Non-Sparse PCA in High Dimensions via Cone Projected Power Iteration

Yufei Yi yy544@cmu.edu
Matey Neykov mneykov@stat.cmu.edu

Department of Statistics and Data Science, Carnegie Mellon University
5000 Forbes Ave, Pittsburgh, PA 15213, U.S.A.

Abstract

In this paper, we propose a cone projected power iteration algorithm to recover the principal eigenvector from a noisy positive semidefinite matrix. When the true principal eigenvector is assumed to belong to a convex cone, the proposed algorithm is fast and has a tractable error. Specifically, the method achieves polynomial time complexity for certain convex cones equipped with fast projection such as the monotone cone. It attains a small error when the noisy matrix has a small cone-restricted operator norm. We supplement the above results with a minimax lower bound of the error under the spiked covariance model. Our numerical experiments on simulated and real data, show that our method achieves shorter run time and smaller error in comparison to the ordinary power iteration and some sparse principal component analysis algorithms if the principal eigenvector is in a convex cone.

Keywords—Power iteration; Dimension reduction; Principal Component Analysis; Convex cone; Monotone cone.

1 Introduction

Principal component analysis was developed by Hotelling (1933) after its origin by Pearson (1901), and is widely used nowadays for dimension reduction. It works by replacing a set of \( p \) variables with a smaller set of principal components which capture the maximal variance. A principal component is a linear combination of the \( p \) variables. The coefficients of such a linear combination depend on the principal eigenvectors of the population covariance matrix, which is often estimated by eigenvectors of the sample covariance matrix. Numerical methods to compute principal eigenvectors include the QR algorithm and power iteration (Watkins, 2004, Chapter 5).

The consistency of principal component analysis is thoroughly studied in the statistical literature (Baik and Silverstein, 2006; Paul, 2007; Nadler et al. 2008; Donoho et al. 2013; Fan and Wang, 2013; Perry et al., 2018; Cai et al., 2020). In low dimensions \( p \ll n \), the sample estimators consistently recover the population principal eigenvectors (Anderson, 2003). However in high dimensions \( p \gg n \), all methods fail to recover the population principal eigenvectors if there is no additional structure imposed. Johnstone and Lu (2009) introduced the spiked covariance model and proved that the sample estimator is asymptotically orthogonal to the population principal eigenvector if \( p/n \) is bounded away from zero. Wainwright (2019, Example 15.19) shows that the minimax risk for estimating the spiked eigenvector is lower bounded by a quantity related to \( p/n \).

With the failure of ordinary principal component analysis in high dimensions, researchers have started to impose additional structure on the eigenvectors. Such a structure is exploited by introducing constraints or penalties. The simplest and most studied structure is that of sparsity. Methods exploiting the sparsity structure are referred to as sparse principal component analysis. As to the origin, Cadima and Jolliffe (1993) first proposed the idea to approximate a given principal component by using only a subset of features. The first computational technique -SCoTLASS- was established by Jolliffe et al. (2003), which maximizes the variance of a principal component under the \( \ell_1 \) constraint of eigenvector, inspired by LASSO (Tibshirani, 1996). Afterwards, Zou et al. (2006) proposed ElasticNet SPCA to regress a principal component on \( p \) variables with elastic net constraint (Zou and Hastie, 2005) to get a sparse eigenvector. Witten et al. (2009) established the connections between SCoTLASS and ElasticNet SPCA. In the seminal work of d’Aspremont et al. (2007), the computation of the first sparse eigenvector is formed as approximating the covariance matrix by a rank-one spike matrix under
Frobenius norm with constraints on the spike vector. Then it’s relaxed to a semidefinite programming problem. 

\cite{Vu et al. (2013a)} generalized the work of \cite{d’Aspremont et al. (2005)} to compute more than one sparse eigenvectors by incorporating Fantope in the constraint function. Besides methods based on the relaxation of the sparsity constraint, there is also a substantial literature on non-optimized relaxation techniques. \cite{Moghaddam et al. (2006)} computed the non-zero elements of a sparse eigenvector by solving an unconstrained optimization on corresponding submatrices. The submatrices are selected by bi-directional greedy search. \cite{Johnstone and Lu (2004)} proposed a method which uses the coordinates with highest variance after wavelet transforms of the data. Years later, some new numerical methods came up, such as the work of \cite{Yuan and Zhang (2013)} which integrated the power iteration algorithm with truncation in each iteration, and \cite{Ma et al. (2013)} which incorporated the QR algorithm with a thresholding step. Theoretical analysis of the convergence rate of sparse principal analysis can be found in \cite{Birnbaum et al. (2013), Cai et al. (2013), Vu et al. (2013b)}.

In this paper, we propose an algorithm to estimate the principal eigenvector in high dimensions, when the true eigenvector is in a convex cone, but is not necessarily sparse. Our proposal generalizes the work of \cite{Deshpande et al. (2014)} who analyzed a noisy quadratic observation model under cone constraints. In the noisy quadratic model one observes \( X = w^T + Z \) for some fixed signal vector \( v \) and symmetrized Gaussian noise matrix \( Z \). In contrast, in this paper we work with all positive semidefinite matrices which can be decomposed into the sum of a target positive semidefinite matrix plus a noise matrix. This generalization leads to a cleaner algorithm which does not require tuning parameters. Our algorithm also tackles the problem where the initial guess vector may have a negative inner product with the true eigenvector. As for the run time, our algorithm stops at a finite time with a stopping criteria inputted by the user. We prove that recovering the true principal eigenvector within finite iterations is possible, given a reasonable noise and a not too small eigengap of the covariance matrix. The estimation error when stopping at finite iterations is of the same order as the estimation error at convergence. Additionally, the algorithm complexity based on the stopping criteria is provided in Section \ref{sec:complexity}

Polynomial complexity is achieved in some situations such as the monotone cone.

We compare our proposed cone projected power iteration algorithm with the ordinary power iteration, Truncated Power Iteration \cite{Yuan and Zhang (2013)}, and ElasticNet SPCA \cite{Zou et al. (2006)} on both simulated and real data sets. Ordinary power iteration doesn’t impose any constraint on the eigenvector, and serves as a control group. The other two methods are representations of sparse principal component analysis. Truncated Power Iteration, as the name suggests, is also a power method, but it truncates a certain proportion of coordinates to zero in every iteration to achieve sparsity. ElasticNet SPCA computes the non-sparse principal component first, and obtains the sparse eigenvector by performing elastic net regression of variables on the principal components. As expected, methods exploiting sparsity don’t perform as well as our algorithm in the non-sparse setting. However, experiments show that even in the sparse setting, our algorithm still improves both convergence time and estimation if the true eigenvector is in a convex cone.

\section{Problem Formulation and Notation}

Suppose \( \hat{A} \) is a \( p \times p \) positive semidefinite matrix with a principal eigenvector \( \bar{x} \in K \) where \( K \subset \mathbb{R}^p \) is a known convex cone. Then \( \bar{x} \) is the solution of

\begin{equation}
\arg \max_{u \in K \cap S^{p-1}} u^T \hat{A} u.
\end{equation}

where \( S^{p-1} \) is the unit sphere in \( \mathbb{R}^p \). Usually, instead of observing \( \hat{A} \) we get to observe a noisy matrix \( A = \hat{A} + E \), where \( E \) is the stochastic noise. The problem of interest becomes to recover \( \bar{x} \) from a noisy observation \( A \), with prior knowledge that \( \bar{x} \in K \) and \( A \) is positive semidefinite. For example, \( A \) could be the empirical covariance matrix of a data set, and \( \hat{A} \) could be its true covariance matrix. Additionally, let \( \lambda, \mu \) be the largest and second largest eigenvalues of \( \hat{A} \) and let \( k := \lambda - \mu \) be the first eigengap of \( \hat{A} \). We now introduce some general definitions and notations which will be useful for the later development.

Given a cone \( C \subset \mathbb{R}^p \), define the cone-restricted operator norm of a \( p \times p \) matrix \( E \) as

\[ \|E\|_C = \sup_{x, y \in C \cap S^{p-1}} |x^T E y|, \]

Notice that \( \|E\|_C \) constitutes a seminorm, and one trivially has \( \|E\|_C \leq \|E\|_{op} \) where \( \|E\|_{op} \) is the operator norm of \( E \).

For a set \( T \subset \mathbb{R}^p \), define the Gaussian complexity of \( T \) as

\[ w(T) = \mathbb{E} \sup_{t \in T} \langle g, t \rangle, \quad \text{where } g \sim \mathcal{N}(0, I_p), \]

2
which is the expectation of maximum magnitude of the canonical Gaussian process on $T$. The Gaussian complexity is a basic geometric property of $T$. It measures the size of $T$ and is related to the metric entropy of $T$ (Vershynin, 2018, Theorem 8.1.13).

The tangent cone of a convex cone $K \subset \mathbb{R}^p$ at $\bar{x} \in K$ consists of all the possible directions from which a sequence in $K$ can converge to $\bar{x}$. It is defined as

$$T_K(\bar{x}) = \{t(v - \bar{x}) : t \geq 0, v \in K\}.$$

Throughout we use $\| \cdot \|$ as a shorthand for the Euclidean norm $\| \cdot \|_2$. The projection of a vector $v \in \mathbb{R}^p$ onto a convex cone $K \subset \mathbb{R}^p$ is defined as

$$\Pi_K v = \arg \min_{x \in K} \|v - x\|.$$

We use $\preceq$ and $\succeq$ to mean $\leq$ and $\geq$ up to positive universal constants. Next we define two constants $c_{-1}$ and $c_1$ which will be used in the statements of our results in the consequent sections. For precise definitions in terms of the eigengap $\lambda$ and $\|E\|_K$ please refer to the Supplementary material. Here we only mention that $c_{-1}$ and $c_1$ are well defined when $\lambda \gtrsim \|E\|_K$ and that $4\|E\|_k \leq c_{-1} \lesssim \frac{\|E\|_k}{k}$ and $c_1 \gtrsim \frac{\lambda}{\lambda_k}$. Finally, we use $\wedge$ and $\vee$ as shorthands for the min and max of two numbers respectively.

## 3 Estimation of the Principal Eigenvector

### 3.1 The Idealized Estimator and Its Error Rate

There exists a natural estimator to problem (1). In particular consider estimating $\bar{x}$ with $v \in \arg \max_{u \in K \cap \mathbb{S}^{p-1}} u^\top Au$ where $A$ is the observed noisy matrix. We refer to an estimator $v$ as an idealized estimator since the above program is non-convex and in general could be NP-hard to solve. Our first result, Theorem 3.1 analyzes the $L_2$ error of the idealized estimator. Notice that the idealized estimator is not generally guaranteed to have a positive dot product with the true eigenvector of $A$, so we must consider both situations: $v^\top \bar{x} \geq 0$ and $v^\top \bar{x} \leq 0$. We have

**Theorem 3.1 (L_2 Error Rate of the Idealized Estimator).** For any $v$ satisfying $v \in \arg \max_{u \in K \cap \mathbb{S}^{p-1}} u^\top Au$, we either have

$$\|v - \bar{x}\| \leq \sqrt{\frac{4 \|E\|_K}{k}} \wedge \frac{8 \|E\|_{T_K(\bar{x})}}{k} \quad \text{for} \quad v^\top \bar{x} \geq 0,$$

$$\|v + \bar{x}\| \leq \sqrt{\frac{4 \|E\|_K}{k}} \wedge \frac{8 \|E\|_{K}}{k} \quad \text{for} \quad v^\top \bar{x} \leq 0.$$

The upper bounds above are naturally related to the operator norm of the noise matrix $E$, which is shown in the following corollary. However, one should keep in mind that the conclusion of Theorem 3.1 could be much tighter than Corollary 3.1 when $T_K(\bar{x})$ and/or $K$ is much smaller than $\mathbb{R}^p$.

**Corollary 3.1 (L_2 Error Rate of the Idealized Estimator).** For the estimate defined in Theorem 3.1 we have

$$\|v - \bar{x}\| \wedge \|v + \bar{x}\| \leq \sqrt{\frac{4 \|E\|_{op}}{k}} \wedge \frac{8 \|E\|_{op}}{k}.$$

### 3.2 Cone Projected Power Iteration

If there were no constraint in (1), it would be equivalent to finding the principal eigenvector of $A$. One way to find the principal eigenvector is to use the power iteration method, which starts with a vector $v_0$, such that $v_0$ has a non-zero dot product with the principal eigenvector of $A$, and iterates the following recursion $v_t = \frac{Av_{t-1}}{\|Av_{t-1}\|}$ for $t = 1, 2, \ldots$. We now suggest a simple modification of this method, to target the constrained problem (1). We modify the power iteration by adding a projection step in each iteration in order to force the algorithm to choose vectors belonging to the set $K \cap \mathbb{S}^{p-1}$.
Algorithm 1 Cone Projected Power Iteration Single Vector Version

Input: $A \in \mathbb{R}^{p \times p}$ positive semidefinite matrix, $\Delta \in \mathbb{R}$ stopping criteria, $K$ a convex cone.

Initialize: $v_0 \in \mathbb{R}^p, v_1 = \frac{\|KAv_0\|}{\|KAv_1\|}, t = 1$.

While $\|v_t - v_{t-1}\| > \Delta$

\[ v_{t+1} = \frac{\|K_Av_t\|}{\|K_Av_{t+1}\|}, t++ \]

Output $v_t$

The following proposition upper bounds the number of iterations needed to achieve $\|v_t - v_{t-1}\| \leq \Delta$, and implies that Algorithm 1 converges.

**Proposition 3.1 (Algorithm 1 Computation Time).** To get $\|v_t - v_{t-1}\| \leq \Delta$, we need at most $\frac{\log(\frac{\lambda + \|E\|_K}{\Delta^2})}{\log(1 + \Delta^2)}$ iterations, assuming that $v_0^TAv_0 > 0$.

The estimation error rate of Algorithm 1 is derived in Theorem 3.2. To achieve the consistency, $v_0^T\bar{x} \geq c_0 > 0$ for some $c_0$ is required. Sometimes it may be more convenient to assume that one can find a vector $v_0$ for which it is only known that $\|v_0^T\bar{x}\| \geq c_0 > 0$, since the sign of $\bar{x}$ is unknown. In order to facilitate this assumption, we suggest a simple modification to Algorithm 1 which runs the procedure two times, once starting with $v_0$ and once with $-v_0$, and returns the vector $v$ corresponding to the larger quadratic form $v^TAv$. The details are given in Algorithm 2 below. The motivation of this idea is clear: for at least one of the two starts $v_0$ or $-v_0$ we will have a dot product with $\bar{x}$ which is bigger than $c_0$. Next, we hope that the output vector $v$ which has a bigger product $v^TAv$ will be the one that has started with a positive dot product with $\bar{x}$. Theorem 3.3 shows that even if this is not the case, the fact that the product $v^TAv$ is larger gives us a leverage on the final output vector $v$. This vector will be close to $\bar{x}$ in either case.

Algorithm 2 Cone Projected Power Iteration Double Vectors Version

Input: $A \in \mathbb{R}^{p \times p}$ covariance matrix, $\Delta \in \mathbb{R}$ stopping criteria, $K$ a convex cone.

$v_+ = \text{Output of Algorithm 1} \text{ initialized with } v_0$

$v_- = \text{Output of Algorithm 1} \text{ initialized with } -v_0$

Output $v_{out} = \text{arg max}_{v \in \{v_+, v_-\}} v^TAv$

Remark. For certain types of convex cones such as the monotone cone, the time complexity of Algorithm 1 and Algorithm 2 can be calculated explicitly. The complexity of computing $Av_0$ is $O(p^2)$. The projection onto a monotone cone can be obtained by isotonic regression [Barlow et al., 1972]. By using the pool adjacent violators algorithm [Mair et al., 2009, Page 9], isotonic regression can be solved with $O(p)$ flops. By (5), the upper bound of $\|E\|^q_K$ will not grow faster than a constant as long as $\sqrt{\frac{\log p}{n}} \leq \frac{\log p}{n} \sim O(1)$. Moreover, $v_0^TAv_0$ is a constant controlled by the user. Thus the order of iteration number is $O(\frac{1}{\log(1 + \Delta^2)})$. The overall time complexity of Algorithm 1 for monotone cone is $O(\frac{\sum^2_{n=1} |\beta|^2}{\log(1 + \Delta^2)})$. Algorithm 2 retains the same order of complexity since it just applies Algorithm 1 twice.

### 3.3 Error Rate of the Cone Projected Power Iteration

We will now state two results which provide bounds on the $L_2$ error rates for the estimates based on Algorithms 1 and 2. We start with Algorithm 1.

**Theorem 3.2 (L2 Error Rate of Algorithm 1).** Let $v_t$ be the output vector of Algorithm 1. Suppose that we start the algorithm with a $v_0$ satisfying $v_0^T\bar{x} \geq c_0 > 0$. Assume further that $k \geq (3 + 2\sqrt{2})|\|E\|_K|$. Then if $\Delta \leq \min \left\{ \frac{8|\|E\|_K|^2}{\|\beta\|^2_{min} \|v_0^T\bar{x}\|}, \frac{41|\|E\|_K|^2}{\|\beta\|^2_{min} \|v_0^T\bar{x}\|} \right\} + 1$, we have

\[ \|v_t - \bar{x}\| \leq \frac{8|\|E\|_K|^2}{(c_0 \wedge c_1)k} + \frac{41|\|E\|_K|^2}{(c_0 \wedge c_1)k} \]

(2)
We note that the above theorem gives rates which coincide exactly with the rates of the idealized estimator if one assumes that its output has a positive dot product with the target vector \( \tilde{x} \). However Algorithm 1 requires a finite time to run as proved in Proposition 3.1. Next, we move on to studying the performance of Algorithm 2.

We have the following result.

**Theorem 3.3 (L₂ Error Rate for Algorithm 2).** Without loss of generality let \( v₁^T \tilde{x} ≥ c₀ \), and suppose that \( c₀ > c₋₁ \). Assume additionally that \( k ≥ (3 + 2\sqrt{2})|E|K \). For Algorithm 2 let \( v_{\text{out}} = v₁ \) be the output estimator started with \( v₀ \) as an initial vector, and \( v_{\text{out}} = \tilde{v} \) be the output estimator with \( -v₀ \) as initial vector. Then, if \( \tilde{v}_1^T A \tilde{v}_1 ≥ v₁^T A v₁ \), the \( L₂ \) error \( \| \tilde{v}_1 - \tilde{x} \| \wedge \| \tilde{v}_1 + \tilde{x} \| \) is bounded as

\[
\| \tilde{v}_1 - \tilde{x} \| \wedge \| \tilde{v}_1 + \tilde{x} \| ≤ B₁ \wedge B₂
\]

where

\[
B₁ = \sqrt{\frac{2λ}{k}} \| v₁ - \tilde{x} \| + \sqrt{\frac{4|E|K}{k}};
\]

\[
B₂ = \frac{8|E|K}{k} \wedge \frac{8|E|T_{κ(δ)}}{k} + \sqrt{\frac{2λ|v₁ - \tilde{x}|^2 + 8\| v₁ - \tilde{x} \| |E|T_{κ(δ)}}{k}}.
\]

**Remark.** Couple of remarks regarding the conclusion of Theorem 3.3 are in order. First, note that if \( \tilde{v}_1^T A \tilde{v}_1 ≤ v₁^T A v₁ \), Algorithm 2 will output \( v₁ \). Theorem 3.2 guarantees that \( \| v₁ - \tilde{x} \| \) is small. Otherwise the output will be \( \tilde{v}_1 \). Theorem 3.3 ensures that \( \| \tilde{v}_1 - \tilde{x} \| \wedge \| \tilde{v}_1 + \tilde{x} \| \) is small based on the fact that \( \| v₁ - \tilde{x} \| \) is small. In fact a simple calculation shows that

\[
\| \tilde{v}_1 - \tilde{x} \| \wedge \| \tilde{v}_1 + \tilde{x} \| ≤ \sqrt{|E|K} \wedge (|E|K \wedge |E|T_{κ(δ)}),
\]

where the sign \( ≥ \) here means \( ≤ \), but ignores constants that may depend on \( λ, μ, c₀ \wedge c₁ \) for simplicity. Observe that this inequality is similar to the one obtained for the idealized estimator.

Next, we mention that the proof of Theorem 3.3 does not rely on the fact that we have two output vectors \( v₁ \) and \( \tilde{v}_1 \) to analyze, but utilizes the fact that the final vector has a bigger product with the matrix \( A \). Hence, the proof immediately extends to situations where one has multiple starting vectors \( v \) such that \( v^T A v ≥ 0 \) to pick from. In particular, suppose that one has a \( δ \) covering set of the set \( K \cap \mathcal{S}^{p-1} \), denoted with \( N₂(δ, K \cap \mathcal{S}^{p-1}) \), for some fixed \( δ > 0 \). Then we know there exists a vector \( v₀ ∈ N₂(δ, K \cap \mathcal{S}^{p-1}) \) such that \( \tilde{v}_0^T \tilde{x} = 1 - \frac{|v₀ - \tilde{x}|^2}{2} ≥ 1 - \frac{δ^2}{4} \). Supposing that \( δ \) is small enough so that \( 1 - \frac{δ^2}{4} > c₋₁ \), we can then run Algorithm 1 on all vectors \( v ∈ N₂(δ, K \cap \mathcal{S}^{p-1}) \) such that \( v^T A v ≥ 0 \) (it can be shown that under the assumptions of Theorem 3.3 \( v₀ \) is necessarily such a vector, see the supplement for a short proof) and output the vector \( v_{\text{out}} \) with the largest \( v^T A v \). Then either inequality (2) or (3) will hold for \( v_{\text{out}} \). In both cases \( v_{\text{out}} \) will be close to \( \tilde{x} \). The caveat with this procedure is that the packing set \( N₂(δ, K \cap \mathcal{S}^{p-1}) \) might be very large so running this method might become computationally prohibitive. We note that in any case it is certainly true that we have \( N₂(δ, K \cap \mathcal{S}^{p-1}) ≤ N₂(δ, \mathcal{S}^{p-1}) ≤ (1 + \frac{δ}{2})^p \) ([Wainwright, 2019, Example 5.2]).

### 4 Upper and Lower Bound under the Spiked Covariance Model

#### 4.1 Lower Bound

In order to appreciate the results of Section 3 we now give a concrete example of a statistical model – the spiked covariance model. Within this model, we show an upper bound on the norm \( |E|K \) in terms of the Gaussian complexity of the set \( K \cap \mathcal{S}^{p-1} \), and complement the upper bound with a lower bound on the minimax risk.

Let us observe \( n \) i.i.d. samples from the model \( X_i \sim \mathcal{N}(0, I + k\tilde{x}\tilde{x}^T) \), where \( \tilde{x} ∈ \mathcal{S}^{p-1} \) and \( i = 1, 2, \ldots, n \). This model is known as the spiked covariance model, and has been intensely studied in the literature. It was first considered by [Johnstone and Lu, 2009]. Construct the sample covariance matrix \( \hat{A} = n⁻¹\sum_{i=1}^n X_i X_i^T \), and the target matrix \( A = I + k\tilde{x}\tilde{x}^T \). Clearly then, the noise matrix equals to \( E = A - \hat{A} \). The parameter \( k \), which is the first eigengap of the matrix \( \hat{A} \), can be understood as signal strength in the model.

We will now show a lower bound of the \( L₂ \) error of the eigenvector estimation under the spiked covariance model. This lower bound is obtained using information theory and is related to the Gaussian complexity of \( K \cap \mathcal{S}^{p-1} \).

**Theorem 4.1 (Minimax Lower Bound under Spiked Covariance Model).** Suppose we have \( n \) i.i.d. observations \( X₁, \ldots, X_n \) where \( X_i \sim \mathcal{N}(0, I + k\tilde{x}\tilde{x}^T) \). Let \( \tilde{x} ∈ K \) where \( K \subset \mathbb{R}^p \) is a convex cone with \( w(K \cap \mathcal{S}^{p-1}) ≥ 2\sqrt{\log 3} \).
There exists $k \leq \left( \frac{n}{4p} \right) \sqrt{\frac{2n}{k}}$ such that the minimax risk of estimations of $\bar{x}$ is bounded below by

$$\inf_{\tilde{v}} \sup_{\bar{x} \in K \cap S^{p-1}} \mathbb{E}[\tilde{v} - \bar{x}]^2 \geq \frac{C'w^2(K \cap S^{p-1})}{(\log p)^2 n(k \wedge k^2)},$$

where $C' > 0$ is some universal constant.

**Remark.** Note that in the above result we set the signal strength $k$ to a particular value in order to obtain the lower bound. Hence this lower bound is not applicable to any possible signal strength. In addition this is a lower bound regarding the quantity $||\tilde{v} - \bar{x}||^2$ and not $||\tilde{v} - \bar{x}||^2 \wedge ||\tilde{v} + \bar{x}||^2$. The latter can be smaller in principle. Carefully inspecting the proof of Theorem 4.1 shows however that if $K' \subseteq K$ is such that all vectors $v, w \in K'$ satisfy $v^Tw \geq 0$ we have that there exists a $k \leq \left( \frac{n}{4p} \right) \sqrt{\frac{2n}{k}}$ such that

$$\inf_{\tilde{v}} \sup_{\bar{x} \in K' \cap S^{p-1}} \mathbb{E}[\tilde{v} - \bar{x}]^2 \wedge ||\tilde{v} + \bar{x}||^2 \geq \frac{C'w^2(K' \cap S^{p-1})}{(\log p)^2 n(k \wedge k^2)}.$$  

### 4.2 Comparison of the Upper and Lower Bound

The first result of this section shows an upper bound on the cone-constrained operator norm $\|E\|_K$ under the spiked covariance model. We have

**Lemma 4.1 (Upper Bound on $\|E\|_K$ under Spiked Covariance Model).** Suppose we have $n$ i.i.d. observations from the model above and the matrix $E = A - \tilde{A}$. Let $\tilde{x} \in K$, where $K \subseteq \mathbb{R}^p$ is a convex cone. Then with probability at least $1 - 6e^{-\frac{w^2(K \cap S^{p-1})}{8n}} - 2e^{-7}$,

$$\|E\|_K \leq 2\sqrt{k} + \left[ \frac{w(K \cap S^{p-1})}{\sqrt{n}} \vee \frac{w^2(K \cap S^{p-1})}{n} \right] \vee \frac{k + 2 - 2\sqrt{k} + 1}{\sqrt{n}}. \tag{4}$$

Lemma 4.1 implies that in the spiked covariance model, the upper bounds on the $L_2$ error rates derived in Section 2 can be translated to purely geometric measures of the size of the set $K$.

We now compare the upper bound in Theorem 3.2 with the lower bound in Theorem 4.1 for $X_i \sim N(0, I + k\tilde{x}\tilde{x}^T)$. Combining Lemma 4.1 with the upper bound using $\|E\|_K$ in Theorem 3.2 we obtain

$$\|v_i - \bar{x}\|^2 \leq \frac{8\|E\|_K}{(c_0 \wedge c_1)k} \leq \frac{1}{(k \wedge \sqrt{k})} \left[ \frac{w(K \cap S^{p-1})}{\sqrt{n}} \vee \frac{w^2(K \cap S^{p-1})}{n} \right] + \frac{1}{\sqrt{n}},$$

where $\leq$ is up to universal constants (here we treat $c_0$ as a constant which does not scale with $n$). Most of the time one would expect to have $\frac{w^2(K \cap S^{p-1})}{n} \leq \frac{w(K \cap S^{p-1})}{\sqrt{n}(k \wedge \sqrt{k})}$, in which case the above upper bound reduces to

$$\|v_i - \bar{x}\|^2 \leq \frac{w(K \cap S^{p-1})}{\sqrt{n}(k \wedge \sqrt{k})} + \frac{1}{\sqrt{n}}.$$

This is up to a log $p$ factor and the term $\frac{1}{\sqrt{n}}$ is of the order of the square root of lower bound of Theorem 4.1.

The punchline of this comparison is that the upper bound is moderately larger than the lower bound. Since the two bounds do not match this suggests several possibilities: 1) the lower bound is loose; 2) the upper bound is loose; 3) there exist better algorithms for estimating $\tilde{x}$; 4) some combination of the previous 3 parts. Since we attempt to solve the problem in a very large generality (for any convex cone $K$), our conjecture is that for specific cones $K$ there might exist algorithms geared towards these cones which exhibit better performance compared to the idealized estimator or the cone projected power iteration. Furthermore, for specific sets $K$ there might exist more efficient packing sets than the one used in the proof of the lower bound (which relies on the reverse Sudakov’s minoration inequality), therefore yielding tighter lower bounds. Nevertheless we will show a specialized example in next subsection where the upper bound for the cone projected power iteration nearly matches the lower bound.

### 4.3 Case Study: Monotone Cone

In this section we will look at a specific example of the set $K$ under the spiked covariance model. Namely, we consider the case when $K = M$ where $M \subseteq \mathbb{R}^p$ is the monotone cone given by $M = \{ (x_1, ..., x_p)^T \in \mathbb{R}^p : x_1 \leq ... \leq x_p \}$. For the monotone cone $M$ there exist explicit formulas for the Gaussian complexity $w(M \cap S^{p-1}) = \sqrt{\log p}$ for large $p$. This is proved in Amelunxen et al. (2014, Section D.4), and also can be calculated numerically through Monte Carlo simulations (Donoho et al. 2013, Lemma 4.2). Furthermore, for a piecewise constant vector $\bar{x} \in M$...
with \( m \) constant pieces, the order of tangent cone of \( M \) at \( \bar{x} \) has an explicit order as \( w(M(\bar{x})) \approx \sqrt{m \log \frac{p}{n}} \) (Bellec et al., 2018, Proposition 3.1).

With the order of Gaussian complexity \( w(M \cap \mathcal{S}^{p-1}) \) and \( w(T_M(\bar{x}) \cap \mathcal{S}^{p-1}) \) shown above, and the result in Lemma 4.1, we are now able to upper bound the restricted operator norms \( \|E\|_{M} \) and \( \|E\|_{T_M(\bar{x})} \) as

\[
\|E\|_{M} \leq 2\sqrt{k+1} \left( \sqrt{\frac{\log p}{n}} \vee \frac{\log p}{n} \right) + \frac{k}{\sqrt{n}}
\]

\[
\|E\|_{T_M(\bar{x})} \leq 2\sqrt{k+1} \left( \sqrt{\frac{m \log p}{m}} \vee \frac{m \log p}{m} \right) + \frac{k}{\sqrt{m}}
\]

Suppose now that the vector \( \bar{x} \) has \( m_0 \) constant pieces where \( m_0 \) is not allowed to scale with \( n \). Combining the identity (6) with the upper bound using \( \|E\|_{T_M(\bar{x})} \) in Theorem 4.1, we obtain

\[
\|v_1 - \bar{x}\|^2 \leq \left( \frac{31 \|E\|_{T_M(\bar{x})}}{(c_0 \land c_1) k} \right)^2 \leq \frac{\log p}{n(k \land k^2)} + \frac{1}{n}
\]

whenever \( \sqrt{\frac{\log p}{n}} > \frac{\log p}{n} \). We will now exhibit a lower bound for piecewise constant vectors which is nearly of the same order of the above upper bound. To this end define the set \( \mathcal{M}_0 \) as the identity (6) with the upper bound using

\[
\|E\|_{T_M(\bar{x})} \leq 2\sqrt{k+1} \left( \sqrt{\frac{m \log p}{m}} \vee \frac{m \log p}{m} \right) + \frac{k}{\sqrt{m}}
\]

Proposition 4.1 (Minimax Lower Bound for Piecewise Constant Monotone Vectors). Suppose we observe \( X = [X_1; \ldots; X_p] \) where \( X_i \) are samples from the spiked covariance model described above. Denote the sample covariance matrix \( \hat{A} = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top \) and the true covariance matrix \( A \). Let \( M_0 \subset M \) be as defined above with \( m_0 \geq 3 \) which does not grow with \( p \) and \( n \). Let the true first eigenvector of \( \hat{A} \) be \( \bar{x} \). The \( L_2 \) estimation of \( \bar{x} \) has a lower bound

\[
\inf_{\hat{v}} \max_{\bar{x} \in \mathcal{M}_0} \mathbb{E}[\|\hat{v} - \bar{x}\|^2 \land \|\hat{v} + \bar{x}\|^2] \geq \frac{\log p}{n(k \land k^2)}
\]

Similarly to Theorem 4.1, the proof of Proposition 4.1 relies on Fano’s inequality. However the construction used to obtain a packing set is quite different. We note that the upper bound in 4 and the lower bound in Proposition 4.1 are different up to a log factor.

5 Experiments

5.1 Simulations

In this section, we gauge the performance of Algorithm 2 by comparing its estimation error and run time with three other algorithms which compute the principal eigenvector on different simulated data. Those algorithms include ordinary power iteration (Mises and Pollaczek-Geiringer, 1929), Truncated Power Iteration (Yuan and Zhang, 2013), and ElasticNet SPCA (Zou et al., 2006).

We first run all the above algorithms on matrices whose principal eigenvector is monotone but not sparse. In this case, our Algorithm 2 is expected to perform much better than algorithms which assume the sparsity of the first eigenvector, such as the Truncated Power Iteration (Yuan and Zhang, 2013) and ElasticNet SPCA (Zou et al., 2006).

The data set is generated from a multivariate Gaussian distribution with zero mean and spiked covariance matrix \( \Sigma = I + k \bar{x} \bar{x}^\top \). Here we use \( k = \log p \), where \( p \) is the dimension of \( \bar{x} \). The first eigenvector \( \bar{x} \) is formulated as \( \bar{x}_i = i \sqrt{\sum_{j=1}^{p} \bar{x}_j^2} \). The experiments are run in different dimensions: \( p = 100, p = 1000 \), and \( p = 10000 \). For each scale of \( p \), we also examine different \( n \) values: \( 10 \log p, 0.3p, p, 5p \) and \( 10p \). For each combination of \( p \) and \( n \), the data matrix \( X \in \mathbb{R}^{n \times p} \) is generated by drawing \( n \) samples from \( N(0, \Sigma) \). Then we compute the first eigenvector of the empirical covariance matrix \( \hat{\Sigma} = \frac{1}{n} X^\top X \). The stopping criteria \( \Delta = 10^{-6} \) is used for all power algorithms. The hyperparameters in ElasticNet SPCA and Truncated Power Iteration are tuned by grid search. Finally we compute the \( L_2 \) distance of the estimated eigenvector to the true eigenvector \( \bar{x} \), and record the CPU time of computing each estimation. For each combination of \( n \) and \( p \), the average \( L_2 \) error and CPU time over 50 repetitions are presented in Fig. 1.

From Fig. 1 we can see that Algorithm 2 has a significantly smaller \( L_2 \) error among all the algorithms under all the combinations of \( n \) and \( p \). Algorithm 2 achieves a small \( L_2 \) error even when \( n \ll p \), while other algorithms are not able to converge. Notice that the \( L_2 \) error of Truncated Power Iteration and ElasticNet SPCA are almost identical with that of ordinary power iteration. That’s because the tuned \( \lambda \) in ElasticNet SPCA is very small and the tuned cardinality in Truncated Power Iteration equals to 1. In this case, the two sparse algorithms are
actually almost equivalent with the ordinary power iteration. In terms of the run time, Algorithm 2 obtains more advantage when \( p \) gets larger. When \( p = 100 \), Algorithm 2 is slower than others, but it’s still very fast given the small scale of all run times in this setting. When \( p = 1000 \) and \( n < p \), the run times of other algorithms are more than twice of that of the Algorithm 2. When \( p = 10000 \) and \( n < p \), other algorithms have run times more than 4 times of that of Algorithm 2. Additionally, one should be aware that the hyperparameter tuning of ElasticNet SPCA and Truncated Power Iteration takes extra time. It’s remarkable that the run time of Algorithm 2 is even smaller than the ordinary power iteration when \( p \) is large. This is due to less iterations Algorithm 2 needs to converge, even thought every iteration of it is more costly than the ordinary one.

Figure 1: The \( L_2 \) Error and Run Time for Different Algorithms on Simulated Matrix with Non-sparse First Eigenvector. ·-·, ordinary power iteration; ·-·, Truncated Power Iteration; ·-·, ElasticNet SPCA; ·-·, Algorithm 2

Figure 2: The \( L_2 \) Error and Run Time for Different Algorithms on Simulated Matrix with Sparse First Eigenvector. ·-·, ordinary power iteration; ·-·, Truncated Power Iteration; ·-·, ElasticNet SPCA; ·-·, Algorithm 2
In the second case, the data generation process is the same as the first experiment, except the first principle eigenvector of the population covariance matrix $\bar{x}$ is formulated as a sparse vector $\{\bar{x}_i = 0 \text{ if } i \leq 0.3p, \bar{x}_i = 1 \text{ if } i > 0.3p\}$. We still use $\Delta = 10^{-6}$ as the stopping criteria and grid search for hyperparameter tuning. Since the first eigenvector is sparse and monotone, algorithms which exploit the sparsity of the first eigenvector should perform well. However, we can see from Fig. 2 that Algorithm 2 still retains similar advantages of $L_2$ error and run time in the sparse case.

5.2 S&P Stock Price Data

In this subsection we use the different PCA algorithms to cluster S&P 500 stocks. It’s common to form clusters based on the principal components. Experiments in this section show that the eigenvector estimated by Algorithm 2 produces good clusters on an S&P 500 stock price time series data set (Yahoo! Finance, 2020).

The cleaned data set contains the daily price of 468 stocks from 2013 to 2018. Each stock has 1259 price observations. It is worth mentioning that the S&P 500 index was generally increasing throughout this time period which motivates the assumption that the principal eigenvector of the data is monotone. We use the first half of data as the train set, and second half of data as the test set. To amplify the effect of price change rather than focus on the absolute price, we normalize every row of the train set and then calculate its empirical covariance matrix $\hat{\Sigma}$. The ordinary power iteration, Truncated Power Iteration, ElasticNet SPCA, and Algorithm 2 with monotone cone are used to estimate the first eigenvector of $\hat{\Sigma}$. Before the eigenvector calculation, we use grid search to tune $\lambda$ of ElasticNet SPCA and the cardinality of Truncated Power Iteration. For every stock in the train set, we project its price vector on four estimated principal eigenvectors to get four principal components. Then we cluster stocks based on four principal components respectively. The clustering method used here is DBSCAN (Ester et al., 1996). Statistics of the clusters can be found in Table 1.

Naturally, we expect stocks in the same cluster to be closely related to each other. For each cluster, we calculate the in-cluster future price correlation (IFPC) defined as the average of all pairwise future price correlations in the cluster. We then use the average in-cluster future price correlation (AIFPC) among all clusters to measure the usefulness of a clustering result. A higher AIFPC implies that stocks in the same cluster relate to each other more closely. Hence a clustering with higher AIFPC may give more informative guidance on decision making such as asset portfolio management. The AIFPCs of the prices in test set for clusters formed by different methods are in Table 2. The average pairwise correlation of future prices without clustering is 0.2964, so the clusters do pick out stocks highly correlated with each other. The clusters formed by Algorithm 2 with monotone cone have the largest AIFPC.

| Cluster Numbers | Max Cluster Size | Min Cluster Size | Average Cluster Size |
|-----------------|------------------|------------------|----------------------|
| Algorithm 2 with Monotone Cone | 14 | 196 | 4 | 22 |
| Ordinary Power Iteration | 13 | 209 | 4 | 24 |
| Truncated Power Iteration | 21 | 59 | 5 | 12 |
| ElasticNet SPCA | 20 | 59 | 5 | 14 |

Table 1: Statistics of Clusters Formed by Four Different Principal Components

| Algorithm 2 with Monotone Cone | Ordinary Power Iteration | Truncated Power Iteration | ElasticNet SPCA |
|--------------------------------|--------------------------|--------------------------|-----------------|
| AIFPC                          | 0.5238                   | 0.4709                   | 0.3914          | 0.4462          |

Table 2: The AIFPCs for Clusters Formed by Different Methods

5.3 Major Cities Temperature and Precipitation Data

We evaluated the performance of Cone Projection Power Iteration on the earth surface temperature data set. The raw data was obtained from Berkeley Earth (2020). The cleaned data set consists of yearly average temperatures from 1881 to 2013 for 97 cities worldwide. Geological and biological research revealed that the climate change will be greater at higher latitudes (Deutsch et al., 2008). In this experiment, we found that the eigenvector estimation obtained by Algorithm 2 explained the most correlation between temperature change and latitude.

Based on the description above, the cleaned data set has 97 cities and 133 temperature observations for each city. Thus it’s a data set with $n$ slightly smaller than $p$. We normalized every row of data in order to remove the effect of the scale of temperature, since we are interested in temperature change rather than the absolute
temperature. Four algorithms – ordinary power iteration, Truncated Power Iteration, ElasticNet SPCA, and Algorithm 2 with monotone cone – are used to estimate the principal eigenvector of the empirical covariance matrix $\hat{\Sigma}$. After getting the estimated eigenvectors, we project each city’s temperature vector onto them to get scores. Finally, we calculated the correlations between each of the city scores and city latitudes. From Table 3 we can see the principal eigenvector estimation obtained by Algorithm 2 explained most correlation between temperature change and latitude. From Figure 3 we can see the distribution of scores obtained by Algorithm 2 over all cities on the world map.

| Algorithm | Correlation |
|-----------|-------------|
| Monotone Cone | 0.7097 |
| Ordinary Power Iteration | 0.5937 |
| Truncated Power Iteration | 0.6233 |
| ElasticNet SPCA | 0.6209 |

Table 3: The Correlation between Score and Latitude

Figure 3: The Distribution of Scores Obtained by Algorithm 2 Over All Cities.

6 Discussion

In this paper we propose a cone projected power iteration method to tackle the problem of finding the principal eigenvector in a positive semidefinite matrix obscured by stochastic noise. Unlike other proposals our algorithm is tuning parameter free, and is guaranteed to converge at a finite time (given reasonable initialization). We provide extensive numerical and real data experiments to validate our theoretical findings.

Despite the pervasiveness of sparse constraints in principal component analysis, methods with other constraints are scarce in the literature. Some relevant works include constraining the principal eigenvector in the non-negative orthant cone (Montanari and Richard, 2013) and a convex cone (Deshpande et al., 2014). Clearly, the non-negative orthant cone is a special case of a convex cone. Principal component analysis with non-negative constraints naturally arises in neural signal processing (Pavlov et al., 2007; Quiroga and Panzeri, 2004), computer vision (Lee and Seung, 1999; Arora et al., 2016), and gene expression (Lazzaroni and Owen, 2002).

We believe that the monotone cone is another example of a practically useful convex cone. For instance, in time series forecasting, where often times more recent observations are more important, imposing monotonicity constraints makes intuitive sense. In addition due to the nature of the monotone cone, vectors can be estimated with high precision even in very high-dimensional settings. This is not true for the non-negative orthant cone which is a much “larger” cone.

We would like to point out a couple of remaining open questions which are worth future investigation. The gap in the lower and upper bounds for the general convex cone setting suggests that for some cones, there might be case specific algorithms which achieve smaller estimation error, or the packing set of the cone can be constructed more efficiently. Closing this gap presents an interesting and challenging future direction.

Another open question is whether we can relax the assumption that the dot product of the initial vector and the true eigenvector is greater than a constant. We proved the consistency of the estimation under this
We have the following simple bounds for $c$.

**Lemma B.1.** Let $\eta \in \{-1, 1\}$.

$$ c_0 = \frac{k - |E|_K + \eta \sqrt{(k - |E|_K)^2 - 4k|E|_K}}{2k}, \text{ for } \eta \in \{-1, 1\}. $$

We have the following simple bounds for $c_{-1}$ and $c_1$.

**Lemma B.1 (Bounds for $c_{\pm 1}$).** Assumption that $k \geq (3 + 2\sqrt{2})|E|_K$ we have the following relationships

$$ |E|_K \leq c_{-1} = \frac{(3 + 2\sqrt{2})|E|_K}{(1 + \sqrt{2})k} \text{ and } \frac{2}{5} < \frac{1}{1 + \sqrt{2}} \leq c_1. $$

**Proof of Lemma B.1.** Note that $c_{-1}, c_1$ are well defined when $k \geq (3 + 2\sqrt{2})|E|_K$. Furthermore the following holds:

$$ |E|_K \leq \frac{|E|_K}{k - |E|_K} \leq \frac{2|E|_K}{k - |E|_K} + \sqrt{(k - |E|_K)^2 - 4|E|_K k} = c_{-1}, $$

$$ \leq \frac{2|E|_K}{k - |E|_K} \leq \frac{(3 + 2\sqrt{2})|E|_K}{(1 + \sqrt{2})k}. $$

In addition

$$ c_1 \geq \frac{k - |E|_K}{2k} > \frac{1 + \sqrt{2}}{3 + 2\sqrt{2}} = \frac{1}{1 + \sqrt{2}}. $$

**Lemma B.2.** Under the assumption of Lemma B.1 if $v_0^T \bar{x} > c_{-1}$ it follows that $v_0^T A_0 > 0$.

**Proof of Lemma B.2.** Consider $\bar{x}^T A_0 \geq \lambda v_0^T \bar{x} - |E|_K > \lambda c_{-1} - |E|_K > 0$ by Lemma B.1 Hence by Cauchy-Schwartz $v_0^T A v_0 > 0$.

The following theorem states that any vector $z \in \mathbb{R}^p$ can be decomposed as the sum of its projection onto $K$, and its projection onto $K$’s polar cone $K^\circ$, where $K^\circ = \{u : \langle u, v \rangle \leq 0, \forall v \in K\}$. It is a fundamental result in convex analysis.

**Theorem B.1.** [Moreau 1962](Moreau’s Decomposition) Let $K \subset \mathbb{R}^p$ be a convex cone and $K^\circ$ be its polar cone.

For $x, y, z \in \mathbb{R}^p$, the following properties are equivalent:

1. $z = x + y, x \in K, y \in K^\circ$, and $\langle x, y \rangle = 0$.
2. $x = \Pi_K z, y = \Pi_{K^\circ} z$.

The following Lemma provides some equations and inequalities which will be referred a lot in consequent sections.

**Lemma B.3.** Suppose that the vectors $v_2$ are recursively defined as $v_t = \frac{\Pi_K Av_{t-1}}{||\Pi_K Av_{t-1}||}$.

1. $||\Pi_K Av_{t-1}|| = v_t^T A v_{t-1}$.
Proof of Lemma B.3.

1. \( v_i^T A v_{l-1} = \frac{\langle \Pi_K A v_{l-1}, A v_{l-1} \rangle}{\| \Pi_K A v_{l-1} \|} = \frac{\langle \Pi_K A v_{l-1}, \Pi_K A v_{l-1} \rangle + \langle \Pi_K A v_{l-1}, \Pi_K^c A v_{l-1} \rangle}{\| \Pi_K A v_{l-1} \|} \) by Moreau's Decomposition.

2. One can write \( x = (\bar{x}^T x) \bar{x} + \sqrt{1 - (\bar{x}^T x)^2} \bar{x}^T x \). Then
   \[ x^T A x = (\bar{x}^T x)^2 \lambda + [1 - (\bar{x}^T x)^2] \bar{x}^T A \bar{x} \leq (\bar{x}^T x)^2 \lambda + [1 - (\bar{x}^T x)^2] \mu \]

3. On one hand
   \[ |v^T E v - \bar{x}^T E \bar{x}| \leq |(v - \bar{x})^T E(v - \bar{x}) + 2 \bar{x}^T E(v - \bar{x})| \leq \|v - \bar{x}\|_K E_{\| \bar{x} \|_K} + 2 \|v - \bar{x}\| \|E\|_{\| \bar{x} \|_K} \leq 4 \|v - \bar{x}\| \|E\|_{\| \bar{x} \|_K} \]

On the other hand,
   \[ |v^T E v - \bar{x}^T E \bar{x}| \leq |2 \bar{x}^T E(v + \bar{x})| + |(v + \bar{x})^T E(v + \bar{x})| \]

Since \( v + \bar{x} \in K \) it follows that \( \frac{\bar{x}^T x}{\|v + \bar{x}\|} \in K \) and therefore
   \[ |v^T E v - \bar{x}^T E \bar{x}| \leq 2 \|v + \bar{x}\| \|E\|_K + \|v + \bar{x}\| \|E\|_K \leq 4 \|v + \bar{x}\| \|E\|_K \]

4. By Cauchy-Schwartz inequality we have
   \[ v_i^T A v_{l-1} \leq \sqrt{(v_i^T A v_i)(v_i^T A v_{l-1})} \leq v_i^T A v_i \]

where \( v_i^T A v_i \geq v_{l-1}^T A v_{l-1} \) comes from the first part of the proof of Proposition 1. \( \square \)

Appendix C  Proof of Theorem 1 and Corollary 1

C.1 Proof of Theorem 1

Suppose first that \( v^T \bar{x} \geq 0 \).

\[ v^T E v - \bar{x}^T E \bar{x} = v^T A v - \bar{x} \bar{x} A \bar{x} - \bar{x} \bar{x} A \bar{x} \geq \bar{x} \bar{x} A \bar{x} - v^T A v \geq \lambda - (v^T x)^2 \lambda - [1 - (v^T \bar{x})^2] \mu \]

By 2. in Lemma B.3

\[ \geq k[1 - (v^T \bar{x})^2] \]

\[ \geq \frac{1}{2} k \|v - \bar{x}\|^2 \]

For the first term, by the definition of \( \|E\|_K \), we can directly get \( v^T E v - \bar{x}^T E \bar{x} \leq 2 \|E\|_K \). For the second term, notice that

\[ v^T E v - \bar{x}^T E \bar{x} \leq 4 \|v - \bar{x}\| \|E\|_{\| \bar{x} \|_K} \]

By 3. in Lemma B.3

so

\[ \frac{1}{2} k \|v - \bar{x}\|^2 \leq 4 \|v - \bar{x}\| \|E\|_{\| \bar{x} \|_K} \quad \Rightarrow \quad \|v - \bar{x}\| \leq \frac{8 \|E\|_{\| \bar{x} \|_K}}{k} \]

12
Next suppose that $v^T \bar{x} \leq 0$. Repeating the same proof as above we observe that

$$v^T E v - \bar{x}^T E \bar{x} \geq \frac{1}{2} \lambda ||v + \bar{x}||^2.$$ 

This and the bound $v^T E v - \bar{x}^T E \bar{x} \leq 2||E||_K$ directly give the proof of one of the inequalities. Next by 3. in Lemma B.3 we have

$$v^T E v - \bar{x}^T E \bar{x} \leq 4||v + \bar{x}|| E_K$$

which implies the second bound.

**C.2 Proof of Corollary 1**

It’s sufficient to show $||E||_{\tau_K(\bar{x})} \lor ||E||_{op} \leq ||E||_K$. This is obvious by the definitions of $||E||_{\tau_K(\bar{x})}$ and $||E||_K$.

**Appendix D Proof of Proposition 1**

1. Show $\frac{v_1^T A v_{t+1}}{v_1^T A v_t} \geq 1 + ||v_{t+1} - v_t||^2$, so $\{v_t^T A v_t\}$ is an increasing series.

   By Moreau’s decomposition and 1. in Lemma B.3

   $$v_{t+1}^T A v_{t+1} = v_{t}^T \Pi_K A v_{t} \geq \frac{v_{t}^T A v_{t}}{v_{t+1}^T A v_{t}}$$

   Then

   $$v_{t+1}^T A v_{t+1} - v_{t}^T A v_{t} = 2\langle A v_t, v_{t+1} - v_t \rangle + (v_{t+1} - v_t)^T A (v_{t+1} - v_t)$$

   $$\geq 2\langle A v_t, v_{t+1} - v_t \rangle$$

   $$\geq 2v_{t+1}^T A v_0 (1 - v_t^2)$$

   By (8)

   $$\geq v_t^T A v_t ||v_{t+1} - v_t||^2$$

   Rearrange the terms to get the result. Since $\{v_t^T A v_t\}$ is strictly increasing, and bounded by the first principal eigenvalue of $A$, the algorithm converges. At convergence, $\{v_t^T A v_t\}$ is stationary.

2. Suppose $||v_t - v_{t-1}|| \geq \Delta$ for $t \leq n$, then

   $$\frac{v_n^T A v_n}{v_0^T A v_0} \geq (1 + \Delta^2)^n \Rightarrow n \leq \frac{\log(1 + ||E||_K)}{\log(1 + \Delta^2)}$$

**Appendix E Proof of Theorem 2 and Theorem 3**

A key to analyze the $L_2$ error of the cone projected power iteration estimator is: If $\bar{x}^T v_0$ is larger than a certain constant, $\bar{x}^T v_t$ is always larger than that constant throughout the iterations. This is rigorously presented in the following lemma B.1. It is a foundation for the proof of Theorem 2.

**Lemma E.1.** Suppose $\bar{x}^T v_0 \geq c_0$ for some $c_0 > c_{-1}$, and the first eigengap of $\bar{A}$ is greater than $(3 + 2\sqrt{2})||E||_K$, then $\bar{x}^T v_t \geq c_0 \land c_1$, $\forall t \in N$.

**Proof of Lemma E.1.** We prove this result by induction.

Given $\bar{x}^T v_{t-1} \geq c_0$, if $\bar{x}^T v_t \geq \bar{x}^T v_{t-1}$, the inequality preserves trivially.

Next suppose that $\bar{x}^T v_t \leq \bar{x}^T v_{t-1}$. We first show that $\bar{x}^T v_t$ is non-negative. We have the identity $\bar{x}^T v_t = \bar{x}^T \Pi_K A v_{t-1} \Pi_K A v_{t-1}$, so it suffices to show that $\bar{x}^T \Pi_K A v_{t-1} \geq \bar{x}^T A v_{t-1} = \lambda \bar{x}^T v_{t-1} - ||E||_K \geq 0$, which holds since $\bar{x}^T v_{t-1} \geq c_0 \land c_1 > c_{-1} \geq \frac{||E||_K}{\lambda}$ (the last inequality is shown in Lemma B.1).
Next observe the identities,
\[
\bar{x}^T v_t = \frac{\bar{x}^T \Pi_K A v_{t-1}}{\|\Pi_K A v_{t-1}\|} \geq \frac{\bar{x}^T A v_{t-1}}{v_1^T A v_{t-1}}
\]
By Moreau’s decomposition and 1. in Lemma \ref{lem:sobolev}
\[
\geq \frac{\bar{x}^T A v_{t-1}}{v_1^T A v_t}
\]
By 4. in Lemma \ref{lem:sobolev}
\[
\geq \frac{\bar{x}^T A v_{t-1} - \|E\|_K}{v_1^T A v_t + \|E\|_K}
\]
By 2. in Lemma \ref{lem:sobolev}
\[
\geq \frac{\lambda \bar{x}^T v_{t-1} - \|E\|_K}{k(\bar{x}^T v_{t-1}) + \mu + \|E\|_K}
\]
By the condition \(0 \leq \bar{x}^T v_t \leq \bar{x}^T v_{t-1}\)

Let \(\alpha = \bar{x}^T v_{t-1}\), and \(f(y) = \frac{\lambda y - \|E\|_K}{(\lambda + |E|_K)}\). Note that the roots of the quadratic equation \(f(y) = y\) are \(c_{-1}\) and \(c_1\), so that for \(y \in (-\infty, c_{-1}] \cup [c_1, \infty)\), we have \(y \geq f(y)\). By the fact that \(\alpha = \bar{x}^T v_{t-1} \geq f(\alpha)\) and \(\alpha \geq c_0 \land c_1 > c_{-1}\), it follows that \(\alpha \geq c_1\). In addition since \(\bar{x}^T v_t \geq f(\alpha)\), it’s sufficient to show \(f(\alpha) \geq c_0 \land c_1\). Notice that \(f(y)\) is increasing, so \(f(\alpha) \geq f(c_1) = c_1 \geq c_0 \land c_1\). Thus the proof is complete. \(\blacksquare\)

E.1 Proof of Theorem 2

For the first term, we can start to derive a lower bound and an upper bound of \(\bar{x}^T A v_{t-1}\) based on some inequalities from Lemma \ref{lem:sobolev}
\[
\bar{x}^T A v_{t-1} = \lambda \bar{x}^T v_{t-1} + \bar{x}^T E v_{t-1} \geq \lambda \bar{x}^T v_{t-1} - \|E\|_K
\]
\[
\bar{x}^T A v_{t-1} = \bar{x}^T \Pi_K A v_{t+1} + \bar{x}^T \Pi_K A v_{t-1} \leq \bar{x}^T \Pi_K A v_{t+1} = \|\Pi_K A v_{t-1}\| \bar{x}^T v_t = (v_1^T A v_{t-1}) \bar{x}^T v_t
\]
Combine the above two inequalities to get
\[
\lambda \bar{x}^T v_{t-1} - \|E\|_K \leq (v_1^T A v_{t-1}) \bar{x}^T v_t = (v_1^T A v_{t-1}) \bar{x}^T v_{t-1} + (v_1^T A v_{t-1}) \bar{x}^T (v_t - v_{t-1})
\](9)
Since \(\|v_t - v_{t-1}\| \leq \Delta\), by Cauchy-Schwartz inequality and the definition of \(\|E\|_K\),
\[
(v_1^T A v_{t-1}) \bar{x}^T (v_t - v_{t-1}) \leq |v_1^T A v_{t-1} + v_1^T E v_{t-1}| |v_t - v_{t-1}|
\leq (\lambda + \|E\|_K) \Delta
\]
Furthermore, with the use of results in Lemma \ref{lem:sobolev}
\[
(v_1^T A v_{t-1}) \bar{x}^T v_{t-1} = (v_1^T A v_{t-1} + v_1^T E v_{t-1}) \bar{x}^T v_{t-1}
\leq \leq (v_1^T A v_{t-1} + v_1^T E v_{t-1}) |v_t - v_{t-1}| + (\|E\|_K) \bar{x}^T v_{t-1}
\leq \leq (v_1^T A v_{t-1} + (\Delta + \|E\|_K) \bar{x}^T v_{t-1}
\leq \leq |\lambda (\bar{x}^T v_t)^2 + \mu (1 - (\bar{x}^T v_t)^2) | \bar{x}^T v_{t-1} + (\Delta + \|E\|_K) \bar{x}^T v_{t-1}
\]
Thus (9) becomes
\[
\lambda \bar{x}^T v_{t-1} - |E|_K \leq \left[ \lambda (\bar{x}^T v_t)^2 + \mu (1 - (\bar{x}^T v_t)^2) \right] \bar{x}^T v_{t-1} + (\Delta + \|E\|_K) \bar{x}^T v_{t-1} + (\lambda + \|E\|_K) \Delta
\Rightarrow \Rightarrow k (1 - (\bar{x}^T v_t)^2) \bar{x}^T v_{t-1} \leq (\Delta + \|E\|_K) \bar{x}^T v_{t-1} + (\lambda + \|E\|_K) \Delta + |E|_K
\]
According to Lemma \ref{lem:kappa}
\[
\bar{x}^T v_t \geq c_0 \land c_1 \geq 0, \text{ so } 1 - (\bar{x}^T v_t) \geq 1 - \bar{x}^T v_t = \left\| \frac{1}{2} (v_t - \bar{x}) \right\|^2, \text{ thus}
\]
\[
\frac{k}{2} \left\| v_t - \bar{x} \right\|^2 \bar{x}^T v_{t-1} \leq (\Delta + \|E\|_K) \bar{x}^T v_{t-1} + (\lambda + \|E\|_K) \Delta + |E|_K
\]
Since \(c_0 \land c_1 \leq \bar{x}^T v_{t-1} \leq 1 \text{ and } \Delta \leq \frac{|E|_K}{\lambda^2} \land 1,
\]
\[
\frac{k}{2} \left\| v_t - \bar{x} \right\|^2 \leq (\Delta + |E|_K) + (\lambda + |E|_K) \Delta + |E|_K
\leq 4 |E|_K
\Rightarrow \Rightarrow \left\| v_t - \bar{x} \right\| \leq \sqrt{\frac{8 |E|_K}{(c_0 \land c_1) k}}
\]
For the second part, first to get a lower bound of $\bar{x}^T A_{v_{t-1}}$,

$$\bar{x}^T A_{v_{t-1}} = \bar{x}^T \bar{A}_{v_{t-1}} + \bar{x}^T E_{v_{t-1}}$$

$$= \lambda \bar{x}^T v_{t-1} + \bar{x}^T E_{v_{t-1}}$$

$$= \lambda \bar{x}^T v_{t-1} + \bar{x}^T E\bar{x} + (v_{t-1} - \bar{x})^T E(v_{t-1} - \bar{x}) + 2\bar{x}^T E(v_{t-1} - \bar{x})$$

$$\geq \lambda \bar{x}^T v_{t-1} + \bar{x}^T E\bar{x} - \|v_{t-1} - \bar{x}\|^2 E_{|\tau_K(x)|} - 2\|v_{t-1} - \bar{x}\|\|E\|_{\tau_K(x)}$$

$$\geq \lambda \bar{x}^T v_{t-1} + \bar{x}^T E\bar{x} - 4\|v_{t-1} - \bar{x}\|\|E\|_{\tau_K(x)}$$

By the fact $\|v_{t-1} - \bar{x}\| \leq 2$

$$\lambda \bar{x}^T v_{t-1} + \bar{x}^T E\bar{x} - 4\|v_{t-1} - \bar{x}\|\|E\|_{\tau_K(x)}$$

$$\geq \Delta \|E\|_{\tau_K(x)} - 4\Delta \|E\|_{\tau_K(x)}$$

\(10\)

And again, use the second and third results in Lemma [3, 3] to get an upper bound of $\bar{x}^T A_{v_{t-1}}$,

$$\bar{x}^T A_{v_{t-1}} \leq \bar{x}^T \Pi_K A_{v_{t-1}} = (v_i^T A_{v_{t-1}}) \bar{x}^T v_i$$

$$\leq (v_i^T A_{v_{t-1}}) \bar{x}^T v_i + (v_i^T A_{v_{t-1}}) |v_1 - v_{t-1}|$$

$$= \left[\lambda (\bar{x}^T v_i)^2 + \mu (1 - (\bar{x}^T v_i)^2)\right] \bar{x}^T v_i - \left[(v_i - v_1)^T A_{v_{t-1}} \bar{x}^T v_i - (v_i^T A_{v_{t-1}}) \Delta\right]$$

\(11\)

Then bound each term in the RHS of the above inequality. First, by the third result in Lemma [3, 3]

$$\left[\lambda (\bar{x}^T v_i)^2 + \mu (1 - (\bar{x}^T v_i)^2)\right] \bar{x}^T v_i$$

$$\leq \left[(\bar{x}^T E\bar{x})^\top \bar{x}^T v_i - \left[(v_i - v_1)^T E(v_i - \bar{x}) + 2\bar{x}^T E(v_i - \bar{x})\right] \bar{x}^T v_i$$

$$\leq \left[(\bar{x}^T E\bar{x})^\top \bar{x}^T v_i - \left[(v_i - v_1)^T E(v_i - \bar{x}) + 2\bar{x}^T E(v_i - \bar{x})\right] \bar{x}^T v_i - 4\|v_i - v_1\|\|E\|_{\tau_K(x)}$$

$$\leq \left[(\bar{x}^T E\bar{x})^\top \bar{x}^T v_i - 4\|v_i - v_1\|\|E\|_{\tau_K(x)}$$

\(12\)

With some modifications and the Cauchy-Schwartz inequality, the second term can be bounded as

$$\left[(v_i - v_1)^T A_{v_i} \bar{x}^T v_i - \left[(v_i - v_1)^T E(v_i - \bar{x}) + (v_i - v_1)^T E\bar{x}\right] \bar{x}^T v_i - 4\|v_i - v_1\|\|E\|_{\tau_K(x)}$$

\(13\)

Use the fact that $\|v_i - v_1\| \leq 2$ for any $i$, the above inequality can be written as

$$\left[(v_i - v_1)^T A_{v_i} \bar{x}^T v_i - \Delta \lambda + 3\|v_i - v_1\|\|E\|_{\tau_K(x)} + 3\|v_i - v_1\|\|E\|_{\tau_K(x)}$$

$$= \Delta \lambda + 3\|v_i - v_1\|\|E\|_{\tau_K(x)} + 3\|v_i - v_1\|\|E\|_{\tau_K(x)}$$

\(14\)
Then we need to pick a suitable $\Delta$ such that
\[
\|v_l - \bar{x}\| \leq \frac{1}{\sqrt{2}}\|v_l - \bar{x}\| = \frac{1}{\sqrt{2}}(\Delta + 2\|v_l - \bar{x}\|) \leq \Delta + 2\|v_l - \bar{x}\|.
\]
Plug in (12), (13), (14) into the RHS of above inequality to get
\[
(\lambda - \mu)\{1 - (\bar{x}^Tv_l)^2\}\|v_l - \bar{x}\| - 20\|E\|_{\|v_l - \bar{x}\|} - 10\Delta\|E\|_{\|v_l - \bar{x}\|} - 2\Delta\leq 0.
\]
By the fact that $1 - (\bar{x}^Tv_l)^2 = \frac{1}{2}\|v_l - \bar{x}\|^2 = \frac{1}{2}(\Delta + 2\|v_l - \bar{x}\|)^2 = \frac{1}{2}(\Delta^2 + 2\|v_l - \bar{x}\|^2) \leq \Delta + 2\|v_l - \bar{x}\|$, the above inequality becomes
\[
(\lambda - \mu)[1 - (\bar{x}^Tv_l)^2]\|v_l - \bar{x}\| - 20\|E\|_{\|v_l - \bar{x}\|} - 10\Delta\|E\|_{\|v_l - \bar{x}\|} - 2\Delta\leq 0.
\]
Use the fact $c_0 \cap c_1 \leq \bar{x}^Tv_l - 1$, and $\lambda - \mu = k$, the above inequality becomes
\[
\frac{k(c_0 \cap c_1)}{2}\|v_l - \bar{x}\|^2 - 20\|E\|_{\|v_l - \bar{x}\|} - 10\Delta\|E\|_{\|v_l - \bar{x}\|} - 2\Delta\leq 0.
\]
The above inequality is a quadratic form of $\|v_l - \bar{x}\|$. The discriminant $D = (20\|E\|_{\|v_l - \bar{x}\|})^2 + 4k\Delta(c_0 \cap c_1)\|E\|_{\|v_l - \bar{x}\|} + \lambda \geq 0$, so there are values of $\|v_l - \bar{x}\|$ to make (15) hold. By calculating the roots we get
\[
\|v_l - \bar{x}\| \leq \frac{20\|E\|_{\|v_l - \bar{x}\|} + \sqrt{D}}{k(c_0 \cap c_1)}.
\]
Then we need to pick a suitable $\Delta$ such that $\sqrt{D}$ is the same order as $\|E\|_{\|v_l - \bar{x}\|}$. When $\Delta \leq \frac{\|E\|_{\|v_l - \bar{x}\|} + 4\|E\|_{\|v_l - \bar{x}\|}}{k(c_0 \cap c_1)}$, we have
\[
20k\Delta(c_0 \cap c_1)\|E\|_{\|v_l - \bar{x}\|} \leq (20\|E\|_{\|v_l - \bar{x}\|})^2 + 4k\Delta(c_0 \cap c_1)\|E\|_{\|v_l - \bar{x}\|} \leq 16\|E\|_{\|v_l - \bar{x}\|}^2.
\]
Thus
\[
D \leq 400\|E\|_{\|v_l - \bar{x}\|}^2 + 30\|E\|_{\|v_l - \bar{x}\|} \leq (21\|E\|_{\|v_l - \bar{x}\|})^2
\]
and
\[
\|v_l - \bar{x}\| \leq \frac{20\|E\|_{\|v_l - \bar{x}\|} + \sqrt{D}}{k(c_0 \cap c_1)} \leq \frac{41\|E\|_{\|v_l - \bar{x}\|}}{k(c_0 \cap c_1)}
\]

### E.2 Proof of Theorem 3

\[
v_l^TAv_l = v_l^T\Lambda v_l + v_l^TEv_l \geq \lambda(v_l^T\bar{x})^2 - \|E\|_K
\]
\[
v_l^T\Lambda\bar{v}_l = v_l^T\Lambda\bar{v}_l + v_l^TE\bar{v}_l \leq \lambda(v_l^T\bar{x})^2 + \mu[1 - (v_l^T\bar{x})^2] + \|E\|_K
\]
By 2. in Lemma [13]

By the above inequalities, when $\Lambda v_l \geq v_l^TAv_l$, we have
\[
\lambda(v_l^T\bar{x})^2 - \|E\|_K \leq \lambda(v_l^T\bar{x})^2 + \mu[1 - (v_l^T\bar{x})^2] + \|E\|_K
\]
\[
\Rightarrow \lambda(v_l^T\bar{x})^2 - \lambda - 2\|E\|_K \leq \lambda(v_l^T\bar{x})^2 - \lambda + \mu[1 - (v_l^T\bar{x})^2]
\]
\[
\Rightarrow (\lambda - \mu)[1 - (v_l^T\bar{x})^2] \leq \lambda[1 - (v_l^T\bar{x})^2] + 2\|E\|_K
\]
Notice that
\[
1 - (v_l^T\bar{x})^2 = (1 + v_l^T\bar{x})(1 - v_l^T\bar{x}) \leq 2(1 - v_l^T\bar{x}) = \|v_l - \bar{x}\|^2
\]

16
Thus the previous inequality becomes
\[
k[1 - (\tilde{v}_i^T \tilde{x})^2] \leq \lambda \|v_i - \tilde{x}\|^2 + 2|E| K
\]
\[
\Rightarrow \frac{k}{2} |\tilde{v}_i - \tilde{x}|^2 \leq \lambda \|v_i - \tilde{x}\|^2 + 2|E| K
\]
\[
\Rightarrow \|\tilde{v}_i - \tilde{x}\| \wedge \|\tilde{v}_i + \tilde{x}\| \leq \sqrt{\frac{2\lambda \|v_i - \tilde{x}\|^2}{k}} + \frac{4|E| K}{k}
\]
\[
\leq \sqrt{\frac{2\lambda}{k} \|v_i - \tilde{x}\| + \frac{4|E| K}{k}}.
\]
\[
(16)
\]
By the result 2 and 3 in Lemma B.3 we get
\[
v_i^T A v_i - \tilde{x}^T E \tilde{x} = v_i^T A v_i + v_i^T E v_i - \tilde{x}^T E \tilde{x} \geq \lambda (v_i^T \tilde{x})^2 - 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x})
\]
\[
(17)
\]
\[
v_i^T A \tilde{v}_i - \tilde{x}^T E \tilde{x} = v_i^T A \tilde{v}_i + v_i^T E \tilde{v}_i - \tilde{x}^T E \tilde{x}
\]
\[
\leq \lambda (\tilde{v}_i^T \tilde{x})^2 + \mu (1 - (\tilde{v}_i^T \tilde{x})^2) + 4\|\tilde{v}_i - \tilde{x}\| |E| T_K(\tilde{x})
\]
\[
(18)
\]
Or
\[
\leq \lambda (\tilde{v}_i^T \tilde{x})^2 + \mu (1 - (\tilde{v}_i^T \tilde{x})^2) + 4\|\tilde{v}_i + \tilde{x}\| |E| K
\]
\[
(19)
\]
If \(v_i^T A v_i \geq 0\) and \(\tilde{v}_i^T \tilde{x} \geq 0\) such that \(1 - (\tilde{v}_i^T \tilde{x})^2 \geq \frac{1}{2} \|\tilde{v}_i - \tilde{x}\|^2\), by (17), (18) we have
\[
(\lambda - \mu) [1 - (\tilde{v}_i^T \tilde{x})^2] - \lambda [1 - (v_i^T \tilde{x})^2] - 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x}) - 4\|\tilde{v}_i - \tilde{x}\| |E| T_K(\tilde{x}) \leq 0
\]
\[
\Rightarrow \frac{k}{2} \|\tilde{v}_i - \tilde{x}\|^2 - 4\|\tilde{v}_i - \tilde{x}\| |E| T_K(\tilde{x}) - \lambda [1 - (v_i^T \tilde{x})^2] - 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x}) \leq 0
\]
Notice that
\[
1 - (v_i^T \tilde{x})^2 = (1 + v_i^T \tilde{x})(1 - v_i^T \tilde{x}) \leq 2(1 - v_i^T \tilde{x}) = \|v_i - \tilde{x}\|^2
\]
Thus the previous inequality becomes
\[
\frac{k}{2} \|\tilde{v}_i - \tilde{x}\|^2 - 4\|E| T_K(\tilde{x})\|\tilde{v}_i - \tilde{x}\| - \lambda \|v_i - \tilde{x}\|^2 - 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x}) \leq 0
\]
The discriminant \(D = 16|E| T_K(\tilde{x})^2 + 2k[\lambda \|v_i - \tilde{x}\|^2 + 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x})] > 0\). Thus
\[
\|\tilde{v}_i - \tilde{x}\| \leq \frac{4|E| T_K(\tilde{x}) + \sqrt{D}}{k}
\]
\[
\leq \frac{8|E| T_K(\tilde{x})}{k} + \sqrt{\frac{2\lambda \|v_i - \tilde{x}\|^2 + 8\|v_i - \tilde{x}\| |E| T_K(\tilde{x})}{k}}
\]
Similarly, if \(\tilde{v}_i^T A \tilde{v}_i \geq 0\) and \(\tilde{v}_i^T \tilde{x} \leq 0\) such that \(1 - (\tilde{v}_i^T \tilde{x})^2 \geq \frac{1}{2} \|\tilde{v}_i + \tilde{x}\|^2\), by (17), (18) we have
\[
\frac{k}{2} \|\tilde{v}_i + \tilde{x}\|^2 - 4\|E| K\|\tilde{v}_i + \tilde{x}\| - \lambda \|v_i - \tilde{x}\|^2 - 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x}) \leq 0
\]
The discriminant \(D = 64|E| K^2 + 2k[\lambda \|v_i - \tilde{x}\|^2 + 4\|v_i - \tilde{x}\| |E| T_K(\tilde{x})] > 0\). Thus
\[
\|\tilde{v}_i + \tilde{x}\| \leq \frac{4|E| K + \sqrt{D}}{k}
\]
\[
\leq \frac{8|E| K}{k} + \sqrt{\frac{2\lambda \|v_i - \tilde{x}\|^2 + 8\|v_i - \tilde{x}\| |E| T_K(\tilde{x})}{k}}
\]
Thus
\[
\|\tilde{v}_i - \tilde{x}\| \wedge \|\tilde{v}_i + \tilde{x}\| \leq \left( \frac{8|E| K}{k} + \frac{8|E| T_K(\tilde{x})}{k} \right) + \sqrt{\frac{2\lambda \|v_i - \tilde{x}\|^2 + 8\|v_i - \tilde{x}\| |E| T_K(\tilde{x})}{k}}
\]
\[
(20)
\]
Therefore \(\|\tilde{v}_i - \tilde{x}\| \wedge \|\tilde{v}_i + \tilde{x}\|\) is bounded by the minimum of RHS of (16) and (20).
Appendix F  Proof of Lemma 1

The proof of Lemma 1 is based on Gordon’s escape through mesh inequality and the properties of convex cones. Gordon’s escape through mesh controls the operator norm of a Gaussian matrix by relating the operator norm to the supremum of a Gaussian process (Plan and Vershynin [2016], Theorem 4.2). It is an application of the classic Gordon’s inequality (Gordon [1988], Theorem A).

Lemma F.1. (Gordon’s Escape Through Mesh Tail Probability Version) Let $K \subseteq \mathbb{R}^n$ be a convex cone and $\hat{X}$ be an $n \times p$ standard Gaussian matrix. Then for every $t \geq 0$,

$$
\mathbb{P}\left\{ \sup_{u \in K \cap S^{p-1}} \|\tilde{X}u\|_2 \geq \sqrt{n} + w(K \cap S^{p-1}) + t \right\} \leq e^{-\frac{t^2}{2}}
$$

$$
\mathbb{P}\left\{ \inf_{u \in K \cap S^{p-1}} \|\tilde{X}u\|_2 \leq \sqrt{n} - w(K \cap S^{p-1}) - t \right\} \leq e^{-\frac{t^2}{2}}
$$

F.1 Proof of Lemma 1

Let $X = [X_1^T; X_2^T; \ldots; X_n^T]$ be an $n \times p$ matrix with rows $X_i^T \sim \mathcal{N}(0, I + k\mathbb{E}^T)$. With the fact $\sqrt{I + kvv^T} = I + (\sqrt{1 + k} - 1)vv^T$, we can rewrite the sample covariance matrix $A$ as

$$
A = \frac{X^TX}{n} = I + kvv^T + (I + (\sqrt{k} - 1)vv^T)(\frac{\tilde{X}^\top \tilde{X}}{n} - I)(I + (\sqrt{k} - 1)vv^T),
$$

where $\tilde{X}$ is a standard Gaussian matrix.

1. Define $M = \frac{\tilde{X}^\top \tilde{X}}{n} - I$. We will show that $\|M\|_K \leq \frac{w(K \cap S^{p-1})}{\sqrt{n}} \sqrt{w^2(K \cap S^{p-1})}$ with probability at least $1 - 6e^{-\frac{w(k \cap S^{p-1})^2}{n}}$.

2. With probability at least $1 - e^{-\frac{w(k \cap S^{p-1})^2}{n}}$,

$$(x + y)^\top M(x + y) = (x + y)^\top (\frac{\tilde{X}^\top \tilde{X}}{n})(x + y) - |x + y|^2
$$

$$
= \frac{|\tilde{X}(x + y)|^2}{n} - |x + y|^2
$$

$$
\leq \frac{(\sqrt{n} + 2w(K \cap S^{p-1}))^2}{n} |x + y|^2 - |x + y|^2 \quad \text{By Lemma}\ [\text{D}] \quad \text{let } t = w(K \cap S^{p-1})
$$

$$
\leq \frac{4w^2(K \cap S^{p-1})}{n} + \frac{4w(K \cap S^{p-1})}{\sqrt{n}} |x + y|^2
$$

$$
\leq \frac{w^2(K \cap S^{p-1})}{n}, \quad \text{if } \frac{w(K \cap S^{p-1})}{\sqrt{n}} \geq 1
$$

$$
\leq \frac{w(K \cap S^{p-1})}{\sqrt{n}}, \quad \text{if } \frac{w(K \cap S^{p-1})}{\sqrt{n}} < 1
$$

Similarly with probability at least $1 - e^{-\frac{w(k \cap S^{p-1})^2}{n}}$,

$$
x^\top Mx = x^\top \frac{\tilde{X}^\top \tilde{X}}{n}x - |x|^2
$$

$$
\geq \frac{(\sqrt{n} - 2w(K \cap S^{p-1}))^2}{n} |x|^2 - |x|^2 \quad \text{By Lemma}\ [\text{D}] \quad \text{let } t = w(K \cap S^{p-1})
$$

$$
\geq \left[ \frac{4w^2(K \cap S^{p-1})}{n} - \frac{4\sqrt{n} - 4w(K \cap S^{p-1}) - \frac{1}{n}}{\sqrt{n}} \right] |x|^2
$$

$$
\geq \begin{cases} 0, & \text{if } \frac{w(K \cap S^{p-1})}{\sqrt{n}} \geq 1 \\ - \frac{w(K \cap S^{p-1})}{\sqrt{n}}, & \text{if } \frac{w(K \cap S^{p-1})}{\sqrt{n}} < 1 \end{cases}
$$
Lemma G.1. Gaussian product of inverse spike matrices, which will be used in calculating the Kullback-Leibler divergence of multivariate

In order to prove Theorem 4, we need to introduce three intermediate results. The first one is about trace of the

Appendix G Proof of Theorem 4

Proof of Lemma G.1. By the Sherman-Morrison formula,

\[ (I + kv'v^\top)^{-1} = I - \frac{k}{k+1} v'v^\top \]

One can get a similar lower bound for \( y^\top M y \). Thus with probability at least \( 1 - 3e^{-\frac{w(K \cap S^{p-1})^2}{n}} \),

\[ (x + y)^\top M(x + y) - x^\top Mx - y^\top My \lesssim \frac{w(K \cap S^{p-1})}{\sqrt{n}} \sqrt{\frac{w^2(K \cap S^{p-1})}{n}} \]

Furthermore we can reverse the above inequalities to see that with probability at least \( 1 - 3e^{-\frac{w(K \cap S^{p-1})^2}{n}} \),

\[ (x + y)^\top M(x + y) - x^\top Mx - y^\top My \gtrsim -\left( \frac{w(K \cap S^{p-1})}{\sqrt{n}} \sqrt{\frac{w^2(K \cap S^{p-1})}{n}} \right) \]

Therefore \( |M|_\kappa \leq \frac{w(K \cap S^{p-1})}{\sqrt{n}} \sqrt{\frac{w^2(K \cap S^{p-1})}{n}} \) with probability at least \( 1 - 6e^{-\frac{w(K \cap S^{p-1})^2}{n}} \).

2. Let \( k_0 = \sqrt{k + 1} - 1 \). Show that \( \|E\|_\kappa \leq (1 + 2k_0)|M|_\kappa + k_0^2 \sqrt{\frac{1}{n}} \) with probability at least \( 1 - 2e^{-7} \).

\[
\|E\|_\kappa = \sup \{ x^\top (I + k_0 vv^\top) (\tilde{X}^\top X - I) (I + k_0 vv^\top) y \mid x, y \} \\
= \sup \{ x^\top (I + k_0 vv^\top) (\tilde{X}^\top X - I) y \mid x, y \} \\
\leq \frac{1}{n} \sum_{i=1}^{n} |X_i|^2 - 1 \]  

By Laurent and Massart (2000) [Lemma 1], let \( Z = v^\top \left( \frac{\tilde{X}^\top X}{n} - I \right) v = \frac{1}{n} \sum_{i=1}^{n} |X_i|^2 - 1 \). The following concentration inequalities hold for \( Z \),

\[
P(Z \geq 2 \sqrt{\frac{x}{n}} + 2 \sqrt{\frac{2}{n}}) \leq e^{-x} \\
P(Z \leq -2 \sqrt{\frac{x}{n}}) \leq e^{-x} 
\]

Since \( n \) is large, we can pick \( x \) such that \( e^{-x} \) is very small and \( O(\frac{1}{n^2}) \sim O(\frac{1}{n}) \). Thus \( Z \sim O(\sqrt{\frac{1}{n}}) \). Specifically, let \( x = 7 \), so \( e^{-x} \approx 0.0009 \). Plug in (21) to see with probability at least \( 1 - 2e^{-7} \) we have

\[
\|E\|_\kappa \leq \frac{1}{n} \sum_{i=1}^{n} |X_i|^2 - 1 \\
\leq (1 + 2k_0)|M|_\kappa + k_0^2 \sqrt{\frac{1}{n}} 
\]

Therefore with probability at least \( 1 - 6e^{-\frac{w(K \cap S^{p-1})^2}{n}} - 2e^{-7} \) we have

\[
\|E\|_\kappa \leq 2\sqrt{k + 1} \frac{w(K \cap S^{p-1})}{\sqrt{n}} \sqrt{\frac{w^2(K \cap S^{p-1})}{n}} + k + 2 - 2\sqrt{k + 1} 
\]

Appendix G Proof of Theorem 4

In order to prove Theorem 4, we need to introduce three intermediate results. The first one is about trace of the product of inverse spike matrices, which will be used in calculating the Kullback-Leibler divergence of multivariate Gaussian.

Lemma G.1. If \( \|v\| = 1, \|v'\| = 1 \),

\[ \text{tr}((I + kv'v'^\top)^{-1} (I + kvv'^\top)) = p + \frac{k^2}{k+1} [1 - \langle v, v' \rangle^2] \]

Proof of Lemma (27) By the Sherman-Morrison formula,
Thus
\[
\text{tr}((I + kv'v^t)^{-1}(I + kvv^t)) = \text{tr}((I - \frac{k}{k+1}v'v^t)(I + kvv^t))
\]
\[
= \text{tr}(I + kvv^t) - \frac{k}{k+1}v'v^t - \frac{k^2}{k+1}v'vv^t
\]
\[
= p + k - \frac{k}{k+1} - \frac{k^2}{k+1}\langle v, v' \rangle^2
\]
\[
= p + \frac{k^2}{k+1}[1 - \langle v, v' \rangle^2]
\]
\[
 \Box
\]

The second intermediate result Lemma G.2 relates Gaussian complexity of a set to its metric entropy. Gaussian complexity is actually a basic geometric property. Lemma G.2 reveals a relation between Gaussian complexity and diameter of a set, which is used in the proof of Lemma G.3.

**Lemma G.2.** (Vershynin, 2018 Proposition 7.5.2) (Gaussian Complexity and Diameter) Let \( T \subset \mathbb{R}^p \), and \( w(T) \) is the Gaussian Complexity of \( T \). Then
\[
\frac{1}{\sqrt{2\pi}} \leq \frac{w(T)}{\text{diam}(T)} \leq \frac{\sqrt{p}}{2}
\]

Recall that the \( \epsilon \)-covering number of a set \( T \) is the cardinality of the smallest \( \epsilon \)-cover of \( T \), and the logarithm of it is the metric entropy. The \( \epsilon \)-packing number is the cardinality of the largest \( \epsilon \)-packing of \( T \). The \( \epsilon \)-covering and \( \epsilon \)-packing of a set are in the same order (Wainwright, 2019, Lemma 5.1). The following lemma gives upper bound of the Gaussian complexity using a chaining constructed by the covering sets.

**Lemma G.3.** (Variation of Reverse Sudakov’s Inequality) Let \( T \) be a subset of \( \mathbb{R}^p \) with finite diameter. Let \( \log N \) be the metric entropy of \( T \) with respect to \( \epsilon \), and \( w(T) \) is the Gaussian complexity of \( T \). There exists a constant \( C \), such that
\[
w(T) \leq 2\sqrt{\log 3} + C\log p \sup_{\epsilon > 0, N \geq 4} \epsilon \sqrt{\log N}
\]

**Proof of Lemma G.3** Let \( X_t = \langle g, t \rangle \) where \( t \in T, g \sim N(0, I_p) \) to be a Gaussian process on \( T \). By definition, \( w(T) = \sup_{t \in K} X_t \).

1. **Construct a chaining \( \{\pi_0(t), \ldots, \pi_n(t)\} \) on \( T \).**

   Let \( \epsilon_i = 2^{-i}, i \in \mathbb{Z} \), and \( T(T, \epsilon_i) \) is the min \( \epsilon_i \)-covering of \( T \), and \( |T(K, \epsilon_i)| = N_i \). For any \( t \in T \), let \( \pi_i(t) \) be a point in \( T(T, \epsilon_i) \) satisfying \( |t - \pi_i(t)|_2 \leq \epsilon_i \).

   Let \( \nu = \max\{i : \epsilon_i \geq \text{diam}(T)\} \), so \( T(T, \epsilon_\nu) = \{\pi_\nu(t)\} \). Furthermore, we choose \( n \) such that \( X_{\pi_\nu(t)} \) is close enough to \( X_t \):

   \[
n = \min_{i \in \mathbb{Z}} \{i : 2^{-i} \leq \frac{w(T)}{2\sqrt{p}}\}
\]

   Then

   \[
   X_t - X_{\pi_\nu(t)} = \sum_{i=\nu}^n (X_{\pi_i(t)} - X_{\pi_{i-1}(t)}) + (X_t - X_{\pi_\nu(t)})
   \]

   Since \( \mathbb{E}X_{\pi_\nu(t)} = 0 \),

   \[
w(T) = \mathbb{E}\sup_{t \in T} X_t \leq \mathbb{E}\sum_{i=\nu}^n \sup_{t \in T} (X_{\pi_i(t)} - X_{\pi_{i-1}(t)}) + \mathbb{E}\sup_{t \in T} (X_t - X_{\pi_\nu(t)}) \tag{22}
\]

2. **Upper bound \( \eqref{eq:22} \)** by the properties of chaining.

   For a given \( t \in T \), it’s easy to see \( X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \) is a Lipschitz function of standard normal vector with Lipschitz factor \( L = |\pi_i(t) - \pi_{i-1}(t)|_2 \leq 2\epsilon_{i-1} \). Thus

   \[
   X_{\pi_i(t)} - X_{\pi_{i-1}(t)} \sim SG(4\epsilon_{i-1}^2)
   \]
By the maxima of sub-Gaussian,
\[
\mathbb{E} \sup_{t \in T} (X_{\pi_n(t)} - X_{\pi_{n-1}(t)}) \leq 2\epsilon_{n-1} \sqrt{2 \log (N_{\pi_{n-1}})}
\]
\[
\leq 4\epsilon_{n-1} \sqrt{\log N_{\pi_{n-1}}}
\]
(23)

For the second term,
\[
\mathbb{E} \sup_{t \in T} (X_{\pi_n(t)} - X_{\pi_{n-1}(t)}) = \mathbb{E} \sup_{t \in T} \langle g, t - \pi_n(t) \rangle
\]
\[
\leq \epsilon_n \mathbb{E} \|g\|_2
\]
\[
\leq \frac{w(T)}{2\sqrt{p}} = \frac{1}{2}w(T)
\]
(24)

Combining (22), (23), (24) get
\[
\frac{1}{2}w(T) \leq \sum_{i=0}^{n} \epsilon_{i-1} \sqrt{\log N_{\pi_i}}
\]
(25)

Notice that \(n - \nu = O(\log p)\) steps should be sufficient to walk from \(X_{\pi_n(t)}\) to \(X_t\):
\[
n - \nu - 1 = \log_2 \frac{\epsilon_{\nu+1}}{\epsilon_0} \leq \log_2 \frac{\text{diam}(T)}{w(T)/4\sqrt{p}}
\]
\[
\leq \log_2 (4\sqrt{2\pi} \sqrt{p})
\]
By the choice of \(n\), we have \(\epsilon_n \geq \frac{wk}{4\sqrt{p}}\)

\[Lemma \ G.2\]

\[Lemma \ G.3\]

Let \(i^* = \min_{i \in [0, [C\log p]]} \{N_{\pi_i} < 4\}\), then (25) can be written as
\[
\frac{1}{2}wk \leq \sum_{i=1}^{i^*} 2^{-(i-1)} \sqrt{\log 3 + (C\log p - i^*)} \sup_{\epsilon_{i-1}, N_{i-1} > 4} \epsilon_0 \sqrt{\log N_{\pi_{i-1}}}
\]
\[
\leq 2\sqrt{\log 3 + C\log p} \sum_{i=1}^{i^*} \epsilon_{i-1} \sqrt{\log N_{\pi_{i-1}}}
\]

The last intermediate result we need is the generalized Fano’s inequality for multi sample setting. Suppose that we know a random variable \(Y\) and want to estimate another random variable \(X\) based on \(Y\). Fano’s inequality quantifies the estimation uncertainty in terms of the conditional entropy of \(Y\) on \(X\). Generalized Fano’s method, derived from the original Fano’s inequality, is widely used in statistics literature to provide lower bound of the estimation error (Yu, 1997, Lemma 3). The reference (Yu, 1997, Lemma 3) proved the generalized Fano’s inequality for a single sample setting. With the same idea, we can get a multi sample version of generalized Fano’s method as

**Lemma G.4.** (Generalized Fano’s Method) Let \(r \geq 2\) be an integer and let \(\mathcal{M}_r \subset \mathcal{P}\) contain \(r\) probability measures indexed by \(j = 1, 2, \ldots, r\). Let \(X_1, \ldots, X_n\) be a collection of i.i.d. random variables with the conditional distribution \(P_j\) when \(j\) is given. \(\theta(P_j)\) is a parameter of \(P_j\), and \(\hat{\theta}\) is an estimation based on the sample \(X_1, \ldots, X_n\). The probability measures in \(\mathcal{M}_r\) also satisfy that for all \(j \neq j'\)
\[
d(\theta(P_j), \theta(P_{j'})) \geq \alpha_r,
\]
and
\[
D(P_j || P_{j'}) \leq \beta_r
\]
Then
\[
\max_j \mathbb{E}[d(\hat{\theta}, \theta(P_j))] \geq \frac{\alpha_r}{2} (1 - \frac{n\beta_r + \log 2}{\log r})
\]

Based on the previous three intermediate results, we are able to analyze the lower bound of the \(L_2\) error rate of the eigenvector estimator in spiked covariance model. The lower bound is related to the Gaussian complexity of \(K \cap S^{p-1}\), where \(K\) is the convex cone containing the true eigenvector \(\tilde{x}\).
G.1 Proof of Theorem 4

1. Let \( \mathcal{P}(K \cap S^{p-1}, \epsilon) \) be the max \( \epsilon \)-packing of \( K \cap S^{p-1} \) such that for any \( v_i, v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon) \) and \( v_i \neq v_j \) we have \( \|v_i - v_j\| \geq \epsilon \). Let the \( \epsilon \)-packing number be \( M_\epsilon = |\mathcal{P}(K \cap S^{p-1}, \epsilon)| \).

2. Let \( P_v \sim \mathcal{N}(0, I + kvv^\top) \). By Lemma [G.1] the Kullback-Leibler distance between \( P_{v_i} \) and \( P_{v_j} \) for any \( v_i, v_j \in \mathcal{P}(K \cap S^{p-1}) \) and \( v_i \neq v_j \) is

\[
D(P_{v_i}, P_{v_j}) = \frac{1}{2} \left[ \text{tr}((I + kv_iv_j^\top)^{-1}(I + kv_i^2v_j^2)) - p \right] = \frac{k^2}{2(k+1)}[1 - \langle v_i, v_j \rangle^2] := D
\]

3. By Lemma [G.4] and \( D = \frac{\epsilon^2}{2(\epsilon^2 + 1)} \leq \frac{k^2}{2(k+1)} \leq \frac{k^2 \epsilon^2}{2(\epsilon^2 + 1)} \), we have

\[
\inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \epsilon^2 \left( 1 - \frac{n(k \wedge k^2)}{2 \log N_s} - \frac{2 \log N_s}{n(k \wedge k^2)} \right)\]

Let \( N_s \) be the \( \epsilon \)-covering number of set \( K \cap S^{p-1} \). By the fact \( N_s \leq M_\epsilon \) (Wainwright, 2019, Lemma 5.1), the above inequality becomes

\[
\inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \epsilon^2 \left( 1 - \frac{n(k \wedge k^2)}{2 \log N_s} - \frac{2 \log N_s}{n(k \wedge k^2)} \right) \tag{26}
\]

4. Let \( \epsilon^* = \arg \max_{\epsilon > 0} \epsilon \sqrt{\log N_s} \). By Lemma [G.4] when \( w(K \cap S^{p-1}) \geq 2\sqrt{\log 3} \), there exists \( C_1 > 0 \) such that

\[
\epsilon^* \geq \frac{w(K \cap S^{p-1})}{C_1 \log p \sqrt{\log N_s}} \tag{27}
\]

Plug in \( \epsilon = \epsilon^* \) into (26). Notice that \( N_s \geq 4 \) so \( \frac{\log^2 N_s}{\log N_s} \leq \frac{1}{4} \), we get

\[
\inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon^*)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \frac{\epsilon^*^2}{2} \left( 1 - \frac{2n(k \wedge k^2)}{\log N_s} \right)
\]

Pick \( k = \frac{\log N_s}{2n} \vee \sqrt{\log N_s \over 2n} \). Then we have \( \log N_s = 2n(k \wedge k^2) \), so that

\[
\frac{n(k \wedge k^2)}{2 \log N_s} = \frac{1}{4}
\]

The lower bound becomes

\[
\inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon^*)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \frac{\epsilon^*^2}{8}
\]

Plug in (27) to get

\[
\inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon^*)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \frac{1}{8C_1 \log p \sqrt{\log N_s}} \frac{w^2(K \cap S^{p-1})}{n(k \wedge k^2)}
\]

By \( \log N_s = 2n(k \wedge k^2) \), the lower bound becomes

\[
\inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon^*)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \frac{1}{16C_1 \log p \sqrt{\log N_s}} \frac{w^2(K \cap S^{p-1})}{2n(k \wedge k^2)}
\]

Therefore \( \inf_{\hat{v}} \sup_{K \cap S^{p-1}} \mathbb{E}[\|\hat{v} - \hat{x}\|^2] \geq \inf_{\hat{v}} \max_{v_j \in \mathcal{P}(K \cap S^{p-1}, \epsilon^*)} \mathbb{E}[\|\hat{v} - v_j\|^2] \geq \frac{C^* w^2(K \cap S^{p-1})}{(\log p)^2 n(k \wedge k^2)} \).

Moreover, by choosing \( k = \frac{\log N_s}{2n} \vee \sqrt{\log N_s \over 2n} \), there is an implicit upper bound of \( k \). By Sudakov’s Inequality (Vershynin, 2018, Theorem 8.1.13), \( \log N_s \leq \left( \frac{w(K \cap S^{p-1})}{\epsilon^*} \right)^2 \), and in the proof of Lemma [G.3] we showed \( \epsilon^* \geq \frac{w(K \cap S^{p-1})}{\sqrt{\log 3}} \).

Thus \( \log N_s \leq 16p \) and \( k = \frac{\log N_s}{2n} \vee \sqrt{\log N_s \over 2n} \leq \frac{\sqrt{2} \epsilon^*}{\sqrt{\log 3}} \).
Appendix H  Proof of Proposition 2

Now we construct a special example of monotone cone to calculate the lower bound of $L_2$ error rate. In the monotone cone of this example, the vectors have three constant pieces, and the pairwise distance between vectors is always greater than $\sqrt{2\epsilon}$.

**Example 1.** Let $K_2 = \{a_i\}_{i=0}^{p-2}$ be a subset of $\mathbb{R}^p$ where

$$a_i = \left(0, \ldots, 0, \frac{\epsilon}{\sqrt{p-1-i}}, \ldots, \frac{\epsilon}{\sqrt{p-1-i}}, \sqrt{1-\epsilon^2}\right)$$

Before investigating the lower bound, we need to introduce a result which tells the order of metric entropy of $K_2$.

**Lemma H.1.** For a monotone cone $K_2$ as defined in Example H the cardinality of the maximum $\frac{\epsilon}{2}$-packing set is of order $O(\log p)$.

**Proof of Lemma H.1.** With out loss of generality, let $i' > i$.

$$||a_i - a_{i'}||^2 = 2 - 2 \alpha_i^T a_{i'}$$

$$= 2 - 2 \left[ \frac{\epsilon^2 \sqrt{p - 1 - i'}}{\sqrt{p - 1 - i}} + 1 - \epsilon^2 \right]$$

$$= 2\epsilon \left[ \frac{p - 1 - i'}{\sqrt{p - 1 - i}} \sqrt{p - 1 - i} \right]$$

$$\geq 2\epsilon (i' - i)$$

To let $||a_i - a_{i'}|| \geq \frac{\epsilon}{2}$, we need $\frac{i' - i}{p - 1 - i} \geq \frac{1}{8} \Rightarrow i' \geq \frac{1}{8} (p - 1) + \frac{7i}{8}$.

Thus the max $\frac{\epsilon}{2}$-packing $\{a_n\}_{n=0}^{\infty}$ can be constructed by a sequence $S_c = \{c_n\}_{n=0}^{\infty}$ where

$$c_0 = 0$$

$$c_{k+1} = \left[ \frac{1}{8} (p - 1) + \frac{7}{8} c_k \right]$$

$$c_n \leq p - 2$$

$$\frac{(p - 2) - c_n}{p - 1 - c_n} < \frac{1}{8} \Rightarrow c_n \geq p - \frac{15}{7}$$

* Lower bound of $|S_c|$: 

In order to get a lower bound of $|S_c|$, construct another sequence $S_b = \{b_n\}_{n=0}^{\infty}$ such that

$$b_0 = 0$$

$$b_{k+1} = \frac{1}{8} (p - 1) + \frac{7}{8} b_k + 1$$

$$b_m \geq p - \frac{15}{7}$$

It’s easy to see $|S_b| = |S_c|$ since $c_k \leq b_k$ for all $k$.

Furthermore one can get $b_{k+2} - b_{k+1} = \frac{7}{8} (b_{k+1} - b_k)$ from the above equations. Then

$$b_m = (b_m - b_{m-1}) + \ldots + (b_1 - b_0)$$

$$= (\frac{1}{8} p + \frac{7}{8}) \left[ (\frac{7}{8})^{m-1} + \ldots + (\frac{7}{8}) + (\frac{7}{8}) \right]$$

$$= (\frac{1}{8} p + \frac{7}{8}) \left[ (\frac{7}{8})^{m-1} + \ldots + (\frac{7}{8}) \right]$$

Notice that $b_m \geq p - \frac{15}{7}$ is required. Thus

$$b_m = (p + 7) \left[ 1 - (\frac{7}{8})^m \right] \geq p - \frac{15}{7} \Rightarrow m \geq \frac{\log (p + 7)}{\log \frac{7}{8}} - \frac{\log \left( \frac{15}{7} + 7 \right)}{\log \frac{7}{8}}$$

Therefore $|S_c| = |S_b| = O(\log p)$.
• Upper bound of \(|S_c|\):
In order to get an upper bound of \(|S_c|\), construct another sequence \(S_d = \{d_k\}_{k=0}^\infty\) such that

\[
d_0 = 0, \\
d_{k+1} = \frac{1}{8}(p - 1) + \frac{7}{8}d_k, \\
d_l \leq p - 2
\]

It’s easy to see \(|S_d| = |S_c|\) since \(c_k \geq d_k\) for all \(k\).

Similar as the proof of lower bound of \(|S_c|\), from \(d_{k+1} = \frac{1}{8}(p - 1) + \frac{7}{8}d_k\) we derive the expression of \(d_l\) as

\[
d_l = (p + 7)[1 - \left(\frac{7}{8}\right)^l]
\]

By \(d_l \leq p - 2\) we have

\[
d_l = (p + 7)[1 - \left(\frac{7}{8}\right)^l] \leq p - 2 \quad \Rightarrow \quad l \leq \frac{\log(p + 7)}{\log \frac{7}{8}} + \frac{\log 9}{\log \frac{7}{8}}
\]

Therefore \(|S_c| = |S_d| = O(\log p)|.

\[\Box\]

H.1 Proof of Proposition 2

Let \(\mathcal{P}(K_2, \hat{\mathcal{P}})\) be the max \(\hat{\mathcal{P}}\)-packing of \(K_2\), and \(M_2 = |\mathcal{P}(K_2, \hat{\mathcal{P}})|\). Define a distance function \(d(v, v') = |v - v'| \wedge |v + v'|\). For any \(v, v' \in \mathcal{P}(K_2, \hat{\mathcal{P}})\) with \(v \neq v'\), we have \(v'v' \geq 0\). Then

\[
d(v, v') = |v - v'|^2 = 2 - 2v\cdot v' = 2 - 2\left[ 2\left(\frac{p - 1}{\sqrt{p - 1}} - 1 + \epsilon^2 \right) \right] = 2^2 \left[ 1 - \frac{p - 1 - \epsilon^2}{\sqrt{p - 1}} \right] \leq 2\epsilon^2
\]

By Lemma [H.4] the lower bound of minimax risk is derived by Fano’s method

\[
\inf_{\hat{P}} \max_{P \in \mathcal{P}(K_2, \hat{\mathcal{P}})} \mathbb{E}[\|\hat{v} - v_j\|^2 \wedge |\hat{v} + v_j|^2] \geq \frac{\epsilon^2}{2} (1 - \frac{nD + \log 2}{\log M_2})
\]

The Kullback-Leibler divergence of \(P, P'\) can be upper bounded as

\[
D = \frac{k^2}{2(k + 1)} \left[ 1 - \langle v, v' \rangle^2 \right] \leq \frac{(k \wedge k^2)^2}{2}
\]

Thus the minimax lower bound becomes

\[
\inf_{\hat{v}} \max_{v \in \mathcal{P}(K_2, \hat{\mathcal{P}})} \mathbb{E}[\|\hat{v} - v_j\|^2 \wedge |\hat{v} + v_j|^2] \geq \frac{\epsilon^2}{2} (1 - \frac{n^2 (k \wedge k^2)^2}{2 \log M_2} - \frac{\log 2}{\log M_2})
\]

By Lemma [H.1] \(\log M_2 \sim O(\log p)\), then

\[
\inf_{\hat{v}} \max_{v \in \mathcal{P}(K_2, \hat{\mathcal{P}})} \mathbb{E}[\|\hat{v} - v_j\|^2 \wedge |\hat{v} + v_j|^2] \geq \frac{\epsilon^2}{2} (1 - \frac{n^2 (k \wedge k^2)^2}{2 \log \log p} - \frac{\log 2}{\log \log p})
\]

Pick \(\epsilon = \sqrt{\frac{\log \log p}{2m(k \wedge k^2)}}\), and plug in to the above inequality get

\[
\inf_{\hat{v}} \max_{v \in \mathcal{P}(K_2, \hat{\mathcal{P}})} \mathbb{E}[\|\hat{v} - v_j\|^2 \wedge |\hat{v} + v_j|^2] \geq \frac{\log \log p}{n(k \wedge k^2)}
\]

Finally,

\[
\inf_{\hat{v}} \max_{v \in \mathcal{P}(K_2, \hat{\mathcal{P}})} \mathbb{E}[\|\hat{v} - v_j\|^2 \wedge |\hat{v} + v_j|^2] \geq \inf_{\hat{v}} \max_{v \in \mathcal{P}(K_2, \hat{\mathcal{P}})} \mathbb{E}[\|\hat{v} - v_j\|^2 \wedge |\hat{v} + v_j|^2] \geq \frac{\log \log p}{n(k \wedge k^2)}
\]
References

Amelunxen, D., Lotz, M., McCoy, M. B., and Tropp, J. A. (2014). Living on the edge: Phase transitions in convex programs with random data. *Information and Inference: A Journal of the IMA*, 3(3):224–294.

Anderson, T. (2003). An introduction to multivariate statistical analysis (wiley series in probability and statistics). *July 11.*

Arora, S., Ge, R., Kannan, R., and Moitra, A. (2016). Computing a nonnegative matrix factorization—provably. *SIAM Journal on Computing*, 45(4):1582–1611.

Baik, J. and Silverstein, J. W. (2006). Eigenvalues of large sample covariance matrices of spiked population models. *Journal of multivariate analysis*, 97(6):1382–1408.

Barlow, R. E., Bartholomew, D. J., Brenner, J. M., and Brunk, H. D. (1972). Statistical inference under order restrictions: The theory and application of isotonic regression. Technical report, Wiley New York.

Bellec, P. C. et al. (2018). Sharp oracle inequalities for least squares estimators in shape restricted regression. *The Annals of Statistics*, 46(2):745–780.

Berkeley Earth (2020). Earth surface temperature data. [http://berkeleyearth.lbl.gov/city-list/](http://berkeleyearth.lbl.gov/city-list/).

Birnbaum, A., Johnstone, I. M., Nadler, B., and Paul, D. (2013). Minimax bounds for sparse pca with noisy high-dimensional data. *Annals of statistics*, 41(3):1055.

Cadima, J. and Jolliffe, I. T. (1995). Loading and correlations in the interpretation of principle components. *Journal of Applied Statistics*, 22(2):203–214.

Cai, T. T., Li, H., and Ma, R. (2020). Optimal structured principal subspace estimation: Metric entropy and minimax rates. *arXiv preprint arXiv:2002.07624.*

Cai, T. T., Ma, Z., Wu, Y., et al. (2013). Sparse pca: Optimal rates and adaptive estimation. *The Annals of Statistics*, 41(6):3074–3110.

Cover, T. M. and Thomas, J. A. (2012). *Elements of information theory*. John Wiley & Sons.

d’Aspremont, A., Ghaoui, L. E., Jordan, M. I., and Lanckriet, G. R. (2005). A direct formulation for sparse pca using semidefinite programming. In *Advances in neural information processing systems*, pages 41–48.

Deshpande, Y., Montanari, A., and Richard, E. (2014). Cone-constrained principal component analysis. In *Advances in Neural Information Processing Systems*, pages 2717–2725.

Deutsch, C. A., Tewksbury, J. J., Huey, R. B., Shelton, K. S., Ghalambor, C. K., Haak, D. C., and Martin, P. R. (2008). Impacts of climate warming on terrestrial ectotherms across latitude. *Proceedings of the National Academy of Sciences*, 105(18):6668–6672.

Donoho, D. L., Gavish, M., and Johnstone, I. M. (2013a). Optimal shrinkage of eigenvalues in the spiked covariance model. *arXiv preprint arXiv:1311.0851.*

Donoho, D. L., Johnstone, I., and Montanari, A. (2013b). Accurate prediction of phase transitions in compressed sensing via a connection to minimax denoising. *IEEE transactions on information theory*, 59(6):3396–3433.

Ester, M., Kriegel, H.-P., Sander, J., Xu, X., et al. (1996). A density-based algorithm for discovering clusters in large spatial databases with noise. In *Kdd*, volume 96(34), pages 226–231.

Fan, J. and Wang, W. (2015). Asymptotics of empirical eigen-structure for ultra-high dimensional spiked covariance model. *arXiv preprint arXiv:1502.04733.*

Gordon, Y. (1988). On milman’s inequality and random subspaces which escape through a mesh in R n. In *Geometric Aspects of Functional Analysis*, pages 84–106. Springer.

Hotelling, H. (1933). Analysis of a complex of statistical variables into principal components. *Journal of educational psychology*, 24(6):417.

Johnstone, I. M. and Lu, A. Y. (2009). On consistency and sparsity for principal components analysis in high dimensions. *Journal of the American Statistical Association*, 104(486):682–693.
Witten, D. M., Tibshirani, R., and Hastie, T. (2009). A penalized matrix decomposition, with applications to sparse principal components and canonical correlation analysis. *Biostatistics*, 10(3):515–534.

Yahoo! Finance (2020). S&p historical data. [https://finance.yahoo.com/quote/%5EGSPC/history/](https://finance.yahoo.com/quote/%5EGSPC/history/).

Yu, B. (1997). Assouad, fano, and le cam. In *Festschrift for Lucien Le Cam*, pages 423–435. Springer.

Yuan, X.-T. and Zhang, T. (2013). Truncated power method for sparse eigenvalue problems. *Journal of Machine Learning Research*, 14(Apr):899–925.

Zou, H. and Hastie, T. (2005). Regularization and variable selection via the elastic net. *Journal of the royal statistical society: series B (statistical methodology)*, 67(2):301–320.

Zou, H., Hastie, T., and Tibshirani, R. (2006). Sparse principal component analysis. *Journal of computational and graphical statistics*, 15(2):265–286.