Quasi-spherical hydrodynamic adiabatic accretion in a Keplerian potential

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ABSTRACT

We present an exact $\gamma = 5/3$ spherical accretion model which modifies the Bondi boundary condition of $\rho \to \text{const.}$ as $r \to \infty$ to one where $\rho \to 0$ as $r \to \infty$. This change allows for simple power law solutions on the density and infall velocity fields, ranging from a cold empty free-fall condition where pressure tends to zero, to a hot hydrostatic equilibrium limit with no infall velocity. As in the case of the Bondi solution, a maximum accretion rate appears. As in the $\gamma = 5/3$ case of the Bondi solution, no sonic radius appears, this time however, because the flow is always characterised by a constant Mach number. This number equals 1 for the case of the maximum accretion rate, diverges towards the cold empty state, and becomes subsonic towards the hydrostatic equilibrium limit. Deviations from sphericity are then explored through an analytic perturbative analysis. The perturbed solution yields a rich phenomenology through density and radial velocity fields in terms of Legendre polynomials, which we begin to explore for simple angular velocity boundary conditions having zeros on the plane and pole. Infall/outflow solutions appear for the sub-sonic parameter region, closely resembling the outputs of recent numerical experiments. The strong density gradients in these cases result in significant pressure gradients which accelerate the polar outflows to many times the local escape velocity, well within the validity range of the perturbative analysis. Our results could complement our understanding of the origin of outflows in a variety of astrophysical settings surrounding infall situations, through purely hydrodynamical physics.

Key words: hydrodynamics–gravitation–accretion, accretion discs–instabilities

1 INTRODUCTION

It is well established that the infall of gas in the central gravitational potential of accreting objects is the mechanism powering a vast range of astrophysical phenomena across many orders of magnitude in mass and scale, e.g. Hawley et al. (2015). From the jets originating about YSOs, the powerful emissions of GRBs to the active galactic nuclei harbouring accreting super massive black holes in their centers, it is accretion that ultimately powers the observed outflows.

Whilst the large-scale collimation and stability of astrophysical jets is understood in terms of the interplay between angular momentum and magnetic fields (e.g. Blanford 1976, Lovelace 1976, Blanford & Payne 1982 in Newtonian scenarios or Banford & Znajek 1977 in a relativistic context), it is interesting to consider purely hydrodynamical processes as complementary ingredients towards an initial focusing of an infall into an outflow as a jet launching mechanism, or as suitable to environments less extreme than super massive black holes where magnetic fields might not be dominant.

The study of the hydrodynamics of astrophysical accretion processes started with the seminal work of Bondi (1952), where spherical symmetry is assumed about a point mass, together with the boundary condition of a density which tends to a constant value for large radii. This model, together with its relativistic extension by Michel (1972), has provided valuable insight into the magnitude of accretion rates onto central objects and the physical scalings of this process with the mass of the accretor and the densities and temperatures of the accreting material. However, the lack of exact analytical expressions for the infall velocity and the density fields make it difficult to explore deviations from sphericity for the Bondi solution.

In Hernandez et al. (2014) one of us explored analytically a highly simplified isothermal non-spherical perturbation on a free-fall unperturbed state where hydrodynamical effects were included only on the non-spherical perturbation, to show that breaking the spherical symmetry of the accreting gas can quite naturally result in bipolar infall/outflow solutions. More recently, fully self-consistent
Newtonian hydrodynamical numerical experiments probing a range of adiabatic indices by Aguayo-Ortiz et al. (2019), and Waters et al. (2020) have confirmed the appearance of such infall/outflow configurations, for a range of accretion configurations within central potentials. These results have been extended numerically to the fully relativistic regime by Tejeda et al. (2020), always ignoring the infalling gas self-gravity, but this time within a Schwarzschild metric and a fully relativistic hydrodynamical treatment.

In this paper we modify the Bondi boundary conditions to a density profile which tends to zero at infinity, which for a \( \gamma = 5/3 \) equation of state allows a simple analytic power law steady-state solution of the problem, highly reminiscent of the Bondi solution, and keeping the same physical scalings. As happens also in the Bondi solution for \( \gamma = 5/3 \), there is no sonic radius for the solution obtained, this time because the Mach number of the flow, \( M \), becomes a constant with radius, with super-sonic cases towards the empty, free fall limit, and sub-sonic ones appearing towards the hydrostatic limit solution. The above two regimes are separated by a critical maximum accretion rate which occurs at \( M = 1 \).

The above solution now allows for a perturbative treatment of non-spherical departures with polar angle for the velocity and density fields of the accreting material. We show that this perturbative solution is consistent with a wide range of infall/outflow configurations modeled through Legendre polynomials, and explore the simplest cases consistent with equatorial infall/polar outflow solutions. It is interesting that for one branch of these solutions, the large density gradients appearing towards the centre result in sufficiently steep pressure gradients to accelerate bipolar outflows to above the local escape velocities, for hot configurations close to the hydrostatic equilibrium limit of the spherically symmetric solution explored. A second branch appears where the non-spherical components of the velocity field are only relevant for small radii, focusing the totality of the infalling material into a tight range of angles about the equatorial plane.

In section 2 we present the analytic spherically symmetric accretion model for \( \gamma = 5/3 \), and explore some of its scalings and implications. This model is then used in section 3 as the basis for a perturbative analysis preserving axial symmetry, but to first order on departures from sphericity with the polar angle, which results in a range of solutions consistent with bipolar infall/outflow models. Lastly, section 4 briefly presents a particular example, and section 5 our conclusions.

### 2 SPHERICAL **\( \gamma = 5/3 \)** ACCRETION MODEL

We begin with the steady-state equations of conservation of mass and radial and angular momentum for a gas distribution in the presence of a Keplerian gravitational potential produced by a point mass \( M \). Assuming axial symmetry and using a spherical coordinate system with \( \theta \) the angle between the positive vertical direction and the position vector \( \vec{r} \) we have:

\[
\frac{1}{r^2} \frac{\partial (r^2 \rho V)}{\partial r} = - \frac{1}{r \sin \theta} \frac{\partial (\sin \theta \rho U)}{\partial \theta},
\]

\( \text{(1)} \)

where \( \rho(r, \theta), P(r, \theta), V(r, \theta) \) and \( U(r, \theta) \) are the gas density and pressure, and the \( r \) and \( \theta \) velocities, respectively e.g. Binney & Tremaine (1987). Assuming a polytropic equation of state \( P = K \rho^\gamma \) eqs. (2) and (3) become:

\[
V \frac{\partial V}{\partial r} + \frac{U}{r} \frac{\partial V}{\partial \theta} = - \frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{GM}{r^2},
\]

\( \text{(2)} \)

\[
V \frac{\partial U}{\partial r} + \frac{U}{r} \frac{\partial U}{\partial \theta} + \frac{V U}{r} = - \frac{1}{\rho} \frac{\partial P}{\partial \theta},
\]

\( \text{(3)} \)

where \( \rho(r, \theta), P(r, \theta), V(r, \theta) \) and \( U(r, \theta) \) are the gas density and pressure, and the \( r \) and \( \theta \) velocities, respectively e.g. Binney & Tremaine (1987). Assuming a polytropic equation of state \( P = K \rho^\gamma \) eqs. (2) and (3) become:

\[
V \frac{\partial V}{\partial r} + \frac{U}{r} \frac{\partial V}{\partial \theta} = - \frac{1}{\rho} \frac{\partial P}{\partial r} - \frac{GM}{r^2},
\]

\( \text{(4)} \)

\[
V \frac{\partial U}{\partial r} + \frac{U}{r} \frac{\partial U}{\partial \theta} + \frac{V U}{r} = - \frac{1}{\rho} \frac{\partial P}{\partial \theta},
\]

\( \text{(5)} \)

We now write the above equations in dimensionless form introducing the variables \( g = \rho/\bar{\rho}, V = V/c, U = U/c \), \( R = r/\bar{r}, \) where \( \bar{\rho} \) is a reference density at a certain point, \( c^2 = K \gamma \bar{\rho}^{\gamma - 1} \), the sound speed at this same reference point, and \( \bar{r} = GM/c^2 \). Equations (1), (4) and (5) now read:

\[
\frac{\partial (R^2 g V)}{\partial R} = - \frac{\bar{R}}{\sin \theta} \frac{\partial (\sin \theta g \rho U)}{\partial \theta},
\]

\( \text{(6)} \)

\[
R V \frac{\partial V}{\partial R} + U \frac{\partial V}{\partial \theta} - U^2 = - R \gamma^2 \frac{\partial \rho}{\partial R} - \frac{1}{\bar{R}},
\]

\( \text{(7)} \)

\[
R U \frac{\partial U}{\partial R} + U \frac{\partial U}{\partial \theta} + V U = - R \gamma^2 \frac{\partial \rho}{\partial \theta}.
\]

\( \text{(8)} \)

Equations (6)-(8) now define the general problem. The first step is to obtain a solution for the unperturbed state, which will be one of spherical accretion, and hence a solution to:

\[
\frac{\partial (R^2 g V)}{\partial R} = 0,
\]

\( \text{(9)} \)

\[
R V \frac{\partial V}{\partial R} = - R \gamma^2 \frac{\partial \rho}{\partial R} - \frac{1}{R},
\]

\( \text{(10)} \)

where at this point we have assumed \( g \) and \( V \) are both functions of \( R \) alone, and \( U = 0 \). Although this problem is generally treated in terms of the Bondi (1952) solution, we relax the boundary condition of \( g = \text{const.} \) as \( R \) tends to infinity to the more natural \( g = 0 \) as \( R \) tends to infinity. This allows a closed analytic solution for the natural choice of \( \gamma = 5/3 \), (e.g. Waxman 2004, who use this value to characterise GRB ejecta during the non-relativistic phase) using only radial power laws described by:

\[
V = \mathcal{V}_0 R^{-1/2},
\]

\( \text{(11)} \)

\[
g = \varrho_0 R^{-3/2},
\]

\( \text{(12)} \)

where \( \mathcal{V}_0 \) and \( \varrho_0 \) are two constants satisfying eq.(10):

\[
\mathcal{V}_0^2 = 2 - 3 \varrho_0^{2/3}.
\]

\( \text{(13)} \)
This coincides with the the Bernoulli equation with constant $\rho = 0$, as in the Bondi solution. It is clear that eq.(9) can be integrated directly to yield the dimensionless mass accretion rate as:

$$\dot{M} = 4\pi \varrho_0 \varrho_0.$$  

(14)

We see from eq.(13) that the solution will only exist for the interval $0 < \varrho_0^{2/3} < 2/3$, over which the $\varrho_0$ constant determining the amplitude of the inward velocity flow, and of which hence implies that this constant is negative, will vary within the range $2 < \varrho_0^2 < 0$. As $\varrho_0 \to 0$ pressure tends to zero and we reach an empty state in Keplerian free-fall. As $\varrho_0 \to 2/3$ we approach a dense hydrostatic equilibrium configuration where the infall velocity tends to zero. The accretion rate is clearly zero at these two limits, and has a maximum at some intermediary value of $\varrho_0^{2/3}$, which is hence seen as the sole parameter of the spherical solution. We can use eq.(14) to write eq.(13) in terms of $\varrho_0^{2/3}$ and $\dot{M}$ only:

$$\dot{M}^2 = (4\pi \varrho_0^2)^2 \left(2 - 3\varrho_0^{2/3}\right).$$  

(15)

The above equation can now be derived w.r.t. $\varrho_0$ to obtain the maximum value of the accretion rate. This will occur at $\varrho_0^{2/3} = \varrho_0^2 = 1/2$ at a value of $\dot{M}_e = \pi$. Fig. 1 gives a plot of $\dot{M}$ as a function of $\varrho_0$ in the range of interest. In physical units we obtain:

$$\dot{M}_{c,ph} = \pi \frac{\bar{\rho}}{\bar{c}^2} (GM)^2,$$  

(16)

which can be written in terms of the polytropic constant of the problem $K$ as:

$$\dot{M}_{c,ph} = \pi (GM)^2 \left(\frac{3}{5K}\right)^{3/2}.$$  

(17)

Thus, we see that in terms of physical units, the problem is fully determined by the constant $K$ which can be fixed by specifying a pair of density and sound speed values at a given radius.

We complete the study of the spherically symmetric adiabatic accretion solution with an evaluation of the Mach number of the flow, which can be derived by writing $V$ from eq. (11) in physical units, through multiplication by $\bar{c}$, and dividing by the local physical sound speed $c^2 = (5K/3)\rho^{1/3}$, with $\rho$ written as $\varrho(R) \times \bar{\rho}$ we obtain:

$$M = \varrho_0^{-1/3}.$$  

(18)

Interestingly, the mach number of the flow is a constant with radius, and is fixed once $\varrho_0$ is chosen, which in turn determines $V_0$ through eq.(13). We see that for the maximum accretion rate at $\varrho_0^{2/3} = \varrho_0^2 = 1/2$, the Mach number of the flow is precisely $M = 1$. In going towards the hotter and denser subsonic configurations towards the $\varrho_0 = 2/3$ of hydrostatic equilibrium, the Mach number drops gradually to zero, while for less dense supersonic cold models with $\varrho_0^{2/3} < 1/2$, the Mach number gradually diverges on approaching $\varrho_0^{2/3} = 0$ as the sound speed goes to zero. A plot of this is given in fig. (2). Whilst in the Bondi solution a sonic radius exists for $\gamma \neq 5/3$, and the monoatomic adiabatic value is a singular point where the sonic radius goes to zero, in the power law solutions presented here there is...
where $C$ can be simplified to:

\[ \frac{d(V_R/R^{1/2})}{dR} = -C_0^{2/3} \frac{d(\rho_0/R)}{dR}. \]

The above equation yields two conditions:

\[ \frac{d(V_R/R^{1/2})}{dR} = C_1 \frac{d(\rho_0/R)}{dR}, \]

and,

\[ \rho_\theta = - \left( \frac{C_2 \rho_0}{\theta^{2/3}} \right) V_\theta, \]

where $C_2$ is a separation constant.

Finally, equation (6) yields:

\[ \epsilon_{\theta} V_\theta \frac{d(V_R R^{1/2})}{dR} + \epsilon_{\rho_0} V_\theta \frac{d\rho_0}{dR} = -\epsilon_{\rho_0} dU_R - \frac{d(\rho_0 \sin \theta)}{d\theta}. \]

Although the above equation is not immediately susceptible to separation of variables, use of eq.(29) allows to replace $V_\theta$ for $\rho_\theta$ and hence yields the following two conditions:

\[ V_0 \frac{d\rho_\theta}{dR} - \frac{\theta^{2/3}}{C_2 V_0} \frac{d(V_R R^{1/2})}{dR} = C_1 U_R, \]

and,

\[ \frac{dU_\theta}{d\theta} + \cot \theta U_\theta = -C_1 \rho_\theta, \]

where $C_1$ is a separation constant. Notice that while eqs.(7) and (8) are solved only to first order in $\epsilon$, equation (6) will be solved exactly, as $\rho_\theta$ appears only linearly this time and no series developments are used.

We are hence left with a purely radial system in exact radial derivatives of eqs.(24), (28) and (31), and a purely angular system in exact angular derivatives of eqs.(25), (29) and (32), which we shall deal with first. Equation (25) shall be used to replace $U_\theta$ in eq.(32) for derivatives of $\rho_\theta$, yielding an equation in this last variable alone:

\[ \frac{d^2 \rho_\theta}{d\theta^2} + \cot \theta \frac{d\rho_\theta}{d\theta} - \left( \frac{C_1 C_0 \rho_0}{\theta^{2/3}} \right) \rho_\theta = 0. \]

This last equation has as solutions Legendre polynomials of $\cos \theta$ of the first and second kinds, and hence opens a very rich phenomenology capable of modeling a wide variety of density configurations, and given the constraints linking $U_\theta$ and $V_\theta$ to $\rho_\theta$, an extensive range of flow patterns. Considering for example odd polynomials, or the addition of odd and even polynomials, will yield asymmetric outflows. This last might even suggest the existence of an intrinsic component to observed morphological and energetic jet asymmetries such as the ones often observed in FR II systems e.g., Hardcastle et al. (1999) or Liu et al. (2020). In this first exploration of the problem, and guided by the simple bipolar infall/outflow geometries observed in the numerical experiments of Aguayo-Ortiz et al. (2019), Tejeda et al. (2020)
and Waters et al. (2020), we choose to consider here only solutions of the type

$$-C_1C_3V_0/\delta_0^{2/3} = 6,$$

which will yield a simple infall/outflow geometry with $U_\theta$ satisfying the boundary conditions $U_\theta(0) = U_\theta(\pi/2) = 0$, and hence,

$$\varrho_\theta = \varrho_1 (3 \cos^2 \theta - 1)_1,$$

$${}V_\theta = -\varrho_1 \left( \frac{\delta_0^{2/3}}{C_1V_0} \right) (3 \cos^2 \theta - 1),$$

$$U_\theta = \varrho_1 \left( \frac{6\delta_0^{2/3}}{C_1V_0} \right) \cos \theta \sin \theta,$$

which solve the angular system in terms of an amplitude constant $\varrho_1$.

We now turn to the radial system of eqs.(24), (28) and (29), which can be solved taking radial power laws for the unknown functions:

$$V_R = R^\alpha, U_R = R^\beta, \varrho_R = R^\delta,$$

where $\alpha$, $\beta$ and $\delta$ are real. Equation (24) becomes:

$$\delta + 1 = C_3,$$

which yields conditions:

$$\beta = \delta - 1/2,$$

and,

$$\beta + 1 = C_3.$$

Equation (28) becomes:

$$(\alpha - 1/2)R^{\alpha - 3/2} = C_2(\delta - 1)R^{\delta - 2},$$

which yields conditions:

$$\alpha = \delta - 1/2 \Rightarrow \alpha = \beta,$$

and,

$$C_2 = 1.$$

Lastly, equation (31) becomes:

$$\delta \varrho_0 R^{\delta - 1} - \frac{\delta_0^{2/3}}{V_0}(\alpha + 1/2)R^{\alpha - 1/2} = C_1 R^{\delta - 1/2}$$

which yields constraints on the power indices consistent with those obtained previously, and:

$$\delta \varrho_0 - \frac{\delta_0^{2/3}}{V_0}(\alpha + 1/2) = C_1.$$
Equation (49) has real roots only for the sub-sonic regime of $\theta_0^{2/3} > 1/2$, where one root is larger than zero, and the small interval $0 < \theta_0^{2/3} < 0.02$, where both roots are smaller than zero. The above ranges of $\theta_0$ hence define validity limits for the perturbative analysis presented. Notice that along the pole, for $\theta = 0$, a critical radius $R_J = [\gamma R_0^2/(2\epsilon_\varphi \theta_0^{2/3})]^{1/3}$ appears at which the radial velocity $V = 0$. This point can now be identified as the reference point at which $\theta = 1$. Also, for values of $\delta > 0$, $\alpha > -1/2$, it marks a critical radius beyond which the non-spherical term in eq.(51) dominates over the unperturbed one and an outflow develops. Whether this feature appears within the validity limit of the solution or not depends on the ratio of $R_J$ and the validity threshold where the non-spherical term in $\theta$ of eq.(51) overtake the spherical one for $\theta = 0$, at $R_V = (1/2\epsilon_\varphi)^{1/2}$:

$$\left(\frac{R_V}{R_J}\right)^{\delta} = \frac{\theta_0^{2/3}}{\gamma R_0^2} = \mathcal{M}^{-2},$$

and hence $R_V/R_J = \mathcal{M}^{-2/\delta}$. Since the factor $\epsilon_\varphi$ cancels from the $R_V/R_J$ ratio in eq.(53), deciding if the critical radius $R_J$ lies within the validity regime of the development or not is independent of the chosen value for $\epsilon_\varphi$, which can then be chosen as $\epsilon_\varphi = 1$ in eqs.(50)-(52). Figure (3) gives a plot of the $\alpha$ radial power-law index as a function of $\theta_0^{2/3}$ for the ‘+’ sign in eq.(49), together with the $R_V/R_J$ ratio, both for the subsonic region of parameter space. We see a mild variation of both quantities, with the exception of highly localised divergences towards $\mathcal{M} = 1$ for $\alpha$ and towards the hydrostatic equilibrium limit for $R_V/R_J$, this last point showing that $R_J$ will only lie significantly within the validity regime of the development close to the hydrostatic equilibrium limit.

Notice that for $\alpha > -1/2$, $\delta > 0$ and hence the spherically symmetric components of the velocity and density fields dominate over the non-spherical ones as $R \to 0$, leaving the accretion rate unchanged, with respect to the one calculated in the previous section which assumed spherical symmetry. The material in the bipolar outflows is hence supplied by an inflow rate above the one of the spherically symmetric solution. As seen also in the choked accretion results of Aguayo-Ortiz et al. (2019) and Tejeda et al. (2020), it is the non-spherically symmetric accretion of material above the spherically symmetric accretion rate, which results in the bipolar outflows obtained. Thus, our present results qualitatively maintain the choked accretion phenomenology described in the two references above.

From eq.(51) we see that as $\lambda_0$ tends to zero towards the hydrostatic equilibrium limit, one can obtain arbitrarily large polar ejection velocities, indeed much larger than the local escape velocities, as shown in the following section, where a particular example of the solutions obtained here will be described.

4 A PARTICULAR EXAMPLE

We finish with a particular example of a model close to the hydrostatic equilibrium limit of $\theta_0^{2/3} = 2/3$, taking $\theta_0^{2/3} = 0.66533$ which results in $\gamma R_0^2 = 0.004$ and $\delta = 2.22$, $\alpha = 1.72$, for the ‘+’ sign in equation (49). For the above choice of $\theta_0$, $R_V/R_J = 10$ and hence, the critical stagnation radius of $R_J = 0.073$ will occur well within the validity range of the perturbative analysis of the previous section of $R_J = 0.73$. © 0000 RAS, MNRAS 000, 000–000
A plot of the velocity flow of this model is shown in the left panel of fig. (4), superimposed on the logarithm of the density field shown as a colour scale plot. The equatorial infall and polar outflow structure is evident at large radii, together with the highly spherical configuration of the density field. This last however, is not exactly spherical, it is the slight oblateness present that sources the non-radial terms in radial and angular velocities.

The velocity flow of the left panel in fig. (4) is qualitatively equivalent to those which arise from the numerical simulations of Aguayo-Ortiz et al. (2019), Waters et al. (2020) and Tejeda et al. (2020), showing an inner region tending to spherical accretion in both density and velocity fields, and a more external equatorial infall and polar outflow one. A more quantitative comparison of our analytic solution to numerical simulations will be the subject of a subsequent study. In the case shown here, a small value of $V_0$ results in strong polar outflows with radial velocities which become larger than the local escape velocities well within the validity regime of the approximation, as shown in the right panel of fig. (4). Indeed, on reaching the outer validity limit at $R = R_J$ given by the vertical dotted line, ejection polar velocities are already more than 7 times larger than the local escape velocity.

It is clear that the velocity outflow described above is far from the narrow collimated jets of astrophysical objects, where other physics such as magnetic fields and rotation clearly play a role. However, the model described could present a complementary ingredient in helping to initially turn a infall into an outflow, or to describe the much wider-angle components inferred sometimes to accompany some jet phenomena (e.g. Sato et al. 2021 or Duque et al. 2022). Also, taking higher order Legendre polynomials with arbitrarily larger powers of $\cos \theta$, will naturally result in much more collimated outflows. This last option, accompanied by (multiple) conical shell outflows at various critical angles.

As can be seen from eq. (51), the radial velocity of the ejected material is a strong function of the angle, decreasing gradually at first for small angles, but then at an increasing rate, reminiscent of the structured jets inferred in some GRB studies (e.g. the Gaussian angular velocity profiles of jets treated in Lamb et al. 2021 or the structured jets of Kathirgamaraju et al. 2018), or of the transition towards an isotropic phase envisioned for GRBs, e.g. Waxman (2004).

We note that the perturbative analysis at no point assumes that the non-radial term in the radial velocity is smaller than the radial one, or that the angular velocity is intrinsically small, as only the density is treated through perturbative series expansions, which defines the outer validity radius of the approximation as $R_V = 0.732$, shown by the dotted vertical line in the right panel of fig. (4), the point where the non-spherically symmetric component of the density overtakes the spherically symmetric one. It is easy to see also that the neglected $\epsilon^2$ terms in the development will also be smaller than the included ones, as long as $R < R_V$. 

Turning to the numerical experiments of Aguayo-Ortiz et al. (2019), Waters et al. (2020) and Tejeda et al. (2020), we see that their results are highly insensitive to the actual values of $\gamma$ assumed at each point, which makes it plausible that our results could be of more general applicability than just to the particular $\gamma = 5/3$ case. It is possible that the analytic solution presented here might also be applicable, to a good accuracy level, to other equations of state, although this point would have to be addressed through detailed numerical simulations which lie beyond the scope of this first presentation of the model.

Lastly, we look at the '-' sign in eq. (49), which now fixes a value of $\delta = -2.72$, for the same choice of $g_{\delta}^{2/3} = 0.66533$ as taken above. This solution is qualitatively different to the previous for a number of reasons. First, we see from the denominator in eq. (52) that the sign of the angular velocity will change, meaning that instead of focusing the flow towards the poles, this will happen towards the equator. Also, as the radial powers of the perturbative terms are now smaller than the ones of the spherically symmetric terms in eqs. (50) and (51), the unperturbed solution will dominate for all radii larger than the stagnation radius of $R_J = 8.4$, and the non-spherical terms become relevant at small, rather than large values. Hence, the validity range is now for all radii larger than $R_V = 1.3$.

In fig. (5) we present the velocity flow of this last example, together with the logarithm of the density field; this last, as in the case of fig. (4), deviates only marginally from spherical symmetry. The dominance of the unperturbed spherical solution of eqs. (11) and (12) is evident towards the outer regions of the figure, as is the change in the inflow solution internal to $R_J = 8.4$. This transition occurs well outside of the inner validity limit of the solution at $R_V = 1.3$, denoted by the exclusion rectangle at the centre. Within $R_J = 8.4$ we see the appearance of a radial outflow of limited extent, and
of the funnelling of the accreted material onto the equatorial plane, at a strongly centrally divergent velocity.

The above results are interesting in light of recent indications of super-Eddington accretion in a variety of astrophysical scenarios. The discovery of over 200 quasars at very early redshifts \( z > 6 \) (e.g. Mortlock et al. 2011, Bailer-Jones et al. 2018) imply black hole growth rates well above the Eddington limit, e.g. Volonteri et al. (2015). At stellar scales, X-ray binaries (e.g. Okuda 2002) or ultra Luminous X-ray sources inferred to contain black holes (e.g. Winter et al. 2006) have also been interpreted as evidence for super-Eddington accretion. At a fundamental level, exceeding the Eddington limit is a natural consequence of the breaking of spherical symmetry in the accretion flow, e.g. Paczynski & Abramowicz (1982), Massonneau et al. (2022). As such, the self-consistent accretion model presented in fig.(5), does precisely that; a large-scale spherically-symmetric accretion flow is redirected into an equatorial inflow covering a very small solid angle, presenting a minimal interaction cross-section to any photon flux coming from the central object.

The examples shown above might seem to be artificially close to the hydrostatic equilibrium \( \rho_0^{2/3} = 2/3 \) limit. However, during sufficiently early phases of massive stellar collapse, for example, the system can be chosen arbitrarily close to hydrostatic equilibrium. As can be seen from eq.(51), as \( \gamma_0 \to 0 \) towards the hydrostatic equilibrium limit, the polar positive outflow at \( \theta = 0 \) diverges. Note also that the model presented is equally applicable to accretion about a black hole, provided \( R_l \) is much larger than the Schwarzschild radius of the problem, and hence the polar outflows originate within the Newtonian region.

5 CONCLUSIONS

We have presented a steady-state hydrodynamic accretion model for \( \gamma = 5/3 \) in a Keplerian potential. It differs from classical Bondi accretion in that the condition at infinity has been changed from \( \rho \to \rho_\infty \) to \( \rho \to 0 \), which allows simple power law solutions for the density and the inflow velocity in the spherically symmetric case. These solutions are characterised by having a Mach number, \( M \), which is constant for all radii, at supersonic values towards the cold, empty limit, and at subsonic ones towards the hot, dense hydrostatic equilibrium one. A maximum accretion rate appears precisely at \( M = 1 \).

Having an exact solution for the spherical accretion problem allows its use to explore deviations from sphericity with the polar angle in velocity and density fields, through an analytic perturbative analysis. This yields an ample spectrum of solutions given by Legendre polynomials for the density field, through which the velocity fields follow.

As a first exploration we show simple bipolar configurations having equatorial inflow and polar outflows, where interestingly, one branch can naturally yield polar outflows for radii larger than a critical value, with velocities well above the local escape velocity, still within the consistency region of the perturbative analysis presented. For the same solution, a second branch alters the spherically symmetric solution only at small radii, and does so by redirecting the large-scale spherical inflow into a narrow wedge along the equatorial plane.

Given the extensive range of behaviours available to the Legendre polynomials solving eq.(33), it is reasonable to expect the framework presented might be of interest in a range of astrophysical settings.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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