OMEGA THEOREMS FOR THE TWISTED DIVISOR FUNCTION

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Abstract. For a fixed $\theta \neq 0$, we define the twisted divisor function
$$\tau(n, \theta) := \sum_{d \mid n} d^{i\theta}.$$In this article we consider the error term $\Delta(x)$ in the following asymptotic formula
$$\sum_{n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta) x \log x + \omega_2(\theta) x \cos(\theta \log x) + \omega_3(\theta) x + \Delta(x),$$where $\omega_i(\theta)$ for $i = 1, 2, 3$ are constants depending only on $\theta$. We obtain
$$\Delta(T) = \Omega \left( T^{\alpha(T)} \right)$$where $\alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}}$ and $c > 0$,
along with an $\Omega$-bound for the Lebesgue measure of the set of points where
the above estimate holds.

1. Introduction

For an arithmetical function $f(n)$ we write
$$\sum_{n \leq x} f(n) = M(x) + \Delta(x),$$where $M(x)$ is the main term and $\Delta(x)$ is the error satisfying $\Delta(x) = o(M(x))$. An
$\Omega$-estimate for $\Delta(x)$ helps us understand the magnitude of fluctuation of error and
thereby measures the sharpness of an upper bound for error.

In [1] and [2], Balasubramanian and Ramachandra introduced a method to obtain
a lower bound for
$$\int_T^{T^b} \frac{\Delta(x)^2}{x^{2\alpha+1}} \, dx$$in terms of the second moment of the corresponding Dirichlet series $D(s)$, for some
$b > 0$ and $\alpha > 0$. A nondecreasing lower bound gives
$$\Delta(x) = \Omega(x^{\alpha-\epsilon})$$for any $\epsilon > 0$.

In these papers, they considered the error terms in asymptotic formulas for partial
sums of certain arithmetic functions such as sum of square-free divisors and counting
function for non-isomorphic abelian groups. This method requires the Riemann
Hypothesis to be assumed in certain cases. Balasubramanian, Ramachandra and
Subbarao [3] modified this technique to apply on error term in the asymptotic
formula for the counting function of $k$-full numbers without assuming Riemann
Hypothesis. This method has been used by several authors including [5] and [8].

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For a fixed $\theta \neq 0$, we consider

\[ \tau(n, \theta) = \sum_{d \mid n} d^\theta. \]

Note that

\[ \sum_{d \mid n} 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ib\theta} - e^{-ia\theta}}{-i\theta} d\theta, \]

where $*\sum_{d \mid n}$ denotes that if $e^a \mid n$ or $e^b \mid n$ then their contribution to the sum is $\frac{1}{2}$. So in principle we can restate questions on distribution of divisors of $n$ in terms of $\tau(n, \theta)$. This function is used in [4] to measure the clustering of divisors. In this paper we will study the Dirichlet series of $|\tau(n, \theta)|^2$, which can be expressed in terms of the Riemann zeta function as

\[ D(s) = \sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} = \frac{\zeta(2s)\zeta(s+i\theta)\zeta(s-i\theta)}{\zeta(2s)} \quad \text{for } \Re(s) > 1. \]

In [4, Theorem 33], Hall and Tenenbaum proved that

\[ \sum_{n \leq x} |\tau(n, \theta)|^2 = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x + \Delta(x), \]

where $\omega_i(\theta)$s are explicit constants depending only on $\theta$ and

\[ \Delta(x) = O_\theta(x^1/2 \log^6 x). \]

Here the main term comes from the residues of $D(s)$ at $s = 1, 1 \pm i\theta$. All other poles of $D(s)$ come from the zeros of $\zeta(2s)$. Using a pole on the line $\Re(s) = 1/4$, Landau’s method gives

\[ \Delta(x) = \Omega_\pm(x^{1/4}). \]

In [6], we show that

\[ \mu(A_j \cap [T, 2T]) = \Omega\left(T^{1/2}(\log T)^{-12}\right) \quad \text{for } j = 1, 2, \]

where

\[ A_1 = \left\{ x : \Delta(x) > (\lambda(\theta) - \epsilon)x^{1/4} \right\} \]

and

\[ A_2 = \left\{ x : \Delta(x) < (-\lambda(\theta) + \epsilon)x^{1/4} \right\}, \]

for any $\epsilon > 0$ and $\lambda(\theta) > 0$. Moreover, under Riemann Hypothesis, we obtained

\[ \mu(A_j \cap [T, 2T]) = \Omega\left(T^{3/4-\epsilon}\right), \quad \text{for } j = 1, 2 \]

and for any $\epsilon > 0$.

Adopting the method of Balasubramanian, Ramachandra and Subbarao in case of this twisted divisor function, we derive the following theorem.

**Theorem 1.1.** For any $c > 0$, there exist constants $K(c) > 0$ and $T(c) > 0$ such that for all $T \geq T(c)$, we get

\[ \int_{x}^{\infty} \frac{|\Delta(x)|^2}{x^2} e^{-2x/y} dx \geq K(c) \exp\left(c(\log T)^{7/8}\right), \]
where
\[ \alpha = \alpha(T) = \frac{3}{8} - \frac{c}{(\log T)^{1/8}} \text{ and } y = T^b \text{ for } b \geq 80. \]

In particular, this implies
\[ \Delta(x) = \Omega\left(x^{3/8} \exp\left(-c(\log x)^{7/8}\right)\right) \]
for some suitable \( c > 0 \).

The following localised version of the above theorem is immediate from its proof.

**Corollary 1.1.** For any \( c > 0 \) and for all sufficiently large \( T \) depending on \( c \), there exists an
\[ X \in \left[ T, \frac{T^b}{2} \log^2 T \right] \]
for which we have
\[ \int_X^{2X} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} \, dx \geq \exp\left((c - \epsilon)(\log X)^{7/8}\right), \]
with \( \alpha \) as in Theorem 1.1 and for any \( \epsilon > 0 \).

Optimality of the above bound is justified in Proposition 4.1. We also prove a ‘measure version’ of this result:

**Theorem 1.2.** For any \( c > 0 \), let
\[ \alpha(x) = \frac{3}{8} - \frac{c}{(\log x)^{1/8}} \]
and \( A = \{x : |\Delta(x)| \gg x^{\alpha(x)}\} \). Then
\[ \mu(A \cap [X, 2X]) = \Omega(X^{2\alpha(X)}), \text{ as } X \to \infty. \]

2. Prerequisites

In order to prove the theorem, we need several lemmas, which form the content of this section. We begin with a fixed \( \delta_0 \in (0, 1/16] \) for which we would choose a numerical value at the end of this section.

**Definition 2.1.** For \( T > 1 \), let \( Z(T) \) be the set of all \( \gamma \) such that

1. \( T \leq \gamma \leq 2T \),
2. either \( \zeta(\beta_1 + i\gamma) = 0 \) for some \( \beta_1 \geq \frac{1}{2} + \frac{\delta_0}{2} \)
or \( \zeta(\beta_2 + 2i\gamma) = 0 \) for some \( \beta_2 \geq \frac{1}{2} + \frac{\delta_0}{2} \).

Let
\[ I_{\gamma,k} = \{T \leq t \leq 2T : |t - \gamma| \leq k \log^2 T\} \text{ for } k = 1, 2. \]

We finally define
\[ J_k(T) = [T, 2T] \setminus \cup_{\gamma \in Z(T)} I_{\gamma,k}. \]

**Lemma 2.1.** With the above definition, we have for \( k = 1, 2 \)
\[ \mu(J_k(T)) = T + O\left(T^{1-\delta_0/8} \log^3 T\right). \]
Proof. We shall use an estimate on the function $N(\sigma, T)$, which is defined as

$$N(\sigma, T) := |\{\sigma' + it : \sigma' \geq \sigma, \, 0 < t \leq T, \, \zeta(\sigma' + it) = 0\}|.$$

Selberg [9, Page 237] proved that

$$N(\sigma, T) \ll T^{1-\frac{\delta}{2}(\sigma - \frac{1}{2})} \log T, \quad \text{for} \quad \sigma > 1/2.$$

Now the lemma follows from the above upper bound on $N(\sigma, t)$, and the observation that

$$\mu\left(\bigcup_{\gamma \in Z(T)} I_{\gamma, k}\right) \ll N\left(1/2 + \frac{\delta_0}{2}, T\right) \log^2 T.$$

□

The next lemma closely follows Theorem 14.2 of [9], but we are including a proof as we could not find a clearly written proof of this version which unlike the original one, does not use Riemann Hypothesis.

Lemma 2.2. For $t \in J_1(T)$ and $\sigma = 1/2 + \delta$ with $\delta_0 < \delta < 1/4 - \delta_0/2$, we have

$$|\zeta(\sigma + it)|^{\pm 1} \ll \exp\left(\log \log t \left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{2\delta_0}}\right)$$

and

$$|\zeta(\sigma + 2it)|^{\pm 1} \ll \exp\left(\log \log t \left(\frac{\log t}{\delta_0}\right)^{\frac{1-2\delta}{2\delta_0}}\right).$$

Proof. We provide a proof of the first statement, and the second statement can be similarly proved.

Let $1 < \sigma' \leq \log t$. We consider two concentric circles centered at $\sigma' + it$, with radius $\sigma' - 1/2 - \delta_0/2$ and $\sigma' - 1/2 - \delta_0$. Since $t \in J_1(T)$ and the radius of the circle is $\ll \log t$, we conclude that

$$\zeta(z) \neq 0 \quad \text{for} \quad |z - \sigma' - it| \leq \sigma' - 1/2 - \frac{\delta_0}{2}$$

and also $\zeta(z)$ has polynomial growth in this region. Thus on the larger circle, $\log |\zeta(z)| \leq c_5 \log t$, for some constant $c_5 > 0$. By Borel-Caratheodory theorem,

$$|z - \sigma' - it| \leq \sigma' - 1/2 - \frac{\delta_0}{2} \quad \text{implies} \quad |\log \zeta(z)| \leq \frac{c_6 \sigma'}{\delta_0} \log t,$$

for some $c_6 > 0$. Let $1/2 + \delta_0 < \sigma < 1$, and $\xi > 0$ be such that $1 + \xi < \sigma'$. We consider three concentric circles centered at $\sigma' + it$ with radius $r_1 = \sigma' - 1 - \xi$, $r_2 = \sigma' - \sigma$ and $r_3 = \sigma' - 1/2 - \delta_0$, and call them $C_1, C_2$ and $C_3$ respectively. Let

$$M_k = \sup_{z \in C_k} |\log \zeta(z)|.$$

From the above bound on $|\log \zeta(z)|$, we get

$$M_3 \leq \frac{c_6 \sigma'}{\delta_0} \log t.$$

Suitably enlarging $c_6$, we see that

$$M_1 \leq \frac{c_6}{\xi}. $$
Hence we can apply the Hadamard’s three circle theorem to conclude that

\[ M_2 \leq M_1^{1-\nu} M_3^\nu, \quad \text{for } \nu = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}. \]

Thus

\[ M_2 \leq \left( \frac{c_6}{\xi} \right)^{1-\nu} \left( \frac{c_6 \sigma' \log t}{\delta_0} \right)^\nu. \]

It is easy to see that

\[ \nu = 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0} + O(1) + O\left( \frac{1}{\sigma'} \right). \]

Now we put

\[ \xi = \frac{1}{\sigma'} = \frac{1}{\log \log T}. \]

Hence

\[ M_2 \leq \frac{c_6 \log^\nu t \log \log t}{\delta_0^{\nu'}} = c_7 \log \log t \left( \log t \right)^{2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0}}, \]

for some \( c_7 > 0 \). We observe that

\[ 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\xi - 2\delta_0} < 2 - 2\sigma + \frac{4\delta_0(1 - \sigma)}{1 + 2\delta_0} = 1 - 2\delta. \]

So we get

\[ |\log \zeta(\sigma + it)| \leq c_7 \log \log t \left( \frac{\log t}{\delta_0} \right)^{\frac{1 - 2\delta}{1 - 2\delta_0}}, \]

and hence the lemma. \( \Box \)

We put \( y = T^b \), for a constant \( b \geq 80 \). Now suppose that

\[ \int_T^\infty \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \geq \log^2 T, \]

for sufficiently large \( T \). Then clearly

\[ \Delta(u) = \Omega(u^\alpha). \]

Our next result explores the situation when such an inequality does not hold.

**Proposition 2.1.** Let \( \delta_0 < \delta < \frac{1}{4} - \frac{\delta}{2} \). For \( 1/4 + \delta/2 < \alpha < 1/2 \), suppose that

\[ \int_T^\infty \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} du \leq \log^2 T, \]

for a sufficiently large \( T \). Then we have

\[ \int_{J_2(T)} \frac{|D(\alpha + it)|^2}{|\alpha + it|^2} dt \ll 1 + \int_T^\infty \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-2u/y} du. \]

Before embarking on a proof, we need the following lemma which is easy to prove using Stirling’s formula for \( \Gamma \)-function.

**Lemma 2.3.** Let \( z \) be a complex number with \( 0 \leq \text{Re}(z) \leq 1 \) and \( |\text{Im}(z)| \geq \log^2 T \). For \( y \) as above, we have

\[ \int_T^\infty e^{-u/y} u^{-z} du = \frac{T^{1-z}}{1-z} + O(T^{-b'}). \]
and
\[(8) \quad \int_T^\infty e^{-u/y} u^{-z} \log u \, du = \frac{T^{1-z}}{1-z} \log T + O(T^{-b'}),\]
where \(b' > 0\) depends only on \(b\).

**Lemma 2.4.** Under the assumption (6), there exists \(T_0\) with \(T \leq T_0 \leq 2T\) such that
\[
\frac{\Delta(T_0)e^{-T_0/y}}{T_0^\alpha} \ll \log^2 T,
\]
and
\[
\frac{1}{y} \int_{T_0}^\infty \frac{\Delta(u)e^{-u/y}}{u^\alpha} \, du \ll \log T.
\]

**Proof.** The assumption (6) implies that
\[
\log^2 T \geq \int_T^{2T} \frac{|\Delta(u)|^2}{u^{2\alpha+1}} e^{-u/y} \, du = \int_T^{2T} \frac{|\Delta(u)|^2}{u^{2\alpha}} e^{-2u/y} e^{-u/y} \, du
\]
\[
\geq \min_{T \leq u \leq 2T} \left( \frac{|\Delta(u)|}{u^{\alpha}} e^{-u/y} \right)^2,
\]
which proves the first assertion. To prove the second assertion, we use the previous assertion and Cauchy-Schwarz inequality along with assumption (6) to get
\[
\left( \int_{T_0}^\infty \frac{\Delta(u)}{u^{\alpha}} e^{-u/y} \, du \right)^2 \leq \left( \int_{T_0}^\infty |\Delta(u)|^2 \frac{e^{-u/y}}{u^{2\alpha}} \, du \right) \left( \int_{T_0}^\infty u e^{-u/y} \, du \right)
\]
\[
\ll y^2 \log^2 T.
\]
This completes the proof of this lemma. \(\square\)

We now recall a mean value theorem due to Montgomery and Vaughan [7].

**Notation.** For a real number \(\theta\), let \(\|\theta\| := \min_{n \in \mathbb{Z}}|\theta - n|\).

**Theorem 2.1** (Montgomery and Vaughan [7]). Let \(a_1, \ldots, a_N\) be arbitrary complex numbers, and let \(\lambda_1, \ldots, \lambda_N\) be distinct real numbers such that
\[
\delta = \min_{m,n \neq n} \|\lambda_m - \lambda_n\| > 0.
\]
Then
\[
\int_0^T \left| \sum_{n \leq N} a_n \exp(i\lambda_n t) \right|^2 \, dt = \left( T + O\left( \frac{1}{\delta} \right) \right) \sum_{n \leq N} |a_n|^2.
\]

**Lemma 2.5.** For \(T \leq T_0 \leq 2T\) and \(\text{Re}(s) = \alpha\), we have
\[
\int_T^{2T} \left| \sum_{n \leq T_0} \frac{|r(n, \theta)|^2}{n^\alpha} e^{-n/y} \right|^2 t^{-2} \, dt \ll 1.
\]
Proof. Using theorem 2.1 we get

\[
\int_{T}^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 t^{-2} dt
\]

\[
\leq \frac{1}{T^2} \left( T \sum_{n \leq T_0} |b(n)|^2 + O\left( \sum_{n \leq T_0} n|b(n)|^2 \right) \right),
\]

where

\[
b(n) = \frac{|\tau(n, \theta)|^2}{n^\alpha} e^{-n/y}.
\]

Thus

\[
\sum_{n \leq T_0} |b(n)|^2 \leq \sum_{n \leq T_0} \frac{d(n)^4}{n^{2\alpha}} \ll T_0^{1-2\alpha+\epsilon}
\]

and

\[
\sum_{n \leq T_0} n|b(n)|^2 \leq \sum_{n \leq T_0} \frac{d(n)^4}{n^{2\alpha-1}} \ll T_0^{2-2\alpha+\epsilon}
\]

for any \( \epsilon > 0 \), since the divisor function \( d(n) \ll n^\epsilon \). As we have \( \alpha > 0 \), this completes the proof. \( \Box \)

Lemma 2.6. For Re(s) = \( \alpha \) and \( T \leq T_0 \leq 2T \), we have

\[
\int_{T}^{2T} \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} dx \left| \right|^2 dt \ll \int_{T}^{\infty} \frac{\Delta(x)^2}{x^{2\alpha+1}} e^{-2x/y} dx.
\]

Proof. Using Cauchy-Schwarz inequality, we get

\[
\left| \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} dx \right|^2 \leq \int_{0}^{1} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 dx.
\]

Hence

\[
\int_{T}^{2T} \left| \int_{0}^{1} \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} dx \right|^2 dt \leq \int_{T}^{2T} \left| \int_{0}^{1} \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} dx \right|^2 dt \leq \int_{T}^{2T} \left| \int_{0}^{1} \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} dx \right|^2 dt \leq \int_{0}^{1} \int_{T}^{2T} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 dx dt \leq \int_{0}^{1} \int_{T}^{2T} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{s+1}} e^{-(n+x+T_0)/y} \right|^2 dx dt.
\]
Hence by convexity, we see that \( \zeta \) growth on the horizontal lines of integration. Therefore the horizontal integrals are 

\[
\sum_{n \geq 0} \frac{|\Delta(n + x + T_0)|^2}{(n + x + T_0)^{2\alpha + 1}} e^{-2(n + x + T_0)/y} \leq 0.
\]

Let \( t \in J_2(T) \), we have 

\[
\sum_{n \geq 0} \frac{|\Delta(n + x + T_0)|^2}{(n + x + T_0)^{2\alpha + 1}} e^{-2(n + x + T_0)/y} \ll \int_T^{2T} \left| \sum_{n \geq 0} \frac{\Delta(n + x + T_0)}{(n + x + T_0)^{2\alpha + 1}} e^{-w(n + x + T_0)/y} \right|^2 dt.
\]

completing the proof.

**Proof of Proposition 2.1** For \( s = \alpha + it \) with \( 1/4 + \delta < \alpha < 1/2 \) and \( t \in J_2(T) \), we have

\[
\sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D(s + w) \Gamma(w) y^w dw = \frac{1}{2\pi i} \int_{2-i\log^2 T}^{2+i\log^2 T} \left[ D(s + 2 + iv) \Gamma(2 + iv) \right] dv + O \left( \gamma^2 \int_{\log^2 T}^{\infty} |D(s + 2 + iv)| \Gamma(2 + iv) |dv| \right).
\]

The above error term is estimated to be \( o(1) \). We move the integral to

\[
\left[ \frac{1}{4} + \frac{\delta}{2} - \alpha - i \log^2 T, \frac{1}{4} + \frac{\delta}{2} - \alpha + i \log^2 T \right].
\]

Let \( \delta' = 1/4 + \delta/2 - \alpha \). In the region to the right side of this line, \( \text{Re}(2s + 2w) \geq 1/2 + \delta \). Writing \( w = u + iv \) we observe that \( t + v \in J_1(T) \) since \( t \in J_2(T) \). So we can apply Lemma 2.2 to conclude that

\[
\zeta(2s + 2w) \gg T^{-1}.
\]

On the above line, we have \( \text{Re}(s + w) = 1/4 + \delta/2 \), Thus

\[
\zeta^2(s + w) \zeta(s + w + i\theta) \zeta(s + w - i\theta) \ll T^{3/2 - \delta} \log^4 T,
\]

where we use the fact that \( \zeta(z) \ll \text{Im}(z)^{1-\text{Re}(z)/2} \log(\text{Im}(z)) \) if \( 0 \leq \text{Re}(z) \leq 1 \). Hence by convexity, we see that \( \zeta^2(s + w) \zeta(s + w + i\theta) \zeta(s + w - i\theta) \) has polynomial growth on the horizontal lines of integration. Therefore the horizontal integrals are \( o(1) \) by exponential decay of \( \Gamma \)-function. Since the only pole inside this contour is at \( w = 0 \), we get

\[
\sum_{n=1}^{\infty} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = D(s) + \frac{1}{2\pi i} \int_{\delta' - i\log^2 T}^{\delta' + i\log^2 T} D(s + w) \Gamma(w) y^w dw + o(1).
\]
For the integral on the right hand side, we have

\[ D(s + w)y^w \ll T^{5/2 - \delta(b/2 + 1)} \]

where the exponent of \( T \) is negative by our choice of \( b \) and \( \delta \). Therefore this integral is also \( O(1) \).

Using \( T_0 \) as in Lemma 2.4 we now divide the sum into two parts:

\[ D(s) = \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + \sum_{n > T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + o(1). \]

To estimate the second sum, we write

\[ \sum_{n > T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \left( \sum_{n \leq x} |\tau(n, \theta)|^2 \right) dx \]

\[ = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(M(x) + \Delta(x)) \]

\[ = \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} M'(x)dx + \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(\Delta(x)). \]

Recall that

\[ M(x) = \omega_1(\theta)x \log x + \omega_2(\theta)x \cos(\theta \log x) + \omega_3(\theta)x, \]

thus

\[ M'(x) = \omega_1(\theta) \log x + \omega_2(\theta) \cos(\theta \log x) - \theta \omega_2(\theta) \sin(\theta \log x) + \omega_1(\theta) + \omega_3(\theta). \]

Observe that

\[ \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \cos(\theta \log x)dx = \frac{1}{2} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^{s+iy}}dx + \frac{1}{2} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^{s-iy}}dx. \]

Applying Lemma 2.3 we conclude that

\[ \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} M'(x) dx = O(1). \]

Integrating the second integral by parts:

\[ \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} d(\Delta(x)) = \frac{e^{-T_0/y} \Delta(T_0)}{T_0^s} \]

\[ + \frac{1}{y} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^s} \Delta(x)dx - \frac{1}{y} \int_{T_0}^{\infty} \frac{e^{-x/y}}{x^{s+1}} \Delta(x)dx. \]

Applying Lemma 2.4 we get

\[ \sum_{n > T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} = s \int_{T_0}^{\infty} \frac{\Delta(x)e^{-x/y}}{x^{s+1}} dx + O(\log T) \]

\[ = s \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} dx + O(\log T). \]

Hence we have

\[ D(s) = \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} + s \sum_{n \geq 0} \int_{0}^{1} \frac{\Delta(n + x + T_0)e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} dx + O(\log T). \]
Squaring both sides, and then integrating on $J_2(T)$, we get

$$\int_{J_2(T)} \left| \frac{D(\alpha + it)}{\alpha + it} \right|^2 dt \ll \int_T^{2T} \left| \sum_{n \leq T_0} \frac{|\tau(n, \theta)|^2}{n^s} e^{-n/y} \right|^2 \frac{dt}{t^2} + \int_T^{2T} \left| \sum_{n \geq 0} \int_0^1 \frac{\Delta(n + x + T_0) e^{-(n+x+T_0)/y}}{(n + x + T_0)^{s+1}} dx \right|^2 dt.$$  

The proposition now follows using Lemma 2.5 and Lemma 2.6. □

3. Proofs of The Main Theorems

3.1. **Proof of Theorem 1.1**. We prove by contradiction. Suppose that (5) does not hold. Then there exists a constant $c > 0$ such that given any $N_0 > 1$, there exists $T > N_0$ for which

$$\int_T^{\infty} \frac{|\Delta(x)|^2}{x^{2\beta + 1}} e^{-2x/y} dx \leq \exp \left( c (\log T)^{7/8} \right).$$

Note that the above statement is weaker than the contrapositive of the statement of theorem. This gives

$$\int_T^{\infty} \frac{|\Delta(x)|^2}{x^{2\beta + 1}} e^{-2x/y} dx \ll 1,$$

where

$$\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}}.$$

We apply Proposition 2.1 to get

$$\int_{J_2(T)} \left| \frac{D(\beta + it)}{\beta + it} \right|^2 dt \ll 1.$$  

Now we compute a lower bound for the last integral over $J_2(T)$. Write the functional equation for $\zeta(s)$ as

$$\zeta(s) = \pi^{1/2-s} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \zeta(1-s).$$

Using the Stirling’s formula for $\Gamma$ function, we get

$$|\zeta(s)| = \pi^{1/2-s} t^{1/2-s} |\zeta(1-s)| \left( 1 + O \left( \frac{1}{T} \right) \right)$$

for $s = \sigma + it$. This implies

$$|D(\beta + it)| = t^{2-4\beta} \frac{|\zeta(1-\beta + it)^2 \zeta(1-\beta - it - i\theta)\zeta(1-\beta - it + i\theta)|}{|\zeta(2\beta + i2t)|}.$$  

Let $\delta_0 = 1/16$, and

$$\beta = \frac{3}{8} - \frac{c}{2(\log T)^{1/8}} = \frac{1}{2} - \delta$$

with

$$\delta = \frac{1}{8} + \frac{c}{2(\log T)^{1/8}}.$$
Then using Lemma 2.2, we get

\[ |\zeta(1 - \beta + it)| = \left| \zeta\left(\frac{1}{2} + \delta + it\right) \right| \gg \exp\left(\log \log t \left(\log t \delta_0\right)^{\frac{1-2\delta}{2\delta_0}}\right). \]

For \( t \in J_2(T) \) we observe that \( t + \theta \in J_1(T) \), and so the same bounds hold for \( \zeta(1 - \beta + it + i\theta) \) and \( \zeta(1 - \beta + it - i\theta) \). Further

\[ |\zeta(2\beta + i2t)| = \left| \zeta\left(\frac{1}{2} + \left(\frac{1}{2} - 2\delta\right) + i2t\right) \right| \ll \exp\left(\log \log t \left(\log t \delta_0\right)^{\frac{1-2\delta}{2\delta_0}}\right). \]

Combining these bounds, we get

\[ |D(\beta + it)| \gg t^{2-4\beta} \exp\left(-5 \log \log t \left(\log t \delta_0\right)^{\frac{1-2\delta}{2\delta_0}}\right). \]

Therefore

\[
\int_{J_2(T)} |D(\beta + it)|^2 dt \gg T^{4-8\beta} \exp\left(-10 \log \log T \left(\log T \delta_0\right)^{\frac{1-2\delta}{2\delta_0}}\right) \mu(J_2(T))
\]

\[
\gg T^{5-8\beta} \exp\left(-10 \log \log T \left(\log T \delta_0\right)^{\frac{1-2\delta}{2\delta_0}}\right),
\]

where we use Lemma 2.1 to show that \( \mu(J_2(T)) \gg T \). Now putting the values of \( \delta \) and \( \delta_0 \) as chosen above, we get

\[
\int_{J_2(T)} \frac{|D(\beta + it)|^2}{|\beta + it|^2} dt \gg \exp\left(3c(\log T)^{7/8}\right),
\]

since \( \frac{1-2\delta}{2\delta_0} < 7/8 \). This contradicts (4), and hence the theorem follows.

3.2. Proof of Theorem 1.2 Suppose that the conclusion does not hold, hence \( \mu(A \cap [X, 2X]) \ll X^{2(\alpha)} \).

Thus for every sufficiently large \( X \), we get

\[
\int_{A \cap [X, 2X]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \ll X^{2\alpha} M(X) \frac{M(X)}{X^{2\alpha+1}} = \frac{M(X)}{X},
\]

where \( \alpha = \alpha(X) \) and \( M(X) = \sup_{X \leq x \leq 2X} |\Delta(x)|^2 \). Using dyadic partition, we can prove

\[
\int_{A \cap [T, y]} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} dx \ll \frac{M_0(T)}{T} \log T, \quad \text{where} \quad M_0(T) = \sup_{T \leq x \leq y} |\Delta(x)|^2
\]

and \( y = T^b \) for some \( b > 0 \) and \( T \) sufficiently large. This gives

\[
\int_T^{\infty} \frac{|\Delta(x)|^2}{x^{2\alpha+1}} e^{-2x/y} dx \ll \frac{M_0(T)}{T} \log T.
\]

Along with (5), this implies

\[
M_0(T) \gg T \exp\left(\frac{c}{2}(\log T)^{7/8}\right).
\]

Thus

\[
|\Delta(x)| \gg x^{\frac{c}{4}} \exp\left(\frac{c}{4}(\log x)^{7/8}\right),
\]
for some \( x \in [T, y] \). This contradicts the fact that \( |\Delta(x)| \ll x^\frac{1}{2}(\log x)^6 \).

4. Optimality of the Omega Bound for the Second Moment

The following proposition shows the optimality of the omega bound in Corollary 1.1.

**Proposition 4.1.** Under Riemann Hypothesis (RH), we have

\[
\int_X^{2X} \Delta^2(x) dx \ll X^{7/4+\epsilon}
\]

for any \( \epsilon > 0 \).

**Proof.** Perron’s formula gives

\[
\Delta(x) = \frac{1}{2\pi i} \int_{-T}^{T} D(3/8 + it) x^{3/8+it} \frac{3/8 + it}{dt} + O(x^\epsilon),
\]

for any \( \epsilon > 0 \) and for \( T = X^2 \) with \( x \in [X, 2X] \). Using this expression for \( \Delta(x) \), we write its second moment as

\[
\int_X^{2X} \Delta^2(x) dx = \frac{1}{(2\pi)^2} \int_X^{2X} \int_{-T}^{T} D(3/8 + it_1)D(3/8 - it_2) \frac{x^{3/4+i(t_1-t_2)} dx}{dt_1 dt_2}
\]

\[
\ll X^{7/4} T \int_{-T}^{T} \frac{D(3/8 + it_1)D(3/8 - it_2)}{(3/8 + it_1)(3/8 - it_2)(7/4 + i(t_1 - t_2))} dt_1 dt_2 + O(X^{3/2+\epsilon}).
\]

In the above calculation, we have used the fact that \( \Delta(x) \ll x^{\frac{1}{2}+\epsilon} \) as in (4). Also note that for complex numbers \( a, b \), we have \( |ab| \leq \frac{1}{2}(|a|^2 + |b|^2) \). We use this inequality with

\[
a = \frac{|D(3/8 + it_1)|}{|3/8 + it_1|\sqrt{|7/4 + i(t_1 - t_2)|}} \quad \text{and} \quad b = \frac{|D(3/8 - it_2)|}{|3/8 - it_2|\sqrt{|7/4 + i(t_1 - t_2)|}}
\]

to get

\[
\int_X^{2X} \Delta^2(x) dx \ll X^{7/4} \int_{-T}^{T} \frac{|D(3/8 - it_2)|^2}{|7/4 + i(t_1 - t_2)|} dt_1 dt_2 + O(X^{3/2+\epsilon})
\]

\[
\ll X^{7/4} \log X \int_{-T}^{T} \frac{|D(3/8 - it_2)|^2}{|3/8 - it_2|} dt_2 + O(X^{3/2+\epsilon}).
\]

Under RH, convexity bound gives \( \zeta(\sigma + it) \ll t^{1/2-\sigma} \) for \( 0 \leq \sigma \leq 1/2 \), hence

\[
|D(3/8 - it_2)| \ll |t_2|^{\epsilon+\epsilon}. \quad \text{So we have}
\]

\[
\int_X^{2X} \Delta^2(x) dx \ll X^{7/4+\epsilon} \quad \text{for any} \ \epsilon > 0.
\]

\[\square\]

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