A Hierarchy of Bounds on Accessible Information and Informational Power

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Quantum theory imposes fundamental limitations to the amount of information that can be carried by any quantum system. On the one hand, Holevo bound rules out the possibility to encode more information in a quantum system than in its classical counterpart, comprised of perfectly distinguishable states. On the other hand, when states are uniformly distributed in the state space, the so-called subentropy lower bound is saturated. How uniform quantum systems are can be naturally quantified by characterizing them as $t$-designs, with $t = \infty$ corresponding to the uniform distribution. Here we show the existence of a trade-off between the uniformity of a quantum system and the amount of information it can carry. To this aim, we derive a hierarchy of informational bounds as a function of $t$ and prove their tightness for qubits and qutrits. By deriving asymptotic formulae for large dimensions, we also show that the statistics generated by any $t$-design with $t > 1$ contains no more than a single bit of information, and this amount decreases with $t$. Holevo and subentropy bounds are recovered as particular cases for $t = 1$ and $t = \infty$, respectively.

I. INTRODUCTION

Quantum theory imposes fundamental limitations to the amount of information that can be encoded into or extracted from any quantum system. Formally, the former case is referred to as the problem of accessible information [1–6] of a quantum ensemble, while the latter as the problem of informational power [7–15] of a quantum measurement. Recently, a duality relation between these two quantities was established [7], that allows to generally refer to the problem of quantifying the information carried by a quantum system.

On the one hand, the well-known Holevo upper bound [1] rules out the possibility to encode more information in a quantum system than in its purely classical counterpart, comprised of perfectly distinguishable states. On the other hand, for a genuinely quantum system whose states are uniformly distributed in the state space, the so-called subentropy lower bound [2] is saturated. Therefore, one might conjecture the existence of a general trade-off between the uniformity of a quantum system and the amount of information it can carry.

A natural mean to quantify how uniform quantum systems are, is provided by their characterization in terms of spherical quantum $t$-designs [16–21], with $t = 1$ corresponding to an arbitrary quantum measurement and $t = \infty$ corresponding to the completely uniform distribution. Other remarkable examples of $t$-designs for the case $t = 2$ are symmetric, informationally complete (SIC) quantum measurements [16, 20] and complete sets of mutually unbiased bases (MUBs) [18, 19, 21]. They play a fundamental role in a plethora of applications such as quantum tomography [22], cryptography [23], information locking [24], quantumness of Hilbert space [25–27], entropic uncertainty relations [27–28], and foundations of quantum theory [33–36].

In this work, for any $t$ we derive an upper bound on the information that can be carried by any quantum $t$-design as a function of the dimension of the system. In this sense, the resulting hierarchy of bounds proves the correctness of the above mentioned conjecture and formally quantifies it. Furthermore, we show the tightness of our bounds for qubits and qutrits. By deriving asymptotic formulae for large dimensions, we also show that the statistics generated by any $t$-design with $t > 1$ contains no more than a single bit of information, and that this amount decreases with $t$. The Holevo upper bound [1] and the subentropy lower bound [2] are recovered as particular cases for $t = 1$ and $t = \infty$, respectively.

The paper is structured as follows. First, we introduce quantum $t$-designs in Section II A, we discuss the relevant figures of merit in Section II B and provide a way to estimate them in Section II C. Then, we introduce our main result, namely a hierarchy of upper bounds on the accessible information and the informational power of $t$-designs in Section III A, we derive close analytic expressions for low values of $t$ and asymptotic formulae for large dimension in Section III B and we prove tightness for qubits and qutrits in Section III C. We conclude by sum-
II. FORMALIZATION

A. Spherical quantum t-designs

In this subsection we recall some basic facts from quantum information theory, and specialize them to the case of spherical quantum t-design.

Any quantum system is associated with an Hilbert space $\mathcal{H}$, and we denote with $L(\mathcal{H})$ the space of linear operators on $\mathcal{H}$. We will only consider finite-dimensional Hilbert spaces.

A quantum state $\rho$ is represented by a positive-semidefinite operator in $L(\mathcal{H})$ such that $\text{Tr}[\rho] \leq 1$. A pure state $\psi$ is such that $\text{rank} \psi = 1$ and in Den denoted in Dirac notation by a vector $|\psi\rangle$ with $\psi = |\psi\rangle\langle\psi|$. Any quantum preparation is represented by an ensemble $\rho_x$, namely a measurable function $\rho_x$ from reals to states such that $\int_x \text{Tr}[\rho_x] dx = 1$. An ensemble of pure states is such that $\rho_x \neq 0$ if and only if $\rho_x$ is a pure state. The uniform ensemble is the ensemble of pure states distributed with uniform (Haar) measure on the unit sphere of $\mathcal{H}$.

A quantum effect $\pi$ is represented by a positive-semidefinite operator in $L(\mathcal{H})$ such that $\pi \leq \mathbb{1}_d$, where $\mathbb{1}_d$ is the $d$-dimensional identity operator. Any quantum measurement is represented by a positive-operator valued measure (POVM) $\pi_y$, namely a measurable function $\pi_y$ from reals to effects such that $\int_y \pi_y dy = \mathbb{1}_d$.

For any ensemble $\rho_x$ and POVM $\pi_y$, the joint probability density $p_{x,y}$ of input $x$ and outcome $y$ is given by the Born rule, i.e. $p_{x,y} = \text{Tr}[\rho_x \pi_y]$.

Definition 1 (Spherical quantum t-design). A spherical quantum t-design is an ensemble $\rho_x$ such that

$$\int \frac{\rho_x^\otimes k}{\text{Tr}[\rho_x]^{k-1}} dx := \int \frac{|\psi_x|^\otimes k}{||\psi_x||^2(k-1)} dx$$

holds for any $k \leq t$, where $\psi_x$ is the uniform ensemble.

Lemma 1. Let $\psi_x$ be the uniform ensemble. Then one has

$$\int \frac{\psi_x^\otimes k}{||\psi_x||^2(k-1)} dx = \left(\frac{d - 1 + k}{k}\right)^{-1} P_{\text{sym}},$$

where $P_{\text{sym}}$ is the projector over the symmetric subspace of $\mathcal{H}^\otimes k$ and $d$ is the dimension of $\mathcal{H}$.

Proof. See Ref. [18].

Remark 1. From Definition 1 and Lemma 1 it immediately follows that any POVM is a 1-design up to a normalization factor of $d$.

Remarkable examples of 2-designs are symmetric informationally complete (SIC) POVMs and $d + 1$ mutually unbiased bases (MUBs).

A concept that will be relevant in the following is the so-called index of coincidence.

Definition 2 (Index of coincidence). For any POVM $\pi_y$ and any unit-trace state $\rho$, the index of coincidence $C_k(\pi_y, \rho)$ is given by

$$C_k(\pi_y, \rho) := \int \frac{\text{Tr}[\pi_y \rho]^k}{\text{Tr}[\pi_y]^{k-1}} dy.$$

The following result characterizes the index of coincidence of t-designs.

Lemma 2. Let $\mathcal{H}$ be a d-dimensional Hilbert space. Let $\pi_y \in L(\mathcal{H})$ be a t-design POVM. Let $|\psi\rangle \in \mathcal{H}$ be a unit-trace pure state. For any $k \leq t$, the index of coincidence $C_k(\pi_y, \psi)$ is independent of $\pi_y$ and $\psi$ and is given by

$$C_k = d \left(\frac{d - 1 + k}{k}\right)^{-1}.$$

Proof. By Definition 2 one has

$$C_k(\pi_y, \psi) = \int \text{Tr} \left[\psi^\otimes k \frac{\pi_y^\otimes k}{\text{Tr}[\pi_y]^{k-1}}\right] dy.$$

By Lemma 1 one has

$$C_k(\pi_y, \psi) = d \left(\frac{d - 1 + k}{k}\right)^{-1} \text{Tr}[\psi^\otimes k P_{\text{sym}}].$$

Since $\psi^\otimes k$ belongs to the symmetric subspace, the statement immediately follows.

B. Informational measures

In this subsection we recall some basic definitions from classical information theory.

Given two probability densities $p_x$ and $q_y$, the relative entropy $D(p_x \parallel q_y)$ is given by

$$D(p_x \parallel q_y) := \int p_x \log \frac{p_x}{q_y} dx$$
is a non-symmetric measure of the distance between the two densities. Given two random variables $X$ and $Y$ distributed according to probability density $p_{x,y}$, the mutual information $I(X;Y)$ given by

$$I(X;Y) := D(p_{x,y} \parallel p_x p_y),$$

is a measure of their correlation. For any ensemble $p_x$ and POVM $\pi_y$, we denote with $I(p_x, \pi_y)$ the mutual information $I(X;Y)$ between random variables $X$ and $Y$ distributed according to $p_{x,y} = \text{Tr}[p_x \pi_y]$.

The accessible information \[1-4\] of an ensemble $\psi_x$ is a measure of how much information can be extracted from the ensemble.

**Definition 3 (Accessible information).** The accessible information $A(\rho_x)$ of an ensemble $\rho_x$ is the supremum over any POVM $\pi_y$ of the mutual information $I(\rho_x, \pi_y)$, namely

$$A(\rho_x) := \sup_{\pi_y} I(\rho_x, \pi_y).$$

The informational power \[7\] of a POVM measures how much information can be extracted by the POVM.

**Definition 4 (Informational power).** The informational power $W(\pi_y)$ of a POVM $\pi_y$ is the supremum over any ensemble $\rho_x$ of the mutual information $I(\rho_x, \pi_y)$, namely

$$W(\pi_y) := \sup_{\rho_x} I(\rho_x, \pi_y).$$

The following Lemma allows us to recast upper bounds on the informational power into bounds for the accessible information. Therefore in the following without loss of generality we focus on the former problem.

**Theorem 1.** For any POVM $\pi_y$, the informational power $W(\pi_y)$ is given by

$$W(\pi_y) = \sup_{\rho_x} A(\rho_x^{1/2} \pi_y \rho_x^{1/2}).$$

**Proof.** See Refs. \[7,12\]. \(\Box\)

In the following it will be convenient to introduce the shorthand notation $\eta(x) := -x \log x$.

**Theorem 2.** For any $d$-dimensional POVM $\pi_y$ one has

$$W(\pi_y) \leq \log d - \inf_{\psi_x} \int ||\psi_x||^2 \text{Tr}[\pi_y] \eta \left( \frac{\langle \psi_x | \pi_y | \psi_x \rangle}{||\psi_x||^2 \text{Tr}[\pi_y]} \right) dx dy$$

where the infimum is over any ensemble $\psi_x$ of pure states.

**Proof.** A Davies-like theorem applies \[7\], so it is sufficient to maximize over ensembles $\psi_x$ of pure states. For any such ensemble one has

$$I(\psi_x, \pi_y) = D( (\langle \psi_x | \pi_y | \psi_x \rangle | \text{Tr}[\rho_y] | \psi_x \rangle^2 )$$

$$= \int \int \langle \psi_x | \pi_y | \psi_x \rangle \log \frac{\langle \psi_x | \pi_y | \psi_x \rangle}{||\psi_x||^2 \text{Tr}[\pi_y]} dx dy$$

$$= \int \int \langle \psi_x | \pi_y | \psi_x \rangle \left[ \log \frac{d \langle \psi_x | \pi_y | \psi_x \rangle}{||\psi_x||^2 \text{Tr}[\pi_y]} - \log \frac{d \text{Tr}[\rho_y]}{\text{Tr}[\pi_y]} \right] dx dy$$

$$= D( (\langle \psi_x | \pi_y | \psi_x \rangle ||\psi_x||^2 \text{Tr}[\pi_y]) - D( (\text{Tr}[\rho_y] ||\pi_y||^2)$$

$$\leq D(\langle \psi_x | \pi_y | \psi_x \rangle ||\psi_x||^2 \text{Tr}[\pi_y])$$

where the final inequality holds due to the positivity of relative entropy. Then the statement follows by direct inspection. \(\Box\)

**C. Polynomial interpolation**

In this subsection we introduce an optimization technique based on Hermite polynomial interpolation. The following Lemma bounds the error made in interpolating a function with a polynomial.

**Lemma 3.** Let $a$ and $b$ be reals and $t$ be a positive integer. Let $f(x)$ be a real function with continuous derivatives up to order $t+1$ on $[a,b]$. Let $\{x_i\}_{i=1}^m$ be reals such that $a \leq x_i \leq b$ and $x_i < x_{i'}$ for any $i < i'$. Let $\{j_i\}_{i=1}^m$ be positive integers such that $\sum_i j_i = t$. Let $p(x)$ be the polynomial of degree $t$ that agrees with $f(x)$ at $x_i$ up to derivative of order $j_i - 1$ for $1 \leq i \leq m$, namely

$$p^{(j_i)}(x_i) = f^{(j_i)}(x_i), \quad 0 \leq i \leq m.$$  

For any $x \in [a,b]$ there exists $x'$ such that $\min(x, x_1) < x' < \max(x, x_m)$ and

$$f(x) - p(x) = \frac{f^{(t+1)}(x')}{(t+1)!} \prod_{i=1}^m (x - x_i)^{j_i}.$$

**Proof.** See Ref. \[39\]. \(\Box\)

The following Lemma, derived in Ref. \[11\], provides a polynomial lower bound to a function. We reproduce here its proof for completeness.

**Lemma 4.** Let $a$ and $b$ be reals and $t$ be a positive integer. Let $f(x)$ be a real function with continuous derivatives up to order $t+1$ on $[a,b]$ such that
$f^{(j)}(x) < 0$ for even $j$ and $f^{(j)}(x) > 0$ for odd $j$, for any $j > 1$ and $x \in [a, b]$. Let $\{x_i\}_{i=1}^{t/2}$ be reals such that $a < x_1 < b$ and $x_i < x_{i'}$ for any $i < i'$. The polynomial $p(x)$ of degree $t$ such that $p(a) = f(a)$, $p(b) = f(b)$ if $t$ is odd, and

$$p^{(j)}(x_i) = f^{(j)}(x_i), \quad \forall 1 \leq i \leq t/2, \quad j = 0, 1,$$

is such that $p(x) \leq f(x)$ for $x \in [a, b]$.

\textbf{Proof.} See Ref. \[\text{[1]}\]. Let us distinguish two cases. If $t$ is odd then

$$(x-a)(x-b) \prod_{i=1}^{t/2} (x-x_i)^2 \leq 0,$$

for $x \in [a, b]$. If $t$ is even then

$$(x-a) \prod_{i=1}^{t/2} (x-x_i)^2 \geq 0,$$

for $x \in [a, b]$. Then the statement immediately follows from Lemma \[\text{[3]}\].

\section{III. INFORMATIONAL BOUNDS}

\subsection{A. Main result}

The informational power problem is formally the optimization of an entropic function over complex vectors under a normalization constraint, therefore it is unfeasible in the majority of cases. However, in this subsection we recast the informational power problem for $t$-design POVMs into an unconstrained optimization over $\{t/2\}$ real variables $\{x_i\}$.

\textbf{Theorem 3.} The informational power $W(\pi_y)$ of any $d$-dimensional $t$-design POVM $\pi_y$ satisfies

$$W(\pi_y) \leq \log d - d \sum_{k=1}^{t} a_k \left( \frac{d+k-1}{k} \right)^{-1},$$

where $a_k$ are the coefficients of the polynomial $p(x) := \sum_{k=1}^{t} a_k x^k$ such that $p(1) = 0$ if $t$ is odd and

$$p^{(j)}(x_i) = \eta^{(j)}(x_i), \quad \forall 1 \leq i \leq t/2, \quad j = 0, 1,$$

for some choice of $\{x_i\}_{i=1}^{t/2}$ such that $0 < x_i < 1$ and $x_i < x_{i'}$ for any $i < i'$.

\textbf{Proof.} By direct inspection one has

$$\eta^{(j)}(x) = (-)^{j-1}(j-2)! x^{-j+1}, \quad \forall j \geq 2,$$

then $\eta^{(j)}(x) < 0$ for even $j$ and $\eta^{(j)}(x) > 0$ for odd $j$, for any $j > 1$ and $x \in [0, 1]$.

Then by Lemma \[\text{[4]}\] one has that $p(x) \leq \eta(x)$ for $x \in [0, 1]$, and by Theorem \[\text{[2]}\] one has

$$W(\pi_y) \leq \log d - \inf_{\psi_x} \int \left( \frac{||\psi_x||^2}{\text{Tr}[\pi_y]} \right)^k d\psi_x.$$

By Definition \[\text{[2]}\] and Lemma \[\text{[2]}\] one has

$$W(\pi_y) \leq \log d - d \sum_{k=1}^{t} a_k \left( \frac{d+k-1}{k} \right)^{-1} \inf_{\psi_x} \int ||\psi_x||^2 dx,$$

so the statement immediately follows.

\textbf{Remark 2.} Since the $a_k$ depend upon the choice of $\{x_i\}$, the tightest bound provided by Theorem \[\text{[3]}\] is

$$W(\pi_y) \leq \log d - d \sup_{\{x_i\}} \sum_{k=1}^{t} a_k \left( \frac{d+k-1}{n} \right)^{-1}. \quad (1)$$

\subsection{B. Applications}

In this subsection we solve the optimization problem in Eq. \[\text{[1]}\] to derive upper bounds on the informational power of $t$-designs as a function of the dimension $d$, for $t \in [1, 5]$ and $t = \infty$, and asymptotic formulae for $d \to \infty$. The case $t = 1$ coincides with the well-known Holevo \[\text{[2]}\] bound; the case $t = 2$ was already derived in Ref. \[\text{[14]}\]; the case $t = \infty$ coincides with the well-known subentropy bound \[\text{[6]}\].

\textbf{Corollary 1} (Informational power of 1-designs). For any 1-design POVM $\pi_y$, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leq W_1(d)$, with $W_1(d) := \log d$.

\textbf{Proof.} There is actually no optimization in this case since $\{t/2\} = 0$ so the set $\{x_i\}_{i=1}^{t/2}$ is empty. The statement follows by direct inspection.

\textbf{Corollary 2} (Informational power of 2-designs). For any 2-design POVM $\pi_y$, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leq W_2(d)$, with

$$W_2(d) := \log \frac{2d}{d+1}.$$
Proof. The supremum in Eq. (1) is achieved by $x_1 = 2/(d+1)$. Then the statement follows by direct inspection.

**Remark 3.** The limit for $d \to \infty$ of the upper bound in Corollary 3 is given by

$$W_2(d) \to \log 2 \approx 0.693 \text{nat}$$

**Corollary 3** (Informational power of 3-designs). For any 3-design POVM $\pi_y$, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leq W_3(d)$, with

$$W_3(d) := \log \frac{2d}{d+2} + 2 \log \frac{d+2}{d(d+1)}.$$

*Proof.* The supremum in Eq. (1) is achieved by $x_1 = 2/(d+1)$. Then the statement follows by direct inspection.

**Remark 4.** The limit for $d \to \infty$ of the upper bound in Corollary 3 is given by

$$W_3(d) \to \log 2 = 1 \text{ bit} \approx 0.693 \text{ nat}$$

**Corollary 4** (Informational power of 4-designs). For any 4-design POVM $\pi_y$, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leq W_4(d)$, with

$$W_4(d) := \log \frac{6d^2}{(d+2)(d+3)} + \frac{(d-3)\sqrt{3d(d+2)}}{6d(d+1)} \log \frac{2d + 3 - \sqrt{3d+2}}{d+3},$$

Then the statement follows by direct inspection.

**Remark 5.** The limit for $d \to \infty$ of the upper bound in Corollary 4 is given by

$$W_4(d) \to \frac{\log 6}{2} + \frac{\log(2 - \sqrt{3})}{2\sqrt{3}} \approx 0.744 \text{ bit} \approx 0.516 \text{ nat}$$

**Corollary 5** (Informational power of 5-designs). For any 5-design POVM $\pi_y$, the informational power $W(\pi_y)$ is upper bounded by $W(\pi_y) \leq W_5(d)$, with

$$W_5(d) := \log d + \frac{(d-1)(d+3)(d^2 + 2d + 4)}{2d(d+1)^2(d+2)} \log \frac{6}{(d+3)(d+4)} + \frac{\sqrt{d+3(d-1)(d^2 - 2d - 12)}}{2\sqrt{3d(d+1)^2(d+2)}} \log \frac{2d + 5 - \sqrt{3(d+1)(d+3)}}{d+4},$$

Then the statement follows by direct inspection.

**Remark 6.** The limit for $d \to \infty$ of the upper bound in Corollary 5 is given by

$$W_5(d) \to \frac{\log 6}{2} + \frac{\log(2 - \sqrt{3})}{2\sqrt{3}} \approx 0.744 \text{ bit} \approx 0.516 \text{ nat}$$

**Corollary 6.** For the continuous $d$-dimensional $\infty$-design POVM $\pi_y$ the informational power $W(\pi_y)$ is
given by

\[ W(\pi_y) = W_\infty(d) := \log d - \sum_{n=2}^{d} n^{-1}. \]

Proof. By expanding \( \eta(x) \) in Taylor series around 1 and applying the binomial theorem one has

\[ \eta(x) = 1 - x - \sum_{n=2}^{\infty} \sum_{k=0}^{n} (n-2)! \frac{(-x)^k}{k!(n-k)!}. \]

Then by Theorem 2 and Lemma 2 one has

\[ W(\pi_y) \leq \log d - d + 1 + d \sum_{n=2}^{\infty} \sum_{k=0}^{n} \frac{(n-2)!(-)^k}{k!(n-k)!} \left( \frac{d-1+k}{k} \right)^{-1}. \]

Then by direct inspection (see e.g. Ref. [40]) one has

\[ W(\pi_y) \leq \log d - \sum_{n=2}^{d} n^{-1}. \]

Since this bound is saturated by any orthonormal ensemble \([3]\), the statement follows. \(\square\)

Remark 7. The limit for \( d \to \infty \) of \( W_\infty(d) \) in Corollary 2 is given by

\[ W_\infty(d) \to 1 - \gamma \simeq 0.610 \text{bit} \simeq 0.423 \text{nat}, \]

where \( \gamma \) represents the Euler-Mascheroni constant.

We conclude this subsection by summarizing the derived bounds in Table 1 and illustrating them in Fig. 1.

| \( t \) | \( W_t(d) \) | \( \lim_{d \to \infty} \) |
|-------|-------------|------------------|
| 1     | \( \log \frac{d+2}{d+1} \) | \( \infty \) |
| 2     | \( \log \frac{d^2}{d+1} + 2 \log \frac{d}{d+1} \) | \( \log 2 \) |
| 3     | \( \log \frac{2d}{d+1} + 2 \log \frac{d+2}{d+1} \) | \( \log 2 \) |
| 4     | \( \frac{1}{2} \log \frac{6d^2}{2d+1} + \frac{(d-3)\sqrt{3d^2(d+2)}}{2d(d+1)} \log \frac{2d+3+2\sqrt{3d(d+2)}}{2d+3} \) | \( \log \frac{6}{2} + \log(2 - \sqrt{3}) \) |
| 5     | \( \log(d) + \frac{(d-1)(d+3)(d^2+2d+4)}{2d(d+1)^2(d+2)} \log \frac{6}{(d+3)(d+4)} + \frac{\sqrt{2d+7(d-1)(d^2-2d-12)}}{2\sqrt{2d(d+1)^2(d+2)}} \log \frac{2d+5+2\sqrt{3d(d+1)(d+3)}}{2d+4} \) | \( \log \frac{6}{2} + \log(2 - \sqrt{3}) \) |
| \( \infty \) | \( \log d - \sum_{n=2}^{d} n^{-1} \) | \( 1 - \gamma \) |

Table 1. Upper bounds \( W_t(d) \) on the informational power \( W(\pi_y) \) of any \( d \)-dimensional \( t \)-design POVM \( \pi_y \) for \( t \in [1, 5] \) and \( t = \infty \), along with their asymptotic formulae.

C. Tightness

The bound in Theorem 3 is of course tight for \( t = 1 \) for any dimension \( d \), where optimal ensembles are given by any orthonormal basis \([2]\). In this subsection we prove tightness for 2, 3, 5-designs in dimension 2, and for 2-designs in dimension 3. For \( d = 2 \) the Bloch-sphere representation provides a natural isomorphism between 2-dimensional POVMs and solids in \( \mathbb{R}^3 \), so we will denote POVMs by the name of the corresponding solid (tetrahedron, octahedron, icosahedron). Formal definitions of each POVM can be found in \([11]\) \([12]\).

The informational power of the 2-dimensional tetrahedral, octahedral, and icosahedral POVMs were derived in Refs. \([2]\) \([11]\) \([14]\). By noticing that these POVMs are 2-, 3- and 5-designs, respectively, their informational power directly follows from Theorem 3.

Corollary 7. The 2-dimensional tetrahedral (SIC) POVM \( \pi_y \) is a 2-design, its informational power is given by

\[ W_2(2) = \log \frac{4}{3}, \]

and the optimal (anti-tetrahedral) ensemble \( \psi_x \) is such that \( \psi_x \pi_x = 0 \) for any \( x \).

Proof. Any SIC POVM is a 2-design \([17]\), and the anti-tetrahedral ensemble saturates the bound in Corollary 2. \(\square\)
Corollary 8. The 2-dimensional octahedral (complete MUB) POVM is a 3-design, its informational power is given by

\[ W_3(2) = \frac{1}{6} \log 4, \]

and the optimal (anti-octahedral) ensemble \( \psi_x \) is such that \( \psi_x \pi_x = 0 \) for any \( x \).

Proof. It follows by direct inspection that the 2-dimensional octahedral POVM is a 3-design, and the anti-octahedral ensemble saturates the bound in Corollary 3.

\[ \square \]

Corollary 9. The 2-dimensional icosahedral POVM is a 5-design, its informational power is given by

\[ W_5(2) = \log 2 - \frac{5}{12} \log 5 - \frac{\sqrt{3}}{12} \log \frac{9}{6 - 3 \sqrt{5}}, \]

and the optimal (anti-icosahedral) ensemble \( \psi_x \) is such that \( \psi_x \pi_x = 0 \) for any \( x \).

Proof. It follows by direct inspection that the 2-dimensional icosahedral POVM is a 5-design, and the anti-icosahedral ensemble saturates the bound in Corollary 3.

\[ \square \]

In Ref. 13 it was shown that the informational power of group covariant 3-dimensional SIC POVMs is given by \( W_3(3) = \log \frac{4}{9} \). By noticing that these POVMs are 2-designs, the optimality of this value immediately follows from Corollary 2.

IV. CONCLUSION AND OUTLOOK

In this work we provided in Theorem 3 an upper bound on the information that can be carried by any quantum \( t \)-design for any \( t \), as a function of the dimension of the system, and in Corollaries 1, 2, 3, 4, 5, and 6 we derived closed analytic expressions for such bound for \( t \in \{1, 5\} \) and \( t = \infty \). The Holevo upper bound \( \chi \) and the subentropy lower bound \( \chi \) were recovered as particular cases for \( t = 1 \) and \( t = \infty \), respectively. In this sense, the resulting hierarchy of bounds represents a trade-off between the uniformity of a quantum system and the amount of information it can carry. By deriving asymptotic formulae for large dimensions, we also showed that the statistics generated by any \( t \)-design contains no more than a single bit of information, and that this amount decreases with \( t \). Furthermore, in Corollaries 7, 8, and 9 we showed the tightness of our bounds for qubits and qutrits. Finally, as a direct consequence of Theorem 3 it immediately follows that all the presented upper bounds on the informational power of \( t \)-design POVMs holds as upper bounds on the accessible information of \( t \)-design ensembles.

Various open problems related to the accessible information and informational power of quantum \( t \)-designs were discussed in Refs. 12, 14. In view of the results presented here, we may add to the list the following questions. The asymptotic formulae for the bounds on the informational power of 2- and 3-designs (Remarks 3 and 4), as well as those for 4- and 5-designs (Remarks 5 and 6), are pairwise identical. Can this be generalized to higher \( t \)? Can this phenomenon be given a physical interpretation? Moreover, for all the \( t \)-design qubit POVMs \( \pi_y \) we explicitly optimized (Corollaries 7 and 8), the optimal ensemble \( \psi_x \) turned out to be such that \( \psi_x \pi_x = 0 \) for any \( x \). Is it always the case for qubit \( t \)-designs? Finally, closed analytic expressions for the bounds provided by Theorem 3 for \( t \geq 6 \) require lengthy calculations, therefore their derivation can be made easier by the use of a symbolic calculation package. This will be done in a forthcoming work 18, where their tightness and asymptotic formulae will also be discussed.
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[1] D. S. Lebedev, L. B. Levitin, Information and Control 9, 1 (1966).
[2] A. S. Holevo, J. Multivariate Anal. 3, 337 (1973).
[3] V. P. Belavkin, Stochastics 1, 315 (1975).
[4] V. P. Belavkin, Radio Engineering and Electronic Physics 20, 39 (1975).
[5] E. B. Davies, IEEE Trans. Inf. Theory 24, 596 (1978).
[6] R. Jozsa, D. Robb, and W. K Wootters, Phys. Rev. A 49, 668 (1994).
[7] M. Dall’Arno, G. M. D’Ariano, and M. F. Sacchi, Phys. Rev. A 83, 062304 (2011).
[8] O. Oreshkov, J. Calsamiglia, R. Muñoz-Tapia, and E. Bagan, New J. Phys. 13, 073032 (2011).
[9] A. S. Holevo, Problems of Information Transmission 48, 1 (2012).
[10] A. S. Holevo, Phys. Scr. 2013, 014034 (2013).
[11] W. Słomczyński and A. Szymusiak, arXiv:1402.0375.
[12] M. Dall’Arno, F. Buscemi, and M. Ozawa, J. Phys. A: Math. Theor. 47, 235302 (2014).
[13] A. Szymusiak, J. Phys. A: Math. Theor. 47, 445301 (2014).
[14] M. Dall’Arno, Phys. Rev. A 90, 052311 (2014).
[15] M. Dall’Arno, in preparation.
[16] G. Zauner, Quantendesigns Grundzuge einer nichtkommutativen Designtheorie. Dissertation, Universität Wien, 1999.
[17] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. 45, 2171 (2004).
[18] A. Ambainis and J. Emerson, “Quantum t-designs: t-wise independence in the quantum world,” in Proceedings of the Twenty-Second Annual IEEE Conference on Computational Complexity, 129 (2007).
[19] A. J. Scott and M. Grassl, Journal of Mathematical Physics 51, 042203 (2010).
[20] A. Klappenecker and M. Roetteler, Proceedings 2005 IEEE International Symposium on Information Theory (ISIT 2005), 1740 (2005).
[21] S. Brierley, S. Weigert, and I. Bengtsson, Quantum Info. & Comp. 10, 0803 (2010).
[22] G. Mauro D’Ariano, M. De Laurentis, M. G. A. Paris, A. Porzio, S. Solimeno, J. Opt. B: Quantum Semiclass. Opt. 4, 127 (2002).
[23] C. H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, in Proceedings of IEEE International Conference on Computers, Systems and Signal Processing 175, 8 (1984).
[24] D. P. DiVincenzo, M. Horodecki, D. W. Leung, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. 92, 067902 (2004).
[25] C. A. Fuchs and M. Sasaki, Quantum Inf. & Comput. 3, 377 (2003).
[26] C. A. Fuchs, Quantum Inf. & Comput. 4, 467 (2004).
[27] J. Sánchez-Ruiz, Phys. Lett. A 201, 125 (1995).
[28] M. A. Ballester and S. Wehner, Phys. Rev. A 75, 022319 (2007).
[29] S. Wehner and A. Winter, New J. Phys. 12, 025009 (2010).
[30] I. Bialynicki-Birula and L. Rudnicki, Statistical Complexity: Applications in Electronic Structure, Ed. K. D. Sen, (Springer, U.K., 2011), chapter 1.
[31] F. Buscemi, M. J. W. Hall, M. Ozawa, and M. M. Wilde, Phys. Rev. Lett. 112, 050401 (2014).
[32] A. E. Rastegin, Eur. Phys. J. D 67, 269 (2013).
[33] C. A. Fuchs and R. Schack, Rev. Mod. Phys. 85, 1693 (2013).
[34] C. A. Fuchs and R. Schack, Foundations of Physics 41, 345 (2011).
[35] D. M. Appleby, Å. Ericsson, and C. A. Fuchs, Foundations of Physics 41, 564 (2011).
[36] C. Fuchs, arXiv:1207.2141
[37] I. L. Chuang and M. A. Nielsen, Quantum Information and Communication (Cambridge, Cambridge University Press, 2000).
[38] T. M. Cover and J. A. Thomas, Elements of Information Theory (Hoboken, Wiley-Interscience, 2006).
[39] J. Stoer and R. Bulirsch, Introduction to numerical analysis (Springer, New York, 2002).
[40] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, C. W. Clark, NIST Handbook of Mathematical Functions, (Cambridge, Cambridge University Press, 2010).