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Minimax state estimation for linear descriptor systems

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Outline. This document is organized as follows. Section 1 presents an extended abstract of the dissertation containing research context, description of the aims of the dissertation, discussion of the new results, their validation, impact and dissemination, and information about financial support. Section 2 contains notation, brief summary of the chapters of the dissertation (definitions, duality concepts and minimax state estimation algorithms), conclusions, list of papers and conference presentations related to the dissertation and list of papers cited in the text.

1 Extended abstract of the dissertation

Research context. Differential-Algebraic equations (DAE) are widely used in engineering. In vehicle dynamics, DAEs represent a convenient and powerful basis for software modeling platforms (see for instance the platform Modelica). DAEs are also used in mathematical economics, robotics, biomechanics, image processing and control theory. In this context, the analysis of DAE is an important issue in the framework of mathematical systems and control theory. One of key problems of mathematical systems theory is so-called state estimation, that is to construct an estimate of the state, given observations of the state of the process being modeled. The dissertation is devoted to the development of the mathematical theory and algorithms for state estimation problems for linear DAE. The main mathematical tool is the Minimax State Estimation (MSE) approach. Main assumptions behind MSE are 1) the model is represented by a system of differential equations (for instance, system of Ordinary Differential Equations (ODE) or Partial Differential Equations (PDE)), 2) links between the model and observed data are represented by observation equation and 3) uncertain parameters (for instance, error in initial condition or model error or noise in the observed data) belong to a given bounding set. In other words, the uncertainties in the model and observed data are described in terms of the bounding set. The main idea behind MSE is to describe how the model propagates uncertain parameters which are consistent with observed data and belong to the given bounding set.

Main notions of MSE are reachability set, minimax estimate and worst-case error. By definition, reachability set contains all states of the model which are consistent with observed data and uncertainty description. Given a point \( P \) within the reachability set one defines a worst-case error as the maximal distance between \( P \) and other points of the reachability set. Then the minimax estimate of the state is defined as a point minimizing the worst-case error (a Tchebysheff center of the RS). Basics of the MSE were developed by Bertsekas and Rhodes [1971],
In the case of linear model, given a bounding set the classical MSE allows to construct the minimax estimate for the state of the model and calculate the worst-case estimation error, provided the model operator is bounded and has a bounded inverse. The classical MSE is based on the Kalman Duality principle which states that the state estimation problem is equivalent to a dual optimization problem, provided the model operator is bounded and has a bounded inverse. However, linear DAEs do not fit this framework as the corresponding model operator may not be invertible or may have unbounded inverse. Therefore, the classical duality concept was not applicable for derivation of the MSE theory for linear DAEs.

Aim of the dissertation. The aim of the dissertation is to develop a generalized Kalman Duality concept applicable for linear unbounded non-invertible operators and introduce the minimax state estimation theory and algorithms for linear differential-algebraic equations. In particular, the dissertation pursues the following goals:

- develop generalized duality concept for the minimax state estimation theory for DAEs with unknown but bounded model error and random observation noise with unknown but bounded correlation operator;
- derive the minimax state estimation theory for linear DAEs with unknown but bounded model error and random observation noise with unknown but bounded correlation operator;
- describe how the DAE model propagates uncertain parameters;
- estimate the worst-case error;
- construct fast estimation algorithms in the form of filters;
- develop a tool for model validation, that is to assess how good the model describes observed phenomena.

New results. The dissertation contains the following new results:

- generalized version of the Kalman duality principle is proposed allowing to handle unbounded linear model operators with non-trivial null-space;
- new definitions of the minimax estimates for DAEs based on the generalized Kalman duality principle are proposed;
• theorems of existence for minimax estimates are proved;

• new minimax state estimation algorithms (in the form of filter and in the variational form) for DAE are proposed.

Validation and impact. In order to validate the Generalized Kalman Duality (GKD) concept I applied it Zhuk [2005b, 2006c] to linear incorrect differential operators (see Tikhonov and Arsenin [1977] for further details on incorrect problems). As a consequence, I constructed new minimax state estimation algorithms for linear DAE and linear Boundary Value Problems (BVP) for ODE. The main impact is that the new minimax estimate for DAE does not require regularity assumptions on DAE structure (regularity of the matrix pencil Hanke [1989] or rank-degree condition Dai [1989]), imposed by the majority of authors; the minimax estimate for BVP works without restricting the structure of the corresponding ODE matrices. In Zhuk [2005a, 2006c] I applied GKD to the case of the linear discrete-time DAE. As a final result, I obtained a new minimax recursive estimator for discrete-time DAE Zhuk [2004b, 2005a,c]. These results demonstrate the impact of the GKD for DAE models. In general, GKD extends the scope of the classical MSE framework, originally designed for linear models with bounded invertible operator and bounded uncertainties, on the linear models with unbounded non-invertible operator and unbounded uncertainties. This brings the following advantages: 1) possibility to construct minimax state estimation algorithms for Differential-Algebraic Equations (DAE) and 2) possibility to address the case of unbounded model errors.

Dissemination of the new results. The generalized Kalman duality concept and minimax state estimation algorithms for linear DAEs were published in the sequence of papers Zhuk [2004b, 2005a,c,b,c] and was reported at the conferences Zhuk [2004d, 2006a,b]. Also the results were presented at the following seminars: “System analysis and decision making theory” and “Modeling and optimization of uncertain systems” at the National Taras Shevchenko University of Kyiv, “Non-smooth optimization” at the Institute for system analysis at Kyiv Polytechnic University, “Optimization of controlled processes” at Glushkov Institute for Cybernetics at National Academy of Science of Ukraine.

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2 Summary of the dissertation

Notation. \( E \) denotes the expected value of the random variable, \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, \( (\cdot,\cdot)_n \) denotes the canonical inner product in \( \mathbb{R}^n \), \( \text{tr}Q \) denotes the trace of the matrix \( Q \), \( y = (y_0 \ldots y_N) \) denotes a vector composed by elements \( y_1, \ldots, y_N \), where \( y_i \) also may be a vector, \( L_2([a,c],\mathbb{R}^n) \) denotes a space of square-integrable vector-functions on \( [a,c] \) with values in \( \mathbb{R}^n \), \( W_2^1([a,c],\mathbb{R}^n) \) denotes a space of absolutely continuous vector functions on \( [a,c] \) with values in \( \mathbb{R}^n \), \( \mathcal{H}^* \) denotes the adjoint of the linear mapping \( \mathcal{H} \), \( B' \) denotes the transposed of the matrix \( B \), \( \mathcal{D}(L) \) denotes the domain of the linear operator \( L \), \( \| \cdot \|_X \) denotes the norm of the normed space \( X \), \( \langle \cdot,\cdot \rangle \) denotes the csnonical inner product in the Hilbert space \( H \).

Outline of the dissertation. The dissertation is composed of three chapters. The total number of pages is 140. The first chapter contains the general description of the dissertation, its aims and basic notions. Also it contains the description of the state of the art in the DAE literature, a brief overview of the state estimation methods and description of the new results. The second chapter describes the generalized Kalman duality concept and minimax state estimation algorithms for linear DAE with discrete time. The third chapter contains the generalized Kalman duality concept and minimax state estimation algorithms for linear DAE with continuous time and conclusions.

Brief summary. Let us consider the contents of the second chapter in brief. Let \( x \in \mathbb{R}^n \) solve a linear algebraic equation

\[
Fx = Bf, \tag{1}
\]

and let the observed data \( y \in \mathbb{R}^l \) verify

\[
y = Hx + \eta, \tag{2}
\]

where \( F \) is \( m \times n \)-matrix, \( B \) is \( m \times p \)-matrix, \( H \) is \( l \times n \)-matrix and \( \eta \) is a realization of the random \( l \)-vector, \( x \) denotes the state of the system (1), \( f \in \mathbb{R}^p \) represents an uncertain element.

We will assume that \( f \) and \( R_\eta \overset{\text{def}}{=} E\eta\eta' \) are uncertain and

\[
f \in \mathcal{G}, R_\eta \in \mathcal{G}_2,
\]
where $\mathcal{G}, \mathcal{G}_2$ are given subsets. In what follows we will be looking for the estimate of the linear function $x \mapsto \ell(x) = (\ell, x)_n, \ell \in \mathcal{F}$. We will look for the estimate in the class of affine functions $y \mapsto (u, y)_i + c$ of observed data $(2)$. We will refer $(u, y)_i + c$ as an estimate. Let us assign to each estimate $u_c$ an estimation error

$$
\sigma(u, c) \overset{\text{def}}{=} \sup_{x, R \eta} \{ E[(\ell, x)_n - (u, y)_i - c]^2 | Fx \in B(\mathcal{G}), R\eta \in \mathcal{G}_2 \},
$$

where $B(\mathcal{G}) = \{ Bf | f \in \mathcal{G} \}$.

**Definition 1.** The estimate $(\hat{\ell}, x) = (\hat{u}, y)_i + \hat{c}$ verifying

$$
\sigma(\hat{u}, \hat{c}) = \inf_{u, c} \sup_{x, R \eta} \{ E[(\ell, x)_n - (u, y)_i - c]^2 | Fx \in B(\mathcal{G}), R\eta \in \mathcal{G}_2 \}
$$

is called a minimax a priori mean-squared estimate (a priori estimate). The number $\hat{\sigma} = \inf_{u, c} \sigma(u, c)$ is called a minimax mean-squared error (a priori error).

Let us consider an a posteriori state estimation method. Let $x \in \mathbb{R}^n$ verify $(1)$ and $y \in \mathbb{R}^l$ is given in the form

$$
y = Hx + g,
$$

where $g \in \mathbb{R}^m$ is a vector. In contrast to the previous considerations, we assume that $(f, g)$ are deterministic and belong to the given set $\mathcal{G} \subset \mathbb{R}^l \times \mathbb{R}^m$. Define

$$
X \overset{\text{def}}{=} \{ x \in \mathbb{R}^n | \exists (f, g) \in \mathcal{G} : Fx = Bf, y - Hx = g \}
$$

**Definition 2.** The estimate $\hat{\ell}(x)$ verifying

$$
\sup_{x \in X} |\ell(x) - \hat{\ell}(x)| = \inf_{\bar{x} \in X} \sup_{x \in X} |\ell(x) - \ell(\bar{x})|
$$

is called a minimax a posteriori estimate (a posteriori estimate). The number $\hat{\sigma} \overset{\text{def}}{=} \sup_{x \in X} |\ell(x) - \hat{\ell}(x)|$

is called minimax a posteriori estimation error (a posteriori error).

The a priori and a posteriori estimates are constructed in the dissertation for the generic convex compact bounding sets $\mathcal{G}, \mathcal{G}_2$. Let us consider the case of ellipsoidal bounding set in more details.

**Theorem 1.** Let

$$
\mathcal{G} = \{ f \in \mathbb{R}^p | (Q, f, f)_p \leq 1 \}, \quad \mathcal{G}_2 = \{ R\eta : \text{tr} Q\eta R\eta \leq 1 \},
$$
where $Q_1, Q_2$ are positive definite symmetric matrices. Then the minimax a priori estimate of the linear function

$$x \mapsto (\ell, x)_n, \ell \in \mathcal{F} \equiv \{ \ell = F'z + H'u, z \in \mathbb{R}^m, u \in \mathbb{R}^l \}$$

of the solution of $Fx = Bf$ has the following form

$$\widehat{(\ell, x)} = (\hat{u}, y), \hat{u} = Q_2Hp,$$

where $p$ solves

$$Fp = BQ_1^{-1}B'\hat{z},
F'\hat{z} = \ell - H'Q_2Hp. \quad (4)$$

The minimax a priori error is given by

$$\sup_{x, R_n} E[(\ell, x)_n - (\widehat{\ell, x})_n]^2 = (\ell, p)_n^2$$

If $\ell \not\in \mathcal{F}$, then the minimax a priori error is infinite.

**Theorem 2.** If $\ell \in \mathcal{F}$ and

$$\mathcal{G} = \{(f, g) : (Q_1f, f)_p + (Q_2g, g)_l \leq 1 \},$$

then the minimax a posteriori estimate is given by

$$\widehat{\ell(x)} = (\ell, \hat{x})_n = (Q_2Hp, y)_l, \hat{\sigma} = [1 - (y - H\hat{x}, Q_2y)_l]^\frac{1}{2}(\ell, p)_n^\frac{1}{2}$$

and the minimax a posteriori error is given by

$$\hat{\sigma} = [1 - (y - H\hat{x}, Q_2y)_l]^\frac{1}{2}(\ell, p)_n^\frac{1}{2},$$

where $p$ solves (4) and $\hat{x}$ solves

$$F\hat{x} = BQ_1^{-1}B'\hat{p},
F'\hat{p} = H'Q_2(y - H\hat{x}).$$

If $\ell \not\in \mathcal{F}$, then the minimax a posteriori error is infinite.

Let us demonstrate one application of Theorem 2 to the state estimation for the linear DAEs with discrete time. Assume $x_0 \ldots x_N$ is a solution of the DAE with discrete time:

$$F_{k+1}x_{k+1} - C_kx_k = B_kf_k, \quad F_0x_0 = Sx_0^q, \quad k = 0, N \quad (5)$$
and the observed data is given by
\[ y_k = H_k x_k + g_k, \quad k = 0, N \tag{6} \]
where \( F_k, C_k, S \) are \( m \times n \)-matrices, \( B_k \) is a \( m \times p \)-matrix, \( f_k \in \mathbb{R}^p, x_0^g \in \mathbb{R}^m \) are some vectors, \( H_k \) is \( l \times n \)-matrix and \( g_k \in \mathbb{R}^l \) stands for a deterministic noise in the observed data. Define a linear function
\[
\ell(x) \overset{\text{def}}{=} \sum_{k=0}^{N+1} (\ell_k, x_k)_n \overset{\text{def}}{=} (\ell, x), \quad \ell_k \in \mathbb{R}^n
\]
where \( \ell := (\ell_1, \ldots, \ell_{N+1}) \) and \( x := (x_0 \ldots x_N) \). Define
\[
F = \begin{bmatrix} F_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -C_0 & F_1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -C_{N-1} & F_N \end{bmatrix}, \quad H = \text{diag}\{H_0 \ldots H_N\}, \quad B = \text{diag}\{B_0 \ldots B_{N-1}\}
\]
and set \( y := (y_0 \ldots y_N), \ f := (x_0^g, f_0 \ldots f_{N-1}), \ g := (g_0 \ldots g_N) \). It is easy to see that \( Fx = Bf \) is equivalent to (5) and \( y = Hx + g \) is equivalent to (6). Now we apply Theorem 2 in order to derive the minimax a posteriori estimate of the linear function \( (\ell, x) \) in the variational form.

**Theorem 3.** Assume that \( \ell = (\ell_1, \ldots, \ell_{N+1}) \) is such that \( F_{N+1}' z_{N+1} = \ell_{N+1} \) and
\[
F_k' z_k - C_k' z_{k+1} + H_k u_k = \ell_k, \quad k = 0, N
\]
for some \( z_k \) and \( u_k, \ k = 0, N \). Assume also that
\[
\mathcal{G} \overset{\text{def}}{=} \{(f, v) : (Q_0 x_0^g, x_0^g)_m + \sum_{k=0}^{N} (Q_{1,k} f_k, f_k)_m + (Q_{2,k} v_k, v_k)_i \leq 1\},
\]
where \( Q_0, Q_{1,k}, Q_{2,k} \) are positive definite symmetric matrices for \( k \in [0, N] \). Let \((\hat{p}_k, \hat{x}_k)\) solve
\[
F_{k+1}' \hat{x}_{k+1} = C_k' \hat{x}_k + B_k Q_{1,k}^{-1} B_k' \hat{p}_k, \quad F_0' \hat{x}_0 = S Q_0^{-1} S' \hat{p}_0, \\
F_k' \hat{p}_k = C_k' \hat{p}_{k+1} + H_k' Q_{2,k} (y_k - H_k \hat{x}_k), \quad F_{N+1}' \hat{p}_{N+1} = 0, \quad k = 0, N
\]
The minimax a posteriori estimate of the linear function \( \sum_{k=0}^{N+1} (\ell_k, x_k)_n \) has the following form
\[
(\ell, \hat{x}) = \sum_{k=0}^{N+1} (\ell_k, \hat{x}_k)_i = \sum_{k=0}^{N} (Q_{2,k} H_k p_k, y_k)_i.
\]
The minimax a posteriori error has the following form
\[
\hat{\sigma} = \left[ 1 - \sum_{k=0}^{N} (y_k - H_k \hat{x}_k, Q_{2,k} y_k)_i \right]^{\frac{1}{2}} \left( \sum_{k=0}^{N+1} (\ell_k, p_k)_i \right)^{\frac{1}{2}},
\]
where \( p_k \) solves
\[
F'_k \hat{z}_k = C'_k \hat{z}_{k+1} - H'_k Q_{2,k} H_k p_k + \ell_k, \quad F'_{N+1} \hat{z}_{N+1} = \ell_{N+1},
\]
\[
F_{k+1} p_{k+1} = C_k p_k + B_k Q_{1,k}^{-1} B'_k \hat{z}_k, \quad F_0 p_0 = S Q_0^{-1} S' \hat{z}_0, \quad k = 0, \ldots, N.
\]

Consider the minimax a posteriori estimates in the form of filters.

**Theorem 4.** Let \( B_k = E \) and assume that the columns of the block matrix \([\begin{array}{c} \begin{array}{c} F_k \\ \hline H_k \end{array} \end{array}\])
are linear independent for any \( k = 0, \ldots, N \). Then for any \( \ell \in \mathbb{R}^n \) the minimax a posteriori estimate of the linear function \((\ell, x_N)\) by observations \( y_0 \ldots y_N \) in the form (6) has the following form \((\ell, x_{N\mid N})\), where \( \hat{x}_{k\mid k} \) can be computed using the following algorithm:
\[
\hat{x}_{k\mid k} = P_{k\mid k} F'_k (Q_{1,k-1}^{-1} + C_{k-1} P_{k-1\mid k-1} C'_{k-1})^{-1} C_{k-1} \hat{x}_{k-1\mid k-1} + P_{k\mid k} H'_k Q_{2,k} y_k,
\]
\[
P_{k\mid k} = (F'_k (Q_{1,k-1}^{-1} + C_{k-1} P_{k-1\mid k-1} C'_{k-1})^{-1} F_k + H'_k Q_{2,k} H_k)^{-1},
\]
\[
P_{0\mid 0} = (F'_0 Q_0 F_0 + H'_0 Q_{2,0} H_0)^{-1}, \quad \hat{x}_{0\mid 0} = P_{0\mid 0} H'_0 Q_{2,0} y_0.
\]

Let us consider the contents of the third chapter in brief. Assume that the state \( x(t) \) verifies the following DAE
\[
\frac{d}{dt} Fx(t) - C(t)x(t) = f(t), \quad (7)
\]
and
\[
Fx(a) = f_0 \quad (8)
\]
where \( F \) is a \( m \times n \)-matrix, \( C(t) \) is a \( m \times n \)-matrix with continuous on \([a, c] \) elements, \( t \mapsto f(t) \in \mathbb{R}^m \) is a vector-valued function from \( \mathbb{L}_2([a, c], \mathbb{R}^m) \). In order to define the solution to (7) let us define a linear mapping \( x \mapsto D x \in \mathbb{L}_2([a, c], \mathbb{R}^m) \times \mathbb{R}^m \) by the following rule
\[
D(D) \overset{\text{def}}{=} \{ x \in \mathbb{L}_2([a, c], \mathbb{R}^n) : Fx \in \mathbb{W}_{2}^1([a, c], \mathbb{R}^m) \}, \quad \mathbb{W}_{2}^1([a, c], \mathbb{R}^n) = \mathbb{W}_{2,f}^1([a, c], \mathbb{R}^n),
\]
\[
Dx \overset{\text{def}}{=} \left( \frac{d}{dt} Fx(t) - C(t)x(t), Fx(a) \right), \quad x \in D(D).
\]
Let \( \tilde{f} = (f, f_0) \in \mathbb{L}_2([a, c], \mathbb{R}^m) \times \mathbb{R}^m \). Then \( x(t) \) is a solution of (7)-(8) if
\[
Dx = \tilde{f}. \quad (9)
\]
It was proved in the dissertation that \( D \) is closed dense defined linear mapping and its adjoin \( D^* \) was calculated. The minimax state estimation theory for linear DAEs in the form (7) can be constructed applying the same ideas as for the linear
algebraic equations (1) presented above to operator equation (9). This approach will be presented below.

Assume that $x$ solves (7) and observed data $t \mapsto y(t) \in \mathbb{R}^l$ on $[a, c]$ is represented by

$$y(t) = H(t)x(t) + \eta(t),$$

(10)

where $H(t)$ is continuous $l \times n$-matrix on $[a, c]$, $t \mapsto \eta(t) \in \mathbb{R}^l$ is a realization of $l$-vector valued random process with zero mean and continuous correlation function $R_\eta(t, s) = E\eta(t)\eta'(s)$. Define a linear mapping $\mathcal{H}$ by the rule $\mathcal{H}x(t) = H(t)x(t)$.

Let us consider a priori minimax estimates. We assume that the initial condition $f_0 \in \mathbb{R}^m$, input $f \in L_2([a, c], \mathbb{R}^m)$ and correlation function $(t, s) \mapsto R_\eta(t, s)$ are uncertain and belong to the given bounded set, that is:

$$\tilde{f} \triangleq [f, f_0] \in \mathcal{G}, R_\eta \in \mathcal{G}_2$$

As above we will look for the estimate of the linear transformation of the solution $x(t)$ of (7):

$$\ell(x) \triangleq \int_c^a (\ell(t), x(t))_n \, dt, \quad \ell \in L_2([a, c], \mathbb{R}^n)$$

by means of the linear function $\mathcal{C}$ of observed data

$$\mathcal{C}(y) \triangleq \int_c^a (\mathcal{C}(t), y(t))_l \, dt + c, \quad \mathcal{C} \in L_2([a, c], \mathbb{R}^l), \quad c \in \mathbb{R}. \quad (11)$$

We will refer the function of observation in the form (11) as an estimate. Let us assign a worst-case estimation error

$$\sigma(u, c) \triangleq \sup_{x, R_\eta} \{E[\ell(x) - \mathcal{C}(y)]^2 | \mathcal{D}x \in \mathcal{G}, R_\eta \in \mathcal{G}_2\}$$

The worst-case estimation error measures the quality of the estimate $u$ and it does not depend on the particular realization of the uncertain parameters $(f_0, f, R_\eta)$.

**Definition 3.** The estimate $\hat{u}_c$ verifying

$$\sigma(\hat{u}, \hat{c}) \leq \sup_{x, R_\eta} \{E[\ell(x) - \mathcal{C}(y)]^2 | \mathcal{D}x \in \mathcal{G}, R_\eta \in \mathcal{G}_2\} = \sigma(u, c), \quad u \in U_l, \quad c \in \mathbb{R}$$

is called a minimax a priori mean-squared estimate. The number $\hat{\sigma} \triangleq \inf_{u, c} \sigma(u, c)$ is called a minimax a priori mean-squared error.

The following propositions present the algorithms for calculation of a priori estimates and errors in the variational form for the case of ellipsoidal bounding sets.
Theorem 5. Let
\[ G \overset{\text{def}}{=} \{ [f, f_0] : (Q_0f_0, f_0)_m + \int_a^c (Q_1f, f)_m \ dt \leq 1 \}, \ G_2 \overset{\text{def}}{=} \{ R_\eta : \int_a^c \text{tr}(Q_2R_\eta) \ dt \leq 1 \} \]

where \( Q_{1,2}(t) \) are symmetric positive definite \( m \times m \)-matrices, \( Q_{1,2}^{-1}(t) \) are continuous on \([a, c] \), \( Q_0 \) is symmetric positive definite \( m \times m \)-matrix. Define a linear operator \( T^+ \) by the rule: \( x \mapsto T^+x = \hat{\omega} \), where \( \hat{\omega} \) is a unique solution of the following optimization problem

\[
\| (z_0, z, u) \|_{Q^{-1}}^2 = (Q_0^{-1}z_0, z_0)_m + \int_a^c (Q_1^{-1}z, z)_m + (Q_2^{-1}u, u), \ dt \to \inf_{z_0, z, u},
\]

\[
(z_0, z, u) \in W \overset{\text{def}}{=} \{ D^*(z_0, z) + H^*u = x \}
\]

Then the minimax a priori mean-squared estimate is given by

\[
\hat{u}_c(y) = \langle T^+\ell, (0, y) \rangle_{L_2^m \times L_2^l} = \langle \hat{u}, y \rangle_{L_2^l},
\]

and minimax a priori mean-squared error may be represented as

\[
\hat{\sigma} = \sigma(\hat{u}) = \| T^+\ell \|_{Q^{-1}}^2
\]

provided

\[
\ell \in R(T) = \{ D^*(z_0, z) + H^*u, (z_0, z) \in D(D^*), u \in L_2([a, c], \mathbb{R}^l) \}
\]

Corollary 1. If the set \( R(T) \) is closed, then

\[
\hat{u}(t) = Q_2(t)H(t)p(t), \quad \sigma(\hat{u}) = \int_a^c (\ell(t), p(t))_R^n \ dt,
\]

(12)

where \( p \) solves the following two-point boundary value problem

\[
\frac{d}{dt}F'z(t) = -C'(t)z(t) + H'(t)Q_2(t)H(t)p(t) - \ell(t), \ F'z(c) = 0
\]

\[
\frac{d}{dt}Fp(t) = C(t)p(t) + Q_1^{-1}(t)z(t), \ Fp(a) = Q_0^{-1}(FF^+z(a) + d), \ F'd = 0
\]

(13)

The next proposition represents a way to approximate the a priori estimate \( \hat{u} \) by means of Tikhonov regularization approach.

Theorem 6. Take \( \alpha_k > 0 \) and let \( p_k, \hat{z}_k, d_k \) denote a unique solution of the following two-point boundary value problem:

\[
\frac{d}{dt}Fp(t) = C(t)p(t) + \alpha_kQ_1^{-1}(t)\hat{z}(t), \ Fp(a) = \alpha_kQ_0^{-1}(FF^+\hat{z}(a) + d),
\]

\[
\frac{d}{dt}F'z(t) = -C'(t)z(t) + (E + \frac{1}{\alpha_k}H'(t)Q_2(t)H(t))p(t) - \ell(t), \ F'z(c) = 0,
\]

(14)
Then
\[ \tilde{\ell} \in R(\mathcal{T}) \iff \|u_k - \hat{u}\|_{L^2}^2 + \|\tilde{z}_k - \hat{z}\|_{L^2_w}^2 + \|FF^+\hat{z}_k(a) + d_k - \tilde{z}_0\|_{R^m}^2 \overset{\alpha_k \downarrow 0}{\to} 0, \]
where \( \hat{z}_0 = FF^+\hat{z}(a) + \hat{d} \), \( \hat{u}_k = \frac{1}{\alpha_k}Q_2H_k \).

Let us present minimax estimates in the form of filters.

**Theorem 7.** Assume that \( t \mapsto K(t) \) solves the following descriptor Riccati equation
\[
\frac{d}{dt}(FK(t)) = C(t)K(t) + K'(t)C'(t) - K'(t)H'(t)Q_2H(t)K(t) + Q_i^{-1},
FK(a) = FF^+Q_0^{-1}FF^+,
\]
on \([a, c]\) and \( t \mapsto \hat{z}(t) \) verify the following differential-algebraic equation
\[
\frac{d}{dt}F'z(t) + C'(t)z(t) = H'(t)Q_2(t)H(t)K(t)z(t), \quad F'z(c) = \ell_0.
\]
The a priori minimax mean-squared estimate may be represented by
\[
(\ell, x(c)) = \int_a^c (Q_2(t)H(t)K(t)\hat{z}(t), y(t))dt,
\]
and the a priori minimax mean-squared error is given by
\[
\hat{\sigma} = (FK(c)F'^{+}\ell_0, F'^{+}\ell_0)_m.
\]
Let \( \hat{x} \) denote a solution of the following linear differential-algebraic equation
\[
\frac{d}{dt}F\hat{x}(t) = C(t)\hat{x}(t) + K'(t)H'(t)Q_2(t)(y(t) - H(t)\hat{x}(t)), \quad F\hat{x}(a) = 0
\]
Then the a priori minimax mean-squared estimate is given by
\[
(\ell_0, x(c)) = (F\hat{x}(c), F'^{+}\ell_0)_m.
\]
Definitions and representations for the minimax a posteriori estimates are given in the dissertation.

**Conclusion.** The dissertation presents a generalized Kalman duality concept and minimax state estimation approach for linear differential-algebraic equations. The key results of the dissertation are as follows:

- generalized Kalman duality concept;
• new definitions of the minimax estimates based on the generalized Kalman duality concept;
• new variational form of minimax state estimation algorithms for linear DAE;
• new minimax state estimation algorithms for DAE in the form of filters;
• efficient description of the reachability set for DAE;
• description of the uncertainty propagation by the DAE;

List of papers and conference presentations related to the dissertation.

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