The Instanton Solution of Forced Burgers Equation in Polyakov’s Approach

M. R. Rahimi Tabar

Dept. of Physics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran.

Institute for Studies in Theoretical Physics and Mathematics, Tehran P.O.Box: 19395-5746, Iran.

Abstract

We calculate the coefficients of the operator product expansion (OPE), in Polyakov’s approach for Burgers turbulence. We show that the OPE has to be generalized and it is shown that the extra term gives us the instanton solution (shock solution) of Burgers equation. We consider the effect of the new-term in the OPE, on the right and left-tail of probability distribution function (PDF). It is shown that the left-tail of PDF, where is dominated by the well-separated shocks behaves as $W(u) \sim u^{-7/2}$. Finally we calculate the asymptotic behaviour of the N-point generating function of the velocity field, using the new OPE.

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1- Introduction

A theoretical understanding of turbulence has eluded physicists for a long time. A statistical theory of turbulence has been put forward by Kolmogorov [1], and further developed by others [2-4]. The approach is to model turbulence using stochastic partial differential equations. In this direction, Polyakov [5] has recently offered a field theoretic method to derive the probability distribution or density of states in (1+1)-dimensions in the problem of randomly driven Burgers equation. The importance of the Burgers equation is that, it is the simplest equation that resembles the analytic structure of the Navier–Stokes equation, at least formally, within the scope of applicability of the Kolmogorov’s arguments [6].

Polyakov formulates a new method to analyse the inertial range correlation functions based on two important ingredients in field theory and statistical physics, the operator product expansion (OPE) and anomalies. He argues that in the limit of high Reynolds number because of existence of singularities at the coinciding point, dissipation remains finite and all subleading contributions vanish in the inertial range. Using the OPE one can find the leading singularities and show that this approach is self-consistent.

Using the OPE Polyakov reduces the problem of computation of the velocity correlation functions to the solution of a certain partial differential equation. In this direction the N-point generating functions have been found in [7]. The two-point functions for different type of correlation for noise has been found in [8] and the perturbative calculation in the presence of pressure has been done in [9].

Here using the results of [7], we calculate the OPE coefficients proposed by Polyakov, and show that we have to modify the OPE. We write the generalized OPE and find the first correction to the PDF which was initially found by Polyakov. It is shown that the extra term
in the OPE gives us the instanton, which has found by Gurarie, Migdal [10] and Bouchaud, Mezard [11].

We find the right and left-tail of the PDF and show that the right-tail remains unchanged and the left-tail, where is dominated by the shocks behaves as \( W(u) \sim u^{-7/2} \), which is in agreement with recent numerical experiments [12]. Finally we find the asymptotic behaviour of the N-point generating function with the new-term (instanton-term) in the OPE.

2- The Instanton Solution of the Burgers Equation

We consider the Burgers equation in one-dimension:

\[
\begin{align*}
    u_t + uu_x &= \nu u_{xx} + f(x, t) \\
    \end{align*}
\]

where \( u \) and \( \nu \) are velocity field and viscosity, respectively. The \( f(x, t) \) is a Gaussian random force with the following correlation:

\[
    \langle f(x, t)f(x', t') \rangle = k(x - x')\delta(t - t')
\]

To investigate the statistical description of eq.(1) following Polyakov [5], we consider the following generating functional

\[
    Z_N(\lambda_1, \lambda_2, \ldots \lambda_N, x_1, \ldots x_N) = \langle \exp\left(\sum_{j=1}^{N} \lambda_j u(x_j, t)\right) \rangle
\]

Noting that the random force \( f(x, t) \) has a Gaussian distribution, \( Z_N \) satisfies a closed differential equation provided that the viscosity \( \nu \) tends to zero:

\[
    \dot{Z}_N + \sum \lambda_j \frac{\partial}{\partial \lambda_j} \left( \frac{1}{\lambda_j} \frac{\partial Z_N}{\partial x_j} \right) = \sum k(x_i - x_j)\lambda_i\lambda_j Z_N + D_N
\]

where \( D_N \) is:

\[
    D_N = \nu \sum \lambda_j < u''(x_j, t) \exp\sum \lambda_k u(x_k, t) >
\]
The first term in the r.h.s. of eq. (4) can be derived by applying the following relation and then using the Gaussian character of the noise term in time and also by giving address to causality.

\[ < f(x, t)e^{\lambda u(x', t')} > = \lambda \int_0^t dt' < f(x, t)u(x', t')e^{\lambda u(x', t')} > \]  

(6)

However to remain in the inertial range we must keep \( \nu \) infinitesimal but non-zero. Polyakov argues that the anomaly mechanism implies that infinitesimal viscosity produces a finite effect. To compute this effect, Polyakov makes a conjecture for existence of an operator product expansion or the fusion rules. The fusion rule is the statement concerning the behaviour of correlation functions, when some subset of points are put close together.

Let us use the following notation;

\[ Z(\lambda_1, \lambda_2, \ldots, x_1, \ldots x_N) =< e^{\lambda_1(x_1)} \ldots e^{\lambda_N(x_N)} > \]  

(7)

where \( e_{\lambda_1}(x_1) = e^{\lambda_1 u(x_1)} \).

The Polyakov’s conjecture is that in this case the OPE has the following form,

\[ e_{\lambda_1}(x + y/2)e_{\lambda_2}(x - y/2) = A(\lambda_1, \lambda_2, y)e_{\lambda_1+\lambda_2}(x) + B(\lambda_1, \lambda_2, y)\frac{\partial}{\partial x}e_{\lambda_1+\lambda_2} + o(y^2) \]  

(8)

This implies that \( Z_N \) fuses into functions \( Z_{N-1} \) as we fuse a couple of points together. This conjecture allows us to evaluate the following anomaly operator (i.e. the \( D_N \)-term in eq. (4)),

\[ a_\lambda(x) = \lim_{\nu \to 0} \nu(\lambda u^{(n)}(x) \exp(\lambda u(x))) \]  

(9)

which can be written as:

\[ a_\lambda(x) = \lim_{\xi, \eta, \nu \to 0} \lambda \nu \frac{\partial^3}{\partial \xi \partial \eta^2} e_{\lambda}(x + y)e_{\lambda}(x) \]  

(10)
As discussed in [5] the possible Galilean invariant expression is:

$$a_\lambda(x) = a(\lambda)e_\lambda(x) + \tilde{\beta}(\lambda)\frac{\partial}{\partial x}e_\lambda(x)$$

(11)

Therefore in steady state the master equation takes the following form,

$$\sum \left( \frac{\partial}{\partial \lambda_j} - \beta(\lambda_j) \right) \frac{\partial}{\partial x_j} Z_N - \sum \bar{k}(x_i - x_j)\lambda_i\lambda_j Z_N = \sum a(\lambda_j)Z_N$$

$$\beta(\lambda) = \tilde{\beta}(\lambda) + \frac{1}{\lambda}$$

(12)

Let us consider following correlation for $f$ in $k$-space as follows [13]:

$$<f(k,t)f(k',t')> = \frac{L}{2\pi}k_0\delta(k^2 - \frac{1}{L^2})\delta(k + k')\delta(t - t')$$

(13)

therefore we obtain: In the inertial range where $x_i - x_j << L$, we find:

$$k(x_i - x_j) = k_0(1 - \frac{(x_i - x_j)^2}{2L^2})$$

(14)

Polyakov has found the following explicit form of $Z_2$ for $k(x_i - x_j)$ given by eq.(14):

$$Z_2(\mu y) = e^{\frac{2}{3}(\mu y)^{3/2}}$$

(15)

and the following expression for density of states as the Laplace transform of $Z_2$:

$$W(u, y) = \int_{c-i\infty}^{c+i\infty} \frac{d\mu}{2\pi i}e^{-\mu u}Z_2(\mu y)$$

(16)

where $\mu = 2(\lambda_1 - \lambda_2)$ and $y = x_1 - x_2$. Now one can write the right and left–tail of $W(u, y)$ as following:

$$W(u, y) = \begin{cases} 
  e^{-\frac{1}{3}(\mu y)^3} & \text{if } (u/y) \to +\infty \\
  y^{3/2}u^{-5/2} + y^{9/2}u^{-11/2} & \text{if } (u/y) \to -\infty 
\end{cases}$$

(17)
In ref.[7] we have found the exact N-point generating function (i.e. the $Z_N$) as follows:

$$Z_N = (\lambda_1 \lambda_2 \cdots \lambda_N)^b_N (\mu_2 \mu_3 \cdots \mu_N)^{2N-1 \over 2(N-1)} e^{2/3(\mu_2 y_2 + \mu_3 y_3 + \cdots + \mu_N y_N)^{3/2}}$$  \hspace{1cm} (18)$$

where $b_N = 2N-1 \over 2N$ and $\mu_i$, $y_i$ are given by:

$$y_1 = \frac{x_1 + x_2 + x_3 + \ldots + x_N}{N}$$
$$y_2 = x_1 - \frac{x_2 + x_3 + \ldots + x_N}{N-1}$$
$$y_3 = x_2 - \frac{x_3 + x_4 + \ldots + x_N}{N-2}$$

and

$$y_N = x_{N-1} - x_N$$  \hspace{1cm} (19)$$

and

$$\mu_1 = \frac{\lambda_1 + \lambda_2 + \ldots + \lambda_N}{N}$$
$$\mu_2 = \frac{N-1}{N} [\lambda_1 - \frac{\lambda_2 + \lambda_3 + \ldots + \lambda_N}{N-1}]$$
$$\mu_3 = \frac{N-2}{N-1} [\lambda_2 - \frac{\lambda_3 + \lambda_4 + \ldots + \lambda_N}{N-2}]$$

and

$$\mu_N = 2(\lambda_{N-1} - \lambda_N)$$  \hspace{1cm} (20)$$

It follows that the N–point correlation function of $v$ is:

$$G^{(N)}(x_1, \ldots, x_N) \sim \lim_{\lambda \to 0} \lambda^{-N} \sum_{k=0}^{N} a_k^{(N)}(\lambda x)^{3k/2}$$  \hspace{1cm} (21)$$

where $a_k^{(N)}$ are some constants.

Now we try to find the OPE coefficients in eq.(8). To do this according [7] we use $Z_3$ and tend $x_2$ close to $x_3$. $Z_2$ is given by eqs.(15) with $\lambda_1 + \lambda_2 = 0$, and $Z_3$ has following form:

$$Z_3 = (\lambda_1 \lambda_2 \lambda_3)^{5/6} [3(\lambda_1 - \frac{\lambda_2 + \lambda_3}{2})(\lambda_2 - \lambda_3)]^{-5/4}$$

$$\cdot e^{2/3[3/2(\lambda_1 - \frac{\lambda_2 + \lambda_3}{2})(x_1 - \frac{x_2 + x_3}{2}) + 2(\lambda_2 - \lambda_3)(x_2 - x_3)]^{3/2}}$$  \hspace{1cm} (22)$$
where $\lambda_1 + \lambda_2 + \lambda_3 = 0$. In eq.(22) we take $x_3 = x_2 - 2\epsilon$ and it is easy to show that in the limit $\epsilon \to 0$ we find:

\[
Z_3 = (\lambda_1\lambda_2\lambda_3)^{5/6}(9/2(\lambda_2^2 - \lambda_3^2))^{-5/4}e^{(2/3)2(\lambda_1-(\lambda_2+\lambda_3))(x_1-x_2)^{3/2}}
\]

\[
\left\{ 1 + (-1 + 7/64 + \cdots)7/8(\lambda_2 + \lambda_3)(-2)[-4(\lambda_2 + \lambda_3)]^{1/2}(x_1-x_2)^{3/2}
\right.
\]
\[
+ (1 - 7/32 + \cdots)\frac{7\lambda_2 - 25\lambda_3}{32(\lambda_2 + \lambda_3)}(-2)[-4(\lambda_2 + \lambda_3)]^{3/2}(x_1-x_2)^{1/2}\epsilon + O(\epsilon^2)\}
\]

Comparing with eq.(8) and using eq.(15) we find that:

\[
A(\lambda_2, \lambda_3, \epsilon) = (-\lambda_2\lambda_3(\lambda_2 + \lambda_3))^{5/6}(\frac{9}{2}(\lambda_2^2 - \lambda_3^2))^{-5/4}
\]

\[
B(\lambda_2, \lambda_3, \epsilon) = (1 - 7/32 + \cdots)\frac{7\lambda_2 - 25\lambda_3}{32(\lambda_2 + \lambda_3)}A(\lambda_2\lambda_3, \epsilon)\epsilon
\]

Indeed we determine the $A(\lambda_2, \lambda_3, \epsilon)$ and $B(\lambda_2, \lambda_3, \epsilon)$ by means of the first and third terms in eq.(23). However the second term in eq.(23) (i.e. which is proportioal to $(x_1 - x_2)^{3/2}$), can be treated as $\frac{\partial}{\partial x} Z_2$ and this will change the OPE (i.e. eq.(8)) to the following form:

\[
e^{\lambda_2(x + \epsilon/2)}e^{\lambda_3(x - \epsilon/2)} = A(\lambda_2, \lambda_3, \epsilon)e_{\lambda_2+\lambda_3}(x) + B(\lambda_2, \lambda_3, \epsilon)\frac{\partial}{\partial x} e_{\lambda_2+\lambda_3}
\]
\[
+ C(\lambda_2, \lambda_3, \epsilon)\frac{\partial}{\partial \lambda} e_{\lambda_2+\lambda_3} + O(\epsilon^2)
\]

where using eq.(54), $C(\lambda_2, \lambda_3, \epsilon)$ can be written in terms of $A(\lambda_2, \lambda_3, \epsilon)$ as follows:

\[
C(\lambda_2, \lambda_3, \epsilon) = (-1 + 7/64 + \cdots)\frac{7}{8}(\lambda_2 + \lambda_3)A(\lambda_2, \lambda_3, \epsilon)
\]

Using eq.(26) one can show that the modified OPE leads to following equations for $a_\lambda(x)$ and $Z_2$ (i.e. eqs.(11) and (15)):

\[
a_\lambda(x) = a(\lambda)e_\lambda(x) + \tilde{\beta}(\lambda)\frac{\partial}{\partial x} e_\lambda(x) + \tilde{\gamma}(\lambda)\frac{\partial}{\partial \lambda} e_\lambda(x) + \cdots
\]
\( (\partial_\mu - \frac{2b}{\mu}) \partial_y Z_2 + c \mu \partial_\mu Z_2 - \mu^2 y^2 Z_2 = 0 \) (29)

where we have used \( \tilde{\beta}(\lambda) = \frac{b-1}{\lambda} \), \( a = 0 \) according [5]. We have used the scaling arguments and show that \( \tilde{\gamma}(\lambda) \) has the following form:

\[ \tilde{\gamma}(\lambda) = c\lambda \] (30)

Therefore we find following asymptotic behavior for \( Z_2 \):

\[ Z_2(\mu y) = e^{\tilde{t}(\mu y)^{3/2} - \tilde{t}(\mu y)} \] (31)

First of all it appears that in the limit \( c \to 0 \), one finds the Polyakov's result for \( Z_2 \).

Also by using of eq.(28), one can show that the 1–point function of the velocity behaves as \( < v(y) > \sim y \), which is related to the well-known shock wave structures [6], and is exactly the instanton solution of the noisy Burgers equation, found by Gurarie and Migdal [10], and later in a different way by Bouchaud and Mezard [11]. The interesting point is that Polyakov's approach generates in self-consistent way the instanton solution.

Let us write the right and the left-tail of the PDF. It is easy to show that the \( W(u, y) \) behaves as follows:

\[
W(u, y) = \begin{cases} 
  e^{-\frac{3}{2}(\frac{u}{y} + \frac{\tilde{t}}{y})^3} & \text{if } (u/y) \to +\infty \\
  y^{3/2}u^{-5/2} + cy^{5/2}u^{-7/2} + c^2y^{7/2}u^{-9/2} + y^{9/2}u^{-11/2} & \text{if } (u/y) \to -\infty 
\end{cases}
\] (32)

It is evident that the right-tail remains somewhat unchanged and the left-tail of the PDF, where is dominated by the shocks behaves as \( W(u) \sim u^{-7/2} \), which is in agreement with the observations [12].
Now let us find the asymptotic behaviour of the N-point generating functions. For investigating the asymptotic behaviour we use the following important property that:

\[ \Sigma_{i=1}^{N} \lambda_i \partial_{\lambda_i} = \Sigma_{i=1}^{N} \mu_i \partial_{\mu_i} \]  \hspace{1cm} (33)

where \( \mu_1 = 0 \), which in turn gives the following asymptotic behaviour of the N-point generating functions in the limit of \( \Sigma_{i=1}^{N} \mu_i y_i \to \infty \):

\[ Z_N = (\lambda_1 \lambda_2 \cdots \lambda_N)^{b_N} (\mu_2 \mu_3 \cdots \mu_N)^{c_N} e^{-\frac{N-1}{2(N-1)} e^{2/3(\mu_2 y_2 + \mu_3 y_3 + \cdots + \mu_N y_N)^{3/2}} - \frac{c}{2} \Sigma_{i=1}^{N} \mu_i y_i} \]  \hspace{1cm} (34)

Now it is easy to show that for un-forced Burgers equation \((k_0 = 0)\), we find,

\[ < u^q(r) > \sim r^q \]  \hspace{1cm} (35)

which is again consistent with other approaches [14].

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