Geometric Origin of Large Lepton Mixing in a Higher Dimensional Spacetime

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Abstract

The large mixing in the lepton sector observed in the recent neutrino-oscillation experiments strongly suggests that nature of left-handed lepton doublets is very different from that of left-handed quark doublets. This means that there is a big disparity between the matter multiplets $5^*$'s and $10$'s in the SU(5) unified theory. We show that this big difference can be explained in a six-dimensional spacetime compactified on the $T^2/Z_3$ orbifold. That is, we propose to put three families of $5^*$'s on three equivalent fixed points of the orbifold and the three $10$'s in the two-dimensional bulk. We construct an explicit model realizing this situation and show that the democratic mass structure in the lepton sector is naturally obtained and hence the model explains the observed bi-large lepton mixing and simultaneously the required small mixing $U_{e3}$. The mass matrices and mixing in the quark sector are also briefly discussed.

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1 Introduction

Bi-large mixing in the lepton sector [1, 2] is one of the most remarkable features in the standard model. Contrary to the quark sector, where the mass-diagonalization matrix of the left-handed up-type quarks almost coincides with that of the left-handed down-type quarks, there is a large discrepancy between the matrix of the left-handed charged leptons and that of the neutrinos. What is more embarrassing is the small value of $\mathcal{U}_{e3}$ [3] in the lepton flavour mixing matrix.

Various attempts have been made to get a better understanding of such flavour structure, yet we do not have a satisfactory picture. The Froggatt–Nielsen structure is able to explain the bi-large mixing assuming less hierarchy for the lepton doublets [4, 5], but an accidental cancellation is necessary between $\mathcal{O}(1)$ coefficients to explain the large mixing angle of the solar neutrino oscillation [1] or the small $\mathcal{U}_{e3}$ [5]. The democratic mass matrix for the charged leptons [6, 7] leads to large-angle rotations in the mass diagonalization, but this does not necessarily mean a large mixing in the W-boson current, unless the mass matrix for the Majorana neutrinos is almost diagonal.

A number of tentative solutions are proposed within the framework of four-dimensional field theories, where one tries to understand the flavour structure as a result of some symmetries. In this letter, we propose a framework in a higher-dimensional spacetime. Here, one can incorporate dynamical effects in the mass textures, which cannot be described by symmetries of four-dimensional field theories.

The model is described in terms of SU(5) unified theories with four-dimensional $\mathcal{N} = 1$ supersymmetry. We start from a six-dimensional SU(5) field theory with a minimal supersymmetry. A hypermultiplet in the SU(5)-adj. representation is introduced to cancel the box anomaly arising from the SU(5) vector multiplet. Three sets of an anomaly-free combination of hypermultiplets, SU(5)-(10(10)⊕3×5(5')), are also introduced. Two extra dimensions are compactified on a $T^2/Z_3\langle\sigma\rangle$ orbifold. Three $\mathcal{N}=$1 SU(5)-10 chiral multiplets survive the orbifold projection, and three 5's (and 1’s) are further introduced at the three fixed points of the $T^2/Z_3\langle\sigma\rangle$ orbifold for theoretical consistency. These multiplets are identified with the three families of quarks, leptons (and right-handed neutrinos). It is quite natural to consider that the three 5’s are equivalent, since the three fixed points are equivalent to one another [8].

Origins of SU(5)-10’s and SU(5)-5’s are totally different in this model. The 10’s propa-
gate in the whole bulk, while the $5^*$’s and $1$’s localize at three fixed points. Then, each $10$ has the same wave function at the three fixed points, and hence the texture of the charged-lepton mass matrix will be democratic. On the contrary, the Dirac Yukawa matrix for the (left- and right-handed) neutrinos and mass matrix of the right-handed Majorana neutrinos would be almost diagonal (hence so is that of the left-handed Majorana neutrinos), since a pair of $5^*$ and $1$ is isolated from other pairs of $5^*$ and $1$ at different fixed points. The diagonal nature of the Majorana neutrino mass matrix is naturally explained, and hence the bi-large mixing and the smallness of the $U_{e3}$ follows in the lepton flavour mixing matrix [7]. The flavour structure of the quark sector is also discussed.

2 The Model

An SU(5) vector multiplet of the minimal supersymmetry gives rise to box anomalies in the six-dimensional spacetime. The Green–Schwarz mechanism [9] cannot be used to cancel these anomalies. They are cancelled by introducing a hypermultiplet in the adj. representation of the SU(5) gauge group. One can add one hypermultiplet in the $10(10^*)$ representation together with three in the $5(5^*)$ without introducing irreducible anomalies [10]. We introduce three sets of this anomaly-free combination in addition to the hypermultiplet in the adj. representation.

Two extra dimensions are compactified. The length scale of the compactified manifold is assumed to be larger than the Planck length, but smaller than the inverse of the GUT scale ($\sim 10^{16}$ GeV). Since the minimal supersymmetry in the six-dimensional spacetime becomes $\mathcal{N} = 2$ supersymmetry in the four-dimensional spacetime, we take an orbifold as the compactified manifold to reduce the supersymmetry. We consider a $\mathbb{T}^2/\mathbb{Z}_3 \langle \sigma \rangle$ orbifold. This orbifold has three equivalent fixed points, and this is why we choose this background geometry [8]. The $\mathbb{Z}_3 \langle \sigma \rangle$ action on the two-dimensional torus $\mathbb{T}^2$ (whose coordinates are described by $z \equiv x_4 + ix_5$) is given by

$$\sigma : z \mapsto \sigma \cdot z \equiv \omega^{-2} z,$$

(1)

where the $\sigma$ is a generator of the $\mathbb{Z}_3 \langle \sigma \rangle$ group and $\omega \equiv e^{2\pi i/3}$. Fields transform under this

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3A hypermultiplet in the $10(10^*)$ representation consists of Kaluza–Klein towers of chiral multiplets in the $10$ and $10^*$ representations of the four-dimensional $\mathcal{N} = 1$ supersymmetry. So does that in the $5(5^*)$ representation.

4One has to introduce two-form field to cancel reducible anomalies through the Green–Schwarz mechanism.
geometrical rotation as follows:

\[ \Psi(x, z, \theta, \Theta) \equiv (\Sigma + \Theta^\alpha W_\alpha)(x, z, \theta) \mapsto \omega^{-2}\Psi(x, \sigma \cdot z, \omega \theta, \omega \Theta) \]  

(2)

for the SU(5) vector multiplet (in terms of four-dimensional \( \mathcal{N} = 2 \) supersymmetry), or equivalently,

\[ W_\alpha(x, z, \theta) \mapsto \omega^{-1}W_\alpha(x, \sigma \cdot z, \omega \theta), \]  

(3)

\[ \Sigma(x, z, \theta) \mapsto \omega^{-2}\Sigma(x, \sigma \cdot z, \omega \theta), \]  

(4)

in terms of four-dimensional \( \mathcal{N} = 1 \) supersymmetry. For hypermultiplets,

\[ \Phi(x, z, \theta) \mapsto \Phi(x, \sigma \cdot z, \omega \theta), \]  

(5)

\[ \bar{\Phi}(x, z, \theta) \mapsto \bar{\Phi}(x, \sigma \cdot z, \omega \theta). \]  

(6)

Here, the chiral multiplets of the four-dimensional \( \mathcal{N} = 1 \) supersymmetry, \( \Phi \) and \( \bar{\Phi} \), stand for the hypermultiplets \( (\Phi(\text{adj.}), \bar{\Phi}(\text{adj.})), (\Phi(10), \bar{\Phi}(10^*)), (\Phi(5), \bar{\Phi}(5^*)), (\Phi'(5), \bar{\Phi}'(5^*)) \) and \( (\Phi''(5), \bar{\Phi}''(5^*)) \).

The orbifold projection selects out states that are invariant under an action of an orbifold group; this group is generated by a symmetry transformation of the extra-dimensional space (i.e. \( Z_3 \), in this case, given in eq.(1)) accompanied by twisting fields by internal discrete symmetries of the action. To keep the consistency of the resulting theories the internal symmetry to be used for the twisting of fields should have the same degree as that of the space symmetry (i.e. \( Z_3 \)).

The first candidate of such an internal symmetry is the SU(2) \( \text{R} \) symmetry. (The SU(2) \( \text{R} \) symmetry of the minimal supersymmetry of the six-dimensional spacetime is what would become an SU(2) \( \text{R} \) symmetry of the \( \mathcal{N} = 2 \) supersymmetry in the four-dimensional spacetime when one would simply take a toroidal compactification.) This SU(2) symmetry does not have any anomaly with gauge groups; both SU(2)[SU(5)]\(^3\) and SU(2)[gravity]\(^3\) box anomalies vanish. Thus, we use a \( Z_3 \) subgroup included in the maximal torus (U(1) subgroup) of the SU(2) \( \text{R} \) symmetry. The SU(5) vector multiplet transforms under the maximal torus as

\[ \Psi(x, z, \theta, \Theta) \mapsto \Psi(x, \sigma \cdot z, e^{-i\alpha \theta}, e^{i\alpha \Theta}), \quad \alpha \in \mathbb{R}, \]  

(7)

or equivalently,

\[ W_\alpha(x, z, \theta) \mapsto e^{i\alpha}W_\alpha(x, \sigma \cdot z, e^{-i\alpha \theta}), \]  

(8)

\[ \Sigma(x, z, \theta) \mapsto \Sigma(x, \sigma \cdot z, e^{-i\alpha \theta}). \]  

(9)
At the same time, the hypermultiplets transform as

\[ \Phi(x, z, \theta) \rightarrow e^{i\alpha} \Phi(x, \sigma \cdot z, e^{-i\alpha} \theta), \]
\[ \Phi^\dagger(x, z, \theta) \rightarrow e^{-i\alpha} \Phi^\dagger(x, \sigma \cdot z, e^{i\alpha} \theta). \]

(10)

Since the \( \mathbb{Z}_3 \) space rotation changes the Grassmann coordinate \( \theta \) into \( \omega \theta \) (eqs. (2–6)), the \( \mathbb{Z}_3 \langle \sigma \rangle \) transformation keeps \( \theta \) invariant if the space rotation is accompanied by a twisting under the SU(2) \( R \) symmetry with \( e^{-i\alpha} = \omega^{-1} \). Then, the orbifold projection conditions do not make any discrimination between bosonic and fermionic components in the same \( \mathcal{N} = 1 \) multiplets, and hence the \( \mathcal{N} = 1 \) supersymmetry is kept unbroken. For the vector multiplet, the orbifold projection conditions are now given by

\[ \mathcal{W}_\alpha(x, z, \theta) = \left( \sigma : \mathcal{W}_\alpha(x, z, \theta) \mapsto \mathcal{W}_\alpha(x, \sigma \cdot z, \theta) \right), \]
\[ \Sigma(x, z, \theta) = \left( \sigma : \Sigma(x, z, \theta) \mapsto \omega^{-2} \Sigma(x, \sigma \cdot z, \theta) \right). \]

(12)

(13)

The SU(2) \( R \) symmetry, however, is not sufficient to yield a phenomenologically interesting model. This is because all the hypermultiplets would be, then, under the conditions

\[ \Phi(x, z, \theta) = \omega \Phi(x, \sigma \cdot z, \theta), \]
\[ \Phi^\dagger(x, z, \theta) = \omega^{-1} \Phi^\dagger(x, \sigma \cdot z, \theta), \]

(14)

(15)

and hence no Kaluza–Klein zero mode would survive. We therefore introduce a twisting by an additional global U(1) symmetry in the orbifold projection conditions. This U(1) symmetry should not be violated even quantum mechanically, and hence the anomaly with gauge fields should vanish. U(1)[gravity]\(^3\) always vanishes, and U(1)[SU(5)]\(^3\) vanishes if one takes the charge assignment of “fiveness”\(^5\) given in Table \( \text{[I]} \) (A suitable linear combination of this U(1)\( _{\text{fiveness}} \) and the U(1)\( _Y \) of the standard model is nothing but the U(1)\(_{B-L} \).) The orbifold projection conditions of the hyper multiplets are taken as

\[ \Phi(x, z, \theta) = (\sigma : \Phi(x, z, \theta) \mapsto \omega^q \omega \Phi(x, \sigma \cdot z, \theta)), \]
\[ \Phi^\dagger(x, z, \theta) = \left( \sigma : \Phi^\dagger(x, z, \theta) \mapsto \omega^q \omega^{-1} \Phi^\dagger(x, \sigma \cdot z, \theta) \right), \]

(16)

(17)

where \( q \) is the fiveness charge.

Kaluza–Klein zero modes that survive this orbifold projection conditions are summarized in terms of four-dimensional \( \mathcal{N} = 1 \) supersymmetry as follows: an SU(5) vector multiplet

\(^5\)“Fiveness” U(1) symmetry can be gauged in the bulk. Reducible anomalies can be cancelled by the Green–Schwarz mechanism.
and three sets of chiral multiplets $\Phi(10)$, $\Phi^\prime(5)$ and $\Phi^\prime\prime(5)$. Then, the $[SU(5)]^3$ triangle anomaly arises at all three fixed points, but this anomaly is cancelled by introducing three chiral multiplets in the $5^*$ representation at each fixed point. We denote these three chiral multiplets at a fixed point as $X(5^*)$, $X'(5^*)$ and $X''(5^*)$. The fiveness charge is assigned to these fields as 3 for $X(5^*)$ and $-2$ for $X'(5^*)$ and $X''(5^*)$.

The three sets of the Kaluza–Klein zero modes $\Phi^\prime(5)$ and $\Phi^\prime\prime(5)$ form vector-like pairs with the $X'(5^*)$ and $X''(5^*)$ at all three fixed points, and hence we expect that their mass terms would be generated. The chiral matter content consists of three Kaluza–Klein zero modes $\Phi(10)$ from the bulk and three $X(5^*)$’s from three fixed points (one $X(5^*)$ from one fixed point). We identify these particles with the three families of quarks and leptons in the SU(5)-unified theories. The observed neutrino masses suggest right-handed neutrinos below the GUT scale. Thus, we also introduce the three families of right-handed neutrinos at the three fixed points. They are denoted as $X(1)$, and we expect that they have the same origin and hence much the same transformation property as the $X(5^*)$. The fiveness charge of this $X(1)$ is $-5$. The U(1) fiveness should be broken below the GUT scale so that the right-handed neutrinos acquire masses.

There should be Higgs particles that give masses to quarks and leptons. We also need a sector that breaks the SU(5) symmetry down to the standard-model gauge group. In order to keep the geometrical equivalence of the three fixed points, we assume that the Higgs particles ($H(5)$ and $\bar{H}(5^*)$) and the SU(5)-breaking sector is localized just at the centre of the three fixed points (see Fig. 1). Such localized sectors in the bulk space should appear without any theoretical inconsistencies. Brane solutions of the supergravity give such examples of dynamical localization of supersymmetric gauge theories, and hence we refer

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6 We also use the same nomenclature $\Phi(repr.)$ for the Kaluza–Klein zero modes, which has been used so far as six-dimensional fields that depend on the coordinates $z$.

7 One might expect that a more fundamental theory would provide these particles (e.g. as twisted sector fields in string theories).

8 The reason of this assignment for the $X'(5^*)$ and $X''(5^*)$ will be clear in the next paragraph. The charge of $X(5^*)$ is determined so that the triangle anomaly $U(1)_{\text{fiveness}}[SU(5)]^2$ vanishes at each fixed point.

9 When the “fiveness” U(1) symmetry is gauged in the bulk, then, the triangle anomaly cancellation of the “fiveness” symmetry at orbifold fixed points also requires the right-handed neutrinos, i.e. right-handed neutrinos are also required at fixed points from theoretical consistencies.

10 We should note here that the triangle anomaly $U(1)_{\text{fiveness}}[\text{gravity}]^2$ is cancelled out by this introduction of the right-handed neutrino at each fixed point.

11 One might then consider that there would be unwanted Nambu–Goldstone modes corresponding to the branes’ motion in the smooth space. However, since the continuous translational symmetry has already been lost in the orbifold geometry, it can be expected that this violation of the symmetry gives masses to the Nambu–Goldstone modes through non-perturbative effects. Discussion on such non-perturbative dynamics,
to these sector as “centre brane”. An SU(5)-breaking model can be found in [12], where coloured Higgs particles are given masses of the order of the GUT scale naturally, while Higgs doublets remain massless. The brane-world realization of this model is also discussed in [13, 14]. The four-dimensional \( \mathcal{N} = 2 \) supersymmetric multiplet structure of this model [13, 13] might be of relevance in such a realization.

Finally, we show that this extra-dimensional model is able to provide a suitable R symmetry. This fact provides quite a non-trivial consistency check for the phenomenological model building in the extra-dimensions. Moreover, an R symmetry (mod 4) is indispensable to the above SU(5)-breaking model [12, 16].

The maximal torus of the SU(2) R symmetry can be preserved in the theory on the orbifold, since it commutes with the orbifold group. Since it rotates the Grassmann coordinates \( \theta \) as in eqs. (8–11), it is an R symmetry of the four-dimensional \( \mathcal{N} = 1 \) supersymmetry. The three chiral multiplets \( \Phi(10)'s \) transform as in eq. (10) and hence it carries R charge 1, as desired phenomenologically. At the same time, \( \Phi'(5) \) and \( \Phi''(5) \) also carry R charge 1. Then, the chiral multiplets \( X'(5^*) \) and \( X''(5^*) \) at the fixed points are required to have R charge 1 so that the mass terms

\[
W \propto \Phi'^{(*)'}(5) X'^{(*)'}(5^*)
\]

are allowed by the R symmetry. Now we assume that the remaining multiplets at the fixed points, namely \( X(5^*) \) and \( X(1) \), also have the same R charge as the fixed-point fields \( X'(5^*) \) and \( X''(5^*) \). Then, this means that quarks and leptons in SU(5)-5* and right-handed neutrinos have R charge 1, which is again the desired assignment [12, 13, 14]. As for the Higgs multiplets and fields in the SU(5)-breaking sector, there is no “top down” way to determine their transformation property under the maximal torus of the SU(2) R symmetry. Thus, we simply expect that their charge assignment is suitably realized. We assume that the maximal torus of the SU(2) R symmetry is broken down to the mod 4 R symmetry as required in [12, 16].

3 Mass Matrices

We consider that all operators are generated non-perturbatively, unless they are forbidden by symmetries. We assume that such symmetries are \( \mathcal{N} = 1 \) supersymmetry, the (mod 4)-R symmetry and flavour symmetries, which we discuss below.

however, is beyond the scope of this paper.
The flavour symmetries for the fixed-point fields and for the fields in the bulk are independent, since their origins are different. That is, \((X(5^*)_i, X(1)_i)\) and \(\Phi(10)_a\), where \(i = 1, 2, 3\) correspond to three fixed points and \(a = 1, 2, 3\) to three hypermultiplets in the bulk.

First of all, we discuss the flavour symmetry of the fields on the fixed points, i.e. \(X(5^*)_i\)'s and \(X(1)_i\)'s. Flavour symmetry for the \(\Phi(10)_a\)'s is briefly discussed later.

One can see in the Fig. 1 that the \(T^2/Z_3\langle \sigma \rangle\) orbifold possesses \(Z_3\langle \tau \rangle\) translational symmetry generated by \(\tau\). The translation results in a cyclic permutation of the three fixed points, under which fixed-point fields transform as

\[
\tau : \begin{align*}
X(5^*)_i &\mapsto X(5^*)_{i+1}, \\
X(1)_i &\mapsto X(1)_{i+1},
\end{align*}
\]

where \(i = 4\) is identical to \(i = 1\). Since the \(\tau\) is nothing but a translation, the bulk Kaluza–Klein zero modes \(\Phi(10)_a\) are invariant under the \(Z_3\langle \tau \rangle\) transformation.

Now let us discuss the neutrino mass textures. The \(Z_3\langle \tau \rangle\) translational symmetry allows the following degrees of freedom in the Dirac Yukawa couplings and Majorana mass terms of neutrinos, respectively:

\[
W = y^D_{ij} X(5^*)_i X(1)_j H(5) \quad y^D_{ij} = c^D \begin{pmatrix} 1 & a & b \\ b & 1 & a \\ a & b & 1 \end{pmatrix}, \tag{21}
\]

\[
W = h^R_{ij} M_R X(1)_i X(1)_j \quad h^R_{ij} = c^R \begin{pmatrix} 1 & a' & a' \\ a' & 1 & a' \\ a' & a' & 1 \end{pmatrix}, \tag{22}
\]

where \(M_R\) is of the order of the mass scale of right-handed neutrinos, and the Majorana mass terms of the left-handed neutrinos is obtained through the see-saw mechanism as

\[
W = \frac{1}{M_R} h^L_{ii} (X(5^*)_i H(5))(X(5^*)_i H(5)), \tag{23}
\]

\[
h^L_{ii} = y^D_{ij} (h^R)_{jk}^{-1} y^D_{lk} = (c^D)^2 (c^R)^{-1} \begin{pmatrix} 1 & \kappa & \kappa \\ \kappa & 1 & \kappa \\ \kappa & \kappa & 1 \end{pmatrix}. \tag{24}
\]

Off-diagonal elements are relatively suppressed as \(a, b, a' \sim e^{-M_* l} \ll 1\), provided \(M_* l \gg 1\), where the \(M_*\) is the fundamental scale of the theory and \(l\) is the typical length scale of the
orbifold geometry \[17\]. This is because three $X(5^*)_i$'s and $X(1)_i$'s are localized at a fixed point distant from others. As a result, the Majorana mass matrix of the left-handed neutrinos is also almost diagonal ($\kappa \ll 1$). Notice that the higher-dimensional configuration ($M_* l \gg 1$) suppresses the off-diagonal terms, which cannot be forbidden by the $S_3$ (or $Z_3$) symmetry in the democratic ansatz in the four-dimensional spacetime \[7\]. This is the most crucial point in this paper. Although there are preceding trials to interpret various properties of mass texture in terms of geometry \[18\], it should be emphasized here that the localization of $X(5^*)_i$'s and $X(1)_i$'s at suitable positions is not a choice by hand but rather an inevitable consequence of theoretical consistencies.

The left-handed Majorana neutrinos have almost diagonal mass matrix, even when the $Z_3 \langle \tau \rangle$ symmetry is slightly broken; the breaking effects also have the extra $e^{-M_* l}$ suppression in off-diagonal matrix elements. Since the mass matrix of the charged leptons is subject to the $Z_3 \langle \tau \rangle$ symmetry (with small breaking effects of this symmetry), large angle rotation between charged leptons is necessary for the mass diagonalization \[7\]. Thus, the large mixing follows, in general, without specifying how the $Z_3 \langle \tau \rangle$ symmetry is broken or without assuming any flavour structure in the $\Phi(10)_a$'s.

Before proceeding further, we briefly comment on the flavour symmetry of the $\Phi(10)_a$'s. Since the bulk Lagrangian\[12\] is restricted by the higher-dimensional Lorentz symmetry and an extended supersymmetry, the leading interaction for the $\Phi(10)_a$'s is \[20\]

$$W = 2 \bar{\Phi}(10^*)^a(z) \left( \partial_z - \frac{g}{\sqrt{2}} \Sigma(z) \right) \Phi(10)_a(z), \tag{25}$$

where $g$ is the SU(5)-gauge coupling constant. This interaction is flavour-universal and hence an SU(3) accidental symmetry exists. The $\Phi(10)_a$'s form a 3 representation of the SU(3) symmetry. This SU(3) flavour symmetry should be broken so that the Yukawa couplings are allowed. When its breaking is encoded\[13\] by a spurion field $v^a$ in a 3* representation of the SU(3) with all three components of order 1, the superpotential \[21\]

$$W = c X(5^*)_i v^a \Phi(10)_a \bar{H}(5^*), \quad W = c' v^a \Phi(10)_a v^b \Phi(10)_b H(5), \tag{26}$$

leads to rank 1 mass matrices of the democratic type. The mixing angles of the CKM matrix vanish, since the mass-diagonalization matrices of the $\Phi(10)_a$'s are exactly the same in the
up-type Yukawa coupling and in the down-type Yukawa coupling in the absence of breakings of the democratic form\textsuperscript{[8, 9]}. 

Finally, let us discuss the breaking effects to these flavour symmetry. First of all, let us assume that the centre brane is displaced slightly toward the fixed point 3 (see Fig. 1). Then, the Majorana mass matrix of the left-handed neutrinos becomes non-degenerate as

\[
M_{\nu} = (c^D)^2(c^R)^{-1} \begin{pmatrix} 1 & \kappa & \kappa \\ \kappa & 1 & \kappa \\ \kappa & \kappa & 1 \end{pmatrix} \rightarrow (c^D)^2(c^R)^{-1} \begin{pmatrix} 1 & \kappa & \kappa' \\ \kappa & 1 & \kappa' \\ \kappa' & \kappa' & 1 + \delta \end{pmatrix}, \tag{27}
\]

where \((\kappa' - \kappa) \sim \delta \kappa\) as discussed above, and we assume that \(\kappa \ll \delta \lesssim 1\). Further breaking \((\gtrsim \kappa)\) will resolve the remaining degeneracy, leading to the \(\Delta m^2\) of the solar neutrino oscillation small compared with the \(\Delta m^2\) of the atmospheric neutrino oscillation. Secondly, we assume, for example\textsuperscript{[10]} that the \(\Phi(10)_3\) is the most sensitive to the displacement of the centre brane among all the \(\Phi(10)_a\)'s. Then, the charged-lepton mass matrix becomes

\[
M_{\ell} = c \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \rightarrow c \begin{pmatrix} v_1 & v_2 & v_3(1 + \epsilon) \\ v_1 & v_2 & v_3(1 + \epsilon) \\ v_1(1 + \delta') & v_2(1 + \delta') & v_3(1 + \delta' + \epsilon') \end{pmatrix}, \tag{28}
\]

where \(\delta' \sim \delta\) and \(\epsilon \sim \epsilon'\), and the muon acquires a mass suppressed by \(\sim \epsilon\delta\) relative to the mass of the tau lepton. If the breaking dynamics still preserves the equality between \(X(5^*)_1\) and \(X(5^*)_2\) as in eq. (28), then the small \(U_{e3}\) is also obtained\textsuperscript{[4]}. At least, the displacement of the centre brane to the fixed point 3 does not make any difference in the distances between the Higgs particle (centre brane) and fields at fixed points, \(X(5^*)_1\) and \(X(5^*)_2\). We assume that this equality holds\textsuperscript{[14]}. 

The mass matrix for the up-type quarks may now be written as

\[
c'v_a v_b = c' \begin{pmatrix} v_1v_1 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2v_2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3v_3 \end{pmatrix} \rightarrow c' \begin{pmatrix} v_1v_1 & v_1v_2 & v_1v_3(1 + \epsilon') \\ v_2v_1 & v_2v_2 & v_2v_3(1 + \epsilon'') \\ v_3v_1(1 + \epsilon') & v_3v_2(1 + \epsilon'') & v_3v_3(1 + \epsilon'''') \end{pmatrix}, \tag{29}
\]

where \(\epsilon'', \epsilon''' \sim \epsilon\) and \(m_c/m_t \sim \epsilon^2\), and that for the down-type quarks is given by eq. (28). Mass hierarchy and mixing in the quark sector are obtained in a similar way to \textsuperscript{[6, 7]}. It

\textsuperscript{14}The mixing angle between the first and the second families does not make sense without the breakings, since the quarks of these two families are massless.

\textsuperscript{15}This particular example is taken just for an illustration of our idea.

\textsuperscript{16}Even if there is a difference between the matrix elements of the \(X(5^*)_1\)'s row and the \(X(5^*)_2\)'s row, some of them can be absorbed by rescaling and rephasing of the \(X(5^*)_1\) and \(X(5^*)_2\), and it is possible that the difference that cannot be absorbed is of order \(\delta^2\). Then, the \(U_{e3}\) is still sufficiently small.
is clear from (28) and (29) that small angles are derived in the CKM matrix, since the diagonalization matrices for the up- and the down-type left-handed quarks almost coincide.

Effective Yukawa coupling of down-type quarks and charged leptons $c$ is suppressed by $e^{-M_*l}$ compared with the up-type Yukawa coupling $c'$. We expect $c \sim 10^{-1} c'$, and hence the tan $\beta$ is not so large. Off-diagonal elements of the left-handed Majorana neutrino mass matrix are also suppressed by $\kappa$ relatively to the diagonal elements. Natural explanation of bi-large mixing, on the other hand, requires $\kappa \lesssim 10^{-2}$. When the off-diagonal elements of Eq. (21) and Eq. (22) are generated by exchanging two massive particles, $\kappa$ is given by a ratio $(e^{-M_*l})(e^{-M_*l})/(e^{-M_*l})$, slight difference between three $M_*$’s in this ratio easily leads to $\kappa \sim 10^{-2}$ rather than $\kappa \sim 10^{-1}$. When the off-diagonal elements are generated with suppression factors $e^{-M_*^2(area)}$, then there is no wonder that the $\kappa$ is smaller than the ratio $c/c'$.

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| Fields          | $\Phi(\text{adj.}),\bar{\Phi}(\text{adj.})$ | $\Phi(10),\bar{\Phi}(10^*)$ | $\Phi(5),\bar{\Phi}(5^*)$ | $\Phi'(5),\bar{\Phi}'(5^*)$ | $\Phi''(5),\bar{\Phi}''(5^*)$ |
|----------------|--------------------------------------------|----------------------------|-----------------------------|----------------------------|----------------------------|
| Charges        | 0                                          | -1                         | -3                          | 2                          | 2                          |

Table 1: Fiveness charge for each hypermultiplet.

Figure 1: A picture of the $T^2/Z_3 \langle \sigma \rangle$ geometry. A unit cell of the $T^2$ torus is described by parallel lines. Three fixed points are described by $\bullet$ labelled 123, on which an SU(5)-$5^*$ and a right-handed neutrino are localized. We assume that the Higgs particles and the SU(5)-breaking sector are at the $\circ$ in the figure, which we call the “centre brane”. Although there are three $\circ$’s within the unit cell of the $T^2$, there is only one in the fundamental domain of the $T^2/Z_3 \langle \sigma \rangle$ orbifold. The $Z_3 \langle \tau \rangle$ translational symmetry that leads to the flavour symmetry of the fields on fixed points is described by straight arrow lines. A tiny arrow on the centre brane shows its displacement toward the fixed point 3 that leads to the $Z_3 \langle \tau \rangle$-symmetry breaking.