Casimir stress on lossy magnetodielectric spheres

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An expression for the Casimir stress on arbitrary dispersive and lossy linear magnetodielectric matter at finite temperature, including left-handed material, is derived and applied to spherical systems. To cast the relevant part of the scattering Green tensor for a general magnetodielectric sphere in a convenient form, classical Mie scattering is reformulated.

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I. INTRODUCTION

Spheres and spherical shells display enough symmetry to allow for a quite explicit solution of the Casimir problem (for the earliest calculation, see [1]). Especially two intriguing but nowadays obsolete ideas have substantially stimulated the study of spherically symmetric setups. Firstly, the idea that the Casimir effect might be related to sonoluminescence [4, 5], i.e., the emission of light flashes by small air bubbles in water under the action of ultrasonic waves. Unfortunately, in the available work on the topic (see, e.g., [6, 7, 8, 9, 10, 11]) lossless materials are assumed, dispersion is ignored, or the influence of matter is summarized in perfect conductor boundary conditions [12, 13]. Needless to say that such approximations should be expected to distort the physical picture more or less drastically, not to mention their tendency to cause ‘unreal’ divergence problems. In what follows, we allow for dispersing and absorbing (causal) sphere material characterized by both a complex frequency-dependent permittivity and a complex frequency-dependent permeability, thus including in the calculations also left-handed materials, which have been of increasing interest. To our knowledge, Casimir forces on magnetodielectric bodies have been studied only for non-dispersing and non-absorbing material [2], commonly subject to the condition that the product of the permittivity and the permeability is also uniform in space [8, 9, 10, 11]. By using the quantization scheme given in Ref. [14] in our calculations, we first extend the recently derived basic formula [15] for the Casimir force for causal dielectric matter to causal magnetodielectric matter. We then apply the theory to spherical structures.

II. QUANTIZATION SCHEME

Within the mentioned quantization scheme, the macroscopic (medium-assisted) electromagnetic field operators are all expressed in terms of suitable bosonic basic fields via the classical Green tensor, which satisfies the equation

\[ \nabla \times \kappa(\mathbf{r}, \omega) \nabla \times G(\mathbf{r}, \mathbf{r}', \omega) - \frac{\omega^2}{c^2} \varepsilon(\mathbf{r}, \omega) G(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r}, \mathbf{r}') \]

(\( \varepsilon \) – complex permittivity, \( \kappa = \mu^{-1} \) – complex reciprocal permeability) and the boundary condition at infinity. In the absence of additional charges and currents, the positive-frequency components of the electric field operator are then given by

\[ \hat{E}(\mathbf{r}, \omega) = i \mu_0 \omega \int d^3 r' G(\mathbf{r}, \mathbf{r}', \omega) \hat{j}_N(\mathbf{r}, \omega), \]

(2)

from which the other field operators can be derived by using Faraday’s law

\[ \hat{B}(\mathbf{r}, \omega) = (i \omega)^{-1} \nabla \times \hat{E}(\mathbf{r}, \omega) \]

(3)

and the constitutive relations [\( \kappa_0 = \mu_0^{-1} \)]

\[ \hat{D}(\mathbf{r}, \omega) = \varepsilon_0 \varepsilon(\mathbf{r}, \omega) \hat{E}(\mathbf{r}, \omega) + \hat{P}_N(\mathbf{r}, \omega), \]

(4)

\[ \hat{H}(\mathbf{r}, \omega) = \kappa_0 \kappa(\mathbf{r}, \omega) \hat{B}(\mathbf{r}, \omega) - \hat{M}_N(\mathbf{r}, \omega). \]

(5)

In the above, the frequency-domain Langevin noise quantities (carrying an index \( N \)) are connected with the fundamental bosonic fields \( \hat{f}_\lambda(\mathbf{r}, \omega) \) (\( \lambda = \epsilon, m \)) by the relations

\[ \hat{j}_N(\mathbf{r}, \omega) = -i \omega \hat{P}_N(\mathbf{r}, \omega) + \nabla \times \hat{M}_N(\mathbf{r}, \omega), \]

(6)

\[ \hat{P}_N(\mathbf{r}, \omega) = i [\hbar \varepsilon \Im \varepsilon(\mathbf{r}, \omega)/\pi]^{1/2} \hat{f}_\epsilon(\mathbf{r}, \omega), \]

(7)

\[ \hat{M}_N(\mathbf{r}, \omega) = [-i \hbar \kappa_0 \kappa(\mathbf{r}, \omega)/\pi]^{1/2} \hat{f}_m(\mathbf{r}, \omega). \]

As a consequence, the electromagnetic field operators satisfy the usual commutation relations of QED. For example, the electric field operator (in the Schrödinger picture) reads as

\[ \hat{E}(\mathbf{r}) = \int_0^\infty d\omega \hat{E}(\mathbf{r}, \omega) + \text{H.c.}, \]

(8)

with \( \hat{E}(\mathbf{r}, \omega) \) from Eq. [2] together with Eqs. [5]–[7]. Writing down the corresponding formulas for the other field operators is straightforward. The consistency of the method relies heavily on the facts that the Green tensor, which is a response function like \( \varepsilon(\mathbf{r}, \omega) \) and \( \kappa(\mathbf{r}, \omega) \), is holomorphic in the upper \( \omega \) half-plane, obeys the reciprocity relation

\[ G(\mathbf{r}, \mathbf{r}', \omega) = G^T(\mathbf{r}', \mathbf{r}, \omega) \]

(9)
(the superscript T denotes matrix transposition), and the integral relation
\[
\int d^3s \left\{ |G(r, s, \omega)| \Im(\kappa(s, \omega)|[\nabla_s \times G^*(s, r', \omega)] + \frac{\omega^2}{c^2} G(r, s, \omega)| \Im(\varepsilon(s, \omega)|G^*(s, r', \omega)) \right\} = \Im G(r, r') \omega^2
\]
(10)

(for details, see [14]). Note that the convention
\[
G(r, r', \omega) \times \nabla' = -\nabla' \times G^T(r, r', \omega)
\]
(11)

has been used.

III. CASIMIR STRESS TENSOR

For a given quantum state, the stress tensor, i.e., the (yet unrenormalized) Casimir stress, can be obtained from the correlation function
\[
T(r, r') = \langle \mathbf{D}(r) \otimes \mathbf{E}(r') \rangle + \langle \mathbf{B}(r) \otimes \mathbf{H}(r') \rangle - \frac{i}{2} \text{Tr} \left[ \langle \mathbf{D}(r) \otimes \mathbf{E}(r') \rangle + \langle \mathbf{B}(r) \otimes \mathbf{H}(r') \rangle \right]
\]
(12)
in the coincidence limit \( r' \to r \). In carrying out this limit, we have to drop – in accordance with the Casimir effect’s very definition – the bulk part of the Green tensor, which is singular but independent of geometry [22], to get a physical (finite) value of the Casimir force per unit area. Note that the resulting force formula rigorously applies only to points outside matter, which is the case under consideration.

Let us calculate the thermal-equilibrium Casimir force at temperature \( T \). Recalling the bosonic character of the fundamental fields \( \mathbf{f}_l(r, \omega) \) and assuming them to be excited in thermal states, one quickly finds, in close analogy to Ref. [15], that
\[
\langle \mathbf{f}_l(r, \omega) \otimes \mathbf{f}_l'(r', \omega') \rangle = \frac{1}{2} \left[ \coth \left( \frac{\hbar \omega}{2k_B T} \right) + 1 \right] \delta_{\lambda \lambda'} \delta(\omega - \omega') \delta(r, r'),
\]
(13)
\[
\langle \mathbf{f}^\dagger_l(r, \omega) \otimes \mathbf{f}^\dagger_{l'}(r', \omega') \rangle = \frac{1}{2} \left[ \coth \left( \frac{\hbar \omega}{2k_B T} \right) - 1 \right] \delta_{\lambda \lambda'} \delta(\omega - \omega') \delta(r, r'),
\]
(14)
\[
\langle \mathbf{f}_l(r, \omega) \otimes \mathbf{f}_{l'}(r', \omega') \rangle = \langle \mathbf{f}^\dagger_l(r, \omega) \otimes \mathbf{f}^\dagger_{l'}(r', \omega') \rangle = 0.
\]
(15)
By using Eq. (8) for the electric field and the related equations for the other fields [together with Eqs. (23)–(27)] and Eqs. (13)–(15), and by employing the Green tensor properties [5] and [10], it follows by a cumbersome but straightforward calculation that Eq. (12) may be rewritten as
\[
T(r, r') = \Theta(r, r') - \frac{1}{2} I \text{Tr} \Theta(r, r'),
\]
(16)
where
\[
\Theta(r, r') = \frac{\hbar}{\pi} \int_0^\infty d\omega \coth \left( \frac{\hbar \omega}{2k_B T} \right)
\times \Im \left[ \frac{\omega^2}{c^2} \varepsilon(r, \omega) G(r, r') \omega^2
\right.
\left. - \left\{ \nabla \times G(r, r', \omega) \times \nabla' \right\} \kappa(r, \omega) \right].
\]
(17)
Note that this result differs – apart from the different analytical form of the Green tensor – from the ‘non-magnetic’ result [15] in the permeability that appears in the second term in the curly brackets. By replacing the full Green tensor with its scattering part, \( G(r, r', \omega) \to G_{scat}(r, r', \omega) \), and taking the coincidence limit \( r' \to r \), Eq. (16) together with Eq. (17) is the sought-after basic formula for the Casimir force per unit area for an arbitrary arrangement of lossy magnetodielectric bodies. With the bulk Green tensor removed and application of the standard rule \( \omega \to \omega + i0 \) to handle free-space regions, no divergence problems arise in performing the integral in Eq. (17).

IV. SPHERE GREEN TENSOR

To apply the theory to spheres, we need a convenient expression for the sphere scattering Green tensor – a purely classical problem, which was first treated by Mie [16] and is now standard textbook material (see, e.g., [17, 18]). Here, we give a formulation that is valid for arbitrary complex permittivities and permeabilities that vary radially in a stepwise fashion. (Note that amplifying materials are in fact also allowed.) The required techniques can be gathered from the mentioned volumes, but see also [10, 21, 22]. According to the method of Debye’s potentials, transverse solutions of the vector wave equation
\[
\nabla \times \nabla \times \mathbf{F}(r) - k^2 \mathbf{F}(r) = 0
\]
(18)
are given, for each Debye potential \( u \) that solves the scalar Helmholtz equation \( \Delta + k^2 u = 0 \), by vector functions \( \mathbf{F} = \nabla \times \mathbf{u} \) and \( \mathbf{F} = \nabla \times \nabla \times \mathbf{u} \). In addition, all longitudinal vector functions \( \mathbf{F} = \nabla \times \mathbf{u} \) solve Eq. (18) for \( k = 0 \). Note that the free-space eigenfunctions of the (semi-bounded) operator \( \nabla \times \nabla \times \) obtained from Eq. (18) under the restriction that they be bounded everywhere (including infinity) have real \( k \) and are complete [21]. In free space, an orthonormalized and complete set of scalar (continuum) eigenfunctions \( u \) reads
\[
u_{lm}(r; k) = \sqrt{\frac{2}{\pi}} k_j(k r) Y_{lm}(\theta, \phi) \quad (k \geq 0),
\]
(19)
with \( l = 0, 1, 2, \ldots, m = -l, \ldots, l \) (\( j_l \) — spherical Bessel function, \( Y_{lm} \) — spherical harmonic \( \{23\} \)). Employing them as Debye potentials and introducing appropriate normalization factors yields the orthonormalized vector eigenfunctions as

\[
L_{lm}(\mathbf{r}; k) = \frac{1}{k} \nabla u_{lm}(\mathbf{r}; k) = \sqrt{\frac{2}{\pi}} [j_l(kr) \mathbf{r} Y_{lm}(\theta, \phi)/r + j_l(kr) \nabla Y_{lm}(\theta, \phi)], \tag{20}
\]

\[
M_{lm}(\mathbf{r}; k) = \left[ l(l+1) \right]^{-\frac{1}{2}} \nabla \times \mathbf{r} u_{lm}(\mathbf{r}; k) = -\sqrt{\frac{2}{\pi l(l+1)}} k j_l(kr) \nabla Y_{lm}(\theta, \phi), \tag{21}
\]

\[
N_{lm}(\mathbf{r}; k) = \frac{1}{k} \nabla \times M_{lm}(\mathbf{r}; k) = \sqrt{\frac{2}{\pi l(l+1)}} \left[ l(l+1) j_l(kr) \mathbf{r} Y_{lm}(\theta, \phi)/r^2 \right. \\
+ \left. [kr j_l(kr) + j_l(kr)] \nabla Y_{lm}(\theta, \phi) \right], \tag{22}
\]

where for the \( \mathbf{M} \) and \( \mathbf{N} \) functions the value \( l = 0 \) has to be excluded from consideration. Throughout the literature it is invariably assumed that the Debye potential method does not miss any eigenfunctions. We also do so and assume the completeness relations

\[
\sum_{lm} \int_0^\infty dk \, \mathbf{L}_{lm}(\mathbf{r}; k) \otimes \mathbf{L}_{lm}^\ast(\mathbf{r}'; k) = \delta^\parallel(\mathbf{r}, \mathbf{r}'), \tag{23}
\]

\[
\sum_{lm} \int_0^\infty dk \, [\mathbf{M}_{lm}(\mathbf{r}; k) \otimes \mathbf{M}_{lm}^\ast(\mathbf{r}'; k) \\
+ \mathbf{N}_{lm}(\mathbf{r}; k) \otimes \mathbf{N}_{lm}^\ast(\mathbf{r}'; k)] = \delta^\perp(\mathbf{r}, \mathbf{r}'), \tag{24}
\]

where the primed sum begins with \( l = 1 \). Note that the \( \mathbf{M} \) and \( \mathbf{N} \) functions represent, respectively, TE and TM (to \( \mathbf{r} \)) partial waves, and \( \mathbf{r} u \) plays the role of a Hertz vector.

To find the sphere scattering Green tensor, we will make use of the vector functions just constructed. For this purpose, let us first consider wave propagation in a homogeneous, isotropic bulk material, so that Eq. \( \{11\} \) simplifies to

\[
\nabla \times \nabla \times \mathbf{G}^{\text{(bulk)}}(\mathbf{r}, \mathbf{r}', \omega) - k^2(\omega) \mathbf{G}^{\text{(bulk)}}(\mathbf{r}, \mathbf{r}', \omega) = \mu(\omega) \delta(\mathbf{r}, \mathbf{r}'), \tag{25}
\]

with arbitrary complex

\[
k^2(\omega) = \frac{\omega^2}{c^2} \varepsilon(\omega) \mu(\omega). \tag{26}
\]

The bulk-material Green tensor can then be expressed straightforwardly in terms of the vector functions \( \{24\} \) according to

\[
\mathbf{G}^{\text{(bulk)}}(\mathbf{r}, \mathbf{r}', \omega) \mu^{-1}(\omega) = -\frac{\delta(\mathbf{r}, \mathbf{r}')}{k^2(\omega)} \\
+ \sum_{lm} \int_0^\infty \frac{dk'}{k'^2 - k^2(\omega)} \left[ \mathbf{M}_{lm}(\mathbf{r}; k) \otimes \mathbf{M}_{lm}^\ast(\mathbf{r}'; k') \\
+ \mathbf{N}_{lm}(\mathbf{r}; k) \otimes \mathbf{N}_{lm}^\ast(\mathbf{r}'; k') \right], \tag{27}
\]

and the integral can be evaluated by means of the residue theorem (with due care of ‘static’ poles in the \( \mathbf{N} \otimes \mathbf{N}^* \) terms \( \{17\} \) to obtain \( k = k(\omega) \)

\[
\mathbf{G}^{\text{(bulk)}}(\mathbf{r}, \mathbf{r}', \omega) \mu^{-1}(\omega) = -\frac{\varepsilon_{r} \otimes \varepsilon_{r}}{k^2} \delta(\mathbf{r}, \mathbf{r}') + \frac{i \pi}{2k} \sum_{lm} \left\{ \left[ \mathbf{M}_{lm}(\mathbf{r}; k) \otimes \mathbf{M}_{lm}^\ast(\mathbf{r}'; k) + \mathbf{N}_{lm}(\mathbf{r}; k) \otimes \mathbf{N}_{lm}^\ast(\mathbf{r}'; k) \right] \theta(\mathbf{r} - \mathbf{r}') \\
+ \left[ \mathbf{M}_{lm}^\ast(\mathbf{r}; k) \otimes \mathbf{M}_{lm}(\mathbf{r}'; k) + \mathbf{N}_{lm}^\ast(\mathbf{r}; k) \otimes \mathbf{N}_{lm}(\mathbf{r}'; k) \right] \theta(\mathbf{r}' - \mathbf{r}) \right\}. \tag{28}
\]

Here, the ‘bullet’ symbol (*) denotes complex conjugation of the \( k \)-independent factors only, and the tilde symbol (') means that in the definitions \( \{24\} \) and \( \{22\} \) the spherical Bessel functions \( j_l \) have to be replaced with the ‘outgoing’ and ‘incoming’ spherical Hankel functions \( h_l' = h_l^{(1)} \) and \( h_l'' = h_l^{(2)} \) for \( \text{Im} \, k > 0 \) and \( \text{Im} \, k < 0 \), respectively, to ensure amplitude decay as \( |\mathbf{r} - \mathbf{r}'| \to \infty \).

The advantage of this notation is that the sign in \( k = \pm \omega \sqrt{\varepsilon \mu/c} \) can be chosen freely. In this way, the formulas obtained for the Green tensor apply also to left-handed and amplifying material.

The terms within the curly brackets in Eq. \( \{28\} \) correspond to TE and TM partial waves that are solutions to the homogeneous version of the differential equation \( \{23\} \) with the dispersion relation \( \{20\} \). In a spherically layered medium (in contrast to the bulk material con-
sidered so far), additional waves that arise from reflection at and transmission through layer interfaces must be taken into account. They just form the scattering part of the Green tensor that we are interested in. The total field must satisfy the well-known continuity conditions at the layer interfaces, the required amplitude matching can be done for each partial wave separately. In fact, only the solutions \( \sim k S_l(\mathbf{kr}) \) of the radial Helmholtz equation \[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - (k^2 - (l+1)^2) S_l(\mathbf{kr}) = 0 \] are involved in this matching process. To be more specific, \( S_l(\mathbf{kr}) \) must be a (bounded) superposition of the different kinds of spherical Bessel functions (for complex \( k \)) in every layer such that

- for TE waves continuity of
  \[ kr S_l(\mathbf{kr}) \quad \text{and} \quad \mu^{-1} k \frac{\partial}{\partial r} [r S_l(\mathbf{kr})] \quad (29) \]
- and for TM waves continuity of
  \[ \varepsilon kr S_l(\mathbf{kr}) \quad \text{and} \quad k \frac{\partial}{\partial r} [r S_l(\mathbf{kr})] \quad (30) \]

is ensured at spherical interfaces \( r = \text{const.} \).

The simplest case is a homogeneous sphere (index 1) of radius \( R \) in a homogeneous environment (index 2). If both \( \mathbf{r} \) and \( \mathbf{r}' \) are outside the sphere and \( R < r < r' \) is valid, it can be concluded from the above mentioned continuity conditions that the reflection coefficients related to the terms with \( \theta(r'-r) \) in Eq. (28) are given by

\[ r_{l,21}^{\text{TE}} = -\frac{\mu_1(k_1 R)}{\mu_2} \frac{\partial}{\partial \mathbf{r}} [R(j_l(k_1 R))], \quad \frac{\partial}{\partial R} \left[ R(j_l(k_1 R)) \right] \quad (31) \]

and

\[ r_{l,21}^{\text{TM}} = -\frac{\mu_1(k_1 R)}{\varepsilon_2} \frac{\partial}{\partial \mathbf{r}} [R(j_l(k_1 R))], \quad \frac{\partial}{\partial R} \left[ R(j_l(k_1 R)) \right] \quad (32) \]

for TE and TM waves, respectively. Introducing the definition

\[ \mathcal{D}[j_l(k_1 R), h_l(k_2 R); \alpha] = \det \left[ \begin{array}{cc} j_l(k_1 R) & h_l(k_2 R) \\ \alpha_1^{-1} \frac{\partial}{\partial R} j_l(k_1 R) & \alpha_2^{2} \frac{\partial}{\partial R} h_l(k_2 R) \end{array} \right], \quad (33) \]

we may write Eqs. (31) and (32) in the compact form of

\[ r_{l,21}^{\text{TE}} = -\frac{\mathcal{D}[R(j_l(k_1 R), R(j_l(k_2 R); \mu)]}{\mathcal{D}[R(j_l(k_1 R), R(h_l(k_2 R); \mu)]} \quad (34) \]

and

\[ r_{l,21}^{\text{TM}} = -\frac{\mathcal{D}[R(j_l(k_1 R), R(j_l(k_2 R); \varepsilon)]}{\mathcal{D}[R(j_l(k_1 R), R(h_l(k_2 R); \varepsilon)]} \quad (35) \]

respectively. Note that the so-called Mie resonances of the sphere can be found by studying these expressions in the complex \( \omega \) plane. The contributions to the scattering Green tensor for the case \( r > r' > R \) [related to the terms with \( \theta(r'-r) \) in Eq. (28)] need not be figured out separately, but may also be found from the condition that the scattering Green tensor has to be continuously differentiable at \( r = r' \) (since the bulk part alone accounts for the necessary singularity). The scattering Green tensor can therefore be given, for the case that both arguments \( r \) and \( r' \) are outside the sphere, in the form of

\[ G^{(22)}(r, r', \omega) = \frac{i\pi \mu_2}{2k_2} \sum_{l m} \left[ r_{l,21}^{\text{TE}} \hat{M}_{lm}(\mathbf{r}; k_2) \otimes \hat{M}^{*}_{lm}(\mathbf{r}'; k_2) \right] + r_{l,21}^{\text{TM}} \hat{N}_{lm}(\mathbf{r}; k_2) \otimes \hat{N}^{*}_{lm}(\mathbf{r}'; k_2). \quad (36) \]

By construction, \( G^{(22)}(r, r', \omega) \) is continuously differentiable at \( r = r' \), vanishes for \( |r| \to \infty \) and/or \( |r'| \to \infty \), and satisfies the condition of reciprocity. Note that in fact \( G^{(22)*} = G^{(22)} \), because of the addition theorem

\[ \sum_{m=-l}^{l} Y_{lm}(\theta, \phi) Y_{lm}^{*}(\theta', \phi') = \frac{2l+1}{4\pi} P_l(\cos \psi) \in \mathbb{R} \quad (37) \]

of the spherical harmonics (\( \psi \) is the angle between the primed and unprimed directions). It is worth noting that the form of Eq. (36) does not change when the sphere is not homogeneous but consists of radial layers. Clearly, the reflection coefficients [34] and [35] must then be replaced by generalized ones that take ‘subsurface’ reflections into account. These can be calculated recursively by standard methods [17], on the basis of the matching conditions [29] and [30].

\[ r_{l,21}^{\text{TE}} = \frac{\mathcal{D}[R(j_l(k_1 R), R(j_l(k_2 R); \mu)]}{\mathcal{D}[R(j_l(k_1 R), R(h_l(k_2 R); \mu)]} \quad (34) \]

and

\[ r_{l,21}^{\text{TM}} = \frac{\mathcal{D}[R(j_l(k_1 R), R(j_l(k_2 R); \varepsilon)]}{\mathcal{D}[R(j_l(k_1 R), R(h_l(k_2 R); \varepsilon)]} \quad (35) \]

V. CASIMIR STRESS ON A SPHERE

With Eq. (36) at hand, we can now evaluate the stress tensor according to Eqs. (16) and (17) [we set \( G \mapsto G^{\text{scat}} \mapsto G^{(22)} \), to approach the sphere surface from outside], on using the symmetry between the \( M \) and \( N \) functions and assuming free space around the sphere, i.e., \( \varepsilon_2 = \mu_2 = 1, k_2 = \omega/c + i0 \):

\[ \Theta(r, r) = \Theta'(r, r) - \frac{1}{2} \text{Tr} \, \Theta(r, r), \quad (38) \]

\[ \Theta(r, r) = \frac{\hbar}{2c} \int_0^\infty d\omega \, \omega \coth \left( \frac{\omega}{2k_2 T} \right) \text{Re} \sum_{l m} \left( r_{l,21}^{\text{TE}} + r_{l,21}^{\text{TM}} \right) \times \left( \hat{M}_{lm}(\mathbf{r}; k_2) \otimes \hat{M}^{*}_{lm}(\mathbf{r}; k_2) + \hat{N}_{lm}(\mathbf{r}; k_2) \otimes \hat{N}^{*}_{lm}(\mathbf{r}; k_2) \right). \quad (39) \]

By symmetry, the relevant stress tensor element for the spherical setup is

\[ \mathbf{e}_r \cdot \mathbf{T} \mathbf{e}_r = \mathbf{e}_r \cdot \Theta \mathbf{e}_r - \text{Tr} / 2 \quad (40) \]
and \( \mathbf{e}_r = \mathbf{r}/r \). By taking into account that the addition theorem \( ^{[17]} \) implies the relation
\[
\sum_{m=-l}^{l} |Y_{lm}(\theta, \phi)|^2 = \frac{2l+1}{4\pi},
\]
and making use of Eqs. \( ^{[21]} \) and \( ^{[22]} \) it is not difficult to see that the relations
\[
\sum_{m} \mathbf{e}_r \tilde{N}_{lm}(r; k) \otimes \tilde{M}_{lm}^*(r; k) \mathbf{e}_r = 0, \tag{42}
\]
\[
\sum_{m} \mathbf{e}_r \tilde{N}_{lm}(r; k) \otimes \tilde{N}_{lm}^*(r; k) \mathbf{e}_r = (2l+1) \frac{\hbar^2(kr)}{2\pi^2} \frac{l(l+1)}{r^2}, \tag{43}
\]
are valid. Substituting Eqs. \( ^{[12]}-^{[14]} \) in Eq. \( ^{[39]} \), we obtain from Eq. \( ^{[40]} \) the Casimir force per unit area on the surface of a sphere of radius \( R \):
\[
T_{RR} = \frac{\hbar}{8\pi^2c} \int_0^\infty d\omega \omega \coth\left( \frac{\hbar\omega}{2k_BT} \right) \times \text{Re} \sum_{l=1}^{\infty} (2l+1) \left( r_{l,21}^{TE} + r_{l,21}^{TM} \right) F_l, \tag{46}
\]
where
\[
F_l = \frac{(l+1)}{R^2} \left( \hat{h}_l^2(k_2R) - k_2^2 \hat{h}_l^2(k_2R) \right)
- \frac{1}{R^2} \left[ k_2 \hat{h}_l'(k_2R) + \hat{h}_l(k_2R) \right]^2. \tag{47}
\]
Note that the frequency integral can be further processed by the standard contour deformation to the imaginary frequency axis, thereby leading to a sum over residues (Matsubara frequencies) when \( T > 0 \).

VI. SUMMARY

We have derived a general expression [Eqs. \( ^{[10]} \) and \( ^{[17]} \)] for the Casimir stress that is valid for arbitrary disperse- ing and absorbing magneto-dielectric material, and have evaluated it for (layered) spheres. In particular, Eq. \( ^{[40]} \) [together with Eq. \( ^{[17]} \)] is applicable to spheres made of left-handed material, where interesting results can be expected when the relevant Mie resonances correspond to spectral regions in which the material behaves left-handed. Since the outlined Green tensor construction also allows for linear amplification, the theory of Casimir forces may also be extended to (linearly) amplifying bodies. For this purpose, the calculations can again be based on the quantization scheme given in Ref. \( ^{[14]} \). If appropriate modifications (of the definitions of the fundamental bosonic fields) are taken into account. Such modifications will effectively add to Eq. \( ^{[17]} \) another integral, over those frequency intervals and spatial regions which can contribute to the amplification.

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[23] We employ standard definitions of these special functions. For definiteness, we refer to Ref. [19].