New Accurate Approximation for Average Symbol Error Probability Under $\kappa - \mu$ Shadowed Fading Channel

Yassine Mouchtak, Faissal El Bouanani
ENSIAS, Mohammed V University in Rabat
e-mails: yassine.mouchtak@um5s.net.ma, faissal.elbouanani@um5.ac.ma

Abstract—This paper proposes new accurate approximations for average symbol error probability (ASEP) of a communication system employing either $M$-phase-shift keying (PSK) or differential quaternary PSK (DQPSK) modulation schemes, with Gray coding over $\kappa - \mu$ shadowed fading channel. Firstly, new accurate approximations of symbol error probability (SEP) of both modulation schemes are derived over additive white Gaussian noise (AWGN) channel. Leveraging the trapezoidal integral method, a tight SEP’s approximation for $M$-PSK modulation is presented, while new upper and lower bounds for Marcum $Q$-function of the first order (MQF), and subsequently those for SEP under DQPSK scheme, are proposed. Next, these bounds are linearly combined to propose a highly accurate SEP’s approximation. The key idea manifested in the decrease property of modified Bessel function $I_\nu$, strongly related to MQF, with its argument $\nu$. Finally, theses approximations are used to tackle ASEP’s approximation under $\kappa - \mu$ shadowed fading. Numerical results show the accuracy of the presented approximations compared to the exact ones.

Index Terms—DQPSK modulation, $M$-PSK modulation, Marcum $Q$-function, SEP, Upper bound, $\kappa - \mu$ shadowed fading.

I. INTRODUCTION

Wireless technologies are becoming part of our daily lives and their utilization increase rapidly due to many advantages such as cost-effectiveness, global coverage and flexibility. Nevertheless, these technologies are infected by many phenomena including shadowing which is relatively slow and gives rise to long-term signal variations and multipath fading which is due to constructive and destructive interferences as a result of delayed, diffracted, reflected, and scattered signal components [1]. A great number of communication channels’ models have been proposed in the literature to describe either the fading or the joint shadowing/fading phenomena [2]-[5]. Recently, $\kappa - \mu$ shadowed fading proposed in [6], has attracted a lot of interest due to its versatility and wide applicability in practical scenarios. For instance, it was used for characterizing signal reception in device-to-device communications, body-to-body communications, underwater acoustic, fifth-generation (5G) communications, and satellite communication systems [7]-[11]. In addition, it was shown that numerous statistical models can be derived from the $\kappa - \mu$ shadowed one by setting the parameters to some specific real positive values [12].

The symbol error probability (SEP) is a fundamental performance evaluation tool in digital communications, quantifying the reliability of an instantaneous received signal. Furthermore, dealing with the average SEP (ASEP) is quite practical in most applications as it states the average performance irrespective of time. Nonetheless, evaluating ASEP in closed form remains a big challenge for numerous communication systems because of the complexity of either the end-to-end fading model or the employed modulation technique. Essentially, depending on the employed modulation scheme, SEP is provided in either complicated integral form [1] or first-order Marcum Q-function (MQF) and the zeroth-order modified Bessel function (MBF) of the first kind [13] for various $M$-ary and differential quadrature phase-shift keying (DQPSK) modulation schemes, respectively. That integral form can be reexpressed also in terms Gaussian $Q$-function (GQF), which is not known in closed form. By its turn, the MQF integral-form involves the MBF with exponential term [1], that can be rewritten appropriately as an upper incomplete upper Fox’s H-function (UIFH), or equivalently, an infinite summation of the product of upper incomplete Gamma functions [14]. Thus, obtaining ASEP requires the averaging of a UIFH over a generalized fading distribution, which is not evident particularly for fading model with probability density function (PDF) involving the product of exponential and Fox’s H-functions (e.g. $\kappa - \mu$ shadowed model). Obviously, deriving accurate bounds or approximations for the ASEP is strongly depending on the SEP’s ones. To this end, several SEP’s bounds and approximations for DQPSK are proposed in the literature relying on the tight MQF’s ones [15]-[19]. In [20] and [21], bounds for SEP are investigated, while in [22], new lower and upper bounds for SEP were proposed, based on which a novel approximation was derived. Despite the good accuracy of the latter’s approximation, they remain useless for ASEP computation because of their forms’ complications.

A. Motivation

The performance of wireless communication systems, with perfect channel state information (CSI) knowledge at the receiver, is widely examined by the scientific community. However, imperfect estimation of channel coefficients is dealt in various practical scenarios, leading to a significant degradation of the system performance. To overcome this limitation, differential modulation (DM) can be considered as an alternative solution particularly for low-power wireless systems, such as wireless sensor networks and relay networks [23]. The main advantage of this scheme is its simplicity of detection due to
the unnecessary channel coefficients estimation and tracking, leading to a significant reduction in the receiver computational complexity [24], [25]. However, this comes at a cost of higher error rate or lower spectral efficiency. As a result, selecting the most suitable modulation scheme depends on the considered application and both coherent and non-coherent detections. To this end, this paper is devoted to analyze the performance of two modulation schemes, namely M-PSK and DQPSK over \( \kappa - \mu \) shadowed fading channel.

B. Contribution

Capitalizing on the above, we aim at this work to propose accurate approximations for ASEP under \( \kappa - \mu \) shadowed fading and aforementioned modulation schemes. Specifically, utilizing the trapezoidal integral method, the SEP integral form for various \( M \)-ary modulation schemes is tightly approximated particularly for M-PSK scheme, while for DQPSK technique, we start by deriving simple lower and upper bounds for SEP by bounding MQF, to be used jointly in finding ASEP for generalized fading models.

Pointedly, our main key contributions can be summarized as follows:

- We propose a new exponential type approximation for the SEP's first form applied to M-PSK modulation by using the trapezoidal technique integral. To the best of the authors' knowledge, such accurate SEP's approximation outperforms those presented in the literature.

- We derive new upper and lower bounds of SEP in the case of DQPSK modulation based on which an accurate approximation of SEP is proposed.

- We provide, relying on the two proposed SEP's approximations, a tight approximate expression for ASEP over \( \kappa - \mu \) shadowed fading channel.

- We provide the asymptotic analysis for both forms of ASEP and we demonstrate that the diversity order over \( \kappa - \mu \) shadowed fading channel remains constant.

Motivated by this introduction, the rest of this paper can be structured as follows. In section II, a new approximation for the first SEP form (i.e., \( M \)-ary modulation) is presented for M-PSK while, new lower and upper bound of SEP in the case of DQPSK are derived, based on which an accurate approximation for the SEP is deduced. In Section III, the expression of ASEP under \( \kappa - \mu \) shadowed fading for both modulation schemes is evaluated. In section VI, the respective results are illustrated and verified by comparison with the exact ones using simulation computing. Section VI summarizes the main conclusions.

II. BOUNDS ON THE SEP

In this section, we propose new approximate expressions for the two potential different forms of SEP, namely (i) complicated integral form, and (ii) MQF form, applied to M-PSK and DQPSK modulations with Gray coding, respectively.

A. SEP with integral form

**Proposition 1.** The SEP for M-PSK modulation can be tightly approximated by

\[
\mathcal{H}_1(\gamma) \simeq \sum_{i=1}^{\kappa-\mu} A_i \exp(-B_i \gamma),
\]

while \( A_i \) and \( B_i \) are given in Table I.

**Proof.** The SEP for M-PSK modulation is given as [1, Eq. (8.22)]

\[
\mathcal{H}_1(\gamma) = \frac{1}{\pi} \int_0^{M-1} \exp\left(-\frac{\phi^2}{\sin^2(\theta)}\right) d\theta,
\]

with

\[
\phi = \log_2(M) \sin^2\left(\frac{\pi}{M}\right),
\]

and \( \gamma \) denotes the signal-to-noise (SNR) ratio per bit. Subsequently, (2) can be written as

\[
\mathcal{H}_1(\gamma) = Q\left(\sqrt{2\phi^2}\right) + \frac{1}{\pi} \int_0^{M-1} \exp\left(-\frac{\phi^2}{\cos^2(t)}\right) dt
\]

where \( Q(.) \) denotes the Gaussian Q-Function [1, Eq. (4.1)].

The integral \( \mathcal{I} \) can be approximated using numerical integration rules. The trapezoidal rule for definite integration of an arbitrary function between \([x_0, x_n]\) is given by

\[
\mathcal{I} = \frac{\phi}{2} \left[ g_0 + g_n + 2 \sum_{i=1}^{n-1} g_i \right],
\]

where \( g_i = f(x_0 + i\phi) \) for \( i = 0 \ldots n \), \( n \) refers to the number of sub-intervals equally spaced trapeziums, and \( \phi \) defines the spacing. Note that greater \( n \) the higher the accuracy’s approximation and the computational complexity as well.

By setting \( n = 3 \) and \( f(t) = \exp\left(-\phi^2/\cos^2(t)\right) \), \( \mathcal{I} \) can be approximated as

\[
\mathcal{I} \approx \frac{M-2}{12M} \left[ \exp\left(-\phi^2 \frac{\phi^2}{\sin^2(\frac{\pi}{M})}\right) + \exp\left(-\phi^2 \frac{\phi^2}{\cos^2(\frac{\pi}{M})}\right) \right] \\
+ \frac{M-2}{6M} \left[ \exp\left(-\phi^2 \frac{\phi^2}{\cos^2(\frac{\pi}{M})}\right) + \exp\left(-\phi^2 \frac{\phi^2}{\cos^2(\frac{\pi}{M})}\right) \right].
\]

By plugging [26, Eq. (8b)] and (5) in (4), (13) is attained.

Table II confirms the accuracy of the proposed approximation. Interestingly, one can ascertain that this tightness can be further improved by increasing \( n \), i.e., by increasing the number of terms in the approximation.

B. SEP with MQF form

The SEP for DQPSK modulation with Gray coding is given by [13]

\[
\mathcal{H}_2(\gamma) = Q_1(a\sqrt{\gamma}, b\sqrt{\gamma}) - \frac{1}{2} I_0(\sqrt{2\gamma}) \exp(-2\gamma),
\]

where \( I_0(.) \) denotes the modified Bessel function of order 0.
with $\gamma$ denotes the SNR per bit, $a = \sqrt{2 (1 - \sqrt{0.5})}$, $b = \sqrt{2 (1 + \sqrt{0.5})}$, $I_v(.)$ is the $v$-th order modified Bessel function of the first kind [14, Eq. (8.431)], and $Q_1(.,.)$ represents the first-order MQF defined as [1, Eq. (4.34)]

$$Q_1(\alpha, \beta) = \int_\beta^\infty t \exp\left(-\frac{t^2 + \alpha^2}{2}\right) I_0(\alpha t) dt. \quad (7)$$

1. New lower bound for SEP:

**Proposition 2.** The SEP for DQPSK modulation with Gray coding can be lower bounded as

$$H_2(\gamma) \geq L(\gamma), \quad (8)$$

with

$$L(\gamma) \triangleq \delta K(a, b, \gamma) - \frac{1}{2} I_0(\sqrt{2\gamma}) \exp(-2\gamma), \quad (9)$$

$$K(a, b, \gamma) = Q((b-a)\sqrt{\gamma}) - Q((b+a)\sqrt{\gamma}), \quad (10)$$

and $Q(.)$ denotes the Gaussian Q-function [1, Eq. (4.1)], and

$$\delta = \sqrt{\frac{b}{a}}.$$

**Proof.** As $I_v$ is a decreasing function with respect to the index $v$ [27], yields

$$Q_1(a\sqrt{\gamma}, b\sqrt{\gamma}) \geq J, \quad (11)$$

with

$$J \triangleq \int_{b\sqrt{\gamma}}^\infty t \exp\left(-\frac{t^2 + a^2}{2}\right) I_{\frac{1}{2}}(a\sqrt{\gamma}t) dt. \quad (12)$$

Now applying [14, Eq. (8.431.4)] for $v = \frac{1}{2}$, one can ascertain

$$I_{\frac{1}{2}}(a\sqrt{\gamma}t) = \frac{2 \sinh(a\sqrt{\gamma}t)}{\sqrt{2\pi a} \sqrt{\gamma}t}, \quad (13)$$

where $\sinh(.)$ accounts for the hyperbolic sine function.

By plugging (13) into (12) along with the following identity

$$\sinh(a\sqrt{\gamma}t) = \frac{\exp(a\sqrt{\gamma}t) - \exp(-a\sqrt{\gamma}t)}{2}, \quad (14)$$

one can obtain

$$J \geq \frac{1}{\sqrt{2\pi}} \int_{b\sqrt{\gamma}}^\infty \left[ \exp\left(-\frac{(\tau-a)\sqrt{\gamma})^2}{2}\right) \right] d\tau.$$

Finally, as $t \geq b\sqrt{\gamma}$, yields

$$J \geq \frac{\delta}{\sqrt{2\pi}} \left[ \int_{b\sqrt{\gamma}}^\infty \exp\left(-\frac{(\tau-a)\sqrt{\gamma})^2}{2}\right) d\tau \right]$$

which concludes the proof.

2. New upper bound for SEP:

**Proposition 3.** For $\gamma \geq 0$, holds

$$H_2(\gamma) \leq U(\gamma), \quad (15)$$

with

$$U(\gamma) \triangleq \frac{1}{\delta} K(a, b, \gamma) + \frac{1}{2} I_0(\sqrt{2\gamma}) \exp(-2\gamma). \quad (17)$$

**Proof.** Relying on (7) and using integration by part by considering $u(t) = t \exp\left(-\frac{t^2}{2}\right)$ and $w = I_0(\sqrt{\gamma}t)$, one can see

$$Q_1(a\sqrt{\gamma}, b\sqrt{\gamma}) = T + I_0(\sqrt{2\gamma}) \exp(-2\gamma), \quad (18)$$

with

$$T = \int_{b\sqrt{\gamma}}^\infty a\sqrt{\gamma} \exp\left(-\frac{t^2 + a^2\gamma}{2}\right) I_1(a\sqrt{\gamma}t) dt. \quad (19)$$
Again, by incorporating the inequality $I_1(t) \leq I_{\frac{1}{2}}(t)$ [27], alongside with (13) and (14) into (18), we get

$$T \leq \sqrt{\frac{a}{2\pi}} \int_{b\sqrt{\gamma}}^{\infty} \frac{1}{\sqrt{t}} \left[ \exp \left( -\frac{(t-a)^2}{2} \right) - \exp \left( -\frac{(t+a)^2}{2} \right) \right] dt. \quad (19)$$

Moreover, as $t \geq b\sqrt{\gamma}$ in the aforementioned integrand, one can check

$$T \leq \frac{1}{\delta} K(a, b, \gamma). \quad (20)$$

Therefore (17) can be inferred from (20) jointly with (10); this completes the proof.

3) Approximate SEP for DQPSK: In this part, a tight approximate expression for the SEP under DQPSK scheme is derived based on the two bounds presented above. In a similar manner to the approach followed in [22], the new proposed approximation is a linear combination of the two aforementioned bounds for $H_2(\gamma)$, namely

$$\bar{H}_2(\gamma) = \tilde{\chi}(\gamma)U(\gamma) + (1 - \tilde{\chi}(\gamma))L(\gamma). \quad (21)$$

**Proposition 4.** The function $\chi(\gamma)$ can be chosen as

$$\tilde{\chi}(\gamma) = C_0 \exp(-D_0\gamma) + C_1 \exp(-D_1\gamma), \quad (22)$$

where $C_i$ and $D_i$ are the best-fit parameters, depending on the SNR interval, summarized in Table III.

**TABLE III**

| SNR range | $C_i$, $D_i$ | $C_0$ | $D_0$ | $C_1$ | $D_1$ |
|-----------|-------------|-------|-------|-------|-------|
| $\gamma < 1$ | 0.1786 | 2.908 | 0.7504 | 0.1307 |
| $1 \leq \gamma < 8$ | 0.3798 | 1.895 | 0.6183 | 7.93 $\times 10^{-4}$ |
| $\gamma > 8$ | 0.005206 | 0.2764 | 0.6146 | 5.593 $\times 10^{-4}$ |

**Proof.** First, note that the following function

$$\chi(\gamma) = \frac{H(\gamma) - L(\gamma)}{U(\gamma) - L(\gamma)}, \quad (23)$$

satisfies the identity

$$H_2(\gamma) = \chi(\gamma)U(\gamma) + (1 - \chi(\gamma))L(\gamma). \quad (24)$$

That is, it is sufficient to look for a tight approximation for (23) so as to approximate $H_2(\gamma)$.

By plotting $\chi(\gamma)$ as shown in Fig. 1, one can clearly notice its exponential behavior. It follows that its approximate expression can be written in the form (22). Furthermore, the optimized coefficients $C_i$ and $D_i$ outlined in Table III, for various SNR intervals can be straightforward obtained using a curve fitting tool (e.g., Matlab Curve Fitting app); this ends the proof.

**Remark 1.** It is worthwhile that the first-order MQF can be approximated, relying on (6) and (21), by

$$\bar{Q}_1(a\sqrt{\gamma}, b\sqrt{\gamma}) = \bar{H}_0(\gamma) + \frac{1}{2} \int_0^\infty \sqrt{2\gamma} \exp(-2\gamma) \quad (25)$$

Both exact and approximate functions $\chi(\gamma)$ and $\tilde{\chi}(\gamma)$ are plotted in Fig. 1. One can observe that there exists a strong matching between the two curves over the entire range of $\gamma$.

![Fig. 1. Comparison between $\chi(\gamma)$ and $\tilde{\chi}(\gamma)$.](image)

Table IV summarizes the accuracy of the proposed approximation compared with the best ones proposed in the literature, namely $\{\bar{H}_i(\gamma)\}_i=3,5$ labeled $\{BER_{i+2}\}_i=3,5$ in [22], respectively. Besides, the relative error corresponding to the aforementioned approximations, namely

$$\varepsilon_i = \frac{\bar{H}_i(\gamma) - H_2(\gamma)}{H_2(\gamma)}, \quad i = 2,5,$$

is depicted in Fig. 2. Obviously, the relative error corresponding to the proposed approximation outperforms those in [22], except for a short interval, i.e., [11.7, 12.45] where $H_2(\gamma)$ is negligible, as outlined in Table II, compared to its values getting for small SNR.

![Fig. 2. Comparison of the relative errors.](image)
Comparison between approximate SER and the exact one.

| \( \gamma \) | \( H_2(\gamma) \), Eq. (21) | \( H_3(\gamma) \), [22] | \( H_4(\gamma) \), [22] | \( H_5(\gamma) \), [22] |
|---|---|---|---|---|
| 0.5 | \( 2.6929 \times 10^{-2} \) | \( 2.6918 \times 10^{-2} \) | \( 2.7792 \times 10^{-4} \) | \( 2.6921 \times 10^{-4} \) |
| 1 | \( 1.63519 \times 10^{-2} \) | \( 1.63851 \times 10^{-2} \) | \( 1.63851 \times 10^{-4} \) | \( 1.6368 \times 10^{-4} \) |
| 1.5 | \( 1.0640 \times 10^{-2} \) | \( 1.0607 \times 10^{-2} \) | \( 1.0655 \times 10^{-4} \) | \( 1.0667 \times 10^{-4} \) |
| 2 | \( 7.1611 \times 10^{-3} \) | \( 7.1685 \times 10^{-2} \) | \( 7.1625 \times 10^{-4} \) | \( 7.1885 \times 10^{-4} \) |
| 2.5 | \( 4.9177 \times 10^{-4} \) | \( 4.9190 \times 10^{-2} \) | \( 4.9174 \times 10^{-4} \) | \( 4.9481 \times 10^{-4} \) |
| 3 | \( 3.4227 \times 10^{-4} \) | \( 3.4228 \times 10^{-4} \) | \( 3.4226 \times 10^{-4} \) | \( 3.4482 \times 10^{-4} \) |
| 4 | \( 1.7013 \times 10^{-4} \) | \( 1.7015 \times 10^{-4} \) | \( 1.7018 \times 10^{-4} \) | \( 1.7144 \times 10^{-4} \) |
| 5 | \( 8.6484 \times 10^{-5} \) | \( 8.6501 \times 10^{-5} \) | \( 8.6501 \times 10^{-5} \) | \( 8.7059 \times 10^{-5} \) |
| 6 | \( 4.4613 \times 10^{-5} \) | \( 4.4624 \times 10^{-5} \) | \( 4.4617 \times 10^{-5} \) | \( 4.4859 \times 10^{-5} \) |
| 7 | \( 2.356 \times 10^{-5} \) | \( 2.356 \times 10^{-5} \) | \( 2.357 \times 10^{-5} \) | \( 2.3363 \times 10^{-5} \) |
| 8 | \( 1.2219 \times 10^{-5} \) | \( 1.2222 \times 10^{-5} \) | \( 1.2219 \times 10^{-5} \) | \( 1.2266 \times 10^{-5} \) |
| 9 | \( 6.595 \times 10^{-6} \) | \( 6.597 \times 10^{-6} \) | \( 6.597 \times 10^{-6} \) | \( 6.594 \times 10^{-6} \) |
| 10 | \( 3.318 \times 10^{-6} \) | \( 3.3427 \times 10^{-6} \) | \( 3.3419 \times 10^{-6} \) | \( 3.3418 \times 10^{-6} \) |
| 11 | \( 1.8307 \times 10^{-6} \) | \( 1.8311 \times 10^{-6} \) | \( 1.8307 \times 10^{-6} \) | \( 1.8307 \times 10^{-6} \) |
| 12 | \( 9.7990 \times 10^{-7} \) | \( 9.8011 \times 10^{-7} \) | \( 9.7990 \times 10^{-7} \) | \( 9.7990 \times 10^{-7} \) |

**Remark 2.** It can be seen clearly that the proposed approximation outperforms the concurrent ones. Moreover, one can check that the three expressions \( \{ \mathcal{H}_i(\gamma) \}_{i=1,2,3} \) are tough, which render them useless in numerous applications such as the ASEP computation under a complicated fading model. Contrarily, the expression of \( \mathcal{H}_2(\gamma) \) is quite simple, making it more appropriate for various fields.

**III. ASEP Analysis**

As mentioned above, the proposed approximate SEP is used to derive an approximate ASEP when communicating over \( \kappa-\mu \) shadowed fading.

The PDF of instantaneous SNR \( \gamma \) under the \( \kappa-\mu \) shadowed fading model can be written as [12, Eq. 4]

\[
f_\gamma(\gamma) = \frac{\lambda}{\Gamma(\mu)} \gamma^{\mu-1} e^{-\gamma} F_1(m; \mu; \omega \gamma),
\]

with

\[
\lambda = \mu \mu_m (1 + \kappa) \nu \omega
\]

\[
\nu = \mu (1 + \kappa) \frac{\omega}{\nu}, \quad \omega = \frac{\mu^2 \kappa (1 + \kappa)}{\nu (\mu + m)}
\]

\[
(27)
\]

where \( \nu \) is the average SNR, \( \kappa \) indicates the power ratio between the dominant waves, \( \mu \) refers to the scattered components, while \( m \) accounts for the shape parameter. Further, \( F_1(\cdot; \cdot; \cdot) \) denotes the Kummer confluent hypergeometric and Euler Gamma functions, respectively.

The ASEP approximation for both M-PSK and DQPSK schemes can be straightforwardly evaluated as

\[
P_s = \int_0^\infty f(\gamma) \mathcal{H}_1(\gamma) \, d\gamma,
\]

by setting \( i = 1 \) and \( i = 2 \), respectively.

**Proposition 5.** In the case of M-PSK modulation, the ASEP over \( \kappa-\mu \) shadowed fading model can be approximated as

\[
P_s \approx \lambda \sum_{i=1}^{7} \mathcal{A}_i (\nu + \mathcal{B}_i)^{-\mu} \left( 1 - \frac{\omega}{\nu + \mathcal{B}_i} \right)^{-m}.
\]

**Proof.** By plugging (1) and (26) in (28), and with the help of [14, 7.522 Eq. (9) and 9.121 Eq. (1)] one can obtain (29).

**Proposition 6.** The ASEP for DQPSK scheme with Gray coding over the \( \kappa-\mu \) shadowed fading channel can be tightly approximated as

\[
P_s \approx \frac{\lambda \eta}{\Gamma(\mu)} \sum_{i=1}^{7} \left[ C_i e^{-D_i \gamma} - C_i e^{-D_i \gamma} + \frac{\mathcal{F}_i}{\eta} \right].
\]

with

\[
\mathcal{F}_i = \sum_{k=0}^{\infty} \phi_{i,k} (\frac{1}{2}, \mu + k; \mu + k + 1; \frac{25i}{(b-a)^2 + 25i})
\]

\[
\phi_{i,k} = \frac{(m)_k \omega^k}{(\mu)_k (\mu + k)!} \left( \frac{2}{b-a)^2 + 2\xi_k} \right)^{\mu+k} \Gamma(\mu + k + \frac{1}{2})
\]

\[
(33)
\]

\[
\xi_k = \nu + \mathcal{D}_i,
\]

\[
\eta = \frac{1 - \delta^2}{\delta},
\]

where \((\cdot)\) represents the pochhammer symbol, \( \mathcal{F}_1(\cdot; \cdot; \cdot) \) denotes the hypergeometric functions [14, Eq. (8.310)]. \( F_0 = C_0, F_1 = C_1, F_2 = -\frac{1}{2}, F_2 = \frac{3}{\delta^2 - \pi}, \) and \( D_2 = 0.\)

**Proof.** First, one can check using (9) and (17) jointly with (10), (21), and (22)

\[
\tilde{\mathcal{H}}_0(\gamma) = \eta \sum_{i=0}^{2} \left[ C_i e^{-D_i \tau} K(a, b, \gamma) + \mathcal{F}_i e^{-D_i \gamma} I_0 \left( \sqrt{2} \gamma \right) \right].
\]

On the other hand, the ASEP for DQPSK can be evaluated as

\[
P_s = \int_0^\infty f(\gamma) \mathcal{H}_2(\gamma) \, d\gamma.
\]

Now, using (26) and (37) in (38), the ASEP can be approximated by
\[ P_s \simeq \frac{\lambda_2}{\Gamma(\mu)} \sum_{i=0}^{2} \left[ C_i M_{1,k}^{(1)}(a) - C_i M_{1,k}^{(1)}(-a) + \frac{F_i}{\eta} M_{1,k}^{(2)} \right] \]  

where

\[ M_{1,k}^{(1)}(a) = \int_0^\infty \gamma^{\mu-1} F_1(m; \mu; \omega \gamma) e^{-(D_i+v)_i+\gamma} \times \gamma (b-a) \sqrt{\gamma} d\gamma, \]

and

\[ M_{1,k}^{(2)} = \int_0^\infty \gamma^{\mu-1} F_1(m; \mu; \omega \gamma) e^{-(2+\xi_i) \gamma} I_0 (\sqrt{2 \gamma}) d\gamma. \]

Utilizing Craig's formula of the Gaussian Q-function [28, Eq. (5)] and [14, 9.14.1], (40) can be written as

\[ M_{1,k}^{(1)}(a) = \sum_{k=0}^\infty \frac{(m_k \omega^k \Gamma(\mu+k))}{\pi (\mu)_k k!} \int_0^\infty \int_0^\infty \gamma^{\mu+k-1} \times \exp \left( -\frac{(b-a)^2}{2 \sin^2(\theta) + \xi_i} \right) d\gamma d\theta. \]

By evaluating the inner integral in (42) with the aid of [14, Eq. (3.81.4)], we get

\[ M_{1,k}^{(1)}(a) = \sum_{k=0}^\infty \frac{(m_k \omega^k \Gamma(\mu+k))}{\pi (\mu)_k k!} \int_0^\infty \int_0^\infty \gamma^{\mu+k-1} \times \exp \left( -\frac{(b-a)^2}{2 \sin^2(\theta) + \xi_i} \right) d\gamma d\theta. \]

A. Asymptotic Analysis

In order to gain further insights into system parameters at high SNR regime, an asymptotic analysis for the SNR is carried out. Firstly, note that for large values of \( \gamma \), one can see that \( \omega \) goes to 0 and thus the term \( k = 0 \) dominates the others, \( v \) also goes to 0 (i.e., \( \xi_i \approx D_i \)). It follows that the ASEP can be asymptotically approximated as

\[ P_s \simeq \lambda \sum_{i=1}^{7} A_i B_i^{-\mu}. \]

and

\[ P_s \simeq \frac{\lambda_2}{\Gamma(\mu)} \sum_{i=0}^{2} \left[ C_i M_{1,k}^{(1,asy)}(a) - C_i M_{1,k}^{(1,asy)}(-a) + \frac{F_i}{\eta} M_{1,k}^{(2,asy)} \right], \]

for M-PSK and DQPSK schemes, respectively, with

\[ M_{1,k}^{(1,asy)}(a) \sim \Delta_2 F_1 \left( \frac{1}{2}, \mu; \mu + 1; \frac{2D_i}{(b-a)^2 + 2D_i} \right), \]

\[ \Delta = \frac{1}{2\sqrt{\pi}} \left( \frac{2}{(b-a)^2 + 2D_i} \right)^\mu \Gamma \left( \frac{\mu + 1}{2} \right), \]

\[ M_{1,k}^{(2,asy)} \sim \Gamma(\mu) (2 + D_i) \mu \frac{1}{2} \left( \frac{2}{2 + 2\xi_i} \right) \]

It is worth mentioning from (44) and (45) alongside with (27) that the diversity order equals \( \mu \).

B. Bound on the truncation error

The above approximate ASEP for DQPSK is expressed in terms of infinite series. Truncating such summation and estimating the truncated error are though of paramount importance for numerical evaluation purposes. In what follows, a closed-form bound for such truncation error is provided.

Using (31), the truncation up to \( L - 1 \) terms of the first summation results to the following error

\[ \epsilon_i^{(1)}(a) = \sum_{k=L}^{\infty} \frac{\phi_k}{2\sqrt{\pi}} \frac{1}{2} \left( \frac{2}{2 + 2\xi_i} \right) \]

By changing the summation index to \( j = k - L \) in (49), then using [29, Eq. (06.10.02.0001.01)], and performing some manipulations, the bound can be expressed as

\[ \epsilon_i^{(1)}(a) \leq \frac{\Theta_{i,L}(a)}{2\sqrt{\pi}} \frac{1}{2} \left( \frac{2}{2 + 2\xi_i} \right) \]

with

\[ \Theta_{i,L}(a) = \left( \frac{2}{(b-a)^2 + 2\xi_i} \right)^\mu \Gamma(\mu) (m + L) \]

In a similar manner, the truncated error of the summation (32) can be upper bounded by

\[ \epsilon_i^{(2)} \leq \left( \frac{2\omega}{2 + 2\xi_i} \right) \left( \frac{\Gamma(\mu) \omega^L}{L!} \left( \frac{2}{(b-a)^2 + 2\xi_i} \right)^\mu \right) \]

Consequently, and having in mind that \( \epsilon_i^{(j)} \) are positives, as can be seen from 50 and 52, the absolute value of the total truncated error can be upper bounded by

\[ |\epsilon_{i_k}| \leq \frac{\lambda_2}{\Gamma(\mu)} \sum_{i=0}^{2} \left[ C_i \epsilon_i^{(1)}(a) + C_i \epsilon_i^{(1)}(-a) + \frac{F_i}{\eta} \epsilon_i^{(2)} \right]. \]
IV. RESULTS AND DISCUSSION

In this section, the proposed approximation for the ASEP versus SNR (in dB) of both $M$-PSK and DQPSK modulation over $\kappa - \mu$ shadowed fading model is evaluated and compared with the exact one for various fading parameters.

Figs. 3 and 7 illustrate, for a fixed value of $m$, the effect of parameter $\mu$ on the ASEP for $M$-PSK and DQPSK modulations, respectively under a weak line of sight (LOS) condition. One can notice that the greater the $\mu$, the better the system's performance.

Figs. 4 and 8 depict the ASEP for both modulation schemes under strong LOS ($\kappa = 10$) for a fixed value of $\mu$. It is observed that countering the effect of shadowing requires the increasing of the parameter $m$.

To show the versatility of the $\kappa - \mu$ shadowed fading, Figs 5, 6, and 9 depict the ASEP for some classical fading models. Noteworthy, the results in all figures are provided for either integer or non-integer values of $\mu$ and $m$. Further, the simulation curves match perfectly with the proposed approximation.

Fig. 10 depicts the absolute value of the truncated error versus the number of limited terms $L$ for DQPSK modulation. It can be shown the greater the $L$, the smaller such error. Interestingly, the truncated error decreases with the increase of SNR.

Lastly, Fig. 11 presents the diversity order for both modulation schemes versus the average SNR is computed by evaluating $-\log P_s \over \log \bar{\gamma}$. It is clearly noticed that such metric goes to $\mu$ as $\bar{\gamma}$ tends to infinity.

V. CONCLUSION

New approximate expressions for the SEP of a communication system employing either $M$-PSK or DQPSK modulation have been derived. The proposed approximations ensure optimal accuracy-analytical tractability trade-off that enables its versatility to contribute to the ASEP computation over generalized fading channel. The resulting accuracy is better than that reached by other existing works relying on more complex mathematical expressions. Furthermore, a new closed-form approximation for the ASEP under $\kappa - \mu$ shadowed fading model has been investigated and it is accurate for all the practical values of the SNRs, and are valid for the entire range of the shaping parameters $\kappa$, $\mu$, and $m$. As far as we know, no previous works dealt with such fading and modulation scheme with such simple approximation. As a future aspect, the authors aim to extend the same approach on more general fading model such as $\alpha - \kappa - \mu$ shadowed [30] and fluctuating Beckmann fading models [31].

ABBREVIATIONS

ASEP: average symbol error probability; AWGN: additive white Gaussian noise; DQPSK: differential quaternary phase-shift keying; LOS: line of sight; MQF: Marcum Q-function of the first order; MBF: modified Bessel function; PDF: probability density function; PSK: phase-shift keying modulation schemes; SEP: symbol error probability; SNR: signal-to-noise ratio; UIFH: upper incomplete upper Fox’s H-function.
Average SNR (dB) | ASEP | M= 4, 8 and 16
---|---|---
µ=1, m=0.5 (Nakagami-q)
µ=1, m=∞ (Rice)
µ= 2.6, m=1.3 (η-µ fading)

Asymptotic ASEP | Approximate ASEP | Simulation

Fig. 6. ASEP for M-PSK over various practical fading models with κ = 5.

Average SNR (dB) | ASEP | m= 0.7, 1.9, 5 and 10
---|---|---
1-κ=0, µ =m=0.5 (One sided)
2-κ=0, µ =m=1 (Rayleigh fading)
3-κ=0, µ =1 and m=0.5 (Nakagami-q)
4-κ=5, µ =1 and m=∞ (Rice fading)
5-κ=0, µ =m=2.7 (Nakagami-m)
6-κ=0, µ =m=2.7 (η-µ fading)

Asymptotic ASEP | Approximate ASEP | Simulation

Fig. 8. ASEP for DQPSK under strong LOS scenario (κ = 10) with different values of m and µ = 2.3.

Average SNR (dB) | ASEP | \(1-κ=0, \mu=m=0.5\) (One sided)
---|---|---
2-\(κ=0, \mu=1\) (Rayleigh fading)
3-\(κ=0, \mu=1\) and \(m=0.5\) (Nakagami-q)
4-\(κ=5, \mu=1\) and \(m=\infty\) (Rice fading)
5-\(κ=0, \mu=2.7\) (Nakagami-m)
6-\(κ=0, \mu=2.7\) (η-µ fading)

Asymptotic ASEP | Approximate ASEP | Simulation

Fig. 9. ASEP for DQPSK over numerous practical fading distributions with \(κ = 5\).

**DECLARATIONS**

**Ethics Approval and Consent to Participate**

The authors declare that this subsection is not applied for this work.

**Consent for Publication**

The authors declare that they wrote completely all scripts associated with the results presented in this work (i.e., figures and tables). No script or data has been imported or used from subsection is not applied for this work.

**Availability of Data and Material**

All scripts related to this work, developed by the two authors, can be found in github.com/FaisalElBouanani/MarcumQfunctionKappaMu/

**Competing interests**

The authors declare that they have no competing interests.

**Funding**

The authors received no specific funding for this work.

**Authors’ contributions**

YM derived new approximate expressions for (i) Marcum Q-function of the first order and (ii) SER integral-form for M-PSK modulation. Based on these two results, YM and FE derived the approximate expressions for ASER under both modulation schemes along with their asymptotic form and the achievable diversity order. YM performed the simulations. YM and FE wrote the paper, analyze and revise the results. All authors read and approved the final manuscript.

**Acknowledgment**

The paper has been developed and written exclusively by the two authors.
Truncated Error Upper Bound

Fig. 10. Upper bound for the truncated error for $\mu = 2.3$ and $m = 4.7$ and various values of $\kappa$ and $L$.

Fig. 11. Diversity order for $\kappa = 5$, $m = 4.7$ and various values of $\mu$ under $M$-PSK and DQPSK modulation schemes.

REFERENCES

[1] M. K. Simon and M.-S. Alouini, *Digital communications over fading channels*, 2nd edition, John Wiley & Sons, 2005.

[2] M. D. Yacoub, “The $\alpha - \mu$ distribution: A physical fading model for the Stacy distribution,” *IEEE Trans. Veh. Technol.*, vol. 56, no. 1, pp. 27–34, Jan. 2007.

[3] M. D. Yacoub, “The $\eta - \mu$ distribution and the $\kappa - \mu$ distribution,” *IEEE Antennas Propag. Mag.*, vol. 49, no. 1, pp. 68–81, Feb. 2007.

[4] G. Fraidenraich and M. D. Yacoub, “The $\alpha - \eta - \mu$ and $\alpha - \kappa - \mu$ fading distributions,” 2006, pp. 16–20, Manaus, Amazon, Brazil.

[5] R. Cogliatti and R. A. A. de Souza, “A near-100% efficient algorithm for generating $\alpha - \kappa - \mu$ and $\alpha - \eta - \mu$ variates,” 2013, pp. 1–5, Las Vegas, NV, USA.

[6] J. F. Paris, “Statistical characterization of $\kappa - \mu$ shadowed fading,” *IEEE Trans. Veh. Technol.*, vol. 63, no. 2, pp. 518–526, Feb. 2014.

[7] S. L. Cotton, “Human body shadowing in cellular device to device communications: Channel modeling using the shadowed $\kappa - \mu$ fading model,” *IEEE J. Sel. Areas Commun.*, vol. 33, no. 1, pp. 111–119, Jan. 2015.

[8] “Shadowed fading in body-to-body communications channels in an outdoor environment at 2.45 GHz,” in *Proc. IEEE-APS Topical Conf. Antennas Propag. Wireless Commun.* (APWC), Palm Beach, FL, USA, Aug. 2014, pp. 249–252.

[9] F. J. Canete, J. Lopez-Fernandez, C. Garcia-Corrales, A. Sanchez, E. Robles, F. J. Rodrigo, and J. F. Paris, “Measurement and modeling of narrowband channels for ultrasonic underwater communications,” *Sensors*, vol. 16, no. 2, Feb. 2016.

[10] Y. J. Chan, S. L. Cotton, H. S. Dhillon, F. J. Lopez-Martinez, J. F. Paris, and S. K. Yoo, “A comprehensive analysis of 5G heterogeneous cellular systems operating over $\kappa - \mu$ shadowed fading channels,” *IEEE Trans. Wireless Commun.*, vol. 16, no. 11, pp. 6995–7010, Nov. 2017.

[11] J. Zhang, X. Li, I. S. Ansari, Y. Liu, and K. A. Qaraqe, “Performance analysis of dual-hop satellite relaying over $\kappa - \mu$ shadowed fading channels,” in *Proc. IEEE Wireless Commun. Networking Conf. (WCNC)*, San Francisco, CA, USA, Mar. 2017, pp. 1–6.

[12] L. Moreno-Pozas, F. J. Lopez-Martinez, J. F. Paris, and E. Martos-Naya, “The $\kappa - \mu$ shadowed fading model: Unifying the $\kappa - \mu$ and $\eta - \mu$ distributions,” *IEEE Trans. Veh. Technol.*, vol. 65, no. 12, pp. 9630–9641, Dec 2016.

[13] J.G. Proakis, *Digital Communications*, 4th edition, New-York: McGraw-Hill, 2001.

[14] A. Jeffrey and D. Zwillinger, *Table of integrals, series, and products*. Elsevier, 2007.

[15] P. Y. Kam and R. Li, “Computing and bounding the first-order marcum q-function: A geometric approach,” *IEEE Trans. Comms.*, no. 7, p. 56, 2008.

[16] P. Y. Kam and R. Li, “Simple tight exponential bounds on the first-order marcum q-function via the geometric approach,” *IEEE Int. Sym. Trans. Info. Theory*, vol. 7, pp. 1090–1094, 2006.

[17] P. Y. Kam and R. Li, “Simple tight exponential bounds on the first-order marcum q-function via the geometric approach,” *IEEE Int. Sym. Trans. Info. Theory*, vol. 7, pp. 1090–1094, 2006.

[18] J. Wang and D. Wu, “Tight bounds for the marcum q-function,” *Wirel. Commun. Mob. Comput.*, 2010.

[19] G. D. Zhao, X. Zhao, and Y. Li, “Tight geometric bound for marcum q-function,” *Electronics Letters*, vol. 44, no. 5, pp. 340–341, Feb. 2008.

[20] G. Ferrari and G.E. Corazza, “Tight bounds and accurate approximations for DQPSK transmission bit error rate,” *Electronics Letters*, vol. 40, no. 20, pp. 1284–1285, Sept. 2004.

[21] Y. Sun, A. Baricz, M. Zhao, X. Xu, and S. Zhou, “Approximate average bit error probability for DQPSK over fading channels,” *Electronics Letters*, vol. 45, no. 23, pp. 1177–1179, Nov. 2009.

[22] A. Baricz, A. Szilárd, and J. Fodor, “New approximations for DQPSK transmission bit error rate,” *IEEE 8th International Symposium on Applied Computational Intelligence and Informatics (SACI)*, pp. 73–77, May 2013.

[23] J. Abouei, K. N. Plataniotis, and S. Pasupathy, “Green modulations in energy-constrained wireless sensor networks,” *IET Communications*, vol. 5, no. 2, pp. 240–251, 2011.

[24] B. Natarajan, C. R. Nassar, and S. Shattil, “CIFSK: bandwidth-efficient multicarrier FSK for high performance, high throughput, and enhanced applicability,” *IEEE Trans. Commun.*, vol. 52, no. 3, pp. 362–367, March 2004.

[25] F. F. Digham, M. S. Alouini, and S. Aroa, “Variable-rate variable-power non-coherent M-FSK scheme for power limited systems,” *IEEE Trans. Wireless Commun.*, vol. 5, no. 6, pp. 1306–1312, June 2006.

[26] Sadhwani, Dharmendra, Ram Narayan Yadav, and Supriya Aggarwal, “Tighter Bounds on the Gaussian $Q$-Function and Its Application in Nakagami-$m$ Fading Channel,” *IEEE Wireless Communications Letters* 6.5 (2017): 574-577.

[27] J. A. Cochran, “The monotonicity of modified Bessel functions with respect to their order,” *Journal of Mathematics and Physics*, vol. 46, no. 1-4, pp. 220–222, 1967.

[28] J. W. Craig, “A New, Simple and Exact Result for Calculating the Probability of Error for Two-Dimensional Signal Constellations,” in *Proc. of the IEEE Military Commun. Conf.*, McLean, USA, pp. 571–575, Nov. 1991.

[29] I. W. Research, Mathematica Edition: version 12.1 Champaign, Illinois: Wolfram Research, Inc., 2020

[30] Ramirez-Espinosa, Pablo, et al. “The $\kappa - \mu$ Shadowed Fading Distribution: Statistical Characterization and Applications,” arXiv preprint arXiv:1904.05587 (2019).

[31] P. Ramirez-Espinosa, F. J. Lopez-Martinez, J. F. Paris, M. D. Yacoub, and E. Martos-Naya, “An extension of the $\kappa - \mu$ shadowed fading model: Statistical characterization and applications,” *IEEE Trans. Veh. Technol.*, vol. 67, no. 5, pp. 3826–3837, May 2018.