HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, the author established Hermite-Hadamard's inequalities for harmonically convex functions via fractional integrals and obtained some Hermite-Hadamard type inequalities of these classes of functions.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). The following inequality

\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}
\]

(1.1)

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping \( f \). Both inequalities hold in the reversed direction if \( f \) is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [1, 2, 3, 4, 5]) and references therein.

In [1], Iscan gave definition of harmonically convexity as follows:

**Definition 1.** Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \rightarrow \mathbb{R} \) is said to be harmonically convex, if

\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(y) + (1-t)f(x)
\]

(1.2)

for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (1.2) is reversed, then \( f \) is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

**Theorem 1 ([1]).** Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold

\[
f \left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}.
\]
Lemma 1 (I). Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a differentiable function on \( I^o \) and \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \) then

\[
(1.3) \quad \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx = \frac{ab(b-a)}{2} \int_0^1 \frac{1 - 2t}{tb + (1-t)a)^2} f'(\frac{ab}{tb + (1-t)a}) dt.
\]

In [1], Iscan proved the following results connected with the right part of (1.2)

Theorem 2. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q \geq 1 \), then

\[
(1.4) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \lambda_1^{1 - \frac{1}{q}} \left[ \lambda_2 |f'(a)|^q + \lambda_3 |f'(b)|^q \right]^\frac{1}{q},
\]

where

\[
\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right),
\]

\[
\lambda_2 = \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right),
\]

\[
\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right) = \lambda_1 - \lambda_2.
\]

Theorem 3. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for \( q > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[
(1.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right) \left( \mu_1 |f'(a)|^q + \mu_2 |f'(b)|^q \right)^\frac{1}{q},
\]

where

\[
\mu_1 = \frac{(a^2-2q + b^1-2q) [(b-a) (1-q) - a]}{2 (b-a)^2 (1-q) (1-2q)} ,
\]

\[
\mu_2 = \frac{(b^2-2q - a^1-2q) [(b-a) (1-q) + b]}{2 (b-a)^2 (1-q) (1-2q)}.
\]

We recall the following special functions and inequality
(1) The Beta function:
\[
\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y > 0,
\]

(2) The hypergeometric function:
\[
_{2}F_{1}(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \quad c > b > 0, \quad |z| < 1 \text{ (see [14]).}
\]

**Lemma 2** ([6, 7]). For \(0 < \alpha \leq 1\) and \(0 \leq a < b\), we have
\[
|a^\alpha - b^\alpha| \leq (b-a)^\alpha.
\]

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

**Definition 2.** Let \(f \in L[a, b]\). The Riemann-Liouville integrals \(J^\alpha_{a+}f\) and \(J^\alpha_{b-}f\) of order \(\alpha > 0\) with \(a \geq 0\) are defined by
\[
J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a
\]
and
\[
J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b
\]
respectively, where \(\Gamma(\alpha)\) is the Gamma function defined by \(\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt\) and \(J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)\).

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [7, 8, 9, 10, 11, 12, 13].

The aim of this paper is to establish Hermite–Hadamard’s inequalities for Harmonically convex functions via Riemann–Liouville fractional integral and some other integral inequalities using the identity is obtained for fractional integrals. These results have some relationships with [1].

2. Main results

Let \(f : I \subseteq (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^0\), the interior of \(I\), throughout this section we will take
\[
I_f (g; \alpha, a, b) = \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2} \left( \frac{ab}{b-a} \right)^\alpha \left\{ J^\alpha_{1/a^-} (f \circ g)(1/b) + J^\alpha_{1/b^+} (f \circ g)(1/a) \right\}
\]
where \(a, b \in I\) with \(a < b\), \(\alpha > 0\), \(g(x) = 1/x\) and \(\Gamma\) is Euler Gamma function.
Hermite–Hadamard’s inequalities for Harmonically convex functions can be represented in fractional integral forms as follows:

**Theorem 4.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a function such that \( f \in \mathcal{L}[a, b] \), where \( a, b \in I \) with \( a < b \). If \( f \) is a harmonically convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
(2.1) \quad f \left( \frac{2ab}{a + b} \right) \leq \frac{\Gamma(\alpha + 1)}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{\alpha/2}^\alpha (f \circ g) (1/b) + J_{\alpha/2}^\alpha (f \circ g) (1/a) \right\} \leq \frac{f(a) + f(b)}{2}
\]

with \( \alpha > 0 \).

**Proof.** Since \( f \) is a harmonically convex function on \([a, b]\), we have for all \( x, y \in [a, b] \) (with \( t = 1/2 \) in the inequality \((1.2)\))

\[
(2.2) \quad f \left( \frac{2xy}{x + y} \right) \leq \frac{f(x) + f(y)}{2}.
\]

Choosing \( x = \frac{ab}{ta + (1-t)b} \), \( y = \frac{ab}{tb + (1-t)a} \), we get

\[
f \left( \frac{2ab}{a + b} \right) \leq f \left( \frac{ab}{tb + (1-t)a} \right) + f \left( \frac{ab}{ta + (1-t)b} \right).
\]

Multiplying both sides of \((2.2)\) by \( t^{\alpha-1} \), then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
f \left( \frac{2ab}{a + b} \right) \leq \frac{\alpha}{2} \left\{ \int_0^1 t^{\alpha-1} f \left( \frac{ab}{tb + (1-t)a} \right) dt + \int_0^1 t^{\alpha-1} f \left( \frac{ab}{ta + (1-t)b} \right) dt \right\}
\]

\[= \frac{\alpha}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ \int_0^1 \left( x - \frac{1}{b} \right)^{\alpha-1} f \left( \frac{1}{x} \right) dx + \int_{1/b}^1 \left( 1 - \frac{1}{x} \right)^{\alpha-1} f \left( \frac{x}{1} \right) dx \right\}
\]

\[= \frac{\alpha\Gamma(\alpha)}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{\alpha/2}^\alpha (f \circ g) (1/b) + J_{\alpha/2}^\alpha (f \circ g) (1/a) \right\}
\]

\[= \frac{\Gamma(\alpha + 1)}{2} \left( \frac{ab}{b - a} \right)^\alpha \left\{ J_{\alpha/2}^\alpha (f \circ g) (1/b) + J_{\alpha/2}^\alpha (f \circ g) (1/a) \right\}, \text{ where } g(x) = 1/x.
\]

and the first inequality is proved.

For the proof of the second inequality in \((2.1)\) we first note that if \( f \) is a harmonically convex function, then, for \( t \in [0, 1] \), it yields

\[
f \left( \frac{ab}{tb + (1-t)a} \right) \leq tf(a) + (1-t)f(b)
\]

and

\[
f \left( \frac{ab}{ta + (1-t)b} \right) \leq tf(b) + (1-t)f(a).
\]

By adding these inequalities we have

\[
(2.3) \quad f \left( \frac{ab}{tb + (1-t)a} \right) + f \left( \frac{ab}{ta + (1-t)b} \right) \leq f(a) + f(b).
\]
Then multiplying both sides of \((2.3)\) by \(t^{\alpha-1}\), and integrating the resulting inequality with respect to \(t\) over \([0, 1]\), we obtain

\[
\int_0^1 f \left( \frac{ab}{tb + (1-t)a} \right) t^{\alpha-1} dt + \int_0^1 f \left( \frac{ab}{ta + (1-t)b} \right) t^{\alpha-1} dt \leq \int_0^1 \left| f(a) + f(b) \right| t^{\alpha-1} dt
\]

i.e.

\[
\Gamma(\alpha + 1) \left( \frac{ab}{b - a} \right)^{\alpha} \left\{ J^\alpha_{1/a} (f \circ g) \left( \frac{1}{b} \right) + J^\alpha_{b} (f \circ g) \left( \frac{1}{a} \right) \right\} \leq f(a) + f(b).
\]

The proof is completed. \(\square\)

**Lemma 3.** Let \(f : I \subseteq (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I\) such that \(f' \in L[a, b]\), where \(a, b \in I\) with \(a < b\). Then the following equality for fractional integrals holds:

\[
I_f (g; \alpha, a, b) = \frac{ab}{2} \int_0^1 \left[ \frac{t^\alpha - (1-t)^\alpha}{(ta + (1-t)b)^{\alpha+1}} \right] f' \left( \frac{ab}{ta + (1-t)b} \right) dt.
\]

**Proof.** Let \(A_t = ta + (1-t)b\). It suffices to note that

\[
I_f (g; \alpha, a, b) = \frac{ab}{2} \int_0^1 \left[ \frac{t^\alpha - (1-t)^\alpha}{A_t^{\alpha+1}} \right] f' \left( \frac{ab}{A_t} \right) dt
\]

\[
= \frac{ab}{2} \int_0^1 \frac{t^\alpha}{A_t^{\alpha+1}} f' \left( \frac{ab}{A_t} \right) dt - \frac{ab}{2} \int_0^1 \frac{(1-t)^\alpha}{A_t^{\alpha+1}} f' \left( \frac{ab}{A_t} \right) dt
\]

(2.5)

\(I_1 + I_2\).

By integrating by part, we have

\[
I_1 = \frac{1}{2} \left[ t^\alpha f \left( \frac{ab}{A_t} \right) \right]_0^1 - \alpha \int_0^1 t^{\alpha-1} f \left( \frac{ab}{A_t} \right) dt
\]

\[
= \frac{1}{2} \left[ f(b) - \alpha \left( \frac{ab}{b - a} \right)^{\alpha-1} \int_0^1 \left( \frac{1}{a} - x \right)^{\alpha-1} f \left( \frac{1}{x} \right) dx \right]
\]

(2.6)

\[
I_2 = \frac{1}{2} \left[ f(b) - \Gamma(\alpha + 1) \left( \frac{ab}{b - a} \right)^{\alpha} J^\alpha_{1/b} (f \circ g) \left( \frac{1}{a} \right) \right]
\]
and similarly we get,

\[ I_2 = \frac{1}{2} \left[ (1-t)^{\alpha} f \left( \frac{ab}{A_t} \right) \right]_0^1 + \alpha \int_0^1 (1-t)^{\alpha-1} f \left( \frac{ab}{A_t} \right) dt \]

\[ = \frac{1}{2} \left[ -f(a) + \alpha \left( \frac{ab}{b-a} \right)^{\alpha} \int_{1/b} (x-\frac{1}{b})^{\alpha-1} f \left( \frac{1}{x} \right) dx \right] \]

(2.7) \[ = \frac{1}{2} \left[ f(a) - \Gamma(\alpha + 1) \left( \frac{ab}{b-a} \right)^{\alpha} J_{1/b}^{0} \left( f \circ g \right) (1/b) \right]. \]

Using (2.6) and (2.7) in (2.5), we get equality (2.4). \qed

**Remark 1.** If Lemma 3 we let \( \alpha = 1 \), then equality (2.4) becomes equality (1.3) of Lemma 2.

Using lemma 3 we can obtain the following fractional integral inequality.

**Theorem 5.** Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^o \) such that \( f' \in L[a, b] \), where \( a, b \in I^o \) with \( a < b \). If \( |f'|^q \) is harmonically convex on \( [a, b] \) for some fixed \( q \geq 1 \), then the following inequality for fractional integrals holds:

\[ |I_f (g ; a, b)| \leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\alpha ; a, b) \left( C_2(\alpha ; a, b) |f'(b)|^q + C_3(\alpha ; a, b) |f'(a)|^q \right)^{1/q}, \]

where

\[ C_1(\alpha ; a, b) = \frac{b^{-2}}{\alpha + 1} \left[ _2 F_1 \left( 2, 1; \alpha + 2; 1 - \frac{a}{b} \right) + _2 F_1 \left( 2, \alpha + 1; \alpha + 2; 1 - \frac{a}{b} \right) \right], \]

\[ C_2(\alpha ; a, b) = \frac{b^{-2}}{\alpha + 2} \left[ \frac{1}{\alpha + 1} _2 F_1 \left( 2, 2; \alpha + 3; 1 - \frac{a}{b} \right) + _2 F_1 \left( 2, \alpha + 2; \alpha + 3; 1 - \frac{a}{b} \right) \right], \]

\[ C_3(\alpha ; a, b) = \frac{b^{-2}}{\alpha + 1} \left[ _2 F_1 \left( 2, 1; \alpha + 3; 1 - \frac{a}{b} \right) + \frac{1}{\alpha + 1} _2 F_1 \left( 2, \alpha + 1; \alpha + 3; 1 - \frac{a}{b} \right) \right]. \]

**Proof.** Let \( A_t = ta + (1-t)b \). From Lemma..., using the property of the modulus, the power mean inequality and the harmonically convexity of \( |f'|^q \), we find

\[ |I_f (g ; a, b)| \leq \frac{ab(b-a)}{2} \left( \int_0^1 \left| \frac{(1-t)^{\alpha} - t^{\alpha}}{A_t^2} \right| f' \left( \frac{ab}{A_t} \right) dt \right)^{1/q} \]

\[ \leq \frac{ab(b-a)}{2} \left( \int_0^1 \left| \frac{(1-t)^{\alpha} - t^{\alpha}}{A_t^2} \right| dt \right)^{1/q} \left( \int_0^1 \left| \frac{(1-t)^{\alpha} - t^{\alpha}}{A_t^2} \right| f' \left( \frac{ab}{A_t} \right)^q dt \right)^{1/q} \]

\[ \leq \frac{ab(b-a)}{2} \left( \int_0^1 \left| (1-t)^{\alpha} + t^{\alpha} \right| \frac{dt}{A_t^2} \right)^{1/q} \left( \int_0^1 \left| (1-t)^{\alpha} + t^{\alpha} \right| \frac{t |f'(b)|^q + (1-t) |f'(a)|^q dt}{A_t^2} \right)^{1/q} \]

(2.9) \[ \leq \frac{ab(b-a)}{2} C_1^{1-1/q}(\alpha ; a, b) \left( C_2(\alpha ; a, b) |f'(b)|^q + C_3(\alpha ; a, b) |f'(a)|^q \right)^{1/q}. \]
calculating \( C_1(\alpha; a, b), C_2(\alpha; a, b) \) and \( C_3(\alpha; a, b) \), we have

\[
C_1(\alpha; a, b) = \int_0^1 \frac{[1-t]^\alpha + t^\alpha}{A_t^2} dt
\]

(2.10) \[
= \frac{b^{-2}}{\alpha + 1} \left[ \binom{2,1}{2,1+\alpha} + 2 \binom{2,1+\alpha+2}{2,1+\alpha+2+\alpha} \right].
\]

\[
C_2(\alpha; a, b) = \int_0^1 \frac{[1-t]^\alpha + t^\alpha}{A_t^2} t dt
\]

(2.11) \[
= \frac{b^{-2}}{\alpha + 2} \left[ \frac{1}{\alpha + 1} \binom{2,2}{2,2+\alpha} + 2 \binom{2,2+\alpha+3}{2,2+\alpha+3+\alpha} \right].
\]

\[
C_3(\alpha; a, b) = \int_0^1 \frac{[1-t]^\alpha + t^\alpha}{A_t^2} (1-t) dt
\]

(2.12) \[
= \frac{b^{-2}}{\alpha + 1} \left[ \binom{2,1}{2,1+\alpha+3} - \frac{1}{\alpha + 1} \binom{2,1+\alpha+3}{2,1+\alpha+3+\alpha} \right].
\]

Thus, if we use (2.10), (2.11) and (2.12) in (2.9), we obtain the inequality of (2.8). This completes the proof. \( \square \)

When \( 0 < \alpha \leq 1 \), using Lemma 2 and Lemma 3 we shall give another result for harmonically convex functions as follows.

**Theorem 6.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^0 \) such that \( f' \in L[a, b] \), where \( a, b \in I^0 \) with \( a < b \). If \( |f'|^q \) is harmonically convex on \([a, b]\) for some fixed \( q \geq 1 \), then the following inequality for fractional integrals holds:

\[
|I_f (g; a, b)| \leq \frac{ab(b-a)}{2} C_1^{-1/q}(\alpha; a, b) \left( C_2(\alpha; a, b) |f'(b)|^q + C_3(\alpha; a, b) |f'(a)|^q \right)^{1/q},
\]

where

\[
C_1(\alpha; a, b) = \frac{b^{-2}}{\alpha + 1} \left[ \binom{2,2}{2,2+\alpha} - 2 \binom{2,2+\alpha+3}{2,2+\alpha+3+\alpha} \right]
\]

\[
+ 2 \binom{2,1+\alpha+3}{2,1+\alpha+3+\alpha} \left[ \frac{1}{\alpha + 1} \binom{2,2+\alpha+3}{2,2+\alpha+3+\alpha} \right].
\]

\[
C_2(\alpha; a, b) = \frac{b^{-2}}{\alpha + 2} \left[ \binom{2,2}{2,2+\alpha+3} - \frac{1}{\alpha + 1} \binom{2,2+\alpha+3}{2,2+\alpha+3+\alpha} \right]
\]

\[
+ \frac{1}{2(\alpha + 1)} \binom{2,2+\alpha+3}{2,2+\alpha+3+\alpha} \left[ \frac{1}{\alpha + 1} \binom{2,2+\alpha+3}{2,2+\alpha+3+\alpha} \right].
\]
\[
\begin{align*}
C_3(\alpha; a, b) &= \frac{b^{-2}}{\alpha + 2} \left[ \frac{1}{\alpha + 1} \cdot 2 F_1 \left( 2, \alpha + 1; \alpha + 3; 1 - \frac{a}{b} \right) - 2 F_1 \left( 2, 1; \alpha + 3; 1 - \frac{a}{b} \right) \\
&\quad + 2 F_1 \left( 2, 1; \alpha + 3; \frac{1}{2} \left( 1 - \frac{a}{b} \right) \right) \right] \\
\text{and } 0 < \alpha \leq 1.
\end{align*}
\]

Proof. Let \( A_t = ta + (1 - t)b \). From Lemma 3 using the property of the modulus, the power mean inequality and the harmonically convexity of \(|f'|^q\), we find

\[
|I_f (g; \alpha, a, b)| \\
\leq \frac{ab (b-a)}{2} \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| f' \left( \frac{ab}{A_t} \right) \left| \frac{|1-t|^\alpha - t^\alpha}{A_t^2} \right| dt \\
\leq \frac{ab (b-a)}{2} \left( \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| dt \right)^{1-1/q} \left( \int_0^1 \left| \frac{|1-t|^\alpha - t^\alpha}{A_t^2} \right| f' \left( \frac{ab}{A_t} \right) |^q \right)^{1/q} \\
\leq \frac{ab (b-a)}{2} K_1^{1-1/q} \left( \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| t \left| f'(b) \right|^q + (1-t) \left| f'(a) \right|^q \right) dt \\
\leq \frac{ab (b-a)}{2} K_1^{1-1/q} \left( K_2 \left| f'(b) \right|^q + K_3 \left| f'(a) \right|^q \right)^{1/q},
\]

where

\[
\begin{align*}
K_1 &= \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| dt, \\
K_2 &= \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| t dt, \\
K_3 &= \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| (1-t) dt.
\end{align*}
\]

Calculating \( K_1, K_2 \) and \( K_3 \), by Lemma 2 we have

\[
\begin{align*}
K_1 &= \int_0^1 \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| dt \\
&= \int_0^{1/2} \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| dt + \int_{1/2}^1 \left| \frac{t^\alpha - (1-t)^\alpha}{A_t^2} \right| dt \\
&= \int_0^{1/2} \left| \frac{t^\alpha - (1-t)^\alpha}{A_t^2} \right| dt + 2 \int_0^{1/2} \left| \frac{(1-t)^\alpha - t^\alpha}{A_t^2} \right| dt
\end{align*}
\]
\[
\begin{align*}
\text{HARMONICALLY CONVEX FUNCTIONS} & \leq \int_0^1 t^{\alpha} A_t^{-2} dt - \int_0^1 (1 - t)^{\alpha} A_t^{-2} dt + 2 \int_0^{1/2} (1 - 2t)^{\alpha} A_t^{-2} dt \\
& = \int_0^1 t^{\alpha} A_t^{-2} dt - \int_0^1 (1 - t)^{\alpha} A_t^{-2} dt + \int_0^1 (1 - u)^{\alpha} b^{-2} \left(1 - u^\frac{1}{2}(1 - \frac{a}{b})\right)^{-2} du \\
& = \frac{b^{-2}}{\alpha + 1} \left[\left. 2 F_1 \left(2, \alpha + 1; \alpha + 2; 1 - \frac{a}{b}\right) \right|_{2} - \left. 2 F_1 \left(2, 1; \alpha + 2; 1 - \frac{a}{b}\right) \right|_{-2} \\
& \quad + \left. 2 F_1 \left(2, 1; \alpha + 2; \frac{1}{2} \left(1 - \frac{a}{b}\right)\right) \right|_{+} \right]
\end{align*}
\]

(2.15) \quad = C_1(\alpha; a, b)

and similarly we get

\[
\begin{align*}
K_2 & = \int_0^1 \frac{\left| (1 - t)^{\alpha} - t^{\alpha} \right|}{A_t^2} dt \\
& \leq \int_0^1 t^{\alpha + 1} A_t^{-2} dt - \int_0^1 (1 - t)^{\alpha} t A_t^{-2} dt + 2 \int_0^{1/2} (1 - 2t)^{\alpha} t A_t^{-2} dt \\
& = \frac{b^{-2}}{\alpha + 2} \left[\left. 2 F_1 \left(2, \alpha + 2; \alpha + 3; 1 - \frac{a}{b}\right) \right|_{2} - \frac{1}{\alpha + 1} \left. 2 F_1 \left(2, 2; \alpha + 3; 1 - \frac{a}{b}\right) \right|_{-2} \\
& \quad + \left. \frac{1}{2(\alpha + 1)} 2 F_1 \left(2, 2; \alpha + 3; \frac{1}{2} \left(1 - \frac{a}{b}\right)\right) \right|_{+} \right]
\end{align*}
\]

(2.16) \quad = C_2(\alpha; a, b)

\[
\begin{align*}
K_3 & = \int_0^1 \frac{\left| (1 - t)^{\alpha} - t^{\alpha} \right|}{A_t^2} dt \\
& \leq \int_0^1 t^{\alpha} (1 - t) A_t^{-2} dt - \int_0^1 (1 - t)^{\alpha + 1} A_t^{-2} dt + 2 \int_0^{1/2} (1 - 2t)^{\alpha} (1 - t) A_t^{-2} dt \\
& = \frac{b^{-2}}{\alpha + 2} \left[\left. \frac{1}{\alpha + 1} 2 F_1 \left(2, \alpha + 1; \alpha + 3; 1 - \frac{a}{b}\right) \right|_{2} - \left. 2 F_1 \left(2, 1; \alpha + 3; 1 - \frac{a}{b}\right) \right|_{-2} \\
& \quad + \left. 2 F_1 \left(2, 1; \alpha + 3; \frac{1}{2} \left(1 - \frac{a}{b}\right)\right) \right|_{+} \right]
\end{align*}
\]

(2.17) \quad = C_3(\alpha; a, b).

Thus, if we use (2.15), (2.16) and (2.17) in (2.14), we obtain the inequality of (2.13). This completes the proof. \qed

Remark 2. If we take \( \alpha = 1 \) in Theorem 6, then inequality (2.13) becomes inequality (1.4) of Theorem 2.
Theorem 7. Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I \) such that \( f' \in L[a, b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically convex on \([a, b]\) for some fixed \( q > 1 \), then the following inequality for fractional integrals holds:

\[
(2.18) \quad |I_f (g; \alpha, a, b)| \\
\leq \frac{ab(b-a)}{2} \left( \frac{1}{\alpha p + 1} \right)^{1/p} \left( \frac{|f'(b)|^q + |f'(a)|^q}{2} \right)^{1/q} \\
\times \left[ 2F_1^{1/p} \left( 2p, 1; \alpha p + 2; 1 - \frac{a}{b} \right) + 2F_1^{1/p} \left( 2p, \alpha p + 1; \alpha p + 2; 1 - \frac{a}{b} \right) \right],
\]

where \( 1/p + 1/q = 1 \).

Proof. Let \( A_t = ta + (1-t)b \). From Lemma 3 using the Hölder inequality and the harmonically convexity of \( |f'|^q \), we find

\[
|I_f (g; \alpha, a, b)| \\
\leq \frac{ab(b-a)}{2} \left( \frac{1}{\alpha p + 1} \right)^{1/p} \left( \int_0^1 \left| f' \left( \frac{ab}{A_t} \right) \right| dt \right) \\
\leq \frac{ab(b-a)}{2} \left\{ \left( \int_0^1 \left( \frac{1}{\alpha p} \right)^{1/p} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
+ \left( \int_0^1 \left( \frac{1}{\alpha p} \right)^{1/p} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \right\} \\
\leq \frac{ab(b-a)}{2} \left( K_4^{1/p} + K_5^{1/p} \right) \left( \int_0^1 \left[ |f'(b)|^q + (1-t) |f'(a)|^q \right] dt \right)^{1/q}.
\]

(2.19)

Calculating \( K_4 \) and \( K_5 \), we have

\[
K_4 = \int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}} dt \\
(2.20) = \frac{b^{-2p}}{\alpha p + 1} 2F_1 \left( 2p, 1; \alpha p + 2; 1 - \frac{a}{b} \right),
\]

\[
K_5 = \int_0^1 \frac{t^{\alpha p}}{A_t^{2p}} dt \\
(2.21) = \frac{b^{-2p}}{\alpha p + 1} 2F_1 \left( 2p, \alpha p + 1; \alpha p + 2; 1 - \frac{a}{b} \right).
\]

Thus, if we use (2.20) and (2.21) in (2.19), we obtain the inequality of (2.18).

This completes the proof. \( \square \)
Theorem 8. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:

$$
|I_f (g; \alpha, a, b)| \leq \frac{b - a}{2(ab)^{1-1/p}} L^{2-2/p}(a, b) \left( \frac{1}{\alpha q + 1} \right)^{1/q} \left( \frac{|f'(b)|^q + |f'(a)|^q}{2} \right)^{1/q},
$$

where $1/p + 1/q = 1$ and $L^{2p-2}(a, b) = \left( \frac{b^{2p-1} - a^{2p-1}}{(2p-1)(b-a)} \right)^{1/(2p-2)}$ is the 2p-2-Logarithmic mean.

Proof. Let $A_t = ta + (1-t)b$. From Lemma 3 and Lemma 2, using the Hölder inequality and the harmonically convexity of $|f'|^q$, we find

$$
|I_f (g; \alpha, a, b)| \leq \frac{ab(b-a)}{2} \int_0^1 \frac{|(1-t)^\alpha - t\alpha|}{A_t^{2p}} \left| f' \left( \frac{ab}{A_t} \right) \right| dt
$$

$$
\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{1}{A_t^{2p}} dt \right)^{1/p} \left( \int_0^1 \left| (1-t)^\alpha - t\alpha \right|^q \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q}
$$

$$
\leq \frac{ab(b-a)}{2} \left( \int_0^1 \frac{1}{A_t^{2p}} dt \right)^{1/p} \left( \int_0^1 \left| 1 - 2t \right|^\alpha \left| t \right| f'(b)|^q + (1-t) |f'(a)|^q \right) dt \right)^{1/q}
$$

$$
\leq \frac{ab(b-a)}{2} K_6^{1/p} (K_7 |f'(b)|^q + K_8 |f'(a)|^q)^{1/q},
$$

where

$$
K_6 = \int_0^1 \frac{1}{A_t^{2p}} dt = b^{-2p} \int_0^1 \left( 1 - t \left( \frac{a}{b} \right) \right)^{-2p} dt
$$

$$
= b^{-2p} \int_0^1 \left( 1 - 2t \right)^{-2p} dt = b^{-2p} \Gamma(2p, 1; 1 - \frac{a}{b}) = \frac{L^{2p-2}(a, b)}{(ab)^{2p-1}},
$$

$$
K_7 = \int_0^1 \left| 1 - 2t \right| \alpha q t dt
$$

$$
= \int_0^{1/2} \left( 1 - 2t \right) \alpha q t dt + \int_{1/2}^1 (2t - 1) \alpha q t dt
$$

$$
= \frac{1}{2 (\alpha q + 1)},
$$
and

\[ K_8 = \int_{0}^{1} |1 - 2t|^{\alpha q} (1 - t) dt \]

(2.26)

\[ = \frac{1}{2(\alpha q + 1)}. \]

Thus, if we use (2.24), (2.25) and (2.26) in (2.23), we obtain the inequality of (2.22). This completes the proof. \( \square \)

**Theorem 9.** Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^\circ \) such that \( f' \in L[a, b] \), where \( a, b \in I^\circ \) with \( a < b \). If \( |f'|^q \) is harmonically convex on \([a, b]\) for some fixed \( q > 1 \), then the following inequality for fractional integrals holds:

\[
|I_f (g; \alpha, a, b)| \leq \frac{ab (b - a)}{2b} \left( \frac{1}{\alpha p + 1} \right)^{1/p} \times \left( \frac{2F_1 \left( 2q, 2; 1 - \frac{a}{b} \right) |f'(b)|^q + 2F_1 \left( 2q, 1; 3; 1 - \frac{a}{b} \right) |f'(a)|^q}{2} \right)^{1/q},
\]

(2.27)

where \( 1/p + 1/q = 1 \).

**Proof.** Let \( A_t = ta + (1 - t)b \). From Lemma 3 and Lemma 2 using the Hölder inequality and the harmonically convexity of \( |f'|^q \), we find

\[
|I_f (g; \alpha, a, b)| \leq \frac{ab (b - a)}{2} \int_{0}^{1} \frac{|(1 - t)^\alpha - t^\alpha|}{A_t^2} \left| f' \left( \frac{ab}{A_t} \right) \right| dt \\
\leq \frac{ab (b - a)}{2} \left( \int_{0}^{1} |(1 - t)^\alpha - t^\alpha|^p dt \right)^{1/p} \left( \int_{0}^{1} \frac{1}{A_t^{2q}} \left| f' \left( \frac{ab}{A_t} \right) \right|^q dt \right)^{1/q} \\
\leq \frac{ab (b - a)}{2} \left( \int_{0}^{1} |2t|^\alpha p dt \right)^{1/p} \left( \int_{0}^{1} \frac{1}{A_t^{2q}} \left[ t |f'(b)|^q + (1 - t) |f'(a)|^q \right] dt \right)^{1/q} \\
\leq \frac{ab (b - a)}{2} K_9^{1/p} (K_{10} |f'(b)|^q + K_{11} |f'(a)|^q)^{1/q},
\]

(2.28)

where

\[ K_9 = \int_{0}^{1} |1 - 2t|^\alpha p dt = \frac{1}{\alpha p + 1} \]

(2.29)

\[ K_{10} = \int_{0}^{1} t A_t^{-2q} dt = b^{-2q} \int_{0}^{1} t \left( 1 - t \left( 1 - \frac{a}{b} \right) \right)^{-2q} dt \]

(2.30)

\[ = \frac{1}{2b^{2q}} 2F_1 \left( 2q, 2; 3; 1 - \frac{a}{b} \right) \]
and

\[ K_{11} = \int_0^1 (1-t)A_t^{-2q} dt = \frac{1}{2b^2q} {}_2F_1 \left( 2q, 1; 1 - \frac{a}{b} \right) \]

Thus, if we use (2.26), (2.27) and (2.31) in (2.28), we obtain the inequality of (2.27). This completes the proof. □

**Remark 3.** If we take \( \alpha = 1 \) in Theorem 9, then inequality (2.27) becomes inequality (1.5) of Theorem 3.

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