ABSTRACT
Recent calculations of the hydromagnetic deformation of a stratified, non-barotropic neutron star are generalized to describe objects with superconducting interiors, whose magnetic permeability $\mu$ is much smaller than the vacuum value $\mu_0$. It is found that the star remains oblate if the poloidal magnetic field energy is $\gtrsim 40$ per cent of the total magnetic field energy, that the toroidal field is confined to a torus which shrinks as $\mu$ decreases and that the deformation is much larger (by a factor of $\approx \mu_0/\mu$) than that in a non-superconducting object. The results are applied to the latest direct and indirect upper limits on the gravitational-wave emission from the Laser Interferometer Gravitational Wave Observatory (LIGO) and radio pulse timing (spin-down) observations of 81 millisecond pulsars, to show how one can use these observations to infer the internal field strength. It is found that the indirect spin-down limits already imply astrophysically interesting constraints on the poloidal–toroidal field ratio and the diamagnetic shielding factor (by which accretion reduces the observable external magnetic field e.g. by burial). These constraints will improve following gravitational-wave detections, with implications for accretion-driven magnetic field evolution in recycled pulsars and the hydromagnetic stability of these objects’ interiors.

Key words: gravitational waves – MHD – stars: interiors – stars: magnetic field – stars: neutron.

1 INTRODUCTION
The external magnetic field of a neutron star is (relatively) easily inferred from its spin-down rate, but its internal magnetic field is not directly observable. The main clue suggesting the existence of strong internal neutron star fields comes from the 1998 August 27 giant flare from the soft gamma-ray repeater (SGR) 1900+14 (Feroci et al. 1999; Hurley et al. 1999; Mazets et al. 1999). The giant flare, which released $\approx 10^{37} \text{ J}$ of energy as X-rays, was accompanied by a 2.3-fold increase in the spin-down rate (Mazets et al. 1999; Woods et al. 1999; Thompson et al. 2000). To explain this, Ioka (2001) proposed that the flare and the enhanced spin-down were caused by a global reconfiguration of the internal magnetic field of $\approx 10^{13} \text{ T}$, well above the external dipole field of 6.4 $\times 10^{10} \text{ T}$.

Corsi & Owen (2011) generalized the Ioka (2001) calculation (by allowing the toroidal field strength to change, as well as the moment of inertia) and concluded that an internal field strength of $\approx 10^{12} \text{ T}$ gave rise to the 1998 August 27 event, lower than the first estimated by Ioka (2001), but still significantly higher than the observed external field.

Stellar ellipticity can also be used to constrain the strength of a star’s internal field (Cutler 2002; Dall’Osso, Shore & Stella 2009; Abbott et al. 2010; Pitkin 2011). It is well known that a strong magnetic field can deform a star (Chandrasekhar & Fermi 1953; Ferraro 1954; Goosens 1972; Katz 1989; Payne & Melatos 2004; Haskell et al. 2008; Mastrano et al. 2011). The ellipticity $\epsilon$ is roughly proportional to the magnetic energy (Cutler 2002; Haskell et al. 2008; Dall’Osso et al. 2009). Neutron stars, with their intense magnetic fields, possess significant ellipticities, making them good candidates for gravitational wave sources (Bonazzola & Gourgoulhon 1996; Melatos & Payne 2004; Stella et al. 2005; Haskell et al. 2008; Dall’Osso et al. 2009). Recent data from the fifth Laser Interferometer Gravitational-Wave Observatory (LIGO) science run set an upper limit of $\epsilon \lesssim 1.4 \times 10^{-4}$ on the Crab Pulsar (Abbott et al. 2008, 2010), translating into an internal magnetic field of $\lesssim 10^{12} \text{ T}$ under standard assumptions. LIGO non-detections of the central compact object (CCO) in the supernova remnant Cassiopeia A (Cas A) have constrained its ellipticity as well. The Cas A CCO has not been detected electromagnetically, making it impossible to infer its external magnetic field from the spin-down rate. However, Wette et al. (2008) and Wette (private communication) constrained its ellipticity as a function of gravitational wave frequency (e.g. $\epsilon \lesssim 3.6 \times 10^{-4}$ for 100 Hz, $\epsilon \lesssim 0.6 \times 10^{-4}$ for 200 Hz and $\epsilon \lesssim 0.38 \times 10^{-4}$ for 300 Hz), implying an internal magnetic field...
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sentative of the field structures observed in numerical simulations (Braithwaite & Nordlund 2006; Braithwaite & Spruit 2006; Braithwaite 2009) and exhibit the following properties:

(i) the field is axially symmetric around the z-axis;
(ii) the poloidal part is continuous with a dipole field outside the star (so there are no surface currents) and vanishes at the circle \((r, \theta) = (0.78R, \pi/2)\) (where \(r\) and \(\theta\) are the radial and polar coordinates and \(R\) is the stellar radius; this locus is called the neutral circle);
(iii) the toroidal component is confined to the region of closed poloidal field lines around the neutral circle;
(iv) the current density remains finite and continuous everywhere in the star.

We write the magnetic field in the form pioneered by Chandrasekhar (1956),

\[
B = B_0[\eta_0 \nabla r(r, \theta) \times \nabla \phi + \eta_1 \beta(\alpha) \nabla \phi],
\]

(1)

where \(\eta_0\) and \(\eta_1\) are dimensionless parameters that define the relative strengths of the poloidal and toroidal components, respectively. The function \(\beta(\alpha)\) takes the form \(\beta(\alpha) = (\alpha - 1)^2\) for \(\alpha \geq 1\) and \(\beta(\alpha) = 0\) elsewhere, confining the toroidal field to the region where \(\alpha\) exceeds unity, the value taken by \(\alpha\) at \((r, \theta) = (1, \pi/2);\) the current density goes continuously to zero at this boundary. The flux function \(\alpha(r, \theta)\) is taken to be \(f(r) \sin^2 \theta\). Note that this particular form of \(\alpha(r, \theta)\) is only applicable when we try to match our field to an external dipole; other multipoles match different \(\theta\)-dependences. The radial dependence of \(\alpha\) is given by

\[
f(r) = \frac{35}{8} \left( r^2 - \frac{6\alpha^4}{5} + \frac{3\alpha^6}{7} \right).
\]

(2)

The function \(f(r)\) is postulated to be of this form to ensure that the field described by equations (1) and (2) is continuous with a dipole field outside the star, that there are no surface currents and that the current density is finite at the origin (for a more thorough derivation, see Akgün et al., in preparation).

A schematic diagram of the field is shown in Fig. 1.\(^2\) It represents just one possibility amongst many, chosen for simplicity and mathematical convenience; possible configurations involving higher multipoles are not ruled out by observations (Arons 1993; Thompson, Lyutikov & Kulkarni 2002).

We now calculate the small changes \(\delta p\) and \(\delta \rho\) to the pressure and density of a star in hydrostatic equilibrium caused by this field, which satisfy the force balance equation (Easson & Pethick 1977; Akgün & Wasserman 2008)

\[
(\nabla \times \mathbf{H}) \times \mathbf{B} = \nabla \delta p + \delta \rho \nabla \Phi.
\]

(3)

In equation (3), \(\mathbf{H} = \mathbf{B}/\mu\) is the magnetic intensity, \(\Phi\) is the gravitational potential (the Cowling approximation has been taken, with \(\Delta \Phi = 0\)), and \(\mu\) is the magnetic permeability. This allows for a superconducting interior (where \(\mu\) is smaller than the vacuum permeability \(\mu_0\)), as analysed in Section 3. The assumption that the changes to density and pressure are small enough that they can be treated as perturbations on the steady state is justified a posteriori. Ellipticity is then calculated from the perturbed density. Again, the non-barotropic assumption is essential. If barotropy is assumed instead, pressure must be a function of density only, to all orders; a restriction is then imposed on the magnetic field configuration,

\(^2\)The configuration of this configuration is examined by Akgün et al. (in preparation).

2 FIELD STRENGTH VERSUS ELLIPTICITY FOR NON-BAROTROPIC STARS

Mastrano et al. (2011) considered a general class of poloidal–toroidal magnetic field configurations, which are broadly repre-
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\[ \rho \times \left( \frac{0}{z} \right) = 10^{5}, \]  

\[ \rho \times \left( \frac{0}{z} \right) = 421, \]  

\[ \text{and} \]  

\[ \text{lies in a polytrope.} \]  

\[ \approx \]  

\[ \geq 760–767 \]  

\[ \text{takes the form} \]  

\[ \mu \times (M \text{m}) = 5 \times 10^{4} \]  

\[ \text{The mass} \]  

\[ \times \left( \frac{0}{z} \right) = \text{over all space. We have} \]  

\[ \text{and} \]  

\[ \delta p \text{ and gravitational-wave upper limits on} \]  

\[ \text{for 81 known millisecond radio pulsars as a function of} \]  

\[ \text{is the mass} \]  

\[ \times \left( \frac{0}{z} \right) = B_{r}^{15} \text{(4)} \]  

\[ \text{times the poloidal field} \]  

\[ \text{in this paper. The surface of the star is represented by the dashed circle. The} \]  

\[ \text{toroidal magnetic component is confined to the shaded region and fills a} \]  

\[ \text{torus around the} z \text{-axis. The poloidal field lines vanish at the neutral circle, i.e. at the dots in the shaded regions.} \]  

\[ \rho \times (\frac{0}{z}) = 10^{5} \]  

\[ \text{because the poloidal and toroidal components must be related in} \]  

\[ \text{such a way that} \]  

\[ \delta p \text{ is purely a function of} \]  

\[ \text{e.g. Haskell et al.} \]  

\[ \text{is the mass} \]  

\[ \times \left( \frac{0}{z} \right) = \text{10} \times 10^{3} \text{ to the total magnetic field} \]  

\[ \text{the resulting ellipticity is within 5 per cent of that obtained using} \]  

\[ \text{the more common} \]  

\[ \text{parabolic density profile} \]

\[ \rho = \rho_{c}(1 - r^{2}), \]  

\[ \text{where} \]  

\[ \rho_{c} = 15M_{s} / (8\pi R_{s}^{3}) \]  

\[ \text{is the density at the core,} \]  

\[ \text{mass of the star and} R_{s} \]  

\[ \text{is the radius of the star. While this is one particular,} \]  

\[ \text{simple choice of density profile, chosen to render the calculations} \]

\[ \text{tractable, we showed in section 3.3 of Mastrano et al. (2011) that} \]

\[ \text{the resulting ellipticity is within 5 per cent of that obtained using} \]

\[ \text{the more common} \]  

\[ n \text{ polytrope.} \]  

\[ \text{We define the parameter} \]  

\[ \Lambda \]  

\[ \text{as the ratio of the poloidal field energy to the total field energy. The} \]

\[ \text{energies are obtained by integrating the squares of the poloidal and total magnetic} \]

\[ \text{intensities over the star; note that this definition is slightly different from that} \]

\[ \text{given by Mastrano et al. (2011), who integrated} \]  

\[ \text{H}^{2} \text{ over all space. We have} \]  

\[ \Lambda = 1 \text{ for a purely poloidal field configuration and} \]  

\[ \Lambda = 0 \text{ for a purely} \]  

\[ \text{toroidal configuration. In terms of} \]  

\[ \Lambda \]  

\[ \text{for a} \text{non-superconducting star} \]  

\[ \text{with} \]  

\[ \mu = \mu_{0}, \]  

\[ \text{the ellipticity} \]  

\[ \epsilon \]  

\[ \text{takes the form} \]

\[ \epsilon = 5.63 \times 10^{-6} \left( \frac{B_{s}}{5 \times 10^{10} \text{T}} \right)^{2} \left( \frac{M_{s}}{1.4M_{\odot}} \right)^{-2} \]

\[ \times \left( \frac{R_{s}}{10^{3} \text{m}} \right)^{4} \left( 1 - \frac{0.351}{\Lambda} \right). \]  

\[ \text{where} \]  

\[ B_{s} \]  

\[ \text{is the surface magnetic field strength at the equator and} \]  

\[ M_{\odot} \]  

\[ \text{is the solar mass. As expected, one finds} \]  

\[ \epsilon \propto B_{s}^{2}. \]  

\[ \text{The mass quadrupole moment vanishes for} \]  

\[ \Lambda = 0.351, \]  

\[ \text{that is, when the} \]  

\[ \text{poloidal field energy is 35.1 per cent of the total magnetic field} \]  

\[ \text{energy. Note that both our model and the generalized ‘twisted torus’} \]  

\[ \text{model of Ciolfi et al. (2010) predict} \]  

\[ \epsilon \sim 4 \times 10^{-6} \text{ for a canonical} \]  

\[ \text{‘magnetar-like’ neutron star with a purely poloidal field of strength} \]

\[ B_{s} = 5 \times 10^{10} \text{ T}. \]  

\section{3 Millisecond Pulsars}

\[ \text{Armed with equation (5), we can use spin-down measurements of} \]

\[ B_{s} \]  

\[ \text{and gravitational-wave upper limits on} \]  

\[ \epsilon \]  

\[ \text{to constrain} \]  

\[ \Lambda \]  

\[ \text{and hence the internal magnetic field for various classes of the object.} \]

\[ \text{In Fig. 2, we plot the most up-to-date LIGO (triangles) and spin-down (dots) gravitational-wave upper limits on the absolute value of} \]

\[ \epsilon \]  

\[ \text{for 81 known millisecond radio pulsars as a function of} \]  

\[ B_{s} \]  

\[ \text{(Abbott et al. 2010). As expected, the indirect spin-down limits are} \]

\[ \text{uniformly tighter than the direct LIGO limits for now, although this} \]

\[ \text{will change in the future; already, LIGO has beaten the spin-down} \]

\[ \text{limits for a handful of non-millisecond pulsars like the Crab} \]  

\[ \text{(Abbott et al. 2008) and PSR J0537–6910 (Abbott et al. 2010). However,} \]

\[ \text{the LIGO limits are included in Fig. 2 to show, in the discussion} \]

\[ \text{below, what constraints can be extracted from gravitational-wave} \]

\[ \text{detections when a spin-down measurement is not available, e.g. for a} \]  

\[ \text{CCO-like Cas A. Fig. 2 also displays a selection of theoretical curves} \]

\[ \text{close to the data without appealing to extreme situations like the one in the previous paragraph. First, all objects in} \]  

\[ \text{are intentionally selected to be recycled. It is likely that the} \]  

\[ \text{actual internal magnetic field is much stronger than the measured} \]

\[ \text{external dipole field in a recycled pulsar, because surface currents} \]

\[ \text{are diamagnetically shielded (Bisnovatyi-Kogan 1974), buried by} \]  

\[ \text{polar accretion (Payne & Melatos 2004; Melatos & Payne 2005) or} \]

\[ \text{resistively dissipated (Romani 1990). Secondly, it is also likely} \]

\[ \text{that the core of the star is a superconductor (Jones 1975; Easson & Pethick} \]

\[ \text{1977; Cutler 2002; Wasserman 2003; Akgün & Wasserman 2008). We now discuss these scenarios in turn.} \]

\subsection*{3.1 Core superconductivity}

\[ \text{If the stellar core} \]  

\[ (\rho_{c} \geq 2.8 \times 10^{17} \text{ kg m}^{-3}; \text{Levenfish, Shibanov} \]  

\[ \text{and Yakovlev 1999) is made of a type II superconductor (Baym,} \]  

\[ \text{Pethick & Pines 1969; Baym \& Pethick 1975; Easson \& Pethick} \]

\[ \text{1977; Elgaroy et al. 1996; Jones 2006; Baldo \& Schulze 2007), the} \]

\[ \text{magnetic permeability} \]  

\[ \mu \]  

\[ \text{drops significantly (Easson \& Pethick 1977; Akgün \& Wasserman} \]

\[ \text{2008), and} \]  

\[ \delta p \]  

\[ \text{and} \]  

\[ \delta p \]  

\[ \text{are magnified by the factor} \]  

\[ \mu_{0}\beta \mu. \]  

\[ \text{Akgün \& Wasserman (2008) constructed} \]  

\[ \text{difficulty of the model.} \]

\[ \text{Difficulty of the model.} \]
The functional dependence of $\xi_R$ in a superconductor. The deformation caused by this field $10^5 \mu / \Lambda_1 \geq H = (10^5 \times r_{\text{mer}} + 10^3 \mu) / \Lambda_1$, respectively. Specifically, we have $\Lambda = 10^{-3}$, $\xi = 10^{-4}$ (the upper bound of the top band, red), $\Lambda = 10^{-3}$, $\xi = 10^{-2}$ (the upper bound of the lower band, blue), and $\Lambda = 1$, $\xi = 10^{-2}$ (the lower band, blue). Note that all the $\Lambda = 1$ curves correspond to oblate stars and all the $\Lambda = 10^{-3}$ curves correspond to prolate stars.

Figure 2. Direct LIGO upper limits (triangles) and indirect radio timing spin-down limits (dots) on the absolute ellipticities $|\epsilon|$ of 81 known millisecond pulsars as a function of their measured surface magnetic field strengths $B_s$ (Abbott et al. 2010). Also shown are theoretical $|\epsilon(B_s)|$ curves for the following structural and evolutionary scenarios: non-superconducting ($\mu = \mu_0$) (the dashed black curves) and superconducting ($\mu = 10^{-3} \mu_0$) with diamagnetic shielding (the solid curves, in red and blue). The non-superconducting curves are for $\Lambda = 10^{-3}$ (realistic; the lower dashed black curve) and $\Lambda = 10^{-10}$ (unrealistic; the upper dashed black curve). The red (upper) and blue (lower) bands correspond to diamagnetic shielding factors $\xi = B_s/\text{observed}/B_s/\text{actual} = 10^{-4}$ and $10^{-2}$, respectively.

First, to ensure that the stellar field still possesses the properties outlined in Section 2, the function $f(r)$ in equation (2) must be modified into

$$f(r) = \frac{1}{8}[(26 + 9 \mu_s)r^2 - (28 + 14 \mu_s)r^4 + (10 + 5 \mu_s)r^6].$$

with $\mu_s = \mu/\mu_0$ [note that this reduces to equation (2) for $\mu_s = 1$]. The magnetic field is modified in two ways: (1) the field lines become more radial just inside the surface and (2) the region which contains the toroidal field squeezes closer to the surface and shrinks. In Fig. 3, we sketch the field lines for $\mu_s = 10^{-1}$ (Fig. 3a) and $10^{-3}$ (Fig. 3c). Recall that the toroidal field is confined to the region where $f(r) \sin^2 \theta \geq 1$ (this is also the region described by the outermost poloidal field line which closes inside the star). As is evident from equation (6), this region shrinks as $\mu_s$ decreases (but does not vanish for $\mu_s = 0$, the case of perfect superconductivity, tending instead to $\sim 7.5 \times 10^{-3}$ of the total volume in the limit $\mu_s \to 0$). We magnify this region in Fig. 3(b) (for $\mu_s = 10^{-1}$) and Fig. 3(d) (for $\mu_s = 10^{-3}$). The volume of the torus is 0.16 (0.04) of the original $\mu_s = 1$ case for $\mu_s = 10^{-1}$ ($\mu_s = 10^{-3}$).

3 Specifically, we must ensure that the normal component of the internal field and the tangential component of the magnetic intensity are continuous with the external dipole field at $r = 1$.

The magnetic intensity $\mathbf{H} = \mathbf{B}/\mu$ in equation (3) is magnified by $\mu/\mu_0$ in a superconductor. The deformation caused by this field is then calculated by the method given by Mastrano et al. (2011). We are unable to derive a simple analytic formula describing the dependence of $\epsilon$ on $\mu_s = \mu/\mu_0$ and $\Lambda$ simultaneously, but we find that the general form of $\epsilon(\Lambda)$ for a given $\mu_s$ is still similar to equation (5), namely

$$\epsilon = c_1 \left( \frac{B_s}{5 \times 10^{10} \text{T}} \right)^2 \left( \frac{M_*}{1.4 M_\odot} \right)^{-2} \left( \frac{R_*}{10^5 \text{m}} \right)^4 \left( 1 - \frac{c_2}{\Lambda} \right).$$

The dimensionless constants $c_{1,2}$ are quoted in Table 1 for $\mu_s = 0.5, 0.1, 10^{-2}, 10^{-3}$ and $10^{-4}$. For all $\mu_s$, the star is oblate for $\Lambda \gtrsim 0.4$ and prolate for $\Lambda \lesssim 0.4$. The functional dependence of $\epsilon$ on $\Lambda$ stays roughly the same as $\mu_s$ changes. The smaller magnetic permeability of the superconducting stellar matter enhances the density perturbation by a factor of $\sim \mu_s^{-1}$, as is evident from the force balance equation (equation 3), and this is embodied in the approximate scaling $c_1 \propto \mu_s^{-1}$ for small $\mu_s$.

We can now ask how much closer the theoretical curves approach the data when superconductivity is included. Looking at Fig. 2 for example, the curve for (say) $\mu_s = 10^{-3}$ and $\Lambda = 10^{-3}$ (not drawn) is higher than the lower black dashed curve by three orders of magnitude, which is still below observational limits. We conclude that superconducting interiors are easily compatible with current observational upper limits, if the external magnetic field is not reduced by accretion in any way.

We caution that, in this first pass, we assume $\mu_s \neq 1$ throughout the star, instead of only in some region. This assumption is physically implausible and is only taken to simplify our (illustrative) calculation. A more thorough calculation where $\mu_s$ is allowed to vary inside the star is needed before definite conclusions can be
Figure 3. Poloidal field lines inside and outside a superconducting star with \( \mu_r = 10^{-1} \) (panels a and b) and \( \mu_r = 10^{-3} \) (panels c and d). The surface of the star is represented by the dashed hemisphere. The right-hand panels (b) and (d) are zoom-in of the boxed region around the neutral circle in the left-hand panels (a) and (c), respectively. Note that all field lines are, in fact, closed, since the magnetic field is divergence-free.

Table 1. Dimensionless constants \( c_1, c_2 \) in equation (7), for some selected values of \( \mu_i = \mu / \mu_0 \).

| \( \mu_i \)    | \( c_1 \)        | \( c_2 \)        |
|---------------|------------------|------------------|
| \( 10^0 \)    | \( 5.62 \times 10^{-6} \) | 0.35             |
| \( 5 \times 10^{-1} \) | \( 9.97 \times 10^{-6} \) | 0.42             |
| \( 10^{-1} \) | \( 3.92 \times 10^{-5} \) | 0.40             |
| \( 10^{-2} \) | \( 3.84 \times 10^{-4} \) | 0.42             |
| \( 10^{-3} \) | \( 3.83 \times 10^{-3} \) | 0.43             |
| \( 10^{-4} \) | \( 3.84 \times 10^{-2} \) | 0.43             |

drawn regarding the effects of core superconductivity on ellipticity. However, we conjecture that the calculation presented in this section sets the upper limit on the effects of a superconducting interior on stellar deformation: if superconductivity is limited to a smaller region in the star, the changes to \( \epsilon \) (relative to \( \mu = \mu_0 \)) will be less than predicted by equation (7) and Table 1.

3.2 Accretion

Recycled pulsars are selected as the subjects of this study for two reasons: (i) their spin-down rates are lower than those of other objects, yielding more stringent indirect gravitational-wave limits; and (ii) their internal magnetic fields may be much greater than their surface fields due to diamagnetic screening or burial, yielding larger hydromagnetic deformations than one might otherwise expect. We now examine point (ii). Let us ask what happens if the actual magnetic field strength just below the surface takes its pre-accretion value (e.g. before diamagnetic screening or burial) \( B_{s,\text{actual}} = B_{s,\text{observed}} / \xi \), where \( \xi \) is some dimensionless ‘shielding factor’. We note in passing that this scenario is more realistic than those considered in Section 3.1: it is unlikely that the core poloidal field is reduced to \( \sim 10^4 \) T in a recycled pulsar like the surface field, given the high electrical conductivity expected in the core (Goldreich & Reisenegger 1992), except in the special situation where the source currents reside exclusively in the crust.

We recalculate \( \epsilon \) using equation (7), substituting \( B_{s,\text{actual}} \) for \( B_s \) and keeping \( \mu_r = 10^{-3} \). We plot the results for \( \xi = 10^{-4} \) as the red curves for \( \Lambda = 1 \) (the bottom solid red curve) and \( \Lambda = 10^{-3} \) (the top solid red curve) in Fig. 2. We also plot \( \epsilon \) for the case of \( \xi = 10^{-2} \) as the blue curves for \( \Lambda = 1 \) (the bottom solid blue curve) and \( \Lambda = 10^{-3} \) (the top solid blue curve). For clarity, the case \( \mu = \mu_0 \) is not presented; it lies three decades lower than the curves with \( \mu_r = 10^{-3} \). Without a better knowledge of the screening/burial process than is currently at hand, it is best simply to bracket the plausible
range $10^{-4} \leq \xi \leq 10^{-2}$ inferred from population synthesis studies (Kiel et al. 2008).

Using the potentially stronger shielded pre-accretion fields, the $\epsilon(B_{\text{r}})$ curves in Fig. 2 come close to the spin-down limits on $\epsilon$. Now we can see that, aside from perhaps one object, namely PSR J1823–3021A, all the pulsars plotted must have some internal toroidal field component, i.e. $\Lambda < 1$, if one has $\xi \leq 10^{-2}$. From the red curves and the dots in Fig. 2, many objects seem to have $\Lambda < 0.1$, even $\Lambda < 10^{-3}$ in the notable case of PSR J1910–5959C. On the other hand, the heavily shielded case $\Lambda = 10^{-3}, \xi = 10^{-4}$ (for example) seems to be ruled out by observations (the top solid red curve in Fig. 2). In general, $\Lambda$ cannot be too small, for several reasons: it is ruled out by observations, the virial limit sets an absolute upper bound on the internal field strength and small $\Lambda$ leads to an unstable field configuration (Braithwaite 2009). In contrast, large $\Lambda$ is not ruled out by observations; the only upper bound ($\Lambda = 0.8$) is set by the stability analysis of Braithwaite (2009). Fig. 2 also tells us that $\xi$ cannot be too small, otherwise the solid red curves exceed the observational upper limits (for $\xi \leq 10^{-4}$).

The data in Fig. 2, together with the stability-based limits on $\Lambda$ set by Braithwaite (2009), allow us to use gravitational wave observations and measurements of surface fields to infer bounds on both $\Lambda$ and the shielded field $B_{\text{r, actual}}$, in principle. We show several possibilities in Fig. 4, where we draw curves of constant $\epsilon = 1.184 \times 10^{-9}$ (corresponding to the current lowest spin-down upper limit; see Fig. 2) for a given value of $B_{\text{r, observed}}$ (solid: $10^4$ T, dashed: $10^5$ T), for $\mu_{\text{T}} = 1$ (blue curves) and $\mu_{\text{T}} = 10^{-3}$ (red curves). To be consistent with the spin-down measurement, an object must lie above the curve relevant to the scenario being considered. The shaded region bordered by thick black dashed lines indicates the theoretical limits on $\Lambda$ from stability; $10^{-3} < \Lambda < 0.8$; Braithwaite 2009) and $\xi$ (from population synthesis; $10^{-4} < \xi < 10^{-2}$; Kiel et al. 2008). In other words, Fig. 4 allows us to determine the allowed combinations of $\Lambda$ and $\xi$ for a given observed surface field and $\epsilon$ limit (here $\epsilon = 1.184 \times 10^{-9}$). For example, a detection from a pulsar with a surface field of $10^4$ T indicates $10^{-3} \leq \Lambda \leq 7 \times 10^{-3}$ and $10^{-4} \leq \xi \leq 2.5 \times 10^{-4}$, if we assume that the star is non-superconducting (the blue dashed curve), or $10^{-3} \leq \Lambda \leq 4 \times 10^{-1}$ and $10^{-4} \leq \xi \leq 8 \times 10^{-3}$ (prolate star) or $4.5 \times 10^{-1} \leq \Lambda \leq 0.8$ and $10^{-4} \leq \xi \leq 2.4 \times 10^{-4}$ (oblate star) if the star is superconducting with $\mu_{\text{T}} = 10^{-3}$ (the red dashed curve). From equation (7), we see that the oblate cases (with $c_2 \leq \Lambda \leq 1$) are weakly deformed relative to the prolate cases ($10^{-3} \leq \Lambda \leq c_2$). Hence, for oblate stars to be readily detectable, we need small $\xi$ and $\mu_{\text{T}}$ to boost $\epsilon$ (red curves).

4 DISCUSSION

This paper makes a first attempt at combining non-barotropic magnetized stellar models (Mastrano et al. 2011) with LIGO non-detections and radio timing data to constrain the ratio $\Lambda$ and hence the internal magnetic field of recycled pulsars. The models resemble the ‘twisted torus’ of Cioffi et al. (2009, 2010) and Lander & Jones (2009), in that they consist of a potentially strong internal toroidal field, which is not observable directly and is confined inside the star, and an external field, which can be measured. The main difference is that we assume that the star is stably stratified but non-barotropic (Reisenegger 2009). Because of this, our models allow the poloidal and toroidal components to be adjusted independently of each other; in barotropic models (such as the twisted torus model), the poloidal and toroidal fields must obey certain relations to ensure that pressure is always a function of density purely (see also e.g. Haskell et al. 2008). This means that our models can easily accommodate field configurations with $0.01 \leq \Lambda \leq 0.8$, the range found to be stable by the numerical simulations of Braithwaite & Nordlund (2006).

As seen in Fig. 2, our calculated $\epsilon$ is much lower than the LIGO and spin-down upper limits when the inferred dipole field strengths are input directly into equation (5).\(^4\) Note that the lower black dashed curve in Fig. 2 already assumes a strong toroidal field.

\(^4\) Recall that the magnetic axis is assumed to be perpendicular to the rotation axis throughout this paper.
(\(\Lambda = 10^{-3}\)); a weaker toroidal field will generate a curve even lower down. Next, we calculate how interior superconductivity (\(\mu_s = 10^{-3}\)) and accretion-induced screening or burial (red and blue bands in Fig. 2) enhance \(\epsilon\). This latter possibility is further explored in Fig. 4, which shows how, in principle, a gravitational wave detection from a recycled pulsar with a certain surface field (measured from spin-down) can be used to infer both \(\Lambda\) and shielding factor \(\xi\). Fig. 4 also tells us the sets of parameters that are ruled out by current observations: for a given pulsar with a measured spin-down field strength, non-detection means that \(\xi\) and \(\Lambda\) lie above that particular curve in Fig. 4. For example, the upper solid curve of Fig. 2, corresponding to \(\xi = 10^{-4}\) and \(\Lambda = 10^{-3}\), lies at the corner of the theoretically allowed shaded region in Fig. 4 and is therefore ruled out even for a non-superconducting star with a relatively weak observed field strength of \(10^3\) T (because it lies below the dashed blue curve in Fig. 4).

Is it possible that the magnetic field of a recycled pulsar is much stronger just below the surface than the inferred dipole field? Krolik (1991) and Arons (1993), motivated by the discrepancy between field strengths inferred from dipole spin-down and from cyclotron line measurements of accreting X-ray pulsars, raised such a possibility. They proposed that higher order multipoles may exist close to the surface. Arons (1993) then showed that surface field strengths of \(10^7\) times higher than the observed dipole field can account for the aforementioned discrepancy, as well as the location of the millisecond pulsars on the \(P\)–\(P\) diagram. The surface field can also be masked by the magnetic field burial, whereby accreted matter compresses the poloidal magnetic flux into a narrow belt around the equator. It has been shown that accretion of \(10^{-5}\) M\(_\odot\) is enough to alter the dipole moment significantly (Lai 1999; Payne & Melatos 2004; Melatos & Payne 2005; Zhang & Kojima 2006; Vigelius & Melatos 2009), depending on the equation of state (Priymak, Melatos & Payne 2011).

Pons & Geppert (2007) raised the possibility that the large-scale magnetic field of a neutron star is supported by long-lived currents in its superconducting core, while small-scale, fast-decaying (decay time \(10^5\)–\(10^7\) yr) magnetic structures also exist near the surface, supported by short-lived currents in the crust. After trying several different initial poloidal and toroidal field strengths, Pons & Geppert (2007) found that all their models eventually (after \(\sim 1\) Myr) reconfigure into a long-term stable configuration comprising a dipolar poloidal component and a quadrupolar/octupolar toroidal component. The decay of the crustal currents is dominated by Hall drift in the first \(10^5\)–\(10^7\) yr if the initial field is \(\gtrsim 10^10\) T, i.e. magnetar strength (Goldreich & Reisenegger 1992; Reinhardt & Geppert 2002; Pons & Geppert 2007). When the external dipole field is of magnetar strength, the internal crustal field can still be \(\sim 10\) times greater.

Pons & Geppert (2007) did not calculate the magnetic deformation of their star, but one can extend their work and that of Mastrano et al. (2011) to investigate the effects of this strong, internal, quadrupolar/octupolar toroidal field on stellar ellipticity. To do so, one must remember that equation (5) applies to an internal field that is continuous with an external dipole, as shown in Fig. 1. When the field is distorted significantly by accretion (into e.g. the ‘equatorial tutu’ shape found by Melatos &phin2001 and Payne & Melatos 2004), then the internal toroidal field described by equation (2) must be recalculated to ensure that the total field still has the desired properties listed at the start of Section 2. A similar adjustment must be carried out to accommodate multipolar external fields. It can be shown, for example, that one needs two toroidal field belts around the new neutral circles at \((0.7R, \pi/4)\) and \((0.7R, 3\pi/4)\) to match a quadrupolar external field (Mastrano 2010).

Lastly, we remind the reader that a strong internal field is not the only possible cause for stellar deformation. Other effects that can deform a star are as follows.

(i) Rotation: ellipticity induced by centrifugal forces is \(\epsilon_i \approx 0.3(\nu/\text{kHz})^2\), where \(\nu\) is the spin frequency of the neutron star (Pines & Shaham 1972; Cutler 2002). Magnetar spin frequencies are \(\sim 0.1\) Hz, giving \(\epsilon_i \sim 10^{-9}\). However, the deformation is aligned with the rotation axis (except for a fraction \(\approx 10^{-4}\) proportional to the shear modulus; Goldreich 1970), so it does not contribute to gravitational wave emission or precession (Melatos 2000).

(ii) Crustal shear stresses: ellipticity supported by the crust’s elasticity (Haskell, Jones & Andersson 2006), up to a maximum of \(\sim 10^{-7}\) for conventional neutron stars, \(\sim 10^{-5}\) for hybrid quark-baryon or meson-condensate stars, or \(\sim 10^{-4}\) for solid strange quark stars (Owen 2005; Horowitz & Kadau 2009).

(iii) Nuclear processes: a sustained, asymmetric temperature step of \(\sim 5\) per cent at the base of the crust drives electron capture reactions which produce compositional asymmetries (Ushomirsky, Cutler & Bildsten 2000).

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