ON ZERO-SUM FREE SEQUENCES CONTAINED IN RANDOM SUBSETS OF FINITE CYCLIC GROUPS

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Abstract. Let $C_n$ be a cyclic group of order $n$. A sequence $S$ of length $\ell$ over $C_n$ is a sequence $S = a_1 \cdot a_2 \cdot \ldots \cdot a_\ell$ of $\ell$ elements in $C_n$, where a repetition of elements is allowed and their order is disregarded. We say that $S$ is a zero-sum sequence if $\sum_{i=1}^{\ell} a_i = 0$ and that $S$ is a zero-sum free sequence if $S$ contains no zero-sum subsequence.

Let $R$ be a random subset of $C_n$ obtained by choosing each element in $C_n$ independently with probability $p$. Let $N_{n-1-k}^R$ be the number of zero-sum free sequences of length $n - 1 - k$ in $R$. Also, let $N_{n-1-k,d}^R$ be the number of zero-sum free sequences of length $n - 1 - k$ having $d$ distinct elements in $R$. We obtain the expectation of $N_{n-1-k}^R$ and $N_{n-1-k,d}^R$ for $0 \leq k \leq \lfloor n/3 \rfloor$. We also show a concentration result on $N_{n-1-k}^R$ and $N_{n-1-k,d}^R$ when $k$ is fixed.

1. Introduction

Let $C_n$ be a cyclic group of order $n$. A sequence $S$ of length $\ell$ over $C_n$ is a sequence

$$S = a_1 \cdot a_2 \cdot \ldots \cdot a_\ell$$

of $\ell$ elements in $C_n$, where a repetition of elements is allowed and their order is disregarded. We say that a sequence $S$ over $C_n$ is contained in $A \subset C_n$ if each element in $S$ is contained in $A$. For $a \in C_n$, let

$$v_a(S) = |\{i \in [1, \ell] \mid a_i = a\}|$$

be the multiplicity of $a$ in $S$. A subsequence $T$ of $S$ is a sequence over $C_n$ satisfying $v_a(T) \leq v_a(S)$ for all $a \in C_n$. We say that $S$ is a zero-sum sequence if $a_1 + a_2 + \ldots + a_\ell = 0$. A sequence is called zero-sum free if it contains no zero-sum subsequence.

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An initial study of zero-sum sequences dates back to 1961 when Erdős, Ginzburg, and Ziv [6] proved that $2n - 1$ is the smallest positive integer $\ell$ such that every sequence of length $\ell$ over $C_n$ has a zero-sum subsequence of length $n$. Since that time, zero-sum sequences over a finite group have actively studied in additive combinatorics. For more details, see a survey paper by Gao and Geroldinger [8]. Although earlier works often focused on finite abelian groups, an application to factorization theory and invariant theory pushed the object forward to non-abelian groups. The reader can refer to Geroldinger, Grynkiewicz, Zhong, and the second author [9] for recent progress with respect to factorization theory and to Cziszter, Domokos, and Szőlősi [3, 5] for connection with invariant theory.

In this paper, we focus on zero-sum free sequences over a cyclic group. Well-known problems about zero-sum free sequences over a finite group are to determine the maximum length of zero-sum free sequences, which is a combinatorial group invariant known as the Davenport constant, and to characterize the structure of zero-sum free sequences. Observe that the maximum length of all zero-sum free sequences over $C_n$ is $n - 1$. Also, we have that $S$ is a zero-sum free sequence of length $n - 1$ over $C_n$ if and only if

$$S = g \cdot g \cdot \ldots \cdot g$$

for a generator $g \in C_n$. Gao [7] proved the following result on the structure of long zero-sum free sequences over $C_n$.

**Theorem 1** (Theorem 4.3 in [8], Lemma 2.5 in [7]). Let $n \geq 2$ and $0 \leq k \leq \left[\frac{n}{3}\right]$. Then $S$ is a zero-sum free sequence of length $n - 1 - k$ over $C_n$ if and only if

$$S = g \cdot g \cdot \ldots \cdot g \cdot (x_1 g) \cdot (x_2 g) \cdot \ldots \cdot (x_k g),$$

where $g$ is a generator of $C_n$ and $x_1, x_2, \ldots, x_k$ are positive integers such that

$$1 \leq x_1 \leq x_2 \leq \ldots \leq x_k \quad \text{and} \quad x_1 + x_2 + \ldots + x_k \leq 2k. \quad (1)$$

Theorem 1 was generalized by Savchev and Chen [17] on the zero-sum free sequences of length at least $(n + 1)/2$ over $C_n$. Theorem 1 and the result by Savchev and Chen were applied to the number of minimal zero-sum sequences of long length by Ponomarenko [15] and Cziszter, Domokos, and Geroldinger [4], respectively.

Remark that Theorem 4.3 in [8] only gives the statement in Theorem 1 from the left-hand side to the right-hand side. The proof from the right-hand side to the left-hand side is obvious since $g$ is a generator of $C_n$ and all subsequences $T$ of $S$ satisfy $\sigma(T) = \ell g \neq 0$ for some integer $0 < \ell < n$, where $\sigma(T)$ is the sum of all elements in $T$.  


In this paper, we are interested in zero-sum free sequences of a given length contained in a random subset of $C_n$. Investigating how classical extremal results in dense environments transfer to sparse settings has become a deep line of research. For example, Roth’s theorem on 3-term arithmetic progressions [16] was generalized for random subsets of integers [13], and there are recent generalizations about various classical extremal results by Schacht [18] and Conlon and Gowers [2].

Let $R$ be a random subset of $C_n$ obtained by choosing each element in $C_n$ independently with probability $p$. Let $N_{n-1-k}$ be the number of zero-sum free sequences of length $n-1-k$ over $C_n$. Also, let $N^R_{n-1-k}$ be the number of zero-sum free sequences of length $n-1-k$ in $R$. The result on the expectation of $N^R_{n-1-k}$ is as follows.

**Theorem 2.** Let $n \geq 2$ and $0 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$. The expected number of zero-sum free sequences of length $n-1-k$ in a random subset $R$ of $C_n$ is

$$E\left(N^R_{n-1-k}\right) = \varphi(n) \left[p + \sum_{d=2}^D p^d \left(\sum_{j=\left\lceil \frac{d-1}{d}\right\rceil}^k q(j, d-1)\right)\right],$$

where

- $D = \left\lceil \frac{1+\sqrt{1+8k}}{2} \right\rceil$,
- $\varphi(n)$ denotes the number of generators in $C_n$, and
- $q(j, d-1)$ is the number of partitions of $j$ having $d-1$ distinct parts.

The number $q(j, d-1)$ can be computed in two ways: The first way is based on its generating function (see Section 2.1 for details). Second, we provide a recursive formula for computing $X_{k,d-1} = \sum_{j=(d-1)d/2}^k q(j, d-1)$ (see Section 2.2).

If $k$ is fixed, then we can obtain a simpler statement as follows.

**Corollary 3.** If $k$ is fixed and $p = o(1)$ as $n \to \infty$, then

$$E\left(N^R_{n-1-k}\right) = p\varphi(n) (1 + O_k(p)),$$

where the constant in $O_k$ depends only on $k$.

Next, we have a concentration result on $N^R_{n-1-k}$ when $k$ is fixed.

**Theorem 4.** Let $k$ be fixed, and let $p$ be such that

$$\frac{(\log n)^{2d} \log \log n}{n} \ll p \ll 1.$$

Then, asymptotically almost surely (a.a.s.)

$$N^R_{n-1-k} = p\varphi(n) + O_k \left(p^2 \varphi(n) + \sqrt{p\varphi(n)} (\log n)^d\right),$$

where the constant in $O_k$ depends only on $k$. 

Moreover, we have a refined result. Let $N_{n-1-k,d}$ be the number of zero-sum free sequences of length $n - 1 - k$ having $d$ distinct elements over $C_n$. Also, let $N_{n-1-k,d}^R$ be the number of zero-sum free sequences of length $n - 1 - k$ having $d$ distinct elements contained in a random subset $R$ of $C_n$. We show a concentration result on $N_{n-1-k,d}^R$.

**Theorem 5.** If $0 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$ and $$p \gg \frac{\log \log n}{n},$$
then we have that a.a.s. $$p\varphi(n) - \omega \sqrt{p\varphi(n)} \leq N_{n-1-k,1}^R \leq p\varphi(n) + \omega \sqrt{p\varphi(n)},$$
where $\omega$ tends to $\infty$ arbitrarily slowly as $n \to \infty$.

Let $d \geq 2$. If $k$ is fixed and $$p \gg \frac{(\log n)^2(\log \log n)^{1/d}}{n^{1/d}},$$
then we have that a.a.s. $$N_{n-1-k,d}^R = p^d\varphi(n) \left( \sum_{j=(d-1)d/2}^{d} q(j, d-1) \right) + O_k \left( \sqrt{p^d\varphi(n)(\log n)^d} \right).$$

The organization of this paper is as follows. In Section 2, we consider expectations and prove Theorem 2 and Corollary 3. Then, we deal with our concentration results and prove Theorems 4 and 5 in Section 3.

### 2. Expectation

In this section, we prove Theorem 2 and Corollary 3. Also, we provide a recursive formula to compute the important value $X_{k,d-1} = \sum_{j=(d-1)d/2}^{d} q(j, d-1)$ given in Theorem 2.

#### 2.1. Proofs of Theorem 2 and Corollary 3

It turns out that the number of distinct elements in a zero-sum free sequence plays an important role since each element in $C_n$ is contained in a random set $R$ with probability $p$. Recall that $N_{n-1-k,d}$ is the number of zero-sum free sequences of length $n - 1 - k$ having $d$ distinct elements over $C_n$, and $N_{n-1-k,d}^R$ is the number of zero-sum free sequences over $C_n$ of length $n - 1 - k$ having $d$ distinct elements contained in a random set $R$.

Clearly, the expectation of $N_{n-1-k,d}^R$ is $$\mathbb{E} \left( N_{n-1-k,d}^R \right) = p^d N_{n-1-k,d}.$$
Based on Theorem 1, the numbers \( N_{n-1-k,d} \) and \( N_{n-1-k,d}^R \) are related to the number of

\[
(x_1, x_2, \ldots, x_k)
\]
satisfying that \( x_1, x_2, \ldots, x_k \) are positive integers such that \( 1 \) holds and the number of distinct \( x_i \neq 1 \) is \( d - 1 \). With \( x'_i := x_i - 1 \), the number can be simplified as follows.

**Definition 6.** Let \( X_{k,d} \) be the number of \((x'_1, x'_2, \ldots, x'_k)\) such that

\[
0 \leq x'_1 \leq x'_2 \leq \cdots \leq x'_k, \quad x'_1 + x'_2 + \cdots + x'_k \leq k,
\]

and the number of distinct positive \( x'_i \) is \( d \).

Theorem 1 and Definition 6 give that

\[
N_{n-1-k,d} = \varphi(n) X_{k,d-1},
\]

where \( \varphi(n) \) is the number of generators in \( C_n \). Therefore, the expectation of \( N_{n-1-k,d}^R \) is

\[
E(N_{n-1-k,d}^R) = p^d \varphi(n) X_{k,d-1}.
\]

From now on, we focus on estimating \( X_{k,d-1} \). To this end, we use the definition of a partition of an integer. A partition of a positive integer \( k \) is a non-decreasing sequence whose sum equals \( k \). A partition \( \lambda \) of \( k \) can be shortly expressed by

\[
1^{r_1} 2^{r_2} \cdots t^{r_t}
\]

meaning that

\[
k = (1 + 1 + \ldots + 1) + (2 + 2 + \ldots + 2) + \ldots + (t + t + \ldots + t).
\]

If \( \lambda \) is a partition of \( k \), then we denote \( \lambda \vdash k \). Let \(|\lambda| = k \) if \( \lambda \vdash k \).

Let \( q(k,d) \) be the number of partitions of \( k \) having \( d \) distinct parts. For example, all partitions of 7 are as follows:

- \( 1^7, 7^1 \),
- \( 1^52^1, 1^32^3, 1^43^1, 1^13^2, 2^23^1, 1^34^1, 3^14^1, 1^25^1, 2^15^1, 1^16^1 \),
- \( 1^22^13^1, 1^12^14^1 \).

We have that \( q(7, 1) = 2, q(7, 2) = 11, q(7, 3) = 2, \) and \( q(7, d) = 0 \) for \( d \geq 4 \).

Recalling Definition 6, we have that \( X_{k,d} \) is the same as the number of partitions \( \lambda \) of at most \( k \) having \( d \) distinct parts. Observe that if \( \lambda \) is counted for \( X_{k,d} \), then

\[
\frac{d(d + 1)}{2} \leq |\lambda| \leq k
\]

(4)
because $\lambda$ contains parts with at least $1, 2, \ldots, d$. Thus, we have

$$X_{k,d} = \sum_{j=d(d+1)}^{k} q(j,d).$$

(5)

Remark that the number $q(j,d)$ can be found in A116608 of the on-line encyclopedia of integer sequences (OEIS), and it can be computed from its generating function

$$Q(x,t) = -1 + \prod_{i=1}^{\infty} \left( 1 + \frac{tx^i}{1-x^i} \right),$$

where

$$Q(x,t) = \sum_{j,d \geq 1} q(j,d)x^jt^d.$$

There are related results on $q(j,d)$. Kim [11] constructed a generating function with one variable for $q(j,d)$ when $d$ is fixed. Also, Goh and Schmutz [10] obtained the asymptotic distribution of the number of distinct part sizes in a random integer partition. On the other hand, $X_{k,d}$ is not found in OEIS.

We are ready to prove Theorem 2.

**Proof of Theorem 2.** Trivially, the expected number of zero-sum free sequences of length $n - 1 - k$ with same elements in $R$ is

$$\mathbb{E}(N_{n-1-k,1}^R) = \varphi(n)p.$$  

Next, for $d \geq 2$, we infer that

$$\mathbb{E}(N_{n-1-k,d}^R) = p^dN_{n-1-k,d} \varnothing p^d\varphi(n)X_{k,d-1} \varnothing p^d\varphi(n) \left( \sum_{j=\frac{(d-1)d}{2}}^{k} q(j,d-1) \right).$$

Next we consider the range of $d$. If $X_{k,d-1}$ is positive, then (4) gives that

$$k \geq \frac{(d-1)d}{2}.$$  

Hence, let $D$ be the largest integer $d$ satisfying $(d-1)d/2 \leq k$, and then,

$$d \leq D = \left\lceil \frac{1 + \sqrt{1 + 8k}}{2} \right\rceil.$$

This completes our proof of Theorem 2.  

□

Now we are ready to prove Corollary 3 using Theorem 2.
Proof of Corollary 3. For a fixed $k$, Theorem 2 gives that

$$\mathbb{E}(N_{n-1-k}^R) = \varphi(n) \left[ p + \sum_{d=2}^{D} p^d \left( \sum_{j=\frac{(d-1)j}{d}}^{k} q(j, d-1) \right) \right]$$

$$= p\varphi(n) \left( 1 + \sum_{d=2}^{D} O_k(p^{d-1}) \right)$$

$$= p\varphi(n) \left( 1 + O_k(p) \right),$$

where the constant in $O_k$ depends only on $k$, which completes the proof of Corollary 3. □

2.2. Recursive formula for $X_{k,d}$. Here, we give another way to compute the important value

$$X_{k,d-1} = \sum_{j=\frac{(d-1)j}{d}}^{k} q(j, d-1)$$

given in Theorem 2 using a recursive formula.

A partition of an integer can be illustrated by a Young diagram (also called a Ferrers diagram), which is a useful way to understand a partition in combinatorics. A Young diagram corresponding to a partition $\lambda \vdash k$ is a collection of left-justified rows of $k$ boxes piled up in non-decreasing order of row lengths from parts. For example, the partition $1^22^13^1 \vdash 7$ corresponds to the Young diagram

\[
\begin{array}{c}
\text{Young Diagram}
\end{array}
\]

Let $Y_{b,c,d}$ be the number of partitions of at most $b$ with at most $c$ parts having $d$ distinct parts. Equivalently, $Y_{b,c,d}$ is the number of Young diagrams with at most $b$ boxes, at most $c$ rows, and $d$ distinct rows. See Figure 1(a).

Note that $Y_{b,c,d} > 0$ if and only if $b \geq \frac{d(d+1)}{2}$ and $c \geq d$, where the first inequality follows from (1). Observe that

$$X_{k,d} = Y_{k,k,d}.$$ 

A recursive formula for $Y_{b,c,d}$ is as follows. (Hence we have a recursive formula for $X_{k,d}$.)

Lemma 7. We have that, for $b \geq \frac{d(d+1)}{2}$ and $c \geq d \geq 2$,

$$Y_{b,c,d} = \sum_{i=1}^{c} \sum_{j=1}^{\left\lceil \frac{b-(d-1)j}{d} \right\rceil} Y_{b-ij,i-1,d-1} \quad (6)$$
and, for \( b \geq 1 \) and \( c \geq 1 \),

\[
Y_{b,c,1} = \sum_{i=1}^{c} \left\lfloor \frac{b}{i} \right\rfloor .
\]

(7)

**Proof.** We first show (6). We delete the gray retangle in Figure 1 from a Young diagram counted for \( Y_{b,c,d} \), and then we have a Young diagram with at most \( b - ij \) boxes, at most \( i - 1 \) rows, and \( d - 1 \) distinct rows.

We consider the ranges of \( i \) and \( j \). Clearly, the range of \( i \) is \( 1 \leq i \leq c \). Then the remaining Young diagram after the deletion has \( d - 1 \) distinct rows, and hence, it has at least \((d - 1)d/2\) boxes. Thus,

\[
ij + \frac{(d - 1)d}{2} \leq b.
\]

So the range of \( j \) is

\[
1 \leq j \leq \left\lfloor \frac{b - (d - 1)d/2}{i} \right\rfloor .
\]

Next, we show (7). The number \( Y_{b,c,1} \) is the same as the number of rectangles with at most \( b \) boxes and at most \( c \) rows. Let \( i \) and \( j \) be the numbers of rows and columns, respectively, of such a rectangle. Clearly, \( 1 \leq i \leq c \). Since \( ij \leq b \), we have \( j \leq \left\lfloor \frac{b}{i} \right\rfloor . \)

\[\square\]

3. **Concentration**

Recall that \( N_{n-1-k,d}^R \) be the number of zero-sum free sequences of length \( n-1-k \) having \( d \) distinct elements in a random subset \( R \). From (3), recall that

\[
E(N_{n-1-k,d}^R) = p^d \varphi(n) X_{k,d-1}.
\]

From now on, we consider a concentration of \( N_{n-1-k,d}^R \) and \( N_{n-1-k}^R \) using a graph theoretical approach called the Kim–Vu polynomial concentration result.
3.1. Kim–Vu polynomial concentration result. Let $\mathcal{H} = (V, E)$ be a weighted hypergraph with $V = [n] := \{0, 1, \ldots, n-1\}$. Recall that $R$ is a random subset of $[n]$ obtained by selecting each $v \in [n]$ independently with probability $p$. Let $\mathcal{H}[R]$ be the sub-hypergraph of $\mathcal{H}$ induced on $R$, and we let $Z$ be the sum of weights of hyperedges in $\mathcal{H}[R]$. Kim and Vu [12] obtained a result that provides a concentration of $Z$ around its mean $E(Z)$ with high probability. For more details, see Alon and Spencer [1]. To state the result, we need some definitions.

**Definition 8.** Let $\ell$ be the maximum size of hyperedges in $\mathcal{H}$, and let $A \subset [n]$ be such that $|A| \leq \ell$. We let

- $Z_A :=$ the sum of weights of hyperedges in $\mathcal{H}[R]$ containing $A$,
- $E_A := E(Z_A | A \subset R)$,
- $E_i :=$ the maximum of $E_A$ for $A \subset [n]$ with $|A| = i$,
- $E' := \max_{1 \leq i \leq \ell} E_i$ and $E^* := \max \{E', E(Z)\}$.

The concentration result by Kim and Vu [12] is as follows.

**Theorem 9 (Kim–Vu polynomial concentration inequality).** With the notation as above, we have that, for each $\lambda > 1$,

$$\Pr \left[ |Z - E(Z)| > a_\ell \sqrt{E \cdot E^* \lambda} \right] < 2e^{-\lambda^2 + 2n\ell - 1},$$

where $a_\ell = 8^\ell (\ell!)^{1/2}$.

3.2. Hypergraph and example. For a given positive integer $k$, we define the hypergraph $H_{n-1-k} = H_{n-1-k}(C_n) = ([n], E)$ such that $a_1 \cdot a_2 \cdot \ldots \cdot a_{n-1-k}$ is a zero-sum free sequence over $C_n$ if and only if the corresponding set $\{b_1, b_2, \ldots, b_\ell\} = \{a_1, a_2, \ldots, a_{n-1-k}\}$, with $1 \leq \ell \leq n - 1 - k$, is contained in $E$. The weight of an hyperedge $\{b_1, b_2, \ldots, b_\ell\}$ of $H_{n-1-k}$ is the number of zero-sum free sequences over $C_n$ consisting of $b_1, b_2, \ldots, b_\ell$.

Then $E_{d,A}$ defined above is the expected number of zero-sum free sequences of length $n - 1 - k$ having $d$ distinct elements that contains $A \subset C_n$ and is contained in $R$ under the condition that $A \subset R$. Also, for $1 \leq i \leq d$, let

$$E_{d,i} = \max \{E_{d,A} | A \subset C_n \text{ with } |A| = i\}.$$ 

We will estimate $E_{d,A}$ and $E_{d,i}$.

For an easier understanding, we give an example in $C_8$ before estimating $E_{d,A}$ and $E_{d,i}$ in a general $C_n$. Let $C_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and we consider the case where $k = 2$. In this case, the length of zero-sum free sequences is $n - 1 - k = 8 - 1 - 2 = 5$. All generators in $C_8$ are $1, 3, 5, 7$, and all possible $(x_1, x_2)$ in Theorem 1 are $(1, 1), (1, 2), (1, 3)$, and $(2, 2)$. Thus, Theorem 1 gives that all zero-sum free
sequences of length 5 over $C_8$ are

\[
\begin{align*}
1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 & \quad 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 & \quad 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 & \quad 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \\
1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 & \quad 3 \cdot 3 \cdot 3 \cdot 3 \cdot 6 & \quad 5 \cdot 5 \cdot 5 \cdot 5 \cdot 2 & \quad 7 \cdot 7 \cdot 7 \cdot 7 \cdot 6 \\
1 \cdot 1 \cdot 1 \cdot 1 \cdot 3 & \quad 3 \cdot 3 \cdot 3 \cdot 3 \cdot 1 & \quad 5 \cdot 5 \cdot 5 \cdot 5 \cdot 7 & \quad 7 \cdot 7 \cdot 7 \cdot 7 \cdot 5 \\
1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 & \quad 3 \cdot 3 \cdot 3 \cdot 6 \cdot 6 & \quad 5 \cdot 5 \cdot 5 \cdot 2 \cdot 2 & \quad 7 \cdot 7 \cdot 7 \cdot 6 \cdot 6.
\end{align*}
\]

Hence, the hypergraph $H_5(C_8)$ has hyperedges as follows:

| Hyperedge | \{1\} | \{3\} | \{5\} | \{7\} | \{1,2\} | \{1,3\} | \{3,6\} | \{5,2\} | \{5,7\} | \{7,6\} |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| Weight    | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2     |

As an example, we estimate $E_{2,1}$ by considering $E_{2,\{a\}}$ for $a \in C_8$. First, let $a = 1$. Note that our goal here is not to get the exact value of $E_{2,\{1\}}$ but to obtain a uniform upper bound of $E_{2,\{a\}}$ for all $a \in C_8$. For a generator $g$, there are several cases we need to deal with:

- **Case 1** $(a = 1 = g)$: Trivially, $g = 1$. Since $(x'_1, x'_2) = (0, 1), (0, 2), \text{ or } (1, 1)$, we have $(x_1, x_2) = (1, 2), (1, 3), \text{ or } (2, 2)$, and hence, all zero-sum free sequences of this case in $C_8$ are

  \[
  1 \cdot 1 \cdot 1 \cdot (2 \cdot 1) \quad 1 \cdot 1 \cdot 1 \cdot (3 \cdot 1) \quad 1 \cdot 1 \cdot 1 \cdot (2 \cdot 1) \cdot (2 \cdot 1).
  \]

  Thus, the expected number of all zero-sum free sequences of this case in $R$ is

  \[X_{2,1} \cdot p.\]

- **Case 2** $(a = 1 = 2g)$: There is no such $g$, but we go forward to get a uniform upper bound. Since $a = 2g$, we have $x'_\ell = 1$ for some $\ell$. Hence, the number of all zero-sum free sequences of this case in $C_8$ is at most $X_{2-1,1} + X_{2-1,0}$, where the first term is from the situation when all other $x'$ are different from $x'_\ell$ and the second term is from the other situation. Thus, the expected number of all zero-sum free sequences of this case in $R$ is at most

  \[(X_{2-1,1} + X_{2-1,0}) \cdot p.\]

- **Case 3** $(a = 1 = 3g)$: We infer that $g = 3$ and $(x'_1, x'_2) = (0, 2)$. Hence, every zero-sum free sequences of this case in $C_8$ is

  \[3 \cdot 3 \cdot 3 \cdot 3 \cdot (3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 1.\]

The number of all zero-sum free sequences of this case over $C_8$ is

\[X_{2-2,1} + X_{2-2,0} \leq X_{2-1,1} + X_{2-1,0}.\]
Thus, the expected number of all zero-sum free sequences of this case in $R$ is at most
\[(X_{2-1,1} + X_{2-1,0}) p.\]

Therefore,
\[\mathbb{E}_{2,\{1\}} \leq (X_{2,1} + 2(X_{2-1,1} + X_{2-1,0})) p.\]

By the same argument, for every $a \in C_8$, we have that $\mathbb{E}_{2,\{a\}}$ has the same upper bound, and hence,
\[\mathbb{E}_{2,1} \leq (X_{2,1} + 2(X_{2-1,1} + X_{2-1,0})) p.\]

In a similar way, one can estimate $\mathbb{E}_2$ in $C_8$, which gives $\mathbb{E}'$ and $\mathbb{E}^*$.  

3.3. Estimating $\mathbb{E}_{d,i}$. We are ready to estimate $\mathbb{E}_{d,i}$ in a general $C_n$. First, we consider the case where $i = 1$.

**Lemma 10.** For $2 \leq d \leq \left\lceil \frac{1 + \sqrt{1 + 8E}}{2} \right\rceil$, we have that
\[
\begin{align*}
\mathbb{E}_{d,1} &\leq p^{d-1} (X_{k,d-1} + k(X_{k-1,d-1} + X_{k-1,d-2})) \quad \text{and} \\
\mathbb{E}_{1,1} &= 1.
\end{align*}
\]

**Proof.** Fix $a \in C_n$. We estimate the expected number of zero-sum free sequences
\[g \cdot \ldots \cdot g \cdot (x_1 g) \cdot \ldots \cdot (x_k g)\]
in $R$ containing $\{a\}$ with two cases separately: for a generator $g$, the first case is when $a = g$, and the second case is when $a = j g$ for $2 \leq j \leq k + 1$.

- **Case 1 ($a = g$):** The number of zero-sum free sequences over $C_n$ containing $a = g$ is $X_{k,d-1}$. Hence, the expected number of zero-sum free sequences in $R$ containing $a = g$ is
\[(X_{k,d-1} \cdot p^{d-1}). \tag{8}\]

- **Case 2 ($a = j g$ for $2 \leq j \leq k + 1$):** We first estimate the number of zero-sum free sequences over $C_n$ containing $a = j g = x_\ell g$ for some $\ell$. Since $x_\ell' = x_\ell - 1 \geq 1$, the remaining $x_\ell', \ldots, x_{\ell-1}', x_{\ell+1}', \ldots, x_k'$ satisfy $\sum_{i \neq \ell} x_i' \leq k - 1$. If $x_1', \ldots, x_{\ell-1}', x_{\ell+1}', \ldots, x_k'$ are different from $x_\ell'$, then the number of zero-sum free sequences over $C_n$ is at most $X_{k-1,d-2}$. Otherwise, the number of zero-sum free sequences over $C_n$ is at most $X_{k-1,d-1}$. Since $2 \leq j \leq k + 1$, the expected number of zero-sum free sequences of this case in $R$ is
\[k(X_{k-1,d-1} + X_{k-1,d-2}) p^{d-1}. \tag{9}\]

From (8) and (9), we have that
\[\mathbb{E}_{d,1} \leq \max_{\{a\}} \mathbb{E}_{d,\{a\}} \leq p^{d-1} (X_{k,d-1} + k(X_{k-1,d-1} + X_{k-1,d-2})), \]
which completes our proof of the lemma. □

Next, we consider a general \( i \) with \( |A| = i \).

**Lemma 11.** For \( 1 \leq i < d \leq \left\lfloor \frac{1 + \sqrt{8i - 1}}{2} \right\rfloor \), we have that

\[
\mathbb{E}_{d,i} \leq p^{d-i} \left[ i \binom{k}{i-1} + \binom{k}{i} \right] \left( \sum_{j=0}^{i} X_{k-i+1,d-1-j} \right) \text{ and }
\]

\[
\mathbb{E}_{d,d} = 1.
\]

**Proof.** Fix \( a_1, a_2, \ldots, a_i \in C_n \). We estimate the number of zero-sum free sequences

\[ g \cdot \cdots \cdot g \cdot (x_1g) \cdot \cdots \cdot (x_kg) \]

containing \( \{a_1, a_2, \ldots, a_i\} \) with two cases separately: for a generator \( g \), the first case is when \( g = a_\ell \) for some \( \ell \), and the second case is when \( g \neq a_\ell \) for all \( \ell \).

- **Case 1** \( \{a_1, a_2, \ldots, a_i\} = \{j_2g, \ldots, j_i g\} \) for \( 2 \leq j_2 < \cdots < j_i \leq k+1 \): For fixed \( g \) and \( j_2, \ldots, j_i \), the number of zero-sum free sequences over \( C_n \) containing \( \{a_1, a_2, \ldots, a_i\} = \{g, j_2g, \ldots, j_i g\} \) is at most \( X_{k-i+1,d-1} + \cdots + X_{k-i+1,d-i} \). The number of choices \( \{g, j_2, \ldots, j_i\} \) such that \( g = a_\ell \) for some \( \ell \) and \( 2 \leq j_2 < \cdots < j_i \leq k+1 \) is at most \( i\binom{k}{i-1} \). Hence, the expected number of zero-sum free sequences in \( R \) containing \( \{a_1, a_2, \ldots, a_i\} = \{g, j_2g, \ldots, j_i g\} \) is

\[
\sum_{j=1}^{i} \binom{k}{i-1} (X_{k-i+1,d-1} + \cdots + X_{k-i+1,d-i}) p^{d-i}.
\]

- **Case 2** \( \{a_1, a_2, \ldots, a_i\} = \{j_1g, j_2g, \ldots, j_i g\} \) for \( 2 \leq j_1 < \cdots < j_i \leq k+1 \): For fixed \( j_1 < \cdots < j_i \), we first consider the number of zero-sum free sequences over \( C_n \) containing \( \{a_1, a_2, \ldots, a_i\} = \{j_1g, j_2g, \ldots, j_i g\} \). Without loss of generality, we let \( x_1 = j_1, \ldots, x_i = j_i \). Since \( x'_\ell = x_\ell - 1 \geq 1 \), the remaining \( x'_{i+1}, \ldots, x'_k \) satisfy \( \sum_{i+1 \leq \ell \leq k} x'_\ell \leq k - i \). The number of distinct \( x'_{i+1}, \ldots, x'_k \) from \( x_1', \ldots, x'_i \) are possibly \( d-1, d-2, \ldots, d-1-i \), and hence, the number of zero-sum free sequences over \( C_n \) containing \( \{a_1, a_2, \ldots, a_i\} = \{j_1g, j_2g, \ldots, j_i g\} \) is at most \( X_{k-i,d-1} + X_{k-i,d-2} + \cdots + X_{k-i,d-1-i} \). From the choices of \( 2 \leq j_1 < \cdots < j_i \leq k+1 \), the expected number of zero-sum free sequences in \( R \) containing \( \{a_1, a_2, \ldots, a_i\} \) is at most

\[
\binom{k}{i} (X_{k-i,d-1} + X_{k-i,d-2} + \cdots + X_{k-i,d-1-i}) p^{d-i}.
\]

From (10) and (11), we have that

\[
\mathbb{E}_{d,i} \leq \max_{\{a_1, \ldots, a_i\}} \mathbb{E}_{d,\{a_1, \ldots, a_i\}} \leq \left[ i \binom{k}{i-1} + \binom{k}{i} \right] \left( \sum_{j=0}^{i} X_{k-i+1,d-1-j} \right) p^{d-i},
\]
which completes our proof of the lemma.

3.4. Proofs of Theorems 4 and 5

Proof of Theorem 4. Let $X = N^R_{n-1-k}$. Under the assumption that $k$ is fixed and $p \ll 1$, Corollary 3 gives that
\[ E(X) = p\varphi(n) \left(1 + O_k(p)\right). \]
Since $k$ is fixed, Lemmas 10 and 11 yield that
\[ E_1 = E_{1,1} + E_{2,1} + \cdots + E_{D,1} = O_k(1), \]
\[ E_2 = E_{2,2} + E_{3,2} + \cdots + E_{D,2} = O_k(1), \]
\[ \vdots \]
\[ E_D = E_{D,D} = 1. \]
Hence,
\[ E' = \max_{1 \leq i \leq D} \{E_i\} = O_k(1) \quad \text{and} \quad \]
\[ E^* = \max \{E', E\} = p\varphi(n), \]
provided that $p\varphi(n) \gg 1$, i.e., $p \gg \frac{\log \log n}{n}$.

Set $\lambda = d\log n$, then $e^{-\lambda}n^{d-1} = 1/n = o(1)$, and hence, the Kim–Vu polynomial concentration result (Theorem 9) gives that a.a.s.
\[ |X - E(X)| = O_k \left(\sqrt{p\varphi(n)(\log n)^d}\right), \]
that is,
\[ X = p\varphi(n) + O_k \left(p^2\varphi(n) + \sqrt{p\varphi(n)(\log n)^d}\right). \]
Note that $p\varphi(n) \gg \sqrt{p\varphi(n)(\log n)^d}$ is equivalent to $p \gg \frac{(\log n)^{2d}\log \log n}{n}$, and hence, our assumption on $p$ is
\[ \frac{(\log n)^{2d}\log \log n}{n} \ll p \ll 1. \]
Thus, we complete the proof of Theorem 4.

For the proof of Theorem 5 we use the following version of Chernoff’s bound.

Lemma 12 (Chernoff’s bound, Corollary 4.6 in [14]). Let $X_i$ be independent random variables such that
\[ \Pr[X_i = 1] = p_i \quad \text{and} \quad \Pr[X_i = 0] = 1 - p_i, \]
and let $X = \sum_{i=1}^n X_i$. For $0 < \lambda < 1$,
\[ \Pr \left[ |X - E(X)| \geq \lambda E(X) \right] \leq 2 \exp \left( -\frac{\lambda^2}{3} E(X) \right). \]
We are ready to prove Theorem 5.

Proof of Theorem 5. Let $X = N_{n-1-k,d}$. First, we consider the case where $d = 1$. Observe that $X \sim \text{Bin}(\varphi(n), p)$, and hence, Chernoff’s bound with $\lambda = \omega/\sqrt{p\varphi(n)}$ implies that a.a.s.

$$|X - p\varphi(n)| < \omega(p\varphi(n))^{1/2},$$

provided that $p\varphi(n) \gg 1$, i.e., $p \gg \frac{\log \log n}{n}$, where $\omega$ tends to $\infty$ arbitrarily slowly as $n \to \infty$.

Next we consider the case when $d \geq 2$. It follows from (3) that

$$E(X) = p^d\varphi(n)X_{k,d-1}.$$ 

Lemma [11] gives that for a fixed $k$,

$$E_{d,i} = O_k(p^{d-i}) \text{ for } 1 \leq i \leq d-1 \text{ and }$$

$$E_{d,d} = 1.$$ 

Hence,

$$E' = \max_{1 \leq i \leq d} E_{d,i} = O_k(1) \text{ and }$$

$$E^* = O_k(\max\{1, p^d\varphi(n)\}) = O_k(p^d\varphi(n)),$$

provided that $p^d\varphi(n) \gg 1$, i.e., $p \gg \left(\frac{\log \log n}{n}\right)^{1/d}$.

Set $\lambda = d\log n$, then $e^{-\lambda n^{d-1}} = 1/n = o(1)$, and hence, the Kim–Vu polynomial concentration result (Theorem 9) implies that a.a.s.

$$|X - E(X)| \leq a_d(E'E^*)^{1/2}\lambda^d = O_k\left(p^{d/2}\varphi(n)^{1/2}(\log n)^d\right).$$

Note that $p^d\varphi(n) \gg \sqrt{p\varphi(n)\log n}d$ is equivalent to $p \gg \frac{(\log n)^2(\log \log n)^{1/d}}{n^{1/d}}$, which is our assumption on $p$. This completes our proof of Theorem 5. \[\square\]

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