LOEWNER THEORY IN ANNULUS II: LOEWNER CHAINS

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Abstract. Loewner Theory, based on dynamical viewpoint, proved itself to be a powerful tool in Complex Analysis and its applications. Recently Bracci et al [6, 7, 9] have proposed a new approach bringing together all the variants of the (deterministic) Loewner Evolution in a simply connected reference domain. This paper is devoted to the construction of a general version of Loewner Theory for the annulus launched in [10]. We introduce the general notion of a Loewner chain over a system of annuli and obtain a 1-to-1 correspondence between Loewner chains and evolution families in the doubly connected setting similar to that in the Loewner Theory for the unit disk. Furthermore, we establish a conformal classification of Loewner chains via the corresponding evolution families and via semicomplete weak holomorphic vector fields. Finally, we extend the explicit characterization of the semicomplete weak holomorphic vector fields obtained in [10] to the general case.

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1. INTRODUCTION

Loewner Theory can be regarded as a theory providing a parametric representation of univalent functions in the unit disk $\mathbb{D} := \{ z : |z| < 1 \}$ based on an infinitesimal description of the semigroup of injective holomorphic self-maps of $\mathbb{D}$. Originating in Loewner’s paper [16] of 1923, this theory gave a great impact in the development of Complex Analysis, in which connection one might recall, e.g., its crucial role in the proof of the famous Bieberbach’s Conjecture (see, e.g., [11, Chapter 17]) given by de Branges [8] in 1984.

From another point of view, Loewner Theory can be seen as an analytic tool to describe monotonic (expanding or contracting) domain dynamics in the plane. A stochastic version of such dynamics (SLE), introduced by Schramm [21] in 2000, is of great importance because of its intrinsic connection to classical lattice models of Statistical Physics such as percolation and the planar Ising model.

We note also that the well celebrated free-boundary Hele-Shaw problem describing 2D filtration processes (see, e.g., [15]) is driven by a non-linear analogue of the classical Loewner–Kufarev PDE, playing one of the central roles in Loewner Theory. Finally, we would like to mention recently discovered relations between classical Loewner Theory, integrable systems and representation of the Virasoro algebra, which appears in a number of fundamental problems in Mathematical Physics, see [12, 17, 20, 18].

According to the new general approach in Loewner Theory [6, 7] introduced recently by Bracci and the first two authors, the essence of the modern Loewner Theory resides in the connection and interplay between three basic notions: evolution families, Loewner chains, and semicomplete weak holomorphic vector fields.

This paper is a sequel of [10] and devoted to the construction of a general version of the Loewner Theory for doubly connected domains. For details concerning the classical and modern Loewner Theory in simply connected case and for the history of its extension to multiply connected case we refer the reader to [10] and references cited therein. A historical survey on Loewner Theory and related references can be also found in [2].

In [10] we introduced a general notion of evolution family over an increasing continuous family of annuli and established a 1-to-1 correspondence between these evolution families and semicomplete weak holomorphic vector fields. In the case of all annuli being non-degenerate we also obtained an explicit representation of the involved semicomplete weak holomorphic vector fields.

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1Which is testified by two Fields Medals, awarded to Wendelin Werner in 2006 and to Stanislav Smirnov in 2010.
In the present paper we will introduce a general notion of Loewner chain in the doubly connected setting and study its relation to evolution families and semicomplete weak holomorphic vector fields. Moreover, we will establish a conformal classification of Loewner chains and obtain an explicit representation of semicomplete weak holomorphic vector fields in a more general case than the one considered in [10], allowing the annuli to degenerate into a punctured disk starting from some point.

1.1. Preliminaries. Now we are going to introduce some definitions and results from [10] necessary for our discussion.

In comparison with the simply connected setting, a new feature in the doubly (and more generally, multiply) connected case is that in order to develop a rich theory, instead of a static reference domain (the unit disk) one has to consider a family of canonical domains \((D_t)_{t \geq 0}\), with evolution families being formed by holomorphic mappings \(\varphi_{s,t} : D_s \to D_t\), \(0 \leq s \leq t\). This explains the reason for the following definition.

Denote \(\mathbb{A}_r,\mathbb{R} := \{z : r < |z| < R\}\), \(\mathbb{A}_r := \mathbb{A}_{r,1}\), \(0 \leq r < R \leq +\infty\), and let \(\mathbb{D}^s\) stand for \(\mathbb{A}_0 = \mathbb{D} \setminus \{0\}\). For \(d \in [1, +\infty]\) by \(AC^d(X,Y), X \subset \mathbb{R}, Y \subset \mathbb{C}\), we denote the class of all locally absolutely continuous functions \(f : X \to Y\) such that \(f' \in L^d_{loc}(X,\mathbb{C})\). Finally, let

\[
\omega(r) := \begin{cases} 
-\pi/\log r, & \text{if } r \in (0,1), \\
0, & \text{if } r = 0. 
\end{cases}
\]

**Definition 1.1 ([10]).** Let \(d \in [1, +\infty]\) and \((D_t)_{t \geq 0}\) be a family of annuli \(D_t := \mathbb{A}_{\omega(t)}\). We will say that \((D_t)\) is a (doubly connected) canonical domain system of order \(d\) (or in short, a canonical \(L^d\)-system) if the function \(t \mapsto \omega(r(t))\) belongs to \(AC^d([0, +\infty), [0, +\infty])\) and does not increase. If \(r(t) \equiv 0\), then the canonical domain system \((D_t)\) will be called degenerate. If on the contrary \(r(t)\) does not vanish, then \((D_t)\) will be called non-degenerate. Finally, if there exists \(T > 0\) such that \(r(t) > 0\) for all \(t \in [0,T]\) and \(r(t) = 0\) for all \(t \geq T\), then we will say that \((D_t)\) is of mixed type.

**Remark 1.2.** The condition that \(t \mapsto \omega(r(t))\) is of class \(AC^d\) implies that \(t \mapsto r(t)\) also belongs to \(AC^d([0, +\infty), [0,1])\). If \(r(t) > 0\) for all \(t \geq 0\), or \(d = 1\), then the converse is also true and we can replace \(\omega(r(t))\) by \(r(t)\) in the above definition. However, in general we do not know whether this is possible, because some proofs in [10] use essentially the requirement that \(t \mapsto \omega(r(t))\) is of class \(AC^d\).

Now we can introduce the definition of an evolution family for the doubly connected setting.

**Definition 1.3 ([10]).** Let \((D_t)_{t \geq 0}\) be a canonical domain system of order \(d \in [1, +\infty]\). A family \((\varphi_{s,t})_{0 \leq s \leq t \leq +\infty}\) of holomorphic mappings \(\varphi_{s,t} : D_s \to D_t\) is said to be an evolution family of order \(d\) over \((D_t)\) (in short, an \(L^d\)-evolution family) if the following conditions are satisfied:

**EF1.** \(\varphi_{s,s} = \text{id}_{D_s}\),
EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \leq s \leq u \leq t < +\infty$,

EF3. for any closed interval $I := [S, T] \subset [0, +\infty)$ and any $z \in D_S$ there exists a non-

negative function $k_{z,I} \in L^d([S, T], \mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,I}(\xi)d\xi$$

whenever $S \leq s \leq u \leq t \leq T$.

Suppressing the language we will refer also to the pair $\mathcal{E} := ((D_t), (\varphi_{s,t}))$ as an evolution

family of order $d$ and apply terms degenerate, non-degenerate, of mixed type to $\mathcal{E}$ whenever they are applicable to the canonical domain system $(D_t)$.

The notion of weak holomorphic vector field, as introduced in [10], in the doubly connected setting can be defined as follows. Let $\text{pr}_\mathbb{R}$ stand for the projection $\mathbb{C} \times \mathbb{R} \ni (z, t) \mapsto t \in \mathbb{R}$.

**Definition 1.4.** Let $d \in [1, +\infty]$ and $(D_t)$ be a canonical domain system of order $d$. A function $G : D \rightarrow \mathbb{C}$, where $D := \{(z, t) : t \geq 0, z \in D_t\}$, is said to be a weak holomorphic vector field of order $d$ over $(D_t)$, if it satisfies the following conditions:

WHVF1. For each $z \in \mathbb{C}$ the function $G(z, \cdot)$ is measurable in $E_z := \{t \geq 0 : z \in D_t\}$.

WHVF2. For each $t \geq 0$ the function $G(\cdot, t)$ is holomorphic in $D_t$.

WHVF3. For each compact set $K \subset D$ there exists a non-negative function $k_K \in L^d(\text{pr}_\mathbb{R}(K), \mathbb{R})$, where $\text{pr}_\mathbb{R}(K) := \{t \geq 0 : \exists z \in \mathbb{C} \ (z, t) \in K\}$, such that

$$|G(z, t)| \leq k_K(t), \quad \text{for all } (z, t) \in K.$$

**Definition 1.5.** A weak holomorphic vector field $G$ over a canonical domain system $(D_t)$ is said to be semicomplete, if for any $s \geq 0$ and any $z \in D_s$ the following initial value problem for the Carathéodory ODE

$$\dot{w} = G(w, t), \quad w|_{t=s} = z,$$

has a solution defined for all $t \geq s$.

In [10] we proved the following statement establishing a 1-to-1 correspondence between evolution families and semicomplete weak holomorphic vector fields.

**Theorem A (10 Theorem 5.1).** The following two assertions hold:

(A) For any $L^d$-evolution family $(\varphi_{s,t})$ over the canonical domain system $(D_t)$ there exists

an essentially unique semicomplete weak holomorphic vector field $G : D \rightarrow \mathbb{C}$ of

order $d$ and a null-set $N \subset [0, +\infty)$ such that for all $s \geq 0$ the following statements hold:

(i) the mapping $[s, +\infty) \ni t \mapsto \varphi_{s,t} \in \text{Hol}(D_s, \mathbb{C})$ is locally absolutely continuous;

(ii) the mapping $[s, +\infty) \ni t \mapsto \varphi_{s,t} \in \text{Hol}(D_s, \mathbb{C})$ is differentiable for all $t \in [s, +\infty) \setminus N$;

(iii) $d\varphi_{s,t}/dt = G(\cdot, t) \circ \varphi_{s,t}$ for all $t \in [s, +\infty) \setminus N$. 
(B) For any semicomplete weak holomorphic vector field $G: \mathcal{D} \to \mathbb{C}$ of order $d$ the formula
\[
\varphi_{s,t}(z) := w^*_s(z, t), \quad t \geq s \geq 0, \quad z \in D_s,
\]
where $w^*_s(z, \cdot)$ is the unique non-extendable solution to the initial value problem
\[
\dot{w} = G(w, t), \quad w(s) = z,
\]
defines an $L^d$-evolution family over the canonical domain system $(D_t)$.

The exact meaning of the notions of absolute continuity and differentiability of the mapping from (i)–(iii) in the above theorem is given by [10] Definitions 2.7 and 2.8.

The characterization of semicomplete weak holomorphic vector fields we established in [10] involves some notions from Function Theory in the annulus. The analogue of the Schwartz kernel $K_0(z) := (1+z)/(1-z)$ for an annulus $A_r := \{z : r < |z| < 1\}$, $r \in (0, 1)$, the so-called Villat kernel, is defined by the following formula (see, e.g., [13] or [41 § V.1]):
\[
K_r(z) := \lim_{n \to \pm \infty} \sum_{\nu = -n}^{n} \frac{1 + r^{2\nu} z}{1 - r^{2\nu} z} = \frac{1 + z}{1 - z} + \sum_{\nu = 1}^{\pm \infty} \left( \frac{1 + r^{2\nu} z}{1 - r^{2\nu} z} + \frac{1 + z/r^{2\nu}}{1 - z/r^{2\nu}} \right).
\]
It is known (see e.g. [22] Theorem 2.2.10]) that for any function $f \in \text{Hol}(A_r, \mathbb{C})$ which is continuous in $A_r$,
\[
f(z) = \int_{\mathbb{T}} K_r(z \xi^{-1}) \text{Re} f(\xi) \frac{|d\xi|}{2\pi} + \int_{\mathbb{T}} [K_r(r \xi/z) - 1] \text{Re} f(r \xi) \frac{|d\xi|}{2\pi} + i \int_{\mathbb{T}} \text{Im} f(\rho \xi) \frac{|d\xi|}{2\pi}
\]
for all $z \in A_r$, $\rho \in [r, 1]$.

**Definition 1.6.** Let $r \in (0, 1)$. By the class $\mathcal{V}_r$ we will mean the collection of all functions $p \in \text{Hol}(A_r, \mathbb{C})$ having the following integral representation
\[
p(z) = \int_{\mathbb{T}} K_r(z/\xi)d\mu_1(\xi) + \int_{\mathbb{T}} [1 - K_r(r \xi/z)]d\mu_2(\xi), \quad z \in A_r,
\]
where $\mu_1$ and $\mu_2$ are positive Borel measures on the unit circle $\mathbb{T}$ subject to the condition $\mu_1(\mathbb{T}) + \mu_2(\mathbb{T}) = 1$.

**Remark 1.7.** From the proof of [23] Theorem 1 it is evident that given $p \in \mathcal{V}_r$, the measures $\mu_1$ and $\mu_2$ in representation (1.5) are unique. (See also the proof of [10] Lemma 5.11.)

**Theorem B ([10] Theorem 5.6]).** Let $d \in [1, +\infty]$ and let $(D_t) = (A_{r(t)})$ be a non-degenerate canonical domain system of order $d$. Then a function $G : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} := \{(z, t) : t \geq 0, \quad z \in D_t\}$, is a semicomplete weak holomorphic vector field of order $d$ if and only if there exist functions $p : \mathcal{D} \to \mathbb{C}$ and $C : [0, +\infty) \to \mathbb{R}$ such that:

(i) $G(w, t) = w[iC(t) + r'(t)p(w, t)/r(t)]$ for a.e. $t \geq 0$ and all $w \in D_t$;
(ii) for each $w \in D := \bigcup_{t \geq 0} D_t$ the function $p(w, \cdot)$ is measurable in $E_w := \{t \geq 0 : w \in D_t\}$;
(iii) for each $t \geq 0$ the function $p(\cdot, t)$ belongs to the class $\mathcal{V}_{r(t)}$;
(iv) $C \in L^d_{\text{loc}}([0, +\infty), \mathbb{R})$. 

1.2. Main results. In this paper we introduce a general notion of Loewner chain for the doubly connected case and establish relationships between Loewner chains and evolution families analogous to that in the Loewner Theory for simply connected domains.

Definition 1.8. Let \( d \in [1, +\infty] \) and \((D_t)\) be a canonical domain system of order \( d \). A family \((f_t)_{t \geq 0}\) of holomorphic functions \( f_t : D_t \to \mathbb{C} \) is called a Loewner chain of order \( d \) (or in short an \( L^d \)-Loewner chain) over \((D_t)\) if it satisfies the following conditions:

LC1. each function \( f_t : D_t \to \mathbb{C} \) is univalent,

LC2. \( f_s(D_s) \subset f_t(D_t) \) whenever \( 0 \leq s < t < +\infty \),

LC3. for any compact interval \( I := [S, T] \subset [0, +\infty) \) and any compact set \( K \subset D_S \) there exists a non-negative function \( k_{K,I} \in L^d([S, T], \mathbb{R}) \) such that

\[
|f_s(z) - f_t(z)| \leq \int_s^t k_{K,I}(\xi)d\xi
\]

for all \( z \in K \) and all \((s,t)\) such that \( S \leq s \leq t \leq T \).

The following theorem shows that every Loewner chain generates an evolution family of the same order.

Theorem 1.9. Let \((f_t)\) be a Loewner chain of order \( d \) over a canonical domain system \((D_t)\) of order \( d \). If we define

\[
\varphi_{s,t} := f_t^{-1} \circ f_s, \quad 0 \leq s \leq t < \infty,
\]

then \((\varphi_{s,t})\) is an evolution family of order \( d \) over \((D_t)\).

This theorem is proved in Section 2. As a consequence we show, see Corollary 2.1, that similarly to the case of the unit disk, any Loewner chain over a canonical system of annuli, satisfies a PDE driven by a semicomplete weak holomorphic vector field.

In Section 3 we prove a converse of Theorem 1.9, saying that for any evolution family \((\varphi_{s,t})\) there exists a Loewner chain \((f_t)\) of the same order such that (1.6) holds and describe possible conformal types of \( \cup_{t \geq 0} f_t(D_t) \). These results can be formulated as follows. Denote by \( I(\gamma) \) the index of the origin w.r.t. a closed curve \( \gamma \subset \mathbb{C}^* \). Similarly to the simply connected case [9], we will say that a Loewner chain \((f_t)\) over \((D_t)\) is associated with an evolution family \((\varphi_{s,t})\) over the same canonical domain system if (1.6) holds.

Theorem 1.10. Let \(((D_t), (\varphi_{s,t}))\), where \( D_t := A_{r(t)} \) for all \( t \geq 0 \), be an evolution family of order \( d \in [1, +\infty] \). Let \( r_\infty := \lim_{t \to +\infty} r(t) \). Then there exists a Loewner chain \((f_t)\) of order \( d \) over \((D_t)\) such that

1. \( f_s = f_t \circ \varphi_{s,t} \) for all \( 0 \leq s \leq t < +\infty \), i.e. \((f_t)\) is associated with \((\varphi_{s,t})\);
2. \( I(f_t \circ \gamma) = I(\gamma) \) for any closed curve \( \gamma \subset D_t \) and any \( t \geq 0 \);
3. If \( 0 < r_\infty < 1 \), then \( \cup_{t \in [0, +\infty)} f_t(D_t) = A_{r_\infty} \);
4. If \( r_\infty = 0 \), then \( \cup_{t \in [0, +\infty)} f_t(D_t) \) is either \( \mathbb{D}^* \), \( \mathbb{C} \setminus \overline{\mathbb{D}} \), or \( \mathbb{C}^* \).
If \((g_t)\) is another Loewner chain over \((D_t)\) associated with \((\varphi_{s,t})\), then there is a biholomorphism \(F : \cup_{t \in [0, +\infty]} g_t(D_t) \rightarrow \cup_{t \in [0, +\infty]} f_t(D_t)\) such that \(f_t = F \circ g_t\) for all \(t \geq 0\).

In general, a Loewner chain associated with a given evolution family is not unique. We call a Loewner chain \((f_t)\) to be standard if it satisfies conditions (2) – (4) from Theorem 1.10. It follows from this theorem that the standard Loewner chain \((f_t)\) associated with a given evolution family, is defined uniquely up to a rotation (and scaling if \(g_t(D_t) = \mathbb{C}^*\)). Furthermore, combining Theorems 1.9 and 1.10 one can easily conclude that for any Loewner chain \((g_t)\) of order \(d\) over a canonical domain system \((D_t)\) there exists a biholomorphism \(F : \cup_{t \in [0, +\infty]} g_t(D_t) \rightarrow L([g_t]),\) where \(L([g_t])\) is either \(\mathbb{D}^*\), \(\mathbb{C} \setminus \overline{\mathbb{D}}\), \(\mathbb{C}^*\), or \(A_p\) for some \(p > 0\), such that the formula \(f_t = F \circ g_t, t \geq 0\), defines a standard Loewner chain of order \(d\) over the canonical domain system \((D_t)\). This motivates the following definition.

**Definition 1.11.** Let \((g_t)\) be a Loewner chain of order \(d\) over a canonical domain system \((D_t) = (A_{r(t)})\). Let \(r_\infty := \lim_{t \to +\infty} r(t)\). We say that

- \((g_t)\) is of (conformal) type I, if \(L([g_t]) = A_p\) for some \(p > 0\);
- \((g_t)\) is of (conformal) type II, if \(L([g_t]) = \mathbb{D}^*\);
- \((g_t)\) is of (conformal) type III, if \(L([g_t]) = \mathbb{C} \setminus \overline{\mathbb{D}}\);
- \((g_t)\) is of (conformal) type IV, if \(L([g_t]) = \mathbb{C}^*\).

By the (conformal) type of an evolution family \((\varphi_{s,t})\) we mean the conformal type of any Loewner chain associated with \((\varphi_{s,t})\).

Suppressing the notation we will also write \(L([\varphi_{s,t}])\) meaning \(L([g_t])\), where \((g_t)\) is any Loewner chain associated with \((\varphi_{s,t})\). We call this domain the Loewner range of \((\varphi_{s,t})\).

**Remark 1.12.** The domains \(\mathbb{D}^*\) and \(\mathbb{C} \setminus \overline{\mathbb{D}}\) are conformally equivalent. However, the conformal types II and III can be distinguished because of the condition (2) from Theorem 1.10. Indeed, \(I(G \circ \gamma) = -I(\gamma)\) for any biholomorphism \(G : \mathbb{D}^* \rightarrow \mathbb{C} \setminus \overline{\mathbb{D}}\) and any closed curve \(\gamma \subset \mathbb{D}^*\).

The following statements, proved in Section 4 characterize the conformal type of a Loewner chain via the properties of the corresponding evolution family. Consider an evolution family \((\varphi_{s,t})\) over a canonical domain system \((D_t) = (A_{r(t)})\), where \(r(t) > 0\) for all \(t \geq 0\). Denote \(r_\infty := \lim_{t \to +\infty} r(t)\). Further for each \(s \geq 0\) and \(t \geq s\), define \(\varphi_{s,t}(z) := r(t)/\varphi_{s,t}(r(s)/z)\). By [10] Example 6.3, \((\varphi_{s,t})\) is also an evolution family over \((D_t)\). Note that at least one of the families \((\varphi_{0,t})\) and \((\varphi_{0,t})\) converges to 0 as \(t \to +\infty\) provided \(r_\infty > 0\) (see Lemma 1.13).

**Theorem 1.13.** Let \(((D_t), (\varphi_{s,t})))\) be a non-degenerate evolution family. In the above notation, the following statements hold:

(i) the evolution family \((\varphi_{s,t})\) is of type I if and only if \(r_\infty > 0\);
(ii) the evolution family \((\varphi_{s,t})\) is of type II if and only if \(r_\infty = 0\) and \(\varphi_{0,t}\) does not converge to 0 as \(t \to +\infty\);

(iii) the evolution family \((\varphi_{s,t})\) is of type III if and only if \(r_\infty = 0\) and \(\tilde{\varphi}_{0,t}\) does not converge to 0 as \(t \to +\infty\);

(iv) the evolution family \((\varphi_{s,t})\) is of type IV if and only if \(r_\infty = 0\) and both \(\varphi_{0,t} \to 0\) and \(\tilde{\varphi}_{0,t} \to 0\) as \(t \to +\infty\).

In the mixed-type or degenerate case the situation is simpler. Namely, we prove following

**Proposition 1.14.** Let \((\varphi_{s,t})\) be an evolution family over a canonical domain system \((D_t) = (A_{r(t)})\). Assume that \(r(T) = 0\) for some \(T \in [0, +\infty)\), i.e. \((D_t)\) is of mixed-type or degenerate. Then \((\varphi_{s,t})\) is of type IV if \(\varphi_{0,t} \to 0\) as \(t \to +\infty\), and of type II otherwise.

Further new results of the present paper are as follows. As we mentioned in Section 1.1 each evolution family is generated by a weak holomorphic vector field. So it is possible to study the limit behavior of an evolution family using the corresponding vector fields. In this way we obtain a necessary and sufficient condition for a non-degenerate evolution family \((\varphi_{s,t})\) to satisfy the condition \(\varphi_{s,t} \to 0\) as \(t \to +\infty\), see Theorem 5.1 in Section 5.

In [10] we obtained an explicit characterization of semicomplete weak holomorphic vector fields over *non-degenerate* canonical domain systems. As an application of general Loewner Theory in the unit disk we also obtained in [10] an analogous result for *degenerate* canonical domain systems. In this paper we include Section 6 devoted to obtaining a characterization of semicomplete weak holomorphic vector fields over canonical domain systems of mixed type.

Finally, in the short Section 7 we combine the above results to obtain the conformal classification of Loewner chains, in doubly connected setting, via the corresponding weak holomorphic vector fields.

### 2. From Loewner Chains to Evolution Families

In this Section we prove Theorem 1.9. The proof is based on the following lemmas. In what follows, using the notation \([a, b]\), we allow \(a\) to be equal to \(b\). In such case \([a, b]\) means the singleton \(\{a\}\).

**Lemma 2.1.** Let \((D_t) = (A_{r(t)})\) be a canonical domain system and \((f_t)\) a Loewner chain over \((D_t)\). Then for any compact set \(K \subset \mathcal{D} := \{(z, t) : t \geq 0, z \in D_t\}\) there exists \(M = M(K) > 0\) such that

\[
|\zeta - z| \leq M|f_t(\zeta) - f_t(z)| \quad \text{whenever } (z, t), (\zeta, t) \in K.
\]

**Proof.** Assume the contrary. Then there exist sequences \((\zeta_n)\), \((z_n)\) and \((t_n)\) such that \((\zeta_n, t_n), (z_n, t_n) \in K\) and

\[
|\zeta_n - z_n| > n|f_{t_n}(\zeta_n) - f_{t_n}(z_n)|
\]

for every \(n \in \mathbb{N}\).
By the compactness of $K$ we may assume that the sequences $(\zeta_n)$, $(z_n)$, and $(t_n)$ converge to some $\zeta_0$, $z_0$, and $t_0$, respectively. Clearly, $\zeta_0, z_0 \in D_{t_0}$, because $(\zeta_0, t_0)$ and $(z_0, t_0)$ belong to $K$. Using continuity of $[0, +\infty) \ni t \mapsto r(t)$ we therefore conclude that there exist $n_1 \in \mathbb{N}$ and $\tau_1 \in [0, t_0)$ such that $K_1 := \{\zeta_n, z_n : n > n_1\} \cup \{\zeta_0, z_0\}$ is a compact subset of $D_{\tau_1}$ and $t_n \geq \tau_1$ for all $n > n_1$.

Now we note that by LC3, $f_t \rightarrow f_{t_0}$ uniformly on compact subsets of $U := D_{\tau_1}$ as $t \rightarrow t_0$, $t \geq \tau_1$. In particular it follows that $f_{t_n}(\zeta_n) \rightarrow f_{t_0}(\zeta_0)$ and $f_{t_n}(z_n) \rightarrow f_{t_0}(z_0)$ as $n \rightarrow +\infty, n > n_1$. Moreover, any compact set $B \subset f_{t_0}(U)$ is also contained in $f_t(U)$ if $t \geq \tau_1$ and $|t - t_0|$ is small enough. Hence we conclude that there exist $n_2 > n_1$, $n_2 \in \mathbb{N}$, and $\tau_2 \in [\tau_1, t_0)$ such that $K_2 := \{f_{t_n}(\zeta_n), f_{t_n}(z_n) : n > n_2\} \cup \{f_{t_0}(\zeta_0), f_{t_0}(z_0)\}$ is a compact subset of $W := f_{t_0}(U)$ and $t_n \geq \tau_2$ for all $n > n_2$.

According to the definition of a Loewner chain over a doubly connected canonical domain system, the functions $g_n := (f_{t_n}^{-1})|_W$ are well-defined and holomorphic in $W$ for all $n > n_2$. Moreover, $g_n(W) \subset \mathbb{D}$ for any $n > n_2$. Hence the family $F := \{g_n : n > n_2\}$ is normal in $W$ and its closure in $\text{Hol}(W, \mathbb{C})$ is compact. Therefore, there exists $M' = M'(K_2, F) > 0$ such that

$$|g_n(w_2) - g_n(w_1)| \leq M'|w_2 - w_1|$$

for any $w_1, w_2 \in K_2$ and any $n > n_2$.

Choosing $w_1 := f_{t_n}(z_n)$ and $w_2 := f_{t_n}(\zeta_n)$ we see that the above inequality contradicts \ref{2.1} for large $n \in \mathbb{N}$. This contradiction completes the proof. \hfill $\Box$

**Lemma 2.2.** Under the conditions of Lemma \ref{2.7}, let $K$ be a compact subset of $U := \{(w, t) : t \geq 0, w \in f_t(D_t)\}$. Then $\hat{K} := \{(f^{-1}_t(w), t) : (w, t) \in K\}$ is a compact subset of $\mathcal{D}$.

**Proof.** Consider an arbitrary sequence $((z_n, t_n)) \subset \hat{K}$. We need to prove that it has a subsequence converging to a point in $K$. To this end write $w_n := f_{t_n}(z_n)$ for any $n \in \mathbb{N}$. According to the compactness of $K$, passing if necessary to a subsequence we may assume that $(t_n)$ converges to some $t_0$ and $(w_n)$ converges to some $w_0 \in f_{t_0}(D_{t_0})$.

It is sufficient to show that $z_n \rightarrow z_0 := f_{t_0}^{-1}(w_0)$ as $n \rightarrow +\infty$. Indeed, in this case $((z_n, t_n))$ converges to $(z_0, t_0) \in \hat{K}$.

To prove that $z_n \rightarrow z_0$ as $n \rightarrow +\infty$ we fix $\varepsilon > 0$ small enough. Denote $B_\varepsilon := \{z : |z - z_0| \leq \varepsilon\}$, $C_\varepsilon := \{z : |z - z_0| = \varepsilon\}$. From the continuity of $t \mapsto r(t)$ and from the fact that $w_n \rightarrow w_0$ as $n \rightarrow +\infty$, it follows that there exists $n_0 \in \mathbb{N}$ such that $B_\varepsilon \subset D_{t_0}$ and $w_n \in f_{t_0}(B_\varepsilon \setminus C_\varepsilon)$ for all $n > n_0$. Let $S := \min\{t_n : n > n_0\} \cup \{t_0\}$, $T := \max\{t_n : n > n_0\} \cup \{t_0\}$. Then from LC3 it follows that $f_{t_n} \rightarrow f_{t_0}$ uniformly on $B_\varepsilon$ as $n \rightarrow +\infty, n > n_0$. Hence there exists $n_1 > n_0$ such that

$$\max_{z \in C_\varepsilon} |f_{t_n}(z) - f_{t_0}(z)| < \min_{z \in C_\varepsilon} |f_{t_0}(z) - w_0| - |w_n - w_0|$$

This statement can be easily obtained in the same way as in the proof of the Carathéodory kernel convergence theorem (see, e.g., \cite{14} §II.5, Theorem 1) or \cite{19} p. 29, Theorem 1.8).
for all \( n > n_1 \). Note that \( |f_{t_0}(z) - w_0| - |w_n - w_0| \leq |f_{t_0}(z) - w_n| \).

Recall that by the construction, the open disk \( B_\varepsilon \setminus C_\varepsilon \) contains the unique solution of \( f_{t_0}(z) - w_n = 0 \) provided \( n > n_0 \). Then by the Rouche theorem for the functions \( f_{t_0} - w_n \), for each \( n > n_1 \), the unique solution to \( f_{t_0}(z) - w_n = 0 \), which is \( z = z_n \), belongs to \( B_\varepsilon \setminus C_\varepsilon \). Therefore, \( |z_n - z_0| < \varepsilon \) for all \( n > n_1 \). Since \( \varepsilon > 0 \) was chosen arbitrarily, this shows that \( z_n \to z_0 \) as \( n \to +\infty \) and hence the proof is complete. \( \Box \)

**Lemma 2.3.** Under the conditions of Lemma 2.1, let \( K \) be a compact subset of \( D \) and

\[
E := \left[ \min_{(z,t) \in K} \max_{(z,t) \in K} t \right].
\]

Then there exists a non-negative function \( k_K \in L^d(E, \mathbb{R}) \) such that

\[
|f_t(z) - f_u(z)| \leq \int_u^t k_K(\xi) \, d\xi
\]

for any \( z \in \mathbb{C} \) and any \( u, t \geq 0 \) satisfying \((z, u), (z, t) \in K \) and \( u \leq t \).

**Proof.** Since \( t \mapsto r(t) \) is continuous, for any \((\zeta, s) \in K\) there exists \( \rho > 0 \) and \( \varepsilon > 0 \) such that \( \{z : |z - \zeta| \leq \rho\} \subset D_S \), where \( S := \max\{0, s - \varepsilon\} \). It follows that

\[
K_{(\zeta, s)} := \{z : |z - \zeta| \leq \rho\} \times \left[ \max\{0, s - \varepsilon\}, T \right] \subset D, \quad T := 1 + \max_{(z,t) \in K} t,
\]

for any \((\zeta, s) \in K\).

Let \( U_{(\zeta, s)} \) stand for the interior of \( K_{(\zeta, s)} \) w.r.t. \( \mathbb{C} \times [0, +\infty) \). Then

\[
K \subset \bigcup_{(\zeta, s) \in K} U_{(\zeta, s)}.
\]

Therefore, by the compactness of \( K \), there exist finite sequences \( \zeta_1, \ldots, \zeta_n \in \mathbb{C}, \ S_1, \ldots, S_n \in [0, T] \), and \( \rho_1, \ldots, \rho_n > 0 \) such that

\[
K \subset \bigcup_{j=1}^n K_j \times I_j,
\]

where \( K_j := \{z : |z - \zeta_j| \leq \rho_j\} \subset D_{S_j} \), and \( I_j := [S_j, T] \subset [0, +\infty) \) for all \( j = 1, \ldots, n \).

Then by LC3, there exist non-negative functions \( k_j := k_{K_j, I_j} \in L^d(I_j, \mathbb{R}) \) such that for each \( j = 1, \ldots, n \) and any \( z \in K_j \),

\[
|f_t(z) - f_u(z)| \leq \int_u^t k_j(\xi) \, d\xi
\]

whenever \( u, t \in I_j \) and \( u \leq t \).

Finally, we notice that by construction, for arbitrary \( z \in \mathbb{C} \) and \( u, t \geq 0 \) satisfying \((z, u), (z, t) \in K \), there exists \( j = 1, \ldots, n \) such that \( z \in K_j \) and \( u, t \in I_j \). Thus the
statement of the lemma holds with
\[ k_K := \sum_{j=1}^{n} \chi_{I_j} k_j, \]
where \( \chi_{I_j} \) stands for the characteristic function of the set \( I_j \). This completes the proof. \( \square \)

**Proof of Theorem 1.9.** It is straightforward to check that for any \( s \geq 0 \) and \( t \geq s \), formula (1.6) defines a holomorphic mapping \( \varphi_{s,t} : D_s \to D_t \) and that the family \((\varphi_{s,t})\) satisfies conditions EF1 and EF2.

To prove EF3, fix \([S, T] \subset [0, +\infty)\) and \( z \in D_S \). From LC2 and LC3 it follows that, the set \( K := \{(f_s(z), t) : S \leq s \leq t \leq T\} \) is a compact subset of \( U \) (as a continuous image of a compact set). Then by Lemma 2.2,
\[
\hat{K} := \{(f_t^{-1}(f_s(z)), t) : S \leq s \leq t \leq T\} = \{(\varphi_{s,t}(z), t) : S \leq s \leq t \leq T\}
\]
is a compact subset of \( D \).

Consider the continuous mapping \( G : D \times [0, 1] \to D \) defined by
\[
G : ((\zeta, u), \lambda) \mapsto (\zeta, (1 - \lambda)u + \lambda T).
\]
Since \( \hat{K} \times [0, 1] \) is compact, the set
\[
K_0 := G(\hat{K}) = \{(\varphi_{s,u}(z), t) : S \leq s \leq u \leq t \leq T\}
\]
is again a compact subset of \( D \). Clearly, \( \hat{K} \subset K_0 \).

Apply now Lemma 2.1 with \( K_0 \) substituted for \( K \). Then there exists \( M > 0 \) such that
\[
|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq M|f_t(\varphi_{s,t}(z)) - f_t(\varphi_{s,u}(z))| = M|f_u(\varphi_{s,u}(z)) - f_t(\varphi_{s,u}(z))|
\]
whenever \( S \leq s \leq u \leq t \leq T \). Since for any such \( s, u, \) and \( t \), the points \((\varphi_{s,u}(z), u)\) and \((\varphi_{s,u}(z), t)\) belong to \( K_0 \), applying Lemma 2.3 with \( K_0 \) substituted for \( K \) completes the proof. \( \square \)

We conclude this section with a corollary relating Loewner chains with PDEs.

**Corollary 2.4.** Let \( d \in [1, +\infty] \). Let \((D_t)\) be a canonical domain system of order \( d \) and \((f_t)\) a Loewner chain of order \( d \in [1, +\infty] \) over \((D_t)\). Then the following statements hold:

(i) There exists a null-set \( N \subset [0, +\infty) \) (not depending on \( z \)) such that for every \( s \in [0, +\infty) \setminus N \) the function
\[
z \in D_s \mapsto \frac{\partial f_s(z)}{\partial s} := \lim_{h \to 0} \frac{f_{s+h}(z) - f_s(z)}{h} \in \mathbb{C}
\]
is a well-defined holomorphic function on \( D_s \).
(ii) There exists an essentially unique weak holomorphic vector field \( G \) of order \( d \) over \( (D_t) \) such that for a.e. \( s \in [0, +\infty) \),

\[
\frac{\partial f_s(z)}{\partial s} = -G(z, s)f'_s(z) \quad \text{for all} \quad z \in D_s.
\]

The evolution family \((\varphi_{s,t})\) of the Loewner chain \((f_t)\) solves for every fixed \( s \geq 0 \) and \( z \in D_s \) the ODE

\[
\frac{d\varphi_{s,t}(z)}{dt} = G(\varphi_{s,t}(z), t), \quad \text{a.e.} \quad t \geq s.
\]

Essential uniqueness means here that any two vector fields satisfying (2.2) can differ only for values of \( s \) forming a null-set on the real line.

The proof of Corollary 2.4 is very similar to that of [9, Theorem 4.1(1)], so we omit it. We call the vector field \( G \) in the second statement the \textit{vector field associated with} the Loewner chain \((f_t)\).

3. From evolution families to Loewner chains

This section is devoted to the proof of Theorem 1.10 establishing the existence, in the doubly connected setting, of a standard Loewner chain of order \( d \) associated with a given evolution family of order \( d \).

Important role in our discussion is played by the class \( \mathbb{M}(r_1, r_2) \) of all functions \( \psi \in \text{Hol}(\mathbb{A}_{r_1}, \mathbb{A}_{r_2}) \), \( 1 > r_1 \geq r_2 \geq 0 \), such that \( I(\psi \circ \gamma) = I(\gamma) \) for every closed curve \( \gamma \subset \mathbb{A}_{r_1} \).

We will make use of the following two lemmas.

Lemma 3.1 ([10, Lemma 4.7]). Suppose \(((D_t), (\varphi_{s,t}))\) is an evolution family of order \( d \in [1, +\infty] \). Let \( s \geq 0 \). Then the following statements are true:

(i) for each \( z \in D_s \) the function \( t \mapsto \varphi_{s,t}(z) \) belongs to \( AC^d([s, +\infty), \mathbb{C}) \);

(ii) the mapping \( t \mapsto \varphi_{s,t} \in \text{Hol}(\mathbb{A}_{r(s)}, \mathbb{D}^*) \) is continuous in \([s, +\infty)\);

(iii) \( \varphi_{s,t} \in \mathbb{M}(r(s), r(t)) \) for any \( t \geq s \);

(iv) \( \varphi_{s,t} \) is univalent in \( D_s \) for any \( t \geq s \).

Lemma 3.2. Let \(((D_t), (\varphi_{s,t}))\) be an evolution family of order \( d \in [1, +\infty] \). Let \((f_t)_{t \geq 0}\) be a family of univalent functions \( f_t : D_t \to \mathbb{C} \). If \( f_s = f_t \circ \varphi_{s,t} \) for any \( s \geq 0 \) and any \( t \geq s \), then \((f_t)\) is a Loewner chain of order \( d \) associated with \(((D_t), (\varphi_{s,t}))\).

Proof. We follow ideas of the 4th step in the proof of [5, Theorem 4.5]. Conditions LC1 and LC2 in Definition 1.8 follow easily from the condition of the lemma: \( f_s(D_s) = f_t(\varphi_{s,t}(D_s)) \subset f_t(D_t) \) for any \( s \geq 0 \) and any \( t \geq s \). So we only need to prove LC3.

First of all, we note that by [10, Theorem 5.1(A)] there exists a semicomplete weak holomorphic vector field \( G : \{(z, t) : t \geq 0, \ z \in D_t\} \to \mathbb{C} \) such that for any \( s \geq 0 \) and \( z \in D_s \), the function \([s, +\infty) \ni t \mapsto w_s^*(z, \cdot) \) solves the initial value problem \( \dot{w} = G(w, t) \),
Let us assume the contrary. Then there exist sequences \((z_n), (w_n)\) and \((t_n)\) such that \((z_n), (w_n), t_n) \in \tilde{K}\) and

\[
|\varphi_{t_n}(w_n) - \varphi_{t_n}(z_n)| > n|w_n - z_n| \quad \text{for all } n \in \mathbb{N}.
\]

By the compactness of \(\tilde{K}\) we may assume that the sequences \((z_n), (w_n)\) and \((t_n)\) converge to some \(z_0, w_0\) and \(t_0\), respectively. Clearly, \(t_0 \in I\) and \(z_0, w_0 \in D_0\). Moreover, the left-hand side of \((3.3)\) is bounded and consequently \(w_0 = z_0\). Since, by the definition of a canonical domain system, \(D_t = \mathbb{A}_{r(t)}\), where \(r : [0, +\infty) \to [0, 1]\) is continuous, there exists \(n_0 \in \mathbb{N}\) and \(\tau \in I\) such that \(t_n \in [\tau, T]\) for all \(n > n_0\) and \(X := \{z_n : n > n_0\} \cup \{w_n : n > n_0\} \cup \{z_0\}\) is a compact subset of \(D_{\tau}\). Since \(\varphi_{t,T}(D_{\tau}) \subset \mathbb{D}\) for any \(t \in [\tau, T]\), the family \(\{\varphi_{t,T} : t \in [\tau, T]\}\) is normal in \(D_{\tau}\) and consequently there exists \(M := M(X, \tau, T) > 0\) such that

\[
|\varphi_{t,T}(w) - \varphi_{t,T}(z)| \leq M|w - z| \quad \text{for any } z, w \in X \text{ and } t \in [\tau, T].
\]

The latter contradicts \((3.3)\). This proves \((3.2)\).

Further, by \([10]\) Proposition 4.5 there exists non-negative function \(k_{K,I} \in L^d(I, R)\) such that

\[
|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \leq \int_u^t k_{K,I}(\xi)d\xi
\]

for any \(z \in K\) and all \(s, u, t \in I\) satisfying \(s \leq u \leq t\).

Thus from \((3.1)\), \((3.2)\) and \((3.4)\) we deduce that for any \(s \in I\), \(t \in [s, T]\), and \(z \in K\),

\[
|f_t(z) - f_s(z)| = |f_T(\varphi_{s,T}(z)) - f_T(\varphi_{s,T}(z))|
\]

\[
\leq C|\varphi_{s,T}(z) - \varphi_{t,T}(z)| = C|\varphi_{t,T}(\varphi_{s,T}(z)) - \varphi_{t,T}(z)|
\]

\[
\leq CC'|\varphi_{s,T}(z) - z| = CC'|\varphi_{s,T}(z) - \varphi_{s,s}(z)| \leq CC' \int_s^t k_{K,I}(\xi)d\xi.
\]

This completes the proof. \(\square\)
Now we recall some basic properties of the module of a doubly connected domain. Given any path-connected topological space $X$, we denote by $\Pi_1(X)$ its fundamental group. By the base point of a closed curve $\gamma : [0, 1] \to X$ we mean the point $\gamma(0) = \gamma(1)$. Let $G \subset \mathbb{C}$ be a doubly connected domain. A closed curve $\gamma \subset G$ is homotopically nontrivial in $G$ if $\gamma$ is not homotopic in $G$ to its base point, i.e. if the equivalence class $[\gamma]_{\Pi(G)}$ is not the neutral element of $\Pi_1(G)$. It is known (see, e.g., [3, Chapter 1.D, Example 3]) that for every doubly connected domain $G \subset \mathbb{C}$ there exists a quantity $M(G) \in (0, +\infty]$ called the module of $G$ having the following properties:

M1. The module is invariant under conformal mappings, i.e. if $G_2 = f(G_1)$ and $f$ is a conformal mapping of $G_1$, then $M(G_2) = M(G_1)$.
M2. If $G_1 \subset G_2$ and any closed curve in $G_1$ homotopically nontrivial in $G_1$ is also homotopically nontrivial in $G_2$, then $M(G_1) \leq M(G_2)$.
M3. We have $M(\mathbb{D}^*) = M(\mathbb{C}^*) = +\infty$.
M4. If $0 < r_1 < r_2$, then $M(\mathbb{A}_{r_1,r_2}) = \frac{1}{2\pi} \log \frac{r_2}{r_1}$.

Finally, we will make use of the following remark without explicit reference.

Remark 3.3. Let $\gamma_j$, $j = 1, 2$, be closed curves in $G := \mathbb{A}_{r_1,r_2}$, $0 \leq r_1 < r_2 \leq +\infty$. The curve $\gamma_1$ is homotopic to $\gamma_2$ if and only if $I(\gamma_1) = I(\gamma_2)$.

**Proof of Theorem 1.10.** We divide the proof into several steps.

**Step 1.** There exists a Riemann surface $N$ and a family of mappings $(g_t : D_t \to N)$ such that

(i) $g_t$ is univalent for any $t \geq 0$;
(ii) $g_s(D_s) \subset g_t(D_t)$ whenever $0 \leq s < t < +\infty$;
(iii) $N = \cup_{t \geq 0} g_t(D_t)$;
(iv) $g_s = g_t \circ \varphi_{s,t}$ whenever $0 \leq s < t < +\infty$.

This statement can be easily established if one follows the proof of [5, Theorem 4.5], bearing in mind that in our case the domains of the functions $\varphi_{s,t}$ depend on $s$. Therefore we omit here the proof.

**Step 2.** The surface $N$ is doubly connected.

**Step 2a.** If, for some $s \geq 0$, a closed curve $\gamma \subset N_s := g_s(D_s)$ is homotopically nontrivial in $N_s$, then it is also homotopically nontrivial in $N$ and in $N_t$ for all $t \geq s$.

Indeed, for any $t \geq s$, a closed curve $\gamma \subset N_s$ is homotopically non-trivial in $N_t$ if and only if $I(g_t^{-1} \circ \gamma) \neq 0$. According to the property (iv) above and Lemma 3.1(iii),

$$I(g_t^{-1} \circ \gamma) = I(\varphi_{s,t} \circ g_s^{-1} \circ \gamma) = I(g_s^{-1} \circ \gamma).$$

Therefore, if $\gamma : [0, 1] \to N_s$, $\gamma(0) = \gamma(1)$, is homotopically nontrivial in $N_s$, then it is also homotopically nontrivial in $N_t$ provided $t \geq s$. Suppose that, at the same time, $\gamma$ is homotopically trivial in $N$. Then there exists a homotopy $H : [0, 1] \times [0, 1] \to N$ such

---

3Denote by $\Gamma$ the set of all closed rectifiable curves $\gamma \subset G$ that are homotopically nontrivial in $G$. By definition, the module of $G$ is the reciprocal of the extremal length of $\Gamma$. 

that $H(0, \cdot) = \gamma$ and $H(1, \cdot) \equiv \text{const.}$ In view of (ii) and (iii), by the compactness of $H([0, 1] \times [0, 1])$ we have $H([0, 1] \times [0, 1]) \subset N_t$ for all $t \geq 0$ large enough and hence $\gamma$ is homotopically trivial in $N_t$ for at least one $t \geq s$. This contradicts the statement we have just proved. Thus $\gamma$ must be also homotopically nontrivial in $N$.

Step 2b. The fundamental group $\Pi_1(N)$ is not trivial.

Fix $z_0 \in D_0$. The fundamental groups $\Pi_1(N_t), t \geq 0$, and $\Pi_1(N)$ can be realized as groups of equivalence classes $[\gamma]_{\Pi_1(N_t)}$ (respectively, $[\gamma]_{\Pi_1(N)}$) of closed curves with the base point at $g_0(z_0)$. Note that since each surface $N_t := g_t(D_t)$ is doubly connected, the fundamental group $\Pi_1(N_t)$ is isomorphic to $\mathbb{Z}$ for any $t \geq 0$.

Consider a closed curve $\alpha : [0, 1] \to D_0$ with the base point at $z_0$ such that $I(\alpha) = 1$. Then the equivalence class $[\gamma_0]_{\Pi_1(N_t)}$ of $\gamma_0 := g_0 \circ \alpha$ generates the fundamental group $\Pi_1(N_t)$. By Step 2a, it follows, in particular, that $\gamma_0$ is not homotopically trivial in $N$, so the fundamental group $\Pi_1(N)$ is not trivial.

Step 2c. The fundamental group $\Pi_1(N)$ is generated by one element.

By (3.5) with $\gamma := \gamma_0$ and $s := 0$, we have $I(g_t^{-1} \circ \gamma_0) = 1$ for all $t \geq 0$. Thus the equivalence class $[\gamma_0]_{\Pi_1(N_t)}$ of $\gamma_0$ also generates the fundamental group $\Pi_1(N_t)$.

We claim that $[\gamma_0]_{\Pi_1(N)}$ generates the fundamental group $\Pi_1(N)$, indeed, take any closed curve $\gamma : [0, 1] \to N$ with the base point at $g_0(z_0)$. Combining (ii), (iii) and the compactness of $\gamma([0, 1])$, we see that there exists $t \geq 0$ such that $\gamma([0, 1]) \subset N_t$. Therefore, there must exists $n \in \mathbb{Z}$ such that $[\gamma]_{\Pi_1(N_t)} = ([\gamma_0]_{\Pi_1(N_t)})^n$. But any homotopy in $N_t$ is also a homotopy in $N$ and this implies that $[\gamma]_{\Pi_1(N)}$ belongs to the subgroup generated by $[\gamma_0]_{\Pi_1(N_t)}$. This proves our claim.

Thus we have showed that $\Pi_1(N)$ is non-trivial and generated by one element. In particular, it is Abelian. According to [1] Theorem 1.129, $\Pi_1(N)$ is isomorphic either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. Since it is generated by one element, $\Pi_1(N)$ has to be isomorphic to $\mathbb{Z}$. That is, $N$ is doubly connected. By [1] Corollary 1.1.30, we conclude that there is a biholomorphism $H$ from $N$ onto either $\mathbb{C}^*$, $\mathbb{D}^*$, or an annulus $A_r$ for some $0 < r < 1$.

Write $h_t = H \circ g_t$. According to (i) – (iv),

(i') $h_t$ is univalent for all $t \geq 0$;
(ii') $h_t(D_s) \subset h_t(D_t)$ whenever $0 \leq s < t < +\infty$;
(iii') $\Omega := \bigcup_{t \geq 0} h_t(D_t)$ is either $\mathbb{C}^*$, $\mathbb{D}^*$, or an annulus $A_r$ for some $0 < r < 1$;
(iv') $h_s = h_t \circ \varphi_{s,t}$ whenever $0 \leq s < t < +\infty$.

Step 3. Let $r_\infty := \lim_{t \to +\infty} r(t)$. If $r_\infty = 0$, then $\Omega \in \{\mathbb{D}^*, \mathbb{C}^*\}$. Otherwise, $\Omega = A_{r_\infty}$.

If $\Omega = A_r$ for some $0 < r < 1$, set $a := r$. Otherwise, put $a := 0$. To simplify the exposition, we will assume as usual that $\frac{1}{a} = +\infty$ and $\log(+\infty) = +\infty$.

Take $\varepsilon \in (0, 1 - a)/2$. Then $A_{a + \varepsilon, 1 - \varepsilon} \subset \Omega$. By (i''), (iii'') and the compactness of $A_{a + \varepsilon, 1 - \varepsilon}$, there is $t_0 \geq 0$ such that $G_\varepsilon := A_{a + \varepsilon, 1 - \varepsilon} \subset h_t(D_t)$ for all $t > t_0$. Moreover, it is clear that any closed curve $\gamma \subset G_\varepsilon$ homotopically nontrivial in $G_\varepsilon$ is also homotopically nontrivial in $\Omega$ and hence must be homotopically nontrivial in $h_t(D_t) \subset \Omega$. Therefore,
from M1 – M4 we get
\[ \frac{1}{2\pi} \log \left( \frac{1 - \varepsilon}{a + \varepsilon} \right) = M(\mathbb{G}_\varepsilon) \leq M(h_t(D_t)) = M(D_t) = \frac{1}{2\pi} \log \left( \frac{1}{r(t)} \right). \]
Thus \((1 - \varepsilon)/(a + \varepsilon) \leq 1/r(t)\). Passing to the limit as \(t \to +\infty\) and then letting \(\varepsilon \to +0\) we get \(r_\infty \leq a\).

On the other hand, by Step 2a, any homotopically nontrivial closed curve in \(N_s\) is also homotopically nontrivial in \(N\). By conformal equivalence, this statement can be translated to the domains \(h_s(D_s)\) and \(\Omega\). Hence, using again M1 – M4, we may conclude that
\[ \frac{1}{2\pi} \log \left( \frac{1}{r(s)} \right) = M(D_s) = M(h_s(D_s)) \leq M(\Omega) = \frac{1}{2\pi} \log \left( \frac{1}{a} \right). \]
Passing to the limit as \(s \to +\infty\), we obtain the inequality \(r_\infty \geq a\). Therefore \(r_\infty = a\).

This means that if \(r_\infty = 0\), then \(M(\Omega) = \infty\) and \(\Omega \in \{ \mathbb{D}^*, \mathbb{C}^* \}\), while for \(r_\infty > 0\) we have \(\Omega = \mathbb{A}_{r_\infty}\).

**Step 4.** There is \(\kappa \in \{-1, 1\}\) such that \(I(h_t \circ \gamma) = \kappa I(\gamma)\) for any \(t \geq 0\) and any closed curve \(\gamma \in D_t\).

Fix \(z_0 \in D_0\). First of all we note that given \(t \geq 0\), any closed curve \(\gamma \subset D_t\) is homotopic in \(D_t\) to some closed curve \(\tilde{\gamma} \subset D_0\) with the base point at \(z_0\). In particular this means that
\[ I(\gamma) = I(\tilde{\gamma}). \]
Moreover, by Lemma 3.1(iii), \(I(\tilde{\gamma}) = I(\varphi_{0,t} \circ \tilde{\gamma})\). This means that \(\varphi_{0,t} \circ \tilde{\gamma}\) and \(\tilde{\gamma}\) are homotopic in \(D_t\). Therefore \(h_t \circ \gamma\) is homotopic to \(h_t \circ \varphi_{0,t} \circ \tilde{\gamma}\). Hence
\[ I(h_0 \circ \tilde{\gamma}) = I(h_t \circ \gamma). \]
From (3.6) and (3.7) it follows that in the proof of Step 4 we may fix \(t := 0\) and assume that \(\gamma\) has the base point at \(z_0\).

Now we claim that the mapping \(g_0\) establishes the isomorphism \(G^{g_0}\) between \(\Pi_1(D_0)\) and \(\Pi_1(N)\) that takes the equivalence class \([\gamma]_{\Pi_1(D_0)}\) of each closed curve \(\gamma \subset D_0\) with the base point at \(z_0\) to the equivalence class \([g_0 \circ \gamma]_{\Pi_1(N)}\) of \(g_0 \circ \gamma\). Indeed, \(G^{g_0}\) is a well-defined group homomorphism. Furthermore, according to the argument of Step 2, both fundamental groups are isomorphic to \(\mathbb{Z}\) and the generator \([\alpha]_{\Pi_1(D_0)}\) of \(\Pi_1(D_0)\) is mapped by \(G^{g_0}\) to the generator \([g_0 \circ \alpha]_{\Pi_1(N)}\) of \(\Pi_1(N)\). Thus \(G^{g_0}\) is an isomorphism.

Further, the biholomorphism \(\mathcal{H} : N \to \Omega\) defines in the canonical way the isomorphism \(G^H : \Pi_1(N) \to \Pi_1(\Omega)\).

Notice now that for the domains \(D_0\) and \(\Omega\) there exist a canonical isomorphisms of their fundamental groups onto \(\mathbb{Z}\), \(G^D : \Pi_1(D) \to \mathbb{Z}\), \(D \in \{ D_0, \Omega \}\), defined in the following way: \(G^D\) takes each \([\gamma]_{\Pi_1(D)}\) to \(I(\gamma)\).

Now consider the isomorphism \(G_Z := G_\Omega \circ G^H \circ G^{g_0} \circ (G_{D_0})^{-1} : \mathbb{Z} \to \mathbb{Z}\). The only two isomorphisms of \(\mathbb{Z} = (\mathbb{Z}, +)\) onto itself are the identity \(G_Z = id_{\mathbb{Z}}\) and \(G_Z : \mathbb{Z} \ni n \mapsto -n\).
In the former case we have \( I(h_0 \circ \gamma) = I(\gamma) \) for any closed curve \( \gamma \subset D_0 \) with the base point at \( z_0 \), while in the latter case we have \( I(h_0 \circ \gamma) = -I(\gamma) \) for all such \( \gamma \)'s.

Set \( f_t := h_t \) for all \( t \geq 0 \) if \( \kappa = 1 \), \( f_t := r_\infty / h_t \) for all \( t \geq 0 \) if \( \kappa = -1 \), \( r_\infty > 0 \), and \( f_t := 1 / h_t \) for all \( t \geq 0 \) if \( \kappa = -1 \), \( r_\infty = 0 \).

**Step 5.** \((f_t)\) is a standard Loewner chain of order \( d \) over \((D_t)\) associated with \((\varphi_{s,t})\).

From \((i')\) and \((iv')\) it follows that \((f_t)\) satisfies the condition of Lemma 3.2 Hence \((f_t)\) is an \( L^d\)-Loewner chain associated with \(((D_t), (\varphi_{s,t}))\).

The fact that \((f_t)\) is a standard Loewner chain follows from \((iii')\), the definition of \( f_t \), and Steps 3 and 4.

**Step 6.** If \((g_t)\) is another Loewner chain associated with \(((D_t), (\varphi_{s,t}))\), then there is a biholomorphism \( F : \cup_{t \geq 0} f_t(D_t) \to \cup_{t \geq 0} g_t(D_t) \) such that \( g_t = F \circ f_t \) for all \( t \geq 0 \).

The proof of this step is similar to an argument from the proof of [5, Theorem 4.9], so we omit it.

Since the statement of the theorem is the combination of Step 5 and Step 6, the proof is now finished. \( \square \)

### 4. Conformal Types of Loewner Chains via Evolution Families

This section is devoted to the classification of Loewner chains in terms of the limit behavior of their evolution families. We will prove Theorem 1.13 and Proposition 1.14 giving such a classification.

The proofs are based on following lemmas.

It is known that given a Jordan curve \( \gamma \subset \mathbb{C} \), there exists \( \kappa \in \{1,-1\} \) such that the index of \( w \) w.r.t. \( \gamma \), \( \text{ind}(\gamma, w) \in \{0,\kappa\} \) for all \( w \in \mathbb{C} \setminus \gamma \). As usual, we denote by \( \text{int}(\gamma) := \{w \in \mathbb{C} \setminus \gamma : \text{ind}(\gamma, w) \neq 0\} \) and \( \text{out}(\gamma) := \{w \in \mathbb{C} \setminus \gamma : \text{ind}(\gamma, w) = 0\} \).

**Lemma 4.1.** Let \( f : \mathbb{A}_r \to \mathbb{C}^* \) be a univalent function such that \( I(f \circ \gamma) = I(\gamma) \) for any closed curve \( \gamma \subset \mathbb{A}_r \). Then \( f(z) \in \text{out}(f(C(0, R))) \) whenever \( r < R < |z| < 1 \).

**Proof.** Consider an arbitrary \( z_0 \) satisfying \( R < |z_0| < 1 \). Let \( w_0 := f(z_0) \). We have to prove that \( \text{ind}(f \circ C^-, w_0) = 0 \), where \( C^- \) is the circle \( C(0, R) := \{z : |z| = R\} \) oriented clockwise.

Fix \( \tilde{R} \in (|z_0|, 1) \). By \( C^+ \) we denote the circle \( C(0, \tilde{R}) \) oriented counter-clockwise.

Since the equation \( f(z) - w_0 = 0 \) has exactly one zero \( z = z_0 \) in the annulus \( \mathbb{A}(R, \tilde{R}) := \{z : R < |z| < \tilde{R}\} \), by the argument principle we have

\[
(4.1) \quad 1 = \frac{1}{2\pi i} \int_{\partial \mathbb{A}(R, \tilde{R})} \frac{f'(z)}{f(z) - w_0} \, dz = \frac{1}{2\pi i} \int_{f \circ C^+} \frac{dw}{w - w_0} + \frac{1}{2\pi i} \int_{f \circ C^-} \frac{dw}{w - w_0}.
\]

By hypothesis, \( 1 = I(C^+) = I(f \circ C^+) \). Therefore,

\[
(4.2) \quad \frac{1}{2\pi i} \int_{f \circ C^+} \frac{dw}{w - w_0} = \text{ind}(f \circ C^+, w_0) \in \{0, 1\}.
\]
Analogously,
\[
\frac{1}{2\pi i} \int_{f \circ C^-} \frac{dw}{w - w_0} = \text{ind}(f \circ C^-, w_0) \in \{0, -1\}.
\]

Clearly, equations (4.1), (4.2), and (4.3) show that \(\text{ind}(f \circ C^-, w_0) = 0\). This completes the proof of the lemma.

**Lemma 4.2.** Let \(f : \mathbb{A}_r \to \mathbb{D}^*\), \(r \in [0, 1]\), be a univalent function such that \(I(f \circ \gamma) = I(\gamma)\) for any closed curve \(\gamma \subset \mathbb{A}_r\). Then
\[
|f(z)| \leq \frac{\pi}{\sqrt{2 \log(1/|z|)}}
\]
for all \(z \in \mathbb{A}_r\).

**Proof.** In this proof we will use again the notion and properties of module of a doubly connected domain, see Section [3].

Fix \(R \in (r, 1)\). Take \(z_0\) such that \(|z_0| = R\) and \(|f(z_0)| = N := \max\{|f(z)| : |z| = R\}\). Write \(w_0 := f(z_0)\). Consider an arbitrary closed rectifiable curve \(\gamma \subset f(\mathbb{A}_R)\) with \(I(\gamma) \neq 0\). Denote by \(L_1\) the ray \(-w_0[0, +\infty)\). Since \(I(\gamma) \neq 0\), we have \(\gamma \cap L_1 \neq \emptyset\).

In a similar way we may conclude that \(\gamma \cap L_2 \neq \emptyset\), where \(L_2\) stands for the ray \(w_0[1, +\infty)\). Indeed, the union of \(E := f(C(0, R)) \cup \text{int}(f(C(0, R))) \cup L_2 \cup \{\infty\}\) contains a Jordan arc connecting the origin with \(\infty\). Hence \(I(\gamma) \neq 0\) implies that \(\gamma \cap E \neq \emptyset\). Moreover, by Lemma [4.1] \(\gamma \subset \text{out}(f(C(0, R)))\). Therefore, \(\gamma \cap L_2 \neq \emptyset\).

Since \(\gamma\) is closed and \(\gamma \cap L_j \neq \emptyset, j = 1, 2\), it follows that the Euclidean length of \(\gamma\) is at least \(2|w_0| = 2N\).

Define \(\rho_0(z) := 1\) for \(z \in \mathbb{D}\) and \(\rho_0(z) := 0\) for \(z \notin \mathbb{D}\). Obviously, \(\rho_0 \in L^2(\mathbb{C})\). We denote by \(\text{len}_{\rho_0}(\gamma)\) the length of \(\gamma\) with respect to the metric \(\rho_0(z)|dz|^2\). Then, by the very definition of the module of a doubly connected domain (see, e.g., [3] Section I.D, Example [3]), we have
\[
\frac{(2N)^2}{\pi} \leq \frac{[\inf \text{len}_{\rho_0}(\gamma)]^2}{\|\rho_0\|^2_{L^2(\mathbb{C})}} \leq \frac{1}{M(f(\mathbb{A}_R))} = \frac{1}{M(\mathbb{A}_R)} = \frac{2\pi}{\log(1/R)},
\]
where the infimum is taken over all closed rectifiable curves \(\gamma \subset f(\mathbb{A}_R)\) with \(I(\gamma) \neq 0\). Hence
\[
N \leq \frac{\pi}{\sqrt{2 \sqrt{\log(1/R)}}}
\]
This finishes the proof.

**Lemma 4.3.** Let \((D_t) = (\mathbb{A}_{r(t)})\) be a canonical domain system of order \(d \in [1, +\infty]\) and \((\varphi_{s,t})\) an evolution family of the same order \(d \in [1, +\infty]\) over \((D_t)\). Then the following statements hold:
(i) For any $s \geq 0$ and any sequence $(t_n) \subset [s, +\infty)$ there exists a subsequence of $\psi_n \equiv \varphi_{s,t_n}$ that converges uniformly on compacta in $D_s$ either to a constant or to a univalent holomorphic function $\psi : D_s \to \mathbb{D}$. In the latter case, $\psi \in M(r(s), 0)$.

(ii) If there exist $s_0 \geq 0$ and a sequence $(t_n) \subset [s_0, +\infty)$ such that $t_n \to +\infty$ and $\varphi_{s_0,t_n}$ converges to a constant as $n \to +\infty$, then for all $s \geq 0$, $\varphi_{s,t} \to 0$ uniformly on compacta in $D_s$ as $t \to +\infty$.

(iii) Either $\rho_{z,s}(t) := |\varphi_{s,t}(z)| \to 0$ for all $s \geq 0$ and all $z \in D_s$, or there exists a positive function $\bar{\delta} : \mathbb{D} \to (0, +\infty)$ such that $\delta(z, s) < \rho_{z,s}(t) < 1 - \delta(z, s)$ whenever $0 \leq s \leq t$ and $z \in D_s$.

Proof. Statement (i) follows from the fact that the sequence $(\psi_n)$ forms a normal family in $D_s$ and from Hurwitz’s theorem. Indeed, $\psi_n(D_s) \subset \mathbb{D}$ for all $n \in \mathbb{N}$. Hence $(\psi_n)$ is normal in $D_s$. Let $(\psi_{n_k})$ be a subsequence converging uniformly on compacta in $D_s$ to a function $\psi$. All the functions $\psi_n$ are univalent in $D_s$. Hence, by the Hurwitz theorem, $\psi$ is either a constant in $\mathbb{D}$, or $\psi$ is univalent in $D_s$ and $\psi(D_s) \subset \mathbb{D}$. Clearly, given a closed curve $\sigma : [0, 1] \to \mathbb{C}^*$, there exists $\varepsilon > 0$ such that $I(\bar{\sigma}) = I(\sigma)$ for any closed curve $\bar{\sigma} : [0, 1] \to \mathbb{C}^*$ satisfying the inequality $|\bar{\sigma}(t) - \sigma(t)| < \varepsilon$ for all $t \in [0, 1]$. Taking $\sigma := \psi \circ \gamma$ and $\bar{\sigma} := \psi_{n_k} \circ \gamma$, where $\gamma$ is an arbitrary closed curve in $D_s$, we therefore conclude that $\psi \in M(r(s), 0)$, unless $\psi \equiv 0$.

The above argument proves (i) and shows that if $\psi = \text{const}$, then $\psi \equiv 0$. Therefore, to prove (ii) we may assume that $\varphi_{s_0,t_n} \to 0$ as $n \to +\infty$. Recall for any $s \geq 0$ and any $t \geq s$, 

$$\varphi_{0,t} = \varphi_{s,t} \circ \varphi_{0,s}. \tag{4.5}$$

Therefore, $\varphi_{0,t_n} \to 0$ in $D_0$ as $n \to +\infty$. The convergence is uniform on compacta because the family $(\varphi_{0,t})_{t \geq 0}$ is normal in $D_0$. Fix now $s \geq 0$. Taking into account that $\varphi_{0,s}$ is non-constant and using again (4.5) and the normality of $(\varphi_{s,t})_{t \geq s}$ in $D_s$, we conclude now that 

$$\varphi_{s,t_n} \to 0 \text{ uniformly on compact subsets of } D_s \text{ as } n \to +\infty. \tag{4.6}$$

Take any $n \in \mathbb{N}$ such that $t_n \geq s$ and let $t \geq t_n$. By Lemma 3.1, the function $f := \varphi_{t_n,t}$ satisfies the condition of Lemma 4.2 with $r := r(t_n)$. Note that $\varphi_{s,t} = \varphi_{t_n,t} \circ \varphi_{s,t_n}$. Hence (4.4) and (4.6) imply that $\varphi_{s,t} \to 0$ as $t \to +\infty$ uniformly on compacta in $D_s$. This proves (ii).

It remains to prove (iii). To this end assume that $\rho_{z_0,s_0}(t_n) \to 1$ or $\rho_{z_0,s_0}(t_n) \to 0$ for some $s_0 \geq 0$, some $z_0 \in D_s$, and some sequence $(t_n) \in [s_0, +\infty)$. Since $0 < \rho_{z_0,s_0}(t) < 1$ for all $t \in [s_0, +\infty)$ and the function $\rho_{z_0,s_0}$ is continuous by Lemma 3.1(i), we have that $t_n \to +\infty$ as $n \to +\infty$. Moreover, $\varphi_{s_0,t_n}(D_{s_0}) \subset \mathbb{D}$ for all $n \geq 0$. Hence passing if necessary to a subsequence, we may conclude that $\varphi_{s_0,t_n}$ converges to a constant as $n \to +\infty$. But then by (ii), for any $s \geq 0$, $\varphi_{s,t} \to 0$ in $D_s$ as $t \to +\infty$, i.e. $\rho_{z,s}(t) \to 0$ as $t \to +\infty$ for any $s \geq 0$ and any $z \in D_s$. This proves (iii). \qed

Now we can prove Theorem 1.13.
Proof of Theorem 1.13. Statement (i) of the theorem is already proved: it is equivalent to the statement of Step 3 in the proof of Theorem 1.10.

Now we assume \( r_\infty = 0 \).

First we prove (ii). So suppose that \( \varphi_{0,t} \) does not converge to 0 as \( t \to +\infty \). We have to prove that the Loewner range \( L[\varphi_{s,t}] \) of \( \varphi_{s,t} \) is \( \mathbb{D}^* \). According to Lemma 4.3, there exists a sequence \((t_n) \subset [0, +\infty)\) diverging to \( +\infty \) such that \( (\varphi_{0,t_n}) \) converges to some univalent function \( \varphi_{0,\infty} \) uniformly on compacta in \( D_0 \). For given \( s \geq 0 \) and all \( n \in \mathbb{N} \) large enough, by EF2 we have \( \varphi_{0,t_n} = \varphi_{s,t_n} \circ \varphi_{0,s} \), with \( \varphi_{0,s} \) being univalent in \( D_0 \) by Lemma 3.1(iv). Using again Lemma 4.3 and taking into account the normality of \( (\varphi_{s,t})_{t \geq s} \) in \( D_s \) we may conclude that \( (\varphi_{s,t_n}) \) also converges to some univalent function \( \varphi_{s,\infty} \) in \( D_s \) and that

\[
(4.7) \quad \varphi_{0,\infty} = \varphi_{s,\infty} \circ \varphi_{0,s}.
\]

By EF2, \( \varphi_{0,s} = \varphi_{s,u} \circ \varphi_{0,u} \) for any \( u \in [0, s] \). In combination with (4.7) this gives \( \varphi_{s,\infty} \circ \varphi_{s,u} \circ \varphi_{0,u} = \varphi_{u,\infty} \circ \varphi_{0,u} \). Since \( \varphi_{0,u} \) is not constant, by the Identity Theorem for holomorphic functions we get \( \varphi_{s,\infty} \circ \varphi_{s,u} = \varphi_{u,\infty} \). This holds for any \( s, u \geq 0 \) with \( u \leq s \). Then by Lemma 3.2, \( (\varphi_{t,\infty})_{t \geq 0} \) is a Loewner chain over \( (D_t) \) associated with \( (\varphi_{s,t}) \).

By Theorem 1.10 there exists a standard Loewner chain \((f_t)\) associated with \( (\varphi_{s,t}) \) and a biholomorphism \( F : L[\varphi_{s,t}] \to \Omega := \bigcup_{t \geq 0} \varphi_{t,\infty}(D_t) \subset \mathbb{D}^* \) such that \( \varphi_{t,\infty} = F \circ f_t \) for all \( t \geq 0 \).

We claim that for any closed curve \( \gamma \subset L[\varphi_{s,t}] \),

\[
(4.8) \quad I(F \circ \gamma) = I(\gamma).
\]

Indeed, fix such a curve \( \gamma \). By the compactness of \( \gamma \), there exists \( t \geq 0 \) such that \( \gamma \subset f_t(D_t) \) and hence \( \gamma = f_t \circ \gamma_t \) for some closed curve \( \gamma_t \subset D_t \). On the one hand, by the definition of a standard Loewner chain \( I(\gamma) = I(f_t \circ \gamma_t) = I(\gamma_t) \). On the other hand, \( \varphi_{t,\infty} \in \mathbb{M}(r(t), 0) \) by Lemma 4.3(i) and hence \( I(\varphi_{t,\infty} \circ \gamma_t) = I(\gamma_t) \). Recall that \( \varphi_{t,\infty} = F \circ f_t \). Now (4.8) follows easily.

Further, we note that \( L[\varphi_{s,t}] \neq \mathbb{C}^* \) because \( \Omega \) is a bounded domain. Moreover, from statement (i) we know that \( L[\varphi_{s,t}] \neq \mathbb{A}_r \) for any \( r \geq 0 \). Hence \( L[\varphi_{s,t}] \in \{ \mathbb{D}^*, \mathbb{C} \setminus \mathbb{D} \} \) and it remains to show that \( L[\varphi_{s,t}] \neq \mathbb{C} \setminus \mathbb{D} \). Assume on the contrary that \( L[\varphi_{s,t}] \neq \mathbb{C} \setminus \mathbb{D} \). The function \( F(1/z) \) is holomorphic and bounded in \( \mathbb{D}^* \). Hence it has a removable singularity at \( z = 0 \). Let \( G \) be its holomorphic extension to \( \mathbb{D} \). Apply the Argument Principle to this function on the circle \( \gamma := C(0, 1/2) \) oriented counterclockwise, so that \( I(\gamma) = 1 \). Then on the one hand, \( I(G \circ \gamma) \geq 0 \) because \( G \) has no poles in \( \mathbb{D} \). But on the other hand \( I(G \circ \gamma) = -1 \) by (4.8). This contradiction proves that \( L[\varphi_{s,t}] = \mathbb{D}^* \).

To complete the proof of statement (ii) we have to show that if \( \varphi_{0,t} \to 0 \), then \( L[\varphi_{s,t}] \neq \mathbb{D}^* \). Assume the contrary and let \( (f_t) \) stand again for a standard Loewner chain associated with \( (\varphi_{s,t}) \). Fix any \( z \in D_0 \). By Lemma 1.2 applied for \( f_t : \mathbb{A}_{r(t)} \to \mathbb{D}^* \), there

with \( t \geq 0 \), we get
\[
|f_0(z)| = |f_t(\varphi_{0,t}(z))| \leq \frac{\pi}{\sqrt{2 \log (1/|\varphi_{0,t}(z)|)}}.
\]
The left-hand side tends to zero as \( t \to +\infty \). Therefore, \( f_0(z) = 0 \) for all \( z \in D_0 \). This contradiction shows that \( L[(\varphi_{s,t})] \neq \mathbb{D}^* \). The proof of (ii) is now finished.

To prove (iii), we only need to apply statement (ii) to \((\tilde{\varphi}_{s,t})\) instead of \((\varphi_{s,t})\) and note that if \((f_t)\) is a standard Loewner chain associated with \((\varphi_{s,t})\) and \( r_\infty = 0 \), then by Lemma 3.2, \((\tilde{f}_t)\) is a standard Loewner chain associated with \((\varphi_{s,t})\), where \( \tilde{f}_t(z) := 1/f_t(r(t)/z) \), \( t \geq 0 \), \( z \in D_t \).

Finally, statement (iv) holds by the exclusion principle: \( L[(\varphi_{s,t})] = \mathbb{C}^* \) if and only if \( L[(\varphi_{s,t})] \notin \{\mathbb{D}^*, \mathbb{C} \setminus \partial \mathbb{D}, \mathbb{A}_r : r \in (0, 1)\} \). The proof is now complete. \( \square \)

At the end of the section we prove Proposition 1.14 giving conformal characterization of a Loewner chain via its evolution family in the degenerate and mixed-type cases.

**Proof of Proposition 1.14.** Let \((f_t)\) be a standard Loewner chain associated with \(((D_t), \varphi_{s,t})\). Notice that \((\varphi_{T+s,T+t})\) is an evolution family over \((D_{T+t})\), whose standard Loewner chain is \((f_{T+t})\). Moreover, it is evident that \( L[(f_{T+t})] = L[(f_t)] \).

Therefore, we may assume that \( T = 0 \) and \(((D_t), \varphi_{s,t})\) is of degenerate type. According to [10, Proposition 5.15], the functions defined by \( \phi_{s,t}(z) := \varphi_{s,t}(z) \) for \( z \in \mathbb{D}^* \), \( \phi_{s,t}(0) = 0 \), \( 0 \leq s \leq t \), form in this case an evolution family in the unit disk \( \mathbb{D} \). By [9, Theorem 1.6] there exist a Loewner chain \((g_t)\) in the unit disk \( \mathbb{D} \) associated with \((\phi_{s,t})\) such that \( g_t(0) = g_0(\phi_{0,t}(0)) = g_0(0) = 0 \) and \( \Omega := \cup_{t \geq 0} g_t(\mathbb{D}) \) is either a Euclidian disk centered at the origin or the whole complex plane \( \mathbb{C} \). Moreover, according to the same theorem, \( \Omega = \mathbb{C} \) if and only if \( \phi'_{0,t}(0) \to 0 \) as \( t \to +\infty \). Clearly, the latter condition is equivalent to the requirement that \( \tilde{\varphi}_{0,t} \to 0 \) as \( t \to +\infty \). Finally, we notice that (up to scaling in case \( \Omega \neq \mathbb{C} \)) the family \((g_t|_{\mathbb{D}^*})\) is a standard Loewner chain associated with \((\varphi_{s,t})\). This finishes the proof, since \( \cup_{t \geq 0} g_t(\mathbb{D}^*) = \Omega \setminus \{0\} \). \( \square \)

5. **Non-degenerate evolution families: convergence to zero**

For any \( r \in [0, 1) \) and any \( f \in \text{Hol}(\mathbb{A}_r, \mathbb{C}) \) we denote by \( \mathcal{N}(f) \) the free term in the Laurent development of \( f \):
\[
\mathcal{N}(f) := \int_T f(\rho \xi) \frac{|d\xi|}{2\pi}, \quad \rho \in (r, 1).
\]

**Theorem 5.1.** Let \((D_t) = (\mathbb{A}_{r(t)})\) be a non-degenerate canonical domain system of order \( d \in [1, +\infty] \) and \((\varphi_{s,t})\) an evolution family of the same order \( d \in [1, +\infty] \) over \((D_t)\). Suppose that \( r_\infty := \lim_{t \to +\infty} r(t) = 0 \). Then the following two statements are equivalent:

(A) For any \( s \geq 0 \), \( \varphi_{s,t} \to 0 \) uniformly on compacta in \( D_s \) as \( t \to +\infty \).
(B) The weak holomorphic vector field $G : D \rightarrow \mathbb{C}$ associated with $(\varphi_{s,t})$ satisfies the condition

$$
\int_{0}^{+\infty} \text{Re} \mathcal{N}(D_{t} \ni w \mapsto G(w,t)/w) \, dt = -\infty.
$$

Remark 5.2. According to Theorem B, the weak holomorphic vector field $G$ in the above theorem has the following representation

$$
G(w,t) = w \left[ iC(t) + \frac{r'(t)}{r(t)} p(w,t) \right], \quad \text{for a.e. } t \geq 0 \text{ and all } w \in D_{t},
$$

where $C \in L^{d}_{\text{loc}}([0, +\infty), \mathbb{R})$, and the function $p$ is measurable in $t$ and belongs, as a function of $w$, to the class $\mathcal{V}_{r}(t)$ for every fixed $t \geq 0$. Denote by $\mu^{t}_{1}$ and $\mu^{t}_{2}$ the measures in representation (1.5) for $p(\cdot, t)$. Then condition (5.1) takes the following form:

$$
- \int_{0}^{+\infty} r'(t) \frac{\mu^{t}_{1}(T)}{r(t)} \, dt = +\infty,
$$

while the negation of (5.1) is equivalent to the convergence of the above integral, because the integrand is non-positive for a.e. $t \geq 0$.

Proof of Theorem 5.1. Fix any $z \in D_{0}$. Denote $w(t) := \varphi_{0,t}(z)$ and $\rho(t) := |w(t)|$ for all $t \geq 0$.

In this proof we use the notation introduced in Remark 5.2. Using this remark, from the equation $\dot{w} = G(w,t)$ we get

$$
\frac{dp(t)}{dt} = r'(t) \frac{\rho(t)}{r(t)} \text{Re} \, p(w(t),t).
$$

Denote $\nu(t) := \mu^{t}_{1}(T)$. Note that $0 \leq \nu(t) \leq 1$ for all $t \geq 0$. From representation (1.5) and properties of the Villat kernel $\mathcal{K}_{r}(t)$ (see, e.g., [10, Remark 5.2]) it follows that

$$
\nu(t) \mathcal{K}_{r}(t)\left( - \rho(t) \right) + (1 - \nu(t)) \left[ 1 - \mathcal{K}_{r}(t) \left( r(t)/\rho(t) \right) \right] \leq \text{Re} \, p(w(t), t) \leq \nu(t) \mathcal{K}_{r}(t)\left( \rho(t) \right) + (1 - \nu(t)) \left[ 1 - \mathcal{K}_{r}(t)\left( - r(t)/\rho(t) \right) \right].
$$

Using the Laurent development of the Villat kernel, we get

$$
\mathcal{K}_{r}(x) - 1 = 2 \sum_{k=1}^{+\infty} \frac{x^{k} - (r^{2}/x)^{k}}{1 - r^{2k}} \leq 2 \sum_{k=1}^{+\infty} \frac{x^{k}}{1 - r^{2k}} \leq \frac{2}{1 - r^{2}} \frac{x}{1 - x}, \quad 0 < r < x < 1,
$$

while from (1.3) it follows that

$$
\mathcal{K}_{r}(-x) \geq \mathcal{K}_{0}(-x) = \frac{1 - x}{1 + x}, \quad 0 < r < x < 1.
$$

Let us first prove that (B) implies (A). Assume that statement (A) does not hold. Then by Lemma 4.3(iii), we have $1 - \delta > \rho(t) > \delta$ for some positive constant $\delta$ and all $t \geq 0$. 


Recall that $r(t) \to 0$ as $t \to +\infty$. Since the functions $t \mapsto r(t)$ and $t \mapsto \rho(t)$ are continuous and satisfy inequality $r(t) < \rho(t)$ for all $t \geq 0$, we can conclude that there exists $\delta_1 > 0$ such that $\rho(t) > r(t) + \delta_1$. Then taking into account that $t \mapsto \rho(t)$ is locally absolutely continuous in $[0, +\infty)$ and that $r'(t) \leq 0$ for a.e. $t \geq 0$, from (5.3) – (5.6) we get that for all $T > 0$,

\begin{align*}
(5.7) \quad \rho(0) - \rho(T) \geq - \int_0^T r'(t) \frac{\rho(t)}{r(t)} \left[ \nu(t) \frac{1 - \rho(t)}{1 + \rho(t)} - (1 - \nu(t)) \frac{2r(t)}{1 - r(t)^2} \frac{1}{\rho(t) - r(t)} \right] dt \\
\quad \quad \geq - \frac{\delta^2}{2} \int_0^T r'(t) \frac{\nu(t)}{r(t)} dt + \frac{2}{\delta_1} \int_0^T \frac{r'(t)}{1 - r(t)^2} dt.
\end{align*}

The left-hand side and the second term in the right-hand side of the above inequality are bounded. Hence the integral

\[- \int_0^T r'(t) \frac{\nu(t)}{r(t)} dt\]

is bounded from above. With the help of Remark 5.2 it follows that statement (B) fails to be true. Thus, (B)⇒(A).

It remains to prove that (A)⇒(B). Assume on the contrary that (B) does not hold, while (A) is true. Then on the one hand, the integral in (5.2) converges, but on the other hand, $\rho(t) \to 0$ as $t \to +\infty$. To obtain a contradiction we need another estimate for $K_r(x)$. Using again the Laurent development of the Villat kernel, we obtain

\begin{align*}
(5.8) \quad K_r(x) - 1 &= 2 \sum_{k=1}^{+\infty} \frac{x^k - (r^2/x)^k}{1 - r^{2k}} \leq 2 \sum_{k=1}^{+\infty} \left( x^k - (r^2/x)^k \right) \\
\quad &= \frac{2}{1 - r^2} \left( \frac{x}{1 - x} - \frac{r^2/x}{1 - r^2/x} \right) = \frac{2}{1 - r^2} \frac{(x^2 - r^2)/x}{(1 - x)(1 - r^2/x)} \\
\quad &\leq \frac{4(x - r)}{(1 - r^2)(1 - x)(1 - r^2/x)} \leq \frac{4(x - r)}{(1 - r^2)(1 - x)(1 - r)}, \quad 0 < r < x < 1.
\end{align*}

Applying (5.4) and (5.6), from (5.3) we get

\[- \left( \frac{1}{\sqrt{\rho(t)}} + \frac{r(t)}{\rho(t)^{3/2}} \right) \frac{d\rho(t)}{dt} \leq - r'(t) \left[ \frac{2}{\sqrt{\rho(t)}} + \frac{\nu(t)}{r(t)} F(t)K_r(t)(\rho(t)) + \frac{2\nu(t)}{\sqrt{\rho(t)}} \left( K_r(t)(\rho(t)) - 1 \right) \right],\]
where $F(t) := \sqrt{\rho(t) - r(t)} / \sqrt{\rho(t)}$ for all $t \geq 0$. Adding $2r'(t)/\sqrt{\rho(t)}$ to both sides and applying estimate (5.8), we finally obtain

\begin{equation}
(5.9) \quad -2 \frac{dF(t)}{dt} \leq -r'(t) \left[ \frac{\nu(t)}{r(t)} F(t) K_{r(t)}(\rho(t)) + \frac{8\nu(t)F(t)}{(1 - r(t)^2)(1 - r(t))(1 - \rho(t))} \right].
\end{equation}

Recall that $\rho(t) > r(t) > 0$ for all $t \geq 0$ and that both $r(t)$ and $\rho(t)$ tend to 0 as $t \to +\infty$. Hence $F(t) > 0$ for all $t \geq 0$ and $F(t) \to 0$ as $t \to +\infty$. In particular, since $F$ is continuous, there exists a sequence $(t_n) \subset [0, +\infty)$ tending to $+\infty$ such that for every $n \in \mathbb{N}$, $F(t_n) \geq F(t)$ whenever $t \geq t_n$. Then, from (5.9) for any $n \in \mathbb{N}$ we obtain

\[ 2F(t_n) \leq -F(t_n) \int_{t_n}^{+\infty} r'(t) \left[ \frac{\nu(t)}{r(t)} K_{r(t)}(\rho(t)) + \frac{8\nu(t)}{(1 - r(t)^2)(1 - r(t))(1 - \rho(t))} \right] dt. \]

Thus, bearing in mind that $F(t_n) > 0$,

\begin{equation}
(5.10) \quad 2 \leq -\int_{t_n}^{+\infty} r'(t) \left[ \frac{\nu(t)}{r(t)} K_{r(t)}(\rho(t)) + \frac{8\nu(t)}{(1 - r(t)^2)(1 - r(t))(1 - \rho(t))} \right] dt.
\end{equation}

By (5.5), $K_{r(t)}(\rho(t))$ is bounded. Recall also that by our assumption, the integral $\int_{0}^{+\infty} r'(t)\nu(t)/r(t) \, dt$ converges. Hence the integrals in the right-hand side of (5.10) (note that they depend on $n$) converge as well and their values tend to 0 as $n \to +\infty$. However, this fact contradicts inequality (5.10) for $n$ large enough, which completes the proof of the theorem. $\Box$

6. **Semicomplete weak holomorphic vector fields in the mixed-type case**

Consider a canonical domain system $(D_t) = (A_{r(t)})$ of some order $d \in [1, +\infty]$. Recall that $(D_t)$ is called *non-degenerate* (degenerate) if $r(t) > 0$ for all $t \geq 0$ ($r(t) \equiv 0$, respectively). If there exists $T \in (0, +\infty)$ such that $r(t) > 0$ for $t \in [0, T)$ and $r(t) = 0$ for $t \geq T$, then we say that $(D_t)$ is of *mixed type*.

In [10] §5.1 we established an explicit characterization of semicomplete weak holomorphic vector fields of order $d$ over a non-degenerate canonical domain system of the same order $d$, similar to the non-autonomous Berkson–Porta representation in Loewner Theory in the unit disk [3 Theorem 4.8]. The degenerate case was shown to be equivalent to the case of the unit disk with the common fixed point at the origin [10] §5.2.

In this section we will combine results mentioned above with Theorem 5.1 to obtain a characterization of semicomplete weak holomorphic vector fields in the mixed-type case. To simplify the formulation of our result we will use notation $\mathcal{V}_0$ for the Carathéodory class consisting, by definition, of all holomorphic functions $p : \mathbb{D} \to \mathbb{C}$ such that $p(0) = 1$ and $\text{Re} \, p(z) > 0$ for all $z \in \mathbb{D}$.
**Theorem 6.1.** Let $d \in [1, +\infty]$ and let $(D_t) = (A_{\tau(t)})$ be an $L^d$-canonical domain system of mixed type with $\mathcal{T} := \inf \{t \geq 0 : \tau(t) = 0\}$. Then a function $G : \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} := \{(z, t) : t \geq 0, z \in D_t\}$, is a semicomplete weak holomorphic vector field of order $d$ if and only if there exist functions $\alpha : [0, +\infty) \to [0, +\infty)$, $C : [0, +\infty) \to \mathbb{R}$, and $p : \mathcal{D} \to \mathbb{C}$ such that:

(i) $G(w, t) = w[iC(t) - \alpha(t)p(w, t)]$ for a.e. $t \geq 0$ and all $w \in D_t$;

(ii) for each $w \in D := \cup_{t \geq 0} D_t$ the function $p(w, \cdot)$ is measurable in $E_w := \{t \geq 0 : (w, t) \in \mathcal{D}\}$;

(iii) for each $t \geq 0$ the function $p(\cdot, t)$ belongs to the class $\mathcal{V}_{\tau(t)}$;

(iv) $C \in L^d_{\text{loc}}([0, +\infty), \mathbb{R})$;

(v) $\alpha(t) = -r'(t)/r(t)$ for a.e. $t \in [0, \mathcal{T})$ and $\alpha|_{[\mathcal{T}, +\infty)} \in L^d_{\text{loc}}([\mathcal{T}, +\infty), [0, +\infty))$;

(vi) the function $t \mapsto P(t) := \alpha(t)N(p(\cdot, t))$ belongs to $L^d_{\text{loc}}([0, +\infty), \mathbb{R})$.

**Remark 6.2.** Condition (vi) in the above theorem is equivalent, provided (ii)–(v) hold, to the requirement that $P|_{[0, \mathcal{T}]} \in L^d([0, \mathcal{T}], \mathbb{R})$. Indeed, from $N(K_r) = 1$ for any $r \in [0, 1)$, it follows that $0 \leq N(p(\cdot, t)) \leq 1$ for all $t \geq 0$. Condition (ii) implies that $t \mapsto N(p(\cdot, t))$ is measurable in $[0, +\infty)$. Now our claim is clear in view of condition (v).

In the proof of Theorem 6.1 we will use a change of variable, which clearly preserves evolution families of any given order. At the same time, the possibility of change of variable might be of some independent interest in principle. The following statement establishes much more general conditions for admissibility of a variable change.

**Proposition 6.3.** Let $[0, +\infty) \ni t \mapsto \tau(t) \in [0, +\infty)$ be increasing and locally absolutely continuous. Suppose that the inverse mapping $\tau^{-1} : [0, +\infty) \to [0, +\infty)$ is also locally absolutely continuous and that $\tau' \in L^\infty_{\text{loc}}([0, +\infty), \mathbb{R})$. Then:

(i) for any $d \in [1, +\infty]$ and any $L^d$-evolution family $((D_t), (\varphi_{s,t}))$, the formulas $\varphi_{s,t} := \varphi_{\tau(s), \tau(t)}$ and $D_t^* := D_{\tau(t)}$, where $0 \leq s \leq t$, define an $L^d$-evolution family $((D_t^*), (\varphi_{s,t}^*))$;

(ii) if $G : \mathcal{D} := \{(z, t) : t \geq 0, z \in D_t\} \to \mathbb{C}$ is a semicomplete weak holomorphic vector field of order $d$, then $G^* : \mathcal{D}^* := \{(z, t) : t \geq 0, z \in D_t^*\} \to \mathbb{C}$ defined by $G^*(z, t) := G(z, \tau(t))\tau'(t)$, $t \geq 0$, is also a semicomplete weak holomorphic vector field of order $d$;

(iii) if the vector field $G$ generates, in the sense of Theorem A, the evolution family $(\varphi_{s,t})$, then the vector field $G^*$ generates, in the same sense, the evolution family $(\varphi_{s,t}^*)$.

**Proof.** First of all note that since both $\tau$ and $\tau^{-1}$ are locally absolutely continuous, $\tau(I)$ and $f \circ \tau$ are measurable for any measurable set $I \subset [0, +\infty)$ and any measurable function $f : \tau(I) \to \mathbb{R}$. Moreover, we claim that

**Claim.** For any $d \in [1, +\infty]$, any measurable set $I \subset [0, +\infty)$, and any $f \in L^d(\tau(I), \mathbb{R})$, the function $(f \circ \tau)\tau'$ belongs to $L^d(I, \mathbb{R})$. 

Indeed, since $\tau^{-1}$ is locally absolutely continuous, $\|f \circ \tau\|_{L^\infty(I)} \leq \|f\|_{L^\infty(\tau(I))}$. This proves our claim for $d = +\infty$. Now assume $d \in [1, +\infty)$. Making change of variable in the integral and using the Hölder inequality, we get

$$
\|(f \circ \tau) \tau'|^d_{L^d(I)} = \int_I |f(\tau(t))|^d (\tau'(t))^d \, dt
= \int_{\tau(I)} |f(\xi)|^d (\tau'(\tau^{-1}(\xi)))^{d-1} \, d\xi
\leq \|f\|_{L^d(\tau(I))}^d \cdot \|\tau' \circ \tau^{-1}\|_{L^\infty(\tau(I))}^{d-1}.
$$

Since $\tau$ is locally absolutely continuous, $\|\tau' \circ \tau^{-1}\|_{L^\infty(\tau(I))} \leq \|\tau\|_{L^\infty(I)} < +\infty$. This completes the proof of the claim for $d \in [0, +\infty)$.

Now let us prove (i). Assume $(D_t) = (A_{\tau(t)})$ is a canonical domain system of order $d$, see Definition 1.1. Then the function $t \mapsto \omega(r(t))$ is of class $AC^d([0, +\infty), \mathbb{R})$. From the above claim it follows that $t \mapsto \omega(r(\tau(t)))$ is also of class $AC^d([0, +\infty), \mathbb{R})$, and hence $(D_t^*)$ is also a canonical domain system of order $d$. Furthermore, it is evident that the family $(\varphi^*_{s,t})$ satisfies conditions EF1 and EF2 from Definition 1.3. So we only need to prove EF3 for $(\varphi^*_{s,t})$. Fix $I := [S, T] \subset [0, +\infty)$ and $z \in D_S^*$. By EF3 for $(\varphi_{s,t})$ with $\tau(S)$ and $\tau(T)$ substituted for $S$ and $T$, respectively, there exists a non-negative function $k_{z,\tau(I)} \in L^d(\tau(I), \mathbb{R})$ such that

$$
(6.1) \quad |\varphi_{\tau(s),\tau(t)}(z) - \varphi_{\tau(s),\tau(t)}(u)| \leq \int_{\tau(u)}^{\tau(t)} k_{z,\tau(I)}(\xi) \, d\xi
$$

whenever $S \leq s \leq u \leq t \leq T$. Now let $k^*_{s,t} := (k_{z,\tau(I)} \circ \tau) \circ \tau'$. By our claim, $k^*_{s,t} \in L^d(I, \mathbb{R})$. Then, using the change of variable $\xi = \tau(\sigma)$, from (6.1) we get

$$
|\varphi^*_{s,t}(z) - \varphi^*_{s,t}(u)| \leq \int_s^t k^*_{s,t}(\sigma) \, d\sigma,
$$

which proves EF3 for $(\varphi^*_{s,t})$.

To prove (ii) we observe first that $G^*$ is a weak holomorphic vector field of order $d$. Indeed, conditions WHVF1 and WHVF2 in Definition 1.4 hold trivially, while WHVF3 holds with $(k_{\tau_s(K)} \circ \tau) \tau'$ substituted for $k_K$, where $\tau_s : (z, t) \mapsto (z, \tau(t))$ and $k_{\tau_s(K)}$ is the function of class $L^d$ from condition WHVF3 for the original vector field $G$. The fact that $G^*$ is semicomplete follows from the fact that $G$ is semicomplete and that if $t \mapsto w(t)$ solves the equation $dw/dt = G(w, t)$ then the function $w^* := w \circ \tau$ is a solution to $dw^*/dt = G^*(w^*, t)$. By the same reason, (iii) takes place. Thus the proof is complete. \(\square\)

**Proof of Theorem 6.1** Let us prove first that conditions of the theorem are necessary for $G$ to be a semicomplete weak holomorphic field of order $d$. By Theorem A any semicomplete weak holomorphic field of order $d$ over $(D_t)$ generates an evolution family $(\varphi_{s,t})$
over \((D_t)\) of the same order \(d\) such that \((d/dt)\varphi_{s,t}(z) = G(\varphi_{s,t}(z), t)\) for any \(s \geq 0\) and a.e. \(t \geq s\).

According to Proposition 6.3(i) the formulas
\[
\varphi_{s,t}^1 := \varphi_{\tau(s), \tau(t)}, \quad \tau(t) := T(1 - e^{-t}), \quad \varphi_{s,t}^2 := \varphi_{s+T, t+T}
\]
define two evolution families of order \(d\): \((\varphi_{s,t}^1)\) is an evolution family over the non-degenerate \(L^d\)-canonical domain system \((D_t^1) := (D_{\tau(t)})\) and \((\varphi_{s,t}^2)\) is an evolution family over the degenerate canonical domain system \((D_t^2) := (D_{t+T})\). Moreover, the semicomplete weak holomorphic vector fields \(G^1\) and \(G^2\) corresponding in the sense of Theorem A to the evolution families \((\varphi_{s,t}^1)\) and \((\varphi_{s,t}^2)\), respectively, are given for a.e. \(t \geq 0\) by
\[
G^1(z, t) = \frac{d\varphi_{s,t}^1(z)}{dt}|_{s=0} = G(z, \tau(t))T e^{-t}, \quad G^2(z, t) = \frac{d\varphi_{s,t}^2(z)}{dt}|_{s=0} = G(z, t + T).
\]

Hence, using Theorem 5.6 and Proposition 5.16 from [10], one can conclude that there exist functions \(\alpha : [0, +\infty) \to [0, +\infty),\ C : [0, +\infty) \to \mathbb{R},\) and \(p : \mathcal{D} \to \mathbb{C}\) satisfying conditions (i) – (iii) and (v).

To prove (iv) we use essentially the same argument as in [10]. Note that \(\mathcal{N}(\mathcal{K}_T) = 1\) for any \(r \in [0, 1].\) Hence \(\text{Im} \mathcal{N}(p(\cdot, t)) = 0\) for all \(t \geq 0\) and consequently \(C(t) = (1/2\pi) \text{Im} \int_T G(\rho \xi, t)/|\rho \xi| \, d\xi,\) where we have fixed some \(\rho \in (r(0), 1).\) Since by definition \(G(\cdot, z)\) is measurable for all \(z \in \mathbb{A}_{r(0)}\) and for every \(T > 0\) there exists a non-negative \(k_T \in L^d([0, T], \mathbb{R})\) such that \(|G(z, t)| \leq k_T(t)\) whenever \(|z| = \rho\) and \(t \in [0, T],\) it follows with the help of the Lebesgue dominated convergence theorem that \(t \mapsto C(t)\) belongs to \(L^d_{\text{loc}}([0, +\infty), \mathbb{R}).\)

To check condition (vi) fix any \(z_0 \in D_0\) and denote \(\rho(t) := |\varphi_{0,t}(z_0)|\) for all \(t \geq 0.\) By continuity of \(t \mapsto \rho(t)\) and \(t \mapsto r(t)\) there exists \(\delta > 0\) such that \(r(t) + \delta < \rho(t) < 1 - \delta\) for all \(t \in [0, \tau]\). Repeating the argument to deduce inequality \((6.4)\) in the proof of Theorem 5.1, we can conclude that for any \(S\) and \(T\) satisfying \(0 \leq S \leq T \leq \tau,\)
\[
\rho(S) - \rho(T) \geq -\frac{\delta^2}{2} \int_S^T r'(t) \frac{\nu(t)}{r(t)} \, dt + \frac{2}{\delta} \int_S^T \frac{r'(t)}{1 - r(t)^2} \, dt,
\]
where \(\nu(t) := \mathcal{N}(p(\cdot, t))\) for all \(t \in [0, \tau]\). From the definition of a canonical domain system of order \(d\) and the definition of an \(L^d\)-evolution family it follows that the functions \(t \mapsto r(t)\) and \(t \mapsto \rho(t)\) belong to \(AC^d([0, \tau], \mathbb{R}).\) Hence from \((6.4)\) it follows that the function \(t \mapsto -r'(t)\nu(t)/r(t)\) belongs to \(L^d([0, \tau], \mathbb{R}).\) In view of Remark 6.2, this means that (vi) holds.

Thus we have proved that conditions of the theorem are necessary for \(G\) to be a semicomplete weak holomorphic field of order \(d.\) Let us now show that they are also sufficient.

Assume that there exist functions \(\alpha : [0, +\infty) \to [0, +\infty),\ C : [0, +\infty) \to \mathbb{R},\) and \(p : \mathcal{D} \to \mathbb{C}\) satisfying conditions (i) – (vi). The first step to prove that \(G\) is a semicomplete weak holomorphic vector field of order \(d,\) is to check conditions WHVF1, WHVF2, and
WHVF3 from Definition 1.4. The first two of these conditions follow directly from (i) – (v) if it is taken into account that $t \mapsto r'(t)/r(t)$ is measurable in $[0, \mathcal{T})$ according to the definition of a canonical domain system.

Let us show that $G$ satisfies also WHVF3. To this end we have to obtain some estimates for the Villat kernel. Using its Laurent development, we in particular get

$$|K_r(z) - 1| = 2 \left| \sum_{k=1}^{+\infty} \frac{z^k - (r^2/z)^k}{1 - r^{2k}} \right| \leq \frac{2}{1 - r^2} \sum_{k=1}^{+\infty} \left( |z|^k + |r^2/z|^k \right)$$

$$\leq \frac{2}{1 - r^2} \left( \frac{|z|}{1 - |z|} + \frac{r^2/|z|}{1 - r^2/|z|} \right), \quad 0 < r < |z| < 1,$$

It follows that

$$|K_r(z)| \leq A(r, |z|) := 1 + \frac{2}{1 - r^2} \left( \frac{|z|}{1 - |z|} + \frac{r}{|z| - r} \right),$$

$$|K_r(r/z) - 1| \leq B(r, |z|) := \frac{2r}{1 - r^2} \left( \frac{1}{1 - |z|} + \frac{1}{|z| - r} \right), \quad 0 < r < |z| < 1,$$

from which we deduce, using Remark 5.2, that for all $t \in [0, \mathcal{T})$,

$$p(z, t)| \leq A(r(t), |z|)\mathcal{N}(p(\cdot, t)) + B(r(t), |z|)\left[1 - \mathcal{N}(p(\cdot, t))\right].$$

The above inequality holds also for $t \geq \mathcal{T}$, because then $r(t) = 0$ and consequently

$$\mathcal{N}(p(\cdot, t)) = 1, \quad A(r(t), |z|) = A(0, |z|) = (1 + |z|)/(1 - |z|),$$

and (6.5) reduces to the well-known estimate for the Carathéodory class, see e.g. [19, ineq. (11) on p. 40].

We proceed as in the proof of [10] Lemma 5.14. Let us fix any compact set $K \subset \mathcal{D}$. Then there exists $\delta > 0$ and $T > 0$ such that $r(t) + \delta \leq |z| \leq 1 - \delta$ and $t \leq T$ for all $(z, t) \in K$. From (i) and (6.5) we deduce that

$$|G(z, t)| \leq |C(t)| + (1 + M)\alpha(t)\mathcal{N}(p(\cdot, t)) + M\alpha(t)r(t) \quad \text{for all} \ (z, t) \in K,$$

where

$$M := \frac{4/\delta}{1 - r(0)^2}.$$

Recall that $t \mapsto r'(t)$ belong to $L^d([0, \mathcal{T}], \mathbb{R})$. Hence from (iv) – (vi) it follows that the right-hand side of (6.6) is a function of $t$ from $L^d_{loc}(\mathcal{T}, \mathbb{R})$. This completes that proof of WHVF3.

It remains to show that the weak holomorphic vector field $G$ is semicomplete. To this end we write:

$$G^1(w, t) := G(w, \tau(t))\mathcal{T}e^{-t}, \quad t \geq 0, \ w \in D_{\tau(t)},$$
where \( \tau(t) := T(1 - e^{-t}) \), and

\[
G^2(w, t) := G(w, t + T), \quad t \geq 0, \ w \in D_{t+T}.
\]

By Theorem 5.6 and Proposition 5.18 from [10], the functions \( G^1 \) and \( G^2 \) are semicomplete weak holomorphic vector fields of order \( d \) in \( D_1 := \{(w, t) : t \geq 0, w \in D_{\tau(t)}\} \) and \( D_2 := \{(w, t) : t \geq 0, w \in D_{t+T}\} \) respectively. Therefore, by [10 Theorem 5.1(B)], there exist unique evolution families \((\varphi_{s,t}^1)\) and \((\varphi_{s,t}^2)\) over the canonical domain systems \((D_{\tau(t)})\) and \((D_{t+T})\) respectively such that \((\partial/\partial t)\varphi_{s,t}^1(z_1) = G^1(\varphi_{s,t}^1(z_1), t)\) and \((\partial/\partial t)\varphi_{s,t}^2(z_2) = G^2(\varphi_{s,t}^2(z_2), t)\) for all \( s \geq 0 \), a.e. \( t \geq s \) and any \( z_1 \in D_{\tau(t)} \), \( z_2 \in D_{t+T} \).

By construction, the equation \( dw/d\tau = G(w, \tau) \) is equivalent to \( dw(\tau(t))/dt = G^1(w(\tau(t)), t) \) when \( \tau \in [0, T) \) and to the equation \( dw(t + T)/dt = G^2(w(t + T), t) \) when \( \tau \geq T \). It now follows from the general theory of Carathéodory ODEs (see, e.g., [10 Theorem 2.3]) that it is sufficient to show that given \( s \geq 0 \) and \( z \in D_{\tau(s)} \) there exists \( \delta = \delta(z, s) > 0 \) such that \( r(\tau(t)) + \delta < |\varphi_{s,t}^1(z)| < 1 - \delta \) for all \( t \geq s \). Recall that \( t \mapsto r(\tau(t)) \) and \( t \mapsto |\varphi_{s,t}^1(z)| \) are continuous functions in \([s, +\infty)\), with \( r(\tau(t)) < |\varphi_{s,t}^1(z)| < 1 \) for all \( t \geq 0 \). Hence according to Lemma 4.3, it remains to see that \( \varphi_{0,t}^1 \not\to 0 \) as \( t \to +\infty \).

The latter follows readily from (vi) and Theorem 5.1. The proof is complete. \( \square \)

7. Conformal types of Loewner chains via vector fields

In this section we combine results of Sections 4 - 6 to get the conformal classification of Loewner chains in terms of the corresponding vector fields.

Let \( d \in [1, +\infty] \). Let \((D_t) = (A_{r(t)})\) be a canonical domain system of order \( d \) and \((f_t)\) a Loewner chain of order \( d \) over \((D_t)\). By Corollary 2.4 there exists a unique vector field \( G \) associated with \((f_t)\), i.e., a weak holomorphic vector field \( G \) of order \( d \) over \((D_t)\) satisfying for a.e. \( s \geq 0 \) equality \( 2.2 \).

Let us assume first that \((D_t)\) is non-degenerate, i.e. \( D_t := A_{r(t)} \) with \( r(t) > 0 \) for each \( t \geq 0 \). The following theorem characterizes the conformal type of \((f_t)\) via the associated vector field \( G \).

Recall (see Remark 5.2) that for a.e. \( t \geq 0 \), the function \( G_t = G(\cdot, t) \) admits the following representation

\[
G_t(z) = z \left( \frac{r'(t)}{r(t)} p_t(z) + iC_t \right),
\]

where \( C_t \in \mathbb{R} \),

\[
p_t(z) := \int_T K_r(z/\xi) d\mu_1^t(\xi) + \int_T [1 - K_r(r\xi/z)] d\mu_2^t(\xi), \quad z \in A_r, \ r := r(t),
\]
Proposition 7.2. In other words, we suppose that \( \varphi \) of \([10, \text{Example 6.3}]\) we conclude that \( \tilde{\varphi} \) if \( \mathcal{I} \). Assume now that the canonical domain system \((\mathcal{D}_t)\) be the evolution family of the Loewner chain \((f_t)\) and let \( \tilde{\varphi}_{s,t}(z) := r(t)/\varphi_{s,t}(r(s)/z) \) for all \( s \geq 0 \), all \( t \geq s \) and all \( z \in D_s \). Then by Theorem 5.1 combined with Remark 6.2 \( \varphi_{0,t} \to 0 \) as \( t \to +\infty \) if and only if \( I_1 = +\infty \). Similarly, with the help of \([10] \text{Example 6.3}\) we conclude that \( \tilde{\varphi}_{0,t} \to 0 \) as \( t \to +\infty \) if and only if \( I_2 = +\infty \).

Now Theorem 7.1 follows immediately from Theorem 1.13.

Assume now that the canonical domain system \((\mathcal{D}_t)\) is either degenerate or of mixed type. In other words, we suppose that \( \{t \geq 0 : r(t) = 0\} \neq \emptyset \). Let us denote
\[
I := - \int_0^{+\infty} \text{Re} \mathcal{N}(w \mapsto G(w)/w) \, dt.
\]

Theorem 7.1. In the above notation, the following statements hold:

(i) the evolution family \((\varphi_{s,t})\) is of type I if and only if \( I_1 + I_2 < +\infty \);
(ii) the evolution family \((\varphi_{s,t})\) is of type II if and only if \( I_1 < +\infty \) and \( I_2 = +\infty \);
(iii) the evolution family \((\varphi_{s,t})\) is of type III if and only if \( I_1 = +\infty \) and \( I_2 < +\infty \);
(iv) the evolution family \((\varphi_{s,t})\) is of type IV if and only if \( I_1 = I_2 = +\infty \).

Proof of Theorem 7.1. Note that \( I_1, I_2 \geq 0 \) and that \( I_1 + I_2 < +\infty \) if and only if \( r_\infty := \lim_{t \to +\infty} r(t) > 0 \).

Choose \( T > 0 \) such that \( r(T) = 0 \). From \([10] \text{Proposition 5.15}\) it follows that the family \((\phi_{s,t})_{0 \leq s \leq t} \) defined by \( \phi_{s,t}(z) := \varphi_{s,T+t+T}(z) \) for all \( z \in \mathbb{D}^* \) and by \( \phi_{s,t}(0) = 0 \), is an \( L^2 \)-evolution family in the unit disk. Hence the vector field \( G(w,t) := G(w,t+T) \), \( t \geq 0 \), \( w \in \mathbb{D}^* \), extended to the origin by \( G(0,t) = 0 \), is a Herglotz vector field in \( \mathbb{D} \) and \( d\phi_{s,t}(z)/dt = G(\phi_{s,t}(z),t) \) for all \( z \in \mathbb{D} \) and a.e. \( t \geq 0 \). It follows that
\[
|\phi_{s,t}'(0)| = \exp \int_0^t \text{Re} G'(0,\xi) \, d\xi = \exp \left( - \int_0^t \text{Re} \mathcal{N}(w \mapsto G(w,\xi)/w) \, d\xi \right).
\]
Thus $\phi_{0,t}(0) \to 0$ as $t \to +\infty$ if and only if $I = +\infty$. This completes the proof. 

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