Abstract

Two types of results are presented for distinguishing pure bipartite quantum states using Local Operations and Classical Communications. We examine sets of states that can be perfectly distinguished, in particular showing that any three orthogonal maximally entangled states in $C^3 \otimes C^3$ form such a set. In cases where orthogonal states cannot be distinguished, we obtain upper bounds for the probability of error using LOCC taken over all sets of $k$ orthogonal states in $C^n \otimes C^m$. In the process of proving these bounds, we identify some sets of orthogonal states for which perfect distinguishability is not possible.

1 Introduction

There is much interest in understanding what can and cannot be achieved using Local Operations and Classical Communications (LOCC) on a composite quantum system, pursued with an eye towards applications in communication and cryptography. One of the first and most basic problems in LOCC is that of distinguishing orthogonal quantum states. While some direct applications of this problem do exist (for instance, data hiding [13] and corrected channels [8, 9]),
these are limited by the usual assumption that no additional entanglement exists between the two parties. However, the problem of LOCC discrimination has proved a fertile area for attempts to better understand the relationship between entanglement and locality, the mysterious interplay that underlies virtually all quantum communication and cryptography protocols. It is in this spirit that the current work is undertaken.

The set-up for bipartite LOCC is quite simple: Two parties (by convention Alice and Bob) are physically separate but share a quantum state. Each may perform local quantum operations on his/her piece of the system, but the two may only communicate through a classical channel. In this paper, we will suppose that Alice and Bob share one of a known set of orthogonal states; their task is to determine the identity of this state (even if it is destroyed in the process). Since the possible states are orthogonal, they clearly could be distinguished and preserved were global operations permitted.

The most fundamental and surprising results in this area are those of Walgate, et al.,[15] that any two orthogonal states can always be locally distinguished; and of Bennett, et al.,[2] that there exists a basis of product states that cannot be distinguished with LOCC. These two facts demonstrate that there is no simple relationship between entanglement and locality, which has led to further exploration, e.g. [10, 14].

Following the definitive result for two states [15], work has been done to identify larger sets of orthogonal states that can and cannot be perfectly distinguished with LOCC. Both [5] and [6] looked at generalized Bell bases in $C^n \otimes C^n$. Fan [5] showed that any $k$ such states can be perfectly distinguished if $n$ is prime and $k(k-1) \leq 2n$, in particular in the case $k = n = 3$. The question was posed in [6] whether any 3 maximally entangled states could be distinguished; we answer this question in the affirmative. We also also give a sufficient condition for perfect distinguishability among maximally entangled states in $C^n \otimes C^n$ using unbiased bases, thus providing an alternative proof of the result in [5].

It not always possible to perfectly distinguish $k$ orthogonal vectors when $k > 2$. For instance, Ghosh, et al., showed that $k$ generalized Bell states in $C^n \otimes C^n$ cannot be distinguished with LOCC if $k > n$. [6, 7] The second part of this paper establishes lower bounds on the effectiveness of probabilistic LOCC discrimination of orthogonal vectors. If Alice and Bob share one of $k$ arbitrary orthogonal vectors in $C^n \otimes C^n$, what is their guaranteed minimal probability of correctly identifying it? And which sets of states achieve this minimum? These questions have an immediate application to a data hiding set-up as described in [13], in which a ‘Boss’ can clear prior entanglement between Alice and Bob.
before giving them pieces of a secret quantum state to work on.

It is shown that for \(2 \leq k \leq 4\), \(k\) arbitrary orthogonal vectors in \(C^m \otimes C^n\) can be correctly identified with probability at least \(\frac{2}{k}\), and this bound is tight. An interesting fact is that this does not depend on the dimension of the overall space—the worst case occurs when the states are embedded in a \(C^2 \otimes C^2\) subspace. Our final result translates these ideas into the more familiar language of mutual information and recovers a bound implied by [1].

The bounds from these propositions identify sets of vectors for which perfect distinguishability is impossible. In particular, we generalize [6] to show that no \(k\) maximally entangled states can be perfectly distinguished if \(k > n\). The bounds also lead to the well-known result of Horodecki, et al., [10] that a complete basis of perfectly distinguishable vectors must be a product basis.

As a final comment, we note the distinction made in [4] between LOCC protocols that have so-called infinite resources and those that use a finite number of rounds of communication and remain in finite-dimensional ancillary spaces. The results in this paper are established under the assumption that all protocols terminate with probability one and that each ancillary system is finite dimensional.

The paper is organized as follows: Section 2 states the results and gives necessary background, and Sections 3 and 4 provide the proofs.

## 2 Statement of Results

Following the result [15], we would like to identify sets of \(k\) orthogonal vectors that can be perfectly distinguished with LOCC for \(k > 2\). For instance, it is immediate that any three orthogonal states can be perfectly distinguished if two of them are product states. Also, from [5], any 3 states of a generalized Bell basis of \(C^n \otimes C^n\) can be distinguished if \(n \geq 3\); the question for general maximally entangled vectors in \(C^3 \otimes C^3\) is noted but not answered in [6].

**Proposition 1** Any three orthogonal maximally entangled states in \(C^3 \otimes C^3\) can be perfectly distinguished using LOCC.

It is not clear whether any 3 orthogonal maximally entangled states are distinguishable in \(C^n \otimes C^n\). However, the following proposition gives a sufficient condition for distinguishing maximally entangled states using the idea of mutually unbiased bases, which arise in several area of quantum information (see,
for instance [11, 16]. The more general notion of a common unbiased basis is not well-studied but is defined here for convenience:

**Definition 2** Let $A_i = \{a_i \rangle : i \in I\}$ be a family of orthonormal bases of $C^n$, with $A_i = \{|a_{i1}\rangle, |a_{i2}\rangle, \ldots, |a_{in}\rangle\}$ and $I$ some indexing set.

A basis $B$ of $C^n$ is a common unbiased basis for $A$ if, for all $|b\rangle \in B$ and for all $i \in I, 1 \leq j \leq n$:

$$|\langle b|a_{ij}\rangle|^2 = \frac{1}{n} \tag{1}$$

So, a set of bases $A$ is mutually unbiased if and only if for all $i \in I, A_i$ is a common unbiased basis for $A - \{A_i\}$.

In the following proposition, we write our states in terms of a (non-canonical) standard maximally entangled state of $C^n \otimes C^n$:

$$|ME_n\rangle := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} |j\rangle |j\rangle \tag{2}$$

**Proposition 3** Let $|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_k\rangle$ be orthogonal, maximally entangled vectors in $C^n \otimes C^n$, with $|\Psi_i\rangle = (I \otimes B_i)|ME_n\rangle$.

For each pair $(i, j)$, let $A_{ij}$ be a basis of eigenvectors of $B_i^\dagger B_j$, and let

$$A = \{A_{ij} : 1 \leq i < j \leq k\}$$

If the family $A$ has a common unbiased basis, then the $k$ states can be perfectly distinguished by LOCC.

The result is actually more general—we do not require that the states be maximally entangled, only that the matrices $B_i^\dagger B_j$ be diagonalizable. For instance, we could use the same proof to show that any simultaneously diagonalizable orthogonal states can be locally distinguished. These are sets of the form

$$\{|\varphi_i\rangle = \sum_{j=0}^{n-1} u_{ij} |jj\rangle, 1 \leq i \leq n\} \tag{3}$$

where $u$ is an $n \times n$ unitary matrix.

The main result of [5] follows from Proposition 3. It involves the generalized Pauli matrices $Z = \sum_j e^{2\pi ij/n} |j\rangle \langle j|$ and $X = \sum_j |j\rangle \langle j+1|$ and the generalized Bell basis

$$BB_n := \{(I \otimes X^m Z^l)|ME_n\rangle : 0 \leq m, l \leq n - 1\} \subset C^n \otimes C^n \tag{4}$$
Corollary 4 (H. Fan) Let $|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_k\rangle$ be orthogonal, maximally entangled vectors in $C^n \otimes C^n$, with $n$ prime and $|\Psi_i\rangle \in BB_n$.

Then if $k(k-1)/2 \leq n$, the $k$ vectors can be perfectly distinguished by LOCC.

Proof: This follows from the fact that for $n$ prime, the eigenbases of $\{X^m Z^l : 0 \leq l, m < n\}$ form a maximum set of $(n+1)$ mutually unbiased bases in $C^n$.[11]

Up to a global phase,

$$(X^m Z^l)\dagger (X^{m_j} Z^{l_j}) \equiv X^{m_j-m_i} Z^{l_j-l_i} \quad (5)$$

so the eigenbases of the pairwise products also belong to the set of mutually unbiased bases. As long as the number of pairs $(i, j)$ is less than the number of mutually unbiased bases, then there exists a common unbiased basis and the proposition can be applied. But this is the condition that $k(k-1)/2 < n + 1$.

It is not always possible to distinguish maximally entangled states ([7]), which raises the question of how bad it can be (or conversely, what minimal level of success is guaranteed). When perfect discrimination is not possible, one possible strategy is unambiguous discrimination, in which either the correct identity of the state is discovered or else a generic error message is returned. Another strategy is minimum error discrimination, in which the protocol always produces one of the possible states but this identification might be incorrect. The challenge in this case is to find a protocol that minimizes the probability of error. It is this problem of minimum error discrimination that we will consider throughout the rest of the paper.

Suppose Alice and Bob share one of the orthogonal vectors $\{|\Psi_i\rangle\}$ with a priori probabilities $\{p_i\}$. They apply an LOCC protocol, which produces a best guess as to the identity of their state. Define $P(\{|\Psi_i\rangle\}, \{p_i\})$ as the probability that Alice and Bob correctly identify which vector they share, assuming an optimal strategy is used. We are interested in the worst case scenario—what ensembles of $k$ orthogonal vectors are hardest to distinguish using LOCC? Initially, we restrict ourselves to maximally entangled states and define

$$f_{me}(k, n) := \min_{\{|\Psi_i\rangle\}, \{p_i\}} P(\{|\Psi_i\rangle\}, \{p_i\}) \quad (6)$$

where the minimum is taken over probability distributions $p_i$ and sets of orthogonal maximally entangled states $\{|\Psi_1\rangle, \ldots, |\Psi_k\rangle\} \subset C^n \otimes C^n$.

We immediately observe that $f_{me}$ is a nonincreasing function in both $k$ and $n$; as $k$ and $n$ increase, the minimum is taken over larger nested sets. We note that
for all $n$, $f_{me}(2, n) = 1$, since two orthogonal states can always be distinguished by LOCC. Proposition 1 is equivalent to the fact that $f_{me}(3, 3) = 1$.

But there are limitations to what can be done if the number of vectors is bigger than the dimension:

**Proposition 5** For all $2 \leq n \leq k \leq n^2$,

$$\frac{2}{k} \leq f_{me}(k, n) \leq \frac{n}{k}$$

In the case $n = 3 \leq k \leq 9$,

$$f_{me}(k, 3) = \frac{3}{k}$$

We can also define a more general function in which we remove the assumption that the states are maximally entangled

$$f(k, n) := \min_{\{|\Psi_i\rangle\}, \{p_i\}} P(\{|\Psi_i\rangle\}, \{p_i\})$$

where the minimum is taken over probability distributions $p_i$ and *all* sets of orthogonal states $\{|\Psi_1\rangle, \ldots, |\Psi_k\rangle\} \subset C^n \otimes C^n$.

Again, $f$ is nonincreasing with respect to $n$ and $k$ and $f(2, n) = 1$. Also, for $k \leq m^2 \leq n^2$, $k$ maximally entangled vectors in $C^m \otimes C^m$ can be embedded in $C^n \otimes C^n$, so $f(k, n) \leq f_{me}(k, m)$. The previous results for $f_{me}$ imply bounds on $f$:

**Proposition 6** For $2 \leq n \leq k \leq n^2$,

$$\frac{2}{k} \leq f(k, n) \leq \frac{\sqrt{k}}{k}$$

In particular,

$$f(3, n) = \frac{2}{3} \quad f(4, n) = \frac{1}{2}$$

The function $f(k, n)$ is defined only when the two spaces have the same dimension. We could just as easily have defined $f(k, m, n)$ for $k$ vectors in $C^m \otimes C^n$ and applied Lemma 8 to that. However, we have discovered no bounds
on this that don’t follow from inclusion; that is, for \( m \leq n \), the best we can say is:

\[
\begin{align*}
  f(k, n) & \leq f(k, m, n) \leq f(k, m) \quad k \leq m^2 \\
  f(k, n) & \leq f(k, m, n) \leq \frac{n}{k} \quad m^2 < k \leq mn
\end{align*}
\]  

We note that for \( k \leq 4 \), \( f(k, n) \) is independent of \( n \); the \( k \) vectors are most difficult to distinguish when they are squeezed into the smallest possible space. It seems entirely possible that \( f(k, n) \) will remain independent of \( n \) even for higher values of \( k \).

Propositions 5 and 6 are proved using the following lemmas. In fact, most of the work goes into the proof of Lemma 8 as it requires us to analyze Alice and Bob’s protocol in detail.

**Lemma 7** For all \( 2 \leq j \leq k \leq n^2 \),

\[
\frac{j}{k}f_{me}(j, n) \leq f_{me}(k, n)
\]  

\[
\frac{j}{k}f(j, n) \leq f(k, n)
\] 

**Lemma 8** Given \( k \) equally probable vectors \( \{|\Psi_1\rangle, \ldots, |\Psi_k\rangle\} \subset C^n \otimes C^n \), \( n \leq k \leq mn \), with the property that for each \( i \), \( |\Psi_i\rangle = (I \otimes U_i)|\Psi_1\rangle \) for \( U_i \) unitary. Then the \( k \) vectors can be distinguished using LOCC with probability at most \( \frac{n}{k} \).

The assumption in Lemma 8 is equivalent to the fact that the \( C^n \) party can unilaterally transform \( |\Psi_i\rangle \) into \( |\Psi_j\rangle \) for any \( i, j \). The lemma includes the special case in which all the states are maximally entangled. Also, note that there is no assumption here that the states are orthogonal, though this is clearly the most interesting case.

**Examples:** Given a basis of 4 orthogonal maximally entangled states in \( C^2 \otimes C^2 \). One naive notion is ignore two of the possible states and perfectly distinguish the remaining two, thus achieving the lower bound in Lemma 7. Lemma 8 states that this, in fact, is an optimal strategy for identifying the given state. Proposition 6 combines the lemmas to say that this is the worst case for trying to distinguish 4 orthogonal states.

Likewise, given \( k > 3 \) orthogonal maximally entangled states in \( C^3 \otimes C^3 \), one can discard all but three of them and then perfectly distinguish those that
remain using Proposition \[1\]. Again, the lemma states that this is optimal. However, for \( k = 4 \) or \( k = 5 \), this succeeds with probability greater than \( \frac{1}{2} \) and so is no longer the worst case in \( C^3 \otimes C^3 \). A worse case would be 4 equally probable maximally entangled states in a \( C^2 \otimes C^2 \) subspace.

Finally, we look at an example using the generalized Bell basis \( BB_n \) defined in \[4\]. Suppose we wish to distinguish the states in a set \( T \subset BB_n \) with \( |T| = k \).

If \( n \) is prime, then the argument in \[5\] implies that Alice and Bob can correctly identify their vectors with probability \( \frac{n}{k} \); Lemma \[8\] shows that this is in fact optimal.

The following modification of Lemma \[8\] establishes a necessary condition to distinguish a set of states:

**Proposition 9** Given \( k \) equally probable vectors \( \{|\Psi_1\rangle, \ldots, |\Psi_k\rangle\} \subset C^m \otimes C^n \) and let \( \lambda_M \) be the largest Schmidt coefficient in any of the \( |\Psi_i\rangle \). Then the \( k \) vectors can be distinguished using LOCC with probability at most \( \lambda_M \frac{mn}{k} \).

In particular, if \( k \) vectors can be perfectly distinguished with LOCC, then \( \lambda_M \geq \frac{k}{mn} \).

It is interesting to note that in the case of perfect distinguishability, this proposition gives a lower bound on the maximal Schmidt coefficient, while the result of Chen and Li \[3\] gives an upper bound on the number of nonzero Schmidt coefficients.

The following generalizes the work of \[6\] by setting \( \lambda_M = \frac{1}{n} \) above.

**Corollary 10** No \( k \) maximally entangled states in \( C^n \otimes C^n \) can be perfectly distinguished with LOCC if \( k > n \).

Both Proposition \[9\] and the result \[3\] imply the fundamental result of Horodecki, et al., that a distinguishable basis must be a product basis \[10\]:

**Corollary 11** (Horodecki, et al.) Let \( \{|\Psi_1\rangle, \ldots, |\Psi_{mn}\rangle\} \) be an orthonormal basis of \( C^m \otimes C^n \), and suppose these vectors can be perfectly distinguished using LOCC. Then each of the vectors is a product vector.

To see this as a consequence of Proposition \[9\] suppose we have have one of the \( |\Psi_i\rangle \) with equal probability. Then clearly \( \lambda_M = k/mn = 1 \). Examining the proof of Proposition \[9\] reveals that if \( |\Psi_i\rangle \) has maximal Schmidt coefficient \( \lambda_i < \lambda_M \), then either \( P(Z = i) = 0 \) or else the inequality on \( P(Z = V) \) is strict. Neither of these is possible with perfect distinguishability, which means \( \lambda_i = \lambda_M = 1 \) and \( |\Psi_i\rangle \) is a product state for all \( i \).
These types of results are useful in that they allow us to identify classes of sets of \( k \) vectors in \( C^m \otimes C^n \) that cannot be perfectly distinguished. Also, they provide an upper bound on the probabilities and allows us to deduce optimal strategies for correct identification.

The function \( f(k, n) \) is one way of assessing how much information Alice and Bob can gain from LOCC measurements on their vectors. Another approach would be to use the classical mutual information between the identity \( V \) of the vector sent and the outcomes of Alice and Bob’s measurements. (This idea was explored, for instance, with reference to the specific 9-state ensemble in [2].) Let \( Y \) represent the outcomes of the first \( r - 1 \) measurements and \( Z \) indicate the final measurement, i.e. the conclusion as to the value of \( V \), and write

\[
I(V; YZ) = H(V) - H(V|YZ)
\]

where \( H \) is the Shannon entropy.

As we defined \( f(k, n) \), we define a function \( g(k, n) \) based on mutual information. Assuming that Alice and Bob use optimal measurements, we can consider \( I(V; YZ) \) to be the optimal mutual information between the input vector \( V \) and the measurement results.

\[
g(k, n) := \min_{\{|\Psi_i\rangle\}} I(V; YZ)
\]

Note that we now assume that all the \( k \) vectors are equally likely; there is no sensible lower bound if the entropy of the a priori probability distribution is allowed to approach zero.

**Proposition 12** The function \( g(k, n) \) defined above for \( 1 < k \leq n^2 \) satisfies the following bounds:

\[
\frac{2}{k} \log 2 \leq g(k, n) \leq \log \left\lceil \sqrt{k} \right\rceil
\]

This proposition is proved as a consequence of Lemma 8. The same upper bound can be seen as a consequence of the following inequality given in [1]:

\[
I_{LOCC}^{acc} \leq S(\rho_A) + S(\rho_B) - \sum_i p_i S(\rho_A^i)
\]

where \( I_{acc}^{LOCC} \) is the classical mutual accessible information using LOCC, \( S \) is von Neumann entropy, \( \rho = \sum p_i |\Psi_i\rangle \langle \Psi_i| \), and \( \rho_A \) and \( \rho_B \) are the partial traces.
Let the $|\Psi_i\rangle$ be maximally entangled states in $C^n \otimes C^n$. Then

\[ \rho_A^i = \rho_A = \rho_B = \frac{1}{n}I_n \quad \forall \ i \]

\[ I_{\text{acc}}^{\text{LOCC}} \leq S(\rho_A) + S(\rho_B) - \sum_i p_i S(\text{Tr}_A(|\Psi_i\rangle\langle\Psi_i|)) \]

\[ \leq \log n + \log n - \sum_i p_i S(\text{Tr}_A(|\Psi_i\rangle\langle\Psi_i|)) = \log n \]

This gives another way to see that $k$ maximally entangled states in $C^n \otimes C^n$ cannot be distinguished if $k > n$.

**Example:** Recall the set $BB_n$ defined in (4); it is a generalized Bell basis for $C^n \otimes C^n$. Suppose Alice and Bob share a state $|\Psi\rangle = (I \otimes X^n Z^l)|ME_n\rangle$, uniformly chosen from $BB_n$. Each measures in the standard basis, allowing them to perfectly determine the value of $m$ but giving no information about $l$.

If at this point, they make a guess as to the value of $l$, they will be correct with probability $\frac{1}{n}$, which saturates the inequality in Lemma 8, and hence is optimal for $P(Z = V)$.

Perhaps more surprising, this protocol is also optimal with respect to classical mutual information, as $I(V;Y Z) = \log n$ and the proof of the upper bound in Proposition 12 shows that this is maximal.

## 3 Proofs of Propositions for Distinguishing Maximally Entangled States

### 3.1 Preliminaries

As has been previously noted (for instance in [12]), there is one-to-one correspondence between states $|\Psi\rangle \in C^n \otimes C^m$ and $m \times n$ complex matrices $B$ given by $|\Psi\rangle = (I \otimes B)|ME_n\rangle$, where $|ME_n\rangle$ is the standard maximally entangled $C^n \otimes C^m$ state defined in (2). Throughout the paper, we will use the following property, which was noted in [12] and implicitly used in [15]:

**Lemma 13** For any $m \times n$ matrix $A$ written in the standard basis,

\[ \sqrt{n}(I \otimes A)|ME_n\rangle = \sqrt{m}(A^T \otimes I)|ME_m\rangle \]
In particular, setting $m = 1$,

$$\sqrt{n}(I \otimes \langle v |)|ME_n\rangle = |\overline{v}\rangle \otimes I$$

(24)

where $|\overline{v}\rangle$ denotes the entrywise complex conjugate of $|v\rangle$ in the standard basis.

We adopt the convention of associating states $|\Psi\rangle$ with $\langle \Psi |\Psi \rangle = 1$ and $m \times n$ matrices $B$ with $\text{Tr} B^\dagger B = n$. This correspondance has the following immediate properties:

1. If $|\Psi_i\rangle = (I \otimes B_i)|ME_n\rangle$ for $i = 1, 2$, then $\langle \Psi_1 |\Psi_2 \rangle = \frac{1}{n} \text{Tr} B_1^\dagger B_2$

2. $||B^\dagger B||_\infty = n\lambda_M$, where $\lambda_M$ is the largest Schmidt coefficient of $|\Psi\rangle$.

3. $|\Psi\rangle = (I \otimes B)|ME_n\rangle \in C^n \otimes C^n$ is maximally entangled if and only if $B$ is unitary.

We will use this correspondance throughout what follows.

### 3.2 Proof of Proposition 1

For $i = 1, 2, 3$, write $|\Psi_i\rangle = (I \otimes B_i)|ME_3\rangle$ with $B_i$ unitary and $\text{Tr} B_i^\dagger B_j = 3\delta_{ij}$. The matrix $B_2^\dagger B_1$ is a traceless $3 \times 3$ unitary matrix, so its eigenvalues are $\{1, \omega, \omega^2\}$, with $\omega = e^{i2\pi/3}$. The same is also true for $B_3^\dagger B_2$. We write these matrices in terms of their eigenvectors:

$$B_2^\dagger B_1 = \sum_{i=0}^2 \omega^i |e_i\rangle \langle e_i| \quad B_3^\dagger B_2 = \sum_{i=0}^2 \omega^i |f_i\rangle \langle f_i|$$

(25)

Given $|\Psi_i\rangle$, for $i$ unknown, choose a unitary $U$ and measure the first system in the basis $\{U|j\rangle : j = 0, 1, 2\}$, where $U$ indicates the entrywise complex conjugate of $U$. If the outcome of the measurement is $x \in \{0, 1, 2\}$, then Lemma 13 implies the state now looks like:

$$\langle U|x\rangle \langle x|U^T \otimes I |\Psi_i\rangle = \langle U|x\rangle \langle x|U^T \otimes B_i|ME_n\rangle$$

(26)

$$= \langle U|x\rangle \otimes B_i (\langle x|U^T \otimes I |ME_n\rangle)$$

(27)

$$= \frac{1}{\sqrt{n}} \langle U|x\rangle \otimes B_i (I \otimes U|x\rangle)$$

(28)

$$= \frac{1}{\sqrt{n}} U|x\rangle \otimes B_i U|x\rangle$$

(29)
In particular, after normalization, the second system is in the state

\[ B_2 U |x\rangle \] (30)

We want to show that for appropriate choice of \( U \), the vectors \( \{ B_1 U |x\rangle, B_2 U |x\rangle, B_3 U |x\rangle \} \) are orthogonal for all \( x \). The proof is constructive and is achieved in 3 steps:

1. Observe that the quantity \( |\langle e_i | f_j \rangle|\) depends only on \((j - i) \mod 3\).
2. Show that we can adjust the phases of the \( |e_i\rangle \) and \( |f_j\rangle \) so that we may assume that \( \langle e_i | f_j \rangle \) depends only on \((j - i) \mod 3\).
3. Let our unitary \( U \) be the Fourier matrix in the basis \( \{ |e_i\rangle \} \) and show that the vectors \( \{ B_1 U |x\rangle, B_2 U |x\rangle, B_3 U |x\rangle \} \) are orthogonal for all \( x \).

The proof of each step is given below. Note that all operations on indices are assumed to be taken modulo 3.

1. Since \( \text{Tr} B_3^\dagger B_1 = 0 \):

\[
0 = \text{Tr} B_3^\dagger B_2 B_2^\dagger B_1 = \sum_{i,j} \omega^{i+j} |\langle e_i | f_j \rangle|^2 = \sum_{i,k} \omega^k |\langle e_i | f_{k-i} \rangle|^2
\] (31)

For any \( a_k \geq 0 \), \( \sum_{k=0}^{2} \omega^k a_k = 0 \) implies that all the \( a_k \) are the same. Therefore,

\[
\sum_i |\langle e_i | f_{-i} \rangle|^2 = \sum_i |\langle e_i | f_{1-i} \rangle|^2 = \sum_i |\langle e_i | f_{2-i} \rangle|^2
\] (32)

Combining with the normalization conditions for any \( i, j \)

\[
\sum_k |\langle e_k | f_j \rangle|^2 = \sum_k |\langle e_i | f_k \rangle|^2 = 1
\] (33)

gives a linear system of 7 independent equations in the 9 unknowns \( |\langle e_i | f_j \rangle|^2 \) whose solutions look like this:

\[
(|\langle e_i | f_j \rangle|^2)_{ij} = \begin{pmatrix} |a|^2 & |c|^2 & |b|^2 \\ |b|^2 & |a|^2 & |c|^2 \\ |c|^2 & |b|^2 & |a|^2 \end{pmatrix} \] (34)

That is, the quantity \( |\langle e_i | f_j \rangle|^2 \) depends only on \((j - i) \mod 3\).
2. Let $V$ be the unitary matrix whose $(i,j)$ entry is given by $\langle e_i|f_j \rangle$. From above, $|V_{i,j}|$ depends only on $(j-i)$ mod 3. We would like to have $V_{i,j}$ itself depend only on $(j-i)$ mod 3. We accomplish this by adjusting the phases of the $|e_i\rangle$ and $|f_j\rangle$, which is equivalent to finding diagonal unitaries $U_1$ and $U_2$ such that

$$V' = U_1 V U_2^\dagger = \begin{pmatrix} a & c & b \\ b & a & c \\ c & b & a \end{pmatrix}$$

(35)

for some $a,b,c \in \mathbb{C}$. Write $m_{ij} = \arg(\langle e_i|f_j \rangle)$ and

$$U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\gamma} & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}$$

Solving a system of 3 linear equations in the phases of the first two columns of $V$ allows us to set:

$$\gamma = \frac{1}{3} \sum_{j=0}^{2} (m_{1j} - m_{0j})$$

(36)

$$\alpha = m_{00} - m_{11} + \gamma$$

(37)

$$\beta = m_{01} - m_{20} - \gamma$$

(38)

Put these values into $U_1$ and $U_2$ and choose $\delta$ to adjust the top right corner, which gets our matrix into the form

$$V' = \begin{pmatrix} a & c & b \\ b & a & ce^{i\delta_1} \\ c & b & ae^{i\delta_2} \end{pmatrix}$$

(39)

The fact that $V'$ is unitary implies its columns are orthogonal, yielding the three equations

$$\begin{pmatrix} 1 & e^{i\delta_1} & 1 \\ e^{-i\delta_1} & 1 & e^{i\delta_2} \\ e^{-i\delta_2} & e^{-i\delta_1} & 1 \end{pmatrix} \begin{pmatrix} \overline{a}c \\ \overline{a}b \\ \overline{b}a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(40)

The determinant of the above matrix cannot be zero unless $e^{i\delta_1} = e^{i\delta_2} = 1$, which means that in fact $V'$ is already in the desired form (35).
Adjusting our matrix \( V \) was equivalent to adjusting the phases of the vectors \(|e_i\rangle\) and \(|f_j\rangle\). Therefore, without loss of generality, we assume that \( \langle e_i | f_j \rangle \) depends only on \((j - i) \bmod 3\) and define

\[
A_k := \langle e_i | f_{k+i} \rangle
\]  

which is independent of \(i\).

3. For \(x \in \{0, 1, 2\}\) define:

\[
U|x\rangle = \frac{1}{3^{\frac{1}{2}}} \sum_{i=0}^{2} \omega^{ix} |e_i\rangle
\]

(42)

Explicit calculation shows that for all \(x\), the vectors \(B_1 U|x\rangle, B_2 U|x\rangle, B_3 U|x\rangle\) are pairwise orthogonal:

\[
3 \langle x | U^\dagger B_1^\dagger B_1 U | x \rangle = \sum_k \omega^{-kx} \omega^k \omega^{lx} = 0
\]

(43)

\[
3 \langle x | U^\dagger B_1^\dagger B_2 U | x \rangle = \sum_{k,i,l} \omega^{-kx} \omega^i \omega^{lx} \langle e_k | f_i \rangle \langle f_l | e_i \rangle
\]

(44)

\[
= \sum_{k,i,l} \omega^{(l-k)x} \omega^i \omega^{lx} A_{i-k} \overline{A_{i-l}}
\]

(45)

\[
= \sum_{k',l'} (\omega^{(k'-l')x} A_{k'} \overline{A_{l'}}) \sum_i \omega^i = 0
\]

(46)

\[
3 \langle x | U^\dagger B_1^\dagger B_3 U | x \rangle = \langle x | U^\dagger B_3^\dagger B_2 B_1 U | x \rangle
\]

(47)

\[
= \sum_{k,i,l} \omega^{(l-k)x} \omega^{j+l} A_{i-k} \overline{A_{i-l}}
\]

(48)

\[
= \sum_{k',l'} (\omega^{(k'-l')x} \omega^{l'-j} A_{k'} \overline{A_{l'}}) \sum_i \omega^{2i} = 0
\]

(49)

This proves that for all \(x\), the vectors \(B_1 U|x\rangle, B_2 U|x\rangle, B_3 U|x\rangle\) are orthogonal and hence can be perfectly distinguished.
3.3 Proof of Proposition 3

Let \( \mathcal{B} = \{ |b_1\rangle, \ldots, |b_n\rangle \} \) be the common unbiased basis. We need to show that for any \( i \neq j \) and any \( k \), the vectors \( B_i|b_k\rangle \) and \( B_j|b_k\rangle \) are orthogonal. Using the eigenbasis \( \mathcal{A}_{ij} \), write

\[
B_i^\dagger B_j = \sum_s \lambda_s |e_s\rangle \langle e_s| \quad (50)
\]

Then for all \( k \),

\[
\langle b_k|B_i^\dagger B_j|b_k\rangle = \sum_s \lambda_s |\langle b_k|e_s\rangle|^2 = \frac{1}{n} \sum_s \lambda_s = \frac{1}{n} \mathrm{Tr} B_i^\dagger B_j = 0 \quad (53)
\]

4 Proofs on the Worst Cases for Distinguishing Orthogonal States

Throughout what follows, let \( V \) be the true identity of the vector \( |\Psi_i\rangle \), and let \( Z \) be Alice and Bob’s best guess of the value of \( V \), which we assume is also the outcome of the final measurement. Their goal, then, is to maximize \( P(Z = V) \).

4.1 Proof of Propositions 5 and 6 Using the Lemmas

Setting \( j = 2 \) in Lemma 7 gives us the desired lower bounds, since \( f_{me}(2, n) = f(2, n) = 1 \). As long as \( k \leq n^2 \), there exist \( k \) orthogonal maximally entangled vectors in \( C^n \otimes C^n \), so Lemma 8 implies that \( f_{me}(k, n) \leq \frac{n}{k} \). In the case \( k \leq 9 \), we know \( f_{me}(3, 3) = 1 \) so

\[
\frac{3}{k} = \frac{3}{k} f_{me}(3, 3) \leq f_{me}(k, 3) \leq \frac{3}{k} \quad (54)
\]

so \( f_{me}(k, 3) = \frac{3}{k} \). Similarly, if \( k \leq m^2 \leq n^2 \), then \( f(k, n) \leq f_{me}(k, m) \), since we can embed maximally entangled \( C^m \otimes C^m \) vectors into \( C^n \otimes C^n \). The minimum value of \( m \) for which we can do this is \( \lceil \sqrt{k} \rceil \), which implies

\[
f(k, n) \leq \frac{\lceil \sqrt{k} \rceil}{k} \quad (55)
\]
In the case $2 \leq k \leq 4$, $\lceil \sqrt{k} \rceil = 2$ and
\[
\frac{2}{k} = \frac{2}{k} f(2, n) \leq f(k, n) \leq \frac{2}{k}
\] (56)
which implies
\[
f(k, n) = \frac{2}{k}
\] (57)

4.2 Proof of Lemma 7

We prove the lemma for the function $f(k, n)$; the proof for $f_{me}$ is identical. Given any orthogonal vectors $|\Psi_i\rangle \in \{|\Psi_1\rangle, \ldots, |\Psi_k\rangle\}$ with probabilities $p_1 \geq p_2 \geq \ldots \geq p_k$. There exists an algorithm that can distinguish the first $j$ of these vectors with probability at least $f(j, n)$. Applying this algorithm to the received vector $|\Psi_i\rangle$ cannot succeed if $i > j$, but clearly:
\[
P(Z = V) \geq P(Z = V, i \leq j) = P(i \leq j)P(Z = V|i \leq j) \geq \frac{j}{k} f(j, n)
\] (58)
which gives the desired lower bound on $f(k, n)$.

4.3 Proof of Lemma 8

For this proof, we will need to examine the measurement process more closely. As mentioned earlier, we will assume that the protocol terminates with probability 1. In fact, through the calculation, we will assume there exists an $r$ such that the protocol terminates after at most $r$ rounds of communication. Completing the argument for arbitrary $r$ is sufficient. Let $R$ be the actual number of rounds needed to complete to protocol and let $p_r$ be the probability that more than $r$ rounds are needed. Then
\[
P(Z = V) = (1 - p_r)P(Z = V|R \leq r) + p_r P(Z = V|R > r) \leq P(Z = V|R \leq r) + p_r
\] (61)
Our proof will show that for any $r$, $P(Z = V|R \leq r) \leq \frac{n}{k}$. Taking the limit as $r \to \infty$, $p_r$ gets arbitrarily small and we can bound $P(Z = V)$ by $\frac{n}{k}$. 16
The actions of Alice and Bob will consist of adding ancilla systems, performing unitary operations, and performing measurements. All of these can be encoded into a POVM. Alice measures first; we write her POVM as \( \mathcal{X}^T = \{X_1^T, X_2^T, \ldots, X_k^T\} \). (Because we will eventually apply Lemma 13 to show the effect of Alice’s POVM on Bob’s system, we write it in terms of the transpose.) Suppose Alice gets the result \( j_1 \); then Bob uses a POVM that depends on \( j_1 \):

\[
\mathcal{E}_{j_1} = \{E(j_1), E(j_1 \otimes B_i)\}
\]

Alice then measures in a POVM that depends on \( j_1 \) and \( j_2 \), and so on. After \( r \) rounds of measurement, Alice and Bob have effectively measured using the POVM

\[
\{\mathcal{X}_{j_1,j_2,\ldots,j_{r-1}}^T \otimes \mathcal{E}_{j_1,j_2,\ldots,j_r} : j_1, j_2, \ldots, j_r \geq 0\}
\]

which are defined recursively as in [2]:

\[
\mathcal{X}_{j_1,j_2,\ldots,j_{r-1}}^T = X^T(j_1, j_2, \ldots, j_{r-1}) \mathcal{X}_{j_1,j_2,\ldots,j_{r-2}}^T
\]

\[
\mathcal{E}_{j_1,j_2,\ldots,j_{r}} = E(j_1, j_2, \ldots, j_{r-1}) \mathcal{E}_{j_1,j_2,\ldots,j_{r-2}}
\]

The subscripts show that each measurement depends on the previous outcomes. Here each \( X^T(m_0) \) and \( E(m_1) \) is a POVM, where \( m_0 \) is a vector encoding an even number of previous outcomes and \( m_1 \) encodes an odd number. This corresponds to the fact that Alice and Bob alternate measurements, so Alice’s action will always depend on an even number of previous results while’s Bob’s will always depend on an odd number. As usual, we have the normalization

\[
\sum_i (X(m_0)_i^T)^\dagger X(m_0)_i^T = I_{d_A(m_0)} = \sum_i X(m_0)_i X(m_0)_i^\dagger
\]

\[
\sum_i E(m_1)_i^\dagger E(m_1)_i = I_{d_B(m_1)}
\]

where \( d_A(m_0) \) and \( d_B(m_1) \) are sufficiently large dimensions to include any ancilla spaces.

Alice and Bob start with the state \( |\Psi_i\rangle = (I \otimes B_i) |ME_n\rangle \) and then apply the POVM above, getting results \( m = (j_1, j_2, \ldots, j_{r-1}) \) and \( j_r \), for \( r \) an even number. Then, using Lemma 13, their state now looks like

\[
(X_m^T \otimes \mathcal{E}_{m,j_r})(I \otimes B_i) |ME_n\rangle = I \otimes (\mathcal{E}_{m,j_r} B_i X_m) |ME_n\rangle
\]

This state is not normalized—it’s magnitude indicates the probability of this outcome. Without loss of generality, we assume that the final measurement
identifies the best guess of the value of $V$. This gives us a more formal definition of our optimal measurement, where we sum over all outcomes with the final output equal to the correct state identity:

$$P(\{\Psi_i\}, \{p_i\}) := \sup_{\mathcal{X}, \mathcal{E}} P(Z = V)$$  \hspace{1cm} (68)

$$P(Z = V) = \sum_i P(Z = V = i)$$  \hspace{1cm} (69)

$$= \sum_{i,m} p_i \langle \Psi_i | (X_m^T \otimes \mathcal{E}_{m,i} | \Psi_i \rangle$$  \hspace{1cm} (70)

$$= \sum_{i,m} p_i \langle ME_n | (I \otimes \mathcal{X}_m^B_i \mathcal{E}_{m,i}^B | ME_n \rangle$$  \hspace{1cm} (71)

$$= \frac{1}{n} \sum_{i,m} p_i \text{Tr}(X_m^B_i \mathcal{E}_{m,i}^B \mathcal{X}_m)$$  \hspace{1cm} (72)

The measurements might make use of ancilla systems, so we write $P_A$ and $P_B$ as the projections back onto our original Alice and Bob spaces; since each $B_i$ maps Alice’s space to Bob’s, we see that $P_B B_i = B_i P_A = B_i$. Recall also that $\text{Tr} B_i^B B_i = n$ by assumption.

We may now turn to the lemma, which assumes that $|\Psi_i\rangle = (I \otimes U_i B) |ME_n\rangle$ with $U_i$ unitary and $B$ fixed. Suppose Alice and Bob make $r$ measurements with the POVMs described in (64). We assume that $r$ is even so Bob measures last--we can always append a trivial measurement to make this so. Suppose that the first $r - 2$ measurement outcomes are contained in the vector $m = (j_1, j_2, \ldots, j_{r-2})$. For simplicity we write $j_{r-1}$ as $j$ and assume that $Z = V$ if and only if Bob’s final measurement $j_r = i$. Plugging this into (72) and setting $p_i = \frac{1}{k}$ yields

$$P(Z = V) = \frac{1}{kn} \sum_{m,j,i} \text{Tr}(X_m^B_i \mathcal{E}_{m,i}^B \mathcal{X}_m)$$  \hspace{1cm} (73)

$$= \frac{1}{kn} \sum_{m,j,i} \text{Tr}(X_m^B_i P_B \mathcal{E}_{m,i}^B \mathcal{X}_m)$$  \hspace{1cm} (74)

$$\leq \frac{1}{kn} \sum_{m,j,i} (\text{Tr} P_B \mathcal{E}_{m,i}^B \mathcal{X}_m) (\text{Tr} B_i \mathcal{X}_m)$$  \hspace{1cm} (75)

$$= \frac{1}{kn} \sum_{m,j,i} (\text{Tr} P_B \mathcal{E}_{m,i}^B \mathcal{X}_m) (\text{Tr} B^B \mathcal{X}_m)$$  \hspace{1cm} (76)

In (75), we use the fact that for matrices $A, B \geq 0$, $\text{Tr} AB \leq (\text{Tr} A)(\text{Tr} B)$, and in
we use the assumption of the lemma that $B_i = U_i B$. The key observation now is that there is no $i$ in the second term of (76); rewriting the first term as in (64) shows that summing the first term over $i$ yields the identity matrix on the inside, allowing us to drop two subscripts, not just one:

$$\text{Tr} \left( \sum_i P_B \mathcal{E}_{m,j,i}^\dagger \mathcal{E}_{m,j,i} \right) = \text{Tr} \left( \sum_i P_B \mathcal{E}_{m,j}^\dagger E(m,j)_i^\dagger E(m,j)_i \mathcal{E}_m \right) = \text{Tr}(P_B \mathcal{E}_{m}^\dagger \mathcal{E}_m)$$

(77)

(78)

This corresponds to the fact that Alice does nothing during Bob’s measurement phase. We now have

$$P(Z = V) \leq \frac{1}{kn} \sum_{m,j} \text{Tr}(P_B \mathcal{E}_{m}^\dagger \mathcal{E}_m)(\text{Tr}(B^\dagger B \mathcal{X}_{m,j}^\dagger \mathcal{X}_{m,j}^\dagger))$$

(79)

Now, there is no $j$ in the first term, only in the second, so we can likewise sum to get the identity on the inner term. Alternating in this way, we can count back through the measurements until they all sum to the identity and we are left with

$$P(Z = V) \leq \frac{1}{kn} \text{Tr}(P_B) \text{Tr}(B^\dagger B) = \frac{1}{kn} (n)(n) = \frac{n}{k}$$

(80)

This shows that even if Alice and Bob add ancilla systems to do their measurements, the relevant bound comes from the dimension of Bob’s system. This proves the lemma.

4.4 Proof of Proposition 9

In equation (75), we insert the projection onto Alice’s space $P_A$ and use Hölder’s Inequality to note that

$$\text{Tr}B_i \mathcal{X}_{m,j}^\dagger \mathcal{X}_{m,j}^\dagger B_i \leq \|B_i^\dagger B_i\|_\infty \text{Tr}P_A \mathcal{X}_{m,j}^\dagger \mathcal{X}_{m,j}^\dagger P_A \leq n\lambda_M \text{Tr}P_A \mathcal{X}_{m,j}^\dagger \mathcal{X}_{m,j}^\dagger P_A$$

(81)

(82)

(83)

since $\|B_i^\dagger B_i\|_\infty \leq \max_i \|B_i^\dagger B_i\|_\infty = n\lambda_M$. Aside from the new factor of $n\lambda_M$, the rest of the calculation from Lemma 8 remains unchanged, inserting $P_A$ for $B$ so that (80) becomes

$$P(Z = V) \leq \frac{1}{kn} (n\lambda_M) \text{Tr}(P_B) \text{Tr}(P_A) = \frac{\lambda_M}{k} (n)(m) = \frac{\lambda_M mn}{k}$$

(84)
4.5 Proof of Proposition 12

The lower bound comes from the idea of tossing out all but two of the vectors and distinguishing them perfectly. At worst, this process gives you $\frac{2}{k} \log 2$ bits of information. The upper bound arises in the case of $k$ states to which Lemma 8 applies. The joint probability distribution on $(V, Y, Z)$ must have two properties. First, that the marginal distribution on $V$ is uniform, since the states are equally likely. Second, by relabeling in Lemma 8 we see that for any permutation $\sigma \in S_k, P(Z = \sigma(V)) \leq \frac{n}{k}$. The set of distributions with these properties is a convex set on which the mutual information is convex. The extreme points of this set are distributions for which $Z$ takes on only $n$ values and $Y$ is a function of $Z$. Hence the maximum happens at an extreme point and

$$I(V; YZ) \leq H(YZ) = H(Z) \leq \log n$$

This implies that the maximum mutual information in this case is $\log n$. Making $n$ as small as possible, we see that

$$g(k, n) \leq \log \lceil \sqrt{k} \rceil$$

5 Conclusion

In summary, we have demonstrated that several classes of maximally entangled states that can be distinguished using LOCC. By examining the measurement process itself, we have explored bounds on both the success probability and the mutual information and shown that the well-understood $C^2 \otimes C^2$ Bell basis provides the worst case of 3 or 4 vectors with respect to either of these measures. In the process, we have identified some sets of states that cannot be perfectly distinguished. It is hoped that through better understanding best and worst cases of the distinguishing problem, we can further our understanding of the interplay between locality and entanglement.

Acknowledgements: Many thanks to Chris King for suggesting this line of inquiry, and to him and Beth Ruskai for helpful discussions. The author was supported in part by National Science Foundation Grant DMS-0400426.

References

[1] P. Badziag, M. Horodecki, A. Sen(De), U. Sen, Phys. Rev. Lett. 91, 117901 (2003), quant-ph/0304040
[2] C. Bennett, D. DiVincenzo, C. Fuchs, T. Mor, E. Rains, P. Shor, J. Smolin, W. Wootters, Quantum Nonlocality without Entanglement, *Phys. Rev. A* **59**, 1070 (1999), quant-ph/9804053.

[3] P.-X. Chen, C.-Z. Li, Orthogonality and Distinguishability: Criterion for Local Distinguishability of Arbitrary Orthogonal States, *Phys. Rev. A* **68**, 062107 (2003), quant-ph/0209048.

[4] S. De Rinaldis, Distinguishability of complete and unextendible product bases, quant-ph/0304027.

[5] H. Fan, Distinguishability and Indistinguishability by LOCC, *Phys. Rev. Lett.* **92**, 177905 (2004), quant-ph/0311026.

[6] S. Ghosh, G. Kar, A. Roy, D. Sarkar, Distinguishability of maximally entangled states, quant-ph/0205105.

[7] S. Ghosh, G. Kar, A. Roy, A. Sen(De), U. Sen, Distinguishability of the Bell States, *Phys. Rev. Lett.* **87**, 277902 (2001), quant-ph/0106148.

[8] M. Gregoratti, R. F. Werner, On quantum error correction by classical feedback in discrete time, quant-ph/0403092.

[9] P. Hayden, C. King, Correcting quantum channels by measuring the environment, quant-ph/0409026.

[10] M. Horodecki, A. Sen(De), U. Sen, K. Horodecki, Local indistinguishability: more nonlocality with less entanglement, *Phys. Rev. Lett.* **90**, 047902 (2003), quant-ph/0301106.

[11] A. Pittenger, M. Rubin, Mutually Unbiased Bases, Generalized Spin Matrices and Separability, quant-ph/0308142.

[12] E. Rains, Entanglement Purification via separable superoperators, quant-ph/9707002.

[13] B. Terhal, D. DiVincenzo, D. Leung, Hiding Bits in Bell States, *Phys. Rev. Lett.* **86**, 5807 (2001), quant-ph/0011042.

[14] J. Walgate, L. Hardy, Nonlocality, Asymmetry, and Distinguishing Bipartite States, *Phys. Rev. Lett.* **89**, 147901 (2002), quant-ph/0202034.
[15] J. Walgate, A. Short, L. Hardy, V. Vedral; Local Distinguishability of Multipartite Orthogonal Quantum States, *Phys. Rev. Lett.* **85**, 4972 (2000), quant-ph/0007098

[16] W. Wootters, Picturing qubits in phase space, quant-ph/0306135