Splitting theorem for sheaves of holomorphic $k$-vectors on complex contact manifolds

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Abstract. A complex contact structure $\gamma$ is defined by a system of holomorphic local 1-forms satisfying the completely non-integrability condition. The contact structure induces a subbundle $\text{Ker} \, \gamma$ of the tangent bundle and a line bundle $L$. In this paper, we prove that the sheaf of holomorphic $k$-vectors on a complex contact manifold splits into the sum of $\mathcal{O}(\bigwedge^k \text{Ker} \, \gamma)$ and $\mathcal{O}(L \otimes \bigwedge^{k-1} \text{Ker} \, \gamma)$ as sheaves of $C$-module. The theorem induces the short exact sequence of cohomology of holomorphic $k$-vectors, and we obtain vanishing theorems for the cohomology of $\mathcal{O}(\bigwedge^k \text{Ker} \, \gamma)$.

1 Introduction

Originating in physics, contact geometry is a mathematical formulation of classical mechanics. Contact geometry describes a geometric structure which appears in any constant energy hypersurface in the even-dimensional phase space of a mechanical system. In mathematics, the concept of contact structure appears explicitly in the work of Sophus Lie, and implicitly perhaps much earlier. By using a sheaf coefficient cohomology theory, Gray developed the idea and introduced a concept of almost contact structure [5]. He considered the deformation of a global contact structure in the terminology of homological algebra. Boothby and Wang studied the homogeneous manifolds associated with the contact transformation group [2]. Furthermore, Kobayashi introduced the complex contact structure and developed several results of complex contact geometry [6]. The complex contact structure is associated with a quaternionic structure with respect to the twistor correspondence [3] [10] [8]. From the beginning of the study of contact structures, it has been known that contact structures have a deep relationship with sheaf cohomology, for example, Gray' work.

Let $M$ be a complex manifold of dimension $2n + 1$. We consider a system $\{(U_i, \gamma_i)\}$ of an open covering $\{U_i\}$ of $M$ and holomorphic 1-forms $\gamma_i$ on $U_i$ such that $\gamma_i$ is a contact 1-form, that is, $(d\gamma_i)^n \wedge \gamma_i \neq 0$ on $U_i$, and $\gamma_i = f_{ij} \gamma_j$ for a holomorphic function $f_{ij}$ on $U_i \cap U_j$. We say that such systems $\{(U_i, \gamma_i)\}$ and $\{(U_i', \gamma_i')\}$ are equivalent if there exists a holomorphic function $g_{i'i}$ on each intersection $U_i \cap U_i'$ so that $\gamma_i = g_{i'i} \gamma_i'$ on $U_i \cap U_i'$, and call an equivalent class of $\{(U_i, \gamma_i)\}$ a complex contact structure on $M$. We denote by $\gamma = \{(U_i, \gamma_i)\}$ the contact structure on $M$. A pair $(M, \gamma)$ is called a complex contact manifold. The contact structure $\gamma$ induces a line bundle $L$ on $M$ by the transition function...

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{f_{ij}}$. The canonical bundle $K_M$ is equal to $L^{-n-1}$ since $L^{n+1} \otimes \bigwedge^{2n+1} T^* \cong \mathbb{C}$ by the global section $(d\gamma_i)^{n+1} \wedge \gamma_i$. The contact structure $\gamma$ is an $L$-valued 1-form on $M$. We regard $\gamma$ as a bundle map from the holomorphic tangent bundle $T$ of $M$ to $L$ and denote by $\text{Ker} \gamma$ the kernel of the map $\gamma$:

$$\text{Ker} \gamma = \{ v \in T \mid \gamma(v) = 0 \}.$$  

There exists a short exact sequence of sheaves:

$$0 \to \mathcal{O}(\text{Ker} \gamma) \to \mathcal{O}(T) \xrightarrow{\gamma} \mathcal{O}(L) \to 0.$$  

Let $X$ be a holomorphic vector field on $M$. The Lie derivative $L_X \gamma_i$ of $\gamma_i$ with respect to $X$ is a 1-form on $U_i$. The set $\{L_X \gamma_i\}$ is not a global form on $M$. However, the restriction $L_X \gamma_i|_{\text{Ker} \gamma}$ to $\text{Ker} \gamma$ defines a global section $\{L_X \gamma_i|_{\text{Ker} \gamma}\}$ of the tensor $L \otimes \text{Ker} \gamma^*$ of $L$ and the dual of $\text{Ker} \gamma$. We call a vector field $X$ a contact vector field of $\gamma$ if $L_X \gamma_i|_{\text{Ker} \gamma} = 0$ for each $i$. The system of equations $\{L_X \gamma_i|_{\text{Ker} \gamma} = 0\}$ is a global and holomorphic equation for $X$. Such a vector field generates a contact automorphism. We define $\text{aut}(M, \gamma)$ to be the set of contact vector fields of $\gamma$. As an analogy of the real contact structures \cite{7}, Nitta and Takeuchi showed that for any element $s$ of $\mathcal{O}(L)$, there exists a unique contact vector field $X$ of $\gamma$ such that $\gamma(X) = s$. Moreover, the correspondence $\mathcal{O}(L) \to \text{aut}(M, \gamma)$ is isomorphic \cite{11}. It means that the holomorphic tangent sheaf $\mathcal{O}(T)$ splits into $\mathcal{O}(\text{Ker} \gamma)$ and $\mathcal{O}(L)$ as sheaves of $\mathbb{C}$-module. LeBrun showed that $\mathcal{O}(T)$ does not split into the sum of $\mathcal{O}(\text{Ker} \gamma)$ and $\mathcal{O}(L)$ as sheaves of $\mathcal{O}$-module on Fano manifolds \cite{5}. Therefore, the splitting of $\mathcal{O}(T)$ does not directly induce that of the sheaf $\mathcal{O}(\bigwedge^k T)$ of $k$-vectors.

In this paper, we show the splitting of $\mathcal{O}(\bigwedge^k T)$ into the sum of $\mathcal{O}(\bigwedge^k \text{Ker} \gamma)$ and $\mathcal{O}(L \otimes \bigwedge^{k-1} \text{Ker} \gamma)$ as sheaves of $\mathcal{O}$-module. We extend the map $\gamma$ to a bundle map from $\bigwedge^k T$ to $L \otimes \bigwedge^{k-1} \text{Ker} \gamma$ by

$$v_1 \wedge \cdots \wedge v_k \mapsto \sum_{j=1}^k (-1)^{j-1} \gamma(v_j) \otimes v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k$$

and denote the map also by $\gamma$ for simplicity. Since the kernel of the map $\gamma$ is just the space $\bigwedge^k \text{Ker} \gamma$, we obtain the short exact sequence of the sheaves:

$$0 \to \mathcal{O}(\bigwedge^k \text{Ker} \gamma) \to \mathcal{O}(\bigwedge^k T) \xrightarrow{\gamma} \mathcal{O}(L \otimes \bigwedge^{k-1} \text{Ker} \gamma) \to 0.$$  

We shall extend to the equation $L_X \gamma_i|_{\text{Ker} \gamma} = 0$ for a 1-vector $X$ to a global and holomorphic equation for $k$-vectors. Let $\nabla$ be a connection of the line bundle $L$ such that $\nabla^{0,1} = \bar{\partial}$. For a 1-vector $X$, the local equation $L_X \gamma_i|_{\text{Ker} \gamma} = 0$ is given by the global equation $(d^c \nabla \gamma)(X) + d^c(\gamma(X))|_{\text{Ker} \gamma} = 0$ which is holomorphic. The direct extension $(\bigotimes^k d^c \nabla \gamma)(X) + d^c(\bigotimes^{k-1} d^c \nabla \gamma(\gamma(X)))|_{\text{Ker} \gamma} = 0$ for a $k$-vector $X$ is global but not holomorphic whenever $\nabla$ is holomorphic. In order to find a global and holomorphic equation for holomorphic $k$-vector fields, we decompose the space $\bigwedge^k \text{Ker} \gamma$ as a sum of primitive parts with respect to the symplectic structure on $\text{Ker} \gamma$. Then we obtain such a equation which induces the following splitting theorem for $\mathcal{O}(\bigwedge^k T)$:

**Theorem 1.1.** Let $k$ be an integer from $1$ to $2n + 1$. The sequence

$$0 \to \mathcal{O}(\bigwedge^k \text{Ker} \gamma) \to \mathcal{O}(\bigwedge^k T) \to \mathcal{O}(L \otimes \bigwedge^{k-1} \text{Ker} \gamma) \to 0$$
splits as sheaves of \( \mathbb{C} \)-module. In particular, the sequence

\[
0 \to H^i(\bigwedge \ker \gamma) \to H^i(\bigwedge \mathbb{T}) \to H^i(L \otimes \bigwedge \ker \gamma) \to 0
\]

is exact for each \( i = 0, \ldots, 2n + 1 \).

We generalize the theorem to the splitting of \( \mathcal{O}(L^m \otimes \bigwedge^k \mathbb{T}) \) and the exact sequence of \( H^i(L^m \otimes \bigwedge^k \mathbb{T}) \) under a condition for \( m \) and \( k \) (see Theorem 3.3). As an application, we obtain the following vanishing theorem for \( H^i(L^m \otimes \bigwedge^k \ker \gamma) \) on compact Kähler manifolds by Kodaira-Akizuki-Nakano vanishing theorem:

**Theorem 1.2.** If \( M \) is a compact Kähler complex contact manifold with \( c_1(M) > 0 \), then \( H^i(M, L^m \bigwedge^k \ker \gamma) = \{0\} \) for \( k \) and \( m \) satisfying one of following three conditions

\[
\begin{align*}
    i &\leq 2n - k, & m &\leq -\left[\frac{k+1}{2}\right] - n - 1, & 1 \leq k \leq 2n + 1, \\
    \forall i, & -n \leq m \leq -k - 1, & 1 \leq k \leq n - 1, \\
    i &\geq k + 1, & m &\geq -\left[\frac{k}{2}\right], & 1 \leq k \leq 2n + 1.
\end{align*}
\]

We also have a similar result for vanishing of cohomology in the case of \( c_1(M) < 0 \) (see Theorem 4.1). Moreover, on \( \mathbb{CP}^{2n+1} \), we show the vanishing theorem for \( H^i(\mathcal{O}(m) \bigwedge^k \ker \gamma) \) by Bott vanishing theorem (see Theorem 4.3).

This paper is organized as follows. In Section 2, we prepare some propositions for complex symplectic vector spaces. The space of \( k \)-vectors is decomposed into a sum of primitive parts with respect to the symplectic structure. In Section 3, the equation \( L_X \gamma_i|_{\ker \gamma} = 0 \) for a 1-vector \( X \) is extended to a global and holomorphic equation for \( k \)-vectors. By using the equation, we prove Theorem 1.1 (see Theorem 3.4). The theorem is generalized to the splitting theorem for \( \mathcal{O}(L^m \otimes \bigwedge^k \mathbb{T}) \) (Theorem 3.5). On \( \mathbb{CP}^{2n+1} \), \( L \) is given by \( \mathcal{O}(2) \) and we also obtain the splitting of \( \mathcal{O}(m) \otimes \bigwedge^k \ker \gamma \). In the last section, we show two kinds of vanishing theorems (Theorem 4.1 and Theorem 4.3).

## 2 Complex symplectic vector spaces

In this section, we prepare some propositions for complex symplectic vector spaces in order to show our main theorems in Section 3.

### 2.1 Symplectic structures

Let \( V \) be a complex vector space of dimension \( 2n \). A **complex symplectic vector space** is a pair \((V, \omega)\) of \( V \) and a non-degenerate skew-symmetric bilinear form \( \omega \) on \( V \). If we regard the symplectic structure \( \omega \) as the isomorphism from \( V \) to the dual space \( V^* \), then the 2-tensor \( \otimes^2 \omega \) of \( \omega \) is the isomorphism

\[
\otimes^2 \omega : \otimes^2 V \to \otimes^2 V^*.
\]

It induces the isomorphism \( \otimes^2 \omega : \wedge^2 V \to \wedge^2 V^* \). A 2-vector \( w_0 \) is defined by

\[
\otimes^2 \omega(w_0) = \omega.
\]

If we take a basis \( e_1, \ldots, e_{2n} \) of \( V \) such that \( \omega = e_1^* \wedge e_2^* + \cdots + e_{2n-1}^* \wedge e_{2n}^* \), then \( w_0 \) is represented as \( w_0 = e_1 \wedge e_2 + \cdots + e_{2n-1} \wedge e_{2n} \) and \( \omega(w_0) = n \).
2.2 Decomposition of the space of $k$-vectors

Let $k$ be an integer from $1$ to $2n$. We define an operator $\mathcal{L} : \wedge^k V \to \wedge^{k+2} V$ by $\mathcal{L}(X) = X \wedge w_0$ for $X \in \wedge^k V$. We consider a 1-form $\theta$ as a map from $\wedge^k V$ to $\wedge^{k-1} V$ by

$$\theta(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k} (-1)^{j-1} \theta(v_j)v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k$$

for $v_1, \ldots, v_k \in V$ and denote the map also by $\theta$ for simplicity. Let $l$ be an integer with $0 \leq l \leq k$. We regard an $l$-form $\theta_l \wedge \cdots \wedge \theta_l$ as a map $\theta_1 \wedge \cdots \wedge \theta_l : \wedge^k V \to \wedge^{k-l} V$ by

$$\theta_1 \wedge \cdots \wedge \theta_l(X) = \theta_l \circ \cdots \circ \theta_2 \circ \theta_1(X)$$

for $X \in \wedge^k V$. Let $\Lambda$ be an operator $\Lambda : \wedge^k V \to \wedge^{k-2} V$ defined by $\Lambda(X) = \omega(X)$ for $X \in \wedge^k V$. Then we obtain the formula $(\Lambda \mathcal{L} - \mathcal{L} \Lambda)(X) = (n-k)X$ for $X \in \wedge^k V$, and inductively,

$$(\Lambda \mathcal{L}^r - \mathcal{L} \Lambda^r)(X) = r(n-k-r+1)\mathcal{L}^{r-1}X$$

where we define $\mathcal{L}^0 = \text{id}$. A $k$-vector $X$ is called primitive if $\Lambda(X) = 0$. If follows from the formula (1) that

$$\Lambda^s \mathcal{L}^r X = \begin{cases} \frac{r!}{(r-s)!}(n-k-r+1)(n-k-r+2) \cdots (n-k-r+s)\mathcal{L}^{r-s}X, & r \geq s, \\ 0, & r < s \end{cases}$$

for a primitive $k$-vector $X$. Let $\wedge^k_e V$ denote the space of primitive $k$-vectors:

$$\wedge^k_e V = \{X \in \wedge^k V | \Lambda(X) = 0\}.$$

Then we have the following decomposition of the space $\wedge^k V$ of $k$-vectors:

**Proposition 2.1.** If $k \leq n$, then

$$\wedge^k V = \wedge^k_e V + \mathcal{L} \wedge^{k-2} V + \cdots + \mathcal{L}^{[\frac{k}{2}]} \wedge^{k-2[\frac{k}{2}]} V.$$

If $k > n$, then

$$\wedge^k V = \mathcal{L}^{k-n} \wedge^{2n-k} V + \mathcal{L}^{k-n+1} \wedge^{2n-k-2} V + \cdots + \mathcal{L}^{[\frac{k}{2}]} \wedge^{k-2[\frac{k}{2}]} V$$

where $[m]$ means the Gauss symbol of $m$. \hfill $\square$

2.3 Transformation associated with the decomposition

We define a linear transformation $T$ on $\wedge^k V$ by

$$T(X) = c_0 X + c_1 \mathcal{L} \Lambda X + c_2 \mathcal{L}^2 \Lambda^2 X + \cdots + c_{[\frac{k}{2}]} \mathcal{L}^{[\frac{k}{2}]} \Lambda^{[\frac{k}{2}]} X = \sum_{i=0}^{[\frac{k}{2}]} c_i \mathcal{L}^i \Lambda^i X$$

for constants $c_0, c_1, c_2, \ldots, c_{[\frac{k}{2}]}$, then we obtain the following:

**Proposition 2.2.** The transformation $T$ is isomorphic if and only if the constants $c_0, c_1, \ldots, c_{[\frac{k}{2}]}$ satisfy

$$\sum_{s=0}^{r} c_s \frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} \neq 0$$

for any $r = 0, \ldots, [\frac{k}{2}]$. 

Proof. In the case $k \leq n$ and $0 \leq r \leq \lfloor \frac{k}{2} \rfloor$, the equation \[ (2) \] implies that

$$
\mathcal{L}^s \Lambda^s \mathcal{L}^r X = \left\{ \begin{array}{ll}
\frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} \mathcal{L}^r X, & s \leq r, \\
0, & s > r
\end{array} \right.
$$

for $X \in \wedge^k \Lambda^r$. It yields that

$$
T(\mathcal{L}^r X) = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} c_s \mathcal{L}^s \Lambda^s \mathcal{L}^r X = \left( \sum_{s=0}^r c_s \frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} \right) \mathcal{L}^r X \tag{3}
$$

for $X \in \wedge^k \Lambda^r$. In the case $k > n$ and $k - n \leq r \leq \lfloor \frac{k}{2} \rfloor$, we also have the same equation \[ (3) \] for $X \in \wedge^k \Lambda^r$. Hence Proposition 2.1 implies that $T$ is an isomorphism from $\wedge^k \Lambda^r$ to itself if and only if $\sum_{s=0}^r c_s \frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} \neq 0$ for each $r = 0, \ldots, \lfloor \frac{k}{2} \rfloor$. \[ \square \]

3 Splitting of sheaves on complex contact manifolds

Let $(M, \gamma)$ be a complex contact manifold of dimension $2n + 1$ and $L$ the line bundle associated with the contact structure $\gamma$. We denote by $D$ the subbundle $\ker \gamma$ of $T$. Let $\nabla$ be a connection of $L \otimes \Lambda^1 \mathbb{T}^*$. The covariant exterior differentiation $d\nabla \gamma$ of $\gamma$ is a smooth section of $L \otimes \Lambda^2 \mathbb{T}^*$.

3.1 The isomorphisms associated with $d\nabla \gamma$

Let $d\nabla \gamma|_D$ denote the restriction of $d\nabla \gamma$ to $D$. Then, $d\nabla \gamma|_D$ is a holomorphic section of $L \otimes \Lambda^2 \Lambda^r \mathbb{T}^*$ which is independent of the choice of a connection $\nabla$. We identify $d\nabla \gamma|_D$ with the holomorphic bundle map from $D$ to $L \otimes \Lambda^r \mathbb{T}^*$. Then the map $d\nabla \gamma|_D : D \to L \otimes \Lambda^r \mathbb{T}^*$ is isomorphic since $d\gamma_i$ is non-degenerate on $D$. In general, the $k$-th tensor $\otimes^k d\nabla \gamma$ of $d\nabla \gamma$ is a smooth section of $L^k \otimes^k \Lambda^2 \mathbb{T}^*$ which is regarded as the smooth bundle map

$$
\otimes^k d\nabla : \Lambda^k \to L^k \otimes^k \mathbb{T}^*.
$$

Let $e$ be a local frame of $L$ and $A$ a connection form of $\nabla$ with respect to $e$. The contact structure $\gamma$ is given by $\gamma = e \otimes \gamma_0$ for a holomorphic 1-form $\gamma_0$. Then $d\nabla \gamma = e \otimes (d\gamma_0 + A \wedge \gamma_0)$. For 1-vectors $v_1, \ldots, v_k$, the $L^k$-valued $k$-form $\otimes^k d\nabla \gamma(v_1 \wedge \cdots \wedge v_k)$ is written by

$$
e^k \otimes \left\{ (d\gamma_0 + A \wedge \gamma_0)(v_1) \wedge \cdots \wedge (d\gamma_0 + A \wedge \gamma_0)(v_k) \right\}
= e^k \otimes \left\{ \otimes^k d\gamma_0(v_1 \wedge \cdots \wedge v_k) + \gamma_0 \wedge (\otimes^{k-1} d\gamma_0)(A(v_1 \wedge \cdots \wedge v_k)) - A \wedge (\otimes^{k-1} d\gamma_0)(\gamma_0(v_1 \wedge \cdots \wedge v_k)) \right\}.
$$

It implies

$$
\otimes^k d\nabla \gamma(X) = e^k \otimes \left\{ \otimes^k d\gamma_0(X) + \gamma_0 \wedge (\otimes^{k-1} d\gamma_0)(A(X)) - A \wedge (\otimes^{k-1} d\gamma_0)(\gamma_0(X)) \right\} \tag{4}
$$

for any $k$-vector $X$. We remark that the map $\otimes^k d\nabla \gamma$ is not holomorphic whenever $\nabla$ is holomorphic. However, the restriction $\otimes^k d\nabla \gamma|_D$ is the holomorphic map from $\Lambda^k D$ to
We fix an integer $k \geq 3$. The map $F : \wedge^k \mathbb{T} \to L^k \otimes \wedge^k \mathbb{T}^*$ is defined by

$$F(X) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-i)!}{k!} \otimes^{k-2i} d\nabla((d\nabla)^i(X)) \wedge (d\nabla)^i$$

for $X \in \wedge^k \mathbb{T}$, where $(d\nabla)^0$ is the identity map and $(d\nabla)^i$ is the $i$-th wedge $d\nabla \wedge \cdots \wedge d\nabla$ of $d\nabla \gamma$. We have

$$(d\nabla)^i(X) = e^i \otimes \{(d\gamma_0)^i(X) - i(d\gamma_0)^{i-1}(A(\gamma_0(X)))\}$$

for each $i = 0, \ldots, \lfloor \frac{k}{2} \rfloor$ since $(d\nabla)^i = e^i \otimes \{(d\gamma_0)^i + iA \wedge \gamma_0 \wedge (d\gamma_0)^{i-1}\}$. The equations (4) and (5) imply that

$$\otimes^{k-2i} d\nabla((d\nabla)^i(X)) = e^{k-i} \otimes \{(d\gamma_0)^i((d\gamma_0)^i(X)) - i \otimes^{k-2i} d\gamma_0((d\gamma_0)^{i-1}(A(\gamma_0(X))))$$

$$+ \gamma_0 \wedge (\otimes^{k-1-2i} d\gamma_0)((d\gamma_0)^i(A(X))) - A \wedge (\otimes^{k-1-2i} d\gamma_0)((d\gamma_0)^i(\gamma_0(X))))\}$$

(6)

where we use $A((d\gamma_0)^{i-1}(A(\gamma_0(X)))) = \gamma_0((d\gamma_0)^{i-1}(A(\gamma_0(X)))) = 0$ in the first line. It yields that

$$\otimes^{k-2i} d\nabla((d\nabla)^i(X))|_D = e^{k-i} \otimes \{(d\gamma_0)^i((d\gamma_0)^i(X)) - i \otimes^{k-2i} d\gamma_0((d\gamma_0)^{i-1}(A(\gamma_0(X))))$$

$$- A \wedge (\otimes^{k-1-2i} d\gamma_0)((d\gamma_0)^i(\gamma_0(X))))\}. \quad (7)$$

The second term in the right hand side of the equation (7) is 0 except for $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$ and the third term is 0 except for $0 \leq i \leq \lfloor \frac{k-1}{2} \rfloor$. Hence

$$F(X)|_D = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-i)!}{k!} \{\otimes^{k-2i} d\nabla((d\nabla)^i(X)) \wedge (d\nabla)^i\}|_D$$

$$= e^k \otimes \{(\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-i)!}{k!} \{\otimes^{k-2i} d\gamma_0((d\gamma_0)^i(X)) \wedge (d\gamma_0)^i\} - \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-i)!}{k!(i-1)!} \otimes^{k-2i} d\gamma_0((d\gamma_0)^{i-1}(A(\gamma_0(X)))) \wedge (d\gamma_0)^i)$$

$$- \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(k-i)!}{k!} A \wedge (\otimes^{k-1-2i} d\gamma_0)((d\gamma_0)^i(\gamma_0(X))) \wedge (d\gamma_0)^i)\}|_D. \quad (8)$$
The restriction $F|_D : \Lambda^k D \to L^k \Lambda^k D^*$ satisfies $F|_D(X) = F(X)|_D$ for $X \in \Lambda^k D$. If $X$ is a holomorphic section of $\Lambda^k D$, then it follows from $\gamma_0(X) = 0$ that

$$F|_D(X) = F(X)|_D = e^k \otimes \left\{ \sum_{i=0}^{\lfloor k \rfloor} \frac{(k-i)!}{k!i!} \otimes^{k-2i} d\gamma_0((d\gamma_0)^i(X)) \wedge (d\gamma_0)^i \right\}|_D$$

is a holomorphic section of $L^k \otimes \Lambda^k D^*$. Hence we obtain the following:

**Lemma 3.1.** An $L^k$-valued $k$-form $F|_D(X)$ restricted to $D$ is holomorphic for any holomorphic section $X$ of $\Lambda^k D$.

The lemma implies that $F|_D$ is regarded as a map from $\mathcal{O}(\Lambda^k D)$ to $\mathcal{O}(L^k \otimes \Lambda^k D^*)$. The map $F|_D$ is written by

$$F|_D(X) = \otimes^k d\nabla \gamma|_D \left( \sum_{i=0}^{\lfloor k \rfloor} \frac{(k-i)!}{k!i!} (d\nabla \gamma|_D)^i(X) \wedge (w)^i \right)$$

for $X \in \mathcal{O}(\Lambda^k D)$ since $(d\nabla \gamma|_D)^i = \otimes^2 d\nabla \gamma|_D(w)^i$. We define the transformation $f : \mathcal{O}(\Lambda^k D) \to \mathcal{O}(\Lambda^k D)$ by

$$f(X) = \sum_{i=0}^{\lfloor k \rfloor} \frac{(k-i)!}{k!i!} (d\nabla \gamma|_D)^i(X) \wedge (w)^i$$

for $X \in \mathcal{O}(\Lambda^k D)$. Then the map $F|_D$ is the composition of $\otimes^k d\nabla \gamma|_D$ and $f$. Proposition 2.2 implies that the following:

**Proposition 3.2.** The map $F|_D : \mathcal{O}(\Lambda^k D) \to \mathcal{O}(L^k \otimes \Lambda^k D^*)$ is isomorphic.

**Proof.** It suffices to show that $f$ is isomorphic since $\mathcal{O}(L^k \otimes \Lambda^k D^*)$ is isomorphic. At each point $x \in M$, the linear map $f_x : \Lambda^k D_x \to \Lambda^k D_x$ is written by

$$f_x = \sum_{i=0}^{\lfloor k \rfloor} \frac{(k-i)!}{k!i!} \mathcal{L}^i \Lambda^i$$

where $\mathcal{L}$ and $\Lambda$ are operators as in the previous section associated with the symplectic structure $(d\nabla \gamma|_D)_x$ on $D_x$. The map $f_x$ is isomorphic by Proposition 2.2 since each coefficient $c_i = \frac{(k-i)!}{k!i!}$ is positive. Hence $F|_D$ is also isomorphic. □

### 3.3 The map $G : \Gamma(L \otimes \Lambda^{k-1} D) \to \Gamma(L^k \otimes \Lambda^k \mathbb{T}^*)$

We define a map $G : \Gamma(L \otimes \Lambda^{k-1} D) \to \Gamma(L^k \otimes \Lambda^k \mathbb{T}^*)$ by

$$G(s) = \sum_{i=0}^{\lfloor k-1 \rfloor} \frac{(k-1-i)!}{k!i!} d\nabla \left( \otimes^{k-1-2i} d\nabla \gamma((d\nabla \gamma)^i(s)) \right) \wedge (d\nabla \gamma)^i$$

for $s \in \Gamma(L \otimes \Lambda^{k-1} D)$. Let $s$ be a section of $L \otimes \Lambda^{k-1} D$. The section $s$ is locally written as $s = e \otimes s_0$ for a section $s_0 \in \Lambda^{k-1} D$. It follows from the equations (9) and $(d\nabla \gamma)^i(s) = e^{i+1} \otimes (d\gamma_0)^i(s_0)$ that

$$\otimes^{k-1-2i} d\nabla \gamma((d\nabla \gamma)^i(s)) = e^{k-i} \otimes \left\{ \otimes^{k-1-2i} d\gamma_0((d\gamma_0)^i(s_0) + \gamma_0 \wedge (\otimes^{k-2-2i} d\gamma_0)((d\gamma_0)^i(A(s_0)))) \right\}.$$  

(9)
We remark that the second term in the right hand side of the equation [9] is 0 except for 
\(0 \leq i \leq \lceil \frac{k}{2} \rceil - 1\). It yields that
\[
d^\nabla \otimes^{k-1-2i} d^\nabla \gamma((d^\nabla \gamma)^i(s))|_D = e^{k-i} \otimes \{d \otimes^{k-1-2i} d\gamma_0((d\gamma)^i(s_0) + d\gamma_0 \wedge (\otimes^{k-2-2i} d\gamma_0)((d\gamma)^i(A(s_0))) + (k-i) A \wedge \otimes^{k-1-2i} d\gamma_0((d\gamma)^i(s_0)))\}|_{\mathcal{D}}.
\]
Hence
\[
G(s)|_D = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-1-i)!}{k! i!} d^\nabla \otimes^{k-1-2i} d^\nabla \gamma((d^\nabla \gamma)^i(s))|_{\mathcal{D}} \wedge (d^\nabla \gamma)^i|_{\mathcal{D}}
\]
\[
e^{k} \otimes \left\{ \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-1-i)!}{k! i!} (\otimes^{k-2-2i} d\gamma_0)((d\gamma)^i-1(A(s_0))) \wedge (d\gamma)^i \right\}
\]
\[
+ \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-i)!}{i! (i-1)!} (\otimes^{k-2i} d\gamma_0)((d\gamma)^{-1}(A(s_0))) \wedge (d\gamma)^i
\]
\[
+ \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-i)!}{k! i!} A \wedge \otimes^{k-1-2i} d\gamma_0((d\gamma)^i(s_0)) \wedge (d\gamma)^i \right\}_D.
\]
Then we have

**Proposition 3.3.** An \(L^k\)-valued \(k\)-form \(\{F(X) + G(\gamma(X))\}|_{\mathcal{D}}\) restricted to \(\mathcal{D}\) is holomorphic for any holomorphic \(k\)-vector \(X\).

**Proof.** If we take \(s\) as the image \(\gamma(X)\) of a \(k\)-vector \(X\), then \(s_0 = \gamma_0(X)\), and it follows from the equations [8] and [10] that
\[
\{F(X) + G(s)\}|_{\mathcal{D}} = e^{k} \otimes \left\{ \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-1-i)!}{k! i!} (\otimes^{k-2i} d\gamma_0((d\gamma)^i(X)) \wedge (d\gamma)^i \right\}
\]
\[
+ \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(k-1-i)!}{k! i!} d(\otimes^{k-1-2i} d\gamma_0((d\gamma)^i(\gamma_0(X)))) \wedge (d\gamma)^i \right\}_D.
\]
Hence \(\{F(X) + G(\gamma(X))\}|_{\mathcal{D}}\) is holomorphic for a holomorphic \(k\)-vector \(X\).

**3.4 The splitting of sheaves \(\mathcal{O}(\bigwedge^k \mathbb{T})\) as \(\mathbb{C}\)-module**

We have the following theorem:

**Theorem 3.4.** Let \(k\) be an integer from 1 to \(2n + 1\). The sequence
\[
0 \rightarrow \mathcal{O}(\bigwedge^k D) \rightarrow \mathcal{O}(\bigwedge^k \mathbb{T}) \rightarrow \mathcal{O}(L \otimes \bigwedge^{k-1} D) \rightarrow 0
\]
splits as sheaves of \(\mathbb{C}\)-module. In particular, the sequence
\[
0 \rightarrow H^i(\bigwedge^k D) \rightarrow H^i(\bigwedge^k \mathbb{T}) \rightarrow H^i(L \otimes \bigwedge^{k-1} D) \rightarrow 0
\]
is exact for each \(i = 0, \ldots, 2n + 1\).
Proof. Let $s$ be a holomorphic section of $L \otimes \bigwedge^{k-1} D$. We take an open set $U$ of $M$ where the bundles $L$ and $D$ are trivial. Then we can take a holomorphic $k$-vector $Y$ on $U$ such that $\gamma(Y) = s$ as follows. We fix a local frame $e$ of $L$ on $U$. The contact form $\gamma$ is given by $\gamma = e \otimes \gamma_0$ for a holomorphic 1-form $\gamma_0$ on $U$. We can choose a local frame $\{e_1, \ldots, e_{2n}\}$ of $D$ and a local section $e_{2n+1}$ of $T$ on $U$ such that $\gamma(e_{2n+1}) = e$. If $s$ is written by $s = \sum s_{i_1 \ldots i_k} e \otimes e_{i_1} \wedge \cdots \wedge e_{i_k-1}$ on $U$, then we take a section $Y$ by $Y = \sum s_{i_1 \ldots i_k} e_{2n+1} \wedge e_{i_1} \wedge \cdots \wedge e_{i_k-1}$.

Now we consider the smooth section $F(Y) + G(s)$ of $L^k \otimes \bigwedge^k T^*$. Proposition 3.3 implies that $\{F(Y) + G(s)\}|_D$ restricted to $D$ is a holomorphic section of $L^k \otimes \bigwedge^k D^*$. Hence, there exists a holomorphic section $h$ of $\bigwedge^k D$ such that

$$\{F(Y) + G(s)\}|_D = F|_D(h)$$

by the isomorphism $F|_D$ in Proposition 3.2. We define $X$ by $X = Y - h$. Then we obtain the holomorphic $k$-vector $X$ on $U$ satisfying the following equations

(i) $\gamma(X) = s$,

(ii) $\{F(Y) + G(s)\}|_D = 0$.

We take such local sections $X_1$ and $X_2$ on open sets $U_1$ and $U_2$, respectively. The first condition (i) implies that the difference $X_1 - X_2$ is in $\bigwedge^k D$ on $U_1 \cap U_2$. We also have the equation $F|_D(X_1 - X_2) = F(X_1 - X_2)|_D = 0$ by the second condition (ii). Then $X_1 = X_2 = 0$ on $U_1 \cap U_2$ since $F|_D$ is isomorphic. Hence the correspondence of $s$ to $X$ provides a right inverse of the map $\gamma : \mathcal{O}(\bigwedge^k T) \to \mathcal{O}(L \otimes \bigwedge^{k-1} D)$ as a $\mathbb{C}$-module map. It induces the splitting of $\mathcal{O}(\bigwedge^k T)$ and the exactness of $H^i(\bigwedge^k T)$ for each $i$. It completes the proof.

Remark 3.5. In the proof, $h$ is independent of the connection $\nabla$ since $\{F(Y) + G(s)\}|_D$ and $F|_D$ do not depend on $\nabla$ by Proposition 3.2. The $k$-vector $X = Y - h$ is also independent of $\nabla$. Hence, the splitting of the sequence in Theorem 3.4 is independent of the choice of the connection.

3.5 The splitting of the sheaves $\mathcal{O}(L^m \otimes \bigwedge^k T)$ as $\mathbb{C}$-module

In this section, we generalize Theorem 3.4 to the splitting of $\mathcal{O}(L^m \otimes \bigwedge^k T)$ under a condition for $m$ and $k$. Let $m$ be an integer such that $m \leq -k - 1$ or $m \geq -\frac{k}{2}$. We define a constant $c_{m,i}$ by $c_{m,0} = 1$ and

$$c_{m,i} = \frac{1}{(k+m)(k+m-1) \cdots (k+m-i+1) i!}$$

for each $i = 1, \ldots, \lfloor \frac{k}{2} \rfloor$. These constants are well-defined since $m + k < 0$ and $m + k - \lfloor \frac{k}{2} \rfloor + 1 \geq 0$ in the cases of $m \leq -k - 1$ and $m \geq -\frac{k}{2}$, respectively. We define a map $F_m : L^m \otimes \bigwedge^k T \to L^{m+k} \otimes \bigwedge^k T^*$ by

$$F_m(X) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} c_{m,i} \otimes^{k-2i} d\nabla(\gamma((d\nabla) i(X)) \wedge (d\nabla) i)$$

for $X \in L^m \otimes \bigwedge^k T$. By the same argument in Lemma 3.1, the restriction $F_m|_D$ induces the map from $\mathcal{O}(L^m \otimes \bigwedge^k D)$ to $\mathcal{O}(L^{m+k} \otimes \bigwedge^k D^*)$. 
Proposition 3.6. The map $F_m|_D : \mathcal{O}(L^m \otimes \bigwedge^k D) \to \mathcal{O}(L^{m+k} \otimes \bigwedge^k D^*)$ is isomorphic if $k$ and $m$ satisfy one of following three conditions

\[
\begin{aligned}
    &m \leq -n - \left[ \frac{k}{2} \right] - 2, & 1 \leq k \leq 2n + 1, \\
    &-n - 1 \leq m \leq -k - 1, & 1 \leq k \leq n, \\
    &m \geq -\left[ \frac{k}{2} \right], & 1 \leq k \leq 2n + 1.
\end{aligned}
\]

Proof. The map $F_m|_D$ is the composition of $\otimes^k d^\nabla \gamma|_D$ and the transformation $f_m$ of $\mathcal{O}(L^m \otimes \bigwedge^k D)$ defined by $f_m(X) = \sum_{i=0}^{\left[ \frac{k}{2} \right]} c_{m,i}(d^\nabla \gamma|_D)^i(X) \wedge w^i$ for $X \in \mathcal{O}(L^m \otimes \bigwedge^k D)$. At each point $x \in M$, the linear map $(f_m)_x : L^m_x \otimes \bigwedge^k D_x \to L^m_x \otimes \bigwedge^k D^*_x$ is written by $(f_m)_x = \sum_{i=0}^{\left[ \frac{k}{2} \right]} c_{m,i} L^i \Lambda^i$. Proposition 2.2 implies that $(f_m)_x$ is isomorphic if $
sum_{s=0}^{r} c_{m,s} \frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} = \frac{(m+n+r+1) \cdots (m+n+2)}{(m+k) \cdots (m+k-r+1)}$ for any $r = 1, \ldots, \left[ \frac{k}{2} \right]$. It yields that $\sum_{s=0}^{r} c_{m,s} \frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} \neq 0$ for each $r$ if $m \leq -n - \left[ \frac{k}{2} \right] - 2$ or $m \geq -n - 1$. By the assumption $m \leq -k - 1$ or $m \geq -\left[ \frac{k}{2} \right]$, the linear map $(f_m)_x$ is isomorphic if

\[
\begin{aligned}
    &m \leq -n - \left[ \frac{k}{2} \right] - 2, & \forall k, \\
    &-n - 1 \leq m \leq -k - 1, & 1 \leq k \leq n, \\
    &m \geq -\left[ \frac{k}{2} \right], & \forall k.
\end{aligned}
\]

Hence, we finish the proof. \qed

We define a constant $c'_{m,i}$ by

\[
    c'_{m,i} = \frac{1}{(k+m)(k+m-1) \cdots (k+m-i)/i!}
\]

for each $i = 0, 1, \ldots, \left[ \frac{k+1}{2} \right]$. We remark that these constants are well-defined since $m+k < 0$ and $m+k - \left[ \frac{k+1}{2} \right] > 0$ in the cases of $m \leq -k - 1$ and $m \geq -\left[ \frac{k}{2} \right]$, respectively. We define a map $G_m : \Gamma(L^{m+1} \otimes \bigwedge^{k-1} D) \to \Gamma(L^{m+k} \otimes \bigwedge^k D^*)$ by

\[
    G_m(s) = \sum_{i=0}^{\left[ \frac{k-1}{2} \right]} c'_{m,i} d^\nabla \left( \otimes^{k-1-i} d^\nabla \gamma((d^\nabla \gamma)^i(s)) \right) \wedge (d^\nabla \gamma)^i
\]

for $s \in \Gamma(L^{m+1} \otimes \bigwedge^{k-1} D)$. Similarly to Proposition 3.3, we have

Proposition 3.7. An $L^{k+m}$-valued $k$-form $\{F_m(X) + G_m(\gamma(X))\}|_D$ restricted to $D$ is holomorphic for any holomorphic $L^m$-valued $k$-vector $X$. \qed

By repeating the proof of Theorem 3.3 with $F_m$ and $G_m$ instead of $F$ and $G$, then we obtain

Theorem 3.8. The sequence

\[
0 \to \mathcal{O}(L^m \otimes \bigwedge^k D) \to \mathcal{O}(L^m \otimes \bigwedge^k \mathbb{T}) \to \mathcal{O}(L^{m+1} \otimes \bigwedge^{k-1} D) \to 0
\]
splits as sheaves of $\mathbb{C}$-module, and the sequence

\[ 0 \to H^i(L^m \otimes \bigwedge^k D) \to H^i(L^m \otimes \bigwedge^k \mathbb{P}^n) \to H^i(L^{m+1} \otimes \bigwedge^{k-1} D) \to 0 \]

is exact for each $i = 0, \ldots, 2n+1$ if $k$ and $m$ satisfy one of following three conditions

\[
\begin{cases}
  m \leq -n - \left\lceil \frac{k}{2} \right\rceil - 2, & 1 \leq k \leq 2n + 1, \\
  -n - 1 \leq m \leq -k - 1, & 1 \leq k \leq n, \\
  m \geq -\left\lceil \frac{k}{2} \right\rceil, & 1 \leq k \leq 2n + 1.
\end{cases}
\]

\[ \square \]

### 3.6 The splitting of the sheaves $\mathcal{O}(l^m \otimes \bigwedge^k \mathbb{P}^n)$ on $\mathbb{C}P^{2n+1}$

On the odd dimensional projective space $\mathbb{C}P^{2n+1}$, there exists a standard contact structure $\gamma$ written by $\gamma = z^0dz^1 - z^1dz^2 + \cdots + z^{2n}dz^{2n+1} - z^{2n+1}dz^{2n}$ in the homogeneous coordinate. Let $l$ denote the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{C}P^{2n+1}$. The associated bundle $L$ is given by $l^2 = \mathcal{O}(2)$ and the contact structure $\gamma$ is regarded as a section of $l^2 \otimes \mathbb{T}^*$. We consider the short exact sequence

\[ 0 \to l^m \otimes \bigwedge^k D \to l^m \otimes \bigwedge^k \mathbb{P}^n \to l^{m+2} \otimes \bigwedge^k D \to 0 \]

for $m \in \mathbb{Z}$ and $1 \leq k \leq 2n+1$. If $m$ is even, then the splitting of $\mathcal{O}(l^m \otimes \bigwedge^k \mathbb{P}^n)$ is induced by Theorem 3.8 since $L = l^2$. From now on, we assume that $m$ is odd. We define a constant $\tilde{c}_{m,i}$ by $\tilde{c}_{m,0} = 1$ and

\[ \tilde{c}_{m,i} = \frac{1}{(k + \frac{m}{2})(k + \frac{m}{2} - 1) \cdots (k + \frac{m}{2} - i + 1)}! \]

for each $i = 1, \ldots, \left\lceil \frac{k}{2} \right\rceil$. These constants are well-defined since $k + \frac{m}{2} - i + 1 \neq 0$ for any $i$. We fix a connection $\nabla$ of $l$ and define a map $\tilde{F}_m : l^m \otimes \bigwedge^k \mathbb{P}^n \to l^{m+2} \otimes \bigwedge^k D^*$ by

\[ \tilde{F}_m(X) = \sum_{i=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \tilde{c}_{m,i} \otimes \nabla^i \gamma((d\nabla \gamma)^i(X)) \wedge (d\nabla \gamma)^i \]

for $X \in l^m \otimes \bigwedge^k \mathbb{P}^n$. The restriction $\tilde{F}_m|_D$ induces the map from $\mathcal{O}(l^m \otimes \bigwedge^k D)$ to $\mathcal{O}(l^{m+2} \otimes \bigwedge^k D^*)$.

**Proposition 3.9.** The map $\tilde{F}_m|_D : \mathcal{O}(l^m \otimes \bigwedge^k D) \to \mathcal{O}(l^{m+2} \otimes \bigwedge^k D^*)$ is isomorphic if $m$ satisfies $m \leq -2n - 2\left\lfloor \frac{k}{2} \right\rfloor - 3$ or $-2n - 3 \leq m$.

**Proof.** If $k \geq 2$, then

\[ \sum_{s=0}^{r} \frac{r!}{(r-s)!} \frac{(n-k+r+s)!}{(n-k+r)!} = \frac{(m+n+r+1) \cdots (m+n+2)}{(m+k) \cdots (m+k+r+1)} \]

for any $r = 1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor$. By the same argument in Proposition 3.6 the map $\tilde{F}_m|_D$ is isomorphic if $m \leq -2n - 2\left\lfloor \frac{k}{2} \right\rfloor - 3$ or $-2n - 3 \leq m$. \[ \square \]
We define a constant \( \tilde{c}^i_{m;i} \) by
\[
\tilde{c}^i_{m;i} = \frac{1}{(k + \frac{m}{2})(k + \frac{m}{2} - 1) \cdots (k + \frac{m}{2} - i) i!}
\]
for each \( i = 0, 1, \ldots, \lfloor \frac{k-1}{2} \rfloor \). These constants are well-defined since \( \frac{m}{2} + k - \lfloor \frac{k-1}{2} \rfloor \neq 0 \). We define a map \( \tilde{G}_m : \Gamma(l^{m+1} \otimes \bigwedge^{k-1} D) \rightarrow \Gamma(l^{m+2k} \otimes \bigwedge^k T^*) \) by
\[
\tilde{G}_m(s) = \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{c}^i_{m;i} d^{\nabla} \left( \otimes^{k-1-2i} d^{\nabla} \gamma((d^{\nabla} \gamma)^i(s)) \right) \wedge (d^{\nabla} \gamma)^i
\]
for \( s \in \Gamma(l^{m+2} \otimes \bigwedge^{k-1} D) \). Then we have

**Proposition 3.10.** An \( l^{2k+m} \)-valued \( k \)-form \( \{ \tilde{F}_m(X) + \tilde{G}_m(\gamma(X)) \} \mid_D \) restricted to \( D \) is holomorphic for any holomorphic \( l^m \)-valued \( k \)-vector \( X \).

**Proof.** We fix a frame \( \tilde{e} \) of \( l \) on \( U \). Then \( \gamma \) is given by \( \gamma = \tilde{e}^2 \otimes \gamma_0 \) for a holomorphic 1-form \( \gamma_0 \) on \( U \). Let \( X \) be a holomorphic \( l^m \)-valued \( k \)-vector \( X \) on \( M \). By the same argument in the proof of Proposition 3.3, we have
\[
\{ \tilde{F}_m(X) + \tilde{G}_m(\gamma(X)) \} \mid_D = \tilde{e}^{2k+m} \otimes \left\{ \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{c}^i_{m;i} \left( \otimes^{k-2i} d\gamma_0((d\gamma_0)^i(X)) \wedge (d\gamma_0)^i \right) + \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \tilde{c}^i_{m;i} d(\otimes^{k-1-2i} d\gamma_0((d\gamma_0)^i(\gamma_0(X)))) \wedge (d\gamma_0)^i \right\} \mid_D.
\]

Hence \( \{ \tilde{F}_m(X) + \tilde{G}_m(\gamma(X)) \} \mid_D \) is holomorphic. \( \square \)

By repeating the proof of Theorem 3.3 with \( \tilde{F}_m \) and \( \tilde{G}_m \) instead of \( F \) and \( G \), then we obtain

**Theorem 3.11.** The sequence
\[
0 \rightarrow \mathcal{O}(l^m \otimes \bigwedge^k D) \rightarrow \mathcal{O}(l^m \otimes \bigwedge^k T) \rightarrow \mathcal{O}(l^{m+2} \otimes \bigwedge^{k-1} D) \rightarrow 0
\]
splits as sheaves of \( \mathbb{C} \)-module, and the sequence
\[
0 \rightarrow H^i(l^m \otimes \bigwedge^k D) \rightarrow H^i(l^m \otimes \bigwedge^k T) \rightarrow H^i(l^{m+2} \otimes \bigwedge^{k-1} D) \rightarrow 0
\]
is exact for each \( i = 0, \ldots, 2n + 1 \) if \( k \) and \( m \) satisfy one of following conditions
\[
\begin{align*}
\begin{cases} m \leq -2n - 2\lfloor \frac{k}{2} \rfloor - 4, & 1 \leq k \leq 2n + 1, \quad m : \text{even}, \\
-2n - 2 \leq m \leq -2k - 2, & 1 \leq k \leq n, \quad m : \text{even}, \\
m \geq -2\lfloor \frac{k}{2} \rfloor, & 1 \leq k \leq 2n + 1, \quad m : \text{even}, \\
m \leq -2n - 2\lfloor \frac{k}{2} \rfloor - 3, & 1 \leq k \leq 2n + 1, \quad m : \text{odd}, \\
m \geq -2n - 3, & 1 \leq k \leq 2n + 1, \quad m : \text{odd}.
\end{cases}
\end{align*}
\]
4 Vanishing theorems for cohomology of $\bigwedge^k D$

In this section, we apply the splitting theorems to the cohomology of $\bigwedge^k D$ and obtain the vanishing theorems. From now on, we denote by $E \bigwedge^k D$ the tensor $E \otimes \bigwedge^k D$ of vector bundles $E$ and $\bigwedge^k D$ for simplicity.

4.1 Vanishing of the cohomology on compact Kähler complex contact manifolds

We have the following vanishing theorem of the cohomology on compact Kähler manifolds:

**Theorem 4.1.** If $M$ is a compact Kähler complex contact manifold with $c_1(M) > 0$, then

$$H^i(M, L^m \bigwedge^k D) = \{0\} \quad \begin{cases} i \leq 2n - k, \quad m \leq -\left[\frac{k+1}{2}\right] - n - 1, & 1 \leq k \leq 2n + 1, \\ \forall i, \quad -n \leq m \leq -k - 1, & 1 \leq k \leq n - 1, \\ i \geq k + 1, \quad m \geq -\left[\frac{k}{2}\right], & 1 \leq k \leq 2n + 1. \end{cases}$$

If $c_1(M) < 0$, then

$$H^i(M, L^m \bigwedge^k D) = \{0\} \quad \begin{cases} i \geq k + 1, \quad m \leq -n - \left[\frac{k}{2}\right] - 2, & 1 \leq k \leq 2n + 1, \\ i \geq k + 2, \quad m = -n - 1, & 1 \leq k \leq n, \\ i < 2n - k - 1, \quad m = -k, & 1 \leq k \leq n, \\ i < 2n - k, \quad m \geq -\left[\frac{k}{2}\right], & 1 \leq k \leq 2n + 1. \end{cases}$$

**Proof.** By Theorem 3.8, the sequence

$$0 \rightarrow H^i(M, L^m \bigwedge^k D) \rightarrow H^i(M, L^m \bigwedge^k \mathbb{T}) \rightarrow H^i(M, L^{m+1} \bigwedge^k D) \rightarrow 0 \quad (11)$$

is exact for each $i$ if $m$ and $k$ satisfy one of following three conditions

$$\begin{cases} \quad m \leq -n - \left[\frac{k}{2}\right] - 2, & 1 \leq k \leq 2n + 1, \\ -n - 1 \leq m \leq -k - 1, & 1 \leq k \leq n, \\ \quad m \geq -\left[\frac{k}{2}\right], & 1 \leq k \leq 2n + 1. \end{cases} \quad (12)$$

It follows from Serre’s duality that

$$H^i(M, L^m \bigwedge^k \mathbb{T})^* \cong H^{2n+1-i}(M, \Omega^k L^{-m} K_M) \cong H^{2n+1-i}(M, \Omega^k L^{-m-n-1}). \quad (13)$$

In the case of $c_1(M) > 0$, the first Chern class $c_1(L^{-m-n-1})$ of the line bundle $L^{-m-n-1}$ is negative if $m > -n - 1$ since $c_1(L) = -\frac{1}{n+1} c_1(K_M) > 0$. By applying the Kodaira-Akizuki-Nakano vanishing theorem [1] to the last cohomology in (13), we have $H^i(M, L^m \bigwedge^k \mathbb{T}) = \{0\}$ for $k + 1 \leq i$ if $m > -n - 1$. Hence, the sequence (11) implies

$$H^i(M, L^m \bigwedge^k D) = \{0\}, \quad (14)$$

$$H^i(M, L^{m+1} \bigwedge^k D) = \{0\} \quad (15)$$

for $k + 1 \leq i \leq 2n + 1$, $m > -n - 1$ and $1 \leq k \leq 2n + 1$. The condition (15) is written by $H^i(M, L^{m'} \bigwedge^{k'} D) = \{0\}$ for $k' + 2 \leq i \leq 2n + 1$, $m' > -n$ and $0 \leq k' \leq 2n$. We only
consider the vanishing for $k' \geq 1$ since the case $k' = 0$ that $H^i(M, L^{m'}) = \{0\}$ for $2 \leq i$ and $m' > -n$ is induced by the Kodaira-Akizuki-Nakano vanishing theorem. The first condition (14) induces the second one (15) for $k \geq 1$. Hence, (12) and (14) imply that $H^i(M, L^m \wedge^k D) = \{0\}$ if $m$ satisfies one of two conditions

\[
\begin{cases}
  i \geq k + 1, & -n \leq m \leq -k - 1, \\
  i \geq k + 1, & m \geq -\left[\frac{k}{2}\right],
\end{cases}
\]

(16)

The Serre’s duality implies that

\[
H^i(M, L^m \wedge^k D)^* \cong H^{2n+1-i}(M, K_M L^{-m} \wedge^k D)^* \cong H^{2n+1-i}(M, L^{-k-m-n-1} \wedge^k D) \tag{17}
\]

since $D^* = L^{-1}D$. We apply the vanishing in (16) to the last cohomology in (17), and obtain $H^i(M, L^m \wedge^k D) = \{0\}$ if $m$ satisfies one of two conditions

\[
\begin{cases}
  i \leq 2n - k, & m \leq -\left[\frac{k+1}{2}\right] - n - 1, \\
  i \leq 2n - k, & -n \leq m \leq -k - 1, \\
  \forall i, & -n \leq m \leq -k - 1, \\
  i \geq k + 1, & m \geq -\left[\frac{k}{2}\right],
\end{cases}
\]

(18)

We remark that the condition $k \leq -n - 1$ implies that $2n - k \geq k + 1$. It follows from (16) and (18) that $H^i(M, L^m \wedge^k D) = \{0\}$ for

\[
\begin{cases}
  i \geq k + 1, & m \leq -n - \left[\frac{k}{2}\right] - 2, \\
  i \geq k + 2, & m = -n - 1,
\end{cases}
\]

(19)

Serre’s duality implies

\[
H^i(M, L^m \wedge^k D) = \{0\} \begin{cases}
  i \leq 2n - k, & m \geq -\left[\frac{k}{2}\right], \\
  i \leq 2n - k - 1, & m = -k,
\end{cases}
\]

(20)

Hence it completes the proof.

\[\square\]

**Remark 4.2.** Salamon proved that any $(p, q)$-cohomology $H^{p,q}(M)$ vanishes for $p \neq q$ if $M$ is the twistor space of a quaternion manifold with a positive scalar curvature [10]. In the proof, he also obtained the vanishing of the cohomology of $\wedge^k D$ on the twistor space (Equation (6.4) in [10]). He used the notation of $L$ and $E$ as $L_T^2$ and $L^{-\frac{1}{2}}D$ in our notation, respectively. These results are improved to the case of compact Kähler complex contact manifolds with $c_1(M) > 0$ by the same argument, and the vanishing is translated into

\[
H^i(M, L^m \wedge^k D) = \{0\} \begin{cases}
  \forall i, & -n \leq m \leq -k - 1, \\
  i \neq k, & m = -k,
\end{cases}
\]

(21)

The first condition in (21) is equal to the second condition in Theorem 4.1. The second one in (21) is independent of our theorem. However, we remark that the first and third conditions in Theorem 4.1 are not induced by the vanishing (21).
4.2 The vanishing of the cohomology on $\mathbb{C}P^{2n+1}$

In this section, we show the vanishing theorem for $H^i(l^m \wedge^k D) = H^i(O(m) \wedge^k D)$ on $\mathbb{C}P^{2n+1}$ by using Bott’s vanishing formula \cite{3}. We have the short exact sequence

$$0 \rightarrow D \rightarrow T \xrightarrow{\gamma} l^2 \rightarrow 0.$$  

It induces the exact sequence

$$0 \rightarrow l^m \wedge^k D \rightarrow l^m \wedge^k T \xrightarrow{\gamma} l^{m+2} \wedge^k D \rightarrow 0$$

for $m \in \mathbb{Z}$. Then we have a vanishing of the cohomology as follows.

**Theorem 4.3.** $H^i(l^m \wedge^k D) = \{0\}$ if, in the case $m$ is even

$$\begin{cases} 
  i \neq 2n + 1, & m \leq -2n - 2 - 2[k+1], \quad 1 \leq k \leq 2n + 1, \\
  i \neq 2n + 1 - k, & m = -2n - 2, \quad 1 \leq k \leq n, \\
  \forall i, & -2n \leq m \leq -2k - 2, \quad 1 \leq k \leq n, \\
  i \neq k, & m = -2k, \quad 1 \leq k \leq n, \\
  i \neq 0, & m \geq -2[k], \quad 1 \leq k \leq 2n + 1 
\end{cases}$$

and, in the case $m$ is odd

$$\begin{cases} 
  i \neq 2n + 1, & m \leq -2n - 3 - k, \quad 1 \leq k \leq 2n + 1, \\
  \forall i, & -2n - 2 - k \leq m \leq -2k + 1, \quad 1 \leq k \leq 2n + 1, \\
  \forall i, & -2n - 3 \leq m \leq -k, \quad 1 \leq k \leq 2n + 1, \\
  i \neq 0, & m \leq -k + 1, \quad 1 \leq k \leq 2n + 1. 
\end{cases}$$

*Proof.* By applying Theorem 3.11 to the sequence (22), we obtain the short exact sequence

$$0 \rightarrow H^i(l^m \wedge^k D) \rightarrow H^i(l^m \wedge^k T) \rightarrow H^i(l^{m+2} \wedge^k D) \rightarrow 0$$

if

$$\begin{cases} 
  m \leq -2n - 2[k] - 4, & 1 \leq k \leq 2n + 1, \quad m : \text{even}, \\
  -2n - 2 \leq m \leq -2k - 2, & 1 \leq k \leq n, \quad m : \text{even}, \\
  m \geq -2[k], & 1 \leq k \leq 2n + 1, \quad m : \text{even}, \\
  m \leq -2n - 2[k] - 3, & 1 \leq k \leq 2n + 1, \quad m : \text{odd}, \\
  m \geq -2n - 3, & 1 \leq k \leq 2n + 1, \quad m : \text{odd}. 
\end{cases}$$

By applying Bott’s vanishing formula to Serre’s duality $H^i(l^m \wedge^k T)^* \cong H^{2n+1-i}(\Omega^k(l^{m-2n-2}))$, $H^i(l^m \wedge^k T) = \{0\}$ holds except for the following cases

$$\begin{cases} 
  i = 2n + 1 - k, & m = -2n - 2, \\
  i = 2n + 1, & m < -2n - 2 - k, \\
  i = 0, & m > -k - 1. 
\end{cases}$$

It follows from (23) and (25) that $H^i(l^m \wedge^k D) = H^i(l^{m+2} \wedge^{k-1} D) = \{0\}$ if

$$\begin{cases} 
  i \neq 2n + 1 - k, & m = -2n - 2, \\
  i \neq 2n + 1, & m \leq -2n - 3 - k, \\
  \forall i, & -2n - 2 - k \leq m \leq -k - 1, \quad (m \neq -2n - 2), \\
  i \neq 0, & m \geq -k. 
\end{cases}$$
In the case that \( m \) is even, \((24)\) and \((26)\) imply that \( H^i(l^m \wedge^k D) = \{0\} \) and \( H^i(l^{m+2} \wedge^{k-1} D) = \{0\} \) if
\[
\begin{align*}
&i \neq 2n + 1 - k, \quad m = -2n - 2, \quad 1 \leq k \leq n, \\
&i \neq 2n + 1, \quad m = -2n - 4 - 2[k_3], \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad -2n - 1 \leq m \leq -2k - 2, \quad 1 \leq k \leq n, \\
&i \neq 0, \quad m \geq -2[k_3], \quad 1 \leq k \leq 2n + 1.
\end{align*}
\]
(27)

We replace the vanishing \( H^i(l^{m+2} \wedge^{k-1} D) = \{0\} \) for \((27)\) by \( H^i(l^n \wedge^k D) = \{0\} \) for
\[
\begin{align*}
&i \neq 2n - k, \quad m = -2n, \quad 0 \leq k \leq n - 1, \\
&i \neq 2n + 1, \quad m = -2n - 2 - 2[k_3], \quad 0 \leq k \leq 2n, \\
&\forall i, \quad -2n + 1 \leq m \leq -2k - 2, \quad 0 \leq k \leq n - 1, \\
&i \neq 0, \quad m \geq -2[k_3] + 2, \quad 0 \leq k \leq 2n.
\end{align*}
\]
(28)

We remark that the case \( k = 0 \) of \((28)\) is contained in Bott’s vanishing formula. We only consider the vanishing of \( H^i(l^m \wedge^k D) \) for \( k \geq 1 \). We summarize \((27)\) and \((28)\) as \( H^i(l^m \wedge^k D) = \{0\} \) for even \( m \) and
\[
\begin{align*}
&i \neq 2n + 1, \quad m \leq -2n - 2 - 2[k_3], \quad 1 \leq k \leq 2n + 1, \\
&i \neq 2n + 1 - k, \quad m = -2n - 2, \quad 1 \leq k \leq n, \\
&\forall i, \quad -2n \leq m \leq -2k - 2, \quad 1 \leq k \leq n, \\
&i \neq 0, \quad m \geq -2[k_3], \quad 1 \leq k \leq 2n + 1.
\end{align*}
\]
(29)

In the case that \( m \) is odd, by repeating the above argument the conditions \((24)\) and \((26)\) imply that \( H^i(l^n \wedge^k D) = \{0\} \) for
\[
\begin{align*}
&i \neq 2n + 1, \quad m \leq -2n - 3 - k, \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad m = -2n - 2 - k, \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad m = -2n - 1 - k, \quad 1 \leq k \leq 2n, \\
&\forall i, \quad -2n - 3 \leq m \leq -k - 1, \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad m = -k, \quad 1 \leq k \leq 2n - 1, \\
&i \neq 0, \quad m = -k, \quad k = 2n + 1, \\
&i \neq 0, \quad m \geq -k + 1, \quad 1 \leq k \leq 2n + 1.
\end{align*}
\]
(30)

Applying \((29)\) and \((30)\) to Serre’s duality \( H^i(l^m \wedge^k D)^* \cong H^{2n+1-i}(l^{-m-2n-2-2k} \wedge^k D) \), we obtain \( H^i(l^m \wedge^k D) = \{0\} \) if, in the case \( m \) is even
\[
\begin{align*}
&i \neq 0, \quad m \geq -2[k_3], \quad 1 \leq k \leq 2n + 1, \\
&i \neq k, \quad m = -2k, \quad 1 \leq k \leq n, \\
&\forall i, \quad -2n \leq m \leq -2k - 2, \quad 1 \leq k \leq n, \\
&i \neq 2n + 1, \quad m \leq -2n - 2 - 2[k_3], \quad 1 \leq k \leq 2n + 1
\end{align*}
\]
(31)

and, in the case \( m \) is odd
\[
\begin{align*}
&i \neq 0, \quad m \geq -k + 1, \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad m = -k, \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad m = -k - 1, \quad 1 \leq k \leq 2n, \\
&\forall i, \quad -2n - 1 - k \leq m \leq -2k + 1, \quad 1 \leq k \leq 2n + 1, \\
&\forall i, \quad m = -2n - 2 - k, \quad 1 \leq k \leq 2n - 1, \\
&i \neq 2n + 1, \quad m = -2n - 2 - k, \quad k = 2n + 1, \\
&i \neq 2n + 1, \quad m \leq -2n - 3 - k, \quad 1 \leq k \leq 2n + 1.
\end{align*}
\]
(32)

The conditions \((29)\) and \((31)\) in the case of even \( m \), and \((30)\) and \((32)\) in the case of odd \( m \) induce the conditions in theorem. Hence it completes the proof. \( \square \)
Remark 4.4. A contact structure on $\mathbb{C}P^{2n+1}$ is unique, up to automorphisms [11]. Hence Theorem 3.11 and 4.3 hold for any contact structure on $\mathbb{C}P^{2n+1}$.

Remark 4.5. In algebraic geometry, the null correlation bundle $N$ on $\mathbb{C}P^{2n+1}$ is defined by the short exact sequence

$$0 \to N \to T(-1) \to \mathcal{O}(1) \to 0$$

where $T(-1) = \mathcal{O}(-1) \otimes T([9])$. It induces to the following :

$$0 \to N(1) \to T \to \mathcal{O}(2) \to 0.$$ 

The bundle $N(1) = \mathcal{O}(1) \otimes N$ is the kernel of a transformation $A : T \to \mathcal{O}(2)$ which is given by $a = \sum a_{ij} z_i dz_j$ in the homogeneous coordinate. If $a_{ij} = -a_{ji}$ and $(a_{ij})$ is non-degenerate, then $A$ induces a contact structure on $\mathbb{C}P^{2n+1}$. By applying Theorem 4.3 to $D = N(1)$, we obtain the vanishing formula for the cohomology $H^1(\wedge^k N(m + k))$.

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References

[1] Akizuki, Y.; Nakano, S. Note on Kodaira-Spencer’s proof of Lefschetz theorems, Proc. Japan Acad. 30 (1954), 266–272.

[2] Boothby, W. M.; Wang, H. C. On contact manifolds, Ann. of Math.(2) 68 (1958), 721–734.

[3] Bott, R. Homogeneous vector bundles, Ann. of Math.(2) 66 (1957), 203–248.

[4] Ishihara, S. Quaternion Kährlerian manifolds. J. Differential Geometry 9 (1974), 483–500.

[5] Gray, J.W. Some global properties of contact structures, Ann. of Math.(2) 69 (1959), 421–450.

[6] Kobayashi, S. Remarks on complex contact manifolds, Proc. Amer. Math. Soc. 10 (1959), no. 3, 164–167.

[7] Kobayashi, S. Transformation groups in differential geometry, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70. Springer-Verlag, New York-Heidelberg, (1972).

[8] LeBrun, C. Fano manifolds, contact structures, and quaternionic geometry, Internat. J. Math. 6 (1995), no. 3, 419–437.

[9] Okonek, C.; Schneider, M.; Spindler, H. Vector bundles on complex projective spaces, Progress in Math. 3 Birkhäuser, Boston, Mass. (1980).

[10] Salamon, S. Quaternionic Kährler manifolds, Invent. Math. 67 (1982), no. 1, 143–171.
[11] Nitta, T.; Takeuchi, M. Contact structures on twistor spaces, *J. Math. Soc. Japan* **39** (1987), no. 1, 139–162.

[12] Ye, Y.-G. A note on complex projective threefolds admitting holomorphic contact structures, *Invent. Math.* **115** (1994), no. 2, 311–314.

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