Conformal orthosymplectic quantum mechanics

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Abstract
We present the most general curvature obstruction to the deformed parabolic orthosymplectic symmetry subalgebra of the supersymmetric quantum mechanical models recently developed to describe Lichnerowicz wave operators acting on arbitrary tensors and spinors. For geometries possessing a hypersurface-orthogonal homothetic conformal Killing vector we show that the parabolic subalgebra is enhanced to a (curvature-obstructed) orthosymplectic algebra. The new symmetries correspond to time-dependent conformal symmetries of the underlying particle model. We also comment on generalizations germane to three dimensions and new Chern–Simons-like particle models.

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The models we consider generalize many well-known theories and have a wide range of applications. For all curved backgrounds, they enjoy an \( \mathfrak{osp}(Q|2p) \) internal symmetry. The series of models \( \mathfrak{osp}(Q|0) \) correspond to \( \mathfrak{o}(Q) \) spinning particle models \([4, 5]\). The lowest cases \( Q = 1, 2 \) are the \( N = 1, 2 \) supersymmetric quantum mechanical models originally employed by Alvarez-Gaumé and Witten in their study of gravitational anomalies and Pontryagin classes \([6]\), and by Witten in an application to Morse theory \([7]\). (Indeed, the Hodge–Lefschetz symmetry algebra of \( N = 4 \) supersymmetric quantum mechanics in Kähler backgrounds \([8, 9]\) plays an analogous role to the \( \mathfrak{osp}(Q|2p) \) algebras studied here.)

The dynamical symmetries (as opposed to internal, \( R \)-symmetries) of our models are sensitive to details of the background geometry. Geometrically, they correspond to linear differential operators such as the gradient, divergence, exterior derivative, codifferential, Dirac operator and generalizations thereof. These operators commute with the Lichnerowicz wave operator when the background is locally symmetric. (In fact, in the case of the \( \mathfrak{osp}(1|0) \) and \( \mathfrak{osp}(2|0) \) models, the Lichnerowicz wave operator is central in any curved background.) In the most general curved backgrounds, however, there is a curvature obstruction to dynamical symmetries. An important new result of this paper is an explicit computation of this obstruction appearing as the result of commutators between the Lichnerowicz wave operator and the linear, dynamical symmetry operators.

Armed with the generalization of the results of \([1]\) to arbitrary backgrounds, we can investigate non-symmetric spaces. Because our philosophy is to develop models which maximize symmetries, we consider spaces which share many features of flat space. Already in \([1]\) it was observed that together, the dynamical and internal symmetries of the symmetric space models formed a parabolic Lie subalgebra of a larger \( \mathfrak{osp}(Q|2p + 2) \) superalgebra if one introduced a new operator which measured the engineering dimension of the existing symmetry charges. This strongly suggests the study of quantum mechanical models with conformal symmetry. The first such model was developed some time ago in \([10]\). In particular, it was shown that the dilation operator corresponded to a time-dependent symmetry of the underlying particle action. There is an extensive literature concerning supersymmetric generalizations of conformal quantum mechanics; in particular, for the case that the dynamical symmetries are supersymmetries, the authoritative study of \([11]\) is invaluable (in fact the current paper could be viewed as the synthesis of \([1]\) and \([11]\)).

In flat backgrounds the dilation operator corresponds to the Euler operator, which generates radial dilations. This property can be mimicked in more general backgrounds by requiring the existence of a hypersurface-orthogonal homothetic conformal Killing vector. Flat space is the only symmetric space satisfying this requirement. Therefore, in flat backgrounds, the \( \mathfrak{osp}(Q|2p) \) models in fact enjoy a larger, time-dependent \( \mathfrak{osp}(Q|2p + 2) \) symmetry algebra that contains an \( \mathfrak{sl}(2, \mathbb{R}) \) subalgebra, which is precisely the conformal algebra of the one-dimensional particle worldline. However, in general backgrounds obeying the hypersurface-orthogonal homothetic conformal Killing vector condition, the \( \mathfrak{osp}(Q|2p + 2) \) algebra remains largely unscathed save for exactly the curvature obstruction discussed above.

The paper is organized as follows. In section 2 we give the relevant mathematical details for manifolds possessing hypersurface orthogonal, homothetic, conformal Killing vectors. In section 3 we consider the theory of symmetric tensors on such backgrounds and reformulate it as a quantum mechanical system with an \( \mathfrak{sp}(4) \) symmetry. Then in section 4 we extend this analysis to tensors of arbitrary type and show how to describe them by a supersymmetric quantum mechanical system with \( \mathfrak{osp}(Q|2p) \) symmetry. In section 5 we specialize to three dimensions where we can extend our algebras by the symmetrized curl operation discovered in \([12]\).
2. Geometry

Our results apply to curved, torsion-free, $d$-dimensional, (pseudo)Riemannian manifolds. We will often specialize to spaces which possess a hypersurface-orthogonal homothetic conformal Killing vector, i.e., a vector field $\xi = \xi^\mu \partial_\mu$ satisfying

$$g_{\mu \nu} = \nabla_\mu \xi_\nu.$$  

(1)

Henceforth we refer to condition (1) as hyperhomothety. To quickly motivate this condition, note that setting $\xi^\mu$ equal to coordinates $x^\mu$, the vector $\xi$ is the Euler vector field and the metric (1) is the flat metric. In this sense, hyperhomothetic spaces closely resemble flat ones.

From $\xi$, we can form the homothetic potential

$$\phi = \frac{\xi^\mu \xi_\mu}{2},$$  

(2)

which satisfies

$$\nabla_\mu \phi = \xi_\mu,$$  

(3)

and consequently

$$g_{\mu \nu} = \nabla_\mu \partial_\nu \phi.$$  

(4)

Under the hyperhomothety condition, it can be shown that the manifold admits coordinates $(r, x^i)$ such that the metric is explicitly a cone over some base manifold [13] with metric $h_{ij}$, where in particular the homothetic potential is simply

$$\phi = r^2,$$  

(5)

and

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = dr^2 + r^2 h_{ij} dx^i dx^j.$$  

(6)

Note that hyperhomothety (1) immediately implies that the contraction of $\xi$ on the Riemann tensor vanishes

$$R_{\mu \nu \rho \sigma} ^\sigma \xi_\sigma = \left[ \nabla_\mu, \nabla_\nu \right] \xi_\rho = 0.$$  

(7)

As mentioned above, the most elementary example of a hyperhomothetic space is flat space

$$ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu \Rightarrow \xi = x^\mu \partial_\mu,$$  

(8)

where the Euler vector field $\xi$ generates dilations. In fact, in many computations the components $\xi^\mu$ of $\xi$ mimic the flat space coordinates $x^\mu$. There are many other interesting physical instances of hyperhomothetic spaces. An important one amongst these are Fefferman–Graham ambient metrics [14] which are used to study conformal geometry. Other examples include Kähler cones over Kähler manifolds and hyperKähler cones over quaternionic Kähler manifolds [15] that appear in many supersymmetric contexts.

Locally symmetric spaces,

$$\nabla_\gamma R_{\mu \nu \rho \sigma} = 0,$$  

(9)

3 We label flat indices with Latin letters $m, n, \ldots$ and curved indices with Greek letters $\mu, \nu, \ldots$. Greek letters $\alpha, \beta, \gamma, \delta, \ldots$ are reserved for the orthosymplectic superindices. The covariant derivative is defined by example as

$$\nabla_\mu v^m = \partial_\mu v^m + \Gamma^m_{\mu \lambda} v^\lambda + \omega^m_\nu v^\nu.$$  

Our Riemann tensor conventions are summarized by

$$R_{\mu \nu \rho \sigma} v^\sigma = \left[ \nabla_\mu, \nabla_\nu \right] v_\sigma = 2 \left( \partial_\nu \Gamma^\alpha_{\mu \lambda} + \Gamma^\alpha_{\rho \lambda} \Gamma^\rho_{\mu \nu} \right) v_\sigma = \epsilon^\nu \rho R_{\mu \nu \rho \sigma} v_\sigma = 2 \left( \epsilon^\mu_{\alpha \beta} \epsilon^\nu_{\gamma \delta} \left( \partial_\sigma \omega^\gamma_{\alpha \beta} + \omega^\gamma_{\alpha \mu} \omega^\mu_{\beta \sigma} + \omega^\gamma_{\beta \sigma} \omega^\mu_{\alpha \nu} \right) v_\nu \right).$$  


enjoy isometries and are, in that sense, also similar to flat space. This condition was explored in [1] to obtain particle models with symmetries subject to a maximal parabolic subalgebra of the superalgebra $\text{osp}$.

These two conditions—hyperhomothety in equation (1), and locally symmetric space from equation (9)—intersect only in flat space, and are otherwise mutually exclusive. To see this, apply equation (1) to obtain the identity

$$\xi^\mu \nabla_\mu R_{\nu\rho\sigma} = -2R_{\nu\rho\sigma}.$$ \hspace{1cm} (10)

Clearly this contradicts the symmetric space condition (9) unless the Riemann tensor vanishes.

In the following sections we will use the geometric conditions just presented to explore algebras of differential geometry operators, which can in turn be reformulated as the quantization of orthosymplectic particle models. Our first example is the theory of symmetric tensors.

3. $\text{sp}(4)$ conformal quantum mechanics

Here we present a set of geometric operators on totally symmetric tensors which form a representation of the $\text{sp}(4)$ Lie algebra. We then show that these operators can be reinterpreted as quantized Noether charges of a particle with intrinsic structure. These charges correspond to rigid, continuous symmetries obeying the same $\text{sp}(4)$ algebra with an $\text{sp}(2)$ subalgebra playing the role of the worldline conformal group. This necessitates both time-independent and dependent symmetries.

3.1. Symmetric tensors

Totally symmetric tensors can be represented as analytic functions by completely contracting all indices with commuting coordinate differentials $dx^\mu$. Given a rank-$n$ symmetric tensor field $\phi_{\nu_1\nu_2\cdots\nu_n}$, we introduce

$$\Phi(x^\mu, dx^\mu) = \phi_{\nu_1\nu_2\cdots\nu_n}(x) \, dx^{\nu_1} \, dx^{\nu_2} \cdots \, dx^{\nu_n}.$$ \hspace{1cm} (11)

Hence we can interpret the original tensor $\phi$ as a function $\Phi$ of the coordinates $x^\mu$ and an analytic function of commuting coordinate differentials $dx^\mu$. Note that it is now possible to add symmetric tensors of different ranks. In this indexless notation, one can then define various important operators. To this end, in addition to coordinate differentials it is useful to introduce dual objects $dx^\ast_\mu$, which mutually commute but obey the Heisenberg algebra

$$[dx^\ast_\mu, dx^\nu] = \delta^\nu_\mu.$$ \hspace{1cm} (12)

Since we now consider coordinate differentials like coordinates, we can represent these dual differentials acting on symmetric tensors by

$$dx^\ast_\mu = \frac{\partial}{\partial(dx^\mu)}.$$  

With these ingredients we can build an operator which is equivalent to the covariant derivative when it acts on a symmetric tensor in the indexless form (11), which we denote $D_\mu$ (distinct from the ordinary covariant derivative $\nabla_\mu$, which acts on tensors with indices)

$$D_\mu = \partial_\mu - T^\nu_\mu dx^\nu \, dx^\ast_\nu.$$ \hspace{1cm} (13)

4 The algebra we present here was first discovered by Lichnerowicz [3] and then formalized in [12]. It was subsequently employed in studies of higher spin theories in [16–18].
where $\partial\mu$ denotes the partial derivative $\partial/\partial x^\mu$. That is,

$$D_\mu \Phi = (\nabla_\mu \phi_{\nu_1 \nu_2 \cdots \nu_\nu}) \, dx^{\nu_1} \, dx^{\nu_2} \cdots \, dx^{\nu_\nu}.$$ 

We can also build the Lorentz/rotation generators

$$M_{\mu\nu} = 2g^{\rho(\nu} \, dx^\mu_{\rho)}, \quad [M_{\mu\nu}, M_{\rho\sigma}] = 4M^{\nu}_{[\sigma} \delta^{\rho]}_{\mu]},$$

where $[\cdot]$ denotes antisymmetrization with unit weight. Note that although $D_\mu$ and $M_{\mu\nu}$ act on tensors contracted with coordinate differentials, their outputs have open indices. For this reason, they will not appear alone in the algebra we will discuss, but only in larger composite operators.

Now we will introduce a set of operators which map symmetric tensors to symmetric tensors (without producing extra indices). First, using just the operators $dx$ and $dx^*$, we can construct three bilinears

$$N = dx^\mu \, dx^*_\mu, \quad g = g_{\mu\nu} \, dx^\mu \, dx^\nu, \quad tr = g^{\mu\nu} \, dx^*_\mu \, dx^*_\nu.$$ (14)

Geometrically, these operators perform the following tasks: $N$ determines tensor rank; $g$ is the symmetrized outer product with the metric tensor; and $tr$ is the trace/contraction with the metric tensor.

The above operators are ‘non-dynamical’, whereas employing the covariant operator $D_\mu$, we can form another three dynamical bilinears

$$\text{grad} = dx^\mu \, D_\mu, \quad \text{div} = g^{\mu\nu} \, dx^*_\nu \, D_\nu, \quad \Box = \Delta + \frac{1}{4} R_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma}.$$ (15)

We are forced to add a ‘quantum ordering’ term\(^5\) to form the Laplacian from the operator $D_\mu$\(^\Delta\):

$$\Delta = g^{\mu\nu} \left(D_\mu D_\nu - \Gamma^{\nu}_{\mu\lambda} D_\lambda\right).$$

We have added a curvature term to $\Delta$ to make the operator $\Box$ for symmetry reasons soon to become apparent. These operators can be interpreted as follows: grad is the symmetrized gradient; div is the symmetrized divergence; and $\Box$ is the Lichnerowicz wave operator [3]. The operators from equations (14) and (15) are discussed more thoroughly in [1].

Lastly, using the hyperhomothetic Killing vector $\xi$, we can form four additional bilinears

$$\text{ord} = -\xi^\mu D_\mu, \quad \iota_\xi = \xi^\mu \, dx^*_\mu, \quad \iota^*_\xi = \xi_\mu \, dx^\mu, \quad \nabla_\xi = \xi^\mu \xi^*_\mu.$$ (16)

These four operators can, of course, be defined whether or not the hyperhomothety condition holds; if it does not, $\xi$ can be an arbitrary vector field, but if $\xi$ obeys (1) they have additional special properties. Indeed, these operators can be interpreted as follows: ord is the covariant derivative $\nabla_\xi$ along $\xi$—subject to hyperhomothety it counts derivatives. The operators $\iota_\xi$ and $\iota^*_\xi$ are, respectively, the symmetrized inner and outer products with the vector field $\xi$. Lastly, $\nabla_\xi$ is multiplication by the scalar field/homothetic potential $\xi^2$.

Thus far, we have simply introduced an operator notation for standard geometric operations on symmetric tensors. Remarkably, subject to combinations of the conditions stipulated in section 2, these operators constitute representations of certain deformations of the symplectic Lie algebra $sp(4)$. To see this we first note that the three operators formed from bilinears in $dx, dx^*$ in equation (14) form a representation of $sp(2)$ in any background (figure 2(i)) with the index-counting operator $N$ as its Cartan generator

$$[N, g] = 2g, \quad [N, tr] = -2tr, \quad [g, tr] = -4(N + d/2).$$ (17)

\(^5\) For a study of operator orderings in supersymmetric quantum mechanics see [19].
Keeping the background (and therefore also the vector field $\xi$) arbitrary, the operators $N, g, \text{tr}, \iota_\xi, \iota^*_\xi$ and $\Box$ form a subalgebra of $\mathfrak{sp}(4)$. To add $\text{ord}$ to the algebra as the other $\mathfrak{sp}(4)$ Cartan generator, we need to invoke hyperhomothety, and then arrive at a maximal parabolic subalgebra (figure 2(ii)).

Next, relaxing the hyperhomothety condition and instead imposing the locally symmetric space condition (9), we have $N, g, \text{tr}, \text{grad}, \text{div}$ and $\Box$ forming a representation of a subalgebra of $\mathfrak{sp}(4)$ (figure 2(iii)), up to a mild curvature obstruction given by

$$[\text{div}, \text{grad}] = \Box - \frac{1}{2} R_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma} \neq \Box. \tag{18}$$

This computation was performed already in [1]. Importantly, note that the operators $N, g, \text{tr}, \text{grad}, \text{div}$ are all symmetries, in the sense that they commute with the operator $\Box$ which can be interpreted as the Hamiltonian. In the case of constant curvature manifolds, the obstruction term $\frac{1}{2} R_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma}$ to $(N, g, \text{tr}, \text{grad}, \text{div}, \Box)$ forming a Lie algebra equals the Casimir operator of the $\mathfrak{sp}(2)$ subalgebra built from $(N, g, \text{tr})$. In this case it is possible to reformulate this deformed Lie algebra as a novel associative Fourier–Jacobi algebra (see [2]).

Our ultimate algebra requires hyperhomothety, and with all ten operators from equations (14)–(16) forms a representation of $\mathfrak{sp}(4)$ (figure 2(iv)), up to obstructions given by equation (18) together with

$$[\Box, \text{grad}] = -\frac{1}{4} (\nabla_{\mathcal{J}} R_{\mu\nu\rho\sigma}) (M^{\mu\nu} \text{d}x^{\rho} M^{\sigma\nu}) \neq 0, \tag{19}$$

as well as a similar result for $\Box$ with $\text{div}$. This result is new and its generalization to more general tensors and spinors is given in section 4. It is actually valid in any (pseudo)Riemannian background (regardless of whether it is hyperhomothetic or not). Unlike the obstruction in the Weitzenbock-type identity (18), it restricts the symmetries of $\Box$—regarded as a Hamiltonian—to $(N, g, \text{tr})$ (and also $\text{ord}, \iota_\xi, \iota^*_\xi, \Box$ if one considers time-dependent conformal symmetries). Note that the right-hand side of equation (19) vanishes in symmetric spaces, and that both curvature obstructions vanish together exclusively in flat space, where $\mathfrak{sp}(4)$ is fully realized as a Lie algebra; any relaxation of flat space to more general backgrounds immediately yields a deformation of $\mathfrak{sp}(4)$. The $\mathfrak{sp}(4)$ root lattice and our identification of roots with differential geometry operators are displayed in figure 1. The various different subalgebras discussed
Figure 2. Root lattices for potential subalgebras of $\mathfrak{sp}(4)$, individually conditional upon hyperhomothety (1) or locally symmetric space (9) conditions.

above are exhibited in figure 2. All possible commutation relations of these operators are tabulated in appendix B.

3.2. Quantum mechanics

Next we would like to interpret the geometric system described in section 3.1 as a quantum mechanical one, with the symmetric tensors becoming wavefunctions of particle model with internal structure

$$\Phi(x^\mu, dx^\mu) \rightarrow |\Psi\rangle.$$  \hspace{1cm} (20)

We moreover interpret the coordinate differentials $dx$ and their duals $dx^*$ from the preceding section as raising and lowering operators, respectively

$$dx^\mu \rightarrow a^\dagger_\mu, \hspace{0.5cm} dx^*_\mu \rightarrow a_\mu.$$  \hspace{1cm} (21)

For example, a vector field $\psi_\nu$ can be written in this way as a wavefunction

$$|\Psi\rangle = \psi_\nu(x) a^\dagger_\nu |0\rangle,$$  \hspace{1cm} (21)

where we have introduced the Fock vacuum state $|0\rangle$ annihilated by $a_\mu$.

The Heisenberg algebra (12) now becomes the standard oscillator one for $a^\dagger$ and $a$

$$[a^\mu, a^\dagger_\nu] = \delta^\mu_\nu,$$  \hspace{1cm} (22)

and similarly the momentum $p_\mu = -i\partial_\mu$ and position $x^\mu$ obey

$$[p_\mu, x^\nu] = -i\delta^\mu_\nu.$$  \hspace{1cm} (23)

Also, just as the partial derivative is replaced by the canonical momentum, so too the operator $D_\mu$ from equation (13) is replaced by the covariant canonical momentum

$$-iD_\mu \rightarrow \pi_\mu = p_\mu + i\Gamma^\sigma_\mu a^\dagger_\nu a_\sigma.$$  \hspace{1cm} (24)
The inner product for our Hilbert space is implied by the normalization \( \langle 0|0 \rangle = 1 \), so for example the norm of a pair of eigenstates of \( N \) is
\[
\langle \Phi|\Psi \rangle = \sqrt{\frac{1}{g}} \int d^d x.
\] (25)

The symmetric-tensor operators from the previous section are now manifested as quantized Noether charge operators acting on quantum states, with the appropriate replacements for \( dx \) obtained above. Note that all orderings of operators are fixed by their geometric ancestors. The algebra satisfied by each charge in this system is exactly identical to that of its geometric \textit{vis-à-vis} from section 3.1. To gain further insight, we next analyze the classical system underlying this quantum mechanical model.

3.3. Classical mechanics

We now work in the classical theory for a particle, where we choose to interpret the coordinate differentials \( dx \) and their duals \( dx^* \) from the geometrical representation in section 3.1, or alternatively the quantum mechanical raising and lowering operators \( a^\dagger, a \) from section 3.2, as comprising a complex-valued vector carried by the particle, i.e.,
\[
a^\dagger \mu \rightarrow \bar{z}^\mu, \quad a \mu \rightarrow z_{\mu}.
\] (26)

In the classical quantum theory, so that the quantum mechanical commutators become instead Poisson brackets
\[
\{p_{\mu}, x^\nu\}_\text{PB} = \delta^\nu_\mu, \quad \{\bar{z}^\mu, z_{\nu}\}_\text{PB} = -i\delta^\mu_\nu.
\] (27)
Throughout this section we assume the hyperhomothety condition (1). At this point it is convenient to define the covariant variation \( D \) and covariant worldline derivative \( \nabla /dt \) as
\[
D v^\mu = \delta x^\sigma \nabla_\sigma v^\mu = \delta v^\mu + \Gamma ^\mu_\nu^\rho v^\nu \delta x^\rho
\] (28)
\[
\nabla v^\mu \frac{dt}{dr} = \dot{x}^\sigma \nabla_\sigma v^\mu = \dot{v}^\mu + \Gamma ^\mu_\nu^\rho v^\rho \dot{x}^\nu.
\] (29)

Since it is central (up to obstructions), we take as our Hamiltonian \( H = -\Box /2 \), dropping the \( g_{\mu\nu} \Gamma ^\sigma_\mu^\nu D_\sigma \) quantum ordering term for the classical system
\[
H = \frac{1}{2} \left( \pi^\mu \pi_\mu - R^\mu_\nu^\rho \bar{z}^\mu z^\nu \bar{z}^\rho z_\sigma \right).
\] (30)

From (27) it is evident that we already have Darboux coordinates \( x, p, z, \bar{z} \), so we can perform a Legendre transformation to obtain a suitable action principle for our particle
\[
S = \int \left( \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + i \bar{z}^\mu \nabla z_{\mu} + \frac{1}{2} R_{\mu\nu}^\rho \bar{z}^\mu z^\nu \bar{z}^\rho z_\sigma \right) dt.
\] (31)

The three terms in this Lagrangian all have interesting geometric interpretations. The first is the usual energy integral, the extremization of which yields simple parametrized geodesic motion. The second ensures parallel transport of the vector \( z_\mu \). The third is in effect a coupling between the first two; including it in our model results in many more symmetries (discussed below) than if it is omitted.

We also note two important identities
\[
D t^\mu = \nabla t^\mu
\] \[
\left[ D, \nabla \right] t^\mu = \delta t^\mu \nabla t^\nu R^\nu_\mu^\rho t^\rho.
\]
Table 1. List of geometric symmetric-tensor operators, complete with root vectors for their corresponding $\mathfrak{sp}(4)$ root lattice displayed in figure 1.

| Name | Root | Operator | Interpretation |
|------|------|----------|----------------|
| $N$  | Cartan | $dx^\mu dx_\mu$ | Counts tensor rank |
| $g$  | $(0, 2)$ | $g_{\mu\nu} dx^\mu dx^\nu$ | Metric outer product |
| $tr$ | $(0, -2)$ | $g^{\mu\nu} dx^\mu_d x^\nu$ | Trace |
| $\text{grad}$ | $(1, 1)$ | $dx^\mu D_\mu$ | Gradient |
| $\text{div}$ | $(1, -1)$ | $g^{\mu\nu} dx^\mu D_\nu$ | Divergence |
| $\Box$ | $(2, 0)$ | $g^{\mu\nu} (D_\mu D_\nu - \Gamma^\sigma_{\mu\nu} D_\sigma)$ | Lichnerowicz wave operator |
| $\text{ord}$ | | | |
| $\iota_\xi$ | $(-1, -1)$ | $\xi^\mu dx_\mu$ | Vector field inner product |
| $\iota^*_\xi$ | $(1, 1)$ | $\xi^\mu dx_\mu$ | Vector field outer product |
| $\blacksquare$ | $(-2, 0)$ | $g_{\mu\nu} \xi^\mu \xi^\nu$ | Scalar field multiplication |

The ten operators forming our representation of $\mathfrak{sp}(4)$ in section 3.1, listed in table 1, correspond via Noether’s theorem to conserved quantities of the above action, with explicit symmetries listed in table 2. There are, however, three interesting caveats to make. First, because there is an obstruction to the algebra between the $\Box$ charge and both the $\text{grad}$ and $\text{div}$ charges (equation (19)), $\text{grad}$ and $\text{div}$ technically are not conserved charges of our action—performing the $\text{grad}$ variation, we find

$$\delta S = i \int \left( \nabla_\lambda R^{\nu\rho}_{\mu\sigma} \bar{z}_\lambda \bar{z}_\mu z_\nu \bar{z}_\rho z_\sigma \right) dt \quad \text{subject to} \quad \delta x^\mu = i \bar{z}^\mu, \quad \bar{D}z_\mu = \dot{x}_\mu. \quad (32)$$

The result of this variation, which is exactly the classical equivalent of equation (19), shows the failure of the $\text{grad}$ and $\text{div}$ symmetries due to the curvature of the underlying manifold.

Second, it is important to note that the symmetries corresponding to the hyperhomothetic operators—$\text{ord}$, $\iota_\xi$, $\iota^*_\xi$ and $\blacksquare$—do not commute with the Hamiltonian $H$; consequently, they are off-shell symmetries and furthermore possess time dependence; it is of course sufficient to set $t = 0$, since they are conserved. However, because of the subalgebra of the $\mathfrak{sp}(4)$ Lie algebra which they satisfy, their time dependence can be computed analytically through the following procedure. First, given a quantized Noether charge operator $Q_0$, its time dependence is determined using the system’s Hamiltonian operator $H$

$$\frac{dQ}{dt} = [Q_0, H], \quad (33)$$

which can be integrated to yield $Q(t)$. For example, to obtain the time dependence of the $\text{ord}$ charge

$$\frac{d}{dt} \text{ord} = [\text{ord}, H] = 2H \quad \Rightarrow \quad \text{ord}(t) = \text{ord} + 2Ht, \quad (34)$$

or for $\blacksquare$,

$$\frac{d}{dt} \blacksquare = [\blacksquare, H] = -4 \text{ord} \quad \Rightarrow \quad \blacksquare(t) = \blacksquare - 4 \text{ord} t - 4\Box t^2. \quad (35)$$

Note that the spectrum-generating algebra obeyed by the charges allows us to explicitly integrate their time dependence. From here, $Q$’s corresponding rigid symmetry for the coordinate $q^\mu$ is then

$$\delta Q q^\mu = [Q(t), q^\mu]. \quad (36)$$

Proceeding in this way one can obtain all the symmetries from table 2.
Table 2. Symmetries and corresponding Noether charges of the action from (31). Unspecified variations are zero.

| Name | Symmetry | Noether charge |
|------|----------|----------------|
| N    | $\delta z_\mu = -z_\mu$ | $g^{\mu\nu}z_\mu$ |
|      | $\delta \bar{z}_\mu = \bar{z}_\mu$ | $g^{\mu\nu}\bar{z}_\mu \bar{z}_\nu$ |
| g    | $\delta z_\mu = g_{\mu\nu} z^\nu$ | $R_{\mu\nu}^{\ \ \ \ \ \ \ \ \ \ \ \} g_{\mu\nu} z_\mu$ |
| tr   | $\delta z_\mu = g^{\mu\nu} z_\nu$ | $R_{\mu\nu}^{\ \ \ \ \ \ \ \ \ \ \ \} g^{\mu\nu} z_\mu z_\nu$ |
| grad | $\delta x_\mu = i\bar{z}_\mu$ | $\lambda^\mu z_\mu$ |
|      | $\delta \bar{z}_\mu = \bar{\lambda}^\mu z_\mu$ | $\lambda^\mu z_\mu$ |
| div  | $\delta x_\mu = i\bar{z}_\mu$ | $\lambda^\mu z_\mu$ |
|      | $\delta \bar{z}_\mu = \bar{\lambda}^\mu z_\mu$ | $\lambda^\mu z_\mu$ |
| □    | $D\delta x_\mu = -iR_{\alpha\beta}^{\ \ \ \ \ \ \ \ \ \ \ \} g_{\alpha\beta} x_\mu$ | $\lambda^\mu z_\mu - \frac{1}{2} R_{\alpha\beta}^{\ \ \ \ \ \ \ \ \ \ \ \} g_{\alpha\beta} z_\mu z_\nu z_\sigma$ |
|      | $D\delta \bar{z}_\mu = iR_{\alpha\beta}^{\ \ \ \ \ \ \ \ \ \ \ \} g_{\alpha\beta} \bar{z}_\mu$ | $\lambda^\mu z_\mu - \frac{1}{2} R_{\alpha\beta}^{\ \ \ \ \ \ \ \ \ \ \ \} g_{\alpha\beta} \bar{z}_\mu \bar{z}_\nu \bar{z}_\sigma$ |
| ord  | $\delta x_\mu = 2t\bar{x}^\mu - \bar{\xi}_\mu$ | $\lambda^\mu \bar{\xi}_\mu - t\lambda^\mu \bar{x}_\mu$ |
|      | $\delta \bar{x}_\mu = -i\lambda^\mu$ | $\lambda^\mu z_\mu - t \lambda^\mu \bar{z}_\mu$ |
| tξ   | $D\delta x_\mu = t\lambda^\mu - \bar{\xi}_\mu$ | $\lambda^\mu z_\mu - t \lambda^\mu \bar{z}_\mu$ |
|      | $D\delta \bar{x}_\mu = t\lambda^\mu - \bar{\xi}_\mu$ | $\lambda^\mu z_\mu - t \lambda^\mu \bar{z}_\mu$ |
| iξ   | $\delta x_\mu = it\lambda^\mu$ | $\lambda^\mu \bar{\xi}_\mu - 2i\lambda^\mu \bar{x}_\mu + t^2 \lambda^\mu \bar{\xi}_\mu$ |
|      | $\delta \bar{x}_\mu = it\lambda^\mu - i\bar{\xi}_\mu$ | $\lambda^\mu \bar{\xi}_\mu - 2i\lambda^\mu \bar{x}_\mu + t^2 \lambda^\mu \bar{\xi}_\mu$ |

4. $\mathfrak{osp}(Q|2p+2)$ conformal quantum mechanics

We now generalize the algebra from section 3.1 to the orthosymplectic Lie superalgebra $\mathfrak{osp}(Q|2p+2)$. We will always assume the hyperhomothety condition from equation (1) in this section and use the spin connection $\omega_{\mu\nu\rho}$ rather than the Christoffel symbols to build covariant derivative operators. In place of the coordinate differentials $d\!^x_\mu$ and $d\!^{x}_\mu$ from before, we now employ an orthosymplectic vector denoted by $X_\alpha$, where $\alpha$ is a superindex taking values $1 \leq \alpha \leq 2p+Q$. For $1 \leq \alpha \leq 2p$, $X_\alpha$ is a bosonic variable, and otherwise it is fermionic. These satisfy the supercommutation relations

$$[X^m_\alpha, X^n_\beta] = \eta^{mn} J_{\alpha\beta}.$$

Here $J_{\alpha\beta}$ is the orthosymplectic bilinear form, given by

$$(J_{\alpha\beta}) = \begin{pmatrix}
\begin{pmatrix}
1_{p \times p} & -1_{p \times p} \\
-1_{p \times q} & 1_{q \times q}
\end{pmatrix}
& Q = 2q \text{ even,}
\begin{pmatrix}
1_{p \times p} & -1_{p \times q} \\
-1_{q \times q} & 1
\end{pmatrix}
\end{pmatrix}$$

$$(38)$$

We denote the inverse of $J_{\alpha\beta}$ by $J^{\alpha\beta}$, so that $J_{\alpha\beta}J^{\gamma\beta} = \delta^\alpha_\gamma$ (while $J^{\alpha\beta}J_{\gamma\beta} = -T_{\alpha\gamma}$; see appendix A).
From $X$’s we can now form the $SO(d)$ rotation/Lorentz generators
\[
M_{mn}^{\mu} = J^{\mu} X_{\mu}^{m} X_{\mu}^{n},
\]
(39)
as well as the covariant derivative operator
\[
D_{\mu} = \partial_{\mu} + \omega_{\mu mn} M_{mn}^{\mu}.
\]
(40)

Next, from the space of orthosymplectic bilinears, we define
\[
f_{\alpha \beta} = \eta_{mn} X_{\alpha}^{m} X_{\beta}^{n} (\alpha X_{\beta}^{n} - \beta X_{\alpha}^{m}),
\]
(41)
where $\alpha$ and $\beta$ are both fermionic indices, and otherwise denotes symmetrization; we again refer the reader to appendix A, where a computational scheme for handling such symmetrization is detailed. The supermatrix $f_{\alpha \beta}$ subsumes $N$, $g$ and $\text{tr}$ from section 3.1, as detailed below.

In turn we introduce dynamical symmetry generators
\[
v_{\alpha} = X_{\mu}^{\alpha} e^{\mu}_{\nu} D_{\nu} \equiv X_{\alpha}^{m} D^{m},
\]
(42)
which subsume $\text{grad}$ and $\text{div}$, and which could be viewed as generalized Dirac operators. We also clearly need a Lichnerowicz wave operator
\[
\Box = D^{m} D_{m} - \omega_{mn}^{\mu} D_{m} + \frac{1}{4} R_{mnrs} M_{mn}^{\mu} M_{rs}^{\nu}.
\]
(43)

To complete our set, we now add the hyperhomothetic operators. The operators $\text{ord}$ and $\Box$ remain unchanged, except of course that $\text{ord}$ now uses the orthosymplectic covariant derivative operator from equation (40)
\[
\text{ord} = -\xi^{m} D_{m}, \quad \Box = \xi^{m} \xi_{m}.
\]
(44)
Lastly, the generalizations of $\iota_{\xi}$ and $\iota^{*}_{\xi}$ are
\[
u_{\alpha} = \xi_{m} X_{\alpha}^{m}.
\]
(45)

The explicit correspondence between the $\mathfrak{osp}(Q|2p + 2)$ operators ($f_{\alpha \beta}, v_{\alpha}, w_{\alpha}$) and their $\mathfrak{sp}(4)$ equivalents is
\[
(f_{\alpha \beta}) \leftrightarrow \begin{pmatrix} g & N + d/2 \\ N + d/2 & \text{tr} \end{pmatrix}, \quad (v_{\alpha}) \leftrightarrow \begin{pmatrix} \text{grad} \\ \text{div} \end{pmatrix}, \quad (w_{\alpha}) \leftrightarrow \begin{pmatrix} \iota_{\xi} \\ \iota^{*}_{\xi} \end{pmatrix}.
\]
(46)

The symmetry algebras which can be formed by our $\mathfrak{osp}(Q|2p + 2)$ operators are exactly analogous to the $\mathfrak{sp}(4)$ case (figure 2), with $\mathfrak{sp}(2) \rightarrow \mathfrak{osp}(Q|2p)$ and $\mathfrak{sp}(4) \rightarrow \mathfrak{osp}(Q|2p + 2)$. Specifically:

(i) $f_{\alpha \beta}$ form $\mathfrak{osp}(Q|2p)$ in any background;
(ii) $f_{\alpha \beta}, v_{\alpha}$ and $\Box$ form a subalgebra of $\mathfrak{osp}(Q|2p + 2)$ in any background, which becomes maximal and parabolic with the addition of $\text{ord}$ and the hyperhomothety condition (equation (1));
(iii) $f_{\alpha \beta}, v_{\alpha}$ and $\Box$ form a subalgebra of $\mathfrak{osp}(Q|2p + 2)$ under the locally symmetric space condition, equation (9), with an obstruction (47);
(iv) together $f_{\alpha \beta}, \text{ord}, v_{\alpha}, \Box, w_{\alpha}$ and $\Box$ form $\mathfrak{osp}(Q|2p + 2)$, subject to obstructions (48) and (47).

We now present the two obstructions to the full $\mathfrak{osp}(Q|2p + 2)$ algebra under hyperhomothety, respectively generalizing their $\mathfrak{sp}(4)$ counterparts in equations (18) and (19)
\[
[v_{\alpha}, v_{\gamma}] = J_{\alpha \gamma} \left( \omega_{m}^{\mu} D_{\mu} + D^{\mu} D_{\mu} \right) + \frac{1}{2} R_{mnrs} X_{\mu}^{m} X_{\nu}^{n} M_{\mu \nu}^{rs} \neq J_{\alpha \gamma} \Box,
\]
(47)
\[
[\Box, v_{\alpha}] = -\frac{1}{4} \left( \nabla_{\nu} R_{mnrs} \right) \left( M_{mn}^{\mu} X_{\alpha}^{m} X_{\nu}^{n} M_{\mu \nu}^{rs} \right) \neq 0.
\]
(48)
Both these obstructions can be removed when \( Q = 0, 1 \) and \( p = 0 \), at which values our models revert to \( N = 1, 2 \) supersymmetric quantum mechanics (see [1] for details). The remainder of the algebra is explicitly presented in appendix B.

The Hilbert space of these othosymplectic models is described in great detail in [1] (various special cases have been studied in [20, 22–27]). Wavefunctions are tensors expanded in terms of multi-forms and multi-symmetric-forms. Moreover, when \( Q \) is odd, wavefunctions are spinor valued so carry a spinor index \( \alpha \). In a Young diagram notation, where rows are totally symmetric and columns antisymmetric, we would write

\[
\Phi^p_{\boxtimes} \quad [Q/2 \text{ times}]
\]

Clearly, although this is not an irreducible basis for tensors and spinors on a manifold, we can generate all such objects this way. Irreducible tensors can be obtained by placing constraints coming from parabolic subgroups of the \( \mathfrak{osp}(Q|2p) \) algebraically acting \( R \)-symmetry algebra generated by \( f_{\alpha\beta} \).

5. Three dimensions

All our results so far have not depended crucially on the dimensionality of the background manifold. In this section we wish to study particle symmetries built from the totally antisymmetric Lévi-Civita symbol. These are dimension dependent. Therefore, in this initial study, we concentrate on three dimensions and the Noether charges that can be built from the three-dimensional Lévi-Civita symbol\(^7\) \( \epsilon^{\mu\nu\rho} \). We further specialize to constant curvature backgrounds with the advantage of maximizing the set of symmetries.

To be completely concrete let us take the three-dimensional hyperbolic metric

\[
d^2 = \frac{dx^2 + dy^2 + dz^2}{z^2},
\]

although none of our algebraic results depend on this choice of coordinates. We focus on the \( sp(2) \) symmetric-tensor model, but the generalization to \( osp(Q|2p) \) spinning degrees of freedom is immediate. Representing \( dx^\mu = \partial/\partial(dx^\mu) \) acting on symmetric tensors viewed as analytic functions of commuting differentials \( dx^\mu \), the covariant derivative operator \( D_\mu \) of (13) is given explicitly by

\[
D_x = \partial_x - \frac{1}{z} \left\{ dx \frac{\partial}{\partial(dx)} - dz \frac{\partial}{\partial(dx)} \right\},
\]

\[
D_y = \partial_y - \frac{1}{z} \left\{ dy \frac{\partial}{\partial(dy)} - dz \frac{\partial}{\partial(dy)} \right\},
\]

\[
D_z = \partial_z + \frac{1}{z} N,
\]

where the index-counting operator is

\[
N = \frac{dx}{\partial(dx)} + dy \frac{\partial}{\partial(dy)} + dz \frac{\partial}{\partial(dz)}.
\]

\(^7\) In our notations the Lévi-Civita symbol is a density so \( \epsilon^{\mu\nu\rho} \) is characterized by \( \epsilon^{123} = 1 \) in any coordinate system.
The \( sp(2) \) algebra of internal symmetries is generated by \( N, g = ds^2 \) as in (50), and the trace operator

\[
\text{tr} = z^2 \left\{ \frac{\partial^2}{\partial (\partial x)^2} + \frac{\partial^2}{\partial (\partial y)^2} + \frac{\partial^2}{\partial (\partial z)^2} \right\}.
\]  

(53)

The quadratic \( sp(2) \) Casimir is

\[
c = g \text{tr} - N(N+1).
\]  

(54)

Then we have the pair of differential operators

\[
\text{grad} = dx D_x + dy D_y + dz D_z,
\]

\[
\text{div} = z^2 \left\{ \frac{\partial}{\partial (\partial x)} D_x + \frac{\partial}{\partial (\partial y)} D_y + \frac{\partial}{\partial (\partial z)} D_z \right\},
\]  

(55)

which form an \( sp(2) \) doublet. In turn, the Lichnerowicz wave operator

\[
\Box = \text{div \, grad} - \text{grad \, div} - 2c
\]  

(56)

is central. So far we have simply written out the results of the previous sections explicitly for this three-dimensional background. Now we introduce a new operator built from the Lévi-Civitá symbol

\[
\text{curl} = \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\rho} dx_{\nu} D_{\rho} = z \left\{ dx \frac{\partial}{\partial (\partial y)} D_z \pm \text{permutations} \right\}.
\]  

(57)

Geometrically, this is the symmetrized curl operation first introduced in [12]. It can be generalized to more spinning degrees of freedom by considering

\[
\text{curl} = \frac{1}{2\sqrt{g}} \epsilon^{\mu\nu\rho} M_{\mu\nu} D_{\rho},
\]  

(58)

where the \( SO(3) \) rotation operators are given in (39). The symmetrized curl operator has various interesting properties. First, it is central, i.e. it commutes with \( (\Box, \text{div}, \text{grad}, \text{tr}, N, g) \). Moreover, since it is central, so is its square which equals

\[
\text{curl}^2 = \Box(g \text{tr} - N^2) + \text{grad}(2N+1)\text{div} - g\text{div}^2 - \text{grad}^2\text{tr} + c(c+2N).
\]  

(59)

This operator is rather interesting: it has been known for quite some time that the Fourier–Jacobi algebra obeyed by \( (\Box, \text{div}, \text{grad}, \text{tr}, N, g) \) in flat backgrounds possessed a quadratic Casimir [28]. The above result shows that this Casimir can be generalized to constant curvature backgrounds (at least in three—and possibly higher—dimensions).

Another interesting consequence follows by considering the quantum mechanical origin of the above operators. In our previous models, we viewed the above operators as Noether charges and took \( \Box \) as the Hamiltonian to ensure a maximal set of symmetries. In three dimensions however, we have the additional possibility of regarding the curl operator as the Hamiltonian. This corresponds to a new particle model with action

\[
S = \int \left( \tau_{\mu} z^{\mu} + i\bar{z}^{\mu} \nabla z_{\mu} \right) dr - \frac{1}{\sqrt{g}} \epsilon^{\mu\nu\rho} z_{\mu} \tau_{\nu} \tau_{\rho} dr,
\]  

(60)

and symmetries corresponding to the above Noether charges. Note that this model does not admit a Legendre transformation with respect to \( \tau_{\mu} \), but if a second-order theory is desired, one can always add an additional term to the Hamiltonian proportional to the Lichnerowicz wave operator. Finally, we note that a gauged version of this model would provide a first-quantized
description of three dimensional topologically massive theories [29]. (Essentially, one should identify the correct first class algebra from the set of generators \(\text{curl, } \Box, \text{ div, } \text{ grad, } \text{ tr, } N, g\) and employ BRST detour quantization techniques along the lines of those described in [30].)

6. Conclusions

In a previous work [1] it was found that Weitzenbock identities for differential operators acting on tensors of general types on symmetric spaces followed from a symmetry algebra that formed the maximal parabolic subalgebra of a larger orthosymplectic algebra. These symmetries could also be realized as generalized supersymmetries and \(R\)-symmetries of an underlying curved space quantum mechanical model. A puzzle remained, however—namely whether the full orthosymplectic algebra could be realized as symmetries of the quantum mechanical model and therefore, in turn, as symmetries of differential operators on manifolds. In this paper we have solved this puzzle by showing that the remainder of the orthosymplectic algebra corresponds to worldline conformal symmetries of the quantum mechanical model when the background geometry possesses a hypersurface-orthogonal conformal Killing vector. In terms of geometry, the new conformal symmetries correspond to all operations on tensors that can be performed using this vector.

Another aspect of the work [1] was that quantum mechanical symmetries were built from all available invariant tensors save the totally antisymmetric Lévi-Civitá symbol. In the original work of [12], in the context of topologically massive theories, it was also realized that a symmetrized curl operation could be added to the algebra of operators acting on symmetric tensors in three dimensions. Clearly this operation can be generalized to tensors of different types. Furthermore, there should be a quantum mechanical model underlying this symmetry. We have constructed exactly this model in the present paper.

An immediate application of our results is to projective geometry. If we consider a model which enjoys the full orthosymplectic algebra as its symmetry group, we can take the quadratic Casimir as Hamiltonian. In flat space this amounts to taking the square of the total angular momentum operator as the Hamiltonian. By examining the square of the orbital angular momentum operator \(M^2 = x^2 p^2 - x \cdot p^2\) we see that the Hamiltonian \(H \sim x^2 \hat{p}^2\) where the covector \(\hat{p}\) is the momentum orthogonal to the homothety \(x\). This model therefore describes motion on the projective space \(\mathbb{R}P^{d-1}\) by projectivizing with respect to dilations. It would be most interesting to generalize this construction to the models considered here and obtain an algebra of operators acting on projective spaces.

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Appendix A. Supercommutation computation

A convenient method for supercommutator and superindex computations is to define a symbol \(T^{\mu\nu}_{\alpha\beta}\) similar to a four-index Kronecker delta, which when contracted over \(\mu\) and \(\nu\) produces a sign depending on the values of the indices \(\alpha\) and \(\beta\). The sign symbol \(T\) allows us to
Figure B1. Explicit commutation algebra of the operators from section 3.1. All results apply to arbitrary curved manifolds, except those labeled †, which are contingent upon application of the hyperhomothety condition from equation (1). Commutators are of the form (left column top row).

| \cdot \cdot \cdot | \text{ord} | g | \text{tr} | \text{grad} | \text{div} | δ^i_j | -δ^i_j | \text{tr} | \text{ord}^I | \text{grad}^I | \text{div}^I | \cdot \cdot \cdot |
|-----------------|----------|-----|---------|----------|--------|--------|--------|--------|---------|---------|---------|----------|
| N               | 0        | 2g  | -2tr   | grad     | -div   | 0      | δ^i_j  | -δ^i_j | 0       | 0       | 0       | 2i_k    |
| ord             | 0        | 0   | 0       | grad^I   | div^I   | 2i_k   | -δ^i_j | -δ^i_j | -2i_k   | 0       | 0       | 0       |
| g               | -4N      | 0   | 0       | 2grad^I  | 2div^I | 0      | 0      | 2i_k   | 0       | 0       | 0       |
| tr              | 2div     | 0   | 0       | 0        | 2i_k   | 0      | 0      | 0      | 0       | 0       | 0       |
| grad            | Eq. 18   | Eq. 19 | g^I   | N + ord^I | tr^I   | 2i_k^I | 0      | 0      | 0       | 0       | 0       |
| div             | Eq. 19   | Eq. 19 | g^I   | N - ord^I | tr^I   | 2i_k^I | 0      | 0      | 0       | 0       | 0       |
| δ^i_j           | -        | 0   | 0       | 0        | 0      | 0      | 0      | 0      | 0       | 0       | 0       |

use standard Einstein notation for tensors and is easily implemented in computer algebra applications. Specifically, we define

\[ T_{\alpha\beta}^{\delta\epsilon} = \begin{cases} 0 & \text{if } \alpha \neq \delta \text{ or } \beta \neq \epsilon, \\ -1 & \text{if } (\alpha = \beta) > 2p, \\ +1 & \text{otherwise.} \end{cases} \]  

(A.1)

The Leibniz rule for supercommutators (here we are working in \(\mathfrak{osp}(Q|2p)\) with generators \(X_{\alpha}\), where indices are orthosymplectic superindices—see section 4) is then simply

\[ [X_{\alpha}, X_{\beta} X_{\gamma}] = [X_{\alpha}, X_{\beta}] X_{\gamma} + T_{\alpha\beta}^{\delta\epsilon} X_{\epsilon}[X_{\delta}, X_{\gamma}], \]  

(A.2)

while for supercommutators of bilinears we find

\[ [A_{\alpha\beta}, X_{\gamma} X_{\delta}] = [A_{\alpha\beta}, X_{\gamma}] X_{\delta} + T_{\alpha\beta}^{\mu\nu} T_{\mu\nu}^{\gamma\delta} X_{\mu}[A_{\mu\nu}, X_{\delta}]. \]  

(A.3)

Symmetry or antisymmetry in two orthosymplectic indices depending on their value reads

\[ A_{(\alpha\beta)} B_{\gamma} = \frac{1}{2} (A_{\alpha\beta} B_{\gamma} + T_{\alpha\beta}^{\gamma\mu} A_{\mu} B_{\gamma}) \]  

(A.4)

\((T\) replaces the standard \((-)_{\alpha\beta}\) symbol). The following set of identities are most useful:

\[
\begin{align*}
T_{\alpha\gamma}^{\mu\nu} T_{\mu\nu}^{\alpha\beta} &= \delta_{\alpha}^{\beta} \delta_{\gamma}^{\delta}, \\
J_{\alpha\beta} T_{\alpha\beta}^{\gamma\delta} &= -J_{\gamma\delta}, \\
T_{\alpha\beta}^{\mu\nu} J_{\alpha\beta} &= -J_{\mu\nu}, \\
J_{\gamma\delta} J_{\beta\gamma} &= -T_{\alpha\beta}^{\mu\nu}, \\
J_{\alpha\beta} T_{\alpha\beta}^{\gamma\delta} J_{\gamma\delta} &= T_{\alpha\beta}^{\gamma\delta} J_{\gamma\delta}, \\
T_{\gamma\delta}^{\mu\nu} T_{\alpha\beta}^{\gamma\delta} J_{\gamma\delta} &= T_{\alpha\beta}^{\mu\nu} J_{\mu\nu}.
\end{align*}
\]

Appendix B. Explicit algebras

We present the explicit commutation algebras of the \(\mathfrak{osp}(4)\) operators from sections 3.1 and the \(\mathfrak{osp}(Q|2p+2)\) operators of section 4 in figures B1 and B2, respectively. Note that the operators \(N\) and \(\text{ord}\) used here are defined by \(N = N + d/2\), \(\text{ord} = \text{ord} - d/2\), where \(d\) represents the dimension of the manifold in question. These operators are Lie algebra elements.
Table B2. Explicit supercommutation algebra of the operators from section 4. All results apply to arbitrary curved manifolds, except those labeled *, which are contingent upon application of the hyperhomothety condition from equation (1). Supercommutators are of the form (left column top row). Note that the symmetrizations on the result for $[f_{αβ}, f_{γδ}]$ implies symmetry of the form $(βα] and [γδ].$

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