On the G-Similarities of two open B-spline curves in $\mathbb{R}^3$

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ABSTRACT

Let $G$ be a transformation group in $\mathbb{R}^3$. Any two vectors $x$ and $y$ in $\mathbb{R}^3$ are called $G$-equivalence vectors if there exist a transformation $g \in G$ such that $y = gx$ satisfies. In this paper the transformation group $G$ will be considered as similarity transformations group or its any subgroup. So if given two vectors $x$ and $y$ in $\mathbb{R}^3$ are $G$-equivalence vectors then these vectors $x$ and $y$ are called $G$-similar. i.e. rotational, reflectional, translational or scaling similarity. B-spline curves are used basically in Computer Aided Design (CAD), Computer Aided Geometric Design (CAGD), Computer Aided Modeling (CAM). In determining the invariants of spline curves and surfaces at any point, it is necessary to find the analytical equation of each curve and surface and calculate its invariants such as curvature, torsion, principal curvatures, mean and Gaussian curvatures at the desired point. However, it can be very difficult to find the curve or surface to be designed analytically. For example, when a car is designed, the aerodynamic curves in the car will be different from the known surface equation of the car. It is very difficult to write this equation exactly. For these curves and surfaces we designed, the way to overcome this difficulty is to design them with spline curves and surfaces. In this paper the $G$-equivalence conditions of given two open B-spline curves are studied in case $G$ is similarity transformations group or its any subgroup.

Keywords: open B-spline curves, similarity groups, G-similar splines

R$^3$ de Açık B-Spline Eğrilerinin G-Benzerlikleri

ÖZ

$G$, $\mathbb{R}^3$ de bir dönüşüm grubu olsun. $\mathbb{R}^3$ te herhangi iki $x$ ve $y$ vektörleri verildiğinde eğer bir $g \in G$ dönüşümü $y = gx$ şartını sağlayacak şekilde bulunabilirse bu iki vektöre G- denk vektörler denir. Bu çalışmada $G$ dönüşüm grubu olarak benzerlik dönüşümlerini grubu ve bu grubun tüm altgrupları dikkate alıncaktır. Böylece $\mathbb{R}^3$ te herhangi iki $x$ ve $y$ vektörleri G- denk vektörler ise bu vektörlere G-benzer denir. Döndürülme, yansıtırma, ötelenme, ya da germe benzerliği gibi. B-spline eğrileri temelde Bilgisayar Destekli Tasarım (BDT), Bilgisayar Destekli Geometrik Tasarım (BDGT) ya da Bilgisayar Destekli Modellerle (BDM) alanlarından kullanılır. Herhangi bir noktada spline eğri ve yüzeylerinin invarıantlarını belirlediğinde eğri ve yüzeyin analitik denkleminini bulmak ve istenen noktada eğrilik torsiyon, asal eğriler, ortalama ve Gauss eğrilerini hesaplamak gerekmedikdir Oysa ki tasarlanan eğri ve yüzeyde bunu analitik olarak bulmak oldukça zordur. Örneğin, bir araç tasarlandığında, onun aerodinamik yapısından dolayı yüzeyin ve onun üzerindeki eğrilerin analitik denkelemi tam olarak bulunmak oldukça zordur. Tasarlanan bu eğri ve yüzeyler için bu zorluğun üstesinden gelmenin yolu bunlara spline eğri ve yüzeyleri ile tasarlamaktır. Bu çalışmada G, benzerlik dönüşümleri grubu ve onun altgrupları olması durumunda verilen iki B-spline eğrilerinin G- denklik koşulları verilmiştir.

Anahtar Kelimeler: Açık B-spline curves, benzerlik grupları, G-benzer splinelar

INTRODUCTION

Invariant rational functions under any transformation have very important roles to determine any properties which can formulated by these rational functions and are independent from this transformation. If the generators of invariant rational functions under any transformation known then any properties which are invariant under this transformation can be formulated by the generator functions. So any invariant properties are the functions of generators. Developments in invariant theory at the end of the 19th century have affected different areas of mathematics.

In the 20th century Bridgman [5], Sedov [6], and Langhaar [7] are some contributors in this area. In 1946, Herman Weyl gave the complete invariant system of points for real $n$ dimensional orthogonal group $O(n)$ in [8]. After him, Dj. Khadjiev and R. Aripov generalized this invariants to all Euclidean motions in [9, 10]. Recently Sagirolgu [11-13], Oren [14-16, 21] Peksen [13, 16], Incesu and Gursoy [17-23] are some contributors in this area. The best examples of points systems are Bézier curves and Bézier surfaces. The invariants of these curves and surfaces under an affine transformation have the same meaning as the invariants of the control points of these curves and surfaces [22], Bézier and B-Spline...
curves has been studied in many different are of CAD and CAM system. Some of these studies by G. Farin [24], R. Farouki [25, 26], J. Hoschek [27], W. Tiller [28], H. Potmann [29], Inceus and Guroy [30, 34], Samanci et al. [31, 33, 36, 37], Bulut and Caliskan [32], Erkan and Yuce [35], Baydas and Karakas [38] can be given exemplarily.

MATERIALS and METHODS

G-invariant Functions

Definition 1. Let (G, *) be a group and X be a nonempty set. Let the transformation ϕ: G × X → X be given. If following conditions

i) ϕ(g; ϕ(g2; x)) = ϕ(g1; g2; x), ∀g1; g2; x ∈ G and ∀x ∈ X
ii) ϕ(e; x) = x; for ∀x ∈ X where e ∈ G is identity satisfies then the transformation ϕ is called Group action of the group G on X.

This action is denoted by G : X and the image (g; x) is stated as gx briefly. [19]

Definition 2. Let G be a group and the group action ϕ = G : X and the subset H ⊆ X be given. the subset H is called "G-invariant subset" if ϕ(g; h) = h for every h ∈ H and for every g ∈ G satisfies [19].

Definition 3. Let G be a transformation group and the action G : X be given. Then The vectors x1, x2 ∈ X are called G-equivalent vectors if there exist any transformation g ∈ G such that x2 = gx1 satisfies. The G-equivalent vectors x1, x2 ∈ X is denoted as x1 G ≈ x2 [23].

Definition 4: Let the point sets {x1, x2, ..., xk} , {y1, y2, ..., yk} ⊆ X and the action G : X be given. Then These sets are called G-equivalent set if there exist any transformation g ∈ G such that y1 = gx1 satisfies for every i = 1, 2, ..., k. These G-equivalent sets are denoted as {x1, x2, ..., xk} G ≈ {y1, y2, ..., yk} [23].

Definition 5: Let G be a group, f : X → ℝ be a function and the action G : X be given. Then the function f is called G- invariant function if f(x) = f(y) satisfies when x ≈ y, satisfied or if f(gx) = f(x) satisfies for ∀g ∈ G and ∀x ∈ X .[23].

Definition 5: Let G be a group, H ⊆ G be a subgroup and f, be a real valued function defined in R³. If there exist a real valued function λ : H → ℝ , such that for ∀h ∈ H , ∀x ∈ R³

\[ f(hx) = λ(h)f(x) \]

satisfies then the function f is called proportional G-invariant function and the function λ is called the factor of the function f.[23]

Definition 6: Let f be a proportional invariant function. The function f is called even O(3)-invariant function if \( \lambda(g) = 1 \) for every g ∈ O(3).

Open B-Spline Curves

Definition: The B- Spline basis functions of degree d, denoted \( N_{i,d}(t) \), defined by the knot vectors \( t_0, t_1, ..., t_m \) are defined recursively as follows

\[ N_{i,0}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \]

and

\[ N_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t} N_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1,d-1}(t) \]

for i = 0, 1, 2, ..., n and d ≥ 1. If the knot vector contains a sufficient number of repeated knot values, then a division of the form \( N_{i,d-1}(t) / t_{i+d} - t \) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that 0/0 = 0 [39].

Definition: The B-spline curve of degree d (or order d + 1) with control points \( b_0, b_1, ..., b_n \) and knots \( t_0, t_m \) is defined on the interval \([a, b] = [t_0, t_m] \) by

\[ B(t) = \sum_{i=0}^{n} b_i N_{i,d}(t) \] (1)

where \( N_{i,d}(t) \) are the B-spline basis functions of degree d. [39]

Theorem 1: The B-spline basis functions \( N_{i,d}(t) \) satisfy the following properties.

i) Positivity: \( N_{i,d}(t) > 0 \) for \( t \in (t_i, t_{i+d+1}) \).

ii) Local Support: \( N_{i,d}(t) = 0 \) for \( t \not\in (t_i, t_{i+d+1}) \).

iii) Piecewise Polynomial: \( N_{i,d}(t) \) are piecewise polynomial functions of degree d.

iv) Partition of Unity: for \( t \in [t_r, t_{r+1}) \)

\[ \sum_{i=r-d}^{r} N_{i,d}(t) = 1 \]

[39].

Theorem 2: The B-spline curves defined as (1) satisfy the following properties.

i) Local Control: Each segment is determined by \( d + 1 \) control points. If \( t \in [t_r, t_{r+1}) \) \( (d \leq r \leq m - d - 1) \), then

\[ B(t) = \sum_{i=r-d}^{r} b_i N_{i,d}(t) \] (5)

ii) Convex Hull: If \( t \in [t_r, t_{r+1}) \)
In general, B-spline curves do not interpolate the first and last control points \( b_0 \) and \( b_n \). For curves of degree \( d \), end point interpolation and an endpoint tangent condition are obtained by open B-splines for which the end knots satisfy \( t_0 = t_1 \ldots = t_d \) and \( t_{m-d} = t_{m-d+1} \ldots = t_m \).

**Theorem 3:** Let the B-spline curves of degree \( d \) with control points with control points \( b_0, b_1 \ldots, b_r \) and \( t_0 = t_1 \ldots = t_d, t_{m-d} = t_{m-d+1} \ldots = t_m \). Be given then,

\[
B(t_d) = b_0 \quad \text{and} \quad B(t_{m-d}) = b_n
\]
satisfy. [39].

**Theorem 4:** Let the B-spline curves of degree \( d \) with control points with control points \( b_0, b_1 \ldots, b_r \) and \( t_0 = t_1 \ldots = t_d, t_{m-d} = t_{m-d+1} \ldots = t_m \). Be given then,

\[
B'(t_d) = \frac{d}{t_{d+1} - t_1}(b_1 - b_0)
\]

and

\[
B'(t_{m-d}) = \frac{d}{t_{d+1} - t_1}(b_n - b_{n-1})
\]

Satisfy [39].

The following chapter is cited from [19].

**The Similarity Group** \( G = S(3) \) and **its all subgroups in** \( \mathbb{R}^3 \)

**Orthogonal Transformations’ Group** \( O(3) \)

This group is a group of all rotations and reections. Orthogonal Transformations’ group is the same of the linear isometries’ group. For any \( g \in O(3) \); the determinant of \( g \) is equal \( \pm1 \). While the rotating the rotating frames \( \{ e_1^*, e_2^*, e_3^* \} \) and fixed frames \( \{ e_1, e_2, e_3 \} \) be given. The rotations are depend on the angles \( \theta_{ij} \) between the axisses \( \{ e_i^*, e_j \} \). So this group can be stated as

\[
O(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = gx; \quad g^T = g^{-1}; \forall x \in \mathbb{R}^3 \}
\]

\[
= \{ g \in M^{3 \times 3} ; detg = \pm1 \} = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{11} & \cos \theta_{21} & \cos \theta_{31} \\
\cos \theta_{12} & \cos \theta_{22} & \cos \theta_{32} \\
\cos \theta_{13} & \cos \theta_{23} & \cos \theta_{33}
\end{pmatrix} : \theta_{ij} \epsilon \mathbb{R}
\]

**Special Orthogonal Transformations’ Group** \( SO(3) \)

This group is a group of only all rotations. For any \( g \in SO(3) \) the determinant of \( g \) is equal to \( +1 \). This group is denoted also as \( O^+(3) \). So this group can be stated as

\[
SO(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = gx; \quad g^T = g^{-1}; \forall x \in \mathbb{R}^3 \}
\]

\[
= \{ g \in M^{3 \times 3} ; g^T = g^{-1} \land detg = 1 \} = \begin{pmatrix}
\cos \theta_{11} & \cos \theta_{21} & \cos \theta_{31} \\
\cos \theta_{12} & \cos \theta_{22} & \cos \theta_{32} \\
\cos \theta_{13} & \cos \theta_{23} & \cos \theta_{33}
\end{pmatrix} : \theta_{ij} \epsilon \mathbb{R}
\]

**Translations’ Group** \( Tr(3) \)

This group is a group of all translations.

\[
Tr(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = x + b; \quad b \in \mathbb{R}^3, \forall x \in \mathbb{R}^3 \}
\]

**Euclid Transformations’ Group** \( E(3) \)

This group is a group of all translations, rotations, rotations and translations, reections, reections and translations. This group is the same of all isometries’ group. Euclid transformations can be stated as a composition of a translation and an orthogonal transformation. So this group can be stated as

\[
E(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = gx + b; \quad b \in \mathbb{R}^3, g \in O(3) \}
\]

**Special Euclid Transformations’ Group** \( SE(3) \)

This group is a group of all translations, rotations, rotations and translations. Special Euclid transformations can be stated as a composition of a translation and a special orthogonal transformation. So this group can be stated as

\[
SE(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = gx + b; \quad b \in \mathbb{R}^3, g \in SO(3) \}
\]

**Linear Homotheties’ Group** \( LH(3) \)

This group is a group of all central dilations or radial transformations. Linear homotheties are the transformations that its center of homothety are origin. So this group can be stated as

\[
LH(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = \lambda x; \quad \lambda \in \mathbb{R}^+ \}
\]

**Homotheties’ Group** \( H(3) \)

The homothety transformation can be defined as

\[
f(x) = a + \lambda(x - a) \quad \text{where} \quad a \quad \text{is called homothety center of} \quad f \quad \text{. This transformation can be stated also as}
\]

\[
f(x) = \lambda x + b \quad \text{for} \quad \lambda \neq 1 \quad \text{where the center of}
\]

\[
\text{homothety is} \quad a = \frac{b}{1-\lambda} \quad \text{. So this group can be stated}
\]

\[
H(3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 ; f(x) = \lambda x + b; \quad \lambda \in \mathbb{R}^+, b \in \mathbb{R}^3 \}
\]

**Linear Similarities’ Group** \( LS(3) \)

Linear similarity transformations can be stated as a composition of a linear homothety transform-mation and
an orthogonal transformation. So linear similarity transformations can be stated as
\[ \text{LS}(3) = \{ f: R^3 \to R^3; f(x) = \lambda g x; \; \lambda \in R^*; \; g \in O(3) \} \]

**Special Linear Similarities' Group SLS(3)**

Special linear similarity transformations can be stated as a composition of a linear homothety transformation and a special orthogonal transformation. So linear similarity transformations can be stated as
\[ \text{SLS}(3) = \{ f: R^3 \to R^3; f(x) = \lambda g x; \; \lambda \in R^*; \; g \in SO(3) \} \]

**Special Similarities' Group SS(3)**

Special similarity transformations can be stated as a composition of a special linear similarity transformation and a translation. In other words, a similarity transformation can be stated as a composition of a linear homothety transformation, a special orthogonal transformation, and a translation. So special similarity transformations can be stated as
\[ \text{SS}(3) = \{ f: R^3 \to R^3; f(x) = \lambda g x + b; \; \lambda \in R^*; \; g \in SO(3), b \in R^3 \} \]

**All Similarities' Group S(3)**

Similarity transformations can be stated as a composition of a linear similarity transformation and a translation. In other words, a similarity transformation can be stated as a composition of a linear homothety transformation, an orthogonal transformation, and a translation. So similarity transformations can be stated as
\[ S(3) = \{ f: R^3 \to R^3; f(x) = \lambda g x + b; \; \lambda \in R^*; \; g \in SO(3), b \in R^3 \} \]

**The Generator Invariants of Points by the group G = S(3) and its Some subgroups in R^3**

Let \( \{x_1, x_2, ..., x_k \} \subseteq R^3 \) be given.

**Theorem 5:** Let \( G = O(3) \subseteq S(3) \) be supposed. In this case;

i) if \( k \leq 3 \) then the system
\[ \langle x_i, x_j \rangle; \; i,j = 1, ..., k; \; i \leq j \]
gever the \( O(3) \)- invariant rational functions.

ii) if \( k > 3 \) then the system
\[ \langle x_i, x_j \rangle; \; i,j = 1,2,3; \; i \leq j \]
\[ \langle x_i, y_p \rangle; \; i,j = 1,2,3; \; p = 4,5,...,k \]
gever the \( O(3) \)- invariant rational functions. [22].

**Theorem 6:** Let \( G = SO(3) \subseteq S(3) \) be supposed. Then;

i) in case \( k < 3 \) the system
\[ \langle x_i, x_j \rangle; \; i,j = 1, ..., k; \; i \leq j \]
gever the \( SO(3) \)- invariant rational functions.

ii) in case \( k \geq 3 \) the system
\[ \det [x_1 \; x_2 \; x_3]; \]
\[ \langle x_i, x_j \rangle; \; i,j = 1,2,3; \; i \leq j ; i + j < 6; \]
\[ \langle x_i, y_p \rangle; \; i,j = 1,2,3; \; p = 4,5,...,k \]
gever the \( SO(3) \)- invariant rational functions [1].

**Theorem 7:** Let \( G = LS(3) \subseteq S(3) \) be supposed. Then;

i) in case \( k \leq 3 \) the system
\[ \langle \frac{x_i x_j}{x_i x_1}; \; i,j = 1, ..., k; \; i \leq j \]
gever the \( LS(3)- \) invariant rational functions.

ii) in case \( k > 3 \) then the system
\[ \langle \frac{x_i x_j}{x_i x_1}; \; i,j = 1,2,3; \; i \leq j \]
\[ \langle \frac{x_i x_p}{x_1 x_1}; \; i,j = 1,2,3; \; p = 4,5,...,k \]
gever the \( LS(3)- \) invariant rational functions [22].

**Theorem 8** [22]: the set of generators of the field of \( G = S(3) \) invariant rational functions for \( k \) vector variables are

1. if \( k \leq 4 \) then
\[ \left\{ \frac{x_i - x_j x_i - x_j}{x_2 - x_1 x_2 - x_1}; \; i,j = 2,3,4; \; i \leq j \right\} \]
2. if \( k \geq 4 \) then
\[ \left\{ \frac{x_i - x_j x_i - x_j}{x_2 - x_1 x_2 - x_1}; \; i,j = 2,3,4; \; i \leq j \right\} \]
\[ \left\{ \frac{x_i - x_j x_i - x_j}{x_2 - x_1 x_2 - x_1}; \; i \geq 2,3,4; \; p = 5,...,k \right\} \]

**Theorem 9:** Let \( G = E(3) \subseteq S(3) \) be given. Then;

i) if \( k=1 \) then \( E(3)- \) invariant rational functions are constant.

ii) if \( 2 \leq k \leq 4 \) then the system
\[ \langle x_i - x_1, x_j - x_1 \rangle; \; i,j = 2,3,4; \; i \leq j \]
gever the \( E(3)- \) invariant rational functions.

iii) if \( k=5 \) then the system
\[ \langle x_i - x_1, x_j - x_1 \rangle; \; i,j = 1,2,3; \; i \leq j \]
\[ \langle x_i - x_1, x_p - x_1 \rangle; \; i,j = 1,2,3; \; p = 4,5,...,k \]
gever the \( E(3)- \) invariant rational functions. [1]

**Theorem 10:** Let \( G = SE(3) \subseteq S(3) \) be given. Then;

i) if \( k=1 \) then \( SE(3)- \) invariant rational functions are constant.

ii) if \( 2 \leq k \leq 4 \) then the system
\[ \langle x_i - x_1, x_j - x_1 \rangle; \; i,j = 2,3,4; \; i \leq j \]
gever the \( SE(3)- \) invariant rational functions.

iii) if \( k=5 \) then the system
\[ \det [x_2 - x_1, x_3 - x_1, x_4 - x_1]; \]
\[ \langle x_i - x_1, x_j - x_1 \rangle; \; i,j = 2,3,4; \; i \leq j ; i + j < 8; \]
\[ \langle x_i - x_1, x_p - x_1 \rangle; \; i,j = 1,2,3; \; p = 5,...,k \]
gever the \( SE(3)- \) invariant rational functions. [1]

**MAIN RESULTS**

The Generator Invariants of Points by the other subgroups of \( G = S(3) \) in \( R^3 \)

**Theorem 11:** Let \( G = Tr(3) \subseteq S(3) \) be given. Then any \( g \in Tr(3)- \) invariant rational function for \( k \) vector variables can be written as
\[ g(x_1, ..., x_k) = f(x_2 - x_1, ..., x_k - x_1) \]
where $x_i \neq x_1$ and $f$ is a linear function.

**Proof**: Since $g$ is a $Tr(3)$-invariant, 
\[ g(x_1 + p, \ldots, x_k + p) = g(x_1, \ldots, x_k) \]
must be satisfied. So 
\[ g(x_1 + p, \ldots, x_k + p) =
\]
\[ = f((x_2 + p) - (x_1 + p), \ldots, (x_k + p) - (x_1 + p))
\]
\[ = f(x_2 - x_1, \ldots, x_k - x_1) = g(x_1, \ldots, x_k) \]
satisfies.

**Theorem 12**: Let $G = LH(3) \subset S(3)$ be given. Then:

The LH(3)-invariant rational function can be written as the proportion of the polynomial functions whose terms have the same degree $[1]$.

Example: For the vector \( (x, y, z) \in R^3 \), the function \( f(x, y, z) = x^2y_2z^2 + 2x_3y_2z^2 \) is a LH(3)-invariant. Because;
\[ f(kx, ky, kz) =
\]
\[ = (k^2x^2y_2z^2 + k^2x_3y_2z^2) =
\]
\[ = k^2f(x, y, z) \]
\[ = f(x, y, z) \]
satisfies. Example 2: For these vectors \( X, Y, Z, W \in R^3 \) the function \( f(X, Y, Z, W) = \det[X \cdot Y \cdot Z \cdot W] \) is a LH(3)-invariant.

**Theorem 13**: Let $G = SLS(3) \subset S(3)$ be given. Then;

i) if $k < 3$ then the system
\[ \frac{(x_i)}{(x_j)}; \quad i, j = 1, \ldots, k; \quad i \leq j \]
generate the SLS(3)-invariant rational functions.

ii) if $k \geq 3$ then the system
\[ \frac{\det[x_j, x_k]}{\det[x_i, x_j, x_k]}; \quad i < j < k \]
\[ \frac{(x_i)}{(x_j)}; \quad i, j = 1, 2, 3; \quad i \leq j; \quad i + j < 6 \]
\[ \frac{(x_i, x_j, x_k)}{(x_j, x_k)}; \quad i, j = 1, 2; \quad i, j, k = 4, 5, \ldots, k \]
generate the SLS(3)-invariant rational functions.

**Proof**: Since the LS(3)-invariant functions are both $O(3)$-invariant and LH(3)-invariant, it is stated as Theorem 7. Similarly, because the SLS(3)-invariant rational functions will be both SO(3)-invariant and LH(3)-invariant, the proof is completed similarly as Theorem 7.

**Theorem 14**: The set of $G = SS(3)$ invariant rational functions for $k$ vector variables

1. Contains only one element “1” which is identity of the field if $k = 1$. Indeed in this case $G = SS(3)$ invariant rational functions are constant.
2. Contains the system
\[ \frac{(x_i - x_j, x_j - x_i)}{(x_i - x_j, x_j - x_i)}; \quad i, j = 2, 3, 4; \quad i \leq j \]
\[ \text{if } 2 \leq k \leq 4 \]

3. Contains the system
\[ \frac{\det[x_i - x_j, x_j - x_k - x_i]}{\det[x_i - x_j, x_j - x_k - x_i]}; \quad i < j < k \]
\[ \frac{(x_i - x_j, x_j - x_k)}{(x_i - x_j, x_j - x_k)}; \quad i = 2, 3, 4; \quad i \leq j; \quad i + j < 8 \]
\[ \frac{(x_i - x_j, x_j - x_k - x_i)}{(x_i - x_j, x_j - x_k - x_i)}; \quad i = 2, 3, 4; \quad p = 5, \ldots, k \]
\[ \text{if } k > 4 \]

**Proof**: As the proof of the Theorem 8 with the help of proofs of Theorem 7 and Theorem 11, this theorem can be proved from theorem 13 and theorem 11 similarly.

**The G- Equivalence Conditions of Points Systems for G= S(3) and its all subgroups in R^3**

Fist of all these propositions can be given for $G = S(3)$ and its all subgroups in $R^3$.

**Proposition 1**: Let \{ $x_1, x_2, \ldots, x_k$ \}, \{ $y_1, y_2, \ldots, y_k$ \} $\subset R^3$ be given and $G \subset S(3)$ be a linear transformation’s subgroup (may be $O(3)$, $SO(3)$, $LH(3)$, $LS(3)$, $SLS(3)$). Then

1. If $x_1 = 0$ and $y_1 \neq 0$ (or vice versa) then these vectors is not G-equivalent, i.e. \{ $x_1$ \} $\cong \neq \{ y_1 \}$

2. If $x_1 = 0$ and $y_1 = 0$ then these vectors is always G-equivalent, i.e. \{ $x_1$ \} $\cong \{ y_1 \}$ and the G-equivalent conditions of two point systems with $k$ vectors \{ $x_1, x_2, \ldots, x_k$ \} and \{ $y_1, y_2, \ldots, y_k$ \} is reduced to G-equivalent conditions of two point systems with $k$-1 vectors \{ $x_2, \ldots, x_k$ \} and \{ $y_2, \ldots, y_k$ \} (The same theorem can be expressed by subtracting the $i$th vectors.)

**Proof 1**: Let $x_1 = 0$ be given. Then for any transformation $g$ in S(3) or its any subgroup $G$, the vector $gx_i$ is equal to zero. So if \{ $x_1$ \} $\cong \neq \{ y_1 \}$ then $y_1 \neq 0$ must be satisfied. So this is a contradiction. Thus in case $x_1 = 0$ and $y_1 \neq 0$ these vectors must be not G-equivalent, i.e. \{ $x_1$ \} $\cong \neq \{ y_1 \}$. The Other case can be proved similarly.

2. Let $x_1 = 0$ and $y_1 = 0$ be given. Then for any transformation $g$ in $G$. Suppose that $gx_1 = 0$ satisfies. So \{ $x_1$ \} $\cong \{ y_1 \}$ be proved. Now, \{ $x_1, x_2, \ldots, x_k$ \} and \{ $y_1, y_2, \ldots, y_k$ \} be given and $x_1 = 0$ and $y_1 = 0$ be supposed. In this case it must be proved that the G-equivalent conditions of two point systems with $k$ vectors \{ $x_1, x_2, \ldots, x_k$ \} and \{ $y_1, y_2, \ldots, y_k$ \} is reduced to G-equivalent conditions of $k$-1 vectors except $x_1$ and $y_1$. 789
Let \( \{x_1, x_2, ..., x_k\} \cong \{y_1, y_2, ..., y_k\} \) be supposed. Then there exist a transformation \( g \in G \) such that \( y_i = gx_i \) is written for \( i = 1, ..., k \). So \( y_i = gx_i \) satisfies for \( i = 2, ..., k \). Thus \( \{x_2, ..., x_k\} \cong \{y_2, ..., y_k\} \).

Conversely \( \{x_2, ..., x_k\} \cong \{y_2, ..., y_k\} \) be supposed. Then there exist a transformation \( g \in G \) such that \( y_i = gx_i \) is written for \( i = 2, ..., k \). Since \( x_1 = 0 \) and \( y_1 = 0 \), \( gx_1 = y_1 \) is satisfied \( (g0 = 0) \) so \( y_i = gx_i \) is written for \( i = 1, ..., k \). And then \( \{x_1, x_2, ..., x_k\} \cong \{y_1, y_2, ..., y_k\} \) is obtained.

**Theorem 15:**

\( \{x_1, x_2, ..., x_k\} \) and \( \{y_1, y_2, ..., y_k\} \) be given. Then \( \{x_1, x_2, ..., x_k\} \cong \{y_1, y_2, ..., y_k\} \) if and only if

- if \( k \leq 3 \) then \( (x_i, y_i) = (y_i, y_j) ; i \leq j ; i,j = 1, ..., k \)
- if \( k > 3 \) then \( (x_i, y_i) = (y_i, y_j) ; i \leq j ; i,j = 1, 2, 3 \) and \( (x_i, y_p) = (y_i, y_p) ; i = 1, 2, 3 ; p = 4, ..., k \)

satisfies for \( G = O(3) \subset S(3) \).

Proof: see [10].

**Theorem 16:**

\( \{x_1, x_2, ..., x_k\} \) and \( \{y_1, y_2, ..., y_k\} \) be given. Then \( \{x_1, x_2, ..., x_k\} \cong \{y_1, y_2, ..., y_k\} \) if and only if

- if \( k \leq 3 \) then \( (x_i, y_i) = (y_i, y_j) ; i \leq j ; i,j = 1, ..., k \)
- if \( k > 3 \) then \( (x_i, y_i) = (y_i, y_j) ; i \leq j ; i,j = 1, 2, 3 \) and \( (x_i, y_p) = (y_i, y_p) ; i = 1, 2, 3 ; p = 4, ..., k \)

satisfies for \( G = SO(3) \subset S(3) \).

Proof: see [10].

**Theorem 17:**

\( \{x_1, x_2, ..., x_k\} \) and \( \{y_1, y_2, ..., y_k\} \) be given. Then \( \{x_1, x_2, ..., x_k\} \cong \{y_1, y_2, ..., y_k\} \) if and only if

- if \( k = 1 \) then always \( x_1 \) \( \cong \{y_1\} \) satisfies.
- if \( 2 \leq k \leq 4 \) then \( (x_1 - x_1, x_j - y_1) = (y_1 - y_1, y_j - y_1) ; i \leq j ; i,j = 2, ..., k \)
- if \( k > 4 \) then \( (x_1 - x_1, x_j - y_1) = (y_1 - y_1, y_j - y_1) ; i \leq j ; i,j = 1, 2, 3, 4 \) and \( (x_1 - x_1, y_p - y_p) = (y_1 - y_1, y_p - y_p) ; i = 1, 2, 3, 4 ; p = 5, ..., k \)

satisfies for \( G = SE(3) \subset S(3) \).

Proof: see [10].

**Theorem 18:**

\( \{x_1, x_2, ..., x_k\} \) and \( \{y_1, y_2, ..., y_k\} \) be given. Then \( \{x_1, x_2, ..., x_k\} \cong \{y_1, y_2, ..., y_k\} \) if and only if

- if \( k = 1 \) then always \( x_1 \) \( \cong \{y_1\} \) satisfies.
- if \( 2 \leq k \leq 4 \) then \( (x_1 - x_1, x_j - y_1) = (y_1 - y_1, y_j - y_1) ; i \leq j ; i,j = 2, ..., k \)
- if \( k > 4 \) then \( (x_1 - x_1, x_j - y_1) = (y_1 - y_1, y_j - y_1) ; i \leq j ; i,j = 1, 2, 3, 4 \) and \( (x_1 - x_1, y_p - y_p) = (y_1 - y_1, y_p - y_p) ; i = 1, 2, 3, 4 ; p = 5, ..., k \)

satisfies for \( G = L(3) \subset S(3) \).

Proof: see [22].
\( \frac{\langle x_i, x_j \rangle}{\langle x_i, x_i \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_i, y_i \rangle} \); \( i \leq j; \ i, j = 1, 2, 3 \) and \( i + j < 6 \)

\( \frac{\langle x_i, x_p \rangle}{\langle x_i, x_1 \rangle} = \frac{\langle y_i, y_p \rangle}{\langle y_i, y_1 \rangle} \); \( i = 1, 2, 3; \ p = 4, \ldots, k \)

\( \frac{\text{det}[x_i \ x_j \ x_m]}{\text{det}[x_1 \ x_2 \ x_3]} = \frac{\text{det}[y_i \ y_j \ y_m]}{\text{det}[y_1 \ y_2 \ y_3]} \); \( i < j < m; \ i, j, m = 1, \ldots, k \)

This equations can be rewritten as
\[
\langle y_i, y_j \rangle = (\lambda x_i, \lambda x_j)
\]

(1)

if \( \frac{\langle y_i, y_j \rangle}{\langle x_i, x_j \rangle} \) is denoted by \( \lambda \). Since \( r = 3 \) then there are three linear independent vectors in each systems. So, without anything in general, these vectors let be chosen as \( x_1, x_2, x_3 \) and \( y_1, y_2, y_3 \). In this case for every \( i,j,m \)

\[
x_i = \alpha_{1i} x_1 + \alpha_{i2} x_2 + \alpha_{i3} x_3
\]

\[
y_j = \alpha_{j1} x_1 + \alpha_{j2} x_2 + \alpha_{j3} x_3
\]

and

\[
y_m = \alpha_{m1} x_1 + \alpha_{m2} x_2 + \alpha_{m3} x_3
\]

(2)

(3)

can be written. After this the proof can be completed as the proof of theorem 19 in [22]. So, \( y_i = \lambda x_i \) (4) is obtained. However, the thing to note here is that the transformation \( g \) satisfying the condition (4) may be from SLS(3) or it may be from reflections whose determinant is -1. The way to ensure that this is from SLS(3) is to satisfy that the third condition, determinant ratios are equal and positive. Now let it be proved. From (2) and (3)

\[
\frac{\text{det}[y_i \ y_j \ y_m]}{\text{det}[x_i \ x_j \ x_m]} = \frac{\text{det}B \cdot \text{det}[y_i \ y_j \ y_m]}{\text{det}A \cdot \text{det}[x_i \ x_j \ x_m]}
\]

is obtained where

\[
A = \begin{pmatrix}
\alpha_{11} & \alpha_{12} & \alpha_{13} \\
\alpha_{j1} & \alpha_{j2} & \alpha_{j3} \\
\alpha_{m1} & \alpha_{m2} & \alpha_{m3}
\end{pmatrix} \quad B = \begin{pmatrix}
\beta_{11} & \beta_{12} & \beta_{13} \\
\beta_{j1} & \beta_{j2} & \beta_{j3} \\
\beta_{m1} & \beta_{m2} & \beta_{m3}
\end{pmatrix}
\]

(5)

As can be seen in the proof of theorem 19 in [22], A and B are equal. So \( \text{det}A = \text{det}B \) is written. Then

\[
\frac{\text{det}[y_i \ y_j \ y_3]}{\text{det}[x_i \ x_2 \ x_3]} = \frac{\text{det}[\lambda x_1, \lambda x_2, \lambda x_3]}{\text{det}[x_1, x_2, x_3]} = \lambda^3 \text{det}g
\]

is obtained. Since \( \lambda > 0 \), the signal of proportion \( \text{det}[y_i \ y_j \ y_3] \) depend on the signal of \( g \). So it must be positive. Then \( \text{det}g = 1 \) and from (4) \( \{x_1, x_2, \ldots, x_k \} \approx \{y_1, y_2, \ldots, y_k \} \) is obtained for the group \( G = \text{SLS}(3) \).

In case \( r = 2 \) or \( r = 1 \), since the determinant of the matrices of any three vectors must be zero, i.e.

\[
\text{det}[x_1 \ x_2 \ x_3] = \text{det}[y_1 \ y_2 \ y_3] = 0
\]

and the proportion of determinant is undefined, the conditions in the first are sufficient. This means that every
two lines (or two planes) passing through origine can be rotated to transform one into the other. So the first conditions are sufficient.

**Theorem 22:**

\[
\begin{align*}
\{x_1, x_2, \ldots, x_k\} \text{ and } \{y_1, y_2, \ldots, y_k\} \text{ be given.} \\
\text{the ranks of matrices of these vectors systems } \{(x_2 - x_1) (x_3 - x_1) \ldots (x_k - x_1)\} \text{ and } \{(y_2 - y_1) (y_3 - y_1) \ldots (y_k - y_1)\} \text{ should be same in order to be equivalent:} \\
\text{if the ranks of matrices, (let denoted it by } r \text{ as above),} \\
\text{then; } \{x_1, x_2, \ldots, x_k\} & \approx \{y_1, y_2, \ldots, y_k\} \text{ if and only if} \\
\frac{(x_1-x_1,x_2-x_1)}{(x_2-x_1,x_2-x_1)} &= \frac{(y_1-y_1,y_2-y_1)}{(y_2-y_1,y_2-y_1)}; i \leq j \text{ satisfies for } k \leq 3 \\
\text{or for arbitrary } k \text{ and } r = 1,2 \\
\left\{ \begin{array}{l}
(x_2 - x_1, x_2 - x_1) = (y_2 - y_1, y_2 - y_1); i = j; i, j = 2,3,4 \text{ and } i + j < k; \\
(x_2 - x_1, x_2 - x_1) = (y_2 - y_1, y_2 - y_1); i = 3,4; p = 5, \ldots, k \\
\det(x_2 - x_1, x_2 - x_1) = \det(y_2 - y_1, y_2 - y_1) > 0; \\
\det(y_2 - y_1, y_2 - y_1) = \det(y_2 - y_1, y_2 - y_1) > 0; \\
i < j < m; i, j, m = 2, \ldots, k \\
satisfies (for } k \geq 4 \text{ for } G = SS(3) \subset S(3). \\
\end{array} \right.
\end{align*}
\]

**Proof:** Since SS(3)- invariant rational function is both SLS(3)- invariant and Tr(3)- invariant, this theorem can easily proved from Theorem 11 and Theorem 21.

**The G- Equivalence Conditions of Open B-Spline Curves for G= S(3) and its all subgroups in R³**

One of the most important properties of bezier and b-spline curves, even their surfaces, is the Invariance under Affine Transformations property expressed in theorem 2. According to this property, when we want to transform a bezier or b-spline curve with any affine transformation, it is sufficient to transform only control points instead of transforming each point of the curve. In other words, when we transform a B-spline curve B(t) of degree d with control points b, by an affine transformation T, the curve we will obtain is also a B-spline curve of degree d whose control points are T(b).

So these theorems can be stated as follows:

**Theorem 23:** Let \( B_1, B_2 \) are open B-spline curves of degree d with control points \( b_0, b_1, \ldots, b_d \) and \( c_0, c_1, \ldots, c_n \) respectively and knot vectors \( t_0 = t_1 = \cdots = t_d; \ t_{d+1}, t_{d+2}, \ldots \ t_{m-d-1}; \ t_{m-d} = t_{m-d+1} = \cdots = t_m \) be given.

\[
B_1 \overset{G}{\cong} B_2 \text{ if and only if} \\
\begin{array}{l}
\langle b_j, b_j \rangle = \langle c_j, c_j \rangle; i \leq j; i, j = 0,1,2; \\
\langle b_j, b_p \rangle = \langle c_j, c_p \rangle; i = 0,1,2; p = 3, \ldots, n \\
satisfies for } G = O(3) \subset S(3). \\
\end{array}
\]

**Proof:** This theorem is a result of theorem 15.

**Theorem 24:** Let \( B_1, B_2 \) are open B-spline curves of degree d with control points \( b_0, b_1, \ldots, b_n \) and \( c_0, c_1, \ldots, c_n \) respectively and knot vectors \( t_0 = t_1 = \cdots = t_d; \ t_{d+1}, t_{d+2}, \ldots \ t_{m-d-1}; \ t_{m-d} = t_{m-d+1} = \cdots = t_m \) be given.

\[
B_1 \overset{G}{\cong} B_2 \text{ if and only if} \\
\begin{array}{l}
\langle b_j, b_j \rangle = \langle c_j, c_j \rangle; i \leq j; i, j = 0,1,2; i + j < 4 \\
\langle b_j, b_p \rangle = \langle c_j, c_p \rangle; i = 0,1,2; p = 3, \ldots, n \\
\det[\langle b_0, b_1 \rangle] = \det[\langle c_0, c_1 \rangle] \\
satisfies for } G = SO(3) \subset S(3). \\
\end{array}
\]

**Proof:** This theorem is a result of theorem 16.

**Theorem 25:** Let \( B_1, B_2 \) are open B-spline curves of degree d with control points \( b_0, b_1, \ldots, b_n \) and \( c_0, c_1, \ldots, c_n \) respectively and knot vectors \( t_0 = t_1 = \cdots = t_d; \ t_{d+1}, t_{d+2}, \ldots \ t_{m-d-1}; \ t_{m-d} = t_{m-d+1} = \cdots = t_m \) be given.

\[
B_1 \overset{G}{\cong} B_2 \text{ if and only if} \\
\begin{array}{l}
\langle b_i - b_j, b_j \rangle = \langle c_i - c_j, c_j \rangle; i \leq j; i, j = 1,2,3; i + j < 6 \\
\langle b_i - b_j, b_j - b_i \rangle = \langle c_i - c_j, c_j - c_i \rangle; i = 1,2,3; p = 4, \ldots, n \\
\det[\langle b_0, b_1 \rangle - \langle b_0, b_2 \rangle] = \det[\langle c_0, c_1 \rangle - \langle c_0, c_2 \rangle] \\
satisfies for } G = SE(3) \subset S(3). \\
\end{array}
\]

**Proof:** This theorem is a result of theorem 17.

**Theorem 26:** Let \( B_1, B_2 \) are open B-spline curves of degree d with control points \( b_0, b_1, \ldots, b_n \) and \( c_0, c_1, \ldots, c_n \) respectively and knot vectors \( t_0 = t_1 = \cdots = t_d; \ t_{d+1}, t_{d+2}, \ldots \ t_{m-d-1}; \ t_{m-d} = t_{m-d+1} = \cdots = t_m \) be given.

\[
B_1 \overset{G}{\cong} B_2 \text{ if and only if} \\
\begin{array}{l}
\langle b_i - b_j, b_j - b_i \rangle = \langle c_i - c_j, c_j - c_i \rangle; i \leq j; i, j = 1,2,3; i + j < 6 \\
\langle b_i - b_j, b_j - b_i \rangle = \langle c_i - c_j, c_j - c_i \rangle; i = 1,2,3; p = 4, \ldots, n \\
\det[\langle b_0, b_1 \rangle - \langle b_0, b_2 \rangle] = \det[\langle c_0, c_1 \rangle - \langle c_0, c_2 \rangle] \\
satisfies for } G = E(3) \subset S(3). \\
\end{array}
\]

**Proof:** This theorem is a result of theorem 18.

**Theorem 27:** Let \( B_1, B_2 \) are open B-spline curves of degree d with control points \( b_0, b_1, \ldots, b_n \) and \( c_0, c_1, \ldots, c_n \) respectively and knot vectors \( t_0 = t_1 = \cdots = t_d; \ t_{d+1}, t_{d+2}, \ldots \ t_{m-d-1}; \ t_{m-d} = t_{m-d+1} = \cdots = t_m \) be given and \( \text{rank}[b_0, b_1, \ldots, b_n] = \text{rank}[c_0, c_1, \ldots, c_n] = r \) be supposed. Then

\[
B_1 \overset{G}{\cong} B_2 \text{ if and only if} \\
\begin{array}{l}
\langle b_j, b_j \rangle = \langle c_j, c_j \rangle; i \leq j; i, j = 0,1,2; \\
\langle b_j, b_p \rangle = \langle c_j, c_p \rangle; i = 0,1,2; p = 3, \ldots, n \\
\end{array}
\]

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satisfies for \( r = 3 \) or
\[
\langle b_i, b_p \rangle = \langle c_{i_0}, c_p \rangle; \quad i = 0, 1; \quad p = 2, \ldots, n
\]
satisfies for \( r = 2 \) or
\[
\langle b_0, b_p \rangle = \langle c_{1_0}, c_p \rangle; \quad p = 1, \ldots, n
\]
satisfies for \( r = 1 \)
for \( G = \text{LS}(3) \subset \text{S}(3) \).

\textbf{Proof:} This theorem is a result of theorem 19.

\begin{equation}
\textbf{Theorem 28:} \text{Let } \mathcal{B}_1, \mathcal{B}_2 \text{ are open B-spline curves of degree } d \text{ with control points } b_0, b_1, \ldots, b_n \text{ and } c_0, c_1, \ldots, c_n \text{ respectively and knot vectors } t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \ldots, t_{m-d}; t_{m-d-1} = \cdots = t_m \text{ be given and rank } \text{rank}[b_1 - b_0 \ldots b_n - b_0] = \text{rank}[c_1 - c_0 \ldots c_n - c_0] = r \text{ be supposed. Then}
\end{equation}

\begin{equation}
\mathcal{B}_1 \overset{G}{\approx} \mathcal{B}_2 \text{ if and only if}
\end{equation}

\begin{equation}
\begin{cases}
\langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; \quad i, j \leq d; i, j = 1, 2, 3; \\
\langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; \quad i = 1, 2, 3; \quad p = 3, \ldots, n
\end{cases}
\end{equation}
satisfies for \( r = 3 \) or
\begin{equation}
\begin{cases}
\langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; \quad i = 1, 2, 3; \quad p = 3, \ldots, n
\end{cases}
\end{equation}
satisfies for \( r = 2 \) or
\begin{equation}
\begin{cases}
\langle b_0, b_p \rangle = \langle c_{1_0}, c_p \rangle; \quad i = 0, 1; \quad p = 2, \ldots, n
\end{cases}
\end{equation}
satisfies for \( r = 1 \)
for \( G = \text{LS}(3) \subset \text{S}(3) \).

\textbf{Proof:} This theorem is a result of theorem 20.

\begin{equation}
\textbf{Theorem 29:} \text{Let } \mathcal{B}_1, \mathcal{B}_2 \text{ are open B-spline curves of degree } d \text{ with control points } b_0, b_1, \ldots, b_n \text{ and } c_0, c_1, \ldots, c_n \text{ respectively and knot vectors } t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \ldots, t_{m-d}; t_{m-d-1} = \cdots = t_m \text{ be given and rank } \text{rank}[b_1 - b_0 \ldots b_n - b_0] = \text{rank}[c_1 - c_0 \ldots c_n - c_0] = r \text{ be supposed. Then}
\end{equation}

\begin{equation}
\mathcal{B}_1 \overset{G}{\approx} \mathcal{B}_2 \text{ if and only if}
\end{equation}

\begin{equation}
\begin{cases}
\langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; \quad i \leq j; i, j = 1, 2, 3; \\
\langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; \quad i = 1, 2, 3; \quad p = 3, \ldots, n
\end{cases}
\end{equation}
satisfies for \( r = 3 \) or
\begin{equation}
\begin{cases}
\langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; \quad i = 1, 2, 3; \quad p = 3, \ldots, n
\end{cases}
\end{equation}
satisfies for \( r = 2 \) or
\begin{equation}
\begin{cases}
\langle b_0, b_p \rangle = \langle c_{1_0}, c_p \rangle; \quad i = 0, 1; \quad p = 2, \ldots, n
\end{cases}
\end{equation}
satisfies for \( r = 1 \)
for \( G = \text{SLS}(3) \subset \text{S}(3) \).

\textbf{Proof:} This theorem is a result of theorem 22.

\begin{equation}
\textbf{Example:}
\end{equation}

Let \( \gamma \) be a cubic B-spline curve of degree 3 with control points \( b_0 = (4, 2, 2), \quad b_1 = (2, 1, 4), \quad b_2 = (3, 4, 1), \quad b_3 = (3, 5, 5) \) and \( t_0 = t_1 = \ldots = t_3 = 0;\ldots; t_4 = t_5 = \cdots = t_7 = 1 \) be given.

The spline basis functions:

\begin{equation}
N_{0,0} = 0, \quad N_{1,0} = 0, \quad N_{2,0} = 0, \quad N_{3,0} = \begin{cases}
1, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{4,0} = 0, \quad N_{5,0} = 0, \quad N_{6,0} = 0
\end{equation}

\begin{equation}
N_{0,1} = 0, \quad N_{1,1} = 0, \quad N_{2,1} = \begin{cases}
1- t, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{3,1} = \begin{cases}
1, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{4,1} = 0, \quad N_{5,1} = 0
\end{equation}

\begin{equation}
N_{0,2} = 0, \quad N_{1,2} = \begin{cases}
(1- t)^2, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{2,2} = \begin{cases}
2(1-t), & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{3,2} = \begin{cases}
\begin{cases}
(1-t)^3, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{cases}
\end{equation}

\begin{equation}
N_{4,2} = 0
\end{equation}

\begin{equation}
N_{0,3} = \begin{cases}
(1-t)^3, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{1,3} = \begin{cases}
3(1-t)^2, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{2,3} = \begin{cases}
3t^2(1-t), & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

\begin{equation}
N_{3,3} = \begin{cases}
t^3, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

Then
\begin{equation}
\gamma(t) = N_{0,3}b_0 + N_{1,3}b_1 + N_{2,3}b_2 + N_{3,3}b_3
\end{equation}

\begin{equation}
= \begin{cases}
(1-t)^3b_0 + 3t(1-t)^2b_1 + 3t^2(1-t)b_2 + t^3b_3, & t \in [0, 1] \\
0, & \text{other case}
\end{cases}
\end{equation}

This means: for \( t \in [0, 1] \).
\[
\gamma(t) = (-4t^3 + 9t^2 - 6t + 4, -6t^3 + 12t^2 - 3t + 2, 12t^3 - 15t^2 + 6t + 2)
\]
can be written.
Let these transformations be choosen as element of S(3) and its subgroubs:
For \(x \in \mathbb{R}^3\),
\(g_1 \in \text{Tr}(3) \Rightarrow g_1(x) = x + (2,3,5)\)
\(g_2 \in \text{O}(3) \Rightarrow g_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \)
\(g_3 \in SO(3) \Rightarrow g_3 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \)
\(g_4 \in SE(3) \Rightarrow g_4(x) = g_3(x) + (2,3,5)\)

\(g_5 \in E(3) \Rightarrow g_5(x) = g_2(x) + (2,3,5)\)
\(g_6 \in LH(3) \Rightarrow g_6 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \)
\(g_7 \in SLS(3) \Rightarrow g_7 = \frac{3\sqrt{2}}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \)
\(g_8 \in LS(3) \Rightarrow g_8 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix} \)
\(g_9 \in SS(3) \Rightarrow g_9(x) = g_7(x) + (2,3,5)\)
\(g_{10} \in S(3) \Rightarrow g_{10}(x) = g_9(x) + (2,3,5)\)

Let the images of the curve \(\gamma\) under the transformations \(g_i\) be denoted by \(\gamma_i\) respectively.
If we denote the subgroup of S(3) containing the transformation \(g_i\) with \(G_i\) then \(\gamma \approx G_i \gamma_i\). (see Fig.1)

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