STATISTICAL PROPERTIES OF DYNAMICS
INTRODUCTION TO THE FUNCTIONAL ANALYTIC
APPROACH.

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Abstract. These are lecture notes for a simple minicourse approaching the satistical properties of a dynamical system by the study of the associated transfer operator (considered on a suitable functions or measures spaces).

The following questions will be addressed:
• existence of a regular invariant measure;
• Lasota Yorke inequalities and spectral gap;
• decay of correlations and some limit theorem;
• stability under perturbations of the system
• linear response
• random systems
• hyperbolic systems

The point of view taken is to present the general construction and ideas needed to obtain these results in the simplest way. For this, some theorem is proved in a form which is weaker than usually known, but with an elementary and simple proof.

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1. Introduction

The term statistical properties of a dynamical system refers to the long time behavior of a trajectory $x_1, \ldots, x_n$, or a set of trajectories of the system: their distribution in the phase space, the average of a given observable along the trajectory (consider a function $f$ with values on $\mathbb{R}$ or $\mathbb{C}$ and the time average $\frac{1}{n}(f(x_1) + \ldots + f(x_n))$, the speed of convergence to those averages, the frequency of a deviation from the average behavior and so on. As we will see in the following, this relates to the properties of the evolution of large sets of trajectories, measures or even distributions by the action of the dynamics.

In chaotic systems the statistical properties of the dynamics are often a better object to be studied than the pointwise behavior of trajectories. Indeed, due to the initial condition sensitivity, the future behavior of initial data can be unstable and unpredictable, but statistical properties are often stable and their description simpler. This is a classical approach to dynamics that has been implemented in the so called Ergodic Theory (the reader may find in any library or even in the references to these notes very good books on this general theory and about its many applications).

In Ergodic Theory often it is supposed that the dynamics preserves a given measure, and some other properties (ergodicity, mixing) from this, deep consequences on the statistical behavior of the system are deduced. But, given a dynamical system, how to show that there are interesting invariant measures, which are their properties and how to show that the resulting measure preserving system is ergodic or mixing and how to estimate the mixing speed? Furthermore, are these findings stable by perturbations of the system? Can, all of this be computed numerically in a reliable way?
These notes focus on some questions of this kind, between dynamics and ergodic theory. We consider dynamical systems having certain geometrical properties and show that they have "good" invariant measures. Sometimes we can prove that they satisfy other finer properties (mixing, fast decay of correlations, spectral gap), so that we can apply many results from Ergodic Theory and Probability to deduce other consequences. We also consider the important problem of the stability of these invariant measures and mixing properties, this allows to have information about whether the statistical properties are stable under small changes in the system or not, or even about the direction of change of these properties when the system changes. (this has of course important applications for the understanding of the behavior of many systems).

As said before, the properties we mean to investigate are related to the evolution of measures by the action of the dynamics. We will see that given a dynamical system it is possible to associate to the system a transfer operator describing the action of the dynamics on suitable functional spaces of measures (or distributions sometime). Many important results can be obtained studying the properties of this transfer operator\(^1\). This is the main subject of the following sections. We will start defining the transfer operator and its basic properties. We will see how it is possible to deduce the existence of a regular invariant measure and the speed of convergence to this measure by the iteration of the dynamics. We will see that under additional assumptions we may have a precise description of the action of the transfer operator on suitable spaces of measures (spectral gap, Section 6), and some of its statistical consequences. We will then consider the problem of stability of all these concepts under perturbation (Section 7).

The general theory and tools shown in the notes are applied to some of the simplest kind of dynamical systems. We enter in details showing how all these concepts can be applied to expanding maps, but we also show how the same approach can be applied to piecewise expanding maps (Section 9) and some class of hyperbolic maps (Section 12). Here the technicalities needed are much more complicated, but we try to give a sketch of the ideas needed to extend the transfer operator approach to this case, in which there is both contraction and expansion in the map generating the dynamics. We also see how the transfer operator approach can be applied to random dynamical systems (Section 10).

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2. Physical measures

In these section we introduce the notion of physical measure for deterministic, discrete time dynamical systems. Let \(X\) be a metric space, \(T : X \to X\) a Borel measurable map. Given some initial condition \(x_0 \in X\) we define the orbit of \(x_0\) by the dynamics \(T\) as \(x_1 = T(x_0), x_2 = T(x_1), \ldots, x_i = T^i(x_0)\). Now let \(M(X)\) be the set of Borel positive measures on \(X\). Lus see how we can apply the dynamics

\(^1\)From an historical point of view, this approach brought important developments and applications starting around the beginning of 21 century (see [42], [38], [42]) and it is still fastly developing.
to measures instead of points, by considering the so called pushforward map $T_* : M(X) \to M(X)$. Given a Borel probability measure $\nu$ on $X$, we define $T^*(\nu)$ as

$$T_*[\nu](A) = \nu(T^{-1}(A)),$$

where $A$ is any measurable subset of $X$.

We say that a Borel probability measure $\mu$ is $T$-invariant if $T_*[\mu] = \mu$, or equivalently, if for each measurable set $A$ it holds $\mu(A) = \mu(T^{-1}(A))$. In this framework, when $T : X \to X$ and $\mu$ is invariant, we call the triple $(X,T,\mu)$ measure preserving transformation.

**Example 1.** For a rotation $x \to x + \alpha \ (\text{mod } 1)$ the Lebesgue measure (the 1-d volume) is invariant.

**Example 2.** For the doubling map $x \to 2x \ (\text{mod } 1)$ the Lebesgue measure (the 1-d volume) is invariant, but there are many other invariant measures.

Invariant measures represent equilibrium states, in the sense that probabilities of events do not change in time. A given a map $T$, may have many of these invariant measures, but some of them is particularly important to describe the statistical properties of the dynamics associated to $T$.

In this section we will define the notion of Physical measure, which is a particularly important kind of invariant measure. In the following, we will see that under suitable assumptions some physical measure is an attractor of many other regular measures by the dynamics induced on $M(X)$ by $T_*$, and the speed of convergence to this equilibrium state has important consequences for the statistical properties of the dynamics.

Given a measure preserving transformation $(X,T,\mu)$, a set $A \subseteq X$ is called $T$-invariant if $T^{-1}(A) = A$ up to zero measure sets. The system $(X,T,\mu)$ is said to be ergodic if each $T$-invariant set has total or null measure. An ergodic system is then a system whose dynamics is indecomposable into different invariant sets (up to zero measure).

The celebrated Birkhoff pointwise ergodic theorem (see any book about ergodic theory) says that in this case, time averages computed along $\mu$-typical orbits coincides with space average with respect to $\mu$. More precisely, in ergodic systems, for any $f \in L^1(X,\mu)$ it holds

$$\lim_{n \to \infty} \frac{S^f_n(x)}{n} = \int f \,d\mu,$$

for $\mu$ almost each $x$, where $S^f_n = f + f \circ T + \ldots + f \circ T^{n-1}$.

Note that the equality in (1) is up to negligible sets according to $\mu$. A given map $T : X \to X$ may hence have many invariant measures corresponding to many possible statistical limit behaviors for its orbits.

It is important to select the physically relevant ones; the ones which come from the time averages of a large set of points. Large according to the natural measure we can consider a priori on our phase space; when $X$ is a manifold this could be the Riemann volume or Lebesgue measure.

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2To help in the understanding of why this is a reasonable definition of the action of the dynamics $T$ on measures, we invite the reader to verify that by the above definition, if $\delta_{x_0}$ is the Dirac measure placed on $x_0$ then $T^*(\delta_{x_0}) = \delta_{T(x_0)}$.

3For example, to every periodic orbit it correspond one invariant measure for the system, but as we will see in the following there can be are many more in the system.
Definition 3. Let $X$ be a manifold with boundary. We say that a point $x$ belongs to the basin of an invariant measure $\mu$ if (1) holds at $x$ for each bounded continuous $f$. A physical measure is an invariant measure whose basin has positive Lebesgue measure.

Often these physical measures also have other interesting features such as:

- they are as regular as possible among the invariant ones;
- they have a certain stability under perturbations of the system;
- they are in some sense limits of iterates of the Lebesgue measure (by iterating the pushforward map or taking suitable Cesaro averages of such iterates).

These measures hence encode important information about the statistical behavior of the system (see [45] for a general survey). In the following we will see some method to select those measures, prove their existence and some of its main statistical properties.

Remark 4. If $\mu$ is ergodic and physical, then the correspondence between time and space averages expressed by (1) also hold for a set which is also relevant for the Lebesgue measure.

Remark 5. In the following we will show techniques to find the physical invariant measures of a given system and to know many interesting properties of them. Although in many systems of interest in the mathematics and in the applications one can show that such invariant measures exist, it is worth to remark that there are deterministic systems having no invariant measures at all. A remarkable general statement about existence of invariant measures (Krylov-Bogoliubov theorem) says that "if $X$ is a compact metric space $T : X \to X$ is continuous then there is at least one invariant measure", we now show a system which is discontinuous and does not have such measures. Consider $T : [0, 1] \to [0, 1]$ defined as

$$T(x) = \begin{cases} 
\frac{1}{2}x + \frac{1}{2} & \text{if } x \neq \frac{1}{2} \\
0 & \text{if } x = \frac{1}{2}
\end{cases},$$

whose graph is represented in the following figure.

This map has no invariant measures, let us briefly explain why. Let $A = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, it holds that $\forall \epsilon \exists n \text{ s.t. } T^n(A) \subseteq (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) - \{\frac{1}{2}\} := S_\epsilon$.

Suppose $\mu$ is an invariant measure, since the map is $1-1$ on $A$ we get $\mu(T^nA) = \mu(A) = \mu([0, 1] - \{\frac{1}{2}\})$. It must also hold that $\mu(S_\epsilon) \to 0$, hence $\mu(A) = 0$. However since $T^{-1}(\frac{1}{2}) = \emptyset$ we also have $\mu(0) = 0$ and thus the whole space has zero measure.

Example 6. We show two examples of maps of the unit interval with a plot of the density of their associated absolutely continuous and hence physical invariant measures, computed with the methods described in Section 8.2.
The first example is a piecewise expanding map (see Section 9) \( T : [0, 1] \to [0, 1] \) defined by
\[
T(x) = \begin{cases} 
\frac{109}{64} (1/2 - x)^{51/64} & x < 1/2 \\
1 - \frac{109}{64} (x - 1/2)^{51/64} & x > 1/2.
\end{cases}
\]
The graph of the map and the associated invariant density is shown in figure 1.

The second example is another map on the interval which has a generally expanding behavior, but also a fixed point where the derivative is 1. This is a slowly repelling fixed point, and this fact forces the invariant measure to concentrate around that point, giving an invariant density which diverges around the point. Again we consider a map \( T : [0, 1] \to [0, 1] \) defined by
\[
(2) \quad T(x) = x + x^{1+\frac{1}{8}} \mod 1,
\]
whose graph is shown in Figure 2.

Figure 2.

and a plot of the invariant density is shown in Figure 3.
3. The transfer operator

We have seen how the dynamics $T$ can be applied to measures by the pushforward map. Let us consider the space $SM(X)$ of Borel measures with sign on $X$ (equivalently complex valued measures can be considered) this is a vector space and the pushforward map $T_*$ can be extended to $SM(X)$ with the same definition to a linear map $L : SM(X) \to SM(X)$. Because of its importance and to emphasize the fact that this is a linear function we will denote it by $L$ or $LT$ and call it the transfer operator associated to $T$. Recalling the definition, we remark that if $\nu \in SM(X)$ then $L[\nu] \in SM(X)$ is such that $L[\nu](A) = \nu(T^{-1}(A))$.

The main theme of these lectures is to see how the understanding of the properties of this operator, applied on suitable normed vector spaces of measures allow to understand many important statistical properties of the dynamics of $T$.

We will see this by dealing first with some of the simplest examples, the expanding maps for which we will prove the existence of physical measures having a smooth density, the convergence to equilibrium and statistical stability properties. To do this we will hence restrict our interest on absolutely continuous measures, and until Section 11 we will only consider this kind of measures and the related densities, also on other more complicated classes of of deterministic and random systems. We then now focus our attention on the understanding of the properties of the transfer operator applied to absolutely continuous measures.

Remark that if the measure we consider is absolutely continuous: $d\nu = f \, dm$ (here we are considering the Lebesgue measure $m$ as a reference measure, note that other measures can be considered) and if $T$ is nonsingular\(^4\) the operator $L$ induces another operator $\tilde{L} : L^1(m) \to L^1(m)$ acting on the measure densities defined by

$$\tilde{L}f = \frac{d(L(f \, m))}{dm}.$$  

By a small abuse of notation we will still indicate by $L$ this operator.\(^5\) We remark that in this point of view, one can see $L^1(m)$ equivalently as a space of integrable functions or a space of measures which are absolutely continuous with respect to $m$, hence measures having some regularity.

Considering hence $L : L^1 \to L^1$, it is easy to verify that this is a positive operator\(^6\) and preserves the integral

$$\int_X Lf \, dm = \int_X f \, dm.$$  

We recall that positive, integral preserving operators are also called Markov Operators. Let us see some other important basic properties

**Proposition 7.** $L : L^1 \to L^1$ is a weak contraction for the $L^1$ norm. If $f$ is a $L^1$ density, then $||Lf||_1 \leq ||f||_1$.

\(^4\)A map is nonsingular (with respect to the Lebesgue measure) when $m(T^{-1}(A)) = 0 \iff m(A) = 0$.

\(^5\)In many applications the transfer operator will be considered as acting on some suitable space of regular measures with sign hence other subspaces of $L^1(m)$ or $SM(X)$.

\(^6\)A positive operator $L$ is an operator for which $f \geq 0 \implies Lf \geq 0$. We remark that by linearity one also has $f \geq g \implies Lf \geq Lg.$
Proof. Since $L$ preserves the integral
\[
|Lf|_1 = \int |Lf| \, dm \leq \int |L(f^+ - f^-)| \, dm \leq \int |L(f^+)| + |L(f^-)| \, dm \\
\leq \int (f^+) + (f^-) \, dm = \int |f| \, dm = |f|_1.
\]
□

**Proposition 8.** Consider $f \in L^1(m)$, and $g \in L^\infty(m)$, then:
\[
\int g \, L(f) \, dm = \int g \circ T \, f \, dm.
\]

Proof. Let us first prove it for simple functions if $g = 1_B$ then
\[
\int g \circ T \, f \, dm = \int 1_B \circ T \, f \, dm = \\
\int 1_{T^{-1}B} \, f \, dm = \int 1_{T^{-1}B \cap A} \, f \, dm.
\]
If $g \in L^\infty$ we can approximate it by a combination of simple functions $\hat{g} = \sum_i a_i 1_{A_i}$ in a way that $||g - \hat{g}||_\infty \leq \epsilon$ and
\[
\int \hat{g} \circ T \, f \, dm = \int \hat{g} \, L(f) \, dm.
\]
Then
\[
\int g \circ T \, f \, dm = \int [g - \hat{g} + \hat{g}] \circ T \, f \, dm \\
= \int [g - \hat{g}] \circ T \, f \, dm + \int \hat{g} \circ T \, f \, dm.
\]
Moreover
\[
\int g \, L(f) \, dm = \int [g - \hat{g} + \hat{g}] \, L(f) \, dm \\
= \int [g - \hat{g}] \, L(f) \, dm + \int \hat{g} \, L(f) \, dm
\]
and since $T$ is nonsingular
\[
|\int [g - \hat{g}] \circ T \, f \, dm| \leq ||g - \hat{g} \circ T||_\infty ||f||_1 \\
\leq ||g - \hat{g}||_\infty ||f||_1 \leq \epsilon,
\]
moreover
\[
|\int [g - \hat{g}] \, L(f) \, dm| \leq ||g - \hat{g}||_\infty ||L(f)||_1 \leq \epsilon
\]
directly leading to the statement. □

**Remark 9.** A similar statement
\[
\int g \, d(L\mu) = \int g \circ T \, d\mu.
\]
applies in the more general case $\mu$ is any Borel measure with sign. The proof is almost the same as above.
Measures which are invariant for $T$ are fixed points of $L$. Since physical measures usually have some "as good as possible" regularity property we will find such invariant measures in some space of "regular" measures. A first example which will be explained in more details below is the one of expanding maps, where we are going to find physical measures in the space of invariant measures having an absolutely continuous density.

4. Expanding maps: regularizing action of the transfer operator and existence of a regular invariant measure

Figure 2. The expanding map $x \rightarrow 4x + 0.01 \sin(8\pi x) \mod 1$ and a plot of its invariant density.

In this section we illustrate one approach which allows to prove the existence of regular invariant measures. The approach is quite general, but we will show it on a class of one dimensional maps where the construction is technically simple. An important step is to find a suitable function space on which the transfer operator has good properties.

Let us consider a map $T$ which is expanding on the circle. i.e.

- $T : S^1 \rightarrow S^1$,
- $T \in C^2$,
- $|T'(x)| > 1 \forall x$.

Let us consider the Banach space $W^{1,1}$ of absolutely continuous density functions with the norm

$$||f||_{W^{1,1}} = ||f||_1 + ||f'||_1.$$  

We will show that the transfer operator is regularizing for the $|| ||_{W^{1,1}}$ norm. This implies that iterates of a starting measure have bounded $|| ||_{W^{1,1}}$ norm, allowing to find a suitable invariant measure (and much more information on the statistical behavior of the system, as it will be described in the following sections).

4.1. Lasota-Yorke inequalities. A main tool to implement this idea is the so called Lasota Yorke inequality (Hö). Let us see what it is about: we consider the operator $L$ restricted to some normed vector space of signed Borel measures $(B_s, || ||_s)$ (often a Banach space) and we consider another space $B_w \supset B_s$ equipped with a weaker norm $|| ||_w$ such that $||L^n||_{B_w \rightarrow B_w} \leq M$ is uniformly bounded (as it is for the $L^1$ norm, see Remark). In this context, if the two spaces are well chosen

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7For which $f(x) = f(0) + \int_0^x f'(t) \, dt$ for some $f' \in L^1$.

8In the probabilistic context, this kind of estimations are often called as Doebelin Fortet inequalities.
it is possible in many interesting cases to prove that there are \( A \geq 0, 0 \leq \lambda < 1 \) such that for each \( n \)
\[
||L^n g||_s \leq A\lambda^n ||g||_s + B||g||_w.
\]
This means that the iterates \( L^n g \) have bounded strong norm \( ||.||_s \) and then by suitable compactness arguments which will be shown in the next paragraphs this inequality may show the existence of an invariant measure in \( B_s \). Similar inequalities can be proved in many systems, and they are a main tool for the study of statistical properties of dynamical systems.

Now let us see how the above inequality can be obtained in the case of expanding maps, were \( W^{1,1} \) and \( L^1 \) are considered as strong and weak space. Let us consider a nonsingular transformation and see its action on densities. In the case of expanding maps we are considering we have an explicit formula for the transfer operator:
\[
(Lf)(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}.
\]
Taking the derivative of (2) \(^{10} \) (remember that \( T'(y) = T'(T'^{-1}(x)) \)) we get
\[
(Lf)' = \sum_{y \in T^{-1}(x)} \frac{1}{|T'(y)|} f'(y) - \frac{T''(y)}{(T'(y))^2} f(y).
\]
Note that
\[
(Lf)' = L\left(\frac{1}{T} f'\right) - L\left(\frac{T''}{(T')^2} f\right)
\]
and
\[
||f||_1 \leq ||f'||_1 + ||\frac{T''}{(T')^2} f||_1 \leq \alpha ||f'||_1 + \frac{||T''||_\infty ||f||_1}{(T')^2}
\]
where \( \alpha = \max\left(\frac{1}{T'}\right) \).
Hence
\[
||f||_1 + ||Lf||_1 \leq \alpha ||f'||_1 + \alpha ||f||_1 + \left(||\frac{T''}{(T')^2}||_\infty + 1\right)||f||_1
\]
and
\[
||f||_{W^{1,1}} \leq \alpha ||f||_{W^{1,1}} + \left(||\frac{T''}{(T')^2}||_\infty + 1\right)||f||_1
\]
Iterating the inequality\(^{\text{10}}\)
\[
||L^n f||_{W^{1,1}} \leq \alpha^n ||f||_{W^{1,1}} + \frac{\left(||\frac{T''}{(T')^2}||_\infty + 1\right)||f||_1}{1 - \alpha}
\]
Hence if we start with \( f \in W^{1,1} \) all the elements of the sequence \( L^n f' \) of iterates of \( f \) are in \( W^{1,1} \) and their strong norms are uniformly bounded.

\(^{10}\)To prove the formula consider a small neighborhood \( B(x, \epsilon) \) of \( x \) and check the amount of measure which is sent there by \( L \). The value of the density \( Lf(x) \) will a.e. be the value of the limit \( \lim_{\epsilon \to 0} \frac{\mu(T^{-1}(B(x, \epsilon)))}{2\epsilon} \) (or see [19] pag. 85 ).
\(^{11}\)||f||_{W^{1,1}} \leq \alpha ||L^n f||_1 + B||Lf||_1 
\leq \alpha^2 ||f||_1 + \alpha B||f||_1 + B||f||_1, ...
4.1.1. A Lipschitz Lasota Yorke inequality. By Equation 6 \( ||(L^n f)||_{W^{1,1}} \) is uniformly bounded, then also \( ||(L^n f)||_{\infty} \) is. Let us remark that since the transfer operator is positive

\[
M := \sup_{n, ||f||_{\infty} \leq 1} ||(L^n f)||_{\infty} = \sup_n ||(L^n 1)||_{\infty}.
\]

Then there is \( n_1 \) such that \( \alpha^{n_1} M < 1 \).

Now consider a new map, \( T_2 = T^{n_1} \). This map still has the same regularity properties as before and is uniformly expanding on the circle. Let \( L_2 \) be its transfer operator. From (5) and the positivity of \( L_2 \) we have

\[
|||(L_2 f)^\prime|||_{\infty} \leq ||L_2(\frac{1}{T_2} f')|||_{\infty} + ||L_2(\frac{T''_2}{T_2^2} f)|||_{\infty}
\]

\[
\leq \alpha^{n_1} M ||f'||_{\infty} + M ||\frac{T''_2}{T_2^2}|||f|||_{\infty}.
\]

Then \( L_2 \) satisfies a Lasota Yorke inequality, with the norms \( || ||_{Lip} \) defined as \( ||f||_{Lip} = ||f'||_{\infty} + ||f||_{\infty} \) and \( || ||_{\infty} \), that is

\[
||L_2^{n_1} f||_{Lip} \leq \lambda^n ||f||_{Lip} + B ||f||_{\infty}
\]

with \( \lambda = \alpha^{n_1} M < 1 \) and \( B \geq 1 \). By this

\[
||L_2^{n_1+\ell} f||_{Lip} \leq \lambda^n ||L_2^\ell f||_{Lip} + B ||L_2^\ell f||_{\infty}
\]

\[
\leq \lambda^n M ||f||_{Lip} + B \lambda^n ||f||_{\infty}
\]

and then also \( L \) satisfies a Lasota Yorke inequality of the type (3) with these norms.

Remark 10. In this section we have shown two examples of regularization estimations on different function spaces. This kind of estimations are possible over many kind of systems having some uniform contracting/expanding behavior. In these cases the choice of the good measure spaces involved is crucial. When the system is expanding, even on higher dimension and with low regularity, spaces of bounded variation functions and absolutely continuous measures are usually considered. If the system has contracting directions, the physical measure usually has fractal support and it is often included in some suitable space on which a Lasota Yorke estimation can be proved (see, e.g. [38],[3],[19],[6],[31]), in these cases it is often useful to include the space of measures in a suitable distribution spaces. In Section 12 we will see an example of spaces adapted to a class of hyperbolic systems.

4.2. Existence of a regular invariant measure. In this section we prove the existence of an absolutely continuous invariant measure for the expanding maps. The idea is to iterate the transfer operator on the space \( W^{1,1} \), the Lasota Yorke inequality will ensure that the iterates of some initial smooth density remain of bounded \( W^{1,1} \) norm. The following theorem provides a compactness argument to prove that from this sequence of iterates there is a subsequence converging in \( L^1 \).

(see [20] for more details and generalizations)

**Proposition 11 (Rellich-Kondrachov).** \( W^{1,1} \) is compactly immersed in \( L^1 \). If \( B \subset W^{1,1} \) is a strongly bounded set: \( B \subset B_{W^{1,1}}(0, K) \) then for each \( \epsilon \), \( B \) has a finite \( \epsilon \)-net for the \( L^1 \) topology.

\[\text{11}\text{The strong ball centered in 0 with radius } K.\]
In particular any bounded subsequence \( f_n \in W^{1,1} \) has a weakly converging subsequence. There is \( f_{n_k} \) and \( f \in L^1 \) such that
\[
 f_{n_k} \to f
\]
in \( L^1 \).

**Proof.** (sketch) Let us consider a subdivision \( x_1, \ldots, x_m \) of \( I \) with step \( \epsilon \). Let \( f \in B_{W^{1,1}}(0, K) \), let us consider \( \pi_n f \) to be the piecewise linear approximation of \( f \) such that
\[
 \pi_n f(x_i) = f(x_i).
\]

Now, if \( x_i \leq \overline{x} \leq x_{i+1} \) then
\[
 |f_n(\overline{x}) - \pi_n f(\overline{x})| \leq \int_{x_i}^{x_{i+1}} |f_n'(t)| \, dt := h_i.
\]

Remark that \( \sum h_i \leq K \). Hence
\[
 ||f_n - \pi_n f||_1 \leq \epsilon \sum h_i \leq \epsilon K. \tag{10}
\]

Since \( \pi_n \) has finite rank it is then standard to construct a finite \( 2\epsilon K \)-net for \( \pi_n(B_{W^{1,1}}(0, K)) \) which by (10) will also cover \( B_{W^{1,1}}(0, K) \). \( \square \)

Now let us prove the existence of an absolutely continuous invariant measure (with density in \( W^{1,1} \)) for the expanding maps. As the reader may notice, the procedure can be generalized to many other cases where a regularization inequality and a compactness statement like Proposition 11 are available.

**Proposition 12.** The transfer operator \( L \) associate to an expanding map has an invariant density \( h \in L^1 \).

**Proof.** Let us consider the sequence
\[
 g_n = \frac{1}{n} \sum_{i=0}^{n-1} L^n 1
\]
where \( 1 \) is the density of the normalized Lebesgue measure. By the Lasota Yorke inequality \( \mathbf{6} \) the sequence has uniformly bounded \( W^{1,1} \) norm and by Proposition 11 has a subsequence \( g_{n_k} \) converging in \( L^1 \) to a limit \( h \).

Now recall that \( L \) is continuous in the \( L^1 \) norm. By this\( \mathbf{12} \)
\[
 Lh = L( \lim_{k \to \infty} g_{n_k} ) = \lim_{k \to \infty} Lg_{n_k} = h.
\]

Then \( h \) is an invariant density. \( \square \)

**Proposition 13.** The density \( h \) found above has the following properties:

- \( h \in W^{1,1} \) and \( ||h|| \leq \frac{(||L^0||_\infty + 1)}{1 - \alpha} ||g_n||_1 \)
- \( ||h|| \leq \frac{(||L^0||_\infty + 1)}{1 - \alpha} \)

**Proof.** Consider \( g_n \) as defined at (11) and \( g_{n,m} = L^m(g_n) \). Remark that \( g_{n,m} \in W^{1,1} \) and the norms are uniformly bounded. By the Lasota Yorke inequality
\[
 ||g_{n_1,a+m} - g_{n_2,b+n}||_{W^{1,1}} \leq \alpha^m ||g_{n_1,a} - g_{n_2,b}||_{W^{1,1}} + B||g_{n_1,a} - g_{n_2,b}||_1 \tag{12}
\]
\( \mathbf{12} \)Remark that by the definition \( g_n = \frac{1}{n}(L^0 + \ldots + L^n 1) \) giving \( ||Lg_{n_k} - g_{n_k}||_1 \leq \frac{2}{n} \).
also remark that if \( \|g_{n_k,0} - h\|_1 \leq \epsilon \) then \( \|g_{n_k,j} - h\|_1 \leq \epsilon \) for all \( j \geq 0 \). Then the sequence \( g_{n_k} \to h \) in \( L^1 \), and by (12) is a Cauchy sequence in the \( W^{1,1} \) norm, indeed, suppose \( k_1 \leq k_2 \)

\[
\|g_{n_{k_1},k_1} - g_{n_{k_2},k_2}\|_{W^{1,1}} \leq \alpha^{k_1}\|g_{n_{k_1},0} - g_{n_{k_2},k_2-k_1}\|_{W^{1,1}} + B\|g_{n_{k_1},0} - g_{n_{k_2},k_2-k_1}\|_1.
\]

Since \( W^{1,1} \) is complete, this implies that it converges in \( W^{1,1} \) to some limit which is forced to be \( h \). Hence \( h \in W^{1,1} \).

By the Lasota Yorke inequality, since \( h \) is invariant then \( \|h\| = \|Lh\| \leq \frac{(\|g\|_{L^\infty})^{\alpha}\|\|\|_{L^\infty} + 1)}{1-\alpha} \).

**Remark 14.** Using the procedure explained in this section, using the Lasota Yorke inequality (remark that starting with the constant density 1 we are constructing at each step a \( C^1 \) function and we are controlling its Lipschitz norm, which is equivalent to the \( C^1 \) norm on \( C^1 \)). It is possible to prove that a \( C^2 \) expanding map has a \( C^1 \) invariant density. In a similar way it is possible to prove that a \( C^3 \) expanding map of the circle has a \( C^2 \) invariant density.

**Remark 15.** The measure \( h_m \), where \( h \) is the smooth invariant density found above is a physical measure for the expanding map \( T \).

In the next section we will also see that such \( h \) is unique.

## 5. Convergence to equilibrium and mixing

In this section we see the concept of convergence to equilbrous, and how to use it to prove that the absolutely continuous invariant measure found for an expanding map in the previous section is unique. We will also see that an expanding map considered with its absolutely continuous invariant measure is a mixing measure preserving transformation.

Let us consider a positive, integral preserving (Markov) operator \( L \) acting on a strong and a weak normed vector spaces \( B_s \subseteq B_w \subseteq L^1 \) as we have seen before.

**Assumptions A.** Let us suppose that:

- \( B_w \) contains the indicator functions of measurable sets
- \( B_s \) is closed under product and there is \( C \geq 1 \) such that for each \( f, g \in B_s \), \( \|fg\|_s \leq C\|f\|_s\|g\|_s \).
- \( B_s \) is dense in \( B_w \) for the \( \| \|_w \) topology
- for each \( f \in B_w \) \( \|f\|_w \geq \int |f| \ dm \) and \( \| \|_{B_s} \geq \| \|_{L^\infty} \).

**Remark 16.** We remark that by a rescaling the assumption \( \|fg\|_s \leq C\|f\|_s\|g\|_s \) can be put in the form \( \|fg\|_s \leq \|f\|_s\|g\|_s \). Considering the new rescaled norm \( \| \|_K = K\| \|_s \) it holds

\[
\|fg\|_K = K\|fg\|_s \leq K^2\|f\|_s\|g\|_s \leq \|f\|_K\|g\|_K.
\]

Let us consider the strong and weak space of zero average densities

\[
V_s = \{g \in B_s \ s.t. \ \int g \ dm = 0\}
\]

and

\[
V_w = \{g \in B_w \ s.t. \ \int g \ dm = 0\}.
\]
If the dynamics has some mixing properties we expect that iterating a zero average density its positive part gets annihilated with the negative part and the iterates will eventually converge to zero in some sense.

This can be expressed more in general for Markov operators and the general definition will be useful to treat deterministic and random dynamical systems with the same tools and general results.

**Definition 17** (Convergence to equilibrium). A Markov operator \( L : B_s \to B_s \) is said to have convergence to equilibrium if for each \( g \in V_s \)

\[
\lim_{n \to \infty} ||L^n g||_w = 0.
\]

We recall the classical definition of mixing

**Definition 18** (Mixing). A measure preserving transformation \((X, T, \mu)\) is said to be mixing if for each measurable \( E, F \subseteq X \)

\[
\lim_{n \to \infty} \mu(E \cap T^{-n}F) = \mu(E) \mu(F).
\]

We now see how a convergence to equilibrium result implies mixing.

**Proposition 19.** Consider \( B_s \) and \( B_w \) satisfying the properties listed at the beginning of the section and the transfer operator \( L : B_w \to B_w \) associated to a map \( T \) having \( \mu = hm \) with \( h \in B_s \) as an invariant probability measure. Suppose that \( h \) is bounded, and that for each \( f \in V_s \)

\[
(14) \quad \lim_{n \to \infty} ||L^n f||_w \to 0
\]

then the system \((T, \mu)\) is mixing.

**Proof.** By the assumptions, given any \( f \in B_s \) with \( \int f \ dm = 1 \) the sequence \( L^n f - h \int f \ dm \to 0 \) in \( B_w \).

Now let us consider two measurable sets \( E \) and \( F \) and the indicator functions \( 1_E, 1_F \in B_w \). We have that

\[
\mu(E \cap T^{-n}F) = \left[ hm \right](E \cap T^{-n}F) = \int 1_E (1_F \circ T^n) h \ dm.
\]

By the density of \( B_s \) in \( B_w \), let us consider \( \epsilon > 0 \) and \( g_\epsilon \in B_s \) such that \( ||1_E - g_\epsilon||_w \leq \epsilon \) thus for each \( n \geq 0 \)

\[
|\mu(E \cap T^{-n}F) - \int g_\epsilon (1_F \circ T^n) h \ dm| \leq ||1_E - g_\epsilon||_w
\]

and

\[
|\mu(E \cap T^{-n}F) - \int 1_F L^n(g_\epsilon) \ dm| \leq \epsilon.
\]

By (14) we have \( h g_\epsilon \to h[\int (h g_\epsilon) \ dm] \) in \( B_w \), then

\[
\int 1_F L^n(g_\epsilon) \ dm \to \int 1_F h[\int (h g_\epsilon) \ dm] \ dm = \int (h1_F) \ dm \cdot \int (h g_\epsilon) \ dm
\]

and since

\[
|\int (h g_\epsilon) \ dm - \int (h1_E) \ dm| \leq \epsilon
\]

we have that for each \( \epsilon \), eventually as \( n \to \infty \)

\[
|\mu(E \cap T^{-n}F) - \mu(E) \mu(F)| \leq 3\epsilon
\]
proving the statement. □

Now we prove that expanding maps have convergence to equilibrium and are mixing. Later we will see that the speed of convergence of this equilibrium is exponential. We consider \(W^{1,1}\) and \(L^1\) as a strong and weak spaces.

**Proposition 20.** For each \(g \in V_s\), it holds

\[
\lim_{n \to \infty} \|L^n g\|_1 = 0.
\]

**Proof.** First let us suppose that \(\|g\|_{Lip} < \infty\). By (9) we know that all the iterates of \(g\) have uniformly bounded \(l\) norm. There is \(\mathcal{M} \geq 0\) such that for each \(n \geq 0\)

\[
\|L^n g\|_{Lip} \leq \mathcal{M}.
\]

Let us denote by \(g^+, g^-\) the positive and negative parts of \(g\). Remark that since \(g \in V_s\), \(\|g\|_1 = 2 \int g^+\ dm\). There is a point \(\bar{x}\) such that \(g^+(\bar{x}) \geq \frac{1}{2}\|g\|_1\). Around this point consider a neighborhood \(N = B(\bar{x}, \frac{1}{4}\|g\|_1^{-1}\mathcal{M}^{-1})\). For each point \(x \in N\), \(g^+(x) \geq \frac{1}{2}\|g\|_1\).

Now let \(d = \min |T'|, D = \max |T'|\). If \(n_1\) is the smallest integer such that \(d^{n_1} (\frac{1}{2}\|g\|_1^{-1}\mathcal{M}^{-1}) > 1\)

\[
(i.e. \frac{\log(2\|g\|_1^{-1}\mathcal{M})}{\log d} + 1 > n_1 > \frac{\log(2\|g\|_1^{-1}\mathcal{M})}{\log d})
\]

then \(T^{n_1}(N) = S^1\) and \(L^{n_1} g^+\) has density

\[
L^{n_1} g^+ \geq \frac{\|g\|_1}{4D^{n_1}} \geq \frac{\|g\|_1}{4De^{\log(2\|g\|_1^{-1}\mathcal{M})/\log d}} = \frac{\|g\|_1}{4D} \left(1 - \frac{1}{2\|g\|_1^{-1}\mathcal{M}^{-1}}\right)^{\frac{\log D}{\log d}}
\]

on \(S^1\). The same is true for \(g^-\) and then, after iterating \(n_1\) times this positive constant part of the density and the corresponding negative one annihilates, and setting \(C = (\frac{1}{4D} \left(1 - \frac{1}{2\|g\|_1^{-1}\mathcal{M}^{-1}}\right)^{\frac{\log D}{\log d}})\), it holds

\[
\|L^{n_1} g\|_1 \leq \|g\|_1 - C\|g\|_1^{\frac{\log D}{\log d} + 1}.
\]

Let us denote \(g_1 = L^{n_1} g\). We can repeat the above construction and obtain \(n_2\) such that

\[
\|g_2\|_1 := \|L^{n_2} g_1\|_1 \leq \|g_1\|_1 - C\|g_1\|_1^{\frac{\log D}{\log d} + 1}
\]

and so on. Continuing, we have a sequence \(g_n\) such that

\[
\|g_{n+1}\|_1 \leq \|g_n\|_1 - C\|g_n\|_1^{\frac{\log D}{\log d} + 1}
\]

and then \(\|g_n\|_1 \to 0\).

If now more generally, \(g \in W^{1,1}\) we can approximate \(g\) with a \(\tilde{g}\) such that \(\|\tilde{g}\|_{Lip} < \infty\) in a way that \(\|g - \tilde{g}\|_1 \leq \epsilon\). Since \(\|L\|_{L^1 \to L^1} \leq 1, \lim_{n \to \infty} \|L^n (g - \tilde{g})\|_1 \leq \epsilon\). And then the statement follows. □

Using Proposition 19 we get
Corollary 21. Expanding maps of the circle, considered with its $W^{1,1}$ invariant measure are mixing.

Corollary 22. For expanding maps of the circle, there is only one invariant measure in $W^{1,1}$.

Proof. If there are two invariant probability densities $h_1, h_2$ then $h_1 - h_2 \in V_s$ and invariant, impossible. □

Corollary 23. The whole sequence $g_n = \frac{1}{n} \sum_{i=0}^{n-1} L^n 1$ converges to $h$.

Remark 24. By Proposition 20 it also follows that, if $g \in W^{1,1}$ is a probability density (and then $g - h \in V_s$) then $L^n g \to h$ in the $L^1$ norm. By the Lasota Yorke inequality we also get the convergence in $W^{1,1}$. Indeed (see the proof of Proposition 13) $|| L^{n+m} (g - h) ||_{w,1} \leq \lambda^n || L^m (g - h) ||_{w,1} + B || L^m (g - h) ||_1$ and letting first $m \to \infty$ and then $n \to \infty$ we get the statement.

5.1. Speed of convergence to equilibrium and decay of correlations. For several applications, it is important to quantify the speed of mixing or convergence to equilibrium.

Let us see how to quantify: consider two vector subspaces of the space of signed (complex) Borel measures on $X$

$B_s \subseteq B_w$

endowed with two norms, the strong norm $|| \cdot ||_s$ on $B_s$ and the weak norm $|| \cdot ||_w$ on $B_w$, such that $|| \cdot ||_s \geq || \cdot ||_w$ on $B_w$.

In the case of operators acting on spaces of measures which are not necessarily absolutely continuous we say that an operator is Markov if it preserves positive measures and for each signed measure $\mu$ it holds $\mu(X) = [L \mu](X)$.

Definition 25. Let us consider $\Phi : N \to \mathbb{R}$ with $\lim_{n \to \infty} \Phi(n) = 0$. We say that a Markov operator $L$ has convergence to equilibrium with speed $\Phi$ with respect to these norms if for any $f \in V_s$.

\begin{equation}
|| L^n f ||_w \leq \Phi(n) || f ||_s.
\end{equation}

We remark that in this case if $\nu$ is a starting probability measure in $B_s$ and $\mu$ is the invariant measure, still in $B_s$, then $\nu - \mu \in V_s$ and then

\begin{equation}
|| L^n \nu - \mu ||_w \leq \Phi(n) || \nu - \mu ||_s.
\end{equation}

and then $L^n \nu$ converges to $\mu$ with a speed $\Phi(n)$. Depending on the strong norm, in certain cases one may prove that for each probability measure $\nu \in B_s$, $|| \nu - \mu ||_s \leq K || \nu ||_s$ where $K$ does not depend on $\nu$, obtaining

\begin{equation}
|| L^n \nu - \mu ||_w \leq K \Phi(n) || \nu ||_s
\end{equation}

5.2. Decay of correlations. Convergence to equilibrium is often estimated or applied in the form of correlation integrals. In this subsection we show an example of how to relate these correlation estimates with the notions of convergence to equilibrium we defined above.

Let us suppose that $X$ is a manifold, let us denote by $m$ the normalized Lebesgue measure, consider non singular transformations, and let us apply the transfer operators to absolutely continuous invariant measures. We consider the transfer operator acting on a strong and a weak space $B_s \subseteq B_w \subseteq L^1$ with norms satisfying the Assumptions $A$ listed at beginning of Section 5.
The reader can easily verify that these assumptions hold for example when $X = S^1$ and $B_s = W^{1,1}$, $B_w = L^1$ and apply the results to expanding maps. Similar arguments apply to other kinds of spaces.

5.2.1. Estimating $\int \psi \circ T^n g \, dm - \int g \, dm \int \psi \, d\mu$. In this subsection we consider a measure preserving transformation $(T, \mu)$ with $\mu = hm$ and we see how the notion of convergence to equilibrium, as in Definition 25 allow the estimate of the following integral

$$\int \psi \circ T^n g \, dm - \int g \, dm \int \psi \, d\mu.$$ 

Lemma 26. Let us consider normed vector spaces $B_s \subseteq B_w$, as described above. Consider the transfer operator $L$ associated to a map having $\mu = hm$ with $h \in B_s$ as an invariant measure. Suppose that there is $\Phi(n) \to 0$ such that for each $f \in V_s$ and $n \geq 0$

$$||L^n f||_w \leq \Phi(n)||f||_{B_s}$$

then for each $\forall \psi \in L^\infty$, $g \in B_s$, $n \geq 0$ we have

$$|\int \psi \circ T^n g \, dm - \int g \, dm \int \psi \, d\mu| \leq 2C\Phi(n)||h||_{B_s}||\psi||_{L^\infty}||g||_{B_s}.$$ 

Proof. Since $\mu$ is invariant and $\psi \in L^\infty$

$$|\int \psi \circ T^n g \, dm - \int g \, dm \int \psi \, d\mu| \leq$$

$$\leq |\int \psi \circ T^n g \, dm - \int g \, dm \int \psi \circ T^n h \, dm|$$

$$\leq |\int \psi \circ T^n (g - h \int g \, dm) \, dm|$$

$$= |\int \psi L^n[g - h \int g \, dm] \, dm|$$

since $g - h \int g \, dm$ is a zero average density in $B_s$

(16) $|\int \psi \circ T^n g \, dm - \int g \, dm \int \psi \, d\mu| \leq ||\psi||_{L^\infty}||L^n[g - h \int g \, dm]||_w$

(17) $\leq \Phi(n)||\psi||_{L^\infty}||g - h \int g \, dm||_{B_s}$

and

$$||(g - \int g \, dm)h||_{B_s} \leq ||gh||_{B_s} + || \int g \, dmh||_{B_s} \leq 2C||h||_{B_s}||g||_{B_s}.$$ 

□

5.2.2. Estimating $\int \psi \circ T^n g \, d\mu - \int g \, d\mu \int \psi \, d\mu$ (a decay of correlation estimate).

Suppose we have a system having invariant measure $\mu$ and convergence to equilibrium with respect to norms $||||_{w}, ||||_{B_s}$ and speed $\Phi$ as above. For many applications it is useful to estimate the speed of decreasing of the following correlation integral

(18) $|\int g \cdot (\psi \circ F^n) \, d\mu - \int \psi \, d\mu \int g \, d\mu|$. 
Lemma 27. Consider the transfer operator $L$ associated to a map having $\mu = h \mu$ with $h \in B_s$ as an invariant measure as above. Suppose that for each $f \in B_s$ such that $\int f \, dm = 0$ and $n \geq 0$

$$||L^nf||_w \leq \Phi(n)||f||_{B_s}$$

then for each $\forall \psi \in L^\infty, g \in B_s$ and $n \geq 0$ it holds

$$\left| \int g \cdot (\psi \circ F^n) \, d\mu - \int \psi \, d\mu \int g \, d\mu \right| \leq 2C[||h||_{B_s} + ||h||^2_{B_s}]\Phi(n)||\psi||_{\infty}||g||_{B_s}.$$  

Proof. First we remark that adding a constant $K$ to $g$ does not change the correlation integral:

$$\int (g + K)(\psi \circ F^n) \, d\mu = \int \psi \, d\mu \int (g + K) \, d\mu - \int \psi \, d\mu \int g \, d\mu.$$  

We then choose $K = -\int g \, d\mu$, we have that $\int g - [\int g \, d\mu] \, d\mu = 0$ and

$$\left| \int g \cdot (\psi \circ F^n) \, d\mu - \int \psi \, d\mu \int g \, d\mu \right| = \left| \int (g - \int g \, d\mu) \cdot (\psi \circ F^n) \, d\mu \right|$$

$$\leq ||\psi||_{\infty}||L^n[(g - \int g \, d\mu)h]||_{w}$$

and since $\int [g - \int g \, d\mu]h \, dm = 0$ the convergence to equilibrium implies

$$\left| \int g \cdot (\psi \circ F^n) \, d\mu - \int \psi \, d\mu \int g \, d\mu \right| \leq ||\psi||_{\infty} \Phi(n)||g - \int g \, d\mu||_{B_s}$$

and

$$||g - \int g \, d\mu|h||_{B_s} \leq ||gh||_{B_s} + ||g||_{B_s} \leq C||h||_{B_s}||g||_{B_s} + \int g \, d\mu ||h||_{B_s} \leq C[||h||_{B_s} + ||h||^2_{B_s}]||g||_{B_s}.$$  

\[\square\]

In the next section we will see how the Lasota Yorke inequality and the properties of the spaces we have chosen allows to prove exponential speed of convergence for our circle expanding maps.
6. Spectral gap and consequences

Now we see a general result that easily implies that the convergence to equilibrium of certain systems is exponentially fast.

We recall some basic concepts on the spectrum of operators. Let $L : B \to B$ be an operator acting on a complex Banach space $(B, \| \cdot \|)$:

- the spectrum of an operator is defined as
  $$\text{spec}(L) = \{ \lambda \in \mathbb{C} : (\lambda I - L) \text{ has no bounded inverse} \}$$

- the spectral radius of $L$ is defined as
  $$\rho(L) = \sup \{|z| : z \in \text{spec}(L)\}.$$

An important connection between the spectral properties of the operator and the asymptotic behavior of its iterates is given by the following formula

**Proposition 28** (Spectral radius formula). Under the above assumptions

$$\rho(L) = \lim_{n \to \infty} \sqrt[n]{\|L^n\|} = \inf_n \sqrt[n]{\|L^n\|}.$$

**Definition 29** (Spectral gap). The operator $L : B \to B$ is said to have spectral gap if

$$L = \lambda P + N$$

where

- $P$ is a projection (i.e. $P^2 = P$) and $\dim(\text{Im}(P)) = 1$;
- the spectral radius of $N$ satisfies $\rho(N) < |\lambda|$;
- $PN = NP = 0$.

The following is an elementary tool to verify spectral gap of $L$ on $B_s$.

**Theorem 30.** Let us consider a Markov operator $L$ acting on two normed vector spaces $(B_s, \| \cdot \|_s)$, $(B_w, \| \cdot \|_w)$, $B_s \subseteq B_w \subseteq CM(X)$ with $\| \cdot \|_s \geq \| \cdot \|_w$ (where $CM(X)$ stands for the set of Borel complex valued measures on $X$). Suppose:

1. (Lasota Yorke inequality). There are $A, B \geq 0$ and $0 \leq \lambda_1 \leq 1$ such that for each $g \in B_s$
   $$\|L^n g\|_s \leq A \lambda_1^n \|g\|_s + B \|g\|_w;$$

2. (Convergence to equilibrium) for each $g \in V_s$, it holds
   $$\lim_{n \to \infty} \|L^n g\|_w = 0;$$

3. (Compact inclusion) the strong zero average space $V_s$ is compactly immersed in the weak one $V_w$ (more precisely, the strong unit ball has a finite $\epsilon$ net in the weak topology for each $\epsilon$);

4. (Weak boundedness) the weak norm of the operator restricted to $V_s$ satisfies
   $$\sup_n \|L^n g\|_w < \infty.$$

Under these assumptions there are $C_2 > 0, \rho_2 < 1$ such that for all $g \in V_s$

$$\|L^n g\|_s \leq C_2 \rho_2^n \|g\|_s.$$
Proof. We first show that assumptions (2) and (3) and (4) imply that $L$ is uniformly contracting from $V_s$ to $V_w$: there is $n_1 > 0$ such that $\forall g \in V_s$

\begin{equation}
\|L^{n_1}g\|_w \leq \lambda_2\|g\|_s
\end{equation}

where $\lambda_2 B < 1$.

Indeed, by (3), for any $\epsilon$ there is a finite set $\{g_i\}_{i \in \{1, \ldots, k\}}$ in the strong unit ball $B$ of $V_s$ such that for each $g \in B$ there is a $g_i \in V_s$ such that $\|g - g_i\|_w \leq \epsilon$.

Hence

$$\sup_{g \in V_s, ||g||_s \leq 1} \sup_{1 \leq i \leq k, v \in \{v \in V_s, s.t. \|v\|_w \leq \epsilon\}} ||L^n(g_i + v)||_w.$$ 

Now, by (4) suppose that $\forall n \ |L^n|_{V_s} \leq M$, then

$$\sup_i ||L^n(g_i + v)||_w \leq \sup_i ||L^n(g_i)||_w + M\epsilon.$$ 

Since $\epsilon$ can be chosen as small as wanted and by (2) for each $i$, $\lim_{n \to \infty} ||L^n(g_i)||_w = 0$ and we have (23) (first fix $\epsilon$ small enough and then choose $i$ big enough).

Let us apply the Lasota Yorke inequality to strengthen (23) to an estimate for the strong norm. For each $f \in V_s$

$$||L^{n_1+m}f||_s \leq A\lambda_1^m ||L^{n_1}f||_s + B||L^{n_1}f||_w$$

then

$$||L^{n_1+m}f||_s \leq A\lambda_1^m \|L^{n_1}f\|_s + B\lambda_2\|f\|_s \leq A\lambda_1^m [A\lambda_1^{n_1}\|f\|_s + B\|f\|_w] + B\lambda_2\|f\|_s.$$ 

If $m$ is big enough

$$||L^{n_1+m}f||_s \leq \lambda_3\|f\|_s$$

with $\lambda_3 < 1$.

This easily implies the statement. Indeed set $n_2 = n_1 + m$, for each $k, q \in \mathbb{N}$, $q \leq n_2$, $g \in V_s$,

$$\|L^{kn_2+q}g\|_s \leq \lambda_3^k ||L^qg||_s \leq \lambda_3^k (\lambda_1^q ||g||_s + B\|g\|_w).$$ 

Implying that for each $g \in V_s$ there are $C_2 > 0, \rho_2 < 1$ such that

\begin{equation}
||L^ng||_s \leq C_2\rho_2^n ||g||_s.
\end{equation}

In the case where $(B_s, ||\cdot||_s)$ is a complex Banach space, by this theorem and the spectral radius formula, the spectral radius of $L$ restricted to $V_s$ is strictly smaller than 1, and the spectral gap as defined in Definition 29 follows.

**Theorem 31.** Under the assumptions of Theorem 29 if $(B_s, ||\cdot||_s)$ is a Banach space then $L$ has spectral gap.

Before the proof we need a preliminary Lemma

**Lemma 32.** Under the assumptions of Theorem 30, if $(B_s, ||\cdot||_s)$ is a Banach space then $L$ has a unique invariant probability measure in $B_s$.

Proof. The proof follows the same construction as in section 4.2 using the compact immersion (assumption 3 of Theorem 30) instead of Proposition 11. □
Proof of Theorem 31 (sketch). Remark that by the Lasota Yorke inequality and the spectral radius formula, the spectral radius of $L$ on $B_s$ is not greater than 1. Since there is an invariant measure in $B_s$ then this radius is 1. By (2) there can be only one fixed probability measure of $L$ in $B_s$ which we denote by $\mu$ (if there were two, consider the difference which is in $V_s$ and iterate...).

Now let us remark that every $g \in B_s$ can be written as follows:

$$g = [g - \mu g(X)] + [\mu g(X)].$$

the function $P : B_s \to B_s$ defined as

$$P(g) = \mu g(X)$$

is a projection. The function $N : B_s \to B_s$ defined as

$$N(g) = L[g - \mu g(X)]$$

is such that $N(B_s) \subseteq V_s$, $N|V_s = L|V_s$, and by (22) it satisfies $\rho(N) < 1$. It holds $L = P + N$ and $PN = NP = 0$. Thus $L$ has spectral gap according to the Definition 29. □

We remark that in several texts the role of Theorem 31 is played by a general result referred to Hennion, Hervé or Ionescu-Tulcea and Marinescu (see e.g. [38], [42]) whose proof is more complicated.

Remark 33. Equation 24 obviously implies exponential convergence to equilibrium with exponential speed.

Remark 34 (spectral gap for expanding maps of the circle). By Proposition 11, Proposition 20 and the Lasota Yorke inequality, the assumptions of Theorem 30 are verified on our expanding maps of the circle for the $W^{1,1}$ norm (with the $L^1$ norm as a weak norm). Then their transfer operator have spectral gap.

6.1. Central limit. We see an application of Theorem 30 to the estimation of the fluctuations of an observable, obtaining a sort of central limit theorem. A proof of the result can be found in [42] (see also Remark 16).

Theorem 35. Let $(X, T, \mu)$ be a mixing nonsingular measure preserving transformation. Consider its associater transfer operator $L$ acting on some Banach spaces $B_s$ and $B_w = L^1$, suppose $B_s$ and $B_w$ satisfy Assumptions A at beginning of Section 5 and that furthermore $B_s$ contains the constant functions. Suppose $L : B_s \to B_s$ has spectral gap.

Let $f \in V_s$. If there is no $\nu \in B$ such that $f = \nu - \nu \circ T$ a.e., then $\exists \sigma > 0$ s.t. for all intervals $[a, b]$,

$$\mu \left\{ x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ T^k \in [a, b] \right\} \to \frac{1}{\sqrt{2\pi}\sigma^2} \int_a^b e^{-t^2/2\sigma^2} dt.$$

We remark, that by Theorem 31 and the general properties we have shown about the spaces $W^{1,1}$ and $L^1$, Theorem 35 applies to expanding maps on $S^1$.

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13Where $g(X)$ stands for the $g$–measure of the whole space.

14Which can be easily adapted to $V_s$, by considering an integral preserving projection $\pi_2 f = \pi f - \int \pi f$. 

7. Stability and response to perturbation

In this section we consider small perturbations of a given system and try to study the dependence of the invariant measure on the perturbation. If the measure varies continuously, we know that many of the statistical properties of the system are stable under perturbation (see [1] and [15] for examples of results in this direction, in several classes of systems) and the system will be said to have statistical stability.

On the other hand it is known that even in relatively simple families of piecewise expanding maps, the physical invariant measure may change discontinuously (see Section 9.2).

We will see that under certain general assumptions related to the convergence to equilibrium of the system and the kind of perturbation, the physical measure changes continuously, and we can estimate quantitatively the modulus of continuity. If stronger, assumptions applies, the dependence can be Lipschitz, or even differentiable.

We remark that with more work, other stability results can be proved for the whole spectral picture of the system and not only for the physical measure (see [38]).

Consider again two vector spaces of measures with sign on $X$ $B_{s} \subseteq B_{w} \subseteq SM(X)$, endowed with two norms, the strong norm $|| ||_{s}$ on $B_{s}$ and the weak norm $|| ||_{w}$ on $B_{w}$, such that $|| ||_{s} \geq || ||_{w}$ as before. Suppose $L_{\delta}(B_{s}) \subseteq B_{s}$ and $L_{\delta}(B_{w}) \subseteq B_{w}$.

**A uniform family of operators.** Let us consider a one parameter family of operators $L_{\delta}$, $\delta \in [0,1)$. Suppose that:

**UF1** (Uniform Lasota Yorke ineq.) There are constants $A, B, \lambda_{1} \geq 0$ with $\lambda_{1} < 1$ such that $\forall f \in B_{s}, \forall n \geq 1, \forall \delta \in [0,1)$ and each operator satisfies a Lasota Yorke inequality.

(25) $|| L_{\delta}^{n} f ||_{s} \leq A \lambda_{1}^{n} || f ||_{s} + B || f ||_{w}.$

**UF2** Suppose that $L_{\delta}$ approximates $L_{0}$ when $\delta$ is small in the following sense: there is $C \in \mathbb{R}$ such that $\forall g \in B_{s}$:

(26) $|| (L_{\delta} - L_{0}) g ||_{w} \leq \delta C || g ||_{s}.$

**UF3** Suppose that $L_{0}$ has exponential convergence to equilibrium, with respect to the norms $|| ||_{w}$ and $|| ||_{s}$.

**UF4** (The weak norm is not expanded) There is $M$ such that $\forall \delta, n, g \in B_{s}$ $|| L_{\delta}^{n} g ||_{w} \leq M || g ||_{w}$.

We will see that under these assumptions we can ensure that the invariant measure of the system varies continuously (in the weak norm) when $L_{0}$ is perturbed to $L_{\delta}$ for small values of $\delta$. We will also provide a quantitative estimate for the modulus of continuity.

**Remark 36.** We remark that UF3, UF4 and UF1 together implies that $L_{0}$ eventually contracts exponentially fast the zero average space $V_{s}$. Indeed let $f \in V_{s}$,
using the inequality and then the convergence to equilibrium
\[ ||L_0^{n+m}f||_s \leq A\lambda_1^n ||L_0^m f||_s + B||L_0^m f||_w \]
\[ \leq A\lambda_1^n ||L_0^m f||_s + BEX_2^m ||f||_s \]
\[ \leq A\lambda_1^n (B + A)||f||_s + BEX_2^m ||f||_s \]
by which there are \( n, m \) big enough that \( ||L_0^{n+m}f||_s \leq \frac{1}{2}||f||_s \).

7.1. Stability of fixed points, a general statement. We state a general result on the stability of fixed points of Markov operators satisfying certain assumptions. This will be a flexible tool to obtain the stability of the invariant measure under small perturbations.

Let us consider two operators \( L_0 \) and \( L_\delta \) preserving spaces \( B_s \subseteq B_w \subseteq \text{SM}(X) \) with norms \( || ||_s, || ||_w \). Let us suppose that \( f_0, f_\delta \in B_s \) are fixed probability measures, respectively of \( L_0 \) and \( L_\delta \).

**Lemma 37.** Suppose that:

a): \( ||L_\delta f_\delta - L_0 f_\delta||_w < \infty \)

b): \( \exists C_i \) s.t. \( \forall g \in B, ||L_0^i g||_w \leq C_i ||g||_w \) (compare with UF4)

Then for each \( N \)
\[ ||f_\delta - f_0||_w \leq ||L_0^N (f_\delta - f_0)||_w + ||L_\delta f_\delta - L_0 f_\delta||_w \sum_{i \in [0,N-1]} C_i. \]

**Proof.** The proof is a direct computation
\[ ||f_\delta - f_0||_w \leq ||L_0^N f_\delta - L_0^N f_0||_w \]
\[ \leq ||L_0^N f_0 - L_\delta^N f_\delta||_w + ||L_\delta^N f_\delta - L_\delta^N f_\delta||_w \]
Hence
\[ ||f_0 - f_\delta||_w \leq ||L_0^N (f_0 - f_\delta)||_w + ||L_\delta^N f_\delta - L_\delta^N f_\delta||_w \]
but
\[ L_0^N - L_\delta^N = \sum_{k=1}^N L_0^{N-k}(L_0 - L_\delta)^k L_\delta^{k-1} \]
and
\[ (L_0^N - L_\delta^N) f = \sum_{k=1}^N L_0^{N-k}(L_0 - L_\delta)^k L_\delta^{k-1} f_\delta \]
\[ = \sum_{k=1}^N L_0^{N-k}(L_0 - L_\delta) f_\delta \]
by item b)
\[ ||(L_0^N - L_\delta^N) f_\delta||_w \leq \sum_{k=1}^N C_{N-k} ||(L_0 - L_\delta) f_\delta||_w \]
\[ \leq ||(L_0 - L_\delta) f_\delta||_w \sum_{i \in [0,N-1]} C_i \]
then
\[ ||f_\delta - f_0||_w \leq ||L_0^N (f_0 - f_\delta)||_w + ||(L_0 - L_\delta) f_\delta||_w \sum_{i \in [0,N-1]} C_i. \]
Now, let us apply the statement to our family of operators satisfying assumptions UF 1,...,4. On can fix $C_i = M$. We have the following

**Proposition 38.** Suppose $L_\delta$ is a uniform family of operators satisfying UF1,...,4. $f_0$ is the unique invariant probability measure of $L_0$, $f_\delta$ is an invariant probability measure of $L_\delta$. Then

$$||f_\delta - f_0||_w = O(\delta \log \delta).$$

**Proof.** We remark that by the uniform Lasota Yorke inequality $||f_\delta||_s \leq M$ are uniformly bounded.

Hence

$$||L_\delta f_\delta - L_0 f_\delta||_w \leq \delta CM$$

(see item a) of Lemma [37]. Moreover by UF4, $C_i \leq M_2$.

Hence

$$||f_\delta - f_0||_w \leq \delta \log \delta ||f_\delta - f_0||_w.$$ 

Now by the exponential convergence to equilibrium of $L_0$

$$||L_0^N (f_\delta - f_0)||_w \leq C_2 \rho_2^N ||(f_\delta - f_0)||_s \leq C_2 \rho_2^N M$$

hence

$$||f_\delta - f_0||_w \leq \delta \log \delta ||f_\delta - f_0||_w \leq \delta \log \delta \log \delta \log \rho_2^2 M + C_2 \rho_2^N M$$

choosing $N = \left\lfloor \frac{\log \delta}{\log \rho_2} \right\rfloor$

$$||f_\delta - f_0||_w \leq \delta \log \delta \log \delta \log \rho_2^2 M + C_2 \rho_2^N M$$

(28)

$$||f_\delta - f_0||_w \leq \delta \log \delta \log \rho_2^2 M + C_2 \rho_2^N M \leq \delta \log \delta \log \rho_2^2 M + C_2 \delta M.$$ 

□

**Remark 39.** We remark that in this statement we did not really use the Lasota Yorke inequality in its full strength. We used it only to get $||f_\delta||_s \leq M$. Moreover the statement could be generalized to slower than exponential convergence to equilibrium (see [24]) obtaining other kinds of continuity relations. In the following sections we apply these statements to some classes of maps, we remark that the modulus of continuity $\delta \log \delta$ is sharp for Piecewise Expanding map (see Section 9).

**Remark 40.** We remark that in UF2 the size of the perturbation is measured in the strong-weak norm, i.e. as an operator: $B_s \rightarrow B_w$. This allows general perturbations (allowing to move discontinuities, like when making small perturbation in the Skorokhod distance, see Eq. [68]) furthermore even in the differentiable case, small perturbations of the map in the $C^k$ norm induce small perturbations of the associated transfer operator in the strong-weak norm (see also Proposition [44]). Measuring the size of the perturbation in the strong-strong norm, will lead to stronger results, like Lipschitz or differentiable stability (see Sections 7, 8). However the typical perturbations one is interested to put on a deterministic dynamical system (perturbing a $C^k$ map slightly in the $C^k$ norm e.g.) are not small in the strong-strong norm.
7.2. Application to expanding maps. In the previous section we considered the stability of the invariant measure under small perturbations of the transfer operator.

There are many kinds of interesting perturbations to be considered. Two main classes are deterministic or stochastic ones.

In the deterministic ones the transfer operator is perturbed by small changes on the underlying dynamics (the map).

The stochastic ones can be of several kinds. The simplest one is the adding of some noise perturbing the result of the deterministic dynamics at each iteration (see [38] for some example and related estimations).

We now consider small deterministic perturbations of our expanding maps on $S^1$. Let us consider an expanding map $T_0$ and a one parameter family $T_\delta$, $\delta \in [0, 1]$ of expanding maps of the circle satisfying the properties stated at beginning of Section 4 and

UFM: $||T_\delta - T_0||_{C^2} \leq K \delta$ for some $K \in \mathbb{R}$.

To each of these maps it is associated a transfer operator $T_\delta$. We now prove that the transfer operators of a uniform family of expanding maps satisfy the general property UF2 and this will allow to apply our general quantitative stability results.

**Proposition 41.** If $L_0$ and $L_\delta$ are transfer operators of expanding maps $T_0$ and $T_\delta$, satisfying UFM, then there is a $C \in \mathbb{R}$ such that $\forall g \in W^{1,1}$:

$$
|| (L_\delta - L_0)f || \leq \delta C ||f||_{W^{1,1}}
$$

and assumption UF2 is satisfied.

**Proof.** We have that the transfer operator is defined by the formula

$$
[L_\delta f](x) = \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T_\delta'(y)|}.
$$

$$
|[L_\delta f](x) - [L_0 f](x)| = | \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T_\delta'(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T_0'(y)|} | \\
\leq | \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T_\delta'(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T_0'(y)|} | + \\
+ | \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T_\delta'(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T_0'(y)|} |.
$$

The first summand can be estimated as follows

$$
| \sum_{y \in T_\delta^{-1}(x)} \frac{f(y)}{|T_\delta'(y)|} - \sum_{y \in T_0^{-1}(x)} \frac{f(y)}{|T_0'(y)|} | \leq \sum_{y \in T_\delta^{-1}(x)} \frac{|f(y)|}{|T_\delta'(y)|} |1 - \frac{|T_\delta'(y)|}{|T_0'(y)|}| \\
\leq D_1(\delta) \sum_{y \in T_\delta^{-1}(x)} \frac{|f(y)|}{|T_\delta'(y)|} \\
\leq D_1(\delta) ||f||_{W^{1,1}}.
$$
where \( D_1(\delta) = \sup_{y \in S^1} |1 - \frac{T^i(\delta)}{T^i(\delta)}| \) and remark that \( D_1 = O(\delta) \). For second summand let us denote \( T_{\delta}^{-1}(x) = \{y_1, \ldots, y_n\} \), \( T_{\delta}^{-1}(x) = \{y_1^0, \ldots, y_n^0\} \). Let \( \Delta_y = O(\delta) \) (see Lemma 57 and its proof or [30] Lemma 3.2 for details)

\[
| \sum_{y \in T_{\delta}^{-1}(x)} \frac{f(y)}{|T_0(y)|} - \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T_0(y)|} | \leq \left| \sum_{i=1}^{n} \frac{f(y_i) - f(y_i^0)}{|T_0(y_i)|} \right| + \left| \sum_{i=1}^{n} f(y_i^0) \left( \frac{1}{|T_0(y_i)|} - \frac{1}{|T_0(y_i^0)|} \right) \right|
\]

\[
\leq \left| \sum_{i=1}^{n} \frac{f(y_i) - f(y_i^0)}{|T_0(y_i)|} \right| + \left| \sum_{i=1}^{n} \frac{f(y_i^0)}{|T_0(y_i^0)|} \cdot \left| T_0(y_i) - 1 \right| \right|
\]

\[
\leq \left| \sum_{i=1}^{n} \frac{f(y_i) - f(y_i^0)}{|T_0(y_i)|} \right| + D_2(\delta) \sum_{i=1}^{n} \frac{f(y_i^0)}{|T_0(y_i^0)|} + D_2(\delta) |L_0 f(x)|
\]

where \( D_2(\delta) := \sup_{x,i} \left| \frac{T^i(y_i^0)}{|T_0(y_i)|} - 1 \right| = O(\delta) \). Hence

\[
|L_\delta f - L_0 f|_1 \leq D_1(\delta) |L_\delta f(x)|_1 + \sum_{i=1}^{n} \frac{\int_{y_i^0} f'(t) dt}{|T_0(y_i)|} |L_0 f(x)|_1
\]

\[
\leq (D_1(\delta) + D_2(\delta)) |f(x)|_1 + \sum_{i=1}^{n} \frac{\int_{y_i^0} f'(t) dt}{|T_0(y_i)|} |f(x)|_1
\]

\[
\leq O(\delta) |f(x)|_1 + \sum_{i=1}^{n} \frac{\int_{y_i^0} f'(t) dt}{|T_0(y_i)|} |f(x)|_1
\]

\[
\leq O(\delta) |f(x)|_1 + \sum_{i=1}^{n} \frac{\int_{0, \Delta_y} |f'(t)| dt}{|T_0(y_i)|} |f(x)|_1
\]

\[
\leq O(\delta) |f(x)|_1 + |L_\delta[1_{0, \Delta_y}] * |f'||_1
\]

\[
\leq O(\delta) |f(x)|_1 + |1_{0, \Delta_y}| |f'|_1
\]

\[
\leq O(\delta) |f(x)|_1 + |1_{0, \Delta_y}| |f'|_1
\]

\[
\leq O(\delta) |f(x)|_1 + |1_{0, \Delta_y}| |f'|_1
\]

where \([1_{0, \Delta_y}] * |f'| \) stands for the convolution function between the characteristic of the interval \([0, \Delta_y]\) (mod 1) and \(|f'| \). And the statement is proved. 

By the above estimates one can get a quantitative statistical stability estimate for expanding maps and small deterministic perturbations.

**Corollary 42.** Let \( T_0 \) a \( C^2 \) expanding map and \( T_\delta \) be a family of expanding maps satisfying the assumption UFM above. Let \( h_0 \) be the family of invariant measures in \( L^1 \) for the maps \( T_\delta \). Then

\[
|h_0 - h_\delta|_1 = O(\delta \log \delta).
\]

**Proof.** It is easy to verify that for \( \delta \) small enough the map \( T_\delta \) satisfy the assumptions at beginning of Section 4 and the associated transfer operators satisfy
UF1, UF3, UF4 uniformly. By Proposition 41 UFM implies UF2. This allows to apply Proposition 38 and obtain the result. □

Remark 43. We will see in the next sections that in the case of expanding maps one can be able to prove more precise estimates on the stability of the physical measure to deterministic perturbations of the system. A stability result similar to Corollary 42 also applies to suitable deterministic perturbations of piecewise expanding maps (see Section 9).

7.2.1. Further small Perturbation estimates. In this subsection we show that small perturbations of an expanding map induces a small perturbation of the associated transfer operator when considered as acting from stronger to weaker Sobolev spaces.

Proposition 44. If $L_0$ and $L_{\delta}$ are transfer operators of $C^4$ expanding maps $T_0$ and $T_{\delta}$, such that for some $K \in \mathbb{R}$

\[ ||T_{\delta} - T_0||_{C^2} \leq K\delta \]

then there is a $C \in \mathbb{R}$ such that $\forall f \in W^{1,1}$:

\[ ||(L_{\delta} - L_0)f||_{W^{1,1}} \leq \delta C ||f||_{W^{2,1}}. \]  

If furthermore

\[ ||T_{\delta} - T_0||_{C^3} \leq K\delta \]

then

\[ ||(L_{\delta} - L_0)f||_{W^{2,1}} \leq \delta C ||f||_{W^{3,1}}. \]

Proof. In Proposition 43 it is shown that if $L_0$ and $L_{\delta}$ are transfer operators of expanding maps $T_0$ and $T_{\delta}$, such that for some $K \in \mathbb{R}$

\[ ||T_{\delta} - T_0||_{C^2} \leq K\delta \]

then there is a $C \in \mathbb{R}$ such that $\forall g \in W^{1,1}$:

\[ ||(L_{\delta} - L_0)f||_{1} \leq \delta C ||f||_{W^{2,1}}. \]

and the first line of (31) is established. From this we can also recover the second line, indeed in our case we have an explicit formula for the transfer operator:

\[ [L_0f](x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'_0(y)|}. \]

Considering that $T'_0(y) = T'_0(T_0^{-1}(y))$ we can compute the derivative of (4)

\[ (L_0f)' = \sum_{y \in T_0^{-1}(x)} \frac{1}{(T'_0(y))^2} f'(y) - \frac{T''_0(y)}{(T'_0(y))^3} f(y). \]

And similarly for $L_{\delta}$. Note that

\[ (L_0f)' = L_0\left(\frac{1}{T_0} f'\right) - L_0\left(\frac{T''_0}{(T_0)^2} f\right). \]
Hence
\[
\|(L_{\delta} - L_0)f\|_{W^{1,1}} \leq \|((L_{\delta} - L_0)f)\|_{L^1} + \|((L_{\delta} - L_0)f)\|_{L^1} \\
\leq \delta C\|f\|_{W^{1,1}} + ||((L_{\delta} - L_0)(\frac{1}{T_0}f') - (L_{\delta} - L_0)(\frac{T''_0}{(T_0)^2} f))\|_{L^1} \\
+ ||(\frac{1}{T_0}f') - (\frac{1}{T_0}f') + (\frac{T''_0}{(T_0)^2} f) - (\frac{T''_0}{(T_0)^2} f)\|_{L^1} \\
\leq \delta C(||f||_{W^{1,1}} + \|\frac{1}{T}f'\|_{W^{1,1}} + \|\frac{T''}{(T)^2} f\|_{W^{1,1}}) \\
+ ||T_{\delta} - T_0||_{C^2}||f||_{W^{1,1}} \\
\leq \delta C_2||f||_{W^{2,1}}
\]
for some $C_2 \geq 0$ depending on $T_0$ but not on $f$. This proves (31).

To prove (32) one has to take a further derivative, applying (5) again, and involving further derivatives of $T_0$ and $f$, but leading to a similar computation and final result.

7.3. Uniform family of operators and uniform $V_s$ contraction. Now we show how a suitable uniform family of nearby operators, not only has a certain stability on the invariant measure as seen above, but also a uniform rate of contraction of the space $V_s$ and hence a uniform convergence to equilibrium and spectral gap (we remark that stability results on the whole spectral picture are known, see [38], [3] e.g.).

Consider two vector subspaces of the space of signed measures on $X$
\[
B_s \subseteq B_w \subseteq SM(X),
\]
endowed with two norms, the strong norm $|| \cdot ||_s$ on $B_s$ and the weak norm $|| \cdot ||_w$ on $B_w$, such that $|| \cdot ||_s \geq || \cdot ||_w$. Denote as before by $V_s, V_w$ the "zero average" strong and weak spaces.

**Proposition 45** (Uniform $V_s$ contraction for the family of operators). Let us consider a one parameter family of operators $L_\delta$, $\delta \in [0,1)$. Suppose that they satisfy UF1,...,UF4, then there are $\lambda_1 < 1$ and $A_2, \delta_0 \geq 0$ such that for each $\delta \leq \delta_0$ and $f \in V_s$
\[
||L_\delta f||_s \leq A_2 \lambda_1^k ||f||_s.
\]

(36)

We remark that the contraction rate of the zero average space for the operator $L_0$ can be obtained simply by applying directly Remark 36. Before the proof of Proposition 45 we need the following

**Lemma 46.** Suppose that $L_0$ satisfies a Lasota Yorke inequality
\[
||L_0^k g||_s \leq A \lambda_1^k ||g||_s + B ||g||_w
\]
and the following holds

- $\forall g \in B_s$, $||(L_\delta - L_0)g||_w \leq C\delta ||g||_s$;
- there is $M \geq 0$ such that $\forall \delta, n, g \in B_s$, $||L_\delta^n g||_w \leq M ||g||_w$;

then $L_\delta^n$ approximates $L_0^n$ in the following sense: there are constants $E, D \geq 0$ such that $\forall g \in B_s, \forall n \geq 0$
\[
||((L_\delta^n - L_0^n)g)||_w \leq \delta (E ||g||_s + nD ||g||_w).
\]

(37)
Proof. Developing $L_0^n - L_0^m$ as a telescopic sum as done before

\[ ||(L_0^n - L_0^m)g||_w \leq \sum_{k=1}^{n} ||L_0^{n-k}(L_0 - L_0)L_0^{k-1}g||_w \leq M \sum_{k=1}^{n} ||(L_0 - L_0)L_0^{k-1}g||_w \]

\[ \leq M \sum_{k=1}^{n} \delta C ||L_0^{k-1}g||_s \]

\[ \leq \delta MC \sum_{k=1}^{n} (A\lambda_1^{-1} ||g||_s + B||g||_w) \]

\[ \leq \delta MC \left( \frac{A}{1 - \lambda_1} ||g||_s + Bn||g||_w \right). \]

\[ \square \]

Proof of Proposition 46. Let us apply the Lasota Yorke inequality

\[ ||L_0^{n+m}f||_s \leq A\lambda_1^n ||L_0^m f||_s + B||L_0^m f||_w \]

by the assumption UF3, and Lemma 10

\[ ||L_0^{n+m}f||_s \leq A\lambda_1^n ||L_0^m f||_s + B[F\lambda_2^m ||f||_s + \delta(\epsilon)||f||_s + mD||f||_w] \]

\[ \leq A\lambda_1^n [A\lambda_2^n ||f||_s + B||f||_w] + B[F\lambda_2^m ||f||_s + \delta(\epsilon)||f||_s + mD||f||_w]. \]

If $n, m$ are big enough suitably chosen and $\delta$ small enough, then we have that there is a $\lambda_3 < 1$ such that for each $f \in V_s$

\[ ||L_0^{n+m}f||_s \leq \lambda_3 ||f||_s \]

thus there are $\lambda_4 < 1, A_2 \in \mathbb{R}$ such that for each $f \in V_s$ and $\delta$ small enough

(38) \[ ||L_0^k f||_s \leq A_2\lambda_4^k ||f||_s. \]

\[ \square \]

The following Lemma establishes the continuity of the resolvent when the operator is perturbed, and will be used in the following.

Lemma 47. Let us suppose that $B_w$ and $B_s$ are Banach spaces. Let us consider a one parameter family of operators $L_\delta$, $\delta \in [0, 1)$. Suppose that they satisfy UF1,...,UF4. Consider the resolvent operator $(Id - L_\delta)^{-1} : V_s \rightarrow V_w$

\[ (Id - L_\delta)^{-1} := \sum_{k=0}^{\infty} L_\delta^k \]

which is well defined and continuous thanks to (38) and the completeness of $B_w$. Under these assumptions we have

\[ \lim_{\delta \rightarrow 0} ||(Id - L_\delta)^{-1} - (Id - L_0)^{-1}||_{V_s \rightarrow V_w} = 0. \]

Proof. Let us fix $\epsilon > 0$ and let us prove that for $\delta$ small enough $||(Id - L_\delta)^{-1} - (Id - L_0)^{-1}||_{V_s \rightarrow V_w} \leq \epsilon$. By (38) there is there is $\delta > 0$ and $n \geq 0$ such that for each $0 \leq \delta \leq \delta$ $\sum_{k=n}^{\infty} ||L_\delta^k||_{V_s \rightarrow V_w} \leq \frac{\epsilon}{2}$. By Lemma 10 there is $\delta > 0$ such for each
\[ k \leq n, \quad \| (L^k_\delta - L^k_0) \|_{B_s \rightarrow B_w} \leq \frac{\delta}{3^k}. \]

Then considering \( g \in V_s \) with \( \| g \|_s \leq 1 \) we have for each \( 0 \leq \delta \leq \delta \)
\[ \| (I - L_\delta)^{-1} g - (I - L_0)^{-1} g \|_w \leq \| \sum_{k=n}^{\infty} L^k_\delta g \|_w + \| \sum_{k=n}^{\infty} L^k_0 g \|_w + \| \sum_{k=0}^{n} (L^k_\delta - L^k_0) g \|_w \leq \epsilon. \]

\[ \square \]

7.4. Lipschitz continuity. Let us suppose \( B_w \) and \( B_s \) are a Banach spaces as above. Now we see that exploiting the uniform contraction rate of \( V_s \) and some further assumptions we can prove Lipschitz dependence of the relevant invariant measure under system perturbations (see [38] for similar reasoning on expanding maps). Further work also lead to differentiable dependence (see next section).

**Proposition 48.** Let us consider a uniform family \( L_\delta, \delta \in [0,1) \) of operators satisfying UF1,...,UF4. Suppose that each operator \( L_\delta \) has a unique invariant probability measure \( h_\delta \) in \( B_s \). Suppose furthermore that there is \( C_{h_0} \) such that for \( \delta \) small enough
\[ \| (L_\delta - L_0) h_0 \|_s \leq \delta C_{h_0} \]

then the map \( \delta \rightarrow h_\delta \) is Lipschitz (with respect to the strong norm)
\[ \| h_0 - h_\delta \|_s \leq O(\delta). \]

**Proof.** Denote \( \Delta h = h_\delta - h_0 \):
\[ (I - L_\delta) \Delta h = (I - L_\delta)(h_\delta - h_0) = h_\delta - L_\delta h_\delta - h_0 + L_\delta h_0 = (L_\delta - L_0) h_0. \]

By the uniform contraction \([39]\) we have that \( (I - L_\delta) \) is invertible on \( V_s \), and \( (I - L_\delta)^{-1} = \sum_{k=0}^{\infty} L^k_\delta \) is uniformly bounded and there is \( M_2 \geq 0 \) such that for \( \delta \) small enough \( \| (I - L_\delta)^{-1} \|_{V_s \rightarrow V_s} \leq M_2. \)

Since \( (L_\delta - L_0) h_0 \in V_s \), then
\[ \Delta h = (I - L_\delta)^{-1} (L_\delta - L_0) h_0. \]

and since \( \| (L_\delta - L_0) h_0 \|_s \leq \delta C_{h_0} \)
\[ \| \Delta h \|_s \leq \delta M_2 C_{h_0}. \]

hence we have the statement. \( \square \)

**Remark 49.** We remark that \([39]\) contains a "small perturbation" estimate similar to UF2, but on the strong topology. The result can be easily applied to a suitable family of expanding maps satisfying UF1,...,UF4 and \([39]\) obtaining Lipschitz statistical stability on the strong norm for this family of maps. In Section [8.1.1] we show a set of easy to be verified conditions on the family implying \([39]\).
8. Some general Linear Response statements

In this section we prove that when a system has fast enough convergence to equilibrium and it is perturbed smoothly in with respect to the strong norm then its invariant measure (and then its statistical properties) changes in a smooth way. This is called Linear Response. We refer to [5] for a general introduction to this kind of problems and a survey of recent results. In the following we show a general and simple result (Theorems 50, 53) allowing to prove linear response for a quite large set of systems and perturbations. A different approach to prove a Linear Response result (even for higher derivatives) was also provided in [43]. Other general results can be found in [8].

Let $X$ be a compact metric space. Let us consider some complete normed vector subspaces $(B_{ss}, \| \cdot \|_{ss}) \subseteq (B_{s}, \| \cdot \|_{s}) \subseteq (B_{w}, \| \cdot \|_{w}) \subseteq SM(X)$ of the space of signed Borel measures on $X$, $SM(X)$, with norms satisfying $\| \cdot \|_{w} \leq \| \cdot \|_{s} \leq \| \cdot \|_{ss}$.

We will assume that the linear form $\mu \to \mu(X)$ is continuous on $B_{i}$, for $i \in \{ss, s, w\}$.

We will consider Markov operators acting on these spaces, the following (closed) spaces $V_{i} := \{ \mu \in B_{i} | \mu(X) = 0 \}$ where $i \in \{ss, s, w\}$, will play an important role. If $A, B$ are two normed vector spaces and $L : A \to B$ we denote the mixed norm $\| L \|_{A \to B}$ as

$$\| L \|_{A \to B} := \sup_{f \in A, \| f \|_{A} \leq 1} \| Lf \|_{B}.$$ 

Let us consider a system having a transfer operator $L_{0}$, some $\delta_{0} > 0$ and a family of "nearby" system $L_{\delta}$ with $\delta \in [0, \delta_{0}]$ and suppose the operators $L_{\delta}$ preserve the spaces $L_{\delta}(B_{ss}) \subset B_{ss}, L_{\delta}(B_{s}) \subset B_{s}$ and $L_{\delta}(B_{w}) \subset B_{w}$.

We now enter more in details about what we mean by linear response and the motivation behind the search of results establishing the differentiable behavior of the invariant measure with respect to a small perturbation. Let us consider a family of dynamical systems $(X, T_{\delta})$ and suppose that $\mu_{\delta}$ is a physical measure for $T_{\delta}$. Suppose that the invariant measure varies in a smooth way, and after a small perturbation $p$ of size $\delta$ we know that in some sense

$$(41) \quad \frac{\mu_{\delta} - \mu_{0}}{\delta} \to \dot{\mu}.$$ 

Consider an observable $f$ and its time average $\lim_{n \to \infty} \frac{S_{f}(x)}{n}$ (see (1)). If the topology in which (41) converges is strong enough, on the basin of $f$ we get

$$\frac{d}{d\delta} \left( \lim_{n \to \infty} \frac{S_{f}(x)}{n} \right) = \int f d\mu_{\delta} - \int f d\mu_{0} \to \int f d\dot{\mu}.$$ 

This shows how the Linear Response controls the behavior of long time averages of observables and the statistical behavior of the system under perturbations.

Let us see an heuristic argument to compute a Linear Response formula, giving $\dot{\mu}$ as a function of the initial system and of the perturbation applied.
By using that \( \mu_0 \) and \( \mu_\delta \) are fixed points of their respective operators \( L_0, L_\delta \) we obtain that
\[
(Id - L_0) \frac{\mu_\delta - \mu_0}{\delta} = \frac{1}{\delta} (L_\delta - L_0) f_\delta.
\]
If the convergence to equilibrium is fast enough, like done in the proof of Proposition 48 one can prove that the resolvent \((Id - L_0)^{-1}\) is well defined and continuous. By applying the resolvent to (42) one gets
\[
(Id - L_0)^{-1}(Id - L_0) \frac{\mu_\delta - \mu_0}{\delta} = (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} \mu_\delta
\]
\[
= (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} \mu_0
\]
\[
+ (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (\mu_\delta - \mu_0)
\]
and then
\[
\frac{\mu_\delta - \mu_0}{\delta} = (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} \mu_0 + (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (\mu_\delta - \mu_0).
\]
if we choose the right topologies:
- \( \frac{\mu_\delta - \mu_0}{\delta} \) tends to the Linear response \( \dot{\mu} \).
- \( (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} f_0 \) tends to \( (Id - L_0)^{-1} \dot{L} f_0 \)
- \( (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (f_\delta - f_0) \) tends to zero

We remark that the first result we are going to see (Theorem 50) applies to systems having less than exponential convergence to equilibrium. Examples of application of this statement in this case are outside the scope of these lectures. In next section we apply the statement to expanding maps.

We recall that the speed of convergence to equilibrium of the system (see Definition 25) is measured by the speed of contraction to 0 of the zero average spaces \( V_s \) and \( V_w \) we suppose that the topology of \( B_s \) is strong enough so that \( V_s \) is a closed subspace of \( B_s \).

**Theorem 50** (Linear Response, summable decay). Let \( L_\delta \) a family of Markov operators preserving \( B_{ss}, B_s \) and \( B_w \) as above. Suppose that for each \( \delta \in [0, \delta] \) there is \( f_\delta \in B_{ss} \) which is a fixed probability measure of \( L_\delta \). Suppose the system satisfy the following:

1. (summable convergence) there is \( \phi \) such that \( \sum \phi(u) < \infty \), such that \( L_0 \) has convergence to equilibrium with respect \( B_s, B_w \) and speed \( \dot{\phi} \). (remark that by this \( f_0 \) is the unique fixed point of \( L_0 \)).
2. (strong statistical stability) \( \lim_{\delta \to 0} ||f_\delta - f_0||_{ss} = 0 \);
3. (derivative operator) suppose there is \( \dot{L} : B_{ss} \to V_s \) continuous such that for each \( f \in B_{ss} \)
\[
\lim_{\delta \to 0} ||\left(\frac{L_\delta - L_0}{\delta}\right)f - \dot{L} f||_s = 0
\]
then
\[
\lim_{\delta \to 0} ||\left(\frac{f_\delta - f_0}{\delta}\right) - (1 - L_0)^{-1} \dot{L} f_0||_w = 0
\]
where \( (1 - L_0)^{-1} := \sum_{\infty} L_0^i \) is a continuous operator \( V_s \to V_w \).
Proof. Let $f \in B_s$. Since $\sum_0^\infty \phi(n) < \infty$, then $(1 - L_0)^{-1} f := \sum_0^\infty L_\delta^n f$ converges in $B_w$ and defines a continuous operator $V_s \to V_w$. It also holds $||(1 - L_0)^{-1}||_{B_s \to B_w} \leq \sum_0^\infty \phi(n)$.

Denote $\Delta f = f_\delta - f_0$

$$\frac{(I - L_0) \Delta f}{\delta} = (I - L_0) \frac{f_\delta - f_0}{\delta}$$

$$= \frac{1}{\delta} (f_\delta - L_0 f_\delta - f_0 + L_0 f_0)$$

$$= \frac{1}{\delta} (L_\delta - L_0) f_\delta.$$

$$\frac{(1 + L_0 + ... + L_0^n) (I - L_0) \Delta f}{\delta} = (1 + L_0 + ... + L_0^n) \frac{L_\delta - L_0}{\delta} f_\delta$$

$$\frac{\Delta f}{\delta} - L_0^{n+1} \frac{\Delta f}{\delta} = (1 + L_0 + ... + L_0^n) \frac{L_\delta - L_0}{\delta} (f_\delta + f_0 - f_0)$$

$$\frac{\Delta f}{\delta} - L_0^{n+1} \frac{\Delta f}{\delta} = (1 + L_0 + ... + L_0^n) \frac{L_\delta - L_0}{\delta} f_0$$

Letting $n \to \infty$, since $\Delta f \in V_s$, by convergence to equilibrium, it holds that $L_0^{n+1} \frac{\Delta f}{\delta} \to 0$ in the weak norm. Thus

$$\frac{\Delta f}{\delta} = (1 - L_0)^{-1} \frac{L_\delta - L_0}{\delta} f_0 + (1 - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (f_\delta - f_0)$$

as elements of $B_w$. Now by the strong statistical stability

$$||(1 - L_0)^{-1} \frac{L_\delta - L_0}{\delta} (f_\delta - f_0)||_w \leq (\sum_I ||L_0^n||_{B_s \to B_w}) ||\tilde{L}||_{B_s \to B_w} ||f_\delta - f_0||_w \to 0$$

then in the weak norm, as $\delta \to 0$

$$\hat{\mu} = \lim_{\delta \to 0} \frac{\Delta f}{\delta} = (1 - L_0)^{-1} \tilde{L} f_0.$$

\[\square\]

Remark 51. Theorem [27] is quite abstract and it is stated for families of operators. In particular it may be adapted both to stochastic or deterministic perturbations of (deterministic of stochastic) systems. One key point is the existence of the derivative operator (assumption 3). The form of this operator is strictly related to the kind of perturbation considered.

In the following section we will compute this operator for smooth perturbations of expanding maps, see [8], [28] or [26] for the derivative operator in a stochastic case.

Remark 52. If $||L_0^n(g)||_s \leq \phi(n)||g||_s$ with $\phi(n)$ summable (prove that this is equivalent to exponential contraction of the zero average space). Then $(1 - L_0)^{-1}$ is defined $B_s \to B_s$ and the conclusion of the theorem is reinforced: $\lim_{\delta \to 0} ||\frac{L_\delta - L_0}{\delta} - (1 - L_0)^{-1} \tilde{L} f_0||_w = 0$. 
Theorem 50 requires a weak assumption on the decay of correlation, which is only assumed to be summable and only checked at the unperturbed transfer operator $L_0$, on the other hand it requires the strong statistical stability of the system.

$$\lim_{\delta \to 0} ||f_\delta - f_0||_{ss} = 0.$$  

Sometimes is easy to verify some uniform convergence to equilibrium for the family of perturbed systems, like in Section 7.3. We then show a linear response result exploiting this uniform estimate in the place of the strong statistical stability.

**Theorem 53.** Suppose that for $\delta \in [0, \bar{\delta})$ there is a probability measure $v_\delta \in B_s$ such that

$$L_\delta v_\delta = v_\delta$$

and that there is $\hat{L}v_0 \in B_s$ such that

$$\lim_{\delta \to 0} \frac{L_\delta - L_0}{\delta} v_0 - \hat{L}v_0||_{s} = 0.$$

Suppose the resolvent operator is defined and bounded from $V_s$ to $V_w$, $||(Id - L_\delta)^{-1}||_{V_s \to V_w} = M < +\infty$ and

$$\lim_{\delta \to 0} ||(Id - L_\delta)^{-1} - (Id - L_0)^{-1}||_{V_s \to V_w} = 0.$$

Then

$$\lim_{\delta \to 0} \frac{v_\delta - v_0}{\delta} - (Id - L_0)^{-1} \hat{L}v_0||_{V_w} = 0.$$

**Proof.** We have that for each $\delta \in [0, \bar{\delta})$, $v_\delta$ is a fixed point of $L_\delta$. Using this we get

$$(Id - L_\delta) \frac{v_\delta - v_0}{\delta} = \frac{v_\delta - v_0}{\delta} - \frac{L_\delta v_\delta - L_\delta v_0}{\delta}$$

$$= \frac{-v_0 + L_\delta v_0}{\delta}$$

$$= \frac{1}{\delta}(L_\delta - L_0)v_0.$$

We remark that for each $\delta$, $L_\delta$ preserves $V_s$. Since $\forall \delta > 0$, $\frac{L_\delta - L_0}{\delta} v_0 \in V_s$ and $(Id - L_\delta)^{-1} : V_s \to V_w$ is a bounded operator, we can apply the resolvent both sides and get

$$\frac{v_\delta - v_0}{\delta} = (Id - L_\delta)^{-1} \frac{L_\delta - L_0}{\delta} v_0$$

$$= (Id - L_\delta)^{-1} \frac{L_\delta - L_0}{\delta} v_0 - (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} v_0$$

$$+ (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} v_0.$$

Since $||(Id - L_\delta)^{-1} - (Id - L_0)^{-1}||_{V_s \to V_w} \to 0$ we have

$$||((Id - L_\delta)^{-1} - (Id - L_0)^{-1}) \frac{L_\delta - L_0}{\delta} v_0||_{w} \leq \frac{1}{\delta} \frac{L_\delta - L_0}{\delta} v_0||_{V_w} \to 0.$$

$$\frac{1}{\delta} \frac{L_\delta - L_0}{\delta} v_0||_{V_w} \to 0.$$
Since \( \lim_{\delta \to 0} \frac{L_\delta - L_0}{\delta} v_0 \) converges in \( V_\epsilon \), then \( \text{(43)} \) implies that in the \( B_w \) topology
\[
\lim_{\delta \to 0} \frac{v_\delta - v_0}{\delta} = \lim_{\delta \to 0} (Id - L_0)^{-1} \frac{L_\delta - L_0}{\delta} v_0 = (Id - L_0)^{-1} \lim_{\delta \to 0} \frac{L_\delta - L_0}{\delta} v_0 = (Id - L_0)^{-1} [\hat{L} v_0].
\]
\[\Box\]

8.1. Applying the general theorems to expanding maps. In this subsection we show how to get a linear response result for small deterministic perturbations of expanding maps, applying Theorem \( \text{53} \) Let \( T_\delta : S^1 \to S^1 \) be a family of \( C^3 \) expanding orientation preserving maps of the circle \( X \) where \( \delta \in (0, \bar{\delta}) \). Let us suppose that the dependence of the family on \( \delta \) is differentiable at 0, hence can be written
\[
T_\delta(x) = T_0(x) + \delta \hat{T}(x) + o_{C^3}(\delta) \text{ for } x \in X
\]
where \( \hat{T} \in C^3(X, \mathbb{R}) \), and \( o_{C^3}(\delta) \) denotes a term whose \( C^3 \) norm tends to zero faster than \( \delta \), as \( \delta \to 0 \).
\[\text{We will see that if } \hat{T} \text{ is small enough, UF1,...,UF4 are satisfied by the associated transfer operators, when applied to suitable Sobolev spaces } W^{k,1} \text{ and then, provided we establish the existence of the derivative operator, Theorem } \text{53} \text{ can be applied.}
\]

**Definition 54.** A set \( A_{M,L} \) of expanding maps is called a uniform \( C^k \) family with parameters \( M \geq 0 \) and \( L > 1 \) if it satisfies uniformly the expansiveness and regularity condition: \( \forall T \in A_{M,L} \)
\[
\|T\|_{C^k} \leq M, \inf_{x \in S^1} |T'(x)| \geq L.
\]

We already proved in Section \( \text{6} \) Lasota Yorke inequalities for the transfer operators associated to such maps, acting on Sobolev spaces, with a similar proof on can obtain a general result (see \( \text{28} \), Lemma 29 and its proof).

**Lemma 55.** Let \( A_{M,L} \) be a uniform \( C^k \) family of expanding maps, the transfer operators \( L_T \) associated to a \( T \in A_{M,L} \) satisfy a uniform Lasota-Yorke inequality on \( W^{1,1}(S^1) \): for each \( 1 \leq i \leq k - 1 \) there are \( \alpha < 1 \), \( A_i, B_i \geq 0 \) such that for each \( n \geq 0 \), \( T \in A_{M,L} \)
\[
\|L^n_T f\|_{W^{i-1,1}} \leq A_i \|f\|_{W^{i-1,1}},
\]
\[
\|L^n_T f\|_{W^{i,1}} \leq \alpha^{in} \|f\|_{W^{i,1}} + B_i \|f\|_{W^{i-1,1}}.
\]

From this last result, and the compact immersion of \( W^{k,1} \) in \( W^{k-1,1} \) it is classically deduced that the transfer operator \( L_T \) of a \( C^k \) expanding map \( T \) has spectral gap on each \( W^{i,1}(S^1) \), with \( 1 \leq i \leq k - 1 \).

Since a family of maps \( T_\delta \) as in \( \text{(45)} \) for \( \delta \) small enough is a uniform family in the sense of Definition \( \text{54} \), then Lemma \( \text{55} \) applies to their transfer operators, and then UF1 and UF4 are verified considering \( B_s = W^{1,1} \) and \( B_w = L^1 \). UF2 is

\[\text{More precisely we say that } T_\delta \text{ is a differentiable family of } C^3 \text{ expanding maps if there exists } \epsilon \in C^3(X, \mathbb{R}) \text{ such that } \| (T_\delta - T_0)/\delta - \epsilon \|_{C^3} \to 0 \text{ as } \delta \to 0, \text{ where}
\]
\[
\|f(x)\|_{C^3} = \sup_{x \in X} |f(x)| + \sup_{x \in X} |f'(x)| + \sup_{x \in X} |f''(x)| + \sup_{x \in X} |f'''(x)|
\]
is the usual norm on \( C^3 \) functions.
also verified by Proposition \[11\] and UF3 is proved in Section \[5\]. Since Proposition \[17\] allow to establish the stability of the resolvent for these perturbations, to apply Theorem \[53\] we only need to verify the existence of the derivative transfer operator.

8.1.1. The derivative operator and linear response for circle expanding maps. In the following we show how to obtain the existence of the derivative operator for a smooth family of expanding maps. Let us consider \( T_\delta : S^1 \to S^1 \) be a family of \( C^3 \) expanding maps as in \[14\].

We hence have a family of \( C^3 \) expanding maps, and each one of them has an invariant density \( f_\delta \) in \( C^2 \) Remark \[13\]. The following proposition present a detailed description of the structure of the operator \( L : C^2 \to W^{1,1} \) in our case.

**Proposition 56.** Let \( w \in C^2(S^1, \mathbb{R}) \). For each \( x \in S^1 \) we can write
\[
\dot{L}w(x) = \lim_{\delta \to 0} \left( \frac{L_\delta w(x) - L_0 w(x)}{\delta} \right)
= -L_0 \left( \frac{\dot{T}^{\ast}}{T_0^{\ast}} \right) w(x) + L_0 \left( \frac{\dot{T} T'}{T_0^{\ast}} w(x) \right)
\]

and the convergence is also in the \( C^1 \) topology.

Before presenting the proof of Proposition \[56\] we state a technical lemma.

**Lemma 57.** Let \( T_\delta : S^1 \to S^1 \), where \( \delta \in (0, \bar{\delta}) \) be a family of \( C^2 \) expanding maps. Let us suppose that the dependence of the family on \( \delta \) is differentiable at 0 in the following sense
\[
(49) \quad T_\delta(x) = T_0(x) + \delta \dot{T}(x) + o_C(\delta)
\]

where \( \dot{T} \in C^2(S^1, \mathbb{R}) \), and \( o_C(\delta) \) denotes a function \( f(x) \in C^k(S^1) \) satisfying
\[
\lim_{\delta \to 0} \frac{|f(\delta)|}{\delta} = 0.
\]

Under these assumptions, if \( y^\delta \in T_\delta^{-1}(x) \) then when \( \delta \to 0 \) we can expand
\[
y^\delta = y^0 + \delta \left( -\frac{\dot{T}(y^0)}{T_0(y^0)} \right) + o_C(\delta).
\]

**Proof of Lemma \[57\].** Let us fix \( x \in S^1 \) and write
\[
(50) \quad y^\delta(x) = y^0(x) + \delta \epsilon(x) + F_\delta(x)
\]

where for each \( x \), \( F_\delta(\delta, x) = o(\delta) \) as \( \delta \to 0 \). We will show that \( \epsilon(x) = -\frac{\dot{T}(y^0(x))}{T_0(y^0(x))} \) for each \( x \), then we will show that \( F_\delta(\delta, x) = o_C(\delta) \).

For the first claim, let us fix \( x \in S^1 \). Substituting \[49\] into the identity \( T_\delta(y^\delta(x)) = x \) we can expand
\[
(51) \quad x = T_\delta(y^\delta(x))
\]
\[
(52) \quad = T_0(y^\delta(x)) + \delta \dot{T}(y^\delta(x)) + E(\delta, x)
\]

where \( E(\delta, x) = o_C(\delta) \) and then by \[50\]
\[
(53) \quad x = T_0(y^0(x) + \delta \epsilon(x) + F_\delta(\delta, x))
\]
\[
(54) \quad + \delta \dot{T}(y^\delta(x)) + \delta \epsilon(x) + F_\delta(\delta, x) + E(\delta, x).
\]

Since \( T_0 \in C^2 \) we can write the first term in the right hand side of \[53\] as
\[
(55) T_0(y^0(x) + \delta \epsilon(x) + F_\delta(\delta, x)) = T_0(y^0(x)) + T_0'(y^0(x))(\delta \epsilon(x) + F_\delta(\delta, x))
\]
\[
(56) \quad + o(\delta \epsilon(x) + F_\delta(\delta, x)).
\]
Since $T \in C^2$ we can write the second term of (53) as

$$
\delta \dot{T}(y_i^0(x)\delta \epsilon_i(x) + F_i(\delta, x)) = \delta \dot{T}(y_i^0(x)) + \delta \dot{T}'(y_i^0(x))(\delta \epsilon_i(x) + F_i(\delta, x)) + \delta o((\delta \epsilon_i(x) + F_i(\delta, x))).
$$

(58)

and use that $T_0(y_i^0(x)) = x$ to cancel terms on either side of (53) to get that

$$
0 = T_0(y_i^0(x))(\delta \epsilon_i(x) + F_i(\delta, x)) + \delta \dot{T}(y_i^0(x)) + \delta \dot{T}'(y_i^0(x))(\delta \epsilon_i(x) + F_i(\delta, x)) + o(\delta \epsilon_i(x) + F_i(\delta, x)).
$$

For each fixed $x$, as $\delta \rightarrow 0$ we can then identify the relation among the first order terms (dividing by $\delta$ and letting $\delta \rightarrow 0$) as

$$
\delta T_0(y_i^0(x))\epsilon_i(x) + \delta \dot{T}(y_i^0(x)) = 0
$$

giving $\epsilon_i(x) = -\frac{T(y_i^0)}{T_0(y_i^0)}$. Now we have $F_i(\delta, x) = y_i^\delta(x) - y_i^0(x) + \delta \frac{T_i(y_i^0(x))}{T_i(y_i^0(x))}$. We remark since each $T_\delta$ is uniformly $C^2$ and $T' > 1$ we have that $|\frac{\partial F_i(\delta, x)}{\partial x}|$ and $|\frac{\partial^2 F_i(\delta, x)}{\partial x^2}|$ are uniformly bounded for each $\delta$, $x$ and $i$. Thus $||\frac{F_i(\delta, x)}{\delta}||_{C^1} \rightarrow 0$ as $\delta \rightarrow 0$ and $F_i(\delta, x) = o_{C^1}(\delta)$. □

We now return to the proof of Proposition 56.

Proof of Proposition 56. Let us again denote by $\{y_i^\delta\}_{i=1}^d := T^{-1}_\delta(x)$ and $\{y_i^0\}_{i=1}^d := T^{-1}_0(x)$ the $d$ preimages under $T_\delta$ and $T_0$, respectively, of a point $x \in X$. Furthermore, we assume that the indexing is chosen so that $y_i^\delta$ is a small perturbation of $y_i^0$, for $1 \leq i \leq d$. We can write

$$
\frac{L_\delta w(x) - L_0 w(x)}{\delta} = \frac{1}{\delta} \left( \sum_{i=1}^d \frac{w(y_i^\delta) - w(y_i^0)}{T_{\delta}(y_i^\delta)} - \sum_{i=1}^d \frac{w(y_i^0)}{T_{0}(y_i^0)} \right) = \frac{1}{\delta} \left( \sum_{i=1}^d \frac{w(y_i^\delta)}{T_{\delta}(y_i^\delta)} \left( \frac{1}{T_{\delta}(y_i^\delta)} - \frac{1}{T_{0}(y_i^\delta)} \right) \right)
$$

$$
+ \frac{1}{\delta} \left( \sum_{i=1}^d \frac{w(y_i^0)}{T_{0}(y_i^0)} \left( \frac{1}{T_{\delta}(y_i^\delta)} - \frac{1}{T_{0}(y_i^0)} \right) \right).
$$

To develop the the first term we differentiate the expansion $T_\delta(x) = T_0(x) + \delta \dot{T}(x) + o_{C^1}(\delta)$ in $x$ to get:

$$
T_\delta(x) = T_0(x) + \delta \dot{T}(x) + o_{C^2}(\delta).
$$
We can then write

\[
(I) = \frac{1}{\delta} \left( \sum_{i=1}^{d} w(y_i^0) \left( \frac{1}{T'_i(y_i^0)} - \frac{1}{T_0(y_i^0)} \right) \right)
\]

\[
= \frac{1}{\delta} \left( \sum_{i=1}^{d} \frac{w(y_i^0)}{T'_i(y_i^0)} \left( 1 - T'_i(y_i^0) \right) \right)
\]

\[
= \frac{1}{\delta} \left( \sum_{i=1}^{d} \frac{w(y_i^0)}{T'_i(y_i^0)} \left( 1 - \left( T'_i(y_i^0) + \delta \hat{T}_i'(y_i^0) + o_{C^2}(\delta) \right) \right) \right)
\]

\[
= \left( - \sum_{i=1}^{d} \frac{w(y_i^0) \hat{T}_i'(y_i^0)}{T'_i(y_i^0)T_0(y_i^0)} \right) + o_{C^2}(1).
\]

Thus we have that

\[
\lim_{\delta \to 0} \frac{1}{\delta} \left( \sum_{i=1}^{d} w(y_i^0) \left( \frac{1}{T'_i(y_i^0)} - \frac{1}{T_0(y_i^0)} \right) \right) = \lim_{\delta \to 0} \left( - \sum_{i=1}^{d} \frac{w(y_i^0) \hat{T}_i'(y_i^0)}{T'_i(y_i^0)T_0(y_i^0)} \right)
\]

\[
= -L_0 \left( \frac{w \hat{T}}{T_0} \right)
\]

and by Lemma $\ref{lemma57}$ the limit also converges in $C^1$.

For the second term of $(8.1.1)$ we remark that by Lagrange theorem, for any small $h$ there is $\xi$ such that $|\xi| \leq |h|$ and

\[
w(y_i^0 + h) = w(y_i^0) + hw'(y_i^0) + \frac{1}{2} h^2 w''(\xi)
\]

considering Lemma $\ref{lemma57}$ and setting $h = y_i^\delta - y_i^0 = \delta \left( - \frac{T(y_i^0)}{T_0(y_i^0)} \right) + o_{C^1}(\delta)$ we get

\[
w(y_i^\delta) = w(y_i^0) + \delta \left( - \frac{\hat{T}(y_i^0)}{T_0(y_i^0)} \right) + o_{C^1}(\delta)
\]

\[
(59)
\]

\[
(60)
\]

\[
(61)
\]

Since $w''$ is uniformly bounded, then

\[
w(y_i^\delta) = w(y_i^0) + w'(y_i^0) \delta \left( - \frac{\hat{T}(y_i^0)}{T_0(y_i^0)} \right) + o_{C^1}(\delta).
\]

Thus

\[
(II) = \frac{1}{\delta} \sum_{i=1}^{d} \frac{w(y_i^\delta) - w(y_i^0)}{T_0(y_i^0)} = \sum_{i=1}^{d} \frac{w'(y_i^0) T(y_i^0)}{T_0(y_i^0)} \left( - \frac{\hat{T}(y_i^0)}{T_0(y_i^0)} \right) + o_{C^1}(1)
\]

\[
= - \sum_{i=1}^{d} \frac{\hat{T}(y_i^0) w'(y_i^0)}{T_0(y_i^0)T_0(y_i^0)} + o_{C^1}(1)
\]
and therefore, both pointwise and in the $C^1$ topology
\[
\lim_{\delta \to 0} \frac{1}{\delta} \sum_{i=1}^{d} \frac{w(y_i^\delta) - w(y_i^0)}{T'_0(y_i^0)} = -L_0 \left( \frac{\hat{T}w'}{T'_0} \right)(x).
\]

Finally, for the third term we can write
\[
T'_0(y_i^\delta) = T'_0(y_i^0) + T''_0(y_i^0) \left( \frac{dy_0^\delta}{d\delta}_{|\delta=0} \delta + o_{C^1}(\delta) \right)
= T'_0(y_i^0) + T''_0(y_i^0) \left( \frac{\hat{T}(y_i^0)}{T'_0(y_i^0)} \delta + o_{C^1}(\delta), \right)
\]

Again using the Lemma\textsuperscript{57} we get
\[
(III) = \frac{1}{\delta} \left( \sum_{i=1}^{d} w(y_i^0) \left( \frac{1}{T'_0(y_i^0)} - \frac{1}{T'_0(y_i^0)} \right) \right)
= \frac{1}{\delta} \left( \sum_{i=1}^{d} w(y_i^0) \left( \frac{T''_0(y_i^0)}{T'_0(y_i^0)} \right) \right)
= \frac{1}{\delta} \left( \sum_{i=1}^{d} w(y_i^0) \left( - \frac{T'_0(y_i^0) + T''_0(y_i^0) \left( \frac{\hat{T}(y_i^0)}{T'_0(y_i^0)} \delta + T'_0(y_i^0) \right)}{T'_0(y_i^0)^2} \right) \right) + o_{C^1}(1)
= \left( \sum_{i=1}^{d} w(y_i^0) \left( \frac{\hat{T}(y_i^0)T''_0(y_i^0)}{T'_0(y_i^0)^2} \right) \right) + o_{C^1}(1)
\]
and thus, finally,
\[
\lim_{\delta \to 0} \frac{1}{\delta} \left( \sum_{i=1}^{d} w(y_i^0) \left( \frac{1}{T'_0(y_i^0)} - \frac{1}{T'_0(y_i^0)} \right) \right) = L_0 \left( \frac{\hat{T}T''}{T'_0^2} w \right)(x)
\]
in $C^1$. \hfill \Box

We remark that the $C^1$ convergence implies the convergence in the $W^{1,1}$ topology. Since we have obtained the existence of the operator $\hat{L} : C^2 \to W^{1,1}$, we know that $T_0$ has spectral gap and then exponential contraction on the space of zero average $W^{1,1}$ densities, since the assumption of Theorem\textsuperscript{53} have been verified in the last paragraph before Section\textsuperscript{5.3.1} we can apply the theorem and get

**Proposition 58.** Let us assume that $T_\delta$ is a $C^1$ family of $C^3$ expanding maps as in (45). Let $f_\delta \in C^2(X, \mathbb{R})$ be the invatiant probability density of $T_\delta$. We have the following
\[
\lim_{\delta \to 0} \left\| \frac{f_\delta - f_0}{\delta} - (1 - L_0)^{-1} \hat{L}f_0 \right\|_{L^1} = 0.
\]
Where $\hat{L}$ is the operator defined in Proposition\textsuperscript{56}. Hence we have a linear response formula
\[
\lim_{\delta \to 0} \frac{f_\delta - f_0}{\delta} = (1 - L_0)^{-1} \hat{L}f_0
\]
where the limit converge is in $L^1$. 

8.2. Rigorous numerical methods for the computation of invariant measures. We briefly mention an application of the transfer operator methods exposed in these notes. It is possible to use the quantitative stability results on the invariant measures here explained to design efficient numerical methods for the approximation of the invariant measure and other important features of the statistical behavior of the system.

The approximation can also be rigorous, in the sense that an explicit bound on the approximation error can be provided (for example it is possible to approximate the absolutely continuous invariant measure of a system up to a small explicitly given error in the $L^1$ distance). We remark that doing in this way, the result of the computation has a mathematical meaning and can be used for computer aided proofs.

This can be done by approximating the transfer operator $L_0$ of the system by a suitable finite rank one $L_\delta$ which is essentially a matrix, of which we can compute fixed points and other properties.

There are many ways to construct a suitable approximating operator $L_\delta$ depending on the system which is considered. The most used one (for $L^1$ approximations) is the so called Ulam discretization.

In the Ulam Discretization method, the phase space $X$ is discretized by a partition $I_\delta = \{ I_i \}$ (where $\delta$ is a resolution parameter, for example the maximum diameter of the elements of the partition) and the system is approximated by a (finite state) Markov Chain. In the deterministic case, supposing the dynamics is generated by a map $T : X \to X$, the transition transition probabilities of the approximating Markov chain are given by

\begin{equation}
P_{ij} = m(T^{-1}(I_j) \cap I_i)/m(I_i)
\end{equation}

(where $m$ is the normalized Lebesgue measure on the phase space). The approximated operator $L_\delta$ can be also seen in the following way: let $F_\delta$ be the $\sigma-$algebra associated to the partition $I_\delta$, let us consider the projection on the step functions supported on the elements of the partition given by

$$
\pi_\delta(f) = \mathbb{E}(f|F_\delta)
$$

($\mathbb{E}$ is the conditional expectation), we define the approximated operator as:

\begin{equation}
L_\delta = \pi_\delta L \pi_\delta.
\end{equation}

In a series of works it was proved that in several cases the fixed probability measure $f_\delta$ of $L_\delta$ converges to the fixed point of $L$ as the accuracy of the approximation gets better and better. Explicit bounds on the error have been given, rigorous methods implemented and experimented in several classes of cases (see e.g. [7], [12], [13], [18], [39] and [22] where several computations on nontrivial systems are also shown). Ulam methods and similar methods have been also used to rigorously compute (up to prescribed errors) other important quantities related to the statistical properties of dynamics, as Linear Response (see [8]), dimension of attractors (see e.g. [23]) or diffusion coefficients (see e.g. [9]). Some more details on the Ulam method will be given in Section 9.3 in the case of Piecewise Expanding maps of the interval.
9. PIECEWISE EXPANDING MAPS

We now consider a class of maps on the interval which are expanding, but allow discontinuities. This class is interesting and was much studied because it presents a quite rich behavior, while being approachable with techniques similar to the ones introduced in the previous sections. In the following we outline the main properties and tools which allow the study of the statistical properties of these maps when seen as deterministic dynamical systems. Since the theory of the statistical behavior of piecewise expanding maps was exposed in several books (see [19] or [44]) we will not enter in the technical details, but we will focus on similarities and differences between the behavior of these systems and the behavior of expanding maps treated in the previous sections.

Definition 59. We call a nonsingular function $T : ([0,1],m) \to ([0,1],m)$ piecewise expanding if

- There is a finite set of points $d_1 = 0, d_2, \ldots, d_n = 1$ such that for each $i$, $T_i := T|_{(d_i,d_{i+1})}$ is $C^2$ and $\sup_{[0,1]} |T''(x)| dx < \infty$.
- $\inf_{x \in [0,1]} |T'(x)| > 1$ on the set where it is defined.

The transfer operator associated to a map of this class has general properties similar to the ones of the expanding maps, if we apply the transfer operator to measures having a density we obtain the following explicit formula (see e.g. [19] chapter 4 for details) describing the action of the associated transfer operator (which we continue denoting with $L$) on measure densities

$$[Lf](x) = \sum_{i \leq n} \frac{f(T_i^{-1}x)1_{T_i(d_i,d_{i+1})}}{|T'(T_i^{-1}x)|}$$

where $1_{T_i(d_i,d_{i+1})}$ is the characteristic function of the interval $T_i(d_i,d_{i+1})$.

In the presence of discontinuities of the map $T$ the transfer operator does not necessarily preserve spaces of continuous densities. For this the introduction of a suitable space of regular densities including discontinuous functions is important.

9.1. Bounded variation and the Lasota Yorke inequality for piecewise expanding maps. Let $\phi : [0,1] \to \mathbb{R}$ a real function. Let $\{x_1, \ldots, x_k\} \subseteq [0,1]$ be a finite sequence of points. Let us define the variation of $\phi$ with respect to $\{x_1, \ldots, x_k\}$ as...
we define the variation of \( \phi \) as the supremum of \( \text{Var}_{\{x_1,\ldots,x_k\}}(\phi) \) over all the finite sequences \( \{x_1,\ldots,x_k\} \)

\[
\text{Var}(\phi) = \sup_{\{x_1,\ldots,x_k\} \subseteq [0,1]} \text{Var}_{\{x_1,\ldots,x_k\}}(\phi).
\]

We say that \( \phi \) has bounded variation if \( \text{Var}(\phi) < \infty \). We call \( BV \) or \( BV[0,1] \) the set of bounded variation functions on the interval. An important property of bounded variation functions is the following

**Theorem 60** (Helly selection principle). Let \( \phi_n \) be a sequence of bounded variation functions on the interval \([0,1]\) such that \( \text{Var}(\phi_n) \leq M \) and \( ||\phi_n||_1 \leq M \) are uniformly bounded.

Then there is \( \phi \in BV \) and subsequence \( \phi_{n_k} \) such that

\[
\phi_{n_k} \to \phi
\]

in \( L^1 \) (and almost everywhere).

Bounded variation functions are preserved by the transfer operator of a piecewise expanding map, moreover the following Lasota Yorke inequality can be proved.

**Theorem 61.** Let \( T \) be a piecewise expanding map with branches \( T_i \) on the intervals \( I_i \). Let \( \phi \) a bounded variation density on an interval \( I_i = [d_i, d_{i+1}] \) and \( T : I_i \to [0,1] \) a piecewise expanding function. Then

\[
\text{Var}(L_T(\phi)) \leq \frac{2}{\inf_{I_i}(T')} \text{Var}(\phi) + (\sup_{[0,1]}(\frac{T''}{T'}) + \frac{2}{\inf_{I_i}|I_i|}) \int |\phi|.
\]

Before the proof we consider the behavior of the operator when acting on the density on a single interval \( I_i \).

**Lemma 62.** Let \( \phi \) a bounded variation density on an interval \( I_i \) and \( T : I_i \to [0,1] \) an invertible expanding function. Then

\[
\text{Var}(L_T(\phi)) \leq \frac{2}{\inf_{I_i}(T')} \text{Var}(\phi) + (\sup_{I_i}(\frac{T''}{T'}) + \frac{2}{|I_i|}) \int |\phi|.
\]

**Proof.** Let us consider \( y_1,\ldots,y_k \in [0,1] \) and let us suppose there are \( h_1, h \in \mathbb{N} \) and \( x_1,\ldots,x_h \in I_i \) such that \( T^{-1}y_i = \{x_{i-h_1+1} if \; h_1 \leq i \leq h_1 + h \}

\[
\text{Var}_{y_1,\ldots,y_k}(L_T(\phi)) = \sum_{i=1}^{k-1} |L_T(\phi(y_i)) - L_T(\phi(y_{i+1}))|
\]

\[
\leq \frac{1}{T'(x_{h_1})} |\phi(x_{h_1})| + \frac{1}{T'(x_{h_1}+h)} |\phi(x_{h_1}+h)| + \sum_{h_1}^{h+b-1} \frac{1}{T'(x_i)} |\phi(x_i) - \frac{1}{T'(x_{i+1})} \phi(x_{i+1})|
\]
Since there must be \( \hat{x} \in I_i \) such that \( \phi(\hat{x}) \leq \frac{1}{|I_i|} \int |\phi| \), then
\[
\frac{1}{T'(x_{i+1})} |\phi(x_{i+1})| + \frac{1}{T'(x_{i+1} + h)} |\phi(x_{i+1} + h)| \leq \frac{1}{\inf_{I_i}(T')} (2\phi(\hat{x}) + |\phi(x_{i+1}) - \phi(\hat{x})| + |\phi(x_{i+1} + h) - \phi(\hat{x})|) \leq \frac{1}{\inf_{I_i}(T')} \Var(\phi) + \frac{2}{|I_i|} \int |\phi|.
\]

The other summand can be bounded by
\[
\sum_{h_i}^{|h_i + h - 1|} \left| \frac{1}{T'(x_i)} \phi(x_i) - \frac{1}{T'(x_{i+1})} \phi(x_{i+1}) \right| \leq \sum_{h_i}^{|h_i + h - 1|} \left| \frac{1}{T'(x_i)} \phi(x_i) - \frac{1}{T'(x_i)} \phi(x_{i+1}) \right| + \left| \frac{1}{T'(x_i)} \phi(x_{i+1}) - \frac{1}{T'(x_{i+1})} \phi(x_{i+1}) \right| \leq \frac{1}{\inf_{I_i}(T')} \Var(\phi) + \sum_{h_i}^{|h_i + h - 1|} \left| \frac{1}{T'(x_i)} \phi(x_{i+1}) - \frac{1}{T'(x_{i+1})} \phi(x_{i+1}) \right| \leq \frac{1}{\inf_{I_i}(T')} \Var(\phi) + \sum_{h_i}^{|h_i + h - 1|} \frac{T''(\xi_i)}{T'(\xi_i)} |x_i - x_{i+1}| \phi(x_{i+1})
\]
by Lagrange theorem, for \( \xi_i \in [x_i, x_{i+1}] \). And
\[
\frac{1}{\inf_{I_i}(T')} \Var(\phi) + \sum_{h_i}^{|h_i + h - 1|} \frac{T''(\xi_i)}{T'(\xi_i)} |x_i - x_{i+1}| \phi(x_{i+1}) \leq \frac{1}{\inf_{I_i}(T')} \Var(\phi) + \sup_{I_i} \left| \frac{T''}{T'^2} \right| \sum_{h_i}^{|h_i + h - 1|} |x_i - x_{i+1}| \phi(x_{i+1})|.
\]

Remarking that for some \( \epsilon > 0 \), if the subdivision \( \{x_i\} \) is fine enough then \( \sum_{h_i}^{|h_i + h - 1|} |x_i - x_{i+1}| \phi(x_{i+1})| \leq \int_{I_i} \phi + \epsilon \) and collecting all summands we have the statement. \( \Box \)

This allow to conclude

Proof. (of Thm.\ref{61}) Let \( \phi_i = \phi|_{I_i} \). We have that \( L_T \phi = \sum_i L_T \phi_i \). Then
\[
\Var(L_T \phi) \leq \sum_i \Var(L_T \phi_i)
\]
by Lemma \ref{62}
\[
\sum_i \Var(L_T \phi_i) \leq \sum_i \frac{2}{\inf_{I_i}(T')} \Var(\phi_i) + \left( \sup_{I_i} \left| \frac{T''}{T'^2} \right| \right) \frac{2}{|I_i|} \int |\phi_i| \leq \frac{2}{\inf_{[0,1]}(T')} \Var(\phi) + \left( \sup_{[0,1]} \left| \frac{T''}{T'^2} \right| \right) \frac{2}{\inf_{[0,1]}|I_i|} \int |\phi| \leq \Var(\phi) + \|f\|_1.
\]

One can define the Bounded Variation norm \( ||f||_{BV} \) as
\[
||f||_{BV} = \Var(f) + ||f||_1.
\]
It is immediate to deduce from (65) the Lasota Yorke inequality for the bounded variation norm: there is $B$ such that
\[ \| L_T \phi \|_{BV} \leq \frac{2}{\inf_{I_i} (T')} \| \phi \|_{BV} + B \| \phi \|_1. \]

Like done before for expanding maps, writing $\lambda = \frac{2}{\inf_{I_i} (T')}$, and iterating the inequality we obtain
\[ \| L^n_T \phi \|_{BV} \leq \lambda^n \| \phi \|_{BV} + B \left( \frac{1}{1 - \lambda} \right) \| \phi \|_1. \]

This inequality holds for maps such that $\frac{2}{\inf_{I_i} (T')} < 1$. The inequality can be applied to the other piecewise expanding maps by previously iterating the map until $\frac{2}{\inf_{I_i} (T')} < 1$.

In this case, as before, a straightforward computation lead to the general Lasota Yorke inequality valid for any piecewise expanding map: there are $A, B \geq 0$ and $\lambda \in [0, 1)$ such that
\[ (66) \quad \| L^n_T \phi \|_{BV} \leq A \lambda^n \| \phi \|_{BV} + B \| \phi \|_1. \]

The first consequence of the inequality is the existence of a bounded variation invariant density for each piecewise expanding map, indeed, by (66) and Theorem 60, repeating the arguments stated in Section 4.2 we get the existence of an absolutely continuous invariant measure with bounded variation density for this kind of maps. By arguments very similar to the ones presented in Section 6 it is also possible to obtain that a piecewise expanding map has spectral gap on the space of bounded variation densities if the map is mixing in some sense, then there will be an exponential convergence to equilibrium of the system.

About the mixing assumption, it is well known that a condition which is sufficient to imply the exponential convergence to equilibrium (and hence all the other statistical consequences) for Piecewise expanding maps is the topological mixing (see [7], assumption E3 and following for the details)

**Definition 63.** We say that a piecewise expanding map $T$ is topologically mixing if there is an interval $I_\ast \subseteq I$ such that $f(I_\ast) = I_\ast$, every orbit $T^n(x), x \in [0, 1]$ eventually enters $I_\ast$ and for every $J \subset I_\ast$ there is $n \geq 1$ such that $T^n(J) = I_\ast$.

Piecewise expanding maps however, have a more complicated behavior than expanding ones with respect to statistical stability on perturbations. We point out that the Lasota Yorke inequality we have proved in Lemma 62 works only if the expansion rate of the map is bigger than 2. To prove the existence of an absolutely continuous invariant measure as sketched above, one has to take an iterate of the map such that $\frac{2}{\inf_{I_i} (T')} < 1$.

When considering the statistical stability of a family of maps $T_\delta$ this is not only a technical point but is substantial, because sometime it is not possible to find a uniform iterate which is suitable for the whole family $T_\delta$. While many of the stability arguments outlined in the previous sections applies also to piecewise expanding maps with expansion rate greater than 2, for the maps $T$ such that $1 \leq \inf_{I_i} (T') \leq 2$ the stability questions are more complicated. We present below some examples of results illustrating the questions.

\[ \text{Using the } BV \text{ and } L^1 \text{ as strong and weak spaces and its Lasota Yorke inequality, Theorem 30 and Theorem 60 to obtain the compact inclusion property.} \]
9.2. The stability under deterministic perturbations. We have seen that mixing piecewise expanding maps associated transfer operators have spectral gap on bounded variation functions.

Now let us consider their statistical stability to perturbations. If we consider $BV$ and $L^1$ as a weak and strong spaces, and we consider perturbations for which UF1, ..., UF4 are satisfied, then Proposition 38 gives us a quantitative statistical stability estimation

$$||h_\delta - h_0||_1 = O(\delta \log \delta)$$

where $h_\delta, h_0$ are the absolutely continuous invariant measures of the perturbed map $T_\delta$ and of the unperturbed one $T_0$. UF1, ..., UF4 are satisfied by a wide variety of stochastic and deterministic perturbations.

In particular let us consider deterministic perturbations so that $L_\delta$ is the transfer operator of a family of maps $T_\delta$ satisfying a uniform Lasota Yorke inequality (UF1) with $BV$ and $L^1$ as strong and weak space (we remark that given a family or a given perturbation of some map, the existence of a uniform Lasota Yorke inequality is something that can be checked easily). In the case $T_0$ is topologically mixing the assumptions UF3, UF4 are easily satisfied. We recall that in this case the assumption UF2 would be $||(L_\delta - L_0)g||_1 \leq \delta C ||g||_{BV}$. If also UF2 is satisfied for some kind of perturbation then we have our quantitative statistical stability estimation (67) established for these systems and perturbations.

It is not much complicated to characterize families of deterministic perturbations for which UF2 hold. The Skorokhod metric defines the distance between two maps $T_1$ and $T_2$ as

$$d_s(T_1, T_2) = \inf \{ \epsilon > 0 : \exists A \subseteq I \text{ and } \exists \sigma : [0, 1] \to [0, 1] \text{ s.t. } m(A) \geq 1 - \epsilon, \sigma \text{ is a diffeomorphism, } T_1|_A = T_2 \circ \sigma|_A \text{ and } \forall x \in A, |\sigma(x) - x| \leq \epsilon, |1 - \frac{1}{\sigma'(x)}| \leq \epsilon \}$$

Families $T_\delta$ where $d_s(T_0, T_\delta) \leq K_\delta$ will satisfy UF2 (see e.g. [19], chapter 11.2) and then one can prove quantitative statistical stability results as in (67).

In this context also uniform contraction can be proved (Proposition 45), but Lipschitz continuity is expected to hold only in particular cases because the difference $||(L_\delta - L_0)h_0||_{BV}$ (see assumptions in Proposition 38) of the initial and perturbed operators applied to the initial invariant measure is not small in the strong norm (the $BV$ norm in this case) for many typical deterministic perturbations one would like to consider (consider perturbations moving discontinuities or values at discontinuities for example). For examples of non Lipschitz behavior of the statistical stability of families of piecewise expanding maps satisfying a uniform Lasota Yorke inequality see [4] or [35].

Even more complicated behavior can be found if we consider the case when the family of maps has not a uniform Lasota Yorke inequality. The simplest case is when the slope of the family $T_\delta$ is not uniformly above 2 (see Theorem 61). In this case we can have a discontinuous behavior of the family of associated invariant measures, as shown by [17] and [33] (see also [21] for further examples).

Consider the 3 parameters family of maps $W_{a,b,r}$ defined by
Figure 4. The map \( W \) is continuous, piecewise linear and expanding with slopes \( a/r \) and \( 2b/(1-2r) \)

\[
W_{a,b,r}(x) = \begin{cases} 
  a(1-x/r) & \text{for } 0 \leq x \leq r \\
  (2b/(1-2r))(x-r) & \text{for } r \leq x \leq 1/2 \\
  W_{a,b,r}(1-x) & \text{for } 1/2 \leq x \leq 1.
\end{cases}
\]

The maps are piecewise expanding, let \( h_{a,d,r} \) denote the unique invariant density for \( W_{a,b,r} \). Now let us consider a sequence \((a_n, d_n, r_n) \to (1/2, 1/2, 1/4)\) and the related densities \( h_{a_n,d_n,r_n} \). In [33] (page 331) is shown that \( h_{1/2,1/2,r} = \frac{3}{2}1_{[0,1/2]} + \frac{1}{2}1_{(1/2,1]} \) and \( h_{b_n,r} = 21_{[0,b_n]} \) while if \( \frac{1}{2} < b_n \leq 1-2r_n \) then \( h_{a_n,d_n,r_n} \to \delta_{1/2} \) weakly. This is due to the fact that for \( \frac{1}{2} \leq b \leq 1 - 2r \) the interval \([1-b,b]\) is sent to itself by the map, and "attracts" all the measure while iterating the map. Hence for \( a = 1/2 \) and \( a = 1 \) the limit measure does not coincide with the invariant absolutely continuous measure of the limit map shown above (by the way the limit map has \( \delta_{1/2} \) as a non absolutely continuous invariant measure since \( 1/2 \) is a fixed point). We remark that for this family of maps we cannot have a uniform Lasota Yorke inequality as in UF1, as the slopes tend to 2. On the other hand if one takes iterates of the maps to increase the slope, the smaller and smaller invariant interval around \( 1/2 \) would let the second coefficient of the Lasota Yorke inequality to converge to \( \infty \).

9.3. The approximation of the invariant measure for piecewise expanding maps. In this section we apply Proposition 38 to show how our stability results can give a quantitative estimate on the error in the numerical approximation of invariant densities of piecewise expanding maps with the Ulam method. We will use \( BV \) and \( L^1 \) as a strong and weak space.

Let us consider a piecewise expanding map \( T \) which is topologically mixing and for which \( \frac{1}{2} \inf_{I_i} ((T')') < 1 \). Let us consider its transfer operator \( L \). Suppose we want to approximate the absolutely continuous invariant probability measure \( h \) by the Ulam method outlined in Section 8.2 \( L \) is then approximated by \( L_\delta \) defined in [63]. We will approximate \( h \) by the invariant density \( h_\delta \) of \( L_\delta \). We consider \( L_\delta \) as a small perturbation of \( L \) and we will use Proposition 38 to estimate the distance between \( h \) and \( h_\delta \). Suppose \( L \) satisfies a Lasota Yorke inequality. Then the approximated operator \( L_\delta \) satisfies the same Lasota Yorke inequality

Lemma 64. If \( L \) is such that \( ||L\phi||_{BV} \leq \lambda ||\phi||_{BV} + B||\phi||_1 \) with \( \lambda < 1 \), then

\[
||L_\delta\phi||_{BV} \leq \lambda ||\phi||_{BV} + B||\phi||_1.
\]
Proof. We remark that \(||E(f_<_\delta F_\delta)||_BV \leq ||f||_BV|\) (see [90] Lemma 4.1). Then

\[
||L_\delta \phi||_BV = ||\pi_\delta L \pi_\delta(\phi)||_BV \leq ||L \pi_\delta(\phi)||_BV \\
\leq \lambda ||\pi_\delta(\phi)||_BV + B ||\pi_\delta \phi||_1 \leq \lambda ||\phi||_BV + B ||\phi||_1.
\]

□

For each \(\delta \geq 0\) we consider on the interval \([0,1]\) a partition \(F\) made of equal intervals \(I_i\) having length \(\frac{1}{3\delta}\) (we consider intervals of size \(\frac{1}{\delta}\) when \(\frac{1}{\delta}\) is integer).

To apply Proposition [38] we need an estimation on the quality of approximation by Ulam discretization like asked in the assumption UF2. This is provided by the following

Lemma 65. If \(L\) is the transfer operator associated to a piecewise expanding map and \(L_\delta\) is given by the Ulam discretization as explained before we have that there is \(C > 0\) such that

\[||L f - L_\delta f||_1 \leq C \delta ||f||_BV\]

Proof. It is not difficult to see that for \(f \in BV\), it holds

\[(69) \quad ||\pi_\delta f - f||_{L^1} \leq \delta \cdot ||f||_{BV}.
\]

Indeed from the definition of the norm we can see that \(||f||_{BV} \geq \sum_i |\sup_{I_i}(f) - \inf_{I_i}(f)|\), where \(I_i\) are the intervals composing the partition \(F\). Since \(\sup_{I_i}(f) \geq E(f|I_i) \geq \inf_{I_i}(f)\), it follows \(\int_{I_i} |E(f|F_\delta) - f| \leq \delta |\sup_{I_i}(f) - \inf_{I_i}(f)|\) leading to (69).

By this it holds

\[||(L - L_\delta)f||_{L^1} \leq ||\pi_\delta L \pi_\delta(f) - \pi_\delta L(f)||_{L^1} + ||\pi_\delta(Lf) - Lf||_{L^1},\]

and

\[||\pi_\delta L \pi_\delta(f) - \pi_\delta L(f)||_{L^1} \leq ||\pi_\delta(Lf) - Lf||_{L^1} \leq \delta ||Lf||_{BV} \leq \delta (\lambda ||f||_{BV} + B ||f||_1) \leq \delta (B + 1) ||f||_{BV}.
\]

\[\square\]

Since \(||\pi_\delta||_{L^1 \rightarrow L^1} \leq 1\) we immediately get that if \(L_\delta\) is given by the Ulam method, for each \(f \in L^1\)

\[||L_\delta f||_{L^1} \leq ||f||_{L^1}.
\]

Hence we proved that UF1,...,UF4 applies to the family of perturbed operators \(L_\delta\) and thus we can apply Proposition [38] concluding that for Piecewise expanding maps which are topologically mixing and such that \(\frac{1}{\inf_{T_i}(T)} < 1\) the Ulam method approximation \(h_\delta\) converge to the real invariant density \(h\). Furthermore, we have a quantitative estimation

\[||h_\delta - h_0||_1 = O(\delta \log \delta).
\]

In [12] it is proved that this rate is the optimal one. There are examples of map for which the approximation rate is asymptotically proportional to \(\delta \log \delta\).
In this section we apply the transfer operator approach to random dynamical systems. A random dynamical system is a dynamical system where the dynamics depend on some random parameter. Instead of iterating a single map $T$ in this case we have a family of maps $\{T_\omega\}_{\omega \in \Omega}$ and the dynamics applies a sequence of maps drawn at random in this family. We will see that in this case too, we can define an associated transfer operator, consider the associated invariant measures, and relevant statistical properties of the system can be studied by these concepts. In the first part of the section we will introduce what is a random dynamical system and what is the associated transfer operator. We restrict our study to a class of dynamical systems in which the maps $\{T_\omega\}_{\omega \in \Omega}$ are drawn independently. The concept of random dynamical system can be considered from a more general point of view (see [2]). In the following subsection we briefly introduce the basic of the ergodic theory of random dynamical systems. The (simple) approach we choose follows Section 5 of [44]. Then we will focus our attention on the class of systems with additive noise. These are system in which at each iteration of the dynamics one applies a deterministic map and the adds a random perturbation (the noise). This class of systems is flexible enough to include models of nontrivial real phenomena (see e.g. [29], [36]) and for this kinds of systems the transfer operator approach allow to obtain several interesting results in a relatively simple way.

10.1. Random dynamics, basic definitions. First let us describe the probability space defining the randomness in our systems Let $\Omega^\mathbb{N}$ be a probability space. Let $(\Omega^\mathbb{N}, \mathcal{A}, \mu)$ the space of sequences on $\Omega$ endowed with the product $\sigma$-algebra $\mathcal{A} = \mathcal{A}^\mathbb{N}$ and the product measure $\mu = p^\mathbb{N}$. Let $s : \Omega^\mathbb{N} \rightarrow \Omega^\mathbb{N}$ be the usual shift map on $\Omega^\mathbb{N}$ defined by $s(\omega) = (\omega_{i+1})_{i \in \mathbb{N}}$ where $\omega = (\omega_i)_{i \in \mathbb{N}}$ (for example $s((\omega_0, \omega_1, \omega_2, ...)) = (\omega_1, \omega_2, \omega_3, ...)$). The measure $\mu$ is invariant for $s$, and $(\Omega^\mathbb{N}, s, \mu)$ is a deterministic ergodic dynamical system which is a model of independent sorting from $\Omega$ with probability $p$. The $\sigma$-algebra $\mathcal{A}$ is indeed generated by the cylinders $C_{i,A} = \{\omega \in \Omega^\mathbb{N} | \omega_i \in A\}$ and for all $i \in \mathbb{N}, A \subseteq \Omega$ we have $\mu(C_{i,A}) = \mu(A) = \mu(s^{-1}(C_{i,A})) = \mu(C_{i+1,A})$.

Now we define a random dynamical system formally as a skew product. Let $(N, \mathcal{B})$ be a measurable space. Let us consider $\Omega^\mathbb{N} \times N$ with the product $\sigma$-algebra $\mathcal{A} \times \mathcal{B}$. Let $\mathcal{F} = \{F_i : N \rightarrow N\}_{i \in \Omega}$ be a set of maps $N \rightarrow N$. A random transformation $F_\omega : N \rightarrow N$ over $s$ is a measurable $F : \Omega^\mathbb{N} \times N \rightarrow \Omega^\mathbb{N} \times N$ defined by

$$F(\omega, v) = (s(\omega), F_{\omega_0}(v))$$

where $F_{\omega_0} \in \mathcal{F}$ depends only on the 0-th coordinate $\omega_0 \in \Omega$ of $\omega \in \Omega^\mathbb{N}$.

**Example 66.** Let $\Omega = \{0, 1\}$ suppose $p(0) = p(1) = \frac{1}{2}$ (and the usual product structure) Let $F_0, F_1$ be maps $F_i : N \rightarrow N$ and

$$F(\omega, v) = (s(\omega), F_{\omega_0}(v))$$

Then $F$ represents the random dynamics in which $F_0, F_1$ are applied independently and with the same probability.
Example 67. Let $\Omega = [-\frac{1}{2}, \frac{1}{2}]$ suppose $p = \text{Leb meas.}$ (and the usual product structure) Let $g : \mathbb{R} \to \mathbb{R}$ be a measurable function and

$$F_\omega(v) = g(v) + \omega_0.$$ Then $F$ represents the random dynamics in which at each step $g$ is applied and some random unif. distributed noise in $[-\frac{1}{2}, \frac{1}{2}]$ is added.

10.2. The transfer operator. Given $F_\omega : N \to N$ the associated (annealed) transfer operator $L : SM(N) \to SM(N)$ is defined by considering in some sense the average transfer operator among all the transfer operators associated to the maps $\{T_\omega\}_{\omega \in \Omega}$. The average will be considered according to the measure $p$ describing the randomness in our system

$$L(\nu) = \int_{x \in \Omega} L_{F_x}(\nu) \, dp(x)$$

i.e. given a measurable $A \subseteq N$

$$[L(\nu)](A) = \int_{x \in \Omega} [L_{F_x}(\nu)](A) \, dp(x) = \int_{x \in \Omega} \nu[F_x^{-1}(A)] \, dp(x).$$

As in the deterministic case: this is a positive operator, and preserves probability measures, hence it is a Markov operator. Furthermore by (7) if almost all the maps $\{T_\omega\}_{\omega \in \Omega}$ are nonsingular and $\nu \in L^1$ then it easily follows

$$||L(\nu)||_1 \leq ||\nu||_1.$$ Associated to $F$ there is another linear map acting of functions, which is in some sense the dual of the transfer operator. This is the analog of the composition operator and it is called as the Koopman operator associated to our system. Let $\phi : N \to \mathbb{R}$ be measurable, let $x \in N$, define $P(\phi) : N \to \mathbb{R}$ as

$$[P(\phi)](x) = \int_{y \in \Omega} \phi(F_y(x)) \, dp(y).$$

As for deterministic dynamical systems we have the following duality relation between the two operators

Lemma 68. Under the above assumptions, suppose that $\phi$ is bounded and measurable, let $\nu \in SM(N)$, then

$$\int \phi \, d[L\nu] = \int [P(\phi)] \, d\nu.$$ Compare this statement with Proposition 8, from which this proposition easily follows.

Now we consider the measures which are in a certain sense invariant for a random dynamical system: a measure $\eta \in PM(N)$ is called stationary for $F_\omega$ if

$$L\eta = \eta.$$ The following proposition shows a link between being stationary for a random system and invariant for the associated skew product $F$, for the proof see [13], Section 5.
Theorem 69. Let $F : \Omega^N \times N \to \Omega^N \times N$ be a one sided random transformation with associated transfer operator $L$. A probability measure $\nu$ on $N$ is stationary for the system $(L\eta = \eta)$ if and only if the probability measure $\mu \times \nu$ is invariant for the skew product map $F$.

10.3. Ergodic stationary measures. In this section we extend the notion of ergodicity to random dynamical systems. The reader will notice the similarity of the definitions with the analogous definitions in the deterministic case. Let $\eta$ be a stationary measure. A bounded measurable function $\phi : N \to \mathbb{R}$ is said to be stationary if it satisfies

$$P\phi = \phi.$$

A set $B \subset N$ is said to be stationary if $1_B$ is stationary. A bounded measurable function $\phi : N \to \mathbb{R}$ is said to be $\eta$-stationary if it satisfies

$$P\phi = \phi \quad \eta \text{ a.e.}$$

A set $B \subset N$ is said $\eta$-stationary if $1_B$ is $\eta$-stationary.

Example 70. The complement of a stationary set is stationary. Consider a stationary set $B$, then $1_B = P(1_B)$. Observe that $1_{B^C} = 1 - 1_B$ and

$$P(1_{B^C}) = \int_{x \in \Omega} (1 - 1_B) \circ F_x \, dp(x) = 1 - \int_{x \in \Omega} (1_B) \circ F_x \, dp(x) = 1 - P1_B = 1_{B^C}$$

hence $B^C$ is stationary.

As in the deterministic case, one has the following equivalence

Theorem 71. Let $\eta$ be a stationary measure. The following conditions are equivalent:

1. every $\eta$-stationary function is constant on some set with full $\eta$-measure;
2. if $B \subset N$ is an $\eta$-stationary set then $\eta(B)$ is either 0 or 1.

Definition 72. $\eta$ is said to be ergodic if 1 and 2 holds.

There is a relation between the concept of ergodicity in the random system, and the same concept for the associated skew product map (for the proof of the statement, again see [44], Section 5).

Theorem 73. Let $F : \Omega^N \times N \to \Omega^N \times N$ be a "one sided" random transformation as defined in Section 10.1. A probability measure $\nu$ on $N$ is ergodic for the random system in the sense of Definition 72 if and only if the probability measure $\mu \times \nu$ is ergodic for the map $F$.

The set of stationary measures is trivially a convex set. As in the deterministic case, every stationary measure is a convex combination of ergodic measures (and the ergodic ones are extremal points, see [34], Appendix A.1 for details).

Next proposition shows another aspect of ergodicity in a random dynamical system. It can be interpreted as the fact that in an ergodic system, given some integrable observable $f : N \to \mathbb{R}$ for a typical random orbit $x_0, ..., x_i \in N$ the time average of the observable will coincide with the space average with respect to the stationary measure, like in the deterministic case.
Proposition 74. Let $F : \Omega^N \times N \to \Omega^N \times N$ be the skew product map associated to a random dynamical system. Let us consider an ergodic stationary measure $\nu$ on $N$. Let us consider an observable $f \in L^1(N, \nu)$ on $N$. Let us consider $\omega_0 \in \Omega^N$ and $x_0 \in N$ as initial conditions. For such conditions we have a random orbit $x_i \in N$ with this initial condition, defined by $x_i := \pi_N(F^i(\omega_0, x_0))$ where $\pi_N : \Omega^N \times N \to N$ is the natural projection. With this notations, for $\mu \times \nu$ almost each initial condition $(\omega_0, x_0)$ it holds:
\[
\int f \, d\nu = \lim_{n \to \infty} \frac{f(x_0) + f(x_1) + \ldots + f(x_n)}{n+1}.
\]

Proof. Let us consider $f \in L^1(N, \nu)$. This function can be extended to $\tilde{f} \in L^1(\Omega^N \times N, \mu \times \nu)$ by setting $\tilde{f}(\omega, x) = f(x)$. By the ergodic theorem applied to $F$ we get
\[
\int \tilde{f} \, d(\mu \times \nu) = \lim_{n \to \infty} \frac{\tilde{f}(\omega, x) + \tilde{f}(F(\omega, x)) + \ldots + \tilde{f}(F^n(\omega, x))}{n+1}
\]
for $\mu \times \nu$ almost each $(\omega, x) \in \Omega^N \times N$. Now $\forall \omega \lim_{n \to \infty} \tilde{f}(\omega, x) + \tilde{f}(F(\omega, x)) + \ldots + \tilde{f}(F^n(\omega, x)) = \lim_{n \to \infty} \frac{f(x_0) + f(x_1) + \ldots + f(x_n)}{n+1}$ the conclusion follows, since trivially
\[
\int \tilde{f} \, d(\mu \times \nu) = \int f \, d\nu.
\]

\[\square\]

10.3.1. Additive noise. We are going to define a class of systems (with additive noise) which allow a simple functional analytic treatment. To start let us consider a random dynamics on $[0, 1]$. Everything we say generalizes easily to other spaces.

We consider a dynamics in which the orbit of a point is generated by a deterministic map, but were at each iterate some (small or not so small) random perturbation is added (the noise). We suppose that the perurbation is independend from the point and always distributed in the same way. For simplicity we will suppose that the random perturbation ranges in $[-1, 1]$. All these assumptions can be generalized, making the noise depending on the point, following a similar construction and similar arguments.

More precisely, consider a Borel map $T : [0, 1] \to [0, 1]$ (the deterministic part). Consider a probability density $\rho \in BV[-1, 1]$ (the noise kernel). For simplicity let us suppose $\rho$ being symmetric ($\rho(x) = \rho(-x)$). Consider the process
\[
x_{n+1} = T(x_n) + \Omega_n
\]
where $\Omega_n$ is an i.i.d. process distributed according to $\rho \in BV$.

This is the idea behind a random dynamical system with additive noise. If we want to define a dynamics on the interval there is a technical point to address: $T(x_n) \in [0, 1]$ but $T(x_n) + \Omega_n$ could be outside, i.e the noise can let the point jump outside the interval. We have to decide where the point goes in this case. For this we can consider "boundary reflecting conditions", "periodic boundary conditions" or other ways to define the dynamics on $[0, 1]$ in this case. We enter in the details of the boundary reflecting conditions. Consider hence the process
\[
x_{n+1} = T(x_n) + \Omega_n
\]
\[17\]Notice that this can be realized with the $n$-th coordinate of a shift as shown before: we can set $\Omega = [-1, 1]$, the process being defined by $\Omega_n(\omega) = \omega_n$, with $\omega \in (\Omega^N, p^N)$ and $\frac{d\Omega}{dx} = \rho$. 

and \( \hat{+} \) is the “reflecting boundaries sum” on \([0, 1]\) defined as follows.

**Definition 75.** Let \( \pi : \mathbb{R} \to [0, 1] \) be a piecewise linear projection on the interval
\[
\pi(x) = \min_{i \in \mathbb{Z}} |x - 2i|.
\]

Let \( a, b \in \mathbb{R} \) then
\[
a \hat{+} b := \pi(a + b)
\]
where \( + \) is the usual sum operator on \( \mathbb{R} \). By this \( a \hat{+} b \in [0, 1] \).

Now let us describe the structure of the transfer operator associated to a system of this kind. By definition
\[
L(\nu) = \int_{x \in \Omega} L_{F_x}(\nu) \, dp(x).
\]

Were on \( \mathbb{R} \), \( F_x(a) = T(a) + x \), hence \( L_{F_x} = \text{Tran}(x) \circ L_T \) where \( \text{Tran}(x) : SM(\mathbb{R}) \to SM(\mathbb{R}) \) is the pushforward of the translation by \( x \).

On \([0, 1]\) with reflecting boundary conditions we have \( F_x(a) = T(a) \hat{+} x \) and then
\[
L_{F_x} = \pi \circ \text{Tran}(x) \circ L_T.
\]

On \( \mathbb{R} \), since \( \rho \) is symmetric and the Lebesgue measure is invariant by translation, if we denote with \( \mu_\rho = \rho m \) the measure on \([-1, 1]\) having density \( \rho \).

\[
L(\nu) = \int_{x \in \Omega} \text{Tran}(x)(L_T(\nu)) \cdot \rho(x) \, dx = \mu_\rho * L_T(\nu)
\]
where \(*\) stands for the ordinary convolution operator between measures.
Lemma 76. Let \( f, g \in L^1 \). We have
\[
\| f \hat{g} \|_1 \leq \| f \|_1 \cdot \| g \|_1.
\]
Furthermore, let \( f \in L^1, g \in BV \)
\[
\| f \hat{g} \|_{BV} \leq 3 \| f \|_1 \cdot \| g \|_{BV}.
\]
Under the above assumptions, if \( \rho \in BV, \nu \in L^1 \) and \( n \geq 0 \)
\[
\| L^n(\nu) \|_{BV} \leq 3 \| \rho \|_{BV} \| \nu \|_1.
\]

Proof. We know that \( \pi \) is a weak contraction with respect to the \( L^1 \) norm. By the classical properties of the convolution we have \( \| f * g \|_1 \leq \| f \|_1 \cdot \| g \|_1 \), then also \( \| f \hat{g} \|_1 \leq \| f \|_1 \cdot \| g \|_1 \).

About Equation (75) on the real line we have the well known estimate \( \| f * g \|_{BV} \leq \| f \|_1 \cdot \| g \|_{BV} \). For the convolution on the interval let us remark that if \( \mu \) is supported on \([-1, 2]\) then \( \| \pi^*(\mu) \|_{BV} \leq 3 \| \mu \|_{BV} \), giving (73).

About (76), since \( L \) is a weak contraction with respect to the \( L^1 \) norm
\[
\| L^n(\nu) \|_{BV} \leq 3 \| \rho \|_{BV} \| L^{n-1}(\nu) \|_1 \leq 3 \| \rho \|_{BV} \| \nu \|_1.
\]

The previous lemma implies that the transfer operator \( L \) can also be seen as \( L : BV[0,1] \to BV[0,1] \). Now, considering \( L \) as an operator acting on a strong and a weak space, \( BV \) and \( L^1 \) and exploiting a strategy very similar to what we have done for expanding and piecewise expanding maps, we prove that a system with additive noise has a stationary measure with density in \( BV \). In this construction the Helly selection principle (see Theorem 60) will provide the compact immersion between the strong and the weak space.

Theorem 77. A random dynamical system with additive noise with \( \rho \in BV \) has an absolutely continuous stationary measure having density \( h \in BV \) and
\[
\| h \|_{BV} \leq 3 \| \rho \|_{BV}.
\]

Proof. Consider the Cesaro averages of iterates of an uniform distribution
\[
f_n = \frac{1}{n} \sum_{i=0}^{n-1} L^n 1
\]
where \( 1 \) is the density of the normalized Lebesgue measure.

By Lemma 76 the sequence has uniformly bounded \( BV \) norm and by Theorem 60 this has a subsequence \( f_{n_k} \) converging in \( L^1 \) to a limit \( h \).

Since \( f_n = \frac{1}{n}(L^n 1 + ... + L^n 1) \) we get \( \| Lf_{n_k} - f_{n_k} \|_1 \leq \frac{2}{n} \). Recalling that \( L \) is continuous in the \( L^1 \) norm we get
\[
Lh = L( \lim_{k \to \infty} f_{n_k} ) = \lim_{k \to \infty} Lf_{n_k} = h.
\]

Then \( h \) is a stationary probability density in \( L^1 \). Applying again Lemma 76 we get
\[
\| h \|_{BV} = \| Lh \|_{BV} \leq 3 \| \rho \|_{BV} \| h \|_1 < \infty
\]
and we proved the statement. \( \square \)
Now we discuss the problem of how to understand when a system is ergodic and stronger notions like the notion of convergence to equilibrium (see Definition 25). In the following proposition we see that in a system with additive noise the convergence to equilibrium implies ergodicity, thus we can apply the results we know to estimate the statistical behavior of observables. In next subsection we will show a general criteria (Lemma 84) to establish the convergence to equilibrium in such systems.

Proposition 78. If a dynamical system with additive noise as above has convergence to equilibrium (using $\text{BV}$ and $L^1$ as a strong and weak space, see (21)) then the system is ergodic.

Proof. Consider a stationary set $B$. Suppose that $\mu$ is a stationary measure for the system. It holds for each $\phi \in \text{BV}$ with $\phi \geq 0$

$$\int 1_B \phi \, d\mu = \int P^n(1_B) \phi \, d\mu = \int 1_B \, d(L^n \phi \mu).$$

Since $\phi \mu \in \text{BV}$, if the system has convergence to equilibrium $L^n \phi \mu \to [\int \phi \, d\mu] \mu$ in $L^1$. We get then that $\forall \phi \in \text{BV}$ with $\phi \geq 0$

$$\int 1_B \phi \, d\mu = \int 1_B \, d\mu \int \phi \, d\mu.$$ 

Thus $1_B$ is a.e constant and $\mu(B) \in \{0,1\}$. The system is hence ergodic. □

We now see that if a map with additive noise has convergence to equilibrium then it also has spectral gap, and then the convergence to equilibrium is exponentially fast among many other consequences.

Proposition 79. Suppose the transfer operator $L : \text{BV} \to \text{BV}$, associated to a map with additive noise as above has convergence to equilibrium using $\text{BV}$ and $L^1$ as a strong and weak space (see (21)) then it has spectral gap.

Proof. The statement directly follows by Theorem 30. We consider $B_s = \text{BV}[0,1]$ and $B_w = L^1[0,1]$ as strong and weak space. Weak boundedness in $L^1$ is granted. We have compact inclusion of the strong space into the weak one (Theorem 60). Furthermore

$$||L^n(\nu)||_{\text{BV}} \leq 3 ||\rho||_{\text{BV}} ||\nu||_1$$

is a Lasota Yorke inequality with $\lambda = 0$. □

Remark 80. Once established the spectral gap on $\text{BV}$, the spectral gap on $L^1$ for this kind of systems follows easily. Indeed, let us consider $\nu \in \{\mu \in L^1 \text{ s.t. } \mu([0,1]) = 0\}$. By (77) $||L(\nu)||_{\text{BV}} \leq 3 ||\rho||_{\text{BV}} ||\nu||_1$ and $L(\nu) \in \{\mu \in \text{BV} \text{ s.t. } \mu([0,1]) = 0\} := V_{\text{BV}}$. By the spectral gap on $\text{BV}$ there is $A \geq 0$, $\lambda \in (0,1)$ such that

$$||L^n(\nu)||_1 \leq ||L^n(\nu)||_{\text{BV}} \leq ||L^{n-1}(L\nu)||_{\text{BV}} \leq A\lambda^{n-1}3||\rho||_{\text{BV}} ||\nu||_1$$

implying spectral gap on $L^1$.

As an example of a first simple application of the previous proposition we state the following
Proposition 81. Let \((x_n)_{n \geq 0}\) be a random orbit starting from the initial condition \(x_0\) and a realization \(\omega_0\) of the noise. Let \(\phi \in L^1(N)\). Let us define the average behavior of the random orbits on such realizations

\[
E(\phi(x_n)) := \int_{\Omega^N} \phi(x_n) \, d\omega_0.
\]

In a map with additive noise having convergence to equilibrium one has that for each \(x_0\)

\[
|E(\phi(x_n)) - \int \phi \, d\mu| = O(e^{-\lambda n}).
\]

Proof. Let us first remark that we can apply the transfer operator to a delta measure \(\delta_{x_0}\) and get a measure with \(BV\) density: indeed \(L(\delta_{x_0}) = \pi_* [\rho * \delta_{T(x_0)}]\), and \(\rho * \delta_{T(x_0)}\) is a measure having \(BV\) density. We remark that given \(\phi \in L^1(N)\) and \(x_0 \in N\) we get

\[
E(\phi(x_1)) = \int_{x \in \Omega} \phi(F(x_0)) \, dp(x)
\]

\[
= [P(\phi)](x_0)
\]

\[
= \int P(\phi) d\delta_{x_0}
\]

\[
= \int \phi \, dL(\delta_{x_0})
\]

and continuing the iteration

\[
\int_{\Omega^N} \phi(x_2) = \int_{\Omega} \int_{\Omega} \phi(F_{\omega_1}(F_{\omega_0}(x_0))) \, d\omega_1 \, d\omega_0 = \int_{\omega_0} [P(\phi)](F_{\omega_0}(x_0)) \ldots
\]

We then get

\[
E(\phi(x_n)) = [P^n(\phi)](x_0)
\]

\[
= \int \phi \, dL^n(\delta_{x_0}).
\]

But by the spectral gap \(||L^n(\delta_{x_0}) - \mu||_{BV} = O(e^{-\lambda n})\). Then for each \(x_0\)

\[
|E(\phi(x_n)) - \int \phi \, d\mu| = O(e^{-\lambda n}).
\]

\[\square\]

Having established that the transfer operators associated to systems with additive noise have good properties as the spectral gap, on \(BV\) or \(L^1\) we can apply the results of Section 7 to suitable families of perturbations of such systems and get quantitative stability estimates. We briefly discuss this direction of work, and leave the details to the reader.

We now consider the easiest way to perturb such a system with additive noise, by perturbing the distribution of the noise. Let us consider the following proposition:

Proposition 82. Let \(f \in L^1\), suppose \(||\rho_1 - \rho_0||_1 \leq C\xi||\) and \(L_{1,T}, L_{0,T}\) are the transfer operators associated to the systems given by the map \(T\) with additive noise distributed as \(\rho_1\) and \(\rho_0\). Then

\[
\|L_{1,T}f - L_{0,T}f\|_1 \leq C\xi\|f\|_1.
\]
Proof. The proof is a direct application of the well known fact \( \| \rho * g \|_1 \leq \| \rho \|_1 \| g \|_1 \).

By this it also holds
\[
\| \hat{\rho}^* g \|_1 = \pi^* (\hat{\rho}^* \hat{g}) \leq \| \rho \|_1 \| g \|_1
\]

and
\[
\| \hat{\rho}_0^* g - \hat{\rho}_0^* \hat{g} \|_1 \leq \| \rho_0 - \rho_0 \|_1 \| g \|_1.
\]

This last proposition can be used as to show that the property UF2 is verified and establish \( \delta \log \delta \) modulus of continuity for the statistical stability of a map with additive noise having convergence to equilibrium, but also Lipschitz statistical stability (using the results of Section 7.4 and the spectral gap on \( L^1 \)). We suggest the interested reader to fill the details.

Similar results can be proved for other kinds of perturbations of the transfer operator associated to maps with additive noise like the Ulam discretization (see Section 7.3 and [29]) or perturbations of the map \( T \) (see [26, 27] for perturbations on the map or linear response results for these systems up to perturbation of the map or noise, including zero-noise limits).

10.3.2. A way to establishing convergence to equilibrium in systems with additive noise. Let us consider a family of nonsingular maps with additive noise. We will suppose that these maps are nonsingular and piecewise \( C^2 \) having good distortion properties. More precisely let us suppose:

A1 \( T_0 \) is a nonsingular piecewise \( C^2 \) map whose associated pushforward operator \( L_{T_0} : BV[0, 1] \to BV[0, 1] \) is continuous: there is \( C \geq 0 \) such that
\[
\| L_{T_0} f \|_{BV} \leq C \| f \|_{BV}.
\]

A2 \( T_0 \) is eventually onto: for each open interval \( I \subseteq [0, 1] \) there is \( n \) such that \( T_0^n(I) = [0, 1] \).

Suppose now that at each iteration of \( T_0 \) a random perturbation is added with noise kernel \( \rho_0 \in BV \), and we suppose that the support of \( \rho_0 \) contains a neighborhood of the origin. We have a system with additive noise. Let us see that under these conditions this system has convergence to equilibrium.

Remark 83. Assumption A2 implies the topological mixing of the map. In the case of piecewise expanding maps this assumption is often taken to get a topologically mixing system (see [14, Section 3.1, Property E3]). This assumption is not the most general one possible to get convergence to equilibrium, but it will keep the exposition simple.

Lemma 84. Under the assumptions above, the transfer operator \( L \) associated to the system has convergence to equilibrium: for each \( f \in BV \) with \( \int f = 0 \) we have \( \| L_0^N f \|_1 \to 0 \).

Proof. By Lemma 76 and Theorem 60 \( L \) is a compact operator \( L^1 \to L^1 \). The spectral radius of \( L \) as an operator on \( L^1 \) is bounded by 1, and the spectrum is discrete. This also hold for \( L \) considered on \( BV \).

The system has convergence to equilibrium if and only if there are no other eigenvalues on the unit circle than the eigenvalue 1 with multiplicity 1.

Let us consider one positive stationary probability measure \( \mu_0 \in BV \) for \( L \) (as was proved to exist in Theorem 77). Suppose that the system has no convergence

---

\[19\] The space \([0, 1]\) can be decomposed into a union of intervals \( I_i \) such that in every set \( T_i \) the map \( T_0 \) can be extended to a \( C^2 \) function \( T_i : [0, 1] \to [0, 1] \).
to equilibrium, then there is a complex measure $\hat{\mu} \in BV$, $\hat{\mu} \neq \mu_0$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $L^i\hat{\mu} = \lambda^i\hat{\mu}$ for each $i \geq 0$. Let $\mu$ be the real part of $\hat{\mu}$. This is a signed measure. Since the transfer operator preserves real valued measures $L^i\mu$ is a real valued measure with bounded variation density and, for each $\epsilon \geq 0$ there are infinitely many $i$ such that $\|L^i\mu - \hat{\mu}\| \leq \epsilon$ (for each $\epsilon$, $|\lambda^i - 1| \leq \epsilon$ for infinitely many $i$). We also have that there is $c \in \mathbb{R}$ such that $\mu_1 = \mu + c\mu_0$ is a zero average measure with density in $BV$ and

$$\limsup_{i \to \infty} \|L^i_0\mu_1\|_1 = \|\mu_1\|_1$$

(indeed $\|L^i_0\mu_1\|_1 \leq \|\mu_1\|_1$ and $\|L^i_0\mu_1\|_1 = \|L^i_0(\mu + c\mu_0)\|_1 = \|L^i_0(\mu) - \mu + \mu + L^i_0(c\mu_0)\|_1 \leq \|L^i_0(\mu) - \mu\|_1 + \|\mu_1\|_1$).

Now let $I$ be an interval for which $\mu_1|_I$ has a strictly positive density. By assumption A2 there is $n$ such that $T^0_n(I) = [0,1]$. Let us consider the measure $\nu = \mu_1|_I$. Suppose $s(\nu)$ is the support of $\nu$ (the set on which $\nu$ has strictly positive density). Since $L_{T^0_n}$ is a positive operator and $T^0_n$ is piecewise $C^2$, then $L_{T^0_n}\nu$ has also a strictly positive density almost everywhere on $T^0_n(I)$. Since the convolution can only increase the support of a positive measure, we get that $s(L^0_{\nu}) \supseteq s(L_{T^0_n}\nu)$ $\supseteq T^0_n(I)$, $s(L^0_{\nu}) \supseteq T^0_n(I)$ and $s(L^0_{\nu}) \supseteq T^0_n(I)$ for $i \geq 1$. Then by assumption A2 we get inductively $s(L^0_{\nu}) = [0,1]$. This contradicts the fact that $\limsup_{i \to \infty} \|L^i_0\mu_1\|_1 = \|\mu_1\|_1$. Indeed recall that any measure $\mu_1$ of zero average can be decomposed in $\mu_1^+ + \mu_1^-$, the positive and negative component of $\mu_1$. We have that $L^0_0\mu_1^-$ is a negative measure having a bounded variation density and the support of $L^0_0\nu$ being the whole space overlaps the support of $L^0_0\mu_1^-$ in this way

$$\|L^0_{\nu}\mu_1\|_1 = \|L^0_{\nu}(\mu_1^+ + \mu_1^-)\|_1$$

$$\leq \|L^0_{\nu}(\mu_1^+ - \nu + \nu + \mu_1^-)\|_1$$

$$\leq \|L^0_{\nu}(\mu_1^+ - \nu)\|_1 + \|L^0_{\nu} + L^0_{\nu}\mu_1^-\|_1$$

$$< \|L^0_{\nu}(\mu_1^+ - \nu)\|_1 + \|\mu_1^-\|_1 + \|\nu\|_1$$

$$= \|\mu_1\|_1.$$ 

Then for each $k \geq 0$ $\|L^{n+k}_0\mu_k\|_1 \leq \|L^n_0\mu_k\|_1 < \|\mu_k\|_1$, contraddicting $\limsup_{i \to \infty} \|L^{i-k}_0\mu_1\|_1 = \|\mu_1\|_1$.

11. Uniformly contracting maps

We have seen how expansion helps the transfer operator to have regularization properties when considered on suitable spaces of regular measures (expressed by the Lasota Yorke inequality). If the dynamics is contracting instead we cannot expect such a regularization on the same spaces. Iterates of absolutely continuous initial measures instead will converge to a Dirac delta measure which is not regular at all according to the norms considered for expanding maps. We will see in this section that considering different spaces, which are essentially the dual of the spaces considered in the expanding case we recover a regularization property, even for the transfer operator associated to a contracting map. This will be source of inspiration to define the suitable space to consider when considering maps having both contracting and expanding directions (hyperbolic systema) as it happens in many models of interesting physical phenomena.
Let \((X, d)\) be a compact metric space, \(g : X \rightarrow \mathbb{R}\) be a Lipschitz function and let \(\text{Lip}(g)\) be its best Lipschitz constant, i.e.

\[
\text{Lip}(g) = \sup_{x, y \in X} \left\{ \frac{|g(x) - g(y)|}{d(x, y)} \right\}.
\]

**Definition 85.** Given two Borel signed measures \(\mu\) and \(\nu\) on \(X\), we define a Wasserstein-Kantorovich Like distance between \(\mu\) and \(\nu\) by

\[
W_1^0(\mu, \nu) = \sup_{g \text{ s.t. } \text{Lip}(g) \leq 1, ||g||_\infty \leq 1} \left| \int g d\mu - \int g d\nu \right|.
\]

From now on we denote \(||\cdot||_W := W_1^0(0, \mu)\).

As a matter of fact, \(|| \cdot ||_W\) defines a norm on the vector space of signed measures defined on a compact metric space.

**Remark 86.** The space of signed measures is not complete with respect to the \(W_1\) distance, and its completion would be a distribution space. However for our purposes it is sometime sufficient to consider sequences of positive measures. The set of positive Borel measures on \(X\) however is complete with respect to the distance \(W_1\) (see [10], [11]).

We show that a contracting map \(F\), such that \(\text{Lip}(F) < 1\) is in some sense regularizing for the above norm

**Lemma 87.** Let \(F : X \rightarrow X\), be a Lipschitz function and \(X\) is a metric space. For every Borel measure with sign \(\mu\) it holds

\[
||L_F \mu||_W \leq \alpha ||\mu||_W + \mu(X).
\]

(where \(\alpha = \text{Lip}(F)\)). In particular, if \(\mu(X) = 0\) then

\[
||L_F \mu||_W \leq \alpha ||\mu||_W.
\]

**Proof:** If \(\text{Lip}(g) \leq 1\) and \(||g||_\infty \leq 1\), then \(g \circ F\) is \(\alpha\)-Lipschitz. Moreover since \(||g||_\infty \leq 1\) then \(||g \circ F, - \theta||_\infty \leq \alpha\) for some \(\theta \leq 1\). This implies

\[
\left| \int g dL_F \mu \right| = \left| \int g \circ F d\mu \right| = \left| \int g \circ F - \theta d\mu \right| + \left| \int \theta d\mu \right| = \alpha \left| \int \frac{g \circ F - \theta}{\alpha} d\mu \right| + \theta \mu(X) = \alpha ||\mu||_W + \mu(X).
\]

And we have \(||L_F \mu||_W \leq \alpha ||\mu||_W + \mu(X)\). In particular, if \(\mu(X) = 0\) we get the second part. \(\square\)

- From Lemma 87 it easily follows that \(L_F\) has spectral gap on the normed vector space of signed measured endowed with the norm \(|| \cdot ||_W\). Since the
space of signed Borel measures is not complete, what we are going to prove precisely is that (Compare with Definition 29)

\[ L_F = P + N \]

where

- \( P \) is a projection and \( \dim(\text{Im}(P)) = 1 \);
- there is \( \alpha, C \geq 0 \) such that \( \alpha < 1 \) and for each \( n \), and each Borel signed measure \( \mu \), \( \| N^n \mu \|_w < \alpha^n C \| \mu \|_w \);
- \( P N = N P = 0 \).

**Proposition 88.** The transfer operator \( L_F \) associated to a contracting map \( F \) has the decomposition (82) as above.

**Proof.** Being a contraction the map \( F \) has a unique fixed point \( x_0 \). Let us consider the Dirac measure \( \delta_{x_0} \) placed on \( x_0 \). This is a fixed point for \( L_F \), generating a one dimensional fixed space \( \mathbb{R}\delta_{x_0} \) and one can define a projection \( P \) to \( \mathbb{R}\delta_{x_0} \) as \( \mu \to \mu(X) \delta_{x_0} \). Now let us consider the \( L_F \) invariant space \( V \) of zero average measures, there is also a projection to this space, defined as \( \mu \to \mu - \mu(X) \delta_{x_0} \), and define \( N \) as \( N(\mu_0) = L_F \mu_0 - \mu_0(X) \delta_{x_0} \). By Lemma 87 we get that for any measure \( \mu \), \( N^n(\mu) \) converges exponentially to 0. \( \square \)

From this proposition one could recover many consequences for the statistical properties of the dynamics of contracting maps, but since this dynamics is quite trivial (every initial condition is attracted by the fixed point of \( F \)) there is no need to do this. This example and the spaces considered to get a spectral gap in this case are on the other hand, illuminating to find a transfer operator approach for systems having both contracting and expanding directions. In fact one idea can be to consider in some sense some measure space endowed with some norm which behaves as the Lipschitz norm or some Sobolev norm in the directions for which the dynamics is expanding and as its dual (i.e. the \( \| \cdot \|_W \) norm) in the contracting directions. This is what we will do in the next section on some simple example of Hyperbolic dynamical systems and in many papers dealing with this kind of dynamical systems (see [16] and [14] for a survey and a detailed treatise on this subject).

**Remark 89.** If \( \mu^+ \) and \( \mu^- \) are positive measures such that \( \mu = \mu^+ - \mu^- \) (the Jordan decomposition of \( \mu \)) then one can define the total variation of \( \mu \) as

\[ \| \mu \|_{TV} := \mu^+(X) + \mu^-(X) \]

then one has from (81)

\[ \| L_F \mu \|_W \leq \alpha \| \mu \|_W + \| \mu \|_{TV} \]

which looks like a Lasota Yorke inequality for the operator \( L_F \) but it is not, since \( \| \cdot \|_{TV} \) is not weaker than \( \| \cdot \|_W \). For an approach based on a real Lasota Yorke inequality and a statement like Theorem 30 to get spectral gap on a weak space like \( \| \cdot \|_W \) we refer to [16].

12. A LOOK AT HYPERBOLIC SYSTEMS

The transfer operator approach we described in the previous sections also works for a large class of systems with uniform expansion and contraction rate when appropriate functional spaces are considered. In this section, we give an example
of this for a class of uniformly hyperbolic solenoidal maps. Following an approach of \cite{[24]}, based on the disintegration along stable manifolds, we show how to define spaces of measures with sign adapted to this system. We show some properties of the transfer operator restricted to these spaces, giving the existence of a physical measure for these systems and a Lasota Yorke inequality allowing to estimate the regularity of iterates of measures. Quantitative statistical stability and spectral gap can be obtained with these kind of construction, but this is outside the scope of these elementary lectures, for more information about this see \cite{[25], [24]}.

A solenoidal map is a $C^2$ map $F : X \to X$ where $X = S^1 \times D^2$ the filled torus, such that $F$ is a skew product

$$F(x, y) = (T(x), G(x, y)), \quad \text{(83)}$$

where $T : S^1 \to S^1$ and $G : X \to D^2$ are differentiable maps. We suppose the map $T : S^1 \to S^1$ to be $C^2$, expanding of degree $q$, giving rise to a map $[0, 1] \to [0, 1]$, which by a small abuse of notation we denote by $T$ and whose branches will be denoted by $T_i, i \in [1, ..., q]$ and we make the following assumptions on $G$:

- Consider the $F$-invariant foliation $\mathcal{F}^s := \{(x) \times D^2\}_{x \in S^1}$. We suppose that $\mathcal{F}^s$ is contracted: there exists $0 < \alpha < 1$ such that for all $x \in S^1$ holds

$$|G(x, y_1) - G(x, y_2)| \leq \alpha |y_1 - y_2| \quad \text{for all} \quad y_1, y_2 \in D^2, \quad \text{(84)}$$

- $\|\frac{\partial G}{\partial y}\|_\infty < \infty$.

We construct now some function spaces which are suitable for the systems we consider. The idea is to consider spaces of measures with sign, with suitable norms constructed by disintegrating measures along the stable foliation. Thus a measure will be seen as a collection (a path, see Remark \ref{96}) of measures on each leaf. In the stable direction (and on the leaves) we will consider a norm which is the dual of the Lipschitz norm. In the expanding direction, since we have an expanding map, we will consider the $L^1$ norm or a suitable Sobolev norm.

Let $SM(X)$ be the set of Borel signed measures on $\Sigma$. Given $\mu \in SM(X)$ denote by $\mu^+$ and $\mu^-$ the positive and negative parts of it ($\mu = \mu^+ - \mu^-$).

Denote by $\mathcal{AB}$ the set of signed measures $\mu \in SM(X)$ such that its associated marginal signed measures, $\mu^\pm_x = \pi^*_x \mu^\pm$ are absolutely continuous with respect to the Lebesgue measure $m$, on $S^1$ i.e.

$$\mathcal{AB} = \{\mu \in SM(X) : \pi^*_x \mu^+ << m \text{ and } \pi^*_x \mu^- << m\} \quad \text{(85)}$$

where $\pi^*_x : X \to S^1$ is the projection defined by $\pi(x, y) = x$ and $\pi^*_x$ is the associated pushforward map.

Let us consider a finite positive measure $\mu \in \mathcal{AB}$ on the space $X$ foliated by the contracting leaves $\mathcal{F}^s = \{\gamma_l\}_{l \in S^1}$ such that $\gamma_l = \pi_x^{-1}(l)$. The Rokhlin Disintegration Theorem describes a disintegration $\{(\mu_x, \gamma, \phi_x) = \phi_x m\}$ by a family $\{\mu_x\}_\gamma$ of probability measures on the stable leaves and a non negative marginal density $\phi_x : S^1 \to \mathbb{R}$ with $||\phi_x||_1 = \mu(X)$.

**Remark 90.** The disintegration of a measure $\mu$ is the $\mu_x$-unique measurable family $\{(\mu_x, \gamma, \phi_x)\}$ such that, for every measurable set $E \subset X$ it holds

$$\mu(E) = \int_{S^1} \mu_x(E \cap \gamma)d\mu_x(\gamma). \quad \text{(86)}$$

\footnote{In the following to simplify notations, when no confusion is possible we will indicate the generic leaf or its coordinate with $\gamma$.}
Definition 91. Let $\pi_{\gamma,y} : \gamma \rightarrow D^2$ be the restriction $\pi_y|_\gamma$, where $\pi_y : X \rightarrow D^2$ is the projection defined by $\pi_y(x,y) = y$ and $\gamma \in \mathcal{F}^s$. Given a positive measure $\mu \in \mathcal{AB}$ and its disintegration along the stable leaves $\mathcal{F}^s$, $\{\mu_\gamma, \mu_y = \phi_y m_1\}$ (where $m_1$ is the Lebesgue measure on $S^1$) we define the restriction of $\mu$ on $\gamma$ as the positive measure $\mu|_\gamma$ on $D^2$ (not on the leaf $\gamma$) defined, for all measurable set $A \subset D^2$, as

$$\mu|_\gamma(A) = \pi_{\gamma,y}^*(\phi_y(\gamma)\mu_\gamma)(A).$$

For a given signed measure $\mu \in \mathcal{AB}$ and its decomposition $\mu = \mu^+ - \mu^-$, define the restriction of $\mu$ on $\gamma$ by

$$\mu|_\gamma = \mu^+|_\gamma - \mu^-|_\gamma.$$ 

Definition 92. Let $\mathcal{L}^1 \subseteq \mathcal{AB}$ be defined as

$$\mathcal{L}^1 = \left\{ \mu \in \mathcal{AB} : \int_{S^1} W_0^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma) < \infty \right\}$$

and define a norm on it, $|| \cdot ||_{1^1} : \mathcal{L}^1 \rightarrow \mathbb{R}$, by

$$||\mu||_{1^1} = \int_{S^1} W_0^0(\mu^+|_\gamma, \mu^-|_\gamma) dm_1(\gamma).$$

The notation we use for this norm is similar to the usual $L^1$ norm.

Remark 93. Indeed this is formally the case of some $L^1$ norm if we associate to $\mu$, by disintegration, a path $G_\mu : S^1 \rightarrow SB(D^2)$ defined by $G_\mu(\gamma) = \mu|_\gamma$. In this case, this will be the $L^1$ norm of the path. For more details about the disintegration and the properties of the restriction, see the appendix of [25].

Later, similarly we will define a norm $|| \cdot ||_{W^{1,1}}$ which will work as a Sobolev norm for these paths (see Definition [101]).

12.1. Transfer operator associated to $F$. Let us now consider the transfer operator $L_F$ associated with $F$, i.e. such that

$$[L_F \mu](E) = \mu(F^{-1}(E))$$

for each signed measure $\mu$ on $X$ and for each measurable set $E \subset X$. Being a pushforward map, the same function can be also denoted by $F^*$ we will use this notation sometime. There is a nice characterization of the transfer operator in our case, which makes it work quite like a one dimensional transfer operator. For the proof see [25].

Proposition 94. For a given leaf $\gamma \in \mathcal{F}^s$, define the map $F_\gamma : D_2 \rightarrow D_2$ by

$$F_\gamma = \pi_y \circ F|_\gamma \circ \pi_{\gamma,y}^{-1}.$$ 

For all $\mu \in \mathcal{L}^1$ and for almost all $\gamma \in S^1$ holds

$$\langle L_F \mu \rangle_\gamma = \sum_{i=1}^q \frac{F^*_i T^{-1}_i(\gamma) \mu|_{T^{-1}_i(\gamma)}}{|T'_i \circ T^{-1}_i(\gamma)|} \text{ for almost all } \gamma \in S^1.$$
12.2. General properties of $L^1$ like norms.

**Remark 95.** If $F$ is a weak contraction : $X \to X$, where $X$ is a metric space, for every Borel measure with sign $\mu$ it holds
\[ ||L_F\mu||_W \leq ||\mu||_W. \]
Indeed, since $F$ is a contraction, if $|g|_{\infty} \leq 1$ and $\text{Lip}(g) \leq 1$ the same holds for $g \circ F$. Then
\[ \left| \int g \, d(L_F(\mu)) \right| = \left| \int g \circ F \, d\mu \right| \leq ||\mu||_W. \]
Taking the supremum over $|g|_{\infty} \leq 1$ and $\text{Lip}(g) \leq 1$ we finish the proof of the inequality.

**Proposition 96 (The weak norm is weakly contracted by $L_F$).** If $\mu \in L^1$ then
\[
||L_F\mu||_{-1^*} \leq ||\mu||_{-1^*}.
\]

**Proof.** In the following we consider, for all $i$, the change of variable $\gamma = T_i(\alpha)$. Thus by Remark 95 and equation (90), we have
\[
||L_F\mu||_{-1^*} = \int_{N_1} \left| (L_F\mu)|_{\gamma} \right| \, dm_1(\gamma)
\leq \sum_{i=1}^{q} \int_{T(\eta_i)} \left| F_{T_i^{-1}(\gamma)}^* \mu^{-1}(\gamma) \right| \, dm_1(\gamma)
= \sum_{i=1}^{q} \int_{\eta_i} \left| F_{\alpha_i}^* \mu|_{\alpha} \right| \, dm_1(\alpha)
= \sum_{i=1}^{q} \int_{\eta_i} \left| \mu|_{\alpha} \right| \, dm_1(\alpha)
= ||\mu||_{-1^*}.
\]

12.2.1. **Convergence to equilibrium.** Now we prove that $F$ in some sense has exponential convergence to equilibrium.

**Proposition 97.** For all signed measure $\mu \in L^1$ it holds
\[
||F^* \mu||_{-1^*} \leq \alpha ||\mu||_{-1^*} + (\alpha + 1) ||\phi_x||_1.
\]

**Proof.** Consider a signed measure $\mu \in L^1$ and its restriction on the leaf $\gamma$, $\mu|_{\gamma} = \pi_{\gamma, x}(\phi_x(\gamma) \mu_{\gamma})$. Set
\[
\overline{\mu}|_{\gamma} = \pi_{\gamma, x}^* \mu_{\gamma}.
\]
If $\mu$ is a positive measure then $\overline{\mu}|_{\gamma}$ is a probability on $D^2$. Moreover $\mu|_{\gamma} = \phi_x(\gamma) \overline{\mu}|_{\gamma}$.

By the above comments and the expression given by remark 94 we have
Let us estimate $I_1 + I_2$

\[
\| F^\ast \mu \|_{1^\ast} \leq \sum_{i=1}^{q} \int_I \left| \frac{F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \phi^+_x(T^{-1}_i(\gamma))}{|T^\prime_i| \circ T^{-1}_i(\gamma)} - \frac{F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \phi^+_x(T^{-1}_i(\gamma))}{|T^\prime_i| \circ T^{-1}_i(\gamma)} \right| \, dm_1(\gamma)
\]

and

\[
I_1 = \sum_{i=1}^{q} \int_I \left| \frac{F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \phi^+_x(T^{-1}_i(\gamma))}{|T^\prime_i| \circ T^{-1}_i(\gamma)} - \frac{F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \phi^+_x(T^{-1}_i(\gamma))}{|T^\prime_i| \circ T^{-1}_i(\gamma)} \right| \, dm_1(\gamma)
\]

\[
I_2 = \sum_{i=1}^{q} \int_I \left| \frac{F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \phi^+_x(T^{-1}_i(\gamma))}{|T^\prime_i| \circ T^{-1}_i(\gamma)} - \frac{F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \phi^+_x(T^{-1}_i(\gamma))}{|T^\prime_i| \circ T^{-1}_i(\gamma)} \right| \, dm_1(\gamma).
\]

Let us estimate $I_1$ and $I_2$.

By Lemma S7 and a change of variable we have

\[
I_1 = \sum_{i=1}^{q} \int_I \left| F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \right|_{W} \frac{\phi^+_x - \phi^+_x}{|T^\prime_i| \circ T^{-1}_i(\gamma)} \, dm_1(\gamma)
\]

\[
= \int_I \left| F^\ast_{T^{-1}_i}(\gamma) \mu^\ast_{T^{-1}_i}(\gamma) \right|_{W} \frac{\phi^+_x - \phi^+_x}{|T^\prime_i|} \, dm_1(\beta)
\]

\[
= \int_I |\phi^+_x - \phi^+_x| \, dm_1(\beta)
\]

\[
= \| \phi^+_x \|_1
\]
Proposition 99. The integrability of $|| | |$ follows by the fact that $|| | |$ converges pointwise to $|| | |$. The uniqueness follow trivially by Proposition 99.

□

Corollary 98. $|| L_F^n \mu ||_{1^r} \leq \alpha^n || \mu ||_{1^r} + \| \phi_x \|_1$.

where $\alpha = \frac{1 + \alpha}{1 - \alpha}$.

Let us consider the set of zero average measures

(93) $\mathcal{V} = \{ \mu \in L^1 : \mu(X) = 0 \}$.

Since $\pi_n^x \mu = \phi_x m_1 (\phi_x = \phi_x^+ - \phi_x^-)$ we have $\int \phi_x dm_1 = 0$. From the last corollary and the convergence to equilibrium for expanding maps with respect to $L^1$ and $W^{1,1}$ norms (see Remarks 33 and 34) it directly follows:

**Proposition 99 (Exponential convergence to equilibrium).** There exist $D \in \mathbb{R}$ and $0 < \beta_1 < 1$ such that, for every signed measure $\mu \in \mathcal{V}$, it holds

$|| L_F^n \mu ||_{1^r} \leq D_2 \beta_1^n (|| \mu ||_1 + || \phi_x ||_{W^{1,1}})$

for all $n \geq 1$.

We now prove the existence of an invariant measure for the solenoidal system in the set $L^1$ it is not difficult to deduce that this should be a physical measure (points in the same stable leaves must have the same long time average for a continuous observable).

**Proposition 100.** There is a unique $\mu \in L^1$ such that $L_F \mu = \mu$.

**Proof.** The base map $T$ is expanding and has an absolutely continuous invariant measure. Let us call it $\varphi_x$. Consider the measure $\nu = \varphi_x \times m$ (the measure having marginal $\varphi_x$ and Lebesgue measure on the stable leaves) and the sequence $\nu_n = L_F^n \nu$. By Proposition 97 $|| \nu_n - \nu_m ||_{1^r} \leq D_2 \beta_1^n$ and $\nu_n$ is a Cauchy sequence. By passing to a subsequence we can find $\nu_{n_k}$ such that for almost each leaf $\gamma$, $\nu_{n_k} |_{\gamma}$ is a Cauchy sequence for the $|| | |$ norm. By Remark 30 this sequence must have a limit which is a positive measure. This defines a limit measure $\mu$ which is invariant.

The integrability of $|| | |$ follows by the fact that $|| | |$ converges pointwise to $|| | |$. The uniqueness follow trivially by Proposition 99.

□
12.3. **Strong norm and Lasota Yorke inequality.** We give here an example of a strong space satisfying a kind of Lasota Yorke inequality which holds for positive measures.

Given $\mu \in L^1$ let us denote by $\phi_\mu$ its marginal density. Let us consider the following space of measures

$$ W^{1,1} = \{ \mu \in L^1 : |\phi_\mu| \in W^{1,1} \} $$

for almost all $\gamma$, $D(\mu, \gamma) := \limsup_{\gamma_2 \to \gamma_1} \| \frac{\mu_{\gamma_2} - \mu_{\gamma_1}}{\gamma_2 - \gamma_1} \|_W < \infty$

**Definition 101.** Let us consider the norm

$$ \|\mu\|_{W^{1,1}} := \|\mu\|_1 + \int |D(\mu, \gamma)| \, d\gamma. $$

This norm will play the role of the strong norm in the solenoid case. Indeed the following Lasota-Yorke-like inequality can be proved

**Proposition 102.** Let $F$ be a solenoidal map, then $L^\prime F W^{1,1} \subseteq W^{1,1}$ and there are $\lambda < 1, B > 0$ s.t $\forall \mu \in W^{1,1}$ such that $\mu \geq 0$.

$$ \|L^\prime F \mu\|_{W^{1,1}} \leq \lambda (\alpha \|\mu\|_{W^{1,1}} + \|\phi'\|_1) + B \|\mu\|_{1}. $$

**Proof.** Since the map is $C^2$ it is obvious that $\lim_{\gamma \to \gamma_1} \|\mu_{\gamma} - \mu_{\gamma_1}\|_W = 0$ and $\phi_{L^\prime F(\mu)} \in W^{1,1}$. Let us estimate

$$ \|D(L^\prime F \mu, \gamma_1)\|_1 = \int \limsup_{\gamma_2 \to \gamma_1} \| \frac{(L^\prime F \mu)_{\gamma_2} - (L^\prime F \mu)_{\gamma_1}}{\gamma_2 - \gamma_1} \|_W \, d\gamma. $$

By Equation (90) we have

$$ (L^\prime F \mu)_{\gamma} = \sum_{i=1}^{q} \frac{F^*_{T_i^{-1}(\gamma)} \mu_{T_i^{-1}(\gamma)}}{|T_i' \circ T_i^{-1}(\gamma)|}, $$

for almost all $\gamma \in N_1$.

Then

$$ \|D(L^\prime F \mu, \gamma_1)\|_1 \leq \sum_{i=1}^{q} \int \limsup_{\gamma_2 \to \gamma_1} \| \frac{1}{\gamma_2 - \gamma_1} \left( \frac{F^*_{T_i^{-1}(\gamma)} \mu_{T_i^{-1}(\gamma)}}{|T_i' \circ T_i^{-1}(\gamma)|} - \frac{F^*_{T_i^{-1}(\gamma_2)} \mu_{T_i^{-1}(\gamma_2)}}{|T_i' \circ T_i^{-1}(\gamma_2)|} \right) \|_W \, d\gamma. $$

$$ \leq \sum_{i=1}^{q} \int \limsup_{\gamma_2 \to \gamma_1} \| \frac{1}{\gamma_2 - \gamma_1} \left( \frac{F^*_{T_i^{-1}(\gamma)} \mu_{T_i^{-1}(\gamma)}}{|T_i' \circ T_i^{-1}(\gamma)|} - \frac{F^*_{T_i^{-1}(\gamma_2)} \mu_{T_i^{-1}(\gamma_2)}}{|T_i' \circ T_i^{-1}(\gamma_2)|} \right) \|_W \, d\gamma + \int \limsup_{\gamma_2 \to \gamma_1} \| \frac{1}{\gamma_2 - \gamma_1} \left( \frac{F^*_{T_i^{-1}(\gamma_2)} \mu_{T_i^{-1}(\gamma_2)}}{|T_i' \circ T_i^{-1}(\gamma_2)|} - \frac{1}{|T_i' \circ T_i^{-1}(\gamma_2)|} \right) \|_W \, d\gamma $$

$$ \leq \sum_{i=1}^{q} \int \frac{1}{|T_i' \circ T_i^{-1}(\gamma)|} \limsup_{\gamma_2 \to \gamma_1} \| \frac{F^*_{T_i^{-1}(\gamma)} \mu_{T_i^{-1}(\gamma)}}{|T_i' \circ T_i^{-1}(\gamma)|} - \frac{F^*_{T_i^{-1}(\gamma_2)} \mu_{T_i^{-1}(\gamma_2)}}{|T_i' \circ T_i^{-1}(\gamma_2)|} \|_W \, d\gamma $$

$$ + \int \| F^*_{T_i^{-1}(\gamma_2)} \mu_{T_i^{-1}(\gamma_2)} \|_W \limsup_{\gamma_2 \to \gamma_1} \frac{1}{\gamma_2 - \gamma_1} \left( \frac{1}{|T_i' \circ T_i^{-1}(\gamma)|} - \frac{1}{|T_i' \circ T_i^{-1}(\gamma)|} \right) \, d\gamma. $$
Hence
\[
\|D(LF\mu, \gamma_1)\|_1 \leq \sum_{i=1}^{q} \int \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \|F^*_{T_i^{-1}(\gamma_2)} \mu|_{T_i^{-1}(\gamma_2)} - F^*_{T_i^{-1}(\gamma_1)} \mu|_{T_i^{-1}(\gamma_2)}\|_W \\
+ \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \|F^*_{T_i^{-1}(\gamma_1)} \mu|_{T_i^{-1}(\gamma_2)} - F^*_{T_i^{-1}(\gamma_2)} \mu|_{T_i^{-1}(\gamma_2)}\|_W d\gamma_1 \\
+ \int \|F^*_{T_i^{-1}(\gamma_1)} \mu|_{T_i^{-1}(\gamma_1)}\|_W \limsup_{\gamma_2 \to \gamma_1} \frac{1}{\gamma_2 - \gamma_1} \left( \frac{1}{|T_i \circ T_i^{-1}(\gamma_2)|} - \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \right) d\gamma_1 \\
= I + II + III
\]

Let us estimate the first summand
\[
I = \sum_{i=1}^{q} \int \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \|F^*_{T_i^{-1}(\gamma_1)} \left( \mu|_{T_i^{-1}(\gamma_1)} - \mu|_{T_i^{-1}(\gamma_2)} \right)\|_W d\gamma_1
\]

We recall that by Lemma 87 \( \|F^* \mu\|_W \leq \alpha \|\mu\|_W + \mu(D^2) \) then
\[
I \leq \sum_{i=1}^{q} \int \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \left\{ \alpha \left\| \frac{\left( \mu|_{T_i^{-1}(\gamma_1)} - \mu|_{T_i^{-1}(\gamma_2)} \right)}{\gamma_2 - \gamma_1} \right\|_W + \left( \frac{\mu|_{T_i^{-1}(\gamma_1)}(D^2) - \mu|_{T_i^{-1}(\gamma_2)}(D^2)}{\gamma_2 - \gamma_1} \right) \right\} d\gamma_1
\]

and
\[
I \leq \sum_{i=1}^{q} \int \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \frac{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)}{\gamma_2 - \gamma_1} \limsup_{\gamma_2 \to \gamma_1} \alpha \left\| \frac{\left( \mu|_{T_i^{-1}(\gamma_1)} - \mu|_{T_i^{-1}(\gamma_2)} \right)}{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)} \right\|_W \\
+ \limsup_{\gamma_2 \to \gamma_1} \left( \frac{\mu|_{T_i^{-1}(\gamma_1)}(D^2) - \mu|_{T_i^{-1}(\gamma_2)}(D^2)}{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)} \right) d\gamma_1 \\
\leq \sum_{i=1}^{q} \sup_{\gamma_2 \to \gamma_1} \frac{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)}{\gamma_2 - \gamma_1} \int_{I_i} \alpha D(\mu, T_i^{-1}(\gamma_1)) + |\phi'_x(T_i^{-1}(\gamma_1))| dT_i^{-1}(\gamma_1).
\]

Then summing the contributions from all branches \( T_i \) and intervals \( I_i \)
\[
I \leq \sup_{\gamma_2 \to \gamma_1} \frac{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)}{\gamma_2 - \gamma_1} \left[ \alpha \|D(\mu, \gamma_1)\|_1 + \|\phi'_x\|_1 \right] \\
\leq \frac{1}{\inf |T|} \left[ \alpha \|D(\mu, \gamma_1)\|_1 + \|\phi'_x\|_1 \right].
\]
Now the other summands,

\[ II \leq \sum_i \int \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \left| \frac{F_{T_i^{-1}(\gamma_1)}^* \mu|_{T_i^{-1}(\gamma_2)} - F_{T_i^{-1}(\gamma_2)}^* \mu|_{T_i^{-1}(\gamma_2)}}{\gamma_2 - \gamma_1} \right| \omega d\gamma_1 \]
\[ \leq \sup_i \limsup_{\gamma_2 \to \gamma_1} \left| \frac{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)}{\gamma_2 - \gamma_1} \right| \]
\[ \times \sum_i \int \frac{1}{|T_i \circ T_i^{-1}(\gamma_1)|} \limsup_{\gamma_2 \to \gamma_1} \left| \frac{F_{T_i^{-1}(\gamma_1)}^* \mu|_{T_i^{-1}(\gamma_2)} - F_{T_i^{-1}(\gamma_2)}^* \mu|_{T_i^{-1}(\gamma_2)}}{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)} \right| \omega d\gamma_1 \]
\[ \leq \sup_i \limsup_{\gamma_2 \to \gamma_1} \left| \frac{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)}{\gamma_2 - \gamma_1} \right| \sum_i \int \limsup_{\gamma_2 \to \gamma_1} \left| \frac{F_{T_i^{-1}(\gamma_1)}^* \mu|_{T_i^{-1}(\gamma_2)} - F_{T_i^{-1}(\gamma_2)}^* \mu|_{T_i^{-1}(\gamma_2)}}{T_i^{-1}(\gamma_2) - T_i^{-1}(\gamma_1)} \right| \omega d\gamma_1 \]
\[ \leq \frac{1}{\inf |T''|} \frac{|\partial G|}{\partial x} \|\mu\|_1. \]

where in the last step we used that \( \mu \geq 0 \). Finally

\[ III \leq \sum_i \int \|T''\| \frac{T''}{(T')^2} \|\mu\|_1. \]

Summarizing

(95)

\[ ||L_F^\mu||_{W^{1,1}} \leq \frac{1}{\inf |T'|} (\alpha ||\mu||_{W^{1,1}} + ||\phi'\mu||_1 + ||\partial G/\partial x||_1 ||\mu||_1 + (1 + ||T''||_1) \frac{T''}{(T')^2} ||\mu||_1. \]

As done before, iterating the inequality, gives

**Corollary 103.** There are \( B > 0, \lambda < 1 \) such that

\[ ||L_F^\mu||_{W^{1,1}} \leq \lambda^n (||\mu||_{W^{1,1}} + ||\phi'\mu||_1) + B ||\mu||_1. \]

This inequality shows that the iteration of a positive measure keeps a bounded regularity in this strong norm.

13. **Bibliographic remarks and acknowledgements**

The books [3], [19], [44], [14] contains detailed introductions to subjects similar to the one considered in these notes (from different points of view). Our point of view is closer to the one given in the notes [38] and [42], from which some definition and statement is taken. We recommend to consult all these texts for many related topics, generalizations, applications and complements we cannot include in these short introductory notes.

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