SURFACE MEASURES IN INFINITE DIMENSION

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Abstract. We construct surface measures associated to Gaussian measures in separable Banach spaces, and we prove several properties including an integration by parts formula.

1. Introduction

Let $X$ be a separable Banach space with norm $\| \cdot \|$, endowed with a nondegenerate centered Gaussian measure $\mu$, with covariance $Q$ and associated Cameron–Martin space $H$.

We will construct surface measures, defined on level sets $\{ x \in X : G(x) = r \}$ for suitable $G : X \mapsto \mathbb{R}$, and prove several properties including an integration by parts formula for Sobolev functions, that involves a surface integral.

Surface measures in Banach spaces are not a novelty. The first steps were made in the case of Hilbert spaces, for instance in the book [21] where a class of smooth surfaces was considered. To our knowledge, the earliest results in Banach spaces are due to Uglanov [22], about surface measures on (unions of) graphs of smooth functions, and Hertle [12], that deals only with hyperplanes and spherical surfaces.

The first systematic treatment for a more general class of surfaces was done by Airault and Malliavin in [1], that concerns level sets of functions $G \in \cap_{k \in \mathbb{N}, p \geq 1} W^{k,p}(X, \mu)$ satisfying a nondegeneracy condition, where $X$ is the classical Wiener space.

The approach of [1] is through the study of the densities of suitable image measures. The same approach was considered in the books [3, 16] in more general contexts, and in [15] in the special case where $X$ is the space of the tempered distributions in $\mathbb{R}$. The aim of this paper is to give an alternative simpler and clearer construction and description of surface measures through the image measures approach, under less regularity assumptions on $G$ with respect to [1, 3, 16].

A completely different approach was followed by Feyel and de La Pradelle in [10], who defined a Hausdorff–Gauss measure $\rho$ on the Borel sets of $X$ by finite dimensional approximations. Another very general surface measure is the perimeter measure, defined as the variation measure of the characteristic function of $\{ x \in X : G(x) < r \}$ in the case that such characteristic function is of bounded variation, see [11] and the following papers [2, 13]. It is known that under some regularity assumption on $G$, the perimeter measure coincides with $\rho$ on the level surfaces of $G$, and they coincide with the Airault–Malliavin surface measure under further assumptions ([5, 4]).

After the construction of the surface measures $\sigma^G_r$, we show several properties of them, under minimal assumptions: they are non trivial (namely, $\sigma^G_r(X) > 0$) for every $r \in (\text{ess inf } G, \text{ess sup } G)$, the support of $\sigma^G_r$ is contained in $G^{-1}(r)$, Borel sets with null Gaussian capacity are negligible with respect to $\sigma^G_r$ for every $r$, and the integration by parts formula

$$\int_{G^{-1}(-\infty,r)} (D_k \varphi - \hat{v}_k \varphi) \, d\mu = \int_{G^{-1}(r)} \varphi D_k G \, d\sigma^G_r, \quad k \in \mathbb{N},$$

holds for functions $\varphi \in C^1_X(\mathbb{R})$. Here we use standard notation: we fix any orthonormal basis $\{v_k : k \in \mathbb{N}\}$ of $H$ contained in $Q(X^*)$, $D_k \varphi$ denotes the derivative of $\varphi$ along $v_k$, and $\hat{v}_k =$
$Q^{-1}v_k \in X^*$. The integration by parts formula holds also for Sobolev functions, provided $\varphi D_k G$ in the surface integral is meant in the sense of traces. Indeed, traces of Sobolev functions on the level hypersurfaces $G^{-1}(r)$ are readily defined through this approach.

At the end of the paper we show that, under suitable assumptions, the measures constructed here coincide with weighted Hausdorff–Gauss surface measures, namely for every $r \in \mathbb{R}$ and for every Borel set $B \subset X$ we have

$$\sigma^G_r (B) = \int_{B \cap G^{-1}(r)} \frac{1}{|DHG|_H} \, d\rho,$$

where $\rho$ is the above mentioned measure of $[10]$, $DHG$ is the generalized gradient of $G$ along $H$, and $| \cdot |_H$ is the $H$-norm, see Sect. 2. This formula clarifies the dependence of $\sigma^G_r$ on $G$. Moreover, more refined properties of surface integrals and traces of Sobolev functions are consequences of the results of [5]. Also, the examples contained in [5] serve as examples here.

2. Notation and preliminaries

We denote by $X^*$ the dual space of $X$, and by $Q : X^* \to X$ the covariance of $\mu$. The Cameron–Martin space is denoted by $H$, its scalar product by $\langle \cdot, \cdot \rangle_H$ and its norm by $| \cdot |_H$. The closed ball in $H$ centered at $h_0$ with radius $r$ is denoted by $B_H(h_0,r)$.

We fix once and for all an orthonormal basis $V = \{v_k : k \in \mathbb{N}\}$ of $H$, contained in $Q(X^*)$. For every $k \in \mathbb{N}$ we set $\hat{v}_k = Q^{-1} (v_k)$.

We recall that if $X$ is a Hilbert space and $X^*$ is canonically identified with $X$, then $Q$ is a compact self-adjoint operator with finite trace, and we can choose a basis $\{e_k : k \in \mathbb{N}\}$ of $X$ consisting of eigenvectors of $Q$, $Q e_k = \lambda_k e_k$. The space $H$ is just $Q^{1/2}(X)$ with the scalar product $\langle h_1, h_2 \rangle_H = \langle Q^{-1/2} h_1, Q^{-1/2} h_2 \rangle_X$, the set $\{v_k := \sqrt{\lambda_k} e_k : k \in \mathbb{N}\}$ is an orthonormal basis of $H$ and we have $\hat{v}_k(x) = \langle x, v_k \rangle_H / \lambda_k$ for each $k \in \mathbb{N}$ and $x \in X$.

Let us recall the definition of Gaussian Sobolev spaces $W^{k,p}(X,\mu)$ for $k = 1, 2, p \geq 1$.

We say that a function $f : X \to \mathbb{R}$ is $H$-differentiable at $x$ if there is $v \in H$ such that $f(x + h) - f(x) = \langle v, h \rangle_H + o(|h|_H)$, for every $h \in H$. In this case $v$ is unique, and we set $D_H f(x) := v$. Moreover for every $k \in \mathbb{N}$ the directional derivative $D_k f(x) := \lim_{t \to 0} (f(x + tv_k) - f(x)) / t$ exists and coincides with $(D_H f(x), v_k)_H$. It is easy to see that if $f$ is Fréchet differentiable at $x$ (as a function from $X$ to $\mathbb{R}$), then it is $H$-differentiable. In particular, the smooth cylindrical functions, namely the functions of the type $f(x) = \varphi(l_1(x), \ldots, l_n(x))$, for some $\varphi \in C^\infty_b (\mathbb{R}^n), l_1, \ldots, l_n \in X^*$, $n \in \mathbb{N}$, are $H$-differentiable at each $x$.

$W^{1,p}(X,\mu)$ and $W^{2,p}(X,\mu)$ are the completions of the smooth cylindrical functions in the norms

$$\|f\|_{W^{1,p}(X,\mu)} := \|f\|_{L^p(X,\mu)} + \left( \int_X \left( \sum_{k=1}^\infty (D_k f(x))^2 \right)^{p/2} \mu(dx) \right)^{1/p},$$

$$\|f\|_{W^{2,p}(X,\mu)} := \|f\|_{W^{1,p}(X,\mu)} + \left( \int_X \left( \sum_{k=1}^\infty \int (D_k f(x))^2 \mu(dx) \right)^{p/2} \mu(dx) \right)^{1/p}.$$

Such spaces are in fact identified with subspaces of $L^p(X,\mu)$, since if $(f_n)$ and $(g_n)$ are Cauchy sequences in the norm of $W^{1,p}(X,\mu)$ (respectively, in the norm of $W^{2,p}(X,\mu)$), and converge to $f$ in $L^p(X,\mu)$, then the sequences $(D_H f_n), (D_H g_n)$ (respectively, $(D_H^2 f_n), (D_H^2 g_n)$) have equal limits in $L^p(X,\mu;H)$ (respectively, in $L^p(X,\mu;\mathcal{H}_2)$, where $\mathcal{H}_2$ is the set of all Hilbert-Schmidt bilinear forms in $H$). In other words, the operators $D_H$ and $D_H^2$, defined in the set of the smooth cylindrical
functions with values in \( L^p(X, \mu; H) \) and in \( L^p(X, \mu; \mathcal{H}_2) \), are closable in \( L^p(X, \mu) \). We still denote by \( D_H \) and \( D^2_H \) their closures, that are called \( H \)-gradient and \( H \)-Hessian.

The spaces \( W^{1,p}(X, \mu; H) \) are defined similarly, using \( H \)-valued, instead of real valued, cylindrical functions. The latter are the elements of the linear span of functions such as \( \psi(x) = \varphi(x)h \), with any smooth cylindrical \( \varphi : X \to \mathbb{R} \) and \( h \in H \).

The Gaussian divergence \( \text{div}_\mu \) is defined in \( W^{1,p}(X, \mu; H) \) by

\[
\text{div}_\mu \Psi(x) = \sum_{k=1}^{\infty} (D_k \psi_k - \hat{v}_k \psi_k),
\]

where \( \psi_k(x) = (\langle \varphi(x), v_k \rangle)_H \), and the series converges in \( L^p(X, \mu) \). See [3, Prop. 5.8.8]. It coincides with (minus) the formal adjoint of the \( H \)-gradient, since we have the integration by parts formula

\[
\int_X \langle D_H \varphi, \Psi \rangle_H d\mu = -\int_X \varphi \text{div}_\mu \Psi d\mu, \quad \varphi \in W^{1,p'}(X, \mu), \; \Psi \in W^{1,p}(X, \mu; H),
\]

with \( p' = p/(p - 1) \). Taking in particular \( \Psi(x) = v_k \) (constant) for any \( k \in \mathbb{N} \), we obtain \( \text{div}_\mu \Psi = Q^{-1}v_k = \hat{v}_k \), and the integration formula reads as

\[
\int_X D_k \varphi d\mu = \int_X \hat{v}_k \varphi d\mu, \quad k \in \mathbb{N}. \tag{2.1}
\]

We refer to [3, Ch.5] for equivalent definitions and properties.

The surfaces taken into consideration are level sets \( \{ x \in X : G(x) = r \} \) of a sufficiently regular function \( G \). Namely, our \( G : X \to \mathbb{R} \) is a \( C_p \)-quasicontinuous function that satisfies

\[
\frac{D_H^2 G}{|D_H G|_H^2} \in W^{1,p}(X, \mu; H). \tag{2.2}
\]

Let us recall that a function \( G : X \to \mathbb{R} \) is \( C_p \)-quasicontinuous if for every \( \varepsilon > 0 \) there exists an open set \( A_\varepsilon \) with Gaussian capacity \( C_p(A_\varepsilon) < \varepsilon \), such that \( G \) is continuous at any \( x \notin A_\varepsilon \). The Gaussian capacity of an open set \( A \subset X \) is defined as

\[
C_p(A) := \inf \{ \| g \|_{W^{1,p}(X, \mu)} : g \geq 1 \; \mu \text{-a.e. on } A, \; g \in W^{1,p}(X, \mu) \}.
\]

We recall that every element of \( W^{1,p}(X, \mu) \) has a \( C_p \)-quasicontinuous version ([3, Thm. 5.9.6]).

In addition to the Sobolev spaces, we shall consider the space \( BUC(X; \mathbb{R}) \) of the uniformly continuous and bounded functions from \( X \) to \( \mathbb{R} \), endowed with the sup norm \( \| \cdot \|_\infty \), and the space \( C^1_b(X; \mathbb{R}) \) of the bounded continuously Fréchet differentiable functions with bounded derivative.

For any Borel function \( f : X \to \mathbb{R} \) we denote by \( \mu \circ f^{-1} \) the image measure on the Borel sets of \( \mathbb{R} \) defined by \( (\mu \circ f^{-1})(I) = \mu(f^{-1}(I)) \). More generally, if \( \varphi : X \to \mathbb{R} \) is another Borel function in \( L^1(X, \mu) \), we define the signed measure \( (\varphi \mu \circ f^{-1})(I) := \int_{f^{-1}(I)} \varphi d\mu \) on the Borel sets \( I \) of \( \mathbb{R} \).

### 3. Construction of surface measures

Throughout the paper, \( G : X \to \mathbb{R} \) is a fixed version of an element of \( W^{1,p}(X, \mu) \) (still denoted by \( G \)), that satisfies (2.2). As pointed out in [19], for \( p = 2 \) an easy sufficient condition for \( G \) to satisfy (2.2) is

\[
G \in W^{2,4}(X, \mu), \quad \frac{1}{|D_H G|_H} \in L^8(X, \mu).
\]

which is generalized to

\[
G \in W^{2,s}(X, \mu), \quad \frac{1}{|D_H G|_H} \in L^q(X, \mu), \quad \frac{1}{s} + \frac{2}{q} \leq \frac{1}{p}
\]

if \( p \) is any number \( > 1 \).
In the following, we shall make some further summability assumptions on the derivatives of $G$. All of them are satisfied if $G$ fulfills next condition (3.20).

3.1. **Continuity of densities.** The starting point is the following well known lemma. See [19, Proposition 2.1.1].

**Lemma 3.1.** $\mu \circ G^{-1}$ is absolutely continuous with respect to the Lebesgue measure $dr$ in $\mathbb{R}$. Moreover the density $q_1 := \frac{d(\mu \circ G^{-1})}{dr}$ is given by

$$q_1(r) = \int_{G^{-1}(-\infty,r)} \text{div}_\mu \left( \frac{DH G}{|DH G|^2_H} \right) d\mu, \quad r \in \mathbb{R},$$

and it is continuous and bounded.

Lemma 3.1 implies that every level surface $G^{-1}(r)$ is negligible, and that for every $\varphi \in L^1(X,\mu)$ the signed measure $\varphi \mu \circ G^{-1}$ is absolutely continuous with respect to the Lebesgue measure. In the following we shall need some properties of the density $q_\varphi$ of $\varphi \mu \circ G^{-1}$ when $\varphi$ belongs to a Sobolev space. They are provided by the following lemma, which is a generalization of Lemma 3.1.

**Lemma 3.2.** Let $\varphi \in W^{1,p'}(X,\mu)$. Then $\varphi \mu \circ G^{-1}$ is absolutely continuous with respect to the Lebesgue measure, and the corresponding density $q_\varphi$ is given by

$$q_\varphi(r) = \int_{G^{-1}(-\infty,r)} \left( \varphi \text{div}_\mu \left( \frac{DH G}{|DH G|^2_H} \right) + \langle DH \varphi, \frac{DH G}{|DH G|^2_H} \rangle_H \right) d\mu, \quad r \in \mathbb{R},$$

and it is continuous and bounded. There is $C > 0$ independent of $\varphi$ such that

$$|q_\varphi(r)| \leq C \|\varphi\|_{W^{1,p'}(X,\mu)}, \quad r \in \mathbb{R}. \quad (3.3)$$

**Proof.** Fix $\alpha < \beta \in \mathbb{R}$ and set

$$f(r) := \mathbb{1}_{[\alpha,\beta]}(r), \quad h(r) = \int_\alpha^r f(s) ds = \begin{cases} 0 & \text{if } r \leq \alpha, \\ r - \alpha & \text{if } \alpha \leq r \leq \beta, \\ \beta - \alpha & \text{if } r \geq \beta. \end{cases}$$

Since $h$ is Lipschitz continuous, then $h \circ G \in W^{1,p}(X,\mu)$ and

$$(h \circ G)'(x) = (f \circ G)'(x) = f'(G(x))DH G.$$
Set as before $\psi = D_H G |D_H G|^2_h$. Then, $\text{div}_\mu(\varphi\psi) = \varphi \text{div}_\mu(\psi) + (D_H\varphi, \psi)_H$. Therefore,

$$\int_{G^{-1}(\alpha, \beta)} \varphi \, d\mu = -\int_X (h \circ G) \text{div}_\mu(\varphi\psi) \, d\mu = -\int_X \left( \int_{\mathbb{R}} 1_{G^{-1}(r, +\infty)} f(r) \, dr \right) \text{div}_\mu(\varphi\psi)(x) \, d\mu$$

$$= -\int_{\mathbb{R}} f(r) \left( \int_X 1_{G^{-1}(r, +\infty)} \text{div}_\mu(\varphi\psi) \, d\mu \right) \, dr = -\int_\alpha^\beta \int_X 1_{G^{-1}(r, +\infty)} \text{div}_\mu(\varphi\psi) \, d\mu \, dr$$

$$= \int_\alpha^\beta \int_X 1_{G^{-1}(-\infty, r)}(\varphi \text{div}_\mu\psi + (D_H\varphi, \psi)_H) \, d\mu$$

(in the last equality we used the fact that the divergence of any vector field in $W^{1,p}(X, \mu; H)$ has zero mean value). Now, let $\varphi \in W^{1,p'}(X, \mu)$ and approach it by a sequence of smooth cylindrical functions $\varphi_n$. After applying the above formula to each $\varphi_n$, we may let $n \to \infty$ in both sides, since $\varphi_n \to \varphi$ and $\varphi_n \text{div}_\mu\psi + (D_H\varphi_n, \psi)_H \to \varphi \text{div}_\mu\psi + (D_H\varphi, \psi)_H$ in $L^1(X, \mu)$. Therefore we get

$$((\varphi \circ G^{-1})(\alpha, \beta)) = \int_\alpha^\beta \int_X 1_{G^{-1}(-\infty, r)}(\varphi \text{div}_\mu\psi + (D_H\varphi, \psi)_H) \, d\mu,$$

namely $\varphi \circ G^{-1}$ has density $q_\varphi$ given by

$$q_\varphi(r) = \int_{G^{-1}(-\infty, r)} (\varphi \text{div}_\mu\psi + (D_H\varphi, \psi)_H) \, \mu(dx),$$

which is continuous and bounded, since $\varphi \text{div}_\mu\psi + (D_H\varphi, \psi)_H \in L^1(X, \mu)$ and $\mu(G^{-1}(r_0)) = 0$ for every $r_0 \in \mathbb{R}$. Estimate $\ref{3.3}$ follows just applying the Hölder inequality. $\square$

### 3.2. Smoothness of densities.

This section is devoted to show that for every uniformly continuous and bounded $\varphi : X \mapsto \mathbb{R}$, the function

$$F_\varphi(r) := \int_{G^{-1}(-\infty, r)} \varphi \, d\mu \quad (\text{3.4})$$

is continuously differentiable.

A useful tool will be the following disintegration formula, whose proof is given for the reader’s convenience in the appendix.

**Theorem 3.3.** Let $X$ be a Polish space endowed with a Borel probability measure $\mu$. Let $\Gamma : X \to \mathbb{R}$ be a Borel function, and set $\lambda = \mu \circ \Gamma^{-1}$. Then there exists a family of Borel probability measures $\{m_s : s \in \mathbb{R}\}$ on $X$ such that

$$\int_X \varphi(x) \, \mu(dx) = \int_{\mathbb{R}} \left( \int_X \varphi(x) \, m_s(dx) \right) \lambda(ds), \quad (\text{3.5})$$

for all $\varphi : X \to \mathbb{R}$ bounded and Borel measurable.

Moreover the support of $m_s$ is contained in $\Gamma^{-1}(s)$ for $\lambda$-almost all $s \in \mathbb{R}$.

**Proposition 3.4.** Let $\varphi \in BUC(X; \mathbb{R})$. Then $F_\varphi \in C^1_b(\mathbb{R})$.

**Proof.** To begin with, let $\varphi : X \mapsto \mathbb{R}$ be Lipschitz continuous. By Lemma $\ref{3.2}$ for each $r \in \mathbb{R}$ we have

$$F_\varphi(r) = \int_{-\infty}^r q_\varphi(s) \, ds,$$

where the function $q_\varphi \in L^1(\mathbb{R})$ is continuous and bounded. Hence, $F_\varphi \in C^1_b(\mathbb{R})$. 

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Taking $\Gamma = G$ and replacing $\varphi$ by $\varphi \mathbb{1}_{G^{-1}(\infty, r)}$ we write the disintegration formula as

$$F_{\varphi}(r) = \int_{-\infty}^{r} \left( \int_X \varphi(x)m_n(dx) \right) q_1(s)ds, \quad r \in \mathbb{R}. \quad (3.6)$$

Therefore, there is a Borel set $I_{\varphi} \subset \mathbb{R}$ such that $(\mu \circ G^{-1})(I_{\varphi}) = 0$ and

$$F'_{\varphi}(r) = q_1(r) \int_X \varphi(x)m_n(dx), \quad r \notin I_{\varphi},$$

so that

$$|F'_{\varphi}(r)| \leq q_1(r)\|\varphi\|_{\infty}, \quad r \notin I_{\varphi}.$$ 

Since both $F'_{\varphi}$ and $q_1$ are continuous,

$$|F'_{\varphi}(r)| \leq q_1(r)\|\varphi\|_{\infty}, \quad r \in \mathbb{R}. \quad (3.7)$$

Let now $\varphi \in \text{BUC}(X; \mathbb{R})$. By [13, Thm.1], there is a sequence of Lipschitz continuous and bounded functions $\varphi_n$ that converge uniformly to $\varphi$ on $X$. Recalling that $|F_{\varphi}(r)| \leq \|\varphi\|_{L^1(X, \mu)} \leq \|\varphi\|_{\infty}$ for every $r \in \mathbb{R}$, estimate (3.7) yields that $(F_{\varphi_n})$ is a Cauchy sequence in $C^b_b(\mathbb{R})$, and the conclusion follows.

For every $\varphi \in \text{BUC}(X; \mathbb{R})$ we still set

$$q_{\varphi}(r) := F'_{\varphi}(r), \quad r \in \mathbb{R}. \quad (3.8)$$

Of course, $q_{\varphi}$ is given by (3.2) only if $\varphi \in W^{1, p'}(X, \mu)$.

3.3. **Surface measures.** Now we are ready to prove the existence of measures on every level surface $G^{-1}(r)$.

**Theorem 3.5.** For every $r \in \mathbb{R}$ there exists a unique Borel measure $\sigma^G_r$ on $\mathcal{B}(X)$ such that

$$q_{\varphi}(r) = \int_X \varphi(x)\sigma^G_r(dx), \quad \varphi \in \text{BUC}(X; \mathbb{R}). \quad (3.9)$$

Moreover, the support of $\sigma^G_r$ is contained in $G^{-1}(r)$, and $\sigma^G_r(X) = q_1(r)$. Therefore, $\sigma^G_r$ is nontrivial iff $q_1(r) > 0$.

**Proof.** Fix $r \in \mathbb{R}$ and set

$$L(\varphi) := q_{\varphi}(r) = F'_{\varphi}(r), \quad \varphi \in \text{BUC}(X; \mathbb{R}) \cup W^{1, p'}(X, \mu). \quad (3.10)$$

Since $F_{\varphi}$ is an increasing function for every $\varphi$ with nonnegative values, then $L(\varphi) \geq 0$ if $\varphi(x) \geq 0$ a.e. Linear positive functionals defined on $\text{BUC}(X; \mathbb{R})$ have not necessarily an integral representation such as (3.9). To show that this is the case, we use the following procedure. We approach $X$ by a sequence of compact sets $K_n$ such that $\lim_{n \to \infty} \mu(K_n) = 1$ and we consider suitable restrictions $L_n$ of $L$ to $C(K_n; \mathbb{R})$, By the Riesz Theorem, such restrictions are represented by measures defined on the Borel sets of $K_n$, readily extended to measures $\lambda_n$ on all Borel sets of $X$. Since $(\lambda_n(B))$ is an increasing sequence for every Borel set $B$, we set $\sigma^G_r(B) := \lim_{n \to \infty} \lambda_n(B)$ and we prove that $\sigma^G_r$ is a measure, that satisfies (3.9).

Let $K$ be a compact subset of $X$ with positive measure. Since the embedding $H \subset X$ is compact, we may assume that $K$ contains $B_H(0, 1)$. Moreover, replacing $K$ by its absolutely convex hull, we may assume that $K$ is symmetric (namely, $K = -K$) and convex. The linear span $E$ of $K$ is a measurable subspace of $X$ with positive measure; by the $0 - 1$ law (e.g., [3, Thm. 2.5.5]) it has measure 1. Therefore, setting

$$K_n := nK,$$
we have

$$\lim_{n \to \infty} \mu(K_n) = 1.$$  

Now we follow a classical procedure in measure theory, see e.g. [6, Ch. 6]. For any \( n \in \mathbb{N} \) we consider the restriction \( L_n \) of \( L \) to \( K_n \), defined for all \( \varphi \geq 0 \) as

$$L_n(\varphi) = \inf \{ L(\psi) : \psi \in BUC(X; \mathbb{R}), \varphi = \psi \text{ on } K_n, \ \psi \geq 0 \text{ on } X \},$$

while if \( \varphi \) takes both positive and negative values, \( L_n \varphi \) is defined by

$$L_n \varphi = L_n \varphi^+ - L_n \varphi^-,$$

where \( \varphi^+ \) and \( \varphi^- \) denote the positive and the negative part of \( \varphi \). The set used in the definition of \( L_n \) is not empty, for instance it contains the extension studied in [17],

$$\psi(x) = \begin{cases} \varphi(x), & x \in K_n, \\ \inf_{u \in K_n} \frac{\varphi(u)}{\|x-u\|} \text{dist}(x, K_n), & x \notin K_n. \end{cases}$$

Then, \( L_n \) is a positive linear functional in \( C(K_n; \mathbb{R}) \). Positivity follows immediately from the positivity of \( L \), linearity is not immediate although elementary, it may be proved as in Lemma 6.4 of [6]. Then, there exists a Borel measure \( \lambda_n \) on \( K_n \) such that

$$L_n(\varphi) = \int_{K_n} \varphi \, d\lambda_n, \quad \varphi \in C(K_n; \mathbb{R}).$$

The obvious extension of \( \lambda_n \) to \( \mathcal{B}(X) \), \( B \mapsto \lambda_n(B \cap K_n) \), is still denoted by \( \lambda_n \).

For every \( \varphi \in BUC(X; \mathbb{R}) \) with nonnegative values the sequence \( (L_n(\varphi)) \) is increasing, since \( \{ L(\psi) : \psi \in BUC(X; \mathbb{R}), \varphi = \psi \text{ on } K_{n+1}, \ \psi \geq 0 \text{ on } X \} \subset \{ L(\psi) : \psi \in BUC(X; \mathbb{R}), \varphi = \psi \text{ on } K_n, \ \psi \geq 0 \text{ on } X \} \) for every \( n \in \mathbb{N} \). It follows that for every \( B \in \mathcal{B}(X) \) the sequence \( (\lambda_n(B)) \) is increasing. Setting

$$\sigma^G_r(B) := \lim_{n \to \infty} \lambda_n(B),$$

we claim that \( \sigma^G_r \) is a measure on \( \mathcal{B}(X) \) and that (3.9) holds.

Note that if \( A, B \) are Borel sets such that \( A \subset B \), then \( \sigma_n(A) \leq \sigma_n(B) \) for every \( n \), and consequently \( \sigma^G_r(A) \leq \sigma^G_r(B) \). Let now \( B, B_m \in \mathcal{B}(X) \) be such that \( B_m \uparrow B \). Then

$$\lim_{m \to \infty} \sigma^G_r(B_m) = \lim_{m \to \infty} (\lim_{n \to \infty} \lambda_n(B_m)) = \sup_{m \in \mathbb{N}} (\sup_{n \in \mathbb{N}} \lambda_n(B_m)) = \sup_{n \in \mathbb{N}} \lambda_n(B) = \sigma^G_r(B).$$

So, \( \sigma^G_r \) is a measure. As a next step, we prove that (3.9) holds for \( \varphi \equiv 1 \). To this aim we construct a sequence of \( W^1(X, \mu) \) functions \( \theta_n \), such that \( \theta_n \equiv 1 \) on \( K_n \) and \( \theta_n \equiv 0 \) outside \( K_{2n} \). The starting point is the Minkowsky functional of \( K \),

$$m(x) := \inf \{ \lambda \geq 0 : x \in \lambda K \}, \quad x \in E,$$

which is positively homogeneous, sub-additive, and \( H \)-Lipschitz since \( K \) contains the unit ball of \( H \). Indeed, for any \( h \in H \) we have

$$\frac{h}{|h|_H} \in B_H(0,1) \subset K,$$

that is, \( h \in |h|_HK \), that implies \( m(h) \leq |h|_H \). As a consequence, for any \( x \in E, \ h \in H \),

$$m(x + h) \leq m(x) + m(h) \leq m(x) + |h|_H,$$

and

$$m(x) = m(x + h - h) \leq m(x + h) + m(-h) \leq m(x + h) + |h|_H,$$
Moreover, \[ \lim_{n \to \infty} \] where the right-hand side converges to \( x \). Then \( \theta_n \in W^{1,q}(X,\mu) \) for every \( q > 1 \) (e.g., [3, Ex. 5.4.10]). Now, let \( \alpha \in C_c^\infty(\mathbb{R}) \) be such that \( \alpha \equiv 1 \) in \([0,1]\), \( \alpha \equiv 0 \) in \([2,+\infty)\), \( 0 \leq \alpha \leq 1 \), and set

\[
\theta_n(x) = \begin{cases} 
\alpha(m(x/n)), & x \in E, \\
0, & x \notin E.
\end{cases}
\]

Then \( \theta_n \in W^{1,q}(X,\mu) \) for every \( q > 1 \). Recalling that for every \( x \in E \) we have \( m(x/n) \leq 1 \) iff \( x \in K_n \), we obtain \( \theta_n \equiv 1 \) in \( K_n \), \( \theta_n \equiv 0 \) outside \( K_{2n} \), and

\[
\lim_{n \to \infty} \|\theta_n - 1\|_{W^{1,q}(X,\mu)} = 0.
\]

Indeed, \( \lim_{n \to \infty} \theta_n(x) = 1 \) for every \( x \in E \) and \( 0 \leq \theta_n(x) \leq 1 \), so that \( \lim_{n \to \infty} \theta_n = 1 \) in \( L^q(X,\mu) \). Moreover,

\[
D_H \theta_n(x) = \frac{1}{n} \alpha'(m(x/n)) D_H m(x/n),
\]

so that \( \lim_{n \to \infty} D_H \theta_n = 0 \) in \( L^q(X,\mu;H) \).

Then,

\[
\sigma_r^G(X) = \lim_{n \to \infty} \lambda_n(X) = \lim_{n \to \infty} \int_X 1 \, d\lambda_n = \lim_{n \to \infty} L_n(1) = \lim_{n \to \infty} L_{2n}(1).
\]

On the other hand, for every \( \psi \in BUC(X) \) such that \( \psi \geq 1 \) in \( K_{2n} \), \( \psi \geq 0 \) in \( X \), we have \( \psi \geq \theta_n \) and therefore \( L \psi \geq L(\theta_n) \), since \( L(\psi) - L(\theta_n) \) is the derivative at \( r \) of the increasing function \( \xi \mapsto \mu(\{x : \psi(x) - \theta_n(x) \leq \xi\}) \). Taking the infimum, we get \( L_{2n}(1) \geq L(\theta_n) \). Since \( \theta_n \) goes to \( 1 \) in \( W^{1,q}(X,\mu) \), by Lemma 3.2 \( L(\theta_n) \) goes to \( L(1) = q_1(r) \) as \( n \to \infty \). This shows that

\[
\sigma_r^G(X) = q_1(r) = L(1). \tag{3.11}
\]

Now we show that (3.9) holds for any \( \varphi \in BUC(X;\mathbb{R}) \). It is sufficient to prove that it holds for every \( \varphi \in BUC(X;\mathbb{R}) \) with values in \([0,1]\). In this case, by definition,

\[
L \varphi \geq L_n(\varphi|_{K_n}) = \int_X \varphi \, d\lambda_n
\]

where the right-hand side converges to \( \int_X \varphi \, d\sigma_r^G \) as \( n \to \infty \), since the sequence \( (\lambda_n) \) weakly converges to \( \sigma_r^G \). Therefore,

\[
L \varphi \geq \int_X \varphi \, d\sigma_r^G
\]

Now we remark that \( 1 - \varphi \) has positive values, and using (3.11) and the above inequality we get

\[
q_1(r) - L \varphi = L(1 - \varphi) \geq \int_X (1 - \varphi) \, d\sigma_r^G = q_1(r) - \int_X \varphi \, d\sigma_r^G
\]

so that

\[
L \varphi \leq \int_X \varphi \, d\sigma_r^G,
\]

and (3.9) follows.

It remains to prove that \( \sigma_r^G \) has support in \( G^{-1}(r) \). To this aim, we remark that for every \( \varepsilon > 0 \) and \( \varphi \in BUC(X;\mathbb{R}) \) with support contained in \( G^{-1}((-\infty,r-\varepsilon) \cup G^{-1}(r+\varepsilon,\infty) \), the function \( F_\varphi \) is constant in \( (r-\varepsilon, r+\varepsilon) \), and therefore \( F_\varphi'(r) = 0 \). By (3.9), \( \int_X \varphi \, d\sigma_r^G = 0 \). So, the support of \( \sigma_r^G \) is contained in \( \cap_{\varepsilon > 0} G^{-1}[r-\varepsilon, r+\varepsilon] = G^{-1}(r) \). \( \square \)
Remark 3.6. Let \( m_r \) be the probability measures given by the disintegration theorem \([3.3]\). If \( X^* \) is separable, then for a.e. \( r \in \mathbb{R} \) such that \( q_1(r) > 0 \) we have

\[
\sigma_r^G = q_1(r)m_r.
\]

Proof. Fix any \( \varphi \in \text{BUC}(X;\mathbb{R}) \). Applying formula \([3.5]\) to the function \( \varphi \mathbb{1}_{G^{-1}(-\infty,r)} \) we obtain

\[
F_\varphi(r) = \int_{-\infty}^{r} \left( \int_{X} \varphi(x)m_s(dx) \right) q_1(s) ds, \quad r \in \mathbb{R}.
\]

On the other hand, by \([3.9]\) we have

\[
F_\varphi(r) = \int_{-\infty}^{r} \left( \int_{X} \varphi(x)\sigma_s^G(dx) \right) ds, \quad r \in \mathbb{R}.
\]

Therefore, there exists a negligible \( I_\varphi \subset \mathbb{R} \) such that for every \( s \not\in I_\varphi \) we have

\[
\int_{X} \varphi \, d\sigma_s^G = q_1(s) \int_{X} \varphi(x) m_s(dx).
\]

Let \( \mathcal{F} = \{f_n: n \in \mathbb{N}\} \) be any dense subset of \( X^* \), and set \( I = \bigcup_{n \in \mathbb{N}} I_{e^{\imath f_n}} \). For every \( r \not\in I \) we have

\[
\int_{X} e^{\imath f_n} \, d\sigma_r^G = q_1(r) \int_{X} e^{\imath f_n(x)} m_r(dx).
\]

Approaching every \( f \in X^* \) by a sequence of elements of \( \mathcal{F} \), and using the Dominated Convergence Theorem, we obtain that if \( q_1(r) \neq 0 \), then the probability measures \( \sigma_r^G/q_1(r) \) and \( m_r \) have the same Fourier transform, so that they coincide. \( \square \)

The following proposition shows a class of sets that are negligible with respect to all measures \( \sigma_r^G \).

Proposition 3.7. Let \( B \subset X \) be a Borel set with \( C_{\rho'}(B) = 0 \). Then \( \sigma_r^G(B) = 0 \), for every \( r \in \mathbb{R} \).

Proof. We partly follow the argument used in \([3]\), Lemma 6.10.1. For every \( \varepsilon > 0 \) let \( O_{\varepsilon} \supset B \) be an open set such that \( C_{\rho'}(O_{\varepsilon}) < \varepsilon \). Then there exists \( f_\varepsilon \in W^{1,\rho'}(X,\mu) \) such that \( \|f_\varepsilon\|_{W^{1,\rho'}(X,\mu)} \leq \varepsilon \) and \( f_\varepsilon \geq 1 \) a.e. in \( O_{\varepsilon} \). Replacing \( f_\varepsilon \) by \( \max\{f_\varepsilon,0\} \) we may assume that \( f_\varepsilon \geq \mathbb{1}_{O_{\varepsilon}}, \mu\text{-a.e.} \)

Let us fix a sequence of \( \text{BUC} \) functions that converge to \( \mathbb{1}_{O_{\varepsilon}} \) pointwise. For instance, we can take

\[
\theta_n(x) = \begin{cases} 0, & x \in X \setminus O_{\varepsilon}, \\ n \operatorname{dist}(x, X \setminus O_{\varepsilon}), & 0 < \operatorname{dist}(x, X \setminus O_{\varepsilon}) < 1/n, \\ 1, & \operatorname{dist}(x, X \setminus O_{\varepsilon}) \geq 1/n. 
\end{cases}
\]

Then, \( \lim_{n \to \infty} \theta_n(x) = \mathbb{1}_{O_{\varepsilon}}(x) \), for every \( x \in X \). Using the Dominated Convergence Theorem, and then formula \([3.9]\), we get

\[
\sigma_r^G(O_{\varepsilon}) = \int_X \mathbb{1}_{O_{\varepsilon}} \, d\sigma_r^G = \lim_{n \to \infty} \int_X \theta_n \, d\sigma_r^G = \lim_{n \to \infty} q_{\theta_n}(r).
\]

On the other hand, \( f_\varepsilon(x) \geq \mathbb{1}_{O_{\varepsilon}}(x) \geq \theta_n(x) \), for \( \mu\text{-a.e.} \) \( x \in X \), so that the function \( F_{f_\varepsilon-\theta_n} \) is increasing. In particular, \( F_{f_\varepsilon-\theta_n}(r) = q_{f_\varepsilon}(r) - q_{\theta_n}(r) \geq 0 \) for every \( r \in \mathbb{R} \). Therefore, for every \( r \in \mathbb{R} \),

\[
\sigma_r^G(O_{\varepsilon}) \leq q_{f_\varepsilon}(r).
\]

On the other hand, by \([3.3]\) we have

\[
\|q_{f_\varepsilon}(r)\| \leq C \|f_\varepsilon\|_{W^{1,\rho'}(X,\mu)} \leq C \varepsilon,
\]

with \( C \) independent of \( \varepsilon \). Therefore, \( \sigma_r^G(O_{\varepsilon}) \leq C \varepsilon \), which implies \( \sigma_r^G(B) = 0 \). \( \square \)
Proposition 3.7 clarifies the dependence of the measures $\sigma^G_\epsilon$ on the version of $G$ that we have fixed. Two versions of $G$ that coincide outside a set with null $C'_\mu$ capacity give rise to the same measures $\sigma^G_\epsilon$.

3.4. Integration by parts formulae. To start with, we establish an integration formula for $C^1_\mu$ functions that is a first step towards an integration by parts formula. The proof follows arguments from [7, 5] (in fact, it is a rewriting of a part of [5, Prop. 4.1] in our setting).

Proposition 3.8. Let $k \in \mathbb{N}$ be such that either $D_kG \in W^{1,p'}(X,\mu)$ or $D_kG \in \text{BUC}(X,\mathbb{R})$. Then for every $\varphi \in C^1_\mu(X,\mathbb{R})$ and for every $r \in \mathbb{R}$ we have

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi) \, d\mu = q_{\varphi D_kG}(r).$$

(3.12)

Moreover, (3.12) holds also for every $\varphi \in W^{1,q}(X,\mu)$ provided $D_kG \in W^{1,s}(X,\mu)$ and

$$\frac{1}{q} + \frac{1}{s} + \frac{1}{p} \leq 1.$$  

(3.13)

Proof. Fix $\varphi \in C^1_\mu(X;\mathbb{R})$. For $\epsilon > 0$ we define a function $\theta_\epsilon$ by

$$\theta_\epsilon(\xi) := \begin{cases} 1, & \xi \leq -\epsilon, \\ -\frac{\epsilon}{\epsilon} \xi, & -\epsilon < \xi < 0, \\ 0, & \xi \geq 0. \end{cases}$$

and we consider the function

$$x \mapsto \varphi(x)\theta_\epsilon(G(x) - r),$$

which belongs to $W^{1,p'}(X,\mu)$, and its derivative along $v_k$ is $\theta'_\epsilon(G(x))D_kG(x)\varphi(x) + \theta_\epsilon(G(x))D_k\varphi(x)$. Applying the integration by parts formula (2.1) we get

$$\int_X (D_k\varphi - \hat{v}_k\varphi)(\theta_\epsilon \circ G) \, d\mu = \frac{1}{\epsilon} \int_{G^{-1}(r-\epsilon,r)} \varphi D_kG \, d\mu, \quad k \in \mathbb{N}. \quad (3.14)$$

As $\epsilon \to 0$, $\theta_\epsilon \circ G$ converges pointwise to $1_{G^{-1}(-\infty,r)}$. Since $0 \leq \theta_\epsilon \circ G \leq 1$, by the Dominated Convergence Theorem the left hand side converges to

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi - \hat{v}_k\varphi) \, d\mu.$$

Concerning the right hand side, for every $\epsilon > 0$ we have

$$\frac{1}{\epsilon} \int_{G^{-1}(r-\epsilon,r)} \varphi D_kG \, d\mu = \frac{1}{\epsilon} \int_{r-\epsilon}^r q_{\varphi D_kG}(\xi) \, d\xi.$$

Since $\varphi D_kG$ belongs to $W^{1,p'}(X,\mu)$ or to $\text{BUC}(X;\mathbb{R})$, by Lemma 3.2 or by Proposition 3.4 the function $q_{\varphi D_kG}$ is continuous. Therefore,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{G^{-1}(r-\epsilon,r)} \varphi D_kG \, d\mu = q_{\varphi D_kG}(r),$$

and (3.12) follows.

Let now $\varphi \in W^{1,q}(X,\mu)$, $D_kG \in W^{1,s}(X,\mu)$, with $q, s$ satisfying (3.13). Let $\varphi_n \in C^1_\mu(X,\mu)$ approach $\varphi$ in $W^{1,q}(X,\mu)$, so that $\varphi_n D_kG$ approaches $\varphi D_kG$ in $W^{1,p'}(X,\mu)$. By (3.12), for every $r \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$\int_{G^{-1}(-\infty,r)} (D_k\varphi_n - \hat{v}_k\varphi_n) \, d\mu = q_{\varphi_n D_kG}(r).$$

(3.12)
Letting $n \to \infty$, the left hand side goes to $\int_{G^{-1}(-\infty,r)} (D_k \varphi_n - \hat{v}_k \varphi_n) \, d\mu$, while the right hand side goes to $q_\varphi D_k G(r)$ by (3.13).

The measures $\sigma_r^G$ constructed in Theorem 3.5 are trivial if $q_1(r) = 0$. So, it is important to know for which values of $r$ we have $q_1(r) > 0$. This question was addressed in the paper [14], where it was proved that under the assumptions of [1], the set $I := \{r \in \mathbb{R} : q_1(r) > 0\}$ is an interval. Here we improve such a result, characterizing $I$ under more general assumptions and with a different simpler proof.

**Lemma 3.9.** Assume that for every $k \in \mathbb{N}$, $D_k G \in W^{1,p'}(X,\mu) \cup BUC(X,\mathbb{R})$. Then $\{r \in \mathbb{R} : q_1(r) > 0\} = (\text{ess inf } G, \text{ess sup } G)$.

**Proof.** By Theorem 3.5 if $q_1(r) = 0$ then $\int_X \varphi \, d\sigma_r^G = 0$ for every $\varphi \in BUC(X,\mathbb{R})$, which implies $q_\varphi(r) = 0$ for every $\varphi \in BUC(X,\mathbb{R})$. Approaching any $\varphi \in W^{1,p'}(X,\mu)$ by a sequence of $C^1_b$ functions $\varphi_n$, it follows that $q_\varphi(r) = 0$ for every $\varphi \in W^{1,p'}(X,\mu)$. Taking $\varphi = D_k \psi D_k G$, with any cylindrical smooth $\psi$, we have $q_\varphi(r) = 0$, and formula (3.12) yields

$$\int_{G^{-1}(-\infty,r)} (D_k \psi - \hat{v}_k D_k \psi) \, d\mu = 0.$$ 

Summing over $k$ and using [3] Thm. 5.8.3, Rem. 5.8.7 we obtain

$$\int_{G^{-1}(-\infty,r)} L \psi \, d\mu = 0,$$

where $L$ is the realization of the Ornstein–Uhlenbeck operator in $L^2(X,\mu)$. We recall (e.g., [3] Thm. 5.7.1]) that the domain of $L$ is $W^{2,2}(X,\mu)$, and the graph norm of $L$ is equivalent to the $W^{2,2}$-norm. This implies that the set of the cylindrical smooth functions is a core for $L$, and then (3.15) holds for every $\psi \in D(L)$. In other words, the characteristic function $1_{G^{-1}(-\infty,r)}$ is orthogonal to the range of $L$. Since $0$ is an isolated simple eigenvalue of $L$, the orthogonal space to the range of $L$ consists of constant a.e. functions. Then, $1_{G^{-1}(-\infty,r)}$ is constant $\mu$-a.e., which implies that either $\mu(G^{-1}(-\infty,r)) = 0$ or $\mu(G^{-1}(-\infty,r)) = 1$. So, $q_1(r) = 0$ implies that $r \in (-\infty, \text{ess inf } G] \cup [\text{ess sup } G, +\infty)$.

Conversely, the function $F_1$ is continuously differentiable, and it is constant in $(-\infty, \text{ess inf } G]$ and in $[\text{ess sup } G, +\infty)$, so that for every $r \in (-\infty, \text{ess inf } G] \cup [\text{ess sup } G, +\infty)$ we have $F_1'(r) = q_1(r) = 0$.

Let us go back to Proposition 3.8. We recall that $q_\varphi(r) = \int_X \varphi \, d\sigma_r^G$ if $\varphi \in BUC(X;\mathbb{R})$. Therefore, if $G$ and $\varphi$ are so smooth that $\varphi D_k G \in BUC(X;\mathbb{R})$, (3.12) yields

$$\int_{G^{-1}(-\infty,r)} (D_k \varphi - \hat{v}_k \varphi) \, d\mu = \int_{G^{-1}(r)} \varphi D_k G \, d\sigma_r^G.$$ 

For more general $G$ and $\varphi$ the above formula still holds, but it is not obvious. For the right hand side of (3.16) to make sense, we need conditions guaranteeing that $\varphi D_k G$ has a trace at $G^{-1}(r)$, belonging to $L^1(X,\sigma_r^G)$. Then, $\varphi D_k G$ in the right hand side integral should be interpreted in the sense of traces.

The starting point is Lemma 3.2 and in particular formula (3.3), applied to the function $|\varphi|$, that together with (3.2) yields

$$\int_{G^{-1}(r)} |\varphi| \, d\sigma_r^G = q_{|\varphi|}(r) \leq C\|\varphi\|_{W^{1,p'}(X,\mu)}, \quad \varphi \in C^1_b(X;\mathbb{R}).$$

Since $C^1_b(X;\mathbb{R})$ is dense in $W^{1,p'}(X,\mu)$, the above estimate is extended to the whole of $W^{1,p'}(X,\mu)$, and it allows to define the traces of such Sobolev functions at $G^{-1}(r)$. Indeed, approaching any
\( \varphi \in W^{1,p'}(X, \mu) \) by a sequence of \( C^1_b \) functions \( \varphi_n \). Implication (3.17) implies that the sequence of the restrictions \( \varphi_n|_{G^{-1}(r)} \) to \( G^{-1}(r) \) is a Cauchy sequence in \( L^1(G^{-1}(r), \sigma_r^G) \), that converges to an element of \( L^1(G^{-1}(r), \sigma_r^G) \). Still by (3.17), such element does not depend on the approximating sequence.

**Definition 3.10.** Let \( \varphi \in W^{1,p'}(X, \mu) \). The trace of \( \varphi \) at \( G^{-1}(r) \) is the limit in \( L^1(G^{-1}(r), \sigma_r^G) \) of the sequence of the restrictions \( \varphi_n|_{G^{-1}(r)} \) to \( G^{-1}(r) \), for every sequence of \( C^1_b \) functions \( \varphi_n \) that converges to \( \varphi \) in \( W^{1,p'}(X, \mu) \). It is denoted by \( \varphi|_{G^{-1}(r)} \).

By definition, \( \varphi|_{G^{-1}(r)} \in L^1(G^{-1}(r), \sigma_r^G) \), and \( \| \varphi|_{G^{-1}(r)} \|_{L^1(G^{-1}(r), \sigma_r^G)} \leq C \| \varphi \|_{W^{1,p'}(X, \mu)} \), where \( C \) is the constant in (3.17). In other words, the trace is a bounded operator from \( W^{1,p'}(X, \mu) \) to \( L^1(G^{-1}(r), \sigma_r^G) \). If \( \varphi \in W^{1,q}(X, \mu) \), with \( q > p' \), then \( \| \varphi|^{q/p'} \in W^{1,p'}(X, \mu) \), and estimate (3.17) applied to \( |\varphi|^{q/p'} \) yields that the trace of \( \varphi \) at \( G^{-1}(r) \) belongs to \( L^{q/p'}(G^{-1}(r), \sigma_r^G) \), and the trace operator is bounded from \( W^{1,q}(X, \mu) \) to \( L^{q/p'}(G^{-1}(r), \sigma_r^G) \).

The trace operator preserves positivity, as the next lemma shows.

**Lemma 3.11.** Let \( \varphi \in W^{1,p'}(X, \mu) \) have nonnegative values, \( \mu \)-a.e. Then for every \( r \in \mathbb{R} \) the trace of \( \varphi \) at \( G^{-1}(r) \) has nonnegative values, \( \sigma_r^G \)-a.e.

**Proof.** Let \( (\varphi_n) \) be a sequence of \( C^1_b \) functions, converging to \( \varphi \) in \( W^{1,p'}(X, \mu) \). Possibly replacing \( (\varphi_n) \) by a subsequence, we may assume that \( (\varphi_n) \) converges to \( \varphi \) pointwise \( \mu \)-a.e.

We claim that the sequence \( (\varphi_n^+) \) (the positive parts of \( \varphi_n \)) still converges to \( \varphi \) in \( W^{1,p'}(X, \mu) \). Indeed, \( \| \varphi_n - \varphi \|_{L^{p'}(X, \mu)} \leq \| \varphi_n - \varphi \|_{L^{p'}(X, \mu)} \), while, recalling that \( D_H\varphi_n^+ = D_H\varphi_n \) in the set \( \{x : \varphi_n(x) > 0\} \), and \( D_H\varphi_n^+ = 0 \) in the set \( \{x : \varphi_n(x) \leq 0\} \), \( D_H\varphi = 0 \) in the set \( \{x : \varphi(x) = 0\} \) (Lemma 5.7.7) we obtain

\[
\int_X |D_H\varphi_n^+ - D_H\varphi|^{p'}_{X, \mu} d\mu = \int_{\{x: \varphi_n(x) > 0\}} |D_H\varphi_n^+ - D_H\varphi|^{p'}_{X, \mu} d\mu + \int_{\{x: \varphi_n(x) \leq 0\}} |D_H\varphi_n^+ - D_H\varphi|^{p'}_{X, \mu} d\mu
\]

\[
= \int_{\{x: \varphi_n(x) > 0\}} |D_H\varphi_n - D_H\varphi|^{p'}_{X, \mu} d\mu + \int_{\{x: \varphi_n(x) \leq 0\}} |D_H\varphi|^{p'}_{X, \mu} d\mu
\]

\[
\leq \|D_H\varphi_n - D_H\varphi\|_{L^{p'}(X, \mu; H)} + \|D_H\varphi\|_{L^{p'}(X, \mu; H)}
\]

Setting \( A_n := \{x : \varphi_n(x) \leq 0, \varphi(x) > 0\} \), then \( \mu(A_n) \) vanishes as \( n \to \infty \). This is because for every \( x \) in the set

\[
A := \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k,
\]

the sequence \( (\varphi_n(x)) \) does not converge to \( \varphi(x) \), and therefore \( 0 = \mu(A) = \lim_{n \to \infty} \mu(\bigcup_{k \geq n} A_k) \geq \limsup_{n \to \infty} \mu(A_n) \). Then, the right hand side of (3.18) vanishes as \( n \to \infty \), and this implies that \( (\varphi_n^+) \) converges to \( \varphi \) in \( W^{1,p'}(X, \mu) \). Consequently, the traces of \( (\varphi_n^+) \) at \( G^{-1}(r) \) converge to the trace of \( \varphi \) at \( G^{-1}(r) \), in \( L^1(G^{-1}(r), \sigma_r^G) \). Since each \( (\varphi_n^+) \) has nonnegative values at every \( x \in G^{-1}(r) \), then their \( L^1 \) limit has nonnegative values, \( \sigma_r^G \)-a.e.

**Formula** (3.3) may now be extended to elements of \( W^{1,p'}(X, \mu) \).

**Lemma 3.12.** For every \( \varphi \in W^{1,p'}(X, \mu) \) and for every \( r \in \mathbb{R} \) we have

\[
q_\varphi(r) = \int_{G^{-1}(r)} |\varphi|_{G^{-1}(r)} d\sigma_r^G.
\]
Proof. It is sufficient to approximate \( \varphi \) in \( W^{1,p'}(X, \mu) \) by a sequence of functions \( \varphi_n \in C^1_b(X; \mathbb{R}) \), and to let \( n \to \infty \) in the equality

\[
q_{\varphi_n}(r) = \int_X \varphi_n \, d\sigma_r^G,
\]

that holds by Theorem 3.3. The left hand side goes to \( q_\varphi(r) \) by estimate (3.3), the right hand side goes to \( \int_{G^{-1}(r)} \varphi|_{G^{-1}(r)} \, d\sigma_r^G \) by the above construction of the trace of \( \varphi \).

With the aid of Proposition 3.7 we can prove that the traces of the elements of \( W^{1,p'}(X, \mu) \) at \( G^{-1}(r) \) coincide with the restrictions of their \( C_{p'} \)-quasicontinuous versions at \( G^{-1}(r) \). In particular, if \( \varphi \) is a continuous version of a Sobolev function, its trace is just the restriction of \( \varphi \) at \( G^{-1}(r) \). This justifies the notation \( \varphi|_{G^{-1}(r)} \) for the trace of \( \varphi \) at \( G^{-1}(r) \).

**Proposition 3.13.** Let \( \varphi \) be a \( C_{p'} \)-quasicontinuous version of an element of \( W^{1,p'}(X, \mu) \). Then the trace of \( \varphi \) at \( G^{-1}(r) \) coincides with the restriction of \( \varphi \) at \( G^{-1}(r) \), \( \sigma_r^G \)-a.e.

**Proof.** We use arguments similar to [5] Prop. 4.8. Let \( (\varphi_n) \) be a sequence of smooth cylindrical functions that converge to \( \varphi \) in \( W^{1,p'}(X, \mu) \). By [3] Thm. 5.9.6(ii), applied with the operator \( T = (I - L)^{-1/2} \), a subsequence \( (\varphi_{nk}) \) converges to \( \varphi(x) \) for every \( x \) except at most on a set with zero Gaussian capacity \( C_{p'} \). By Proposition 3.7 such a subsequence converges \( \sigma_r^G \)-a.e to \( \varphi \). On the other hand, by the definition of the trace, the restrictions of \( \varphi_n \) to \( G^{-1}(r) \) converge to \( \varphi|_{G^{-1}(r)} \) in \( L^1(G^{-1}(r), \sigma_r^G) \). In particular, a subsequence of \( (\varphi_{nk}) \) converges to \( \varphi|_{G^{-1}(r)} \), \( \sigma_r^G \)-a.e. Therefore, \( \varphi|_{G^{-1}(r)} = \varphi, \sigma_r^G \)-a.e.

To extend the integration by parts formula (3.16) to Sobolev functions we need some further assumptions on \( G \).

**Corollary 3.14.** Let \( \varphi \in W^{1,q}(X, \mu) \) and \( D_kG \in W^{1,s}(X, \mu) \), with \( q, s \) satisfying (3.13). Then formula (3.16) holds, with \( \varphi D_kG \) replaced by \( (\varphi D_kG)|_{G^{-1}(r)} \).

**Proof.** Note that \( \varphi D_kG \in W^{1,p'}(X, \mu) \). By Lemma 3.12 we have

\[
\int_{G^{-1}(r)} (\varphi D_kG)|_{G^{-1}(r)} \, d\sigma_r^G = q_{\varphi D_kG}(r).
\]

On the other hand, Proposition 3.8 yields

\[
q_{\varphi D_kG}(r) = \int_{G^{-1}(-\infty, r)} (D_k\varphi - \hat{v}_k \varphi) \, dm,
\]

and the statement follows.

**Corollary 3.14** yields an integration by parts formula for \( \varphi \in W^{1,s}(X, \mu) \) for any \( s > 1 \), provided \( G \) is good enough. In particular, if next assumption (3.20) holds, then \( G \) satisfies (2.2) and the conditions of Corollary 3.14 with any \( s > 1 \), and (3.16) holds for \( \varphi \in W^{1,q}(X, \mu) \) for any \( q > 1 \).

### 3.5 Dependence on \( G \), and relationship with other surface measures.

Now we are ready to compare the measures \( \sigma_r^G \) defined in (3.3) with the perimeter measure and with the Hausdorff-Gauss surface measure \( \rho \) of Feyel and de La Pradelle [10].

We use the notation of [2]. We recall that a subset \( B \subset X \) is said to have finite perimeter if \( 1_B \) is a bounded variation function, namely there exists a \( H \)-valued measure \( \Gamma \) such that for every \( k \in \mathbb{N} \) and for every smooth cylindrical function \( \varphi \) we have

\[
\int_B (D_k\varphi - \hat{v}_k \varphi) \, dm = \int_X \varphi \, d\gamma_k,
\]

with \( \gamma_k = \langle \Gamma, v_k \rangle_H \). In this case, \( \Gamma \) is unique, it is called perimeter measure, and denoted by \( D_\mu 1_B \).
If $G \in W^{2,p'}(X, \mu)$, then for every $r \in \mathbb{R}$ the set $B = G^{-1}(\mathbb{R}^+, r)$ satisfies the above condition, with $D_\mu \mathbb{I}_{G^{-1}(\mathbb{R}^+, r)} = D_H G |_{G^{-1}(r)} \sigma^G_r$. Indeed, by formulae (3.12) (applied to $\varphi$) and (3.19) (applied to $\varphi D_k G$), for every for smooth cylindrical $\varphi$ and for every $k \in \mathbb{N}$ we have

$$\int_{G^{-1}(\mathbb{R}^+, r)} (D_k \varphi - \check{\varphi}_k \varphi) \, d\mu = \int_X (\varphi D_k G) |_{G^{-1}(r)} \, d\sigma^G_r.$$ 

On the other hand, for smooth cylindrical $\varphi$ the trace of $\varphi D_k G$ at the support $G^{-1}(r)$ of $\sigma^G_r$ coincides $\sigma^G_r$-a.e. with the restriction of $\varphi D_k G$ at $G^{-1}(r)$, where $\hat{D}_k G$ is any $C_p'$-quasicontinuous version of $D_k G$, by Proposition 3.15.

Let us now recall the assumptions of Feyel [9],

$$G \in \bigcap_{p > 1} W^{2,p}(X, \mu), \quad \frac{1}{|D_H G|^H} \in \bigcap_{p > 1} L^p(X, \mu),$$

(3.20)

under which it was proved that for every $\varphi \in W^{1,q}(X; \mathbb{R})$ for some $q > 1$, the density $q_\varphi$ of the signed measure $\varphi \mu \circ G^{-1}$ with respect to the Lebesgue measure is given by

$$q_\varphi(r) = \int_{G^{-1}(r)} \varphi \, \frac{1}{|D_H G|^H} \, d\rho, \quad r \in \mathbb{R}.$$ (3.21)

In the right hand side, $\varphi$ and $|D_H G|^H$ are quasicontinuous versions of the respective Sobolev elements. More precisely, $\varphi$ is $C_q$-quasicontinuous and $|D_H G|^H$ is $C_p'$-quasicontinuous for every $p > 1$. See [9–11].

**Proposition 3.15.** Let $G$ satisfy (3.20). Then for every Borel set $B \subset X$ we have

$$\sigma^G_r(B) = \int_{B \cap G^{-1}(r)} \frac{1}{|D_H G|^H} \, d\rho, \quad r \in \mathbb{R}.$$ 

**Proof.** Note that if (3.20) holds, then $G$ satisfies (2.2). Comparing (3.21) with (3.9) yields

$$\int_X \varphi \, d\sigma^G_r = \int_{G^{-1}(r)} \varphi \, \frac{1}{|D_H G|^H} \, d\rho, \quad \varphi \in C_b^1(X; \mathbb{R}),$$

and the statement holds. $\square$

**Corollary 3.16.** Let $G_1, G_2$ satisfy (3.20). Assume that for some $r_1, r_2 \in \mathbb{R}$ we have $G^{-1}_1(r_1) = G^{-1}_2(r_2) := \Sigma$. Then

$$|D_H G_1(r_1)|H \, d\sigma^G_{r_1} = |D_H G_1(r_2)|H \, d\sigma^G_{r_2} = \rho|\Sigma|.$$ 

We recall that the assumptions of [1] are

$$G \in \bigcap_{k \in \mathbb{N}, p > 1} W^{k,p}(X, \mu), \quad \frac{1}{|D_H G|^H} \in \bigcap_{p > 1} L^p(X, \mu),$$

(3.22)

and that the measures $\nu_r$ constructed in [1–8] under such assumptions, for all $r \in \mathbb{R}$ such that $q_1(r) > 0$, coincide with the restriction of $\rho$ to $G^{-1}(r)$. This is because they satisfy the same integration by parts formula,

$$\int_{G^{-1}(r)} (D_k \varphi - \check{\varphi}_k \varphi) \, d\mu = \int_{G^{-1}(r)} \varphi \frac{D_k G}{|D_H G|^H} \, d\nu_r = \int_{G^{-1}(r)} \varphi \frac{D_k G}{|D_H G|^H} \, d\rho, \quad \varphi \in \bigcap_{k \in \mathbb{N}, p > 1} W^{k,p}(X, \mu),$$

(3.23)

and replacing $\varphi$ by $\varphi \frac{D_k G}{|D_H G|^H}$ and summing up, we obtain

$$\int_{G^{-1}(r)} \varphi \, d\nu_r = \int_{G^{-1}(r)} \varphi \, d\rho, \quad \varphi \in \bigcap_{k \in \mathbb{N}, p > 1} W^{k,p}(X, \mu).$$
which implies the statement.

**APPENDIX A. PROOF OF THE DISINTEGRATION THEOREM**

We follow here [20]. For any $A \in \mathcal{B}(X)$ we consider the conditional expectation $\mathbb{E}[\mathbb{1}_A|\Gamma]$, which may be expressed as $f_A \circ \Gamma$ for some Borel function $f_A : \mathbb{R} \to \mathbb{R}$. So, for any $I \in \mathcal{B}(\mathbb{R})$ we have

$$\int_{\Gamma^{-1}(I)} \mathbb{1}_A \, d\mu = \int_{\Gamma^{-1}(I)} (f_A \circ \Gamma) \, d\mu = \int_I f_A(r) (\mu \circ \Gamma^{-1})(dr)$$

which we rewrite as

$$\mu(A \cap \Gamma^{-1}(I)) = \int_I f_A(r) \lambda(dr). \quad (\text{A.1})$$

Since $X$ is separable, there exists $K \subset \mathbb{R}$ with $\lambda(K) = 0$ and for any $r \notin K$ a Borel measure $m_r$ on $\mathbb{R}$ such that

$$f_A(r) = m_r(A), \quad \forall \ r \notin K.$$ 

See e.g. [3], Theorem 10.2.2. Replacing in (A.1) we obtain

$$\mu(A \cap \Gamma^{-1}(I)) = \int_I m_r(A) \lambda(dr), \quad I \in \mathcal{B}(\mathbb{R}). \quad (\text{A.2})$$

It is enough to prove that (A.1) holds for $\varphi = \mathbb{1}_{\Gamma^{-1}(J)}$ with $J \in \mathcal{B}(X)$. In this case,

$$\int_X \varphi(x) m_r(dx) = m_r(\Gamma^{-1}(J))$$

and integrating with respect to $\lambda$ over $\mathbb{R}$ and taking into account of (A.2), with $I = \mathbb{R}$, yields

$$\int_{\mathbb{R}} \left( \int_X \varphi(x) m_r(dx) \right) \lambda(dr) = \int_{\mathbb{R}} m_r(\Gamma^{-1}(J)) \lambda(dr) = \mu(\Gamma^{-1}(J)),$$

as claimed.

Let us show that for $\lambda$-a.e. $r_0 \in \mathbb{R}$, the support of $m_{r_0}$ is contained in $\Gamma^{-1}(r_0)$. If $I$ is any interval, setting $A = \Gamma^{-1}(\mathbb{R} \setminus I)$ in (A.2), we find

$$0 = \int_I m_r(\Gamma^{-1}(\mathbb{R} \setminus I)) \lambda(dr),$$

so that $m_r(\Gamma^{-1}(\mathbb{R} \setminus I)) = 0$ for $\lambda$-almost all $r \in I$, say for every $r \in I \setminus J_I$ with $\lambda(J_I) = 0$. Now, let us consider all the open intervals with rational endpoints, $I = (a_n, b_n)$ with $a_n < b_n \in \mathbb{Q}$. The set $J := \bigcup_{n \in \mathbb{N}} J_{(a_n,b_n)}$ is still $\lambda$-negligible, and we have

$$m_r(\Gamma^{-1}(\mathbb{R} \setminus J_{(a_n,b_n)})) = 0, \quad n \in \mathbb{N}, \ r \in (a_n, b_n) \setminus J. \quad (\text{A.3})$$

For every $r_0 \in \mathbb{R} \setminus J$, fix two subsequences $(a_{n_k}), (b_{n_k})$ such that $a_{n_k} < r_0, b_{n_k} > r_0$ and $\lim_{k \to \infty} a_{n_k} = \lim_{k \to \infty} b_{n_k} = r_0$. Taking $r = r_0$ and replacing $a_n, b_n$ by $a_{n_k}, b_{n_k}$ in (A.3), we obtain that $m_{r_0}$ has support contained in $(a_{n_k}, b_{n_k})$ for every $k \in \mathbb{N}$, and the statement follows.

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