ON SCALES OF SOBOLEV SPACES
ASSOCIATED TO GENERALIZED HARDY OPERATORS

KONSTANTIN MERZ

Abstract. We consider the fractional Laplacian with Hardy potential and study
the scale of homogeneous $L^p$ Sobolev spaces generated by this operator. Besides
generalized and reversed Hardy inequalities, the analysis relies on a Hörmander mul-
tiplier theorem which is crucial to construct a basic Littlewood–Paley theory. The
results extend those obtained recently in $L^2$ but do not cover negative coupling
constants in general due to the slow decay of the associated heat kernel.

1. Introduction & Result

Introduction. The classical, sharp Hardy inequality
\[
\int_{\mathbb{R}^d} |\nabla f|^p \, dx - \left( \frac{|d-p|}{p} \right)^p \int_{\mathbb{R}^d} \frac{|f(x)|^p}{|x|^p} \, dx \geq 0
\]
is one of the longest known inequalities relating the weighted $L^p$ norm of a decaying
function with the $L^p$ norm of its gradient and plays an important role in fields such
as mathematical physics, non-linear PDEs, and harmonic analysis. This inequality
holds for all $f \in C^\infty_c(\mathbb{R}^d)$ if $1 \leq p < d$ and for all $f \in C^\infty_c(\mathbb{R}^d \setminus \{0\})$ if $p > d$.
Herbst considered a generalization of the above inequality related to the fractional
Laplacian $(-\Delta)^{\alpha/2}$. Here, and in the following we restrict ourselves to $\alpha \in (0, 2)$. For
$1 < p < 2d/\alpha$ the inequality states
\[
\|(-\Delta)^{\alpha/4} f\|_{L^p(\mathbb{R}^d)} - \frac{\Gamma\left(\frac{d/p+\alpha/2}{2}\right)}{\Gamma\left(\frac{d/p-\alpha/2}{2}\right)} \frac{\Gamma\left(\frac{d}{2p}\right)}{\Gamma\left(\frac{d}{2p}+\alpha/2\right)} \|x^{-\alpha/2} f\|_{L^p(\mathbb{R}^d)} \geq 0 \quad \text{for all } f \in C^\infty_c(\mathbb{R}^d)
\]
(1.1)
where the constant on the right side is sharp, see [17, Theorem 2.5]. We emphasize
that, for $p \neq 2$, $\|(-\Delta)^{\alpha/4} f\|_p$ is not proportional to
\[
\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{d+\alpha p/2}} \, dx \, dy \right)^{1/p}.
\]

Date: November 4, 2020.
2010 Mathematics Subject Classification. 35A23, 46E35.
Key words and phrases. Fractional Laplacian, Hardy inequality, Hardy operator, spectral multiplier theorem.

Deutsche Forschungsgemeinschaft grant SI 348/15-1 is gratefully acknowledged.
Instead, there is a one-sided inequality depending on whether $1 < p < 2$ or $p > 2$, see, e.g., [30, Chapter V] and also Frank and Seiringer [13] concerning the sharp fractional Hardy inequality involving this expression. If 

$$-a_* := \frac{2^a \Gamma((d + \alpha)/4)^2}{\Gamma((d - \alpha)/4)^2}$$

denotes the squared sharp constant for $p = 2$, we define the generalized Hardy operator

$$\mathcal{L}_{a,\alpha} := (-\Delta)^{\alpha/2} + a|x|^{-\alpha}$$

in $L^2(\mathbb{R}^d)$ as the Friedrichs extension of the corresponding quadratic form on $C_c^\infty(\mathbb{R}^d)$ for $a \geq a_*$. For $d = 3$ and $\alpha = 1$, the optimal $a_*$ was already known to Kato [21, Chapter 5, Equation (5.33)]. For general $d$ and $\alpha$ it was first computed by Herbst, but see also [26, 33, 11, 13] for alternative proofs of the inequality with sharp constant.

Due to the homogeneity of $\mathcal{L}_{a,\alpha}$ it is natural to ask whether the operators with $a = 0$ and $a \neq 0$ are in some sense equivalent to each other. For instance, one may ask whether they generate scales of homogeneous Sobolev spaces which are comparable with each other, i.e., whether there are $0 < A < A'$ such that

$$\|(-\Delta)^{\alpha s/4} f\|_{L^p(\mathbb{R}^d)} \leq A \|\mathcal{L}_{a,\alpha}^{s/2} f\|_{L^p(\mathbb{R}^d)} \leq A' \|(-\Delta)^{\alpha s/4} f\|_{L^p(\mathbb{R}^d)}$$

holds for certain $\alpha, a, s, p$. For the Schrödinger operator, i.e., $\alpha = 2$ and $d \geq 3$, Killip et al. [22] proved that the norms are in fact equivalent to each other for certain $a, s, p$.

Our results may be useful to study non-linear PDEs involving $\mathcal{L}_{a,\alpha}$ in order to reduce problems to those involving only $|p|^\alpha$, i.e., without the Hardy potential. For $\alpha = 2$, the corresponding result was used, e.g., by Killip et al. [21, 23] to determine the threshold between scattering and finite-time blowup of the focusing cubic nonlinear Schrödinger equation with Hardy potential, or the well-posedness of the energy-critical nonlinear Schrödinger equation with Hardy potential.

Before proceeding to the main result, we introduce some notation that is used throughout the rest of this paper.

(1) We write $X \lesssim Y$ for non-negative quantities $X$ and $Y$, whenever there is a positive constant $A$ such that $X \leq A \cdot Y$. If $A$ depends on some parameter $r$, we sometimes write $X \lesssim_r Y$. Moreover, $X \sim Y$ means $Y \lesssim X \lesssim Y$ and in this case, we say that $X$ is equivalent to $Y$.

(2) We define $X \wedge Y := \min\{X, Y\}$ and $X \vee Y := \max\{X, Y\}$.

(3) The integer part of $x \in \mathbb{R}$ is denoted by $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$. The positive part is denoted by $x_+ := \max\{x, 0\}$.

(4) For $1 \leq p \leq \infty$, we abbreviate $\|f\|_p = \|f\|_{L^p(\mathbb{R}^d)}$ and denote Hölder conjugate exponents by primes, i.e., $p^{-1} + p'^{-1} = 1$.

(5) The $s$-th $L^2$ potential space ($s \in \mathbb{R}$) is denoted by $H^s(\mathbb{R}^d)$. It consists of all
functions $f$ such that the norm $\|f\|_{H^s} := \|(1 - \Delta)^{s/2} f\|_{L^2}$ is finite where $(1 - \Delta)^{s/2}$ denotes the operator which is given by multiplication with $(1 + 4\pi^2 \alpha)^{s/2}$ in Fourier space. Moreover, $f \in H^s_{loc}(\mathbb{R}^d)$ if and only if $\|\varphi f\|_{H^s} < \infty$ for all $\varphi \in C^\infty_c(\mathbb{R}^d)$.

(6) We abbreviate $|p| = \sqrt{-\Delta}$.

**Main result and strategy of the proof.** Let us recall the following parameterization of the coupling constant in terms of the power of the formal ground state of $\mathcal{L}_{a,\alpha}$, namely

$$
\Psi_{a,d}(\sigma) := -2^\alpha \frac{\Gamma(\frac{\sigma + \alpha}{2})}{\Gamma(\frac{d-\sigma}{2})} \frac{\Gamma(\frac{d-\sigma}{2})}{\Gamma(\frac{\sigma}{2})} \quad \text{if } \sigma \in (-\alpha, (d - \alpha)/2) \setminus \{0\}
$$

and $\Psi_{a,d}(0) = 0$. According to [11] Lemma 3.2] and [20] p. 8], the function $\sigma \mapsto \Psi_{a,d}(\sigma)$ is continuous and strictly decreasing in $(-\alpha, (d - \alpha)/2]$ with

$$
\lim_{\sigma \to -\alpha} \Psi_{a,d}(\sigma) = \infty \quad \text{and} \quad \Psi_{a,d}\left(\frac{d - \alpha}{2}\right) = a_*.
$$

Consequently, for any $a \geq a_*$, we may define

$$
\delta := \Psi_{a,d}^{-1}(a)
$$

which allows us to formulate our main theorem on the equivalence of $L^p$ Sobolev norms generated by powers of $\mathcal{L}_{a,\alpha}$.

**Theorem 1.1 (Equivalence of Sobolev norms on $L^p(\mathbb{R}^d)$).** Let $d \in \mathbb{N}$, $0 < \alpha < 2 \wedge d$, and $s \in (0, 2]$. Let $a \geq a_*$ if $s = 2$ and $a \geq 0$ if $s \in (0, 2)$. Let furthermore $\delta$ be defined by (1.3).

1. If $1 < p < \infty$ satisfies $\alpha s/2 + \delta < d/p < \min\{d, d - \delta\}$, then

$$
\|p|^{\alpha s/2} f\|_{L^p(\mathbb{R}^d)} \lesssim_d p, a, s \|\mathcal{L}^{s/2}_{a,\alpha} f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in C^\infty_c(\mathbb{R}^d).
$$

2. If $\alpha s/2 < d/p < d$ (which already ensures $1 < p < \infty$), then

$$
\|\mathcal{L}^{s/2}_{a,\alpha} f\|_{L^p(\mathbb{R}^d)} \lesssim_d p, a, s \|p|^{\alpha s/2} f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in C^\infty_c(\mathbb{R}^d).
$$

We remark that for $p = 2$, an equivalence of Sobolev norms for some $s \in (0, 2]$ and $a \geq a_*$ yields, by the spectral theorem and the operator monotonicity of positive roots, an equivalence of norms for any $0 < t < s$ with the same $a$, see also [12] Remarks 1.2 and 1.3. If $p \neq 2$, this assertion is far from obvious.

Let us now outline the strategy of the proof. For $s = 2$, the assertion follows immediately from the ordinary Hardy inequality ([11]) and a generalized Hardy inequality which is why we can also handle $a < 0$ in this case.

**Proposition 1.2 (Generalized Hardy inequality).** Let $1 < p < \infty$, $\alpha \in (0, 2 \wedge d)$, $a \geq a_*$, $\delta$ be defined by (1.3), and $\alpha s/2 \in (0, d)$. If $s$ and $p$ satisfy $\alpha s/2 + \delta < d/p < d - \delta$, then

$$
\|\langle x \rangle^{\alpha s/2} f\|_p \lesssim_d p, a, s, p \|\mathcal{L}^{s/2}_{a,\alpha} f\|_p \quad \text{for all } f \in C^\infty_c(\mathbb{R}^d).
$$

(1.4)
Conversely, if \( \alpha s/2 \in \left(0, \min\{d, d-2\delta\}\right) \) and the above estimate holds, then \( \alpha s/2 + \delta < d/p < d - \delta \).

**Proof.** The assertion is equivalent to the \( L^p \)-boundedness of the operator \( |x|^{-\alpha s/2} \mathcal{L}_{a,\alpha}^{-s/2} \). Using the pointwise bounds on the Riesz kernel of \( \mathcal{L}_{a,\alpha} \) (see [12, Theorem 1.6]), i.e.,

\[
\mathcal{L}_{a,\alpha}^{-s/2}(x, y) \sim_{d,\alpha, a, s} |x - y|^{\frac{d}{2} - \frac{d}{p}} \left(1 \wedge \frac{|x|}{|x - y|} \wedge \frac{|y|}{|x - y|}\right)^{-\delta} \quad \text{for} \quad \frac{\alpha s}{2} \in \left(0, d \wedge d - 2\delta\right),
\]

this follows from the \( L^p \)-boundedness of the operator whose integral kernel is the above kernel multiplied by \( |x|^{-\alpha s/2} \) which in turn is proven by performing a Schur test. As the involved computations are analogous those in [22, Proposition 3.2] (with \( \alpha \) instead of \( \alpha \) and \( \delta \) instead of \( \sigma \)), respectively [12, Proposition 1.4], we omit a proof.

The fact that (1.4) fails for \( d/p \leq \alpha s/2 + \delta \) or \( d/p \geq d - \delta \) follows from the lower bound in (1.5) and the same counterexamples as in [22, Proposition 3.2]. \( \square \)

If \( s < 2 \), the proof is a bit more laborious. Still, the idea is to use the triangle inequality, obtain an estimate like

\[
\|p|^{\alpha s/2}f\|_p \leq \|L_{a,\alpha}^{s/2} - |p|^{\alpha s/2}f\|_p + \|L_{a,\alpha}^{s/2}f\|_p \lesssim \|\mathfrak{R}|^{-\alpha s/2}f\|_p + \|L_{a,\alpha}^{s/2}f\|_p,
\]

and then apply the generalized Hardy inequality. For \( p = 2 \), the second inequality in (1.6) was called a reversed Hardy inequality (cf. [12, Proposition 1.5]), because it yields a lower bound on the norm of \( |x|^{-\alpha s/2}f \) in terms of the difference \( (L_{a,\alpha}^{s/2} - |p|^{\alpha s/2})f \).

There, (1.6) was proven using the spectral theorem, i.e.,

\[
\Gamma \left(\frac{s}{2}\right) \|L_{a,\alpha}^{s/2}f\|_2 = \left\| \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} (1 - e^{-tL_{a,\alpha}}) f \right\|_2 \leq \left\| \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} (e^{-t|p|^\alpha} - e^{-tL_{a,\alpha}}) f \right\|_2 + \left\| \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} (1 - e^{-t|p|^\alpha}) f \right\|_2,
\]

thereby rewriting the difference \( \|(L_{a,\alpha}^{s/2} - |p|^{\alpha s/2})f\|_2 \) directly in terms of the difference of the associated heat kernels. However, due to the lack of a spectral theorem in \( L^p \), we will first express \( \|L_{a,\alpha}^{s/2}f\|_p \) and \( \|p|^{\alpha s/2}f\|_p \) in terms of Littlewood–Paley square functions employing two-sided square function estimates, Theorem 4.3. The corresponding Littlewood–Paley projections will be defined via the heat kernels of \( \mathcal{L}_{a,\alpha} \) and \( |p|^{\alpha} \), because we have good pointwise bounds on the individual kernels and on their difference, Theorem 2.1 and Lemma 5.1. The latter bounds then allow us to prove a reversed Hardy inequality expressed in terms of these square functions, thereby yielding an analog of (1.6), see Proposition 5.2.

The proof of the square function estimates, however, crucially depends on the \( L^p \)-boundedness of certain functions of \( \mathcal{L}_{a,\alpha} \). In \( L^2 \) it follows from the spectral theorem that measurable, bounded functions of self-adjoint operators are bounded on \( L^2 \). The \( L^p \)-boundedness of functions of such operators (which may initially be defined by the \( L^2 \) functional calculus), however, relies on much stronger regularity assumptions on
the multiplier and specific knowledge of the operator in question. Here, we discuss two instances of such spectral multiplier theorems which differ in the conditions on the multiplier. On the one hand, Mikhlin multiplier theorems \[27\] require that the multiplier \( m \) is at least \( s \) times continuously differentiable and satisfies the Mikhlin condition

\[|\lambda^j \partial^j_x m(\lambda)| \lesssim j \quad \text{for all } j = 0, \ldots, s.\]

On the other hand, Hörmander multiplier theorems rely on the condition that the multiplier \( F \) belongs to \( H^s_{\text{loc}}(\mathbb{R}) \) for some sufficiently large \( s > 0 \). Moreover, for a non-zero \( \varphi \in C^\infty_c(\mathbb{R}^+), \) the Hörmander condition

\[\sup_{t > 0} \| \varphi(\cdot) F(t \cdot) \|_{H^s} < \infty\]

must be satisfied. This reveals in particular that Hörmander multiplier theorems imply Mikhlin multiplier theorems. It is known that \( s > d/2 \) suffices to prove a Mikhlin or a Hörmander multiplier theorem for Fourier multipliers, see \[30, \text{Chapter IV, §3, Theorem 3}\] and \[18\].

There is a broad literature on the derivation of spectral multiplier theorems. However, these usually rely on the assumption that the corresponding heat kernel satisfies pointwise Gaussian estimates \[15, 14, 10, 4\] or so-called generalized Gaussian estimates \[2\]. The kernel may even have local singularities, like the one of \(-\Delta + a|x|^{-2}\) for \( a < 0 \), see, e.g., Milman and Semenov \[28\]. For a survey on spectral multiplier theorems for operators with Gaussian heat kernel bounds, we refer to Duong et al. \[9\] and the references therein. Using the maximum principle and the exponential decay of \( \exp(\Delta) \), Hebisch \[15\] derived a multiplier theorem for Schrödinger operators \(-\Delta + V\) in \( L^2(\mathbb{R}^d) \) when \( V \geq 0 \). Unlike in an earlier work \[14\] where the heat kernel needed to satisfy a certain Hölder condition, the proof relies on decent \( L^2 \) estimates and is based on a clever dyadic decomposition of the multiplier. Naturally, the maximum principle can also be invoked for \( \exp(-(|p|^\alpha + V)) \) with \( \alpha \in (0, 2) \) and \( V \geq 0 \). However, due to the slow, i.e., algebraic decay of \( \exp(-|p|^\alpha) \) it is considerably more difficult to show a multiplier theorem also in this case. Using similar techniques as in \[15\], it is however possible to prove a Hörmander multiplier theorem for \(|p|^\alpha + V\), at least in the special case \( d = 1 \) and \( \alpha > 1 \), see \[16, \text{Theorem 3.8}\]. The reason for this restriction is the slow decay of the heat kernel which makes it difficult to deduce radial, integrable upper bounds of functions of \( \mathcal{L}_{\alpha, \alpha} \), even if these functions are smooth and compactly supported. The existence of such upper bounds is, however, vital to make use of a well-known property of the Hardy–Littlewood maximal function in order to conclude the proof. Let us also point out to a recent work of Chen et al. \[4\] who proved multiplier theorems for abstract self-adjoint operators whose methods do not rely on a-priori heat kernel bounds. In particular, they obtain a multiplier theorem for \(|p|^\alpha + V\) with \( V \geq 0 \), however again, only in \( d = 1 \) and with \( \alpha > 1 \), see \[5\] Section
5.3] and their Theorem 3.1 and the subsequent corollary. In any case, these results are however not applicable in our situation since we are requiring $\alpha < d$.

Nonetheless, it is possible to establish a spectral multiplier theorem associated to $\mathcal{L}_{a,\alpha}$ in two special cases. On the one hand, a simple computation using the pointwise bounds on $e^{-\mathcal{L}_{a,\alpha}(x,y)}$ shows that the heat kernel is bounded on $L^p$ for all $a \geq a_*$, see Lemma 4.2. On the other hand, based on an abstract result by Hebisch [16], we prove a Hörmander multiplier theorem for $\mathcal{L}_{a,\alpha}$ if $a \geq 0$. In fact, we obtain the following result.

**Theorem 1.3.** Let $d \in \mathbb{N}$, $\tilde{a} \geq a > 0$, $\alpha \in (0, 2 \wedge d)$, and $c \in (0, \alpha)$. Moreover, let $V$ be a measurable function on $\mathbb{R}^d$ satisfying

$$\frac{a}{|x|^\alpha} \leq V(x) \leq \frac{\tilde{a}}{|x|^\alpha}.$$  \tag{1.7}

If $F \in H^s_{\text{loc}}(\mathbb{R})$ with

$$s > \frac{d}{2(\alpha + c)} \left[ \frac{d}{2} \left( 1 + \frac{1}{c} \right) + 1 \right] + \frac{1}{2}$$

and for $a \neq \varphi \in C_c^\infty(\mathbb{R}_+)$, one has $\sup_{t>0} \|\varphi F(t \cdot)\|_{H^s} < \infty$, then $F(|p|^\alpha + V)$, initially defined via the $L^2$ functional calculus, has weak type $(1,1)$ and is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.

Immediate consequences of this result are, e.g., the $L^p$-boundedness of Riesz means $R_\beta(\mathcal{L}_{a,\alpha})$ where $R_\beta(\lambda) := (1 - \lambda)_+^{\beta}$ whenever $\beta > s - 1/2$, and imaginary powers $\mathcal{L}_{a,\alpha}^{i\tau}$, $\tau \in \mathbb{R}$. In the context of the present work, the main importance of this result is that it allows us to construct a basic Littlewood–Paley theory by deriving Bernstein estimates, Lemma 4.2, and the crucial square function estimates, Theorem 4.3.

We would like to emphasize that the above strategy is inspired by [22, 12]. The idea to formulate the norms $\|\mathcal{L}_{a,\alpha}^{s/2}f\|_p$ in terms of square functions, which are in turn expressed via the heat kernel, is borrowed from [22]. The construction of Littlewood–Paley theory based on heat kernels is, e.g., exhaustively treated in [31]. On the other hand, we are fortunate to invoke the key estimates on the Riesz kernel of $\mathcal{L}_{a,\alpha}$ and on the difference of the heat kernels of $\mathcal{L}_{a,\alpha}$ and $|p|^\alpha$ which were obtained in [12]. In order to make the paper self-contained, we have, however, decided to review and present the involved arguments for the reader’s convenience.

**Organization.** In the next section we recall the crucial bounds on the heat kernel of $\mathcal{L}_{a,\alpha}$ and state simple but important weighted and ultracontractive estimates for $e^{-(|p|^\alpha + V)}$ for non-negative functions $V$ on $\mathbb{R}^d$. These estimates play a major role in the subsequent section where we prove a Hörmander multiplier theorem for $|p|^\alpha + V$ with $V$ as in (1.7). Afterwards, we discuss difficulties arising in the case of negative coupling constants. In the fourth section we derive Bernstein estimates and square function estimates which are crucial to express the $L^p$ norms generated by powers of $\mathcal{L}_{a,\alpha}$. In the fifth section, we prove a reversed Hardy inequality expressed in terms of
square functions and give the proof of Theorem 1.1. In the last section we present a simple generalization of the main result when the Hardy potential is replaced by a function $V$ which satisfies (1.7).

2. Heat kernel associated to $\mathcal{L}_{a, \alpha}$

We recall recent two-sided bounds on the heat kernel of $\mathcal{L}_{a, \alpha}$ by Bogdan et al. [3] for $a < 0$ and Cho et al. [6] or Jakubowski and Wang [20] for $a > 0$. For $a = 0$ these bounds were already proven by Blumenthal and Getoor [1]. Moreover, for $a = 0$ and $\alpha = 1$, the heat kernel is just the Poisson kernel, see also [32, Theorem 1.14].

**Theorem 2.1** (Heat kernels of generalized Hardy operators). Let $\alpha \in (0, 2 \wedge d)$, $a \geq a_*$ and $\delta$ be defined by (1.3). Then the heat kernel of $\mathcal{L}_{a, \alpha}$ satisfies for all $x, y \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$,

$$e^{-t \mathcal{L}_{a, \alpha}(x, y)} \sim \left(1 \lor \frac{t^{1/\alpha}}{|x|}\right)^{\delta} \left(1 \lor \frac{t^{1/\alpha}}{|y|}\right)^{\delta} t^{-d/\alpha} \left(1 \lor \frac{t^{1+d/\alpha}}{|x - y|^{d+\alpha}}\right).$$

The following bounds are going to be vital in the proof of the spectral multiplier theorem for $|p|^\alpha + V$ with $V$ as in (1.7) and follow immediately from Theorem 2.1.

**Lemma 2.2.** Let $\alpha \in (0, 2 \wedge d)$ and $V$ be a non-negative, measurable function on $\mathbb{R}^d$. Then, for all $t > 0$ and $c < \alpha$,

$$\sup_{y \in \mathbb{R}^d} \int |e^{-t(|p|^\alpha + V)(x, y)}(1 + t^{-1/\alpha}|x - y|)^c| \, dx < \infty \quad \text{and} \quad (2.1a)$$

$$\sup_{y \in \mathbb{R}^d} t^{d/\alpha} \int |e^{-t(|p|^\alpha + V)(x, y)}|^2 \, dx < \infty. \quad (2.1b)$$

**Proof.** By Trotter’s formula, it suffices to prove (2.1a) and (2.1b) where the kernel $e^{-t(|p|^\alpha + V)(x, y)}$ is replaced by $e^{-t|p|^\alpha(x, y)}$. Moreover, the substitution $x \mapsto t^{1/\alpha}x$ shows that it suffices to consider $t = 1$. Since

$$1 \lor \frac{1}{|x - y|^{d+\alpha}} \sim \frac{1}{(1 + |x - y|)^{d+\alpha}},$$

the integral

$$\int_{\mathbb{R}^d} \frac{(1 + |x - y|)^c}{(1 + |x - y|)^{d+\alpha}} \, dx$$

is finite for all $y \in \mathbb{R}^d$, if $c < \alpha$ which shows (2.1a). On the other hand, Plancherel’s theorem implies

$$\int_{\mathbb{R}^d} |e^{-|p|^\alpha(x, y)}|^2 \, dx \sim \int_{\mathbb{R}^d} e^{-2|p|^\alpha} \, dp = \text{const}$$

which yields the finiteness of the left side of (2.1b). \qed
3. A multiplier theorem for $L_{a,\alpha}$

In [16] Hebisch proved a Hörmander multiplier theorem for self-adjoint operators if the associated heat kernel satisfies weighted and ultracontractive estimates and a certain Hölder condition. The proof is inspired by the one of Zo [34], see also [14, Section 4-6]. Although the result holds in $L^2(M,d\mu)$ where $M$ is some metric space and $\mu$ is some Borel measure, we will only state it for $M = \mathbb{R}^d$ and $\mu$ being the Lebesgue measure.

**Theorem 3.1** (Hebisch [16, Theorem 3.1]). Let $A$ be a non-negative, self-adjoint operator in $L^2(\mathbb{R}^d)$ and assume there exist positive numbers $c,b,m$ such that for all $t > 0$, the bounds

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} |e^{-tA}(x,y)|(1 + t^{-1/m}|x-y|)^c \, dx < \infty,$$

$$\sup_{y \in \mathbb{R}^d} t^{d/m} \int_{\mathbb{R}^d} |e^{-tA}(x,y)|^2 \, dx < \infty,$$

and

$$\int_{\mathbb{R}^d} |e^{-tA}(x,y) - e^{-tA}(x,z)| \, dx \lesssim t^{-b/m} |y-z|^b \quad \text{for all } y, z \in \mathbb{R}^d \quad (3.1)$$

hold. If $F \in H^s_{\text{loc}}(\mathbb{R})$ with

$$s > \frac{2^{d/(2c)}}{2} \left( \frac{d}{2} \left( 1 + \frac{1}{c} \right) + 1 \right) + \frac{1}{2}$$

and for a $0 \neq \varphi \in C_0^\infty(\mathbb{R}_+)$, one has $\sup_{t>0} \|\varphi F(t\cdot)\|_{H^s} < \infty$, then $F(A)$, initially defined via the $L^2$ functional calculus, has weak type $(1,1)$ and is bounded on $L^p$ for all $p \in (1,\infty)$.

The rest of this section is devoted to the verification of the assumptions of this theorem for $A = |p|^\alpha + V$ with $V$ as in (1.7), thereby proving Theorem 1.3. Since $V \geq 0$, the first two conditions follow immediately from Lemma 2.2. Verifying the third condition is more delicate since the heat kernel bounds of Theorem 2.1 can only be used after resolving the absolute value. However, after resolving it, cancellations are not expected anymore due to the different constants in front of the heat kernel bounds. Nonetheless, one can first verify the condition for $e^{-|p|^\alpha}$. Afterwards, using Duhamel’s formula and the heat kernel bounds of Theorem 2.1, we verify the condition also for $e^{-(|p|^\alpha + V)}$.

**Proposition 3.2.** Let $d \in \mathbb{N}$, $\alpha \in (0,2)$, and $b \in (0,1]$. Then (3.1) holds for $A = |p|^\alpha$ and $m = \alpha$.

**Proof.** Translating $x \mapsto x + z$ and scaling $x \mapsto t^{1/\alpha}x$ shows that it suffices to prove

$$\int_{\mathbb{R}^d} |e^{-|p|^\alpha}(x,0) - e^{-|p|^\alpha}(x,w)| \, dx \lesssim |w|^b \quad (3.2)$$

$$\int_{\mathbb{R}^d} |e^{-|p|^\alpha}(x,0) - e^{-|p|^\alpha}(x,w)| \, dx \lesssim |w|^b \quad (3.2)$$
where \( w = (y - z)/t^{1/\alpha} \). Since \( e^{-|w|^\alpha}(x) \) is integrable by (2.1a), it suffices to consider \(|w| \leq 1/2\). We split the integral over \( x \) at \(|x| = 3|w|\) and consider first \(|x| \leq 3|w|\). Since the heat kernel is uniformly bounded in \( x \) by Theorem 2.1, the triangle inequality yields
\[
\int_{|x| \leq 3|w|} |e^{-|p|^\alpha}(x, 0) - e^{-|p|^\alpha}(x, w)| \, dx \leq 2 \int |e^{-|p|^\alpha}(x, 0)| \, dx \lesssim |w|^d.
\]
For \(|x| \geq 3|w|\), we use the mean value theorem to estimate the left side of (3.2) by a constant times
\[
|w| \int_{|x| \geq 2|w|} \left| \nabla_x \int_{\mathbb{R}^d} e^{ipx} e^{-|p|^\alpha} \, dp \right|.
\]
Using the Fourier–Bessel transform (see, e.g., Stein and Weiss [32, Chapter IV]) and the formulas for derivatives of Bessel functions [29, Formula 9.1.30], namely
\[
\frac{d}{dz}(z^{-\nu} J_{\nu}(z)) = -z^{-\nu} J_{\nu+1}(z) \quad \text{for } z > 0, \ \nu \in \mathbb{R}, \quad (3.3)
\]
we obtain for \( r = |x| \),
\[
\left| \nabla \int_{\mathbb{R}^d} e^{ipx} e^{-|p|^\alpha} \, dp \right| = \left| \partial_r \int_0^\infty k^{d-1} e^{-k^\alpha} (kr)^{-(d-2)/2} J_{(d-2)/2}(kr) \, dk \right|
= \left| \int_0^\infty k^d e^{-k^\alpha} (kr)^{1-d/2} J_{d/2}(kr) \, dk \right|. \quad (3.4)
\]
We split the integral over \( x \) once more at \(|x| = 2\) and first show that the right side of (3.4) is integrable for \(|x| \geq 2\). To this end, we integrate by parts, using once more (3.3), and obtain
\[
\int_0^\infty k^d e^{-k^\alpha} (kr)^{1-d/2} J_{d/2}(kr) \, dk = -r^{-1} \int_0^\infty e^{-k^\alpha} k^d \partial_k \left[ (kr)^{1-d/2} J_{d/2-1}(kr) \right] \, dk
= r^{-1} \int_0^\infty e^{-k^\alpha} k^{d-1} (d - \alpha k^\alpha) \cdot (kr)^{1-d/2} J_{d/2-1}(kr) \, dk.
\]
The integral over \( k \) obviously exists for large \( k \) due to the \( e^{-k^\alpha} \) factor. However, we must be careful with the behavior of the integrand for small \( k \). Integrating \( n - 1 \) more times by parts shows that the right side of the last equation is equal to
\[
r^{1-d/2-n} \int_0^\infty k^{1+d/2-n} J_{d/2-n}(kr) \sum_{j=0}^{n} a_j e^{-k^\alpha} k^{j\alpha} \, dk.
= r^{-d} \int_0^\infty (kr)^{1+d/2-n} J_{d/2-n}(kr) \sum_{j=0}^{n} a_j e^{-k^\alpha} k^{j\alpha} \, dk \quad (3.5)
\]
where \( a_j = a_j(d, \alpha) \in \mathbb{R} \) and the \( k^{j\alpha} \) arise from differentiating \( e^{-k^\alpha} \). The boundary terms vanish at \( k = \infty \) due to the \( e^{-k^\alpha} \) factor. We will momentarily explain why the boundary terms also vanish at \( k = 0 \).
We distinguish now between odd and even \( d \). If \( d \) is even, we choose \( n = d/2 \). Using (3.3) and \( J_{-m}(z) = (-1)^m J_m(z) \) for \( m \in \mathbb{N} \) (see [29, Formula 9.1.5]), the \( j \)-th summand on the right side of (3.5) becomes
\[
a_j r^{-d} \int_0^\infty (kr) J_0(kr) k^{j\alpha} e^{-k^\alpha} \, dk = -a_j r^{-d-1} \int_0^\infty \partial_k ((kr) J_{-1}(kr)) k^{j\alpha} e^{-k^\alpha} \, dk,
\]
where the boundary term of the partial integration vanished at \( k = 0 \) quadratically. Using the bound \( |J_1(z)| \lesssim \min\{z, z^{-1/2}\} \) (see [29, Formula 9.1.7 and 9.2.1]), the absolute value of the right side of the last formula can be bounded by a constant times
\[
r^{-d-1} \int_0^\infty k^{-1} \cdot kr J_1(kr) (\alpha j^{\alpha} - \alpha k^{j\alpha+\alpha}) e^{-k^\alpha} \, dk,
\]
The second summand is bounded by a constant times \( r^{-d-1/2} \) whereas the first summand is bounded by \( r^{-d-1-j\alpha} + r^{-d-1-(j+1)\alpha} \). Thus, the contribution of even \( d \) is integrable for \( |x| = r \geq 2 \) in \( \mathbb{R}^d \). Note that for \( n = d/2 - 1 \), the integrand of (3.5) is
\[
(kr)^2 J_1(kr) \sum_{j=0}^n a_j k^{j\alpha} e^{-k^\alpha} = -rk^2 \partial_k (J_0(kr)) \sum_{j=0}^n a_j k^{j\alpha} e^{-k^\alpha},
\]
i.e., the boundary terms of the partial integration always vanished at least quadratically.

If on the other hand \( d \) is odd, we choose \( n = (d+1)/2 \) and use [8, Formula 10.16.1], i.e., \( J_{-1/2}(kr) = \sqrt{2/\pi} (kr)^{-1/2} \cos(kr) \). Thus, (3.5) becomes
\[
\sqrt{\frac{2}{\pi}} r^{-d} \int_0^\infty e^{-k^\alpha} \cos(kr) \sum_{j=0}^n a_j k^{j\alpha} \, dk
\]
\[
= \frac{a_0}{\sqrt{2\pi}} r^{-d} \int_\mathbb{R} e^{-|k|^\alpha} e^{ikr} \, dk + \sqrt{\frac{2}{\pi}} r^{-d-1} \int_0^\infty r \cos(kr) \sum_{j=1}^n a_j e^{-k^\alpha} k^{j\alpha} \, dk.
\]
The first integral over \( k \) is just the one-dimensional heat kernel \( e^{-|p|^\alpha}(r) \) which, by Theorem 2.1 decays like \( r^{-1-\alpha} \). Integrating the second summand once more by parts yields
\[
-\sqrt{\frac{2}{\pi}} r^{-d-1} \sum_{j=1}^n a_j \int_0^\infty (j\alpha k^{j\alpha-1} - \alpha k^{j\alpha+\alpha-1}) e^{-k^\alpha} \sin(kr) \, dk.
\]
This shows that both the integral over \( k \), as well as the subsequent integral over \( \{x \in \mathbb{R}^d : |x| \geq 2\} \) exist. Finally, we mention why the boundary terms at \( k = 0 \) also vanished in this case. If \( n = (d-1)/2 \), the integrand of (3.5) is
\[
k r \sin(kr) \sum_{j=0}^n a_j k^{j\alpha} e^{-k^\alpha} = -\left( \partial_k \cos(kr) \right) \sum_{j=0}^n a_j k^{j\alpha+1} e^{-k^\alpha}\]
by [8, Formula 10.16.1]. This shows that the boundary terms vanish at least linearly at $k = 0$.

Combining the cases of even and odd $d$ thus shows

$$|w| \int_{|x| \geq 2} |\nabla \int_{\mathbb{R}^d} e^{ipx} e^{-|p|^\alpha} \, dp| \, dx \lesssim |w|.$$ 

If $2|w| \leq |x| \leq 2$, we use [29, Formula 9.1.60], i.e., $|J_{d/2}(kr)| \leq 1$, to estimate the right side of (3.3) by

$$\int_0^\infty k^d e^{-k^\alpha} (kr)^{1-d/2} \, dk \lesssim r^{1-d/2}.$$ 

This shows that the integral over $2|w| \leq |x| \leq 2$ exists uniformly in $|w|$ and thus

$$|w| \int_{2|w| \leq |x| \leq 2} |\nabla \int_{\mathbb{R}^d} e^{ipx} e^{-|p|^\alpha} \, dp| \, dx \lesssim |w|,$$

too.

We will now use perturbation theory to generalize this result to $|p|^\alpha + V$ with $V$ as in (1.7).

**Proof of Theorem 1.3.** We merely need to check the H"older condition (3.1) in Theorem 3.1 (with $m = \alpha$) since the first two conditions were already verified in Lemma 2.2. As in the proof of Proposition 3.2, the fact that for any $t > 0$,

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|e^{-t(|p|^\alpha + V)}(x,y)\right| \, dx \leq \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left|e^{-t|p|^\alpha}(x,y)\right| \, dx \lesssim 1 \quad (3.6)$$

holds, shows that it suffices to prove

$$\int_{\mathbb{R}^d} \left|e^{-t(|p|^\alpha + V)}(x,w) - e^{-t(|p|^\alpha + V)}(x,y)\right| \, dx \lesssim t^{-b/\alpha} |w - y|^b \quad (3.7)$$

for some $b > 0$ and $t^{-1/\alpha} |w - y| \leq 1/2$. By the Duhamel formula

$$e^{-t(|p|^\alpha + V)}(x,w) = e^{-t|p|^\alpha}(x,w) - \int_0^t ds \int_{\mathbb{R}^d} dz \ e^{-(t-s)(|p|^\alpha + V)}(x,z)V(z) e^{-s|p|^\alpha}(z,w),$$

the triangle inequality, and the maximum principle, the left side of (3.7) is bounded by

$$\int_{\mathbb{R}^d} \left|e^{-t|p|^\alpha}(x,w) - e^{-t|p|^\alpha}(x,y)\right| \, dx$$

$$+ \tilde{a} \int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} dz \ e^{-(t-s)L_{a,\alpha}}(x,z) |z|^{-\alpha} \left|e^{-s|p|^\alpha}(z,w) - e^{-s|p|^\alpha}(z,y)\right|.$$ 

The relation $e^{-tL_{a,\alpha}}(x,y) = t^{-d/\alpha} e^{-L_{a,\alpha}(t^{-1/\alpha} x, t^{-1/\alpha} y)}$ (see [20, Lemma 2.1]) shows that it suffices to consider $t = 1$ again. The assertion for the first summand was
already shown in Proposition 3.2 and any \( b \in (0, 1] \). For \( \gamma \in (0, 1) \) and \( |z| \geq |w - y|^{\gamma} \), the second summand can be estimated using (3.6),

\[
e^{-s|p|^\alpha}(x) \geq e^{-s|p|^\alpha}(y) \quad \text{for all } |x| \leq |y| \text{ and } s > 0
\]

(see, e.g., [1] Formula (5.1))), the formula for the Riesz kernel of \( |p|^\alpha \), i.e.,

\[
|p|^{-\alpha}(x, y) = \int_0^\infty e^{-s|p|^\alpha}(x, y) \, ds = \frac{\Gamma((d - \alpha)/2)}{\pi^{d/2}2^{\alpha}\Gamma(\alpha/2)}|x - y|^{-d+\alpha}
\]

(see, e.g., [30] Chapter V, §1.1], and [19] Theorem 4.5.10) (with \( 1 \leq p \leq \infty \) such that \( 0 < d - d + \alpha - d/p < 1 \), i.e., \( \alpha - 1 < d/p < \alpha \), and \( |z|^{-\alpha} \in L^p(|z| \geq |w - y|^{\gamma}) \) for \( p \in (d/\alpha, \infty) \) by

\[
\int_0^1 ds \int_{\mathbb{R}^d} dx \; e^{-(1-s)L_{a,\alpha}(x, z)} \int_{|z| \geq |w - y|^{\gamma}} dz \; |z|^{-\alpha}|e^{-s|p|^\alpha}(z, w) - e^{-s|p|^\alpha}(z, y)|
\]

\[
\lesssim \int_{|z| \geq |w - y|^{\gamma}} dz \; |z|^{-\alpha} \int_0^\infty ds \; \left| e^{-s|p|^\alpha}(z, w) - e^{-s|p|^\alpha}(z, y) \right|
\]

\[
\times \left( 1_{\{|z-w| \leq |z-y|\}} - 1_{\{|z-w| \geq |z-y|\}} \right)
\]

\[
= \frac{\Gamma((d - \alpha)/2)}{\pi^{d/2}2^{\alpha}\Gamma(\alpha/2)} \int_{|z| \geq |w - y|^{\gamma}} dz \; |z|^{-\alpha} \left| |z - w|^{-d+\alpha} - |z - y|^{-d+\alpha} \right|
\]

\[
\lesssim_{d,\alpha} |w - y|^{d-d+\alpha-d/p} \| |z|^{-\alpha} \|_{L^p(|z| \geq |w - y|^{\gamma})} = A|w - y|^{1 - \gamma(\alpha p - d)}.
\]

Thus, we are left to examine the case \( |z| \leq |w - y|^{\gamma} \) with the above \( \gamma < 1 \). As in the proof of Proposition 3.2, we do not expect any further cancellations in this region anymore. Therefore, using the triangle inequality, it suffices to estimate the contributions from \( e^{-s|p|^\alpha}(z, w) \), respectively \( e^{-s|p|^\alpha}(z, y) \) separately. Without loss of generality, we only treat the summand with \( e^{-s|p|^\alpha}(z, w) \) and examine closer the behavior for \( s \leq |w - y|^{\gamma} \) and \( |z - w| \leq |w - y|^{\gamma} \) with \( 0 < \varepsilon < \gamma \). On the one hand, for \( s \geq |w - y|^{\gamma} \) and arbitrary \( |z - w| \), one estimates (using the maximum principle to perform the integration over \( x \) and using \( \exp(-s|p|^\alpha)(z, w) \leq s^{-d/\alpha} \))

\[
\int_{|w - y|^{\alpha \varepsilon}}^1 ds \int_{|z| \leq |w - y|^{\gamma}} dz \; |z|^{-\alpha} e^{-s|p|^\alpha}(z, w)
\]

\[
\leq |w - y|^{(d-\alpha)} \int_{|w - y|^{\varepsilon}}^1 ds \; s^{-d/\alpha} \leq |w - y|^{(\gamma - \varepsilon)(d-\alpha)}.
\]
On the other hand, if \(|z - w| \geq |w - y|^{\varepsilon}\) and \(s \in (0, 1)\), one uses again the maximum principle to perform the integration over \(x\) and obtains
\[
\int \frac{dz}{|z|^{\alpha}}|z - w|^{d - \alpha} \int_0^1 ds + \int_0^1 \frac{dz}{|z|^{\alpha}} \int_0^1 ds s^{-d/\alpha}
\]

Thus, we are left with the region where \(|z - w| \leq |w - y|^{\varepsilon}\) and \(s \leq |w - y|^{\alpha \varepsilon}\) with \(\varepsilon < \gamma\). At this stage, we invoke the heat kernel bounds for \(a > 0\) of Theorem 2.1. Using \(1 - s \geq 1 - (1/2)^{\alpha \varepsilon}\), \(\delta \in (-\alpha, 0)\), translating \(x \mapsto x + z\), and applying Hölder’s inequality, we obtain
\[
\int dx \int_0^{\varepsilon} ds \int_0^1 \frac{dz}{|z|^{\alpha \varepsilon}} \left(1 \vee \frac{(1 - s)^{1/\alpha}}{|x|}\right)^{\delta/\alpha} \left(1 \wedge \frac{(1 - s)^{1 + d/\alpha}}{|z|^{d + \alpha}}\right) e^{-s|p|^\alpha(z, w)}
\]

Both integrals converged since \(p < d/(\alpha + \delta)\) and \(p' < d/(d - \alpha)\), i.e., \(p \in (d/\alpha, d/(\alpha + \delta))\) which is not an empty interval since \(\delta < 0\). Moreover, this shows that the exponent \(\gamma(d/p - \alpha - \delta) + \varepsilon(\alpha - d/p)\) is positive which concludes the proof of Theorem 1.3. \(\Box\)

Theorem 3.1 is certainly not applicable if \(a < 0\) since already the simple bounds of Lemma 2.2 do not hold due to the singularity of \(e^{-\mathcal{L}_{a,\alpha}(x, y)}\) for \(|x|, |y| \lesssim 1\). In \[22\], Killip et al. proved a Mikhlin multiplier theorem associated to \(-\Delta + a|x|^{-2}\) where they used the fact that the associated wave equation has the finite speed of propagation property. This follows from a Paley–Wiener argument and the fact that the heat
kernel satisfies a Davies–Gaffney estimate. In fact, this estimate is also a necessary condition for the finite speed of propagation property, see, e.g., Coulhon and Sikora [7] for further details. If $\alpha < 2$, the distributional support of $\cos(\sqrt{L_{a,\alpha}})$ is not compact anymore which is the main reason why it seems non-trivial to adapt their proof (cf. [22, p. 1286f]), even if the coupling constant is positive.

We conclude with the observation that $\exp(-L_{a,\alpha})$ is unbounded on $L^p$ for $a < 0$ if $p \notin (d/(d - \delta), d/\delta)$. For this purpose, consider $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp} \varphi \subseteq B(0, 1)$ such that $\varphi(x) = 1$ for $|x| \leq 1/2$. By the lower bound on the heat kernel of Theorem 2.1 one obtains for $|x| \leq 1$,

$$
(e^{-L_{a,\alpha}}\varphi)(x) \gtrsim |x|^{-\delta} \int_{|y| \leq 1/2} dy \frac{1}{|x - y|^{d+\alpha}} \gtrsim |x|^{-\delta}.
$$

Hence, $e^{-L_{a,\alpha}}\varphi \notin L^p$ for any $p \geq d/\delta$ and by self-adjointness and the duality of $L^p$ spaces, it follows that the $L^p$-boundedness also fails if $p \leq d/(d - \delta)$. This indicates that $p \in (d/(d - \delta), d/\delta)$ seems to be a “reasonable” necessary condition for a multiplier theorem if $a < 0$.

4. Littlewood–Paley theory

We define two families of Littlewood–Paley projections associated to $L_{a,\alpha}$ and apply the multiplier theorem to infer their $L^p$-boundedness. Afterwards, we derive Bernstein estimates and square function estimates.

**Definition 4.1** (Littlewood–Paley projections). Let $\Phi : [0, \infty) \to [0, 1]$ be a smooth function such that $\Phi(\lambda) = 1$ for $0 \leq \lambda \leq 1$ and $\Phi(\lambda) = 0$ for $\lambda \geq 2$.

For each dyadic number $N \in 2^\mathbb{Z}$, let

$$
\Phi_N(\lambda) = \Phi(\lambda/N^{\alpha/2}) \quad \text{and} \quad \Psi_N(\lambda) = \Phi_N(\lambda) - \Phi_{N/2}(\lambda)
$$

such that $\sum_{N \in 2^\mathbb{Z}} \Psi_N(\lambda) = 1$ for $\lambda \in \mathbb{R}_+$. We define the standard Littlewood–Paley projections as

$$
\tilde{P}^{\alpha}_{N} := \Psi_N(\sqrt{L_{a,\alpha}}) \quad \text{and} \quad \tilde{P}^0_N := \tilde{P}^{0,\alpha}_{N}
$$

and a second set of Littlewood–Paley projections via the heat kernel as

$$
P^{\alpha}_{N} := e^{-L_{a,\alpha}/N^\alpha} - e^{-L_{a,\alpha}/(N^\alpha/2^\alpha)} \quad \text{and} \quad P^0_N := P^{0,\alpha}_{N}.
$$

We will now prove Bernstein estimates for these projections which show in particular that $e^{-L_{a,\alpha}}$ is bounded on $L^p$ for all $a \geq a_*$. In general, these inequalities are useful when the spectral parameter $\lambda$ is localized, because low Lebesgue integrability can be upgraded to high Lebesgue integrability at the cost of some powers of $N$. In fact, this cost is a gain for low $N$ which improves the inequality.
Lemma 4.2 (Bernstein estimates). Let $1 < p \leq q < \infty$ when $a \geq 0$ and let $d/(d-\delta) < p \leq q < d/\delta$ when $0 > a \geq a_*$. Then the following assertions hold.

1. If $a \geq 0$, then \( \| (\mathcal{L}_{a,\alpha} / N^a)^{\frac{1}{2}} \tilde{P}_{N}^{a,\alpha} f \|_p \sim \| \tilde{P}_{N}^{a,\alpha} f \|_p \), i.e., \( N^{\frac{a}{2}} \| \tilde{P}_{N}^{a,\alpha} f \|_p \sim \| \mathcal{L}_{a,\alpha}^{\frac{1}{2}} \tilde{P}_{N}^{a,\alpha} f \|_p \) for all $f \in C_c^\infty (\mathbb{R}^d)$ and all $s \in \mathbb{R}$.

2. The projections $\tilde{P}_{N}^{a,\alpha}$ and, if $a \geq 0$, $\tilde{P}_{N}^{a,\alpha}$ are bounded from $L^p$ to $L^q$ with norm $O(N^{d(\frac{1}{p} - \frac{1}{q})})$.

Proof. The first assertion follows immediately from Theorem 1.3.

We focus now on the second assertion and begin with the observation that $\tilde{e}^{a,\alpha}$ can be written as a product of bounded multipliers due to Theorem 1.3 and the $L^p \to L^q$-boundedness of $e^{-L_{a,\alpha}/N^a}$. More precisely, we have for some $r \in (p,q)$

\[
\| \tilde{P}_{N}^{a,\alpha} f \|_{L^q} \\
\lesssim \| e^{-L_{a,\alpha}/N^a} \|_{L^r \to L^p} \| e^{L_{a,\alpha}/N^a} \tilde{P}_{N}^{a,\alpha} e^{-L_{a,\alpha}/N^a} \|_{L^r \to L^p} \| e^{-L_{a,\alpha}/N^a} \|_{L^p \to L^q} \| f \|_{L^p}
\lesssim N^{d(1/p - 1/r + 1/r - 1/q)} \| f \|_{L^p}.
\]

Thus, it suffices to prove the second assertion for $e^{-L_{a,\alpha}/N^a}$. If $a \geq 0$, applying the maximum principle shows that it suffices to compute the $L^p \to L^q$-norm of the heat kernel associated to $|p|^a$. Scaling $x \mapsto N^{-1} x$ and applying Young’s inequality with $r = (1 + 1/q - 1/p)^{-1} \geq 1$ yields

\[
\| e^{-L_{a,\alpha}/N^a} f \|_q \lesssim N^d \left\| 1 \wedge \frac{N^{-a-d}}{|x|^{d+a}} \| f \|_p \lesssim N^{d(\frac{1}{p} - \frac{1}{q})} \| f \|_p.
\]

For $0 > a \geq a_*$, we employ the heat kernel bounds of Theorem 2.1 to estimate

\[
\| e^{-L_{a,\alpha}/N^a} f \|_q \\
\lesssim N^d \left\| \left( 1 \vee \frac{N^{-1}}{|x|} \right)^{\delta} \int_{\mathbb{R}^d} \left( 1 \vee \frac{N^{-1}}{|y|} \right)^{\delta} \left( 1 \wedge \frac{N^{-a-d}}{|x-y|^{d+a}} \right) |f(y)| \ dy \right\|_q.
\]

To handle the right side, we distinguish between the following four cases.

Case 1: $|x| \leq N^{-1}$, $|y| \leq N^{-1}$. Using Hölder’s inequality and recalling $d/(d-\delta) < p \leq q < d/\delta$, one can estimate the right side of (4.1) by

\[
N^{d-2\delta} \left\| x^{-\delta} \int_{|y| \leq N^{-1}} |y|^{-\delta} |f(y)| \ dy \right\|_{L^q(|x| \leq N^{-1})} \\
\lesssim N^{d-2\delta} \left\| x^{-\delta} \right\|_{L^q(|x| \leq N^{-1})} \left\| |y|^{-\delta} \right\|_{L^p(|y| \leq N^{-1})} \left\| f \right\|_p \lesssim N^{d(\frac{1}{p} - \frac{1}{q})} \| f \|_p.
\]
Case 2: $|x| \leq N^{-1}$, $|y| > N^{-1}$. Using Hölder’s inequality, the right side of (4.1) can be estimated by
\[
N^{d-\delta} \left\| \frac{1}{|x-y|^{d+\alpha}} \right\|_{L^q(|x| \leq N^{-1})} \left\| \frac{N^{\alpha-d}}{|x|} |f(y)| \right\|_{q} \lesssim N^{d-\delta} \left\| \frac{1}{|x-y|^{d+\alpha}} \right\|_{L^q(|x| \leq N^{-1})} \left\| \frac{N^{\alpha-d}}{|y|} |f(y)| \right\|_{p} \lesssim N^d \left( \frac{1}{p} - \frac{1}{q} \right) \|f\|_p.
\]

Case 3: $|x| > N^{-1}$, $|y| \leq N^{-1}$. Using Minkowski’s inequality and then Hölder’s inequality, the right side of (4.1) can be bounded by
\[
N^{d-\delta} \left\| \int_{|y| \leq N^{-1}} \frac{1}{|y|^{\delta}} \left( 1 \wedge \frac{N^{\alpha-d}}{|x-y|^{d+\alpha}} \right) |f(y)| \, dy \right\|_q \lesssim N^{d-\delta} \left\| \frac{1}{|x|^{d+\alpha}} \right\|_q \left\| \frac{N^{\alpha-d}}{|y|} |f(y)| \right\|_{L^q(|y| \leq N^{-1})} \|f\|_p \lesssim N^d \left( \frac{1}{p} - \frac{1}{q} \right) \|f\|_p.
\]

Case 4: $|x| > N^{-1}$, $|y| > N^{-1}$. As in the case of non-negative couplings, we employ Young’s inequality to estimate the last contribution of the right side of (4.1) by
\[
N^d \left\| \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d+\alpha}} |f(y)| \, dy \right\|_q \lesssim N^d \left\| \frac{1}{|x|^{d+\alpha}} \right\|_q \|f\|_p \lesssim N^d \left( \frac{1}{p} - \frac{1}{q} \right) \|f\|_p \]
where $1 + 1/q = 1/r + 1/p$. 

The spectral multiplier theorem, a randomization argument involving Khintchine’s inequality, and a duality argument yield the following two-sided square function estimates. Since the same arguments already appear in the proof of [25] Theorem 4.3, we skip the proof.

**Theorem 4.3 (Square function estimates).** Let $\alpha \in (0, 2 \wedge d)$, $a \geq 0$, $1 < p < \infty$, and $s > 0$. If $k \in \mathbb{N}$ satisfies $k > s/2$, then
\[
\left\| \left( \sum_{n \in \mathbb{Z}} |N^{\alpha/2} \tilde{P}_N a \cdot f|^2 \right)^{1/2} \right\|_p \sim \left\| \mathcal{L}_{a,\alpha}^{s/2} f \right\|_p \sim \left\| \left( \sum_{N \in \mathbb{Z}} |N^{\alpha/2} (P_N a \cdot f)|^2 \right)^{1/2} \right\|_p
\]
for all $f \in C_c^{\infty}(\mathbb{R}^d)$.

Although we apply these estimates only for $k = 1$ (since $s \in (0, 2)$), we remark that $k > s/2$ guarantees that $(N^{\alpha/2} \lambda^{-s/2}) \cdot (e^{-\lambda/N^\alpha} - e^{-\lambda/(N/2)^\alpha})^k$ is actually a Hörmander multiplier.

We conclude this section by noting that the spectral theorem in $L^2$ yields an expansion of the identity of $L^2$ functions in terms of eigenfunctions of $\mathcal{L}_{a,\alpha}$ as in [25] [22].
More precisely, one has the $L^2$-convergence (noting that zero is not an eigenvalue of $\mathcal{L}_{a,\alpha}$)
\[
\sum_{N\in\mathbb{Z}} (\tilde{P}_N^{a,\alpha} f)(x) = f(x) = \sum_{N\in\mathbb{Z}} (P_N^{a,\alpha} f)(x)
\]
for all $a \geq a_*$ and $f \in L^2(\mathbb{R}^d)$. By the spectral multiplier theorem, respectively the $L^p$-boundedness of the heat kernel, the first equality continues to hold in $L^p$ for $a \geq 0$ and $1 < p < \infty$ and the second one for any $1 < p < \infty$ if $a \geq 0$ and for any $d/(d-\delta) < p < d/\delta$ if $a < 0$. This is because partial sums are $L^p$-bounded for the above $p$ which allows one to conclude via a density and interpolation (via Hölder’s inequality) argument.

5. A REVERSE HARDY INEQUALITY AND PROOF OF THEOREM 1.1

The key tool to prove the reversed Hardy inequality is a pointwise bound on the difference of the heat kernels of $\mathcal{L}_{a,\alpha}$ and $|p|^{\alpha}$, i.e.,
\[
K_t^{a}(x, y) := e^{-t|p|^\alpha}(x, y) - e^{-t\mathcal{L}_{a,\alpha}}(x, y).
\]
In (12) Lemma 3.1 it was shown that there is an effective cancellation in the region $(|x| \lor |y|)^\alpha \geq t$ and $|x| \sim |y|$. There, the bound was formulated in terms of the functions
\[
L_t^{a,\delta}(x, y) := 1_{\{|x|\lor|y|\alpha \leq t\}} t^{-\frac{d}{\alpha}} \left( \frac{t^{2/\alpha}}{|x| |y|} \right)^\delta + 1_{\{|x|\lor|y|\alpha \geq t\}} \left( \frac{t^{1/\alpha}}{|x| \lor |y|} \right)^\delta \]  
and
\[
M_t^{a}(x, y) := 1_{\{|x|\lor|y|\alpha \leq t\}} 1_{\{1/2 |x| \leq |y| \leq 2|x|\}} \left( \frac{t^{1-\frac{d}{\alpha}}}{|x| \lor |y|} \right) \left( 1 \lor \frac{t^{1+\frac{d}{\alpha}}}{|x-y|^{d+\alpha}} \right).
\]
Recall that $\delta_+ = 0$ if $a \geq 0$ and $\delta_+ = \delta$ if $a < 0$.

**Lemma 5.1** (Difference of kernels). Let $\alpha \in (0, 2 \land d)$, $\alpha \in [a_*, \infty)$ and $\delta$ be defined by (13). Then for all $x, y \in \mathbb{R}^d \setminus \{0\}$ and $t > 0$,
\[
|K_t^{a}(x, y)| \lesssim L_t^{a,\delta_+}(x, y) + M_t^{a}(x, y).
\]  

Using this lemma, we formulate and prove a reversed Hardy inequality for the difference $\mathcal{L}_{a,\alpha}^{\alpha/2} - |p|^{\alpha/2}$, expressed in terms of square functions.

**Proposition 5.2** (Reverse Hardy inequality in $L^p$). Let $s \in (0, 2)$, $\alpha \in (0, 2 \land d)$, $a \geq a_*$, $\delta$ be defined by (13), $p \in (1, \infty)$ if $a \geq 0$, and $p \in (d/(d-\delta), d/\delta)$ if $a < 0$. Then,
\[
\left\| \left( \sum_{N \in \mathbb{Z}} |N^{\alpha s/2} P_N^{a,\alpha} f|^2 \right)^{1/2} - \left( \sum_{N \in \mathbb{Z}} |N^{\alpha s/2} P_N^{a,\alpha} f|^2 \right)^{1/2} \right\|_p \lesssim_{d, a, a, s} \| |x|^{-\alpha s/2} f \|_p
\]
for all $f \in C_0^\infty(\mathbb{R}^d)$. 
Proof. By the triangle inequality in \( \ell^2 \), the \( \ell^1 \rightarrow \ell^2 \)-embedding, and Lemma 5.1, we estimate
\[
\left\| \left( \sum_{N \in 2^\mathbb{Z}} |N^{\alpha s/2} P_N^0 f|^2 \right)^{1/2} - \left( \sum_{N \in 2^\mathbb{Z}} |N^{\alpha s/2} P_N^{0,0} f|^2 \right)^{1/2} \right\|_p \\
\leq \left\| \left( \sum_{N \in 2^\mathbb{Z}} |N^{\alpha s/2} (P_N^0 - P_N^{0,0}) f|^2 \right)^{1/2} \right\|_p \lesssim \int_{\mathbb{R}^d} dy \sum_{N \in 2^\mathbb{Z}} N^{\alpha s} |K_N^{\alpha} (x, y)| |f(y)| \\
\leq \int_{\mathbb{R}^d} dy \left( \sum_{N \in 2^\mathbb{Z}} N^{\alpha s} |L_N^{\alpha,\delta_+} (x, y)||y|^{\alpha s/2} \right) \frac{|f(y)|}{|y|^{\alpha s/2}} \\
\quad + \int_{\mathbb{R}^d} dy \left( \sum_{N \in 2^\mathbb{Z}} N^{\alpha s} |M_N^{\alpha} (x, y)||y|^{\alpha s/2} \right) \frac{|f(y)|}{|y|^{\alpha s/2}} .
\]
\[(5.3)\]

Thus, it suffices to show that the right side is bounded by \( \| |x|^{-\alpha s/2} f \|_p \) for all \( f \in C^\infty_0 (\mathbb{R}^d) \). To simplify notation, let \( g(x) := |x|^{-\alpha s/2} |f(x)| \).

As in the proof of [12, Proposition 1.5], we use Schur tests to prove the assertion. We begin by estimating the first summand and obtain
\[
\sum_{N \in 2^\mathbb{Z}} N^{\alpha s} L_N^{\alpha,\delta_+} (x, y) = (|x||y|)^{-\delta_+} \sum_{N \leq (|x|\vee |y|)^{-1}} N^{\alpha s + d - 2\delta_+} \\
+ \sum_{N \geq (|x|\vee |y|)^{-1}} N^{\alpha s} \frac{N^{-\alpha}}{(|x| \vee |y|)^{d+\alpha}} \left( 1 \vee \frac{N^{-1}}{|x| \wedge |y|} \right)^{\delta_+} \\
\simeq (|x||y|)^{-\delta_+} (|x| \vee |y|)^{-\alpha s/2 - d + 2\delta_+} + \frac{1}{(|x| \vee |y|)^{\alpha s + d}} \left( \frac{|x| \vee |y|}{|x| \wedge |y|} \right)^{\delta_+}
\]
where the summability relied on \( s < 2 \) and \( \alpha s/2 + d - 2\delta_+ > 0 \) which in turn follows from \( \delta \leq (d - \alpha)/2 \). Observe that the two summands are equal since
\[
\frac{|x| \vee |y|}{|x| \wedge |y|} = \frac{1}{|x| \wedge |y|} .
\]

Thus,
\[
\left\| \int_{\mathbb{R}^d} dy \sum_{N \in 2^\mathbb{Z}} N^{\alpha s} L_N^{\alpha,\delta_+} (x, y)|y|^{\alpha s/2} g(y) \right\|_p \lesssim \left\| \int_{\mathbb{R}^d} dy \left( \frac{|x| \vee |y|}{|x| \wedge |y|} \right)^{2\delta_+-d} \frac{g(y)}{|x|^{\delta_+}} \right\|_p .
\]

For any \( (p \vee p')\delta_+ < \beta < (p \wedge p')(d - \delta_+) \) (such \( \beta \) exist since \( d - 2\delta \geq \alpha > 0 \) and \( d/(d - \delta_+) < p, p' < d/\delta_+ \), we have
\[
\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx \left( \frac{|y|}{|x|} \right)^{\beta} \frac{(|x| \vee |y|)^{2\delta_+-d}}{(|x| \wedge |y|)^{\delta_+}} = \int_{\mathbb{R}^d} \frac{1 \vee |z|^{2\delta_+-d}}{|z|^{\delta_+ + 2}} \frac{d|z|}{p} < \infty
\]
for all \( p \) and \( p' \).
and similarly, since the integral kernel is symmetric in $x$ and $y$,

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} dy \left( \frac{|x|}{|y|} \right) \frac{d}{d^+} \left( \frac{(|x| \vee |y|)^{2d} - d}{(|x| |y|)^{d+}} \right) = \int_{\mathbb{R}^d} \frac{(1 \vee |z|)^{2d} - d}{|z|^{d+}} \frac{d}{d^+} dz < \infty.$$ 

Thus, by a weighted Schur test,

$$\left\| \int_{\mathbb{R}^d} dy \left( \frac{|x| \vee |y|)^{2d} - d}{(|x| |y|)^{d+}} g(y) \right) \right\|_p \lesssim \|g\|_p$$

which shows that the first term in (5.3) satisfies the claimed bound.

Since $|x| \sim |y|$ on the support of $M_{N-\alpha}^\alpha$, the second term in (5.3) is bounded by a constant times

$$\sup_{y \in \mathbb{R}^d} \left| \frac{\alpha}{2} - \frac{\alpha}{2} \right| \int_{\frac{1}{2}|y| \leq |x| \leq 2|y|} dx \sum_{N \geq (|x| \vee |y|)^{d+}} N^{\frac{\alpha}{2} - \alpha} \frac{N^{\alpha - d} \alpha \left( 1 \wedge \frac{N^{\alpha - d}}{|x - y|^{d+}} \right) (|x| |y|)^{\frac{\alpha}{2} - \alpha}}.$$ 

Interchanging the order of integration and summation shows that the right side is bounded by

$$\sup_{y \in \mathbb{R}^d} \left| \frac{\alpha}{2} - \alpha \right| \sum_{N \geq (2|y|)^{-1}} N^{\frac{\alpha}{2} - \alpha + d} \int_{\frac{1}{2}|y| \leq |x| \leq 2|y|} dx \left( 1 \wedge \frac{N^{\alpha - d}}{|x - y|^{d+}} \right) \lesssim 1.$$ 

Thus, the Schur test implies

$$\left\| \int_{\mathbb{R}^d} dy \sum_{N \in 2^2} N^{\frac{\alpha}{2} - \alpha} M_{N-\alpha}^\alpha (x, y) (|x| |y|)^{\frac{\alpha}{2}} g(y) \right\|_p \lesssim \|g\|_p$$

which shows the asserted inequality.
We now show that Theorem 1.1 is an immediate consequence of Propositions 1.2 and 5.2 and the Littlewood–Paley theory from the last section.

**Proof of Theorem 1.1** In the following, we always assume $1 < p < \infty$. If $s \in (0, 2)$ and $a \geq 0$ (i.e., $\delta \leq 0$), the assertion

$$\| |p|^{\alpha s/2} f \|_p \lesssim_{d, p, \alpha, s} \| L^{s/2}_{a, \alpha} f \|_p$$

follows from Theorem 4.3, the triangle inequality, Proposition 1.2 (requiring $\alpha s/2 + \delta < d/p$), and Proposition 5.2. More precisely,

$$\| |p|^{\alpha s/2} f \|_p \sim \left\| \left( \sum_{N \in 2^Z} |N^{\alpha s/2} P_N^a f|^2 \right)^{1/2} \right\|_p$$

$$\leq \left\| \left( \sum_{N \in 2^Z} |N^{\alpha s/2} P_N^{a, \alpha} f|^2 \right)^{1/2} \right\|_p + \left\| \left( \sum_{N \in 2^Z} |N^{\alpha s/2} P_N^{a, \alpha} f|^2 \right)^{1/2} \right\|_p$$

$$\lesssim \| L^{s/2}_{a, \alpha} f \|_p \sim \left\| |x|^{-\alpha s/2} f \right\|_p \lesssim \| L^{s/2}_{a, \alpha} f \|_p.$$  

Note that the condition $\alpha s/2 < d$ in Proposition 1.2 is automatically satisfied since we assumed $s \leq 2$ and $\alpha < d$.

The other inequality, i.e.,

$$\| L^{s/2}_{a, \alpha} f \|_p \lesssim_{d, p, \alpha, s} \| |p|^{\alpha s/2} f \|_p$$

is proven analogously, but employs (1.1) (with $\alpha s$ instead of $\alpha$ and requiring $\alpha s/2 < d/p$) instead of Proposition 1.2.

If $s = 2$ and $a \geq a^*$, the inequality $\| L_{a, \alpha} f \|_p \lesssim \| |p|^{\alpha} f \|_p$ follows from the triangle inequality and the ordinary Hardy inequality (1.1) (with $2\alpha$ instead of $\alpha$ and requiring $\alpha < d/p$). The other inequality follows from

$$\| |p|^{\alpha} f \|_p \leq \| (|p|^{\alpha} - L_{a, \alpha}) f \|_p + \| L_{a, \alpha} f \|_p = \| |x|^{-\alpha} f \|_p + \| L_{a, \alpha} f \|_p$$

and the generalized Hardy inequality, Proposition 1.2 (requiring $\alpha + \delta < d/p < d - \delta$). □

6. **Non-power-like potentials**

As in [12], it is possible to generalize Theorem 1.1 to the operator $|p|^{\alpha} + V$ where $V$ is a function on $\mathbb{R}^d$ satisfying

$$\frac{a}{|x|^\alpha} \leq V(x) \leq \frac{\tilde{a}}{|x|^\alpha}$$

(6.1)

with $a^* \leq a \leq \tilde{a} < \infty$. We prove the following result.
Theorem 6.1. Let $\alpha \in (0, 2 \wedge d)$ and $s \in (0, 2]$. Let $a_* \leq a \leq \tilde{a} < \infty$ if $s = 2$ and $0 < a \leq \tilde{a} < \infty$ if $s \in (0, 2)$. Let furthermore $\delta = \delta(a)$ be defined by (1.3).

1. If $1 < p < \infty$ satisfies $\alpha s / 2 + \delta < d / p < \min \{d, d - \delta\}$, then for any $V$ satisfying (6.1),
   \[ \| |p|^{s/2} f\|_{L^p(\mathbb{R}^d)} \lesssim_{d, a, a, s} \| (|p|^\alpha + V)^{s/2} f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \] (6.2)

2. If $\alpha s / 2 < d / p < d$ (which already ensures $1 < p < \infty$), then for any $V$ satisfying (6.1),
   \[ \| (|p|^\alpha + V)^{s/2} f\|_{L^p(\mathbb{R}^d)} \lesssim_{d, a, a, s} \| |p|^{s/2} f\|_{L^p(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \] (6.3)

The proof of this theorem is akin to the one of Theorem 1.1. If $s = 2$, we merely use the triangle inequality, the ordinary Hardy inequality, and a modification of the generalized Hardy inequality, Proposition 1.2. If $s \in (0, 2)$, we apply the Littlewood–Paley theory of Section 4, i.e., square function estimates adapted to $|p|^\alpha + V$, and a modification of the reversed Hardy inequality, Proposition 5.2. These modifications are summarized in the following propositions whose proofs are analogous to those in [12], respectively Theorem 4.3, i.e., [25, Theorem 4.3]. We begin with the modified, generalized Hardy inequality.

Proposition 6.2. Let $1 < p < \infty$, $\alpha \in (0, 2 \wedge d)$, $a_* \leq a \leq \tilde{a} < \infty$, $\delta = \delta(a)$ be defined by (1.3), and $\alpha s / 2 \in (0, d)$. If $s$ and $p$ satisfy $\alpha s / 2 + \delta < d / p < d - \delta$, then for any $V$ satisfying (6.1),
   \[ \| x^{-\alpha s / 2} f\|_p \lesssim_{d, a, a, s} \| (|p|^\alpha + V)^{s/2} f\|_p \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \]

Proof. By Trotter’s formula, we have for all $x, y \in \mathbb{R}^d$ and $t > 0$,
   \[ 0 \leq e^{-t(|p|^\alpha + V)}(x, y) \leq e^{-t\mathcal{L}_{a, a}(x, y)}. \]

By the spectral theorem, i.e.,
   \[ \mathcal{L}_{a, a}^{s/2}(x, y) = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-t\mathcal{L}_{a, a}(x, y)} t^{s/2} \frac{dt}{t}, \]
and analogously for $(|p|^\alpha + V)^{-s/2}(x, y)$, it follows that
   \[ (|p|^\alpha + V)^{-s/2}(x, y) \leq \mathcal{L}_{a, a}^{-s/2}(x, y). \]

Therefore, the upper bound (1.5) on $\mathcal{L}_{a, a}^{-s/2}(x, y)$ continues to hold for $(|p|^\alpha + V)^{-s/2}(x, y)$, so the claim follows as in Proposition 1.2.

As in Section 4, we define Littlewood–Paley projections associated to $|p|^\alpha + V$ via the heat kernel as
   \[ P_N^V := e^{-(|p|^\alpha + V)/N^\alpha} - e^{-(|p|^\alpha + V)/(N^\alpha/2^\alpha)}. \]

By Theorem 1.3 and the arguments of [25, Theorem 4.3], we obtain the following square function estimates.
Proposition 6.3. Let $\alpha \in (0, 2 \land d)$, $a_s \leq a \leq \bar{a} < \infty$, $\delta = \delta(a)$ be defined by (1.3), $p \in (1, \infty)$ if $a \geq 0$, and $p \in (d/(d - \delta), d/\delta)$ if $a < 0$. Let furthermore $s \in (0, 2)$. Then, for any $V$ satisfying (6.1),

$$\left\| (|p|^\alpha + V)^{s/2} f \right\|_p \sim \left\| \left( \sum_{N \in \mathbb{Z}} |N^{\alpha s/2} (P_N^V)^k f|^2 \right)^{1/2} \right\|_p$$

for all $f \in C^c_c(\mathbb{R}^d)$.

Finally, we have the following modified reversed Hardy inequality.

Proposition 6.4. Let $\alpha \in (0, 2 \land d)$, $a_s \leq a \leq \bar{a} < \infty$, $\delta = \delta(a)$ be defined by (1.3), $p \in (1, \infty)$ if $a \geq 0$, and $p \in (d/(d - \delta), d/\delta)$ if $a < 0$. Let furthermore $s \in (0, 2)$. Then, for any $V$ satisfying (6.1),

$$\left\| \left( \sum_{N \in \mathbb{Z}} |N^{\alpha s/2} P_N^V f|^2 \right)^{1/2} - \left( \sum_{N \in \mathbb{Z}} |N^{\alpha s/2} P_N^V f|^2 \right)^{1/2} \right\|_p \lesssim_{d, a, a_s} \|x|^{-\alpha s/2} f\|_p$$

for all $f \in C^c_c(\mathbb{R}^d)$.

Proof. Denoting

$$\tilde{K}^\alpha_t(x, y) := e^{-t|p|^\alpha} (x, y) - e^{-t(|p|^\alpha + V)}(x, y),$$

Trotter’s formula yields for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$e^{-t|p|^\alpha} (x, y) - e^{-t\mathcal{L}_a}(x, y) \leq \tilde{K}^\alpha_t(x, y) \leq e^{-t|p|^\alpha} (x, y) - e^{-t\mathcal{L}\bar{\alpha}}(x, y).$$

Since $\bar{\delta} := \Psi_{a, d}^{-1}(\bar{a}) \leq \delta$, Lemma 5.1 with $a$ and $\tilde{a}$ implies

$$|\tilde{K}^\alpha_t(x, y)| \lesssim L_t^{\alpha, \bar{\delta}_+}(x, y) + L_t^{\alpha, \delta_+}(x, y) + M_t^\alpha(x, y) \lesssim L_t^{\alpha, \delta_+}(x, y) + M_t^\alpha(x, y).$$

Using this estimate, the assertion follows in the same way as Proposition 5.2. \qed

References

[1] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. Trans. Amer. Math. Soc., 95:263–273, 1960.

[2] Sönke Blunck. A Hörmander-type spectral multiplier theorem for operators without heat kernel. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2(3):449–459, 2003.

[3] Krzysztof Bogdan, Tomasz Grzywny, Tomasz Jakubowski, and Dominika Pilarczyk. Fractional Laplacian with Hardy potential. Communications in Partial Differential Equations, 44(1):20–50, 2019.

[4] Peng Chen, El Maati Ouhabaz, Adam Sikora, and Lixin Yan. Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. J. Anal. Math., 129:219–283, 2016.

[5] Peng Chen, El Maati Ouhabaz, Adam Sikora, and Lixin Yan. Spectral multipliers without semigroup framework and application to random walks. J. Math. Pures Appl. (9), 143:162–191, 2020.

[6] Soobin Cho, Panki Kim, Renming Song, and Zoran Vondraček. Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings. J. Math. Pures Appl. (9), 143:208–256, 2020.
Thierry Coulhon and Adam Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. Proc. Lond. Math. Soc. (3), 96(2):507–544, 2008.

[8] NIST Digital Library of Mathematical Functions. [http://dlmf.nist.gov/] Release 1.0.21 of 2018-12-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

Xuan Thinh Duong, El Maati Ouhabaz, and Adam Sikora. Spectral multipliers for self-adjoint operators. In Geometric analysis and applications (Canberra, 2000), volume 39 of Proc. Centre Math. Appl. Austral. Nat. Univ., pages 56–66. Austral. Nat. Univ., Canberra, 2001.

Xuan Thinh Duong, El Maati Ouhabaz, and Adam Sikora. Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal., 196(2):443–485, 2002.

Rupert L. Frank, Elliott H. Lieb, and Robert Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. J. Amer. Math. Soc., 21(4):925–950, 2008.

Rupert L. Frank, Konstantin Merz, and Heinz Siedentop. Equivalence of Sobolev norms involving generalized Hardy operators. Int. Math. Res. Not. IMRN, 07 2019.

Waldemar Hebisch. Almost everywhere summability of eigenfunction expansions associated to elliptic operators. Studia Math., 96(3):263–275, 1990.

Waldemar Hebisch. A multiplier theorem for Schrödinger operators. Colloq. Math., 60/61(2):659–664, 1990.

Waldemar Hebisch. Functional calculus for slowly decaying kernels. preprint, 1995.

Ira W. Herbst. Spectral theory of the operator \((p^2 + m^2)^{1/2} - Ze^2/r\). Comm. Math. Phys., 53:285–294, 1977.

Lars Hörmander. Estimates for translation invariant operators in \(L^p\) spaces. Acta Math., 104:93–140, 1960.

Lars Hörmander. The Analysis of Linear Partial Differential Operators. I. Springer Study Edition. Springer-Verlag, Berlin, second edition, 1990. Distribution theory and Fourier analysis.

Tomasz Jakubowski and Jian Wang. Heat kernel estimates of fractional Schrödinger operators with negative Hardy potential. Potential Anal., 53(3):997–1024, 2020.

Tosio Kato. Perturbation Theory for Linear Operators, volume 132 of Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin, 1 edition, 1966.

R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng. Sobolev spaces adapted to the Schrödinger operator with inverse-square potential. Math. Z., 288(3-4):1273–1298, 2018.

Rowan Killip, Changxing Miao, Monica Visan, Junyong Zhang, and Jiqiang Zheng. The energy-critical NLS with inverse-square potential. Discrete Contin. Dyn. Syst., 37(7):3831–3866, 2017.

Rowan Killip, Jason Murphy, Monica Visan, and Jiqiang Zheng. The focusing cubic NLS with inverse-square potential in three space dimensions. Differential Integral Equations, 30(3-4):161–206, 2017.

Rowan Killip, Monica Visan, and Xiaoyi Zhang. Riesz transforms outside a convex obstacle. Int. Math. Res. Not. IMRN, (19):5875–5921, 2016.

V. F. Kovalenko, M. A. Perelmuter, and Ya. A. Semenov. Schrödinger operators with \(L^1_w(\mathbb{R}^l)\)-potentials. J. Math. Phys., 22:1033–1044, 1981.

S. G. Mihlin. On the multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR (N.S.), 109:701–703, 1956.

Pierre D. Milman and Yu. A. Semenov. Global heat kernel bounds via desingularizing weights. J. Funct. Anal., 212(2):373–398, 2004.
[29] F. W. J. Olver. Bessel functions of integer order. In Milton Abramowitz and Irene A. Stegun, editors, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, chapter 9, pages 355–433. Dover Publications, New York, 5 edition, 1968.

[30] Elias M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[31] Elias M. Stein. *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*. Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970.

[32] Elias M. Stein and Guido Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, New Jersey, 2 edition, 1971.

[33] D. Yafaev. Sharp constants in the Hardy-Rellich inequalities. *Journ. Functional Analysis*, 168(1):212–144, October 1999.

[34] Felipe Zo. A note on approximation of the identity. *Studia Math.*, 55(2):111–122, 1976.

(Konstantin Merz) Mathematisches Institut, Ludwig-Maximilians Universität München, Theresienstr. 39, 80333 München, Germany. Address as of October 2019: Institut für Analysis und Algebra, Carolo-Wilhelmina, Universitätsplatz 2, 38106 Braunschweig, Germany

Email address: k.merz@tu-bs.de