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Self-avoiding walks and amenability. (English) Zbl 1376.05068
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Summary: The connective constant $\mu(G)$ of an infinite transitive graph $G$ is the exponential growth rate of the number of self-avoiding walks from a given origin. The relationship between connective constants and amenability is explored in the current work.

Various properties of connective constants depend on the existence of so-called ‘unimodular graph height functions’, namely: (i) whether $\mu(G)$ is a local function on certain graphs derived from $G$, (ii) the equality of $\mu(G)$ and the asymptotic growth rate of bridges, and (iii) whether there exists a terminating algorithm for approximating $\mu(G)$ to a given degree of accuracy.

In the context of amenable groups, it is proved that the Cayley graphs of infinite, finitely generated, elementary amenable (and, more generally, virtually indicable) groups support unimodular graph height functions, which are in addition harmonic. In contrast, the Cayley graph of the Grigorchuk group, which is amenable but not elementary amenable, does not have a graph height function.

In the context of non-amenable, transitive graphs, a lower bound is presented for the connective constant in terms of the spectral bottom of the graph. This is a strengthening of an earlier result of the same authors. Secondly, using a percolation inequality of A. Benjamini et al. [Probab. Theory Relat. Fields 149, No. 1–2, 261–269 (2011; Zbl 1230.60099)], it is explained that the connective constant of a non-amenable, transitive graph with large girth is close to that of a regular tree. Examples are given of non-amenable groups without graph height functions, of which one is the Higman group.

The emphasis of the work is upon the structure of Cayley graphs, rather than upon the algebraic properties of the underlying groups. New methods are needed since a Cayley graph generally possesses automorphisms beyond those arising through the action of the group.

MSC:
05C25 Graphs and abstract algebra (groups, rings, fields, etc.)
05C30 Enumeration in graph theory
20F65 Geometric group theory
60K35 Interacting random processes; statistical mechanics type models; percolation theory
82B20 Lattice systems (Ising, dimer, Potts, etc.) and systems on graphs arising in equilibrium statistical mechanics

Keywords:
self-avoiding walk; connective constant; Cayley graph; amenable group; elementary amenable group; indicable group; Grigorchuk group; Higman group; Baumslag-Solitar group; graph height function; group height function; harmonic function; unimodularity; spectral radius; spectral bottom

Full Text: arXiv Link

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