Canonical reduction for dilatonic gravity in 3+1 dimensions

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We generalize the 1 + 1-dimensional gravity formalism of Ohta and Mann to 3 + 1 dimensions by developing the canonical reduction of a proposed formalism applied to a system coupled with a set of point particles. This is done via the Arnowitt-Deser-Misner method and by eliminating the resulting constraints and imposing coordinate conditions. The reduced Hamiltonian is completely determined in terms of the particles’ canonical variables (coordinates, dilaton field and momenta). It is found that the equation governing the dilaton field under suitable gauge and coordinate conditions, including the absence of transverse-traceless metric components, is a logarithmic Schrödinger equation. Thus, although different, the 3 + 1 formalism retains some essential features of the earlier 1 + 1 formalism, in particular the means of obtaining a quantum theory for dilatonic gravity.

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I. INTRODUCTION

Two of the oldest and most notoriously vexing problems in gravitational theory (which are possibly related to each other) are (i) obtaining a quantum gravity theory which is renormalizable and therefore amenable to meaningful physical predictions, and (ii) determining the (self-consistent) motion of $N$ bodies and the resultant metric they collectively produce under their mutual gravitational influence. In the latter case, lower-dimensional theories such as 1 + 1 dimensional gravity, (meaning one spatial dimension and one time dimension) have been examined in large part because problems in quantum gravity become much more mathematically tractable in this context. However, the problematic issue for General Relativity (GRT) is that the Einstein tensor is topologically trivial in 1 + 1 dimensions and cannot yield the correct Newtonian limit. Through the addition of an auxiliary field corresponding to a particle known as a dilaton, this problem can be addressed and yields a successful many-body gravity theory.

Regarding the first issue, in lower dimensions, namely 1 + 1, a normalizable quantum theory combining gravity, quantum mechanics and even an electromagnetic interaction was found through the addition of a dilaton. It also reduces to the Newtonian $N$-body gravitational action in the nonrelativistic limit. The action for the
gravitational scalar-tensor formulation in 1 + 1 dimen-
sions coupled to $N$ particles is, in the presence of a cosmological constant $\Lambda$

\[
\mathcal{L}_M \text{ being the matter Lagrangian given by the second term in brackets on the right-hand side of (1). The trace of Eq. (2) is}
\]

\[
\Box \Psi = \nabla^\mu \nabla_\mu \Psi = \kappa T^{\mu \mu} + \Lambda \ ,
\]

which yields

\[
R = \kappa T^{\mu \mu} + \Lambda \ .
\]

(hence, this is called $R = T$ theory). If the stress-energy tensor of the particles is absent the above equation reduces to that considered in earlier work of Jackiw \[3,10\] and Teitelboim \[11\]. Although GRT yields trivial field equations in 1 + 1 dimensions, incorporating a dilaton in the manner shown in \[11\] ensures a nontrivial set of field equations with the correct Newtonian limit \[1\].

It was later found that in 1 + 1 dimensions the above $N$-body problem could be mapped onto the quantum-mechanical problem of an $N$-body generalization of the problem of the $\text{H}_2^*$ molecular ion in one dimension, combining into a normalizable theory represented by the Schrödinger equation \[4\]. The formalism could also be extended to include electromagnetic charges. However, since our world is in 3 + 1 dimensions, the impact of this work is not yet clear.

Our proposed generalization of the action \[1\] is simply the outcome of the $d + 1$ generalization of Eq. (1) as written in Sec. 9 of Ref. \[1\] i.e. \[\int d^{d+1}x \mathcal{L}\] where [Eq. (6.1) of \[12\], Eq. (2.1) of \[13\]]

\[
\mathcal{L} = \mathcal{L}_F + \mathcal{L}_M = \frac{\kappa^2}{\kappa^2 - 2y} \left\{ \Psi R + \frac{1}{2} g^{\mu \nu} \nabla_\mu \Psi \nabla_\nu \Psi - 2A + \frac{1}{4} \sum_a \sqrt{-g} \left( g^{\mu \nu} \varphi_a \cdot \varphi_a + m_a^2 \varphi^2_a \right) \right\}
\]

\[
\text{where } \kappa^2 = 32\pi G/c^4 \text{ (note the redefinition of } \kappa \text{ compared to the 1 + 1 theory), and where } F_{\mu \nu} = A_{\mu \nu} - A_{\nu \mu} \text{ is an electromagnetic field strength tensor density whose} \\text{gauge potential is } A_\mu. \text{ Note that } p_{i \mu} \text{ is the mechanical} \\
momentum, e_i \text{ is the charge and } \lambda_i \text{ is the Lagrange multiplier of the } i\text{th particle. We have included coupling to} \\
N \text{ neutral massive scalar fields } \varphi_a, \text{ which for certain purposes can be used instead of point particles in studying the} \\
N\text{-body problem.}
\]

The reasons for using \[8\] are as follows. As shown in Ref. \[1\], the $d + 1$ generalization of Eq. (1) which includes the dilaton guarantees the correct Newtonian li-
mit in $d + 1$ dimensions. This was proven for $d = 1, 2, 3$ in Sec. 9 of [1] and, since it is a vital cornerstone in our proposed generalization, the proof is reproduced here in Appendix A with more detail.

Note that dimensional scaling pioneered by Hershbach et al. [14] has provided much insight in quantum theory and is suggestive of sound theories. For example, dimensional scaling helped establish that the mathematical structure of the energy eigenvalues for the three-dimensional hydrogen molecular ion was a generalized Lambert $W$ function [15] from its simpler one-dimensional counterpart, the double-well Dirac-delta function model [16].

An obvious criticism is that in $3 + 1$ dimensions, the scalar-tensor theory of Eq. (8) is clearly a departure from GRT. However, if we let $\kappa = 1$, (9) reduces to the familiar Einstein-Hilbert action of GRT. The results for Appendix A prove that both GRT and our scalar-tensor theory of Eq. (8) yield the same correct Newtonian limit in $3 + 1$ dimensions. Thus in $3 + 1$ dimensions, the effect of the dilaton is very small; this is salutary as we do not, a priori, expect it to contradict experiments vindicating GRT to known accuracies as given by, e.g., the Gravity Probe A and B experiments [17].

Another reason for retaining the dilaton is the observation of the unusual resemblance reported between the dilaton (a particle whose origin can be traced to Kaluza-Klein theory) and the Higgs boson (from the standard model) to the extent that a number of authors have wondered if they represent two different signatures for the same particle and so might even be the same particle (e.g. see the work of Bellazzini et al. [18] and references therein). Of course, it will take time for experiments to sort out this issue, but it becomes tantalizing to consider that perhaps the dilaton is closer to being discovered experimentally than the graviton. Thus retaining the dilaton becomes timely and instructive.

This paper is intended as a first in a series to flesh out the proposed $3 + 1$ scalar-tensor theory as a possible foundation for dilatonic quantum gravity through a canonical reduction of Eq. (8). The goal of this work is to isolate the effective field equation governing the dilaton field. This is done as follows. After obtaining the field equations, we apply the Arnowitt-Deser-Misner (ADM) method [12] to our scalar-tensor theory as it proven useful in decoupling the field equations for the $1 + 1$ case. Next, we then eliminate variables while trying to retain the greatest generality. We examine the behavior in the far field and, in general, under suitable gauge and coordinate conditions. Finally, we obtain the essential partial differential equations (PDEs) governing the dilaton field, the canonical momenta and the metric, and see how the outcome relates to that of the $1 + 1$ case. Concluding remarks are made at the end.

Throughout the paper, we use the Greek alphabet for spacetime indices, the latin alphabet $a; b; c; \ldots$, for spatial indices, and $i, j, k; \ldots$, for internal or particle indices.

### II. HAMILTONIAN ANALYSIS

To obtain the field equations, we rely on the results of Zhang and Ma (Sec. II of [19]). Although the latter work was aimed towards loop quantum gravity (which is not the intent of the current work), the initial derivations of their Hamiltonian analysis use the ADM approach in the context of $f(R)$ gravity and thus their field equations (obtained before the injection of the Ashtekar variables in Sec. III of [19]) can be extracted (we only use Sec. II of their work).

#### A. Field equations

In terms of the their own notation, the settings for their coupling parameter $\omega$ and potential $\xi$ are [19]

$$\omega(\Psi) = \frac{1}{2} \Psi, \quad \xi(\Psi) = -\Lambda \quad \text{and} \quad 8\pi G = 1$$

Variation with respect to $g_{\mu\nu}$ yields (in $n = d + 1$ space-time dimensions) [Eq. (2.2) of [19]]

$$\Psi G_{\mu\nu} - \nabla_\mu \nabla_\nu \Psi + g_{\mu\nu} \left( \Box \Psi - \frac{1}{4} (\nabla \Psi)^2 + \Lambda \right)$$

$$+ \frac{1}{2} \left( \partial_\mu \Psi \right) \partial_\nu \Psi = \frac{\kappa^2}{4} T_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

which has the same functional form of Eq. (2) apart from a nonzero Einstein tensor $G_{\mu\nu}$ and a slight rescaling of the definitions for $T_{\mu\nu}$ and the gravitational constant (and of course the realization that a covariant derivative on a scalar is just the partial derivative $\nabla_\mu \Psi = \partial_\mu \Psi$). In the limit $\Psi \to 1$, Eq. (10) reduces to the standard Einstein field equations. Variation with respect to $\Psi$ yields [Eq. (2.3) of [19]]

$$R = \Box \Psi$$

as in Eq. (3). However this is not $R = T$ theory. Rather the trace of Eq. (10) yields, using Eq. (11)

$$(n - 1)R + (\Psi R + \frac{1}{2} (\nabla \Psi)^2) \left( 1 - \frac{n}{2} \right) = \frac{\kappa^2}{4} T_{\mu\nu} - n \Lambda$$

When $n = 2$, Eq. (12) does reduce to $R = \text{const} \times T_{\mu\nu}$ for $\Lambda = 0$. However, for $n = 4$, it becomes

$$(3 - \Psi) R - \frac{1}{2} (\nabla \Psi)^2 = \frac{\kappa^2}{4} T_{\mu\nu} - 4\Lambda$$

Zhang and Ma set $8\pi G = 1$ [19]. We will do something very similar and henceforth multiply the Lagrangian of Eq. (8) by $\kappa^2/2$. 

B. ADM method

Our derivation will follow the general ideas from the original ADM method [20, 21]. In this formalism, the metric is defined as

$$ds^2 = -N^2 dt^2 + g_{ab}(dx^a + N^a dt)(dx^b + N^b dt)$$

(14)

where $N = (g^{00})^{-1/2}$ and $N_0 = g_{0b}$ are the lapse function and the shift covector. Here $\gamma_{ab}$ is the 3-metric for the spatial coordinates of $g_{ab}$ and $\sqrt{\gamma}$ is the square root of the determinant of $\gamma_{ab}$ where $\sqrt{-g} = N \sqrt{\gamma}$. By doing a $3 + 1$ decomposition of the spacetime, the four-dimensional scalar curvature can be expressed as [Eq. (2.4) of 19]

$$K = K_{ab} K^{ab} - K^2 + 3R + \frac{2}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} n^\mu K)$$

(15)

where $K_{ab}$ is the extrinsic curvature of a spatial hypersurface $\Sigma$. $K = K_{ab} \gamma^{ab}$, $3R$ denotes the scalar curvature of the 3-metric $\gamma_{ab}$ induced on $\Sigma$ and $n^\nu$ is the unit normal of $\Sigma$. The canonical momenta are respectively [Eq. (2.5) of 19]

$$\pi^{ab} = \frac{\partial L}{\partial (\partial_t \gamma_{ab})} = \frac{\partial L}{\partial \gamma_{ab}} = \frac{\sqrt{\gamma}}{2} \left[ \Psi (K_{ab} - K \gamma_{ab}) - \gamma_{ab} \right] (\Psi - N^c \partial_c \Psi)$$

(16)

and [Eq. (2.6) of 19]

$$\Pi = \frac{\partial L}{\partial (\partial_t \Psi)} = \frac{\partial L}{\partial \Psi} = -\sqrt{\gamma} \left( K + \frac{1}{2N} (\Psi - N^c \partial_c \Psi) \right)$$

(17)

where $N^c$ is again the shift vector. Combining the trace of Eqs. (16) and (17) gives [Eq. (2.7) of 19]

$$(3 - \Psi) (\Psi - N^c \partial_c \Psi) = \frac{2N}{\sqrt{\gamma}} (\Psi \Pi - \pi)$$

(18)

where $\pi = \pi^{ab} \gamma_{ab}$. Note that we can write

$$n^\mu \nabla_\mu \Psi = \frac{1}{N} (\Psi - N^c \partial_c \Psi)$$

(19)

using $n^0 = \frac{1}{N}$ and $n^a = -\frac{N^a}{N}$. The total Hamiltonian can be derived as a linear combination of constraints as

$$H_{total} = \int_{\Sigma} d^3x (N^a V_a + NH)$$

(20)

where the smeared diffeomorphism and Hamiltonian constraints read, respectively [Eqs. (2.8) and (2.9) of 19]

$$V(N) = \int_{\Sigma} d^3x N^a V_a$$

(21)

$$= \int_{\Sigma} d^3x N^a (-2D^b (\pi_{ab}) + \Pi \partial_a \Psi - \bar{\xi}_a)$$

and

$$H(N) = \int_{\Sigma} d^3x N H$$

(22)

$$= \int_{\Sigma} d^3x N \left( \pi^{ab} H_{ab} + \Pi \Phi - L \right)$$

$$= \int_{\Sigma} d^3x N \left( \frac{2}{\sqrt{\gamma}} \left( \frac{\pi^{ab} \pi_{ab} - \frac{1}{4} \pi^2}{\Psi} + \frac{(\pi - \Psi \Pi)^2}{2} \Psi (3 - \Psi) \right) \right)$$

$$+ \frac{1}{2} \sqrt{\gamma} \left(-3 R \Psi - \frac{1}{2} (D_a \Psi) D^a \Psi + 2D_a D^a \Psi - 2 \Lambda \right) - \bar{\xi}_0$$

where $D_a$ is the covariant derivative with respect to the 3-metric $\gamma_{ab}$. Note that $D_a \Psi = \partial_a \Psi$ because $\Psi$ is a scalar and so [Eq. (A.4) of 19]

$$(D_a \Psi) D^a \Psi = \gamma^{ab} (D_a \Psi) D_b \Psi = \gamma^{ab} (\partial_a \Psi) \partial_b \Psi$$

and

$$D_a D^a \Psi = \frac{1}{\sqrt{\gamma}} D_a (\sqrt{\gamma} \gamma^{ab} D_b \Psi) = \frac{1}{\sqrt{\gamma}} D_a (\sqrt{\gamma} \gamma^{ab} \partial_b \Psi)$$

(23)

where $\Gamma^c_{ab}$ are the Christoffel symbols of the second kind [22] (see Appendix F).

The terms $\bar{\xi}_{\mu}$ are contributions from the matter Lagrangian, such as that which appears in [5]. According to the ADM approach (Sec. 6.2 of 20), the contributions to the constraint equations (21) and (22) can be obtained via

$$\bar{\xi}_0 = \frac{\partial M}{\partial N}$$

(24)

$$\bar{\xi}_a = \frac{\partial M}{\partial N^a}$$

upon using (13) in the matter Lagrangian $L_M$. Although we formally include these terms, we will not explicitly resolve their effect on the metric. Rather, the focus of the present work is on the treatment of the free Lagrangian contribution $L_F$ [Eqs. (21) and (22) but without $\bar{\xi}_{\mu}$].

The approach to solving the constraint equations in 1+1 dimensions was to obtain coordinate conditions that would set the conjugate momenta $\Pi$ and $\gamma_{ab}$ to fixed numerical values [1] and inject them into both the shift covector constraint equation and the Hamiltonian equation. However, as we shall see, in 3+1 dimensions the conjugate momenta do not generally collapse into fixed numbers but are functions of the dilaton field $\Psi$. Furthermore, the $1/\Psi$ and $3 - \Psi$ terms in Eq. (22) render a solution to the constraints somewhat unwieldy. We therefore use other important relationships determined by Zhang and Ma, including relating the curvature $K_{ab}$ to the momentum $\pi_{ab}$ and $\Pi$ Eq. (2.21) of 19, i.e.

$$K_{ab} = \frac{2}{(3 - \Psi) \sqrt{\gamma}} \left( \frac{(2 - \Psi)}{2(3 - \Psi)} \pi_{ab} \right) - \frac{\Pi \gamma_{ab}}{(3 - \Psi) \sqrt{\gamma}}$$

(25)

whose trace with respect to the 3-metric $\gamma_{ab}$ is

$$K = \frac{2\pi}{(3 - \Psi) \sqrt{\gamma}} - \frac{n_a \Pi}{(3 - \Psi) \sqrt{\gamma}}$$

(26)
where \( n, \gamma = 3 \) in \( 3 + 1 \) dimensions. Furthermore \( \gamma^{ab} \) and \( K^{ab} \) are related to each other by

\[
\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 = \frac{\gamma}{4} \left[ \Psi^2 (K^{ab} K_{ab} - K^2) - 3 \left( n^\mu \nabla_\mu \Psi \right)^2 - 2 \Psi K n^\mu \nabla_\mu \Psi \right]
\]  

(27)

where we discern the first two terms of Eq. (15) for \( R \). These relationships allow us to rewrite the constraint (22) in terms of \( K_{ab} \) [Eq. (2.17) of [19]] (valid in \( 3 + 1 \) dimensions),

\[
0 = \frac{\sqrt{\gamma}}{2} \left( K_{ab} K^{ab} - K^2 - 3R \right)
\]  

(28)

and

\[
\frac{\sqrt{\gamma}}{2} \left( 2D_a D^a \Psi - \frac{1}{2} (D_a \Psi) D^a \Psi \right)
\]

\[
- \sqrt{\gamma} \left[ (n^\mu \partial_\mu \Psi) \left( K + \frac{1}{4} n^\mu \partial_\mu \Psi \right) + \Lambda \right] - \tilde{\delta}_0
\]

upon using [19]. The structural form of Eq. (28) is easier to deal with than Eq. (22) since the problematic \( 3 - \Psi \) term of Eq. (22) is embedded into the curvature \( K_{ab} \). It is now a matter of eliminating variables where possible to isolate the equation governing the dilaton field under general conditions. Equation (28) will prove most useful in this regard. The remainder of this work serves to eliminate many of the variables of the formulations of Ref. [19] to obtain the final equation governing the dilaton field.

### III. ELIMINATION OF VARIABLES

#### A. Elimination of \( N \) and \( N^a \)

An important simplification results from setting [24]

\[
N = 1 \quad N^a = 0
\]  

(29)

which is allowed at the cost of abandoning explicit four-dimensional general covariance [24-26]. The time derivatives of \( N \) and \( N^a \) are also taken as zero. These settings are often used when applying standard ADM to GRT, and are justified with more detail in Appendix C. This section includes any restrictions to the class of solutions from these settings. Though different, these settings nonetheless agree respectively with Eqs. (94) and (95) of Ref. [3] of the values for a gauge choice in the limit \( \kappa \rightarrow 0 \) and approximately agree with the results \( N \neq 0 \) and \( N^a = 0 \) for a different gauge choice in the 1+1 case [Eq. (99) of [3]]. Note that the canonical theory of GRT does not directly determine the \( N^a \) (Sec. 4 of [19]); the latter are obtained later by the time evolution of the system, through e.g. [Eq. (16)] of [19]

\[
\dot{\gamma}_{ab} = 2NK_{ab} + D_a N_b + D_b N_a
\]  

(30)

and the consistency of the coordinate and gravitational equations. Equation (30) is nothing other than the definition of the extrinsic curvature \( K_{ab} \). Here, for the \( 3 + 1 \) case, \( n^\mu \partial_\mu \Psi = \frac{1}{\sqrt{\gamma}} (\dot{\Psi} - N^c \partial_c \Psi) = \dot{\Psi} \). The d’Alembertian of Eq. (4) reduces to

\[
\Box \Psi = - (\ddot{\Psi} + \dot{\Psi}) + D_a D^a \Psi
\]  

(31)

where

\[
\dot{\epsilon} = \partial_t (\ln \sqrt{\gamma})
\]

when \( \partial_t N^a = 0 \). Note that under our choice of coordinate conditions, the coefficient of \( \dot{\Psi} \) in Eq. (31) will add nothing to the chosen class of solutions for the dilaton field.

#### B. Curvature \( K \)

Another important equation is the left-hand side of Eq. (2.15) in Ref. [19]

\[
\dot{\Pi} - \partial_a (N^a \Pi) - \partial_\mu (\sqrt{-gn} \n^\mu \Pi) = - \frac{1}{2} \partial_\mu \left( \sqrt{-gn} n^\mu \partial_\nu \Pi \right)
\]  

(32)

which in light of (24) becomes

\[
\frac{\partial}{\partial t} \left[ \Pi - K \sqrt{\gamma} \right] = - \frac{1}{2} \partial_t (\sqrt{\gamma} \dot{\Pi})
\]  

(33)

This also assumes that the spatial derivatives of \( N \) and \( N^a \) are also zero (note that the result obtained here will be reiterated further in Sec. III C). The combination of Eqs. (18), (26) and (29) yields an expression for \( \Pi \),

\[
\Pi = - \sqrt{\gamma} K - \frac{1}{2} \sqrt{\gamma} \dot{\Pi}
\]  

(34)

and implicit differentiation of Eq. (34) above with respect to the time \( t \) also yields a result for \( \partial_t \Pi \) and is consistent with Eq. (35) if

\[
\partial_t (K \sqrt{\gamma}) = 0 \quad \Rightarrow \quad \dot{K} = - \frac{K}{2} \partial_t \ln(\gamma)
\]  

(35)

The expression above will be examined further but under the assumptions made, we can see that \( \dot{K} \) is a constant of the motion ( \( \dot{K} = 0 \) if \( \dot{\gamma} \) is zero.

#### C. Coordinate Conditions

We begin by noting that any given symmetric second rank tensor \( f_{ab} \) has the orthogonal decomposition [Eqs. (4.7a) of Ref. [12], Eqs. (2.10) – (2.12) of [27]]

\[
f_{ab} = f_{ab}^{TT} + f_{ab}^T + f_{ab} + f_{b,a}
\]

where

\[
f_{ab}^{TT} = \frac{1}{2} \left( f_{ab}^{T} \delta_{ab} - \frac{1}{\Delta} f_{ab}^{T} \right)
\]

\[
f_{ab}^T = f_{aa} - \frac{1}{\Delta} f_{ab,a,b}
\]

\[
f_a = \frac{1}{\Delta} \left( f_{ab,b} - \frac{1}{\Delta} f_{bc,bca} \right)
\]
and \( f_{ab}^{TT} = f_{ab}^{T} - \frac{1}{2} f^{T} \delta_{ab} \) is the transverse-traceless (TT) part of \( f_{ab} \) and \( \triangle \) is the Laplacian for the 3-metric. We apply this to \( g_{ab} \) and \( \gamma^{ab} \). Next, we define

\[
\pi_{ab} \rightarrow \pi_{ab}^{GRT} \quad (36)
\]

and then make an orthogonal decomposition

\[
\begin{align*}
h_{ab} &= h_{ab}^{TT} + h_{ab}^{T} + h_{ab} + h_{b,a} \\
\pi^{ab} &= \pi^{abTT} + \pi^{abT} + \pi^{a,b} + \pi^{b,a}
\end{align*}
\quad (37)
\]

The definition of \( h_{ab} \) in Eq. (36) is that of Kimura [Eq. (2.4a) of [14]] and especially Ohta [Eqs. (2.10) – (2.19) of [27]], and not that of ADM. This is essential for our discussion. Also bear in mind that \( \pi^{ab} \) in Eqs. (37) is the GRT quantity. The coordinate conditions and the generator are worked out in Appendix D. Equation (16) can be rewritten as

\[
\pi^{ab} = -\frac{1}{2} \Psi \pi^{ab}^{GRT} - \sqrt{\gamma} \pi^{ab} \left( \Psi - N^{c} \partial_{c} \Psi \right) \quad (38)
\]

where \( \pi^{ab}^{GRT} = -\sqrt{\gamma} (K^{ab} - K g^{ij}) \) is the standard definition in ADM [Eq. (3.3) of [12]]. This suggests treating \( \pi^{ab}^{GRT} \) as a function of the coordinates only (for a given time, as is usually the most desirable scenario in standard ADM applied to GRT) and recasting our scalar-tensor theory into the mold of standard ADM (or nearly so). To this end, the ADM generator \( G \) is developed as shown in Appendix D together with the orthogonal decomposition of Eqs. (37) applied to \( \pi^{ab}^{GRT} \). This yields the following coordinate conditions for \( \pi^{ab} \) and \( g_{ab} \)

\[
\begin{align*}
\gamma_{ab} &= g_{ab} = \delta_{ab}(1 + \frac{1}{2} h^{T}) + h_{ab}^{TT} \\
\pi^{aa} &= -\sqrt{\gamma} \pi^{aa} \left( \Psi \neq 0 \right) \quad (40)
\end{align*}
\]

since \( N^{c} = 0 \). This reduces to the standard result \( \pi^{aa} = 0 \) in the GRT limit as \( \Psi \to 1 \) as expected, or simply if \( \Psi = 0 \). Equation (39) is the familiar result obtained by standard ADM applied to GRT. In the absence of gravitons, or generally for \( g_{ab}^{TT} = 0 \), the metric \( g_{ab} \) reduces to the isotropic form [Eq. (4.7) of [12], i.e.,

\[
g_{ab} = \gamma_{ab} = \delta_{ab} h = \delta_{ab}(1 + \frac{1}{2} h^{T}) \quad \rightarrow \quad \sqrt{\gamma} = h^{3/2} \quad (41)
\]

with the far-field boundary condition

\[
\lim_{r \to \infty} h^{T}(r) = 0.
\]

In isotropic coordinates \( \delta_{ab} \gamma^{ab} = 3/h \) and, from Eq. (40)

\[
\pi = \frac{3}{2} \sqrt{\gamma} \Psi = -\frac{3}{2} \sqrt{\gamma} \Psi
\]

\( \Rightarrow K = 0 \)

\[
\Pi = -\frac{1}{2} \sqrt{\gamma} \Psi = \frac{1}{3} \pi
\]

Substituting \( \Pi \) from Eq. (12) into an implicit differentiation of Eq. (13) with respect to the time \( t \) yields the same equation, Eq. (35), relating \( \dot{K} \) to \( K \) and thus vindicating it [this is because this recent derivation did not require explicit assumptions about the spatial derivatives of \( N \) and \( N^{a} \) being zero, but resulted rather from Eq. (29) and the coordinate conditions].

Thus the simplifications of Eq. (29) with Eqs. (35) and (42) lead to \( \dot{K} = K = 0 \). Consequently, the Ricci scalar in Eq. (15) with (11) simplifies to

\[
R = K_{ab}K^{ab} - K^{2} + 3R = 0 \quad (43)
\]

Note that there remains the term \( \partial_{a} (\sqrt{\gamma} \gamma^{ab} \partial_{b} N) \) on the far-right side Eq. (15) but even if \( \partial_{b} N \) is not taken as zero, it can be “absorbed” in the Ricci scalar term \( 3R \) and added on as discussed later. Moreover, if \( \Psi \) is time independent, then \( \Psi, \pi, \) and \( \Pi \) are all zero. As mentioned in Appendix A the effect of the graviton can be treated as the (TT) part of the metric and can be handled thanks to e.g. Eq. (1.55) of [28],

\[
\sqrt{-g} = \sqrt{-\eta} \left[ 1 + \frac{1}{2} h_{\mu}^{T} + \frac{1}{8} h_{\mu}^{N} h_{\nu}^{T} - \frac{1}{4} h_{\mu}^{N} h_{\nu}^{N} \right]...
\]

which is valid for any perturbation \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), where the raising and lowering of indices is done with respect to any arbitrary background with metric \( \eta \). A (TT) perturbation would consequently only affect \( \sqrt{-g} \) at second order. In a number of cases, the (TT) contribution is not used (e.g. Kimura’s treatment of the two-body problem within the Einstein-Infeld-Hoffman approximation [P. 159 of [13]]). Therefore, in what follows we work mostly with isotropic coordinates in which the (TT) contribution vanishes to leading order perturbatively.

In general, from the generator of Eq. (13), the action can be rewritten in the tradition of ADM [Eq. (4.17) of [12]],

\[
G = G_{M} + \int dx^{3} \left\{ \pi^{abTT} \delta h_{ab}^{TT} + T_{0}^{a} \delta x^{a} \right\} \quad (44)
\]

where \( G_{M} \) is the generator from the matter Lagrangian. Therefore

\[
T_{0}^{a} = H = \triangle h^{T} \quad (45)
\]

\[
T_{a}^{0} = -2\partial_{b} \pi^{ab} \quad (46)
\]

and the Hamiltonian density is in terms of the metric just as in standard GRT, and not \( \Delta \Psi \) as it would be in the 1 + 1 case. In the latter, the dilaton field represents most of the dynamics (because conventional GRT yields nothing in 1 + 1 whereas in 3 + 1 dimensions, the dilaton field is treated as a small departure from GRT. (Appendix A shows that conventional GRT is sufficient to ensure the correct nonrelativistic limit). Usually such simplifications would allow us to address the constraint equations (24) and (22) in terms of the two unknowns \( \pi^{a b} \) and \( h^{T} \) (related to the metric). However, we have additionally the dilaton field \( \Psi \). This is addressed in the next sections.
IV. SHIFT-COVARIANT CONSTRAINT EQUATION

Here, we highlight the possibilities of reduction and simplification to make Eq. (21) more solvable. Given Eq. (45), we consider dividing $\pi^{ab}$ into two parts, where the second part depends explicitly on $\Psi$; expressing the momentum in lower indices,

$$\pi_{ab} = -\frac{1}{2}\Psi \pi_{ab} G + p_{ab} \quad \text{where} \quad p_{ab} = -\frac{\sqrt{T} \gamma_{ab}}{2} \Psi$$ (47)

which allows us to simplify the left side of the shift-vector equation with

$$D^b(p_{ab}) = -\frac{\sqrt{T} \gamma_{ab}}{2} D^b(\Psi) = -\frac{\sqrt{T}}{2} D_a(\Psi) = -\frac{\sqrt{T}}{2} \partial_a \Psi$$

From Eq. (18)

$$\Pi \partial_b \Psi = \frac{\sqrt{T}}{2\Psi} (3 - \Psi) \partial_a \Psi + \frac{\partial_b \Psi}{\partial_a \ln \Psi}$$ (48)

We see that if $\Psi = 0$ then both $\pi = 0$ [from Eq. (40)] and $p_{ab} = 0$ [from Eq. (17)]. Consequently, $\Pi \partial_b \Psi = 0$ in Eq. (48). In such a case, the shift-vector constraint (21) reduces to that of standard GRT and $\pi^{ab}$ can be readily calculated by existing methods. For a nonzero $\Psi$, Appendix E shows that for these particular cases of separability for $\Psi$, i.e.,

$$\Psi = F(x) G(t) \quad \text{product} \quad (49)$$

$$\Psi = F(x) + G(t) \quad \text{sum} \quad (50)$$

the first term on the right-side of Eq. (48) is a divergence; i.e. it yields a vanishing surface term to the integral of the shift-vector constraint equation over spatial coordinates. Consequently it does not contribute to the Euler-Lagrange equations and can therefore be discarded. In the appendix, we make use of the “densitized” lapse function or “Taub function”, which was introduced by York as a means of improving the ADM approach [Eq. (41) of 29],

$$\alpha \equiv \frac{N}{\sqrt{T}}$$ (51)

and which often appears in the context of setting an initial value problem. In ADM, the lapse function tells how the proper time moves along from spatial slice to spatial slice as the coordinate time moves. Its setting is a matter of choice, and is consequently an additional coordinate (“gauge”) freedom which does not change the physical solution, but will change how well posed the problem is 30, 32 (and what happens if the initial data conditions are slightly perturbed). The Taub function appears in boundary-value problems 32 and for stabilizing numerical relativity 34, 35. In our case, clearly $\alpha \to 1$ in the far-field limit.

Thus for $\Psi \neq 0$ and for the separable cases of Eqs. (49) or (50), the first term of Eq. (15) can be discarded, leaving only the term proportional $\pi \partial_a \ln(\Psi)$, with $\pi$ given by Eq. (42). In isotropic coordinates this is also a divergence and can therefore be discarded, as explained in Appendix E.

These particular separable solutions of $\Psi$ therefore eliminate the $\pi \partial_a \ln(\Psi)$ term, and bring the vector equation of Eq. (21) much closer to the GRT result. Consequently we can make use of existing GRT results to obtain solutions to the vector constraint Eq. (21).

We now focus exclusively on the first term of Eq. (21) and since the indices of individual components can be raised and lowered with the metric, e.g. $D^b(\pi_{ab}) = D_b(\pi^a_b)$, we make use of an important result in ADM [Sec. 3 of 21], which allows us to convert the covariant derivative into a simple partial derivative, i.e.,

$$D^b(\pi_{ab}) \to \partial_b \pi_a^b \to \partial_b \pi^{ab}$$ (52)

In this regard, a useful identity is

$$D_b(\pi^a_a) = \partial_b \pi^a_a - \frac{1}{2} \left[ \pi^{bc} \partial_a \gamma_{bc} \right] + \frac{1}{2} \pi^a_a \operatorname{Tr}(\partial_c \ln(\gamma))$$ (53)

as shown in Appendix E. Under the orthogonal decomposition for $\pi^{ab}$ and $\gamma_{ab}$, the term in the square brackets will contain a divergence which can be completely eliminated, apart from a $(TT)$ contribution (which is zero in our choice of isotropic coordinates) [Eqs. (3.10) – (3.11) of 21]. The last term in Eq. (53) involves the logarithmic derivative of the determinant of the metric. It will vanish if the volume (whose element is proportional to this term) is fixed within ADM. Although this is the case in many applications of ADM, this could be in doubt in e.g. cosmological studies of an expanding universe. However, it can be justified if e.g. the Taub function $\alpha$ of Eq. (51) is unit or a constant. Moreover, for the last part of Eq. (52), the difference between $\partial_b \pi^a_a$ and $\partial_b \pi^{ab}$ is also a divergence under this orthogonal decomposition [Eq. (3.12) of 21]. Kimura apparently uses this result in the transition from his Eq. (2.3b) to Eq. (3.5b) in Ref. 13.

$$-2D_b(\pi^a_a) \to -2\partial_b \pi^{ab}$$ (modus the raising/lowering operation of indices with the metric). However, Ohta et al. does not, and instead computes the explicit Christoffel symbol for the covariant derivative [Eq. (3.5) of 27]. Yet, for a matter Lagrangian of Eq. 9 without external fields, both obtain the same solutions using an iterative approximation scheme for the lead term of the metric [Eq. (3.9) of 13 and Eq. (3.7) of 27],

$$h^T \approx \sum_i \frac{m_i}{4\pi r_i}$$ (54)

where $r_i = |r - z_i|$ and for the momenta [Eq. (3.13) of 13, 27, and Eq. (3.10) of 27],

$$\pi^i \approx \frac{1}{8\pi} \sum_i \left\{ p_{in} \left( \frac{1}{r_i} \right) - \frac{1}{4} p_{ib} \partial_b \partial_i r_i \right\}$$ (55)
(though only cited the solution for \( \pi^{ij} \) is relevant here.) Here we have only cited the results (without rederiving them) which give us confidence in Eq. (33) in its use via the reductions of Ref. 21, and the iterative methods for obtaining solutions by Kimura and Ohta. Therefore, Eqs. (54) and (55) can serve as initial solutions of the metric using the anzatz (9) for the case \( \Psi \) (or equivalently small \( \kappa \)) in a perturbative scheme for small \( \kappa \). Ohta’s solutions include a transverse-traceless contribution \( g_{ab}^{TT} \) [Eq. (3.11) of 22]. As noted above, the \( (TT) \) contribution only affects the metric \( \gamma_{ab} \) at second order as shown by Ohta et al. and adds linearly at this order [Eq. (4.1) of 22]. We have thus identified conditions for which the field \( \Psi \) need not be injected into the shift-vector equation (or, alternatively, conditions for eliminating most or all of the terms involving \( \Pi_0 \Psi \)), or for which the Christoffel symbols need not be included in the covariant derivative, thus making the task of solving for the momenta \( \pi^{ij} \) tractable in terms of known methods.

V. HAMILTONIAN CONSTRAINT

Taking all the simplifications of Sec. II into account including Eq. (13), \( K \) in Eq. (12), the isotropic metric of Eq. (11) with \( K = 0 \), and, especially the combination of Eqs. (11) and (13) which allows us to rewrite \( R \), the Hamiltonian constraint of Eq. (28) can be rewritten entirely in terms of the metric and \( \Psi \) and its time derivatives as

\[
\frac{\sqrt{\tilde{h}}}{2} \left( 5 \nabla^2 \Psi - (\nabla \Psi)^2 \right) + \frac{5}{8\sqrt{\tilde{h}}} \nabla \Psi \cdot \nabla h^T
- \frac{h^{3/2}}{2} \left\{ \frac{3}{2} \frac{\partial^2 \Psi}{\partial t^2} + \frac{1}{4} \left( \frac{\partial \Psi}{\partial t} \right)^2 \right\} + \frac{1}{2} \sqrt{\tilde{g}} \nabla \cdot (3 \nabla h^T)
- \frac{h^{3/2}}{2} \left( \nabla \Psi \cdot \nabla h^T \right) - \tilde{S}_0 = 0
\]

where

\[
\chi_1 = \frac{1}{c} = \frac{\partial (\ln \sqrt{\gamma})}{\partial t} - \partial_a N^a = \frac{\partial (\ln \sqrt{\gamma})}{\partial t}
\]

\[
\chi_2 = -\frac{2}{\sqrt{\tilde{g}}} \partial_a \left( \sqrt{\gamma} \gamma^{ab} \partial_b N \right) = 0
\]

and where the gradients are now with respect to the Euclidean 3-metric \( \nabla_\theta = \partial_\theta \) and \( \tilde{S} \) is the right side of Eq. (13) i.e. \( \tilde{S} = \frac{\kappa}{\sqrt{\tilde{g}}} T_{\mu}^\mu \) - 4\Lambda. The dot product of the gradients of \( \Psi \) and \( h^T \) i.e. \( \nabla \Psi \cdot \nabla h^T \) results from the Christoffel symbols of the covariant Laplacian of Eq. (23).

The second derivative of \( \Psi \) with respect to time i.e. \( \frac{\partial^2 \Psi}{\partial t^2} \) appears in Eq. (55) because the first term in brackets on the right-hand side of Eq. (28) for the Hamiltonian constraint, expressed in terms of the extrinsic curvature \( K_{ab} \) and the Ricci scalar \( 3R \), is rewritten in terms of the d’Alembertian of Eq. (53) using the simplification of Eq. (13).

Note that we have rewritten the constraint (22) in terms of time derivatives of \( \Psi \) for reasons of convenience: our goal here is to obtain a self-contained solvable differential equation for \( \Psi \). Equation (50) can be recast into canonical form by employing Eqs. (42) and (31) to eliminate \( \dot{\Psi} \) and \( \dot{\Psi} \) in terms of \( \Pi_0 \) and other fields. Moreover, the term \( \chi_1 \) is the coefficient \( \epsilon \) of \( \dot{\Psi} \) for the d’Alembertian in Eq. (51), and \( \chi_2 \), is the remaining term of the four-dimensional Ricci scalar of Eq. (17) and is neglected as mentioned in the simplification of Eq. (13).

Using the separability of Eq. (50) for \( \Psi \) by which the term \( \Pi_0 \Psi \) can be eliminated in the shift-vector equation of Sec. IV we eliminate \( (\nabla \Psi)^2 \) using the same approach as in Eq. (29) of 7, i.e.

\[
\Psi(t, x, y, z) = F(t) - c \ln(|\psi(x, y, z)|)
\]

With \( c = 5 \) Eq. (50) becomes

\[
\frac{25\sqrt{\tilde{g}}}{\psi^2} \frac{\nabla^2 \psi}{2} - \frac{25}{8\sqrt{\tilde{g}}} \nabla \psi \cdot \nabla h^T
- \frac{h^{3/2}}{2} \left\{ \frac{3}{2} \frac{\partial^2 F(t)}{\partial t^2} + \frac{1}{4} \left( \frac{\partial F(t)}{\partial t} \right)^2 + \frac{3}{2} \chi_1 \frac{\partial F(t)}{\partial t} \right\}
- \frac{h^{3/2}}{2} \left( \tilde{S} + 3R \left( 2F(t) - 10 \ln(|\psi|) \right) + 2\Lambda \right) - \tilde{S}_0 = 0
\]

If we divide the above by \( h^{3/2} \) and ignore \( 3R \), Eq. (59) divides into the sum of a pure function of \( t \) only and a function of the spatial coordinates only where each term is forscibly a constant (for all time and spatial positions) depending of course on how the matter Lagrangian term depends on spacetime coordinates.

For further simplicity, let us consider a matter Lagrangian term depending only on the spatial coordinates. If we let \( F(t) = F \) be a constant, Eq. (58) fits into the pattern of Eq. (46) and \( \phi \) is time independent [and terms like \( \Psi \) do not appear in Eq. (58)]. Also, the term in \( \chi_1 \) of Eq. (57) from the coefficient of \( \Psi \) in Eq. (51) drops out, as mentioned before. Note that in the 1+1 case, \( c = 4 \) and the difference is caused by the first term coupled to \( \Psi \) in Eq. (28), i.e. \( (K_{ab} \kappa^{ab} - K^2 - 3\tilde{S}) \), which does not appear in the 1+1 case. The term \( \nabla \Psi \cdot \nabla h^T \) vanishess in the far field; it can be eliminated via the transformation

\[
\Phi = h^{1/4} \psi
\]

and Eq. (59) can be rewritten as

\[
- \frac{1}{2} \nabla^2 \Phi + V \Phi + S \Phi \ln(|\Phi|) - E \Phi = 0
\]

where

\[
V = \frac{1}{16h} \nabla^2 \tilde{h}^T - \frac{3}{128h^2} (\nabla \tilde{h}^T)^2 - \frac{h}{50} \tilde{S}
- \frac{h}{20} \left( \frac{\ln(h)}{25} + \frac{F}{25} \right) 3R - \frac{h}{50} \frac{\tilde{S}_0}{25 \sqrt{h}},
\]

\[
E = -\frac{\Lambda}{25} \quad \text{and} \quad S = \frac{h}{5} 3R.
\]
Eq. (61) has the functional form of a logarithmic Schrödinger equation. As mentioned before, the term $\chi_2$ of Eq. (57) resulting from the 4-dimensional scalar curvature of Eq. (15) merely adds to the the Ricci scalar of the 3-metric. The Ricci scalar is

$$3R = \frac{1}{h^2} \nabla^2 h^T - \frac{3}{8h^4} (\nabla h^T)^2$$  \hspace{1cm} (62)

Thus, the potential $V$ in Eq. (61) is made of gradients of the metric and the matter Lagrangian term as well as the gravitational constant (common to the “eigenenergy” $E$).

### A. Equation for the 3-metric

There remains the matter of obtaining $h^T$ for the isotropic coordinates themselves. The relationship between $\rho^{ab}$ and $K^{ab}$ in Eq. (27) with Eq. (13) and Eq. (13), the latter resulting from Eq. (29), becomes

$$\Psi^2 R = \Psi^2 \Box \Psi = \Psi^2 (K^{ab} K_{ab} - K^2 + 3R) \hspace{1cm} (63)$$

$$= \frac{4}{\gamma} \left( \pi_{ab} \pi^{ab} - \frac{1}{2} \pi^2 \right) + \frac{3}{2} \frac{\Psi^2}{2} + 2 \Psi K \dot{\Psi} + 3 \frac{R}{\Psi} \frac{\Psi^2}{2}$$

$$= \frac{4}{\gamma} \left( \pi_{ab} \pi^{ab} - \frac{1}{2} \pi^2 \right) + \frac{3}{2} \frac{\Psi^2}{2} + 3 R \frac{\Psi^2}{2}$$

since $K = \dot{K} = 0$. Using the same simplifications as before, we obtain

$$5h^2 \left( \nabla^2 \psi - \left( \frac{\nabla \psi}{\psi} \right)^2 \right) + \frac{4}{25} \left( \frac{\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2}{\ln(\psi)^2} \right)$$

$$+ \hbar \left( \nabla^2 h^T + \frac{5 \nabla \psi \cdot \nabla h^T}{4 \psi} \right) - \frac{3}{8} (\nabla h^T)^2 = 0$$  \hspace{1cm} (64)

with $\psi$ is related to $\Phi$ via Eq. (60), where Eq. (62) for the Ricci scalar of the 3-metric was used, and again the gradients in Eq. (61) are with respect to to Euclidean 3-metric. Eq. (61) looks complicated, but once $\pi_{ab}$ is obtained from the shift-vector equation and $\Phi$ from the Hamiltonian, it is a matter of injecting these quantities into Eq. (64) through which we can solve for the metric term $h^T$. In the far field, the coefficient of the log potential term $S \to 0$ and consequently Eq. (61) reduces to the standard linear Schrödinger equation and is completely decoupled from the metric terms $\hbar$ and $h^T$ that also go to zero, and thus asymptotically, the equation governing the dilaton field is completely self-contained with the matter Lagrangian (subject to regularization with respect to the metric term $\hbar$) being the case also for the $1+1$ problem.

### VI. DISCUSSION

After deriving the field equations for the scalar-tensor theory represented by Eq. (5), a $d+1$-dimensional version of the $1+1$ action of Eq. (1) was proposed and shown to yield the correct nonrelativistic limit in $d$ dimensions. We then applied the Arnowitt-Deser-Misner method with the gauge settings of Eq. (28) and the orthogonal decompositions of the momentum $\pi_{ab}$ and the metric $g_{ab}$ according to Eq. (37). We found essentially three coupled PDEs: the shift-covector equation governing the momentum $\pi_{ab}$ as given by Eq. (24) and discussed in sec. 15, the Hamiltonian constraint governing the dilaton field $\Psi$ as given by Eq. (22) or equivalently Eq. (25), and a relationship by which the metric can be obtained from the solutions of $\pi_{ab}$ and $\Psi$ namely Eqs. (11) and (15) with Eq. (27).

We found that under the right choice of coordinate and gauge conditions, i.e. in isotropic coordinates and when $g_{ab} = h_{ab} = 0$, the PDE governing the dilaton field in Eq. (28) is a logarithmic Schrödinger equation, the nonlinear logarithmic term being directly proportional to the Ricci scalar of the 3-metric which becomes zero in the far-field limit (Minkowski flat-space) as given by Eq. (40). In the latter regime, the PDE becomes a linear Schrödinger equation in the far field. Equation (64) then allows one to solve for the metric in terms of $\psi$.

Thus, we find that our proposed $3+1$ scalar-tensor theory holds similar properties to those of the $1+1$ formulation, very much what would be expected from sound dimensional scaling. The main difference is that unlike the $1+1$ case where the reduced Hamiltonian is expressed as a form of spatial integral of the second derivative of the scalar field, the $3+1$ formulation uses the second derivative of the metric function $h^T$, just as in standard GRT. This can be understood from the fact that in $1+1$ dimensions, the dilaton field contributes most of the physics (GRT in $1+1$ dimensions yields nothing). However, in $3+1$ dimensions, the dilaton field represents a small departure from standard GRT.

This outcome is interesting as the logarithmic Schrödinger equation finds applications in quantum mechanics, the theory of superfluidity and Bose-Einstein condensates [37], and even nuclear physics [38]. As in the previous $1+1$ case, since we already know the Lagrangian density whose Euler-Lagrange equations are the (linear) Schrödinger wave equations, we can obtain a Hamiltonian density and quantize the system. This procedure, often called second quantization, allows transitions between states, with the dilaton itself acting as the agent of transition. In effect, this yields a theory of quantum gravity which is normalizable because only the dilaton field is quantized. (Note that our treatment of the one-graviton exchange in Appendix A is still relevant because the effective potentials $V_2$ are obtained in the lowest order and have a well-known correspondence with classical mechanics in the limit $\hbar \to 0$.)

From here, many directions are possible. An obvious next step is to explicitly solve Eq. (15) for the momentum $\pi_{ab}$ according to sec. 15, Eq. (61) for the wave function $\Phi$, and Eq. (64) for the metric term $h^T$ to obtain the Hamiltonian of Eq. (15) using the explicit terms of $\Phi$ from Eq. (24) i.e. the various components of the
matter Lagrangian in Eq. (6). Analytical solutions are desirable, and we anticipate that the generalized Lambert W function will be useful as it was for the 1 + 1 lineal gravity problem [13, 33, 40]. Departures from the gauge conditions of Eq. (29) and the addition of a nonzero transverse-traceless component for the metric (to model gravitational radiation, perhaps) can be explored by iterative schemes like those mentioned in Sec. A1.

It has been hypothesized that superfluid vacuum theory (SVT) might be responsible for the mass mechanism (in contradistinction to the Higgs boson or perhaps working in tandem with it). Some versions of SVT favor a logarithmic Schrödinger equation [41]. Given the apparent resemblance between the Higgs boson and the dilaton mentioned earlier in the Introduction, the formulation herein could be fruitful in investigating this direction. However, we wish to emphasize that regardless of whether the dilaton and Higgs boson are related to each other or not, the results of the present work and their implications concerning quantum gravity stand on their own.

The very fact of the dilaton field being governed by an energy-balancing quantum mechanical wave equation suggests that the wave function itself might also be a “geometrical” quantity, apart from its usual interpretation; this would be very much as spin is treated in GRT, but such a notion needs further investigation. Granted, we have found a particular class of solutions to the scalar-tensor gravitational theory proposed in Eq. (5); nonetheless, the class found here shares similar essential properties with its simpler 1 + 1 counterpart of Eq. (11).

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Appendix A: Correspondence with Newtonian gravity in d + 1 dimensions

In this section we illustrate how a Newtonian limit generically arises in d + 1 dimensions for the scalar-tensor theory of Eq. (5). We shall compute the Newtonian limit(s) by considering the one graviton exchange potential (keeping in mind that there are no propagating gravitons in two or three spacetime dimensions). Gravitons can only propagate in 3 + 1 dimensions (so the treatment for n = d + 1 < 4 is formal).

Specifically, we calculate the T-matrix element of the one graviton exchange diagram in d + 1 Einstein gravity in the framework of the conventional quantum field theory, and we determine the classical potential as the Fourier transformation of the T-matrix element in the limit of h → 0. Here the word “conventional” means that we do not touch on the Faddeev-Popov (FP) ghosts etc . . . As far as we treat the lowest order and static contributions, this causes no problem.

We begin by extending the theory in Eq. (1) to d + 1 = n dimensions and coupling N scalar fields. We introduce the neutral scalar fields with mass m_{a} instead of the point particles. This yields the Lagrangian density: in 1 + 1 dimensions the \tilde{g}_{\mu\nu} = \sqrt{-\tilde{g}} g^{\mu\nu} is not an appropriate variable for developing the quantum field theory, because its components are not independent due to the identity \det(\tilde{g}_{\mu\nu}) = -1. We define the graviton field \tilde{h}_{\mu\nu} and the dilaton field \psi from Eq. (9) via

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa \tilde{h}_{\mu\nu} \]  

(A1)

\[ g_{\mu\nu} \approx \eta_{\mu\nu} + \kappa h_{\mu\nu} \]  

(A2)

where \eta_{\mu\nu} is the metric for flat (Minkowski) space. The reason why we defined the dilaton field not by \Psi = \kappa \psi but by \eta_{\mu\nu} is to introduce the correct kinematical part of the graviton field and also formally ensure the Einstein-Hilbert action as \kappa \rightarrow 0 for \Psi.

Though this separation in Eq. (A1) is well known in the weak-field approximation or “linearized gravity” [12], it is exact and can be done without any loss of generality. However the counterpart in terms of upper indices is not exactly separable. To first order in \kappa, it is given by

\[ g^{\mu\nu} \approx \eta^{\mu\nu} + \kappa h^{\mu\nu} \]

where \tilde{h}_{\mu\nu} = \eta^{\alpha\beta} \tilde{g}^{\mu\nu} h_{\alpha\beta} and we treat \tilde{h}_{\mu\nu} as the graviton field. In general, it is not possible to separate the metric into a static background and a perturbation in the form of radiation. However, if such a separation can be made the perturbation is transverse (perpendicular to the direction of motion) and traceless relative to the background. Thus the physical graviton is the transverse-traceless part of \tilde{h}_{\mu\nu}, we refer here to all components as graviton. Here,
the Lagrangian density of Eq. (8) becomes

$$L_0 = -\frac{1}{2} \left\{ \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} - \partial^\lambda h^{\mu\nu} \partial_\lambda h_{\mu\nu} - 2 \partial_\mu h^{\lambda\nu} \partial_\lambda h_{\mu\nu} \right. $$

$$\left. + 2 \partial_\mu h^{\lambda\nu} \partial_\lambda h_{\mu\nu} \right\} + 2 \partial^{\nu} (\partial^\rho h_{\mu\nu} - \partial_\mu h^{\rho\nu}) \psi + \partial^\mu \psi \partial_\mu \psi $$

$$- \frac{1}{2} \sum_a (\varphi_{a\mu} \varphi_{a\mu} + m_a^2 \varphi_{a\mu}^2) \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
where
\[ \bar{\eta}_{\mu \nu} \equiv \eta_{\mu \nu} + \eta_{\mu 0} \eta_{\nu 0} . \] (A19)

The proof of this relation is given in the Appendix of ref. [1]. The solution to (A14) is
\[ h_{\mu \nu}(x) = - \int d^{n-1}z \, D^{(n)}(x-z) \partial_0^2 h_{\mu \nu}(z) \]
\[ - \int d^{n-1}z \, \tilde{D}^{(n)}(x-z) \partial_0^2 \Box h_{\mu \nu}(z) \]
\[ - \int d^{n-1}z \, \tilde{\tilde{D}}^{(n)}(x-z) \partial_0^2 \Box^2 h_{\mu \nu}(z) \] (A20)

where \( D^{(n)} \), \( \tilde{D}^{(n)} \) and \( \tilde{\tilde{D}}^{(n)} \) are defined via
\[ D^{(n)}(x) = - \frac{i}{(2\pi)^{n-1}} \int d^n k \, e^{i k \cdot x} \delta(k^2) \]
\[ \tilde{D}^{(n)}(x) = - \frac{i}{(2\pi)^{n-1}} \int d^n k \, e^{i k \cdot x} \delta'(k^2) \]
\[ \tilde{\tilde{D}}^{(n)}(x) = - \frac{i}{(2\pi)^{n-1}} \int d^n k \, e^{i k \cdot x} \delta''(k^2) \]
as elaborated in appendix [2]. We next need to express all of \( h_{\mu \nu}, \partial_0 h_{\mu \nu}, \Box \partial_0 h_{\mu \nu}, \Box^2 h_{\mu \nu} \) and \( \Box^2 \partial_0 h_{\mu \nu} \) in terms of the canonical variables and calculate commutators at equal-time. This rather lengthy and complicated calculation is given in the Appendix of ref. [1].

From (A20) and the equal-time commutators, the commutator among the components of \( h_{\mu \nu} \) at two arbitrary space-time points can be calculated
\[ \langle 0| T(h_{\mu \nu}(x) h_{\lambda \rho}(y)) |0\rangle = - \frac{i}{2(2\pi)^{n}} \int d^n k \, e^{i k \cdot (x-y)} \frac{X_{\mu \nu, \lambda \rho}}{k^2 - i\epsilon} \] (A22)

where
\[ X_{\mu \nu, \lambda \rho} = \eta_{\mu \lambda} \eta_{\nu \rho} + \eta_{\mu \rho} \eta_{\nu \lambda} - \frac{2}{n} \eta_{\mu \nu} \eta_{\lambda \rho} \]
\[ + \frac{1}{2k^2} \left\{ - \eta_{\lambda \rho} k_\mu k_\nu - \eta_{\mu \rho} k_\lambda k_\nu - \eta_{\mu \lambda} k_\rho k_\nu - \eta_{\nu \rho} k_\lambda k_\mu \right\} \]
\[ + \left( 1 - \frac{2}{n} \right) \frac{1}{k^2} \delta_{\mu \lambda} \delta_{\nu \rho} . \] (A23)

We turn now to the scalar fields. Since \( \sqrt{-g} = 1 + \frac{2}{n} \sqrt{g} + \mathcal{O}(\kappa^2) \), the interaction Lagrangian in the lowest order is
\[ L_{\text{int}} = - \frac{1}{2} \left\{ \frac{1}{2} \eta^{\mu \nu} \left( \phi^\alpha \phi_\alpha + m^2 \phi^2 \right) - \phi^\mu \phi^\nu \right\} h_{\mu \nu} . \] (A24)

Using the propagator (A22), the S-matrix element of the one graviton exchange between two scalar particles is calculated as
\[ S_n = \frac{4\pi i G_n}{(2\pi)^{n-2}} (p_1^0 p_2^0 q_1^0 q_2^0)^{-1/2} \left[ \frac{p_1^\eta q_1^\eta}{2} - \frac{1}{2} \eta^{\mu \nu} (p_1 \cdot q_1 + m_1^2) \right] \left[ p_2^\lambda q_2^\lambda - \frac{1}{2} \eta^{\rho \sigma} (p_2 \cdot q_2 + m_2^2) \right] \times \frac{X_{\mu \nu, \lambda \rho}}{k^2} \delta^{(n)}(p_1 + p_2 - q_1 - q_2) \]
\[ S_2 = 4\pi i G \left( p_1^0 p_2^0 q_1^0 q_2^0 \right)^{-1/2} \frac{1}{k^2} \left[ (p_1 \cdot p_2)(q_1 \cdot q_2) + (p_1 \cdot q_2)(q_1 \cdot p_1) - (p_1 \cdot q_1)(p_2 \cdot q_2) \right] \times \delta^{(2)}(p_1 + p_2 - q_1 - q_2) . \] (A25)

In the lowest order, the static approximation T-matrix element is
\[ T_n = -4 \left( 1 - \frac{1}{n} \right) \frac{G_n}{(2\pi)^{n-2}} \frac{m_1 m_2}{k^2} . \] (A28)

whose associated potential is \( V = \int d^n k e^{-i k \cdot r} T(k) \) in \( n \) dimensions.

**Remark:** \( k_\mu \) in the propagator yields the term as \( (k \cdot p_1) = -k^0 p_1^0 + k \cdot p_1 \). This term does not contribute
to the static potential, because \( k^0 \) is momentum-dependent. The reason is as follows.

Let’s consider the energy conservation at the vertices of the one-graviton exchange diagram, which leads to

\[
k^0 = p^0_1 - q^0_1 \quad \text{and} \quad k^0 = q^0_2 - p^0_2.
\]

The choice between these two \( k^0 \) leads to the different result of momentum sector of the potential. This was pointed out by Y. Nambu in 1950 [14 15]. T. Ohta investigated this problem many years ago and found the consistent choice is to take an average of two expressions

\[
k^0 = \frac{1}{2}(p^0_1 - q^0_1 + q^0_2 - p^0_2).
\]

More generally a parameter can be introduced in the above expression, which was proved to be identical with a gauge parameter. At any rate \( k_0 \) does not contribute to the static potential.

The \( T \)-matrix elements for \( n = 2, 3 \) and 4 are

\[
\begin{align*}
T_2 &= -\frac{2G_2m_1m_2}{k^2} \quad (A29) \\
T_3 &= -\frac{8}{3} \cdot \frac{G_3}{(2\pi)^2} \frac{m_1m_2}{k^2} \quad (A30) \\
T_4 &= -\frac{3G_4}{(2\pi)^2} \frac{m_1m_2}{k^2} \quad (A31)
\end{align*}
\]

and the corresponding potentials are

\[
\begin{align*}
V_2 &= 2\pi G_2m_1m_2 r \quad (A32) \\
V_3 &= 2 \left( \frac{4}{3} G_3 \right) m_1m_2 \log r \quad (A33) \\
V_4 &= -\frac{3}{2} \left( \frac{4}{3} G_4 \right) m_1m_2 \quad (A34)
\end{align*}
\]

By identifying the gravitational constants as

\[
G_{N,2} \equiv G_2 \quad G_{N,3} \equiv \frac{4}{3} G_3 \quad G_{N,4} \equiv \frac{3}{2} G_4 \quad (A35)
\]

we get the correct Newtonian potentials in each dimension. The result for \( V_4 \) tells this method has been established and led to the exact potential to the post-post-Newtonian order.

The results above are in strong contrast with \( d + 1 \)-dimensional GRT, whose free Lagrangian density is

\[
L = \frac{2}{\kappa^2} \sqrt{-g}R - \frac{1}{2} \sum_a \sqrt{-g} (g^{\mu\nu} \phi_a,_{\mu} \phi_a,_{\nu} + m_a^2 \phi_a^2) \quad (A36)
\]

from which the free Lagrangian of the graviton is

\[
L_{0g} = -\frac{1}{2} \left\{ \partial_\lambda h_{\mu\nu} \partial^\lambda h_{\mu\nu} - \partial^\lambda h_{\mu} \partial_\lambda h_{\mu\nu} - 2 \partial_\mu h_{\mu\nu} \partial^\lambda h_{\lambda\nu} + 2 \partial_\mu h_{\mu\nu} \partial_\nu h_{\lambda} \right\} + \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h^\lambda \right) B^\mu + \frac{1}{4} B_\mu B^\mu \quad (A37)
\]

where gauge fixing terms have been added. A computation analogous to the one above gives the following. There are ways of eliminating the gauge fields

1. Define a new field \( C_\mu \) by

\[
C_\mu \equiv B_\mu + 2 \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h^\lambda \right)
\]

The gauge fixing term of \( A37 \) becomes

\[
- \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h^\nu \right) \left( \partial^\lambda h_{\lambda} - \frac{1}{2} \partial_\mu h_{\lambda} \right) + \frac{1}{4} C_\mu C^\mu
\]

The \( C_\mu \) field is completely separated form the graviton’s world and has no contribution to physics.

2. Eliminate \( B_\mu \) directly in terms of the field equation

\[
\partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h^\nu + \frac{1}{2} B_\mu = 0
\]

Either way, the gauge fixing term becomes

\[
- \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h^\nu \right) \left( \partial^\lambda h_{\lambda} - \frac{1}{2} \partial_\mu h_{\lambda} \right)
\]

and the Lagrangian density becomes

\[
L_{0g} = -\frac{1}{2} \left\{ \partial_\lambda h_{\mu\nu} \partial^\lambda h_{\mu\nu} - \partial^\lambda h_{\mu} \partial_\lambda h_{\mu\nu} - 2 \partial_\mu h_{\mu\nu} \partial^\lambda h_{\lambda\nu} + 2 \partial_\mu h_{\mu\nu} \partial_\nu h_{\lambda} \right\} - \left( \partial^\nu h_{\nu\mu} - \frac{1}{2} \partial_\mu h^\nu \right) \left( \partial^\lambda h_{\lambda} - \frac{1}{2} \partial_\mu h_{\lambda} \right) \quad (A38)
\]

We obtain the field equation

\[
\Box h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \Box h_{\lambda} = 0 \quad (A39)
\]

The trace of \( A39 \) is

\[
\frac{2 - n}{2} \Box h_{\lambda} = 0 \quad \Rightarrow \quad \Box h_{\lambda} = 0 \quad (n > 2)
\]

Then

\[
\Box h_{\mu\nu} = 0 \quad (A40)
\]

The conjugate momentum is

\[
\pi_{\mu\nu} = \partial_\mu h_{\mu\nu} - \eta_{\mu\nu} \left( \partial_\lambda h_{\lambda} + \partial_\lambda h^\lambda_0 + \frac{1}{2} B^\nu \right)
\]

\[
+ \eta^{\nu\rho} \left( \partial_\lambda h^\lambda_{\rho} - \frac{1}{2} \partial^\rho h_{\lambda} + \frac{1}{2} B_{\rho} \right)
\]

\[
+ \eta^{\nu\rho} \left( \partial_\mu h_{\lambda} - \frac{1}{2} \partial^\rho h_{\lambda} + \frac{1}{2} B_{\rho} \right)
\]

\[
\approx \partial_\mu h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \partial_\lambda h_{\lambda} \quad (A42)
\]
The \( n \) dimensional commutation relations among the components of \( h_{\mu \nu} \) at two arbitrary space-time points

\[
[h_{\mu \nu}(x), h_{\alpha \beta}(y)] = \frac{i}{2} \bigl\{ \eta_{\mu \alpha} \eta_{\nu \beta} + \eta_{\mu \beta} \eta_{\nu \alpha} 
- \frac{2}{n-2} \eta_{\mu \nu} \eta_{\alpha \beta} \bigr\} D^{(n)}(x-y)
\]

The equal-time commutation relation are

\[
[h_{\mu \nu}(x), \pi^{\mu \nu}(y)]_{eq} = \frac{i}{2} \left( \delta^{\alpha \beta}_{\mu \nu} + \delta^{\alpha \beta}_{\nu \mu} \right) \delta^{(n-1)}(x-y)
\]

\[
[h_{\mu \nu}(x), \pi^{\lambda \rho}(y)]_{eq} = \left[ \pi^{\mu \nu}(x), \pi^{\lambda \rho}(y) \right]_{eq} = 0
\]

(A43)

\[
S_n = \frac{4\pi G_n}{(2\pi)^{n-2}} (p_1^0 q_2^0) \eta_{\mu \nu} \eta_{\alpha \beta}^{-1/2} \left[ p_2^0 q_2^0 - \frac{1}{2} \eta^{\mu \nu} (p_1 \cdot q_1 + m_1^2) \right] \times \frac{X_{\mu \nu, \alpha \beta}}{k^2} \delta^{(n)}(p_1 + p_2 - q_1 - q_2)
\]

(A44)

which in turn yields the \( T \)-matrix element

\[
T_n = -\frac{4G_n}{(2\pi)^{n-2}} \frac{n-3}{n-2} \frac{m_1 m_2}{k^2} \quad (A45)
\]

in the static approximation in \( n \) dimensions. The reader should not be fooled by the apparent sameness of \( S_n \) from GRT in Eq. (A44) and the \( S_n \) of dilaton gravity in Eq. (A26): the \( X_{\mu \nu, \lambda \rho} \) matrix elements are distinct! Thus for \( n = 3 \), \( T_3(k) = 0 \) in the static approximation. Then there exists no static potential in the lowest order in \( 2+1 \)-dimensional Einstein gravity. The potentials in \( n = 3, 4 \) are

\[
V_3 = 0 \quad V_4 = \frac{-G_4 m_1 m_2}{r} \quad (A46)
\]

The potential \( V_4 \) is in agreement with (A34). However the potential for \( n = 3 \) vanishes, and the potential for \( n = 2 \) diverges. This latter situation can be dealt with by setting \( G_n = (1 - \frac{3}{2}) G_2 \) and taking the \( n \to 2 \) limiting method of Mann and Ross [2], which yields the two-dimensional \( T \)-matrix element

\[
T_2 = -\frac{2G_2 m_1 m_2}{k^2} \quad (A47)
\]

and the potential is calculated from Eq. (A27) as

\[
V_2 = -2G_2 m_1 m_2 \int dk \frac{e^{-ikr}}{k^2} = 2\pi G_2 m_1 m_2 r
\]

(A48)

Note that for \( n = 2 \), we cannot get the consistent quantization of the theory based on the free-graviton Lagrangian derived from \( \frac{2}{\sqrt{\eta}} \sqrt{-gR} \). For example, the propagator cannot be defined in the case of \( n = 2 \). In the dilaton theory the dilaton contributes to the Newtonian potential “indirectly” through the mixing with the graviton.

The graviton’s propagator is

\[
\langle 0 \mid T(h_{\mu \nu}(x)h_{\lambda \rho}(y)) \mid 0 \rangle = -\frac{i}{2(2\pi)^n} \int d^m k \ e^{ik(x-y)} \frac{X_{\mu \nu, \lambda \rho}}{k^2 - i\epsilon}
\]

where

\[
X_{\mu \nu, \lambda \rho} = \eta_{\mu \lambda} \eta_{\nu \rho} + \eta_{\mu \rho} \eta_{\nu \lambda} - \frac{2}{n-2} \eta_{\mu \nu} \eta_{\lambda \rho}
\]

The \( S \)-matrix element of the one graviton exchange diagram is

\[
\langle \Phi(x) \Phi(y) \rangle = \frac{1}{\sqrt{2(2\pi)^{n-1}/2}} \int_{k_0>0} \frac{d^{n-1}k}{k_0} \left( e^{ikx a_k} + e^{-ikx a_k^\dagger} \right)
\]

\[
\left[ a_k, a_{k'}^\dagger \right] = k_0 \delta^{n-1}(k - k')
\]

\[
[\varphi(x), \varphi(y)] = \frac{\sqrt{2}}{2(2\pi)^{n-1}/2} \int_{k_0>0} \frac{d^{n-1}k}{k_0} \left\{ e^{ik(x-y)} - e^{-ik(x-y)} \right\}
\]

\[
= i D^{(n)}(x-y)
\]

\[
D^{(n)}(x) = -\frac{i}{2(2\pi)^{n-1}/2} \int_{k_0>0} \frac{d^{n-1}k}{k_0} \left\{ e^{ikx} - e^{-ikx} \right\}
\]

\[
= -\frac{1}{(2\pi)^{n-1}/2} \int_{k_0>0} \frac{d^{n-1}k}{k_0} e^{ikx} \sin k_0 x_0
\]

\[
= -\frac{i}{(2\pi)^{n-1}/2} \int d^n k \ c(k_0) \delta(k^2) e^{ikx}
\]

Appendix B: Invariant function for massless field in \( n \) dimensions

This section elaborates \( D^{(n)} \) which is used in appendix A. Consider the following massless field,

\[
\varphi(x) = \frac{1}{\sqrt{2(2\pi)^{n-1}/2}} \int_{k_0>0} \frac{d^{n-1}k}{k_0} \left( e^{ikx a_k} + e^{-ikx a_k^\dagger} \right)
\]

\[
\left[ a_k, a_{k'}^\dagger \right] = k_0 \delta^{n-1}(k - k')
\]

\[
[\varphi(x), \varphi(y)] = \frac{1}{2(2\pi)^{n-1}/2} \int_{k_0>0} \frac{d^{n-1}k}{k_0} \left\{ e^{ik(x-y)} - e^{-ik(x-y)} \right\}
\]

\[
= i D^{(n)}(x-y)
\]

\[
D^{(n)}(x) = -\frac{i}{2(2\pi)^{n-1}/2} \int_{k_0>0} \frac{d^{n-1}k}{k_0} \left\{ e^{ikx} - e^{-ikx} \right\}
\]

\[
= -\frac{1}{(2\pi)^{n-1}/2} \int_{k_0>0} \frac{d^{n-1}k}{k_0} e^{ikx} \sin k_0 x_0
\]

\[
= -\frac{i}{(2\pi)^{n-1}/2} \int d^n k \ c(k_0) \delta(k^2) e^{ikx}
\]
Appendix C: Setting of metric terms $g_{\mu\nu}$

Concerning the setting the lapse and shift functions, DeWitt wrote

\[ D^{(n)}(-x) = -D^{(n)}(x) \]
\[ D^{(n)}(\Delta x) = D^{(n)}(x) \]
\[ \Box D^{(n)}(x) = 0 \]
\[ D^{(n)}(x, 0) = 0 \]
\[ \frac{\partial}{\partial x_0} D^{(n)}(x) \bigg|_{x_0=0} = -\delta^{(n-1)}(x) \]

Yet, Kiriushcheva admits:

\[ \text{If desired, one can always assign definite values to } N \text{ and } N_a (\alpha \text{ and } \beta_\alpha \text{ in } \text{DeWitt’s notations}) \text{ which may be purely numerical or may depend on } \gamma_{ab} \text{ and } \pi^{ab}. \text{ Each choice corresponds to the imposition of certain conditions on the space-time coordinates. For example, one may choose } N = 1, N_a = 0 \ldots . \]

(And consequently $N^a = g^{ab} N_b = 0.$) Since the Wheeler-DeWitt Quantum Geometrodynamics is supposed to be gauge invariant, a concrete choice of $N, N_a, N^a$ is believed not to have any significance. York [P. 8 of 24] points out:

\[ \text{“In this "canonical"-like } 3 + 1 \text{ form, there are no time derivatives of } N \text{ or of } N^a \ldots \text{ we have an easy way of seeing that } N \text{ and } N^a \text{ are dynamically irrelevant".} \]

However, this notion has been challenged to some extent by Tatyana Shestakova who advocates an “extended phase space” approach and insists that the choice of $N, N_a,$ is actually a choice of gauge conditions and affects the resulting physical picture. Natalia Kiriushcheva criticized the work of Shestakova and points out:

\[ \text{“Dirac made an additional assumption that } g_{0a} = 0 \text{ which noticeably simplified his calculations but also led him to the conclusion that this simplification can be achieved only at the expense of abandoning four-dimensional symmetry.} \]

Yet, Kiriushcheva admits:

\[ \ldots \text{we show that his assumption } g_{0a} = 0, \text{ used to simplify his calculation of different contributions to the secondary constraints, is unwarranted; yet, remarkably his total Hamiltonian is equivalent to the one computed without the assumption } g_{0a} = 0. \]

Shestakova responded to the criticism by Kiriushcheva and her collaborators. In the opinion of Natalia Kiriushcheva, the ADM approach already contains the loss of physics as this representation restricts possible coordinate transformations: a space-like hypersurface remains space-like. Kiriushcheva claims it contradicts the main principle of GRT, the principle of general covariance (according to which any possible transformation of a coordinate system should be acceptable).

However, the ADM formulation was used for the Hamiltonian analysis where, e.g. a time coordinate is singled out and different treatment of time and space coordinates does not lead to contradictions. It proved invaluable in decoupling the field equations for the 1+1 case. If, in addition, the lapse and shift are set to constants, Kiriushcheva believes this will produce further “destruction of physics”, specifically the general class of physics solutions, because such an operation will mean a substitution of the $3 + 1$ picture of the world, which is the essence of General and Special Relativity, by a 3-dimensional description, i.e. Newtonian mechanics.

Notwithstanding the contention of either Kiriushcheva or Shestakova with what is now conventional wisdom (and even the disagreement between themselves), we take the point of view that Dirac was essentially correct. The ADM approach and the assumptions $g_{0a} = 0$ do restrict the class of solutions obtainable. However, we are not seeking e.g. Kerr metric solutions with a manifest 4 × 4 covariance where the $g_{0\nu}$ are nonzero and vary in time. For an interacting system of point-particles, these assumptions can potentially yield a realistic class of solutions with departures from these assumptions addressed subsequently.

In practice, $g_{0a} = 0$ often serves as initial conditions with departures obtained from the time evolution of the given system, as is often the case in numerical relativity where the equations determining $N$ and $N_a$ are obtained by taking the time derivatives of the coordinate conditions. Note that departures from these starting assumptions for a matter Lagrangian of Eq. (12) (last term of Eq. 21) in the absence of an external magnetic field has been worked out by Kimura with analytical solutions [Eq. (3.18) − (3.20) of 13].

These arguments were made with respect to GRT but also apply to our scalar-tensor theory in particular since the latter becomes GRT in the limit $\Psi \rightarrow 1$ and deviations from GRT are small. Similarly, they also apply to a range of matter Lagrangians (e.g. individual terms of Eq. 0 and in totality).

Appendix D: On the generator and coordinate conditions

In $3+1$ dimensional GRT, it is commonly known that one of the coordinate conditions appropriate for particle dynamics is [Eqs. (4.22a) and (4.22b) of 12]

\[ t = -\frac{1}{2\triangle} \left( \pi^T + \frac{1}{\triangle} \delta^{cd} \pi_{,cd} \right) = -\frac{1}{2\triangle} \pi^{\alpha a} \quad (D1) \]
\[ x^a = h_a - \frac{1}{4\triangle} h^T_{,a} \quad (D2) \]
where $\Delta = \nabla^2$ is the Laplacian in 3-space. The total generator is

$$G = G_M + \int d^3x \pi^{ab} \delta g_{ab}$$  \hspace{1cm} (D3)$$

$$G = G_M + \int d^3x \left\{ \pi^{ab} \delta h_{ab}^{TT} + \left[ \frac{1}{2} (\delta_{ab} - \frac{1}{\Delta} \partial_a \partial_b) \pi^T + \pi^a_b + \pi^b_a \right] \delta \left[ \frac{1}{2} \delta_{ab} h^T - \frac{1}{2\Delta} \partial_a \partial_b h^T + h_{a,b} + h_{b,a} \right] \right\}$$

$$= G_M + \int d^3x \left\{ \pi^{ab} \delta h_{ab}^{TT} + \frac{1}{2} \pi^T + \pi^{a}_{a} \right\} \delta h^T + (\pi^{ab} - \pi^{ab} TT) \delta \left[ -\frac{1}{2\Delta} \partial_a \partial_b h^T + h_{a,b} + h_{b,a} \right]$$

$$= G_M + \int d^3x \left\{ \pi^{ab} \delta h_{ab}^{TT} - \Delta h^T \delta \left[ \frac{1}{2\Delta} (\pi^T + \frac{1}{\Delta} \pi^{ca}_{ca}) \right] - 2\pi^{ab} \delta \left[ h_{a} - \frac{1}{4} \Delta h^T \right] \right\}$$

In this transformation the surface terms are discarded. Of course the vanishing of the surface terms has been checked. (Variations at spatial infinity are consistently set to zero.) From this expression we set the coordinate conditions of (D1) and (D2) are [Eqs (48)] of [12]. The differential form of the conditions of (D1) and (D2) are [Eqs (4.22c) and (4.22d)] of [12].

$$\Delta g_{ab,b} - \frac{1}{4} g_{abc,bc} - \frac{1}{4} \Delta g_{bb,a} = 0 \hspace{1cm} (D4)$$

$$\pi^{aa} = \pi^{aa} \text{GRT} = 0 \hspace{1cm} (D5)$$

The solution of the metric tensor is [Eq. (2.12) of [13] and Eq. (2.19) of [27]]

$$\gamma_{ab} = g_{ab} = \delta_{ab}(1 + \frac{1}{2} h^T) + h_{ab}^{TT} \hspace{1cm} (D6)$$

**Appendix E: Divergences of Eq. (48)**

Holding the Taub function $\alpha = N/\sqrt{h}$ of Eq. (61) constant and expanding Eq. (48) and excluding the term in $\pi$, we obtain

$$\Pi \partial_a \Psi \rightarrow \frac{3}{2} \partial_a \Psi \partial_a \Psi - \frac{1}{2} \partial_a \Psi \partial_a \Psi$$  \hspace{1cm} (E1)$$

Substituting $\Psi(t,x,y,z) = G(t) F(x,y,z)$ into the first part i.e. the term (1) of Eq. (E1) yields

$$\partial_t \Psi \partial_a \Psi = \frac{dG(t)}{dt} \frac{dF(x,y,z)}{dx_a}$$

This is clearly a divergence. For example, set $x_a = x$

$$\int d^3x \frac{dG(t)}{dt} \frac{dF(x,y,z)}{dx} = \frac{dG(t)}{dt} \int dydz F(x,y,z) \bigl|_{-\infty}^{\infty} = 0$$

for any function of the form $F = 1 + \phi(x,y,z)$, as in Eq. (9) where $\phi$ vanishes at infinity, and similarly for

where $G_M$ refers to the part dependent on the matter Lagrangian; using the orthogonal decomposition of Eqs. (37), $G$ is transformed as:

$$x_a = y \text{ and } z \text{. In view of the functional form of Eq. (48), consider now a sum i.e. } \Psi(t,x,y,z) = G(t) + F(x,y,z) \text{, into the (1) term of Eq. (E1)}$$

$$\partial_t \Psi \partial_a \Psi = \frac{dG(t)}{dt} \frac{dF(x,y,z)}{dx_a} / [G(t) + F(x,y,z)]$$

Here mixed terms in $t$ and spatial coordinates appear. However, $G(t)$ and $G'(t)$ depend only on $t$ and are therefore constant relative to the integration over spatial coordinates. Again we have a divergence for the $d/dx_a$ term.

We repeat this exercise for term (2) of Eq. (E1) with $\Psi(t,x,y,z) = G(t) F(x,y,z)$

$$\partial_t \Psi \partial_a \Psi = \frac{dG(t)}{dt} \frac{dF(x,y,z)}{dx_a} F(x,y,z) = \frac{1}{4} \frac{d(G(t)^2)}{dt} \frac{d(F(x,y,z)^2)}{dx_a}$$

This is again a divergence similar to the second case of (1). Finally, for $\Psi(t,x,y,z) = G(t) + F(x,y,z)$, we have

$$\partial_t \Psi \partial_a \Psi = \frac{dG(t)}{dt} \frac{dF(x,y,z)}{dx_a}$$

which is also clearly a divergence just like in the first case of (1).

The term of Eq. (48), with $\pi$ given by Eq. (12) for isotropic coordinates, is

$$\pi = -\frac{3}{2} \sqrt{\gamma} \Psi \partial_a [\ln(\Psi)] \hspace{1cm} (E2)$$

The Taub condition with the simplifications of Eq. (29), i.e. $N = 1$, implies that $\gamma = \text{const.}$, and therefore the term in square brackets is clearly a divergence.

**Appendix F: Derivation of Eq. (53)**

$\Gamma^a_{nb}$ is the Christoffel symbol of the second kind,

$$\Gamma^a_{nc} = \gamma^{am} \Gamma_{mnc} = \frac{1}{2} \gamma^{am} (\gamma_{mn,e} + \gamma_{mc,n} - \gamma_{nc,m})$$
and is symmetric in the two lower indices. We have
\[ D_\alpha (\pi^a_b) = \partial_\alpha \pi^a_b + \Gamma^c_{ab} \pi^c_b - \Gamma^c_{ac} \pi^a_b \]
\[ = \partial_\alpha \pi^a_b + \frac{1}{2} \gamma^{ac} (\partial_a \gamma_{cc} + \partial_c \gamma_{ca} - \partial_e \gamma_{ac}) \pi^c_b \]
\[ - \frac{1}{2} \gamma^{cc} (\partial_a \gamma_{eb} + \partial_b \gamma_{ae} - \partial_e \gamma_{ab}) \pi^c_a \]
\[ = \partial_\alpha \pi^a_b + \frac{1}{2} \gamma^{ac} \left( \partial_a \gamma_{cc} - \partial_e \gamma_{ac} + \partial_c \gamma_{ea} \right) \pi^c_b \]
\[ - \frac{1}{2} \left( \partial_a \gamma_{eb} - \partial_e \gamma_{ab} + \partial_b \gamma_{ae} \right) \pi^c_a \]
\[ = \partial_\alpha \pi^a_b - \frac{1}{2} \gamma^{ac} \partial_b \gamma_{ac} + \frac{1}{2} \gamma^{ac} \partial_c \gamma_{ea} \pi^c_b \]
\[ \text{Tr}(\partial \ln(\gamma)) \ (F1) \]

where we have used the symmetry of $\gamma_{ab}$ and $\pi_{ab}$. Equation (F1) is the result of Eq. (53) with $a$ and $b$ interchanged. The last term in Eq. (F1) involves the logarithmic derivative of the determinant of the metric. It goes to zero if $\gamma = 1$ or if the volume (whose element is proportional to this term) is fixed within ADM. Although this is the case for many applications of ADM, this could be in doubt in e.g., cosmological studies of an expanding universe. However, it can be justified if, for example, the Taub function $\alpha$ of Eq. (51) is unit or a constant.

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