THE CHARACTERISTIC 2 ANISOTROPICITY OF SIMPLICIAL SPHERES

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Abstract. Assume $D$ is a simplicial sphere, and $k_1$ is a field. We say that $D$ is generically anisotropic over $k_1$ if, for a certain purely transcendental field extension $k$ of $k_1$, a certain Artinian reduction $A$ of the Stanley-Reisner ring $k[D]$ has the following property: All nonzero homogeneous elements $u \in A$ of degree less or equal to $(\dim D + 1)/2$ have nonzero square. We prove, using suitable differential operators, that, if the field $k_1$ has characteristic 2, then every simplicial sphere $D$ is generically anisotropic over $k_1$. As an application, we give a second proof of a recent result of Adiprasito, known as McMullen’s $g$-conjecture for simplicial spheres. We also prove that the simplicial spheres of dimension 1 are generically anisotropic over any field $k_1$.

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1. INTRODUCTION

The motivation for the present work was McMullen’s $g$-conjecture for simplicial spheres [4, 15, 18]. Recently, a proof of the conjecture, due to Adiprasito, has appeared [1, 2].
Our approach is based on the well-known result that to prove the g-conjecture for a simplicial sphere $D$ it is enough to find a field $k$ such that the Stanley-Reisner ring (also known as face ring) $k[D]$ has the Weak Lefschetz Property.

Instead of working directly with the Weak Lefschetz Property, we introduce in Definition 3.2 the notion of a simplicial sphere $D$ being generically anisotropic over a field $k_1$. This means that for a certain purely transcendental field extension $k$ of $k_1$, a certain Artinian reduction $A$ of the Stanley-Reisner ring $k[D]$ has the following property: All nonzero homogeneous elements $u \in A$ of degree less or equal to $(\dim D + 1)/2$ have nonzero square. We remark that $A$ is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 2.2.

Using some ideas and results of Swartz [17], we show in Theorem 9.1 that if the suspension $S(D)$ of $D$ is generically anisotropic over $k_1$, then the Stanley-Reisner ring $k[D]$ has the Weak Lefschetz Property.

We establish two results related to generic anisotropicity. In Theorem 10.7, we prove that a simplicial sphere of dimension 1 is generically anisotropic over any (finite or infinite) field $k_1$. In Theorem 3.3, we prove that over any (finite or infinite) field of characteristic 2, every simplicial sphere is generically anisotropic. The question of generic anisotropicity of simplicial spheres of dimension $\geq 2$ over a field of characteristic not equal to 2 remains open.

For both theorems, an important step is to understand (part of) the multiplicative structure of $A$ in terms of rational functions on a certain transcendence basis of the field extension $k_1 \subset k$. The key results here are Proposition 10.4, which works in all characteristics but only for simplicial spheres of dimension 1, and Theorem 4.14 which is valid in any dimension but only in characteristic 2. An interesting open question is to establish a version of Theorem 4.14 in all characteristics.

Using Proposition 10.4, we establish generic anisotropicity in all characteristics for simplicial spheres of dimension 1 by an initial terms argument, see Proposition 10.10.

We now describe the way we use Theorem 4.14 to prove generic anisotropicity in characteristic 2 and all dimensions. We introduce, in Section 5, $(\dim D + 1)$-th order differential operators $\partial_\sigma$ and $\partial_{p,\sigma}$, associated to certain faces $\sigma, \sigma \cup \{p\}$ of $D$. In Sections 6, 7 and 8, we study the differential operators in some detail, and prove Theorem 8.6 which states identities related to the differentiation of the product of the maximal minors of certain matrices. The theorem is used to prove the key Propositions 5.1 and 5.7. The propositions imply Corollaries 5.6 and 5.13, and the corollaries imply Theorem 3.3. We remark that even though the differential operators can be defined in any characteristic, most of their properties we need are true only in characteristic 2.

Finally, combining Theorem 3.3 with Theorem 9.1 we get a second proof of McMullen’s g-conjecture for simplicial spheres in Theorem 9.2.

One key question is where and how one gets the positivity or, maybe better, nondegeneracy needed for anisotropicity and the Lefschetz Properties. In the present work, we get it from the well-known nondegenerate pairings, valid in any Gorenstein Artinian graded algebra, described in Remark 2.1. The actual places where we use the nondegeneracy of the pairings to achieve anisotropicity are in the proofs of Corollaries 5.6 and 5.13.

It is well-known that anisotropicity is not well-behaved under field extensions. For example, if $m \geq 2$, then any $m$-dimensional positive definite symmetric bilinear form over the real numbers is anisotropic, but, if we make a field extension to the complex numbers, anisotropicity is lost. In addition, over an algebraically closed field any symmetric bilinear form on a vector space of dimension $\geq 2$ is not anisotropic. This (partially) explains the need to introduce the field extension $k$ of $k_1$ and the Artinian $k$-algebra $A$. As Section 10 suggests, this way we enter into a setting with an arithmetic over a function field flavour.

Since $A$ is a graded Artinian Gorenstein algebra canonically associated with the pair $(D, k_1)$, it is, perhaps, not surprising that it contains useful information. On the other hand, we found really surprising that the peculiarities of characteristic 2, such as the property that partial derivatives are linear over the subfield $(k_1)^2$ of $k_1$ (see Remark 8.1), play a key role in our arguments. Maybe this hints to a closer connection between the combinatorial
properties of a simplicial complex $D$ and the properties of the generic Artinian reduction of the Stanley-Reisner ring of $D$ over the finite field with two element $\mathbb{Z}/(2)$.

The file [http://users.uio.no/rippel/transfer/2002code1.txt](http://users.uio.no/rippel/transfer/2002code1.txt) contains Macaulay2 [8] code related to the present paper.

2. Notation

Assume $k$ is a field of arbitrary characteristic. All graded $k$-algebras will be commutative, Noetherian and of the form $G = \oplus_{i \geq 0} G_i$ with $G_0 = k$ and $\dim_k G_i < \infty$ for all $i$. The $k$-algebra $G$ is called standard graded if it is generated, as a $k$-algebra, by $G_1$. We denote by $\dim G$ the Krull dimension of $G$.

Assume $F$ is an Artinian graded $k$-algebra. There exists a largest integer $d$ such that the $d$-th graded part $F_d$ is nonzero, and we call $d$ the socle degree of $F$. An element $\omega \in F_1$ is called a Weak Lefschetz element if, for all $\omega \in F_1$, the multiplication by $\omega$ map $F_i \to F_{i+1}$ is of maximal rank, which means that it is injective or surjective (or both). We say that $F$ has the Weak Lefschetz Property if there exists a Weak Lefschetz element $\omega \in F_1$.

Assume $F$ is an Artinian Gorenstein graded $k$-algebra of socle degree $d$. An element $\omega \in F_1$ is called a Strong Lefschetz element if, for all $i$ with $0 \leq 2i \leq d$, the multiplication by $\omega^{d-2i}$ map $F_i \to F_{d-i}$ is bijective. We say that $F$ has the Strong Lefschetz Property if there exists a Strong Lefschetz element $\omega \in F_1$.

We say that a standard graded $k$-algebra $G$ with positive Krull dimension has the Weak Lefschetz Property if it is Cohen-Macaulay, the field $k$ is infinite and for Zariski general homogeneous degree 1 elements $f_1, \ldots, f_{\dim G}$ of $G$ the Artinian $k$-algebra $G/(f_1, \ldots, f_{\dim G})$ has the Weak Lefschetz Property.

We say that a standard graded $k$-algebra $G$ with positive Krull dimension has the Strong Lefschetz Property if it is Gorenstein, the field $k$ is infinite and for Zariski general homogeneous degree 1 elements $f_1, \ldots, f_{\dim G}$ of $G$ the Artinian $k$-algebra $G/(f_1, \ldots, f_{\dim G})$ has the Strong Lefschetz Property. Good general references for the Weak and Strong Lefschetz Properties are [9, 13].

Remark 2.1. We will use the following well-known fact, see for example [9] Theorem 2.79. Assume $F = \oplus_{i=0}^d F_i$ with $F_d \neq 0$ is a standard graded Gorenstein Artinian $k$-algebra. Then $F_d$ is 1-dimensional, and, for all $i$ with $0 \leq i \leq d$, the multiplication map $F_i \times F_{d-i} \to F_d \cong k$ is a perfect pairing. As a consequence, given $i, j$ with $0 \leq i \leq j \leq d$ and a nonzero element $u \in F_i$, there exists $w \in F_{j-i}$ such that $uw \neq 0$. The reason is that by the perfect pairing property there exists $w_1 \in F_{d-i}$ such that $uw_1 \neq 0$, and since $F$ is standard graded, $w_1$ is a sum of products of elements of $F_{j-i}$ with elements of $F_{d-j}$.

We will use the definitions of [5, Section 5.1] for the notions of simplicial complex, vertices, faces, facets and dimension of a simplicial complex, and the notion of Stanley-Reisner ideal (also known as face ideal) and Stanley-Reisner ring (also known as face ring) of a simplicial complex over the field $k$. If $n \geq 1$ is an integer, a simplicial sphere of dimension $n$ is a simplicial complex $D$ of dimension $n$ such that there exists a geometric realization of $D$, in the sense of [5, Definition 5.2.8], which is homeomorphic to the sphere $S^n$. An additional important general reference is [16].

2.1. The generic Artinian reduction of an algebra. The following is a useful construction that will appear a number of times in the present paper.

Assume $m \geq 1$ and $k_1$ is a field. We consider the polynomial ring $k_1[x_1, \ldots, x_m]$, where the degree of the variable $x_i$ is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset k_1[x_1, \ldots, x_m]$ is a homogeneous ideal. We denote by $d$ the Krull dimension of the quotient ring $k_1[x_1, \ldots, x_m]/I$. We assume $d \geq 1$, and denote by $k$ the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m].$$
For $1 \leq i \leq d$, we set
\[ f_i = \sum_{j=1}^{m} a_{i,j} x_j. \]

**Definition 2.2.** We define the generic Artinian reduction of $k_1[x_1, \ldots, x_m]/I$ to be the Artinian $k$-algebra
\[ k[x_1, \ldots, x_m]/((I) + (f_1, \ldots, f_d)), \]
where $(I)$ denotes the ideal of $k[x_1, \ldots, x_m]$ generated by $I$.

3. Statement of the main theorem

Assume $n \geq 1$ is an integer and $D$ is a simplicial sphere of dimension $n$ with vertex set \{1, \ldots, m\}. Assume $k_1$ is any field and denote by $k$ the field of fractions of the polynomial ring
\[ k_1[a_{i,j} : 1 \leq i \leq n+1, 1 \leq j \leq m]. \]

We define the polynomial ring $R = k[x_1, \ldots, x_m]$, where we put degree 1 for all variables $x_i$. We denote by $I_D \subset R$ the Stanley-Reisner ideal of $D$ and we set $k[D] = R/I_D$. For $i = 1, \ldots, n+1$, we set
\[ f_i = \sum_{j=1}^{m} a_{i,j} x_j, \]
and we define $A = k[D]/(f_1, \ldots, f_{n+1})$. Hence, $A$ is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 2.2. We denote by $\pi : R \to A$ the natural projection $k$-algebra homomorphism.

**Remark 3.1.** By [5] Section 5, the $k$-algebra $k[D]$ is standard graded and Gorenstein with Krull dimension equal to $n+1$. Since $a_{i,j}$ are independent variables that do not appear in the minimal monomial generating set for $I_D$, the sequence $f_1, \ldots, f_{n+1}$ is a regular sequence for $k[D]$, see [5] Proposition 1.5.12. Hence, $A$ is a Gorenstein Artinian standard graded $k$-algebra. It has socle degree equal to $n+1$ by [5] Lemma 5.6.4. Consequently, $A_i = 0$ for all $i \geq n+2$ and $\dim_k A_{n+1} = 1$. In particular, $\dim_k A_1 \geq 1$, which implies that $m \geq n+2$.

**Definition 3.2.** We call $D$ generically anisotropic over $k_1$, if for all integers $j$ with $1 \leq 2j \leq n+1$ and all nonzero elements $u \in A_j$ we have $u^2 \neq 0$.

The main result of the present paper is the following theorem, whose proof will be given in Subsection 5.3.

**Theorem 3.3.** Assume that the field $k_1$ has characteristic 2, $n \geq 1$ is an integer, and $D$ is a simplicial sphere of dimension $n$. Then $D$ is generically anisotropic over $k_1$.

4. The Artinian reduction of the Stanley-Reisner ring

We keep using the notations and assumptions defined in Section 3. In particular, we allow the field $k_1$ to be of arbitrary characteristic.

If $\sigma = (b_1, \ldots, b_q)$ is a sequence of integers, with $1 \leq b_i \leq m$ for all $i$, we set
\[ x_\sigma = \prod_{i=1}^{q} x_i \in R. \]
Whenever $q = n + 1$, we also use the notation
\[ [\sigma] = [b_1, \ldots, b_{n+1}] \in k, \]
where, by definition, $[b_1, \ldots, b_{n+1}]$ is the determinant of the $(n+1) \times (n+1)$ matrix with $(i,j)$-entry equal to $a_{i,b_j}$.

We denote by $F(D)$ the set of facets of $D$. We define an ordered facet of $D$ to be a sequence $(b_1, b_2, \ldots, b_{n+1})$ of positive integers such that the set $\{b_1, b_2, \ldots, b_{n+1}\}$ is a facet.
of $D$. For $0 \leq i \leq n$, we define a codimension $i$ face $\sigma$ of $D$ to be a face of dimension $n - i$. This is equivalent to $\# \sigma = n + 1 - i$.

Assume $g = \prod_{i=1}^{m} x_i^{a_i} \in R$ is a monomial. We define the complexity $c(g)$ of $g$ by

$$c(g) = \sum_{i=1}^{m} a_i - \# \{i : a_i > 0\}.$$  

It is clear that $c(g) \geq 0$ and that $c(g) = 0$ if and only if $g$ is square-free.

The following proposition is well-known, but we provide a proof for completeness.

**Proposition 4.1.** Assume $1 \leq r \leq n + 1$. We have that the $r$-th graded piece $A_r$ of $A$ is spanned, as a $k$-vector space, by the image under $\pi$ of the set of square-free monomials of $R$ of degree $r$.

**Proof.** By finite induction, it is enough to show that if $g \in R$ is a nonzero monomial of degree $r$ and complexity $\geq 1$, then there exists $q \in R$ homogeneous of degree $r$, such that $\pi(q) = \pi(g)$ and $q$ is a linear combination of monomials of complexity $c(g) - 1$.

Assume $g = \prod_{i=1}^{m} x_i^{a_i}$. Since $c(g) \geq 1$, by rearranging indices we can assume that $a_1 \geq 2$. Since $r \leq n + 1$, by rearranging indices we can assume that $a_i = 0$ for all $i \geq n + 2$.

By Proposition [11.1], we have

$$\sum_{t=1}^{m} [2, 3, \ldots, n + 1, t] \pi(x_t) = 0.$$  

Hence,

$$[2, 3, \ldots, n + 1, 1] \pi(x_1) = - \sum_{t=n+2}^{m} [2, 3, \ldots, n + 1, t] \pi(x_t).$$

As a consequence, multiplying by $\pi(g/x_1)$ we get

$$\pi(g) = - (\sum_{t=n+2}^{m} [2, 3, \ldots, n + 1, t] \pi(x_t g/x_1)) /[2, 3, \ldots, n + 1, 1].$$

Since, for all $t \geq n + 2$, we have $c(x_t g/x_1) = c(g) - 1$, the result follows. $\square$

**Remark 4.2.** For a strengthening of Proposition [4.1] see Proposition [5.9].

**Remark 4.3.** We will use the following two facts, see [6], p. 111, Remark before Corollary 7.19]. Each codimension 1 face of $D$ is contained in exactly two facets of $D$. Moreover, if $\sigma_1$ and $\sigma_2$ are two facets of $D$, then there exists a finite sequence

$$\tau_0, \tau_1, \ldots, \tau_q$$

of facets of $D$ such that $\tau_0 = \sigma_1$, $\tau_q = \sigma_2$, and, for all $0 \leq i \leq q - 1$, the intersection $\tau_i \cap \tau_{i+1}$ is a codimension 1 face of $D$.

**Proposition 4.4.** Assume

$$\sigma_1 = (b_1, \ldots, b_n, d_1), \quad \sigma_2 = (b_1, \ldots, b_n, d_2),$$

are two ordered facets of $D$ having codimension 1 intersection. We then have the following equality in the ring $A$

$$[\sigma_1] \pi(x_{\sigma_1}) = - [\sigma_2] \pi(x_{\sigma_2}).$$

**Proof.** We set $\tau = \sigma_1 \cap \sigma_2$. Hence, $\tau = \{b_1, \ldots, b_n\}$. By Proposition [11.1] we have that

$$\sum_{j=1}^{m} [b_1, b_2, \ldots, b_n, j] \pi(x_j) = 0.$$  

Hence,

$$\sum_{j=1}^{m} [b_1, b_2, \ldots, b_n, j] \pi(x_j x_\tau) = 0.$$
If \( j \in \tau \), we have \([b_1, b_2, \ldots, b_n, j] = 0\). By Remark 4.3, \( \sigma_1 \) and \( \sigma_2 \) are the only facets of \( D \) which contain the codimension 1 face \( \tau \). Hence, the only terms of the last sum that are nonzero are for \( j = d_1 \) and \( j = d_2 \). The result follows. \( \square \)

**Corollary 4.5.** Assume \( \sigma_1 \) and \( \sigma_2 \) are two ordered facets of \( D \). Then there exists \( \varepsilon \in \{ -1, 1 \} \), such that
\[
[\sigma_1] \pi(x_{\sigma_1}) = \varepsilon[\sigma_2] \pi(x_{\sigma_2}).
\]

**Proof.** By Remark 4.3, there exists a finite sequence \( \tau_0, \tau_1, \ldots, \tau_q \) of facets of \( D \) such that \( \tau_0 = \sigma_1, \tau_q = \sigma_2 \), and, for all \( 0 \leq i \leq q - 1 \), the intersection \( \tau_i \cap \tau_{i+1} \) is a codimension 1 face of \( D \). Using Proposition 4.4, we have that, for all \( 0 \leq i \leq q - 1 \), there exists \( \varepsilon_i \in \{ -1, 1 \} \), such that we have the following equality in the ring \( A \)
\[
[\tau_i] \pi(x_{\tau_i}) = \varepsilon_i[\tau_{i+1}] \pi(x_{\tau_{i+1}}).
\]
The result follows. \( \square \)

We fix an ordered facet \( e = (e_1, \ldots, e_{n+1}) \) of \( D \). By Remark 3.1, \( \dim_k A_{n+1} = 1 \). Using Proposition 4.4, \( A_{n+1} \) is spanned, as a \( k \)-vector space, by the square-free monomials that correspond to the facets of \( D \). Corollary 4.5 implies that any of them spans \( A_{n+1} \). As a consequence, \( \pi(x_e) \neq 0 \) and \( \pi(x_e) \) is a \( k \)-basis of \( A_{n+1} \). Hence, there exists a unique set-theoretic map \( \Psi_e : A_{n+1} \to k \) with the property that
\[
(1) \quad u = \Psi_e(u)[e] \pi(x_e)
\]
for all \( u \in A_{n+1} \). It is clear that \( \Psi_e \) is an isomorphism of \( k \)-vector spaces. In addition, if \( n \) is odd, we set \( p_1 = (n + 1)/2 \) and define the symmetric bilinear form
\[
(2) \quad \rho_e : A_{p_1} \times A_{p_1} \to k
\]
by
\[
\rho_e(u, w) = \Psi_e(uw)
\]
for all \( u, w \in A_{p_1} \).

**Remark 4.6.** If we change the ordered facet \( e \) of \( D \) to another ordered facet \( \sigma \), Corollary 4.5 implies that either \( \Psi_\sigma = \Psi_e \) or \( \Psi_\sigma = -\Psi_e \). Hence, if the field \( k_1 \) has characteristic 2 the map \( \Psi_e \) is canonical, in the sense that it is independent of the choice of the facet \( e \) of \( D \), and we will denote it by \( \Psi \).

**Remark 4.7.** Assume \( n \) is odd. Recall that a symmetric bilinear form \( \delta : A_{p_1} \times A_{p_1} \to k \) is called anisotropic if \( \delta(u, u) \neq 0 \) for all nonzero elements \( u \in A_{p_1} \). Using Remark 2.1, it follows that \( \rho_e \) is anisotropic if and only if for all integers \( j \) with \( 1 \leq 2j \leq n + 1 \) and all nonzero elements \( u \in A_j \) we have \( u^2 \neq 0 \). This (partially) explains the use of the term generic anisotropicity in Definition 3.2.

**Remark 4.8.** The proof of Proposition 4.4 gives that for all \( u \in A_{n+1} \) the element \( \Psi_e(u) \) of \( k \) is a rational function in the set of all bracket polynomials
\[
\{ [i_1, \ldots, i_{n+1}] : 1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq m \}.
\]
In addition, combined with the proof of Corollary 4.5, it provides an algorithm for computing \( \Psi_e(u) \).

**Proposition 4.9.** Assume \( k_1 \) is a field of characteristic 2 and \( \sigma = (b_1, \ldots, b_{n+1}) \) is a facet of \( D \). We have
\[
(\Psi \circ \pi)(x_\sigma) = 1/[b_1, \ldots, b_{n+1}].
\]

**Proof.** By Corollary 4.5, we have
\[
[\sigma] \pi(x_\sigma) = [e] \pi(x_e).
\]
The result follows from the definition of \( \Psi \). \( \square \)
The following proposition allows the computation of $\Psi_e(u)$ in more cases.

**Proposition 4.10.** Assume $\sigma = (b_1, \ldots, b_{n-1}, c)$ is a codimension 1 ordered face of $D$. Denote by $\tau_1 = (b_1, \ldots, b_{n-1}, c, d_1)$ and $\tau_2 = (b_1, \ldots, b_{n-1}, c, d_2)$ the two ordered facets of $D$ that contain $\sigma$. We then have the following two equalities

$$[b_1, \ldots, b_{n-1}, c, d_1][b_1, \ldots, b_{n-1}, c, d_2] \pi(x^2_c \prod_{i=1}^{n-1} x_{b_i}) = -[b_1, \ldots, b_{n-1}, d_1, d_2][\tau_1] \pi(x_{\tau_1})$$

$$= [b_1, \ldots, b_{n-1}, d_1, d_2][\tau_2] \pi(x_{\tau_2}).$$

**Proof.** We set $S = \{1, \ldots, m\} \setminus \{c\}$. By Proposition 4.10, we have that

$$\sum_{j=1}^{m} [b_1, b_2, \ldots, b_{n-1}, d_1, j] \pi(x_j) = 0.$$

Hence,

$$[b_1, b_2, \ldots, b_{n-1}, d_1, c] \pi(x_c) = -\sum_{j \in S} [b_1, b_2, \ldots, b_{n-1}, c, j] \pi(x_j).$$

Consequently,

$$[b_1, b_2, \ldots, b_{n-1}, d_1, c] \pi(x^2_c \prod_{i=1}^{n-1} x_{b_i}) = -\sum_{j \in S} [b_1, b_2, \ldots, b_{n-1}, d_1, j] \pi(x_j x_c \prod_{i=1}^{n-1} x_{b_i}).$$

Arguing for the last sum as in the proof of Proposition 4.4, we get

$$[b_1, b_2, \ldots, b_{n-1}, d_1, c] \pi(x^2_c \prod_{i=1}^{n-1} x_{b_i}) = -[b_1, b_2, \ldots, b_{n-1}, d_1, d_2] \pi(x_{d_2} x_c \prod_{i=1}^{n-1} x_{b_i}).$$

Using that, by Proposition 4.10, $[\tau_1] \pi(x_{\tau_1}) = -[\tau_2] \pi(x_{\tau_2})$, the result follows. \hfill $\square$

The following corollary is an immediate consequence of Proposition 4.10.

**Corollary 4.11.** Assume $k_1$ is a field of characteristic 2 and $\sigma = (b_1, \ldots, b_{n-1}, c)$ is a codimension 1 face of $D$. Denote by $(b_1, \ldots, b_{n-1}, c, d_1)$ and $(b_1, \ldots, b_{n-1}, c, d_2)$ the two facets of $D$ that contain $\sigma$. We have

$$(\Psi \circ \pi)(x^2_c \prod_{i=1}^{n-1} x_{b_i}) = \frac{[b_1, \ldots, b_{n-1}, d_1, d_2]}{[b_1, \ldots, b_{n-1}, c, d_1][b_1, \ldots, b_{n-1}, c, d_2]}.$$

**FURTHER ASSUMPTION.** For the rest of this section we make the additional assumption that the field $k_1$ has characteristic 2.

We set $Z = m + 2n$ and denote by $M$ the $(n+1) \times Z$ matrix whose $(i, j)$-entry is equal to the variable $a_{i,j}$, for $1 \leq i \leq n+1$ and $1 \leq j \leq Z$. Given a subset $A$ of the set $\{1, 2, \ldots, Z\}$ of cardinality $n+1$, we denote by $M(A)$ the determinant of the $(n+1) \times (n+1)$ submatrix of $M$ obtained by keeping the columns of $M$ specified by the set $A$.

We denote by $k_2$ the field of fractions of the polynomial ring $k_1[a_{i,j} : 1 \leq i \leq n+1, 1 \leq j \leq Z]$.

It follows that $k$ is a subfield of $k_2$.

**Proposition 4.12.** (Recall that the field $k_1$ has characteristic equal to 2.) Assume $n$ is odd. We set $l = (n+1)/2$. Assume $D$ is the boundary complex of the $(n+1)$-dimensional simplex with vertex set $\tau = \{c_1, \ldots, c_l, g_1, \ldots, g_{l+1}\}$. We then have the following equality in the field $k_2$

$$(\Psi \circ \pi)(\prod_{i=1}^{l} x^2_{c_i}) = \frac{\prod_{i=1}^{l} M(\tau \setminus \{c_i\})}{\prod_{i=1}^{l+1} M(\tau \setminus \{g_i\})}.$$
Proof. We set \( c = \{ c_1, \ldots, c_l \}, g = \{ g_1, \ldots, g_{l+1} \} \). Assume \( 1 \leq i \leq l \). By Proposition 4.11 we have that
\[
\sum_{t=1}^{l} [c \setminus \{ c_t \}, g \setminus \{ g_t \}, c_t] \pi(x_{c_t}) + \sum_{t=1}^{l+1} [c \setminus \{ c_t \}, g \setminus \{ g_t \}, g_{t+1}] \pi(x_{g_{t+1}}) = 0.
\]

Hence,
\[
[c \setminus \{ c_t \}, g \setminus \{ g_t \}, c_t] \pi(x_{c_t}) = [c \setminus \{ c_t \}, g \setminus \{ g_t \}, g_{t+1}] \pi(x_{g_{t+1}}),
\]
since the field has characteristic 2 and the other terms in the two sums are zero.

Multiplying the above equations for \( 1 \leq i \leq l \), we get
\[
(\prod_{i=1}^{l} [c \setminus \{ c_i \}, g \setminus \{ g_i \}, c_i]) u_1 = (\prod_{i=1}^{l} [c \setminus \{ c_i \}, g \setminus \{ g_i \}, g_i]) u_2,
\]
where
\[
u_1 = \prod_{i=1}^{l} \pi(x_{c_i}), \quad u_2 = \prod_{i=1}^{l} \pi(x_{g_i}).
\]
The result follows by multiplying both sides of Equality (3) by \( u_1 \) and using that, by Corollary 4.9,
\[
\Psi(u_1 u_2) = 1/[c, g \setminus \{ g_{l+1} \}].
\]

Proposition 4.13. (Recall that the field \( k_1 \) has characteristic equal to 2.) Assume \( n \) is even. We set \( l = n/2 \). Assume \( D \) is the boundary complex of the \((n+1)\)-dimensional simplex with vertex set \( \tau = \{ c_1, \ldots, c_l, b, g_1, \ldots, g_{l+1} \} \). We then have the following equality in the field \( k_2 \)
\[
(\Psi \circ \pi)(x_b \prod_{i=1}^{l} x_{c_i}^2) = \prod_{i=1}^{l+1} M(\tau \setminus \{ c_i \}) \prod_{i=1}^{l+1} M(\tau \setminus \{ g_i \}).
\]

Proof. We set \( c = \{ c_1, \ldots, c_l \}, g = \{ g_1, \ldots, g_{l+1} \} \). Assume \( 1 \leq i \leq l \). By Proposition 4.11 we have that
\[
\sum_{t=1}^{l} [b, c \setminus \{ c_t \}, g \setminus \{ g_t \}, c_t] \pi(x_{c_t}) + \sum_{t=1}^{l+1} [b, c \setminus \{ c_t \}, g \setminus \{ g_t \}, g_{t+1}] \pi(x_{g_{t+1}}) = 0.
\]

Hence,
\[
[b, c \setminus \{ c_t \}, g \setminus \{ g_t \}, c_t] \pi(x_{c_t}) = [b, c \setminus \{ c_t \}, g \setminus \{ g_t \}, g_{t+1}] \pi(x_{g_{t+1}}),
\]
since the field has characteristic 2 and the other terms in the two sums are zero.

Multiplying the above equalities for \( 1 \leq i \leq l \), we get
\[
(\prod_{i=1}^{l} [b, c \setminus \{ c_i \}, g \setminus \{ g_i \}, c_i]) u_1 = (\prod_{i=1}^{l} [b, c \setminus \{ c_i \}, g \setminus \{ g_i \}, g_i]) u_2,
\]
where
\[
u_1 = \prod_{i=1}^{l} \pi(x_{c_i}), \quad u_2 = \prod_{i=1}^{l} \pi(x_{g_i}).
\]
The result follows by multiplying both sides of Equality (4) by \( \pi(x_b)u_1 \) and using that, by Corollary 4.9,
\[
\Psi(\pi(x_b)u_1 u_2) = 1/[b, c, g \setminus \{ g_{l+1} \}].
\]

We fix an integer \( r \) with \( m+1 \leq r \leq Z \). Assume \( l \) is an integer with \( 2 \leq 2l \leq n + 1 \). We set \( s = n + 1 - 2l \). Assume
\[
\tau_1 = \{ c_1, \ldots, c_l \}, \quad \tau_2 = \{ b_1, \ldots, b_s \}
\]
are two subsets of the vertex set \( \{ 1, \ldots, m \} \) of \( D \), such that \( \tau_1 \cup \tau_2 \) has cardinality \( l + s \) and is a face of \( D \). We set \( \tau = \tau_1 \cup \tau_2 \).
Assume $\sigma \in F(D)$ is a facet of $D$. We define the rational function $H(\tau_1, \tau_2, \sigma)$ as follows:

1. If $\tau$ is not a subset of $\sigma$ we set $H(\tau_1, \tau_2, \sigma) = 0$.
2. If $\tau$ is a subset of $\sigma$, we denote the elements of $\sigma \setminus \tau$ by $f_1, \ldots, f_k$ and we set

$$H(\tau_1, \tau_2, \sigma) = \frac{\prod_{l=1}^m M(\sigma \cup \{f_l\})}{M(\sigma) \prod_{l=1}^m M(\sigma \cup \{f_l\})}.$$ 

Clearly,

$$H(\tau_1, \tau_2, \sigma) = \frac{\prod_{j \in \tau} M((\sigma \cup \{r\}) \setminus \{j\})}{M(\sigma) \prod_{j \in (\sigma \setminus (\tau \cup \tau_2))} M((\sigma \cup \{r\}) \setminus \{j\})}.$$ 

The proof of the following theorem will be given in Subsection 4.1.

**Theorem 4.14.** (Recall that the field $k_1$ has characteristic equal to 2.) We have the following equality in the field $k_2$

$$(\Psi \circ \pi)((\prod_{i=1}^l x_i^2) \prod_{i=1}^m x_i) = \sum_{\sigma \in F(D)} H(\tau_1, \tau_2, \sigma).$$

**Remark 4.15.** It is interesting to notice the similarities in the statement and proof of Theorem 4.14 with the results obtained by Lee in [11, Section 6].

**Remark 4.16.** Using the definition of the function $H$, it is clear that the nonzero terms of the sum in Equation (5) are exactly those where $\sigma$ contains $\tau$. Hence, the sum can also be considered as a sum over the facets of the link (3, Definition 5.3.4) of the face $\tau$ in $D$.

**Remark 4.17.** Even though the left hand side in Equation (5) is completely independent of $r$, each nonzero term $H(\tau_1, \tau_2, \sigma)$ on the right hand side does depend on $r$. Hence, provided no denominator vanishes, we are allowed to specialise the variables $a_i, r$, for $1 \leq i \leq n + 1$. This observation will be used in Corollaries 6.3 and 7.3.

**Example 4.18.** Assume $k_1$ is a field of characteristic 2, $m \geq 3$ and $D$ is the $m$-gon with consecutive vertices $1, 2, \ldots, m$. By Corollary 4.11, we have

$$(\Psi \circ \pi)(x_2^2) = \frac{[1,3]}{[1,2][2,3]},$$

while, by Theorem 4.14 we have

$$(\Psi \circ \pi)(x_2^2) = H(\{2\}, \emptyset, \{1,2\}) + H(\{2\}, \emptyset, \{2,3\}) = \frac{[1,r]}{[1,2][2,r]} + \frac{[3,r]}{[2,3][2,r]}.$$ 

**Example 4.19.** Assume $k_1$ is a field of characteristic 2, and $D$ is the simplicial complex with vertex set $\{1,2,\ldots,7\}$ and Stanley-Reisner ideal equal to $I_D = (x_1x_2, x_3x_4x_5, x_6x_7)$. Then $D$ is a simplicial sphere of dimension 3.

We set $\tau_1 = \{1,3\}, \tau_2 = \emptyset$. Clearly $\tau_1$ is a face of $D$. Since $I_D : (x_1x_3) = (x_2, x_4x_5, x_6x_7)$, the link of $\tau_1$ in $D$ is the 4-gon with consecutive vertices $4, 6, 5, 7$. By Theorem 4.14

$$(\Psi \circ \pi)(x_1^2x_3^2) = H_{4,6} + H_{6,5} + H_{5,7} + H_{7,4},$$

where

$$H_{a,b} = H(\tau_1, \tau_2, \tau_1 \cup \{a,b\}) = \frac{[1,a,b,r]}{[1,3,a,b][1,3,a,r]}.$$ 

**Remark 4.20.** We expect that with the correct sign adjustments there should be a version of Theorem 4.14 valid over a field $k_1$ of arbitrary characteristic. We do not pursue this direction further in the present work.
4.1. The proof of Theorem 4.14 We now give the proof of Theorem 4.14 by induction on $l \geq 1$.

Assume $l = 1$. We have $s = n - 1$ and

$$\tau_1 = \{c_1\}, \quad \tau_2 = \{b_1, \ldots, b_{n-1}\}.$$  
Recall that $\tau = \tau_1 \cup \tau_2$. Hence, $\tau$ is a codimension 1 face of $D$. Using Remark 4.3, $\tau$ is contained in exactly two facets of $D$. We denote them by

$$\sigma_1 = \{b_1, \ldots, b_{n-1}, c_1, d_1\}, \quad \sigma_2 = \{b_1, \ldots, b_{n-1}, c_1, d_2\}.$$  

We use the notation

$$[\tau_2, i, j] = [b_1, \ldots, b_{n-1}, i, j].$$

By Corollary 4.11

$$(\Psi \circ \pi)(x_{c_1}^2 \prod_{i=1}^{n-1} x_{b_i}) = \frac{[\tau_2, d_1, d_2]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2]}.$$  

We have $H(\tau_1, \tau_2, \sigma) = 0$ if $\sigma \in F(D) \setminus \{\sigma_1, \sigma_2\}$. Using the Plücker relation (10, Theorem 5.2.3)]

$$[\tau_2, d_1, d_2][\tau_2, c_1, r] = [\tau_2, d_1, r][\tau_2, c_1, d_2] + [\tau_2, d_1, c_1][\tau_2, d_2, r]$$

and taking into account that the field $k_1$ has characteristic 2, we have

$$(\Psi \circ \pi)(x_{c_1}^2 \prod_{i=1}^{n-1} x_{b_i}) = \frac{[\tau_2, d_1, d_2]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2]} = \frac{[\tau_2, d_1, d_2][\tau_2, c_1, r]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2][\tau_2, c_1, r]} = \frac{[\tau_2, d_1, r][\tau_2, c_1, d_2] + [\tau_2, d_1, c_1][\tau_2, d_2, r]}{[\tau_2, c_1, d_1][\tau_2, c_1, d_2][\tau_2, c_1, r]} = H(\tau_1, \tau_2, \sigma_1) + H(\tau_1, \tau_2, \sigma_2).$$

We assume now that $l \geq 1$ with $2(l+1) \leq n + 1$ and that Theorem 4.14 is true for $l$. We will prove that Theorem 4.14 is true for the value $l + 1$. We set $s = n + 1 - 2(l + 1)$. Assume

$$\tau_1 = \{c_1, \ldots, c_{l+1}\}, \quad \tau_2 = \{b_1, \ldots, b_s\}$$

such that $\tau_1 \cup \tau_2$ has cardinality $l + s + 1$ and is a face of $D$.

We fix integers $p_1, \ldots, p_l$, such that $m + 1 \leq p_i \leq Z$, for all $1 \leq i \leq l$, and the set $\{r, p_1, p_2, \ldots, p_l\}$ has cardinality equal to $l + 1$. We set $B = \{1, \ldots, m\} \setminus (\tau_1 \cup \tau_2)$ and

$$u = \left(\prod_{i=1}^{l+1} x_{c_i}^2\right)\left(\prod_{i=1}^s x_{b_i}\right).$$

For $1 \leq i \leq Z$, we set

$$[[i]] = [b_1, \ldots, b_s, c_1, \ldots, c_l, r, p_1, \ldots, p_l, i].$$

Using Proposition 11.1

$$\sum_{i=1}^m [[[i]]] \pi(x_i) = 0.$$  

Hence

$$\pi(x_{cl+1}) = \sum_i \frac{[[i]]}{[[c_{l+1}]]} \pi(x_i),$$
with the sum for $1 \leq i \leq m$ and $i \neq c_{l+1}$. Since $[[i]] = 0$ when $i \in \{c_1, \ldots, c_l, b_1, \ldots, b_s\}$, we have that

$$
\pi(x_{c_{l+1}}) = \sum_{i \in B} [[i]] \pi(x_i).
$$

Multiplying this equality by

$$
\pi(x_{c_{l+1}}) \prod_{i=1}^l \pi(x_{c_i}^2) \prod_{i=1}^s \pi(x_b_i)
$$

we get

$$
\pi(u) = \sum_{i \in B} [[i]] \pi(E_i),
$$

where

$$
E_i = x_i x_{c_{l+1}} (\prod_{i=1}^l x_{c_i}^2) (\prod_{i=1}^s x_{b_i}).
$$

Hence,

$$
(\Psi \circ \pi)(u) = \sum_{i \in B} [[i]] (\Psi \circ \pi)(E_i).
$$

Since, for all $i \in B$, the expression for $E_i$ has $l$ squares, we can use the inductive hypothesis for $(\Psi \circ \pi)(E_i)$ to get

$$
(\Psi \circ \pi)(E_i) = \sum_{\sigma \in F(D)} H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{i, c_{l+1}\}, \sigma).
$$

As a consequence,

$$
(\Psi \circ \pi)(u) = \sum_{i \in B} \sum_{\sigma \in F(D)} V_{i,\sigma} = \sum_{\sigma \in F(D)} \sum_{i \in B} V_{i,\sigma},
$$

where

$$
V_{i,\sigma} = \frac{[[i]]}{[[c_{l+1}]}} H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{i, c_{l+1}\}, \sigma).
$$

Therefore, to finish the proof it is enough to show that for all $\sigma \in F(D)$ it holds

$$
(6) \sum_{i \in B} V_{i,\sigma} = H(\tau_1, \tau_2, \sigma).
$$

For $i \in B$ we set

$$
\eta_i = (\tau_1 \setminus \{c_{l+1}\}) \cup (\tau_2 \cup \{i, c_{l+1}\}),
$$

therefore $\eta_i = \tau \cup \{i\}$.

We first assume that $\sigma \in F(D)$ does not contain $\tau$ as a subset. Hence $H(\tau_1, \tau_2, \sigma) = 0$. Assume $i \in B$. Since $\tau \subset \eta_i$, it follows that $\eta_i$ is not a subset of $\sigma$. This implies that $H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{i, c_{l+1}\}) = 0$, therefore $V_{i,\sigma} = 0$. As a consequence, Equality (6) is true.

Assume now that $\sigma \in F(D)$ contains $\tau$ as a subset. We set $C = \sigma \setminus \tau$ and denote the elements of $C$ by $g_1, \ldots, g_{l+1}$. We set $\sigma' = \sigma \cup \{r\}$. If $i \in B \setminus C$, it follows that $\eta_i$ is not a subset of $\sigma$, therefore $V_{i,\sigma} = 0$. As a consequence,

$$
\sum_{i \in B} V_{i,\sigma} = \sum_{i \in \mathcal{C}} V_{i,\sigma} = \sum_{i=1}^{l+1} V_{g_i,\sigma}.
$$
We have
\[ V_{g_i,\sigma} = \frac{\left[ [g_i] \right]}{\left[ [c_{l+1}] \right]} H(\tau_1 \setminus \{c_{l+1}\}, \tau_2 \cup \{g_i, c_{l+1}\}, \sigma) \]
\[ \frac{[[g_i]]}{[[c_{l+1}]]} \prod_{i=1}^{l} M(\sigma^r \setminus \{c_i\}) \frac{M(\sigma^r \setminus \{g_i\}) \prod_{i=1}^{l} M(\sigma^r \setminus \{g_i\})}{M(\sigma) \prod_{i=1}^{l} M(\sigma^r \setminus \{g_i\})} \]
\[ = \Gamma [[g_i]] M(\sigma^r \setminus \{g_i\}), \]
where
\[ \Gamma = \frac{\prod_{i=1}^{l} M(\sigma^r \setminus \{c_i\})}{[[c_{l+1}]] M(\sigma) \prod_{i=1}^{l} M(\sigma^r \setminus \{g_i\})}. \]
Hence,
\[ \sum_{i=1}^{l+1} V_{g_i,\sigma} = \Gamma \sum_{i=1}^{l+1} [[g_i]] M(\sigma^r \setminus \{g_i\}). \]

By the Plücker relation ([10, Theorem 5.2.3]),
\[ \sum_{i=1}^{l+1} [[g_i]] M(\sigma^r \setminus \{g_i\}) = [[c_{l+1}]] M(\sigma^r \setminus \{c_{l+1}\}). \]

Therefore,
\[ \sum_{i=1}^{l+1} V_{g_i,\sigma} = \Gamma [[c_{l+1}]] M(\sigma^r \setminus \{c_{l+1}\}) \]
\[ = \frac{\prod_{i=1}^{l+1} M(\sigma^r \setminus \{c_i\})}{M(\sigma) \prod_{i=1}^{l+1} M(\sigma^r \setminus \{g_i\})} H(\tau_1, \tau_2, \sigma). \]

As a consequence, Equality (6) is true, which finishes the proof of Theorem 4.14.

5. Using the Differential Operators to Establish Anisotropicity

We keep using the notations introduced in Sections 3 and 4. Moreover, we assume that the field \( k_1 \) has characteristic 2.

5.1. Case \( n \) is odd. Assume \( n \geq 1 \) is odd. We set \( l = (n+1)/2 \). We assume \( \sigma \in D \) is a face of dimension \( l - 1 \). We denote, in increasing order, the elements of \( \sigma \) by \( \sigma(1), \sigma(2), \ldots, \sigma(l) \).

We define \( \partial_\sigma : k_2 \to k_2 \) to be the \( (n+1) \)-th order differential operator which is differentiation with respect to the variables in the set
\[ \{a_i, \sigma(j) : 1 \leq i \leq n+1, j = [(i+1)/2]\}, \]
where \([x]\) denotes the integral part of the real number \( x \).

Proposition 5.1. Assume \( \tau \) is a face of \( D \) of dimension \( l - 1 \). We then have
\[ (7) \quad (\partial_\sigma \circ \Psi \circ \pi)(x_\tau^2) = ((\Psi \circ \pi)(x_\sigma x_\tau))^2. \]

Proof. We define the sets
\[ \mathcal{K}_1 = \{\eta \in F(D) : \tau \subset \eta\}, \quad \mathcal{K}_2 = \{\eta \in F(D) : \tau \cup \sigma \subset \eta\}. \]

We set \( \gamma_1 = \tau \cap \sigma \), \( \gamma_2 = (\tau \cup \sigma) \setminus \gamma_1 \). Using Theorem 4.14 we get
\[ (\Psi \circ \pi)(x_\tau^2) = \sum_{\eta \in \mathcal{K}_1} H(\tau, \varnothing, \eta) \quad \text{and} \quad (\Psi \circ \pi)(x_\tau x_\sigma) = \sum_{\eta \in \mathcal{K}_2} H(\gamma_1, \gamma_2, \eta). \]
Clearly $K_2 \subset K_1$. If $\eta \in K_1 \setminus K_2$, we have that $\sigma \setminus (\eta \cap \sigma) \neq \emptyset$, which implies that $\partial_\sigma(H(\tau, \varnothing, \eta)) = 0$. Hence

$$\partial_\sigma (\Psi \circ \pi)(x^2) = \sum_{\eta \in K_2} \partial_\sigma(H(\tau, \varnothing, \eta)).$$

Since the field $k_1$ has characteristic 2, we get

$$((\Psi \circ \pi)(x_\tau x_\sigma))^2 = \sum_{\eta \in K_2} (H(\gamma_1, \gamma_2, \eta))^2.$$

Assume that $\eta \in K_2$. Using Corollary 6.3 we have

$$\partial_\sigma(H(\tau, \varnothing, \eta)) = \partial_\sigma\left(\prod_{i \in \tau} M((\eta \cup \{r\}) \setminus \{i\}) \cdot \prod_{i \in \eta \setminus \tau} M((\eta \cup \{r\}) \setminus \{i\})\right) \frac{1}{\eta \setminus \tau} \left(\prod_{i \in \eta \setminus \tau} M((\eta \cup \{r\}) \setminus \{i\})\right)^2 = \left(\frac{\prod_{i \in \eta \setminus \tau} M((\eta \cup \{r\}) \setminus \{i\})}{\eta \setminus \tau}\right)^2 = (H(\gamma_1, \gamma_2, \eta))^2,$$

which finishes the proof.

\textbf{Remark 5.2.} Conjecture 14.1 contains a conjectural statement generalising Proposition 5.1.

\textbf{Corollary 5.3.} Assume $u$ is a homogeneous element of $R$ of degree $l$. We then have

$$(\partial_\sigma \circ \Psi \circ \pi)(u^2) = ((\Psi \circ \pi)(x_\sigma u))^2.$$  

\textbf{Proof.} Using Proposition 5.1 there exist $s > 0$, faces $\tau_1, \ldots, \tau_s$ of $D$ of dimension $l - 1$ and elements $\lambda_1, \ldots, \lambda_s$ in $k$ such that

$$\pi(u) = \pi\left(\sum_{i=1}^{s} \lambda_i x_{\tau_i}\right).$$

Taking into account that the field $k_1$ has characteristic 2 and combining Proposition 5.1 with Remark 5.1 we have

$$((\Psi \circ \pi)(x_\sigma u))^2 = \left(\sum_{i=1}^{s} \lambda_i^2 (x_{\tau_i})^2\right)^2 = \left(\sum_{i=1}^{s} \lambda_i ((\Psi \circ \pi)(x_{\sigma x_{\tau_i}}))^2\right)^2 = \left((\Psi \circ \pi)(\sum_{i=1}^{s} \lambda_i x_{\tau_i x_\sigma})\right)^2 = ((\Psi \circ \pi)(x_\sigma u))^2.$$

\textbf{Remark 5.4.} If we abuse the notation by avoiding writing down the maps $\Psi$ and $\pi$, Equations (7) and (8) take the simpler form

$$\partial_\sigma(x^2) = (x_\tau x_\sigma)^2 \quad \text{and} \quad \partial_\sigma(u^2) = (x_\sigma u)^2$$

respectively.

\textbf{Example 5.5.} We use the assumptions of Example 4.15 and the notational convention described in Remark 5.4. We have

$$x^2 = \frac{[1,3]}{[1,2][2,3]}, \quad \partial_{\{1\}}(x^2) = \frac{1}{[1,2]^2} = (x_1 x_2)^2, \quad \partial_{\{2\}}(x^2) = \frac{[1,3]^2}{[1,2]^2[2,3]^2} = (x_2)^2.$$  

Assume, in addition, that $m \geq 4$. Then

$$\partial_{\{4\}}(x^2) = 0 = (x_4 x_2)^2.$$  

\textbf{Corollary 5.6.} Assume $u$ is a homogeneous element of $R$ of degree less or equal than $l$ such that $\pi(u) \neq 0$. We then have that $(\pi(u))^2 \neq 0$.
Using Theorem 4.14, we get a face of dimension $K$. Clearly Proposition 5.7.

5.2. Case $n$ is even. Assume $n \geq 2$ is even. We set $l = n/2$. We assume $\sigma \in D$ is a face of dimension $l - 1$ and that $p$ is vertex of $D$ such that $\sigma \cup \{p\}$ is a face of $D$ of dimension $l$. We denote, in increasing order, the elements of $\sigma$ by $\sigma(1), \sigma(2), \ldots, \sigma(l)$. We define $\partial_{p,\sigma} : k_2 \to k_2$ to be the $(n + 1)$-th order differential operator which is differentiation with respect to the variables in the set

$$\{a_{1,p} \} \cup \{a_{i,\sigma(j)} : 2 \leq i \leq n + 1, j = [i/2]\},$$

where $[x]$ denotes the integral part of the real number $x$.

**Proposition 5.7.** Assume $\tau$ is a face of $D$ of dimension $l - 1$ which does not contain $p$. We then have

$$\partial_{p,\sigma} \circ (x^2_{\tau} x_p) = ((\Psi \circ \pi)(x_{\sigma} x_p))^2.$$

**Proof.** We set $\tau_1 = \tau \cup \{p\}$. If $\tau_1$ is not a face of $D$, we have $\pi(x_{\sigma} x_p) = 0$ and the proposition is true.

Hence, we can assume that $\tau_1$ is a face of $D$. We define the sets

$$\mathcal{K}_1 = \{\eta \in F(D) : \tau_1 \subset \eta\}, \quad \mathcal{K}_2 = \{\eta \in F(D) : \tau_1 \cup \sigma \subset \eta\}.$$

We set $\gamma_1 = \tau_1 \cap \sigma$, $\gamma_2 = (\tau_1 \cup \sigma) \setminus \gamma_1$. Since $p$ is not an element of $\sigma$, we have $\gamma_1 = \tau \cap \sigma$. Using Theorem 4.14 we get

$$(\Psi \circ \pi)(x^2_{\tau} x_p) = \sum_{\eta \in \mathcal{K}_1} H(\tau, \{p\}, \eta) \quad \text{and} \quad (\Psi \circ \pi)(x_{\sigma} x_p) = \sum_{\eta \in \mathcal{K}_2} H(\gamma_1, \gamma_2, \eta).$$

Clearly $\mathcal{K}_2 \subset \mathcal{K}_1$. If $\eta \in \mathcal{K}_1 \setminus \mathcal{K}_2$, we have that $\sigma \setminus (\eta \cap \sigma) \neq \emptyset$, which implies that $\partial_{p,\sigma}(H(\tau, \{p\}, \eta)) = 0$. Hence

$$(\partial_{p,\sigma} \circ (x^2_{\tau} x_p) = \sum_{\eta \in \mathcal{K}_2} \partial_{p,\sigma}(H(\tau, \{p\}, \eta)).$$

Since the field $k_1$ has characteristic 2

$$(\Psi \circ \pi)(x_{\sigma} x_p))^2 = \sum_{\eta \in \mathcal{K}_2} (H(\gamma_1, \gamma_2, \eta))^2.$$

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Assume that $\eta \in \mathcal{K}_2$. Using Corollary 7.3, we have

$$\partial_{p,\sigma}(H(\tau, \{p\}, \eta)) = \partial_{p,\sigma}(\frac{\prod_{i \in \tau} M((\eta \cup \{r\}) \setminus \{i\})}{\prod_{i \in \tau \cap \sigma} M((\eta \cup \{r\}) \setminus \{i\})}) = \frac{(M(\eta))^2 \cdot \prod_{i \in \eta \cup \{r\} \cap \sigma} M((\eta \cup \{r\}) \setminus \{i\})^2}{(M(\eta))^2 \cdot \prod_{i \in \eta \cup \{r\} \cup \{s\} \setminus \sigma} M((\eta \cup \{r\}) \setminus \{i\})^2}.$$

which finishes the proof. □

**Remark 5.8.** Conjecture 14.1 contains a conjectural statement generalising Proposition 5.7.

We will need the following strengthening of Proposition 4.1.

**Proposition 5.9.** Assume $1 \leq d \leq n + 1$ and $u \in R_d$. Then there exist $s > 0$, faces $\tau_1, \ldots, \tau_s$ of $D$ dimension $d - 1$ and elements $\lambda_1, \ldots, \lambda_s$ in $k$ such that

$$\pi(u) = \pi(\sum_{i=1}^s \lambda_i x_{\tau_i}).$$
and, moreover, \( p \) is not an element of \( \tau_i \) for all \( 1 \leq i \leq s \).

**Proof.** Using Proposition 4.11, it is enough to assume that \( u = x_\eta \), where \( \eta \) is a face of \( D \) of dimension \( d - 1 \). If \( p \) is not an element of \( \eta \), the result is obvious by setting \( s = 1, \tau_1 = \eta, \lambda_1 = 1 \).

Assume now \( p \in \eta \). Without loss of generality, we can assume \( p = 1, \eta = \{1, 2, \ldots, d\} \). By Proposition 5.11, we have

\[
\sum_{t=1}^{m} [2, 3, \ldots, n + 1, t] \pi(x_t) = 0.
\]

Hence,

\[
\pi(x_1) = - \sum_{t=n+2}^{m} \frac{[2, 3, \ldots, n + 1, t]}{[2, 3, \ldots, n + 1, 1]} \pi(x_t),
\]

which implies that

\[
\pi(x_\eta) = - \sum_{t=n+2}^{m} \frac{[2, 3, \ldots, n + 1, t]}{[2, 3, \ldots, n + 1, 1]} \pi(x_t \prod_{i=2}^{d} x_i).
\]

The result follows.

**Corollary 5.10.** Assume \( u \) is a homogeneous element of \( R \) of degree \( l \). We then have

\[
(\partial_{p,\sigma} \circ \Psi \circ \pi)(u^2 x_p) = (\pi \circ \pi)(x_\sigma u x_p)^2.
\]

**Proof.** Using Proposition 4.11, there exist \( s > 0 \), faces \( \tau_1, \ldots, \tau_s \) of \( D \) of dimension \( l - 1 \) and elements \( \lambda_1, \ldots, \lambda_s \) in \( k \) such that

\[
\pi(u) = \pi(\sum_{i=1}^{s} \lambda_i x_{\tau_i})
\]

and, moreover, \( p \) is not an element of \( \tau_i \) for all \( 1 \leq i \leq s \).

Taking into account that the field \( k_1 \) has characteristic 2 and combining Proposition 5.11 with Remark 5.11, we have

\[
(\partial_{p,\sigma} \circ \Psi \circ \pi)(u^2 x_p) = (\partial_{p,\sigma} \circ \Psi \circ \pi)(\sum_{i=1}^{s} \lambda_i^2 x_{\tau_i}^2 x_p) = \sum_{i=1}^{s} \lambda_i^2 (\partial_{p,\sigma} \circ \Psi \circ \pi)(x_{\tau_i}^2 x_p)
\]

\[
= \sum_{i=1}^{s} \lambda_i^2 ((\Psi \circ \pi)(x_\sigma x_{\tau_i} x_p))^2 = \left( \sum_{i=1}^{s} \lambda_i ((\Psi \circ \pi)(x_\sigma x_{\tau_i} x_p))^2 \right)
\]

\[
= (\pi \circ \pi)(\sum_{i=1}^{s} \lambda_i x_{\tau_i} x_\sigma x_p)^2 = (\pi \circ \pi)(x_\sigma u x_p)^2.
\]

**Remark 5.11.** If we abuse the notation by avoiding writing down the maps \( \Psi \) and \( \pi \), Equations (9) and (10) take the simpler form

\[
\partial_{p,\sigma}(x_{\tau_i}^2 x_p) = (x_\sigma x_{\tau_i} x_p)^2 \quad \text{and} \quad \partial_{p,\sigma}(u^2 x_p) = (x_\sigma u x_p)^2
\]

respectively.

**Example 5.12.** Assume \( D \) is the boundary complex of the 3-simplex with vertex set \( \{1, 2, 3, 4\} \). We set \( p = 1, \tau = \{2\} \). Using Corollary 4.11 and the notational convention described in Remark 5.11, we have

\[
x_{\tau_2} x_p = \frac{[1, 3, 4]}{[1, 2, 3][1, 2, 4]}, \quad \partial_{p,\{2\}}(x_{\tau_2}^2 x_p) = \frac{[1, 3, 4]^2}{[1, 2, 3][1, 2, 4]^2} = (x_2 x_{\tau_2} x_p)^2
\]

and

\[
\partial_{p,\{3\}}(x_{\tau_3}^2 x_p) = \frac{1}{[1, 2, 3]^2} = (x_3 x_{\tau_3} x_p)^2, \quad \partial_{p,\{4\}}(x_{\tau_4}^2 x_p) = \frac{1}{[1, 2, 4]^2} = (x_4 x_{\tau_4} x_p)^2.
\]
Corollary 5.13. Assume $u$ is a homogeneous element of $R$ of degree less or equal than $l$ such that $\pi(u) \neq 0$. We then have that $(\pi(u))^2 \neq 0$.

Proof. Using Remark 5.1, $A$ is Artinian, Gorenstein and standard graded with socle degree equal to $n + 1$. It follows, by Remark 2.1, that there exists a homogeneous element $h \in R$ of degree $l - \deg(u)$ such that $\pi(uh) \neq 0$. Combining Proposition 4.1 with Remark 2.1, there exists a face $\sigma_1$ of $D$ of dimension $l$ such that $\pi(x_{\sigma_1}uh) \neq 0$.

We fix an element $p$ of $\sigma_1$, and set $\sigma = \sigma_1 \setminus \{p\}$. Therefore, $\pi(x_{\sigma_1}uh) \neq 0$ implies that $(\Psi \circ \pi)(x_{\sigma}uhxp) \neq 0$. Using Corollary 5.10 it follows that $(\Psi \circ \pi)((uh)^2x_p) \neq 0$. Since $\pi$ is a $k$-algebra homomorphism, we get $(\pi(u))^2 \neq 0$. \(\square\)

5.3. Proof of Theorem 3.3 We now prove Theorem 3.3. If $n$ is odd, it follows from Corollary 5.6 while if $n$ is even, it follows from Corollary 5.13.

6. The differential operator for $n$ odd

The aim of the present section is to establish, in conjunction with the following two Sections 7 and 8, the results about the differential operators that were used in Section 5.

In the present section we work over a field $k_1$ of characteristic 2.

Assume $n \geq 1$ is odd and $m$ is an integer with $m \geq n + 1$. We set $Z = m + 2n$ and denote by $M$ the $(n + 1) \times Z$ matrix whose $(i,j)$-entry is equal to the variable $a_{i,j}$, for $1 \leq i \leq n$ and $1 \leq j \leq Z$. Given a subset $A$ of the set $\{1,2,\ldots,Z\}$ of cardinality $n + 1$, we denote by $M(A)$ the determinant of the $(n + 1) \times (n + 1)$ submatrix of $M$ obtained by keeping the columns of $M$ specified by the set $A$.

We denote by $k_2$ the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n + 1, 1 \leq j \leq Z].$$

We set $l = (n + 1)/2$.

Assume

$$\tau_1 = \{c_1,\ldots,c_l\}, \quad \tau_2 = \{g_1,\ldots,g_{l+1}\}$$

are two subsets of the set $\{1,2,\ldots,Z\}$ such that $\tau_1 \cup \tau_2$ has cardinality $2l + 1$.

We set $\tau = \tau_1 \cup \tau_2$ and

$$G(\tau_1,\tau_2) = \frac{\prod_{i=1}^{l} M(\tau \setminus \{c_i\})}{\prod_{i=1}^{l+1} M(\tau \setminus \{g_i\})}.$$

For the rest of this section we make the assumption that $\tau$ is a subset of the set $\{1,2,\ldots,m\}$.

We fix $r$ with $m + 1 \leq r \leq Z$ and set, for $1 \leq i \leq l + 1$,

$$G_i(\tau_1,\tau_2,\{r\}) = G(\tau_1,\tau_2 \cup \{r\} \setminus \{g_i\}).$$

We denote by $G_{i,sp}(\tau_1,\tau_2)$ the result of substituting in $G_i(\tau_1,\tau_2,\{r\})$ the value 1 for the variable $a_{1,r}$ and the value 0 for the variables $a_{j,r}$, for $2 \leq j \leq n + 1$. We remark that $G_{i,sp}(\tau_1,\tau_2)$ is well-defined, since the denominator of $G_i(\tau_1,\tau_2,\{r\})$ does not vanish when we perform the substitution.

Moreover, we denote by $T_{n+1} : k_2 \rightarrow k_2$ the $(n + 1)$-th order differential operator which is differentiation with respect to the set of variables

$$\{a_{1,c_1}, a_{2,c_1}, a_{3,c_2}, a_{4,c_2}, a_{5,c_3}, a_{6,c_3}, \ldots, a_{n,c_1}, a_{n+1,c_1}\}.$$

Remark 6.1. This set of variables can also be described as the set

$$\{a_{i,c_j} : 1 \leq i \leq n + 1, j = [(i + 1)/2]\},$$

where $[x]$ denotes the integral part of the real number $x$. For example, if $n = 3$, then

$$T_{n+1} = \frac{\partial^4}{\partial a_{1,c_1} \partial a_{2,c_1} \partial a_{3,c_2} \partial a_{4,c_2}}.$$

We remind the reader that the field $k_1$ has characteristic 2.
Proposition 6.2. We have the following equality in the field \( k_2 \)

\[
G(\tau_1, \tau_2) = \sum_{i=1}^{l+1} G_i(\tau_1, \tau_2, \{r\}).
\]

Proof. Denote by \( D \) the boundary complex of the simplex of dimension \( n+1 \) with vertex set \( \tau \). By Proposition 4.12 we have

\[
(\Psi \circ \pi)(\prod_{i=1}^{l} x_i^{2}) = G(\tau_1, \tau_2).
\]

Since \( G_i(\tau_1, \tau_2, \{r\}) = H(\tau_1, \emptyset, \{g_i\}) \), by Theorem 4.14 we have

\[
(\Psi \circ \pi)(\prod_{i=1}^{l} x_i^{2}) = \sum_{i=1}^{l+1} G_i(\tau_1, \tau_2, \{r\}).
\]

The result follows. \( \square \)

For an example related to the above Proposition 6.2 see Example 4.18.

The following corollary follows immediately from Proposition 6.2, by taking into account that, for all \( 1 \leq j \leq n+1 \), the variable \( a_{i,r} \) does not appear in \( G(\tau_1, \tau_2) \).

Corollary 6.3. We have the following equality in the field \( k_2 \)

\[
G(\tau_1, \tau_2) = \sum_{i=1}^{l+1} G_{\text{sp}}(\tau_1, \tau_2).
\]

The following proposition is an immediate corollary of Part 1 of Theorem 8.6. For simplicity of notation, for \( i \in \tau \) we set \( M_i = M(\tau \setminus \{i\}) \).

Proposition 6.4. We have the following equality in the field \( k_2 \)

\[
T_{n+1} \left( \prod_{i \in \tau} M_i \right) = \prod_{i \in \tau_1} (M_i)^2.
\]

Corollary 6.5. Assume \( S \) is a subset of \( \tau \). We then have the following equality in the field \( k_2 \)

\[
T_{n+1} \left( \frac{\prod_{i \in S} M_i}{\prod_{i \in \tau \setminus S} M_i} \right) = \frac{\prod_{i \in S \cap \tau_1} (M_i)^2}{\prod_{i \in \tau_2 \setminus S} (M_i)^2}.
\]

Proof. Using Proposition 6.3 and Remark 8.1, we have

\[
T_{n+1} \left( \frac{\prod_{i \in S} M_i}{\prod_{i \in \tau \setminus S} M_i} \right) = T_{n+1} \left( \frac{\prod_{i \in S} M_i}{\prod_{i \in \tau \setminus S} M_i} \right)^2 = \frac{T_{n+1}(\prod_{i \in S} M_i)}{(\prod_{i \in \tau \setminus S} M_i)^2} = \frac{\prod_{i \in S \cap \tau_1} (M_i)^2}{\prod_{i \in \tau \setminus S} (M_i)^2} = E \cdot \frac{\prod_{i \in S \cap \tau_1} (M_i)^2}{E \cdot \prod_{i \in \tau_2 \setminus S} (M_i)^2},
\]

where \( E = \prod_{i \in \tau_1 \setminus S} (M_i)^2 \). The result follows. \( \square \)

7. The Differential Operator for \( n \) Even

The aim of the present section is to establish, in conjunction with the previous Section 6 and the following Section 8, the results about the differential operators that were used in Section 5.

In the present section we work over a field \( k_1 \) of characteristic 2.

Assume \( n \geq 1 \) is even and \( m \) is an integer with \( m \geq n+1 \). We set \( Z = m+2n \) and denote by \( M \) the \((n+1) \times Z\) matrix whose \((i,j)\)-entry is equal to the variable \( a_{i,j} \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq Z \). Given a subset \( A \) of the set \( \{1,2,\ldots,Z\} \) of cardinality \( n+1 \), we denote by \( M(A) \) the determinant of the \((n+1) \times (n+1)\) submatrix of \( M \) obtained by keeping the columns of \( M \) specified by the set \( A \).
We denote by $k_2$ the field of fractions of the polynomial ring
\[ k_1[a_{i,j} : 1 \leq i \leq n + 1, 1 \leq j \leq Z]. \]

We set $l = n/2$. Assume
\[ \tau_1 = \{c_1, \ldots, c_l\}, \quad \tau_2 = \{b\}, \quad \tau_3 = \{g_1, \ldots, g_{i+1}\} \]
are three subsets of the set \{1, 2, \ldots, Z\} such that $\bigcup_{i=1}^3 \tau_i$ has cardinality $2l + 2$. We set $\tau = \bigcup_{i=1}^3 \tau_i$ and
\[ G(\tau_1, \tau_2, \tau_3) = \prod_{i=1}^l M(\tau \setminus \{c_i\}) \prod_{i=1}^{l+1} M(\tau \setminus \{g_i\}) \]

For the rest of this section we make the assumption that $\tau$ is a subset of the set \{1, 2, \ldots, m\}. We fix $r$ with $m + 1 \leq r \leq Z$ and set, for $1 \leq i \leq l + 1$, \[ G_i(\tau_1, \tau_2, \tau_3, \{r\}) = G(\tau_1, \tau_2, (\tau_3 \cup \{r\}) \setminus \{g_i\}). \]

We denote by $G_{i}^{sp}(\tau_1, \tau_2, \tau_3)$ the result of substituting in $G_i(\tau_1, \tau_2, \tau_3, \{r\})$ the value 1 for the variable $a_{1,r}$ and the value 0 for the variables $a_{j,r}$, for $2 \leq j \leq n + 1$. We remark that $G_{i}^{sp}(\tau_1, \tau_2, \tau_3)$ is well-defined, since the denominator of $G_i(\tau_1, \tau_2, \tau_3, \{r\})$ does not vanish when we perform the substitution.

Moreover, we denote by $T_{n+1} : k_2 \to k_2$ the $(n+1)$-th order differential operator which is differentiation with respect to the set of variables
\[ \{a_{1,b}, a_{2,c_1}, a_{3,c_1}, a_{4,c_2}, a_{5,c_2}, \ldots, a_{n,c_1}, a_{n+1,c_i}\}. \]

**Remark 7.1.** This set of variables can also be described as the set
\[ \{a_{1,b}\} \cup \{a_{i,c_j} : 2 \leq i \leq n + 1, j = [i/2]\}, \]
where $[x]$ denotes the integral part of the real number $x$. For example, if $n = 2$, then
\[ T_{n+1} = \frac{\partial^3}{\partial a_{1,b} \partial a_{2,c_1} \partial a_{3,c_1}}. \]

We remind the reader that the field $k_1$ has characteristic 2.

**Proposition 7.2.** We have the following equality in the field $k_2$
\[ G(\tau_1, \tau_2, \tau_3) = \sum_{i=1}^{l+1} G_i(\tau_1, \tau_2, \tau_3, \{r\}). \]

**Proof.** Denote by $D$ the boundary complex of the simplex of dimension $n + 1$ with vertex set $\tau$. By Proposition 4.13 we have
\[ (\Psi \circ \pi)(x_b \prod_{i=1}^l x_{c_i}^2) = G(\tau_1, \tau_2, \tau_3). \]

Since $G_i(\tau_1, \tau_2, \tau_3, \{r\}) = H(\tau_1, \tau_2, \tau \setminus \{g_i\})$, by Theorem 4.14 we have
\[ (\Psi \circ \pi)(x_b \prod_{i=1}^l x_{c_i}^2) = \sum_{i=1}^{l+1} G_i(\tau_1, \tau_2, \tau_3, \{r\}). \]

The result follows. \qed

The following corollary follows immediately from Proposition 7.2 by taking into account that, for all $1 \leq j \leq n + 1$, the variable $a_{j,r}$ does not appear in $G(\tau_1, \tau_2, \tau_3)$.

**Corollary 7.3.** We have the following equality in the field $k_2$
\[ G(\tau_1, \tau_2, \tau_3) = \sum_{i=1}^{l+1} G_i^{sp}(\tau_1, \tau_2, \tau_3). \]
The following proposition is an immediate corollary of Part 2 of Theorem 8.6. For simplicity of notation, for \( i \in \tau_1 \cup \tau_3 \) we set \( M_i = M(\tau \setminus \{i\}) \).

**Proposition 7.4.** We have the following equality in the field \( k_2 \)

\[
T_{n+1} \left( \prod_{i \in \tau_1 \cup \tau_3} M_i \right) = \prod_{i \in \tau_1} (M_i)^2.
\]

**Corollary 7.5.** Assume \( S \) is a subset of \( \tau_1 \cup \tau_3 \). We then have the following equality in the field \( k_2 \)

\[
T_{n+1} \left( \prod_{i \in S} M_i \prod_{i \in (\tau_1 \cup \tau_3) \setminus S} M_i \right) = \prod_{i \in \tau_1 \cap \tau_3 \setminus S} (M_i)^2.
\]

**Proof.** We set \( w = \tau_1 \cup \tau_3 \). Using Proposition 7.4 and Remark 8.1 we have

\[
T_{n+1} \left( \prod_{i \in S} M_i \right) = \frac{T_{n+1} \left( \prod_{i \in w \setminus S} M_i \right)}{\prod_{i \in (\tau_1 \cup \tau_3) \setminus S} (M_i)^2} = \frac{\prod_{i \in \tau_1} (M_i)^2}{\prod_{i \in w \setminus S} (M_i)^2} = \frac{E \cdot \prod_{i \in \tau_1 \cap \tau_3 \setminus S} (M_i)^2}{E \cdot \prod_{i \in \tau_1 \setminus S} (M_i)^2}.
\]

where \( E = \prod_{i \in \tau_1 \setminus S} (M_i)^2 \). The result follows. \( \square \)

8. Some useful characteristic 2 identities

The aim of the present section is to establish, in conjunction with the previous two Sections 6 and 7, the results about the differential operators that were used in Section 5.

In the present section we work over a field \( k_1 \) of characteristic 2.

Assume \( h \geq 2 \) is an integer. We denote by \( k \) the field of fractions of the polynomial ring

\[ k_1[a_{i,j} : 1 \leq i < h + 2, 1 \leq j < h + 1] \]

We denote by \( M^{big} \) the \((h + 2) \times (h + 1)\) matrix whose \((i, j)\)-entry is equal to the variable \( a_{i,j} \), for \( 1 \leq i < h + 2 \) and \( 1 \leq j < h + 1 \).

Assume \( h \geq 2 \) is even. We denote by \( N^{(h)} \) the \( h \times (h + 1) \) submatrix of \( M^{big} \), obtained by keeping the rows indexed by \( 1, 2, \ldots, h \). We denote by \( P^{(h)} \) the \( h \times (h + 1) \) submatrix of \( M^{big} \), obtained by keeping the rows indexed by \( 3, 4, \ldots, h + 2 \). We define the following two sets of variables

\[ \mathcal{A}_{N,h} = \{a_{i,j} : 1 \leq i \leq h, j = [(i + 1)/2]\}, \quad \mathcal{A}_{P,h} = \{a_{i,j} : 3 \leq i \leq h + 2, j = [(i + 1)/2]\} \]

where \([x]\) denotes the integral part of the real number \( x \). For \( S \in \{N, P\} \), we denote by \( T_{S,h} \) the \( h \)-th order differential operator which is partial differentiation with respect to the variables in the set \( \mathcal{A}_{S,h} \).

Assume \( h \geq 3 \) is odd. We denote by \( Q^{(h)} \) the \( h \times (h + 1) \) submatrix of \( M^{big} \), obtained by keeping the rows indexed by \( 2, 3, \ldots, h + 1 \). We define the following set of variables

\[ \mathcal{A}_{Q,h} = \{a_{2,1}\} \cup \{a_{i,j} : 3 \leq i \leq h + 1, j = [(i + 1)/2]\}. \]

We denote by \( T_{Q,h} \) the \( h \)-th order differential operator which is partial differentiation with respect to the variables in the set \( \mathcal{A}_{Q,h} \).

In the present section we will use the following notational convention. Assume \( l \geq 1 \), \( S \) is an \( l \times (l + 1) \) matrix and \( 1 \leq i \leq l + 1 \). We will denote by \( S_i \) the determinant of the \( l \times l \) submatrix of \( S \) obtained by deleting the \( i \)-th column of \( S \).

**Remark 8.1.** We will use that, since the field \( k_1 \) has characteristic 2, we have

\[ T_{S,h}(f^2 g) = f^2 T_{S,h}(g) \]

for all \( f, g \in k \), \( S \in \{N, P, Q\} \) and \( h \geq 2 \) as above (that is, \( h \) even if \( S = N \) or \( S = P \) and \( h \) odd if \( S = Q \)). Indeed, by the Leibnitz Rule,

\[
\frac{\partial}{\partial a_{i,j}} (f^2 g) = g \frac{\partial}{\partial a_{i,j}} (f^2) + f^2 \frac{\partial}{\partial a_{i,j}} (g) = 2gf \frac{\partial}{\partial a_{i,j}} (f) + f^2 \frac{\partial}{\partial a_{i,j}} (g) = f^2 \frac{\partial}{\partial a_{i,j}} (g),
\]
and $T_{S,h}$ is a composition of such operators. Consequently, if $f, g \in k$ with $g \neq 0$, then
\[
T_{S,h}(\frac{f}{g}) = T_{S,h}(\frac{fg}{g^2}) = \frac{T_{S,h}(fg)}{g^2}.
\]

**Proposition 8.2.** Assume that $h \geq 2$ is even and that
\[
T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{(h/2)} (N_i^{(h)})^2.
\]

We then have
\[
T_{P,h}(\prod_{i=1}^{h+1} P_i^{(h)}) = \prod_{i=2}^{(h+2)/2} (P_i^{(h)})^2.
\]

**Proof.** We denote by $N^{mod}$ the matrix obtained from $N^{(h)}$ by putting the last column of $N^{(h)}$ first. Taking into account that the field $k_1$ has characteristic 2, we get $N_1^{mod} = N_{h+1}^{(h)}$ and that
\[
N_i^{mod} = N_{i-1}^{(h)},
\]
for all $2 \leq i \leq h + 1$. Hence, using the assumption we have
\[
(11) \quad T_{N,h}(\prod_{i=1}^{h+1} N_i^{mod}) = T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2 = \prod_{i=1}^{h/2} (N_i^{mod})^2 = \prod_{i=2}^{(h+2)/2} (N_i^{(h)})^2.
\]

We have that both $N^{mod}$ and $P^{(h)}$ are $h \times (h + 1)$ matrices. The entries of each matrix are independent indeterminates. For $1 \leq i \leq h$ and $1 \leq j \leq h + 1$, we denote by $n_{i,j}$ the $(i,j)$-entry of $N^{mod}$ and by $p_{i,j}$ the $(i,j)$-entry of $P^{(h)}$. By definition, $T_{N,h}$ is differentiation with respect to the variables in the set
\[
\{n_{i,j} : 1 \leq i \leq h, j = 1 + [(i + 1)/2]\},
\]
while $T_{P,h}$ is differentiation with respect to the variables in the set
\[
\{p_{i,j} : 1 \leq i \leq h, j = 1 + [(i + 1)/2]\}.
\]

There exists a unique isomorphism of $k_1$-algebras $\phi : k_1[n_{i,j}] \rightarrow k_1[p_{i,j}]$ such that $\phi(n_{i,j}) = p_{i,j}$ for all $1 \leq i \leq h$ and $1 \leq j \leq h + 1$. As a consequence, the result follows from Equation (11). \qed

**Proposition 8.3.** Assume $h = 2$. We have
\[
T_{N,2}(N_1^{(2)} N_2^{(2)} N_3^{(2)}) = (N_1^{(2)})^2 \quad \text{and} \quad T_{P,2}(P_1^{(2)} P_2^{(2)} P_3^{(2)}) = (P_2^{(2)})^2.
\]

**Proof.** Using Proposition 8.2 it is enough to prove only the first equality. We have
\[
T_{N,2} = \frac{\partial^2}{\partial a_{1,1} \partial a_{2,1}}
\]
and
\[
N_1^{(2)} = \det \begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix}, \quad N_2^{(2)} = \det \begin{pmatrix} a_{1,1} & a_{1,3} \\ a_{2,1} & a_{2,3} \end{pmatrix}, \quad N_3^{(2)} = \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}.
\]
The result follows by an easy direct computation, taking into account that the field $k_1$ has characteristic 2. \qed

**Remark 8.4.** It is easy to see that the assumption that the field $k_1$ has characteristic 2 is crucial in order to have the equalities in the statement of Proposition 8.3.

**Proposition 8.5.** 1) Assume $h \geq 4$ is even and that
\[
T_{P,h-2}(\prod_{i=1}^{h-1} P_i^{(h-2)}) = \prod_{i=2}^{h/2} (P_i^{(h-2)})^2.
\]
We then have
\[ T_{Q,h-1}(\prod_{i=2}^{h} Q_i^{(h-1)}) = \prod_{i=2}^{h/2} (Q_i^{(h-1)})^2. \]

2) Assume \( h \geq 4 \) is even and that
\[ T_{Q,h-1}(\prod_{i=2}^{h} Q_i^{(h-1)}) = \prod_{i=2}^{h/2} (Q_i^{(h-1)})^2. \]

We then have
\[ T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2. \]

**Proof.** We fix an even integer \( h \geq 4 \). For simplicity of notation, we set
\[ N = N^{(h)}, \quad T_N = T_{N,h}, \quad Q = Q^{(h-1)}, \quad T_Q = T_{Q,h-1}, \quad P = P^{(h-2)}, \quad T_P = T_{P,h-2}. \]

We first prove Part 1). We set
\[ W = \frac{\prod_{i=2}^{h-1} Q_i}{Q_h \prod_{i=2}^{h/2} Q_i}. \]

Using Remark 8.1, it is enough to prove that
\[ T_Q(W) = 1/(Q_h)^2. \]

Denote by \( r_1 \) the transpose of the \( 1 \times (h-1) \) matrix \( (1, 0, 0, \ldots, 0) \). For \( s \in \{2, 3, \ldots, h/2\} \cup \{h\} \) we denote by \( U^{<s>} \) the \( (h-1) \times h \) matrix obtained by replacing the \( s \)-th column of \( Q \) with \( r_1 \).

We set, for \( s \in \{2, 3, \ldots, h/2\} \cup \{h\} \),
\[ W^{<s>} = \frac{\prod_{j=(h+2)/2}^{h-1} U_j^{<s>}}{U_h^{<s>} \prod_{j=2}^{h/2} U_j^{<s>}}. \]

Since \( U_j^{<s>} \neq 0 \), for all \( 1 \leq j \leq h \), we have that \( W^{<s>} \) is well-defined. By Corollary 7.3 we have
\[ W = W^{<h>} + \sum_{s=2}^{h/2} W^{<s>}. \]

For simplicity, we set \( B = U^{<h>} \). Assume \( s \in \{2, 3, \ldots, h/2\} \). We have \( T_Q(W^{<s>}) = 0 \), since the variable \( a_{2s,s} \) is an element of \( A_{Q,h-1} \) but does not appear in \( W^{<s>} \). As a consequence, we have
\[ T_Q(W) = T_Q(W^{<h>}). \]

Hence, using that \( Q_h = B_h \), to prove Equation (12) it is enough to prove that
\[ T_Q(W^{<h>}) = 1/(B_h)^2. \]

Taking into account Remark 8.1 it follows that to prove Equation (12) it is enough to prove that
\[ T_Q(\prod_{i=2}^{h} B_i) = \prod_{i=2}^{h/2} (B_i)^2. \]

Using the definition of \( r_1 \), we get that \( B_i = P_i \) for all \( 1 \leq i \leq h-1 \). We set
\[ K = B_h - a_{2,1}P_1. \]

By the well-known formula for the development of the determinant \( B_h \) using the row containing the element \( a_{2,1} \) it follows that the variable \( a_{2,1} \) does not appear in \( K \). Consequently,
the differential operator $\frac{\partial}{\partial a_{j}}$ annihilates $K$. Since the same operator annihilates $P_j$ for all $1 \leq j \leq h$, and $T_Q = T_P \circ \frac{\partial}{\partial a_{j}}$, we get

$$T_Q \left( \prod_{i=2}^{h} B_i \right) = T_Q \left( B_h \prod_{i=2}^{h-1} B_i \right) = T_Q \left( a_{2,1} P_1 \prod_{i=2}^{h-1} P_i \right) = T_P \left( P_1 \prod_{i=2}^{h-1} P_i \right) = \prod_{i=2}^{h} (P_i)^2,$$

with the last equality by the assumption for Part 1). Since, for $1 \leq i \leq h - 1$, we have $B_i = P_i$, Equality (13) follows, which finishes the proof of Part 1).

**EXAMPLE** (to help understand the above proof of Part 1): If $h = 4$, we have

$$Q = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}, \quad B = U^{<h>} = \begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} & 1 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}$$

and

$$T_Q = \frac{\partial^2}{\partial a_{2,1} \partial a_{3,2} \partial a_{4,2}}, \quad T_P = \frac{\partial^2}{\partial a_{3,2} \partial a_{4,2}}.$$

We now prove Part 2) using similar arguments to the ones used in the proof of Part 1). We set

$$W = \frac{\prod_{i=2}^{h} N_i}{N_{h+1} \prod_{i=1}^{h/2} N_i}.$$

Using Remark 8.1, it is enough to prove that

$$(14) \quad T_N(W) = 1/(N_{h+1})^2.$$

Denote by $r_2$ the transpose of the $1 \times h$ matrix $(1, 0, 0, \ldots, 0)$. For $s \in \{1, 2, \ldots, h/2\} \cup \{h + 1\}$ we denote by $X^{<s>}$ the $h \times (h + 1)$ matrix obtained by replacing the $s$-th column of $N$ with $r_2$.

We set, for $s \in \{1, 2, \ldots, h/2\} \cup \{h + 1\}$,

$$W^{<s>} = \frac{\prod_{j=2}^{h} X^{<s>}}{X^{<s>} \prod_{j=1}^{h/2} X^{<s>}^{<j>}}.$$

Since $X^{<j>}_{<s>} \neq 0$, for all $1 \leq j \leq h + 1$, we have that $W^{<s>}$ is well-defined. By Corollary 6.3 we have

$$W = W^{<h+1>} + \sum_{s=1}^{h/2} W^{<s>}.$$

For simplicity, we set $C = X^{<h+1>}$. Assume $s \in \{1, 2, \ldots, h/2\}$. We have $T_N(W^{<s>}) = 0$, since the variable $a_{2,s}$ is an element of $A_{N,h}$ but does not appear in $W^{<s>}$. As a consequence, we have

$$T_N(W) = T_N(W^{<h+1>}).$$

Hence, using that $N_{h+1} = C_{h+1}$, to prove Equation (14) it is enough to prove that

$$T_N(W^{<h+1>}) = 1/(C_{h+1})^2.$$
Taking into account Remark 8.1, it follows that to prove Equation (14) it is enough to prove that
\[
T_N\left(\prod_{i=1}^{h+1} C_i\right) = \prod_{i=1}^{h/2} (C_i)^2.
\]

Using the definition of \(r_2\), we get that \(C_i = Q_i\) for all \(1 \leq i \leq h\). We set
\[
K = C_{h+1} - a_{1,1}Q_1.
\]

By the well-known formula for the development of the determinant \(C_{h+1}\) using the row containing the element \(a_{1,1}\) it follows that the variable \(a_{1,1}\) does not appear in \(K\). Consequently, the differential operator \(\frac{\partial}{\partial a_{1,1}}\) annihilates \(K\). Since the same operator annihilates \(Q_j\) for all \(1 \leq j \leq h\), and \(T_N = T_Q \circ \frac{\partial}{\partial a_{1,1}}\), we get
\[
T_N\left(\prod_{i=1}^{h+1} C_i\right) = T_N\left(C_{h+1} \prod_{i=1}^{h} C_i\right) = T_N\left(a_{1,1}Q_1 \prod_{i=1}^{h} Q_i\right) = T_N\left(a_{1,1}(Q_1)^2 \prod_{i=2}^{h} Q_i\right)
\]
\[
= T_Q((Q_1)^2 \prod_{i=2}^{h} Q_i) = (Q_1)^2 \prod_{i=2}^{h} (Q_i)^2.
\]

EXAMPLE (to help understand the above proof of Part 2): If \(h = 4\), we have
\[
N = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5}
\end{pmatrix}, \quad C = X^{<h+1>} = \begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & 1 \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & 0 \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & 0 \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & 0
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4}
\end{pmatrix}
\]

and
\[
T_N = \frac{\partial^4}{\partial a_{1,1} \partial a_{2,1} \partial a_{3,2} \partial a_{4,2}}, \quad T_Q = \frac{\partial^3}{\partial a_{2,1} \partial a_{3,2} \partial a_{4,2}}.
\]

\[\square\]

**Theorem 8.6.**

1) Assume \(h \geq 2\) is even. We have
\[
T_{N,h}(\prod_{i=1}^{h+1} N_i^{(h)}) = \prod_{i=1}^{h/2} (N_i^{(h)})^2.
\]

2) Assume \(h \geq 3\) is odd. We have
\[
T_{Q,h}(\prod_{i=2}^{h+1} Q_i^{(h)}) = \prod_{i=2}^{(h+1)/2} (Q_i^{(h)})^2.
\]

**Proof.** It is obvious that Part 2) is equivalent to the statement that for all even integers \(h \geq 4\) we have
\[
T_{Q,h-1}(\prod_{i=2}^{h} Q_i^{(h-1)}) = \prod_{i=2}^{h/2} (Q_i^{(h-1)})^2.
\]

Using induction on the even integer \(h \geq 2\), the proof of the present theorem follows by combining Proposition 8.3 which provides the starting case \(h = 2\), and Propositions 8.2 and 8.5 which provide the inductive step. \[\square\]
Remark 8.7. Conjecture 14.1 contains a conjectural statement generalising Theorem 8.6.

9. Anisotropicity implies the Lefschetz Properties

In the present section we investigate the relations between generic anisotropicity and the Lefschetz properties. As an application, in Theorem 9.1 we give a second proof of McMullen’s g-conjecture for simplicial spheres.

Assume $k_1$ is a field of arbitrary characteristic, $n \geq 1$ is an integer, and $D$ is a simplicial sphere of dimension $n$ and vertex set $\{1, 2, \ldots, m\}$.

We denote by $S(D)$ the suspension of $D$. More precisely, it is the simplicial complex with vertex set $\{1, 2, \ldots, m+2\}$ and set of facets equal to

$$\{\sigma \cup \{x_{m+1}\} : \sigma \in F(D)\} \cup \{\sigma \cup \{x_{m+2}\} : \sigma \in F(D)\},$$

where $F(D)$ denotes the set of facets of $D$. It is well-known that $S(D)$ is a simplicial sphere of dimension $n+1$. Moreover, we denote by $k$ the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n+2, 1 \leq j \leq m+2].$$

The proof of the following theorem will be given in Subsection 9.1.

Theorem 9.1. Assume that $S(D)$ is generically anisotropic over the field $k_1$. Then the graded $k$-algebra $k[D]$ has the Weak Lefschetz Property.

All three statements in the following theorem are results originally due to Adiprasito [1]. The proof of the theorem will be given in Subsection 9.2.

Theorem 9.2. (Adiprasito) Assume $D$ is a simplicial sphere of dimension $n$, with $n \geq 1$. Then

i) McMullen’s g-conjecture is true for $D$.

ii) Assume $k_1$ is an infinite field of characteristic 2. Then the Stanley-Reisner ring $k_1[D]$ has the Weak Lefschetz Property.

iii) Assume $k_1$ is an infinite field of characteristic 2. Then the Stanley-Reisner ring $k_1[D]$ has the Strong Lefschetz Property.

Remark 9.3. It is well-known that iii) implies ii). We state both ii) and iii), since in our approach we first prove ii) and then use it to establish iii). Notice also that the paper [1] contains the stronger result that for any infinite field $k_1$ of arbitrary characteristic the Stanley-Reisner ring $k_1[D]$ has the Strong Lefschetz Property.

9.1. The proof of Theorem 9.1. The aim of the present subsection is to prove Proposition 9.3, since it immediately implies Theorem 9.1. We use some key ideas and results of Swartz, which were developed in [17] Section 4.

We keep using the notations defined in Section 9. In addition, we set $R_{sm} = k[x_1, \ldots, x_m]$ and $R = R_{sm}[x_{m+1}, x_{m+2}]$. We denote by $I_D \subset R_{sm}$ the Stanley-Reisner ideal of $D$ over the field $k$ and by $I_{S(D)} \subset R$ the Stanley-Reisner ideal of $S(D)$ over the same field $k$. It is clear that

$$I_{S(D)} = (I_D) + (x_{m+1}x_{m+2}).$$

We denote by $k[D] = R_{sm}/I_D$ and $k[S(D)] = R/I_{S(D)}$ the corresponding Stanley-Reisner rings over $k$.

For $1 \leq i \leq n+2$, we set

$$f_i = \sum_{j=1}^{m+2} a_{i,j}x_j \in R.$$

We use the notation $A = k[S(D)]/(f_1, \ldots, f_{n+2})$, and denote by $\pi_A : R \to A$ the natural projection $k$-algebra homomorphism. Therefore, $A$ is the generic Artinian reduction of $k_1[S(D)]$ in the sense of Definition 2.2.

We set $J = I_{S(D)} : (x_{m+1}) \subset R$. In other words,

$$J = \{u \in R : ux_{m+1} \in I_{S(D)}\}.$$
It is clear that \( J = (I_D) + (x_{m+2}) \). We use the notation
\[
B = \frac{R}{J + (f_1, f_2, \ldots, f_{n+2})},
\]
and we denote by \( \pi_B : R \to B \) the natural projection \( k \)-algebra homomorphism.

For \( 2 \leq i \leq n + 2 \) and \( 1 \leq j \leq m \), we set
\[
c_{i,j} = \det \begin{pmatrix} a_{1,j} & a_{1,m+1} \\ a_{i,j} & a_{i,m+1} \end{pmatrix} \in k.
\]
In addition, for \( 2 \leq i \leq n + 2 \), we set
\[
g_i = \sum_{j=1}^{m} c_{i,j} x_j \in R_{sm}.
\]
Since, for all \( 2 \leq i \leq n + 2 \), it holds
\[
g_i = a_{i,m+1} f_1 - a_{1,m+1} f_i + x_{m+2}(a_{1,m+1} a_{i,m+2} - a_{i,m+1} a_{1,m+2}),
\]
we get the following equality of ideals of \( R \)
\[
(f_1, f_2, \ldots, f_{n+2}) + (x_{m+2}) = (f_1) + (g_2, g_3, \ldots, g_{n+2}) + (x_{m+2}).
\]

We use the notation \( C = k[D]/(g_2, g_3, \ldots, g_{n+2}) \), and denote by \( \pi_C : R_{sm} \to C \) the natural projection \( k \)-algebra homomorphism. We set
\[
\omega = -\sum_{i=1}^{m} \frac{a_{1,i}}{a_{1,m+1}} x_i \in R_{sm}.
\]
It is clear that \( \pi_B(\omega) = \pi_B(x_{m+1}) \). We consider the unique \( k \)-algebra homomorphism \( \phi_{\text{mod}} : R \to C \), such that \( \phi_{\text{mod}}(x_i) = \pi_C(x_i) \) for all \( 1 \leq i \leq m \), \( \phi_{\text{mod}}(x_{m+1}) = \pi_C(\omega) \) and \( \phi_{\text{mod}}(x_{m+2}) = 0 \). From the definition of \( \omega \) it follows that \( f_1 \in \ker \phi_{\text{mod}} \). Hence, Equation (16) implies that the ideal \( J + (f_1, \ldots, f_{n+2}) \) is contained in the kernel of \( \phi_{\text{mod}} \). Consequently, there exists an induced \( k \)-algebra homomorphism \( \phi : B \to C \) such that \( \phi \circ \pi_B = \phi_{\text{mod}} \).

**Proposition 9.4.** The map \( \phi \) is an isomorphism of graded \( k \)-algebras.

**Proof.** It is clear from the definition that \( \phi \) preserves degrees. We consider the unique \( k \)-algebra homomorphism \( R_{sm} \to B \), that sends \( x_i \) to \( \pi_B(x_i) \), for all \( 1 \leq i \leq m \). Using Equation (16), it follows that the ideal \( I_D + (g_2, g_3, \ldots, g_{n+2}) \) of \( R_{sm} \) is inside its kernel, hence there exists an induced \( k \)-algebra homomorphism \( \psi : C \to B \). It follows from the definitions that \( \psi \) is the inverse map of \( \phi \).

**Proposition 9.5.** i) The \( k \)-algebra \( A \) is graded, Artinian and Gorenstein with socle degree equal to \( n + 2 \).

ii) The \( k \)-algebras \( B \) and \( C \) are graded, Artinian and Gorenstein with socle degree equal to \( n + 1 \).

**Proof.** We first remark that, by Proposition 9.4, the graded \( k \)-algebras \( B \) and \( C \) are isomorphic.

By Remark 3.1, the \( k \)-algebra \( k[S(D)] \) is graded and Gorenstein with Krull dimension equal to \( n + 2 \). Moreover, by the same remark \( A \) is Artinian and Gorenstein with socle degree equal to \( n + 2 \).

Since \( I_D \subseteq J \), there exists a unique surjective homomorphism of \( k \)-algebras \( \pi_{\text{new}} : A \to B \), such that \( \pi_{\text{new}} \circ \pi_A = \pi_B \). Since \( \pi_{\text{new}} \) is surjective and \( A \) is Artinian, it follows that \( B \) is Artinian. Since \( C \) is isomorphic to \( B \) we get that \( C \) is also Artinian. By Remark 3.1 the \( k \)-algebra \( k[D] \) is graded and Gorenstein with Krull dimension equal to \( n + 1 \). It follows that the sequence \( g_2, \ldots, g_{n+2} \) is a regular sequence for \( k[D] \). This implies that the \( k \)-algebra \( C \) is Gorenstein and, using again Remark 3.1 that the socle degree of \( C \) is equal to \( n + 1 \). □
We consider the homomorphism of \( R \)-modules \( R \to A \), that sends \( u \) to \( \pi_A(x_{m+1}u) \), for all \( u \in R \). It is clear that the ideal \( J + (f_1, \ldots, f_{n+2}) \) of \( R \) is inside its kernel. Hence, we get an induced homomorphism of \( R \)-modules \( m_{x_{m+1}} : B \to A \), such that
\[
m_{x_{m+1}}(\pi_B(u)) = \pi_A(x_{m+1}u)
\]
for all \( u \in R \). The following proposition is a special case of \cite[Proposition 4.24]{17}.

**Proposition 9.6.** (Swartz) The homomorphism \( m_{x_{m+1}} \) is injective.

**Proof.** Recall the map \( \psi \) defined in the proof of Proposition 9.4. We set
\[
\delta = m_{x_{m+1}} \circ \psi : C \to A.
\]
Since \( \psi \) is an isomorphism, it is enough to prove that \( \delta \) is injective.

Since, for all \( j \geq 0 \), we have \( \delta(C_j) \subset A_{j+1} \), to prove that \( \delta \) is injective it is enough to assume that \( 0 \leq j \leq n+1 \) and \( u \in (R_{sm})_j \) is a homogeneous element of degree \( j \) such that \( \pi_C(u) \neq 0 \), and prove that \( \delta(\pi_C(u)) \neq 0 \). In order to get a contradiction, we assume that \( \delta(\pi_C(u)) = 0 \).

By Proposition 9.5, \( C \) is a graded Artinian Gorenstein \( k \)-algebra with socle degree \( n+1 \). Therefore, by Remark 2.1 there exists \( w \in (R_{sm})_{n+1-j} \) such that \( \pi_C(uw) \neq 0 \). Using Equation \( 17 \)
\[
\delta(\pi_C(uw)) = \pi_A(x_{m+1}uw) = \pi_A(x_{m+1}u)\pi_A(w) = \delta(\pi_C(u))\pi_A(w) = 0.
\]

We fix a facet \( \{a_1, \ldots, a_{n+1}\} \) of \( D \) and consider the facet \( \{a_1, \ldots, a_{n+1}, m+1\} \) of \( S(D) \). We set
\[
z_C = \prod_{r=1}^{n+1} x_{ar} \in R_{sm}, \quad z_A = x_{m+1} \prod_{r=1}^{n+1} x_{ar} \in R.
\]
Using the discussion after the proof of Corollary \( 15 \), \( \pi_A(z_A) \) is a nonzero element of \( A_{n+2} \).

By the same discussion, \( \pi_C(z_C) \) is nonzero, hence is a basis of the 1-dimensional \( k \)-vector space \( C_{n+1} \). Therefore, there exists a nonzero element \( \lambda \in k \) such that
\[
\pi_C(uw) = \lambda \pi_C(z_C).
\]
Consequently,
\[
\delta(\pi_C(uw)) = \delta(\lambda \pi_C(z_C)) = \lambda \delta(\pi_C(z_C)) = \lambda \pi_A(x_{m+1}z_C) = \lambda \pi_A(z_A) \neq 0,
\]
which contradicts Equation \( 18 \). \( \square \)

The following corollary follows immediately from Proposition 9.6.

**Corollary 9.7.** Assume \( u \in R \). Then the following are equivalent
\begin{enumerate}[i)]
\item We have \( \pi_B(u) = 0 \).
\item We have \( \pi_A(x_{m+1}u) = 0 \).
\end{enumerate}

**Proposition 9.8.** Assume \( S(D) \) is generically anisotropic over the field \( k_1 \). Then the element \( \pi_C(\omega) \) is a Weak Lefschetz element for the Artinian \( k \)-algebra \( C \). As a consequence, the graded \( k \)-algebra \( k[D] \) has the Weak Lefschetz Property.

**Proof.** We denote by \( p \) the integral value of the rational number \( n/2 \).

Using that \( \pi_B(\omega) = \pi_B(x_{m+1}) \) and Proposition 9.4, it is enough to prove that the element \( \pi_B(x_{m+1}) \) is a Weak Lefschetz element for the Artinian \( k \)-algebra \( B \). By Proposition 9.5, \( B \) is a graded Artinian Gorenstein \( k \)-algebra with socle degree \( n+1 \). Using \cite[Remark 2.1]{14}, it is enough to prove that the multiplication by \( \pi_B(x_{m+1}) \) map from \( B_p \) to \( B_{p+1} \) is injective.

Assume \( u \in R_p \) has the property
\[
\pi_B(x_{m+1}u) = 0.
\]
Using Corollary 9.7, we have \( \pi_A(x^2_{m+1}u) = 0 \), hence \( \pi_A(x^2_{m+1}u^2) = 0 \). Using that the socle degree of \( A \) is \( n+2 \) and the assumption that \( S(D) \) is generically anisotropic over the field \( k_1 \), we get \( \pi_A(x_{m+1}u) = 0 \). Corollary 9.7 implies that \( \pi_B(u) = 0 \). \( \square \)
Proposition 9.9. Assume the dimension of $D$ is even and $S(D)$ is generically anisotropic over the field $k_1$. Then the element $\pi_C(\omega)$ is a Strong Lefschetz element for the Artinian $k$-algebra $C$. As a consequence, the graded $k$-algebra $k[D]$ has the Strong Lefschetz Property.

Proof. We set $z = \pi_B(\omega)$. Using Proposition 9.4 it is enough to prove that the element $z$ is a Strong Lefschetz element for the Artinian $k$-algebra $B$. By Proposition 9.5 $B$ is a graded Artinian Gorenstein $k$-algebra with socle degree $n + 1$. Hence, to finish the proof it is enough to prove that, for all $i$ with $0 \leq 2i \leq n + 1$, the multiplication by $z^{n+1-2i}$ map $B_i \to B_{n+1-i}$ is injective.

Assume $0 \leq 2i \leq n + 1$ and $u \in R_i$ has the property

$$z^{n+1-2i}\pi_B(u) = 0.$$ 

Using that $z = \pi_B(x_{m+1})$ and Corollary 9.7 we get $\pi_A(x_{m+1}^{n+2-2i}u) = 0$, which implies that $\pi_A(x_{m+1}^{n+2-2i}u^2) = 0$.

Since $n$ is even, the socle degree of $A$ is $n + 2$ and we assumed that $S(D)$ is generically anisotropic over the field $k_1$, we get $\pi_A(x_{m+1}^{(n+2)/2-i}u) = 0$. Corollary 9.7 implies that $\pi_B(x_{m+1}^{(n+2)/2-i-1}u) = 0$, therefore

$$z^{(n+2)/2-i-1}\pi_B(u) = 0.$$ 

By the proof of Proposition 9.8, $z$ is a Weak Lefschetz element for $B$. Hence, the multiplication by $z$ map $B_{n/2} \to B_{n/2+1}$ is injective. Using Proposition 13.7 we have that, for all $t$ with $0 \leq t \leq n/2$, the multiplication by $z$ map $B_t \to B_{t+1}$ is injective. Consequently, Equation (19) implies that $\pi_B(u) = 0$. \hfill $\square$

9.2. The proof of Theorem 9.2. We start the proof of Theorem 9.2. We denote by $k_{mod}$ the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n + 1, 1 \leq j \leq m].$$

For $1 \leq i \leq n + 1$, we set

$$f_{mod,i} = \sum_{j=1}^{m} a_{i,j}x_j.$$ 

We use the notation

$$A_{mod} = k_{mod}[D]/(f_{mod,1}, \ldots, f_{mod,n+1}).$$

Hence, $A_{mod}$ is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 2.2.

We first prove Part i). We denote by $k_1$ the field $\mathbb{Z}/(2)$ with two elements. By Theorem 5.3 $S(D)$ is generically anisotropic over the field $k_1$. Hence, by Theorem 9.1 $k[D]$ has the Weak Lefschetz Property. It is well-known ([15]) that this implies that McMullen’s g-conjecture is true for $D$.

We now prove Part ii). Assume $k_1$ is an infinite field of characteristic 2. By Theorem 5.3 $S(D)$ is generically anisotropic over the field $k_1$. Hence, by Theorem 9.1 $k[D]$ has the Weak Lefschetz Property. Using Proposition 13.3 $k_1[D]$ also has the Weak Lefschetz Property.

We now prove Part iii). Assume $k_1$ is an infinite field of characteristic 2. By Theorem 5.3 $S(D)$ is generically anisotropic over the field $k_1$. If the dimension $n$ of $D$ is even, Proposition 13.4 implies that $k[D]$ has the Strong Lefschetz Property. Using Proposition 13.6 $k_1[D]$ also has the Strong Lefschetz Property.

Assume now that $n$ is odd. By Part ii), $k_1[D]$ has the Weak Lefschetz Property. Using Proposition 13.4 the Artinian $k_{mod}$-algebra $A_{mod}$ has the Weak Lefschetz Property. By Theorem 5.3 $D$ is generically anisotropic over the field $k_1$. Hence, for all $i$ with $0 \leq i \leq (n + 1)/2$ and all $0 \neq u \in (A_{mod})_i$, we have $u^2 \neq 0$. Proposition 13.8 now implies that $A_{mod}$ has the Strong Lefschetz Property. Since $k_1$ is infinite, Proposition 13.6 implies that $k_1[D]$ has the Strong Lefschetz Property. This finishes the proof of Theorem 9.2.
Corollary 9.10. Assume $D$ is a simplicial sphere of dimension $n \geq 1$, and $k_1$ is a (finite or infinite) field of characteristic 2. Then the $k_{mod}$-algebra $A_{mod}$ has the Strong Lefschetz Property.

Proof. The field $k_{mod}$ is infinite and has characteristic 2. Hence, Theorem 9.2 implies that the $k_{mod}$-algebra $k_{mod}[D]$ has the Strong Lefschetz Property. Using Proposition 10.2, the result follows. $\square$

10. Anisotropicity in dimension 1

In this section $k_1$ denotes a field of arbitrary characteristic.

We assume that $m \geq 3$ and $D$ is the boundary of the $m$-gon with vertex set $\{1, \ldots, m\}$. We also assume the following: the vertex 1 is connected to the vertices $m$ and 2, the vertex $i$ is connected to the vertices $i-1$ and $i+1$ when $2 \leq i \leq m-1$, and the vertex $m$ is connected to the vertices $m-1$ and 1.

We denote by $S_{sp}$ the polynomial ring

$$S_{sp} = k_1[a_{i,j} : 1 \leq i \leq 2, 1 \leq j \leq m]$$

and by $k$ the field of fractions of $S_{sp}$. We define the polynomial ring $R = k[x_1, \ldots, x_m]$. We denote by $I_D \subset R$ the Stanley-Reisner ideal of $D$, and we set $k[D] = R/I_D$. For $1 \leq i \leq 2$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and we define $A = k[D]/(f_1, f_2)$. Therefore, $A$ is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 2.2.

If $m \geq 4$ we have

$$I_D = (x_i x_j : 3 \leq j \leq m-1) + (x_i x_j : 2 \leq i \leq m-2, i+2 \leq j \leq m),$$

while if $m = 3$, we have $I_D = (x_1 x_2 x_3)$.

We fix the ordered facet $(1,2)$ of $D$. Following Equations (1) and (2), we set

$$\Psi = \Psi_{(1,2)} : A_2 \to k \quad \text{and} \quad \rho = \rho_{(1,2)} : A_1 \times A_1 \to k.$$

Proposition 10.1. For $1 \leq i \leq m-1$, we have

$$(\Psi \circ \pi)(x_i x_{i+1}) = \frac{1}{[i,i+1]}.$$

Moreover, we have

$$(\Psi \circ \pi)(x_1 x_m) = \frac{1}{[m,1]}, \quad (\Psi \circ \pi)(x_1^2) = -\frac{[m,2]}{[m,1][1,2]}, \quad (\Psi \circ \pi)(x_m^2) = -\frac{[m-1,1]}{[m-1,m][m,1]},$$

and

$$(\Psi \circ \pi)(x_i^2) = -\frac{[i-1,i+1]}{[i-1,i][i,i+1]}$$

for $2 \leq i \leq m-1$.

Proof. Combining Proposition 10.4 with Proposition 10.10, the result is immediate. $\square$

Proposition 10.2. We have $\dim_k A_1 = m-2$. If $S$ is any subset of $\{1, \ldots, m\}$ of cardinality $m-2$, then the set $\{\pi(x_i) : i \in S\}$ is a $k$-basis of $A_1$.

Proof. We denote by $M$ the $2 \times m$ matrix with $(i,j)$-entry equal to $a_{i,j}$. The determinant of every $2 \times 2$ submatrix of $M$ is a nonzero element of the field $k$. Since $A = k[D]/(f_1, f_2)$ and $I_D$ is a homogeneous ideal with generators of degrees $\geq 2$, the result follows. $\square$

For $1 \leq i \leq m-2$, we set $e_i = \pi(x_{i+1})$. By Proposition 10.2, the finite sequence

$$e_1, e_2, \ldots, e_{m-2}$$

is an ordered basis of $A_1$. We denote by $N_m$ the $(m-2) \times (m-2)$ symmetric matrix, with $(i,j)$-entry equal to $\rho(e_i, e_j)$. We call $N_m$ the matrix of $\rho$ with respect to the ordered basis.
Remark 10.3. Assume $a, b, c, d \in \{1, \ldots, m\}$. Then, we have the well-known Plücker identity

$$[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0,$$

see [10] Theorem 5.2.3.

Proposition 10.4. We have

$$\det(N_m) = (-1)^m \frac{[1, m]}{\prod_{i=1}^{m-1}[i, i+1]}.$$

Proof. We use induction on $m \geq 3$. For $m = 3$, it follows from Proposition 10.1.

Assume $m = 4$. Then we have to compute the determinant of the matrix

$$N_4 = \left( \begin{array}{cc}
-\frac{[1,3]}{1} & \frac{1}{[2,3]} \\
\frac{[1,2][2,4]}{[2,3]} & -\frac{1}{[2,3][3,4]}
\end{array} \right).$$

It is equal to

$$\frac{[1,3][2,4]}{[2,3][2,3][3,4]} - \left( \frac{1}{[2,3]} \right)^2 = \frac{[1,3][2,4] - [1,2][3,4]}{[2,3][2,3][3,4]}.$$

Using the Plücker identity $[1, 2][3, 4] - [1, 3][2, 4] + [1,4][2,3] = 0$ (see Remark 10.3) the result follows.

Assume now $m \geq 5$ and that the result holds for all previous values up to $m - 1$. Using Proposition 10.1 we have that $N_m$ has the block format

$$N_m = \left( \begin{array}{cc}
N_{m-1} & v^t \\
v & \frac{1}{[m-2,m-1][m-1,m]}
\end{array} \right),$$

where $v$ is the $(m-3) \times 1$ matrix

$$v = \left( 0 \ 0 \ \ldots \ 0 \ \
\frac{1}{[m-2,m-1]} \right).$$

Moreover, a similar block format statement holds for the matrix $N_{m-1}$.

Developing the determinant of $N_m$ using the last column, and using the inductive hypothesis together with the Plücker identity (see Remark 10.3)

$$[1, m - 2][m - 1, m] - [1, m - 1][m - 2, m] + [1, m][m - 2, m - 1] = 0,$$

we get

$$\det(N_m) = -\frac{[m-2, m]}{[m-2, m-1][m-1, m]} \det(N_{m-1}) - \frac{1}{[m-2, m-1][m-1, m]} \det(N_{m-2})$$

$$= (-1)^{m-1} \left( \frac{[m-2, m]}{[m-2, m-1][m-1, m]} \prod_{i=1}^{m-2}[i, i+1] - \frac{1}{[m-2, m-1][m-1, m]} \prod_{i=1}^{m-3}[i, i+1] \right)$$

$$= (-1)^{m-2} \left( \frac{[1, m - 1][m - 2, m]}{[m - 2, m - 1][m - 1, m]} \prod_{i=1}^{m-2}[i, i+1] - \frac{1}{[m-2, m-1][m-1, m]} \prod_{i=1}^{m-3}[i, i+1] \right)$$

$$= (-1)^m \left( \frac{[1, m]}{\prod_{i=1}^{m-1}[i, i+1]} \right).$$



Remark 10.5. Assume $1 \leq c < d \leq m$. It is well-known that $[c, d]$ is an irreducible element of $S_{sp}$. Hence, there exists an induced valuation map

$$\text{val}_{[c,d]} : k \setminus \{0\} \to \mathbb{Z}.$$

Recall that if $f, g \in S_{sp} \setminus \{0\}$, then $\text{val}_{[c,d]}(f)$ is the largest integer $s$ such that $[c, d]^s$ divides $f$ in $S_{sp}$, and

$$\text{val}_{[c,d]}(f/g) = \text{val}_{[c,d]}(f) - \text{val}_{[c,d]}(g).$$
Remark 10.6. Assume that \( h \) is any ordered basis of \( A_1 \). We denote by \( H \) the matrix of \( \rho \) with respect to \( h \). By the basic theory of bilinear forms, there exists an invertible matrix \( P \) with entries in \( k \) such that
\[
H = P^t N_m P.
\]
As a consequence, using Proposition 10.4,
\[
\det(H) = (-1)^m (\det P)^2 \prod_{i=1}^{m-1} i, i + 1 [1, m]
\]
Taking into account Remark 10.5, we conclude that we can recover the simplicial complex \( D \) from (the determinant of) \( \rho \), since the set of facets of \( D \) is exactly the set of ordered pairs \( (c, d) \) such that \( 1 \leq c < d \leq m \) and \( \text{val}_{[c,d]}(\det(H)) \) is an odd integer. An interesting question is whether this holds for all simplicial spheres of odd dimension. In other words, assume \( E \) is a simplicial sphere of odd dimension \( \geq 3 \) and \( e \) is an ordered facet of \( E \). Is it possible to recover \( E \) from (the determinant of) the symmetric bilinear form \( \rho_e \)?

The proof of the following theorem will be given in Subsection 10.1.

Theorem 10.7. The simplicial sphere \( D \) is generically anisotropic over \( k_1 \).

10.1. The proof of Theorem 10.7. We keep using the notations of Section 10. Using Remark 4.7, to prove Theorem 10.7 it is enough to prove that the symmetric bilinear form \( \rho : A_1 \times A_1 \rightarrow k \) is anisotropic.

We define a second basis of \( A_1 \), by using the Gram-Schmidt orthogonalization. We set 
\[
\tilde{e}_1 = e_1,
\]
and we inductively define
\[
\tilde{e}_i = e_i + \sum_{t=2}^{i} \frac{[1, i]}{[1, i + 1]} \pi([1, i + 1], [1, i + 2]) \pi(x_t),
\]
for \( 2 \leq i \leq m - 2 \).

Proposition 10.8. For all \( 1 \leq i \leq m - 2 \), we have
\[
\rho(\tilde{e}_i, \tilde{e}_i) = -\sum_{t=2}^{i} \frac{[1, i]}{[1, i + 1]} \pi([1, i + 1], [1, i + 2]) \pi(x_t).
\]

Proof. We prove Equation (20) using induction on \( i \). For \( i = 1 \) it is true by the definition of \( \tilde{e}_1 \). Assume \( 1 \leq i \leq m - 3 \) and that Equation (20) is true for the value \( i \). We have
\[
\tilde{e}_{i+1} = e_{i+1} + \sum_{t=2}^{i+1} \frac{[1, i]}{[1, i + 1]} \pi([1, i + 1], [1, i + 2]) \pi(x_t)
\]
\[
\rho(\tilde{e}_{i+1}, \tilde{e}_{i+1}) = \rho(e_{i+1}, e_{i+1}) + \sum_{t=2}^{i+1} \frac{[1, i]}{[1, i + 1]} \pi([1, i + 1], [1, i + 2]) \pi(x_t)
\]
\[
= \rho(e_{i+1}, e_{i+1}) + \sum_{t=2}^{i+1} \frac{[1, i]}{[1, i + 2]} \pi(x_t) = \sum_{t=2}^{i+2} \frac{[1, i]}{[1, i + 2]} \pi(x_t).
\]

Proposition 10.9. For all \( 1 \leq i \leq m - 2 \), we have
\[
\rho(\tilde{e}_i, \tilde{e}_j) = -\frac{[1, i + 2]}{[1, i + 1] [i + 1, i + 2]} \pi([1, i + 1], [i + 1, i + 2]) \pi(x_t).
\]

Moreover, if \( 1 \leq j \leq m - 2 \) and \( j \neq i \), we have
\[
\rho(\tilde{e}_i, \tilde{e}_j) = 0.
\]

Proof. Assume \( 1 \leq i \leq m - 2 \). We set \( u = \sum_{t=2}^{i+1} \pi(x_t) \). By Proposition 11.1
\[
\sum_{t=2}^{m} \pi(x_t) = 0.
\]
Hence, if \( 1 \leq r \leq i \), taking into account that \( \pi(x_r, x_t) = 0 \) when \( r + 2 \leq t \leq m \), we get
\[
\rho(\tilde{e}_i, \tilde{e}_j) = 0.
\]
Assume $1 \leq j < i$. Using Proposition 10.8, Equation (21) implies that $\rho(\tilde{e}_i, \tilde{e}_j) = 0$. Moreover, Equation (21) also implies that

$$
\Psi(u^2) = \Psi(u (\sum_{t=2}^{i+1} [1, t] \pi(x_t))) = \Psi([1, i + 1]u \pi(x_{i+1}))
$$

$$
= \Psi([1, i + 1][1, i] \pi(x_i x_{i+1}) + [1, i + 1]^2 \pi(x_{i+1}^2))
$$

$$
= [1, i + 1]\left(\frac{[1, i]}{[i, i + 1]} + \frac{[1, i + 1]}{[i, i + 1][i + 1, i + 2]}\right)
$$

$$
= [1, i + 1]\frac{[1, i][i + 1, i + 2] - [1, i + 1][i, i + 2]}{[i, i + 1][i + 1, i + 2]}
$$

$$
= -[1, i + 1]\frac{[1, i + 2][i, i + 1]}{[i, i + 1][i + 1, i + 2]} = -[1, i + 1]\frac{[1, i + 2]}{[i, i + 1, i + 2]},
$$

where we used Proposition 10.1 and Remark 10.3. Since $\tilde{e}_i = u/[1, i + 1]$, this proves the formula for $\rho(\tilde{e}_i, \tilde{e}_i)$. \hfill \Box

We set

$$
L = \prod_{s=2}^{m-1} [1, s][s, s + 1]
$$

and, for $1 \leq t \leq m - 2$, we define $L_t = L/([1, t + 1][t + 1, t + 2]) \in S_{sp}$.

Using Proposition 10.9, it is clear that to prove Theorem 10.7 it is enough to prove that, if $d_t \in \mathbb{k}$ satisfy

$$
\sum_{t=1}^{m-2} \frac{[1, t + 2]}{[1, t + 1][t + 1, t + 2]} = 0,
$$

we then have $d_t = 0$ for all $1 \leq t \leq m - 2$. By clearing denominators, it is enough to prove the following proposition.

**Proposition 10.10.** Assume $d_1, \ldots, d_{m-2} \in S_{sp}$ satisfy

$$
\sum_{t=1}^{m-2} d_t^2 [1, t + 2] L_t = 0.
$$

Then, we have $d_t = 0$, for all $1 \leq t \leq m - 2$.

**Proof.** We give to the polynomial ring $S_{sp}$ the lexicographic ordering $> with

$$
a_{1,1} > a_{1,2} > \cdots > a_{1,m} > a_{2,1} > a_{2,2} > \cdots > a_{2,m}.
$$

Using Corollary 12.3, it is enough to prove that if $i, j$ have the properties $1 \leq i < j \leq m - 2$, $d_i \neq 0$ and $d_j \neq 0$, we then have

$$
in_>(d_i^2[1, i + 2]L_i) \neq in_>(d_j^2[1, j + 2]L_j).
$$

Using the definitions of $L_i$ and $L_j$ and Remark 12.1, we have

$$
in_>(d_i^2[1, i + 2]L_i) = (in_>(d_i))^2 \cdot (a_{1,1})^{m-2} \cdot \prod_{s=1}^{i} a_{1,s} \cdot \prod_{s=i+2}^{m-1} a_{1,s} \cdot Q_i,
$$

and

$$
in_>(d_j^2[1, j + 2]L_j) = (in_>(d_j))^2 \cdot (a_{1,1})^{m-2} \cdot \prod_{s=1}^{j} a_{1,s} \cdot \prod_{s=j+2}^{m-1} a_{1,s} \cdot Q_j,
$$

where $Q_i$ and $Q_j$ are monomials in the variables $a_{2,1}, \ldots, a_{2,m}$. Therefore, the variable $a_{1,j+1}$ appears in the monomial $in_>(d_i^2[1, i + 2]L_i)$ with an odd power, and in the monomial $in_>(d_j^2[1, j + 2]L_j)$ with an even power. Hence, Inequality (23) is true. \hfill \Box
Example 10.11. Assume \( m = 6 \). Equation (22) becomes
\[
d_1^2[1, 3]L_1 + d_2^2[1, 4]L_2 + d_3^2[1, 5]L_3 + d_4^2[1, 6]L_4 = 0,\]
where
\[
L_1 = \frac{L}{[1, 2][2, 3]}, \quad L_2 = \frac{L}{[1, 3][3, 4]}, \quad L_3 = \frac{L}{[1, 4][4, 5]}, \quad L_4 = \frac{L}{[1, 5][5, 6]}
\]
and
\[
L = [1, 2][1, 3][1, 4][1, 5][2, 3][3, 4][4, 5][5, 6].
\]

11. A GENERAL PROPOSITION RELATED TO ELIMINATION

In this section we describe a specific form of Gauss elimination that is used in the present paper.

Assume \( R \) is a commutative ring with unit, and \( n, m, Z \) are positive integers with \( n < m \leq Z \). Assume that, for \( 1 \leq j \leq m \), \( x_j \) is an elements of \( R \) and that for \( 1 \leq i \leq n \) and \( 1 \leq j \leq Z \), \( a_{ij} \) is an element of \( R \). We denote by \( M \) the \( n \times Z \) matrix with \( (i, j) \)-entry equal to \( a_{ij} \).

Assume \( b_1, \ldots, b_n \) are \( n \) integers, with \( 1 \leq b_i \leq Z \), for all \( i \). We denote by \( [b_1, \ldots, b_n] \) the determinant of the \( n \times n \) matrix, whose \( i \)-th column is equal to the \( b_i \)-th column of \( M \). For \( 1 \leq i \leq n \), we set
\[
f_i = \sum_{t=1}^{m} a_{i,t} x_t,
\]
and we denote by \( I = (f_1, \ldots, f_n) \) the ideal of \( R \) generated by the \( f_i \).

**Proposition 11.1.** Assume \( c_1, \ldots, c_{n-1} \) are integers, with \( 1 \leq c_i \leq Z \) for all \( i \). We have
\[
\sum_{t=1}^{m} [c_1, c_2, \ldots, c_{n-1}, t] x_t \in I.
\]

**Proof.** Denote by \( N \) the \( n \times (n - 1) \) matrix, whose \( i \)-th column is the \( c_i \)-th column of \( M \). For \( 1 \leq j \leq n \), we denote by \( N_j \) the determinant of the submatrix of \( N \) obtained by deleting the \( j \)-th row of \( N \). We claim that
\[
\sum_{t=1}^{m} [c_1, c_2, \ldots, c_{n-1}, t] x_t = \sum_{j=1}^{n} (-1)^{j+n} N_j f_j.
\]
Indeed, on the left hand side, the coefficient of \( x_t \) is \([c_1, c_2, \ldots, c_{n-1}, t] \), while on the right hand side the coefficient is equal to \( \sum_{j=1}^{n} (-1)^{j+n} N_j a_{j,t} \). The two quantities are equal, by developing the determinant \([c_1, c_2, \ldots, c_{n-1}, t] \) using the last column. \( \square \)

12. A GENERAL TECHNIQUE FOR PROVING A POLYNOMIAL IS NONZERO

Here we discuss a well-known general method which is useful for proving that certain sums of products of bracket polynomials are nonzero. We use it in the proof of Proposition 10.10.

Assume \( m \geq 1 \), \( k \) is a field and \( R = k[x_i : 1 \leq i \leq m] \). We denote by \( A_R \) the set of all monomials of \( R \). In other words,
\[
A_R = \{ x_1^{a_1} \cdots x_n^{a_n} : a_i \geq 0 \text{ for all } i \}.
\]
Following [7 Section 15.2], a monomial order on \( R \) is a total order \( > \) on \( A_R \) such that if \( u_1, u_2, w \in A_R \) with \( u_1 > u_2 \) and \( w \neq 1 \), we then have \( wu_1 > wu_2 > u_2 \). In addition, by the same reference, the lexicographic order on \( R \) with \( x_1 > x_2 > \cdots > x_m \) is the total order \( > \) on \( A_R \) defined by \( x_1^{a_1} \cdots x_m^{a_m} > x_1^{b_1} \cdots x_m^{b_m} \) if and only if \( a_i > b_i \) for the first index \( i \) such that \( a_i \neq b_i \). It is a monomial order on \( R \).
Assume now $\succ$ is a monomial order on $R$. It induces the initial monomial map, $\text{in}_\succ: R \setminus \{0\} \to \mathcal{A}_R$, defined as follows. Assume $f \in R \setminus \{0\}$. Then, there exist (unique) $s > 0$, $g_1, \ldots, g_s \in \mathcal{A}_R$ and $\lambda_1, \ldots, \lambda_s \in k \setminus \{0\}$ such that

$$f = \sum_{i=1}^{s} \lambda_i g_i \quad \text{and} \quad g_1 > g_2 > g_3 > \cdots > g_s.$$ 

By definition, $\text{in}_\succ(f) = g_1$.

**Remark 12.1.** By the definition of a monomial ordering, we have

$$\text{in}_\succ(f_1 f_2) = (\text{in}_\succ(f_1))(\text{in}_\succ(f_2))$$

for all $f_1, f_2 \in R \setminus \{0\}$.

Moreover, by the definition of a monomial ordering we have the following proposition.

**Proposition 12.2.** Assume $f_1, f_2, \ldots, f_t \in R \setminus \{0\}$. Assume there exists a with $1 \leq a \leq t$ such that

$$\text{in}_\succ(f_a) > \text{in}_\succ(f_b)$$

for all $b$ with $1 \leq b \leq t$ and $b \neq a$. Then $\sum_{i=1}^{t} f_i \neq 0$ and $\text{in}_\succ(\sum_{i=1}^{t} f_i) = \text{in}_\succ(f_a)$.

**Corollary 12.3.** Assume $f_1, f_2, \ldots, f_t \in R \setminus \{0\}$ satisfy

$$\text{in}_\succ(f_i) \neq \text{in}_\succ(f_j)$$

for all $1 \leq i, j \leq t$ with $i \neq j$. Then $\sum_{i=1}^{t} f_i \neq 0$.

**Proof.** Since $\text{in}_\succ(f_i) \neq \text{in}_\succ(f_j)$ for all $1 \leq i, j \leq t$ with $i \neq j$, there exists a unique integer $a$ such that $1 \leq a \leq t$ and $\text{in}_\succ(f_a) > \text{in}_\succ(f_b)$ for all $b$ with $1 \leq b \leq t$ and $b \neq a$. The result follows by Proposition 12.2.

13. Lefschetz Properties and base change

We believe that the statements in the present section, with the likely exception of Proposition 13.8, are well-known. We include them for completeness.

**Proposition 13.1.** Assume $E$ is an infinite field, $f \in E[x_1, \ldots, x_m]$ is a nonzero polynomial and, for $1 \leq i \leq m$, $Z_i$ is an infinite subset of $E$. Then, there exists a point $p$ in the set $Z_1 \times Z_2 \times \cdots \times Z_m$ such that $f(p) \neq 0$.

**Proof.** We use induction on $m$. If $m = 1$, it is well-known that the polynomial $f$ has a finite number of roots in the field $E$, and the result follows.

Assume $m \geq 2$ and that the result is true for $m - 1$. There exist $s > 0$ and, for $0 \leq i \leq s$, a polynomial $g_i \in E[x_1, \ldots, x_{m-1}]$, such that

$$f = \sum_{i=0}^{s} g_i x_i^m.$$ 

Since $f$ is nonzero, there exists $c$, with $0 \leq c \leq s$, such that $g_c$ is nonzero. Hence, by the inductive hypothesis, there exists an element $(a_1, \ldots, a_{m-1}) \in Z_1 \times Z_2 \times \cdots \times Z_{m-1}$ such that $g_c(a_1, \ldots, a_{m-1}) \neq 0$. Consequently, the polynomial $h \in E[x_m]$, with

$$h = \sum_{i=0}^{s} g_i(a_1, \ldots, a_{m-1}) x_i^m,$$

is nonzero. By the case $m = 1$, there exists $a_m \in Z_m$ such that $h(a_m) \neq 0$. This implies that $f(a_1, \ldots, a_m) \neq 0$. \qed
Corollary 13.2. Assume $E$ is an infinite field, $m \geq 1$ is an integer and $f \in E[x_1, \ldots, x_m]$ is a nonzero polynomial. Assume $k_1$ is an infinite subfield of $E$. Then

i) There exists a point $p \in k_1^m$ such that $f(p) \neq 0$.

ii) Endow the set $E^m$ with the Zariski topology. Then the subset $k_1^m$ of $E^m$ is Zariski dense.

Proof. Part i) follows from Proposition 13.1 by setting $Z_i = k_1$ for all $1 \leq i \leq m$.

Part ii) follows immediately from Part i).

We first assume that $k_1 \subset E$ is a field extension. We consider the polynomial ring $k_1[x_1, \ldots, x_m]$, where the degree of the variable $x_i$ is equal to 1, for all $1 \leq i \leq m$. Assume $I \subset k_1[x_1, \ldots, x_m]$ is a homogeneous ideal such that the quotient $G = k_1[x_1, \ldots, x_m]/I$ is Cohen-Macaulay. We denote by $d$ the Krull dimension of $G$.

We set $G_E = E[x_1, \ldots, x_m]/(I)$, where $(I)$ is the ideal of $E[x_1, \ldots, x_m]$ generated by $I$. By Theorem 2.1.10, $G_E$ is also Cohen-Macaulay. Since, for all $i \geq 0$, $(G_E)_i = G_i \otimes_{k_1} E$, the Hilbert function of $G$ as a graded $k_1$-algebra is equal to the Hilbert function of $G_E$ as a graded $E$-algebra. Consequently, the Krull dimension of $G_E$ is $d$.

Proposition 13.3. Assume that the field $k_1$ is infinite. Then the following are equivalent:

i) The graded $k_1$-algebra $G$ has the Weak Lefschetz Property.

ii) The graded $E$-algebra $G_E$ has the Weak Lefschetz Property.

Proof. We first assume that $G$ has the Weak Lefschetz Property. Then, there exist elements $g_1, \ldots, g_d, \omega, g \in G_1$ such that $g_1, \ldots, g_d$ is a regular sequence for $G$ and $\omega$ is a Weak Lefschetz element for $G/(g_1, \ldots, g_d)$. Clearly, $g_1, \ldots, g_d$ is a regular sequence also for $G_E$ and $\omega$ is a Weak Lefschetz element also for $G_E/(g_1, \ldots, g_d)$. Hence, the $k$-algebra $G_E$ has the Weak Lefschetz Property.

For the opposite direction, we assume that $G_E$ has the Weak Lefschetz Property. By taking the coefficients of $f_1$ and $\omega$, we can identify the set

$$S = \{(g_1, \ldots, g_d, \omega) : g_i \in (G_E)_1, \omega \in (G_E)_1\}$$

with the affine space $(G_E)_1^{d+1}$. We denote by $U$ the subset of $S$ consisting of the elements $(g_1, \ldots, g_d, \omega)$ such that $g_1, \ldots, g_d$ is a regular sequence for $G_E$ and $\omega$ is a Weak Lefschetz element for $G_E/(g_1, \ldots, g_d)$.

By the assumption that $G_E$ has the Weak Lefschetz Property, the set $U$ is nonempty. Hence, by Lemma 4.1, $U$ is a nonempty Zariski open subset of $S$. Using that the field $k_1$ is infinite, Corollary 13.2 implies that $G_1^{d+1}$ is Zariski dense in $(G_E)_1^{d+1}$, hence $G_1^{d+1} \cap U \neq \emptyset$. Let $(g_1, \ldots, g_d, \omega) \in G_1^{d+1} \cap U$. Then $g_1, \ldots, g_d$ is a regular sequence for $G$ and $\omega$ is a Weak Lefschetz element for $G/(g_1, \ldots, g_d)$. Hence, the $k_1$-algebra $G$ has the Weak Lefschetz Property.

We denote by $k$ the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m].$$

We set $G_k = k[x_1, \ldots, x_m]/(I)$, where $(I)$ is the ideal of $k[x_1, \ldots, x_m]$ generated by $I$. For $1 \leq i \leq d$, we set $f_i = \sum_{j=1}^m a_{i,j} x_j$. Hence, the Artinian $k$-algebra $G_k/(f_1, \ldots, f_d)$ is the generic Artinian reduction of the $k_1$-algebra $G$ in the sense of Definition 2.2.

Proposition 13.4. Assume $d \geq 1$. Then the following are equivalent:

i) The Artinian $k$-algebra $G_k/(f_1, \ldots, f_d)$ has the Weak Lefschetz Property.

ii) If $E$ is an infinite field containing $k_1$ as a subfield, then the $E$-algebra $G_E$ has the Weak Lefschetz Property.

iii) There exists an infinite field $F$ containing $k_1$ as a subfield such that the $F$-algebra $G_F$ has the Weak Lefschetz Property.

Proof. We first prove that i) implies iii). Since the $k$-algebra $G_k/(f_1, \ldots, f_d)$ has the Weak Lefschetz Property, it follows that the $k$-algebra $G_k$ has the Weak Lefschetz Property. Assume $E$ is an infinite field containing $k_1$ as a subfield. We denote by $E_1$ the field of fractions
of the polynomial ring

\[ E[a_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m]. \]

Since \( k \) is a subfield of \( E_1 \), Proposition [13.3] implies that the \( E_1 \)-algebra \( G_{E_1} \) has the Weak Lefschetz Property. Since \( E \) is an infinite subfield of \( E_1 \), the same proposition gives that the \( E \)-algebra \( G_E \) has the Weak Lefschetz Property.

We now prove that ii) implies iii). It is clear.

We now prove that ii) implies i). We denote by \( E \) the field of fractions of the polynomial ring in one variable \( k[T] \) over \( k \). We denote by \( F_1 \) the field of fractions of the polynomial ring

\[ F[T, a_{i,j} : 1 \leq i \leq d, 1 \leq j \leq m]. \]

Since \( F \) is a subfield of \( F_1 \), both fields are infinite, and, by the assumption, the \( F \)-algebra \( G_F \) has the Weak Lefschetz Property, it follows, by Proposition [13.3], that the \( E \)-algebra \( G_E \) has the Weak Lefschetz Property. Since \( E \) is an infinite subfield of \( F_1 \), the same proposition implies that the \( E \)-algebra \( G_E \) has the Weak Lefschetz Property.

We denote by \( I^r \) the ideal of \( E[x_1, \ldots, x_m] \) generated by \( I \), and by \( V \) the \( m \)-dimensional \( E \)-vector subspace of \( E[x_1, \ldots, x_m] \) consisting of homogeneous degree 1 polynomials. For \( 1 \leq i \leq d \), \( 1 \leq j \leq m \), we define the infinite subset

\[ Z_{i,j} = \{ a_{i,j} + T^r : r \geq 1 \} \]

of \( E \). We denote by \( Z \) the Cartesian product, for \( 1 \leq i \leq d \), \( 1 \leq j \leq m \), of the sets \( Z_{i,j} \). By Corollary [13.2] \( Z \) is Zariski dense in the affine space \( E^{dm} \).

Since \( G_E \) has the Weak Lefschetz Property, the set \( U \) consisting of all \((g_1, \ldots, g_d) \in V^d \) such that \( g_1, \ldots, g_d \) is a regular sequence for \( G_E \) and \( G_E/(g_1, \ldots, g_d) \) has the Weak Lefschetz Property, is a nonempty Zariski open subset of the affine space \( V^d \).

We now discuss the corresponding statements of the last two propositions for the Strong Lefschetz Property.

**Proposition 13.5.** Assume that the field \( k_1 \) is infinite and \( G \) is Gorenstein. Then the following are equivalent:

i) The graded \( k_1 \)-algebra \( G \) has the Strong Lefschetz Property.

ii) The graded \( k \)-algebra \( G_E \) has the Strong Lefschetz Property.

**Proof.** With the obvious modifications, the arguments in the proof of Proposition [13.3] also work here. \( \square \)
Proposition 13.6. Assume that $G$ is Gorenstein and $d \geq 1$. Then the following are equivalent:

i) The Artinian $k$-algebra $G_E/(f_1, \ldots, f_d)$ has the Strong Lefschetz Property.

ii) If $E$ is an infinite field containing $k_1$ as a subfield, then the $E$-algebra $G_E$ has the Strong Lefschetz Property.

iii) There exists an infinite field $F$ containing $k_1$ as a subfield such that the $F$-algebra $G_F$ has the Strong Lefschetz Property.

Proof. With the obvious modifications, the arguments in the proof of Proposition 13.4 also work here.

We also need the following two propositions. The first is a special case of Part (b) of [12, Proposition 2.1].

Proposition 13.7. Assume $k_1$ is a field and $A$ is a standard graded Artinian Gorenstein $k_1$-algebra of socle degree $d$. Assume $s$ is an integer with $1 \leq s < d$. Assume $\omega \in A_1$ has the property that the multiplication by $\omega$ map $A_s \to A_{s+1}$ is injective. Then, for all $t$ with $0 \leq t \leq s$, we have that the multiplication by $\omega$ map $A_t \to A_{t+1}$ is injective.

Proof. Assume $0 \leq t \leq s$ and $0 \neq u \in A_t$. By Remark 2.1 there exists $z \in A_{s-t}$ such that $uz \neq 0$. Hence $\omega(uz) \neq 0$, which implies that $\omega u \neq 0$.

Proposition 13.8. Assume $k_1$ is a field and $A$ is a standard graded Artinian Gorenstein $k_1$-algebra of even socle degree $d$. We assume that $A$ has the Weak Lefschetz Property and that, for all $i$ with $0 \leq i \leq d/2$ and all $0 \neq u \in A_i$, we have $u^2 \neq 0$. Then $A$ has the Strong Lefschetz Property.

Proof. We fix $\omega \in A_1$ such that, for all $t \geq 0$, the multiplication by $\omega$ map $A_t \to A_{t+1}$ has maximal rank. Since $A$ is Gorenstein of even socle degree $d$, it follows that the multiplication by $\omega$ map form $A_{d/2-1} \to A_{d/2}$ is injective. By the definition of the Strong Lefschetz Property, and using that $A$ is Gorenstein, to prove the proposition it is enough to show that for all $i$, with $0 \leq i < d/2$, the multiplication by $\omega^{d-2i}$ map from $A_i$ to $A_{d-i}$ is injective.

Assume $0 \leq i < d/2$ and that $z \in A_i$ has the property

$$\omega^{d-2i}z = 0.$$  

As a consequence, $\omega^{d/2-i}z^2 = 0$. Using the assumption, it follows that $\omega^{d/2-i}z = 0$. Proposition 13.7 implies that $z = 0$.

14. A CONJECTURE ABOUT DIFFERENTIATION

Assume $k_1$ is a field of characteristic $2$. Assume $n \geq 1$ is an integer and $D$ is a simplicial sphere of dimension $n$ with vertex set $\{1, 2, \ldots, m\}$. We denote by $k$ the field of fractions of the polynomial ring

$$k_1[a_{i,j} : 1 \leq i \leq n + 1, 1 \leq j \leq m].$$

We define the polynomial ring $R = k[x_1, \ldots, x_m]$, where we put degree 1 for all variables $x_i$. We denote by $I_D \subset R$ the Stanley-Reisner ideal of $D$ and we set $k[D] = R/I_D$. For $i = 1, \ldots, n + 1$, we set

$$f_i = \sum_{j=1}^{m} a_{i,j} x_j,$$

and we define $A = k[D]/(f_1, \ldots, f_{n+1})$. Hence, $A$ is the generic Artinian reduction of $k_1[D]$ in the sense of Definition 2.2. We denote by $\pi : R \to A$ the natural projection $k$-algebra homomorphism, and by $\Psi : A_{n+1} \to k$ the vector space isomorphism defined in Remark 1.6.

For a finite sequence $\delta = (\delta_1, \ldots, \delta_{n+1})$ such that $1 \leq \delta_i \leq m$ for all $1 \leq i \leq n + 1$ we set

$$x_\delta = \prod_{i=1}^{n+1} x_{\delta_i} \in R.$$
Assume $\sigma = (\sigma_1, \ldots, \sigma_{n+1})$ and $\tau = (\tau_1, \ldots, \tau_{n+1})$ are two finite sequences such that $1 \leq \sigma_i, \tau_i \leq m$, for all $1 \leq i \leq n+1$. We denote by $\partial^\sigma \mod : k \to k$ the $(n+1)$-th order differential operator which is differentiation with respect to the variables in the set $\{a_{i, \sigma_i} : 1 \leq i \leq n+1\}$.

The following conjecture, if true, will generalise Theorem 8.6 and Propositions 5.1 and 5.7.

**Conjecture 14.1.** 1) Assume the monomial $x_{\sigma} x_\tau$ is not the square of a monomial in $R$. We then have

$$(\partial^\sigma \mod \circ \Psi \circ \pi)(x_\tau) = 0.$$  

2) Assume $x_{\sigma} x_\tau$ is the square of a monomial in $R$. Assume $\delta = (\delta_1, \ldots, \delta_{n+1})$ is a finite sequence such that $1 \leq \delta_i \leq m$, for all $1 \leq i \leq n+1$, and $x_{\sigma} x_\tau = (x_\delta)^2$. We then have

$$(\partial^\sigma \mod \circ \Psi \circ \pi)(x_\tau) = ((\Psi \circ \pi)(x_\delta))^2.$$  

**Remark 14.2.** We note that Conjecture [14.1] implies the following interesting equality

$$(\partial^\sigma \mod \circ \Psi \circ \pi)(x_\tau) = (\partial^\tau \mod \circ \Psi \circ \pi)(x_\sigma).$$

**Remark 14.3.** Assume $i \geq 1$ and $t \geq 2$ are two integers such that $ti \leq n+1$. Assume $0 \neq u \in A_i$. Theorem [3.3] implies that if $t$ is a power of $2$ then $u^t \neq 0$. Is this also true for all values of $t$?

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