Distance $r$-domination number and $r$-independence complexes of graphs

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Abstract

For $r \geq 1$, the $r$-independence complex of a graph $G$, denoted $\text{Ind}_r(G)$, is a simplicial complex whose faces are subsets $A \subseteq V(G)$ such that each component of the induced subgraph $G[A]$ has at most $r$ vertices. In this article, we establish a relation between the distance $r$-domination number of $G$ and (homological) connectivity of $\text{Ind}_r(G)$. We also prove that $\text{Ind}_r(G)$, for a chordal graph $G$, is either contractible or homotopy equivalent to a wedge of spheres. Given a wedge of spheres, we also provide a construction of a chordal graph whose $r$-independence complex has the homotopy type of the given wedge.

Keywords: Independence complex, higher independence complex, distance $r$-domination number, chordal graphs

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1 Introduction

The independence complex, $\text{Ind}(G)$, of a graph $G$ is the simplicial complex whose simplices are those subsets $I$ of vertices of $G$ such that the induced subgraph $G[I]$ does not have any edge. Independence complexes have applications in several areas. Study of topological properties of independence complexes has been an active direction of research. For example, Babson and Kozlov [2] used the topology of independence complexes of cycles to prove a conjecture by Lovász. Properties of independence complexes have also been used to study the Tverberg graphs [12] and the independent system of representatives [1]. For more on these complexes, interested reader is referred to [3, 5, 6, 10, 11, 14].

Let $r$ be a positive integer and $G$ be a graph. A set $A \subseteq V(G)$ is called $r$-independent if each connected component of the induced subgraph $G[A]$ has at most $r$ vertices. In [17], Paolini and Salvetti generalized the concept of independence complex by defining $r$-independence complex of a graph $G$ for any $r \geq 1$. In the same paper, they gave a relation between twisted homology of classical braid groups and the homology of $r$-independence complexes of associated Coxeter graphs. The $r$-independence complex of $G$, denoted $\text{Ind}_r(G)$, is the simplicial complex whose simplices are $r$-independent sets of vertices of $G$. Observe that $\text{Ind}_1(G)$ is same as $\text{Ind}(G)$. First and third authors [8] initiated the study of these complexes and gave a closed form formula for the homotopy

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type of \( r \)-independence complexes for certain families of graphs including complete \( s \)-partite graphs, fully whiskered graphs, cycle graphs and perfect \( m \)-ary trees.

Let \( \Gamma(G) \) and \( \Gamma_0(G) \) denote the domination number and strong domination number of a graph \( G \) respectively. For \( k \geq 0 \), let \( \tilde{H}_k(X) \) denotes the \( k \)th reduced homology of a topological space \( X \) with integer coefficients. In [10], Meshulam proved the following.

**Theorem 1.** [10, Theorem 1.2]

(i) If \( \Gamma_0(G) > 2k \), then \( \tilde{H}_{k-1}(\text{Ind}_1(G)) = 0 \).

(ii) If \( G \) is chordal and \( \Gamma(G) > k \), then \( \tilde{H}_{k-1}(\text{Ind}_1(G)) = 0 \).

It is natural to ask, whether we can relate some topological properties of \( \text{Ind}_r(G) \) to some graph theoretic invariants of \( G \). One of the main motivations of this article is to establish similar results for \( r \)-independence complexes.

**Definition 1.** A set \( D \subseteq V(G) \) is said to be a distance \( r \)-dominating set of \( G \) if the distance \( d(u, D) \) between each vertex \( u \in V(G) \setminus D \) and \( D \) is at most \( r \). The minimum cardinality of a distance \( r \)-dominating set in \( G \) is the distance \( r \)-domination number of \( G \), denoted by \( \gamma_r(G) \).

The distance \( r \)-dominating number of graphs is a well studied notion in graph theory. For more about this invariant see, for example [7, 13, 18].

**Definition 2.** Let \( S_r \) be a collection of connected subgraphs of \( G \) of cardinality at most \( r \). The collection \( S_r \) is called a dominating \( r \)-collection if for each \( v \in V(G) \) there exists an element \( S \in S_r \) such that \( d(v, S) \) is at most 1. Given \( r \geq 1 \), the \( r \)-set domination number of \( G \), denoted by \( \omega_r(G) \) is the minimum \( m \) such that there exists dominating \( r \)-collection of cardinality \( m \), i.e.,

\[ \omega_r(G) := \min\{|S_r| : S_r \text{ is a dominating } r\text{-collection}\} \]

Clearly, \( \gamma_1(G) = \omega_1(G) = \Gamma(G) \) and \( \omega_r(G) \geq \gamma_r(G) \) for each \( r \geq 1 \). The following example shows that the gap between \( \omega_r(G) \) and \( \gamma_r(G) \) can be arbitrarily large for any \( r \geq 2 \).

**Example 1.** Let \( G \) be the graph shown in Figure 1. Since \( d(v_1, w) \) is at most 2 for all vertices in \( G \), \( \gamma_2(G) = 1 \). Further, it is easy to see that \( \omega_2(G) = 5 \). We can attach more paths of length 3 with vertex \( v_1 \) to increase the number \( w_2 \) by keeping \( \gamma_2 \) constant. A similar construction can be done for any \( r > 2 \).

![Figure 1](image)

The main results of this article are following.

**Theorem 2** (See Theorem 11). Let \( G \) be a graph and \( r \geq 1 \).
(iii) If $\gamma_r(G) > 2k$, then $\tilde{H}_j(\text{Ind}_r(G)) = 0$ for all $j \leq k + r - 2$.

(iii) If $\omega_r(G) > 2k$, then $\tilde{H}_j(\text{Ind}_r(G)) = 0$ for all $j \leq k - 1$.

**Theorem 3** (See Theorem 16). Let $G$ be a chordal graph and $r \geq 1$.

(i) $\text{Ind}_r(G)$ is either contractible or homotopy equivalent to wedge of spheres.

(ii) If $\omega_r(G) > k$, then $\tilde{H}_i(\text{Ind}_r(G)) = 0$ for each $i \leq rk - 1$.

(iii) If $\text{Ind}_r(G) \simeq \bigvee S^k$, then for each $i_k$ there exists a positive integer $s_k$ such that $i_k = rs_k - 1$.

**Theorem 4** (See Theorem 17). Let $r \geq 2$. Let $(d_1, \ldots, d_n)$ and $(k_1, \ldots, k_n)$ be two sequences of positive integers. There exists a chordal graph $G$ such that $\text{Ind}_r(G) \simeq \bigvee_{i=1}^n \bigvee d_i S^{k_i-1}$.

This article is organized as follows: In Section 2, we give basic definitions and results which are used in the remaining sections. Section 3 is dedicated to the proof of Theorem 2. Proof of Theorem 3 and Theorem 4 is given in Section 4.

## 2 Preliminaries

A **graph** is an ordered pair $G = (V(G), E(G))$ where $V(G)$ is called the set of vertices and $E(G) \subseteq V(G) \times V(G)$, the set of unordered edges of $G$. If $G$ is a graph on $n$ vertices, then we also say that $G$ is of cardinality $n$. The vertices $v_1, v_2 \in V(G)$ are said to be adjacent, if $(v_1, v_2) \in E(G)$. This is also denoted by $v_1 \sim v_2$. For a subset $U \subseteq V(G)$, the **induced subgraph** $G[U]$ is the subgraph whose set of vertices $V(G[U]) = U$ and the set of edges $E(G[U]) = \{(a, b) \in E(G) \mid a, b \in U\}$. We also denote the graph $G[V(G) \setminus U]$ by $G - U$. For a graph $G$ and $S \subseteq V(G)$, let $N(S) := \{v \in V(G) : v \sim s \text{ for some } s \in S\}$ and $N[S] = N(S) \cup S$.

For two distinct vertices $u$ and $v$, the **distance** $d(u, v)$ between $u$ and $v$ is the length of a shortest path between $u$ and $v$. Here, the length of a path is the number of edges in that path. If $X$ and $Y$ are two disjoint subsets of $V(G)$, then the distance between $X$ and $Y$ is defined as $d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$.

A subset $S \subseteq V(G)$ is called a **dominating set** if for each vertex $v \in V(G) \setminus S$, there exists a $s \in S$ such that $v \sim s$. The **domination number** $\Gamma(G)$ is the minimum cardinality of a dominating set. Set $S$ is called a **strong dominating set** if each vertex $v \in V(G)$ is adjacent to some vertex of $S$. The **strong domination number** $\Gamma_0(G)$ is the minimum cardinality of a strong dominating set.

**Definition 3.** For $r \geq 0$, $v \in V(G)$, $S \subseteq V(G)$ is called an $r$-**support** of $v$ in $G$, if $v \notin S$, $|S| = r$ and $G[S \cup \{v\}]$ is a connected graph. Let $\text{Supp}_r(v, G)$ denote the collection of all $r$-supports in $G$, i.e.,

$$\text{Supp}_r(v, G) = \{S : S \text{ is an } r\text{-support of } v \text{ in } G\}.$$  

Clearly, $\text{Supp}_0(v, G) = \{\emptyset\}$. We say that $\text{Supp}_r(v, G)$ is **connected**, if $G[S]$ is connected for all $S \in \text{Supp}_r(v, G)$. Whenever the underlying graph is clear, we simply denote it by $\text{Supp}_r(v)$.

A **finite abstract simplicial complex** $K$ is a collection of finite sets such that if $\tau \in K$ and $\sigma \subseteq \tau$, then $\sigma \in K$. The elements of $K$ are called **simplices** of $K$. The dimension of a simplex $\sigma$ is equal to $|\sigma| - 1$, here $|\cdot|$ denote the cardinality. The dimension of an abstract simplicial complex is the
maximum of the dimensions of its simplices. The 0-dimensional simplices are called vertices of $K$. If $\sigma \subseteq \tau$, we say that $\sigma$ is a face of $\tau$. If a simplex has dimension $k$, it is said to be $k$-dimensional or $k$-simplex. The boundary of a $k$-simplex $\sigma$ is the simplicial complex, consisting of all faces of $\sigma$ of dimension $\leq k - 1$ and it is denoted by $\text{Bd}(\sigma)$. The star of a simplex $\sigma \in K$ is the subcomplex of $K$ defined as

$$\text{st}_K(\sigma) := \{ \tau \in K \mid \sigma \cup \tau \in K \}.$$  

Whenever the underlying space $K$ is clear, we write $\text{st}(\sigma)$ to denote $\text{st}_K(\sigma)$.

**Definition 4.** Let $K_1$ and $K_2$ be two simplicial complexes whose vertices are indexed by disjoint sets. The join of $K_1$ and $K_2$ is the simplicial complex $K_1 \ast K_2$, whose simplices are those subset $\sigma \subseteq V(K_1) \cup V(K_2)$ such that $\sigma \cap V(K_1) \subseteq K_1$ and $\sigma \cap V(K_2) \subseteq K_2$.

In this article we consider a simplicial complex as a topological space, namely its geometric realization. For definition of geometric realization and details about simplicial complexes, we refer to the book [15] by Kozlov.

A topological space $X$ is said to be $k$-connected if the homotopy groups $\pi_m(X)$ are trivial for each $m \in \{0, 1, \ldots, k\}$.

For a space $X$, let $\Sigma^r(X)$ denote its $r$-fold suspension, where $r \geq 1$ is a natural number. Recall that, there is a homotopy equivalence

$$\Sigma^{r-1} \ast X \simeq \Sigma^r(X).$$  

(1)

If $X$ is empty, then $\Sigma(X) \simeq \mathbb{S}^0$. The following results will be used repeatedly in this article.

**Lemma 5.** [4. Lemma 10.4 (ii)] Let $K = K_0 \cup K_1 \cup \ldots \cup K_n$ be a simplicial complex with subcomplexes $K_i$ and assume that $K_i \cap K_j \subseteq K_0$ for all $1 \leq i < j \leq n$. If $K_i$ is contractible for all $0 \leq i \leq n$, then

$$K \simeq \bigvee_{i=1}^n \Sigma(K_i \cap K_0).$$

The nerve of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $N = N(\{A_i\})$ defined on the vertex set $I$ so that a finite subset $\sigma \subseteq I$ is in $N$ precisely when $\bigcap_{i \in \sigma} A_i \neq \emptyset$.

**Theorem 6.** [4. Theorem 10.6(ii)] Let $K$ be a simplicial complex and $(K_i)_{i \in I}$ be a family of subcomplexes such that $K = \bigcup_{i \in I} K_i$. Suppose every nonempty finite intersection $K_{i_1} \cap \ldots \cap K_{i_t}$ is $(k - t + 1)$-connected. Then $K$ is $k$-connected if and only if $N(\{K_i\})$ is $k$-connected.

### 3 Proof of Theorem 2

Throughout this section, we fix the graph $G$ and an edge $e = \{u, v\} \subseteq E(G)$. Let $G - e$ denote the graph obtained from $G$ by removing $e$, i.e., $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. For a set $A$, let $\Delta$ denote the simplex on vertex set $A$. For $0 \leq i, j \leq r - 1$, define

$$\Sigma_{i,j}^{u,v} := \{(S, T) : S \in \text{Supp}_i(u, G), T \in \text{Supp}_j(v, G) \text{ and } (G - e)[S \cup T \cup \{u, v\}] \text{ is not connected}\}$$

and

$$\Delta_{u,v} := \bigcup_{i,j \leq r-1} \bigcup_{(S,T) \in \Sigma_{i,j}^{u,v}} \Sigma(S \cup T) \ast \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).$$
Observe that, for \((S, T) \in S_{i,j}^{u,v}\) we have that \((G - e)[S \cup T \cup \{u, v\}] = G[S \cup u] \cup G[T \cup \{v\}]\) which implies \(S \cap T = \emptyset\) and \(G[S \cup T] = G[S] \cup G[T]\).

Lemma 7. For all \(r \geq 1\), we have

(i) \(\text{Ind}_r(G - e) = \text{Ind}_r(G) \cup (\bar{e} \ast \Delta_{u,v}).\)

(ii) \(\text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}) = \bigcup_{i+j \leq r-2}(S,T) \in S_{i,j}^{u,v} \bar{e} \ast (S \cup T) \ast \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]) \cup \bigcup_{i,j \leq r-1}(S,T) \in S_{i,j}^{u,v} \text{Bd}(\bar{e}) \ast (S \cup T) \ast \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).\)

Proof. (i) Clearly, \(\text{Ind}_r(G) \subseteq \text{Ind}_r(G - e).\) Since, \(|S \cup \{u\}| \leq r, |T \cup \{v\}| \leq r\) and \(S \cup T \cup \{u, v\}\) is not connected, we conclude that \(\bar{e} \ast \Delta_{u,v} \subseteq \text{Ind}_r(G - e).\) So, \(\text{Ind}_r(G) \cup (\bar{e} \ast \Delta_{u,v}) \subseteq \text{Ind}_r(G - e).\)

Now, let \(\sigma \in \text{Ind}_r(G - e).\) If \(u \notin \sigma\) or \(v \notin \sigma\), then clearly \(\sigma \in \text{Ind}_r(G).\) So, assume that \(\{u, v\} \subseteq \sigma.\) Let \(C_u\) and \(C_v\) be the connected components of \((G - e)[\sigma]\) containing \(u\) and \(v\) respectively. Observe that either \(C_u = C_v\) or \((V(C_u) \cap V(C_v)) = \emptyset).\) If \(C_u = C_v\), then since \(|V(C_u)| = |V(C_v)| \leq r\), we conclude that \(\sigma \in \text{Ind}_r(G).\) If \(C_u \neq C_v\), then take \(S = V(C_u) \setminus \{u\}\) and \(T = V(C_v) \setminus \{v\}.\) Clearly, \(\sigma \setminus (V(C_u) \cup C_v)) \cap N[V(C_u \cup C_v)] = \emptyset\), which implies that \(\sigma = V(C_u) \cup C_v) \cup \tau = \{u, v\} \cup (S \cup T) \cup \tau\) for some \(\tau \in \text{Ind}_r(G - N[V(C_u \cup C_v)] = \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).\)

(ii) For simplicity of notation, let

\[
Z_1 = \bigcup_{i+j \leq r-2}(S,T) \in S_{i,j}^{u,v} \bar{e} \ast (S \cup T) \ast \text{Ind}_r(G - N[S \cup T \cup \{u, v\}] \text{ and}

Z_2 = \bigcup_{i,j \leq r-1}(S,T) \in S_{i,j}^{u,v} \text{Bd}(\bar{e}) \ast (S \cup T) \ast \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).
\]

Since \(S \cap T = \emptyset, i + j \leq r - 2\) implies that \(|S \cup T| \leq r - 2\). Hence, \(\bar{e} \ast (S \cup T) \in \text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}).\) Therefore, \(Z_1 \subseteq \text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}).\) Now, let \(i, j \leq r - 1\) and \((S,T) \in S_{i,j}^{u,v}.\) Since \(G[S \cup T \cup \{u\}] = G[S \cup \{u\}] \cup G[T]\) and \(G[S \cup T \cup \{v\}] = G[S] \cup G[T] \cup \{v\}\), we get that \(\{u\} \cup S \cup T \in \text{Ind}_r(G).\) Therefore, we conclude that \(Z_2 \subseteq \text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}).\)

To show the other way inclusion, let \(\sigma \in \text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}).\) Then exist \(i, j \leq r - 1\) and \((S,T) \in S_{i,j}^{u,v}\) such that \(\sigma \setminus \{u, v\} \in \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).\)

Case 1. \(\{u, v\} \not\subseteq \sigma.\)

Write \(\sigma = \{u, v\} \cup \tau \cup \gamma,\) where \(\tau = \sigma \cap (S \cup T)\) and \(\gamma = \sigma \setminus (\{u, v\} \cup \tau) \in \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).\) Let \(C\) be the connected component of \(G[\sigma]\) containing \(u\) and \(v\) (hence \(v \in C).\) Observe that, \(C \setminus \{u, v\} \subseteq \tau.\) Now, let \(S_1 = V(C) \cap S\) and \(T_1 = V(C) \cap T.\) Clearly, \((S_1, T_1) \in S_{i,j}^{u,v}\) for some \(i_1 \leq i, j_1 \leq j.\) Since \(V(C) \subseteq \sigma \in \text{Ind}_r(G),\) \(|V(C)| \leq r\) and therefore \(i_1 + j_1 \leq r - 2.\) Further, since \(C\) is a component, \(\sigma \setminus V(C) = \emptyset\) and therefore \(\sigma \setminus V(C) \in \text{Ind}_r(G - N[V(C)]) = \text{Ind}_r(G - N[S_1 \cup T_1 \cup \{u, v\}]).\) Thus, we conclude that \(\sigma \in Z_1.\)

Case 2. \(\{u, v\} \not\subseteq \sigma.\)

Since \(\sigma \in \bar{e} \ast \Delta_{u,v}\) and \(\{u, v\} \not\subseteq \sigma,\) we get that \(\sigma \in \text{Bd}(\bar{e}) \ast (S \cup T) \ast \text{Ind}_r(G - N[S \cup T \cup \{u, v\}])\) and therefore \(\sigma \in Z_2.\)
Proposition 8. Let $t$ be a positive integer and for each $1 \leq l \leq t$, let $S_l$ and $T_l$ be supports of $u$ and $v$ respectively. For each $1 \leq l \leq t$, there exists $i_l, j_l \leq r - 1$, such that $(S_{i_l}, T_{j_l}) \in S_{i,j}^u v$. Let $L = \left( \bigcup_{l=1}^{t} (S_l \cup T_l) \right) \setminus \left( \bigcup_{l=1}^{t} (N[L_l] \setminus L_l) \right)$, where $L_l = S_l \cup T_l \cup \{u, v\}$. Then $\text{Bd}(\bar{e}) * \overline{\mathcal{Z}} \subseteq \text{Ind}_r(G)$.

Proof. Let $\sigma \in \text{Bd}(\bar{e}) * \overline{\mathcal{Z}}$. Observe that for any $l$ and $w \in N[L_l] \setminus L_l$, $w \notin \sigma$. Recall that, $(G - e)[L_l] = (G - e)[S_l \cup \{u\}] \cup (G - e)[T_l \cup \{v\}] = G[S_l \cup \{u\}] \cup G[T_l \cup \{v\}]$. Therefore, any connected component of $G[\sigma]$ must be a subset of either $S_l \cup \{u\}$ for some $1 \leq l \leq t$ or $T_j \cup \{v\}$ for some $1 \leq j \leq t$. Since $|S_l|, |T_j| \leq r - 1$, the result follows. 

For $0 \leq i, j \leq r - 1$ and for any $S \in \text{Supp}_i(u, G)$ and $T \in \text{Supp}_j(v, G)$, let

\[
W_{S,T} = \partial(\bar{e}) * S \cup T * \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]) \\
Y_{S,T} = \bar{e} * S \cup T * \text{Ind}_r(G - N[S \cup T \cup \{u, v\}]).
\]

Using Lemma 7(ii), we can write $\text{Ind}_r(G) \cap (\bar{e} \Delta u, v) = \bigcup_l X_l$, where each $X_l$ is of the form either $W_{S_l, T_l}$ or $Y_{S_l, T_l}$ for some $(S_l, T_l) \in S_{i,j}^u v$. We first understand the structure of arbitrary $t$-intersection of $X_l$'s, i.e., of $X_i \cap \ldots \cap X_t$. For each $1 \leq l \leq t$, there exists $(S_l, T_l)$ such that either $X_{i_l} = W_{S_l, T_l}$ or $X_{i_l} = Y_{S_l, T_l}$.

Lemma 9. For $1 \leq l \leq t$, let $L_l$ and $L$ be as in Proposition 8. Then $\bigcap_{l=1}^{t} X_{i_l}$ is either $\text{Bd}(\bar{e}) \ast \text{Ind}_r(G - N[\bigcup_{l=1}^{t} L_l]) \ast \overline{\mathcal{Z}}$ or a cone over $\bar{e}$.

Proof. If each $X_{i_l}$ is of the form $W_{S_l, T_l}$, then clearly since $X_i \cap \ldots \cap X_t$ is a cone over $\bar{e}$ and therefore it is contractible. So assume that there exists at least one $1 \leq k \leq t$ such that $X_{i_k}$ is of the form $Y_{S_k, T_k}$. In this case, we show the following,

\[
\bigcap_{l=1}^{t} X_{i_l} = \text{Bd}(\bar{e}) * \text{Ind}_r(G - N[\bigcup_{l=1}^{t} L_l]) * \overline{\mathcal{Z}}.
\]

Let $\sigma \in \bigcap_{l=1}^{t} X_{i_l}$. Define $\sigma = \tau_1 \sqcup \tau_2 \sqcup \tau_3$, where $\tau_1 = \sigma \cap \{u, v\}$, $\tau_2 = \sigma \cap V(G - N[\bigcup_{l=1}^{t} L_l])$ and $\tau_3 = \sigma \setminus (\tau_1 \sqcup \tau_2)$. To prove that $\sigma \in \text{Bd}(\bar{e}) * \text{Ind}_r(G - N[\bigcup_{l=1}^{t} L_l]) * \overline{\mathcal{Z}}$, it is enough to show that $\tau_3 \subseteq L$.

Let $L = \bigcup_{l=1}^{t} L_l$. Observe that $\tau_3 \subseteq N[L] \setminus \{u, v\}$. Suppose there exists $w \in \tau_3$ such that $w \in N[L] \setminus L$. Then there exist $x \in L$ such that $x \sim w$. Firts, if $x \in \{u, v\}$, then $w \in N[u, v] \setminus L$, which implies $w \in N[S_1 \cup T_1 \cup \{u, v\}] \setminus L \implies \{w\} \notin X_{i_1}$, a contradiction. Secondly, if $x \in \bigcup_{l=1}^{t} (S_l \cup T_l)$, then without loss of generality we assume that $x \in S_1 \cup T_1$. Here, $w \in N[L_1] \setminus L \implies \{w\} \notin X_{i_1}$, again a contradiction. Therefore $\tau_3 \subseteq L \setminus \{u, v\} = \bigcup_{l=1}^{t} (S_l \cup T_l)$. We now show that $\tau_3 \cap \bigcup_{l=1}^{t} (N[L_l] \setminus L_l) = \emptyset$.

Let $z \in \tau_3 \cap \bigcup_{l=1}^{t} N[L_l] \setminus L_l$. If $z \in N[L_l] \setminus L_l$, then $\{z\} \notin X_{i_l}$, which is a contradiction. Hence, $\tau_3 \subseteq L$.  

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To show the other way inclusion, let \( \sigma \in \text{Bd}(\bar{e}) \ast \text{Ind}_r(G - N[\bigcup_{l=1}^{t} L_l]) \ast \bar{\omega} \). From Proposition \( \text{S} \) \( \sigma \in \text{Ind}_r(G) \). Write \( \sigma = \sigma_1 \sqcup \sigma_2 \sqcup \sigma_3 \), where \( \sigma_1 = \sigma \cap \{u, v\} \), \( \sigma_2 = \sigma \cap V(G - N[\bigcup_{l=1}^{t} L_l]) \) and \( \sigma_3 \subseteq L \).

Again, write \( \sigma_3 = \sigma_3^1 \sqcup \sigma_3^2 \), where \( \sigma_3^1 = \sigma_3 \cap (S_1 \cup T_1) \) and \( \sigma_3^2 = \sigma_3 \setminus \sigma_3^1 \). Clearly, \( \sigma_3^2 \subseteq \bigcup_{l=2}^{t} (S_l \cup T_l) \) and \( \sigma_3^2 \cap N[L_1] = \emptyset \), which implies that \( \sigma_3^2 \in \text{Ind}_r(G - N[L_1]) \).

Since \( \sigma_3^2 \subseteq \bigcup_{l=2}^{t} (S_l \cup T_l) \cap N[L_1] \) and \( \sigma_2 \cap N[\bigcup_{l=1}^{t} L_l] = \emptyset \), we conclude that \( \sigma_3^2 \cup \sigma_2 \in \text{Ind}_r(G - N[L_1]) \).

Thus, \( \sigma \) can be written as follows: \( \sigma = \sigma_1 \sqcup \sigma_2 \sqcup \sigma_3^1 \sqcup \sigma_3^2 = \sigma_1 \sqcup \sigma_3^1 \sqcup (\sigma_2 \sqcup \sigma_3^2) \in X_{i_{\text{max}}} \). By similar arguments, we see that \( \sigma \in X_{i_l} \) for each \( 1 \leq l \leq t \).

\[ \square \]

**Proposition 10.** (The Hurewicz Theorem)

If a space \( X \) is \((n - 1)\) connected, \( n \geq 2 \), then \( \tilde{H}_i(X) = 0 \) for \( i < n \) and \( \pi_n(X) \cong H_n(X) \).

We are now ready to prove the main result of this section.

**Theorem 11.** Let \( G \) be a graph and \( r \geq 1 \).

(i) If \( \gamma_r(G) > 2k \), then \( \tilde{H}_j(\text{Ind}_r(G)) = 0 \) for all \( j \leq k + r - 2 \).

(ii) If \( \omega_r(G) > 2k \), then \( \tilde{H}_j(\text{Ind}_r(G)) = 0 \) for all \( j \leq k - 1 \).

**Proof.** If \( r = 1 \), then since \( \gamma_1(G) = \omega_1(G) = \Gamma_0(G) \), result follows from Theorem \( \text{II} \)(i). So, assume that \( r \geq 2 \). From Lemma \( \text{II}(i) \) and using Mayer–Vietoris sequence, we have

\[ \cdots \to \tilde{H}_i(\text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v})) \to \tilde{H}_i(\text{Ind}_r(G)) \oplus \tilde{H}_i(\bar{e} \ast \Delta_{u,v}) \to \tilde{H}_i(\text{Ind}_r(G - e)) \to \cdots \]

Since \( \bar{e} \ast \Delta_{u,v} \) is contractible, we get the following sequence.

\[ \cdots \to \tilde{H}_i(\text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v})) \to \tilde{H}_i(\text{Ind}_r(G)) \to \tilde{H}_i(\text{Ind}_r(G - e)) \to \cdots \] \hspace{1cm} (2)

The remaining proof is now by induction on \( k \) and the number of edges of \( G \). Using Lemma \( \text{II}(ii) \), we get that \( \text{Ind}_1(G) \cap (\bar{e} \ast \Delta_{u,v}) = \text{Bd}(\bar{e}) \ast \text{Ind}_r(G - N[\{u, v\}]) = \Sigma(\text{Ind}_r(G - N[\{u, v\}])) \)

(i) Since \( \text{Ind}_r(G) \) is always \((r - 2)\)-connected for any non-empty graph, base case of induction follows. So, assume that \( k \geq 1 \). Clearly, \( \gamma_r(G - e) \geq \gamma_r(G) \) and hence by induction \( \tilde{H}_{k+r-2}(\text{Ind}_r(G - e)) = 0 \).

**Claim 1.** \( \text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}) \) is \((k + r - 2)\)-connected.

**Proof of Claim 1.** Using Lemma \( \text{II}(ii) \), we can write \( \text{Ind}_r(G) \cap (\bar{e} \ast \Delta_{u,v}) = \bigcup X_i \), where each \( X_i \) is of the form either \( W_{S,T} \) or \( Y_{S,T} \) for some \( (S, T) \in S_{u,v}^{i,j} \). Consider the intersection \( X_{i_1} \cap \cdots \cap X_{i_l} \). For each \( 1 \leq l \leq t \), there exists \( (S_{i_l}, T_{i_l}) \) such that either \( X_{i_l} = W_{S_{i_l}, T_{i_l}} \) or
Again, the base case is straightforward. Since $\omega_r(G) > 2$, $G$ is non-empty, using Lemma 9 we conclude $\tilde{N}(\{X_i\})$ is a simplex and hence contractible. Therefore, result follows from Theorem 6.

Since $\tilde{N}(\{X_i\})$ is path-connected, the join of a path connected space with an non empty space is always simply connected, using Lemma 9 we conclude that $\bigcap_{l=1}^t X_i$ is simply connected. Hence, from equations Equation (3) and Lemma 9 and Proposition 10, we get that $\tilde{H}_j(\text{Ind}_r(G - e)) = 0$ for all $j \leq k + r - 2$.

(ii) Again, the base case is straightforward. Since $\omega_r(G - e) \geq \omega_r(G)$, by induction $\tilde{H}_j(\text{Ind}_r(G - e)) = 0$ for all $j \leq k - 1$.

**Claim 2.** $\text{Ind}_r(G) \cap (\tilde{e} \ast \Delta_{u,v})$ is $(k-1)$-connected.

**Proof of Claim 2.** The proof here is similar to that of Claim 1. Again, write $\text{Ind}_r(G) \cap (\tilde{e} \ast \Delta_{u,v}) = \bigcup_{l} X_l$ and consider the intersection $X_{i_1} \cap \ldots \cap X_{i_t}$, where each $X_i$ is either $W_{S_{u,v}}$ or $Y_{S_{u,v}}$. For each $1 \leq l \leq t$, let $L_{i_l} = S_{i_l} \cup T_{i_l} \cup \{u,v\}$. Observe that if $\mathcal{D}_r$ is a dominating $r$-collection (see Definition 2) for $G - N[\bigcup_{l} L_{i_l}]$, then $\mathcal{D}_r \cup \{S_{i_l} \cup \{u\}, T_{i_l} \cup \{v\} : 1 \leq l \leq t\}$ is a dominating $r$-collection for $G$. Hence, $\omega_r(G - N[\bigcup_{l} L_{i_l}]) \geq \omega_r(G - e) > 2(k - t)$. Hence, by induction

$$\tilde{H}_j(\text{Ind}_r(G - N[\bigcup_{l} L_{i_l}])) = 0$$

for all $j \leq k - t - 1$. (4)

If $G - N[\bigcup_{l} L_{i_l}]$ is non-empty graph, then $r \geq 2$ implies that $\text{Ind}_r(G - N[\bigcup_{l} L_{i_l}])$ is path-connected. Therefore, from Lemma 9 and Proposition 10, we get that $\bigcap_{l=1}^t X_{i_l}$ is $(k - t)$-connected. Hence, from Lemma 9 and Proposition 10, we get that $(\bigcap_{l=1}^t X_{i_l})$ is $(k - t)$-connected.
If $G - N[\bigcup_{i=1}^{t} L_{ij}]$ is empty then observe that $\omega_r(G) \leq 2t$ and therefore $k - t < 0$. Since $\text{Bd}(\bar{e}) \subseteq \bigcap_{l=1}^{t} X_{il}$, we see that $\bigcap_{l=1}^{t} X_{il} \neq \emptyset$. Thus, we conclude that $\bigcap_{l=1}^{t} X_{il}$ is $k - t \leq -1$ connected.

Since the nerve $N(\{X_i\})$ is contractible and arbitrary $t$ intersection $\bigcap_{l=1}^{t} X_{il}$ is $k - t$ connected, Theorem 6 implies that $\text{Ind}_r(G) \cap (\bar{e} * \Delta_{u,v})$ is $(k - 1)$-connected. □

Since $\tilde{H}_j(\text{Ind}_r(G - e)) = 0 \forall j \leq k - 1$, from Claim 2 and Equation 2, we get that $\tilde{H}_j(\text{Ind}_r(G)) = 0$ for all $j \leq k - 1$. □

4 Chordal graphs

In this section, we study $r$-independence complexes of chordal graphs. However, first we present a general result that relates the $r$-independence complex of a graph $G$ with $r$-independence complexes of its certain proper subgraphs (cf. Theorem 13).

Lemma 12. Let $G$ be a graph, $v \in V(G)$ and let $\text{Supp}_r(v, G) = \{S_1, \ldots, S_n\}$. Then

$$\text{Ind}_r(G) = \text{st}(v) \cup \bigcup_{i=1}^{n} \text{st}(S_i).$$

Proof. Since $\text{st}(v), \text{st}(S_i) \subseteq \text{Ind}_r(G)$ for all $i \in \{1, \ldots, n\}$, $\text{st}(v) \cup \bigcup_{i=1}^{n} \text{st}(S_i) \subseteq \text{Ind}_r(G)$. To show the other way inclusion, let $\sigma \in \text{Ind}_r(G)$. If $\sigma \notin \text{st}(v)$, then $G[\sigma \cup \{v\}]$ has a connected component with at least $r + 1$ vertices. Let $H$ be a such connected component of $G[\sigma \cup \{v\}]$. Since $\sigma \in \text{Ind}_r(G)$, $v \in V(H)$. Choose a subset $S \subset V(H) \setminus \{v\}$, such that $|S| = r$ and $G[S \cup \{v\}]$ is connected. Then $S \in \text{Supp}_r(v, G)$ and $\sigma \in \text{st}(S)$. □

Recall that for a vertex $v \in V(G)$, $\text{Supp}_r(v, G)$ is called connected if $G[S]$ is connected for all $S \in \text{Supp}_r(v, G)$.

Theorem 13. Let $G$ be a connected graph, $v \in V(G)$ and $\text{Supp}_r(v, G) = \{S_1, S_2, \ldots, S_n\}$. If $\text{Supp}_r(v, G)$ is connected and $N(v) \subseteq N[S_i]$ for each $i \in \{1, \ldots, n\}$, then

$$\text{Ind}_r(G) \simeq \bigvee_{i=1}^{n} \Sigma^r(\text{Ind}_r(G - N[S_i])).$$

Proof. From Lemma 12 $\text{Ind}_r(G) = \text{st}(v) \cup \bigcup_{i=1}^{n} \text{st}(S_i)$.

Claim 3. For all $1 \leq i < j \leq n$, $\text{st}(S_i) \cap \text{st}(S_j) \subseteq \text{st}(v)$.

Proof of Claim 3. Fix $i \neq j$ and let $\sigma \in \text{st}(S_i) \cap \text{st}(S_j)$. Clearly, $v \notin \sigma$. Since $\text{Supp}_r(v, G)$ is connected, we see that $\sigma \cap (N[S_i] \setminus S_i) = \emptyset$ and $\sigma \cap (N[S_j] \setminus S_j) = \emptyset$. Hence, $N(v) \subseteq N[S_i] \cap N[S_j]$ implies that $\sigma \cap N(v) \subseteq S_i \cap S_j$. Suppose there exists a subset $\tau \in \text{st}(v)$ such that $\tau = \sigma \cap N(v)$ and $H$ be a connected
component of $G[\sigma \cup \{v\}]$ of cardinality at least $r + 1$. Clearly, $(V(H) \setminus \{v\}) \nsubseteq (S_i \cap S_j)$ (since $|S_i \cap S_j| < r$). Let $w \in (V(H) \setminus \{v\}) \setminus (S_i \cap S_j)$ and $P$ be a path of minimal length from $w$ to $v$ in $H$. Without loss of generality, we can assume that $w \notin S_i$. Note that, $G[S_i]$ and $G[(V(P) \setminus \{v\})]$ are connected subgraphs. Further, $N(v) \subseteq N[S_i]$ implies that $N[S_i] \cap (V(P) \setminus \{v\}) \neq \emptyset$. Therefore, $G[S_i \cup (V(P) \setminus \{v\})]$ is connected subgraph of cardinality more than $r$. Hence, $V(P) \setminus \{v\} \notin st(S_i)$ implying that $\sigma \notin st(S_i)$. Which is a contradiction to our assumption that $\sigma \in st(S_i) \cap st(S_j)$. $\square$

For each $1 \leq i \leq n$, let $\Delta^{S_i}$ be the simplex on vertex set $S_i$.

**Claim 4.** For each $i \in \{1, \ldots, n\}$, $st(S_i) \cap st(v) = Ind_r(G - N[S_i]) \ast Bd(\Delta^{S_i})$.

**Proof of Claim 4.** Let $\tau = \sigma \cup \delta$, where $\sigma \in Ind_r(G - N[S_i])$ and $\delta \subseteq S_i$. Since $\sigma \cap N[S_i] = \emptyset$ and $N(v) \subseteq N[S_i]$, we get that $\tau \in st(S_i)$ and $v \notin N(\sigma)$. Therefore, $N(\tau) \cap (\{v\} \cup \delta) = \emptyset$ which implies that $\tau \in st(v)$. Hence, $Ind_r(G - N[S_i]) \ast Bd(\Delta^{S_i}) \subseteq st(S_i) \cap st(v)$.

To show the other way inclusion, let $\sigma \in st(S_i) \cap st(v)$. Since $\sigma \in st(v)$ and $S_i$ is an $r$-support, we see that $S_i \nsubseteq \sigma$. Further, $\sigma \in st(S_i)$ implies that $\sigma \cap (N[S_i] \setminus S_i) = \emptyset$. Let $\tau = \sigma \cap (V(G) \setminus N[S_i])$ and $\delta = \sigma \cap S_i$. Then $\sigma = \tau \cup \delta$ and therefore $\sigma \in Ind_r(G - N[S_i]) \ast Bd(\Delta^{S_i})$. $\square$

Observe that $Ind_r(G - N[S_i]) \ast Bd(\Delta^{S_i}) \simeq \Sigma^{-1}(Ind_r(G - N[S_i]))$ and $st(v)$, $st(S_i)$ is contractible. Therefore, Theorem 13 follows from Claim 4, Claim 4, Lemma 5 and Lemma 12. $\square$

For $n \geq 3$, a wheel graph, denoted $W_n$, is a graph constructed from the cycle graph $C_n$ by adding a new vertex $w$ and an edge $(w, v)$ for each vertex $v \in V(C_n)$. As an immediate consequence of Theorem 13 we get the following results.

**Corollary 14.** For $n \geq 3$, $Ind_{n-1}(W_n) \simeq \bigvee_n S^{n-2}$.

**Proof.** It is easy to observe that $Supp_{n-1}(w, W_n)$ is connected, $|Supp_{n-1}(w, W_n)| = n$ and $W_n - N[S] = \emptyset$ for each $S \in Supp_{n-1}(w, W_n)$. Therefore, $Ind_{n-1}(W_n) \simeq \bigvee_n S^{n-1}(\emptyset) = \bigvee_n S^{n-2}$. $\square$

For a vertex $v$ of graph $G$, let $N_2(v)$ denotes the set of vertices of $G$ whose distance from $v$ is exactly 2, i.e., $N_2(v) = \{w \in V(G) : d(v, w) = 2\}$.

**Corollary 15.** Let $v \in V(G)$ such that $N(v) \times N_2(v) \subseteq E(G)$. If $r > |N(v)|$, then $Supp_r(v)$ is connected. In particular, $H_i(Ind_r(G)) = 0$ for all $i < r - 1$ and $H_j(Ind_r(G))$ is torsion-free for $j = r - 1, r$.

A vertex $v$ of a graph $G$ is called simplicial, if the induced subgraph $G[N(v)]$ is a complete graph. It is a classical result of Dirac [3] that every chordal graph has a simplicial vertex. For a simplicial vertex $v$ and $S \in Supp_r(v, G)$, since each connected component of $G[S]$ has a vertex from $N(v)$, we get the following.

**Remark 1.** If $v \in V(G)$ is a simplicial vertex, then $Supp_r(v, G)$ is connected and $N(v) \subseteq N[S]$ for each $S \in Supp_r(v, G)$.

It is easy to see that, $Supp_1(v, G)$ is always connected and $Supp_2(v, G)$ is connected if and only if $v$ is a simplicial vertex. When $r > 2$, we can construct examples satisfying the condition given in Corollary 15 such that $v$ is not simplicial yet $Supp_r(v, G)$ is connected. Thus, the condition being $v$ simplicial is sufficient but not necessary for $Supp_r(v, G)$ to be connected.

We now compute the homotopy type of $r$-independence complexes of chordal graphs.
Theorem 16. Let $G$ be a chordal graph and $r \geq 1$.

(i) $\text{Ind}_r(G)$ is either contractible or homotopy equivalent to wedge of spheres.

(ii) If $\omega_r(G) > k$, then $\bar{H}_i(\text{Ind}_r(G)) = 0$ for each $i \leq rk - 1$.

(iii) If $\text{Ind}_r(G) \simeq \bigvee S^k$, then for each $i_k$ there exists a positive integer $s_k$ such that $i_k = rs_k - 1$.

Proof. Let $v$ be a simplicial vertex of $G$ and let $\text{Supp}_r(v,G) = \{S_1, \ldots, S_n\}$.

(i) From Remark 1 and Theorem 13, we have

$$\text{Ind}_r(G) \simeq \bigvee_{i=1}^n \Sigma^r(\text{Ind}_r(G - N[S_i])).$$

Hence, the proof follows from induction on number of vertices of graph and the fact that induced subgraph of a chordal graph is also chordal.

(ii) Remark 1 implies that $G[S_i]$ is connected for each $i \in \{1, \ldots, n\}$ and using Theorem 13, we get $\text{Ind}_r(G) \simeq \bigvee_{i=1}^n \Sigma^r(\text{Ind}_r(G - N[S_i])).$ If each $\text{Ind}_r(G - N[S_i])$ is contractible, then so is $\text{Ind}_r(G)$. Suppose there exists $i$ such that $\text{Ind}_r(G - N[S_i])$ is not contractible. Let $\text{Ind}_r(G - N[S_i]) \simeq \bigvee S^{i_t}$. Then $\Sigma^r(\text{Ind}_r(G - N[S_i])) \simeq \bigvee S^{i_t+r}$. For each dominating $r$-collection $D_r$ of $G - N[S_i]$, the set $D_r \cup \{S_i\}$ is also a dominating $r$-collection of $G$. This implies $\omega_r(G) \leq \omega_r(G - N[S_i]) + 1$. By induction, we have $i_t \geq r\omega_r(G - N[S_i]) - 1$. Hence, $i_t + r \geq r(\omega_r(G) - 1) - 1 + r = r\omega_r(G) - 1$ for each $i_t$.

(iii) If $|V(G)| \leq r$, then $\text{Ind}_r(G)$ is contractible and result follows trivially. Now, let $|V(G)| \geq r + 1$ and $v$ be a simplicial vertex of $G$. If $\text{Supp}_r(v,G) = \emptyset$, then the connected component $H_v$ of $G$ containing $v$ is of cardinality at most $r$. In this case $\text{Ind}_r(G) \simeq \text{Ind}_r(G - H_v) \ast \text{Ind}_r(H_v)$. Since, $\text{Ind}_r(H_v)$ is contractible, $\text{Ind}_r(G)$ is also contractible. So, assume that $\text{Supp}_r(v,G) \neq \emptyset$. Using Theorem 13, we get

$$\text{Ind}_r(G) \simeq \bigvee_{S \in \text{Supp}_r(v,G)} \Sigma^r(\text{Ind}_r(G - N[S])).$$

By induction, if $\text{Ind}_r(G - N[S]) \simeq \bigvee S^{i_t}$ then for each $i_t$ there exists an $s_t \in \mathbb{N}$ such that $i_t = rs_t - 1$. Hence, we get that $i_t + r = r(s_t + 1) - 1$.

The following result is the converse part of Theorem 16 (iii).

Theorem 17. Let $r \geq 2$. Let $(d_1, \ldots, d_n)$ and $(k_1, \ldots, k_n)$ be two sequences of positive integers. There exists a chordal graph $G$ such that $\text{Ind}_r(G) \simeq \bigvee_{i=1}^n \bigvee_{j=1}^{d_i} S^{r k_i - 1}$.

Before proving Theorem 17, we illustrate the construction of desired graph by an example. Fix $r = 2$, $(d_1, d_2) = (1, 2)$ and $(k_1, k_2) = (1, 2)$. For $n \geq 1$, let $P_n$ denote the path graph on $n$ vertices. The $r$-independence complexes of path graphs have been computed by Paolini and Salvetti. They proved the following.
Proposition 18 ([17] Proposition 3.7). For \( r \geq 1 \), we have

\[
\text{Ind}_r(P_n) \cong \begin{cases} S^{r-1}, & \text{if } n = (r+2)k \text{ or } n = (r+2)k - 1; \\ \{\text{point}\}, & \text{otherwise}. \end{cases}
\]

Let \( G \) be the graph given in Figure 2. Clearly, \( G \) is a chordal graph and \( v_1 \) is a simplicial vertex. Here, \( \text{Supp}_2(v, G) = \{\{v_2, a_1\}, \{v_2, b_1\}, \{v_2, b_2\}\} \) and \( G - N[\{v_2, a_1\}] = \emptyset \), \( G - N[\{v_2, b_1\}] \cong P_4 \cong G - N[\{v_2, b_2\}] \). Thus, using Theorem 13 and Proposition 18, we get the following.

\[
\text{Ind}_2(G) \cong \Sigma^2(\text{Ind}_2(G - N[\{v_2, a_1\}])) \lor \Sigma^2(\text{Ind}_2(G - N[\{v_2, b_1\}])) \lor \\
\Sigma^2(\text{Ind}_2(G - N[\{v_2, b_2\}])) \\
\cong \Sigma^2(\emptyset) \lor \Sigma^2(\text{Ind}_2(P_3)) \lor \Sigma^2(\text{Ind}_2(P_3)) \\
\cong S^1 \lor S^3 \lor S^3.
\]

**Figure 2**

**Proof of Theorem** [17] Let \( P_r \) be a path graph on vertex set \( \{v_1, v_2, \ldots, v_r\} \). For each \( i \in \{1, \ldots, n\} \), let \( W_i \) be a set of cardinality \( d_i \). Let \( W = \bigcup_{i=1}^{n} W_i \) and \( K_{W+1} \) be the complete graph on vertex set \( W \cup \{v_r\} \). For each \( x \in W_i \), let \( P_x^i \) be a path graph on \( (r+2)(k_i - 1) + 1 \) vertices with \( x \) as an end vertex. Here, \( V(P_x^i \cap (P_r \cup K_{W+1})) = \{x\} \) for each \( x \in W \) and \( V(P_x^i \cap P_y^j) = \emptyset \) whenever \( x \neq y \). Let

\[
G = P_r \cup K_{W+1} \bigcup_{i=1}^{n} (\bigcup_{x \in W_i} P_x^i).
\]

Let \( \tilde{G} \) be the graph with vertex set \( V(G) \) and \( E(\tilde{G}) = E(G) \cup \bigcup_{1 \leq i, j \leq n} \{(a, b) : a \in W_j, b \in P_x^i, \text{where either } i \neq j \text{ or } b \neq x\} \). Clearly, \( v_1 \) is a simplicial vertex in \( \tilde{G} \) as \( N_{\tilde{G}}(v_1) = \{v_2\} \) and \( \text{Supp}_r(v_1, \tilde{G}) = \{\{v_2, \ldots, v_r, x\} : x \in W\} \). For each \( x \in W \), let \( S_x = \{v_1, \ldots, v_r, x\} \). Then by
Theorem \[ \text{Ind}_r(\tilde{G}) \simeq \bigvee_{x \in W} \Sigma^r \left( \text{Ind}_r(\tilde{G} - N[S_x]) \right). \] Observe that for each \( x \in W_i \), \( \tilde{G} - N[S_x] \) is isomorphic to a path on \((r + 2)(k_i - 1) - 1\) vertices and therefore \( \text{Ind}_r(\tilde{G} - N[S_x]) \simeq S^{r(k_i - 1) - 1} \) by Proposition 18. Hence, \( \text{Ind}_r(\tilde{G}) \simeq \bigvee_{1 \leq i \leq n} \Sigma^r \left( S^{r(k_i - 1) - 1} \right) = \bigvee_{1 \leq i \leq n} S^{rk_i - 1} = \bigvee_{i=1}^n \bigvee_{d_i} S^{rk_i - 1}. \) □

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