WHAT DOES THE AUTOMORPHISM GROUP OF A FREE ABELIAN GROUP A KNOW ABOUT A?

VLADIMIR TOLSTYKH

Abstract. Let $A$ be an infinitely generated free abelian group. We prove that the automorphism group $\text{Aut}(A)$ first-order interprets the full second-order theory of the set $|A|$ with no structure. In particular, this implies that the automorphism groups of two infinitely generated free abelian groups $A_1, A_2$ are elementarily equivalent if and only if the sets $|A_1|, |A_2|$ are second-order equivalent.

Introduction

In his paper [6] of 1976 Shelah proved that the elementary theories of the endomorphism semi-groups of free algebras of ‘large’ infinite ranks had very strong expressive power. More precisely, let $V$ be an arbitrary variety of algebras and $F_{\kappa}(V)$ be a free algebra from $V$ with $\kappa \geq \aleph_0$ free generators. Then the endomorphism semi-group $\text{End}(F_{\kappa}(V))$ first-order interprets the full-second theory $\text{Th}_{2}(\kappa)$ of the cardinal $\kappa$ (viewed as a set with no structure), provided that $\kappa$ is greater than the cardinality of the language of $V$.

That remarkable result naturally leads to the following problem: what are the varieties of algebras for which the automorphism groups of free algebras are logically strong in a similar sense? Shelah himself formulated this problem in the cited paper [6] and then after more than 20 years mentioned it again in his survey [7]: Problem 3.14 from [7] suggested to classify the varieties of algebras $V$ such that the automorphism groups $\text{Aut}(F_{\kappa}(V))$ first-order interpret the theory $\text{Th}_{2}(\kappa)$ for all (or all sufficiently large) infinite cardinals $\kappa$.

The results on symmetric groups obtained by Shelah before the publication of the paper [6] implied that, for instance, the variety of all sets with no structure and the variety of all semi-groups were the examples of, say, ‘negative’ kind. Indeed, according to [5], the symmetric group of an infinite cardinal $\kappa$, in other words, the automorphism group of the set $\kappa$ with no structure, first-order interprets the theory $\text{Th}_{2}(\kappa)$ only if the cardinal $\kappa$ is ‘small’ (namely, at most $2^{\aleph_0}$).

The author found in [8]—as a byproduct of his study of the elementary types of infinite-dimensional classical groups—that for any variety of vector spaces the automorphism groups of free algebras are as logically strong as the endomorphism semi-groups. A bit informally, one of the results from [8] can be quoted in the following form: if $\kappa$ is an infinite cardinal, then the general linear group $\text{GL}(\kappa, D)$...
over a division ring $D$ first-order interprets $\text{Th}_2(\kappa)$, provided that $\kappa > |D|$. Thus varieties of vector spaces give examples of ‘positive’ kind as to Shelah’s problem.

In the papers \cite{9} and \cite{10} the author studied Shelah’s problem for classical group varieties. It turned out that the variety of all groups and any variety $\mathfrak{V}$ of nilpotent groups of class $c \geq 2$ meet the requirements of Shelah’s problem: if $F$ is an infinitely generated free or free nilpotent group, then the group $\text{Aut}(F)$ first-order interprets the theory $\text{Th}_2(|F|) (= \text{Th}_2(\text{rank } F))$. In the present paper we examine the case of the variety of all abelian groups. The main result of the paper states that the variety in question also meets requirements of Shelah’s problem.

Let $A$ denote an infinitely generated free abelian group; clearly, $A$ can be considered as a free $\mathbb{Z}$-module. One of the standard approaches to understanding of the nature of the automorphism groups of modules is an investigation of possibility of generalization for these groups of the methods developed for general linear groups, the automorphism groups of vector spaces. In the first section of the paper we, like in \cite{8}, work to reconstruct by means of first-order logic in $\text{Aut}(A)$ some geometry of the $\mathbb{Z}$-module $A$. Namely, we interpret in $\text{Aut}(A)$ the family $D^1(A)$ consisting of all direct summands of $A$ having rank or corank one. To make comparison, the first-order interpretation in the general linear group $\text{GL}(V)$ of an infinite-dimensional vector space $V$ of the family of all lines and hyperplanes of $V$ done in \cite{8} is much longer. However, both interpretations have much in common and both originated from the well-known works on classical groups.

In principle, the reconstruction of $D^1(A)$ can be extended to the reconstruction in $\text{Aut}(A)$ of the family $D(A)$ of all direct summands of $A$ followed by the first-order interpretation in the structure $(\text{Aut}(A), D(A))$ of the endomorphism semigroup $\text{End}(A)$ of $A$ (similarly to \cite{8}). We, however, prefer a shorter way, making in Section 2 an effort to reconstruct in $\text{Aut}(A)$ the general linear group of some vector space of dimension $|A|$. Namely, using the action of $\text{Aut}(A)$ on $D^1(A)$ we prove $\emptyset$-definability in $\text{Aut}(A)$ of the principal congruence subgroup $\Gamma_2(A)$ of level two. The quotient subgroup $\text{Aut}(A)/\Gamma_2(A)$ is isomorphic to the general linear group of the vector space $A/2A$ over the field $\mathbb{Z}_2$. Thus the group $\text{Aut}(A)$ first-order interprets the group $\text{GL}(|A|, \mathbb{Z}_2)$. The latter group, as it has been said above, first-order interprets the theory $\text{Th}_2(|A|)$. As a consequence, we have that the automorphism groups $\text{Aut}(A_1)$ and $\text{Aut}(A_2)$, where $A_1, A_2$ are infinitely generated free abelian groups, are elementarily equivalent if and only the cardinals $|A_1|$ and $|A_2|$ are second-order equivalent as sets.

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1. Definable geometric properties of automorphisms

Let $A$ denote a free abelian group of infinite rank. As it has been said in the Introduction, our aim in this section is a first-order reconstruction in $\text{Aut}(A)$ of the family of direct summands of $A$ of rank or corank one (we say that a direct summand $B$ of $A$ has corank $m$, if any direct complement of $B$ to $A$ is of rank $m$.)

We shall essentially exploit the structure of involutions (the elements of the order two) in the group $\text{Aut}(A)$ given by the following theorem.

**Theorem 1.1.** Let $G$ be a free abelian group. Every involution $\varphi \in \text{Aut}(G)$ has a basis $\mathcal{B}$ of $G$ such that for any $b \in \mathcal{B}$ either $\varphi b = \pm b$, or $\varphi b \in \mathcal{B}$.
The theorem was first established for the groups of finite rank by Hua and Reiner [3, Lemma 1]; in general, the result is proven in [11]. Let us call a basis of \( A \) on which \( \varphi \) acts in a way described in the Theorem a canonical basis for \( \varphi \).

Let \( 2A \) denote the group of even elements of \( A \):
\[
2A = \{2a : a \in A\}.
\]

The natural homomorphism \( A \to A/2A \) induces the homomorphism of the automorphism groups \( \text{Aut}(A) \to \text{Aut}(A/2A) \) which we will denote by \( \hat{\to} \). The fact that the group \( A/2A \) can be viewed as a vector space over \( \mathbb{Z}_2 \) will be extensively used in this paper.

**Remark 1.2.** Take an involution \( \varphi \in \text{Aut}(A) \) and some its canonical basis \( B \). The (cardinal) number \( p(B) \) of unordered pairs \( \{b, \varphi b\} \), where \( b \in B \) and \( \varphi b \neq \pm b \) is an invariant of \( \varphi \). Indeed, \( p(B) \) equals the residue of the induced linear transformation \( \hat{\varphi} \) of the vector space \( A/2A \) over \( \mathbb{Z}_2 \):
\[
p(B) = \text{res}(\hat{\varphi}) = \dim \text{Res}(\hat{\varphi}).
\]

(here \( \text{Res}(\hat{\varphi}) \) is the image of the linear transformation \( 1 - \hat{\varphi} \), see [4]). This implies that if \( (\varphi_1, B_1), (\varphi_2, B_2) \) are pairs similar to the pair \( (\varphi, B) \) and \( \varphi_1, \varphi_2 \) are conjugate in \( \text{Aut}(A) \), then \( p(B_1) = p(B_2) \).

Let \( \varphi \) be an involution in \( \text{Aut}(A) \); we let \( A_{\varphi}^+ \) and \( A_{\varphi}^- \) denote the subgroups
\[
\{a : \varphi a = a\} \quad \text{and} \quad \{a : \varphi a = -a\}
\]
respectively; clearly, \( \varphi \) is diagonalizable if and only
\[
A = A_{\varphi}^+ \oplus A_{\varphi}^-.
\]

It is helpful to remember that two diagonalizable involutions from \( \text{Aut}(A) \) are commuting if and only if there is a basis of \( A \) in which they both diagonalizable.

We shall call a diagonalizable involuton \( \varphi \) a \( \gamma \)-involution, where \( \gamma \) is a cardinal, if
\[
\gamma = \text{rank} A_{\varphi}^- < \text{rank} A_{\varphi}^+.
\]

1-involutions, like in linear group theory, will be called extremal involutions.

A number of facts on definability of certain families of involutions in the automorphism groups of infinitely generated free abelian groups has been proved implicitly in the author’s paper [11]. Because of that we shall give only sketches of proofs for the next two statements, Lemma 1.3 and Lemma 1.4; the reader is referred to the proof of Proposition 2.4 in [11] to find there the omitted details.

For an involution \( \varphi \) in the group \( \text{Aut}(A) \) we shall denote by \( K(\varphi) \) the conjugacy class of \( \varphi \) in \( \text{Aut}(A) \). The set \( K^2(\varphi) = K(\varphi)K(\varphi) \) is the family of all products \( \varphi_1\varphi_2 \), where \( \varphi_1, \varphi_2 \in K(\varphi) \).

**Lemma 1.3.** The family of all diagonalizable involutions is \( \emptyset \)-definable in \( \text{Aut}(A) \).

**Proof.** We claim that \( \varphi \) is diagonalizable if and only if the set \( K^2(\varphi) \) contains no elements of order three.

Using Theorem 1.3 one checks that the diagonalizable involutions are exactly involutions in the kernel of the homomorphism \( \hat{\to} : \text{Aut}(A) \to \text{Aut}(A/2A) \). On the other hand, the images under \( \hat{\to} \) of all elements of order three from \( \text{Aut}(A) \) are non-trivial. This implies that if \( \varphi \) is diagonalizable, then there are no elements of order three in \( K^2(\varphi) \).
Conversely, for any non-diagonalizable involution $\psi \in \text{Aut}(A)$ we can easily find a conjugate $\psi'$ of $\psi$ such that the automorphism $\psi\psi'$ is of order three. \qed

**Lemma 1.4.** The families of extremal involutions (1-involutions), 2-involutions and 4-involutions are all $\emptyset$-definable in $\text{Aut}(A)$.

**Proof.** A diagonalizable involution $\varphi$ is an extremal involution if and only if all involutions in $K^2(\varphi)$ are conjugate and $\varphi$ is not a square in $\text{Aut}(A)$.

Indeed, if $\varphi$ is an extremal involution, then the only involutions in the set $K^2(\varphi)$ are 2-involutions. In particular, all involutions in $K^2(\varphi)$ are conjugate. Applying Theorem 1.1 we can demonstrate that the latter property holds also only for diagonalizable involutions $\rho$ such that

$$\text{rank } A^+_\rho = 1.$$  

But any such an involution is a square in $\text{Aut}(A)$, whereas any 1-involution is not.

The 2-involutions are the only involutions from $K^2(\varphi)$, where $\varphi$ is an arbitrary 1-involution. Let $\theta$ be a 2-involution. Then 4-involutions are those involutions in $K^2(\theta)$ that are not conjugate to $\theta$. \qed

We need also a family of non-diagonalizable involutions $\{\pi\}$ whose elements satisfy the condition

$$(1.1) \quad \text{rank } A^+_\pi = 1 \text{ or } \text{rank } A^-_\pi = 1.$$  

For any canonical basis $B$ for a non-diagonalizable involution $\pi$ with (1.1) we have that

(a) $B$ contains exactly one pair of distinct elements, say, $b, c$ taken by $\pi$ to one another (Remark 1.2);

(b) $\pi$ either inverts all elements in $B \setminus \{b, c\}$, or fixes all these elements (otherwise, both subgroups $A^+_\pi$ and $A^-_\pi$ were of rank $> 1$).

Thus either $\pi \sim \pi'$, or $\pi \sim -\pi'$ for every pair of non-diagonalizable involutions $\pi, \pi'$ with (1.1), where $\sim$ denotes the conjugacy relation. Keeping in mind (a), we shall call non-diagonalizable involutions with (1.1) by 1-permutations.

**Lemma 1.5.** The following statements are equivalent:

(i) $\pi$ is a 1-permutation;

(ii) $\pi$ is not diagonalizable and the set $K^2(\pi)$ contains no 4-involutions.

In particular, the family of 1-permutations is $\emptyset$-definable in $\text{Aut}(A)$.

**Proof.** Let $\pi$ be a non-diagonalizable involution, which is not a 1-permutation, and let $B$ be a canonical basis for $\pi$. One then can readily find $\pi'$, a conjugate of $\pi$, whose product with $\pi$ is a 4-involution. Indeed, suppose first that $p(B) > 1$ (the notation was introduced in Remark 1.2). Then $B$ contains distinct elements $b_1, b_2, b_3, b_4$ such that $\pi b_1 = b_2$ and $\pi b_3 = b_4$.

The second case is the case when $p(B) = 1$. Here $\pi b_1 = b_2$ for some distinct $b_1, b_2 \in B$ and, since $\pi$ is not a 1-permutation, two such elements $b_3$ and $b_4$ can be found in $B$ that

$$\pi b_3 = b_3 \text{ and } \pi b_4 = -b_4.$$  

Then, for both of the cases under consideration, we construct $\pi'$ as follows: $\pi' b_i = -\pi b_i$ for $i = 1, \ldots, 4$ and $\pi' b = \pi b$ for all $b \in B \setminus \{b_1, b_2, b_3, b_4\}$. 

Conversely, suppose \( \pi \) is a 1-permutation. We may assume that \( \text{rank } A_\pi^- = 1 \). Let \( \pi_1, \pi_2 \) be conjugates of \( \pi \). Then \( \text{Im}(1 - \pi_1) \) and \( \text{Im}(1 - \pi_2) \) are subgroups of rank 1. Since
\[
1 - \pi_1 \pi_2 = (1 - \pi_2) + (1 - \pi_1) \pi_2,
\]
we have
\[
\text{Im}(1 - \pi_1 \pi_2) \subseteq \text{Im}(1 - \pi_1) + \text{Im}(1 - \pi_2),
\]
and so \( \text{rank } \text{Im}(1 - \pi_1 \pi_2) \leq 2 \). Then \( \pi_1 \pi_2 \) is not a 4-involution because for any 4-involution \( \psi \) we have \( \text{rank } \text{Im}(1 - \psi) = 4 \).

Until the end of this section we fix some 2-involution \( \theta^* \). In order to mark somehow one special type of commutativity with \( \theta^* \), we say that an extremal involution \( \psi \) (resp. a 1-permutation \( \psi \)) commutes with \( \theta^* \) properly, if \( \psi \sim \theta^* \psi \).

We fix also an extremal involution \( \varphi^* \) and a 1-permutation \( \pi^* \) both properly commuting with \( \theta^* \) such that
\[
(\pi^* \varphi^*)^2 = \theta^*.
\]
Let \( B \) denote the subgroup \( A_\theta^- \). Since both \( \varphi^* \) and \( \pi^* \) commute with \( \theta^* \), they both preserve \( B \):
\[
\varphi^* B = \pi^* B = B.
\]
Since, further, \( \varphi^* \) and \( \pi^* \) commute with \( \theta^* \) properly, their restrictions to \( B \) are an extremal involution and a 1-permutation of \( \text{Aut}(B) \), respectively. Let
\[
f^* = \varphi^*|_B \text{ and } p^* = \pi^*|_B.
\]
We have that
\[
f^* p^* f^* p^* = -\text{id}_B
\]
and then \( p^* f^* p^* = -f^* \). This implies that \( p^* \) takes to each other the subgroups \( A^+_f \) and \( A^-_f \):
\[
p^* A^+_f = A^-_f.
\]

If then \( e_1 \) is a basis element of \( A^+_f \), then \( e_2 = p^* e_1 \) is a basis element of \( A^-_f \). Summing up, we see that in the basis \( \{e_1, e_2\} \) of \( B \) the automorphisms \( f^* \) and \( p^* \) have the matrices
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
respectively.

Next is the proof of \( \emptyset \)-definability of certain transvections. Recall that a unimodular element of \( A \) (a primitive element in a more general context) is an element of \( A \) that can be included in some basis of \( A \). Let \( \delta : A \to \mathbb{Z} \) be a non-zero homomorphism of abelian groups; in this case the \( \ker \delta \) is a direct summand of \( A \) of rank 1. Fix a unimodular element \( x \) in \( \ker \delta \). Then the mapping \( \tau a = a + \delta(a)x \) is an automorphism of \( A \), which is called a transvection. If \( \tau \) is a transvection determined by a homomorphism \( \delta \), then one may correctly associate with \( \tau \) a natural number defining it via
\[
m(\tau) = |\delta(y)|
\]
where \( y \in A \) satisfies \( A = \langle y \rangle \oplus \ker \delta \). It can be easily seen that for every pair \( \tau_1, \tau_2 \) of transvections \( m(\tau_1) = m(\tau_2) \) if and only if \( \tau_1 \) and \( \tau_2 \) are conjugate. We shall call a transvection \( \tau \) an \( m \)-transvection, if \( m(\tau) = m \).
Lemma 1.6. (i) Among the conjugates $\rho$ of $\pi^*$ properly commuting with $\theta^*$ there are exactly four ones different from $\pi^*$ that satisfy the equation

$$(\pi^*\rho)^3 = \text{id}_A;$$

(ii) The automorphisms $(\varphi^*\rho)^2$, where $\rho$ is any of 1-permutations described in (i), are all $2$-transvections.

Proof. Let $\rho$ be a 1-permutation satisfying the conditions from (i). First note that due to the proper commutativity with $\theta^*$, the restriction of $\rho$ to $A_{\theta^*}$ must be equal to that one of $\pi^*$. We denote by $R$ the matrix of the restriction of $\rho$ on $B = A_{\theta^*}$ in the above described basis $\{e_1, e_2\}$.

Since the condition $(\pi^*\rho)^3 = \text{id}$ can be rewritten as

$$\pi^*\rho\pi = \rho\pi\rho,$$

we have

$$R = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $a, b, c \in \mathbb{Z}$ (the trace of $R$ should be equal to zero like the trace of any non-central involution in $\text{GL}(2, \mathbb{Z})$, Theorem [1.1]). It follows from (1.2) that

$$-a = a(b + c),$$

$$c = b^2 - a^2,$$

$$b = c^2 - a^2.$$

According to (1.3), there are two cases for study: $a = 0$ and $b + c + 1 = 0$.

In the first case we have that $b = c = 1$ and then $\rho = \pi^*$, which is impossible.

The second case: we use the condition $\det R = -1$ ($\rho$ is a conjugate of $\pi^*$). Then

$$\det R = -1 = -a^2 - bc = -a^2 - b(-b - 1)$$

or

$$a^2 = b^2 + b + 1.$$

The only $b \in \mathbb{Z}$ for which the number $b^2 + b + 1$ is a square are $b = 0, -1$.

Thus, there are indeed at most four possibilities for $R$:

$$R = \begin{pmatrix} e & 0 \\ -1 & -e \end{pmatrix}, \begin{pmatrix} e & -1 \\ 0 & -e \end{pmatrix},$$

where $e = \pm 1$. One easily verifies that for all four 1-permutations $\rho$ that correspond to the matrices in (1.4) and such that $\pi^*c = \rho c$ for all $c \in A_{\theta^*}$, the conditions from (i) of the Lemma are true.

The statement in (ii) is now a consequence of the following observations:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 2e & 0 \\ 0 & e^2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} e^2 & 2e \\ 0 & e^2 \end{pmatrix},$$

where $e = \pm 1$. □
Lemma 1.7. The family of all $2m$-tranvections (where $m$ runs over $\mathbb{N}$) is $\emptyset$-definable in $\text{Aut}(A)$.

Proof. We shall continue to use the parameters picked up above. One more parameter will be serviceable, however: a 2-transvection $\tau^*$, one of the four 2-transvections described in Lemma 1.6 (ii).

Let us consider the set $S$ of automorphisms $\{\varphi^*\rho\}$, where $\rho$ is an extremal involution or a 1-permutation properly commuting with $\theta^*$. If the matrix of the restriction of $\tau^*$ on $B$ is, for instance,

$$
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
$$

then only those elements from $S$ commute with $\tau^*$ whose restrictions on $B$ have matrices

$$(1.5)~
\begin{pmatrix}
e & b \\
0 & e
\end{pmatrix}
$$

where $e = \pm 1$ and $b \in \mathbb{Z}$. The squares of the matrices of the form $(1.5)$ are matrices

$$
\begin{pmatrix}
1 & 2b \\
0 & 1
\end{pmatrix}.
$$

Thus the set consisting of squares of elements of $S$ is a set that for each natural number $m$ contains a $2m$-transvection. This implies that a suitably chosen existential formula defines the $2m$-transvections. $\Box$

Lemma 1.8. Two distinct extremal involutions $\varphi_1, \varphi_2$ have the mutual (eigen) subgroup, that is, either

$$
A^+_{\varphi_1} = A^+_\varphi_2, \text{ or } A^-_{\varphi_1} = A^-_{\varphi_2},
$$

if and only if the product $\varphi_2\varphi_1$ is a $2m$-transvection for some non-zero natural $m$.

Proof. Assume that subgroups $A^-_{\varphi_1}$ and $A^-_{\varphi_2}$ are generated by unimodular elements $x_1$ and $x_2$ respectively, and write $B_1$ and $B_2$ for $A^+_{\varphi_1}$ and $A^+_{\varphi_2}$. Let also $\tau$ denote the product $\varphi_2\varphi_1$.

(\Leftarrow). Suppose $B_1 \neq B_2$. Then the intersection $B_1 \cap B_2$ is of corank 2. The fixed-point subgroup $C$ of $\tau$ has corank 1 and contains (a direct summand of $A$) $B_1 \cap B_2$; then there is a unimodular element $y \in A$ such that

$$
C = \langle y \rangle \oplus (B_1 \cap B_2).
$$

We have $\varphi_2 \varphi_1 y = y$ and then

$$\varphi_1 y - y = \varphi_2 y - y.$$

The above element is non-zero, since otherwise $y \in B_1 \cap B_2$. Thus $\langle x_1 \rangle \cap \langle x_2 \rangle \neq 0$, or $\langle x_1 \rangle = \langle x_2 \rangle$, since both $x_1, x_2$ are unimodular.

(\Rightarrow). (i) Suppose that $B_1 \neq B_2$, but $\langle x_1 \rangle = \langle x_2 \rangle$. Since $B_1 \cap B_2$ is a direct summand of $A$ of corank 2, then for some unimodular $z$

$$
B_1 = \langle z \rangle \oplus (B_1 \cap B_2);
$$

the element $z$ can be expressed as $mx_2 + b_2$, where $m \in \mathbb{Z}$ and $b_2 \in B_2$. We then have

$$
\tau z = \varphi_2 \varphi_1 z = \varphi_2(mx_2 + b_2) = -mx_2 + b_2 = mx_2 + b_2 - 2mx_2 = z - 2mx_2.
$$

Taking into account that $\tau x_2 = x_2$, we see that $\tau$ is a $2m$-transvection.
(ii) Suppose that $B = B_1 = B_2$ and $\langle x_1 \rangle \neq \langle x_2 \rangle$. The element $x_1$ can be then written as

$$x_1 = ex_2 + b = ex_2 + mc,$$

where $b = mc$ is an element of $B$ and $c$ is a unimodular. Hence

$$\tau x_1 = \varphi_2\varphi_1 x_1 = \varphi_2(-ex_2 - mc) = ex_2 - mc = x_1 - 2mc$$

and $\tau$ is a $2m$-transvection. □

**Proposition 1.9.** Let $D^1(A)$ be the family of all direct summands of $A$ having rank or corank one. Then the action of the group $\text{Aut}(A)$ on the family $D^1(A)$ is first-interpretable in $\text{Aut}(A)$ without parameters.

**Proof.** In view of Lemma 1.4, Lemma 1.7 and Lemma 1.8 all we have to do is to explain when two pairs of extremal involutions $(\varphi_1, \varphi_2)$ and $(\psi_1, \psi_2)$ both having mutual subgroups determine the same direct summand of $A$. It is easy: we just say that for all $i, j$ either $\varphi_i = \psi_j$, or $\varphi_i \psi_j$ is a $2m$-transvection. □

In the conclusion of the section we present a purely algebraic observation due to Oleg Belegradek who had found it while reading the first draft of the paper.

**Proposition 1.10.** Let $A_1, A_2$ be infinitely generated free abelian groups. The groups $\text{Aut}(A_1)$ and $\text{Aut}(A_2)$ are isomorphic if and only if the cardinals rank $A_1$ and rank $A_2$ are equal.

**Proof.** Let $A$ be an infinitely generated free abelian group. It is easy to show that the cardinality of any maximal family of pairwise commuting 1-involutions in $\text{Aut}(A)$ is equal to rank of $A$. Since, by Lemma 1.4, the 1-involutions are $\varnothing$-definable in $\text{Aut}(A)$ uniformly in $A$, and isomorphisms preserve first-order formulae, the result follows. □

### 2. Definability of the Congruence Subgroup of Level Two

Let $m > 1$ be a natural number. Write $\Gamma_m(A)$ for the subgroup of $\text{Aut}(A)$ consisting of the automorphisms of $A$ that act trivially (in the natural way) on the group $A/mA$. The subgroups $\Gamma_m(A)$ are natural analogues of the principal congruence subgroups of the groups $\text{SL}(n, \mathbb{Z})$.

We are going to prove $\varnothing$-definability of the subgroup $\Gamma_2(A)$, the principal congruence subgroup of $\text{Aut}(A)$ of level two. As it has been said in the Introduction this will imply a possibility of first-order interpretation in $\text{Aut}(A)$ of the general linear group of the vector space $A/2A$ over the field $\mathbb{Z}_2$.

**Theorem 2.1.** The subgroup $\Gamma_2(A)$ is $\varnothing$-definable in $\text{Aut}(A)$.

**Proof.** We shall use properties of the group $\text{SL}(3, \mathbb{Z})$ and with this idea in mind we are going to fix somehow some three direct summands of rank one in $A$. To achieve that we use certain definable parameters. First, we take three pairwise commuting extremal involutions $\varphi^*_1, \varphi^*_2, \varphi^*_3$ in $\text{Aut}(A)$ such that any product $\varphi^*_i \varphi^*_j$, where $i \neq j$ is a 2-involution. There exists a basis $B$ of $A$ in which $\varphi^*_1, \varphi^*_2, \varphi^*_3$ are all diagonalizable. Let $e_i$ denote the element of $B$ that $\varphi^*_i$ ($i = 1, 2, 3$) sends to the opposite.

Second, we need two 1-permutations $\pi^*_1$ and $\pi^*_2$ to provide a suitable action on $\{e_1, e_2, e_3\}$; our requirements on $\pi^*_1$ and $\pi^*_2$ are therefore as follows:

(i) $\pi^*_1 \varphi^*_1 \pi^*_1 = \varphi^*_2$ and $\pi^*_1$ commutes with $\varphi^*_2$.
(ii) $\pi_2^\ast \pi_3^\ast = \varphi_3^\ast$ and $\pi_1^\ast$ commutes with $\varphi_3^\ast$;
(iii) $\pi_1^\ast$ and $\pi_2^\ast$ are conjugate and their product is of order three.

In the following statement we simultaneously introduce and characterize some transvections we are going to deal with.

Claim 1. The elementary transvections which act trivially on $B \setminus \{e_1, e_2, e_3\}$ and whose matrices in $\{e_1, e_2, e_3\}$ (more precisely, matrices of the corresponding restrictions) are of the form $E + nE_{ij}$, where $1 \leq i, j \leq 3$, $i \neq j$ and $E_{ij}$ are the matrix units, are definable with parameters $\varphi_1^\ast, \varphi_2^\ast, \varphi_3^\ast$ and $\pi_1^\ast, \pi_2^\ast$.

We choose a 2-transvection $\tau_1^\ast$, one of the four 2-transvections that satisfy the condition (ii) of Lemma 1.6 for the 2-involution $\theta_1^\ast = \varphi_1^\ast \varphi_2^\ast$ and the 1-permutation $\pi_1^\ast$. Without loss of generality we may suppose that the matrix of $\tau_1^\ast$ in $\{e_1, e_2, e_3\}$ is

$$
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

It is easy to see that among the automorphisms $\varphi_i^\ast \rho$, where $\rho$ is either an extremal involution, or a 1-permutation properly commuting with $\theta_1^\ast$ there are exactly four automorphisms whose square is $\tau_1^\ast$. The reason is that there are two solutions to the matrix equation

$$
X^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
$$

in $\text{SL}(2, \mathbb{Z})$, namely,

$$
X = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

and that any automorphism properly commuting with $\theta_1^\ast$ must act on its fixed-point subgroup, say, $C$, either as the id$_C$, or $-\text{id}_C$. Let us denote the said four automorphisms by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and let us further agree that $\sigma_1$ is the only transvection among the automorphisms $\sigma_i$.

The matrices of the automorphisms $\sigma_i$ in the basis $\{e_1, e_2, e_3\}$ are

$$
\pm \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

(the reader may as well imagine the diagonals of the matrices stretched up to infinity filled with units, but there is actually no need in that, since already three coordinates do the job.)

Let $\sigma$ be one of our automorphisms $\sigma_i$. We consider the conjugate $\sigma' = \pi \sigma \pi^{-1}$ of $\sigma$ by the automorphism $\pi = \pi_2^\ast \pi_1^\ast$. Then the matrix of the commutator $[\sigma, \sigma'] = \sigma \sigma' \sigma^{-1} \sigma'^{-1}$ is either the matrix

$$
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Thus only in the case when $\sigma = \sigma_1$ we have the commutator $[\sigma, \sigma']$ conjugate to $\sigma$. Really, as to the automorphisms $\sigma_2, \sigma_3, \sigma_4$ they all have eigen value $-1$, while none of the commutators $[\sigma_i, \sigma_i']$ with $i = 2, 3, 4$ has this eigen value. Summing up, we see that $\sigma_1$, a 1-transvection, is definable over the chosen parameters.
Like in the proof of Lemma 1.7 we see that the elementary transvections whose matrices in \(\{e_1, e_2, e_3\}\) are

\[
\begin{pmatrix}
1 & 2m & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

are definable with parameters \(\varphi_1^*, \varphi_2^*, \pi_1^*\) and \(\tau_1^*\). Then elementary transvection with the matrices

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

are also definable with the parameters \(\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*, \tau_2^*\), since they are none the other than either the transvections with matrices (2.1), or the products of the transvections with (2.1) and the elementary transvection with the matrix

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

which is now known to be definable over \(\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*, \tau_2^*\). The other required elementary transvections are conjugates of the tranvections with (2.1) by suitable automorphisms acting on \(\{e_1, e_2, e_3\}\) as permutations, definable products of \(\pi_1^*\) and \(\pi_2^*\). Claim 1 is proved.

Let us note in passing that definability of 1-transvections with definable parameters we have just proved immediately implies the following proposition.

**Proposition 2.2.** Let \(A\) be an infinitely generated free abelian group. Then

(i) the family of all transvections is \(\emptyset\)-definable in \(\text{Aut}(A)\);

(ii) Let \(m \geq 1\) be a natural number. The family of all \(m\)-transvections is \(\emptyset\)-definable in \(\text{Aut}(A)\).

Next is the construction of some set which is contained in \(\Gamma_2(A)\) and which is definable with our parameters.

**Claim 2.** There is a set \(D\) definable with parameters \(\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*, \tau_2^*\) such that

(i) the automorphisms from \(D\) act trivially on \(B \setminus \{e_1, e_2, e_3\}\) and their matrices in \(\{e_1, e_2, e_3\}\) are congruent modulo 2 to the identity matrix;

(ii) \(D\) contains all automorphisms with (i) whose matrices in \(\{e_1, e_2, e_3\}\) are of the form

\[
\begin{pmatrix}
a & b & 0 \\
c & d & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where

\[a \equiv d \equiv 1 \pmod{2}\] and \(b \equiv c \equiv 0 \pmod{2}\).

The argument is based upon the remarkable observation made in the paper [2] by Carter and Keller:

each matrix of the form

\[
\begin{pmatrix}
a & b & 0 \\
c & d & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
from the (matrix) group $\text{SL}(3, \mathbb{Z})$ is a product of at most 41 elementary transvections.

Suppose that $t_1, \ldots, t_{41}$ are elementary transvections, matrices from $\text{SL}(3, \mathbb{Z})$. One corresponds to the product

$$t_1 t_2 \ldots t_{41}$$

a sequence

(2.2) $$(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{41})$$

where $\bar{t}$ is the image of $t$ in $\text{SL}(3, \mathbb{Z}_2)$ under the natural homomorphism $\text{SL}(3, \mathbb{Z}) \to \text{SL}(3, \mathbb{Z}_2)$. There are of course finitely many sequences of the form (2.2). Some of them determine the identity matrix in $\text{SL}(3, \mathbb{Z}_2)$, some do not; we appreciate the former sequences, say ‘good’ ones. Clearly, the image $\bar{t}$ of an elementary transvection $t$ is trivial in $\text{SL}(3, \mathbb{Z}_2)$ if and only if $t$ is a square of an elementary transvection in $\text{SL}(3, \mathbb{Z})$. So the fact that a sequence $(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{41})$ is ‘good’ can be translated into a disjunction of statements each of which says for every $i = 1, \ldots, 41$ that the $i$th transvection $t_i$ is or is not a square.

Having the elementary transvections with respect to the basis $\{e_1, e_2, e_3\}$ (this time automorphisms of $A$) definable in $\text{Aut}(A)$ with the parameters introduced above, we may realize the above considerations for the group $\text{Aut}(A)$. This completes the proof of Claim 2.

Let now $\chi(\overline{\varphi})$ be a first-order formula that describes the parameters $\varphi_1^*, \varphi_2^*, \pi_1^*, \pi_2^*, \tau_1^*, \tau_2^*$. Suppose that $\overline{\varphi}$ is any tuple of elements of $\text{Aut}(A)$ that satisfies $\chi$; we then denote by $D(\overline{\varphi})$ the family of automorphisms constructed over $\overline{\varphi}$ in the same way as $D$ is constructed over our parameters.

**Claim 3.** The following are equivalent:

(a) $\sigma \in \text{Aut}(A)$ is an element of $\Gamma_2(A)$;

(b) there is a direct summand $B$ of $A$ of rank or corank 1 such that for every direct summand $C$ isomorphic via some automorphism from $\text{Aut}(A)$ to $B$ there exist a tuple $\overline{\varphi}$ satisfying $\chi$ and $\rho \in D(\overline{\varphi})$ with

$$\sigma C = \rho C.$$

Let consider the implication (b) $\Rightarrow$ (a). Suppose that the direct summand $B$ mentioned in (b) is of rank one and $e$ a unimodular element of $A$. Then for suitable parameters $\overline{\varphi}$ there is $\rho \in D(\overline{\varphi})$ with

$$\sigma \langle e \rangle = \rho \langle e \rangle.$$ 

By Claim 2 the set $D(\overline{\varphi})$ is contained in $\Gamma_2(A)$ and hence

$$\sigma e = \pm pe \equiv \pm e \equiv e (\text{mod } 2A).$$

It then follows that $\sigma \in \Gamma_2(A)$.

Suppose now that $B$ is of corank 1. Let $e$ be a unimodular element of $A$ and let $\{e, e_0, e_1, \ldots, e_n, \ldots\}$ be a basis of $A$. According to the condition $\sigma$ moves the direct summand

$$C_0 = \langle e, e_1, e_2, \ldots, e_n, \ldots \rangle$$

exactly as some $\rho \in \Gamma_2(A)$ does:

$$\sigma C_0 = \rho C_0.$$
This implies that $\sigma e$ is congruent modulo $2A$ to some element of $C_0$:

$$\sigma e \equiv ke + k_1e_1 + k_2e_2 + \ldots + k_ne_n + \ldots \pmod{2A}.$$  

(2.3)

The same argument can be applied to the subgroup

$$C_1 = \langle e, e_0, e_2, \ldots, e_n, \ldots \rangle$$

of which $e$ is also a member; this leads to

$$\sigma e \equiv le + l_0e_0 + l_2e_2 + \ldots + l_ne_n + \ldots \pmod{2A}.$$  

One deduces then that

$$(k - l)e - l_0e_0 + k_1e_1 + (k_2 - l_2)e_2 + \ldots + (k_n - l_n)e_n + \ldots \equiv 0 \pmod{2A}.$$  

The images of $e, e_0, e_1, e_2, \ldots$ under the natural homomorphism $A \to A/2A$ must be linearly independent over $\mathbb{Z}_2$ and therefore

$$l_0 \equiv k_1 \equiv 0 \pmod{2}.$$  

Continuing in a similar fashion, we see that all (non-zero) coefficients $k_i$ in (2.3) are even; the coefficient $k$ must therefore be odd. Thus $\sigma$ is in $\Gamma_2(A)$, as required.

The implication (a) $\Rightarrow$ (b). Suppose that $\sigma \in \Gamma_2(A)$ and $e$ is a unimodular element of $A$. Then for a basis $\{e, e_0, e_1, e_2, \ldots\}$ of which $e$ forms a part we have

$$\sigma e = e + 2(ke + \sum_i k_i e_i).$$

Suppose that $s$ is the greatest common divisor of non-zero elements $k_i$. Then

$$\sigma e = (1 + 2k)e + 2s\left(\sum_i k_i' e_i\right).$$

Clearly, $\gcd(1 + 2k, 2s) = 1$ (since $\sigma e$ is unimodular) and the element $g = \sum_i k_i' e_i$ is unimodular. If so, there are $b, d \in \mathbb{Z}$ such that the matrix

$$\begin{pmatrix} 1 + 2k & b & 0 \\ 2s & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

from $\text{SL}(3, \mathbb{Z})$ is congruent to the identity matrix modulo 2. This implies that there exist a tuple $\mathfrak{v}$ satisfying $\chi$ and some $\rho \in D(\mathfrak{v})$ such that

$$\sigma\langle e \rangle = \rho\langle e \rangle.$$  

Claim 3 is proved.

Since we know how to interpret in $\text{Aut}(A)$ by means of first-order logic the direct summands of $A$ of rank/corank 1, the conditions in (ii) of Claim 3 are easily translated into first-order formulae. The proof of Theorem 2.1 is now completed.  

□

Remark 2.3. Very recently Bardakov proved that the principal congruence subgroups of the groups $\text{SL}(n, \mathbb{Z})$, where $n \geq 3$ all have finite width with respect to elementary transvections (unpublished; personal communication). Recall that the width of a group $G$ relative to a generating set $S$ with $S^{-1} = S$ is either the minimal natural number $k$ such that every element of $G$ is a product of at most $k$ elements of $S$, or $\infty$ otherwise.

The result by Bardakov could be used then to simplify the proof of Theorem 2.1.
Theorem 2.4. Let $A$ be an infinitely generated free abelian group. Then the group $\text{Aut}(A)$ first-order interprets the second-order theory $\text{Th}_2(|A|)$, uniformly in $A$.

Proof. The proof is based on Theorem 2.1 and the following important theorem from the paper [1] by Bryant and Macedonska.

Theorem. Let $F$ be a free group of infinite rank and let $V$ be a characteristic subgroup of $F$ such that $F/V$ is nilpotent. Then every automorphism of $F/V$ is induced by an automorphism of $F$.

Let $A$ stand for the free abelian group $F/[F,F]$. As a corollary of the result by Bryant–Macedonska we have that the natural homomorphism $\mu: \text{Aut}(A) \to \text{Aut}(A/2A)$ (induced by the natural homomorphism $A \to A/2A$) is surjective. Indeed, according to the Theorem, the natural homomorphisms $\mu_1: \text{Aut}(F) \to \text{Aut}(A)$ and $\mu_2: \text{Aut}(F) \to \text{Aut}(A/2A)$ are both surjective. On the other hand,

$$\mu_2 = \mu \circ \mu_1,$$

and then $\mu$ must be surjective, too.

Adding this to the fact that $\Gamma_2(A)$, the kernel of $\mu$, is $\emptyset$-definable in $\text{Aut}(A)$, we get that the group $\text{Aut}(A)$ first-order interprets the group $\text{Aut}(A/2A)$:

$$\text{Aut}(A)/\ker \mu = \text{Aut}(A)/\Gamma_2(A) \cong \text{Aut}(A/2A).$$

The group $\text{Aut}(A/2A)$ is the general linear group of the vector space $A/2A$ over the field $\mathbb{Z}_2$. On the other hand, the general linear group $\text{GL}(V)$ of a infinite-dimensional vector space $V$ over a field $D$ first-order interprets $\text{Th}_2(\dim_D V)$, see [8, Theorem 11.4]. Therefore the elementary theory of the group $\text{Aut}(A/2A)$ first-order interprets the second-order theory

$$\text{Th}_2(\dim_2 A/2A) = \text{Th}_2(|A|),$$

and the result follows.

Corollary. Let $A_1, A_2$ be infinitely generated free abelian groups. The groups $\text{Aut}(A_1)$ and $\text{Aut}(A_2)$ are elementarily equivalent if and only if the cardinals $|A_1|$ and $|A_2|$ (viewed as sets with no structure) are second-order equivalent.

Proof. The necessity part is a consequence of Theorem 2.4. To prove the converse, one syntactically interprets in the second-order theory $\text{Th}_2(\kappa)$, where $\kappa$ is an infinite cardinal, the elementary theory of the automorphism group of a free abelian group with $\kappa$ as the domain (rather easy; cf. [9, Theorem 4.1] where a similar interpretation is done in quite full detail for the case of the elementary theory of the automorphism group a free group over $\kappa$).

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Department of Mathematics, Yeditepe University, 34755 Kayışdağlı, Istanbul, Turkey