Compact Quantum Groupoids

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Abstract

Quantum groupoids are a joint generalization of groupoids and quantum groups. We propose a definition of a compact quantum groupoid that is based on the theory of \( C^* \)-algebras and Hilbert bimodules. The essential point is that whenever one has a tensor product over \( C \) in the theory of quantum groups, one now uses a certain tensor product over the base algebra of the quantum groupoid.

1 Introduction

Quantum groupoids are relatively novel objects in mathematics, for which a number of distinct definitions have been proposed in the literature. The “algebraic” definition of Maltsiniotis \[13\] is relevant to Tannakian categories \[5, 3\], the “measurable” definition of Vallin \[19\] plays a role in subfactors \[6\], and the “smooth” definition of Lu \[12\] has applications to deformation quantization \[21\] and to quantum dynamical Yang–Baxter equations \[7\]. Moreover, weak \( C^* \) Hopf algebras \[4\], which occur in the algebraic theory of superselection sectors in quantum field theory \[16\], are finite quantum groupoids in disguise \[14\].

For reasons that originate in algebraic quantum field theory, we are here going to define compact quantum groupoids. Our general strategy is borrowed from noncommutative geometry \[4, 20\]. Given some geometric or topological object of “classical” mathematics, one attempts to describe this object in terms of commutative \( C^* \)-algebras endowed with additional structure, and subsequently tries to define the corresponding “quantum” object by dropping commutativity. The first step usually involves the passage from the object in question to its “dual” description in terms of the space of continuous functions on it. Ideally, this should lead to an anti-equivalence of categories, also known as a duality.

In the simplest example, the object is a compact\(^2\) space \( X \), which is dually and completely characterized by the commutative unital \( C^* \)-algebra \( C(X) \) of continuous functions on it. It is

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\(^1\)Dedicated to Professor H.-D. Doebner on the occasion of his formal retirement. To appear in ‘Quantum Theory and Symmetries’ (Goslar, 18-22 July 1999), eds. H.-D. Doebner, V.K. Dobrev, J.-D. Hennig and W. Luecke (World Scientific, 2000).

\(^2\)We include the Hausdorff property in the definition of compactness; in this terminology a topological space satisfying the finite covering property is called quasicompact.
interesting to note that the algebraic operation of pointwise multiplication in \( C(X) \) may be seen as a map
\[
\mu : C(X) \otimes C(X) \simeq C(X \times X) \to C(X),
\]
given by \( \mu(F) : x \to F(x,x) \), which is dual to the diagonal map \( \delta : x \to x \times x \) from \( X \to X \times X \); see [8]. Note that the tensor product is not quite the algebraic one over \( \mathbb{C} \), but a \( C^* \)-algebraic one, which is a certain completion of it. The latter is uniquely defined on commutative \( C^* \)-algebras, and satisfiess \( C(X) \otimes C(Y) \simeq C(X \times Y) \). The unit \( I \) in \( C(X) \), which is the function identically equal to 1, should be seen as a map \( \eta : \mathbb{C} \to C(X) \), given by \( \eta(z) = zI \). This map is dual to the map \( X \to e \), where \( e = \{ e \} \) is any set with one element.

The Gelfand-Neumark lemma on the structure of commutative \( C^* \)-algebras actually implies that the category of compact Hausdorff spaces is dual to the category of commutative unital \( C^* \)-algebras [8]. Hence a quantum compact space is nothing but a general \( C^* \)-algebra with unit. In this case, the additional structure alluded to above merely consist of the presence of a unit.

In this paper we extend these ideas to increasingly involved algebraic structures, still combined with the topological structure of compactness; this culminates in the definition of compact quantum groupoids.

2 Dual descriptions

The following diagram suggests how we may continue to add structure:

\[
\text{groups} \subset \text{unital semigroups} \subset \text{semigroups} \quad \cap \quad \cap
\text{groupoids} \subset \text{small categories} \subset \text{sets}
\]

From ‘sets’ in the lower right corner we may evidently proceed in two directions. In one direction, one has the compact semigroups, whose dual description is due to Hofmann [8]. Given that a compact semigroup \( \Sigma \) is dually described by \( C(\Sigma) \) as a topological space, the additional structure encoding the semigroup law on \( \Sigma \) is now given by a coproduct \( \Delta : C(\Sigma) \to C(\Sigma) \otimes C(\Sigma) \). Explicitly, the coproduct is given by
\[
\Delta f(x,y) = f(xy).
\]

Associativity of the semigroup multiplication law is expressed by the coassociativity property imposed on the coproduct, viz.
\[
(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta.
\]

The presence of a unit \( e \) in a semigroup \( \Sigma \) is dually encoded by a counit \( \epsilon : C(\Sigma) \to \mathbb{C} \), given by \( \epsilon(f) = f(e) \). The axiom \( xe = ex = x \) for all \( x \in \Sigma \) is dually expressed by
\[
(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id.
\]

The singleton \( e \) to which the unit in \( C(\Sigma) \) is dual may now be identified with the unit in \( \Sigma \).

This leads to an anti-equivalence between the category of compact semigroups (with unit) and the category of commutative unital \( C^* \)-algebras with coproduct (and counit). Although this clearly suggests a definition, a general theory of quantum (unital) semigroups remains to be developed.

\[3\] Here and in what follows, any map between two unital \( C^* \)-algebras is a unital \( \ast \)-homomorphism as a standing assumption (except when some conflicting property is explicitly stated, as in the case of a coinverse). These are the arrows in the category of all unital \( C^* \)-algebras.
The passage from unital semigroups $\Sigma$ to groups $G$ is effected by adding an inverse $x \to x^{-1}$. In the dual description, the inverse is encoded by the coinverse (inferiorly called antipode) $S : C(G) \to C(G)$, given by

$$Sf(x) = f(x^{-1}).$$

Here, in the commutative case, this map happens to be an involutive $^*$-homomorphism, but in general a coinverse is by definition an algebra anti-homomorphism satisfying

$$(S \circ s)^2 = \text{id}. \quad (5)$$

Further properties of the inverse are dually encoded by the axioms

$$\mu \circ (\text{id} \otimes S) \circ \Delta = \mu \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon. \quad (6)$$

Since these axioms cannot be generalized to groupoids, for later use we here list two well-known consequences of (6), viz.

$$S \circ \eta = \eta; \quad (7)$$

$$\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S. \quad (8)$$

Here $\tau$ is the flip map $\tau(f \otimes g) = g \otimes f$.

Thus one is led to a categorical duality between compact groups and commutative unital $C^*$-algebras with coproduct, counit, and coinverse satisfying (6) and a counit for a coinverse satisfying (7) and (8) and a Haar measure. Hence one can trade a coinverse is a consequence of these axioms. However, as a warmup for the dual description of a compact group, we assume that $C$ consists of an arrow (or morphism) space $C_1$, an object space $C_0$, source and target projections $s : C_1 \to C_0$ and $t : C_1 \to C_0$, an object inclusion map $\iota : C_0 \to C_1$, and a multiplication $m : C_1 \to C_1$.

Here

$$C_2 = C_1 \times C_1 \ni \{ (x, y) \in C_1 \times C_1 \mid s(x) = t(y) \}. \quad (10)$$

These are subject to axioms discussed below. In order to pass to the dual description of a category, we assume that $C_1$ and $C_0$ are compact. In that case, $C(C_0)$ and $C(C_1)$ are commutative $C^*$-algebras with unit. Instead of a single unit $\eta : C \to C(X)$, we now have two maps $\eta_t : C(C_0) \to C(C_1)$ and $\eta_s : C(C_0) \to C(C_1)$, given by the pullbacks $\eta_t = t^*$ and $\eta_s = s^*$.

We dualize the diagram $C_0 \overset{\iota}{\leftarrow} C_1 \overset{\iota^*}{\to} C_0$, by constructing a bimodule $C(C_0) \to C(C_1) \leftarrow C(C_0)$. The left action $\lambda$ and the right action $\rho$ of $C(C_0)$ on $C(C_1)$ are given, for $f \in C(C_0)$, $g \in C(C_1)$, by

$$\lambda(f)g = \eta_t(f)g; \quad \rho(f)g = \eta_s(f)g. \quad (11)$$

These are often called (commutative) Hopf $C^*$-algebras. Woronowicz’s definition of a compact quantum group is not based on the above axioms, but on a rather subtle reformulation of them.\footnote{In their recent definition of a locally compact quantum group, Kustermans and Vaes require the existence of a Haar system as an axiom, too.}

A category is called small when the space of arrows is a set.\footnote{An $A \otimes B$ bimodule, where $A$ and $B$ are complex algebras, is a vector space $E$ with a left action of $A$ and a right action of $B$ which commute. We write $A \to E \leftarrow B$.}
We may, of course, dualize \( m \) as a map \( \tilde{\Delta} : C(C_1) \to C(C_2) \), defined by \( \tilde{\Delta} f(x, y) = f(xy) \), where \((x, y) \in C_2\). This is rather awkward, and doesn’t suggest a noncommutative generalization. However, we note the \( C^* \)-algebraic isomorphism
\[
C(C_1) \otimes_{C(C_0)} C(C_1) \simeq C(C_2),
\]
(12)
where the left-hand side is the completed bimodule tensor product. This is defined by first taking the \( C^* \)-algebraic tensor product of \( C(C_1) \) with itself, which is isomorphic to \( C(C_1 \times C_1) \), and subsequently taking the quotient of the latter by the closed ideal generated by \( \eta^{(1)}_t(C(C_0)) - \eta^{(2)}_t(C(C_0)) \); the notation means that \((\eta^{(1)}_t \tilde{f})(x, y) = \tilde{f}(s(x)) \) and \((\eta^{(1)}_t \tilde{f})(x, y) = \tilde{f}(t(y)) \). The isomorphism (12) is easily established with the aid of the Stone–Weierstrass theorem. Accordingly, we may dualize groupoid multiplication by a “coproductoid” \( \Delta : C(C_1) \to C(C_1) \otimes_{C(C_0)} C(C_1) \),
\[
\Delta : C(C_1) \to C(C_1) \otimes_{C(C_0)} C(C_1),
\]
(13)
defined by combining \( \tilde{\Delta} \) with the isomorphism (12). Being essentially the pullback of the continuous map \( m \), this map is a morphism of unital \( C^* \)-algebras. In addition, note that \( C(C_1) \otimes_{C(C_0)} C(C_1) \) is a \( C(C_0) - C(C_0) \) bimodule in the obvious way (inheriting the left action of \( C(C_0) \) on the first factor and the right action on the second). Now the groupoid axioms \( t(fg) = t(f) \) and \( s(fg) = s(g) \) correspond to \( \Delta \) being a morphism of bimodules. As in the case of semigroups, associativity of the groupoid product is equivalent to coassociativity of \( \Delta \), now expressed by
\[
(id \otimes_{C(C_0)} \Delta) \circ \Delta = (\Delta \otimes_{C(C_0)} id) \circ \Delta.
\]
(14)
Here the bimodule tensor product of two maps is well defined, since each is a morphism of bimodules.

The object inclusion map \( \epsilon \) is evidently dualized by the counit \( \epsilon : C(C_1) \to C(C_0) \), given by \( \epsilon = \epsilon^* \). Regarding \( C(C_0) \) as a \( C(C_0) - C(C_0) \) bimodule in the obvious way, the groupoid axiom \( t \circ t = s \circ t = id \) is equivalent to \( \epsilon \) being morphism of bimodules. The axiom \( x t(s(x)) = t(t(x)) x = x \) for all \( x \in C_1 \) is dually expressed by
\[
(id \otimes_{C(C_0)} \epsilon) \circ \Delta = (\epsilon \otimes_{C(C_0)} id) \circ \Delta = id;
\]
(15)
cf. the comment following (12).

Combining these structures, one easily obtains a duality between the category of compact categories (with functors as arrows) and the category of pairs of unital commutative \( C^* \)-algebras \((A, B)\), equipped with injections \( \eta_{s,t} : B \to A \), a counit \( : A \to B \), and a coproductoid \( \Delta : A \to A \otimes_B A \).

A groupoid \( \Gamma \) is a small category in which every arrow is invertible; it follows that \( xx^{-1} = t(t(x)) \) and \( x^{-1} x = t(s(x)) \). The inverse is dually described by a coinverse \( S : C(\Gamma_1) \to C(\Gamma_1) \), still given by (4). We now have the property \( S \circ \eta_{s,t} = \eta_{s,t} \), stating that \( S \) is an anti-morphism of the bimodule \( C(\Gamma_0) \to C(\Gamma_1) \). Since condition (3) cannot be written down for groupoids, we will generalize (8), and, instead of (3), impose
\[
\tau \circ (S \otimes_{C(\Gamma_0)} S) \circ \Delta = \Delta \circ S.
\]
(16)
Because of the presence of \( \tau \), the left-hand side is well defined.

There exists an analogue of Haar measure for (locally) compact groupoids; for each \( q \in \Gamma_0 \) one now has a measure \( \mu^q \) on \( \Gamma_1 \) with support \( t^{-1}(q) \), such that the family is left-invariant in the obvious sense (7) 13. Such a “Haar system”, whose existence, in contradistinction to the group case, needs to be postulated (except for Lie groupoids, where it is automatic (11)), leads to a map.

\[\text{For a related concept in pure algebra, see Deligne [2]; also cf. [13] for applications to algebraic quantum groupoids, and [14] for a similar coproduct occurring in the definition of a measurable quantum groupoid.}\]
$P : C(\Gamma_1) \to C(\Gamma_0)$, given by $P(f) : q \to \int_{t^{-1}(q)} d\mu_q(x) f(x)$. This map intertwines the left action $\lambda$ of $C(\Gamma_0)$ on $C(\Gamma_1)$ with the natural left action of $C(\Gamma_0)$ on itself. This renders the property

$$(\text{id} \otimes_{C(\Gamma_0)} P) \circ \Delta = P,$$

which replaces (16), well defined; strictly speaking, the right-hand side should be preceded by the canonical isomorphism $C(\Gamma_1) \otimes_{C(\Gamma_0)} C(\Gamma_0) \to C(\Gamma_1)$.

We eventually arrive at a duality between compact groupoids and pairs of unital commutative $C^*$-algebras $(A, B)$, equipped with injections $\eta_{s, t} : B \to A$, a coproduct $\Delta : A \to A \otimes_B A$, a coinverse satisfying (16), and a Haar measure. The latter replaces the counit in the axiomatic setup.

3 Intermezzo on Hilbert bimodules

As we have seen, the dual description of categories and groupoids, as compared with unital semi-groups and groups, respectively, is that tensor products over $\mathbb{C}$ tend to be replaced by tensor products over $C(\mathcal{G})$ or $C(\Gamma)$.

One requires, for example, a Hilbert space is a Hilbert module over a given $C^*$-algebra $B$. One may construct a space with a $C^*$-norm in $B$ coincides with its norm as a Hilbert module because of the $C^*$-axiom $\|A^*A\| = \|A\|^2$.

A map $A : \mathcal{E} \to \mathcal{E}$ for which there exists a map $A^* : \mathcal{E} \to \mathcal{E}$ such that $\langle \Psi, A\Phi \rangle_B = \langle A^*\Psi, \Phi \rangle_B$ is called adjointable. An adjointable map is automatically $\mathcal{C}$-linear, $B$-linear, and bounded. The adjoint of an adjointable map is unique, and the map $A \mapsto A^*$ defines an involution on the space $\mathcal{L}_B(\mathcal{E})$ of all adjointable maps on $\mathcal{E}$. This space thereby becomes a $C^*$-algebra.

An $\mathcal{A} - \mathcal{B}$ Hilbert bimodule, where $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras, is now defined as a Hilbert module $\mathcal{E}$ over $\mathcal{B}$, along with an $\ast$-homomorphism of $\mathcal{A}$ into $\mathcal{L}_B(\mathcal{E})$. Hence one has a space with a $\mathcal{B}$-valued inner product and compatible left $\mathcal{A}$ and right $\mathcal{B}$-actions where it should be remarked that the left and right compatibility conditions are quite different from each other.

A Hilbert bimodule over $\mathcal{B}$ is a $\mathcal{B} - \mathcal{B}$ Hilbert bimodule. For example, a Hilbert space is a Hilbert bimodule over $\mathcal{C}$, and a $C^*$-algebra $\mathcal{B}$ is a Hilbert bimodule over itself. One may construct a tensor product $\mathcal{E}_1 \otimes_B \mathcal{E}_2$ of two Hilbert bimodules over $\mathcal{B}$, yielding an object of the same kind. The definition is a special case of the following construction, which goes back to Rieffel (13) also see (14).

Given $\mathcal{A} - \mathcal{B}$ Hilbert bimodule $\mathcal{E}_1$ and a $\mathcal{B} - \mathcal{C}$ Hilbert bimodule $\mathcal{E}_2$, define a $\mathcal{C}$-valued inner product on the algebraic tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ by

$$\langle \Psi_1 \otimes \Psi_2, \Phi_1 \otimes \Phi_2 \rangle_{\mathcal{E}} = \langle \Psi_2, (\Psi_1, \Phi_1)_{\mathcal{B}} \Phi_2 \rangle_{\mathcal{E}}.$$  

In the context of finite groupoids this possibility was first mentioned in (12) also see (13).

Note that an $\mathcal{A} - \mathcal{B}$ Hilbert bimodule is an $\mathcal{A} - \mathcal{B}$ bimodule, since $\mathcal{L}_{\mathcal{B}}(\mathcal{E})$ commutes with the right $\mathcal{B}$-action.

One sometimes calls $\mathcal{E}$ a $C^*$-correspondence between $\mathcal{A}$ and $\mathcal{B}$. Correspondences between von Neumann algebras are a special case of $C^*$-correspondences; see (14).
This is positive semidefinite; the completion of the quotient of $\mathcal{E}_1 \otimes \mathcal{E}_2$ by the null space of $\langle \cdot, \cdot \rangle_\mathcal{E}$ is $\mathcal{E}_1 \hat{\otimes} \mathcal{E}_2$ as a vector space. The crucial point is that $\mathcal{E}_1 \hat{\otimes} \mathcal{E}_2$ inherits the left action of $\mathfrak{A}$ on $\mathcal{E}_1$, the right action of $\mathfrak{C}$ on $\mathcal{E}_2$, and also the $\mathfrak{C}$-valued inner product $\langle \cdot, \cdot \rangle_\mathcal{C}$, so that $\mathcal{E}_1 \hat{\otimes} \mathcal{E}_2$ itself becomes a $\mathfrak{A} - \mathfrak{C}$ Hilbert bimodule. For Hilbert bimodules over $\mathfrak{B}$, simply take $\mathfrak{A} = \mathfrak{C} = \mathfrak{B}$.

4 Corepresentations

As a first application, we employ the topological bimodule tensor product $\hat{\otimes}_\mathfrak{B}$ to formulate a theory of corepresentations of compact groupoids. Recall [15] that an action of a groupoid $\Gamma$ on a fibered space $S \xrightarrow{p} \Gamma_0$ is a map $\Gamma_1 \rtimes_p S \to S$, such that $p(x\sigma) = t(x)$, $\iota(p(\sigma))\sigma = \sigma$, and $x(y\sigma) = (xy)\sigma$ whenever defined. Hence $x \in \Gamma_1$ maps $p^{-1}(s(x))$ into $p^{-1}(t(x))$. For example, groupoid multiplication is seen to be an action of $\Gamma$ on $\Gamma \xrightarrow{\tau} \Gamma_0$. Or choose $S = \Gamma_0$ and $p = \text{id}$. The product of two actions on $S_i \xrightarrow{\tau_i} \Gamma_0$, $i = 1, 2$, is defined on $S_1 \rtimes_{p_1} S_2 \xrightarrow{\tau_1} \Gamma_0$, and is given by $x : (\sigma_1, \sigma_2) \mapsto (x\sigma_1, x\sigma_2)$.

A unitary representation of $\Gamma$ is an action for which $S$ is a topological tensor product in the sense of Grothendieck. $\delta$ is precisely defined on $E$.

Finally, the tensor product of two unitary representations of $\Gamma$ on $\mathcal{H}$, in category language, this is simply a functor from $\Gamma$ to the category of Hilbert spaces with partial isometries as arrows.

$\delta$ is defined on $\mathcal{H}$ (here seen as a field of Hilbert spaces over a point) is dualized by a so-called corepresentation $\mathcal{C}$. Accordingly, we may define a corepresentation of $\Gamma$ as a map $\delta : C(\Gamma_0, \mathcal{H}) \to C(\Gamma_1, \Gamma_{1t} \rtimes_p \mathcal{H})$ that dualizes $U$, given by

$$\delta(\Psi)(x) = U(x)\Psi(s(x)).$$

For a neat description, first note that, for $X$ compact, the class of continuous fields $\mathcal{H}$ of Hilbert spaces over $X$ precisely corresponds to the class of full Hilbert bimodules $\mathcal{C}(X, \mathcal{H})$ over $\mathcal{C}(X)$; indeed, this $C^*$-algebraic Serre–Swan theorem is a good way to define a continuous fields of Hilbert spaces [15]. Secondly, we may turn $C(\Gamma_1)$ into a $C(\Gamma_0) - C(\Gamma_1)$ Hilbert bimodule by putting $\lambda(f)g = t(f)g$, $\rho(f)g = fg$, and (canonically) $(f, g)_{C(\Gamma_1)} = \overline{fg}$ for $f, g \in C(\Gamma_1)$ and $\overline{f} \in C(\Gamma_0)$. Hence we may form the tensor product of the $C(\Gamma_0) - C(\Gamma_0)$ Hilbert bimodule $C(\Gamma_0, \mathcal{H})$ with the $C(\Gamma_0) - C(\Gamma_1)$ Hilbert bimodule $C(\Gamma_1)$, which yields the isomorphism

$$C(\Gamma_0, \mathcal{H}) \hat{\otimes}_{C(\Gamma_0)} C(\Gamma_1) \simeq C(\Gamma_1, \Gamma_{1t} \rtimes_p \mathcal{H})$$

as $C(\Gamma_0) - C(\Gamma_1)$ Hilbert bimodules. Accordingly, we may define a corepresentation of $\Gamma$ as a map $\delta : \mathcal{E} \to \mathcal{E} \hat{\otimes}_{C(\Gamma_0)} C(\Gamma_1)$, where $\mathcal{E}$ is some full Hilbert bimodule over $C(\Gamma_0)$. The defining conditions of a groupoid actions may be dualized by the axioms

$$\langle \delta(\Psi), \delta(\Phi) \rangle_{C(\Gamma_1)} = s^*(\overline{\Psi}, \Phi)_{C(\Gamma_0)};$$

$$\langle \text{id} \otimes_{C(\Gamma_0)} \Delta \rangle \circ \delta = \langle \delta \otimes_{C(\Gamma_0)} \text{id} \rangle \circ \delta;$$

$$\langle \epsilon \otimes_{C(\Gamma_0)} \epsilon \rangle \circ \delta = \text{id}.$$

Note that [21] implies that $\delta(\overline{f}) = s^*(\overline{f})\delta(\Psi)$ for $\overline{f} \in C(\Gamma_0)$, which renders [23] well defined.
5 A definition of compact quantum groupoids

We are now in a position to define a compact quantum groupoid. The basic structure is a pair of unital $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$ that generalize $C(\Gamma_1)$ and $C(\Gamma_0)$, respectively. The injective unital maps $\eta_\mathcal{A}: \mathcal{B} \to \mathcal{A}$ and $\eta_\mathcal{B}: \mathcal{B} \to \mathcal{A}$ are now required to be an $\ast$-homomorphism and an $\ast$-anti-homomorphism, respectively, whose images in $\mathcal{A}$ commute. The noncommutative Haar measure, whose existence is now taken as an axiom replacing the counit, is defined as a faithful completely positive map $P: \mathcal{A} \to \mathcal{B}$ satisfying $P(\eta_\mathcal{A}(\mathcal{B})) = P(\mathcal{A})B$. In other words, $E = \eta_\mathcal{B} \circ P: \mathcal{A} \to \eta_\mathcal{B}(\mathcal{B}) \subset \mathcal{A}$ is a faithful conditional expectation.

This setup enables us to define a certain Hilbert bimodule $\mathcal{A}^−$ over $\mathcal{B}$, as follows. We first define a left $\mathcal{B}$-action $\lambda$ and a right $\mathcal{B}$-action $\rho$ on $\mathcal{A}$ by $\lambda(B)A = A\eta_\mathcal{B}(B)$ and $\rho(B)A = \eta_\mathcal{B}(B)A$. Subsequently, we put a $\mathcal{B}$-valued inner product on $\mathcal{A}$ by $(A, C)_\mathcal{B} = P(A^*C)$. Since $P$ is faithful, we may define a new norm on $\mathcal{A}$ by $\|A\|^2 = \|P(A^*A)\|_\mathcal{B}$. The completion of $\mathcal{A}$ in this norm is $\mathcal{A}^−$, which is a Hilbert module over $\mathcal{B}$. If the technical condition $\lambda(\mathcal{B}) \subseteq \mathcal{L}_\mathcal{B}(\mathcal{A}^−)$ is satisfied\(^{16}\), then the given data even define $\mathcal{A}^−$ as a Hilbert bimodule over $\mathcal{B}$.

The next step is to form the tensor product $\mathcal{A}^− \otimes_\mathcal{B} \mathcal{A}^−$. By the general theory, this is a Hilbert bimodule over $\mathcal{B}$. In addition, it is possible to define a coproductoid $\Delta: \mathcal{A} \to \mathcal{A} \otimes_\mathcal{B} \mathcal{A}$ as the $C^*$-algebra in $\mathcal{L}_\mathcal{B}(\mathcal{A}^− \otimes_\mathcal{B} \mathcal{A}^−)$ that is generated by $\varphi^1(\mathcal{A})$ and $\varphi^2(\mathcal{A})$. The definition of a compact quantum groupoid is completed by postulating a coproductoid $\Delta: \mathcal{A} \to \mathcal{A} \otimes_\mathcal{B} \mathcal{A}$, as well as a coinverse $S: \mathcal{A} \to \mathcal{A}$ that is an algebra and bimodule anti-homomorphism. The compatibility axioms are

\[
\begin{align*}
(id \otimes_\mathcal{B} \Delta) \circ \Delta &= (\Delta \otimes_\mathcal{B} id) \circ \Delta; \\
(id \otimes_\mathcal{B} P) \circ \Delta &= P; \\
\tau \circ (S \otimes_\mathcal{B} S) \circ \Delta &= \Delta \circ S;
\end{align*}
\]

(24) (25) (26)

cf. (18), (20), and (17), respectively. One may check that in the commutative case these axioms reduce to the dual description of a compact groupoid\(^3\).

The reformulation of the representation theory of a compact groupoid $\Gamma$ as a theory of corepresentations of $C(\Gamma_1)$ on full Hilbert bimodules over $C(\Gamma_0)$ in the preceding section has been motivated by the possibility of setting up a corepresentation theory of compact quantum groupoids. Indeed, one may now define a corepresentation as a map $\delta: \mathcal{E} \to \mathcal{E} \otimes_\mathcal{B} \mathcal{A}$, where $\mathcal{E}$ is a Hilbert bimodule over $\mathcal{B}$, satisfying axioms resembling those stated for the commutative case. The collection of all corepresentations of a given compact quantum groupoid over $\mathcal{B}$ then becomes a tensor category under $\otimes_\mathcal{B}$.

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\(^{15}\)This “localizes” the canonical Hilbert module over $\mathcal{A}$ with respect to $P$; see [18].
\(^{16}\)This is true for all von Neumann algebras and all unital commutative $C^*$-algebras. The condition may well be superfluous.
\(^{17}\)Here $\mathcal{A} \otimes \mathcal{A}$ is the algebraic tensor product over $\mathcal{C}$.
\(^{18}\)One might define a compact quantum category by omitting the coinverse from our definition of a compact quantum groupoid, but in the commutative case one wouldn’t necessarily recover a category.
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