FUNDAMENTAL SOLUTION TO 1D DEGENERATE DIFFUSION EQUATION
WITH LOCALLY BOUNDED COEFFICIENTS

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ABSTRACT. In this work we study the degenerate diffusion equation \( \partial_t u_f(x,t) = x^\alpha a(x) \partial_x^2 u_f(x,t) + b(x) \partial_x u_f(x,t) \) for \((x,t) \in (0,\infty)^2\), equipped with a Cauchy initial data and the Dirichlet boundary condition at 0. We assume that the order of degeneracy at 0 of the diffusion operator is \( \alpha \in (0,2) \), and both \( a(x) \) and \( b(x) \) are only locally bounded. We adopt a combination of probabilistic approach and analytic method: by analyzing the behaviors of the underlying diffusion process, we give an explicit construction to the fundamental solution \( p(x,y,t) \) and prove several properties for \( p(x,y,t) \); by conducting a localization procedure, we obtain an approximation for \( p(x,y,t) \) for \( x,y \) in a neighborhood of 0 and \( t \) sufficiently small, where the error estimates only rely on the local bounds of \( a(x) \) and \( b(x) \) (and their derivatives).

There is a rich literature on such a degenerate diffusion in the case of \( \alpha = 1 \). Our work extends part of the existing results to cases with more general order of degeneracy, both in the analysis context (e.g., heat kernel estimates on fundamental solutions) and in the probability view (e.g., wellposedness of stochastic differential equations).

1. INTRODUCTION

In this article we consider the following Cauchy initial value problem with the Dirichlet boundary condition:

\[
\begin{aligned}
\partial_t u_f(x,t) &= x^\alpha a(x) \partial_x^2 u_f(x,t) + b(x) \partial_x u_f(x,t) \quad \text{for } (x,t) \in (0,\infty)^2, \\
\lim_{t \to 0} u_f(x,t) &= f(x) \quad \text{for } x \in (0,\infty) \quad \text{and} \quad \lim_{x \to 0} u_f(x,t) = 0 \quad \text{for } t \in (0,\infty),
\end{aligned}
\]

where \( f \in C_c((0,\infty)) \) and \( \alpha \in (0,2) \). Set \( L := x^\alpha a(x) \partial_x^2 + b(x) \partial_x \). We further impose the following assumptions on \( a(x) \) and \( b(x) \):

(H1): \begin{align*}
  a &\in C\left((0,\infty)\right) \cap C^2\left((0,\infty)\right), \quad a(x) > 0 \quad \text{for every } x \in [0,\infty) \quad \text{and} \quad a(0) = 1. \\
  b &\in C\left((0,\infty)\right) \cap C^1\left((0,\infty)\right), \quad b(0) \in (0,1) \quad \text{when } \alpha \leq 1, \quad \text{and} \quad b(0) = 0 \quad \text{when } \alpha > 1. \\
  a(x), a'(x), a''(x), b(x) \text{ and } b'(x) &\text{ are all bounded on } (0,1] \quad \text{for every } I > 0.
\end{align*}

(H2): There exists \( C > 0 \) such that for every \( x \in [0,\infty) \),

\[ a(x) \leq C \left(1 + x^{2-\alpha}\right) \quad \text{and} \quad |b(x)| \leq C \left(1 + x\right). \]

Our goal is to construct and to study the fundamental solution \( p(x,y,t) \) to (1.1), with a particular emphasis on the behaviors of \( p(x,y,t) \) when \( x,y \) are near the boundary and when \( t \) is small. Since \( L \) is degenerate at 0, standard methods on strictly parabolic equation no longer apply in this case, and the degeneracy of \( L \) does have an impact on the regularity of \( p(x,y,t) \). Moreover, the assumptions (H1) and (H2) only guarantee local boundedness of \( a(x) \), \( b(x) \) and their derivatives, so we need to conduct our analysis of \( p(x,y,t) \) only relying on the local bounds of the coefficients.
1.1. Background and motivation. Our work is primarily motivated by two previous works \[8\] and \[7\] on related problems. \[8\] treats the initial/boundary value problem for a degenerate diffusion equation similar to the one in \[1.1\], but under stronger conditions on the coefficients. To interpret the hypotheses adopted in \[8\] in terms of \(\alpha, a(x)\) and \(b(x)\) in our setting, we define the following two functions for \(x > 0\):

\[
\phi(x) := \frac{1}{4} \left( \int_0^x \frac{ds}{s^\alpha a(s)} \right)^2 \quad \text{and} \quad \theta(x) := \frac{1}{2} - \nu + \frac{2b(x) - (x^\alpha a(x))'}{2x^\alpha a(x)} \sqrt{\phi(x)},
\]

where \(\nu\) is the constant such that \(\lim_{x \to 0} \theta(x) = 0\). It is assumed in \[8\] that \(\nu < 1\), as well as

\[
\sup_{x \in (0, \infty)} \frac{\theta(x)}{\phi(x)} < \infty \quad \text{and} \quad \sup_{x \in (0, \infty)} \frac{\theta'(x)}{\phi'(x)} < \infty.
\]

Under the assumption \(1.3\), through a series of transformations and perturbations, \[8\] completes a construction of the fundamental solution \(p(x, y, t)\) to \(1.1\), conducts a careful analysis of the regularity properties of \(p(x, y, t)\) near the boundary, and derive an approximations for \(p(x, y, t)\) in terms of explicitly formulated functions. In particular, if \(p^{\approx}\) denotes the approximation for \(p(x, y, t)\), then it is proven in \[8\] that there exists a constant \(C > 0\), universal in all \(x, y\) in a neighborhood of 0 and all \(t\) sufficiently small, such that

\[
\left| \frac{p(x, y, t)}{p^{\approx}(x, y, t)} - 1 \right| \leq Ct.
\]

Such an estimate is useful in multiple ways. First, while one expects \(p(x, y, t)\) to resemble the fundamental solution to a strictly parabolic equation for \(x, y\) away from the boundary, \(1.4\) captures accurately the asymptotics of \(p(x, y, t)\) when \(x, y\) are close to the boundary, and demonstrates the influence of the degeneracy of \(L\) on \(p(x, y, t)\). Second, if one could apply the general heat kernel estimates (see, e.g., \$4\) of \[32\]) to \(p(x, y, t)\), then one would get that for every \(\delta\) and \(t\) sufficiently small, there is constant \(C_{\delta,t} > 1\) such that

\[
C_{\delta,t}^{-1} \exp \left( -\frac{d(x, y)^2}{2(1 - \delta) t} \right) \leq p(x, y, t) \leq C_{\delta,t} \exp \left( -\frac{d(x, y)^2}{2(1 + \delta) t} \right),
\]

where \(d(x, y)\) is the distance between \(x\) and \(y\) under the Riemannian metric corresponding to \(L\); it is clear that \(1.4\) is a sharper estimate than \(1.5\) for small \(t\), and hence \(p^{\approx}\) is a more accurate short-term approximation for \(p(x, y, t)\) compared with the general heat kernel approximation. In addition, in \[8\], \(p^{\approx}(x, y, t)\) is presented in an explicit formula (in terms of special functions) and “\(Ct\)” in \(1.4\) can be replaced by an exact expression; therefore, \(1.3\) is easily accessible in computational applications that involve the fundamental solution to any degenerate diffusion equation in the form of \(\partial_t - L = 0\).

We aim to generalize the results in \[8\], particularly the construction of \(p(x, y, t)\) and the short-term near-boundary approximation \(p^{\approx}(x, y, t)\), to a more general family of degenerate diffusion equations. The hypotheses \((H1)\) and \((H2)\) proposed above are more relaxed compared with the assumption \(1.3\) adopted in \[8\]. For example, it can be checked with direct computations that in general \(1.3\) does not hold if \(b(0) \neq 0\), which means that, given \((H1)\) and \((H2)\), \(1.3\) is only satisfied when \(1 \leq \alpha < 2\). Moreover, \(1.3\) clearly imposes strong global conditions on \(a(x)\) and \(b(x)\), but with \((H1)\) and \((H2)\), we have to find an access to \(p(x, y, t)\) without relying on any global bound on the coefficients. To tackle this issue, we invoke a “localization” procedure, as inspired by \[7\].

\[7\] studies the following well known Wright-Fisher diffusion equation, which has its origin in population genetics:

\[
\begin{align*}
\partial_t u_f(x, t) &= x(1 - x) \partial_x^2 u_f(x, t) \quad \text{for} \quad (x, t) \in (0, 1) \times (0, \infty), \\
\lim_{t \to 0} u_f(x, t) &= f(x) \quad \text{for} \quad x \in (0, 1) \quad \text{for some} \quad f \in C_0((0, 1)), \\
\text{and} \quad \lim_{x \to 0^+} u_f(x, t) &= \lim_{x \to 1^-} u_f(x, t) = 0 \quad \text{for} \quad t \in (0, \infty).
\end{align*}
\]
Different from (1.1), (1.6) has two-sided Dirichlet boundaries at 0 and 1, and the diffusion operator degenerates linearly at both boundaries. Set $L_{WF} := x(1 - x)\partial_x^2$ and let $p_{WF}(x, y, t)$ be the fundamental solution to (1.6). Since (1.6) is symmetric on $[0, 1]$, to study $p_{WF}(x, y, t)$ near the boundaries, it is sufficient to only consider the left boundary 0. In [7], a “localization” method is devised to construct $p_{WF}(x, y, t)$ near 0: since $p_{WF}(x, y, t)$ can be viewed as the density of the underlying diffusion process corresponding to $L_{WF}$, we can acquire information on $p_{WF}(x, y, t)$ by studying the behaviors of the process near 0; in particular, by tracking the excursions of the diffusion process near 0, we can “localize” $L_{WF}$ and $p_{WF}(x, y, t)$ within a neighborhood of 0 where only the degeneracy at 0 has a substantial impact. Heuristically speaking, when restricted near 0, $L_{WF}$ is close to the operator $x\partial_x^2$, and hence it is natural to expect that $p_{WF}(x, y, t)$ with $x, y$ near 0 is close to the fundamental solution $p_0(x, y, t)$ to $\partial_t - x\partial_x^2 = 0$ (with Dirichlet boundary 0). Indeed, it is established in [7] that, not only can $p_{WF}(x, y, t)$ be constructed in an explicit way via $p_0(x, y, t)$, $p_{WF}(x, y, t)$ is also well approximated by $p_0(x, y, t)$ in the sense that $p_{WF}(x, y, t)/p_0(x, y, t)$ satisfies (1.4) for $x, y$ near 0 and $t$ sufficiently small. In our work we want to adopt a similar localization procedure and start our investigation of (1.1) on a bounded set where the local bounds of the coefficients would be sufficient for our purposes.

In addition to treating directly the fundamental solutions, degenerate diffusion equations in the form of $\partial_t - L = 0$ have also been discussed in many other contexts, with most of the existing literature concerning the case when $\alpha = 1$. For example, Epstein et al ([15, 16, 17, 18, 19]) conduct an comprehensive study of the generalized Kimura operators, which can be viewed as a generalization of $L$ with $\alpha = 1$ in the manifold setting, obtaining results such as the Hölder space of the solutions, the maximum principle and the Harnack inequality. Related works on generalized Kimura diffusions include [18, 19, 30, 31]. From a probabilistic view, there are abundant theories on existence and uniqueness of solutions to stochastic differential equations with degenerate diffusion coefficients (see, e.g., [10, 20, 27, 28, 34, 36, 39] and the references therein); when $\alpha = 1$, a series of works (see, e.g., [1, 3, 4, 9, 14, 38]) provide conditions on $a(x)$ and $b(x)$ that are sufficient for the stochastic differential equation corresponding to $L$ to be well posed, and some of the results will also be used later in our discussions.

Degenerate diffusions have also been treated in the context of the measure-valued process (see, e.g., [6, 12, 13, 21, 22, 29]), as well as via the semigroup approach (see, e.g., [2, 5, 11, 23, 24, 37]).

1.2. Our main results. Our strategy in solving (1.1) and getting $p(x, y, t)$ is to combine the ideas and the techniques from [8] and [7], and tackle the two challenges we face: general order of degeneracy and hence it is natural to expect that $p(x, y, t)$ satisfies (1.4) for $x, y$ near 0 and $t$ sufficiently small. In our work we want to adopt a similar localization procedure and start our investigation of (1.1) on a bounded set where the local bounds of the coefficients would be sufficient for our purposes.

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1. Localization and transformation (§2.1). Since the coefficients in $L$ are locally bounded, we first consider a “localized” version of (1.1). Given $I > 0$, we study the diffusion equation in (1.1) on $(0, I)$ with an extra Dirichlet boundary at $I$, i.e.,

$$\partial_t u(x, t) = Lu(x, t) \quad \text{for } (x, t) \in (0, I) \times (0, \infty)
\text{with } u(x, t) \to 0 \text{ as } x \searrow 0 \text{ or } x \nearrow I \text{ for } t \in (0, \infty).$$  (**)

Let $p_I(x, y, t)$ be the fundamental solution to (**). To solve (**), we carry out a transformation that turns $L$ into a diffusion operator that degenerates linearly at 0. In fact, with a change of variable $x \mapsto z$, solving (**) becomes equivalent to solving the following problem:

$$\partial_t v^c(z, t) = \left(z \partial_z^2 + \nu \partial_z + V(z)\right)v^c(z, t) \quad \text{for } (z, t) \in (0, J) \times (0, \infty)
\text{with } v^c(z, t) \to 0 \text{ as } z \searrow 0 \text{ or } z \nearrow J \text{ for } t \in (0, \infty),$$  (†)

where $J$ is the image of $I$ after the change of variable, $\nu < 1$ is a constant, and $V(z)$ is a function on $(0, J)$ ($J, \nu$ and $V$ will be specified in §2.1). If we can find the fundamental solution to (†), denoted by $q^I_J(z, w, t)$, then $p_I(x, y, t)$ can be obtained through $p^I(x, y, t)$ via the transformation (and its inverse) between (** and (†).
2. Model equation (§2.2). Our strategy for solving (†) is to treat the operator \( z\partial_z^2 + \nu \partial_z + V(z) \) as a perturbation of \( z\partial_z^2 + \nu \partial_z \). We temporarily return to the “global” view, omit the potential \( V(z) \) and the right boundary \( J \), and consider the following model equation on the entire \((0, \infty)\):
\[
\partial_t v(z, t) = (z\partial_z^2 + \nu \partial_z) v(z, t) \quad \text{for every } (z, t) \in (0, \infty)^2
\]
with \( v(z, t) \to 0 \) as \( z \to 0 \) for \( t \in (0, \infty) \).

This model equation has the advantage that its fundamental solution \( q(z, w, t) \) has an explicit formula in terms of a Bessel function (§3), and properties of the solutions to the model equation are already known to us (Proposition 2.4). With \( q(z, w, t) \) in hand, we return to the local view of the model equation (with the Dirichlet boundary condition “restored” at \( J \)) and derive the fundamental solution \( q_J(z, w, t) \) to the localized model equation on \((0, J)\) (Proposition 2.0).

3. Solving the localized equation (§3). Upon getting \( q_J(z, w, t) \), we can start the construction of the fundamental solutions to (†) and (⋆). Viewing \( z\partial_z^2 + \nu \partial_z + V(z) \) as a perturbation of \( z\partial_z^2 + \nu \partial_z \) with a potential function \( V(z) \), we invoke Duhamel’s perturbation method to construct \( q_J^Y(z, w, t) \) using \( q_J(z, w, t) \) as the “building block” (Proposition 3.2). Although in general \( q_J^Y(z, w, t) \) does not have a closed-form formula and our representation of \( q_J^Y(z, w, t) \) is in the form of a series, by focusing on the first term of the series expression we can show that \( q_J^Y(z, w, t) \) is well approximated by \( q(z, w, t) \) for sufficiently small \( t \) (Proposition 3.4).

4. Solving the global equation (§4). We finally return to (1.1) and produce \( p(x, y, t) \) from \( p_I(x, y, t) \) by “reversing” the localization procedure. More specifically, we establish the relation between (1.1) and its localized version (⋆) with the help of the underlying diffusion process corresponding to \( L \). By analyzing the excursions of the diffusion process over \((0, I)\), \( p(x, y, t) \) is achieved as the limit of \( p_I(x, y, t) \) as \( I \) increases to infinity (Theorem 4.3). Again, although \( p(x, y, t) \) does not have a closed-form formula, we find an approximation \( p^{\text{approx}}(x, y, t) \) for \( p(x, y, t) \) such that \( p^{\text{approx}}(x, y, t) \) has an explicit and relatively simple expression, and \( p^{\text{approx}}(x, y, t) \) is more accurate than the standard heat kernel estimate for \( p(x, y, t) \) (Theorem 4.5).

\[
\begin{array}{ccc}
\text{localized equations:} & \quad p_I(x, y, t) & \quad \leftrightarrow \quad q_I^Y(z, w, t) \\
\quad (\text{transformation}) & \quad \downarrow \quad (\text{convergence}) & \quad \uparrow \quad (\text{localization}) & \quad \uparrow \quad (\text{perturbation}) \\
\text{global equations:} & \quad p(x, y, t) & \quad \uparrow \quad (\text{approximation}) \\
\quad (\text{transformation}) & \quad p^{\text{approx}}(x, y, t) & \quad \leftrightarrow \quad q(z, w, t) \\
\end{array}
\]

Table 1. Relation among the fundamental solutions.

In each of the steps above, in addition to the standard analytic methods from the study of parabolic equations, we also rely on a probabilistic point of view towards diffusion equations. Whenever applicable, we treat the fundamental solution as the transition probability density function of the underlying diffusion process corresponding to the concerned operator. In fact, the localization procedure (and the reverse of it) proposed above is possible because of the (strong) Markov properties of the diffusion process. We also invoke some classical tools in the study of stochastic processes, e.g., Itô’s formula and Doob’s stopping time theorem, in deriving probabilistic interpretations of the (fundamental) solutions to the involved diffusion equations. In §1.3 we give a brief overview of the probabilistic components involved in this work.

4
In §5 we consider a generalization of the classical Wright-Fisher equation (1.6), where we assume that the diffusion operator vanishes with a general order at both of the degenerate boundaries 0 and 1. In particular, for \( f \in C_\varepsilon ((0,1)) \) and \( \alpha, \beta \in (0, 2) \), we consider the equation

\[
\begin{align*}
\partial_t u_f (x,t) &= x^\alpha \left( 1 - x \right)^2 \partial_x^2 u_f (x,t) \quad \text{for} \quad (x,t) \in (0,1) \times (0, \infty), \\
\lim_{x \uparrow 0} u_f (x,t) &= f(x) \quad \text{for} \quad x \in (0,1) \quad \text{and} \\
\lim_{x \downarrow 0} u_f (x,t) &= 0 \quad \text{for} \quad t \in (0, \infty).
\end{align*}
\]

Although this problem is in a different setting from (1.1), our methods and results still apply. We can follow the same steps as above to study its fundamental solution \( p(x,y,t) \) and obtain similar estimates for \( p(x,y,t) \) near either of the boundaries (Proposition 5.1).

1.3. Stochastic differential equation, underlying diffusion process. This subsection gives a brief overview of the probabilistic foundation needed for our investigation. We start with the stochastic differential equation corresponding to the operator \( L = x^\alpha a(x) \partial_x^2 + b(x) \partial_x \), and that is, given \( x > 0 \),

\[
(1.7) \quad dX(x,t) = \sqrt{2x^\alpha (x,t)} a(X(x,t)) dB(t) + b(X(x,t)) \, dt \text{ for every } t \geq 0 \text{ with } X(0,0) \equiv x.
\]

For a general stochastic differential equation, there are two notions of existence/uniqueness of a solution: strong existence/uniqueness and weak existence/uniqueness. Our work only requires the existence of a weak solution to (1.7) and the solution being unique in the weak sense. We will not expand on the general theory and refer interested readers to [25, 35] for a comprehensive exposition on these topics.

We say that (1.7) has a (weak) solution if, on some filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P}) \), there exist two adapted processes \( \{B(t) : t \geq 0\} \) and \( \{X(x,t) : t \geq 0\} \) such that, (i) \( \{B(t) : t \geq 0\} \) is a standard Brownian motion; (ii) \( \{X(x,t) : t \geq 0\} \) has continuous sample paths; (iii) almost surely \( \{X(x,t) : t \geq 0\} \) satisfies that

\[
X(x,t) = x + \int_0^t \sqrt{2x^\alpha(s)} \, a(X(x,s)) \, dB(s) + \int_0^t b(X(x,s)) \, ds \text{ for every } t \geq 0.
\]

In this case, we also refer to \( \{X(x,t) : t \geq 0\} \) as the underlying diffusion process corresponding to \( L \) starting from \( x \). We say that a solution \( \{X(x,t) : t \geq 0\} \) is unique (in law), if whenever (i)-(iii) are satisfied by another triple \( (\Omega', \mathcal{F}', \{\mathcal{F}'_t : t \geq 0\}, \mathbb{P}') \), \( \{B'(t) : t \geq 0\} \) and \( \{X'(x,t) : t \geq 0\} \), it must be that the distribution of \( \{X(x,t) : t \geq 0\} \) under \( \mathbb{P} \) is identical with that of \( \{X'(x,t) : t \geq 0\} \) under \( \mathbb{P}' \).

We say that the stochastic differential equation (1.7) is well posed if a solution exists and is unique.

In later discussions we will use an important corollary of the wellposedness property, and that is, if (1.7) is well posed and \( \{X(x,t) : t \geq 0\} \) is the unique solution, then \( \{X(x,t) : t \geq 0\} \) is a strong Markov process and for every \( H \in C ([0, \infty)^2) \cap C^{1,1} ([0, \infty)^2) \),

\[
\left\{ H(X(x,t),t) - \int_0^t (\partial_x + L) H(X(x,s),s) \, ds : t \geq 0 \right\}
\]

is a local martingale.

Now let us examine the wellposedness of (1.7) under the hypotheses (H1) and (H2). There is a rich literature on the wellposedness of a degenerate stochastic differential equation with a diffusion operator that degenerates linearly. While the diffusion coefficient in (1.7) may have nonlinear degeneracy, we can convert it into a linear degeneracy case simply through a change of variable. To be specific, we consider the following diffeomorphism on \( (0, \infty) \) and its inverse:

\[
(1.8) \quad \xi = \xi(x) := \frac{x^{2-\alpha}}{(2-\alpha)^{\alpha}} \text{ and } x = x(\xi) := (2-\alpha)\sqrt{\xi} \text{ for } x, \xi > 0.
\]

One can easily verify that \( u_f (x,t) \in C^{2,1} ((0, \infty)) \) is a solution to (1.1) if and only if \( w_g (\xi,t) := u_f (x(\xi),t) \) is the solution to

\[
\begin{align*}
\partial_t w_g (\xi,t) &= \xi c(\xi) \partial^2_x w_g (\xi,t) + d(\xi) \partial_x w_g (\xi,t) \quad \text{for} \quad (\xi,t) \in (0, \infty)^2, \\
\lim_{\xi \searrow 0} w_g (\xi,t) &= g(\xi) \quad \text{for} \quad \xi \in (0,1) \quad \text{and} \quad \lim_{\xi \searrow 0} w_g (\xi,t) = 0 \quad \text{for} \quad t \in (0, \infty),
\end{align*}
\]

\[\quad \text{for} \quad \xi \equiv x(\xi) \quad \text{and} \quad x \equiv x(\xi).\]
where \( g(x) := f(x) \), \( c(x) := a(x) \) and

\[
d(\xi) := \frac{1 - \alpha}{2 - \alpha} a(x(\xi)) + \frac{(x(\xi))^{1 - \alpha}}{2 - \alpha} b(x(\xi)) .
\]

The stochastic differential equation corresponding to (1.9) is that, given \( \xi > 0 \),

\[
dZ(\xi, t) = \sqrt{2Z(\xi, t)} c(Z(\xi, t)) dB(t) + d(Z(\xi, t)) dt \text{ for every } t \geq 0 \text{ with } Z(\xi, 0) = \xi .
\]

Assuming (H1) and (H2), we get down to verifying the wellposedness of (1.10) where the diffusion operator degenerates linearly at \( 0 \). First, when \( \alpha \in (1, 2) \), by (H1), (1.8) and direct computations, we see that both \( c(\xi) \) and \( d(\xi) \) are Lipschitz continuous on any bounded subset of \( (0, \infty) \). Furthermore, (H2) and (1.8) imply that there exists constant \( C > 0 \) such that for every \( \xi \in [0, \infty) \),

\[
|c(\xi)| + |d(\xi)| \leq C \left( 1 + (x(\xi))^{2 - \alpha} \right) \leq C (1 + \xi) .
\]

It follows from classical results (e.g., Yamada-Watanabe \[38\], Stroock-Varadhan \[39\], Engelbert-Schmidt \[44\], Cherny \[9\]) that (1.10) is well posed for every \( \xi > 0 \) in this case. Next, when \( \alpha \in (0, 1] \), we note that

\[
\lim_{\xi \searrow 0} \frac{d(\xi)}{\xi} = \lim_{x \searrow 0} \left( \frac{1 - \alpha}{2 - \alpha} a(x) + \frac{x^{1 - \alpha}}{2 - \alpha} b(x) \right) \geq 0 .
\]

This time (H1) and (1.8) guarantee that \( c(\xi) \) and \( d(\xi) \) are both Hölder continuous on any bounded subset of \( (0, \infty) \); meanwhile, the growth control (1.11) on \( c(\xi) \) and \( d(\xi) \) still applies. Thus, the results of Bass-Perkins \[3\] lead to the wellposedness of (1.10) for every \( \xi > 0 \). Therefore, for every \( \alpha \in (0, 2] \), (H1) and (H2) are sufficient for (1.10) to be well posed. Assume that \( \{ Z(\xi, t) : t \geq 0 \} \) is the unique solution to (1.10). By setting

\[
X(x, t) := x(\xi(x), t) \text{ for } t \geq 0 ,
\]

we immediately get the following conclusion.

**Lemma.** The stochastic differential equation (1.10) is well posed for every \( x > 0 \), \( \{ X(x, t) : t \geq 0 \} \) defined above is the unique solution to (1.10), and \( \{ X(x, t) : t \geq 0 \} \) is a strong Markov process. Moreover, if \( u(x, t) \in C^{2,1}((0, \infty)^2) \) is a solution to \( \partial_t u(x, t) = Lu(x, t) \), then given any \( x, t \in (0, \infty)^2 \), \( \{ u(X(x, s), t - s) : s \in [0, t] \} \) is a local martingale.

So far there is no constraint on the behavior of \( X(x, t) \) at the boundary \( 0 \). Returning to the original problem (1.11), to incorporate the Dirichlet boundary condition, we only need to focus on \( X(x, t) \) up to the time it hits \( 0 \). Intuitively speaking, if we set

\[
\zeta_0^X(x) := \inf \{ s \geq 0 : X(x, s) = 0 \} \text{ for } x > 0 ,
\]

then the probability density function of the conditional distribution of \( X(x, t) \) given \( \{ t < \zeta_0^X(x) \} \) should coincide with the fundamental solution to (1.11).

**Notations.** For \( \alpha, \beta \in \mathbb{R} \), we write \( \alpha \vee \beta := \max \{ \alpha, \beta \} \) and \( \alpha \wedge \beta := \min \{ \alpha, \beta \} \).

For every \( \Gamma \subseteq [0, \infty) \), \( I_\Gamma \) denotes the indicator function of \( \Gamma \).

Let \( (\Omega, \mathcal{F}, \{ \mathcal{F}_t : t \geq 0 \}, \mathbb{P}) \) be a filtered probability space. For an integrable random variable \( X \) on \( \Omega \) and a set \( A \subseteq \mathcal{F} \), we write \( E[X; A] := \int_A Xd\mathbb{P} \). For an adapted process \( \{ W(t) : t \geq 0 \} \) with non-negative continuous sample paths, we set

\[
\zeta_y^W(x) := \inf \{ t \geq 0 : W(t) = y | W(0) = x \} \text{ for every } x, y \geq 0 ,
\]

i.e., \( \zeta_y^W(x) \) is the hitting time at \( y \) conditioning on the process starting from \( x \); for \( y_1, y_2 \geq 0 \), we set

\[
\zeta_{y_1, y_2}^W(x) := \zeta_{y_1}^W(x) \wedge \zeta_{y_2}^W(x) .
\]
2. Model Equation

In this section, we will carry out the first three steps outlined in §1.2. Although we use similar transformations as those in [8], we need to adapt the method so that it applies to \( a(x) \) and \( b(x) \) that are under weaker conditions.

2.1. Localization and Transformation. Let \( \alpha \in (0, 2) \) and \( I > 0 \) be fixed throughout this section. Our first step is to introduce an extra Dirichlet boundary to the equation \( \partial_t - L = 0 \) at \( x = I \) and to consider a localized version of (1.1) on \((0, I)\). Namely, given \( f \in C_c((0, I)) \), we look for \( u_{f,(0,I)}(x,t) \in C^{2,1}((0, I) \times (0, \infty)) \) such that

\[
\begin{align*}
\partial_t u_{f,(0,I)}(x,t) &= L u_{f,(0,I)}(x,t) \quad \text{for} \quad (x,t) \in (0, I) \times (0, \infty), \\
\lim_{t \to 0^+} u_{f,(0,I)}(x,t) &= f(x) \quad \text{for} \quad x \in (0, I), \\
\lim_{x \to I^-} u_{f,(0,I)}(x,t) &= 0 \quad \text{for} \quad t \in (0, \infty),
\end{align*}
\]

(2.1)

Once restricted on \((0, I)\), the coefficients (and their derivatives) in (2.1) are all bounded.

We want to find the fundamental solution \( p_I(x,y,t) \) to (2.1). Given \( x > 0 \), let \( \{X(x,t) : t \geq 0\} \) be the unique solution to (1.1), as found in §1.3. We expect that \( y \mapsto p_I(x,y,t) \) coincides with the probability density function of \( \{X(x,t) : t \geq 0\} \), conditioning on \( \{t < \zeta^{X}_{0,I}(x)\} \). This probabilistic interpretation of \( p_I(x,y,t) \) is indeed correct and will be justified later. For now, let us conduct an analysis of (2.1) via standard perturbation methods.

As mentioned in §1.2, we will transform (1.1) into a diffusion equation that has linear degeneracy at 0. For \( x \in (0, I] \), let \( \phi(x) \) and \( \theta(x) \) be defined as in (1.2). It is clear that \( \phi \in C^2((0, I]), \phi \) is strictly increasing, and \( \theta \in C^1((0, I)) \). The constant \( \nu \) in the definition of \( \theta(x) \) is chosen such that

\[
\nu = 1 + \lim_{x \to 0^+} \frac{2b(x) - (\alpha \alpha x^{\alpha-1} a(x) + x^\alpha a'(x))}{2x^\alpha \sqrt{a(x)}} \sqrt{\phi(x)}.
\]

Under (H1) and (H2), it is easy to verify that

\[
\nu = \frac{1 - \alpha}{2 - \alpha} \mathbb{1}_{(0,2)\setminus(1)}(\alpha) + b(0) \mathbb{1}_{(1)}(\alpha),
\]

and hence \( \nu < 1 \). Let \( J := \phi(I), \psi : (0, J] \to (0, I] \) be the inverse function of \( \phi \) and \( \tilde{\theta} := \theta \circ \psi \). We introduce two more functions on \((0, J)\):

\[
(2.2) \quad \Theta : z \in (0, J] \mapsto \Theta(z) := \exp \left( - \int_0^z \frac{\tilde{\theta}(u)}{2u} du \right),
\]

and

\[
(2.3) \quad V : z \in (0, J] \mapsto V(z) := z \frac{\Theta''(z)}{\Theta(z)} + \left( \nu + \tilde{\theta}(z) \right) \frac{\Theta'(z)}{\Theta(z)},
\]

or equivalently,

\[
V(z) = - \frac{\tilde{\theta}'(z)}{4z} - \frac{\Theta''(z)}{2} + (1 - \nu) \frac{\tilde{\theta}(z)}{2z}.
\]

Now we are ready to state the result on the transformation.

**Proposition 2.1.** Given \( f \in C_c((0, I)) \), we define

\[
h(z) := \frac{f \circ \psi(z)}{\Theta(z)} \quad \text{for} \quad z \in (0, J).
\]

Then, \( u_{f,(0,I)}(x,t) \in C^{2,1}((0, I) \times (0, \infty)) \) is a solution to (2.1) if and only if

\[
u \quad u_{f,(0,I)}(x,t) = \Theta(\phi(x)) V_h(\phi(x), t) \quad \text{for} \quad (x,t) \in (0, I) \times (0, \infty),
\]

(2.4)
where \( v_{h,0}(z, t) \in C^{2,1}((0, J) \times (0, \infty)) \) is a solution to the following problem:

\[
\begin{align*}
\partial_t v_{h,0}(z, t) &= (z \partial_z^2 + \nu \partial_z + V(z)) v_{h,0}(z, t) \quad \text{for } (z, t) \in (0, J) \times (0, \infty), \\
\lim_{z \searrow 0} v_{h,0}(z, t) &= h(z) \quad \text{for } z \in (0, J) \quad \text{and} \\
\lim_{z \nearrow 0} v_{h,0}(z, t) &= \lim_{z \nearrow J} v_{h,0}(z, t) = 0 \quad \text{for } t \in (0, \infty).
\end{align*}
\]

We omit the proof of Proposition 2.1 since it can be verified by direct computations. If \( q^V(z, w, t) \) is the fundamental solution to (2.5), then \( q_{t}(x, y, t) \) is connected with \( q^V(z, w, t) \) following the same relation as the one in (2.4). Set \( L^V := z \partial_z^2 + \nu \partial_z + V(z) \). Compared with \( L \), \( L^V \) has a simpler structure consisting of a linear diffusion, a constant drift and a potential. In the next subsection we will solve (2.5) by treating \( L^V \) as a perturbation of \( L_0 := z \partial_z^2 + \nu \partial_z \) and invoking Duhamel’s perturbation method. As a preparation, we state below some technical results on \( \Theta(z) \) and \( V(z) \).

**Lemma 2.2.** Let \( \Theta(z) \) be defined as in (2.2). Then, for every \( z \in (0, J) \),

\[
\Theta(z) = \begin{cases} 
(\psi(z))^2 + (4z)^{-\frac{a(\psi(z))}{2b(\psi(z))}} \exp \left(- \int_0^{\psi(z)} \frac{h(w)}{2\omega(a(w))} dw \right) & \text{if } \alpha \neq 1, \\
(\psi(z))^2 + (4z)^{-\frac{a(\psi(z))}{2b(\psi(z))}} \exp \left(- \int_0^{\psi(z)} \frac{1}{b(\psi(z))} dw \right) & \text{if } \alpha = 1.
\end{cases}
\]

Hence, there exists constant \( \Theta_J > 0 \) that can be made explicit (see (6.1) in the Appendix) such that

\[
\sup_{z \in (0, J)} \left( \Theta(z) \vee \frac{1}{\Theta(z)} \right) \leq \Theta_J.
\]

Let \( V(z) \) be defined as in (2.3). Then, there exists constant \( V_J > 0 \) such that for every \( z \in (0, J) \),

\[
|V(z)| \leq \begin{cases} 
V_J \cdot z^{-\frac{1}{\alpha}} & \text{if } \alpha \in (0, 1) \text{ and } b(0) \neq 0, \\
V_J \cdot z^{-\frac{1}{\alpha}} & \text{if } \alpha \in (0, 1) \text{ and } b(0) = 0, \\
V_J & \text{if } \alpha \in [1, 2).
\end{cases}
\]

The proof of Lemma 2.2 is left in the Appendix since it is based on straightforward computations that are lengthy and not crucial to our work. We note that when \( \alpha \in (0, 1) \), the potential function \( V(z) \) may be singular at 0. This is a generalization of the case considered in [8] where \( V(z) \) is assumed to be bounded near 0.

**2.2. From \( q(z, w, t) \) to \( q_{J}(z, w, t) \).** Let \( I \) and \( J \) be the same as above. As mentioned in the previous subsection, to solve (2.5), we will first consider the analogous problem with \( L^V \) replaced by \( L_0 \). Namely, given \( g \in C_{c}((0, J)) \), we look for \( v_{g,0}(z, t) \in C^{2,1}((0, J) \times (0, \infty)) \) such that

\[
\begin{align*}
\partial_t v_{g,0}(z, t) &= L_0 v_{g,0}(z, t) \quad \text{for every } (z, t) \in (0, J) \times (0, \infty) \\
\lim_{z \searrow 0} v_{g,0}(z, t) &= g(z) \quad \text{for } z \in (0, J) \quad \text{and} \\
\lim_{z \nearrow 0} v_{g,0}(z, t) &= \lim_{z \nearrow J} v_{g,0}(z, t) = 0 \quad \text{for } t \in (0, \infty).
\end{align*}
\]

Let \( q_{J}(z, w, t) \) be the fundamental solution to (2.9). We consider \( \partial_t - L_0 = 0 \) as our model equation. To solve (2.9), we temporarily return to the “global” view and study the model equation on \((0, \infty)\) instead of \((0, J)\). That is, for \( g \in C_{c}((0, \infty)) \), we consider the following problem:

\[
\begin{align*}
\partial_t v_{g}(z, t) &= L_0 v_{g}(z, t) \quad \text{for every } (z, t) \in (0, \infty)^2 \\
\lim_{z \searrow 0} v_{g}(z, t) &= g(z) \quad \text{for } z \in (0, \infty) \quad \text{and} \\
\lim_{z \nearrow 0} v_{g}(z, t) &= 0 \quad \text{for } t \in (0, \infty).
\end{align*}
\]

Let \( q(z, w, t) \) be the fundamental solution to (2.10). In fact, \( q(z, w, t) \) is the starting point of our “journey”, and from \( q(z, w, t) \) we will derive the (fundamental) solutions to all the concerned equations.

The stochastic differential equation corresponding to the model equation is that, given \( z > 0 \),

\[
dY(z,t) = \sqrt{2Y(z,t)} dB(t) + \nu dt \quad \text{for } t \geq 0 \quad \text{with } Y(0,0) = z.
\]

It follows from the discussions in §1.2 that (2.11) is well posed, and hence there exists a unique solution \( \{Y(z,t) : t \geq 0\} \) that is also a strong Markov process.
Remark 2.3. We want to remark that, independent of the Dirichlet boundary condition imposed in (2.10), the constant \( \nu \) determines the attainability of the boundary 0. Under (H1) and (H2), we have that \( \nu < 1 \), and hence 0 is either an exit boundary or a regular boundary. This is to say that, no matter what \( z \) is, \( \{ Y (z, t) : t \geq 0 \} \) hits 0 with a positive probability in finite time. For more details on the topic of boundary classification, we refer readers to §15 of [20].

The operator \( L_0 \), as well as (2.10) and (2.10), has been well studied in [8]. Below we will review some useful facts about \( q (z, w, t) \), \( \nu_g (z, t) \) and their connections to \( \{ Y (z, t) : t \geq 0 \} \). The details can be found in §2 of [8].

Proposition 2.4. (Proposition 2.1, 2.3 of [8]) The fundamental solution to (2.10) is

\[
q (z, w, t) := \frac{1}{\sqrt{2 \pi t}} e^{-\frac{z^2}{2t}} I_{1-\nu} \left( \frac{2 \sqrt{zw}}{t} \right) = \frac{z^{1-\nu}}{t^{2-\nu}} e^{-\frac{z^2}{2t}} \sum_{n=0}^{\infty} \frac{(zw)^n}{t^{2n} n!} (n+2-\nu)
\]

for \((z, w, t) \in (0, \infty)^3\), where \( I_{1-\nu} \) is the modified Bessel function. \( q (z, w, t) \) is smooth on \((0, \infty)^3\), and for every \((z, w, t) \in (0, \infty)^3\),

\[
\frac{z^{1-\nu}}{t^{2-\nu}} e^{-\frac{z^2}{2t}} \leq q (z, w, t) \leq \left( \frac{z^{1-\nu}}{t^{2-\nu}} \right) e^{-\frac{(\sqrt{\nu^2} - \nu)^2}{4}}
\]

and

\[
w^{1-\nu} q (z, w, t) = z^{1-\nu} q (w, z, t).
\]

Given \( g \in C_c ((0, \infty)) \), if

\[
v_g (z, t) := \int_0^{\infty} g (w) q (z, w, t) \, dw \text{ for } (z, t) \in (0, \infty),
\]

then \( v_g (z, t) \) is the unique solution in \( C^{2,1} ((0, \infty)^2) \) to (2.10), and \( v_g (z, t) \) is smooth on \((0, \infty)^2\).

Moreover,

\[
v_g (z, t) = \mathbb{E} \left[ g (Y (z, t)) ; t < \zeta^Y_0 (z) \right] \text{ for every } (z, t) \in (0, \infty)^2,
\]

which implies that for every Borel set \( \Gamma \subseteq (0, \infty) \),

\[
\int_{\Gamma} q (z, w, t) \, dw = \mathbb{P} \left( Y (z, t) \in \Gamma, t < \zeta^Y_0 (z) \right).
\]

Finally, \( q (z, w, t) \) satisfies the Chapman-Kolmogorov equation, i.e., for every \( z, w > 0 \) and \( t, s > 0 \),

\[
q (z, w, t+s) = \int_0^\infty q (z, \xi, t) q (\xi, w, s) \, d\xi.
\]

It is clear from (2.10) that, for every \((z, t) \in (0, \infty)^2\), \( w \mapsto q (z, w, t) \) is the probability density function of \( Y (z, t) \), provided that \( t < \zeta^Y_0 (z) \). Now we turn our attention to \( q_J (z, w, t) \), the fundamental solution to (2.5) which has an extra Dirichlet boundary at \( J \). Intuitively speaking, to get \( q_J (z, w, t) \), we need to remove from \( q (z, w, t) \) the “contribution” of \( Y (z, t) \) once \( Y (z, t) \) exists the interval \((0, J)\).

Based on this idea combined with the fact that \( \{ Y (z, t) : t \geq 0 \} \) is a strong Markov process, we define

\[
q_J (z, w, t) := q (z, w, t) - \mathbb{E} \left[ q (J, w, t - \zeta^Y_J (z)) ; \zeta^Y_J (z) \leq t \wedge \zeta^Y_0 (z) \right]
\]

for every \((z, w, t) \in (0, J)^2 \times (0, \infty)\). Again, by (2.10), we see that for every Borel set \( \Gamma \subseteq (0, J) \),

\[
\int_{\Gamma} q_J (z, w, t) \, dw = \mathbb{P} \left( Y (z, t) \in \Gamma, t < \zeta^Y_0 (z) \right) - \mathbb{P} \left( Y (z, t) \in \Gamma, \zeta^Y_J (z) \leq t < \zeta^Y_0 (z) \right)
\]

In other words, \( q_J (z, w, t) \) is the probability density function of \( Y (z, t) \) provided that \( t < \zeta^Y_0 \).
Lemma 2.5. For every \( z \in (0, J) \),

\[
\mathbb{P} \left( \zeta^Y_j(z) \leq \zeta^Y_0(z) \right) = \frac{\nu^{1-\nu}}{J^{1-\nu}}.
\]

For \( t > 0 \) and \( J - z \geq |\nu| t \), we have that

\[
\mathbb{P} \left( \zeta^Y_j(z) \leq t \right) \leq \exp \left( -\frac{(J - z - t|\nu|)^2}{4tJ} \right).
\]

Furthermore, almost surely

\[
\lim_{J \to \infty} \zeta^Y_j(z) = \infty \quad \text{and} \quad \lim_{z \to J} \zeta^Y_j(z) = 0.
\]

**Proof.** Based on (2.11), one can apply Itô’s formula (see, e.g., Proof. Lemma 2.5). For every \( z \in (0, J) \), \( \{Y(z, t)\}_{t \geq 0} \) is a local martingale, and hence by Doob’s stopping time theorem (see, e.g., §8 of [33]), \( \{Y(z, t \wedge \zeta^Y_j(z))\}_{t \geq 0} \) is a bounded martingale. Thus,

\[
z^{1-\nu} = \mathbb{E} \left[ \left( Y(z, \zeta^Y_0(z)) \right)^{1-\nu} \right] = \mathbb{P} \left( \zeta^Y_j(z) \leq \zeta^Y_0(z) \right) \lambda^{1-\nu}.
\]

To show (2.21), we check that for every \( z \in (0, J) \) and every \( \lambda > 0 \), if

\[
E(z, t) := \exp \left( \lambda Y(z, t) - \lambda \nu t - \lambda^2 t \int_0^t Y(z, s) \, ds \right)
\]

then \( \{E(z, t \wedge \zeta^Y_j(z)) : t \geq 0\} \) is a martingale. By a similar argument as above and Fatou’s lemma, we get that

\[
e^{\lambda J} \mathbb{E} \left[ e^{-(\lambda \nu + \lambda^2 J) \zeta^Y_j(z)} ; \zeta^Y_j(z) < \infty \right] \leq \liminf_{t \to \infty} \mathbb{E} \left[ E(z, t \wedge \zeta^Y_j(z)) \right] = e^{\lambda z}.
\]

Set \( \lambda := \frac{\nu}{2} + \frac{\lambda^2}{2} \). Since \( J > z + t |\nu| \), we have that \( \lambda > 0 \) and \( \lambda \nu + \lambda^2 J > 0 \). Therefore, a simple application of Markov’s inequality leads to

\[
\mathbb{P} \left( \zeta^Y_j(z) \leq t \right) = \mathbb{P} \left( e^{-(\lambda \nu + \lambda^2 J) \zeta^Y_j(z)} \geq e^{-(\lambda \nu + \lambda^2 J) t} \right) \\
\leq e^{(\lambda \nu + \lambda^2 J) t} \mathbb{E} \left[ e^{-(\lambda \nu + \lambda^2 J) \zeta^Y_j(z)} ; \zeta^Y_j(z) < \infty \right] \\
\leq \exp \left( \lambda^2 t J - \lambda (J - z - t|\nu|) \right).
\]

Plugging the value of \( \lambda \) into the right hand side yields (2.21). The fact that \( \zeta^Y_j(z) \) converges to \( \infty \) as \( J \to \infty \) almost surely follows from (2.21) and the monotonicity of \( \zeta^Y_j(z) \) in \( J \).

Finally, we observe that \( \zeta := \lim_{z \to J} \zeta^Y_j(z) \) exists almost surely. Take \( \lambda \in \mathbb{R} \) such that \( \lambda \nu \geq 0 \). It follows from (2.22) and the reverse Fatou’s lemma that

\[
e^{\lambda J} \mathbb{E} \left[ e^{-(\lambda \nu \xi)} \right] \geq \mathbb{E} \left[ \limsup_{z \to J} \mathbb{E} \left( E(z, \zeta^Y_j(z)) \right) \right] \\
\geq \limsup_{z \to J} \mathbb{E} \left( E(z, t \wedge \zeta^Y_j(z)) \right) = e^{\lambda J},
\]

which implies that \( \mathbb{E} \left[ e^{-(\lambda \nu \xi)} \right] = 1 \) and hence \( \zeta = 0 \) almost surely. \( \square \)

**Proposition 2.6.** Let \( q_j(z, w, t) \) be defined as in (2.18). Then, \( q_j(z, w, t) \) is continuous on \( (0, J)^2 \times (0, \infty) \), and for every \( (z, w, t) \in (0, J)^2 \times (0, \infty) \), we have that

\[
w^{1-\nu} q_j(z, w, t) = z^{1-\nu} q_j(w, z, t).
\]

\( q_j(z, w, t) \) satisfies the Chapman-Kolmogorov equation, i.e., for every \( z, w \in (0, J) \) and \( t, s > 0 \),

\[
q_j(z, w, t + s) = \int_0^t q_j(z, \xi, t) q_j(\xi, w, s) \, d\xi.
\]

10
For every $w \in (0, J)$, $(z, t) \mapsto q_J(z, w, t)$ is a smooth solution to the Kolmogorov backward equation corresponding to $L_0$:

$$\tag{2.25} \partial_t q_J(z, w, t) = L_0 q_J(z, w, t);$$

for every $z \in (0, J)$, $(w, t) \mapsto q_J(z, w, t)$ is a smooth solution to the Kolmogorov forward equation corresponding to $L_0$:

$$\tag{2.26} \partial_w q_J(z, w, t) = L_0^* q_J(z, w, t)$$

where $L_0^* = w \partial_w^2 + (2 - \nu) \partial_w$ is the formal adjoint of $L_0$.

Moreover, $q_J(z, w, t)$ is the fundamental solution to (2.24). Given $g \in C_c((0, J))$, if

$$\tag{2.27} v_{g,(0,J)}(z,t) := \int_0^J g(w) q_J(z,w,t) \, dw \text{ for } (z,t) \in (0, J) \times (0, \infty),$$

then $v_{g,(0,J)}(z,t)$ is the unique solution in $C^{2,1}((0, J) \times (0, \infty))$ to (2.24), and $v_{g,(0,J)}(z,t)$ is smooth on $(0, J) \times (0, \infty)$.

Proof. We start with (2.24) since its proof is straightforward. Given $z, w \in (0, J)$, $t, s > 0$ and Borel set $\Gamma \subseteq (0, J)$, by (2.19) and the strong Markov property of $Y(z, t)$, we can write

$$\int_{\Gamma} q_J(z, w, t + s) \, dw = \mathbb{P}(Y(z, t + s) \in \Gamma, t + s < \zeta^Y_{0,J}(z)) = \int_{\Gamma} E[q_J(Y(z, t), w, s); t < \zeta^Y_{0,J}(z)] \, dw = \int_{\Gamma} \int_0^J q_J(z, \xi, t) q_J(\xi, w, s) \, d\xi \, dw,$$

which implies (2.24). Next, given $t > 0$, we take any $m \in \mathbb{N}$, any $0 = s_0 < s_1 < s_2 < \cdots < s_{m-1} < s_m = t$ and $\varphi_0, \varphi_2, \ldots, \varphi_m \in C_c((0, J))$. By (2.19) and, again, the Markov property of $Y(z, t)$, we have that

$$\int_0^J E\left[\prod_{k=0}^m \varphi_k(Y(z, s_k)); t < \zeta^Y_{0,J}(z)\right] \frac{dz}{z^{1-\nu}} = \int_0^J \cdots \int_{(0, J)^m} \varphi_0(z) \prod_{k=1}^m \varphi_k(\xi_k) q_J(z, \xi_1, s_1) q_J(\xi_1, \xi_2, s_2 - s_1) \cdots q_J(\xi_{m-1}, \xi_m, t - s_{m-1}) \, d\xi_m \cdots d\xi_1 \frac{dz}{z^{1-\nu}} = \int_0^J \cdots \int_{(0, J)^m} \varphi_0(z) \prod_{k=1}^m \varphi_k(\xi_k) q_J(\xi_1, z, t - (t - s_1)) q_J(\xi_2, \xi_1, (t - s_1) - (t - s_2)) \cdots q_J(\xi_{m-1}, \xi_m, t - s_{m-1}) \, d\xi_m \cdots d\xi_1 \, dz$$

Since $t \mapsto Y(z, t)$ is almost surely continuous and $s_0, \ldots, s_m, \varphi_0, \ldots, \varphi_m$ are chosen arbitrarily, the relation above implies that for every measurable functional $F$ on $C([0, t])$,

$$\int_{\mathbb{R}^+} \mathbb{E}\left[F\left(Y(z, \cdot)|_{[0,t]}\right); t < \zeta^Y_{0,J}(z)\right] \frac{dz}{z^{1-\nu}} = \int_{\mathbb{R}^+} \mathbb{E}\left[F\left(Y(w, \cdot)|_{[0,t]}\right); t < \zeta^Y_{0,J}(w)\right] \frac{dw}{w^{1-\nu}}.$$
where \( \overline{Y}(z,s) := Y(z,t-s) \) for every \( s \in [0,t] \). In particular, for arbitrary \( \varphi, \varphi^* \in C_c((0,J)) \), if \( F \) is chosen such that for every \( y(\cdot) \in C([0,t]) \),

\[
F(y(\cdot)) = \begin{cases} 
\varphi(y(0)) \varphi^*(y(t)), & \text{if } 0 < y(s) < J \text{ for every } s \in [0,t], \\
0, & \text{otherwise},
\end{cases}
\]

then we have that

\[
\int_0^1 \int_0^1 \varphi(z) \varphi^*(w) q_J(z,w,t) \frac{dwdz}{z^{1-\nu}} = \int_0^1 \int_0^1 \varphi(z) \varphi^*(w) q_J(w,z,t) \frac{dwdz}{w^{1-\nu}}.
\]

This is sufficient for us to conclude (2.23).

Now we turn attention to (2.25) and (2.26). By (2.23), it suffices to prove only one of them, say, (2.25). To this end, we take \( \varphi \in C_c^\infty((0,J)) \) and consider \( v_{\varphi,(0,J)}(z,t) \), which, according to (2.19), can be written as

\[
v_{\varphi,(0,J)}(z,t) = E \left[ \varphi(Y(z,t)); t < \zeta^J_{0,J}(z) \right] \text{ for every } (z,t) \in (0,J) \times (0,\infty).
\]

As reviewed in §1.2, for every \( z \in (0,J) \),

\[
\left\{ \varphi(Y(z,t \land \zeta^J_{0,J}(z))) - \int_0^t \chi_{0,J}^\nu(z) (L_0 \varphi)(Y(z,s)) \, ds : t \geq 0 \right\}
\]

is a bounded martingale. Thus,

\[
\varphi(z) = E \left[ \varphi(Y(z,t)); t < \zeta^J_{0,J}(z) \right] - \int_0^t E \left[ L_0 \varphi(Y(z,s)); s < \zeta^J_{0,J}(z) \right] \, ds
\]

\[
= v_{\varphi,(0,J)}(z,t) - \int_0^t \int_0^J L_0 \varphi(w) q_J(z,w,s) \, dwds,
\]

and hence

\[
\partial_t \left( \int_0^J \varphi(w) q_J(z,w,t) \, dw \right) = \int_0^J L_0 \varphi(w) q_J(z,w,t) \, dw.
\]

This means that for every \( z \in (0,J) \), \( (w,t) \mapsto q_J(z,w,t) \) solves the equation \( (\partial_t - L_0^*) q_J(z,w,t) = 0 \) in the sense of distribution. Since \( \partial_t - L_0^* \) is a hypoelliptic operator (see, e.g., §7.3 of [32]), \( (w,t) \mapsto q_J(z,w,t) \) is a smooth solution to (2.26).

For \( (z,w,t) \in (0,J)^2 \times (0,\infty) \), we set

\[
(2.28) \quad r(z,w,t) := q_J(z,w,t) - q_J(z,w) = E \left[ q(J,w,t - \zeta^J_{0,J}(z)); \zeta^J_{0,J}(z) \leq w \right] \text{ for every } (z,w,t) \in (0,J) \times (0,\infty).
\]

Then, for every \( w \in (0,J) \), \( (z,w) \mapsto r(z,w,t) \) is smooth on \( (0,J) \times (0,\infty) \). It is easy to see that \( w \mapsto r(z,w,t) \) is equicontinuous in \( (z,t) \) from any bounded subset of \( (0,J) \times (0,\infty) \), which implies that \( r(z,w,t) \), as well as \( q_J(z,w,t) \), is continuous on \( (0,J)^2 \times (0,\infty) \).

We proceed to the proof of the last statement. Again, by the hypoellipticity of \( \partial_t - L_0 \), to show that \( v_{\varphi,(0,J)}(z,t) \) is a smooth solution to the model equation, we only need to show that it solves the equation as a distribution. Let us take \( \varphi \in C_c^\infty((0,J)) \) and consider, for every \( t > 0 \),

\[
\langle \varphi, v_{\varphi,(0,J)}(\cdot,t) \rangle := \int_0^J \varphi(z) v_{\varphi,(0,J)}(z,t) \, dz = \int_0^J \int_0^J \varphi(z) q_J(z,w,t) \, dzg(w) \, dw.
\]

By (2.25), we have that

\[
\frac{d}{dt} \langle \varphi, v_{\varphi,(0,J)}(\cdot,t) \rangle = \int_0^J \left( \int_0^J \varphi(z) (L_0 q_J(\cdot,w,t))(z) \, dz \right) g(w) \, dw
\]

\[
= \int_0^J \int_0^J L_0^* \varphi(z) q_J(z,w,t) g(w) \, dwdz = \langle L_0^* \varphi, v_{\varphi,(0,J)}(\cdot,t) \rangle.
\]
The only remaining thing to do is to verify that \( v_{g, (0, J)} (z, t) \) satisfies the initial value and the boundary value conditions in (2.9). Given \( g \in C_{c} ((0, J)) \), by (2.13) and (2.14), we have that for every \( z \in (0, I) \),

\[
|v_{g, (0, J)} (z, t) - v_{g} (z, t)| \leq \mathbb{E} [ |g(Y(z, t))| ; \zeta_{J}^{Y} (z) \leq t < \zeta_{0}^{Y} (z) ] \leq \|g\|_{1} \mathbb{P} (\zeta_{J}^{Y} (z) \leq t)
\]

which, according to (2.21), goes to 0 as \( t \searrow 0 \), and the convergence is uniformly fast for \( z \) on any compact subset of \((0, J)\). Therefore,

\[
\lim_{t \searrow 0} v_{g, (0, J)} (z, t) = \lim_{t \searrow 0} v_{g} (z, t) = 0.
\]

To verify that \( v_{g, (0, J)} (z, t) \) satisfies the boundary condition, it is sufficient to show that

\[
\lim_{s \searrow 0} r(z, w, t) = 0 \text{ and } \lim_{s \searrow J} r(z, w, t) = q(J, w, t) \text{ for every } (w, t) \in (0, J) \times (0, \infty).
\]

We observe that, by (2.13), \( q(J, w, t - \zeta_{J}^{Y} (z)) \) is bounded uniformly in \( z \) by

\[
J^{1-\nu} \left( \sqrt{J} - \sqrt{w} \right)^{2(\nu-2)} \left( \frac{2 - \nu}{e} \right)^{2 - \nu}
\]

where we used the fact that

\[
\sup_{s > 0} s^{2 - \nu} e^{-s} = \left( \frac{2 - \nu}{e} \right)^{2 - \nu}.
\]

Therefore, (2.20) implies that

\[
\lim_{s \searrow 0} r(z, w, t) \leq J^{1-\nu} \left( \frac{2 - \nu}{e} \right)^{2 - \nu} \lim_{s \searrow 0} \mathbb{P} (\zeta_{J}^{Y} (z) \leq \zeta_{0}^{Y} (z)) = 0.
\]

Finally, the last statement in Lemma 2.5 and the dominated convergence theorem lead to

\[
\lim_{s \searrow J} r(z, w, t) = q(J, w, t).
\]

□

We will close this subsection with a result on the comparison between \( q_{J}(z, w, t) \) and \( q(z, w, t) \). Intuitively speaking, given \( z \in (0, J) \) sufficiently far from the boundary \( J \) and \( t \) sufficiently small, \( Y(z, t) \) would not have exited \((0, J)\) by time \( t \), which means that \( q(z, w, t) \) and \( q_{J}(z, w, t) \) should be close to each other. We will make this statement rigorous by proving that, as \( t \searrow 0 \), \( q_{J}(z, w, t) / q(z, w, t) \) converges to 1 uniformly fast in \((z, w)\) away from \( J \).

**Corollary 2.7.** Set \( t_{J} := \frac{4J}{9(2 - \nu)} \). Then, for every \( t \in (0, t_{J}) \),

\[
(2.29) \quad \sup_{(z, w) \in (0, 1/J)^{2}} \left| \frac{q_{J}(z, w, t)}{q(z, w, t)} - 1 \right| \leq \exp \left( \frac{-4J}{9t} \right).
\]

**Proof.** It is easy to verify that \( t_{J} \) is chosen such that the function \( s \mapsto s^{\nu-2} \exp \left( -\frac{4J}{9s} \right) \) is increasing on \((0, t_{J})\). By (2.13) and (2.20), we have that for every \( (z, w) \in (0, 1/J) \) and \( t \in (0, t_{J}) \),

\[
\left| \frac{q_{J}(z, w, t)}{q(z, w, t)} - 1 \right| = \frac{|r(z, w, t)|}{q(z, w, t)} \leq J^{1-\nu} \mathbb{P} (\zeta_{J}^{Y} (z) \leq \zeta_{0}^{Y} (z)) \cdot \sup_{s \in (0, t)} s^{\nu-2} \exp \left( -\frac{(\sqrt{J} - \sqrt{w})^{2}}{s} \right)
\]

\[
\leq \sup_{s \in (0, t)} s^{\nu-2} \exp \left( -\frac{4J}{9s} \right) \leq \exp \left( -\frac{4J}{9t} + \frac{z + w}{t} \right) \leq \exp \left( \frac{-2J}{9t} \right).
\]

□
3. Localized Equation

3.1. From \( q_J (z, w, t) \) to \( q^V_J (z, w, t) \). Now we get down to solving (2.3) by the perturbation method of Duhamel. First we want to find a function \( q^V_J (z, w, t) \) on \((0, J)^2 \times (0, \infty)\) that solves the integral equation

\[ q^V_J (z, w, t) = q_J (z, w, t) + \int_0^t \int_0^J q_J (z, \xi, t-s) q^V_J (\xi, w, s) V (\xi) d\xi ds \]  

for every \((z, w, t) \in (0, J)^2 \times (0, \infty)\), and then verify that \( q^V_J (z, w, t) \) is the fundamental solution to (2.3). To this end, for every \((z, w, t) \in (0, J)^2 \times (0, \infty)\) and \( n \in \mathbb{N} \), we define

\[ q_{J,0} (z, w, t) := q_J (z, w, t) \quad \text{and} \quad q_{J,n+1} (z, w, t) := \int_0^t \int_0^J q_J (z, \xi, t-s) q_{J,n} (\xi, w, s) V (\xi) d\xi ds. \]

To state the technical results on \( \{ q_{J,n} (z, w, t) : n \geq 0 \} \), we need to introduce more notations. Set

\[ b := \begin{cases} \nu & \text{if } \alpha \in (0, 1) \text{ and } b (0) \neq 0, \\ 1-\nu & \text{if } \alpha \in (0, 1) \text{ and } b (0) = 0, \\ 1 & \text{if } \alpha \in [1, 2). \end{cases} \]

We have that \( 0 < b \leq 1 \), and if \( V_J \) is the constant found in Lemma 2.2, then (2.8) can be rewritten as

\[ |V (z)| \leq V_J \cdot z^{b-1} \text{ for every } z \in (0, J). \]

For \( n \in \mathbb{N} \) and \( t > 0 \), we define

\[ m_n (t) := \frac{\Gamma^{n+1} (b) (ct^b V_J)^n}{\Gamma ((n+1) b)} \quad \text{and} \quad M (t) := \sum_{n=0}^{\infty} m_n (t). \]

It follows from a simple application of Stirling's formula that \( m_n (t) \) is summable in \( n \in \mathbb{N} \), and hence \( M (t) \) is well defined.

**Lemma 3.1.** There exists a universal constant \( c \geq 1 \) such that for every \( n \in \mathbb{N} \) and \((z, w, t) \in (0, J)^2 \times (0, \infty)\),

\[ |q_{J,n} (z, w, t)| \leq m_n (t) q (z, w, t), \]

and hence

\[ q^V_J (z, w, t) := \sum_{n=0}^{\infty} q_{J,n} (z, w, t) \]

is well defined as an absolutely convergent series. Moreover, for every \((z, w, t) \in (0, J)^2 \times (0, \infty)\),

\[ |q^V_J (z, w, t)| \leq M (t) q (z, w, t), \]

and \( q^V_J (z, w, t) \) satisfies (3.7).

**Proof.** Without causing any substantial change, we will assume that \( V (z) \) is defined on \((0, \infty)\) with \( V (z) \equiv 0 \) for \( z \geq J \). When \( 1 \leq \alpha < 2 \), since \( V (z) \) is bounded on \((0, \infty)\) with \( V_J = \| V \|_\infty \), (3.5)-(3.7) can be derived in exactly the same way as in [8] (Lemma 3.4) with

\[ m_n (t) = \frac{(tV_J)^n}{n!} \quad \text{and} \quad M (t) = e^{tV_J}. \]

There is nothing we need to do in this case. Hence, we will assume \( \alpha \in (0, 1) \) for the rest of the proof, and only treat the case when \( V (z) \) has a singularity at \( 0 \).

First, we claim that there exists a universal constant \( c > 0 \) such that

\[ \int_0^\infty q (z, \xi, t) q (\xi, w, s) \xi^{b-1} d\xi \leq c \left( \frac{t+s}{ts} \right)^{1-b} q (z, w, t+s) \]
for every \( z, w \in (0, J)^2 \) and \( t, s > 0 \). To see this, we use (2.12) and (2.14) to write the integral in (3.8) as

\[
\frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \int_0^\infty e^{-\frac{(t+s)z}{ts} \xi} \xi^{b-\nu} \left( \sum_{n=0}^\infty \frac{(z\xi)^n}{n! \Gamma(n + 2 - \nu) t^{2n}} \right) \left( \sum_{n=0}^\infty \frac{(w\xi)^n}{n! \Gamma(n + 2 - \nu) s^{2n}} \right) d\xi
\]

where for every \( n \in \mathbb{N}, \)

\[
\omega_n(z, w, t, s) := \sum_{k=0}^n \frac{z^k w^{n-k}}{k! (n-k)! \Gamma(k + 2 - \nu) \Gamma(n - k + 2 - \nu) t^{2k}s^{2(n-k)}}.
\]

Interchanging the order of summation and integration yields

\[
\frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \int_0^\infty e^{-\frac{(t+s)z}{ts} \xi} \xi^{2n+b-\nu} d\xi
\]

\[
= \frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \int_0^\infty e^{-\frac{(t+s)z}{ts} \xi} \xi^{2n+b-\nu} d\xi
\]

\[
= \left( \frac{t+s}{ts} \right)^{1-b} \frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \sum_{n=0}^\infty \left( \frac{t+s}{ts} \right)^{\nu-2-2n} \Gamma(2n + 1 + b - \nu) \omega_n(z, w, t, s).
\]

Since \( 0 < b \leq 1 \), we have that for \( n \in \mathbb{N}, \)

\[
\frac{\Gamma(2n + 1 + b - \nu)}{\Gamma(2n + 2 - \nu)} = \frac{B(2n + 1 + b - \nu, 1-b)}{\Gamma(1-b)} \leq \frac{1}{(1-b) \Gamma(1-b)} = \frac{1}{\Gamma(2-b)} \leq \epsilon,
\]

where \( B(u, v) \) (with \( u, v > 0 \)) is the beta function and

\[
(3.9) \quad \epsilon := \min_{s \in [1,2]} \Gamma(s) \approx 1.12917.
\]

Therefore, we have that

\[
\left( \frac{t+s}{ts} \right)^{1-b} \frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \sum_{n=0}^\infty \left( \frac{t+s}{ts} \right)^{\nu-2-2n} \Gamma(2n + 1 + b - \nu) \omega_n(z, w, t, s)
\]

\[
\leq \epsilon \left( \frac{t+s}{ts} \right)^{1-b} \frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \sum_{n=0}^\infty \left( \frac{t+s}{ts} \right)^{\nu-2-2n} \Gamma(2n + 2 - \nu) \omega_n(z, w, t, s)
\]

\[
= \epsilon \left( \frac{t+s}{ts} \right)^{1-b} \frac{z^{1-\nu}}{(ts)^{2-\nu}} e^{-\frac{\nu + 1}{\nu \xi}} \sum_{n=0}^\infty \xi^{2n+1-\nu} \omega_n(z, w, t, s) d\xi
\]

\[
= \epsilon \left( \frac{t+s}{ts} \right)^{1-b} \int_0^\infty q(z, \xi, t) q(\xi, w, s) d\xi
\]

\[
= \epsilon \left( \frac{t+s}{ts} \right)^{1-b} q(z, w, t + s),
\]

which confirms the claim (3.8).
To proceed, we notice that by Lemma 2.7 and (3.2),
\[ |q_{J,1}(z, w, t)| \leq V_J \int_0^t \int_0^\infty q_J(z, \xi, t-s) q_J(\xi, w, s) |V(\xi)| d\xi ds \]
\[ \leq V_J \int_0^t \int_0^\infty q(z, \xi, t-s) q(\xi, w, s) \xi^{b-1} d\xi ds \]
\[ \leq c V_J \int_0^t \frac{t^{1-b}}{s^{1-b} (t-s)^{1-b}} ds \cdot q(z, w, t) \]
\[ = ct^b V_J B(b, b) q(z, w, t) \]
for every \((z, w, t) \in (0, \infty)^3\). Assume that up to some \(n \geq 1\), for every \((z, w, t) \in (0, \infty)^3\),
\[ |q_{J,n}(z, w, t)| \leq (ct^b V_J)^n \left( \prod_{j=1}^n B(b, jb) \right) q(z, w, t). \]

For \(n+1\), we have that
\[ |q_{J,n+1}(z, w, t)| \leq V_J \int_0^t \int_0^\infty q(z, \xi, t-s) |q_{J,n}(\xi, w, s)| \xi^{b-1} d\xi ds \]
\[ \leq c^n V_J^{n+1} \left( \prod_{j=1}^n B(b, jb) \right) \int_0^t s^{n\beta} \int_0^\infty q(z, \xi, t-s) q(z, w, t) \xi^{b-1} d\xi ds \]
\[ \leq (c V_J)^{n+1} \left( \prod_{j=1}^n B(b, jb) \right) \int_0^t s^{n\beta} \frac{t^{1-b}}{s^{1-b} (t-s)^{1-b}} ds \cdot q(z, w, t) \]
\[ = (ct^b V_J)^{n+1} \left( \prod_{j=1}^{n+1} B(b, jb) \right) q(z, w, t). \]

Upon rewriting \(\prod_{j=1}^n B(b, jb)\) as \((\Gamma(b))^{n+1} \Gamma((n+1)\beta)\), we immediately obtain (3.5)-(3.7). Finally, (3.1) can be verified by plugging the series representation of \(q_J^V(z, w, t)\) into the right hand side of (3.1) and integrating term by term.

We are now ready to solve (2.5).

**Proposition 3.2.** Let \(q_J^V(z, w, t)\) be defined as in (3.6). Then, \(q_J^V(z, w, t)\) is continuous on \((0, J)^2 \times (0, \infty)\), and for every \((z, w, t) \in (0, J)^2 \times (0, \infty)\), we have that
\[ w^{-\nu} q_J^V(z, w, t) = z^{-\nu} q_J^V(w, z, t). \]

\(q_J^V(z, w, t)\) also satisfies the following integral equation:
\[ q_J^V(z, w, t) = q_J(z, w, t) + \int_0^t \int_0^\infty q_J^V(z, \xi, t-s) q_J(\xi, w, s) V(\xi) d\xi ds. \]

Moreover, \(q_J^V(z, w, t)\) is the fundamental solution to (2.5). Given \(h \in C_c((0, J))\),
\[ v_{h,(0,J)}^V(z, t) := \int_0^J q_J^V(z, w, t) h(w) dw \text{ for } (z, t) \in (0, J) \times (0, \infty) \]
is a smooth solution to (2.5).

**Proof.** To prove (3.10), we first note that if, for \((z, w, t) \in (0, J)^2 \times (0, \infty)\) and \(n \in \mathbb{N}\), we define
\[ \tilde{q}_{J,0}(z, w, t) := q_J(z, w, t) \text{ and } \tilde{q}_{J,n+1}(z, w, t) := \int_0^t \int_0^\infty \tilde{q}_{J,n}(z, \xi, t-s) q_J(\xi, w, s) V(\xi) d\xi ds, \]

\[ \tilde{q}_{J,n}(z, w, t) := q_J(z, w, t) \]
then \( \hat{q}_{J,n}(z, w, t) = q_{J,n}(z, w, t) \). In other words, (3.13) is an equivalent recursive relation to (3.2). To see this, one can expand both the right hand side of (3.2) and that of (3.13) into two respective \( 2n \)-fold integrals, and confirm that the two integrals are identical. Next, we will show by induction that for every \((z, w, t) \in (0, J)^2 \times (0, \infty) \) and \( n \in \mathbb{N} \),

\[
w^{1 - \nu} q_{J,n}(z, w, t) = z^{1 - \nu} q_{J,n}(w, z, t).
\]

When \( n = 0 \), this relation is simply (2.23). Assume that this relation holds up to some \( n \in \mathbb{N} \). By (2.23) and the equivalence between (3.2) and (3.13), we have that

\[
w^{1 - \nu} q_{J,n+1}(z, w, t) = z^{1 - \nu} \int_0^t \int_0^J q_J(\xi, z, t - s) q_{J,n}(w, \xi, s) V(\xi) \, d\xi \, ds
\]

\[
= z^{1 - \nu} \int_0^t \int_0^J \hat{q}_{J,n}(w, \xi, s) q_J(\xi, z, t - s) V(\xi) \, d\xi \, ds
\]

\[
= z^{1 - \nu} \int_0^t \int_0^J \hat{q}_{J,n+1}(w, z, t) = z^{1 - \nu} \hat{q}_{J,n+1}(w, z, t).
\]

(3.10) follows immediately. To establish (3.11), we write its right hand side as

\[
q_J(z, w, t) + \int_0^t \int_0^\infty q_J^V(z, \xi, t - s) q_J(\xi, w, s) V(\xi) \, d\xi \, ds
\]

\[
= q_J(z, w, t) + \sum_{n=0}^\infty \int_0^t \int_0^\infty \hat{q}_{J,n}(z, \xi, t - s) q_J(\xi, w, s) V(\xi) \, d\xi \, ds
\]

\[
= q_J(z, w, t) + \sum_{n=0}^\infty \hat{q}_{J,n+1}(z, w, t)
\]

where, again, we used the equivalence between (3.2) and (3.13). By (3.1) and (3.7), \( (z, t) \mapsto q_J^V(z, w, t) \) is continuous for every \( w \in (0, J) \), and by (3.11), \( w \mapsto q_J^V(z, w, t) \) is equicontinuous in \( (z, t) \) from any bounded subset of \((0, J) \times (0, \infty) \). From here one can easily derives the continuity of \( q_J^V(z, w, t) \) in \( (z, w, t) \) on \((0, J)^2 \times (0, \infty) \).

Given \( h \in C_c((0, J)) \), for every \((z, t) \in (0, J) \times (0, \infty) \), let \( v_{h,(0,J)}^V(z, t) \) and \( v_{h,(0,J)}(z, t) \) be defined as in (3.12) and (2.27) respectively. It follows from (3.1) that

\[
(3.14) \quad v_{h,(0,J)}^V(z, t) = v_{h,(0,J)}(z, t) + \int_0^t \int_0^J q_J(z, \xi, t - s) v_{h,(0,J)}^V(\xi, s) V(\xi) \, d\xi \, ds.
\]

Let \( b \) and \( c \) be as in (3.3) and (3.9) respectively. By (3.7) and (3.3), we have that

\[
\left| v_{h,(0,J)}^V(z, t) - v_{h,(0,J)}(z, t) \right| = \left| \int_0^t \int_0^J q_J(z, \xi, t - s) v_{h,(0,J)}^V(\xi, s) V(\xi) \, d\xi \, ds \right|
\]

\[
\leq \| h \|_u \int_0^t \int_0^J \int_0^J q_J(z, \xi, t - s) \left| q_J^V(\xi, u, s) \right| |V(\xi)| \, d\xi \, ds \, du
\]

\[
\leq \| h \|_u M(t) \int_0^J \int_0^J \int_0^J q_J(z, \xi, t - s) q(\xi, u, s) |V(\xi)| \, d\xi \, ds \, du
\]

\[
\leq \| h \|_u M(t) c b V_J B(b, b) \int_0^J q(z, u, t) \, du.
\]

Since \( v_{h,(0,J)}(z, t) \) is a solution to (2.9), the second last inequality implies that

\[
\lim_{z \to 0} v_{h,(0,J)}^V(z, t) = \lim_{z \to 0} v_{h,(0,J)}^V(z, t) = 0,
\]

and the last inequality leads to \( \lim_{z \to 0} v_{h,(0,J)}^V(z, t) = h(z) \).
The only thing that remains to be proven is that \( v_h^V(0, J) (z, t) \) is a smooth solution to the equation in (2.29), which, by the hypoellipticity of the operator \( \partial_t - L^V \), can be reduced to showing that \( v_h^V(0, J) (z, t) \) is a solution in the sense of distribution. We take \( \varphi \in C^\infty_c ((0, J)) \) and consider

\[
\left\langle \varphi, v_h^V(0, J) (\cdot, t) \right\rangle := \int_0^J v_h^V(0, J) (z, t) \varphi(z) \, dz \quad \text{for} \quad t \geq 0,
\]

and use (3.14) to write it as

\[
\left\langle \varphi, \psi_{\varphi,h}^V(0, J) (\cdot, t) \right\rangle = \left\langle \varphi, v_h(0, J) (\cdot, t) \right\rangle + \int_0^t \int_0^J (\varphi, \psi_{\varphi,J} (\cdot, u, t-s)) v_h^V(0, J) (u, s) V(u) \, du \, ds.
\]

Therefore,

\[
\frac{d}{dt} \left\langle \varphi, v_h(0, J) (\cdot, t) \right\rangle = \frac{d}{dt} \left\langle \varphi, v_h(0, J) (\cdot, t) \right\rangle + \left\langle V \varphi, v_h(0, J) (\cdot, t) \right\rangle
\]

\[
+ \int_0^t \int_0^J \frac{d}{dt} (\varphi, \psi_{\varphi,J} (\cdot, u, t-s)) v_h^V(0, J) (u, s) V(u) \, du \, ds.
\]

Hence, for some constant \( C > 0 \) uniformly in \( t \in (0, t_J) \) (\( C \) may depend on \( J \) and \( \alpha \)),

\[
\sup_{z, w \in (0, t_J)^2} \left| \frac{q_f^V(z, w, t)}{q(z, w, t)} - 1 \right| \leq \sup_{z, w \in (0, t_J)^2} \left( \left| \frac{q_f^V(z, w, t)}{q(z, w, t)} - q_J(z, w, t) \right| + \left| \frac{q_J(z, w, t)}{q(z, w, t)} - 1 \right| \right)
\]

\[
\leq M(t) - 1 + \exp \left( -\frac{2J}{9t} \right).
\]

Hence, for some constant \( C > 0 \) uniformly in \( t \in (0, t_J) \) (\( C \) may depend on \( J \) and \( \alpha \)),

\[
\sup_{z, w \in (0, t_J)^2} \left| \frac{q_f^V(z, w, t)}{q(z, w, t)} - 1 \right| \leq Ct^b.
\]

(3.15) confirms that when \( t \) is small, \( q_f^V(z, w, t) \) is indeed well approximated by \( q(z, w, t) \). However, viewing from (3.6), \( q(z, w, t) \) is only the “first order” approximation to \( q_f^V(z, w, t) \), since the error \( t^b \) in (3.15) is generated by keeping only the first term in the series in (3.4). It is possible to derive a more general “\( k \)–th order” approximation for \( q_f^V(z, w, t) \) with \( k \in \mathbb{N} \), and obtain an analog of (3.15) with the error being \( t^{kb} \). To achieve this purpose, we introduce a new sequence of functions. For \( (z, w, t) \in (0, \infty)^3 \) and \( n \in \mathbb{N} \), we set

\[
q_0(z, w, t) := q(z, w, t) \quad \text{and} \quad q_n(z, w, t) := \int_0^t \int_0^\infty q(z, \xi, t-s) V(\xi) q_n(\xi, w, s) \, d\xi \, ds,
\]

(3.16)
where, again, we assume that \( V(z) \equiv 0 \) for \( z > J \). By following the proof of (3.3) line by line with \( q_{J,n}(z,w,t) \) replaced by \( q_n(z,w,t) \), we also get that for every \((z,w,t) \in (0,\infty)^3 \) and \( n \in \mathbb{N} \),

\[
(3.17) \quad |q_n(z,w,t)| \leq m_n(t) q(z,w,t).
\]

Clearly, \( q_n(z,w,t) \) is the “global” counterpart of \( q_{J,n}(z,w,t) \), and we will justify that \( q_{J,n}(z,w,t) \) is close to \( q_n(z,w,t) \) when \( t \) is sufficiently small.

**Lemma 3.3.** For every \((z,w,t) \in (0,J)^2 \times (0,\infty) \) and \( n \in \mathbb{N} \),

\[
(3.18) \quad |q_{J,n}(z,w,t) - q_n(z,w,t)| \leq (2ct^n B(b,b)V_J)^n r(z,w,t),
\]

where \( r(z,w,t) \) is as in (2.28).

**Proof.** When \( n = 0 \), (3.18) simply becomes (2.28). Assume that (3.18) holds up to some \( n \geq 0 \). Following (3.2) and (3.16), we write

\[
q_n(z,w,t) = \int_0^t \int_0^\infty r(z,\xi,t-s) V(\xi) q_n(\xi,w,s) d\xi ds
\]

\[
+ \int_0^t \int_0^\infty q_J(\xi,t-s) V(\xi) (q_n(\xi,w,s) - q_{J,n}(\xi,w,s)) d\xi ds.
\]

We use Fubini’s theorem and (2.28) to rewrite the first term on the right hand side of (3.19) as

\[
(3.20) \quad \mathbb{E} \left[ \int_0^{t-c_J^Y(z)} \int_0^\infty q(J,\xi,t-s-c_J^Y(z)) V(\xi) q_n(\xi,w,s) d\xi ds ; c_J^Y(z) \leq t \wedge \zeta_0^Y(z) \right],
\]

which, by (3.17), is bounded by

\[
\frac{\Gamma^{n+1}(b)}{\Gamma((n+1)b)} \mathbb{E} \left[ \int_0^{t-c_J^Y(z)} \int_0^\infty q(J,\xi,t-s-c_J^Y(z)) V(\xi) |s|^b q(\xi,w,s) d\xi ds ; c_J^Y(z) \leq t \wedge \zeta_0^Y(z) \right].
\]

By (3.8) and the fact that

\[
\frac{\Gamma^{n+1}(b)}{\Gamma((n+1)b)} = \prod_{j=1}^n B(b,jb) \leq B^n(b,b),
\]

we can further bound (3.20) from above by

\[
c^n B^n(b,b)V_J^{n+1} \mathbb{E} \left[ \int_0^{t-c_J^Y(z)} |s|^b \int_0^\infty q(J,\xi,t-s-c_J^Y(z)) c_J^{b-1} q(\xi,w,s) d\xi ds ; c_J^Y(z) \leq t \wedge \zeta_0^Y(z) \right]
\]

\[
\leq c^{n+1} B^n(b,b)V_J^{n+1} \mathbb{E} \left[ q(J,w,t-c_J^Y(z)) \int_0^{t-c_J^Y(z)} \frac{(t-c_J^Y(z))^{1-b}}{(t-s-c_J^Y(z))^{1-b}} s^n q(\xi,w,s) d\xi ds ; c_J^Y(z) \leq t \wedge \zeta_0^Y(z) \right]
\]

\[
\leq (ct^n B(b,b)V_J)^{n+1} r(z,w,t).
\]

According to the inductive assumption, the second term on the right hand side of (3.19) is bounded by

\[
2^n c^n B^n(b,b)V_J^{n+1} \int_0^t \int_0^\infty q(z,\xi,t-s) \xi^{b-1} s^n r(\xi,w,s) d\xi ds,
\]

which, by (2.14) and (2.23), is equal to

\[
\frac{2^n c^n B^n(b,b)V_J^{n+1}}{w^{1-\nu} z^{\nu-1}} \int_0^t \int_0^\infty q(\xi,z,t-s) \xi^{b-1} s^n r(\xi,w,s) d\xi ds.
\]
We use Fubini’s theorem again to rewrite the expression above as

\[
\frac{2^n e^n B^n (b, b) V_j^{n+1}}{w^{1-\nu} z^{n+1}} \mathbb{E} \left[ \int_0^t \int_0^t \int_0^t q (\xi, z, t-s) e^{s-n} q (J, \xi, s-\zeta (w)) d\xi ds; \zeta (w) \leq t \wedge \zeta (w) \right]
\]

\[
\leq \frac{2^n e^{n+1} B^n (b, b) V_j^{n+1}}{w^{1-\nu} z^{n+1}} \mathbb{E} \left[ \int_0^t \int_0^t (t-s) e^{s-n} (s-\zeta (w))^{1-n} ds; \zeta (w) \leq t \wedge \zeta (w) \right]
\]

\[
\leq \frac{2^n (ct^m B (b, b) V_j)^{n+1}}{w^{1-\nu} z^{n+1}} r (w, z, t) = 2^n (ct^m B (b, b) V_j)^{n+1} r (z, w, t).
\]

Thus, combining the estimates of the two terms on the right hand side of (3.19), we obtain that

\[
|q_{J, n+1} (z, w, t) - q_{n+1} (z, w, t)| \leq (1 + 2^n) (ct^m B (b, b) V_j)^{n+1} r (z, w, t)
\]

\[
\leq (2ct^m B (b, b) V_j)^{n+1} r (z, w, t).
\]

\[
\square
\]

**Proposition 3.4.** Let \( t_J := \frac{4 J}{\nu (2-\nu)} \). Then, for every \( t \in (0, t_J) \) and \( k \in \mathbb{N} \setminus \{0\} \),

\[
\sum_{z, w \in (0, 4 J)^2} \left| q_{J} (z, w, t) - \sum_{n=0}^{k-1} q_{n} (z, w, t) \right| \leq m_k (t) M (t) + D_k (t) \exp \left( -\frac{2J}{\nu t} \right),
\]

where

\[
D_k (t) := \sum_{n=0}^{k-1} (2ct^m B (b, b) V_j)^n.
\]

In particular, there exists \( C > 0 \) uniformly in \( t \in (0, t_J) \) and \( k \in \mathbb{N} \setminus \{0\} \) (\( C \) may depend on \( J \) and \( \nu \)) such that

\[
\sup_{z, w \in (0, 4 J)^2} \left| q_{J} (z, w, t) - \sum_{n=0}^{k-1} q_{n} (z, w, t) \right| \leq C t^k.
\]

**Proof.** Only (3.21) requires proof, since (3.23) follows from (3.21) trivially. By (3.5) and (3.6), we know that for every \( (z, w, t) \in (0, J)^2 \times (0, \infty) \) and \( k \in \mathbb{N} \setminus \{0\} \)

\[
\left| q_{J} (z, w, t) - \sum_{n=0}^{k-1} q_{J,n} (z, w, t) \right| \leq \sum_{n=k}^{\infty} m_n (t)
\]

and we further derive that

\[
\sum_{n=k}^{\infty} m_n (t) = \sum_{n=k}^{\infty} \frac{\Gamma^{n+1} (b) (ct^m V_j)^n}{\Gamma ((n+1) b)}
\]

\[
= \frac{\Gamma^k (b) (ct^m V_j)^k}{\Gamma (k b)} \sum_{l=0}^{\infty} \frac{\Gamma^{l+1} (b) (ct^m V_j)^l}{\Gamma ((l+1) b)} B ((l+1) b, kb)
\]

\[
\leq \frac{\Gamma^{k+1} (b) (ct^m V_j)^k}{\Gamma ((k+1) b)} \sum_{l=0}^{\infty} \frac{\Gamma^{l+1} (b) (ct^m V_j)^l}{\Gamma ((l+1) b)}
\]

\[
= m_k (t) M (t),
\]

20
where we again used the fact that $B ((l + 1) b, kb) \leq B (b, kb)$ for every $l \in \mathbb{N}$. Meanwhile, by (3.18), we have that for every $(z, w, t) \in (0, J)^2 \times (0, \infty),
\begin{align*}
\sum_{n=0}^{k-1} q_{I,n} (z, w, t) - \sum_{n=0}^{k-1} q_n (z, w, t) \leq D_k (t) r (z, w, t),
\end{align*}
which, combined with (2.29), leads to (3.21).

3.3. From $q^V (z, w, t)$ to $p_I (z, w, t)$. Now we are ready to return to the localized equation (2.1). Recall that $I > 0$, $\phi (x)$ and $\theta (x)$ are functions on $(0, I)$ defined by (1.2), and $I$ and $J$ are related by $J = \phi (I)$; with $z \in (0, J)$, $\psi (z)$ is the inverse function of $\phi$, $\theta (z) = \theta (\psi (z))$ and $\Theta (z)$ is as defined in (2.2). Guided by Proposition 2.1, we define
\begin{align*}
p_I (x, y, t) := q^V (\phi (x), \phi (y), t) \frac{\Theta (\phi (x))}{\Theta (\phi (y))} \phi' (y).
\end{align*}
for every $(x, y, t) \in (0, I)^2 \times (0, \infty)$. We immediately obtain several results on $p_I (x, y, t)$ based on Proposition 2.1 and Proposition 3.2. In addition, we can establish the connection between $p_I (x, y, t)$ and \{ $X (x, t) : t \geq 0$ \} the unique solution to (1.7) and underlying diffusion process corresponding to $L = x^n a (x) \partial^2_x + b (x) \partial_x$.

**Proposition 3.5.** Let $p_I (x, y, t)$ be defined as in (3.24). Then, $p_I (x, y, t)$ is continuous on $(0, I)^2 \times (0, \infty)$ and
\begin{align*}
\frac{(\phi (y))^{1-\nu}}{\phi' (y)} \Theta^2 (\phi (y)) p_I (x, y, t) = \frac{(\phi (x))^{1-\nu}}{\phi' (x)} \Theta^2 (\phi (x)) p_I (y, x, t)
\end{align*}
for every $(x, y, t) \in (0, I)^2 \times (0, \infty).

$p_I (x, y, t)$ is the fundamental solution to (2.1). Given $f \in C_c ((0, I))$
\begin{align*}
u_f, (0, I) (x, t) := \int_0^t f (y) p_I (x, y, t) dy
\end{align*}
is the unique solution in $C^{2,1} ((0, I) \times (0, \infty))$ to (2.1), and $\nu_f, (0, I) (x, t)$ is smooth on $(0, I) \times (0, \infty)$.
Moreover,
\begin{align*}
u_f, (0, I) (x, t) = \mathbb{E} \left[ f (X (x, t)) ; t < \zeta^X_{0, I} (x) \right],
\end{align*}
and hence for every Borel set $\Gamma \subseteq (0, I),
\begin{align*}
\int_{\Gamma} p_I (x, y, t) dy = \mathbb{P} \{ X (x, t) \in \Gamma, t < \zeta_{0, I}^X (x) \}.
\end{align*}

Finally, $p_I (x, y, t)$ satisfies the Chapman-Kolmogorov equation, i.e., for every $x, y \in (0, I)$ and $t, s > 0$,
\begin{align*}
p_I (x, y, t + s) = \int_0^t p_I (x, \xi, t) p_I (\xi, y, s) d\xi.
\end{align*}

**Proof.** (3.26) follows directly from (3.10). Given $f \in C_c ((0, I))$, we set $h (z) := \frac{f (\psi (z))}{\Theta (\psi (z))}$ for $z \in (0, J)$. By (3.12), it is straightforward to check that
\begin{align*}
u_f, (0, I) (x, t) = \Theta (\phi (x)) \int_0^t f (y) q^V (\phi (x), \phi (y), t) \frac{\phi' (y)}{\Theta (\phi (y))} dy
= \Theta (\phi (x)) \int_0^t f (\psi (w)) q^V (\phi (x), w, t) \frac{d w}{\Theta (w)}
= \Theta (\phi (x)) v^V_{\psi, (0, I)} (\phi (x), t),
\end{align*}
and hence it follows from Proposition 3.2 that $\nu_f, (0, I) (x, t)$ is a smooth solution to (2.1). Since
\begin{align*}
\{ \nu_f, (0, I) (X (x, s \wedge \zeta_{0, I}^X (x)), t - s \wedge \zeta_{0, I}^X (x)) ; 0 \leq s \leq t \}
\end{align*}
is a bounded martingale, by equating its expectation at \( s = 0 \) and \( s = t \), we obtain \( \Theta(x,t) \), which further leads to \( \Theta(x,t) \). Since \( \{ X(x,t) : t \geq 0 \} \) is the unique solution to \( \Theta(x,t) \), \( u_{f,(0,t)}(x,t) \) is the unique \( C^{2,1}((0, I) \times (0, \infty)) \) solution to \( \Theta(x,t) \). Finally, \( \Theta(x,t) \) follows from \( \Theta(x,t) \) and the strong Markov property of \( \{ X(x,t) : t \geq 0 \} \). 

**Remark 3.6.** Note that the properties developed above for \( p_{f}(x,y,t) \) and \( u_{f,(0,t)}(x,t) \) also lead to corresponding results on \( q_{f}^{V}(z,w,t) \) and \( v_{h,(0,t)}^{V}(z,t) \). For example, we see from \( \Theta(x,t) \) that \( q_{f}^{V}(z,w,t) \) also satisfies the Chapman-Kolmogorov equation, i.e., for every \( z, w \in (0, J) \) and \( t, s > 0 \),

\[
q_{f}^{V}(z,w,t+s) = \int_{0}^{t} q_{f}^{V}(z,\xi,t) q_{f}^{V}(\xi,w,s) \, d\xi,
\]

and the uniqueness of \( u_{f,(0,t)}(z,t) \) implies that, given \( h \in \mathcal{C}_{c}((0, J)) \), \( v_{h,(0,t)}^{V}(z,t) \) is the unique \( C^{2,1}((0, J) \times (0, \infty)) \) solution to \( \Theta(x,t) \).

The approximations we obtained in Proposition \( \Theta(x,t) \) for \( q_{f}^{V}(z,w,t) \) can also be “transported” to \( p_{f}(x,y,t) \) in a straightforward way. To see this, we define, for \((x,y,t) \in (0, I)^2 \times (0, \infty)\),

\[
p^{\text{approx}}(x,y,t) := q(\phi(x),\phi(y),t) \Theta(\phi(x)) \Theta(\phi(y)) \phi'(y),
\]

and more generally for \( k \in \mathbb{N} \setminus \{0\}\),

\[
p^{k-\text{approx}}(x,y,t) := \sum_{n=0}^{k-1} q_{n}(\phi(x),\phi(y),t) \Theta(\phi(x)) \Theta(\phi(y)) \phi'(y).
\]

Then Proposition \( \Theta(x,t) \) can be rewritten as follows.

**Corollary 3.7.** There exists \( t_{I} > 0 \) such that for every \( t \in (0, t_{I}) \) and \( k \in \mathbb{N} \setminus \{0\}\),

\[
\sup_{(x,y) \in (0, \phi(I))^{2}} \left| \frac{p_{I}(x,y,t) - p^{k-\text{approx}}(x,y,t)}{p^{\text{approx}}(x,y,t)} \right| \leq m_{k}(t) M(t) + D_{k}(t) \exp \left( -\frac{2\phi(I)}{9t} \right)
\]

where \( D_{k}(t) \) is as in \( \Theta(x,t) \). In particular,

\[
\sup_{(x,y) \in (0, \phi(I))^{2}} \left| \frac{p_{I}(x,y,t)}{p^{\text{approx}}(x,y,t)} - 1 \right| = M(t) - 1 + D_{k}(t) \exp \left( -\frac{2\phi(I)}{9t} \right).
\]

**Remark 3.8.** We want to point out that, from now on, whenever \( J = \phi(I) \), the constants \( \Theta, V, t_{J} \) and \( t_{I} \) that were introduced in \( \S 3 \) will also be written as \( \Theta, V, t_{I} \) and \( t_{I} \) respectively. In addition, by plugging \( \phi(x) \) into \( \Theta(x,t) \) and \( \Theta(x,t) \), we get that

\[
V(\phi(x)) = -\frac{\theta^{2}(x)}{4\phi(x)} - \frac{\theta'(x)}{2\phi'(x)} + \frac{1 - \nu}{2} \frac{\Theta(x)}{\phi(x)}
\]

and

\[
\Theta(\phi(x)) = \begin{cases} \frac{1}{2^{\frac{\alpha}{2}} - \frac{\nu}{2}} \exp \left( -\int_{0}^{x} \frac{b(w)}{2\phi'(w)} \, dw \right) & \text{if } \alpha \neq 1, \\ \frac{1}{2^{\frac{\alpha}{2}} - \frac{\nu}{2}} \exp \left( -\int_{0}^{x} \frac{1}{2\phi'(w)} \left( b(w) - b(0) \right) \, dw \right) & \text{if } \alpha = 1. \end{cases}
\]

With \( \Theta(x,t) \) and \( \Theta(x,t) \), it is possible to rewrite some of the expressions that appeared above (e.g., \( \Theta(x,t) \) and \( \Theta(x,t) \)) in a more explicit way, see, e.g., \( \Theta(x,t) \) and \( \Theta(x,t) \) in the Appendix. Especially when \( b(x) \equiv 0 \), these expressions take much simpler forms than in the general case, as we will see with a concrete example in \( \S 5 \).
4. Global Equation

In the previous section we have solved the localized equation (2.1) and obtained its fundamental solution \( p_t (x, y, t) \). Now we proceed with the last step to complete our project, which is to build the “link” between (2.1) and the original problem (1.1). To achieve this goal, we rely on the strong Markov property of \( \{ X(x, t) : t \geq 0 \} \) and the probabilistic interpretations of the solutions found in the previous sections.

4.1. From \( p_t (x, y, t) \) to \( p(x, y, t) \). We introduce two more notations for this section: given \( I > 0 \),

\[
 a_t := \max_{x \in [0,I]} \left\{ \frac{1}{a(x)}, a(x) \right\} \quad \text{and} \quad b_t := \max_{x \in [0,I]} |b(x)|.
\]

Our first task is to derive probability estimates for the hitting times of \( \{ X(x, t) : t \geq 0 \} \).

**Lemma 4.1.** We define, for \( x \geq 0 \),

\[
 S(x) := \int_0^{\phi(x)} e^{-\nu w} \Theta^2(w) \, dw.
\]

Then, for every \( 0 < x < y \leq I \),

\[
 \mathbb{P} (\zeta^X_y (x) < \zeta^X_y (x)) = \frac{S(x)}{S(y)};
\]

if \( \Theta_I \) is the constant found in Lemma 2.2 (upon identifying \( \Theta_I \) with \( \Theta_I \) for \( J = \phi(I) \)), then

\[
 \mathbb{P} (\zeta^X_y (x) < \zeta^X_y (x)) \leq \Theta_I^2 \left( \frac{\phi(x)}{\phi(y)} \right)^{1-\nu}.
\]

Moreover, for every \( G \in (0, I), x \in (0, G) \) and \( t > 0 \) such that \( I - G > t b_I \), we have that

\[
 \mathbb{P} (\zeta^X_I (x) \leq t) \leq \exp \left( - \frac{(I - x - t b_I)^2}{4 I^\alpha a_I} \right).
\]

**Proof.** We use Itô’s formula to verify that, for every \( y > x \), \( \{ S(X(x, t) \wedge \zeta^X_I (x)) \} \) is a bounded martingale, and hence (4.2) follows immediately. Further, by (2.7), we have that for every \( x \in (0, I) \),

\[
 \Theta_I^2 \left( \frac{\phi(x)}{\phi(y)} \right)^{1-\nu} \leq S(x) \leq \Theta_I^2 \left( \frac{\phi(x)}{\phi(y)} \right)^{1-\nu},
\]

which leads to (4.3).

Now we get down to proving (4.4), and the proof is similar to that of Lemma 2.2. For every \( \lambda \geq 0 \) and \( t \geq 0 \),

\[
 \left\{ \exp \left( \lambda X(x, t \wedge \zeta^X_I(x)) - \lambda \int_0^t \zeta^X_I(x) \right) - \lambda \int_0^t \zeta^X_I(x) b(X(x, s)) \, ds - \lambda^2 \int_0^t \zeta^X_I(x) X^a(x, s) a(X(x, s)) \, ds : t \geq 0 \right\},
\]

is a bounded martingale, from where we get that

\[
 \mathbb{E} \left[ \exp \left( - \lambda^2 \int_0^{\zeta^X_I(x)} X^a(x, s) a(X(x, s)) \, ds \right) ; \zeta^X_I(x) < \infty \right] \leq \exp (\lambda x - \lambda I).
\]

Since

\[
 \lambda \int_0^{\zeta^X_I(x)} b(X(x, s)) \, ds + \lambda^2 \int_0^{\zeta^X_I(x)} X^a(x, s) a(X(x, s)) \, ds \leq (\lambda b_I + \lambda^2 I^\alpha a_I) \zeta^X_I(x),
\]

we further have that

\[
 \mathbb{E} \left[ \exp \left( - (\lambda b_I + \lambda^2 I^\alpha a_I) \zeta^X_I(x) ; \zeta^X_I(x) < \infty \right) \right] \leq \exp (\lambda x - \lambda I).
\]

By Markov’s inequality,

\[
 \mathbb{P} (\zeta^X_I(x) \leq t) = \mathbb{P} \left( e^{- (\lambda b_I + \lambda^2 I^\alpha a_I) t} \zeta^X_I(x) \geq e^{- (\lambda b_I + \lambda^2 I^\alpha a_I) t} \right) \leq e^{\lambda^2 t I^\alpha a_I - \lambda (I - t - t b_I)}.
\]
is obtained by minimizing the right hand side above over $\lambda \geq 0$.

Next, we consider $\{p_I(x, y, t) : I > 0\}$ as a family parametrized by $I$, and for every $0 < I < H$, we want to find out the link between $p_I(x, y, t)$ and $p_H(x, y, t)$, i.e., the fundamental solutions to (2.1) with the right boundary at $I$ and $H$ respectively. To this end, we choose a third constant $G \in (0, I)$ and define for each $x \in (0, G)$ a sequence of hitting times of $\{X(t) : t \geq 0\}$ where $\eta_0(x) := 0$ and for $n \in \mathbb{N}\backslash\{0\},$

$$\eta_{2n-1}(x) := \inf \{s \geq \eta_{2n-2}(x) : X(s, s) \geq I\}, \eta_{2n}(x) := \inf \{s \geq \eta_{2n-1}(x) : X(s, s) \leq G\}.$$  

In other words, the sequence $\{\eta_n(x) : n \in \mathbb{N}\}$ records the downward crossings of $\{X(t) : t \geq 0\}$ from $I$ to $G$. With the help of $\{\eta_n(x) : n \in \mathbb{N}\}$ and the strong Markov property of $\{X(t) : t \geq 0\}$, we are able to connect $p_H(x, y, t)$ and $p_I(x, y, t)$ as follows.

**Proposition 4.2.** For $(x, y, t) \in (0, G)^2 \times (0, \infty),$

$$p_H(x, y, t) = p_I(x, y, t) + \sum_{n=1}^{\infty} \mathbb{E} \left[ p_I(G, y, t - \eta_{2n}(x)) : \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_H^X(x) \right].$$

**Proof.** Given $f \in C_c((0, G))$, we use (3.24) to write

$$\int_0^G f(y) p_H(x, y, t) dy = \mathbb{E} \left[ f(X(t)) : t < \zeta_{0,H}^X(x) \right].$$

According to the number of downward crossings (from $I$ to $G$) completed by $\{X(s, s) : 0 \leq s \leq t\}$, we further decompose $\mathbb{E} \left[ f(X(t)) : t < \zeta_{0,H}^X(x) \right]$ as

$$\mathbb{E} \left[ f(X(t)) : t < \zeta_{0,I}^X(x) \right] + \sum_{n=1}^{\infty} \mathbb{E} \left[ f(X(t)) : \eta_{2n}(x) \leq t < \eta_{2n+1}(x) \wedge \zeta_{0,H}^X(x), \eta_{2n}(x) < \zeta_{0,H}^X(x) \right].$$

By the strong Markov property of $X(x, t)$, we have that for each $n \geq 1,$

$$\mathbb{E} \left[ f(X(t)) : \eta_{2n}(x) \leq t < \eta_{2n+1}(x) \wedge \zeta_{0,H}^X(x), \eta_{2n}(x) < \zeta_{0,H}^X(x) \right] = \mathbb{E} \left[ \int_0^G f(y) p_I(G, y, t - \eta_{2n}(x)) : \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_{0,H}^X(x) \right].$$

On one hand, by (2.13), (3.7) and (3.24),

$$p_I(G, y, t - \eta_{2n}(x)) \leq M(t) \frac{(\phi(G))^{1-\nu}}{(t - \eta_{2n}(x))^2} \exp \left( -\frac{(\sqrt{\phi(G)} - \sqrt{\phi(y)})^2}{t - \eta_{2n}(x)} \right) \frac{\Theta(\phi(G))}{\Theta(\phi(y))} \phi'(y).$$

On the other hand, if $\eta_{2n}(x) < \zeta_{0,H}^X(x)$, then it must be that (i) $\zeta_{I}^X(x) < \zeta_{0}^X(x)$, (ii) during the time interval $[\zeta_{I}^X(x), \eta_{2n}(x)]$, the process starts from $G$ and hits $I$ before 0, and (iii) for each $j = 0, \cdots, n-1$, during the time interval $[\eta_{2j}(x), \eta_{2j+1}(x)]$, the process starts from $G$ and hits $I$ before 0. Hence, by (4.2) and the strong Markov property of $X(x, t)$, we have that

$$\mathbb{P} \left( \eta_{2n}(x) < \zeta_{0,H}^X(x) \right) \leq \mathbb{P} \left( \zeta_{I}^X(x) < \zeta_{0}^X(x) \right) \left( \mathbb{P} \left( \zeta_{I}^X(G) < \zeta_{0}^X(G) \right) \right)^n = \frac{S(x)}{S(G)} \left( \frac{S(G)}{S(I)} \right)^n.$$

Combining the above, we obtain that for every $(x, y, t) \in (0, G)^2 \times (0, \infty),$

$$\sum_{n=1}^{\infty} \mathbb{E} \left[ p_I(G, y, t - \eta_{2n}(x)) : \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_{0}^X(x) \right] \leq M(t) \frac{(2 - \nu)}{e} \frac{(\phi(G))^{1-\nu}}{(t - \eta_{2n}(x))^2} \exp \left( -\frac{(\sqrt{\phi(G)} - \sqrt{\phi(y)})^2}{t - \eta_{2n}(x)} \right) \frac{\Theta(\phi(G))}{\Theta(\phi(y))} \phi'(y) \frac{S(x)}{S(I) - S(G)}.$$
With Lemma 4.1 and Proposition 4.2 we are ready to prove our main result.

**Theorem 4.3.** For every \((x, y, t) \in (0, \infty)^3\), we set

\[
p(x, y, t) := \lim_{t \to \infty} p_I(x, y, t).
\]

Given \(0 < G < I < H\), let \(\{\eta_n(x) : n \in \mathbb{N}\}\) be the sequence of hitting times defined as in (4.7) (for the downward crossings of \(\{X(x, t) : t \geq 0\}\) from \(I\) to \(G\)). Then, for every \((x, y, t) \in (0, G)^2 \times (0, \infty)\),

\[
p(x, y, t) = p_I(x, y, t) + \sum_{n=1}^{\infty} \mathbb{E} \left[ p_I(G, y, t - \eta_{2n}(x)) : \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta^X_0(x) \right].
\]

\(p(x, y, t)\) is continuous on \((0, \infty)^3\), and for every \((x, y, t) \in (0, \infty)^3\),

\[
\frac{(\phi(y))^{1-\nu}}{\phi'(y)} \Theta^2(\phi(y)) p(x, y, t) = \frac{(\phi(x))^{1-\nu}}{\phi'(x)} \Theta^2(\phi(x)) p(y, x, t).
\]

For every \(y > 0\), \((x, t) \mapsto p(x, y, t)\) is a smooth solution to the Kolmogorov backward equation corresponding to \(L\), i.e.,

\[
\partial_t p(x, y, t) = x^a(x) \partial^2_x p(x, y, t) + b(x) \partial_x p(x, y, t);
\]

for every \(x > 0\), \((y, t) \mapsto p(x, y, t)\) is a smooth solution to the Kolmogorov forward equation corresponding to \(L\), i.e.,

\[
\partial_t p(x, y, t) = \partial^2_y (x^a(y) p(x, y, t)) - \partial_y (b(y) p(x, y, t)).
\]

\(p(x, y, t)\) is the fundamental solution to (4.7). Given \(f \in C_c((0, \infty))\),

\[
u_f(x, t) := \int_0^\infty f(y) p(x, y, t) \, dy \text{ for } (x, t) \in (0, \infty)^2
\]

is the unique solution in \(C^{2,1}([0, \infty)^2)\) to (4.7), and \(\nu_f(x, t)\) is smooth on \((0, \infty)^2\). Moreover, for every \((x, t) \in (0, \infty)^2\),

\[
u_f(x, t) = \mathbb{E} \left[ f(X(x, t)) : t < \zeta^X_0(x) \right],
\]

and hence for every Borel set \(\Gamma \subseteq (0, \infty)\),

\[
\int_{\Gamma} p(x, y, t) \, dy = \mathbb{P} \left( X(x, t) \in \Gamma, t < \zeta^X_0(x) \right).
\]

Finally, \(p(x, y, t)\) satisfies the Chapman-Kolmogorov equation, i.e., for every \(x, y > 0\) and \(t, s > 0\),

\[
p(x, y, t + s) = \int_0^\infty p(x, \xi, t) p(\xi, y, s) \, d\xi.
\]

**Proof.** It is clear from (4.8) that for every \((x, y, t) \in (0, \infty)^3\), by taking \(G > x \vee y\), we know that the family \(I \in (G, \infty) \mapsto p_I(x, y, t)\) is non-decreasing, so \(p(x, y, t)\) as the limit of \(p_I(x, y, t)\) (as \(I \nearrow \infty\)) is well defined. Since \(\{X(x, t) : t \geq 0\}\) is the unique solution to (4.7), \(\zeta^X_0(x) \mapsto \zeta^X_0(x)\) almost surely as \(H \nearrow \infty\) (see, e.g., §10 of [35]). Thus, (4.11) follows from (4.8) by sending \(H\) to infinity, and (4.12) follows from (4.2). Now we examine the continuity of \(p(x, y, t)\). First, (4.9) and (4.10) guarantee that the series in the right hand side of (4.11) converges uniformly on any bounded subset of \((0, G)^2 \times (0, \infty)\), from where it is easy to see that for every \(x \in (0, G), (y, t) \mapsto p(x, y, t)\) is continuous on \((0, G) \times (0, \infty)\). Furthermore in the proof of Proposition 3.2 we have seen that \(x \mapsto p_I(G, x, s)\) is equicontinuous in \(s\) from any bounded subset of \((0, \infty)\), which, combined with (4.12), leads to the continuity of \(p(x, y, t)\) in all three variables.

Next, we turn our attention to \(\nu_f(x, t)\) for \(f \in C_c((0, \infty))\). It is clear that

\[
\nu_f(x, t) = \lim_{I \nearrow \infty} \int_0^I f(y) p_I(x, y, t) \, dy = \lim_{I \nearrow \infty} \nu_{f,(I)}(x, t),
\]
and further by (3.27),
\[ u_f(x,t) = \lim_{I \to \infty} \mathbb{E} \left[ f(X(x,t)) ; t < \zeta_0^X(x) \right] = \mathbb{E} \left[ f(X(x,t)) ; t < \zeta_0^X(x) \right], \]
which means that \( p(x,y,t) \) is indeed the probability density function of \( X(x,t) \) provided that \( t < \zeta_0^X(x) \). (4.13) follows from the strong Markov property of \{\( X(x,t) ; t \geq 0 \}). Furthermore, by (4.11), if \( G \) and \( I \) are sufficiently large such that \( x \in (0,G) \) and \( \text{supp} \, f \subseteq (0,I) \), then
\[ u_f(x,t) = u_{f,(0,t)}(x,t) + \sum_{n=1}^{\infty} \mathbb{E} \left[ u_{f,(0,t)}(G,t - \eta_{2n})(x) ; \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0^X(x) \right]. \]
Let us re-examine the event \( \{ \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0^X(x) \} \) involved in the series above. If \( \eta_{2n}(x) \leq t \), then we must have that \( \zeta_0^X(x), \eta_n(x) - \zeta_0^X(x), \) and for each \( j = 0, \ldots, n-1, \eta_{2j+1}(x) - \eta_{2j}(x) \leq t \). Thus,
\[ \mathbb{P} (\eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0^X(x)) \leq \mathbb{P} (\zeta_0^X(x) < \zeta_0^X(x)) (\mathbb{P} (\zeta_0^X(G < t))^n. \]
By (4.12) and (4.13), we have that when \( I - G > tb_1 \),
\[ \mathbb{P} (\eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0^X(x)) \leq \frac{S(x)}{S(G)} \exp \left( -\frac{(I - G - tb_1)^2}{4tI^aI_1} \right). \]
Therefore, when \( t \) is sufficiently small,
\[
\left| \sum_{n=1}^{\infty} \mathbb{E} \left[ u_{f,(0,t)}(G,t - \eta_{2n})(x) ; \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0^X(x) \right] \right|
\leq \| f \| \sum_{n=1}^{\infty} \mathbb{P} (\eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0^X(x))
\leq \| f \| \frac{S(x)}{S(G)} \exp \left( -\frac{(I - G - tb_1)^2}{4tI^aI_1} \right) \frac{4tI^aI_1}{(I - G - tb_1)^2}
\]
which tends to 0 as \( t \to 0 \) or as \( x \to 0 \). Therefore, we have that
\[ \lim_{x \to 0} u_f(x,t) = \lim_{x \to 0} u_{f,(0,t)}(x,t) = 0 \quad \text{and} \quad \lim_{t \to 0} u_f(x,t) = \lim_{t \to 0} u_{f,(0,t)}(x,t) = f(x). \]

The only remaining thing is to prove the statement on \( p(x,y,t) \) and \( u_f(x,t) \) being smooth solutions to the concerned equations, which, again, by the hypoellipticity of \( \partial_t - L \), is reduced to showing that they are distribution solutions. Take \( u_f(x,t) \) for instance. We observe that for any \( \varphi \in C^\infty([-0\infty,0)) \),
\[ \langle \varphi, u_f(\cdot, t) \rangle = \lim_{t \to \infty} \int_0^\infty \varphi(x) u_{f,(0,t)}(x,t) \, dx \]
\[ = \langle \varphi, f \rangle + \lim_{t \to \infty} \int_0^\infty \varphi(x) Lu_{f,(0,t)}(x,s) \, dsds \]
\[ = \langle \varphi, f \rangle + \lim_{t \to \infty} \int_0^\infty \int_0^\infty (L^* \varphi)(x) u_{f,(0,t)}(x,s) \, dxdxds \]
\[ = \langle \varphi, f \rangle + \int_0^t \int_0^\infty (L^* \varphi)(x) u_f(x,t) \, dxdxds, \]
which implies that
\[ \frac{d}{dt} \langle \varphi, u_f(\cdot, t) \rangle = (L^* \varphi, u_f(\cdot, t)). \]
This confirms that \( u_f(x,t) \) is a solution to (4.14) as a distribution. The statements on \( p(x,y,t) \) follow from similar arguments. \( \square \)
In particular, we find explicitly defined approximations to $p$ on $G$. Then, when $\lim_{x \to \infty} S(x) = \infty$, in which case (4.2) implies that $p_G = 0$; when $\lim_{x \to \infty} S(x) < \infty$, $\infty$ is attracting and $p_G > 0$.

4.2. Approximation of $p(x, y, t)$. In the previous sections, for the fundamental solutions that do not have explicit formulas, we provide approximations that are accessible and of high accuracy, at least for small time. These approximations can be useful in computational applications of degenerate diffusion equations studied in this work. Below we will present an approximation for $p(x, y, t)$ in the same spirit. In particular, we find explicitly defined approximations to $p(x, y, t)$ such that (i) these approximations are more accurate than the standard heat kernel estimates, and (ii) when $t$ is sufficiently small, these approximations are “close” to $p(x, y, t)$ uniformly in $(x, y)$ in any compact set. Note that this result is a generalization of [8] for that the error estimates we derive here only depend on the local bounds of $a(x)$ and $b(x)$.

**Theorem 4.5.** Let $p^{approx.}(x, y, t)$ and $p^{k-approx.}(x, y, t)$, $k \in \mathbb{N} \setminus \{0\}$, be defined as in (3.4) and (3.22) respectively. For any $G > 0$, set $t_G := \frac{4\phi(G)}{9(1-\nu)}$. Then, for every $t \in (0, t_G)$, $I > G$ and $k \in \mathbb{N} \setminus \{0\}$,

$$
\sup_{(x, y) \in (0, \psi(\phi(G)))^2} \left| \frac{p(x, y, t) - p^{k-approx.}(x, y, t)}{p^{approx.}(x, y, t)} \right| 
\leq m_k(t) M(t) + D_k(t) + \frac{\Theta_k^2 M(t)(\phi(G))^{1-\nu}}{(1-\nu)(S(I) - S(G))} \exp \left( -\frac{2\phi(G)}{9t} \right),
$$

(4.16)

where $m_k(t)$, $M(t)$ and $D_k(t)$ are as in (3.3) and (3.22) respectively.

In particular, there exists constant $C > 0$ uniformly in $t \in (0, t_G)$ and $k \in \mathbb{N} \setminus \{0\}$ ($C$ may depend on $G$ and $\nu$) such that

$$
\sup_{(x, y) \in (0, \psi(\phi(G)))^2} \left| \frac{p(x, y, t) - p^{k-approx.}(x, y, t)}{p^{approx.}(x, y, t)} \right| \leq C t^k b,
$$

where $b$ is the constant defined in (3.3).

**Proof.** Only (4.16) requires proof. For every $(x, y, t) \in (0, G)^2 \times (0, \infty)$, we have that

$$
\left| \frac{p(x, y, t) - p^{k-approx.}(x, y, t)}{p^{approx.}(x, y, t)} \right| 
\leq \left| \frac{p(x, y, t) - p_I(x, y, t)}{p^{approx.}(x, y, t)} \right| + \left| \frac{p_I(x, y, t) - p^{k-approx.}(x, y, t)}{p^{approx.}(x, y, t)} \right|.
$$

By (3.32), we have that for every $t \in (0, t_G)$, the second term on the right hand side above is bounded uniformly in $(x, y) \in \left(0, \psi\left(\phi(G)\right)\right)^2$ by

$$
m_k(t) M(t) + D_k(t) \exp \left( -\frac{2\phi(G)}{9t} \right).
$$

We define hitting times $\{\eta_n(x) : n \in \mathbb{N}\}$ as in (4.7) (for the downward crossings from $I$ to $G$). Then, according to (4.11),

$$
p(x, y, t) - p_I(x, y, t) = \sum_{n=1}^{\infty} E \left[ p_I(G, y, t - \eta_{2n}(x)) : \eta_{2n}(x) \leq t, \eta_{2n}(x) < \zeta_0(X) \right].
$$
It follows from (4.13), (3.7), (4.24) and (4.10) that for every $t \in (0, t_G)$ and $(x, y) \in \left(0, \psi \left(\frac{\phi(G)}{9}\right)\right)^2$, 
\[
|p(x, y, t) - p_t(x, y, t)| 
\leq M(t) \left( \sup_{s \in (0,t)} s^{\nu - 2} e^{-4 \phi(G) - \phi(y)} (\phi(G))^{1 - \nu} \frac{\Theta(\phi(G))}{\Theta(\phi(y))} \phi'(y) \exp \left( - \frac{4 \phi(G)}{9t} \right) \frac{S(x)}{S(I) - S(G)} \right),
\]
and further by (3.30) and (4.5) we have that
\[
|\frac{p(x, y, t) - p_t(x, y, t)}{p^{\text{approx.}}(x, y, t)}| \leq M(t) \left( \Theta(\phi(G)) \phi(x) \left( \frac{\phi(G)}{\phi(x)} \right)^{1 - \nu} \exp \left( - \frac{2 \phi(G)}{9t} \right) \frac{S(x)}{S(I) - S(G)} \right) 
\leq \frac{\Theta_2 M(t) (\phi(G))^{1 - \nu}}{(1 - \nu)(S(I) - S(G))} \exp \left( - \frac{2 \phi(G)}{9t} \right).
\]

We close this section with two variations of (4.16). First, by (4.5), we note that
\[
\frac{1}{S(I) - S(G)} = \frac{1}{S(G)/S(I)} \leq \frac{\Theta_2}{(\phi(G))^{1 - \nu}} \frac{S(G)/S(I)}{1 - S(G)/S(I)}.
\]
Therefore, by sending $I$ to $\infty$ in (4.16), we get the following estimate.

**Corollary 4.6.** For every $G > 0$, let $t_G > 0$ be the same as in Theorem 4.3, and $p_G$ be defined as in (4.15). Then, for every $t \in (0, t_G)$,
\[
\sup_{(x, y) \in \left(0, \psi \left(\frac{\phi(G)}{9}\right)\right)^2} \left| \frac{p(x, y, t) - p^{k-\text{approx.}}(x, y, t)}{p^{\text{approx.}}(x, y, t)} \right| \leq m_k(t) M(t) + \left(D_k(t) + \Theta_2 M(t) \frac{p_G}{1 - p_G} \right) \exp \left( - \frac{2 \phi(G)}{9t} \right).
\]

Second, by making $t_G$ in Theorem 4.5 smaller if necessary, we can derive an estimate analogous to (4.10) but independent of $p_G$. Intuitively speaking, when $t$ is sufficiently small, how well $p^{\text{approx.}}(x, y, t)$ approximates $p(x, y, t)$ should not depend on the probability of the process escaping to infinity. To make it rigorous, we first observe that (H2) guarantees the existence of $t'_G > 0$ such that
\[
(4.17) \quad I - G - t'_G b_I > 2 \sqrt{I_G^{\nu_1} \alpha_I} \quad \text{for every } I > 2G;
\]
then, by using (4.14) instead of (4.10) in the proof of (4.10), we get that for every $t \in (0, t'_G)$ and $(x, y) \in \left(0, \psi \left(\frac{\phi(G)}{9}\right)\right)^2$, $|p(x, y, t) - p_t(x, y, t)|$ is bounded from above by
\[
M(t) t^{\nu - 2} (\phi(G))^{1 - \nu} \frac{\Theta(\phi(G))}{\Theta(\phi(y))} \phi'(y) \exp \left( - \frac{4 \phi(G)}{9t} \right) \frac{S(x)}{S(G)} \frac{4 \nu_1 \alpha_I}{S(G)(I - G - b_I)^{2}}.
\]
It follows that for every $(x, y, t) \in \left(0, \psi \left(\frac{\phi(G)}{9}\right)\right)^2 \times (0, t'_G)$,
\[
|\frac{p(x, y, t) - p_t(x, y, t)}{p^{\text{approx.}}(x, y, t)}| \leq M(t) \frac{\Theta(\phi(G))}{\Theta(\phi(x))} \left( \frac{\phi(G)}{\phi(x)} \right)^{1 - \nu} \frac{S(x)}{S(G)} \exp \left( - \frac{2 \phi(G)}{9t} \right),
\]
\[
\leq \Theta_2 M(t) \exp \left( - \frac{2 \phi(G)}{9t} \right).
\]
Therefore, we have the following estimate on the error between $p^{k-\text{approx.}}(x,y,t)$ and $p(x,y,t)$, which is a potential improvement of (4.16) for small $t$.

**Corollary 4.7.** For every $G > 0$, let $t'_G > 0$ be such that (4.17) holds. Then, for every $t \in (0, t'_G)$,

$$
\sup_{(x,y) \in (0,\phi(\frac{1}{G}))} \left| \frac{p(x,y,t) - p^{k-\text{approx.}}(x,y,t)}{p^{\text{approx.}}(x,y,t)} \right| \\
\leq m_k(t) M(t) + (D_k(t) + \Theta G M(t)) \exp\left(\frac{2\phi(G)}{9t}\right).
$$

5. **Generalized Wright-Fisher Diffusion**

As reviewed in §1.1, the classical Wright-Fisher diffusion equation given by (1.1) has two degenerate boundaries at 0 and 1, and the localization method was adopted in [7] so that one only needs to focus on one boundary at a time. Although in our setting only degenerate diffusions with one-sided boundary are concerned, the framework developed in the previous sections can also be applied to degenerate diffusions with two-sided boundaries. In this section we discuss a variation of the Wright-Fisher diffusion where the diffusion operator has general order of degeneracy at both boundaries 0 and 1.

For two constants $\alpha, \beta \in (0, 2)$, we consider the following Cauchy problem with two-sided boundaries on $(0,1)$, where, given $f \in C_b((0,1))$, we look for $u_f(x,t) \in C^2((0,1) \times (0,\infty))$ such that

$$
\partial_t u_f(x,t) = x^\alpha (1-x)^\beta \partial_x^2 u_f(x,t) \quad \text{for every } (x,t) \in (0,1) \times (0,\infty),
$$

(5.1)

$$
\lim_{x \to 0} u_f(x,t) = f(x) \quad \text{for every } x \in (0,1),
$$

$$
\lim_{x \to 1} u_f(x,t) = \lim_{x \to 1} u_f(x,t) = 0 \quad \text{for every } t \in (0,\infty).
$$

Set $L_{\alpha,\beta} := x^\alpha (1-x)^\beta \partial_x^2$. We want to apply the method developed in the previous sections to construct and study the fundamental solution $p(x,y,t)$ to (5.1). $L_{\alpha,\beta}$ has two degenerate boundaries 0 and 1 with (possibly distinct) general order of degeneracy, and both boundaries are attainable according to the boundary classification mentioned in Remark 2.3.

Although having a second degenerate boundary at 1, $L_{\alpha,\beta}$ has the advantage that its coefficient $x^\alpha (1-x)^\beta$ is bounded on $(0,1)$. Therefore, for every $x \in (0,1)$, the stochastic differential equation

$$
\text{d}X(x,t) = \sqrt{2X^\alpha(x,t)(1-X(x,t))}\partial_t X(x,t) \, dB(t) \quad \text{with } X(x,0) = x
$$

always has a solution in the sense described in §1.3 (see, e.g., of [32]). Although we are not yet ready to claim the uniqueness of this solution, we can follow the theory in §12 of [32] to extract a solution to (5.2) that has strong Markov property. In other words, (5.2) always has a solution $\{X(x,t) : t \geq 0\}$ that is a strong Markov process.

The existence of a strong Markovian solution to (5.2) enables us to follow the steps in §2–§4 to tackle (5.1). In particular, with the localization procedure, we have the option of placing our “focal point” in the neighborhood of either 0 or 1 while constructing $p(x,y,t)$. We will see that these two views are consistent and will lead to the same $p(x,y,t)$.

Let us start with the construction of $p(x,y,t)$ with a focus only on the left boundary 0, and we will follow the steps in the previous sections with $a(x) = (1-x)^{\beta}$ and $b(x) \equiv 0$. Here we only state the results of each step but leave the computational details in the Appendix (i.e., (6.1)–(6.6)). We add a superscript $^u(L)^n$ to relevant quantities and functions to indicate that only the left boundary 0 is “effective” in this construction.

We take $I \in (0,1)$ and localize (5.1) onto $(0,I)$. All the functions involved in the transformation are as follows:

$$
\phi^{(L)}(x) = \frac{1}{4} \left(b^{(L)}(x)\right)^2 \quad \text{and} \quad \theta^{(L)}(x) = \frac{\alpha}{\beta - \alpha} - \frac{\alpha - \alpha x - \beta x}{2(\beta - \alpha)} \left(1-x\right)^{\frac{\beta-2}{2}} b^{(L)}(x),
$$

\(\theta^{(L)}(x)\)
where \( b^{(L)}(x) := \int_0^x s^{-\alpha/2} (1-s)^{-\beta/2} \, ds \) is the incomplete beta function; furthermore,
\[
\Theta \left( \phi^{(L)}(x) \right) = \frac{x^\alpha (1-x)^\beta}{(b^{(L)}(x))^{\frac{4-\alpha}{2}}} \quad \text{with} \quad \Theta^{(L)}_y = \left( \frac{2}{2-\alpha} \right)^{\frac{\alpha}{2-\alpha}} (1-I)^{-\frac{\beta}{2(2-\alpha)}};
\]
in addition, for every \( x \in (0, I) \),
\[
V \left( \phi^{(L)}(x) \right) = -\frac{\alpha (\alpha - 4)}{4(2-\alpha)^2 (b^{(L)}(x))^2} + x^{\alpha - 2} (1-x)^{\beta - 2} \left( \frac{(\alpha - \alpha x - \beta x)^2}{16} - \frac{\alpha (1-x)^2 + \beta x^2}{4} \right),
\]
and hence
\[
\left| V \left( \phi^{(L)}(x) \right) \right| \leq V^{(L)}_y \left( \phi^{(L)}(x) \right)^{-\frac{\alpha}{2-\alpha}} \quad \text{with} \quad V^{(L)}_y = \frac{\beta}{16} (4 - \beta + 2\alpha) (1-I)^{\frac{\beta}{2-\alpha} - 2}.
\]

This confirms that the statement in Lemma \( \text{[2.2]} \) still holds in this case.

Next, for the model equation discussed in §2.2, we plug in \( \nu^{(L)} := \frac{1-\alpha}{2-\alpha} \) and obtain \( q^{(L)}(z, w, t) \) as in \( \text{[2.12]} \) and \( q^{(L)}_{\partial^{(L)}(1)}(z, w, t) \) as in \( \text{[2.18]} \) accordingly. We then follow exactly the same steps as in §3.1 to derive \( q^{(L)}_{\partial^{(L)}(1)}(z, w, t) \) based on \( q^{(L)}_{\partial^{(L)}(1)}(z, w, t) \), and to obtain \( p^{(L)}_y(x, y, t) \) through reversing the transformation \( z = \phi^{(L)}(x) \), i.e.,
\[
p^{(L)}_y(x, y, t) = q^{(L)}_{\partial^{(L)}(1)} \left( \phi^{(L)}(x), \phi^{(L)}(y), t \right) \frac{x^\alpha (1-x)^\beta}{2y^\alpha (1-y)^\beta} \left( \frac{b^{(L)}(y)}{b^{(L)}(x)} \right)^{\frac{4-\alpha}{2-\alpha}}.
\]

To proceed, we follow the arguments in §4 to obtain the fundamental solution to \( 5.1 \) as
\[
op(x, y, t) = \lim_{t \to 1} p^{(L)}_y(x, y, t) \quad \text{for} \quad (x, y, t) \in (0, 1)^2 \times (0, \infty).
\]

By \( \text{[3.2]} \), \( \{ X(x, t) : t \geq 0 \} \) itself is a martingale, and as in Lemma \( \text{[4.1]} \) we can derive probability estimates for the hitting times of \( X(x, t) \) as
\[
\mathbb{P} \left( \zeta^X_y(x) < \zeta^X_0(x) \right) = \frac{x}{y} \quad \text{and} \quad \mathbb{P} \left( \zeta^X_1(x) \leq t \right) \leq \exp \left( -\frac{(I - x)^2}{4M_{\alpha, \beta} t} \right)
\]
for every \( 0 < x < y < I \) and \( t > 0 \), where
\[
M_{\alpha, \beta} := \max_{x \in [0, 1]} x^\alpha (1-x)^\beta = \frac{\alpha^\alpha \beta^\beta}{(\alpha + \beta)^{\alpha + \beta}}.
\]

For every \( 0 < G < I < H < 1 \), if \( \{ \eta_n(x) : n \in \mathbb{N} \} \) is the sequence of hitting times as in \( \text{[4.7]} \) (for the downward crossings of \( X(x, t) \) from \( I \) to \( G \)), then for every \( (x, y, t) \in (0, G)^2 \times (0, \infty) \),
\[
\mathbb{P} \left( \zeta^X_y(x) < \zeta^X_{\eta_{2n}}(x) \right) \leq \mathbb{P} \left( \zeta^X_{\eta_{2n}}(x) \leq t, \eta_{2n}(x) < \zeta^X_{\eta_{2n}}(x) \right) \leq \exp \left( -\frac{(I - x)^2}{4M_{\alpha, \beta} t} \right)
\]
where the series on the right hand side is absolutely convergent.

Let us rewrite the results in Theorem \( \text{[4.3]} \) for \( p(x, y, t) \) found above.

**Proposition 5.1.** \( p(x, y, t) \) is smooth on \((0, 1)^2 \times (0, \infty)\), and for every \((x, y, t) \in (0, 1)^2 \times (0, \infty)\),
\[
y^\alpha (1-y)^\beta p(x, y, t) = x^\alpha (1-x)^\beta p(y, x, t).
\]
For every \( y \in (0, 1) \), \( (x, t) \mapsto p(x, y, t) \) is a smooth solution to the Kolmogorov backward equation corresponding to \( L_{\alpha, \beta} \), i.e.,
\[
\partial_t p(x, y, t) = x^\alpha (1-x)^\beta \partial_x^2 p(x, y, t); 
\]
for every \( x \in (0, 1) \), \( (y, t) \mapsto p(x, y, t) \) is a smooth solution to the Kolmogorov forward equation corresponding to \( L_{\alpha, \beta} \), i.e.,
\[
\partial_t p(x, y, t) = \partial_y^2 \left( y^\alpha (1-y)^\beta p(x, y, t) \right).
\]
\( p(x, y, t) \) is the fundamental solution to (5.1). Given \( f \in C_c((0,1)) \),

\[
(5.6) \quad u_f(x,t) := \int_0^\infty f(y) p(x,y,t) \, dy \text{ for } (x,t) \in (0,1) \times (0,\infty)
\]
is a smooth solution to (5.1). Moreover, for every \( (x,t) \in (0,1) \times (0,\infty) \),

\[
(5.7) \quad u_f(x,t) = \mathbb{E} [ f(X(x,t)) ; t < \zeta^X_{0,1}(x) ],
\]
and hence for every Borel set \( \Gamma \subseteq (0,1) \),

\[
(5.8) \quad \int_\Gamma p(x,y,t) \, dy = \mathbb{P} (X(x,t) \in \Gamma, t < \zeta^X_{0,1}(x))
\]

Finally, \( p(x,y,t) \) satisfies the Chapman-Kolmogorov equation, i.e., for every \( x,y \in (0,1) \) and \( t,s > 0 \),

\[
(5.9) \quad p(x,y,t+s) = \int_0^\infty p(x,\xi,t) p(\xi,y,s) \, d\xi.
\]

**Proof.** The only thing that requires proof is the smoothness of \( p(x,y,t) \) on \( (0,1)^2 \times (0,\infty) \). By Theorem 4.3, we know that \( (y,t) \mapsto p(x,y,t) \) is smooth, and at the same time \( (x,t) \mapsto p(x,y,t) \) solves the equation \( (\partial_t - L_{\alpha,\beta}) p(x,y,t) = 0 \). It is easy to see from here that \( p(x,y,t) \) has all the partial derivatives in \( (x,y,t) \) of all orders. \( \square \)

The proposition above also leads to the wellposedness of the stochastic differential equation associated with \( L_{\alpha,\beta} \).

**Corollary 5.2.** The stochastic differential equation (5.2) is well posed for every \( x \in (0,1) \) up to the hitting time at either 0 or 1 in the sense that if \( \{ \tilde{X}(x,t) ; t \geq 0 \} \) is another solution to (5.2), then the distribution of \( X(x,t) \) conditioning on \( t < \zeta^X_{0,1}(x) \) is identical with that of \( \tilde{X}(x,t) \) given \( t < \zeta^X_{0,1}(x) \).

**Proof.** It is sufficient to observe that, for every \( f \in C_c((0,1)) \), if \( u_f(x,t) \) is defined as in (5.6), then \( \{ u_f(\tilde{X}(s),t-s) ; s \in [0,t] \} \) is a martingale, which, by (5.7), implies that

\[
\mathbb{E} [ f(X(x,t)) ; t < \zeta^X_{0,1}(x) ] = u_f(x,t) = \mathbb{E} [ f(\tilde{X}(x,t)) ; t < \zeta^X_{0,1}(x) ].
\]

\( \square \)

Next we briefly discuss the other way of constructing \( p(x,y,t) \), which is to start with the localization of (5.1) in a neighborhood of the right boundary 1. It is easy to see that, by exchanging \( x \) and \( 1-x \), and at the same time exchanging \( \alpha \) and \( \beta \), we can follow the same steps as above to develop another construction of the fundamental solution to (5.1). We will not repeat the details but only specify quantities and functions that are necessary for the statement of the results. For example, in this case the transformation is given by

\[
z = \phi^{(R)}(x) = \frac{1}{4} \left( b^{(R)}(x) \right)^2 \text{ where } b^{(R)}(x) := \int_x^1 s^{-\alpha/2} (1-s)^{-\beta/2} \, ds;
\]

\( q^{(R)}(z,w,t) \) is the fundamental solution to the model equation with \( \nu^{(R)} = \frac{1-\beta}{2-\beta} \), and given \( I \in (0,1) \),

\( q^{(R)}_{(I)}(z,w,t) \) is the fundamental solution to the localization of the model equation on \( (0,\phi^{(R)}(I)) \); furthermore, we have that

\[
\Theta^{(R)}(\phi^{(R)}(x)) = \frac{x^{2/3}(1-x)^{2/3}}{(b^{(R)}(x))^{2/3}} \text{ with } \Theta^{(R)}(x) = \left( \frac{2}{2-\beta} \right)^{2/\nu^{(R)}} I^{-\nu^{(R)}};
\]
and for every \( x \in (I,1) \),

\[
\left| V^{(R)}(\phi^{(R)}(x)) \right| \leq \frac{4}{16} \left( \frac{\alpha}{\nu^{(R)}} \right)^{2/\nu^{(R)}} \text{ with } V^{(R)}(x) = \frac{\alpha}{16} (4-\alpha+2\beta) I^{\nu^{(R)}},
\]

where
we can develop approximations for \( q_{\mu{}^{(R)}(I)} (z, w, t) \) from \( q_{\mu{}^{(R)}(I)} (z, w, t) \) via Duhamel's method, and obtain the fundamental solution to (5.1) localized on \((I, 1)\) as the definition of 

\[
\hat{p}^{(R)} (x, y, t) = q_{\mu{}^{(R)}(I)} (\phi^{(R)} (x), \phi^{(R)} (y), t) \frac{x^\alpha (1 - x)^\beta}{2y^\alpha (1 - y)^\beta \left( b^{(R)} (y) \right)^{\frac{1 - \beta}{\gamma_\alpha}}}.
\]

finally, if \( \tilde{\eta}_n (x) : n \in \mathbb{N} \) is the sequence of hitting times that records the upward crossings of \( X (x, t) \) from \( I \) to \( H \). Then, (5.3) and the strong Markov property of \( \{X(x, t) : t \geq 0\} \) are sufficient for us to obtain another version of the fundamental solution, denoted by \( \hat{p} (x, y, t) \) temporarily, as

\[
\hat{p} (x, y, t) = \lim_{G \searrow 0} \hat{p}^{(R)} (x, y, t)
\]

(5.10)

for \( (x, y, t) \in (H, 1)^2 \times (0, \infty) \). It is easy to see that \( \hat{p} (x, y, t) \) also satisfies (5.3), (5.8) and (5.9), which implies that \( \hat{p} (x, y, t) = p (x, y, t) \) almost everywhere on \((0, 1)^2 \times (0, \infty)\), i.e., the two constructions of the fundamental solution to (5.1) are consistent and \( p (x, y, t) \) satisfies both (5.9) and (5.10).

Depending on near which boundary we are conducting our analysis, we can choose either (5.4) or (5.10) as the definition of \( p (x, y, t) \). For example, when both \( x \) and \( y \) are close to one of the boundaries, we can develop approximations for \( p (x, y, t) \) similarly as in §4.2.

Corollary 5.3. For \((x, y, t) \in (0, 1)^2 \times (0, \infty)\), set

\[
p^{(L) - \text{approx}} (x, y, t) := q^{(L)} (\phi^{(L)} (x), \phi^{(L)} (y), t) \frac{x^\alpha (1 - x)^\beta}{2y^\alpha (1 - y)^\beta \left( b^{(L)} (y) \right)^{\frac{1 - \beta}{\gamma_\alpha}}}
\]

and

\[
p^{(R) - \text{approx}} (x, y, t) = q^{(R)} (\phi^{(R)} (x), \phi^{(R)} (y), t) \frac{x^\alpha (1 - x)^\beta}{2y^\alpha (1 - y)^\beta \left( b^{(R)} (y) \right)^{\frac{1 - \beta}{\gamma_\alpha}}}.
\]

Let \( M^{(L)} (t) \) and, respectively, \( M^{(R)} (t) \) be defined as in (5.4) with \( \nu = \nu^{(L)} \), \( V_I = V_I^{(L)} \) and, respectively, \( \nu = \nu^{(R)} \), \( V_I = V_I^{(R)} \). Fix \( 0 < G < I < H < 1 \), and set

\[
t^{(L)} := \left( \frac{4(2 - \alpha)}{9(3 - \alpha)} \phi^{(L)} (G) \right) \wedge \left( \frac{(I - G)^2}{4M_{\alpha, \beta}} \right) \text{ and } t^{(R)} := \left( \frac{4(2 - \beta)}{9(3 - \beta)} \phi^{(R)} (H) \right) \wedge \left( \frac{(H - I)^2}{4M_{\alpha, \beta}} \right).
\]

Then, for every \( t \in (0, t^{(L)}) \) and \( (x, y) \in (0, G)^2 \) such that \( \phi^{(L)} (x) \vee \phi^{(L)} (y) \leq \frac{1}{4} \phi^{(L)} (G) \),

\[
\left| \frac{p (x, y, t)}{p^{(L) - \text{approx}} (x, y, t)} - 1 \right| \leq M^{(L)} (t) - 1
\]

\[
+ \left[ 1 + \left( \frac{2}{2 - \alpha} \right)^\frac{2\alpha}{\gamma_\alpha} \frac{M^{(L)} (t)}{(1 - G)^{\frac{1 - \alpha}{\gamma_\alpha}}} \left( \frac{G}{1 - G} \wedge 1 \right) \right] \exp \left( - \frac{2\phi^{(L)} (G)}{9t} \right).
\]

Similarly, for every \( t \in (0, t^{(R)}) \) and \( (x, y) \in (H, 1)^2 \) such that \( \phi^{(R)} (x) \vee \phi^{(R)} (y) \leq \frac{1}{4} \phi^{(R)} (H) \),

\[
\left| \frac{p (x, y, t)}{p^{(R) - \text{approx}} (x, y, t)} - 1 \right| \leq M^{(R)} (t) - 1
\]

\[
+ \left[ 1 + \left( \frac{2}{2 - \beta} \right)^\frac{2\beta}{\gamma_\beta} \frac{M^{(R)} (t)}{(1 - H)^{\frac{1 - \beta}{\gamma_\beta}}} \left( \frac{1 - H}{1 - H} \wedge 1 \right) \right] \exp \left( - \frac{2\phi^{(R)} (H)}{9t} \right).\]

Proof. We only need to look at the statement involving \( p^{(L) - \text{approx}} (x, y, t) \). There is not much to be done since a similar estimate (4.10) has been proven in Theorem 4.3. We notice that \( t^{(L)} \) is chosen such that the function \( s \mapsto s^{\nu - 2} \exp \left( - \frac{2\phi^{(L)} (G)}{9s} \right) \) is increasing on \((0, t^{(L)})\), and \((I - G)^2 \geq 4M_{\alpha, \beta} t^{(L)}\).
Furthermore, in this case \( S(x) = x \) for every \( x \in (0, 1) \), and hence \( p_G = G \). Combining the proof of Corollary 4.6 and Corollary 4.7, as well as (6.4) and (6.5) in the Appendix, we get that for every \((x, y, t) \in (0, G)^2 \times (0, \ell(L))\) as described in the statement,

\[
\left| \frac{p(x, y, t) - p^{(L)}_t(x, y, t)}{p^{(L)}_t(x, y, t)} \right| \leq M(L) \frac{\Theta \left( \phi^{(L)}(G) \right)}{\Theta \left( \phi^{(L)}(g) \right)} \left( \frac{\phi^{(L)}(G)}{\phi^{(L)}(g)} \right)^{\frac{1}{2} - \alpha} \frac{x}{G} \exp \left( - \frac{2\phi^{(L)}(G)}{9t} \right) \frac{G}{1 - G} \leq \left( \frac{2}{2 - \alpha} \right)^{\frac{1}{2} - \alpha} (1 - G)^{-\frac{2}{2 - \alpha}} M(L) \frac{\Theta \left( \phi^{(L)}(G) \right)}{\Theta \left( \phi^{(L)}(g) \right)} \left( \frac{\phi^{(L)}(G)}{\phi^{(L)}(g)} \right)^{\frac{1}{2} - \alpha} \frac{x}{G} \exp \left( - \frac{2\phi^{(L)}(G)}{9t} \right) \frac{G}{1 - G} \land \right) .
\]

□

6. Appendix

This Appendix contains detailed derivations involving \( \Theta(z) \) and \( V(z) \) for \( z \in (0, J) \). Assuming that \( J = \phi(I) \), it is sufficient for us to look at \( \Theta(\phi(x)) \) and \( V(\phi(x)) \) for \( x \in (0, I) \), where the notations become simpler. Recall that \( a_I := \max_{x \in [0, I]} \left\{ \frac{1}{a(x)}, a(x) \right\} \) and \( b_I := \max_{x \in [0, I]} |b(x)| \).

We also introduce two more notations:

\( a_I' := \max_{x \in [0, I]} |a'(x)| \) and \( b_I' := \max_{x \in [0, I]} |b'(x)| \).

According to (122) and (222), we have that for every \( x \in (0, I) \),

\[
\Theta(\phi(x)) = \exp \left( - \int_0^x \frac{\theta(w)}{2\phi(w)} \phi'(w) dw \right) = \exp \left( - \int_0^x \left( \frac{\frac{1}{2} - \nu}{2\sqrt{\phi(w)} w^\alpha a(w)} - \frac{2w^\alpha a(w)}{4w^\alpha a(w)} + \frac{b(w)}{2w^\alpha a(w)} \right) dw \right).
\]

Notice that

\[
\frac{\frac{1}{2} - \nu}{2\sqrt{\phi(w)} w^\alpha a(w)} = \frac{(w^\alpha a(w))'}{4w^\alpha a(w)} = \left( \ln \left( \frac{2\sqrt{\phi(w)}}{(w^\alpha a(w))^{\frac{1}{2}}} \right) \right)',
\]

and further, if \( \alpha = 1 \), then

\[
\frac{b(w)}{2w^\alpha a(w)} = \left( \ln \left( \frac{w^{0.5}}{2w^\alpha a(w)} \right) \right)' + \frac{1}{2w} \left( \frac{b(w)}{a(w)} - b(0) \right).
\]

Plugging these two expressions back into the right hand side of \( \Theta(\phi(x)) \) leads to

\[
\Theta(\phi(x)) = \begin{cases} 
  x^{\frac{1}{4}(\frac{4\phi(x)}{w^\alpha a(w)} - \frac{b(w)}{2w^\alpha a(w)})} \exp \left( - \int_0^x \frac{b(w)}{2w^\alpha a(w)} dw \right) & \text{if } \alpha \neq 1, \\
  \left( \frac{4\phi(x)}{w^\alpha a(w)} - \frac{b(w)}{2w^\alpha a(w)} \right)^{\frac{1}{4}} \exp \left( - \int_0^x \frac{1}{2w} \left( \frac{b(w)}{a(w)} - b(0) \right) dw \right) & \text{if } \alpha = 1,
\end{cases}
\]

which is exactly (240). Given (H1) and (H2), the integral in the exponential function above is well defined in both cases (when \( \alpha = 1 \) and \( \alpha \neq 1 \)).

With the notations introduced above, we have that when \( \alpha \neq 1 \), for every \( x \in (0, I) \),

\[
\left( 1 - \frac{\alpha}{2} \right)^{\frac{\alpha}{\alpha - 1}} \frac{a_I}{a_I} - \frac{1}{x^{\frac{1}{\alpha - 1}}} \leq x^{\frac{1}{4}(\frac{4\phi(x)}{w^\alpha a(w)} - \frac{b(w)}{2w^\alpha a(w)})} \leq \left( 1 - \frac{\alpha}{2} \right)^{\frac{\alpha}{\alpha - 1}} \frac{a_I}{a_I} - \frac{1}{x^{\frac{1}{\alpha - 1}}},
\]

and

\[
\exp \left( \int_0^x \frac{\left| b(w) \right|}{2w^\alpha a(w)} dw \right) \leq \mathbb{I}_{(0,1)}(\alpha) \cdot e^{\frac{\alpha}{2(1 - \alpha)}} + \mathbb{I}_{(1,2)}(\alpha) \cdot e^{\frac{\alpha}{2(1 - \alpha)}};
\]

33
when \( \alpha = 1 \), for every \( x \in (0, I) \),
\[
2^{b(0)-\frac{1}{2}a_I}\frac{b(0)}{4\phi(x)} \leq \left( \frac{x}{a_I} \right)^{\frac{1}{4}} \frac{1}{4\phi(x)} (a(x)) \leq 2^{b(0)-\frac{1}{2}a_I}\frac{b(0)}{4\phi(x)}
\]
and
\[
\exp \left( \int_0^x \frac{1}{2w} \left| \frac{b(w)}{a(w)} - b(0) \right| dw \right) \leq e^{\frac{1}{2}a_I^2(a_I b_I + a_I b_I) t}.
\]
Hence, if we set
\[
A_I := \mathbb{I}_{(0,1)}(\alpha) \cdot e^{2b(1-a)} + \mathbb{I}_{(1)}(\alpha) \cdot e^{\frac{1}{2}a_I^2(a_I b_I + a_I b_I) t} + \mathbb{I}_{(1,2)}(\alpha) \cdot e^{\frac{1}{2}a_I^2(2-\alpha)}.
\]
then for every \( x \in (0, I) \),
\[
\left( 1 - \alpha \frac{a_I^2}{2} \right)^{\frac{a_I^2}{a_I^2} - \frac{1}{a_I^2} A_I} \leq \Theta(\phi(x)) \leq \left( 1 - \alpha \frac{a_I^2}{2} \right)^{\frac{a_I^2}{a_I^2} - \frac{1}{a_I^2} A_I}.
\]
(6.1) follows from here by setting
(6.1)
\[
\Theta_I := \left( 1 - \alpha \frac{a_I^2}{2} \right)^{\frac{a_I^2}{a_I^2} - \frac{1}{a_I^2} A_I} A_I.
\]
For every \( x, y \in (0, I) \), we can follow the arguments above to get that
\[
a_I^{\frac{a_I^2}{a_I^2} - \frac{1}{a_I^2} A_I} \leq \frac{\Theta(\phi(x))}{\Theta(\phi(y))} \leq a_I^{\frac{a_I^2}{a_I^2} - \frac{1}{a_I^2} A_I}.
\]
Moreover, if \( S(x) \) is as defined in (4.1), then
\[
S(x) = 2^{2\alpha-1} \int_0^x \exp \left( - \int_0^u \frac{b(w)}{w^\alpha a(w)} dw \right) du.
\]
In addition, from (3.3a), we can easily derive that, for every \( x, y \in (0, I) \),
\[
\frac{\Theta(\phi(x))}{\Theta(\phi(y))} \cdot \phi'(y) = \left( \frac{\phi(y)}{\phi(x)} \right)^{\frac{a_I^2}{a_I^2} - \frac{1}{a_I^2}} \cdot \frac{\phi'(y)}{y^{\frac{a_I^2}{a_I^2}} a_I^{\frac{a_I^2}{a_I^2}}(y)} \exp \left( - \int_y^x \frac{b(w)}{w^\alpha a(w)} dw \right),
\]
and
\[
\frac{(\phi(x))^{1-\nu}}{\phi'(x)} \Theta^2(\phi(x)) = \frac{(\phi(x))^{\frac{1}{2} - \nu} x^\alpha a(x)}{(4\phi(x))^{\frac{a_I^2}{a_I^2}} - \frac{1}{a_I^2}} \exp \left( - \int_0^x \frac{b(w)}{w^\alpha a(w)} dw \right)
\]
\[
= 2^{2\alpha-1} x^\alpha a(x) \exp \left( - \int_0^x \frac{b(w)}{w^\alpha a(w)} dw \right) .
\]
Now we move onto \( V(z) \) and recall from (3.3a) that for every \( x \in (0, I) \),
\[
V(\phi(x)) = \frac{\theta(x)}{4\phi(x)} (-\theta(x) + 2 - 2\nu) - \frac{\theta'(x)}{2\phi'(x)} \quad \text{for every } x \in (0, I).
\]
By (1.2), the choice of \( \nu \) and (H1) and (H2), it is straightforward to verify that when \( x \in (0, I) \),
\[
\phi(x) = \frac{x^{2-\alpha}}{(2-\alpha)^2} (1 + O(x)) \quad \text{and} \quad \theta(x) = \frac{b(0) x^{1-\alpha}}{2 - \alpha} \mathbb{I}_{(0,1)}(\alpha) + O(x^{2-\alpha}),
\]
which implies that
\[
\frac{\theta(x)}{\phi(x)} = \frac{(2 - \alpha) b(0)}{x} \mathbb{I}_{(0,1)}(\alpha) + O(1).
\]
In addition, we also have that
\[
\frac{\theta'(x)}{\phi'(x)} = b'(x) - \frac{(x^\alpha a(x))''}{2} + \frac{2b(x) - (x^\alpha a(x))'}{2} \left( \frac{1}{2\sqrt{\phi(x)x^\alpha a(x)}} - \frac{(x^\alpha a(x))'}{2x^\alpha a(x)} \right)
\]
\[
= b'(x) + \frac{b(x)}{2\sqrt{\phi(x)x^\alpha a(x)}} - \frac{b(x)(x^\alpha a(x))'}{2x^\alpha a(x)} - \frac{(x^\alpha a(x))''}{2} - \frac{(x^\alpha a(x))'}{4\sqrt{\phi(x)x^\alpha a(x)}} + \left( \frac{(x^\alpha a(x))'}{4x^\alpha a(x)} \right)^2.
\]
We notice that
\[
- \frac{(x^\alpha a(x))''}{2} - \frac{(x^\alpha a(x))'}{4\sqrt{\phi(x)x^\alpha a(x)}} + \left( \frac{(x^\alpha a(x))'}{4x^\alpha a(x)} \right)^2 = \mathcal{O}(x^{\alpha-1}),
\]
and
\[
\frac{b(x)}{2\sqrt{\phi(x)x^\alpha a(x)}} - \frac{b(x)(x^\alpha a(x))'}{2x^\alpha a(x)} = \mathcal{I}(0,1) (a) \left( \frac{b(0)}{x} + \mathcal{O}(x^{\alpha-1}) \right) + \mathcal{O}(1).
\]
Putting all the above together yields that when \( \alpha \in (0,1) \),
\[
V(\phi(x)) = \frac{(1-\nu)(2-\alpha)b(0)}{2x} - \frac{(1-\alpha)b(0)}{2x} + \mathcal{O}(x^{\alpha-1}) = \frac{ab(0)}{2x} + \mathcal{O}(x^{\alpha-1});
\]
when \( \alpha \in [1,2) \), \( V(\phi(x)) \) is bounded for \( x \in (0,1) \). Thus, we have proven all the claims in Lemma \( \text{2.2} \).

Next, we look at the case when \( b(x) \equiv 0 \), where most of the expressions above take simpler forms. For example,
\[
\Theta(\phi(x)) = \frac{x^\alpha a^\frac{1}{4}(x)}{2^{3/4}a^\frac{1}{4}(x)}, \quad V(\phi(x)) = -\frac{\alpha(\alpha-4)}{16(2-\alpha)^2} \phi(x) + \frac{(x^\alpha a(x))''}{4} - \frac{3((x^\alpha a(x))')^2}{16},
\]
\[
\frac{\Theta(\phi(x))}{\Theta(\phi(y))} \phi'(y) = \left( \frac{\phi(y)}{\phi(x)} \right)^{\frac{\nu}{1-\nu}} \frac{\phi^\frac{1}{4}(y)x^\frac{\alpha}{4}a^\frac{1}{4}(x)}{y^\frac{\alpha}{4}a^\frac{1}{4}(y)} + \frac{\phi(x)^{1-\nu}}{\phi'(x)} \Theta^2(\phi(x)) = 2^{-\frac{\alpha}{1-\nu}} x^\alpha a(x).
\]
In particular, if \( a(x) = (1-x)^\beta \) as in §5, then
\[
(6.4) \quad \Theta(\phi(x)) = \frac{x^\alpha (1-x)^{\frac{\beta}{2}}}{2^{\frac{3}{4}\alpha-\beta}} \left( \int_0^x \frac{ds}{\sqrt{s^\alpha (1-s)^\beta}} \right)^{\frac{1-\alpha}{1-\beta}} \Theta_I = \frac{2^{-\frac{\alpha}{1-\beta}}}{(1-I)^{\frac{1}{2}\alpha-\beta}}.
\]
Furthermore,
\[
V(\phi(x)) = \frac{\alpha(\alpha-4)}{4(2-\alpha)^2} \left( \int_0^x \frac{ds}{\sqrt{s^\alpha (1-s)^\beta}} \right)^{-2} + \frac{\alpha(\alpha-4)}{16} x^{\alpha-2}(1-x)^\beta
\]
\[
- \frac{\alpha\beta}{8} x^\alpha (1-x)^{\beta-1} + \frac{\beta(\beta-4)}{16} x^\alpha (1-x)^{\beta-2}.
\]
Since
\[
(6.5) \quad \frac{x^{2-\alpha}}{(2-\alpha)^2} \leq \phi(x) \leq (1-x)^{-\beta} \frac{x^{2-\alpha}}{(2-\alpha)^2},
\]
we see that
\[
V(\phi(x)) \geq x^{\alpha-2}(1-x)^{\beta-2} \left( -\frac{\alpha\beta x(1-x)}{8} + \frac{\beta(\beta-4)}{16} x^2 \right)
\]
and

\[ V(\phi(x)) \leq \frac{\alpha(4-\alpha)}{16}x^{\alpha-2}\left[1-(1-x)^\beta\right] + x^{\alpha-2}(1-x)^{\beta-2}\left(-\frac{\alpha\beta x(1-x)}{8} + \frac{\beta(\beta-4)}{16}x^2\right) \leq x^{\alpha-1}(1-x)^{\beta-2}\left(\frac{\alpha(4-\alpha)\beta}{16} - \frac{\alpha\beta}{8}(1-x) + \frac{\beta(\beta-4)}{16}x^2\right), \]

where in the last line we used the fact that for every \(x \in (0,1)\),

\[ 1 - (1-x)^\beta \leq \beta x(1-x)^{\beta-2}. \]

Combining the upper bound and the lower bound of \(V(\phi(x))\) leads to

\[ |V(\phi(x))| \leq x^{\alpha-1}(1-x)^{\beta-2}\frac{\beta}{16}(4-\beta+2\alpha) \quad \text{for every } x \in (0,1), \]

which, by (6.5), implies that when \(\alpha \in (0,1)\),

\[ |V(\phi(x))| \leq \frac{\beta}{16}(4-\beta+2\alpha)(2-\alpha)^{\frac{\beta}{4}}(1-I)^{\frac{\beta}{4}}(\phi(x))^{\frac{\beta}{4}} \quad \text{for every } x \in (0,1). \]

Therefore, with this specific case of \(a(x) = (1-x)^\beta\), we see that the constant \(V_I\) as introduced Lemma 2.2 (identified with \(V_I\) in for \(J = \phi(I)\)) can be taken as

\[ V_I = \frac{\beta}{16}(4-\beta+2\alpha)(1-I)^{\frac{\beta}{4}}. \]

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