Status of the Hadronic $\tau$ Decay Determination of $|V_{us}|$

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We update the hadronic $\tau$ determination of $|V_{us}|$, showing that current strange branching fractions produce results $2-3\sigma$ lower than 3-family unitarity expectations. Issues related to the size of theoretical uncertainties and results from an alternate, mixed $\tau$-electroproduction sum rule determination are also considered.

1. Introduction and Background

The determination of $|V_{us}|$ from hadronic $\tau$ decay data rests on the finite energy sum rule (FESR) relation,

$$\int_0^{s_0} w(s) \rho(s) \frac{d}{ds} = -\frac{1}{2\pi i} \oint_{|s|=s_0} w(s) \Pi(s) \frac{d}{ds}$$

valid for any analytic $w(s)$ and kinematic-singularity-free correlator, $\Pi$, having spectral function, $\rho(s)$. To obtain $|V_{us}|$, Eq. (1) is applied to the flavor-breaking (FB) correlator difference

$$\Delta \Pi(s) = [\Pi^{(0+1)}_{V/A;ud}(s) - \Pi^{(0+1)}_{V/A;us}(s)],$$

where $\Pi^{(J)}_{V/A;ij}$ are the spin $J = 0, 1$ components of the flavor $ij$, vector (V) or axial vector (A) current two-point functions, and $(0 + 1)$ denotes the sum of $J = 0$ and $1$ components. The OPE is to be employed on the RHS for sufficiently large $s_0$.

The spectral functions, $\rho^{(J)}_{V/A;ij}$, are related to the differential distributions, $dR_{V/A;ij}/ds$, of the normalized flavor $ij$ V or A current induced decay widths, $R_{V/A;ij} \equiv \Gamma(\tau^- \to \nu_\tau \text{hadrons} \gamma/\gamma_{V/A;ij}(\gamma))/\Gamma(\tau^- \to \nu_\tau e^- \nu_\tau(\gamma))$, by

$$\frac{dR_{V/A;ij}}{ds} = c_{\tau}^{EW} |V_{ij}|^2 \left[w^{(00)}_{L+T}(y_\tau)\rho^{(0+1)}_{V/A;ij}(s) - w^{(00)}_{L}(y_\tau)\rho^{(0)}_{V/A;ij}(s)\right]$$

with $y_\tau = s/m_\tau^2$, $w^{(00)}_{L+T}(y_\tau) = (y_\tau - 1 + 2y_\tau)$, $w^{(00)}_{L}(y_\tau) = 2y_\tau(1 + y_\tau)^2$, $V_{ij}$ the flavor $ij$ CKM matrix element, and, with $S_{EW}$ a short-distance electroweak correction $[2]$, $c_{\tau}^{EW} \equiv 12\pi^2 S_{EW}/m_\tau^2$.

Use of the $J = 0 + 1$, FB difference $\Delta \Pi_{\tau}$, rather than the analogous difference involving the linear combination of $J = 0, 1$ spectral functions appearing in Eq. (2), is a consequence of the extremely bad behavior of the integrated $J = 0$ (longitudinal) $D = 2$ OPE series $[3]$. Fortunately, apart from the accurately known $\pi$ and $K$ pole terms, contributions to $\rho^{(0)}_{V/A;ij}$ are $\propto [(m_i^2 - m_j^2)^2]$, making $ud$ continuum contributions negligible. Once the small continuum $us$ $J = 0$ contributions are determined phenomenologically using dispersive $[4]$ and sum rule $[5]$ analyses of the strange scalar and pseudoscalar channels, respectively, the $J = 0$ contributions can be subtracted, bin-by-bin, from $dR_{V/A;ij}/ds$, allowing one to construct the re-weighted $J = 0 + 1$ spectral integrals, $R_{W_{V/A;ij}}^{(\tau)}(s_0)$, defined by

$$\frac{R_{W_{V/A;ij}}^{(\tau)}(s_0)}{c_{\tau}^{EW} |V_{ij}|^2} = \int_0^{s_0} ds \, w(s) \rho^{(0+1)}_{V/A;ij}(s),$$

and, from these, the FB combinations,

$$\delta R_{W_{V/A}}^{(\tau)}(s_0) = \frac{R_{W_{V/A;ud}}^{(\tau)}(s_0)}{|V_{sd}|^2} - \frac{R_{W_{V/A;us}}^{(\tau)}(s_0)}{|V_{us}|^2}$$

with $|V_{sd}|$, and any parameters in

$$\delta R_{W_{V/A}}^{(\tau, \text{OPE})}(s_0) = c_{\tau}^{EW} \left[\frac{1}{2\pi i} \oint_{|s|=s_0} ds \, w(s) \Delta \Pi_{\tau}(s)\right]$$
from other sources, Eq. 1 yields [6]

$$|V_{us}| = \frac{R_{w,A;us}^w(s_0)}{R_{w,A;udd}^w(s_0)} \frac{\delta R_{w,\text{OPE}}^w(s_0)}{\delta R_{w,A;us}^w(s_0)}.$$  

(5)

$\delta R_{w,\text{OPE}}^w(s_0)$ is typically << $R_{w,A;udd,us}^w(s_0)$ (usually at the few-to-several-% level) for $s_0 \gtrsim 2$ GeV$^2$, making a high precision $|V_{us}|$ determination possible with only modest OPE precision [6].

It turns out (see also below) that the convergence of the integrated $J = 0 + 1, D = 2$ OPE series may also be somewhat problematic. As a result, it is also of interest to consider FESRs based on the alternate FB correlator difference,

$$\Delta \Pi_M \equiv 9\Pi_{EM} - 5\Pi_{V;ud}^{(0+1)} + \Pi_{A;ud}^{(0+1)} - \Pi_{V;A;us}^{(0+1)};$$  

(6)

where $\Pi_{EM}$ is the scalar part of the electromagnetic (EM) current two-point function. $\Delta \Pi_M$ shares with $\Delta \Pi_T$ the vanishing of $D = 0$ contributions to all orders but, by construction, has strongly suppressed $D = 2$ contributions [7].

$D = 4$ contributions turn out also strongly suppressed compared to those of $\Delta \Pi_T$. This suppression does not, however, persist beyond $D = 4$ [7]. The EM spectral function, $\rho_{EM}(s)$, required on the LHS of the $\Delta \Pi_M$ FESR, is given by $\rho_{EM}(s) = \frac{s_0(s)}{4\pi s_0(s) + m_0^2}$, with $s_0(s)$ the bare inclusive hadronic electroproduction cross-section. The $\Delta \Pi_M$ FESR yields a solution for $|V_{us}|$ of the form Eq. (5), with the RHS denominator replaced by $9R_{w,EM}^w(s_0) - 5 \frac{R_{w,A;us}^w(s_0)}{\sqrt{|V_{ud}|^2}} + \frac{R_{w,A;udd}^w(s_0)}{|V_{ud}|^2} - \delta R_{w,M}^w(s_0)$, where $R_{w,EM}^w(s_0) = c_{EM}^w \int_{s_0}^{s} ds w(s) \rho_{EM}(s)$ and

$$\delta R_{w,M}^w(s_0) = -c_{EM}^w \int_{s_0}^{s} ds w(s) \Delta \Pi_M(s).$$  

(7)

### 2. Spectral and OPE Input

#### 2.1. Spectral Input

We compute $R_{w,A;udd}^w(s_0)$ and $R_{w,A;us}^w(s_0)$ using the publicly available ALEPH $ud$ and $us$ [8] and $ud$ [9] spectral data and covariances. Separate $ud$ and A analogous, $R_{w,V/A;udd}^w(s_0)$, required for the mixed $\tau$-electroproduction FESRs implement the improved $K^{\pm}K^{\mp}$ V/A $ud$ separation [10] provided by CVC and the BaBar determination of $I = 1 K^{\mp}K^{\mp}$ electroproduction cross-sections [11].

A small global rescaling of the continuum $ud$ V+A distribution accounts for recent changes in $S_{EW}$, $R_{V+A;us}$ and $B_c$. We employ $|V_{ud}| = 0.97425(23)$ [12] and current values [13] for $B_c$, $R_{V+A;us}$ and $R_{V+A;ud}$. Since BaBar and Belle have not yet completed their remeasurements of $dR_{V+A;us}/ds$, we work with an interim partial update obtained by rescaling the 1999 ALEPH distribution [8] mode-by-mode by the ratio of new to old world averages for the branching fractions [14]. The new world averages, based on the results of Refs. [15,16,17,18,19,20,21,22], are given in Table 1 [13]. The $us$ V+A covariance matrix cannot yet be analogously updated, so the improved precision on the $us$ branching fractions translates into an improved $us$ spectral integral error only for $w = w_{L,T}^{(00)} = w_{L,T}^{(00)}$ and $s_0 = m_\tau^2$.

Details of the treatment of the EM spectral data, required for the spectral integral side of the $\Delta \Pi_M$ FESR, are omitted here because of space constraints, but may be found in Ref. [7].

Table 1

| $X_{us}$ | $B_{WA,2008}$ (%) | Refs. |
|---------|-----------------|-------|
| $K^-$ | 0.690(10) | [13,18] |
| $K^-\pi^0$ | (0.715(4)) | |
| $K^0\pi^-$ | 0.835(22) | (S = 1.4) [17,22] |
| $K^-\pi^0\pi^0$ | 0.058(24) | |
| $K^0\pi^0\pi^-$ | 0.360(40) | |
| $K^-\pi^-\pi^+$ | 0.290(18) | (S = 2.3) [16,21] |
| $K^\eta$ | 0.016(1) | [22] |
| $(K3\pi)^-$ (est’d) | 0.074(30) | |
| $K^-\phi$ | 0.067(21) | |
| $(K4\pi)^-$ (est’d) | 0.011(7) | |
| $K^*\eta$ | 0.014(1) | [22] |
| $K^\phi$ | 0.0037(3) | (S = 1.3) [10,19] |
| **TOTAL** | 2.845(69) | (2.870(68)) |
2.2. OPE input

To keep OPE-breaking contributions from the vicinity of the timelike point on \( |s| = s_0 \) sufficiently suppressed, we restrict our attention to \( w(s) \) having at least a double zero at \( s = s_0 \), and to \( s_0 \gtrsim 2 \text{ GeV}^2 \).

The leading, \( D = 2 \), OPE contribution to \( \Delta \Pi \) is known to 4 loops [24]:

\[
[\Delta \Pi_r(Q^2)]_{D=2}^{\text{OPE}} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[ 1 + \frac{7}{3} \bar{a} + 19.93 \bar{a}^2 + 208.75 \bar{a}^3 + \cdots \right],
\]

with \( \bar{a} = \alpha_s(Q^2)/\pi \), and \( \alpha_s(Q^2) \) and \( m_s(Q^2) \) the running coupling and strange quark mass in the \( \overline{MS} \) scheme. Since independent determinations of \( \alpha_s \) imply \( \bar{a}(m_s^2) \approx 0.1 \), convergence at the spacelike point on \( |s| = s_0 \) is marginal at best. With such slow convergence, conventional prescriptions for assessing the \( D = 2 \) truncation uncertainty may lead to significant underestimates.

To deal with the potential \( D = 2 \) convergence problem, one may either work with \( \Delta \Pi \) and \( w(s) \) chosen to emphasize regions of the complex \( s = -Q^2 \)-plane away from the spacelike point, where \( |\alpha_s(Q^2)| \) is smaller and convergence improved [25], or switch to the alternate \( \Delta \Pi_M \) FESRs where \( D = 2 \) contributions are suppressed already at the correlator level [7]. In the latter case, the \( D = 2 \) contribution becomes

\[
[\Delta \Pi_{r,EM}(Q^2)]_{D=2}^{\text{OPE}} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[ \frac{1}{3} \bar{a} + 4.384 \bar{a}^2 + 44.94 \bar{a}^3 + \cdots \right],
\]

more than an order of magnitude smaller than in the \( \Delta \Pi \) case. Since \( \alpha_s(s_0) \) grows with decreasing \( s_0 \), making higher order terms relatively more important at lower scales, extracted \( |V_{us}| \) results will display an unphysical \( s_0 \)-dependence if neglected, higher order \( D = 2 \) terms are, in fact, not negligible. \( s_0 \)-stability studies thus provide a handle on the impact of the potentially slow integrated \( D = 2 \) convergence on \( |V_{us}| \).

As \( D = 4 \) OPE contributions to \( \Delta \Pi_{r}(Q^2) \) and \( \Delta \Pi_M(Q^2) \) are determined by \( \langle m_s s \rangle \) and \( \langle m_t \ell \ell \rangle \), up to negligible \( O(m_s^2) \) corrections. The relevant expressions, as well as those for the \( D = 6 \) four-quark condensate contributions, are easily constructed from the results of Ref. [26], and given in Ref. [7]. If one works with weights \( w(s) = \sum_{m=0}b_m y^m \), with \( y = s/s_0 \), integrated \( D = 2k + 2 \) OPE terms scale as \( 1/s_0^k \), allowing contributions of different \( D \) to be distinguished by their differing \( s_0 \)-dependences.

As \( D = 2 \) OPE input, we employ \( m_s(2 \text{ GeV}) = 96(10) \text{ MeV} \) [27] and \( \alpha_s(m_s^2) = 0.323(9) \), the latter obtained from an average, \( \alpha_s(M_Z) = 0.1190(10) \), of various recent determinations (including lattice [28] and \( \tau \) [29] results, which are now in very good agreement) via the standard combination of 4-loop running and 3-loop matching at the flavor thresholds [30].

At \( D = 4 \), we employ the GMOR relation for \( \langle m_t \ell \ell \rangle \) and evaluate \( \langle m_s s \rangle \) using the ChPT determination of \( m_s/m_t \) [31] and \( \langle m_s s \rangle/\langle m_t \ell \ell \rangle = 1.2(3) \), the latter obtained by updating Ref. [32] using the average of recent \( n_f = 2 + 1 \) lattice determinations of \( f_{B_s}/f_B \) as input [33].

As \( D = 6 \) contributions are estimated using the vacuum saturation approximation (VSA), rescaled by \( \rho_{VSA} = 1(5) \), while \( D > 6 \) contributions are neglected. Since integrated \( D \geq 6 \) OPE contributions scale as \( 1/s_0^N \) (\( N \geq 2 \)), if \( D > 4 \)
Figure 2. $|V_{us}|$ versus $s_0$ from the $\Delta \Pi_\tau$ FESRs for, from top to bottom, $w_{20}$, $\hat{w}_{10}$, $w_{10}$ and $w_{(00)}$, with the spectral input modified by rescaling up by $3\sigma$ the branching fraction of the large, but not yet remeasured, $\bar{K}^0\pi^-\pi^0$ mode.

contributions are, in fact, not small, and these input assumptions are unreliable, an unphysical $s_0$-dependence of $|V_{us}|$ will result, again making $s_0$-stability tests important.

3. Results and discussion

The results for $|V_{us}|$ obtained using the inputs specified above for the $\Delta \Pi_\tau$ FESRs based on the $J = 0 + 1$ kinematic weight $w_{(00)}(y)$, and three weights, $w_{10}(y)$, $w_{20}(y)$, $\hat{w}_{10}(y)$, constructed in Ref. [25] specifically to improve the poor integrated $J = 0 + 1$, $D = 2$ convergence, are displayed in Fig. 1. The $s_0$-instability of the $w_{(00)}$ results is much greater than the theoretical uncertainty $\sim \pm 0.0005$ often quoted for the $s_0 = m^2_\tau$ version of this analysis in the literature. The results corresponding to $\hat{w}_{10}$, in contrast, display a very good window of $s_0$-stability. A positive feature of the $\Delta \Pi_\tau$ analysis is the fact that the results for all four weights appear to be converging towards the stable $\hat{w}_{10}$ value as $s_0 \to m^2_\tau$.

The $s_0 = m^2_\tau$ versions of the various analyses are

$$|V_{us}| = \begin{cases} 0.2180(32)(15) & (\hat{w}_{10}) \\ 0.2188(29)(22) & (w_{20}) \\ 0.2172(34)(11) & (w_{10}) \\ 0.2160(26)(8) & (w_{(00)}) \end{cases}$$

where the first error is experimental (dominated by $us$ spectral errors) and the second the nominal theoretical error. The nominal theory error is obviously much smaller than the observed $s_0$-instability in the $w_{(00)}$ case, and hence unrealistically small. Comparison to the results of earlier $\Delta \Pi_\tau$ FESR analyses [6,27,34,35] shows the significant impact of recent, improved $us$ experimental results on the $|V_{us}|$ central values. The decreases represented by the remeasured $us$ branching fractions, lead to $|V_{us}|$ results $2 - 3\sigma$ below the 3-family-unitarity expectation, 0.2255(1) [12].

It should be stressed that several important strange decay modes have yet to be remeasured.
by either BaBar or Belle, and that the level of consistency of the $s_0 = m_{\tau}^2$ results for different weights could be significantly affected by such future remeasurements. As an illustration, we show, in Figure 2, the impact on $|V_{us}|$ as a function of $s_0$ of rescaling upward by $3\sigma$ the as-yet-unmeasured $K^0\pi^-\pi^0$ branching fraction, and hence also the $K^0\pi^-\pi^0$ component of the $us$ spectral distribution employed above. The issue of whether plausible shifts in the as-yet-unmeasured branching fractions are capable of restoring agreement with 3-family-unitarity expectations is less clear. In fact, it would take simultaneous $3\sigma$ upward rescalings of all currently unreasured $us$ branching fractions to restore agreement. Such a rescaling, moreover, does not produce a convincing $s_0$-stability plateau for any of the weights considered, as shown in Figure 2.

The results for $|V_{us}|$ for the $\Delta\Pi_M$ FESRs based on $w_{(00)}(y)$, the weight $\hat{w}_{10}(y)$ displaying the best $s_0$-stability for the $\Delta\Pi_s$, FESR, and the weights $w_2$, $w_3$, and $w_4$, where $w_N = 1 - \frac{N}{s_0} y + \frac{y^2}{s_0^2}$, are displayed in Fig. 1. The weight $w_N$ produces a single surviving integrated $D = 2N + 2 > 4$ OPE contribution suppressed by the coefficient $1/(N-1)$ and scaling as $1/s_0^N$, making it a useful choice in this case, where the slow integrated $D = 2$ convergence found for the $w_N$ versions of the $\Delta\Pi_s$ FESRs is no longer relevant.

If it was poor $D = 2$ convergence which was responsible for the $s_0$-instability of the $w_{(00)}$ $\Delta\Pi_s$ FESR results, one would expect to see a much improved stability plateau for the corresponding $\Delta\Pi_M$ FESR, as is indeed found. The very good stability for the $w_N$ results also indicates that the integrated $D = 2N + 2$ contributions relevant to these cases become negligible in the upper part of the $s_0$ window displayed in the Figure. Since, however, $D \geq 6$ contributions increase in going from $\Delta\Pi_s$ to $\Delta\Pi_M$, one would expect the instability for weights like $\hat{w}_{10}$, which do not suppress these to the same extent as do the other weights considered, to be enhanced, as is indeed found to be the case. Even so, the $\hat{w}_{10}$ results converge well to the stable results for the other weights as $s_0 \to m_{\tau}^2$.

Figure 4. $|V_{us}|$ as a function of $s_0$ from the mixed $\tau$-electroproduction FESRs for, from top to bottom at the left, $\hat{w}_{10}$, $w_{(00)}$, $w_3$, $w_4$ and $w_2$.

Given the very good stability of the $w_{(00)}$ results, it is possible to quote a final result based on the $s_0 = m_{\tau}^2$ version of the $w_{(00)}$ FESR, which allows us to take advantage of the improvements in the $us$ branching fraction errors. The result is

$$|V_{us}| = 0.2208(27)(28)(5)(2) \quad (11)$$

where the first three errors are due to the uncertainties on the $us$ $V+A$, residual $I = 0$ EM and residual $ud$ $V/A$ spectral integrals, respectively, and the fourth is due to the $D = 2$ and 4 OPE uncertainties (see Ref. [7] for further details).

We conclude by stressing that, for both the $\Delta\Pi_s$ and $\Delta\Pi_M$ FESRs, improved errors on $dR_{V+A;us}/ds$ are crucial. This requires both remeasurements of as-yet-unmeasured strange mode branching fractions, pursuit of higher multiplicity modes with branching fractions down to the few$\times 10^{-5}$ level, and, in particular, a full investigation of the $K3\pi$ and $K4\pi$ modes, which were not in fact measured, but rather estimated, in the earlier experimental analyses.
REFERENCES

1. Y.-S. Tsai, Phys. Rev. D4 (1971) 2821.
2. J. Erler, Rev. Mex. Fis. 50 (2004) 200.
3. K. Maltman, Phys. Rev. D58 (1998) 093015; K. Maltman and J. Kambor, Phys. Rev. D64 (2001) 093014.
4. M. Jamin, J.A. Oller and A. Pich, Nucl. Phys. B587 (2000) 331; ibid. B622 (2002) 279; and Phys. Rev. D74 (2006) 074009.
5. K. Maltman and J. Kambor, Phys. Rev. D65 (2002) 074013.
6. E. Gamiz et al., JHEP 0301 (2003) 060; Phys. Rev. Lett. 94 (2005) 011803.
7. K. Maltman, arXiv:0811.1590.
8. R. Barate et al. (ALEPH Collaboration), Eur. Phys. J. C11 (1999) 599.
9. S. Schael et al. (The ALEPH Collaboration), Phys. Rep. 421 (2005) 191; M. Davies, A. Hocker and Z.Q. Zhang, Rev. Mod. Phys. 78 (2006) 1043.
10. M. Davies, arXiv:0811.1429.
11. B. Aubert et al. (The BaBar Collaboration), Phys. Rev. D77 (2008) 092002.
12. I. Towner, CKM2008 talk, Rome, Italy, Sep. 9-13, 2008; I.S. Towner and J.C. Hardy, Phys. Rev. C77 (2008) 025501; T. Eronen, et al., Phys. Rev. Lett. 100 (2008) 132502.
13. S. Banerjee (for the BaBar Collaboration), arXiv:0811.1211.
14. S. Chen et al., Eur. Phys. J. C22 (2001) 31.
15. B. Aubert et al. (BaBar Collaboration), Phys. Rev. D76 (2007) 051104.
16. B. Aubert et al. (BaBar Collaboration), Phys. Rev. Lett. 100 (2008) 011801.
17. B. Aubert et al. (BaBar Collaboration), arXiv:0808.1121.
18. See I. Nugent’s talk at τ 2008, Novosibirsk, Sep. 22-25, 2008.
19. K. Inami et al. (Belle Collaboration), Phys. Lett. B643 (2006) 5.
20. D. Epifanov et al. (Belle Collaboration), Phys. Lett. B654 (2007) 65.
21. K. Inami, K. Hayasaka, Y. Usuki and M.J. Lee (for the Belle Collaboration), arXiv:0810.3464.
22. K. Inami et al. (Belle Collaboration), arXiv:0811.0088
23. E.C. Poggio, H.R. Quinn, S. Weinberg, Phys. Rev. D13 (1976) 1958; K. Maltman, Phys. Lett. B440 (1998) 367, Nucl. Phys. Proc. Suppl. 123 (2003) 149; B.V. Geshkenbein, B.L. Ioffe, K.N. Zybalyuk, Phys. Rev. D64 (2001) 093009; V. Cirigliano, J.F. Donoghue, E. Golowich, K. Maltman, Phys. Lett. B555 (2003) 71 (2003) and Phys. Lett. B522 (2001) 245; V. Cirigliano, E. Golowich, K. Maltman, Phys. Rev. D68 (2003) 054013.
24. K.G. Chetyrkin and A. Kwiatkowski, Z. Phys. C59 (1993) 525 and hep-ph/9805232.
25. P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 95 (2005) 012003.
26. V. Cirigliano, E. Golowich, K. Maltman, Phys. Rev. D68 (2003) 054013.
27. E. Gamiz et al., PoS KAON 2007 (2008) 008; A. Pich, Nucl. Phys. Proc. Suppl. 181-182 (2008) 300.
28. C.T.H. Davies et al. (The HPQCD Collaboration), arXiv:0807.1687.
29. P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, Phys. Rev. Lett. 101 (2008) 012002; M. Beneke, M. Jamin, JHEP 0809 (2008) 044; K. Maltman, T. Yavin, Phys. Rev. D78 (2008) 094020.
30. K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, Phys. Rev. Lett. 79 (1997) 2184.
31. H. Leutwyler, Phys. Lett. B378 (1996) 313.
32. M. Jamin and B.O. Lange, Phys. Rev. D65 (2002) 056005; M. Jamin, Phys. Lett. B538 (2002) 71.
33. A. Gray et al. (The HPQCD Collaboration), Phys. Rev. Lett. 95 (2005) 212001; C. Bernard et al. (The FNAL and MILC Collaborations) PoS LATTICE 2007 (2007) 370; V. Lubicz and C. Tarantino, arXiv:0807.4605.
34. K. Maltman and C.E. Wolfe, Phys. Lett. B639 (2006) 283; ibid. B650 (2007) 27; hep-ph/0703314.
35. K. Maltman, C.E. Wolfe, S. Banerjee, J.M. Roney and I. Nugent, arXiv:0807.3195.