Abstract. In this article, we investigate the existence of positive solutions to the following class of quasilinear Schrödinger equations involving Stein-Weiss type convolution:

\[
\begin{cases}
-\Delta_N u - \Delta_N(u^2)u + V(x)|u|^{N-2}u = \left(\int_{\mathbb{R}^N} \frac{F(y,u)}{|y|^\beta|x-y|^\mu} \, dy\right) \frac{f(x,u)}{|x|^\beta}
\end{cases}
\]

in \( \mathbb{R}^N \),

where \( N \geq 2 \), \( 0 < \mu < N \), \( \beta \geq 0 \), \( 2\beta + \mu < N \).

The nonlinearity \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a continuous function with critical exponential growth in the sense of the Trudinger-Moser inequality and fulfills some appropriate hypotheses, described later and \( F(x,s) = \int_0^s f(x,t) \, dt \) is the primitive of \( f \).

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1. INTRODUCTION AND THE MAIN RESULTS

This article concerns the existence of positive solutions for the following family of quasilinear Schrödinger equations with Stein-Weiss type convolution:

\[
\begin{cases}
-\Delta_N u - \Delta_N(u^2)u + V(x)|u|^{N-2}u = \left(\int_{\mathbb{R}^N} \frac{F(y,u)}{|y|^\beta|x-y|^\mu} \, dy\right) \frac{f(x,u)}{|x|^\beta}
\end{cases}
\]

where \( \beta, \mu \) satisfy the following:

\[ N \geq 2, \, \beta \geq 0, \, 0 < \mu < N, \text{ and } 2\beta + \mu < N. \]  

The nonlinearity \( f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a continuous function with critical exponential growth in the sense of the Trudinger-Moser inequality and fulfills some appropriate hypotheses, described later and \( F(x,s) = \int_0^s f(x,t) \, dt \) is the primitive of \( f \). The potential function \( V : \mathbb{R}^N \to \mathbb{R} \) is continuous and is assumed to satisfy some suitable assumptions which are stated afterwards.
The problems driven by the quasilinear operator $-\Delta u - \Delta (u^2)u$, have an ample amount of applications in the modeling of the physical phenomenon such as dissipative quantum mechanics [24], plasma physics and fluid mechanics [7], etc. Solutions of such problems are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iu_t = -\Delta u + V(x)u - h_1(|u|^2)u - C\Delta h_2(|u|^2)h'_2(|u|^2)u, \quad x \in \mathbb{R}^N, \quad (1.2)$$

where $V$ is a continuous potential, $C$ is some real constant, $h_1$ and $h_2$ are some real valued functions with some suitable assumptions. For different types of $h_2$, the quasilinear equations of the form (1.2) represent different phenomenon in the mathematical physics. If $h_2(s) = s$ (see [24]), then (1.2) models the superfluid film equation in plasma physics and in case of $h_2 = \sqrt{1+s^2}$ (see [41]), (1.2) describes the self-channeling of a high-power ultra short laser in matter. Observe that the term $\Delta_N (u^2)u$, present in problem $[P_s^0]$, restrains the natural energy functional corresponding to the problem $[P_s^0]$ to be well defined for all $u \in W^{1, N}(\mathbb{R}^N)$ (defined in Section 2). Hence, the variational method can’t be applied directly for such problems of type $[P_s^0]$. To deal with this inconvenience, researchers have developed several methods and arguments, such as a constrained minimization technique (see for e.g., [29, 30, 42]), the perturbation method (see for e.g., [28, 31]) and a change of variables (see for e.g., [12, 16, 17, 22]).

The nonlinearity in the problem $[P_s^0]$ is nonlocal in nature. It is basically driven by the doubly weighted Hardy-Littlewood-Sobolev inequality (also called Stein-Weiss type inequality) and the Trudinger-Moser inequality in $\mathbb{R}^N$. Let us first recall the following doubly weighted Hardy-Littlewood-Sobolev inequality (see [16]).

**Proposition 1.1. (Doubly Weighted Hardy-Littlewood-Sobolev inequality)** Let $t, s > 1$ and $0 < \mu < N$ with $\vartheta + \beta \geq 0$, $\frac{1}{t} + \frac{t}{s} + \frac{1}{\vartheta} + \frac{1}{\beta} = 2$, $\vartheta < \frac{N}{s}$, $\beta < \frac{N}{t}$, $g_1 \in L^t(\mathbb{R}^N)$ and $g_2 \in L^s(\mathbb{R}^N)$, where $t'$ and $s'$ denote the Hölder conjugate of $t$ and $s$, respectively. Then there exists a constant $C(N, \mu, \vartheta, \beta, t, s)$, independent of $g_1, g_2$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g_1(x)g_2(y)}{|x-y|^\beta |y|^\vartheta} \, dx \, dy \leq C(N, \mu, \vartheta, \beta, t, s) \|g_1\|_{L^t(\mathbb{R}^N)} \|g_2\|_{L^s(\mathbb{R}^N)}. \quad (1.3)$$

For $\vartheta = \beta = 0$, it is reduced to the Hartree type (also called the Choquard type) nonlinearity, which is driven by the classical Hardy-Littlewood-Sobolev inequality (see [20]). So, the problem $[P_s^0]$ is closely related to the Choquard type equations. The study of Choquard equations started with the seminal work of S. Pekar [39], where the author considered the following nonlinear Schrödinger-Newton equation:

$$-\Delta u + V(x)u = (K_\mu * u^2)u + \lambda f_1(x, u) \quad \text{in} \ \mathbb{R}^N, \quad (1.4)$$

where $\lambda > 0$, $K_\mu$ denotes the Riesz potential, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $f_1 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with some appropriate growth assumptions. Researchers pay serious attention to studying such equations due to the rich applications in Physics, such as the Bose-Einstein condensation (see [13]), the self gravitational collapse of a quantum mechanical wave function (see [40]), etc. To describe the quantum theory of a polaron at rest and for modeling the phenomenon when an electron gets trapped in its own hole in the Hartree-Fock theory, P. Choquard (see [25]) used such elliptic equations of type (1.4). For more rigorous study of Choquard type equations, the readers can refer to [26, 27, 31, 35, 36, 37] and the references therein.

One of the main features of problem $[P_s^0]$ is that the nonlinear term $f(x, s)$ has the maximal growth on $s$, that is, critical exponential growth in the sense of Trudinger–Moser inequality in $\mathbb{R}^N$. For some bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) the Trudinger–Moser inequality is established in [38]. That form of the Trudinger–Moser inequality is not valid in unbounded domains. Hence, for the whole of $\mathbb{R}^N$, Cao [11] proposed an alternative version for the case $N = 2$. Later, for general $N \geq 2$ such inequality is established by J. M. do Ó [19].

**Theorem 1.2. (Trudinger–Moser inequality in $\mathbb{R}^N$)** If $\alpha > 0$, $N \geq 2$ and $u \in W^{1,N}(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} \left[ \exp(\alpha |u|^{\frac{N}{N-\alpha}}) - S_{N-2}(\alpha, u) \right] \, dx < \infty,$$
where

\[ S_{N-2}(\alpha, u) = \sum_{m=0}^{N-2} \alpha^m |u|^{\frac{mN}{N-2}}. \]

Moreover, if \( \alpha < \alpha_N \), and \( \|u\|_{L^N(\mathbb{R}^N)} \leq K \), then there exists a positive constant \( C = C(\alpha, K, N) \) such that

\[ \sup_{\|\nabla u\|_{L^N(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \left[ \exp(\alpha |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u) \right] \, dx < C, \]

where \( \alpha_N = N \omega_{N-1}^{\frac{1}{N-1}} \) and \( \omega_{N-1} = (N - 1) \)-dimensional surface area of \( \mathbb{S}^{N-1} \).

Here \( W^{1,N}(\mathbb{R}^N) \) is the Sobolev space, defined in Section 2 and \( L^N(\mathbb{R}^N) \) denotes the classical Lebesgue space. The critical growth non-compact problems associated to this inequality in bounded domains are initially studied by Adimurthi [1] and de Figueiredo et al. [14]. We would like to point out the fact that in the case of critical exponential problem involving \( N \)-Laplacian, the critical exponential growth is equivalent to \( \exp(|u|^{N/(N-1)}) \). But in problem \( P_4 \), due to the term \( \Delta_N(u^2)u \), the nature of the critical exponential growth is of the form \( \exp(|s|^{2N/(N-1)}) \). In the case \( 1 < p < N \), the nonlinearity is of polynomial growth and the critical growth is equivalent to \( |u|^{2p^*} \), where \( p^* = Np/(N-p) \) (see e.g. [15, 17, 33]). In the same spirit, in the critical dimension \( N = 2 \), the critical nonlinearity is expected to behave like \( \exp(\alpha s^\beta) \) as \( s \to \infty \) (see [16, 48]). For more development on this topic, we refer to some recent contemporary works [11, 15], where the authors studied the equations of type \( P_4 \) with critical exponential nonlinearity for \( N = 2 \), without the convolution term \( \int_{\mathbb{R}^N} F(y,u)|x-y|^{-\mu} \, dy \).

For the case \( \beta = 0 \) and \( N = 2 \), without the quasilinear term \( \Delta_N(u^2)u \) in problem \( P_4 \), Alves et al. [1] studied the following Choquard equation:

\[ -\epsilon^2 \Delta u + V(x)u = \left( \int_{\mathbb{R}^2} \frac{F(y,u)}{|x-y|^\mu} \, dy \right) f(x,u), \quad x \in \mathbb{R}^2, \]

where \( \epsilon > 0, 0 < \mu < 2, V: \mathbb{R}^2 \to \mathbb{R} \) is a continuous potential function with some suitable properties and the continuous function \( f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \) enjoys the critical exponential growth, in the sense of Trudinger-Moser inequality in \( \mathbb{R}^2 \). In case of higher critical dimension, that is for \( N \geq 2 \) and in bounded domain \( \Omega \subset \mathbb{R}^N \), Giacomoni et al. [5] investigated the Kirchhoff-Choquard problems involving the \( N \)-Laplacian (\( N \geq 2 \)) with critical exponential growth. Furthermore, the authors in [8] discussed the existence result for the problem \( P_4 \) in case of \( \beta = 0, V \equiv 0 \) with critical exponential growth on the nonlinearity in a bounded domain \( \Omega \subset \mathbb{R}^N, N \geq 2 \).

On the other hand, for the problems involving Stein-Weiss type nonlinearity, we can find very few works on that topic. In [6], Giacomoni et al. studied the polyharmonic Kirchhoff equations involving the singular weights with critical Choquard (that is, critical Stein-Weiss) type exponential nonlinearity in a bounded domain in \( \mathbb{R}^N \) for \( N \geq 2 \). Then, in [20], the authors discussed the following equation in the whole of \( \mathbb{R}^N \):

\[ -\Delta u = \frac{1}{|x|^\gamma} \left( \int_{\mathbb{R}^N} \frac{|u(y)|^{2^{*}_{\beta,\mu}}}{|x-y|^\mu} \, dy \right) |u|^{2^{*}_{\beta,\mu}-2} u, \quad x \in \mathbb{R}^N, \]

where the critical exponent \( 2^{*}_{\beta,\mu} := \frac{2N-(\mu+\beta)}{N-2} \) is due to the weighted Hardy-Littlewood-Sobolev inequality [1,3] (with \( \beta = \vartheta \)) and Sobolev embedding. For more works on this type of nonlinearity, we refer to [9, 32, 17, 49] and references there in without attempting to provide the complete list.

Inspired by all the aforementioned works, in this article, we investigate the existence results for the problem \( P_4 \), involving the Stein-Weiss type convolution with critical exponential nonlinearity. We would like to highlight that, to the best of our knowledge, there is no such existence result in the literature for problem \( P_4 \) even for the case when the dimension \( N = 2 \). Moreover, our result in this article is new in some sense without the presence of the convolution term in the right hand side in problem \( P_4 \). Furthermore, the equation of type \( P_4 \) without the
term $\Delta_N(u^2)u$ has not been addressed yet. All these three cases are covered in this article for the first time.

Now we consider the following hypotheses on the continuous function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$:

1. $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ such that for a.e. $x \in \mathbb{R}^N$, $f(x, s) = 0$, if $s \leq 0$ and $f(x, s) > 0$, if $s > 0$.

2. $\lim_{s \to 0^+} \frac{f(x, s)}{s^{N-1}} = 0$ uniformly in $x \in \mathbb{R}^N$.

3. There exists some $\alpha_0 > 0$ such that,

$$\lim_{s \to +\infty} f(x, s) \exp\left(-\alpha|s|^{\frac{2N}{N-2}}\right) = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0. \end{cases}$$

4. There exist positive constants $s_0$, $M_0$ and $m_0$ such that

$$0 < s^{m_0}f(x, s) \leq M_0f(x, s),$$

for all $(x, s) \in \mathbb{R}^N \times [s_0, +\infty)$.

5. There exists $\ell > N$ such that $0 < \ell F(x, s) \leq f(x, s)s$, for all $s > 0$.

6. We assume that

$$\lim_{s \to +\infty} \frac{sf(x, s)F(x, s)}{\exp\left(2|s|^{\frac{2N}{N-2}}\right)} = +\infty,$$

uniformly in $x \in \mathbb{R}^N$. \hfill (1.5)

**Remark 1.3.** It is important to mention that the condition [1.5] can be weakened by replacing $\infty$ by some real number $C$, whose optimum value can be estimated (see [3, 18], etc). But we would like to convey that it doesn’t bring any effective changes in this context of the main result except for the Lemma 5.2.

Next, we consider the following assumptions on the potential function $V$:

1. $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $\inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0$.

2. The potential function $V : \mathbb{R}^N \to \mathbb{R}$ satisfies $\lim_{|x| \to +\infty} V(x) = +\infty$.

3. $V(x) \leq \lim_{|x| \to +\infty} V(x) = V_\infty < +\infty$, with $V \neq V_\infty$.

Finally, we state the main theorems in this article. The first theorem in this article reads as:

**Theorem 1.4.** Let $\beta, \mu$ satisfy [1.1]. Suppose that the hypotheses $(V_1)$, $(V_2)$ and $(f_1)$-$(f_6)$ are satisfied. Then the problem $(P_6)$ has a nontrivial positive solution.

**Remark 1.5.** In Theorem 1.4, due to the presence of Stein-Weiss type nonlinearity, the corresponding energy functional is not translation invariant. Therefore, to show the existence of nontrivial solution, we are unable to use Lions type concentration compactness lemma, when we consider the potential function $V$ to be satisfying the condition $(V_1)$, $(V_2)$. We propose this case as an open question even for the problem $(P_6)$ without the quasilinear term $\Delta_N(u^2)u$. So to tackle this issue, we consider the assumption $(V'_2)$ instead of $(V_2)$ in the above theorem.

In the next theorem, we deal with the Choquard type nonlinearity considering $(V'_2)$ which is a weaker assumption on $V$ and creates a non compact situation while investigating a positive solution to the problem $(P_6)$.

**Theorem 1.6.** Let $\beta = 0$ in [1.1]. Assume that the hypotheses $(V_1)$, $(V'_2)$ and $(f_1)$-$(f_6)$ hold. Then the problem $(P_6)$ has a nontrivial positive solution.

half of the critical level when the principal operator does not contain $\Delta_N(u^2)u$. The main contributions in this work are as follows:

- Finding the suitable first critical energy level (which is exactly half of the critical level when the principal operator does not contain $\Delta_N(u^2)u$) to describe the convergence of the Cerami sequences below this level.
- For the case $\beta = 0$, constructing a suitable path between two energy functionals $J$ (see (2.2)) and $I_\infty$ (see (5.24)) due to the lack of compactness occurred in the embedding $W^{1,N}(\mathbb{R}^N)$ into $L^N(\mathbb{R}^N)$ (see Section 2) because of the condition $(V'_2)$ as well as, due to the presence of the term $\Delta_N(u^2)u$ in our problem.
- Establishing a general Pohozaev type identity related to our problem which is an important tool to prove Theorem 1.6.
To analyze these, we need to carry out very delicate and crucial estimates.

Remark 1.7. Following the similar idea, we can establish the results as in Theorem 1.6 under any of the following conditions imposed on the potential functions $V$:

1. (Compact-coercive case): $V$ satisfies $(V_1)$ and $(V_2)$. In this case, we obtain the compact embedding from $W^{1,N}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ for $N \leq q < \infty$.

2. (Radially symmetric case): $V$ satisfies $(V_1)$ and $V(x) = V(|x|)$ for all $x \in \mathbb{R}^N$. In this case, we have the compact embedding from $W^{1,N}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ for $N < q < +\infty$.

3. (Asymptotic case of a periodic function): $V$ satisfies $(V_1)$ with $\lim_{|x| \to +\infty} V(x) = V_m(x)$, where $V_m$ is a 1-periodic continuous function. Also, $V(x) \leq V_m(x)$ for all $x \in \mathbb{R}^N$ and $V(x) < V_m(x)$ on a positive Lebesgue measure set of $\mathbb{R}^N$.

Notation. Throughout this paper, we make use of the following notations:

- $C_1, C_2, \cdots, \tilde{C}_1, \tilde{C}_2, \cdots, C$ and $\tilde{C}$ denote (possibly different from line to line) positive constants.
- For any exponent $p > 1$, $p'$ denotes the conjugate of $p$ and is given as $p' = \frac{p}{p-1}$.
- $B_r(x)$ denotes the ball of radius $r$ centered at $x \in \mathbb{R}^N$.
- If $S$ is a measurable set in $\mathbb{R}^N$, we denote the Lebesgue measure of $S$ by $|S|$.
- The arrows $\rightarrow$ and $\Rightarrow$ denote the weak convergence and strong convergence, respectively.

2. Variational Frame-work

In this section we recall some preliminary results. For $1 \leq p < \infty$, the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is defined as

$$W^{1,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p dx < \infty \right\}$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p} := \left( \int_{\mathbb{R}^N} |\nabla u|^p dx + \int_{\mathbb{R}^N} |u|^p dx \right)^{1/p}.$$ 

When $p = N$, we denote $\|\cdot\|_{1,N}$. The embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for $N \leq q < +\infty$. This embedding is compact for $N < q < +\infty$. The space $(W^{1,N}(\mathbb{R}^N))^*$ is the topological dual of $W^{1,N}(\mathbb{R}^N)$.

Next, when the potential function $V$ satisfies the assumptions $(V_1)$ and $(V_2)$, we define a subspace of $W^{1,N}(\mathbb{R}^N)$

$$E := \left\{ u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_{E} := \left( \int_{\mathbb{R}^N} |\nabla u|^N dx + \int_{\mathbb{R}^N} V(x)|u|^N dx \right)^{1/N}.$$ 

Note that $(E, \|\cdot\|_E)$ is a Banach space and since $V(x) \geq V_0 > 0$, it follows that $E$ is continuously embedded in $W^{1,N}(\mathbb{R}^N)$. Moreover, there exist continuous embedding and compact embedding from $W^{1,N}(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for $N \leq q < +\infty$. The topological dual of the space $E$ is denoted by $E^*$. When $V$ satisfies $(V_1)$ and $(V_2)$ then $\|\cdot\|_E$ is equivalent to $\|\cdot\|$, that is, $E = W^{1,N}(\mathbb{R}^N)$. Also, note that $C_c^\infty(\mathbb{R}^N)$ is dense in $E$ and in $W^{1,N}(\mathbb{R}^N)$ with respect to corresponding norms.

Due to the presence of the singular weight $|x|^{-\beta}$ in the problem (P4), we recall the following version of the Trudinger-Moser inequalities studied by Adimurthi and Sandeep [2] for bounded domains $\Omega \subset \mathbb{R}^N$.

Theorem 2.1. Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a smooth bounded domain. Then for $u \in W_0^{1,N}(\Omega)$ and for any $\alpha > 0$, $0 < \beta < N$,

$$\frac{\exp\left(\alpha |u|^\frac{N}{N-\beta}\right)}{|x|^{\beta}} \in L^1(\Omega).$$
Moreover,
\[ \sup_{\|u\|_{W_{0}^{1,\cdot}(\Omega)} \leq 1} \int_{\Omega} \frac{\exp(\alpha|u|^{\frac{N}{N-\tau}})}{|x|^\beta} \, dx < \infty \]
if and only if \(\alpha/\alpha + \beta/N \leq 1\), where
\[ W_{0}^{1,\cdot}(\Omega) = \{ u \in W^{1,\cdot}(\Omega) : u = 0 \text{ on } \partial \Omega \} \]
with the norm
\[ \|u\|_{W_{0}^{1,\cdot}(\Omega)} := \left( \int_{\Omega} |\nabla u|^{N} \, dx \right)^{1/N} . \]
The analogous version of this result in the whole of \(\mathbb{R}^{N}\) was established by Adimurthi and Yang (2.2) and we state that result by taking \(\tau = 1\).

**Theorem 2.2.** For any \(\alpha > 0\), \(0 \leq \beta < N\) and \(u \in W^{1,\cdot}(\mathbb{R}^{N})\), there holds
\[ \frac{\exp \left( \alpha|u|^{\frac{N}{N-\tau}} - S_{N-2}(\alpha,u) \right)}{|x|^\beta} \in L^{1}(\mathbb{R}^{N}) . \]
Moreover
\[ \sup_{\|u\| \leq 1} \int_{\mathbb{R}^{N}} \frac{\exp(\alpha|u|^{\frac{N}{N-\tau}}) - S_{N-2}(\alpha,u)}{|x|^\beta} \, dx < \infty \]
if and only if \(\alpha/\alpha + \beta/N \leq 1\).

The natural energy functional associated to the problem \([P]\) is the following:
\[ I(u) = \frac{1}{N} \int_{\mathbb{R}^{N}} (1 + 2^{N-1}|u|^{N})|\nabla u|^{N} \, dx + \frac{1}{N} \int_{\mathbb{R}^{N}} V(x)|u|^{N} \, dx - \frac{1}{2} \int_{\mathbb{R}^{N}} \left( \int_{\mathbb{R}^{N}} \frac{F(y,u(y))}{|y|^\beta|x-y|^\alpha} \, dy \right) F(x,u(x)) \, dx . \]
Note that the term \(\int_{\mathbb{R}^{N}} u^{N}|\nabla u|^{N} \, dx\) is not finite for all \(u \in W^{1,\cdot}(\mathbb{R}^{N})\). To overcome this difficulty, we employ the following change of variables which was introduced in [12], namely, \(w := h^{-1}(u)\), where \(h\) is defined by
\[
\begin{aligned}
    h'(s) &= \frac{1}{\left(1 + 2^{N-1}|h(s)|^{N}\right)^{\frac{N}{N-1}}} \text{ in } [0, \infty), \\
    h(s) &= -h(-s) \text{ in } (-\infty, 0].
\end{aligned}
\]
Now we state some important and useful properties of \(h\). For the detailed proofs of such results, one can see [12] [16] and references there in.

**Lemma 2.3.** The function \(h\) satisfies the following properties:

- \((h_1)\) \(h\) is uniquely defined, \(C^\infty\) and invertible;
- \((h_2)\) \(h(0) = 0\);
- \((h_3)\) \(0 < h'(s) \leq 1\) for all \(s \in \mathbb{R}\);
- \((h_4)\) \(\frac{1}{2}h(s) \leq sh'(s) \leq h(s)\) for all \(s > 0\);
- \((h_5)\) \(|h(s)| \leq |s|\) for all \(s \in \mathbb{R}\);
- \((h_6)\) \(|h(s)| \leq 2^{1/(2N)}(|s|^{1/2} \text{ for all } s \in \mathbb{R}\);
- \((h_7)\) \(\lim_{s \to +\infty} h(s) \frac{s}{s^\frac{1}{2}} = 2^{-\frac{N}{2}}\);
- \((h_8)\) \(|h(s)| \geq h(1)|s|\) for \(|s| \leq 1\) and \(|h(s)| \geq h(1)|s|^{1/2}\) for \(|s| \geq 1\);
- \((h_9)\) \(h''(s) \leq 0\) when \(s > 0\) and \(h''(s) > 0\) when \(s < 0\).
- \((h_{10})\) \(\lim_{s \to 0} h(s)/s = 1\).

**Example 2.4.** One of the examples of such functions is given in
implies that Lemma 2.5. For any \( h \).

Next, we have the following two technical results for the later consideration.

This together with the properties of \( J \)

Thus, in light of the Sobolev embedding, for any \( w \in W^{1,N}(\mathbb{R}^N) \),

Now by (2.1), if \( w \in W^{1,N}(\mathbb{R}^N) \), then \( h(w) \in W^{1,N}(\mathbb{R}^N) \). Therefore, Proposition 1.1 with \( t = s \) and \( \vartheta = \beta(\geq 0) \) implies that

Moreover, from the assumptions (f2)-(f3), we obtain that for any \( \varepsilon > 0, r \geq N \), there exist positive constants \( C(r, \varepsilon) > 0, \alpha > \alpha_0 > 0 \) such that

Thus, in light of the Sobolev embedding, for any \( v \in W^{1,N}(\mathbb{R}^N) \),

Now by (2.1), if \( w \in W^{1,N}(\mathbb{R}^N) \), then \( h(w) \in W^{1,N}(\mathbb{R}^N) \). Therefore, Proposition 1.1 with \( t = s \) and \( \vartheta = \beta(\geq 0) \) implies that

This together with the properties of \( f \) and \( h \), (2.3) and (2.5) yields that \( J \) is well defined and \( J \) is a \( C^1 \) functional. A function \( w \in W^{1,N}(\mathbb{R}^N) \) is a critical point of the functional \( J \), if for every \( v \in W^{1,N}(\mathbb{R}^N) \), we have

Therefore, \( w \) is a weak solution to the following problem:

As in [43], it can be verified that the transformed problem (2.8) is equivalent to the problem (P'), which takes \( u = h(w) \) as its solution. Thus, it is enough to show the existence of solution to (2.8) by finding the critical point of \( J \), and then apply the transformation \( h \) on that solution, which will serve as the solution to the problem (P').

Next, we have the following two technical results for the later consideration.

Lemma 2.5. For any \( p \geq 1, \alpha > 0 \), it holds that

\[
\left( \exp \left( \alpha |s|^{\frac{2N}{N-\alpha}} \right) - S_{N-2}(\alpha, s) \right)^p \leq \exp \left( p\alpha |s|^{\frac{N}{N-\alpha}} \right) - S_{N-2}(p\alpha, s), \quad \text{for all} \quad s \in \mathbb{R}.
\]
Lemma 2.6. Let the function \( h \) be defined as in (2.1). Then for any \( w \in W^{1,N}(\mathbb{R}^N) \) and \( p > 0 \), we have
\[
\exp \left( p |h(w)|^{\frac{2N}{N-2}} \right) - S_{N-2} \left( \frac{p}{2} |w|^{\frac{2N}{N-2}} \right) \leq \exp \left( p |h(w)|^{\frac{2N}{N-2}} \right) - S_{N-2} \left( \frac{p}{2} |w|^{\frac{2N}{N-2}} \right).
\]

Proof. Recalling Lemma 2.3-(\( h_6 \)), we obtain
\[
\exp \left( p |h(w)|^{\frac{2N}{N-2}} \right) - S_{N-2} \left( p |h(w)|^{2} \right) = \sum_{m=N-1}^{\infty} \frac{p^m |h(w)|^{\frac{2mN}{N-2}}}{m!} \leq \sum_{m=N-1}^{\infty} \left( \frac{p |w|^{\frac{2mN}{N-2}}}{m!} \right).
\]
Thus, the result is proved. \( \square \)

### 3. Analysis of the Cerami sequence and convergence results

In this section, we discuss the behaviour of the Cerami sequence of \( J \) and prove some convergence results. The sequence \( \{w_k\} \) in \( W^{1,N}(\mathbb{R}^N) \) is called a Cerami sequence of \( J \) at level \( c \in \mathbb{R} \), if
\[
J(w_k) \to c; \quad \text{and} \quad (1 + \|w_k\|) J'(w_k) \to 0 \quad \text{in} \quad (W^{1,N}(\mathbb{R}^N))^* \quad \text{as} \quad k \to +\infty.
\]

Lemma 3.1. Let (1.1) hold. Suppose the function \( h \) is defined in (2.1) and let the hypotheses \((V_1), (V'_2)\) on \( V \) and \((f_1)\) and \((f_5)\) on \( f \) be satisfied. Then every Cerami sequence of \( J \) is bounded in \( W^{1,N}(\mathbb{R}^N) \).

Proof. Let \( \{w_k\} \subset W^{1,N}(\mathbb{R}^N) \) be a Cerami sequence of \( J \) at level \( c \in \mathbb{R} \). Then, as \( k \to +\infty \), we have
\[
J(w_k) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w_k|^N dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x) |h(w_k)|^N dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w_k))}{|y|^\beta |x-y|^{\mu}} dy \right) \frac{f(x,h(w_k))}{|x|^\beta} dx \to c, \quad (3.1)
\]
and for any \( \phi \in W^{1,N}(\mathbb{R}^N) \),
\[
(1 + \|w_k\|) \|J'(w_k), \phi\| = \int_{\mathbb{R}^N} |\nabla w_k|^{N-2} \nabla w_k \nabla \phi dx + \int_{\mathbb{R}^N} V(x) |h(w_k)|^{N-2} h(w_k) h'(w_k) \phi dx
\]
\[
- \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w_k))}{|y|^\beta |x-y|^{\mu}} dy \right) \frac{f(x,h(w_k))}{|x|^\beta} h'(w_k) \phi dx \leq \epsilon_k \|\phi\|, \quad (3.2)
\]
where \( \epsilon_k \to 0 \) as \( k \to +\infty \). In the last relation, taking \( \phi = w_k \), we get
\[
\left| \int_{\mathbb{R}^N} |\nabla w_k|^N dx + \int_{\mathbb{R}^N} V(x) |h(w_k)|^N dx - \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w_k))}{|y|^\beta |x-y|^{\mu}} dy \right) \frac{f(x,h(w_k))}{|x|^\beta} h'(w_k) w_k dx \right| \leq \epsilon_k. \quad (3.3)
\]
Now set
\[
v_k := \frac{h(w_k)}{h'(w_k)}.
\]
By Lemma 2.3-(\( h_4 \)), we have \( |v_k| \leq 2 |w_k| \). Moreover, using (2.1) and Lemma 2.3-(\( h_5 \)), we get
\[
|\nabla v_k|^N = \left( 1 + \frac{2^{N-1} |h(w_k)|^N}{1 + 2^{N-1} |h(w_k)|^N} \right) |\nabla w_k|^N \leq 2 |\nabla w_k|^N.
\]
Therefore,
\[
\|v_k\| \leq 2 \|w_k\|.
\]
Therefore, \( v_k \in W^{1,N}(\mathbb{R}^N) \) and hence, in \((3.2)\), taking \( \phi = v_k \), it follows that
\[
|\langle J'(w_k), v_k \rangle| = \left| \int_{\mathbb{R}^N} \left( 1 + \frac{2^{N-1}|h(w_k)|^N}{1 + 2^{N-1}|h(w_k)|^N} \right) |\nabla w_k|^N dx + \int_{\mathbb{R}^N} V(x)|h(w_k)|^N dx \right. \\
- \left. \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta|x-y|^\mu} dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h(w_k) dx \right| \\
\leq \epsilon_k \left\| v_k \right\| \left( 1 + \left\| w_k \right\| \right) \leq 2\epsilon_k. \tag{3.4}
\]

Recalling \((3.1)\), \((3.4)\) and \((f_5)\), we get
\[
c + \frac{\epsilon_k}{\ell} \geq J(w_k) - \frac{1}{2\ell} \langle J'(w_k), v_k \rangle \\
\geq \int_{\mathbb{R}^N} \left[ \frac{1}{N} - \frac{1}{2\ell} \left( 1 + \frac{2^{N-1}|h(w_k)|^N}{1 + 2^{N-1}|h(w_k)|^N} \right) \right] |\nabla w_k|^N dx + \int_{\mathbb{R}^N} \left( \frac{1}{N} - \frac{1}{2\ell} \right) V(x)|h(w_k)|^N dx \\
- \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta|x-y|^\mu} dy \right) \left[ \frac{F(x, h(w_k))}{|x|^\beta} - \frac{1}{\ell} \frac{f(x, h(w_k))}{|x|^\beta} h(w_k) \right] dx \\
\geq \int_{\mathbb{R}^N} \left( \frac{1}{N} - \frac{1}{2\ell} \right) |\nabla w_k|^N dx + \int_{\mathbb{R}^N} \left( \frac{1}{N} - \frac{1}{2\ell} \right) V(x)|h(w_k)|^N dx. \tag{3.5}
\]

This yields that there is a constant \( C \) (independent of \( k, w_k, v_k \)) such that
\[
\int_{\mathbb{R}^N} |\nabla w_k|^N dx + \int_{\mathbb{R}^N} V(x)|h(w_k)|^N dx < C, \tag{3.6}
\]
since \( \ell > N \). Using \((3.6)\) together with Lemma \(2.3\) \((h_5), (h_8), (V_1)\) and the Gagliardo-Nirenberg interpolation inequality, we deduce
\[
\| w_k \|_{L^N(\mathbb{R}^N)}^N = \int_{\mathbb{R}^N} |w_k|^N dx = \int_{\{w_k \leq 1\}} |w_k|^N dx + \int_{\{w_k > 1\}} |w_k|^N dx \\
\leq \frac{1}{V_0(h(1))} \int_{\{w_k \leq 1\}} V(x)|h(w_k)|^N dx + \frac{1}{(h(1))^N} \int_{\{w_k > 1\}} |h(w_k)|^{2N} dx \\
\leq C + C\| \nabla h(w_k) \|_{L^N(\{w_k > 1\})}^N \| h(w_k) \|_{L^N(\{w_k > 1\})}^N \\
\leq C + \frac{C}{V_0}\| h'(w_k) \nabla w_k \|_{L^N(\mathbb{R}^N)}^N \int_{\mathbb{R}^N} V(x)|h(w_k)|^N dx \leq C.
\]

From the last estimate, we infer that the sequence \( \{w_k\} \) is bounded in \( W^{1,N}(\mathbb{R}^N) \). This concludes the lemma. \( \square \)

Let \( \{w_k\} \subset W^{1,N}(\mathbb{R}^N) \) be a Cerami sequence for \( J \). Then Lemma \(3.1\) yields that \( \{w_k\} \) is bounded in \( W^{1,N}(\mathbb{R}^N) \). Thus, there exists \( w \in W^{1,N}(\mathbb{R}^N) \) such that up to a subsequence, still denoted by \( \{w_k\} \), as \( k \to +\infty \)
\[
\begin{align*}
\{w_k\} \to w & \quad \text{weakly in } W^{1,N}(\mathbb{R}^N), \\
\{w_k\} \rightharpoonup w & \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N), \quad \text{for all } q \in [1, \infty), \\
w_k(x) \to w(x) & \quad \text{point-wise a.e. in } \mathbb{R}^N.
\end{align*}
\tag{3.7}
\]

Then from \((3.2)\) and \((3.3)\), we deduce
\[
\begin{align*}
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta|x-y|^\mu} dy \right) \frac{F(x, h(w_k))}{|x|^\beta} dx & \leq C, \tag{3.8} \\
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta|x-y|^\mu} dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k dx & \leq C, \tag{3.9}
\end{align*}
\]
where \( C \) is some positive constant. Now in the next lemmas, we consider the Cerami sequence \( \{w_k\} \) to be satisfying all the facts stated above.
Lemma 3.2. Assume that (1.1) holds. Let the function $h$ be defined as in (2.1) and the assumptions (f1)-(f5) hold. Suppose $\{w_k\} \subset W^{1,N}(\mathbb{R}^N)$ is a Cerami sequence for $J$. Then for any $\Omega \Subset \mathbb{R}^N$, it follows that

$$
\lim_{k \to +\infty} \int_{\Omega} \left( \int_{\mathbb{R}^N} F(y, h(w_k)) \frac{F(x, h(w_k))}{|x|^\beta} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx = \int_{\Omega} \left( \int_{\mathbb{R}^N} F(y, h(w)) \frac{F(x, h(w))}{|x|^\beta} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx.
$$

Proof. Since $\{w_k\}$ is a Cerami sequence, it is bounded in $W^{1,N}(\mathbb{R}^N)$ by Lemma 3.1 and it verifies (3.7). On the other hand, from (2.6), we have

$$
\left( \int_{\mathbb{R}^N} F(y, h(w)) \frac{F(\cdot, h(w))}{|x|^\beta} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \in L^1(\mathbb{R}^N).
$$

Therefore, for any $\delta > 0$, we can choose $R > \max \left\{ 1, \left( \frac{2CM}{\delta} \right)^{\frac{m+1}{m}} , s_0 \right\}$, where $C$ is defined in (3.8) and (3.9), such that

$$
\int_{\{h(w) \geq R\}} \left( \int_{\mathbb{R}^N} F(y, h(w)) \frac{F(x, h(w))}{|x|^\beta} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \leq \delta. \tag{3.10}
$$

Now using (f4), Lemma 2.3 (h4) and (3.9), we obtain

$$
\int_{\{h(w_k) \geq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx \\
\leq M_0 \int_{\{h(w_k) \geq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{f(x, h(w_k)) h(w_k)}{|x|^\beta (h(w_k))^{m+1}} \, dx \\
\leq \frac{2M_0}{R^{m+1}} \int_{\{h(w_k) \geq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{f(x, h(w_k)) h'(w_k) w_k}{|x|^\beta} \, dx < \delta. \tag{3.11}
$$

Gathering (3.10) and (3.11), we get

$$
\left| \int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx - \int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \right| \\
\leq 2\delta + \left| \int_{\Omega \setminus \{h(w_k) \leq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx \right| \\
- \left| \int_{\Omega \setminus \{h(w_k) \leq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \right|.
$$

So, now it is enough to prove that

$$
\int_{\Omega \setminus \{h(w_k) \leq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx \to \int_{\Omega \setminus \{h(w) \leq R\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx
$$
as $k \to +\infty$. Since

$$
\left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \in L^1(\mathbb{R}^N),
$$
by Fubini’s theorem, we infer that

$$
\lim_{\Lambda \to +\infty} \int_{\Omega \setminus \{h(w) \leq \Lambda\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx
= \lim_{\Lambda \to +\infty} \int_{\{h(w) \geq \Lambda\}} \left( \int_{\Omega \setminus \{h(w) \leq \Lambda\}} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx = 0.
$$

Thus, we can fix $\Lambda > \max \left\{ \left( \frac{2CM}{\delta} \right)^{\frac{m+1}{m}} , s_0 \right\}$, where $C$ is defined in (3.8) and (3.9), such that

$$
\int_{\Omega \setminus \{h(w) \leq \Lambda\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|y|^{3} |x-y|^{\mu}} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \leq \delta.
$$
Moreover, from (3.9), (f4) and Lemma 2.3 (h4), we deduce
\[
\int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx
\]
\[
\leq \frac{M_0}{\Lambda^{m_0+\frac{3}{2}}} \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{f(y, h(w_k)) h(w_k)(y)}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx
\]
\[
\leq 2 \frac{M_0}{\Lambda^{m_0+\frac{3}{2}}} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \leq \delta.
\]
Therefore, using the last two relations, it follows that
\[
\left| \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx
\]
\[- \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx \right| \leq 2 \delta.
\]
Now we claim that for fixed positive real numbers \(R\) and \(\Lambda\), the following holds:
\[
\lim_{k \to +\infty} \left| \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx
\]
\[- \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \right| = 0.
\]
(3.12)

It can be easily verified that as \(k \to +\infty\),
\[
\left( \int_{\{h(w) \leq \Lambda\}} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \chi_{\Omega \cap \{h(w) \leq R\}}(x)
\]
\[\to \left( \int_{\{h(w) \leq \Lambda\}} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \chi_{\Omega \cap \{h(w) \leq R\}}(x)
\]
(3.13)

point-wise a.e. Now taking \(r = N\) in [2.4], using Lemma 2.3 (h5) and (1.3), we get a constant \(C_{K, \Lambda} > 0\) depending on \(K\) and \(\Lambda\) such that
\[
\int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx
\]
\[\leq C_{R, \Lambda} \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{|h(w_k)(y)|}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{|h(w_k)(x)|}{|x|^\beta} \, dx
\]
\[\leq C_{R, \Lambda} \int_{\Omega \cap \{h(w) \leq R\}} \left( \int_{\{h(w) \geq \Lambda\}} \frac{|w_k(y)|}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{|w_k(x)|}{|x|^\beta} \, dx
\]
\[\leq C_{R, \Lambda} \int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{|w_k(y)|}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{|w_k(x)|}{|x|^\beta} \, dx
\]
\[\leq C_{R, \Lambda} C(N, \mu, \beta) \|w_k\|^N \|w\|^{2N} \|L^{\frac{2N}{2N+2\mu}}(\Omega) \to C_{R, \Lambda} C(N, \mu, \beta) \|w\|^{2N} \|L^{\frac{2N}{2N+2\mu}}(\Omega) \text{ as } k \to +\infty.
\]

Hence, by Theorem 4.9 in [10] there exists some function \(F \in L^1(\mathbb{R}^N)\) such that up to a subsequence, still denoted by \(\{w_k\}\), for each \(k \in \mathbb{N}\), we have
\[
\left| \left( \int_{\{h(w) \leq \Lambda\}} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \chi_{\Omega \cap \{h(w) \leq R\}} \right| \leq |F(x)|.
\]
So, using (3.13) and applying the Lebesgue dominated convergence theorem, we achieve (3.12). This concludes the proof. \(\square\)
Next, we prove the following result regarding the regularity of the convolution term with the singular weight, which we will use in the later context.

**Lemma 3.3.** Let \( f_{11} \) be satisfied and let the function \( h \) be defined in \([2.1]\). Assume that \((f_1)-(f_3)\) hold. Then for any \( w \in W^{1,N}(\mathbb{R}^N) \), we have

\[
\int_{\mathbb{R}^N} \frac{F(y,h(w))}{|y|^\mu} dy \in L^\infty(\mathbb{R}^N). \tag{3.14}
\]

**Proof.** To prove this result, we follow the idea as in [20, Theorem 2.7]. For any \( r > 0 \), we can write

\[
\int_{\mathbb{R}^N} \frac{F(y,h(w))}{|y|^\mu} dy \leq \int_{B_r(0)} F(y,h(w)) \frac{|y|^\mu}{|y|^\mu} dy + \int_{\mathbb{R}^N \setminus B_r(0)} F(y,h(w)) \frac{|y|^\mu}{|y|^\mu} dy. \tag{3.15}
\]

Now for \( x \in \mathbb{R}^N \setminus B_{2r}(0) \), we have \(|x-y| > |y|\), which together with Hölder’s inequality and the fact \( F(\cdot, h(w)) \in L^q(B_r(0)) \) for any \( q > 1 \), implies that

\[
\int_{B_r(0)} F(y,h(w)) \frac{|y|^\mu}{|y|^\mu} dy < \int_{B_r(0)} \frac{F(y,h(w))}{|y|^\mu} dy \leq \|F(\cdot, h(w))\|_{L^q(B_r(0))} \int_{B_r(0)} \frac{1}{|y|^\mu} dy < +\infty,
\]

provided \( 1 < p < \frac{N}{\mu+\beta} \).

For \( x \in B_{2r}(0) \), using the arguments as above, we deduce

\[
\int_{B_{2r}(0)} \frac{F(y,h(w))}{|y|^\mu} dy < +\infty.
\]

On the other hand, we have

\[
\int_{\mathbb{R}^N \setminus B_r(0)} \frac{F(y,h(w))}{|y|^\mu} dy = \int_{(\mathbb{R}^N \setminus B_r(0)) \cap B_r(x)} \frac{F(y,h(w))}{|y|^\mu} dy + \int_{(\mathbb{R}^N \setminus B_r(0)) \cap (\mathbb{R}^N - B_r(x))} \frac{F(y,h(w))}{|y|^\mu} dy
\]

Again, following the above estimates, we obtain

\[
\int_{(\mathbb{R}^N \setminus B_r(0)) \cap B_r(x)} \frac{F(y,h(w))}{|y|^\mu} dy \leq \frac{1}{r^\beta} \int_{(\mathbb{R}^N \setminus B_r(0)) \cap B_r(x)} \frac{F(y,h(w))}{|x-y|^\beta} dy
\]

\[
\leq \frac{1}{r^\beta} \int_{B_r(x)} \frac{F(y,h(w))}{|x-y|^\beta} dy < +\infty.
\]

Similarly, using \( F(\cdot, h(w)) \in L^q(\mathbb{R}^N) \), for any \( q \geq N \), we have the below estimate

\[
\int_{(\mathbb{R}^N \setminus B_r(0)) \cap (\mathbb{R}^N \setminus B_r(x))} \frac{F(y,h(w))}{|y|^\beta} dy \leq \frac{1}{r^\mu} \int_{\mathbb{R}^N \setminus B_r(0)} \frac{F(y,h(w))}{|y|^\beta} dy
\]

\[
\leq \frac{1}{r^\mu} \|F(\cdot, h(w))\|_{L^\infty(\mathbb{R}^N \setminus B_r(0))} \int_{\mathbb{R}^N \setminus B_r(0)} \frac{1}{|y|^\beta} dy < +\infty,
\]

provided \( p > \frac{N}{\beta} \). Thus,

\[
\int_{\mathbb{R}^N \setminus B_r(0)} \frac{F(y,h(w))}{|y|^\beta} dy < +\infty. \tag{3.17}
\]

Combining \([3.15], [3.16], [3.17]\), we can obtain our result. \(\square\)

**Lemma 3.4.** Assume that \([1.1]\) holds. Let the assumptions \((f_1)-(f_3)\) be satisfied and the function \( h \) be defined in \([2.1]\). If \( \{w_k\} \subset W^{1,N}(\mathbb{R}^N) \) is a Cerami sequence for \( J \), then \( \nabla w_k(x) \to \nabla w(x) \) a.e. in \( \mathbb{R}^N \). Moreover,

\[
|\nabla w_k|^N - 2 \nabla w_k \rightharpoonup |\nabla w|^N - 2 \nabla w \text{ weakly in } L^{\frac{N}{N-2}}(\mathbb{R}^N)^N \text{ as } k \to +\infty. \tag{3.18}
\]

**Proof.** Since \( \{w_k\} \) is a Cerami sequence for \( J \), by Lemma \([3.1]\), \( \{w_k\} \) is bounded in \( W^{1,N}(\mathbb{R}^N) \) and it satisfies \([3.7]\). Thus, the sequence \( \{|\nabla w_k|^N - 2 \nabla w_k\} \) is bounded in \( \left(L^{\frac{N}{N-2}}(\mathbb{R}^N)\right)^N \). This implies that there exists \( u \in \left(L^{\frac{N}{N-2}}(\mathbb{R}^N)\right)^N \)
such that,

$$|\nabla w_k|^{N-2} \nabla w_k \to u \text{ weakly in } (L^{\frac{N}{\lambda}}(\mathbb{R}^N))^N \text{ as } k \to +\infty.$$ 

Also we have, \(|\nabla w_k|^N + |w_k|^N\) is bounded in \(L^1(\mathbb{R}^N)\), which yields that there exists a non-negative radon measure \(\sigma\) such that, up to a subsequence, we have

$$|\nabla w_k|^N + |w_k|^N \to \sigma \text{ in } (C(\Omega))^* \text{ as } k \to +\infty,$$

for any \(\Omega \subset \subset \mathbb{R}^N\). We show that \(u = |\nabla w|^{N-2}\nabla w\). For any fixed \(\nu > 0\) we define the energy concentration set

$$X_\nu := \left\{ x \in \mathbb{R}^N : \lim_{l \to 0} \lim_{k \to +\infty} \int_{B_l(x)} \left( |\nabla w_k|^N + |w_k|^N \right) dx \geq \nu \right\}.$$ 

Thus, \(X_\nu\) is a finite set. Indeed, if not, then there exists a sequence of distinct points \(\{z_k\}\) in \(X_\nu\) such that, \(\sigma(B_l(z_k)) \geq \nu\) for all \(l > 0\) and for all \(k \in \mathbb{N}\). This gives that \(\sigma(\{z_k\}) \geq \nu\) for all \(k\). Therefore, \(\sigma(X_\nu) = +\infty\). But this is a contradiction to the fact that

$$\sigma(X_\nu) = \lim_{k \to +\infty} \int_{X_\nu} \left( |\nabla w_k|^N + |w_k|^N \right) dx \leq C.$$

Thus, we can take \(X_\nu = \{z_1, z_2, \ldots, z_n\}\).

Next we claim that, we can choose sufficiently small \(\nu > 0\), with \(\nu \frac{1}{2} < \frac{1}{2} \frac{2N-2\beta-\mu}{2N} \left( 1 - \frac{\beta}{N} \right) \frac{\alpha \nu}{\alpha_0} \) such that

$$\lim_{k \to +\infty} \int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{\beta}} dy \right) \left( \frac{f(x, h(w_k))}{|x|^\beta} \right) h'(w_k) w_k dx = \int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^{\beta}} dy \right) \frac{f(x, h(w))}{|x|^\beta} h'(w) w dx,$$

where \(\Omega\) is any compact subset of \(\mathbb{R}^N \setminus X_\nu\).

Let us choose \(z_0 \in \Omega\) and \(l_0 > 0\) be such that \(\sigma(B_{l_0}(z_0)) < \nu\). Hence \(z_0 \notin X_\nu\). Furthermore, we consider \(\phi \in C^\infty_c(\mathbb{R}^N)\) with \(0 \leq \phi(x) \leq 1\) for \(x \in \mathbb{R}^N\), \(\phi \equiv 1\) in \(B_{l_0}(z_0)\) and \(\phi \equiv 0\) in \(\mathbb{R}^N \setminus B_{l_0}(z_0)\). Then

$$\lim_{k \to +\infty} \int_{B_{l_0}(z_0)} \left( |\nabla w_k|^N + |w_k|^N \right) dx \leq \lim_{k \to +\infty} \int_{B_{l_0}(z_0)} (|\nabla w_k|^N + |w_k|^N) \phi dx \leq \sigma(B_{l_0}(z_0)) < \nu.$$

Therefore, for sufficiently large \(k \in \mathbb{N}\) and sufficiently small \(\epsilon > 0\), we have

$$\int_{B_{l_0}(z_0)} \left( |\nabla w_k|^N + |w_k|^N \right) dx \leq \nu(1 - \epsilon). \quad (3.20)$$

From \((f_2), (f_3)\), for \(\alpha > \alpha_0\), very close to \(\alpha_0\), there exists some constant \(C > 0\) such that \(|f(x, s)| \leq C \exp \left( \alpha |s| \frac{N}{\lambda} \right)\) in \(B_{l_0}(z_0)\). Using this, together with \((3.20)\) and Lemma 2.3 (h), we deduce

$$\int_{B_{l_0}(z_0)} \frac{|f(x, h(w_k))|^q}{|x|^{\beta}} dx \leq C \int_{B_{l_0}(z_0)} \frac{\exp \left( \alpha q \frac{N}{\lambda} |w_k|^\frac{N}{\lambda} \right)}{|x|^{\beta}} dx$$

$$\leq C \int_{B_{l_0}(z_0)} \frac{1}{|x|^{\beta}} \exp \left( \alpha q \frac{N}{\lambda} \nu \frac{N}{\lambda} (1 - \epsilon) \frac{\alpha \nu}{\alpha_0} \left( \int_{B_{l_0}(z_0)} \left( |\nabla w_k|^N + |w_k|^N \right) dx \right)^{\frac{N}{\lambda}} \right) dx$$

Hence, we can choose \(q > 1\) such that \(\alpha q \frac{N}{\lambda} \nu \frac{N}{\lambda} (1 - \epsilon) \frac{\alpha \nu}{\alpha_0} < \left( 1 - \frac{\beta}{N} \right) \frac{\alpha \nu}{\alpha_0}\) and using Theorem 2.1 in the last relation, we get

$$\int_{B_{l_0}(z_0)} \frac{|f(x, h(w_k))|^q}{|x|^{\beta}} dx \leq C. \quad (3.21)$$
Next, we consider
\[ \int_{B_{|x|}(z_0)} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k)) - F(y, h(w))}{|y|^\beta|x - y|^\mu} \, dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \leq \int_{B_{|x|}(z_0)} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta|x - y|^\mu} \, dy \right) \frac{f(x, h(w))}{|x|^\beta} h'(w) w \, dx \]
\[ + \int_{B_{|x|}(z_0)} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k)) - F(y, h(w))}{|y|^\beta|x - y|^\mu} \, dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \]
\[ := L_1 + L_2. \]

Now using (3.14), we get the estimate
\[ L_1 \leq C \int_{B_{|x|}(z_0)} \left| \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k - \frac{f(x, h(w))}{|x|^\beta} h'(w) w \right| \, dx, \tag{3.22} \]
where \( C > 0 \) is some constant. Moreover, the asymptotic growth assumptions on \( f \) gives us
\[ \lim_{s \to +\infty} \frac{f(x, s)}{(f(x, s))^r} = 0 \text{ uniformly in } x \in \mathbb{R}^N, \text{ for all } r > 1. \]

This together with (3.21) and Lemma 2.3 \( (h_4) \) implies that
\[ \int_{B_{|x|}(z_0)} \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \leq \int_{B_{|x|}(z_0)} \frac{f(x, h(w))}{|x|^\beta} h(w) \, dx \leq \int_{B_{|x|}(z_0)} \left( \frac{f(x, h(w_k))}{|x|^\beta} \right) \, dx \leq C. \]
That is \( \left\{ \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \right\} \) is an equi-integrable family of functions over \( B_{|x|}(z_0) \). Also, by the continuity of \( f \) and \( h \) we get, \( \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k(x) \to \frac{f(x, h(w))}{|x|^\beta} h'(w) w(x) \) a.e. in \( \mathbb{R}^N \), as \( k \to +\infty \). Hence, we obtain \( L_1 \to 0 \) as \( k \to +\infty \), thanks to Vitali’s convergence theorem. Next, we show that \( L_2 \to 0 \) as \( k \to +\infty \).

For that, first we get the following estimate with the help of the semigroup property of the Riesz Potential and the Cauchy-Schwartz inequality:
\[ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k)) - F(y, h(w))}{|y|^\beta|x - y|^\mu} \, dy \right) \chi_{B_{|x|}(z_0)}(y) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \]
\[ \leq C \left( \int_{B_{|x|}(z_0)} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k)) - F(y, h(w))}{|y|^\beta|x - y|^\mu} \, dy \right) \frac{F(x, h(w_k)) - F(x, h(w))}{|x|^\beta} \, dx \right)^{\frac{1}{2}} \]
\[ \times \left( \int_{\mathbb{R}^N} \left( \int_{B_{|x|}(z_0)} (y) \frac{f(x, h(w_k))}{|y|^\beta|x - y|^\mu} h'(w_k) w_k \, dy \right) \chi_{B_{|x|}(z_0)}(x) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \right)^{\frac{1}{2}}, \]
where \( C \) is some constant, independent of \( k \). By Theorem 1.1 with \( \beta = \theta \) and gathering (3.21), (3.22), and choosing \( \nu \) sufficiently small such that \( \nu \frac{|x|}{|y|} < \frac{1}{2N-2-\mu} \left( 1 - \frac{2}{N} \right) \frac{\alpha}{\nu_0} \), we find that
\[ \left( \int_{\mathbb{R}^N} \chi_{B_{|x|}(z_0)}(y) \frac{f(x, h(w_k))}{|y|^\beta|x - y|^\mu} h'(w_k) w_k \, dy \right) \chi_{B_{|x|}(z_0)}(x) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \]
\[ \leq \| \chi_{B_{|x|}(z_0)}(y) \frac{f(x, h(w_k))}{|y|^\beta|x - y|^\mu} h'(w_k) w_k \|_{L^{\frac{2N-2-\mu}{N}}(\mathbb{R}^N)} \leq C. \tag{3.23} \]

Now we claim that
\[ \lim_{k \to +\infty} \int_{B_{|x|}(z_0)} \left( \int_{\mathbb{R}^N} \frac{|F(y, h(w_k)) - F(y, h(w))|}{|y|^\beta|x - y|^\mu} \, dy \right) \frac{|F(x, h(w_k)) - F(x, h(w))|}{|x|^\beta} \, dx = 0. \tag{3.24} \]
Using the similar arguments used in Lemma 3.2, we get as $\Lambda \to +\infty$

\[
\int_{B_{\frac{1}{2}}(z_0)} \int_{\{|h(w)| \geq \Lambda\}} \frac{F(y, h(w))}{|y|^{\beta}|x-y|^\mu} \frac{F(x, h(w))}{|x|^{\beta}} \, dy \, dx = o(\Lambda),
\]  

(3.25)

\[
\int_{B_{\frac{1}{2}}(z_0)} \int_{\{|h(w_k)| \geq \Lambda\}} \frac{F(y, h(w_k))}{|y|^{\beta}|x-y|^\mu} \frac{F(x, h(w_k))}{|x|^{\beta}} \, dy \, dx = o(\Lambda),
\]  

(3.26)

\[
\int_{B_{\frac{1}{2}}(z_0)} \int_{\{|h(w)| \geq \Lambda\}} \frac{F(y, h(w))}{|y|^{\beta}|x-y|^\mu} \frac{F(x, h(w))}{|x|^{\beta}} \, dy \, dx = o(\Lambda),
\]  

(3.27)

and

\[
\int_{B_{\frac{1}{2}}(z_0)} \int_{\{|h(w_k)| \geq \Lambda\}} \frac{F(y, h(w_k))}{|y|^{\beta}|x-y|^\mu} \frac{F(x, h(w))}{|x|^{\beta}} \, dy \, dx = o(\Lambda).
\]  

(3.28)

So,

\[
\int_{B_{\frac{1}{2}}(z_0)} \left( \int_{\mathbb{R}^N} \frac{|F(y, h(w_k)) - F(y, h(w))|}{|y|^{\beta}|x-y|^\mu} \frac{|F(x, h(w_k)) - F(x, h(w))|}{|x|^{\beta}} \, dy \right) \, dx
\]

\leq \frac{2}{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\chi\{|h(w_k)| \geq \Lambda\}(y) F(y, h(w_k))}{|y|^{\beta}|x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^{\beta}} \, dx
\]

+ 4 \int_{B_{\frac{1}{2}}(z_0)} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{\beta}|x-y|^\mu} \, dy \right) \chi\{|h(w)\geq \Lambda\}(x) \frac{F(x, h(w))}{|x|^{\beta}} \, dx
\]

+ 4 \int_{B_{\frac{1}{2}}(z_0)} \left( \int_{\mathbb{R}^N} \frac{\chi\{|h(w_k)| \geq \Lambda\}(y) F(y, h(w_k))}{|y|^{\beta}|x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^{\beta}} \, dx
\]

+ 2 \int_{B_{\frac{1}{2}}(z_0)} \left[ \left( \int_{\mathbb{R}^N} \frac{|F(y, h(w_k))| \chi\{|h(w_k)| \leq \Lambda\}(y) - F(y, h(w)) \chi\{|h(w)\leq \Lambda\}(y)|}{|y|^{\beta}|x-y|^\mu} \, dy \right) \frac{F(x, h(w_k))}{|x|^{\beta}} \, dx
\]

\[
\frac{|F(x, h(w_k))| \chi\{|h(w_k)| \leq \Lambda\}(x) - F(x, h(w)) \chi\{|h(w)\leq \Lambda\}(x)|}{|x|^{\beta}} \right] \, dx.
\]

Then the Lebesgue dominated convergence theorem yields that the last integration tends to 0 as $k \to +\infty$. Recalling (3.25)-(3.28), we get (3.24), which together with (3.23) implies that $L_2 \to 0$ as $k \to +\infty$. This yields that

\[
\lim_{k \to +\infty} \int_{B_{\frac{1}{2}}(z_0)} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^{\beta}|x-y|^\mu} \, dy \right) \frac{f(x, h(w_k))}{|x|^{\beta}} h'(w_k) \, w_k
\]

\[
- \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^{\beta}|x-y|^\mu} \, dy \right) \frac{f(x, h(w))}{|x|^{\beta}} h'(w) \, w \right] \, dx = 0.
\]

Since $\Omega$ is compact, by applying standard finite covering lemma, we obtain (3.19). Rest of the proof of (3.18) can be concluded by following the classical arguments as in the proof of Lemma 4 in [3] and Lemma 4.4 in [3].

**Lemma 3.5.** Assume that (1.1) is satisfied. Suppose that $(V_1)$, $(V_2)$ and $(f_1)$-(f$_6$) hold and $h$ is defined as in (2.1). Let $\{w_k\} \subset W^{1,N}(\mathbb{R}^N)$ be a Cerami sequence for $J$. Then

\[
\int_{\mathbb{R}^N} V(x)|h(w_k)|^{N-2} h(w_k) h'(w_k) \phi \, dx \to \int_{\mathbb{R}^N} V(x)|h(w)|^{N-2} h(w) h'(w) \phi \, dx \ 	ext{for any} \ \phi \in C_c^\infty(\mathbb{R}^N) \ 	ext{as} \ k \to +\infty.
\]

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\text{supp}(\varphi) = \Omega$. Then using Lemma 2.3(h$_3$), (h$_5$), we get

\[
|V(x)|h(w_k)|^{N-2} h(w_k) h'(w_k) \varphi| \leq \|V\|_{L^\infty(\Omega)}|w_k|^{N-1} |\varphi|.
\]
Now Hölder’s inequality and the continuous embedding $E \hookrightarrow L^N(\mathbb{R}^N)$ imply
\[
\int_{\Omega} |V|_{L^\infty(\Omega)} |w_k|^{N-1} |\varphi| \, dx \leq |V|_{L^\infty(\Omega)} |w_k|^{N-1} |\varphi|_{L^\infty(\Omega)} \leq |V|_{L^\infty(\Omega)} |w_k|^{N-1} |\varphi|_{L^N(\Omega)} \leq C,
\]
for some positive constant $C$, independent in $k$, where in the last line we used the fact that $\{w_k\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$ by Lemma 3.1. Finally by applying the Lebesgue dominated convergence theorem, we conclude the proof. □

**Lemma 3.6.** Assume that \((1.1)\) and \((f_1)-(f_5)\) hold and the function $h$ is defined as in \((2.1)\). Let $\{w_k\} \subset W^{1,N}(\mathbb{R}^N)$ be a Cerami sequence for $J$. Then for all $\psi \in C_c^\infty(\mathbb{R}^N)$, as $k \to +\infty$, we have

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) h'(w_k) \psi \, dx \to \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w)) h'(w) \psi \, dx.
\]

**Proof.** Let $\Omega$ be any compact subset of $\mathbb{R}^N$. Let $\psi \in C_c^\infty(\mathbb{R}^N)$ with compact support $\Omega' \supset \Omega$ such that $\psi \equiv 1$ in $\Omega$ and $0 \leq \psi \leq 1$ in $\Omega'$. It can be easily computed that

\[
\left\| \frac{\psi}{1+w_k} \right\|^N = \int_{\mathbb{R}^N} \left\| \frac{\nabla \psi}{1+w_k} - \frac{\nabla w_k}{(1+w_k)^2} \right\|^N \, dx + \int_{\mathbb{R}^N} \frac{\psi}{1+w_k} |\nabla w_k|^N \, dx \leq 2(\|\psi\|^N + \|w_k\|^N),
\]

that is $\frac{\psi}{1+w_k} \in W^{1,N}(\mathbb{R}^N)$. Now in \((3.2)\), choosing $\phi := \frac{\psi}{1+w_k}$ as a test function and using Lemma 2.3-(h3), \((h_5)\), Hölder’s inequality and \((3.29)\), we obtain

\[
\int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) h'(w_k) \, dx \\
\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) h'(w_k) \psi \, dx \\
= \varepsilon_k \left\| \frac{\psi}{1+w_k} \right\|^N + \int_{\mathbb{R}^N} |\nabla w_k|^{N-2} \nabla w_k \nabla \left( \frac{\psi}{1+w_k} \right) \, dx + \int_{\mathbb{R}^N} V(x) |h(w_k)|^{N-2} h(w_k) h'(w_k) \left( \frac{\psi}{1+w_k} \right) \, dx \\
\leq \varepsilon_k 2^{N-1} (\|\psi\|^N + \|w_k\|^N) + \int_{\mathbb{R}^N} |\nabla w_k|^{N-2} \nabla w_k \left( \frac{\nabla \psi}{1+w_k} - \frac{\nabla w_k}{(1+w_k)^2} \right) \, dx + \int_{\mathbb{R}^N} V(x) |w_k|^{N-2} w_k \left( \frac{\psi}{1+w_k} \right) \, dx \\
\leq \varepsilon_k 2^{N-1} (\|\psi\|^N + \|w_k\|^N) + \int_{\mathbb{R}^N} |\nabla \psi|_{L^\infty(\Omega')} |\nabla w_k|_{L^N(\Omega')} + \|w_k\|_{L^{N-1}(\Omega')} \leq C_1,
\]

where $C_1 := C_1(\Omega)$ is a positive constant and in the last line we used the facts that sequence $\{w_k\}$ is bounded in $W^{1,N}(\mathbb{R}^N)$ by Lemma 3.1 and $w_k \to w$ strongly in $L^{N-1}(\Omega')$. Gathering \((3.29)\) and \((3.9)\), we deduce

\[
\int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) \, dx \\
\leq 2 \int_{\Omega \cap \{|w_k| < 1\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) h'(w_k) \, dx \\
+ \int_{\Omega \cap \{|w_k| \geq 1\}} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) w_k h'(w_k) \, dx \\
\leq 2 \int_{\Omega} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) h'(w_k) \, dx \\
+ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x-y|^{\mu}} \, dy \right) f(x, h(w_k)) h'(w_k) w_k \, dx \\
\leq 2C_1 + C := C_2.
\]
Thus, the sequence \( \{ v_k \} := \left\{ \left( \int_{\mathbb{R}^N} \frac{F(y,h(w_k))}{|y|^p |y - y'|^p} \, dy \right) \frac{f(x,h(w_k))}{|x|^\beta} h'(w_k) \right\} \) is bounded in \( L^1_{\text{loc}}(\mathbb{R}^N) \). Hence, there is a radon measure \( \zeta \) such that, up to a subsequence, \( w_k \to \zeta \) in the weak*-topology as \( k \to +\infty \). Therefore, we have

\[
\lim_{k \to +\infty} \int_{\Omega} v_k \varphi = \lim_{k \to +\infty} \int_{\Omega} \int_{\mathbb{R}^N} \left( \frac{F(y,h(w_k))}{|y|^\beta |x - y'|^\mu} \, dy \right) \frac{f(x,h(w_k))}{|x|^\beta} h'(w_k) \varphi \, dx = \int_{\mathbb{R}^N} \varphi \, d\zeta, \quad \forall \varphi \in C_c(\Omega).
\]

Since \( w \) satisfies \( (3.2) \), we achieve

\[
\int_{\Omega} \varphi \, d\zeta = \lim_{k \to +\infty} \int_{\mathbb{R}^N} |\nabla w_k|^{N-2} \nabla w_k \nabla \varphi \, dx, \quad \text{for all } \varphi \in C_c(\Omega),
\]

which together with Lemma 3.3 yields that the Radon measure \( \zeta \) is absolutely continuous with respect to the Lebesgue measure. So, by Radon-Nikodym theorem, since \( \Omega \subset \mathbb{R}^N \) is an arbitrary compact set, there exists a function \( g \in L^1_{\text{loc}}(\mathbb{R}^N) \) such that for any \( \psi \in C_c(\mathbb{R}^N) \), we have \( \int_{\mathbb{R}^N} \psi \, d\zeta = \int_{\mathbb{R}^N} \psi \, g \, dx \). Therefore, we get

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w_k))}{|y|^\beta |x - y'|^\mu} \, dy \right) \frac{f(x,h(w_k))}{|x|^\beta} h'(w_k) \psi(x) \, dx = \int_{\mathbb{R}^N} \psi \, g \, dx.
\]

This completes the proof.

\[ \square \]

4. Mountain pass geometry

In this section first we study the mountain pass structure to the energy functional \( J : W^{1,N}(\mathbb{R}^N) \to \mathbb{R} \) associated to the problem \( (2.8) \). Now we obtain the following necessary result to prove the next lemma.

Lemma 4.1. Let the function \( h \) be defined as in \( (2.1) \). Let \( V \) satisfy \( (V_1), (V_2') \). Then for any \( w \in W^{1,N}(\mathbb{R}^N) \), there exist constants \( C, \lambda_* > 0 \) such that

\[
\int_{\mathbb{R}^N} |\nabla w|^N \, dx + \int_{\mathbb{R}^N} V(x) |h(w)|^N \, dx \geq C \|w\|^N, \quad \text{whenever } \|w\| < \lambda_*.
\]

Proof. Suppose \( (4.1) \) does not hold. Then for each \( k \in \mathbb{N} \), there exists \( w_k(\neq 0) \in W^{1,N}(\mathbb{R}^N) \), such that \( w_k \to 0 \) strongly in \( W^{1,N}(\mathbb{R}^N) \) as \( k \to +\infty \) and

\[
\int_{\mathbb{R}^N} |\nabla w_k|^N \, dx + \int_{\mathbb{R}^N} V(x) |h(w_k)|^N \, dx < \frac{1}{k} \|w_k\|^N.
\]

Let us set \( v_k := \frac{w_k}{\|w_k\|} \). Using this in the last relation, we deduce

\[
\int_{\mathbb{R}^N} |\nabla v_k|^N \, dx + \int_{\mathbb{R}^N} V(x) |v_k|^N \, dx + \int_{\mathbb{R}^N} V(x) \left( \frac{|h(w_k)|^N}{|w_k|^N} - 1 \right) |v_k|^N \, dx < \frac{1}{k}.
\]

Since \( w_k \to 0 \) in \( W^{1,N}(\mathbb{R}^N) \), we have \( w_k \to 0 \) strongly in \( L^q_{\text{loc}}(\mathbb{R}^N) \) for \( 1 \leq q < \infty \) and \( w_k(x) \to 0 \) a.e. in \( \mathbb{R}^N \). Therefore, for any given \( \epsilon > 0 \), as \( k \to +\infty \), the measure

\[
\left\{ x \in \mathbb{R}^N : |w_k| > \epsilon \right\} \to 0.
\]

Hence, for \( q > N \), using Hölder’s inequality and continuous embedding \( W^{1,N}(\mathbb{R}^N) \to L^q(\mathbb{R}^N) \) along with the last limit, we obtain

\[
\int_{\left\{ x \in \mathbb{R}^N : |w_k| > \epsilon \right\}} |v_k|^N \, dx \leq C \left\{ x \in \mathbb{R}^N : |w_k| > \epsilon \right\} \frac{q-N}{q} \|v_k\|^N \to 0.
\]

Now using \( (4.3) \) and Lemma 2.3 \( (h_{10}) \), as \( k \to +\infty \), applying Vitali’s convergence theorem, we get

\[
\int_{\mathbb{R}^N} V(x) \left( \frac{|h(w_k)|^N}{|w_k|^N} - 1 \right) |v_k|^N \, dx \to 0.
\]

Therefore, from \( (4.2) \) and \( (V_1) \) we get \( \|v_k\| \to 0 \) which contradicts the fact that \( \|w_k\| = 1 \). Thus, the lemma is proved.

\[ \square \]
Lemma 4.2. Let (1.1) hold and the function \( h \) be defined in (2.1). Suppose that the conditions (V_1), (V_2) and (f_1)-(f_5) hold. Then there exist \( \lambda_\beta > 0 \) and \( \tau_\beta > 0 \) such that

\[ J(w) \geq \tau_\beta > 0 \quad \text{for any } w \in W^{1,N}(\mathbb{R}^N) \text{ with } \|w\| = \lambda_\beta. \]

Proof. Let \( w \in W^{1,N}(\mathbb{R}^N) \). Then using (2.6), (2.4), Lemma 2.3(h_5), Lemma 2.6, Hölder’s inequality, and the Sobolev embedding, we deduce

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \\
\leq C(N, \mu, \beta) \left\| \epsilon |h(w)|^N + C|h(w)|^N \left[ \exp \left( \alpha |h(w)|^\frac{2N}{2N-2\beta-\mu} \right) - S_{N-2}(\alpha, |h(w)|^2) \right] \right\|^2_{L^{\frac{2N}{2N-2\beta-\mu}}(\mathbb{R}^N)} \\
\leq C(N, \mu, \beta) \left[ 2^{\frac{2N}{2N-2\beta-\mu}} \left\{ \epsilon 2^{\frac{2N}{2N-2\beta-\mu}} \int_{\mathbb{R}^N} |h(w)|^{\frac{2N^2}{2N-2\beta-\mu}} \, dx + C 2^{\frac{2N}{2N-2\beta-\mu}} \int_{\mathbb{R}^N} |h(w)|^{\frac{2N^2}{2N-2\beta-\mu}} \, dx \right\} \right]^{\frac{2N}{2N-2\beta-\mu}} \\
\leq 4C(N, \mu, \beta) \left[ \epsilon^2 \int_{\mathbb{R}^N} |w|^{\frac{2N^2}{2N-2\beta-\mu}} \, dx \right]^{\frac{2N}{2N-2\beta-\mu}} \\
+ C^2 \epsilon \left\{ \int_{\mathbb{R}^N} |w|^{\frac{2N^2}{2N-2\beta-\mu}} \left( \exp \left( \frac{2N\alpha 2^{\frac{1}{N-\tau}}}{2N-2\beta-\mu} |w|^{\frac{N}{N-\tau}} \right) \right. \right. \\
- S_{N-2} \left( \frac{2N\alpha 2^{\frac{1}{N-\tau}}}{2N-2\beta-\mu} |w|^{\frac{N}{N-\tau}} \right) \left. \right\} \right\}^{\frac{2N}{2N-2\beta-\mu}} \\
\leq C_1(N, \mu, \beta, \epsilon) \left[ \|w\|^{2N} + \|w\|^{2p} \int_{\mathbb{R}^N} \left( \exp \left( \frac{4N\alpha 2^{\frac{1}{N-\tau}}}{2N-2\beta-\mu} |w|^{\frac{N}{N-\tau}} \right) \right. \right. \\
- S_{N-2} \left( \frac{4N\alpha 2^{\frac{1}{N-\tau}}}{2N-2\beta-\mu} |w|^{\frac{N}{N-\tau}} \right) \left. \right) \int_{\mathbb{R}^N} \left. \right\}^{\frac{2N}{2N-2\beta-\mu}} \\
\leq C_2(N, \mu, \beta, \epsilon) \left[ \|w\|^{2N} + \|w\|^{2^p} \int_{\mathbb{R}^N} \left( \exp \left( \frac{4N\alpha 2^{\frac{1}{N-\tau}} \|\nabla w\|_{L^N(\mathbb{R}^N)}^{\frac{N}{N-\tau}}}{2N-2\beta-\mu} \right) \right. \right. \\
- S_{N-2} \left( \frac{4N\alpha 2^{\frac{1}{N-\tau}} \|\nabla w\|_{L^N(\mathbb{R}^N)}^{\frac{N}{N-\tau}}}{2N-2\beta-\mu} \right) \left. \right) \int_{\mathbb{R}^N} \left. \right\}^{\frac{2N}{2N-2\beta-\mu}}. \tag{4.4} \]

Now we choose \( w \) with \( \|w\| \) sufficiently small such that

\[
\frac{4N\alpha 2^{\frac{1}{N-\tau}} \|\nabla w\|_{L^N(\mathbb{R}^N)}^{\frac{N}{N-\tau}}}{2N-2\beta-\mu} < \alpha_N.
\]

Then employing Theorem 1.2 in (4.4), we obtain

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx \leq C_2(N, \mu, \beta, \epsilon) (\|w\|^{2N} + \|w\|^{2^p}). \tag{4.5}
\]
Using (2.2), (4.5), (V1) and Lemma 4.1 we have
\[
J(w) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \, dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x)|h(w)|^N \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx
\]
\[
\geq \frac{1}{N} C \|w\|^N - \frac{1}{2} C_2(N, \mu, \beta, \epsilon) (\|w\|^{2N} + \|w\|^{2\ell}).
\]
Now by taking \( r > 0 \) such that \( r > N \), we can choose \( 0 < \lambda_\beta < \min\{1, \lambda_0\} \) (where \( \lambda_0 \) is defined in Lemma 4.1) sufficiently small so that, we finally obtain \( J(w) \geq \tau_\beta > 0 \) for all \( w \in W^{1,N}(\mathbb{R}^N) \) with \( \|w\| = \lambda_\beta \) and for some \( \tau_\beta > 0 \) depending on \( \lambda_\beta \).

\[ \text{Lemma 4.3.} \quad \text{Let (1.1) hold and the function } h \text{ be as in (2.1). Assume that the conditions (V1), (V2) and (f1)-(f5) hold. Then there exists a } w_\beta \in W^{1,N}(\mathbb{R}^N) \text{ with } \|w_\beta\| > \lambda_\beta \text{ such that } J(w_\beta) < 0 \text{, where } \lambda_\beta \text{ is given as in Lemma 4.2.} \]

\[ \text{Proof.} \quad \text{The condition (f5) implies that there exist some positive constant } C_1, C_2 > 0 \text{ such that}
\]
\[
F(x, s) \geq C_1 s^\ell - C_2 \text{ for all } (x, s) \in \mathbb{R}^N \times [0, \infty).
\]

Let \( \phi(\geq 0) \in W^{1,N}(\mathbb{R}^N) \) such that \( \|\phi\| = 1 \) and \( \text{supp}(\phi) = \Omega \). Now by Lemma 2.3, (h6),(h8) and (4.6), for large \( t > 1 \), we obtain
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, h(t\phi))F(y, h(t\phi))}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy
\]
\[
\geq \int_{\Omega} \int_{\Omega} \left( C_1(h(t\phi(y)))^\ell - C_2(C_1(h(t\phi(x)))^\ell - C_2 \right) \frac{dxdy}{|x-y|^\mu}
\]
\[
= C_1^2 \int_{\Omega} \int_{\Omega} \frac{(h(t\phi(y)))^\ell (h(t\phi(x)))^\ell}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy
\]
\[
- 2C_1C_2 \int_{\Omega} \int_{\Omega} \frac{(h(t\phi(x)))^\ell}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy + C_2^2 \int_{\Omega} \int_{\Omega} \frac{1}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy
\]
\[
\geq C_1^2(1)^{2\ell} \int_{\Omega} \int_{\Omega} \frac{(t\phi(x))^{\ell}}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy
\]
\[
- 2C_1C_2 \int_{\Omega} \int_{\Omega} \frac{t^{\ell}}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy + C_2^2 \int_{\Omega} \int_{\Omega} \frac{1}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy.
\]

Using the last relation together with (2.2) and applying Lemma 2.3, we obtain
\[
J(t\phi) \leq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla t\phi|^N \, dx + \frac{1}{N} \|V\|_{L^\infty(\Omega)} \int_{\mathbb{R}^N} |h(t\phi)|^N \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, h(t\phi))F(y, h(t\phi))}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy
\]
\[
\leq \frac{1}{N} \max\{1, \|V\|_{L^\infty(\Omega)}\} \|t\phi\|^N - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, h(t\phi))F(y, h(t\phi))}{|x|^{\beta}|y|^{\beta}|x-y|^\mu} \, dxdy
\]
\[
\leq C_1 t^N - C_2 t^\ell + C_2 t^\ell - C_6,
\]
where \( C_1's \) are positive constants for \( i = 3, 4, 5, 6 \). From (4.7), we infer that \( J(t\phi) \to -\infty \) as \( t \to +\infty \), since \( \ell > N \). Thus, there exists \( t_\beta(>0) \in \mathbb{R} \) so that \( w_\beta := t_\beta \phi \in W^{1,N}(\mathbb{R}^N) \) with \( \|w_\beta\| > \lambda_\beta \) and \( J(w_\beta) < 0 \).

5. Existence of positive weak solutions

In this section, we give the detailed proofs of Theorem 1.4 and Theorem 1.6.

5.1. Compact-Coercive case. Here we concentrate upon proving Theorem 1.4 with the potential function \( V \) following the assumptions (V1) and (V2). For that we define the energy functional \( E := J|_E : E \to \mathbb{R} \) associated to (2.8) as
\[
E(w) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \, dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x)|h(w)|^N \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta |x-y|^\mu} \, dy \right) \frac{F(x, h(w))}{|x|^\beta} \, dx
\]
for any \( w \in E \). We know that \( E \subset C^1(E, \mathbb{R}) \). The following lemma shows that \( E \) has the mountain pass geometry.
Lemma 5.1. Let \([1.1] \) hold and the function \(h\) be defined as in \([1.5] \). Suppose that the conditions \((V_1), (V_2)\) and \((f_1)\)-(\(f_5\)) hold. Then we have the following assertions:

(i) there exist \(\lambda_\beta^* > 0\) and \(\tau_\beta^* > 0\) such that \(\mathcal{E}(w) \geq \tau_\beta^* > 0\) for any \(w \in E\) with \(\|w\|_E = \lambda_\beta^*\).

(ii) there exists a \(w_\beta^* \in E\) with \(\|w_\beta^*\|_E > \lambda_\beta^*\) such that \(\mathcal{E}(w_\beta^*) < 0\).

Proof. With some minute modifications in the proof of lemmas \(4.2\) and \(4.3\) by replacing \(\|\|\) by \(\|\|_E\) the proofs of (i) and (ii) follows respectively.

Next, we define the mountain pass level

\[
\theta_* := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{E}(\gamma(t)),
\]

where \(\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \mathcal{E}(\gamma(1)) < 0\}\). By the previous lemma \(\theta_* > 0\). Now for some fixed \(\delta > 0\) we define the sequence of Moser functions \(\{\mathcal{M}_k\}\) as:

\[
\mathcal{M}_k(x, \delta) = \begin{cases} 
\frac{1}{\omega_{N-1}} \left( \frac{\log k}{(\log k)^{\frac{1}{N}}} \right), & \text{if } 0 \leq |x| \leq \frac{\delta}{k}, \\
\frac{1}{\omega_{N-1}} \left( \frac{\frac{\delta}{2^k}}{(\log k)^{\frac{1}{N}}} \right), & \text{if } \frac{\delta}{k} \leq |x| \leq \delta, \\
0, & \text{if } |x| \geq \delta. 
\end{cases}
\]

Then, \(\text{supp}(\mathcal{M}_k) \subset B_\delta(0)\) with \(\mathcal{M}_k(\cdot, \delta) \in W^{1,N}(\mathbb{R}^N)\) and . Moreover, \(\int_{\mathbb{R}^N} |\nabla \mathcal{M}_k(x, \delta)|^N dx = 1\) and \(\int_{\mathbb{R}^N} |\nabla \mathcal{M}_k(x, \delta)|^N dx = O\left(\frac{1}{\log k}\right)\) as \(k \to +\infty\). Also, \(\int_{\mathbb{R}^N} \mathcal{M}_k(x, \delta) |V(x)|^N dx = \frac{1}{\log k}\) as \(k \to +\infty\). Set

\[
\mathcal{M}_k(x, \delta) := \frac{\mathcal{M}_k(x, \delta)}{\|\mathcal{M}_k\|_E}.
\]

By a simple computation, we can derive that

\[
\mathcal{M}_k(x, \delta) = \frac{1}{\omega_{N-1}} \log k + a_k, \quad \text{for all } |x| \leq \frac{\delta}{k},
\]

where \(\{a_k\}\) is a bounded sequence of non-negative real numbers.

Lemma 5.2. Let \([1.1] \) hold and the function \(h\) be defined in \(2.1\). If \(V\) satisfies \((V_1), (V_2)\) and \(f\) satisfies the assumptions \((f_1)\)-(\(f_6\)), then there exists some \(k \in \mathbb{N}\) such that

\[
0 < \theta_* < \frac{1}{2N} \left( \frac{2N - 2\beta - \mu}{2N} \cdot \frac{\alpha_N}{\alpha_0} \right)^{N-1},
\]

where \(\theta_*\) is defined in \(5.2\).

Proof. We argue this proof by contradiction. Suppose \(5.4\) doesn’t hold. Then for given any \(k \in \mathbb{N}\), there exists \(t_k > 0\) such that

\[
\max_{t \in [0,\infty)} \mathcal{E}(t \mathcal{M}_k) = \mathcal{E}(t_k \mathcal{M}_k) \geq \frac{1}{2N} \left( \frac{2N - 2\beta - \mu}{2N} \cdot \frac{\alpha_N}{\alpha_0} \right)^{N-1}.
\]

Since by \((f_1)\), \(F(x, h(t_k \mathcal{M}_k)) \geq 0\) for all \(k \in \mathbb{N}\), using \(5.1\), \(5.5\), Lemma \(2.3\) \((h_5)\) and the fact that \(\|\mathcal{M}_k\|_E = 1\), we get

\[
t_k^N \geq \frac{1}{2} \left( \frac{2N - 2\beta - \mu}{2N} \cdot \frac{\alpha_N}{\alpha_0} \right)^{N-1} > 0.
\]
Also, from (5.5), it follows that $\frac{d}{dt}(J(tM_k))|_{t=k} = 0$. Combining this with Lemma 2.3(h), we obtain

\[
t_k^N = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(tM_k))}{|y|^\beta |x-y|\mu} \right) \frac{f(x, h(tM_k))}{|x|^{\beta}} h'(tM_k) tM_k \, dx
\]

Then by (1.5), for each $p > 0$ there exists a constant $R_p$ such that

\[
\text{s.f.}(x,s)F(x,s) \geq p \exp \left(2\alpha_0 |s| \frac{2N}{\pi^\mu} \right), \quad \text{whenever } s \geq R_p.
\]

The relation (5.6) gives that $tM_k \to +\infty$ as $k \to +\infty$ in $B_{\delta/k}(0)$. Then by Lemma 2.3(h7), for any given $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

\[
|h(tM_k)|^{\frac{2N}{\pi^\mu}} \geq (tM_k)^{\frac{N}{\pi^\mu}} (2\pi^\mu - \epsilon).
\]

Moreover, Lemma 2.3(h8) yields that $h(tM_k) \to +\infty$ as $k \to +\infty$, uniformly in $B_{\delta/k}(0)$. Hence, we can choose $s_p \in \mathbb{N}$ such that in $B_{\delta/k}(0)$, it holds that

\[
h(tM_k) \geq R_p, \quad \text{for all } k \geq s_p.
\]

On the other hand, using the same idea as in 4 (see equation (2.11)), we can deduce

\[
\int_{B_{\delta/k}(0)} \int_{B_{\delta/k}(0)} \frac{dxdy}{|x|^{\beta} |y|^\mu |x-y|^{\alpha}} \geq C_{\mu,\beta,N} \left( \frac{\delta}{k} \right)^{2N-2\beta - \mu},
\]

Using all these above estimates, from (5.7), for sufficiently large $p > 0$ and sufficiently large $k \in \mathbb{N}$, we get

\[
t_k^N \geq \frac{p}{2} \int_{B_{\delta/k}(0)} \int_{B_{\delta/k}(0)} \exp \left(2\alpha_0 (h(tM_k))^{\frac{2N}{\pi^\mu}} \right) \frac{dxdy}{|x|^{\beta} |y|^\mu |x-y|^{\alpha}}
\]

\[
\geq \frac{p}{2} \int_{B_{\delta/k}(0)} \int_{B_{\delta/k}(0)} \exp \left(2\alpha_0 (tM_k)^{\frac{N}{\pi^\mu}} (2\pi^\mu - \epsilon) tK_k a_k \right) C_{\mu,\beta,N} \left( \frac{\delta}{k} \right)^{2N-2\beta - \mu}
\]

\[
= \frac{p}{2} \exp \left( \log k \frac{2\alpha_0 (2\pi^\mu - \epsilon) tK_k^{\frac{N}{\pi^\mu}}}{\omega_{N-1}} \right) + 2\alpha_0 (2\pi^\mu - \epsilon) tK_k^{\frac{N}{\pi^\mu}} a_k \right) C_{\mu,\beta,N} \left( \frac{\delta}{k} \right)^{2N-2\beta - \mu}
\]

\[
= \frac{p}{2} \exp \left( \log k \left( \frac{2\alpha_0 (2\pi^\mu - \epsilon) tK_k^{\frac{N}{\pi^\mu}}}{\omega_{N-1}} \right) - (2N - 2\beta - \mu) \right) + 2\alpha_0 (2\pi^\mu - \epsilon) tK_k^{\frac{N}{\pi^\mu}} a_k \right) C_{\mu,\beta,N} \delta^{2N-2\beta - \mu}.
\]

Therefore, we have

\[
1 \leq \frac{p}{2} \delta^{2N-2\beta - \mu} C_{\mu,\beta,N} \exp \left( \log k \left( \frac{2\alpha_0 (2\pi^\mu - \epsilon) tK_k^{\frac{N}{\pi^\mu}}}{\omega_{N-1}} \right) - (2N - 2\beta - \mu) \right) + 2\alpha_0 (2\pi^\mu - \epsilon) tK_k^{\frac{N}{\pi^\mu}} a_k - N \log t_k,
\]

which implies that the sequence $\{t_k\}$ must be bounded. If not, then $t_k \to +\infty$, which implies that the right hand side of the last inequality goes to $\infty$, as $k \to +\infty$, which is absurd. So, up to a subsequence, still denoted by $\{t_k\}$, $t_k \to \tilde{t}$ as $k \to +\infty$, for some $\tilde{t} \in \mathbb{R}$. Now we claim that as $k \to +\infty$,

\[
t_k^{\frac{N}{\pi^\mu}} \to \frac{1}{2\pi^\mu} \left( \frac{2N - 2\beta - \mu}{2N} \right) \frac{\alpha_N}{\alpha_0}.
\]
Indeed, if not, then there exists some \( \eta > 0 \) such that for sufficiently large \( k \in \mathbb{N} \), we have
\[
\frac{t_k^N}{\eta} \geq \frac{1}{2\pi^2} \left( \frac{2N - 2\beta - \mu}{2N} \right) \left( \frac{\alpha_N \omega_{N-1}}{\alpha_0} \right).
\]
By plugging the last relation in (5.8), and using the fact that \( \{a_k\} \) is a bounded sequence of non-negative real numbers, we obtain
\[
t_k^N \geq \frac{p}{2} C_{\mu, \beta, N} \delta^{2N - 2\beta - \mu} \exp \left( \log k \left[ \left( \frac{2\alpha_0 (2\pi^{-1} - \epsilon) \eta}{(2N - 2\beta - \mu) \omega_{N-1}} + \frac{2\pi^{-1} - \epsilon}{2\pi^{-1}} \right) - 1 \right] (2N - 2\beta - \mu) \right)
\]
and thus, we achieve
\[
t_k^N \geq \frac{p}{2} C_{\mu, \beta, N} \delta^{2N - 2\beta - \mu} \exp \left( \log k \left[ \left( \frac{2\alpha_0 (2\pi^{-1} - \epsilon) \eta}{(2N - 2\beta - \mu) \omega_{N-1}} + \frac{2\pi^{-1} - \epsilon}{2\pi^{-1}} \right) - 1 \right] (2N - 2\beta - \mu) \right)
\]
\[
\rightarrow +\infty
\]
as \( k \rightarrow +\infty \), which is a contradiction. Therefore, (5.9) holds. Now letting limit \( \epsilon \rightarrow 0^+ \) in the relation (5.8), we get
\[
t_k^N \geq \frac{p}{2} \exp \left( \log k \left[ \left( \frac{2\alpha_0 (2\pi^{-1} - \epsilon) \eta}{(2N - 2\beta - \mu) \omega_{N-1}} + \frac{2\pi^{-1} - \epsilon}{2\pi^{-1}} \right) - 1 \right] (2N - 2\beta - \mu) \right)
\]
which together with (5.6), and the fact that \( \{a_k\} \) is a non-negative bounded sequence yields that
\[
t_k^N \geq \frac{p}{2} C_{\mu, \beta, N} \delta^{2N - 2\beta - \mu},
\]
Taking \( k \rightarrow +\infty \) in the last relation, and using (5.9), we deduce
\[
\frac{1}{2\pi^2} \left( \frac{2N - 2\beta - \mu}{2N} \right) \left( \frac{\alpha_N \omega_{N-1}}{\alpha_0} \right) \geq \frac{p}{2} C_{\mu, \beta, N} \delta^{2N - 2\beta - \mu},
\]
which is a contradiction, since \( p > 0 \) can be chosen arbitrarily. This completes the proof of the lemma. \( \square \)

Proof of Theorem 1.4. Since \( \mathcal{E} \) satisfies Lemma 5.1 by mountain pass theorem, there exists a Cerami sequence \( \{w_k\} \) in \( E(\subset W^{1, N}(\mathbb{R}^N)) \) for \( \mathcal{E} \) at level \( \theta_* \), that is,
\[
\mathcal{E}(w_k) \rightarrow \theta_*; \text{ and } (1 + \|w_k\|_E) \mathcal{E}'(w_k) \rightarrow 0 \text{ in } E^* \text{ as } k \rightarrow +\infty. \quad (5.10)
\]
Also, Lemma 5.1(ii) yields that \( \theta_* > 0 \). Now by Lemma 3.1, we have that \( \{w_k\} \) is bounded in \( W^{1, N}(\mathbb{R}^N) \). Hence, up to a subsequence, still denoted by \( \{w_k\} \), there exists some \( w \in W^{1, N}(\mathbb{R}^N) \) such that \( w_k \rightharpoonup w \) weakly in \( W^{1, N}(\mathbb{R}^N) \) as \( k \rightarrow +\infty \). Now using Lemma 3.4-3.6, we deduce that \( w \) satisfies (2.7) for all \( v \in W^{1, N}(\mathbb{R}^N) \) since \( C_\infty(\mathbb{R}^N) \) is dense in \( W^{1, N}(\mathbb{R}^N) \). So, \( w \) is a weak solution to (2.8).
Next, we show that \( w \) is nontrivial. Suppose on the contrary, the weak solution we obtain \( w \in W^{1, N}(\mathbb{R}^N) \) is trivial,
that is \( \psi = 0 \). Then, from Lemma 3.2, we have

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x - y|^\mu} dy \right) \frac{F(x, h(w_k))}{|x|^\beta} \, dx \to 0 \text{ as } k \to +\infty,
\]

(5.11)

where we used the fact that \( h(w_k) \to 0 \) strongly in \( L^{2,2N-2\beta-\mu}_r(\mathbb{R}^N) \), as \( \frac{2N^2}{2N-2\beta-\mu} > N \). Since \( E \) is compactly embedded into \( L^N(\mathbb{R}^N) \), we have \( \int_{\mathbb{R}^N} V(x)|w_k|^N \, dx \to 0 \text{ as } k \to +\infty \). This together with (5.11) imply that

\[
\theta_* = \lim_{k \to +\infty} \mathcal{E}(w_k) = \frac{1}{N} \lim_{k \to +\infty} \int_{\mathbb{R}^N} |\nabla w_k|^N,
\]

that is,

\[
\lim_{k \to +\infty} \|\nabla w_k\|_{L^N(\mathbb{R}^N)}^N = \theta_* N.
\]

Therefore, there exists a real number \( l \in (0, 1) \), and corresponding to that \( l \), there exists \( k_0 \in \mathbb{N} \) such that

\[
\|\nabla w_k\|_{L^N(\mathbb{R}^N)}^N < \frac{2N - 2\beta - \mu}{2N} \cdot \frac{\alpha_N}{\alpha_0} (1 - l), \text{ for all } k \geq k_0.
\]

(5.12)

Next, we show that

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x - y|^\mu} dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx \to 0 \text{ as } k \to +\infty.
\]

(5.13)

By the assumptions, (f2)-(f3), for any \( \epsilon > 0 \), \( r \geq N \), there exist constants \( C > 0 \), \( \alpha > \alpha_0 > 0 \) such that

\[
|f(x, s)| \leq \epsilon |s|^N + C(r, \epsilon) |s|^r \exp \left( \alpha |s|^{\frac{2N}{2N - 2\beta - \mu}} \right) - S_{N-2}(\alpha, s^2), \text{ for all } (x, s) \in \mathbb{R}^N \times \mathbb{R}.
\]

(5.14)

Now using (f5), Proposition 1.1 with \( t = s, \beta = \nu, \) (5.14), Lemma 2.6 Lemma 2.5 Lemma 2.3(4) and Hölder’s inequality, we deduce

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta |x - y|^\mu} dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) w_k \, dx
\]

\[
\leq \ell \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{f(y, h(w_k)) h'(w_k)}{|y|^\beta |x - y|^\mu} dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h(w_k) \, dx
\]

\[
\leq \ell C(\ell, N, \mu, \beta) \left( \int_{\mathbb{R}^N} |f(x, h(w_k)) h'(w_k)|^{\frac{2N}{2N - 2\beta - \mu}} \, dx \right)^{\frac{2N - 2\beta - \mu}{2N}}
\]

\[
\leq C(\ell, N, \mu, \beta, \epsilon) \left[ \|h(w_k)\|_{L^{2N}_r(\mathbb{R}^N)}^{2N} + \|h(w_k)\|_{L^{2N-2\beta-\mu}_r(\mathbb{R}^N)}^{2N-2\beta-\mu} \left( \int_{\mathbb{R}^N} \exp \left( \frac{2N \alpha p}{2N - 2\beta - \mu} |h(w_k)|^{\frac{2N}{2N - 2\beta - \mu}} \right) \, dx \right)^{\frac{2N - 2\beta - \mu}{2N}} \right]
\]

\[
\leq C(\ell, N, \mu, \beta, \epsilon, r, p) \left[ \|h(w_k)\|_{L^{2N}_r(\mathbb{R}^N)}^{2N} + \|h(w_k)\|_{L^{2N-2\beta-\mu}_r(\mathbb{R}^N)}^{2N-2\beta-\mu} \left( \int_{\mathbb{R}^N} \exp \left( \frac{2N \alpha p}{2N - 2\beta - \mu} \|\nabla w_k\|_{L^N(\mathbb{R}^N)}^N \right) \, dx \right)^{\frac{2N - 2\beta - \mu}{2N}} \right] - S_{N-2} \left( \frac{2N \alpha p}{2N - 2\beta - \mu}, \|h(w_k)\|^2 \right)
\]

(5.15)

Recalling (5.12) and by choosing \( p > 1 \) sufficiently close to 1 and choosing \( \alpha > \alpha_0 \), very close to \( \alpha_0 \), we can have

\[
\frac{2N \alpha p}{2N - 2\beta - \mu} \|\nabla w_k\|_{L^N(\mathbb{R}^N)}^N < p \alpha N \alpha_0 (1 - l) < \alpha_N.
\]
Therefore, in view of Theorem 1.2 and using (5.22), for sufficiently large \( k \in \mathbb{N} \), from (5.15), we get
\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) y_k \, dx < C,
\]
where the constant \( C \) is independent of \( k \). Thus, by employing Vitali’s convergence theorem, we obtain
\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_k))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w_k))}{|x|^\beta} h'(w_k) y_k \, dx \rightarrow 0 \quad \text{as} \quad k \to +\infty.
\]
Since, \( \{w_k\} \) is a Cerami sequence for \( \mathcal{E} \), we have \( \lim_{k \to +\infty} \langle \mathcal{E}'(w_k), y_k \rangle_E = 0 \) which together with (5.13) implies that
\[
\int_{\mathbb{R}^N} |\nabla w_k|^N \, dx + \int_{\mathbb{R}^N} V(x)|h(w_k)|^N \, dx \to 0 \quad \text{as} \quad k \to +\infty.
\]
Invoking this and (5.11) in (5.10), we obtain \( \theta_* = \lim_{k \to +\infty} \mathcal{E}(w_k) = 0 \) which contradicts the fact that \( \theta_* > 0 \). Thus, \( w \not\equiv 0 \). Since \( w \) is a weak solution to (2.8), it satisfies
\[
\int_{\mathbb{R}^N} |\nabla w|^{N-2} \nabla w \nabla v \, dx + \int_{\mathbb{R}^N} V(x)|h(w)|^{N-2} h(w) h'(w) v \, dx = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w))}{|x|^\beta} h'(w) v \, dx,
\]
for all \( v \in W^{1,N}(\mathbb{R}^N) \). Now choose \( v = w^- := \max\{-w, 0\} \in W^{1,N}(\mathbb{R}^N) \) in the last equation. Then (f1), (V1) and Lemma 2.3 (h5) yield that \( \|w^-\| \leq 0 \) and hence, \( w^- = 0 \) a.e. in \( \mathbb{R}^N \). Therefore, \( w \geq 0 \) a.e. in \( \mathbb{R}^N \). Now we can rearrange the equation (2.8) as
\[
-\Delta_N w + V(x)|w|^{N-2} w = V(x)\left(|w|^{N-2} w - |h(w)|^{N-2} h(w) h'(w)\right) + \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \right) F(x, h(w)) \, dx.
\]
By (V1), Lemma 2.3 (h3), (h5) and (f1) we infer that the right hand side of this last equation is non negative. We claim that \( w > 0 \). Suppose not, then there exists at least one point, say \( x_0 \in \mathbb{R}^N \) such that \( w(x_0) = 0 \). Now Lemma 3.3 yields that
\[
\int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \in L^\infty(\mathbb{R}^N).
\]
So, using the fact that \( h'(w) \leq 1 \), and by recalling standard regularity results for the elliptic equations, we infer that \( \tilde{w} \in L^\infty_{loc}(\mathbb{R}^N) \cap C^0_{loc}(\mathbb{R}^N) \) for some \( \xi \in (0, 1) \). Therefore, strong maximum principle implies that \( w = 0 \) in \( B_r(x_0) \subset \mathbb{R}^N \) for all \( r > 0 \). So, \( w \equiv 0 \) in \( \mathbb{R}^N \) and this contradicts the fact that \( w \) is non trivial. Thus, \( w > 0 \) and consequently, \( h(w) > 0 \) which is a positive solution to (P*). Hence, the proof of Theorem 1.1 is complete.

5.2. Non-compact case. Here we aim to prove Theorem 1.6. So, we consider the problem (P*’), and the transformed equation (2.8), for \( \beta = 0 \). The potential function \( V \) satisfies (V1) and (V2’). The condition (V2) is a weaker condition compared to (V2’) and induces lack of compactness in the embedding from \( W^{1,N}(\mathbb{R}^N) \) into \( L^N(\mathbb{R}^N) \). Therefore, we need to carry out technically involved analysis to ensure the existence of nontrivial positive solution to (2.8). For the brevity, we still denote the energy functional associated to (2.8) with \( \beta = 0 \) as \( J : W^{1,N}(\mathbb{R}^N) \to \mathbb{R} \), which is defined as
\[
J(w) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \, dx + \frac{1}{N} \int_{\mathbb{R}^N} V(x)|h(w)|^N \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \right) F(x, h(w)) \, dx. \tag{5.16}
\]
Clearly \( J \) follows mountain pass geometry (that is, Lemma 4.2 and Lemma 1.3 with \( \beta = 0 \)) near \( 0 \).

Let \( \Gamma = \{ \gamma \in C([0,1], W^{1,N}(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0 \} \). Define the mountain pass level
\[
\theta_* := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)). \tag{5.17}
\]
Then by the mountain pass theorem, there exists a Cerami sequence \( \{w_k\} \subset W^{1,N}(\mathbb{R}^N) \) for \( J \) at level \( \theta_* \), that is, as \( k \to +\infty \)
\[
J(w_k) \to \theta_*; \text{ and } (1 + \|w_k\|)J'(w_k) \to 0 \quad \text{in} \quad (W^{1,N}(\mathbb{R}^N))^*. \tag{5.18}
\]
Also by Lemma 4.2 we get \( \theta_c > 0 \). Moreover Lemma 3.1 ensures that \( \{ w_k \} \) is bounded in \( W^{1,N}(\mathbb{R}^N) \) and hence, up to a subsequence, still denoted by \( \{ w_k \} \), \( w_k \to w \) weakly in \( W^{1,N}(\mathbb{R}^N) \) as \( k \to +\infty \) for some \( w \in W^{1,N}(\mathbb{R}^N) \). Now Lemma 3.4−3.6 yield that \( w \) satisfies (2.7) for all \( v \in W^{1,N}(\mathbb{R}^N) \). So, \( w \) is a weak solution to (2.8). Our next goal is to show that \( w \) is non-trivial and positive. For that, we construct the arguments in the next lemmas, by assuming that \( w \equiv 0 \) and finally, we arrive at some contradiction.

We begin with analyzing the critical mountain pass level associated with the following newly defined functional

\[
J_\infty(w) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \, dx + \frac{1}{N} \int_{\mathbb{R}^N} V_\infty |w|^N \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w))}{|x-y|^\mu} \, dy \right) F(x,h(w)) \, dx
\]

(5.19)

in the next lemma. There we consider the following equivalent norm on \( W^{1,N}(\mathbb{R}^N) \), still denoted by \( \| \cdot \| \), defined as

\[
\| v \| = \left( \int_{\mathbb{R}^N} |\nabla v|^N \, dx + V_\infty \int_{\mathbb{R}^N} |v|^N \, dx \right)^{\frac{1}{N}}, \quad \text{for} \ v \in W^{1,N}(\mathbb{R}^N).
\]

One can easily see that \( J_\infty \in C^1(W^{1,N}(\mathbb{R}^N),\mathbb{R}) \). Also, we can show that \( J_\infty \) satisfies the mountain pass geometry as similar to Lemma 4.2 and Lemma 4.3 with some minute modifications in proofs of both the lemmas.

For some fixed \( \delta > 0 \), let us set

\[
\tilde{\mathcal{M}}_k(x,\delta) := \| \tilde{M}_k(x,\delta) \|,
\]

where \( \{ \tilde{M}_k \} \) is the sequence of the Moser functions as defined in (5.3).

**Lemma 5.3.** Let the conditions in Theorem 1.6 hold. Then, then exists some \( k \in \mathbb{N} \) such that

\[
\max_{t \in [0,\infty)} J_\infty(t\tilde{M}_k) < d_c := \frac{1}{2N} \left( \frac{2N-\mu}{2N} \cdot \frac{\alpha N}{\alpha_0} \right)^{N-1}.
\]

(5.20)

**Proof.** The proof of the lemma follows in a similar manner as in the proof of Lemma 5.2 by taking \( \beta = 0 \). \( \square \)

**Remark 5.4.** The above lemma gives that \( \theta_c < d_c := \frac{1}{2N} \left( \frac{2N-\mu}{2N} \cdot \frac{\alpha N}{\alpha_0} \right)^{N-1} \), where \( \theta_c \) is defined in (5.17). Indeed, using (\( V_2' \)) we get that \( J(w) \leq J_\infty(w) \) for all \( w \in W^{1,N}(\mathbb{R}^N) \). Then by using Lemma 5.3 we get the desired estimate.

In the next lemma, we show the non-vanishing behaviour of the Cerami sequence \( \{ w_k \} \), which is defined in (5.18).

**Lemma 5.5.** Let the conditions in Theorem 1.6 hold. Then, there exist positive constants \( b, R \) and a sequence \( \{ z_k \} \subset \mathbb{R}^N \) such that

\[
\lim_{k \to +\infty} \int_{B_R(z_k)} |h(w_k)|^N \geq b > 0.
\]

(5.21)

**Proof.** Suppose on the contrary, (5.21) does not hold. Then we have

\[
\lim_{k \to +\infty} \sup_{z \in \mathbb{R}^N} \int_{B_R(z)} |h(w_k)|^N = 0.
\]

This, together with Lions compactness lemma (see Lemma I.1 in [27]) implies that, as \( k \to +\infty \)

\[
h(w_k) \to 0 \quad \text{strongly in} \quad L^q(\mathbb{R}^N), \quad \text{for all} \quad q \in (N,\infty).
\]

(5.22)

Therefore, \( h(w_k(x)) \to 0 \) a.e. in \( \mathbb{R}^N \). Using this together with (\( f_2 \)), for sufficiently large \( k \in \mathbb{N} \), we have

\[
|F(x,h(w_k))| \leq C|h(w_k)|^N
\]

for some positive constant \( C > 0 \). So, by (2.6) with \( \beta = 0 \), we get

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w_k))}{|x-y|^\mu} \, dy \right) F(x,h(w_k)) \, dx \leq C(N,\mu,\beta)\| h(w_k) \|^2_{L^{\frac{2N^2}{2N-\mu}}(\mathbb{R}^N)} \to 0 \quad \text{as} \quad k \to +\infty,
\]

(5.23)

in the last line we used the fact that \( h(w_k) \to 0 \) strongly in \( L^{\frac{2N^2}{2N-\mu}}(\mathbb{R}^N) \), since \( \frac{2N^2}{2N-\mu} > N \). This yields that

\[
\theta_c = \lim_{k \to +\infty} J(w_k) = \frac{1}{N} \lim_{k \to +\infty} \left( \| \nabla w_k \|^N_{L^N(\mathbb{R}^N)} + \int_{\Omega} V(x)|h(w_k)|^N \, dx \right) \geq \frac{1}{N} \lim_{k \to +\infty} \| \nabla w_k \|^N_{L^N(\mathbb{R}^N)}.
\]
That is,  
\[ \lim_{k \to +\infty} \|\nabla w_k\|_{L^N(\mathbb{R}^N)}^N \leq \theta_c N. \]

Now using the similar arguments, we get (5.13) for \( \beta = 0 \). Using this and the fact that \( \lim_{k \to +\infty} \langle J(w_k), w_k \rangle = 0 \), we obtain
\[ \int_{\mathbb{R}^N} |\nabla w_k|^N \, dx + \int_{\mathbb{R}^N} V(x)|h(w_k)|^N \, dx \to 0 \quad \text{as} \quad k \to +\infty. \]

Plugging this and (5.23) in (5.18), it follows that \( \theta_c = \lim_{k \to +\infty} J(w_k) = 0 \) which is a contradiction to the fact that \( \theta_c > 0 \). The proof is completed. \( \square \)

Next, we define the functional \( I_\infty : W^{1,N}(\mathbb{R}^N) \to \mathbb{R} \) as
\[ I_\infty(w) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \, dx - \frac{1}{N} \int_{\mathbb{R}^N} V(x)|h(w)|^N \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y,h(w))}{|x-y|^\mu} \, dy \right) F(x,h(w)) \, dx. \] (5.24)

Clearly \( I_\infty \in C^1(W^{1,N}(\mathbb{R}^N), \mathbb{R}) \).

**Lemma 5.6.** Let the conditions in Theorem 1.6 hold. Then, the sequence \( \{w_k\} \) is a Palais-Smale sequence for \( I_\infty \), where \( \{w_k\} \) is defined as in (5.18).

**Proof.** By (V2), for any given \( \epsilon > 0 \), there exists a constant \( l > 0 \) such that for all \( |x| \geq l \), we have  
\[ |V(x) - V_\infty| < \epsilon, \]

which implies
\[ |I_\infty(w_k) - J(w_k)| = \frac{1}{N} \int_{B_{c}(0)} |V(x) - V_\infty||h(w_k)|^N \, dx + \frac{1}{N} \int_{\mathbb{R}^N \setminus B_{c}(0)} |V(x) - V_\infty||h(w_k)|^N \, dx \]
\[ \leq \frac{1}{N} \max_{x \in B_l(0)} |V(x) - V_\infty| \int_{B_l(0)} |h(w_k)|^N \, dx + \frac{1}{N} \epsilon \int_{\mathbb{R}^N \setminus B_l(0)} |h(w_k)|^N \, dx \]
\[ \leq o_k(1), \]

where in the last inequality we used the fact that the embedding \( W^{1,N}(B_l(0)) \hookrightarrow L^N_{\infty}(B_l(0)) \) is compact. Thus
\[ I_\infty(w) \to \theta_c \quad \text{as} \quad k \to +\infty. \]

Arguing similarly, we get
\[ \sup_{\zeta \in W^{1,N}(\mathbb{R}^N), \|\zeta\| \leq 1} |\langle I'_\infty(w_k) - J'(w_k), \zeta \rangle| = \sup_{\zeta \in W^{1,N}(\mathbb{R}^N), \|\zeta\| \leq 1} \left| \int_{\mathbb{R}^N} (V(x) - V_\infty)|h(w_k)|^{N-2}h(w_k)h'(w_k) \zeta \, dx \right| \]
\[ = o_k(1). \]

Hence \( I'_\infty \to 0 \) in \( (W^{1,N}(\mathbb{R}^N))^* \) as \( k \to +\infty \). This proves the lemma. \( \square \)

Now we define \( \tilde{w}_k(x) = w_k(x + z_k) \), where the sequences \( \{z_k\} \) and \( \{w_k\} \) are defined in Lemma 5.5 and in (5.18), respectively. Then \( \{\tilde{w}_k\} \) is bounded in \( W^{1,N}(\mathbb{R}^N) \) and hence, there is some \( \tilde{w} \) such that up to a subsequence, \( \tilde{w}_k \to \tilde{w} \) as \( k \to +\infty \). Also, it can easily be computed that, as \( k \to +\infty \),
\[ I_\infty(\tilde{w}_k) = I_\infty(w_k) \to \theta_c \quad \text{and} \quad I'_\infty(\tilde{w}_k) \to 0 \quad \text{in} \quad (W^{1,N}(\mathbb{R}^N))^*. \]

Therefore, following the previous arguments, we get that \( \tilde{w} \) is a critical point of the functional \( I_\infty \). Thus, using Lemma 5.5 and Lemma 2.3 (h5), we get
\[ \int_{B_{c}(0)} |\tilde{w}|^N \, dx = \lim_{k \to +\infty} \int_{B_{c}(0)} |\tilde{w}_k|^N \, dx \leq \lim_{k \to +\infty} \int_{B_{c}(0)} |w|^N \, dx \geq \lim_{k \to +\infty} \int_{B_{c}(0)} |h(w)|^N \, dx \geq b > 0. \]

Hence, \( \tilde{w} \neq 0 \).
Lemma 5.7. Suppose the conditions in Theorem 1.6 hold. Let \( \theta_\infty := \inf \max \Gamma_\infty(\gamma(t)) \) and \( \Gamma_\infty := \{ \gamma \in C([0, 1], W^{1,N}(\mathbb{R}^N)) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0 \} \). Then,
\[
\theta_\infty \leq I_\infty(\tilde{w}) \leq \theta_c,
\]
where \( \theta_c \) is defined in (5.17).

Proof. First, we show that \( I_\infty(\tilde{w}) \leq \theta_c \). For that, using Lemma 2.3 \((h_4)\) and \((f_5)\), we obtain
\[
\frac{1}{N} f(x, h(\tilde{w}_k)) h'(\tilde{w}_k) \tilde{w}_k - \frac{1}{2} F(x, h(\tilde{w}_k)) \geq \frac{1}{2N} f(x, h(\tilde{w}_k)) h(\tilde{w}_k) - \frac{1}{2} F(x, h(\tilde{w}_k)) \geq 0.
\]
This together with Lemma 2.3 \((h_8)\) and Fatou’s lemma implies that
\[
\theta_c = \lim_{k \to +\infty} \inf \left( I_\infty(\tilde{w}_k) - \frac{1}{N} I'_\infty(\tilde{w}_k), \tilde{w}_k \right)
\]
\[
= \lim_{k \to +\infty} \inf \left( \int_{\mathbb{R}^N} \frac{1}{N} V_\infty(h(\tilde{w}_k)) |N - h(\tilde{w}_k)|^{N-2} h(\tilde{w}_k) h'(\tilde{w}_k) \tilde{w}_k \, dx 
\right.
\]
\[
+ \left. \int_{\mathbb{R}^N} \frac{1}{N} F(y, h(\tilde{w}_k)) \frac{|x - y|^{1-\mu}}{y^{1-\mu}} \, dx \right)
\]
\[
\geq \int_{\mathbb{R}^N} \frac{1}{N} V_\infty(h(\tilde{w})) |N - h(\tilde{w})|^{N-2} h(\tilde{w}) h'(\tilde{w}) \tilde{w} \, dx
\]
\[
+ \int_{\mathbb{R}^N} \frac{1}{N} F(y, h(\tilde{w})) \frac{|x - y|^{1-\mu}}{y^{1-\mu}} \, dx
\]
\[
= I_\infty(\tilde{w}) - \frac{1}{N} I'_\infty(\tilde{w}), \tilde{w}
\]
\[
= I_\infty(\tilde{w}),
\]
since \( \tilde{w} \) is a critical point of the functional \( I_\infty \). Hence, \( I_\infty(\tilde{w}) \leq \theta_c \).

Next, we use the approach as in [16] for showing \( I_\infty(\tilde{w}) \geq \theta_\infty \). For that, we construct the following arguments to achieve a suitable path \( \gamma : [0, 1] \to W^{1,N}(\mathbb{R}^N) \) such that
\[
\left\{ \begin{array}{l}
\gamma(0) = 0, \quad I_\infty(\gamma(1)) < 0, \quad \tilde{w} \in \gamma([0, 1]), \\
\max_{t \in [0, 1]} I_\infty(\gamma(t)) = I_\infty(\tilde{w}).
\end{array} \right.
\]
So, we define
\[
\tilde{w}_t(x) := \begin{cases} 
\tilde{w}(x/t), & \text{if } t > 0, \\
0, & \text{if } t = 0.
\end{cases}
\]
Our next aim is to choose the real numbers \( 0 < t_1 < 1 < t_2 < s_0 \) such that the path \( \gamma \) defined by three pieces in the below is turned out to be our desired path:
\[
\gamma(s) = \begin{cases} 
s\tilde{w}_{t_1}(x), & \text{if } s \in [0, t_1], \\
s\tilde{w}_t(x), & \text{if } s \in [t_1, t_2], \\
s\tilde{w}_{t_2}(x), & \text{if } s \in [t_2, s_0].
\end{cases}
\]
Let us define the function \( g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) as
\[
g(x, s) := h'(s) \left( -V_\infty h(s)|N-2 h(s) + \left( \int_{\mathbb{R}^N} \frac{F(y, h(s))}{|x - y|^{1-\mu}} \, dy \right) f(x, h(s)) \right).
\]
Since \( \tilde{w} \) is a critical point of \( I_\infty \), \( \tilde{w} \) is solution to the following problem
\[
-\Delta_N w = g(x, w) \quad \text{in} \quad \mathbb{R}^N
\]
and \( \tilde{w} \neq 0 \). Thus, we have
\[
\int_{\mathbb{R}^N} g(x, \tilde{w}) \tilde{w} \, dx = \| \nabla \tilde{w} \|^N_{L^N(\mathbb{R}^N)} > 0.
\]
Thus, there exists some $s_0 > 1$ such that
\[ \int_{\mathbb{R}^N} g(x, s\tilde{w})\tilde{w}dx > 0, \quad \forall s \in [1, s_0]. \]
Set $\Phi(x, s) := g(x, s)/s^{N-1}$. Then, it is easy to see that $\lim_{|s| \to 0} \Phi(x, s) = -c < 0$ uniformly in $x \in \mathbb{R}^N$ for some $c > 0$ and $\Phi \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and
\[ \int_{\mathbb{R}^N} \Phi(x, s\tilde{w})|\tilde{w}|^Ndx > 0, \quad \forall s \in [1, s_0]. \]
On the other hand,
\[ \frac{d}{ds} I_\infty(s\tilde{w}_t) = (I'(s\tilde{w}_t), \tilde{w}_t) \]
\[ = s^{N-1} \left( \left\| \nabla \tilde{w}_t \right\|_{L^N(\mathbb{R}^N)}^N - \int_{\mathbb{R}^N} g(x, s\tilde{w}_t)\tilde{w}_tdx \right) \]
\[ = s^{N-1} \left( \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N - t^N \int_{\mathbb{R}^N} \Phi(x, s\tilde{w})|\tilde{w}|^Ndx \right). \] (5.25)
Therefore, we can choose $t_1 \in (0, 1)$ sufficiently small and $t_2 > 1$ sufficiently large such that from the last relation, we obtain
\[ \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N - t_1^N \int_{\mathbb{R}^N} \Phi(x, s\tilde{w})|\tilde{w}|^Ndx > 0, \quad \text{for all } s \in [1, s_0] \]
and
\[ \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N - t_2^N \int_{\mathbb{R}^N} \Phi(x, s\tilde{w})|\tilde{w}|^Ndx \leq - \frac{1}{s_0 - 1} \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N, \quad \text{for all } s \in [1, s_0]. \] (5.26)
Thus from (5.25), it follows that $I_\infty(s\tilde{w}_t)$ is increasing on $[0, t_1]$ and one can check that $I_\infty(s\tilde{w}_t)$ takes its maximum value at $s = 1$. In addition, from Theorem 1.2, we have $f(\cdot, h(w))h(w)$, $F(x, h(w)) \in L^q_{\text{loc}}(\mathbb{R}^N)$, for $1 \leq q < \infty$.

Since by Lemma 3.3 for $\beta = 0$, $\int_{\mathbb{R}^N} \frac{F(y, h(w))}{|x - y|^\alpha}dy \in L^\infty(\mathbb{R}^N)$, using $h'(w) \leq 1$, we obtain
\[ \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|x - y|^\alpha}dy \right) f(x, h(w))h'(w) \in L^q_{\text{loc}}(\mathbb{R}^N), \]
for $1 \leq q < \infty$. Now by recalling regularity results for the elliptic equations, from (5.27), we infer that $\tilde{w} \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap C^1_{\text{loc}}(\mathbb{R}^N)$ for some $\xi \in (0, 1)$. Therefore, using Proposition 6.1 by taking $p = N$ and $\beta = 0$, we have
\[ \int_{\mathbb{R}^N} G(x, \tilde{w})dx = 0, \quad \text{where } G(x, t) = \int_0^t g(x, s)ds \text{ is the primitive of } g. \]
Therefore, we get
\[ I_\infty(\tilde{w}_t) = I_\infty(\tilde{w}) = \frac{1}{N} \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N, \]
This together with (5.26) yields that
\[ I_\infty(s_0\tilde{w}_{t_2}) = I_\infty(\tilde{w}_{t_2}) + \int_1^{s_0} \frac{d}{ds} I_\infty(s\tilde{w}_{t_2})ds \]
\[ = \frac{1}{N} \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N - \int_1^{s_0} \frac{1}{s_0 - 1} \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^Nds \]
\[ < \left( \frac{1}{N} - 1 \right) \left\| \nabla \tilde{w} \right\|_{L^N(\mathbb{R}^N)}^N < 0. \]
Thus we have achieved our desired path, which together with the definition of $\theta_\infty$ gives that
\[ \theta_\infty \leq \max_{t \in [0, 1]} I_\infty(\gamma(t)) = I_\infty(\tilde{w}). \]
Hence, the relation (5.25) hold. This completes the lemma.
Proof of Theorem 1.6 Consider the path $\gamma$ defined in (5.26). Since $\gamma \in \Gamma_\infty \subset \Gamma$, $\gamma(t)(x) > 0$ and by (V’), we have $V(x) \leq V_\infty$ with $V \neq V_\infty$, from (5.17), (5.25), and (5.26), we obtain

$$
\theta_c \leq \max_{t \in [0,1]} J(\gamma(t)) = J(\gamma(t_{max}))
$$

$$
< I_\infty(\gamma(t_{max})) \leq \max_{t \in [0,1]} I_\infty(\gamma(t)) = I_\infty(\tilde{\gamma}) \leq \theta_c.
$$

This gives a contradiction. Thus $w$ is non-trivial weak solution to (2.8). Next, we prove that $w > 0$ in $\mathbb{R}^N$. Since $w$ is a weak solution to (2.8), it satisfies

$$
\int_{\mathbb{R}^N} |\nabla w|^{N-2} \nabla w \nabla \varphi \, dx + \int_{\mathbb{R}^N} V(x)|h(w)|^{N-2} h(w)h'(w)\varphi \, dx = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|x-y|^{\mu}} \, dy \right) \frac{f(x, h(w))h'(w)}{|x-y|^{\mu}} \varphi \, dx,
$$

for all $\varphi \in W^{1,N}(\mathbb{R}^N)$. Now in particular, taking $\varphi = w^- := \max\{-w, 0\}$ in the last equation, and using (f1), (V1) and Lemma 2.3-(h3), we obtain $\|w^-\| \leq 0$ which implies that $w^- = 0$ a.e. in $\mathbb{R}^N$. Therefore, $w \geq 0$ a.e. in $\mathbb{R}^N$. Now rearranging the equation (2.8), we get

$$
-\Delta_N w + V(x)|w|^{N-2}w = V(x) \left( |w|^{N-2}w - |h(w)|^{N-2}h(w)h'(w) \right) + \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|x-y|^{\mu}} \, dy \right) \frac{f(x, h(w))h'(w)}{|x-y|^{\mu}}
$$

and right hand side this equation is non negative, thanks to (V1), Lemma 2.3-(h3), (h3) and (f1). Recall that in the proof of Lemma 5.7 we obtain $w \in L^\infty_{loc}(\mathbb{R}^N) \cap C^{1,\theta}_{loc}(\mathbb{R}^N)$ for some $\xi \in (0, 1)$. Therefore, using the strong maximum principle we get $w \equiv 0$ in $\mathbb{R}^N$. Hence, $h(w) > 0$ which serves as a positive solution to (P). This completes the proof of Theorem 1.6.

6. Appendix

Here, we prove the generalized Pohozaev identity for the following quasilinear equation:

$$\left\{ \begin{array}{l}
-\Delta_p w + |h(w)|^{p-2}h(w)h'(w) = \left( \int_{\Omega} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w))}{|x|^\beta}h'(w) \quad \text{in } \mathbb{R}^N, \quad (Q_a),
\end{array} \right.$$

where $1 < p \leq N$, $0 < \mu < N, \beta \geq 0$, and $2\beta + \mu \leq N$. The nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $F(x,s) = \int_0^s f(x,t) \, dt$ is the primitive of $f$. Similar results for the semilinear Choquard equations with Laplacian operator, readers are referred to [20, 36].

Proposition 6.1. Let $w \in W^{1,p}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ be a weak solution to problem (Qa). Then

$$
\frac{(N-p)}{p} \int_{\mathbb{R}^N} |\nabla w|^p \, dx + \frac{N}{p} \int_{\mathbb{R}^N} |h(w)|^p \, dx + \frac{(2N - \mu - 2\beta)}{2} \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w))}{|x|^\beta} \, dx = 0.
$$

Proof. Using the idea of [21], let $w_\varepsilon$ be the classical solution in $C^{3}_{loc}(\mathbb{R}^N \setminus \{0\})$ of

$$\left\{ \begin{array}{l}
-\text{div}(\varepsilon + |\nabla w|^2) \frac{|w|^{p-2}}{2} w + h(w)w' = \left( \int_{\Omega} \frac{F(y, h(w))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w))}{|x|^\beta}w' \quad \text{in } \mathbb{R}^N.
\end{array} \right.$$

Then $w_\varepsilon$ is bounded in $C^{1,\theta}(\mathbb{R}^N \setminus \{0\})$ independently of $\varepsilon \in (0, 1]$ and converges to $w$ in $C^{1,\theta}(\mathbb{R}^N \setminus \{0\})$ for any $0 < \theta < \bar{\theta}$ as $\varepsilon \to 0$. For $0 < r < R$, define $\varphi_{r,R} \in C^\infty_c(\mathbb{R}^N)$ with $0 \leq \varphi_{r,R} \leq 1$, $\varphi_{r,R}(x) = 0$ in $|x| < \frac{r}{2}$ and $\varphi_{r,R}(x) = 1$ on $r < |x| < R$ with $|\nabla \varphi_{r,R}(x)| < \frac{1}{2}$. By testing the equation against the function $\psi_{r,R}(x) = \varphi_{r,R}(x \cdot \nabla w_\varepsilon(x))$ and integrate over $\mathbb{R}^N$, we have

$$
\int_{\mathbb{R}^N} \left( \varepsilon + |\nabla w_\varepsilon|^2 \right) \frac{|w_\varepsilon|^{p-2}}{2} w \cdot \nabla \psi_{r,R} \, dx + \int_{\mathbb{R}^N} |h(w_\varepsilon)|^{p-2}h(w_\varepsilon)h'(w_\varepsilon)\psi_{r,R} \, dx = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_\varepsilon))}{|y|^\beta|x-y|^\mu} \, dy \right) \frac{f(x, h(w_\varepsilon))}{|x|^\beta}h'(w_\varepsilon)\psi_{r,R} \, dx.
$$
Now we compute, for every $R > r > 0$,
\[
\int_{\mathbb{R}^N} |h(w_\epsilon)|^{p-2} h(w_\epsilon) h'(w_\epsilon) \psi_{r,R}(x) dx = \int_{\mathbb{R}^N} |h(w_\epsilon)|^{p-2} h(w_\epsilon) h'(w_\epsilon) \phi_{r,R}(x) x \nabla w_\epsilon(x) dx
= \int_{\mathbb{R}^N} \phi_{r,R}(x) x \nabla \left( \frac{|h(w_\epsilon)|}{p} \right) (x) dx
= - \int_{\mathbb{R}^N} (N \phi_{r,R}(x) + \lambda x \cdot \nabla \phi_{r,R}) \frac{|h(w_\epsilon)|}{p} dx.
\]
Letting $\epsilon \to 0$, $r \to 0$ and $R \to \infty$, we obtain
\[
\int_{\mathbb{R}^N} |h(w_\epsilon)|^{p-2} h(w_\epsilon) h'(w_\epsilon) \psi_{r,R}(x) dx \to - \frac{N}{p} \int_{\mathbb{R}^N} |h(w)|^p dx,
\]
thanks to Lebesgue’s dominated convergence Theorem. Next, we have
\[
\int_{\mathbb{R}^N} (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} \nabla w_\epsilon(x) \cdot \nabla \psi_{r,R}(x) dx
= \int_{\mathbb{R}^N} (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} \nabla w_\epsilon(x) \nabla (\phi_{r,R}(x) x \nabla w_\epsilon(x)) dx
= \int_{\mathbb{R}^N} \phi_{r,R}(x) (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} |\nabla w_\epsilon(x)|^p - \frac{N}{p} \int_{\mathbb{R}^N} \phi_{r,R}(x) (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} dx - \int_{\mathbb{R}^N} x \nabla \phi_{r,R}(x) (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} dx.
\]
Similarly, taking $\epsilon \to 0$ in the left hand side of the last relation,
\[
\int_{\mathbb{R}^N} (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} \nabla w_\epsilon(x) \cdot \nabla \psi_{r,R}(x) dx \to - \int_{\mathbb{R}^N} (N - p) \phi_{r,R}(x) + x \cdot \nabla \phi_{r,R}(x) \frac{|\nabla w_\epsilon(x)|^p}{p} dx
\]
and using this,
\[
\lim_{r \to 0} \lim_{R \to \infty} \int_{\mathbb{R}^N} (\epsilon + |\nabla w_\epsilon(x)|^2)^{\frac{p-2}{2}} \nabla w_\epsilon(x) \cdot \nabla \psi_{r,R}(x) dx = - \frac{N - p}{p} \int_{\mathbb{R}^N} |\nabla w|^p dx.
\]
Finally, we have
\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_\epsilon))}{|y|^\beta |x-y|^\mu |x|^\beta} dy \right) \frac{f(x, h(w_\epsilon))}{|x|^\beta} \psi_{r,R}(x) h'(w_\epsilon) dx
= \int_{\mathbb{R}^N} \frac{F(y, h(w_\epsilon)) f(x, h(w_\epsilon))}{|y|^\beta |x-y|^\mu |x|^\beta} \psi_{r,R}(x) h'(w_\epsilon) dx
+ \frac{1}{2} \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) f(y, h(w_\epsilon)) h'(w_\epsilon) \phi_{r,R}(y) y \cdot \nabla w_\epsilon(y)}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy
= \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) F(y, h(w_\epsilon)) \phi_{r,R}(x)}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy
+ \frac{\mu}{2} \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) F(y, h(w_\epsilon)) (x-y) \cdot (x \phi_{r,R}(x) - y \phi_{r,R}(y))}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy
+ \beta \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) F(y, h(w_\epsilon)) \phi_{r,R}(x)}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy.
\]
Taking $r \to 0$ and $R \to \infty$, by Lebesgue’s dominated convergence Theorem,
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) F(y, h(w_\epsilon)) (x-y) \cdot (x \phi_{r,R}(x) - y \phi_{r,R}(y))}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy \to \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) F(y, h(w_\epsilon))}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy;
\]
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(x, h(w_\epsilon)) F(y, h(w_\epsilon)) x \cdot \nabla \phi_{r,R}(x)}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy \to 0.
\]
Hence, combining the above estimates and taking $\epsilon \to 0$, we obtain
\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{F(y, h(w_\epsilon))}{|y|^\beta |x-y|^\mu |x|^\beta} dy \right) \frac{f(x, h(w_\epsilon))}{|x|^\beta} \psi_{r,R}(x) dx \to - \frac{2N - \mu - 2\beta}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(y, h(w)) F(x, h(w))}{|y|^\beta |x-y|^\mu |x|^\beta} dx dy.
\]
This completes the proof pf the proposition. \qed
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