On the construction of global models
describing isolated rotating charged bodies;
uniqueness of the exterior gravitational field

Raúl Vera
Dublin City University, Ireland.

A relatively recent study by Mars and Senovilla provided us with a uniqueness result for the exterior vacuum gravitational field of global models describing finite isolated rotating bodies in equilibrium in General Relativity (GR). The generalisation to exterior electrovacuum gravitational fields, to include charged rotating objects, is presented here.

1 Introduction

The description of astrophysical self-gravitating isolated, rotating bodies in equilibrium in GR is still poorly understood. To describe such bodies we would require global models for finite isolated rotating objects in equilibrium, but we still lack complete non-spherically symmetric models for a finite body together with its exterior.

The global models one is interested in consist of stationary spacetimes, to account for the equilibrium state, being also asymptotically flat, to account for the isolation of the body. Furthermore, it has been always argued that for the model to be in a final equilibrium state, the spacetime has to be axisymmetric. Now, the framework used here lies on the construction of such global models by means of the matching of spacetimes: one spacetime \((V_I, g_I)\) describing the interior of the body and another describing the exterior \((V_E, g_E)\), matched across a hypersurface that describes the surface of the body at all times. The global model is then theoretically attacked by solving the corresponding Einstein field equations at both the interior and the exterior taking the common boundary data at the matching (timelike) hypersurface \(\sigma\), which is assumed to preserve the symmetry \([1]\). Regarding the vacuum exterior problem, uniqueness for a stationary axially symmetric asymptotically flat vacuum solution with boundary conditions at \(\sigma\) coming from the matching with an interior region was solved in \([2]\). The aim of this work has been to generalize the uniqueness results for the vacuum exterior to exteriors containing electromagnetic fields without sources, i.e. electrovacuum solutions of the Einstein-Maxwell equations, in order to describe the exterior gravitational field of finite (and spatially simply-connected) charged bodies. As a natural further assumption, the exterior electromagnetic (e-m) field is supposed to be also stationary and axisymmetric.

In the following, \(*\) denotes the Hodge dual, and \(\wedge\) the exterior product.
2 The electrovac exterior; the exterior problem

Let us consider a strictly stationary (free of ergoregions and/or Killing horizons) and axisymmetric electrovacuum spacetime \((V_E, g_E)\) (containing a stationary and axisymmetric e-m field). There exists a coordinate system \(\{t, \phi, \rho, z\}\), the so-called Weyl coordinates, in which the corresponding line-element reads globally

\[
ds^2_E = -e^{2U} (dt + A d\phi)^2 + e^{-2U} \left[ e^{2k} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right],
\]

where \(U, A, k\) are functions of \(\rho\) and \(z\). The axial Killing vector is given by \(\vec{\eta} = \partial_\phi\) and the axis of symmetry is located at \(\rho = 0\). The coordinate \(t\) can be chosen to measure proper time of an observer at infinity, and hence the Killing vector \(\vec{\xi} = \partial_t\) is unit at infinity. With this choice, the remaining coordinate freedom in (1) consists only of constant shifts of \(t, \phi\) and \(z\).

Denoting by \(F_{\alpha\beta}\) the e-m tensor and by \(\kappa\) a (non-null) Killing vector (will be either \(\vec{\xi}\) or \(\vec{\eta}\)), since \(\mathcal{L}(\kappa)F_{\alpha\beta} = 0\) (stationary and axisymmetric e-m field) there exist two scalar e-m potentials \(E_\kappa, B_\kappa\) with respect to \(\kappa\) such that the corresponding e-m fields \(\vec{E}(\kappa)\) and \(\vec{B}(\kappa)\) are given by \(E(\kappa)_{\alpha\beta} = \partial_\alpha E_\kappa = B(\kappa)_{\alpha\beta} = \partial_\alpha B_\kappa\). For convenience, one defines \(\Lambda_\kappa \equiv E_\kappa + i B_\kappa\), which, given \(\kappa\), contains all the information of the e-m field.

It is well known (see e.g. [3]) that the Einstein-Maxwell equations imply the existence of the so-called Ernst (complex) potential with respect to \(\kappa\)

\[
E_\kappa = -N_\kappa - \Lambda_\kappa \bar{\Lambda}_\kappa + i \Omega_\kappa,  \tag{2}
\]

where \(N_\kappa \equiv \kappa_\alpha \kappa^\alpha\), the bar indicates the complex conjugate, and the real potential \(\Omega_\kappa\) is such that

\[
d\Omega_\kappa = w_\kappa - i \left( \Lambda_\kappa d\bar{\Lambda}_\kappa - \bar{\Lambda}_\kappa d\Lambda_\kappa \right), \tag{3}
\]

where \(w_\kappa \equiv * (\kappa \wedge d\kappa)\) is the twist 1-form of \(\kappa\). The Einstein-Maxwell equations reduce then to an elliptic system of equations for \((E, \Lambda)\) known as the Ernst equations, plus a quadrature in terms of \(E\) and \(\Lambda\) for the function \(k\). Dropping the \(\kappa\) indices, the Ernst equations read

\[
N \delta^{ij} \partial_i (\rho \partial_j E) + \rho \delta^{ij} \partial_i E \left( \partial_j E + 2 \bar{\Lambda} \partial_j \Lambda \right) = 0, \tag{4}
\]
\[
N \delta^{ij} \partial_i (\rho \partial_j \Lambda) + \rho \delta^{ij} \partial_i \Lambda \left( \partial_j E + 2 \bar{\Lambda} \partial_j \Lambda \right) = 0, \tag{5}
\]

where \(N = -(E + \bar{E} + 2A\Lambda)/2\), \(i, j : \{\rho, z\}\), and \(\delta_{ij}\) is the \(2 \times 2\) identity. Given \((E, \Lambda)\) solution of the Ernst equations, all the information of the exterior electrovacuum solution is recovered. Choosing the form \([4]\), apart from \(k\), the metric for \((V_E, g_E)\) can be obtained by taking \(\kappa = \vec{\xi} = \partial_t\) so that \(U\) is obtained from \(N_\xi = -e^{2U}\), and \(A\) is determined, up to a constant, by the quadrature \(dA = -\rho N_\xi^{-2} * (h) w_\xi\), where \(* (h) d\rho = dz, * (h) dz = -d\rho\) and \(w_\xi\) has been obtained from [4].
3 The boundary conditions on $\sigma$

The boundary for the exterior problem consists, in principle, of the boundary associated to the surface of the body ($\sigma$, whose spacelike cuts are topologically spheres) plus the surface at infinity, where the flat asymptoticity assumed on the exterior ($V_E, g_E$) determines the behaviour of the potentials ($\mathcal{E}, \Lambda$) there. This section is devoted to the boundary conditions on the matching hypersurface $\sigma$ that result from the matching conditions with a given stationary and axisymmetric interior.

Given an interior, it has been proven [2, 4] that the matching conditions fix $\sigma$ in the general case, together with the values of the metric functions $U$ and $A$ and their normal derivatives on $\sigma$:

$$U|_\sigma, \quad \vec{n}(U)|_\sigma, \quad A|_\sigma, \quad \vec{n}(A)|_\sigma, \quad (6)$$

where $\vec{n}$ denotes the normal to $\sigma$. On the other hand, the continuity of $F_{\alpha\beta}$ fixes

$$d\Lambda|_\sigma, \quad (7)$$

for either $\vec{k}$, so that in particular, $\Lambda|_\sigma$ is fixed up to an additive arbitrary complex constant $\lambda$.

It is now straightforward to show that the data given by (6) translates onto data for $\mathcal{E}$ as follows. Using (6), and taking into account $\lambda$ as given above, the matching conditions fix $\Omega|_\sigma$ and $d\Omega|_\sigma$ up to transformations of the form

$$\Omega|_\sigma \to \Omega|_\sigma + c_\Omega - i(\lambda\Lambda - \Lambda\lambda)|_\sigma, \quad \text{and} \quad d\Omega|_\sigma \to d\Omega|_\sigma - i(\lambda d\Lambda - \Lambda d\lambda)|_\sigma, \quad (8)$$

where $c_\Omega$ is an arbitrary real constant. From (6), given an interior, the matching conditions fix

$$N|_\sigma, \quad \vec{n}(N)|_\sigma. \quad (9)$$

The data for $\mathcal{E}|_\sigma$ and $d\mathcal{E}|_\sigma$ can be obtained now from (8) and (9).

Notice that due to the elliptic character of the Ernst equations system for $(\mathcal{E}, \Lambda)$, the data coming from the matching conditions (Cauchy data) overdetermines the problem. Nevertheless, there is still three degrees of freedom driven by $\lambda$ and $c_\Omega$ on that data. The questions to address at this point are then: (a) uniqueness of the exterior solution $(\mathcal{E}, \Lambda)$ given Dirichlet data on $\sigma$, i.e. $\mathcal{E}|_\sigma, \Lambda|_\sigma$, and (b) if a solution $(\mathcal{E}, \Lambda)$ exists, are $\lambda$ and $c_\Omega$ determined, and thus the Dirichlet data? Of course, the affirmative answer to both questions solves the issue of the uniqueness of the exterior field.

---

1The additive constants in $A$ and $k$ are then determined, see [4]. It has been assumed that the identification of the exterior and interior across $\sigma$ has been prescribed, in order to fix two extra degrees of freedom introduced by the matching procedure, see [2, 4].

2It is assumed that no surface charges are present on the surface of the body.
Uniqueness given Dirichlet data on $\sigma$

The proof, presented here schematically, follows closely those used in the uniqueness theorems of black holes (see [6]), making use of the very rich intrinsic structure of the Ernst equations. In what follows, the $E$ suffix (for 'exterior') and the $\kappa$ indices will be omitted for simplicity. Equations (4)-(5) can be interpreted as the Euler-Lagrange equations for the action [6]

$$ S = 4 \int \left( \frac{1}{4N^2} |d\xi|^2 + 2\Lambda d\Lambda |^2 + \frac{1}{N} |d\Lambda|^2 \right) \rho \rho d\rho dz, $$

where $|\theta|^2 \equiv g^{\alpha\beta} \theta_\alpha \bar{\theta}_\beta$. Taking $\Phi$ to be a hermitian $su(2,1)$ matrix defined by $(a, b : 1, 2, 3)$

$$ \Phi_{ab} \equiv \eta_{ab} + 2 \text{sign}(N) \bar{v}_a v_b $$

where $\eta_{ab} = \text{diag}(-1, 1, 1)$ and $v_a = (2\sqrt{|N|})^{-1} (\xi - 1, \xi + 1, 2\Lambda)$, one can define a $su(2,1)$-valued 1-from by

$$ J \equiv \Phi^{-1} \cdot d\Phi. $$

The pairs of solutions $(\xi, \Lambda)$ have been now translated onto $\Phi$. In terms of $J$, the above action is rewritten as [6]

$$ S = \int \frac{1}{2} \eta_{\alpha\beta} \text{Tr}(J^\alpha \cdot J^\beta) \rho \rho d\rho dz, $$

for which the variational equation reads [6]

$$ \nabla_\alpha J^\alpha = 0. \quad (10) $$

The key property for the proofs of the uniqueness theorems is the positivity of $\Phi$ (and the action), which is ensured by taking $\vec{\kappa} = \vec{\eta}$. Let us define $X \equiv N_\eta = e^{-2U} \rho^2 - e^{2U} A^2 (> 0)$, $Y \equiv \Omega_\eta$ and $A_\eta \equiv -A e^{2U} X^{-1}$. The line-element (1) can be cast as

$$ ds^2 = -X^{-1} \rho^2 dt^2 + X(d\phi + A_\eta dt)^2 + X^{-1} e^{2k}(d\rho^2 + dz^2). $$

The problem in this choice is that $N$, i.e. $X$, vanishes on the axis, and therefore a careful analysis there is needed (see [6]). It is convenient to change to prolate spheroidal coordinates $\{x, y\}$, $x > 1, |y| \leq 1$ by $\rho^2 = \nu^2 (x^2 - 1)(1 - y^2)$, $z = \nu xy$ ($\nu > 0$) so that $d\rho^2 + dz^2 = \nu^2 (x^2 - y^2) ds^2$ where $ds^2 = \frac{1}{x^2 - 1} dx^2 + \frac{1}{1 - y^2} dy^2 \equiv h_{ij} dx^i dx^j$. The boundaries to the domain $D$ for the solutions are now given as follows: (i) infinity, located at $x = \infty$, (ii) the two parts of the axis at $y = \pm 1$, (iii) the boundary corresponding to $\sigma$, a curve $\Sigma$ with $x > 1$ joining $y = 1$ with $y = -1$.

The core of the proof consists in considering two sets of solutions $\Phi^{(1)}$ and $\Phi^{(2)}$, with corresponding pairs $J^{(1)}, J^{(2)}$, $X^{(1)}, X^{(2)}$, etc, with common Dirichlet data
on \( \Sigma \). Defining \( \mathcal{I} \equiv f_1 - f_2 \) and \( \Psi \equiv \Phi_1 \cdot \Phi_2^{-1} - \mathbb{1} \), where \( \mathbb{1} \) is the identity, and taking into account that \( \mathcal{J} \) has only components in \( \rho, z \) \((i, j)\), the Mazur identity follows (see [5, 6]):

\[
(\rho \text{Tr} \Psi^{ij})_{;i} = \rho h_{ij} \text{Tr}\{\mathcal{J}_i^{(j)} \cdot \Psi^{(2)} \cdot \mathcal{J}^j \cdot \Psi^{(1)}\}.
\]

If \( \mathcal{J} \neq 0 \), and since \( \rho, h_{ij}, \Phi \) are positive definite, then \( (\rho \text{Tr} \Psi^{ij})_{;i} > 0 \).

The point here lies on showing that the data on the boundary of our domain, \( \partial D \), implies that the integral over \( \partial D = (i) + (ii) + \Sigma \) of \( \rho \text{Tr} \Psi^{ij} dS_i \) vanish, and use then the Stokes theorem. Explicitly, defining \( \bar{\mathcal{I}} \equiv f_1 + f_2 \), one has [6]

\[
\text{Tr} \Psi = \frac{1}{X(1)X(2)} \left[ X^2 + 2X| \mathcal{A} |^2 + | \mathcal{A} |^4 + \left[Y + \text{Im}(\mathcal{A}^\ddagger)\right]^2 \right].
\]

(11)

Following the black hole theorems, taking the values of \( X, Y \) and \( \Lambda \) on the axis and the limits at infinity as were computed by Carter (see [5, 6]), expression (11) leads to the vanishing of \( \rho \text{Tr} \Phi^{ij} \) on (i) and (ii). Now, since the numerator of \( \text{Tr} \Psi \) is quadratic in \( \mathcal{A} \) etc, and by assumption \( \Phi|\Sigma = 0 \), one infers \( \text{Tr} \Psi|_{\Sigma} = 0 \).

Therefore \( \mathcal{J} = 0 \). Since by construction \( \rho \Psi^{ij} = \Phi_2 \cdot \mathcal{J}^{ij} \cdot \Phi_1^{-1} \), \( \Psi \) is constant all over \( D \), and thus \( \Psi = 0 \) because \( \Psi|\Sigma = 0 \) by assumption. This ends the proof showing that given Dirichlet data on \( \sigma \), \( (\mathcal{E}_\eta|\sigma, \Lambda_\eta|\sigma) \), the solution \( (\mathcal{E}_\eta, \Lambda_\eta) \) of the Ernst equations in the exterior region \( (\mathcal{V}_E, g_E) \) is unique. Finally, the correspondence between conjugate solutions \( (\mathcal{E}_\eta, \Lambda_\eta) \) and \( (\mathcal{E}_\xi, \Lambda_\xi) \) (see [2]) leads to the same result for \( \bar{k} = \xi \).

## 5 Fixing the Dirichlet data

The purpose now is to show how the full set of Cauchy data, i.e. taking into account \( (\bar{n}(\mathcal{E})|\sigma, \bar{n}(\Lambda)|\sigma) \), fixes the values of \( c_\Omega, \lambda \), provided that the solution exists.

Following [2], the proof makes use of the divergence free fields \( \mathcal{J} \) choosing \( \bar{k} = \xi \). Decomposing \( \mathcal{J} \) on a basis of \( su(2,1) \), 8 conserved 1-forms are obtained, which are used to define a divergence free real 1-form in \( (\mathcal{V}_E, g_E) \) depending on 8 real constants \( W(\Lambda, \mathcal{E}; c_e, c_k, c_d, c_{h_1}, c_{h_2}, c_{a_1}, c_{a_2}, c_3) \equiv c_e \mathcal{J}^e + c_k \mathcal{J}^k + c_d \mathcal{J}^d + c_{h_1} \mathcal{J}^{h_1} + c_{h_2} \mathcal{J}^{h_2} + c_{a_1} \mathcal{J}^{a_1} + c_{a_2} \mathcal{J}^{a_2} + c_3 \mathcal{J}^3 \), where all \( \mathcal{J} = \mathcal{J}(\mathcal{E}, \Lambda) \) and such that the only non-vanishing surface integrals at infinity are \( \int_{S_\infty} \mathcal{J}_\alpha^d dS^\alpha = 8\pi M \), \( \int_{S_\infty} \mathcal{J}_\alpha^h dS^\alpha = -4\pi q \) and \( \int_{S_\infty} \mathcal{J}_\alpha^a dS^\alpha = -4\pi q \), where \( M(\mathcal{E}, \Lambda) \) and the complex \( q(\mathcal{E}, \Lambda) \) relative to a given solution \( (\mathcal{E}, \Lambda) \) correspond to the mass and the e-m charge of the configuration, respectively. As a consequence of \( W^{\alpha}_{;\alpha} = 0 \), we have

\[
\int_{\Sigma} W_\alpha dS^\alpha = \int_{S_\infty} W_\alpha dS^\alpha = 8\pi M c_d - 4\pi [\bar{q}(c_{h_1} + c_{a_1}) + \bar{q}(c_{h_2} + c_{a_2})].
\]

(12)
Now, we assume two exterior solutions for the same interior exist \((\mathcal{E}_1, \Lambda_1), (\mathcal{E}_2, \Lambda_2)\) such that their Cauchy boundary data differs by \(\lambda\) and \(c_\Omega\), i.e. \(\Lambda_1|_\sigma = \Lambda_2|_\sigma + \lambda\), \(d\Lambda_1|_\sigma = d\Lambda_2|_\sigma\), \(N_1|_\sigma = N_2|_\sigma\), \(dN_1|_\sigma = dN_2|_\sigma\), \(\Omega_1|_\sigma = \Omega_2|_\sigma - c_\Omega + i(\bar{\lambda}\Lambda_2 - \lambda\bar{\Lambda}_2)|_\sigma\), \(d\Omega_1 = d\Omega_2|_\sigma + i(\bar{\lambda}d\Lambda_2 - \lambda d\bar{\Lambda}_2)|_\sigma\). From this, the following equality holds on \(\sigma\)

\[
W(\mathcal{E}_1, \Lambda_1; c_e, \ldots, c_s)|_\sigma = W(\mathcal{E}_2, \Lambda_2; \hat{c}_e, \ldots, \hat{c}_s)|_\sigma,
\]

for a set of 8 certain explicit relations

\[
\hat{c}_e = \hat{c}_e(c_e, \ldots, c_s; c_\Omega, \lambda), \quad \ldots, \quad \hat{c}_s = \hat{c}_s(c_e, \ldots, c_s; c_\Omega, \lambda).
\]

The integration of (12) over \(\sigma\) using (12) leads to

\[
8\pi M(1)c_d - 4\pi \left[q(1)(c_{h_1} + c_{a_1}) + \bar{q}(1)(c_{h_2} + c_{a_2})\right] = 8\pi M(2)\hat{c}_d - 4\pi \left[q(2)(\hat{c}_{h_1} + \hat{c}_{a_1}) + \bar{q}(2)(\hat{c}_{h_2} + \hat{c}_{a_2})\right],
\]

which has to hold for arbitrary choices of the 8 constants \(\{c_e, \ldots, c_s\}\). A straightforward calculation leads to the fact that

\[M(1) + M(2) \neq 0 \Rightarrow \lambda = c_\Omega = 0.\]

For physically well behaved solutions the total mass should be positive, and thus this result implies that both \(\lambda\) and \(c_\Omega\) must vanish, and hence the uniqueness of the exterior solution generated by a given interior distribution of matter in stationary and axially symmetric rotation follows.

I wish to thank Brien Nolan for reading this manuscript. This work was produced while I was in Queen Mary, University of London, and funded by the EPSRC grant GR/R53685/01. I also thank the IRCSET for grant PD/2002/108.

References

[1] R. Vera Class. Quantum Grav. 19, 5249 (2002)
[2] M. Mars, J.M.M. Senovilla Mod. Phys. Lett. A13, 1509 (1998)
[3] H. Stephani, D. Kramer, M.A.H. MacCallum, C. Hoenselaers, E. Herlt Exact solutions of Einstein’s field equations. Second Edition, Cambridge University Press 2003
[4] R. Vera Class. Quantum Grav. 20, 2785 (2003)
[5] B. Carter Commun. Math. Phys. 99, 563 (1985)
[6] M. Heusler Black Hole Uniqueness Theorems, Cambridge lecture notes in Physics, Cambridge University Press (1996)
[7] P. Breitenlohner, D. Maison Commun. Math. Phys. 209, 785 (2000)