A Proof for Delta Conjecture

Pedro Díaz Navarro

Junio, 2018

Abstract

By finding orthogonal representation for a family of simple connected called \(\delta\)-graphs it is possible to show that \(\delta\)-graphs satisfy delta conjecture. An extension of the argument to graphs of the form \(P_{\Delta(G)+2} \sqcup G\) where \(P_{\Delta(G)+2}\) is a path and \(G\) is a simple connected graph it is possible to find an orthogonal representation of \(P_{\Delta(G)+2} \sqcup G\) in \(\mathbb{R}^{\Delta(G)+1}\). As a consequence we prove delta conjecture.

Key words: delta conjecture, simple connected graphs, minimum semidefinite rank, \(\delta\)-graph, \(C-\delta\) graphs, orthogonal representation.

DOI: 05C50, 05C76, 05C85, 68R05, 65F99, 97K30.

1 Introduction

A graph \(G\) consists of a set of vertices \(V(G) = \{1, 2, \ldots, n\}\) and a set of edges \(E(G)\), where an edge is defined to be an unordered pair of vertices. The order of \(G\), denoted \(|G|\), is the cardinality of \(V(G)\). A graph is simple if it has no multiple edges or loops. The complement of a graph \(G(V, E)\) is the graph \(\overline{G} = (V, \overline{E})\), where \(\overline{E}\) consists of all those edges of the complete graph \(K|G|\) that are not in \(E\).

A matrix \(A = [a_{ij}]\) is combinatorially symmetric when \(a_{ij} = 0\) if and only if \(a_{ji} = 0\). We say that \(G(A)\) is the graph of a combinatorially symmetric matrix \(A = [a_{ij}]\) if \(V = \{1, 2, \ldots, n\}\) and \(E = \{\{i, j\} : a_{ij} \neq 0\} \). The main diagonal entries of \(A\) play no role in determining \(G\). Define \(S(G, \mathbb{F})\) as the set of all \(n \times n\) matrices that are real symmetric if \(\mathbb{F} = \mathbb{R}\) or complex Hermitian if \(\mathbb{F} = \mathbb{C}\) whose graph is \(G\). The sets \(S_+(G, \mathbb{F})\) are the corresponding subsets of positive semidefinite (psd) matrices. The smallest possible rank of any matrix \(A \in S(G, \mathbb{F})\) is the minimum rank of \(G\), denoted \(\text{mr}(G, \mathbb{F})\), and the smallest possible rank of any matrix \(A \in S_+(G, \mathbb{F})\) is the minimum semidefinite rank of \(G\), denoted \(\text{msr}(G)\) or \(\text{msr}(G)\).

In 1996, the minimum rank among real symmetric matrices with a given graph was studied by Nylen [28]. It gave rise to the area of minimum rank problems which led to the study of minimum rank among complex Hermitian matrices and positive semidefinite matrices associated with a given graph. Many results can be found for example in [1, 20, 24, 25, 28].

During the AIM workshop of 2006 in Palo Alto, CA, it was conjectured that for any graph \(G\) and infinite field \(F\), \(\text{mr}(G, \mathbb{F}) \leq |G| - \delta(G)\) where \(\delta(G)\) is the minimum degree of \(G\). It was shown that for if \(\delta(G) \leq 3\) or \(\delta(G) \geq |G| - 2\) this inequality holds. Also it can be verified that if \(|G| \leq 6\) then \(\text{mr}(G, F) \leq |G| - \delta(G)\). Also it was proven that any bipartite graph satisfies this conjecture. This conjecture is called the Delta Conjecture. If we restrict the study to consider matrices in \(S_+(G, \mathbb{F})\) then delta conjecture is written as \(\text{msr}(G) \leq |G| - \delta(G)\). Some results on delta conjecture can be found in [7, 13, 27, 31] but the general problem remains unsolved. In this
paper, by using a generalization of the argument in [15], we give an argument which prove that delta conjecture is true for any simple and connected graph which means that delta conjecture is true.

2 Graph Theory Preliminaries

In this section we give definitions and results from graph theory which will be used in the remaining chapters. Further details can be found in [8][9][14].

A graph \( G(V, E) \) is a pair \((V(G), E(G))\), where \( V(G) \) is the set of vertices and \( E(G) \) is the set of edges together with an incidence function \( \psi(G) \) that associate with each edge of \( G \) an unordered pair of (not necessarily distinct) vertices of \( G \). The order of \( G \), denoted \(|G|\), is the number of vertices in \( G \). A graph is said to be simple if it has no loops or multiple edges. The complement of a graph \( G(V, E) \) is the graph \( \overline{G} = (V, \overline{E}) \), where \( \overline{E} \) consists of all the edges that are not in \( E \). A subgraph \( H = (V(H), E(H)) \) of \( G = (V, E) \) is a graph with \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). An induced subgraph \( H \) of \( G \), denoted \( G[V(H)] \), is a subgraph with \( V(H) \subseteq V(G) \) and \( E(H) = \{\{i, j\} \in E(G) : i, j \in V(H)\} \). Sometimes we denote the edge \( \{i, j\} \) as \( ij \).

We say that two vertices of a graph \( G \) are adjacent, denoted \( v_i \sim v_j \), if there is an edge \( \{v_i, v_j\} \) in \( G \). Otherwise we say that the two vertices \( v_i \) and \( v_j \) are non-adjacent and we denote this by \( v_i \not\sim v_j \). Let \( N(v) \) denote the set of vertices that are adjacent to the vertex \( v \) and let \( N(v) = \{v\} \cup N(v) \). The degree of a vertex \( v \) in \( G \), denoted \( d_G(v) \), is the cardinality of \( N(v) \). If \( d_G(v) = 1 \), then \( v \) is said to be a pendant vertex of \( G \). We use \( \delta(G) \) to denote the minimum degree of the vertices in \( G \), whereas \( \Delta(G) \) will denote the maximum degree of the vertices in \( G \).

Two graphs \( G(V, E) \) and \( H(V', E') \) are identical denoted \( G = H \), if \( V = V', E = E' \), and \( \psi_G = \psi_H \). Two graphs \( G(V, E) \) and \( H(V', E') \) are isomorphic, denoted by \( G \cong H \), if there exist bijections \( \theta : V \to V' \) and \( \phi : E \to E' \) such that \( \psi_G(v) = \{u, v\} \) if and only if \( \psi_H(\phi(v)) = \{\theta(u), \theta(v)\} \).

A complete graph is a simple graph in which the vertices are pairwise adjacent. We will use \( nG \) to denote \( n \) copies of a graph \( G \). For example, \( 3K_1 \) denotes three isolated vertices \( K_1 \) while \( 2K_2 \) is the graph given by two disconnected copies of \( K_2 \).

A path is a list of distinct vertices in which successive vertices are connected by edges. A path on \( n \) vertices is denoted by \( P_n \). A graph \( G \) is said to be connected if there is a path between any two vertices of \( G \). A cycle on \( n \) vertices, denoted \( C_n \), is a path such that the beginning vertex and the end vertex are the same. A tree is a connected graph with no cycles. A graph \( G(V, E) \) is said to be chordal if it has no induced cycles \( C_n \) with \( n \geq 4 \). A component of a graph \( G(V, E) \) is a maximal connected subgraph. A cut vertex is a vertex whose deletion increases the number of components.

The union \( G \cup G_2 \) of two graphs \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \) is the union of their vertex set and edge set, that is \( G \cup G_2(V_1 \cup V_2, E_1 \cup E_2) \). When \( V_1 \) and \( V_2 \) are disjoint their union is called disjoint union and denoted \( G_1 \cup G_2 \).

3 The Minimum Semidefinite Rank of a Graph

In this section we will establish some of the results for the minimum semidefinite rank (msr) of a graph \( G \) that we will be using in the subsequent chapters.

A positive definite matrix \( A \) is an Hermitian \( n \times n \) matrix such that \( x^*Ax > 0 \) for all nonzero \( x \in \mathbb{C}^n \). Equivalently, \( A \) is a \( n \times n \) Hermitian positive definite matrix if and only if all the eigenvalues of \( A \) are positive ([21], p.250).
A $n \times n$ Hermitian matrix $A$ such that $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ is said to be **positive semidefinite** (psd). Equivalently, $A$ is a $n \times n$ Hermitian positive semidefinite matrix if and only if $A$ has all eigenvalues nonnegative (\cite{2}, p.182).

If $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^m$ is a set of column vectors then the matrix $A^T A$, where $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $A^T$ represents the transpose matrix of $A$, is a psd matrix called the **Gram matrix** of $V$. Let $G(V, E)$ be a graph associated with this Gram matrix. Then $V_G = \{v_1, \ldots, v_n\}$ correspond to the set of vectors in $V$ and $E(G)$ correspond to the nonzero inner products among the vectors in $V$. In this case $V$ is called an **orthogonal representation** of $G(V, E)$ in $\mathbb{R}^m$. If such an orthogonal representation exists for $G$ then $\text{msr}(G) \leq m$.

Some results about the minimum semidefinite rank of a graph are the following:

**Result 3.1.** \cite{20} If $T$ is a tree then $\text{msr}(T) = |T| - 1$.

**Result 3.2.** \cite{17} The cycle $C_n$ has minimum semidefinite rank $n - 2$.

**Result 3.3.** \cite{17} If a connected graph $G$ has a pendant vertex $v$, then $\text{msr}(G) = \text{msr}(G - v) + 1$ where $G - v$ is obtained as an induced subgraph of $G$ by deleting $v$.

**Result 3.4.** \cite{19} If $G$ is a connected, chordal graph, then $\text{msr}(G) = \text{cc}(G)$.

**Result 3.5.** \cite{20} If a graph $G(V, E)$ has a cut vertex, so that $G = G_1 \cdot G_2$, then $\text{msr}(G) = \text{msr}(G_1) + \text{msr}(G_2)$.

## 4 Delta-Graphs and the Delta Conjecture

In \cite{15} is is defined a family of graphs called $\delta$-graphs and show that they satisfy the delta conjecture.

**Definition 4.1.** Suppose that $G = (V, E)$ with $|G| = n \geq 4$ is simple and connected such that $\overline{G} = (V, E)$ is also simple and connected. We say that $G$ is a $\delta$-**graph** if we can label the vertices of $G$ in such a way that

1. the induced graph of the vertices $v_1, v_2, v_3$ in $G$ is either $3K_1$ or $K_2 \sqcup K_1$, and
2. for $m \geq 4$, the vertex $v_m$ is adjacent to at most $\left\lfloor \frac{m}{2} - 1 \right\rfloor$ vertices.

A second family of graphs also defined in \cite{15} contains the complements of $\delta$-graphs.

**Definition 4.2.** Suppose that a graph $G(V, E)$ with $|G| = n \geq 4$ is simple and connected such that $\overline{G} = (V, \overline{E})$ is also simple and connected. We say that $G(V, E)$ is a C-$\delta$ **graph** if $\overline{G}$ is a $\delta$-graph.

In other words, $G$ is a C-$\delta$ **graph** if we can label the vertices of $G$ in such a way that

1. the induced graph of the vertices $v_1, v_2, v_3$ in $G$ is either $K_3$ or $P_3$, and
2. for $m \geq 4$, the vertex $v_m$ is adjacent to at most $\left\lfloor \frac{m}{2} - 1 \right\rfloor$ of the prior vertices $v_1, v_2, \ldots, v_{m-1}$.

**Example 4.3.** The cartesian product $K_3 \square P_3$ is a C-$\delta$ graph and its complement is a $\delta$-graph. By labeling as the following picture we can verified the definition for both graphs.
Figure 1: The Graph $K_3 \square P_4$ and its complement $\overline{K_3 \square P_4}$

Note that we can label the vertices of $K_3 \square P_4$ clockwise $v_1 = (1, 1), v_2 = (1, 2), v_3 = (1, 3), \ldots v_{12} = (3, 4)$. The graph induced by $v_1, v_2, v_3$ is $P_3$. The vertex $v_4$ is adjacent to a prior vertex which is $v_3$ in the induced subgraph of $K_3 \square P_4$ given by $\{v_1, v_2, v_3, v_4\}$. Also, the vertex $v_5$ is adjacent only to vertex $v_1$ in the induced subgraph of $K_3 \square P_4$ given by $\{v_1, v_2, v_3, v_4, v_5\}$. Continuing the process through vertex $v_{12}$ we conclude that $K_3 \square P_4$ is a C-$\delta$ graph. In the same way we conclude that its complement $\overline{K_3 \square P_4}$ is a $\delta$-graph.

**Lemma 4.4.** Let $G(V, E)$ be a $\delta$-graph. Then the induced graph of $\{v_1, v_2, v_3\}$ in $G$ denoted by $H$ has an orthogonal representation in $\mathbb{R}^{\Delta(G)+1}$ satisfying the following conditions:

(i) the vectors in the orthogonal representation of $H$ can be chosen with nonzero coordinates, and

(ii) $\vec{v} \not\in \text{Span}(\vec{u})$ for each pair of distinct vertices $u, v$ in $H$.

**Theorem 4.5.** Let $G(V, E)$ be a $\delta$-graph then

$$\text{msr}(G) \leq \Delta(G) + 1 = |G| - \delta(G)$$

The proof of these two results can be found in [15] and [16]. The argument of the proof is based on the construction of an orthogonal representation of pairwise linear independent vectors for a $\delta$ graph $G$ at $\mathbb{R}^{\Delta(G)+1}$. Since $\text{msr}(G)$ is the minimum dimension in which we can get an orthogonal representation for a simple connected graphs the result is a direct consequence of this construction.

5 A survey of $\delta$-graphs and upper bounds their minimum Semidefinite rank

The theorem 4.5 give us a huge family of graph which satisfies delta conjecture. Since, the complement of a C-$\delta$ graphs is usually a $\delta$-graph, it is enough to identify a C-$\delta$-graph and therefore we know that its complement is a $\delta$-graph satisfying delta conjecture if it is simple and connected.

Some examples of C-$\delta$ graphs that we can find in [15] are the Cartesian Product $K_n \square P_m, n \geq 3, m \geq 4$, Mobius Lader $ML_{2n}, n \geq 3$, Supertriangles $T_n, n \geq 4$, Coronas $S_n \circ P_m, n \geq 2, m \geq 1$ where $S_n$ is a star and $P_m$ a path, Cages like Tutte’s (3,8) cage, Headwood’s (3,6) cage and many others, Blanusa Snarks of type 1 and 2 with 26, 34, and 42 vertices, and Generalized Petersen Graphs $Gp1$ to $Gp16$. 

4
5.1 Upper bounds for the Minimum semidefinite rank of some families of Simple connected graphs

From the definition of C-$\delta$ graph and the Theorem 4.5 we can obtain upper bounds for the graph complement of a C-$\delta$ graphs. It is enough to label the vertices of $G$ in such a way that the labeled sequence of vertices satisfies the definition. That is, if we start with the induced graph of $\{v_1, v_2, v_3\}$, the newly added vertex $v_m$ is adjacent to at most $\lfloor \frac{m}{2} - 1 \rfloor$ of the prior vertices $v_1, v_2, \ldots, v_{m-1}$. Then $G$ is a C-$\delta$ graph and its graph complement $\overline{G}(V, E)$ will have an orthogonal representation in $\mathbb{R}^{\Delta(G)+1}$ any time it is simple and connected.

In order to show the technique used in the proved result consider the following examples

**Example 5.1.** If $G$ is the Robertson’s (4,5)-cage on 19 vertices then it is a 4-regular C-$\delta$ graph. Since $\Delta(G) = 4$, the $\text{msr}(G) \leq 5$. To see this is a C-$\delta$ graph it is enough to label its vertices in the way shown in the next figure:

![Figure B.2 Robertson’s (4,5)-cage (19 vertices)](image)

**Example 5.2.** If $G$ is the platonic graph Dodecahedron then it is a 3-regular C-$\delta$ graph. Since $\Delta(G) = 3$, the $\text{msr}(G) \leq 4$. To see this is a C-$\delta$ graph it is enough to label its vertices in the way shown in the next figure:

![Figure 3. Dodecahedron](image)

The next table contains C-$\delta$ graphs $G$ taken from [30] and upper bounds for $\text{msr}(\overline{G})$ given by $\Delta(G) + 1$ are found in [15].
Table 1: Table of C-δ graphs $G$ taken from [30] and upper bounds for $\text{msr}(G)$ given by $\Delta(G) + 1$.

| Family                  | Name of Graph $G$ | $|G|$ | $\text{msr}(G)$ \(\leq \Delta(G) + 1\) |
|-------------------------|-------------------|-----|---------------------------------|
| Archimedean Graphs      |                   |     |                                 |
|                         | Cuboctahedron     | 12  | 4                               |
|                         | Icosidodecahedron | 30  | 5                               |
|                         | Rhombicuboctahedron | 24  | 5                               |
|                         | Rhombicosidodecahedron | 60  | 6                               |
|                         | Snub cube         | 24  | 6                               |
|                         | Snub dodecahedron | 60  | 6                               |
|                         | Truncated cube    | 24  | 4                               |
|                         | Truncated Cuboctahedron | 48  | 4                               |
|                         | Truncated dodecahedron | 60  | 4                               |
|                         | Truncated icosahedron | 60  | 4                               |
|                         | Truncated icosidodecahedron | 120 | 6                               |
|                         | Truncated Tetrahedron | 12  | 4                               |
|                         | Truncated octahedron | 24  | 4                               |
| Antiprisms              |                   |     |                                 |
|                         | $2n, n \in \mathbb{N}, n \geq 3$ | $2n, n \geq 3$ | 5 |
|                         | 4-antiprism       | 8   | 5                               |
|                         | 5-antiprism       | 10  | 5                               |
| Cages                   |                   |     |                                 |
|                         | Balaban’s (3, 10) cage | 70  | 4                               |
|                         | Foster (5, 5) cage | 30  | 6                               |
|                         | Harries’s (3, 10) cage | 70  | 4                               |
|                         | Headwood’s (3, 6) cage | 14  | 4                               |
|                         | MacGee’s (3, 7) cage | 24  | 4                               |
|                         | Petersen’s (3, 5) cage | 10  | 4                               |
|                         | Robertson’s (5, 5) cage | 30  | 6                               |
|                         | Robertson’s (4, 5) cage | 19  | 5                               |
|                         | The Harries-Wong (3, 10) cage | 70  | 4                               |
|                         | The (4, 6) cage | 26  | 5                               |
|                         | Tutte’s (3, 8) cage | 30  | 4                               |
|                         | Wongs’s (5, 5) cage | 30  | 4                               |
|                         | The Harries-Wong (3, 10) cage | 70  | 4                               |
|                         | The (4, 6) cage | 26  | 5                               |
|                         | Tutte’s (3, 8) cage | 30  | 4                               |
|                         | Wongs’s (5, 5) cage | 30  | 4                               |
| Family                  | Name of the Graph $G$ | $|G|$ | $\text{msr}(G)$ | $\Delta(G)+1$ |
|------------------------|-----------------------|------|----------------|----------------|
| **Blanusa Snarks**     |                       |      |                |                |
| Type 1: 26 vertices    | 26                    | 4    |                |                |
| Type 2: 26 vertices    | 26                    | 4    |                |                |
| Type 1: 34 vertices    | 34                    | 4    |                |                |
| Type 2: 34 vertices    | 34                    | 4    |                |                |
| Type 1: 42 vertices    | 42                    | 4    |                |                |
| Type 2: 34 vertices    | 42                    | 4    |                |                |
| **Generalized Petersen Graphs** |                     |      |                |                |
| Gp1                    | 10                    | 4    |                |                |
| Gp2                    | 12                    | 4    |                |                |
| Gp3                    | 14                    | 4    |                |                |
| Gp4                    | 16                    | 4    |                |                |
| Gp5                    | 16                    | 4    |                |                |
| Gp6                    | 18                    | 4    |                |                |
| Gp7                    | 18                    | 4    |                |                |
| Gp8                    | 20                    | 4    |                |                |
| Gp9                    | 20                    | 4    |                |                |
| Gp10                   | 20                    | 4    |                |                |
| Gp11                   | 22                    | 4    |                |                |
| Gp12                   | 22                    | 4    |                |                |
| Gp13                   | 24                    | 4    |                |                |
| Gp14                   | 24                    | 4    |                |                |
| Gp15                   | 24                    | 4    |                |                |
| Gp16                   | 24                    | 4    |                |                |
| **Non-Hamiltonian Cubic** |                     |      |                |                |
| Grinberg’s Graph       | 44                    | 4    |                |                |
| Tutte’s Graph          | 46                    | 4    |                |                |
| (38 vertices)          | 38                    | 4    |                |                |
| (42 vertices)          | 42                    | 4    |                |                |
| **Platonic Graphs**    |                       |      |                |                |
| Cube                   | 8                     | 4    |                |                |
| Dodecahedron           | 20                    | 4    |                |                |
| **Prisms**             |                       |      |                |                |
| $n$-prism, $n \geq 4$  | $2n$                  | 4    |                |                |
| 4-prism                | 8                     | 4    |                |                |
| 5-prism                | 10                    | 4    |                |                |
| **Snarks**             |                       |      |                |                |
| Celmins-Swarf snark 1  | 26                    | 4    |                |                |
| Celmins-Swarf snark 2  | 26                    | 4    |                |                |
| Double Star snark      | 30                    | 4    |                |                |
| Flower snark $J_7$     | 28                    | 4    |                |                |
| Flower snark $J_9$     | 36                    | 4    |                |                |
| Flower snark $J_{11}$  | 44                    | 4    |                |                |
| Family                      | Graph $G$ | $|G|\leq\Delta(G)+1$ |
|-----------------------------|-----------|---------------------|
| Hypercube                   | $2^4$     | 5                   |
| Loupekine’s snark 1 (Sn28) | 22        | 4                   |
| Loupekine’s snark 2 (Sn29) | 22        | 4                   |
| The Biggs-Smith             | 102       | 4                   |
| The Greenwood-Gleason       | 16        | 6                   |
| The Szekeres snark          | 50        | 4                   |
| Watkin’s snark              | 50        | 4                   |
| **Miscellaneous Regular Graphs** |           |                     |
| Chvatal’s graph             | 12        | 5                   |
| Cubic Graph with no perfect matching | 16 | 4               |
| Cubic Identity graphs       | 12        | 4                   |
| Folkman’s graph             | 20        | 5                   |
| Franklin’s graph            | 12        | 4                   |
| Herschel’s graph            | 11        | 5                   |
| Hypercube                   | 16        | 4                   |
| Meredith’s graph            | 70        | 4                   |
| Mycielski’s graph           | 11        | 6                   |
| The Greenwood-Gleason graph | 16        | 6                   |
| The Goldner-Harary dual     | 18        | 4                   |
| (the truncated Prism)       |           |                     |
| Tietze’s graph              | 11        | 4                   |

## 6 Proof of Delta Conjecture

In this section we give an argument which prove that Delta Conjecture is true for any simple graph not necessarily connected as a generalization of the result given in [15]. For that purpose we define a generalization of C-\(\delta\) graphs called **extended C-\(\delta\) graph**.

Previously, we establish that Delta Conjecture holds for \(\delta\)-Graphs. The condition \(2 \leq \Delta(G) \leq |G|-2\) in the proof of [4.5] was given as a sufficient condition to obtain that the graph complement of a C-\(\delta\) graph is connected. We will see that the condition of connectivity of a C-\(\delta\) graphs is not necessary in order to proof Delta Conjecture when using the result [4.5].

Hence, we can define a generalization of C-\(\delta\) graphs in the following way.

**Definition 6.1.** A **extended C-\(\delta\) graph** $G'$ is a simple graph which is the disjoint union of a simple connected graph $G$, $|G| \geq 4$ (not necessarily connected) and a path $P_n$, where $n = 2\Delta(G) + 2$, $n \geq 4$. That is

$$G' = P_n \cup G; n = 2\Delta(G) + 2.$$ 

All vertices of $G$ are connected with all vertices of $P_n$ in $G'$. As a consequence $G'$ is a simple and connected graph.
Definition 6.2. A graph $G$, $|G| \geq 4$ has a C-δ construction if it can be constructed starting with $K_3$ or $P_3$ and by adding one vertex at a time in such a way that the newest vertex $v_m$, $m \geq 4$ is adjacent to at most $\left\lfloor \frac{m^2}{2} - 1 \right\rfloor$ of the prior vertices $v_1, v_2, \ldots, v_{m-1}$.

![Figure 2: Extended C-δ Graph](image)

Proposition 6.3. Let $G' = P_n \sqcup G$ be an extended C-δ graph. Then $\overline{G'}$ has an orthogonal representation in $\mathbb{R}^{\Delta(G)+1}$.

PROOF:

Let $G'(V', E')$ be an extended C-δ graph. Then $G' = P_n \sqcup G$; $n = 2\Delta(G) + 2$ and $G(V, E)$ is a simple graph. Since $P_n$ is a C-δ graph we know that we can label its vertices in such a way that if $v_2, \ldots, v_n$ are its vertices then $\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}$ is an orthogonal representation of its graph complement $\overline{P_n}$ in $\mathbb{R}^3$. But since $\Delta(G) \geq 2$ because $G$ is connected and $|G| \geq 4$ then we can also obtain an orthogonal representation of $P_n$ in $\mathbb{R}^{\Delta(G)+1}$ getting $2\Delta(G) + 2$ vectors for $\overline{G'}$ in $\mathbb{R}^{\Delta(G)+1}$ using the C-δ construction.

Thus, in $G'$, $v_{2\Delta(G)+2}$ is adjacent with all prior vertices $v_1, \ldots, v_{2\Delta(G)+1}$ but at most

$$\left\lfloor \frac{2\Delta(G) + 2}{2} - 1 \right\rfloor = \Delta(G) \geq 2$$

vertices. Actually to all of them but one.

Now, choose a vertex $v'$ in $G$ and label it as $v' = v_{2\Delta(G)+3}$. In $\overline{G'}$ $v_{2\Delta(G)+3}$ is adjacent with all of the vertices of $P_n$. As a consequence, $v_{2\Delta(G)+3}$ satisfies the delta construction in $G'$.

If $Y_{2\Delta(G)+4}$ is the induced graph of $G'$ given by $v_1, \ldots, v_{2\Delta(G)+2}$ then $Y_{2\Delta(G)+3} = Y_{2\Delta(G)+2} \cup \{v_{2\Delta(G)+3}\}$ is simple and connected and $Y_{2\Delta(G)+4}$ can be constructed by using $\delta$-construction because $v_{2\Delta(G)+3}$ is adjacent with all previous vertices $v_1, v_2, \ldots, v_{2\Delta(G)+2}$ in $\overline{G'}$. Then it is adjacent with all previous vertices in $Y_{2\Delta(G)+2}$ but at most

$$\left\lfloor \frac{2\Delta(G) + 3}{2} - 1 \right\rfloor \geq \Delta(G).$$

Now, by labeling the remaining vertices in $G'$ which are vertices in $G$ in any random sequence to obtain $v_{2\Delta(G)+4}, v_{2\Delta(G)+5}, \ldots$ we get a sequence of induced subgraph of $\overline{G'}$

$$Y_{2\Delta(G)+4} \subseteq Y_{2\Delta(G)+5} \subseteq \cdots Y_{2\Delta(G)+2+|G|} = \overline{G'}$$

All of these induced subgraphs can be constructed using $\delta$-construction. As a consequence $Y_{2\Delta(G)+2+|G|} = \overline{G'}$ can be constructed using $\delta$-construction which implies that there is an orthogonal representation $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_{2\Delta(G)+2+|G|}}$ of the vertices of $\overline{G'}$ at $\mathbb{R}^{\Delta(G')+1}$. 

9
But $\Delta(G') \geq \Delta(G)$ since $|G| \geq 4$, $\overline{G}$ is simple and connected, and $G$ is an induced graph of $G'$.

Then we can get the orthogonal representation of $\overline{G}$ in $\mathbb{R}^{\Delta(G)+1}$.

Finally, if $\overrightarrow{v}_1, \overrightarrow{v}_2, \ldots, \overrightarrow{v}_{2\Delta(G)+2+|G|}$ is the orthogonal representation of $G'$ in $\mathbb{R}^{\Delta(G)+1}$ take the vectors $\overrightarrow{v}_{2\Delta(G)+3}, \overrightarrow{v}_{2\Delta(G)+4}, \ldots, \overrightarrow{v}_{2\Delta(G)+2+|G|}$. These vectors satisfy all the adjacency conditions and orthogonal conditions of $\overline{G}$ because $\overline{G}$ is an induced subgraph of $\overline{G}'$. As a consequence, $\overrightarrow{v}_{2\Delta(G)+3}, \overrightarrow{v}_{2\Delta(G)+4}, \ldots, \overrightarrow{v}_{2\Delta(G)+2+|G|}$ is an orthogonal representation of $\overline{G}$ in $\mathbb{R}^{\Delta(G)+1}$.

\begin{proof}
\end{proof}

\begin{theorem}
If $G$ is a simple connected graph, $|G| \geq 4$ then $G$ satisfies Delta conjecture.
\end{theorem}

\begin{proof}
Let $G$ be a simple connected graph. Since $G$ can be seen as a component of a extended C-$\delta$ graph $G' = P_{2\Delta(G)+2} \cup G$ by the proposition proved above $G$ has an orthogonal representation in $\mathbb{R}^{\Delta(G)+1} = \mathbb{R}^{G-\delta(G)}$ which implies that $\text{msr}(G) \leq |G| - \delta(G)$. As a consequence delta conjecture holds for any simple connected graph $G$ with $|G| \geq 4$.

Finally, by using extended C-$\delta$ graphs $G' = P_{2\Delta(G)+2} \cup G$ for all $G, |G| \leq 3$ and the technique described in the proof of the proposition above or any other way it is easy to check that all of simple connected graphs with $|G| \leq 3$ satisfies Delta conjecture. As a consequence we have the following theorem:

\begin{corollary}
Let $G$ be a simple and connected graph. Then $G$ satisfies delta conjecture.
\end{corollary}

\begin{proof}
From \ref{theorem}, we know that delta conjecture hold for any simple graph $G, |G| \geq 4$. Checking all cases for all simple connected graph $G, |G| \leq 3$ we complete the proof for delta conjecture.
\end{proof}

\section{Conclusion}

In this paper we proved the delta conjecture as a main result. Also we applied the technique for finding the minimum semidefinite rank of a C-$\delta$ to give a table of upper bounds of a large amount of families of simple connected graphs. These upper bounds will be usefull in the study of the minimum semidefinite rank of a graph.

In the future, the techniques applied in this paper could be useful to solve other problems related with simple connected graphs and minimum semidefinite rank.

\section{Acknowledgment}

I would like to thanks to my advisor Dr. Sivaram Narayan for his guidance and suggestions of this research. Also I want to thank to the math department of University of Costa Rica and Universidad Nacional Estatal a Distancia because their sponsorship during my dissertation research and specially thanks to the math department of Central Michigan University where I did the research for C-\(\text{delta}\) graphs which was a paramount research to proof delta conjecture.

\section*{References}

[1] AIM Minimum Rank-Special Graphs Work Group (Francesco Barioli, Wayne Barrett, Steven Butler, Sebastian M. Cioaba, Shaun M. Fallat, Chris Godsil, Willem Haemers, Leslie Hogben, Rana Mikkelson, Sivaram Narayan, Olga Pryporova, Irene Sciriha, Dragan Stevanovic, Hein Van Der Holst, Kevin Van Der Meulen, and Amy Wangsness), Zero Forcing Sets and the Minimum Rank of Graphs, Linear Algebra and its Applications, 428 (2008) 1628-1648.
[2] Francesco Barioli, Wayne Barrett, Shaun M. Fallat, H. Tracy Hall, Leslie Hogben, and Hein van der Holst, On the Graph Complement Conjecture for Minimum Rank, Linear Algebra and its Applications, in press, doi:10.1016/j.laa.2010.12.024.

[3] Francesco Barioli, Wayne Barrett, Shaun M. Fallat, H. Tracy Hall, Leslie Hogben, Bryan Shader, P. van den Driessche, and Hein van der Holst, Zero Forcing Parameters and Minimum Rank Problems, Linear Algebra and its Applications, 433 (2010) 401-411.

[4] Francesco Barioli, Shaun M. Fallat, Lon H. Mitchell, and Sivaram Narayan, Minimum Semidefinite Rank of Outerplanar Graphs and the Tree Cover Number, Electronic Journal of Linear Algebra, 22 (2011) 10-21.

[5] Wayne Barrett, Hein Van Der Holst, and Raphael Loewy, Graphs Whose Minimal Rank is Two, Electronic Journal of Linear Algebra, 11 (2004) 258-280.

[6] Jonathan Beagley, Sivaram Narayan, Eileen Radzwion, Sara Rimer, Rachel Tomasino, Jennifer Wolfe, and Andrew Zimmer , On the Minimum Semidefinite Rank of a Graph Using Vertex Sums, Graphs with $\text{msr}(G) = |G| - 2$, and the $\text{msrs}$ of Certain Graphs Classes ,in NSF-REU Report from Central Michigan University (Summer 2007).

[7] Avi Berman, Shmuel Friedland, Leslie Hogben, Uriel G. Rothblum, and Bryan Shader, An Upper Bound for the Minimum Rank of a Graph, Linear Algebra and its Applications, 429 (2008) 1629-1638.

[8] Béla Bollobás. Modern Graph Theory, Springer, Memphis, TN, 1998.

[9] John Adrian Bondy, Uppaluri Siva Ramachandra Murty, Graph Theory, Springer, San Francisco, CA, 2008.

[10] Matthew Booth, Philip Hackney, Benjamin Harris, Charles Johnson, Margaret Lay, Terry Lenker, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, and Brian D. Sutton, On the Minimum Semidefinite Rank of a Simple Graph, Linear and Multilinear Algebra, 59 (2011) 483-506.

[11] Matthew Booth, Philip Hackney, Benjamin Harris, Charles R. Johnson, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, Amanda Pascoe, Kelly Steinmetz, Brian D. Sutton, and Wendy Wang, On the Minimum Rank Among Positive Semidefinite Matrices with a Given Graph, SIAM Journal on Matrix Analysis and Applications, 30 (2008) 731-740.

[12] Andreas Brandstdt, Van Bang Le, Jeremy Spinrad, Graph Classes: A Survey, SIAM Monographs on Discrete Mathematics and Applications, ISBN 0-89871-432-X. (1999) p 169.

[13] Richard Brualdi, Leslie Hogben, Bryan Shader; AIM Workshop Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns Final Report: Mathematical Results (Revised*) Available at http://www.aimath.org/pastworkshop/matrixspectrumrep.pdf, (2007).

[14] Gary Chartrand, Linda Lesniak, and Ping Zhang. Graphs & Digraphs, Taylor & Francis Group. Boca Raton, FL, 2011.

[15] Pedro, Diaz. On the delta Conjecture and the Graph Complement Conjecture for Minimum Semidefinite rank of a Graph, Ph.D Dissertation, Central Michigan University, July (2014).

[16] Pedro, Diaz. On Delta Graphs and Delta Conjecture, Revista de matemática Teoría y Aplicaciones, 25 (2018) 1-28.
[17] Jason Ekstrand, Craig Erickson, H. Tracy Hall, Diana Hay, Leslie Hogben, Ryan Johnson, Nicole Kingsley, Steven Osborne, Travis Peters, Jolie Roat, Arianne Ross, Darren D. Row, Nathan Warnberg, and Michael Young, Positive Semidefinite Zero Forcing, Linear Algebra and its Applications, 439 (2013) 1862-1874.

[18] Shaun M. Fallat, Leslie Hogben, The Minimum Rank of Symmetric Matrices Described by a Graph: A Survey, Linear Algebra and its Applications, 426 (2007) 558-582.

[19] Philip Hackney, Benjamin Harris, Margaret Lay, Lon H. Mitchell, Sivaram K. Narayan, and Amanda Pascoe, Linearly Independent Vertices and Minimum Semidefinite Rank, Linear Algebra and its Applications, 431 (2009) 1105-1115.

[20] Hein van der Holst, Graphs whose Positive Semidefinite Matrices have Nullity at Most Two, Linear Algebra and its Applications, 375 (2003) 1-11.

[21] Roger Horn, Charles Johnson, Matrix Analysis, Cambridge University Press, 1985.

[22] Leslie Hogben, Minimum Rank Problems, Linear Algebra and its Applications, 432 (2009) 1961-1974.

[23] Leslie Hogben, Orthogonal Representations, Minimum Rank, and Graph Complements, Linear Algebra and its Applications, 428 (2008) 2560-2568.

[24] Yunjiang Jiang, Lon H. Mitchell, and Sivaram K. Narayan, Unitary Matrix Digraphs and Minimum Semidefinite Rank, Linear Algebra and its Applications, 428 (2008) 1685-1695.

[25] Lon Mitchell, On the Graph Complement Conjecture for Minimum Semidefinite Rank, Linear Algebra and its Applications (2011), in press, doi:10.1016/j.laa.2011.03.011.

[26] Lon Mitchell, Sivaram K. Narayan, and Andrew M. Zimmer, Lower Bounds in Minimum Rank Problems, Linear Algebra and its Applications, 432 (2010) 430-440.

[27] Sivaram K. Narayan, Yousra Sharawi, Bounds on Minimum Semidefinite Rank of Graphs, Linear and Multilinear Algebra (2014). Retrieved from http://dx.doi.org/10.1080/03081087.2014.898763 on June 13, 2014.

[28] Peter Nylen, Minimum-Rank Matrices with Prescribed Graph, Linear Algebra and its Applications, 248 (1996) 303-316.

[29] Travis Peters, Positive Semidefinite Maximum Nullity and Zero Forcing Number, Electronic Journal of Linear Algebra, 23 (2012)828-829.

[30] Ronald Read, Robin Wilson. An Atlas of Graphs, Oxford University Press., 1998, 125-305.

[31] Yousra Sharawi, Minimum Semidefinite Rank of a Graph Ph.D. Dissertation. Central Michigan University, December (2011).

[32] Douglas B. West, Introduction to Graph Theory, Prentice Hall Inc., 1996.