Kinematic Self-Similar Cylindrically Symmetric Solutions

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Abstract

This paper is devoted to find out cylindrically symmetric kinematic self-similar perfect fluid and dust solutions. We study the cylindrically symmetric solutions which admit kinematic self-similar vectors of second, zeroth and infinite kinds, not only for the tilted fluid case but also for the parallel and orthogonal cases. It is found that the parallel case gives contradiction both in perfect fluid and dust cases. The orthogonal perfect fluid case yields a vacuum solution while the orthogonal dust case gives contradiction. It is worth mentioning that the tilted case provides solution both for the perfect as well as dust cases.

Keywords: Cylindrical symmetry, Self-similar variable.

1 Introduction

Due to the mathematical complexity of Einstein field equations (EFEs)

\[ R_{ab} - \frac{1}{2}g_{ab}R = 8\pi GT_{ab}, \]

we are frequently forced to impose some symmetry on the concerned system. Self-similarity is very helpful in simplifying the field equations. In Newtonian

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gravity or General Relativity (GR), there does not exist any characteristic scale. A set of field equations remains invariant under a scale transformation for an appropriate matter field. This indicates that there exist scale invariant solutions to the EFEs. These solutions are known as self-similar solutions. The special feature of self-similar solutions is that, by a suitable coordinate transformations, the number of independent variables can be reduced by one and hence reduces the field equations. This variable is a dimensionless combination of the independent variables, namely the space coordinates and the time. In other words, self-similarity refers to an invariance which simply allows the reduction of a system of partial differential equations to ordinary differential equations.

In order to obtain realistic solutions of gravitational collapse leading to star formation, self-similar solutions have been investigated by many authors in Newtonian gravity [1]. However, in GR, these solutions were first studied by Cahill and Taub [2]. They studied these solutions in the cosmological context and under the assumption of spherically symmetric distribution of a self-gravitating perfect fluid. In GR, self-similarity is defined by the existence of a homothetic vector (HV) field. Such similarity is called the first kind (or homothety or continuous self similarity (CSS)). There exists a natural generalization of homothety called kinematic self-similarity, which is defined by the existence of a kinematic self-similar (KSS) vector field. The basic condition characterizing a manifold vector field $\xi$ as a self-similar generator is given by

$$\mathcal{L}_\xi A = \lambda A,$$

where $\lambda$ is a constant, $A$ is independent physical field and $\mathcal{L}_\xi$ denotes the Lie derivative along $\xi$. This field can be scalar (e.g., $\mu$), vector (e.g., $u_a$) or tensor (e.g., $g_{ab}$). In GR, the gravitational field is represented by the metric tensor $g_{ab}$, and an appropriate definition of geometrical self-similarity is necessary.

The self-similar idea of Cahill and Taub corresponds to Newtonian self-similarity of the homothetic class. Carter et al. [3,4] defined the other kinds of self-similarity namely second, zeroth and infinite kind. In the context of kinematic self-similarity, homothety is considered as the first kind. Several authors [5-12] have explored KSS perfect fluid solutions. The only barotropic equation of state which is compatible with self-similarity of first kind is

$$p = k\rho.$$
Carr [5] has classified the self-similar perfect fluid solutions of the first kind for the dust case \((k = 0)\). The case \(0 < k < 1\) has been studied by Carr et al. [6]. Coley [7] has shown that the FRW solution is the only spherically symmetric homothetic perfect fluid solution in the parallel case. McIntosh [8] has discussed that a stiff fluid \((k = 1)\) is the only compatible perfect fluid with the homothety in the orthogonal case. Benoit et al. [9] have studied analytic spherically symmetric solutions of the EFEs coupled with a perfect fluid and admitting a KSS vector of the first, second and zeroth kinds.

Carr et al. [10] have considered the KSS associated with the critical behavior observed in the gravitational collapse of spherically symmetric perfect fluid with equation of state \(p = k\rho\). They showed for the first time the global nature of these solutions and showed that it is sensitive to the value of \(\alpha\) (given in Eq.(15)). Carr et al. [11], further, investigated solution space of self-similar spherically symmetric perfect fluid models and physical aspects of these solutions. They combine the state space description of the homothetic approach with the use of the physically interesting quantities arising in the co-moving approach. Coley and Goliath [12] have investigated self-similar spherically symmetric cosmological models with a perfect fluid and a scalar field with an exponential potential.

Gravitational collapse is one of the fundamental problems in GR. Self-similar gravitational collapse and critical collapse provides information about the collapse. The collapse generally has three kinds of possible final states. First is simply the halt of the processes in a self-sustained object or the description of a matter field or gravitational field. The second is the formation of black holes with outgoing gravitational radiation and matter, while the third is the formation of naked singularities. Critical collapse in the context of self-similarity gives the information about the mass of black holes formed as a result of collapse.

Recently, Maeda et al. [13,14] investigated the KSS vector of the second kind in the tilted case. They assumed the perfect fluid spacetime obeying a relativistic polytropic equation of state. Further, they assumed two kinds of polytropic equation of state and showed that such spacetimes must be vacuum in both cases. They studied the case in which a KSS vector is not only tilted to the fluid flow but also parallel or orthogonal. In the recent paper [15], the same authors discussed the classification of the spherically symmetric KSS perfect fluid and dust solutions. This analysis has provided some interesting solutions. In this paper, we shall use the same procedure to calculate self-similar solutions for the cylindrically symmetric spacetimes.
The governing equations for perfect fluid cosmological models are introduced and a set of integrability conditions for the existence of a KSS solutions are derived.

The paper can be outlined as follows. In section 2, we shall discuss KSS of different kinds for the cylindrically symmetric spacetimes. Section 3 is devoted to titled perfect fluid case. In section 4, we shall find out the titled dust solutions. Sections 5 and 6 are used to discuss the orthogonal perfect fluid and dust solutions respectively. Finally, we shall summarise and discuss all the results.

2 Cylindrically Symmetric Spacetime and Kinematic Self-Similarity

The general cylindrically symmetric spacetime is given by the line element [16]

\[ ds^2 = -e^{2\phi(t,r)} dt^2 + dr^2 + e^{2\mu(t,r)} d\theta^2 + e^{2\nu(t,r)} dz^2, \]  

(4)

where \( \phi, \mu \) and \( \nu \) are arbitrary functions of \( t \) and \( r \). The energy-momentum tensor for a perfect fluid is given by

\[ T_{ab} = \left[ \rho(t,r) + p(t,r) \right] u_a u_b + p(t,r) g_{ab}, \quad (a, b = 0, 1, 2, 3). \]  

(5)

where \( \rho \) is the density, \( p \) is the pressure and \( u_a \) is the four velocity of the fluid element in the co-moving coordinate system given as \( u_a = (-e^\phi, 0, 0, 0) \). The EFEs reduce to the following form [17]

\[ 8\pi G \rho = e^{-2\phi} \nu_t \mu_t - \nu_r \mu_r - \mu_r^2 - \nu_r^2 - \nu_{rr}, \]  

(6)

\[ 8\pi G p = e^{-2\phi} (-\nu_t + \mu_t + \nu_t \nu_t - \mu_t^2 - \nu_t \nu_t - \nu_t^2) + \phi_r \nu_r \]  

+ \phi_r \nu_r + \nu_r \mu_r, \]  

(7)

\[ 8\pi G p = e^{-2\phi} (-\nu_t + \nu_t \nu_t - \nu_t^2) + \nu_r^2 + \nu_{rr} + \phi_r \nu_r + \phi_r \nu_r + \phi_r^2 + \phi_{rr}, \]  

(8)

\[ 8\pi G p = e^{-2\phi} (-\mu_t + \phi_t \mu_t - \mu_t^2) + \mu_r^2 + \mu_{rr} + \phi_r \nu_r + \phi_r \nu_r + \phi_r^2 + \phi_{rr}, \]  

(9)

\[ 0 = \mu_t + \mu_t \nu_t + \nu_t \nu_t + \nu_t \nu_t - \phi_t \nu_t. \]  

(10)

The conservation of energy-momentum tensor \( T^{ab}_{\ ;b} = 0 \) yields the following equations

\[ \mu_t = -\frac{\rho_t}{(\rho + p)} - \nu_t, \]  

(11)
\[ \phi_r = -\frac{\rho_r}{(\rho + p)}. \]  

(12)

For a cylindrically symmetric spacetime, the general form of a vector field \( \eta \) can be written as

\[ \eta^a \frac{\partial}{\partial x^a} = h_1(t, r) \frac{\partial}{\partial t} + h_2(t, r) \frac{\partial}{\partial r}, \]  

(13)

where \( h_1 \) and \( h_2 \) are arbitrary functions. When \( \eta \) is parallel to the fluid flow, \( h_2 = 0 \) while \( h_1 = 0 \) indicates that \( \eta \) is orthogonal to the fluid flow. When \( \eta \) is tilted to the fluid flow, both \( h_1 \) and \( h_2 \) are non-zero.

A KSS vector \( \xi \) satisfies the following two conditions

\[ \mathcal{L}_\xi h_{ab} = 2\delta h_{ab}, \]  

(14)

\[ \mathcal{L}_\xi u_a = \alpha u_a, \]  

(15)

where \( h_{ab} = g_{ab} + u_a u_b \) is the projection tensor, \( \alpha \) and \( \delta \) are constants. The similarity transformation is characterized by the scale independent ratio, \( \alpha/\delta \), which is known as the similarity index. The similarity index gives rise to the following two cases according as:

1. \( \delta \neq 0 \),
2. \( \delta = 0 \).

**Case 1:** If \( \delta \neq 0 \) it can be chosen as unity and the KSS vector for the tilted case can take the following form

\[ \xi^a \frac{\partial}{\partial x^a} = (\alpha t + \beta) \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}. \]  

(16)

For this case, the similarity index, \( \alpha/\delta \), further yields the following three different possibilities.

(i) \( \delta \neq 0, \ \alpha = 1 \ (\beta \ can \ be \ taken \ to \ be \ zero) \),

(ii) \( \delta \neq 0, \ \alpha = 0 \ (\beta \ can \ be \ taken \ to \ be \ unity) \),

(iii) \( \delta \neq 0, \ \alpha \neq 0,1 \ (\beta \ can \ be \ taken \ to \ be \ zero) \).

The case 1(i) corresponds to the self-similarity of the first kind. In this case \( \xi \) is a homothetic vector and the self-similar variable \( \xi \) turns out to be \( r/t \).
For the second case 1(ii), it is termed as the self-similarity of the \textit{zeroth kind} and the self-similar variable follows

$$\xi = re^{-t}.$$  

In the last case 1(iii), it is called the self-similarity of the \textit{second kind} and the self-similar variable becomes

$$\xi = \frac{r}{(\alpha t)^{\frac{1}{\alpha}}}.$$  

It turns out that for the case (1), when \(\delta \neq 0\), with the self-similar variable \(\xi\), the metric functions become

$$\phi = \phi(\xi), \quad e^{\mu} = re^{\mu(\xi)}, \quad e^{\nu} = re^{\nu(\xi)}. \quad (17)$$

The case (2), in which \(\delta = 0\) and \(\alpha \neq 0\) (\(\alpha\) can be unity and \(\beta\) can be re-scaled to zero), the self-similarity is known as \textit{infinite kind}. In this case, the KSS vector \(\xi\) turns out to be

$$\xi^a \frac{\partial}{\partial x^a} = t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \quad (18)$$

and the self-similar variable will become

$$\xi = e^{\xi}/t,$$

where \(c\) is an arbitrary constant. The metric functions will take the following form

$$\phi = \phi(\xi), \quad \mu = \mu(\xi), \quad \nu = \nu(\xi). \quad (19)$$

It is mentioned here that for the cylindrically symmetric spacetime, the self-similar variable of the first, second and zeroth kinds are the same as for the spherically symmetric spacetime but for the infinite kind we obtain different self-similar variable. Further, it is noted that for \(\delta = 0 = \alpha\), the KSS vector \(\xi\) becomes Killing vector.

If the KSS vector \(\xi\) is parallel to the fluid flow, it follows that

$$\xi^a \frac{\partial}{\partial x^a} = f(t) \frac{\partial}{\partial t}, \quad (20)$$

where \(f(t)\) is an arbitrary function and the self-similar variable is \(r\). In this case, we obtain contradictory results for the cylindrically symmetric metric.
in all kinds. This implies that there does not exist any solution for $\xi$ parallel to the fluid flow. If the KSS vector $\xi$ is orthogonal to the fluid flow, it takes the following form

$$\xi^a \frac{\partial}{\partial x^a} = g(r) \frac{\partial}{\partial r},$$

(21)

where $g$ is an arbitrary function and the self-similar variable is $t$.

We assume the following two types of polytropic equation of states (EOS). We denote the first equation of state by EOS(1) and is given by

$$p = k\rho^\gamma,$$

where $k$ and $\gamma$ are constants. The other EOS can be written as [12]

$$p = k n^\gamma,$$

$$\rho = m_b n + \frac{p}{\gamma - 1},$$

where $m_b$ is a constant and corresponds to the baryon mass, and $n(t, r)$ corresponds to baryon number density. This equation is called second equation of state written as EOS(2). For EOS(1) and EOS(2), we take $k \neq 0$ and $\gamma \neq 0, 1$. The third equation of state, denoted by EOS(3), is the following

$$p = k \rho.$$

Here we assume that $-1 \leq k \leq 1$ and $k \neq 0$.

For different values of $\gamma$, EOS(1) and EOS(2) have different properties. Thermodynamical instability of the fluid is shown for $\gamma < 0$. For $0 < \gamma < 1$, both EOS(1) and EOS(2) are approximated by a dust fluid in high density regime. For $\gamma > 1$, EOS(2) is approximated by EOS(3) with $k = \gamma - 1$ in high density regime. The cases $\gamma > 2$ for EOS(2) and $\gamma > 1$ for EOS(2) shows that the dominant energy condition can be violated in high density regime which is physically not interesting [14].

3 Tilted Case for Perfect Fluid

3.1 Self-similarity of the second kind

Here we discuss the self-similarity of the second kind for the tilted perfect fluid case. In this case, it follows from the EFEs that the energy density $\rho$
and pressure $p$ must take the following form

\begin{align}
8\pi G \rho &= \frac{1}{r^2} [\rho_1(\xi) + \frac{r^2}{t^2} \rho_2(\xi)], \quad (22) \\
8\pi G p &= \frac{1}{r^2} [p_1(\xi) + \frac{r^2}{t^2} p_2(\xi)], \quad (23)
\end{align}

where the self-similar variable is $\xi = r/(\alpha t)^\frac{1}{\gamma}$. If the EFEs and the equations of motion for the matter field are satisfied for $O[(\xi)^0]$ and $O[(\xi)^2]$ terms separately, we obtain a set of ordinary differential equations. Thus Eqs.(6)-(12) reduce to the following

\begin{align}
\dot{\rho}_1 &= 1 + \dot{\mu} \dot{\nu} + \ddot{\mu} + \mu^2 + 2\dot{\nu} + \nu^2 + 2\nu, \quad (24) \\
\alpha^2 \dot{\rho}_2 &= e^{-2\phi} \dot{\mu} \nu, \quad (25) \\
\rho_1 &= 1 + 2\dot{\phi} + \dot{\phi} \dot{\nu} + \dot{\phi} \dot{\mu} + \dot{\nu} + \ddot{\mu}, \quad (26) \\
-\alpha^2 e^{2\phi} \rho_2 &= \ddot{\nu} + \dot{\nu}^2 + \alpha \dot{\nu} + \ddot{\mu} + \ddot{\mu} + \ddot{\mu} - \dot{\phi} \dot{\nu} - \dot{\phi} \ddot{\mu} + \dot{\phi} \dot{\mu}, \quad (27) \\
0 &= \ddot{\nu} + \dot{\nu}^2 + \dot{\nu} + \ddot{\mu} + \mu^2 + \mu - \dot{\phi} \dot{\nu} - \dot{\phi} \ddot{\mu}, \quad (28) \\
\dot{\phi}(\rho_1 + p_1) &= 2\rho_1 - \dot{\rho}_1, \quad (29) \\
\dot{\phi}(\rho_2 + p_2) &= -\dot{\rho}_2, \quad (30) \\
-(\dot{\mu} + \dot{\nu})(\rho_1 + p_1) &= \dot{\rho}_1, \quad (31) \\
-(\dot{\mu} + \dot{\nu})(\rho_2 + p_2) &= \dot{\rho}_2 + 2\alpha \rho_2, \quad (32)
\end{align}

where dot (.) represents derivative with respect to $\ln(\xi)$. Adding Eqs.(24) and (26) and using Eq.(28), we get

\begin{equation}
\rho_1 + p_1 = 2\dot{\phi}, \quad (33)
\end{equation}

Using Eq.(33) in Eq.(29) and Eq.(30), we have

\begin{equation}
(\rho_1 + p_1)^2 = 4\rho_1 - \dot{\rho}_1, \quad (34)
\end{equation}

and

\begin{equation}
(\rho_2 + p_2)(\rho_1 + p_1) = -2\dot{\rho}_2 \quad (35)
\end{equation}

respectively.

### 3.1.1 Equations of State (1) and (2)

If a perfect fluid satisfies EOS(1) for $k \neq 0$ and $\gamma \neq 0, 1$, Eqs.(22) and (23) imply that

\begin{equation}
\alpha = \gamma, \quad p_1 = \rho_2 = 0, \quad p_2 = \frac{k}{(8\pi G)^{(\gamma-1)/\gamma^2}} \xi^{-2\gamma} \rho_1 \gamma, \quad [Case \ I] \quad (36)
\end{equation}
\[\alpha = \frac{1}{\gamma}, \quad p_2 = \rho_1 = 0, \quad p_1 = \frac{k}{(8\pi G)^{(\gamma-1)}\gamma^2}\xi^2 \rho_2 \gamma. \quad [\text{Case II}] \quad (37)\]

If a perfect fluid obeys EOS(2) for \(k \neq 0\) and \(\gamma \neq 0, 1\), we find from Eqs.(22) and (23) that

\[\alpha = \gamma, \quad p_1 = 0, \quad p_2 = \frac{k}{m_b \gamma (8\pi G)^{(\gamma-1)}\gamma^2}\xi^{-2}\rho_1 \gamma = (\gamma - 1)\rho_2, \quad [\text{Case III}] \quad (38)\]

or

\[\alpha = \frac{1}{\gamma}, \quad p_2 = 0, \quad p_1 = \frac{k}{m_b \gamma (8\pi G)^{(\gamma-1)}\gamma^2}\xi^2 \rho_2 \gamma = (\gamma - 1)\rho_1. \quad [\text{Case IV}] \quad (39)\]

In the cases I and III, Eq.(34) gives \(\rho_1 = 0\) and ultimately we have a vacuum spacetime. For the cases II and IV, using Eq.(35), it can also be shown that the spacetime is vacuum. Thus we can conclude that the spacetime must be vacuum in all these cases.

### 3.1.2 Equation of State (3)

When a perfect fluid satisfies EOS(3), it follows from Eqs.(22) and (23) that

\[p_1 = k\rho_1, \quad p_2 = k\rho_2. \quad [\text{Case V}] \quad (40)\]

For \(k = -1\), we use Eqs.(29), (30) and (32) which ultimately yields a vacuum spacetime. When \(k \neq -1\), we assume that \(\rho_1 \neq 0\) and \(\rho_2 \neq 0\). In this case, using Eqs.(29) and (30), it follows that \(2 - \frac{\dot{\rho}_1}{\rho_1} = -\frac{\dot{\rho}_2}{\rho_2}\), and from Eqs.(31) and (32), we obtain \(2\alpha + \frac{\dot{\rho}_2}{\rho_2} = \frac{\dot{\rho}_1}{\rho_1}\). These two expressions imply that \(\rho_1\rho_2 = 0\) as \(\alpha \neq 1\). For the case when \(\rho_1 = 0 = p_1\) and \(\rho_2 \neq 0\), we have a contradiction.

In the case when \(\rho_2 = 0 = p_2\) and \(\rho_1 \neq 0\), we subtract Eq.(28) from Eq.(27) and using Eq.(25) so that

\[(\alpha - 1)(\dot{\mu} + \dot{\nu}) = 0. \quad (41)\]

This implies that \(\dot{\mu} = -\dot{\nu}\) as \(\alpha \neq 0\). Using this in Eq.(31) and making use of EOS(III), we have \(\dot{p} = \dot{0}\) which means \(\rho_1 = \text{const} = w_0\) and this gives \(p_1 = p_0 = \text{constant}\). Using these results in Eq.(29), we have

\[\dot{\phi} = \frac{2k}{k+1}, \quad (42)\]
and this gives
\[ e^\phi = c_0 \xi^{\frac{2k}{k+1}}. \]  
(43)

From Eqs.(41) and Eq.(28) we find \( \mu \) to be a constant and since \( \dot{\mu} = -\dot{\nu} \), therefore we can say \( \nu = constant \). The resulting solution becomes
\[ e^\phi = c_0 \xi^{\frac{2k}{k+1}}, \quad e^\mu = a_0, \quad e^{\nu} = b_0, \]
\[ w_0 = -1, \quad p_1 = -k, \quad k = -3 \pm 2\sqrt{2}. \]  
(44)

where \( a_0, b_0, c_0 \) are constants.

3.2 Self-similarity of the zeroth kind

In this section we shall attempt self-similar solutions of the zeroth kind. For this case, the EFEs indicate that the quantities \( \rho \) and \( p \) should be of the form
\[ 8\pi G \rho = \frac{1}{r^2} [\rho_1(\xi) + r^2 \rho_2(\xi)], \]  
(45)
\[ 8\pi G p = \frac{1}{r^2} [p_1(\xi) + r^2 p_2(\xi)], \]  
(46)

where the self-similar variable is \( \xi = \frac{r}{e^{\nu - r}} \). If it is assumed that the EFEs and the equations of motion for the matter field are satisfied for \( O[(r)^0] \) and \( O[(r)^2] \) terms separately, we obtain the following set of ordinary differential equations.

\[ \dot{\rho}_1 = 1 + \dot{\mu} \dot{\nu} + \dot{\mu} + \mu^2 + 2\dot{\mu} \dot{\nu} + \nu^2 + 2\nu, \]  
(47)
\[ \dot{\rho}_2 = e^{-2\phi} \dot{\mu} \dot{\nu}, \]  
(48)
\[ p_1 = 1 + 2\dot{\phi} + \dot{\phi} \dot{\nu} + \dot{\phi} \dot{\mu} + \dot{\nu} + \mu \dot{\mu}, \]  
(49)
\[ e^{2\phi} p_2 = \dot{\nu} + \nu^2 + \mu + \mu^2 - \dot{\phi} \dot{\nu} - \dot{\phi} \dot{\mu} \dot{\mu}, \]  
(50)
\[ 0 = \ddot{\nu} + \nu^2 + \dot{\nu} + \dot{\mu} + \mu^2 + \mu - \dot{\phi} \dot{\nu} - \dot{\phi} \dot{\mu}, \]  
(51)
\[ \dot{\phi}(\rho_1 + p_1) = 2p_1 - \ddot{\rho}_1, \]  
(52)
\[ \dot{\phi}(\rho_2 + p_2) = -\ddot{\rho}_2, \]  
(53)
\[ (\dot{\mu} + \dot{\nu})(\rho_1 + p_1) = \dot{\rho}_1, \]  
(54)
\[ (\dot{\mu} + \dot{\nu})(\rho_2 + p_2) = \ddot{\rho}_2, \]  
(55)
\[ p_1 = \ddot{\phi} + \dot{\phi}^2 + \dot{\phi} \dot{\nu} + \dot{\nu}^2 + \dot{\nu}, \]  
(56)
\[ e^{2\phi} p_2 = \dot{\phi} \dot{\nu} - \ddot{\nu} - \dot{\nu}^2. \]  
(57)
Here again dot (.) represents derivative with respect to \( \ln(\xi) \). If we add Eqs. (47) and (49) and use Eq. (51), we get
\[
\rho_1 + p_1 = 2 \dot{\phi}.
\] (58)
Substituting Eq. (58) in Eqs. (52) and (53), it follows that
\[
(\rho_1 + p_1)^2 = 4p_1 - 2 \dot{p}_1,
\] (59)
and
\[
(\rho_2 + p_2)(\rho_1 + p_1) = -2 \dot{p}_2
\] (60)
respectively.

### 3.2.1 EOS(1) and EOS(2)

For these two EOS, we obtain a contradiction and consequently we do not have any solution.

### 3.2.2 EOS(3)

When a perfect fluid satisfies EOS(3), it follows from Eqs. (45) and (46) that
\[
p_1 = k \rho_1, \quad p_2 = k \rho_2. \quad [Case \text{III}]
\] (61)
First we assume that \( \rho_1 \neq 0 \) and \( \rho_2 \neq 0 \). For this case, we make use of Eqs. (52) and (53) so that we get \( 2\rho_1 \rho_2 - \dot{\rho}_1 \rho_2 + \dot{\rho}_2 \rho_1 = 0 \). Also from Eqs. (55) and (56), we obtain \( -\dot{\rho}_1 \rho_2 + \dot{\rho}_2 \rho_1 = 0 \). Subtracting these two expressions we can conclude that \( \rho_1 \rho_2 = 0 \) which gives a contradiction. If we assume that \( \rho_1 = 0 = p_1, \rho_2 \neq 0 \), we again have a contradiction.

In the third case, we assume that \( \rho_2 = 0 = p_2, \rho_1 \neq 0 \). It follows from Eq. (48) that \( \dot{\mu} \dot{\nu} = 0 \) and also from Eqs. (50) and (51) we get \( \dot{\mu} + \dot{\nu} = 0 \). Eq. (54) requires that \( \rho_1 = w_0 = constant \) which implies that \( p_1 = kw_0 \) and ultimately Eq. (49) gives the value of \( \phi \). Here we require that \( k \neq -1 \). The resulting solution is
\[
e^\phi = c_0 \xi^{-(1+k)/2}, \quad e^{\mu} = 1 = e^\nu,
\]
\[
w_0 = -1, \quad p_0 = -k, \quad k = -3 \pm 2\sqrt{2}.
\] (62)
This corresponds to the solution already found in the second kind with EOS(3).
3.3 Self-similarity of the infinite kind

This section is devoted to discuss the self-similar solution of the infinite kind. In this case, the EFEs imply that the quantities $\rho$ and $p$ must be of the form

\begin{align}
8\pi G\rho &= \frac{1}{t^2} \rho_1(\xi) + \rho_2(\xi), \\
8\pi Gp &= \frac{1}{t^2} p_1(\xi) + p_2(\xi),
\end{align}

where $\xi = \frac{e^\xi}{t}$. Now if we require that the EFEs and the equations of motion for the matter field are satisfied for $O[(t)^{0}]$ and $O[(t)^{-2}]$ terms separately, we obtain a set of ordinary differential equations. For a perfect fluid, Eqs.(6)-(12) takes the following form

\begin{align}
\rho_1 &= e^{-2\phi} \dot{\mu} \dot{\nu}, \\
-c^2 p_2 &= \dot{\mu} \dot{\nu} + \mu^2 + \dot{\nu}^2, \\
-e^{2\phi} p_1 &= \ddot{\nu} + \dot{\nu}^2 + \dot{\nu} + \ddot{\mu} + \dot{\mu}^2 + \dot{\phi} \dot{\nu} - \dot{\phi} \dot{\mu} + \ddot{\mu} \dot{\nu}, \\
c^2 p_2 &= \dot{\phi} \dot{\nu} + \ddot{\phi} \dot{\mu} + \ddot{\mu} \dot{\nu}, \\
0 &= \ddot{\nu} + \dot{\nu}^2 + \dot{\nu} + \ddot{\mu} - \dot{\phi} \dot{\nu} - \dot{\phi} \dot{\mu}, \\
\dot{\phi}(p_1 + p_1) &= -\dot{p}_1, \\
\dot{\phi}(p_2 + p_2) &= -\dot{p}_2, \\
(\dot{\mu} + \dot{\nu})(p_1 + p_1) &= -\dot{p}_1, \\
(\dot{\mu} + \dot{\nu})(p_2 + p_2) &= \dot{p}_2, \\
-e^{2\phi} p_1 &= \ddot{\nu} + \dot{\nu}^2 + \dot{\nu} - \dot{\phi} \dot{\nu}, \\
c^2 p_2 &= \ddot{\phi} + \ddot{\phi} + \dot{\phi} \ddot{\nu} + \ddot{\nu} + \dot{\nu}^2, \\
-e^{2\phi} p_1 &= \ddot{\mu} + \dot{\mu}^2 + \ddot{\mu} - \dot{\phi} \ddot{\mu}, \\
c^2 p_2 &= \ddot{\phi} + \ddot{\phi} + \dot{\phi} \ddot{\mu} + \ddot{\mu} + \dot{\mu}^2,
\end{align}

where dot (.) represents derivative with respect to $ln(\xi)$. Now if we subtract Eq.(67) from Eq.(65) and use Eq.(69), and also subtract Eq.(66) from Eq.(68) and use Eq.(69), it follows that

\begin{align}
-e^{2\phi}(\rho_1 + p_1) &= \dot{\mu} + \dot{\nu}, \\
\rho_2 + p_2 &= 0
\end{align}

respectively. From Eq.(71) we can write $p_2 = p_0 = constant$, and Eq.(73) gives $\rho_2 = w_0 = constant$. This implies that $p_0 = -w_0$. 

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3.3.1 EOS(1) and EOS(2)

When a perfect fluid satisfies EOS(1), it can be seen from Eq.(63) and Eq.(64) that

\[ p_1 = \rho_1 = 0, \]
\[ p_2 = k(8\pi G)^{(1-\gamma)} \rho_2^\gamma. \quad [\text{Case I}] \] (80)

For the condition given by EOS(2), it turns out that

\[ p_1 = 0 = \rho_1, \]
\[ p_2 = \frac{k}{m_b^\gamma (8\pi G)^{\gamma-1}} \left( \rho_2 - \frac{p_2}{\gamma-1} \right)^\gamma. \quad [\text{Case II}] \] (81)

In both cases, we have \( w_0 = 0 = p_0 \) and consequently spacetime turns out to be vacuum.

3.3.2 EOS(3)

For this equation of state, it follows from Eqs.(63) and (64) that

\[ p_1 = k\rho_1, \quad p_2 = k\rho_2. \quad [\text{Case III}] \] (82)

Eq.(77) implies that \( p_2 = -\rho_2 \), and from Eq.(71) with Eq.(73), we have \( p_2 = p_0, \rho_0 = w_0 \) which yield \( p_0 = -w_0 \). This gives rise to the following two cases either \( k = -1 \) or \( p_0 = 0 = w_0 \). In the first case, we have

\[ p_1 + \rho_1 = 0. \] (83)

If we make use of Eqs.(70) and (72) with Eq.(83), it follows that \( \rho_1 = \text{constant} \) and \( p_1 = \text{constant} \). Eqs.(72) and (66) together give \( \dot{\mu} = -\dot{\nu} \) and on using this in Eq.(69), it follows that \( \dot{\mu} = 0 = \dot{\nu} \). The resulting solution is

\[ e^\phi = ln(\xi), \quad e^\mu = 1, \quad e^\nu = 1, \]
\[ \rho_1 = \rho_2 = 0, \quad p_1 = p_2 = 0, \quad k = -1. \] (84)

This corresponds to Minkowski spacetime.

For the second case, if we use Eqs.(74) and (76) in Eq.(67), we get

\[ -e^{2\phi} p_1 = \dot{\mu}\dot{\nu}. \] (85)
Also, use of Eqs.(74) and (76) in Eq.(69) yield
\[- e^{2\phi} p_1 = \dot{\mu} + \dot{\nu}. \tag{86}\]

Using these two in Eq.(68), we further have two possibilities either \( p_1 = 0 \) or \( \dot{\phi} = \frac{1}{2} \). For the first possibility we have the same spacetime as given above but for the second possibility we get a contradiction.

4 Tilted Dust Case

4.1 Self-similarity of the second kind

If we set \( p_1 = 0 = p_2 \) in the basic Eqs.(14)-(22) for the tilted perfect fluid case (self-similarity of the second kind), Eqs.(19) and (20) immediately gives \( \dot{\phi} = 0 \) and we can take \( e^{\phi} = c_0 \). The rest of the equations reduce to
\[- \rho_1 = 1 + \dot{\mu} \dot{\nu} + \ddot{\mu} + \ddot{\nu} + 2 \dot{\mu} + \dot{\nu}^2 + 2 \dot{\nu}, \tag{87}\]
\[\alpha^2 \rho_2 c_0^2 = \dddot{\mu} \dot{\nu}, \tag{88}\]
\[0 = 1 + \dot{\nu} + \dddot{\mu} + \dddot{\nu}, \tag{89}\]
\[-\alpha^2 c_0^2 \rho_2 = \dddot{\nu} + \dot{\nu}^2 + \alpha \dot{\nu} + \dddot{\mu} + \dddot{\nu} + \dddot{\mu} \dot{\nu}, \tag{90}\]
\[0 = \dddot{\nu} + \dot{\nu}^2 + \dot{\nu} + \dddot{\mu} + \dddot{\mu} \dot{\nu} + \dddot{\mu}, \tag{91}\]
\[-(\dot{\mu} + \dot{\nu}) \rho_1 = \dot{\rho}_1, \tag{92}\]
\[-(\dot{\mu} + \dot{\nu}) \rho_2 = \dot{\rho}_2 + 2 \alpha \rho_2, \tag{93}\]
\[0 = \dddot{\nu} + 2 \dot{\nu}, \tag{94}\]
\[0 = \dddot{\nu} + \dot{\nu}^2 + \alpha \dot{\nu}, \tag{95}\]
\[0 = \dddot{\mu} + 2 \dddot{\mu}, \tag{96}\]
\[0 = \dddot{\mu} + \dddot{\mu}^2 + \alpha \dddot{\mu}. \tag{97}\]

Eqs.(91) and (89) with Eq.(87) gives \( \rho_1 = 0 \). Now making use of Eqs.(94) and (97) in Eq.(90), we have \( \rho_2 = 0, \ \dot{\mu} \dot{\nu} = 0, \ \rho_2 = 0 \). This leads to contradiction to our assumption that \( \alpha \neq 1 \).

4.2 Self-similarity of zeroth kind

If we choose \( p_1 = 0 = p_2 \) in Eqs.(51)-(57) for the tilted perfect fluid case (self-similarity of zeroth kind), it follows from Eqs.(52) and (53) that \( \dot{\phi} = 0 \)
and we can take $e^{\phi} = c_0$. Now Eqs.(56) and (57) show that $\nu$ is a constant but then Eqs.(49) and (50) give contradiction. Thus we can conclude that there is no solution in this case.

4.3 Self-similarity of infinite kind

When we set $p_1 = 0 = p_2$ in Eqs.(65)-(77) for the tilted perfect fluid case (self-similarity of infinite kind), Eqs.(70) and (71) imply that $\dot{\phi} = 0$ and we set $e^{\phi} = c_0$. Now Eqs.(65) and (68) show that $\rho_1 = 0$, and also the two equations Eqs.(69) and (66) with Eq.(68) give $\rho_2 = 0$. This case yields vacuum spacetime.

5 Orthogonal Case for Perfect Fluid

5.1 Self-similarity of the second kind

Here we discuss self-similar solution for the orthogonal perfect fluid case. First, we consider the self-similarity of the second kind. For this case, the self-similar variable can be written as

$$\xi^a \frac{\partial}{\partial x^a} = r \frac{\partial}{\partial r} \quad (98)$$

The cylindrically symmetric spacetime takes the form

$$ds^2 = -r^{2\alpha} dt^2 + dr^2 + r^2 e^{2\mu(t)} d\theta^2 + r^2 e^{2\nu(t)} dz^2. \quad (99)$$

EFEs imply that the quantities $\rho$ and $p$ must be of the form

$$8\pi G\rho = r^{-2} \rho_1(\xi) + r^{-2\alpha} \rho_2(\xi), \quad (100)$$
$$8\pi Gp = r^{-2} p_1(\xi) + r^{-2\alpha} p_2(\xi), \quad (101)$$

where the self-similar variable is $\xi = t$. We note that the solution is always singular at $r = 0$ which corresponds to the physical center. When the EFEs and the equations of motion for the matter field are satisfied for $O[(r)^0]$ and $O[(r)^{2-2\alpha}]$ terms separately, we obtain a set of ordinary differential equations. These are given as

$$\rho_1 = -1, \quad (102)$$
\[ \rho_2 = \rho_2 = e^{-2\phi} \mu' \nu', \tag{103} \]
\[ p_1 = 1 + 2\alpha, \tag{104} \]
\[ e^{2\phi} p_2 = -\nu'' - \nu'^2 - \mu'' - \mu'^2 + \phi' \nu' + \phi' \mu' - \mu' \nu', \tag{105} \]
\[ 0 = (1 - \alpha)(\nu' + \mu'), \tag{106} \]
\[ (2 - \alpha)p_1 = \alpha \rho_1, \tag{107} \]
\[ \rho_2 = p_2, \tag{108} \]
\[ (\mu' + \nu')[p_1 + p_1] = 0, \tag{109} \]
\[ (\mu' + \nu')[p_2 + p_2] = -\rho_2'. \tag{110} \]

Here prime (') denotes derivative with respect to \( t \). Since \( \rho_1 = 0 \) contradicts Eq.(102), a vacuum spacetime is not compatible with this case.

5.1.1 EOS(1) and EOS(2)

For a perfect fluid satisfying EOS(I), it follows from Eqs.(100) and (101) that

\[ \alpha = \gamma, \quad p_1 = p_2 = 0, \]
\[ p_2 = \frac{k}{(8\pi G)(\gamma - 1)} \rho_1^\gamma, \quad [\text{Case I}] \tag{111} \]

or

\[ \alpha = \frac{1}{\gamma}, \quad p_2 = p_1 = 0, \]
\[ p_1 = \frac{k}{(8\pi G)(\gamma - 1)} \rho_2^\gamma, \quad [\text{Case II}] \tag{112} \]

For EOS(2), Eqs.(100) and (101) imply that

\[ \alpha = \gamma, \quad p_1 = 0, \]
\[ p_2 = \frac{k}{m_0^\gamma(8\pi G)(\gamma - 1)} \rho_1^\gamma = (\gamma - 1) \rho_2, \quad [\text{Case III}] \tag{113} \]

or

\[ \alpha = \frac{1}{\gamma}, \quad p_2 = 0, \]
\[ p_1 = \frac{k}{m_0^\gamma(8\pi G)(\gamma - 1)} \rho_2^\gamma = (\gamma - 1) \rho_1. \quad [\text{Case IV}] \tag{114} \]
Case II directly gives contradiction to Eq.(102). Also, when we make use of Eq.(108) with Eq.(114), case IV gives contradiction to Eq.(102). In case I, again using Eqs.(108) and (111), we have a contradiction. For case III, we have contradiction to Eq.(107) and hence no solution.

5.1.2 EOS(3)

In this case Eqs.(100) and (101) yield that

\[ p_1 = k\rho_1, \quad p_2 = k\rho_2. \]  \[\text{[Case V]}\]  \hspace{1cm} (115)

Now if we use Eqs.(102) and Eq.(108) in Eq.(115), we obtain \( p_1 = -1 \). Using this value in Eq.(104), it follows that \( \alpha = 0 \) which contradicts our assumption that \( \alpha = 1 \). Thus there is no self-similar solution of the second kind for the orthogonal perfect fluid case.

5.2 Self-similarity of the zeroth kind

In the case of self-similarity of the zeroth kind, the basic equations for perfect fluid gives us a contradiction and hence we have no solution in this case.

5.3 Self-similarity of the infinite kind

For the self-similarity of the infinite kind, EFEs imply that the quantities \( \rho \) and \( p \) must be of the form

\[ 8\pi G \rho = e^{-2r} \rho_1(\xi) + \rho_2(\xi), \]  \hspace{1cm} (116)
\[ 8\pi G p = e^{-2r} p_1(\xi) + p_2(\xi), \]  \hspace{1cm} (117)

where \( \xi = t \). A set of ordinary differential equations is obtained if EFEs and the equations of motion for the matter field are satisfied for \( O[(r)^0] \) and \( O[(r)^{-2}] \) terms separately. In this case, Eqs.(2.3)-(2.9) take the following form

\[ \rho_1 = e^{-2\phi} \mu' \nu', \]  \hspace{1cm} (118)
\[ \rho_2 = 0, \]  \hspace{1cm} (119)
\[ -e^{2\phi} p_1 = \nu'' + \nu'^2 + \mu'' + \mu'2 - \phi' \nu' - \phi' \mu' + \mu' \nu', \]  \hspace{1cm} (120)
\[ p_2 = 0, \]  \hspace{1cm} (121)
\[
0 = \nu' + \mu', \quad (122)
\]
\[
p_1 = \rho_1, \quad (123)
\]
\[
(\mu' + \nu')(\rho_1 + p_1) = -\rho_1', \quad (124)
\]
where prime (') represents derivative with respect to \( t \). Eqs.(122) and (124) yield that \( \rho_1 \) is a constant and assume that \( \rho_1 = \rho_0 \). Also, Eq.(122) implies that \( \mu' = -\nu' \).

For EOS(1) and EOS (2), we have a contradiction if we take \( p_2 = 0 = \rho_2 \) (as given by the above equations). If we take \( p_1 = 0 = \rho_1 \), this gives a vacuum spacetime. If we consider \( p_1 = 0 = \rho_1 \) or \( p_2 = 0 = \rho_1 \), we also get a vacuum spacetime. EOS (3) gives us imaginary complex results.

### 6 Orthogonal Dust case

For the dust case, we substitute \( p_1 = 0 = p_2 \) in the basic equations for the orthogonal perfect fluid case. We obtain contradiction in all the cases, i.e., second, zeroth and infinite kinds. Hence there does not exist any self-similar solution for the orthogonal dust case.

### 7 Conclusion

We have attempted to find out KSS perfect fluid and dust solutions for the cases when KSS vector is tilted, parallel or orthogonal to the fluid flow with either EOS(1) or EOS(2) or EOS(3). The parallel case gives a contradiction and hence there is no self-similar cylindrical symmetric solution for this case (i.e. second, zeroth or infinite kind).

For the tilted perfect fluid case (self-similarity of the second kind), EOS(1) and EOS(2) give only vacuum spacetimes. The EOS(3), for \( k = -1 \), yields a contradiction and also the cases \( \rho_1 = 0 = p_1 \) and \( \rho_2 \neq 0 \neq p_2 \) give a contradiction. We obtain only one solution for EOS(3) when \( \rho_2 = 0 = p_2 \). For the tilted perfect fluid case with self-similarity of the zeroth kind, EOS(1) and EOS(2) give contradiction. However, EOS(3) provides one solution and the remaining possibilities yield either a contradiction or a vacuum solution. In the tilted perfect fluid case (self-similarity of the infinite kind), we obtain that the spacetime must be vacuum for EOS(1) and EOS(2). EOS(3) for \( k = -1 \) provides one solution which corresponds to Minkowski spacetime. The remaining case yields a contradiction.
When we solve the tilted dust case with the self-similarity of the infinite kind, it follows a vacuum solution while the other kinds yield contradiction. In the orthogonal perfect fluid case with self-similarity of the infinite kind, we have vacuum solution for EOS(1) and EOS(2) while EOS(3) do not give a solution. For zeroth kind we have a contradiction in basic equations. For the orthogonal dust case we obtain contradiction in all the cases. The summary of the results can be given below in the form of tables.

**Table 1.** Perfect fluid kinematic self-similar solutions for the EOS(1).

| Self-similarity       | Solution |
|-----------------------|----------|
| Second kind (tilted)  | Vacuum   |
| Second kind (parallel)| None     |
| Second kind (orthogonal)| None    |
| Zeroth kind (tilted)  | None     |
| Zeroth kind (parallel)| None     |
| Zeroth kind (orthogonal)| None    |
| Infinite kind (tilted)| Vacuum   |
| Infinite kind (parallel)| None    |
| Infinite kind (orthogonal)| Vacuum |

**Table 2.** Perfect fluid kinematic self-similar solutions for the EOS(2).

| Self-similarity       | Solution |
|-----------------------|----------|
| Second kind (tilted)  | Vacuum   |
| Second kind (parallel)| None     |
| Second kind (orthogonal)| None    |
| Zeroth kind (tilted)  | None     |
| Zeroth kind (parallel)| None     |
| Zeroth kind (orthogonal)| None    |
| Infinite kind (tilted)| Vacuum   |
| Infinite kind (parallel)| None    |
| Infinite kind (orthogonal)| Vacuum |
Table 3. Perfect fluid kinematic self-similar solutions for the EOS(3).

| Self-similarity         | Solution                      |
|-------------------------|-------------------------------|
| Second kind (tilted)    | solution given by Eq.(44)     |
| Second kind (parallel)  | None                          |
| Second kind (orthogonal)| None                          |
| Zeroth kind (tilted)    | solution given by Eq.(62)     |
| Zeroth kind (parallel)  | None                          |
| Zeroth kind (orthogonal)| None                          |
| Infinite kind (tilted)  | Minkowski                     |
| Infinite kind (parallel)| None                          |
| Infinite kind (orthogonal)| None                     |

It is to be noted that there is only vacuum solution in the case of dust fluid for the tilted infinite kind. In the remaining cases, we do not have solution.
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