Self-Consistent Approximations to Non-Equilibrium Many-Body Theory

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Abstract

Within the non-equilibrium Green’s function technique on the real time contour, the $\Phi$-functional method of Baym is reviewed and generalized to arbitrary non-equilibrium many-particle systems. The scheme may be closed at any desired order in the number of loops or vertices of the generating functional. It defines effective theories, which provide a closed set of coupled classical field and Dyson equations, which are self-consistent, conserving and thermodynamically consistent. The approach permits to include unstable particles and therefore unifies the description of resonances with all other particles, which obtain a mass width by collisions, decays or creation processes in dense matter. The inclusion of classical fields enables the treatment of soft modes and phase instabilities. The method can be taken as a starting point for adequate and consistent quantum improvements of the in-medium rates in transport theories.

1 Introduction

Non-equilibrium Green’s function technique, developed by Schwinger, Kadanoff, Baym and Keldysh \cite{1-4}, is the appropriate concept to study the space–time evolution of many-particle quantum systems. This formalism finds now applications in various fields, such as quantum chromodynamics \cite{5}, nuclear physics \cite{6-12}, astrophysics \cite{10,13,14}, cosmology \cite{15}, spin systems \cite{16}, lasers \cite{17}, physics of plasma \cite{18,19}, physics of liquid $^3$He \cite{20}, critical phenomena, quenched random systems and disordered systems \cite{21}, normal metals and super-conductors \cite{13,22,23}, semiconductors \cite{24}, tunneling and secondary emission \cite{25}, etc.

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For actual calculations certain approximation steps are necessary. In many cases perturbative approaches are insufficient, as for physical systems with strong couplings, e.g., like those treated in nuclear physics, in physics of liquids $^3$He and $^4$He, or high-temperature super-conductivity, etc. In such cases, one has to re-sum certain sub-series of diagrams in order to obtain a reasonable approximation scheme. In contrast to perturbation theory for such re-summations one frequently encounters the complication that the resulting equations of motion, even though self-consistent, may no longer comply with the conservation laws, e.g., of charge, energy and momentum. This problem has first been considered in two pioneering papers by Baym and Kadanoff [26,27] discussing the response to an external perturbation of quantum systems in thermodynamic equilibrium. Baym, in particular, showed [27] that any approximation, in order to be conserving, must be so-called $\Phi$-derivable. Thereby, he exploited the properties of an auxiliary functional, the $\Phi$-functional, introduced by Luttinger and Ward [28] a year earlier for the formulation of the thermodynamic potential (see also [29]). Thereby the $\Phi$-functional is determined in terms of full, i.e. re-summed, Green’s functions and free vertices. The scaling parameter of the vertices can be considered as an expansion parameter of a given approximation level. In the non-equilibrium formalism the problem of conserving approximations could be even more severe than in the case of the systems response to an external perturbation close to thermal equilibrium, since the system may exercise a rather violent evolution. Apart from transport models, mostly based on the quasi-particle approximation like Landau’s Fermi liquid theory, there were only few attempts to discuss the issue of conserving approximations in the context of the non-equilibrium field theory (see, e.g., [2,6,9]), which mainly considered Hartree-Fock and T-matrix approximations. However, the general problem of constructing conserving approximations in the non-equilibrium case and, in particular, beyond the quasi-particle limit has not explicitly been addressed yet.

Alongside, the question of thermodynamic consistency is vital. If, as a result of a non-equilibrium evolution, a system arrives at an equilibrium state, the non-equilibrium Green’s functions should properly describe thermodynamic quantities and potentials, such that thermodynamic relations between them are preserved. This problem is also relevant to the thermodynamic Green’s function technique, as already considered by Baym [27]. Baym demonstrated that any $\Phi$-derivable approximation is at the same time thermodynamically consistent.

In this paper we re-address the above problems and extend the concept i) to the genuine non-equilibrium case formulated on the closed real-time contour, ii) to relativistic field-theory Lagrangians in principle of arbitrary type (not just two-body fermion interactions) and iii) to the inclusion of classical fields, i.e. non-vanishing expectation values of the field operators. The generalized scheme permits to construct self-consistent, approximate, coupled dynamical equations of motion for the classical fields and Green’s functions of the system on the closed real-time contour. This set of equations is conserving and thermodynamically consistent. Thereby, the inclusion of classical fields permits to account for the phase-transition phenomena or to describe the coherent dynamics of soft modes, much in the spirit of hard-thermal-loop re-summations [30,31,12]. Avoiding the quasi-particle limit enables us to appropriately consider the finite mass-width of all constituents in the dense
matter environment. The latter aspect unifies the description of resonances, which have already a mass-width in vacuum, with all particles, which acquire a dynamical width during the collision processes in the dense matter. The proper account of the finite width of the particles is also vital for a non-singular treatment of the soft-mode problem [12], where space-time coherence effects, like the Landau–Pomeranchuk–Migdal effect, defer the use of zero width quasi-particles and require non-perturbative re-summations. In this paper we confine the presentation to the derivation of the closed set of self-consistent couple Kadanoff-Baym and classical field equations. This constitutes the basis for various further steps towards classical-type transport schemes through the gradient approximation, which will be presented in a forthcoming paper[38].

For the sake of clarity, we restrict the presentation to systems of relativistic scalar bosons. This allows us to formulate the basic ideas in a simple and transparent form. The resulting relations can directly be generalized to multi-component systems of relativistic bosons and fermions. In sect. 2 we introduce the general equations of motion and the expressions for the conserved quantities on the operator level. The equations of motion of the corresponding expectation values are formulated within the real-time closed contour formalism (sect. 3). Thereby, it is advantage to formulate the concepts in terms of generating functionals on the non-equilibrium contour, where the special functional \(\Phi\) plays a central role (sect. 4). This generating functional takes the same status in the space of Green’s functions (two-point functions) and classical fields (one-point functions), as the original Lagrangian for the field operators. Subsequently, we formulate the diagrammatic representation for \(\Phi\) (sect. 5). We show that any approximation, where all classical field sources and self-energies are \(\Phi\)-derivable in the sense of a variational principle, has the following properties: (i) it is conserving, (ii) it provides conserved current and energy-momentum tensor, which are identical to the corresponding Noether quantities (sect. 6), and (iii) it is at the same time thermodynamically consistent (sect. 7). In the summary, we formulate the main results and briefly discuss extensions and applications of the derived formalism. The list of diagrammatic rules is deferred to the Appendix A, while Appendix B contains some helpful equilibrium relations.

2 Energy-Momentum Tensor and Conserved Currents

We consider a system of relativistic scalar bosons, specified by the free Klein-Gordon Lagrangians

\[
\hat{\mathcal{L}}^0 = \begin{cases} 
\frac{1}{2} \left( \partial_{\mu} \hat{\phi} \cdot \partial^{\mu} \hat{\phi} - m^2 \hat{\phi}^2 \right) & \text{for neutral bosons,} \\
\partial_{\mu} \hat{\phi}^{\dagger} \cdot \partial^{\mu} \hat{\phi} - m^2 \hat{\phi}^{\dagger} \hat{\phi} & \text{for charged bosons,} 
\end{cases}
\]  

(2.1)

where \(\hat{\phi}(x)\) and \(\hat{\phi}^{\dagger}(x)\) are bosonic field operators. The convention of units is such that \(\hbar = c = 1\). The interaction Lagrangians \(\hat{\mathcal{L}}^{\text{int}}\{\hat{\phi}\}\) (for neutral bosons) and \(\hat{\mathcal{L}}^{\text{int}}\{\hat{\phi}, \hat{\phi}^{\dagger}\}\)
(for charged bosons) are assumed to be local, i.e. without derivative coupling. Under these conditions the Lagrangians are charge symmetric. Charges are understood as the electric charge, strangeness, iso-spin, etc.

The variational principle of stationary action leads to the Euler–Lagrange equations of motion for the field operators

\[ - S_x \hat{\phi}(x) = \hat{J}(x) = \frac{\partial \hat{\mathcal{L}}^{\text{int}}}{\partial \hat{\phi}^\dagger}, \quad \text{where} \quad S_x = -\partial_\mu \partial^\mu - m^2, \]  

(2.2)

and similarly for the corresponding adjoint equation. Thereby, the \( \hat{J}(x) \) operator is a local source current of the field \( \hat{\phi} \), while \( S_x \) is the differential operator of the free evolution with the free propagator \( \Delta^0(y, x) \) as resolvent.

The standard canonical energy-momentum tensor [32] has some undesired features, as it is non-symmetric in the Lorentz indices, for example. Alternatively, using the Euler–Lagrange equations of motion and the definition of the source current (2.2), one can show that the following form also defines a conserving energy momentum tensor

\[
\hat{\Theta}_{\mu\nu}^{(\text{alt.})}(x) = -\frac{1}{2} \left[ \left( \frac{1}{2} \right)_{\text{neut.}} \left( \partial_x^\nu - \partial_y^\nu \right) \left( \frac{\partial \hat{\mathcal{L}}^0(x)}{\partial \hat{\phi}} \hat{\phi}(y) - \hat{\phi}^\dagger(x) \frac{\partial \hat{\mathcal{L}}^0(y)}{\partial \hat{\phi}^\dagger} \right) \right]_{x=y} + \eta^{\mu\nu} \left( \hat{\mathcal{E}}^{\text{int}}(x) - \hat{\mathcal{E}}^{\text{pot}}(x) \right) \]  

(2.3)

with \( \partial_\mu \hat{\Theta}_{\mu\nu}^{(\text{alt.})}(x) = 0. \)

For notational simplicity, expression (2.3) and similar expressions below, which appear symmetric in \( \hat{\phi} \) and \( \hat{\phi}^\dagger \), are written in such a way that they directly apply to complex fields. The symbol \( (1/2)_{\text{neut.}} \) implies that for real fields the corresponding expressions are obtained from those for complex fields by multiplying by the factor \( 1/2 \) upon equating \( \hat{\phi}^\dagger = \hat{\phi} \). Above, we have introduced operators of the interaction energy density \( \hat{\mathcal{E}}^{\text{int}}(x) \) of the system, which accounts for the total interaction part of the energy density, and the potential energy density \( \hat{\mathcal{E}}^{\text{pot}}(x) \), both given as

\[
\hat{\mathcal{E}}^{\text{int}}(x) = -\hat{\mathcal{L}}^{\text{int}}(x),
\]

\[
\hat{\mathcal{E}}^{\text{pot}}(x) = -\frac{1}{2} \left( \frac{1}{2} \right)_{\text{neut.}} \left( \frac{\partial \hat{\mathcal{L}}^{\text{int}}}{\partial \hat{\phi}} \hat{\phi}(x) + \hat{\phi}^\dagger(x) \frac{\partial \hat{\mathcal{L}}^{\text{int}}}{\partial \hat{\phi}^\dagger} \right) 
= -\frac{1}{2} \left( \frac{1}{2} \right)_{\text{neut.}} \left( \hat{J}^\dagger(x) \hat{\phi}(x) + \hat{\phi}^\dagger(x) \hat{J}(x) \right). \]  

(2.4)

For a multi-component system, the latter defines the sum of the potential energy densities of any field \( \hat{\phi}(x) \) with the currents \( \hat{J}(x) \) induced by the other fields in the system.

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In terms of the differential operator $\hat{p}_x^\mu = i \partial_x^\mu$, the alternative energy–momentum tensor of eq. (2.3) can be written in a charge symmetric form as

$$\hat{\Theta}^{\mu\nu}_{\text{(alt.)}}(x) = \frac{1}{4} \frac{1}{2} \left( \left( (\hat{p}_x^\nu)^* + \hat{p}_y^\nu \right) \left( (\hat{p}_x^\mu)^* + \hat{p}_y^\mu \right) \left( \hat{\phi}^\dagger(x)\hat{\phi}(x) + \hat{\phi}(y)\hat{\phi}^\dagger(x) \right) \right)_{x=y} + g^{\mu\nu} \left( \hat{E}^{\text{int}}(x) - \hat{E}^{\text{pot}}(x) \right).$$

(2.5)

This form of the energy–momentum tensor is equal to the metric one, which results from variation of the action over the metric tensor $g_{\mu\nu}$ rather than over the fields. All terms in eq. (2.5) are evidently symmetric in $\mu\nu$.

For specific interactions, eq. (2.4) provides simple relations between $\hat{E}^{\text{int}}(x)$ and $\hat{E}^{\text{pot}}(x)$. If all vertices of $\hat{L}^{\text{int}}$ have the same number $\gamma$ of field operators attached, one simply deduces

$$\hat{E}^{\text{int}}(x) = \frac{2}{\gamma} \hat{E}^{\text{pot}}(x).$$

(2.6)

For instance, for the $\phi^4$-theory, where $\gamma = 4$, the interaction energy is half of the potential energy.

If the Lagrangian is invariant under some global transformation of charged fields (with the charge $e$), e.g.,

$$\hat{\phi}(x) \Rightarrow e^{-ie\Lambda} \hat{\phi}(x); \quad \hat{\phi}^\dagger(x) \Rightarrow e^{ie\Lambda} \hat{\phi}^\dagger(x),$$

(2.7)

there exists a Noether current defined as [32]

$$\hat{j}^{\mu}_{\text{(Noether)}} = \frac{\partial \hat{L}(\Lambda(x), \partial_\mu \Lambda(x))}{\partial (\partial_\mu \Lambda(x))} \bigg|_{\Lambda(x)=0},$$

(2.8)

which is conserved, i.e. $\partial_\mu \hat{j}^{\mu}_{\text{(Noether)}} = 0$. Formally, it is derived by applying the local transformation of the form (2.7) and using the stationary condition of the action around physical solutions.

From the Euler–Lagrange equations of motion (2.2), one obtains

$$\partial_\mu \hat{j}^{\mu}_{\text{(Noether)}} = -ie \left( \hat{j}^\dagger(x)\hat{\phi}(x) - \hat{\phi}^\dagger(x)\hat{j}(x) \right),$$

(2.9)

which vanishes, since for the symmetry (2.7) the interaction Lagrangian has to consist of terms, where $\hat{\phi}(x)$ and $\hat{\phi}^\dagger(x)$ appear pairwise. In terms of the $\hat{p}_x^\mu = i \partial_x^\mu$ operator, the
\( \tilde{j}_{\text{(Noether)}}^\mu \) current can again be written in a charge symmetric form as

\[
\tilde{j}_{\text{(Noether)}}^\mu (x) = e \frac{1}{2} \left[ \left( (\hat{p}_x^\mu)^* + \hat{p}_y^\mu \right) \left( \hat{\phi}^\dagger (x) \hat{\phi} (y) + \hat{\phi} (y) \hat{\phi}^\dagger (x) \right) \right]_{x=y},
\]

which naturally vanishes for the neutral particles \( (e = 0) \).

One may also define the tensor \( M^{\mu\nu\rho} \), which is associated with the Lorentz invariance of the Lagrangian and provides the angular momentum conservation. However, we do not treat this tensor in this paper, since it is of no common use in kinetics.

### 3 Real-Time Contours

In the non-equilibrium case, one assumes that the system has been prepared at some initial time \( t_0 \) described in terms of a given density operator \( \hat{\rho}_0 = \sum_a P_a |a\rangle \langle a| \), where the \( |a\rangle \) form a complete set of eigenstates of \( \hat{\rho}_0 \). All observables can be expressed through \( n \)-point Wightman functions of Heisenberg operators \( \hat{A}(t_1), \ldots, \hat{O}(t_n) \) at some later times

\[
\left\langle \hat{O}(t_n) \ldots \hat{B}(t_2) \hat{A}(t_1) \right\rangle =: \text{Tr} \left( \hat{O}(t_n) \ldots \hat{B}(t_2) \hat{A}(t_1) \hat{\rho}_0(t_0) \right) = \sum_a P_a \langle a| \hat{O}(t_n) \ldots \hat{B}(t_2) \hat{A}(t_1) |a\rangle. \tag{3.1}
\]

Note the fixed operator ordering for Wightman functions.

Figure 1: Closed real-time contour with two external points \( x, y \) on the contour.

The non-equilibrium theory can entirely be formulated on a \emph{special} contour—the \emph{closed real-time contour} (see figure 1) with the time argument running from \( t_0 \) to \( \infty \) along the \emph{time-ordered} branch and back to \( t_0 \) along the \emph{anti-time-ordered} branch. Contour-ordered multi-point functions are defined as expectation values of contour ordered products of operators

\[
\left\langle \mathcal{T}_c \hat{A}(x_1) \hat{B}(x_2) \ldots \right\rangle = \left\langle \mathcal{T}_c \hat{A}_1(x_1) \hat{B}_1(x_2) \ldots \exp \left\{ i \int_c \hat{L}_1^{\text{int}} \, dx \right\} \right\rangle, \tag{3.2}
\]
where $T_C$ denotes the special time-ordering operator, which orders the operators according to a time parameter running along the time contour $C$. The l.h.s. of eq. (3.2) is written in the Heisenberg representation, whereas the r.h.s. is given in the interaction (I) representation. Here and below, the subscript "I" indicates the interaction picture. Functions with $n$ points can be expressed in terms of products of other multi-point functions contour integrated over internal coordinates. Ultimately, one likes to express the multi-point functions of interest in terms of one- and two-point functions (Wick’s linked cluster expansion), i.e. in terms of classical fields, classical source-currents, propagators and self-energies. Note that at this level the contour is not a contour in the complex plane, as the figure may suggest, but rather it runs along real time arguments. It is through the placement of external points on the contour that the contour ordering obtains its particular sense.

In certain calculations, e.g., in those that apply the Fourier and Wigner transformations, it is necessary to decompose the full contour into its two branches—the time-ordered and anti-time-ordered branches. One then has to distinguish between the physical space-time coordinates $x, \ldots$ and the corresponding contour coordinates $x^C$ which for a given $x$ take two values $x^- = (x^\mu_\mu)$ and $x^+ = (x^\mu_\mu)$ ($\mu \in \{0, 1, 2, 3\}$) on the time ordered and anti-time ordered branches, respectively (see figure 1). Closed real-time contour integrations can then be decomposed as

$$
\int_C dx^C \ldots = \int_{t_0}^\infty dx^- \ldots + \int_{t_0}^\infty dx^+ \ldots = \int_{t_0}^\infty dx^- \ldots - \int_{t_0}^\infty dx^+ \ldots,
$$

(3.3)

where only the time limits are explicitly given. Thus, the anti-time-ordered branch acquires an extra minus sign if integrated over physical times. For any two-point function $F$, the contour values are defined as

$$
F^{ij}(x, y) := F(x^i, y^j), \quad i, j \in \{-, +\},
$$

(3.4)

on the different branches of the contour. The contour $\delta$-function is defined as

$$
\delta_C(x^i, y^i) = \sigma^{ij} \delta^4(x - y), \quad \sigma^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(3.5)

where, $\sigma_{ik}$ is the $\sigma_3$-Pauli matrix. For any multi-point function, the external point $x_{max}$, which has the largest physical time, can be placed on either branch of the contour without changing the value, since the contour-time evolution from $x_{max}^-$ to $x_{max}^+$ provides unity. Therefore, one-point functions have the same value on both sides of the contour.

Due to the change of operator ordering, genuine multi-point functions are discontinuous in general, when two contour coordinates become identical. In particular, two-point functions
like $iF(x,y) = \langle T_c \hat{A}(x) \hat{B}(y) \rangle$ become

$$iF(x,y) = \begin{pmatrix} iF^{--}(x,y) & iF^{-+}(x,y) \\ iF^{-+}(x,y) & iF^{++}(x,y) \end{pmatrix} = \begin{pmatrix} \langle T \hat{A}(x) \hat{B}(y) \rangle & \langle \hat{B}(y) \hat{A}(x) \rangle \\ \langle \hat{A}(x) \hat{B}(y) \rangle & \langle T^{-1} \hat{A}(x) \hat{B}(y) \rangle \end{pmatrix}, \quad (3.6)$$

where $T$ and $T^{-1}$ are the usual time and anti-time ordering operators. Since there are altogether only two possible orderings of the two operators, in fact given by the Wightman functions $F^{-+}$ and $F^{++}$, which are both continuous, not all four components of $F$ are independent. From eq. (3.6) follow relations between non-equilibrium and usual retarded and advanced functions

$$F^R(x,y) = F^{--}(x,y) - F^{-+}(x,y) = F^{++}(x,y) - F^{++}(x,y)$$
$$:= \Theta(x_0 - y_0) \left( F^{--}(x,y) - F^{-+}(x,y) \right),$$
$$F^A(x,y) = F^{--}(x,y) - F^{-+}(x,y) = F^{++}(x,y) - F^{++}(x,y)$$
$$:= -\Theta(y_0 - x_0) \left( F^{--}(x,y) - F^{++}(x,y) \right), \quad (3.7)$$

where $\Theta(x_0 - y_0)$ is the step function of the time difference.

Discontinuities of a two-point function may cause problems for differentiations, in particular, since they often occur simultaneously in products of two or more two-point functions. The proper procedure is, first, with the help of eq. (3.7) to represent the discontinuous parts in $F^{--}$ and $F^{++}$ by the continuous $F^{--}$ and $F^{++}$ times $\Theta$-functions, then to combine all discontinuities, e.g. with respect to $x_0 - y_0$, into a single term proportional to $\Theta(x_0 - y_0)$, and finally to apply the differentiations. One can easily check that in the following particularly relevant cases

$$\int_C dz \left( F(x^i,z)G(z,x^j) - G(x^i,z)F(z,x^j) \right), \quad (3.8)$$
$$\frac{\partial}{\partial x_\mu} \int_C dz \left( F(x^i,z)G(z,x^j) + G(x^i,z)F(z,x^j) \right), \quad (3.9)$$
$$\left[ \left( \frac{\partial}{\partial x_\mu} - \frac{\partial}{\partial y_\mu} \right) \int_C dz \left( F(x^i,z)G(z,y^j) - G(x^i,z)F(z,y^j) \right) \right]_{x=y} \quad (3.10)$$

_all discontinuities exactly cancel._ Thereby, these values are independent of the placement

1 Quite commonly, like in refs. [2,6], the notation $F = \begin{pmatrix} F^c & F^< \\ F^> & F^a \end{pmatrix}$ is used for two-point functions instead of (3.6). We prefer the more flexible \{-+\} labeling of contour points.
of $x^i$ and $x^j$ on the contour, i.e. the values are only a function of the physical coordinate $x$.

Boson fields may take non-vanishing expectation values of the field operators $\phi(x) = \langle \hat{\phi} \rangle$, called classical fields. The corresponding equations of motion are provided by the ensemble averaging of the operator equations of motion (2.2)

$$S_x \phi(x) = -J(x), \quad \text{or} \quad \phi(x) = \phi^0(x) - \int \delta y \Delta^0(x, y) J(y). \quad (3.11)$$

Here $J(x) = \langle \hat{J}(x) \rangle$, while $\phi^0(x) = \langle \hat{\phi}_1(x) \rangle$ is the freely evolving classical field which starts from $\phi(t_0, x)$ at time $t_0$. Thereby, $\Delta^0(x, y)$ is the free contour Green’s function

$$i\Delta^0(x, y) = \langle T_c \hat{\phi}_1(x) \hat{\phi}_1^\dagger(y) \rangle - \phi^0(x)\phi^0(y)^* \quad (3.12)$$

The reader can easily verify that the contour form (3.11) is equivalent to the standard retarded form of the classical field equations due to eq. (3.7) and the fact that $J(x)$ and $\phi(x)$ are one-point functions, which have identical values on both sides of the contour. The free propagator is resolvent of equation

$$S_x \Delta^0(x, y) = \delta_c(x, y) \quad (3.13)$$
on the contour, where $\delta_c(x^i, y^j)$ is the contour $\delta$-function (3.5). Subtracting the classical fields via

$$\hat{\phi} = \phi + \hat{\varphi}, \quad (3.14)$$
we define the full propagator in terms of quantum-fluctuating parts $\hat{\varphi}$ of the fields

$$i\Delta(x, y) = \langle T_c \hat{\varphi}(x) \hat{\varphi}^\dagger(y) \rangle = \langle T_c \hat{\phi}(x) \hat{\phi}^\dagger(y) \rangle - \phi(x)\phi^*(y) = \langle T_c \hat{\phi}(x) \hat{\phi}^\dagger(y) \rangle_c. \quad (3.15)$$

Here and below, the sub-label "c" indicates that uncorrelated parts are subtracted. In terms of diagrams it implies, that the corresponding expectation values are given by sums of entirely connected diagrams.

Averaging the operator equations of motion (2.2) multiplied by $\hat{\phi}^\dagger(x)$ and subtracting classical-field parts, one obtains the equation of motion for the propagators as

$$S_x \Delta(x, y) = \delta_c(x, y) + i \langle T_c \hat{J}(x) \hat{\varphi}^\dagger(y) \rangle_c, \quad (3.16)$$
where the contour $\delta$-function appears due to the contour ordering of the operators in $\Delta(x, y)$. 

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Eq. (3.16) is still exact and accounts for the full set of initial correlations contained in $\hat{\rho}_0$. In order to proceed, one may suggest that the typical interaction time $\tau_{\text{int}}$ for the change of the correlation functions is significantly shorter than the typical relaxation time $\tau_{\text{rel}}$, which determines the system evolution, when one neglects those initial correlations. Then, describing the system at times $t - t_0 \gg \tau_{\text{int}}$, one can neglect the initial correlations which are supposed to be dying at the time $\sim \tau_{\text{int}}$ in accordance with the Bogolyubov’s principle of the weakening of initial correlations. As a result one can apply the standard Wick decomposition dropping higher order correlations for the driving terms on the r.h.s. of both equations of motion (3.11) and (3.16). Then both driving terms can be expressed solely as functionals of the classical fields and one-particle propagators rather than on higher order correlations. In particular the driving term of (3.16) can then be expressed through the proper self-energy $\Pi$

$$
\langle T_C \hat{J}(x) \hat{\phi}^\dagger(y) \rangle_c = i \langle T_C \left[ \exp \left\{ i \int \frac{d\hat{L}^\text{int}}{\hat{\phi}_1(x)} \right\} \hat{J}_1(x) \hat{\phi}_1^\dagger(y) \right] \rangle_c
$$

$$
= i \int \frac{dz}{c} \left[ \frac{\partial}{\partial \hat{\phi}_1(z)} \right] \left[ \exp \left\{ i \int \frac{d\hat{L}^\text{int}}{\hat{\phi}_1(x)} \right\} \hat{J}_1(x) \right] \langle T_C \hat{\phi}(z) \hat{\phi}^\dagger(y) \rangle_{\text{cl}}
$$

$$
= \int \frac{dz}{c} \Pi(x, z) \Delta(z, y).
$$

The second line of eq. (3.17) results from the fact that the expectation value is connected and, therefore, in the first line, $\hat{\phi}_1^\dagger(y)$ has to be contracted with one of the operators $\hat{\phi}_1(z)$ occurring in $\exp \left\{ i \int \frac{d\hat{L}^\text{int}}{\hat{\phi}_1(x)} \right\} \hat{J}_1(x)$. The expression for the self-energy $\Pi$ deserves attention, since we have separated the full propagator in (3.17). In order to achieve the proper counting of terms, $\Pi$ has to be one-particle irreducible (label $c1$), i.e. the corresponding diagram cannot be split into two pieces, which separate $x$ from $z$ by cutting a single propagator line. Obviously, $\Pi$ may have singular (\(\delta\)-functional) one-point parts and genuine two-point parts (the latter are given by all connected one-particle irreducible diagrams of the current–current correlator)

$$
- i\Pi(x, y) = \left[ T_C \left( \frac{\partial^2 \hat{\phi}^\text{int}(x)}{\partial \hat{\phi} \partial \hat{\phi}^\dagger} \right) \right] \delta_C(x, y) - \left. \langle T_C \hat{J}(x) \hat{J}^\dagger(y) \rangle \right|_{\text{cl}},
$$

in the Heisenberg picture.

With the help of eqs (3.16) and (3.17), one recovers the Dyson equation in the differential form

\footnote{Actually, considering a dilute gas limit, Bogolyubov suggested the weakening of all the correlations, whereas we use a weaker assumption on the weakening of only short-range ($\sim \tau_{\text{int}}$) correlations, cf. [33].}
\[ S_x \Delta(x, y) = \delta c(x, y) + \int dz \Pi(x, z) \Delta(z, y), \quad (3.19) \]
\[ (S_y)^* \Delta(x, y) = \delta c(x, y) + \int dz \Delta(x, z) \Pi(z, y). \quad (3.20) \]

Using the resolvent property of the free propagator, we can write down these equations also in the integral form.

In diagrams, free and full classical fields are represented by "pins" with cross and "o-cross" as heads

\[ \phi^0(x) = \begin{array}{c} \times \\ \times \end{array} \quad \phi(x) = \begin{array}{c} \otimes \\ \otimes \end{array}, \quad (3.21) \]

while free and full propagators are given by thin and thick lines, respectively,

\[ i \Delta^0(x, y) = \begin{array}{c} \leftrightarrow \\ \leftrightarrow \end{array}_{x \rightarrow y} \quad i \Delta(x, y) = \begin{array}{c} \leftrightarrow \\ \leftrightarrow \end{array}_{x \rightarrow y} \quad (3.22) \]

Thereby, complex fields carry a sense, the arrow always pointing towards the \( \hat{\phi} \) in the contour ordered expressions. In diagrammatic representation, the classical-field equations (3.11) and Dyson equations (3.19) are given by

\[ \begin{array}{c} \otimes = \begin{array}{c} \times \\ \times \end{array} + \begin{array}{c} \leftrightarrow \\ \leftrightarrow \end{array} \; \i J \end{array}, \quad (3.23) \]
\[ \begin{array}{c} \leftrightarrow = \begin{array}{c} \leftrightarrow \\ \leftrightarrow \end{array} + \begin{array}{c} \leftrightarrow \\ \leftrightarrow \end{array} \; \i \Pi \end{array} \quad (3.24) \]

with the one- and two-point functions \( i J(x) \) and \( -i \Pi(x, y) \) as driving terms.

The averaged values of conserved quantities can be expressed in terms of the one- and two-point functions introduced so far. Averaging the operator value of the energy–momentum tensor of eq. (2.5), we arrive at

\[ \Theta^{\mu\nu}(x) =: \langle \Theta^{\mu\nu}_{(\text{alt.})} \rangle = \frac{1}{2} \left( \frac{1}{2} \right) \text{neut.} \left[ \left( (\hat{p}^\nu_x)^* + \hat{p}^\nu_y \right) \left( (\hat{p}^\mu_x)^* + \hat{p}^\mu_y \right) \left( \phi^*(x) \phi(y) + i \Delta^{\text{sym}}(y, x) \right) \right]_{x=y} \]
\[ + g^{\mu\nu} \left( \mathcal{E}^{\text{int}}(x) - \mathcal{E}^{\text{pot}}(x) \right), \quad (3.25) \]

where \( \mathcal{E}^{\text{int}}(x) = -\langle \hat{\mathcal{L}}^{\text{int}}(x) \rangle \), and the potential energy density becomes
\[ E_{\text{pot}}(x) = \left\langle \hat{E}_{\text{pot}}(x) \right\rangle = \frac{1}{2} \left( \frac{1}{2} \right)_{\text{neut.}} \left\{ -[J^*(x)\phi(x) + J(x)\phi^*(x)] \right. \\
+ i \int_C \text{d}z \left[ \Pi(x,z)\Delta(z,x) + \Delta(x,z)\Pi(z,x) \right] \bigg\} . \tag{3.26} \]

due to eqs (2.4) and (3.17). Note that we do not prescribe any contour indices to \( x \) in the integral term of \( E_{\text{pot}} \), since actually this term is independent of the contour placement of \( x \) due to discontinuity property (3.9). The Noether current (2.10) takes the form

\[ j^\mu(x) =: \langle \hat{j}^\mu_{(\text{Noether)}}, \rangle = e \left[ \left( \hat{p}_x^\mu + \hat{p}_y^\mu \right) \left( \phi^*(x)\phi(y) + i\Delta^{\text{sym}}(y,x) \right) \right]_{x=y} . \tag{3.27} \]

In order to keep the expressions \( \Theta^{\mu\nu} \) and \( j^\mu \) charge symmetric we have introduced the symmetric quantities

\[ F^{\text{sym}}(x,y) = \frac{1}{2} \left( F(x^-,y^+) + F(x^+,y^-) \right) , \tag{3.28} \]

where \( F(x,y) \) is any two-point function on the real-time contour. This is not automatically provided by the variational methods leading to the Noether energy–momentum tensor and current, as they only provide the vanishing of the corresponding divergence. The integrated form of the conserved quantities has to be adjusted such that charge symmetry is maintained, thus describing contributions of both, particles and anti-particles, on equal footing. This way one properly accounts for the modification of the vacuum polarization in the medium, since the vacuum-polarization energy coincides with the zero point energies of the field oscillations. The corresponding divergence has still to be appropriately renormalized.

4 Functionals \( W, \Gamma \) and \( \Phi \)

The standard generating functional for connected \( n \)-point functions is given by the logarithm of the expectation value of the time-evolution operator with external one-point sources \( \eta(x) \) and \( \eta^*(x) \)

\[ W \{ \eta, \eta^* \} = -i \ln \left\langle \mathcal{T} \exp \left[ i \int \text{d}x \left[ \eta(x)\hat{\phi}^\dagger(x) + \eta^*(x)\hat{\phi}(x) \right] \right] \right\rangle \]
\[ = -i \ln \left\langle \exp \left[ -i \int \text{d}t \hat{H}_1^0 \right] \mathcal{T} \exp \left[ i \int \text{d}x \left( \hat{\mathcal{L}}^\text{int}_1 + [\eta(x)\hat{\phi}^\dagger_1(x) + \eta^*(x)\hat{\phi}_1(x)] \right) \right] \right\rangle , \]

where \( \hat{H}_1^0 \) is the free Hamiltonian of the system in the interaction representation. Since the functional dependence concerns only the external sources \( \eta(x) \), the operator part can
be cast into different pictures, such as the Heisenberg or interaction ones. The latter establishes the perturbation expansion. Here and below, $\langle \ldots \rangle$ denotes a trace over all states, which includes the ensemble average over the density operator $\hat{\rho}_0 = \hat{\rho}(t_0)$ at initial time $t_0$ (cf. eq. (3.1)).

It is advantageous to introduce a scale function $\lambda(x)$, which scales interaction vertices of the interaction Lagrangian density $\hat{L}^\text{int}$ at space-time coordinate $x$ and defines a $\lambda(x)$ dependent interaction Lagrangian density

$$\hat{L}^\text{int}_\lambda = \lambda(x) \hat{L}^\text{int} \left\{ \hat{\phi}^\dagger(x), \hat{\phi}(x) \right\}. \quad (4.1)$$

This scaling provides the clue to the proof of the diagrammatic representation of the auxiliary functional $\Phi$ in terms of closed diagrams.

We generalize the above $W$ functional to the real-time contour $\mathcal{C}$. Following Luttinger and Ward [28] and Baym [27], we also extend it to include external bilinear sources $K(x, y)$, besides the interaction scale $\lambda(x)$,

$$W\{\eta, K, \lambda\} = \left. -\text{Im} \left\{ \exp \left[ -i \int_{\mathcal{C}} dx \hat{H}_1^0 \right] \mathcal{T}_\mathcal{C} \exp \left\{ i \int_{\mathcal{C}} dx \left( \lambda \hat{L}^\text{int}_\lambda \right) + \left( \frac{1}{2} \right)_{\text{neut.}} \left[ \eta(x) \hat{\phi}^\dagger_1(x) + \eta^*(x) \hat{\phi}_1(x) + i \int_{\mathcal{C}} dy \hat{\phi}_1(x) K(x, y) \hat{\phi}^\dagger_1(y) \right] \right\} \right| \right| \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
\[ \Delta \] rather than on the external sources \( \eta \) and \( K \) is provided by the double Legendre transformation of \( W \) to \( \Gamma \{ \phi, \phi^*, \Delta, \lambda \} \) given as

\[
\Gamma \{ \phi, \phi^*, \Delta, \lambda \} = W \{ \eta, K, \lambda \} - \left( \frac{1}{2} \right)_{\text{neut.}} \left[ \int_{\mathcal{C}} dx \left[ \eta^*(x) \phi(x) + \eta(x) \phi^*(x) \right] \right.
\]

\[
+ \int_{\mathcal{C}} dy \left[ \phi(y) \phi^*(x) + \Delta(y, x) \right] K(y, x) \right].
\] (4.4)

Here, the sources \( \eta \) and \( K \) have to be expressed through \( \phi \) and \( \Delta \). Apart from the \( \delta \lambda \) dependences, the functional variation

\[
\delta \Gamma \{ \phi, \phi^*, \Delta, \lambda \} = - \left( \frac{1}{2} \right)_{\text{neut.}} \left[ \int_{\mathcal{C}} dx \left[ \eta^*(x) \delta \phi(x) + \eta(x) \delta \phi^*(x) \right] \right.
\]

\[
+ \int_{\mathcal{C}} dy \left[ \delta \phi(y) \phi^*(x) + \phi(y) \delta \phi^*(x) + \delta \Delta(y, x) \right] K(y, x) \right] - \int_{\mathcal{C}} dx \mathcal{E}^{\text{int}}(x) \delta \lambda(x) / \lambda(x)
\] (4.5)

vanishes at vanishing external sources \( \eta \) and \( K \). The latter together with the condition \( \lambda = 1 \) corresponds to the physical solution. Note also that for the physical solution, i.e. at \( \eta = K = 0 \), the values of the two functionals \( \Gamma \) and \( W \) are identical. Indeed, the variations

\[
\delta \Gamma / \delta \phi = 0, \quad \delta \Gamma / \delta \phi^* = 0, \quad \delta \Gamma / \delta \Delta = 0
\] (4.6)

provide us with equations of motion for classical fields (3.11) and the complex conjugated one, as well as Dyson’s equation (3.19).

Following Luttinger, Ward [28], and later by Cornwell et al. [34], who used path-integral methods for case of the imaginary-time formulation of equilibrium systems, we represent our \( \Gamma \) functional related to the real-time quantities in the form

\[
\Gamma \{ \phi, \phi^*, \Delta, \lambda \} = \Gamma^0 + \int_{\mathcal{C}} dx \mathcal{L}^0 \{ \phi, \partial_\mu \phi \}
\]

\[
+ i \left( \frac{1}{2} \right)_{\text{neut.}} \left[ \ln \left( 1 - \otimes \Delta^0 \otimes \Pi \right) + \otimes \Delta \otimes \Pi \right] + \Phi \{ \phi, \phi^*, \Delta, \lambda \}, \quad (4.7)
\]

this way defining the auxiliary functional \( \Phi \{ \phi, \phi^*, \Delta, \lambda \} \). Constructing \( \Phi \) solely in terms of the fields and one-particle propagators complies with the assumption of ignoring higher

\[3\] Please note that our \( \Phi \) is different from the auxiliary quantity \( \Gamma_2 \) defined by Cornwell et. al. [34] to the extent that their \( \Gamma_2 \) is void of zero- and one-loop terms. These terms are rather
order correlations. The $\Gamma^0$ and $\mathcal{L}^0$ parts, where $\mathcal{L}^0$ is the free classical Lagrangian function, represent the non-interacting parts of $\Gamma$. Thereby, the $\Gamma^0$ term solely depends on the unperturbed propagator $\Delta^0$ and hence is treated as a constant with respect to functional variations of $\Gamma$. The $\ln(\ldots)$ is understood in the functional sense, i.e. by a series of $n$-folded contour convolutions, denoted by the $\odot$-symbol, formally resulting from the Taylor expansion of the $\ln(1+x)$ at $x = 0$. This ln-term accounts for the change of $\Gamma$ due to the self-energies of the particles. The $\Gamma^0$, $\mathcal{L}^0$ and ln terms in eq. (4.7) account for the one-body components in the $\Gamma$. The remaining $\Delta \odot \Pi$ and $\Phi$ terms correct for the true interaction energy part of $\Gamma$. As shown in the next section, the form (4.7) presents a re-summation of the corresponding perturbative expansion of the value of $\Gamma$ (cf. eq. (4.4)) in terms of full classical fields and propagators.

The specific form (4.7) has important functional properties, which provide us with a number of useful relations. Functional variation of $\Gamma \{\phi, \phi^*, \Delta, \lambda\}$ in the form of eq. (4.7) leads to

$$
\delta \Gamma \{\phi, \phi^*, \Delta, \lambda\} = \left(\frac{1}{2}\right)_{\text{neut.}} \int \mathcal{C} \text{d}x \left[ \delta \phi(x)(S_x)^* \phi^*(x) + \delta \phi^*(x)S_x \phi(x) \right] - \frac{1}{1 - \odot \Delta^0 \odot \Pi} \odot \Delta^0 - \odot \Delta \odot \delta \Pi + i \int \mathcal{C} \text{d}x \text{d}y \Pi(x, y) \delta \Delta(y, x) + \delta \Phi \{\phi, \phi^*, \Delta, \lambda\}. \tag{4.8}
$$

Here $\delta \Pi$ is understood as a variation induced by $\delta \Delta$, $\delta \phi$, $\delta \phi^*$, and $\delta \lambda$, respectively. Eqs (4.5) and (4.6) imply the following variational rules for the auxiliary $\Phi$ functional

$$
\delta \Phi \{\phi, \phi^*, \Delta, \lambda\} = \left(\frac{1}{2}\right)_{\text{neut.}} \int \mathcal{C} \text{d}x \left[ J^*(x) \delta \phi(x) + J(x) \delta \phi^*(x) \right] - i \int \mathcal{C} \text{d}x \text{d}y \Pi(x, y) \delta \Delta(y, x) - \int \mathcal{C} \text{d}x \mathcal{E}^{\text{int}}(x) \delta \lambda(x), \tag{4.9}
$$

or

$$
i J(x) = \frac{\delta i \Phi}{\delta \phi^*(x)}, \quad \tag{4.10}
$$

$$
- i \Pi(x, y) = \frac{\delta i \Phi}{\delta i \Delta(y, x)} \times \begin{cases} 2 & \text{for neutral bosons,} \\ 1 & \text{for charged bosons,} \end{cases} \tag{4.11}
$$

included in defining a full classical Lagrangian that depends on the classical fields and tad-poles, while we have placed all interaction parts into $\Phi$. 

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The virtue of the functional form (4.7) is that these requirements can be met simultaneously and that there exists a unique form of \( \Phi \), for which the three derived quantities—the one-body source current \( J(x) \), the two-body self-energy \(-i\Pi(x, y)\) and the interaction energy density \( \mathcal{E}^{\text{int}}(x) \)—take their physical values at the physical solutions of the equations of motion (3.19) and (3.11). This will become clear in more detail in the next section, where we discuss the diagrams defining the various functionals. In its turn, \( \Phi \) can be seen as a generating functional for the source terms \( J \) of classical fields and self-energies \( \Pi \) for the set of Dyson equations. Therefore, approximation schemes can be defined through a particular approximation to \( \Phi \). Thereby, the invariance properties of \( \Phi \) play a central role to define conservation laws for the approximate dynamics.

It is important to emphasize that we do all functional variations independently of any place on the contour. Thus, different contour times are considered as independent, even though they may refer to the same physical time\(^4\). In principle, all variational considerations given in this section apply to any kind of time contour, even to non-closed and complex ones as well as to any operator \( \hat{\rho}_0 \) defining the averaging \( \langle \ldots \rangle = \text{Tr} \{ \ldots \hat{\rho}_0 \} \) including the unit operator, as used in Matsubara’s imaginary-time formalism. For a particular choice of \( \hat{\rho}_0 \) and of a contour the physical values of \( W \) and \( \Gamma \) are identical for the corresponding physical solutions along this contour. In the imaginary-time method the value \( \Gamma = W \) takes the meaning of the thermodynamic partition sum. In this paper we concentrate on the non-equilibrium closed real-time formalism for which \( \text{Tr} \hat{\rho}_0 = 1 \), and therefore the physical values of \( W \) and \( \Gamma \) trivially vanish, i.e.

\[
W = \Gamma = \Phi = 0 \quad \text{for physical solutions of } \Delta, \phi, \phi^* \quad \text{on the contour.}
\]

An important comment must be given at this stage. One should clearly distinguish between the functional form of a functional, which acquires its meaning through variational methods, and the physical value that functional takes once the physical solutions of the equations of motion are inserted. For instance, two functionals \( W \) and \( \Gamma \) are completely different in their functional meaning, while they take the same physical value for the physical solution. Therefore, for all functionals the functional dependences are explicitly given in braces. Our strategy below will be first to perform general variations of \( \Gamma \) and \( \Phi \), allowing non-physical values of \( \Delta, \phi, \phi^* \) and \( \lambda \), and only then to put them to their physical values. This way, a number of important relations between Green’s functions, self-energies and mean fields will be obtained.

\(^4\) The fact that for the physical solutions the components of \( \Delta \) on the different branches of the contour are not independent (cf. (3.7)), has no importance for the variational procedure. The reason is that rules (3.7) only apply to the physical \( \Delta \) and \( \phi \), which are provided by the stationary “points” of the variational principle, i.e. solving the equations of motion (3.11), (3.19) and (3.20).
5 Diagrams for $\Gamma$, $\Phi$ and $\xi^\text{int}_\lambda(x)$

According to eqs (4.5) and (4.12), we have

$$- \int_c dx \xi^\text{int}(x) = \left[ \lambda \frac{d}{d\lambda} \Gamma\{\phi(\lambda), \phi^*(\lambda), \Delta(\lambda), \lambda\} \right]_{\lambda=1} = \left[ \lambda \frac{\partial}{\partial\lambda} \Phi\{\phi, \phi^*, \Delta, \lambda\} \right]_{\lambda=1}$$ (5.1)

where now $\lambda$ is treated as a global scale parameter (note that only a partial derivative is applied to $\Phi$, i.e. the $\phi$, $\phi^*$ and $\Delta$ values are kept constants.). In the perturbation theory, the diagrammatic rules to calculate the one-point function $\xi^\text{int}(x)$ are straightforward

$$- i \xi^\text{int}(x) = i \left\langle Tc \hat{L}^\text{int}_1(x) \exp \left[ i \int_c dx' \hat{L}^\text{int}_1(x') \right] \right\rangle = \sum_{n\lambda} \bigcirc \cdot c. $$ (5.2)

Here the diagram symbolically denotes all connected (label $c$) closed perturbation-theory diagrams generated by expanding the exponential function in (5.2). The full dot denotes the external point $x$ which is not integrated out. Integrating (5.1) with respect to $\lambda$, we define the quantity $i \bar{\Gamma}$ given by the following perturbative diagrammatic representation

$$i \bar{\Gamma}\{\phi^0, \phi^{0*}, \Delta^0, \lambda\} = i \Gamma^0\{\Delta^0\} + i \int_c dx \mathcal{L}^0\{\phi^0, \partial_\mu \phi^0\} + \sum_{n\lambda} \frac{1}{n\lambda} \bigcirc \cdot c, $$ (5.3)

where the integration constants have been chosen such that for physical solutions $\bar{\Gamma} = \Gamma$. One can see that each diagram contributing to $\bar{\Gamma}$ has to be weighted with its inverse number of vertices $1/n\lambda$, due to the formal $\lambda$-integration of (5.2). It is important to realize that due to these global factors such a set of diagrams is not resumable in the standard diagrammatic sense. Also $\bar{\Gamma}$ in the form of eq. (5.3) is a functional of $\phi^0, \phi^{0*}, \Delta^0, \lambda$ rather than of $\phi, \phi^*, \Delta, \lambda$, as needed for $\Gamma$ and $\Phi$ as discussed in the previous section. However, we can arrive at the required functional dependence of $\Gamma$ as follows. The expression (5.2) for $-i \xi^\text{int}(x)$ can be re-summed and entirely expressed in terms of full classical fields and full propagators. The re-summed diagrams are then void of any self-energy insertions and therefore have to be two-particle irreducible

$$- i \xi^\text{int}(x) = \sum_{n\lambda} \bigcirc \cdot c_{2}, $$ (5.4)

Diagrammatic re-summation implies that sub-diagrams with the same external structure (i.e. same number of external points and types of propagators to be attached at each external point) can be summed up to give a total re-summed expression that can then be embedded into more complicated diagrams, e.g. self-energy insertions can be re-summed to full Green’s functions.
Diagrams of class $c2$ cannot be decomposed into two pieces by cutting two propagator lines. The formal integration of the last equality in (5.1) with respect to $\lambda$ keeping $\phi$ and $\Delta$ constant provides the diagrammatic expression for $\Phi$ in terms of full Green’s functions and classical fields. Therefore, $i\Gamma \{\phi, \phi^*, \Delta, \lambda\}$ can be expressed in terms of the following diagrams (cf. eq. (4.7))

$$i\Gamma \{\phi, \phi^*, \Delta, \lambda\} = i\Gamma^0 \{\Delta^0\} + i \int \frac{dxL^0}{c} \{\phi, \partial_\mu \phi\} + \left(\frac{1}{2}\right)_{\text{neut.}} \sum \frac{1}{n_\Pi} \left\{ \begin{array}{c}
-\imath \Pi \\
-\imath \Pi \\
-\imath \Pi \\
\ldots \\
-\imath \Pi
\end{array} \right\} + \sum \frac{1}{n_\lambda} \left\{ \begin{array}{c}
\mathcal{I}\mathcal{I} \\
\mathcal{I}\mathcal{I}
\end{array} \right\}_{c2} + i\Phi. \quad (5.5)$$

Here $n_\Pi$ counts the number of $\Pi$ insertions in the ring diagrams providing the $\ln$-terms, while for the closed diagrams of $\Phi$ the value $n_\lambda$ counts the number of vertices building up the functional $\Phi$. Contrary to the perturbative diagrams of $i\bar{\Gamma}$, cf. eq. (5.3), here the diagrams contributing to $\Phi$ are given in terms of full propagators $\Delta$ and full time-dependent classical fields $\phi$. As a consequence, these diagrams have to be two-particle irreducible (label $c2$). The latter property is required because of the re-summations of $\mathcal{E}^\text{int}(x)$. This also matches the diagrammatic rules for the re-summed self-energy $\Pi(x, y)$, which results from functional variation of $\Phi$ with respect to any propagator $\Delta(y, x)$. In graphical terms, this variation is realized by opening a propagator line in all diagrams of $\Phi$. The resulting set of thus opened diagrams must then be that of proper skeleton diagrams of $\Pi$ in terms of full propagators, i.e. void of any self-energy insertion.

The diagrammatic rules for $\Phi$, $\mathcal{E}^\text{int}(x)$, $J$ and $\Pi$ are determined by the following steps:

(a) For all bosonic fields in $i\hat{\mathcal{L}}^\text{int}$, replace $\hat{\phi}$ by $\phi + \hat{\varphi}$ in order to account for the classical fields (cf. (3.14));

(b) consider all possible pair contractions of the field operator $\hat{\varphi}(x)$ with $\hat{\varphi}^\dagger(y)$ in the formal expressions (5.6)–(5.9) given below and replace them by $i\Delta(x, y)$;

(c) keep only those terms that correspond to two-particle irreducible diagrams for $\Phi$, i.e. which cannot be split into two pieces by cutting two different propagator lines.

Further details are given in Appendix A. The diagrams of $i\Phi$, $-i\mathcal{E}^\text{int}(x)$, $iJ(x)$ and $-i\Pi(x, y)$ are then generated by applying the above general rules to the following formal expressions

$$i\Phi = \left\langle \mathcal{T}_c \exp \left( i \int \frac{dx'}{c} \hat{\mathcal{L}}^\text{int}(x') \right) \right\rangle_{2c[\Delta]}$$

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\[
\begin{align*}
= \sum_{n} \frac{1}{n!} \int dx_{1} \ldots dx_{n} \left\langle T_{c} \mathcal{L}^{\text{int}}(x_{1}) \ldots i \mathcal{L}^{\text{int}}(x_{n}) \right\rangle_{2c\{\Delta\}},
\end{align*}
\]

\[
-i \mathcal{E}^{\text{int}}(x) = \left\langle T_{c} i \mathcal{L}^{\text{int}}(x) \exp \left( i \int c d x' \mathcal{L}^{\text{int}}(x') \right) \right\rangle_{2c\{\Delta\}},
\]

\[
i J(x) = \left\langle T_{c} \frac{\delta}{\delta \phi^{*}(x)} \exp \left( i \int c d x' \mathcal{L}^{\text{int}}(x') \right) \right\rangle_{2c\{\Delta\}},
\]

\[
-i \Pi(x, y) = \left\langle T_{c} \frac{\delta^{2}}{\delta \phi^{*}(x) \delta \phi(y)} \exp \left( i \int c d x' \mathcal{L}^{\text{int}}(x') \right) \right\rangle_{2c\{\Delta\}},
\]

where the sub-label \(2c\{\Delta\}\) refers to the above point (c).

As an example, we quote the diagrams in neutral scalar \(g \hat{\phi}^{4}/4!\) theory. Up to two vertices, the functional \(\Phi\) is given by the following expressions

\[
i \Phi = \frac{-ig}{4!} \int c \left( \phi^{4}(x) + 6 \phi^{2}(x) \langle \hat{\phi}(x) \hat{\phi}(x) \rangle_{c} + 3 \langle \hat{\phi}(x) \hat{\phi}(x) \rangle_{c}^{2} \right)
\]

\[
+ \frac{1}{2} \left( \frac{-ig}{4!} \right)^{2} \int c \int d y \left( 4 \cdot 4! \phi(x) \phi(y) \langle \hat{\phi}(x) \hat{\phi}(y) \rangle_{c}^{3} + 4! \langle \hat{\phi}(x) \hat{\phi}(y) \rangle_{c}^{4} + \ldots \right),
\]

or in terms of diagrams (cf. also Appendix A)

\[
i \Phi = \begin{pmatrix}
\frac{1}{4!} + \frac{1}{2 \cdot 2!} + \frac{1}{2^{2} \cdot 2!} + \frac{1}{3!} + \frac{1}{4!} \end{pmatrix} + \frac{1}{3} \ldots
\]

The \(1/n_{\lambda}\) factors are explicitly given, while the combinatorial factors according to rule (vii) in Appendix A are given in square brackets below each diagram. Functional derivatives with respect to \(\phi\) (pins) and propagators (full lines), cf. eq.(4.9), determine the source \(J(x)\) of the classical field and the self-energy \(\Pi(x, y)\), respectively.
Small full dots define vertices which are to be integrated over, while big full dots specify the external points \( x \) or \( y \); the first two diagrams of \( \Pi(x, y) \) give the singular \( \delta_C(x, y) \) parts arising from classical fields and tad-poles.

### 6 Φ-Derivable Approximations and Invariances of Φ

The expressions for \( W, \Gamma \) and \( \Phi \) given so far are exact and represent a convenient formulation of the theory in terms of full propagators and self-energies. However, for any practical calculation one needs certain truncated approximate schemes. In the weak-coupling limit, one can restrict the perturbation series for the Green’s function to a certain order. Then, as far as conservation laws are concerned, one encounters no particular problems, as conserved quantities are conserved order by order in perturbation theory. On the other hand, such perturbative expansion may not be adequate, as, for example, in the strong coupling limit, where re-summation concepts have to be applied, which re-sum certain sub-series of diagrams to any order. For such re-summation schemes, the situation with conservation laws is not that obvious.

We consider so-called Φ-derivable approximations, first introduced by Baym [27] within the imaginary time method. Such approximations are constructed by confining the infinite diagrammatic series for \( \Phi \) either to a set of a few diagrams or to some sub-series of diagrams. Note that the approximate \( \Phi^{(\text{appr.})} \) itself is constructed in terms of “full” Green’s functions and “full” classical fields, where “full” now implies that we have to self-consistently solve the classical-field and Dyson equations with the driving terms derived from this \( \Phi^{(\text{appr.})} \) through relations (4.10) and (4.11). It means that even restricting ourselves to a single diagram in \( \Phi^{(\text{appr.})} \), in fact, we deal with a whole sub-series of diagrams in perturbation theory. Thereby, the term “full” takes the sense of the sum of this whole sub-series. Thus, a Φ-derivable approximation offers a natural way of introducing closed, and therefore self-consistent approximation schemes based on summation of diagrammatic sub-series. In order to preserve the symmetry of the exact Φ with respect to permutations among \( i\mathcal{L}^{\text{int}}(x_1) \ldots i\mathcal{L}^{\text{int}}(x_n) \) (see eq. (5.6)), we postulate that a Φ-derivable
approximation either takes into account all the diagrams up to a certain order \( n \) (in terms of “full” Green’s functions and classical fields) or confines the treatment to only those diagrams with topologically equivalent vertices. As a consequence, approximate forms of \( \Phi^{(\text{appr.})} \) define effective theories, where \( \Phi^{(\text{appr.})} \) serves as a generating functional for the approximate source currents \( J^{(\text{appr.})}(x) \) and self-energies \( \Pi^{(\text{appr.})}(x, y) \) (see eqs (4.10) and (4.11))

\[
i J^{(\text{appr.})}(x) = \frac{\delta i \Phi^{(\text{appr.})}}{\delta (\phi^{(\text{appr.})}* (x))},
\]

\[
-i \Pi^{(\text{appr.})}(x, y) = \frac{\delta i \Phi^{(\text{appr.})}}{\delta i \Delta^{(\text{appr.})}(y, x)} \times \begin{cases} 2 & \text{for neutral fields}, \\ 1 & \text{for charged fields}, \end{cases}
\]

which then are the driving terms for the equations of motion for the classical fields and propagators. The approximate \( \Phi \) also provides the corresponding expression for \( \mathcal{E}^{\text{int}} \) (see eq. (4.12)). Below, we omit the superscript “appr.”.

We now like to demonstrate that \( \Phi \)-derivable approximations possess a number of remarkable properties. For such approximations, the invariances of \( \Phi \) play as central a role as the invariances of the Lagrangian for the full theory. Thereby, the variational principle, where the interaction strength \( \lambda(x) \), the classical fields \( \phi(x) \), and propagators \( \Delta(x, y) \) can be varied independently, provides a set of useful identities and relations.

A general invariance of \( \Phi \) is provided by the substitution \( x \Rightarrow x + \xi(x) \) for all integration variables in the contour integrations defining \( \Phi \). The Jakobi determinant required for each integration variable can be accommodated by a simultaneous change of the scale function \( \lambda(x) \) at each vertex. Thus, the simultaneous variation

\[
\phi(x) \Rightarrow \phi(x + \xi(x)), \\
\Delta(x, y) \Rightarrow \Delta(x + \xi(x), y + \xi(y)), \\
\lambda(x) = 1 \Rightarrow \lambda(x) = \det \left( \delta^\nu_\mu + \frac{\partial \xi_\mu}{\partial x_\nu} \right), \quad \text{i.e.} \quad \delta \lambda(x) = \frac{\partial \xi_\mu}{\partial x_\nu},
\]

leaves \( \Phi \) invariant. This way, one deduces

\[
\delta \Phi = \left( \frac{1}{2} \right)_{\text{neut.}} \left\{ \int_C dx \left[ J^*(x) \frac{\partial \phi(x)}{\partial x_\mu} + J(x) \frac{\partial \phi^*(x)}{\partial x_\mu} \right] \xi_\mu(x) \\
-i \int_C dx \Pi(x, y) \left[ \frac{\partial \Delta(y, x)}{\partial x_\mu} \xi_\mu(x) + \frac{\partial \Delta(y, x)}{\partial y_\mu} \xi_\mu(y) \right] \\
- \int_C dx \mathcal{E}^{\text{int}}(x) \frac{\partial \xi_\mu}{\partial x_\mu} = 0.
\]

\[ (6.4) \]
Interchanging $x$ and $y$ in the second $\Pi$ term in squared brackets, using partial integration and that the transformation $\xi(x)$ can be chosen arbitrarily, one obtains the following relation

$$\frac{\partial}{\partial x_\mu} E^{\text{int}}(x) + \left( \frac{1}{2} \right)_{\text{neut.}} \left[ J^*(x) \frac{\partial \phi(x)}{\partial x_\mu} + J(x) \frac{\partial \phi^*(x)}{\partial x_\mu} \right] - i \left( \frac{1}{2} \right)_{\text{neut.}} \int dy \left[ \Pi(x, y) \frac{\partial \Delta(y, x)}{\partial x_\mu} + \frac{\partial \Delta(x, y)}{\partial x_\mu} \Pi(y, x) \right] = 0. \quad (6.5)$$

This is the key relation to prove energy-momentum conservation. It has features similar to a Ward identity, as it links derivatives of one-point functions with those of two-point functions. The two-point function contribution to this expression is of type of eq. (3.10), so that in eq. (6.5) the differentiations of the discontinuities, indeed, cancel out.

With the help of the equations of motion (3.11), (3.19) and (3.20), the divergence of the kinetic term of the energy-momentum tensor $\Theta^{\mu \nu}$ (3.25) can be cast into

$$\frac{1}{2} \left( \frac{1}{2} \right)_{\text{neut.}} \partial_\mu \left[ \left( \tilde{p}^*_x \right)^* + \tilde{p}^*_y \right] \left[ \left( \tilde{p}^\mu_x \right)^* + \tilde{p}^\mu_y \right] \left( \phi^*(x) \phi(y) + i \Delta^\text{sym}(y, x) \right)_{x=y}$$

$$= \partial_\mu g^{\mu \nu} E^\text{pot}(x) - \left( \frac{1}{2} \right)_{\text{neut.}} \left\{ \frac{1}{2} \right\} = [J(x) \partial_\nu \phi^*(x) + J^*(x) \partial_\nu \phi(x)]$$

$$+ i \int dz \left[ \Pi(x, z) \cdot \partial_\nu \Delta(z, x) + \partial_\nu \Delta(x, z) \cdot \Pi(z, x) \right]. \quad (6.6)$$

The remaining non-full-derivative terms on the r.h.s. of this equation are undesired, since they prevent us from presenting of the entire equation in the form of a full divergence. However, they can also be transformed into a full derivative form by means of identity (6.5). Adding this identity to eq. (6.6) cancels out the undesired last braced terms. Gathering all surviving terms of (6.6), we recognize it as the energy-momentum conservation law $\partial_\mu \Theta^{\mu \nu}(x) = 0$ with the energy-momentum tensor given by the Noether expression (3.25). Hence, the existence of a conserved energy-momentum tensor is proven for any $\Phi$-derivable approximation.

Along similar lines charge conservation can be proven, assuming that $\Phi$ is invariant under the following simultaneous variation of classical fields and propagators

$$\phi(x) \Rightarrow e^{-i e \Lambda(x)} \phi(x), \quad \phi^*(x) \Rightarrow e^{ie \Lambda(x)} \phi^*(x), \quad \Delta(x, y) \Rightarrow e^{-i e \Lambda(x)} \Delta(x, y) e^{ie \Lambda(y)}. \quad (6.7)$$

Applying the rule (4.9) of the $\Phi$ variation, to linear order in the phase $\Lambda$, one obtains

$$\delta \Phi = e \int \frac{dx}{c} [J^*(x) \phi(x) - J(x) \phi^*(x)] [-i \Lambda(x)]$$
Here we have used the variation rules for complex fields. Equating $\delta \Phi = 0$ for arbitrary $\Lambda$, we arrive at the identity

$$e \left[ J^*(x)\phi(x) - J(x)\phi^*(x) \right] + ie \int_C dy \left[ \Pi(x, y)\Delta(y, x) - \Delta(x, y)\Pi(y, x) \right] = 0. \quad (6.9)$$

Note that the integral term of this identity is independent of the contour placement of the $x$ variable due to discontinuity relation (3.8) and, therefore, it is only a function of the physical value of $x$.

By means of equations of motion (3.11), (3.19) and (3.20), the divergence of the Noether current of eq. (3.27) is seen to vanish

$$i\partial_\mu j^\mu = e \left[ J^*(x)\phi(x) - J(x)\phi^*(x) \right] + ie \int_C dy \left[ \Pi(x, y)\Delta(y, x) - \Delta(x, y)\Pi(y, x) \right] = 0, \quad (6.10)$$

according to eq. (6.9). Thus, we have arrived at the current conservation for any $\Phi$-derivable approximation, which is invariant under (6.7).

Similarly, one may derive the relation, resulting from the Lorentz invariance of the $\Phi$ functional, which permits to demonstrate the conservation of the angular momentum. However, we do not consider it here, since the angular-momentum conservation is not of practical use in kinetics.

Further invariances generally depend on the properties of the interaction vertices in the theory considered. For instance, other invariances result from the property of fixed number of field operators attached to the interaction vertices in a theory. This property provides relations between densities of the interaction $E^{\text{int}}(x)$ and potential $E^{\text{pot}}(x)$ energies. If the number $\gamma$ of field operators per vertex is fixed, the functional $\Phi$ is invariant under the simultaneous local scaling of the interaction vertices by $\lambda(x)$, and the classical fields and propagators by $(\lambda(x))^{-1/\gamma}$ and $(\lambda(x)\lambda(y))^{-1/\gamma}$, respectively. This invariance provides relation (2.6) at the level of expectation values.

7 Thermodynamic Consistency

In the thermal equilibrium the density matrix is explicitly known, cf. [35],

$$\hat{\rho}^{\text{eq}} = \frac{\exp \left( -\beta \hat{H} \{\mu\} \right)}{Z}, \quad (7.1)$$
where \( \beta = 1/T \) is the inverse temperature, and \( Z \) is the partition function which is directly related to the thermodynamical potential,

\[
\Omega = -T \ln Z. \tag{7.2}
\]

Since we deal now with thermodynamics, we have introduced the chemical potential \( \mu \) in the conventional way, i.e. by adding to the Hamiltonian the relevant term

\[
\hat{H}\{\mu\} = \hat{H} - \int d^3x \mu \hat{J}_{(\text{Noether})}(x), \tag{7.3}
\]

where \( \hat{J}_{(\text{Noether})} \) is the time-component of the charged current eq. (2.10), now with \( e = 1 \).

We can use the same trick as that in the Matsubara technique, i.e. use the fact that the equilibrium density matrix formally coincides with evolution operator in the imaginary time. In the definition of the \( W \) functional (4.2) we explicitly write \( \text{Tr} \hat{\rho}_{\text{eq}} \) instead of \( \langle \ldots \rangle \). Thus, taking into account that \( \Gamma = W \) at vanishing external sources, we arrive at the following form of \( \Gamma \) functional in equilibrium

\[
\Gamma_{\text{eq}}\{\phi, \phi^*, \Delta, \lambda, \mu\} = -i \ln \left( \frac{1}{Z} \text{Tr} \exp \left[ -i \int_{C_{\text{eq}}} dt \hat{H}_{1}^{0}\{\mu\} \right] \mathcal{T} \exp \left[ i \int_{C_{\text{eq}}} dx \lambda \hat{L}_{1}^{\text{int}}\{\mu\} \right] \right), \tag{7.4}
\]

with the integration contour \( C_{\text{eq}} \) now being the sum of the real-time Schwinger-Keldysh contour (see figure 1) and the imaginary-time Matsubara contour, i.e. it starts from an initial time \( t_0 \) goes to infinity, then back to this initial time and after that, to \( t_0 - i\beta \). Taking into account the fact that \( \Gamma = 0 \) for the physical values of \( \phi, \phi^*, \Delta, \) and \( x \)-independent \( \lambda \), we obtain for the value of the thermodynamic potential (7.2)

\[
\Omega\{\phi, \phi^*, \Delta, \lambda, \mu\} = -T \ln \left\{ \text{Tr} \left[ \exp \left[ -i \int_{C_{\text{eq}}} dt \hat{H}_{1}^{0}\{\mu\} \right] \mathcal{T} \exp \left[ i \int_{C_{\text{eq}}} dx \lambda \hat{L}_{1}^{\text{int}}\{\mu\} \right] \right] \right\}, \tag{7.5}
\]

where the integral over the real-time section of the contour gives zero. Hence, in eq. (7.5) we can make the replacement

\[
\int_{C_{\text{eq}}} dt \ldots = \int_0^{-i\beta} dt \ldots. \tag{7.6}
\]

Thus, we have arrived at the proper thermodynamic representation of the thermodynamic potential originally proposed by Luttinger and Ward [28]. Indeed, since all quantities under the integral are analytically continued from the Schwinger–Keldysh contour to the
Matsubara contour, $\Omega$ is determined by the same expression as the $\Gamma$ functional (4.7) but in terms of the Matsubara Green’s functions with the thermodynamic $\Phi_T$ functional represented by the same set of closed diagrams. Thus, in the momentum representation from eq. (7.5) we arrive at

$$\Omega \{\phi, \phi^*, \Delta, \lambda, \mu\} = -\int d^3x L_0^0 \{\phi, \partial_\mu \phi\} + T \left(\frac{1}{2}\right)_{\text{neut.}} \sum_{\omega_n} \int d^3x \frac{d^3p}{(2\pi)^3} \exp(i\omega_n \eta)$$

$$\times \left( -\ln[-\Delta(\omega_n, p)] + \Pi(\omega_n, p)\Delta(\omega_n, p) \right) + \Phi_T, \quad \eta \to 0, \quad (7.7)$$

where $\Phi_T = -iT\Phi$, $\omega_n = 2\pi nT$, and summation runs over Matsubara frequencies. In the standard way (see, e.g., ref. [29]) by converting the $\omega_n$-sum in eq. (7.7) into the energy integral expressed in terms of the real-time quantities of eqs (B.2) and (B.3), this thermodynamic potential is also easily expressed in terms of the real-time quantities

$$\Omega \{\phi, \phi^*, \Delta, \lambda, \mu\} = -\int d^3x L_0^0 \{\phi, \partial_\mu \phi\} + \left(\frac{1}{2}\right)_{\text{neut.}} \int d^3x \frac{d^4p}{(2\pi)^4} n(\varepsilon)$$

$$\times \left( -2\text{Im} \ln[-\Delta^R(\varepsilon + i0, p)] - \Re\Delta^R\Gamma - A\Re\Pi^R \right) + \Phi_T, \quad (7.8)$$

where

$$n(\varepsilon) = \left[\exp(\varepsilon/T) - 1\right]^{-1} \quad (7.9)$$

is the thermal Bose–Einstein occupation number. Thus, the problem of the thermodynamic consistency can immediately be re-addressed from the Schwinger–Keldysh approach to the Matsubara one. Within the Matsubara formalism, this problem was considered by Baym [27]. He has shown that any $\Phi$-derivable approximation to the thermodynamic potential is thermodynamically consistent. Hence, we have proved that our $\Phi$ derivable approximations to the $\Gamma$-functional are also thermodynamically consistent.

The stationary property of the $\Gamma$ functional (and, hence, of $\Omega$) with respect to variations in full Green’s functions and classical fields, eq. (4.6), is the key feature of $\Gamma$ that provides the thermodynamic consistency. It implies that any derivative of the thermodynamic potential to any thermodynamic parameter like $\beta$ or $\mu$ is then given by accounting only the explicit dependence of $\Omega$ on these parameters, since the implicit dependences through $\Delta$ and $\phi$ drop out due to the stationarity property. Therefore $\Phi$-derivable approximations preserve the corresponding thermodynamic relations as for the exact partition sum, and thus provide thermodynamic consistency.
8 Conclusion

Our aim was to develop a regular way of constructing self-consistent approximations to quantum transport theory within the formalism of non-equilibrium Green’s functions on the real-time contour, developed by Schwinger, Kadanoff, Baym and Keldysh [1–3].

We have employed functional methods for Green’s functions on the real-time contour. Our main result is the definition of a generating functional $\Phi$ which is determined by the sum of all closed (i.e. without external points) skeleton diagrams in terms of classical fields and full Green’s functions on the real-time contour. The main feature of this $\Phi$ functional is that it plays a similar role as the interaction Lagrangian but in the space of classical fields and full Green’s functions on the contour. This means that all important quantities of a system (such as sources of classical fields, self-energies, interaction energy, etc.) can be derived by respective variations of the $\Phi$ functional. Within thermodynamics, the $\Phi$ functional was introduced by Luttinger and Ward [28] and later used by Baym [27]. Our treatment extends the definition of the $\Phi$ functional to any non-equilibrium system. Following the thermodynamic treatment of Cornwell et al. [34], we have also extended the non-equilibrium $\Phi$ functional to the case of non-vanishing classical bosonic fields. This last generalization allows us to self-consistently describe the dynamics of both the order parameter (the classical field) and fluctuations on equal footing, e.g. in the theory of phase-transition phenomena.

The advantage of the $\Phi$ functional is that we may formulate various approximations at the level of $\Phi$, thus defining so called $\Phi$-derivable approximations. In particular, we may construct effective theories right at the level of Green’s functions and effective vertices. These approximations possess some important features: they respect exact conservation laws on the level of expectation values (with the Noether values for the conserved quantities) and have a proper thermodynamic limit. Note that other approximation schemes, e.g. at the level of self-energies, far not always possess such properties.

The question of consistency becomes especially important for a multi-component system, where the properties of one species can change due to the presence of interaction with the others and vice versa. The ”vice versa” is very important and corresponds to the principle of actio = re-actio. This implies that the self-energy of one species cannot be changed through the interaction with other species without affecting the self-energies of the latter ones also. The $\Phi$-derivable scheme offers a natural and consistent way to account for this principle. Within thermodynamic considerations this has recently been considered for the interacting pion - nucleon -delta-resonance system, where the coupling to the delta resonance leads to a softening of the pion modes below the resonance mass [36] and for a relativistic QED plasma in [37].

For the relativistic scheme considered here we argue that a careful construction of conserved quantities requires symmetric expressions in terms of $\Delta^{-+}$ and $\Delta^{+-}$ Green’s functions ($\Delta^<$ and $\Delta^>$ in the Kadanoff–Baym notation, respectively). This is in contrast to
expressions involving only $\Delta^{-+}$ Green’s function, which are often used in the literature. These symmetric expressions describe contributions of both particles and anti-particles on equal footing, as well as take a proper account of modification of the vacuum polarization in the medium. Of course, these symmetric expressions still require a proper vacuum renormalization to be done in any actual calculation.

We have done our consideration on the example of relativistic bosons, since they allow us to demonstrate the basic features of this approach — inclusion of both Green’s functions and classical fields into the scheme — without extra complications. The generalization to multi-component systems is straightforward. Here, we expect a consistent description of chiral $\sigma$, $\pi$-condensates together with fluctuations, as an immediate application. The generalization to vector mesons and/or fermions, though more technically involved due to the resulting tensor structure of the propagators and self energies is also straightforward. The formalism can also be generalized to theories with derivative couplings in the interaction Lagrangians. While the expressions for the equations of motion and the energy-momentum tensor receive additional terms from the derivative couplings, the diagrammatic rules are the same as given here.

The developed scheme of constructing self-consistent approximations provides a suitable basis for the derivation of kinetic equations beyond the limitations of the quasi-particle approximation. Such generalized transport schemes respect parts of the quantum nature of the particles and, in particular, take account of their finite mass-widths. The finite mass-width may be either inherent in a particle already from its vacuum properties (e.g., resonances) or may be acquired by a stable particle in a dense environment due to frequent interactions. In the case of nuclear collisions at intermediate ($\sim 1$ GeV/nucleon) to ultra-relativistic energies, one encounters mean single-particle energies in the range of the typical temperature of $T = 50 - 200$ MeV. Important resonances, like the delta-resonance or the rho-meson, have decay widths beyond $\sim 100$ MeV, while typical collision rates estimated from presently used quasi-particle transport schemes are typically in the order of $T$. These circumstances definitely prevent quasi-particle based transport codes from providing reliable results for such collisions. The main steps in the derivation of self-consistent and tractable transport equations for particles with finite width will be published in a forthcoming paper [38].

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A Diagram Rules for \( \Phi, J, \) and \( \Pi \)

The interaction vertex function \( V(x) \) entering the diagram is normalized in the standard way, c.f. [32], i.e. with factors \( n! \) relative to \( \hat{L}^\text{int}(x) \) for each type of operator occurring with multiplicity \( n \) in the vertex. E.g., the vertex function simply becomes \( -iV(x_k) = -ig \) for \( \hat{L}^\text{int} = -g\phi^4/4! \) (4 identical operators) and for \( \hat{L}^\text{int} = -g(\phi^*\phi)^2/(2! \cdot 2!) \) (twice two identical operators). The diagrammatic rules to calculate \( i\Phi, iJ(x) \) and \( -i\Pi \) for a given theory are as follows

(i) Draw all topologically distinct, closed and entirely connected diagrams with \( N \) internal vertices \( x_1, x_2, \ldots x_N \), where classical field pins and propagator lines saturate the valences of all vertices in the diagram, c.f. (5.11) above. Closed diagrams for \( i\Phi \) have no external points, while \( iJ(x) \) has one external point, and \( -i\Pi \) has two external points. For charged bosons, pins and propagator lines have an arrows sense, distinguishing \( \phi \) from \( \phi^\dagger \) at the vertices, the sense direction pointing towards \( \phi \).

(ii) For \( i\Phi, iJ(x) \) and \( -i\Pi \) keep only those diagrams that are two-particle irreducible, i.e. which cannot be split into two pieces by cutting two different propagator lines. For \( -i\Pi \) keep only those diagrams which result from \( \Phi \) by opening one propagator line.

(iii) To each line, connecting \( x_i \rightarrow x_k \), assign the factor \( i\Delta(x_k, x_i) \).

(iv) To each pin attached to \( x_k \), assign the factor \( \phi(x_k) \) or \( \phi^*(x_k) \) depending on the sense.

(v) To each vertex \( x_k \) assign the vertex factor \( -iV(x_k) \) as determined by \( \hat{L}^\text{int}(x) \).

(vi) Integrate all internal \( x_1, x_2, \ldots x_N \) over the contour.

(vii) Multiply the result by the symmetry factor \( S \), which is calculated as follows

1. \( 1/N_G! \) factor for every \( N_G \) equivalent internal lines,
2. \( 1/N_\phi! \) factor for every \( N_\phi \) classical fields entering each vertex,
3. \( 1/2 \) factor for every self-closed line loop (tad-poles) for real fields.

(viii) Sum all diagrams. For \( i\Phi \) (contrary to \( iJ(x) \) and \( -i\Pi \)), put the extra factor \( 1/n_\lambda \) for each diagram, where \( n_\lambda \) counts the number of vertices in the diagram.

In many cases like in transport treatments, it is advantageous to consider the diagrams decomposed into the two contour sections at each vertex, e.g., to calculate quantities like \( \Pi^- \) and \( \Pi^{+-} \) self-energies in terms of exact Green’s functions. Therefore, ”physical”-time diagrammatic rules in the matrix scheme are also required. Here we present only those rules which differ from the above ones on the real-time contour, bearing in mind that all other rules remain valid:

(iii’) To each internal vertex \( x_k \) first assign a sign \( i_k \in \{+, -\} \) defining the contour placement \( x_k^i \). To each line, connecting \( x_i^j \rightarrow x_k^i \), assign the factor \( i\Delta_{x_k^i}^{i_j}(x_i, x_k) \), \( i_k, i_l \in \{+, -\} \).

(vi’) For all internal points integrate all \( x_1, x_2, \ldots x_N \) over the real-time axis and space, for each internal ”+” vertex multiply by \( (-1) \) and finally sum over all internal contour placements \( i_1, i_2, \ldots i_N \) \( (i_k \in \{+, -\}) \).
B Equilibrium Relations

For completeness of the thermodynamic consideration, we explicitly present here equilibrium relations between quantities on the real-time contour. Basically, they follow from the Kubo–Martin–Schwinger condition \[39\]

\[
\Delta^{-+}(p) = \Delta^{+-}(p)e^{-\varepsilon/T}, \quad \Pi^{-+}(p) = \Pi^{+-}(p)e^{-\varepsilon/T},
\]

where \(\varepsilon = p_{\nu}U^\nu - \mu\) with \(U^\nu\) and \(\mu\) being a global 4-velocity of the system and a chemical potential related to the charge, respectively. All the Green’s functions can be expressed through retarded or advanced Green’s functions:

\[
\begin{pmatrix}
\Delta^{ij}(p) \\
\Pi^{ij}(p)
\end{pmatrix} =
\begin{pmatrix}
[1 + n(\varepsilon)] \Delta^R(p) - n(\varepsilon) \Delta^A(p) & -i n(\varepsilon) A(p) \\
-i [1 + n(\varepsilon)] A(p) & -[1 + n(\varepsilon)] \Delta^A(p) + n(\varepsilon) \Delta^R(p)
\end{pmatrix}
\]

\(i, j \in \{+, -\}\), and the self-energies take a similar form

\[
\begin{pmatrix}
\Pi^{ij}(p)
\end{pmatrix} =
\begin{pmatrix}
\Pi^R(p) - i n(\varepsilon) \Gamma(p) & -i n(\varepsilon) \Gamma(p) \\
-i [1 + n(\varepsilon)] \Gamma(p) & -\Pi^A(p) - i n(\varepsilon) \Gamma(p)
\end{pmatrix}
\]  

Here \(n(\varepsilon)\) is the thermal Bose–Einstein occupation number defined in eq. (7.9).

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