The fundamental form of a second-order homogeneous Lagrangian in two variables

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Abstract

We construct, for a second-order homogeneous Lagrangian in two independent variables, a differential 2-form with the property that it is closed precisely when the Lagrangian is null. This is similar to the property of the ‘fundamental Lepage equivalent’ associated with first-order Lagrangians defined on jets of sections of a fibred manifold. We show that this form may be defined on a fourth-order frame bundle but is not, in general, projectable to a bundle of contact elements.

Keywords: homogeneous Lagrangian, Lepage equivalent, Euler-Lagrange form

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1 Introduction

The ‘Lepage equivalents’ of a Lagrangian are important tools for use when studying variational problems on fibred manifolds: they are differential forms having the same extremals as the Lagrangian form, with a further property ensuring that their differentials give rise to the Euler-Lagrange form. If \( \pi : E \to M \) is the fibred manifold with \( \dim M = m \), and if \( l \in \Omega^m J^k \pi \) is the Lagrangian form, then any Lepage equivalent \( \theta \) of \( l \) will be defined on a jet manifold \( J^l \pi \) (with, in general, \( l \geq k \)) and will satisfy the conditions that \( \theta - \pi^*_{l,k} \) should be contact, and that for any vector field \( Z \in \mathfrak{X}(J^l \pi) \) vertical over \( E \) the contraction \( i_Z d\theta \) should also be contact. The Euler-Lagrange form \( \varepsilon \) is then the 1-contact part of \( d\theta \). The basic example of a Lepage equivalent is the Poincaré form from

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classical mechanics: in coordinates \((t, q^a, \dot{q}^a)\), if \(l = L dt\) then
\[
\theta = L dt + \frac{\partial L}{\partial \dot{q}^a} (dq^a - \dot{q}^a dt).
\]

Global Lepage equivalents may always be found for a given Lagrangian, and if \(m = 1\) then they are unique. They are never unique when \(m > 1\), because adding an arbitrary non-zero 2-contact form to any Lepage equivalent will give a different Lepage equivalent, although such a modification will not affect the Euler-Lagrange form. Nevertheless, when the order \(k\) of the Lagrangian is no more than 2 then it is possible to make a canonical choice of Lepage equivalent; this cannot, however, be done when \(k \geq 3\) without the specification of some additional structure in the problem.

A particularly important question concerns the relationship between Lepage equivalents and null Lagrangians: that is, Lagrangians whose Euler-Lagrange forms vanish. Clearly if a Lagrangian has a closed Lepage equivalent then it will be null; and when \(m = 1\) then the unique Lepage equivalent of a null Lagrangian is closed. But when \(m > 1\) then a choice of Lepage equivalent would be needed, and it is not immediately obvious how this choice should be made.

An answer to this question for first-order Lagrangians was found by Krupka [4], and also subsequently by Betounes [1]. In coordinates \(x^i\) on \(M\) and fibred coordinates \((x^i, u^a)\) on \(E\), the Lepage equivalent
\[
\theta = L \omega + \sum_{r=0}^{\min\{m,n\}} \frac{1}{(r!)} \frac{\partial^r L}{\partial u_{i_1}^{a_1} \cdots \partial u_{i_r}^{a_r}} \theta^{a_1} \wedge \cdots \wedge \theta^{a_r} \wedge \omega_{i_1 \cdots i_r},
\]
of a Lagrangian \(L \omega\) (where \(\omega = dx^1 \wedge \cdots \wedge dx^m\) and \(\omega_{i_1 \cdots i_r} = i_{\partial/\partial x^{i_r}} \omega_{i_1 \cdots i_{r-1}}\), and where \(\theta^a = du^a - u^a_i dx^i\) is closed precisely when \(L \omega\) is null. A similar formula for second-order (or higher-order) Lagrangians has not yet been found, and the existence of Lepage equivalents having this additional property has not been firmly established.

The idea of a Lepage equivalent is not directly appropriate for homogeneous variational problems. These are problems defined on a manifold \(E\) without any given fibration over a space of independent variables, where the solution to the variational problem is a submanifold with an orientation but without any preferred parametrization. Instead of using jet bundles for these problems, the Lagrangian is defined instead on the bundle of \(k\)-th order \(m\)-frames \(F^k_{\{m\}} E\) in the manifold [2] (this is also called the bundle of regular \(k\)-th order \(m\)-velocities). The Lagrangian is a function \(L\) rather than an \(m\)-form, and is required to satisfy a certain homogeneity condition. Factoring the bundle of \(m\)-frames by the vector fields used to specify the homogeneity condition gives rise to the bundle \(J^k_\pi(E, m)\) of \(k\)-th order oriented contact elements of dimension \(m\); a Lagrangian \(m\)-form \(l\) on this bundle gives rise to a homogeneous function \(L\) on the frame bundle. If a fibration \(\pi : E \to M\) is given then there is an inclusion \(J^k \pi \subset J^k_\pi(E, m)\), and a Lagrangian form on \(J^k \pi\) gives rise to a homogeneous function \(L\) on an open subset of the frame bundle.
It was shown in [2] that for every Lagrangian function \( L \) on \( F^k_{(m)}E \) it is possible to construct an \( m \)-form on \( F^{2k-1}_{(m)}E \) called the **Hilbert-Carathéodory form** having the same extremals as \( L \) and giving rise to a suitable Euler-Lagrange form. The Hilbert-Carathéodory form is projectable to the bundle of contact elements when \( m = 1 \) or \( k \leq 2 \).

It was subsequently shown in [3] that for a first-order Lagrangian function there is another \( m \)-form on \( F^1_{(m)}E \) having the property that it is closed precisely when the Lagrangian is null. This second \( m \)-form is projectable to the bundle of contact elements, and if there is a fibration of \( E \) over some \( m \)-dimensional manifold then the restriction to the corresponding jet bundle takes the coordinate form shown above.

The present paper is a preliminary report on a project to generalize the latter construction to Lagrangians of arbitrary order: we describe a method of constructing, for a second-order homogeneous Lagrangian in two independent variables, a 2-form that has the same extremals as the Lagrangian, and is closed precisely when the Lagrangian is null. This will be the **fundamental form** of the Lagrangian. Although the restriction on order and dimension suggests that this is a rather small advance, it is nevertheless significant because the corresponding construction for second-order Lagrangians on jets of fibrations has not yet been found.

In Section 2 of the paper we summarise the results from [2] that will be needed. Section 3 contains our main theorems, and in Section 4 we investigate the projectability of the fundamental form to a lower-order frame bundle, and to the bundle of contact elements. Finally, Section 5 puts this work in the context of the project as a whole, where a new calculus of vector-valued forms [5] is likely to be a useful tool in generalizing the construction to higher orders and more variables.

### 2 Homogeneous variational problems

We consider a smooth manifold \( E \) of dimension \( n \), and its bundles \( \tau^k_{(2)}E : F^k_{(2)}E \to E \) of \( k \)-th order 2-frames. Important operators on these bundles are the total derivatives and the vertical endomorphisms. The former are vector fields \( T_i \) along the map \( \tau^{k+1}_{(2)}E : F^{k+1}_{(2)}E \to F^k_{(2)}E \) given in coordinates by

\[
T_i = \sum_{s=0}^{k} \frac{1}{\#(i_1 \ldots i_s)} u^\alpha_{i_1 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}}
\]

and the latter are type \((1,1)\) tensor fields \( S^j \) on \( F^{k+1}_{(2)}E \) given by

\[
S^j = \sum_{s=0}^{k} \frac{s+1}{\#(i_1 \ldots i_s)} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}} \otimes du^\alpha_{i_1 \ldots i_s}
\]

Here and subsequently we take local coordinates \((u^\alpha)\) on \( E \) and corresponding jet coordinates \((u^\alpha_{i_1 \ldots i_s})\) on \( F^k_{(m)}E \) where the indices \( i_1, \ldots, i_s \) take the values \((1,2)\). The symbol
#(i_1 \cdots i_s) denotes the number of distinct rearrangements of the indices (i_1, \ldots, i_s), and is needed because the jet coordinates (u^{a, i_1 \cdots i_s}) are totally symmetric in their subscripts. Intrinsic definitions of the operators $T_i$ and $S^j$ may be found in [2, 5].

We also need to use the fundamental vector fields $\Delta^{i_1 \cdots i_s}_i$ defined by

$$\Delta^{i_1 \cdots i_s}_i = S^{i_1 \cdots i_s}(T_i)$$

where the tensor fields $S^i$ and $S^j$ commute, so that $S^{i_1 \cdots i_s}$ may be defined by iteration; these vector fields are well-defined on the manifold $\mathcal{F}^{k+1}_2 E$ (rather than along the map $\tau^{(k+1,k)}_2 E$).

We shall be interested in the actions of these objects as derivations on differential forms. The contractions corresponding to the vector fields $T_i$ and $\Delta^{i_1 \cdots i_s}_i$ will be denoted $i_i$ and $i^{i_1 \cdots i_s}_i$, and the Lie derivatives will be denoted $d_i$ and $d^{i_1 \cdots i_s}_i$, so that $d_i = di_i + i_i d$ and $d^{i_1 \cdots i_s}_i = d^{i_1 \cdots i_s}_i + i^{i_1 \cdots i_s}_i d$. We shall retain the symbol $S^{i_1 \cdots i_s}$ for an iterated vertical endomorphism, and we shall also use the symbol $d^{j_1 \cdots j_s}_j$ for an iterated total derivative, using the property that $d_i$ and $d_j$ commute. Note that we often omit the pull-back maps when they would clutter up the formulæ.

We shall make considerable use of the commutation properties of these operators, and so we list these properties for the action on an $r$-form; the proofs are straightforward.

**Lemma 1**

$$i_i d_j = d_j i_i$$

$$d_j S^i = S^r d_j - r \delta_j^i$$

$$d^{j_1 \cdots j_s}_j S^i = S^r d^{j_1 \cdots j_s}_j - \delta^i_ j S^{i_1 \cdots j_s}$$

$$i^{j_1 \cdots j_s}_j S^i = S^{i_1 \cdots j_s} d^{j_1 \cdots j_s}_j + i^{j_1 \cdots j_s}_j$$

$$d^k_i d_j = d_j d^k_i + \delta^k_j d_i$$

$$d^{i_1 \cdots i_s}_i d_j = d_j d^{i_1 \cdots i_s}_i + \sum_{r=1}^{s} \delta^i_j d^{i_1 \cdots \hat{i} \cdots i_s}_i$$

A second-order homogeneous variational problem in two independent variables is given by a Lagrangian function $L$ on $\mathcal{F}^{3}_2 E$ satisfying the homogeneity properties

$$d^i_j L = \delta^i_j L , \quad d^{i k}_j L = 0 .$$

Associated with such a Lagrangian are its two Hilbert forms. These are the 1-forms $\vartheta^i$ on $\mathcal{F}^{3}_2 E$ defined by

$$\vartheta^i = (S^i - \frac{1}{2} d_j S^{j i}) dL$$

which are used to construct the Euler-Lagrange form

$$\varepsilon = dL - di \vartheta^i$$
on $\mathcal{F}^4_2 E$. In coordinates

$$\varepsilon = \left( \frac{\partial L}{\partial u^\alpha} - d_i \left( \frac{\partial L}{\partial u_i^\alpha} \right) \right) + \frac{1}{\#(ij)} d_{ij} \left( \frac{\partial L}{\partial u_{ij}^\alpha} \right) du^\alpha,$$

incorporating the Euler-Lagrange equations for the variational problem defined by $L$. More details of this construction may be found in [2].

3 The fundamental form

Let $L : \mathcal{F}^2_2 E \to \mathbb{R}$ be a second-order homogeneous Lagrangian function. We define the fundamental form of $L$ to be the 2-form

$$\Theta = P_{(1)}^i d\vartheta^i - P_{(1)}^i d\theta^2 \in \Omega^2 \mathcal{F}^5_2 E$$

where the operator $P_{(1)}^i : \Omega^2 \mathcal{F}^3_2 E \to \Omega^2 \mathcal{F}^5_2 E$ is defined by

$$P_{(1)}^i = \frac{1}{4} S^i - \frac{1}{24} d_j S^{ij} + \frac{1}{192} d_{jk} S^{jki}.$$

Our tasks in this section will be to show that $\Theta$ has the same extremals as $L$, and that $d\Theta = 0$ precisely when the Euler-Lagrange form $\varepsilon = dL - d_i \vartheta^i$ vanishes.

We shall carry out the first task by demonstrating that a homogeneous Lagrangian can be recovered from its fundamental form by contracting with total derivatives. There are two stages to the argument; the first, where we obtain the Lagrangian from the Hilbert forms, is straightforward.

**Proposition 1** If the Lagrangian $L$ is homogeneous then

$$L = \frac{1}{2} i_i \vartheta^i.$$

**Proof** We use the formula for $P_{(2)}^i$ in the definition $\vartheta^i = P_{(2)}^i dL$, and the commutation rules. Starting with

$$i_k \vartheta^i = i_k (S^i - \frac{1}{2} d_j S^{ji}) dL$$

we have

$$i_k S^i dL = S^i i_k dL + i_k dL = \delta_k^i L$$

using the homogeneity property $i_k dL = d_k L = \delta_k^i L$ and the fact that the contraction $i_k dL$ is a function so that $S^i i_k dL$ vanishes. We also have

$$i_k d_j S^{ji} dL = d_j i_k S^{ji} dL = d_j (S^j i_k + S^j i_k + S^j i_k + i_k i_k) dL = 0$$

using the homogeneity property $i_k i_k dL = d_k^2 L = 0$ and the fact that the other three contractions are functions and are annihilated by $S^i$. We conclude that $i_i \vartheta^i = \delta_i^i L = 2L$.

\[\square\]
The second stage of this argument, that we can obtain the Hilbert forms from the fundamental form, requires considerably more work. We shall first assemble some preliminary results.

**Lemma 2** If the Lagrangian \( L \) is homogeneous then

\[
 i^l_1 \vartheta^m = i^l_1 \vartheta^m = i^l_1 \vartheta^m = 0.
\]

**Proof** We give the proof for \( i^l_1 \vartheta^m \); the remaining arguments are similar. We have

\[
i^l_1 \vartheta^m = i^l_1 (S^m - \frac{1}{2} d_n S^{nm}) dL;
\]

and then both

\[
i^l_1 S^m dL = S^m i^l_1 dL + i^m_1 dL = 0
\]

and

\[
i^l_1 d_n S^{nm} dL = d_n i^l_1 S^{nm} dL = d_n (S^{nm} i^l_1 + S^m i^n_1 + S^n i^m_1 + i^{nm}_1) dL = 0,
\]

as \( i^m_1 dL = i^m_1 dL = 0 \) by homogeneity, and \( S^m i^l_1 dL = 0 \) because \( i^l_1 dL \) is a function.

**Lemma 3** If the Lagrangian \( L \) is homogeneous then

\[
i^l_1 \vartheta^m = \delta^m_l L.
\]

**Proof** We have

\[
i^l_1 \vartheta^m = i^l_1 (S^m - \frac{1}{2} d_n S^{nm}) dL;
\]

and then

\[
i^l_1 S^m dL = S^m i^l_1 dL + i^m_1 dL = d^m_l dL = \delta^m_l L
\]

using the homogeneity of \( L \) and the fact that \( i^l_1 dL \) is a function so that \( S^m i^l_1 dL \) vanishes, whereas

\[
i^l_1 d_n S^{nm} dL = d_n i^l_1 S^{nm} dL = d_n (S^{nm} i^l_1 + S^m i^n_1 + S^n i^m_1 + i^{nm}_1) dL = 0
\]

for similar reasons.

**Lemma 4** If the Lagrangian \( L \) is homogeneous then

\[
d^l_1 \vartheta^m = d^l_1 \vartheta^m = 0.
\]

**Proof** We give the proof for \( d^l_1 \vartheta^m \). We have

\[
d^l_1 \vartheta^m = d^l_1 (S^m - \frac{1}{2} d_n S^{nm}) dL;
\]
and then
\[ d_{ij}^m S^m dL = S^m d_{ij}^m dL - \delta^m_i \delta^m_j dL = -\delta^m_i S^i j dL \]
because \( d_{ij}^m dL = 0 \) by homogeneity, whereas
\[
d_{ij} d_n S^{nm} dL &= (d_n d_{ij}^m + \delta_{nj}^m d_{ij}^n + \delta_{ni}^m d_{ij}^n)S^{nm} dL \\
&= (d_n (S^{nm} d_{ij}^m - \delta_{nj}^m S^{jim} - \delta_{ni}^m S^{imi}) + (S^{im} d_{ij}^m - \delta^m_i S^{jim} - \delta^m_i S^{jim})) dL \\
&= (S^{im} \delta_i^j - \delta^m_i S^{jim} - \delta^m_i S^{jim}) dL \\
&= -2 \delta^m_i S^{ij} dL
\]
where \( d_i^m dL = \delta_i^m dL \), \( d_{ij}^m dL = 0 \) by homogeneity and \( S^{ijm} dL = 0 \) because \( dL \) is a second-order 1-form. Consequently \( d_{ij}^m \vartheta^m = 0 \).

The proof for \( d_{ijk}^m \vartheta^m \) is similar, but simpler because (for instance) \( S^{ij} dL \) is replaced by \( S^{ijk} dL \), so that all the terms vanish individually.

**Lemma 5** If the Lagrangian \( L \) is homogeneous then
\[ d_{ij}^m \vartheta^m = \delta_i^m \vartheta^m - \delta^m_i \vartheta^m. \]

**Proof** Using
\[ d_{ij}^m \vartheta^m = d_{ij}^m (S^m - \frac{1}{2} d_j S^{jm}) dL \]
we have
\[
d_{ij}^m S^m dL &= (S^m d_{ij}^m - \delta^m_i S^j) dL \\
&= (\delta^m_i S^j - \delta^m_i S^j) dL
\]
and
\[
d_{ij}^m d_j S^{jm} dL &= (d_j d_{ij}^m + \delta_{ji}^m d_{ij}^m) S^{jm} dL \\
&= (d_j (S^{jm} d_{ij}^m - \delta^m_i S^{jim} - \delta^m_i S^{jim}) + \delta^m_j d_{ij}^m S^{jm}) dL \\
&= (\delta^m_j d_{ij} S^{jm} - \delta^m_i d_j S^{jim}) dL
\]
from which the result follows.

**Lemma 6** The Hilbert forms \( \vartheta^i \) satisfy
\[ S^i \vartheta^m = \frac{1}{2} S^{im} dL \]
so that
\[ S^i \vartheta^m = S^m \vartheta^i. \]
Proof We have
\begin{align*}
S^i \vartheta^m &= S^i(S^m - \frac{1}{2} d_j S^{jm}) dL \\
&= \left(S^m - \frac{1}{2} (d_j S^i + \delta_j^i) S^{jm}\right) dL \\
&= \frac{1}{2} S^m dL
\end{align*}
because \(dL\) is a second-order 1-form.

\[ \text{Lemma 7} \] The Hilbert forms \( \vartheta^i \) satisfy
\[ S^i d_i \vartheta^m - S^m d_i \vartheta^i = \vartheta^m. \]

Proof We have
\begin{align*}
S^i d_i \vartheta^m - S^m d_i \vartheta^i &= (d_i S^i \vartheta^m + \delta_i^m \vartheta^m) - (d_i S^m \vartheta^i + \delta_i^m \vartheta^i) \\
&= d_i (S^i \vartheta^m - S^m \vartheta^i) + \vartheta^m \\
&= \vartheta^m.
\end{align*}

We now return to the relationship between the fundamental form and the Hilbert forms.

\[ \text{Proposition 2} \] If the Lagrangian \( L \) is homogeneous then
\[ \vartheta^1 = i_2 \Theta, \quad \vartheta^2 = -i_1 \Theta. \]

Proof We use the formula for \( P^i_{(1)} \) in the definition \( \Theta = P^2_{(1)} d\vartheta^1 - P^1_{(1)} d\vartheta^2 \), and the commutation rules. Starting with
\begin{align*}
i_1 P^i_{(1)} d\vartheta^m &= i_1 \left( \frac{1}{4} S^i - \frac{1}{24} d_j S^{ji} + \frac{1}{192} d_{jk} S^{jki} \right) d\vartheta^m
\end{align*}
we again consider the terms separately. For the first term, we get
\begin{align*}
\frac{1}{4} i_1 S^i d\vartheta^m &= \frac{1}{4} (S^i i_1 + i_1^i) d\vartheta^m \\
&= \frac{1}{4} \left( S^i d_i \vartheta^m - S^i d_i \vartheta^m + d_i \vartheta^m - d_i \vartheta^m \right);
\end{align*}
but we know that \( i_1^i \vartheta^m = 0 \) from Lemma 2, \( d_i \vartheta^m = \delta_i^m dL \) from Lemma 3 and \( d_i \vartheta^m = \delta_i^m dL \) from Lemma 5, so we get
\begin{align*}
\frac{1}{4} i_1 S^i d\vartheta^m &= \frac{1}{4} \left( S^i d_i \vartheta^m - \delta_i^m S^i dL + \delta_i^m \vartheta^m - \delta_i^m \vartheta^i \right).
\end{align*}
But we also have
\[ S^i d_i \vartheta^m - S^m d_i \vartheta^i = \vartheta^m. \]
Thus from Lemma 7, so that

\[ \frac{1}{2}(i_S^i d\theta^m - i_i S^m d\theta^i) = \frac{1}{2} \left( S^i d_i \theta^m - \delta_i^m S^i dL + \delta_i^m \theta^m - \delta_i^m \theta^i \right) \]

\[ - \frac{1}{2} \left( S^m d_i \theta^i - \delta_i^m S^m dL + \delta_i^m \theta^i - \delta_i^m \theta^m \right) \]

\[ = \frac{1}{2} \left( S^i d_i \theta^m - S^m d_i \theta^i + S^m dL + 2\theta^m \right) \]

\[ = \frac{1}{2} \left( S^m dL + 3\theta^m \right). \]

For the second term, we get

\[ - \frac{1}{24} i_L d_j S^{ji} d\theta^m = - \frac{1}{24} d_j i_S^{ji} d\theta^m \]

\[ = - \frac{1}{24} d_j \left( S^i d_i + i_i^j \right) S^i d\theta^m \]

\[ = - \frac{1}{24} d_j \left( S^j (S^i d_i + i_i^j) + (S^i i_i^j + i_i^{ji}) \right) d\theta^m \]

\[ = \frac{1}{24} \left( -d_j S_{ji} d_i \theta^m + d_j S^{ji} d_i \theta^m - d_j S^i d_i \theta^m - d_j S^{ji} d_i \theta^m \right) \]

\[ + \frac{1}{24} \left( -d_j S^i d_i \theta^m + d_j S^i d_i \theta^m - d_j d_i^{ji} \theta^m + d_j d_i^{ji} \theta^m \right); \]

but we know that \( i^i \theta^m = 0 \) from Lemma 2, \( i_i \theta^m = \delta_i^m L \) from Lemma 3, \( d_i \theta^m = 0 \) from Lemma 4 and \( d_i \theta^m = \delta_i^m \theta^m - \delta_i^m \theta^i \) from Lemma 5, so that

\[ - \frac{1}{24} i_L d_j S^{ji} d\theta^m = \frac{1}{24} \left( -d_j S^{ji} d_i \theta^m + d_j S^{ji} \delta_i^m dL - d_j S^{ji} (\delta_i^m \theta^m - \delta_i^m \theta^i) \right) \]

\[ -d_j S^i (\delta_i^m \theta^m - \delta_i^m \theta^i). \]

Thus

\[ \frac{1}{24} \left( -i_L d_j S^{ji} d\theta^m + i_L d_j S^{jm} d\theta^i \right) \]

\[ = \frac{1}{24} \left( -d_j S^{ji} d_i \theta^m + d_j S^{ji} \delta_i^m dL - d_j S^{ji} (\delta_i^m \theta^m - \delta_i^m \theta^i) \right) \]

\[ -d_j S^i (\delta_i^m \theta^m - \delta_i^m \theta^i) \]

\[ - \frac{1}{24} \left( -d_j S^{jm} d_i \theta^i + d_j S^{jm} \delta_i^m dL - d_j S^{jm} (\delta_i^m \theta^i - \delta_i^m \theta^m) \right) \]

\[ -d_j S^m (\delta_i^m \theta^i - \delta_i^m \theta^m) \]

\[ = \frac{1}{24} \left( d_j S^{jm} d_i \theta^i - d_j S^{ji} d_i \theta^m - d_j S^{jm} dL - 3d_j S^i \theta^m \right); \]

but

\[ d_j S^{jm} d_i \theta^i - d_j S^{ji} d_i \theta^m = d_j S^i \left( S^m d_i \theta^i - S^i d_i \theta^m \right) \]

\[ = -d_j S^i \theta^m \]

from Lemma 7, so that

\[ \frac{1}{24} \left( -i_L d_j S^{ji} d\theta^m + i_L d_j S^{jm} d\theta^i \right) = \frac{1}{24} \left( -d_j S^{jm} dL - 4d_j S^i \theta^m \right). \]
Finally, for the third term we get

\[ \frac{1}{192} i_id_{jk} S^{jki} d\vartheta^m, \]

so we need to consider

\[ i_id_{jk} S^{jki} d\vartheta^m = d_{jk} i_i S^{jki} d\vartheta^m \]

\[ = d_{jk} \left( S^{jki} \dot{i}_i + S^{jik} \dot{i}_i + S^{kji} \dot{i}_i + S^{ijk} \dot{i}_i + S^{i;kj} \dot{i}_i \right) d\vartheta^m \]

where, for instance,

\[ S^{jki} \dot{i}_i d\vartheta^m = S^{jik} \dot{i}_i d\vartheta^m = S^{jki} \dot{i}_i d\vartheta^m. \]

Now \( i_i \dot{\vartheta}^m = i_i \dot{\vartheta}^m = 0 \), and

\[ S^{jki} \dot{i}_i d\vartheta^m = -\delta_i^m S^{jki} dL = 0 \]

because \( dL \) is a second-order 1-form. Also, \( d_{ij} \dot{\vartheta}^m = 0 \), and we have

\[ d_{jk} S^{jki} \dot{i}_i d\vartheta^m = 0 \]

because \( d_{ij} \vartheta^m \) is horizontal over \( F^2_{(2)} E \), and

\[ d_{jk} S^{jki} \dot{i}_i d\vartheta^m = d_{jk} S^{ij} \dot{i}_i d\vartheta^m = d_{jk} S^{ik} \dot{i}_i d\vartheta^m = 0 \]

because the \( \dot{i}_i d\vartheta^m \) are horizontal over \( F^1_{(2)} E \). Thus

\[ \frac{1}{192} (i_id_{jk} S^{jki} d\vartheta^m - i_id_{jk} S^{jkm} d\vartheta^i) = 0. \]

Putting all three terms together, we now have

\[ i_i P^i_1 d\vartheta^m - i_i P^m_1 d\vartheta^i = \frac{1}{4} (S^m dL + 3\vartheta^m) + \frac{1}{24} \left( -d_j S^{jm} dL - 4d_j S^j \dot{\vartheta}^m \right); \]

but

\[ S^i \dot{\vartheta}^m = \frac{1}{2} S^i S^m dL \]

from Lemma 6, so we obtain

\[ -4d_j S^j \dot{\vartheta}^m = -2d_j S^{jm} dL \]

and consequently

\[ i_i P^i_1 d\vartheta^m - i_i P^m_1 d\vartheta^i = \frac{1}{4} (S^m dL + 3\vartheta^m) - \frac{1}{8} d_j S^{jm} dL = \vartheta^m. \]

We conclude, using \( \Theta = P^i_1 d\vartheta^1 - P^m_1 d\vartheta^2 \), that \( \vartheta^1 = i_2 \Theta \) and \( \vartheta^2 = -i_1 \Theta. \)

With the help of these two propositions, we can show that a homogeneous Lagrangian has the same extremals as its fundamental form.
Theorem 1  For any map $\phi : \mathbb{R}^2 \to E$,

$$(j^2 \phi)^* L dt^1 \wedge dt^2 = (j^5 \phi)^* \Theta$$

where $t^1, t^2$ are the standard coordinates on $\mathbb{R}^2$; thus the two variational problems

$$\delta \int L \, dt^1 dt^2 = 0, \quad \delta \int \Theta = 0$$

have the same extremals.

Proof We have

$$L = \frac{1}{2} (i_1 \vartheta^1 + i_2 \vartheta^2) = \frac{1}{2} (i_1 i_2 \Theta - i_2 i_1 \Theta) = i_1 i_2 \Theta$$

modulo pullback maps, and the result follows immediately from the properties of contraction with total derivatives.

We now move on to our second task, to show that $d \Theta = 0$ precisely when the Lagrangian is null. We consider the two implications separately.

Theorem 2 If $d \Theta = 0$ then $L$ is a null Lagrangian.

Proof From

$$\vartheta^1 = i_2 \Theta, \quad \vartheta^2 = -i_1 \Theta.$$

we have

$$d \vartheta^1 = di_2 \Theta = d_2 \Theta, \quad d \vartheta^2 = -di_1 \Theta = -d_1 \Theta$$

using $d \Theta = 0$. Then from $L = \frac{1}{2} i_i \vartheta^i$ we have

$$dL = \frac{1}{2} \left( di_1 \vartheta^1 + di_2 \vartheta^2 \right)$$

$$= \frac{1}{2} \left( d_1 \vartheta^1 - i_1 d_1 \vartheta^1 + d_2 \vartheta^2 - i_2 d_2 \vartheta^2 \right)$$

$$= \frac{1}{2} \left( d_1 \vartheta^1 + d_2 \vartheta^2 \right) - \frac{1}{2} (i_1 d_2 \Theta - i_2 d_1 \Theta)$$

$$= \frac{1}{2} \left( d_1 \vartheta^1 + d_2 \vartheta^2 \right) - \frac{1}{2} \left( -d_2 \vartheta^2 - d_1 \vartheta^1 \right)$$

$$= d_1 \vartheta^1 + d_2 \vartheta^2$$

so that $\varepsilon = dL - d_i \vartheta^i = 0$ and the Lagrangian is null.

To show the converse, we must examine the relationship between the total derivatives $d_i$ and the operators $P_{(1)}^i$. We need the following Lemma.

Lemma 8 The Hilbert forms $\vartheta^m$ satisfy

$$S^{ijkl} d \vartheta^m = 0.$$
Proof We note first that, as \( dL \) is a second-order 1-form, both \( S^{ijkl}dS^m dL \) and \( S^{ijk}dS^{lm} dL \) vanish; a coordinate proof of this is straightforward. But then

\[
S^{ijkl}dS^m + S^{ijkl}dS^m = S^{ijkl}(d_n S^l + \delta^l_n)S^{nm} dL = 0
\]

so that \( S^{ijkl}d\vartheta^m = S^{ijkl}dP^m_{(2)} dL = 0. \)

We now introduce two new operators,

\[
Q^i_{(2)} : \Omega^2 \mathcal{F}^4 E \to \Omega^2 \mathcal{F}^7 E , \quad Q^i_{(1)} : \Omega^3 \mathcal{F}^6 E \to \Omega^3 \mathcal{F}^{10} E ,
\]

by the formulæ

\[
Q^i_{(2)} = \frac{1}{2} S^i - \frac{1}{8} d_j S^{ji} + \frac{1}{48} d_{jk} S^{jki} - \frac{1}{384} d_{jkl} S^{ijkl}.
\]

and

\[
Q^i_{(1)} = \frac{1}{6} S^i - \frac{1}{72} d_j S^{ji} + \frac{1}{96} d_{jk} S^{jki} - \frac{1}{384} d_{jkl} S^{ijkl} + \frac{1}{17280} d_{jklm} S^{ijklm}.
\]

Lemma 9 The operators \( P^i_{(1)} \) and \( Q^i_{(2)} \), when acting on total derivatives of the 2-form \( d\vartheta^m \), satisfy

\[
\begin{align*}
(Q^1_{(2)}d_2 - d_2 P^1_{(1)})d\vartheta^m &= 0 \quad (1) \\
(Q^2_{(2)}d_1 - d_1 P^2_{(1)})d\vartheta^m &= 0 \quad (2) \\
(Q^1_{(2)}d_1 + d_2 P^2_{(1)})d\vartheta^m &= d\vartheta^m \quad (3) \\
(Q^2_{(2)}d_2 + d_1 P^1_{(1)})d\vartheta^m &= d\vartheta^m. \quad (4)
\end{align*}
\]

Proof As \( d\vartheta^m \) is a 2-form, the commutation relation to use is

\[
S^i d_j - d_j S^i = 2\delta^i_j.
\]

We prove formula (1) as an example: we have

\[
Q^1_{(2)}d_2 = \left( \frac{1}{2} S^1 - \frac{1}{8} d_j S^{j1} + \frac{1}{48} d_{jk} S^{jkl} - \frac{1}{384} d_{jkl} S^{ijkl} \right) d_2
\]

and so

\[
\begin{align*}
\frac{1}{2} S^1 d_2 &= \frac{1}{2} d_2 S^1 , \\
-\frac{1}{8} d_j S^{j1} d_2 &= -\frac{1}{8} d_j (d_2 S^j + 2\delta^j_2) S^1 \\
&= -\frac{1}{8} d_2 d_j S^{j1} - \frac{1}{4} d_2 S^1 , \\
\frac{1}{48} d_{jk} S^{jkl} d_2 &= \frac{1}{48} d_{jk} (d_2 S^{jk} + 2\delta^j_2 S^k + 2\delta^k_2 S^j) S^1 \\
&= \frac{1}{48} d_2 d_{jk} S^{jkl} + \frac{1}{12} d_2 d_j S^{j1} , \\
-\frac{1}{384} d_{jkl} S^{ijkl} d_2 &= -\frac{1}{384} d_{jkl} (d_2 S^{jkl} + 2\delta^j_2 S^{kl} + 2\delta^k_2 S^{jl} + 2\delta^l_2 S^{jk}) S^1 \\
&= -\frac{1}{384} d_2 d_{jkl} S^{jkl} + \frac{1}{12} d_2 d_{jkl} S^{jkl}.
\end{align*}
\]
because $S^{jkl} d\theta^m = 0$ by Lemma 8. Thus
\[
Q_{(2)}^i d_2 = \frac{1}{4} d_2 S^i - \frac{1}{24} d_2 d_j S^{ij} + \frac{1}{144} d_2 d_{jk} S^{jkl} = d_2 P_{(1)}^i .
\]

The other formulæ may be obtained by similar calculations. 

**Lemma 10** The operator $Q_{(1)}^i$, when acting on total derivatives of the 3-form $d\Theta$, satisfies
\[
Q_{(1)}^i d_i d\Theta = d\Theta .
\]

**Proof** As $d\Theta$ is a 3-form, the commutation relation to use is now
\[
S^i d_j - d_j S^i = 3\delta^i_j .
\]

We have
\[
Q_{(1)}^i d_i = \left( \frac{1}{6} S^i j - \frac{1}{54} d_j S^{ji} + \frac{1}{648} d_{jk} S^{jki} - \frac{1}{9720} d_{jkl} S^{jkl} + \frac{1}{174960} d_{jkln} S^{jklni} \right) d_i ,
\]
and so
\[
\frac{1}{6} S^i d_i = \frac{1}{6} (d_i S^i + 3\delta^i_i) = \frac{1}{6} d_i S^i + 1 ,
\]
\[
- \frac{1}{54} d_j S^{ji} d_i = - \frac{1}{54} d_j (d_i S^{ji} + 3\delta^i_j S^i + 3\delta^i_i S^j) = - \frac{1}{54} d_j S^{ji} - \frac{1}{6} d_j S^j
\]
and so on, giving a collapsing series; the final term, involving $S^{jklni} d\Theta$, vanishes owing to the properties of $\Theta$. 

We remark that, although we have not specified in detail the properties of $\Theta$ which result in $S^{jklni} d\Theta$ vanishing (it is similar to Lemma 8), this doesn’t really matter: we could instead have specified a series with 16 terms when defining $Q_{(1)}^i$ and then, as $d\Theta$ is a 3-form on a 5th-order frame bundle, the final term, involving $d_i S^{ji} - j=0^i S^i$ $d\Theta$, would be guaranteed to vanish. This affects only the omitted pull-back maps, not the final conclusion.

**Theorem 3** If $L$ is a null Lagrangian then $d\Theta = 0$.

**Proof** If $L$ is a null Lagrangian then $dL = d_i \theta^i$, and therefore
\[
Q_{(2)}^k d_2 d\theta^i = Q_{(2)}^k d d_i \theta^i = Q_{(2)}^k d dL = 0 .
\]
But
\[
\Theta = P_{(1)}^2 d\theta^1 - P_{(1)}^1 d\theta^2
\]
so that
\[ d_1 \Theta = d_1 P^2_{(1)} d\theta^1 - d_1 P^1_{(1)} d\theta^2 \]
\[ = Q^2_{(2)} d_1 d\theta^1 - (1 - Q^2_{(2)} d_2) d\theta^2 \]
\[ = Q^2_{(2)} d_1 d\theta^1 - d\theta^2 \]
\[ = -d\theta^2 ; \]
similarly \( d_2 \Theta = d\theta^1 \). Thus
\[ d\Theta = Q^{(1)} d_i d\Theta = Q^1_{(1)} d_i d\theta^1 - Q^1_{(1)} d_i d\theta^2 \]
\[ = 0. \]

4 Further properties of the fundamental form

In this section we consider whether or not the fundamental form is projectable, first to a lower-order frame bundle, and then to a bundle of oriented contact elements.

Our first result is positive. Although the fundamental form has been defined on a fifth-order frame bundle, it is always projectable to the fourth-order bundle, and furthermore it is horizontal over the second-order bundle. We shall demonstrate projectability in coordinates, using the Lie derivative action of the locally-defined vector fields
\[ \partial_{ijk}^\alpha = \frac{\partial}{\partial u_{ijk}^\alpha} , \quad \partial_{ijkl}^\alpha = \frac{\partial}{\partial u_{ijkl}^\alpha} , \quad \partial_{ijklm}^\alpha = \frac{\partial}{\partial u_{ijklm}^\alpha} ; \]
the result will then follow from the connectedness of the fibres of \( F^5_2 E \to F^4_2 E \). We start with a lemma.

**Lemma 11** The Lie derivatives \( \partial_{ijk}^\alpha \), \( \partial_{ijkl}^\alpha \), \( \partial_{ijklm}^\alpha \) commute with the contractions \( S^p \).

They also satisfy
\[ \partial_{ijklm}^\alpha d_q = \delta_q^i \partial_{ijkm}^\alpha + \delta_q^j \partial_{iklm}^\alpha + \delta_q^k \partial_{ijlm}^\alpha + \delta_q^l \partial_{ijkm}^\alpha + \delta_q^i \partial_{ijkl}^\alpha \]
\[ \partial_{ijkl}^\alpha d_q = \delta_q^i \partial_{ijkl}^\alpha + \delta_q^j \partial_{ikjl}^\alpha + \delta_q^k \partial_{ijl}^\alpha + \delta_q^l \partial_{ijkl}^\alpha \]
when acting on forms or functions on \( F^4_2 E \) or \( F^5_2 E \) respectively.

**Proof** The first assertion holds because, when writing a form in coordinates in terms of the basis forms, the Lie derivatives by \( \partial_{ijk}^\alpha \), \( \partial_{ijkl}^\alpha \), \( \partial_{ijklm}^\alpha \) affect only the coefficient functions, whereas the contractions by \( S^p \) affect only the basis forms.

The second assertion is a straightforward computation using the coordinate expression for the total derivative \( d_q \).
Theorem 4 The fundamental form $\Theta$ is projectable to $\mathcal{F}^4(2)E$ and is horizontal over $\mathcal{F}^2(2)E$.

Proof We prove the second assertion first, by showing that $S^{pqrs}\Theta$ vanishes. We have

$$S^{pqrs}S^i d\bar{\vartheta}^m = 0$$

by Lemma 8; for the same reason

$$S^{pqrs}d_j S^{ji} d\bar{\vartheta}^m = \left( d_j S^{pqrs} + \delta^p_j S^{qr} + \delta^q_j S^{pr} + \delta^r_j S^{pq} \right) S^{ji} d\bar{\vartheta}^m = 0$$

and

$$S^{pqrs}d_{jk} S^{jki} d\bar{\vartheta}^m = \left( d_{jk} S^{pqrs} + d_j (\delta^p_k S^{qr} + \delta^q_k S^{pr} + \delta^r_k S^{pq}) + \delta^p_j (d_k S^{pr} + \delta^r_k S^r + \delta^r_k S^q) 
+ \delta^q_j (d_k S^{pr} + \delta^r_k S^q + \delta^k_s S^p) + \delta^r_j (d_k S^{pq} + \delta^k_s S^q + \delta^k_s S^p) \right) S^{jki} d\bar{\vartheta}^m = 0,$$

so that $S^{pqrs}P^{ij}_{(1)} d\bar{\vartheta}^m = 0$ and hence $S^{pqrs}\Theta = 0$. Thus $\Theta$ is horizontal over $\mathcal{F}^2(2)E$.

To show that $\Theta$ is projectable to $\mathcal{F}^4(2)E$ it is now sufficient to take a local basis of vector fields on $\mathcal{F}^5(2)E$ vertical over $\mathcal{F}^4(2)E$ and show that the Lie derivatives of $\Theta$ all vanish (the contractions vanish as a consequence of the part of the theorem just proved). But, using Lemma 11, we have

$$\partial_{\alpha}^{pqrs} S^i d\bar{\vartheta}^m = S^i \partial_{\alpha}^{pqrs} d\bar{\vartheta}^m = 0$$

because $d\bar{\vartheta}^m$ is pulled back from $\mathcal{F}^3(2)E$, and

$$\partial_{\alpha}^{pqrs}d_j S^{ji} d\bar{\vartheta}^m = \left( \delta^p_j \partial_{\alpha}^{qr} + \delta^q_j \partial_{\alpha}^{pr} + \delta^r_j \partial_{\alpha}^{pq} + \delta_j \partial_{\alpha}^{pqrs} \right) S^{ji} d\bar{\vartheta}^m = 0$$

for a similar reason. Finally,

$$\partial_{\alpha}^{pqrs}d_{jk} S^{jki} d\bar{\vartheta}^m = \left( \delta^p_j \partial_{\alpha}^{qr} + \delta^q_j \partial_{\alpha}^{pr} + \delta^r_j \partial_{\alpha}^{pq} + \delta_{jk} \partial_{\alpha}^{pqrs} \right) S^{jki} d\bar{\vartheta}^m = 0,$$

but if we use the coordinate representation of the Hilbert forms $\bar{\vartheta}^m$ as

$$\bar{\vartheta}^m = \left( \frac{\partial L}{\partial u^m} - \frac{1}{\#(mn)} d_n \frac{\partial L}{\partial u_{mn}} \right) du^\gamma + \frac{1}{\#(mn)} \frac{\partial L}{\partial u_{mn}} du^\gamma_n,$$

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(see [2]) then the only non-zero terms in the expansion of $S^{jki} d\vartheta^m$ are

$$S^{jki} \left( -\frac{1}{\#(mn)} \frac{\partial^2 L}{\partial u^\beta_{pl} \partial u^\gamma_{mn}} d\vartheta^\beta_p \wedge d\vartheta^\gamma_m + \frac{1}{\#(mn)} \frac{\partial^2 L}{\partial u^\beta_{pl} \partial u^\gamma_{mn}} d\vartheta^\beta_p \wedge d\vartheta^\gamma_m \right) ,$$

and these are second-order; thus each term of the form $\partial^{pqr}_a S^{jki} d\vartheta^m$ vanishes, so that $\partial^{pqr}_a d_{jk} S^{jki} d\vartheta^m = 0$. We conclude that $\partial^{pqr}_a \vartheta = 0$, so that $\vartheta$ is indeed projectable to $F^4_{(2)} E$.

We shall henceforth regard $\vartheta$ as being defined on $F^4_{(2)} E$, rather than on $F^5_{(2)} E$. Our second result is negative: it is not in general the case that $\vartheta$ is projectable to the manifold $J^4_1(E,2)$ of oriented fourth-order 2-dimensional contact elements. Projectability here would require that the contractions $\delta^i_1 \vartheta$, $\delta^i_2 \vartheta$, $\delta^i_3 \vartheta$ and $\delta^i_4 \vartheta$, and the Lie derivatives $d^i \vartheta$, $d^i \vartheta$, $d^i \vartheta$ and $d^i \vartheta$, should all vanish. In principle, therefore, it would be sufficient to choose a suitable Lagrangian, substitute into the coordinate formula for $\vartheta$, and show that at least one of the above conditions does not hold. However the coordinate formula for $\vartheta$ is already quite complicated, and it is necessary to use a homogeneous Lagrangian, so the calculations would be rather lengthy. We shall, instead, take a more indirect route.

**Lemma 12** The Lie derivative $d^i S^{pqr}$ satisfies

$$d^i S^{pqr} = \frac{1}{2} \left( \delta^i_1 S^{pqr} + \delta^i_2 S^{pqr} + \delta^i_3 S^{pqr} + \delta^i_4 S^{pqr} \right) d\vartheta^1 + \frac{1}{2} \delta^i_1 \left( S^{pqr} d\vartheta^1 + S^{pqr} d\vartheta^2 + S^{pqr} d\vartheta^4 \right)$$

$$+ \frac{1}{2} \delta^i_2 \left( S^{1pq} d\vartheta^1 + S^{1qr} d\vartheta^2 + S^{1rp} d\vartheta^4 \right)$$

$$+ \frac{1}{2} \delta^i_3 \left( S^{1pq} d\vartheta^2 + S^{1qr} d\vartheta^3 + S^{1rp} d\vartheta^4 \right)$$

$$+ \frac{1}{2} \delta^i_4 \left( S^{1pq} d\vartheta^3 + S^{1qr} d\vartheta^4 + S^{1rp} d\vartheta^4 \right).$$

**Proof** Once again we use the commutation relations from Lemma 1, together with some of our other lemmas. We have

$$d^i S^{pqr} S^i d\vartheta^m = (S^i d^i S^{pqr} - \delta^i S^{pq} r) d\vartheta^m$$

$$= -\delta^i S^{pq} r d\vartheta^m$$

using Lemma 4, and

$$d^i S^{pqr} d^i S^{pq} r d\vartheta^m = (d^i d^i S^{pqr} + \delta^i d^i S^{q} r + \delta^i d^i S^{q} r + \delta^i d^i S^{q} r) S^i S^i d\vartheta^m$$

$$= \left( d_j (S^i S^{pqr} - \delta^i S^{pq} r - \delta^i S^{pq} r) \right) S^i d\vartheta^m$$

$$= \left( d_j (S^i S^{pqr} - \delta^i S^{pq} r - \delta^i S^{pq} r) \right) S^i d\vartheta^m$$

$$= \left( (S^i d^i S^{pqr} - \delta^i S^{pq} r - \delta^i S^{pq} r) \right) d\vartheta^m$$

$$= -(3 \delta^i S^{pq} r + \delta^i S^{pq} r + \delta^i S^{pq} r + \delta^i S^{pq} r) d\vartheta^m$$
using Lemmas 4 and 8. We also have

\[
d_s^{pq} d_{jk} S^{ijk} d\vartheta^m = (d_s^{pq} + \delta_s^{pq} + \delta_s^{p} + \delta_s^{q}) d_{jk} S^{ijk} d\vartheta^m
\]

\[
= \left( d_j (d_s^{pq} + \delta_s^{pq} + \delta_s^{p} + \delta_s^{q}) + \delta_j (d_s^{pq} + \delta_s^{p} + \delta_s^{q}) + \delta_j (d_s^{pq} + \delta_s^{q} + \delta_s^{p}) + \delta_j (d_s^{pq} + \delta_s^{q} + \delta_s^{p}) \right) S^{ijk} d\vartheta^m ;
\]

but this simplifies considerably because, for instance,

\[
d_s^{pq} S^{ijk} d\vartheta^m = (\delta_s^{pq} - \delta_s^{pq} - \delta_s^{p} - \delta_s^{p}) S^{ijk} d\vartheta^m
\]

\[
= 0
\]

using Lemmas 4 and 8 again, and for the same reason we have \(d_s^{pq} S^{ijk} d\vartheta^m = 0\). We are left with

\[
d_s^{pq} d_{jk} S^{ijk} d\vartheta^m = \left( (\delta_s^{pq} + \delta_s^{pq} + \delta_s^{p} + \delta_s^{q}) d_s^{pq} + (\delta_s^{pq} + \delta_s^{p} + \delta_s^{q}) \right) S^{ijk} d\vartheta^m
\]

\[
= 2 \left( S^{pq} d_s^{pq} + S^{pq} d_s^{p} + S^{pq} d_s^{q}
\right.
\]

\[
- 3\delta_s^{pq} S^{pq} - 2\delta_s^{p} S^{pq} - 2\delta_s^{q} S^{pq} - 2\delta_s^{pq} \right) d\vartheta^m
\]

\[
= 2 \left( S^{pq} d_s^{pq} + S^{pq} d_s^{p} + S^{pq} d_s^{q}
\right)
\]

\[
- 3\delta_s^{pq} S^{pq} - 2\delta_s^{p} S^{pq} - 2\delta_s^{q} S^{pq} - 2\delta_s^{pq} \right) d\vartheta^m
\]

\[
= 2 \left( S^{pq} d_s^{pq} + S^{pq} d_s^{p} + S^{pq} d_s^{q}
\right)
\]

\[
- 2(3\delta_s^{pq} S^{pq} - \delta_s^{p} S^{pq} - \delta_s^{q}) S^{pq} \right) d\vartheta^m
\]

using Lemma 5. Putting all this together, we obtain

\[
d_s^{pq} P^{i}_{(1)} d\vartheta^m = \frac{1}{24} S^{pq} d_s^{pq} d\vartheta^m + \frac{1}{24} (3\delta_s^{pq} S^{pq} + \delta_s^{p} S^{pq} + \delta_s^{q} S^{pq}) d\vartheta^m
\]

\[
+ \frac{1}{24} (S^{pq} d_s^{pq} + S^{pq} d_s^{p} + S^{pq} d_s^{q}) d\vartheta^m
\]

\[
- \frac{1}{24} (3\delta_s^{pq} S^{pq} - \delta_s^{pq} - \delta_s^{p} S^{pq} - \delta_s^{q} S^{pq}) d\vartheta^m
\]

\[
= \frac{3}{16} S^{pq} d_s^{pq} d\vartheta^m + \frac{5}{24} (S^{pq} d_s^{pq} + S^{pq} d_s^{p} + S^{pq} d_s^{q}) d\vartheta^m
\]

\[
+ \frac{1}{24} (3\delta_s^{pq} S^{pq} - \delta_s^{pq} - \delta_s^{p} S^{pq} - \delta_s^{q} S^{pq}) d\vartheta^m
\]

from which the result follows.

**Lemma 13** If \(\Theta\) is projectable to \(J^1_+(E, 2)\) then \(S^{112} d\vartheta^1 = S^{111} d\vartheta^2 = 0\).

**Proof** If \(\Theta\) is projectable to \(J^1_+(E, 2)\) then in particular we must have \(d_1^{111} = d_2^{112} = 0\); but from the previous Lemma we find that

\[
d_1^{111} \Theta = \frac{3}{16} S^{112} d\vartheta^1 - \frac{5}{24} S^{111} d\vartheta^2
\]

\[
d_2^{112} \Theta = \frac{1}{12} S^{112} d\vartheta^1 - \frac{1}{16} S^{111} d\vartheta^2
\]
Corollary If $\Theta$ is projectable to $J_+^4(E, 2)$ then

$$\frac{\partial^2 L}{\partial u_{11}^\beta \partial u_{21}^\alpha} du^\beta \wedge du^\alpha = 0.$$ 

Proof We again use the coordinate representation of the Hilbert forms $\vartheta^m$. The only non-zero terms in the expansion of $S^{111} d\vartheta^2$ are

$$S^{111} \left( -\frac{1}{\#(2n)} \frac{\partial^2 L}{\partial u_{ij}^\beta \partial u_{2n}^\alpha} du_{ij}^\beta \wedge du^\alpha + \frac{1}{\#(2n)} \frac{\partial^2 L}{\partial u_{ij}^\beta \partial u_{2n}^\alpha} du_{ij}^\beta \wedge du^\alpha \right),$$

and if this expression is to vanish then the condition of the Corollary must hold.

Our task is now to find a Lagrangian $L$ that is homogeneous, but does not satisfy the condition in the Corollary above. First-order homogeneous Lagrangians are easy to find: for instance, any determinant $u_1^\alpha u_2^\beta - u_2^\alpha u_1^\beta$ is a (null) homogeneous Lagrangian. But second-order homogeneous Lagrangians are rather more complicated, and so we shall use the result (see [2]) that a Lagrangian 2-form $l$ on $J_+^2(E, 2)$ horizontal over $J_+^1(E, 2)$ gives rise to a homogeneous Lagrangian function $L = i_{2i} \rho^* l$ on $F_{(2)}^2 E$, where $\rho : F_{(2)}^2 E \to J_+^2(E, 2)$ is the projection. The fact that $l$ is horizontal means that the contraction $i_{2i}$ with the total derivatives is well-defined.

Take $E = \mathbb{R}^4$ with coordinates $(u^1, u^2, u^3, u^4)$, and let

$$D^{12} = u_1^1 u_2^2 - u_2^1 u_1^2, \quad D^{23} = u_1^2 u_2^3 - u_2^2 u_1^3, \quad D^{34} = u_1^3 u_2^4 - u_2^3 u_1^4$$

be the three determinants on $F_{(2)}^2 E$, so that the functions

$$F_1 = \frac{D^{23}}{D^{12}}, \quad F_2 = \frac{D^{34}}{D^{12}}$$

(defined on a suitable open submanifold) are projectable to $J_+^2(E, 2)$. We then construct the 2-form $dF_1 \wedge dF_2$, which is also projectable to $J_+^2(E, 2)$. The projection of this 2-form is certainly horizontal over $J_+^1(E, 2)$, so we may define a homogeneous Lagrangian function $L$ by

$$L = i_{2i} (dF_1 \wedge dF_2).$$

Of course we obtain a null Lagrangian, but this has no bearing on the argument.

We now use this Lagrangian function in a MAPLE calculation. The coordinate expression of $L$ involves over a page of MAPLE output, and is of no particular interest. But the calculations confirm, as we expect, that $d^2_j L = \delta^j_i L$ and that $d^3_k L = 0$, so that $L$ is indeed homogeneous; they also give

$$\frac{\partial^2 L}{\partial u_{11}^\beta \partial u_{12}^\alpha} - \frac{\partial^2 L}{\partial u_{21}^\beta \partial u_{12}^\alpha} = \frac{4 u_1^2 u_2^3 D^{34}}{(D^{12})^3} = \frac{4 u_1^2 u_2^3 (u_1^3 u_2^2 - u_2^3 u_1^2)}{(u_1^3 u_2^2 - u_2^3 u_1^2)^3} \neq 0,$$

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showing that the condition of the Corollary is not satisfied, and therefore that the corresponding fundamental form Θ is not projectable to $J_4^4(E, 2)$.

**Theorem 5**  If $L$ is a homogeneous Lagrangian defined on an open submanifold of $\mathcal{F}_{(2)}^2 E$ then its fundamental form $\Theta$, defined on the corresponding open submanifold of $\mathcal{F}_{(2)}^4 E$, will not in general be projectable to the bundle $J_{1+4}(E, 2)$ of oriented fourth-order 2-dimensional contact elements.

5 Further developments

As remarked in the Introduction, a construction for second-order Lagrangians in two independent variables is a rather small advance; a generalization to higher orders and more variables would be desireable. The problem, of course, is that the calculations rapidly become unmanageable without the use of more complicated machinery.

The extension to higher orders can be aided by the use of a simple multi-index notation for jet variables, but including extra independent variables requires a more sophisticated tool. This has been developed in [5], and involves the use of certain vector-valued forms on frame bundles, namely those taking their values in $\wedge^s \mathbb{R}^m$ for the case of $m$ independent variables. The total derivatives can be combined into a coboundary operator $d_T$ on the spaces of these forms, and this is (modulo pull-backs) globally exact: in fact the various operators $P^i_1$, $P^i_2$, $Q^{i(1)}_1$, $Q^{i(2)}_1$ used above are truncated components of the homotopy operator for $d_T$, and Lemmas 9 and 10 are special cases of the homotopy formula. The coboundary operator may be combined with the exterior derivative to give, for each order, a homogeneous variational bicomplex, and the step from the Lagrangian to the Hilbert forms involves a diagonal move across one square of the bicomplex together with an increase in the order. The step from the Hilbert forms to the fundamental form, in both the general first-order case and the two-variable second-order case, involves further diagonal moves and further increases in the order until the edge of the appropriate bicomplex has been reached.

We can therefore see what the answer ought to be in the general higher-order case: we simply carry out the procedure above. The proof that the resulting fundamental form is closed for a null Lagrangian is then an easy consequence of the homotopy formula. The remaining task is to demonstrate that the Lagrangian can be reconstructed from the fundamental form by contraction with total derivatives. This involves a computation of significant complexity, and work on the project continues.

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