ON SIMPLE SHAMSUDDIN DERIVATIONS IN TWO VARIABLES

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Abstract. We study the subgroup of $k$-automorphisms of $k[x, y]$ which commute with a simple derivation $D$ of $k[x, y]$. We prove, for example, that this subgroup is trivial when $D$ is a Shamsuddin simple derivation. In the general case of simple derivations, we obtain properties for the elements of this subgroup.

1. Introduction

Let $k$ be an algebraically closed field of characteristic zero and the ring $k[x, y]$ of polynomials over $k$ in two variables.

A $k$-derivation $d : k[x, y] \to k[x, y]$ of $k[x, y]$ is a $k$-linear map such that

$$d(ab) = d(a)b + ad(b)$$

for any $a, b \in k[x, y]$. Denoting by $\text{Der}_k(k[x, y])$ the set of all $k$-derivations of $k[x, y]$. Let $d \in \text{Der}_k(k[x, y])$. An ideal $I$ of $k[x, y]$ is called $d$-stable if $d(I) \subset I$. For example, the ideals 0 and $k[x, y]$ are always $d$-stable. If $k[x, y]$ has no other $d$-stable ideal it is called $d$-simple. Even in the case of two variables, a few examples of simple derivations are known (see for explanation [BLL2003], [Cec2012], [No2008], [BP2015], [KM2013] and [Leq2011]).

We denote by $\text{Aut}(k[x, y])$ the group of $k$-automorphisms of $k[x, y]$. Let $\text{Aut}(k[x, y])$ act on $\text{Der}_k(k[x, y])$ by:

$$(\rho, D) \mapsto \rho^{-1} \circ D \circ \rho = \rho^{-1} D \rho.$$ 

Fixed a derivation $d \in \text{Der}_k(k[x, y])$. The isotropy subgroup, with respect to this group action, is

$$\text{Aut}(k[x, y])_D := \{\rho \in \text{Aut}(k[x, y]) / \rho D = D \rho\}.$$ 

We are interested in the following question proposed by I.Pan (see [B2014]):

**Conjecture 1.** If $d$ is a simple derivation of $k[x, y]$, then $\text{Aut}(k[x, y])_d$ is finite.

Research of R. Baltazar was partially supported by CAPES.
At a first moment, in the §2, we show that the conjecture is true for a family of derivations, named Shamsuddin derivations (Theorem 6). For this, we use a theorem of the Shamsuddin [Sh1977], mentioned in [No1994, Theorem 13.2.1.], that determines a condition that would preserve the simplicity by extending, in some way, the derivation to $R[t]$, with $t$ an indeterminate. The reader may also remember that Y.Lequain [Leq2011] showed that these derivations check a conjecture about the $A_n$, the Weyl algebra over $k$.

In order to understand the isotropy of a simple derivation of the $k[x, y]$, in §3, we analysed necessary conditions for an automorphism to belong to the isotropy of a simple derivation. For example, we prove that if such an automorphism has a fixed point, then it is the identity (Proposition 7). Following, we present the definition of dynamical degree of a polynomial application and thus proved that in the case $k = \mathbb{C}$, the elements in $\text{Aut}(\mathbb{C}[x, y])_{d}$, with $d$ a simple derivation, has dynamical degree 1 (Corollary 9). More precisely, the condition dynamical degree $> 1$ corresponds to exponential growth of degree under iteration, and this may be viewed as a complexity of the automorphism in the isotropy (see [FM1989]).

2. Shamsuddin derivation

The main aim of this section is study the isotropy group of the a Shamsuddin derivation in $k[x, y]$. In [No1994, §13.3], there are numerous examples of these derivations and also shown a criterion for determining the simplicity; furthermore, Y.Lequain [Leq2008] introduced an algorithm for determining when an Shamsuddin derivation is simple. However, before this, the following example shows the isotropy of an arbitrary derivation can be complicated.

**Example 1.** Let be $d = \partial_x \in \text{Der}_k(k[x, y])$ and $\rho \in \text{Aut}(k[x, y])_{d}$. Note that $d$ is not a simple derivation; indeed, for any $u(y) \in k[y]$, the ideal generated by $u(x)$ is always invariant. Consider

$$\rho(x) = f(x, y) = a_0(x) + a_1(x)y + \ldots + a_t(x)y^t$$

$$\rho(y) = g(x, y) = b_0(x) + b_1(x)y + \ldots + b_s(x)y^s.$$

Since $\rho \in \text{Aut}(k[x, y])_{d}$, we obtain two conditions:

1) $\rho(d(x)) = d(\rho(x))$.

Thus,

$$1 = d(a_0(x) + a_1(x)y + \ldots + a_t(x)y^t) = d(a_0(x)) + d(a_1(x))y + \ldots + d(a_t(x))y^t.$$

Then, $d(a_0(x)) = 1$ and $d(a_j(x)) = 0$, $j = 1, \ldots, t$. We conclude that $\rho(x)$ is of the type $\rho(x) = x + c_0 + c_1y + \ldots + c_t y^t$, $c_i \in k$.

2) $\rho(d(y)) = d(\rho(y))$. 
Analogously,

\[ 0 = d(b_0(x) + b_1(x)y + \ldots + b_s(x)y^s) = d(b_0(x)) + d(b_1(x))y + \ldots + d(b_s(x))y^s. \]

That is, \( b_i(x) = d_i \) with \( d_i \in k \). We conclude also that \( \rho(y) \) is of the type

\[ \rho(y) = d_0 + d_1y + \ldots + d_s y^s, \quad d_i \in k. \]

Thus, \( \text{Aut}(k[x, y])_d \) contains the affine automorphisms

\[ (x + uy + r, uy + s), \]

with \( u, r, s \in k \). In particular, \( \text{Aut}(k[x, y])_d \) is not finite.

Notice that \( \text{Aut}(k[x, y])_d \) contains also the automorphisms of the type \( (x + p(y), y) \), with \( p(y) \in k[y] \).

Now, we determine indeed the isotropy. Using only the conditions 1 and 2,

\[ \rho = (x + p(y), q(y)) \]

with \( p(y), q(y) \in k[y] \). However, \( \rho \) is an automorphism, in other words, the determinant of the Jacobian matrix must be a nonzero constant. Thus, \( |J_\rho| = q'(y) = c, \quad c \in k^*. \)

Therefore, \( \rho = (x + p(y), ay + c) \), with \( p(y) \in k[y] \) and \( a, b \in k \). Consequently, \( \text{Aut}(k[x, y])_d \) is not finite and, more than that, the first component has elements with any degree.

The following lemma is a well known result.

**Lemma 2.** Let \( R \) be a commutative ring, \( d \) a derivation of \( R \) and \( h(t) \in R[t] \), with \( t \) an indeterminate. Then, we can also extend \( d \) to a unique derivation \( \tilde{d} \) of the \( R[t] \) such that \( \tilde{d}(t) = h(t) \).

We will use the following result of Shamsuddin [Sh1977].

**Theorem 3.** Let \( R \) be a ring containing \( \mathbb{Q} \) and let \( d \) be a simple derivation of \( R \). Extend the derivation \( d \) to a derivation \( \tilde{d} \) of the polynomial ring \( R[t] \) by setting \( \tilde{d}(t) = at + b \) where \( a, b \in R \). Then the following two conditions are equivalent:

1. \( \tilde{d} \) is a simple derivation.
2. There exist no elements \( r \in R \) such that \( d(r) = ar + b \).

**Proof.** See [No1994, Theorem 13.2.1.] for a demonstration in details. \( \square \)

A derivation \( d \) of \( k[x, y] \) is said to be a **Shamsuddin derivation** if \( d \) is of the form

\[ d = \partial_x + (a(x)y + b(x))\partial_y, \]

where \( a(x), b(x) \in k[x] \).
Example 4. Let $d$ be a derivation of $k[x, y]$ as follows

$$d = \partial_x + (xy + 1)\partial_y.$$ 

Writing $R = k[x]$, we know that $R$ is $\partial_x$-simple and, taking $a = x$ and $b = 1$, we are exactly the conditions of Theorem 3. Thus, we know that $d$ is simple if, and only if, there exist no elements $r \in R$ such that $\partial_x(r) = xr + 1$; but the right side of the equivalence is satisfied by the degree of $r$. Therefore, by Theorem 3, $d$ is a simple derivation of $k[x, y]$.

Lemma 5. ([No1994, Proposition. 13.3.2]) Let $d = \partial_x + (a(x)y + b(x))\partial_y$ be a Shamsuddin derivation, where $a(x), b(x) \in k[x]$. Thus, if $d$ is a simple derivation, then $a(x) \neq 0$ and $b(x) \neq 0$.

Proof. If $b(x) = 0$, then the ideal $(y)$ is $d$-invariante. If $a(x) = 0$, let $h(x) \in k[x]$ such that $h' = b(x)$, then the ideal $(y - h)$ is $d$-invariante. □

One can determine the simplicity of the a Shamsuddin derivation according the polynomials $a(x)$ and $b(x)$ (see ([No1994 §13.3])).

Theorem 6. Let $D \in \text{Der}_k(k[x, y])$ be a Shamsuddin derivation. If $D$ is a simple derivation, then $\text{Aut}(k[x, y])_D = \{\text{id}\}$.

Proof. Let us denote $\rho(x) = f(x, y)$ and $\rho(y) = g(x, y)$. Let $D$ be a Shamsuddin derivation and

$$D(x) = 1,$$

$$D(y) = a(x)y + b(x),$$

where $a(x), b(x) \in k[x]$

Since $\rho \in \text{Aut}(k[x, y])_D$, we obtain two conditions:

(1) $\rho(D(x)) = D(\rho(x)).$

(2) $\rho(D(y)) = D(\rho(y)).$

Then, by condition (1), $D(f(x, y)) = 1$ and since $f(x, y)$ can be written in the form

$$f(x, y) = a_0(x) + a_1(x)y + \ldots + a_s(x)y^s,$$

with $s \geq 0$, we obtain

$$D(a_0(x)) + D(a_1(x))y + a_1(x)(a(x)y + b(x)) + \ldots$$

$$+ D(a_s(x))y^s + sa_s(x)y^{s-1}(a(x)y + b(x)) = 1$$

Comparing the coefficients in $y^s$,

$$D(a_s(x)) = -sa_s(x)a(x),$$
which can not occur by the simplicity. More explicitly, the Lemma implies that \( a(x) = 0 \). Thus \( s = 0 \), this is \( f(x, y) = a_0(x) \). Therefore \( D(a_0(x)) = 1 \) and \( f = x+c \), with \( c \) constant.

Using the condition (2),
\[
D(g(x, y)) = \rho(a(x)y + b(x)) = \rho(a(x))\rho(y) + \rho(b(x)) = a(x+c)g(x, y) + b(x+c)
\]

Writing \( g(x, y) = b_0(x) + b_1(x)y + \ldots + b_i(x)y^i \); wherein, by the previous part, we can suppose that \( t > 0 \), because \( \rho \) is a automorphism. Thus
\[
a(x+c)g(x, y) + b(x+c) = D(b_0(x)) + D(b_1(x))y + D(b_i(x))y^i + b(x)c + b(x)\]
\[
= a(x)(b_0(x) + b_1 y) + b(x).
\]

Comparing the coefficients in \( y^i \), we obtain
\[
D(b_i(x)) + b(x)c + b(x) = a(x+c)b_i(x)
\]

Then \( D(b_i(x)) = b_i(x)(-ta(x) + a(x+c)) \). In this way, \( b_i(x) \) is a constant and, consequently, \( a(x+c) = ta(x) \). Comparing the coefficients in the last equality, we obtain \( t = 1 \) and then \( b_1(x) = b_1 \) a constant. Moreover, if \( a(x) \) is not a constant, since \( a(x+c) = a(x) \), is easy to see that \( c = 0 \). Indeed, if \( c \neq 0 \) we obtain that the polynomial \( a(x) \) has infinite distinct roots. If \( a(x) \) is a constant, then \( a(x) D \) is not a simple derivation (a consequence of [Leq2008, Lemma.2.6 and Theorem.3.2]); thus, we obtain \( c = 0 \).

Note that \( g(x, y) = b_0(x) + b_1 y \) and, using the condition (2) again,
\[
D(g(x, y)) = D(b_0(x)) + b_1(a(x)y + b(x)) = a(x)(b_0(x) + b_1 y) + b(x).
\]

Considering the independent term of \( y \),
\[
D(b_0(x)) = b_0(x)a(x) + b(x)(1 - b_1) \tag{1}
\]

If \( b_1 \neq 1 \), we consider the derivation \( D' \) such that
\[
D'(x) = 1, \quad D'(y) = a(x)y + b(x)(1 - b_1).
\]

In [No1994, Proposition. 13.3.3], it is noted that \( D \) is a simple derivation if and only if \( D' \) is a simple derivation. Furthermore, by the Theorem 3 there exist no elements \( h(x) \) in \( K[x] \) such that
\[
D(h(x)) = h(x)a(x) + b(x)(1 - b_1) :\]

what contradicts the equation (1). Then, \( b_1 = 1 \) and \( D(b_0(x)) = b_0(x)a(x) \), since \( D \) is a simple derivation we know that \( a(x) \neq 0 \), consequently \( b_0(x) = 0 \). This shows that \( \rho = id. \)
3. On the isotropy of the simple derivations

The purpose of this section is to study the isotropy in the general case of a simple derivation. More precisely, we obtain results that reveal some characteristics of the elements in $\text{Aut}(k[x,y])_D$. For this, we use some concepts presented in the previous sections and also the concept of dynamical degree of a polynomial application.

In [BP2015], which was inspired by [BLL2003], we introduce and study a general notion of solution associated to a Noetherian differential $k$-algebra and its relationship with simplicity.

The following proposition geometrically says that if an element in the isotropy of a simple derivation has fixed point then it is the identity automorphism.

**Proposition 7.** Let $D \in \text{Der}_k(k[x_1, ..., x_n])$ be a simple derivation and $\rho \in \text{Aut}(k[x_1, ..., x_n])_D$ an automorphism in the isotropy. Suppose that there exist a maximal ideal $\mathfrak{m} \subset k[x_1, ..., x_n]$ such that $\rho(\mathfrak{m}) = \mathfrak{m}$, then $\rho = \text{id}$.

*Proof.* Let $\varphi$ be a solution of $D$ passing through $\mathfrak{m}$ (see [BP2015 Definition 1]). We know that $\frac{\partial}{\partial t} \varphi = \varphi D$ and $\varphi^{-1}(t) = \mathfrak{m}$. If $\rho \in \text{Aut}(k[x_1, ..., x_n])_D$, then

$$\frac{\partial}{\partial t} \varphi \rho = \varphi D \rho = \varphi \rho D.$$  

In other words, $\varphi \rho$ is a solution of $D$ passing through $\rho^{-1}(\mathfrak{m}) = \mathfrak{m}$. Therefore, by the uniqueness of the solution ([BP2015 Theorem 7(c)]), $\varphi \rho = \varphi$. Note that $\varphi$ is one to one, because $k[x_1, ..., x_n]$ is $D$-simple and $\varphi$ is a nontrivial solution. Then, we obtain that $\rho = \text{id}$.  

□

F. Lane, in [Lane75], proved that every $k$-automorphism $\rho$ of $k[x,y]$ leaves a nontrivial proper ideal $I$ invariant, over an algebraically closed field; this is, $\rho(I) \subseteq I$. Em [Sh1982], A. Shamsuddin proved that this result does not extend to $k[x, y, z]$, proving that the $k$-automorphism given by $\chi(x) = x + 1$, $\chi(y) = y + xz + 1$ e $\chi(z) = y + (x + 1)z$ has no nontrivial invariant ideal.

Note that, in addition, $\rho$ leaves a nontrivial proper ideal $I$ invariant if and only if $\rho(I) = I$, because $k[x,y]$ is Noetherian. In fact, the ascending chain

$$I \subset \rho^{-1}(I) \subset \rho^{-2}(I) \subset \ldots \subset \rho^{-l}(I) \subset \ldots$$

must stabilize; thus, there exists a positive integer $n$ such that $\rho^{-n}(I) = \rho^{-n-1}(I)$, then $\rho(I) = I$. 
Suppose that $\rho \in \text{Aut}(k[x,y])_D$ and $D$ is a simple derivation of $k[x,y]$. If this invariant ideal $I$ is maximal, by the Proposition 7, we have $\rho = id$.

Suppose that $I$, this invariant ideal, is radical. Let $I = (m_1 \cap \ldots \cap m_s) \cap (p_1 \cap \ldots \cap p_t)$ be a primary decomposition where each ideal $m_i$ is a maximal ideal and $p_j$ are prime ideals with height 1 such that $p_j = (f_j)$, with $f_j$ irreducible (by AM1969 Theorem 5.]). If $\rho(m_1 \cap \ldots \cap m_s) = m_1 \cap \ldots \cap m_s$, we claim that $\rho^N$ leaves invariant one maximal ideal for some $N \in \mathbb{N}$: suppose $m_1$ this ideal. Indeed, we know that $\rho(m_1) \supseteq m_1 \cap \ldots \cap m_s$, since $\rho(m_1)$ is a prime ideal, we deduce that $\rho(m_1) \supseteq m_i$, for some $i = 1, \ldots, s$ (AM1969 Prop.11.1.(ii)). Then, $\rho(m_1) = m_i$; that is, $\rho^N$ leaves invariant the maximal ideal $m_1$ for some $N \in \mathbb{N}$. Thus follows from Proposition 7 that $\rho^N = id$.

Note that $\rho(p_1 \cap \ldots \cap p_t) = p_1 \cap \ldots \cap p_t$. In fact, writing $p_1 \cap \ldots \cap p_t = (f_1 \ldots f_t)$, with $f_i$ irreducible, we can choose $h \in m_1 \cap \ldots \cap m_s$ such that $\rho(h) \notin p_1$. We observe that there exists $h$. Otherwise, we obtain $m_1 \cap \ldots \cap m_s \subseteq p_1$, then $p_1 \supseteq m_i$, for some $i = 1, \ldots, s$ (AM1969 Prop.11.1.(ii)): a contradiction. Thus, since $hf_1 \ldots f_t \in I$, we obtain $\rho(h)\rho(f_1) \ldots \rho(f_t) \in I \subseteq p_1$. Therefore, $\rho(f_1 \ldots f_t) \in p_1$. Likewise, we conclude the same for the other primes $p_i$, $i = 1, \ldots, t$. Finally, $\rho(p_1 \cap \ldots \cap p_t) = p_1 \cap \ldots \cap p_t$.

With the next corollary, we obtain some consequences on the last case.

**Corollary 8.** Let $\rho \in \text{Aut}(k[x,y])_D$, $D$ a simple derivation of $k[x,y]$ and $I = (f)$, with $f$ reduced, a ideal with height 1 such that $\rho(I) = I$. If $V(f)$ is singular or some irreducible component $C_i$ of $V(f)$ has genus greater than two, then $\rho$ is a automorphism of finite order.

**Proof.** Suppose that $V(f)$ is not a smooth variety and let $q$ be a singularity of $V(f)$. Since the set of the singular points is invariant by $\rho$, then there exist $N \in \mathbb{N}$ such that $\rho^N(q) = q$. Using that $\rho \in \text{Aut}(k[x,y])_D$, we obtain, by Proposition 7, $\rho^N = id$.

Let $C_i$ be a component irreducible of $V(f)$ that has genus greater than two. Note that there exist $M \in \mathbb{N}$ such that $\rho^M(C_i) = C_i$. By FK1992 Thm. Hunwitz, p.241], the number of elements in $\text{Aut}(C_i)$ is finite; in fact, $\#(\text{Aut}(C_i)) < 84(g_i - 1)$, where $g_i$ is the genus of $C_i$. Then, we deduce that $\rho$ is a automorphism of finite order.

We take for the rest of this section $k = \mathbb{C}$.

Consider a polynomial application $f(x, y) = (f_1(x, y), f_2(x, y)) : \mathbb{C}^2 \to \mathbb{C}^2$ and define the degree of $f$ by $\deg(f) := \max(\deg(f_1), \deg(f_2))$. Thus we may define the dynamical degree (see BD2012, FM1989, Silv12) of $f$ as

$$\delta(f) := \lim_{n \to \infty} (\deg(f^n))^{\frac{1}{n}}.$$
Corollary 9. If $\rho \in \text{Aut}(\mathbb{C}[x,y])_D$ and $D$ is a simple derivation of $\mathbb{C}[x,y]$, then $\delta(\rho) = 1$.

Proof. Suppose $\delta(\rho) > 1$. By [FM1989, Theorem 3.1.], $\rho^n$ has exactly $\delta(\rho)^n$ fix points counted with multiplicities. Then, by Proposition 7 $\rho = id$, which shows that dynamical degree of $\rho$ is 1. □

Acknowledgements. I would like to thank Ivan Pan for his comments and suggestions.

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