Multi-valued hyperelliptic continued fractions of generalized Halphen type

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Abstract

We introduce and study higher genera generalizations of the Halphen theory of continued fractions. The basic notion we start with is hyperelliptic Halphen (HH) element

\[
\frac{\sqrt{X_{2g+2}} - \sqrt{Y_{2g+2}}}{x-y}
\]

depending on parameter \( y \), where \( X_{2g+2} \) is a polynomial of degree \( 2g + 2 \) and \( Y_{2g+2} = X_{2g+2}(y) \). We study regular and irregular HH elements, their continued fraction development and some basic properties of such developments such as: even and odd symmetry and periodicity.
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1 Introduction

Modern algebraic approximation theory with continued fraction theory was established by Tchebycheff and his Sankt Petrsburg school in the second half of the XIX century. Tchebycheff motivation for this studies was his interest in practical problems: in the mechanism theory as an important part of mechanical engineering of that time and ballistics. Steam engines were fundamental tool in technological revolution and their kernel part was the Watt’s complete parallelogram, a planar mechanism to transform linear motion into circular.

![Figure 1: Watt’s complete parallelogram](image)

Fundamental problem was to estimate error of the mechanism in execution of that transformation.

Starting point of Tchebycheff investigation ([12]) was work on the theory of mechanisms of French military engineer, professor of mechanics and academician Jean Victor Poncelet [10]. In his study of mistakes of mechanisms, Poncelet came to the question of rational and linear approximation of the function

$$\sqrt{x^2 + 1}.$$

In other words he studied approximation of the functions $\sqrt{X_2(x)}$ of the form of square root of polynomials of the second degree, and he gave two approaches to the posed problems, one based on the analytical arguments and the second one based on geometric consideration.

Although Poncelet was described by Tchebycheff as ”well-known scientist in practical mechanics” (see [11]), nowadays J. V. Poncelet is known first of all as one of the biggest geometers of the XIX century. The Great Poncelet Theorem
(GPT) is considered as one of the nicest and deepest results in projective geometry: Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed about the other one. Then, GPT states that infinitely many such closed polygonal lines exist – every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides.

Figure 2: Poncelet theorem

In his Traité des propriétés projectives des figures, Poncelet proved even more general result and used only purely geometric, synthetic arguments.

(The case when the two ellipses are confocal has clear mechanical interpretation as billiard system within outer ellipse as boundary and having inner ellipse as the caustic of a given billiard trajectory. GPT in this case describes periodic billiard trajectories. For a modern account of GPT see and references therein.)

Figure 3: billiard system and confocal conics
However, nowadays it is almost forgotten that there also exists amazing connection between Great Poncelet Theorem and continued fractions and approximation theory of the functions of the form 

$$\sqrt{X_4(x)},$$

where $X_4(x)$ denotes general polynomial of the fourth degree. This connection of continued fractions and approximations of functions of the form of square root of polynomials of fourth degree with the Poncelet configuration and GPT was indicated by Halphen [7].

![Figure 4: the Poncelet configuration](image)

Theory of continued fractions of square roots of polynomials of degree up to four $\sqrt{X_4(x)}$ started with Abel and Jacobi. Meeting problems with very complicated algebraic formulae in algorithm, Jacobi [8] turned to an approach based on elliptic function theory. Further development of that approach has been done by Halphen [7]. Halphen studied, instead of square root of a polynomial, the following, more general, Halphen element

$$\frac{\sqrt{X_4} - \sqrt{Y_4}}{x - y},$$

where $Y_4 = X_4(y)$ is the value of the polynomial $X$ at a given point $y$.

In this paper we are going to study more general theory of continued fractions of hyperelliptic Halphen elements

$$\frac{\sqrt{X_{2g+2}} - \sqrt{Y_{2g+2}}}{x - y},$$

where $X_{2g+2}$ is a polynomial of degree $2g + 2$ and $Y_{2g+2} = X_{2g+2}(y)$. It is obviously related to the theory of functions of the hyperelliptic curve

$$\Gamma : z^2 = X_{2g+2}$$
of genus $g$. We are also going to refer to this theory as HH continued fractions. We hope that this theory will quickly find its way to concrete applications in modern technology. As a possibility we can mention development of branched, multivalued algorithms to be used in future cryptology.

2 Basic Algebraic Lemma

Given a polynomial $X$ of degree $2g+2$ in $x$. We suppose that $X$ is not a square of a polynomial. Assuming that the values of $y$ and $\epsilon$ are finite and fixed, we are going to study HH elements in a neighborhood of $\epsilon$. Then, $X$ can be considered as a polynomial of degree $2g+2$ in $s$, where $s = x - \epsilon$ is chosen as a variable in a neighborhood of $\epsilon$.

Lemma 1 [Basic Algebraic Lemma] Let $X$ be a polynomial of degree $2g+2$ in $x$ and $Y = X(y)$ its value at a given fixed point $y$. Then, there exists a unique triplet of polynomials $A, B, C$ with $\deg A = g + 1$, $\deg B = \deg C = g$ in $x$ such that

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} - C = \frac{B(x - \epsilon)^{g+1}}{\sqrt{X} + A}. \quad (3)$$

Proof. Put $s = x - \epsilon$ and $t = y - \epsilon$ and denote $X = X'(s) = \sum_{i=0}^{2g+2} p_i s^i$ and

$$A = \sum_{i=0}^{g+1} A_i s^i, \quad B = \sum_{i=0}^{g} B_i s^i, \quad C = \sum_{i=0}^{g} C_i s^i.$$  

We are going to determine the coefficients $A_i, B_i, C_i$ in the way that the equation (3) is satisfied. The last equation can be separated into two equations taking into account that $\sqrt{X}$ is irrational:

$$A - \sqrt{Y} - C(s - t) = 0;$$

$$X - A\sqrt{Y} - AC(s - t) - Bs^{g+1}(s - t) = 0. \quad (4)$$

We also add the next equation which is a consequence of the last two:

$$X - A^2 = Bs^{g+1}(s - t). \quad (5)$$

We obtain from equation (3) for $s = 0$

$$C_0 = \frac{\sqrt{Y} - \sqrt{p_0}}{t}$$

and again for $s = 0$ from equation (4)

$$A_0 = -C_0 t + \sqrt{Y} = \sqrt{p_0}.$$
Then we calculate $A_i$, $i = 1, \ldots, g$ from equation (5). From the first of equations (4), by putting $s = t$, we get

$$A(t) = \sqrt{Y}.$$  

From the first equation (4) for $s = t$ we see that $\deg C = \deg A - 1$ and we compute all the coefficients $C_i$ as functions of the coefficients of the polynomial $A$. For example, $C_g = A_{g+1}$.

The last step is to compute the polynomial $B$. Observe that the coefficients $A_0, \ldots A_g$ of the polynomial $A$ are obtained in a manner such that

$$s^{g+1} | X - A^2.$$  

The leading coefficient $A_{g+1}$ is such that $X - A^2 = 0$ for $s = t$. Thus, there is a unique polynomial $B$, $\deg B = g$ such that

$$X - A^2 = Bs^{g+1}(s - t).$$

□

3 Hyperelliptic Halphen-type continued fractions (HH c.f.)

Let us start with the factorization of the polynomial $B$:

$$B(s) = B_g \prod_{i=1}^{g} (s - t_{i1})$$

and denote $A(t_{11}) = -\sqrt{Y_{11}}$. Then we have

$$\frac{A + \sqrt{X}}{s - t_{11}} = P_A^{g}(t_{11}, s) + \frac{\sqrt{X} - \sqrt{Y_{11}}}{x - y_{11}},$$

with certain polynomial $P_A^{g}$ of degree $g$ in $s$ and with coefficients depending on the coefficients of $A$ and $t_{11}$.

Denote

$$Q_0 = \frac{\sqrt{X} - \sqrt{Y}}{x - y} - C.$$  

Then we have

$$Q_0 = \frac{B_g \prod_{i=1, j \neq i}^{g} (s - t_{i1})s^{g+1}}{P_A^{g}(t_{11}, s) + \frac{\sqrt{X} - \sqrt{Y_{11}}}{x - y_{11}}}.$$  

Now, by applying the Lemma 4 we obtain the polynomials $A^{(1,i)}, B^{(1,i)}, C^{(1,i)}$ of degree $g + 1, g, g$ respectively, such that

$$\frac{\sqrt{X} - \sqrt{Y_{11}}}{x - y_{11}} - C^{(1,i)} = \frac{B^{(1,i)}(x - e)^{g+1}}{\sqrt{X} + A^{(1,i)}}.$$
Denote
$$\alpha_1^{(i)} := P_A^g(t_i, s), \quad \beta_1^{(i)} := B_g \prod_{j=1, j \neq i}^g (s - t_j^1)s^{g+1},$$
and introduce $Q_1^{(i)}$ by the equation
$$Q_0 = \frac{\beta_1^{(i)}}{\alpha_1^{(i)} + Q_1^{(i)}}.$$

Observe that $\deg \alpha_1^{(i)} = g$ and $\deg \beta_1^{(i)} = 2g$.

Now, one can go further, step by step: to factorize $B^{(1,i)}$, to choose one of its zeroes $t_2^1$ and to denote by $B^{i,j} := B^{(1,i)}/(s - t_2^1)$. Further, we denote
$$\alpha_2^{(i,j)} := P_A^{j-1,j}(t_2^1, s), \quad \beta_2^{(i,j)} := B^{i,j}s^{g+1},$$
and calculate $Q_2^{(i,j)}$ from the equation
$$Q_1^{(i)} = \frac{\beta_2^{(i,j)}}{\alpha_2^{(i,j)} + Q_2^{(i,j)}}.$$

Thus we have
$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = C + \frac{\beta_1^{(i)}}{\alpha_1^{(i)} + \frac{\beta_2^{(i,j)}}{\alpha_2^{(i,j)} + Q_2^{(i,j)}}}.$$

Following the same scheme, in the $i$-th step we introduce polynomials
$$A^{(i,j_1,\ldots,j_i)}, \quad B^{(i,j_1,\ldots,j_i)}, \quad C^{(i,j_1,\ldots,j_i)}$$
of degrees $g + 1$, $g$, $g$ respectively. They satisfy the equations

\begin{equation}
\begin{aligned}
A^{(i,j_1,\ldots,j_i)} &= C^{(i,j_1,\ldots,j_i)}(s - t_1^{j_1,\ldots,j_i}) + \sqrt{Y_1^{j_1,\ldots,j_i}}, \\
X - A^{(i,j_1,\ldots,j_i)^2} &= B^{(i,j_1,\ldots,j_i)}s^{g+1}(s - t_1^{j_1,\ldots,j_i}).
\end{aligned}
\end{equation}

We see that in the case $g > 1$ the formulae of the $i + 1$-th step depend on the choice of one of the roots of the polynomial $B^{(i)}$ and of the choices from the previous steps. To avoid abuse of notations we are going to omit many times in future formulae the indexes $j_1, \ldots, j_i$, which indicate the choices done in the first $i$ steps, although we assume all the time that the choice has been done.

According to our notation we have
$$s - t_i | B^{(i-1)}$$
and
$$B^{(i-1)} = \frac{\beta_1^{(i-1)}}{s^{g+1}}(s - t_i)$$
or
$$B^{(i)} = \hat{\beta}_{i+1}(s - t_{i+1}),$$
where \( \hat{\beta}_i = \beta_i / s^{g+1} \). From the equations (6) we have

\[
X - A^{(i-1)^2} = \hat{\beta}_i (s - t_{i-1}) s^{g+1} (s - t_i),
\]
\[
X - A^{(i)^2} = \hat{\beta}_{i+1} (s - t_{i+1}) s^{g+1} (s - t_i)
\]
together with

\[
A^{(i)}(t_i) = \sqrt{Y_i},
\]
\[
A^{(i-1)}(t_i) = -\sqrt{Y_i}.
\]

We introduce \( \lambda_i \) by the relation

\[
A^{(i)}(s) = \sqrt{p_0} \lambda_i.
\]

**Theorem 1** If \( \lambda_i \) is fixed, then \( t_i \) and \( \{t^{(1)}_{i+1}, \ldots, t^{(g)}_{i+1}\} \) are the roots of polynomial equation of degree \( g + 1 \) in \( s \)

\[
Q_X(\lambda_i, s) = 0.
\]

The proof follows from the equations (7). On the same way we get

**Theorem 2** If \( t_i \) is fixed, then \( \lambda_i \) and \( \lambda_{i-1} \) are the roots of the polynomial equation of degree 2 in \( \lambda \):

\[
Q_X(\lambda_{i-1}, t_i) = 0, \quad Q_X(\lambda_i, t_i) = 0.
\]

One can easily calculate

\[
B_i^{(s)} = p_{2g+2} - p_0 \lambda_i^2,
\]
thus

\[
\beta_{i+1} = (p_{2g+2} - p_0 \lambda_i^2) \prod_{j=2}^{g} (s - t_j^i) s^{g+1}.
\]

We also have

\[
A^{(i)} = \sqrt{p_0} \left( 1 + q_1 s + \cdots + \lambda_i s^{g+1} \right),
\]
\[
C^{(i)} = \sqrt{p_0} \left( q_1 + \cdots + \lambda_i (s^g + s^{g-1} t_i + \cdots + t_i^g) \right),
\]
and

\[
\alpha_i = \sqrt{p_0} \left( 2q_1 + \cdots + (\lambda_{i-1} + \lambda_i) (s^g + s^{g-1} t_i + \cdots + t_i^g) \right).
\]

In the last equation the sum \( \lambda_{i-1} + \lambda_i \) can be expressed through the coefficients of the polynomial \( Q_X(\lambda, t_i) \) as a polynomial of the second degree in \( \lambda \) according to the Theorem 2.
4 Basic examples: genus one case

The genus one case, or the elliptic case, has been studied by Halphen. Here we reproduce some of his formulae. See [7] for more details. The elliptic curve is given by a polynomial \( X \) of degree 4, in variable \( s \) in a neighborhood of \( \epsilon \)

\[
X = S(s) = \sum_{i=0}^{4} p_i s^i
\]

The development around the point \( \epsilon \) of its square root has the form

\[
\sqrt{X} = \sqrt{p_0} (1 + q_1 s + q_2 s^2 + q_3 s^3 + \ldots ),
\]

with relations between \( q \)'s and \( p \)'s:

\[
q_1 = \frac{p_1}{2p_0}, \\
q_2 = \frac{1}{8p_0^2} (4p_0 p_2 - p_1^2), \\
q_3 = \frac{1}{4p_0^2} \left( 2p_0 p_3 - p_0 p_1 p_2 + \frac{p_1^3}{4} \right), \\
q_4 = \frac{1}{2p_0} (p_4 - 2q_1 q_3 p_0 - q_2^2 p_0).
\]

Here we have

\[
\frac{X}{p_0} = (1 + q_1 s + q_2 s^2)^2 + 2q_3 s^3 + 2(q_1 q_3 + q_4) s^4.
\]

From the Basic Algebraic Lemma, applied in the case \( g = 1 \), we get the polynomials \( A = A_0 + A_1 s + A_2 s^2 \), \( B = B_0 + B_1 s \), \( C = C_0 + C_1 s \) which satisfy

\[
A - \sqrt{Y} - C(s - t) = 0; \\
X - A \sqrt{Y} - AC(s - t) - Bs^2(s - t) = 0; \\
X - A^2 = Bs^2(s - t).
\]  

From the equations (8) one gets the formulae for the polynomials \( A, B, C \):

\[
A_0 = \sqrt{p_0}, \quad A_1 = q_1 \sqrt{p_0}, \quad A_2 = \sqrt{\frac{Y}{t^2} - \left(1 + q_1 t\right) \sqrt{p_0}}, \\
C_0 = \frac{\sqrt{Y} - \sqrt{p_0}}{t}, \quad C_1 = A_2, \\
B_0 = \frac{2}{t^3} \left( \sqrt{Y} - \sqrt{p_0} (1 + q_1 t + q_2 t^2) \right), \\
B_1 = \frac{2}{t^4} \left( (1 + q_1 t) \sqrt{Y} - \sqrt{p_0} (1 + 2q_1 t + (q_1^2 + q_2) t^2 + (q_1 q_2 + q_3) t^3) \right).
\]
If we denote 

\[ P_A^{(1)}(t, s) := A_1 + A_2(s + t), \]

then we have 

\[ Q_0 = \frac{B_1 s^2}{P_A^{(1)}(t_1, s) + \sqrt{X - Y}}. \]

and, step by step

\[ A^{(i)}(t_i) = \sqrt{Y_i - C^{(i)}}, \]

\[ X - A^{(i)} = B^{(i)} s^2(s - t_i), \]

where

\[ B^{(i-1)}(t_i) = 0, \]

\[ A^{(i)}(t_{i+1}) = -\sqrt{Y_{i+1}}. \]

Finally, one gets

\[ \beta_i = B_1^{(i-1)} s^2, \]

\[ \alpha_i = P_A^{(1)}(t_{i-1}, s) + C^{(i)}. \]

From the equation (9) we get

\[ X - A^{(i-1)} = m s^2(s - t_{i-1})(s - t_i); \]

\[ X - A^{(i+1)} = n s^2(s - t_i)(s - t_{i+1}); \]

\[ A^{(i)}(t_i) = \sqrt{Y_i}; \]

\[ A^{(i-1)}(t_i) = -\sqrt{Y_i}; \]

\[ A_2^{(i)} = \sqrt{p_0 q_2}. \]

with some constants \( m, n \), and then we get

\[ \lambda_i = \frac{1}{t_i^2} \left( \sqrt{Y_i} \left( \frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1 + q_i t_i) \right) \right), \]

\[ \lambda_{i-1} = \frac{1}{t_i^2} \left( -\sqrt{Y_i} \left( \frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1 + q_i t_i) \right) \right). \]

From the last equations one gets

**Proposition 1** If \( \lambda_i \) is fixed, then \( t_i, t_{i+1} \) are roots of the polynomial \( Q_X(\lambda_i, s) \) quadratic in \( s \):

\[ Q_X(\lambda_i, s) := (p_4 - p_0 \lambda_i^2) s^2 + (p_3 - p_1 \lambda_i) s + 2p_0(q_2 - \lambda_i) = 0. \]

**Corollary 1** Product of two consecutive \( t_i \) and \( t_{i+1} \) is

\[ t_i t_{i+1} = \frac{2p_0(\lambda_i - q_2)}{p_0 \lambda_i^2 - p_4}, \]
and the sum is

\[ t_i + t_{i+1} = \frac{p_1 \lambda_i - p_3}{p_4 - p_0} \lambda_i. \]

The last Proposition can be reformulated giving relation between two consecutive \( \lambda_{i-1} \) and \( \lambda_i \):

**Proposition 2** If \( t_i \) is fixed, then \( \lambda_{i-1}, \lambda_i \) are solutions of quadratic equation:

\[ \lambda^2 (p_0 t_i^2) + \lambda (p_1 t_i + 2p_0) - (p_4 t_i^2 + p_3 t_i + 2p_0 q_2) = 0. \]

For the normal form of the elliptic HH c. f. we consider the case where

\[ \alpha'_i = 1 + u_is, \quad \beta'_i = v_is^2. \]

Then we have

\[ t_i = -\frac{1}{q_1 + u_i}, \quad \lambda_i = q_2 - 2v_{i+1}. \]

The recurrent relations are given with

\[ u_i + u_{i-1} = -q_1 + \frac{q_2}{2v_i}, \]
\[ v_i + u_i u_{i-1} = q_2 + \frac{q_4}{2v_i}. \] (13)

The second set of recurrent equations is done by

\[ v_i + v_{i+1} = q_2 + q_1 u_i + u_i^2, \]
\[ 2v_i v_{i+1} = -q_4 + q_3 u_i. \] (14)

5 Basic examples: genus two case

5.1 Notation

We start with a polynomial \( X \) of degree 6 in \( x \) and rewrite it as a polynomial in \( s \) in a neighborhood of \( \epsilon \)

\[ X = S(s) = \sum_{i=0}^{6} p_i s^i \]

and its square root developed around \( \epsilon \) as

\[ \sqrt{X} = \sqrt{p_0}(1 + q_1 s + q_2 s^2 + q_3 s^3 + q_4 s^4 + q_5 s^5 + q_6 s^6 + q_7 s^7 + \ldots). \]
Then, the relations between coefficients \( p_i \) and \( q_j \) are
\[
\begin{align*}
p_1 &= 2p_0q_1, \\
p_2 &= p_0(2q_2 + q_1^2), \\
p_3 &= 2p_0(q_3 + q_1q_2), \\
p_4 &= p_0(2q_4 + q_2^2 + 2q_1q_3), \\
p_5 &= p_0(q_5 + 2q_3 + q_1q_4), \\
p_6 &= p_0(2q_6 + 2q_1q_5 + 2q_2q_4 + q_3^2),
\end{align*}
\]
with relations between \( q_i \) such as:
\[
0 = q_7 + 2q_1q_6 + 2q_2q_5 + 2q_3q_4.
\]
Conversely, \( q_i \)’s can be expressed through \( p_i \)’s:
\[
\begin{align*}
q_1 &= \frac{p_1}{2p_0}, \\
q_2 &= \frac{1}{2p_0} \left( p_2p_0 - \frac{p_1^2}{4} \right), \\
q_3 &= \frac{1}{2p_0} \left( p_3p_0^2 - \frac{p_1p_2p_0}{2} + \frac{p_1^3}{8} \right), \\
q_4 &= \frac{1}{2p_0} \left( p_4p_0^3 - \frac{4p_2p_0^2 - p_1^2}{8} \right), \\
q_5 &= \frac{p_5}{2p_0} - q_2q_3 - q_1q_4, \\
q_6 &= \frac{1}{2p_0} (p_6 - 2q_1q_5p_0 - 2q_3q_4p - 0 - q_3^2p_0).
\end{align*}
\]
The initial polynomial \( X \) can be expressed through \( q_i \)’s:
\[
X_{p_0} = (1 + q_1s + q_2s^2 + q_3s^3)^2 + 2q_4s^4 + 2(q_1q_4 + q_5)s^5 + 2(q_1q_5 + q_2q_4 + q_6)s^6.
\]

5.2 The case of the Basic Algebraic Lemma

We are going to determine polynomials \( A, B, C \) of degree \( \deg A = 3, \ \deg B = \deg C = 2 \). Denote \( A = A(s) = A_0 + A_1s + A_2s^2 + A_3s^3, \ B = B(s) = B_0 + B_1s + B_2s^2, \ C = C(s) = C_0 + C_1s + C_2s^2 \). Then the equations (1) and (2) become
\[
\begin{align*}
A - \sqrt{Y} - C(s - t) &= 0; \\
X - A\sqrt{Y} - AC(s - t) - Bs^3(s - t) &= 0; \\
X - A^2 &= Bs^3(s - t).
\end{align*}
\]
We obtain for \( s = 0 \)
\[
C_0 = \frac{\sqrt{Y} - \sqrt{p_0}}{t}, \quad A_0 = \sqrt{p_0}.
\]
Then we calculate $A_i, i = 1, 2$ from the last equation of (15) by comparing polynomials $X$ and $A$ term by term up to the second degree:

$$A_1 = \frac{p_1}{2\sqrt{p_0}}, \quad A_2 = \frac{1}{2\sqrt{p_0}} \frac{4p_2p_0 - p_1^2}{4p_0},$$

thus

$$A = \sqrt{p_0}(1 + q_1s + q_2s^2 + \lambda_3 s^3).$$

From the relation $A(t) = \sqrt{Y}$ we get

$$A_3 = \frac{1}{t^3} \left[ \sqrt{Y} - \left( \sqrt{p_0} + \frac{p_1}{2\sqrt{p_0}} t + \frac{4p_0p_2 - p_1^2}{8p_0^{3/2}} t^2 \right) \right].$$

The coefficients of $C$ are $C_1 = A_2$ and $C_2 = A_3$. The coefficients of the polynomial $B$ are

$$B_2 = p_6 - A_3^2,$$
$$B_1 = B_2t + p_5 - 2A_2A_3,$$
$$B_0 = B_1t + p_4 - (2A_1A_3 + A_2^2).$$

We factorize it

$$B = B_2(s - t_1)(s - t_1^0),$$

and denote

$$A(t_1^0) = -\sqrt{Y_{1^0}}, \quad A(t_1) = -\sqrt{Y_1}.$$ 

Now, we have

$$\frac{A + \sqrt{X}}{s - t_1^0} = \frac{A + \sqrt{Y_{1^0}}}{s - t_1^0} + \frac{\sqrt{X} - \sqrt{Y_{1^0}}}{x - y_{1^0}^0}$$

$$= \frac{A(s) - A(t_1^0)}{s - t_1^0} + \frac{\sqrt{X} - \sqrt{Y_{1^0}}}{x - y_{1^0}^0}$$

$$= A_1 + A_2(s + t_1^0) + A_3(s^2 + st_1^0 + t_1^0) + \frac{\sqrt{X} - \sqrt{Y_{1^0}}}{x - y_{1^0}^0}.$$ 

Denote

$$P_A^{(2)}(t, s) := A_1 + A_2(s + t) + A_3(s^2 + st + t^2).$$

Then we have finally

$$Q_0 = \frac{B_2(s - t_1^0)^3}{P_A^{(2)}(t_1^0, s) + \frac{\sqrt{X} - \sqrt{Y_{1^0}}}{x - y_{1^0}^0}}.$$ 

Step by step we get

$$A^{(i)} = \sqrt{Y_i} - C^{(i)}(s - t_i);$$
$$X - A^{(i)2} = B^{(i)} s^3(s - t_i),$$

(16)
where
\[ B^{(i-1)}(t_i) = 0, \quad t_i := t_i^0, \]
\[ A^{(i)}(t_{i+1}) = -\sqrt{Y_{i+1}}. \]

Now, we have
\[ \beta_i = B_2^{(i-1)}(s - t_i^1)s^3, \]
\[ \alpha_i = P_{A^{(i-1)}}^{(2)}(t_i, s) + C^{(i)}. \]

We can represent the HH continued fraction in the following manner
\[ \sqrt{X - \sqrt{Y}} \xrightarrow{x-y} = C + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \cdots + \frac{\beta_i}{\alpha_i + Q_i}, \]
where
\[ Q_i = \sqrt{X - \sqrt{Y_i}} - C^{(i)} = \frac{B^{(i)}s^3}{\sqrt{X + A^{(i)}}} \]
and
\[ Q_1 = -\frac{\beta_{i+1}}{\alpha_{i+1} + Q_{i+1}}. \]

### 5.3 Relations between \( \lambda_i \) and \( t_i \)

From the equation (18) we get
\[ X - A^{(i-1)} = B_2^{(i-1)}(s - t_i^1)s^3(s - t_{i-1})(s - t_i); \]
\[ X - A^{(i)} = B_2^{(i)}(s - t_{i+1})s^3(s - t_{i})(s - t_{i+1}); \]
\[ A^{(i)}(t_i) = \sqrt{Y_i}; \]
\[ A^{(i-1)}(t_i) = -\sqrt{Y_i}; \]
\[ A^{(i)} = \sqrt{p_0}\lambda_i. \]

From equations (18) we get
\[ \lambda_i = \frac{1}{t_i^2} \left( \frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1 + q_1t_i + q_2t_i^2) \right), \]
\[ \lambda_{i-1} = \frac{1}{t_i^2} \left( -\frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1 + q_1t_i + q_2t_i^2) \right), \]
and thus
\[ t_i^3\sqrt{Y_{i+1} + q_3^1\sqrt{Y_i}} = \sqrt{p_0(t_{i+1} - t_i)[t_{i+1}^2 + t_{i+1}t_i + q_1t_it_{i+1}(t_{i+1} + t_i) + q_2t_i^2t_{i+1}]]. \]

From equations (18) we also get
\[ \lambda_{i-1} + \lambda_i = -\frac{2}{t_i^2}(1 + q_1t_i + q_2t_i^2), \]
\[ \lambda_{i-1} \lambda_i = -\frac{1}{t_i^2} \left[ (1 + q_1t_i + q_2t_i^2)^2 - \frac{Y_i}{p_0} \right]. \]
Finally we have

**Proposition 3** If $\lambda_i$ is fixed, then $t_i$, $t_{i+1}$, $t_{i+1}^1$ are roots of the polynomial $Q_X(\lambda_i, s)$ of degree 3 in $s$:

$$Q_X(\lambda_i, s) := (p_6 - p_0 \lambda_i^2) s^3 + (p_5 - 2p_0 q_2 \lambda_i) s^2 + (p_4 - 2p_0 q_1 \lambda_i - q_2^2 p_0) s + (p_3 - 2p_0 \lambda_i - 2p_0 q_1 q_2) = 0.$$ 

**Corollary 2** Product of two consecutive $t_i$ and $t_{i+1}$ is

$$t_i t_{i+1} = \frac{p_3 - 2p_0 \lambda_i - 2p_0 q_1 q_2}{t_{i+1}^1 (p_6 - p_0 \lambda_i^2)}.$$ 

The last Proposition can be reformulated giving relation between two consecutive $\lambda_{i-1}$ and $\lambda_i$:

**Proposition 4** If $t_i$ is fixed, then $\lambda_{i-1}, \lambda_i$ are solutions of quadratic equation:

$$\lambda^2 (-p_0 t_i^3) + \lambda (-2p_0 q_2 t_i - 2p_0 q_1 t_i - 2p_0) + (p_6 t_i^3 + p_5 t_i^2 + (p_4 - q_2^2 p_0) t_i + p_3 - 2p_0 q_1 q_2) = 0.$$

### 5.4 Normal form of genus 2 HH c. f. Recurrent relations

Using equations (20) and (17) we get formulae for $\alpha_i$:

$$\alpha_i = \sqrt{p_0} \left( -\frac{2}{t_i} + \frac{2q_2 + \lambda_{i-1} t_i}{t_i^2} s - \frac{2}{t_i^3} (1 + q_1 t_i + q_2 t_i^2) s^2 \right).$$

Given HH c. f. with $\alpha_i, \beta_i$, it can be transformed to the equivalent with

$$\alpha'_i = c_i \alpha_i, \quad \beta'_i = c_{i-1} c_i \beta_i.$$ 

Here we chose coefficients

$$c_i = \frac{-t_i}{2\sqrt{p_0}}$$

and get

$$\alpha'_i = 1 + w_i s + u_i s^2,$$

$$\beta'_i = \frac{s - t_i^1}{t_i^1} s^3,$$ 

where

$$u_i = \frac{1 + q_1 t_i + q_2 t_i^2}{t_i^4},$$

$$w_i = -(q_2 t_i + \lambda_{i-1} t_i^2),$$

$$v_i = -\frac{\lambda_{i-1}}{2} + q_3.$$ 

16
We are going to call *normal form* of the given HH c. f. the form given with equation \(21\).

From the equations \(21\) we get
\[
\lambda_i - 1 = -2v_i + q_3
\]  \(23\)
and
\[
(q_2 - u_i)t_i^2 + q_1t_i + 1 = 0,
\]
\[
\frac{\lambda_i - 1}{2}t_i^2 + q_2t_i + w_i = 0.
\]  \(24\)
From the equations \(24\) we get
\[
t_i = \frac{(u_i - q_2)w_i + (v_i - \frac{q_3}{2})}{q_2(q_2 - u_i) - q_1(\frac{q_3}{2} - v_i)}.
\]
From Proposition 4 and equation \(24\) we get
\[
\lambda^2\left(-\frac{t_i}{2}\right) - u_i\lambda + [q_6t_i + q_5(q_1t_i + 1) + q_3u_i + q_4u_it_i] = 0,
\]
having two zeroes \(\lambda_{i-1}\) and \(\lambda_i\). From the last equation we get
\[
t_i = \frac{u_i(\lambda - q_3) - q_5}{-\frac{\lambda^2}{2} + q_6 + q_5q_1 + q_4u_i}.
\]
By using the second of equations \(24\) and equating the right sides of the last equation for \(\lambda_{i-1}\) and \(\lambda_i\) we get

**Lemma 2** The relation between \(u_i\) and \(v_i, v_{i+1}\) is
\[
-\frac{1}{2}(2u_iv_i + v_5)(q_5 - 2v_i)^2 + 2u_i(q_6 + q_5q_1 + q_4u_i)(v_{i+1} - v_i) + \\
+ \frac{1}{2}(2u_iv_i + q_5)(q_5 - 2v_{i+1})^2 = 0.
\]  \(25\)
From the last Lemma, we get

**Corollary 3** If \(v_i \neq v_{i+1}\) then
\[
0 = q_3^2u_i - 4u_iv_iv_{i+1} - 2q_5(v_i + v_{i+1}) + 2q_5q_3 - 2u_i(q_6 + q_5q_1 + q_4u_i).
\]
From the equations \(20\), \(23\), \(22\) we get

**Proposition 5** The recurrence equations connecting \(v_i\) and \(v_{i+1}\) for fixed \(u_i\) and \(t_i\) are:
\[
\begin{align*}
v_i + v_{i+1} &= \frac{u_i}{t_i} + q_3, \\
4v_i^2v_{i+1} &= (-2q_6 - 2q_1q_5 - 2q_4u_i) + q_5^2 - 2q_5.
\end{align*}
\]  \(26\)
We rewrite the polynomial \( Q_X(\lambda_i, s) \) in the form
\[
Q_X(\lambda_i, s) = Q_3s^3 + Q_2s^2 + Q_1s + Q_0,
\]
where
\[
Q_3 = q_6 + q_1q_5 + q_4q_2 + \frac{q_2^2}{2} - \frac{\lambda_i}{2},
\]
\[
Q_2 = q_5 + q_1q_4 + q_2(q_3 - \lambda_i),
\]
\[
Q_1 = q_4 + q_1(q_3 - \lambda_i),
\]
\[
Q_0 = q_3 - \lambda_i.
\]
Summing the relations \( Q_X(\lambda_i, t_i) = 0 \) and \( Q_X(\lambda_i, t_{i+1}) = 0 \) we get
\[
(u_i + u_{i+1})(q_3 - \lambda_i) = (t_i + t_{i+1}) \left( \frac{\lambda_i^2}{2} - q_6 - q_5q_1 - q_4q_2 \right) - q_4 \frac{t_i + t_{i+1}}{t_it_{i+1}} - 2q_5 - 2q_1q_4.
\]
From the last equation by using the Viete formulae for the polynomial \( Q_X(s) \) and equation (23) we get

**Proposition 6**

\[
u_i + u_{i+1} = \frac{1}{2} v_{i+1} \left( \frac{Q_2}{Q_3} - t_{i+1}^1 \right) \left( \frac{(-2v_{i+1} + q_3)^2}{2} - q_6 - q_5q_1 - q_4q_2 \right) - q_4 \left( \frac{Q_2}{Q_3} - t_{i+1}^1 \right) \frac{Q_3}{Q_0} t_{i+1}^1 - 2q_5 - 2q_1q_4.
\]

(27)

### 6 Periodicity and symmetry

#### 6.1 Definition and the first properties

According to the Theorem \( \Box \) in the case
\[
t_h = t_k
\]
for some \( h, k \) there are two possibilities:
\[
(I) \quad \lambda_{h-1} = \lambda_{k-1}, \quad \lambda_h = \lambda_k;
\]
\[
(II) \quad \lambda_{h-1} = \lambda_k, \quad \lambda_h = \lambda_{k-1}.
\]
The first possibility leads to periodicity:
\[
t_{h+s} = t_{k+s}, \quad \lambda_{h+s} = \lambda_{k+s}
\]
for any \( s \) and with appropriate choice of roots. If \( p = h - k \) and \( r \equiv s(\text{mod}p) \) then
\[
\alpha_r = \alpha_s, \quad \beta_r = \beta_s.
\]
The second possibility leads to symmetry:

\[ t_{h+s} = t_{k-s}, \quad \lambda_{h+s} = \lambda_{k-s-1} \]

for any \( s \). More precisely, we introduce

**Definition 1**

- (i) If \( h + k = 2n \) we say that HH c. f. is **even symmetric** with
  \[ \alpha_{n-i} = \alpha_{n+i}, \quad \beta_{n-i} = \beta_{n+i-1}. \]
  for any \( i \) and with \( \alpha_n \) as the centre of symmetry.

- (ii) If \( h + k = 2n + 1 \) we say that HH c. f. is **odd symmetric** with
  \[ \alpha_{n-i} = \alpha_{n+i-1}, \quad \beta_{n-i} = \beta_{n+i}. \]
  for any \( i \) and with \( \beta_n \) as the centre of symmetry.

Now we can formulate some initial properties connecting periodicity and symmetry.

**Proposition 7**

- (A) If a HH c. f. is periodic with the period of \( 2r \) and even symmetric with respect \( \alpha_n \), then it is also even symmetric with respect \( \alpha_{n+r} \).

- (B) If a HH c. f. is periodic with the period of \( 2r \) and odd symmetric with respect \( \beta_n \), then it is also odd symmetric with respect \( \beta_{n+r} \).

- (C) If a HH c. f. is periodic with the period of \( 2r - 1 \) and even symmetric with respect \( \alpha_n \), then it is also odd symmetric with respect \( \beta_{n+r} \). The converse is also true.

**Proposition 8** If a HH c. f. is double symmetric, then it is periodic. Moreover:

- (A) If a HH c. f. is even symmetric with respect \( \alpha_m \) and \( \alpha_n \), \( n < m \) then the period is \( 2(n - m) \).

- (B) If a HH c. f. is odd symmetric with respect \( \beta_m \) and \( \beta_n \), \( n < m \) then the period is \( 2(m - n) \).

- (C) If a HH c. f. is even symmetric with respect \( \alpha_n \) and \( \beta_m \), then the period is \( 2(n - m) + 1 \) in the case \( m \leq n \) and the period is \( 2(m - n) - 1 \) when \( m > n \).

**Observations:**

- (i) A HH c. f. can be at the same time even symmetric and odd symmetric.

- (ii) If \( \lambda_i = \lambda_{i-1} \) then the symmetry is even; if \( t_i = t_{i+1} \) then the symmetry is odd.
6.2 Further results

**Theorem 3** An H. H. c. f. is even-symmetric with the central parameter \( y \) if \( X(y) = 0 \).

The proof follows from the fact that even-symmetry is equivalent to the condition \( \lambda_p = \lambda_{p-1} \), which is equivalent to the condition \( Y_p = 0 \). For odd-symmetry let us start with the example of genus two case. From the relations

\[
Q_X(\lambda, s) = 0, \quad \frac{d}{ds} Q_X(\lambda, s) = 0
\]

we get the system

\[
3Q_3s^2 + 2Q_2s + Q_1 = 0 \\
Q_2s^2 + 2Q_1s + 3Q_0 = 0.
\]

From the last system we get

\[
v_{i+1} = -\frac{s(q_5 + q_1q_4)s + q_4}{2q_2s^2 + 4q_1s + 6}
\]

or, equivalently

\[
\lambda_i = \frac{p_5s^2 + 2(p_4 - q_2^2p_0)s + 3(p_3 - 2p_0q_1q_2)}{2p_0q_2s^2 + 4p_0q_2s + 6p_0}.
\]

By replacing any of the last two relations in the first equation of the relations (29) we get the equation of the sixth degree in \( s \). On the other hand, from the relations (29) we get

\[
s = \frac{9Q_0Q_3 - Q_1Q_2}{2Q_2^2 - 6Q_1Q_3}
\]

Now, by replacing the last formula in the first equation of the relations (29) we get the equation of the eight degree in \( \lambda_i \).

7 General case

7.1 Invariant approach

Now we pass to the general case, with a polynomial \( X \) of degree \( 2g + 2 \). Relation

\[
Q_X(\lambda, s) = 0
\]

defines a basic curve \( \Gamma_X \). Denote its genus as \( G \) and consider its projections \( p_1 \) to the \( \lambda \)-plane, and \( p_2 \) to the \( s \)-plane. The ramification points of the second projection we denote \( R_e \) and call them even-symmetric points of the basic curve. The ramification points of the first projection, denoted as \( R_{o+r} \), are union of
the odd-symmetric points and of the gluing points. The gluing points represent situation where some of the roots of the polynomial $B^{(i)}$ coincide. For example in genus 2 case the gluing points correspond to the condition $t_{i+1} = t'_{i+1}$. From the last theorem we get

$$\deg R_e = 2g + 2.$$  

By applying the Riemann-Hurwitz formula we have

$$2 - 2G = 4 - \deg R_e;$$
$$2 - 2g = 2(g + 1) - \deg R_{o+r}.$$  

Thus

$$\text{genus}(\Gamma_X) = G = g$$

and

$$\deg R_{o+r} = 4g.$$  

We get a birational morphism

$$f : \Gamma \to \Gamma_X$$

by the formulae

$$f : (x, s) \mapsto (t, \lambda),$$

where

$$t = x, \quad \lambda = \frac{1}{t^{g+1}} \left( \frac{s}{\sqrt{p_0}} - Q_g(t) \right),$$

$$Q_g(t) = 1 + q_1 t + \cdots + q_g t^g.$$  

The function $f$ satisfies commuting relation

$$f \circ \tau_\Gamma = \tau_{\Gamma_X} \circ f,$$

where $\tau_\Gamma$ and $\tau_{\Gamma_X}$ are naturale involutions on the hyperelliptic curves $\Gamma$ and $\Gamma_X$ respectively.

### 7.2 Multi-valued divisor dynamics

The inverse image of a value $z$ of the function $\lambda$ is a divisor of degree $g + 1$:

$$\lambda^{-1}(z) =: D(z), \quad \deg D(z) = g + 1.$$  

Now, the HH-continued fractions development can be described as a multi-valued discrete dynamics of divisors $D_j^k = D(z_j^k)$. Here the lower index $k$ denotes the $k$-th step of the dynamics and the upper index $j$ goes in the range from 1 to $(g+1)k$ denoting branches of multivaluedness. More precisely, the discrete divisor dynamics which governs HH-continued fraction development can be described
as follows. Suppose the development has started with a point \( P_0 = P_0^1 \). It leads to the divisor

\[
D_0 := D(\lambda(P_0)) = P_0^1 + P_0^2 + \cdots + P_0^{g+1},
\]

with \( \lambda(P_0^i) = \lambda(P_0^j) \). In the next step we get \( g + 1 \) divisors of degree \( g + 1 \):

\[
D_i^j := D(\lambda(\tau_1(P_0^i))),
\]

And we continue like this. In each step, a divisor from the previous step

\[
D_{k-1}^j = P_{k-1}^{(j,1)} + \cdots + P_{k-1}^{(j,(g+1))}
\]

gives \( g + 1 \) new divisors

\[
D_{k-1}^{(j-1)(g+1)+l} := D(\lambda(\tau_1(P_{k-1}^{(j,l)}))), \quad l = 1, \ldots, g + 1.
\]

In the case of genus one, this dynamics can be traced out from the \( 2 - 2 \) - correspondence \( Q_\Gamma(\lambda, t) = 0 \). According to [7], for example, there exist constants \( a, b, c, d, T \) such that for every \( i \) we have

\[
\lambda_i = \frac{ax(u_i + T) + b}{cx(u_i + T) + d},
\]

where \( u \) is an uniformizing parameter on the elliptic curve. The involution is symmetry at the origin and since the function \( x \) is even, the two parameters corresponding to the fixed value \( \lambda_i \) are \( u_i \) and \( \bar{u}_i = -u_i - 2T \). Thus

\[
u_{i+1} = u_i + 2T,
\lambda_{i+1} = \frac{ax(u_i + 3T) + b}{cx(u_i + 3T) + d}.
\]

In the cases of higher genera the dynamics is much more complicated. Thus we pass to the consideration of generalized Jacobians.

### 7.3 Generalized Jacobians

A natural environment for consideration divisors of degree \( g + 1 \) on the curve \( \Gamma \) of genus \( g \) is generalized Jacobian of \( \Gamma \) obtained by gluing a pair of points \( Q_1, Q_2 \) of \( \Gamma \), denoted as \( \text{Jac}(\Gamma, \{Q_1, Q_2\}) \) (see [?]).

It can be understood as a set of classes of relative equivalence among the divisors on \( \Gamma \) of certain degree. Two divisors of the same degree \( D_1 \) and \( D_2 \) are called equivalent relative to the points \( Q_1, Q_2 \), if there exists a function \( f \) meromorphic on \( \Gamma \) such that \( (f) = D_1 - D_2 \) and \( f(Q_1) = f(Q_2) \).

The generalized Abel map is defined with

\[
\tilde{A}(P) = (A(P), \mu_1(P), \mu_2(P)), \quad \mu_i(P) = \exp \int_{P_0}^P \Omega_{Q_i, Q_0}, i = 1, 2,
\]
and $A(P)$ is the standard Abel map. Here $\Omega_{Q_iQ_0}$ denotes the normalized differential of the third kind, with poles at the point $Q_i$ and at arbitrary fixed point $Q_0$.

Here we consider the case where $Q_1 = +\infty$ and $Q_2 = -\infty$ on the curve $\Gamma$ of genus $g$. The divisors we are going to consider are those of degree $g + 1$ of the form $D_i = D(z_i)$ where usually $z_i = \lambda(P_i)$. The divisors of degree $g + 1$ up to the equivalence relative to the points $Q_1$ and $Q_2$ are uniquely determined by their generalized Abel image on the generalized Jacobian.

Thus, in order to measure the distance between relative classes of $D_1 = D(z_1) = D(\lambda(P_1))$ and of $D_2 = D(z_2) = D(\lambda(P_2))$ we introduce the following index

$$I(D_1, D_2) = I(z_1, z_2) = I(P_1, P_2) := \lim_{P \to +\infty} \frac{\lambda(P) - z_1}{\lambda(P) - z_2} \frac{\lambda(P) - z_2}{\lambda(P) - z_1}.$$ 

We are interested in the case $P_2 = \tau(P_1)$ and we have

$$I(P_1) := I(P_1, \tau(P_1)) = \lim_{P \to +\infty} \frac{\lambda(P) - \lambda(P_1)}{\lambda(P) - \lambda(\tau(P_1))} \frac{\lambda(\tau(P_1)) - \lambda(P_1)}{\lambda(\tau(P_1)) - \lambda(\tau(P_1))}.$$

After some calculations we get

**Lemma 3** The index of the point is given by the formula

$$I(P_1) = 1 + \frac{2\sqrt{p_{2g+2}(\lambda(\tau(P_1)) - \lambda(P_1))}}{p_{2g+2} - \sqrt{p_{2g+2}(\lambda(\tau(P_1)) - \lambda(P_1))} - \lambda(P_1)\lambda(\tau(P_1))}.$$

### 8 Irregular terms

If some of the parameters $t$ appear to be infinite or zero, we are going to call them *irregular*.

#### 8.1 $t_h$ - infinite

Suppose $t_0 = \infty$. We start from the following

$$X - A^2 = B_{g+1}.$$ 

Then, HH continued fraction is based on the relation

$$\sqrt{X} - \sqrt{p_{2g+2}g_1} = C + \frac{B_{g+1}}{\sqrt{X} + A}.$$ 

**Proposition 9** Irregular HH c. f. with $t_h = \infty$ is even symmetric if and only if $p_{2g+2} = 0$. 


8.2 \( t_h = 0 \)

Let \( t_0 = 0 \). In that case the basic relation of HH continued fraction is
\[
\frac{\sqrt{X} - \sqrt{p_0}}{x - \epsilon} - C = \frac{B(x - \epsilon)^{g+1}}{\sqrt{X} + A}.
\]
Then we have also
\[
A - \sqrt{p_0} = Cs,
\]
\[
X - A^2 = Bs^{g+2}.
\]
An HH continued fraction is developed through the following relations
\[
\sqrt{X} = A + \frac{Bs^{g+2}}{\sqrt{X} + A},
\]
\[
\sqrt{X} = A + \frac{Bs^{g+2}}{P_A^{(g)}} + \frac{\sqrt{X} - \sqrt{Y}}{x - y_1}
\]

**Proposition 10** The condition \( t_h = 0 \) is equivalent to \( v_{h+1} = \infty \). Such an HH c. f. is odd symmetric with respect to \( \beta_{h+1} \).

8.3 \( \epsilon \)-infinite

The starting relation in the case \( \epsilon = \infty \) is
\[
X - A^2 = B(x - y).
\]
Changing the variables \( x = 1/s, y = 1/t \) we come to
\[
X' - A'^2 = \frac{1}{t} Bs^{g+1}(s - t).
\]
The HH c. f. takes the form
\[
\frac{\sqrt{X} - \sqrt{Y}}{x - y} = C + \frac{B_0}{|A_1|} + \frac{B_0^{(1)}}{|A_2|} + \cdots + \frac{B_0^{(i-1)}}{|A_i + Q_i|},
\]
where \( \deg B_0^{(i)} = g - 1, \deg B_0^{(i)} = B^{(i)}/(x - t_0^i), \deg C = g, \deg A_i = g \). Appropriate HH c. f. is obtained from the last one after the change of variables.

**Lemma 4** The identity holds
\[
y^{g+1} \frac{\sqrt{X} - \sqrt{Y}}{x - y} = (x^g + x^{g-1}y + \cdots + xy^{g-1} + y^g) \sqrt{Y} + \frac{y^{g+1} \sqrt{X} - x^{g+1} \sqrt{Y}}{x - y}.
\]

**Proposition 11** The HH element \( (\sqrt{X} - \sqrt{Y})/(x - y) \) around \( x = \infty \) has the same coefficient as \( (\sqrt{X'} - \sqrt{Y'})/(s - t) \) around \( s = 0 \).
9 Remainders, continuants and approximation

We consider an HH c. f. of an element \( f \)

\[
f = C + \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} + \ldots.
\]

Together with the remainder of rank \( i \) \( Q_i \), where

\[
Q_i = \frac{B^{(i)s+1}}{\sqrt{X} + A^{(i)}},
\]

we consider the continuants \( (G_i) \) and \( (H_i) \) and the convergents \( G_i/H_i \) such that

\[
\begin{bmatrix}
G_m & G_{m-1} \\
H_m & H_{m-1}
\end{bmatrix} = T_C T_1 \cdots T_m.
\]

Here

\[
T_i = \begin{bmatrix}
\alpha_i & 1 \\
\beta_i & 0
\end{bmatrix}
\]

and

\[
T_C = \begin{bmatrix}
C & 1 \\
1 & 0
\end{bmatrix}.
\]

By taking the determinant of the above matrix relation we get

\[
G_m H_{m-1} - G_{m-1} H_m = (-1)^{m-1} \beta_1 \beta_2 \ldots \beta_m = \delta_m s^{(g+1)m}
\]

\[
\deg \delta_m = (g - 1)m.
\]

(30)

We also have the following relations

\[
f = \frac{(\alpha_m + Q_m)G_{m-1} + \beta_m G_{m-2}}{(\alpha_m + Q_m)H_{m-1} + \beta_m H_{m-2}} = \frac{G_m + Q_m G_{m-1}}{H_m + Q_m H_{m-1}}.
\]

and

\[
Q_m = -\frac{G_m - H_m f}{G_{m-1} - H_{m-1}}.
\]

**Proposition 12** The degree of the continuants is \( \deg G_m = g(m+1), \deg H_m = gm \).

Let us introduce

\[
\hat{G}_m = G_m + \frac{H_m}{s-l} \sqrt{Y}
\]

\[
\hat{H}_m = \frac{H_m}{s-l}.
\]
Then we have
\[ Q_m = -\frac{\dot{G}_m - \dot{H}_m \sqrt{X}}{G_{m-1} - H_{m-1} \sqrt{X}} \]
and also
\[ \dot{G}_m A^{(m)} + \dot{G}_{m-1} B^{(m)} s^{g+1} = \dot{H}_m X \]
\[ \dot{H}_m A^{(m)} + \dot{H}_{m-1} B^{(m)} s^{g+1} = \dot{G}_m X. \]  \( (31) \)

From the last equations we get
\[ \delta_m s^{(g+1)m} A^{(m)} = P_1(s) \]
\[ \delta_m s^{(g+1)(m+1)} B^{(m)} = P_2(s), \]
with
\[ P_1(s) := H_m H_{m-1} \frac{X Y}{x - y} - (G_m H_{m-1} - G_{m-1} H_m) \sqrt{Y} - G_m G_{m-1} (s - t) \]
\[ P_2(s) := G_m^2 (s - t) + 2G_m H_m \sqrt{Y} - H_m^2 \frac{X - Y}{x - y}. \]

**Theorem 4**

- (A) The polynomial \( G_m H_{m-1} - H_m G_{m-1} \) is of degree \( 2gm \). The first \((g + 1)m\) coefficients are zero.

- (B) The polynomial \( P_1 \) is of degree \( 2mg + g + 1 \). Its first \((g + 1)m\) coefficients are zero.

- (C) The polynomial \( P_2 \) is of degree \( 2mg + 2g + 1 \) and its \((g + 1)(m + 1)\) first coefficients are zero.

**Lemma 5** The following relations hold
\[ \frac{G_{m-1}(t_m)}{H_{m-1}(t_m)} = -A^{(m)}(t_m) = A^{(m-1)}(t_m) \]
\[ \dot{G}_m - \dot{H}_m \sqrt{X} = (-1)^{m+1} Q_0 Q_1 Q_2 \ldots Q_m. \]

**Theorem 5** If \( X(\epsilon) \neq 0 \) and \( \epsilon \neq y \), then the element
\[ \dot{G}_m - \dot{H}_m \sqrt{X} = G_m - H_m \frac{\sqrt{X} - \sqrt{Y}}{x - y} \]
has a zero of order \((g + 1)(m + 1)\) at \( s = 0 \). If \( H(0) \neq 0 \) then the differences
\[ \frac{\sqrt{X} - \sqrt{Y}}{x - y} - \frac{G_m}{H_m}, \quad \sqrt{X} - \frac{\dot{G}_m}{\dot{H}_m} \]
have developments starting with the order of \( s^{(g+1)(m+1)}. \)
Now, we consider $\sqrt{X}$ and its development as HH c. f. In that case, starting from
\[ \frac{\sqrt{X} - \sqrt{p_0}}{x - \epsilon}, \]
we have
\[ \text{deg} G_0 = g + 1, \quad H_0 = 1, \quad H_1 = \alpha_1, \quad G_1 = \alpha_1 G_0 + \beta_1 s^{g+2} \]
and
\[ G_m = \alpha_m G_{m-1} + \beta_m G_{m-2}, \]
\[ H_m = \alpha_m H_{m-1} + \beta_m H_{m-2}. \]
From the last relation we have

**Theorem 6**

- (A) The degree of the continuants in this case is $\text{deg} G_m = g(m + 1) + 1$, $\text{deg} H_m = gm$.
- (B) If $y = \epsilon$ then the development of the difference
  \[ \sqrt{X} - \frac{\hat{G}_m}{\hat{H}_m} \]
  starts with the order $s^{(g+1)(m+1)+1}$.

**Theorem 7**

- (A) The polynomial $G_m H_{m-1} - H_m G_{m-1}$ is of degree $2gm + 1$ in $s$. The first $(g + 1)m + 1$ coefficients are 0.
- (B) The polynomial $H_m H_{m-1}X - G_m G_{m-1}$ is of degree $2mg + g + 2$. Its first $(g + 1)m + 1$ coefficients are zero.
- (C) The polynomial $G_m^2 - H_m^2 X$ is of degree $2mg + 2g + 2$ and its $(g + 1)(m + 1) + 1$ coefficients are zero.

There are infinite ways to calculate $\sqrt{X}$ in the neighborhood of $\epsilon$, depending on choice of the parameter $y$. The best approximation one obtains for the choice $y = \epsilon$.

To conclude the last observation we need to check the case $y = \infty$. In this case we have:
\[ G_0 = \sqrt{p_0}(1 + q_1 s + \cdots + q_g s^g), \quad H_0 = 1, \quad G_1 = \alpha_1 G_0 + \beta_1, \quad H_1 = \alpha_1, \]
and we denote
\[ \hat{G}_m = G_m + \sqrt{a_0} H_m s^{g+1}, \quad \hat{H}_m = H_m. \]
Then we have
Proposition 13  

(A) The degree of continuants is \( \text{deg}G_m = g(m + 1) \), \( \text{deg}H_m = gm \).

(B) If \( X(\epsilon) \neq 0 \) and suppose the parameters \( t_1, \ldots, t_{m+1} \) are finite and different from zero, then

\[
\sqrt{X} - \frac{G_m}{H_m}
\]

has the development starting with order \( 2m + 2 \) in \( s \).

10 Conclusion: Polynomial growth and integrability

Due to the well-known facts, the Pade approximants of a hyperelliptic functions are unique up to the scalar factors. The approximants discussed in the previous section in the case of genus higher than 1 are neither unique nor of Padé type. By construction they have an exponential growth on the first sight. But careful analysis of their degrees comparing to their degrees of approximation done in the previous section indicates their real polynomial growth. After Veselov, we can consider a discrete multi-valued dynamics to be integrable one if it has polynomial growth instead of an exponential one. Thus, in that sense, we can say that the multi-valued discrete dynamics associated with HH-continued fractions is an integrable dynamics.

In the case of genus one, it can be seen as multi-valued discrete dynamics associated with the Euler-Chasles 2-2 correspondence, which has been studied by Veselov (see [13]) and Veselov and Buchstaber (see [3]). It would be quite interesting to consider higher genus dynamics from the point of view of \( n \)-valued groups and their actions, following Buchstaber (see [2]).

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