On the Transition of the Rayleigh-Taylor Instability in 2d Water Waves with Point Vortices

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Abstract
In this paper, by considering 2d water waves with a pair of point vortices, we prove the existence of water waves with sign-changing Taylor sign coefficients. That is, the strong Taylor sign condition holds initially, while it breaks down at a later time. Such a phenomenon can be regarded as the transition between the stable and unstable regime in the sense of Rayleigh-Taylor of water waves. As a byproduct, we prove the wellposedness of 2d water waves in Gevrey-2 spaces.

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1 Introduction

1.1 The Background and the Main Result

We consider the motion of an inviscid and incompressible ideal fluid with a free surface in two space dimensions (that is, the interface separating the fluid and the vacuum is one dimensional), such as the surface waves in the ocean. We refer to such fluid as water waves. Denote the fluid region by \( \Omega(t) \), with a free interface \( \Sigma(t) \). The equations of motion are Euler’s equations, coupled to the motion of the boundary and with vanishing boundary conditions for the pressure. Assume that the fluid region is below the air region and the density of the fluid is 1, the gravitational field is \(-\vec{k}\), where \( \vec{k} \) is the unit vector pointing in the upward vertical direction. In two dimensions, if the surface tension is zero, then the motion of the fluid is described by

\[
\begin{cases}
    v_t + v \cdot \nabla v = -\nabla P - (0, 1) & \text{in } \Omega(t), \\
    \text{div } v = 0, & \text{in } \Sigma(t), \\
    P \equiv 0, & \text{on } \Omega(t), \\
    (1, v) \text{ is tangent to the free surface } (t, \Sigma(t)). & \end{cases}
\]

where \( v \) is the fluid velocity, \( P \) is the fluid pressure.

This system, and many variants and generalizations, has been extensively studied in the literature. The so-called Taylor sign condition \( -\frac{\partial P}{\partial n} > 0 \) on the pressure is an important stability condition for the water waves problem. We shall call \( -\frac{\partial P}{\partial n} \) the Taylor sign coefficient. If the Taylor sign condition fails, the system is, in general, unstable, see, for example, \([7, 9, 21, 46]\). We refer to such instability as the Rayleigh-Taylor instability. In the irrotational case and without a bottom the validity of the Taylor sign condition was shown by Wu \([52, 53]\), and was the key to obtaining the first local-in-time existence results for large data in Sobolev spaces. In the case of non-trivial

\[1\] Also referred as Rayleigh-Taylor sign condition in many literature.
vorticity or with a rough bottom it is widely believed that the Taylor sign condition can fail and it has to be part of the assumptions for the initial data. In the irrotational case, Nalimov [38], Yosihara [58] and Craig [18] proved local well-posedness for 2d water waves equation for small initial data. In S. Wu’s breakthrough works [52, 53] she proved local-in-time well-posedness without smallness assumption. Since then, a lot of interesting local well-posedness results were obtained, see for example [1, 2, 5, 6, 14, 15, 30, 33, 34, 37, 39, 42, 59], and the references therein. See also [41, 50, 51, 57] for water waves with non-smooth interface. Regarding the local-in-time wellposedness with regular vorticity, see [14, 29, 34, 39, 40, 59], and [45] for water waves with point vortices.

Once the initial data problem of the dynamics of the system (1.1) is shown to be well posed for short time intervals, the following questions arise:

- Do these solutions exist for all times?
- Can finite time singularities be developed?

In the irrotational case with small and localized initial data (small slope of the interface and small velocity), almost global and global well-posedness for water waves were proved in [4, 23, 31, 54, 55], and see also [26, 27, 48, 60]. In the rotational case, see [8, 24, 28], and [45] for some long time existence results.

The formation of finite time singularities is much less understood. In [10], Castro, Córdoba, Fefferman, Gancedo, and Gómez-Serrano proved the formation of splash and splat singularities. See also [11, 16, 17]. All these works assume that the initial interface is non-graph, then at some later time, the interface becomes self-intersecting. However, their initial interfaces are very close to the singularities. The more interesting case is still open:

**Question** Can an initially flat interface become self-intersecting in finite time?

Also, although the existence of water waves with angle crests are well-known (e.g., the extreme Stokes waves and a class of water waves with angle crests constructed by Wu [57]), it is still not known how these angle crests are formed from the initially smooth interfaces. To construct water waves which blow up in finite time, it should be advantageous to consider the rotational water waves. Indeed, in [47], J. Telste considered the problem of calculating nonlinear two-dimensional free surface potential flow about a pair of counter-rotating point vortices rising under their influence towards a free surface. In Telste’s numerical simulation, two vortices are inserted into the still water. He showed that in an appropriate regime, wave breaking occurs. The free surface develops from a flat line into a mushroom-like shape, and then splash singularity forms. The blowup of curvature is also observed in [47].

In order to rigorously justify the observation of Telste\(^2\), one needs to solve the water waves near a class of singular solutions\(^3\). Therefore, we need to find an ansatz that allows the water waves with a pair of counter-rotating point vortices which travels toward the free interface. Interestingly, this is closely related to the Rayleigh-Taylor

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\(^2\) The immediate consequence of such rigorous justification is the proof of the formation of finite time singularities.

\(^3\) Singularities of splash type or blow up of curvature of the free interface.
instability. As observed by the formal calculation in [45], the Taylor sign condition fails if the vorticity is sufficiently close to the free interface. Indeed, the formal calculation in [45] suggests that the Taylor sign transit can transition from hold to fail. Therefore, such water wave is subject to the transition of Rayleigh-Taylor instability and cannot be solved in Sobolev spaces.

Shinbrot [43], Kano, and Nishida [32] justify the Friedrichs expansion for the water-waves equations in terms of the shallowness parameter using the Cauchy-Kowalevski theorem. They take analytic initial data, and the Taylor-sign condition is not assumed. Very recently, Alazard, Burq, and Zuily [3] revisited the analysis of the water wave problem with analytic data, using tools and methods that they developed previously to study the Cauchy problem with rough initial data. Their result allows a bottom with Sobolev regularity. Nevertheless, none of these results proved the existence of solutions with the transition of the Taylor sign condition. It is natural to ask the following question:

**Question 1** Is there a solution to the system (1.1) such that the strong Taylor sign condition holds at \( t = 0 \), while failing at some \( t_0 > 0 \)?

Besides its relation to the study of the formation of finite time singularities, the rigorous mathematical analysis of the transition of the Rayleigh-Taylor instability is an interesting yet less-understood subject on its own. From the technical point of view, the breakdown of the Taylor sign condition corresponds to the loss of derivatives of the solution. In [12], Castro, Cordoba, Fefferman, and Gancedo showed that the Rayleigh-Taylor condition for the Muskat problem might hold initially but break down in finite time. In their case, the solution has instant analyticity, that is, even if the initial data has finite smoothness, the solution will instantaneously become real analytic after the initial time. Such instant analyticity can compensate for the loss of derivatives caused by the breakdown of the Rayleigh-Taylor condition. For the water waves, there is no such instant analyticity, which makes the problem more difficult. To the best of our knowledge, the rigorous study of the transition of the Rayleigh-Taylor instability/stability was not known for the water waves. Of course, for infinite depth and irrotational water waves, the Taylor sign condition always holds, as long as the free interface is smooth and non-self-intersecting. In order that (1.1) admits a solution with the Taylor sign breaking down, we must assume that the initial data has nontrivial vorticity.

The interaction of the free surface and the vorticity is very complicated. To control the motion of the vorticity, we assume that the initial vorticity is highly concentrated in a small region. In idealized cases, we can assume that the vorticity is a linear combination of point vortices, that is, the vorticity distribution \( \omega := \text{curl} \, v \) is of the form

\[
\omega(\cdot, t) = \sum_{j=1}^{N} \lambda_j \delta_{z_j(t)}(\cdot), \tag{1.2}
\]

where \( \{z_j(t)\} \subset \Omega(t) \), and \( \lambda_j \in \mathbb{R} \). Since the vorticity of the 2d Euler is transported following the fluid, if the initial vorticity of (1.1) is a Dirac delta mass, then \( \omega \) remain
as such a point vortex for all \( t \). By the Biot-Savart Law, we know that the velocity field generated by these point vortices is \( \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \nabla \perp \log |z - z_j(t)| \). So the velocity \( v \) of the water waves can be decomposed as

\[
v(\cdot, t) = u(\cdot, t) + \sum_{j=1}^{N} \frac{\lambda_j}{\pi} \nabla \perp \log |\cdot - z_j(t)|,
\]

(1.3)

where \( u \) is regular with \( \text{div} u = 0 \) and \( \text{curl} u = 0 \). For each given \( j \), the motion of \( z_j(t) \) satisfies

\[
\dot{z}_j(t) = u(z_j(t), t) + \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_k}{\pi} \nabla \perp \log |z_k(t) - z_j(t)|.
\]

(1.4)

This well-known result is rigorously established by considering a family of vortex patch solutions with initial vorticity limiting (weakly) to a Dirac delta mass (see [35] Theorem 4.1, 4.2 for more details).

The system (1.1)-(1.2)-(1.4) is a model for the motion of submerged bodies (see e.g. [13, 20]) and it is believed to give some insight into the problem of turbulence ([35], chap 4, §4.6). There have been many numerical studies on this system (see for example, [19, 22, 25, 36, 47, 49]) and studies on the local in time well-posedness of the Cauchy problem in the irrotational case (that is, no vorticity), and for regular vortex distributions. In [45], the author proved local wellposedness of (1.1) with point vortices under the strong Taylor sign assumption, and obtained extended lifespan with small initial data when there is a pair of counter-rotating point vortices travelling downward.

In the case of point vortices analyzed in this paper, we give an affirmative answer to Question 1. The main result of this paper is

**Theorem 1** For any given constants \( 0 < \eta_0 < 1 \), and \( \eta_1 > 0 \), there exist \( T_0 > 0 \) and a nonempty open set \( \mathcal{O} \) in a category of infinite smoothness such that for each \( u_0 \in \mathcal{O} \), the system (1.1)-(1.2)-(1.4) with initial value \( u_0 \) admits a unique smooth solution on \([0, T_0]\). Moreover, the Taylor sign coefficient \( -\frac{\partial P}{\partial n} \) of the corresponding solution satisfies:

1. At \( t = 0 \), \( \inf_{\alpha \in \mathbb{R}} (-\frac{\partial P}{\partial n}(\alpha, 0)) \geq \eta_0 \);
2. At \( t = T_0 \), \( \inf_{\alpha \in \mathbb{R}} (-\frac{\partial P}{\partial n}(\alpha, T_0)) \leq -\eta_1 \).

In other words, the infimum of the Taylor sign coefficient can transition from positive to an arbitrary large yet negative number.

**Remark 1.1** The category of infinite smoothness in Theorem 1 is indeed the Gevrey-2 space, see Definition 2.4 for the definition of such spaces, and Theorem 4 for the precise quantitative version of Theorem 1.

Such a result is our first step to investigate systematically the transition of the Rayleigh-Taylor instability (from stable regime to unstable regime, and vice versa) in
water waves. Since (1.1)-(1.2)-(1.4) is reversible in time, Theorem 1 implies that there exists a nonempty open set of initial data in a category of infinite smoothness with a pair of point vortices such that \( \inf_{\alpha \in \mathbb{R}} \left( -\frac{\partial P}{\partial n}(\alpha, 0) \right) < -\eta_1 \), while \( \inf_{\alpha \in \mathbb{R}} \left( -\frac{\partial P}{\partial n}(\alpha, T_0) \right) \geq \eta_0 \) at the finite time \( 0 < T_0 < \infty \). More importantly, we believe that Theorem 1 provides us the right framework to rigorously justify the numerical observation of Telste.

1.2 The Difficulty and Strategy

Let \( \Lambda_{1} \) be the Fourier multiplier defined by \( \hat{\Lambda}_{1}(\xi) = |\xi| \hat{f}(\xi) \). In Riemann mapping variables (See § 2.1.2 for the precise derivations), the water waves system can be written in the form

\[
\begin{align*}
D_t U &= -\Re\{D_t Q\} + A \Lambda_{1} W, \\
D_t W &= -U + \Re\{Q\} - \Re\{[D_t Z, \mathbb{H}]\} \left( \frac{1}{Z_\alpha} - 1 \right) - 2\Re\{(I - \mathbb{H})Q\}, \\
Z - \alpha &= (I + \mathbb{H})W, \\
F &= (I + \mathbb{H})U.
\end{align*}
\]

(1.5)

where \( Z \) is the free interface in Riemann variable; \( \tilde{Q} \) is the velocity field along the free boundary generated by the point vortices, \( \tilde{Q} + \tilde{F} \) is the velocity of the free interface, \( U \) is the real part of \( F \) and \( W \) is the real part of \( Z - \alpha \). \( D_t = \partial_t + b\partial_\alpha \), \( A \) and \( b \) are real-valued functions depending on \( U \), \( W \) and \( Q \), and \( Q = -\sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{1}{Z_\alpha(z_j(t) - z_j(t))} \).

\( \mathbb{H} \) is the standard Hilbert transform. \( A \) plays the role of the Taylor-sign coefficient. If \( \inf_{\alpha \in \mathbb{R}} A \geq 0 \), then the Taylor-sign condition holds; otherwise, the Taylor-sign condition fails.

1.2.1 A Mechanism for Solutions with A Changing Sign: The Interaction of Vorticity and the Free Interface

Let \( A_1 := A[Z_\alpha]^2 \). It suffices to construct solutions to the water waves with a sign-changing \( A_1 \). In [44], the author derived a formula for \( A_1 \), which we record as follows.

\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta \\
- \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \Re\left\{ \left( \frac{Z_\alpha}{Z(\alpha, t) - z_j(t)} \right)^2 \right\} \left( D_t Z - \dot{z}_j(t) \right) \right\}.
\]

(1.6)

Here, \( D_t Z \) is the trace of the velocity field along the free boundary. We split \( A_1 \) as

\[
A_1 = A_1^{pu} + A_1^{int},
\]

where \( A_1^{pu} \) represents the unperturbed case, i.e., corresponding to \( Z(\alpha, t) = \alpha, F = 0 \); \( A_1^{int} \) represents the contribution due to the interaction of the wave \( F \) and the point
vortices. Our idea is to find a solution to the water wave system such that, for given \( \eta_0 \in (0, 1) \) and \( \eta_1 > 0 \),

\[
\begin{align*}
(R1) \quad \inf_{\alpha \in \mathbb{R}} A_1^{\nu} (\alpha, 0) &> \eta_0, \\
(R2) \quad \inf_{\alpha \in \mathbb{R}} A_1^{\nu} (\alpha, T_0) &< -\eta_1, \text{ for some } T_0 > 0, \\
(R3) \quad \sup_{\alpha \in \mathbb{R}} |A_1^{int} (\alpha, t)| &\ll \left| \inf_{\alpha \in \mathbb{R}} A_1^{\nu} (\alpha, t) \right|, \quad \text{for } t = 0, T_0.
\end{align*}
\]

(R1) and (R2) can be guaranteed by analyzing the motion of a pair of counter-rotating point vortices. Indeed, we have the following observation.

**The Taylor sign of the 2d water waves with a vortex pair**

**Idealized case: the motion of the water waves is completely given by the motion of the point vortices.** Suppose there exists a smooth solution to the water waves such that at time \( t \), the free interface \( \Sigma_t = \mathbb{R} \), and the velocity field is generated by a pair of counter-rotating point vortices only, that is, the initial vorticity \( \omega := \lambda \delta z_1(t) - \lambda \delta z_2(t) \), with \( z_1(t) = -x(t) + iy(t) \) and \( z_2(t) = x(t) + iy(t) \) symmetric about the vertical axis. In this case, \( A_1^{\nu} = A_1 \). We take \( x(t) \sim 1 \), then we have

\[
\begin{align*}
(C1) \quad \text{For } \lambda \text{ fixed, if } |y(t)| \text{ is sufficiently large, then } A_1 &= 1 + O\left( \frac{\lambda^2}{|y(t)|^4} \right), \\
(C2) \quad \text{Let } \frac{\lambda^2}{|y(t)|^4} \to \infty \text{ and require } |y(t)| \gg 1, \text{ then } A_1 \to -\infty.
\end{align*}
\]

(C1) and (C2) will be justified from the calculation in § 7 and the appendix.

**Another observation:** If \( \lambda > 0 \), then the vortex pair \( z_1(t), z_2(t) \) travel toward the free interface (so \( y(t) \) becomes smaller).

**A natural idea:** Construct water waves with a symmetric pair of point vortices far away from the free interface initially and traveling toward the free interface rapidly. Then we expect \( A_1 \) to satisfy (R1)-(R2)-(R3).

To guarantee (R3), the main difficulty is to justify that the water waves remain a small perturbation of the motion of the vortex pair (for a period of time that allows the vortex pair to travel sufficiently close to the free interface such that the Taylor sign fails), which can be guaranteed if we can control the growth of \( U \) and \( W \). This forces us to prove the local existence in some reasonable spaces of such water waves. Nevertheless, even in the irrotational case, all the existing local wellposedness results for the water wave system with finite smoothness require the strong Taylor sign condition to hold. As we mentioned earlier, if the Taylor sign condition fails, the water waves can be subject to the Rayleigh-Taylor instability and a local wellposedness is not expected, at least in spaces with finite smoothness.

Roughly speaking, if the Taylor sign condition fails, the system (1.5) loses \( \frac{1}{2} \) derivatives. In Sobolev spaces \( H^s \), such a loss of derivative can be overcome if the strong Taylor sign condition holds. If the Taylor sign condition fails, then the Fourier mode can grow exponentially, which prevents a local wellposedness in Sobolev spaces.

Instead of seeking water waves with finite regularity (e.g., Sobolev spaces), we perturb the vortex pair in a category with infinite smoothness. Such an idea is certainly not new. Even in the context of water waves, by expanding the solution as a power series in time, Shinbrot [43], Kano and Nishida’s [32] obtained local existence in time...
for the general initial value problem with real analytic data. Their tool is the Cauchy-Kowalevski theorem. In [3], Alazard, Burq, and Zuily proved the local wellposedness of water waves in real analytic spaces via the energy method. Their main analytical tool is the paradifferential calculus. In this paper, working in Riemann mapping variables, we prove the local wellposedness in Gevrey-2 space (The Gevrey-2 space is a space of infinite smoothness that extends the real-analytic space. See Definition 2.4 for the precise definition) by using the energy method directly. The key to prove Theorem 1 is the following.

**Theorem 2** The water wave system (1.5) is locally wellposed in Gevrey-2 spaces.

Such a local wellposedness is expected to be sharp, in the sense that if the initial data is in a space rougher than the Gevrey-2 space, then the system (1.5) is illposed. See Remark 3.2 for more quantitative discussions.

To illustrate our strategy, we consider the following toy model:

\[
\begin{aligned}
\partial_t U &= A \Lambda W, \\
\partial_t W &= -U, \\
(U(t=0), W(t=0)) &= (U_0, W_0),
\end{aligned}
\]  

(1.7)

for some real-valued function \(A\) that might depend on \(u\) and \(W\).

### 1.2.2 The Gevrey Framework

Assume that \(A(\alpha_0, 0) < 0\) for some given \(\alpha_0 \in \mathbb{R}\). Heuristically, by considering solutions \(U\) and \(W\) concentrating near \(\alpha_0\), we can then without loss of generality (although it is not trivial to rigorously justify this) assume that \(A(\alpha, t) \equiv A(\alpha_0, 0) < 0\). In Fourier variables, we have

\[
\partial_t \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix} = \begin{bmatrix} 0 & A|\xi| \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}.
\]  

(1.8)

The matrix \(\begin{bmatrix} 0 & A|\xi| \\ -1 & 0 \end{bmatrix}\) has real eigenvalues \(\lambda_1 = \sqrt{-A|\xi|}\) and \(\lambda_2 = -\sqrt{-A|\xi|}\), with eigenvectors \(V_1 = \begin{bmatrix} -\sqrt{-A|\xi|} \\ 1 \end{bmatrix}\) and \(V_2 = \begin{bmatrix} \sqrt{-A|\xi|} \\ 1 \end{bmatrix}\), respectively. Let \(S = [V_1, V_2]\), and let \(Y = S^{-1} \begin{bmatrix} \hat{U} \\ \hat{W} \end{bmatrix}\), then

\[
\partial_t Y = \begin{bmatrix} \sqrt{-A|\xi|} & 0 \\ 0 & -\sqrt{-A|\xi|} \end{bmatrix} Y.
\]  

(1.9)

\[^{4}\text{The quantities } D_t Q, Q \text{ are not the main trouble of proving the local wellposedness in Gevrey spaces.}\]
So
\[ Y(\xi, t) = \left[ e^{t\sqrt{-A |\xi|}} 0 \\ 0 e^{-t\sqrt{-A |\xi|}} \right] Y(\xi, 0), \]  
which implies that if
\[ |Y(\xi, 0)| \gtrsim e^{-\delta |\xi|^{1/2}}, \]
for all \( \xi \in \mathbb{R} \) and for some \( \delta < \sqrt{-A} \), then we have
\[ \int_{\mathbb{R}} e^{2\delta |\xi|} |Y(\xi, t)|^2 d\xi = \infty. \]  
Therefore, it is natural to impose the assumption \( |Y(\xi, 0)| \lesssim e^{-\sqrt{-A |\xi|}} \). To guarantee
\[ \int_{\mathbb{R}} e^{t\sqrt{-A |\xi|}} |Y(\xi, 0)|^2 d\xi < \infty, \]
we require \( Y(\cdot, 0) \) lies in the Gevrey-2 spaces. See Definition 2.4 for the more precise definition. Denote
\[ \|f\|_{X_{L_0}}^2 := \sum_{n=0}^{\infty} \frac{L_0^{2n}}{(n!)^4} \int_{\mathbb{R}} |\xi|^{2n} |\hat{f}(\xi)|^2 d\xi \quad (= \sum_{n=0}^{\infty} \frac{L_0^{2n}}{(n!)^4} \|\partial_\alpha^nf\|_{L_2}^2). \]
\( L_0 \) is called the \textit{radius of analyticity}. We seek for solution \((U, W)\) such that the initial data \((U_0, W_0)\) satisfies
\[ \|\Lambda^{1/2}U_0\|_{X_{L_0}} < \infty, \quad \|\partial_\alpha W_0\|_{X_{L_0}} < \infty. \]

Of course, in general, \( A \) is not a constant. In particular, in the context of water waves, \( A \) depends on \( U \) and \( W \). Moreover, in the first equation of (1.7), \( W \) loses one half derivative, due to the term \( \Lambda U \). Using \( W_t = -U \), taking one time derivative on both sides of the equation \( \partial_t U = \Lambda \Lambda W \), we have
\[ U_{tt} = -\Lambda \Lambda U + A_t \Lambda W. \]
If \( A \) has a positive lower bound, then one can use the energy method to obtain closed energy estimates for the quantity \( \|U_t\|_{H^k} + \|\Lambda^{1/2}U\|_{H^k} \) (for some \( k > 0 \)), and moreover one can show that the quantity \( A_t \Lambda W \) can be control by \( \|U_t\|_{H^k} + \|\Lambda^{1/2}U_t\|_{H^k} \). However, if the lower bound of \( A \) is zero or even negative, then one loses the control of \( \|\Lambda^{1/2}U\|_{H^k} \), and therefore one cannot bound \( -\Lambda \Lambda U + A_t \Lambda W \) in Sobolev spaces.
To compensate for this loss, we allow the radius of analyticity of the solution to decay linearly in time, that is, we seek solution \((U, W)\) to (1.7) with initial data \((U_0, W_0)\) such that

\[
\|U(\cdot, t)\|_{X_{t_0-\delta t}} < \infty, \quad \|\Lambda^{1/2}W\|_{X_{t_0-\delta t}} < \infty, \quad (1.12)
\]

where \(\delta > 0\) is a constant. Then using \(U_t = AA W\), one has

\[
\frac{d}{dt} \|U(\cdot, t)\|_{X_{t_0-\delta t}}^2 = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{(L_0 - \delta t)^{2n}}{(n!)^4} \|\partial^n_{\alpha} U\|_{L^2}^2
\]

\[
= -2\delta \sum_{n=1}^{\infty} n(L_0 - \delta t)^{2n-1} \|\partial^n_{\alpha} U\|_{L^2}^2 + 2 \sum_{n=0}^{\infty} \frac{(L_0 - \delta t)^{2n}}{(n!)^4} \delta^n_{\alpha} U \frac{\partial^n_{\alpha} W}{Lt}
\]

\[
= -2\delta \sum_{n=1}^{\infty} n(L_0 - \delta t)^{2n-1} \|\partial^n_{\alpha} U\|_{L^2}^2 + 2 \sum_{n=0}^{\infty} \frac{(L_0 - \delta t)^{2n}}{(n!)^4} \delta^n_{\alpha} W (A \Lambda W)
\]

\[
:= I + II.
\]

Note that \(I\) is negative, which can be used to gain \(\frac{1}{4}\) derivatives. Similarly, using \(W_t = -U\) and therefore \((\Lambda^{1/2}W) _t = -\Lambda^{1/2}U\), we have

\[
\frac{d}{dt} \|\Lambda^{1/2}W(\cdot, t)\|_{X_{t_0-\delta t}}^2 = -2\delta \sum_{n=1}^{\infty} n(L_0 - \delta t)^{2n-1} \|\partial^n_{\alpha} \Lambda^{1/2}W\|_{L^2}^2
\]

\[
- 2 \sum_{n=0}^{\infty} \frac{(L_0 - \delta t)^{2n}}{(n!)^4} \delta^n_{\alpha} \Lambda^{1/2}W (\partial^n_{\alpha} (AA W))
\]

\[
:= III + IV.
\]

Again, \(III\) can be used to gain \(\frac{1}{4}\) derivatives. Note that

\[
\delta^n_{\alpha} (AA W) = AA \delta^n_{\alpha} W + \Lambda W \delta^n_{\alpha} A + G,
\]

where \(G\) consists of terms which can be handled easily. It is not difficult to estimate \(\|A - 1\|_{X_{t_0-\delta t}}\). So one can obtain

\[
\left| \sum_{n=0}^{\infty} \frac{(L_0 - \delta t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} U, \Lambda W \partial^n_{\alpha} A \rangle \right| \lesssim \|U\|_{X_{t_0-\delta t}} \|\Lambda W\|_\infty \left( \|A - 1\|_{X_{t_0-\delta t}} + 1 \right).
\]

---

5 Note that \(\sum_{n=1}^{\infty} \frac{\delta^{2n}}{(n!)^4} \|\partial^n_{\alpha} f\|_{L^2}^2 = \sum_{n=0}^{\infty} \frac{\delta^{2(n+1)}}{(n+1)!)^4} \|\partial^{n+1}_{\alpha} f\|_{L^2}^2 = \delta^2 \sum_{n=0}^{\infty} \frac{\delta^{2n}}{(n!)^4} \|\partial^n_{\alpha} f\|_{L^2}^2\). The denominator in the last summation has an extra factor \((n + 1)^4\), which corresponds to one derivative. Therefore, the quantity \(\sum_{n=0}^{\infty} \frac{n\delta^{2n}}{(n!)^4} \|\partial^n_{\alpha} f\|_{L^2}^2\) can control \(\frac{1}{4}\) more derivatives than \(\sum_{n=0}^{\infty} \frac{\delta^{2n}}{(n!)^4} \|\partial^n_{\alpha} f\|_{L^2}^2\).
And
\[
\sum_{n=0}^{\infty} \frac{(L_0 - \delta)^{2n}}{(n!)^4} \langle \partial_{\alpha}^n U, A A \partial_{\alpha}^n W \rangle 
\lesssim \left\| L^{1/4} U \right\|_{X_{L_0^{-\delta t}}} \left\| A L^{3/4} W \right\|_{X_{L_0^{-\delta t}}} + \text{good terms},
\]
where the "good terms" involve estimates for commutators, which are not difficult to estimate. We can choose \( \delta \) large so that \( \left\| L^{1/4} U \right\|_{X_{L_0^{-\delta t}}} \) can be bounded by \( |I| \), and \( \left\| L^{3/4} W \right\|_{X_{L_0^{-\delta t}}} \) can be bounded by \( |III| \). So we can obtain closed energy estimates.

The system (1.5) involves more complicated nonlinear interactions, yet the idea of compensating the loss of derivatives is the same as for the toy model. Therefore, we obtain closed energy estimates in Gevrey spaces. See §4 and §5 for the details.

1.2.3 The Lifespan of the Perturbation.

In this subsection we briefly discuss the lifespan for \( A_1 \) to transit from \( \eta_0 \) to \( -\eta_1 \), the details can be found in §7.

Consider the water waves with a pair of symmetric vortex pair. The initial interface and velocity satisfy the following:

(E1) The initial interface is flat, i.e., \( Z(\alpha, 0) = \alpha \), for all \( \alpha \in \mathbb{R} \);

(E2) A pair of counter-rotating vortex pair \( \lambda \delta z_1(0) - \lambda \delta z_2(0) \), with \( \lambda > 0 \) and
\[
z_1(0) = -x_0 + i y_0, \quad z_2(0) = x_0 + i y_0,
\]
where \( x_0 > 0 \) and \( y_0 < 0 \). We normalize \( x_0 = 1 \).

(E3) Assume that the initial velocity is generated by the point vortices, that is,
\[
v(z, 0) = \frac{\lambda i}{2\pi} \frac{1}{z - z_1(t)} - \frac{\lambda i}{2\pi} \frac{1}{z - z_2(t)},
\]
where \( v(z, 0) \) is the velocity of the water waves at position \( z \) and at time \( t = 0 \).

(E4) Assume the initial Taylor sign condition holds:
\[
\eta_0 := \inf_{\alpha \in \mathbb{R}} A_1(\alpha, 0) > 0.
\]

(E5) We choose \( |y_0| \gg 1 \). In order that \( \eta_0 > 0 \), we also assume \( |y_0| \geq \lambda^{2/3} \).

Denote \( Z(\alpha, t), D_t Z(\alpha, t) \) the free interface and the velocity field along the free interface at time \( t \). Denote
\[
Q(\cdot, t) = \frac{\lambda i}{2\pi} \frac{1}{Z(\cdot, t) - z_1(t)} - \frac{\lambda i}{2\pi} \frac{1}{Z(\cdot, t) - z_2(t)}.
\]
Denote\[\Gamma(t) := \inf_{\alpha \in \mathbb{R}} A_1(\alpha, t).\]

Let $T_0 > 0$. Suppose the water waves remains a small perturbation of the motion of the vortex pair on $[0, T_0]$, that is, for $0 \leq t \leq T_0$,

(P1) $\|Z(\alpha, t) - \alpha\|_{H^4} \ll 1$ (we solve the water waves in Gevrey spaces, in particular, it is in $H^4$.)

(P2) $\|D_t Z(\cdot, t) - D_t Q(\cdot, t)\|_{H^4} \ll 1$.

The reason that we choose $|y_0| \gg 1$ is due to the following estimates: assuming (P1)-(P2), if $|y(t)| \gg 1$, then by Corollary 5.1, we have

$$\|Q(\cdot, t)\|_{\dot{H}^1} \sim \left(\frac{\lambda^2}{|y(t)|^3}\right)^{1/2}, \quad \|D_t Q(\cdot, t)\|_{\dot{H}^1} \sim \frac{\lambda^2}{|y(t)|^{5/2}},$$

and the higher order derivatives of $Q$ and $D_t Q$ have better estimates:

$$(\ref{1.13})$$

With (1.13) and (1.14), we show that the minimum of the Taylor-sign coefficients evolves almost linearly: By Lemma 7.1, one has

$$A_1(\alpha, t) = 1 - \frac{\lambda^2}{\pi^2 |y(t)|^3} g(k) + O\left(\frac{\lambda^2}{|y(t)|^3}\right),$$

where $k = \frac{\alpha}{|y(t)|}$ and

$$g(k) = \frac{3k^4 + 2k^2 - 1}{(k^2 + 1)^4}.$$

Since $g(k)$ has maximum $\frac{1}{4}$, we have

$$\Gamma(t) = \inf_{\alpha} A_1(\alpha, t) = 1 - \frac{\lambda^2}{4\pi^2 |y(t)|^3} + O\left(\frac{\lambda^2}{|y(t)|^3}\right).$$

Note that the vortex pair moves almost vertically upwards at a speed approximately $\frac{\lambda}{4\pi x(t)} \approx \frac{\lambda}{4\pi}$ (we take $x(0) = 1$, and $x(t)$ is close to 1 for $0 \leq t \leq T_0$), so

$$y(t) = y_0 + \frac{\lambda t}{4\pi} + \text{small error}.\]
So we have
\[
\frac{\lambda^2}{4\pi^2 |y(t)|^3} = \frac{\lambda^2}{4\pi^2 |y_0|^3} \frac{1}{1 - \frac{\lambda}{4\pi |y_0|}} t \tag{1.17}
\]

For \(\frac{\lambda}{|y_0|} t \ll 1\), i.e., \(t \ll \frac{|y_0|}{\lambda}\), we have
\[
\frac{\lambda^2}{4\pi^2 |y(t)|^3} = \frac{\lambda^2}{4\pi^2 |y_0|^3} \frac{\lambda}{4\pi |y_0|} t + \text{small error}.
\]

So we obtain
\[
\Gamma(t) = 1 - \frac{\lambda^3}{16\pi^3 |y_0|^4} t, \tag{1.18}
\]
up to a negligible error.

Let’s make some comments on the following cases.

**Case 1.** \(|y_0|^3 \gg \lambda^2\). In particular, we can choose \(|y_0|\) so large that it looks as if the point vortices are coming from infinity. To avoid too many technical details, and also we think that this is of independent interest, we will not consider this case here.

So we consider \(|y_0|^3 \ll \lambda^2\). If we want \(\eta_0 > 0\), by (1.16), we need also to assume \(|y_0|^3 \gg \lambda^2\). So \(\lambda^2 = O(|y_0|^3/2)\) (e.g., \(|y_0|^3 = 100\lambda^2\)).

**Case 2.** \(\eta_1 + \eta_0 \ll 1\). In this case, both \(\eta_0\) and \(\eta_1\) are small. We need only to solve the water waves on \(O((\eta_0 + \eta_1)|y_0|^{-1/2})\) lifespan, which is not difficult.

**Case 3.** \(\eta_0 \sim 1\), and \(\Gamma_1 = -\eta_1\) with \(\eta_1 > 0\) is an arbitrary given number. If \(\eta_1\) is large, if (1.16) still holds, in order that \(A_1(T_0)\) takes a large negative number, we need \(\frac{\lambda^2}{|y(t)|^3}\) to be sufficiently large, which suggests that \(|y(T_0)| \ll |y_0|\). For simplicity, we take \(y(T_0) \approx -|y_0|^{9/10}\).

In a perturbative regime, the velocity of the point vortex pair is \(\frac{\lambda}{4\pi x(0)}\), plus a negligible error term (the details will be provided in §7). So we have
\[
T_0 = \frac{|y_0 - y(T_0)|}{\frac{\lambda}{4\pi x(0)}}. \tag{1.19}
\]

In particular, \(T_0 \gg \frac{1}{|y_0|^{1/2}}\).

Such a desired lifespan can be obtain by a careful analysis of the water wave system in Gevrey spaces. The main idea is as long as \(|y(t)| \gg 1\), the higher order derivatives of \(Q, D_t Q\) are small (see the estimates (1.13) and (1.14)). Therefore, the worst case is when both \(D_t Q\) and \(Q\) are estimated in \(L^2\). From (1.5), keeping the “worst terms”, we consider the following toy model:6

---

6 In §1.2.2, we have addressed the issue of wellposedness in the Gevrey spaces. Now our focus is on extending the lifespan of the solution. The main trouble is the terms involving the points vortices: \(Q\) and \(D_t Q\). Therefore, for the following toy model, we keep the terms with \(Q\) and \(D_t Q\).
\[
\begin{align*}
\partial_t U &= -\Re\{D_t Q\} \\
\partial_t W &= \Re\{Q\} - \Re\{[D_t Z, \mathbb{H}](\frac{1}{\omega^2} - 1)\}, \\
Z - \alpha &= (I + \mathbb{H})W, \\
F &= (I + \mathbb{H})U.
\end{align*}
\] (1.20)

From the structure of \(D_t Q\), the real part of \(D_t Q\) has better estimates, indeed we have

\[
\|\Re\{D_t Q(\cdot, t)\}\|_{L^2} \lesssim \frac{\lambda^2}{|y(t)|^{7/2}}.
\] (1.21)

For \(|y(t)| \lesssim |y_0|^{9/10}\), one has

\[
\partial_t \|U(\cdot, t)\|_{L^2} \lesssim \frac{\lambda^2}{|y_0|^{3/20}} \ll |y_0|^{-3/20} \ll 1.
\] (1.22)

So we can obtain

\[
\|U(\cdot, t)\|_{L^2} \ll 1,
\] (1.23)

A similar argument can be applied to control \(W\). The same idea is applied to study the system (1.5), so that we can solve the water wave equation with point vortices in a perturbative regime, and therefore by (1.16), one can show the transition of \(A_1\) from \(\eta_0\) to \(-\eta_1\).

**Remark 1.2** Another reason that we obtain a precise bound on the lifespan of the solution is for future investigation. We wish to refine our estimates in future work and extend the lifespan near the time when singularities occur.

### 1.3 Outline of the Paper

In § 2, we formulate the water wave system in Riemann variables, define the function spaces, and derive some estimates in Gevrey spaces. In § 3, we state a quantitative version of the main results. In § 4, using the energy method, we prove the local wellposedness in Gevrey spaces of a specific quasilinear system. In § 5, we derive a priori estimates. In § 6, we use the Picard iteration method to prove the existence and uniqueness of solutions of the system (2.29) and therefore conclude the proof of Theorem 3. In § 7, using Theorem 3, we prove Theorem 4. In the appendix, we provide the details of some calculations for the Taylor sign coefficient.

### 2 Preliminaries

Throughout this paper, we denote the fluid region at time \(t\) by \(\Omega(t)\), with a nonself-intersect free surface \(\Sigma(t)\). We identify a point \((x, y) \in \mathbb{R}\) as a point \(x + iy \in \mathbb{C}\). For a point \(x + iy \in \Sigma(t)\), we assume \(|y| \to 0\) as \(|x| \to \infty\). That is, \(\Sigma(t)\) approaches...
the real axis at \( \pm \infty \). Define \( \langle u, v \rangle := \int_R u(\alpha) \bar{v}(\alpha) \). Let \( z = x + iy \in \mathbb{C}, \Re\{z\} := x, \Im\{z\} := y \).

### 2.1 Governing Equation for the Free Boundary

The system (1.1)-(1.2)-(1.4) is completely determined by the free surface \( \Sigma(t) \), the trace of the velocity \( \bar{v} \) along the free surface, and the position of the point vortices. We shall first formulate the system (1.1)-(1.2)-(1.4) in Lagrangian coordinates. Because of the moving boundary, it is convenient to use the Riemann mapping variables to study the Taylor sign and construct special solutions.

#### 2.1.1 Lagrangian Formulation

We parametrize the free surface by Lagrangian coordinates, i.e., let \( \alpha \) be such that

\[
\frac{\partial}{\partial x}(\alpha, t) = v(z(\alpha, t), t).
\]

(2.1)

\( P \big|_{\Sigma(t)} \equiv 0 \) implies that \( \nabla P \big|_{\Sigma(t)} \) is along the normal direction, so we can write \( \nabla P \) as \( -ia z_{\alpha} \), where \( a = -\frac{\partial P}{\partial n} \frac{1}{|az_{\alpha}|} \) is real valued. Here, \( n = i \frac{z_{\alpha}}{|az_{\alpha}|} \) is the unit outward normal to \( \Omega(t) \). So the trace of the momentum equation \( \frac{\partial}{\partial t} v + v \cdot \nabla v = -\nabla P - (0, 1) \) can be written as

\[
z_{tt} - ia z_{\alpha} = -i.
\]

(2.2)

**Definition 2.1** (Hilbert transform). Assume that \( z(\alpha) \) satisfies

\[
\beta_0 |\alpha - \beta| \leq |z(\alpha) - z(\beta)| \leq \beta_1 |\alpha - \beta|, \quad \forall \, \alpha, \beta \in \mathbb{R},
\]

(2.3)

where \( 0 < \beta_0 < \beta_1 < \infty \) are constants. The Hilbert transform associated to the curve \( z(\alpha) \) is defined as

\[
\mathcal{H}_f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{z_{\beta}(\beta)}{z(\alpha) - z(\beta)} f(\beta) d\beta.
\]

(2.4)

The standard Hilbert transform is the Hilbert transform associated to \( z(\alpha) = \alpha \), that is,

\[
\mathbb{H}_f(\alpha) := \frac{1}{\pi i} p.v. \int_{-\infty}^{\infty} \frac{1}{\alpha - \beta} f(\beta) d\beta.
\]

(2.5)

We can use the Hilbert transform to characterize the boundary value of holomorphic functions. Such characterization is classical; the reader can see for example, Proposition 2.1 in [56].

**Lemma 2.1** Let \( \Omega \subset \mathbb{C} \) be a domain with \( C^1 \) boundary \( \Sigma : z = z(\alpha), \alpha \in \mathbb{R} \), oriented clockwise. Let \( \mathcal{H} \) be the cto \( \Omega \).
(a.) Let $g \in L^p$ for some $1 < p < \infty$. Then $g$ is the boundary value of a holomorphic function $G$ on $\Omega$ with $G(z) \to 0$ at infinity if and only if
\[(I - \mathcal{H})g = 0.\] (2.6)

(b.) Let $f \in L^p$ for some $1 < p < \infty$. Then $\frac{1}{2}(I + \mathcal{H})f$ is the boundary value of a holomorphic function $\mathcal{G}$ on $\Omega$, with $\mathcal{G}(z) \to 0$ as $|z| \to \infty$.

(c.) $\mathcal{H}1 = 0$.

We decompose $\bar{z}_t$ as $\bar{z}_t = f + p$, where $p = -\sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{z(\alpha,t) - z_j(t)}$. Note that $f$ is holomorphic in $\Omega(t)$ with the value at the boundary $\Sigma(t)$ given by $\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}$. Let $f \in L^2(\mathbb{R})$, then $f$ is the boundary value of a holomorphic function in $\Omega(t)$ if and only if
\[(I - \mathcal{H})f = 0,\] (2.7)
where $\mathcal{H}$ is the Hilbert transform associated with the curve $z(\alpha, t)$, i.e.,
\[\mathcal{H}f(\alpha) := \frac{1}{\pi i} \text{p.v.} \int_{-\infty}^{\infty} \frac{z_{\beta}}{z(\alpha,t) - z(\beta,t)} f(\beta) d\beta.\] (2.8)

Because of the singularity of the velocity at the point vortices, we don’t have $(I - \mathcal{H})\bar{z}_t = 0$. However, the following lemma asserts that $\bar{z}_t$ is almost holomorphic, in the sense that $(I - \mathcal{H})\bar{z}_t$ consists of lower order terms.

**Lemma 2.2** (Almost holomorphicity). Assume that $z(\cdot, t) \in L^2(\mathbb{R})$ satisfies (2.3) and $\bar{z}_t$ is the boundary value of a velocity field $\bar{v}$ in $\Omega(t)$ such that $\bar{v} + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z - z_j(t))}$ is holomorphic in $\Omega(t)$. Then we have
\[(I - \mathcal{H})\bar{z}_t = -\frac{i}{\pi} \sum_{j=1}^{N} \frac{\lambda_j}{z(\alpha,t) - z_j(t)}.\] (2.9)

**Proof** Since $\bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}$ is the boundary value of a holomorphic function in $\Omega(t)$, by lemma 2.1,
\[(I - \mathcal{H}) \left( \bar{z}_t + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))} \right) = 0,
\] we have
\[(I - \mathcal{H})\bar{z}_t = -\sum_{j=1}^{N} (I - \mathcal{H}) \frac{\lambda_j i}{2\pi(z(\alpha,t) - z_j(t))}.\] (2.10)

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Since \( \frac{1}{z(\alpha,t) - z_j(t)} \) is boundary value of the holomorphic function \( \frac{1}{z - z_j(t)} \) in \( \Omega(t)^c \), by lemma 2.1 again, we have

\[
(I - \mathcal{H}) \frac{1}{z(\alpha,t) - z_j(t)} = \frac{2}{z(\alpha,t) - z_j(t)}. \tag{2.11}
\]

(2.10) together with (2.11) complete the proof of the lemma.

So the system (1.1) is reduced to a system of equations for the free boundary coupled with the dynamic equation for the motion of the point vortices:

\[
\begin{cases}
  z_{tt} - i a z_{\alpha} = -i \\
  \frac{d}{dt} z_j(t) = (v - \lambda j i 2\pi (z - z_j(t))) |_{z=z_j} \\
  (I - \mathcal{H}) f = 0.
\end{cases} \tag{2.12}
\]

Note that \( v \) can be recovered from (2.12). Indeed, we have

\[
\tilde{u}(z,t) + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi (z - z_j(t))} = \frac{1}{2\pi i} \int \frac{z\beta}{z - z(\beta)} \left( z_t(\beta,t) + \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi (z(\beta,t) - z_j(t))} \right) d\beta. \tag{2.13}
\]

So the system (1.1) and the system (2.12) are equivalent. The quantity \( a|z_{\alpha}| \) plays an important role in the study of water waves\(^7\).

**Definition 2.2** (The Taylor sign condition and the strong Taylor sign condition)

1. If \( a|z_{\alpha}| \geq 0 \) pointwisely, then we say the Taylor-sign condition holds.
2. If there is some positive constant \( c_0 > 0 \) pointwisely, then we say the strong Taylor sign condition holds.

In order to derive a useful formula for the Taylor sign coefficient and use the iteration method to construct solutions to (2.12), we use the Riemann mapping formulation.

**2.1.2 The Riemann Mapping Formulation.**

Let \( \mathbb{P}_- = \{ z \in \mathbb{C} : \Im(z) < 0 \} \). Let \( \Phi(\cdot,t) : \Omega(t) \to \mathbb{P}_- \) be the Riemann mapping such that \( \Phi_z \to 1 \) as \( z \to \infty \). Denote

\[ a|z_{\alpha}| = \frac{\partial P}{\partial t} \bigg|_{\Sigma(t)}. \]

\(^7\) Indeed, \( a|z_{\alpha}| = -\frac{\partial P}{\partial t} \bigg|_{\Sigma(t)}. \)
\[
\begin{align*}
  h(\alpha, t) &:= \Phi(z(\alpha, t), t), \\
  Z(\alpha, t) &:= z \circ h^{-1}(\alpha, t), \\
  b &:= h_t \circ h^{-1}, \\
  D_t &:= \partial_t + b \partial_\alpha, \\
  A &:= (ah_\alpha) \circ h^{-1}, \\
  F &:= f \circ h^{-1}.
\end{align*}
\] (2.14)

In Riemann mapping variables, the system (2.12) becomes
\[
\begin{align*}
  (D_t^2 - iA \partial_\alpha)Z &= -i \\
  \frac{d}{dt}z_j(t) &= (v - \frac{\lambda_j i}{2\pi(z - z_j)}) \bigg|_{z=z_j} \\
  (I - \mathbb{H})F &= 0.
\end{align*}
\] (2.15)

Denote
\[
A_1 := A|Z_\alpha|^2.
\] (2.16)

Since \((a|z_\alpha|) \circ h^{-1} = A|Z_\alpha| = \frac{A_1}{Z_\alpha^2}\), it’s clear that the Taylor-sign condition holds if and only if
\[
\inf_{\alpha \in \mathbb{R}} \frac{A_1}{|Z_\alpha|} \geq 0,
\] (2.17)

and the strong Taylor sign condition holds if and only if
\[
\inf_{\alpha \in \mathbb{R}} \frac{A_1}{|Z_\alpha|} > 0.
\] (2.18)

**Definition 2.3** We call \(A_1\) the Taylor sign coefficient corresponding to the solution \((Z, F, \{z_j\})\) of the water waves.

**Remark 2.1** Note that \(A_1 = -|Z_\alpha| \frac{\partial P}{\partial n}\).

**Formula for \(b\):** Recall that \(h(\alpha, t) = \Phi(z(\alpha, t), t)\), where \(\Phi\) is the Riemann mapping. So we have
\[
h_t = \Phi_t + \Phi_z z_t, \quad \Phi_z = \frac{h_\alpha}{z_\alpha}.
\] (2.19)

Precomposite with \(h^{-1}\) on both sides of the above,
\[
b = h_t \circ h^{-1} = \Phi_t \circ Z + \frac{D_t Z}{Z_\alpha} = \Phi_t \circ Z + D_t Z(\frac{1}{Z_\alpha} - 1) + D_t Z
\]
\[
= \Phi_t \circ Z + D_t Z(\frac{1}{Z_\alpha} - 1) + \tilde{Q} + \tilde{F}.
\]
Applying $I - H$, using $(I - H)\Phi_r \circ Z = 0$ and $(I - H)\left(\frac{1}{Z_\alpha} - 1\right) = 0$, $(I - H)\bar{F} = 2F$, then taking real part, we obtain

\[
b = \Re\{(D_t Z, \bar{F})\left(\frac{1}{Z_\alpha} - 1\right)\} + 2\Re\{(D_t Z)\}
\]

Decomposing $b$ as $b = b_0 + b_1$, where

\[
b_0 = 2\Re\{F\} + \Re\{(\bar{F}, \bar{H})\left(\frac{1}{Z_\alpha} - 1\right)\},
\]

and

\[
b_1 = \Re\{(\bar{Q}, \bar{H})\left(\frac{1}{Z_\alpha} - 1\right)\} + 2\Re\{(I - H)\bar{Q}\}.
\]

Note that $b_1$ is more regular than $b_0$.

**Formula for $A_1$:** In §4 of [44], the author derived a formula for $A_1$, which we record as follows:

\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta
\]

\[
- \sum_{j=1}^N \frac{\lambda_j}{2\pi} \Re\left\{\left((I - H)\frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2}\right)(D_t Z - \hat{z}_j(t))\right\}. \quad (2.23)
\]

Let $u = D_t \bar{Z} = F + Q$, recall that $F(\alpha, t) = \mathcal{U}(Z(\alpha, t), t)$ for some holomorphic function $\mathcal{U}$ in $\mathbb{P}_-$, and $Q$ is given by

\[
Q = -\sum_{j=1}^N \frac{\lambda_j}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)},
\]

We have

\[
\mathcal{U}(z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z_\beta}{z - Z(\beta, t)} F(\beta, t) d\beta.
\]

So we obtain a system in Riemann variables:

\[
\begin{cases}
D_t F = -D_t Q - iA \bar{Z}_\alpha + i, \\
D_t (\bar{Z} - \alpha) = F + Q - b, \\
F, Z - \alpha \text{ holomorphic}.
\end{cases}
\]

\[
\odot \text{ Springer}
\]
(2.26) is equivalent to (2.12). Let \( U = \Re \{ F \} \), \( W = \Re \{ Z - \alpha \} \). Since \( F \) and \( Z - \alpha \) are holomorphic, \( U \) and \( W \) determine \( F \) and \( Z - \alpha \), respectively. Moreover, we have
\[
- i A \bar{Z} + i = \Lambda (\bar{Z} - \alpha) - i (A - 1),
\]
where \( \Lambda = |\partial_\alpha| \). Taking real parts on both sides of the first and the second equation of (2.26), we obtain
\[
\begin{align*}
\begin{cases}
D_t U &= \Lambda \Lambda W - \Re \{ D_t Q \}, \\
D_t W &= U + \Re \{ Q \} - b, \\
Z - \alpha &= (I - \mathbb{H}) W, \\
F &= (I - \mathbb{H}) U.
\end{cases}
\end{align*}
\]
Using (2.20), and decompose \( b = b_0 + b_1 \) for \( b_0, b_1 \) given by (2.21) and (2.22), respectively, we obtain
\[
\begin{align*}
\begin{cases}
D_t U &= -\Re \{ D_t Q \} + \Lambda \Lambda W \\
D_t W &= -U + \Re \{ \bar{Q} \} - \Re \{ [D_t \bar{F}, \mathbb{H}](\frac{1}{Z_\alpha} - 1) \} - b_1, \\
Z - \alpha &= (I + \mathbb{H}) W, \\
F &= (I + \mathbb{H}) U.
\end{cases}
\end{align*}
\]
The motion of \( z_j(t) \) is given by
\[
\frac{d}{dt} z_j(t) = \bar{U}(z_j(t), t) + \sum_{1 \leq k \leq N} \frac{\lambda_k i}{2\pi} \frac{1}{z_j(t) - z_k(t)},
\]
where \( \bar{U} \) is given by (2.25). Now we obtain a system (2.29)-(2.24)-(2.30), which is equivalent to (1.1). Let’s denote
\[
\begin{align*}
G &:= -\Re \{ D_t Q \}, \\
R &:= \Re \{ Q \} - b_1 = \Re \{ Q \} - 2\Re \{ (I - \mathbb{H}) \bar{Q} \} - \Re \{ [\bar{Q}, \mathbb{H}](\frac{1}{Z_\alpha} - 1) \}.
\end{align*}
\]
(2.29) then becomes
\[
\begin{align*}
\begin{cases}
D_t U &= \Lambda \Lambda W + G \\
D_t W &= -U - \Re \{ [\bar{F}, \mathbb{H}](\frac{1}{Z_\alpha} - 1) \} + R, \\
Z - \alpha &= (I - \mathbb{H}) W, \\
F &= (I - \mathbb{H}) U.
\end{cases}
\end{align*}
\]
\[8\] We need to estimate \( \| W_\alpha \|_{X_{\phi(t)}} \) (see Definition 2.4 for the definition of the Gevrey-2 norm \( \| \cdot \|_{X_\sigma} \)), so we need to estimate \( \| \partial_\alpha b_1 \|_{X_{\phi(t)}} \). We can prove that \( \| \partial_\alpha b_1 \|_{X_{\phi(t)}} < \infty \). However, \( \| \partial_\alpha b_0 \|_{X_{\phi(t)}} \) is not necessarily finite. So we need to treat \( \partial_\alpha b_0 \) carefully.
Note that $G$ is determined by $Z$, $F$ and $\{z_j\}$ (and therefore determined by $W$, $U$, and $\{z_j\}$), so we can write $G$ as $G(W, U, \{z_j(t)\})$. Here, $\{z_j\}$ means $\{z_1(t), \cdots, z_N(t)\}$. For the same reason, $D_t F$, $b_1$, $A$, $R$ are determined by $W$, $U$ and $\{z_j\}$. Therefore, we write

$$G = G(W, U, \{z_j(t)\})$$
$$R = R(W, U, \{z_j(t)\})$$
$$b_1 = b_1(W, U, \{z_j(t)\})$$
$$A = A(W, U, \{z_j(t)\})$$

### 2.1.3 Some Notations

For the reader’s convenience, we list some notations as follows

- $\Sigma(t)$: The free surface at time $t$,
- $\Omega(t)$: The fluid region at time $t$, $\Omega(t)$ is bounded above by $\Sigma(t)$,
- $\Phi(\cdot, t)$: The Riemann mapping from $\Omega(t)$ to $\mathbb{P}_-$,
- $Z(\alpha, t)$: The free interface in Riemann variables,
- $h(\alpha, t)$: The trace of $\Phi(\cdot, t)$, that is, $h(\alpha, t) = \Phi(Z(\alpha, t), t)$,
- $b(\alpha, t)$: $b(\cdot, t) = h_t \circ h^{-1}$,
- $D_t$ $D_t = \partial_t + b \partial_\alpha$,
- $D_t Z$: The trace of the velocity field,
- $A|Z_\alpha|$: The Taylor sign coefficient in Riemann mapping variable,
- $z_j(t)$: The coordinates of the $j$-th point vortex,
- $\dot{z}_j$: The time derivative of the point vortex,
- $d_I(t)$: The distance between the point vortices and the free interface at time $t$, that is, $d_I(t) := \inf_{\alpha \in \mathbb{R}} \min_{1 \leq j \leq N} |Z(\alpha, t) - z_j(t)|$,
- $Q(\alpha, t)$: The conjugate of the velocity field generated by the point vortices,
- $Q(\alpha, t) = -\sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)}$
- $F(\alpha, t)$: The wave part of the conjugate of the velocity field, that is, $F = D_t$ $\bar{Z} - Q$,
- $U(\cdot, t)$: The holomorphic extension of $F(\cdot, t)$, that is, $F(\alpha, t) = U(Z(\alpha, t), t)$,
- $A_1(\alpha, t)$: $A_1(\alpha, t) = A|Z_\alpha|^2$,
- $U$: The real part of the velocity field, that is, $U = \Re\{D_t Z\}$,
- $W(\alpha, t)$: The real part of $Z(\alpha, t) - \alpha : W = \Re\{Z - \alpha\}$.
We parametrize \( \Sigma_0 \) by Riemann mapping, that is, \( h_0(\alpha) = \alpha \), so \( z_0(\alpha) = Z_0(\alpha) \). Let \( U_0, W_0, z_j, Q_0, Q_1 \) denote the initial value of \( U, W, z_j, Q, D_t Q \), respectively.

Using (2.24), we obtain

\[
D_t Q = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{D_t Z - z_j(t)}{(Z(\alpha, t) - z_j(t))^2}.
\]

Using (2.30), we have

\[
\dot{z}_{j,0} = \sum_{1 \leq k \leq N, k \neq j} \frac{\lambda_j i}{2\pi} \frac{1}{z_{k,0} - z_{j,0}} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial \beta Z_0(\beta)}{z_{j,0} - Z_0(\beta)} F_0(\beta) d\beta. 
\] (2.34)

\[
Q_1 = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{\bar{u}_0 - \dot{z}_{j,0}}{(Z_0(\alpha) - z_{j,0})^2},
\] (2.35)

\[
A_{1,0} = 1 + \frac{1}{2\pi} \int |\bar{u}_0(\alpha) - \bar{u}_0(\beta)|^2 d\beta
\]

\[
- \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \Re \left\{ \left( I - \mathbb{H} \right) \frac{\partial \alpha Z_0(\alpha)}{(Z_0(\alpha) - z_j(t))^2} \right\} (\bar{u}_0(\alpha) - \dot{z}_{j,0}).
\] (2.36)

The initial interface \( Z_0 \) satisfies

\[
|Z_0(\alpha) - Z_0(\beta)| \geq C_0|\alpha - \beta|, \quad \forall \, \alpha, \beta \in \mathbb{R}.
\] (2.37)

### 2.2 Function Spaces

We define the Gevrey-2 spaces as follows.

**Definition 2.4** Let \( \sigma > 0 \), we define

\[
\dot{X}_\sigma = \{ f \in C^\infty(\mathbb{R}) \mid \| f \|_{\dot{X}_\sigma}^2 := \sum_{j=1}^{\infty} \frac{\sigma^{2j}}{(j!)^4} \| \partial_\alpha^j f \|_{L^2}^2 < \infty \}. 
\] (2.38)

\[
X_\sigma = \{ f \in C^\infty(\mathbb{R}) \mid \| f \|_{X_\sigma}^2 := \sum_{j=0}^{\infty} \frac{\sigma^{2j}}{(j!)^4} \| \partial_\alpha^j f \|_{L^2}^2 < \infty \}. 
\] (2.39)

\[
\dot{Y}_\sigma = \{ f \in C^\infty(\mathbb{R}) \mid \| f \|_{\dot{Y}_\sigma}^2 := \sum_{j=1}^{\infty} \frac{j^2 \sigma^{2j}}{(j!)^4} \| \partial_\alpha^j f \|_{L^2}^2 < \infty \}. 
\] (2.40)

\[
Y_\sigma = \{ f \in C^\infty(\mathbb{R}) \mid \| f \|_{Y_\sigma}^2 := \| f \|_{L^2}^2 + \sum_{j=0}^{\infty} \frac{j^2 \sigma^{2j}}{(j!)^4} \| \partial_\alpha^j f \|_{L^2}^2 < \infty \}. 
\] (2.41)
It is not difficult to verify that $\dot{X}_\sigma$, $X_\sigma$, $\dot{Y}_\sigma$ and $Y_\sigma$ are Banach spaces (actually, Hilbert spaces.), we shall call these spaces the Gevrey spaces, or Gevrey-2 spaces, and call $\sigma$ the radius of convergence. Moreover, we have the Sobolev type embedding.

Lemma 2.3  1. Let $f \in X_\sigma$ and $n \geq 0$ be an integer, then there is an absolute constant $C > 0$ such that

$$\|\partial_\alpha^n f\|_{L^\infty} \leq C \left( \frac{(n+1)!^2}{\sigma^{n+1}} + \frac{(n!)^2}{\sigma^n} \right) \|f\|_{X_\sigma}. \quad (2.42)$$

2. Let $f \in Y_\sigma$ and $n \geq 1$ be an integer, then

$$\|\partial_\alpha^n f\|_{L^\infty} \leq C \left( \frac{(n+1)!^2}{(n+1)! \sigma^{n+1}} + \frac{(n!)^2}{n \sigma^n} \right) \|f\|_{Y_\sigma}. \quad (2.43)$$

Proof The proof follows from the standard Sobolev embedding $\|f\|_{L^\infty} \leq C \|f\|_{H^1}$. 

The Hilbert transform is a unitary operator on these spaces.

$$\mathbb{H} f(\alpha) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{1}{\alpha - \beta} f(\beta) d\beta. \quad (2.44)$$

Lemma 2.4 Let $\sigma > 0$ and $f \in X_\sigma$, we have

$$\|\mathbb{H} f\|_{X_\sigma} = \|f\|_{X_\sigma}, \quad (2.45)$$
$$\|\mathbb{H} f\|_{\dot{X}_\sigma} = \|f\|_{\dot{X}_\sigma}, \quad (2.46)$$
$$\|\mathbb{H} f\|_{Y_\sigma} = \|f\|_{Y_\sigma}. \quad (2.47)$$

and

$$\|\mathbb{H} f\|_{\dot{Y}_\sigma} = \|f\|_{\dot{Y}_\sigma}. \quad (2.48)$$

Proof We prove (2.45) only. (2.47) and (2.48) are proved similarly. Using the fact that $\|\partial_\alpha^n \mathbb{H} f\|_{L^2} = \|\mathbb{H} \partial_\alpha^n f\|_{L^2} = \|\partial_\alpha^n f\|_{L^2}$, we have

$$\|\mathbb{H} f\|_{\dot{X}_\sigma}^2 = \sum_{n \geq 0} \frac{\sigma^{2n}}{(n!)^4} \|\partial_\alpha^n \mathbb{H} f\|_{L^2}^2 = \sum_{n \geq 0} \frac{\sigma^{2n}}{(n!)^4} \|\partial_\alpha^n f\|_{L^2}^2 = \|f\|_{\dot{X}_\sigma}^2. \quad \square$$

Lemma 2.5 (Product estimates). Let $\sigma \geq \sigma_0 > 0$. We have

$$\|fg\|_{X_\sigma} \leq C \|f\|_{X_\sigma} \|g\|_{X_\sigma}, \quad (2.49)$$
$$\|fg\|_{Y_\sigma} \leq C \|f\|_{Y_\sigma} \|g\|_{Y_\sigma}. \quad (2.50)$$
\[ \|fg\|_{X_\sigma} \leq C \|f\|_{X_\sigma} \|g\|_{X_\sigma} + C \|f\|_{X_\sigma} \|g\|_{\dot{X}_\sigma}, \quad (2.51) \]
\[ \|fg\|_{\dot{Y}_\sigma} \leq C \|f\|_{\dot{Y}_\sigma} \|g\|_{Y_\sigma} + C \|f\|_{Y_\sigma} \|g\|_{\dot{Y}_\sigma}, \quad (2.52) \]

for some constant \( C > 0 \) depends linearly on \( 1 + \frac{1}{\sigma_0} \).

**Proof** We prove (2.49) only.

First, note that
\[
\left( \sum_{n=0}^{10} \frac{\sigma^{2n}}{(n!)^2} \|\partial^n_s (fg)\|_{L^2}^2 \right)^{1/2} \leq C \|f\|_{X_\sigma} \|g\|_{X_\sigma},
\]

(2.53)

Second, for \( n > 10 \), we estimate \( \|\partial^n_s (fg)\|_{L^2} \) by
\[
\|\partial^n_s (fg)\|_{L^2} \leq \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \|f^{(k)} g^{(n-k)}\|_{L^2} \leq \sum_{k=0}^\lfloor n/2 \rfloor \frac{n!}{k!(n-k)!} \|f^{(k)} g^{(n-k)}\|_{L^2} + \sum_{k=\lfloor n/2 \rfloor + 1}^{n} \frac{n!}{k!(n-k)!} \|f^{(k)} g^{(n-k)}\|_{L^2}
\]
\[
:= I + II.
\]

(2.54)
\[
(2.55)
\]

Here, \( f^{(k)} \) and \( g^{(n-k)} \) represent the \( k \)-th and \( (n-k) \)-th derivative of \( f \) and \( g \), respectively.

For \( I \), we estimate \( f^{(k)} \) in \( L^\infty \) and estimate \( g^{(n-k)} \) in \( L^2 \), and we have
\[
\|f^{(k)} g^{(n-k)}\|_{L^2} \leq \|f^{(k)}\|_{L^\infty} \|g^{(k)}\|_{L^2} \leq C \left( \frac{(k!)^2}{\sigma^k} + \frac{(k+1)!^2}{\sigma^{k+1}} \right) \|f\|_{X_\sigma} \|g\|_{X_\sigma} \quad \text{for } 0 \leq k \leq 5
\]
\[
= C \left( 1 + \frac{(k+1)!^2}{\sigma^k} \right) \frac{(n-k)!^2}{\sigma^n} \|f\|_{X_\sigma} \|g\|_{X_\sigma}.
\]

(2.57)

Moreover, we further decompose \( I \) as
\[
I = \sum_{k=0}^{4} \frac{n!}{k!(n-k)!} \|f^{(k)} g^{(n-k)}\|_{L^2} + \sum_{k=5}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-k)!} \|f^{(k)} g^{(n-k)}\|_{L^2} := I_1 + I_2.
\]

(2.58)

For \( n > 10 \) and \( 0 \leq k \leq 4 \), we have
\[
\frac{\sigma^n}{(n!)^2} I_1 \leq C n^{-2} \|f\|_{X_\sigma} \|g\|_{X_\sigma}.
\]

(2.59)
For $k \geq 5$ and $n > 10$, we have

$$\frac{n!}{k!(n-k)!} = \frac{(n-k)! \prod_{j=0}^{k-1} (n-j)}{(n-k)! \frac{(n-k)!}{k!}} \geq C n^4.$$  \hspace{1cm} (2.60)$$

Using (2.57) and (2.60), we obtain

$$\frac{\sigma^n}{(n!)^2} I_2 \leq C \sum_{k=5}^{[n/2]} \frac{k!(n-k)!}{n!} \frac{1}{(k!)^2((n-k)!)^2} \frac{(k!)^2((n-k)!)^2}{\sigma^n} \|f\|_{X_{\sigma}} \|g\|_{X_{\sigma}}$$

$$\leq C \sum_{k=5}^{[n/2]} \frac{1}{n^4} (1 + \frac{k^2}{\sigma}) \|f\|_{X_{\sigma}} \|g\|_{X_{\sigma}}$$

$$\leq C n^{-1} \|f\|_{X_{\sigma}} \|g\|_{X_{\sigma}}.$$

So we obtain

$$\frac{\sigma^n}{(n!)^2} I \leq C n^{-1} \|f\|_{X_{\sigma}} \|g\|_{X_{\sigma}}.$$

Similarly,

$$\frac{\sigma^n}{(n!)^2} II \leq C n^{-1} \|f\|_{X_{\sigma}} \|g\|_{X_{\sigma}}.$$

Then we obtain

$$\|fg\|_{X_{\sigma}}^2 \leq \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^2} \|\partial^n f g\|_{L^2}^2$$

$$\leq \sum_{n=0}^{\infty} (1 + n^2)^{-1} \|f\|_{X_{\sigma}}^2 \|g\|_{X_{\sigma}}^2$$

$$\leq C \|f\|_{X_{\sigma}}^2 \|g\|_{X_{\sigma}}^2.$$

We have also the following estimates.

**Lemma 2.6** Let $\sigma \geq \sigma_0 > 0$ and $f, g \in X_{\sigma}$ and $g$ is real-valued. If in addition

$$\sum_{n=1}^{\infty} \frac{n^3 \sigma^{2n}}{(n!)^4} \|\partial^n f\|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n \sigma^{2n}}{(n!)^4} \|\partial^n g\|_{L^2}^2 < \infty,$$

We have also the following estimates.
then

\[
\left| \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \langle \partial_{\alpha}^{n+1} (fg), \partial_{\alpha}^{n} g \rangle \right| \\
\leq d_1 \|f\|_{L_\infty} \|g\|_{L_2}^2 + d_1 \|g\|_{L_\infty} \left( \sum_{n=1}^{\infty} \frac{n^3 \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} f\|_{L_2}^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{n \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} g\|_{L_2}^2 \right)^{1/2},
\]

(2.61)

where \(d_1 = d(\sigma_0^{-1} + 1)\) for some absolute constant \(d > 0\). In particular, if \(\sigma_0 \geq 1\), then \(d_1\) is an absolute constant.

**Proof** We have

\[
\partial_{\alpha}^{n+1} (fg) = g \partial_{\alpha}^{n+1} f + f \partial_{\alpha}^{n+1} g + \sum_{k=1}^{n+1} \frac{(n + 1)!}{k! (n + 1 - k)!} \partial_{\alpha}^{k} f \partial_{\alpha}^{n+1-k} g.
\]

Using the same proof as in Lemma 2.5, we obtain

\[
\left| \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left( \sum_{k=1}^{n+1} \frac{(n + 1)!}{k! (n + 1 - k)!} \partial_{\alpha}^{k} f \partial_{\alpha}^{n+1-k} g, \partial_{\alpha}^{n} g \right) \right| \leq d_1 \|f\|_{L_\infty} \|g\|_{L_2}^2.
\]

By Cauchy-Schwarz inequality, we have

\[
\left| \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \langle g \partial_{\alpha}^{n+1} f, \partial_{\alpha}^{n} g \rangle \right| \\
\leq \left( \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{n(n!)^4} \|g \partial_{\alpha}^{n+1} f\|_{L_2}^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{n \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} g\|_{L_2}^2 \right)^{1/2} \\
\leq \|g\|_{L_\infty} \left( \sum_{n=1}^{\infty} \frac{\sigma^{2(n+1)}}{(n+1)!^4} \|\partial_{\alpha}^{n+1} f\|_{L_2}^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{n \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} g\|_{L_2}^2 \right)^{1/2} \\
= \|g\|_{L_\infty} \left( \sum_{n=1}^{\infty} \frac{\sigma^{2(n+1)}}{(n+1)!^4} \|\partial_{\alpha}^{n+1} f\|_{L_2}^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{n \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} g\|_{L_2}^2 \right)^{1/2} \\
\leq d_1 \|g\|_{L_\infty} \left( \sum_{n=1}^{\infty} \frac{n^3 \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} f\|_{L_2}^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{n \sigma^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} g\|_{L_2}^2 \right)^{1/2},
\]

where we can take \(d_1 = 2 \sigma^{-1} + K\), for some absolute constant \(K > 0\). Integration by parts, we obtain

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\[
\left| \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left( f \partial_{\alpha}^{n+1} g, \partial_{\alpha}^n g \right) \right| = \left| \sum_{n=1}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left( f_{\alpha} \partial_{\alpha}^n g, \partial_{\alpha}^n g \right) \right| \\
\leq \| f_{\alpha} \|_{L^\infty} \| g \|_{X_\sigma}^2 \leq d_1 \| f \|_{\dot{X}_\sigma} \| g \|_{X_\sigma}^2.
\]

Using \( \| g \|_{L^\infty} \leq 2(1 + \sigma^{-1}) \| g \|_{X_\sigma} \), we conclude the proof of the lemma. \( \square \)

### 2.3 Commutator Estimates in Gevrey Spaces

Let \( f, g \in X_\sigma \). Define
\[
S_1(g, f)(\alpha, t) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{g(\alpha, t) - g(\beta, t)}{(\alpha - \beta)^2} f(\beta, t) d\beta.
\]
\[
S_2(g, f)h(\alpha, t) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{(g(\alpha, t) - g(\beta, t))(f(\alpha, t) - f(\beta, t))}{(\alpha - \beta)^2} \frac{h_\beta}{Z_\beta} d\beta.
\]

The following commutator estimates are the Gevrey version of the Sobolev counterpart.

**Lemma 2.7** 1. Let \( f \in Y_\sigma \), \( g \in X_\sigma \), then
\[
\|[f, \mathbb{H}] g_\alpha \|_{X_\sigma} \leq C \| f \|_{\dot{X}_\sigma} \| g \|_{X_\sigma}.
\]  \( (2.64) \)
\[
\|[f, \mathbb{H}] g_\alpha \|_{Y_\sigma} \leq C \| f \|_{\dot{Y}_\sigma} \| g \|_{X_\sigma}.
\]  \( (2.65) \)

2. Let \( f, g, h \in X_\sigma \), \( Z - \alpha \in \dot{Y}_\sigma \). There holds
\[
\| S_2(g, f)h \|_{X_\sigma} \leq C \| f \|_{X_\sigma} \| g \|_{X_\sigma} \| h \|_{X_\sigma}(1 + \| Z - \alpha \|_{\dot{Y}_\sigma}).
\]  \( (2.66) \)

3. If in addition \( \sum_{n=1}^{\infty} \frac{n^3 \sigma^{2n}}{(n!)^4} \| \partial_{\alpha}^n f \|_{L^2}^2 < \infty \), and \( \sum_{n=1}^{\infty} \frac{n^2 \sigma^{2n}}{(n!)^4} \| \partial_{\alpha}^n g \|_{L^2}^2 < \infty \), then
\[
\|[f_{\alpha}, \mathbb{H}] g \|_{X_\sigma} \leq C \left( \sum_{n=1}^{\infty} \frac{n^3 \sigma^{2n}}{(n!)^4} \| \partial_{\alpha}^n f \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n^2 \sigma^{2n}}{(n!)^4} \| \partial_{\alpha}^n g \|_{L^2}^2 \right) + C \| g \|_{L^2}^2.
\]  \( (2.67) \)

**Proof** We prove \( (2.64) \) only. It suffices to notice that
\[
\|[f, \mathbb{H}] g_\alpha \|_{L^2} \leq C \min\{ \| f \|_{L^2}, \| f' \|_{L^2}, \| f'' \|_{L^2}, \| g \|_{L^2} \}. \quad (2.68)
\]
\( (2.68) \) follows easily from Fourier analysis, we omit the proof. Using \( (2.68) \), we obtain

(1) For \( n = 0 \):
\[
\|[f, \mathbb{H}] g_\alpha \|_{L^2} \leq C \| f' \|_{L^\infty} \| g \|_{L^2} \leq C \| f \|_{\dot{X}_\sigma} \| g \|_{X_\sigma}.
\]  \( (2.69) \)
(2) For \( n \geq 1 \), we have

\[
\| \partial^n \alpha [f, H] g_\alpha \|_{L^2} \leq C(\| \partial^n \alpha \|_{L^2} \| g \|_{X_\sigma}). \tag{2.70}
\]

Indeed, let \( n \geq 1 \), using \( |\xi|^n |\text{sgn}(\xi) - \text{sgn}(\eta)| \leq |\xi - \eta|^n \), we have

\[
\left| \widehat{\partial^n \alpha [f, H] g_\alpha}(\xi) \right| \leq |\xi|^n \int |\text{sgn}(\eta) - \text{sgn}(\xi)| |\eta| \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \\
\leq \int |\widehat{\partial^n \alpha f}(\xi - \eta) \partial_\alpha \hat{g}(\eta) d\eta \\
= |\widehat{\partial^n \alpha f}| \| \partial_\alpha \hat{g} \|_{(\xi)}
\]

So

\[
\| \partial^n \alpha [f, H] g_\alpha \|_{L^2} \leq \| |\widehat{\partial^n \alpha f}| \|_{L^2} \| \partial_\alpha \hat{g} \|_{L^1} \leq \| \partial^n \alpha f \|_{L^2} \| \partial_\alpha \hat{g} \|_{L^1}.
\]

By Plancherel Theorem, \( \| |\widehat{\partial^n \alpha f}| \|_{L^2} = \| \partial^n \alpha f \|_{L^2} \). Using

\[
\hat{g}_\alpha(\xi) = \frac{1}{|\xi|} \hat{g}_\alpha(\xi) = \frac{1}{|\xi|} \partial_\alpha \hat{g},
\]

we have

\[
\| \partial_\alpha \hat{g} \|_{L^1} \leq \int_{|\xi| \leq 1} |\partial_\alpha \hat{g}(\xi)| d\xi + \int_{|\xi| \geq 1} \frac{1}{|\xi|} |\partial_\alpha ^2 \hat{g}(\xi)| d\xi \leq \| g_\alpha \|_{H^1} \leq C \| g \|_{X_\sigma}.
\]

So we obtain (2.70), and therefore conclude the proof of the lemma. \( \square \)

**Lemma 2.8** Let \( H \in X_\sigma \) be such that \( 1 + H \geq c_0 \) for some constant \( c_0 > 0 \). Then

(1) \( \frac{1}{H + 1} - 1 \in X_\sigma \), and

\[
\left\| \frac{1}{H + 1} - 1 \right\|_{X_\sigma} \leq C(c_0) \| H \|_{X_\sigma} \tag{2.71}
\]

(2) If in addition \( \sum_{n=1}^{\infty} \frac{n!}{n!} \| \partial^n H \|_{L^2}^2 < \infty \), then

\[
\sum_{n=1}^{\infty} \frac{n!}{n!} \| \partial^n \left( \frac{1}{H + 1} - 1 \right) \|_{L^2}^2 \leq C(c_0) \sum_{n=1}^{\infty} \frac{n!}{n!} \| \partial^n H \|_{L^2}^2 \tag{2.72}
\]

Here \( C(c_0) \) is a constant depending on \( c_0 \).

The proof of Lemma 2.8 is similar to that of Lemma 2.5, so we omit the proof.
Lemma 2.9 Let \( w \in \mathbb{P}^- \) and define \( h(\alpha) = \frac{1}{(\alpha - \omega)^2} \). Then \( h \in X_\sigma \) for any \( \sigma \in \mathbb{R} \), and

\[
\|h\|_{X_\sigma} \leq C \frac{1}{\sqrt{|Im\{w\}|}} e^{\frac{\sigma}{4\pi|Im\{w\}|}}
\]  

(2.73)

**Proof** The Fourier transform of \( h \) is

\[
\hat{h}(\xi) = -2\pi i (i\xi) e^{-w|\xi|}.
\]

(2.74)

For \( \sigma \in \mathbb{R} \),

\[
\|h\|_{X_\sigma}^2 = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \|\partial_\alpha^n h\|_{L^2}^2
\]

\[
= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \|\xi^n \hat{h}(\xi)\|_{L^2}^2
\]

\[
= \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \int |\xi|^{2n+2} e^{2\pi i|Im\{w\}||\xi^2|} d\xi
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \frac{1}{|2\pi|Im\{w\}|^{2n+1}} \int_0^{\infty} |\xi|^{2n} e^{-|\xi^2|} d\xi
\]

\[
= 2 \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \frac{1}{|2\pi|Im\{w\}|^{2n+1}} (2n+1)!
\]

\[
\leq \frac{1}{2\pi|Im\{w\}|} \sum_{n=0}^{\infty} \left( \frac{\sigma}{2\pi|Im\{w\}|} \right)^{2n} \frac{2^n}{(n!)^2}
\]

\[
\leq C \frac{1}{|Im\{w\}|} e^{\frac{\sigma}{2\pi|Im\{w\}|}},
\]

for some absolute constant \( C > 0 \). \( \square \)

Lemma 2.10 Let \( \sigma > 0 \) and let \( f \in X_\sigma, g, h \in Y_\sigma \). Then

\[
\sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left\| \partial_\alpha^n (f \wedge g), \partial_\alpha^n h \right\| \leq d_0 \|f\|_{X_\sigma} \|g\|_{Y_\sigma} \|h\|_{Y_\sigma},
\]

(2.75)

\[
\sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left\| \partial_\alpha^n (f \partial_\alpha g), \partial_\alpha^n h \right\| \leq d_0 \|f\|_{X_\sigma} \|g\|_{Y_\sigma} \|h\|_{Y_\sigma},
\]

(2.76)

\[
\sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left\| \partial_\alpha^n (f \partial_\alpha g), \partial_\alpha^n g \right\| \leq d_0 \|f\|_{X_\sigma} \|g\|_{X_\sigma}^2,
\]

(2.77)

where \( d_0 = K (1 + \sigma^{-1}) \) for some absolute constant \( K > 0 \). In particular, if \( \sigma \geq 1 \), then \( d_0 \) is an absolute constant.
(2) If in addition \( \sum_{n=1}^{\infty} \frac{n^2 \sigma^2 n}{(n!)^2} \| \partial^n_{\alpha} g \|_{L^2}^2 < \infty \), and \( \sum_{n=1}^{\infty} \frac{n^2 \sigma^2 n}{(n!)^2} \| \partial^n_{\alpha} h \|_{L^2}^2 < \infty \), then

\[
\sum_{n=1}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \left| \langle \partial^n_{\alpha} (f \Lambda g), \partial^n_{\alpha} h \rangle \right| \leq d_0 \| f \|_{X_{\sigma}} \left( \sum_{n=1}^{\infty} \frac{n^2 \sigma^2 n}{(n!)^4} \| \partial^n_{\alpha} g \|_{L^2}^2 \right)^{1/2} \left( \| h \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n^2 \sigma^2 n}{(n!)^4} \| \partial^n_{\alpha} h \|_{L^2}^2 \right)^{1/2}.
\]

(2.78)

**Proof** Let \( n \geq 1 \), we have

\[
\left| \langle \partial^n_{\alpha} (f \Lambda g), \partial^n_{\alpha} h \rangle \right| \leq \left| \langle f \partial^n_{\alpha} \Lambda g, \partial^n_{\alpha} h \rangle \right| + \left| \langle \partial^n_{\alpha} (f \Lambda g) - f \partial^n_{\alpha} \Lambda g, \partial^n_{\alpha} h \rangle \right|
\]

:= \( I_1 + I_2 \).

For \( I_1 \), we have

\[
|I_1| = \left| \int f (\partial^n_{\alpha} \Lambda g) \overline{\partial^n_{\alpha} h} \right| \\
\leq \int \left| f (\partial^n_{\alpha} \Lambda^{1/2} g) \Lambda^{1/2} \overline{\partial^n_{\alpha} h} \right| + \int \left| g \right| \left| \Lambda^{1/2} \overline{\partial^n_{\alpha} h} \right| \\
\leq d_0 \| f \|_{X_{\sigma}} \| \Lambda^{1/2} \partial^n_{\alpha} g \|_{L^2} \| \Lambda^{1/2} \partial^n_{\alpha} h \|_{L^2}.
\]

So we obtain

\[
\sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} I_1 \leq C \| f \|_{X_{\sigma}} \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} \| \Lambda^{1/2} \partial^n_{\alpha} g \|_{L^2} \| \Lambda^{1/2} \partial^n_{\alpha} h \|_{L^2} \\
\leq d_0 \| f \|_{X_{\sigma}} \| g \|_{Y_{\sigma}} \| h \|_{Y_{\sigma}}.
\]

The term \( I_2 \) can be handled in the same way as in the proof of Lemma 2.5, we omit the details. We have

\[
\sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^4} I_2 \leq d_0 \| f \|_{X_{\sigma}} \| g \|_{Y_{\sigma}} \| h \|_{Y_{\sigma}}.
\]

So we complete the proof of (2.75). The proof of (2.76) is the same as that of (2.75). The proof of (2.77) is similar:

\[
\left| \langle \partial^n_{\alpha} (f \Lambda g), \partial^n_{\alpha} g \rangle \right| \leq \left| \langle f \partial^n_{\alpha} \Lambda g, \partial^n_{\alpha} g \rangle \right| + \left| \langle \partial^n_{\alpha} (f \Lambda g) - f \partial^n_{\alpha} \Lambda g, \partial^n_{\alpha} g \rangle \right|
\]

:= \( II_1 + II_2 \).
The treatment of $II_2$ is the same as that for $I_2$. For $II_1$, we have

$$II_1 = \left| \Re \int f \partial^n_\alpha \Lambda g \overline{\partial^m_\alpha g} \right| = \left| \Re \int |\partial^n_\alpha g|^2 \Lambda f \right| \leq d_0 \|\Lambda f\|_{L^\infty} \|\overline{\partial^m_\alpha g}\|_{L^2}^2 \leq d_0 \|f\|_{\dot{X}_\sigma} \|\partial^m_\alpha g\|_{L^2}^2.$$

So \( \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^2} I_1 \leq C \|f\|_{\dot{X}_\sigma} \|\overline{\partial^m_\alpha g}\|_{\dot{X}_\sigma}. \) So we obtain (2.77).

For the proof of (2.78), it suffices to notice that

$$\left| \int f (\partial^m_\alpha \Lambda g) \overline{\partial^m_\alpha h} \right| \leq \int \left| f (\partial^m_\alpha \Lambda^{3/4} g) \overline{\Lambda^{1/4} \partial^m_\alpha h} \right| + \int \left| g[f, \Lambda^{1/4}] \overline{\partial^m_\alpha h} \right| \leq d_0 \|f\|_{\dot{X}_\sigma} \|\Lambda \overline{\partial^m_\alpha g}\|_{L^2} (1 + \Lambda^{1/4}) \|\partial^m_\alpha h\|_{L^2}.$$

\[ \square \]

3 The Main Results: Quantitative Statements

Throughout the rest of this paper, for brevity, we consider a pair of symmetric and counter-rotating point vortices embedded in the water waves, the general situation can actually be treated similarly. We assume

$$\omega(\cdot, t) = \lambda \delta_{z_1(t)} - \lambda \delta_{z_2(t)},$$

where \( \lambda \in \mathbb{R} \), and \( z_1(t) = -x(t) + iy(t), z_2(t) = x(t) + iy(t) \), with \( x(t) > 0 \) and \( z_1(t), z_2(t) \in \Omega(t) \). We assume also that \( \Omega(t) \) is symmetric about the vertical axis. Without loss of generality, we assume \( x(0) = 1. \) Moreover, we assume \( |y(0)| \gg 1 \). It should be clear from the proof that regarding the local wellposedness in Gevrey spaces, these assumptions are unnecessary.

3.1 The Initial Data

Let the initial fluid region be given by \( \Omega_0 \), with a nonself-intersecting smooth free surface \( \Sigma_0 \). We parametrize \( \Sigma_0 \) by Riemann variable, that is, we choose \( \alpha \) such that \( h_0(\alpha) = \alpha. \)

Let \( Z_0(\alpha) := Z(\alpha, 0), U_0 = \Re \{F_0\}, \) and \( W_0 = \Re \{Z_0 - \alpha\}. \) Recall that

$$d_I(t) = \inf_{\alpha \in \mathbb{R}} \min_{1 \leq j \leq 2} |Z(\alpha, t) - z_j(t)|,$$

and \( d_{I,0} := d_I(0), x_0 := x(0) = 1. \) Without loss of generality, we assume

$$d_{I,0} = \inf_{\alpha \in \mathbb{R}} \min_{j=1,2} |\Re\{Z(\alpha, 0) - z_j(0)\}|.$$

9 Indeed, such an assumption can be removed. We assume it for convenience.
Let $x_0, y_0, \lambda$, and $L_0 \geq 4$ be given constants. Denote $\phi(t) = L_0 - \delta_0 t$. We assume the following:

(H1) $(W_0, U_0)$ is given such that $(\partial_\alpha W_0, U_0) \in X_{L_0} \times Y_{L_0}$. Without loss of generality, we assume for simplicity that

$$\|(U_0, \partial_\alpha W_0)\|_{Y_{L_0} \times X_{L_0}} \leq 1/2. \quad (3.2)$$

(H2) $\delta_0$ is chosen such that

$$\frac{\delta_0}{L_0} - 4 - 2d_0(\|A(\cdot, 0)\|_{L^\infty} + \|A(\cdot, 0)\|_{\dot{Y}_2})$$

$$- 4d_1(\|U_0, \partial_\alpha W_0\|_{Y_{\phi(0)} \times X_{\phi(0)}}) \geq 0. \quad (3.3)$$

Here, $A(\cdot, 0) = \frac{A_{1, 0}}{[\delta_0, Z_0]^2}$, with $A_{1, 0}$ given by (2.36), and $d_0$ and $d_1$ are the constants given in Lemma 2.10 and Lemma 2.6, respectively.

(H3) $z_{1,0} = -x_0 + iy_0, z_{2,0} = x_0 + iy_0$, where $x_0$ and $y_0$ are constants, satisfying $x_0 = 1, y_0 < 0$, and $|y_0| \gg 1$.

(H4) $|Z_0(\alpha) - Z_0(\beta)| \geq C_0 |\alpha - \beta|$, for some absolute constant $C_0 > 0$.

(H5) $Z_0(\alpha) := \alpha + (I + \mathbb{H})W_0, F_0 = (I + \mathbb{H})U_0$.

(H6) $\Omega_0$ is symmetric about the $y$-axis, $W_0$ is an odd function. $U_0$ is also an odd function.

For $0 \leq t \leq \frac{L_0}{2\delta_0}$, we have $\phi(t) \geq \frac{1}{2} L_0$. Denote

$$\tau_0 := \frac{1}{4 + 2d_0(\|A(\cdot, 0)\|_{L^\infty} + \|A(\cdot, 0)\|_{\dot{Y}_2}) + 4d_1(\|U_0, \partial_\alpha W_0\|_{Y_{\phi(0)} \times X_{\phi(0)}}). \quad (3.4)$$

In particular, $\delta_0 \geq 4L_0 \geq 16$.

**Remark 3.1** The assumption that $|y_0| \gg 1$ is not essential in terms of the local well-posedness in Gevrey-2 spaces for the system (2.33). We can certainly remove it. Nevertheless, for the purpose of proving Theorem 1, do need to assume $|y_0| \gg 1$. Therefore, for convenience, we make such an assumption.

### 3.2 The Main Theorems

**Theorem 3** Let $W_0, U_0, \{z_j, 0\}, L_0, \delta_0, Z_0$, and $F_0$ be given such that (H1)-(H6) hold. Then there exists a constant $T > 0$, such that the system (2.33) with initial data $(W_0, U_0, \{z_j, 0\})$ admits a unique solution $(W, U, \{z_j\})$ satisfying

(a) $(W_\alpha, U) \in C([0, T]; X_{\phi(t)} \times Y_{\phi(t)}))$, and $z_j \in C^1([0, T]; \Omega(t)), \quad j = 1, 2$. Moreover,

$$\sup_{0 \leq t \leq T} \left( \|U(\cdot, t)\|^2_{Y_{\phi(t)}} + \|W_\alpha\|^2_{\dot{Y}_{\phi(t)}} \right)$$

\[10\] In application, we will take $\|(U_0, \partial_\alpha W_0)\|_{Y_{L_0} \times X_{L_0}}$ to be small.
Theorem 4

Let \( \delta \) constructed in Theorem 3 with initial data \( \delta \), \( \lambda > 0 \), \( \lambda < 0 \)

\[
\begin{aligned}
\| (U_0, \partial_\alpha W_0) \|^2_{Y_{\phi(t)} \times X_{\phi(t)}} &+ C|\lambda|(d_{1,0} - \frac{|\lambda|}{4\pi x(0)} T)^{-5/2}, \\
\| (U_0, \partial_\alpha W_0) \|^2_{Y_{\phi(t)} \times X_{\phi(t)}} &+ C|\lambda|d_{1,0}^{-5/2},
\end{aligned}
\]

and

\[
\sup_{0 \leq t \leq T} \left\| U(\cdot, t) \right\|_{L^2}^2 \leq 1, \quad \sup_{0 \leq t \leq T} \| U(\cdot, t) \|_{L^\infty} \leq C d_{1,0}^{-1/3}.
\]

Here, the constant \( C \) depends on \( C_0 \) and \( \frac{1}{L_0} \).

(b) \( |Z(\alpha, t) - Z(\beta, t)| \geq \frac{1}{2} C_0 |\alpha - \beta| \), \( \alpha, \beta \in \mathbb{R}, \ t \in [0, T] \).

(c) \( d_1(t) \geq \frac{1}{2} |y_0|^{9/10}, \quad \frac{1}{2} x(0) \leq x(t) \leq 2x(0), \quad t \in [0, T] \).

(d) \( \Omega(t) \) is symmetric about the \( y \)-axis. For each fixed \( t \in [0, T] \), \( W(\cdot, t) \) and \( U(\cdot, t) \) are odd functions. \( \Re \{z_1(t)\} = -\Re \{z_2(t)\}, \ \Im \{z_1(t)\} = \Im \{z_2(t)\} \).

Here \(^{11}\),

\[
T = \min\{T_1(C_0, \| (U_0, \partial_\alpha W_0) \|_{Y_{L_0} \times X_{L_0}}), \frac{L_0}{2L_0}, \tau_0, \frac{4\pi x(0)\|y_0\| - |y_0|^{9/10}}{\| \lambda \|}, \lambda > 0, \lambda < 0, \}
\]

with \( T_1, T_2 \) depend continuously on its parameters. In particular, if \( |y_0| \) is sufficiently large, then if \( \lambda > 0 \), we can take \( T = \frac{4\pi x(0)\|y_0\| - |y_0|^{9/10}}{\| \lambda \|}; \) if \( \lambda < 0 \), we can take \( T = O(1) \).

Remark 3.2 (1) Using the same proof as in [44], we can prove that the solutions to the water waves preserve the symmetries in (H6). We shall omit the proof of (d) and refer the readers to Theorem 5 in [44].

(2) From the discussion in § 1.2.2, in particular, the equation (1.11), our wellposedness result is sharp in the Gevrey spaces. That is, if we define

\[
X_{\sigma,k} := \{ f \in C^\infty : \| f \|^2_{X_{\sigma,k}} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{(n!)^{2k}} \| \partial_\alpha^n f \|_{L^2}^2 \},
\]

and \( \hat{X}_{\sigma,k}, Y_{\sigma,k}, \hat{Y}_{\sigma,k} \) similarly. Then (2.33) is illposed in space \( (U, W_\alpha) \in Y_{\phi(t),k} \times X_{\phi(t),k} \) for \( k > 2 \). We shall prove this in a separate paper.

Using Theorem 3, we are able to prove the following.

Theorem 4 Let \( W, U, \{z_1(t), z_2(t)\} \) be the unique solution to (2.33) on \([0, T]\) constructed in Theorem 3 with initial data \( W_0, U_0, \{z_1(0), z_2(0)\} \). For any given constants \( 0 < \eta_0 < 1, \eta_1 > 0, \gamma > 0 \) and \( 0 < \epsilon_0 \ll 1, \) there exist constants \( N_0 \gg 1, \delta_0, \delta_1, \delta_2 \ll 1 \) such that for all

\[
y(0) \leq -N_0, \quad \lambda = \gamma |y_0|^{3/2 - \epsilon_0}, \quad |x(0) - 1| \leq \epsilon_1, \quad \| (U_0, \partial_\alpha W_0) \|_{Y_{L_0} \times X_{L_0}} \leq \epsilon_2,
\]

\(^{11}\) The power \( \frac{9}{10} \) for \( |y_0|^{9/10} \) is not optimal. We can use any \( |y_0|^{1-\epsilon} \), where \( \epsilon \in (0, 1) \).
there holds,

\[ \inf_{\alpha \in \mathbb{R}} A_1(\alpha, 0) \geq \eta_0, \quad \inf_{\alpha \in \mathbb{R}} A_1(\alpha, T) < -\eta_1, \]

For example, we can take \( \epsilon_1 = \frac{1}{N_0}, \epsilon_2 = \frac{1}{N_0}, \epsilon_0 = \frac{1}{10}, L_0 = 10, \delta_0 = 1000. \)

**Remark 3.3** The choice of the parameters \( \epsilon_0, \epsilon_1, \epsilon_2, \) etc are certainly not optimal. It is not our primary goal in this paper to obtain those sharp bounds.

### 4 A Quasilinear System

Given \( b_1, G, R, A, U_0 \) and \( W_0, \) we consider the following quasilinear system (with \( G \) and \( R \) as external forces):

\[
\begin{aligned}
    D_t U &= A \Lambda W + G, \\
    D_t W &= -U - \Re\{(\bar{F}, H)(\frac{1}{Z_\alpha} - 1)\} + R, \\
    (U, W)(\cdot, 0) &= (U_0, W_0).
\end{aligned}
\]

(4.1)

Here, \( Z_\alpha = (I + \mathbb{H})W_\alpha + 1. \) \( D_t = \partial_t + b \partial_\alpha, \) with

\[
b = b_0 + b_1, \quad b_0 = 2U + \Re\left\{(I - \mathbb{H})U, \mathbb{H}\right\}\left(\frac{1}{Z_\alpha} - 1\right),
\]

where \( b_1 \) is an a priori given real valued function. Using \( \partial_\alpha [f, \mathbb{H}]g = [f_\alpha, \mathbb{H}]g + [f, \mathbb{H}]g_\alpha, \) and by (2.64) and (2.67), we have

\[
\|\partial_\alpha \Re\{(I - \mathbb{H})U, \mathbb{H}\}\left(\frac{1}{Z_\alpha} - 1\right)\|_{X_\phi(t)} \leq C \left( \|U\|_{X_\phi(t)}^2 + \|\frac{1}{Z_\alpha} - 1\|_{Y_\phi(t)}^2 \right)
\]

\[
+ C \left( \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^2}{(n!)^4} \|\partial_\alpha^n U\|_{L_2}^2 + \sum_{n=1}^{\infty} \frac{n \phi(t)^{2n}}{(n!)^4} \|\partial_\alpha^n (\frac{1}{Z_\alpha} - 1)\|_{L_2}^2 \right),
\]

(4.2)

for some constant \( C > 0 \) depending on \( \frac{1}{\phi(t)}. \)

Denote

\[ H := (I + \mathbb{H})W_\alpha. \]

Let \( \phi(t) = \phi_{\delta_0}, L_0(t) := L_0 - \delta_0 t. \) Let

\[ E_{L_0, \delta_0}(t) := \frac{1}{2} \|\partial_\alpha W(\cdot, t)\|_{Y_\phi(t) \times X_\phi(t)}^2.
\]

(4.3)
By Lemma 2.8, if $\inf_{\alpha \in \mathbb{R}} (H + 1) \geq c_0 / 2$, then
\[
\left\| \frac{1}{H + 1} - \right\|_{X_{\phi(t)}} \leq C(c_0) \| H \|_{X_{\phi(t)}} \leq C(c_0) \| W_{\alpha} \|_{X_{\phi(t)}},
\] (4.4)
and
\[
\sum_{n=1}^{\infty} \frac{n\sigma^{2n}}{(n!)^4} \left\| \partial_{n}^n \left( \frac{1}{H + 1} - \right) \right\|_{L^2}^2 \leq C(c_0) \sum_{n=1}^{\infty} \frac{n\sigma^{2n}}{(n!)^4} \| \partial_{n}^n W_{\alpha} \|_{L^2}^2.
\] (4.5)

**Theorem 5** Let $c_0 > 0$ and $L_0 \geq 4$ be given constants. Assume that $b_1 \in C([0, T]; \mathcal{Y}_{\phi(t)}), A - 1, R_{\alpha} \in C([0, T]; \mathcal{X}_{\phi(t)}), G \in C([0, T]; \mathcal{Y}_{\phi(t)})$ and $(U_0, \partial_{\alpha} W_0) \in \mathcal{Y}_{\phi(0)} \times \mathcal{X}_{\phi(0)}$. Assume further that $\inf_{\alpha \in \mathbb{R}} (1 + H(\alpha, 0)) \geq c_0$. Let $d_0$ and $d_1$ be the constant given in Lemma 2.10 and Lemma 2.6, respectively. Assume that
\[
\sup_{0 \leq t \leq T} \left( \frac{\delta_0}{L_0} - 4 - 2d_0(\| A \|_{L^\infty} + \| A \|_{\dot{Y}_{\phi(t)}}) - 4d_1(\| U_0, \partial_{\alpha} W_0 \|_{\mathcal{Y}_{\phi(0)} \times \mathcal{X}_{\phi(0)}}) \right) \geq 0.
\] (4.6)

Then there exists $0 < T_0 \leq T$ such that (4.1) admits a unique solution $(U, \partial_{\alpha} W) \in C^0([0, T]; \mathcal{Y}_{\phi(t)} \times \mathcal{X}_{\phi(t)})$. Moreover,
\[
E_{L_0, \delta_0}(t) \leq E_{L_0, \delta_0}(0)e^{B_{T_0}t} + \int_0^{t} e^{B_{T_0}\tau} \mathcal{N}(\tau) d\tau,
\] (4.7)
\[
\| U(\cdot, t) \|_{L^2}^2 \leq U(\cdot, 0)\|_{L^2}^2e^{\gamma(t)} + \int_0^{t} e^{\gamma(t) - \gamma(s)} \| G(s) \|_{L^2}^2
\]
\[
+ \| A(s) \|_{L^\infty} E_{L_0, \delta_0}(s) ds,
\] (4.8)
where
\[
B_{T_0} := C(1 + \| b_1 \|_{C(\cdot; \mathcal{Y}_{\phi(t)})}),
\] (4.9)
and
\[
\mathcal{N}(t) := \| R_{\alpha} \|_{\mathcal{X}_{\phi(t)}}^2 + \| G \|_{\mathcal{Y}_{\phi(t)}}^2 + C \| b_1 \|_{\mathcal{Y}_{\phi(t)}}^2,
\] (4.10)
\[
\gamma(t) := \int_0^{t} C(E_{L_0, \delta_0}(\tau) + \| b_1(\cdot, \tau) \|_{\dot{Y}_{\phi(t)}} + \| A(\cdot, t) \|_{L^\infty}) d\tau.
\]

Here, $T_0 = \min\{ T, \frac{L_0}{2\delta_0}, \tau_0 \}$, with $\tau_0$ given by (3.4); and $C$ is a constant depending on $c_0$ and $\frac{1}{L_0}$.
We prove Theorem 5 by the energy method. We provide closed a priori energy estimates only. Note that $T_\alpha \leq \frac{T_0}{2}$ implies that $\phi(t) \geq \frac{T_0}{2} \geq 2$ for $t \in [0, T_0]$, so $d_0$ and $d_1$ are absolute constants. By Lemma 2.7 and (4.5), under the assumption of Theorem 5, we have the a priori estimate

$$\|b_0\|_{Y_{\phi(t)}}^2 = \sum_{n=1}^{\infty} \frac{n^2 \phi(t) 2n}{(n!)^4} \left\| \partial_{\alpha}^n \left( 2U + \Re\{(I - \mathbb{H})U, \mathbb{H}\} \left( \frac{1}{Z_{\alpha}} - 1 \right) \right) \right\|_{L^2}^2 \leq C \|U\|_{Y_{\phi(t)}}^2 + C \|U\|_{Y_{\phi(t)}} \|\partial_{\alpha} W\|_{X_{\phi(t)}} \leq C E L_{0, \delta_0}$$

(4.11)

for some constant $C > 0$ depending on $c_0$ and $\frac{1}{L_\alpha}$.

**Proof** Applying $\partial_{\alpha}$ on both sides of $D_t W = -U - \Re\{[\bar{F}, \mathbb{H}] \left( \frac{1}{Z_{\alpha}} - 1 \right) \} + R$, we obtain

$$D_t W_{\alpha} = -U_{\alpha} + R_{\alpha} - b_{\alpha} W_{\alpha} - \partial_{\alpha} \Re\{[\bar{F}, \mathbb{H}] \left( \frac{1}{Z_{\alpha}} - 1 \right) \}.$$  

(4.12)

Using (4.12) and $U_t = A \Lambda W - b U_{\alpha} + G$, we have

$$\frac{d}{dt} E_{L_{0, \delta_0}(t)}$$

$$= \frac{1}{2} \frac{d}{dt} \left( \sum_{n=1}^{\infty} \frac{n^2 \phi(t) 2n}{(n!)^4} \| \partial_{\alpha}^n U (\cdot, t) \|_{L^2}^2 \right) + \frac{1}{2} \frac{d}{dt} \sum_{n=0}^{\infty} \frac{\phi(t) 2n}{(n!)^4} \| \partial_{\alpha}^n W_{\alpha} (\cdot, t) \|_{L^2}^2$$

$$= -\delta_0 \sum_{n=1}^{\infty} \frac{n^3 \phi(t) 2n - 1}{(n!)^4} \| \partial_{\alpha}^n U (\cdot, t) \|_{L^2}^2 + \Re \sum_{n=0}^{\infty} \frac{n^2 \phi(t) 2n}{(n!)^4} \langle \partial_{\alpha}^n U_t, \partial_{\alpha}^n U \rangle$$

$$- \delta_0 \sum_{n=1}^{\infty} \frac{n^2 \phi(t) 2n - 1}{(n!)^4} \| \partial_{\alpha}^n W_{\alpha} (\cdot, t) \|_{L^2}^2 + \Re \sum_{n=0}^{\infty} \frac{\phi(t) 2n}{(n!)^4} \langle \partial_{\alpha}^n \partial_{\alpha} W_{\alpha}, \partial_{\alpha}^n W_{\alpha} \rangle$$

$$= -\delta_0 \sum_{n=1}^{\infty} \frac{n^3 \phi(t) 2n - 1}{(n!)^4} \| \partial_{\alpha}^n U (\cdot, t) \|_{L^2}^2 + \Re \sum_{n=1}^{\infty} \frac{n^2 \phi(t) 2n}{(n!)^4} \langle \partial_{\alpha}^n (A \Lambda W - b U_{\alpha} + G), \partial_{\alpha}^n U \rangle$$

$$- \delta_0 \sum_{n=1}^{\infty} \frac{n^2 \phi(t) 2n - 1}{(n!)^4} \| \partial_{\alpha}^n W_{\alpha} (\cdot, t) \|_{L^2}^2 + \Re \sum_{n=0}^{\infty} \frac{\phi(t) 2n}{(n!)^4} \langle \partial_{\alpha}^n (-U_{\alpha} + R_{\alpha} - b_{\alpha} W_{\alpha} - \partial_{\alpha} \Re\{[\bar{F}, \mathbb{H}] \left( \frac{1}{Z_{\alpha}} - 1 \right) \} - b \partial_{\alpha} W_{\alpha}), \partial_{\alpha}^n W_{\alpha} \rangle$$
Using Lemma 2.10, we have
\[
\left| \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} (A \Lambda W), \partial^n_{\alpha} U \rangle \right|
\leq d_0 \left( \| A \|_{L^\infty} + \| A \|_{\mathcal{X}_\phi(t)} \right) \left( \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} \Lambda W \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} U \|_{L^2}^2 \right),
\]
for some absolute constant \( d_0 > 0 \). Using Lemma 2.10 again, we obtain
\[
\left| \Re \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} (A \Lambda W - bU_\alpha + G), \partial^n_{\alpha} U \rangle \right|
\leq C \left( \| b \|_{L^\infty} + \| b \|_{\mathcal{X}_\phi(t)} \right) \left( \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} \Lambda W \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} U \|_{L^2}^2 \right)
+ d_0 \left( \| A \|_{L^\infty} + \| A \|_{\mathcal{X}_\phi(t)} \right) \left( \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} \Lambda W \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} U \|_{L^2}^2 \right)
+ d_0 \left( \| A \|_{L^\infty} + \| A \|_{\mathcal{X}_\phi(t)} \right) \left( \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} \Lambda W \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} U \|_{L^2}^2 \right).
\]
Cauchy-Schwarz implies
\[
\left| \Re \sum_{n=1}^{\infty} \frac{n^2 \phi(t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} U_\alpha, \partial^n_{\alpha} W_\alpha \rangle \right|
\leq \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} U_\alpha \|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n \phi(t)^{2n}}{(n!)^4} \| \partial^n_{\alpha} W_\alpha \|_{L^2}^2.
\]
Decomposing \( b = b_0 + b_1 \), we obtain
\[
\left| \Re \sum_{n=1}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} (b_0 W_\alpha + b_1 W_\alpha), \partial^n_{\alpha} W_\alpha \rangle \right|
\leq \left| \sum_{n=1}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} (b_0 W_\alpha), \partial^n_{\alpha} W_\alpha \rangle \right|
+ \left| \sum_{n=1}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial^n_{\alpha} (b_1 W_\alpha), \partial^n_{\alpha} W_\alpha \rangle \right|.
\]
12 It turns out that if we choose \( |\gamma_0| \) large and \( |\lambda| \sim |\gamma_0|^{3/2} \), then \( \| A - 1 \|_{L^\infty} \sim 1 \), while \( \| A - 1 \|_{L^2} \sim |\gamma_0|^{1/2} \), which is large. So we avoid to bound \( \Lambda - 1 \) in \( L^2 \).
Since we can estimate \( \partial_\alpha b_1 \) in \( X_{\phi(t)} \), expressing \( \partial_\alpha^{n+1}(b_1 W_\alpha) = \partial_\alpha^n(\partial_\alpha b_1 W_\alpha + b_1 \partial_\alpha W_\alpha) \) and using (2.77), we obtain

\[
\left| \sum_{n=1}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial_\alpha^{n+1}(b_1 W_\alpha), \partial_\alpha^n W_\alpha \rangle \right| \leq \|\partial_\alpha b_1\|_{X_{\phi(t)}} \|\partial_\alpha W\|_{X_{\phi(t)}}^2.
\]

The formula for \( b_0 \) implies

\[
\left| \sum_{n=1}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial_\alpha^{n+1}(b_0 W_\alpha), \partial_\alpha^n W_\alpha \rangle \right| \leq \left| \sum_{n=1}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial_\alpha^{n+1}(2U W_\alpha), \partial_\alpha^n W_\alpha \rangle \right|
\]

\[
i + II.
\]

Using Lemma 2.6, we have

\[
I \leq d_1 \|U\|_{X_\alpha} \|W_\alpha\|_{X_\alpha}^2 + d_1 \|W_\alpha\|_{X_\alpha} \left( \sum_{n=1}^{\infty} \frac{n^3 \sigma_{2n}^2}{(n!)^4} \|\partial_\alpha^n U\|_{L^2}^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{n \sigma_{2n}^2}{(n!)^4} \|\partial_\alpha^n W_\alpha\|_{L^2}^2 \right)^{1/2}
\]

\[
\leq d_1 E_{L_0, \delta_0}^{3/2} + d_1 E_{L_0, \delta_0}^{1/2} \left( \sum_{n=1}^{\infty} \frac{n^3 \sigma_{2n}^2}{(n!)^4} \|\partial_\alpha^n U\|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n \sigma_{2n}^2}{(n!)^4} \|\partial_\alpha^n W_\alpha\|_{L^2}^2 \right),
\]

where \( d_1 \) is the constant in Lemma 2.6. Similarly, using Lemma 2.6, (4.4) and (4.5), and \( \partial_\alpha[f, H]g = [f, H]g + [f, H]g_\alpha \), we obtain

\[
II \leq d_1 E_{L_0, \delta_0}^{3/2} + d_1 E_{L_0, \delta_0}^{1/2} \left( \sum_{n=1}^{\infty} \frac{n^3 \sigma_{2n}^2}{(n!)^4} \|\partial_\alpha^n U\|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n \sigma_{2n}^2}{(n!)^4} \|\partial_\alpha^n W_\alpha\|_{L^2}^2 \right).
\]

By Cauchy-Schwarz,

\[
\left| \sum_{n=0}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \langle \partial_\alpha^n R_\alpha, \partial_\alpha^n W_\alpha \rangle \right| \leq \|R_\alpha\|_{X_{\phi(t)}}^2 + \|W_\alpha\|_{X_{\phi(t)}}^2.
\]

Combining the above estimates together with (4.11), bounding \( \|b_1\|_{L^\infty} \) by \( C \|b_1\|_{Y_{\phi(t)}} \), we obtain

\[
\frac{d}{dt} E_{L_0, \delta_0}(t) + \left( \frac{\delta_0}{\phi(t)} - 2 - d_0(\|A\|_{L^\infty} + \|A\|_{X_{\phi(t)}}) - d_1 E_{L_0, \delta_0}^{1/2} \right)
\]

\[
	imes \left( \sum_{n=1}^{\infty} \frac{n^3 \phi(t)^{2n}}{(n!)^4} \|\partial_\alpha^n U\|_{L^2}^2 + \sum_{n=1}^{\infty} \frac{n \phi(t)^{2n}}{(n!)^4} \|\partial_\alpha^n W_\alpha\|_{L^2}^2 \right)
\]
\[ \leq C (1 + \|b\|_{L^\infty} + \|b\|_{Y_{\phi(t)}}) E_{L_0, \delta_0}(t) + \|R_a\|^2_{X_{\phi(t)}} + d_1 E^{3/2}_{L_0, \delta_0} + \|G\|^2_{Y_{\phi(t)}} + C \|b\|^2_{Y_{\phi(t)}} \]

\[ \leq C (1 + \|b_1\|_{Y_{\phi(t)}} + E_{L_0, \delta_0}(t) / 2) E_{L_0, \delta_0}(t) + \|R_a\|^2_{X_{\phi(t)}} + d_1 E^{3/2}_{L_0, \delta_0} + \|G\|^2_{Y_{\phi(t)}} + C \|b_1\|^2_{Y_{\phi(t)}} + CE_{L_0, \delta_0}(t). \]

We choose \( T_0 = \min\{T, L_0 / \delta_0, \tau_0\} \). With this choice, \( \phi(t) \geq L_0 / 2 \geq 2 \). Using (4.6), by a bootstrap argument,

\[ \frac{\delta_0}{\phi(t)} - 2 - d_0(\|A\|_{L^\infty} + \|A\|_{X_{\phi(t)}}) - \|E\|_{L_0, \delta_0} \geq 0, \quad \forall \ t \in [0, T_0]. \]

By choosing \( d_1 \) larger if necessary, we obtain

\[ \frac{d}{dt} E_{L_0, \delta_0}(t) \leq C (1 + \|b_1\|_{Y_{\phi(t)}}) E_{L_0, \delta_0}(t) + d_1 E^{3/2}_{L_0, \delta_0} + \|R_a\|^2_{X_{\phi(t)}} + \|G\|^2_{Y_{\phi(t)}} + C \|b_1\|^2_{Y_{\phi(t)}}, \] (4.13)

By the method of continuity, we obtain

\[ E_{L_0, \delta_0}(t) \leq E_{L_0, \delta_0}(0) e^{B\tau_0} + \int_0^t e^{B\tau_0} \mathcal{N}(\tau) d\tau, \] (4.14)

where \( \mathcal{N} \) is defined in (4.10).

To estimate \( \|U\|_{L^2} \), using energy estimates, we have

\[ \frac{d}{dt} \|U(\cdot, t)\|^2_{L^2} = 2 \langle U_1, U \rangle = 2(-b \partial_a U + G + A \Lambda W, U) \]

\[ \leq 2 \|b_1\|_{L^\infty} \|U\|^2_{L^2} + \|G\|^2_{L^2} + \|U\|^2_{L^2} + \|A\|_{L^\infty} \|\Lambda W\|^2_{L^2} + \|A\|_{L^\infty} \|U\|^2_{L^2} \]

\[ \leq C(E_{L_0, \delta_0}^{1/2} + \|b_1\|_{Y_{\phi(t)}}) \|U\|^2_{L^2} + \|G\|^2_{L^2} + \|U\|^2_{L^2} + \|A\|_{L^\infty} E_{L_0, \delta_0}(t) \]

\[ + \|A\|_{L^\infty} \|U\|^2_{L^2}. \]

So we obtain

\[ \|U(\cdot, t)\|^2_{L^2} \leq U(\cdot, 0) \|e^{\gamma(t)}\|^2_{L^2} + \int_0^t e^{\gamma(t) - \gamma(s)} \|G(s)\|^2_{L^2} ds, \]

(4.15)

where

\[ \gamma(t) := \int_0^t C(E_{L_0, \delta_0}^{1/2}(\tau) + \|b_1(\cdot, \tau)\|_{Y_{\phi(t)}} + \|A(\cdot, t)\|_{L^\infty}) d\tau. \]

\[ \square \]
5 Estimates

In § 6, we use the Picard iteration to prove Theorem 3. For each iteration, we need to solve the quasilinear system (4.1) with \( G, R, b_1, A \) constructed in the previous iteration. Hence we need to derive estimates for these quantities. Most of the estimates in this section hold for the general case: without symmetry, and applies to an arbitrary number of point vortices. The only place that the symmetry plays a role is the estimates for the velocity of the point vortices, which is in Lemma 5.6.

5.1 A Priori Assumptions

We’ll derive estimates under the following a priori assumptions. These assumptions are verified during the process of the Picard iteration in the next section. Without loss of generality, we assume \( 0 < C_0 < C_1 \). Denote

\[
\frac{1}{4}m_0^2 := \|U_0\|_{Y_{\phi(0)}}^2 + \|\partial_\alpha W_0\|_{X_{\phi(0)}}^2. \tag{5.1}
\]

Let’s denote

\[
M_{\lambda, h}^2 := \begin{cases} 
m_0^2 + C|\lambda|(|y_0| - \frac{|\lambda|}{4\pi x(0)}T)^{-5/2}, & \lambda > 0; \\
m_0^2 + |\lambda|d_{I,0}^{-5/2}, & \lambda < 0.
\end{cases} \tag{5.2}
\]

\[
M_\infty = (d_{I,0} + C|\lambda|(|y_0| - \frac{|\lambda|}{4\pi x(0)}T)^{-5/2})^{-1/2}. \tag{5.3}
\]

Here, \( C > 0 \) is a constant depending on \( C_0 \) and \( \frac{1}{L_0} \) only.\(^{14}\) We will choose \( T = \frac{4\pi x(0)(|y_0| - |y_0|^{9/10})}{|\lambda|} \) if \( \lambda > 0 \) and \( T = O(1) \) if \( \lambda < 0 \). So we can take

\[
M_{\lambda, h}^2 = \begin{cases} 
m_0^2 + C|y_0|^{-3/4}, & \lambda > 0 \\
m_0^2 + C|y_0|^{-1}, & \lambda < 0.
\end{cases} \tag{5.4}
\]

Without loss of generality, we assume

\[
\begin{cases}
m_0^2 \leq |y_0|^{-3/4}, & \lambda > 0; \\
m_0^2 \leq |y_0|^{-1}, & \lambda < 0.
\end{cases} \tag{5.5}
\]

Definition 5.1 Given \( W, U, z_j(t) \), we say that \( (U, W, \{z_j(t)\}) \) satisfies AS, if the following hold:

\( \text{(AS1)} \ W \in C([0, T]; X_{\phi(t)}), U \in C([0, T]; Y_{\phi(t)}), z_j(t) \in C^1([0, T]; \Omega(t)) \).

\(^{13}\) In the definition of \( M_{\lambda, h} \), we will choose \( d_{I,0} \) large. We do not claim that \( |d_{I,0}|^{3/2} \) is the optimal choice. The letter \( h \) in \( M_{\lambda, h} \) refers to homogeneous.

\(^{14}\) Since we choose \( L_0 \geq 4 \), the constant can be chosen such that it depends on \( C_0 \) only. At the moment let’s keep track of its dependence on \( \frac{1}{L_0} \).
\( (AS2) \sup_{0 \leq t \leq T} \left( \| U(\cdot, t) \|_{Y_\phi(t)}^2 + \| W_\alpha \|_{X_\phi(t)}^2 \right) \leq M_{k,h}^2 \leq 1, \text{ and} \)

\[
\sup_{0 \leq t \leq T} \| U(\cdot, t) \|_{L^2}^2 \leq 1, \quad \sup_{0 \leq t \leq T} \| U(\cdot, t) \|_{L^\infty} \leq M_\infty.
\]

\( (AS3) \frac{1}{2} C_0 |\alpha - \beta| \leq |Z(\alpha, t) - Z(\beta, t)|, \quad \forall \ t \in [0, T]. \)

\( (AS4) \ d_I(t) \geq \frac{1}{2} d_{I,0}^{9/10} \geq 1, \ x(t) \geq \frac{1}{2} x(0), \ t \in [0, T]. \)

\( (AS5) \sup_{t \in [0, T]} \phi(t) \geq \frac{1}{2} L_0. \)

We derive a priori estimates under the a priori assumptions (AS1)-(AS5). Note that (AS2) implies

\[
\sup_{0 \leq t \leq T} \left( \| F \|_{Y_\phi(t)}^2 + \| Z_\alpha - 1 \|_{X_\phi(t)}^2 \right) \leq 4 M_{k,h}^2.
\]  

(5.6)

**Convention:** If not specified, in this section, a constant \( C \) will depend on \( C_0, C_1, M_\lambda, h, d_I(t), \) and \( \frac{1}{L_0}. \)

### 5.2 Estimates

**Lemma 5.1** Assume that \((W, U, \{z_j(t)\})\) satisfies AS. Then any \( k \geq 2, \) there holds

\[
\int_{-\infty}^{\infty} \frac{1}{|Z(\beta, t) - z_j(t)|^k} d\beta \leq \frac{8}{C_0} d_I(t)^{-k+1} + \frac{8}{(k-1)C_0} \left( \frac{4C_1}{d_I(t)C_0} \right)^{k-1}. \quad (5.7)
\]

In particular, we have

\[
\int_{-\infty}^{\infty} \frac{1}{|Z(\beta, t) - z_j(t)|^k} d\beta \leq \frac{16}{C_0} \left( \frac{4C_1}{d_I(t)C_0} \right)^{k-1}. \quad (5.8)
\]

**Proof** We may assume that \( d_I(t) = d(z_j(t), z(0, t)). \)

\[
\begin{align*}
\int_{-\infty}^{\infty} \frac{1}{|z_j(t) - Z(\beta, t)|^k} d\beta \\
= & \int_{|Z(0,t) - Z(\beta,t)| \leq 2d_I(t)} \frac{1}{|z_j(t) - Z(\beta, t)|^k} d\beta \\
& + \int_{|z(0,t) - Z(\beta,t)| \geq 2d_I(t)} \frac{1}{|z_j(t) - Z(\beta, t)|^k} d\beta \\
& := I + II.
\end{align*}
\]

Denote

\[
E := \{ \beta : |Z(0, t) - Z(\beta, t)| \leq 2d_I(t) \}.
\]
Since
\[ \frac{1}{2} C_0 |\beta - 0| \leq |Z(\beta, t) - Z(0, t)|, \]
we have for \( \beta \in E \),
\[ |\beta - 0| \leq \frac{4}{C_0} d_I(t). \]
Therefore
\[ I \leq 8 C_0^{-1} d_I(t)^{-k+1}. \]

For \( \beta \in E^c \), we have
\[
|Z(\beta, t) - Z(0, t) - d_I(t)| \geq |Z(\beta, t) - Z(0, t)| - d_I(t) \\
\geq \frac{1}{2} |Z(\beta, t) - Z(0, t)| \geq \frac{C_0}{4} |\beta - 0|. \quad (5.9)
\]
Also, we have
\[
2C_1 |\beta - 0| \geq |Z(\beta, t) - Z(0, t)| \geq 2d_I(t). \quad (5.10)
\]
So
\[ |\beta| \geq \frac{1}{C_1} d_I(t) \quad (5.11) \]
Therefore, for \( II \), we have
\[
II \leq \frac{4^k}{C_0^k} \int_{|\beta| \geq \frac{1}{C_1} d_I(t)} |\beta|^{-k} d\beta \\
= 2 \frac{4^k}{(k-1) C_0^k} C_1^{k-1} d_I(t)^{-k+1} = \frac{8}{(k-1) C_0} \left( \frac{4C_1}{d_I(t) C_0} \right)^{k-1}. \]

\[ \square \]

**Lemma 5.2** Denote \( \tilde{Q}_j := \frac{1}{Z(\alpha, t) - z_j(t)} \). Assume that \( (W, U, \{z_j(t)\}) \) satisfies AS. Then
\[ \| \partial_\alpha \tilde{Q}_j \|_{X_{\phi(t)}} \leq C d_I(t)^{-3/2}, \quad (5.12) \]
for some constant \( C > 0 \) depending on \( M_{\lambda, h}, C_0, C_1, \) and \( \frac{1}{L_0} \).
**Proof**

We have

\[
\frac{\partial \alpha Q_j}{\partial \alpha} = -\int \frac{Z_\alpha - 1}{(Z(\alpha, t) - z_j(t))^2} \, d\phi(t)
\]

Since

\[
\left\| \frac{Z_\alpha - 1}{(Z(\alpha, t) - z_j(t))^2} \right\|_{X_\phi(t)} \leq C \left\| Z_\alpha - 1 \right\|_{X_\phi(t)} \left\| \frac{1}{(Z(\alpha, t) - z_j(t))^2} \right\|_{X_\phi(t)},
\]

it suffices to prove

\[
\left\| \frac{1}{(Z(\alpha, t) - z_j(t))^2} \right\|_{X_\phi(t)} \leq C d(t)^{-3/2}.
\]

Denote \( Q_j := \frac{1}{(Z(\alpha, t) - z_j(t))^2} \), and \( g(\alpha) := \frac{1}{(\alpha - z_j(t))^2} \). The Faa di Bruno formula implies

\[
\partial^n_\alpha Q_j(\alpha, t) = \sum_{k=1}^{n} \sum_{\Lambda_{n,k}} \frac{n!}{k_1! k_2! \cdots k_n!} g^{(k)}(Z(\alpha, t), t) \prod_{j=1}^{n} \left( \frac{\partial^j_\alpha Z(\alpha, t)}{j!} \right)^{k_j},
\]

where

\[
\Lambda_{n,k} := \{(k_1, \ldots, k_n) \in (\mathbb{N} \cup \{0\})^n : \sum_{j=1}^{n} k_j = k, \sum_{j=1}^{n} jk_j = n\}.
\]

Using (AS2) and (AS5), by Sobolev embedding, we estimate \( Z_\alpha \) by

\[
|Z_\alpha(\alpha, t)| \leq 1 + |Z_\alpha - 1| \leq 1 + \|Z_\alpha - 1\|_{H^1}
\]

\[
\leq 1 + (1 + \frac{2}{\phi(t)}) 2 M_{\lambda, h}
\]

\[
\leq 1 + 6 M_{\lambda, h} := K_0.
\]

For \( 2 \leq j \leq n - 3 \), we estimate \( \partial^j_\alpha Z(\alpha, t) \) by

\[
|\partial^j_\alpha Z(\alpha, t)| \leq \|\partial^j_\alpha (Z - \alpha)\|_{H^1}
\]

\[
\leq \frac{(j!)^2}{\phi^j(t)} (1 + \frac{(j + 1)^2}{\phi(t)}) \|Z - \alpha\|_{H^1} \frac{1}{\phi^j(t)}
\]

\[
\leq 4 \frac{(j + 1)!^2}{\phi(t)^j} M_{\lambda, h} \leq \frac{K_0 ((j + 1)!)^2}{\phi(t)^j}.
\]

So we obtain

\[
\prod_{j=1}^{n-1} \left( \frac{\partial^j_\alpha Z(\alpha, t)}{j!} \right)^{k_j} \leq \prod_{j=1}^{n} \left( \frac{K_0 j!}{\phi(t)^{j-1}} \right)^{k_j} = K_0 \sum_{j=1}^{n} k_j \phi(t)^{-\sum_{j=1}^{n} jk_j} \prod_{j=1}^{n} (j!)^{k_j}
\]

\[
= K_0^k \phi(t)^{-n + k} \prod_{j=1}^{n} (j!)^{k_j}.
\]
For $n - 2 \leq j \leq n$, we use
\[
\|\partial_{\alpha}^{j} Z(\alpha, t)\|_{L^2} = \|\partial_{\alpha}^{j-1}(Z\alpha - 1)\|_{L^2} \leq \frac{(j - 1)!}{\phi(t)^{j-1} - M_{\lambda, h}}.
\] (5.14)

For $n \leq 5$, there holds
\[
\sum_{n=0}^{5} \frac{\phi(t)^{2n}}{(n!)^4} \|\partial_{\alpha}^{n} Q_j\|_{L^2}^2 \leq Cd_t(t)^{-3/2},
\] (5.15)
for some constant $C$ depending on $C_0$, $C_1$, $M_{\lambda, h}$, and $\frac{1}{L_0}$.

Note that for $n \geq 6$ and $k \geq \lceil n/2 \rceil$, we must have $j \leq n - 3$. So for $n \geq 6$,
\[
\|\partial_{\alpha}^{n} Q_j(\alpha, t)\|_{L^2}
\leq \sum_{k=3}^{\lceil n/2 \rceil} \sum_{\Lambda_{n,k}} \frac{n!k!}{k_1! \cdots k_n!} K_0^k \phi(t)^{-n+k} \prod_{j=1}^{n} (j!)^{k_j} \frac{1}{|Z(\alpha, t) - z_j(t)|^{k+1}} \|L^2
+ \sum_{k=\lceil n/2 \rceil + 1}^{n} \sum_{\Lambda_{n,k}} \frac{n!k!}{k_1! \cdots k_n!} K_0^k \phi(t)^{-n+k} \prod_{j=1}^{n} (j!)^{k_j} \frac{1}{|Z(\alpha, t) - z_j(t)|^{k+1}} \|L^2
+ \sum_{k=1}^{\lceil n/2 \rceil} \sum_{\Lambda_{n,k}} \frac{n!}{k_1! k_2! \cdots k_n!} g^{(k)}(Z(\alpha, t), t) \prod_{j=1}^{n} \left(\frac{\partial_{\alpha}^{j} Z(\alpha, t)}{j!}\right)^{k_j} \|L^2
:= II_1 + II_2 + II_3.
\]

Assume that $n \geq 6$ and $k \geq 3$. Let $(k_1, ..., k_n) \in \Lambda_{n,k}$. Note that
\[
\prod_{j=2}^{n} (j!)^{k_j} = \frac{(n!)^{-1} \prod_{2 \leq j \leq n} (j!)^{k_j}}{n^2 \prod_{j=0}^{k-1} (n-j)^{k_j}} \leq \frac{C}{n^2 \prod_{j=0}^{k-1} (n-j)^{k_j}},
\] (5.16)
for some absolute constant $C > 0$. For example, we can take $C = 100$. Note that $n \geq 6$ and $1 \leq j \leq \lfloor n/2 \rfloor$ imply $n - j \geq \frac{1}{2} n$. Therefore, for $3 \leq k \leq \lfloor n/2 \rfloor$, \n
\[
k^{-1} \prod_{j=0}^{k-1} (n-j) \geq 2^{-k} n^k.
\] (5.17)

For $k \geq \lceil n/2 \rceil + 1$, we have
\[
k^{-1} \prod_{j=0}^{k-1} (n-j) \geq 2^{-\lceil n/2 \rceil} n^{\lceil n/2 \rceil}.
\] (5.18)
Lemma 5.1 and (5.17) imply

\[
II_1 \leq C \frac{(n!)^2}{n^2 \phi(t)^n} \sum_{k=3}^{[n/2]} \frac{k!}{\Lambda_{n,k}!} \frac{4K_0 C_1 \phi(t)}{nC_0 d_I(t)}^k
\]

\[
\leq C \frac{(n!)^2}{n^2 \phi(t)^n} \sum_{k=1}^{n} \frac{k!}{\Lambda_{n,k}!} \frac{4K_0 C_1 \phi(t)}{nC_0 d_I(t)}^k
\]

\[
= C \frac{(n!)^2}{n^2 \phi(t)^n} \frac{4K_0 C_1 \phi(t)}{nC_0 d_I(t)} \left(1 + \frac{4K_0 C_1 \phi(t)}{nC_0 d_I(t)}\right)^n
\]

\[
\leq Cd_I(t)^{-1} \frac{(n!)^2}{n^2 \phi(t)^n},
\]

for some constant \( C > 0 \) depending on \( C_0, C_1, M_{\lambda,h}, \frac{1}{L_0} \). Here we’ve used the identity

\[
\sum_{k=1}^{n} \sum_{\Lambda_{n,k}} \frac{k!}{(k_1)! \cdots (k_n)!} R^k = R(1 + R)^{n-1},
\]

and the estimate

\[
\sup_{n \geq 1} \left(1 + \frac{4K_0 C_1 \phi(t)}{nC_0 d_I(t)}\right)^n \leq e^{\frac{4K_0 C_1 \phi(t)}{C_0 d_I(t)}} \leq e^{\frac{4K_0 C_1 L_0}{C_0 d_I(t)}}.
\]

So we have

\[
\sum_{n=0}^{\infty} \frac{(1 + n^2 \phi(t))^{2n}}{(n!)^4} II_1^2 \leq Cd_I(t)^{-1}.
\]

for some constant \( C > 0 \) depending on \( K_0, C_0, C_1, M_{\lambda,h}, \frac{1}{L_0} \).

For \( II_2 \), using Lemma 5.1 and (5.18), we obtain

\[
II_2 \leq C \frac{(n!)^2}{\phi(t)^n} \sum_{k=[n/2]+1}^{n} \frac{k!}{\Lambda_{n,k}!} \frac{2K_0 C_1 \phi(t)}{C_0 d_I(t)}^k \prod_{j=0}^{k-1} \frac{1}{(n-j)}
\]

\[
\leq C \frac{(n!)^2}{\phi(t)^n} \sum_{k=[n/2]+1}^{n} \frac{k!}{\Lambda_{n,k}!} \frac{2K_0 C_1 \phi(t)}{C_0 d_I(t)}^k \left(\frac{n/2}{n/2}\right)^{n/2}
\]

\[
\leq Cd_I(t)^{-1} \frac{(n!)^2}{\phi(t)^n} \frac{1 + 2K_0 C_1 \phi(t)}{C_0 d_I(t)^n} \left(\frac{n/2}{n/2}\right)^{n/2}
\]

\[
\leq Cd_I(t)^{-1} \frac{(n!)^2}{n^2 \phi(t)^n},
\]
for some constant $C > 0$ depending on $K_0, C_0, C_1, M_{\lambda, h}, \frac{1}{L_0}$. Here, we’ve used the identity

$$\sum_{k=1}^{n} \frac{k!}{(k_1)! \cdots (k_n)!} R^k = R(1 + R)^{n-1},$$

(5.22)

and the estimate

$$\sup_{n \geq 1} \frac{1}{n^{[n/2]} - 2} \left(1 + \frac{2K_0 C_1 \phi(t)}{C_0 d_I(t)}\right)^n \leq C,$$

(5.23)

for some constant $C > 0$ depending on $K_0, C_0, C_1, M_{\lambda, h}$ and $\frac{1}{L_0}$.

So we have

$$\sum_{n=1}^{\infty} \left(\phi(t)\right)^{2n} \frac{(n!)^2}{(n!)^4} I_{I_2}^2 \leq C d_I(t)^{-1} \sum_{n=0}^{\infty} \left(\phi(t)\right)^{2n} \frac{(n!)^4}{\phi(t)^{2n} n^4} \leq C d_I(t)^{-3/2}.$$  

(5.24)

For $I_{I_3}$, when $k = 1$, we must have $\Lambda_{n, 1} = (0, ..., 0, 1)$. In this case,

$$\sum_{n=0}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \sum_{\Lambda_{n, 1}} \frac{n!}{k_1! k_2! \cdots k_n!} g^{(k)}(Z(\alpha, t), t) \prod_{j=1}^{n} \left(\frac{\partial_j Z(\alpha, t)}{j!}\right)^{k_j} \leq \sum_{n=0}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} \frac{1}{d_I(t)^2} \frac{\phi(t)^{2(n-1)}}{(n-1)!} \frac{\phi(t)^{2(n-1)}}{(n-1)!} \frac{\|\partial^{n-1}_{\alpha} (Z_\alpha - 1)\|^2_{L^2}}{L^2} \leq C d_I(t)^{-2}.$$  

for some constant $C > 0$ depending on $C_0, C_1, M_{\lambda, h}$, and $\frac{1}{L_0}$.

The same argument applies to $k = 2$. We obtain

$$\sum_{n=0}^{\infty} \frac{\phi(t)^{2n}}{(n!)^4} I_{I_3}^2 \leq C d_I(t)^{-2}.$$  

(5.25)

This concludes the proof of the lemma. \hfill $\Box$

Similarly, we can prove

**Lemma 5.3** Assume that $(W, U, \{z_j(t)\})$ satisfies AS, then

$$\left\|\frac{1}{(Z(\alpha, t) - z_j(t))^m}\right\|_{Y_{\phi(t)}} \leq C d_I(t)^{-m+1/2},$$

(5.26)
for some constant $C > 0$ depending on $C_0$, $C_1$, $M_{\lambda, h}$, and $\frac{1}{L_0}$.

As a consequence,

**Corollary 5.1** Assume that $(W, U, \{z_j(t)\})$ satisfies AS. Let $g \in X_{\phi(t)}$, then

1. For arbitrary $N$, let $\lambda_0 := N \max_{1 \leq j \leq N} |\lambda_i|$, we have

\[
\left\| \sum_{j=1}^{N} \frac{\lambda_j g}{Z(\alpha, t) - z_j(t)} \right\|_{X_{\phi(t)}} \leq C \lambda_0 d_I(t)^{-m} \|g\|_{X_{\phi(t)}},
\]

for some constant $C > 0$ depending on $C_0$, $C_1$, $M_{\lambda, h}$, and $\frac{1}{L_0}$.

**Proof** For (5.29), we estimate

\[
\left\| \sum_{j=1}^{N} \frac{\lambda_j g}{Z(\alpha, t) - z_j(t)} \right\|_{L^2} \leq C \left\| g \right\|_{L^2} \left\| \sum_{j=1}^{N} \frac{\lambda_j g}{Z(\alpha, t) - z_j(t)} \right\|_{L^\infty} \leq C d_I(t)^{-m} \|g\|_{X_{\phi(t)}}.
\]

By Lemma 2.5 and Lemma 5.3,

\[
\left\| \sum_{j=1}^{N} \frac{\lambda_j g}{Z(\alpha, t) - z_j(t)} \right\|_{\dot{X}_{\phi(t)}} \leq C \lambda_0 d_I(t)^{-m-1/2} \|g\|_{X_{\phi(t)}},
\]

We will construct solutions with $d_I(t)$ large. Hence larger $k$ gives smaller quantity $d_I(t)^{-k}$. 
So
\[
\left\| \sum_{j=1}^{N} \frac{\lambda_j g_j}{(Z^{(\alpha, t)} - z_j(t))^m} \right\|_{X_{\phi(t)}} \leq C \lambda_0 d_I(t)^{-m-1/2} \|g\|_{X_{\phi(t)}} + Cd_I(t)^{-m} \|g\|_{X_{\phi(t)}} \leq Cd_I(t)^{-m} \|g\|_{X_{\phi(t)}}.
\]

Here, we use the assumption that \(d_I(t) \geq 1\). So we obtain (5.29). (5.30)-(5.32) follow from (5.29) and Lemma 5.3. □

Using the same proof as for Lemma 5.2, we obtain the following.

\textbf{Lemma 5.4} Assume that \((W, U, \{z_j(t)\})\) satisfies AS. Then
\[
\left\| \frac{1}{Z^{(\alpha, t)}} - 1 \right\|_{X_{\phi(t)}} \leq CM_{\lambda, h},
\] (5.34)

for some constant \(C > 0\) depending on \(C_0\) and \(M_{\lambda, h}\).

\textbf{Lemma 5.5} Assume that \((W, U, \{z_j(t)\})\) satisfies AS, then
\[
|U(z, t)| \leq C \min\{M_{\infty}, (1 + d(z, \Sigma(t)))^{-1/2}\},
\] (5.35)
\[
|U_{\phi}(z, t)| \leq C \min\{M_{\lambda, h}, (1 + d(z, \Sigma(t)))^{-3/2}\},
\] (5.36)

for some constant \(C > 0\) depending on \(C_0, C_1\). Here, \(d(z, \Sigma(t)) := \inf_{\beta \in \mathbb{R}} |z - Z^{(\beta, t)}|\).

\textbf{Proof} Recall that \(U\) is holomorphic in \(\Omega(t)\) with boundary value \(F\). So we have for \(z \in \Omega(t)\),
\[
U(z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z^{(\beta, t)}}{z - Z^{(\beta, t)}} F^{(\beta, t)} d\beta.
\]
If \(d(z, \Sigma(t)) \leq 1\), then by (AS2)
\[
|U(z, t)| \leq \|U(\cdot, t)\|_{L^\infty(\Omega(t))} \leq M_{\infty}.
\]
If \(d(z, \Sigma(t)) \geq 1\), we have
\[
|U(z, t)| \leq \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \frac{1}{|z - Z^{(\beta, t)}|^2} d\beta \right)^{1/2} \|Z^{(\alpha, t)}\|_{L^\infty} \|F\|_{L^2} \leq CM_{\lambda, h} z^{-1/2}.
\]
So we obtain (5.35). Notice that \(U_{\phi}\) has boundary values \(\frac{E^{(\alpha, t)}}{Z^{(\alpha, t)}}\). Using the same proof as for (5.35), we obtain (5.36). □

\(\Box\) Springer
Lemma 5.6 Assume that \((W, U, \{z_j(t)\})\) satisfies AS, then

\[
|\dot{z}_j(t)| \leq C(d_I(t)^{-1/2} + \frac{|\lambda|}{x(t)}), \quad j = 1, 2. \tag{5.37}
\]

\[
|\Re\{\dot{z}_j(t)\}| \leq C d_I(t)^{-3/2}. \tag{5.38}
\]

\[
|\dot{z}_1(t) - \dot{z}_2(t)| \leq C d_I(t)^{-3/2} x(t), \tag{5.39}
\]

Proof The main tool is the maximum principle for holomorphic functions. Recall that for \(j = 1, 2,\)

\[
\dot{z}_j(t) = \sum_{1 \leq k \leq 2, k \neq j} \frac{\lambda_k i}{2\pi (z_j(t) - z_k(t))} + \bar{U}(z_j(t), t),
\]

where \(U(Z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{Z_\beta(\beta, t)}{Z - Z(\beta, t)} F(\beta, t) d\beta\) is holomorphic in \(\Omega(t)\) with boundary value \(F\) on \(\Sigma(t)\).

Estimate \(\dot{z}_j\): Clearly,

\[
\frac{\lambda_2}{2\pi (z_1(t) - z_2(t))} \leq \frac{|\lambda|}{4\pi x(t)}.
\]

By (5.35) and the assumption that \(d_I(t) \geq 1\), we have

\[
|\bar{U}(z_j(t), t)| \leq C (1 + d(z_j(t), \Sigma(t)))^{-1/2} = C d_I(t)^{-1/2}.
\]

So we obtain

\[
|\dot{z}_j(t)| \leq C(d_I(t)^{-1/2} + \frac{|\lambda|}{x(t)}). \tag{5.40}
\]

Estimate \(|\dot{z}_1(t) - \dot{z}_2(t)|\): By (5.36) of Lemma 5.5 we have

\[
|\dot{z}_1(t) - \dot{z}_2(t)| = |U(z_1(t), t) - U(z_2(t), t)\|
\leq \|U\|_{L^\infty(\Omega(t))}|z_1(t) - z_2(t)| \leq C d_I(t)^{-3/2} x(t).
\]

(5.38) follows immediately from

\[
\Re\{\dot{z}_j(t)\} = \Re\{\bar{U}(z_j(t), t)\} \leq |U(x(t) + iy(t), t)| x(t) \leq C d_I(t)^{-3/2}.
\]

\square

The following lemma follows by direct calculation.
Lemma 5.7 Assume that \((W, U, \{z_j(t)\})\) satisfies AS, then for \(t \in [0, T]\),

\[
\left| \frac{d}{dt} x(t) \right| \leq C d_I(t)^{-3/2} x(t),
\]

(5.41)

1. If \(\lambda < 0\), then

\[
\frac{d}{dt} y(t) \leq - \frac{|\lambda|}{8\pi x(0)},
\]

(5.42)

2. If \(\lambda > 0\), then

\[
\frac{d}{dt} y(t) \geq \frac{|\lambda|}{8\pi x(0)},
\]

(5.43)

Lemma 5.8 Assume that \((W, U, \{z_j(t)\})\) satisfies AS, then

\[
\|A_1 - 1\|_{X(t)} \leq C(1 + |\lambda|^2 d_I(t)^{-5/2}).
\]

(5.44)

\[
\|A_1 - 1\|_{\tilde{X}(t)} \leq C(M_{\lambda,h}^2 + |\lambda|^2 d_I(t)^{-7/2}),
\]

(5.45)

\[
\|A_1 - 1\|_{L^\infty} \leq C(1 + |\lambda|^2 d_I(t)^{-3}).
\]

(5.46)

Proof We prove (5.44) only. We have

\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta
\]

\[
- \sum_{j=1}^2 \frac{\lambda_j}{2\pi} Re \left\{ \left( (I - \mathbb{H}) \frac{Z_{\alpha}}{(Z(\alpha, t) - z_j(t))^2} \right) (D_t Z - \dot{z}_j(t)) \right\}.
\]

Splitting \(D_t Z = \tilde{F} + \tilde{Q}\). Using (AS2), (5.35), Lemma 2.5, Lemma 2.7 and (5.30), we have

\[
\left\| \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta \right\|_{X(t)}
\]

\[
\leq \left\| \int \frac{|F(\alpha, t) - F(\beta, t)|^2}{(\alpha - \beta)^2} d\beta \right\|_{X(t)}
\]

\[
+ \left\| \int \frac{|Q(\alpha, t) - Q(\beta, t)|^2}{(\alpha - \beta)^2} d\beta \right\|_{X(t)}
\]

\[
\leq C(\|F\|_{X(t)}^2 + \|Q\|_{\tilde{X}(t)}^2)
\]

\[
\leq C(1 + M_{\lambda,h}^2 + |\lambda|^2 x(t)^2 d_I(t)^{-3}).
\]
Similarly, we obtain

\[ \left\| \sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} D_t Z \right\} \right\|_{X_{\phi(t)}} \leq C(1 + M^2_{\lambda, h} + |\lambda|^2 x(t)^2 d_I(t)^{-3}). \]

For \( \left\| \sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \dot{z}_j(t) \right\} \right\|_{X_{\phi(t)}} \), we use \( \dot{z}_j(t) = \frac{\lambda_j t}{4\pi x(t)} + \tilde{U}(z_j(t), t) \). We have

\[
\sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \frac{\lambda_j t}{4\pi x(t)} \right\} = -\frac{\lambda^2}{4\pi^2} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_1(t))^2(Z(\alpha, t) - z_2(t))} \right. \\
+ \frac{Z_\alpha}{(Z(\alpha, t) - z_1(t))^2(Z(\alpha, t) - z_2(t))} \right\}
\]

Using Lemma 5.3 (and similar proofs, if necessary), we obtain

\[
\sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \frac{\lambda_j t}{4\pi x(t)} \right\} \right\|_{X_{\phi(t)}} \leq C|\lambda|^2 d_I(t)^{-5/2},
\]

\[
\sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \frac{\lambda_j t}{4\pi x(t)} \right\} \right\|_{L^\infty} \leq C|\lambda|^2 d_I(t)^{-3},
\]

and

\[
\sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \frac{\lambda_j t}{4\pi x(t)} \right\} \right\|_{X_{\phi(t)}} \leq C|\lambda|^2 d_I(t)^{-7/2},
\]

Using (5.35), we obtain

\[
\sum_{j=1}^{2} \frac{\lambda_j t}{2\pi i} \Re \left\{ (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \tilde{U}(z_j(t), t) \right\} \right\|_{X_{\phi(t)}} \leq C|\lambda| d_I(t)^{-2}.
\]

So we conclude the proof of the lemma. \( \square \)

Since \( A - 1 = \frac{A_1}{|Z_\alpha|^2} - 1 = \frac{A_1 - 1}{|Z_\alpha|^2} - \frac{|Z_\alpha|^2 - 1}{|Z_\alpha|^2} \), a direct consequence of Lemma 5.8 and Lemma 2.8 yields the following estimates for \( A - 1 \).
Corollary 5.2 Assume that \((W, U, \{z_j(t)\})\) satisfies \(\text{AS}\), then
\[
\|A - 1\|_{L^\infty} \leq C(1 + M_{\lambda, h}^2 + |\lambda|^2 d_I(t)^{-3}). \tag{5.47}
\]
\[
\|A - 1\|_{X_{\phi(t)}} \leq C(1 + M_{\lambda, h}^2 + |\lambda|^2 d_I(t)^{-5/2}). \tag{5.48}
\]
\[
\|A - 1\|_{\dot{X}_{\phi(t)}} \leq C(M_{\lambda, h}^2 + |\lambda|^2 d_I(t)^{-4}). \tag{5.49}
\]

In particular, if \(m_0 \leq C|y_0|^{-1/2}\), then
\[
\sup_{0 \leq t \leq T} \|A - 1\|_{L^\infty} \leq \begin{cases}
C(1 + |y_0|^3), & \lambda > 0, \\
C, & \lambda < 0.
\end{cases} \tag{5.50}
\]
and
\[
\sup_{0 \leq t \leq T} \|A - 1\|_{\dot{X}_{\phi(t)}} \leq \begin{cases}
C|y_0|^{-3/8}, & \lambda > 0, \\
C|y_0|^{-1/2}, & \lambda < 0.
\end{cases} \tag{5.51}
\]

Note that \(\|A - 1\|_{L^2} \leq C|d_I(t)|^{3/4}\), which is significantly larger than \(\|A - 1\|_{L^\infty}\).

Lemma 5.9 Assume that \((W, U, \{z_j(t)\})\) satisfies \(\text{AS}\), then
\[
\|D_t Q\|_{Y_{\phi(t)}} \leq C \left( M_{\lambda, h} \frac{|\lambda|}{d_I(t)} + C \frac{\lambda^2}{d_I(t)^{5/2}} \right), \tag{5.52}
\]
\[
\|D_t Q\|_{\dot{Y}_{\phi(t)}} \leq C M_{\lambda, h} \frac{|\lambda|}{d_I(t)^2} + C \frac{\lambda^2}{d_I(t)^{7/2}}, \tag{5.53}
\]
\[
\|D_t Q\|_{L^\infty} \leq C M_{\lambda, \infty} \frac{|\lambda|}{d_I(t)^2} + C \frac{\lambda^2}{d_I(t)^3}, \tag{5.54}
\]
for some constant \(C > 0\) depending on \(C_0, C_1, M_{\lambda, h}, \frac{1}{L_0}\).

Proof To prove (5.52), note that
\[
D_t Q = \sum_{j=1}^2 \frac{\lambda_j i}{2\pi} \frac{D_t Z - \dot{z}_j(t)}{(Z(\alpha, t) - z_j(t))^2}. \tag{5.55}
\]
Decomposing \(D_t Z = \bar{F} + \bar{Q}\), we estimate \(D_t Q\) by
\[
\|D_t Q\|_{Y_{\phi(t)}} \leq \sum_{j=1}^2 \frac{\lambda_j}{2\pi} \left\{ \left\| \frac{\bar{F}}{(Z(\alpha, t) - z_j(t))^2} \right\|_{Y_{\phi(t)}} + \left\| \frac{\bar{Q}}{(Z(\alpha, t) - z_j(t))^2} \right\|_{Y_{\phi(t)}} \right\} + \sum_{j=1}^2 \left\| \frac{\lambda_j \dot{z}_j(t)}{(Z(\alpha, t) - z_j(t))^2} \right\|_{Y_{\phi(t)}} := I_1 + I_2 + I_3.
\]
Note that Lemma 5.3, Lemma 5.6 and (AS2) imply that

\[ I_1 + I_2 \leq C \frac{|\lambda|}{d_I(t)^2} + C \frac{\lambda^2}{d_I(t)^{7/2}}. \]

For \( I_3 \), using \( \dot{z}_j(t) = \frac{\lambda_i}{4\pi x(t)} + \tilde{U}(z_j(t), t) \), we obtain

\[ I_3 \leq \sum_{j=1}^{2} \frac{\lambda_j}{2\pi} \frac{1}{(Z(\alpha, t) - z_1(t))^2} + \frac{1}{(Z(\alpha, t) - z_1(t))^2} \| \frac{\lambda_j}{2\pi} \frac{\dot{z}_j(t)}{(Z(\alpha, t) - z_2(t))^2} \|_{Y_{\varphi(t)}} \]

\[ \leq C\lambda^2 d_I(t)^{-5/2}, \]

and

\[ I_{32} \leq C|\lambda|^2 d_I(t)^{-3}. \quad (5.56) \]

We obtain (5.52) by combining the estimates for \( I_1, I_2, I_{31}, I_{32} \). Similarly,

\[ \| D_t Q \|_{L^\infty} \leq \sum_{j=1}^{2} \frac{\lambda_j}{2\pi} \left\{ \left\| \frac{\dot{F}}{(Z(\alpha, t) - z_1(t))^2} \right\|_{L^\infty} + \left\| \frac{\dot{Q}}{(Z(\alpha, t) - z_1(t))^2} \right\|_{L^\infty} \right\} \]

\[ + \sum_{j=1}^{2} \frac{\lambda_j}{2\pi} \frac{\dot{z}_j(t)}{(Z(\alpha, t) - z_1(t))^2} \|_{L^\infty} \]

\[ \leq CM_{\lambda, h} \frac{|\lambda|}{d_I(t)^2} + C \frac{\lambda^2}{d_I(t)^3}. \]

To prove (5.53), it suffices to notice that

\[ \left\| \sum_{j=1}^{2} \frac{\lambda_j}{2\pi} \frac{\dot{z}_j(t)}{(Z(\alpha, t) - z_1(t))^2} \right\|_{\dot{Y}_{\varphi(t)}} \leq C \frac{\lambda^2}{d_I(t)^{7/2}}. \]

So we obtain (5.53).

\[ \square \]

**Lemma 5.10** Assume that \( (W, U, \{z_j(t)\}) \) satisfies AS, then

\[ \| b_1 \|_{Y_{\varphi(t)}} \leq C(1 + M_{\lambda, h})|\lambda|d_I(t)^{-5/2}), \quad (5.57) \]
\[ \|b_1\|_{Y_{\phi(t)}} \leq CM_{\lambda,h}|\lambda|d_1(t)^{-5/2}, \quad (5.58) \]
\[ \|b_1\|_{L^\infty} \leq C(1 + M_{\lambda,\infty})|\lambda|d_1(t)^{-3}, \quad (5.59) \]

for some constant \( C > 0 \) depending on \( C_0, C_1, M_{\lambda,h}, \) and \( \frac{1}{L_0} \).

**Proof** Recall that

\[ b_1 = \Re\{[\hat{Q}, \mathbb{H}](\frac{1}{Z_\alpha} - 1)\} + 2\Re\{(I - \mathbb{H})\hat{Q}\}. \quad (5.60) \]

We prove (5.57) only. The proof of (5.58) is similar. By Lemma 2.7, Corollary 5.1, Lemma 5.4, we have

\[ \|b_1\|_{Y_{\phi(t)}} = \left\| \Re\{[\hat{Q}, \mathbb{H}](\frac{1}{Z_\alpha} - 1)\} + 2\Re\{(I - \mathbb{H})\hat{Q}\} \right\|_{Y_{\phi(t)}} \]
\[ \leq \left\| [\hat{Q}, \mathbb{H}](\frac{1}{Z_\alpha} - 1) \right\|_{Y_{\phi(t)}} + 2\| (I + \mathbb{H})\hat{Q} \|_{Y_{\phi(t)}} \]
\[ \leq C\|Q\|_{Y_{\phi(t)}} \left\| \frac{1}{Z_\alpha} - 1 \right\|_{X_{\phi(t)}} + 2\| (I + \mathbb{H})\hat{Q} \|_{Y_{\phi(t)}} \]
\[ \leq C(1 + M_{\lambda,h})|\lambda|d_1(t)^{-5/2}, \]

for some constant \( C > 0 \) depending on \( C_0, C_1, M_{\lambda,h}, \frac{1}{L_0} \). \( \square \)

The following two lemmas are consequence of the previous estimates.

**Lemma 5.11** Assume that \((W, U, \{z_j(t)\})\) satisfies AS, then

(a) \( G = G(W, U, \{z_j\}) \in C([0, T]; Y_{\phi(t)}) \), and

\[ \|G(W, U, \{z_j\})\|_{Y_{\phi(t)}} \leq CM_{\lambda,h}\frac{|\lambda|}{d_1(t)^2} + C\frac{\lambda^2}{d_1(t)^{3/2}}, \quad (5.61) \]
\[ \|G(W, U, \{z_j\})\|_{Y_{\phi(t)}} \leq C\frac{|\lambda|}{d^2_{I,t}} + C\frac{\lambda^2}{d_1(t)^{5/2}}, \quad (5.62) \]

(b) \( R = R(W, U, \{z_j\}) \in C([0, T]; X_{\phi(t)}) \), and

\[ \|\partial_\alpha R(W, U, \{z_j\})\|_{X_{\phi(t)}} \leq C(1 + M_{\lambda,h})|\lambda|d_1(t)^{-5/2}. \quad (5.63) \]

**Proof** Recall that \( G = -\Re\{D_t Q\} \). Invoking Lemma 5.9, we obtain

\[ \|G\|_{Y_{\phi(t)}} \leq C\frac{|\lambda|}{d_1(t)^2} + C\frac{\lambda^2}{d_1(t)^{3/2}}. \]

Recall that \( R = \Re\{Q\} - 2\Re\{(I - \mathbb{H})\hat{Q}\} - \Re\{[\hat{Q}, \mathbb{H}](\frac{1}{Z_\alpha} - 1)\} \). Then (5.63) follows immediately from Lemma 2.7, Lemma 5.10 and Corollary 5.3. \( \square \)
Lemma 5.12 Assume that $(W, U, \{z_j(t)\})$ satisfies AS.

1. If $\lambda < 0$, then

$$d_I(t) \geq d_{I,0} + \frac{\lambda}{8\pi} t.$$  \hspace{1cm} (5.64)

2. If $\lambda > 0$, then

$$d_I(t) \geq d_{I,0} - |\lambda| t.$$  \hspace{1cm} (5.65)

**Proof** If $\lambda < 0$, then we have

$$\Im\{Z(\alpha, t) - z_j(t)\} = -|\Im\{Z(\alpha, 0) - z_j(0)\}| - \int_0^t |\Im\{\frac{d}{d\tau}(Z(\alpha, \tau) - z_j(\tau))\}| d\tau$$

$$\leq -|\Im\{Z(\alpha, 0) - z_j(0)\}| - \frac{|\lambda|}{8\pi x(0)} t.$$  \hspace{1cm} (5.66)

Therefore

$$d_I(t) \geq |\inf_{\alpha \in \mathbb{R}} \Im\{Z(\alpha, t) - z_1(t)\}| \geq d_{I,0} + \frac{|\lambda|}{2\pi x(0)}.$$  \hspace{1cm} (5.67)

If $\lambda > 0$, then we have

$$\Im\{Z(\alpha, t) - z_j(t)\} = |\Im\{Z(\alpha, 0) - z_j(0)\}| - \int_0^t |\Im\{\frac{d}{d\tau}(Z(\alpha, \tau) - z_j(\tau))\}| d\tau$$

$$\geq d_{I,0} - \frac{|\lambda|}{x(0)} t.$$  \hspace{1cm} (5.68)

Then we have

$$d_I(t) \geq d_{I,0} - \frac{|\lambda|}{x(0)} t.$$  \hspace{1cm} (5.69)

Recall that we assume $x(0) = 1$. So we conclude the proof of the lemma. \hfill $\square$

6 Proof of Theorem 3

6.1 The wellposedness of the quasilinear system (2.29)

In this subsection we prove Theorem 3. Without loss of generality, we assume $d_{I,0} \geq 4$.

$^{16}$ Recall that we assume $d_{I,0} = \inf_{\alpha \in \mathbb{R}} \min_{j=1,2} |\Im\{Z(\alpha, 0) - z_j(0)\}|$. 

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The proof of Theorem 3 Let $T > 0$ to be determined, define

$$S_T = \left\{ (W_\alpha, U, \{z_j\}) \middle| \text{(AS1)-(AS5) hold for } (W, U, \{z_j\}) \right\}.$$  

So

$$S_T \subset C([0, T]; X_{\phi(t)}) \times C([0, T]; Y_{\phi(t)}) \times \{ C^1([0, T]; \mathbb{C}) \}.$$  

We denote $D^{(n)}_t$ by $\partial_t + b^n \partial_\alpha$, where $b^n$ is the $n$-th approximation of $b$, which will be constructed shortly.

The zero-th approximation. We take $U^0 = \Re\{F_0\}, \{z^0_j\} = \{z_{j, 0}\}, Z^0 = Z_0, W^0 := \Re\{Z_0 - \alpha\}, G^0 := G(W^0, U^0, \{z^0_j\}), R^0 := R(W^0, U^0, \{z^0_j\}), b^0_1 := b_1(W^0, U^0, \{z^0_j\}), A^0 := A(W^0, U^0, \{z^0_j\}).$ For arbitrary $T > 0$,

$$(W^0, U^0, \{z^0_j\}) \in S_T.$$  

The $n$-th approximation. Assume we have constructed $(W^n, U^n, \{z^n_j\})$ such that $(W^n, U^n, \{z^n_j\}) \in S_T$.

Define $Z^n$ and $F^n$ by

$$Z^n(\alpha, t) = \alpha + (I + \mathbb{H})W^n, \quad F^n = (I + \mathbb{H})U^n. \tag{6.1}$$  

The $(n + 1)$-th approximation. Let’s construct $(W^{n+1}, U^{n+1}, \{z^{n+1}_j\})$ as follows.

**Step 1.** Define

$$\begin{cases}
G^n := G(W^n, U^n, \{z^n_j(t)\}), \\
R^n := R(W^n, U^n, \{z^n_j(t)\}), \\
b^n_1 := b_1(W^n, U^n, \{z^n_j(t)\}), \\
b^n_0 := \Re\{\tilde{F}^{n+1}, \mathbb{H}\}(\frac{1}{Z^n_\alpha} - 1) + 2F^{n+1}, \\
b^n = b^n_1 + b^n_0, \\
A^n := A(W^n, U^n, \{z^n_j(t)\}), \\
D^{(n)}_t := \partial_t + b^n \partial_\alpha, \\
Q^n := -\sum_{j=1}^{2} \frac{\lambda_{j,i}}{2\pi} \frac{1}{Z^n(\alpha, t) - z^n_j(t)}, \\
d_{1,n}(t) := \inf_{\alpha \in \mathbb{R}} \min_{j=1,2} |Z^n(\alpha, t) - z_j(t)|, \end{cases} \tag{6.2}$$  

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and define \( \mathcal{U}^n \) by

\[
\mathcal{U}^n(Z, t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial_\beta Z^n(\beta, t)}{Z - Z^n(\beta, t)} F^n(\beta, t) d\beta.
\]

Let \( \Sigma^n(t) \) be the curve parametrized by \( Z^n(\alpha, t) \), and \( \Omega(t)^n \) the region bounded above by \( Z^n \). Note that \( b^0_0 \) depends on the unknowns \( F^{n+1} \) and \( Z^{n+1} \).

**Step 2.** \((U^{n+1}, W^{n+1})\) is defined as the solution of

\[
\begin{cases}
D_t(U^n+1) = A\Lambda W^{n+1} + G^n, \\
D_t(W^{n+1}) = -U^{n+1} - \Re(F^{n+1}, \mathbb{H})(\frac{1}{\alpha Z^n} - 1)) + R^n, \\
F^{n+1} = (I + \mathbb{H})U^{n+1}, \\
W^{n+1}(., 0) = W_0, \quad U^{n+1}(., 0) = U_0.
\end{cases}
\]

(6.3)

Define \( Z^{n+1} \) and \( \{z^{n+1}_j\} \) by

\[
\begin{cases}
Z^{n+1}(\alpha, t) := \alpha + (I + \mathbb{H})W^{n+1}, \\
\frac{d}{dt}z^{n+1}_j(t) = \mathcal{U}^n(z^n_j(t), t) + \sum_{1 \leq k \leq \lambda / 2} \frac{1}{\pi i} \frac{\lambda_k}{z_k(t) - \lambda_j(t)} , \\
z^{n+1}_j(0) = z_j(0).
\end{cases}
\]

(6.4)

(6.5)

We show that \((W^{n+1}, U^{n+1}, \{z^{n+1}_j\})\) satisfies (AS1)-(AS5).

We choose \( T \) such that

\[
T = \begin{cases}
\min \{ T_1(C_0, \|(U_0, \partial_\alpha W_0)\|_{Y_{L_0 \times X_{L_0}}}), \frac{L_0}{2\delta_0}, \tau_0, 4\pi x(0)(|y_0| - |y_0|^{9/10})/|\lambda| \}, & \lambda > 0, \\
\min \{ T_2(C_0, \|(U_0, \partial_\alpha W_0)\|_{Y_{L_0 \times X_{L_0}}}), \tau_0, \frac{L_0}{2\delta_0} \}, & \lambda < 0.
\end{cases}
\]

Here, \( T_1, T_2 \) depend continuously on its parameters.

By Lemma 5.2, Lemma 5.10, Lemma 5.11, we have \( A^n - 1 \in C([0, T]; X_{\phi(t)}), \), \( b^n \in C([0, T]; Y_{\phi(t)}), \) \( G^n \in C([0, T]; Y_{\phi(t)}) \) and \( \partial_\alpha R^n \in C([0, T]; X_{\phi(t)}) \). Moreover, for \( t \in [0, T] \),

\[
\begin{align*}
\|b^n_1\|_{Y_{\phi(t)}} \leq C(1 + M_{\lambda, h})|\lambda|d_{I, n}(t)^{-5/2} \leq C|\lambda|d_{I, n}^{-5/2} \leq \begin{cases} C|y_0|^{-3/4}, & \lambda > 0, \\
C|y_0|^{-1}, & \lambda < 0. \end{cases}
\end{align*}
\]

\[
\begin{align*}
\|A^n - 1\|_{X_{\phi(t)}} \leq C(1 + |\lambda|^2d_{I, n}(t)^{-5/2}) \leq \begin{cases} C|y_0|^{3/4}, & \lambda > 0, \\
C|y_0|^{1/2}, & \lambda < 0. \end{cases}
\end{align*}
\]

(6.6)

\[
\begin{align*}
\|A^n - 1\|_{L^\infty} \leq C(1 + |\lambda|^2d_{I, n}(t)^{-3}) \leq \begin{cases} C(1 + |y_0|^3)^\frac{3}{10}, & \lambda > 0, \\
C, & \lambda < 0. \end{cases}
\end{align*}
\]

\[
\begin{align*}
\|A^n - 1\|_{X_{\phi(t)}} \leq C(M_{\lambda, h}^2 + |\lambda|^2d_{I, n}(t)^{-4}) \leq \begin{cases} C|y_0|^{-3/4}, & \lambda > 0, \\
C|y_0|^{-1}, & \lambda < 0. \end{cases}
\end{align*}
\]

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for some constant $C$ depending only on $C_0$, $C_1$, $L_0^{-1}$ and $M_{\lambda,h}$. Since $M_{\lambda,h} \leq 1$, and $C_1 \leq \|Z_0^n\|_{L^\infty} \leq 1 + \|Z_0^n\|_{X_{\phi(0)}} \leq 1$, $L_0^{-1} \leq 1$, we can chose $C$ depending only on $C_0$. Since we require

$$\sup_{0 \leq t \leq T} \left( \frac{\delta_0}{L_0} - 4 - 2d_0(\|A\|_{L^\infty} + \|A\|_{\dot{Y}_{\phi(0)}}) - 4d_1\|U_0, \partial_\alpha W_0\|_{Y_{\phi(0)} \times X_{\phi(0)}} \right) \geq 0,$$

we choose $\delta_0 = \left\{ \begin{array}{ll} |\gamma_0|^{7/16}, & \lambda > 0 \\ M_2, & \lambda < 0 \end{array} \right.$ Here, $M_2$ is a large but absolute constant. Basing on these arguments, we can take

$$T = \left\{ \begin{array}{ll} \tau, & \lambda < 0 \\ T = \frac{4\pi x(0)(|\gamma_0|-|\gamma_0|^{9/10})}{|\lambda|}, & \lambda > 0 \end{array} \right.$$

where $\tau > 0$ is a small but absolute constant.

By analyzing (6.5), using Lemma 5.6, we obtain

$$|x^{n+1}(t) - x(0)| \leq C \int_0^t |\mathbb{H}[x^{n+1}(\tau)]| d\tau \leq C T d_I(t)^{-3/2} \leq C |\gamma_0|^{-1}, \quad (6.7)$$

and

$$|y^{n+1}(t) - y(0) - \frac{\lambda}{4\pi x(0)}| \leq C T d_I(t)^{-3/2} \leq C |\gamma_0|^{-1}. \quad (6.8)$$

By Theorem 5, there is a unique solution $(U^{n+1}, \partial_\alpha W^{n+1}) \in C([0, T]; Y_{\phi(t)}) \times C([0, T]; X_{\phi(t)})$, to the system (6.3), such that

$$\sup_{t \in [0, T]} (\|U^{n+1}\|_{Y_{\phi(t)} \times X_{\phi(t)}}^2) \leq \frac{1}{4} M_{\lambda,h}^2 e^{B^nT} + \int_0^T e^{B^n(T-\tau)} N^n(\tau) d\tau, \quad (6.9)$$

where by (6.6),

$$B^n := C (1 + \|b^n_1\|_{C([0,T];Y_{\phi(t)})}) \leq C + C \sup_{0 \leq t \leq T} |\lambda| d_I^{(n)}(t)^{-5/2} := \gamma^n(T),$$

and

$$N^n(t) := \|R^n_\alpha\|_{X_{\phi(t)}}^2 + \|G^n\|_{Y_{\phi(t)}}^2 + C \|b^n_1\|_{Y_{\phi(t)}}^2 \leq CM_{\lambda,h}^2 + C\lambda^2 d_{I,n}(t)^{-3} \leq Cm_0^2 + C\lambda^2 d_{I,n}(t)^{-7/2} := \beta^n(t).$$
Case 1: \( \lambda < 0 \). By Lemma 5.12, we have \( d_{I,n}(t) \geq \frac{1}{2} (d_{I,0} + \frac{|\lambda|}{4 \pi \lambda(0)} t) \). So
\[
\gamma^n(t) \leq Cm_0 + C\lambda^2 (d_{I,0} + |\lambda| t)^{-3} \leq Cm_0 + C\lambda^2 d_{I,0}^{-3},
\]
\[
\beta^n(t) \leq Cm_0^2 + C\lambda^2 (d_{I,0} + |\lambda| t)^{-7/2}.
\]

Denote
\[
\gamma_0 := Cm_0 + C\lambda^2 d_{I,0}^{-3}.
\]

So we have
\[
\sup_{t \in [0,T]} \| (U^{(n+1)} , \partial_\alpha W^{n+1}) \|_{\gamma_{\phi(t)} \times X_{\phi(t)}}^2 \leq \frac{1}{4} m_0^2 e^{\gamma_0 T} + C \int_0^T e^{\gamma_0 (T - \tau)} \frac{\lambda^2}{(d_{I,0} + |\lambda| t)^{7/2}} d\tau 
\leq \frac{1}{4} M_{\lambda,0}^2 e^{\gamma_0 T} + C e^{\gamma_0 T} \frac{|\lambda|}{d_{I,0}^{7/2}}.
\]

Choosing \( T \leq \frac{1}{\gamma_0} \) and \( |\lambda|^2 \approx |d_{I,0}|^{3/2} \gg 1 \). Then \( Ce^{\gamma_0 T} \frac{|\lambda|}{d_{I,0}^{7/2}} \leq \frac{C}{d_{I,0}} \). So we have
\[
\sup_{t \in [0,T]} \| (U^{(n+1)} , \partial_\alpha W^{n+1}) \|_{\gamma_{\phi(t)} \times X_{\phi(t)}}^2 \leq m_0^2 + C \frac{1}{d_{I,0}}.
\] (6.10)

Similarly, we obtain
\[
\sup_{0 \leq t \leq T} \| U^{n+1}(\cdot, t) \|_{L^2}^2 \leq 1, \quad \sup_{0 \leq t \leq T} \| U^{n+1}(\cdot, t) \|_{L^\infty} \leq M_{\infty}.
\]

Therefore, \( (W^{n+1}, U^{n+1}, \{z_j^{n+1}\}) \) satisfies (AS2) and (AS5).

Case 2: \( \lambda > 0 \). Let’s again consider the situation when \( |y_0| \gg 1 \). So we take \( T = \frac{4 \pi \lambda(0)(|y_0| - |y_0|^{9/10})}{|\lambda|} \).

Using the similar argument, we have
\[
\sup_{t \in [0,T]} d_I(t) \geq \frac{1}{2} d_{I,0}^{9/10},
\] (6.11)

and
\[
\sup_{t \in [0,T]} \| (U^{(n+1)} , \partial_\alpha W^{n+1}) \|_{\gamma_{\phi(t)} \times X_{\phi(t)}}^2 \leq m_0^2 + C \frac{1}{d_{I,0}^{3/4}}.
\] (6.12)

So \( (W^{n+1}, U^{n+1}, \{z_j^{n+1}\}) \) satisfies (AS2) and (AS5).
Let \( \Sigma^{n+1}(t) \) be the curve parametrized by \( Z^{n+1}(\alpha, t) \), and let \( \Omega^{n+1}(t) \) be the region bounded above by \( \Sigma^{n+1}(t) \). Define \( U^{n+1} \) by

\[
U^{n+1}(Z, t) = \frac{1}{2\pi i} \int_{\infty}^{-\infty} \frac{Z^{n+1}_\beta(\beta, t)}{Z - Z^{n+1}(\beta, t)} F^{n+1}(\beta, t) d\beta.
\]

**Estimate** \(|Z^{n+1}(\alpha, t) - Z^{(n+1)}(\beta, t)|\): Since \( Z^{n+1}(\alpha, t) = \alpha + (I + \mathbb{H})U^{n+1} \), we have

\[
\partial_t(Z^{n+1}(\alpha, t) - Z^{n+1}(\beta, t)) = [(\bar{F}^n(\alpha, t) - \bar{F}^n(\beta, t)) + (\bar{Q}^n(\alpha, t) - \bar{Q}^n(\beta, t))] - (b^n(\alpha, t)Z^{n+1}_\alpha(\alpha, t) - b^n(\beta, t)Z^{n+1}_\beta(\beta, t)).
\]

Note that

\[
\left| [(\bar{F}^n(\alpha, t) - \bar{F}(\beta, t)) + (\bar{Q}^n(\alpha, t) - \bar{Q}(\beta, t))] - (b^n(\alpha, t)Z(\alpha, t) - b^n(\beta, t)Z(\beta, t)) \right| \leq CM_{\lambda, h}|\alpha - \beta|.
\]

(6.13)

So we obtain

\[
|Z_0(\alpha) - Z_0(\beta)| - CM_{\lambda, h}|\alpha - \beta| t \leq |Z^{n+1}(\alpha, t) - Z^{n+1}(\beta, t)|
\]

\[
\leq |Z_0(\alpha) - Z_0(\beta)| t + CM_{\lambda, h}|\alpha - \beta|.
\]

(6.14)

So we obtain

\[
\frac{1}{2} C_0|\alpha - \beta| \leq |Z^{n+1}(\alpha, t) - Z^{n+1}(\beta, t)| \quad \forall \alpha, \beta \in \mathbb{R}, \ t \in [0, T].
\]

(6.15)

So \((Z^{n+1}, F^{n+1}, \{z_j^{n+1}\})\) satisfies (AS3).

By (6.7), (6.8), and (6.15), \((W^{n+1}, U^{n+1}, \{z_j^{n+1}\})\) satisfies (AS4). So \((W^{n+1}, U^{n+1}, \{z_j^{n+1}\})\) is a Cauchy sequence in some Banach space. Let

\[
V^{k+1} := -\Re\{[\bar{F}^{k+1} + \mathbb{H}](\frac{1}{\partial_\alpha Z^k} - 1)] + R^k.
\]

(6.16)

Denote

\[
\begin{align*}
\hat{U} &= U^{k+1} - U^k, \\
\hat{G} &= -(b^k - b^{k-1})\partial_\alpha U^k - (A^k - A^{k-1})\Lambda W^k + G^k - G^{k-1}, \\
\hat{V} &= -(b^k - b^{k-1})\partial_\alpha W^k + V^{k+1} - V^k, \\
\hat{Z} &= Z^{k+1} - Z^k, \\
\hat{W} &= W^{k+1} - W^k, \\
\hat{z}_j &= \hat{z}_{j}^{k+1} - \hat{z}_j^k.
\end{align*}
\]

(6.17)
Then $\hat{U}$ and $\hat{W}$ satisfy

$$
\begin{cases}
D_t^{(k)} \hat{U} = -A^k \Lambda \hat{W} + \hat{G}, \\
D_t^{(k)} \hat{W} = -\hat{U} + \hat{V}, \\
\hat{U} (\cdot, 0) = 0, \quad \hat{W} (\cdot, 0) = 0.
\end{cases}
$$

(6.18)

And $\hat{z}_j$ satisfies

$$
\begin{cases}
\frac{d}{dt} \hat{z}_j = \mathcal{U}^k_{\text{error}} + \mathcal{S} \mathcal{P}^k_{\text{error}}, \\
\hat{z}_j (0) = 0.
\end{cases}
$$

(6.19)

Here,

$$\mathcal{U}^k_{\text{error}} = \overline{\mathcal{U}^k (z^k_j (t), t) - \mathcal{U}^{k-1} (z^{k-1}_j (t), t)},$$

and

$$\mathcal{S} \mathcal{P}^k_{\text{error}} = \sum_{1 \leq l \leq 2} \lambda_{il} \frac{1}{2\pi} \left( \frac{1}{z^l_k (t) - z^l_j (t)} - \frac{1}{z^{k-1}_l (t) - z^{k-1}_j (t)} \right).$$

Denote

$$\mathcal{E}^k (t) := \| (\hat{F}, \hat{W}) \|^2_{H^4 \times H^4} + |\{ \hat{z}_j \}|^2.$$

(6.20)

It’s elementary to check that

$$|\hat{z}_j (t)|^2 \leq CM^2 \lambda_{t, h} t \mathcal{E}^{k-1} (t).$$

(6.21)

Using Lemma 5.11, we have

$$\| \hat{V} \|_{H^4} + \| \hat{G} \|_{H^4} \leq CM_{\lambda, h} \mathcal{E}^{k-1}.$$

(6.22)

Using energy estimates, we obtain

$$\| (\hat{F}, \hat{H}) \|^2_{H^4 \times H^4} \leq CM^2_{\lambda, h} t \mathcal{E}^{k-1}.$$

(6.23)

To estimate $\hat{Z}$, using (6.4), we obtain

$$\| \hat{Z} \|_{H^4} \leq CM^2_{\lambda, h} t \mathcal{E}^{k-1}.$$

(6.24)

So we obtain

$$\mathcal{E}^k (t) \leq CM^2_{\lambda, h} t \mathcal{E}^{k-1}.$$

(6.25)
By (5.4) and (5.5) and the assumption that \(|y_0|\) large, we have
\[
\sup_{0 \leq t \leq T} \mathcal{E}^k(t) \leq c \sup_{0 \leq t \leq T} \mathcal{E}^{k-1}(t),
\]
for some constant \(0 < c < 1\). So \((W^n, U^n, \{z^n_j\})\) and therefore \((Z^n - \alpha, F^n, \{z^n_j\})\) is a Cauchy sequence in \(C([0, T]; H^4 \times H^4 \times \mathbb{C}^2)\). So
\[
(W^n, U^n, \{z^n_j\}) \to (W, U, \{z_j\})
\]
in \(C([0, T]; H^4 \times H^4 \times \mathbb{C}^2)\). Since \((\partial_\alpha W^n, U^n)\) is bounded in \(C([0, T]; X\phi(t) \times Y\phi(t))\), we have
\[
(W_\alpha, U) \in C([0, T]; X\phi(t) \times Y\phi(t)).
\]
Also,
\[
Q^n \to - \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{Z(\alpha, t) - z_j(t)} \quad \text{in } C([0, T]; Y\phi(t)).
\]
\[
b^n \to b = \Re\{[D_t Z, H]\} \frac{1}{Z_\alpha} - 1\} + 2\Re\{Q\} + 2\Re\{F\} \quad \text{in } C([0, T]; Y\phi(t)).
\]

Let \(\Sigma(t)\) be the curve parametrized by \(Z(\alpha, t)\) and \(\Omega(t)\) the region bounded above by \(\Sigma(t)\). Then we have
\[
\mathcal{U}^n(z, t) \to \frac{1}{2\pi i} \int \frac{Z_\beta(\beta, t)}{z - Z(\beta, t)} F(\beta, t) d\beta \quad \text{uniformly in } \Omega(t).
\]
So we can verify that
\[
\dot{z}_j(t) = \hat{\mathcal{U}}(z_j(t), t) + \sum_{\substack{1 \leq k \leq 2 \\ k \neq j}} \frac{\lambda_k i}{2\pi} \frac{1}{z_k(t) - z_j(t)},
\]
and \(\{z_j(t)\} \in C^2([0, T]; \Omega(t))\). So \((W, U, \{z_j\})\) is the unique solution to (2.29) on \([0, T]\). So we complete the proof of the theorem. \(\Box\)

### 7 Applications: sign changing of the Taylor sign coefficient

In this section, we apply Theorem 3 to prove Theorem 4.

Given appropriate initial data \((W_0, U_0, \{z_{1,0}, z_{2,0}\})\) and \(\lambda > 0\), we solve the water waves backward and forward in time. Note that the backward evolution is equivalent to
solving the forward in time water waves with the same initial data \((W_0, U, \{z_{1,0}, \{z_{2,0}\})\) but with \(\lambda\) replaced by \(-\lambda\). We choose \(|\lambda| = O(|\gamma_0|^{3/2})\).

(1) If \(\lambda > 0\), the point vortices travel toward the free interface. At time \(T_0 = \frac{4\pi x(0)|y_0|^{-1}}{|\lambda|^{9/10}}\), the distance between the point vortices and the free interface is \(\approx|y_0|^{9/10}\). This distance is significantly smaller than the initial one, so we can show that \(\inf_{\alpha \in \mathbb{R}} A_1(\alpha, T_0) \leq -\eta_1\), for arbitrary large \(\eta_1 > 0\), provided that we choose \(|y_0|\) sufficiently large.

(2) If \(\lambda < 0\), the point vortices travel away from the free interface. At time \(T_0 = O(1)\), the distance between the point vortices and the free interface is \(\approx|y_0|^{3/2}\). This distance is significantly larger than the initial one, so we can show that \(\inf_{\alpha \in \mathbb{R}} A_1(\alpha, T_0) \geq 1 - \eta_0\), for arbitrary small \(\eta_0 > 0\), provided that we choose \(|y_0|\) sufficiently large.

### 7.1 The Taylor sign coefficient

Let \(A_1 := A|Z_\alpha|^2\) we have

**Proposition 1** (Corollary 4.2 in [44]) Let \((Z, F, \{z_j\})\) be a solution to the water waves system such that \((Z_\alpha - 1, D_t Z) \in C([0, T_0]; H^2 \times H^2)\), then we have

\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta - \text{Im} \left\{ \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \left( (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \right) (D_t Z - \dot{z}_j(t)) \right\}. \tag{7.1}
\]

**Corollary 7.1** Let \((Z, F, \{z_j\})\) be a solution to the water waves system such that \((Z_\alpha - 1, D_t Z) \in C([0, T_0]; H^2 \times H^2)\). Fix a time \(t \in [0, T_0]\). Assume that \(Z(\alpha, t) = \alpha, D_t Z = \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)}\). Assume \(z_1(t) = -x(t) + iy(t), z_2(t) = x(t) + iy(t)\), with \(x(t) > 0, y(t) < 0\), and \(\lambda_1 = -\lambda_2 := \lambda\). We simply write \(x(t), y(t)\) as \(x, y\). Then

\[
A_1(\alpha, t) = 1 + \frac{\lambda^2}{\pi^2} \frac{3y\alpha^4 + (x^2 + y^2)y(3x^2 - y^2 + 2\alpha^2)}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2)^2)} + \frac{\lambda^2}{4\pi^2} \frac{\alpha^2x^2 + x^4 + 5\alpha^2y^2}{((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2)(x^2 + y^2)|y|}. \tag{7.2}
\]

The detail calculation for (7.2) is in § A.1.

To simplify (7.2), we take \(x \approx 1\) and \(|y| \gg 1\). Furthermore, we assume that

\[
|y(t)| \gtrsim \lambda^{1/2}, \quad 0 \leq t \leq T. \tag{7.3}
\]

Direct calculation yields the following lemma.
Lemma 7.1 Let $x \lesssim 1$ and $|y| \gg 1$. Then

$$
\sup_{\alpha \in \mathbb{R}} \frac{\lambda^2}{4\pi^2} \frac{\alpha^2 x^2 + x^4 + 5x^2 y^2}{((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2)(x^2 + y^2)|y|} \leq \frac{C' \lambda^2}{y^5},
$$

(7.4)

$$
\sup_{\alpha \in \mathbb{R}} \frac{\lambda^2}{\pi^2} \frac{3y\alpha^4 + (x^2 + y^2)y(3x^2 - y^2 + 2\alpha^2)}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2} - \frac{3y\alpha^4 + y^3(-y^2 + 2\alpha^2)}{(\alpha^4 + y^4 + 2\alpha^2y^2)^2}
\leq \frac{C' \lambda^2}{y^5},
$$

(7.5)

for some constant $C'$ depending on $C$ only. In particular, we have

$$
|A_1(\alpha, 0) - G(\alpha; y, \lambda)| \leq \frac{C' \lambda^2}{y^5},
$$

(7.6)

where

$$
G(\alpha; y, \lambda) := 1 + \frac{\lambda^2}{\pi^2} \frac{3y\alpha^4 + y^3(-y^2 + 2\alpha^2)}{(\alpha^4 + y^4 + 2\alpha^2y^2)^2}.
$$

(7.7)

Denote $\gamma := \frac{\lambda^2}{\pi^2|y|^3}$. Assume $\alpha = k|y|$, we have \(^{17}\)

$$
f(\gamma, k) := G(\alpha; y, \lambda) = 1 - \gamma g(k),
$$

(7.8)

where

$$
g(k) = \frac{3k^4 + 2k^2 - 1}{(k^2 + 1)^4}.
$$

By routine calculus, we have

1. $g(k)$ obtains its absolute maximum $\frac{1}{4}$ at $k = \pm 1$.
2. $g(k)$ obtains its absolute minimum $-1$ at $k = 0$.

So we conclude that

i. For $\gamma < 4$, we have $f(\gamma, k) > 0$ for all $k \in \mathbb{R}$.
ii. For $\gamma > 4$, we have $f(\gamma, \pm 1) < 0$.
iii. $f(4, \pm 1) = 0$, which is equivalent to $G(\pm |y|; -|y|, 2\pi |y|^{3/2}) = 0$.

\(^{17}\) Recall that $y < 0$. 

\[ Springer \]
7.2 The case \( \lambda_1 = -\lambda_2 > 0 \) (travel upward)

7.2.1 Initial data

Fix \( \gamma_0 \gg 1, x_0 = 1, \) and \( \gamma > 0 \) (independent of \( \gamma_0 \)). Assume that \( \lambda > 0 \) and let \( \lambda = \gamma^{1/2} \pi |\gamma_0|^{3/2} \). Let

\[
    z_1(0) = -x_0 + i(y_0 - 1), \quad z_2(0) = x_0 + i(y_0 - 1). \tag{7.9}
\]

Choosing the initial \( W_0 \) and \( U_0 \) such that

\[
    \|(\partial_\alpha W_0, U_0)\|_{\mathcal{X}_{10} \times \mathcal{Y}_{10}} + \|W_0\|_{L^2} \leq \frac{1}{|\gamma_0|}. \tag{7.10}
\]

We assume that \( U_0 \) and \( W_0 \) are odd functions. Denote \( Q(\alpha, t) = \frac{\lambda}{2\pi i} \frac{1}{Z(\alpha, t) - z_1(t)} - \frac{\lambda}{2\pi i} \frac{1}{Z(\alpha, t) - z_2(t)} \). We use \( L_0 = 10 \) and \( \delta_0 = 1000 \). In this case, \( d_{t,0} > |\gamma_0| \).

7.2.2 Some basic estimates

By Theorem 3, there exists \( T_0 > 0 \) such that (2.15) admits a unique solution \( (W, U, \{z_1(t), z_2(t)\}) \) on \([0, T_0]\) such that

(1) \( T_0 = \frac{|\gamma_0| - |\gamma_0|^{9/10}}{4\pi \gamma_0} \).

(2) Define \( Z = \alpha + (I + \mathbb{H})W \) and \( F = (I + \mathbb{H})U \). For all \( t \in [0, T_0] \),

\[
    \|(Z_\alpha(\cdot, t) - 1, F(\cdot, t))\|_{C([0,T_0]; H^4 \times \hat{H}^4)} \leq \frac{C}{|\gamma_0|^{3/8}}, \quad \|F\|_{C([0,T_0]; L^2)} \leq 1. \tag{7.11}
\]

Here, \( \hat{H}^4 \) represents the homogeneous Sobolev space \( \hat{H}^4 \).

(3) For each fixed \( t \in [0, T_0] \), \( W(\cdot, t) \) and \( U(\cdot, t) \) are odd functions, and \( \Re\{z_1(t)\} = -\Re\{z_2(t)\} \), \( \Im\{z_1(t)\} = \Im\{z_2(t)\} \).

(4) \( \sup_{0 \leq t \leq T_0} d_{t}(t) \geq \frac{\gamma_0}{2} |\gamma_0|^{9/10} \).

(5) For \( 0 \leq t \leq T_0 \),

\[
    \|Q\|_{L^\infty} = \left\| \frac{\lambda}{2\pi} \left( \frac{1}{Z(\alpha, t) - z_1(t)} - \frac{1}{Z(\alpha, t) - z_2(t)} \right) \right\|_{L^\infty} \leq C|\lambda| |d_1(t)|^{-2} \leq C|\gamma_0|^{-3/10}.
\]

We simply bound \( \|D_t Z(\cdot, t)\|_{L^\infty} \) by

\[
    \sup_{t \in [0,T_0]} \|D_t Z(\cdot, t)\|_{L^\infty} \leq \sup_{0 \leq t \leq T_0} (\|F(\cdot, t)\|_{L^\infty} + \|Q(\cdot, t)\|_{L^\infty}) \leq 1, \tag{7.12}
\]
and simply bound $b$ by
\[
\sup_{0 \leq t \leq T_0} \| b(\cdot, t) \|_{L^\infty} \leq 1. \tag{7.13}
\]

Using
\[
Z(\alpha, t) - \alpha = \int_0^t \partial_\tau (Z(\alpha, \tau) - \alpha) d\tau = \int_0^t D_\tau Z(\alpha, \tau) - b(\alpha, \tau) Z_\alpha(\alpha, \tau) d\tau,
\]
we obtain
\[
\| Z(\alpha, t) - \alpha \|_{H^4} \leq C \sup_{t \in [0, T_0]} (\| D_\tau Z(\cdot, t) \|_{L^\infty} + \| b(\cdot, t) \|_{L^\infty}) t \\
\leq C |y_0|^{-1/2} \leq C |y_0|^{-1/2}. \tag{7.14}
\]

7.2.3 The velocity of the point vortices

We have
\[
\begin{align*}
\frac{d}{dt} z_1(t) &= -\frac{\lambda i}{2\pi} \frac{1}{z_1(t) - z_2(t)} + \mathcal{U}(z_1(t), t) = \frac{\lambda i}{4\pi x(t)} + \mathcal{U}(z_1(t), t), \\
\frac{d}{dt} z_2(t) &= \frac{\lambda i}{2\pi} \frac{1}{z_2(t) - z_1(t)} + \mathcal{U}(z_2(t), t) = \frac{\lambda i}{4\pi x(t)} + \mathcal{U}(z_2(t), t).
\end{align*} \tag{7.15}
\]

(5.35) implies
\[
|\mathcal{U}(z_j(t), t)| \leq C d_I(t)^{-1/2} \leq C |y_0|^{-9/20}. \tag{7.16}
\]

An application of (5.38) yields
\[
|\frac{d}{dt} x(t)| \leq C |d_I(t)|^{-3/2} \leq C |y_0|^{-27/20}, \quad \text{for } 0 \leq t \leq T_0. \tag{7.17}
\]

So
\[
1 - C |y_0|^{-27/20} t \leq x(t) \leq 1 + C |y_0|^{-27/20} t, \quad 0 \leq t \leq T_0. \tag{7.18}
\]

Integrating (7.15) in time, using (7.16) and (7.18), we obtain
\[
\begin{align*}
z_1(t) &= -x_0 + i (y_0 + \frac{\lambda}{4\pi x_0} t) + O(t |y_0|^{-9/20}), \\
z_2(t) &= x_0 + i (y_0 + \frac{\lambda}{4\pi x_0} t) + O(t |y_0|^{-9/20}).
\end{align*} \tag{7.19}
\]
In particular, at $T_0 = \frac{4\pi x_0(|y_0|^{9/10})}{|y_0|^{3/2}}$,

\[
  z_1(T_0) = -|x_0| - i|y_0|^{9/10} + O\left(\frac{1}{|y_0|^{19/20}}\right),
\]

\[
  z_2(t_0) = |x_0| - i|y_0|^{9/10} + O\left(\frac{1}{|y_0|^{19/20}}\right)
\]  

(7.20)

7.2.4 The Taylor sign at $T_0$

By (7.1), at $T_0$,

\[
  A_1(\alpha, T_0) = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, T_0) - D_t Z(\beta, T_0)|^2}{(\alpha - \beta)^2} \, d\beta
\]

\[
  - 3 \left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( I - \mathbb{H}(Z(\alpha, T_0) - z_j(t_0))^2 \right) Z_\alpha \right\}.
\]

Denote

\[
  G_1(\alpha; x, y, \lambda) := \frac{\lambda^2 3y\alpha^4 + (x^2 + y^2)y(3x^2 - y^2 + 2\alpha^2)}{\pi^2 (\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2}.
\]  

(7.21)

\[
  G_2(\alpha; x, y, \lambda) := \frac{\lambda^2}{4\pi^2 ((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2)(x^2 + y^2)|y|}.
\]  

(7.22)

Proposition 2 Choosing the initial data as in § 7.2.1, then

\[
  A_1(\alpha, T_0) = 1 + G_1(\alpha; 0, -|y_0|^{9/10}, \lambda) + O\left(\frac{1}{|y_0|^{1/2}}\right).
\]  

(7.23)

Decomposing the integral term in (7.1) as follows:

\[
  \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} \, d\beta
\]

\[
  = \frac{1}{2\pi} \int \frac{|Q(\alpha, t) - Q(\beta, t)|^2}{(\alpha - \beta)^2} \, d\beta
\]

\[
  + \frac{1}{2\pi} \int \frac{|F(\alpha, t) - F(\beta, t)|^2}{|\alpha - \beta|^2} \, d\beta
\]

\[
  + \frac{1}{\pi} \int \Re \frac{(F(\alpha, t) - F(\beta, t))Q(\alpha, t) - Q(\beta, t)}{(\alpha - \beta)^2} \, d\beta
\]

\[
  := I_1(\alpha, t) + I_2(\alpha, t) + I_3(\alpha, t).
\]

First, direct calculation yields the following estimates for $I_1, I_2, and I_3$. 
Lemma 7.2 For $0 \leq t \leq T_0$, we have

$$|I_1(\alpha, t)| + |I_2(\alpha, t)| + |I_3(\alpha, t)| = O\left(\frac{1}{|y_0|^{1/2}}\right). \quad (7.24)$$

Proof For $I_1$, we have

$$I_1 = \int_{|\alpha| \leq |y_0|^{1/2}} \frac{|Q(\alpha, t) - Q(\beta, t)|^2}{(\alpha - \beta)^2} d\beta + \int_{|\alpha| \geq |y_0|^{1/2}} \frac{|Q(\alpha, t) - Q(\beta, t)|^2}{(\alpha - \beta)^2} d\beta$$

$$:= I_{11} + I_{12}.$$

We use (5.30) to bound $\frac{Q(\alpha, t) - Q(\beta, t)}{\alpha - \beta}$ by

$$\left|\frac{Q(\alpha, t) - Q(\beta, t)}{\alpha - \beta}\right| \leq \|\partial_\alpha Q(\cdot, t)\|_{L^\infty} \leq C|\lambda|d_1(t)^{-5/2} \leq C|y_0|^{-3/4}.$$

Therefore,

$$I_{11} \leq \int_{|\alpha| \leq |y_0|^{1/2}} \|\partial_\alpha Q(\cdot, t)\|_{L^\infty}^2 d\alpha \leq C|y_0|^{1/2}|y_0|^{-3/2} = C|y_0|^{-1}.$$

For $I_{12}$,

$$I_{12} \leq 2\|Q\|_{L^\infty}^2 \int_{|\alpha - \beta| \geq |y_0|^{1/2}} \frac{1}{(\alpha - \beta)^2} d\beta \leq C|y_0|^{-1}.$$

So we obtain $I_1(\alpha, t) \leq C|y_0|^{-1}$. Using the same argument, we obtain

$$I_2 \leq C|y_0|^{-1/2}.$$

For $I_3$, using Cauchy-Schwarz inequality, we have $I_3 \leq 2(I_1 + I_2)$. So we conclude the proof of the lemma.

Second, we manipulate $-\text{Im} \left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( I - \mathbb{H} \frac{Z\alpha}{Z(\alpha, t) - z_j(t)^2} \right) (D_t Z - \dot{z}_j(t)) \right\}$.

Lemma 7.3 For $0 \leq t \leq T_0$, we have

$$-\text{Im} \left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( I - \mathbb{H} \frac{Z\alpha}{Z(\alpha, t) - z_j(t)^2} \right) (D_t Z - \dot{z}_j(t)) \right\}$$

$$= G_1(\alpha; 0, y(t), \lambda) + O\left(\frac{1}{|y_0|^{1/2}}\right).$$
Proof Indeed, by Corollary A.2,

\[-Im\left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( (I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} \right)(D_t Z - \dot{z}_j(t)) \right\} \]

\[= - \sum_{j=1}^{2} \frac{\lambda_j}{\pi} Re\left\{ \frac{D_t Z - \dot{z}_j}{c_0^j(\alpha - w_0^j)^2} \right\}, \tag{7.25} \]

where

\[c_0^j = (\Phi^{-1})_\alpha(\omega_0^j), \quad \omega_0^j = \Phi(z_j). \tag{7.26} \]

We claim that

\[c_0^j = 1 + O\left( \frac{1}{|y_0|} \right), \quad \omega_0^j = z_j(t) + O\left( \frac{1}{|y_0|^{19/20}} \right), \quad t \in [0, T_0]. \tag{7.27} \]

Indeed, since \(\Phi^{-1}\) has boundary value \(Z(\alpha, t)\), we have

\[\Phi^{-1}(z, t) - z = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{z - \beta} (Z(\beta, t) - \beta) d\beta. \tag{7.28} \]

Expanding \(\Phi(z, t)\) about \(z_j(t)\), we obtain

\[\omega_0^j - z_j(t) = \Phi(z_j(t), t) - z_j(t) = \Phi(z_j(t), t) - \Phi^{-1}(z_j(t), t, t) \]

\[\leq \|\Phi_z\|_{L^\infty} |\Phi^{-1}(z_j(t), t) - z_j(t)| \]

\[\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{|z_j(t) - \beta|} |Z(\beta, t) - \beta| d\beta \]

\[\leq C d_1(t)^{-1/2} \left( \int_{\mathbb{R}} |Z(\beta, t) - \beta|^2 d\beta \right)^{1/2}. \]

Using (7.14) and (7.19), we obtain

\[|\omega_0^j - z_j(t)| \leq C \left( |y_0|^{9/10} \right)^{-1/2} |y_0|^{-1/2} \leq C |y_0|^{-19/20}. \tag{7.29} \]

Taking \(\partial_z\) on both sides of (7.28) yields

\[(\Phi^{-1})_\alpha(z, t) = 1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{(z - \beta)^2} (Z(\beta, t) - \beta) d\beta. \]

By (7.14), Cauchy-Schwarz and \(\omega_0^j = z_j(t) + O\left( \frac{1}{|y_0|^{19/20}} \right), \) we obtain

\[|\Phi^{-1}_\alpha(\omega_0^j, t) - 1| \leq \left| \int_{-\infty}^{\infty} \frac{1}{(\omega_0^j(t) - \beta)^2} (Z(\beta, t) - \beta) d\beta \right| \leq C |y_0|^{-1}. \]
So we obtain \(|c^j_0 - 1| \leq C|y_0|^{-1}\) and therefore verify (7.27).
Decomposing \(D_t Z = \tilde{F} + \tilde{Q}\). We use the following rough estimate
\[
\left| \sum_{j=1}^{2} \lambda_j \frac{\tilde{Q}}{c^j_0(\alpha - \omega^j_0)^2} \right| \leq C|y|^{-1}.
\]
Using the symmetry \(\lambda_1 = -\lambda_2\), the estimate \(\|F\|_\infty \leq C|y_0|^{-1/3}\) (from Theorem 3), (7.26), we obtain
\[
\left| \sum_{j=1}^{2} \lambda_j \frac{\tilde{F}}{c^j_0(\alpha - \omega^j_0)^2} \right| \leq C|\lambda|d_I(t)^{-3}\|F\|_\infty \leq C|y_0|^{-1}.
\]
The same argument gives
\[
\left| \sum_{j=1}^{2} \lambda_j \frac{\mathcal{U}(z_j(t), t)}{c^j_0(\alpha - \omega^j_0)^2} \right| \leq C|y_0|^{-1}.
\]
Using (7.27) and the symmetry \(\lambda_1 = -\lambda_2\), it is straightforward to verify that
\[
\left| \sum_{j=1}^{2} \lambda_j \frac{\mathcal{U}(z_j(t), t)}{c^j_0(\alpha - \omega^j_0)^2} \right| \leq C|y_0|^{-1/2},
\]
using \(\lambda = O(|y_0|^{3/2})\) and \(d_I(t) \geq C|y_0|^{9/10}\), we have
\[
\sum_{j=1}^{2} \lambda_j \frac{\mathcal{U}(z_j(t), t)}{c^j_0(\alpha - \omega^j_0)^2} \right| \leq C|y_0|^{-1/2}.
\]
Since \(z_j(t) = \frac{\lambda_j}{4\pi x(t)} + \tilde{U}(z_j(t), t)\), we obtain
\[
- \text{Im} \left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{2\pi} \left( I - \mathbb{H} \right) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} (D_t Z - \dot{z}_j(t)) \right\}
\]
\[
= - \sum_{j=1}^{2} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{-\dot{z}_j}{c^j_0(\alpha - \omega^j_0)^2} \right\} + O(|y_0|^{-1})
\]
\[
= - \sum_{j=1}^{2} \frac{\lambda_j}{\pi} \text{Re} \left\{ \frac{-\frac{\lambda_j}{4\pi x(t)}}{(\alpha - z_j(t))^2} \right\} + O(|y_0|^{-1})
\]
\[
= \frac{\lambda^2}{\pi^2} \frac{3y(t)\alpha^4 + (x(t)^2 + y(t)^2)y(t)(3x(t)^2 - y(t)^2 + 2\alpha^2)}{\alpha^4 + (x(t)^2 + y(t)^2)^2 + 2\alpha^2(y(t)^2 - x(t)^2)^2} + O(|y_0|^{-1}),
\]
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where the last equality follows from

\[
- \sum_{j=1}^{\lambda} \frac{\lambda}{\pi} \left\{ \frac{-\frac{\lambda}{4\pi(x(t))}}{(\alpha - z_j(t))^2} \right\} = \frac{\lambda^2}{\pi^2} \frac{3y(t)\alpha^4 + (x(t)^2 + y(t)^2)\alpha(t)(3x(t)^2 - y(t)^2 + 2\alpha^2)}{(\alpha^4 + (x(t)^2 + y(t)^2)^2 + 2\alpha^2(y(t)^2 - x(t)^2))^2}.
\]

(7.30)

For the calculation of (7.30), see § A.1. So we obtain

\[
- \text{Im} \left\{ \sum_{j=1}^{\lambda} \frac{\lambda}{2\pi} \left( (I - \mathbb{H}) \frac{Z_{\alpha}}{(Z(\alpha, t) - z_j(t))^2} \right)(D_t Z - \dot{z}_j(t)) \right\} = G_1(\alpha; x(t), y(t), \lambda) + O(|y_0|^{-1}).
\]

Finally, using (7.5) and \(|y(t)| \geq C|y_0|^{9/10}\), we obtain

\[
G_1(\alpha; x(t), y(t), \lambda) = G_1(\alpha; 0, y(t), \lambda) + O \left( \frac{1}{|y_0|^{1/2}} \right).
\]

(7.31)

So we conclude the proof of the lemma. \(\square\)

**Proof of Proposition 2** Proposition 2 follows from Lemma 7.2, \(y(T_0) = -|y_0|^{9/10}\), and Lemma 7.3. \(\square\)

Now we are able to calculate \(A_1(\cdot, T_0)\). Note that

\[
1 + G_1(|y_0|^{9/10}; 0, -|y_0|^{9/10}, \lambda) = G(|y_0|^{9/10}; -|y_0|^{9/10}, \lambda)
= 1 - \frac{\lambda^2}{\pi^2(|y_0|^{9/10})^3} \times \frac{1}{4}.
\]

Therefore, (7.31) and Proposition 2, we have

\[
A_1(|y_0|^{9/10}, T_0) = 1 + G_1(|y_0|^{9/10}; 0, -|y_0|^{9/10}, \lambda) + O \left( \frac{1}{|y_0|^{1/2}} \right)
= 1 - \frac{\lambda^2}{\pi^2(|y_0|^{9/10})^3} \times \frac{1}{4} + O \left( \frac{1}{|y_0|^{1/2}} \right)
= 1 - \frac{\gamma|y_0|^3}{4\pi^2|y_0|^{27/10}} + O \left( \frac{1}{|y_0|^{1/2}} \right)
= 1 - \frac{\gamma|y_0|^{3/10}}{4\pi^2} + O \left( \frac{1}{|y_0|^{1/2}} \right).
\]

So for any given \(\eta_1\), choosing \(|y_0| \geq \left( \frac{4\pi^2(\eta_1 + 1)}{\gamma} \right)^{10/3}\), we have

\[
A_1 \leq -\eta_1.
\]
7.3 The case $\lambda_1 = -\lambda_2 < 0$ (Travel downward)

To distinguish from the case that $\lambda > 0$, we write the solution as $(W_-, U_-, \{z_j, -\})$, and denote the corresponding Taylor sign coefficient by $A_{1, -}$. Denote the strength of the $j$-th point vortex by $\lambda_j, j = 1, 2$. We take

$$\lambda_{1, -} = -|\lambda|, \quad \lambda_{2, -} = |\lambda|,$$

so the point vortices travel downward. $Z_-$ and $F_-$ are defined as

$$Z_-(\alpha, t) = \alpha + (I + \mathbb{H})W_-, \quad F_- = (I + \mathbb{H})U_-.$$

7.3.1 The initial data

Choosing the same initial data as in § 7.2.1: Let $x_0, y_0, U_0, W_0$ and $\gamma$ be the same as in § 7.2.1. Let

$$z_{1, -}(0) = -x_0 + i(y_0 - 1), \quad z_{2, -}(0) = x_0 + i(y_0 - 1). \quad (7.32)$$

We use $L_0 = 10$ and $\delta_0 = 1000$.

By Theorem 3, there exists $T_{0, -} > 0$ such that (2.15) admits a unique solution

$$(\partial_\alpha W_-, U_-, \{z_{1, -}(0), z_{2, -}(0)\})$$

satisfying

(1) $T_{0, -} = O(1)$. In particular, $T_{0, -}$ does not depend on $|y_0|$.

(2) At $T_{0, -}$,

$$d_I(T_{0, -}) = O(|\lambda|) = O(|y_0|^{3/2}).$$

(3) $W$ and $U$ are odd functions, and $\Re \{z_1(t)\} = -\Re \{z_2(t)\}$, $\Im \{z_1(t)\} = \Im \{z_2(t)\}$.

(4) For $0 \leq t \leq T_{0, -}$, we have $\frac{1}{2} \leq x(t) \leq 2$, and $y(T_{0, -}) \geq |y_0| + \frac{|\lambda|}{16\pi} T_{0, -}$. Then we have

$$Q(T_{0, -}) \leq C \frac{|\lambda|}{d_I(T_{0, -})^2} \leq C \frac{|\lambda|}{|\lambda|^2} \leq C |y_0|^{-3/2}.$$

We simply bound $D_t Z_-$ by the rough estimate

$$\|D_t Z_-(\cdot, T_{0, -})\|_{L^\infty} \leq \|F_-(\cdot, T_{0, -})\|_{L^\infty} + \|Q(\cdot, T_{0, -})\|_{L^\infty} \leq 1.$$

Using

$$|\dot{z}_{j, -}(t)| = \frac{|\lambda|}{4\pi x_0} + O(1),$$
we obtain
\[
\left| \text{Im} \left\{ 2 \sum_{j=1}^{\infty} \lambda_j^{-1} \left( (I - \mathbb{H}) \frac{\partial_\alpha Z_-}{(Z_-(\alpha, t) - z_j, - (t))^2} \right) (D_t Z_- - \dot{z}_j, - (t)) \right\} \right| 
\leq \left| \text{Im} \left\{ 2 \sum_{j=1}^{\infty} \lambda_j^{-1} \left( (I - \mathbb{H}) \frac{\partial_\alpha Z_-}{(Z_-(\alpha, t) - z_j, - (t))^2} \right) \right\} \right| \sum_{j=1}^{\infty} |D_t Z_- - \dot{z}_j, - (t)|
\]
\[
\leq C \frac{\lambda^2}{d_1(T_{0,-})^3} (\| F_- \|_{L^\infty} + \| Q \|_{L^\infty} + |\lambda|) 
\leq C \frac{1}{|\lambda|T_{0,-}^3}.
\]

Since $T_{0,-}$ does not depend on $|\lambda|$, for given $\eta_0 \in (0, 1)$, we can take $|\lambda|$ sufficiently large such that $C \frac{1}{|\lambda|T_{0,-}^3} \leq \eta_0$. So we have

\[
A_{1,-}(\alpha, T_{0,-}) 
\geq 1 - \left| \text{Im} \left\{ 2 \sum_{j=1}^{\infty} \lambda_j^{-1} \left( (I - \mathbb{H}) \frac{\partial_\alpha Z_-}{(Z_-(\alpha, t) - z_j, - (t))^2} \right) (D_t Z_- - \dot{z}_j, - (t)) \right\} \right| 
\geq 1 - \eta_0.
\]

(7.33)

### 7.4 Conclude the proof of Theorem 4

Let $\lambda > 0$. Let $(W, U, \{z_j\})$ and $(W_-, U_-, \{z_j, -\})$ be the solution constructed in § 7.2 and § 7.3, respectively. Let $(\tilde{W}, \tilde{U}, \{\tilde{z}_j\})$ be the solution to (2.33) with initial data

\[
\begin{align*}
\tilde{W}(\cdot, -T_{0,-}) &:= W_-(\cdot, T_{0,-}), \\
\tilde{U}(\cdot, -T_{0,-}) &:= U_-(\cdot, T_{0,-}), \\
\tilde{z}_j(-T_{0,-}) &:= z_j, - (T_{0,-}).
\end{align*}
\]

(7.34)

Since (2.33) is time reversible and translation invariant in time, by the uniqueness of solutions, we have

\[
\tilde{W}(\cdot, t) = W_-(\cdot, T_{0,-} - t), \quad \tilde{U}(\cdot, t) = U_-(\cdot, T_{0,-} - t), \quad \tilde{z}_j(t) = z_j, - (T_{0,-} - t).
\]

(7.35)

In particular,

\[
\tilde{W}(\cdot, T_{0,-}) = W_-(\cdot, 0), \quad \tilde{U}(\cdot, T_{0,-}) = U_-(\cdot, 0), \quad \tilde{z}_j(T_{0,-}) = z_j, - (0).
\]
By the uniqueness of solution again, we have
\[ \tilde{W}(\cdot, t) = W(\cdot, t), \quad \tilde{U}(\cdot, t) = U(\cdot, t), \quad \tilde{z}_j(t) = z_j(t), \quad t \in [0, T_0]. \] (7.36)

Denote \( \tilde{A}_1 \) the Taylor sign coefficient corresponding to \((\tilde{W}, \tilde{U}, \{\tilde{z}_j\})\), then

1. \( \inf_{\alpha \in \mathbb{R}} \tilde{A}_1(\alpha, -T_0, -) \geq 1 - \eta_0 \).
2. \( \inf_{\alpha \in \mathbb{R}} \tilde{A}_1(\alpha, T_0) \leq -\eta_1 \).

Up to a time translation \( t \mapsto t + T_0, - \), we conclude the proof of Theorem 4.

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Appendix A. The Taylor sign

Lemma A.1 Let \( w_1, w_2 \in \mathbb{P}_- \). Then
\[
\int_{-\infty}^{\infty} \frac{1}{(\beta - w_1)(\beta - w_2)} d\beta = \frac{2\pi i}{w_2 - w_1} \quad (A.1)
\]

Proof \( w_2 \) is the only residue of \( \frac{1}{(\beta - w_1)(\beta - w_2)} \) in \( \mathbb{P}_+ \). By residue Theorem,
\[
\int_{-\infty}^{\infty} \frac{1}{(\beta - w_1)(\beta - w_2)} d\beta = \frac{2\pi i}{w_2 - w_1}. \]

Corollary A.1 Assume further that \( Z(\alpha, t) = \alpha, \tilde{Q} = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{1}{\alpha - z_j(t)} \). Then we have
\[
\frac{1}{2\pi} \int \frac{|\tilde{Q}(\alpha, t) - \tilde{Q}(\beta, t)|^2}{(\alpha - \beta)^2} d\beta = \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{i}{z_k - z_j}. \quad (A.2)
\]

Proof We have
\[
\tilde{Q}(\alpha, t) - \tilde{Q}(\beta, t) = \sum_{j=1}^{N} \frac{\lambda_j i}{2\pi} \frac{\beta - \alpha}{(\alpha - z_j)(\beta - z_j)}. \quad (A.3)
\]
So we have
\[
\left| \frac{\tilde{Q}(\alpha, t) - \tilde{Q}(\beta, t)}{\alpha - \beta} \right|^2 = \left| \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \frac{1}{(\alpha - z_j)(\beta - z_j)} \right|^2 = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\beta - z_j)(\alpha - z_k)(\beta - z_k)}. \quad (A.4)
\]
Apply lemma A.1, we have

\[ \int_{-\infty}^{\infty} \frac{1}{(\alpha - z_j)(\beta - z_j)(\alpha - z_k)(\beta - z_k)} d\beta = \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{2\pi i}{z_k - z_j}. \]

So we have

\[ \frac{1}{2\pi} \int \frac{|\bar{Q}(\alpha, t) - \bar{Q}(\beta, t)|^2}{(\alpha - \beta)^2} d\beta = \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^3 (\alpha - z_j)(\alpha - z_k)} \frac{2\pi i}{z_k - z_j} \]

\[ = \sum_{1 \leq j, k \leq N} \frac{\lambda_j \lambda_k}{(2\pi)^2 (\alpha - z_j)(\alpha - z_k)} \frac{i}{z_k - z_j}. \]

\[ \square \]

**Lemma A.2** Let \( z_0 \in \Omega(t) \). Assume that \( Z(\alpha, t) = \Phi^{-1}(\alpha, t) \), where \( \Phi : \Omega(t) \to \mathbb{P}_- \) is the Riemann mapping, we have

\[ (I - \mathbb{H}) \frac{1}{Z(\alpha, t) - z_0} = \frac{2}{c_1(\alpha - w_0)}, \quad c_1 = (\Phi^{-1})_z(w_0), \quad w_0 = \Phi(z_0, t). \]  

\[ \text{(A.5)} \]

**Proof** Note that \( Z(\alpha, t) = \Phi^{-1}(\alpha, t) \). So \( Z(\alpha, t) - z_0 \) is the boundary value of \( \Phi^{-1}(z, t) - z_0 \) in the lower half plane. Since \( \Phi^{-1} \) is 1-1 and onto, \( \Phi^{-1}(z, t) - z_0 \) has a unique zero \( w_0 \) in \( \mathbb{P}_- \) has a exactly one pole of multiplicity one. For \( z \) near \( w_0 \), we have

\[ \Phi^{-1}(z, t) - z_0 = c_1(z - w_0) + \sum_{n=2}^{\infty} c_n(z - w_0)^n, \quad \text{where} \quad c_1 = (\Phi^{-1})_z(w_0) \neq 0. \]

\[ \text{(A.6)} \]

Therefore, we have \( \frac{1}{Z(\alpha, t) - z_0} - \frac{1}{c_1(\alpha - w_0)} \) is holomorphic in \( \mathbb{P}_- \), and hence

\[ (I - \mathbb{H}) \left( \frac{1}{Z(\alpha, t) - z_0} - \frac{1}{c_1(\alpha - w_0)} \right) = 0. \]  

\[ \text{(A.7)} \]

Since \( \frac{1}{c_1(\alpha - w_0)} \) is holomorphic in \( \mathbb{P}_+ \), we have

\[ (I - \mathbb{H}) \frac{1}{Z(\alpha, t) - z_0} = (I - \mathbb{H}) \frac{1}{c_1(\alpha - w_0)} = \frac{2}{c_1(\alpha - w_0)}. \]  

\[ \text{(A.8)} \]

\[ \square \]
Corollary A.2  Let \( z_j(t) \in \Omega(t) \). Assume that \( Z(\alpha, t) = \Phi^{-1}(\alpha, t) \), where \( \Phi : \Omega(t) \to \mathbb{P}_- \) is the Riemann mapping, then
\[
(I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} = \frac{2}{(\Phi^{-1})_z(\Phi(z_j(t)))(\alpha - \Phi(z_j(t)))^2}
\] (A.9)

**Proof**  We have
\[
(I - \mathbb{H}) \frac{Z_\alpha}{(Z(\alpha, t) - z_j(t))^2} = -\partial_\alpha (I - \mathbb{H}) \frac{1}{Z(\alpha, t) - z_j(t)} = -\partial_\alpha \left( \frac{2}{(\Phi^{-1})_z(\Phi(z_j(t)))(\alpha - \Phi(z_j(t)))} \right)
\]
\[
= \frac{2}{(\Phi^{-1})_z(\Phi(z_j(t)))(\alpha - \Phi(z_j(t)))^2}.
\]

\[\square\]

Corollary A.3  Assume that \( Z(\alpha, t) = \Phi^{-1}(\alpha, t) \), where \( \Phi : \Omega(t) \to \mathbb{P}_- \) is the Riemann mapping
\[
A_1 = 1 + \frac{1}{2\pi} \int \frac{|D_t Z(\alpha, t) - D_t Z(\beta, t)|^2}{(\alpha - \beta)^2} d\beta - \sum_{j=1}^{N} \frac{\lambda_j}{\pi} Re \left\{ \frac{D_t Z - \dot{z}_j}{c_0^j(\alpha - w_j)^2} \right\},
\]
(A.10)

where
\[
c_0^j = (\Phi^{-1})_z(\omega_0^j), \quad \omega_0^j = \Phi(z_j).
\] (A.11)

A.1 The proof of Corollary 7.1

**Proof**  we calculate \(-Im\{\sum_{j=1}^{2} \frac{\lambda_j}{\pi} \frac{1}{(\alpha - z_j)^2} (D_t Z(\alpha) - \dot{z}_j)\}\). We have
\[
\dot{z}_j = \frac{\lambda i}{4\pi x}.
\]

We have
\[
\sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2} = \frac{\lambda i}{\pi} \left( \frac{1}{(\alpha - z_1)^2} - \frac{1}{(\alpha - z_2)^2} \right) = \frac{\lambda i}{\pi} \frac{(\alpha - z_2)^2 - (\alpha - z_1)^2}{(\alpha - z_1)^2(\alpha - z_2)^2}
\]
\[
= \frac{\lambda i}{\pi} \frac{(z_1 - z_2)(2\alpha - z_1 - z_2)}{(\alpha - z_1)^2(\alpha - z_2)^2}
\]

\[\square\] Springer
We have
\[
z_t(\alpha) = \frac{\lambda i}{2\pi} \left( \frac{1}{\alpha - z_1} - \frac{1}{\alpha - z_2} \right) = \frac{\lambda i}{2\pi} \frac{z_1 - z_2}{(\alpha - z_1)(\alpha - z_2)}
\]
So we have
\[
\sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2} z_t(\alpha) = -\frac{\lambda}{2\pi^2} \frac{|z_1 - z_2|^2(2\alpha - 2yi)}{|\alpha - z_1|^2|\alpha - z_2|^2(\alpha - z_1)(\alpha - z_2)}
\]
\[
= -\frac{\lambda}{2\pi^2} \frac{8x^2(\alpha - yi)(\alpha - z_1)(\alpha - z_2)}{|\alpha - z_1|^4|\alpha - z_2|^4}
\]
\[
= -\frac{4\lambda^2 x^2}{\pi^2} \frac{(\alpha - yi)(\alpha^2 - x^2 - y^2 + 2yi)}{((\alpha + x)^2 + y^2)^2((\alpha - x)^2 + y^2)^2},
\]
here, we've used
\[
(\alpha - z_1)(\alpha - z_2) = \alpha^2 - \alpha(z_1 + z_2) + z_1 z_2 = \alpha^2 - x^2 - y^2 - 2yi.
\]
Use also that
\[
((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2) = (\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2,
\]
we have
\[
-\text{Im}\left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2} z_t(\alpha) \right\}
\]
\[
= \text{Im}\left\{ \frac{4\lambda^2 x^2}{\pi^2} \frac{(\alpha - yi)(\alpha^2 - x^2 - y^2 + 2yi)}{((\alpha + x)^2 + y^2)^2((\alpha - x)^2 + y^2)^2} \right\}
\]
\[
= \frac{4\lambda^2 x^2}{\pi^2} \frac{2y\alpha^2 - y(\alpha^2 - x^2 - y^2)}{((\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2)
\]
\[
= \frac{4\lambda^2 x^2}{\pi^2} \frac{y\alpha^2 + y(x^2 + y^2)}{((\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2)
\]
We have
\[
\sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2} (-z_j)
\]
\[
= -\frac{\lambda i}{4\pi x} \sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2}
\]
\[
= -\frac{\lambda i (z_1 - z_2)(2\alpha - z_1 - z_2)}{\pi (\alpha - z_1)^2(\alpha - z_2)^2} \frac{\lambda i}{4\pi x}
\]
On the other hand, we have

\[
\begin{align*}
\sum_{1 \leq j, k \leq 2} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{i}{\overline{z_k} - z_j} & = -\frac{\lambda^2}{\pi^2} \frac{\alpha - yi}{(\alpha - z_1)^2(\alpha - z_2)^2} \\
& = -\frac{\lambda^2}{\pi^2} \frac{(\alpha - yi)(\alpha - z_1)^2(\alpha - 2z_2)^2}{|\alpha - z_1|^4|\alpha - z_2|^4} \\
& = -\frac{\lambda^2}{\pi^2} \frac{(\alpha - y)(\alpha^2 - x^2 - y^2)^2 - 4y^2\alpha^2 + 4y\alpha(\alpha^2 - x^2 - y^2)i}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2}
\end{align*}
\]

So

\[
-Im\left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2}(\overline{\dot{z}_j}) \right\}
= \frac{\lambda^2}{\pi^2} \frac{4y\alpha^2(\alpha^2 - x^2 - y^2) - y((\alpha^2 + x^2 - y^2)^2 - 4y^2\alpha^2)}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2}
\]

So we obtain

\[
-Im\left\{ \sum_{j=1}^{2} \frac{\lambda_j i}{\pi} \frac{1}{(\alpha - z_j)^2}(D_t Z(\alpha) - \dot{z}_j) \right\}
= \frac{4\lambda^2 x^2}{\pi^2} \frac{y\alpha^2 + y(x^2 + y^2)}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2}
+ \frac{\lambda^2}{\pi^2} \frac{4y\alpha^2(\alpha^2 - x^2 - y^2) - y((\alpha^2 + x^2 - y^2)^2 - 4y^2\alpha^2)}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2}
+ \frac{\lambda^2}{\pi^2} \frac{3y\alpha^2 + (x^2 + y^2)y(3x^2 - y^2 + 2\alpha^2)}{(\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2(y^2 - x^2))^2}
\]

On the other hand, we have

\[
\begin{align*}
\sum_{1 \leq j, k \leq 2} \frac{\lambda_j \lambda_k}{(2\pi)^2} \frac{1}{(\alpha - z_j)(\alpha - z_k)} \frac{i}{\overline{z_k} - z_j} & = -\frac{\lambda^2}{4\pi^2} \frac{1}{|\alpha - z_1|^2} \frac{1}{\overline{z_1} - z_1} + \frac{\lambda^2}{4\pi^2} \frac{1}{|\alpha - z_2|^2} \frac{1}{\overline{z_2} - z_2} \\
& = -\frac{\lambda^2}{4\pi^2} \frac{1}{(\alpha - z_1)(\alpha - z_2)} \frac{1}{\overline{z_2} - z_1} \\
& = \frac{\lambda^2}{4\pi^2} \frac{1}{(\alpha + x)^2 + y^2 - 2y} + \frac{\lambda^2}{4\pi^2} \frac{1}{(\alpha - x)^2 + y^2 - 2y}
\end{align*}
\]
\[
\begin{align*}
\lambda^2 & \frac{(\alpha^2 - x^2 + y^2)(\alpha^2 - x^2 + y^2)(x^2 + y^2)}{4\pi^2 ((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2)(x^2 + y^2)} y - 2x^2 y \\
= \lambda^2 & \frac{\alpha^2 x^2 + x^4 + 5x^2 y^2}{4\pi^2 ((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2)(x^2 + y^2)} |y|
\end{align*}
\]

So we obtain

\[
A_1(\alpha) = 1 + \frac{\lambda^2}{4\pi^2} \frac{3y\alpha^4 + (x^2 + y^2) y (3x^2 - y^2 + 2\alpha^2)}{\alpha^4 + (x^2 + y^2)^2 + 2\alpha^2 (y^2 - x^2)^2}
\]

\[
+ \frac{\lambda^2}{4\pi^2} \frac{\alpha^2 x^2 + x^4 + 5x^2 y^2}{((\alpha + x)^2 + y^2)((\alpha - x)^2 + y^2)(x^2 + y^2)} |y|.
\]

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