Clouds of string in 4D novel Einstein-Gauss-Bonnet black holes

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Abstract

Recently it has been shown that the Einstein-Gauss-Bonnet (EGB) gravity, by rescaling the coupling constant as $\alpha/(D - 4)$ and taking the limit $D \to 4$ at the level of the equations of motion, becomes nontrivially ghost-free in 4D - namely the 4D novel EGB gravity. We present an exact charged black hole solution to the theory surrounded by clouds of string (CS), and also analyse their thermodynamic properties to calculate exact expressions for the black hole mass, temperature, and entropy. Owing to the corrected black hole due to the background CS, the thermodynamic quantities have also been corrected except for the entropy, which remains unaffected by a CS background. However as a result of the 4D novel EGB theory, the Bekenstein-Hawking area law turns out to be corrected by a logarithmic area term. The heat capacity $C_+$ diverges at a critical radius $r = r_C$, where incidentally the temperature has a maximum, and the Hawking-Page transitions even in absence of the cosmological term and $C_+ > 0$ for $r_+ < r_C$ allowing the black hole to become thermodynamically stable. In addition, the smaller black holes are globally stable with positive heat capacity $C_+ > 0$ and negative free energy $F_+ < 0$. Our solution can also be identified as a 4D monopole-charged EGB black hole. We regain results of spherically symmetric black hole solutions of general relativity and that of 4D novel EGB, respectively, in the limits $\alpha \to 0$ and $a = 0$. 

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I. INTRODUCTION

Lovelock gravity is one of the natural generalizations of Einstein’s general relativity (GR) to higher dimensions (HD), introduced by David Lovelock [1], the action of which contains series of higher order curvature terms. While well motivated, higher curvature gravities introduce a number of hurdles making their investigation difficult, e.g., the equations of motion, in such theory, are fourth order or higher, and linear perturbations disclose that the graviton is a ghost. However, Lovelock theories are distinct, among general higher curvature theories, in having field equations involving not more than second derivatives of the metric and theories are free from many of the problems that affect other higher derivative gravity theories. In the Lovelock action [1], apart from the cosmological constant ($\Lambda$) and Einstein GR scalar ($\mathcal{R}$) as the first two terms, the third term is a combination of the second order curvature term, namely Gauss-Bonnet [2]. The simplest case of Lovelock theory that departs from GR is the EGB theory in which the Einstein-Hilbert action is supplemented with the quadratic curvature GB term given by

$$\mathcal{L}_{GB} = \mathcal{R}^2 - 4\mathcal{R}_{cd}\mathcal{R}^{cd} + \mathcal{R}_{cdef}\mathcal{R}^{cdef},$$

where a new constant $\alpha$ that can be identified as the inverse of the string tension. This special case of Lovelock gravity has received the most significant attention and is called Einstein-Gauss-Bonnet (EGB) gravity [2], which naturally appears in the low energy effective action of heterotic string theory [3]. The spherically symmetric static black hole solution for the EGB gravity was first obtained by Boulware and Deser [4], A cascade of subsequent interesting work analysed black hole solutions in EGB gravity [5, 6] for various sources including clouds of strings (CS) [7–11]. Some of the EGB black holes have been shown to exhibit Hawking-Page type transitions [5, 9].

As a HD member of Einstein’s GR family, EGB gravity allows us to explore several conceptual issues in a broader setup. However, the GB invariant is a topological invariant in $4D$ as its contribution to all components of Einstein’s equations are in fact proportional to $(D-4)$, and one requires $D \geq 5$ for non-trivial gravitational dynamics. However, it was shown that by rescaling the GB coupling constant as $\alpha \rightarrow \alpha/(D-4)$ the GB invariant, in the limit $D \rightarrow 4$ when finding equations, makes a non-trivial contribution to the gravitational dynamics even in $D = 4$ [12]. The theory preserves the number of degrees of freedom and remains free from Ostrogradsky instability [12]. Further, this extension of Einstein’s gravity
bypasses conditions of Lovelock’s theorem [13] and for definiteness, the effective gravity is called the 4D novel EGB theory, which admit spherically symmetric black hole solutions generalizing the Schwarzschild black holes and is also free from singularity [12]. Such black hole solutions have been considered earlier in the semi-classical Einstein’s equations with conformal anomaly [14], gravity theories with quantum corrections [15], and also recently in regularized Lovelock gravity [16].

However, the 4D gravity was formulated recently and little is known about the theory, which deserves to be understood better. Nevertheless, recently interesting measures have been taken to investigate the 4D novel EGB gravity, including generalizing the black hole solution to include electric charge in an anti-de Sitter space [17], to the radiating or nonstatic black hole solution in Ref. [18], which explores some of their properties. The generalization of these static black holes of the 4D novel EGB gravity to the axially symmetric or rotating case, Kerr-like, was also addressed [19, 20]. In particular, it is was shown that the rotating black holes solutions for the 4D novel EGB gravity can be derived starting from exact spherically symmetric spacetime by Newman and Janis [21], and they also demonstrated that the 4D novel EGB gravity is consistent with the inferred features of M87* black hole shadow. Other probes in the theory include studies of the innermost stable circular orbit (ISCO) [22], its stability and quasi-normal modes [23], relativistic star solution [24] and it’s generalisation to Lovelock gravity [25].

The main purpose of this paper is to obtain an exact spherically symmetric black hole solution, in the 4D novel EGB gravity, endowed with a clouds of string (CS). In particular, we explicitly bring out how the effect of a background CS can modify black hole solutions and their thermodynamics. We will examine how GB corrections and background CS alter the qualitative features we know from our experience with black holes in CS, e.g., we shall analyse GB corrections on thermodynamic properties of the black holes and also on local and global stability. The intense level of activity in string theory has led to the idea that static Schwarzschild black hole (point mass), may have atmospheres composed of a CS, which is the one-dimensional analogue of a cloud of dust. Further, it could describe a globular cluster with components of dark matter. Strings may have been present in the early universe for the seeding of density inhomogeneities [26, 27]. The CS for the Schwarzschild black hole was initiated by by Letelier [28] to show that the event horizon gets enlarged with radius $r_H = 2M/(1 - a)$ with $0 < a < 1$ being the string cloud parameter [28], thereby enlarging
the Schwarzschild radius of the black hole by the factor \((1 - a)^{-1}\).

II. CS FOR EGB

Lovelock demonstrated that Einstein gravity can be extended by a series of higher curvature terms with the resulting equations of motion still remaining second order \([29]\). The Lovelock theory is an extension of the GR to higher dimensions with first and second order terms, respectively, corresponding to the Ricci scalar and a combination of quadratic curvatures - Gauss-Bonnet. Action of the Einstein-Gauss-Bonnet (EGB) gravity, which is motivated by the heterotic string theory \([2, 30]\), by rescaling the GB coupling constant \(\alpha \rightarrow \alpha/(D - 4)\), yields \([12]\)

\[
\mathcal{I} = \frac{1}{2} \int d^Dx \sqrt{-g} \left[ \mathcal{R} + \frac{\alpha}{D - 4} \mathcal{L}_{GB} - F_{ab} F^{ab} \right] + \mathcal{I}_M, \tag{2}
\]

where \(\mathcal{R}\) is the scalar curvature, \(\alpha\) is the GB coupling constant constant, \(F_{ab} = \partial_a A_b - \partial_b A_a\) is the electromagnetic field tensor, and \(A_b = -Q/r dt\) is the vector potential. \(\mathcal{I}_M\) is the action of the matter source which in the present case is a string cloud \([6]\). Varying the action \((2)\), we obtain the equations of motion \([31, 32]\)

\[
G_{ab} + \alpha H_{ab} = T_{ab}, \tag{3}
\]

where \(G_{ab}\) and \(H_{ab}\), respectively, are the Einstein tensor and the Lanczos tensor:

\[
G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R,
\]

\[
H_{ab} = 2 \left[ RR_{ab} - 2 R_{ac} R^c_b - 2 R^{cd} R_{acbd} + R_a^{cde} R_{bcde} \right] - \frac{1}{2} g_{ab} L_{GB}, \tag{4}
\]

and \(T_{ab} = T_{ab} + T_{ab}^{EM}\). We wish to obtain static spherically symmetric black hole solutions of Eq. \((2)\). We assume the metric to be of the following form \([18]\)

\[
\mathfrak{d}s^2 = -f(r) \mathfrak{d}t^2 + \frac{1}{f(r)} \mathfrak{d}r^2 + r^2 d\Omega_{D-2}, \tag{5}
\]

where \(d\Omega_{D-2}\) is the metric of a \((D - 2)\)-dimensional constant curvature space and \(T_{ab}\) is the energy momentum tensor of matter that we consider as a cloud of strings. The Nambu-Goto action \([28]\) of a string evolving in spacetime is given by

\[
S_S = \int_{\Sigma} m(\gamma)^{-1/2} d\lambda^0 d\lambda^1 = \int_{\Sigma} m \left[ -\frac{1}{2} \Sigma^{\mu\nu} \Sigma_{\mu\nu} \right]^{1/2} d\lambda^0 d\lambda^1, \tag{6}
\]
where $\gamma$ is the determinant of the $\gamma_{ab}$, the $\gamma_{ab}$ is given by

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b},$$  

(7)

where $m$ is a positive constant, $\lambda^0$ and $\lambda^1$ being timelike and spacelike parameters [27]. The bivector associated with the string world sheet $\Sigma$ is given by [28]

$$\Sigma^{\mu\nu} = \epsilon^{ab} \frac{\partial x^\mu}{\partial \lambda^a} \frac{\partial x^\nu}{\partial \lambda^b},$$  

(8)

where $\epsilon^{ab}$ is the Levi-Civita tensor in two dimensions, which is anti-symmetric in $a$ and $b$ given by $\epsilon^{01} = -\epsilon^{10} = 1$. Further, since $T^{\mu\nu} = 2\partial\mathcal{L}/\partial g^{\mu\nu}$, and finally adapting to parametrization, we get

$$\partial_\mu(\sqrt{-g}\rho\Sigma^{\mu\sigma}) = 0.$$  

(9)

Here the density $\rho$ and the bivector $\Sigma_{\mu\nu}$ are the functions of $r$ only as we seek static spherically symmetric solutions. The only surviving component of the bivector $\Sigma$ is $\Sigma^{tr} = -\Sigma^{rt}$. Thus, $T_t^t = T_r^r = -\rho\Sigma^{tr}$, and from Eq. (9), we obtain $\partial_r(\sqrt{r^{D-2}T_t^t}) = 0$, which implies [28]

$$T_t^t = T_r^r = \frac{a}{r^{D-2}},$$  

(10)

for some real constant $a$. The stress-energy momentum of CS is same that for the global monopole [33]. The monopoles topological defects like cosmic strings and domain walls which were originated during the cooling phase of the early universe [33, 34], and they play significant role while investigating the black holes [35].

$T_{ab}^{EM}$ is related to the electromagnetic tensor $F_{ab}$ by

$$T_{ab}^{EM} = \frac{1}{4} \left( F_{ac}F^c_b - \frac{1}{4}g_{ab}F_{cd}F^{cd} \right),$$  

(11)

which satisfies Maxwell’s field equations.

**A. Black hole solution for the 4D novel EGB**

Many authors generalized the pioneering work of Letelier [28], for instance, in GR [36], for EGB models [37], and in Lovelock gravity [38]. We are interested in an exact black hole in the 4D novel EGB endowed with a CS. Let us consider the metric (5) with stress tensor and apply the procedure in [12]. Now, in the limit $D \to 4$, the $(r, r)$ equation of (3) reduces to

$$r^5 - 2r^3\alpha(f(r) - 1)f'(r) + r^4(f(r) - 1) + r^2\alpha(f(r) - 1)^2 = a r^4 + \frac{Q^2}{r^4},$$  

(12)
which can be integrated to give

\[ f_\pm(r) = 1 \pm \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + 4\alpha \left( \frac{2M}{r^3} - \frac{Q^2}{r^4} + \frac{a}{r^2} \right)} \right), \]  

(13)

by appropriately relating \( M \) with integrating constants. Solution (13) is an exact solution of the field equation (3) for stress-energy tensor \( T_{ab} \) which in absence of CS and charge \( a = Q = 0 \) reduces to the Glavan and Lin [12] EGB black hole solution, and for \( Q = 0 \) the charged EGB black hole due to Fernandes [17]. The two branches of the solution (13), in the limit \( \alpha \to 0 \) or large \( r \), behaves asymptotically as

\[
\begin{align*}
    f(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - a, \\
    f(r) &= 1 + \frac{2M}{r} - \frac{Q^2}{r^2} + a + \frac{r^2}{\alpha}.
\end{align*}
\]

(14)

Thus, the \(-\)ve branch corresponds to the Reissner-Nordstrom solution surrounded by the CS with positive gravitational mass and real charge, whereas the \(+\)ve branch reduces to the Reissner-Nordstrom solution with negative gravitational mass and imaginary charge. To study the general structure of the solution (13), we take the limit \( r \to \infty \) or \( M = Q = 0 \) in solution (13) to obtain

\[
\begin{align*}
    \lim_{r \to \infty} f_+(r) &= 1 + \frac{r^2}{\alpha}, \\
    \lim_{r \to \infty} f_-(r) &= 1 + a.
\end{align*}
\]

(15)

This means that the plus (+) branch of the solution (13) is asymptotically de Sitter (anti-de Sitter) depending on the sign of \( \alpha \) (±), whereas the minus branch of the solution (13) is asymptotically flat. With appropriate choice of the functions \( M \) and \( Q \), and parameter \( \alpha \), one can generate other solutions. However, we shall confine ourself to the \(-\)ve branch of the solution (13). The solution (13) can be characterised by the mass \( M \), charge \( Q \), CS parameter \( a \) and GB coupling constant \( \alpha \), which is assumed to be positive and for definiteness we call it 4D Charged EGB black hole surrounded by CS. Interestingly, the semi-classical Einstein’s equations with conformal anomaly [14], gravity theory with quantum corrections [15], and the regularized Lovelock gravity [16], have the same form of the black holes solutions as 4D novel EGB gravity given by the metric (5) with (13). Since the stress-energy tensor of the CS is same as that of the global monopole [33], the solution (13) is also recognized as 4D monopole-charged EGB black hole.
III. BLACK HOLE THERMODYNAMICS

Next we shall discuss the thermodynamical properties of $4D$ Charged EGB black hole surrounded by CS. The event horizon is the largest root of $g^r = 0$ of $f(r) = 0$, which admits the simple solution

$$r_{\pm} = M \pm \sqrt{M^2 - (1 - a)(Q^2 + \alpha)/(1 - a)},$$

(16)

where $r_+$ corresponds to the event horizon while $r_-$ is the Cauchy horizon. Elementary analysis of the zeros of $f(r) = 0$ reveals a critical mass

$$M_c = \sqrt{(1 - a)(Q^2 + \alpha)},$$

(17)

such that, $f(r) = 0$ has no zeros if $M > M_c$, one double zero if $M = M_c$, and two simple zeros if $M < M_c$, (Fig. 2). These cases therefore describe, respectively, $4D$ Charged EGB black hole surrounded by CS with degenerate horizon, and a non-extreme black hole with both event and Cauchy horizons. Clearly the two horizons coincide with the critical radius

$$r_c = \sqrt{(Q^2 + \alpha)/(1 - a)}.$$  

(18)

It is clear that the critical value of $M_c$ and $r_c$ depend upon $\alpha$ and $a$.

Now we turn our attention to analyse the thermodynamic quantities associated with the $4D$ Charged EGB black hole surrounded by CS. We note that the gravitational mass of a black hole is determined by $f(r_+) = 0$ [5], which reads

$$M_+ = \frac{r_+}{2} \left( 1 + \frac{Q^2 + \alpha}{r_+^2} - a \right).$$

(19)

Eq. (19) reduces to the black hole mass

$$M_+ = \frac{r_+}{2} \left( 1 + \frac{Q^2 + \alpha}{r_+^2} \right),$$

for the $4D$ Charged EGB black hole [17] when $a = 0$, to the $4D$ Reissner-Nordstrom black hole for $\alpha \to 0, a = 0$, and in the absence of charge ($Q = 0$) it reduces to the mass of $4D$ EGB black hole surrounded by CS, and we obtain mass for the Schwarzschild black hole surrounded by CS as $M_+ = ((1 - a)r_+)/2$ [9]. It is evident from the Eq. (17) that black hole enrich with CS has higher critical mass and so is the event horizon radius with increase in
FIG. 1: Plot of metric function $f(r)$ vs $r$ for different values of CS parameter $a$ with GB coupling constant $\alpha = 0.1$ and $0.2$ for $4D$ EGB black hole surrounded by CS: neutral black hole (upper plots) and charged black hole (lower plots).

the CS parameter ($a$) and increases with the Gauss-Bonnet coupling constant $\alpha$ (cf. Fig. 1 and Table 1).

The Hawking temperature of the black hole is defined to be proportional to the surface gravity $\kappa$ by $T = \kappa/2\pi$, where $\kappa$ is given by

$$
\kappa = \frac{1}{2\pi} \left( -\frac{1}{2} \nabla_\mu \xi_\nu \nabla^\mu \xi^\nu \right)^{1/2},
$$

and $\xi^\mu = \partial/\partial t$ is a Killing vector.

The black hole has a Hawking temperature defined by $T = \kappa/2\pi$ [5], where $\kappa$ is the surface gravity given. Using the metric function (13), the Hawking temperature for the $4D$ EGB black hole with CS is

$$
T_+ = \frac{1}{4\pi r_+} \left( r_+^2 - a r_+^2 - (Q^2 + \alpha) \right) \left( r_+^2 + 2\alpha \right),
$$

(21)
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
 & $\alpha = 0.1$ & & & $\alpha = 0.2$ & & \\
$M$ & $r_-$ & $r_+$ & $\delta$ & $M$ & $r_-$ & $r_+$ & $\delta$ \\
\hline
$Q = 0$ & $a = 0.1$ & & & $Q = 0$ & $a = 0$ & \\
0.36 & 0.2459 & 0.866 & 0.6201 & 0.5090 & 0.2459 & 0.865 & 0.6201 \\
0.33 & 0.3229 & 0.7927 & 0.4698 & 0.4642 & 0.3229 & 0.7927 & 0.4698 \\
$M_\star = 0.3$ & 0.5394 & 0.5394 & 0 & $M_\star = 0.4242$ & 0.5394 & 0.5394 & 0 \\
\hline
$Q \neq 0$ & $a = 0$ & & & $Q \neq 0$ & $a = 0.1$ & \\
2.2585 & 0.2198 & 0.7804 & 0.5606 & 1.3145 & 0.2198 & 0.7804 & 0.5606 \\
1.1536 & 0.2930 & 0.7047 & 0.4147 & 1.2049 & 0.2930 & 0.7804 & 0.04147 \\
$M_\star = 1.0488$ & 0.5017 & 0.5017 & 0 & $M_\star = 1.0954$ & 0.5017 & 0.5017 & 0 \\
\hline
$Q \neq 0$ & $a = 0.1$ & & & $Q \neq 0$ & $a = 0.3$ & \\
1.1928 & 0.2498 & 0.865 & 0.6155 & 1.2468 & 0.2498 & 0.865 & 0.6155 \\
1.0934 & 0.3229 & 0.7885 & 0.4659 & 1.1429 & 0.3229 & 0.7885 & 0.04659 \\
$M_\star = 0.994$ & 0.5524 & 0.5524 & 0 & $M_\star = 1.039$ & 0.5524 & 0.5524 & 0 \\
\hline
$Q \neq 0$ & $a = 0.3$ & & & $Q \neq 0$ & $a = 0$ & \\
1.053 & 0.3196 & 1.108 & 0.7884 & 1.098 & 0.3196 & 1.108 & 0.7884 \\
0.9652 & 0.4161 & 1.005 & 0.5889 & 1.008 & 0.4161 & 1.005 & 0.5889 \\
$M_\star = 0.8775$ & 0.6987 & 0.6987 & 0 & $M_\star = 0.9165$ & 0.6987 & 0.6987 & 0 \\
\hline
\end{tabular}
\caption{Cauchy ($r_-$) and event ($r_+$) horizons, and $\delta = r_+ - r_-$ for the 4D EGB black hole surrounded by CS.}
\end{table}

The temperature of 4D charged EGB black hole with CS (21) reduces to the temperature of 4D charged EGB black hole [17] when $a = 0$, 4D EGB black hole [12] in the limit of $a = 0$, $Q = 0$, 4D black hole with CS parameter when $Q = 0$ and $\alpha \to 0$ [9], and also to Schwarzschild black hole surrounded by CS: $T_+ = (1 - a)/4\pi r_+$ when $Q = 0$ and $\alpha \to 0$ [9].

In Fig. 4, we have shown the Hawking temperature of the 4D charged EGB surrounded by CS grows to a maximum $T_{max}$ then drops to zero temperature. A maximum of the Hawking temperature occurs at the critical radius shown in Table II. It turns out that the maximum value of the Hawking temperature decreases with increase in the values of the CS parameter $a$ and GB coupling constant $\alpha$ for both charged and uncharged black holes (cf.
Table II: The maximum Hawking temperature \( T_{+}^{\text{max}} \) at critical radius \( r_{+}^{T} \) the 4D EGB black hole surrounded by CS.

![Table](image.png)

Fig. 4 and Table II).

We calculate another useful quantity associated with the black hole, in terms of horizon radius \( r_{+} \), known as entropy. The black hole behaves as a thermodynamic system; quantities associated with it must obey the first law of thermodynamics

\[
dM_{+} = T_{+} dS_{+} + \phi dQ, \tag{22}
\]

where \( \phi \) is the potential of the black hole and \( dM = T dS \). Hence the entropy [5, 39] can be obtained by integrating Eq. (22), with \( Q = \text{constant} \), as

\[
S_{+} = \int \frac{1}{T_{+}} \frac{\partial M_{+}}{\partial r_{+}} dr_{+} = \frac{A}{4} + 2\pi \alpha \log(r_{+}^{2}), \tag{23}
\]

with \( A = 4\pi r_{+}^{2} \). This is the standard area law with logarithmic corrections known as the Bekenstein-Hawking area law [14]. It is interesting to note that the entropy (23) is independent of the string cloud background and charge \( Q \) [5].

Finally, we analyse how the background CS affects the thermodynamic stability of the 4D charged EGB black hole by investigating the heat capacity \( C_{+} \). The stability of the black hole is related to sign of the heat capacity \( C_{+} \). When \( C_{+} > 0 \) the black hole is stable and \( C_{+} < 0 \) means it’s unstable. The heat capacity of the black hole is given [40]

\[
C_{+} = \frac{\partial M_{+}}{\partial T_{+}} = \left( \frac{\partial M_{+}}{\partial r_{+}} \right) \left( \frac{\partial r_{+}}{\partial T_{+}} \right). \tag{24}
\]
FIG. 2: Plot of temperature $T_+$ vs horizon radius $r$ both neutral (upper) and charged (lower) 4D EGB black hole surrounded by CS for different values of GB coupling constant $\alpha$.

Substituting the values of mass and temperature from Eqs. (19) and (21) in Eq. (24), we obtain the heat capacity of the 4D charged EGB surrounded by CS as

$$C_+ = -\frac{2\pi r_+^2 (r_+^2 + 2\alpha)^2 \left(\frac{Q^2}{r_+^2} + \frac{\alpha}{r_+^2} - (1 - a)\right)}{(5 - 2\alpha) r_+^2 \alpha + 2\alpha^2 + Q^2(3r_+^2 + 2\alpha) - (1 - a)r_+^4}.$$

The heat capacity (25), depends on the Gauss-Bonnet coefficient $\alpha$, a string cloud parameter.
$a$, and the charge $q$ and, in the limit $\alpha \to 0$, one regains the analogous GR case, i.e., the Eq. (25) becomes
\[ C_+ = \frac{-2\pi r_+^2 [Q^2 - (1 - a) r_+^2]}{3Q^2 - (1 - a)r_+^2}. \] (26)
The heat capacity (25), in the absence of charge ($Q = 0$), reduces for the 4D EGB surrounded by CS
\[ C_+ = \frac{2\pi r_+^2 (r_+^2 + 2\alpha)^2 \left( \frac{\alpha}{r_+^2} - (1 - a) \right)}{(5 - 2\alpha) r_+^2 \alpha + 2\alpha^2 - (1 - a)r_+^2}. \] (27)
To further analyse, we plot the heat capacity in Fig. 3 for different values of CS parameter $a$

![Fig. 3: Plot of heat capacity $C_+ \text{ vs } r_+$ both neutral (upper) and charged (lower) 4D EGB black hole surrounded by CS](image)

and GB coupling constant $\alpha$, which clearly exhibits that the heat capacity, for a given value of $a$ and $\alpha$, is discontinuous exactly at the critical radius $r_c$. Further, we note that there is a flip of sign in the heat capacity around $r_c$. Thus, 4D EGB black holes (both charged and uncharged) are thermodynamically stable for $r_+ < r_c$, whereas it is thermodynamically unstable for $r_+ > r_c$, and there is a phase transition at $r_+ = r_c$ from the stable to unstable
phases. Further, a divergence of the heat capacity at critical $r_+ = r_C$ signals a second order phase transition occurs [41, 42]. The heat capacity is discontinuous at $r_+ = 2.013$, at which the Hawking temperature has the maximum value $T_+ = 0.0243$ for $\alpha = 0.1$ and $a = 0.0909$ (Fig. 3). The phase transition occurs from the higher to lower mass black holes corresponding to negative to positive heat capacity. The critical radius $r_C$ increases with parameter $\alpha$ (cf. Fig. 3 and Table II). The stable phase can be seen at the large value of CS parameter $a$.

In order to obtain more detail of the thermodynamical equilibrium of this black hole, we are interested to study the behaviour of the Gibbs free energy [5, 32]. To understand the global stability of black hole thermodynamics, we turn to study its free energy ($F_+ = M_+ - T_+ S_+$) [5, 32] given by

$$F_+ = \frac{Q^2 + (1 - a)r_+^2 + \alpha}{4r_+(r_+^2 + 2\alpha)} \left[ 3r_+^2 + 4\alpha \left( 1 + \frac{\log(r_+^2)}{2} \right) \right]. \tag{28}$$

In the limit $Q \to 0$ the free energy reduces to

$$F_+ = \frac{(1 - a)r_+^2 + \alpha}{4r_+(r_+^2 + 2\alpha)} \left[ 3r_+^2 + 4\alpha \left( 1 + \frac{\log(r_+^2)}{2} \right) \right]. \tag{29}$$

The Gibbs free energy for various values of CS parameter $a$ is shown in Fig. 4 which suggests that it is mostly positive for larger $r_+$ and for smaller values of the CS parameter.

We finally comment on the black hole remnant which is a source for dark energy [43] and also one of the candidates to resolve the information loss puzzle [44]. The double root $r = r_c$ of $f(r) = 0$ corresponds to the extremal black hole with degenerate horizon. Hence

$$f'(r) = 0, \tag{30}$$

on substituting $f(r)$ from Eq. (13) to (30), we obtain the critical radius as (18) corresponding to the critical mass (17), We can see clearly that the two horizons coincide when $r_- = r_+$ and the temperature decreases with increasing $r_-$ and vanishes. Thus we find that the temperature vanishes at the degenerate horizon leaving a regular double-horizon remnant with $M = M_c$.

IV. CONCLUSION

EGB gravity is a natural extension of Einstein’s GR to HD ($D \geq 5$) that has several additional nice properties than GR and is the first nontrivial term of low energy limit of
string theory. But EGB gravity is topological in $D \leq 4$ and does not make a contribution to the gravitational dynamics. This has been addressed in the 4D novel EGB gravity in which the quadratic GB term in the action makes a non-trivial contribution to the gravitational dynamics in $4D$ and in contrast to the Schwarzschild black hole solution of GR, a black hole in this theory is free from the singularity pathology. However, the quadratic curvature in the theory causes complications in the calculation and hence, in general, investigation of this theory is a bit tedious.

Hence, we have obtained an exact $4D$ static spherically symmetric black hole solution surrounded by the CS to the $4D$ novel EGB which encompasses the known black holes of Glavan and Lin [12] and of Fernandes [17]. In turn, we have analyzed thermodynamics of the $4D$ charged EGB black hole with CS to calculate exact expressions for the thermodynamic quantities like the black hole mass, Hawking temperature, entropy, specific heat and analyzed the thermodynamical stability of black holes. The thermodynamical quantities
get corrected owing to the background CS, except for the entropy which does not depend on the background CS. The entropy of a black hole has the logarithmic correction to the Bekenstein-Hawking area law. The heat capacity increase indefinitely at critical horizon radius $r_+^C$, which depends on both GB coupling constant $\alpha$ and CS parameter $a$, where the black hole is extremal and incidentally local maxima of the Hawking temperature also occur at $r_+^C$, and that the heat capacity is positive for $r < r_+^C$ implying the stability of small black holes against perturbations in the region, and the phase transition exists at $r_+^C$. While the black hole is unstable for $r > r_+^C$ with negative heat capacity. Further, the smaller black hole are globally stable with positive heat capacity $C_+ > 0$ and negative free energy $F_+ < 0$. Finally, we have also shown that the black hole evaporation results in a stable black hole remnant with zero temperature $T_+ = 0$ and positive specific heat $C_+ > 0$. It would be important to understand how these black holes with positive specific heat ($C > 0$) would emerge from thermal radiation through a phase transition.

There are many interesting avenues that are amenable for future work, it will be intriguing to analyse accretion onto the black holes using the obtained. Since, we find that the background CS makes profound influence as the horizon radius of the black hole under consideration becomes larger which may have several astrophysical consequences, like on wormholes and accretion onto. Some of the results presented here are generalization of the previous discussions, on the 4D EGB \cite{12, 17} and GR black holes \cite{9}, in a more general setting, and the possibility of a further generalization of these results to Lovelock gravity \cite{25} is an interesting problem for future. One can also think, in the spirit of the no-hair conjectures \cite{46}, how two different matters viz. CS and global monopole can generate the same spacetime \cite{13} or have same stress-energy tensor.

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