Insight into the dynamics of second grade hybrid radiative nanofluid flow within the boundary layer subject to Lorentz force

Muhammad Jawad1, Anwar Saeed1, Asifa Tassaddiq1, Arshad Khan3, Taza Gul4, Poom Kumam5,6 & Zahir Shah7

The magnetohydrodynamic hybrid second-grade nanofluid flow towards a stretching/shrinking sheet with thermal radiation is inspected in current work. Main concern of current investigation is to consider hybrid Al2O3 − Cu nanofluid which is perceived by hanging two dissimilar kinds of nanoparticles known as alumina and copper within the base fluid. The fluid motion is produced by non-linear stretching/shrinking sheet. The modeled equations which comprise of energy, motion and continuity equations are changed into dimensionless form using group of similar variables. To determine the solution of transformed problem, the Homotopy Analysis technique is used. The findings of this work revealed that the magnetic parameter improves the heat transfer rate. This work also ensures that there are non-unique solutions of modeled problem for shrinking case and a unique solution for stretching case. Higher values of Re results in declining of flow field. Rise in M agrees to a decline in velocity distributions. Higher values of second order fluid parameter reduces the viscosity of fluid and accordingly velocity increases. Velocity profile is also a decreasing function of volume friction.

Abbreviations

u, v Velocity components
x, y Cartesian coordinates
T H Hybrid nanofluid temperature
λ Constant parameter
T∞ Ambient temperature
M Magnetic parameter
Tw Wall temperature
β1 Dimensionless parameter
θw Temperature ratio parameter
ρhnf Density of hybrid nanofluid
νhnf Kinematic viscosity of hybrid nanofluid
αhnf Thermal diffusivity of hybrid nanofluid
(ρcp)hnf Heat capacity of hybrid nanofluid
S1, S2 Transpiration parameters
Ec Eckert number

1Department of Mathematics, Abdul Wali Khan University, Mardan 23200, Khyber, Pakhtunkhwa, Pakistan. 2Department of Basic Sciences and Humanities, College of Computer and Information Sciences, Majmaah University, Al-Majmaah 11952, Saudi Arabia. 3College of Aeronautical Engineering, National University of Sciences and Technology (NUST), Sector H-12, Islamabad 44000, Pakistan. 4Department of Mathematics, City University of Science and Information Technology, Peshawar 25000, Khyber, Pakhtunkhwa, Pakistan. 5Fixed Point Research Laboratory, Fixed Point Theory and Applications Research Group, Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khu, Bangkok 10140, Thailand. 6Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khu, Bangkok 10140, Thailand. 7Department of Mathematical Sciences, University of Lakki Marwat, Lakki Marwat 28420, Khyber Pakhtunkhwa, Pakistan. *email: poom.kum@kmutt.ac.th; zahir@ulm.edu.pk
Nanofluid receives an enormous amount of attention over the last two decades by the researchers because of its high thermal conductivity and novel applications in different branches of science, engineering and technology. The conventional liquids consume low thermal conductivity thus it becomes inadequate for several heat transfer issues. The study of nanofluid is very important for the reason of its unique application that enhances the transfer of heat. That’s why scientists take interest to use nanofluid instead of regular fluids. Nanofluid shows a vital role at manufacturing level such as production of foods, electronics, biomedicines, transportation and cooling of nuclear reactors. Nanoparticles are very small in size (1–100 nm). The structure of nanoparticles contains a metal oxide, nitride, carbide and carbon tubes (SWCNTs and MWCNTs) etc. The quantity of nanoparticles in a base fluid was introduced first by Choi1 for increasing the thermal characteristics of such fluids. These fluids with heightened thermophysical properties were named as nanofluids. There are different types of nanofluids like fermium oxide, graphene oxide, and carbon nanotubes etc. which are discussed in the literature. In the setting of medical flows, Akbar et al.2 utilized Buonjornio’s model to explore systematically the peristaltic hydrodynamics of nanofluids with wall slip impacts. Sayed et al.3 studied the influence of thermal transportation and alternative current on the peristaltic flow of a viscous dielectric fluid. Nakhchi and Esfahani4 examined a mathematical investigation for Cu–water nanoliquid flow through a spherical cylinder implanted with cross-cut warped tape with an alternative axis (CCTA). Furthermore, the reader can study about nanofluid in Refs.5–7.

Carbon nanotubes (CNTs) are round and hollow formed cylinders with indispensable qualities like great thermal conductivity and huge power makes them exceptionally appealing constituents in fluctuated applications for instance enhancer, drug delivery, optics and semiconductors etc. CNTs can be single or multi wall. Homogeneous carbon nanotube/polymer composites utilizing non-covalently functionalized, solvent single-walled carbon nanotubes (SWNTs) were created by Ramasubramaniam et al.8. Xue9 offered the carbon nanotubes (CNTs) alignment dissemination a new model of active thermal conductivity of CNTs. The study of SWCNT on peristaltic transportation of nanofluid in an inclined cylinder with penetrable walls is conferred by Nadeem et al.10. Homogeneous-heterogeneous reactions in the 3D flow of water-based nanoliquid soaking a permeable medium are demonstrated by Hayat et al.11. Nadeem et al.10 observed the transfer of heat by the influence of SWNTC and MWNTC with the state of oscillation. Late examinations featuring nanofluid impacts in different situations might be found in Ref. 13,14. Various mathematical models have been used by the researcher over the linear stretched surfaces while limited work has been carried out in nonlinear or quadratic stretching.

The fluid of second grade is actually a subclass of non-Newtonian fluid for which flow field has a relationship up to second order derivative in terms of stress strain tensor. On the other hand, this relationship is of first order in case of the Newtonian fluids. Due to its importance, many researchers have diverted their attention towards the flow of second grade fluid. The study of thermal flow regarding this type of fluid is of more importance for researchers these days. The second grade liquid flow past a quadratic stretched surface analyzed by Cortell15,16. This idea further comprehended by Mahapatra and Sidui17. Gul et al.18 considered steady, axisymmetric and incompressible hybrid Nano liquid flow over an unending impermeable gyrating disk affected by a magnetic field, which has a few engineering and industrial applications. Sheikholeslami et al.19 discovered unsteady squeezing liquid flow amid corresponding surfaces. They reported that heat transfer escalates for nanoparticle concentration.

A unique type of nanofluid that formed small metallic particles is called hybrid nanofluid. Hybrid nanofluid shares great applications in the field of engineering, agriculture, biological and applied sciences. Hybrid nanofluid increases the thermal efficiency at a very low cost. The electromagnetic radiation caused by the thermal flow of particles in the matter is termed as thermal radiation. These types of radiations are emitted by those matters which have a temperature higher than absolute zero. The motion of particles results in a charge acceleration that causes electromagnetic radiations. The use of hybrid nanofluid is more fruitful because their applications involve diesel engine oil, hybrid power engines, chillers improvement etc. Due to the importance of this class of fluid many researchers have carried out numerous investigations in this area by using different flow conditions and geometries. The exploratory works by Turcu et al.20 were the formerly contemplates that using the hybrid nanoparticles. Suresh et al.21,22 led the exploratory effort utilizing Al2O3–Cu to examine the upgrade of the liquid thermal conductivity. Afterwards numerous investigators carried various studies for hybrid nanofluid by considering stretching/shrinking surfaces such as Waini et al.24,25, Zainal et al.26 and Khashi’ie et al.27,28. Relationships of thermal conductivity for ordinary nanofluids are completely examined and this study emphasizes planning, complications and challenges of hybrid nanofluids by Das29. Free convection of nanofluid in an inclined open cavity with a heat producing strong component is contemplated by Miroshnichenko et al.30.

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Rostami31 mathematically inspected the natural convection heat move of non-Newtonian water-nanoliquid inside a tube shaped annulus with a concentric round heat source secured with a conductive permeable layer. Huminic and Huminic32 presented a research review of the current results used in different heat exchangers, concerning the thermophysical properties and the characteristics flow of the heat transfer in hybrid nanofluids. The review denotes that the hybrid nanofluids may significantly increase the performance of heat exchangers. However, lots of research efforts are still required considering the hybrid nanoparticles combinations, the exact mixing ratio and its stability. Dinarv33 have presented torpor point limit layer flow of CuO–Ag/water hybrid nanoliquid. Dinarvand and Rostami34 have calculated the radiative heat flux for the versatile surface 3-D limit layer flow of a watery hybrid nanoliquid over an impervious turning plate with the steady spiral extending rate. Tayebi and Chamkha35 mathematically reviewed the heat transfer in an annulus between two confocal elliptic squares loaded up with hybrid Al2O3 – Cu/water nanoliquid. A completely evolved laminar convective heat transfer and weight drop attributes through a consistently heated round cylinder utilizing Al2O3 – Cu/water hybrid nanoliquid was introduced by Suresh et al.36. Jena et al.37 explained the combination of nano-composites utilizing hydrogen reduction methods from synthetically articulated mixtures. Volume concentration and temperature exerted on a particle due to the electric field is electric force and is given by the magnitude $\mathbf{F} = \varepsilon_0 \mathbf{E}$. The force exhibited by a particle due to magnetic and electric fields is termed as the Lorentz force. The force $\mathbf{F}$ can be given as $\mathbf{F} = \mathbf{F}_E + \mathbf{F}_B$.

**Mathematical formulation**

We assume here the steady MHD flow of an incompressible, electrically conducting Al2O3-Cu hybrid nanofluid towards a two-dimensional extending/shrinking sheet. Let $x$-direction is along the surface and $y$-axis being normal to the sheet along which magnetic field ($B_0$) is applied. The schematic diagram is depicted in Fig. 1. The flow is likely to be formed by a quadratic (non-linearly) extending/contracting sheet, which are employed along $x$-axis and in this way shear velocity is $u_w(x) = ax + bx^2$. By connecting with the assessments of the common limit layer, we can compose the hybrid nanofluid governing equations15–17,55,56.

$$u_x + v_y = 0, \quad (1)$$

$$\begin{align*}
(u_x^2 + v_y^2) + u_{xx} + v_{yy} &= u_{hff} u_{yy} + u_{hv} \left( u_{x} v_{xx} + u_{y} v_{yy} + v_{hyy} \right) - \frac{\sigma_{hff} B_0^2 u}{\rho_{hff}}, \quad (2)
\end{align*}$$

$$u_{Tx} + v_{Ty} = \frac{k_{hff}}{(\rho c_p)_{hff}} T_{yy} + \frac{16\sigma^*}{(\rho c_p)_{hff} k^*} \left[ T^3 T_{yy} + 3T^2 (T_x)^2 \right], \quad (3)$$

Together with the boundary conditions15–17,55,56.
Thermophysical properties of hybrid nanofluids are shown in Eq. (5):26

\[ \rho(C_p)_{\text{nbf}} = (1 - \phi_2) \left[ (1 - \phi_1) (\rho C_p)_{\text{f}} + \phi_1 (\rho C_p)_{\text{s}} \right] \]

\[ \mu_{\text{nbf}} = \frac{\mu_f}{(1 - \phi_1)^{2.5} (1 - \phi_2)^{0.5}} \]

\[ k_{\text{nbf}} = k_s + 2k_{\text{nbf}} - 2\phi_2 (k_{\text{nbf}} - k_{\text{s}}), \quad \frac{k_{\text{nbf}}}{k_f} = \frac{k_s + 2k_f - 2\phi_1 (k_f - k_{\text{s}})}{k_{\text{nbf}} + 2\phi_1 (k_f - k_{\text{s}})} \]

\[ S_{\text{nbf}} = \frac{(\sigma_{s2} - \sigma_{f}) 3\phi_{s2}}{(\sigma_{s2} + 2\sigma_{f}) + (\sigma_{f} - \sigma_{s2}) \phi_{s2}}, \quad S_{\text{bf}} = \frac{(\sigma_f - \sigma) 3\phi_s}{(\sigma_s - \sigma_f) - (\sigma_s + 2\sigma_f) + 1} \]

\[ \rho_{\text{nbf}} = (1 - \phi_2) \left[ (1 - \phi_1) (\rho)_{\text{f}} + \phi_1 (\rho)_{\text{s}} \right] + \phi_2 (\rho)_{\text{s}} \]

Considering Takhar et al.44, the group of similar variables is defined as:

\[ u = axf'(\eta) + bx^2g'(\eta), \quad v = -\sqrt{\frac{a}{b}} S_1 f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad \eta = y \sqrt{\frac{a}{b}} \]

Hence that:

\[ v_w = -\sqrt{\frac{a}{b}} S_1 - \frac{2b}{\sqrt{b}} S_2 \]

Here, \( S_1, S_2 \) are the transpiration parameters with \( (S_1, S_2) > 0 \) for suction and \( (S_1, S_2) < 0 \) for blowing and injection constraint. By using the Eq. (5), Eqs. (2, 3) are changed to:
\[ f''' - (1 - \phi_1) \phi_2^2 \phi_2 (1 - \phi_2) \left[ (1 - \phi_1) \rho \phi_1 (\rho) + \phi_1 (\rho) \right] + \phi_2 (\rho) \right) = 0, \tag{8} \]

\[ g''' - fg''' + 3f'g' - 2gf''' - (1 - \phi_1) \phi_2^2 (1 - \phi_2) \left[ (1 - \phi_1) \rho \phi_1 (\rho) + \phi_1 (\rho) \right] + \phi_2 (\rho) \right) = 0, \tag{9} \]

Quantities of engineering interest. The engineering quantities of interest are \( C_{fs} \) and \( Nu_x \) that is denoted as: \( C_{fs} = \frac{\tau_w}{\rho(ax)^2} \), \( Nu_x = \frac{xq_w}{k_f(T_w - T_\infty)} \). \( \tag{14} \)

The surface heat flux \( q_w \) and \( \tau_w \) are written as:

\[ \tau_w = \frac{\mu_h}{\rho_f} (T_y)_{y=0} \] \( q_w = k_m f (T_y)_{y=0} + (q_r)w, \) \( \tag{15} \)

By the use of Eqs. (13) and (14), then we have:

\[ C_{fs} Re_x^{-\frac{1}{2}} = \frac{\mu_h}{\rho_f} f''(0) + \beta_1 x \chi''(0) \], \( \frac{Nu_x Re_x^{-\frac{1}{2}}}{k_f} = - \frac{k_m}{k_f} (1 + R\theta_3^3) \theta'(0) \). \( \tag{16} \)

Solution by HAM The problem is explained through the HAM method. The detail of HAM for current modeled problem describes as follows:

\[ L_\phi \phi = \phi''' \tag{17} \]

Linear operators \( L_{-\phi} \) and \( L_\phi \) are signified as:

\[ L_{-\phi} (e_1 + e_2 \eta + e_3 \eta^2) = 0, \] \( L_\phi (e_4 + e_5 \eta) = 0, \) \( \tag{18} \)

The constant non-linear operators are \( N_{\phi} \) and \( N_{-\phi} \) with

\[ N_{\phi} f \phi (\eta; \xi) = f_{\eta \eta} - (1 - \phi_1) \phi_2 \phi_2 (1 - \phi_2) \left[ (1 - \phi_1) \rho \phi_1 (\rho) + \phi_1 (\rho) \right] + \phi_2 (\rho) \right) = \beta \left( 2f_f f_{\eta \eta \eta} + f_{\eta \eta} f_{\eta \eta} - f_{\eta \eta} - f_{\eta \eta} + \left( e_4 + e_5 \eta \right)^2 - (1 - \phi_1) \phi_2 \phi_2 (1 - \phi_2) \right) \] \( \tag{19} \)
\[
N_{g} \left[ \tilde{g}(\eta; \zeta) \right] = \tilde{g}_{qq} - (1 - \phi_{1})^{2.5}(1 - \phi_{2})^{2.5}(1 - \phi_{3}) \left[ \left(1 - \phi_{1}(\rho)_{1} + \phi_{1}(\rho)_{2} \right) + \phi_{2}(\rho)_{3} \right]
\]
\[
\beta \left( 3f_{n} \tilde{g}_{qq} + 3f_{n} \tilde{g}_{qq} \right) - \tilde{f}_{n} \tilde{g}_{qq} - 2f_{n} \tilde{g}_{qq} \tilde{g} - \tilde{f}_{n} \tilde{g}_{qq} + 3f_{n} \tilde{g}_{qq} - 2f_{n} \tilde{g}_{qq} 
\]
\[
= - (1 - \phi_{1})^{2.5}(1 - \phi_{2})^{2.5} \left[ \frac{\sigma_{h}}{\sigma_{T}} \right] M \tilde{g}_{qq},
\]
\[
\frac{h_{n} \tilde{g}_{qq} - \frac{1}{\eta^{2}}}{\sigma_{T}} \left[ 1 - \frac{1}{\eta} \right] \left( 1 + (\theta_{u} - 1) \theta \right)^{3} \tilde{\theta}_{qq} + 3 \left( 1 + (\theta_{u} - 1) \theta \right)^{2} (\theta_{u} - 1) \theta \right)^{2} + (f + g) \tilde{\theta}_{q},
\]
\[
(1 - \eta) L_{\eta} \left[ \tilde{g}(\eta; \zeta) - \tilde{g}_{0}(\eta) \right] = ph_{n} N_{g} \left[ \tilde{g}(\eta; \zeta) \right],
\]
\[
(1 - \eta) L_{\eta} \left[ \tilde{\theta}(\eta; \zeta) - \tilde{\theta}_{0}(\eta) \right] = ph_{n} N_{\eta} \left[ \tilde{\theta}(\eta; \zeta) \right],
\]
whereas BCs are:
\[
\left. \tilde{f}(\eta; \zeta) \right|_{\eta=0} = f_{S}, \quad \left. \tilde{g}(\eta; \zeta) \right|_{\eta=0} = g_{S}, \quad \left. \frac{\partial \tilde{f}(\eta; \zeta)}{\partial \eta} \right|_{\eta=0} = \lambda, \quad \left. \frac{\partial \tilde{g}(\eta; \zeta)}{\partial \eta} \right|_{\eta=0} = \lambda, \quad \left. \tilde{\theta}(\eta; \zeta) \right|_{\eta=0} = 1.
\]
\[
\left. \frac{\partial \tilde{f}(\eta; \zeta)}{\partial \eta} \right|_{\eta=\infty} = 0, \quad \left. \frac{\partial \tilde{g}(\eta; \zeta)}{\partial \eta} \right|_{\eta=\infty} = 0, \quad \left. \tilde{\theta}(\eta; \zeta) \right|_{\eta=\infty} = 0, \quad \left. \tilde{\phi}(\eta; \zeta) \right|_{\eta=\infty} = 0.
\]
While the implanting constraint is \( \zeta \in [0, 1] \), to adjust for the solution convergence \( h_{f} \), \( h_{g} \) and \( h_{\theta} \) are utilized. When \( \zeta = 0 \) and \( \zeta = 1 \) we have:
\[
\tilde{f}(\eta; 1) = \tilde{f}(\eta), \quad \tilde{g}(\eta; 1) = \tilde{g}(\eta), \quad \tilde{\theta}(\eta; 1) = \tilde{\theta}(\eta)
\]
Enlarging the \( \tilde{f}(\eta; \zeta), \tilde{g}(\eta; \zeta) \) and \( \tilde{\theta}(\eta; \zeta) \) over Taylor’s series for \( \zeta = 0 \)
\[
\tilde{f}(\eta; \zeta) = \tilde{f}_{0}(\eta) + \sum_{n=1}^{\infty} \tilde{f}_{n}(\eta) \zeta^{n}
\]
\[
\tilde{g}(\eta; \zeta) = \tilde{g}_{0}(\eta) + \sum_{n=1}^{\infty} \tilde{g}_{n}(\eta) \zeta^{n}
\]
\[
\tilde{\theta}(\eta; \zeta) = \tilde{\theta}_{0}(\eta) + \sum_{n=1}^{\infty} \tilde{\theta}_{n}(\eta) \zeta^{n}
\]
\[
\tilde{f}(\eta) = \frac{1}{n!} \frac{\partial \tilde{f}(\eta)}{\partial \eta} \bigg|_{p=0}, \quad \tilde{g}(\eta) = \frac{1}{n!} \frac{\partial \tilde{g}(\eta)}{\partial \eta} \bigg|_{p=0}, \quad \tilde{\theta}(\eta) = \frac{1}{n!} \frac{\partial \tilde{\theta}(\eta)}{\partial \eta} \bigg|_{p=0}.
\]
where BCs are:
\[
\tilde{f}(0) = S_{1}, \quad \tilde{g}(0) = S_{2}, \quad \tilde{f}(0) = \lambda, \quad \tilde{g}(0) = \lambda, \quad \tilde{\theta}(0) = 1, \quad \tilde{f}(\infty) = 0, \quad \tilde{g}(\infty) = 0, \quad \tilde{\theta}(\infty) = 0.
\]
where
\[
\chi_{n} = \begin{cases} 
0, & \text{if } n \leq 1 \\
1, & \text{if } n > 1.
\end{cases}
\]

**Results and discussions**

In this section, we now deliberate the different outcomes of the existing study and displayed graphically in Figs. 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11. Figure 1 shows the Schematic representation of the flow problem. The influence of magnetic factor \( M \) on these velocity distributions is reflected in Figs. 2 and 3. It is witnessed in Fig. 2 that axial velocity distributions is declining with amassled estimations of magnetic factors. The intensification in \( M \) indicates
to expanding Lorentz force which is because of the interface of electric and magnetic fields in motion of electrically directed liquid. We can also say that bigger Lorentz force deals more resistance to the transportation phenomenon, that is why upsurge in $M$ agrees to a decline in velocity distributions. On the other hand a tangential component $g(\eta)$ is also declines with an expansion in estimations of $M$ as presented in Fig. 3. Figures 4 and 5 depict the influence of $\beta$ on $f'(\eta)$ and $g(\eta)$. We see from Figs. 4 and 5 that the $f'(\eta)$ and $g(\eta)$ decline against $\beta$. It is evident that for higher assessment of $\beta$ the viscosity of fluid improves and accordingly velocity $f'(\eta)$ and $g(\eta)$ increase. In fact, the second grade parameter $\beta$ improves the non-Newtonian behavior for its larger vales and consequently declines the velocity profile. The flow profiles in Fig. 6 suggest that augmentation in momentum

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**Figure 2.** Influence of $M$ on $f'(\eta)$ for $Re_x = 5, \beta = 0.2, \phi_2 = 0.02$.

**Figure 3.** Influence of $M$ on $g(\eta)$ for $Re_x = 5, \beta = 0.2, \phi_2 = 0.02$.

**Figure 4.** Influence of $\beta$ on $f'(\eta)$ for $M = 0.9, \phi_2 = 0.02$. 

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limit layer viscosity declines the flow and gradient in flow of fluid. Moreover, the thicker the momentum limit layer proposes the little wall shear stress as a result of which $f'(\eta)$ reduces with a corresponding increase in $\phi_1$, $\phi_2$. Figure 7 shows the impact of $\phi_2$ on $g(\eta)$. The larger magnitude of the volume fraction $\phi_1$ and $\phi_2$ improve the resistive force and consequently declines the velocity field $g(\eta)$.

**Temperature profile.** Figure 8 indicates that $\theta(\eta)$ reduces with Pr impact. It is additionally certain that expansion in the Pr prompts diminishing the limit of thermal layer thickness. Actually Prandtl number is inversely proportional to diffusivity of thermal boundary layer; hence increase in the values of Pr results in
decline of thermal characteristics. So that \( \theta(\eta) \) for extending sheet reduces by growing Pr. Figure 9 detects that the distribution \( \theta(\eta) \) increases as \( R \) upsurges with the datum that the rate of energy transmission jumps up due to increase in thermal radiations. Figure 10 observes that the temperature profiles \( \theta(\eta) \) increase as \( \theta_w \) increases because \( \theta_w \) is the ratio of the hybrid nanofluid temperature at the sheet surface to the ambient temperature of the liquid. Figure 11 observes that the temperature profiles \( \theta(\eta) \) increase as \( \phi_1, \phi_2 \) increase. The larger magnitude of the parameters \( \phi_1, \phi_2 \) enhancing the thermal efficiency of the base fluid and consequently the temperature profile increases.

Figure 8. Influence of Pr on \( \theta(\eta) \) for \( R = 0.3, \theta_w = 1.2 \).

Figure 9. Influence of \( R \) on \( \theta(\eta) \) for \( Pr = 10.2, \theta_w = 1.2 \).

Figure 10. Impact of \( \theta_w \) on \( \theta(\eta) \) for \( R = 0.3, Pr = 10.2 \).
Table discussion. Table 2 shows the influence of various physical constraints on skin friction of the hybrid nanofluid flow. From Table 2 it is realized that the increase of the nanoparticle percentage in the base fluid boosts the skin friction of the base fluid. The parameters $S_1$ and $S_2$ are respiration parameters or injection and suction parameters of the nanofluid if we inject more fluid, then skin friction enhances. When we apply the orthogonal magnetic field to the flow of the hybrid nanofluid; this magnetic field attracts the metallic nanoparticles due to which the skin fraction increases. Skin friction enhances with the larger magnitude of the second grade parameter $\beta$.

Table 3 displays the influence of various physical factors on the Nusselt number. Nusselt number means the flow rate of the heat. From Table 3 we clearly see that if we enhance the Prandtl number then by physical definition of the Prandtl number when it enhances, the thermal conductivity of the fluid decreases, therefore the heat transfer rate declines with the enhancement of the Prandtl number. The addition in the thermal radiation factor improves the heat transfer rate. From Table 3 we clearly see that with the enhancement of the parameters $\phi_1, \phi_2, \theta_w$ improve the heat transfer rate. It has been observed that the hybrid nanofluids are the most efficient to improve the heat transfer rate as compared to the traditional fluids. Tables 4 and 5 are displayed to validate the obtained results with the existing literature. The Nusselt number and skin friction of the present study are compared and closed agreement is obtained.
Conclusions

The magnetohydrodynamic second-grade hybrid nanofluid flow towards an extending/shrinking sheet with thermal radiation is inspected in this investigation. The main concern of this research work is to consider the hybrid nanofluid which is perceived by hanging two distinctive nanoparticles known as alumina and copper within the second grade fluid while the fluid motion is formed from the non-linearly stretching/shrinking sheet.

The important observations are given below.

• Second-grade fluid used as a base fluid for the solid nanoparticles and the influence of the second grade parameter $\beta$ observed versus the velocity field.
• $Al_2O_3$ and $Cu$ are used as the solid nanoparticles. The increments in the volume fraction $\phi_1$ and $\phi_2$ of the nanoparticles increase the thermal efficiency of the fluid.
• For greater values of $M$ the velocity $f'(\eta)$ and $g(\eta)$ decrease.
• The heat transfer rate upsurges for bigger $Rd$ and $\theta_w$.
• The heat transfer rate diminishes with augmentation of Prandtl number.
• The velocity profile declines for the larger magnitude of the Reynolds number.
• It has been observed that the hybrid nanofluids are most efficient to enhance the thermal conductivity of the second grade fluids as compared to the traditional fluids.

| $Pr$ | $R$ | $\theta_w$ | $\phi_1$ | $\phi_2$ | $Nu_xRe_x^{-\frac{1}{2}}$ |
|------|-----|------------|----------|----------|--------------------------|
| 10   | 0.3 | 0.2        | 0.05     | 0.03     | 0.4827149               |
| 11   |     |            |          |          | 0.4773147               |
| 12   |     | 0.3        |          |          | 0.4946365               |
|      |     | 0.6        |          |          | 0.4957187               |
|      |     | 0.9        |          |          | 0.4971463               |

Table 3. Influence of dissimilar physical constraints over Nusselt number $Nu_xRe_x^{-\frac{1}{2}}$.

| $C_{Nu}Re_x^{-\frac{1}{2}}$ [Results of Ref.15] | $C_{Nu}Re_x^{-\frac{1}{2}}$ [Results of Ref.16] | $C_{Nu}Re_x^{-\frac{1}{2}}$ [Present results] |
|--------------------------------------------------|--------------------------------------------------|---------------------------------------------|
| 0.4                                              | 0.4186600                                        | 0.4189483                                   |
| 0.5                                              | 0.4437324                                        | 0.4438372                                   |
| 0.6                                              | 0.4726413                                        | 0.4727261                                   |
| 0.7                                              | 0.5015302                                        | 0.5015150                                   |

Table 4. Comparison of the present work with the published work15,16.

| $Nu_xRe_x^{-\frac{1}{2}}$ [Results of Ref.17] | $Nu_xRe_x^{-\frac{1}{2}}$ [Present results] |
|--------------------------------------------------|---------------------------------------------|
| 10                                               | 0.4889312                                   |
| 11                                               | 0.4867101                                   |
| 12                                               | 0.4745012                                   |

Table 5. Comparison of the present work with the published work17.
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Author contributions

M.J., A.S. and A.K. modeled and solved the problem. M.J. and A.S. wrote the manuscript. Z.S., T.G. and A.T. contributed in the numerical computations and plotting the graphical results. Z.S., P.K., A.S. and T.G. work in the revision of the manuscript. All the corresponding authors finalized the manuscript after its internal evaluation.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to P.K. or Z.S.

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