MODULI SPACES OF PRINCIPAL $F$-BUNDLES

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Abstract. In this paper we construct certain moduli spaces, which we call moduli spaces of (principal) $F$-bundles, and study their basic properties. These spaces are associated to triples consisting of a smooth projective geometrically connected curve over a finite field, a split reductive group $G$, and an irreducible algebraic representation $\varpi$ of $(\hat{G})^n/Z(\hat{G})$. Our spaces generalize moduli spaces of $F$-sheaves, studied by Drinfeld and Lafforgue, which correspond to the case $G = GL_r$ and $\varpi$ is the tensor product of the standard representation and its dual. The importance of the moduli spaces of $F$-bundles is due to the belief that Langlands correspondence is realized in their cohomology.

1. Introduction

Let $X$ be a smooth projective curve geometrically connected over a finite field $\mathbb{F}_q$, $F = \mathbb{F}_q(X)$ the field of rational functions on $X$, $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of $F$, $\Gamma_F$ the absolute Galois group of $F$, and $l$ a fixed prime, not dividing $q$.

Recall that the Langlands correspondence for $GL_r$, proved by Lafforgue ([La2]), associates an irreducible $\ell$-adic representation $\rho_{\pi} : \Gamma_F \to GL_r(\mathbb{Q}_l)$ to every cuspidal representation $\pi$ of $GL_r(\mathbb{A})$, whose central character is of finite order. As a result, to each pair consisting of $\pi$ and an algebraic representation $\omega$ of $GL_r$, Langlands correspondence associates an $\ell$-adic representation $\rho_{\pi,\omega} := \omega \circ \rho_{\pi}$ of $\Gamma_F$.

Let $G$ be a split reductive group over $\mathbb{F}_q$, hence over $F$, and let $\hat{G}/\mathbb{Q}_l$ be the dual group of $G$. Then Lafforgue theorem together with Langlands functoriality conjecture predicts that to every pair $(\pi, \omega)$ consisting of a tempered cuspidal representation $\pi$ of $G(\mathbb{A})$ with finite order central character and an algebraic representation $\omega$ of $\hat{G}$, one can associate an $\ell$-adic representation $\rho_{\pi,\omega}$ of $\Gamma_F$, whose $L$-function equals that of $(\pi, \omega)$.

More generally, for each $n \in \mathbb{N}$ let $F^{(n)} = \mathbb{F}_q(X^n)$ be the field of rational functions of $X^n$. Then $\pi$ together with an $n$-tuple $\varpi = (\omega_1, \ldots, \omega_n)$ of representations of $\hat{G}$ give rise to an $\ell$-adic representation $\rho_{\pi,\varpi}$ of $\Gamma_{F^{(n)}}$, defined as the composition of the natural restriction map $\Gamma_{F^{(n)}} \to (\Gamma_F)^n$ with representation $\otimes_{i=1}^n \rho_{\pi,\omega_i}$ of $(\Gamma_F)^n$. In

This research was supported by ISF (grant No. 38/01-1).
particular, the Langlands conjecture associates to every pair consisting of $\pi$ and an irreducible representation $\omega$ of $(\hat{G})^n$ a certain $\ell$-adic representation $\rho_{\pi,\omega}$ of $\Gamma_F(n)$.

Furthermore, it is generally believed that the corresponding $\rho_{\pi,\omega}$ is “motivic”, that is, there exists an algebraic variety $X_{\pi,\omega}/F(n)$ such that $\rho_{\pi,\omega}$ is a subquotient of its cohomology. Moreover, for certain $\omega$’s one hopes to find an algebraic “space” $X_{\omega}$ which “realizes all of $\rho_{\pi,\omega}$”. By this we mean that $X_{\omega}$ is equipped with an action $G(\mathbb{A})$, and $H^*_c(X_{\omega}, \text{IC}(\mathbb{Q}_l))$ has a “motivic” subquotient isomorphic to the direct sum of the $(\pi \boxtimes \rho_{\pi,\omega})$’s, taken with certain multiplicities.

When $G = GL_r$, $n = 2$ and $\omega$ is the product of the standard representation and its dual, the existence of $X_{\omega}$ was proved by Lafforgue ([La2]), generalizing an earlier work of Drinfeld ([Dr1, Dr2]). On the other hand, the required space cannot exist in the case $G = GL_2$, $n = 1$ and $\omega$ is the standard representation. Indeed, the subquotient should be defined over $\mathbb{Q}_l$, but the direct sum $\bigoplus_{\pi} (\pi \boxtimes \rho_{\pi,\omega})$ is not (see [Ka]). Thus $X_{\omega}$ can exist only for certain $\omega$’s.

The goal of this paper is to construct a candidate of $X_{\omega}$ for each irreducible representation $\omega$ of $(\hat{G})^n/\mathbb{Z}((\hat{G}))$ (where $\mathbb{Z}((\hat{G}))$ is the center of $\hat{G}$, embedded diagonally in $(\hat{G})^n$) and to study its basic properties. By analogy with $F$-sheaves, introduced by Drinfeld, we will call our spaces moduli spaces of (principal) $F$-bundles.

More precisely, for each $n \in \mathbb{N}$ we construct a “space” $X_n$ over $F(n)$, equipped with an action of $G(\mathbb{A})$. Next for each irreducible representation $\omega$ of $(\hat{G})^n/\mathbb{Z}((\hat{G}))$ we construct a $G(\mathbb{A})$-invariant closed “subspace” $X_{\omega}$. Then we construct a “cuspidal” subquotient of $H^*_c(X_{\omega}, \text{IC}(\mathbb{Q}_l))$, in which Langlands correspondence “should be realized”. This is especially plausible in the case $G = GL_r$ (see Conjecture 2.35), when Langlands correspondence is known, so the question is well posed. As evidence, we show that our conjecture holds in the Drinfeld’s case and holds “up to $r$-negligibles” in the Lafforgue case.

Roughly speaking, our construction can be described as follows: the space $X_n$ classifies triples consisting of an $n$-tuple $(x_1, \ldots, x_n) \in X^n$, a $G$-bundle $\mathcal{G}$ on $X$, and an isomorphism $\phi$ between the restrictions of $G$ and its Frobenius twist $\tau \mathcal{G}$ to the complement of (the graphs of) the $x_i$’s.

To define $X_{\omega}$’s, observe that each irreducible representation $\omega$ of $(\hat{G})^n$ (hence of $(\hat{G})^n/\mathbb{Z}((\hat{G}))$) corresponds to a certain $n$-tuple $(\omega_1, \ldots, \omega_n)$ of dominant coweights of $G$ (see Remark 2.17). Then we define $X_{\omega}$ to be the closed substack of $X_n$, consisting of those triples $(\mathcal{G}; x_1, \ldots, x_n; \phi)$, for which the relative position of $\phi(\mathcal{G})$ and $\tau \mathcal{G}$ at $x_i$ is less than or equal to $\omega_i$ for each $i$.

By construction, $X_n$ is just a twisted version of the global affine grassmannian over $X^n$. Moreover, if we denote by $\mathcal{F}_{\omega}/X_n$ the extension by zero of the IC-sheaf
of $X_\omega$, then the correspondence $\omega \mapsto F_\omega$ is just a twisted version of the geometric Satake correspondence (see Theorem 2.20 and Corollary 2.21).

Finally, to get the required subquotient of $H^0_c(X_\omega, IC(Q_l))$ we proceed in two steps: first, we consider its maximal pure quotient of weight zero $H^0_c, pure(X_\omega, IC(Q_l))$, and then take the subspace of $H^0_c, pure(X_\omega, IC(Q_l))$, consisting of all elements, vanishing on the locus of reducible $F$-bundles (i.e., those $F$-bundles which has a $\phi$-invariant parabolic structure).

We would like to note that our construction is a rather straightforward combination of the original Drinfeld construction of the moduli of $F$-sheaves, Beilinson–Drinfeld construction of the Hecke stacks, and geometric Satake correspondence. In particular, it was known to Drinfeld and some others.

Notation and conventions

1) Let $G$ be a split reductive group over a finite field $\mathbb{F}_q$, let $G^{der}$ be the derived group of $G$, $G^{sc}$ the simply-connected cover of $G^{der}$, $G^{ab} := G/G^{der}$ the abelianization of $G$, and $G^{ad}$ the adjoint group of $G$. Let $B \supset T \subset Z$ be a Borel subgroup, a maximal torus, and the center of $G$, respectively. We denote by $B^{sc} \supset T^{sc} \subset Z^{sc}$ the corresponding objects of $G^{sc}$, and similarly for $G^{der}$ and $G^{ad}$.

2) Let $\rho$ be the half-sum of all positive coroots of $G$.

3) By a quasi-fundamental weight of $G$ we mean the smallest positive multiple of a fundamental weight of $G^{sc}$, which belongs to $X^*(T^{ad}) \subset X^*(T)$.

4) Let $X^*_+(T)$ and $X^{*_+}(T)$ be the sets of dominant weights and coweights of $G$, respectively.

5) Weights (resp. coweights) of $G$ we equip with (standard) ordering: $\lambda_1 \leq \lambda_2$ if and only if the difference $\lambda_2 - \lambda_1$ is a positive integral linear combination of simple roots (resp. coroots) of $G$.

6) For an algebraic group $H$, an $H$-bundle $\mathcal{H}$ on $Y$, and a representation $V$ of $G$, we denote the vector bundle $H \backslash [\mathcal{H} \times V]$ on $Y$ by $\mathcal{H}_V$.

7) For a dominant weight $\lambda$ of $G$, we denote by $V_\lambda$ the Weyl module of $G$ with the highest weight $\lambda$. Also for a $G$-bundle $G$ on $Y$, we denote $G_{V_\lambda}$ by $G_\lambda$.

8) For a finite scheme $D$ over a field $k$, put $\mathcal{O}_D := k[D]$ and $|D| := \dim_k \mathcal{O}_D$.

9) For a finite scheme $D$ over a field $k$, denote the Weil restriction of scalars $R_{D/k}G$ by $G_D$. In particular, $G_D(k) = G(\mathcal{O}_D)$. More generally, for every closed embedding between finite schemes $D_1 \hookrightarrow D_2$, we denote by $G_{D_1,D_2}$ the kernel of the natural homomorphism $G_{D_2} \to G_{D_1}$.

10) For a closed point $v$ of a curve $X$, let $\mathcal{O}_v$ and $F_v$ be the completions at $v$ of the stalk at $v$ of the structure sheaf and the field of its fractions, respectively.

11) For an $S$-point $x$ of a scheme $X$, let $\Gamma_x \subset X \times S$ be the graph of $x$. 


12) By $\Delta \subset X^n$ we denote the set of all $n$-tuples $(x_1, \ldots, x_n)$ for which there exist $i \neq j$ with $x_i = x_j$.

13) By an IC-sheaf on a stack $Y$, we will mean the intermediate extension of the constant perverse $\mathbb{Q}_l$-sheaf on a open dense substack $Y^0$ of $Y$ such that the corresponding reduced stack $(Y^0)_{\text{red}}$ is smooth. The IC-sheaf is normalized so that it is pure of weight zero. The IC-sheaf on $Y$ will denote by $\text{IC}_Y$ or simply by IC.

14) For a stack $Y$ over a finite field $\mathbb{F}_q$, we denote by $\text{Frob}_q : Y \to Y$ the absolute Frobenius morphism over $\mathbb{F}_q$.

Acknowledgments

First, the author thanks V. Drinfeld, who suggested to study moduli spaces of $F$-sheaves and mentioned that this construction can be extended to arbitrary groups. Secondly, the present work would not be possible without numerous conversations with D. Gaitsgory, who explained many things to me about $B$-structures, affine grassmannians and geometric Satake correspondence. He also read preliminary drafts of the paper, and his suggestions significantly improved the exposition. Also I thank J. Arthur, J. Bernstein and D. Kazhdan for their interest, stimulating conversations and useful remarks.

Different parts of the work were done while the author visited the University of Toronto, IHP and IHES, which I thank for stimulating atmosphere and financial support.

2. Main constructions and results

Notation 2.1. a) Let $X$ be a smooth projective curve geometrically connected over a field $k$, and let $\text{Bun} = \text{Bun}_G$ be the stack classifying $G$-bundles on $X$, i.e., $\text{Bun}_G(S) = \{G \text{- bundles on } X \times S\}$ for each scheme $S$ over $k$.

More generally, for each finite subscheme $D \subset X$, let $\text{Bun}_D = \text{Bun}_{G,D}$ be the stack over $\text{Bun}$ classifying $G$-bundles on $X$ with $D$-level structures, i.e.,

$$\text{Bun}_{G,D}(S) = \{G \in \text{Bun}_G(S), \psi : G \mid_{D \times S} \sim G \times D \times S\}.$$  

b) For each $\mu \in X_*(T^{\text{sc}}) \otimes \mathbb{Q} = X_*(T^{\text{ad}}) \otimes \mathbb{Q}$, let $\text{Bun}_{G}^{\leq \mu}$ be a substack of $\text{Bun}_G$, consisting of $G$-bundles, whose degree of instability is bounded by $\mu$, i.e.,

$$\text{Bun}_G^{\leq \mu}(S) = \{G \in \text{Bun}_G(S) \mid \text{ for each geometric point } s \in S, \text{ each } B \text{- structure } B \text{ of } G_s \text{ and each } \lambda \in X_+^*(T^{\text{ad}}) : \deg B_\lambda \leq \langle \mu, \lambda \rangle\},$$  

where $B_\lambda$ is the corresponding line bundle. A substack $\text{Bun}_G^{\leq \mu}$ is open in $\text{Bun}_G$ (see Lemma A.3). More generally, for every $D$ we will denote by $\text{Bun}_{G,D}^{\leq \mu}$ the preimage of $\text{Bun}_G^{\leq \mu}$ in $\text{Bun}_{G,D}$.  

The proof of the following basic fact will be recalled in \( \Box \).

**Lemma 2.2.** The set of connected components \( \pi_0(Bun_G) \) of \( Bun_G \) is canonically isomorphic to \( \pi_1(G) := X_*(T)/X_*(T^{\text{sc}}) \) (which in turn is canonically isomorphic to the group of characters of \( Z(\hat{G}) \)).

**Notation 2.3.** Denote by \( \pi_0 \) the canonical map \( Bun_G \to \pi_0(Bun_G) = \pi_1(G) \), and denote by \( [\omega] \in \pi_1(G) \) the class of \( \omega \in X_*(T) \).

Next will introduce affine grassmannians and Hecke stacks following Beilinson and Drinfeld ([BD]).

**Definition 2.4.** a) For each \( n \in \mathbb{N} \) and each finite (not necessary non-empty) subscheme \( D \subset X \), let \( Hecke_{D,n} \) be the stack, which for each scheme \( S \) over \( k \), classifies triples:

i) \((\mathcal{G}, \psi), (\mathcal{G}', \psi') \in Bun_D(S)\);

ii) \( n \) points \( x_1, x_2, \ldots, x_n \in (X \setminus D)(S) \);

iii) isomorphism \( \phi : G_{|(X \times S) \setminus (\Gamma_1 \cup \cdots \cup \Gamma_n)} \cong G'_{|(X \times S) \setminus (\Gamma_1 \cup \cdots \cup \Gamma_n)} \), preserving \( D \)-level structures (that is satisfying \( \psi' \circ \phi|_{D \times S} = \psi \)).

As in the case of \( Bun_D \), we will omit \( D \) from the notation when \( D = \emptyset \).

b) For each \( n \)-tuple of dominant coweights \( \bar{\omega} = (\omega_1, \ldots, \omega_n) \) of \( G \), let \( Hecke_{D,n,\bar{\omega}} \) be the closed substack of \( Hecke_{D,n} \) defined by the condition that "the relative position of \( \phi(\mathcal{G}) \) and \( \mathcal{G}' \) at \( x_j \) is less than or equal to \( \omega_i \) for each \( i \)" in the following sense:

i) \( \phi(\mathcal{G}_\lambda) \subset \mathcal{G}'_\lambda((\sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i})) \) for each dominant weight \( \lambda \) of \( G \);

ii) \( \pi_0(\mathcal{G}_s) - \pi_0(\mathcal{G}'_s) = [\sum_{i=1}^n \omega_i] \) for each geometric point \( s \in S \).

By Lemma 3.1, \( Hecke_{D,n,\bar{\omega}} \) is an algebraic stack locally of finite type over \( k \).

**Remark 2.5.** a) Condition iii)\( \bar{\omega} \) implies that for each character \( \lambda \in X^*(G) \) we get \( \phi(\mathcal{G}_\lambda) = G'_\lambda((\sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i})) \). Indeed, apply iii)\( \bar{\omega} \) to both \( \lambda \) and \( \lambda^{-1} \).

b) If \( G_{\text{def}} \) is simply connected, then iii)\( \bar{\omega} \) is a consequence of iii)\( \bar{\omega} \). Indeed, in this case, \( \pi_1(G) = X_*(G^{\text{ab}}) \). Hence condition iii)\( \bar{\omega} \) is equivalent to the equality \( \text{deg}(\mathcal{G}_\lambda) = \text{deg}(G'_\lambda + \sum_{i=1}^n \langle \lambda, \omega_i \rangle \Gamma_{x_i}) \) for every \( \lambda \in X^*(G) = X^*(G^{\text{ab}}) \) (compare the proof of Lemma 2.2 in \( \Box \)). Thus the statement follows from a).

c) Stack \( Hecke_n \) has a natural involution which interchanges \( \mathcal{G}, \phi \) with \( \mathcal{G}', \phi^{-1} \). This involution sends \( Hecke_{n,\bar{\omega}} \) into \( Hecke_{n,-w_0(\bar{\omega})} \), where \( w_0 \) is the longest element of the Weyl group of \( G \), acting on \( X^*(T)^n \) diagonally. To see this, note that \( V_{-w_0(\lambda)} \) is the dual of the Weyl module \( V_\lambda \) and that the projection \( X^*(T) \to \pi_1(G) \) is constant on the orbits of the Weyl group.

d) One may consider a variant of the definition of \( Hecke_{n,\bar{\omega}} \), in which iii)\( \bar{\omega} \) is replaced by an a priori stronger condition: \( \phi(\mathcal{G}_V) \subset \mathcal{G}'_V((\sum_{i=1}^n \langle \xi, \omega_i \rangle \Gamma_{x_i})) \) for each weight \( \xi \) of \( G \) and each representation \( V \) of \( G \) all of whose weights are less than
or equal to $\xi$. Though the stack defined by this condition might be smaller than the original one, it follows from the Cartan decomposition that the corresponding reduced stacks coincide. However we do not know whether the same is true for stacks themselves (compare Remark A.10).

Following [BD], we also consider iterated Hecke stacks.

**Definition 2.6.** a) For each $n \in \mathbb{N}$ and each finite subscheme $D \subset X$, let $\text{Hecke}'_{D,n}$ be the stack which for each scheme $S$ over $k$ classifies triples consisting of the following:

i) $n + 1$ elements $(G, \psi) = (G_0, \psi_0), (G_1, \psi_1), \ldots, (G_n, \psi_n) = (G', \psi')$ of $\text{Bun}_D(S)$;

ii) $n$ points $x_1, x_2, \ldots, x_n \in (X \setminus D)(S)$;

iii) isomorphisms $\phi_i : G_{i-1}|_{(X \times S) \setminus \Gamma x_i} \sim G_i|_{(X \times S) \setminus \Gamma x_i}$ preserving $D$-level structures, for each $i = 1, \ldots, n$.

b) For each $n$-tuple of dominant coweights $\underline{\omega} = (\omega_1, \ldots, \omega_n)$ of $G$, let $\text{Hecke}'_{D,n,\underline{\omega}}$ be the closed substack of $\text{Hecke}'_{D,n}$ defined by the condition that each $[(G_{i-1}, \psi_{i-1}), (G_i, \psi_i); x_i; \phi_i] \in \text{Hecke}_{D,1}(S)$ belongs to $\text{Hecke}_{D,1,\omega_i}(S)$.

**Remark 2.7.** a) Alternatively, $\text{Hecke}'_{D,n,\underline{\omega}}$ can be defined as a fiber product

$$\text{Hecke}_{D,1,\omega_1} \times_{\text{Bun}_D} \text{Hecke}_{D,1,\omega_2} \times_{\text{Bun}_D} \ldots \times_{\text{Bun}_D} \text{Hecke}_{D,1,\omega_n}.$$  

b) We have a natural forgetful map $\pi : \text{Hecke}'_{D,n,\underline{\omega}} \to \text{Hecke}_{D,n,\underline{\omega}}$, which forgets $(G_1, \psi_1), \ldots, (G_{n-1}, \psi_{n-1})$ and replaces the $\phi_i$’s by their composition. Furthermore, $\pi$ is projective (see Lemma 3.1) surjective and small (see Lemma A.12).

c) More generally, for each partition $n = k_1 + \ldots + k_l$ one can similarly consider a partially iterated Hecke stack

$$\text{Hecke}_{D,k_1} \times_{\text{Bun}_D} \text{Hecke}_{D,k_2} \times_{\text{Bun}_D} \ldots \times_{\text{Bun}_D} \text{Hecke}_{D,k_l}.$$  

The previously defined stacks $\text{Hecke}_{D,n}$ and $\text{Hecke}'_{D,n}$ correspond to the trivial partition $n = n$ and the maximal partition $n = 1 + \ldots + 1$, respectively.

**Notation 2.8.** Denote by $p$ and $p'$ forgetful morphisms $\text{Hecke}_{D,n} \to \text{Bun}_D$ sending the triple to $(G, \psi)$ and $(G', \psi')$ respectively, and define by $\text{Hecke}^{\leq \mu}_{D,n}$ and $\text{Hecke}^{\leq \mu}_{D,n,\underline{\omega}}$ the preimages of $\text{Bun}^{\leq \mu}_D$ under $p$. Similarly we define $\text{Hecke}^{\leq \mu}{'}_{D,n}$ and $\text{Hecke}^{\leq \mu}{'}_{D,n,\underline{\omega}}$.

**Remark 2.9.** The space $\text{Hecke}_{D,n}$ is canonically isomorphic to the restriction to $(X \setminus D)^n$ of the fiber product $\text{Hecken} \times_{\text{Bun}} \text{Bun}_D$, where the map $\text{Hecken} \to \text{Bun}$ is either $p$ or $p'$.
Definition 2.10. Let $Gr_n$ (resp. $Gr_n$, $Gr'_n$, $Gr'_n$, $Gr''_n$, $Gr''_n$) be the stacks classifying the same data as $Hecke_n$ (resp $Hecke_n$, $Hecke'_n$, $Hecke'_n$, $Hecke''_n$, $Hecke''_n$), together with a trivialization of $G'$. These spaces are called global affine grassmannians (over $X^n$).

Now we are ready to introduce our main object. From now on $k$ will be a finite field $\mathbb{F}_q$.

Notation 2.11. For a scheme $S/\mathbb{F}_q$ and an $S$-point $A$ of a stack $\mathcal{X}$ over $\mathbb{F}_q$, we denote the $S$-point $Frob_q^*(A)$ by $^\tau A$. In particular, for a coherent sheaf or a $G$-bundle $\mathcal{F}$ over $X \times S$, we will write $^\tau \mathcal{F}$ instead of $(Id_X \times Frob_q)^*(\mathcal{F})$.

Definition 2.12. For each $n \in \mathbb{N}$ and finite subscheme $D \subset X$, let $FBun_{D,n}$ (resp. $FBun_{D,n}$, $FBun'_{D,n}$, $FBun''_{D,n}$) be the stack classifying the same data i)–iii) as $Hecke_{D,n}$ (resp. $Hecke_{D,n}$, $Hecke'_{D,n}$, $Hecke''_{D,n}$) together with an isomorphism $G' \sim ^\tau G$, preserving $D$-level structures.

We will call these spaces moduli spaces of (principal) $F$-bundles.

Remark 2.13. a) Explicitly, a stack $FBun_{D,n}$ classify triples consisting of a pair $(\mathcal{G}, \psi) \in Bun_D(S)$, an $n$-tuple $(x_1, \ldots, x_n) \in (X \setminus D)(S)$, and an isomorphism

$$\phi : \mathcal{G}|_{(X \times S) \setminus (\Gamma_{x_1} \cup \ldots \cup \Gamma_{x_n})} \sim ^\tau \mathcal{G}|_{(X \times S) \setminus (\Gamma_{x_1} \cup \ldots \cup \Gamma_{x_n})}$$

such that $^\tau \psi \circ \phi|_{D \times S} = \psi$. In particular, $FBun_{D,n}$ is equipped with a natural action of the group $G(\mathcal{O}_D)$, which replaces $\psi$ by $g \circ \psi$ for each $g \in G(\mathcal{O}_D)$ and does not change all the other data.

b) In the case $G = GL_r$, $\omega_1 = (0, \ldots, 0, -1)$ and $\omega_2 = (1, 0, \ldots, 0)$, the space of $F$-bundles $FBun'_{D,2,\omega_1,\omega_2}$ (resp. $FBun'_{D,2,\omega_2,\omega_1}$) is the moduli space of right (resp. left) $F$-sheaves $FSh_{D,r}$ (resp. $d_u FSh$) studied by Drinfeld and Lafforgue.

Definition 2.14. An $n$-tuple $\overline{\omega} = (\omega_1, \ldots, \omega_n) \in X_*(T)^n$ is called admissible if the sum of the $\omega_i$’s belongs to $X_*(T^{\Sigma})$.

Remark 2.15. As $Frob_q$ acts trivially on $\pi_0(Bun_G)$, we get from Lemma 2.2 that condition iii')$_{\overline{\omega}}$ of Definition 2.4 implies that $\overline{\omega}$ is admissible if $FBun_{n,\overline{\omega}}$ is non-empty. Conversely, if $\overline{\omega}$ is admissible, then condition iii')$_{\overline{\omega}}$ in the definition of $FBun_{n,\overline{\omega}}$ holds automatically.

The following proposition, whose proof will be given in [3.2], summarizes basic properties of the moduli spaces of $F$-bundles, generalizing [Dr1 Prop. 2.3 and 3.2].

Proposition 2.16. a) $FBun_{D,n}$ is a Deligne–Mumford stack over $(X \setminus D)^n$, locally of finite type. Moreover, connected components of $FBun_{D,n}^\mu$ are quotients of quasi-projective schemes over $(X \setminus D)^n$ by finite groups. Furthermore, these components are quasi-projective schemes, if $|D|$ is sufficiently large relative to $\mu$. 

b) Every $\text{FBun}_{D,n,\varpi}$ is a finite (étale) Galois cover of $\text{FBun}_{n,\varpi} \times X^n (X \setminus D)^n$ with Galois group $G_D(\mathbb{F}_q)$. In particular, for each $D_1 \subset D_2$, $\text{FBun}_{D_1,n,\varpi}$ is a finite Galois cover of $\text{FBun}_{D_1,n,\varpi} \times (X \setminus D_1)^n (X \setminus D_2)^n$ with Galois group $G_{D_1,D_2}(\mathbb{F}_q)$.

c) If $\omega_1 = \ldots = \omega_n = 0$, then $\text{FBun}_{D,n,\varpi}$ is canonically isomorphic to the product of $(X \setminus D)^n$ with a discrete stack $\text{Bun}_D(\mathbb{F}_q)$.

d) $\text{FBun}_{D,n,\varpi}$ is non-empty if and only if $\varpi$ is admissible.

e) The forgetful morphism $\pi : \text{FBun}_{D,n,\varpi} \to \text{FBun}_{D,n,\varpi}$ is projective. Moreover, $\pi$ is an isomorphism over $X^n \setminus \Delta$.

Remark 2.17. $n$-tuples of dominant coweights of $G$ are in canonical bijection with dominant weights of $(\hat{G})^n$ and hence with irreducible representations of $(\hat{G})^n$. Under this bijection, admissible $n$-tuples correspond to representations, trivial on $Z(\hat{G})$.

Notation 2.18. Denote by $[n]$ the set $\{1, \ldots, n\}$, and for each set $A$ we will identify $A^n$ with the set of functions on $[n]$ with values in $A$. In particular, every map $\eta : [n] \to [k]$ induces a map $\eta^* : A^k \to A^n$. Moreover, if $A$ is an abelian group, then $\eta$ induces also a map $\eta : A^n \to A^k$ (defined as $\eta(f)(i) := \sum_{j \in \eta^{-1}(i)} f(j)$).

2.19. Stratification. All of the above stacks have natural stratifications:

a) The stratification of $\text{Hecke}_n'$ (and hence of $\text{FBun}_n'$ and $\text{Gr}_n'$) is indexed by equivalence classes of triples $(k, \eta, \varpi)$, where $k \leq n$ is a positive integer, $\eta : [n] \to [k]$ is a surjection, $\varpi$ is an element of $X^+_*(T)^k$, and the equivalence relation is given by the rule $(k, \sigma \circ \eta, \varpi') \sim (k, \eta, \sigma^*(\varpi))$ for each $\sigma \in S_k$. The set of equivalence classes has a natural partial order defined by the rule $[(k', \eta', \varpi')] \leq [(k'', \eta'', \varpi'')]$ if and only if there is a surjection $\eta' : [k'] \to [k'']$ such that $\eta' = \eta \circ \eta''$ and $\varpi' \leq \eta(\varpi'')$.

For each $\mathcal{T} = [(k, \eta, \varpi)]$, let $\text{Hecke}_{n,\mathcal{T}} \subset \text{Hecke}_n$ be the image of $\text{Hecke}_{k,\eta(\varpi)}$ under the closed embedding $\text{Hecke}_k \hookrightarrow \text{Hecke}_n$, induced by $\eta^* : X^k \hookrightarrow X^n$. Then $\text{Hecke}_{n,\mathcal{T}}$ is contained in $\text{Hecke}_{n,\mathcal{T}'}$ if $\mathcal{T}' \leq \mathcal{T}$. Denote by $\text{Hecke}_{0,n,\mathcal{T}}$ the complement in $\text{Hecke}_{n,\mathcal{T}}$ of the union of all $\text{Hecke}_{n,\mathcal{T}'}$'s with $\mathcal{T}' < \mathcal{T}$. Then $\{\text{Hecke}_{0,n,\mathcal{T}}\}_{\mathcal{T}}$ gives us a required stratification of $\text{Hecke}_n'$, and we denote by $\{\text{FBun}_{0,n,\mathcal{T}}\}_{\mathcal{T}}$ and $\{\text{Gr}_{0,n,\mathcal{T}}\}_{\mathcal{T}}$ the induced stratifications of $\text{FBun}_n$ and $\text{Gr}_n$, respectively.
For each \( \omega \in X_1^+(T) \), we denote \( T_\omega := \{(n, \text{Id}, \omega)\} \) simply by \( \omega \). This will not lead to confusion, since each \( \text{Hecke}_{n, \omega} \) coincides with \( \text{Hecke}_{n, \overline{\omega}} \) and since we have \( T_\omega \leq T_{\overline{\omega}} \) if and only if \( \omega \leq \overline{\omega} \).

The following theorem and its corollary, which will be proved in \( \ref{4.2} \) and \( \ref{4.4} \) respectively, imply that locally in the étale topology, \( \text{FBun}_{D,n,\omega} \) (resp. \( \text{FBun}'_{D,n,\omega} \)) is isomorphic to \( \text{Gr}_{n,\omega} \) (resp. \( \text{Gr}'_{n,\omega} \)). This result generalizes the corresponding result of Drinfeld (\[\text{Dr1}, \text{Prop. 3.3}\]), asserting that the moduli space of \( F \)-sheaves is smooth.

**Theorem 2.20.** \( \text{Gr}_{n,\omega} \) is a local model of \( \text{FBun}_{D,n,\overline{\omega}} \). In other words, for every point \( y \in \text{FBun}_{D,n,\omega} \), there exists an étale neighborhood \( p_1 : U_y \to \text{FBun}_{D,n,\omega} \) of \( y \) and an étale morphism \( p_2 : U_y \to \text{Gr}_{n,\omega} \). Moreover, \( p_1 \) and \( p_2 \) induce the same stratification of \( U_y \) and the same morphism \( U_y \to X^n \). Furthermore, \( p_2 \) lifts to an étale morphism \( U_y \times_{\text{FBun}_{D,n,\omega}} \text{FBun}'_{D,n,\omega} \to \text{Gr}'_{n,\omega} \), compatible with stratifications. In particular, \( \text{Gr}'_{n,\omega} \) is a local model of \( \text{FBun}'_{D,n,\omega} \).

**Corollary 2.21.**

a) The open stratum \( \text{FBun}^0_{D,n,\omega} \) of \( \text{FBun}_{D,n,\omega} \) (resp. \( \text{FBun}'^0_{D,n,\omega} \)) is dense. It is non-empty if and only if \( \omega \) is admissible.

b) The reduced stacks \( (\text{FBun}^0_{D,n,\omega})_{\text{red}} \) and \( (\text{FBun}'^0_{D,n,\omega})_{\text{red}} \) are smooth over \( (X \setminus D)^n \) of relative dimension \( \sum_{i=1}^n (2\rho, \omega_i) \). Furthermore, both \( \text{FBun}^0_{D,n,\omega} \) and \( \text{FBun}'^0_{D,n,\omega} \) are reduced, unless \( \text{char } \mathbb{F}_q = 2 \), and \( G \) has a direct factor isomorphic to \( \text{PGL}_2 \) or \( \text{PO}_{2m+1} \).

c) The IC-sheaf of \( \text{FBun}_{D,n,\omega} \) (resp. \( \text{FBun}'_{D,n,\omega} \)) is the restriction (up to a homological shift and Tate twist) of that of \( \text{Hecke}_{D,n,\omega} \) (resp. \( \text{Hecke}'_{D,n,\omega} \)). In particular, its restriction to each stratum is a direct sum of complexes of the form \( \mathbb{Q}(k+n/2)[2k+n] \) with \( k \in \mathbb{Z} \).

d) The forgetful morphism \( \pi : \text{FBun}'_{D,n,\omega} \to \text{FBun}_{D,n,\omega} \) is projective, surjective, and small. In particular, the intersection cohomology (with compact support) of \( \text{FBun}_{D,n,\omega} \) coincides with that of \( \text{FBun}'_{D,n,\omega} \).

**Remark 2.22.** The second assertion of b) says that in most cases the open strata \( \text{FBun}_{D,n,\omega} \) and \( \text{FBun}'^0_{D,n,\omega} \) are reduced. However, we do not know whether the same is true for the full stacks \( \text{FBun}_{D,n,\omega} \) and \( \text{FBun}'_{D,n,\omega} \) (compare Remark \[\text{A.10}\]).

**Definition 2.23.** We will call an \( F \)-bundle \( (\mathcal{G}; x_1, \ldots, x_n; \phi) \) reducible if there exists a maximal parabolic subgroup \( P \subset G \) and a \( P \)-structure \( \mathcal{P} \) of \( \mathcal{G} \) such that \( \phi \) induces a rational isomorphism between \( \mathcal{P} \) and \( \tau \mathcal{P} \).

The following result, proved in \[\text{5.3}\] shows that “at infinity” all \( F \)-bundles are reducible.

**Notation 2.24.** Let \( d(\overline{\omega}) \) be the maximum of the \( \langle \sum_{k=1}^n (\omega_k + 4g\rho, \lambda_i) \rangle \)'s taken over the set of all fundamental weights \( \lambda_i \) of \( G^{sc} \), where \( g \) is the genus of \( X \).
Theorem 2.25. Every $F$-bundle from $FBun_{n,\varpi} \setminus FBun_{n,\varpi,\leq d(\varpi)\rho}$ is reducible.

Remark 2.26. Using methods and results of K. Behrend ([Be2]), one can show that Theorem 2.25 still remains true if $d(\varpi)$ is replaced by the maximum of the $\langle \sum_{k=1}^{n} \omega_k, \lambda_i \rangle$'s. In particular, the bound $d(\varpi)$ can be made independent of the curve $X$. However, Theorem 2.25 seems to be sufficient for all the applications, and the proof of a better bound is much more involved.

Notation 2.27. Let $FBun_{*,n}$ be the generic fiber over $X^n$ of the inverse limit of the $FBun_{D,n}$'s, and let $FBun_{*,n,\varpi}$ be the corresponding closed substack.

2.28. For each maximal parabolic $P$, let $FBun_{P,n}$ be the stack classifying the data consisting of an $F$-bundle $(G; x_1, \ldots, x_n; \phi)$ and a $P$-structure $\mathcal{P}$ of $G$ such that $\phi$ induces a rational isomorphism between $\mathcal{P}$ and $\tau^\mathcal{P}$. We have a natural forgetful map $FBun_{P,n} \rightarrow FBun_{n}$, whose image is the set of all reducible $F$-bundles, corresponding to $P$. More generally, define $FBun_{P,D,n}$, $FBun_{P,n,\varpi}$ be the fiber product of $FBun_{P,n}$ over $FBun_{D,n}$ with $FBun_{n,\varpi}$, respectively.

Definition 2.29. By an orispheric substack we will call the image in $FBun_{D,n,\varpi}$ (resp. $FBun_{*,n,\varpi}$) of an irreducible component of $FBun_{P,D,n,\varpi}$ (resp. $FBun_{P,n,\varpi}$).

A more precise version (Proposition 5.7) of the following result generalizes the corresponding results of Drinfeld ([Dr1, Prop. 4.3]) and Lafforgue ([La1, II, Thm. 5]).

Proposition 2.30. Every orispheric substack of $FBun_{*,n,\varpi}$ is closed.

The following simple result, proven in 3.7, provides us with a space over $F^{(n)}$, equipped with an action of $G(\mathbb{A})$.

Proposition 2.31. a) The group $Z(\mathbb{A})/Z(F)$ acts naturally on $FBun_{D,n}$ and preserves each $FBun_{\leq \mu}$.

b) For each cocompact lattice $J \subset Z(\mathbb{A})/Z(F)$, the quotient $J \backslash FBun_{D,n,\varpi}$ a Deligne–Mumford stack, which is a quotient of a quasi-projective scheme by a finite group. Furthermore, it is a quasi-projective scheme if $J$ is torsion-free, and $|D|$ is sufficiently large relative to $\mu$.

c) The induced (from a) actions of $Z(\mathbb{A})/Z(F)$ on $FBun_{*,n}$ and $FBun_{*,n,\varpi}$ naturally extend to continuous right actions of $G(\mathbb{A})/Z(F)$.

From now on fix a cocompact lattice $J \subset Z(\mathbb{A})/Z(F)$ (which we may assume to be torsion-free) and an admissible $n$-tuple $\varpi$. We are going to define for each $i \in \mathbb{Z}$ the intersection cohomology with compact support $H^i = H^i_J(\varpi) = H^i_J(FBun_{*,n,\varpi}, IC)$ and its pure quotient $H^i_{\text{pure}} = H^i_{\text{pure},J}(\varpi)$, both being $\Gamma_{F^{(n)}}$-modules over $\mathbb{Q}_l$. 
2.32. By Proposition 2.31 b), every $J \setminus FBun_{D,n,\varpi}$ is a quotient of a quasi-projective scheme over $X^n$ by finite group. Therefore we can consider the intersection cohomology $H^i_D := H^i(J \setminus FBun_{D,n,\varpi}) \times X^n F^{(n)} IC$ and its pure quotient $H^i_{\text{pure},D}$ (see Notation 6.1, 6.3 and Remark 6.4).

For each $\mu \leq \mu'$ and $D \subset D'$, we have a natural morphism $H^i_{\mu'} \rightarrow H^i_{\mu'}$ (using Proposition 2.16 b)), which by Remarks 6.2 and 6.4 induces an embedding $H^i_{\text{pure},D} \hookrightarrow H^i_{\text{pure},D'}$. Thus we can form direct limits $H^i$ and $H^i_{\text{pure}}$ of the $H^i_{\mu}$'s and the $H^i_{\text{pure},D}$'s, respectively. Both spaces are equipped with a continuous action of the product $G(A) \times \Gamma_{F(n)}$.

Finally, let $H^i_{\text{pure}}$ and $H^i_{\text{pure},D}$ be subspaces of $H^i_{\text{pure}}$, obtained as direct limits of the $H^i_{\text{pure},D}$'s taken over $D$'s and $\mu$'s, respectively.

Notation 2.33. Let $H^i_{\text{cusp}} = H^i_{\text{cusp},J}(\varpi)$ be the subspace of $H^i_{\text{pure}}$, consisting of all elements vanishing on all orispheric substacks $C \subset J \setminus FBun_{n,\varpi}$ (recall that orispheric substacks are closed by Proposition 2.30, and see Remark 6.2 c) for the definition of the restriction map).

The main advantage of $H^i_{\text{cusp}}$ over all previously defined spaces is due to part b) of the following result, proved in 6.5.

Proposition 2.34. a) For each dominant coweight $\mu \geq d(\varpi)\rho$, we have

$$H^i_{\text{cusp},J}(\varpi) = \cap_{g \in G(A)} g(H^i_{\text{pure},J}(\varpi)).$$

b) $H^i_{\text{cusp}}$ is an admissible representation of $G(A)$.

Conjecture 2.35. If $G = GL_r$, then the representation $H^0_{\text{cusp}}(\varpi)$ of $GL_r(A) \times \Gamma_{F(n)}$ is isomorphic to the direct sum $\bigoplus_\pi (\pi \boxtimes \rho_{\pi,\varpi})$, where $\pi$ runs over the set of all cuspidal representations of $GL_r(A)$ with $\pi(J) = \text{Id}$, and $\rho_{\pi,\varpi}$ is the same as in the introduction. (Here we identify $\varpi$ with the corresponding representation of $(\hat{G})^n$ as in Remark 2.17.)

Remark 2.36. When $G$ is arbitrary, we also expect that cuspidal tempered part of Langlands correspondence can be realized in $H^0_{\text{cusp}}$, but one has to take into account the contribution of endoscopic groups as well.

To provide an evidence to our conjecture, we will show in Section 7 the following result.

Theorem 2.37. a) In the Lafforgue case (that is, for $G = GL_r$, $n = 2$ and $\varpi$ is the tensor product of the standard representation of $\hat{G} = GL_r$ and its dual), Conjecture 2.35 holds up to $r$-negligibles (see Notation 7.1). More precisely, there exists an
Proof of Proposition 2.16. 

a) Conjecture 2.35 holds in Drinfeld’s case (that is, in the Lafforgue case with \( r = 2 \)).

b) Conjecture 2.35 holds in Drinfeld’s case (that is, in the Lafforgue case with \( r = 2 \)).

Remark 2.38. Finally, let us introduce objects, described in the introduction. Let \( X_n \) and \( X_{\overline{\omega}} \) be the inverse limits of the \( J/FBun_{s,n} \)'s and the \( J/FBun_{s,n,\overline{\omega}} \)'s, respectively, taken over all cocompact lattices in \( \mathbb{Z}(\mathbf{A})/\mathbb{Z}(F) \). Then \( H^i_c(X_{\overline{\omega}}, IC) \) and its required subquotient \( H^i_{cusp}(X_{\overline{\omega}}, IC) \) are the direct limits of the \( H^i_c(J(\overline{\omega})) \)'s and the \( H^i_{cusp,J}(\overline{\omega}) \)'s, respectively. In particular, in the case \( G = GL_r \), our Conjecture 2.35 for all \( J \)'s is equivalent to the assertion that \( H^0_{cusp}(X_{\overline{\omega}}, IC) \) is isomorphic to the direct sum \( \bigoplus_{\pi}(\pi \otimes \rho_{s,\overline{\omega}}) \), where \( \pi \) runs over the set of all cuspidal representations with finite order central characters.

3. Basic properties of \( F \)-bundles

In this section we will prove Propositions 2.16 and 2.31. For each \( \nu \in \pi_1(G) = \pi_0(Bun_G) \), let us denote by \( Bun_{G,D}^{\leq \mu, \nu} \) the preimage in \( Bun_{G,D}^{\leq \mu} \) of the connected component of \( Bun_G \), corresponding to \( \nu \), and similarly for other spaces such as Hecke and \( FBun \). We will use the following lemma, whose proof will be sketched in \( \S 4 \).

Lemma 3.1. a) If \( |D| \) sufficiently large relative to \( \mu \), then \( Bun_{G,D}^{\leq \mu, \nu} \) is a smooth quasi-projective scheme for each \( \nu \in \pi_1(G) \).

b) Stack Hecke_{G,D}^{\leq \mu, \nu} is projective over \( Bun \times X^n \).

c) The forgetful map \( \pi : Hecke_{G,D,n,\overline{\omega}} \to Hecke_{G,D,n,\overline{\omega}} \) is projective. Moreover, \( \pi \) is an isomorphism over \( X^n \setminus \Delta \).

3.2. Proof of Proposition 2.16. b) The map \( (G, \phi) \mapsto (G_{|D \times S}, \phi_{|D \times S}) \) gives a morphism from \( FBun_n \times X^n (X \setminus D)^n \) to the stack \( \mathcal{Y}_D \) classifying pairs consisting of a \( G \)-bundle \( \tilde{G} \) on \( D \times S \) and an isomorphism \( \tilde{\gamma} \tilde{G} \cong \tilde{G} \). Since \( G \) and therefore \( G_D \) are geometri-cally connected, Lang’s theorem implies that \( \mathcal{Y}_D \) is a classifying space of the discrete group \( G_D(\mathbb{F}_q) = G(O_D) \) (use Lemma 3.3 below). Now the statement follows from the fact that \( FBun_{D,n} \) is canonically isomorphic to the fiber product of \( FBun_n \times X^n (X \setminus D)^n \) and \( Spec \mathbb{F}_q \) over \( Y_D \cong G(O_D) \setminus Spec \mathbb{F}_q \).

a) By b), we can replace \( D \) by its multiple, so we can assume that \( |D| \) is sufficiently large to satisfy a) of Lemma 3.1. Then \( Hecke_{G,D,n,\overline{\omega}} \), being the restriction of \( Bun_{G,D}^{\leq \mu, \nu} \times Bun Hecke_{n,\overline{\omega}} \) to \( (X \setminus D)^n \), is a quasi-projective scheme. Now the statement follows from the fact that \( FBun_{G,D,n,\overline{\omega}} \) is a closed substack of \( Hecke_{G,D,n,\overline{\omega}} \).
Indeed, $FBun_{D,n,[\omega]}^{\leq \mu_\nu}$ is the preimage in $Hecke_{D,n,[\omega]}^{\leq \mu_\nu}$ of the graph of the Frobenius morphism in $Bun_{D}^{\leq \mu_\nu} \times Bun_{D}^{\leq \mu_\nu}$.

c) We have to check that $Bun_{D}$ satisfies the conclusion of Lemma 3.3(b) below. Instead of checking that $Bun_{D}$ satisfies the assumption of Lemma 3.3(b), we can argue as follows. As the question is local for the Zariski topology, we may replace $Bun_{D}$ by its open substack $Bun_{D}^{\leq \mu_\nu}$. By Lemma 3.1(a), there exists a finite subsheaf $D \subset X$ containing $D$ such that $Bun_{D}^{\leq \mu_\nu}$ is a scheme. Then $Bun_{D}^{\leq \mu_\nu}$ clearly satisfies Lemma 3.3 so the statement for $Bun_{D}^{\leq \mu_\nu}$ follows from b) together with the fact that groupoid $Bun_{D}^{\leq \mu_\nu}(\mathbb{F}_q)$ is isomorphic to the quotient of $Bun_{D}^{\leq \mu_\nu}(\mathbb{F}_q)$ by $G_{D,D}(\mathbb{F}_q)$ (use again Lang’s theorem).

d) The “only if” statement was explained in Remark 2.15. Assume now that $\omega$ is admissible. Then $FBun_{D,n,[\omega]}$ contains a substack consisting of $F$-bundles for which $x_1 \ldots = x_n$ and $\phi$ is an isomorphism. As this substack obviously contains (and actually is isomorphic by c) to) a non-empty stack $Bun_{D}(\mathbb{F}_q) \times (X \setminus D)$, the statement follows.

e) follows immediately from statement c) of Lemma 3.1. \hfill \Box

**Lemma 3.3.** a) Let $\mathcal{X}$ be an algebraic stack locally of finite type over $\mathbb{F}_q$, and let $\mathcal{Y}$ be a stack over $\mathbb{F}_q$ such that $\mathcal{Y}(S) = \{(A, \Phi) | A \in \mathcal{X}(S), \Phi \in \text{Isom}_{\mathcal{X}(S)}(\tau A, A)\}$. Then $\mathcal{Y}$ is a Deligne–Mumford stack, étale over $\mathbb{F}_q$, containing the discrete stack $\mathcal{X}(\mathbb{F}_q)$ as an open and closed substack.

b) If, in the notation of a), all geometric fibers of the diagonal morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ are connected, then $\mathcal{Y}$ is canonically isomorphic to $\mathcal{X}(\mathbb{F}_q)$.

**Proof.** a) As $\mathcal{Y}$ is a fiber product over $\mathcal{X} \times \mathcal{X}$ of the diagonal $\Delta_{\mathcal{X}}$ and the graph of Frobenius morphism, $\mathcal{Y}$ is an algebraic stack locally of finite type over $\mathbb{F}_q$. One checks that the canonical morphism $\mathcal{Y}(\mathbb{F}_q[t]/(t^2)) \to \mathcal{Y}(\mathbb{F}_q)$ is an equivalence of categories, therefore the diagonal morphism $\Delta_{\mathcal{Y}}$ is unramified. Hence $\mathcal{Y}$ is a Deligne–Mumford stack (see [LMB] Thm. 8.1]), and $\mathcal{Y}$ is étale over $\mathbb{F}_q$.

It remains to check that the natural functor $i : \mathcal{X}(\mathbb{F}_q) \to \mathcal{Y}(\mathbb{F}_q)$ is fully faithful. But this follows from (actually is equivalent to) the first axiom of a stack (sheaf axiom for Isom$(x, y)$) applied to étale covers $\text{Spec} \mathbb{F}_q^m \to \text{Spec} \mathbb{F}_q$ for all $m \in \mathbb{N}$.

b) We have to show that the functor $i : \mathcal{X}(\mathbb{F}_q) \to \mathcal{Y}(\mathbb{F}_q)$ from a) is essentially surjective. Let $(A, \Phi)$ be any object of $\mathcal{Y}(\mathbb{F}_q)$, and we want to find an object $B$ of $\mathcal{X}(\mathbb{F}_q)$ such that $i(B)$ is isomorphic to $(A, \Phi)$. Choose $m$ such that $A$ is (a pull-back of) an object of $\mathcal{X}(\mathbb{F}_q^m)$, and $\Phi$ belongs to $\text{Isom}_{\mathcal{X}(\mathbb{F}_q^m)}(\tau A, A)$. Then

$$\Phi^{(m)} := \Phi \circ \tau \Phi \circ \ldots \circ \tau^{m-1} \Phi : A \Rightarrow \tau^mA \Rightarrow A$$
defines an $F_q^m$-point of an algebraic group $H := \text{Isom}_X(A, A)$. If $\Phi^{(m)}$ is the identity, then the existence of $B$ is equivalent to the second axiom of a stack applied to the étale cover $\text{Spec} \, F_q^m \to \text{Spec} \, F_q$. The general case easily reduces to this one. Indeed, our assumption about $\Delta_X$ implies that $H$ is a connected group over $F_q^m$. Therefore Lang's theorem implies the existence of $h \in H(Fq^m)$ such that $\tau^m h = h \circ \Phi^{(m)}$. Then $h$ induces an isomorphism between $(A, \Phi)$ and $(A, \Phi' := h \circ \Phi \circ h^{-1})$. By construction, $\Phi'^{(m)}$ is the identity, completing the reduction. □

**Remark 3.4.** a) Lemma 3.3 gives an explanation of the Drinfeld's lemma (see for example [La1, Ch.1, 3, Lem. 3]) used by Drinfeld and Lafforgue.

b) In the proof of Proposition 2.16 we used Lemma 3.3 only in two particular cases: when $X$ is a scheme and when $X$ is a classifying space of a connected algebraic group. In both cases the proof can be simplified.

3.5. Before beginning the proof of Proposition 2.31, recall that $G(A)$ acts naturally (on the right) on the inverse limit $\text{Bun}_*$ of the $\text{Bun}$'s. Any element of $\text{Bun}_*(S)$ consists of a $G$-bundle $G$ over $X \times S$ equipped with trivializations $\phi_v : G|_{O_v \times S} \sim G \times O_v \times S$ for all closed points $v$ of $X$. For each $g = (g_v), v \in G(A)$, choose a finite set of closed points $T$ of $X$ such that $g_v \in G(O_v)$ for all $v \notin T$. We claim that there exists a unique $G$-bundle $\tilde{G}$ over $X \times S$ such that $\tilde{G}|_{(X \setminus T) \times S} = G|_{(X \setminus T) \times S}$ and $\tilde{G}|_{O_v \times S} = \phi_v^{-1}(g_v(G \times O_v \times S))$ for each $v \in T$ (the last equality we consider inside $G|_{F_v \times S}$). Indeed, by [BL], the corresponding statement holds for vector bundles, so by Tannakian formalism it holds in general.

Since $\tilde{G}$ is clearly independent of $T$, the rule $(G, \{\phi_v\}_v) : (\tilde{G}, \{g_v^{-1} \circ \phi_v\}_v)$ defines the required group action. Moreover, it follows from the construction that $Z(F)$ acts trivially, and that the induced action of $Z(A)/Z(F)$ on $\text{Bun}_*$ gives an action on each $\text{Bun}_D$. Furthermore, since the open substack $\text{Bun}^\mu_G$ was defined as the preimage of $\text{Bun}^\mu_G \subset \text{Bun}^\mu_G$, the group $Z(A)/Z(F)$ preserves each $\text{Bun}^\mu_G$.

**Remark 3.6.** The above argument actually shows that $\text{Bun}_*$ is equipped with an action of a huge ind-pro-algebraic group, whose group of $F_q$-points is $G(A)$.

3.7. **Proof of Proposition 2.31** a) and c) follow from the fact that the action of $G(A)/Z(F)$ on $\text{Bun}_*$ (resp. $Z(A)/Z(F)$ on $\text{Bun}^\mu_D$), defined in 3.3, naturally lifts to the actions on $\text{FBun}_{*,n}$ (resp. $\text{FBun}^\mu_{D,n}$) and leaves $\text{FBun}_{*,n,\pi}$ (resp. $\text{FBun}^\mu_{D,n,\pi}$) invariant.

b) By Proposition 2.16 b), the statement would follow if we show that the induced action of $J$ on $\pi_0(\text{FBun}_{D,n,\pi})$ has finitely many orbits and has finite stabilizers. As we proved in 3.2 that the projection $\pi_0(\text{FBun}_{D,n,\pi}) \to \pi_0(\text{Bun})$ has finite
fibers, it will suffice to prove the corresponding statement for \( \pi_0(Bun_G) \) instead of \( \pi_0(FBun_{D,n,\mathbb{A}}) \).

Notice that \( Z(\mathbb{A}) \) acts on \( \pi_0(Bun_G) = \pi_1(G) \) via the continuous homomorphism \( \Pi : Z(\mathbb{A}) \to T(\mathbb{A}) \to X_*(T) \to \pi_1(G) \), whether \( \pi \) is given by the rule \( \langle \pi(t), \nu \rangle = \log q |\nu(t)| \) for each \( \nu \in X^*(T) \) and \( t \in T(\mathbb{A}) \). Indeed, as the natural surjection \( \pi_0(Bun_T) = X_*(T) \to \pi_1(G) = \pi_0(Bun_G) \) is induced by the inclusion \( T \hookrightarrow G \) (see the proof of Lemma 2.2 in A.1), it will suffice to show the corresponding statement for \( G = T \), hence for \( G = \mathbb{G}_m \), in which case it is clear.

Note that \( \Pi \) factors through \( Z(\mathbb{A})/Z(F) \), and that the induced homomorphism \( \Pi' : Z(\mathbb{A})/Z(F) \to \pi_1(G) \) has a compact kernel and a finite cokernel. Since \( J \subset Z(\mathbb{A})/Z(F) \) is a cocompact lattice, both kernel and cokernel of the restriction of \( \Pi' \) to \( J \) are therefore compact and discrete, hence finite. This implies the assertion. \( \square \)

4. Local model of \( FBun_{n,\mathbb{A}} \)

The goal of this section is to prove Theorem 2.20 and Corollary 2.21. Our strategy will be to decompose locally \( Hecke_{n,\mathbb{A}} \) as a product \( Gr_{n,\mathbb{A}} \times Bun \) and then to use the fact that the Frobenius morphism has a zero differential.

Lemma 4.1. Let \( G_0 \) be a \( G \)-bundle on \( X \times S \), locally trivial in the Zariski topology, and let \( \pi : S \to Bun_G \) be the morphism, corresponding to \( G_0 \).

Then the fiber product \( Hecke_{n,\mathbb{A}} \times_{Bun} S \) (taken with respect to the projection \( p' \) (see Notation 2.20)) and the product \( Gr_{n,\mathbb{A}} \times S \) are locally Zariski isomorphic fibrations over \( X^n \times S \). Moreover, the isomorphism preserves the stratifications induced by those of \( Hecke_{n,\mathbb{A}} \) and \( Gr_{n,\mathbb{A}} \).

Furthermore, this local isomorphism lifts to a stratification preserving local isomorphism between \( Hecke'_{n,\mathbb{A}} \times_{Bun} S \) and \( Gr'_{n,\mathbb{A}} \times S \).

Proof. Let \( (x_1', \ldots, x_n'; s') \) be a closed point of \( X^n \times S \). We want to find its open neighborhood, whose inverse images in \( Hecke_{n,\mathbb{A}} \times_{Bun} S \) and \( Gr_{n,\mathbb{A}} \times S \) are isomorphic. We are going to prove the statement by induction on \( n \).

Assume first that \( x_1' = x_2' = \ldots = x_n' \) (this condition holds automatically for \( n = 1 \)). By our assumption, there exists an open neighborhood \( V \subset X \times S \) of \( (x_1', s') \) and a trivialization \( \psi \) of the restriction of \( G_0 \) over \( V \). Consider the open subscheme \( U \) of \( X^n \times S \), consisting of points \( (x_1, \ldots, x_n, s) \) such that \( (x_i, s) \in V \) for each \( i = 1, \ldots, n \). We claim that \( U \) is a required neighborhood. Let \( U'' \subset Hecke_{n,\mathbb{A}} \times_{Bun} S \) and \( U'' \subset Gr_{n,\mathbb{A}} \times S \) be the inverse images of \( U \). We are going to find an isomorphism \( U' \sim U'' \) over \( U \).

Let \( (\mathcal{G}, \mathcal{G}'; y_1, \ldots, y_n; \phi) \) be the pullback to \( U' \) of the universal object over \( Hecke_{n,\mathbb{A}} \).
Let \( V' \subset X \times U' \) be the preimage of \( V \), and let \( \psi' \) be the trivialization of \( \mathcal{G}' \) over
V' induced by ψ. The composition of φ and ψ' defines a trivialization φ' of \( \mathcal{G} \) over \( V' \setminus (\Gamma_{y_1} \cup \ldots \cup \Gamma_{y_n}) \). As \( V' \) contains each \( \Gamma_{y_i} \subset X \times U' \), there exists a unique G-bundle \( \tilde{\mathcal{G}} \) over \( X \times U' \), trivial over \( (X \times U') \setminus (\Gamma_{y_1} \cup \ldots \cup \Gamma_{y_n}) \) such that \( \tilde{\mathcal{G}}_{|V'} = \mathcal{G}_{|V'} \) and the gluing is done by means of \( \phi' \). By the definition of the affine grassmannian, \( \tilde{\mathcal{G}} \) defines the required morphism \( U' \to U'' \). The construction of the inverse map and rest of the statements is now straightforward and is therefore omitted (compare the proof of Lemma A.8).

Assume now that \( x_i' \neq x_j' \) for some \( i \) and \( j \); then after renumbering of indexes there exists a positive integer \( k < n \) such that \( x_i' \neq x_j' \) for each \( i \leq k < j \). Using the second statement of Lemma A.8(a), the assertion now follows by induction hypothesis. □

4.2. Proof of Theorem 2.20. Choose \( \mu \) such that \( y \in FBun_{D,n,\overline{\mu}}^\leq \). By Proposition 2.16(b) and Lemma 3.1(a), we can enlarge \( D \) (and replace \( y \) by one of its preimages) so that \( Bun_{D}^{\leq \mu} \) is a scheme. By the theorem of Drinfeld–Simpson ([DS]), there exists a surjective étale morphism \( \pi : S \to Bun_{D}^{\mu} \) such that the pull-back to \( X \times S \) of the universal G-bundle on \( X \times Bun_{D} \) is locally trivial in the Zariski topology.

Choose a preimage \( y' \in FBun_{D,n,\overline{\mu}} \times Bun_{D} S \) of \( y \), and let \( y'' \in X^n \times S \) be the image of \( y' \). By Lemma 4.1, \( y'' \) has an open neighborhood \( U \subset X^n \times S \), whose inverse images \( U' \subset Hecke_{n,\overline{\mu}} \times Bun_{D} S \) and \( U'' \subset Gr_{n,\overline{\mu}} \times S \) are isomorphic over \( U \). We claim that \( U_y := U' \times_{Hecke_{D,n}} FBun_{D,n} \) is the required étale neighborhood of \( y \).

Since the natural projection \( U_y \to FBun_{D,n} \) is étale (since \( \pi \) is so), it will suffice to show that the composition map \( U_y \hookrightarrow U' \xrightarrow{\sim} U'' \hookrightarrow Gr_{n,\overline{\mu}} \times S \to Gr_{n,\overline{\mu}} \) is étale.

Denote by \( f \) the restriction to \( U'' \) of the projection \( p : Hecke_{D,n} \to Bun_{D} \) from Notation 2.8. Then if we identify \( U' \) and \( U'' \) by means of isomorphism \( U' \xrightarrow{\sim} U'' \), chosen above, the statement follows from Lemma 4.3 below, applied to \( Y := S, Z := Bun_{D}^{\mu}, T := Gr_{n,\overline{\mu}} \) and \( W := f^{-1}(Bun_{D}^{\mu}) \) (hence \( V = U_y \)).

Furthermore, by the last assertion of Lemma 4.4, \( U_y \to Gr_{n,\overline{\mu}} \) lifts to a morphism \( U_y \times_{FBun_{D,n,\overline{\mu}}} FBun_{D,n,\overline{\mu}} \to Gr_{n,\overline{\mu}} \), which is again étale by Lemma 4.3 below. □

Lemma 4.3. Let \( Y, Z \) and \( T \) be schemes locally of finite type over \( \mathbb{F}_q \), let \( W \subset Y \times T \) be an open subscheme, let \( \pi : Y \to Z \) be an étale morphism, let \( f : W \to Z \) be any morphism, and let \( V \) be given by equation

\[
V = \{(y, t) \in W \subset Y \times T | \text{Frob}_q(f(y, t)) = \pi(y)\}.
\]

Assume that \( Z \) is smooth over \( \mathbb{F}_q \). Then the canonical map \( \Pi : V \hookrightarrow W \to T \) is étale.

Proof. Assume first that \( T \) is smooth over \( \mathbb{F}_q \) (compare [DF1] Prop. 3.3)). In this case \( Y \) and \( W \) are smooth as well. Since the Frobenius morphism has a zero differential, \( V \) inside \( W \) is locally given by \( \text{dim} \ Z \) equations with linearly independent differentials.
Therefore $V$ is smooth. Moreover, the projection $\Pi : V \to T$ induces an isomorphism on tangent spaces, hence $\Pi$ is étale, as claimed.

In the general case, we may assume that $Z = \mathbb{A}^m$. Indeed, as the question is local on $V$, we may shrink all the schemes in question so that all of them are affine and there exists an étale map $\phi : Z \to \mathbb{A}^m$. Then $V$ is an open and closed subscheme of the scheme

$$V' = \{(y,t) \in W \subset Y \times T | \text{Frob}_q(\phi \circ f(y,t)) = \phi \circ \pi(y)\}.$$ 

In particular, we may replace $Z, \phi$ and $f$ by $\mathbb{A}^m, \phi \circ \pi$ and $\phi \circ f$, respectively.

Next choose a closed embedding of $T$ into an affine space $\tilde{T} = \mathbb{A}^k$. Then there exists an étale map $\tilde{f} : \tilde{W} \to Z = \mathbb{A}^m$ of $f$ to $\tilde{W}$. Consider

$$\tilde{V} := \{(y,t) \in \tilde{W} \subset Y \times \tilde{T} | \text{Frob}_q(\tilde{f}(y,t)) = \pi(y)\}.$$ 

As $\tilde{T}$ is smooth, we have seen before that the natural projection $\tilde{\Pi} : \tilde{V} \to \tilde{T}$ is étale. But $\Pi : V \to T$ is just the restriction of $\tilde{\Pi}$ to $V = \tilde{\Pi}^{-1}(T)$. Therefore it is also étale, as claimed.

4.4. Proof of Corollary 2.21 b) and the first statement of a) are reduced by Theorem 2.20 to the corresponding questions about affine grassmannians, which will be shown in Proposition A.9. As for the second statement of a), the “only if” part was explained in Remark 2.13 b), while the “if” part for $\text{FBun}^0_{D,n,\omega}$ follows from Proposition 2.16 d). Moreover, since $\pi : \text{FBun}^0_{D,n,\omega} \to \text{FBun}_{D,n,\omega}$ induces an isomorphism $\text{FBun}^0_{D,n,\omega} \times X^n(X^n \setminus \Delta) \sim \text{FBun}^0_{D,n,\omega}$ (by Proposition 2.16 e)), the “if” part for $\text{FBun}^0_{D,n,\omega}$ follows as well.

c) Let $m$ be the dimension of $\text{Bun}_D$, and let $\mathcal{F}(m/2)[m]$ be the restriction of the IC-sheaf of $\text{Hecke}_{D,n,\omega}$ to $\text{FBun}^0_{D,n,\omega}$. We want to show that $\mathcal{F}$ is the IC-sheaf. Since the statement clearly holds for the restriction of $\mathcal{F}$ to the smooth open dense stratum $\text{FBun}^0_{D,n,\omega}$, it remains to show that $\mathcal{F}$ is an irreducible perverse sheaf.

As the map from the disjoint union of the $U_y$’s (from Theorem 2.20) to $\text{FBun}_{D,n,\omega}$ is étale and surjective, it will suffice to show the corresponding statement for the restriction of $\mathcal{F}$ to each $U_y$. Consider the commutative diagram,

\[
\begin{array}{ccc}
U_y & \xrightarrow{i} & U' \\
\downarrow & & \downarrow \pi_2 \\
\text{FBun}^0_{D,n,\omega} & \xrightarrow{\pi_1} & \text{Hecke}_{D,n,\omega}
\end{array}
\]
constructed in the course of the proof of Theorem 2.20. As \( \pi_2 \) is étale, \( \pi_1 \) is smooth of relative dimension \( m \), and \( \pi_1 \circ i \) is étale, we get that

\[
F_{|U_y} = i^* \pi_2^*(IC_{Hecke,D,n,\varpi}(-\frac{m}{2})[-m]) = i^*(IC_{U,(-\frac{m}{2})[-m]}) = i^* \pi_1^*(IC_{Gr,n,\varpi}) = IC_{U_y},
\]

as claimed.

The last assertion follows immediately from the corresponding statement for \( Hecke_{n,\varpi} \) shown in Proposition A.13 and the observation that the difference

\[
\dim \text{FBun}_{D,n,\varpi} - n = 2\langle \sum_{i=1}^n \omega_i, \rho \rangle
\]

is even.

Finally, the assertion for \( \text{FBun}'_{D,n,\varpi} \) follows from the (proof of) the corresponding assertion of Theorem 2.20 by precisely the same argument.

d) By Proposition 2.16 e), \( \pi : \text{FBun}'_{D,n,\varpi} \to \text{FBun}_{D,n,\varpi} \) is projective and induces an isomorphism \( \text{FBun}'_{D,n,\varpi} \times_{X^n} (X^n - \Delta) \simeq \text{FBun}_{D,n,\varpi} \). Hence the surjectivity statement follows from a). Finally, by Theorem 2.20 and Lemma 4.1, the smallness of \( \pi \) is equivalent to the smallness of the forgetful morphism \( \text{Hecke}'_{n,\varpi} \to \text{Hecke}_{n,\varpi} \), which will be shown in Lemma A.12.

5. Reducible F-bundles

In order to prove Theorem 2.25, we first need some preparations. Fix a \( G \)-bundle \( G \) over a smooth connected projective curve \( X \) over an algebraically closed field \( k \).

Equip the set of all \( B \)-structures of \( G \) with a following partial order: we say that \( B' \leq B'' \) if \( \deg(B'_\lambda) \leq \deg(B''_\lambda) \) for each dominant (or equivalently quasi-fundamental) weight \( \lambda \) of \( G \).

Lemma 5.1. a) For every \( B \)-structure \( B \) of \( G \) there exists a \( B \)-structure \( B' \) such that \( B \leq B' \) and \( B' \) is maximal with respect to the above order.

b) Let \( B \) be a \( B \)-structure of \( G \), which is maximal with respect to the above order, and let \( \mathcal{P} \) be any maximal parabolic structure of \( G \), corresponding to a certain simple root \( \alpha \) of \( G \) with the corresponding quasi-fundamental weight \( \lambda \). Then either \( \mathcal{P} \) contains \( B \), or \( \deg(\mathcal{P}_\lambda) \leq \deg(B_\lambda) - \deg(B_{\alpha}) \langle \lambda, \alpha \rangle + 4g \langle \rho, \lambda \rangle \).

Remark 5.2. The inequality is very far from being optimal.

Let us first show how Lemma 5.1 implies Theorem 2.25.

5.3. Proof Theorem 2.25. Let \( (G; x_1, \ldots, x_n; \phi) \) be any geometric point of the complement \( FBun_{n,\varpi} - FBun_{n,\varpi}^{\leq d(\varpi)\rho} \). Then there exists a \( B \)-structure \( B \) of \( G \) and a dominant weight \( \lambda' \) of \( G \) such that \( \deg(B_{\lambda'}) > \langle d(\varpi)\rho, \lambda' \rangle \). By Lemma 5.1 a), we may assume that \( B \) is maximal with respect to the above order. Hence there exists a simple root \( \alpha \) of \( G \) such that \( \deg(B_{\alpha}) > \langle d(\varpi)\rho, \alpha \rangle = d(\varpi) \). Let \( \mathcal{P} \supset B \) be the maximal parabolic structure of \( G \), corresponding to \( \alpha \). We want to show that \( \phi \)
induces a rational isomorphism between $\mathcal{P}$ and $^r\mathcal{P}$. Clearly, $\phi^{-1}$ induces a rational isomorphism between $^r\mathcal{P}$ and a certain parabolic structure $\mathcal{P}'$ of $\mathcal{G}$, so it remains to show that $\mathcal{P}'$ contains $\mathcal{B}$.

Let $\lambda$ be the quasi-fundamental weight of $G$, corresponding to $\alpha$. By Lemma A.2 $\mathcal{P}$ and $\mathcal{P}'$ define line subbundles $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{G}_\Lambda$, respectively, satisfying Plücker relations. As $\mathcal{L} \cong \mathcal{B}_\lambda$ and $\mathcal{L}' \cong \mathcal{P}'_{\lambda}$, the statement will follow from Lemma 5.1 applied to $\mathcal{P}'$ if we check that $\deg(\mathcal{L}') \geq \deg(\mathcal{L}) - d(\overline{w})\langle \lambda, \alpha \rangle + 4g\langle \rho, \lambda \rangle$.

By Remark 2.5 c), $\phi^{-1}(\overline{\mathcal{G}}_{\lambda})$ is contained in $\mathcal{G}_\Lambda(\sum_{k=1}^{n}\langle \omega_k, -w_0(\lambda) \rangle x_k)$. Therefore $\phi^{-1}(\overline{\mathcal{L}}) \subset \mathcal{L}'(\sum_{k=1}^{n}\langle \omega_k, -w_0(\lambda) \rangle x_k)$. It follows that

$$\deg(\mathcal{L}') \geq \deg(\mathcal{L}) - \sum_{k=1}^{n}\omega_k, -w_0(\lambda);$$

thus it remains to check that $\langle \sum_{k=1}^{n}\omega_k, -w_0(\lambda) \rangle \leq d(\overline{w})\langle \lambda, \alpha \rangle - 4g\langle \rho, \lambda \rangle$. Since $-w_0(\lambda)$ is the quasi-fundamental weight of $G$ corresponding to the simple root $-w_0(\alpha)$, the statement follows from the definition of $d(\overline{w})$ and the equality $\langle \rho, \lambda \rangle = -\langle w_0(\rho), \lambda \rangle = \langle \rho, -w_0(\lambda) \rangle$. \hfill $\Box$

5.4. Proof of Lemma 5.1 a) It will suffice to show that for each quasi-fundamental weight $\lambda$ of $G$, the set $\{\deg(\mathcal{B}_\beta)\}_{\beta}$ is bounded from above. Since every $\mathcal{B}_\lambda$ is canonically a line subbundle of $\mathcal{G}_\Lambda$, the statement follows.

b) Let $\mathcal{B}$ be any maximal $B$-structure of $\mathcal{G}$. First we claim that $\deg(\mathcal{B}_\beta) \geq -2g$ for every simple root $\beta$ of $G$. Assume first that $G = GL_2$. In this case, our claim asserts that every rank two vector bundle $\mathcal{E}$ contains a line subbundle of degree at least $\frac{1}{2}\deg(\mathcal{E}) - g$, so it follows immediately from the Riemann–Roch theorem. The general case reduces to that of $GL_2$. Indeed, fix any $\beta$. Let $P_\beta \supset B$ be the parabolic subgroup $G$ such that $\beta$ is the only simple root of its Levi subgroup, and let $R(P_\beta)$ be the radical of $P_\beta$. Consider Borel structure $\mathcal{B}' := R(P_\beta)\setminus B$ of the $P_\beta/R(P_\beta)$-bundle $R(P_\beta)\setminus [P_\beta \times_B \mathcal{B}]$. Then $\deg(\mathcal{B}_\beta') = \deg(\mathcal{B}_\beta)$, and $\mathcal{B}'$ is maximal. [If not, then there exists a Borel structure $\mathcal{B}_0'$ larger than $\mathcal{B}'$. Hence the preimage of $\mathcal{B}_0'$ in $P_\beta \times_B \mathcal{B}$ would give us a $B$-structure of $\mathcal{G}$, larger than $\mathcal{B}$, contradicting the maximality of $\mathcal{B}$]. Since $P_\beta/R(P_\beta) \cong PGL_2$, we are thus reduced to the case of $PGL_2$, hence to that of $GL_2$, as claimed.

Now we are ready to prove the assertion. Observe first that we can replace $G$ by $G^{sc}$ and $\lambda$ by the corresponding fundamental weight. By Lemma A.2 $\mathcal{B}$ and $\mathcal{P}$ define line subbundles $\mathcal{L}$ and $\mathcal{L}'$ of $\mathcal{G}_\Lambda$, respectively, and we have to show that either $\mathcal{L}' = \mathcal{L}$ or $\deg(\mathcal{L}') \leq \deg(\mathcal{L}) - \deg(\mathcal{B}_\alpha) + 4g\langle \rho, \lambda \rangle$. For this we will show that the latter inequality holds for every line subbundle $\mathcal{L}' \neq \mathcal{L}$ of $\mathcal{G}_\Lambda$. Fix any $B$-invariant complete flag $0 = V_0 \subset V_1 \subset \ldots \subset V_M = V_\lambda$ of $V_\lambda$. Then $\mathcal{B}$ defines a complete flag $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_M = \mathcal{G}_\lambda$ of $\mathcal{G}_\lambda$ with $\mathcal{E}_i = B\setminus [\mathcal{B} \times V_i]$. By construction,
each quotient $E_i/E_{i-1}$ is isomorphic to $B_{\lambda_i}$, where $\lambda_i$ is the weight of $V_i/V_{i-1}$. In particular, $E_1 \cong L$.

For each line subbundle $L' \neq L$, let $i \geq 2$ be the smallest integer such that $L'$ is contained in $E_i$. Then $L'$ embeds into some $E_i/E_{i-1} \cong B_{\lambda_i}$, so it will suffice to show that $\deg(B_{\lambda'}) \leq \deg(B_{\lambda}) - 4g(\rho, \lambda)$ for every weight $\lambda' \neq \lambda$ of $V_\lambda$ or, equivalently, that $\deg(B_{\lambda-\lambda'}) \geq -4g(\rho, \lambda)$. But $\lambda - \lambda' - \alpha$ is of the form $\sum \beta n_\beta\beta$, where $\beta$ runs over the set of all simple roots of $G$ and each $n_\beta$ is non-negative. Since $\deg(B_{\beta}) \geq -2g$ for each $\beta$, it remains to show that $\sum n_\beta \leq 2(\rho, \lambda)$. As

$$\sum_{\beta} n_\beta = \langle \lambda - \lambda' - \alpha, \rho \rangle < \langle \lambda - \lambda', \rho \rangle \leq \langle \lambda - w_0(\lambda), \rho \rangle = \langle \lambda, \rho - w_0(\rho) \rangle = 2(\rho, \lambda),$$

we get the assertion. $\square$

To formulate a more precise version of Proposition 2.30, we will need the following result.

**Claim 5.5.** Fiber product $FBun_{P,D,n} \times X^n (X^n \setminus \Delta)$ naturally decomposes as a disjoint union of open and closed substacks $FBun_{P,D,n}^{d; \bar{k}; [g]}$, indexed by triples $(d; \bar{k}; [g])$, where $d$ is an integer, $\bar{k}$ is an $n$-tuple of integers with zero sum, and $[g]$ is an element of $G(O_D)/P(O_D)$.

**Proof.** Let $G$ be the universal $G$-bundle on $X \times FBun_{P,n}$, and let $\lambda$ be the quasi-fundamental weight of $G$ corresponding to $P$. By Lemma A.2, the universal $P$-bundle on $FBun_{P,n}$ defines a line subbundle $L \subset G_\lambda$, satisfying Plücker relations and such that the rational isomorphism $\phi_\lambda$ between $G_\lambda$ and $^rG_\lambda$ induces that between $L$ and $^rL$.

Now we claim that for each pair $(d, \bar{k})$ as in the assertion, there exist an open and closed substack $FBun_{P,D,n}^{d; \bar{k}}$ of $FBun_{P,n} \times X^n (X^n \setminus \Delta)$ consisting of geometric points $s$ such that $\deg(L_s) = d$ and $^rL_s = \phi(L_s)(\sum_{i=1}^{[g]} k_i x_i)$. The first condition is clearly open and closed. For the second, consider the unique line bundle $L_i$ on $X \times [FBun_{P,n} \times X^n (X^n \setminus \Delta)]$, whose restriction to the complement of $\Gamma_{x_i}$ is $^rL$ and whose restriction to the complement of $\cup_{j \neq i} \Gamma_{x_j}$ is $\phi(L)$. Since the second condition is equivalent to $\deg(L_i) - \deg(L_s) = k_i$ for each $i$, it is open and closed as well.

Finally, observe that the map $(G, P, \psi, ... ) \mapsto \psi(F_{P \times S})$ defines a morphism from $FBun_{P,D,n}$ to the stack classifying Frobenius-equivalent $P_D$-structures of the trivial $G_D$-structure. As the latter stack is isomorphic to the discrete stack $G(O_D)/P(O_D)$ (compare Lemma 3.3), we get the required decomposition by the $[g]$'s.

$\square$

**Remark 5.6.** Passing to the limit over $D$'s, we get a decomposition of $FBun_{P,n}$ indexed by the triples as above but with $[g]$'s belonging to $G(O_D)/P(O_D)$, where $O_D$ is the ring of integral adeles of $F$. 

**Proposition 5.7.** For each triple \((d; k; [g])\) as in Claim [A.3], the restriction of the projection \(\Pi : FBun^{d,k;[g]}_{P,D,n} \to FBun_{D,n}\) to the generic fiber over \(X^n\) is finite and unramified. Furthermore, each projection \(FBun^{d,k;[g]}_{P^*;n} \to FBun_{*,n}\) is a closed embedding. In particular, every orispheric substack of \(FBun_{*,n}\) is closed.

**Proof.** As \(G(\mathbb{O})\) acts transitively on the set of \([g]\)'s, we may and will assume that \([g] = [1]\). Since Plücker relations are closed, Lemma [A.2] implies that it is enough to show the statement in the case \(G = GL_m\) and \(P\) is the maximal parabolic corresponding to the standard representation. Also the statement is clearly local on the base, and the first assertion is independent of \(D\) (by Proposition [2.16] (b)). Thus it will suffice to check that for each quasi-compact open substack \(V\) of \(Bun_{GL_m}\), each sufficiently large \(D\) and each sufficiently small open subscheme \(U \subset X^n\) (both depending on \(V\)), the restriction of \(\Pi\) to the preimage of \(V \times U \subset Bun_{GL_m} \times X^n\) is a closed embedding.

Given \(V\), let \(l_1\) and \(l_2\) be two integers such that for every geometric point of \(V\), the corresponding vector bundle does not have line (resp. rank two) subbundles of degree greater than \(l_1\) (resp. \(l_2\)). Let \(D \subset X\) be a finite subscheme such that \(|D| > \max\{l_1 - d, l_2 - 2d\}\) and \(D\) contains all points of \(X\) of degree \(\leq (l_1 - d)\) over \(\mathbb{F}_q\). Finally let \(U \subset X^n\) be an open subscheme such that each \((x_1, \ldots, x_n) \in U\) satisfies \(x_i \neq \tau^r x_j\) for each \(i, j\) and each \(r = 1, \ldots, l_1 - d\). Denote the preimages of \(V \times U \subset Bun_{GL_m} \times X^n\) in \(FBun_{D,n}\) and \(FBun^{d,k;[g]}_{P,D,n}\) by \(A\) and \(B\), respectively, and we going to check that the projection \(B \to A\) is a closed embedding.

Consider the natural morphism \(\nu : B \to FBun^d_{GL_1,D,n,k} \times (X - D)^n A\), where the first projection \(B \to FBun^d_{GL_1,D,n,k}\) was defined during the proof of Claim [5.3]. Note that \(FBun^d_{GL_1,D,n,k} \times (X - D)^n A\) classifies pairs consisting of a line bundle \(\mathcal{L}\) and a rank \(m\) vector bundle \(\mathcal{E}\) on \(X \times S\), equipped with \(D\)-level structures and \(F\)-structures (that is, rational isomorphisms from \(\mathcal{L}\) and \(\mathcal{E}\) to \(\tau \mathcal{L}\) and \(\tau \mathcal{E}\), respectively). The fiber of \(\nu\) over \((\mathcal{L}, \mathcal{E})\) classifies embeddings of vector bundles \(\eta : \mathcal{L} \hookrightarrow \mathcal{E}\), commuting with \(F\)-structures and preserving \(D\)-level structures. In particular, this means that the \(D\)-level structure of \(\mathcal{E}\) induces an isomorphism between \(\mathcal{L}|_{D \times S}\) and the first summand of \(\mathcal{O}_{D \times S}^n\). (Here we use the assumption that \([g] = 1\).)

Consider first an a priori slightly bigger stack \(B'\) equipped with a morphism \(\nu' : B' \to FBun^d_{GL_1,D,n,k} \times (X - D)^n A\), whose fibers classify the same data as \(\nu\), but \(\eta(\mathcal{L})\) is just a subsheaf of \(\mathcal{E}\) and not necessary a subbundle. As \(|D| > l_1 - d\), our choice of \(l_1\) implies that such an \(\eta\) is at most unique, therefore \(\nu'\) is a closed embedding. Since \(FBun^d_{GL_1,D,n,k}\) is finite and étale over \((X - D)^n\), this shows that \(B'\) is finite and unramified over \(A\).
Next we will show that an open embedding $B \hookrightarrow B'$ is actually an isomorphism. Assuming the contrary, there exists an $\overline{\mathbb{F}_q}$-point of $B'$ such that the corresponding $\mathcal{L}$ is not a subbundle of $\mathcal{E}$ (here we identify $\mathcal{L}$ with $\eta(\mathcal{L})$). Let $\mathcal{L}'$ be the line subbundle of $\mathcal{E}$, containing $\mathcal{L}$. Then $\mathcal{L}' = \mathcal{L}(D')$ for certain finite non-empty subscheme $D'$ of $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. As $\mathcal{E} \in \mathcal{A}(\mathbb{F}_q)$, we get $|D'| \leq l_1 - d$. Also we know that $\phi$ induces an isomorphism between restrictions of $\mathcal{L}'$ and $\tau \mathcal{L}'$ to $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} - \{x_1, \ldots, x_n\}$. Hence there exist integers $r_1, \ldots, r_n$ such that the divisor $\tau(D') - D'$ is equal to $\sum_{i=1}^n r_i x_i$.

As $\eta$ preserves $D$-level structures, $D'$ is disjoint from $D$. Hence each point of $D'$ has a degree at least $l_1 - d + 1$ over $\mathbb{F}_q$. Then for each $y \in D'$, all points $y, \tau y, \ldots, \tau^{l_1-d} y$ are distinct. Since $|D'| \leq l_1 - d$, we thus get that $\tau(D') \neq D'$. Choose a point $y \in D'$, which does not lie in $\tau(D')$. Let $r$ be the smallest positive integer such that $\tau^r y \notin D'$. Then $r \leq l_1 - d$ and $\tau^{r-1} y \in D'$, hence $\tau^r y \in \tau(D') \setminus D'$. Then the equality $\tau(D') - D' = \sum_{i=1}^n r_i x_i$ implies that both $y$ and $\tau^r y$ belong to $\{x_1, \ldots, x_n\}$, contradicting our assumption.

It remains to show that the projection $B \to \mathcal{A}$ is injective. If not, then there exists a geometric point of $\mathcal{A}$ and two different degree $d$ line subbundles $\mathcal{L}_1$ and $\mathcal{L}_2$ of the corresponding vector bundle $\mathcal{E}$, whose restrictions to $D$ coincide. Let $\tilde{\mathcal{E}}$ be the rank two subbundle of $\mathcal{E}$, generated by $\mathcal{L}_1$ and $\mathcal{L}_2$. Then we get a contradiction $\deg \tilde{\mathcal{E}} \geq \deg \mathcal{L}_1 + \deg \mathcal{L}_2 + |D| = 2d + |D| > l_2$. \hfill $\Box$

6. CUSPIDAL PART OF THE COHOMOLOGY

**Notation 6.1.** a) Let $L/\mathbb{F}_q$ be a finitely generated field extension. We will say that a $\text{Gal}(\overline{L}/L)$-module $\mathcal{F}/\mathbb{Q}_L$ is **mixed of weight** $\leq i$ (resp. **pure of weight** $i$) if there is a scheme of finite type $Z/\mathbb{F}_q$ with $\mathbb{F}_q(Z) = L$ such that $\mathcal{F}$ is the restriction of a mixed of weight $\leq i$ (resp. pure of weight $i$) $\mathbb{Q}_L$-sheaf on $Z$. In particular, if $\mathcal{F}$ is a mixed module of weight $\leq i$, it has a finite weight filtration $0 = \mathcal{F}_j \subset \mathcal{F}_{j+1} \subset \ldots \subset \mathcal{F}_i = \mathcal{F}$ such that each graded piece $\text{gr}_k(\mathcal{F}) := \mathcal{F}_k/\mathcal{F}_{k-1}$ is pure of weight $i$.

b) Let $Y/L$ be a scheme of finite type, and let $\mathcal{F}$ be an object of $D^b(Y, \mathbb{Q}_L)$. We will say that $\mathcal{F}$ is a **mixed complex of weight** $\leq 0$ if there is a morphism $\pi : \tilde{Y} \to Z$ of schemes of finite type over $\mathbb{F}_q$ such that $\mathbb{F}_q(Z) = L$, $Y$ is the generic fiber of $\pi$, and $\mathcal{F}$ is the restriction of a mixed complex on $\tilde{Y}$ of weight $\leq 0$.

c) Combining Deligne's theorem (\cite[6.3.3]{De}) with proper base change theorem, we obtain that if $\mathcal{F}$ is a mixed complex on $Y$ of weight $\leq 0$, then each $H^i_c(Y, \mathcal{F})$ is a mixed $\text{Gal}(\overline{L}/L)$-module of weight $\leq i$. We denote by $H^i_{c, \text{pure}}(Y, \mathcal{F})$ the corresponding graded piece $\text{gr}_i(H^i_c(Y, \mathcal{F}))$ and will call it pure cohomology with compact support.
Remark 6.2. Let \( Y \) be as in Notation 6.1 and let \( \mathcal{F} \) be a mixed complex on \( Y \) of weight \( \leq 0 \).

a) If \( f: Y' \rightarrow Y \) is a morphism of schemes of finite type over \( L \), then \( f^*(\mathcal{F}) \) is mixed of weight \( \leq 0 \). Therefore pure cohomology \( H^i_{c,\text{pure}}(Y', f^*(\mathcal{F})) \) is defined.

b) Every open embedding \( U \hookrightarrow Y \) induces an embedding \( H^i_{c,\text{pure}}(U, \mathcal{F}_U) \hookrightarrow H^i_{c,\text{pure}}(Y, \mathcal{F}) \). Indeed, the kernel of the canonical map \( H^i_{c}(U, \mathcal{F}|_U) \rightarrow H^i_{c}(Y, \mathcal{F}) \) is a quotient of \( H^{i-1}_{c}(Y \setminus U, \mathcal{F}|_{Y \setminus U}) \), hence by Deligne’s theorem, it is mixed of weight \( \leq (i - 1) \).

c) Every proper morphism \( f: Y' \rightarrow Y \) induces a morphism

\[ f^*: H^i_{c,\text{pure}}(Y, \mathcal{F}) \rightarrow H^i_{c,\text{pure}}(Y', f^*(\mathcal{F})). \]

In particular, we have a restriction map \( H^i_{c,\text{pure}}(Y, \mathcal{F}) \rightarrow H^i_{c,\text{pure}}(Y', \mathcal{F}|_{Y'}) \) for each closed subscheme \( Y' \) of \( Y \). Moreover, an argument, similar to b) shows that if \( Y_1, \ldots, Y_k \) are the set of all irreducible components of \( Y \), then the natural restriction map \( H^i_{c,\text{pure}}(Y, \mathcal{F}) \rightarrow \bigoplus_{j=1}^k H^i_{c,\text{pure}}(Y_j, \mathcal{F}|_{Y_j}) \) is an embedding.

d) If \( Y/L \) is proper, then \( H^i_{c}(Y, IC_Y) = H^i(Y, IC_Y) \) is pure of weight \( i \), and thus \( H^i_{c,\text{pure}}(Y, IC_Y) = H^i_{c}(Y, IC_Y) \). More generally, if \( Y \) is an open subscheme of a proper scheme \( Y'/L \), then by b), \( H^i_{c,\text{pure}}(Y, IC_Y) \) is the image of the natural morphism \( H^i_{c}(Y, IC_Y) \rightarrow H^i(Y', IC_{Y'}) \).

Notation 6.3. Let a Deligne–Mumford stack \( Y \) be the quotient of a quasi-projective scheme \( X \) by a finite group \( G \), let \( Z \) be the quotient \( X \) by \( G \) in the category of schemes, and let \( q: Y \rightarrow Z \) be the natural map. Following [La2, App. A] we define \( H^i_{c}(Y, \mathcal{F}) \) to be \( H^i_{c}(Z, q_*(\mathcal{F})) \) for every \( \mathcal{F} \in D^b(Y, IC) \).

Remark 6.4. In the notation of 6.3 we have

a) \( q_*(IC_Y) = IC_Z \) (compare [La2] proof of Prop. A.5] and c) below), therefore \( H^i_{c}(Y, IC_Y) = H^i_{c}(Z, IC_Z) \).

b) If \( \mathcal{F} \) is a mixed complex on \( Y \) of weight \( \leq 0 \), then \( q_*(\mathcal{F}) \) is mixed of weight \( \leq 0 \) as well, so we can define \( H^i_{c,\text{pure}}(Y, \mathcal{F}) := H^i_{c,\text{pure}}(Z, q_*(\mathcal{F})) \) which automatically satisfies all the properties of Remark 6.

c) \( H^i_{c}(Y, IC_Y) = H^i_{c}(X, IC_X)^G \). Indeed, let \( \pi: X \rightarrow Z \) be the quotient map. Then \( \pi \) is finite, thus \( \pi_*(IC_X) \) is a pervers sheaf, and \( IC_Z = \pi_*(IC_X)^G \). Since \( H^i_{c}(X, IC_X) = H^i_{c}(Z, \pi_*(IC_X)) \), the statement follows from the fact that any additive functor (e.g., \( H^i_{c}(Z, \cdot) \)) between \( \mathbb{Q} \)-linear abelian categories commutes with taking invariants with respect to a finite group.

6.5. Proof of Proposition 2.34 a) First we will show that the space \( H^i_{cusp} \) is contained in \( H^i_{pure}(\overline{\mathcal{M}}) \). Let \( h \) be any element of \( H^i_{pure}(\overline{\mathcal{M}}) \). Choose sufficiently large
\( \mu \) and \( D \) such that \( h \) belongs to \( H_{\text{pure},D}^{i,\mu} \setminus H_{\text{pure},D}^{i,d(\varnothing)\rho} \). Hence \( h \) does not vanish on
\[ J\setminus [FBun_{D,n,\varnothing}^{\leq \mu} \setminus FBun_{D,n,\varnothing}^{d(\varnothing)\rho}] \]. By Remark 6.2 c), \( h \) therefore does not vanish on
a certain irreducible component \( C^0 \) of \( J\setminus [FBun_{D,n,\varnothing}^{\leq \mu} \setminus FBun_{D,n,\varnothing}^{d(\varnothing)\rho}] \). By Remark 6.2 b), \( h \) then does not vanish on the closure \( C \) of \( C^0 \) in \( J\setminus FBun_{D,n,\varnothing} \). As \( C \) is an
orispheric substack (use Theorem 2.22 and Proposition 5.1), we see that \( h \notin H_{\text{cusp}}^i \).
This shows that \( H_{\text{cusp}}^i \subset H_{\text{pure}}^{i,d(\varnothing)\rho} \).

As the set of orispheric substacks is \( G(\mathbb{A}) \)-invariant, \( H_{\text{cusp}}^i \) is a \( G(\mathbb{A}) \)-invariant
subspace of \( H_{\text{pure}}^i \). Therefore \( H_{\text{cusp}}^i \) is contained in \( \cap_{g \in G(\mathbb{A})} g(H_{\text{pure}}^{i,d(\varnothing)\rho}) \), hence in
\( \cap_{g \in G(\mathbb{A})} g(H_{\text{pure}}^{i,\mu}) \) for each \( \mu \geq d(\varnothing)\rho \).

Conversely, assume that some \( h \in H_{\text{pure}}^i \) does not lie in \( H_{\text{cusp}}^i \). Then the restriction of
\( h \) to some orispheric substack \( C \subset J\setminus FBun_{s,n,\varnothing} \) is non-trivial. Let \( s \) be the
geometric generic point of \( C \). It remains to show that for every \( \mu \) there exists
\( g \in G(\mathbb{A}) \) such that \( g(s) \notin J\setminus FBun_{s,n,\varnothing}^{\leq \mu} \). Indeed, this would imply that \( g(C) \cap
J\setminus FBun_{s,n,\varnothing}^{\leq \mu} = \emptyset \), hence \( g^{-1}(h) \notin H_{\text{pure}}^{i,\mu} \), thus \( h \notin g(H_{\text{pure}}^i) \), as claimed.

By Proposition 5.7 s can be considered as a geometric point of \( FBun_{P,s,n} \) for
certain maximal parabolic \( P \) of \( G \). Moreover, replacing \( s \) by its \( G(\mathbb{O}) \)-translate, we
may assume that it lies in \( FBun_{P,s,n}^{d,k,[g]} \) with \( g = 1 \). Let \( \lambda \) be the quasi-fundamental
weight of \( G \) corresponding to \( P \). If for each \( g \in P(\mathbb{A}) \) we denote by \( P_g \) the fiber at
\( g(s) \) of the universal \( P \)-bundle, then \( \deg(P_g) \lambda = \deg(P_1)\lambda + \log_q |\lambda(g)| \). In particular,
\( g(s) \) does not belong to \( J\setminus FBun_{s,n,\varnothing}^{\leq \mu} \) if \( |\lambda(g)| \) is sufficiently large.

b) Let \( U \subset G(\mathbb{O}) \subset G(\mathbb{A}) \) be any compact open subgroup. By a), the space of invariants \( (H_{\text{cusp}}^i)^U \) is contained in \( (H_{\text{pure}}^{i,d(\varnothing)\rho})^U \). By Remark 6.4 c), the latter space
is contained in some \( H_{\text{pure},D}^{i,d(\varnothing)\rho} \). Since this space is clearly finite dimensional, the
statement follows. \( \square \)

7. Drinfeld–Lafforgue case

Proof of Theorem 2.3.4. During the proof we will use Drinfeld’s notation \( FSh_{\tau} \)
instead of the generic fiber of \( FBun_{GL_\tau} \). Since \( FSh_{\tau} \) is smooth of relative dimension
\( 2(r-1) \) over \( F^{(2)} \), we have \( H^0_c(FSh_{\tau},\mathbb{C}) = H^2_{c}((FSh_{\tau},\mathbb{C})(r-1)) \).

Notation 7.1. Following Lafforgue ([La2]), we will say that a representation \( V \)
of \( \Gamma_{F^{(2)}} \) is \( r \)-negligible if each irreducible subquotient \( V' \) of \( V \) is isomorphic to a
subquotient of the exterior product \( V'_1 \boxtimes V'_2 \) of \( (\Gamma_{F})^2 \) (composed with the projection
\( \Gamma_{F^{(2)}} \rightarrow (\Gamma_{F})^2 \)) for some representations \( V'_1 \) and \( V'_2 \) of \( \Gamma_{F} \) of dimensions strictly less
than \( r \).
Lemma 7.2. For each \(d(\pi)\rho \leq \mu \leq \mu'\) and each \(D\), the quotient \(H^{0,\mu'}_{\text{pure},D}/H^{0,\mu}_{\text{pure},D}\) is \(r\)-negligible.

**Remark 7.3.** a) Lafforgue proved this assertion under the assumption that \(\mu\) is sufficiently large as a function of \(D\) (see [La2, Cor. VI.21]).

b) Though Lafforgue [La2] and Drinfeld [Dr2] worked only in the case when \(J \subset \mathbb{A}^x\) is a cyclic group generated by an element of degree one, the general case applies without any changes.

First we will show the theorem, assuming the lemma.

a) Lemma 7.2 implies that \(H^0_{\text{pure}}/H^{0,d(\pi)\rho}_{\text{pure}}\) is \(r\)-negligible. By Proposition 2.34 a), \(H^0_{\text{pure}}/H^0_{\text{cusp}}\) is a subrepresentation of \(\bigoplus_{g \in G(\mathbb{A})} H^0_{\text{pure}}/g(H^{0,d(\pi)\rho}_{\text{pure}})\). Since the latter space is isomorphic as a \(\Gamma\)-space to the direct sum of \(G(\mathbb{A})\) copies of \(H^0_{\text{pure}}/H^0_{\text{cusp}}\), we get that \(H^0_{\text{pure}}/H^0_{\text{cusp}}\) is \(r\)-negligible as well.

By (one of) the Main result(s) of Lafforgue (see [La2, Lem. IV.25 and Thm. VI.27]), there exists an exhausting \(G(\mathbb{A}) \times \Gamma\)-invariant filtration \(0 = \tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \subset \ldots \subset H^0\) such that each quotient \(\tilde{V}_{2i+1}/\tilde{V}_{2i}\) is \(r\)-negligible, and \(\bigoplus_j \tilde{V}_{2j}/\tilde{V}_{2j-1}\) is isomorphic to \(\bigoplus_\pi (\pi \boxtimes \rho_\pi \boxtimes \hat{\rho}_\pi)\) (compare Remark 7.3 b)). Actually Lafforgue has shown that the last isomorphism holds after semi-simplification. However, it was observed by Drinfeld that there are no non-trivial extensions between non-isomorphic cuspidal representations \(\pi_1\) and \(\pi_2\) of \(GL_r(\mathbb{A})\), therefore the former representation is automatically semi-simple. (Indeed, choose a compact and open subgroup \(U\) of \(GL_r(\mathbb{A})\) such that \((\pi_1)^U \neq 0\) and \((\pi_2)^U \neq 0\). It remains to show that there are no non-trivial extensions between \((\pi_1)^U\) and \((\pi_2)^U\) as representations of the Hecke algebra \(\mathcal{H}_U := \mathcal{H}(GL_r(\mathbb{A}), U)\). Since the center of \(\mathcal{H}_U\) contains \(\mathcal{H}(GL_r(F_v), GL_r(O_v))\) for almost all points \(v\) of \(X\), the strong multiplicity one theorem for \(GL_r\) implies that \((\pi_1)^U\) and \((\pi_2)^U\) has non-equal central characters of \(\mathcal{H}_U\). This implies the assertion).

Since we know that \(\rho_\pi \boxtimes \hat{\rho}_\pi\) is pure (Ramanujan conjecture [La2, Thm. VI.10]) and that \(H^0_{\text{pure}}/H^0_{\text{cusp}}\) is \(r\)-negligible, the induced filtration of \(H^0_{\text{cusp}}\) satisfies the required property. (By an induced filtration we mean the filtration \(\{V_j\}_j\) such that each \(V_j\) is the intersection of \(H^0_{\text{cusp}}\) with the image of \(\tilde{V}_j\) in \(H^0_{\text{pure}}\).)

b) In the case \(r = 2\), Drinfeld ([Dr2]) constructed a compactification \(\overline{J/FSh^\leq_{1,2D}}\) of each \(J/FSh^\leq_{2,D}\) with \(\mu\) sufficiently large (which is the coarse moduli space of the compactification considered by Lafforgue [La2, III]). Moreover, for each \(\mu_1 \leq \mu_2\) and \(D_1 \subset D_2\), there exists a canonical morphism \(\overline{J/FSh^\leq_{2,D}} \rightarrow \overline{J/FSh^\leq_{1,2D}}\), and the inverse limit \(\overline{J/FSh^\leq_{2,D}}\) of the \(J/FSh^\leq_{2,D}\)'s is equipped with an action of the adelic group \(G(\mathbb{A})\).
Though compactifications \( \overline{J \setminus FSh^≤_2} \) are not smooth, their middle cohomology 
\( H^2(\overline{J \setminus FSh^≤_2}, \mathbb{Q}_l(1)) \) is pure of weight zero and satisfies Poincaré duality. As a result, their direct limit \( H^2(J \setminus FSh^≤_2, \mathbb{Q}_l(1)) \) is equipped with a non-degenerate pairing. Furthermore, Drinfeld’s main result says that the orthogonal complement \( H_{Drin} \subset H^2(J \setminus FSh^≤_2, \mathbb{Q}_l(1)) \) of the set of all Chern classes of orispheric curves is isomorphic to the direct sum \( \bigoplus_\pi (\pi \otimes \rho_\pi \otimes \hat{\rho}_\pi) \). By the Poincaré duality, \( H_{Drin} \) can be described as a set of all elements of \( H^2(J \setminus FSh^≤_2, \mathbb{Q}_l(1)) \), vanishing on all orispheric curves.

Since \( H^2(J \setminus FSh^≤_2, \mathbb{Q}_l(1)) \) is pure of weight zero, we get a canonical embedding \( H^0_{cusp} \hookrightarrow H^2(J \setminus FSh^≤_2, \mathbb{Q}_l(1)) \), which therefore induces an embedding \( H^0_{cusp} \hookrightarrow H_{Drin} \). Combining this with the result of a), we thus conclude that \( H^0_{cusp} = H_{Drin} \).

7.4. Proof of Lemma [7.2] By Remark [6.2 c), it suffices to check that \( H^{2(r-1)}_{c\text{pure}}(S, \mathbb{Q}_l) \) is \( r \)-negligible for every irreducible component \( S \) of \( J \setminus FSh^≤_{r,D} \setminus FSh^≤_{m,D} \). Our strategy will be to reduce the statement to the case, where \( S \) is an \( \mathbb{A}^{r-m} \)-bundle over an open substack \( FSh^≤_{m,D} \) of \( FSh^≤_{m,D} \) for certain \( m < r \). Indeed, in this case \( H^{2(r-1)}_{c\text{pure}}(S, \mathbb{Q}_l) \cong H^{2(m-1)}_{c\text{pure}}(FSh^≤_{m,D}, \mathbb{Q}_l) \) would be isomorphic to a subspace of \( H^{2(m-1)}_{c\text{pure}}(FSh^≤_{m,D}, \mathbb{Q}_l) \) (use Remark [6.2 b)). Hence by [La2] Prop. VI.15 it is \((m+1)\)-negligible, hence \( r \)-negligible.

By Theorem [2.23] and Proposition [5.7], \( S \) is an open substack of an orispheric substack. Let \( P \) be the parabolic subgroup of \( G \), corresponding to \( S \). Since the statement for large \( D \)'s implies that for small ones, we may increase \( D \) during the proof. Hence using Proposition [5.7] and Remark [6.2 b), we may replace \( S \) by the generic fiber of \( FSh^≤_{r,P,D} \). In other words it will suffice to show the statement for each connected component of the) stack \( S \) classifying pairs consisting of an \( F \)-sheaf of rank \( r \) with \( D \)-level structure \((\mathcal{E}, \psi; x_1, x_2; \phi) \) and a subbundle \( \mathcal{A} \) of \( \mathcal{E} \) of rank \( m \) (determined by \( P \)) such that \( \phi(\mathcal{A}) \subset \mathcal{T} \mathcal{A}(x_1) \) and \( D \)-level structure \( \psi \) maps \( \mathcal{A}_D \) into the first \( m \)-coordinates.

Applying if necessary transformation sending an \( F \)-sheaf to its dual, we may assume that \( S \) classifies pairs for which \( \mathcal{B} := \mathcal{E}/\mathcal{A} \) is a pullback of a vector bundle over \( X \), and \( \phi \) induces a canonical isomorphism \( \mathcal{B} \cong \mathcal{T} \mathcal{B} \). Thus we can replace \( S \) with its open and closed substack, over which both the quotient \( \mathcal{B} = \mathcal{E}/\mathcal{A} \) and its induced \( D \)-level structure are constant. Moreover, we may fix a coweight \( \nu \) of \( SL_m \) and an integer \( d \) and to replace \( S \) by its open substack such that \( \mathcal{A} \) belongs to \( FSh_{m,D}^{≤\nu,v} \). Enlarging \( |D| \), we may assume that \( \text{Hom}(\mathcal{B}, \mathcal{A}((-D))) = 0 \) for each \((\mathcal{T} \mathcal{A} \hookrightarrow \mathcal{A}' \hookrightarrow \mathcal{A}) \in FSh_{m,D}^{≤\nu,v} \).
Choose a complete flag of subbundles \(0 = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_{k-m-1} \subset \mathcal{B}_{k-m} = \mathcal{B}\) defined over \(\mathbb{F}_q\), and put \(M := d - 2g + 1 - |D| - \max_j \deg(\mathcal{B}_j/\mathcal{B}_{j-1})\). By induction, we may assume the statement for smaller \(m\)'s (the assumption is vacuously true for \(m = 1\)), and thus to replace \(S\) by an open substack \(S'\) characterized by the condition that \(\mathcal{A}\) does not have \(F\)-subsheaves of rank \((m - 1)\) and degree larger than \(M\). We claim that the natural forgetful map \(\pi : S' \to \mathcal{F}Sh_{m,D}^{\leq \nu,d}\) is smooth, and all its non-empty fibers are isomorphic to \(\mathbb{A}^{r-m}\).

As the smoothness statement was shown in [La1, II, Thm. 11] (and easily follows from our arguments below), it remains to show the statement about fibers. Denote the image of \(\pi\) by \(\mathcal{F}Sh_{m,D}'\), and let \(s\) be any geometric point of \(\mathcal{F}Sh_{m,D}'\). We claim that the fiber \(\pi^{-1}(s)\) is canonically isomorphic to the space of extensions \(\tilde{\mathcal{E}} \in \text{Ext}(\mathcal{B}, \mathcal{A}_s(-D))\) whose image in \(\text{Ext}(\mathcal{B}, \mathcal{A}'_s(-D))\), induced by the embedding \(\mathcal{A}_s \hookrightarrow \mathcal{A}'_s\), coincides with the image of \(\tau \tilde{\mathcal{E}} \in \text{Ext}(\mathcal{B}, \tau \mathcal{A}_s(-D))\), induced by the embedding \(\tau \mathcal{A}_s \hookrightarrow \mathcal{A}'_s\). Indeed, given \((\mathcal{E}_s, \psi)\) in \(\pi^{-1}(s)\) we can get \(\tilde{\mathcal{E}}\) as the kernel of the composition of the projection \(\mathcal{E}_s \to \mathcal{E}_s|_D\), the level structure \(\psi\), and the projection to the first \(m\) factors. Conversely, given \(\tilde{\mathcal{E}}\), we can define \(\mathcal{E}_s\) as \((\tilde{\mathcal{E}} \oplus \mathcal{A}_s)/\mathcal{A}_s(-D)\), and the first \(m\) (resp. the last \(r - m\)) coordinates of \(\psi\) to be the composition of the projection \(\mathcal{E}_s \to \mathcal{E}_s/\mathcal{A}_s(-D) = \mathcal{A}_s|_D\) (resp. \(\mathcal{E}_s \to \tilde{\mathcal{E}}|_D \to \mathcal{B}|_D\)) and \(D\)-level structure of \(\mathcal{A}\) (resp. \(\mathcal{B}\)).

Put \(U := \text{Ext}(\mathcal{B}, \mathcal{A}_s(-D))\) and \(V := \text{Ext}(\mathcal{B}, \mathcal{A}'_s(-D))\). Then embeddings \(\mathcal{A}_s \hookrightarrow \mathcal{A}'_s\) and \(\tau \mathcal{A}_s \hookrightarrow \mathcal{A}'_s\) define a linear homomorphism \(\lambda : U \to V\) and a \(\tau\)-linear homomorphism \(\psi : U \to V\), respectively. As \(\mathcal{A}'/\mathcal{A}\) is supported at one point, \(\lambda\) is surjective, and our assumption \(\text{Hom}(\mathcal{B}, \mathcal{A}'_s(-D)) = 0\) implies that its kernel is of dimension \(\text{rk}(\mathcal{B}) = k - m\). As \(\pi^{-1}(s)\) is isomorphic to \(\text{Ker}(\lambda - \psi)\), [La1, II, Lem. 18 iii] and its proof (compare [Dr1 §4, Lem. 1]) means that in order to show the statement, it remains to check that there is no linear functional \(L\) on \(V\) such that \(L(\psi(u)) = L(\lambda(u))^q\) for each \(u \in U\).

Assume that such an \(L\) exist. By Serre duality, \(\tilde{V} \cong H^0(X, K_X \otimes \mathcal{B} \otimes \mathcal{A}'(D))\), so \(L\) defines a non-trivial \(\phi\)-equivariant section of \(K_X \otimes \mathcal{B} \otimes \mathcal{A}'(D)\). Hence the exists \(j\) such that \(L\) defines a non-trivial \(\phi\)-equivariant section of \(K_X \otimes (\mathcal{B}_j/\mathcal{B}_{j-1}) \otimes \mathcal{A}'(D)\), thus a non-trivial \(\phi\)-equivariant morphism \(\mathcal{A} \to K_X \otimes (\mathcal{B}_j/\mathcal{B}_{j-1})(D)\). Its kernel is a \(F\)-subsheaf of \(\mathcal{A}'\) rank \(l - 1\) and degree at least \(d - (2g - 2) - \deg(\mathcal{B}_j/\mathcal{B}_{j-1}) - |D|\) > \(M\), contradicting our definition of \(S'\). This completes the proof of the claim, which (as it was observed in the beginning of the proof) implies the lemma.
Appendix A.

In the appendix we give proofs (or sketches of proofs) of basic properties of $G$-bundles, affine grassmannians and the stacks of Hecke, used in the paper. Though at least some of these facts are considered well known among experts, we were not able to find any reference in the literature, so we sketch the argument for completeness.

A.1. Proof of Lemma 2.2. In the case $G = G_m$, the required isomorphism between $\pi_0(Bun_{G_m}) = \pi_0(Pic)$ and $X_*(G_m) = Z$ is given by the degree map. This implies the statement in the case of torus. In general, the embedding $T \hookrightarrow G$ defines a surjective map $\pi : X_*(T) = \pi_0(Bun_T) \to \pi_0(Bun_G)$ (see [DS App.]). Moreover, using standard reduction to the $GL_2$-case (see [DS App., where the semi-simple simply-connected case is treated, and compare the proof of Proposition A.9 b)), we get that $\pi$ factors through $\pi_1(G)$. Thus it remains to show the injectivity of the resulting canonical map $\pi_1: \pi_1(G) \to \pi_0(Bun_G)$.

When $G^{der}$ is simply connected, the injectivity of $\pi_1$ follows from the fact that the composition map $\pi_1(G) \to \pi_0(Bun_G) \to \pi_0(Bun_{G^{ab}}) = \pi_1(G^{ab})$ is an isomorphism.

For a general $G$, choose a central extension $\nu : H \to G$ such that $H^{der}$ is simply-connected, and $S := Ker(\nu)$ is a split torus (see [MS, Prop. 3.1]). Then we have the following commutative diagram, induced by $\nu$:

\[
\begin{array}{ccc}
\pi_1(H) & \xrightarrow{\pi_H} & \pi_0(Bun_H) \\
\downarrow \pi_1(\nu) & & \downarrow \pi_0(\nu) \\
\pi_1(G) & \xrightarrow{\pi_G} & \pi_0(Bun_G)
\end{array}
\]

As $Bun_S$ acts transitively on all geometric fibres of the projection $Bun_H \to Bun_G$, $\pi_0(Buns) = \pi_1(S)$ acts transitively on all fibers of $\pi_0(\nu)$. Since $\pi_H$ is injective and $\pi_1(S)$-equivariant, and since $\pi_1(G) = \pi_1(S) \setminus \pi_1(H)$, the injectivity of $\pi_G$ follows.

Lemma A.2. Let $P$ be a parabolic subgroup of $G$, and let $G$ be a $G$-bundle over a scheme $S$. Then there exists a canonical bijection between:

(i) $P$-structures of $G$;

(ii) families of line subbundles $\{A_\lambda \subset G_\lambda\}$ (indexed by characters of $P$ which are dominant weights of $G$) such that $A_{\lambda_1 + \lambda_2} = A_{\lambda_1} \otimes A_{\lambda_2}$ (considered as subbundles of $G_{\lambda_1 + \lambda_2} \subset G_{\lambda_1} \otimes G_{\lambda_2}$) for each $\lambda_1$ and $\lambda_2$;

(iii) families of line subbundles $\{A_\lambda \subset G_\lambda\}$ (indexed by characters of $P$ which are quasi-fundamental weights of $G^{ad}$) satisfying the Plücker relations: for each tuple of non-negative integers $(k_i)_i$, the line subbundle $\otimes_i (A_\lambda)^{k_i} \subset \otimes_i (G_\lambda)^{\otimes k_i}$ is contained in $G_{\sum_i k_i \lambda_i}$. 
Proof. (i) \implies (ii). Let \( \mathcal{P} \) be a \( P \)-structure of \( \mathcal{G} \). If a dominant weight \( \lambda \) of \( G \) is a character of \( P \), then the highest weight line \( l_\lambda \) of \( V_\lambda \) is a \( P \)-subrepresentation of \( V_\lambda \). Moreover, line subbundles \( \mathcal{A}_\lambda := \mathcal{P}[\mathcal{P} \times l_\lambda] \subset \mathcal{P}[\mathcal{P} \times V_\lambda] = \mathcal{G}_\lambda \) satisfy (ii).

(ii) \implies (iii) is clear.

(iii) \implies (i). Suppose that we are given a family \( \{\mathcal{A}_\lambda \subset \mathcal{G}_\lambda\}_i \), satisfying the Plücker relations. We claim that this family defines a canonical \( P \)-structure of \( \mathcal{G} \) or what is the same a canonical section of the fibration \( P\mathcal{G} \to S \). By the uniqueness assertion, the statement is local in the étale topology on \( S \), so we may assume that \( \mathcal{G} \) is trivial. Hence \( P\mathcal{G} \cong (P/G) \times S \), thus we want to construct an \( S \)-point of \( P\mathcal{G} \).

We have a canonical embedding \( P\mathcal{G} \hookrightarrow \prod_i \mathbb{P}(V_\lambda) \). Each \( \mathcal{A}_\lambda \subset \mathcal{G}_\lambda \) corresponds to an \( S \)-point of \( \mathbb{P}(V_\lambda) \), and the fact that the \( \{\mathcal{A}_\lambda \subset \mathcal{G}_\lambda\}_i \) satisfy the Plücker relations means precisely that the corresponding \( S \)-point of the product \( \prod_i \mathbb{P}(V_\lambda) \) defines an \( S \)-point of \( P\mathcal{G} \).

Proof of Lemma A.3. \( \mathcal{B} \cap \mathcal{G} \) is an open substack of \( \mathcal{B} \).

Proof. As any dominant weight of \( G^{\text{ad}} \) is a linear combination of quasi-fundamental weights with rational non-negative coefficients, it is enough to check the condition \( \deg(B_\lambda) \leq \langle \mu, \lambda \rangle \) only when \( \lambda \) is a quasi-fundamental weight. Since the number of quasi-fundamental weights is finite, it remains to check the openness of this condition for a given quasi-fundamental weight \( \lambda_i \).

Let \( P_i \supset B \) be the maximal parabolic subgroup corresponding to \( \lambda_i \); then \( (B_\lambda)_{\lambda_i} \) depends only on \( P_i \times_B B \). By Lemma A.2, condition \( \deg(B_\lambda)_{\lambda_i} \leq \langle \mu, \lambda_i \rangle \) is equivalent to the assertion that \( (\mathcal{G}_s)_{\lambda_i} \) has no line subsheaves of degree \( \langle \mu, \lambda_i \rangle + 1 \), satisfying Plücker relations. Since Plücker relations are closed, and since line subsheaves of given degree of a given vector bundle are represented by a projective scheme, our condition \( \deg(B_\lambda)_{\lambda_i} \leq \langle \mu, \lambda_i \rangle \) is therefore open, as claimed.

Proof of Lemma A.4. a) First we consider the case of \( \mathcal{G} = GL_n \). Fix an ample line bundle \( \mathcal{O}(1) \) on \( X \). Then given \( \mu \), there exists \( m \in \mathbb{N} \) such that for every \( S/\mathbb{P}_q \) and every \( \mathcal{E} \in \mathcal{B}_\mu(S) \), we have:

i) the direct image \( \text{pr}_*(\mathcal{E}(m)) \) is a vector bundle on \( S \),

ii) \( R^1 \text{pr}_*(\mathcal{E}(m)) = 0 \), and

iii) \( \mathcal{E} \) is a quotient of \( \text{pr}^*([\text{pr}_*(\mathcal{E}(m))])(-m) \).

Next given \( (\mu, m) \), we get that \( \text{pr}_*(\mathcal{E}(m)) \) is a subbundle of \( \text{pr}_*\mathcal{E}(m)|_{D \times S} \) \( \mathcal{O}_D^\mu \mathcal{E}_D \) for each \( |D| \) sufficiently large, where the last isomorphism is induced by the \( D \)-level structure. Thus the functor \( \mathcal{F} \hookrightarrow \mathcal{G}(\mathcal{F}, \psi) \) embeds \( \mathcal{B}_\mu \) into a quasi-projective scheme classifying pairs consisting of a subbundle \( \mathcal{H} \) of \( \mathcal{O}_D^\mu \mathcal{E}_D \) of given rank, and a locally free quotient of \( \text{pr}^*\mathcal{H}(-m) \) of rank \( n \) and degree \( \nu \). This implies the representability of our functor in the \( G L_n \)-case.
For the general case, we can proceed as in ([Be1 Sec.4]): Choose an embedding of $G$ into $GL_n$. Since $Bun_{GL}^{e,w}$ is quasi-compact, the map $G \to GL_n \times G$ maps $Bun_{GL}^{e,w}$ into some $Bun_{GL_n}^{e,w}$. Next, using the fact that $G$ is reductive and, therefore, $GL_n/G$ is affine, we conclude that the induced map $Bun_{GL}^{e,w} \to Bun_{GL_n}^{e,w}$ is affine, implying the representability assertion. Finally, since coherent sheaves on curves have no second cohomology, the smoothness assertion follows from deformation theory.

b) First, we claim that there exists a faithful representation $V$ of $G$, which is a direct sum of Weyl modules. Indeed, our claim is equivalent to the assertion that the intersection of kernels of all Weyl modules is trivial. As $G$ is reductive, this intersection is obviously central, so it is contained in the maximal torus $T$ of $G$. But dominant weights of $G$ generate the group of all characters of $T$, so their kernels have a trivial intersection, as claimed.

Now our strategy will be very similar to that of [Ga A.5], where the corresponding statement for $Gr_w$, defined below, is shown. Choose $N \in \mathbb{N}$ such that $-N \leq \langle \omega_i, \lambda \rangle \leq N$ for each $i = 1, \ldots, n$ and each weight $\lambda$ of $V$. Consider a substack $Hecke'$ of $Hecke_n$, consisting of tuples $(\mathcal{G}, \mathcal{G}'; x_1, \ldots, x_n; \phi) \in Hecke_n$ satisfying

i) $\mathcal{G}'_V(-N(\sum_{i=1}^n \Gamma_{x_i})) \subset \phi(\mathcal{G}_V) \subset \mathcal{G}'_V(N(\sum_{i=1}^n \Gamma_{x_i}))$ and

ii) $\det(\mathcal{G}_V) = \det(\mathcal{G}'_V)(\sum_{i=1}^n (\det(V), \omega_i) \Gamma_{x_i})$.

Then $Hecke_{n,\mathfrak{F}}$ is a closed substack of $Hecke'$ (use Remark 2.5c), so it remains to show the representability and projectivity of $Hecke' \to Bun \times X^n$.

Consider another stack $Hecke''$ classifying data $(E, \mathcal{G}'; x_1, \ldots, x_n; \phi)$, where $(\mathcal{G}'; x_1, \ldots, x_n) \in (Bun \times X^n)(S)$, $E$ is a vector bundle over $X \times S$ of rank $\dim V$, and $\phi$ is an isomorphism between the restrictions of $E$ and $\mathcal{G}_V'$ to $(X \times S) \setminus (\Gamma_{x_1} \cup \ldots \cup \Gamma_{x_n})$ satisfying conditions i) and ii) as above with $\mathcal{G}_V$ replaced by $E$. Then $Hecke''$ is represented by a closed substack of a relative grassmannian over $Bun \times X^n$, hence $Hecke'' \to Bun \times X^n$ is representable and projective.

Let $\widetilde{\mathcal{G}}$ be the universal $GL(V)$-bundle over $Hecke'' \times X$, corresponding to $E$. Then the quotient $G/\widetilde{\mathcal{G}}$ has a canonical section $l$ over $(Hecke'' \times X) \setminus (\Gamma_{x_1} \cup \ldots \cup \Gamma_{x_n})$, corresponding to the universal isomorphism of $GL(V) \times G$ with $\widetilde{\mathcal{G}}$ over this set. Moreover, $Hecke'$ is the largest substack $\mathcal{A} \subset Hecke''$ such that $l$ extends to a regular section on all of $\mathcal{A} \times X$. We want to show that $Hecke'$ is a closed substack of $Hecke''$.

As $G$ is reductive, $G/\widetilde{\mathcal{G}}$ is affine. Thus $G/\widetilde{\mathcal{G}}$ is affine over $Hecke'' \times X$. Since the question is local in the Zariski topology on both $Hecke''$ and $X$, we are thus reduced to the following assertion: Suppose we are given a scheme $S$, $n$ points $x_1, \ldots, x_n \in X(S)$ and a regular function $f$ on $U := (S \times X) \setminus (\Gamma_{x_1} \cup \ldots \cup \Gamma_{x_n})$. Then the functor $\mathcal{F}_i/S$, which for every scheme $T$ classifies morphisms $f \in \text{Hom}(T, S)$
such that \( f^*(l) \) extends to a regular function on \( T \times X \), is represented by a closed subscheme of \( S \).

Let \( j : U \hookrightarrow S \times X \) be the open embedding, let \( s \in \Gamma(X \times S, j_*(\mathcal{O}_U)/\mathcal{O}_{X \times S}) \) be the image of \( l \), and for each \( i = 1, \ldots, n \), let \( s_i \) be the restriction of \( s \) to \( \Gamma_{x_i} \). Then \( f \in \text{Hom}(T, S) \) belongs to \( \mathcal{F}_l(T) \) if and only if \( f^*(s_i) = 0 \) for each \( i \). Therefore \( \mathcal{F}_l \) is represented by the intersection of the schemes of zeros of the \( s_i \)'s.

- Since \( \text{Hecke}^a_{n,\overline{\omega}} = \text{Hecke}^a_{1,\overline{\omega}_1} \times_{\text{Bun}} \text{Hecke}^a_{1,\overline{\omega}_2} \times_{\text{Bun}} \ldots \times_{\text{Bun}} \text{Hecke}^a_{1,\overline{\omega}_n} \), it is projective over \( X^n \times \text{Bun} \) (by b)) hence over \( \text{Hecke}^a_{n,\overline{\omega}} \). The second assertion follows from Lemma A.8(a) below applied \( (n-1) \)-times.

\[ \square \]

**Notation A.5.** Consider a formal disc \( \mathcal{D} := \text{Spec } k[[x]] \) (resp. punctured formal disc \( \mathcal{D}^* := \text{Spec } k((x)) \)) over \( k \). For each scheme \( X/k \), we denote by \( X_\mathcal{D} \) (resp. \( X_{\mathcal{D}^*} \)) be a scheme (resp. functor) over \( k \) such that \( X_\mathcal{D}(A) = X(A \otimes_k k[[x]]) \) (resp. \( X_{\mathcal{D}^*}(A) = X(A \otimes_k k((x))) \)) for each \( k \)-algebra \( A \). In particular, \( X_\mathcal{D} \) is just the inverse limit of the \( X_{\mathcal{D}^*_i} \)'s, where \( \mathcal{D}_i := \text{Spec } k[[x]]/(x^i) \), and \( X_{\mathcal{D}_i} = R_{\mathcal{D}_i/k}X \) is the Weil restriction of scalars.

**Definition A.6.** Let \( \text{Gr} \) be an ind-scheme, which classifying \( G \)-bundles over a formal disc \( \mathcal{D} \), trivialized over a punctured formal disc \( \mathcal{D}^* \) (compare [GA]). \( \text{Gr} \) is called a local affine grassmannian. As in the global case, for each \( \omega \in X^+_*(T) \) we define a closed subscheme \( \text{Gr}_\omega \) of \( \text{Gr} \) and a locally closed subscheme \( \text{Gr}_\omega^0 \).

**Remark A.7.** \( \text{Gr}_{\mathcal{D}^*} \) is the group of automorphisms of the trivial \( G \)-bundle on \( \mathcal{D}^* \). In particular, \( \text{Gr}_{\mathcal{D}^*} \) acts naturally on \( \text{Gr} \), making \( \text{Gr} \) a homogeneous space. Moreover, \( \text{Gr}_\omega \) and \( \text{Gr}_\omega^0 \) are \( \text{Gr}_{\mathcal{D}^*} \)-invariant subschemes of \( \text{Gr} \).

**Lemma A.8.** a) For each two positive integers \( k < n \), put

\[
U_{k,n-k} := \{(x_1, \ldots, x_n) \in X^n | x_i \neq x_j \text{ for each } i \leq k < j \}.
\]

Then for each \( \overline{\omega}_1 \in X^+_k(T)^k \) and \( \overline{\omega}_2 \in X^+_k(T)^{n-k} \), the forgetful morphism

\[ \pi : \text{Hecke}^a_{k,\overline{\omega}_1} \times_{\text{Bun}} \text{Hecke}^a_{n-k,\overline{\omega}_2} \to \text{Hecke}^a_{n,(\overline{\omega}_1, \overline{\omega}_2)} \]

is an isomorphism over \( U_{k,n-k} \). Also over \( U_{k,n-k} \), \( \text{Hecke}^a_{n,(\overline{\omega}_1, \overline{\omega}_2)} \) is canonically isomorphic to the fiber product

\[
\text{Hecke}^a_{(k,\overline{\omega}_1), (n-k,\overline{\omega}_2)} := (\text{Hecke}^a_{k,\overline{\omega}_1} \times \text{Hecke}^a_{n-k,\overline{\omega}_2}) \times_{\text{Bun} \times \text{Bun}} \text{Bun},
\]

taken with respect to the projection \( p^t \times p^t : \text{Hecke}^a_{k,\overline{\omega}_1} \times \text{Hecke}^a_{n-k,\overline{\omega}_2} \to \text{Bun} \times \text{Bun} \) (see Notation [2A]) and the diagonal morphism \( \text{Bun} \to \text{Bun} \times \text{Bun} \).

b) Over \( X^n \setminus \Delta \), \( \text{Gr}_{n,\overline{\omega}} \) is canonically isomorphic to the product \( \prod_{i=1}^n \text{Gr}_{1,\overline{\omega}_i} \).

c) There exists a smooth surjective morphism \( S \to \text{Bun} \times X \) with connected fibers such that \( \text{Hecke}^a_{1,\overline{\omega}} \times_{X \times \text{Bun}} S \) is isomorphic over \( S \) to \( \text{Gr}_{1,\overline{\omega}} \times_{X} S \).

d) Each \( \text{Gr}_{1,\overline{\omega}} \) is a Zariski locally trivial fibration over \( X \) with fiber \( \text{Gr}_\omega \).
Proof. a) Let \((\mathcal{G}, \mathcal{G}'; y_1, \ldots, y_\ell; \phi)\) be an \(S\)-point of \(\text{Hecke}_{n,\bar{\omega}} \times X^n U_{k,n-k}\). Define \(\mathcal{G}_1\) (resp. \(\mathcal{G}_2\)) to be the \(G\)-bundle over \(X \times S\) whose restriction to the complement of \(\cup_{i \leq k} \Gamma_{y_i}\) is that of \(\mathcal{G}'\) (resp. \(\mathcal{G}\)), restriction to the complement of \(\cup_{i > k} \Gamma_{y_i}\) is that of \(G\) (resp. \(\mathcal{G}'\)), and the gluing is done with help of \(\phi\). Then \((\mathcal{G}, \mathcal{G}'\ldots) \mapsto ((\mathcal{G}_1, \mathcal{G}'\ldots), (\mathcal{G}_2, \mathcal{G}'\ldots))\) gives us the inverse (of the restriction) of \(\pi\), while the map

\[
(\mathcal{G}, \mathcal{G}'\ldots) \mapsto ((\mathcal{G}_1, \mathcal{G}'\ldots), (\mathcal{G}_2, \mathcal{G}'\ldots))
\]
gives us the required isomorphism

\[
\text{Hecke}_{n,\bar{\omega}} \times X^n U_{k,n-k} \to \text{Hecke}_{n,\bar{\omega}} \times X^n U_{k,n-k}.
\]

b) follows from the second part of a) applied \((n - 1)\)-times.

c) The space \(S\), classifying pairs \((x, \mathcal{G}) \in X \times \text{Bun}\) together with a trivialization of \(\mathcal{G}\) over the completion of the graph of \(x\) (or sufficiently large (depending on \(\bar{\omega}\)) nilpotent neighborhood of \(x\)), satisfies the required property by [BL].

d) As explained in [Ga, 2.1.2], the fibration \(\text{Gr}_{1,\omega} \to X\) becomes trivial over a certain principal bundle over \(X\), whose structure group \(A\) is the inverse limit of the \(A_k\)'s with \(A_k(R) = \text{Aut}_R(R[t]/(t^{k+1}))\). As every \(A\)-bundle is locally trivial in the Zariski topology, the statement follows.

\[\Box\]

**Proposition A.9.** a) The reduced schemes \((\text{Gr}_{n,\omega}^0)^{\text{red}}\) and \((\text{Gr}_n^0)^{\text{red}}\) are smooth over \(X^n\) of relative dimension \(2\rho, \sum_{i=1}^n \omega_i\). Moreover, \(\text{Gr}_{n,\omega}^0\) and \(\text{Gr}_n^0\) are reduced, unless the characteristic of \(k\) is two, and \(G\) has a direct factor isomorphic to \(\text{PGL}_2\) or \(\text{PO}_{2m+1}\).

b) Both \(\text{Gr}_{n,\omega}^0\) and \(\text{Gr}_n^0\) are irreducible.

**Proof.** a) By Lemma 3.1(c), \(\text{Gr}_{n,\omega}^0 \cong \text{Gr}_n^0 \times X^n (X^n \setminus \Delta)\). Therefore it remains to show the statement for \(\text{Gr}_n^0\). Secondly, as

\[
\text{Gr}_n^0 = \text{Hecke}_{1,\omega_1}^0 \times \text{Bun} \cdots \times \text{Bun} \text{Hecke}_{1,\omega_{n-1}}^0 \times \text{Bun} \text{Gr}_1^0
\]

Lemma 4.1 and Lemma 4.8 imply that it will suffice to show that \((\text{Gr}_{\bar{\omega}}^0)^{\text{red}}\) is non-singular of dimension \(2\rho, \omega\) and that \(\text{Gr}_{\omega}^0\) is reduced unless \(\text{char } k = 2\), and \(G\) has a direct factor isomorphic to \(\text{PGL}_2\) or \(\text{PO}_{2m+1}\).

By the Cartan decomposition, \((\text{Gr}_{\bar{\omega}}^0)^{\text{red}}\) is a homogeneous space for the action of \(G_D\) of dimension \(2\rho, \omega\). Thus \((\text{Gr}_{\omega}^0)^{\text{red}}\) is non-singular, and the smoothness of \(\text{Gr}_{\omega}^0\) is equivalent to the smoothness at some point. Assume now either that \(\text{char } k \neq 2\) or that \(G\) does not have a direct factor isomorphic to \(\text{PGL}_2\) or \(\text{PO}_{2m+1}\).

Let \((\mathcal{G}, \phi : \mathcal{G}_{(D^\times \times \text{Gr}_0^0) \sim G \times \text{D}^* \times \text{Gr}_0^0})\) be the universal object over \(\text{Gr}_{\omega}^0\). Then for each Weyl module \(V_\lambda\), \(\phi\) gives an embedding of \(\mathcal{G}_\lambda\) into \(\text{triv}_\lambda(\langle \lambda, \omega\rangle_\times x)\), where we write \(\text{triv}_\lambda\) instead of \(V_\lambda \times \text{D} \times \text{Gr}_\omega^0\). Let \(\tilde{\mathcal{B}}\) be the stack over \(\text{Gr}_{\omega}^0\) classifying
$B$-structures of $\mathcal{G}$, and let us replace the universal object over $Gr^0_\omega$ by its pullback to $\tilde{B}$. Then each $b \in \tilde{B}$ defines a line subbundle $\mathcal{L}_\omega$ of the fiber of $\mathcal{G}_\lambda$ over $X \times \{b\}$.

Consider a substack $\mathcal{B}'$ of $\tilde{B}$ consisting of those points $b$ such that for each $\lambda \in X^+_\ast(T)$, the corresponding line subbundle $\mathcal{L}_\lambda$ is not contained in $\text{triv}_\lambda(\langle \lambda, \omega \rangle - 1)x)$. Since the conditions for $\lambda_1$ and $\lambda_2$ appearing in the definition of $\mathcal{B}'$ imply that for $\lambda_1 + \lambda_2$, $\mathcal{B}'$ is defined inside $\tilde{B}$ by finitely many open conditions, so $\mathcal{B}'$ is open in $\tilde{B}$. Also since $\tilde{B}$ is pro-smooth over $Gr^0_\omega$, it will suffice to show that $\mathcal{B}'$ is reduced.

For each $b \in \mathcal{B}'$, the corresponding line subsheaf $\mathcal{L}_\lambda(-\langle \lambda, \omega \rangle x)$ of $\text{triv}_\lambda$ is a subbundle. Furthermore, these subbundles satisfy the Plücker relations, so by Lemma A.2, the rule $b \mapsto \mathcal{L}_\lambda(-\langle \lambda, \omega \rangle x)$ defines a morphism $f$ from $\mathcal{B}'$ to the reduced scheme $\mathcal{B} := (G/B)_D$, classifying $B$-structures of the trivial $G$-bundle on $D$. Hence it will suffice to show that $f$ is an isomorphism. Note that both $\mathcal{B}'$ and $\mathcal{B}$ are equipped with a natural action of $G_D$, that $f$ is $G_D$-equivariant, and that $\mathcal{B}$ is a homogeneous space for the action of $G_D$. Therefore it will suffice to check that the schematic preimage $C := f^{-1}(o)$ of the point $o \in \mathcal{B}$, corresponding to the standard $B$-structure, consists of one reduced point.

By Cartan decomposition, $C_{\text{red}}$ consists of one point $y_\omega \in Gr(k)$. Explicitly, $y_\omega = g_\omega(y_0)$, where $y_0 \in Gr(k)$ is a point corresponding to the trivial $G$-bundle on $D$, and $g_\omega \in T(k((x))) \subset G(k((x))) = G_{D^\ast}(k)$ is the image of $x^{-1} \in k((x))^\times = \mathbb{G}_m(k((x)))$ under $\omega : \mathbb{G}_m \to T$. So it remains to show that the tangent space $T_{y_\omega}(C)$ is trivial.

For each dominant weight $\lambda$ of $G$, denote by $\mathcal{L}_{0,\lambda} \subset \text{triv}_\lambda$ be the line subbundle corresponding to the standard $B$-structure. Then $C$ is a schematic intersection inside $Gr$ of $Gr^0_\omega$ with an ind-subscheme $\mathcal{N}_\omega$, consisting of points $(\mathcal{G}, \phi)$ such that $\mathcal{L}_{0,\lambda}(\langle \lambda, \omega \rangle x)$ is a line subbundle of $\mathcal{G}_\lambda$ for each $\lambda$. Let $N$ be the unipotent radical of $B$. Then $\mathcal{N}_\omega$ is a homogeneous space for the action of the group $N_{D^\ast}$. Therefore we get a surjective map $p : N_{D^\ast} \to \mathcal{N}_\omega$ sending $u$ to $u(y_\omega)$ which induces a surjection $dp : \text{Lie } N(k((x))) \to T_{y_\omega}(\mathcal{N}_\omega)$. Hence we have to check that $dp(u) \notin T_{y_\omega}(Gr_\omega)$ for each $u \in \text{Lie } N(k((x))) \setminus \text{Ker}(dp)$.

For each $u \in \text{Lie } N(k((x)))$, denote by $\mathcal{G}_u \subset G \times D^\ast \times \text{Spec } k[t]/(t^2)$ the $G$-bundle on $D \times \text{Spec } k[t]/(t^2)$, corresponding to $dp(u) \in T_{y_\omega}(Gr) \subset Gr(k[t]/(t^2))$. Then our assertion is equivalent to the fact that for each $u \in \text{Lie } N(k((x))) \setminus \text{Ker}(dp)$, there exists a dominant weight $\lambda$ of $G$ such that $(\mathcal{G}_u)_\lambda \subset \mathcal{V}_\lambda \times D^\ast \times \text{Spec } k[t]/(t^2)$ is not contained in $(V_\lambda \times D \times \text{Spec } k[t]/(t^2))(-\langle \lambda, \omega \rangle x)$.

Note that $\text{Lie } N(k((x)))$ decomposes as the direct sum $\bigoplus \alpha k((x))$, where $\alpha$ runs over the set $\Delta_+$ of all positive roots of $G$, and that $\text{Ker}(dp) = \bigoplus \alpha x^{-\langle \omega, \alpha \rangle} k[[x]]$. For each $u \in \text{Lie } N(k((x)))$ and $\alpha \in \Delta_+$, denote by $u_\alpha \in k((x))$ the $\alpha$-component of $u$. Choose a basis $\{v_i\}_i$ of $\mathcal{V}_\lambda$ consisting of $T$-eigenvectors and denote by $\mu_i \in X^\ast(T)$ the weight of $v_i$. Then $\{x^{-\langle \mu_i, \omega \rangle} (v_i + t \sum \alpha u_\alpha(v_i))\}_i$ generate the vector bundle $(\mathcal{G}_u)_\lambda$.
Therefore we have to check that for each $u \in \text{Lie } N(k((x))) \setminus \text{Ker}(dp)$ there exists $\lambda \in X^*_+(T)$, a $T$-eigen vector $v$ of $V_{\lambda}$ of weight $\mu$, and $\alpha \in \Delta_+$ such that $\alpha(v) \neq 0$ and $u_\alpha \notin x^{-(\lambda-\mu,\omega)} k[[x]]$. As $u \notin \text{Ker}(dp)$, there exists $\alpha$ such that $u_\alpha \notin x^{-(\alpha,\omega)} k[[x]]$. Thus it remains to show that for each $\alpha \in \Delta_+$, there exists $\lambda \in X^*_+(T)$ and a $T$-eigen vector $v$ of $V_{\lambda}$ of weight $\lambda - \alpha$ such that $\alpha(v) \neq 0$.

Using the representation theory of $SL_2$, the last statement is equivalent to the assertion that for each $\alpha$ there exists $\lambda$ such that $\langle \tilde{\alpha}, \lambda \rangle$ (where $\tilde{\alpha}$ is the coroot corresponding to $\alpha$) is not divisible by char $k$. Assume that this is not the case. Then for each $\mu \in X^*_+(T)$, the product $\langle \tilde{\alpha}, \mu \rangle$ is divisible by char $k$. Since $\langle \tilde{\alpha}, \alpha \rangle = 2$, we thus get that char $k = 2$. Let $G_1, \ldots, G_l$ be all the simple factors of $G^\text{ad}$ numbered in a way that $\alpha$ is a root of $G_1$. Replacing $\alpha$ by a Weyl group conjugate, we can assume that $\alpha$ is simple. As for every other simple root $\beta$ of $G_1$, we have $\langle \tilde{\alpha}, \beta \rangle$ is even, we see from the classification of simple groups that $G_1$ is either $PGL_2$ or $PO_{2m+1}$.

It remains to show that $G_1$ is a direct factor of $G$. Since $G_1$ is adjoint, and the center of $G_1^{\text{sc}}$ consists of two elements, it remains to show that the canonical homomorphism $G_1^{\text{sc}} \to G$ is not injective. To see this, denote by $T_1 \subset G_1^{\text{sc}}$ the preimage of $T$. Then $X^*_+(T_1^{\text{sc}})$ is generated by coroots of $G_1^{\text{sc}}$, hence there exists $\mu \in X^*(T_1^{\text{sc}})$ such that $\langle \tilde{\alpha}, \mu \rangle = 1$. Therefore our assumption on $\alpha$ implies that the restriction map $X^*_+(T) \to X^*(T_1^{\text{sc}})$ is not surjective, or, what is the same, the homomorphism $G_1^{\text{sc}} \to G$ is not injective, as claimed.

**Remark A.10.** a) By the above argument, the condition in the proposition is not only sufficient but also necessary, that is, $Gr^0_{n,\omega}$ is not reduced if $k$ is a field of characteristic two, and $G$ has a direct factor isomorphic to $PGL_2$ or $PO_{2m+1}$.

b) It would be interesting to check whether the full stack $Gr_\omega$ is reduced. This seems to be the case at least in some simple cases (e.g. for $G = SL_2$). A positive answer to this question would imply that both $FBun^\Gamma_{n,\omega}$ and $Hecke_{n,\omega}$ are reduced. In particular, a variant of $Hecke_{n,\omega}$, considered in Remark 2.3(d), would then coincide with the original one.

b) We start from showing that $Gr_\omega$ is irreducible. As $Gr^0_\omega$ is a homogeneous space for the action of a connected group scheme $G_D$, it is irreducible. Hence it will suffice to show that for every two dominant coweights satisfying $\lambda_1 < \lambda_2$, the orbit corresponding to $\lambda_1$ lies in the closure of that of $\lambda_2$. For this we may assume that the difference $\lambda_2 - \lambda_1$ is a positive coroot $\alpha$ of $G$. Indeed, by the lemma of Stembridge (see e.g., [Ra] Lem 2.3), there exists a sequence of dominant coweights $\lambda_1 = \mu_0 < \mu_1 < \ldots < \mu_r = \lambda_2$ such that any two neighboring $\mu$'s differ by a positive coroot. Next we may assume that $G$ is of semi-simple rank one. Indeed, choose a maximal torus $T$ of $G$, and let $G'$ be the subgroup of $G$ generated by $T$ together
with the image of the canonical morphism \( SL_2 \to G \), corresponding to \( \alpha \). Then \( G' \) is a reductive group of semi-simple rank one, and the statement for \( G' \) implies that for \( G \). As any reductive group of semi-simple rank one is a product of a torus with either \( GL_2 \), \( SL_2 \) or \( PGL_2 \), it remains to check the statement for \( GL_2 \). In this case the irreducibility of \( Gr_\omega \) easily follows from explicit resolution of singularities.

**Remark A.11.** The irreducibility of \( Gr_\omega \) is essentially equivalent (using Bott–Samelson resolution of singularities) to the fact that the standard order on coweights is induced by the Bruhat order on the affine Weyl group.

Next, as in the beginning of a), the assertion for \( Gr'_n,\omega \) follows immediately from that for \( Gr_\omega \). In particular, we get the statement for \( Gr_1,\omega \). The assertion for \( Gr_n,\omega \) will be shown by induction on \( n \). First by Lemma A.8 c) we get the irreducibility of \( Gr_n,\omega \times X_n \ (X^n \setminus \Delta) \). Denote now by \( \overline{Gr}_n,\omega \) the closure of \( Gr_n,\omega \times X_n \ (X^n \setminus \Delta) \) in \( Gr_n,\omega \). For each \( i \neq j \), consider the closed subscheme \( (\overline{Gr}_n,\omega)_{x_i = x_j} \) of \( \overline{Gr}_n,\omega \). As it is given locally by one equation in \( \overline{Gr}_n,\omega \), it is of codimension one. On the other hand, by induction hypothesis, \( (\overline{Gr}_n,\omega)_{x_i = x_j} \) is irreducible and has the same dimension as \( (\overline{Gr}_n,\omega)_{x_i = x_j} \). Thus \( (\overline{Gr}_n,\omega)_{x_i = x_j} \) is contained in \( Gr_n,\omega \) for each \( i \neq j \).

Hence \( \overline{Gr}_n,\omega = Gr_n,\omega \), as claimed. □

**Lemma A.12.** The forgetful morphism \( \pi : Hecke'_n,\omega \to Hecke_n,\omega \) is small.

**Proof.** As the statement is well known to experts, we will just sketch the argument for the convenience of the reader. Observe first that it follows from basic properties of Coxeter groups that the Bott–Samelson resolution \( \overline{Gr}_\omega \to Gr_\omega \) is semi-small. Our statement is a formal consequence of this fact. Indeed, consider the stratification of \( X^n \), given by diagonals \( x_i = x_j \). As the restriction of \( \pi \) to the open stratum of \( X^n \) is an isomorphism (by Lemma 3.1), it will suffice to show that the restriction of \( \pi \) to each stratum is semi-small. Moreover, by Lemma A.8 a) and the induction hypothesis, we have to check the assertion only for the closed stratum \( x_1 = \ldots = x_n \). Furthermore, by Lemma A.8 c),d), it will suffice to show that for each \( (x,\mathcal{G}) \in X \times Bun \) the restriction \( \pi_y \) of \( \pi \) to \( y := (x,\ldots,x;\mathcal{G}) \in X^n \times Bun \) is semi-small. The last assertion follows from the observations that the fiber of \( Hecke_n,\omega \) over \( y \) is isomorphic to \( Gr_{\omega_1 + \ldots + \omega_n} \), and the (semi-small) Bott–Samelson resolution of \( Gr_{\omega_1 + \ldots + \omega_n} \) factors through \( \pi_y \). □

**Proposition A.13.** The restriction of the IC-sheaves of \( Hecke_n,\omega \) and \( Hecke'_n,\omega \) to each stratum isomorphic to a direct sum of complexes of the form \( \mathbb{Q}(k/2)[k] \) with the parity of \( k \) is the same as that of \( \dim Hecke_n,\omega = \dim Hecke'_n,\omega \).

**Proof.** First we will show the corresponding statement for \( Gr_\omega \). Let (the Iwahori subgroup) \( I \subset G_D \) be the preimage of \( B \subset G \) under the natural projection \( G_D \to G \).
Since the IC-sheaf of $Gr_\omega$ is $G_\mathcal{D}$-equivariant and since each stratum of $Gr_\omega$ has an open $I$-orbit, it remains to show the corresponding statement for the restriction of the IC-sheaf to each $I$-orbit. For this we will use the same strategy as in [Ga A.7]. Consider the Bott–Samelson resolution $\pi: \widetilde{Gr}_\omega \to Gr_\omega$. By the decomposition theorem, $IC_{Gr_\omega}$ is a direct summand of $\pi_!((Q_i)(\dim Gr_\omega)[2\dim Gr_\omega])$. Therefore it will suffice to show that the restriction of $\pi_!((Q_i))$ to each $I$-orbit is a direct sum of complexes of the form $\mathbb{Q}(k)[2k]$. Consider the stratification of $\widetilde{Gr}_\omega$ by $I$-orbits. As $I$ is pro-unipotent, each stratum of $\widetilde{Gr}_\omega$ is an $\mathbb{A}^N$-bundle (for some $N$) over the corresponding stratum of $Gr_\omega$. By the proper base change theorem, it will therefore suffice to show the statement for each fiber.

Thus we are reduced to showing that if $\rho : X \to \mathbb{F}_q$ has a stratification by affine spaces, then $\rho_!(\mathbb{Q}_l)$ decomposes as a direct sum of complexes of the form $\mathbb{Q}(k)[2k]$. Let $N$ be the dimension of $X$, and let $U$ be the disjoint union of (open) strata of $X$ of top dimension. By induction, we may assume that the statement holds for $X \setminus U$. As the statement clearly holds for affine spaces, hence for $U$, it would suffice to show that $\rho_!(\mathbb{Q}_l)$ decomposes as a direct sum $\rho_!(\mathbb{Q}_l|_U) \oplus \rho_!(\mathbb{Q}_l|_{X \setminus U})$. As the statement is equivalent to splitting of the canonical distinguished triangle $\rho_!(\mathbb{Q}_l|_U) \to \rho_!(\mathbb{Q}_l) \to \rho_!(\mathbb{Q}_l|_{X \setminus U}) \to$, the assertion for $Gr_\omega$ now follows from the fact that $\rho_!(\mathbb{Q}_l) = \tau_{\geq 2N}\rho_!(\mathbb{Q}_l)$.

In the global case, we will show our assertion by induction on $n$. For $n = 1$, it is an immediate consequence of the case of $Gr_\omega$ was proved above (use Lemma [A.8 c),d]). As $Hecke_{n,\omega} = Hecke_{1,\omega_1} \times_{Bun} Hecke_{1,\omega_2} \times_{Bun} \ldots \times_{Bun} Hecke_{1,\omega_n}$, the assertion for $Hecke_{1,\omega}$ implies that for $Hecke_{n,\omega}$. For $n > 1$, take any stratum $S$ of $Hecke_{n,\omega}$. We have two cases: either $S$ lies over $X^n \setminus \Delta$, or $S$ is a stratum of some $(Hecke_{n,\omega})|_{x_i = x_j}$. In the first case, the statement follows from that for $Hecke_{n,\omega}$ and Lemma 3.11.

In the second case, the statement will follow from the induction hypothesis if we show that the restriction of the IC-sheaf of $Hecke_{n,\omega}$ to $(Hecke_{n,\omega})|_{x_i = x_j} \subset Hecke_{n-1}$ is a direct sum of IC-sheaves of closed strata. By Lemma [A.12] $\pi_!$ maps the IC-sheaf of $Hecke_{n,\omega}$ to that of $Hecke_{n-1}$. By the proper base change theorem, we are therefore reduced to the case $n = 2$, which in its turn reduces to the case of $Gr_{2,\omega}$. In this case, the assertion over $\mathbb{F}_q$ is shown in the proof of [Ga Prop 1], and the decomposition over $\mathbb{F}_q$ easily follows from the fact that each fiber of $Gr_{2,\omega} \to Gr_{2,\omega}$ has a stratification by affine spaces (compare the proof of Lemma [A.12]).

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