Abstract

We show that the existence of a well-known type of ideals on a regular cardinal \( \lambda \) implies a compactness property concerning the specialisability of a tree of height \( \lambda \) with no cofinal branches. We also use Neeman’s method of side conditions to show that the existence of such ideals is consistent with stationarily many appropriate guessing models. These objects suffice to extend the main theorem of [13]: one can generically specialise any branchless tree of height \( \kappa^+ \) with a \( <\kappa \)-closed, \( \kappa^+ \)-proper, and \( \kappa^+ \)-preserving forcing, which has the \( \kappa^+ \)-approximation property.

Keywords. Tree, Special tree, Guessing model, Densely closed ideal

MSC. 03E05, 03E35

1 Introduction

This paper is somewhat a continuation of [13] where the interested reader can find our motivations. A tree \( T \) of height \( \kappa^+ \), for an infinite cardinal \( \kappa \), is called special if there is a function \( f : T \to \kappa \) such that for every \( t, s \in T \) with \( t < T s \), we have \( f(t) \neq f(s) \). Assuming \( T \) has no cofinal branches, a standard way to generically specialise \( T \) is to use partial specialising functions.

Let \( T \) be a tree of uncountable height. For infinite cardinals \( \mu \leq \lambda < \text{ht}(T) \) with \( \mu \) regular, we let \( S_\mu(T, \lambda) \) denote the forcing notion whose underlying set consists of partial functions \( p : T \to \lambda \) with \( |p| < \mu \) such that if \( s, t \in \text{dom}(p) \) and \( s < T t \), then \( p(s) \neq p(t) \), and whose order is reversed inclusion. It is clear that \( S_\mu(T, \lambda) \) is \( <\mu \)-closed and turns all cardinals in \( (\lambda, \text{ht}(T)) \) to be of size \( |\lambda| \). If \( \omega = \mu = \lambda \) and \( \text{ht}(T) = \omega_1 \), then \( S_\omega(T, \omega) \) has the countable chain condition, as proved by Baumgartner, Malitz and Reinhardt [1] and by Baumgartner [2]. The analogous situation for trees of height beyond \( \omega_1 \) is dramatically different, as then \( S_\mu(T, \lambda) \) always collapses all cardinals in the interval \( [\mu, \text{ht}(T)) \) onto \( \mu \), and collapses \( \gamma^{<\mu} \) onto \( \mu \), for every \( \gamma \in [\mu, \text{ht}(T)) \). However, even under the GCH, for a branchless tree \( T \), \( S_\mu(T, \lambda) \) may not be able to preserve the height of \( T \), since it may - opposed to branchless trees of height \( \omega_1 \) - not be ever specialisable and - similar to trees with cofinal branches - collapses the height of \( T \)!

The \( \text{ht}(T) \)-chain condition of \( S_\mu(T, \mu) \) has been extensively explored by Lücke [10]. His theorem can be restated as follows.
Theorem 1.1 (Lücke [10, Theorem 1.9]). Assume that $T$ is a Hausdorff tree\(^2\) of height $\kappa$. Suppose that $\omega \leq \mu \leq \text{cof}(\lambda) \leq \lambda < \kappa$ with $\mu$ and $\kappa$ regular such that $\gamma^< \mu < \kappa$, for every $\gamma < \kappa$. Then the following are equivalent.

1. There is no ascent path of width less than $\mu$ through $T$.\(^3\)

2. The forcing $S_\mu(T, \lambda)$ has the $\kappa$-chain condition.

3. The forcing $S_\mu(T, \lambda)$ preserves the regularity of $\kappa$.

In [13], we showed that under a guessing model principle, which follows from PFA, one could generically specialise a tree $T$ of height $\omega_2$ via a proper and $\omega_2$-preserving forcing with finite conditions. That forcing avoids “countable conditions” in a strong sense as it was shown to have the $\omega_1$-approximation property. Note that if $T$ is of height $\omega_2$, then $S_{\omega_1}(T, \omega_1)$ collapses $\omega_2$ under PFA (because of $\neg\text{CH}$) and $S_\omega(T, \omega_1)$ collapses $\omega_1$. Therefore, in order to specialise a tree of height $\omega_2$ with finite conditions, we attached finite chains of models as side conditions to $S_{\omega}(T, \omega_1)$. In this paper, we shall discuss a similar problem for trees of height beyond $\omega_2$.

The paper is organized as follows. Section 2 includes our basic definitions and notation. Section 3 investigates the compactness of ascent paths under the existence of the so-called densely closed ideals. In Section 4, we will prepare the ground via Neeman forcing to obtain suitable guessing models and appropriate densely closed ideals together. In Section 5, we sketch how to generalise our previous work [13] using the appropriate circumstances obtained in Section 4.

2 Preliminaries

We shall follow standard conventions and notation. By $p \leq q$ in a forcing ordering $\leq$, we mean $p$ is stronger than $q$; for a cardinal $\theta$, $H_\theta$ denotes the collection of sets whose hereditary size is less than $\theta$; for a set $X$, we let $\mathcal{P}(X)$ denote the power-set of $X$, and if $\kappa$ is a cardinal, we let $\mathcal{P}_\kappa(X) := \{A \in \mathcal{P}(X) : |A| < \kappa\}$; recall that a set $S \subseteq \mathcal{P}_\kappa(H_\theta)$ is stationary, if for every function $F : \mathcal{P}_\omega(H_\theta) \to \mathcal{P}_\kappa(H_\theta)$, there is $M \prec H_\theta$ in $S$ with $M \cap \kappa \in \kappa$ such that $M$ is closed under $F$.

For a set $M$, we say that a set $x \subseteq M$ is bounded in $M$ if there is $y \in M$ such that $x \subseteq y$. We say $x$ is guessed in $M$ if there is some $x^* \in M$ such that $x^* \cap M = x \cap M$. We also let $\kappa_M := \min\{\alpha \in M \cap \text{ORD} : \alpha \not\in M\}$. We denote the collapsing map of $M$ by $\pi_M$.

Trees

Recall that a tree is a partially ordered set $(T, \prec_T)$ such that for every $t \in T$, the set $b_t := \{s \in T : s \prec_T t\}$ is well ordered with respect to $\prec_T$. Suppose $T = (T, \prec_T)$ is a tree. For every $t \in T$, the height of $t$, denoted by $\text{ht}_T(t)$, is the order type of $b_t$. The height of $T$, denoted by $\text{ht}(T)$, is

\(^2\)see Section 2.

\(^3\)see Definition 2.2.
sup\{ht_T(t) + 1 : t \in T\}. For every ordinal \( \alpha \leq ht(T) \), \( T_\alpha \) denotes the set of nodes of height \( \alpha \). \( T_{\leq \alpha} \) and \( T_{< \alpha} \) have obvious meanings. By a branch \( b \) through \( T \), we mean \((b, < T)\) is a (maximal) downward-closed, linearly ordered set. A branch is a cofinal branch if its order type is the height of \( T \).

A tree \( T \) is called Hausdorff if for every two incomparable nodes \( t, s \) in \( T \), there is a largest \( u < T t, s \). The following is well-known.

**Lemma 2.1** (Folklore). In order to specialise a tree \( T \) (of height \( \kappa^+ \), for some infinite cardinal \( \kappa \)), one may assume, without loss of generality, that \( T \) is a Hausdorff tree.

**Proof.** See [13, Lemma 2.5] for a proof.

**Definition 2.2** (ascent path). Suppose \((T, < T)\) is a tree and that \( \mu \) is a nonzero cardinal. An ascent path of width \( \mu \) through \( T \) is a sequence \( \langle \vec{t}_\alpha : \alpha < ht(T) \rangle \) such that

- for every \( \alpha < ht(T) \), \( \vec{t}_\alpha : \mu \to T_\alpha \) is a function, and
- for every \( \alpha < \beta < ht(T) \), there are \( \zeta, \eta < \mu \) such that \( \vec{t}_\alpha(\zeta) < T \vec{t}_\beta(\eta) \).

We believe trees without \( \mu \)-ascent paths deserve to have a name.

**Definition 2.3** (unclimbable tree). Suppose \( \mu \) is a non-zero cardinal. A tree \( T \) is called \( \mu \)-unclimbable if \( T \) does not have any ascent path of width \( \mu \). It is called \( < \mu \)-unclimbable if it is \( \mu^\prime \)-unclimbable, for every non-zero cardinal \( \mu^\prime < \mu \).

Thus \( T \) is 1-unclimbable if and only if there is no cofinal branch through \( T \). Also \( T \) is \( 1 \)-unclimbable if and only if it is \( < \omega \)-unclimbable (see [10, Lemma 1.4]). We say \( T \) is \( \mu \)-climbable if it is not \( \mu \)-unclimbable. Thus \( T \) is always \( |T| \)-climbable. Lücke [10, Proposition 1.5] showed that \( T \) is \( ht(T) \)-climbable.

**Strong properness and the approximation property**

Recall that if \( M \prec H_\theta \) contains a forcing \( \mathbb{P} \), then a condition \( p \in \mathbb{P} \) is called \((M, \mathbb{P})\)-generic if for every dense subset \( D \) of \( \mathbb{P} \) in \( M \), \( D \cap M \) is pre-dense below \( p \).

**Definition 2.4.** Assume that \( \mathbb{P} \) is a forcing, and \( \theta \) is a sufficiently large regular cardinal. Suppose \( S \subseteq \mathcal{P}_\kappa(H_\theta) \) consists of elementary submodels. Then, \( \mathbb{P} \) is said to be proper for \( S \), if for every \( M \in S \) and every \( p \in \mathbb{P} \cap M \), there is an \((M, \mathbb{P})\)-generic condition \( q \leq p \).

The following is well-known.

**Lemma 2.5** (Folklore). Let \( \kappa \) be a regular cardinal. Assume that \( \mathbb{P} \) is a forcing, and \( \theta > \kappa \) is a sufficiently large regular cardinal. Suppose \( S \subseteq \mathcal{P}_\kappa(H_\theta) \) is a stationary set of elementary submodels. If \( \mathbb{P} \) is proper for \( S \), then \( \mathbb{P} \) preserves the regularity of \( \kappa \).

**Proof.** See [13, Lemma 2.7] for a proof.
Let us now recall the following closely related definitions from [12] and [6], respectively.

**Definition 2.6** (strong properness). Suppose $\mathbb{P}$ is a forcing notion.

1. Let $X$ be a set. A condition $p \in \mathbb{P}$ is said to be strongly $(X, \mathbb{P})$-generic, if for every $q \leq p$, there is some $q \upharpoonright X \in X \cap \mathbb{P}$ such that every condition $r \in \mathbb{P} \cap X$ extending $q \upharpoonright X$ is compatible with $q$.

2. For a collection of sets $S$, we say $\mathbb{P}$ is strongly proper for $S$, if for every $X \in S$ and every $p \in \mathbb{P} \cap X$, there is a strongly $(X, \mathbb{P})$-generic condition extending $p$.

**Remark 2.7.** It is easily seen that if $p$ is strongly $(X, \mathbb{P})$-generic and $M \prec H_\theta$ is such that $M \cap \mathbb{P} = X \cap \mathbb{P}$, then $p$ is strongly $(M, \mathbb{P})$-generic, and hence $(M, \mathbb{P})$-generic. It turns out that if a forcing notion is strongly proper for some stationary set $S \subseteq \mathcal{P}_\kappa(H_\theta)$, then $\mathbb{P}$ is $S$-proper, and hence it preserves $\kappa$, by Lemma 2.5.

**Definition 2.8** ($\kappa$-approximation property). Assume that $(M, N)$ is a pair of transitive models of of a sufficiently strong fragment of ZFC with $M \subseteq N$. Suppose $\kappa$ is an uncountable regular cardinal in $M$. The pair $(M, N)$ has the $\kappa$-approximation property, if every $A \in N$ which is bounded in $M$ and satisfies $a \cap A \in V$, for every $a \in V$ of $V$-cardinality $<\kappa$, must belong to $V$.

**Definition 2.9.** Suppose $\kappa$ is an uncountable regular cardinal. A forcing notion $\mathbb{P}$ has the $\kappa$-approximation property, if for every $V$-generic filter $G$, the pair $(V, V[G])$ has the $\kappa$-approximation property.

**Remark 2.10.** Note that it is well-known that if a forcing notion is strongly proper for stationarily many models in $\mathcal{P}_\kappa(H_\theta)$, then it has the $\kappa$-approximation property.

**Guessing models**

We recall the definition of a guessing model from [17].

**Definition 2.11** (guessing model). Assume $\theta$ is an uncountable regular cardinal. Let $M \prec H_\theta$. Suppose that $\gamma \in M$ is a regular cardinal with $\gamma \leq \kappa_M$. Then $M$ is said to be a $\gamma$-guessing model if the following are equivalent for any $x$ which is bounded in $M$.

1. $x$ is $\gamma$-approximated in $M$, i.e., $x \cap a \in M$, for all $a \in M$ of size less than $\gamma$.

2. $x$ is guessed in $M$.

We shall use the following characterisation of guessing models.

**Proposition 2.12** (Cox–Krueger [3]). Assume that $M$ is an elementary submodel of $H_\theta$. Then $M$ is $\gamma$-guessing if and only if $(\pi_M[M], V)$ has the $\pi_M(\gamma)$-approximation property.
**Definition 2.13** (GM∗). Assume that κ is a regular cardinal. We let GM∗(κ++ ) state that for every sufficiently large regular cardinal θ, the set \( G^*(\kappa^+, H_\theta) \) consisting of \( \kappa^+ \)-guessing models \( M \) with the following properties is stationary in \( P_{\kappa^{++}}(H_\theta) \).

1. \( M = \bigcup_{\alpha<\kappa} M_\alpha \), where
2. each \( M_\alpha \prec H_\theta \) is of size \( \kappa \) and <\( \kappa \)-closed with \( \kappa \cup \{ \kappa^+, \kappa^{++} \} \subseteq M_\alpha \),
3. if \( \alpha < \beta \), then \( M_\alpha \in M_\beta \), and
4. if \( \text{cof}(\alpha) = \kappa \), then \( M_\alpha = \bigcup_{\beta<\alpha} M_\beta \).

In [17], Viale and Weiβ showed that GM∗(\( \omega_2 \)) holds under PFA.

**Magidor models**

We need the following characterisation of a supercompact cardinal.

**Theorem 2.14** (Magidor [11]). The following are equivalent for a regular cardinal κ.

1. κ is supercompact.
2. For every γ > κ and \( x \in V_\gamma \) there exist \( \kappa < \xi < \kappa \), and an elementary embedding \( j : V_\gamma \to V_\xi \) with critical point \( \kappa \) such that \( j(\kappa) = \kappa \) and \( x \in j[V_\gamma] \).

**Definition 2.15.** We say that a model \( M \) is a κ-Magidor model if, letting \( \tilde{M} \) be the transitive collapse of \( M \) and \( \pi \) the collapsing map, \( \tilde{M} = V_\gamma \), for some \( \gamma < \kappa \) with \( \text{cof}(\gamma) \geq \pi(\kappa) \), and \( V_{\pi(\kappa)} \subseteq M \).

The following is a straightforward corollary of Magidor’s theorem.

**Corollary 2.16.** Suppose κ is supercompact and \( \mu > \kappa \) with \( \text{cof}(\mu) \geq \kappa \). Then the set of κ-Magidor models is stationary in \( P_\kappa(V_\mu) \).

**Ideals**

By a proper ideal \( I \) on a set \( Z \), we always mean \( Z \notin I \subseteq \mathcal{P}(Z) \). A set \( X \subseteq Z \) is an \( I \)-positive set, if \( X \notin I \). The collection of \( I \)-positive sets is denoted by \( I^+ \).

**Definition 2.17** (densely \( \kappa \)-closed ideal). Assume κ is an infinite cardinal. We call an ideal \( I \) on \( \lambda > \kappa \) a densely \( \kappa \)-closed ideal if it is proper, it contains all bounded subsets of \( \kappa^+ \), and that \( (I^+, \subseteq) \) has a <\( \kappa \)-closed dense subset, i.e., there is \( D \subseteq I^+ \) which is closed under decreasing sequences of length less than \( \kappa \), and for every \( A \in I^+ \), there is \( B \in D \) with \( B \subseteq A \).
Note that the ideal of bounded subset of an infinite cardinal $\kappa$ is a densely $\omega$-closed ideal. Laver and independently, Galvin, Jech and Magidor [5] showed the existence of densely $\mu$-closed ideals on $\mu^+$ by Levy collapsing a measurable cardinal (the optimal large cardinal assumption) onto $\mu$. Laver also was apparently the one who used these ideals to show that one can obtain a model of the Continuum Hypothesis and the $\aleph_2$-Suslin Hypothesis, a result which later appeared in a joint work with Shelah [9] using a weakly compact cardinal, and so wiping out densely $\omega_1$-closed ideals on $\omega_2$ from Laver’s original proof. It is also not hard to give a game-theoretic reformulation, similar to the one in [5], of the existence of a densely $\kappa$-closed ideal

**Foreman’s Duality Theorem**

We explain Foreman’s duality theorem [4, Theorem 17]. Suppose that $\mathcal{I}$ is a precipitous ideal on a set $Z$, and that $G \subseteq \mathcal{P}(Z)/\mathcal{J}$ is a $V$-generic filter. Assume

$$j : V \rightarrow M \cong V[Z/G] \subseteq V[G],$$

with $M$ transitive, is the corresponding ultrapower embedding.

Let $\mathbb{P}$ be a forcing in $V$, and let $\dot{H}$ be a canonical name for $\mathbb{P}$-generic filters. Suppose that there is a $P(Z)/\mathcal{J}$-name $\dot{m}$ for a master condition in $j(\mathbb{P})$ such that the embedding

$$e : \mathbb{P} \rightarrow \mathcal{P}(Z)/\mathcal{J} * j(\mathbb{P})/\dot{m},$$

defined by $e(p) = (1. j(p))$ is a regular embedding. If $[A] \in \mathcal{P}(Z)/\mathcal{J}$ forces that “$\dot{m} = [f]$” and $\mathcal{M} \in V^\mathbb{P}$ is defined as $\{ z \in A : f(z) \in \dot{H} \}$, then there is a canonical isomorphism $\iota$ extending $e$ that witnesses:

$$B(\mathbb{P} * \dot{\mathbb{P}}(Z)/\dot{\mathcal{J}} \upharpoonright \mathcal{M}) \cong B(\mathcal{P}(Z)/(\mathcal{J} \upharpoonright A) * j(\mathbb{P})/\dot{m}),$$

where $B(-)$ denotes the Boolean completion of a poset and $\dot{\mathcal{J}}$ is the ideal generated by $\mathcal{J}$ in $V^\mathbb{P}$.

**Lemma 2.18.** Let $\mathcal{I}$ be a $\lambda$-complete ultrafilter on $\lambda$, and assume that $j$ is the corresponding ultrapower embedding. Let $\kappa < \lambda$ be an infinite cardinal, and let $\mathbb{P} \subseteq V_\lambda$ be a $<\kappa$-closed $\lambda$-c.c. forcing such that $\mathbb{P}$ is a regular suborder of $j(\mathbb{P})$ and $\mathbb{1}_\mathbb{P}$ forces $j(\mathbb{P})/\dot{G}_\mathbb{P}$ to be $<\kappa$-closed. Let $G$ be a $V$-generic filter, and let $\overline{\mathcal{I}}$ be the ideal generated by $\mathcal{I}$ in $V[G]$. Then $\overline{\mathcal{I}}$ is a densely $\kappa$-closed ideal on $\lambda$.

**Proof.** (Foreman [4, Example 27])

In Foreman’s duality theorem, take $Z = \lambda$, $\mathcal{J} = \mathcal{I}$, $A = \lambda$ and $\dot{m} = \dot{1}_{j(\mathbb{P})}$. On the other hand $\mathcal{P}(\lambda)/\mathcal{I}$ is the trivial Boolean algebra. It follows that $\mathcal{M} = \lambda$ and

$$B(\mathbb{P} * \mathcal{P}^G(\lambda)/\overline{\mathcal{I}}) \cong B(j(\mathbb{P})).$$

Thus if $G \subseteq \mathbb{P}$ is a $V$-generic filter, then

$$B(\mathcal{P}^G(\lambda)/\overline{\mathcal{I}}) \cong B(j(\mathbb{P})/G).$$

It is easily seen that $\overline{\mathcal{I}}^+$ has a $<\kappa$-closed dense subset, as $j(\mathbb{P})/G$ is $<\kappa$-closed. Clearly $\overline{\mathcal{I}}$ is proper and contains all bounded subsets. Hence a densely $\kappa$-closed ideal on $\lambda$. 2.18
3 The compactness of unclimbableness

The following is our key lemma.

Lemma 3.1. Suppose \( \mathcal{I} \) is an ideal on a set \( Z \). Let \( M \prec H_\theta \) with \( Z, \mathcal{I} \in M \). Let \( f : M \cap W \to 2 \) be a function which is not guessed in \( M \), where \( W \in M \). Suppose that \( \phi \in M \) is a function such that:

1. \( A := \text{dom}(\phi) \in \mathcal{I}^+ \),
2. For every \( z \in A \), \( \phi(z) \) is a function, and
3. For every \( w \in W \), \( \{ z \in A : w \in \text{dom}(\phi(z)) \} \in \mathcal{I}^+ \).

Then there is \( A^* \in M \cap \mathcal{I}^+ \) with \( A^* \subseteq A \) such that for every \( z \in A^* \), \( \phi(z) \not\subseteq f \).

Proof. For each \( z \in A \), set \( \phi(z) := f_z \), and let
\[
A^*_\epsilon = \{ z \in A : f_z(w) = \epsilon \}, \quad \text{for } \epsilon = 0, 1.
\]
Notice that the sequence
\[
\langle A^*_\epsilon : w \in W, \epsilon \in \{0, 1\} \rangle
\]
belongs to \( M \). The 3rd item above implies that for every \( w \in W \), \( A^*_0 \cup A^*_1 \) belongs to \( \mathcal{I}^+ \). We are done if there is some \( w \in W \) such that both \( A^*_0 \) and \( A^*_1 \) are in \( \mathcal{I}^+ \), as then by elementarity one can find such a \( w \in M \cap W \), and pick \( A^* := A^*_1 \cdot f(w) \). Therefore, let us assume that this is not the case. Thus for every \( w \in W \), there is a unique \( \epsilon \in \{0, 1\} \) such that \( A^*_\epsilon \) is in \( \mathcal{I}^+ \). Now, define \( h \) on \( W \) by letting \( h(w) \) be \( \epsilon \) if and only if \( A^*_\epsilon \) is in \( \mathcal{I}^+ \). Clearly \( h \) is in \( M \), and \( h \upharpoonright M \neq f \), since \( f \) is not guessed in \( M \). Thus, there exists \( w \in M \cap W \) such that \( h(w) \neq f(w) \), which in turn implies that \( A^*_1 \cdot f(w) \) is in \( \mathcal{I}^+ \) and belongs to \( M \). Then \( A^* := A^*_1 \cdot f(w) \) is as required. \(\blacksquare\)

Lemma 3.2. Let \( \mathcal{I}, M, \theta, Z, W, A \) be as in Lemma 3.1. Let also \( \mu', \mu'' < \mu \) be non-zero cardinals. Moreover, assume the following.

- \( \mathcal{I} \) is a densely \( \mu \)-closed ideal on \( Z \), where \( \mu \) is a regular cardinal.
- \( M \) is closed under \( < \mu \)-sequences, and
- \( (f^\xi : \xi < \mu') \) and \( (\phi^\xi : \xi < \mu'') \) are sequence so that for each \( \xi < \mu' \) and \( \zeta < \mu'' \), \( f^\xi \) and \( \phi^\xi \) are as in Lemma 3.1

Then there is \( A^* \in M \cap \mathcal{I}^+ \) with \( A^* \subseteq A \) such that for every \( z \in A^* \), for every \( \xi < \mu' \) and \( \zeta < \mu'' \), \( \phi^\xi(z) \not\subseteq f^\xi \).

Proof. Observe that \( \mu \subseteq M \). By elementarity, there is a \( < \mu \)-closed dense subset \( \mathcal{D} \) of \( \mathcal{I}^+ \) in \( M \). Let \( \mu^* = \mu' \cdot \mu'' \). Fix a bijection \( \rho : \mu^* \to \mu' \times \mu'' \) in \( M \). We shall inductively construct a sequence \( \langle A_\eta : \eta < \mu^* \rangle \in M \) such that for every \( \eta, \iota < \mu^* \),
\[ A_{\eta'} \in \mathcal{D}, \text{ for every } \eta' < \eta, \]

\[ \eta < \iota \Rightarrow A_{\iota} \subseteq A_{\eta}, \]

\[ \text{for every } z \in A_{\eta}, \phi^\xi(z) \notin f^\xi, \text{ where } \rho(\eta) = (\xi, \zeta) \]

It is then clear that \( A^\ast = \bigcap_{\eta < \mu} A_{\eta} \) is as required.

Thus assume that the sequence has been constructed up to \( \eta \), and in particular, we assume that

\[ (A_{\eta'} : \eta' < \eta) \in M. \]

Let \( \rho(\eta) = (\xi, \zeta) \) and set \( A''_{\eta} := \bigcap_{\iota < \eta} A_{\iota} \) with \( A''_{\eta} \in M \cap \mathcal{I}^+ \) such that for every \( z \in A''_{\eta} \), \( \phi^\xi(z) \notin f^\xi \). Now pick, by the density of \( \mathcal{D} \), a set \( A_{\eta} \in \mathcal{D} \cap M \) with \( A_{\eta} \subseteq A''_{\eta} \). It is easy to check that \( A_{\eta} \) is as required. This shows that the inductive construction works as we want.

**Lemma 3.3.** Let \( T \) be a 1-unclimbable tree. Assume that \( M \prec H_\theta \) contains \( T \). Suppose that \( \delta := \sup(M \cap \text{ht}(T)) < \text{ht}(T) \). If \( t \in T_\delta \), then \( b_t \) is not guessed in \( M \).

**Proof.** Suppose \( b_t \) is guessed in \( M \), and let \( b^* \in M \) be a witness. Therefore, by elementarity \( b^* \) is a cofinal branch through \( T \), a contradiction!

A theorem proved by Lambie-Hanson [7] and Lücke [10], independently, states that \( \text{GM}^\ast(\kappa^{++}) \) implies that a tree \( T \) of height \( \kappa^{++} \) is 1-unclimbable if and only if \( T \) is \( \kappa \)-unclimbable. Laver-Shelah’s work [9] shows that the latter statant obtains after Levy collapsing a weakly compact cardinal onto \( \kappa^+ \) with conditions of size \( \kappa \), provided that \( \kappa^{<\kappa} = \kappa \). On the other hand, Lambie-Hanson and Lücke [8] showed that the consistency strength of the above compactness phenomenon is that of a weakly compact. We introduce another principle which is equiconsistent with a measurable cardinal.

**Proposition 3.4.** Let \( T \) be a Hausdorff tree of height \( \kappa \). Assume that \( \mu < \kappa \) are regular cardinals such that for every \( \gamma < \kappa, \gamma^{<\mu} < \kappa \). Suppose that there is a densely \( \mu \)-closed ideal on \( \kappa \). Let \( \lambda \) be a cardinal with \( \mu \leq \lambda < \kappa \). Then \( S_{\mu}(T, \lambda) \) has the \( \kappa \)-chain condition.

**Proof.** For the sake of simplicity, let us denote \( S_{\mu}(T, \lambda) \) by \( S \). Fix a densely \( \mu \)-closed ideal \( \mathcal{I} \) on \( \kappa \) and a \( <\mu \)-closed dense set \( \mathcal{D} \subseteq \mathcal{I}^+ \). Let \( \theta \) be a sufficiently large regular cardinal. It is enough to show that for unboundedly many models \( M \prec H_\theta \) of size \( <\kappa \), the maximal condition of \( S \) is \((M, S)\)-generic. Thus fix such an \( M \prec H_\theta \) with

\[ \mu, \lambda, \kappa, T, \mathcal{I}, \mathcal{D} \in M, \]

\[ M^{<\mu} \subseteq M, \text{ and} \]

\[ M \cap \kappa \in \kappa. \]
Note that by our cardinal arithmetic assumption, the set of such models is stationary in $\mathcal{P}_\kappa(H_\theta)$.

For every $t \in T \setminus M$, let

$$O_M(t) = \sup \{ s \in T \cap M : s <_T t \}.$$  

**Claim 3.5.** For every $t \in T \setminus M$, either $O_M(t) \in M$ or $\text{cof}(\text{ht}(O_M(t))) \geq \mu$.

**Proof.** Assume that for some $t \in T \setminus M$, neither $O_M(t) \in M$ nor $\text{cof}(\text{ht}(O_M(t))) \geq \mu$. Thus, the height of $O_M(t)$ is a limit ordinal with $\text{cof}(\text{ht}(O_M(t))) < \mu$. Since $M$ is closed under sequences of length less than $\mu$, $T$ is Hausdorff, and that $\mu \subseteq M$, we can read off $O_M(t)$ from $\{ s \in T \cap M : s <_T t \}$ which belongs to $M$. Thus $O_M(t)$ is in $M$, a contradiction! \[\text{3.5}\]

Set $\delta = M \cap \kappa$. Observe that the cofinality of $\delta$ is at least $\mu$. Let $p \in S$. By extending $p$, we may assume that there are nodes $t$ in $\text{dom}(p)$ with $\text{ht}(O_M(t)) = \delta$ and that for each node $t$ in $\text{dom}(p) \cap T_{>\delta}$, there is some node $s \in \text{dom}(p)$ of height $\delta$ with $s <_T t$. Since $M^{<\mu} \subseteq M$ and $p \cap M$ is of size less than $\mu$, we have $p \restriction M := p \cap M \in M \cap S$. Let $E \in M$ be an open dense subset of $S$. Our aim is to find "$\mathbb{T}^{+}$-many" conditions in $E \cap M$ which are compatible with $p$. We may also assume without loss of generality that $p \in E$. We consider two categories of nodes in $T$ with respect to their interaction with $M$.

**C$^1$-nodes.** Fix an enumeration $\langle O_\xi : \xi < \gamma < \mu \rangle$ of

$$\{O_M(t) : t \in \text{dom}(p) \setminus M \} \cap T_{<\delta}.$$  

Now for each $\xi < \gamma$, pick $O_\xi^*$ so that

- $O_\xi^* \in T \cap M$ and $O_\xi^* \leq_T O_\xi$, and
- if $s \in \text{dom}(p)$, then $\text{ht}(s) \notin (\text{ht}(O_\xi^*), \text{ht}(O_\xi))$.

This is possible by Claim 3.5, as either $O_\xi \in M$, in which case one can pick it as $O_\xi^*$, or the cofinality of the height of $O_\xi$ is at least $\mu$, in which case we can find $O_\xi^*$ by the fact that $p$ is of size less than $\mu$.

**C$^2$-nodes.** Fix a sufficiently large ordinal $\delta^* \in M \cap \delta$ with

$$\delta^* > \sup \{ \text{ht}(t) : t \in \text{dom}(p) \cap T_{<\delta} \},$$  

and let $\langle t_\xi : \xi < \mu' \rangle$ enumerate

$$\{ t \in \text{dom}(p) \cap T_\delta : t = O_M(t) \}.$$  

Now for every $\xi < \mu'$, pick $O_\xi^* \in M \cap T_{=\delta^*}$ with $O_\xi^* <_T t_\xi$.

We are now about to reflect $p$ to $M$ and carefully define $\kappa$-many reflections of $p$.

**Claim 3.6.** Let $\alpha \in M \cap \kappa$. Then there is $(p_\alpha, \delta_\alpha) \in M$ such that:

1. (trivial requirement) $p \in E$ and $p_\alpha \leq p \cap M$,
2. (interval avoiding $C^1$) for each $\xi < \gamma$, if $s \in \text{dom}(p_\alpha)$, then
\[
\text{ht}(s) \notin (\text{ht}(O_\xi^s), \text{ht}(O_\xi^t)),
\]
3. (colour avoiding $C^1$) for each $\xi < \gamma$, if $s \in \text{dom}(p_\alpha)$ and $s \leq T \ O_\xi^s$, then
\[
p_\alpha(s) \notin \{p(u) : O_M(u) = O_\xi\},
\]
4. (interval avoiding $C^2$) $\delta_\alpha \in \kappa \setminus \delta^*$ satisfies $\text{dom}(p_\alpha) \cap T_{<\delta_\alpha} \subseteq T_{<\delta^*}$, and for every $t \in \text{dom}(p_\alpha) \cap T_{\geq \delta_\alpha}$, there is $s \in \text{dom}(p) \cap T_{\delta_\alpha}$ with $s \leq T \ t$.
5. (colour avoiding $C^2$) if $t \in \text{dom}(p_\alpha) \cap T_{<\delta^*}$ and $t \leq O_\xi^s$ for some $\xi < \mu'$, then
\[
p_\alpha(t) \notin \{p(u) : u \in \text{dom}(p) \text{ and } O_M(u) = t_\xi\}, \text{ and}
\]
6. (size freezing) there are $\mu''$ points in $\text{dom}(p_\alpha) \cap T_{\delta_\alpha}$, where $\mu'' = |\text{dom}(p) \cap T_\delta|$.

**Proof.** Observe that the above properties are definable in $M$. Note that for every ordinal $\alpha \in M \cap \kappa$, $(p_\alpha, \delta_\alpha) := p(\alpha, \delta)$ satisfy the above properties. Thus by elementarity, there are $p_\alpha$ and $\delta_\alpha$ as above.

Now by the Axiom of Choice, there is a mapping $\Phi : \kappa \rightarrow \mathbb{S} \times \kappa$ in $M$ such that for each $\alpha \in \kappa$, $\Phi(\alpha) = (p_\alpha, \delta_\alpha)$ is as in Claim 3.6. Fix such a mapping $\alpha \mapsto (p_\alpha, \delta_\alpha)$.

**Definition 3.7.** For every $t \in T$, let $\chi_t$ be the function defined on $T$ by $\chi_t(s) = 1$ if and only if $s \leq T \ t$.

We want to apply Lemma 3.2. Thus fix an enumeration $\langle t_\xi^\alpha : \xi < \mu''\rangle$ of $\text{dom}(p_\alpha) \cap T_{\delta_\alpha}$. Now for each $\xi < \mu'$, let
\[
f_\xi := \chi_{t_\xi} | T_{<\delta},
\]
and for each $\zeta < \mu''$, let $\phi_\zeta$ be defined on $\kappa$ by
\[
\phi_\zeta(\alpha) := \chi_{t_\xi^\alpha} | T_{<\text{ht}(t_\xi^\alpha)}.
\]

**Claim 3.8.** The objects $\langle f_\xi : \xi < \mu'\rangle, \langle \phi_\zeta : \zeta < \mu''\rangle, \mathcal{I}, Z := T, A := \kappa, M, \theta, \mu, \mu', \text{ and } \mu''$ satisfy the premises of Lemma 3.2.

**Proof.** We only need to show if $\xi < \mu'$ and $\zeta < \mu''$, then
1. $f_\xi$ is not guessed in $M$, and
2. for every $s \in T$, $\{\alpha \in \kappa : s \in \text{dom}(\phi_\zeta(\alpha))\} \in \mathcal{I}^+.$

The first item follows from Lemma 3.3. To see the second item, observe that
\[
\{\alpha \in \kappa : s \in \text{dom}(\phi_\zeta(\alpha))\} \supseteq \kappa \setminus (\text{ht}_T(s) + 1).
\]
Since $\mathcal{I}$ is proper and contains all bounded subsets of $\kappa$, we have $\kappa \setminus (\text{ht}_T(s) + 1) \in \mathcal{I}^+$. Therefore, $\{\alpha \in \kappa : s \in \text{dom}(\phi_\zeta(\alpha))\} \in \mathcal{I}^+.$
By Claim 3.8, we can apply Lemma 3.2 to obtain a set \( A^* \in M \cap I^+ \) such that for every \( \xi < \mu' \) and \( \zeta < \mu'' \) and every \( \alpha \in A^* \cap M \), we have
\[
\phi^\xi(\alpha) \not\subseteq f^\xi.
\]
We have the following straightforward corollary.

**Corollary 3.9.** For every \((\xi, \zeta) \in \mu' \times \mu''\) and every \( \alpha \in A^* \cap M \), we have
\[
t^\alpha \zeta \not\leq_T t^\xi.
\]

3.9

The following claim finishes the proof. Recall that if \( \alpha \in M \cap \kappa \), then \( p_\alpha \in M \cap E \).

**Claim 3.10.** For each \( \alpha \in A^* \cap M \), \( p_\alpha \) is compatible with \( p \)

**Proof.** Fix \( \alpha \in A^* \cap M \). Suppose that \( t \in \text{dom}(p) \) and \( s \in \text{dom}(p_\alpha) \) are comparable in \( T \). We shall show that \( p_\alpha(s) \neq p(t) \). Since \( p_\alpha \leq p \cap M \) (by Item 1) and \( M \cap \kappa \in \kappa \), we may assume that \( t \) is not in \( M \). Thus the only possibility is \( s <_T t \). Observe that if \( \text{ht}(O_M(t)) < \delta \), then we have \( s \leq O_M(t) \) (by Item 2) and hence \( p_\alpha(s) \neq p(t) \) by Item 3. Thus let us assume that \( \text{ht}(O_M(t)) = \delta \). By our assumption, we may also assume that \( t \in T_\delta \). Now if \( \text{ht}(s) \geq \delta^* \), then we may assume also \( s \in T_{\delta^*} \) (by Item 4). Thus for some \( \xi < \mu' \) and some \( \zeta < \mu'' \), \( t = t^\xi \) and \( s = t^\zeta \). On the other hand, if \( \text{ht}(s) < \delta^* \), then by Item 5, we have \( p_\alpha(s) \neq p(t) \). And by Corollary 3.9, we have \( s \not\leq_T t \), a contradiction.

3.10

3.4

Letting \( \kappa = \mu^+ \) in the above proposition, \( S_{\mu}(T, \mu) \) is the standard forcing to specialise a tree \( T \) of height \( \mu^+ \) without cofinal branches. In particular, if \( \omega = \mu = \lambda \), we can use the ideal of bounded subsets of \( \omega_1 \) to show that \( S_{\omega}(T, \omega) \) has the countable chain condition. Notice that it may be possible to use Baumgartner’s idea [2] to prove Proposition 3.4 with a (possibly) shorter argument, but we believe that the above argument is of independent interest. It is the same idea we used in the main theorem of [13]. Also it resembles the recent analysis of the forcing \( S_{\omega}(T, \omega) \) by Switzer in [15]. More importantly, the above proof can be easily modified to show that \( S_{\mu}(T, \lambda) \) has the \( \mu^+ \)-approximation property.

**Theorem 3.11.** Assume there is a densely \( \mu \)-closed ideal on \( \kappa \), where \( \mu < \kappa \) are regular infinite cardinals such that for every \( \gamma < \kappa \), \( \gamma < \mu < \kappa \). Let \( T \) be a Hausdorff tree of height \( \kappa \). Then \( T \) is \( <\mu \)-unclimbable if and only if it is 1-unclimbable.

**Proof.** The “only if” part is obvious. Consider \( S_{\mu}(T, \mu) \). By Lücke’s theorem (Theorem 1.1), the “if” part follows from Proposition 3.4.
4 A preparation via Neeman forcing

Throughout this section, we fix an infinite cardinal \( \kappa \) with \( \kappa^\kappa = \kappa \) and a supercompact cardinal \( \lambda > \kappa \). We let \( S := S_\lambda \) be the collection of \( \kappa \)-closed and \( \kappa \)-sized elementary submodels of \( V_\lambda \). Thus \( S \) is stationary in \( \mathcal{P}_{\kappa^+}(V_\lambda) \), by our assumption on \( \kappa \). Observe that for each \( M \in S \), we have \( M \cap \kappa^+ \in \kappa^+ \) and that \( \text{cof}(M \cap \kappa^+) = \kappa \). Let also

\[
T := \{ V_\alpha < V_\lambda : \text{cof}(\alpha) \geq \kappa^+ \}.
\]

Let \( \mathbb{P} := \mathbb{P}_{\kappa, S_\lambda, T_\lambda} \) be Neeman forcing based on \( S \cup T \) with decorated conditions of size less than \( \kappa \), i.e., \( p \in \mathbb{P} \) if and only if \( p = (M_p, d_p) \), where

- \( M_p \subseteq S \cup T \) is an \( \in \)-chain closed under intersections such that \( |M_p| < \kappa \), and
- \( d_p : M_p \rightarrow \mathcal{P}_\kappa(V_\lambda) \) is a function such that if \( M \in N \), then \( d_p(M) \in N \).

Notice that we abused language in the definition of a condition, where by an \( \in \)-chain, we mean that it is an \( \in^* \)-chain, where \( \in^* \) is the transitive closure of the binary relation \( \in \cap M_p \times M_p \).

The order on \( \mathbb{P} \) is defined by letting \( p \leq q \) if and only if \( p \supseteq q \) and for every \( M \in M_q \), \( d_p(M) \supseteq d_q(M) \), see [14, Definition 2.35]. There are two important constructions of forcing conditions which are used in the proof of the following proposition. The first one is that when \( p \) is a condition and \( M \in M_p \), then \( p \cap M \) is a condition belonging to \( M \), which is also weaker than \( p \). Moreover, any condition in \( M \) extending \( p \cap M \) is compatible with \( M \). The second one is that when \( p \) is a condition belonging to a model \( M \in \mathcal{S} \cup \mathcal{T} \), \( p^M \) obtained by adding \( M \) to \( M_p \) and close it under intersections, and \( d_{p^M} \) is the same as \( d_p \), except for the new models, we let the value to be \( \varnothing \). Notice that \( p^M \leq p \). The other important fact about this poset is that if \( p \) is a condition and \( M \in \mathcal{T} \), then we can always form \( p^M \leq p \). We use these properties freely in our arguments often without mentioning.

Proposition 4.1 ([14]). \( \mathbb{P} \) has the following properties.

1. \( \mathbb{P} \) is \( \kappa \)-closed, and hence preserves all cardinals \( \leq \kappa \).
2. \( \mathbb{P} \) is \( S \cup \mathcal{T} \)-strongly proper, and hence it preserves \( \kappa^+ \) and \( \lambda \), as \( S \) is stationary and \( \lambda \) is supercompact.
3. \( \mathbb{P} \) collapses all cardinals in the interval \( (\kappa^+, \lambda) \) onto \( \kappa^+ \), and hence forces \( \lambda = \kappa^{++} \).
4. \( \mathbb{P} \) has the \( \lambda \)-chain condition, and hence preserves all cardinal \( \geq \lambda \).

Proof. 1. This is straightforward, see [14, Claim 4.2].

2. See [14, Section 2, from Definition 2.35 on].

3. As in [14, Page 284].
4. It is enough to show that for some sufficiently large ordinal $\lambda^*$ with $\text{cof}(\lambda^*) \geq \lambda$, the maximal condition is $(M, \mathbb{P})$-generic for unboundedly many models $M \prec V_{\lambda^*}$ with $|M| < \kappa$. Pick such a $\lambda^*$. Let

$$U^* := \{M \prec V_{\lambda^*} : \mathbb{P} \in M, M \cap V_{\lambda} \in \mathcal{T}, \text{ and } M \text{ is a } \lambda\text{-Magidor model}\}$$

It follows from Corollary 2.16 that $U^*$ is stationary in $\mathcal{P}_\lambda(V_{\lambda^*})$. Fix $M \in U^*$ and let $V_\alpha = M \cap V_{\lambda}$. Let $p \in \mathbb{P}$. As in [14, Claim 5.7], we may extend $p$ to a condition $q$ with $V_\alpha \in \mathcal{M}_q$. By [14, Claim 4.2], $q$ is $(V_\alpha, \mathbb{P})$-strongly generic, and hence $(M, \mathbb{P})$-strongly generic. Thus $1_\mathbb{P}$ is $(M, \mathbb{P})$-generic, as desired!

Proposition 4.2. $\mathbb{P}$ forces $\text{GM}^*(\kappa^{++})$.

Proof. Assume that $\theta > \lambda$ is a sufficiently large regular cardinal $\kappa$ with $\text{cof}(\theta) \geq \lambda$. Let $G$ be a $\mathbb{P}$-generic filter, and set

$$\mathcal{M}_G := \bigcup \{ \mathcal{M}_p : p \in G \}$$

and

$$\mathcal{S}_G := \mathcal{S} \cap \mathcal{M}_G.$$ 

Let $\mathcal{U}_\theta^\lambda$ be the set of $\lambda$-Magidor elementary submodels $M$ of $(H_{\theta^*}, \in, \kappa, \lambda)$ with $M \cap V_{\lambda} \in \mathcal{T}$. By Corollary 2.16, $\mathcal{U}_\theta^\lambda$ is stationary in $\mathcal{P}_\kappa(H_{\theta^*})$. Let also

$$\mathcal{G} = \{ M[G] : M \in \mathcal{U}_\theta^\lambda \text{ and } M \cap V_{\lambda} \in \mathcal{M}_G \}$$

Claim 4.3. $\mathcal{G}$ is stationary in $\mathcal{P}_\lambda(H_{\theta^*}[\mathcal{G}])$.

Proof. Assume that $p$ forces that $\hat{F} : \mathcal{P}_\omega(H_{\theta}[\mathcal{G}]) \to \mathcal{P}_\lambda(H_{\theta}[\mathcal{G}])$ is a function. Pick $M \in \mathcal{U}_\theta^\lambda$ with $p, \hat{F}, \theta \in M$, where $\theta^* > \theta$ is a sufficiently large regular cardinal. Note that $M \cap H_{\theta} \in \mathcal{U}_\theta^\lambda$. Let $M^\gamma = V_{\gamma}$ be the transitive collapse of $M$. Then $p^{V_{\gamma}}$ forces that $(M \cap H_{\theta})[\mathcal{G}]$ is in $\mathcal{M}_{\hat{F}}$ and is closed under $\hat{F}$.

The following claim finishes the main proof.

Claim 4.4. $\mathcal{G} \subseteq \mathcal{G}^*(\kappa^{++}, H_{\theta})$.

Proof. Let $M[G] \in \mathcal{G}$. Assume that $\pi : M \to V_{\gamma}$ is the collapsing map of $M$. Then $\pi$ defines an isomorphism between $\mathbb{P} \cap M$ and $\pi(\mathbb{P})$. As $G$ contains a strongly master condition for $M$, it follows that $G \cap M$ is a $V$-generic filter on $\mathbb{P} \cap M$. Thus $\hat{G} := \pi[G \cap M]$ is a $V$-generic filter on $\mathbb{P}$. Then in $V[G \cap M]$, we have

$$V_{\gamma}[\hat{G}] = V_{\gamma}^{V[G \cap M]}.$$ 

On the other hand, letting $j := \pi^{-1}$, since $G$ contains a strongly master condition for $M$, we can lift, in $V[G]$, the embedding $j : V_{\gamma} \to H_{\theta}$ to

$$j^+ : V_{\gamma}[\hat{G}] \to H_{\theta}[G] \text{ with range}(j^+) = M[G].$$
Therefore, the transitive collapse of \( \overline{M[G]} \) is \( V_\gamma[\dot{G}] \).

Working in \( V[G \cap M] \), we can form the poset
\[
\mathcal{Q} = \{ p \in \mathbb{P} : V_\gamma \in p \text{ and } p \cap V_\gamma \in G \cap M \}.
\]

Let also
\[
S^* = \{ N[G \cap M] : V_\gamma \in N \in \mathcal{S} \text{ and } V_\gamma \cap N \in G \cap M \}.
\]

It is easy to see that \( H := G \cap \mathcal{Q} \) is a \( V \)-generic filter on \( \mathcal{Q} \) and \( V[H] = V[G] \). A standard argument as in Neeman’s paper shows that \( \mathcal{Q} \) is \( S^* \)-strongly proper. On the other hand, \( S^* \) is stationary in \( V[G \cap M] \). To see this, let \( \dot{F} \) be a \( \mathbb{P} \cap M \)-name so that \( p \in \mathbb{P} \cap M \) forces to be a function from \( P_\lambda(\dot{G}[\dot{G} \cap M]) \) into \( P_\kappa(\dot{G}[\dot{G} \cap M]) \). Pick \( N \in \mathcal{S} \) with \( \dot{F}, p, V_\gamma \in N \). It is easy to see that \( p^{N \cap V_\gamma} \Vdash \dot{N}[G] \in S^* \). Therefore, \( \mathcal{Q} \) has the \( \kappa \)-approximation property by Remark 2.10. On the other hand, \( (V_\gamma[\dot{G}], V[G \cap M]) \) has the \( \kappa \)-approximation property, since \( V_\gamma V[G \cap M] = V_\gamma[\dot{G}] \). Since \( \mathcal{Q} \) is proper for a stationary set, it has the \( \kappa \)-covering property\(^4\), and hence the pair \( (V_\gamma[\dot{G}], V[G]) \) has the \( \kappa \)-approximation property. Therefore, \( M[G] \) is a \( \kappa^+ \)-guessing model, by Proposition 2.12. Note that \( |M[G]| = \kappa^+ \) and \( \lambda = (\kappa^+)V[G] \).

To finish the proof, it remains to show that every \( M[G] \in \mathcal{G} \) satisfies the properties listed in Definition 2.13. Observe that \( \kappa \) is a regular cardinal in \( V[G] \). Let \( \delta \geq \gamma \) be the least ordinal with \( V_\delta \in M[G] \). A standard density argument shows that
\[
V_\gamma = \bigcup \{ N \cap V_\gamma : \gamma, \lambda \in \mathcal{S} \text{ and } N \cap V_\delta \in M[G] \}.
\]

Let us denote the right-hand side of the above equality by \( X \), which is a (transitive) \( \in \)-chain of \( < \kappa \)-closed models of size \( \kappa \). Note that in \( V[G] \), we have \( |V_\gamma| = \kappa^+ \). Let us enumerate \( X \) as an \( \in \)-increasing sequence
\[
\bar{X} := (M_\alpha : \alpha < \kappa^+).
\]

Thus \( V_\gamma[\dot{G}] = \bigcup_{\alpha < \kappa^+} M_\alpha[\dot{G}] \). It is straightforward to see that \( \bar{X}^+ := (j^+(M_\alpha[\dot{G}]) : \alpha < \kappa^+) \) is an \( \in \)-increasing sequence of \( < \kappa \)-closed and \( \kappa \)-sized elementary submodels of \( M[G] = \bigcup_{\alpha < \kappa^+} j^+(M_\alpha[\dot{G}]) \).

To prove that \( \bar{X}^+ \) is \( \subseteq \)-continuous at ordinals of cofinality \( \kappa \), it is sufficient to prove that so is the original sequence \( \bar{X} \). This is where we use decorations. The proof is similar to Neeman’s related arguments in [14]. Thus let \( \check{X} \) be a \( \mathbb{P} \)-name for \( X \). Suppose that \( \alpha \) is of cofinality \( \kappa \). There is \( p \in G \) forcing \( \check{M}_\alpha \) to be \( \alpha \)-th element of \( \check{X} \). We may extend \( p \) further so that \( M_\alpha \in M_p \). Thus we may assume that \( M_\alpha \in M_p \). Fix \( x \in M_\alpha \). Since \( |p| < \kappa \), we can find \( \beta < \alpha \) with \( p \cap M_\alpha \in M_\beta \). Then \( (p \cap M_\alpha)^{M_\beta} \in M_\alpha \cap G \). Thus, without loss of generality, we may assume that \( M_p \cap M \) has a last node, say \( P \). Now, let \( q = (M_p, d_q) \), where \( d_q \) is the same as \( d_p \) possibly except at \( P \), where we

\(^4\)A forcing has a \( \kappa \)-covering property, if every set of size less than \( \kappa \) in the generic extension is a subset of a ground model set of size less than \( \kappa \)
let $d_q(P) = d_p(P) \cup \{x\}$. Thus $q$ is a conditions and $q \leq p$. It is now easy to see that $q$ forces $x$ to belong to some model below $\hat{X}(\alpha)$. Therefore, $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$.

**Proposition 4.5.** $\mathbb{P}$ forces that there is a densely $\kappa$-closed ideal on $\kappa^{++}$.

**Proof.** Fix a $\kappa$-complete ultrafilter $U$ on $\kappa$ and let $j : V \to \mathcal{M} \cong \text{Ult}(V, U)$ be the ultrapower embedding with $\mathcal{M}$ a transitive inner model of $V$.

Then

$$\mathcal{M} \models \text{"}j(\mathbb{P}) = \mathbb{P}^{\text{dec}, \lambda, j(S), j(T), j(\lambda)} = \mathbb{P}^{\text{dec}, \kappa, \mathcal{S}, \mathcal{T}, j(\lambda)}\text{"}.$$ 

Thus $\mathcal{M}$ thinks that $j(\mathbb{P})$ is Neeman forcing based on $S_{j(\lambda)} \cup T_{j(\lambda)}$ with decorated conditions of size less than $\kappa$. Also, in $V$, $j(\mathbb{P})$ is a Neeman-type forcing, since $V$ thinks that

$$j(\mathbb{P}) = \mathbb{P}^{\text{dec}, \mathcal{S}, \mathcal{T}, \check{K}}.$$

where $K = V_{j(\lambda)} \cap \mathcal{M}$, which is definable in $V$.

$$\mathcal{S}^* = \{M \in K : M \prec K, M^{<\kappa} \subseteq M, \text{ and } |M| = \kappa\}, \text{ and}$$

$$\mathcal{T}^* = \{V_\alpha \cap K : V_\alpha \cap K \prec K \text{ and } \text{cof}(\alpha) \geq \kappa^+\}.$$ 

We have also $V_\lambda \subseteq \mathcal{M}$ and $V_\lambda \prec \mathcal{M} \cap V_{j(\lambda)} = K$. Thus $V_\lambda \in \mathcal{T}^*$, which in turn implies that $\mathcal{T} \subseteq \mathcal{T}^*$. It is easily seen that $\mathbb{P}$ is a regular suborder of $j(\mathbb{P})$, as $V_\lambda \in \mathcal{T}^*$ and $j|\mathbb{P} = \mathbb{P}$. Note that $\mathcal{M}$ is closed under $\lambda$-sequences. Therefore, $j(\mathbb{P})$ is a $<\kappa$-closed forcing in $V$. Assume that $G \subseteq \mathbb{P}$ is a $V$-generic filter. It is easy to see that the quotient $j(\mathbb{P})/G$ is $<\kappa$-closed. Let $\mathcal{I}$ be the dual ideal of $\mathcal{U}$, and let $\bar{\mathcal{I}}$ be the ideal generated by $\mathcal{I}$ in $V[G]$. By Lemma 2.18, $\bar{\mathcal{I}}$ is a densely $\kappa$-closed ideal on $\lambda = \kappa^{++}$.

Notice that in Proposition 4.5, it suffices to assume $\lambda$ is measurable.

## 5 The main theorems

**Theorem 5.1.** Assume $\kappa$ is an infinite regular cardinal below a supercompact cardinal. Then, in a forcing extension, $\text{GM}^*(\kappa^{++})$ holds and there are densely $\kappa$-closed ideals on $\kappa^{++}$.

**Proof.** Let $\mu > \kappa$ be a supercompact cardinal. Using a small forcing, we can assume, without loss of generality, that $\kappa^{<\kappa} = \kappa$ holds in the ground model. Let $\mathbb{P} := \mathbb{P}_\mu^{\kappa}$ be Neeman forcing. By Proposition 4.2, $\text{GM}^*(\kappa^{++})$ holds in $V[\mathbb{P}]$, and by Proposition 4.5, there is a densely $\kappa$-closed ideal on $\kappa^{++}$ in $V[\mathbb{P}]$.

The following generalises our main forcing construction and theorem in [13].
**Theorem 5.2.** Let $\kappa$ be an infinite cardinal with $\kappa^{<\kappa} = \kappa$. Assume that $\text{GM}^\kappa(\kappa^{++})$ holds, and that there are densely $\kappa$-closed ideals on $\kappa^{++}$. Let $T$ be a tree of height $\kappa^{++}$ without cofinal branches. Then there is a $<\kappa$-closed and $\kappa^+$, $\kappa^{++}$-preserving forcing notion $\mathbb{P}_T$ with the $\kappa^+$-approximation property such that $\mathbb{P}_T$ specialises $T$.

**Proof.** By Lemma 2.1, we may assume that $T$ is a Hausdorff tree. Pick a sufficiently large regular cardinal $\theta$ with $\mathcal{P}(T) \in H_\theta$. The main forcing is defined as in [13]. Let $\mathcal{E}^0$ be the set of $<\kappa$-closed and $\kappa$-sized elementary submodels $M$ of $(H_\theta, \in, T)$ with $M \cap \kappa^+ \in \kappa^+$. Note that $\mathcal{E}^0$ is stationary in $\mathcal{P}_{\kappa^+}(H_\theta)$, as $\kappa^{<\kappa} = \kappa$. Let also

$$\mathcal{E}^1 = \{ M \in \mathcal{G}^*(\kappa^+, H_\theta) : T \in M \},$$

which is stationary by $\text{GM}^\kappa(\kappa^{++})$. Observe that if $M \in \mathcal{E}^0$ and $N \in \mathcal{E}^1 \cap M$, then $N \cap M \in \mathcal{E}^0$.

Let $M := M(\mathcal{E}^0, \mathcal{E}^1)$ consist of $p = \mathcal{M}_p$ such that $\mathcal{M}_p \subseteq \mathcal{E}^0 \cup \mathcal{E}^1$ is of size $<\kappa$ and is closed under intersections. The order on $M$ is the reversed inclusion. This forcing is a variant of Neeman forcing we used earlier. This presentation is due to Veličković [16]. Although Veličković stated the lemmas for $\kappa = \omega$, the same proofs work here due to our cardinal arithmetic assumption. Let us review the most important features of this forcing. For every $M \in \mathcal{E}^0 \cup \mathcal{E}^1$ and every $p \in M \cap M$, one can set $p^M$ to be the closure of $\mathcal{M}_p \cup \{ M \}$ under intersections. Then $p^M$ is a condition in $M$ and $p^M \leq p$. Also, if $p \in M$ and $M \in \mathcal{M}_p$, then $p \upharpoonright M := \mathcal{M}_p \cap M$ is a condition and $p \leq p \upharpoonright M$.

It is also proved that if $q \in M$ extends $p \upharpoonright M$, then, letting $p \land q$ be the closure of $\mathcal{M}_p \cup \mathcal{M}_q$ under intersections, $p \land q$ is a condition which extends both $p$ and $q$.

**The forcing construction**

As in [13], let us abuse language and say a node $t \in T$ is guessed in $M \in \mathcal{E}^0 \cup \mathcal{E}^1$ if there is a branch $b \in M$ through $T$ such that $t \in b$.

**Definition 5.3 ($\mathbb{P}_T$).** A condition in $\mathbb{P}_T$ is a pair $p = (\mathcal{M}_p, f_p)$ satisfying the following items.

1. $\mathcal{M}_p \in M$.
2. $f_p \in S_\kappa(T, \kappa^+)$.
3. For every $M \in \mathcal{E}^0 \cap \mathcal{M}_p$, if $t \in \text{dom}(f_p) \cap M$, then $f_p(t) \in M$.
4. For every $M \in \mathcal{E}^0 \cap \mathcal{M}_p$ and every $t \in \text{dom}(f_p)$ with $f_p(t) \in M$, if $t$ is guessed in $M$, then $t \in M$.

We say $p$ is stronger than $q$ if and only if $\mathcal{M}_p \supseteq \mathcal{M}_q$ and $f_p \supseteq f_q$.

It is clear that $\mathbb{P}_T$ is $<\kappa$-closed. It follows that $\mathbb{P}_T$ is proper for $\mathcal{E}^0$ and $\mathcal{E}^1$, and hence it preserves both $\kappa^+$ and $\kappa^{++}$, since $\mathcal{E}^0$ and $\mathcal{E}^1$ are stationary. The complicated part is the $\mathcal{E}^0$-properness and $\kappa^+$-approximation property of $\mathbb{P}_T$. However, these two properties were proved similarly. The crucial part is the proof of [13, Proposition 4.35] which is more or less the same as Proposition 3.4
in this paper. The new thing here is that one needs to use positive sets in a densely $\kappa$-closed ideal on $\kappa^{++}$ while in [13], we used the cofinal subsets of $(P_{\omega_1}(T), \subseteq)$. Nevertheless, with a slight modification, one can use a densely $\kappa$-closed ideal on $\kappa^{++}$.

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