The Relation between Hölder Continuous Function of Order $\alpha \in (0,1)$ and Function of Bounded Variation

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Abstract. The Cantor ternary function is the most famous example of a continuous function of bounded variation for which it satisfies the Hölder continuous function of order $\alpha = \log_3 2$, but does not satisfy for order $\alpha = 1$. In this paper, based on previous work of Hölder continuous function of order $\alpha \in (0,1)$ and using $F^\alpha$ - calculus on fractal set $F$, we show the relation between the Hölder continuous function of order $\alpha \in (0,1)$ and function of bounded variation. In particular, we give the necessary and sufficient condition for the variation of function satisfies Hölder condition or bi-Hölder condition.

1 Introduction

Hölder continuous function on $F \cap [a,b]$ of order $\alpha \in (0,1)$ defined as follows. A function $f: F \subset [a,b] \to R$ satisfies the Hölder condition, or is Hölder continuous (Lipschitz continuous) of order $\alpha \in (0,1]$ if there exists a number $c > 0$ such that

$$|f(y) - f(x)| \leq c|y - x|^\alpha$$

for all $x, y \in F \cap [a,b]$ [13]. An interesting example of a function that is continuous and bounded variation for which it satisfies the Hölder continuous function of order $\alpha = \log_3 2$, but does not satisfy for order $\alpha = 1$ is the Cantor ternary function, also called the Lebesgue-Cantor staircase function. The integral staircase function $S^\alpha f(x)$ for any $x \in F$ of order (exponent) $\alpha \in (0,1)$ is a generalization of the Lebesgue-Cantor staircase function $S^\alpha E(x)$ for any $x \in \mathcal{C}$, where $\mathcal{C}$ is the triadic Cantor set (Cantor ternary set) which is created by repeatedly deleting the open middle thirds of a set of line segments [2].

Recently, $F^\alpha$ -calculus was formulated in a seminal paper as a framework by Parvate and Gangal, it is a generalization of ordinary calculus which applies in cases where standard calculus is not applicable (see [11], [12] and [7]). The integral staircase function plays a key role in fractal $F^\alpha$ -calculus. Many results regarding the $F^\alpha$ -calculus and its application on Cantor set for example on sub-and super-diffusion, diffraction from fractal grating, new heat and Maxwell’s equations, and diffusion on middle-$\xi$ (see, e.g., [1], [3], [4], [5], [6]).

In [14], Wibowo et al. discussed the integral staircase function that is defined on the $\gamma$ -dimensional compact and $F$ -perfect sets $F$ which satisfies the bi-Lipschitz condition of order $\alpha \in (0,1)$. Furthermore, it is shown with using $F^\alpha$ -calculus that the images of the integral staircase function does not preserve $\gamma$ -dimensions of $F$. Wibowo et al. also shown that using $F^\alpha$ -calculus the image of the function $f$ satisfy bi-Lipschitz condition do not preserve $\gamma$ -dimensions of $F$ [15].

Based on previous work of Hölder continuous function of order $\alpha \in (0,1)$ and using $F^\alpha$ calculus, in this work, we investigate the relation between Hölder continuous function of order $\alpha \in (0,1)$ and...
function of bounded variation. In particular, we are interested in determining the necessary and sufficient condition for the variation of function satisfies a Hölder condition or bi-Hölder condition.

2 Preliminaries

2.1 The $F^\alpha$ -Calculus

In ([11] and [12]), the author gave the definitions of coarse-grained mass, $\gamma$ –dimension, the integral staircase function, $F$ –limits, $F$ –continuity, $F^\alpha$ –derivative, and their properties as follows.

**Definition 2.1** A subdivision $P_{[a,b]}$ (or just $P$) of the interval $[a, b]$ is a finite set of points $\{a = x_0, x_1, ..., x_m = b\}$, $x_j < x_{j+1}$. Any interval of the form $[x_j, x_{j+1}]$ is called a component interval or just a component of the subdivision $P$. If $Q$ is any subdivision of $[a, b]$ and $P \subseteq Q$, then we say that $Q$ is a refinement of $P$. If $a = b$, then the set $\{a\}$ is the only subdivision of $[a, b]$.

**Definition 2.2** For a set $F$ and a subdivision $P_{[a,b]}$, $a < b$,

$$\sigma^\alpha[F, P] = \sum_{j=0}^{m-1} \frac{(x_{j+1} - x_j)^\alpha}{\Gamma(\alpha + 1)} \theta(F, [x_j, x_{j+1}]).$$

Given $\delta > 0$ and $a \leq b$, the coarse-grained mass $\gamma^\alpha F(a, b)$ of $F \cap [a, b]$ is given by

$$\gamma^\alpha F(a, b) = \inf_{P_{[a,b]}} \sigma^\alpha[F, P],$$

where $|P| = \max_{0 \leq j \leq m} (x_{j+1} - x_j)$, for a subdivision $P$, and the infimum in (1) is taken over all subdivisions $P$ of $[a, b]$ satisfying $|P| \leq \delta$.

The mass function $\gamma^\alpha F(a, b)$ of $F \cap [a, b]$ is given by

$$\gamma^\alpha F(a, b) = \lim_{\delta \to 0} \gamma^\alpha F(a, b),$$

where $\Gamma(\cdot)$ is gamma function and $\theta(F, [x_j, x_{j+1}]) = 1$ if $F \cap [x_j, x_{j+1}] \neq \emptyset$, $\theta(F, [x_j, x_{j+1}]) = 0$ otherwise.

The mass function can be used to define a fractal dimension, we call this number the $\gamma$–dimension of $F$.

**Definition 2.3** The $\gamma$-dimension of $F \cap [a, b]$, denoted by $\dim_\gamma (F \cap [a, b])$, is

$$\dim_\gamma (F \cap [a, b]) = \inf \{ \beta : \gamma^\beta (F, a, b) = 0 \}$$

so that

$$\gamma^\beta (F, a, b) = \left\{ \begin{array}{ll} \infty, & 0 < \beta < \dim_\gamma (F \cap [a, b]) \\ 0, & \dim_\gamma (F \cap [a, b]) < \beta \leq 1 \end{array} \right.$$

If $\alpha = \dim_\gamma (F \cap [a, b])$, then $\gamma^\beta (F, a, b)$ may be zero or infinite, or may satisfy $0 < \gamma^\beta (F, a, b) < \infty$.

The integral staircase function $S^\alpha_F (x)$ of order $\alpha \in (0, 1)$ for a fractal set $F$ is defined as follows.

**Definition 2.4** Let $a_0$ be an arbitrary but fixed real number. The integral staircase function $S^\alpha_F (x)$ of order $\alpha \in (0, 1)$ for a set $F$ is given by

$$S^\alpha_F (x) = \begin{cases} \gamma^\alpha (F, a_0, x), & x \geq a_0 \\ -\gamma^\alpha (F, a_0, a_0), & x < a_0 \end{cases}.$$

The number $a_0$ can be chosen according to convenience.

**Definition 2.5** A function $f$ is said to be bounded on $F$ if there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in F$.

We also define the concepts of $F$ –limits and $F$ –continuity, which will be used in the next section.

**Definition 2.6** Let $F \subset R$, $f : R \to R$ and $x \in F$. A number $l$ is said to be the limit of $f$ through the points of $F$, or simply $F$ –limit, as $y \to x$, if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$y \in F \text{ and } |y - x| < \delta \Rightarrow |f(y) - l| < \varepsilon.$$  

If such a number exists, then it is denoted by

$$l = \lim_{y \to x} f(y).$$

This definition does not involve values of the function at $y$ if $y \notin F$. Also, $F$ –limit is not defined at points $x \notin F$. All points of change of $x$ is named these to $f$ change of $f(x)$ for all $x \in F$ and is denoted by $Sch (f)$. If $Sch (S^\alpha_F)$ is a closed set and every point in it is a limit point, then $Sch (S^\alpha_F)$ is called $\alpha$ – perfect.
Definition 2.7 A function $f: R \to R$ is said to be $F$-continuous at $x \in F$ if
\[ f(x) = \lim_{y \to x} f(y). \]
We note that the notion of $F$-continuity is not defined at $x \notin F$.

2.2 The Hölder Continuous Function
In this section, we provide some properties of Hölder or bi-Hölder continuous function of order $\alpha \in (0,1)$. Properties that are listed but not proven have proofs in Wibowo’s \textit{et al.} \cite{14} and \cite{15}.

Definition 2.8 If $F \subset R$ and $f: F \to R$ satisfies a bi-Hölder condition, or is bi-Hölder continuous on $F$ of order $\alpha \in (0,1)$, then there exists positive real numbers $c_1, c_2, 0 < c_1 \leq c_2 < \infty$ such that
\[ c_1|y - x|^\alpha \leq |f(y) - f(x)| \leq c_2|y - x|^\alpha \]
for all $x, y \in F$.

Theorem 2.9 If $F$ be a compact and $\alpha$-perfect sets and let $S_F^{\alpha}: F \to R, \alpha \in (0,1)$ be an integral staircase function, then there exists real numbers $c_1, c_2, 0 < c_1 \leq c_2 < \infty$ such that
\[ c_1|y - x|^\alpha \leq |S_F^{\alpha}(y) - S_F^{\alpha}(x)| \leq c_2|y - x|^\alpha \]
for all $x, y \in F$.

2.3 The Bounded Variation of Function
In this section, we provide definitions and some properties of bounded variation. Properties that are listed but not proven have proofs in Gordon’s text \cite{8}.

Definition 2.11 The total variation of function $f: [a, b] \to R$ over $[a, b]$ defined by
\[ \text{Var}_f[a, b] = \sup \{ \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| \mid a = x_0, x_1, ..., x_n = b \text{ is partition of } [a, b] \}. \]
If $\text{Var}_f[a, b] < \infty$ then we say that $f$ has bounded variation on $[a, b]$.

Theorem 2.12 Let $f$ and $g$ be functions of bounded variation on $[a, b]$ and let $k$ be a constant. Then
i. $f$ is bounded on $[a, b]$.
ii. $f$ is of bounded variation on every closed subinterval of $[a, b]$.
iii. $kf$ is of bounded variation on $[a, b]$.
iv. $f + g$ and $f - g$ are of bounded variation on $[a, b]$.
v. $fg$ is of bounded variation on $[a, b]$.
vi. if $1/g$ is bounded on $[a, b]$, then $f/g$ is of bounded variation on $[a, b]$.

3 Main Results
In this section, we will present an investigation of the relationship between the Hölder continuous of order $\alpha \in (0,1)$ and function of bounded variation.

Theorem 3.1 Let $F \subset R$ be a compact set, $\alpha$-perfect set, and $\text{Sch}(f) \subset F$ with $\text{dim}_p(F) = \alpha, \alpha \in (0,1)$. If $f$ is Hölder continuous on $F \cap [a, b]$ of order $\alpha \in (0,1)$ then $f$ is bounded variation and there is constant $K > 0$ satisfies $\text{Var}_f[a, b] \leq K|b - a|\alpha$.

Proof. Because $f$ is Hölder continuous on $F \cap [a, b]$ of order $\alpha \in (0,1)$, i.e. there is $c > 0$ such that
\[ |f(y) - f(x)| \leq c|y - x|^\alpha \]
for all $x, y \in F \cap [a, b]$. Take $P = \{a = x_0, x_1, ..., x_n = b\}$ is any subdivision of $[a, b]$, then $V$ variation of $f$ over $[a, b]$,
\[ V = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \leq \sum_{i=0}^{n-1} c|x_{i+1} - x_i|^\alpha \leq c(S_F^{\alpha}(b) - S_F^{\alpha}(a)). \]
By using Theorem 2.9 and substitute in (1), we get
\[ V \leq ck|b - a|\alpha = K|b - a|\alpha \]
if put $ck = K$. Since $P$ was arbitrary the inequality above is valid for any subdivision in $[a, b]$, which means that the sums $V$ are bounded above by $K|b - a|\alpha$ whence it follows that $f$ is bounded variation and satisfies
\[ \text{Var}_f[a, b] \leq K|b - a|\alpha. \]

The proof is complete.

Example 3.2 Given the function $c^{\log_3 2}(x), x \in C$ is integral staircase function or sometimes called Lebesgue-Cantor staircase function defined on compact set, $(\log_3 2)\text{-perfect set, and}$
$\mathcal{S}_c^{\log_3 2} \subset \mathcal{C}$ with $\dim_\gamma(\mathcal{C}) = \log_3 2$ with $\mathcal{C}$ is Cantor middle third set. This function is not Hölder continuous function of order $\alpha = 1$ on $\mathcal{C}$, but Hölder continuous function of order $\alpha = \log_3 2$ on $\mathcal{C}$. Since

$$|S_c^{\log_3 2}(y) - S_c^{\log_3 2}(x)| \leq 2c|y - x|^{\log_3 2} \quad \text{for all } 0 \leq x < y \leq 1,$$

for some $c > 0$. We see that $S_c^{\log_3 2}$ Hölder continuous function of order $\alpha = \log_3 2$ on $\mathcal{C}$. Based on the Theorem 3.1, this function is bounded variation on $[0, 1]$.

Next, we give an example of a function $f$ satisfying a Hölder continuous function of order $\alpha \in (0, 1)$ on $[a, b]$ but not of bounded variation on $[a, b]$. Define a function $f$ as follows. We partition $[0, 1]$ into infinitely many subintervals. Consider

$$x_0 = 0, x_1 - x_0 = \frac{1}{M^1}, x_2 - x_1 = \frac{1}{M^2}, ..., x_n - x_{n-1} = \frac{1}{M^n}, ...$$

We define the function by $f(x) = \left| x - \frac{x_i + x_{i+1}}{2} \right|^\alpha$ for every subinterval $[x_i, x_{i+1}], i = 0, 1, ...$. In order to show that $f$ satisfies Hölder continuous function of order $\alpha \in (0, 1)$, we consider three cases.

$i$. If $x, y \in [x_i, x_{i+1}]$ and $x, y \in \left[ \frac{x_i + x_{i+1}}{2}, x_{i+1} \right]$, then

$$|f(x) - f(y)| = \left| x - \frac{x_i + x_{i+1}}{2} \right|^\alpha \leq |x - y|^\alpha.$$

$ii$. If $x, y \in [x_i, x_{i+1}]$ and $x \in \left[ \frac{x_i + x_{i+1}}{2}, x_{i+1} \right]$, then there is a $z \in \left[ \frac{x_i + x_{i+1}}{2}, x_i \right]$ such that $f(z) = f(x)$. So

$$|f(x) - f(y)| = |f(x) - f(z)| \leq |x - z|^\alpha \leq |x - y|^\alpha.$$

$iii$. If $x \in [x_i, x_{i+1}]$ and $y \in \left[ \frac{x_i + x_{i+1}}{2}, x_{i+1} \right]$ where if $i > j$.

If $x \in \left[ \frac{x_i + x_{i+1}}{2}, x_i \right]$, then there is a $z \in \left[ \frac{x_i + x_{i+1}}{2}, x_i \right]$ such that $f(y) = f(x)$. So

$$|f(x) - f(y)| = |f(x) - f(z)| \leq |x - z|^\alpha \leq |x - y|^\alpha.$$

Similarily for $x \in \left[ \frac{x_i + x_{i+1}}{2}, x_{i+1} \right]$.

To show that the $f$ is not bounded variation on $[0, 1]$, let $\beta > 1$ such that $\alpha \beta < 1$ with $\alpha \in (0, 1)$ and let $M = \sum_{k=1}^{\infty} \frac{1}{k^\beta}$ since the series converges. So, we have $1 = \frac{1}{M} \sum_{k=1}^{\infty} \frac{1}{k^\beta}$. Since the series

$$\sum_{k=1}^{\infty} \left( \frac{1}{M^1} \right)^\alpha = \frac{1}{(2M)^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^\beta}$$

is diverges, we conclude $f$ is not bounded variation on $[0, 1]$. Theorem 3.1 cannot be applied to function $f$, because the sufficiency conditions are not fullfilled, i.e. $\dim_\gamma([0,1]) = 1$ which is different from $\alpha$ where $\alpha \in (0, 1)$ is order of Hölder continuous function.

**Lemma 3.3** Let $F \subset R$ be a compact set, $\alpha \rightarrow$ perfect set and $\mathcal{S}(F) \subset F$ with $\dim_\gamma(F) = \alpha, \alpha \in (0, 1)$. If $f$ and $g$ are Hölder continuous functions on $F \cap [a, b]$ of order $\alpha$ then $f \pm g$ are Hölder continuous functions of order $\alpha$.

**Proof.** Suppose $f$ and $g$ are Hölder continuous functions on $F \cap [a, b]$ of order $\alpha \in (0, 1)$, i.e. there exists $c_f, c_g > 0$ such that

$$|f(y) - f(x)| \leq c_f|y - x|^\alpha$$

and

$$|g(y) - g(x)| \leq c_g|y - x|^\alpha$$

for all $x, y \in F \cap [a, b]$. We get

$$|(f \pm g)(y) - (f \pm g)(x)| = |(f(y) - f(x)) \mp (g(y) - g(x))| \leq |f(y) - f(x)| + |g(y) - g(x)| \leq c_f|y - x|^\alpha + c_g|y - x|^\alpha = (c_f + c_g)|y - x|^\alpha$$

if put $(c_f + c_g) = c$, then

$$|(f \pm g)(y) - (f \pm g)(x)| \leq c|y - x|^\alpha$$

for all $x, y \in F \cap [a, b]$. Which shows that $f \pm g$ are Hölder continuous functions on $F \cap [a, b]$ of order $\alpha \in (0, 1)$.

**Theorem 3.4** Let $F \subset R$ be a compact set, $\alpha \rightarrow$ perfect set and $\mathcal{S}(f) \subset F$ with $\dim_\gamma(F) = \alpha, \alpha \in (0, 1)$. If $f$ is Hölder continuous on $F \cap [a, b]$ of order $\alpha$ then the function $\nu(x) = \text{Var}_f[a, x], x \in F$. Then

$i$. $\nu$ is nondecreasing function and satisfies
\[ |f(y) - f(x)| \leq v(y) - v(x) \text{ for all } x, y \in F \text{ and } a \leq x < y \leq b \]

ii. \( v + f, \ v - f \) are Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \in (0,1) \) and nondecreasing functions.

Proof. i. Suppose \( \{a = x_0, x_1, ..., x_n = x\} \) is any subdivision of \([a, x]\) and \( \{a = x_0, x_1, ..., x_n = x, y\} \) is any subdivision of \([a, y]\) with \( a \leq x < y \leq b \), so
\[
\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| + |f(y) - f(x)| \leq v(y).
\]

Since this valid for any subdivision of \([a, x]\),
\[
v(x) + |f(y) - f(x)| \leq v(y) \quad |f(y) - f(x)| \leq v(y) - v(x).
\]

This shows in particular that \( v(x) \leq v(y) \), and thus that \( v \) is nondecreasing.

ii. For \( a \leq x < y \leq b \),
\[
f(y) - f(x) \leq |f(y) - f(x)| \leq v(y) - v(x).
\]

Thus
\[
v(x) - f(x) \leq v(y) - f(y).
\]

By using Theorem 3.3, showing that \( v(x) - f(x) \) are Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \in (0,1) \) and nondecreasing. Likewise,
\[
f(x) - f(y) \leq |f(y) - f(x)| \leq v(y) - v(x)
\]

thus
\[
f(x) + v(x) \leq f(y) + v(y),
\]

it is showing that \( v(x) + f(x) \) are Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \in (0,1) \) and nondecreasing.

The following theorem shows the necessary and sufficient condition for any variation function defined on a fractal set satisfies the Hölder continuous and bi-Hölder continuous functions with the order \( \alpha \in (0,1) \).

**Theorem 3.5** Let \( F \subset R \) be a compact set, \( \alpha \) -perfect set and \( Sch \ (f) \subset F \) with \( \dim_{F} (F) = \alpha, \alpha \in (0,1) \). \( f \) is Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \) if and only if the function \( v(x) = Var_f[x, a] \), \( x \in F \cap [a, b] \) is Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \).

Proof. (sufficient conditions) Let \( f \) be a Hölder continuous of order \( \alpha \in (0,1) \), i.e. there is \( c > 0 \) such that
\[
|f(y) - f(x)| \leq c|y - x|^{\alpha}
\]

for all \( x, y \in F \cap [a, b] \). Suppose \( \{a = x_0, x_1, ..., x_n = x\} \) is any subdivision of \([a, x]\) and \( \{a = x_0, x_1, ..., x_n = x, y\} \) is any subdivision of \([a, y]\) with \( a \leq x < y \leq b \), so
\[
\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| + |f(y) - f(x)| \leq v(y).
\]

Since this valid for any subdivision of \([a, x]\),
\[
v(x) + |f(y) - f(x)| \leq v(y) \quad |f(y) - f(x)| \leq v(y) - v(x).
\]

By using Theorem 3.1 and apply to \( |Var_f[x, y]| \), we get
\[
|f(y) - f(x)| \leq |v(y) - v(x)| \leq k|y - x|^{\alpha}
\]

for some real constant \( K > 0 \) and for all \( x, y \in F \cap [a, b] \), which shows that \( v \) is Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \in (0,1) \).

(necessary condition) Take \( P = \{x = x_0, x_1, ..., x_n = y\} \) is any subdivision of \([x, y]\), then variation of \( f \) over \([x, y]\), we have a relation
\[
|f(y) - f(x)| \leq \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \leq |v(y) - v(x)|
\]

Hence \( v \) is a Hölder continuous on \( F \cap [a, b] \) of order \( \alpha \in (0,1) \), i.e. there exists \( k > 0 \) such that
\[
|v(y) - v(x)| \leq k|y - x|^{\alpha}
\]

for all \( x, y \in F \cap [a, b] \). Substitute (4) in (3), we get
\[
|f(y) - f(x)| \leq k|y - x|^{\alpha}
\]

for all \( x, y \in F \cap [a, b] \).
We conclude that $f$ is Hölder continuous function on $F \cap [a, b]$ of order $\alpha \in (0,1)$.

**Theorem 3.6** Let $F \subset \mathbb{R}$ be a compact set, $\alpha$ - perfect set and $\text{Sch}(f) \subset F$ with $\dim_{\text{f}}(F) = \alpha, \alpha \in (0,1)$. If $f$ are bi-Hölder continuous and nondecreasing functions on $F \cap [a, b]$ of order $\alpha$ if and only if the function $v(x) = \text{Var}_f[a,x], x \in F \cap [a, b]$ are bi-Hölder continuous and nondecreasing functions on $F \cap [a, b]$ of order $\alpha$.

**Proof.** (sufficient conditions) Let $f$ be a bi-Hölder continuous on $F \cap [a, b]$ of order $\alpha \in (0,1)$, i.e. there exists positive real numbers $c_1, c_2, 0 < c_1 \leq c_2 < \infty$ such that

$$c_1|y - x|^\alpha \leq |f(y) - f(x)| \leq c_2|y - x|^\alpha$$

for all $x, y \in F \cap [a, b]$. Substitute inequality (3) in (5), we have

$$c_1|y - x|^\alpha \leq |f(y) - f(x)| \leq \nu(y) - \nu(x) = |v(y) - v(x)|$$

$$c_1|y - x|^\alpha \leq |v(y) - v(x)|.$$  

Next, by using Theorem 3.5 (inequality (2)) and substitute in (6), we get

$$c_1|y - x|^\alpha \leq |v(y) - v(x)| \leq K|y - x|^\alpha$$

for some constant $c_1, K$, $0 < c_1 < K < \infty$ and for all $x, y \in F \cap [a, b]$. We see that $v$ is bi-Hölder continuous function on $F \cap [a, b]$ of order $\alpha \in (0,1)$.

(necessary condition) Take $P = \{x = x_0, x_1, \ldots, x_n = y\}$ is any subdivision of $[x, y]$. Since $f$ is nondecreasing we have $|f(x_{i+1}) - f(x_i)| = |f(x_{i+1}) - f(x_i)|$ and hence variation $V$ of $f$ over $[x, y]$, we have relation

$$V = \sum_{i=0}^{n-1}|f(x_{i+1}) - f(x_i)| = \sum_{i=0}^{n-1}|f(x_{i+1}) - f(x_i)| = f(y) - f(x).$$

Since the sum $V$ is independent of the partition $P$, we conclude that

$$\nu(y) - \nu(x) = f(y) - f(x).$$

Because $v$ is a bi-Hölder continuous on $F \cap [a, b]$ of order $\alpha \in (0,1)$, i.e. there is $k_1, k_2, 0 < k_1 \leq k_2 < \infty$ such that

$$k_1|y - x|^\alpha \leq |v(y) - v(x)| \leq k_2|y - x|^\alpha$$

for all $x, y \in F \cap [a, b]$. Substitute (7) in (8), we get

$$k_1|y - x|^\alpha \leq |f(y) - f(x)| \leq k_2|y - x|^\alpha$$

for all $x, y \in F \cap [a, b]$. We conclude that $f$ is bi-Hölder continuous function on $F \cap [a, b]$ of order $\alpha \in (0,1)$.

4 Conclusions

In this paper, we have given necessary and sufficient conditions for the variation of function to satisfy the Hölder condition that their function must satisfy the Hölder condition too.

Open Problem. Based on previous work by Indrati and Aryati (see [9] and [10]), on the countably Lipschitz condition or countably Hölder condition of order $\alpha = 1$, the research can be continued to the countably Hölder condition of order $\alpha \in (0,1)$ which is a generalization of the Hölder condition and investigate the relation between this function with function of bounded variation.

Acknowledgment

The authors would like to thank the Institute for Research and Community Services of Sebelas Maret University for funding this research in the academic year of 2019.

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