Two Proofs of Fine’s Theorem

J.J.Halliwell*

Blackett Laboratory
Imperial College
London SW7 2BZ
UK

Abstract

Fine’s theorem concerns the question of determining the conditions under which a certain set of probabilities for pairs of four bivalent quantities may be taken to be the marginals of an underlying probability distribution. The eight CHSH inequalities are well-known to be necessary conditions, but Fine’s theorem is the striking result that they are also a sufficient condition. It has application to the question of finding a local hidden variables theory for measurements of pairs of spins for a system in an EPRB state. Here we present two simple and self-contained proofs of Fine’s theorem in which the origins of this non-obvious result can be easily seen. The first is a physically motivated proof which simply notes that this matching problem is solved using a local hidden variables model given by Peres. The second is a straightforward algebraic proof which uses a representation of the probabilities in terms of correlation functions and takes advantage of certain simplifications naturally arising in that representation. A third, unsuccessful attempt at a proof, involving the maximum entropy technique is also briefly described.

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*Electronic address: j.halliwell@imperial.ac.uk
Consider the following simple but non-trivial problem in probability theory. We suppose we are given a system described by four variables $s_1, s_2, s_3, s_4$ which may take values $\pm 1$, and which for convenience we call spins. We suppose also that we are given the pair probabilities, $p(s_1, s_3), p(s_1, s_4), p(s_2, s_3)$ and $p(s_2, s_4)$. Under what conditions are these pair probabilities the marginals of an underlying probability for all four variables, $p(s_1, s_2, s_3, s_4)$?

This question is of course very closely linked to the Clauser-Horne-Shimony-Holt (CHSH) analysis of an entangled pair of spin states \[1\] and in this connection it is well-known that a necessary set of conditions is the eight CHSH inequalities,

\[
\begin{align*}
-2 &\leq C_{13} + C_{14} + C_{23} - C_{24} \leq 2 \quad (1.1) \\
-2 &\leq C_{13} + C_{14} - C_{23} + C_{24} \leq 2 \quad (1.2) \\
-2 &\leq C_{13} - C_{14} + C_{23} + C_{24} \leq 2 \quad (1.3) \\
-2 &\leq -C_{13} + C_{14} + C_{23} + C_{24} \leq 2 \quad (1.4)
\end{align*}
\]

where $C_{13}, C_{14}, C_{23}, C_{24}$ denote the correlation functions

\[ C_{ij} = \sum_{s_1, s_2, s_3, s_4} s_i s_j p(s_1, s_2, s_3, s_4) \] (1.5)

and

\[ \sum_{s_1, s_2, s_3, s_4} p(s_1, s_2, s_3, s_4) = 1 \] (1.6)

The CHSH inequalities are easily derived by assuming that a probability $p(s_1, s_2, s_3, s_4)$ exists and then summing it with simple inequalities of the form

\[ -2 \leq s_1 s_3 + s_1 s_4 + s_2 s_3 - s_2 s_4 \leq 2 \] (1.7)

plus three more similar ones, thereby obtaining Eqs.\[\text{(1.1)}-\text{(1.4)}.\]

However, an important result due to Fine is that the CHSH inequalities are also a sufficient condition for the existence of a probability matching the given marginals. This intriguing result goes by the name of Fine’s theorem. Its proof is not as immediate or obvious as the proof of necessity. Fine gave a direct proof in Refs.\[\text{[2, 3]}\] by showing by purely algebraic means how to flesh out the given set of marginals into a full probability distribution. (A simple proof of the much easier problem involving three bivalent variables, involving Bell’s
original inequalities [4], was given by Suppes and Zanotti [5]. Pitowski [6] gave a very different proof using the geometry of polytopes. Garg and Mermin considered a general class of problems of this type [7], using properties of convex sets, and gave a proof of Fine’s theorem as an example. Generalizations of these ideas to \(N\) qubits have been considered by Zukowski and Brukner [8].

Since this result is far from obvious, it is of interest to find alternative proofs which are clearer and more immediate. The purpose of the present paper is therefore to give two self-contained proofs of Fine’s theorem which are different and perhaps simpler than those cited above. The idea is not to give a general solution to this matching problem, but to give simple pedagogical pictures in which it is not hard to see why the theorem is true.

The first proof is a physically-motivated one involving an explicit local hidden variables model given by Peres [9]. Clearly if a local hidden variables theory exists matching the given marginals, then an underlying probability \(p(s_1, s_2, s_3, s_4)\) exists so the CHSH inequalities must be satisfied. The point here is to show that this statement is logically reversible for this model – if the CHSH inequalities are satisfied then the parameters of the local hidden variables model may be chosen to match the given marginals so the sought-after probability solving the matching problem is that supplied by the local hidden variables theory. In essence, we make a strategic guess as to the form of the underlying probability and confirm that it solves the problem.

The second proof is a direct algebraic one, which takes advantage of a particularly useful representation of the underlying probability in terms of its correlation functions,

\[
p(s_1, s_2, s_3, s_4) = \frac{1}{16} \left( 1 + \sum_i B_i s_i + \sum_{i<j} C_{ij} s_i s_j + \sum_{i<j<k} D_{ijk} s_i s_j s_k + E s_1 s_2 s_3 s_4 \right)
\]  

(1.8)

where the indices \(i, j, k\) run over the values 1, 2, 3, 4. The correlation functions \(C_{ij}\) are given by Eq.(1.5) and the remaining correlators are given by

\[
B_i = \sum_{s_1 s_2 s_3 s_4} s_i p(s_1, s_2, s_3, s_4)
\]

(1.9)

\[
D_{ijk} = \sum_{s_1 s_2 s_3 s_4} s_i s_j s_k p(s_1, s_2, s_3, s_4)
\]

\[
E = \sum_{s_1 s_2 s_3 s_4} s_1 s_2 s_3 s_4 p(s_1, s_2, s_3, s_4)
\]

The marginals are then easily constructed by summing out some of the \(s_i\)’s. So for example

\[
p(s_1, s_3) = \frac{1}{4} (1 + B_1 s_1 + B_3 s_3 + C_{13} s_1 s_3)
\]  

(1.10)
Note that we are using the mathematically incorrect but commonly employed notation in which functions, such as \( p(s_1, s_3) \) are identified by their arguments. Also, we are assuming that the single spin probabilities are consistent with the specified two-spin probabilities, so for example, we assume that

\[
\sum_{s_1} p(s_1, s_3) = p(s_3) = \sum_{s_2} p(s_2, s_3) \tag{1.11}
\]

Eq. (1.10), plus three similar relations, mean that fixing the given four marginals is equivalent to fixing the values of \( B_i \) for \( i = 1, 2, 3, 4 \) and the values of the four correlation functions \( C_{13}, C_{23}, C_{14} \) and \( C_{24} \). The question of finding a probability matching the given marginals is then the question of whether the remaining unfixed correlation functions, \( C_{12}, C_{34}, D_{ijk} \) and \( E \) can be chosen in such a way that the probability Eq. (1.8) is positive. However, as stated, we are not looking for the most general solution to the problem, but instead seeking to show that some solution exists as long as the CHSH inequalities are satisfied. This allows us to make a number of simplifications, based on symmetries of the CHSH inequalities, as we shall see, and the algebraic solution then turns out to be very straightforward.

We begin in Section 2 by briefly describing the related quantum problem from which this question arises and we show from this how to argue that we may set the average spins, \( B_i \), to zero. In Section 3 we describe the proof of Fine’s theorem using a local hidden variable model.

Turning to the second algebraic proof, in Section 4 we solve algebraically a simpler problem involving three variables, and in this case the necessary and sufficient conditions are the four Bell inequalities. We give the algebraic solution to the main problem, finding the conditions under which Eq. (1.8) is positive, in Section 5. We summarize and conclude in Section 6. We also briefly describe a third attempt at proving Fine’s theorem using an ansatz for the probability supplied by the maximum entropy technique, but this turns out to be unsuccessful.

This work arose directly from an early work about the use and misuse of quasi-probabilities and their relation to Fine’s theorem \[10\]. In particular, the formula Eq. (1.8) was introduced there in the context of quasi-probabilities but has found particular use here as a genuine probability. This formula was also written down earlier by Klyshko \[11\], who showed that a number of different problems involving quantum “paradoxes” can reduce to a problem in probability theory of matching given marginals. Fine’s theorem appears to
have had very wide impact and many applications, with his original paper receiving a very
large number of citations, far too many to discuss here in any detail. However, it is clearly
very relevant to the questions concerning the existence and interpretation of hidden variable
theories (see, for example Refs. [12, 13]) and to generalizations of quantum theory [14]. It
may also have some role in the Leggett-Garg (or “temporal Bell”) inequalities [15, 16], since
they have the same form as the CHSH inequalities, but this does not seem to have been
explored.

II. THE QUANTUM PROBLEM AND A SIMPLIFICATION

Some background and insight into the Fine problem may be obtained by considering
some aspects of the quantum-mechanical problem from which it arose. The situation is the
standard EPRB set up, in which we consider a pair of particles \( A \) and \( B \) whose spins are in
an entangled state. (For general reviews of the Bell and CHSH inequalities in this area see
for example Refs. [17, 18]). The most famous example is of course the EPRB state

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle),
\]

where \(|\uparrow\rangle\) denotes spin up in the \( z \)-direction, but we do not restrict attention to this choice of
state. Measurements are made on particle \( A \) in the directions characterized by unit vectors
\( a_1 \) and \( a_2 \) and on particle \( B \) in directions \( a_3 \) and \( a_4 \). The probabilities for pairs of such
measurements, one on \( A \), one on \( B \) is of the form,

\[
p(s_1, s_3) = \langle \Psi | P_{s_1}^{a_1} \otimes P_{s_3}^{a_3} |\Psi\rangle
\]

plus three similar expressions. The measurements are described by projection operators of
the form

\[
P_s^a = \frac{1}{2} (1 + sa \cdot \sigma)
\]

where \( \sigma_i \) denotes the Pauli spin matrices.

The EPRB state has the property that \( \langle a \cdot \sigma \rangle = 0 \) for all four direction vectors and
this simplifies the analysis considerably since it means that \( B_i = 0 \) in Eq. (1.8). This is
not true for more general states but it can be arranged by a simple unitary transformation
on the initial state. It is easy to find a unitary transformation operator which carries out
independent rotations on subsystems \( A \) and \( B \) and this has the effect of performing a rotation
on the average Pauli spin matrices for each subsystem, $\langle \sigma^A_i \rangle$ and $\langle \sigma^B_i \rangle$. We may choose the rotation on $A$ so that $\langle \sigma^A_i \rangle$ becomes orthogonal to $a_1$ and $a_2$, and the rotation on $B$ so that $\langle \sigma^B_i \rangle$ becomes orthogonal to $a_3$ and $a_4$. This then sets all four average spins are zero, as required. This rotation will in general change the correlation functions. However, since it acts independently on systems $A$ and $B$ it will not change the degree of entanglement, so it should not affect whether or not the correlation functions satisfy the CHSH inequalities.

This argument shows that if analyzing the quantum problem, we can without loss of generality work with a state for which the average spins are zero. However, in the most general case, the probabilities are not of quantum-mechanical form. It is clearly very plausible that the probabilities may be invertibly transformed into a set with zero average spin, but we have not proved this. This will be addressed in more detail elsewhere.

III. SOLUTION USING A HIDDEN VARIABLE MODEL

We now give a simple proof of Fine’s theorem by writing down an explicit local hidden variable model for the probabilities. This model is essentially that given by Peres to illustrate the CHSH inequalities [9]. The model consists of a classical particle which splits into two with equal and opposite angular momenta, $\pm J$, and measurements of the sign of the angular momentum of each particle are made along directions characterized by unit vectors $a_1, a_2$ for one particle and $a_3, a_4$ for the other. We focus on the signs of the variables of the form $a \cdot J$ where $J$ is assumed to be uniformly distributed. The probability for all four spins is given by

$$p(s_1, s_2, s_3, s_4) = \langle (1 + s_1 \text{ sgn}(a_1 \cdot J)) (1 + s_2 \text{ sgn}(a_2 \cdot J))$$
$$\times (1 - s_3 \text{ sgn}(a_3 \cdot J)) (1 - s_4 \text{ sgn}(a_4 \cdot J)) \rangle$$

which is clearly non-negative, where the average is over $J$ with a uniform distribution.

The average spins are zero in this model and the correlation functions are all then of the form

$$C_{13} = -\langle \text{ sgn} (a_1 \cdot J) \text{ sgn} (a_3 \cdot J) \rangle$$

$$= -1 + \frac{2\theta_{13}}{\pi}$$

where $\theta_{13}$ is the angle between the two vectors [9] and lies in the range $0 \leq \theta_{13} \leq \pi$, and similarly for the other three correlation functions. Hence the correlation functions in this
model reduce to a simple geometric feature, namely the angle between two vectors. The CHSH inequalities take the form

\[
0 \leq \theta_{13} + \theta_{23} + \theta_{24} - \theta_{14} \leq 2\pi \tag{3.3}
\]
\[
0 \leq \theta_{13} + \theta_{23} - \theta_{24} + \theta_{14} \leq 2\pi \tag{3.4}
\]
\[
0 \leq \theta_{13} - \theta_{23} + \theta_{24} + \theta_{14} \leq 2\pi \tag{3.5}
\]
\[
0 \leq -\theta_{13} + \theta_{23} + \theta_{24} + \theta_{14} \leq 2\pi \tag{3.6}
\]

These are of course satisfied for any orientation of the four vectors since there exists a probability Eq.(3.1) for this model. One can also confirm geometrically that these inequalities hold in this model by examining all the possible orientations of the four vectors (this is set as an exercise in Peres’ book [9]).

However, here, we are interested in the converse to this problem: can we match the non-negative probability Eq.(3.1) to any given set of four marginals satisfying the CHSH inequalities? Or in other words, can we always choose the four vectors in Eq.(3.1) to match any given set of the four angles satisfying the CHSH inequalities Eqs.(3.3)-(3.6)? It is not hard to see geometrically that this is indeed possible, thereby providing a proof of sufficiency in Fine’s theorem.

In essence the hidden variables model provides a sensible guess for the underlying probability solving the matching problem and our goal is to show that it actually does the job. Note that this is not guaranteed – a particular guess for the probability may have a set of correlation functions which do not explore the full range of possible values satisfying the CHSH inequalities, and indeed we will see an example of this in Section 6.

We need to show that we can choose the four vectors \( \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \) to match a given set of angles, \( \theta_{13}, \theta_{23}, \theta_{24}, \theta_{14} \) satisfying the CHSH inequalities Eqs.(3.3)-(3.6), with \( \theta_{12} \) and \( \theta_{14} \) unspecified. We first let the four vectors lie in a plane and adjust them so that three of the angles are fixed to the given values, say \( \theta_{13}, \theta_{23} \) and \( \theta_{24} \). This is shown for a particular orientation of vectors in Figure 1 which shows three triangles whose edges radiating from the origin \( O \) are the four vectors, with the third side of the triangle completed for illustrative convenience. We then need to adjust these vectors, by moving them out of the plane, to match the fourth angle \( \theta_{14} \), but preserving the three fixed already. To do this, we imagine that the outer two triangles in the plane figure shown in Figure 1 are allowed to fold inwards along the edges \( \mathbf{a}_2 \) and \( \mathbf{a}_3 \), thereby varying the angle \( \theta_{14} \) until it reaches its prescribed value.
FIG. 1: A plane figure showing an orientation of the four vectors $a_1, a_2, a_3, a_4$ which matches given values for the three angles $\theta_{13}, \theta_{23}$ and $\theta_{24}$. The angle $\theta_{14}$ can be adjusted, with the first three fixed, by folding in the outer two triangles along the edges $a_2$ and $a_3$, subject to the upper bound Eq. (3.7).

There are limits to the range of values that can be reached. The largest possible angle is achieved when all four vectors lie in the plane, as shown in Figure 1, so the upper limit is

$$\theta_{14} \leq \theta_{13} + \theta_{23} + \theta_{24} \quad \text{(3.7)}$$

which we know is satisfied since it is one of the CHSH inequalities, Eq. (3.3).

A small value of $\theta_{14}$ can be reached by folding in the outer two triangles as far as possible but it is not always possible to reach $\theta_{14} = 0$ if one of the three fixed angles is sufficiently large. For example, suppose $\theta_{23}$ is much larger than the other two fixed angles, as depicted in Figure 2. Then the smallest possible value for $\theta_{14}$ is obtained by folding in the outer two triangles so they lie flat in the inner triangle, with the vectors $a_1$ and $a_4$ lying on the dotted lines. There is therefore a lower bound on the possible values of $\theta_{14}$ which is easily seen to be

$$\theta_{14} \geq \theta_{23} - \theta_{13} - \theta_{24} \quad \text{(3.8)}$$
FIG. 2: The smallest possible value of $\theta_{14}$, when non-zero, is obtained by folding in the outer two triangles until they lie flat on the inner triangle, as denoted by the dotted lines, and thus satisfy the lower bound Eq. (3.8).

which is again seen to be one of the CHSH inequalities, Eq. (3.5), so will be satisfied.

The orientations depicted in Figures 1 and 2 explore the lower bound on the CHSH inequalities, Eqs. (3.3)-(3.6). The upper bound becomes relevant when some of the specified angles are close to $\pi$. One such case is when the unfixed angles $\theta_{12}$ and $\theta_{34}$ are very small and $\mathbf{a}_1, \mathbf{a}_2$ point in the direction approximately opposite to $\mathbf{a}_3, \mathbf{a}_4$. In this case, we may work with a different set of vectors in which either the pair $\mathbf{a}_1, \mathbf{a}_2$ or the pair $\mathbf{a}_3, \mathbf{a}_4$ are reflected in the origin. This has the effect that all four angles in the CHSH inequalities Eqs. (3.3)-(3.6) are changed according to $\theta \to \pi - \theta$ and as a consequence the upper and lower bounds are interchanged. Hence we are back to the situation depicted in Figures 1 and 2.

Another case is when three of the vectors, say $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, have small angles between them and a large angle with $\mathbf{a}_4$. In this case, we replace $\mathbf{a}_4$ with its reflection in the origin which causes two of the angles $\theta_{14}$ and $\theta_{24}$ to undergo the transformation $\theta \to \pi - \theta$. This creates a more complicated transformation of the CHSH inequalities Eqs. (3.3)-(3.6) in which again the upper and lower bounds are interchanged, but also some of the CHSH inequalities are interchanged with each other. Again we get back to situations similar to those depicted in Figures 1 and 2.
These arguments can be repeated for other orientations of the vectors and this will involve the other CHSH inequalities. We thus establish in a simple geometric way that the parameters of this local hidden variables model, the four vectors, may be chosen in such a way that any set of values of the four correlation functions may be matched, as long as the correlation functions satisfy the CHSH inequalities, and Eq. (3.1) is the solution to the matching problem. This therefore proves sufficiency in Fine’s theorem.

IV. THE BELL CASE

Turning now to the algebraic proof, we consider first a simpler example, namely that in which we seek a probability \( p(s_1, s_2, s_3) \) matching the three marginals \( p(s_1, s_2), p(s_2, s_3) \) and \( p(s_1, s_3) \). The probability in this case may be written

\[
p(s_1, s_2, s_3) = \frac{1}{8} \left( 1 + \sum_i B_is_i + \sum_{i<j} C_{ij}s_is_j + Ds_1s_2s_3 \right),
\]

(4.1)

where \( i, j, k \) runs over values 1, 2, 3. The marginals are obtained by summing out one of the \( s_i \) variables and they have the form Eq. (1.10). Since these marginals are, by assumption, non-negative, this imposes certain restrictions on the coefficients \( B_i \) and \( C_{ij} \). For example, one obtains a restriction of the form

\[
1 + B_1 - B_2 - C_{12} \geq 0
\]

(4.2)

We are therefore assuming that all such restrictions are satisfied by \( B_i \) and \( C_{ij} \). As noted one can argue that \( B_i \) may be set to zero, but it is not difficult to maintain a non-zero value in this proof, so we will do so, thereby seeing explicitly that it plays essentially no role.

In this example the necessary and sufficient conditions for the existence of a probability are the inequalities [5, 6]

\[
C_{12} + C_{13} - C_{23} \leq 1
\]

(4.3)

\[
C_{12} - C_{13} + C_{23} \leq 1
\]

(4.4)

\[
-C_{12} + C_{13} + C_{23} \leq 1
\]

(4.5)

\[
-C_{12} - C_{13} - C_{23} \leq 1
\]

(4.6)

which are a form of Bell’s original inequalities [4]. Necessity is easy to establish, along the lines of Eq. (1.7). To prove sufficiency, since the three marginals fix the six quantities \( B_i \) and
$C_{ij}$, the only free parameter is $D$ so we need to show that the Bell inequalities ensure that the constant $D$ can be chosen in such a way that Eq.(4.1) is non-negative.

Eq.(4.1) is non-negative if

$$A(s_1, s_2, s_3) = 1 + \sum_i B_i s_i + \sum_{i<j} C_{ij} s_i s_j \geq -Ds_1 s_2 s_3$$  \hspace{1cm} (4.7)

For the four values of $s_1, s_2, s_3$ for which $s_1 s_2 s_3 = -1$, this gives four upper bounds on $D$,

$$A(s_1, s_2, s_3) \geq D,$$  \hspace{1cm} (4.8)

and for the values with $s_1 s_2 s_3 = 1$, this give four lower bounds on $D$

$$-A(s_1, s_2, s_3) \leq D$$  \hspace{1cm} (4.9)

Hence a value of $D$ exists as long as all four upper bounds are greater that the all four lower bounds:

$$A(-, -, -), A(+, +, -), A(+, -, +), A(-, +, +)$$
$$\geq -A(+, +, +), -A(-, -, +), -A(-, +, -), -A(+, -, -)$$  \hspace{1cm} (4.10)

Of the sixteen resultant inequalities, there are four of the form

$$A(s_1, s_2, s_3) + A(-s_1, -s_2, -s_3) \geq 0$$  \hspace{1cm} (4.11)

in which the dependence on $B_i$ drops out and these are easily seen to be precisely the Bell inequalities. The remaining twelve are simply the restrictions on $B_i$ and $C_{ij}$ of the form Eq.(4.2) which ensure that the marginals are non-negative. (This is proved by simply writing them all out, but it is essentially straightforward). The proves the result.

V. THE CHSH CASE

We now turn to our main algebraic problem, which is proving sufficiency in Fine’s theorem for the CHSH case. We thus seek to show that the CHSH inequalities are a sufficient condition for the probability Eq.(1.8) to be non-negative, for given values of $B_i, C_{13}, C_{23}, C_{14}$ and $C_{24}$. We use three simplifications. The first is to restrict attention to the case of zero average spins, $B_i = 0$, as discussed. The second is to note that the CHSH inequalities are unchanged under the transformation

$$(s_1, s_2, s_3, s_4) \rightarrow (-s_1, -s_2, -s_3, -s_4)$$  \hspace{1cm} (5.1)
This indicates that a solution to the problem exists which possesses this symmetry. The probability Eq.(1.8) with \( B_i = 0 \) will have this symmetry if in addition \( D_{ijk} = 0 \), so we make this choice. This means that we will not obtain the most general solution to the problem, but our aim is to find a reasonably quick way of showing that a solution exists, which in fact turns out to be not much more complicated than the Bell case considered in the previous section. Our third simplification is to note that the CHSH inequalities have another symmetry, namely

\[(s_1, s_2) \rightarrow (-s_1, -s_2) \tag{5.2}\]

which, via the symmetry Eq.(5.1), is equivalent to

\[(s_3, s_4) \rightarrow (-s_3, -s_4) \tag{5.3}\]

This symmetry is equivalent to changing the signs of the four fixed correlation functions, \( C_{13}, C_{14}, C_{23} \) and \( C_{24} \) whilst preserving the signs of the unfixed ones \( C_{12} \) and \( C_{34} \). This symmetry leads to further simplifications as we shall see.

With the choices \( B_i = 0 = D_{ijk} \), the requirement that Eq.(1.8) is non-negative may be written

\[1 + \sum_{i<j} C_{ij}s_is_j \geq -Es_{1}s_2s_3s_4 \tag{5.4}\]

which may be written out more explicitly as

\[1 + s_1s_3C_{13} + s_1s_4C_{14} + s_2s_3C_{23} + s_2s_4C_{24} \geq -s_1s_2C_{12} - s_3s_4C_{34} - Es_{1}s_2s_3s_4 \tag{5.5}\]

We determine the conditions under which these inequalities have a solution.

With the above choices there are only eight inequalities to check, rather than sixteen and we can select eight independent ones by setting \( s_1 = +1 \). Choosing \( s_1, s_2, s_3, s_4 \) to be the four sets of values \((++++)\), \((++--)\), \((+-++)\), \((--++)\) yields, respectively

\[1 + C_{13} + C_{14} + C_{23} + C_{24} \geq -C_{12} - C_{34} - E \tag{5.6}\]
\[1 - C_{13} - C_{14} - C_{23} - C_{24} \geq -C_{12} - C_{34} - E \tag{5.7}\]
\[1 + C_{13} - C_{14} - C_{23} + C_{24} \geq C_{12} + C_{34} - E \tag{5.8}\]
\[1 - C_{13} + C_{14} + C_{23} - C_{24} \geq C_{12} + C_{34} - E \tag{5.9}\]
Choosing the four values \((+ + + +), (+ + -), (+ - +), (+ - - -)\) yields

\[ 1 + C_{13} - C_{14} + C_{23} - C_{24} \geq -C_{12} + C_{34} + E \]  
(5.10)
\[ 1 - C_{13} + C_{14} - C_{23} + C_{24} \geq -C_{12} + C_{34} + E \]  
(5.11)
\[ 1 + C_{13} + C_{14} - C_{23} - C_{24} \geq C_{12} - C_{34} + E \]  
(5.12)
\[ 1 - C_{13} - C_{14} + C_{23} + C_{24} \geq C_{12} - C_{34} + E \]  
(5.13)

Now note that the eight inequalities occur in successive pairs differing only by a reversal of signs of the correlation functions on the left-hand side, so each pair has the form \(1 \pm G \geq L\) which are written more concisely as a single relation \(1 - |G| \geq L\). This is a consequence of the symmetry Eq. (5.2). The eight inequalities therefore reduce to the four inequalities

\[ 1 - |G_1| \geq -C_{12} - C_{34} - E \]  
(5.14)
\[ 1 - |G_2| \geq C_{12} + C_{34} - E \]  
(5.15)
\[ 1 - |G_3| \geq -C_{12} + C_{34} + E \]  
(5.16)
\[ 1 - |G_4| \geq C_{12} - C_{34} + E \]  
(5.17)

where

\[ G_1 = C_{13} + C_{14} + C_{23} + C_{24} \]  
(5.18)
\[ G_2 = C_{13} - C_{14} - C_{23} + C_{24} \]  
(5.19)
\[ G_3 = C_{13} - C_{14} + C_{23} - C_{24} \]  
(5.20)
\[ G_4 = C_{13} + C_{14} - C_{23} - C_{24} \]  
(5.21)

The inequalities Eqs. (5.14)-(5.17) are now easily solved. Eqs. (5.14), (5.15) gives an upper and lower bound on \(C_{12} + C_{34}\),

\[ 1 - |G_2| + E \geq C_{12} + C_{34} \geq -1 + |G_1| - E \]  
(5.22)

which has a solution as long as

\[ 2 - |G_1| - |G_2| \geq -2E \]  
(5.23)

Similarly, Eqs. (5.16), (5.17) yield

\[ 1 - |G_4| - E \geq C_{12} - C_{34} \geq -1 + |G_3| + E \]  
(5.24)
which has a solution as long as

$$2 - |G_3| - |G_4| \geq 2E$$  \hspace{1cm} (5.25)

Finally, Eqs. (5.23), (5.25) give an upper and lower bound on $E$ and have a solution for $E$ as long as

$$|G_1| + |G_2| + |G_3| + |G_4| \leq 4$$  \hspace{1cm} (5.26)

This single, simple inequality is a sufficient condition to ensure the non-negativity of the probability $p(s_1, s_2, s_3, s_4)$. Written out in full, it reads

$$|C_{13} + C_{14} + C_{23} + C_{24}| + |C_{13} - C_{14} - C_{23} + C_{24}| + |C_{13} - C_{14} + C_{23} - C_{24}| + |C_{13} + C_{14} - C_{23} - C_{24}| \leq 4$$  \hspace{1cm} (5.27)

This is the main result of this section. Eq. (5.26) or (5.27) is equivalent to sixteen inequalities corresponding to all the different possible sign choices for $G_1, G_2, G_3, G_4$. It is not immediately obvious but these sixteen inequalities are, in fact, the eight CHSH inequalities and the eight restrictions of the form $|C_{ij}| \leq 1$ on the four fixed correlation functions, thus proving Fine’s theorem.

We briefly outline this last step. We have

$$G_1 + G_2 + G_3 + G_4 = 4C_{13}$$  \hspace{1cm} (5.28)
$$G_1 - G_2 - G_3 + G_4 = 4C_{14}$$  \hspace{1cm} (5.29)
$$G_1 - G_2 + G_3 - G_4 = 4C_{23}$$  \hspace{1cm} (5.30)
$$G_1 + G_2 - G_3 - G_4 = 4C_{24}$$  \hspace{1cm} (5.31)

so Eq. (5.26) implies $C_{ij} \leq 1$. The restrictions $C_{ij} \geq -1$ are easily found by taking the opposite set of signs for the $G_i's$. Similarly

$$G_1 + G_2 + G_3 - G_4 = 2(C_{13} - C_{14} + C_{23} + C_{24})$$  \hspace{1cm} (5.32)
$$G_1 + G_2 - G_3 + G_4 = 2(C_{13} + C_{14} - C_{23} + C_{24})$$  \hspace{1cm} (5.33)
$$G_1 - G_2 + G_3 - G_4 = 2(C_{13} + C_{14} + C_{23} - C_{24})$$  \hspace{1cm} (5.34)
$$G_1 - G_2 - G_3 - G_4 = 2(-C_{13} + C_{14} + C_{23} + C_{24})$$  \hspace{1cm} (5.35)

Eq. (5.26) then gives upper bound half of the set of CHSH inequalities. The lower bound half is easily obtained by taking the opposite set of signs. (This possibility is another consequence of the symmetry Eq. (5.2)).
We have therefore shown algebraically that the CHSH inequalities are a sufficient condition for the non-negativity of the probability Eq. (1.8), thereby proving Fine’s theorem. The proof hinges on identifying the possibility of setting the average spins to zero and with making use of the symmetries of the CHSH inequalities. These simplifications reduce the algebraic solution of the inequalities on the probabilities to just a few lines. A side product is an unusual form of the CHSH inequalities, written as a single inequality, Eq. (5.27). This does not appear to have been written down previously although is closely related to a formula written down in Ref. [8]. A different but closely related form of the CHSH inequalities was also obtained by Parrott [19].

VI. SUMMARY AND CONCLUSIONS

In this paper two proofs of Fine’s theorem were presented, with the aim to be simple and pedagogical. The first is based on an explicit local hidden variables model and the essence of this proof is the simple observation that this model not only satisfies the CHSH inequalities, as it should, but also provides a complete solution to the CHSH inequalities, in the sense that the parameters of the model may be chosen to match any values of the correlation functions satisfying the inequalities, hence the model’s probability solves the matching problem.

The second proof is based on a representation of the underlying probability in terms of correlation functions. This representation highlights a number of simplifying features, namely the symmetries of the CHSH inequalities. The solution obtained for the probability is not the most general one matching the given marginals, since it involves setting the triple correlator \( D_{ijk} \) to zero. General solutions have been given in previous proofs. The essence of this work is to find the simplest and clearest way to see why the CHSH inequalities are a sufficient condition for the positivity of the underlying probability matching the given marginals. A side-product of the investigation is a novel form of the CHSH inequalities.

In both of these proofs, we assumed that one may set the average spins \( B_i \) to zero. We argued that this is easily achieved in the quantum case, but we have not proved it in general. This will be addressed in future publications.

Finally, we briefly mention a natural but unsuccessful attempt to prove Fine’s theorem, using a maximum entropy approach [20]. As stated, the local hidden variables model used here was essentially a guess as to the form of the probability solving the matching problem.
Once in the realm of guessing, it seems reasonable to ask what sort of form for the probability might be the least-biased guess. The maximum entropy method answers this question. The idea is to find the probability which extremizes the entropy

$$S = - \sum_{s_1 s_2 s_3 s_4} p(s_1, s_2, s_3, s_4) \ln p(s_1, s_2, s_3, s_4)$$

subject to the constraints the probability is normalized and the four correlation functions Eq.(1.5) are fixed. We assume the average spins are zero. The extremization problem is easily solved, with solution

$$p(s_1, s_2, s_3, s_4) = N \exp (\lambda_1 s_1 s_3 + \lambda_2 s_1 s_4 + \lambda_3 s_2 s_3 + \lambda_4 s_2 s_4)$$

where $N$ and the four Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are to be determined using the normalization condition and expression for the correlation functions Eq.(1.5). In effect the maximum entropy method provides a particular ansatz for the solution to the problem. However, it falls short of solving the problem. At some length, one can show that the algebraic equations for $N$ and the Lagrange multipliers can be solved for sufficiently small $C_{ij}$, but they cannot be solved for the full range of values of the $C_{ij}$ satisfying the CHSH inequalities. For example, when the four $C_{ij}$ are close to 1, i.e. close to equality in the CHSH inequalities, there is no solution.

This shows that not every reasonable guess leads to a solution to the problem. Although it may be that a modified version of this problem, perhaps with more quantities fixed, may yield a solution.

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