Geometric approach to $p$-singular Gelfand-Tsetlin $\mathfrak{gl}_n$-modules

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Abstract

We give an elementary construction of a $p \geq 1$-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-module in terms of local distributions. This is a generalization of the universal 1-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-module obtained in [FGR1]. We expect that the family of new Gelfand-Tsetlin modules that we obtained will lead to a classification of all irreducible $p > 1$-singular Gelfand-Tsetlin modules. So far such a classification is known only for singularity $n = 1$.

1 Introduction

In classical Gelfand-Tsetlin Theory one constructs explicitly an action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{C})$ in a basis forming by Gelfand-Tsetlin tableaux. Let $V$ be the vector space of all Gelfand-Tsetlin tableaux of fixed order, see the main text for details, and $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. In [FO] it was proved that there exists a ring structure on the vector space $\mathcal{R} := H^0(V, \mathcal{M} \star \Gamma)$, where $\mathcal{M} \star \Gamma$ is the sheaf of meromorphic functions on $V$ with values in a certain group $\Gamma$ acting on $V$, such that the classical Gelfand-Tsetlin formulas define a homomorphism of rings $\Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{R}$. In the case when $\text{Im} \, \Phi$ is holomorphic on a certain orbit $\Gamma(o)$ of a point $o \in V$ the converse statement is also true: any homomorphism of rings $\Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{R}$ defines a Gelfand-Tsetlin like formulas with basis forming by elements of $\Gamma(o)$. The study of the case when $\text{Im} \, \Phi$ is not holomorphic in $\Gamma(o)$, but one singular, was initiated in [FGR1]. For instance the authors [FGR1] constructed the universal 1-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-module using pure algebraic methods. Another version of the construction from [FGR1] of 1-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-module can be found in [Z].

In the present paper we study the case of singularity $p \geq 1$. In fact, we give a new elementary geometric construction of $p$-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-modules. In the case $p = 1$ our construction gives another version of the constructions of the universal 1-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-modules obtained in [FGR1] and [Z]. The universal 1-singular Gelfand-Tsetlin $\mathfrak{gl}_n(\mathbb{C})$-modules was
used in [FGR1, FGR2] to classify all irreducible Gelfand-Tsetlin modules with 1-singularity. We expect that our construction of p-singular Gelfand-Tsetlin \( gl_n(\mathbb{C}) \)-modules will lead to a classification of all irreducible \( p > 1 \)-singular Gelfand-Tsetlin modules.

Moreover, our approach leads to a geometric explanation of the formulas from [FGR1, Z] for the \( gl_n(\mathbb{C}) \)-action in the universal 1-singular Gelfand-Tsetlin basis. On the other side, our approach may be used for other homomorphisms \( \Psi : U(\mathfrak{h}) \to \mathcal{R} \), where \( \mathfrak{h} \) is any Lie algebra. In fact, we describe necessary conditions for the image of the homomorphism \( \Psi : U(\mathfrak{h}) \to \mathcal{R} \) such that certain local distributions supported on the elements of the orbit \( \Gamma(o) \) form a basis for an \( \mathfrak{h} \)-module.

## 2 Preliminaries

Let \( V \) be the vector space of Gelfand-Tsetlin tableaux \( V \cong \mathbb{C}^{n(n+1)/2} = \{(x_{ki}) \mid 1 \leq i \leq k \leq n\} \), where \( n \geq 2 \), and \( \Gamma \cong \mathbb{C}^{n(n-1)/2} \) be the free abelian group generated by \( \sigma_{st} \), where \( 1 \leq t \leq s \leq n-1 \). We fix the following action of \( \Gamma \) on \( V \): \( \sigma_{st}(x) = (x_{ki} + \delta_{kt}^{st}) \), where \( x = (x_{ki}) \in V \) and \( \delta_{kt}^{st} \) is the Kronecker delta. We put \( G = S_1 \times S_2 \times \cdots \times S_n \), where \( S_i \) is the symmetric group of degree \( i \). The group \( G \) acts on \( V \) in the following way \( (s(x))_{ki} = x_{ks(i)} \), where \( s = (s_1, \ldots, s_n) \in G \).

Denote by \( \mathcal{M} \) and by \( \mathcal{O} \) the sheaves of meromorphic and holomorphic functions on \( V \), respectively. Let us take \( f \in H^0(V,\mathcal{M}) \), \( s \in G \) and \( \sigma \in \Gamma \). We set

\[
\begin{align*}
  s(f) &= f \circ s^{-1}, \\
  \sigma(f) &= f \circ \sigma^{-1}, \quad \text{and} \quad s(\sigma) = s \circ \sigma \circ s^{-1}.
\end{align*}
\]

Denote by \( \mathcal{M} \star \Gamma := \bigoplus_{\sigma \in \Gamma} \mathcal{M} \sigma \) the sheaf of meromorphic functions on \( V \) with values in \( G \). In other words, \( \mathcal{M} \star \Gamma \) is the sheaf of meromorphic sections of the trivial bundle \( V \times \bigoplus_{\sigma \in \Gamma} \mathbb{C} \sigma \to V \). An element in \( \mathcal{M} \star \Gamma \) is a finite linear combination of \( f \sigma \), where \( f \in \mathcal{M} \) and \( \sigma \in \Gamma \). There exists a natural structure of a skew group ring on \( H^0(V,\mathcal{M} \star \Gamma) \), see [FO]. Indeed,

\[
\sum_i f_i \sigma_i \circ \sum_j f'_j \sigma'_j := \sum_{ij} f_i \sigma_i(f'_j) \sigma_i \circ \sigma'_j.
\]

Here \( f_i, f'_j \in H^0(V,\mathcal{M}) \) and \( \sigma_i, \sigma'_j \in \Gamma \). This skew ring we will denote by \( \mathcal{R} \). To simplify notations we use \( \circ \) for the multiplication in \( \mathcal{R} \) and for the product in \( \Gamma \). We will also consider the multiplication \( A \star B := B \circ A \) in \( H^0(V,\mathcal{M} \star \Gamma) \). The ring \( \mathcal{R} \) possesses the following action of the group \( G \)

\[
\begin{align*}
  s\left( \sum_i f_i \sigma_i \right) &= \sum_i s(f_i) s(\sigma_i).
\end{align*}
\]
It is easy to see that this action preserves the multiplication in \( \mathcal{R} \). Hence the vector space \( \mathcal{R}^G \) of all \( G \)-invariant elements is a subring in \( \mathcal{R} \).

The classical Gelfand-Tsetlin formulas have the following form in terms of generators:

\[
E_{k,k+1}(T(v)) = -\sum_{i=1}^{k} \frac{\prod_{j=1}^{k+1} (x_{ki} - x_{k+1,j})}{\prod_{j \neq i} (x_{ki} - x_{kj})} T(v + \delta_{ki});
\]

\[
E_{k+1,k}(T(v)) = \sum_{i=1}^{k} \frac{\prod_{j=1}^{k} (x_{ki} - x_{k-1,j})}{\prod_{j \neq i} (x_{ki} - x_{kj})} T(v - \delta_{ki});
\]

\[
E_{k,k}(T(v)) = \left( \sum_{i=1}^{k} (x_{ki} + i - 1) - \sum_{i=1}^{k-1} (x_{k-1,i} + i - 1) \right) (T(v)),
\]

see for instance [FGR1], Theorem 3.6. Here \( E_{st} \in \mathfrak{gl}_n(\mathbb{C}) \), \( T(v) \in V \) is a point in \( V \) with coordinates \( v = (x_{ki}) \) and \( T(v \pm \delta_{ki}) \in V \) is the tableau obtained by adding \( \pm 1 \) to the \( (k,i) \)-th entry of \( T(v) \). A Gelfand-Tsetlin tableau is called generic if \( x_{rt} - x_{rs} \notin \mathbb{Z} \) for any \( r \) and for any \( s \neq t \). In the case when \( T(v) \) is a generic Gelfand-Tsetlin tableau Formulas (1) define a \( \mathfrak{gl}_n(\mathbb{C}) \)-module structure on the vector space spanned by the elements of the orbit \( \Gamma(T(v)) \), see for instance Theorem 3.8 in [FGR1] and references therein. Note that the action of \( \Gamma \) in \( V \) is free. Hence, \( \Gamma(T(v)) \simeq \Gamma \) and the elements of the orbit \( \Gamma(T(v)) \) form the Gelfand-Tsetlin basis. Another observation is that the coefficients in Formulas (1) are holomorphic in sufficiently small neighborhood of \( \Gamma(T(v)) \) for a generic \( T(v) \).

There is a natural action of \( \mathcal{R} \) on \( H^0(V, \mathcal{M}) \) that is given by

\[
F \mapsto (f \sigma)(F) := f \sigma(F) = fF \circ \sigma^{-1}.
\]

Let us identify \( T(v) \in V \) with the corresponding evaluation map \( ev_v : H^0(V, \mathcal{O}) \to \mathbb{C}, \ F \mapsto F(v) \). (Note that \( V \) is a Stein manifold, so this identification exists.) Assume that \( R = f_i \sigma_i \in \mathcal{R} \) is holomorphic in a neighborhood of \( v \). Then we have \( ev_v \circ f_i \sigma_i(F) = f_i(v) F(\sigma_i^{-1}(v)) \), where \( F \in H^0(V, \mathcal{O}) \). We put \( R(ev_v) := ev_v \circ R \). In these notations we have \( R_1(R_2(ev_v)) = ev_v \circ (R_2 \circ R_1) = (R_1 * R_2)(ev_v) \), where \( R_1, R_2 \in \mathcal{R} \). Now we can rewrite Formulas (1) in the following form

\[
\Phi(E_{st})(ev_v) = ev_v \circ \Phi(E_{st}). \tag{2}
\]

Here \( \Phi(E_{st}) \in \mathcal{R} \) is defined by Formulas (1). For example,

\[
\Phi(E_{k,k+1}) = -\sum_{i=1}^{k} \frac{\prod_{j=1}^{k+1} (x_{ki} - x_{k+1,j})}{\prod_{j \neq i} (x_{ki} - x_{kj})} \sigma_{ki}^{-1}. \tag{3}
\]
In this formula we interpret $x_{ki}$ as the coordinate functions on $V$. Now we see that the statement of Theorem 3.8 in [FGR1] is equivalent to

$$
(\Phi(X) \ast \Phi(Y))(ev_v) - (\Phi(Y) \ast \Phi(X))(ev_v) = \Phi([X,Y])(ev_v),
$$

where $X, Y \in \mathfrak{g}$ and $v$ is generic. In [FO] the following theorem was proved.

**Theorem 1.** [Futorny-Ovsienko] The classical Gelfand-Tsetlin formulas (1) define a homomorphism of rings $\Phi : U(\mathfrak{g}_-) \to \mathcal{R}$, where $\Phi(X), X \in \mathfrak{g}$, is as in (2) and $\mathfrak{g}_- = \mathfrak{gl}_n(\mathbb{C})$ with the multiplication $[X,Y]_\mathfrak{g} = -[X,Y] = Y \circ X - X \circ Y$.

**Proof.** Let us give a proof of this theorem for completeness using complex analysis. First of all define a homomorphism of the (free associative) tensor algebra $\mathcal{T}(\mathfrak{g})$ to $\mathcal{R}$ using (1). Such a homomorphism always exists because $\mathcal{T}(\mathfrak{g})$ is free. We need to show that the ideal generated by the relation $\Phi(Y) \circ \Phi(X) - \Phi(X) \circ \Phi(Y) - \Phi([X,Y])$ maps to 0 for any $X, Y \in \mathfrak{g}$. In fact we can rewrite Formulas (3) in the following form for any generic $v \in V$, any $X, Y \in \mathfrak{g}$ and any $F \in H^0(V, \mathcal{O})$:

$$
ev_v \circ (\Phi(Y) \circ \Phi(X)(F) - \Phi(X) \circ \Phi(Y)(T)) = ev_v \circ \Phi([X,Y])(F).
$$

Since generic points $v$ are dense in $V$, the following holds for any holomorphic $F$:

$$
(\Phi(Y) \circ \Phi(X)(F) - \Phi(X) \circ \Phi(Y)(F)) = \Phi([X,Y])(F).
$$

It is remaining to prove that if $R \in \mathcal{R}$ such that $R(F) = 0$ for any $F \in H^0(V, \mathcal{O})$, then $R = 0$. Indeed, let $R = \sum_{i=1}^S f_i \sigma_i$ and $U \subset V$ be a sufficiently small open set.

Clearly it is enough to prove a local version of our statement for any such $U$: from $R(F)|_U = 0$ for any $F \in H^0(V, \mathcal{O})$ and $R \in \mathcal{R}|_U$, it follows that $R|_U = 0$.

Firstly assume that all functions $f_i$ are holomorphic in $U$ and $x_0 \in U$. Let us fix $i_0 \in \{1, \ldots, s\}$ and let us take $F \in H^0(V, \mathcal{O})$ such that $F(\sigma_{i_0}^{-1}(x_0)) \neq 0$ and $F(\sigma_{i_0}^{-1}(x_0)) = 0$ for $i \neq i_0$. Then $R(F)(x_0) = (\sum_{i=1}^S f_i \sigma_i(F))(x_0) = f_{i_0}(x_0)F(\sigma_{i_0}^{-1}(x_0)) = 0$. Hence $f_{i_0}(x_0) = 0$. Therefore, $f_{i_0}|_U = 0$ for any $i_0$ and $R|_U = 0$.

Further, by induction assume that our statement holds for $(q-1)$ non-holomorphic in $U$ coefficients $f_1, \ldots, f_{q-1}$, where $q - 1 < s$. Consider meromorphic in $U$ functions $f_1 = g_1/h_1, \ldots, f_q$, where $g_1, h_1$ are holomorphic in $U$ without common non-invertible factors. Then $h_1R \in \mathcal{R}|_U$ satisfies the equality $h_1R(F)|_U = 0$ for any $F$ and it has $(q - 1)$ non-holomorphic summands. Therefore, $h_1R|_U = 0$ and in particular $g_1 = 0$. $\square$

**Remark.** It is well-known that the image $\Phi(U(\mathfrak{g}_-))$ is $G$-invariant. This fact can be also verified directly.

The interpretation of a point $T(v)$ as an evaluation map $ev_v$ suggests a possibility to define a $\mathfrak{gl}_n(\mathbb{C})$-module structure on local distributions, i.e. on linear
maps $D_v : \mathcal{O}_v \to \mathbb{C}$ with $m^*_v \subset \text{Ker}(D_v)$, where $s > 0$ and $m_v$ is the maximal ideal in the local algebra $\mathcal{O}_v$. In [FGR1] the authors consider formal limits $\lim_{v \to v_0} (T(v + z) - T(v + \tau(z)))/(x_{ki} - x_{kj})$, where $\tau \in G$ is a certain involution, $i \neq j$, $v_0 \in V$ is an 1-singular tableau, see Section 5, and $v$ is a generic tableau. In fact this limit may be interpreted as a sum of local distributions, see Section 5. However geometric interpretations were not given in [FGR1]. The idea to use local distributions we develop in the present paper. In more details, let $R \in R$ and $D_v$ be a local distribution. We have an action of $(R, \ast)$ on local distributions defined by $R(D_v) = D_v \circ R$. Indeed, $(R_1 \ast R_2)(D_v) = D_v \circ R_1 \circ R_2 = R_1(R_2(D_v)), R_i \in R$.

By Theorem 1 we have $(\Phi(X) \ast \Phi(Y))(D_v) - (\Phi(Y) \ast \Phi(X))(D_v) = \Phi([X, Y])(D_v)$, if this expression is defined, and $D_v \mapsto \Phi(X)(D_v)$ gives a structure of a $\mathfrak{gl}_n(\mathbb{C})$-module on local distributions. Our goal now is to find orbits $\Gamma(o)$ and describe local distributions at points of $\Gamma(o)$ such that this formula is defined.

3 Gelfand-Tsetlin modules

In this section we follow [FGR1, Z]. Consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. We have the following sequence of subalgebras

$$\mathfrak{gl}_1(\mathbb{C}) \subset \mathfrak{gl}_2(\mathbb{C}) \subset \cdots \subset \mathfrak{gl}_n(\mathbb{C}).$$

This sequence induces the sequence of the corresponding enveloping algebras

$$\mathcal{U}(\mathfrak{gl}_1(\mathbb{C})) \subset \cdots \subset \mathcal{U}(\mathfrak{gl}_n(\mathbb{C})).$$

Denote by $Z_m$ the center of $\mathcal{U}(\mathfrak{gl}_m(\mathbb{C}))$, where $1 \leq m \leq n$. The subalgebra $\Upsilon$ in $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$ generated by elements of $Z_m$, where $1 \leq m \leq n$, is called the Gelfand-Tsetlin subalgebra of $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$. This subalgebra is the polynomial algebra with $n(n+1)/2$ generators ($c_{ij}$), where $1 \leq j \leq i \leq n$, see [FGR1], Section 3. Explicitly these generators are given by

$$c_{ij} = \sum_{(s_1, \ldots, s_j) \in \{1, \ldots, i\}^j} E_{s_1s_2}E_{s_2s_3} \cdots E_{s_js_1},$$

where $E_{st}$ form the standard basis of $\mathfrak{gl}_n(\mathbb{C})$.

**Definition.** [Definition 3.1, [FGR1]] A finitely generated $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))$-module $M$ is called a Gelfand-Tsetlin module with respect to $\Upsilon$ if $M$ splits into a direct sum of $\Upsilon$-submodules:

$$M = \bigoplus_m M(m),$$

5
where the sum is taken over all maximal ideals \( m \) in \( \Upsilon \). Here
\[
M(m) = \{ v \in M \mid m^q(v) = 0 \text{ for some } q \geq 0 \}.
\]

For the following theorem we refer [FGR1], Section 3 and [Z], Theorem 2. Compare also with Theorem 1.

**Theorem 2.** The image \( \Phi(\Upsilon) \) coincides with the subalgebra of polynomials in \( H^0(V, O^G) \). In other words, for any \( X \in \Upsilon \) we have \( \Phi(X) = F \text{id} \), where \( F \) is a \( G \)-invariant polynomial.

**Corollary.** All modules corresponding to the homomorphism \( \Phi \) with a basis forming by local distributions on \( V \) are Gelfand-Tsetlin modules.

**Proof.** Indeed, let \( D_v \) be a local distribution on \( V \). Then for any \( X \in \Upsilon \) we have
\[
\Phi(X)(D_v) = D_v \circ \Phi(X) = D_v \circ (F \text{id}).
\]
By definition of a local distribution, \( D_v \) annihilates \( m_q \) for some \( q > 0 \), where \( m_v \) is the maximal ideal in \( H^0(V, O) \). In particular, \( D_v \) annihilates a degree of the corresponding to \( m_v \) maximal ideal in \( \Phi(\Upsilon) \).

4 Alternating holomorphic functions

An alternating polynomial is a polynomial \( f(x_1, \ldots, x_n) \) such that
\[
f(\tau(x_1), \ldots, \tau(x_n)) = (-1)^\tau f(x_1, \ldots, x_n),
\]
for any \( \tau \in S_n \). An example of an alternating polynomial is the Vandermonde determinant
\[
V_n = \prod_{1 \leq i < j \leq n} (x_j - x_i).
\]
In fact this example is in some sense unique. More precisely, we need the following property of alternating polynomials.

**Proposition 1.** Any alternating polynomial \( f(x_1, \ldots, x_n) \) can be written in the form \( f = V_n \cdot g \), where \( g = g(x_1, \ldots, x_n) \) is a symmetric polynomial.

**Proof.** The proof follows from the following facts. Firstly every alternating polynomial \( f \) vanishes on the subvariety \( x_i = x_j \), where \( i \neq j \). Hence \( (x_i - x_j) \) is a factor of \( f \) and therefore \( V_n \) is also a factor of \( f \). Secondly it is clear that the ratio \( g = f/V_n \) is a symmetric polynomial.

**Corollary.** Let \( F = F(x_1, \ldots, x_n) \) be a holomorphic alternating function, i.e \( F(\tau(x_1), \ldots, \tau(x_n)) = (-1)^\tau F(x_1, \ldots, x_n) \) for any \( \tau \in S_n \). Then \( F = V_n \cdot G \), where \( G = G(x_1, \ldots, x_n) \) is a symmetric holomorphic function.
We will use this Corollary in Gelfand-Tsetlin Theory. Let \( o = (x_{0ij}^o) \in V \) be a Gelfand-Tsetlin tableau such that \( x_{0k1}^o = \cdots = x_{0k_p}^o \), where \( p \geq 2 \). Denote by \( W \) a sufficiently small neighborhood of the orbit \( \Gamma(o) \). For simplicity we put \( x_j := x_{kij} \). Let \( V_p = \mathcal{V}(x_1, \ldots, x_p) \) and \( S_p \subset G \) be the permutation group of \( (x_1, \ldots, x_p) \). We need the following proposition.

**Proposition 2.** Let \( A_j = \sum (H_j^i) / \mathcal{V}_p \sigma_i \in R \), where \( j = 1, \ldots, q \), and \( H_j^i \) are holomorphic in \( W \), be \( S_p \)-invariant elements in \( R \). Then \( A_1 \circ \cdots \circ A_q = \sum (G_i / \mathcal{V}_p) \sigma_i \), where \( G_i \) are holomorphic at \( o \).

**Proof.** Assume by induction that for \( k = q - 1 \) our statement holds. In other words, assume that \( A_1 \circ \cdots \circ A_{q-1} = \sum (G_i / \mathcal{V}_p) \sigma_i \), where \( G_i \) are holomorphic at \( o \). We have

\[
A_1 \circ \cdots \circ A_q = \sum_{i,j} G_i \sigma_i (H_j^i) / \mathcal{V}_p \sigma_i (\mathcal{V}_p) \sigma_i \circ \sigma_j.
\]

Assume that \( (G_i / \mathcal{V}_p) \sigma_i (H_j^i) / \mathcal{V}_p ) \sigma_i (\mathcal{V}_p) \sigma_i \) is singular at \( o \). Note that \( G_i \sigma_i (H_j^i) \) is holomorphic at \( o \). Let \( \sigma_i(x_1, \ldots, x_p) = (x_1 + m_1, \ldots, x_p + m_p) \), where \( m_i \in \mathbb{Z} \). Hence,

\[
\sigma_i (\mathcal{V}_p) = \prod_{1 \leq i < j \leq p} (x_j - x_i + m_j - m_i).
\]

If \( (G_i / \mathcal{V}_p) \sigma_i (H_j^i) / \mathcal{V}_p \) is singular at \( o \), we have \( m_{i_1} = \cdots = m_{i_r} \), where \( r \leq p \). For instance, \( \tau (\sigma_{i_0}) = \sigma_{i_0} \) for any \( \tau \in S_r \), where \( S_r \) is the permutation group of \( (x_{i_1}, \ldots, x_{i_r}) \). The product \( \sum (G_i / \mathcal{V}_p) \sigma_i \) is \( S_p \)-invariant since \( A_s \) are \( S_p \)-invariant by assumption. Since \( \sigma_{i_0} \) is \( S_r \)-invariant and the decomposition \( \sum (G_i / \mathcal{V}_p) \sigma_i \in R \) is unique, the function \( G_{i_0} / \mathcal{V}_p \) is \( S_r \)-invariant. Therefore the holomorphic function \( G_{i_0} \) is \( S_r \)-alternating. By Corollary of Proposition 1, we have \( G_{i_0} = \mathcal{V}_r (x_{i_1}, \ldots, x_{i_r}) G_{i_0} \). Hence, \( (G_i / \mathcal{V}_p) \sigma_i (H_j^i) / \mathcal{V}_p ) \sigma_i (\mathcal{V}_p) \) is holomorphic at \( o \). \( \square \)

## 5 Main results

Let \( o = (x_{ij}^o) \) be a Gelfand-Tsetlin tableau such that \( x_{0k1}^o = \cdots = x_{0k_p}^o \), where \( p \geq 2 \), and such that \( x_{st}^0 - x_{sr}^0 \notin \mathbb{Z} \) otherwise. We will call the orbits \( \Gamma(o) \) of such points \( p \)-singular and corresponding modules \( p \)-singular Gelfand-Tsetlin \( \mathfrak{gl}_n(\mathbb{C}) \)-modules. The stabilizer \( G_o \subset G \) of \( o \) is isomorphic to the permutation group \( G_o \simeq S_p \). We put \( z_{rt} = x_{kri} - x_{kjt} \), where \( r \neq t \). Then \( \frac{\partial}{\partial z_{rt}} = \frac{1}{2} (\frac{\partial}{\partial x_{kri}} - \frac{\partial}{\partial x_{kjt}}) \) are the corresponding
derivations. Let us fix a sufficiently small neighborhood \( W \) of the orbit \( \Gamma(o) \) such that \( W \) is \( \Gamma \) and \( G \)-invariant. We put

\[
\mathcal{L} := ev_o \circ \frac{\partial}{\partial z_{12}} \circ \cdots \circ \frac{\partial}{\partial z_{1p}} \circ \cdots \circ \frac{\partial}{\partial z_{p-1,p}} \cdot z_{12} \cdots z_{1p} z_{23} \cdots z_{p-1,p}.
\]

Clearly \( \mathcal{L} \) is \( S_p \)-invariant. Indeed, \( z_{12} \cdots z_{1p} z_{23} \cdots z_{p-1,p} \) is equal in fact to the Vandermonde determinant \( V_p \) in \( x_{ki_1}, \ldots, x_{ki_p} \). Hence it is alternating, see Section 4. On the other side, \( \tau(\frac{\partial}{\partial z_{1i}}) = \tau \circ \frac{\partial}{\partial z_{1i}} \circ \tau^{-1} = \frac{\partial}{\partial z_{1i}(\tau(o))} \). Therefore the sequence of derivations in \( \mathcal{L} \) is also alternating. Further we put

\[
T := \{(12), \ldots, (1p), (23), \ldots, (p-1, p)\}
\]

and \( I, J \) are two subsets in \( T \) such that \( I \cup J = T \) and \( I \cap J = \emptyset \). Then the elements \( z_T, z_I, z_J \) and the elements \( \frac{\partial}{\partial z_T}, \frac{\partial}{\partial z_I}, \frac{\partial}{\partial z_J} \) are the product and the composition of the corresponding \( z_{ij} \), respectively. In this notations \( \mathcal{L} = ev_o \circ \frac{\partial}{\partial z_T} \cdot z_T \). Note that \( z_T = V_p \). For any subset \( I \subset T \) and \( \sigma_i \in \Gamma \), consider the following sum of local distributions

\[
D_{I,\sigma_i} := \mathcal{L} \circ \sum_{\tau \in S_p} (-1)^{\tau} \tau(z_I \sigma_i)/z_T.
\]

Clearly, we have the following relations

\[
D_{I,\sigma_i} = (-1)^{\tau'} D_{\tau'(I),\tau'(\sigma_i)}, \quad \tau' \in S_p.
\] (4)

And we do not have other relations here. We need the following proposition.

**Proposition 3.** Let us take \( \sum_i h_i \sigma_i \in \mathcal{R} \) an \( S_p \)-invariant element that satisfies conditions of Proposition 2 at \( o \). Then we have the following equality of holomorphic operators

\[
\mathcal{L} \circ \left( \sum_i h_i \sigma_i \right) = \frac{1}{|S_p|} \sum_{I,i} \frac{\partial g_i}{\partial z_I} (o) D_{I,\sigma_i},
\] (5)

where \( g_i = z_T h_i \) and \( T \) and \( I \) are as above. In other words Formula (5) means that \( (\sum_i h_i \sigma_i)(\mathcal{L}) \) is a linear combination of the local distributions \( D_{I,\sigma_i} \).

**Proof.** We have

\[
\mathcal{L}(h_i \sigma_i) = ev_o \circ \frac{\partial}{\partial z_T} \circ g_i \sigma_i = ev_o \circ \left( \sum_I \frac{\partial g_i}{\partial z_I} \frac{\partial}{\partial z_J} \circ \sigma_i \right) = \\
\sum_I \frac{\partial g_i}{\partial z_I} (o) \left( ev_o \circ \frac{\partial}{\partial z_J} \frac{\partial}{\partial z_I} \circ z_i \sigma_i \right) = \sum_I \frac{\partial g_i}{\partial z_I} (o) \left( ev_o \circ \frac{\partial}{\partial z_T} \cdot z_T \circ \sigma_i \right) = \\
\sum_I \frac{\partial g_i}{\partial z_I} (o) \left( \mathcal{L} \circ \frac{\sigma_i}{z_J} \right).
\]
Since $L(h_\sigma_i)$ is $S_p$-invariant, we have:

$$|S_p|L(h_\sigma_i) = \sum \frac{\partial g_i}{\partial z_I}(o) \mathcal{L} \circ \left( \sum_{\tau \in S_p} \frac{(-1)^\tau(z_I\sigma_i)}{z_T} \right) = \sum \frac{\partial g_i}{\partial z_I}(o) \mathcal{L} \circ D_{I,\sigma_i}.$$ 

The proof is complete. □

**Theorem 3.** [Main result 1] Let $\mathfrak{g}$ be any Lie algebra and $\Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{R}$ be a homomorphism of rings. If $\Phi(\mathfrak{g})$ is generated by elements satisfying the conditions of Proposition 2, then the vector space spanned by the elements $D_{I,\sigma_i}$, where $I \subset T$ is a subset and $\sigma_i \in \Gamma$, up to relations (4) form a basis for the $\mathfrak{g}$-module.

**Proof.** Assume that $\Phi(\mathfrak{g})$ is generated by $A_i$ as in Proposition 2. By this proposition we see that any product of such generators has the form $\sum (G_i/\mathcal{V}_p)\sigma_i$, where $G_i$ are holomorphic at $o$. We need to prove that $\sum (G_i/\mathcal{V}_p)\sigma_i(D_{I,\sigma_j})$ is a linear combination of $D_{I',\sigma_{j'}}$. We have

$$\sum (G_i/\mathcal{V}_p)\sigma_i(D_{I,\sigma_j}) = D_{I,\sigma_j} \circ \sum (G_i/\mathcal{V}_p)\sigma_i =$$

$$\mathcal{L} \circ \left( \sum_{\tau \in S_p} \frac{(-1)^\tau(z_I\sigma_j)}{\mathcal{V}_p} \right) \circ \sum_i (G_i/\mathcal{V}_p)\sigma_i.$$ 

Now we apply Proposition 2 to the composition in the last line. The result follows from Proposition 3. □

Note that the classical Gelfand-Tsetlin generators (1) satisfy conditions of Proposition 2.

**Theorem 4.** [Main result 2] The vector space spanned by the elements $D_{I,\sigma_i}$, where $I \subset T$ is a subset and $\sigma_i \in \Gamma$, up to relations (4) form a basis for the $\mathfrak{gl}_n(\mathbb{C})$-module. By Corollary of Theorem 2, this module is a Gelfand-Tsetlin module.

**Remark.** The basis of elements $D_{I,\sigma_i}$, where $I \subset T$ is a subset and $\sigma_i \in \Gamma$, up to relations (4), can be simplified in some cases. In other words sometimes we can find a natural submodules in the corresponding module. It depends on the singularity type of $\Phi(\mathcal{U}(\mathfrak{g}))$ at $o$. It is required in Theorem 3 than the singularity type is not more than $\mathcal{V}_p$. However, if the singularity type of $\Phi(\mathcal{U}(\mathfrak{g}))$ at $o$ is less than type of $\mathcal{V}_p$, Formula (5) says that we can reduce our basis.

## 6 Case of singularity 1

The result of this section is published in [Vi]. Let us fix an 1-singular point $o = (x^0_{k_j}) \in V$ such that $x^0_{k_i} - x^0_{k_j} \in \mathbb{Z}$. We put $z_1 = x_{k_i} - x_{k_j}$. Let $W$ be a
sufficiently small neighborhood of the orbit $\Gamma(o) = \Gamma(x^0)$ that is invariant with respect to the group $\Gamma$ and with respect to $\tau \in G$, where $\tau \in G$ is defined by $\tau(z_1) = -z_1$ and $\tau(z_i) = z_i$, $i > 1$. In this case we have

$$\mathcal{L} := ev_o \circ \frac{\partial}{\partial z_1} \cdot z_1.$$ 

In this case our basis has the following form

$$D_1^\sigma := \mathcal{L} \circ (\sigma + \tau(\sigma)), \quad D_2^{\sigma'} := \mathcal{L} \circ \frac{\sigma' - \tau(\sigma')}{z_1}, \quad \sigma, \sigma' \in \Gamma.$$ 

(6)

Here $D_1^\sigma$ is the sum of two local distributions $\mathcal{L} \circ \sigma$ at the point $\sigma^{-1}(o)$ and $\mathcal{L} \circ \tau(\sigma)$ at the point $(\tau(\sigma))^{-1}(o)$. The same holds for $D_2^{\sigma'}$. We have the following equalities

$$D_1^{\tau(\sigma)} = D_1^\sigma \quad \text{and} \quad D_2^{\tau(\sigma')} = -D_2^{\sigma'}.$$ 

(7)

From Theorem 4 it follows that the local distributions $D_1^\sigma, D_2^{\sigma'}$ up to relations (7) form a basis for a $\mathfrak{gl}_n(\mathbb{C})$-module. This $\mathfrak{gl}_n(\mathbb{C})$-module was constructed in [FGR1] and it was called the universal 1-singular Gelfand-Tsetlin module. We can reformulate Theorem 4 in this case as follows.

**Theorem 5.** The vector space spanned by $(D_1^\sigma, D_2^{\sigma'})$ up to relations (7), where $\sigma, \sigma' \in \Gamma$, is a $\mathfrak{gl}_n(\mathbb{C})$-module. The action is given by Formulas (5).

So we reproved Theorem 4.11 from [FGR1]. The explicit correspondence between notation in [FGR1] and our notations can be deduced from

$$\mathcal{D}T(v + z) := D_2^{\sigma'}(T(v)), \quad \mathcal{D}^{\sigma}(F) = \frac{\partial}{\partial z_1} \bigg|_o (F),$$

where $\sigma'(v) = v + z$ and $T(v)$ and $F$ are holomorphic function on $W$.

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