The Curve Shortening Flow in the Metric-Affine Plane

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Abstract: We investigated, for the first time, the curve shortening flow in the metric-affine plane and prove that under simple geometric condition (when the curvature of initial curve dominates the torsion term) it shrinks a closed convex curve to a “round point” in finite time. This generalizes the classical result by M. Gage and R.S. Hamilton about convex curves in a Euclidean plane.

Keywords: curve shortening flow; affine connection; curvature; convex

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1. Introduction

The one-dimensional mean curvature flow is called the curve shortening flow (CSF), because it is the negative $L^2$-gradient flow of the length of the interface, and it is used in modeling the dynamics of melting solids. The CSF deals with a family of closed curves $\gamma$ in the plane $\mathbb{R}^2$ with a Euclidean metric $g = \langle \cdot, \cdot \rangle$ and the Levi-Civita connection $\nabla$, satisfying the initial value problem (with parabolic partial differential equation)

$$\frac{\partial \gamma}{\partial t} = kN, \quad \gamma|_{t=0} = \gamma_0. \quad (1)$$

Here, $k$ is the curvature of $\gamma$ with respect the unit inner normal vector $N$ and $\gamma_0$ is an embedded plane curve, see survey in [1,2]. The flow defined by (1) is invariant under translations and rotations. Recall that the curvature of a convex plane curve is positive. The next theorem by M. Gage and R.S. Hamilton [3] describes this flow of convex curves.

**Theorem 1.** (a) Under the CSF (1), a convex closed curve in the Euclidean plane smoothly shrinks to a point in finite time. (b) Rescaling in order to keep the length constant, the flow converges exponentially fast to a circle in $C^\infty$.

This theorem and further result by M.A. Grayson, [4] (that the flow moves any closed embedded in the Euclidean plane curve in a finite time to a convex curve) have many generalizations and applications in natural and computer sciences. For example, the anisotropic curvature-eikonal flow (ACEF) for closed convex curves in a Euclidean plane, see ([1] Section 3.4),

$$\frac{\partial \gamma}{\partial t} = (\Phi(\theta)k + \lambda \Psi(\theta))N, \quad \gamma|_{t=0} = \gamma_0, \quad (2)$$

where $\Phi > 0$ and $\Psi$ are $2\pi$-periodic functions of the normal to $\gamma(\cdot, t)$ angle $\theta$ and $\lambda \in \mathbb{R}$, generalizes the CSF. Anisotropy of the flow (2), studied when $\Psi$ is also positive, is indispensable in dealing with phase transition, crystal growth, frame propagation, chemical reaction, and mathematical biology. The particular case of ACEF, when $\Phi$ and $\Psi$ are positive constants, serves as a model for essential biological processes, see [5]. On the other hand, (2) is a particular case of the flow in a Euclidean plane $\mathbb{R}^2(x_1, x_2)$,

$$\frac{\partial \gamma}{\partial t} = F(\gamma, \theta, k)N, \quad \gamma|_{t=0} = \gamma_0.$$
where \( \Psi \) we have

\[
\text{where } \bar{\Sigma} \text{ is the contorsion tensor related to the spin tensor of matter.}
\]

In recent decades, many results have appeared in the differential geometry of a manifold with an affine connection \( \nabla \) (which is a method for transporting tangent vectors along curves), e.g., collective monographs \([6,7]\). The difference \( \Sigma = \nabla - \nabla \) (of \( \nabla \) and the Levi-Civita connection \( \nabla \) of \( g \)), is a \((1,2)\)-tensor, called contorsion tensor. Two interesting particular cases of \( \nabla \) (and \( \Sigma \)) are as follows.

1. **Metric compatible connection**: \( \nabla g = 0 \), i.e., \( \langle \Sigma(X,Y), Z \rangle = -\langle \Sigma(X,Z), Y \rangle \). Such manifolds appear in almost Hermitian and Finsler geometries and are central in Einstein-Cartan theory of gravity, where the contorsion tensor is related to the spin tensor of matter.

2. **Statistical connection**: \( \nabla \) is torsionless and the rank 3 tensor \( \nabla g \) is symmetric in all its entries, i.e., \( \langle \Sigma(X,Y), Z \rangle \) is fully symmetric. Statistical manifold structure, which is related to geometry of a pair of dual affine connections, is central in Information Geometry, see \([8]\); affine hypersurfaces in \( \mathbb{R}^{n+1} \) are a natural source of statistical manifolds.

There are no results about the CSF in metric-affine geometry. In this paper, the **metric-affine plane** is \( \mathbb{R}^2 \) (a two-dimensional real vector space) endowed with a Euclidean metric \( g \) and an affine connection \( \nabla \). Our objective is to study the CSF in the metric-affine plane and to generalize Theorem 1 for convex curves in \((\mathbb{R}^2, g, \nabla)\). Thus, we replace (1) by the following initial value problem:

\[
\frac{\partial \gamma}{\partial t} = \bar{k} N, \quad \gamma|_{t=0} = \gamma_0, \tag{3}
\]

where \( \bar{k} \) is the curvature of a curve \( \gamma \) with respect to \( \nabla \) and \( \gamma_0 \) is a closed convex curve. Such flow can be interesting as a geometrically natural analogue of the CSF, which could perhaps show some different behavior and it is not clear that it is a gradient flow for length and is variational in nature. Note that (3) is the particular case (when \( \Phi = 1 = \lambda \)) of the ACEF, where \( \Psi \) may change its sign. Put

\[
k_0 := \min\{k(x) : x \in \gamma_0\} > 0.
\]

Let \( \{e_1, e_2\} \) be the orthonormal frame in \((\mathbb{R}^2, g, \nabla)\). In the paper we assume the following

**Condition 1**: \( \Sigma \) has constant components \( \Sigma^k_{ij} = \langle \Sigma(e_i, e_j), e_k \rangle \) (that is the contorsion tensor \( \Sigma \) is \( \nabla \)-parallel) and constant norm \( \|\Sigma\| = c \geq 0 \).

Let \( \gamma : S^1 \to \mathbb{R}^2 \) be a closed curve in the metric-affine plane with the arclength parameter \( s \). Then \( T = \partial \gamma / \partial s \) is the unit vector tangent to \( \gamma \). In this case, \( k = \langle \nabla_T T, N \rangle \) and the curvature of \( \gamma \) with respect to an affine connection \( \nabla \) is \( \bar{k} = \langle \nabla_T T, N \rangle \), we obtain

\[
\bar{k} = k + \Psi, \tag{4}
\]

where \( \Psi \) is the following function on \( \gamma \):

\[
\Psi = \langle \Sigma(T,T), N \rangle. \tag{5}
\]

By assumptions \( \|\Sigma\| := \sup_{X,Y \in \mathbb{R}^2 \setminus \{0\}} \|\Sigma(X,Y)\| = c \), see **Condition 1**, and \( \|T\| = \|N\| = 1 \), we have

\[
|\Psi| \leq c. \tag{6}
\]

As far as we are aware such variations to the CSF have not been considered before. In this novel case, the new term defining the difference from the classical case is of a different order of homogeneity with respect to the curvature itself, so the extra term becomes weaker and weaker where the curvature is large, unlike the Finsler/anisotropic setting, where the anisotropic effects remain at all scales. The convergence of the ACEF (2) when \( \Phi \) and \( \Psi \) are positive has been studied in ([1] Chapter 3). However, our function \( \Psi \) in (4) takes both positive and negative values, and ([1] Theorem 3.23) is not
applicable to our flow of (3). By this reason, we independently develop the geometrical approach to prove the convergence of (3) to a “round point”.

The main result of the paper states that a “sufficiently convex” closed curve (i.e., with curvature bounded below by a positive constant chosen to bound the size of the extra term), the curve shrinks to a point under our new flow, and is asymptotic to a shrinking circle solution of the classical CSF.

The following theorem generalizes Theorem 1(a).

**Theorem 2.** Let $\gamma_0$ be a closed convex curve in the metric-affine plane with condition $k_0 > 2c$. Then (3) has a unique solution $\gamma(\cdot, t)$, and it exists at a finite time interval $[0, \omega)$, and as $t \uparrow \omega$, the solution $\gamma(\cdot, t)$ converges to a point. Moreover, if $k_0 > 3c$ then $\omega \leq \frac{A(\gamma_0)}{2\pi c} \cdot \frac{k_0 - 2c}{k_0 - c}$, where $A(\gamma_0)$ is the area enclosed by $\gamma_0$.

Nonetheless, the approach of [1] to the normalized flow of (2) in the contracting case still works without the positivity of $\Psi$, see ([1] Remark 3.14). Based on this result and Theorem 2, we obtain the following result, generalizing Theorem 1(b).

**Theorem 3.** Consider the normalized curves $\tilde{\gamma}(\cdot, t) = (2(\omega - t))^{1/2} \gamma(\cdot, t)$, see Theorem 2, and introduce a new time variable $\tau = -(1/2) \log(1 - \omega^{-1} t) \in [0, \infty)$. Then the curves $\tilde{\gamma}(\cdot, \tau)$ converge to the unit circle smoothly as $\tau \to \infty$.

In Section 2, we prove Theorem 2 in several steps, some of them generalize the steps in the proof of ([2] Theorem 1.3). In Section 3, we prove Theorem 3 about the normalized flow (3), following the proof of convergence of the normalized flow (2) in the contracting case.

Theorem 2 can be easily extended to the case of non-constant contorsion tensor $\text{T}$ of small norm, but we can not now reject the **Condition 1** for Theorem 3, since its proof is based on the result for the normalized ACEF, see [1], where $\Psi$ depends only on $\theta$.

### 2. Proof of Theorem 2

Recall the axioms of affine connections $\nabla : \mathfrak{X}_M \times \mathfrak{X}_M \to \mathfrak{X}_M$ on a manifold $M$, e.g., [7]:

$$\nabla_{fX_1 + X_2} Y = f \nabla_X Y + \nabla_X Y_2, \quad \nabla_X (fY_1 + Y_2) = f \nabla_X Y_1 + X(f) \cdot Y_1 + \nabla_X Y_2$$

for any vector fields $X, Y, X_1, X_2, Y_1, Y_2$ and smooth function $f$ on $M$.

Let $\theta$ be the normal angle for a convex closed curve $\gamma : S^1 \to \mathbb{R}^2$, i.e., $\cos \theta = -\langle N, e_1 \rangle$ and $\sin \theta = -\langle N, e_2 \rangle$. Hence,

$$N = -[\cos \theta, \sin \theta], \quad T = [-\sin \theta, \cos \theta]. \quad (7)$$

**Lemma 1.** The function $\Psi$ given in (5) has the following view in the coordinates:

$$\Psi = a_{30} \sin^3 \theta + a_{03} \cos^3 \theta + a_{12} \sin \theta + a_{21} \cos \theta, \quad (8)$$

where $a_{ij}$ are given by

$$a_{12} = \Sigma_{12}^2 + \Sigma_{12}^1 - \Sigma_{11}^1, \quad a_{21} = \Sigma_{12}^1 + \Sigma_{12}^2 - \Sigma_{22}^2, \quad a_{03} = \Sigma_{22}^2 - \Sigma_{12}^2 - \Sigma_{12}^1 - \Sigma_{21}^1, \quad a_{30} = \Sigma_{11}^1 - \Sigma_{11}^2 - \Sigma_{12}^1 - \Sigma_{22}^1. \quad (9)$$

**Proof.** Using (7), we find

$$\text{T}(T, T) = \left[\text{T}(e_1, e_1), \text{T}(e_2, e_2) \right] \sin^2 \theta - \left(\text{T}(e_1, e_2) + \text{T}(e_2, e_1) \right) \sin \theta \cos \theta + \left[\text{T}(e_2, e_2), \text{T}(e_2, e_2) \right] \cos^2 \theta,$$

$$\langle \text{T}(T, T), N \rangle = - \left(\text{T}(e_1, e_2), e_2 \right) \sin^3 \theta + \left(\text{T}(e_1, e_2), e_2 \right) + \left(\text{T}(e_2, e_1), e_2 \right) - \left(\text{T}(e_1, e_1), e_1 \right) \sin^2 \theta \cos \theta + \left(\text{T}(e_2, e_2), e_1 \right) + \left(\text{T}(e_2, e_1), e_1 \right) - \left(\text{T}(e_2, e_2), e_2 \right) \sin \theta \cos^2 \theta - \left(\text{T}(e_2, e_2), e_1 \right) \cos^2 \theta.$$
From this and the definition $\Sigma_{ij} = \sum_k \xi_{ij}^k e_k$ and $\Sigma_{ij}^\theta = \langle \Sigma(e_i, e_j), e_k \rangle$ the equalities (8) and (9) follow.

**Remark 1.** By equalities $T_\theta = N$, $N_\theta = -T$ and (5), we obtain the following:

$$|\Psi_{\theta\theta} + \Psi| = |2\langle\Sigma(N, N), N \rangle - 4\langle\Sigma(N, T), T \rangle - 3\langle\Sigma(T, T), N \rangle| \leq 9c.$$

**Example 1.** Recall the Frenet–Serret formulas (with the $\nabla$-curvature $k$ of $\gamma$):

$$\nabla_T T = k N, \quad \nabla_T N = -k T. \quad (10)$$

For the affine connection $\nabla$, using (10) we obtain

$$\nabla_T T = k N + \Sigma(T, T), \quad \nabla_T N = -k T + \Sigma(T, N). \quad (11)$$

By (11), the Frenet–Serret formulas $\nabla_T T = \bar{k} N$ and $\nabla_T N = -\bar{k} T$ (with the $\nabla$-curvature $\bar{k}$ of $\gamma$) hold for any curve $\gamma$ if and only if

$$\langle\Sigma(T, T), N \rangle = -\langle\Sigma(T, N), T \rangle, \quad \langle\Sigma(T, N), N \rangle = 0 = \langle\Sigma(T, T), T \rangle.$$

In this case, we have in coordinates the following symmetries:

$$\xi_{12}^1 = -\xi_{11}^2, \quad \xi_{21}^1 = 0 = \xi_{22}^1, \quad \xi_{21}^2 = -\xi_{12}^1, \quad \xi_{12}^2 = 0 = \xi_{11}^2,$$

$$a_{12} = -a_{22}, \quad a_{21} = -a_{11}, \quad a_{03} = -a_{30} = \xi_{11}^2 - \xi_{22}^1,$$

and the formula $\Psi = a_{30}(\sin 3\theta - \cos 3\theta) + a_{12} \sin \theta + a_{21} \cos \theta$.

The support function $S$ of a convex curve $\gamma$ is given by, e.g., [2],

$$S(\theta) = \langle\gamma(\theta), -N \rangle = \gamma^1(\theta) \cos \theta + \gamma^2(\theta) \sin \theta. \quad (12)$$

For example, a circle of radius $\rho$ has $S(\theta) \equiv \rho$. Since $\langle \partial \gamma / \partial \theta, N \rangle = 0$, the derivative $S_\theta$ is

$$S_\theta(\theta) = -\gamma^1(\theta) \sin \theta + \gamma^2(\theta) \cos \theta,$$

and $\gamma$ can be represented by the support function and parameterized by $\theta$, see [2],

$$\gamma^1 = S \cos \theta - S_\theta \sin \theta, \quad \gamma^2 = S \sin \theta + S_\theta \cos \theta. \quad (13)$$

This yields the following known formula for the curvature of $\gamma(\theta)$:

$$k = (S_{\theta\theta} + S)^{-1}. \quad (14)$$

Then, according to (4) and (14),

$$\bar{k} = (S_{\theta\theta} + S)^{-1} + \Psi. \quad (15)$$

Let $\bar{\gamma}(u, t) : S^1 \times [0, t_1) \rightarrow \mathbb{R}^2$ be a family of closed curves satisfying (3). We will use the normal angle $\theta$ to parameterize each curve: $\gamma(\theta, t) = \bar{\gamma}(u(\theta, t), t)$.

**Proposition 1.** The support function $S(\cdot, t) = \langle\gamma(\cdot, t), -N \rangle$ of $\gamma(\cdot, t)$ satisfies the following PDE:

$$S_t = -(S_{\theta\theta} + S)^{-1} - \Psi. \quad (16)$$
Then there exists a unique family of convex closed curves \( \gamma_\tilde{\gamma} \) with the initial value \( \Psi \). Let \( \gamma(\cdot, t) \) be the maximal time interval for the solution \( \gamma(\cdot, 0) = \gamma_0 \) satisfying (3).

Proof. Using (3) and that \( \partial \tilde{\gamma} / \partial u \) is orthogonal to \( N \), we get

\[
\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial u} \cdot \frac{\partial u}{\partial t} + k \cdot N.
\]

Using this, (12), (3) and equality \( N_\gamma = 0 \), see (7), we obtain

\[
S_t = \frac{\partial}{\partial t} \langle \gamma(\theta, t), -N \rangle = \langle \frac{\partial \gamma}{\partial t}, -N \rangle = -k.
\] (17)

Then we apply (15). \( \square \)

By the theory of parabolic equations we have the following.

Proposition 2 (Local existence and uniqueness). Let \( \gamma_0 \) be a convex closed curve in the metric-affine plane. Then there exists a unique family of convex closed curves \( \gamma(\cdot, t) \), \( t \in [0, t_0) \) with \( t_0 > 0 \), and \( \gamma(\cdot, 0) = \gamma_0 \) satisfying (3).

Proof. We will show that (16) is parabolic on \( S(\theta, t) \). To approximate (16) linearly, consider the second order partial differential equation \( \partial_t S = f \) for \( S(\theta, t) \), where \( f(S, \theta, \Psi) = -(S_{\theta \theta} + S)^{-1} - \Psi \). Take the initial point \( \hat{S} = (\hat{S}, \hat{S}_{\theta \theta}, \hat{\Psi}) \) and set \( h = S - \hat{S} \) for the difference of support functions. Then

\[
f(S, \theta, \Psi) = f(\hat{S}, \hat{S}_{\theta \theta}, \hat{\Psi}) + \frac{\partial f}{\partial S}|_{\hat{S}} \cdot h + \frac{\partial f}{\partial S_{\theta \theta}} |_{\hat{S}} \cdot h_{\theta \theta} + \frac{\partial f}{\partial \Psi} |_{\hat{S}} \cdot (\theta - \hat{\Psi}).
\]

where \( \frac{\partial f}{\partial S}|_{\hat{S}} = (\hat{S}_{\theta \theta} + \hat{S})^{-2}, \frac{\partial f}{\partial S_{\theta \theta}} |_{\hat{S}} = (\hat{S}_{\theta \theta} + \hat{S})^{-2} \) and \( \frac{\partial f}{\partial \Psi} |_{\hat{S}} = -\Psi_{\theta} \). Hence, the linearized partial differential equation for \( h \) is

\[
\partial_t h = (\hat{S}_{\theta \theta} + \hat{S})^{-2} (h_{\theta \theta} + h) - \Psi_{\theta} (\theta - \hat{\Psi}).
\] (18)

The coefficient \( (\hat{S}_{\theta \theta} + \hat{S})^{-2} \) of \( h_{\theta \theta} \) is positive, therefore, (18) is parabolic. \( \square \)

Proposition 3 (Containment principle). Let convex closed curves \( \gamma_1 \) and \( \gamma_2 : S^1 \times [0, t_0) \rightarrow \mathbb{R}^2 \) in the metric-affine plane be solutions of (3) and \( \gamma_2(\cdot, t) \) lie in the domain enclosed by \( \gamma_1(\cdot, t) \) for all \( t \in [0, t_0) \).

Proof. Let \( S_i(\theta, t) \) be the support function of \( \gamma_i(\cdot, t) \) for \( 0 \leq t < t_0 \) and \( i = 1, 2 \). These \( \gamma_i \) satisfy (3) with the same function \( \Psi \). Denote \( \tilde{S} = S_2 - S_1 \). Since \( \gamma_1 \) and \( \gamma_2 \) are convex for all \( t \), their curvatures \( k_i \) are positive. Using (14) and (16), we get the parabolic equation

\[
\tilde{S}_t = k_1k_2(\tilde{S}_{\theta \theta} + \tilde{S})
\]

with the initial value \( \tilde{S}(\theta, 0) \geq 0 \). Applying the scalar maximum principle of parabolic equations, e.g., ([1] Section 1.2), we deduce that \( \tilde{S}(\theta, t) \geq 0 \). Hence, \( \gamma_2(\cdot, t) \) lies in the domain enclosed by \( \gamma_1(\cdot, t) \) for all \( t \in [0, t_0) \). \( \square \)

Proposition 4 (Preserving convexity). Let \( [0, \omega) \) be the maximal time interval for the solution \( \gamma(\cdot, t) \) of (3) in the metric-affine plane, and let the curvature of \( \gamma_0 \) obey condition \( k_0 > 2\varepsilon \). Then the solution \( \gamma(\cdot, t) \) remains convex on \( [0, \omega) \) and its curvature has a uniform positive lower bound \( k_0 - 2\varepsilon \) for all \( t \in [0, \omega) \).

Proof. By Proposition 2, \( \gamma(\cdot, t) \) is convex (i.e., \( k > 0 \)) on a time interval \( [0, \tilde{\omega}) \) for some \( \tilde{\omega} \leq \omega \), and its support function satisfies (16) for \( (\theta, t) \in S^1 \times [0, \omega) \). Taking derivative of \( k_1 \) in \( t \), see (15), we get:

\[
k_1 = ((S_{\theta \theta} + S)^{-1})_t = -2(S_{\theta \theta} + S)^{-2}(S_{\theta \theta} + S_t) = k^2(k_{\theta \theta} + k).
\]
Thus, \( \tilde{k}(\theta, t) \) satisfies the following parabolic equation:

\[
\tilde{k}_t = k^2(\tilde{k}_{\theta\theta} + \tilde{k}).
\] (19)

Applying the maximum principle to (19), we find \( \min_{\theta \in S^1} \tilde{k}(\theta, t) \geq \min_{\theta \in S^1} \tilde{k}(\theta, 0) = \tilde{k}_0 \) for \( t \in [0, \omega) \). By conditions and (6),

\[
\tilde{k} = k + \Psi \geq k - |\Psi| \geq k_0 - c > 0.
\]

This and equality (4) imply that the curvature \( k(t) \) of \( \gamma(\cdot, t) \) has a uniform positive lower bound \( k_0 - 2c \) for all \( t \in [0, \omega) \). \( \square \)

**Lemma 2.** Let \( \gamma_1 \) be a solution of (3) in the metric-affine plane with \( \Psi \) given in (5). Then in the coordinates, \( \gamma_t = \gamma_1 + t[a_{21}, a_{12}] \) is a solution of (3) with the \( \nabla^-\)-curvature \( k = k + \Psi \) and \( \Psi = a_{30} \sin^3 \theta + a_{03} \cos^3 \theta \).

**Proof.** By (12), the support function \( \tilde{S}(t, \cdot) \) of the curve \( \gamma_t \), obtained by parallel translation from the curve \( \gamma_t \), thus, having the same curvature \( \bar{k} = k \), satisfies

\[
\tilde{S}(\theta, t) = S(\theta, t) + t[a_{21} \cos \theta + a_{12} \sin \theta].
\]

This, (4) and (17) yield \( \tilde{S}_1 = -k - \Psi \), where \( \Psi = \nabla^- (\bar{T}, \bar{\bar{T}}) \) is defined for \( \bar{T} \) and has the view \( \Psi = a_{01} \sin \theta - a_{21} \cos \theta \). Using (8) for \( \Psi \), completes the proof. \( \square \)

From Lemma 2 we conclude the following.

**Proposition 5.** If \( a_{30} = a_{03} = 0 \), see (8) and (9), then the problem (3) in the metric-affine plane reduces to the classical problem (1) in the Euclidean plane for modified by parallel translation of \( \gamma_1 \) curves \( \gamma_t = \gamma_1 + t[a_{21}, a_{12}] \).

**Example 2.** One may show that

\[
S(\theta, t) = \rho(t) - \epsilon_1(t) \sin \theta - \epsilon_2(t) \cos \theta
\]

with

\[
\rho(t) = \sqrt{\rho^2(0) - 2t}, \quad \epsilon_1(t) = a_{12} t, \quad \epsilon_2(t) = a_{21} t, \quad 0 \leq t \leq \frac{\rho^2(0)}{2},
\]

(20)
is the support function of a special solution of (3) with \( a_{30} = a_{03} = 0 \). We claim that the solution is a family of round circles of radius \( \rho(t) \) \( t \geq 0 \) shrinking to a point at the time \( t_0 = \frac{1}{2} \rho^2(0) \). Indeed, by (13), \( S(\theta, t) \) corresponds to a family of circles

\[
\gamma_t = [\rho(t) \cos \theta - \epsilon_2(t), \rho(t) \sin \theta - \epsilon_1(t)]
\]

with centers \( \left(-\epsilon_2(t), -\epsilon_1(t)\right) \) and the curvature \( k = -1/\rho(t) \). We then calculate

\[
S_{\theta\theta} = \rho'(t) \sin \theta - \epsilon_1'(t) \cos \theta, \quad S_{\theta\theta} = \epsilon_1(t) \sin \theta + \epsilon_2(t) \cos \theta.
\]

Thus, \( S_{\theta\theta} = 1 = \rho(t) \) holds, and (16) reduces to \( \rho' - \epsilon_1' \sin \theta - \epsilon_2' \cos \theta = -1/\rho - \Psi \), where \( \theta \) is arbitrary. We get the system of three ODEs:

\[
\rho' = -1/\rho, \quad \epsilon_1' = a_{12}, \quad \epsilon_2' = a_{21}.
\]

Its solution with initial conditions \( \epsilon_1(0) = 0 \) is (20).
Example 3. (a) The projective connections $\nabla = \nabla + \mathcal{T}$ are defined by the condition

$$\mathcal{T}_X Y = \langle U, Y \rangle X + \langle U, X \rangle Y,$$

where $U$ is a given vector field, e.g., \cite{7}. Then $\Psi = \langle \mathcal{T}_T, N \rangle = 0$, see (5). Thus, (3) in the metric-affine plane with a projective connection is equal to (1) in the Euclidean plane.

(b) The semi-symmetric connections $\nabla = \nabla + \mathcal{S}$ are defined by the condition

$$\mathcal{S}_X Y = \langle U, Y \rangle X - \langle X, Y \rangle U,$$

where $U$ is a given vector field, e.g., \cite{9}. Such connections are metric compatible, and for them the Formulas (11) are valid. The definition (5) reads

$$\Psi = -\langle U, N \rangle = -\langle U, e_1 \rangle \cos \theta - \langle U, e_2 \rangle \sin \theta.$$

Then, see (9), $a_{30} = \langle U, e_2 - e_1 \rangle = -a_{03}$. Let $U$ be a constant vector field on $\mathbb{R}^2$, then we can take the orthonormal frame $\{e_1, e_2\}$ in $(\mathbb{R}^2, g, \nabla)$ such that $U$ is orthogonal to $e_1 - e_2$. Thus, see Proposition 5, the problem (3) in the metric-affine plane with a semi-symmetric connection and constant $U$ reduces to the problem (1) in the Euclidean plane.

Proposition 6 (Finite time existence). Let a convex closed curve $\gamma_0$ in the metric-affine plane with condition $k_0 > 2c$ be evolved by (3). Then, the solution $\gamma_t$ must be singular at some time $\omega > 0$.

Proof. By Lemma 2 and Example 2, using translations we can assume the equalities $a_{21} = a_{12} = 0$, see (9). Hence, $\Psi = a_{30} \sin^3 \theta + a_{03} \cos^3 \theta$, where $a_{03} = \mathcal{T}_{12}^2$ and $a_{30} = \mathcal{T}_{11}^4$. Then we calculate

$$a_{30} \sin^3 \theta + a_{03} \cos^3 \theta = -\frac{1}{4} \sqrt{a_{30}^2 + a_{03}^2} (\cos (\theta - \theta_0) - \cos (3\theta + \theta_0)) + a_{30} \sin \theta + a_{03} \cos \theta$$

for some $\theta_0$. By Lemma 2 again and using the rotation $\theta \rightarrow \theta - \theta_0$, the underlined terms can be canceled, and the retained expression will be $\frac{1}{4} \sqrt{a_{30}^2 + a_{03}^2} \cos (3\theta + \theta_0)$, which can be reduced to simpler form $\tilde{a} \sin^3 \theta$ for some $\tilde{a} \in \mathbb{R}$, using the identity $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

Thus, we may assume $\Psi = \tilde{a} \sin^3 \theta$ with $\tilde{a} < 0$. Let $\gamma_0$ lies in a circle $\Gamma_0$ of radius

$$\rho(0) \geq \max_{t \in S^1} S_t(0, 0) / (1 - 2c / k_0)$$

and centered at the origin $O$. Let evolve $\Gamma_0$ by (3) to obtain a solution $\Gamma(\cdot, t)$ with support function $S_\Gamma$. By Proposition 3, $\gamma_t$ lies in the domain enclosed by $\Gamma(\cdot, t)$, thus, $S \leq S_\Gamma$. Consider two families of circles, see Example 2,

$$\Gamma_t^\pm = [\rho(t) \cos \theta, \rho(t) \sin \theta \pm t\tilde{a}], \quad \rho(t) = \sqrt{\rho^2(0) - 2t},$$

being solutions of (3), hence, having support functions satisfying (16),

$$S_t^\pm = (S_{\rho_0}^\pm + S_{\tilde{a}}^\pm)^{-1} \mp t\tilde{a} \sin \theta = \rho(t) \mp t\tilde{a} \sin \theta.$$

By Proposition 3, $S_t \leq S_\Gamma$ holds, and since $|\sin^3 \theta| \leq |\sin \theta|$, we also have

$$S_t^\Gamma \leq \begin{cases} S_t^\pm, & 0 \leq \theta \leq \pi, \\ S_\tilde{a}, & \pi \leq \theta \leq 2\pi. \end{cases}$$

Hence, $\Gamma_t$ lies (in $\mathbb{R}^2$) below any tangent line to the upper semicircle $\Gamma_t^+$ and above any tangent line to the lower semicircle $\Gamma_t^-$. Thus, $\Gamma_t \subset \text{conv}(\Gamma_t^{+} \cup \Gamma_t^{-})$. The solution $\Gamma^\pm(\cdot, t)$ exists only at a finite
time interval \([0, \tau]\) with \(\tau = \rho^2(0)/2\), and \(\Gamma^\pm(t)\) converges, as \(t \to \tau\), to a point \(\Gamma^\pm_t = [0, \pm \delta \tau]\). Hence, the convex hull of \(\Gamma^+_t \cup \Gamma^-_t\) shrinks to the line segment with the endpoints \((0, \delta \tau)\) and \((0, -\delta \tau)\). We conclude that the solution \(\gamma_t\) must be singular at some time \(\omega \leq \tau\). \(\square\)

Note that a point or a line segment are the only compact convex sets of zero area in \(\mathbb{R}^2\). Using Proposition 6 and Blaschke’s selection theorem given below, we can directly deduce that \(\gamma(\cdot, t)\) converges to a (maybe degenerate and nonsmooth) weakly convex curve \(\gamma(\cdot, \omega)\) in the Hausdorff metric.

**Blaschke Selection Theorem** (e.g., ([2] p. 4)). Let \(\{K_i\}\) be a sequence of convex sets, which lie a bounded set. Then there exists a subsequence \(\{K_{i_k}\}\) and a convex set \(K\) such that \(K_{i_k}\) converges to \(K\) in the Hausdorff metric.

**Lemma 3** (Enclosed area). Let a convex closed curve \(\gamma_0\) in the metric-affine plane with condition \(k_0 > 2c\) be evolved by (3). Then the area enclosed by \(\gamma(\cdot, \omega)\) must be zero, i.e., \(\gamma(\cdot, \omega)\) is either a point or a line segment.

**Proof.** Suppose the lemma is not true. We may assume the origin is contained in the interior of the region enclosed by \(\gamma(\cdot, \omega)\). We can draw a small circle, with radius \(2\rho\) and centered at the origin, in the interior of the region enclosed by \(\gamma(\cdot, \omega)\).

Since the solution \(\gamma(\cdot, t)\) becomes singular at the time \(t = \omega\), we know that the curvature \(k(\cdot, t)\) becomes unbounded as \(t \to \omega\). (Indeed, if \(k\) is bounded at \(t = \omega\), then \(\bar{k}\) is also bounded at \(t = \omega\), and due to the evolution equation (17), the solution \(\gamma(\cdot, t)\) can be extended for a larger interval \([0, \omega']\) for some \(\omega' > \omega\).) To derive a contradiction, we only need to get a uniform bound for the curvature. Consider

\[
\phi = \frac{-S_i}{S - \rho} \quad (16) \quad \bar{k} \quad \frac{S_i}{S - \rho}.
\]

For any \(\omega < \omega\), we can choose \((\theta_0, t_0)\) such that

\[
\phi(\theta_0, t_0) = \max \{ \phi(\theta, t) : (\theta, t) \in S^1 \times [0, \omega] \}.
\]

Without loss of generality, we may assume \(t_0 > 0\). Then at \((\theta_0, t_0)\),

\[
0 = \phi_\theta = \frac{-S_{i\theta}}{S - \rho} + \frac{S_i S_\theta}{(S - \rho)^2},
\]

\[0 \leq \phi_t = \frac{-S_{i\theta}}{S - \rho} + \frac{S_i^2}{(S - \rho)^2},\]

\[0 \geq \phi_{\theta\theta} = \frac{-S_{i\theta\theta}}{S - \rho} + \frac{S_i S_{i\theta\theta}}{(S - \rho)^2}.
\]

(21)

On the other hand,

\[S_{i\theta} = -k_{i\theta} = -k^2(k_{\theta\theta} + k),\]

\[0 \leq \phi_t = \frac{k^2(k_{\theta\theta} + k)}{S - \rho} + \frac{k^2}{(S - \rho)^2} \]

\[= \frac{k^2(-S_{i\theta}\theta + k)}{S - \rho} + \frac{k^2}{(S - \rho)^2} = \frac{k^2}{S - \rho}(-S_{i\theta\theta}) + \frac{k^2 k}{S - \rho} + \frac{k^2}{(S - \rho)^2}.
\]

By the above,

\[0 \leq \phi_t \leq \frac{k^2}{S - \rho} - \frac{S_{i\theta\theta}}{S - \rho} + \frac{k^2 k}{S - \rho} + \frac{k^2}{(S - \rho)^2} \]

\[= \frac{k^2 k}{S - \rho} (S_{\theta\theta} + S - \rho) + \frac{k^2}{(S - \rho)^2} \quad (14) \quad k(k + k - \rho k^2).
\]
Since \( \tilde{k} = k + \Psi(\theta) \) with \( \tilde{k} > 0 \), see the proof of Proposition 4, and using (6), we obtain
\[
\rho k^2 \leq k + \tilde{k} = 2k + \Psi(\theta) \leq 2k + c.
\] (22)

From quadratic in \( k \) inequality (22) we conclude that
\[
0 \leq k \leq (1 + \sqrt{1+c\rho})\rho^{-1} < \infty \quad \text{on} \quad S^1 \times [0, \omega).
\]

Thus, \( k \) is bounded as \( t \uparrow \omega \) – a contradiction. Thus, the area enclosed by \( \gamma(,t) \) tends to zero as \( t \uparrow \omega \). \( \square \)

To complete the proof of Theorem 2, we note that if the flow (3) does not converge to a point as the enclosed by \( \gamma(,t) \) area tends to zero (see Lemma 3), then \( \min_{\theta \in S^1} k(\theta, t) \) tends to zero as \( t \uparrow \omega \). But in Proposition 4 we have shown that the curvature of \( \gamma(,t) \) has a uniform positive lower bound. So the flow must converge to a point.

The area enclosed by the convex curve \( \gamma(,t) \subset \mathbb{R}^2 \), e.g., ([2] p. 6), is calculated by
\[
A(t) = -\frac{1}{2} \int_{\gamma(,t)} \langle \gamma(,t), N \rangle \, ds = \frac{1}{2} \int_0^{2\pi} \frac{S}{k} \, d\theta.
\] (23)

We will estimate the maximal time \( \omega \) under rather stronger condition to control convexity.

**Proposition 7.** Let a convex closed curve \( \gamma_0 \) in the metric-affine plane be evolved by (3). If \( k_0 > 3c \) then the maximal time \( \omega \) is estimated by
\[
\omega \leq \frac{A(0)}{2\pi} \cdot \frac{k_0 - 2c}{k_0 - 3c}.
\] (24)

**Proof.** Using (17), (19) and the identity \( \int_0^{2\pi} (S_{\theta\theta} + \tilde{k}) \, d\theta = \int_0^{2\pi} (S_{\theta\theta} + S) \tilde{k} \, d\theta \), we get
\[
\frac{d}{dt} A(t) = \frac{1}{2} \int_0^{2\pi} \frac{S_{\theta\theta} - S_{\theta} \tilde{k}}{k^2} \, d\theta = -\frac{1}{2} \int_0^{2\pi} \left[ 1 + \frac{\Psi}{k} + S_{\theta\theta} + \tilde{k} \right] \, d\theta = -2\pi - \int_0^{2\pi} \frac{\Psi(\theta)}{k(\theta, t)} \, d\theta.
\]

Using the inequality \( k(\theta, t) \geq k_0 - 2c \), see Lemma 4, and \( |\Psi| \leq c \), see (6), we get
\[
\frac{d}{dt} A(t) \leq -2\pi + \frac{2\pi c}{k_0 - 2c}.
\]

By this, we have \( A(0) \geq 2\pi \frac{k_0 - 3c}{k_0 - 2c} \omega \). Hence, the inequality (24) holds when \( k_0 > 3c \). \( \square \)

**Question:** can one estimate \( \omega \) when \( 2c < k_0 \leq 3c \)?

**3. Proof of Theorem 3**

Here, we study the normalized flow (3). From (23) we have
\[
\frac{A(t)}{2(\omega - t)} = \pi + \frac{1}{2(\omega - t)} \int_t^\omega \int_{\gamma} \Psi(\theta) \, ds \, dt,
\]
hence, \( \lim_{t \to \omega} \frac{A(t)}{2(\omega - t)} = \pi \). Without loss of generality, we may assume that the flow shrinks at the origin. Thus, we rescale the solution \( \gamma(,t) \) of (3) as
\[
\tilde{\gamma}(,t) = \left(2(\omega - t)\right)^{1/2} \gamma(,t).
\]
The corresponding support function and curvature are given by
\[ \tilde{S}(\cdot, t) = (2(\omega - t))^{1/2}S(\cdot, t), \quad \tilde{k}(\cdot, t) = (2(\omega - t))^{-1/2}k(\cdot, t). \]

Introduce a new time variable \( \tau \in [0, \infty) \) by
\[ \tau = -(1/2) \log(1 - \omega^{-1} t). \]

Using the above definitions, we find the partial differential equation for \( \tilde{S} \),
\[ \tilde{S}_\tau = -(\tilde{k} + \sqrt{2} \omega e^{-\tau} \Psi) + \tilde{S}, \]
and that the normalized curvature, i.e., of the curves \( \tilde{\gamma}(\cdot, \tau) \), satisfies the equation
\[ \tilde{k}_\tau = \tilde{k}^2(\tilde{k}_{\theta\theta} + \tilde{k}) - \tilde{k} + \sqrt{2} \omega e^{-\tau} \tilde{k}^2(\Psi_{\theta\theta} + \Psi). \]

The following steps for the ACEF, see [1], are applicable to the normalized flow (3):
1. The entropy for the normalized flow,
\[ E(\tilde{\gamma}(\cdot, \tau)) = \frac{1}{2\pi} \int_0^{2\pi} \log \tilde{k} \, d\theta, \]
is uniformly bounded for \( \tau \in [0, \infty) \), see ([1] pp. 63–68). The bound on the entropy yields upper bounds for the diameter and length of the normalized flow, and also that \( \tilde{k} \) and its gradient are uniformly bounded.
2. \( e^{-\tau} \tilde{k}_{\text{max}}(\tau) \to 0 \) as \( \tau \to \infty \), see ([1] Lemma 3.15).
3. The normalized curvature \( \tilde{k} \) has a positive lower bound, see ([1] pages 70–71).
4. With two-sided bounds for \( \tilde{k} \), the convergence of the normalized flow (3), as \( \tau \to \infty \), follows.

Based on the fact [10] that the only embedded solution of (26) is the unit circle, we conclude (similarly as in ([1] p. 73) for ACEF with \( \Phi = 1 \)) that \( \tilde{\gamma}(\cdot, \tau) \) converges, as \( \tau \to \infty \), to the unit circle in \( C^\infty \), that completes the proof of Theorem 3.

4. Conclusion

The main contribution of this paper is a geometrical proof of convergence of the flow (3) for convex closed curves. The paper carries out the initial step of the investigation of interest, when the curvature of initial curve dominates the torsion term so that the behavior is essentially the same as for the classical CSF. Since the condition used to control the convexity is rather rough, it is natural that it can be refined. For example, is it possible to prove the result in the same way under the assumption that \( \tilde{k} \) is strictly positive on the initial curve? This would be especially interesting because it would allow some non-convexity in the classical sense. Another interesting question is whether there exist ancient solutions for the metric-affine flows, and what they might look like: for large negative times one expects the torsion term to dominate, so a natural question is whether there are contracting ‘self-similar’ solutions under the flow driven by the torsion term alone? If so then perhaps there should be ancient solutions for the flow (3) asymptotic to these. It is this large-scale regime which might provide interesting differences with the classical CSF. Finally, the question, is there any obstruction to an analogue of Grayson’s theorem (see also [11]) for the flow (3)?

In the future, one may study these questions and several related problems on convergence of flows in metric-affine geometry, for example:
(1) The flow (3) for non-constant contorsion tensor $\mathcal{T}$ and for not just convex and embedded $\gamma_0$.

(2) Anisotropic CSF in metric-affine geometry,

(3) Behavior of solutions for (3) (and numerical experiments as in [10]) when $\gamma_0$ is immersed.

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