MANIFEST DUALITY IN BORN-INFELD THEORY

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Abstract

Born-Infeld theory is formulated using an infinite set of gauge fields, along the lines of McClain, Wu and Yu. In this formulation electromagnetic duality is generated by a fully local functional. The resulting consistency problems are analyzed and the formulation is shown to be consistent.

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1. INTRODUCTION.

The present paper has three ingredients: Electromagnetic duality, a novel formulation of field theories that uses an infinite set of gauge fields to describe a finite number of degrees of freedom per spatial point, and Born-Infeld theory.

Of the many "dualities" now being studied in mathematical physics electromagnetic duality stands almost alone in that the duality transformation is known explicitly as a function of the basic fields of the theory. Even so, it is a non-local transformation which is — as it were — easily studied only because an abelian gauge theory without matter is so simple. This is where the second ingredient of the paper enters: If one adopts a remarkable suggestion due to McClain, Wu and Yu [1] electromagnetic duality is in fact generated by a fully local functional. The price that one has to pay for this is that an infinite set of gauge fields are brought in to describe the two degrees of freedom per spatial point that are present in the Maxwell theory, and one encounters new consistency problems which are akin to those which arise when one makes the transition from analytical mechanics to field theory in the first place. These problems can be handled however [2] [3]. Now gauge fields were indeed invented to ensure manifest locality of otherwise apparently non-local systems. To use an infinite set of gauge fields seems a drastic measure, but it may be that such measures will prove useful elsewhere — perhaps to give explicit expression to more interesting dualities. Unfortunately, a generalization of the McClain-Wu-Yu proposal to Yang-Mills theory is not possible without a non-trivial insight as input. This we do not have, but it nevertheless seems to be of some interest to study the consistency problems of such a formulation for a genuinely non-linear theory. This is why we turn to the non-linear theory of electrodynamics first studied by Born and Infeld in the thirties [4]. Needless to say the Born-Infeld theory is no longer a viable theory of electromagnetism, but it arises as a part of an effective action derived from string theory as well as in certain worldsheet actions for "branes". It also has some remarkable mathematical properties, notably its causal behaviour, which sets it apart from other non-linear versions of electrodynamics [5]. For these reasons it retains some intrinsic interest. This theory also exhibits electromagnetic duality, and as we shall see it admits a formulation along the lines of McClain, Wu and Yu. The purpose of this paper is to show that this formulation is indeed a consistent one.
In section 2 we discuss the electromagnetic duality of the Maxwell and Born-Infeld theories, in section 3 we discuss the McClain-Wu-Yu proposal as applied to linear electrodynamics, and in section 4 we give and analyze the corresponding formulation for the Born-Infeld theory. About the first two topics we have nothing new to contribute, but we spend some time on them because we want to present them in such a way that the discussion in section 4 — which is new — can be kept brief. Our conclusion is stated in section 5.

2. MAXWELL, BORN-INFELD, AND DUALITY.

The first thing we have to do is to introduce a vector potential and a field strength in the usual way. Then we form the two Lorentz scalars

\[ I_1 = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \quad I_2 = \frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta}, \]

where the star denotes the Hodge dual. (We keep spacetime flat throughout the paper, since the generalization to curved spacetime is straightforward.) If we define electric and magnetic fields as

\[ E_a \equiv F_{ta} \quad B_a \equiv \frac{1}{2} \epsilon_{abc} F_{bc}, \]

the two Lorentz scalars become

\[ I_1 = \frac{1}{2}(E^2 - B^2) \quad I_2 = E \cdot B. \]

These scalars can now be used as building blocks to construct Lorentz invariant actions. In general we consider a Lagrangian density

\[ L = L(I_1, I_2). \]

Two choices of special interest are

\[ L = I_1 \quad \text{(Maxwell)} \]

\[ L = 1 - \sqrt{1 - 2I_1 - I_2^2} \quad \text{(Born-Infeld)}. \]
The first one is familiar. The second is is the Lagrangian of the Born-Infeld theory, and can be rewritten in the elegant form

\[ L = 1 - \sqrt{-\det (g_{\alpha\beta} + F_{\alpha\beta})} \quad \text{(Born-Infeld)} . \]  

(7)

We now consider the action

\[ S = \int L . \]  

(8)

We wish to perform a Legendre transformation and find a phase space action of the form

\[ S = \int F_{ta} D_a - \mathcal{H}(I_1, I_2) , \]  

(9)

where the invariants \( I_1 \) and \( I_2 \) are to be expressed as functions of the canonical variables \( A_a \) and \( D_a \) (and variation of the action with respect to \( A_t \) gives Gauss’ law as a constraint equation, as usual). For the Born-Infeld theory this was first done by Dirac \[6\]. The canonical momentum is the vector

\[ D_a = \frac{\partial L}{\partial I_1} E_a + \frac{\partial L}{\partial I_2} B_a . \]  

(10)

For the two cases that we have singled out for attention we find that

\[ D_a = E_a \quad \text{(Maxwell)} \]  

(11)

\[ D_a = \frac{1}{\sqrt{1 - 2I_1 - I_2^2}} (E_a + I_2 B_a) \quad \text{(Born-Infeld)} . \]  

(12)

To actually express \( \mathcal{H} \) as a function of \( A_a \) and \( D_a \) leads, in general, to a non-trivial calculation. For the Maxwell and Born-Infeld theories the required calculations show that

\[ \mathcal{H} = \frac{1}{2} (D^2 + B^2) \quad \text{(Maxwell)} \]  

(13)

\[ \mathcal{H} = \sqrt{1 + D^2 + B^2 + D^2 B^2 - (D \cdot B)^2} - 1 \quad \text{(Born-Infeld)} . \]  

(14)

When we give the McClain-Wu-Yu formulation of the Born-Infeld theory (in section 4) we will do so directly in the Hamiltonian formalism, without
going through any Lagrangian preparations. It is therefore useful to know that the equation that shows that the Born-Infeld theory is Poincaré invariant is the current algebra relation

$$\{ \mathcal{H}[N], \mathcal{H}[M] \} = \int (N \partial_a M - M \partial_a N) \epsilon_{abc} D_b B_c ,$$

where the notation $\mathcal{H}[N]$ denotes smearing with a test function. (It is not difficult to derive this relation provided that one sets about it in the right way, as we will in section 4.) On the right hand side we see the Poynting vector smeared with a particular test function formed from $N$ and $M$. For suitable choices of test functions the Poynting vector is the canonical generator of spatial translations and rotations. Since these are manifestly realized by the vector notation only the algebra of the Lorentz boosts needs to be checked explicitly; this algebra can be obtained by a suitable choice of the test functions in the relation that we just derived, which is why the latter is enough to ensure Poincaré invariance of the theory.

We now turn to electromagnetic duality in these theories (following Deser and Teitelboim [7]). It is well known that the Chern-Simons functional

$$\omega = \frac{1}{2} \int \epsilon_{abc} A_a \partial_b A_c = \frac{1}{2} \int A_a B_a$$

(which is gauge invariant up to a surface term) generates the canonical transformation

$$\delta A_a = 0 \quad \delta D_a = \{ D_a, \omega \} = -B_a .$$

This is not, however, a symmetry of the action. Before we remedy this we rewrite the Chern-Simons functional in a non-local form. It is not difficult to show that

$$\omega = \frac{1}{2} \int \epsilon_{abc} B_a \frac{1}{\Delta} \partial_b B_c ,$$

where $\Delta$ denotes the Laplace operator. Now, on the constraint surface of our electromagnetic theories the divergence of $D_a$ vanishes, and hence it can be expressed as the curl of a vector field, just as it is possible to do so for $B_a$ which is divergence free by definition. We can therefore introduce an "electric" Chern-Simons functional as well. Since $D_a$ is one of our canonical
variables we prefer to use the non-local form of this functional; we choose to consider

\[ \Omega = \frac{1}{2} \int \epsilon_{abc} (A_a \partial_b A_c - D_a \frac{1}{\Delta} \partial_b D_c) . \]  

(19)

This functional generates the canonical transformations

\[ \delta A_a = -\epsilon_{abc} \frac{1}{\Delta} \partial_b D_c \quad \delta B_a \approx D_a \quad \delta D_a = -B_a , \]  

(20)

where the weak equality sign \( \approx \) denotes equality modulo Gauss’ law.

It is easy to show that

\[ \{ \Omega, H_C \} = 0 \]  

(21)

(where \( H_C \) is the canonical Hamiltonian) for both the Maxwell and the Born-Infeld theory, hence \( \Omega \) generates a symmetry of these theories. This symmetry is called electromagnetic duality. For a generic choice \( L(I_1, I_2) \) of the Lagrangian density the requirement that \( \Omega \) shall generate a symmetry leads to a differential equation for \( L \) whose space of solutions has been studied by Gibbons and Rasheed [8]. The duality of the Born-Infeld theory was first noted by Schrödinger [9]. Note that the whole discussion rests on the constraint equation

\[ \partial \cdot D \approx 0 . \]  

(22)

Hence we have no right to expect this symmetry to survive the introduction of sources or charged matter; there is no known analogue in Yang-Mills theory [7].

There is another way in which to view electromagnetic duality. By definition of \( B_a \) and \( E_a \) we have the equations

\[ \partial \cdot B = 0 \quad \partial_t B_a = \epsilon_{abc} \partial_b E_c . \]  

(23)

If we define

\[ H_a \equiv \frac{\delta H_C}{\delta B_a} \]  

(24)

we can write the remaining field equations in the form
\[ \partial \cdot D = 0 \quad \partial_t D_a = -\epsilon_{abc} \partial_b H_c . \] (25)

Evidently the whole set of equations is left invariant by the duality transformation supplemented by

\[ \delta H_a \approx E_a \quad \delta E_a \approx -H_a . \] (26)

But these equations follow from the vanishing of \( \{ \Omega, H C \} \). This is in fact a common way in which to view electromagnetic duality.

Our aim, which is to express the canonical generator of electromagnetic duality in local form, will be reached in the next section; before we come to it it will be useful to give the two-potentials formulation of the Maxwell theory \[10\]. (The two-potentials formulation of the Born-Infeld theory can be easily derived from the equations given in section 4, and will be omitted here.)

The introduction of two vector potentials to describe electromagnetism in fact a rather natural thing to try, since the obstruction to extending duality to a theory with sources has to do with the fact that the electric field is no longer divergence free and consequently can not be written as the curl of a vector potential. On the other hand (taking a four dimensional point of view) it is true that an arbitrary two-form can always be written as the sum of the ”curls” of two independent vector potentials \[5\]. A direct way to arrive at a model with two vector potentials is to consider the action for an anti-symmetric tensor in six space-time dimensions, and decompose it into four dimensional fields

\[ (A_{a\beta}, A_{a4}, A_{\beta5}, A_{45}) \equiv \frac{1}{2\sqrt{2}}(\phi_{ab}, A_{a1}, A_{a2}, \phi) . \] (27)

If we perform dimensional reduction to four dimensions of the standard action for the anti-symmetric tensor in six dimensions, and then throw away the terms involving \( \phi_{ab} \) and \( \phi \), we arrive at

\[ S = \frac{1}{2} \int (F_{ta1}D_{a1} + F_{ta2}D_{a2} - \frac{1}{2}(E_{a1}E_{a1} + D_{a2}D_{a2} + B_{a1}B_{a1} + B_{a2}B_{a2}) . \] (28)

This action has an obvious symmetry generated by the weakly gauge invariant functional
This is a perfectly local functional, whose effect is to rotate the two vector potentials into each other.

We can recover the Maxwell theory by constraining the two-potentials theory suitably. Again this has a natural six dimensional interpretation; what one has to do is to restrict the theory to anti-symmetric tensors whose field strengths are self-dual under six dimensional Hodge duality [11]. (Such objects are called chiral p-forms.) After dimensional reduction, Hodge self-duality of the anti-symmetric tensor implies that

\[ E_{a_1} = B_{a_2} \quad E_{a_2} = -B_{a_1} \]  

(30)  

In the two-potentials theory this is equivalent to the constraints

\[ \Phi_{a_1} = D_{a_1} - B_{a_2} \approx 0 \quad \Phi_{a_2} = D_{a_2} + B_{a_1} \approx 0 \]  

(31)  

These constraints can be implemented by means of Lagrange multiplier terms added to the action, if one desires to do so. Note that

\[ \{\Phi_{a_1}, H\} = -2\epsilon_{abc}\partial_b\Phi_{c_2} \approx 0 \quad \{\Phi_{a_2}, H\} = 2\epsilon_{abc}\partial_b\Phi_{c_1} \approx 0 \]  

(32)  

Preservation of the constraints under time evolution then forces the transverse parts of their corresponding Lagrange multipliers to vanish in a solution. This means that the equations of motion are unchanged by this manoeuvre; the constraints select a subspace of the space of solutions, but leave the solutions themselves unchanged. The non-zero part of the constraint algebra is

\[ \{\Phi_{a_1}(x), \Phi_{b_2}(y)\} = -4\epsilon_{abc}\partial_c\delta(x,y) \]  

(33)  

We see that the constraints are a mixture of first and second class constraints. They contain a first class component because they imply Gauss’ law; we can get rid of this complication by setting the longitudinal parts of the vector potentials to zero, so that Gauss’ law holds strongly. Assume that this has been done. The remaining constraints are purely second class, and can be solved by
\[ E_{a2}^T = -B_{a1} \quad A_{a2}^T = -\epsilon_{abc} \frac{1}{\Delta} B_{c2} = -\epsilon_{abc} \frac{1}{\Delta} D_{c1}^T, \]  

(34)

where \( T \) denotes the transverse part. When we insert this result in the expression for the generator \( \tilde{\Omega} \) the latter becomes non-local. This non-locality is not really due to the Coulomb gauge — it is in fact impossible to solve the second class component of the constraints in a local manner.

When we insert the solution of the constraints into the action for the two-potentials theory we obtain

\[ S = \int \dot{A}_{a1}^T D_{a1}^T - \frac{1}{2} (D_{a1}^T D_{a1}^T + B_{a1} B_{a1}) . \]  

(35)

But this is precisely the action for Maxwell’s theory in the Coulomb gauge. Hence the constrained version of the two-potentials theory is equivalent to Maxwell’s. Unsurprisingly, we also find that

\[ \tilde{\Omega} = \Omega . \]  

(36)

Thus the local symmetry generator \( \tilde{\Omega} \) becomes the non-local generator of electromagnetic duality after constraining the theory.

The advantage of the two-potentials formulation is that it makes electromagnetic duality manifest - the latter is now generated by a local functional. The way in which this advantage was gained may strike the reader as a fake. Indeed it is a fake in a sense, because second class constraints implying a non-local symplectic structure were included in the bargain, and we do not have a consistent Hamiltonian system until these have been solved for. But in another sense it is not, because the two-potentials formulation is the germ of the fully local as well as manifestly duality invariant formulation to be reviewed in the next section.

3. LOCAL DUALITY IN THE MAXWELL CASE.

As we have seen the locality of the two-potentials formulation of electrodynamics is spoilt by the presence of second class constraints. Now the McClain-Wu-Yu formulation was originally obtained by following Batalin’s and Fradkin’s algorithm [12] for replacing second class constraints with gauge symmetries and new degrees of freedom. It was first applied to chiral bosons
in $1 + 1$ dimensions \[1\], then to electrodynamics (by Martin and Restuccia \[13\]) and finally to chiral $p$-forms in twice odd dimensions \[14\], where it solves the problem \[11\] of giving a manifestly covariant formulation for such fields. It is interesting to observe that the McClain-Wu-Yu formulation emerges naturally from string field theory, where the infinite set of gauge fields arises because of the presence of a bosonic ghost zero mode in the Ramond-Ramond sector \[15\]. (For completeness we mention that there exists an alternative approach to manifestly covariant chiral $p$-forms \[16\], but this will play no role here.)

We devote this section to a brief but careful review of these matters for the Maxwell theory. The canonical variables are an infinite set of pairs of vector potentials and their conjugate momenta, indexed by $(n)$ and $i$, where the index $i$ takes the values 1 and 2. The phase space action is

$$S = \frac{1}{2} \sum_{n=0}^{\infty} \int \dot{A}_{ai}^{(n)} D_{ai}^{(n)} - \frac{(-1)^n}{2} (E_{ai}^{(n)} E_{ai}^{(n)} + B_{ai}^{(n)} B_{ai}^{(n)}) - \Lambda_{ai}^{(n+1)} \Psi_{ai}^{(n+1)} - \Lambda_{i}^{(n)} g_{(n)}^{(n)} ,$$

(37)

where summation over $i$ (and the vector indices) is understood. This action can be derived from the manifestly covariant Lagrangian for a chiral two-form in six dimensions \[2\] by dimensional reduction followed by a Legendre transformation. Varying the action with respect to the Lagrange multipliers gives rise to an infinite set of constraints, which by definition are

$$g_{i}^{(n)} \equiv \partial_{a} D_{ai}^{(n)} \approx 0 \quad \text{(38)}$$

$$\Psi_{ai}^{(n+1)} \equiv \Pi_{ai}^{-{(n)}} + \Pi_{ai}^{+(n+1)} \approx 0 , \quad \text{(39)}$$

where we use the useful further definitions

$$\Pi_{ai}^{\pm{(n)}} \equiv E_{ai}^{(n)} \pm \epsilon_{ij} B_{aj}^{(n)} . \quad \text{(40)}$$

It is straightforward to check that all the constraints are first class and that

$$\{\Psi_{ai}^{(n+1)}, H_{C}\} = (-1)^{n+1} \epsilon_{ij} \epsilon_{abc} \partial_{b} \Psi_{cj}^{(n+1)} \approx 0 ,$$

(41)

where $H_{C}$ is the canonical Hamiltonian. It follows that this may be a consistent Hamiltonian system. However, before we can conclude that this is
indeed the case we must give a more careful definition of its phase space, to ensure that all the relevant infinite sums converge [2].

By the way we observe that if one attempts to treat a non-abelian gauge theory in the same way one finds that the constraint algebra does not close — the bracket between Gauss’ law and the ”new” constraints will not behave itself. If there is a generalization of all this to Yang-Mills theory, it is a quite non-trivial one.

Returning to electrodynamics we require that its canonical Hamiltonian exists, which is ensured if for any set of fields there exists an \( N \) such that

\[
 n > N \quad \Rightarrow \quad |\Pi_{ai}^{\pm(n)}| \leq \frac{f}{n}, \tag{42}
\]

where \( f(x) \) is some square integrable function. (A convenient choice is \( f(x) = 0 \) [3]). We will only allow initial data obeying this condition, and we then have to show that it is preserved under time evolution and gauge transformations. This will be so only if we restrict the allowed gauge transformations in a suitable manner. The precise statement is that the formal expression

\[
\sum_{n=0}^{\infty} \Psi_{ai}^{(n+1)} [\Lambda_{ai}^{(n+1)}] \equiv \sum_{n=0}^{\infty} \int \Lambda_{ai}^{(n+1)} \Psi_{ai}^{(n+1)} \tag{43}
\]

is a generator of allowed gauge transformations, and hence a first class constraint, only if there exists an \( N \) such that

\[
 n > N \quad \Rightarrow \quad |\Lambda_{ai}^{\pm(n+1)}| \sim \frac{1}{n}. \tag{44}
\]

When this requirement is kept firmly in mind various ”traps” are avoided; the McClain-Wu-Yu formulation is indeed a consistent one.

In particular we can now show that an allowed set of gauge choices is

\[
 A_{ai}^{L(n+1)} = 0 \quad \Rightarrow \quad E_{ai}^{L(n+1)} = 0 \tag{45}
\]

\[
 \Pi_{ai}^{+(n+1)} = 0 \quad \Rightarrow \quad \Pi_{ai}^{-(n)} = 0, \tag{46}
\]

where \( L \) denotes the longitudinal part. When these conditions are inserted into the phase space action we recover the two-potentials formulation of the Maxwell theory, including its second class constraints, and the equivalence to the original Maxwell theory follows.
Before gauge fixing, the generator of electromagnetic duality transformations is the local functional

\[ \tilde{\Omega} = -\frac{1}{2} \sum_{n=0}^{\infty} \int \epsilon_{ij} A^{(n)}_{ai} D^{(n)}_{aj} . \]  

(Which exists.) One can check that

\[ \{ \Psi_{ai}^{(n+1)}, \tilde{\Omega} \} = \frac{1}{2} \epsilon_{ij} \Psi_{aj}^{(n+1)} \approx 0 \]  

\[ \{ \tilde{\Omega}, H_C \} = 0 . \]

Hence \( \tilde{\Omega} \) is a weakly gauge invariant symmetry generator, and it easy to see that it reduces to the non-local symmetry generator \( \Omega \) when all the gauges have been fixed. We can therefore conclude that the McClain-Wu-Yu formulation indeed leads to a formulation of the Maxwell theory in which electromagnetic duality is realized as a fully local symmetry.

4. LOCAL DUALITY IN THE BORN-INFELD CASE.

Finally we are prepared to deal with the subject of our paper, that is manifest duality of the Born-Infeld theory. This theory differs from Maxwell’s only in the choice of the Hamiltonian, so we use the same phase space — including the definition of allowed gauge transformations and the generator \( \tilde{\Omega} \) of duality transformations — as was introduced in the previous section. It is interesting to ask whether one can derive the Born-Infeld Hamiltonian by dimensional reduction from a suitably constrained action for a two-form in six dimensions, but it is not easy to construct such an action \[17\] — the analogue of the invariant \( I_2 \) does not exist in twice odd dimensions. For this reason we will instead make a reasonable guess for the Hamiltonian, verify that it leads to a consistent and Poincaré invariant theory, and check that the ordinary Born-Infeld theory can be derived from it by gauge fixing.

The obvious guess for the Hamiltonian density is

\[ \mathcal{H} = \sqrt{1 + 2 \mathcal{H}^M + \mathcal{H}_a^M \mathcal{H}_a^M} - 1 , \]  

where
\[
H^M = \sum_{n=0}^{\infty} \frac{(-1)^n}{4} (E_{ai}^{(n)} E_{ai}^{(n)} + B_{ai}^{(n)} B_{ai}^{(n)}), \tag{51}
\]

\[
H^M_a = \sum_{n=0}^{\infty} \frac{1}{2} \epsilon_{abc} E_{bi}^{(n)} B_{ci}^{(n)}, \tag{52}
\]

and the superscript \(M\) may stand for Maxwell, McClain-Wu-Yu, or Martin and Restuccia according to preferences. Given our definition of the phase space these objects, and hence the Hamiltonian, clearly exist.

To see whether time evolution preserves the phase space, and to see whether Poincaré invariance is present, it is convenient to define the functional derivatives

\[
\frac{\delta H[N]}{\delta A_{ai}^{(n)}} = \frac{1}{2} \epsilon_{abc} \partial_b \left( \frac{N}{\mathcal{H} + 1} \left( (-1)^n B_{ci}^{(n)} - \epsilon_{cde} D_{di}^{(n)} H^M_e \right) \right) \tag{53}
\]

\[
\frac{\delta H[N]}{\delta D_{ai}^{(n)}} = \frac{1}{2} \frac{N}{\mathcal{H} + 1} \left( (-1)^n D_{ai}^{(n)} + \epsilon_{abc} B_{bi}^{(n)} H^M_c \right). \tag{54}
\]

We can now see by inspection that the phase space is preserved by time evolution. Poincaré invariance is a little bit more subtle. Given the functional derivatives it is straightforward to write down the crucial current algebra relation

\[
\{ H[N], H[M] \} = H^M_a \left[ N \partial_a M - M \partial_a N \right] -
\]

\[-\sum_{n=0}^{\infty} \int (N \partial_a M - M \partial_a N) (-1)^n (D_{ai}^{(n)} D_{bi}^{(n)} + B_{ai}^{(n)} B_{bi}^{(n)}) H^M_b. \tag{55}\]

The last term on the right hand side ought not to be there. However, we can show that

\[
\sum_{n=0}^{\infty} (-1)^n (D_{ai}^{(n)} D_{bi}^{(n)} + B_{ai}^{(n)} B_{bi}^{(n)}) = \frac{1}{2} \Pi_{ai}^{+(0)} \Pi_{bi}^{-(0)} +
\]

\[
+ \frac{1}{4} \sum_{n=0}^{\infty} (-)^n \left( (\Pi_{ai}^{-(n)} - \Pi_{ai}^{+(n+1)}) \Psi_{bi}^{(n+1)} + \Psi_{ai}^{(n+1)} (\Pi_{bi}^{-(n)} - \Pi_{bi}^{+(n+1)}) \right). \tag{56}\]

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By inspection of the terms that multiply the constraints on the right hand side we see that their behaviour for increasing \( n \) is that of the parameters for an allowed gauge transformation. Therefore — but only therefore — we may conclude that

\[
\sum_{n=0}^{\infty} (-1)^n (D^{(n)}_{ai} D^{(n)}_{bi} + B^{(n)}_{ai} B^{(n)}_{bi}) \approx \frac{1}{2} \Pi^{+(0)}_{ai} \Pi^{+(0)}_{bi} .
\]  

(57)

Proceeding in a similar way, we see that

\[
\mathcal{H}_M \approx -\frac{1}{8} \epsilon_{abc} \epsilon_{ij} \Pi^{+(0)}_{bi} \Pi^{+(0)}_{cj} ,
\]

(58)

and finally that

\[
\{ \mathcal{H}[N], \mathcal{H}[M] \} \approx \mathcal{H}^M_a [N \partial_a M - M \partial_a N] .
\]

(59)

The conclusion that the Hamiltonian that we have defined indeed leads to a Poincaré invariant theory follows.

The rest of the required arguments follow quickly. It is clear that the duality generator \( \tilde{\Omega} \) of the previous section generates a symmetry also of this theory. That the theory is equivalent to the Born-Infeld theory is evident when we fix the gauges, again as in the previous section. Having done so we find that

\[
\Pi^{+(0)}_{a1} = 2D^{(0)}_{a1} \quad \Pi^{+(0)}_{a2} = 2B^{(0)}_{a2} .
\]

(60)

The unconstrained pair \( (A^{(0)}_{a1}, D^{(0)}_{b1}) \) obey the canonical Dirac brackets of the Maxwell theory. Moreover we have already shown that

\[
\mathcal{H}^M \approx \frac{1}{8} \Pi^{+(0)}_{ai} \Pi^{+(0)}_{ai} \approx \frac{1}{2} (D^{(0)}_{a1} D^{(0)}_{a1} + B^{(0)}_{a1} B^{(0)}_{a1})
\]

(61)

\[
\mathcal{H}^M_a \approx \epsilon_{abc} D^{(0)}_{b1} B^{(0)}_{c1} .
\]

(62)

Hence our Hamiltonian agrees with the Born-Infeld Hamiltonian on the constraint surface, and the two theories are indeed equivalent.

5. CONCLUSION.
Through the introduction of an infinite set of gauge fields it is possible to formulate the Maxwell theory in such a way that electromagnetic duality is generated by a local functional. We showed — paying attention to the precise definition of the phase space — that the same formulation works for the non-linear Born-Infeld theory.

We did not treat sources (but see the papers by Berkovits [3] for this). Whether there are other and perhaps more interesting dualities that can be treated in a similar way is an open question, but this is an issue that may be worth thinking about.
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