Nuclei Theory

Nuclear Shape Transition, Triaxiality and Energy Staggering of Υ-Band States for Even–Even Xenon Isotopic Chain

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Abstract—The positive-parity states of even–even Xe nuclei are inspected within the framework of modified $O(6)$ limit of the interacting boson model (IBM). The effective three-body interaction $[QQQ]^{(0)}$ where $Q$ is the IBM $O(6)$ quadrupole operator is introduced to exhibit the triaxiality nature. The shape of nuclear surface is quantified by the deformation parameters $\beta, \gamma$ by using the intrinsic coherent state. For each nucleus, the potential energy surfaces (PES) of the transition $U(5)$—Triaxiality—$O(6)$ are resolved and analyzed and the critical phase transition points are identified. For each nucleus a fitting procedure is adopted to get the best model parameters by fitting some selected calculated energy levels and $B(E2)$ transition rate ratios with experimental ones. These ratios are analyzed because they serve as effective order parameters in the shape phase transition. The nuclei in Xe isotopic chain evolve from spherical vibrator $U(5)$ to $\gamma$-soft rotor $O(6)$ by increasing the boson number from $N = 3$ (heavy isotope $^{132}$Xe) to $N = 10$ (light isotope $^{120}$Xe) and the isotope $^{126}$Xe represents the critical nucleus. The nucleus $^{126}$Xe has triaxial nature. To transact with high spin states in $\gamma$-band in $^{118,120}$Xe isotopic chain to investigate and exhibit the odd—even—spin energy staggering, we introduce the two-parameter collective nuclear softness rotor model (CNS2). Three different staggering indices are considered depending on the dipole transitions linking the two families of spins while the quadrupole transitions are within each spin family. Strong odd—even—spin energy staggering has been seen. As a link between the IBM and CNS2 models we observed that the energy difference $[E (I^+_{\gamma}) - E (I^+_{\gamma})]$ between the $\gamma$-band and ground state band normalized to $[E (2^+_\gamma) - E (2^+_\gamma)]$ decreases with increasing the mass number.

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1. INTRODUCTION

Structures, electromagnetic transition rates and shape phase transitions (which describe how one geometric shape evolve into another) of several collective nuclei in different mass regions were systematically considered and analyzed in the framework of the algebraic interacting boson model (IBM1) [1–12]. In IBM, collective excitations in nuclei are studied in terms of a system of $N$ interacting monopole $(s)$ and quadrupole $(d_m)$ with $m = 0, \pm 1, \pm 2$ bosons with angular momentum $L = 0$ and $L = 2$ respectively. In its simplest version the model does not distinguish between protons and neutrons and have $U(6)$ group. There are three possible phases known as $U(5), SU(3)$ and $O(6)$ corresponding to the three limiting symmetries of the geometric collective model (GCM) [13–17] namely: spherical vibrator, axially symmetric deformed rotor and $\gamma$-unstable respectively.

The critical point symmetries $E(5), X(5), Y(5)$ and $Z(5)$ were introduced [18–21] to depict the critical point of the phase transition from spherical vibrator to $\gamma$-unstable rotor, from spherical vibrator to axial symmetric rotor, from axially deformed shapes to triaxial deformed shapes and for the prolate to oblate nuclear shape transition respectively. The study of collective Bohr–Hamiltonian near the critical point symmetries in nuclei is an important topic of nuclear structure [22–25]. Also the critical point symmetries are needed to describe the transition between two different shapes [26–28].

The connection between the IBM, potential energy surfaces (PES), geometric shapes and phase transitions can be acquired by introducing the intrinsic coherent state [29]. The shape of deformed nuclei can be labeled into prolate, oblate and triaxial nuclei [30] according to the ratios between the three

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where the pairing, the angular momentum, the even-spin staggering in also the CNS2 model to investigate the Xenon isotopic chain. Furthermore, we present here of PES will give the evolution of shape transition in SU$\text{\textit{stable rotational and triaxial nuclear states without the model contains only two parameters: the ground state moment of inertia and the nuclear softness parameter. The nuclear softness (NS) model was noteworthy successful in the description of the ground and $\gamma$ bands of many medium and heavy even–even nuclei. The triaxiality and softness in Xe, Ba and Te isotopic chains were studied experimentally [43–47] and interpreted by different nuclear models [48–52].

In the present paper the effects of adding the three $[QQQ]^{(0)}$ interaction (where $Q$ is the $O(6)$ symmetric quadrupole operator) to the dynamical symmetry $O(6)$ of the IBM1 Hamiltonian is calculated to generate rotational and triaxial nuclear states without the use of the rotational $SU(3)$ limit to IBM1. Analysis of PES will give the evolution of shape transition in Xenon isotopic chain. Furthermore, we present here also the CNS2 model to investigate the $\Delta I = 1$ odd–even–spin staggering in $\gamma$-bands of the even–even Xe isotopic chain. There is poor information about the $\gamma$-bands of very light and very heavy Xe nuclei. This pays attention to estimate the available measured $\gamma$-band energy levels of the intermediate isotopes $^{120–120}$Xe by CNS2 model, while the extended IBM is used for the calculations of ground state bands $^{120–134}$Xe.

2. OUTLINE OF THE IBM APPROACH

In the $sd$ version of IBM, the Hamiltonian takes different forms depending on the region of $U(5)$, $SU(3)$ and $O(6)$. Often the Hamiltonian is expressed in multipole form as:

$$
\hat{H} = \varepsilon_d \hat{n}_d + a_0 \hat{P} \hat{P} + a_1 \hat{L} \hat{L} + a_2 \hat{Q} \hat{Q} + a_3 T^{(3)} T^{(3)} + a_4 T^{(4)} T^{(4)},
$$

where $\hat{n}_d$, $\hat{P}$, $\hat{L}$, $\hat{Q}$, $T^{(3)}$ and $T^{(4)}$ are the $d$-boson number, the pairing, the angular momentum, the quadrupole, the octupole and the hexadecapole operators respectively defined by:

$$
\hat{n}_d = (d^d \hat{d})^{(0)},
$$

$$
\hat{P} = \frac{1}{2} (\hat{d} \hat{d}) - \frac{1}{2} (ss),
$$

$$
\hat{L} = \sqrt{10} (d^l \hat{d})^{(1)},
$$

$$
\hat{Q} = \left[ s^s \times \hat{d} + d^s \times \hat{s} \right]^{(2)} + \chi (d^l \hat{d})^{(2)},
$$

$$
T^{(3)} = (d^l \hat{d})^{(3)},
$$

$$
T^{(4)} = (d^l \hat{d})^{(4)},
$$

where $(s^s, d^l)$ and $(\hat{s}, \hat{d})$ are the creation and annihilation operators.

Hamiltonian (1) includes set of six coefficients $\varepsilon_d$ and $a_i$ ($i = 0$ to 4) represent the model parameters measured in units of energy. For a tensor product of $t^{l_1}$ and $t^{l_2}$ the square brackets $[i \times \hat{i}]^{(l)}$ indicate

$$
[t^{l_1} \times t^{l_2}]^L_M = \sum_{l_1 m_2} \langle l_1 m_1 l_2 m_2 |LM \rangle t^{(l_1)}_{m_1} t^{(l_2)}_{m_2},
$$

where the symbol $\langle l_1 m_1 l_2 m_2 |LM \rangle$ denotes Clebsch–Gordan coefficients. A particular case of tensor product is the scalar or dot product $(\hat{i} \hat{i})^{(0)}$.

We choose the IBM Hamiltonian operator of the $O(6)$ dynamical symmetry which emerges from the group chain $U(6) \supset O(6) \supset O(5) \supset O(3)$ in terms of multipole operators as:

$$
H [O(6)] = a_0 \hat{P} \hat{P} + a_1 \hat{L} \hat{L} + a_3 \hat{T}_3 \hat{T}_3.
$$

The symbol dot (.) stands for the scalar product defined as

$$
\hat{T}_L \cdot \hat{T}_L = \sum_{M} (-1)^M \hat{T}_{L, M} \hat{T}_{L, -M},
$$

where $\hat{T}_{L, M}$ corresponds to the component of the operator $\hat{T}_L$. The operators $\hat{d}_m = (-1)^m \hat{d}_{-m}$ and $\hat{s} = \hat{s}$ are introduced to include the correct tensorial character under spatial rotations.

The corresponding energy eigenvalues of the Hamiltonian (1) are

$$
E_{O(6)} (N, \sigma, \tau, I) = \frac{1}{4} A (N - \sigma) (N + \sigma + 4) + B \tau (\tau + 3) + C I (I + 1)
$$
with $A = a_0$, $B = a_3$, $C = a_1 - (1/10)a_3$, with $a_0$, $a_1$, $a_2$ and $a_3$ being the original parameters of Hamiltonian Eq. (8) measured in units of energy and where $[N], \sigma, \tau, I$ are the quantum numbers which distinguish $U(6), O(6), O(5)$ and $O(3)$ respectively and for ground state band $N = \sigma$.

The IBM being an algebraic model, it does not consist of shape variables, so that the geometrical interpretation of the IBM Hamiltonian can be obtained by introducing the intrinsic coherent state formalism [29] in terms of shape deformation parameters $\beta, \gamma$ by considering the state:

$$|N\beta\gamma\rangle = \frac{1}{\sqrt{N!}} (b_c^\dagger)^N |0\rangle,$$  \quad (11)

where $|0\rangle$ is the boson vacuum and $b_c^\dagger$ is the boson creation operator given by

$$b_c^\dagger = \frac{1}{\sqrt{1 + \beta^2}} \left[ s^\dagger + \beta \cos \gamma a_0^\dagger + \frac{1}{\sqrt{2}} \beta \sin \gamma (d_2^\dagger + d_{-2}^\dagger) \right].$$  \quad (12)

Here, $\beta \geq 0$ and $0 \leq \gamma \leq \frac{\pi}{3}$ are intrinsic shape parameters.

The expectation value of a Hamiltonian in the intrinsic coherent state, gives the bosonic PES of the nucleus as a function of shape parameters $\beta$ and $\gamma$.

$$E(N\beta\gamma) = \frac{\langle N\beta\gamma|H|N\beta\gamma\rangle}{\langle N\beta\gamma|N\beta\gamma\rangle}. \quad (13)$$

Using the Hamiltonian Eq. (1), the PES becomes

$$E[O(6)] = \frac{1}{4} N (N - 1) \left( 1 - \beta^2 \right) \left( 1 + \beta^2 \right)^2$$

$$+ \frac{6 N \beta^2}{1 + \beta^2} + \frac{7 N \beta^4}{51 + \beta^2},$$

$$E[O(6)] = \frac{A_2 \beta^2 + A_4 \beta^4}{(1 + \beta^2)^2} + A_0,$$

where $A_2 = [\lambda - a_0 (N - 1)] N$, $A_4 = \lambda N$, $A_0 = \frac{1}{4} a_0 (N - 1)$ with $\lambda = 6a_1 + \frac{7}{5} a_3$.

Since the behavior of the octupole tensor term $a_3 \hat{T}^{(3)}$ is similar to the angular momentum term $a_1 \hat{L}$ (each gives $\beta^2/(1 + \beta^2)$) we replaced them by $\lambda$ the parameter $\lambda$ is linear combination of the original parameters $a_1$ and $a_3$ which are the coefficients of the angular momentum and octupole operators respectively. So that instead of the two parameters $a_1$ and $a_3$ we used only one parameter $\lambda$ to be compared with the pairing coefficients $a_0$.

For any stable equilibrium state the first derivative of the PES with respect to the deformation parameter $\beta$ must be zero and the second derivative must be positive, thus we obtain:

$$A_2 + (2A_4 - A_2) \beta^2 = 0, A_2 + (6A_4 - 8A_2) \beta^2 - (6A_4 - 3A_2) \beta^4 > 0.$$  \quad (15)

The PES Eq. (10) is $\gamma$-independent. Therefore, the equilibrium value of $\beta$ is given by

$$\beta_c = \pm \sqrt{\frac{a_0 (N - 1) - \lambda}{a_0 (N - 1) + \lambda}} \quad (16)$$

for $\lambda < a_0 (N - 1)$.

The antispinodal point is found at $\lambda = a_0 (N - 1)$ and the PES at this point is given by

$$E_{\text{critical}} = \frac{\lambda N \beta^4}{(1 + \beta^2)^2} + \frac{1}{4} a_0 N (N - 1).$$  \quad (17)

For pure $O(6)$ limit we take only the pairing term (i.e. $\lambda = 0$) and the equilibrium deformation parameter $\beta$ becomes $\beta_c = \pm 1$.

In Fig. 1 we illustrated the behavior of the PES corresponding to the $O(6)$ Hamiltonian Eq. (1) for fixed $a_0$ at 101.2 keV and varying $\lambda (\lambda = 831.2$ keV for $N = 4$, $\lambda = 672.2$ keV for $N = 7$, $\lambda = 513.2$ keV for $N = 10$) to produce critical point at $N = 7$. It is clear that the number of bosons $N$ is important.

To remove the dependence on the boson number $N$ for classical limit, we rewrite Hamiltonian (1) in the form

$$\hat{H} = \frac{a_0}{N^2} \hat{P}^\dagger \hat{P} + \frac{a_1}{N} \hat{L} \hat{L} + \frac{a_3}{N} T_3 T_3. \quad (18)$$

The corresponding PES reads

$$E(\beta) = \frac{(\lambda - a_0) \beta^2 + \lambda \beta^4}{(1 + \beta^2)^2} + \frac{1}{4} a_0. \quad (18)$$

The antispinodal critical point occurs when PES becomes flat at $\left. \left( \frac{\partial^2 E}{\partial \beta^2} \right) \right|_{\beta=0} = 0$ or $\beta = 0$ and in this case the coefficient of $\beta_2$ vanishes ($A_2 = 0$), yielding $\lambda - a_0 = 0$. One can see that for $a_0 \leq \lambda$, the global minimum is at $\beta = 0$, for $a_0 > \lambda$ we reach the deformed $\gamma$-soft shape, while for $a_0 = \lambda$ a shape transition from spherical to $\gamma$-unstable occurs. Figure 2 illustrates the behavior of the PES at $N = 4$ and $a_0$ values $a_0 = 0, 2, 4, 8, 12$. A shape phase transition occurs at $a_0 = \lambda = 4$.

3. EXTENDED $O(6)$ IBM

The purpose of the present paper is to exhibit the shape phase transitions and nuclear triaxiality from $O(6)$ dynamical symmetry of the IBM by adding three-body interaction. The transition between the
spherical $U(5)$ and $\gamma$-unstable $O(6)$ shape can be studied by adding the term $\varepsilon \hat{n}_d$ to the Hamiltonian (1), where $\hat{n}_d$ is the $d$-boson number operator defined as $\hat{n}_d = d^\dagger \tilde{d}$.

$$H = H\{O(6)\} + \varepsilon \hat{n}_d. \quad (19)$$
By using the intrinsic coherent state Eq. (8), the corresponding PES as a function of deformation parameter $\beta$ is given by

$$E(N, \beta) = \frac{A_2' \beta^2 + A_4' \beta^4}{(1 + \beta^2)^2} + A_0,$$  

(20)

where $A_2' = [\lambda + \varepsilon - a_0 (N - 1)] N$, $A_4' = (\lambda + \varepsilon) N$.

where $A_2'$ and $A_4'$ are parameters of extended model, notice that we used the symbols $A_2$ and $A_4$ in Eq. (14).

In order to exhibit the triaxiality, we will add to the $O(6)$ IBM Hamiltonian the cubic interaction $[QQQ]^{(6)}$, where $Q$ is the $O(6)$ quadrupole operator defined as

$$Q^{x=0} = [d^T \hat{S} + S^T \hat{a}]^{(2)},$$  

(21)

where $x$ is the structure parameter of the quadrupole operator and $x = 0$ for $O(6)$ limit. The Hamiltonian corresponding to the cubic quadrupole operator with coupling parameter $k$ is given by

$$H_Q = -k \left[ \langle \bar{Q} Q \rangle^{(0)} \right].$$  

(22)

The square brackets $[\bar{Q} Q]^{(0)}$ denote the tensor product of three $O(6)$ quadrupole operators $Q$ (defined in Eq. (21)) coupled to total angular momentum $L = 0$.

The expectation value of the Hamiltonian $H_Q$ is obtained by using the intrinsic coherent state (8) to yield

$$E_Q(N, \beta, \gamma) = -k \sqrt{\frac{8}{35}} \left[ \frac{3N (N-1)}{(1 + \beta^2)^2} + \frac{4N (N-1) (N-2)}{(1 + \beta^2)^3} \beta^3 \cos 3\gamma \right].$$  

(23)

The rigid triaxiality occurs at $\gamma = 30^\circ$, which predicts relations between the first three excited state energies as

$$\Delta E = E_{3+} - E_{2+} - E_{2g} = 0$$  

(24)

or

$$E(3) = E(2^+) + E(2^+).$$  

(25)

To remove the dependence on the boson number $N$ in classical limit, we rewrite the Hamiltonian $H_Q$ in the form

$$H_Q = \frac{K}{N^3} [QQQ]^{(0)}.$$  

(26)

The corresponding PES reads

$$E_Q(N, \beta, \gamma) = -4K \sqrt{\frac{8}{35}} \left[ \frac{\beta^3 \cos 3\gamma}{(1 + \beta^2)^3} \right].$$  

(27)

Adding the eigenvalue $E_Q$ to the eigenvalue $E(\beta)$ Eq. (14) for the $O(6)$ dynamical symmetry yields

$$E(\beta, \gamma) = \frac{1}{(1 + \beta^2)^3} \left[ (\lambda - a_0) \beta^2 + (2\lambda - a_0) \beta^4 - 4K \sqrt{\frac{8}{35}} \beta^3 \cos 3\gamma + \lambda \beta^6 \right] + \frac{1}{4} a_0.$$  

(28)

It is clear that the resultant PES depends on $\gamma$ and leads to triaxiality.

For $\lambda = 1$

1. If $a_0 = 0$, the minimum of PES is at $k > 0 (k = 0.5, 1, 2, 3)$ and $\gamma = 0^\circ$ or $k < 0, \gamma = 60^\circ$ and the PES exhibit spherical and deformed shapes as illustrated in Fig. 3a and the critical value of $k$ is about $k = \pm 1.5$ for spherical–prolate and spherical–oblate.

2. If $a = 0.5 (< \lambda)$ and $k = 0.5, 1, 2, 3$ a spherical shape occurs as seen in Fig. 3b and deformed shape only occurs for $a_0 = 2 (> \lambda)$ as seen in Fig. 3c.

3. The critical point occurs at $a_0 = \lambda = 1$ and $k = 0.5, 1, 2, 3$ and the spherical shape disappears as seen in Fig. 3d.

The angle $\gamma$ gives the degree of axial symmetry and can be determined from rigid triaxial rotor model [53, 54].

Besides the excitation energies and the PES, the $B(E2)$ transition probabilities can be calculated using the $O(6)$ IBM electric quadrupole operator $Q^{x=0}$,

$$T(E2) = \alpha Q^{x=0}.$$  

(29)

The reduced $E2$ transition probabilities read

$$B(E2, I_i \rightarrow I_f) = \frac{|\langle I_f | T(E2) | I_i \rangle|^2}{2I_i + 1}.$$  

(30)

4. OUTLINE OF THE COLLECTIVE NUCLEAR SOFTNESS MODEL (CNSM)

The energy expression formula for axially symmetric rigid rotor is given by

$$E(I) = \frac{\hbar^2}{2J} I (I + 1).$$  

(31)

This however, always predicts states lying higher than that given by experiments. If we attribute this effect to the variation of the moment of inertia $J$ with angular momentum $I$, then we can write Eq. (27) after introducing the variable moment of inertia $J_f$ as

$$E(I) = \frac{\hbar^2}{2J_f} I (I + 1).$$  

(32)
If we make the Taylor series expansion of the variable moment of inertia $J_I$ about the ground state $J$ for $I = 0$, then it yields

$$J_I = J_0 \left( 1 + \sigma_1 I + \sigma_2 I^2 + \sigma_3 I^3 + \ldots \right),$$

(33)

where the nuclear softness $\sigma$ is an appropriate parameter to study the variation in moment of inertia. It is defined as the relative increase of the moment of inertia with angular momentum.

$$\sigma_n = \frac{1}{n!J_I} \left( \frac{\partial^n J_I}{\partial I^n} \right)_{I=0},$$

(34)

where $\sigma_1, \sigma_2, \sigma_3, \ldots$ are the first second, third, . . . nuclear softness. The inverse of $J_I$ is written as

$$\frac{1}{J_I} = \frac{1}{J_0} \frac{1}{1 + \sigma_1 I}$$

(35)
\[ E(I) = \frac{\hbar^2}{2J_0} \frac{I(I+1)}{(1+\sigma I)} \cdot \left[ 1 + \frac{\sigma_2 I^2 + \sigma_3 I^3 + \ldots}{1 + \sigma_1 I} \right]^{-1}. \]

Keeping the nuclear softness to only first order, that is putting \( \sigma_2, \sigma_3, \ldots \) equal to zero, yields
\[ \frac{1}{J_I} = \frac{1}{J_0(1+\sigma I)}. \] (36)

The equation of energy reduced to a two-parameter expression

This expression is denoted as the two-parameter collective nuclear softness model [41, 42, 55, 56] where \( J \) and \( \sigma \) are the constant parameters.

5. ODD–EVENSPIN ENERGY STAGGERING IN \( \gamma \)-BAND

Energy staggering patterns between odd and evenspin sequences were examined and interpreted in several literatures [57–62]. In this paper, to exhibit this staggering phenomenon in \( \gamma \)-bands, we used the CNS2 model and suggested three different dimensionless staggering indices. The first two staggering

\[ 134\text{Xe} \ (N = 3) \quad 132\text{Xe} \ (N = 4) \quad 130\text{Xe} \ (N = 5) \]

\[ 128\text{Xe} \ (N = 6) \quad 126\text{Xe} \ (N = 7) \]

\[ 124\text{Xe} \ (N = 8) \quad 122\text{Xe} \ (N = 9) \quad 120\text{Xe} \ (N = 10) \]

Deformation parameter, \( \beta \)

Fig. 4. The calculated PES’s versus deformation parameter \( \beta \) before adding the cubic term interaction. The critical points in the transition from \( U(5) \) (spherical case) to \( O(6) \) (\( \gamma \)-unstable deformed case) are at \( N = 6, 7 \) for \( 128\text{Xe}, 126\text{Xe} \).
Fig. 5. The calculated PES’s versus deformation parameter $\beta$ after adding the cubic term interaction to exhibit the degree of triaxiality for the $^{120-134}$Xe isotopic chain. The triaxiality $\gamma$ is given at the upper part of each curve.

indices $S(I)$ and $\Delta E(I)$ depend on the dipole transitions linking the two families of spins, while the third one $Y(I)$ depends on the dipole transitions between the two families and the quadrupole transitions within each spin family, namely

$$S(I) = 1 - \frac{(I+1)E(I-1) + IE(I+1)}{(2I+1)E(I)},$$  \hspace{1cm} (38)$$

$$\Delta E(I) = \frac{3}{E(2I)} \left[ E(I) - 2E(I-1) + E(I-2) \right],$$  \hspace{1cm} (39)$$

$$Y(I) = \frac{4I-2}{2I} \left( \frac{E(I) - E(I-1)}{E(I) - E(I-2)} - 1 \right),$$  \hspace{1cm} (40)$$

where $E(I)$ is the energy of the state of angular momentum $I$.

We multiplied the $\Delta E(I)$ index by 3 to give $\Delta E(I)$ equal one for pure rotor instead of 1/3. Also we divided $S(I)$ index [58] by $E(I)$ to give $S(3) = 0$ for pure rotator.

The expression $S(I)$ can be explained as a measure of the dynamical moment of inertia. Its sign indicates the clustering. A positive sign results if the levels with even spins are depressed in energy with respect to the odd-spin states, indicating $\gamma$-softness. The expression $\Delta E(I)$ is used to distinguish between $\gamma$-soft and $\gamma$-rigid rotor. The term $(4I-2/2I)$ in
the expression $Y(I)$ is corresponding to the ratio $[E(I) - E(I - 2)]/[E(I) - E(I - 1)]$ for axial pure rotator. Evidently, for ideal rigid rotor $Y(I) = 0$ for all values of $I$.

Notice that each point in the three staggering indices includes three consecutive energies. For axially symmetric pure rotator $E(I) = AI(I + 1)$, the two indices $S(I)$ and $Y(I)$ equal zero while the $\Delta E(I)$ index is equal to one. We say that odd–even staggering is observed if the staggering index exhibits a zigzag curve with increasing spin. The $S(3)$, $S(5)$ and $\Delta E(4)$ are indices of special interest

$$S(3) = 1 - \frac{4E(2) + 3E(4)}{7E(3)},$$  \hspace{0.5cm} (41)

$$S(5) = 1 - \frac{6E(4) + 5E(6)}{11E(5)},$$  \hspace{0.5cm} (42)

$$\Delta E(4) = \frac{3}{E(2^+_3)} \left[ E(4) - 2E(3) + E(2) \right].$$  \hspace{0.5cm} (43)

The $\Delta E(4)$ reflects the displacement of even-spin members of $\gamma$-band relative to odd-spin members it equals (1) for axially symmetric rotor and (5) for rigid triaxial rotor. For $\gamma$-soft rotor or $O(6)$, $\Delta E(4) = -6$ and for spherical vibrator it is equal to ($-3$).

6. NUMERICAL CALCULATIONS AND DISCUSSION APPLIED TO XENON ISOTOPIC CHAIN

The Xe isotopes have proton number $Z = 54$ with 4 protons above the magic number 50 given 2 proton
The IBM calculations have been done using the code PHINT [63] and a simulated search program while the rootmean square (rms) fits of energies have been performed using the standard $\chi$ measure

$$\chi = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( \frac{E_{\text{exp}}^i - E_{\text{cal}}^i}{E_{\text{exp}}^i} \right)^2},$$

where $N$ is the number of energy levels fitted for the Xe isotopes. The same procedure for $B(E2)$ transition rates has been utilized.

The values of energy ratios $R_{I/2}$ and the electric quadrupole transition rates $B_{1f/1i}$ are evaluated for our $^{120-134}$Xe isotopic chain and compared to those of the available experimental data [64] and the results are listed in Tables 1, 2. The rms deviation $\chi$ is between 0.020 and 0.044. The ratio $R_{4+^2/2^+_1}$ is the best signature to exhibit the shape transition; it has a limiting value 2 for quadrupole vibrator and the value 2.5 for a
Table 1. Comparison of the IBM calculations with those of the experimental ones for the energy ratios $R_{1/2}$ of low-lying levels in the $^{120-134}$Xe isotopic chain

| Nuclide | $R_{4^+_1/2^+_1}$ | $R_{6^+_1/2^+_1}$ | $R_{2^+_2/2^+_1}$ | $R_{3^+_3/2^+_1}$ | $R_{4^+_4/2^+_1}$ | $R_{6^+_0/2^+_1}$ |
|---------|------------------|------------------|------------------|------------------|------------------|------------------|
| $^{120}$Xe | Exp | 2.4678 | 4.3312 | 2.7156 | 3.9418 | 4.3437 | 2.8135 |
| | Cal | 2.5199 | 4.0699 | 2.7920 | 4.0546 | 4.2711 | 3.0253 |
| $^{122}$Xe | Exp | 2.5009 | 4.4283 | 2.5450 | 3.6655 | 4.2341 | 3.4688 |
| | Cal | 2.5647 | 4.1774 | 2.5793 | 3.9501 | 4.2006 | 3.4644 |
| $^{124}$Xe | Exp | 2.4826 | 4.3737 | 2.3910 | 3.5240 | 4.0616 | 3.5841 |
| | Cal | 2.5564 | 4.2602 | 2.4212 | 3.7931 | 4.0135 | 3.4269 |
| $^{126}$Xe | Exp | 2.4238 | 4.2070 | 2.2640 | 3.3905 | 3.8298 | 3.3807 |
| | Cal | 2.4913 | 4.1803 | 2.2994 | 3.5667 | 3.8761 | 3.2569 |
| $^{128}$Xe | Exp | 2.3326 | 3.9224 | 2.1888 | 3.2276 | 3.6203 | 3.5740 |
| | Cal | 2.3799 | 4.1090 | 2.2470 | 3.3505 | 3.5776 | 3.2941 |
| $^{130}$Xe | Exp | 2.2471 | 3.6266 | 2.0932 | 3.0454 | 3.3730 | 3.3456 |
| | Cal | 2.3379 | 3.8207 | 2.0449 | 3.0619 | 3.3649 | 3.2979 |
| $^{132}$Xe | Exp | 2.1570 | 3.1628 | 1.9438 | 2.1294 | 2.3174 | 2.3174 |
| | Cal | 2.2369 | 3.2999 | 1.9109 | 2.1701 | 2.2955 | 2.2955 |
| $^{134}$Xe | Exp | 2.0437 | 2.5224 | 1.9051 | 2.2955 | 2.3174 | 2.3174 |
| | Cal | 2.1399 | 2.7999 | 1.8801 | 2.2955 | 2.3174 | 2.3174 |

$U(5)$ | 2 | 3 |
$O(6)$ | 2.5 | 4.5 |
$E(5)$ | 2.2 | 3.59 |

Table 2. Comparison of the IBM calculations with those of the available experimental ones for the electric quadrupole transition rates $B_{1/2}$, the prediction of the $O(6)$ limit is also given

| Nuclide | $B_{4/2}$ | $B_{6/4}$ | $B_{8/6}$ | $B_{10/8}$ |
|---------|-----------|-----------|-----------|-----------|
| $^{124}$Xe | Cal | 1.501 | 1.991 | 2.490 | 3.354 |
| | Exp | 1.34(24) | 1.59(71) | 0.63(29) | 0.29(8) |
| $N_B = 8$ | $O(6)$ | 1.354 | 1.458 | 1.420 | 1.282 |
| $^{128}$Xe | Cal | 1.790 | 2.853 | 4.232 | 6.001 |
| | Exp | 1.47(2) | 1.94(26) | 2.39(4) | 2.74(114) |
| $N_B = 6$ | $O(6)$ | 1.309 | 1.333 | 1.181 | 0.897 |
| $^{132}$Xe | Cal | 2.990 | 2.032 | 17.492 |
| | Exp | 1.24(18) | 1.205 | 1.666 | 0.625 |

The factors $a$ and $\lambda$ are fixed at $a = 101.2$ keV and $\lambda = 156.2$ keV for all isotopes, while the values of the parameter $\epsilon$ are listed in Table 3. We see that the nuclei in Xe isotopic chain evolve from spherical vibrator $U(5)$ to $\gamma$-soft rotor $O(6)$ by non-axial $\gamma$-soft rotator. As is seen from Table 1 ratio $R_{4^+_1/2^+_1}$ increases from about 2.1 for $^{135}$Xe to about 2.5 for $^{120}$Xe. That is, for Xe isotopes, the $R_{4^+_1/2^+_1}$ values decrease gradually from $\gamma$-soft values to vibration values by increasing the mass number $A$. Our calculated energy levels reflect the triaxiality and softness in Xe isotopes because they verify the conditions $\Delta E_1 = E(3^+_1) - [E(2^+_1) + E(2^+_2)]$ for triaxial nucleus, $\Delta E_2 = E(3^+_1) - [2E(2^+_1) + E(4^+_1)]$ for $\gamma$-soft nucleus. The difference $\Delta E_1$ is low for $^{126,128}$Xe while the difference $\Delta E_2$ is large which reflects the triaxial nature. In $^{122}$Xe, $\Delta E_1$ is large and $\Delta E_2$ is small which shows $\gamma$-soft nature.

Figure 4 illustrates the PES as a function of deformation parameter $\beta$ for the $^{120-134}$Xe isotopic chain calculated from Eq. (14) before adding the threebody interaction. The parameters $a$ and $\lambda$ are fixed at $a = 101.2$ keV and $\lambda = 156.2$ keV for all isotopes, while the values of the parameter $\epsilon$ are listed in Table 3. We see that the nuclei in Xe isotopic chain evolve from spherical vibrator $U(5)$ to $\gamma$-soft rotor $O(6)$ by
Table 3. The values of the parameter \( \varepsilon (\alpha_0 = 101.2 \text{ keV, } \lambda = 156.2 \text{ keV}) \) for the \( ^{120–134} \text{Xe} \) isotopic chain

| \( ^{125} \text{Xe} \) | \( ^{132} \text{Xe} \) | \( ^{130} \text{Xe} \) | \( ^{128} \text{Xe} \) | \( ^{126} \text{Xe} \) | \( ^{124} \text{Xe} \) | \( ^{122} \text{Xe} \) | \( ^{120} \text{Xe} \) |
|---|---|---|---|---|---|---|---|
| \( E, \text{ keV} \) | 727.4 | 674.2 | 621 | 567.8 | 514.6 | 461.4 | 408.2 | 355 |
| \( N \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

Table 4. The adopted best parameters \( J_0 \) and \( \sigma \) of the collective soft rotor model resulted from the fitting procedure for odd- and even-spin values for \( \gamma \)-bands in \( ^{120–126} \text{Xe} \) nuclei

| Nucleus | Sequence, \( I \) | \( J_0, \hbar^2 \text{ MeV}^{-1} \) | \( \sigma \) |
|---|---|---|---|
| \( ^{120} \text{Xe} \) | even | 0.45238 | 3.60855 |
| | odd | 0.18220 | 8.64394 |
| \( ^{122} \text{Xe} \) | even | 1.35523 | 1.02893 |
| | odd | 0.94440 | 1.53418 |
| \( ^{124} \text{Xe} \) | even | 1.04045 | 1.35970 |
| | odd | 0.73187 | 1.95725 |
| \( ^{126} \text{Xe} \) | even | 0.88975 | 1.56287 |
| | odd | 0.73592 | 1.85669 |

increasing the boson number from \( N = 3 \) (heavy isotope \( ^{135} \text{Xe} \)) to \( N = 10 \) (light isotope \( ^{120} \text{Xe} \)). The nucleus \( ^{122} \text{Xe} \) is deformed and has \( \gamma \)-unstable–like character. The isotopes \( ^{126,128} \text{Xe} \) are good candidates for secondorder critical nuclei as indicated in [65].

The PES are calculated after adding the cubic term interaction \( [QQQ]^{(6)} \) to introduce a degree of triaxiality. The result is illustrated in Fig. 5. The calculated asymmetric (triaxiality) parameter \( \gamma \) is given at the upper part of each curve. We see that by increasing the value of the three-body interaction, the spectrum depends on the boson number \( N \), approaching those of triaxiality at \( N = 7 \) for isotope \( ^{126} \text{Xe} \) beginning from the behavior of the isotope \( ^{134} \text{Xe} \) \( (N = 3) \).

The Xe isotopes are triaxial in their ground states with shallow minima at the asymmetry parameter \( \gamma = 27^\circ \), and these nuclei present a \( \gamma \)-unstable behavior with critical point at \( ^{130} \text{Xe} \) isotope. Figure 6 shows the triaxiality parameter \( \gamma \) for \( ^{120–134} \text{Xe} \) nuclei.

The energy spectra of \( \gamma \)-soft bands in \( ^{120–126} \text{Xe} \) isotopes are determined using fitting the experimental energy levels to the calculated ones extracted from the suggested CNS2 model using a second simulated search program to minimize the rms deviation \( \chi \) for the energy levels of \( \gamma \) band. The optimized best fit parameters \( J_0 \) and \( \sigma \) corresponding to odd and even-spin families for each nucleus are listed in Table 4. A comparison between the experimental spectra of \( \gamma \)-bands in \( ^{120–126} \text{Xe} \) and the corresponding CNS2 model are illustrated in Fig. 7, the agreement between them is excellent. Using the values of energy levels of \( \gamma \)bands calculated in the framework of CNS2 model, the three staggering indices \( S(I), \Delta E(I) \) and \( Y(I) \) between odd-spin family are calculated and illustrated in Fig. 8 as functions of spin. A clear staggering pattern is observed. The energy levels with odd angular momentum \( I = 3, 5, 7, 9, \ldots \) are displaced relative to the levels with even \( I = 2, 4, 6, 8, \ldots \) that is, the odd levels do not lie at the energies predicted by pure rotator fit to the even levels, but all of them lie above or below the predicted energies. The odd–even indices \( S(4), \Delta E(4) \) and \( Y(I) \) are large for the transition nucleus \( ^{126} \text{Xe} \) and are reduced for deformed nucleus \( ^{120} \text{Xe} \). That is these staggering indices fall with increasing the number of valence neutrons \( ^{120} \text{Xe} \) \( (N_n = 16) \) and \( ^{126} \text{Xe} \) \( (N_n = 10) \).

7. CONCLUSIONS

Modified IBM \( O(6) \) limit is used including three-body introduction to generate shape transitions and triaxiality in Xenon nuclei and nuclear softness model to exhibit odd–even staggering in gamma bands. In the present paper, we have presented two models: the modified interacting boson model and the collective nuclear softness model. The IBM to study the shape phase transition \( U(5) \)–triaxiality–\( O(6) \) in ground state bands of \( ^{120–134} \text{Xe} \) isotopic chain, while the CNS model to exhibit the \( \Delta I = 1 \) odd–even spin energy staggering in \( \gamma \)-bands of \( ^{120–126} \text{Xe} \) isotopes. We added to the \( O(6) \) IBM Hamiltonian the effective three-body interaction \( \hat{Q} \hat{Q} \hat{Q}^{(6)} \), where \( \hat{Q} \) is the \( O(6) \) IBM quadrupole operator. By using a computer simulated search programs and the code PHINT-IBM a fitting procedure for each nucleus is performed to get the best parameters of our suggested models, in order to obtain a minimum root–mean–square deviation between the experimental and calculated energies and \( B(E2) \) transition rates for some selected low–lying levels. The potential energy surfaces are calculated and analyzed by using the method of intrinsic coherent states and the critical points are identified. The \( \gamma \)-soft rotation phase arose in light Xe isotopes like the nucleus \( ^{122} \text{Xe} \) with energy ratio \( R_{4/2} \) near 2.5, while the spherical vibrator phase arose in heavy isotopes like the nucleus \( ^{134} \text{Xe} \) with \( R_{4/2} = 2.04 \). A degree of triaxiality has been observed in \( ^{126,128} \text{Xe} \) isotopes. In the framework of CNS model, three staggering indices have been suggested. We
proved that the evenspin number \( I = 2, 4, 6, 8, 10 \) of \( \gamma \)-bands are depressed with respect to the odd-spin ones \( I = 3, 5, 7, 9 \) for the four isotopes \( ^{120-126}\text{Xe} \).

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