Cosmic Microwave Background Anisotropy numerical solution (CMBAns) I: An introduction to $C_l$ calculation

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Abstract. Cosmological Boltzmann codes are often used by researchers for calculating the CMB angular power spectra from different theoretical models, for cosmological parameter estimation, etc. Therefore, the accuracy of a Boltzmann code is of utmost importance. Different Markov Chain Monte Carlo based parameter estimation algorithms typically require $10^3 - 10^4$ iterations of Boltzmann code. This makes the time complexity of such codes another critical factor. In the last two decades, several Boltzmann packages, such as CMBFAST, CAMB, CMBEasy, CLASS etc., have been developed. In this paper, we present a new cosmological Boltzmann code, CMBAns, that can be used for accurate calculation of the CMB power spectrum. At present, CMBAns is developed for a flat background matrix. It is mostly written in the C language. However, we borrowed the concept of class from C++. This gives researchers the flexibility to develop their own independent package based on CMBAns, without an in-depth understanding of the source code. We also develop multiple stand-alone facilities which can be directly compiled and run on a given parameter set. In this paper, we discuss all the mathematical formulation, approximation schemes, integration methods etc., that are used in CMBAns. The package will be made available through github for public use in the near future.
# 1 Introduction

Since the discovery of the Cosmic Microwave Background (CMB) by Penzias and Wilson, the CMB has become an invaluable probe for understanding the physics in the early universe. Several cosmological theories, proposed in the past, failed to explain the origin of CMB. Hence, they were rejected as feasible cosmological theories. Others like Big Bang cosmology, with some assumptions, provide a more complete explanation of the origin of CMB radiation. These models later, after several
theoretical modifications, were accepted as the standard cosmological models. Hence, the discovery of CMB marked the path for the birth of standard cosmology.

The precision of the CMB observation has improved over the years. In the past decade, several ground-based and space-based experiments like WMAP, Planck, BICEP, ACT etc. have measured the CMB temperature to an exquisite precision. Future experiments like SPT-3G and Simons Array will provide even better measurements of CMB temperature and polarization (Abazajian et al., 2016). To analyze this influx of data and to test different cosmological models, we also need more accurate Boltzmann codes.

The theory of cosmological perturbations for standard model cosmology was first developed by Lifshitz (Lifshitz, 1946) and later was reviewed by many others (Lifshitz and Khalatnikov, 1963). The subsequent research works are summarized in review articles (Kodama and Sasaki, 1984; Mukhanov et al., 1992), in books and in theses (Tassev, 2011; Weinberg, 2008). Lifshitz used the synchronous gauge for formulating the linear perturbation theory. Later, Bardeen and others developed the perturbation theory in the conformal gauge due to some complications with the synchronous gauge, such as the appearance of the coordinate singularity (Bardeen, 1980; Kodama and Sasaki, 1984) etc. The conformal gauge is more frequently used for analytical calculations of the cosmological perturbation equations. However, the synchronous gauges are preferred for the numerical calculations due to the stability issues (G. Montani, 2011; Hu, 2004).

The Boltzmann codes have been in use in cosmology for a long time to calculate the CMB angular power spectrum. The first of such code provided in the public domain was COSMICS (Ma and Bertschinger, 1995), written by Ma and Bertschinger. Later, Seljak and Zaldarriaga developed CMBFAST (Matias Zaldarriaga and Hui, 2001; Seljak and Zaldarriaga, 1996), in which the line-of-sight integration method was used to make the power spectrum calculation faster. Since then, several packages utilizing Boltzmann codes, such as CAMB (Antony Lewis), CMBeasy (Doran, 2005), CLASS (Lesgourgues, 2011), etc., have come into existence. In this paper, we describe a new Boltzmann code, called CMBAns (Cosmic Microwave Background Anisotropy numerical solution). The package is based on CMBFAST and was initially developed in 2010 for a variety of CMB studies (Das and Souradeep, 2014a; Das et al., 2013; Das Santanu, 2010).

There are three principal motivations behind developing the CMBAns. First of all, in future CMB missions, the precision of the CMB measurements will improve drastically. Hence, the Boltzmann packages should be able to calculate the CMB power spectrum very accurately up to high multipoles. Secondly, different Markov Chain Monte Carlo (MCMC) packages, such as CosmoMC (Lewis, 2013; Lewis and Bridle, 2002), SCoPE (Das and Souradeep, 2014b), AnalyzeThis (Doran and Mueller, 2004), etc., which are often used to estimate the cosmological parameters, typically require $10^3 - 10^4$ evaluations of Boltzmann codes. Therefore, the Boltzmann code should be able to calculate the CMB power spectrum fast and efficiently. Thirdly, most of the present Boltzmann codes follow a monolithic architecture design and are not modular. Therefore, it is difficult to add any new feature in the package and the functions cannot be used independently. Users cannot write their own packages and use existing functions without an extensive knowledge of the entire source code. To overcome this limitation, CMBAns consists of stand-alone codes, as well as some user-defined functions that users can use to write their codes.

CMBAns solves the linear Boltzmann equations for different constituents of the universe and thereafter uses the line-of-sight integration approach to calculate the source terms and the brightness fluctuations. These are then convolved with the primordial power spectrum to get the CMB angular power spectrum. Apart from the standard model power spectrum, CMBAns can calculate the cosmological power spectrum for different dark energy models (both perturbed and unperturbed), two-field inflation model, etc. CMBAns also comes with a MATLAB GUI, where the Hubble parameter of the universe can be visually modified as a function of redshift. CMBAns translates the modified Hubble parameter into the dark energy equation of state (EOS) and then computes the CMB power spectrum for that particular model (Das and Souradeep, 2014a).

In this paper, we describe all the mathematical equations and the discretization techniques that are used in CMBAns. We also briefly describe the cosmological perturbation theory in the synchronous gauge and discuss the equations and approximation schemes used in developing CMBAns. The paper
is divided into eight sections. In the second section, we describe the conformal time calculations between any two eras in the universe. In the third section, we describe different recombination processes, calculating the baryon temperature, sound speed etc. The fourth section discusses CMB perturbation calculations and different approximation schemes used in CMAAns, and how they affect the power spectrum calculations. Different scalar and tensor initial conditions are discussed in the fifth section. The sixth section is for line-of-sight integration and calculating the power spectra. The numerical techniques, time and wave number space grid etc. are discussed in the seventh section. The final section is for discussion and conclusion.

2 Conformal time calculation

In cosmology, the redshift $z$ is often used for measuring time. However, for numerical calculation of the perturbation equations, line-of-sight integration, etc. the conformal time plays an important role. It is straightforward to calculate the conformal time under the assumption of a matter-dominated or dark energy-dominated universe. However, in the presence of all the components of the universe, the calculations can be complicated and an analytical solution may not exist.

In this paper, we denote the conformal time as $\tau$. The Hubble parameter $H(\tau)$ is defined as

$$
H^2(\tau) = \left(\frac{1}{a^2} \frac{da}{d\tau}\right)^2 ,
$$

where $a$ is the scale factor. From the FLRW equation, we can write the Hubble parameter as

$$
\frac{H(\tau)^2}{H_0^2} = \Omega_{0,m} a^{-3} + \Omega_{0,\gamma} a^{-4} + \Omega_{0,\nu} a^{-4} + \Omega_{\nu_m} + \Omega_d .
$$

The above two equations give

$$
\frac{da}{d\tau} = \sqrt{a^4 H_0^2 \left( \Omega_{0,m} a^{-3} + \Omega_{0,\gamma} a^{-4} + \Omega_{0,\nu} a^{-4} + \Omega_{\nu_m} + \Omega_d \right)} ,
$$

where $\Omega_{0,m}, \Omega_{0,\gamma},$ and $\Omega_{0,\nu}$ are the density parameters for present-day matter (which include both cold dark matter and baryonic matter), photon, and massless neutrinos respectively. The density parameter of massive neutrinos and dark energy at a scale factor $a$ are $\Omega_{\nu_m}$ and $\Omega_d$, respectively. The density parameters are defined as the ratios of the respective densities over the critical density:

$$
\Omega_{0,m} = \frac{\rho_{0,m}}{\rho_{cr}}, \quad \Omega_{0,\gamma} = \frac{\rho_{0,\gamma}}{\rho_{cr}}, \quad \Omega_{0,\nu} = \frac{\rho_{0,\nu}}{\rho_{cr}}, \quad \Omega_{\nu_m} = \frac{\rho_{\nu_m}}{\rho_{cr}}, \quad \Omega_d = \frac{\rho_d}{\rho_{cr}}
$$

where, the critical density $\rho_{cr}$ is given by $\rho_{cr} = \frac{3H_0^2}{8\pi G}$. The densities of matter and radiation at any era are scaled as $a^{-3}, a^{-4}$, respectively with their densities at the present era. For calculating the density of the massive neutrinos, we need to use the Fermi-Dirac statistics. For $\Lambda$ dark energy model, the density of the dark energy will be constant. However, for any other dark energy model, we need to calculate the density variation from its equation of state (eos).

2.1 Matter density

The first term in Eq. 2.3 can be calculated by evaluating $aH_0^2\Omega_{0,m}$. As the CDM and baryon density parameters, $\Omega_{0,c}$ and $\Omega_{0,b}$ are the input parameters, we can calculate $\Omega_{0,m} = \Omega_{0,c} + \Omega_{0,b}$. $H_0$ is also an input parameter, but its unit is km/sec/Mpc. In CMAAns, we use Mpc as the unit for both the spatial and temporal dimensions. In order to convert the Hubble parameter in Mpc$^{-1}$, we multiply $H_0$ with $1/c^2 = 1.11265 \times 10^{-11}(\text{km/sec})^{-2}$. 

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2.2 Radiation density for photons

The radiation density consists of two components: photon density \( \rho_\gamma \) and the massless (relativistic) neutrino density \( \rho_\nu \). The photon number density as a function of frequency can be derived from the Planck radiation law:

\[
n_\gamma(\nu) \, d\nu = \frac{8 \pi \nu^2 \, d\nu}{e^{\nu/k_B T_0} - 1},
\]

(2.4)

where \( k_B \) is the Boltzmann constant, \( h \) is the Planck constant, and \( T_0 \) is the current CMB temperature. The photon energy density can be calculated as

\[
\rho_{0,\gamma} c^2 = \int_0^\infty h n_\gamma(\nu) \, d\nu = a_B T_0^4,
\]

(2.5)

where \( a_B = \frac{8 \pi^5 k_B^4}{15 h^3 c^3} = 7.56577 \times 10^{-16} \text{ J m}^{-3} \text{K}^{-4} \) is the radiation constant. We also know that

\[
\rho_{cr} = \frac{3 H_0^2}{8 \pi G} = 1.87847 \times 10^{-30} H_0^2 \text{ kg m}^{-3} \text{(km/sec/Mpc)}^{-2}.
\]

(2.6)

Therefore, the second term in Eq. 2.3 can be calculated by evaluating \( H_0^2 \Omega_{0,\gamma} \) as follows

\[
\Omega_{0,\gamma} H_0^2 = \frac{\rho_{0,\gamma} H_0^2}{\rho_{cr}} = \frac{a_B}{c^2 \rho_{cr}} T_0^4 = 4.98613 \times 10^{-14} \times T_0^4 \text{ Mpc}^{-2}.
\]

(2.7)

2.3 Radiation density for massless neutrinos

Massless neutrinos follow Fermi-Dirac statistics with neutrino temperature \( T_\nu \). The distribution function is given by

\[
n_\nu(\nu) \, d\nu = \frac{8 \pi \nu^2 \, d\nu}{e^{\nu/k_B T_\nu} + 1}.
\]

(2.8)

We can calculate the radiation density of the massless neutrinos as

\[
\rho_{0,\nu} c^2 = \int_0^\infty h n_\nu(\nu) \, d\nu = \left( \frac{7}{8} \right) a_B T_\nu^4.
\]

(2.9)

For relating the temperatures between photon and neutrinos, consider the era before neutrino and photon decoupling. In that ultra high energy regime, as photon and neutrino were coupled, the medium in which they existed had a fixed temperature. Other species in the medium were electrons (2 spin states), positrons (2 spin states), neutrinos (1 spin state for each of the three generations), and antineutrinos (1 spin state for each of the three generations). Shortly after the photon and neutrino decoupling, the temperature drops below the electron mass, and the forward reaction \( e^+ + e^- \leftrightarrow \gamma + \gamma \) (annihilation) becomes strongly favored. This heats up the photons. We can assume that this entropy transfer did not affect the neutrinos because they were already completely decoupled. Using entropy conservation of the electromagnetic plasma, we can calculate the change in the photon temperature before and after \( e^\pm \) annihilation. This gives (Dodelson, 2003)

\[
\frac{T_\nu}{T_0} = \left( \frac{4}{11} \right)^{1/3}.
\]

The neutrino density is related to the photon density by

\[
\rho_{0,\nu} = N_{\text{eff}} \left( \frac{7}{8} \right) \left( \frac{4}{11} \right)^{4/3} \rho_{0,\gamma}.
\]
where $N_{\text{eff}}$ is the effective number of neutrinos. Theoretically, there are 3 neutrino families. However, due to non-instantaneous decoupling and QED effects etc., the effective neutrino density will be slightly higher than this value. This can be accounted for by considering $N_{\text{eff}} > 3$. Considering a general framework for neutrino decoupling, it can be shown that for non-instantaneous neutrino decoupling, $N_{\text{eff}} \approx 3.034$. In addition, the QED effects contribute about $\Delta N_{\text{eff}} \approx 0.011$. Assuming these two effects can be added linearly, the final value of $N_{\text{eff}} \approx 3.045$ (de Salas and Pastor, 2016; Dolgov et al., 1997, 1998; Escudero, 2018; Grohs et al., 2015; Mangano et al., 2001).

Therefore, the third term in Eq. 2.3 can be calculated as

$$\Omega_{0,\nu} h_0^2 = \frac{\rho_{0,\nu}}{\rho_{cr}} H_0^2 = N_{\text{eff}} \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} \frac{a_B}{e^2 \rho_{cr}} T_0^4 = 1.1324 \times N_{\text{eff}} \times 10^{-14} \times T_0^4 \text{Mpc}^{-2} \quad (2.10)$$

### 2.4 Radiation density for massive neutrinos

In the standard model of particle physics, the neutrinos are massless. However, different experiments point toward a small nonzero mass for the neutrinos. For massive neutrinos, the Fermi-Dirac distribution function contains the mass term, and it is not analytically integrable. Therefore, to get the density $\rho_{\nu_m}$ at any given redshift, the distribution function must be integrated numerically.

Assuming that all the neutrino species have equal mass, the mass of the neutrinos is given by

$$m_{\nu_m} = \frac{\rho_{0,\nu_m}}{N_{\text{eff}} \rho_{0,\nu_m}} = \frac{\rho_{0,\nu_m}}{N_{\text{eff}} n_{0,\nu_m}} (2.11)$$

where $N_{\text{eff}}$ is the effective number of neutrinos. $\rho_{0,\nu_m}$ and $n_{0,\nu_m}$ are the massive neutrino density and number density at the present time respectively. $\rho_{cr}$ is the critical density.

The neutrino number density can be calculated by integrating the Fermi-Dirac distribution function:

$$n_{\nu_m} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(p\sqrt{2}\epsilon^2 + m^2 c^4/k_B T_{\nu_m}) + 1}. \quad (2.12)$$

For neutrinos $pc \gg mc^2$, and we can ignore the term $mc^2$ in the above equation. This simplifies to

$$n_{\nu_m} = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp(pc/k_B T_{\nu_m}) + 1} = \frac{8\pi}{h^3} \frac{k_B^3 T_{\nu_m}^3}{c^4} \int_0^\infty \frac{\xi^2 d\xi}{\epsilon^2 + 1} = \frac{8\pi c^3}{h^3} k_B^3 T_{\nu_m}^3 \zeta(3) \Gamma(3), \quad (2.13)$$

where $\zeta(3)$ is the Riemann Zeta function and $\Gamma(3)$ is the Gamma function. $\Gamma(3) = 2! = 2$.

The density and pressure of massive neutrinos at any given redshift can be written as

$$\rho = \frac{8\pi}{h^3} k_B^4 T_{\nu_m}^4 \int_0^\infty q^2 f(q) \epsilon(q) dq, \quad (2.14)$$

$$P = \frac{8\pi}{h^3} k_B^4 T_{\nu_m}^4 \int_0^\infty q^2 f(q) \frac{q^2}{3\epsilon} dq, \quad (2.15)$$

where $q = apc$ and,

$$\epsilon = \frac{a}{k_B T_{\nu_m}} \sqrt{m_{\nu_m}^2 c^4 + (pc)^2}. \quad (2.16)$$

Here, in Eq. 2.15 the factor of 3 comes because we consider 3 spatial dimensions.$^1$ Simple rearrangements of the above equations give us the massive neutrino density and pressure in terms of massless neutrino density, as

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$^1$For an ideal gas, the pressure can be found by $nmv^2/3$. $n$ is the number density of the gas molecule, $v$ is the velocity, and $m$ is the mass of each gas molecules. The factor of 3 arises because we have considered 3 spatial dimensions and we consider that the velocity distribution of the gas is isotropic, i.e. $v_x^2 = v_y^2 = v_z^2 = v^2/3$. Eq. 2.15 can also be derived in a similar way, where $q$ corresponds to the momentum.
Figure 1. Dimensionless neutrino density $\rho_{DL}$ and $P_{DL}$, given by Eq. 2.19 and Eq. 2.20, for different massive neutrino density parameters. The massless neutrinos are shown in dark blue curve for a reference.

Figure 2. Plot of $\frac{da}{dt}$ for different $\Omega_c$ (left) and $H_0$ (right). The radiation dominated era, matter dominated era and the dark energy dominated era are clearly shown in the left plot. As shown in Eq. 2.3, $\frac{da}{dt}$ is constant in the radiation dominated era (orange), varies as $a^2$ in the matter dominated era (blue) and varies as $a^3$ in the dark energy dominated era (gray).

$$\rho = \left(\frac{7}{8}\right) a_B T^4_{\nu} \rho_{dl} = \left(\frac{7}{8}\right) a_B T^4_{\nu,0} a^{-4} \rho_{dl} = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} a_B T^4_{0} a^{-4} \rho_{dl}, \quad (2.17)$$

$$P = \left(\frac{7}{8}\right) a_B T^4_{\nu} p_{dl} = \left(\frac{7}{8}\right) a_B T^4_{\nu,0} a^{-4} P_{dl} = \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} a_B T^4_{0} a^{-4} P_{dl}. \quad (2.18)$$

Here $\rho_{dl}$ and $P_{dl}$ are dimensionless density and pressure and are expressed as

$$\rho_{dl} = \frac{1}{Y} \int_0^\infty q^2 f(q) \epsilon(q) \, dq, \quad (2.19)$$

$$P_{dl} = \frac{1}{Y} \int_0^\infty q^2 f(q) \frac{\tilde{q}^2}{3 \epsilon} \, dq. \quad (2.20)$$

where $Y = \frac{7}{8} a^{4/3}$. In Fig. 1, we plot the dimensionless density and pressure for massive neutrinos for different density parameters, $\Omega_{\nu m}$ (note that $\sum m_{\nu m}/97.04 eV = \Omega_{\nu m} h^2$, where $h$ is the Hubble parameter in units of 100 km/s/Mpc). In the early universe, where the temperature is high, $p c \ll mc^2$, the neutrinos behave like massless particles and $\rho_{dl} \rightarrow 1$ and $P_{dl} \rightarrow \frac{3}{2}$. However, later, where $mc^2$ dominates, the massive neutrinos start behaving like matter particles and $P_{dl} \rightarrow 0$ and $\rho_{dl} \propto a$, i.e. the actual density of the massive neutrinos goes as $a^{-3}$.

\footnote{Note that for calculating $Y$ we need the Bose-Einstein integration formula, $\int_0^\infty \frac{q^3 dq}{e^{q/T} + 1} = \frac{a^4}{4\pi}$.}
2.5 Contribution from dark energy

The last term in Eq. 2.3 is the contribution from the dark energy. We can use the approximation $\Omega_{0, d} \approx 1 - \Omega_{0, m}$ (since $\Omega_{0, \gamma}, \Omega_{0, c}, \Omega_{\nu_m}$ are of the order of $10^{-5}$). For a $\Lambda$CDM model, the equation of state for dark energy is $w_d = -1$. However, for other fluid approximations, the equation of state for dark energy may vary as a function of scale factor, i.e. $w_d(a)$. In such cases we can write the generalized form of $\Omega_d$ as

$$\Omega_d = \Omega_{0, d} \exp \left( -3 \int_1^a \frac{da}{a} [1 - w(a)] \right).$$

(2.21)

CMBAns is capable of handling both the constant $w_d$ or a varying equation of state, $w_{DE}(a)$, models of dark energy.

Fig. 2 shows the variation of $\frac{dn}{d\tau}$ as a function of scale factor for different values of $\Omega_c$ and $H_0$. The conformal time between two given redshifts can be calculated by numerically integrating Eq. 2.3.

3 Recombination and Reionization

For calculating the baryon sound speed, optical depth, and visibility function, we need to calculate the recombination and the reionization process very accurately. CMBAns provides functions for calculating the recombination using the Saha equation, Peebles equation, or recfast method.

3.1 Saha Equation

Saha equation provides a very rough estimate of the recombination epoch. It assumes the recombination reaction $p + e^- \leftrightarrow H + \gamma$ is fast enough that it proceeds near thermal equilibrium, i.e. it ignores the expansion of the universe. According to the Saha equation,

$$n_H \frac{x_e^2}{1 - x_e} = \left( \frac{k_B m_e T_b}{2\pi \hbar^2} \right)^{3/2} e^{-B_1/k_B T_b},$$

(3.1)
where \( x_e \) is the hydrogen ionization fraction. \( n_H \) is the number density of the hydrogen atoms, i.e. \( n_H = n_{1s} + n_p \), where \( n_{1s} \) and \( n_p \) are number density of neutral hydrogen and ionized hydrogen, respectively. \( B_1 = m_e e^4/(2\hbar^2) = 13.6 \text{ eV} \) is the ionization potential of the hydrogen atom. \( T_b \) is the baryon temperature.

The hydrogen number density can be calculated as

\[
\frac{n_H}{n_b} = \frac{m_p}{m_H} (1 - Y_{\text{He}}) = \frac{3}{8\pi G} \Omega_{0, b} a_0^3 H_0^2 (1 - Y_{\text{He}}) / m_H
\]

where \( Y_{\text{He}} \) is the helium fraction after the Big Bang nucleosynthesis. In Fig. 3 we show the recombination result using the Saha equation. The plot shows that the recombination of the Hydrogen is almost instantaneous. For the helium recombination, we separately use the Saha equation given by Eq. 3.9.

### 3.2 Peebles’ Recombination

Peebles’ equation provides a very accurate estimate of the recombination history of hydrogen. The calculations are done using effective three-level atom calculations. Peebles’ formalism is based on the assumptions that

- Direct recombinations to the ground state of hydrogen are very inefficient: each such event leads to a photon with energy greater than 13.6 eV, which almost immediately re-ionizes a neighboring hydrogen atom. Electrons therefore only efficiently recombine to the excited states of hydrogen, from which they cascade very quickly down to the first excited state, with principal quantum number \( n = 2 \).

- From the first excited state, electrons can reach the ground state \( n = 1 \) through two pathways:
  1. Decay from the 2p state by emitting a Lyman-\( \alpha \) photon. This photon will almost always be reabsorbed by another hydrogen atom in its ground state. However, cosmological red-shifting systematically decreases the photon frequency, and hence there is a small chance that it escapes reabsorption if it gets redshifted far enough from the Lyman-\( \alpha \) line resonant frequency before encountering another hydrogen atom.
  2. Decay from the 2s to 1s state, which is only possible using an electron double transition. The rate of this transition is very slow, \( 8.22 \text{ s}^{-1} \). It is however competitive with the slow rate of Lyman-\( \alpha \) escape in producing ground-state hydrogen.

- Atoms in the first excited state may also be re-ionized by the ambient CMB photons before they reach the ground state, as if the recombination to the excited state did not happen in the first place. To account for this possibility, Peebles defines the factor \( C \) as the probability that an atom in the first excited state reaches the ground state through either of the two pathways described above before being photo-ionized.

Accounting for these processes, the recombination history is then described by the differential effect (Peebles, 1968)

\[
\frac{dx_e}{dt} = -aC \left( \alpha^{(2)}(T_b) n_p x_e - 4(1 - x_e) \beta(T_b) e^{-E_{21}/T} \right)
\]

where

\[
\beta(T_b) = \left( \frac{m_e k_B T_b}{2\pi\hbar^2} \right)^{3/2} e^{-B_1/k_B T_b} \alpha^{(2)}(T_b)
\]

The recombination rate to excited states (Ma and Bertschinger, 1995) is taken as

\[
\alpha^{(2)}(T_b) = \frac{64\pi}{(27\pi)^{1/2} m_e^2 c^3} \left( \frac{k_B T_b}{B_1} \right)^{-1/2} \phi_2(T_b), \quad \phi_2(T_b) \approx 0.448 \ln \left( \frac{B_1}{k_B T_b} \right).
\]

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This expression for \( \phi_2(T_b) \) provides a good approximation at low temperature. At high temperature this expression underestimates \( \phi_2 \), but the amount is negligible. For \( T_b > B_0/k_B = 1.58 \times 10^5 \)K, we set \( \phi_2 = 0 \).

\[
C = \frac{\Lambda_\alpha + \Lambda_{2s-1s}}{\Lambda_\alpha + \Lambda_{2s-1s} + \beta^{(2)}(T_b)}
\]

where

\[
\beta^{(2)}(T_b) = \beta(T_b) e^{+\hbar c/\lambda_\alpha k_BT_b}, \quad \Lambda_\alpha = \frac{8\pi \dot{a}}{a^2 \lambda_\alpha^3 n_{1s}}.
\]

\( \lambda_\alpha = \frac{8\pi \hbar c}{3k_BT_b} = 1.21567 \times 10^{-7} \)m, is the wavelength for Lyman-\( \alpha \) emission. Over-dot represents the derivative with respect to the conformal time. \( \Lambda_{2s-1s} \) is the rate of hydrogen double transition from \( 2s \) to \( 1s \). \( \Lambda_{2s-1s} = 8.227 s^{-1} = 8.4678 \times 10^4 \)Mpc\(^{-1}\).

\[
\Lambda_{2s-1s}/\Lambda_\alpha = \frac{\Lambda_{2s-1s} \lambda_\alpha^3 a^2 n_{1s}}{8\pi \dot{a}} = \frac{\Lambda_{2s-1s} \lambda_\alpha^3 (1 - x_c) a^3 n_H}{8\pi \dot{a} a} = (1 - x_c)(1 - Y_{He}) \frac{\Lambda_{2s-1s} \lambda_\alpha^3 a^3 \rho_m}{8\pi \dot{a} a m_H} = \frac{(1 - x_c)}{\dot{a} a} (1 - Y_{He}) \frac{\Omega_m H_0^2}{\Omega_{m0} H_0^2}
\]

Similarly, \( \beta^{(2)}(T_b)/\Lambda_\alpha \) can be calculated using

\[
\frac{\beta^{(2)}(T_b)}{\Lambda_\alpha} = T_b \phi_2(T_b) \mathcal{K} e^{-0.25 T_{ion}/T_b} (8.0230194 \times 10^{-26}) \frac{(1 - x_c)}{\dot{a} a} (1 - Y_{He}) \frac{\Omega_m H_0^2}{\Omega_{m0} H_0^2}
\]

where \( \mathcal{K} = \left( \frac{64\pi}{(2\pi)^{3/2}} \frac{\varepsilon^8}{m_e^4 c^4} \left( \frac{k_B}{m_H} \right)^{3/2} \right)^{1/2} = 5.13 \times 10^{18} \). Here, \( H_0 \) is in \( \text{km/sec/Mpc} \) unit, and \( \dot{a} \) has units of \( \text{Mpc}^{-1} \) unit. The numerical values are converted to match these units.

### 3.2.1 Helium Recombination

For calculating the He recombination, we use the Saha Equation (Ma and Bertschinger, 1995).

\[
\frac{n_e x_{n+1}}{x_n} = 2 \frac{g_{n+1}}{g_n} \left( \frac{m_e k_B T_b}{2\pi \hbar^2} \right)^{3/2} e^{-\chi_n/k_BT_b},
\]

where \( n \in (0, 1) \), and \( x_0 = 1 - x_1 - x_2 \). The helium ionization fractions \( x_1 = n(\text{He}^+) / n(\text{He}) \) and \( x_2 = n(\text{He}^{++}) / n(\text{He}) \), where \( n(\text{He}) \) is the total number density of helium nuclei. \( n_e \) is the free electron number density. \( g_0 = g_1 = 1 \) and \( g_2 = 2 \). \( \frac{n_e}{k_B} = T_1^{\text{ion}} = 2.855 \times 10^5 \)K and \( \frac{n_e}{k_B} = T_2^{\text{ion}} = 6.313 \times 10^5 \)K are the first and second ionization temperature of He.

### 3.3 Recfast, CosmoRec

Peebles’ three-level atom model accounts for the most important physical processes. However, these approximations may lead to errors on the predicted recombination history at a level as high as 10%. This can also alter the temperature and polarization power spectra up to 3 – 5% at high multipoles. Several research groups have revisited the details and proposed different models like recfast\(^3\) (S. Seager and Scott, 1999; Seager et al., 1999), CosmoRec\(^4\) (Ali-HaÅrmond and Hirata, 2010a; Chluba and Thomas, 2010; Chluba et al., 2010; Grin and Hirata, 2010; RubiÅs-MartÃ­n)

\(^3\)https://www.cfa.harvard.edu/~sasselov/rec/
\(^4\)http://www.jb.man.ac.uk/~jchluba/Science/CosmoRec>Welcome.html
Figure 4. Comparison between the ionization fractions from different modern recombination routine \textsc{recfast}, \textsc{recfast++} and \textsc{CosmoRec}. For \textsc{CosmoRec}, we choose the dark matter annihilation efficiency to be $10^{-24}$ eV/sec and all the other parameters are set to default settings. Left: Ionization fraction is plotted with a linear scale to show the He$^+$ recombination. Right: Ionization fraction is plotted with a log scale to amplify the effect at low redshift after the H$^+$ recombination.

et al., 2010; Switzer and Hirata, 2008), HyRec$^5$(Ali-HaÅ‘moud and Hirata, 2010b) etc. These packages can calculate the recombination history up to 0.1% accuracy. We use the available \textsc{recfast} code in \textsc{CMBAns}. The other packages can also be easily added in the \textsc{CMBAns} or run separately. In the later case the ionization fraction, and baryon temperature can be stored in a file as a function of scale factor and use in \textsc{CMBAns}.

In Fig. 3, we show the ionization fraction from different recombination methods. We use a smooth reionization, where we join an ionization fraction before and after the reionization using a \text{tanh}(...)$ function. In Fig. 4, we show the differences between \textsc{recfast}, \textsc{recfast++} and \textsc{CosmoRec} recombination. This small change in the ionization fraction can change the $C_l$ at high multipoles.

### 3.4 Calculating baryon temperature

For calculating ionization fraction during the recombination, we need the baryon temperature at each scale factor. The rate of change of the baryon temperature can be calculated as (check Appendix A)

$$
\dot{T}_b = -2 \left( \frac{\dot{a}}{a} \right) T_b + \frac{8\pi^2}{45} \frac{k_B}{c^4 h^3} \frac{\sigma_T T_\gamma^4}{m_e} f_e (T_\gamma - T_b),
$$

where $\sigma_T$ is the Thomson scattering cross section. $f_e$ is given by

$$
f_e = \frac{(1 - Y_{He}) x_e^{\text{tot}}}{1 - \frac{3}{4} Y_{He} + (1 - Y_{He}) x_e^{\text{tot}}},
$$

$x_e^{\text{tot}}$ is the total ionization fraction and is given by

$$
x_e^{\text{tot}} = x_e + \frac{1}{4} Y_{He} \frac{x_1 + 2 x_2}{(1.0 - Y_{He})}.
$$

The constant term in Eq. 3.10 is given by

$$
\frac{8\pi^2}{45} \frac{k_B}{c^4 h^3} \frac{\sigma_T}{m_e} = 5.0515 \times 10^{-8} \text{K}^{-4} \text{Mpc}^{-1}.
$$

We can see that the baryon temperature depends on the ionization fraction of the electrons. Therefore, we need to jointly evaluate the baryon temperature and ionization fraction. The temperature of the photons at any era is $T_\gamma = a^{-1} T_{0\gamma}$. We can consider $T_b = T_\gamma$ before recombination (in the tight coupling era), and we can use it as the initial condition for solving Eq. 3.10.

\footnote{\url{https://cosmo.nyu.edu/yacine/hyrec/hyrec.html}}
3.5 Baryon sound speed, optical depth and visibility

Calculating the baryon acoustic oscillations require the speed of sound in the plasma, $c_s$. If we consider the plasma as a single fluid, then the pressure, density and the temperature of the fluid will be related as $P_b = \frac{k_B}{m} \rho_b T_b$. We can calculate the sound speed in the plasma as

$$c_s^2 = \left. \frac{dP_b}{d\rho_b} \right|_{adiabatic} = \frac{k_B T_b}{m} \left( 1 - \frac{1}{3} \frac{d\ln T_b}{d\ln a} \right)$$

$$= \frac{k_B T_b}{m_p} \left[ 1.0 - \frac{3}{4} Y_{He} + (1.0 - Y_{He}) x_e^{tot} \right] \left( 1 - \frac{1}{3} \frac{d\ln T_b}{d\ln a} \right).$$ (3.13)

Here $m$ is the mean molecular weight of the fluid, and $m_p$ is the mass of a proton $^6$. The mean molecular weight is calculated assuming the fluid contains free electrons, H, H+, He, He+, He++. The optical depth from the present time ($\tau_0$) to any conformal time $\tau$ is given by

$$\kappa = \int_{\tau}^{\tau_0} an_e \sigma_T d\tau = \int_{\tau}^{\tau_0} \left( \frac{H_0^2 c^2}{8\pi G} \right) \left( \frac{\Omega_b}{m_H a^2} \right) \sigma_T (1 - Y_{He}) d\tau.$$ (3.14)

The visibility function at any conformal time $\tau$ can be calculated as $g = \kappa \exp(-\kappa)$. In Fig. 5 we show the visibility function vs the scale factor. The visibility function is nonzero only during the recombination and reionization process. The change in the visibility function is significantly smaller during reionization, than recombination. To show both on the same plot, we multiply the reionization part by 100.

4 A brief overview of the cosmological perturbations

CMBAns is developed only for the flat background metric, i.e. $\Omega_k = 0$. We can represent a completely isotropic and homogeneous expanding universe using FLRW metric, $ds^2 = a^2 g_{\mu\nu} dx^\mu dx^\nu$, where $g_{\mu\nu} = a(\tau)^2 \eta_{\mu\nu}$, and $\eta_{\mu\nu}$ is the Minkowski metric. Since our universe is not completely homogeneous and isotropic, we need to add some perturbation to the metric. The line element in the perturbed metric is given by

$$ds^2 = a^2(\tau) \left( \eta_{\mu\nu} + h_{\mu\nu} \right) dx^\mu dx^\nu.$$ (4.1)

$^6$For all our calculations, we consider the mass of H and H$^+$ = m$_p$, and mass of He, He$, He^{++}$ = 4m$_p$, i.e. we consider that the mass of electron is negligible and the mass of proton and neutron are the same.
Here \( h_{\mu\nu} \) is the perturbation in the metric. For simplicity of calculation, the metric perturbation is expanded in the spatial and temporal parts. This can be done by breaking \( h_{\mu\nu} \) in (1 + 3) dimensional format as (Weinberg, 2008)

\[
\begin{align*}
  h_{00} &= E \\
  h_{i0} &= \frac{\partial F}{\partial x^i} + G_i \\
  h_{ij} &= A \delta_{ij} + \frac{\partial^2 B}{\partial x^i \partial x^j} + \frac{\partial C_i}{\partial x^j} + D_{ij}
\end{align*}
\]  

(4.2)

where \( A, B, E, F, C_i, G_i \) are \( D_{ij} \) are the perturbation variables and

\[
\begin{align*}
  \frac{\partial C_i}{\partial x^i} &= 0, \quad \frac{\partial D_{ij}}{\partial x^j} = 0, \quad D_{ii} = 0.
\end{align*}
\]  

(4.3)

From Einstein’s equation, we get \( G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu} \), where \( G_{\mu\nu} \) is the Einstein tensor, and \( T_{\mu\nu} \) is the stress-energy tensor. Perturbing the equations up to the first order, we get \( G_{\mu\nu} + \delta G_{\mu\nu} = \frac{8\pi G}{c^2} (T_{\mu\nu} + \delta T_{\mu\nu}) \). We can use the above perturbation variables to calculate the perturbations in the Christoffel symbols. The perturbation to the Einstein’s tensor, \( \delta G_{\mu\nu} \) can then be computed from the Christoffel symbols. For calculating the perturbation in \( \delta T_{\mu\nu} \), we need to know the perturbation in the pressure and density of the different components in the Universe, i.e. baryons, photons, neutrinos, dark matter, and dark energy, etc. For a perfect fluid, the stress-energy tensor is given by

\[
T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) u^\mu u^\nu.
\]  

(4.4)

Similar to the metric tensor, the perturbation in the stress-energy tensor can also be expanded into spatial and temporal parts

\[
\begin{align*}
  \delta T_{00} &= -p + h_{00} \delta \rho \\
  \delta T_{0i} &= ph_{i0} - (p + \rho) \left( \frac{\partial \delta u_i}{\partial x^0} + \delta u_i^V \right) \\
  \delta T_{ij} &= ph_{ij} + \left[ \delta_{ij} \delta p + \frac{\partial^2 \pi^S}{\partial x^i \partial x^j} + \frac{\partial \pi^V_i}{\partial x^j} + \frac{\partial \pi^V_j}{\partial x^i} + \pi_i^T \right]
\end{align*}
\]  

(4.5)

in which

\[
\begin{align*}
  \frac{\partial \pi^V_i}{\partial x^j} &= \frac{\partial \delta u_i^V}{\partial x^j} = 0, \quad \frac{\partial \pi^T_i}{\partial x^j} = 0, \quad \pi_i^T = 0.
\end{align*}
\]  

(4.6)

We can match both the sides in \( \delta G_{\mu\nu} = \frac{8\pi G}{c^2} (\delta T_{\mu\nu}) \) and separate out:

- The terms containing \( A, B, E, F, \delta \rho, \delta p, \pi^S \) and \( \delta u \). These involve all the scalar quantities and are called the scalar perturbations.

- The terms containing \( C_i, G_i, \pi_i^V \) and \( \delta u_i^V \). These involve all the vector quantities in the spatial dimension and are called the vector perturbations. These vector modes decay and hence have a small contribution to the CMB power spectrum (Weinberg, 2008).

- The terms involving \( D_{ij} \) and \( \pi_i^T \). These terms behave as tensor quantities in 3-spatial dimensions and are called the tensor perturbations.

### 4.1 Theory of scalar perturbations

We can obtain the scalar perturbation equations by separating out the terms involving the scalar perturbation variables, i.e. \( A, B, E, F, \delta \rho, \delta p, \pi^S \) and \( \delta u \). However, these terms are not all independent (Weinberg, 2008). Also, there can be unphysical modes due to the choice of the coordinate system. These problems can be resolved by fixing a proper coordinate system, and adopting suitable
conditions on the full perturbed metric and energy-momentum tensor. This process is called gauge fixing (Ma and Bertschinger, 1995; Weinberg, 2008). In 
\textsc{ cmbans}, we do all the calculations involving the scalar perturbation in synchronous gauge (Lifshitz, 1946).

In synchronous gauge, the scalar component of the perturbed metric can be written as \( ds^2 = a^2(\tau)(-dt^2 + (h_{ij} + \delta_{ij})dx^idx^j) \), where \( \delta_{ij} \) is the Kronecker delta, and \( h_{ij} \) is the scalar part of the metric perturbation in synchronous gauge. \( h_{ij} \) can be represented using only two scalar fields \( h(\hat{k}, \tau) \) and \( \eta(\hat{k}, \tau) \), in \((\hat{k}, \tau)\) space, where \( \hat{k} = \hat{k} \), \( \hat{k} \) is the unit vector along direction of vector \( \hat{k} \) and \( k \) is its amplitude. We can relate \( h(\hat{k}, \tau) \) and \( \eta(\hat{k}, \tau) \) with the real space perturbation variables \( A, B, E, F \). This gives \( E = 0, F = 0 \), because in synchronous gauge we take constant time hyperspace, \( A(\vec{x}, \tau) = \int \left( 2\eta(\vec{k}, \tau) \right) d^3k e^{i\vec{k} \cdot \vec{x}} \) and \( B(\vec{x}, \tau) = \int \left( h(\vec{k}, \tau) + 6\eta(\vec{k}, \tau) \right) d^3k e^{i\vec{k} \cdot \vec{x}} \). The growth of these perturbation variables can be computed using the perturbed Einstein equations \( \delta G_{\mu\nu} = \frac{8\pi G}{c^2} \delta T_{\mu\nu} \). The perturbation in \( G_{\mu\nu} \) can be calculated using the metric perturbation variables (Ma and Bertschinger, 1995) as

\[
\begin{align*}
    k^2 \eta - \frac{1}{2} \frac{\dot{a}}{a} \hat{h} &= \frac{4\pi G}{c^2} a^2 \delta T^0_0, \quad (4.8) \\
    k^2 \dot{\eta} &= \frac{4\pi G}{c^2} a^2 (\dot{\rho} + \dot{P}) \theta, \quad (4.9) \\
    \dot{h} + 2 \frac{\dot{a}}{a} \hat{h} - 2k^2 \eta &= -8\pi G a^2 \delta T^i_i, \quad (4.10) \\
    \hat{h} + 6\dot{\eta} + 2 \frac{\dot{a}}{a} (\hat{h} + 6\dot{\eta}) - 2k^2 \eta &= -24\pi G a^2 (\dot{\rho} + \dot{P}) \sigma. \quad (4.11)
\end{align*}
\]

Here \( \theta \) is defined as \( \theta(\vec{k}, \tau) = \frac{ik^j \delta T^0_j}{\rho + P} \). For a fluid, this is simply the divergence of its velocity field, i.e. \( \theta = ik^j v_j \). The \( \sigma \) in Eq. 4.11 can be defined as \( \sigma(\vec{k}, \tau) = \frac{\frac{d}{a}(\vec{\rho}, \vec{P})}{3(\rho + P)} \), where \( \Pi \) is the anisotropic stress term, corresponding to its real space quantity \( \pi^S \) shown in Eq. 4.5.

We also need to conserve the perturbation in the stress-energy tensor, i.e. \( \delta T_{\mu\nu} = 0 \). For non-relativistic perfect fluid, these conservation equations give us

\[
\begin{align*}
    \dot{\delta} &= -(1 + w) \left( \theta + \frac{\dot{h}}{2} \right) - 3 \frac{\dot{a}}{a} \left( \frac{\delta P}{\delta \rho} - w \right) \delta, \\
    \dot{\eta} &= \frac{\dot{a}}{a} (1 - 3w) \theta - \frac{w}{1 + w} \theta + \frac{\delta P}{\delta \rho} \delta \rho \frac{\delta \delta}{\delta \sigma} k^2 \delta - k^2 \sigma, \quad (4.12)
\end{align*}
\]

where \( \delta = \frac{\delta \rho}{\rho} \) is the perturbation in the density of the fluid. \( w \) is the equation of state of the fluid. Non-relativistic fluids like the CDM and baryon should obey these equations.

To calculate the stress-energy tensor for the relativistic particles, we need to use the distribution function. Photons follow the Bose-Einstein distribution, whereas the neutrinos follow the Fermi-Dirac distribution. Considering \( f \) as the phase space distribution function, the stress-energy tensor is given by

\[
\begin{align*}
    T_{\mu\nu} &= \int dP_1 dP_2 dP_3 (\vec{q} \cdot \vec{x})^{-\frac{1}{2}} \frac{P_{\mu} P_{\nu}}{P^0} f(x^i, P_j, \tau). \quad (4.13)
\end{align*}
\]

\( P \) is the four-momentum. \( P_0 = -\epsilon = a(q^2 + m^2)^{\frac{1}{2}} = (q^2 + a^2 m^2)^{\frac{1}{2}} \). \( P_j = aq_j = qn_j \). \( q, \ n_i \) are the magnitude and the direction of the momentum and \( n^i n_i = 1 \). It is convenient to write the distribution function as

\[
\begin{align*}
    f(x^i, P_j, \tau) &= f_0(q) \left[ 1 + \Psi(x^i, q, n_j, \tau) \right]. \quad (4.14)
\end{align*}
\]
Here \( \Psi(x', q, n_j, \tau) \) in the first order perturbation in the distribution function. In synchronous gauge \((-g)^2 = a^{-4}(1 - \frac{1}{2}\dot{a}) \), and \( dP_1 dP_2 dP_3 = (1 + \frac{1}{2}\dot{a})q^2 dq d\Omega \). The phase-space distribution evolves as the Boltzmann equation. If we convert everything to Fourier space \((\vec{k} \text{ space})\), then the first-order perturbations of the Boltzmann equation in the synchronous gauge can be written as

\[
\frac{\partial \Psi}{\partial \tau} + \frac{q}{c_s} (\vec{k} \hat{n}) \Psi + \frac{d(\ln f_0)}{d(ln q)} \left[ \dot{n} - \frac{\ddot{a}}{2} (\vec{k} \hat{n})^2 \right] = \frac{1}{f_0} \left( \frac{\partial f}{\partial \tau} \right)_c \tag{4.15}
\]

Photons and neutrinos will follow this equation. The term on the right-hand side is the collision term. Neutrinos are collisionless in our domain as they decouple long back in the radiation dominated era. However, before decoupling, the photon and baryon collisions will provide some contribution to this term.

\( \delta \) and \( \theta \) for photons and neutrinos can be calculated in terms of \( \Psi \), the detail of which is discussed in Sec. 4.1.3.

### 4.1.1 Scalar perturbations in the metric

In the previous section, we describe the equations for the metric perturbation in terms of \( T_{\mu\nu} \). However, there are the different components of the universe, i.e. CDM, baryon, photon, neutrino, DE, etc. and we need the metric perturbations in terms of perturbations of these individual components. We use the subscript \( c, b, \gamma, \nu, \nu_m \) and \( d \) for representing cold dark matter, baryonic matter, photon, massless neutrinos, massive neutrinos, and dark energy respectively. Following Eq. 4.8 and Eq. 4.9, the equations for the metric perturbation can be written in terms of density (\( \delta \)) and velocity perturbations (\( \theta \)) of the individual components as

\[
\frac{\dot{a}}{a} \ddot{h} = 2k^2 \eta + \frac{8 \pi G}{c^2} \left[ \rho_c \left( \Omega_c \frac{\delta_c}{a} + \Omega_b \frac{\delta_b}{a} + \Omega_d \delta_d a^2 \right) \right. \\
+ \left. \left( \frac{4 \sigma_B T_4}{c^4} \right) \frac{\delta_\gamma}{a^2} + N_{\text{eff}} \left( \frac{7}{8} \right) \left( \frac{4}{11} \right)^{4/3} \frac{\delta_\nu}{a^2} + \left( \frac{43}{11} \right) \left( \frac{1}{g_A} \right)^{4/3} \frac{\delta_{\nu_m}}{a^2} \right] \tag{4.16} \\
2k^2 \dot{\eta} = \frac{8 \pi G}{c^2} \left[ \rho_c \left( \frac{\Omega_b}{a^2} + \Omega_d \delta_d a^2 \right) + \left( \frac{4 \sigma_B T_4}{c^4} \right) \left( \frac{\theta_\gamma}{a^2} + N_{\text{eff}} \left( \frac{7}{8} \right) \left( \frac{4}{11} \right)^{4/3} \frac{\theta_\nu}{a^2} \right) \right. \\
+ \left. \left( \frac{43}{11} \right) \left( \frac{1}{g_A} \right)^{4/3} \times \left( \frac{\theta_{\nu_m}}{a^2} \right) \right] \tag{4.17}
\]

where, over-dot \((\dot{\cdot})\) represents the derivative with respect to the conformal time and \( g_A \) is the effective number of spin states before neutrino decoupling. For \( g_A = 10.75 \), we get \( \left( \frac{43}{11} \frac{1}{g_A} \right) = \frac{4}{11} \). However, in presence of any other particle in the early universe, the above expression will correct for that.

Instead of the last two equations (Eq. 4.10 and Eq. 4.11), we take the conservation equations \((\delta T_{\mu\nu} = 0)\) for different components of the universe, which evolve independently except before decoupling, when the baryon and photon evolve together as a single fluid. The conservation equations for different components in terms of their density and velocity perturbations \((\delta \text{ and } \theta)\) are given below.

### 4.1.2 Conservation equation for Cold Dark Matter (CDM)

CDM can be treated as a pressure-less perfect fluid and it interacts with other particles only through gravity. Therefore, for CDM \( P = w = 0 \) and the conservation equations, i.e Eq. 4.12, for CDM gives

\[
\dot{\delta}_c = -\theta_c - \frac{1}{2} \dot{\theta}_c, \quad \dot{\theta}_c = -\frac{\dot{a}}{a} \theta_c. \tag{4.18}
\]

As CDM does not interact with other particles, if \( \theta_c = 0 \) is fixed as the initial condition then the values of \( \theta_c \) will remain 0 through the entire era. Setting \( \theta_c = 0 \) also solves another purpose by removing the extra gauge mode in the synchronous gauge (Sec. 5.1). Thus the CDM equation becomes as \( \delta_c = -\frac{1}{2} \dot{h} \). In Fig. 6 we show the evolution of the CDM density perturbation for different \( k \) modes.
4.1.3 Conservation equations for massless neutrinos

According to the standard model of particle physics the neutrinos are massless. However, neutrino oscillation provides evidence for non-zero mass, but only small values. Therefore, cosmologists are interested in checking the consequences of both the massive and massless neutrinos in the cosmic fluid. The perturbation equation for the massive and the massless neutrinos differ significantly.

For massless neutrinos, the density and pressure are related as $\rho_\nu = a^{-4} \int q^2 f_{FD}(x^i, q, n_j, \tau) dq d\Omega = 3 P_\nu$, where, $f_{FD}(x^i, q, n_j, \tau)$ is the Fermi-Dirac distribution function and $q$ is the momentum in the co-moving frame. Unlike CDM, neutrinos have pressure. Therefore, solving the growth equations for neutrinos is difficult as there are several direction-dependent variables. However, for solving Eq. 4.8 – Eq. 4.11, we only need $\delta_\nu$, $\theta_\nu$ and $\sigma_\nu$ and these can be simplified by defining a new variable

$$F_\nu(\vec{k}, \hat{n}, \tau) = \frac{\int q^2 f_0(q) q\Psi dq}{\int q^2 f_0(q) dq},$$

where $\Psi$ is the perturbation in the Fermi-Dirac distribution function i.e. $f_{FD} = f_0(1 + \Psi)$, $f_0$ being...
the 0th order term of the Fermi-Dirac distribution function.

According to Eq. 4.15, $k$ and $\hat{n}$ always appear together as $(k, \hat{n})$, showing that only the magnitude and the angle between them are important, not the individual values of $k$ and $\hat{n}$. We can expand $F_\nu(k, \hat{n}, \tau)$ in terms of Legendre Polynomials as

$$F_\nu(k, \hat{n}, \tau) = \sum_{l\nu=0}^{\infty} (-i)^{l\nu}(2l\nu + 1)F_{l\nu\nu}(k, \tau)P_{l\nu}(k, \hat{n})$$  \hspace{1cm} (4.20)$$

where $P_{l\nu}(k, \hat{n})$ are the Legendre polynomials and $F_{l\nu\nu}(k, \tau)$ are their coefficients. The term $(-i)^{l\nu}(2l\nu + 1)$ is taken out to simplify the perturbation equations. For calculating the perturbation equations in terms of these variables, we can use Eq. 4.15 and Eq. 4.19 and expand them in the Legendre polynomials. The perturbation equations for massless neutrinos take the form (Ma and Bertschinger, 1995)

$$\dot{F}_{l\nu0} = -\frac{4}{3}\theta_{l\nu} - \frac{2}{3}\hat{h} = \delta_{l\nu},$$  \hspace{1cm} (4.21)$$

$$\dot{F}_{l\nu1} = k\left(\frac{1}{3}\delta_{l\nu} - \frac{4}{3}\sigma_{l\nu}\right) = \frac{4}{3k}\theta_{l\nu},$$  \hspace{1cm} (4.22)$$

$$\dot{F}_{l\nu2} = \frac{8}{15}\theta_{l\nu} - \frac{3}{5}kF_{l\nu3} + \frac{4}{15}\hat{h} + \frac{8}{5}\hat{\eta} = 2\sigma_{l\nu},$$  \hspace{1cm} (4.23)$$

$$\dot{F}_{l\nu l\nu} = \frac{k}{2l\nu + 1} \left[ l\nu F_{l\nu(l\nu-1)} - (l\nu + 1)F_{l\nu(l\nu+1)} \right], \hspace{1cm} l\nu \geq 3.$$  \hspace{1cm} (4.24)$$

Here $l\nu$ goes up to $\infty$. The truncation condition can be taken as

$$F_{l\nu(l\nu_{\max}+1)} \approx \frac{2l\nu_{\max} + 1}{k\tau}F_{l\nu_{\max}} - F_{l\nu(l\nu_{\max} - 1)}.$$  \hspace{1cm} (4.25)$$

The Bessel functions follow a similar recursion relation as Eq. 4.24. Therefore, if we assume that $F_{l\nu\nu} \propto \tilde{j}_{l\nu}(k)$ (which is just an approximation) then we can use the other recursion relations that Bessel functions follow. One of such recursion relations for Bessel function is similar to Eq. 4.25. Therefore, the relation shown in Eq. 4.25 is used as the truncation condition (Ma and Bertschinger, 1995). However, such a choice of truncation condition can lead to a large error in the final power spectrum calculation, due to the propagation error if the Boltzmann hierarchy is truncated after some low $l\nu_{\max}$ For truncating the set of equations, in CMBAns we use $l\nu_{\max} = 7$ (which the user can change to any higher value). In Fig. 7 we show the percentage errors $\left(\frac{C_{l\nu_{\max}} - C_{l\nu_{\max} = 8}}{C_{l\nu_{\max} = 8}} \times 100\%ight)$ involved in the CMB power spectrum calculation for truncating the equations at different $l\nu_{\max}$, considering $l\nu_{\max} = 8$ to be standard. The plots show that the error introduced due to the truncation of the neutrino equations slowly decreases with the increase in $l\nu_{\max}$.

In Fig. 6, we show the growth of the density fluctuations in neutrino. As they don’t interact with any other species, their evolution is completely independent of others. The modes $\delta_{l\nu}$ evolve as $\propto \tau$ before horizon crossing. After horizon crossing, when they come in causal contact they start oscillating.

### 4.1.4 Conservation equations for massive neutrinos

For the massive neutrinos $\epsilon = (q^2 + a^2m_{\nu m}^2)^{1/2}$, where $q$ is the momentum in the co-moving frame and $m_{\nu m}$ is the mass of the neutrino. Hence, unlike massless neutrinos, the expressions $\int q^2 f_0(q)^{1/2} \Psi dq$ cannot be integrated analytically. Thus, we expand the perturbation $\Psi$ directly in terms of Legendre polynomials as (Ma and Bertschinger, 1995):

$$\Psi(k, \hat{n}, q, \tau) = \sum_{l\nu m=0}^{\infty} (-i)^{l\nu m}(2l\nu m + 1)\Psi_{l\nu m}(k, q, \tau)P_{l\nu m}(k, \hat{n}).$$  \hspace{1cm} (4.26)$$
Substituting this in Eq. 4.15, the Boltzmann equations for the massive neutrinos become

\begin{align}
\dot{\Psi}_0 &= -\frac{qk}{\epsilon} \psi_1 + \frac{1}{6} \dot{\hbar} \frac{d(\ln f_0)}{d(\ln q)}, \\
\dot{\psi}_1 &= \frac{qk}{3\epsilon} (\psi_0 - 2\psi_2), \\
\dot{\psi}_2 &= \frac{qk}{5\epsilon} (2\psi_1 - 3\psi_3) - \left(\frac{1}{15} \dot{\hbar} + \frac{2}{5} \eta\right) \frac{d(\ln f_0)}{d(\ln q)}, \\
\dot{\psi}_{l\nu m} &= \frac{qk}{(2l\nu m + 1)\epsilon} \left[ l\nu m \Psi_{l\nu m - 1} - (l\nu m + 1)\Psi_{l\nu m + 1} \right], \quad l\nu m \geq 3
\end{align}

For truncating the series we use the condition

\begin{equation}
\Psi_{(l\nu m + 1)} \approx \frac{(2l\nu m + 1)\epsilon}{qkT} \Psi_{l\nu m} - \Psi_{(l\nu m - 1)}.
\end{equation}

Here, \(\Psi_l\) are the functions of \(k, \tau\) and \(q\). The integration over \(q\) is performed numerically after calculating different massive neutrino multipoles for different \(q\). For the massive neutrinos the modes are truncated after \(l\nu m = 5\).

### 4.1.5 Photons

Photons evolve differently before recombination, when they were tightly coupled with baryons and after recombination, when they were free streaming. The evolution of the photons can be treated in the same way as the massless neutrinos except the collision terms will be present. Thompson scattering by a density perturbation can introduce polarization in an unpolarized photon field and in a density perturbation. Therefore, along with the total intensity, we also have to consider the polarization component for the photons. The details of the polarization is discussed are Appendix B.

Defining the intensity perturbations as \(\Delta_T\) and the polarization as \(\Delta_P\), the Boltzmann equations for photons in the synchronous gauge can be written as (Ma and Bertschinger, 1995)
The evolution of the photons is shown in Fig. 6. Before recombination, the photons are tightly coupled with the baryons, and they evolve together as a single fluid. However, after decoupling, the photons follow a similar pattern as that of the neutrinos. For $\ell^\gamma = \ell^\gamma_{\text{max}}$ we can replace the $\Delta_{TT}$ and the $\Delta_{P\gamma}$ by the following equations

\[
\begin{align*}
\Delta_{TT}^{\ell^\gamma_{\text{max}}} &= k \Delta_T (\ell^\gamma_{\text{max}} - 1) - \frac{\ell^\gamma_{\text{max}} + 1}{\tau} \Delta_{TT}^{\ell^\gamma_{\text{max}}}, \\
\Delta_{P\gamma}^{\ell^\gamma_{\text{max}}} &= k \Delta_P (\ell^\gamma_{\text{max}} - 1) - \frac{\ell^\gamma_{\text{max}} + 1}{\tau} \Delta_{P\gamma}^{\ell^\gamma_{\text{max}}},
\end{align*}
\]

The evolution of the photons is shown in Fig. 6. Before recombination, the photons are tightly coupled with the baryons, and they evolve together as a single fluid. However, after decoupling, the photons follow a similar pattern as that of the neutrinos.

For photons we choose the truncation at $\ell^\gamma_{\text{max}} = 12$. The error involved due to the truncation of the photon multipole equations, i.e. $\left(\frac{C_{l_{\text{max}}^\gamma} - C_{l_{\text{max}}^\gamma=13}^{l_{\text{max}}^\gamma}}{C_{l_{\text{max}}^\gamma=13}}\right) \times 100\%$ is shown in Fig. 8. The errors are calculated considering $\ell^\gamma_{\text{max}} = 13$ to be standard. We can see that if we truncate the evaluation in low $\ell^\gamma_{\text{max}}$, such as $\ell^\gamma_{\text{max}} = 4$ or 6, then the error involved can be as high as 0.5%. However, if we increase $\ell^\gamma_{\text{max}}$, the error decreases drastically. At $\ell^\gamma_{\text{max}} = 12$, the error in the final $C_l$ is less than 0.01%.

Figure 8. The error $\left(\frac{C_{l_{\text{max}}^\gamma} - C_{l_{\text{max}}^\gamma=13}^{l_{\text{max}}^\gamma}}{C_{l_{\text{max}}^\gamma=13}}\right) \times 100\%$ involved in the power spectrum calculation for truncating the photon multipole equations at different $\ell^\gamma_{\text{max}}$ are shown here. The left and right plot shows the error involved in $C_{TT}^\gamma$ and $C_{EE}^\gamma$ calculations. Here, $\Omega_b h^2 = 0.022068$, $\Omega_m h^2 = 0.14236$, $h = 0.6711$, optical depth ($\kappa$) = 0.0925, $n_s = 0.9624$.
4.1.6 Baryons

Before recombination, the baryons and photons were tightly coupled and they evolved together as a single fluid. However, after recombination, the baryons decouple from the photons and evolve separately. The baryon density and the velocity perturbations are governed by the following equations (Ma and Bertschinger, 1995)

\[
\begin{align*}
\dot{\delta}_b &= -\theta_b - \frac{1}{2} \dot{h}, \\
\dot{\theta}_b &= -\frac{\dot{a}}{a} \theta_b + c_s^2 k^2 \delta_b + \frac{4 \rho_{\gamma 0}}{3 \rho_{b 0}} a n_e \sigma_T (\theta_\gamma - \theta_b),
\end{align*}
\]

(4.31)

where, \(\rho_{\gamma 0}\) and \(\rho_{b 0}\) are the background density for photons and baryons respectively. \(c_s\) is the baryon sound speed shown in Eq 3.13.

In the tight coupling era, the photon and baryon equations cannot be solved independently using the Runge-Kutta methods, as this may lead to large numerical errors. Therefore, a separate set of equations is used for solving the baryon perturbation in the tight coupling era.

4.1.7 Recombination and the Tight Coupling Approximation

Before recombination the opacity \(\dot{\mu} = an_e \sigma_T\) was very large. Hence, the photons and baryons evolve together during the tight coupling approximation. The conformal time scale for the photon baryon interaction is \(\tau_c = \frac{1}{an_e \sigma_T}\). In the tight coupling era, this interaction time scale is much smaller than the timescale on which the modes in the super-Hubble scale evolve, i.e. \(\tau_H = a/\dot{a}\) and the modes for the sub-Hubble scale i.e. \(\tau_k = 1/k\). The standard numerical integration is not efficient to integrate the baryon and photon perturbation equations independently in this era. Therefore, Peebles and Yu (Ma and Bertschinger, 1995; Peebles and Yu, 1970) developed a new set of equations for the baryon perturbations which is valid in the tight coupling regime i.e. where \(\tau_c/\tau_H\) or \(\tau_c/\tau_k\) \(\ll\) 1. Instead of using the standard baryon and photon equations, the idea is to integrate a coupled form of differential equation, given by

\[
(1 + R) \dot{\theta}_b + \frac{\dot{a}}{a} \theta_b - c_s^2 k^2 \delta_b - k^2 R \left( \frac{1}{4} \delta_\gamma - \sigma_\gamma \right) + R (\dot{\theta}_\gamma - \dot{\theta}_b) = 0,
\]

(4.32)

where \(R = (4/3) \rho_{\gamma 0}/\rho_{b 0}\), and \((\dot{\theta}_\gamma - \dot{\theta}_b)\) is given by the following equation
\[ \theta_b - \dot{\theta}_\gamma = \frac{2R}{1 + R} \dot{a} (\theta_b - \dot{\theta}_\gamma) + \frac{\tau_c}{1 + R} \left[ -\frac{\dot{a}}{a} \theta_b - \frac{\dot{a}^2}{a} \left( \frac{1}{2} \dot{\delta}_\gamma \right) \right] + \frac{\tau_c}{1 + R} \left[ -\frac{\dot{a}}{a} \theta_b - \frac{\dot{a}^2}{a} \left( \frac{1}{2} \dot{\delta}_\gamma \right) \right] + \frac{\tau_c}{1 + R} \left[ -\frac{\dot{a}}{a} \theta_b - \frac{\dot{a}^2}{a} \left( \frac{1}{2} \dot{\delta}_\gamma \right) \right] + O(\tau_c^2). \] (4.33)

For obtaining \( \dot{\theta}_\gamma \), we can use the following equation (Ma and Bertschinger, 1995)

\[ \dot{\theta}_\gamma = -R^{-1} \left( \frac{\dot{a}}{a} \theta_b - c_s^2 k^2 \theta_b \right) + \frac{1}{4} k^2 \delta_\theta - \dot{\sigma}_\gamma \] (4.34)

In the tight coupling limit the higher multipoles of the photon distribution, i.e. \( F_{\gamma 3}, F_{\gamma 4}, \) ... and \( G_{\gamma 0}, G_{\gamma 1}, G_{\gamma 2}, \) ... can be taken as 0. We also consider \( \dot{\sigma}_\gamma = 0 \), which from Eq. 4.30 gives

\[ \dot{\sigma}_\gamma = \frac{\tau_c}{9} \left( \frac{8}{3} \theta_\gamma + \frac{4}{3} \dot{h} + 8 \dot{\eta} \right). \] (4.35)

In Fig. 10, we show in light red color the region where we use the tight coupling approximation using light red color.

### 4.1.8 DE perturbation

In the standard cosmological model, dark energy is considered to be the cosmological constant term (\( \Lambda \)). However, there are several other dark energy models available, in which dark energy is considered to be a single scalar field (e.g. quintessence) or a mixture of multiple scalar fields (e.g. k-essence). Thus, in these dark energy models, dark energy will behave in the same manner as the other components and can be perturbed.

A more general closed-form solution (Bean and Dore, 2004; Hannestad, 2005; Weller and Lewis, 2003) for the density and the velocity perturbation for any general form of dark energy is given by

\[ \dot{\delta}_d = -(1 + w_d) \left\{ \left[ k^2 + 9H^2 (c_s^2 - c_a^2) \right] \theta_d + \frac{\dot{h}}{2} \right\} - 3H (c_s^2 - w_d) \delta_d, \]

\[ \frac{\dot{\theta}_d}{k^2} = -H (1 - 3c_s^2) \frac{\theta_d}{k^2} + \frac{c_s^2}{1 + w_d} \delta_d. \] (4.36)

Here \( \delta_d \) and \( \theta_d \) are used in their usual meaning, i.e. \( \delta_d \) is the density perturbation and \( \theta_d \) is the velocity perturbation of the dark energy. \( c_a \) is known as the adiabatic sound speed and given by

\[ c_a^2 = w_d - \frac{\dot{w}_d}{3H(1 + w_d)}, \] (4.37)

where \( w_d \) is the dark energy equation of state and \( c_s \) is the speed of sound in the dark energy and is given by \( c_s^2 = \frac{\dot{\rho}_d}{\dot{\rho}_d}. \) For a perfect fluid \( c_s^2 = c_a^2 \), in which case the DE perturbation equation is the same as the matter perturbation equation given by Eq. 4.12.

In the case of dark energy perturbation, we evolve these equations along with the other perturbation equations.

### 4.2 Theory of tensor perturbations

The tensor perturbations are gauge-invariant quantities. According to Eq. 4.2 and Eq. 4.5, Einstein’s equation for the tensor perturbation can be written as

\[ \ddot{D}_{ij} + 2H \dot{D}_{ij} + \nabla^2 D_{ij} = \frac{16\pi G}{c^2} a^2 \pi_{ij}^T \] (4.38)
Figure 10. The plot shows different epoch of the universe. The $x$ axis shows the wavenumbers and the $y$ axis shows conformal time. The blue dotted line shows the beginning of the reionization epoch and the dashed line shows the beginning of the recombination epoch in the universe. The green line depicts the Hubble crossing of the modes. The light red region shows the era where the tight coupling approximation equations are used. The initial conditions in CMBAns are specified along the brown line at the interface of the light red and yellow region.

Thus, the tensor metric perturbation can be described by two gravity wave polarization modes, $h_+$ and $h_\times$. Here we have chosen the perturbations in the $x-y$ plane, which corresponds to an implicit choice of the $z$ axis to be in the direction of the wave propagation. As the perturbation equations concerning both polarization modes are the same, the metric perturbation can be denoted by a single variable, $h_q$, where $q \in (+, \times)$. Cold dark matter and baryon, being non-relativistic, do not contribute to any anisotropic stress and as a consequence are not involved in tensor perturbation. (for details check Appendix B)

In Fourier space, the tensor perturbation equations take the form

$$
\dddot{h}_q = -2\frac{\dot{a}}{a}\dot{h}_q - k^2 h_q + \frac{16\pi G}{c^2}a^2\Pi^t,
$$

where, $\Pi$ is the anisotropic stress term and is given by $\Pi^t = 2\left(\rho_\gamma S^t_\gamma + \rho_\nu S^t_\nu\right)$ with (Baskaran et al.; Weinberg, b, 2008)

$$
S^t_\gamma = \frac{\delta^t_\gamma}{15} + \frac{\Delta^t_\gamma}{21} + \frac{\Delta^t_\gamma}{35},
$$

$$
S^t_\nu = \frac{\delta^t_\nu}{15} + \frac{F^t_\nu}{21} + \frac{F^t_\nu}{35}.
$$

4.2.1 Photons

The photons behave differently before and after recombination. In the tightly coupled era, the photons and baryons evolve as a single fluid. However, after the epoch of decoupling, the photons and baryons evolve independently. During the time when the photons and baryons are not tightly coupled, the photon perturbations are governed by the equations (Baskaran et al.; Weinberg, b, 2008)
\[ \delta_t^\gamma = \Delta_t^{\gamma 0} = -k^2 \delta_t^\gamma - a n_c \sigma_T \delta_t^\gamma + a n_c \sigma_T \Psi^t_e - \dot{h}_q, \]  
\[ \Delta_t^{P_0} = -k^2 \Delta_t^{P_1} - a n_c \sigma_T \Delta_t^{P_0} - a n_c \sigma_T \Psi^t_e, \]  
(4.42)

For \( l \geq 1 \) the equations are the same as those of the scalar case and are given by

\[ \dot{\Delta}_{tl} = \frac{k}{2l+1} \left[ l \Delta_{(l-1)} - (l+1) \Delta_{(l+1)} \right] - a n_c \sigma_T \Delta_{tl}, \]  
\[ \Delta_{P_0} = \frac{k}{2l+1} \left[ l \Delta_{P_0}^{(l-1)} - (l+1) \Delta_{P_0}^{(l+1)} \right] - a n_c \sigma_T \Delta_{P_0}, \]  
(4.43)

where,

\[ \Psi^t_e = \delta^t_0 + \frac{\Delta^t_2}{70} + \frac{3 \Delta^t_4}{70} + \frac{3 \Delta^t_{P_0}}{70} + \frac{6 \Delta^t_{P_2}}{70} - \frac{3 \Delta^t_{P_u}}{70}. \]  
(4.44)

The truncation condition is the same as Eq. 4.30 used for scalar cases.

### 4.2.2 Tight coupling approximation

In the tight coupling limit, we take \( \Delta_t^\gamma = \Delta_t^P = 0 \) for the \( l \geq 1 \). Thus the equations for \( l = 0 \) mode take the form

\[ \delta^t_\nu = \frac{4}{3} \frac{\dot{h}_q}{a n_c \sigma_T}, \quad \Delta_{P_0} = \frac{1}{3} \frac{\dot{h}_q}{a n_c \sigma_T}. \]  
(4.47)

### 4.2.3 Massless neutrino

The perturbation equations for the massless neutrinos are given by

\[ \delta^t_\nu = \dot{F}^t_\nu = -k^2 \theta^t_\nu - \dot{h}_q, \]  
(4.48)

and for \( l \geq 1 \)

\[ \dot{F}^t_\nu = \frac{k}{2l+1} \left[ l F_{\nu (l-1)} - (l+1) F_{\nu (l+1)} \right]. \]  
(4.49)

The truncation condition is the same as Eq. 4.25. The contribution from the massive neutrinos will be very small. Therefore, we ignore their contribution to massive neutrinos in the tensor perturbations.

### 5 Initial conditions

In the previous section, we discuss the perturbation equations for the scalar and the tensor perturbation. However, for solving the set of differential equations, we need the initial conditions.

#### 5.1 Scalar perturbation

The initial conditions in the universe are assumed to be set by inflation. In single field inflation models, the inflationary field later decays to produce all the constituents of the universe, i.e. baryons (including leptons), photons, neutrinos, CDM, etc. It can be shown that the perturbations produced during inflation stay only in the modes that existed at the end of inflation, for as long as the perturbations remain outside the horizon. The wavelengths that interest us are far outside the horizon during the era of reheating after inflation. Therefore, the perturbations in those modes will remain the same irrespective of whatever the constituents of the universe may become. If the scalar perturbations are adiabatic at the end of inflation, then reheating cannot generate entropic perturbations (Weinberg, a, 2008).
However, there are different multifield inflationary models such as double inflation, where one field decays to CDM and the other fields produce other constituents like baryons, photons, and neutrinos. Such a scenario can produce both the adiabatic and the isocurvature modes. In this particular case, it will produce the CDM isocurvature modes.

In the present work, we want to calculate the CMB power spectrum for different initial conditions and are not interested in the theoretical details of the production of different isocurvature modes. There can be different types of isocurvature initial conditions, such as baryon isocurvature modes, CDM isocurvature modes, neutrino density isocurvature modes, and neutrino velocity isocurvature modes (Bucher et al., 2000).

In scalar perturbation, the metric is perturbed by the perturbations in the primordial plasma. Therefore, if there is no plasma fluid perturbation, then metric perturbations should be 0. We also assume that in the very early universe, even before the decoupling of neutrinos, everything was tightly coupled and hence the anisotropic stress terms were zero. Hence, we just need to set the initial values for \( \delta \) and \( \theta \) for all the 5 components of the universe. In total, we will have 10 different modes.

As we are working with a set of linear differential equations, we can set all these 10 parameters one by one, keeping the rest of them as 0 and evolve the equations independently. Finally, we can add all these solutions to get the final solution. Some authors studied dark energy perturbations (Gordon and Hu, 2004; Liu et al., 2010), but in this work we do not consider any dark energy isocurvature modes, i.e. we set \( \delta_{DE} = 0 \) and \( \theta_{DE} = 0 \) at \( \tau = 0 \). Thus, we are left with a total of 8 degrees of freedom.

We are using the synchronous gauge, and it can be shown that the metric perturbation cannot be eliminated even when there is no density or velocity contrast. Therefore, an extra initial condition is required, making the total number of degrees of freedom to 9. This extra mode is not a physical mode, but a gauge mode and is often eliminated (as we have done in this work) by setting the velocity of the cold dark matter to 0. Thus, we will finally have a total of 8 degrees of freedom corresponding to 8 different modes. It can also be shown that three of these modes will be decaying modes, and hence will not contribute to the final power. As a result, we have a total of 5 degrees of freedom to choose. One of the decaying modes decays, as baryons and photons were tightly coupled and so \( \theta_\gamma \) and \( \theta_b \) cannot be independently chosen. The other two modes decay because of total nonzero velocity perturbation and density perturbation (Bucher et al., 2000).

For defining these 5 modes we can define 5 variables, which are \( \delta_c, \delta_b, \delta_\nu, \theta_\nu \) and \( \eta \). We are choosing \( \eta \) instead of the photon variables \( \delta_\gamma \) because this is standard in the literature. In such a scenario, we can choose

\[
\delta_b = \delta_c = \frac{3}{4} \delta_\nu = \frac{3}{4} \delta_\gamma
\]

which is known as the adiabatic initial condition.

We must set the initial conditions deep inside the radiation dominated era, after neutrino decoupling where the physics is known. In the radiation dominated era, we can get the initial conditions for 5 different modes as: adiabatic initial conditions, baryon isocurvature initial conditions, CDM isocurvature initial conditions, neutrino density isocurvature and neutrino velocity isocurvature model.

5.1.1 Adiabatic initial conditions

In this case \( \delta_b = \delta_c = \frac{3}{4} \delta_\nu = \frac{3}{4} \delta_\gamma \). At \( \tau = 0 \), we can set the density perturbations to zero and set \( \eta \) to a nonzero quantity, i.e.

\[
\delta_b|_{\tau=0} = \delta_c|_{\tau=0} = \delta_\nu|_{\tau=0} = \theta_\nu|_{\tau=0} = 0
\]

Definable variable : \( \eta \)

A few straight forward calculations show that deep inside the radiation dominated era at time \( \tau \), the values of the perturbation variables are
The initial perturbations can be written as baryon isocurvature model (Bucher et al., 2000; Carrilho and Malik, 2018; Langlois, 2003). For the baryon isocurvature model was first proposed by Peebles (Peebles, 1987a,b) for explaining the galaxy peculiar velocity field. Here $R_{\gamma}$ and $R_{\nu}$ are the fractional contribution of the photon and neutrinos respectively in the early radiation dominated universe. For $N_{\nu}$ number of neutrino species we can define $R = \frac{7}{8} N_{\nu} \left( \frac{1}{11} \right)^{\frac{3}{4}}$, $R_{\gamma} = (1 + R)^{-1}$ and $R_{\nu} = R(1 + R)^{-1}$.

### 5.1.2 Baryon isocurvature initial conditions

The baryon isocurvature model was first proposed by Peebles (Peebles, 1987a,b) for explaining the galaxy peculiar velocity field (Bucher et al., 2000; Carrilho and Malik, 2018; Langlois, 2003). For the baryon isocurvature model

\[
\delta_c|_{r=0} = \delta_{\nu}|_{r=0} = \theta|_{r=0} = \eta|_{r=0} = 0
\]

Definable variable : $\delta_b$

The initial perturbations can be written as

\[
\begin{align*}
  h &= C(k\tau)^2, \quad \eta = 2C - \frac{5 + 4R_{\nu}}{6(15 + 4R_{\nu})} C(k\tau)^2, \quad \delta_{\gamma} = -\frac{2}{3}h, \quad \delta_c = \delta_b = \frac{3}{4} \delta_{\nu} = \frac{3}{4} \delta_{\gamma}, \\
  \theta_c &= 0; \quad \theta_{\gamma} = \theta_b = \frac{1}{18} C(k^4 \tau^3), \quad \theta_{\nu} = \frac{23 + 4R_{\nu}}{15 + 4R_{\nu}} \theta_{\gamma}, \quad \sigma_{\nu} = \frac{4C}{3(15 + 4R_{\nu})} (k\tau)^2. 
\end{align*}
\]

(5.3)

Here $R_{\gamma}$ and $R_{\nu}$ are the fractional contribution of the photon and neutrinos respectively in the early radiation dominated universe. For $N_{\nu}$ number of neutrino species we can define $R = \frac{7}{8} N_{\nu} \left( \frac{1}{11} \right)^{\frac{3}{4}}$, $R_{\gamma} = (1 + R)^{-1}$ and $R_{\nu} = R(1 + R)^{-1}$.

\[
\begin{align*}
  h &= \chi_{\nu} \times \left( \frac{1}{1 + \Omega_{b,c}/\Omega_{b,b}} - \frac{\chi_{\nu}}{2} \right), \quad \delta_{\nu} = 1 - \frac{1}{2}h, \quad \delta_{\gamma} = -\frac{2}{3}h, \quad \delta_c = -\frac{1}{2}h, \quad \delta_b = \delta_{\gamma} \\
  \theta_c &= 0; \quad \theta_{\gamma} = \theta_b = \theta_{\nu} = -\frac{h}{12} k^2 \tau, \quad \eta = -\frac{1}{6}h.
\end{align*}
\]

(5.5)
where $Y_r = \frac{\rho_m}{\rho_r}$, the ratio of matter density to radiation density at that particular epoch. In the second row of Fig. 11, we show the temperature and the polarization power spectrum from the baryon isocurvature model.

5.1.3 CDM isocurvature initial conditions

As we have discussed before, the CDM velocity mode is set to 0 to remove the extra gauge mode. CDM density isocurvature modes can arise in different inflationary models such as two field inflation or double inflation etc. In the CDM isocurvature mode (Bucher et al., 2000; Carrilho and Malik, 2018; Langlois, 2003)

$$
\delta_b|_{\tau=0} = \delta_\nu|_{\tau=0} = \theta|_{\tau=0} = \eta|_{\tau=0} = 0
$$

Definable variable : $\delta_c$ (5.6)

Therefore, substituting these values in the perturbation equations we can get

$$
h = Y_r \times \left( \frac{1}{1 + \frac{\Omega_{b,0}}{\Omega_{c,0}} - \frac{Y_r}{2}} \right) \quad \delta_c = 1 - \frac{1}{2} h, \quad \delta_\gamma = -\frac{2}{3} h, \quad \delta_b = -\frac{1}{2} h
$$

$\delta_\nu = \delta_\gamma$, $\theta_c = 0$, $\theta_\gamma = \theta_b = \theta_\nu = -\frac{h}{12} k^2 \tau$, $\eta = -\frac{1}{6} h$ (5.7)

5.1.4 Neutrino density isocurvature

For neutrinos we will have both the neutrino velocity and density isocurvature modes.

$$
\delta_c|_{\tau=0} = \delta_b|_{\tau=0} = \theta|_{\tau=0} = \eta|_{\tau=0} = 0
$$

Definable variable : $\delta_\nu$ (5.8)

For the density modes we can start with an uniform energy density, with the total photon and neutrino density unperturbed. When the modes enter the horizon, the photon behaves as a perfect fluid while the neutrinos freestream. Solving the set of perturbation equations with the above initial conditions...
we get
\[
\begin{align*}
  h &= \frac{\Omega_{0, b} R}{10} k^2 \tau^3 \\
  \delta_c &= -\frac{1}{2} h \\
  \delta_b &= \frac{1}{8} R k^2 \tau^2 \\
  \delta_\gamma &= -R + \frac{4}{3} \delta_b \\
  \delta_\nu &= -\frac{1}{R} \delta_\gamma \\
  \theta_\nu &= 1/4 k^2 \tau \\
  \eta &= -\frac{R_\nu}{6(15 + 4R_\nu)} k^2 \tau^2 \\
  \sigma_\nu &= \frac{1}{2(15 + 4R_\nu)} k^2 \tau^2 
\end{align*}
\] (5.9)

5.1.5 Neutrino velocity isocurvature model

Unlike the other terms, neutrinos can have velocity isocurvature modes. However, we need to start with a total of 0 momentum, or otherwise the mode will decay. This can be done by carefully choosing to match the momentum of the neutrinos and photons in the early universe. For this particular case the definable variables are

\[
\delta_c|_{\tau=0} = \delta_\nu|_{\tau=0} = \delta_b|_{\tau=0} = \eta|_{\tau=0} = 0
\]
Definable variable : \( \theta_\nu \)

Solving the perturbation equations with the above initial conditions gives us
\[
\begin{align*}
  h &= \frac{3\Omega_{0, b} R}{2} k^2 \tau^2 \\
  \delta_c &= -\frac{h}{2} \\
  \delta_b &= R k \tau - \frac{(3 + 2R)}{2} h \\
  \delta_\gamma &= \frac{4}{3} \delta_b \\
  \delta_\nu &= \frac{4}{3} k \tau - \frac{2}{3} h \\
  \theta_\nu &= 0 \\
  \theta_\gamma &= \frac{-Rk + 3\Omega_b R(1 + R)k \tau + 2(1 + R)(1 - 3\Omega_{0, b}(1 + R))h}{3(5 + 4R_\nu)} \\
  \theta_\tau &= k - \frac{(9 + r R)}{3(5 + 4R_\nu)} k^3 \tau^2 \\
  \sigma_\nu &= \frac{4}{3(5 + 4R_\nu)} k^3 \tau^2 \\
  \eta &= -\frac{4R_\nu}{3(5 + 4R_\nu)} k \tau + \frac{16R_\nu}{(5 + 4R_\nu)(15 + 4R_\nu)} k^2 \tau^2
\end{align*}
\] (5.11)

The neutrino velocity and density modes are shown in Fig. 12. Both these modes decay at high \( l \). However, the decay rate is much slower from the CDM or baryon isocurvature modes.

5.2 Tensor perturbation

For the tensor perturbation, the anisotropic stress term in Eq. 4.39 has a very small contribution, and the modes from the anisotropic stress terms decays (Weinberg, 2008). The only initial condition for tensor power spectra can be written as

\[
\begin{align*}
  h &= 1, \\
  \dot{h} &= 0, \\
  \delta_t^i &= \delta_\gamma^i = \delta_\nu^i = 0 
\end{align*}
\] (5.12)

We show the tensor power spectrum in Fig. 13. All the power spectra are plotted in the log-log scale except the \( C_l^{TE} \) which is plotted in the log-linear scale, as it contains the negative values.

5.3 Setting the initial conditions

We set the initial conditions when the wave is far outside the horizon by taking \( \tau^h_{i, \text{hor}}(k) = 0.001/k \), where \( \tau^h_{i, \text{hor}}(k) \) is the point where the initial condition is set for the wave number \( k \) from the horizon crossing cut off. Secondly, we set the initial conditions well within the tight coupling era and the radiation dominated universe. This is confirmed by considering \( \tau_i(k) = \min(\tau^h_{i, \text{hor}}(k), 0.1) \). Fig. 10 shows the positions where the initial conditions are set for different \( k \) modes (brown line). It can be seen that for smaller \( k \) modes, \( \tau_i(k) \) is 0.1, and for higher \( k \), which crosses the horizon earlier, the initial conditions are set at \( \tau^h_{i, \text{hor}}(k) \). If massive neutrinos are present, then we set the initial conditions in an era where the massive neutrinos are highly relativistic, which is given by \( \tau_h = (1 - 3.0/m_\nu) / \dot{a}_{\text{rad}} \). In that case, we choose \( \tau_i(k) = \min(\tau_h, \tau^h_{i, \text{hor}}(k), 0.1) \) as the initial conditions of mode \( k \).
Figure 13. The plot shows the unlensed CMB tensor power spectrum ($C_l$). We use $\Omega_B h^2 = 0.0223$, $\Omega_c h^2 = 0.1188$, $h = 67.74 \text{ km/sec/Mpc}$, $n_t = 0.04$, $\kappa = 0.08$. As $C_l^{TT}$ has negative values we plot the $y$-axis in linear scale.

6 Calculating the CMB power spectrum

In Sec. 4.1.5, we calculate the photon multipole functions, $\Delta_T(k, \tau_0)$ and $\Delta_P(k, \tau_0)$. We can obtain the CMB scalar power spectrum just by convolving the multipole brightness functions with the primordial inflationary power spectrum, i.e.

$$ C_l^{TT} = (4\pi)^2 \int k^2 \frac{dk}{k^2} P_s(k) \left[ \Delta_T(k, \tau_0) \right]^2, \quad (6.1) $$

Analogous expressions can also be obtained for the polarization and the tensor perturbations (Ma and Bertschinger, 1995). However, if we use the expression for $\Delta_T$ from Sec. 4.1.5, then for calculating $C_l$ up to $l_{\text{max}}$, we need to solve $2 \times l_{\text{max}}$ coupled differential equations for photons up to the present era, which will be highly time consuming. Therefore, Seljak and Zaldarriaga proposed a method in Seljak and Zaldarriaga (1996); Zaldarriaga (1998), which analytically integrates the CMB perturbation terms. This is known as the line-of-sight integration method.

6.1 Scalar power spectrum

The Boltzmann equations for the perturbation in photon intensity and polarization are given by (for details check Appendix B)

$$ \frac{\partial \Delta_T}{\partial \tau} + i k \mu \Delta_T + \frac{2}{3} \dot{h} + \frac{4}{3} (\dot{\lambda} + 6 \dot{\eta}) P_\lambda(\mu) = \left( \frac{\partial \Delta_T}{\partial \tau} \right)_C \quad (6.2) $$

$$ \frac{\partial \Delta_P}{\partial \tau} + i k \mu \Delta_P = \left( \frac{\partial \Delta_P}{\partial \tau} \right)_C \quad (6.3) $$
The terms on the right hand side are the collision terms due to the Compton scattering and are given by

$$\left(\frac{\partial \Delta_T}{\partial \tau}\right)_C = an_e\sigma_T \left[-\Delta_T + \Delta_{T0} - 4\frac{\dot{\theta}_b}{k} P_1(\mu) - \frac{1}{2} (\Delta_{T2} + \Delta_{P0} + \Delta_{P2}) P_2(\mu)\right]$$  \hspace{1cm} (6.4)$$

and

$$\left(\frac{\partial \Delta_P}{\partial \tau}\right)_C = an_e\sigma_T \left[-\Delta_P + \frac{1}{2} (\Delta_{T2} + \Delta_{P0} + \Delta_{P2}) (1 - P_2(\mu))\right]$$  \hspace{1cm} (6.5)$$

where \(n_e\) is the proper mean number density of free electrons and \(\mu = \hat{v}_e \cdot \hat{k}\). Both the Eq. 6.2 and Eq. 6.3 are of the form

$$\frac{\partial Y}{\partial \tau} + (an_e\sigma_T + i\mu) Y = \varOmega(\tau), \quad Y \in (\Delta_T, \Delta_P)$$  \hspace{1cm} (6.6)$$

which can be solved as

$$Y = e^{-\int \varPi(\tau) \, d\tau} \int e^{\int \varPi(\tau) \, d\tau} \varOmega(\tau) \, d\tau$$  \hspace{1cm} (6.7)$$

where \(\varPi = an_e\sigma_T + i\mu\). By solving the temperature and the polarization perturbations, we get

$$\Delta_T(\tau_0, k, \mu) = \int_{0}^{\tau_0} d\tau e^{ik\mu(\tau-\tau_0)} e^{-\kappa} \left[\frac{1}{k} \left(\Delta_{T0} - 4\frac{\dot{\theta}_b}{k} P_1(\mu) - \frac{1}{2} \Pi P_2(\mu)\right) - \frac{2}{3}\dot{\kappa} - \frac{4}{3}(\dot{\theta}_b + 6i\mu) P_2(\mu)\right]$$

$$\Delta_P(\tau_0, k, \mu) = \int_{0}^{\tau_0} d\tau e^{ik\mu(\tau-\tau_0)} e^{-\kappa} \frac{\dot{\kappa}}{2\Pi} (1 - P_2(\mu)).$$  \hspace{1cm} (6.8)$$

Here \(\kappa = \int_{\tau_0}^{\infty} an_e\sigma_T d\tau\) is the optical depth at time \(\tau\). \(\Pi\) is the anisotropic stress term and is given by

$$\Pi = \Delta_{T2} + \Delta_{P2} + \Delta_{P0}.$$  \hspace{1cm} (6.9)$$

In the above expression, i.e. Eq. 6.8, the terms with \(\mu\) can be eliminated by integration by parts and considering that the boundary terms can be dropped, because they will vanish as \(\tau \to 0\) and are unobservable at \(\tau = \tau_0\). We can replace every occurrence of \(\mu\) with \(\frac{1}{k} \frac{d}{\kappa} \varPi\). This gives us the scalar source terms as

$$S_T(k, \tau) = -g \left(\Delta_{T0} + 2\dot{\alpha} + \frac{\dot{\theta}_b}{k^2} + \frac{\Pi}{4} + \frac{3\varPi}{4k^2}\right) + e^{-\kappa} (\dot{\kappa} + \dot{\alpha}) + \dot{\alpha} \left(\frac{\theta_b}{k^2} + \frac{3\varPi}{4k^2}\right) + \frac{3g\varPi}{4k^2};$$

$$S_P(k, \tau) = \frac{3g\varPi(\tau, k)}{4k^2(\tau_0 - \tau)^2}. \hspace{1cm} (6.10)$$

Here \(\alpha = (\dot{\theta} + 6i\mu)/2k^2\) (Zaldarriaga, 1998; Zaldarriaga et al., 1997). For obtaining the polarization term, we need to use the recursive property of the spherical Bessel function and \(\lim_{x \to 0} j_l(x)/x^2 = \frac{1}{l_!}\).

Expanding the \(\Delta_T\) and \(\Delta_P\) in Legendre polynomial and using the property

$$\int_{-1}^{1} \frac{d\mu}{2} P_l(\mu) e^{ik\mu(\tau-\tau_0)} = \frac{1}{(-i)^l} j_l(k(\tau - \tau_0))$$ \hspace{1cm} (6.12)$$

we can express the brightness fluctuation functions for temperature and \(E\) mode polarization as

$$\Delta_T(l) = \int_{0}^{\tau_0} d\tau S_T(k, \tau) j_l(x), \quad \Delta_{El}(l) = \sqrt{\frac{(l + 2)!}{(l - 2)!}} \int_{0}^{\tau_0} d\tau S_P(k, \tau) j_l(x)$$ \hspace{1cm} (6.13)$$

The temperature and the polarization power spectra are given by

$$C_T^{XX} = (4\pi)^2 \int k^2 \, dk \, P^s(k)[\Delta^X_T(k)]^2, \quad C_T^{E} = (4\pi)^2 \int k^2 \, dk \, P^s(k) \Delta^X_T \, \Delta^E_T,$$ \hspace{1cm} (6.14)$$

where, \(X\) can be \(T\) or \(E\). \(P^s(k)\) is the scalar primordial power spectrum set by inflation.
Figure 14. A typical grid for calculating the source functions. For small wave-numbers \( (k) \) we use a logarithmic grid and for large wave numbers we use linear grid. The actual time grid is 10 times denser than the grid shown in the plot. \( \tau_i \) is the point where we set the initial conditions. It is a function for \( k \). However, as \( \tau_{\text{min}} \) is very small, it gives an impression that the \( \tau_{\text{min}} \) is constant. We use denser linear grid during the recombination and reionization. Elsewhere, we use a logarithmic grid.

### 6.2 Tensor power spectrum

The power spectrum for the tensor perturbation can be calculated in a similar manner. However, due to the absence of the reflection symmetry, we have a nonzero \( B \) mode polarization (check Appendix B).

The source functions for the tensor polarization, i.e. \( S_T^T(\tau, k) \), \( S_E^T(\tau, k) \), \( S_B^T(\tau, k) \) are given by (Lin and Wandelt, 2006)

\[
S_T^T(\tau, k) = \left( -\dot{h}_q e^{-\alpha_n \sigma_T} + g \dot{\Psi}_T^i \right) / x^2, \tag{6.15}
\]
\[
S_E^T(\tau, k) = g(\tau) \left( -\dot{\Psi}_T^i + \frac{\ddot{\Psi}_T^i}{k^2} + \frac{6\dot{\Psi}_T^i}{x^2} + \frac{4\dot{\Psi}_T^i}{k x} \right) + \ddot{g}(\tau) \left( \frac{2\dot{\Psi}_T^i}{k^2} + \frac{4\dot{\Psi}_T^i}{k x} \right) + \dot{g}(\tau) \frac{\dot{\Psi}_T^i}{k^2}, \tag{6.16}
\]
\[
S_B^T(\tau, k) = g(\tau) \left( \frac{4\dot{\Psi}_T^i}{x} + \frac{2\dot{\Psi}_T^i}{k} \right) + 2\ddot{g}(\tau) \frac{\dot{\Psi}_T^i}{k}, \tag{6.17}
\]

where, \( x = k(\tau_0 - \tau) \). Once we get the source terms, we can calculate the brightness fluctuation functions. However, as the tensor fluctuations are spin 2 quantities, we get an extra \( \frac{1}{\sqrt{l(l+2)}} \) term in the brightness fluctuation functions. The brightness fluctuation functions for the tensor perturbation are

\[
\Delta_T^l(k) = \sqrt{\frac{(l-2)!}{(l+2)!}} \int_0^{\tau_0} d\tau S_T^l(k, \tau) j_l(x), \quad \Delta_{E,B}^l(k) = \int_0^{\tau_0} d\tau S_{E,B}^l(k, \tau) j_l(x) \tag{6.18}
\]
Figure 15. Plots of some of the scalar perturbation variables, $\delta_c$, $\delta_b$, $\theta_b$, $\delta_\gamma$, $\theta_\gamma$ and $\sigma_\gamma$. The $k-\tau$ grid, used for calculating the source terms, is plotted in the background. CDM is not coupled to any other constituents of the universe, and at late time $\delta_c$ grows exponentially. While plotting we truncate the values of $\delta_c > 50$ for a better visualization. Baryons are coupled to photons during recombination, so we can see the acoustic oscillations in the early epoch. However, after decoupling, baryons follow the CDM. We truncate $\delta_b > 30.0$. We can also see the similar oscillatory features in $\theta_b$, $\delta_\gamma$, $\theta_\gamma$ and $\sigma_\gamma$ during recombination. A smoother grid is used during recombination and reionization to capture these oscillating features in the source terms. For this analysis, we use adiabatic initial conditions with $\Omega_b h^2 = 0.0223$, $\Omega_b h^2 = 0.1188$, $h = 67.74 \text{ km/sec/Mpc}$, $n_s = 0.9667$, $\kappa = 0.08$.

The brightness fluctuation functions can be convolved with the primordial tensor power spectrum to get

$$C^{XX}_l = (4\pi)^2 \int k^2 dk P_t(k) [\Delta_X^l(k)]^2, \quad C^{XY}_l = (4\pi)^2 \int k^2 dk P_t(k) \Delta_X^l \Delta_Y^l,$$

where $(X,Y) \in (T,E,B)$.

7 Numerical calculations

In the previous sections, we discussed all the mathematical equations used in CMBAns. Briefly, the calculation of the power spectrum is briefly done through four steps.

1. Calculate the perturbation variables for different wave numbers at different time. This is done by integrating the set of linear differential equations, given by Eq. 4.18 - Eq. 4.36 (for scalar) and Eq. 4.39 - Eq. 4.49 (for tensor) using a numerical integrator. For integration, we require
the initial conditions which are given by Eq. 5.3, Eq. 5.5, Eq. 5.7, Eq. 5.9, Eq. 5.11 (for different types of scalar initial conditions) and Eq. 5.12 (for tensor initial conditions).

2. Calculate the temperature and polarization source functions given by Eq. 6.10 - Eq. 6.11 (for the scalar case) and Eq. 6.15 - Eq. 6.17 (for the tensor case), using the perturbation variables.

3. Calculate the brightness fluctuation functions by convolving the source terms with the spherical Bessel functions (Eq. 6.13 for scalar and Eq. 6.18 for tensor).

4. Convolve the square of the brightness fluctuation function with the primordial power spectrum to get the $C_l$'s.

For scalar perturbations, there are total $8 + 2 \times (1 + L^e_{\text{max}}) + (1 + L^\nu_{\text{max}}) + N^\nu_q \times (1 + L^\nu_{\text{max}})$ perturbation variables (gravity, CDM, baryon, DE each has 2 equations), where $N^\nu_q$ is the number of discretizations of the momentum of massive neutrinos. For the tensor perturbations, we have $2 + 2 \times (1 + L^t_{\text{max}}) + (1 + L^\nu_{\text{max}})$ perturbation variables (2 comes from gravity $h$ and $\dot{h}$). The initial conditions are set deep inside the radiation dominated era, as discussed in Sec. 5.3.

For integrating the set of perturbation equations, we use a C version of the dverk integrator, originally used in CAMBFAST and CAMB. It is a Runge-Kutta (RK) subroutine based on Verner’s fifth and sixth order pairs of formulae\(^7\) for finding approximations to the solution of a system of first order ordinary differential equations with initial conditions. The integrator can solve non-stiff equations very efficiently.

We calculate the perturbation equations and store the source functions in a $k - \tau$ grid, shown in Fig. 14. The choice of the grid is important for speed and the accuracy of the calculation. For calculating the power spectrum, we need to integrate the brightness fluctuation functions for $k$ from 0 to $\infty$. However, numerically we can’t integrate up to $k \rightarrow \infty$, and thus we take $(k\tau)_{\text{max}}$ as an input to the program. For calculating $C_l$ up to $l_{\text{max}}$, the typical value of $(k\tau)_{\text{max}} \approx 2l_{\text{max}}$. The reason is that if we assume that all the fluctuations occur at the last scattering surface, the angle corresponding to $(k\tau)_{\text{max}}$ at last scattering surface will be approximately $\frac{2\pi}{(k\tau)_{\text{max}}}$, which is roughly equal to $\pi/l$. We choose $k_{\text{max}} = (k\tau)_{\text{max}}/\tau_0$.

At low $k$, we take smaller logarithmic grid spacing. For calculating only the scalar perturbations we use the logarithmic grid spacing, $\delta(\ln k) = 0.2$. However, if tensor perturbations are requested, we use a smaller logarithmic grid size at low $k$, $\delta(\ln k) = 0.1$. This logarithmic grid is smoothly matched with a linear grid with grid spacing $\delta k = 0.8/\tau^r$, where $\tau^r$ is the conformal time difference between the present era and the last scattering surface (the point where the visibility function, $g$, is maximum). The smooth matching can be done by taking a logarithmic grid for $k < \frac{\delta k}{\tau^r_{\text{hor}}}$, and a linear grid otherwise. For $k > \frac{5\pi}{\tau_{\text{iss}}}$, we use an even bigger grid spacing $\delta k = 1.5/\tau^r$, $\tau_{\text{iss}} = \int_0^{\tau_{\text{iss}}} c_\text{d}\tau$ is the sound horizon at the last scattering surface. $\tau_{\text{iss}}$ is the conformal time at the last scattering surface.

The equations for the source terms, i.e. Eq. 6.10 - Eq. 6.17, show that apart from the ISW term, all the other terms are either multiplied with the visibility function $g$ or its temporal derivatives. As the visibility function is nonzero only during recombination and reionization, the source terms are also nonzero only in those eras. The ISW term is non oscillatory and is important throughout the expansion history of the universe.

For specifying the temporal grid, we need a fine grid during the recombination and reionization era. However, in the rest of the universe, we can use a larger grid. The grid is shown in Fig. 14. In Sec. 5.3, we have specified the points, $\tau_i = \min(\tau_h, \tau^{\text{hor}}(k), 0.1)$, where the initial conditions are set. For different $k$ the initial conditions are set at different $\tau_i$. Therefore, for specifying the grid we take $\tau_{\text{min}} = \min(\tau_h, \tau^{\text{hor}}(k_{\text{min}}), 0.1)$. We mark $\tau_i$ with a red-line in Fig. 14.

\(^7\)Runge-Kutta pairs: For the solution of initial value problems, the step-size is allowed to vary by estimating the error produced in each step. To achieve this, it is standard practice to build method pairs, based on the same stages which produce an output answer of order $p$ and a second approximation of order $q$, where $q > p$. The difference of these two approximations will give an asymptotically correct estimate of the error in the output value. As for small $h$, the actual local error is approximately proportional to $h^{p+1}$, and the step size in the following step can be chosen to give a value close to that specified as a user tolerance.
The recombination start time is calculated by checking when $\int g'(\tau) d\tau > 10^{-12}$. To avoid any error, we consider $\frac{9}{10}$ of that time as the starting time for the recombination grid, i.e. $\tau_{\text{start}} = 9\tau_f - 10^{12}$. $\tau_{\text{stop}}$ is shown in the figure using a thick black line. We define the visibility function $g'(\tau)$ as $g'(\tau) = \kappa \exp(\int_{\tau_{\text{min}}}^{\tau_{\text{start}}} \kappa d\tau)$. The end of the recombination epoch is defined as the point when $\int g'(\tau) d\tau > 0.9999$, i.e. $\tau_{\text{stop}} = \tau_f^g g'(\tau) d\tau > 0.9999$. $\tau_{\text{stop}}$ is shown using a thin black line. After recombination, we want to smoothly change the grid to a logarithmic grid. This is done by redefining the end point of the recombination grid as $\tau^g = \max(\tau_{\text{stop}}, \delta \tau/\delta \ln \tau)$. $\delta \tau$ is the linear grid spacing during the recombination era and $\delta \ln \tau$ is the logarithmic grid spacing after recombination. There may be cases where the reionization starts before $\tau^g$. To account for those models, we have taken $\tau^g = \min(\tau^g, \tau_{ri}^{\text{start}})$. For obtaining a smooth reionization, we join the ionization fraction before and after reionization using a hyperbolic tangent function. Given the optical depth to the last scattering surface, we find the reionization start redshift by calculating $\kappa = \int_{\tau^{\text{start}}}^{\tau^g} a \sigma_T d\tau$. The reionization end redshift is taken as $0.82\tau^{\text{start}} - 1$. At the end of reionization, we again smoothly match the grid with the logarithmic grid and make proper adjustment at the reionization end redshift to smoothly transform the grid to the logarithmic grid. The reionization grid start and end points are shown using the green line in Fig. 14.

In Fig. 15, we show some of the scalar perturbation variables, plotted over the grid. All the oscillatory features are located near the low $k\tau$. At high $k$ or high $\tau$, fluctuations in the perturbation variables are very small. In Fig. 16, we show the scalar source terms. Most of the structures in the source terms are concentrated near recombination and the reionization. In the rest of the places, the source terms are almost 0, except that the ISW term will be present in the scalar temperature source term. As the ISW effect is non-oscillatory and much smaller in comparison with the SW and the velocity term, it is not clearly visible in the plot. This justifies the choice of the smooth grid during recombination and reionization.

After calculating the source functions, we convolve the source terms with the spherical Bessel functions for calculating the brightness fluctuation functions, $\Delta T_l(k)$ and $\Delta P_l(k)$. Instead of calculating the brightness fluctuation functions for each and every multipole $l$, we choose some specific multipoles, and calculate the brightness fluctuation functions and then the $C_l$’s in those specific $l$’s. Later we interpolate the $C_l$’s to get the power spectrum at every multipole. We pre-compute the spherical Bessel functions in these particular $l$’s in a suitable linear grid. For calculating the brightness fluctuation functions, we interpolate the source functions for each $k$ into the spherical Bessel function grid using spline interpolation. Then we integrate it over $\tau$ using the trapezoidal rule.

The square of the brightness fluctuations are multiplied with the primordial power spectrum, $P(k)$ for calculating the $C_l$’s in the specific multipoles, which are then interpolated to all the $C_l$’s using spline interpolation. We use COBE normalization techniques for normalizing the $C_l$ (Bunn...
8 Conclusion

We develop a cosmological Boltzmann package for fast and accurate calculation of the CMB power spectrum for a flat ($\Omega_k = 0$) background cosmology. In this paper, we discuss all the equations and the approximation schemes and different truncation conditions used in \texttt{CMBAns}. The lensing calculations and the comparison of the results with other Boltzmann packages like \texttt{CAMB}, \texttt{CLASS} etc. are not discussed in this paper and are left for future papers in this series.

\texttt{CMBAns} was initially written in 2010 and used as an internal software package. However the program has been restructured and several new features have been added since then. We have tested \texttt{CMBAns} for a wide range of different initial conditions, Hubble’s parameters, $\Omega_b$, $\Omega_c$ etc. to check for the robustness of the program.

In this paper, we show that if the truncation conditions for neutrinos and the photons are not chosen properly, then they may lead to a large (up to a few percent) propagation error in the CMB power spectrum calculation. Uncontrolled error in the power spectrum calculations can finally lead to huge errors in the parameter estimation. In the tight coupling era, it’s difficult to integrate the baryon and photon equations separately and may introduce high numerical error. Therefore, we have taken a separate tight coupling equation.

\texttt{CMBAns} can calculate the CMB power spectrum for perturbed and unperturbed dark energy models etc. \texttt{CMBAns} is tested for different dark energy models. How different dark energy models can affect the ISW effect are discussed in detail in Das and Souradeep (2014a); Das et al. (2013).

In \texttt{CMBAns}, we use a power law for the primordial power spectra. Users can change the scalar spectral index, $n_s$, running of scalar spectral index, $\alpha_s$ and its running, $\frac{d\alpha_s}{d \ln k}$. However, the nature of the primordial power spectra can be modified easily. It has been tested for two field inflation, double inflation, perturbed power law model etc. (Mukherjee et al., 2015) Default tensor spectral index, and scalar to tensor ratio are taken as $n_t = n_s - 1$ and $r = 7 \times (1 - n_s)$, which can be changed to any values or modify the model of power spectrum model.

We use the C programming language for \texttt{CMBAns}. However, to make the program object oriented, we use the concept of class from C++. Several stand alone codes, such as calculating the recombination history, power spectra evolution with different cosmological parameters, Bessel function calculation etc. are provided with the package. However, users are not limited to what already come with the program. The influx of precision CMB data means that CMB modeling tools must quickly evolve. Modularity, an important feature of \texttt{CMBAns}, offers a way to solve this problem. The modularity of \texttt{CMBAns} offers a lot of flexibility and let users quickly expand the functionalities of the package to include new cosmological models, by simply writing a new module or classes using the functionality provided in \texttt{CMBAns}.

Appendices

A Baryon temperature calculation

The baryons and the photons were coupled in the early universe, mostly due to Compton scattering. Therefore, in the very early universe, the temperature of baryon and photons were equal, i.e. $T_b = T_\gamma$. After decoupling, the baryon’s temperatyre slowly fall. The baryon in the universe was subjected to various sources of heating and cooling (Hirata, 2008). However, in the context of CMB, the important heating and cooling mechanisms are adiabatic cooling and Compton heating of baryons.
A.1 Adiabatic cooling

Due to the expansion of the universe, the photons and the baryons both undergo the adiabatic cooling. For photons the wavelength will increase as the universe expands, i.e. \( \lambda_\gamma \propto a \). As for photons \( T_\gamma \propto \frac{1}{\lambda_\gamma} \), we have \( T_\gamma \propto \frac{1}{a} \).

However, for baryons, \( T_b \propto \frac{E_K}{m_b} \), where \( E_K \) is the kinetic energy of the baryons. The de Broglie wavelength of the baryons varies in proportion to \( a \). Therefore, for baryons \( T_b \propto \frac{1}{a^2} \) provided there is no external heating or cooling.

Therefore, in an adiabatic condition, we get

\[
\dot{T}_b = -2 \left( \frac{\dot{a}}{a} \right) T_b
\]

(A.1)

A.2 Compton heating

In the early universe, the main source of external heating of baryons is Compton scattering.

\[
e^- + \gamma \rightarrow e^- + \gamma
\]

(A.2)

A.2.1 Heating the electrons:

Let’s assume that a photon with momentum \( \omega \hat{i} \) hits an electron at rest and is deflected to \( \omega \cos(\theta) \hat{i} + \omega \sin(\theta) \hat{j} \). Hence, the momentum transfer to the electron is \( q = \omega \sqrt{1 - \cos^2 \theta} \hat{i} + \omega \sin \theta \hat{j} \). Here we assume that the change in energy of the photons is very small and the absolute value of the momentum remains almost the same before and after the collision.

The energy delivered to the electron is

\[
\Delta E = \frac{q^2}{2m_e} = \frac{\omega^2}{m_e} (1 - \cos \theta) \quad \Rightarrow \quad \langle \Delta E \rangle = \frac{\langle \omega^2 \rangle}{m_e}.
\]

Here, we assume that the electron is stationary at the beginning and the photons are isotropic around it. Therefore, the collision of photons with the electrons will be equally likely from all directions and the \( \cos \theta \) term will vanish.

The heating rate of the electrons will be

\[
\Gamma = n_e n_\gamma \sigma_T \langle \Delta E \rangle = n_e n_\gamma \sigma_T \frac{\langle \omega^2 \rangle}{m_e},
\]

(A.4)

where \( \sigma_T \) is the Thomson scattering cross section and \( n_e \) and \( n_\gamma \) are the number density of the electrons and photons. For simplifying the expressions, we take \( c = \hbar = k_B = 1 \).

A.2.2 Energy loss by the electron:

In the above calculation, we assume that the electrons are at rest. However, as the electrons have a temperature they cannot be at rest. Due to their motion, they will give away some energy to the photons via Compton drag.

If an electron is moving at a speed \( v_e \) along the x-direction, then in its rest frame the photons will have some net momentum. In a comoving frame, the photons stress energy tensor is \( T^{\mu\nu} = \text{diag} \left( \rho_\gamma, \frac{1}{3} \rho_\gamma, \frac{1}{3} \rho_\gamma, \frac{1}{3} \rho_\gamma \right) \).

In the comoving frame the electron’s 4-velocity is \( u^\mu = \frac{1}{\sqrt{1-v_e^2}} (1, v_e, 0, 0) \) and it carries three spatial vectors

\[
(e_1)^\mu = \frac{1}{\sqrt{1-v_e^2}} (v_e, 1, 0, 0), \quad (e_2)^\mu = \frac{1}{\sqrt{1-v_e^2}} (0, 0, 1, 0), \quad (e_3)^\mu = \frac{1}{\sqrt{1-v_e^2}} (0, 0, 0, 1).
\]

(A.5)

Therefore, in the electron’s frame, the momentum density of the photons is

\[
\dot{j}_\gamma = -T_{\mu\nu} u^\mu (e_1)^\nu = -\frac{4v_e}{3(1-v_e^2)} \approx -\frac{4}{3} v_e.
\]

(A.6)
Due to the overall velocity of the photons, the electrons will feel some force

\[ F = n_\gamma \sigma_T \langle p_\gamma \rangle, \quad (A.7) \]

where \( \langle p_\gamma \rangle \) is the average momentum of the photons with respect to electron. However, \( n_\gamma \langle p_\gamma \rangle = j_\gamma \), is the photon momentum density. This gives,

\[ F = \sigma_T j_\gamma = -\frac{4}{3} \sigma_T \rho_\gamma v_e = -\frac{4}{3} \sigma_T n_\gamma \langle \omega \rangle v_e. \quad (A.8) \]

As the energy loss by the electron is given by \( -F \cdot v_e \), the net energy loss rate is

\[ \Lambda = -n_e \langle F \cdot v_e \rangle = 4 \frac{\sigma_T}{3} n_e n_\gamma \langle \omega \rangle \langle v_e^2 \rangle \quad (A.9) \]

where \( \langle v_e^2 \rangle = \frac{3}{2} T_b/m_e \), assuming that the electrons follow a Maxwell distribution.

### A.2.3 Heating from stimulated Compton effect

The third process that will heat the photons is the stimulated Compton effect. In any radiative process, the ratio of the stimulated to the spontaneous transition rate is equal to the ambient phase density of photons in the final stage, \( f(\omega) \). A scattering process can be thought of as absorption and emission of a photon (Dreicer, 1964; Gould, 1972). We have seen that the total amount of energy that the photons are emitting is given by

\[ \Gamma = n_e n_\gamma \sigma_T \langle \omega^2 \rangle = n_e \sigma_T \int \omega f(\omega) d\omega \quad (A.10) \]

The total amount of stimulated radiation will be

\[ \Gamma_{stim} = \frac{n_e \sigma_T}{h} \int \omega^2 f(\omega) f(\omega) d\omega = n_e n_\gamma \sigma_T \langle \omega^2 f(\omega) \rangle \quad (A.11) \]

Here \( f(\omega) \) is the phase space distribution of photons. For a blackbody it will follow a Plankian distribution.

### A.3 Total heating of electrons

We assume that the universe behaves as a perfect blackbody. Therefore, the expectation values of the previous expressions are be given by

\[ n_\gamma = \frac{2\zeta(3)}{\pi^2} T_\gamma^3 \quad \langle \omega \rangle = \frac{\pi^4}{30\zeta(3)} T_\gamma \quad \langle \omega^2 \rangle = \frac{12\zeta(5)}{\zeta(3)} T_\gamma^2 \quad \langle \omega^2 f(\omega) \rangle = \frac{4T_\gamma^2}{30\zeta(3)} \left[ \pi^4 - 90\zeta(5) \right] . \]

Replacing all these values we can get the energy that the electrons will receive from photon

\[ \Gamma + \Gamma_{stim} - \Lambda = \frac{4\pi^2}{15} n_e T_\gamma^4 \sigma_T \frac{T_\gamma - T_b}{m_e} \quad (A.12) \]

The specific heat for mono-atomic gas at constant volume is \( C_v = \frac{5}{2} n \), where \( n \) is the total number density of the particles containing free electron, H, H+, He, He++, H+++. Therefore, the rate of change of the temperature can be calculated by \( \Gamma + \Gamma_{stim} - \Lambda = C_v \dot{T}_b \).
B CMB Polarization Calculation

The equation for the photons perturbation are more complicated than the neutrinos due to their scattering. The scattering can change the polarization of photons, so we can’t write separate equation for different helicity states of photons. Instead we need to consider the perturbation in the $2 \times 2$ density matrix of the photons, i.e. $\Delta \rho_{ij}(\hat{n})$. Here $\Delta \rho_{ij}(\hat{n})$ is considered to be normalized over the mean intensity $\rho_0$. In Eq. 4.15, we show the first order perturbation of the Boltzmann equation for photons and neutrinos. However, instead of $\Phi$ we will now have $\Delta \rho_{ij}$. This complicates the equation for the photon perturbation (Bond and Efstathiou, 1984; Kamionkowski et al., 1997a,b; Kosowsky, 1995; Zaldarriaga, 1998; Zaldarriaga and Seljak, 1997).

The density perturbation matrix is a function of the direction on the sky ($\hat{n}$) and its two perpendicular direction $\hat{e}_x$ and $\hat{e}_y$. We can rewrite these perturbations in terms of the perturbation in the four Stokes parameters $\Delta I(\hat{n})$, $\Delta Q(\hat{n})$, $\Delta U(\hat{n})$ and $\Delta V(\hat{n})$. They can be related as

$$\Delta I(\hat{n}) = \frac{1}{2} [\Delta \rho_{11}(\hat{n}) + \Delta \rho_{22}(\hat{n})] \quad \Delta Q(\hat{n}) = \frac{1}{2} [\Delta \rho_{11}(\hat{n}) - \Delta \rho_{22}(\hat{n})]$$
$$\Delta U(\hat{n}) = \frac{1}{2} [\Delta \rho_{12}(\hat{n}) + \Delta \rho_{21}(\hat{n})] \quad \Delta V(\hat{n}) = \frac{1}{2i} [\Delta \rho_{12}(\hat{n}) - \Delta \rho_{21}(\hat{n})]$$

(B.1)

If we rotate the coordinate perpendicular to $\hat{n}$ by an angle $\phi$, then $\Delta Q$ and $\Delta U$ will transform as

$$\Delta Q'(\hat{n}) = \Delta Q(\hat{n}) \cos(2\phi) + \Delta U(\hat{n}) \sin(2\phi)$$
$$\Delta U'(\hat{n}) = -\Delta Q(\hat{n}) \sin(2\phi) + \Delta U(\hat{n}) \cos(2\phi)$$

(B.2)

However, $\Delta I$ and $\Delta V$ remains invariant under such rotation. The above equations also show that $\Delta^2 Q + \Delta^2 U$ remains invariant under such coordinate transformation. Also, for unpolarized light $\Delta \rho_{11} = \Delta \rho_{22}$ and $\Delta \rho_{12} = \Delta \rho_{21} = 0$.

From Eq. B.2, we can see that the $Q$ and $U$ components are direction-dependent quantities, i.e. they depend on $\hat{e}_x$ and $\hat{e}_y$ direction of space. Instead of taking $Q$ and $U$ component independently, if we form two complex quantities $(\Delta Q \pm i \Delta U)$, then they will transform as spin $\pm 2$ quantities, i.e.

$$(\Delta Q \pm i \Delta U)(\hat{n}) = \exp(\pm i2\phi)(\Delta Q \pm i \Delta U)(\hat{n}) .$$

(B.3)

In general, the intensity (temperature) field being a scalar field can be expanded in terms of the spherical harmonics as

$$\Delta I(\hat{n}) = \sum_{lm} a'_{lm} Y_{lm}(\hat{n}) .$$

(B.4)

However, as the $\Delta Q \pm \Delta U$ behave as spin $\pm 2$ quantities, we have to expand them in spin-weighted spherical harmonics. This can be written as

$$(\Delta Q + i \Delta U)(\hat{n}) = \sum_{lm} 2a_{lm} Y_{lm}(\hat{n})$$
$$(\Delta Q - i \Delta U)(\hat{n}) = \sum_{lm} -2a_{lm} Y_{lm}(\hat{n}) .$$

(B.5)

We can use the spin raising ($\Theta$) and lowering ($\bar{\Theta}$) operators to construct some spin 0 quantities as

$$\Theta^2 (\Delta Q + i \Delta U)(\hat{n}) = \sum_{lm} 2a_{lm} \Theta^2 Y_{lm}(\hat{n}) = \sum_{lm} \left( \frac{(l + 2)!}{(l - 2)!} \right) \frac{1}{2} 2a_{lm} Y_{lm}(\hat{n})$$
$$\bar{\Theta}^2 (\Delta Q - i \Delta U)(\hat{n}) = \sum_{lm} -2a_{lm} \bar{\Theta}^2 Y_{lm}(\hat{n}) = \sum_{lm} \left( \frac{(l + 2)!}{(l - 2)!} \right) \frac{1}{2} -2a_{lm} Y_{lm}(\hat{n})$$

(B.7)
In the CMB literature, conventionally people use the two scalar fields $E$ and $B$ to represent the polarization, given by

$$\Delta_E(\hat{n}) = -\frac{1}{2} \left[ \vec{\phi}^2 (\Delta_Q + i \Delta_U)(\hat{n}) + \vec{\phi}^2 (\Delta_Q - i \Delta_U)(\hat{n}) \right]$$

$$= \sum_{lm} a_{lm}^{E} Y_{lm}(\hat{n}) = \sum_{lm} \left( -\frac{1}{2i} \right) (2a_{lm} + 2a_{lm}) Y_{lm}(\hat{n})$$

(B.9)

$$\Delta_B(\hat{n}) = -\frac{1}{2} \left[ \vec{\phi}^2 (\Delta_Q + i \Delta_U)(\hat{n}) - \vec{\phi}^2 (\Delta_Q - i \Delta_U)(\hat{n}) \right]$$

$$= \sum_{lm} a_{lm}^{B} Y_{lm}(\hat{n}) = \sum_{lm} \left( -\frac{1}{2i} \right) (2a_{lm} - 2a_{lm}) Y_{lm}(\hat{n}) .$$

Scalar Components

We can calculate the perturbation equations for $\Delta_{\rho ij}$ and then use Eq. B.1 to calculate the perturbation in the Stokes parameters. If we convert the scalar part of these perturbations in Stokes parameters in Fourier space, then we can get (Kosowsky, 1995)

$$\frac{\partial \Delta_I}{\partial \tau} + i k \mu \Delta_I + \frac{2}{3} h^2 + \frac{4}{3} (h + 6i) P_2(\mu) = -a_{\sigma T} \left[ \Delta_I - \Delta_{I0} - 4 \frac{i \theta h}{k} P_3(\mu) - \frac{1}{2} (\Delta_I - \Delta_Q0 + \Delta_Q2) P_2(\mu) \right]$$

$$\frac{\partial \Delta_Q}{\partial \tau} + i k \mu \Delta_Q = -a_{\sigma T} \left[ \Delta_Q + \frac{1}{2} (1 - 2 P_2(\mu)) (\Delta_I + \Delta_Q2 - \Delta_Q0) \right]$$

$$\frac{\partial \Delta_U}{\partial \tau} + i k \mu \Delta_U = -a_{\sigma T} \Delta_U$$

$$\frac{\partial \Delta_V}{\partial \tau} + i k \mu \Delta_V = -a_{\sigma T} \left[ \Delta_V - \frac{3}{2} \mu \Delta V1 \right] .$$

(B.11)

The $\Delta$s are the functions of $(k, \tau)$. In the early universe, the photons were tightly coupled, so we can consider that the photons were not polarized at that early time. The above equation also shows that the $\Delta_U$ and $\Delta_V$ term don’t have any source terms. If they were zero in the early universe, then they remain 0 afterwards. As the $\Delta_U$ term is related to $\Delta_Q$ by coordinate transformation, we get nonzero $\Delta_U$ in late universe. However, $\Delta_V$ still remains 0. This is also valid for the tensor perturbation.

There is another important property of the scalar perturbations. Let us consider a particular $k$ mode in the scalar perturbation in $\hat{k}$ direction. Now the density field produced by the single mode will have two important symmetry - azimuthal symmetry and reflection symmetry. The azimuthal symmetry implies that neither the temperature nor the stokes parameters depend on rotation around $\hat{k}$.

Let’s consider $\hat{e}_\theta$ and $\hat{e}_\phi$ as two perpendicular directions along $\hat{k}$. Under reflection $\hat{e}_\theta \rightarrow -\hat{e}_\theta$ and $\hat{e}_\phi \rightarrow -\hat{e}_\phi$. Eq. B.2 implies that under reflection, $\Delta_Q \rightarrow \Delta_Q$, i.e. it remains unchanged while $\Delta_U$ changes sign. Therefore, in linear theory, the scalar polarization can’t have the $\Delta_U$ component, and the only polarization component that we can have is $\Delta_Q$. In terms of the $E$ and $B$ field, Eq. B.10 shows that $B$ will be zero and we can have only the $E$ field. As $\Delta_P = \Delta_Q$ ($\Delta_U$ being 0), $\Delta_P$ will be given by Eq. B.11, and $\Delta_{EI} = \left( \frac{(l+2)!}{(l-2)!} \right)^{\frac{1}{2}} \Delta_{Pl}$

Tensor Components

The tensor components will have two polarisation, $\Delta^+$ and $\Delta^\times$. As we have done for the scalar part, we can also calculate the perturbation equations for $\Delta_{\rho ij}$ and then if we convert the tensor part of these perturbations in Stokes parameters in Fourier space, we can get the components of the tensor perturbations. The evaluation equations for the tensor perturbations can be simplified if we apply the following variable transformation (Basko and Polnarev, 1980; Crittenden et al., 1993; Crittenden et al., 1993; Polnarev, 1985)
\[
\begin{align*}
\Delta^+_I &= (1 - \mu^2) \cos(2\phi) \tilde{\Delta}^+_I & \Delta^+ = (1 - \mu^2) \sin(2\phi) \tilde{\Delta}^+_I \\
\Delta^+_Q &= (1 - \mu^2) \cos(2\phi) \tilde{\Delta}^+_Q & \Delta^+_Q = (1 - \mu^2) \sin(2\phi) \tilde{\Delta}^+_Q \\
\Delta^+_U &= 2 \mu \sin(2\phi) \tilde{\Delta}^+_U & \Delta^+_U = 2 \mu \cos(2\phi) \tilde{\Delta}^+_U
\end{align*}
\] (B.12)

In terms of these new variables, the evaluation equations takes the form

\[
\begin{alignat}{2}
\frac{\partial \tilde{\Delta}^+_I}{\partial \tau} + ik\mu \tilde{\Delta}^+_I - 2 \frac{\partial h^+_I}{\partial \tau} &= -an_e \sigma_T (\tilde{\Delta}^+_I + \tilde{\Lambda}^+) & \quad (B.13) \\
\frac{\partial \tilde{\Delta}^+_Q}{\partial \tau} + ik\mu \tilde{\Delta}^+_Q - 2 \frac{\partial h^+_Q}{\partial \tau} &= -an_e \sigma_T (\tilde{\Delta}^+_Q - \tilde{\Lambda}^+) & \quad (B.14) \\
\frac{\partial \tilde{\Delta}^+_U}{\partial \tau} + ik\mu \tilde{\Delta}^+_U &= -an_e \sigma_T \tilde{\Lambda}^+ & \quad (B.15) \\
\frac{\partial \tilde{\Delta}^+}{\partial \tau} + ik\mu \tilde{\Delta}^+ &= -an_e \sigma_T \tilde{\Delta}^+ & \quad (B.16)
\end{alignat}
\]

where

\[
\tilde{\Lambda}^+ = -\frac{3}{70} \tilde{\Delta}^+_{14} + \frac{1}{7} \tilde{\Delta}^+_{12} - \frac{1}{10} \tilde{\Delta}^+_{10} + \frac{3}{70} \tilde{\Delta}^+_{24} + \frac{6}{7} \tilde{\Delta}^+_{22} + \frac{3}{5} \tilde{\Delta}^+_{20}
\] (B.17)

The \times\ polarization also give the similar equations. Here also we can see that the \( V \) perturbations don’t have any source terms. Therefore, they will remain 0 throw out the history of the universe. Also for tensor case, both the \( \Delta^{+}_{Q,\times} \) and \( \Delta^{+}_{U,\times} \) will be nonzero due to the lack of reflection symmetry.

We can convert the \( Q \) and \( U \) components to \( E \) and \( B \) modes using same technique as discussed in Eq. B.10. The contribution from both the \(+\) and \( \times \) mode will be same and the total contribution will be the sum of the quantities. However, each of \( \Delta^{+}, \Delta^{\times} \) quantities are spin ±2 quantities. Therefore, we have to multiply with the spin raising and lowering operators, which eventually multiply all the \( \Delta^l \)’s with a factor of \( \left( \frac{(l+2)!}{(l-2)!} \right)^{1/2} \). This gives the source terms used in Sec. 6.1.

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