PROPORTION OF ORDINARITY IN SOME FAMILIES OF CURVES OVER
FINITE FIELDS

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Abstract. A curve over a field of characteristic $p$ is called ordinary if the $p$-torsion of its Jacobian as large as possible, that is, an $\mathbb{F}_p$ vector space of dimension equal to its genus. In this paper we consider the following question: fix a finite field $\mathbb{F}_q$ and a family $\mathcal{F}$ of curves over $\mathbb{F}_q$. Then, what is the probability that a curve in this family is ordinary? We answer this question when $\mathcal{F}$ is either the Artin-Schreier family in any characteristic or the superelliptic family in characteristic 2.

1. Introduction

Let $C$ be a smooth curve of genus $g$ over a field $k$. Then its Jacobian $\text{Jac}(C)$ is an abelian variety of dimension $g$. For each $n \in \mathbb{Z}_{>0}$, the $n$-torsion group scheme $\text{Jac}[n]$ is a finite flat group scheme. When $(n,p) = 1$, this group scheme is étale, and as an abelian group, is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$ over $\overline{k}$. When $n$ is not invertible in $k$, this group scheme is never étale and its isomorphism class over $\overline{k}$ depends significantly on the curve. In this paper we study the variation of this group scheme in some families of curves. In order to do so, we recall the definitions the following invariants:

Let $G$ be a finite flat group scheme killed by $p$ over a field $k$ of characteristic $p$.

**Definition 1.1.** We define the $a$-number of $G$ as:

$$a(G) = \dim_k \text{Hom}(\alpha_p, G)$$

where $\alpha_p$ is the affine group scheme $\text{Spec}(k[\![x]\!]^{p})$, and the $\text{Hom}$ is in the category of $k$-group schemes.

**Definition 1.2.** The $p$-rank of $G$ is defined as $r(G)$ where:

$$G(\overline{k}) \cong (\mathbb{Z}/p\mathbb{Z})^{r(G)}$$

as abelian groups.

For the purpose of this paper, we will only be interested in $G = \text{Jac}(C)[p]$. In this case, it is well known that $0 \leq r(G) \leq g(C)$ and $0 \leq a(G) + r(G) \leq g$. The Jacobian is called ordinary if $r(G) = g$ or equivalently, when $a(G) = 0$. By abuse of notation, we will denote the $a(C)$ and $r(C)$ to be the corresponding invariants of $\text{Jac}(C)[p]$.

Let $\mathcal{M}_g$ denote the moduli space of smooth curves of genus $g$. The study of $\mathcal{M}_g$ can take two different directions. One is to understand $\mathcal{M}_g(k)$, i.e. to gain a geometric understanding of $\mathcal{M}_g$ for a fixed $g$. A lot of work has been done in this area, some of which we will list later in this section. The second is to understand $\mathcal{M}_g(k)$ for a given $k$. For instance, one might ask if for a fixed $q$, the limit

$$\lim_{X \to \infty} \frac{\#\{C \in \mathcal{M}_g(\mathbb{F}_q) \mid C \text{ ordinary, } q^g < X\}}{\#\{C \in \mathcal{M}_g(\mathbb{F}_q) \mid q^g < X\}}$$

exists. And if so, what is it? Very little is known about this question. In this paper, we ask what happens to the limit when $\mathcal{M}_g$ is replaced by some other families of curves. Fix a family $\mathcal{F}$ of curves over $\mathbb{F}_q$ of arbitrary genus. Note that by a family, we mean a set of curves through the family.
satisfying a particular property, which is not necessarily a family in any geometric sense. A

typical example of a family is $\cup_{g \geq 0} \mathcal{M}_g(q)$. Let $\mathcal{F}_a = \{ C \in \mathcal{F} \mid a(C) = a \}$. We wish to study the probability that a randomly chosen $C \in \mathcal{F}$ lies in $\mathcal{F}_0$ or in other words, what proportion of curves in the family $\mathcal{F}$ is ordinary. More precisely, we define:

- $N(\mathcal{F}, X) = \# \{ C \in \mathcal{F} \mid q^g < X \}$
- $N(\mathcal{F}, a, X) = \# \{ C \in \mathcal{F}_a \mid q^g < X \}$.

**Question 1.3.** For a family $\mathcal{F}$ of curves over a fixed finite field $\mathbb{F}_q$, does the limit:

$$\lim_{X \to \infty} \frac{N(\mathcal{F}, 0, X)}{N(\mathcal{F}, X)}$$

exist, and if so, what is its value?

In section 2, we describe two different families for which we will answer Question 1.3, namely:

- Artin Schreier curves in arbitrary positive characteristic.
- Superelliptic curves over a finite field of characteristic 2.

In each of the above cases, the criteria for ordinarity can be described combinatorially in terms of the ramification invariants of the curves in question.

In section 3, we prove the following results:

**Theorem 1.4** (Corollary 3.5). The probability that an Artin-Schreier curve (under the assumptions of §2) over $\mathbb{F}_q$, $q$ a power of $p$, is ordinary is non-zero for $p = 2$ and zero for all odd primes.

For $p = 2$ we calculate the probability explicitly and give some values for various $q$ in §4. For the family of superelliptic curves, we prove:

**Theorem 1.5** (Theorem 3.17). The probability that a superelliptic curve of prime degree over a finite field of characteristic 2 is ordinary is zero.

This paper is inspired by a paper of Cais, Ellenberg and Zureick-Brown [4], which gives a heuristic for the behaviour of the $p$-divisible group of an abelian variety by proving the distribution for a random (principally quasi-polarized) Dieudonné module. They show that the probability that such a module is ordinary (here the $p$-rank and $a$-number of a Dieudonné module $D$ are defined as those of $D/pD$) is:

$$\prod_{i=1}^{\infty} (1 + q^{-i})^{-1}$$

Further, they ask if the Dieudonné module associated to the Jacobian of a curve behaves like a randomly chosen one, i.e. whether the limit in (1.1) equals (1.2). They find, via numerical experiments, that hyperelliptic curves in odd characteristic do not appear to obey their heuristics, while plane curves do. The families considered in this paper are the first known cases whose behavior provably diverges significantly from the heuristics of [4].

Note that theorems 1.4 and 1.5 show that the Artin-Schreier and superelliptic families do not obey the heuristic (1.2) and therefore do not behave randomly in the sense of [4]. We explain this in greater detail in section 3.

We emphasize that theorems 1.4 and 1.5 are both ‘large $g$-limit’ results, in that they study the behavior of curves as $g \to \infty$, with $q$ fixed. The ‘large $g$-limit’ behavior, i.e. the geometry of families of curves of a fixed genus, is usually incomparable to the large $g$-limit behavior. However, studying the former can provide some insight into it latter. To illustrate our point, we list some results here that show how different the geometry of the Artin-Schreier locus is from that of some other families of curves. It is known that the locus of ordinary curves is a non-empty Zariski open subset of $\mathcal{M}_g$ [12]. Thus for a fixed genus $g$, ‘most’ curves of genus $g$
tend to be ordinary. Let \( \mathcal{V}_{g,r} \) denote the sublocus of \( \mathcal{M}_g \) of curves of \( p \)-rank at most \( r \). In [9] Faber and van der Geer prove that \( \mathcal{V}_{g,r} \) has codimension \( g - r \). A result of Glass and Pries [10] states that \( \mathcal{V}_{g,r} \) intersects the hyperelliptic locus, \( \mathcal{H}_g \), inside \( \mathcal{M}_g \) in a set of dimension \( g - 1 + r \). Since \( \mathcal{H}_g \) has dimension \( 2g - 1 \), this implies that the ordinary locus is dense in \( \mathcal{H}_g \). We compare this to results about \( \mathcal{A}_g \), the Artin-Schreier locus inside \( \mathcal{M}_g \). In [13], Pries and June Zhu prove that for \( p \geq 3 \), the codimension of \( \mathcal{V}_{g,r} \cap \mathcal{A}_g \) inside \( \mathcal{A}_g \) is less than \( g - r \). This indicates that for \( p \geq 3 \), the image under the Torelli morphism of \( \mathcal{A}_g \) in \( \mathcal{A}_g \) is not in general position with respect to the \( p \)-rank stratification. Further, from results in [13], it follows that the ordinary locus intersects only one irreducible component of \( \mathcal{A}_g \). As \( g \to \infty \), the number of components of \( \mathcal{A}_g \) increases except when \( p = 2 \) (in which case \( \mathcal{A}_g \) is \( \mathcal{H}_g \)). This gives a heuristic reason for why one might expect a statement like Theorem 1.4. A similar heuristic explains Theorem [1.5] as well, as we elaborate in remark 3.18.

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## 2. Setup and Background

In this section, we provide the setup and background for each of the families that we will consider.

### 2.1. Artin-Schreier Curves

Let \( k \) be a perfect field of characteristic \( p > 0 \). We now recall some facts about Artin-Schreier curves and covers. An Artin-Schreier curve \( C \) over \( k \) is a smooth \( \mathbb{Z}/p\mathbb{Z} \)-cover of \( \mathbb{P}^1_k \). Such a curve has an affine model

\[
y^p - y = f(x)
\]

where \( f(x) \in k(x) \), and is equipped with a \( \mathbb{Z}/p\mathbb{Z} \) action generated by \( y \mapsto y + 1 \). An Artin-Schreier cover is an Artin-Schreier curve along with a choice of map \( \iota : \mathbb{Z}/p\mathbb{Z} \to \text{Aut}(C) \) and a choice of isomorphism \( C/(\iota(\mathbb{Z}/p\mathbb{Z})) \cong \mathbb{P}^1 \). This amounts to picking a model of the form (2.1).

Let \( B \subset \mathbb{P}^1(\bar{k}) \) be the set of poles of \( f \). Then, the cover above is ramified precisely at the points in \( B \) [14]. For \( \alpha \in B \), let

\[
x_\alpha = \begin{cases} 
\frac{1}{x - \alpha} & \alpha \neq \infty \\
x & \alpha = \infty
\end{cases}
\]

Then, using a partial fraction decomposition one can write

\[
f(x) = \sum_{\alpha \in B} f_\alpha(x_\alpha)
\]

where \( f_\alpha \in \bar{k}[x] \).

**Remark 2.1.** We now make a few helpful observations about the partial fraction decomposition above.

1. We assume that for \( \alpha \neq \infty \), \( f_\alpha \) has no constant term.
2. By a transformation of the form \( y \mapsto y + z \), one can assume that in \( f_\alpha(x) \), the coefficient of \( x^p \) is zero for any \( 0 \leq i \leq \lfloor d_\alpha/p \rfloor \). In particular, we can take \( d_\alpha \neq 0 \mod p \).
3. If \( \alpha, \beta \in B \) are Galois conjugate, then \( d_\alpha = d_\beta \).
4. Let \( Q \) be an irreducible polynomial in \( k[x] \) whose zeroes are ramified in the Artin-Schreier cover under consideration. Then we will denote \( d_Q \) as the degree of any \( f_\alpha, \alpha \) a zero of \( Q \). This is well defined by the above remark.
By the Riemann-Hurwitz theorem for wildly ramified covers, we know that the genus of such a curve is given by:

\[(2.3)\]

\[g = \left( \frac{p-1}{2} \right) \left( -2 + \sum_{\alpha \in B} (d_{\alpha} + 1) \right) = \left( \frac{p-1}{2} \right) \left( -2 + \sum_{Q \text{ irreduc. ramified}} \deg(Q)(d_Q + 1) + (d_\infty + 1) \right)\]

The following criterion for the ordinarity of an Artin-Schreier curve is well known ([6], [2], [15]):

**Proposition 2.2.** The Artin-Schreier cover \(y^p - y = f(x)\) is ordinary if and only if \(f\) has only simple poles.

This is equivalent to the condition that \(d_\alpha = 1\) for each \(\alpha\) in the partial fraction decomposition \((2.2)\).

Let \(S\) be the set of rational functions \(f(x) \in k(x)\) such that the partial fraction decomposition of \(f\) satisfies the conditions (1-3) from remark 2.1. For simplicity, we will assume that \(\infty \notin B\). This assumption is harmless, as we explain in remark 3.6 and makes the computations in §3 much cleaner. We now restrict our attention to \(k = \mathbb{F}_q\) and define the families for this section as follows:

- \(\mathcal{F} = \text{Set of Artin-Schreier covers } y^p - y = f(x), \text{ where } f(x) \in S \text{ has no poles over } \infty \in \mathbb{P}^1.\)
- \(\mathcal{F}_0 = \text{Set of all ordinary Artin-Schreier covers } y^p - y = f(x) \text{ with } f(x) \in S, \text{ unramified over } \infty \in \mathbb{P}^1.\)

2.1.1. **Counting curves versus counting covers.** In our proof of the main theorem in §3, we calculate the limit from question 1.3 by counting polynomials in the set \(S\) defined above. Note that while this is not exactly the same as counting Artin-Schreier curves, it does not significantly affect the proportion of ordinarity. In particular, it does not change whether the limit 1.3 is non-zero. We explain this below.

By remark 2.1, we see that every Artin-Schreier curve over \(\mathbb{F}_q\) is isomorphic to a curve with model:

\[y^p - y = f(x)\]

with \(f(x) \in S\). We let \(C_f\) denote the curve \(y^p - y = f(x)\). Further, let \(f(x), g(x) \in S\). Then we claim that any isomorphism: \(\phi : C_f \cong C_g\) over \(\mathbb{F}_q\) is essentially induced by an isomorphism \(\mathbb{P}^1 \cong \mathbb{P}^1\). This is probably well known, but since we couldn’t find the explicit statement in the literature, we recall the proof here.

**Claim 2.3.** Maintaining the same notation as above, let \(C_f\) and \(C_g\) be two Artin-Schreier covers that are isomorphic as curves over \(\mathbb{F}_q\). Then \(f(x) = ug(\gamma x)\) for some \(u \in \mathbb{Z}/p\mathbb{Z}^\times\) and \(\gamma \in \text{PGL}_2(\mathbb{F}_q)\).

**Proof.** Let \(\phi : C_f \to C_g\) be the map realizing the isomorphism between the two curves. Since \(\phi\) commutes with the \(\mathbb{Z}/p\mathbb{Z}\) action, it descends to an automorphism of \(\mathbb{P}_q\). Therefore we have the following commutative diagram:

\[
\begin{array}{ccc}
C_f & \xrightarrow{\phi} & C_g \\
\downarrow & & \downarrow \\
\mathbb{P}_q^1 & \xrightarrow{\tilde{\phi}} & \mathbb{P}_q^1
\end{array}
\]

where the vertical maps are the quotients by \(\mathbb{Z}/p\mathbb{Z}\) actions and \(\tilde{\phi}\) is induced by \(\phi\). Thus \(\tilde{\phi}\) is induced by some \(\gamma \in \text{PGL}_2(\mathbb{F}_q)\).
Let $D_f$ and $D_g$ denote the ramification divisors of $C_f$ and $C_g$ respectively. By Artin-Schreier theory, these are determined by the poles of $f$ and $g$ respectively. Note that since the curves are defined over $\mathbb{F}_q$, so are their ramification divisors. Since $\phi$ must preserve the ramification invariants (namely, the number of ramified points and the ramification groups at each of these points), we must have that $\phi^*(D_g) = D_f$. Thus $C_{f \circ \gamma}$ and $C_g$ are isomorphic curves with the same ramification divisor.

Now, two Artin-Schreier covers:

$$y^p - y = f_1(x) \quad \text{and} \quad y^p - y = f_2(x)$$

with the same genus and ramification divisor are isomorphic if and only if $N^x$ (see, for example \cite{13}, Remark 3.9) with $u$ with the same genus and ramification divisor are isomorphic if and only if

Now, the map

$$\phi : C \to \mathbb{A}^1$$

sending $f(x) \in S$ to the curve with model $y^p - y = f(x)$. Remark \cite{21} shows that this map is surjective, and claim \cite{23} proves that the fibers of this map are bounded by $(p-1) | \text{PGL}_2(\mathbb{F}_q) |$. In particular, the probability:

$$\lim_{X \to \infty} \frac{\left| \{ f \in S | q^q < X, a(f) = 0 \} \right|}{\left| \{ f \in S | q^q < X \} \right|}$$

exists and is zero if and only if the quantity:

$$\lim_{X \to \infty} \frac{\left| \{ f \in S | q^q < X, a(f) = 0 \} \right|}{\left| \{ f \in S | q^q < X \} \right|}$$

exists and is zero. If either of them is non-zero, so is the other. Thus theorem \cite{35} follows from making the following analogous claim about $S$:

**Proposition 2.4** (Theorem \cite{14} rephrased). The quantity \cite{2.4} is non-zero if $p = 2$ and is zero for odd $p$.

2.2. **Superelliptic curves.** Let $k$ be an arbitrary field. A superelliptic curve over $k$ is a curve defined by the affine equation:

$$y^n = f(x)$$

where $f(x) \in k[x]$ and $n$ is coprime to the characteristic of $k$. Then, this curve has an action of $\mu_n$ ($n$-th roots of unity) on it, namely the map:

$$(x, y) \mapsto (x, \zeta_n y)$$

where $\zeta_n$ is a primitive $n$-th root of unity. One can make a transformation to write

$$f(x) = \prod_{i=1}^{n-1} (f_i(x))^i$$

where each $f_i(x)$ is a squarefree polynomial. The quotient $C/\mu_n$ gives a map to $\mathbb{P}^1$, sending $(x, y) \mapsto x$. We let $N := \sum_{i=1}^{n-1} i \deg(f_i)$. Then the curve $C$ is unramified over $\infty \in \mathbb{P}^1$ if and only if $N \equiv 0 \mod n$. In the case that $N \not\equiv 0 \mod n$, we let $n_\infty$ be the smallest positive integer such that $N + n_\infty \equiv 0 \mod n$.

Now, the map $C \to \mathbb{P}^1$ is ramified at the zeros of $f$ and possibly at $\infty$. The ramification indices at each of the ramified points $\alpha \in \mathbb{P}^1(\bar{k})$ are given by \cite{11}:

$$e(\alpha) = \begin{cases} \frac{n}{(n,n)} & \text{if } f_i(\alpha) = 0 \\ \frac{n}{(n,n_\infty)} & \text{if } \alpha = \infty \end{cases}$$
The genus of this curve is given by:

\[ g = -n + 1 + \frac{1}{2} \sum_{i=1}^{n-1} \deg(f_i)(n - (n, i)) + \frac{1}{2}\epsilon(n - (n, n_{\infty})) \]

where \( \epsilon \) is 0 if the map \( \mathcal{C} \to \mathbb{P}^1 \) is unramified over \( \infty \) and 1 otherwise.

**Remark 2.5.** Since the techniques of this paper are based on counting polynomials, it is necessary to separate the case when the map is ramified over \( \infty \in \mathbb{P}^1 \), even though that seems unnatural.

We now specialize to the case where \( n \) is an odd prime. Let \( B \subset \mathbb{P}^1(k) \) be the set of points ramified in the cover \( y^n = f(x) \). Let \( |B| = m \). If \( \epsilon = 0 \), then \( m = \sum_{i=1}^{n-1} \deg(f_i) \) and if \( \epsilon = 1 \), then \( m = \sum_{i=1}^{n-1} \deg(f_i) + 1 \).

In either case, we have:

\[ g = \frac{1}{2}(n - 1)(m - 2) \]

Thus with regard to superelliptic curves, we will be interested in the family \( \mathcal{F} \) of covers \( y^n = f(x) \), where:

- \( n \) is prime.
- The curve is defined over \( \mathbb{F}_q \), where \( q \) is a power of 2.
- \( f(x) \in \mathbb{F}_q[x] \) is \( n \)-th power-free.

2.2.1. Aside on counting curves versus counting covers. One might wonder, as in the Artin-Schreier case in §2.1, what the difference is between counting superelliptic curves and covers of the form \( y^n = f(x) \). We choose to restrict our attention to covers, i.e. to equations of the form \( y^n = f(x) \) with \( f(x) \in \mathbb{F}_q[x] \) \( n \)-th power-free, and make the claim that this does not significantly affect our results.

We first introduce some notation for this section alone. For any \( u \in \mathbb{Z}/n\mathbb{Z}^\times \), let \([u]\) be the map that takes \( \prod_i (f_i(x))^i \) to \( \prod_i (f_i(x))^{(ui \mod n)} \). By a straightforward sequence of transformations, one can see that if \( f_i \) is squarefree for each \( i \), the two curves:

\[ y^n = \prod_i (f_i(x))^i \quad \text{and} \quad y^n = \prod_i (f_i(x))^{(ui \mod n)} \]

are indeed isomorphic. By abuse of notation, we also call this isomorphism of curves \([u]\). We claim that up to an automorphism of \( \mathbb{P}^1_{\mathbb{F}_q} \), the only isomorphisms between superelliptic covers are of the form \([u]\), with \( u \in \mathbb{Z}/n\mathbb{Z} \). This is a standard Kummer theory argument, whose proof we recall here.

**Claim 2.6.** For \( n \) an odd prime, let \( f(x) = \prod_{i=1}^{n-1} (f_i(x))^i \) and \( g(x) = \prod_{i=1}^{n-1} (g_i(x))^i \) be two monic \( n \)-th power-free polynomials in \( \mathbb{F}_q[x] \) such that:

- For each \( i \), \( f_i(x) \) and \( g_i(x) \) are squarefree,
- \( \text{div}_0(f) = \text{div}_0(g) \).

Suppose that \( C_f : y^n = f(x) \) and \( C_g : y^n = g(x) \) are isomorphic as curves via an isomorphism \( \phi \). Then there is a \( u \in \mathbb{Z}/n\mathbb{Z}^\times \) such that \( \phi = \zeta_n \circ [u] \).

Here \( \zeta_n \) is an \( n \)-th root of unity that acts as an automorphism of the curve sending \( (x, y) \to (x, \zeta_n y) \).

**Proof.** Let \( K = \mathbb{F}_q(\mathbb{P}^1) \) and \( L = \mathbb{F}_q(C_f) \cong \mathbb{F}_q(C_g) \). Note \( L(\zeta_n)/K(\zeta_n) \) is a Galois extension. Let \( \varphi : \text{Gal}(K(\zeta_n)/K(\zeta_n)) \to \mu_n \) be the homomorphism corresponding to \( L(\zeta_n) \). Any other field \( L' \) that is isomorphic to \( L(\zeta_n) \) corresponds to the homomorphism \( \varphi^u \) for some \( u \in \mathbb{Z}/n\mathbb{Z}^\times \). Therefore, if \( [a] \in K(\zeta_n)^\times / (K(\zeta_n)^\times)^n \) is the class corresponding to \( \varphi \) via the Kummer map, then there is a \( u \in \mathbb{Z}/n\mathbb{Z}^\times \) such that the isomorphism \( \mathbb{F}_q(\zeta_n)(C_f) \cong \mathbb{F}_q(\zeta_n)(C_g) \) corresponds to the class \([a^u]\). This proves the claim. \( \square \)
For \( n \) an odd prime, let \( \mathcal{T}_n \) denote the set of \( n \)-th power free polynomials in \( \mathbb{F}_q[x] \). Let \( \mathcal{S}_{g,n}(\mathbb{F}_q) \) denote the set of superelliptic curves of degree \( n \) and genus \( g \) over \( \mathbb{F}_q \). Then the above claim shows that the fibers of the map:
\[
\mathcal{T}_n \to \bigcup_{g \geq 0} \mathcal{S}_{g,n}(\mathbb{F}_q)
\]
\[f(x) \mapsto (y^n = f(x))\]
have size bounded by \( n \mid \mathbb{Z}/n\mathbb{Z}^\times \mid \text{ PGL}_2(\mathbb{F}_q) \mid \). As in \( \S 2.1 \) this proves that understanding the proportion of ordinarity in \( \mathcal{S} \) is the same as understanding it for the family of superelliptic curves of a fixed degree over \( \mathbb{F}_q \).

### 2.2.2. \( a \)-number of Superelliptic Curves in characteristic 2

We now give a combinatorial criterion for the ordinary for superelliptic curves in characteristic 2. The discussion in this section is based on a paper by Elkin [8]. Let \( C \) be a smooth proper superelliptic curve over \( \mathbb{F}_q \), \( q \) a power of 2, with affine model: \( y^n = f(x) \), where \( n \) is an odd prime. We maintain the same notation as before. The space \( H^0(C, \Omega_C^1) \) inherits the action of \( \mu_n \) and decomposes into eigenspaces as follows:

\[
H^0(C, \Omega_C^1) = \bigoplus_{i=1}^{n-1} D_i.
\]

The Cartier operator \( \mathcal{C} \) acts on \( H^0(C, \Omega_C^1) \) and it is well known that the \( a \)-number, \( a(C) \), equals \( g(C) - \text{rank}(\mathcal{C}) \). To state the result in Elkin’s paper, we first describe some notation. Let \( d_i = \dim(D_i) \). Let \( \sigma \) be the permutation of \( \{1, 2, \ldots n-1\} \) defined by:

\[
p \sigma(i) \equiv i \mod n.
\]

By bounding the rank of the Cartier operator, Elkin proves the following:

**Proposition 2.7.** Let \( C \) be as above. Then:

\[
g(C) - a(C) = \sum_{i=1}^{n-1} \min(d_i, d_{\sigma(i)})
\]

where the \( d_i = \dim(D_i) \) can be computed explicitly from the ramification invariants of the curve and \( \sigma \) is the permutation of the set \( \{1, 2, \ldots n-1\} \) described above.

For any rational number \( r \), let \( \langle r \rangle = r - \lfloor r \rfloor \). Elkin proves that the \( d_i \)'s are given by the formula:

\[
d_i = \sum_{j=1}^{n-1} \deg(f_j) \left\langle \frac{ij}{n} \right\rangle + \left\langle \frac{in}{n} \right\rangle - 1
\]

Recall that the ordinarity of an abelian variety is equivalent to the condition that its \( a \)-number is 0. Proposition 2.7 tells us that \( a(C) = 0 \) implies that \( g(C) = \sum_{i=1}^{n-1} \min(d_i, d_{\sigma(i)}) \). We now give a condition for ordinarity in terms of the degrees of \( f_i \). We will treat the case \( n = 3 \) separately from the case of a general odd prime.

### 2.2.3. The case \( n = 3 \):

In this subsection, we consider curves of the form \( C : y^3 = f(x) \). The equation for the genus simplifies to:

\[
g = m - 2
\]

**Proposition 2.8.** A curve of the form \( y^3 = f_1 f_2^2 \), with \( f_1, f_2 \) squarefree is ordinary if and only if one of the following is true:

1. \( n_\infty = 0 \) and \( \deg(f_1) = \deg(f_2) \)
2. \( n_\infty = i \) for some \( i \in \{1, 2\} \) and \( \deg(f_i) + 1 = \deg(f_{3-i}) \)

**Proof.** Note that \( \sigma = (1 \ 2) \). So, \( g = 2 \min(d_1, d_2) \), which in turn implies \( g = 2d_1 \) or \( g(C) = 2d_2 \). We prove case (1) here. The other cases follow by a similar calculation.
In this case:
\[ d_1 = \frac{1}{3} \deg(f_1) + \frac{2}{3} \deg(f_2) - 1 \]
and
\[ d_2 = \frac{2}{3} \deg(f_1) + \frac{1}{3} \deg(f_2) - 1 \]

Therefore, \( \deg(f_1) = \deg(f_2) \). For case (2), we just replace \( \deg(f_i) \) by \( \deg(f_i) + 1 \) in the expression for each \( d_j \).

\[ \Box \]

2.2.4. The case of a general odd prime.

**Proposition 2.9.** A curve defined by \( y^n = \prod_{i=1}^{n-1} (f_i(x))^i \) as in section 2.2, with \( n \) an odd prime, is ordinary if and only if one of the following is true:

1. \( n_\infty = 0 \) and \( \deg(f_i) = \deg(f_{n-i}) \)
2. \( n_\infty = i \) for some \( i \in \{1, 2 \ldots n-1\} \), and \( \deg(f_i) + 1 = \deg(f_{n-i}) \), and for \( j \neq i, n-i \), \( \deg(f_j) = \deg(f_{n-j}) \)

**Proof.** As before, we only prove case (1) and the other case follows from a modified, but similar calculation. The condition for ordinarity gives:
\[ \sum_i d_i = \sum_i \min(d_i, d_{n(i)}) \]
This automatically implies that \( d_i = d_j \) for all \( 1 \leq i, j \leq n \). Since we are considering the case where \( n_\infty = 0 \),
\[ d_i = \sum_{j=1}^{n-1} \deg(f_j) \left\lfloor \frac{ij}{n} \right\rfloor - 1. \]

Define the matrix \( A \), with \( A_{ij} = \left\lfloor \frac{ij}{n} \right\rfloor \). Thus, the degrees of \( f_i \)'s must satisfy:

\[ (2.9) \]

\[ A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} d + 1 \\ d + 1 \\ \vdots \\ d + 1 \end{pmatrix} \]

for some \( d \geq 0 \). Let \( V \) denote the space of \( n-1 \times 1 \) vectors whose coordinates are all equal. We are interested in (the integral points of) the space of \( x = (x_1, x_2 \ldots x_{n-1})^T \) such that \( Ax \in V \).

**Lemma 2.10.** The space \( \{ x \in \mathbb{Z}^{n-1} \mid Ax \in V \} \) consists of vectors \( x \) for which \( x_k = x_{n-k} \) for all \( k = 1, 2 \ldots n-1 \).

**Proof of Lemma.** We prove this lemma by constructing an explicit basis for the kernel of \( A \), \( \text{Ker}(A) \). Let \( x^{(k)} \) denote the \( n-1 \times 1 \) vector which has 1’s in the \( k \)th and \( n-k \)th positions and -1’s in the \( \frac{n-1}{2} \)th and \( \frac{n+1}{2} \)th positions. We claim that \( \{ x^{(k)} \mid k = 1, 2 \ldots \frac{n-1}{2} \} \) is a basis for \( \text{Ker}(A) \).

\[ (Ax^{(k)})_i = \left( \frac{ik}{n} - \left\lfloor \frac{ik}{n} \right\rfloor \right) + \left( \frac{i(n-k)}{n} - \left\lfloor \frac{i(n-k)}{n} \right\rfloor \right) - \left( \frac{i(n-1)}{2n} - \left\lfloor \frac{i(n-1)}{2n} \right\rfloor \right) \]

\[ - \left( \frac{i(n+1)}{2n} - \left\lfloor \frac{i(n+1)}{2n} \right\rfloor \right) \]

\[ = \left\lfloor \frac{i(n-1)}{2n} \right\rfloor + \left\lfloor \frac{i(n+1)}{2n} \right\rfloor - \left\lfloor \frac{ik}{n} \right\rfloor - \left\lfloor \frac{i(n-k)}{n} \right\rfloor \]

\[ = 0 \]
Thus, it only remains to prove that $A$ has rank at least $\frac{n+1}{2}$. Now, $nA$ can be row reduced such that the top left $\frac{n+1}{2} \times \frac{n+1}{2}$ submatrix looks like:

$$
\begin{pmatrix}
1 & 2 & 3 & \ldots & \frac{n-1}{2} & \frac{n+1}{2} \\
0 & 0 & 0 & \ldots & 0 & * \\
0 & 0 & 0 & \ldots & * & * \\
\vdots \\
0 & 0 & * & \ldots & * & * \\
0 & * & * & \ldots & * & * 
\end{pmatrix}
$$

where each of the entries immediately below the anti-diagonal is necessarily non zero. Such a matrix has non-zero determinant.

Thus, any element in the $\text{Ker}(A)$ looks like:

$$(x_1, x_2, \ldots - \sum_{i=1}^{\frac{n-1}{2}} x_i, - \sum_{i=1}^{\frac{n+1}{2}} x_1, x_2)$$

This proves the lemma and hence the proposition.

\[\square\]

**Remark 2.11.** Perhaps a more natural way to interpret Propositions 2.8 and 2.9 is to say that for a curve $y^n = f(x)$ (n prime) has ordinary Jacobian if and only if the same number of points are ramified to degree $i$ and $n-i$ for any $i \in \{1, 2, \ldots n-1\}$. Here we say that a point $P$ is ‘ramified to degree $i$’ if the curve locally looks like $y^n = ux^i$, where $x_P$ is a uniformizer at $P$ and $u$ is unit.

### 3. Main Results

In this section we describe the main results obtained from counting each of the families described above. Our main tool will be the following Tauberian theorem:

**Theorem 3.1** (See [5], Appendix A). Let $\{\lambda_n\}_{n \in \mathbb{Z}_{\geq 0}}$ be strictly increasing sequence of positive integers. Let $f$ be the Dirichlet series:

$$f(s) = \sum_{n=1}^{\infty} c_n \lambda_n^{-s}$$

Further, assume the following:

1. $f(s)$ converges for $\text{Re}(s) > a > 0$.
2. $f$ admits a meromorphic continuation to $\text{Re}(s) > a - \delta_0 > 0$ for some $\delta_0 > 0$.
3. The right-most pole of $f$ is at $s = a$, with multiplicity $b \in \mathbb{N}$. Let $\Theta = \lim_{s \to a} f(s)(s-a)^b$.
4. (Technical assumption) $\exists a \kappa > 0$ such that for $\text{Re}(s) > a - \delta_0$,

$$\left| \frac{f(s)(s-a)^b}{s^b} \right| = O((1 + \text{Im}(s))^\kappa)$$

Then there exists a (monic) polynomial $P$ of degree $b-1$ such that for any $\delta < \delta_0$, we have:

$$\sum_{\lambda_n < X} c_n = \frac{\Theta}{a(b-1)!} X^a P(\log(X)) + O(X^{a-\delta})$$

We will henceforth use the notation $|Q|$ to denote $q^{\deg(Q)}$, where $Q$ is an irreducible polynomial over $\mathbb{F}_q$. We will denote by $\zeta(s)$, the zeta function of $\mathbb{A}_{\mathbb{F}_q}^1$. Thus $\zeta(s) = \prod_Q (1 - |Q|^{-s})^{-1}$, where the product is over monic irreducible polynomials over $\mathbb{F}_q$. 


3.1. Artin-Schreier curves. To recall, the family \( \mathcal{F} \) that we are interested in in this section is that of covers \( y^p - y = f(x) \), with \( f(x) \in S \), such that the corresponding map \( C \to \mathbb{P}^1 \) is unramified over \( \infty \).

We first set up some notation in order to calculate \( N(\mathcal{F}, X) \) and \( N(\mathcal{F}, 0, X) \):

- Define a new invariant: \( m = \frac{2g}{p-1} + 2 \). By equation (2.3), this is an integer and is equal to:
  \[
  \sum_Q \deg(Q)(d_Q + 1)
  \]

- For any \( m \geq 2 \), let \( a(m) \) be the number of Artin-Schreier covers \( C \) with the above invariant equal to \( m \). Let \( b(m) \) be the number of such covers with \( a \)-number 0.
- Define
  \[
  N^*(\mathcal{F}, X) = \sum_{q^m < X} a(m) \quad \text{and} \quad N^*(\mathcal{F}, 0, X) = \sum_{q^m < X} b(m)
  \]

We will calculate these as an intermediate step towards finding \( N(\mathcal{F}, X) = N^*(\mathcal{F}, q^2X^{2/(p-1)}) \) and \( N(\mathcal{F}, 0, X) = N^*(\mathcal{F}, 0, q^2X^{2/(p-1)}) \).

For this section, define the zeta function:

\[
Z(s) = \sum_{C \in \mathcal{F}} q^{-m(C)s} = \sum_m a(m)q^{-ms}
\]

**Lemma 3.2.** \( Z(s) \) converges for \( \Re(s) > 1 \) and has a pole of order \( p-1 \) at \( s = 1 \).

**Proof.** Note that:

\[
m = \sum_Q \deg(Q)(d_Q + 1)
\]

where the sum is over monic irreducible polynomials \( Q \in \mathbb{F}_q[x] \). Here, we set \( d_Q = -1 \) if the map \( C \to \mathbb{P}^1 \) is unramified over the divisor \( \text{div}(Q) \). Since this is a sum of local factors, we factor \( Z(s) \) as a product of local functions, i.e. \( Z(s) = \prod_Q Z_Q(s) \), where \( Q \) varies over monic irreducible polynomials in \( \mathbb{F}_q[x] \). We can write \( Z_Q(s) = \sum_{k \geq 0} c(k) |Q|^{-ks} \). Recall from §2.1 that if \( \alpha \in B \) and \( Q(\alpha) = 0 \), then \( d_Q = \deg(f_\alpha) \) as in the partial fraction decomposition of \( f(x) \). Further recall that in each \( f_\alpha \), the coefficient of \( x^{ip} \) is 0 for each \( 0 \leq i \leq \lfloor d_Q/p \rfloor \). Since \( k = d_Q + 1 \),

\[
c(k) = \# \{ f_\alpha \in \mathbb{F}_q[x] \mid \deg(f_\alpha) = k - 1, \ \text{coefficient of } x^{ip} = 0 \}
\]

where \( k \not\equiv 1 \mod p \) (since \( d_Q \not\equiv 0 \mod p \)). We write \( d_Q = np + i \), with \( 1 \leq i \leq p - 1 \). The above discussion gives us that for \( k = np + i + 1 \),

\[
c(k) = (|Q| - 1) |Q|^{-i-1}|Q|^{n(p-1)}
\]

For convenience, we distinguish the cases where \( p = 2 \) and \( p \geq 3 \).

For \( p = 2 \):

\[
Z_Q(s) = 1 + \sum_{n=0}^{\infty} (|Q|^{-1} |Q|^{n}|Q|^{-s(2n+2)}) = \frac{1 - |Q|^{-2s}}{1 - |Q|^{1-2s}}
\]
For \( p \geq 3 \),
\[
Z_Q(s) = 1 + \sum_{i=1}^{p-3} \sum_{n=0}^{\infty} (|Q| - 1) |Q|^{n(p-1)} |Q|^{-s(np+i+1)}
\]
\[
= 1 + \left( \frac{(|Q| - 1) |Q|^{-2s}}{1 - |Q|^{p-1-ps}} \right) \sum_{i=1}^{p-1} |Q|i^{1-s}
\]
\[
= 1 + \sum_{i=0}^{p-3} |Q|^{i(s+1)-(i+2)s} - \sum_{i=0}^{p-2} |Q|^{i-(i+2)s}.
\]

For \( p \geq 3 \), let
\[
\psi_{p,Q}(s) = \left( 1 + \sum_{i=0}^{p-3} |Q|^{i(s+1)-(i+2)s} - \sum_{i=0}^{p-2} |Q|^{i-(i+2)s} \right) \prod_{i=0}^{p-3} (1 - |Q|^{(i+1)-(i+2)s})
\]

Define:
\[
\psi_p(s) = \begin{cases} 
\zeta(2s)^{-1} & \text{if } p = 2 \\
\prod_Q \psi_{p,Q}(s) & \text{if } p \geq 3
\end{cases}
\]

Then by a straightforward calculation, \( \psi_p(s) \) converges for \( Re(s) > \frac{p-1}{p} \).

Therefore
\[
\prod_Q Z_Q(s) = \psi_p(s) \prod_{i=0}^{p-2} \zeta(s(i+2) - (i+1))
\]

This converges for \( Re(s) > 1 \) and has a pole of order \( p - 1 \) at \( s = 1 \). Further, the residue at \( s = 1 \) is given by:
\[
\lim_{s \to 1} Z(s)(s-1)^{p-1} = \frac{\psi_p(1)}{\log(q)^{p-1}}
\]

To count the number of ordinary curves, we define:
\[
Z_0(s) = \sum_{C \in X_0} q^{-m(C)s} = \sum_m b(m) q^{-ms}
\]

Recall that for such curves, \( d_\alpha = 1 \) for all \( \alpha \). Therefore, the \( Z_0(s) = \prod_Q Z_{0,Q}(s) \), where the local factors are:
\[
Z_{0,Q}(s) = 1 + (|Q| - 1) |Q|^{-2s}
\]

**Lemma 3.3.** \( Z_0(s) \) converges for \( Re(s) > 1 \) and has a simple pole at \( s = 1 \).

**Proof.** Note that
\[
(1 + |Q|^{-2s} - |Q|^{-2s})(1 - |Q|^{-2s}) = 1 - |Q|^{-2s} - |Q|^{-4s} + |Q|^{-4s}
\]

and
\[
\phi(s) := \prod_Q (1 - |Q|^{-2s} - |Q|^{-4s} + |Q|^{-4s})
\]

converges for \( Re(s) > 3/4 \). Therefore \( Z_0(s) = \phi(s) \zeta(2s - 1) \) converges for \( Re(s) > 1 \) and has a simple pole at \( s = 1 \). Further, the residue at \( s = 1 \) is:
\[
\lim_{s \to 1} Z_0(s)(s-1) = \frac{\phi(1)}{\log(q)}
\]

\( \square \)
Proposition 3.4. For any \( \delta > 0 \),
\[
N^*(\mathcal{F}, X) = \frac{\psi_p(1)}{\log(q)} X(\log_q(X))^{p-2} + O(X^{1-\delta})
\]
\[
N^*(\mathcal{F}, 0, X) = \frac{\phi(1)}{\log(q)} X + O(X^{1-\delta})
\]

Proof. This follows from the Tauberian theorem \[\text{3.1}\] applied to the results of lemmas \[\text{3.2} \text{ and } 3.3\] since \( \zeta(s) \) has a meromorphic continuation to the entire complex plane. \( \square \)

Corollary 3.5. For any \( \delta > 0 \),
\[
N(\mathcal{F}, X) = \frac{\psi_p(1)}{\log(q)} q^2 X^{2/(p-1)}(\log_q(X^{2/(p-1)}))^{p-2} + O(X^{\frac{2}{p-\delta}})
\]
\[
N(\mathcal{F}, 0, X) = \frac{\phi(1)}{\log(q)} q^2 X^{2/(p-1)} + O(X^{\frac{2}{p-\delta}})
\]

In particular, the probability that an Artin-Schreier curve is ordinary is:
\[
\phi(1)\zeta(2) \quad \text{if } p = 2
\]
\[
0 \quad \text{if } p \geq 3
\]

Proof. \( N(\mathcal{F}, X) = N^*(\mathcal{F}, q^2 X^{2/(p-1)}) \). \( \square \)

Remark 3.6. Some values of the above probability are calculated in table \[\text{I}\].

1. Calculating the first few terms of the product \( \phi(1)\zeta(2) \) gives:
\[
\phi(1)\zeta(2) = 1 - q^{-1} + q^{-2} - 2q^{-3} + O(q^{-4})
\]

2. If we modify \( \mathcal{F} \) to include the covers ramified over \( \infty \), we must modify the partial fraction decomposition in \[\text{2.2} \] to:
\[
f(x) = \sum_{\alpha \in \mathbb{N}} f_\alpha(x_\alpha) + g(x).
\]

Here \( g(x) \in \mathbb{F}_q[x] \) is a polynomial that, like the other \( f_\alpha \)’s, has degree coprime to \( p \) and the coefficients of \( x^{ip} \) in \( g(x) \) are 0, for all \( 0 \leq i \leq \lfloor \deg(g)/p \rfloor \). This manifests as a change in the zeta functions \( Z(s) \) and \( Z_0(s) \) defined in the above discussion by factors that we will call \( Z_\infty(s) \) and \( Z_{0,\infty}(s) \) respectively. That is, we write \( Z(s) = Z_\infty(s) \prod_Q Z_Q(s) \) and \( Z_0(s) = Z_0,\infty(s) \prod_Q Z_0,0_Q(s) \). Both these factors only affect the residues of \( Z(s) \) and \( Z_0(s) \), which means that for \( p \geq 3 \), the probability of ordinarity for the modified family is still 0. For \( p = 2 \),
\[
Z_\infty(s) = 1 + q^{-1} \quad \text{and} \quad Z_{0,\infty}(s) = 1 - q^{-1} + q^{-2}
\]

Therefore the probability of ordinarity in the modified family is:
\[
(\frac{1 - q^{-1} + q^{-2}}{1 + q^{-1}}) \phi(1)\zeta(2) = 1 - 3q^{-1} + 6q^{-2} + O(q^{-3})
\]

3. Recall that if the Jacobian of a curve behaves randomly in the sense of \[\text{H}\], the heuristics predict that the probability of that a curve is ordinary is:
\[
\prod_{i=1}^{\infty} (1 + q^{-i})^{-1}.
\]

Corollary \[\text{3.5}\] and the above remark prove that the Jacobian of an Artin-Schreier curve does not behave randomly in the sense of \[\text{H}\]. For \( p \geq 3 \) this is clear. For \( p = 2 \), elementary calculations show that the constants are not equal. In fact,
\[
\prod_{i=1}^{\infty} (1 + q^{-i})^{-1} = 1 - q^{-1} - q^{-3} + q^{-4} + O(q^{-5})
\]
3.1.1. Note about irreducibility. For \( p = 2 \) the Artin-Schreier locus \( \mathcal{S}_g \) coincides with the hyperelliptic locus \( \mathcal{H}_g \). However, in general, \( \mathcal{S}_g \) is not irreducible. In [13], the authors give the following characterization for the irreducibility of \( \mathcal{S}_g \):

**Proposition 3.7** ([13] Corollary 1.2). The moduli space \( \mathcal{S}_g \) is irreducible in exactly the following cases: (a) \( p = 2 \), or (b) \( g = 0 \) or \( g = \frac{p-1}{2} \), or (c) \( p = 3 \) and \( g = 2, 3, 5 \)

In particular, for \( p \geq 3 \) and \( g \) large enough, the Artin-Schreier locus is reducible. Further, the dimension of each irreducible component of \( \mathcal{S}_g \) is \( \frac{2g}{p-1} - 1 \). It is interesting to ask whether the reducibility of \( \mathcal{S}_g \) completely explains the probability obtained in theorem 3.5. That is, for each \( g \), let \( \mathcal{L}_{g,g} \) denote the closure of the ordinary locus inside \( \mathcal{S}_g \). Then, is:

\[
\lim_{X \to \infty} \frac{\# \{ C \in \mathcal{L}_{g,g} \mid q^g < X \}}{\# \{ C \in \mathcal{S}_{g,g}(\overline{\mathbb{F}}_q) \mid q^g < X \}}
\]

positive?

For a given \( p \)-rank \( r = (s-1)(p-1) \), the irreducible components of the corresponding \( p \)-rank stratum \( \mathcal{S}_{g,r} \subset \mathcal{S}_g \) are in bijection with partitions of the integer \( \frac{2g}{p-1} + 2 \) into \( s \) integers that are \( \not\equiv 1 \mod p \). For example, the ordinary locus corresponds to the partition \( \{2, 2, \ldots, 2\} \).

If \( \vec{E} = \{e_1, \ldots, e_s\} \) is one such partition, then the dimension of the corresponding Artin-Schreier stratum is \( \frac{2g}{p-1} - 1 - \sum_{i=1}^s \left\lfloor \frac{e_i}{p} \right\rfloor \) ([13]). In [7], the author gives a characterization of all the partitions \( E' \) such that \( \mathcal{S}_{g,E'} \) lies in the closure of \( \mathcal{S}_{g,E} \) for a given partition \( E \). Using this characterization and similar techniques as above, we can show that for \( p = 3 \), the quantity \( \text{(3.3)} \) is indeed positive. This is trickier for larger primes since the combinatorial description of the closure gets more involved as \( p \) increases. It would be interesting to explore this question for large primes, especially if one could find a uniform proof for all odd primes.

3.2. Superelliptic Curves in characteristic 2. For this section, we refer back to the notation of Section 2.2. We are interested in counting covers in the family \( \mathcal{F} \) of covers \( y^n = f(x) \) over a field \( \mathbb{F}_q \) of characteristic 2, where:

- \( n \) is prime.
- \( f(x) \in \mathbb{F}_q[x] \) is \( n \)-th power free.

For convenience, we count by \( q^m \) instead of \( q^g \). Recall that:

\[
m = \frac{2g}{n-1} + 2
\]

is the number of points in \( \mathbb{P}^1(\overline{\mathbb{k}}) \) over which the curve given by \( y^n = f(x) \) is ramified. Since \( n \) is fixed in the entire discussion, this will not change the order of counting significantly. Define \( N^*(\mathcal{F}, X) \) as the set of curves in \( \mathcal{F} \) with \( q^m < X \) and \( N^*(\mathcal{F}, 0, X) \) similarly. We have:

\[
N(\mathcal{F}, X) = N^*(\mathcal{F}, q^2 X^{2/(n-1)}) \quad \text{and} \quad N(\mathcal{F}, 0, X) = N^*(\mathcal{F}, 0, q^2 X^{2/(n-1)})
\]

We define:

\[
\mathcal{F}_{e_1, e_2, \ldots, e_r} = \{ F_1 F_2^2 \cdots F_r^r \mid F_i \in \mathbb{F}_q[x] \text{ monic, squarefree and mutually coprime, } \deg(F_i) = e_i \}
\]

When we write \( m = \sum_{i=1}^r e_i \), we will be interested in the case when there are \( e_i \) points ramifying to degree \( i \). This is the same as the notion defined in Remark 2.11. To express this concretely in terms of polynomials, it is best to use an example. For instance, for a curve given by \( y^3 = F_1(x)(F_2(x))^2 \), where \( F_1(x)(F_2(x))^2 \in \mathcal{F}_{2,4} \), there are 2 points that occur with degree 1 and 4 that occur with degree 2. If on the other hand, the curve is given by \( y^3 = F_1(x)(F_2(x))^2 \), where \( F_1(x)(F_2(x))^2 \in \mathcal{F}_{3,2} \), there are 3 points that occur with degree 1 and 3 that occur with degree 2 (since \( n = 2 \), the curve is ramified over \( \infty \in \mathbb{P}^1 \) to degree 2).
Proposition 3.8. Consider the set $S_m$ of superelliptic curves with the number of ramified points $m = \sum_{i=1}^{n-1} e_i$, such that there are $e_i$ points that ramify to degree $i$. Then the size of $S_m$ is:

$$|F_{e_1, e_2, \ldots, e_{n-1}}| + \sum_{i=1}^{n-1} |F_{e_1, \ldots, e_i, 1, \ldots, e_{n-1}}|$$

Proof. Let $C \in \mathcal{F}$, such that $C \to \mathbb{P}^1$ is ramified over $m$ points in $\mathbb{P}^1(\mathbb{F}_q)$. If the map is not ramified over $\infty$, then $C \in F_{e_1, e_2, \ldots, e_{n-1}}$. If it is ramified over $\infty$ and $n_\infty = i$, then $C \in F_{e_1, \ldots, e_i, 1, \ldots, e_{n-1}}$.

In the above proposition, imposing the condition $m = \sum_{i=1}^{n-1} e_i$, with $e_i$ points occurring with degree $i$, implies that $\sum_{i=1}^{n-1} ie_i \equiv 0 \mod n$. Therefore we are interested in the quantity:

$$\sum_{q^m < X} \left( |F_{e_1, e_2, \ldots, e_{n-1}}| + \sum_{i=1}^{n-1} |F_{e_1, \ldots, e_i, 1, \ldots, e_{n-1}}| \right)$$

where the sum is over $(e_1, e_2 \ldots e_{n-1})$ such that $\sum_{i=1}^{n-1} ie_i \equiv 0 \mod n$. Further, observe that

$$\sum_{\sum ie_i \equiv 0 \mod n} |F_{e_1, \ldots, e_i, 1, \ldots, e_{n-1}}| = \sum_{\sum id_i \equiv 1 \mod n} |F_{d_1, \ldots, d_i, \ldots, d_{n-1}}|$$

Therefore (3.5) can be rewritten as:

$$\left( \sum_{q^1 + e_2 \ldots e_{n-1} < X/q} |F_{e_1, e_2, \ldots, e_{n-1}}| \right) + \sum_{X/q < q^1 + \ldots + e_{n-1} < X} |F_{e_1, e_2, \ldots, e_{n-1}}|$$

where the first sum is over all tuples $(e_1, e_2 \ldots e_{n-1})$ with $q^\sum e_i < X/q$. From now on, a sum of the form $\sum_{q^1 + e_2 \ldots e_{n-1} < X}$ will denote a sum over all tuples of non-negative integers $(e_1, e_2 \ldots e_r)$, with $q^\sum e_i < X$. The second sum, which we will denote $E(X, q)$ is $O((\log(X))^{n-1})$ and will contribute only to the error term of our result.

For any non-negative integer $m$, let $a(m) = \sum_{e_1+e_2\ldots e_{n-1}=m} |F_{e_1, e_2, \ldots, e_{n-1}}|$

Define:

$$Z(s) = \sum_{m \geq 0} a(m)q^{-ms}$$

One way to think about an element of $F_{e_1, e_2, \ldots, e_{n-1}}$ is to say that we are considering a polynomial:

$$f(x) = \prod_{i=1}^{n-1} (F_i(x))^j$$

such that $H := \prod_{i=1}^{n-1} F_i(x)$ is squarefree. We will use this characterization to calculate $Z(s)$.

Now consider a squarefree polynomial $H$. Let $H = \prod h_j$ be its factorization into irreducible polynomials. We want to count the number of ways in which $H$ can be written as a product of squarefree polynomials $\prod_{i=1}^{n-1} F_i$. For each factor $h_j$, there are $n - 1$ choices of squarefree polynomial that it could divide. Therefore, the number of factorizations $H = \prod_{i=1}^{n-1} F_i$ is

$$(n - 1)^{\omega(H)}$$

where $\omega(H)$ is the number of distinct irreducible factors of $H$. Therefore, $Z(s) = \sum_{H \text{ square free}} (n - 1)^{\omega(H)} |H|^{-s} = \prod_{Q} (1 + (n - 1) |Q|^{-s})$.

The second sum, which we will denote $E(X, q)$ is $O((\log(X))^{n-1})$ and will contribute only to the error term of our result.
Note 3.9. Let $\Phi_k(s) = \prod_{Q} (1 + k \mid Q^{-s})$. Then $\Phi_k(s)\zeta(s)^{-k}$ is a function that converges for $\Re(s) > 1/2$. We will denote $\Phi_k(s)\zeta(s)^{-k}$ by $\phi_k(s)$.

Proposition 3.10. As $X \to \infty$, 
\[ N^*(\mathcal{F}, X) = \frac{\phi_{n-1}(1)}{q \log(q) (n-2)!} X (\log_q(X))^{n-2} + O(X(\log(X))^{n-3}) \]

Proof. Note that: 
\[ Z(s) = \Phi_{n-1}(s) = \zeta(s)^{n-1} \phi_{n-1}(s) \]
This function has a pole of order $n - 1$ at $s = 1$. Thus, the Tauberian theorem implies that: 
\[ N^*(\mathcal{F}, X) = \frac{\phi_{n-1}(1)}{q \log(q) (n-2)!} X (\log_q(X))^{n-2} + O(X(\log(X))^{n-3}) \]
As noted before, $E(X, q)$ is absorbed into the error term above. \hfill \Box

Corollary 3.11. The number of superelliptic covers with invariant $\mathbf{m}$ such that $q^1 < X$ is: 
\[ N(\mathcal{F}, X) = \kappa_n(q) X^{2/(n-1)} \log_q(X^{2/(n-1)})^{n-2} + O(X^{2/(n-1)} \log(X)^{n-3}) \]
where 
\[ \kappa_n(q) = \frac{q \phi_{n-1}(1)}{\log(q) (n-2)!} \]
Proof. This follows from $N(\mathcal{F}, X) = N^*(\mathcal{F}, q^2 X^{2/(n-1)})$. \hfill \Box

3.2.1. Upper bounds for $N^*(\mathcal{F}, 0, X)$. In this subsection, we find an upper bound for the quantity $N^*(\mathcal{F}, 0, X)$, as defined in (3.4). We will maintain the notation of §2.2.

Suppose we consider covers with $\mathbf{m} = \sum_{i=1}^{n-1} e_i$ ramification points, $e_i$ points occurring with degree $i$. Using the criterion for ordinarity in Proposition 2.3 we can derive the following conditions on $\mathcal{F}_{e_1, e_2, \ldots, e_{n-1}}$:

1. If $n_{\infty} = 0$, $\deg(f_i) = \deg(f_{n-i})$. Note that in this case, the cover belongs to $\mathcal{F}_{e_1, e_2, \ldots, e_{n-1}}$ with $\deg(f_{n-i}) = e_i$. Therefore the condition for ordinarity implies that there are:
\[ |\mathcal{F}_{e_1, \ldots, e_{n-1}, e_1} | \]
curves of such kind over $\mathbb{F}_q$.

2. If $n_{\infty} = i$, then $\mathcal{C} \in \mathcal{F}_{e_1, \ldots, e_{n-1}}$ with $\deg(f_{j}) = e_j$ for $j \neq i$ and $\deg(f_{i}) = e_{i} - 1$. Further, the condition for ordinarity gives: for $j \neq i, n - i$, $\deg(f_{j}) = \deg(f_{n-j})$ and therefore $e_j = e_{n-j}$. Also, $\deg(f_{i}) + 1 = \deg(f_{n-i})$ implies $e_i = e_{n-i}$. Therefore, the number of such curves is:
\[ |\mathcal{F}_{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n-1}, e_1} | \]
if $i \leq \frac{n-1}{2}$, and
\[ |\mathcal{F}_{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n-1}, e_1} | \]
if $i > \frac{n-1}{2}$.

As in (3.3), we are interested in the size
\[ N^*(\mathcal{F}, 0, X) = \sum_{q < X} \left( |\mathcal{F}_{e_1, \ldots, e_{n-1}} | + \sum_{i=1}^{n-1} |\mathcal{F}_{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{n-1}} | \right) \]
where the sum is now over tuples $(e_1, e_2, \ldots, e_{n-1})$ that satisfy the ordinarity criterion $e_i = e_{n-i}$. Note that for such a tuple, the condition $\sum_{i=1}^{n-1} i e_i \equiv 0 \mod n$ is satisfied automatically. We
now proceed to find an upper bound on this quantity, using a result of Bucur et. al. in \cite{BucurDL19} that we will recall below. Let:

\[
L_{n-2} = \prod_{j=1}^{n-2} \prod_{Q} \left( 1 - \frac{j}{(|Q|+1)(|Q|+j)} \right),
\]

where the product is over monic irreducible polynomials \( Q \in \mathbb{F}_q[x] \).

**Theorem 3.12** (\cite{BucurDL19}, Prop 4.3). Fix a tuple of positive integers \((e_1, e_2)\). Then, for any \( \epsilon > 0 \) and as \( q \) gets large,

\[
|\mathcal{F}_{e_1, e_2}| = \frac{L_1 q^{e_1+e_2}}{\zeta(2)^2} \left( 1 + O(q^{-e_2(1-\epsilon)} + q^{-(e_1/2)}) \right)
\]

**Remark 3.13.** The number of monic polynomials of degree \( d \) in \( \mathbb{F}_q[x] \) is \( q^d \) and the proportion of these that are squarefree is \( (1-1/q) \). One might expect, similarly, that the proportion of pairs of monic polynomials of degrees \((e_1, e_2)\) that are squarefree and coprime, also form a positive proportion of the total number of pairs of monic polynomials, \( q^{e_1+e_2} \). The above theorem shows that this is indeed the case. The next proposition shows that the same is true for \((e_1, e_2 \ldots e_{n-1})\) for any odd prime \( n \), although with a weaker error term.

For the next proposition, we refer the reader to \cite{BucurDL19}, Corollary 7.2.

**Proposition 3.14.** Fix a tuple of positive integers \((e_1, e_2 \ldots e_{n-1})\). Fix an \( \epsilon > 0 \). Then, as \( q \) gets large,

\[
|\mathcal{F}_{(e_1, e_2 \ldots e_{n-1})}| = \frac{L_{n-2} q^{e_1+e_2 \ldots e_{n-1}}}{\zeta(2)^{n-1}} \left( 1 + O(q^{(e_2 + \ldots e_{n-1} + q) + (1-\epsilon)q^{q^{e_2 + \ldots e_{n-1}} + q^{-(e_1-3q)/2}}}) \right)
\]

**Proof.** Consider the expression given in \cite{BucurDL19}, Corollary 7.2. Summing the expression over all possible partitions \( m = k_1 + k_2 \ldots k_{n-1} \) gives:

\[
\frac{L_{n-2} q^{e_1+e_2 \ldots e_{n-1}}}{\zeta(2)^{n-1}} \left( \frac{n-1}{q + n - 1} \right)^m \left( \frac{q}{q + n - 1(q - 1)} \right)^{q-m} \times (1 + O(q^{(e_2 + \ldots e_{n-1} + q) + (1-\epsilon)q^{q^{e_2 + \ldots e_{n-1}} + q^{-(e_1-3q)/2}}}))
\]

Summing over all possibilities of \( m \) now gives the result. \( \square \)

Parsing these propositions tells us that for large enough \( q \),

\[
|\mathcal{F}_{(e_1, e_2 \ldots e_{n-1})}| \leq K_1 q^{e_1+e_2 \ldots e_{n-1}} + K_2 q^{e_1/2+e_2 \ldots e_{n-1}} + \sum_{i=2}^{n-1} K_{3,i} q^{e_1+\ldots e_i \ldots + e_{n-1}}
\]

where \( K_1, K_2 \) and the \( K_{3,i} \)'s depend on \( \epsilon, q \) and \( n \), but are independent of the \( e_j \)'s. Since for \( \epsilon < 1 \) the first term in the above expression is the largest, we let \( K = \max(K_1, K_2, K_{3,2} \ldots K_{3,n-1}) \) and so for large enough \( q \):

\[
|\mathcal{F}_{(e_1, e_2 \ldots e_{n-1})}| \leq K q^{e_1+e_2 \ldots e_{n-1}}.
\]

Thus \( \text{(3.6)} \) implies:

\[
N^*(\mathcal{F}, 0, X) \leq K \left( \frac{q + n - 1}{q} \right) \left( \sum_{q^{2(e_1+e_2 \ldots e_{n-1}/2)} < X} q^{2(e_1+e_2 \ldots e_{n-1}/2)} \right)
\]

(3.7)

The following lemma will be used to find an upper bound for the expression above.

**Lemma 3.15.** As \( X \) gets large,

\[
\sum_{q^{e_1+e_2 \ldots e_r} < X} q^{e_1+e_2 \ldots e_r} = O(X \log(X)^{r-1}).
\]

Here, the implied constants depend on \( q \) and \( r \).
Proof. Consider the expression:

\[
\left( \frac{1}{1 - qT} \right)^r
\]

The coefficient of \( T^m \) in this expression is \( \sum_{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r = m} q^{\varepsilon_1 + \varepsilon_2 + \varepsilon_r} \). On the other hand, by the binomial theorem, the coefficient of \( T^m \) in \( (1 - qT)^{-r} \) is: \( \binom{r + m - 1}{r - 1} \). Further,

\[
\left( \frac{r + m - 1}{r - 1} \right) \leq \frac{(m + r)^{r-1}}{(r-1)!}
\]

Therefore, we have:

\[
\sum_{q^{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r} < X} q^{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r} = \sum_{q^n < X} \sum_{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r = m} q^{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r}
\]

\[
= \sum_{q^n < X} \binom{r + m - 1}{r - 1} q^m
\]

\[
\leq \frac{(2r)^{r-1}}{(r-1)!} \sum_{q^m < X} q^m + \sum_{q^m > X} \frac{(2m)^{r-1}}{(r-1)!} q^m
\]

\[
= D_r X \log(X)^{r-1} + O(X \log(X)^{r-2})
\]

where the last step follows by Euler Summation. This proves the lemma.

**Proposition 3.16.** For large enough \( q \),

\[
N^*(\mathcal{F}, 0, X) = O(X \log(X)^{\frac{n-3}{2}})
\]

Hence, \( N(\mathcal{F}, 0, X) = O(X^{2/(n-1)} \log(X)^{\frac{n-3}{2}}) \), where the implied constants depend on \( q \) and \( n \).

**Proof.** To obtain the first statement, we use Equation (3.7):

\[
N^*(\mathcal{F}, 0, X) \leq K \left( \frac{q + n - 1}{q} \right) \left( \sum_{q^2^{e_1 + e_2 + \cdots + e_{(n-1)/2}} < X} q^{2(e_1 + e_2 + \cdots + e_{(n-1)/2})} \right)
\]

and Lemma 3.15, with \( q \) replaced by \( q^2 \). The second part of the statement follows from the fact that \( N(\mathcal{F}, 0, X) = N^*(\mathcal{F}, 0, q^2 X^{2/(n-1)}) \).

We remind the reader here that for the quantity that we are interested in, namely the probability that a superelliptic curve is ordinary, \( q \) and \( n \) are fixed. Therefore, the fact that the implied constants above depend on \( q \) and \( n \) will make no difference to the theorem below.

**Theorem 3.17.** The probability that a superelliptic curve \( y^n = f(x) \) over \( \mathbb{F}_q \) with \( n \) prime and \( q \) large enough, is ordinary, is zero. That is,

\[
\lim_{X \to \infty} \frac{N(\mathcal{F}, 0, X)}{N(\mathcal{F}, X)} = 0
\]

**Proof.** By Proposition 3.16 the numerator, \( N(\mathcal{F}, 0, X) \) is bounded above by \( X^{2/(n-1)} \log(X)^{\frac{n-3}{2}} \). By Corollary 3.11 the denominator grows like \( X^{2/(n-1)} \log(X)^{n-2} \). This proves the theorem.

**Remark 3.18.** It is interesting to note that for a given \( q \), the space of superelliptic curves of degree \( n \) and genus \( g \) decomposes over \( \mathbb{F}_q \) into irreducible components that correspond to partitions of \( m = \sum_{i=1}^{n-1} e_i \) such that \( \sum_{i=1}^{n-1} i e_i \equiv 0 \mod n \). The ordinary locus intersects exactly one of these components. A similar thing was true for the Artin-Schreier locus \( \mathcal{A}_q \). For fixed \( p \)-rank \( r \), one can obtain a combinatorial description of the components contained in the stratum \( \mathcal{A}_q^{g,r} \). One can ask if a similar result holds for superelliptic curves in even characteristic.
4. Numerical Data

4.1. Artin-Schreier curves in characteristic $p$. Here we list some values of constants calculated in §3. Recall that $\phi(1)\zeta(2)$ is the probability that an Artin-Schreier curve in characteristic 2 as in §3 is ordinary (Corollary 3.5). For brevity, we let $\varphi(q) = \prod_{i=1}^{\infty} (1 + q^{-i})^{-1}$ the constant predicted in [4].

| $p$ | $q$ | $\phi(1)$ | $\phi(1)\zeta(2)$ | $\varphi(q)$ |
|-----|-----|-----------|--------------------|-------------|
| 2   | 2   | 0.314148  | 0.1570745          | 0.419422    |
| 2   | 4   | 0.593976  | 0.4454805          | 0.737512    |
| 2   | 8   | 0.776577  | 0.6795058          | 0.873264    |
| 2   | 16  | 0.882162  | 0.8270268          | 0.937270    |
| 2   | 32  | 0.939367  | 0.9100118          | 0.968720    |

Table 1. Constants for Artin-Schreier curves

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