t)}{(r-r')^2 + (d + t_U + t_L)^2} \right) d^2r' + Q_L(r, t) \left] + \frac{A_L}{M_L} \nabla^2 m_L \right\},$$n

with the thermal noise properties:

$$\langle Q_U(r, t) \rangle = 0 \quad \langle Q_U(r, t) Q_U(r', t') \rangle = 2k_B T \frac{\alpha_U}{\gamma M_U} \delta(t - t') \delta(r - r'),$$n

$$\langle Q_L(r, t) \rangle = 0 \quad \langle Q_L(r, t) Q_L(r', t') \rangle = 2k_B T \frac{\alpha_L}{\gamma M_L} \delta(t - t') \delta(r - r').$$n

(29)
2.2. Small thickness approximation and useful limits

Notice that the interaction energy expression can be evaluated in some simple limiting cases.

Very useful to this aim is the limit in which two films of the same material ($M_U = M_L = M$) and thickness ($t_U = t_L = t_F$) are brought in close contact ($d = 0$), i.e. one obtains a single film of thickness $2t_F$. In fact, the films being identical, their magnetization must behave in the same way. With $m_U = m_L = m$ the stray field energy terms contained in (20) and (21) cancel out with part of the interaction term and one is left with:

$$\mathcal{H}_{\text{int}} + \mathcal{H}_{\text{stray}} = \frac{\mu_0 M_U^2}{4\pi} \int \left( \frac{m(r', t)m(r, t)}{|r - r'|} - \frac{m(r', t)m(r, t)}{\sqrt{(r - r')^2 + (2t_F)^2}} \right) d^2r \ d^2r',$$

which is exactly (9) for a film with with thickness $2t_F$ q.e.d.. In practice, when $d \to 0$, the magnetization of the two films is exactly the same even if, as discussed in section 1.4, an isolated film with larger thickness should display smaller domains with a larger stray field. This happens because the film–film interaction compensates for the stray field energy difference of the isolated films. An analytical demonstration is possible in the small thickness limit $t_L, t_U \to 0$ where it is possible to write:

$$\mathcal{H}_{\text{stray}} + \mathcal{H}_{\text{int}} = \frac{\mu_0 M_U^2 t_U^2}{8\pi} \int \frac{m_U(r', t)m_U(r, t)}{|r - r'|^3} d^2r \ d^2r' + \frac{\mu_0 M_U^2 t_L^2}{8\pi} \int \frac{m_L(r', t)m_L(r, t)}{|r - r'|^3} d^2r \ d^2r' + \frac{\mu_0 M_U M_L}{4\pi} t_U t_L \int \frac{m_U(r, t)m_L(r', t)[(r - r')^2 - 2d^2]}{[(r - r')^2 + d^2]^{5/2}} d^2r \ d^2r',$$

and from the functional derivatives one obtains the fields:

$$B_U = -\frac{\mu_0 M_U t_U}{4\pi} \int \frac{m_U(r', t)}{|r - r'|^3} d^2r' - \frac{\mu_0 M_L t_L}{4\pi} \int \frac{m_L(r', t)[(r - r')^2 - 2d^2]}{[(r - r')^2 + d^2]^{5/2}} d^2r',$$

$$B_L = -\frac{\mu_0 M_L t_L}{4\pi} \int \frac{m_L(r', t)}{|r - r'|^3} d^2r' - \frac{\mu_0 M_U t_U}{4\pi} \int \frac{m_U(r, t)[(r - r')^2 - 2d^2]}{[(r - r')^2 + d^2]^{5/2}} d^2r',$$

note now that for $d = 0$ the two expressions become identical, the total field felt by the two films is thus the same and, even if $t_U \neq t_L$, their domains must behave in the same way and display the same patterns, i.e. $m_U = m_L$. This finding is in agreement with the experimentally observed domain behavior in the limit of $d \to 0$ [4].

Another important consistency check of the derived equations can be performed in the saturation limit $m_U(x, y) = m_U(x, y) \equiv \pm 1 \forall x, y$: as previously discussed, when a FF is uniformly magnetized its outer field must vanish. In this limit the expression for the outer field generated by the lower film (25) becomes:

$$B_U = -\frac{\mu_0 M_L}{4\pi t_U} \int \left( \frac{1}{\sqrt{(r - r')^2 + d^2}} + \frac{1}{\sqrt{(r - r')^2 + (d + t_U)^2}} \right) d^2r' + \frac{1}{\sqrt{(r - r')^2 + (d + t_U + t_L)^2}} d^2r' - \frac{1}{\sqrt{(r - r')^2 + (d + t_U + t_L)^2}} d^2r',$$
Figure 5. A diagram of the reference frame adopted and the vectors involved in the calculation of (a) $F_U$ and (b) $F_L$. In these two cases the origin of the upper frame $O_U$ moves forward together with the upper film. (c) is the equivalent of (b) in the opposite picture in which $O_U$ is immobile and always aligned with $O_L$ and the magnetization is shifted forward with respect to $O_U$.

as in (16) the translational invariance allows the calculation of the field at the convenient point $r = 0$ in polar coordinates:

$$B_U = -\frac{\mu_0 M_L}{2t_U} \lim_{t \to \infty} \int_0^\ell \left( -\frac{r'}{\sqrt{r'^2 + d^2}} + \frac{r'}{\sqrt{r'^2 + (d + t_U)^2}} \right) d^2r'$$

$$= -\frac{\mu_0 M_L}{2t_U} \lim_{t \to \infty} \left( -\sqrt{t^2 + d^2} + \sqrt{t^2 + (d + t_U)^2} \right)$$

$$= 0 \quad \text{q.e.d.}$$

(35)

3. Mobile interacting films

3.1. Force calculation

The equations derived thus far describe the domain evolution into the two FFs, now it is necessary to study the dynamics of the two FF, i.e. the motion of the two coated bodies. To this end one needs to calculate the magnetic force that each film exerts on the other at every time $t$ given the magnetization distributions $m_U(r, t)$ and $m_L(r, t)$. At this point it is convenient to define a vector $s(t)$ that connects a reference point in the lower film $O_L$ to a reference point in the upper film $O_U$. Looking at figure 5(a) it is easy to see that $s(t) = [s_x(t), s_y(t), d(t)] = [s_x(t), d(t)]$ represents the relative displacement between the two films, note that $O_L$ is also the center of the reference frame. Due to the presence of the lower film, a field $B_U$ exists in the upper film volume, and it exerts a force on each infinitesimal dipole moment. The total force on the upper film $F_U$ is thus obtained by integrating over all this infinitesimal contribution. Being the force on a single dipole moment proportional to $(m \cdot \nabla)B$ (see the appendix), it is found that:

$$F_U(t) = M_U \int m_U(r, t) \left. \frac{\partial B_U[r + s_{||}, m_L(t)]}{\partial z} \right|_{z = h, s + dh} d^3r$$

$$= M_U \left( \int m_U(r, t) B_U[r + s_{||}, m_L(t)] d^3r \right)_{z = h, s + dh}$$

(36)

where the field due to the lower film is calculated in the displaced position of the upper film $r + s_{||}$ as depicted in figure 5(a), again the last step comes from a simple integration by parts.
Note also that to study the domain evolution only the $z$ component of $B_U$ is needed, namely (25), now one instead needs the full vector that can be calculated from the gradient of the potential (7):

$$B_U(r') = \frac{\mu_0 M_L}{4\pi} \int m_L(r', t) \left( \frac{r - r' + (z - t_L) \hat{z}}{[(r - r')^2 + (z - t_L)^2]^{3/2}} - \frac{r - r' + z \hat{z}}{[(r - r')^2 + z^2]^{3/2}} \right) d^2 r'. \quad (37)$$

Substituting into (36) the following force is obtained:

$$F_U = -\frac{\mu_0 M_U M_L}{4\pi} \int m_U(r, t) m_L(r', t) \left( \frac{r - r' + s_1 + d \hat{z}}{[(r - r' + s_1)^2 + d^2]^{3/2}} - \frac{r - r' + s_1 + (d + t_L) \hat{z}}{[(r - r' + s_1)^2 + (d + t_L)^2]^{3/2}} \right) d^2 r' \quad (38)$$

Conversely, to calculate the force acting on the lower film, it is necessary to start from:

$$F_L(t) = M_L \left( \int m_L(r, t) B_L[r - s_1, m_U(t)] \ d^2 r \right) \bigg|_{z = 0}^{z = \ell_L}, \quad (39)$$

now the field felt by the lower film is given by:

$$B_L(r) = \frac{\mu_0 M_U M_L}{4\pi} \int m_U(r', t) \left( \frac{r - r' + (z - d - t_U - t_L) \hat{z}}{[(r - r')^2 + (z - d - t_U - t_L)^2]^{3/2}} - \frac{r - r' + (z - d - t_U) \hat{z}}{[(r - r')^2 + (z - d - t_U)^2]^{3/2}} \right) d^2 r', \quad (40)$$

and it must be calculated considering the upper film displaced with respect to the axes origin, i.e. in the points $r' + s_1$, according to figure 5(b) Substituting into the force expression:

$$F_L = -\frac{\mu_0 M_U M_L}{4\pi} \int m_U(r', t) m_L(r, t) \left( \frac{r - r' - s_1 - d \hat{z}}{[(r - r' - s_1)^2 + d^2]^{3/2}} - \frac{r - r' - s_1 - (d + t_U) \hat{z}}{[(r - r' - s_1)^2 + (d + t_U)^2]^{3/2}} \right) d^2 r' \quad (41)$$

renaming $r$ by $r'$ it is immediate to see that $F_L = -F_U$ as required by Newton’s third law.

Expressions (38) and (41) can be simplified in the limit of small thickness $t_U, t_L \to 0$:

$$F_U = \frac{3\mu_0 M_U M_L}{4\pi} t_U t_L \int m_U(r, t) m_U(r', t) \left[ \left( r' - r + s_1 \right)^2 - 4d^2 \right] \left[ \left| r' - r + s_1 \right|^2 - \frac{2}{3} d \right] d^2 r' d^2 r. \quad (42)$$

In section 2.2 it has been demonstrated that the outer field of the FFs goes to zero when they saturate, for this reason the total force between them is expected to vanish as well. This can be easily proved putting $m_U = m_L \equiv \pm 1 \ \forall \ x, y$ in (38) or (41).

Before moving on it is necessary to stress that the LLGEs (27) and (28) have been derived for two perfectly aligned parallel FFs, when letting the upper film move one has to introduce the
displacement vector $s(t)$ in the calculation of the interaction term. In the lower film equation one has to use the field $B_L(r)$ exerted by the upper one (it is still possible to use (26) which is valid when $O_U$ is perfectly aligned with $O_L$) and account for the negative displacement of $O_L$ with respect to $O_U$ replacing $r$ with $r - s_l$:

$$\frac{\partial m_L}{\partial t} = \gamma \alpha_L \left\{ (1 - m_L^2) \left[ \frac{K_L}{M_L} m_L + \mu_0 H_{ext} - \frac{\mu_0 M_L}{2\pi t_L} \int \left| \frac{m_L(r', t)}{|r - r'|} \right|^2 \right] d^2 r' - \frac{m_L(r', t)}{\sqrt{(r - r')^2 + t_L^2}} \int \left( - \frac{m_U(r', t)}{\sqrt{(r - r' - s_l)^2 + d^2}} \right) d^2 r' + Q_L(r, t) \right\} + \frac{A_L}{M_L} \nabla^2 m_L. \tag{43}$$

The same can be done for the upper film, considering the field (25) calculated in the position $r + s_l$ to account for the forward shift of $O_U$ with respect to $O_L$:

$$\frac{\partial m_U}{\partial t} = \gamma \alpha_U \left\{ (1 - m_U^2) \left[ \frac{K_U}{M_U} m_U + \mu_0 H_{ext} - \frac{\mu_0 M_U}{2\pi t_U} \int \left| \frac{m_U(r', t)}{|r - r'|} \right|^2 \right] d^2 r' - \frac{m_U(r', t)}{\sqrt{(r - r')^2 + t_U^2}} \int \left( - \frac{m_L(r', t)}{\sqrt{(r - r' + s_l)^2 + d^2}} \right) d^2 r' + Q_U(r, t) \right\} + \frac{A_U}{M_U} \nabla^2 m_U. \tag{44}$$

As will be shown in section 4.2, the properties of the Fourier transforms allow the transformation of the displacement in the interaction term into a shift of the magnetization, one will thus recover the old expressions (27) and (28) but with a modified magnetization.

3.2. Equation of motion

As illustrated in figure 1, the possible practical set-ups to measure the magnetic interaction between the FFs consist of a rigid substrate and a mobile slider, a single equation of motion is thus needed to evolve the displacement vector $s(t)$ of the upper film, while the lower one is kept fixed. In most of the sliding systems of interest for micro-mechanics and tribology, the slider can be driven at a constant force $F_\parallel$ or at a constant velocity, the latter case is typically modeled by driving the slider through a spring which represents the elastic stiffness of the driving apparatus. For instance, an AFM tip is typically modeled by a spring $k_\parallel$ along the sliding direction, accounting for the torsional stiffness of the cantilever, and a spring $k_\perp$ perpendicular to the sliding plane, representing the vertical bending stiffness of the cantilever. When the slider and the substrate are kept in contact, a force $F_\perp$ can be added to load the slider and modify the contact properties. This force can also be added to effectively model the adhesion force between the two bodies. From these considerations, a very general form of the
equation for motion of the slider (upper film), is given by:

\[ m \frac{\partial^2 s(t)}{\partial t^2} = F_U[s(t)] + F_{\text{driving}} - \zeta m \frac{\partial s(t)}{\partial t}, \]  

(45)

where \( m \) is the slider mass and \( \zeta \) a damping coefficient. The driving force \( F_{\text{driving}} = (F_\parallel, 0, F_\perp) \) has a component along the sliding direction \( x \) and a component perpendicular to it along \( z \). Analogously to the term \(-\eta \partial M/\partial t\) in (1), the viscous damping disposes off the energy with a characteristic time \( 1/\zeta \) representing the dissipation through microscopic mechanical degrees of freedom of the slider and driving apparatus. This last equation, coupled with (27) and (28), completely describes the dynamics of the two sliding bodies and their magnetization.

4. Numerical implementation

4.1. Unit system

For the numerical implementation of the equations of motion it is worth choosing a dimensionless unit system. The single film LLGE (14) can be rewritten in dimensionless units factorizing \( \mu_0 M_s \) in the rhs and defining the film thickness \( t_F \) as the unit length, so that \( \tilde{r} = r/t_F \), and \( 1/\gamma \alpha \mu_0 M_s \) as a unit time, so that \( \tilde{t} = t \gamma \alpha \mu_0 M_s \). With this substitution one finally finds:

\[ \frac{\partial m}{\partial \tilde{t}} = (1 - m^2) \left[ a m + h_{\text{ext}} - \frac{1}{2\pi} \int \left( \frac{m(\tilde{r}')}{|\tilde{r} - \tilde{r}'|} - \frac{m(\tilde{r})}{\sqrt{(\tilde{r} - \tilde{r}')^2 + 1}} \right) \tilde{r}' q(\tilde{r}, \tilde{t}) \right] + b \nabla^2 m, \]  

(46)

where \( a = K_u/\mu_0 M_s^2 \) and \( b = A/\mu_0 M_s^2 t_F^2 \) are the dimensionless uniaxial anisotropy and exchange stiffness respectively, while \( h_{\text{ext}} = H_{\text{ext}}/M_s \) and \( q = Q/\mu_0 M_s \). In this dimensionless system the statistical properties of the thermal fluctuations become:

\[ \langle q(\tilde{r}, \tilde{t}) \rangle = 0 \quad \langle q(\tilde{r}, \tilde{t}) q(\tilde{r}', \tilde{t}') \rangle = 2K_B T a^2 \delta(\tilde{t} - \tilde{t}') \delta(\tilde{r} - \tilde{r}'). \]  

(47)

with \( K_B T = K_B T/\mu_0 M_s^2 t_F^3 \) the dimensionless temperature.

One has to proceed in the same way for the coupled LLGEs (43) and (44) ruling the domain dynamics in two interacting FFs, in this case however, one of the two films must be chosen as the reference, expressing all the fields in units of its saturation magnetization, and all the distances in units of its thickness. As a consequence of this choice the dimensionless equations become asymmetric. Choosing the lower film as a reference it is necessary to define:

\[ \tilde{t} = t \gamma \alpha \mu_0 M_L, \quad \tilde{r} = \frac{r}{t_L}, \quad h_{\text{ext}} = \frac{H_{\text{ext}}}{M_L}, \]

\[ a_L = \frac{K_L}{\mu_0 M_L^2}, \quad b_L = \frac{A_L}{\mu_0 M_L^2 t_L^2}, \quad q_L = \frac{Q_L}{\mu_0 M_L}, \]

\[ a_U = \frac{K_U}{\mu_0 M_U^2}, \quad b_U = \frac{A_U}{\mu_0 M_U^2 t_U^2}, \quad q_U = \frac{Q_U}{\mu_0 M_U}, \]

\[ \xi = \frac{M_U}{M_L}, \quad \tilde{t}_U = \frac{t_U}{t_L}, \quad \tilde{d} = \frac{d}{t_L}, \quad \nu = \frac{a_U}{a_L}. \]  

(48)
with these definitions one has:

\[
\frac{\partial m_U}{\partial t} = \nu \left\{ (1 - m_U^2) \left[ a_U m_U + \frac{h_{ext}}{\xi} - \frac{\xi}{2\pi \tau_U} \int \left( \frac{m_U(\vec{r}', \vec{t})}{|\vec{r} - \vec{r}'|} - \frac{m_L(\vec{r}', \vec{t})}{\sqrt{(\vec{r} - \vec{r}')^2 + \vec{r}_U^2}} \right) d^2r' \right] - \frac{m_U(\vec{r}', \vec{t})}{\sqrt{(\vec{r} - \vec{r}')^2 + \vec{r}_U^2}} + \frac{\xi}{4\pi \tau_U} \int \left( \frac{m_U(\vec{r}', \vec{t})}{|\vec{r}' - \vec{r} + \vec{r}_U^2 + \vec{r}_U^2|} + \frac{m_L(\vec{r}', \vec{t})}{\sqrt{(\vec{r}' - \vec{r} + \vec{r}_U^2 + \vec{r}_U^2)}^2 + \vec{r}_U^2} \right) \right\}.
\]

\[
\frac{\partial m_L}{\partial t} = (1 - m_L^2) \left[ a_L m_L + h_{ext} - \frac{1}{2\pi} \int \left( \frac{m_L(\vec{r}', \vec{t})}{|\vec{r} - \vec{r}'|} - \frac{m_U(\vec{r}', \vec{t})}{\sqrt{(\vec{r} - \vec{r}')^2 + \vec{r}_U^2}} + \frac{\xi}{4\pi \tau_U} \int \left( \frac{m_U(\vec{r}', \vec{t})}{|\vec{r}' - \vec{r} + \vec{r}_U^2 + \vec{r}_U^2|} + \frac{m_L(\vec{r}', \vec{t})}{\sqrt{(\vec{r}' - \vec{r} + \vec{r}_U^2 + \vec{r}_U^2)}^2 + \vec{r}_U^2} \right) \right) \right\}.
\]

In the dimensionless unit system the thermal noise becomes:

\[
\langle q_U(\vec{r}, \vec{t}) \rangle = 0 \quad \langle q_U(\vec{r}, \vec{t}) q_U(\vec{r}', \vec{t}') \rangle = 2K_B T \frac{a_U a_L}{\xi} \delta(\vec{r} - \vec{r}'),
\]

\[
\langle q_L(\vec{r}, \vec{t}) \rangle = 0 \quad \langle q_L(\vec{r}, \vec{t}) q_L(\vec{r}', \vec{t}') \rangle = 2K_B T \alpha_L^2 \delta(\vec{r} - \vec{r}'),
\]

with \( K_B T = K_B T/\mu_0 M_L^2 t_U^2 \).

To conclude, the Newton equation for the slider motion (45) must be put in the same unit system defined by (48):

\[
\tilde{m} \frac{\partial^2 \tilde{s}(t)}{\partial t^2} = -\frac{\xi}{4\pi} \int m_U(\vec{r}, \vec{t}) m_L(\vec{r}', \vec{t}) \left( \frac{\vec{r}' - \vec{r} + \vec{r}_U^2 + \vec{r}_U^2}{[(\vec{r}' - \vec{r} + \vec{r}_U^2 + \vec{r}_U^2)^2 + \vec{r}_U^2]} \right) d^2r' + f_{\text{driving}} - \tau \tilde{m} \frac{\partial \tilde{s}(t)}{\partial t},
\]

where \( \tilde{m} = m \gamma^2 \alpha_L^2 \mu_0 / t_U \) is the dimensionless mass, \( f_{\text{driving}} = F_{\text{driving}} / \mu_0 M_L^2 t_U^2 \) the dimensionless driving and \( \tau = \zeta / \gamma a_L \mu_0 M_L \) is the ratio between the characteristic times of the slider and domain dynamics.
4.2. Equations in reciprocal space

Both the stray field and the interactions terms make (49), (50) and (52) non-local and practically numerically unaffordable in real space. However, assuming that the two FFs are infinitely extended in the $xy$ plane, the equations of motion can be rewritten in reciprocal space, where the non-locality of the Hamiltonian disappears, significantly reducing the computational cost compared to ordinary micromagnetic calculations. Naturally, with the infinite extension of the system the tilde notation used in the previous section will be dropped and any variable or coefficient is intended to be dimensionless. Applying a two-dimensional Fourier transform to the stray field term of the lower film equation (50) one has:

\[
\mathcal{F} \left[ -\frac{1}{2\pi} \int \frac{m_L(r', t)}{|r-r'|} - \frac{m_L(r', t)}{\sqrt{(r-r')^2 + 1}} \right] \, d^2r' = 2\pi \, m_L(k, t) \mathcal{F} \left[ -\frac{1}{2\pi} \left( \frac{1}{|r|} - \frac{1}{\sqrt{r^2 + 1}} \right) \right] = -\frac{m_L(k, t)}{k} (1 - e^{-k}),
\]

(53)

the second step comes from the convolution theorem that, with my choice of Fourier parameter, reads $\mathcal{F}[f * g] = 2\pi \mathcal{F}[f] \mathcal{F}[g]$, $m_L(k)$ is thus the Fourier transform of the magnetization. In the last step the Fourier transform $\mathcal{F}[1/\sqrt{r^2 + a^2}] = e^{-ak}/k$ has been used, with $k = |k| = \sqrt{k_x^2 + k_y^2}$. Analogously, for the interaction term one has:

\[
\mathcal{F} \left[ -\frac{\xi}{4\pi} \int \left( -\frac{m_U(r', t)}{\sqrt{(r-r'-s_i)^2 + d^2}} - \frac{m_U(r', t)}{\sqrt{(r-r'-s_i)^2 + (d + iU + 1)^2}} \right) \, d^2r' \right] = 2\pi \, m_U(k, t) \mathcal{F} \left[ -\frac{\xi}{4\pi} \left( -\frac{1}{\sqrt{(r-s_i)^2 + d^2}} - \frac{1}{\sqrt{(r-s_i)^2 + (d + iU + 1)^2}} \right) \right]
\]

\[
+ \frac{1}{\sqrt{(r-s_i)^2 + (d + iU + 1)^2}} = \frac{\xi \, m_U(k, t)}{2} e^{-d \cdot k} e^{i\xi \cdot k} (1 - e^{-k})(1 - e^{-\xi \cdot k}),
\]

(54)

In the last step the Fourier transform property $\mathcal{F}[f(r + \alpha)] = e^{i\alpha \cdot k} \mathcal{F}[f(r)]$ has been used. Notice that the phase factor $e^{i\xi \cdot k}$ can in principle be moved to the magnetization $m_U$ and, returning to real space, this would lead to a situation in which the displacement between $O_U$ and $O_L$ is always zero, however, the magnetization $m_U$ is shifted with respect to $O_U$ by a quantity $s_i$, see figure 5(c). In this picture of a shifted magnetization, the last Fourier transform can be rewritten as:

\[
\frac{\xi}{2} \frac{S[m_U(k, t)]}{k} e^{-d \cdot k} (1 - e^{-k})(1 - e^{-\xi \cdot k}),
\]

(55)

18
where $\mathcal{S}[f(k)] = e^{i\xi k} f(k)$ stands for the shift operation. The transforms of the stray field and interaction terms can be now substituted into the full transform of (50):

$$\frac{\partial m_L}{\partial t} = \mathcal{F} \left[ (1 - m_L(r, t)^2) \left( a_L m_L(r, t) + h_{\text{ext}} + q_L(r, t) \right) \right] - \mathcal{F} \left[ (1 - m_L(r, t)^2) \mathcal{F}^{-1} \left[ \frac{(1 - e^{-\xi k})}{k} \left( m_L(k, t) - \frac{\xi}{2} \mathcal{S}[m_U(k, t)] e^{-d k} \right) \times (1 - e^{-\nu b}) \right] \right] - b_L k^2 m_L(k, t). \quad (56)$$

The equation for the upper magnetization is obtained with the same procedure:

$$\frac{\partial m_U}{\partial t} = \nu \mathcal{F} \left[ (1 - m_U(r, t)^2) \left( a_U m_U(r, t) + \frac{h_{\text{ext}}}{\xi} + q_U(r, t) \right) \right] - \nu \mathcal{F} \left[ (1 - m_U(r, t)^2) \mathcal{F}^{-1} \left[ \frac{(1 - e^{-\nu b})}{\xi m_U(k, t)} \right] \right] - \frac{1}{2} \mathcal{S}[m_L(k, t)] e^{-d k} (1 - e^{-\xi k}) \right] - \nu b_U k^2 m_U(k, t). \quad (57)$$

From a practical point of view the transform in the first term of the two equations can be performed numerically with some fast Fourier transform (FFT) algorithm at each time step. The second term is obtained by performing a FFT of the magnetization, constructing the new function in reciprocal space and applying an inverse FFT to return to real space. The last term comes from the transform of the Laplacian of the magnetization and is a simple algebraic term. The same inverse transform method used for the non-local part of the LLGEs can be employed in the calculation of the $f_U$ term in the Newton equation (52):

$$m \frac{\partial^2 s(t)}{\partial t^2} = -\frac{\xi}{2} \int m_U(r, t) \mathcal{F}^{-1} \left[ \mathcal{S}[m_L(k, t)] e^{-d k} (1 - e^{-\xi k}) \times (1 - e^{-\nu b}) \left( \frac{k_x^2 + k_y^2 + k_z^2 + k^2}{k} \right) \right] \, \text{d}^2 r + f_{\text{driving}} - \tau m \frac{\partial s(t)}{\partial t}. \quad (58)$$

with $i$ a complex unit. Here as well, after performing the FFT of the lower film magnetization, one can construct the new function in reciprocal space and apply an inverse FFT to return to real space to evaluate the total force. Applying many direct and inverse FFTs at every time step might sound computationally expensive, however, $\ell$ being the size of the system, the FFT algorithm scales as $\ell \log(\ell)$, while a summation in real space would scale as $\ell^2$, thus performing many repeated FFTs is still more convenient that working in real space.

It is important to stress that using FFTs one is implicitly applying periodic boundary conditions (PBCs) to the simulation cell, as illustrated in figure 6(a). The Fourier series of a function defined in the interval $\ell$ in fact requires that function to have at least periodicity $\ell$. Every magnetic dipole in the simulation cell feels a local field due to the rest of the infinite FF, thus interacting with all the cell replicas. This is another advantage with respect to three-dimensional micromagnetic calculations for which an analytical expression for the stray field in reciprocal space is not available and Ewald-like summations must be performed to properly account for the long range interaction [23]. Performing the shift operation on the magnetizations $m_U$ and $m_L$ in the presence of PBC, the portion of magnetization exiting from
Figure 6. (a) A diagram of the simulation cell with periodic boundary conditions. (b) Displacement of the upper film magnetization $m_U$ through the periodic boundary conditions.

one mesh side is restored on the opposite one. This is illustrated in figure 6(b) where the same magnetization pattern is flowing through the boundary of the simulation cell.

4.3. Time evolution algorithm

Equations (56), (57) and (58) are now a set of simple differential equations in time and can be solved using finite difference methods on a squared mesh of spacing $\Delta x = \Delta y = \Delta$ with a time step $\Delta t$. For the time integration of the LLGEs one can use the semi-implicit first order algorithm described in reference [9], the lower film equation becomes:

$$m_L(k_{ij}, t + \Delta t) = m_L(k_{ij}, t) + \Delta t \left\{ \mathcal{F} \left[ 1 - m_L(r, t)^2 \right] \right. $$

$$\times \left( a_L m_L(r, t) + h_{ext} + q_L(r, t) \right)_{ij} - \mathcal{F} \left[ 1 - m_L(r, t)^2 \right] $$

$$\times \mathcal{F}^{-1} \left\{ \left( 1 - e^{-k} \right) \left( m_L(k, t) - \frac{\xi}{2} S[m_U(k, t)] e^{-\theta(t)} \right) \right\}_{ij} $$

$$\left/ \left( 1 + \Delta t h_L k_{ij}^2 \right) \right. \}$$

(59)

To calculate the magnetization $m_L(k_{ij}, t + \Delta t)$ in the reciprocal space mesh point $(i, j)$ one simply needs the magnetizations $m_U(k_{ij}, t)$ and $m_L(k_{ij}, t)$ in the same point at the previous time instant and their real space counterparts $m_U(r, t)$ and $m_L(r, t)$. Notice that the upper film magnetization which appears in the interaction term has been shifted through the PBC, for this operation it is necessary to know $s_k(t)$. An analogous equation can be obtained for the upper film:

$$m_U(k_{ij}, t + \Delta t) = m_U(k_{ij}, t) + v \Delta t \left\{ \mathcal{F} \left[ 1 - m_U(r, t)^2 \right] \right. $$

$$\times \left( a_U m_U(r, t) + \frac{h_{ext}}{\xi} + q_U(r, t) \right)_{ij} - \mathcal{F} \left[ 1 - m_U(r, t)^2 \right] $$

$$\left/ \left( 1 + \Delta t h_U k_{ij}^2 \right) \right. \}$$
\[
\times F^{-1} \left[ \left( 1 - e^{-t_f k} \right) \text{i.m}_U(k, t) - \frac{1}{2} S[m_L(k, t)] e^{-d(t) k \left( 1 - e^{-k} \right)} \right] \right]_{ij} \right) \bigg/ \left( 1 + \nu/dt b_U k_i^2 \right).
\] (60)

This equation is written in the upper film reference frame, thus, before calculating the interaction term, the lower magnetization must be shifted through the PBC, this shift is opposite to that performed in (59). For the Newton equation of the upper film one can use the Verlet velocity algorithm:

\[
s(t + \Delta t) = s(t) + \Delta t \left( 1 - \frac{\Delta t^2}{2} \tau \right) \dot{s}(t) + \frac{\Delta t^2}{2m} \left( f_U(t) + f_{\text{driving}}(t) \right),
\] (61)

\[
\dot{s}(t + \Delta t) = \left[ \left( 1 - \frac{\Delta t^2}{2} \tau \right) \dot{s}(t) + \frac{\Delta t^2}{2m} \left( f_U(t) + f_{\text{driving}}(t) + f_U(t + \Delta t) + f_{\text{driving}}(t + \Delta t) \right) \right] \bigg/ \left( 1 + \frac{\Delta t^2}{2} \tau \right).
\] (62)

with \( f_U(t) = F_U(t)/\mu_0 M_U^2 L \). To calculate the new positions \( s(t + \Delta t) \) one needs to know the magnetic force \( f_U(t) \) and thus the magnetizations \( m_U(r, t) \) and \( m_L(k, t) \) at the previous time instant, whereas for the velocity calculation \( \dot{s}(t + \Delta t) \) the new forces \( f_U(t + \Delta t) \) are needed, this requires solving the LLGEs to get \( m_U(r, t + \Delta t) \) and \( m_L(k, t + \Delta t) \). Note that also in the calculation of \( f_U(t) \) one has to shift the upper film magnetization. The two integration algorithms can thus be combined in the following way:

(i) with \( m_U(r, t) \) and \( m_L(k, t) \) the force \( f_U(t) \) is readily calculated;
(ii) now with \( f_U(t) \), \( \dot{s}(t) \) and \( s(t) \) the new upper film displacement \( s(t + \Delta t) \) can be evolved with (61);
(iii) with \( s(t + \Delta t) \) the magnetizations can be shifted and introduced in (59) and (60) to compute \( m_U(r, t + \Delta t), m_L(r, t + \Delta t) \);
(iv) the evolved magnetizations allow the computation of the new force \( f_U(t + \Delta t) \);
(v) with both \( f_U(t) \) and \( f_U(t + \Delta t) \) the new velocity \( \dot{s}(t + \Delta t) \) can be calculated with (62);
(vi) positions, velocities and forces are updated and finally one can return to point (i).

In this way one has a single integration of the LLGEs and of the Newton equation per time step. \( \Delta t \) must be chosen in such a way as to sample with sufficient accuracy the slowest between the domain and film dynamics. The same applies for the mesh spacing which must be small enough to sample the steepest magnetization variation, i.e. smaller than the domain wall thickness.

5. Conclusions

I have set up a system of equations to describe the dynamics of two bodies coated with thin ferromagnetic films with perpendicular anisotropy below the Curie temperature. It is now possible to simultaneously simulate the dynamics of the two bodies influenced by the magnetic domain interaction and the domain dynamics in each film influenced by the relative motion of the two bodies. Using a phase-field approach one can simulate the domain dynamics over large length scales, up to hundreds of \( \mu m^2 \), at very low computational cost. The downsides of this are the absence of edge effects and the lack of generality of the model, which applies only to perpendicular anisotropy films.
This new tool enables the investigation of how the domain properties can influence the sliding motion of the two bodies, with potential applications in the control and actuation of micro- and nano-scale mechanical devices. On the other hand, one can also study how the body motion influences the domain properties. This can be of great interest for the design of new domain writing and domain manipulation techniques. Finally, note that the theory developed in this paper for ferromagnetic films applies to ferroelectric films as well, allowing the evolving in time of the dimensionless polarization $p(x, y)$. To this aim it is enough to substitute the magnetization $m$ with the polarization $p$, the saturation magnetization $M_s$ with the saturation polarization $P_s$ and the vacuum permeability $\mu_0$ with the inverse of the vacuum permittivity $1/\epsilon_0$, naturally $K_u$ and $A$ will take different values as they are now related to the elastic properties of the materials [24–26].

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Appendix

The force $F$ exerted by a magnetic field $B = (B_x, B_y, B_z)$ on a magnetic dipole moment $d$ depends on the nature of the dipole itself [27]. If the dipole moment is induced by a current, the force must be calculated as:

$$F_c = \nabla (d \cdot B),$$

(A.1)

whereas, in case of a permanent dipole:

$$F_p = (d \cdot \nabla) B.$$

(A.2)

The two definitions are related by:

$$F_c = F_p + d \times (\nabla \times B),$$

(A.3)

and they coincide if $B$ is irrotational, a condition certainly valid for the field (37). Having a dipole moment aligned along the $z$-axis, i.e. $d \equiv d\hat{z}$, the expressions for $F_c$ and $F_p$ simplify to:

$$F_c = d \nabla B_z,$$

(A.4)

$$F_p = d \frac{\partial B}{\partial z},$$

(A.5)

and the two expressions coincide because from $\nabla \times B(r) = 0$ follows $\partial B_j/\partial r_i = \partial B_i/\partial r_j$.

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