Population fragmentation and party dynamics in an evolutionary political game

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We examine kinetic symmetry breaking phenomena in an evolutionary political game in which voters, inhabiting a multidimensional ideological space, cast ballots via selection mechanisms subject to the competing social forces of conformity and dissent. Our understanding of the spatiotemporally complex population dynamics is informed by a system of nonlinear replicator equations, discrete deterministic cousin of the original stochastic Seceder Model.

Fragmentation, dispersal & coalescence are crucial processes in a multitude of highly correlated, dynamically complex many-body systems, particularly those far from equilibrium where stochasticity, symmetry, and self-organization can conspire to generate rich pattern formation phenomena [1]. Natural examples abound- the rings of Saturn, its Cassini and lesser divisions, consist of a gravitationally bound myriad of ever-colliding rocky chunks, macroscopically stable within an intricately detailed set of circular bands [2]. Alternatively, consider the coat of the leopard, zebra, or giraffe- each a glorious product of competing reaction-diffusion biochemistry [3]. In these instances, the dynamical forces are physicochemical in nature, but movement can be inspired by equally compelling, though fundamentally different mechanisms, especially in biological, economic and sociological contexts. Visual/verbal cues (i.e., information propagation) within a school of fish [4] or flock of birds [5] lead to dynamic domain formation & destruction. Volatile populations of informed, interacting stock market traders can condense, exhibiting herd-like behavior [6]. Rich stochastic dynamics, as well as phase transition phenomena, are evident in various evolutionary minority (e.g., El Farol Bar) [7], public-goods [8], and other societal selection games, such as the Seceder Model [9,10], which introduces a novel dynamical frustration via competing tendencies of being distinct, yet part of the group. Here, we consider the seceder mechanism within an evolving political populace, uncovering an interwoven set of strange attractors whose reign is governed by kinetic symmetry-breaking within an ideological plane. We find that the conflicting tendencies toward conformity and dissent yield rich spatiotemporal party population dynamics. The seceder Catch-22 is that dissenting parties provide alternatives, but frequently grow in popularity themselves, inevitably suffering rebellion from within. Clearly, these simple models are not intended to capture all important features; the goal, however, is to gain appreciation of robust, universal aspects, as was done, e.g., for Ising systems in the case of ferromagnetic, liquid-gas, and binary alloy critical phenomena [11], or quadratic maps & Lotka-Volterra coupled ODEs, in the matter of chaotic dynamics [12].

For ease of presentation, we discuss the illustrative case of 3-party dynamics driven by a triplet-selection mechanism; i.e., voters poll three individuals to inform their political decision. We are lead, initially, to study the fixed points (FPs) associated with the following set of symmetry-broken replicator equations [9,10,13]:

\[ R = R^3 + 3R(G^2 + B^2) + \alpha RGB - R \]
\[ G = G^3 + 3G(R^2 + B^2) + \beta RGB - G \]
\[ B = B^3 + 3B(R^2 + G^2) + \gamma RGB - B \]

where \( R, G, B \leq 1 \) represent the relative populations of the three parties, Red, Green & Black, while probability conservation demands \( \alpha + \beta + \gamma = 6 \). The rate variables are dictated by the competing tendencies of, resp., homogeneity, distinction, and the democratic dilemma, equal representation of the three parties. More specifically, the purely cubic term results when the triplet selection group has uniform representation of one party; the decision maker switches to (or remains in, as the case may be...) that party. In the event that the polled group has an unequal distribution; i.e., a minority party, outnumbered 2 to 1, the underdog triumphs- thus, the seceder linear-quadratic terms. Finally, the trilinear contribution arises when the opinion group has one individual from each of the three parties. We consider successive levels of symmetry breaking- first, if \( \alpha = \beta \), we have effectively, an isosceles geometry, \( R \& G \) are equivalent in their relation to \( B \) and we find the following population FPs: \( (R,G,B) = (\frac{3}{2}, \frac{1}{2}, 0) \& (\frac{5}{8}, \frac{5}{8}, \frac{1}{4}) \). For \( \alpha < 4 \), the 3-party solution is stable, 2-party dynamics unstable; for \( \alpha \geq 4 \) vice versa. Single party domination, e.g., \((0,0,1)\), is never stable. Clearly, \( \alpha = 2 \) would represent a fully symmetrized situation with the three groups equidistant from each other, an equilateral (EQ) arrangement with evenly distributed populations \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \). By contrast, \( \alpha = 3 \) corresponds to the standard Seceder Model in one dimension [9], with the odd party, \( B \), poised at the midpoint of the line segment connecting ideologically opposed groups, \( R \& G \), at the ends; in this case, the stable population FP is the familiar \((\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\). In fact, any short isosceles (SI) triangle arrangement of the parties, with sides \( RG > RB = GB \), is drawn to a chaotic orbit about that fixed point; see later. Alternatively, the geometric arrangement could be tall isosceles (TI), with \( RG < RB = GB \), so that there is effectively a remote party equidistant from the others; here, connection to the stochastic Seceder Model suggests \( \alpha = 0 \); the dynamical
flows are drawn to the attractor \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). We reveal, in Figure 1, the full phase-diagram of this nonlinear replicator system in the \(\alpha\beta\)-plane, relaxing the isosceles geometric constraint. Surprisingly, a stable 3-party dynamic persists for quite a range of parameter values, occupying the triangular region bounded by the lines \(\alpha = -2, \beta = -2, \alpha + \beta = 8\) (i.e., \(\gamma = -2\)). The unique, full symmetry EQ FP, \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\), shown as a red diamond at the point \(\alpha = \beta = \gamma = 2\), is surrounded by a trio of permutation-related TI (SI) FPs, indicated by blue circles (green squares). Oddly enough, 3-party politics exists even for modestly negative values of \(\alpha, \beta, \gamma\); i.e., explicit biasing against a particular party. For sufficiently great forcing, however, the dynamics switches from 3 to 2-party politics; e.g., with \(\gamma < -2\), all flows are driven to the same \((R, G, B) = (\frac{1}{3}, \frac{1}{3}, 0)\) FP, with a 50-50 split in the population between the two surviving parties. Along special lines of symmetry, such as \(\alpha = \beta < -2\), there is, numerically, a first-order coexistence between competing relevant 2-party FPs; in the mentioned instance, \((0, 1, \frac{1}{2})\) and \((\frac{1}{2}, 0, \frac{1}{2})\). All this is in strong contrast to the 3-party region of the phase diagram, where there is a unique FP associated with each choice of \(\alpha\beta\gamma\) values.

Having investigated the role of symmetry-breaking within the context of the well-mixed, deterministic replicator equations, we now explore its manifestation for a related spatially-extended, stochastic model in which, one might imagine, voters occupy specific positions in a 2D ideological plane, with axes corresponding to a pair of compelling political issues, e.g., taxation and international cooperation, running the gamut from extreme leftist to ultra-right wing conservative positions. We’ll see shortly that for this stochastic Seceder Model, kinetic symmetry breaking (KSB) arises in an entirely intrinsic manner via fluctuations built-up in this far-from-equilibrium system. We make long-time runs, \(t=5\times10^5\) generations, with substantial population sizes, \(N=10^4\), running simulations in 2d with the entire population initialized in a single party at the origin. As discovered previously [10], the population rapidly fragments and disperses but within a thousand generations coalesces into three divergent groups, heading radially away from the origin at a slightly sublinear speed [14] with equally-sized party populations. In fact, if one carefully monitors party membership over time, there are impressive fluctuations about (as well as away from!) the neighborhood of this dynamical \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) EQ FP. In Figure 2a, we show the entire time series for the three parties- here indicated by the colors red, green and black. We see immediately that for much of \(t < 1\times10^5\), the average party population is indeed \(N/3\) but in due course, one of the parties, in this case the Greens \(@t\approx78000\), suffers a severe drop in membership, suddenly falling to 200; simultaneously, the Red and Black Parties rise, fluctuating about a mean of 400. In Figure 2b, we show an expanded detail of the subsequent, especially active epoch. Apparently, the political system has dynamically migrated to a chaotic stay about the \((\frac{1}{2}, 0, \frac{1}{2})\) SI FP, where two parties, Red & Black, equally dominate the third, Green. This state of affairs persists until \(t\approx91000\), when a second abrupt dynamical transition occurs- this time to the \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\) TI FP, in which there is a single popular party, here the Reds, with twice the membership of two equal minority parties, Black and Green. Finally, just beyond \(t=100000\), the system reverts to the original, unique EQ
FP. Overlays to Figure 2b reveal relevant geometric aspects, including evolution of various pairwise separations between parties (blue), as well as radial distances from the origin (party colors). We notice, for example, when the Green Party membership plummets, the party actually suffers a stochastic stall (green trace plateaus...), its radial motion ceasing as population redistribution takes place. The SI→TI transition is always marked by the simultaneous stall of all three parties (red, green, and black traces all horizontal), while the triangular geometry switches to one short and two (nearly) equal long sides. The return the EQ FP for \(t>100000\) is, of course, temporary- see, again, Figure 2a. Note, in particular, the rather long run, \(t=2.5-3.9\times10^5\), during which the political dynamic is controlled by the \((\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\) SI FP, though even that impressive interlude is punctuated by intermittent, though typically short-lived (\(\approx 1-2000\) generations) TI episodes, three of which are clearly visible in the figure.

It is informative to study the time evolution of the party populations from the vantage point of a Poincare type section; we show in Figure 3 the fractional populations of the Black and Red Parties for the entirety of the original simulation. For the time interval \(t=2.0-4.0\times10^5\), an epoch strongly dominated, recall Figure 2a, by the \((R, G, B) = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\) FP, indicated by leftmost green square, a symbol shared by its SI FP permuted cousins. As previously, the TI FPs are shown by blue circles, and the solitary, but evenly distributed, EQ FP is the red diamond. The time interval was chosen specifically to show governing role of a particular SI FP, which gives rise to a compelling crescent-shaped feature in the Poincare trace. Note, especially, the position of supplementary blue TI FPs in the wings of the crescent, which faithfully track the relatively less frequent excursions of the system to states in which a single majority party strongly dominates the political dynamics; here, Red or Green Parties with memberships in the neighborhood of 500- in such instances, the Black Party moves up momentarily to 250. It is apparent that the interval \(t=2.0-2.4\times10^5\) is responsible for the faint black cloud centered about the EQ FP, complementary to the solid crescent. For the given interval, the system spends little time with either the Green or Red Parties in the weak minority, so those sections of the Poincare plot are rather sparse, but in the infinite time limit, or even if one considers the full 1/2 million generations implicit in Figure 2a, the Poincare section (black and gray dots taken together) has an easily perceived elliptical form resulting from the superposition of three symmetrically permuted cusps pinned at each of the green square, SI FPs. In this way, wings overlap, where the crescents share a common, blue circle TI FP. Of course, this ellipse (or ellipsoid, in the full 3d plot with \((R,G,B)\) on the three axes...), does not have a sharp boundary, though the fall-off is rather rapid. The soft asymptotic ellipse, as well as the short-timescale crescent constitute the signature details of the Seceder Model Poincare section.

We can decipher these matters a bit by studying the party membership PDFs, obtained by isolating intervals of time in which a single FP dominates the dynamics. In Figure 4, the population PDFs associated with the EQ, SI, TI attractors are shown, resp., in red, green, and blue. We construct the red \((\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\) EQ PDF by sampling over some 200000 noncontiguous generations, primarily \(t<80000\) and \(t>420000\) and find three overlapped, essentially identical, asymmetric, nongaussian, skewed distributions, peaked near \(N/3=333\), low slung with very substantial widths. By contrast, focussing on the 150000 generation interval commencing at \(t=240000\), where \((\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\) FP reigns, we uncover the membership PDF associated with the green SI FP. Clearly a different beast altogether, it is bimodal, with 2 superposed, short, but broad components (mean@400), long tails on the low side- still substantial even at 250, and a single sharp, nearly symmetric peak centered at 200, considerably narrower, with rather small probability to rise above 250 or fall below 150. These features, esp. the latter, are evident in the crescent boundaries of Figure 3. Finally, the TI PDF, in blue, also bimodal, is characterized by a very narrow peak for the solitary majority party @500,
and two identical overlapped broader contributions with mean at 250 for the equal pair of minority parties. The nature of the various tails, as well as the mixed population size, go far in explaining why certain dynamical transitions, such as EQ→SI or SI→TI, are considerably more common than others; e.g., the system rarely tunnels EQ→TI. We seek a deeper understanding of the transition probabilities associated with these important global shifts in the population dynamics and have taken a step in this direction, studying the decay modes and lifetime distribution of an initial EQ state—see Figure 5. We discover that the branching ratio is highly biased in favor of the EQ→SI decay route; indeed, 97.6%(2.4%) of the time the transition is to the SI(TI) attractor.

Interestingly, the essential aspects of KSB can be analyzed via the replicator equations; in particular, we consider the case of d + 1 symmetrically arrayed parties in d dimensional space (i.e. 2D equilateral triangle, 3D tetrahedron, etc.) and examine what happens if a single party, with fractional membership z, let’s say, kinetically breaks the symmetry, moving closer/farther from the remaining parties, each of whom possess equal membership x = (1 − z)/d. For this characteristic situation, the rate equation for the anomalous party reads: ̇z = z^3 + 3dz(1 + δ(d − 1))z^2 − z, where the symmetry-breaking parameter δ = 0, 1/2, 1, in SI, EQ, TI geometries, with associated fixed point values z* = 3/(3 + 2δ), 1/(1 + δ), 1/2. The δ = 1/2 result is expected, the total population distributed evenly amongst the d + 1 groups. If, however, the anomalous party distances itself from the rest moving farther away, this subpopulation grows to a dimension-independent percentage of 50%, the remaining parties sharing equally the other half of the population; of course, for the 2D case we’ve focussed on thus far, this corresponds to the TI FP ((1/4, 1/4, 1/2)). Although fluctuations can permit higher fractions for this dominant party—our earlier simulations indicated, see Figures 2&4, upwards of 3/5, there are apparently dynamical constraints that strongly limit the maximum membership of this favored party. Should the party in power become too fashionable, the seceder mechanism inceases the likelihood of desertion from within. Finally, in the situation where the stochasticity draws the errant party closer to the others, a stable SI FP exists only stable below the upper critical dimension d = 3! Consequently, fluctuations in d ≥ 3 bring about the successive demise of all but three of the remaining parties, the system suffering cascadel collapse & dimensional reduction to a 2D ideological hyperplane [10].

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[1] for an inspired, if brutally idiosyncratic, entirely algorithmic perspective, see S. Wolfram, *A New Kind of Science* (Wolfram Media, Champaign IL 2002).
[2] J. Schmidt et al., Phys. Rev. Lett. 90, 061102 (2003).
[3] J. D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin 1993).
[4] J. Parrish et al., Science 284, 99 (1999).
[5] J. Toner and Y. Tu, Phys. Rev. Lett. 75, 4326 (1995); E.V. Albano, ibid, 77, 2129 (1996).
[6] V. Eguíluz et al., Phys. Rev. Lett. 85, 5659 (2000).
[7] D. Challet and Y.-C. Zhang, Physica A 246, 407 (1997); N.F. Johnson et al., Phys. Rev. Lett. 82, 3360 (1999); S. Hod and E. Nakar, ibid, 88, 238702 (2002).
[8] G. Szabó et al., Phys. Rev. Lett. 84, 118101 (2002); C. Hauert et al., Science 296, 1129 (2002).
[9] P. Dittrich, et al., Phys. Rev. Lett. 84, 3205 (2000).
[10] A. Soulier and T. Halpin-Healy, cond-mat/0209451, Phys. Rev. Lett. 90, in press.
[11] K. Huang, *Statistical Mechanics,* (J. Wiley, NY 1987); C. Domb, *The Critical Point,* (Taylor & Francis, NY 1996).
[12] J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos,* (J. Wiley, NY 2002).
[13] J. Hofbauer and K. Sigmund, *Evolutionary Games & Population Dynamics* (Cambridge Press, 1998).
[14] A. Soulier, N. Arkus, and T. Halpin-Healy; submitted.