Thermoodynamics of $d$–Dimensional Charged AdS (Anti-de Sitter) Black Holes: Hamiltonian Approach and Clapeyron Equation

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Abstract

The study of thermodynamics in the view of the Hamiltonian approach is a newest tool to analyze the thermodynamic properties of the black holes. In this letter, we investigate the thermodynamics of $d$-dimensional ($d > 3$) asymptotically Anti-deSitter black holes. A thermodynamic representation based on symplectic geometry is introduced in this letter. We extend the thermodynamics of $d$–dimensional charged Anti-deSitter black holes in the views of a Hamiltonian approach. Firstly, we study the thermodynamics in reduced phase space and correlate with the Schwarzschild solution. Then we enhance it in the extended phase space. In an extended phase space the thermodynamic equations of state are stated as constraints. We apply the canonical transformation to analyze the thermodynamics of said type of black holes. We plot $P$-$v$ diagrams for different dimensions $d$ taking the temperatures $T < T_c, T = T_c$ and $T > T_c$ and analyze the natures of the graphs and the dependences on $d$. In theses diagrams, we point out the regions of coexistence. We also examine the phase transition by applying “Maxwell’s equal area law” of the said black holes. Here we find the regions of coexistence of two phases which are also depicted graphically.

Finally, we derive the “Clapeyron equation” and investigate the latent heat of isothermal phase transition.

Keywords : Maxwell’s equal area law; Clapeyron equation and latent heat.

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1 Introduction :

Hawking put forward his idea that the radiation can be found to come out of black holes (BHs hereafter) via quantum tunneling [1, 2]. This idea provides a real connection with quantum mechanics to gravity. The main consideration of BHs is that they have a certain geometrical boundary the name of which is coined as event horizon. Actually the thermodynamics of BHs is the construction of analogous thermodynamic parameters which are obtained on the event horizon. In the references [3] the authors suggest that the phase transition called Hawking-Page phase transition. These kind of transition be explained as the confinement/deconfinement of phase transition of gauge field in the AntideSitter (AdS hereafter)/ conformal field theory (CFT) correspondence [4, 5, 6]. In broader sense, the AdS/ CFT correspondence is associated with gravity physics in an asymptotically AdS geometry with field theory in the boundary surface of the AdS space. The authors of the reference [6] suggest extended gauge/ gravity dualities in context with string theory.

In recent years the thermodynamic aspects of AdS space-times are considered in several references like [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. The thermodynamics of the BHs with cosmological constant $\Lambda$ as a thermodynamical variable has been studied in the references [26, 27, 28]. The physical interpretation of the conjugate variable associated with $\Lambda$ is a point of interest till now. Though in the references [29, 30] the authors suggest that this conjugate variable could be thermodynamic volume and due to this proposal $\Lambda$ would be interpreted as a pressure and the mass of the BHs would be identified with the total gravitational enthalphy and not with the internal energy of the system in the extended phase space.

After development of string theory or M-theory, the study of the higher dimensional BHs comes in interest. This theory plays a very important role to build the quantum theory of gravity. The study of dynamics of BHs in higher dimensions gives the interesting result even in studying the compactified mechanisms. In [31, 32] the authors has established the statistical- mechanical calculations of the Bekenstein-Hawking entropy for a class of supersymmetric
BHs in 5-dimensions. This is one of the remarkable results in string theory. Brane world scenarios also motivate to study the BHs in higher dimensions. This is a new fundamental scale of quantum gravity. As the dimensions of space-time affect the thermodynamic properties of BHs [33, 34, 35, 36], the analyze of BHs in higher dimensions is very much significant.

In this letter, we consider the thermodynamic aspects of AdS space-time. Here we choose the $d$-dimensional($d > 3$) charged AdS BHs whose metric is the spherically symmetric solution of Einstein field equations (EFE) with a negative cosmological constant [37]. We introduce the cosmological constant as thermodynamic variable as other existing works viz., [26, 27, 28]. Only deference with these articles is that they have taken $\Lambda$ to be the function of coordinates. Only difference with these articles is that they have taken $\Lambda$ to be the function of coordinates and respects the laws of BHs thermodynamics. We may construct a new equation of state which completely characterizes the thermodynamics and we realize that the thermodynamic equation of states are constraints on phase space. This new treatment which is introduced here is consistent because it resembles with the new thermodynamic potential and respects the laws of BHs thermodynamics. We may construct a new equation of state which completely characterizes the thermodynamics of charged $d$-dimensional charged AdS BHs only assuming homogeneity of the thermodynamic variables.

Using “Maxwell’s equal area law”, we analyze the phase transition of the said BHs. Here we notice that there is the regions of coexistence of two phases. We will also depict the coexistence region of two phases by plotting the $P-v$ diagrams for different dimensions $d$, for the temperatures $T < T_c$, $T = T_c$ and $T > T_c$ respectively. Finally, we derive the “Clapeyron equation” and investigate the latent heat of isothermal phase transition by studying the phase transition process.

The letter is organized as follows. In section 2 we establish the thermodynamic results of the $d$-dimensional charged AdS (Anti-de Sitter) BHs in brief. We also study the thermodynamics of that BHs in reduced as well as extended phase space in the view of a Hamiltonian approach in the same section. In section 3 we extend ‘Maxwell’s’ equal area law for this BHs. We compute the ‘Clapeyron equation’ and investigate the latent heat of isothermal phase transition in section 4. Finally we briefly discuss about the outcome of this letter and conclude this letter in section 5. Through out this letter we use the Planck units, i.e., $G = c = h = \kappa_B = 1$ and signature ($- + +.....+$).

2 The Brief Review of Thermodynamics of $d$-dimensional Charged AdS Black Holes:

The $d$-dimensional ($d > 3$) charged AdS BHs metric can be obtained as the spherically symmetric solution of $d$-dimensional Einstein field equation (EFE) of the form: $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$ [39]:

$$ds^2 = -g_{tt}(r)dt^2 + g_{rr}dr^2 + r^2\left\{(d\theta_1)^2 + \sin^2\theta_1(d\theta_1)^2 + \ldots + \sin^2\theta_1\ldots\sin^2\theta_{d-2}(d\theta_{d-2})^2\right\},$$

(1)

where

$$-g_{tt}(r) = g_{rr}(r)^{-1} = 1 - \frac{\tilde{M}}{r^{d-3}} + \frac{\tilde{Q}^2}{r^{2(d-3)}} - \tilde{\Lambda}r^2$$

(2)

and $(d\theta_1)^2 + \sin^2\theta_1(d\theta_1)^2 + \ldots + \sin^2\theta_1\ldots\sin^2\theta_{d-2}(d\theta_{d-2})^2$ represents the canonical volume associated with the induced metric [11] and the constants $\tilde{M}$, $\tilde{Q}$ and $\tilde{\Lambda}$ in the equation (2) are expressed in terms of the mass $M$, the electrical charge and the cosmological constant $\Lambda$ respectively [40] as:

$$\tilde{M} = \frac{16\pi M}{(d-2)B_{d-2}}, \quad B_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \quad \tilde{Q} = \frac{8\pi Q}{B_{d-2}\sqrt{2(d-3)(d-2)}} \quad \text{and} \quad \tilde{\Lambda} = \frac{2\Lambda}{(d-2)(d-1)}.$$  

(3)

On the event horizon $r = r_h$, the mass parameter $\tilde{M}$ can be expressed as:

$$\tilde{M} = \tilde{Q}_{r_h}^2 + \left(1 - \tilde{\Lambda}r_h^2\right)r_h^{d-3}.$$  

(4)

Assuming that the event horizon is a Killing horizon, the surface gravity may be defined as the magnitude of the norm of horizon generating Killing field $\chi^a = \zeta^a + \Omega \psi^a$, evaluated at the horizon and is expressed by using equations (2) and (4) as:
The Killing horizon area $A$ may be written as a function of event horizon’s radius as:

$$A = r_h^{d-2} B_{d-2}. \quad (6)$$

Stationary BH solutions of EFE relate the first law of thermodynamics as:

$$dM = TdS + \Omega dJ + \phi dQ \quad (7)$$

and the Smarr formula for $d$–dimensional charged AdS BHs reads

$$M = \frac{d}{d-3} TS + \phi dQ - \frac{2}{d-3} PV. \quad (8)$$

The classical BH thermodynamics [43, 44, 45] shows that the mass $M$ and surface gravity $\kappa$ of the BHs related to internal energy $U$ and temperature $T$ as:

$$\begin{align*}
M &= U \quad \text{and} \quad T = \frac{\kappa}{2\pi}. \\
\end{align*} \quad (11)$$

More generally the mass $M$ of the BHs is equal to the total gravitational enthalpy $H$ and is expressed as:

$$M = H = U + PV, \quad (12)$$

where $P$ is the thermodynamic pressure and $V$ is the naive geometric volume and they are defined as:

$$P = - \frac{\hat{A}(d-2)(d-1)}{16\pi} \quad \text{and} \quad V = \frac{1}{d-1} \left( \frac{A S}{B_{d-2}} \right)^{\frac{d-3}{2}} B_{d-2}. \quad (13)$$

Again

$$dH = dM = TdS + VdP, \quad (14)$$

which gives,

$$V = \left( \frac{\partial H}{\partial P} \right)_S = \left( \frac{\partial M}{\partial P} \right)_S \quad \text{and} \quad T = \left( \frac{\partial H}{\partial S} \right)_P = \left( \frac{\partial M}{\partial S} \right)_P. \quad (15)$$

This volume $V$ is known as thermodynamic volume. Generally thermodynamic volume is not equal to the naive geometric volume of the BHs.

Using equations (3), (4) and (6) one can obtain the mass $M$ as:

$$M = \frac{(d-2)B_{d-2}}{16\pi} \left( \frac{A}{B_{d-2}} \right)^{\frac{d-3}{2}} \left[ \frac{\hat{Q}^2}{\left( \frac{A}{B_{d-2}} \right)^{\frac{2d-3}{2}}} + 1 - \hat{\Lambda} \left( \frac{A}{B_{d-2}} \right)^{\frac{2d-3}{2}} \right]. \quad (16)$$

Moreover, using equations (5) and (6) one can construct the surface gravity $\kappa$ as:

$$\kappa = \frac{1}{2} \left( \frac{A}{B_{d-2}} \right)^{\frac{d-3}{2}} \left[ \frac{d-3}{\left( \frac{A}{B_{d-2}} \right)^{\frac{d-3}{2}}} - \hat{\Lambda}(d-1) - \frac{\hat{Q}^2(d-3)}{\left( \frac{A}{B_{d-2}} \right)^2} \right]. \quad (17)$$
From equations (10) and (17) one can obtain the following relation,

\[
8\pi M \left( \frac{d-1}{d-2} \right) - \kappa A = A \left( \frac{B_{d-2}}{A} \right) \frac{d-3}{d-2} + \frac{(d-2)(d-1)B_{d-2} \tilde{Q}^2}{\left(\frac{A}{B_{d-2}}\right)^{d-3}}.
\]

(18)

It is shown that in equation (15) the cosmological constant \( \Lambda \) does not appear explicitly. Hence from equation (18) we have the equation of state given as:

\[
U \left( \frac{d-1}{d-2} \right) - TS = \frac{1}{2\pi} \left( \frac{B_{d-2}}{4} S^{d-3} \right)^{\frac{d-3}{d-2}} - \frac{(d-2)(d-1)B_{d-2} \tilde{Q}^2}{8\pi \left( \frac{4S}{B_{d-2}} \right)^{\frac{d-3}{d-2}}}.
\]

(19)

Equation (12) gives the internal energy as:

\[
U = \frac{(d-2)B_{d-2}}{16\pi} \left( \frac{4S}{B_{d-2}} \right)^{\frac{d-3}{d-2}} \left[ 1 + \frac{\tilde{Q}^2}{\left( \frac{4S}{B_{d-2}} \right)^{\frac{d-3}{d-2}}} \right].
\]

(20)

Equation (20) is not compatible with the thermodynamic relation given by

\[
T = \frac{\kappa}{2\pi} + \frac{\partial U}{\partial S},
\]

(21)

This implies that the thermodynamics defined by the relations (9), (11) and (13) are not consistent. Since in this problem \( V \) does not depend upon \( P \), this problem is a consequence of the singularity of the Legendre transformation of the pair \( (P, V) \).

Using the equations (11), (13) and (17), one can obtain the equation of state for a \( d \)-dimensional charged AdS BH as:

\[
P = \frac{(d-2)T}{4 \left( \frac{A}{B_{d-2}} \right)^{\frac{d-3}{d-2}}} - \frac{(d-3)(d-2)}{16\pi \left( \frac{A}{B_{d-2}} \right)^{\frac{d-3}{d-2}}} + \frac{\tilde{Q}^2(d-3)(d-2)}{16\pi \left( \frac{A}{B_{d-2}} \right)^{\frac{d-3}{d-2}}}.
\]

(22)

With the specific volume \( v = \frac{4(\frac{\kappa_4}{\pi} - \frac{1}{v^{d-2}})}{d-2} \), where \( l_p = \sqrt{\frac{\hbar c}{G}} \), we can compute the equation of state for this BH as:

\[
P = \frac{T}{v} - \frac{C}{v^2} + \frac{D\tilde{Q}^2}{v^{2(d-2)}},
\]

(23)

where \( C = \frac{d-3}{\pi(d-2)} \) and \( D = \frac{24(d-3)(d-2)}{\pi(d-2)^2d^2} \).

We have plotted the variations of \( P \) with respect to specific volume \( v \) for different dimensions \( d \) in Fig.-1a-g for equation (22) for different dimensions \( d \) for \( T < T_c, T = T_c \) and \( T > T_c \) respectively. Here we have seen that \( P \) is decreased as \( v \) is increased and reaches to a local minimum for \( T < T_c \). This means that there is a small-large BH phase transition in the system. It is clearly shown in the curves that there are thermodynamic unstable regimes for \( \frac{dT}{dP} > 0 \) for \( T < T_c \). For higher dimensions the thermodynamic unstable regimes also higher, i.e., for \( T < T_c, T = T_c \) and for some extent of \( T > T_c \).

We also depict the \( P-T \) diagrams in Fig.-2. Here we do not restrict the curves in critical points \( (P_c \) and \( T_c) \). Here we notice that due to increase of temperature, pressure also increases but this increment is not linear as expected. It is also found that there are negative temperatures for \( T < T_c \) which implies that there are thermodynamic unstable regimes for \( T < T_c \). Again for higher dimensions this regimes also higher that also shown in Fig.-1a-g.

### 2.1 Hamiltonian Approach to Thermodynamics of \( d \)-dimensional Charged AdS Black Holes:

Symplectic structure which involves thermodynamic variables are studied in the references [31, 32]. For some chosen phase space if the thermodynamic variables can be treated as the coordinates and momentum, the concerned thermodynamic variables’ integrability conditions take place of the analogous Poission brackets and henceforth the duality
Fig.-1a-g represent the variations of $P$ with respect to specific volume $v$ for different dimensions $d$ for $T < T_c, T = T_c$ and $T > T_c$.

Fig.-2 represents the variations of $P$ with respect to the temperature $T$ for different dimensions $d$.

between the mechanics and thermodynamics can be established. However, as it is not clear how to translate Hamiltonian trajectories in phase-space the analogy can not be taken too far. Translating Hamiltonian trajectories requires a clear idea of the Hamiltonian equations of motion $\dot{q} = \{q, H\}$, $\dot{p} = \{p, H\}$. In thermodynamics, the integrability conditions depend on the concerned potential chosen and they are related by Legendre transformations. Hamiltonian system is composed of the triple $(M, \omega, X_H)$, $M$ is a smooth manifold, $\omega$ is the canonical symplectic form on $M$ and $X_H$ the Hamiltonian vector field.

All the equations of state of a thermodynamic system as constraints on phase space can be realised in the Hamiltonian approach to thermodynamics. On the constraints surface ($Hamiltonian (H) = 0$) with the relation $d\tau = dt$, the tautological form in extended phase space degenerates to Poincaré-Cartan form in another phase space with reduced Hamiltonian, $h = h(q, p, \tau)$ as: $p_i dq^i + \zeta d\tau|_{H=0} = p_i dq^i - h dt$, where $(\zeta, \tau)$ is a canonical pair by which one
can extend the phase space [38]. In this condition, one can simplify a Hamiltonian system with Hamiltonian function
\[ H = \zeta + h(q, p, \tau) \]
to the system with \( h \) in the reduced phase space \( (q, p) \). The differential of thermodynamic variables is given by the tautological form as:
\[ pdq + \zeta d\tau = dU \]
on the constraints surface. Thus in this way, all the thermodynamic variables are to be related by the canonical transformation and gives equivalent representations.

Here we introduce the canonical transformations as:
\[ q = \frac{A}{B_{d-2}} = \frac{4S}{B_{d-2}} \quad \text{and} \quad p = \pi T = \frac{\kappa}{2} \]
(24)

The equations (16) and (17) may be written in terms of the canonical transformations (24) as:
\[ M = \frac{(d-2)B_{d-2} \frac{\zeta}{q} + \frac{1}{2} + \frac{2\Lambda}{(d-2)(d-1)}}{4q^{d-3}} \]
(25)
and
\[ M = \frac{(d-3)\frac{\zeta}{q} - \frac{2\Lambda}{d-2} \pi}{q} + \frac{\zeta^2(d-3)}{q^2} \]
respectively.

### 2.1.1 Thermodynamics in Reduced Phase Space:

In this section we have studied the Thermodynamics of \( d \)-dimensional charged AdS BHs in reduced phase space and for this we follow the technique [38] given below:

Differentiating equation (25) and using equation (26), we obtain
\[ dM = \frac{B_{d-2}}{4\pi} \left\{ p - \frac{(d-3)\frac{\zeta}{q} - \frac{2\Lambda}{d-2} \pi}{4q} \right\} dq - \frac{1}{2(d-1)} q^{\frac{d-1}{2}} \frac{d\Lambda}{dq} \]
(27)

If we assume \( \Lambda = \Lambda(q) \), i.e., \( \Lambda \) is a function of \( q \) only, the equation (24) may be expressed as:
\[ dM = \wp dq = \frac{B_{d-2}}{4\pi} \left\{ p - \frac{(d-3)\frac{\zeta}{q} - \frac{2\Lambda}{d-2} \pi}{4q} \right\} dq - \frac{1}{2(d-1)} q^{\frac{d-1}{2}} \frac{d\Lambda}{dq} \]
(28)

The above expression is analogous with the tautological form \( \alpha = \wp dq \) and is restricted to the constraint surface given by the equation (27), \( dM = \alpha |_{\Theta=0} \), where
\[ \Theta = p - \frac{(d-3)\frac{\zeta}{q} - \frac{2\Lambda}{d-2} \pi}{4q} q^{\frac{d-1}{2}} + \frac{\Lambda}{2(d-2)} q^{\frac{d-1}{2}} \]
(29)

Here we introduce a symplectic form of \( \omega \) to represent the expression (28) more precisely given by
\[ \omega_{\tau} = dp \wedge \left( \frac{\partial \wp}{\partial p} \right) dq = dp \wedge \frac{B_{d-2}}{4\pi} dq. \]
(30)

This equation (30) implies that the transformation \( (p, q) \mapsto (\wp, q) \) is canonical. equation (27) may also be written as:
\[ dM = \Omega' dq' - \frac{B_{d-2}}{8\pi(d-1)} \left( q^{\frac{d-1}{2}} \Lambda \right)' = \frac{B_{d-2}}{4\pi} \left\{ p - \frac{(2d-5)(d-3)\frac{\zeta}{q} - \frac{2\Lambda}{d-2} \pi}{4(d-2)q^{\frac{d-1}{2}}} \right\} dq - \frac{B_{d-2}}{8\pi(d-1)} \left( q^{\frac{d-1}{2}} \Lambda \right)' \]
(31)

where \( ' \) stands for the first order derivative. The expression (31) shows that there is a time-dependent, i.e., \( \tau \)-dependent canonical transformation \( q \mapsto \tilde{q}, \wp \mapsto \tilde{\wp} \) mapping the reduced phase space with coordinates \( (\wp, q) \) to a reduced phase space with coordinates \( (\tilde{\wp}, \tilde{q}) \), see detail in reference [38]. Hence the tautological form is represented as:
\[ d\mathcal{M} = \Omega' dq' - \frac{B_{d-2}}{8\pi(d-1)} \left( q^{\frac{d-1}{2}} \Lambda \right)', \]
(32)
where \( \mathcal{M} = M + V \Delta \). The factor \( ' - V \Delta \) represents the space time energy. For the coordinate transformation in reduced phase space \( (\wp, q) \mapsto (\tilde{\wp}, \tilde{q}) \), we consider the generating function of second kind as:
\[ F = \Omega' q' - \frac{B_{d-2}}{8\pi(d-1)} \left( q^{\frac{d-1}{2}} \Lambda \right) = \Omega' q + g(q, \tau). \]
(33)
Differentiating the equation (33) with respect to $q$ and substituting $\Omega'$ from equation (31), we have
\[
\frac{B_{d-2}}{4\pi} \left\{ p - \frac{(d-3)\bar{Q}^2}{4q^{d+2}} \right\} - \frac{1}{2(d-1)}q^{\frac{2}{d-3}} \frac{\partial \Lambda}{\partial q},
\]
(34)
which is equal to $\varpi$, already obtained in equation (28). This gives
\[
\varpi = \frac{\partial F}{\partial q}.
\]
(35)
Moreover from equation (33), we can show
\[
q' = \frac{\partial F}{\partial \Omega'} = q.
\]
(36)
The canonical transformation (33) obey the Bekenstein’s prediction on BH entropy and such transformation enforces the second law of BH Thermodynamics. By using the canonical transformation (33) one can obtain all the above thermodynamic results directly from Schwarzschild solution without solving the EFE, since in the Schwarzschild mass $\mathcal{M}$ the space time energy term which is defined with negative cosmological constant $\Lambda$ is included.

2.1.2 Thermodynamics in Extended Phase Space:

One can expand the expression (30) in extended phase space with a symplectic form given as:
\[
\omega_e = dp \wedge \left( \frac{\partial \varpi}{\partial q} \right) dq + d\tau \wedge \left( \frac{\partial \varpi}{\partial \tau} \right) dq + d\zeta \wedge d\tau.
\]
(37)
If $\Lambda$ is taken to be a function of $q$ and $\tau$, i.e., $\Lambda = \Lambda(q, \tau)$, equation (28) can be written as:
\[
dM = \varpi dq - \frac{B_{d-2}}{8\pi(d-1)}q^{\frac{d-1}{d-2}} \frac{\partial \Lambda}{\partial \tau} d\tau.
\]
(38)
Combining the equations (12) and (14) with equation (32) we have the tautological form
\[
dU = pdq + \zeta d\tau.
\]
(39)
This implies that the symplectic form expressed in equation (31) is true.
In coordinates, the Poisson brackets are given as:
\[
\{f, g\} = \mathcal{L}_{\tilde{\xi}}g - \mathcal{L}_{\tilde{\eta}}f = \frac{\partial f}{\partial q} \frac{\partial \varpi}{\partial \tilde{\eta}} - \frac{\partial f}{\partial \tilde{\eta}} \frac{\partial \varpi}{\partial q} + \frac{\partial f}{\partial \tau} \frac{\partial \varpi}{\partial \zeta} - \frac{\partial f}{\partial \zeta} \frac{\partial \varpi}{\partial \tau},
\]
(40)
where $f$ and $g$ are functions on phase space. Hence the non-zero canonical Poisson brackets are
\[
\{q, p\} = \{\tau, \zeta\} = 1
\]
(41)
Use equations (15) and (24) to combine with the expression (28), we obtain the thermodynamic volume and thermodynamic temperature as:
\[
V = -\frac{B_{d-2}}{8\pi(d-1)} \left( \frac{4S}{B_{d-2}} \right)^{\frac{d-1}{d}} \frac{\partial \Lambda}{\partial F}
\]
(42)
and
\[
T_{\text{eff}} = T - \frac{B_{d-2}}{8\pi(d-1)} \left( \frac{4S}{B_{d-2}} \right)^{\frac{d-1}{d}} \frac{\partial \Lambda}{\partial S}.
\]
(43)
Euler’s theorem for homogeneous functions $g(x, y)$ such that $g(\lambda^ax, \lambda^by) = \lambda^\gamma g(x, y)$ turns out to be
\[
\gamma g(x, y) = \alpha x \left( \frac{\partial g}{\partial x} \right) + \beta y \left( \frac{\partial g}{\partial y} \right).
\]
(44)
Taking $g$ as Schwarzschild mass $\mathcal{M}$, $x = q$ and $y = 0$, Euler’s theorem gives
\[
(d-3)\mathcal{M} = (d-2)\Omega' q'.
\]
(45)
In extended space this relation may be expressed as:
\[
(d-3)\mathcal{M} = (d-2) \left( T_{\text{eff}} S + cVP \right),
\]
(46)
for details see the reference [40].
3 Maxwell’s Equal Area Law And its Extension:

For Van der Waals equation of state there are also unstable state with $\frac{\partial P}{\partial V} > 0$ as charged topological BHs. But they are resolved by using Maxwell’s equal area law \[\text{[47]}\] given as:

$$\int_{v_1}^{v_2} P dv.$$ \[\text{(47)}\]

Now, we extend this equal area law for $d$–dimensional Charged AdS (Anti-de Sitter) BHs to compute a phase transition of the BH, considered as thermodynamics system. Using equations \[\text{[23]}\] and \[\text{[47]}\] one can have

$$Tln \left( \frac{v_2}{v_1} \right) - C \left( \frac{1}{v_1} - \frac{1}{v_2} \right) + \frac{D\tilde{Q}^2}{2d-3} \left( \frac{1}{v_1^{2d-3}} - \frac{1}{v_2^{2d-3}} \right) = P(v_2) - P(v_1).$$ \[\text{(48)}\]

Assuming

$$P(v_1) = \frac{T}{v_1} - C \left( \frac{1}{v_1^2} + \frac{D\tilde{Q}^2}{v_1^{2(d-2)}} \right) \quad \text{and} \quad P(v_2) = \frac{T}{v_2} - C \left( \frac{1}{v_2^2} + \frac{D\tilde{Q}^2}{v_2^{2(d-2)}} \right) \quad \text{[49]}$$

one can get the right hand side of equation \[\text{(48)}\] as:

$$P(v_2) - P(v_1) = T \left( \frac{1}{v_2} - \frac{1}{v_1} \right) - C \left( \frac{1}{v_2^2} - \frac{1}{v_1^2} \right) + D\tilde{Q}^2 \left( \frac{1}{v_2^{2(d-2)}} - \frac{1}{v_1^{2(d-2)}} \right) \quad \text{[50]}$$

Equating the relations \[\text{(48)}\] and \[\text{(50)}\] we can establish

$$C(xv_2)^{2(d-1)} \left[ (1 + x)lnx \right] = D\tilde{Q}^2(xv_2)^{2(d-2)} \left[ \frac{2(d-2)(1 - x^{2d-5})}{2d-5} + \frac{(1 - x^{2(2d-2)})lnx}{1 - x} \right], \quad \text{[51]}$$

where we take $x = \frac{v_1}{v_2} (0 < x < 1)$. Again for $P(v_1) = P(v_2)$, one can calculate from equation \[\text{(50)}\]

$$T(xv_2)^{2d-1} = C(xv_2)^{2(d-1)}(1 + x) - D\tilde{Q}^2(xv_2)^{2(d-2)} \left( \frac{1 - x^{2(d-2)}}{1 - x} \right) \quad \text{[52]}$$

If we relate the temperature $T$ with the critical temperature $T_c$ as $T = \xi T_c$, the equation \[\text{[52]}\] takes the form as:

$$\xi T_c(xv_2)^{2d-1} = C(xv_2)^{2(d-1)}(1 + x) - D\tilde{Q}^2(xv_2)^{2(d-2)} \left( \frac{1 - x^{2(d-2)}}{1 - x} \right), \quad \text{[53]}$$

where $T_c = \frac{4(d-3)^2}{8V_c(2d-2)}$ with the critical volume $V_c = \frac{1}{4\pi} \left[ \tilde{Q}^2(d - 2)(2d - 5) \right]^{\frac{1}{d-3}} \quad \text{[48]}$. It is clear that $x = 1$ and $\xi = 1$ correspond the critical state of the charged AdS BHs.

3.1 Approximation Solution:

We now compute the zeroth order approximation solution at first then substituting this we obtain the first order approximation solution and so on by applying the same method.

The zeroth order approximation solution is calculated as:

$$v_{2,0}^2 = F(x)^2 = \frac{D\tilde{Q}^2 \left[ \frac{2(d-2)(1 - x^{2d-5})}{2d-5} + \frac{(1 - x^{2(2d-2)})lnx}{1 - x} \right]}{Cx^2 \left[ (1 + x)lnx \right]} \quad \text{[54]}$$

and that of first order approximation solution as:

$$v_{2,1}^{2d} = F(x)^{2d} = \frac{D\tilde{Q}^2 F(x)^{2d-2} \left[ \frac{2(d-2)(1 - x^{2d-5})}{2d-5} + \frac{(1 - x^{2(2d-2)})lnx}{1 - x} \right]}{Cx^2 \left[ (1 + x)lnx \right]} \quad \text{[55]}$$
For $x \to 1$, $v_2 = V_c$, the critical volume of that BH, then from equations (54) and (55) we can have

$$V_c^{2d} = V_c^{2(d-1)} \lim_{x \to 1} (2d-2)(1-x^{2(d-2)})^\frac{1}{2d-5} + \frac{(1-x^{2(d-2)})lnx}{1-x}$$

(56)

Meanwhile, from equation (23) we can compute

$$P = \frac{T}{F(x)} - \frac{C}{F(x)^2} + \frac{D\tilde{Q}^2}{F(x)^{2(d-2)}}$$

(57)

4 Clapeyron Equation and The Phase Change:

Clapeyron equation is the direct experimental verification of some phase changes and is expressed as:

$$\frac{dP}{dT} = \frac{L}{T(v_\beta - v_\alpha)}$$

(58)

where $L = T(s_\beta - s_\alpha)$ is the latent heat of phase change form phase $\alpha$ to phase $\beta$ in which the molar volume and molar entropy are $v_\alpha$, $s_\alpha$ and $v_\beta$, $s_\beta$ respectively.

Here we examine the two phase equilibrium coexistence $P$-$T$ curves and the slope $\frac{dP}{dT}$ of that curves for the charged AdS BHs. We may write the equation (52) in terms of $F(x)$ as:

$$T = G(x) = C \left\{1 + \frac{1 - x^{2(d-2)}}{xF(x)}\right\} - D\tilde{Q}^2 \left\{\frac{x^2}{F(x)^3(1-x)}\right\}$$

(59)

And equation (57) in terms of $G(x)$ as:

$$P = H(x) = \frac{G(x)}{F(x)} - \frac{C}{F(x)^2} + \frac{D\tilde{Q}^2}{F(x)^{2(d-2)}}$$

(60)

Hence one can calculate the slope of two phase equilibrium coexistence $P$-$T$ curves from the equations (36) and (60) as:

$$\frac{dP}{dT} = \frac{H(x)'}{G(x)'}$$

(61)

where $H(x)' = \frac{dH}{dx}$ and $G(x)' = \frac{dG}{dx}$. We can have the latent heat of phase change as a function of $x$ for $d$–dimensional charged AdS BH by using the the equations (58), (59), (60) and (61) as:

$$L = (1-x) \frac{H(x)'}{G(x)'} G(x) F(x).$$

(62)

The rate of change of latent heat of phase transition with temperature for $d + 1$ dimensional charged AdS BH is obtained from equations (58), (59) and (62) as:

$$\frac{dL}{dT} = \frac{dL}{dx} \frac{dx}{dT} = \frac{1}{G(x)'} \frac{dL}{dx}$$

(63)

But for some thermodynamic systems, the rate of change of latent heat of phase transition with temperature is given as (17)

$$\frac{dL}{dT} = C_{P \beta} - C_{P \alpha} + \frac{L}{T} \left[\left(\frac{\partial v_\beta}{\partial T}\right)_P - \left(\frac{\partial v_\alpha}{\partial T}\right)_P\right] \frac{v_\beta - v_\alpha}{L}$$

(64)

5 Brief Discussion and Conclusion

In this letter, we have investigated the thermodynamics of $d$–dimensional $(d > 3)$ AntideSitter black holes. Here we have considered the thermodynamic aspects of space-times. We have reconstructed the thermodynamic result by using the Hamiltonian approach developed in the reference [38]. Initially we have studied the thermodynamics in
naked singularity is likely to occur. In the thermodynamic description of the charged Anti-de-Sitter black holes the cosmological constant $\Lambda$ can be introduced as a result of canonical transformation of the Schwarzschild problem. If we considered $\Lambda$ to be the function of coordinates in phase space through, a new thermodynamic equation of state occurs and this new equation of state exhibits the generalized thermodynamic volume. Due to this new approach a new equation of state which completely characterizes the thermodynamics of charged Anti-de-Sitter black holes has been established only assuming homogeneity of the thermodynamic variables. We have assumed that $\Lambda$ is a constant in the space-time manifold, though here we considered $\Lambda$ to be the function of coordinates in phase space. It is to be noted that if we vary $\Lambda$, the thermodynamic results which have been developed here remains consistent. In that case $\Lambda$ consider as a dynamical geometric variable [49] rather than the thermodynamic variable.

When we have plotted the variations of $P$ with respect to specific volume $v$ for different dimensions $d$ in Fig.-1a-g for equation (23) for different dimensions $d$ for $T < T_c, T = T_c$ and $T > T_c$, we find that there is a small-large black hole phase transition in the system and one can also notice that there are thermodynamic unstable regimes for $\partial P/\partial v > 0$ for $T < T_c$. For higher dimensions, the thermodynamic unstable regimes also higher, i.e., for $T < T_c, T = T_c$ and for some extent of $T > T_c$. The thermodynamic unstable regimes also shown in $P - T$ diagrams in Fig.-2. The negative temperatures for $T < T_c$ as shown in Fig.-2 signifies it. Using “Maxwell’s equal area law”, we analyze the phase transition of the said black holes and the problems mentioned above may be removed. From Fig.-2, we derive the slope of the curves which serve the information about latent heat of phase change by Clapeyron equation which helps to find some thermodynamic properties of thermodynamic system such as black holes and also provide the theoretical background for experimental research on analogous black holes.

In this letter the main motivations were to establish a thermodynamic study of the $d$-dimensional black holes. Use of Hamiltonian approach helped us to construct the thermodynamic extended and reduced phases. We observe that for black hole thermodynamics also, these phase spaces have the same results as the classical thermodynamic considerations. Two phase equilibrium coexistence is examined and the corresponding latent heat formula if evaluated in Clapeyron equation. As we increase the dimensions, latent heat has more prominent local maxima or minima which indicates the nature of stabilities around the phase transition points. As we increase the position coordinate $x$, i.e., the entropy $S$ to shift from one phase to another more latent heat is to be given if $x$ is low. For medium level of $x$ quantity of latent heat required reduces. But if $x$ is high, monotonically increasing latent heat is required to give. So it is different to change the phase for smaller or larger black holes. The phases are stable then. But for an intermediate site black hole, it will be quite easier to transmute from one phase to another. Some time even we need not to incorporate latent heat from outside. Rather it will itself radiate energy (due to negative latent heat) outward and transmit to another phase. This signifies highly unstable phase. Here the black hole may turn into a hot AdS space. For higher dimension, this unstable region or range of phase space is more prominent. This, on a ather way, depicts that as we increase dimension the one sidal membrane of event horizon becomes more permeable (i.e., unstable) and we are likely to have an unwrapped naked singularity rather than a black hole. This result supports the previously stated outcome of the higher dimensional gravitational collapse [50][51][52][53], where it was found that if a catastrophic collapse takes place in the background of higher dimensions, naked singularity is likely to occur.

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