New $\mathcal{N}=2$ superspace Calogero models

Sergey Krivonos$^a$, Olaf Lechtenfeld$^b$ and Anton Sutulin$^a$

$^a$ Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia
$^b$ Institut für Theoretische Physik and Riemann Center for Geometry and Physics, Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany

krivonos@theor.jinr.ru, lechtenf@itp.uni-hannover.de, sutulin@theor.jinr.ru

Abstract

Starting from the Hamiltonian formulation of $\mathcal{N}=2$ supersymmetric Calogero models associated with the classical $A_n$, $B_n$, $C_n$ and $D_n$ series and their hyperbolic/trigonometric cousins, we provide their superspace description. The key ingredients include $n$ bosonic and $2n(n-1)$ fermionic $\mathcal{N}=2$ superfields, the latter being subject to a nonlinear chirality constraint. This constraint has a universal form valid for all Calogero models. With its help we find more general supercharges (and a superspace Lagrangian), which provide the $\mathcal{N}=2$ supersymmetrization for bosonic potentials with arbitrary repulsive two-body interactions.

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1 Introduction

Recent progress in the construction of supersymmetric extensions of Calogero models [1, 2] (see [3] for a review) was achieved by adding to the system more fermions as compared to the standard supersymmetrization [4, 5, 6]. It was inspired by supersymmetric extensions of the matrix models which, upon reduction or gauge fixing, give rise to the familiar bosonic systems. The approach developed in [7, 8, 9] for the rational spin-Calogero models with \( \mathcal{N} = 2, 4 \) supersymmetry was recently extended to \( \mathcal{N} = 2, 4 \) supersymmetric hyperbolic Calogero models [10, 11]. However, it suffers from an unclear structure of the bosonic matrix model one has to start from.

In the series of papers [12, 13, 14] we developed a different approach. Mainly working in the Hamiltonian formulation, we worked out an ansatz for the supercharges which accommodates all Calogero models associated with the classical \( A_n, B_n, C_n \) and \( D_n \) Lie algebras and their trigonometric/hyperbolic extensions [14]. Having at hands the Hamiltonian description of \( \mathcal{N} \)-extended supersymmetric Calogero models, it is of interest to gain also its superfield formulation, at least in the simplest case of \( \mathcal{N} = 2 \) supersymmetry.\(^1\) The superspace picture will help in understanding the general supersymmetry structure and clarify the role played by the additional matrix fermions. This is the main goal of this work.

The plan of the paper is as follows. In Section 2 we review the Hamiltonian description of the supersymmetric Calogero models. Their superfield treatment is performed in Section 3 (\( A_1 \oplus A_{n-1} \) models) and in Section 4 (\( B_n, C_n, D_n \) models). The most remarkable result here is a universal nonlinear fermionic chiral supermultiplet which collects all matrix fermions occurring in all super-extended Calogero models. In Section 5 we present more general supercharges (and a superspace Lagrangian), which provide an \( \mathcal{N} = 2 \) supersymmetrization for a bosonic potential \( \frac{2\varphi^2}{2} \sum_{i<j} f(x_i - x_j)^2 \) with an arbitrary function \( f \). We conclude with a short summary and possible extensions.

2 Hamiltonian description of \( \mathcal{N} = 2 \) supersymmetric Calogero models

In the Hamiltonian approach the \( n \)-particle supersymmetric Calogero model with \( \mathcal{N} = 2 \)-extended supersymmetry [14, 12, 13] features the following degrees of freedom:

- \( n \) bosonic coordinates \( x_i \) and momenta \( p_i \) with \( i = 1, \ldots, n \),
- \( 2n \) fermions \( \psi_i \) and \( \bar{\psi}_i \),
- \( 2n(n-1) \) fermions \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) with \( \xi_{ii} = \bar{\xi}_{ii} = 0 \).

Their non-vanishing Poisson brackets are

\[
\{x_i, p_j\} = \delta_{ij}, \quad \{\psi_i, \bar{\psi}_j\} = -i \delta_{ij}, \quad \{\xi_{ij}, \bar{\xi}_{km}\} = -i(1-\delta_{ij})(1-\delta_{km})\delta_{im}\delta_{jk}. \tag{2.1}
\]

A central role of our construction take the composite objects \( \Pi_{ij} \) and \( \bar{\Pi}_{ij} \) defined as

\[
\Pi_{ij} = (\psi_i - \bar{\psi}_j)\xi_{ij} + (\bar{\psi}_i - \psi_j)\xi_{ij} + \sum_{k=1}^n (\xi_{ik}\bar{\xi}_{kj} + \bar{\xi}_{ik}\xi_{kj}),
\]

\[
\bar{\Pi}_{ij} = 2\delta_{ij}\psi_i\bar{\psi}_i + (\psi_i + \bar{\psi}_j)\xi_{ij} - (\bar{\psi}_i + \psi_j)\xi_{ij} + \sum_{k=1}^n (\xi_{ik}\bar{\xi}_{kj} - \bar{\xi}_{ik}\xi_{kj}). \tag{2.2}
\]

One may easily check that \( \Pi_{ij} \) and \( \bar{\Pi}_{ij} \) together form an \( su(n) \oplus u(n) \) algebra,\(^2\)

\[
\{\Pi_{ij}, \Pi_{km}\} = i(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}), \quad \{\Pi_{ij}, \bar{\Pi}_{km}\} = i(\delta_{im}\bar{\Pi}_{kj} - \delta_{kj}\Pi_{im}),
\]

\[
\{\bar{\Pi}_{ij}, \Pi_{km}\} = i(\delta_{im}\Pi_{kj} - \delta_{kj}\Pi_{im}). \tag{2.4}
\]

Our \( \mathcal{N} = 2 \) supersymmetric Calogero models of \( A \)-type [14, 13] are defined by a generic form of their supercharges,

\[
Q = \sum_{i=1}^n p_i\psi_i - i\sum_{i \neq j}^n \left[ (g + \Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]\xi_{ij},
\]

\[
\bar{Q} = \sum_{i=1}^n p_i\bar{\psi}_i - i\sum_{i \neq j}^n \left[ (g + \Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]\bar{\xi}_{ij}. \tag{2.5}
\]

\(^1\)A superspace description of the \( \mathcal{N} \)-extended supersymmetric Euler-Calogero-Moser system has been provided in [15].

\(^2\)We remind that \( \sum_{i} \Pi_{ii} = 0 \).
with some function $f$, to be specified in a moment. Note that $\tilde{\Pi}_{ij}$ does not appear here. These supercharges form an $\mathcal{N}=2$ super-Poincaré algebra,

$$\{Q, \bar{Q}\} = -2i \mathcal{H} \quad \text{and} \quad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0,$$  

(2.6)

together with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^{n} p_{i}^{2} + \frac{1}{2} \sum_{i \neq j}^{n} \left[ (g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[ (g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] + \frac{\alpha}{2} \sum_{i,j} \Pi_{ij} \Pi_{ji},$$  

(2.7)

Here, we abbreviated

$$z_{ij} = x_{i} - x_{j},$$  

(2.8)

and the constant parameter $\alpha$ and the function $f$ are given as follows,

- **rational Calogero model** $\alpha = 0$, $f(z_{ij}) = \frac{1}{z_{ij}} = \frac{1}{x_{i} - x_{j}}$,
- **hyperbolic Calogero-Moser model** $\alpha = -1$, $f(z_{ij}) = \frac{1}{\sinh(z_{ij})} = \frac{1}{\sinh(x_{i} - x_{j})}$,
- **trigonometric Calogero-Moser model** $\alpha = 1$, $f(z_{ij}) = \frac{1}{\sin(z_{ij})} = \frac{1}{\sin(x_{i} - x_{j})}$.

For the $B$, $C$ and $D$-type models, the supercharges take a more complicated generic form (including $\tilde{\Pi}_{ij}$),

\[
\begin{align*}
Q &= \sum_{i=1}^{n} p_{i} \psi_{i} - i \sum_{i \neq j}^{n} \left[ (g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji} + i \sum_{i \neq j}^{n} \left[ (g + \Pi_{ii}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \right] \xi_{ji} \\
&\quad + i \sum_{i}^{n} \left[ (g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \Pi_{ii} \right] \psi_{i}, \\
\bar{Q} &= \sum_{i=1}^{n} p_{i} \bar{\psi}_{i} - i \sum_{i \neq j}^{n} \left[ (g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji} - i \sum_{i \neq j}^{n} \left[ (g + \Pi_{ii}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji} \\
&\quad - i \sum_{i}^{n} \left[ (g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \Pi_{ii} \right] \bar{\psi}_{i}.
\end{align*}
\]

(2.10)

Here,

$$y_{ij} = x_{i} + x_{j},$$  

(2.11)

and the function $f$ is the same as in (2.9). The supercharges (2.10) form the same $\mathcal{N}=2$ super-Poincaré algebra (2.6) together with the Hamiltonian

\[
\begin{align*}
\mathcal{H} &= \frac{1}{2} \sum_{i=1}^{n} p_{i}^{2} + \frac{1}{2} \sum_{i \neq j}^{n} \left[ (g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[ (g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] \\
&\quad + \frac{1}{2} \sum_{i \neq j}^{n} \left[ (g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \right] \left[ (g + \Pi_{ii}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ji} \right] \\
&\quad + \frac{1}{2} \sum_{i}^{n} \left[ (g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \Pi_{ii} \right] \left[ (g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \Pi_{ii} \right].
\end{align*}
\]

(2.12)

Its bosonic sector reads

$$\mathcal{H}_{bos} = \frac{1}{2} \sum_{i=1}^{n} p_{i}^{2} + \frac{g^{2}}{2} \sum_{i \neq j}^{n} \left( f^{2}(z_{ij}) + f^{2}(y_{ij}) \right) + \frac{g'^{2}}{2} \sum_{i}^{n} f^{2}(y_{ii}).$$  

(2.13)

Due to the presence of only two coupling constants, $g$ and $g'$, we may describe $B$, $C$ and $D$-type models in the rational case and $C$ and $D$ (but not $B$)-type models in the hyperbolic/trigonometric case.
3 \( \mathcal{N}=2 \) superspace \( A_1 \oplus A_{n-1} \) Calogero models

To provide a superspace description of \( \mathcal{N}=2 \) supersymmetric Calogero models one has, firstly, to assemble the physical components \( x_i, \psi_i, \bar{\psi}_i, \xi_{ij} \) and \( \bar{\xi}_{ij} \) into \( \mathcal{N}=2 \) superfields.

It immediately follows from the structure of the supercharges \( Q \) and \( \bar{Q} \) (2.5) that under \( \mathcal{N}=2 \) supersymmetry the coordinates \( x_i \) transform into the fermions \( \psi_i \) and \( \bar{\psi}_i \):

\[
\delta x_i \equiv \{ x_i, i\epsilon Q + i\epsilon \bar{Q} \} = i\epsilon \bar{\psi}_i + i\epsilon \psi_i .
\]

Thus, one is let to \( n \) bosonic \( \mathcal{N}=2 \) superfields \( x_i \) with the components \(^3\)

\[
x_i = x_i|, \quad \psi_i = -iDx_i|, \quad \bar{\psi}_i = -iDx_i|, \quad A_i = \frac{1}{2} [D, D] x_i|.
\]

Concerning the fermionic components \( \xi_{ij}, \bar{\xi}_{ij} \), we have no other possibility than to put them into 2n(\( n \)-1) new fermionic superfields \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) with vanishing diagonal parts, i.e.

\[
\xi_{ii} = \bar{\xi}_{ii} = 0 \quad \forall \ i.
\]

As \( \mathcal{N}=2 \) superfields the \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) contain a lot of components. Hence, they have to be constrained somehow. The appropriate constraints derive from the explicit form of the supercharges \( Q \) and \( \bar{Q} \) (2.5), which leads to the following supersymmetry transformations of the leading components \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) of these superfields,

\[
\delta Q \xi_{ij} \sim i\epsilon \left[ -\frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \bar{\psi}_j) \xi_{ij} + \xi_{ij} \left( \sum_{k \neq i} f(z_{ik}) \xi_{ik} - \sum_{k \neq j} f(z_{jk}) \xi_{jk} \right) - \sum_{k \neq i,j} \left( \frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{jk})}{f(z_{jk})} \right) \xi_{ik} \xi_{jk} \right],
\]

\[
\delta \bar{Q} \bar{\xi}_{ij} \sim i\epsilon \left[ -\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \psi_j) \bar{\xi}_{ij} + \bar{\xi}_{ij} \left( \sum_{k \neq i} f(z_{ik}) \bar{\xi}_{ik} - \sum_{k \neq j} f(z_{jk}) \bar{\xi}_{jk} \right) - \sum_{k \neq i,j} \left( \frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{jk})}{f(z_{jk})} \right) \bar{\xi}_{ik} \bar{\xi}_{jk} \right].
\]

To realize this transformation property we are forced to impose a nonlinear chirality condition on the superfields \( \xi_{ij} \) and \( \bar{\xi}_{ij} \),

\[
D \xi_{ij} = \frac{i}{2} [D, D] \xi_{ij} = \frac{i}{2} \sum_{i,j} \left( \xi_{ij} \bar{\xi}_{ji} - \xi_{ji} \bar{\xi}_{ij} \right),
\]

\[
D \bar{\xi}_{ij} = \frac{i}{2} [D, D] \bar{\xi}_{ij} = \frac{i}{2} \sum_{i,j} \left( \bar{\xi}_{ij} \bar{\xi}_{ji} - \bar{\xi}_{ji} \bar{\xi}_{ij} \right).
\]

This condition leaves in the superfields \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) only the components

\[
\xi_{ij} = \xi_{ij}|, \quad B_{ij} = \bar{D} \xi_{ij}|, \quad \bar{\xi}_{ij} = \bar{\xi}_{ij}|, \quad \bar{B}_{ij} = D \bar{\xi}_{ij}|.
\]

Finally, to obtain the correct brackets (2.1) for \( (\psi_i, \bar{\psi}_i) \) and \( (\xi_{ij}, \bar{\xi}_{ij}) \) after passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must have the form

\[
\mathcal{L}_k^{\psi} = \frac{i}{2} \sum_{i=1}^{n} \left( \psi_i \bar{\psi}_i - \psi_i \bar{\psi}_i \right) \quad \text{and} \quad \mathcal{L}^{\xi}_{km} = \frac{i}{2} \sum_{i,j} \left( \xi_{ij} \bar{\xi}_{ji} - \xi_{ji} \bar{\xi}_{ij} \right).
\]

In \( \mathcal{N}=2 \) superspace, this amounts to the free action \((g = 0)\)

\[
S_0 = \int dt d^2 \theta \left[ -\frac{1}{2} \sum_{i=1}^{n} D x_i D x_i + \frac{1}{2} \sum_{i,j} \xi_{ij} \bar{\xi}_{ij} \right] \quad \text{with} \quad d^2 \theta \equiv D \bar{D}.
\]

More interesting is the construction of the interaction terms. Again, some hints come from the transformation properties of the fermions \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) under \( Q \) and \( \bar{Q} \) supersymmetry, respectively,

\[
\delta Q \xi_{ij} \sim i\epsilon g (z_{ij}) + \ldots, \quad \delta \bar{Q} \bar{\xi}_{ij} \sim i\epsilon g (z_{ij}) + \ldots.
\]

\(^3\)We use the \( \mathcal{N}=2 \) spinor covariant derivatives \( D \) and \( \bar{D} \) obeying \( \{ D, D \} = 2i\partial_t \) and \( \{ D, \bar{D} \} = \{ \bar{D}, D \} = 0 \). We denote by \( \mathcal{A} \) the \( \theta = \theta = 0 \) limit of a superspace expression \( \mathcal{A} \).
To reproduce such terms in superspace, the unique possibility is to add to the action $S_0$ (3.8) a term

$$S_{\text{int}} = i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^{n} f(z_{ij}) \xi_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^{n} f(z_{ij}) \bar{\xi}_{ij}.$$  (3.10)

To be supersymmetrically invariant, the integrands in (3.10) must be chiral and antichiral, respectively. It is not too hard to check that this is indeed so: the nonlinear chirality constraint (3.5) implies that

$$D \left( \sum_{i \neq j}^{n} f(z_{ij}) \xi_{ij} \right) = 0 \quad \text{and} \quad \bar{D} \left( \sum_{i \neq j}^{n} f(z_{ij}) \bar{\xi}_{ij} \right) = 0.$$  (3.11)

Combining all these facts together, we conclude that the superfield action reads

$$S = \int dt d^2 \theta \left[ - \frac{1}{2} \sum_{i=1}^{n} Dx_i D\bar{x}_i + \frac{1}{2} \sum_{i,j}^{n} \xi_{ij} \bar{\xi}_{ij} \right] + i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^{n} f(z_{ij}) \xi_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^{n} f(z_{ij}) \bar{\xi}_{ij},$$

(3.12)

where the superfields $\xi_{ij}$ and $\bar{\xi}_{ij}$ are subject to the nonlinear chirality constraint (3.5).

It is important to note that, after passing to new fermionic superfields

$$\lambda_{ij} \equiv f(z_{ij}) \xi_{ij} \quad \text{and} \quad \bar{\lambda}_{ij} \equiv f(z_{ij}) \bar{\xi}_{ij},$$

(3.13)

the nonlinear constraint (3.5) is slightly simplified to

$$D \lambda_{ij} = i \left[ \lambda_{ij} \sum_{k \neq i}^{n} \lambda_{jk} - \lambda_{ij} \sum_{k \neq j}^{n} \lambda_{kj} + (1 - \delta_{ij}) \sum_{k \neq i,j}^{n} \lambda_{ik} \lambda_{kj} \right],$$

$$\bar{D} \bar{\lambda}_{ij} = i \left[ \bar{\lambda}_{ij} \sum_{k \neq i}^{n} \bar{\lambda}_{jk} - \bar{\lambda}_{ij} \sum_{k \neq j}^{n} \bar{\lambda}_{kj} + (1 - \delta_{ij}) \sum_{k \neq i,j}^{n} \bar{\lambda}_{ik} \bar{\lambda}_{kj} \right].$$

(3.14)

In this form, the constraint has lost any $f$-dependence, which however will reappear in the action,

$$S = \int dt d^2 \theta \left[ - \frac{1}{2} \sum_{i=1}^{n} Dx_i D\bar{x}_i + \frac{1}{2} \sum_{i,j}^{n} \lambda_{ij} \bar{\lambda}_{ij} \right] + i \frac{g}{2} \int dt d\bar{\theta} \sum_{i \neq j}^{n} \lambda_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j}^{n} \bar{\lambda}_{ij}.$$  (3.15)

Also, the component Lagrangian, Hamiltonian and Poisson brackets will be more complicated in terms of the composite superfields $\lambda_{ij}$ and $\bar{\lambda}_{ij}$.

Despite the extremely simple form of the superfield action (3.12), its component version looks quite complicated due to the constraint (3.5). Indeed, after integration over $\theta$ in (3.12) we get the off-shell Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{pot}},$$

(3.16)

where

$$\mathcal{L}_0 = \frac{1}{2} \sum_{i=1}^{n} \left( \dot{x}_i \dot{x}_i + A_i A_i \right) + \frac{1}{2} \sum_{i} \left( \dot{\psi}_i \dot{\psi}_i - \psi_i \dot{\psi}_i \right) + \frac{1}{2} \sum_{i,j} \left( \dot{\xi}_{ij} \dot{\xi}_{ij} - \xi_{ij} \dot{\xi}_{ij} \right) + \frac{1}{2} \sum_{i,j} \left( \dot{\bar{\xi}}_{ij} \dot{\bar{\xi}}_{ij} - \bar{\xi}_{ij} \dot{\bar{\xi}}_{ij} \right),$$

$$\mathcal{L}_{\text{pot}} = - \frac{g}{2} \sum_{i,j} f'(z_{ij}) \left[ (\psi_i - \psi_j) \xi_{ij} + (\bar{\psi}_i - \bar{\psi}_j) \bar{\xi}_{ij} \right] + i \frac{g}{2} \sum_{i,j} f(z_{ij}) \left( B_{ij} + \bar{B}_{ij} \right).$$

(3.17)

To eliminate the auxiliary fields $A_i$ and $B_{ij}$ one firstly has to evaluate the terms in the second line of (3.17) by using the constraint (3.5). This is a straightforward but rather tedious calculation. After employing the equations of motion we finally obtain the desired result,

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \frac{1}{2} \sum_{i=1}^{n} \left( \dot{\psi}_i \dot{\psi}_i - \psi_i \dot{\psi}_i \right) + \frac{1}{2} \sum_{i,j} \left( \dot{\xi}_{ij} \dot{\xi}_{ij} - \xi_{ij} \dot{\xi}_{ij} \right) - \frac{1}{2} \sum_{i,j} \left[ (g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \left[ (g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] - \frac{g}{2} \sum_{i,j} \Pi_{ii} \Pi_{jj}.$$  (3.18)

Thus, the superfield action (3.12) with the superfields $\xi_{ij}$ and $\bar{\xi}_{ij}$ subject to the nonlinear chirality constraint (3.5) indeed describes all ${\mathcal{N}=2}$ supersymmetric $A_1 \oplus A_{n-1}$ Calogero models.
4 N=2 superspace B_n, C_n and D_n Calogero models

The supercharges of the N=2 supersymmetric B, C and D-type Calogero models (2.10) have a more complicated structure than those in (2.5). Therefore, it is expected that the nonlinear chirality constraint for the superfields ξ_{ij} and ξ'_{ij} are more intricate as well. Indeed, the explicit structure of the supercharges (2.10) uniquely fixes this constraint to be

\[
D\xi_{ij} = i \left\{ -\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \psi_j) \xi_{ij} - \frac{f'(y_{ij})}{f(y_{ij})} (\bar{\psi}_i + \psi_j) \xi_{ij} + \{ \frac{f'(y_{ii})}{f(y_{ii})} - f(y_{ii}) \} \xi_{ij} + \xi_{ij} \sum_{k \neq i} \frac{f'(z_{ik})}{f(z_{ik})} \xi_{ik} - \sum_{k \neq i} \frac{f'(z_{ik})}{f(z_{ik})} \xi_{ik} \xi_{kj} \right\},
\]

(4.1)

\[
\bar{D}\xi_{ij} = i \left\{ -\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \psi_j) \xi_{ij} - \frac{f'(y_{ij})}{f(y_{ij})} (\bar{\psi}_i + \psi_j) \xi_{ij} + \{ \frac{f'(y_{ii})}{f(y_{ii})} + f(y_{ii}) \} \xi_{ij} + \xi_{ij} \sum_{k \neq i} \frac{f'(z_{ik})}{f(z_{ik})} \xi_{ik} - \sum_{k \neq i} \frac{f'(z_{ik})}{f(z_{ik})} \xi_{ik} \xi_{kj} \right\}.
\]

The complicated form of this constraint disappears after passing to the composite superfields

\[
\lambda_{ij} = (f(z_{ij}) + f(y_{ij})) \xi_{ij} \quad \text{and} \quad \lambda'_{ij} = (f(z_{ij}) - f(y_{ij})) \xi_{ij},
\]

(4.2)

in which it acquires its familiar form (3.14),

\[
D\lambda_{ij} = i \left\{ \lambda_{ij} \sum_{k \neq i} \lambda_{ik} - \lambda_{ij} \sum_{k \neq j} \lambda_{jk} + (1 - \delta_{ij}) \sum_{k \neq i,j} \lambda_{ik} \lambda_{kj} \right\},
\]

\[
\bar{D}\lambda_{ij} = i \left\{ \lambda'_{ij} \sum_{k \neq i} \lambda'_{ik} - \lambda'_{ij} \sum_{k \neq j} \lambda'_{jk} + (1 - \delta_{ij}) \sum_{k \neq i,j} \lambda'_{ik} \lambda'_{kj} \right\}.
\]

Finally, the superfield action reads

\[
S = \int dt d^2 \theta \left[ -\frac{1}{2} \sum_{i=1}^{n} D\xi_{i} \bar{D}\xi_{i} + \frac{1}{2} \sum_{i,j} \xi_{ij} \bar{\xi}_{ij} - \frac{1}{2} g' h(y_{ii}) \right] + i \frac{g}{2} \int dt d\theta \sum_{i \neq j} (f(z_{ij}) + f(y_{ij})) \xi_{ij} + i \frac{g}{2} \int dt d\theta \sum_{i \neq j} (f(z_{ij}) + f(y_{ij})) \xi'_{ij},
\]

(4.3)

where

\[
h'(y_{ii}) = f(y_{ii}).
\]

(4.4)

Compared to the action of the A_1 ⊕ A_{n-1} Calogero models (3.15), only the term \(\frac{1}{2} g' \int dt d^2 \theta \ h(y_{ii})\) carrying the new coupling constant \(g'\) appears in the action (4.3). All other terms just mimic those in (3.15).

It is a matter of straightforward but tedious calculations to check that, after excluding the auxiliary fields by
their equations of motion, the final Lagrangian acquires the expected form

\[ \mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i \dot{x}_i + \frac{i}{2} \sum_{i=1}^{n} \left( \dot{\psi}_i \bar{\psi}_i - \psi_i \dot{\bar{\psi}}_i \right) + \frac{i}{2} \sum_{i,j} \left( \dot{\xi}_{ij} \bar{\xi}_{ij} - \xi_{ij} \dot{\bar{\xi}}_{ij} \right) \]

\[ \left( g + \Pi_{ij} \right) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \left[ \left( g + \Pi_{ij} \right) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \]

\[ \frac{1}{2} \sum_{i,j} \left( g + \Pi_{ij} \right) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \left[ \left( g + \Pi_{ij} \right) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \right] \]

\[ \frac{1}{2} \sum_{i,j} \left( (g' + \Pi_{ij}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \right) \left[ \left( (g' + \Pi_{ij}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \Pi_{ij} \right] \right. \]

Thus, the superfield action (4.3) with the superfields \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) subject to the nonlinear chirality constraint (4.1) indeed describes the \( \mathcal{N}=2 \) supersymmetric \( B, C \) and \( D \)-type Calogero models.\(^4\)

5 \textbf{New \( \mathcal{N}=2 \) supersymmetric \( n \)-particle models}

Our point of departure was the explicit form of the \( \mathcal{N}=2 \) supercharges for the Calogero models (2.5) and (2.10) constructed in [14, 13]. In all cases we considered the function \( f \) to be not arbitrary but to be chosen from the list (2.9). Indeed, for generic \( f \) the supercharges (2.5) do not form the closed algebra (2.6). As a consequence, the nonlinear chirality condition (3.5) is not self-consistent for an arbitrary function \( f \) (i.e. \( D \) acting on the r.h.s. of (3.5) does not vanish). On the other hand, the universal chirality constraint (3.14) is self-consistent. This implies a weaker chirality condition on \( \xi_{ij} \) and \( \bar{\xi}_{ij} \):\(^5\)

\[ D \xi_{ij} = i \left[ -\frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \xi_{ij} + \xi_{ij} \left( \sum_{k \neq i} f(z_{ik}) \xi_{ik} - \sum_{k \neq j} f(z_{jk}) \xi_{jk} \right) + \sum_{k \neq i, j} \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \xi_{ik} \xi_{kj} \right], \]

\[ \bar{D} \xi_{ij} = i \left[ -\frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \bar{\xi}_{ij} + \bar{\xi}_{ij} \left( \sum_{k \neq i} f(z_{ik}) \bar{\xi}_{ik} - \sum_{k \neq j} f(z_{jk}) \bar{\xi}_{jk} \right) + \sum_{k \neq i, j} \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \bar{\xi}_{ik} \bar{\xi}_{kj} \right]. \]

One may check that this nonlinear chirality constraint is perfectly self-consistent for an arbitrary function \( f \).

Now, let us again start from the superspace action (3.8), but where the fermionic superfields \( \xi_{ij} \) and \( \bar{\xi}_{ij} \) must obey the constraint (5.1). Passing to the components and eliminating the auxiliary components one arrives at

\[ Q = \sum_{i=1}^{n} p_i \psi_i - i \sum_{i,j} \left[ (g + \Pi_{ij}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \xi_{ij} + \sum_{k \neq i, j} \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \xi_{ik} \xi_{kj} \right] \xi_{ij}, \]

\[ Q = \sum_{i=1}^{n} p_i \bar{\psi}_i - \sum_{i \neq j} \left[ (g + \Pi_{ij}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \bar{\xi}_{ij} + \sum_{k \neq i, j} \frac{f(z_{ik}) f(z_{kj})}{f(z_{ij})} \bar{\xi}_{ik} \bar{\xi}_{kj} \right] \bar{\xi}_{ij}. \]

For the functions listed in (2.9) these supercharges coincide with the ones in (2.5). However, the supercharges (5.2) generate the \( \mathcal{N}=2 \) super-Poincaré algebra (2.6) for an arbitrary function \( f \). It is not too hard to find the bosonic part of the new Hamiltonian,

\[ \mathcal{H}_{\text{bos}} = \frac{1}{2} \sum_{i} p_i^2 + \frac{g^2}{2} \sum_{i \neq j} f^2(z_{ij}). \]

Thus, the supercharges (5.2) provide an \( \mathcal{N}=2 \) supersymmetrization of a wide class of multi-particle systems with a bosonic Hamiltonian of the type (5.3). A detailed analysis of such models will be given elsewhere.

\(^4\)Except again for \( B \)-type models in the trigonometric/hyperbolic case.

\(^5\)For the sake of simplicity we consider here the \( A \)-type case, in which the \((\lambda_{ij}, \bar{\lambda}_{ij})\) and \((\xi_{ij}, \bar{\xi}_{ij})\) are related as in (3.13).
6 Conclusions

In this paper we have provided a superspace description of the $\mathcal{N}=2$ supersymmetric Calogero models, rational as well as trigonometric/hyperbolic, associated with the classical $A_n, B_n, C_n$ and $D_n$ Lie algebras. We presented a minimal superfield content accommodating the $2n^2$ fermions for the $\mathcal{N}=2$ supersymmetric $n$-particle model. As $2n$ fermions accompany the $n$ bosonic coordinates in general bosonic $\mathcal{N}=2$ superfields, the remaining $2n(n-1)$ fermions must be put into additional fermionic $\mathcal{N}=2$ superfields, which have to be constrained such as to describe those fermions alone. The nonlinear chirality condition (3.14) written in terms of composite superfields solves this task. These composite fermionic $\mathcal{N}=2$ superfields make the constraint look simple and universal but complicate the Lagrangian. In terms of the fundamental fermions it is the other way around. We finally presented more general supercharges (and the superspace Lagrangian) which provides an $\mathcal{N}=2$ supersymmetrization of bosonic $n$-particle systems with an arbitrary repulsive two-body interaction.

One might criticize that the approach presented here is unnecessarily complicated, because all $\mathcal{N}=2$ supersymmetric Calogero models can be more or less straightforwardly formulated in the standard fashion employing the minimum of $2n$ fermions [4]. This, however, is no longer the case with $\mathcal{N}>2$ supersymmetric Calogero models, where our treatment with additional fermions becomes essential. Hence, we consider our results here as a preparation for attacking Calogero systems with more supersymmetry in a superspace setting. The main excuse for presenting of our $\mathcal{N}=2$ results is the universal form of the nonlinear chirality constraint together with the almost trivial generalization of the supercharges to $\mathcal{N}=4$ supersymmetry [14] which will further the superspace construction of $\mathcal{N}\geq 4$ Calogero models.

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A Appendix

The nonlinear chirality constraint reads

\[ \frac{1}{n} D\lambda_{ij} = \lambda_{ij} \sum_{k \neq i} \lambda_{ik} - \lambda_{ij} \sum_{k \neq j} \lambda_{jk} + (1-\delta_{ij}) \sum_{k \neq i, j} \lambda_{ik} \lambda_{kj}, \quad \lambda_{ii} = 0, \]  
(A.1) \[ \frac{1}{n} D\overline{\lambda}_{ij} = \overline{\lambda}_{ij} \sum_{k \neq i} \overline{\lambda}_{ik} - \overline{\lambda}_{ij} \sum_{k \neq j} \overline{\lambda}_{jk} + (1-\delta_{ij}) \sum_{k \neq i, j} \overline{\lambda}_{ik} \overline{\lambda}_{kj}, \quad \overline{\lambda}_{ii} = 0. \]  
(A.2)

Let us prove that (A.1) is self-consistent. For this we act with $D$ on it. The l.h.s. is identically zero (since $D^2 = 0$), while the r.h.s. should vanish by virtue of the constraint itself.

\[ \text{r.h.s.} = D\lambda_{ij} \sum_{k \neq i} \lambda_{ik} - \lambda_{ij} \sum_{k \neq j} \lambda_{jk} + \lambda_{ij} \sum_{k \neq j} \lambda_{jk} + (1-\delta_{ij}) \sum_{k \neq i, j} \lambda_{ik} \lambda_{kj} \]

\[ = (\lambda_{ij} \sum_{n \neq i} \lambda_{in} - \lambda_{ij} \sum_{n \neq j} \lambda_{jn}) [\sum_{k \neq i} \lambda_{ik} - \sum_{k \neq j} \lambda_{jk}] \]

\[ - \lambda_{ij} \sum_{k \neq i} \left( \lambda_{ik} \sum_{n \neq i} \lambda_{in} - \lambda_{ik} \sum_{n \neq i, k} \lambda_{in} \lambda_{nk} \right) \]

\[ + \lambda_{ij} \sum_{k \neq j} \left( \lambda_{jk} \sum_{n \neq j} \lambda_{jn} - \lambda_{jk} \sum_{n \neq j, k} \lambda_{jn} \lambda_{nk} \right) \]

\[ + (1-\delta_{ij}) \sum_{k \neq i, j} \left( \lambda_{ik} \sum_{n \neq i} \lambda_{in} - \lambda_{ik} \sum_{n \neq k} \lambda_{in} \lambda_{nk} \right) \lambda_{kj} \]

\[ - (1-\delta_{ij}) \sum_{k \neq i, j} \left( \lambda_{kj} \sum_{n \neq k} \lambda_{kn} - \lambda_{kj} \sum_{n \neq j} \lambda_{jn} \lambda_{nk} \right) \lambda_{ik}. \]  
(A.3)

Let us first collect all terms proportional to $(1-\delta_{ij})$,

\[ \sum_{n \neq i, j} \lambda_{in} \lambda_{nj} \sum_{k \neq i} \lambda_{ik} - \sum_{n \neq i, j} \lambda_{in} \lambda_{nj} \sum_{k \neq j} \lambda_{jk} + \sum_{n \neq i} \lambda_{ik} \sum_{n \neq k} \lambda_{in} \lambda_{kj} - \sum_{n \neq i} \lambda_{ik} \sum_{n \neq k} \lambda_{kn} \lambda_{kj} \]

\[ + \sum_{n \neq i, k} \sum_{n \neq i, k} \lambda_{in} \lambda_{nk} \lambda_{kj} - \sum_{n \neq i, k} \lambda_{kj} \sum_{n \neq k} \lambda_{kn} \lambda_{ik} + \sum_{n \neq j} \lambda_{kj} \sum_{n \neq j} \lambda_{jn} \lambda_{ik} - \sum_{n \neq j} \lambda_{kj} \sum_{n \neq k} \lambda_{kn} \lambda_{nj} \lambda_{ik}. \]  
(A.4)
The fourth and sixth terms cancel each other out. Then, replacing the indices in the first and second terms as $k \leftrightarrow n$ makes them cancel with the third and seventh terms. Thus, after replacing the indices in the first term as $k \leftrightarrow n$, we remain with
\[
\sum_{n \neq i,j} \sum_{k \neq i,n} \lambda_{ik} \lambda_{kn} \lambda_{nj} = \sum_{k \neq i,j} \sum_{n \neq k,j} \lambda_{ik} \lambda_{kn} \lambda_{nj} .
\] (A.5)

A sum over $k$ in the first term can be split into two pieces, corresponding to $k = j$ or $k \neq j$, yielding
\[
\sum_{n \neq i,j} \sum_{k \neq i,n} \lambda_{ik} \lambda_{kn} \lambda_{nj} = \sum_{n \neq i,j} \lambda_{ij} \lambda_{jn} \lambda_{nj} + \sum_{n \neq i,j} \sum_{k \neq i,j,n} \lambda_{ik} \lambda_{kn} \lambda_{nj} .
\] (A.6)

Performing the analogous splitting for the index $n$ in the second term of (A.5), $n = i$ or $n \neq i$, one finds
\[
\sum_{k \neq i,j} \sum_{n \neq i,j} \lambda_{ik} \lambda_{kn} \lambda_{nj} = \sum_{k \neq i,j} \lambda_{ik} \lambda_{ki} \lambda_{ij} + \sum_{k \neq i,j} \sum_{n \neq i,j,k} \lambda_{ik} \lambda_{kn} \lambda_{nj} .
\] (A.7)

After relabeling $k \leftrightarrow n$, the second terms on the r.h.s. of (A.6) and (A.7) become identical. Hence, their contribution to (A.5) cancels,
\[
\sum_{k \neq i,j} \sum_{n \neq i,j,k} \left( \lambda_{in} \lambda_{nk} \lambda_{kj} - \lambda_{ik} \lambda_{kn} \lambda_{nj} \right) = 0 ,
\] (A.8)

so we are left with the first terms on the r.h.s. of (A.6) and (A.7),
\[
\lambda_{ij} \sum_{n \neq i,j} \left( \lambda_{jn} \lambda_{nj} - \lambda_{in} \lambda_{ni} \right) .
\] (A.9)

Let us secondly analyze all terms in (A.3) multiplying $\lambda_{ij}$ and add to them (A.9), which is of the same form,
\[
\sum_{n \neq i} \lambda_{in} \sum_{k \neq i} \lambda_{ik} - \sum_{n \neq i} \lambda_{in} \sum_{k \neq j} \lambda_{jk} - \sum_{n \neq j} \sum_{k \neq i} \lambda_{ik} + \sum_{n \neq j} \sum_{k \neq j} \lambda_{jk} - \sum_{n \neq j} \sum_{k \neq i} \lambda_{in} \lambda_{nk} + \sum_{n \neq j} \sum_{k \neq j} \lambda_{kn} \lambda_{nk} + \sum_{k \neq j} \sum_{n \neq j,k} \lambda_{jk} \lambda_{nj} - \sum_{k \neq j} \sum_{n \neq j,k} \lambda_{ik} \lambda_{ki} .
\] (A.10)

It remains to show that these twelve terms combine to zero. The sum of the first five term cancels with the eighth term. Rewriting the sixth term as a sum of two pieces, separating $n = i$ and $n \neq i$,
\[
\sum_{k \neq i} \lambda_{ik} \sum_{n \neq k} \lambda_{kn} = \sum_{k \neq i} \lambda_{ik} \lambda_{ki} + \sum_{k \neq i} \sum_{n \neq k,i} \lambda_{kn} ,
\] (A.11)

one may observe it to cancel with the seventh and the twelveth term. Analogously, the ninth, tenth and eleventh term add to zero. Nothing remains. This completes the proof.
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