A note on hierarchy of universal relations for neutron stars in terms of multipole moments

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Abstract
Recent studies of the analytical and numerical models of neutron stars suggest that their exterior field can be well approximated by only four arbitrary parameters of the 2-soliton solution of Einstein’s equations, which gives rise to the so-called no-hair conjecture for neutron stars proposed by Yagi et al.

By assuming that the latter conjecture is correct, we show that there exists an infinite hierarchy of universal relations for neutron stars in terms of multipole moments that arises as a series of the degeneration conditions for generic soliton solutions. The analysis of the simplest of these relations involving the hexadecapole mass moment is able to reveal which of the known equations of state for the stellar interior are most consistent with the Yagi et al conjecture.

Keywords: multipole moments, universal relations, neutron stars, solitonic solutions

1. Introduction

In recent years much attention was paid to the study of the universal properties of neutron stars (NSs) with the aid of both the numerical and analytical approaches. A remarkable\ I-Love-Q\ relation between the NS’s moment of inertia, the tidal Love number and the quadrupole moment was first discovered by Yagi and Yunes [1] via a numerical analysis of the Hartle–Thorne slow-rotation approximation [2] and then extended to arbitrary rotation and some new universal properties by Pappas and Apostolatos [3] and by Chakrabarti et al [4]. The exact solutions approach to the analysis of various phenomena around NSs was introduced.
by Sibgatullin and Sunyaev [5] who demonstrated that a 3-parameter quadrupole solution [6] fitted very well the extensive numerical data of the well-known Cook et al paper [7]; they also observed that in terms of the dimensionless multipole moments the properties of NSs independent of the equations of state (EoSs) can be better seen. Comparison of the analytical and numerical models of NSs was performed by Berti and Stergioulas [8] with the aid of the RNS code [9, 10], and this subsequently led, via Ryan’s method [11], to the revision of multipole moments in numerical solutions [12]. A better understanding of the multipole structure of NSs made it possible, on the one hand, to put the universal relations for NSs into the language of multipole moments and, on the other hand, to establish [3] that the above structure is generically determined by only four multipole moments, thus being universal for all the physically realistic EoSs known in the literature. Furthermore, in the paper [13] Yagi et al conjectured that, similar to black holes, NSs are likely to verify their own ‘no-hair’ theorem according to which the higher multipoles could be inferred from the form of the four meaningful lowest multipoles, and they discussed the numerical evaluation of the NS’s mass-hexadecapole moment in the light of their conjecture. In [13] it was observed in particular that the 4-parameter 2-soliton solution of Einstein’s equations [14–16], regarded by Pappas and Apostolatos as a possible analytical model describing the geometry around a universal NS, possesses a hexadecapole moment whose spin dependence starts at quadratic order, whereas, according to Yagi et al, this moment is expected to be strictly quartic in angular momentum.

The objective of the present note is to demonstrate that the Yagi et al no-hair hypothesis, combined with the aforementioned 2-soliton solution (henceforth referred to as the MMR solution), gives rise to a hierarchy of the universal relations for NSs in terms of multipole moments. The simplest relation from this hierarchy yields the expression for the hexadecapole moment $M_4$ of the MMR solution which we will compare, for two known EoSs, with the empirical formulas of [13] to see which of these EoSs has a better consistency with the Yagi et al conjecture.

2. Multipole moments and the universal relations

The multipole structure of stationary axially symmetric vacuum spacetimes is well known thanks to the fundamental papers of Geroch [17], Hansen [18] and Thorne [19]. The technical calculation of the moments, describing the distributions of mass and angular momentum, is facilitated by the Fodor–Hoenselaers–Perjés (FHP) procedure [20] which makes use of the Ernst complex potential formalism [21] in order to find the coefficients $m_n$ arising in the expansion of the function

$$X(z) \equiv \frac{1 - e(z)}{1 + e(z)} = \sum_{n=0}^{\infty} m_n z^{-n}$$

when $z \to \infty$. The above $e(z)$ denotes the axis ($\rho = 0$) value of the Ernst complex potential $E(\rho, z)$ of a particular solution, $\rho$ and $z$ being the Weyl–Papapetrou coordinates. The first four quantities $m_n$ coincide with the Geroch–Hansen (GH) complexified multipole moments $P_{n_0}$, $n = 0, 3$, while other $m_n$, $n \geq 4$, are equal to $P_n$ only up to certain combinations of the lower-order $m_l$, $l < n$ (see [20] for the explicit form of those combinations). It is important to note that the equilibrium models of NSs are described by the solutions which, in addition to being stationary and axisymmetric, are also symmetric about the equatorial plane [10], and the latter symmetry imposes restrictions on the form of the corresponding axis data $e(z)$ and coefficients $m_n$ in (1). As was shown in [22, 23], the function $e(z)$ of an equatorially symmetric spacetime satisfies the condition $e(z)e^*(-z) = 1$ (the star symbol denotes complex conjugation);
consequently, all even quantities $m_n$ of such a spacetime are real, and all odd $m_n$ are pure imaginary [22]. The same is true for the corresponding multipoles $P_n$: in the reflection-symmetric case we have $P_{2k} = M_{2k}$ and $P_{2k+1} = iJ_{2k+1}$, $k = 0, 1, \ldots$, where $M_{2k}$ and $J_{2k+1}$ are, respectively, the mass and angular momentum GH multipole moments.

Noteworthily, the general extended vacuum $N$-soliton solution admits parametrization exclusively in terms of the ‘multipoles’ $m_n$ [24]:

\begin{equation}
    e(z) = \frac{e_-}{e_+}, \quad e_{\pm} = (L_N)^{-1}
    \begin{vmatrix}
        z^N \pm \sum_{n=0}^{N-1} m_n z^{N-1-n} & m_N \ldots m_{2N-1} \\
        z^{N-1} \pm \sum_{n=2}^{N-2} m_n z^{N-2-n} & m_{N-1} \ldots m_{2N-2} \\
        \vdots & \vdots & \ddots & \vdots \\
        z \pm m_0 & m_1 \ldots m_N \\
        1 & m_0 \ldots m_{N-1}
    \end{vmatrix}, \quad (2)
\end{equation}

where the $n \times n$ determinant $L_n$ has the form

\begin{equation}
    L_n =
    \begin{vmatrix}
        m_{n-1} & m_n & \ldots & m_{2n-2} \\
        m_{n-2} & m_{n-1} & \ldots & m_{2n-3} \\
        \vdots & \vdots & \ddots & \vdots \\
        m_1 & m_2 & \ldots & m_n \\
        m_0 & m_1 & \ldots & m_{n-1}
    \end{vmatrix}.
\end{equation}

Restricting ourselves to the equatorially symmetric configurations, we see that the Kerr solution [25] is contained in (2) as the $N=1$ case, with $m_0 = M$ and $m_1 = iJ$, $M$ being the total mass and $J$ the total angular momentum [26]. The next, $N=2$ specialization of formulas (2), determines the MMR solution with the axis data

\begin{equation}
    e_\pm(z) = (L_2)^{-1}
    \begin{vmatrix}
        z^2 \pm m_0 z \pm m_1 & m_2 & m_3 \\
        z \pm m_0 & m_1 & m_2 \\
        1 & m_0 & m_1
    \end{vmatrix}, \quad L_2 =
    \begin{vmatrix}
        m_1 & m_2 \\
        m_0 & m_1
    \end{vmatrix}, \quad (4)
\end{equation}

that was recently regarded and advocated as describing the exterior field of a universal NS [3, 15, 27]; it has four arbitrary parameters corresponding to four arbitrary multipole moments:

\begin{equation}
    m_0 = M_0 \equiv M, \quad m_1 = iJ_1 \equiv iJ, \quad m_2 = M_2, \quad m_3 = iJ_3,
\end{equation}

where $M_2$ is the mass quadrupole moment and $J_3$ is the angular momentum octupole moment (the explicit form of the MMR solution in two different parametrizations can be found in [16]). If we now assume that the results of Pappas and Apostolatos [3] obtained on the basis of a variety of very convincing arguments are correct and the geometry around NSs is indeed determined by only four multipole moments (5), then, bearing in mind the no-hair hypothesis for NSs put forward by Yagi et al [13], we inevitably arrive at the MMR spacetime as the simplest and hence most suitable model for the exterior of a NS complying with the conditions of papers [3, 13] (see [15, 16] for details). The fact that the MMR solution is the simplest one possessing the required four moments is very important in itself because it makes this solution in a sense similar to the Kerr spacetime whose unique property is that it is the simplest possible solution among infinite number of the 2-parameter solutions defined by the parameters of mass and angular momentum [28]. Clearly, the higher GH multiple moments of the MMK solution will then be some well-defined functions of the above four parameters that can be found from the corresponding axis data by means of the FHP procedure.
The explicit expressions of the multipoles $M_{2n}$ and $J_{2n+1}$, $n \geq 2$, as functions of the moments (5), in the case of the MMR solution would give us the simplest hierarchy of the universal relations for NSs. Obviously, each relation from this hierarchy determines how a higher multipole $M_{2n}$ or $J_{2n+1}$, with a specific $n$, depends on the first four lower moments (5); however, since such relations involve only one higher multipole, they do not actually provide any information about possible interrelations between the higher multipole moments themselves. So it is remarkable that there does exist a more sophisticated hierarchy of the universal relations for NSs directly connecting different higher multipoles with each other. This new hierarchy arises in (2) as a series of the degeneration conditions of the solitonic solutions with $N > 2$ to the $N = 2$ case. Indeed, as was shown in [24], the general $N$-soliton solution degenerates to the $(N - 1)$-soliton case when the determinant $L_N$ defined by (3) becomes equal to zero; then further degeneration would require zero values of the determinants $L_{N-1}$, $L_{N-2}$ and so on, until we finally arrive at the 2-soliton solution by means of the conditions $L_3 = 0$, $L_2 \neq 0$, the latter nonequality being needed to stop the degeneration process. By inverting this reasoning, we can say that the higher multipole moments of the MMR 2-soliton solution must be such that the conditions

$$L_n = 0 \quad \text{for all} \quad n > 2$$

(6)

are satisfied. It is easy to see from (3) that the above conditions (6) establish how the moments $m_{2n-2}$ depend on the moments $m_{2n-3}$, or, roughly speaking and taking into account the equatorial symmetry of the MMR solution, how the GH mass multipoles $M_{2n-2}$ depend on the spin multipoles $J_{2n-3}$.

The simplest of the relations (6), accounting for (5), takes the form

$$L_3 = \begin{vmatrix} M_2 & iJ_3 & m_4 \\ iJ & M_2 & iJ_3 \\ M & iJ & M_2 \end{vmatrix} = 0,$$

(7)

whence we get

$$m_4 = \frac{M_2^2 + 2JJ_3M_2 - MJ_4^2}{MM_2 + J^2}.$$ 

(8)

This formula permits us to compare the dependence of the hexadecapole moment $M_4$ on the angular momentum $J$ in the MMR solution and in the known numerical models for NSs. In the paper [13] it was found, with the aid of the quartic-order slow-rotation approximation and numerical solutions, that similar to the Kerr spacetime, the multipoles $M_2$, $J_3$ and $M_4$ of NSs are proportional, respectively, to $J^2$, $J^3$ and $J^4$, so that, according to Yagi et al, the hexadecapole moment $M_4$ is free of the terms proportional to $J^2$. Supposing that $M_2 \propto J^2$ and $J_3 \propto J^3$, one can readily see that the quantity $m_4$ in (8) is proportional to $J^4$. Nonetheless, the relation of $m_4$ to the GH hexadecapole moment $M_4$ is defined by the formula [20]

$$m_4 = M_4 + \frac{1}{7}M(J^2 + MM_2),$$

(9)

and it is clear that the second term on the right-hand side of (9) is proportional to $J^2$, so that the expression of $M_4$ of the MMR solution necessarily contains terms proportional both to $J^2$ and $J^4$. Moreover, the condition for $M_4$ to be proportional strictly to $J^4$ implies $M_2 = -J_4^2/M$, which is exactly the value of the mass-quadrupole moment of the Kerr solution\(^4\). Therefore,

\(^4\)Mention that the situation will be the same if we opt to use the multipole moments constructed according to Thorne’s definition [19], since these are known [29] to be proportional to the GH multipoles.
the structure of the mass hexadecapole moment in the MMR solution and in the solutions analyzed by Yagi et al is not identical, and the difference seems to be determined by the specific properties of the interior solutions in the approximate and numerical models considered in [13]. We shall return to this point later on.

As was remarked in [3], the universal relations for NSs must be independent of the total mass $M$ when these are rewritten in terms of the rescaled, dimensionless moments. Then, bearing this in mind and introducing the rescaled moments via the formulas

$$J = jM^2, \quad M_2 = qM^3, \quad J_3 = sM^4, \quad M_4 = \mu M^5,$$

it is possible, taking into account (9), to rewrite formula (8) in the ‘$M$ free’ form

$$\mu = -\frac{1}{17}(j^2 + q) + \frac{q^3 + 2jqs - s^2}{j^2 + q},$$

thus demonstrating that the relation $L_3 = 0$ safely passes the additional test of universality.

Remarkably, the next relation from the hierarchy (6), $L_4 = 0$, which involves the multipoles $M_6$ and $J_5$, becomes independent of $M_6$ on account of (8) and yields directly the expression for the spin multipole $J_5$; written in terms of the dimensionless moments, with $J_5 = \chi M^5$, it takes the form

$$\chi = \frac{1}{21}(j^3 + 8jq - 7s) + \frac{jqs^2 - q^2s}{j^2 + q} - \frac{jq^4 - 3jqs^2 - 3q^3s + s^3}{(j^2 + q)^2}. \quad (12)$$

In principle, it is not difficult to show that, after an appropriate rescaling, $M$ always cancels out from the generic relation $L_n = 0$ in the equatorially symmetric case under consideration.

### 3. An application

The fact that the mass hexadecapole moment of the MMR solution is not strictly quartic in spin does not actually lead immediately to the conclusion that it contradicts the Yagi et al analysis, because $\mu$ in (10) and (11) can be always formally put into the form $\mu = \alpha_0 j^4$, permitting a trivial estimation of the coefficient $\alpha_0$ for any concrete model of a NS. Therefore, for getting more information, it is likely to compare the values of the moment $M_4$ calculated, for some available numerical models of NSs, with the aid of the Yagi et al approach [13] based on Ryan’s method [11], on the one hand, and by means of our formula (8) after the substitution in it of (9), on the other hand. To obtain the values of the first type, which we denote as $M_4^{(n)}$, we have used the hints left in [13] for the evaluation of $M_4$ in the case of EoSs AU and L, i.e. formulae (93)–(94) and table II of [13]; the concrete models (originally due to Berti and Stergioulas [8]) are taken from tables II and VI of the supplement to the paper [3]. For the same models we then estimate the mass-hexadecapole moments $M_4^{(a)}$ using our analytical formula. The results are summarized in tables 1 and 2, which also include the ratios $M_4^{(a)}/M_4^{(n)}$ for convenience. One can see that the correspondence between $M_4^{(a)}$ and $M_4^{(a)}$ for all three sequences of EoS AU from table 1 is quite good, though ranging from almost full coincidence to an appreciable difference of nearly 40%. As for table 2, it appears that the difference between the values $M_4^{(a)}$ and $M_4^{(a)}$ calculated for the models with the EoS L is rather significant for almost all instances, whence we conclude that the latter EoS shows less agreement with the Yagi et al conjecture than the EoS AU. Therefore, the Yagi et al no-hair hypothesis may be considered

\footnote{As was pointed out by one of the referees, our conclusion seems to be corroborated by the results reported recently in [30] because the EoS L determines a very large radius and maximum mass of a stellar object.}
as an instrument of the advanced selection among the known EoSs (and among the specific parameter sets within each EoS) for the highly accurate modeling of NSs.

It would probably be worth remarking in conclusion that the scope of applicability of the universal relations considered in the present note actually goes beyond the NSs only. Thus, for

| $M$ | $J/M^2$ | $M_2$ | $J_3$ | $M_4^{(n)}$ | $M_4^{(a)}$ | $M_4^{(a)}/M_4^{(n)}$ |
|-----|---------|-------|-------|-------------|-------------|---------------------|
| 2.072 | 0.201 | −1.45 | −1.14 | 2.217 | 1.947 | 0.878 |
| 2.087 | 0.414 | −6.08 | −10.0 | 40.21 | 25.51 | 0.634 |
| 2.097 | 0.529 | −9.96 | −21.2 | 107.7 | 71.38 | 0.662 |
| 2.108 | 0.616 | −13.6 | −34.1 | 199.1 | 121.0 | 0.608 |
| 2.112 | 0.661 | −15.7 | −42.7 | 264.4 | 161.4 | 0.610 |

| $M$ | $J/M^2$ | $M_2$ | $J_3$ | $M_4^{(n)}$ | $M_4^{(a)}$ | $M_4^{(a)}/M_4^{(n)}$ |
|-----|---------|-------|-------|-------------|-------------|---------------------|
| 3.164 | 0.194 | −1.68 | −1.37 | 2.202 | 1.825 | 0.829 |
| 3.207 | 0.406 | −8.08 | −14.5 | 41.97 | 30.23 | 0.720 |
| 3.253 | 0.550 | −16.1 | −41.4 | 140.3 | 116.2 | 0.828 |
| 3.291 | 0.645 | −23.9 | −75.1 | 263.7 | 253.1 | 0.960 |
| 3.318 | 0.706 | −30.3 | −107. | 376.7 | 403.0 | 1.07 |

| $M$ | $J/M^2$ | $M_2$ | $J_3$ | $M_4^{(n)}$ | $M_4^{(a)}$ | $M_4^{(a)}/M_4^{(n)}$ |
|-----|---------|-------|-------|-------------|-------------|---------------------|
| 2.071 | 0.194 | −2.76 | −2.28 | 7.405 | 5.730 | 0.774 |
| 2.080 | 0.417 | −12.2 | −22.0 | 159.3 | 90.39 | 0.567 |
| 2.087 | 0.543 | −20.3 | −47.9 | 460.9 | 243.6 | 0.529 |
| 2.095 | 0.650 | −28.6 | −81.3 | 952.9 | 478.5 | 0.502 |
| 2.097 | 0.698 | −32.9 | −100. | 1269. | 628.6 | 0.495 |

| $M$ | $J/M^2$ | $M_2$ | $J_3$ | $M_4^{(n)}$ | $M_4^{(a)}$ | $M_4^{(a)}/M_4^{(n)}$ |
|-----|---------|-------|-------|-------------|-------------|---------------------|
| 4.012 | 0.178 | −3.80 | −4.23 | 16.8 | 8.943 | 0.532 |
| 4.051 | 0.375 | −18.5 | −45.4 | 338.5 | 138.6 | 0.410 |
| 4.098 | 0.528 | −40.3 | −144. | 1367. | 599.8 | 0.439 |
| 4.139 | 0.635 | −62.6 | −279. | 2930. | 1415. | 0.483 |
| 4.167 | 0.700 | −79.8 | −401. | 4399. | 2276. | 0.518 |

| $M$ | $J/M^2$ | $M_2$ | $J_3$ | $M_4^{(n)}$ | $M_4^{(a)}$ | $M_4^{(a)}/M_4^{(n)}$ |
|-----|---------|-------|-------|-------------|-------------|---------------------|
| 4.321 | 0.479 | −29.5 | −90.2 | 1059. | 308.0 | 0.291 |
| 4.325 | 0.489 | −31.9 | −101. | 1153. | 358.4 | 0.311 |
| 4.355 | 0.555 | −45.2 | −170. | 1950. | 709.4 | 0.364 |
| 4.396 | 0.641 | −66.0 | −299. | 3560. | 1488. | 0.418 |
| 4.420 | 0.686 | −79.4 | −394. | 4742. | 2140. | 0.451 |
instance, the whole hierarchy (6) is eligible in the case of the Kerr solution too, and besides
should be supplemented with the relation $L_2 = 0 \iff M_2 = -J^2/M$ which shows that a NS
collapses to a black hole when its quadruple moment becomes that of the Kerr solution, inde-
pendently of the value of its spin-octupole moment $J_3$. Moreover, other stellar objects with a
richer structure than that of NSs and hence requiring more than four arbitrary real parameters
for its description, could be analytically approximated by the $N = 3, 4, \ldots$ extended soliton
solutions, in which case the inequality in (6) starts, respectively, from $3, 4, \ldots$ The degenera-
tion conditions then would reflect in particular the evolution of stars from one type to another.

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