Title: Super achromatic wide-angle quarter-wave plates using multi-twist retarders

Authors: lingshan li, Michael Escuti

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Super achromatic wide-angle quarter-wave plates using multi-twist retarders: supplementary information

This document provides supplementary information to “Super achromatic and wide-angle quarter-wave plates using multi-twist retarders”.

1. VALIDATION OF EXTENDED JONES MATRIX

A. Case study of cross/parallel polarizers

We first consider the transmission of crossed polarizers. If birefringent plate complex refractive index is set as \( n = n_e + \kappa_e \) for e-wave and \( n = n_o + \kappa_o \) for o-wave, with approximation of small birefringence (\( n_o \approx n_e \)), the sheet polarizers can be characterized by \( \kappa_o = 0 \), \( 0 < \kappa_e \ll 1 \) and thickness \( d \) such that \( 1 \ll 2\pi\kappa_e d/\lambda \).

To simplify the problem, the light source is monochromatic with wavelength \( \lambda = 550 \text{ nm} \), and the two O-type polarizer the small birefringence approximation is satisfied. For the cross polarizers setup, the front polarizer transmission axis is set as \( p_1 = 45^\circ \), and the back polarizer transmission axis at \( p_2 = 135^\circ \). For the parallel polarizers setup, both front/back polarizer is set as \( p_1 = p_2 = 0^\circ \). The calculated transmission between a pair of polarizers is depicted in Fig. S1.

![Fig. S1.](image)

**Fig. S1.** Polar pattern of transmission after (a) cross and (b) parallel ideal O-type polarizers. Light source is set as monochromatic light with wavelength \( \lambda = 550 \text{ nm} \). polarizers refractive index \( n_p = 1.5 \). Polar angle range is \((0^\circ, 90^\circ)\).

B. Case study of uniaxial plates

In this subsection, three most common liquid crystal alignment formats, the uniaxial \( a \)-plate and \( c \)-plate will be simulated.

First we start with uniaxial \( a \)-plate, which represents a kind of birefringent material with optic axis lies in the plane perpendicular to the incident plane (Fig. S2(a)). We can assume the material discussed is of positive birefringence. When it’s oblique incident, the birefringence is angle-dependent as \( \Delta n(\theta, \phi) = n(\theta, \phi) - n_o \), in which \( n(\theta, \phi) \) is angle dependent. The transmission between cross polarizers (front polarizer transmission axis \( p_1 = 45^\circ \), back polarizer transmission axis \( p_2 = 135^\circ \)) of single \( a \)-plate is depicted in Fig. S2(d). Here the transmission calculation includes the Fresnel reflection and transmission on the surface of the polarizer. It can be seen that the correct transmission of \( a \)-plate between cross polarizers has alternate constructive interference and destructive interference, representing the full wave retardation and half-wave retardation respectively.

In the case of uniaxial \( c \)-plate, it has optic axis perpendicular to the birefringent plane (Fig. S2(b)). From the following polar pattern of \( c \)-plate, two things are worth to notice. Firstly, the center of the plate is dark fringes. Considering the thickness \( d = 2.5 \mu m \) and the monochromatic light source is \( \lambda = 500 \mu m \), indicating this is a half-wave plate. Secondly, we can visualize the interference is...
center-symmetric, indicating that the birefringence and the retardation is also center-symmetric.
Finally, we also validate a third type of birefringent plate, $\sigma$-plate. The configuration of $\sigma$-plate (Fig. S2(c)) has optic axis with polar angle $\theta_p$ between $(0^\circ, 90^\circ)$. The polar pattern of $\sigma$-plate with thickness $d = 5\mu m$, $\theta_p = 15^\circ$, $\phi_p = 20^\circ$, is depicted in Fig. S2(d). It can be seen that the central dark area is shifted towards $(\theta_p, \phi_p) = (15^\circ, 20^\circ)$.

![Fig. S2. The configuration of conoscopy of (a) $\alpha$-plate (b) $c$-plate (c) $\sigma$-plate between cross polarizers. Polar pattern of transmission of (d) $\alpha$-plate (optic axis $\phi_p = 45^\circ$, thickness $d = 52.5\mu m$), (e) $c$-plate (thickness $d = 2.5\mu m$) (f) $\sigma$-plate (thickness $d = 5\mu m$. tilt polar angle $\theta_p = 15^\circ$, tilt azimuth angle $\phi_p = 20^\circ$) between cross polarizers. The index and birefringence $\Delta n$ of the LC is $(n_o, n_e, \Delta n) = (1.5, 1.6, 0.1)$. All polarizers are ideal O-type polarizers with refractive index $n_p = 1.5$, with front polarizer transmission axis $p_1 = 0^\circ$ and back polarizer transmission axis $p_2 = 90^\circ$. Light source is set as monochromatic light with wavelength $\lambda = 510$ nm.

**C. Case Study of Single uniaxial HWP/QWP**

To validate the HWPs and QWPs design, we first use the extended Jones matrix to simulate single uniaxial $\alpha$-plate HWP and QWP. With the central wavelength set as $\lambda_c = 550$ nm and birefringence $\Delta n = 0.1$, the thickness of HWP ($d_H$) can be written as $d_H = 0.5\lambda_c/\Delta n$, similarly thickness of QWP $d_Q$ can be written as $d_Q = 0.25\lambda_c/\Delta n$.

In this subsection, both transmission and ellipticity will be validated. For the transmission validation, the uniaxial HWP or QWP will be placed between cross polarizers. Generally, despite some Fresnel reflection loss, the ideal transmission of HWP in this polarizer setting is around 50%, while the QWP case, the ideal transmission is around 25%.

For the ellipticity validation, the HWP or QWP will be placed behind the front polarizer, and the output intensity is used to calculate the ellipticity. To find the ellipticity $\chi$ and orientation angle $\Psi$ of the output light and the retardation of the material $\Phi$, we can represent the Stokes vector by Jones vector at every local point of the polar surface as:

$$S_0 = E_x E_x^* + E_y E_y^*; \quad (S1)$$

$$S_1 = E_x E_y^* - E_y E_x^*; \quad (S2)$$

$$S_2 = E_x E_y^* - E_y E_x^*; \quad (S3)$$

$$S_3 = i(E_y E_x^* - E_x E_y^*); \quad (S4)$$

The ellipticity and orientation angle can be calculated by:

$$\chi = \frac{1}{2} \arcsin \frac{S_3}{S_0} \quad (S5)$$
\[ \Psi = \frac{1}{2} \arctan \frac{S_2}{S_1} \]  

(S6)

In Fig. S3, it can be seen that the center transmission of HWP between cross-polarizers is almost half of the incidence energy, while the QWP gives around 25\%, which meets our expectation. Regarding to the ellipticity, the central ellipticity of HWP is around 0°, and QWP is around 45°. In both HWP and QWP cases, the half-wave or quarter-wave performance degrades as polar angle increases.

Fig. S3. Transmission polar pattern of (a) single uniaxial HWPs and (b) single QWPs between cross polarizers, with \( p_1 = 0° \), and \( p_2 = 90° \). Birefringence \( \Delta n = 0.1 \) and central wavelength \( \lambda_c = 550 \) nm. The ellipticity polar pattern of (c) single uniaxial HWP and (d) QWP. Polarizer is O-type ideal with index \( n_o = 1.52 \).

D. Case study of TN-cell

The case of TN-cell is similar to the chiral layer in MTR structure. Therefore, if we want to use the extended Jones matrix to simulate the whole MTR structure, it’s crucial to prove that the TN-cell simulation is correct. As an example, we analyze the transmission properties of the TN-LC with 20 layers, with 2° pretilt, \( \theta_p = 88° \), \( \phi_p = 45° \) twist angle, with other LC thickness \( d = 5.9 \) \( \mu m \), \( n_o = 1.487 \), \( n_e = 1.568 \) and \( \lambda = 0.55 \) \( \mu m \) for the incident light. When there is no drive voltage \( U = 0V \), the tilt \( \theta_p(z/d) = 0° \), while the twist \( \phi_p(z/d) = 100(z/d)(°) \) in Fig. S4(a), resulting the polar pattern transmission in Fig. S4(b). When there is a drive voltage \( U = 8V \) across TN-LC cell, the TN cell twist and tilt angle will be changed by the voltage (Fig. S4(c)), making TN-LC cell at its OFF state and the incident light is blocked on normal incidence, while some leakage at oblique incidence (Fig. S4(d)).

We find that our simulation of polar intensity pattern approximately matches the experimental pattern[1], therefore we can say that we successfully proved the validity of the Matlab code of the extended Jones matrix. It should be noticed that the slight difference of the OFF case between the experimental pattern and the simulation might originate to the finite sampling of the twist and tilt.

E. Case Study of Prior MTR Waveplate Design

Although the previous work[2] is not simulated by Extended Jones matrix, we assume that we can still reproduce the similar result in normal incidence. In this section we try to use the extended Jones matrix to validate the transmission, equivalent retardation, equivalent optical axis, ellipticity of MTR HWPs and QWPs designs in prior work in the case of normal incidence. The design variables of 5 achromatic HWPs and QWPs design are summarized in the Table. S1.
Fig. S4. TN-cell simulation with 20-layer LC thickness. Each layer with $d = 5.9\,\mu m$, $n_o = 1.487$, $n_e = 1.568$ and $\lambda = 0.55\,\mu m$ for the incident light. (a) Distribution of the tilt and twist angles of the LC directors inside the LC cell (ON state, 0-V applied voltage). (b) Intensity transmission pattern of the NW TN-LCD in its ON state. (c) Distribution of the tilt and twist angles of the LC directors inside the LC cell (OFF state, 8-V applied voltage). (d) Intensity transmission pattern of the NW TN-LCD in its OFF state.

Table S1. Prior Design of MTR waveplate

| Design  | $\phi_0$ ($^\circ$) | $\phi_1$ ($^\circ$) | $d_1$ ($\mu m$) | $\phi_2$ ($^\circ$) | $d_2$ ($\mu m$) | $\phi_3$ ($^\circ$) | $d_3$ ($\mu m$) |
|---------|---------------------|---------------------|-----------------|---------------------|-----------------|---------------------|-----------------|
| 2TR HW-A | -18.5               | 171                 | 1.94            | -62.4               | 3.20            | -                   | -               |
| 2TR HW-B | -3.4                | 69.7                | 1.56            | -69.7               | 1.56            | -                   | -               |
| 2TR QW-A | 14.3                | 0                   | 1.18            | 83.5                | 1.13            | -                   | -               |
| 3TR QW-A | 6.4                 | 0                   | 1.05            | 43.1                | 1.95            | 83                  | 0.83            |

Firstly, we investigate the HWPs design 2TR HW-A. Generally, for the requirement of half-wave retardation with linear input, the initial Stokes vector is $S_i = (1, 1, 0, 0)^T$, and the ideal output should be $S_i = (1, -1, 0, 0)^T$. Such half-wave retardation may varies by wavelength because of the different optical path and wavelength dispersion of different wavelength. In the case of HWPs (2TR HW-A), the variation of $S_1$ is depicted in Fig. S5(a). It can be seen that the $S_1 \in [-1, -0.98]$ from wavelength $\lambda \in [440 \,nm, 680 \,nm]$, suggesting the half-wave performance is satisfactory within this wavelength range.

Another HWPs design 2TR HW-B with circular input is also investigated. Contrary to the 2TR HW-A, The 2TR HW-B is optimized with the input Stokes vector is $S_i = (1, 0, 0, 1)^T$, and the ideal output should be $S_i = (1, 0, 0, -1)^T$. That is the half-wave retardation brings the right-hand-circular (RHC) input to left-hand-circular (LHC) output. It can be seen that the $S_3 \in [-1, -0.98]$ from wavelength $\lambda \in [457 \,nm, 650 \,nm]$, suggesting the half-wave performance is satisfactory within this wavelength range. This wavelength range is slightly narrower than the its linear input counterpart 2TR HW-A (Fig. S5(b)).

Finally, we investigate the QWPs design 2TR QW-A. The ellipticity is plotted to compare with the result in prior work. In this case, horizontal linear input of light is incident onto the 2TR QW-A. It can be seen that the ellipticity $e \in [0.98, 1]$ from wavelength $\lambda \in [450 \,nm, 620 \,nm]$, suggesting satisfactory quarter-wave retardation is imposed within this wavelength range (Fig. S5(c)).

To summarize, the result in this sections are identical to the curved in the prior work[2].
Fig. S5. (a) Stokes vector $S_1$ of MTR design 2TR HW-A. Unity horizontal linear input is incident onto the MTR HWPs. (b) MTR HW $S_3$ with design 2TR HW-B. Circular input $J_{in} = [1; i]$ and circular output. Perfect right-hand-circular input $S_i = (1, 0, 0, 1)^T$ is incident onto the MTR HWPs. (c) MTR QWPs (2TR QW-A) ellipticity at wavelength $\lambda = 550\text{nm}$. Unity horizontal linear input is incident onto the MTR HWPs.

manifesting the validity of extended Jones matrix.

2. RETARDATION PATTERN OF QWPS

This section provides retardation contour and the average retardation in off-axis of four types of QWPs. The average retardation Table. S2 for polar angle $\theta \in (0^\circ, 30^\circ)$ and Table. S3 for $\theta \in (0^\circ, 45^\circ)$ are presented to confirm that from purple to red wavelength range, the 3TR QW-D has the least variation of average retardation ($\bar{\Gamma}$) off quarter wave thus has the highest average ellipticity angle among these four.

| QWPs     | $\bar{\Gamma}(\lambda_P)$ | $\Delta \bar{\Gamma}(\lambda_P)$ | $\bar{\Gamma}(\lambda_B)$ | $\Delta \bar{\Gamma}(\lambda_B)$ | $\bar{\Gamma}(\lambda_C)$ | $\Delta \bar{\Gamma}(\lambda_C)$ | $\bar{\Gamma}(\lambda_R)$ | $\Delta \bar{\Gamma}(\lambda_R)$ |
|----------|--------------------------|----------------------------------|---------------------------|----------------------------------|---------------------------|----------------------------------|---------------------------|----------------------------------|
| HQ       | -80.3                    | 20.2                             | -84.6                     | 15.8                             | -86.6                     | 12.8                             | -85.2                     | 14.8                             |
| 3TR QW-A | -83.3                    | 11.7                             | -82.2                     | 12.2                             | -86.1                     | 8.4                              | -88.2                     | 7.5                              |
| 3TR QW-C | -77.7                    | 22                               | -86.1                     | 11.5                             | -83.5                     | 13.2                             | -85.9                     | 12.3                             |
| 3TR QW-D | -87.3                    | 5.8                              | -86.5                     | 6.7                              | -86.5                     | 8.2                              | -88                      | 8.2                              |

| QWPs     | $\bar{\Gamma}(\lambda_P)$ | $\Delta \bar{\Gamma}(\lambda_P)$ | $\bar{\Gamma}(\lambda_B)$ | $\Delta \bar{\Gamma}(\lambda_B)$ | $\bar{\Gamma}(\lambda_C)$ | $\Delta \bar{\Gamma}(\lambda_C)$ | $\bar{\Gamma}(\lambda_R)$ | $\Delta \bar{\Gamma}(\lambda_R)$ |
|----------|--------------------------|----------------------------------|---------------------------|----------------------------------|---------------------------|----------------------------------|---------------------------|----------------------------------|
| HQ       | -76.8                    | 35.4                             | -80.3                     | 31.6                             | -82.3                     | 28.8                             | -82                      | 30.1                             |
| 3TR QW-A | -81.3                    | 22.7                             | -80.6                     | 25.9                             | -84.4                     | 19.8                             | -86.6                     | 15.1                             |
| 3TR QW-C | -75.4                    | 37.2                             | -82.3                     | 24.7                             | -81.5                     | 20.2                             | -83.9                     | 24.5                             |
| 3TR QW-D | -86.5                    | 14.7                             | -84.5                     | 16.9                             | -84.3                     | 20.2                             | -85.9                     | 19.2                             |

Besides, in Fig. S6 we find that the retardation polar pattern of these four types of QWPs has the similar distribution as the ellipticity pattern. This is because the QWPs retardation is the cause of the ellipticity change from linear polarization to circular polarization.
3. REFLECTION LEAKAGE OF QWPS

We first simulated the HQ design with perfect polarizer, ignoring the Fresnel reflection, and assuming the input of polarizer is 1. We compared the result of the HQ design simulated using commercial software TechWiz LCD in [3], and found that the HQ design matches the polar pattern of the result simulated by TechWiz LCD within $\theta \in (0^\circ, 90^\circ)$ (Fig. S7). This validation further proves that both the extended Jones matrix for the elements and the whole reflection system modelling is correct.

Fig. S6. Retardation of four types QWPs in four central wavelength. Polar angle range is $(0^\circ, 45^\circ)$.
Fig. S7. Polar lumen pattern of the reflection leakage composed by a polarizer, HQ design QWPs, and an ideal mirror. with polar range $\theta \in (0^\circ, 90^\circ)$. The reflection leakage is defined as $L = L_5 / L_1$, where $L_1$ and $L_5$ as the lumen of input Jones vector $E_1$ and final reflection Jones vector $E_5$ in the reflection system.

Fig. S8. Iso-luminance pattern of 3TR QW-D with Polar angle range is $\theta \in (0^\circ, 90^\circ)$.

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