Abstract. The main objective of the present paper is to establish a new uncertainty principle (UP) for the two-sided quaternion Fourier transform (QFT). This result is an extension of a result of Benedicks, Amrein and Berthier, which states that a nonzero function in $L^1(\mathbb{R}^2, \mathbb{H})$ and its two-sided QFT cannot both have support of finite measure.

1. Introduction

In harmonic analysis, it is known that if the supports of a function $f \in L^1(\mathbb{R}^d)$ and its Fourier transform $\mathcal{F}\{f\}$ are contained in bounded rectangles, then $f$ vanishes almost everywhere.

Benedicks [2] relaxed the requirements for this conclusion by showing that the supports of $f$ and $\mathcal{F}\{f\}$ need only have finite measure, whose proof, first circulated as a preprint in 1974 but not formally published for another decade, Amrein and Berthier [1] later gave a different proof.

The classical Fourier transform, defined over the real line, is given by

$$\mathcal{F}\{f(x)\}(y) = \int_{-\infty}^{\infty} e^{-2\pi i xy} f(x) \, dx,$$

for $f \in L^1(\mathbb{R})$.

The Benedicks-Amrein-Berthier’s theorem is the following:

Let $f \in L^1(\mathbb{R})$, and $\mathcal{F}\{f\}$ be its Fourier transform.

Suppose that the sets $\{x \in \mathbb{R} : f(x) \neq 0\}$ and $\{\xi \in \mathbb{R} : \mathcal{F}\{f\}(\xi) \neq 0\}$ both have finite Lebesgue measure. Then $f = 0$.

To the best of our knowledge, a systematic work on the investigation of Benedicks-Amrein-Berthier’s UP using the two-sided QFT has not been carried out yet.

In this paper, we extend the validity of the Benedicks-Amrein-Berthier’s theorem to the two-sided QFT.

Other theorems of the UPs: Hardy, Beurling, Cowling-Price, Gelfand-Shilov, Miyachi have been extended for this transform (e.g. [6, 7, 9]).

We further emphasize that our generalization is non-trivial because the multiplication of quaternions and the quaternion Fourier kernel are both non-commutative.

Our paper is organized as follows: In the next section, we collect some basic concepts in quaternion algebra. In section 3, we recall the definition and some results for the two-sided QFT useful in the sequel. In section 4, we prove Benedicks-Amrein-Berthier’s theorem for the two-sided QFT.

Key words and phrases. Quaternion Fourier transform, Benedicks–Amrein–Berthier’s theorem.
For the sake of further simplicity, we will use the real vector notations
\[ x = (x_1, x_2) \in \mathbb{R}^2, \quad dx = dx_1 dx_2 \]
the Lebesgue-measure, \(|A|\) is the normed Lebesgue
measure of a measurable set \(A \subseteq \mathbb{R}^n\).

2. The algebra of quaternions

Throughout the paper, let
\[ \mathbb{H} = \{ q = q_0 + iq_1 + jq_2 + kq_3; \ q_0, q_1, q_2, q_3 \in \mathbb{R} \} \]
be the quaternion algebra over \(\mathbb{R}\), denoted by \(\mathbb{H}\), is an associative noncommutative
four-dimensional algebra, it was invented by W. R. Hamilton in 1843.

where the elements \(i, j, k\) satisfy the Hamilton’s multiplication rules:
\[ ij = -ji = k; \ jk = -kj = i; \ ki = -ik = j; \ i^2 = j^2 = k^2 = -1. \]
Quaternions are isomorphic to the Clifford algebra \(Cl(O, 2)\) of \(\mathbb{R}^{0, 2}\):
\[ \mathbb{H} \cong Cl(O, 2) \quad (2.1) \]

The scalar part of a quaternion \(q \in \mathbb{H}\) is \(q_0\) denoted by \(\text{Sc}(q)\), the non scalar part(or
pure quaternion) of \(q\) is \(iq_1 + jq_2 + kq_3\) denoted by \(\text{Vec}(q)\).

We define the conjugation of \(q \in \mathbb{H}\) by :
\[ q^* = q_0 - iq_1 - jq_2 - kq_3. \]
The quaternion conjugation is a linear anti-involution
\[ \overline{pq} = \overline{p} \overline{q}, \quad \overline{p + q} = \overline{p} + \overline{q}, \quad \overline{p} = p. \]

The modulus of a quaternion \(q\) is defined by:
\[ |q|_Q = \sqrt{q q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \]

In particular, when \(q = q_0\) is a real number, the module \(|q|_Q\) reduces to the ordinary
Euclidean module \(|q| = \sqrt{q_0^2}\). Also, we observe that for \(x \in \mathbb{R}^2, \ |x|_Q = |x|, \) where \(||\) is
the Euclidean norm \(|(x_1, x_2)|^2 = x_1^2 + x_2^2\)
we have :
\[ |pq|_Q = |p|_Q |q|_Q. \]

It is easy to verify that \(0 \neq q \in \mathbb{H}\) implies :
\[ q^{-1} = \frac{\overline{q}}{|q|_Q^2}. \]

Any quaternion \(q\) can be written as \(q = |q|_Q e^{i\mu}\) where \(e^{i\mu}\) is understood in accord-
ance with Euler’s formula \(e^{i\theta} = \cos (\theta) + \mu \sin (\theta)\) where \(\theta = \text{arctan} \frac{|\text{Vec}(q)|_Q}{\text{Sc}(q)}\), \(0 \leq \theta \leq \pi\)
and \(\mu := \frac{|\text{Vec}(q)|_Q}{|\text{Sc}(q)|_Q}\) verifying \(\mu^2 = -1\).

In this paper, we will study the quaternion-valued signal \(f : \mathbb{R}^2 \rightarrow \mathbb{H}\), which can be
expressed as
\[ f = f_0 + if_1 + jf_2 + kf_3, \]
with \(f_m : \mathbb{R}^2 \rightarrow \mathbb{R}\) for \(m = 0, 1, 2, 3\).
We introduce the space $L^p(\mathbb{R}^2, \mathbb{H})$, $1 \leq p \leq \infty$, as the Banach space of all quaternion-valued functions $f : \mathbb{R}^2 \to \mathbb{H}$ satisfying

$$
|f|_{p,Q} = \left( \int_{\mathbb{R}^2} |f(x)|_Q^p \, dx \right)^{\frac{1}{p}} < \infty \quad \text{for} \quad 1 \leq p < \infty,
$$

and $|f|_{\infty,Q} = \text{ess sup}_{x \in \mathbb{R}^2} |f(x)|_Q < \infty$.

### 3. Quaternion Fourier transforms

The quaternion Fourier transform which has been defined by Ell [8], is a generalization of the classical Fourier transform (CFT) using quaternion algebra. Several known and useful properties, and theorems of this extended transform are generalizations of the corresponding properties, and theorems of the CFT with some modifications (e.g., [3, 4, 8, 11]).

The QFT has an important role in the representations of signals due to transforming a real 2D signal into a quaternion-valued frequency domain signal, QFT belongs to the family of Clifford Fourier transformations because of (2.1).

There are three different types of QFT, the left-sided QFT, the right-sided QFT, and two-sided QFT [12]. In this paper we only treat the two-sided QFT.

We now review the definition and some properties of the two-sided QFT.

**Definition 3.1.** Let $f$ in $L^1(\mathbb{R}^2, \mathbb{H})$. Then two-sided quaternion Fourier transform of the function $f$ is given by

$$
\mathcal{F}\{f(x)\}(\xi) = \int_{\mathbb{R}^2} e^{-i2\pi \xi_1 x_1} f(x) e^{-i2\pi \xi_2 x_2} \, dx,
$$

(3.1)

Where $\xi, x \in \mathbb{R}^2$.

**Lemma 3.2.** If $f \in L^1(\mathbb{R}^2, \mathbb{H})$

then $\mathcal{F}\{f\}$ is continuous function.

Proof. See [4, Proposition 3.1(ix)].

**Lemma 3.3.** Inverse QFT [3, Thm 2.5]

If $f \in L^1(\mathbb{R}^2, \mathbb{H})$ and $\mathcal{F}\{f\} \in L^1(\mathbb{R}^2, \mathbb{H})$, then the two-sided QFT is an invertible transform and its inverse is given by

$$
f(x) = \int_{\mathbb{R}^2} e^{i2\pi \xi_1 x_1} \mathcal{F}\{f(x)\}(\xi) e^{i2\pi \xi_2 x_2} d\xi, \quad d\xi = d\xi_1 d\xi_2.
$$

**Lemma 3.4.** (Dilation property), see example 2 on page 50 [3]

Let $k_1, k_2$ be a positive scalar constants, we have

$$
\mathcal{F}\{f(t_1, t_2)\} \left( \frac{u_1}{k_1}, \frac{u_2}{k_2} \right) = k_1 k_2 \mathcal{F}\{f(k_1 t_1, k_2 t_2)\} (u_1, u_2).
$$

### 4. Benedicks–Amrein–Berthier type theorem

In this section, we prove Benedicks–Amrein–Berthier’s theorem for the two-sided QFT.

**Definition 4.1.** The quaternion-valued signal $f : \mathbb{R}^2 \to \mathbb{H}$ is said to be periodic of period $K = (2L, 2L)$, $L > 0$ if it verifies

$$
f(x + K) = f(x) \quad \forall x \in \mathbb{R}^2.
$$

The Fourier series in the quaternion case may be written as :

$$
f(x) \cong \sum_{(r,s) \in \mathbb{Z}^2} \cos \left( \frac{\pi r x_1}{L} \right) \cos \left( \frac{\pi s x_2}{L} \right) A_{rs} + \sum_{(r,s) \in \mathbb{Z}^2} \sin \left( \frac{\pi r x_1}{L} \right) \cos \left( \frac{\pi s x_2}{L} \right) B_{rs} + \sum_{(r,s) \in \mathbb{Z}^2} \cos \left( \frac{\pi r x_1}{L} \right) \sin \left( \frac{\pi s x_2}{L} \right) C_{rs} + \sum_{(r,s) \in \mathbb{Z}^2} \sin \left( \frac{\pi r x_1}{L} \right) \sin \left( \frac{\pi s x_2}{L} \right) D_{rs},
$$

where $A_{rs}, B_{rs}, C_{rs}, D_{rs}$ are quaternion-valued coefficients.
with $a_0, A_r, B_r, C_r, D_r \in \mathbb{H}$.

we have according to the quaternion Euler formula

\[
\cos\left(\frac{n\pi x}{L}\right) = \frac{e^{jn\pi x/L} + e^{-jn\pi x/L}}{2}, \quad \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{jn\pi x/L} - e^{-jn\pi x/L}}{2j}, \quad r=1,2.
\]

then, we can rewrite the Fourier series in the exponential notations:

\[
f(x) = \sum_{(r,s) \in \mathbb{Z}^2} e^{j\pi r/L} C_{rs} e^{j\pi s/L}, \quad \text{with } C_{rs} \in \mathbb{H}
\]

(4.1)

**Lemma 4.2.** The quaternion Fourier coefficients are given by:

\[
C_{rs} = \frac{1}{(4L)^2} \int_{-L}^{L} \int_{-L}^{L} e^{-j\pi x/L} f(x) e^{-j\pi r/L} dx
\]

Proof. we have by (4.1):

\[
\frac{1}{(4L)^2} \int_{-L}^{L} \int_{-L}^{L} e^{-j\pi x/L} f(x) e^{-j\pi r/L} dx = \frac{1}{(4L)^2} \int_{-L}^{L} \int_{-L}^{L} e^{-j\pi x/L} \sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{j\pi m_1/L} C_{m_1m_2} e^{j\pi m_2/L} e^{-j\pi r/L} dx_1 dx_2
\]

\[
= \frac{1}{(4L)^2} \sum_{(m_1, m_2) \in \mathbb{Z}^2} \int_{-L}^{L} \int_{-L}^{L} e^{-j\pi x/L} e^{j\pi m_1/L} dx_1 C_{m_1m_2} \int_{-L}^{L} e^{j\pi m_2/L} e^{-j\pi r/L} dx_2
\]

\[
= \frac{1}{(4L)^2} \sum_{(m_1, m_2) \in \mathbb{Z}^2} C_{m_1m_2} 2L \delta_{m_1} 2L \delta_{m_2} = C_{rs}
\]

The next two lemmas will be needed to prove the main theorem.

**Lemma 4.3.** Let $T = [0,1]$. If $f \in L^1(\mathbb{R}^2, \mathbb{H})$, Then $\varphi(t) = \sum_{k \in \mathbb{Z}^2} f(t + k)$ converges in $L^1(T^2, \mathbb{H})$, and the quaternion Fourier series of $\varphi$ is

\[
\sum_{k=(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i t_1 k_1} \mathcal{F}\{f\}(k) e^{2\pi i t_2 k_2}
\]

Proof. As $f \in L^1(\mathbb{R}^2, \mathbb{H})$, we have

\[
\int_{[0,1]^2} | \sum_{|k| \leq N} f(t + k) |^2 dt \leq \int_{[0,1]^2} \sum_{|k| \leq N} |f(t + k)|^2 dt
\]

with $|k| = |k_1| + |k_2|$ and $N \in \mathbb{N}^2$ big enough.

\[
= \sum_{|k| \leq N} \int_{[0,1]^2} |f(x + k)|^2 dt
\]

\[
= \sum_{|k| \leq N} \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} |f(t)|^2 dt
\]

\[
\leq \int_{\mathbb{R}^2} |f(t)|^2 dt < \infty
\]

Thus we have shown that $\varphi \in L^1(T^2, \mathbb{H})$.

Since $\varphi$ is periodic with period $= 1$, according to lemma (4.2) The quaternion Fourier coefficients are

\[
f_{t_1}\int_{-L}^{L} \int_{-L}^{L} e^{-2\pi i t_1 m_1} \varphi(t) e^{-2\pi i t_2 m_2} dt = \int_{-L}^{L} \int_{-L}^{L} e^{-2\pi i t_1 m_1} \sum_{k \in \mathbb{Z}^2} f(t + k) e^{-2\pi i t_2 m_2} dt
\]

\[
= \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} e^{-2\pi i (t_1 + k_1)m_1} \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} f(t + k) e^{-2\pi i (t_2 + k_2)m_2} dt
\]

\[
= \sum_{k_2 \in \mathbb{Z}} \int_{k_2}^{k_2+1} \int_{-L}^{L} e^{-2\pi i m_1 y_1} f(y) e^{-2\pi i m_2 y_2} dy_1 dt
\]

\[
= \mathcal{F}\{f(y)\}(m_1, m_2)
\]

**Lemma 4.4.** Let $f : [0,1]^2 \to \mathbb{H}$ be a measurable function.

If $\int_{[0,1]^2} |f(x)|^2 dx < \infty$

then there is a set $E \subseteq [0,1]^2$ with $|E| = 1$ and $|f(a)|^2 < \infty \ \forall a \in E$.

Proof. Let $E = \{ x \in [0,1]^2 : |f(x)|^2 < \infty \}$ and $F = [0,1]^2 \setminus E$. 

\[
\text{Proof. Let } E = \{ x \in [0,1]^2 : |f(x)|^2 < \infty \} \text{ and } F = [0,1]^2 \setminus E.
\]
By absurd, if $|F| > 0$ then $\int_{E} |f(x)| q dx = \infty$
Contradiction with $\int_{[0,1]^2} |f(x)| q dx < \infty$
Then necessarily
$|F| = 0$, therefore $|E| = 1$.
will also need the following result of the zeros of an analytic function.

**Lemma 4.5.** Let $f$ be an analytic function on $\mathbb{C}^2$, and $Z(f) := \{ z \in \mathbb{C}^2 : f(z) = 0 \}$.

One of these properties can occur
(i) $Z(f)$ is of zero measure on $\mathbb{C}^2$.
(ii) $Z(f) = \mathbb{C}^2$.

**Proof.** The lemma holds for a function of 1 variable, see [13, Thm. 10.18].
Let $\mu_r$ be the Lebesgue-measure on $\mathbb{C}^r$, $r = 1, 2$ and Let $f$ be a function of two complex variables $z_1, z_2$.
We define $f(z_1, z_2) = f(z_1, z_2)$, and $f(z_1, z_2) = f(z_1, z_2)$.

We have
$Z(f) = Z(f_{z_1}) \otimes Z(f_{z_2})$ And $\mu_2 (Z(f)) = \mu_1(Z(f_{z_2})) \mu_1(Z(f_{z_1}))$
Hence, if either $\mu_1(Z(f_{z_2})) = 0$ or $\mu_1(Z(f_{z_1})) = 0$
then $\mu_2 (Z(f)) = 0$.
If not, then
$Z(f_{z_2}) = \mathbb{C}$ and $Z(f_{z_1}) = \mathbb{C}$ such that $Z(f) = \mathbb{C}^2$. $\square$

**Theorem 4.6.** Let $f \in L^1 (\mathbb{R}^2, \mathbb{H})$.
We define $\Sigma(f) := \{ x \in \mathbb{R}^2 : f(x) \neq 0 \}$ and $\Sigma(\mathcal{F}\{f\}) := \{ \xi \in \mathbb{R}^2 : \mathcal{F}\{f\} (\xi) \neq 0 \}$.
If
$|\Sigma(f)| \leq |\Sigma(\mathcal{F}\{f\})| < \infty$
then $f = 0$.

Proof. Since $|\Sigma(f)|$ is finite, by composing $f$ with a dilation, lemma 3.4, we can assume that
$|\Sigma(f)| < 1$ without loss of generality.
We have
$\int_{[0,1]^2} \sum_{k \in \mathbb{Z}^2} 1_{\Sigma(\mathcal{F}\{f\})} (y + k) dy = \sum_{k \in \mathbb{Z}^2} \int_{[0,1]^2} 1_{\Sigma(\mathcal{F}\{f\})} (y + k) dy$
$= \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} \int_{-k_1}^{1-k_1} \int_{-k_2}^{1-k_2} 1_{\Sigma(\mathcal{F}\{f\})} (z) dz$
$= \int_{\mathbb{R}^2} 1_{\Sigma(\mathcal{F}\{f\})} (z) dz$
$= |\Sigma(\mathcal{F}\{f\})| < \infty$ (by assumption) (4.2)

Also $\int_{[0,1]^2} \sum_{k \in \mathbb{Z}^2} 1_{\Sigma(f)} (x + k) dx = \int_{\mathbb{R}^2} 1_{\Sigma(f)} (x) dx$
$= |\Sigma(f)| < 1$ (4.3)

According to (4.2) and lemma (4.4),
there exists an $E \subseteq [0,1]^2$, with $|E| = 1$
and $\sum_{k \in \mathbb{Z}^2} 1_{\Sigma(\mathcal{F}\{f\})} (a + k) < \infty$ for all $a \in E$.
Let $F = \{ x \in [0,1]^2 : \sum_{k \in \mathbb{Z}^2} 1_{\Sigma(f)} (x + k) = 0 \}$
We have
$|F| > 0$ ( otherwise we would have $\int_{[0,1]^2} \sum_{k \in \mathbb{Z}^2} 1_{\Sigma(f)} (x + k) dx \geq 1$ contradiction with(4.3))
Given \(a = (a_1, a_2) \in E\), We define
\[
\varphi_a(t) = \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} e^{-2\pi i a_1 (t_1 + k_1)} f(t + k) e^{-2\pi i a_2 (t_2 + k_2)}
\]
for \(t = (t_1, t_2) \in \mathbb{R}^2\).

As \(f \in L^1(\mathbb{R}^2, \mathbb{H})\) by assumption, applying lemma (4.3) gives
\[
\varphi_a \in L^1(\mathbb{T}^2, \mathbb{H}),
\]
Now, Let \(g(t) = e^{-2\pi i a_1 t_1} f(t) e^{-2\pi i a_2 t_2}\).

Straightforward computations show that
\[
\mathcal{F}\{g\}(y) = \mathcal{F}\{f\}(y + a)
\]
(4.5)

Therefore, the quaternion Fourier series of \(\varphi_a\) is
\[
\sum_{k=(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i t_1 k_1} \mathcal{F}\{f\}(a + k) e^{2\pi i t_2 k_2}
\]
Indeed,
\[
\varphi_a(t) = \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} g(t + k)
= \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i t_1 k_1} \mathcal{F}\{g\}(k) e^{2\pi i t_2 k_2}
= \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} e^{2\pi i t_1 k_1} \mathcal{F}\{f\}(a + k) e^{2\pi i t_2 k_2}.
\]
The second equality follows from lemma 4.3, the last by (4.5).

The complexifying the variable \(t = a + iC b; a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2\) (we note by
\(iC\) the complex number checking \(iC^2 = -1\).

Since \(a \in E\), \(\varphi_a(t)\) is well defined for all \(t \in \mathbb{C}^2\), and it is a trigonometric polynomial
(with quaternion coefficients).

Hence \(\varphi_a\) is analytic.

So, by lemma (4.5) either \(\varphi_a = 0\) everywhere, or \(\varphi_a \neq 0\) a.e.

On the other hand,
\[
|\varphi_a(t)| \leq \sum_{k=(k_1,k_2) \in \mathbb{Z}^2} |f(t + k)|< \infty
\]
and \(F\) has measure greater than zero, thus \(\varphi_a = 0\) everywhere for all \(a \in E\).

Whence \(\mathcal{F}\{f\}(a + k) = 0\) for all \(a \in E\) and \(k \in \mathbb{Z}^2\).

Then \(\mathcal{F}\{f\} = 0\) a.e.

Since \(\mathcal{F}\{f\}\) is continuous (lemma 3.2), We obtain \(\mathcal{F}\{f\} = 0\).

Then by lemma (3.3), we conclude that \(f = 0\).

This completes the proof of Benedicks–Amrein–Berthier type theorem. \(\square\)

Corollary 4.7. Let \(f \in L^p(\mathbb{R}^2, \mathbb{H})\) for \(p > 1\),
If
\[
|\Sigma(f)|, |\Sigma(\mathcal{F}\{f\})| < \infty
\]
then \(f = 0\).

Proof. Suppose \(f \in L^p(\mathbb{R}^2, \mathbb{H})\) with \(p > 1\), since \(|\Sigma(f)| < \infty\), then by Hölder inequality we obtain \(f \in L^1(\mathbb{R}^2, \mathbb{H})\).
So by the previous theorem we get \(f = 0\).

which was to be proved. \(\square\)
5. Conclusions

In this paper, based on quaternionic Fourier series representation, a generalization of the Benedicks-Amrein-Berthier’s UP associated with the QFT is proposed. The extension of this qualitative UP to the quaternionic algebra framework shows that a quaternionic 2D signal and its QFT cannot both simultaneously have support of finite measure.

In the future work, we will consider Beurling’s UP, Hardy’s UP, Cowling-Price’s UP and Gelfand-Shilov’s UP for the two-dimensional quaternionic windowed Fourier transform.

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