Nonlinear Scale Invariance in Local Disk Flows

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Abstract. An exact nonlinear scaling transformation is presented for the local three-dimensional dynamical equations of motion for differentially rotating disks. The result is relevant to arguments that have been put forth claiming that numerical simulations lack the necessary numerical resolution to resolve nonlinear instabilities that are supposedly present. We show here that any time dependent velocity field satisfying the local equations of motion and existing on small length scales, has an exact rescaled counterpart that exists on arbitrarily larger scales as well. Large scale flows serve as a microscope to view small scale behavior. The absence of any large scale instabilities in local numerical simulations of Keplerian disks suggests that the equations in this form have no instabilities at any scale, and that finite Reynolds number suppression is not the reason for the exhibited stable behavior. While this argument does not rule out the possibility of global hydrodynamical instability, it does imply that differential rotation per se is not unstable in a manner analogous to shear layers or high Reynolds number Poiseuille flow. Analogies between the stability behavior of accretion disks and these flows are specious.

Key words. Accretion disks – hydrodynamic instabilities – turbulence

1. Introduction

Disks with Keplerian rotation profiles are linearly stable by the Rayleigh criterion of outwardly increasing specific angular momentum, but are extremely sensitive to the presence of magnetic fields. A weakly magnetized disk is linearly unstable if its angular velocity decreases outward, a condition met by Keplerian and almost all other astrophysical rotation profiles (Balbus & Hawley 1991). The underlying physics behind this magnetorotational instability (MRI) is well-understood, and the breakdown of the flow into fully developed turbulence has been convincingly demonstrated in a large series of numerical simulations (Balbus 2003 for a review).

Not all astrophysical disks need have the requisite minimum ionization level to sustain magnetic coupling, however. Protostellar disks, for example, may have an extended “dead zone” near the midplane on radial scales from ~ 0.1 to ~ 10 AU (Gammie 1996; Fromang, Terquem, & Balbus 2001). This, along with other similar cases (e.g. CV disks, cf. Gammie & Menou 1998), has led to speculation that there are also hydrodynamical mechanisms by which Keplerian flow is destabilized (Gammie 1996).

Before the advent of the MRI, such reasoning was orthodox. The pioneering work of Shakura & Sunyaev (1973), for example, invoked nonlinear, high Reynolds number shear instabilities as a likely destabilizing mechanism that would lead to turbulence (see also Crawford & Kraft 1956). Since, for nonaxisymmetric disturbances, there is still no proof either of linear or nonlinear stability, this mechanism continues to attract adherents (Dubrulle 1993, Richard & Zahn 1999, Richard 2003).

Theoretical analysis may have hit an impasse, but the intervening years have in fact seen a stunning rise in the capabilities of numerical simulation. These have shown no indication of local nonlinear rotational instabilities (Hawley, Balbus, & Winters 1999), in
Keplerian disks. They do, however, reveal nonlinear shear instabilities when Coriolis forces are absent, or when the disk is marginally stable (constant specific angular momentum). Indeed, even linear instability is possible in some Rayleigh-stable disks, provided that global physics is introduced (Goldreich, Goodman, & Narayan 1986; Blaes 1987), a result that has been numerically confirmed (Hawley 1991).

The numerical stability findings have been criticized on the grounds that the effective Reynolds number of the codes is too low, and that this damps the nonlinear instabilities: the latter require yet-to-be resolved spatial scales in order to reveal themselves (Richard & Zahn 1999). In this paper, we show that if nonlinear hydrodynamical instabilities must involve dynamics beyond the local approximation, and are not an inevitable nonlinear outcome of differential rotation.

2. The Local Approximation

In cylindrical coordinates \((R, \phi, z)\), the fundamental equations of motion for a flow in which viscous effects are negligible are mass conservation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}
\]

and the dynamical equations,

\[
\frac{\partial v_R}{\partial t} + v_R \nabla v_R - \frac{v^2_\phi}{R} = -\frac{1}{\rho} \frac{\partial P}{\partial R} - \frac{\partial \Phi}{\partial R}, \tag{2}
\]

\[
\frac{\partial v_\phi}{\partial t} + v_R \nabla v_\phi + \frac{v_R v_\phi}{R} = -\frac{1}{\rho} \frac{\partial P}{\partial \phi}, \tag{3}
\]

\[
\frac{\partial v_z}{\partial t} + v_R \nabla v_z = -\frac{1}{\rho} \frac{\partial P}{\partial z} - \frac{\partial \Phi}{\partial z}. \tag{4}
\]

Our notation is standard: \(v\) is the velocity field, \(\rho\) the mass density, \(P\) the gas pressure, and \(\Phi\) is the Newtonian point mass potential for central mass \(M\):

\[
\Phi = -\frac{GM}{(R^2 + z^2)^{1/2}}. \tag{5}
\]

\(G\) is the gravitational constant.

The local limit consists of the following series of approximations. First, we assume that \(R\) is large and \(z \ll R\), so that

\[
\Phi \approx -\frac{GM}{R} \left( 1 - \frac{z^2}{2R^2} \right). \tag{6}
\]

and

\[
\frac{\partial \Phi}{\partial R} \approx \frac{GM}{R^2}, \quad \frac{\partial \Phi}{\partial z} \approx \frac{GMz}{R^3}. \tag{7}
\]

Choose a fiducial value of \(R\), say \(R_0\). Denote the angular velocity as \(\Omega(R)\) (we assume a dependence only upon \(R\)), and let \(\Omega_0 = \Omega(R_0)\). We next erect local Cartesian coordinates

\[
x = R - R_0, \quad y = R_0(\phi - \Omega_0 t), \tag{8}
\]

which corotate with the disk at \(R = R_0\). Let

\[
w \equiv v - R\Omega_0 t e_\phi \tag{9}
\]

be the velocity relative to uniform rotation at \(\Omega = \Omega_0\). In the local approximation, the magnitude of \(w\) is assumed to be small compared with \(R\Omega_0\).

The undisturbed angular velocity is Keplerian,

\[
\Omega^2 = \frac{GM}{R^3}. \tag{10}
\]

Substitution of equations (6-10) into equations (2-4) and retaining leading order, yields the so-called local or Hill equations (e.g., Balbus & Hawley 1998):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) = 0, \tag{11}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla \right) w_R - 2\Omega w_\phi = -x \frac{d\Omega^2}{d\ln R} - \frac{1}{\rho} \frac{\partial P}{\partial x} \tag{12}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla \right) w_\phi + 2\Omega w_R = -\frac{1}{\rho} \frac{\partial P}{\partial y} \tag{13}
\]

\[
\left( \frac{\partial}{\partial t} + \mathbf{w} \cdot \nabla \right) w_z = -z\Omega^2 - \frac{1}{\rho} \frac{\partial P}{\partial z}. \tag{14}
\]

The “0” subscript has been dropped in the 2Ω terms in equations (12) and (13), and in the derivative term
on the right of equation (12). The time derivative is taken in the corotating frame, viz.:

$$\frac{\partial}{\partial t} (\text{inertial}) = \frac{\partial}{\partial t} (\text{corotating}) + \Omega_0 \frac{\partial}{\partial \phi}$$  \hfill (15)

Equations (11–14) are well known, and have been used extensively in both numerical and analytical studies. The fundamental approach dates from nineteenth century treatments of the Earth-moon system (Hill 1878).

3. Scale symmetry in the Hill Equations

The local equations of motion incorporate an important symmetry in their structure. Let

$$w(r, t), \quad \rho(r, t), \quad P(r, t),$$  \hfill (16)

where $r = (x, y, z)$, be an exact solution to the Hill equations (11–14). Then, if $\alpha$ is an arbitrary constant,

$$(1/\alpha)w(\alpha r, t), \quad \rho(\alpha r, t), \quad (1/\alpha^2)P(\alpha r, t)$$  \hfill (17)

is also an exact solution to the same equations. The proof is a simple matter of direct substitution.

An equivalent formulation of the scaling symmetry is

$$w(r/\epsilon, t) \leftrightarrow (1/\epsilon)w(r, t)$$  \hfill (18)

$$\rho(r/\epsilon, t) \leftrightarrow \rho(r, t)$$  \hfill (19)

$$P(r/\epsilon, t) \leftrightarrow (1/\epsilon^2)P(r, t).$$  \hfill (20)

In this form, with $\epsilon \ll 1$, we see that any solution of the Hill equations that involves very small length scales has a rescaled counterpart solution with exactly the same time dependence. In particular, any solution corresponding to a breakdown into turbulence must be present on both large and small scales.

The implications of this scaling symmetry are of particular importance for understanding and testing the possible existence of local nonlinear instabilities in Keplerian disks. The key point is that any such instability would have to exist not just at small scales, but at all scales. Finite difference numerical codes would find such instabilities, if they existed. Indeed, a constant specific angular momentum profile is nonlinearly unstable, and is found to be so even at resolutions as crude as $32^3$. By way of contrast, local Keplerian profiles show no evidence of nonlinear instability at resolutions up to $256^3$, instead converge to the same stable solution in codes with completely different numerical diffusion properties (Hawley, Balbus, & Winters 1999). The argument that small scale flow structure is somehow being repressed is simply untenable.

To see how the Reynolds number changes with scale, assume that a flow is characterized by an effective kinematic viscosity $\nu$. The scaling argument we have just given applies to inviscid equations, so we should not expect it to hold in the presence of viscosity. The Reynolds number associated with the small scale solution is

$$Re_s = \frac{w \times l}{\nu}$$  \hfill (21)

The Reynolds number associated with the large scale solution is

$$Re_l = \frac{w \times l}{\alpha^2 \nu}$$  \hfill (22)

where $w$ here means $w(l, t)$, the value of the velocity function evaluated at a fiducial value length $l$ and time $t$. $Re_l = Re_s/\epsilon^2 \ll Re_s$ because at larger scales both the velocity and the length scales increase by a factor of $1/\epsilon$. In a numerical simulation, strict scaling invariance is not obeyed. Instead, the large scale solutions approach the inviscid limit, while their sufficiently small scale counterparts are damped. But by behaving nearly inviscidly, the large scale solutions capture the behavior of the Hill system at all scales.

4. What this Result Does Not Show

Obviously, scale invariance does not constitute a proof of nonlinear stability in any Keplerian flow. There are several points we have not covered.

First, the local approximation ignores boundary conditions. In laboratory flows, the fluid is always bounded by hard walls, and boundary layers form. A recent laboratory confirmation of the MRI also finds finite amplitude velocity fluctuations in a magnetically stable flow, for example. But the source of such disturbances are boundary layers (Sisan et al. 2004).

The Hill equations emerge in the limit $R \to \infty$, and therefore curvature terms drop out of the analysis. Instabilities that depend, for example, upon inflection points or vorticity maxima in the background rotation profile would not appear in this limit. Nothing precludes them from forming in the $w$ velocity profile, however, and if such instabilities were present they should manifest on large scales as well as small. In any case, the criticism of the numerical simulations is that extremely small structure is being
lost, and that high Reynolds number differential rotation is supposedly intrinsically unstable. It is very difficult to see how large scale curvature could play an essential destabilizing role here. In these equations, the curvature terms are nonsingular perturbations. Planar Couette and Poiseuille flows break down into turbulence without assistance from geometrical curvature.

Our Hill analysis together with numerical simulations would also suggest that a non-Keplerian disk with, say, $\Omega \propto R^{-1.8}$ is nonlinearly stable. But an annulus supporting such a profile is in fact linearly unstable (Goldreich, Goodman, & Narayan 1986), transporting angular momentum outward even in its linear phase. The point is that the annulus supports edge modes that become unstable, and these global modes do not exist in the local approximation. The existence of a similar instabilities in disks found in nature cannot be ruled out, though to date none afflicting Keplerian disks have been found.

The disk thermal structure could also be unstable, at least in principle. Nothing presented in this work bears on these types of instabilities.

Finally, there are technical loopholes to the argument presented in this paper. What if the unstable solution required not just some small scales to be resolved, but very disparate scales? Why this should be so is far from clear, but this possibility cannot be ruled out a priori. Indeed, one could imagine that a fractal structure is required down to infinitesimal scales. Rescaling would not bring such a solution to larger characteristic length scales, by definition. This solution is obviously not characterized by a critical Reynolds number, above which it is necessary to be seen. The critical Reynolds number would be infinity! This is not the argument made by proponents of nonlinear high Reynolds number instability. Such a solution may remain a mathematical possibility, but not one that can be realized in nature.

5. Conclusion

The local dynamics of Keplerian or other astrophysical disk profiles can be captured by a an established formalism known as the local, or Hill, approximation. The resulting system of equations has an exact scale invariance, so that any flow characterized by very small scales has an exact large scale counterpart with same stability properties. This feature of the Hill equations implies that finite difference codes at available resolutions are sufficient to explore the possibility of local nonlinear shear instabilities in astrophysical disks. If simulations accurately describe the large scale behavior of the Hill system, there is nothing more to uncover at small scales; it is simply renormalized large scale behavior. The absence of any observed instabilities in Keplerian numerical studies, coupled with the ready manifestation of such instabilities in local shear layers and constant specific angular momentum systems, suggests that any putative nonmagnetic disk instability would have to incorporate physics beyond simple differential rotation.

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