Zero-free regions near a line

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Abstract
We analyze metrics for how close an entire function of genus one is to having only real roots. These metrics arise from truncated Hankel matrix positivity-type conditions built from power series coefficients at each real point. Specifically, if such a function satisfies our positivity conditions and has well-spaced zeros, we show that all of its zeros have to (in some explicitly quantified sense) be far away from the real axis. The obvious interesting example arises from the Riemann zeta function, where our positivity conditions yield a family of relaxations of the Riemann hypothesis. One might guess that as we tighten our relaxation, the zeros of the zeta function must be close to the critical line. We show that the opposite occurs: any potential non-real zeros are forced to be farther and farther away from the critical line.

Keywords  Zero-free regions · Entire functions of finite order · Laguerre–Pólya class

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1 Introduction

1.1 Motivation and setup

An entire function of genus one is a function of the form

\[ f(z) = z^\ell e^{d_1 z + d_2 z} \prod_{j=1}^{\infty} \left( 1 - \frac{z}{\lambda_j} \right) e^{\epsilon z / \lambda_j}, \tag{1} \]

where \( \ell \) is a non-negative integer and \( \sum_j \frac{1}{|\lambda_j|} < \infty \). If both \( d_2 \) is real and all of the roots \( \lambda_j \) are real, then \( f \) is in the Laguerre–Pólya class, the class of entire functions that are locally, uniformly limits of sequences of polynomials with real zeros [7]. As an aside, every function \( h \) in the Laguerre–Pólya class actually satisfies \( h(z) = e^{d_2 z} f(z) \), where \( f \) is of form (1) with \( d_2 \) and the \( \lambda_j \) real and \( d \leq 0 \). The goal of this investigation, broadly speaking, is to detect how close an entire function of genus one is to being real rooted by developing a hierarchy of relaxations of the Laguerre–Pólya class (when \( d = 0 \)).

To motivate our Laguerre–Pólya relaxations, first note that given \( f \) in (1), its negative log derivative has formula

\[ g(z) := -\frac{d}{dz} \log f(z) = -\frac{\ell}{z} - d_2 + \sum_{j=1}^{\infty} \frac{z}{\lambda_j (\lambda_j - z)}. \]

Elementary calculations show that \( g(z) + d_2 \) maps the upper half plane to itself and the lower half plane to itself if and only if all of the \( \lambda_j \) are real. Expanding the Laurent series for the logarithmic derivative at 0 gives

\[ g(z) = \frac{-\ell}{z} - d_2 + \sum_{k=1}^{\infty} a_k z^k, \]

where \( a_k = \sum \frac{1}{\lambda_j^{k+1}} \). To connect this to a matrix condition, define the measure \( \mu = \sum \frac{1}{\lambda_j} \delta_1/\lambda_j \) on \( \mathbb{C} \) and note that the measure \( \mu \) is positive if and only if all of the \( \lambda_j \) are real. Moreover, its moments satisfy

\[ a_k = \int w^{k-1} d\mu(w). \]

Write the infinite Hankel matrix

\[ A = \begin{bmatrix} a_1 & a_2 & a_3 & \ldots \\ a_2 & a_3 & a_4 & \ldots \\ a_3 & a_4 & a_5 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{m,n} = \left[ \int w^{m+n-2} d\mu(w) \right]_{m,n}. \tag{2} \]

Nevanlinna’s solution to the Hamburger moment problem [24] implies that the infinite matrix \( A \) in (2) is positive semidefinite if and only if \( \mu \) is positive if and only if all of the \( \lambda_j \) are real.

For each \( x \in \mathbb{R} \setminus \{\lambda_j\} \), one can similarly expand \( g(z + x) = \sum_{k=0}^{\infty} a_k(x) z^k \) with \( a_k(x) = \sum \frac{1}{(\lambda_j - x)^{k+1}} \) for \( k > 0 \) and use analogues of the previous arguments to deduce that all of the \( \lambda_j \) are real if and only if the infinite matrix \( A(x) \) with entries \( A(x)_{m,n} = a_{m+n-1}(x) \) is positive semidefinite. Thus, \( f \) is in the Laguerre–Pólya class if and only if \( d_2 \in \mathbb{R} \) and any (or every) such \( A(x) \) is positive semidefinite.
Our relaxations of the Laguerre–Pólya class involve truncations of (2) and the more general $A(x)$, as in the classical results of Dobsch–Donoghue [1, 11, 12] on Löwner’s theorem and $N$-matrix monotonicity. Specifically, for each $N \in \mathbb{N}$, we say $f$ is in the $N$-th order Laguerre–Pólya class (denoted $N$-LP) if $d_2 \in \mathbb{R}$ and the matrix inequality

$$\begin{bmatrix}
a_1(x) & a_2(x) & \ldots & a_N(x) \\
\vdots & \ddots & \ddots & \vdots \\
a_N(x) & \ldots & a_{2N-1}(x)
\end{bmatrix} \geq 0$$

holds for all $x \in \mathbb{R} \setminus \{\lambda_j\}$. Here, the notation $A_N(x) \geq 0$ means $A_N(x)$ is a positive semidefinite matrix. Also, it is worth noting that these $N$-LP classes are nested and if $f$ is in 1-LP, then any non-real zeros of $f$ must come in complex-conjugate pairs (so that $f$ is real on the real line), see Lemma 4.1 for details.

As an aside, it is worth noting that all functions whose negative logarithmic derivative is a self-map of the upper half plane have a continuous version of the Hadamard factorization [27]. Moreover, such functions satisfy the determinantal isoperimetric inequality

$$(\text{det } f(A)) f(C) \leq \text{det } f(B) f(D)$$

whenever $A \leq B \leq C$ and $D = A + B - C$, where $A$, $B$, $C$ are self-adjoint matrices of the same arbitrary size with spectrum contained in some interval in $\mathbb{R}$ where $f$ does not vanish and $f$ is evaluated on matrices via the functional calculus. While we do not belabor to prove the point as it is irrelevant to our current aims, those familiar with the literature will see that functions in $N$-LP preserve the inequality (4) on $N$ by $N$ matrices via the classical Dobsch–Donaghue theorem [12] combined with [27, Theorem 3.3].

We examine entire functions, and particularly those in the $N$-LP whose zeros $\{\lambda_j\}$ satisfy reasonable spacing conditions. To that end, define the spacing constant $c$ of a Hadamard product $f$ as in (1) as $c = 0$ if $f$ has a repeated zero and

$$c = \inf_{j \neq k} \left\{ |\Re(\lambda_j - \lambda_k)| : \lambda_j \neq \bar{\lambda}_k \right\},$$

if $f$ has only simple zeros. If the spacing constant is non-zero, then we call the function spaced. Define the height of a Hadamard product to be

$$b = \inf_j \left\{ |\Im(\lambda_j)| : \Im(\lambda_j) \neq 0 \right\}.$$  

If the infimum is taken over an empty set, we say the height is infinite. We define the aperture of a Hadamard product to be $\kappa = b/c$ if $c \neq 0$. Otherwise we define $\kappa = \infty$. If our function is spaced, then $\kappa = \infty$ implies the zeros are real.

### 1.2 Main results

Our main results are of the following flavor: we assume a function is in $N$-LP and then conclude that the aperture $\kappa$ grows with $N$. Note that if $\kappa$ is large, there can be no non-real zeros near the real axis. For example, we obtain the following for functions in 1-LP.

**Theorem 1.1** Let $f$ be an entire function of genus one such that $f(0) \neq 0$. If $f$ is in the first order Laguerre–Pólya class (1-LP), then

$$\kappa \geq \frac{\sqrt{3}}{\pi}.$$
This appears in Sect. 4 as Theorem 4.2. More generally, we obtain the following, which appears later as Theorem 4.4:

**Theorem 1.2** Let $f$ be an entire function of genus one such that $f(0) \neq 0$. If $f$ is in the $N$-th order Laguerre–Pólya class ($N$-LP) then

$$N \leq \frac{\ln \left( \frac{4\pi^2}{3} + 4 \right)}{\ln 2} + \frac{\pi^3 \sqrt{3}}{\ln 2} \left( \kappa + \sqrt{1 + \kappa^2} \right)^6 \left( 1 + \kappa^2 \right)^{3/2} \approx \kappa^9.$$  

Based on our proof, we conjecture that this 9-th order behavior is optimal. However, it is possible that a different proof would yield a stronger relationship.

To prove our results, we use the following idea, which is similar to one arising in the $N$-matrix monotonicity literature [12, 18, 19]. We test the Hankel matrix $A_N(x)$ on a vector $[q_1 \quad q_2 \quad \vdots \quad q_N]^* A_N(x) [q_1 \quad q_2 \quad \vdots \quad q_N] = \sum_j q \left( \frac{1}{\lambda_j - x} \right)^2,$

where $q(z) = \sum_{n=1}^N q_n z^n$, $q_n \in \mathbb{R}$. Then to detect a non-real zero $\lambda_k$, we try to simultaneously make both $q \left( \frac{1}{\lambda_k - x} \right)^2$ big and negative and $q \left( \frac{1}{\lambda_j - x} \right)^2$ small for $\lambda_j \in \mathbb{R}$. If one wants a $q$ that is small on $\mathbb{R}$ and big off the real line, one might guess that the optimal things to try are Taylor polynomials for $\sin(rz)$. Specifically, our crucial observation is a “sine recovery lemma”, which is perhaps a remarkable result on its own.

**Lemma 1.3** (Sine recovery lemma) Let $f$ be an entire function of genus one such that $f(0) \neq 0$. If $\sum_j \sin(t(\lambda_j - x))^2 \geq 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}\{\lambda_j\}$, then $\kappa = \infty$.

This appears at the end of Sect. 4 as Lemma 4.5.

We also briefly consider functions that satisfy a stronger spacing condition. We call these functions **strongly spaced**, which means that there are constants $\gamma, C > 0$ such that if the zeros $\lambda_k$ are ordered with increasing, positive real part, then

$$\Re(\lambda_{k+1} - \lambda_k) \geq C |\Re(\lambda_k)|^\gamma$$

for all $k$ where $\lambda_{k+1} \neq \bar{\lambda}_k$. Under this assumption, we prove:

**Theorem 1.4** Let $f$ be in the first order Laguerre–Pólya class (1-LP) with $f(0) \neq 0$ and strongly spaced constants $C, \gamma > 0$. If $\lambda_k$ is a zero of $f$ with $\Im(\lambda_k) \neq 0$, $\lambda_k + 1 = \bar{\lambda}_k$, and $k > 1$, then

$$|\Im(\lambda_k)| \geq C \frac{\sqrt{3}}{2\pi} \left[ |\Re(\lambda_{\lfloor k/2 \rfloor})| \right]^\gamma,$$

where $\lfloor k/2 \rfloor$ denotes the greatest integer less than or equal to $k/2$.

This appears in Sect. 5 as Theorem 5.1 (which also mentions the $k = 1$ case). As the strongly spaced setting was outside of the bounds of our original investigation, we only analyze the behavior of strongly spaced functions in 1-LP. However, we encourage the ambitious reader to explore strongly spaced functions in the more general $N$-LP.

For the remainder, we connect our results to the Riemann hypothesis and then prove our main results. In Sect. 2, we use our Laguerre–Pólya relaxations to obtain a family of relaxations of the Riemann hypothesis and assuming certain spacing conditions, use our main results to conjecture certain zero-free regions of the zeta function. Regarding spacing,
we propose a related spacing constant in Sect. 2.1 and discuss two alternative non-constant spacing regimes and their applications in Sect. 2.2. In our final three sections, we establish preliminary facts about Hankel matrices associated to Hadamard products and then prove our main (previously-stated) results.

2 Connections to the Riemann hypothesis

In this section, we further motivate our Laguerre–Pólya relaxations by considering their implications for the Riemann hypothesis. First, let the sequence \( \{ \rho_k \} \) denote the zeros of the Riemann zeta function with positive imaginary part, listed according to multiplicity. To connect our setting to the Riemann hypothesis, we need to change variables slightly. To that end, for each \( k \), define \( \eta_k \in \mathbb{C} \) by

\[
\eta_k = \Im(\rho_k) + i\left( \frac{1}{2} - \Re(\rho_k) \right)
\]

Then \( \rho_k = \Re(\rho_k) + i\Im(\rho_k) = \frac{1}{2} + i\eta_k \). (7)

Set \( \lambda_k = \eta_k^2 \) and let \( \Lambda \) be the complex analytic function

\[
\Lambda(z) = \prod_k \left( 1 - \frac{z}{\lambda_k} \right) = e^{-\left( \sum_k 1/\lambda_k \right)z} \prod_k \left( 1 - \frac{z}{\lambda_k} \right) e^{z/\lambda_k}.
\]

The function \( \Lambda(z) = \Xi(\sqrt{z}) \), where \( \Xi \) is a standard function defined from the Riemann zeta function by factoring out its trivial zeros and pole. The functional equation of the zeta function is equivalent to the assertion that \( \Xi \) is real on the real line. It follows that the Riemann hypothesis holds if and only if each \( \eta_k \) (or equivalently, each \( \lambda_k \)) is on the real line.

2.1 Relaxation hierarchy for the Riemann hypothesis

The key observation motivating these investigations is the following: \( \Lambda \) is in the \( N \)-LP for all \( N \) if and only if the Riemann hypothesis holds, see [27]. Based on that, we propose the following set of relaxations of the Riemann hypothesis for each \( N \in \mathbb{N} \).

Conjecture 2.1 (N-th Hankel relaxation Riemann hypothesis, N-HRRH) The function \( \Lambda \) is in the \( N \)-th order Laguerre–Pólya class (\( N \)-LP).

This condition is equivalent to saying that the matrix \( A_N(x) \) in (3) is positive semi-definite for \( x \in \mathbb{R} \setminus \{ \lambda_j \} \). Here, the \( mn \)-th entry of \( A_N(x) \) is

\[
a_{m+n-1}(x) = \sum_j \frac{1}{(\lambda_j - x)^{m+n}},
\]

see Sect. 3. It follows immediately that each \( A_N(x) \) is self-adjoint, since properties of the Riemann zeta function imply that the zeros of \( \Lambda \) are real or occur in complex conjugate pairs. Then, checking the positivity of \( A_N(x) \) is equivalent to showing that the principle minors of \( A_N(x) \) (the determinants of the square submatrices obtained by removing a (possibly empty) set of rows and their corresponding columns from \( A_N(x) \) are positive or zero. Because the entries of \( A_N(x) \) come from the log derivatives of \( \Lambda \) to order \( 2N \), these determinantal conditions are equivalent to a set of inequalities involving the derivatives of \( \Lambda \) to order \( 2N \).
For example, if \( N = 1 \), this conjecture says that for \( x \in \mathbb{R} \setminus \{ \lambda_j \} \),
\[
0 \leq a_1(x) := -\frac{d^2}{dz^2} (\log \Lambda(z)) \bigg|_{z=x} = \frac{\Lambda'(x)^2 - \Lambda''(x)\Lambda(x)}{\Lambda(x)^2}.
\]

By continuity, the numerator is non-negative for every \( x \in \mathbb{R} \), which implies the following simple form for 1-HRRH:

**Conjecture 2.2** (1-HRRH) We have \( \Lambda'(x)^2 - \Lambda''(x)\Lambda(x) \geq 0 \) for real \( x \).

If \( N = 2 \), our conjecture says that \( a_1(x) \geq 0 \), \( a_3(x) \geq 0 \), and \( a_1(x)a_3(x) - a_2(x)^2 \geq 0 \). One can rewrite these inequalities in terms of derivatives of \( \Lambda \), but the result is not particularly illuminating. Instead, for \( N \geq 2 \), we suggest that the reader think about our order-\( N \) relaxations in terms of the matrix conditions or the polynomial criterion given in Lemma 3.1.

We also note that 1-HRRH was originally conjectured by Csordas in [4]. Moreover, Csordas made some conjectures on higher order Turán inequalities which together would prove the Riemann hypothesis, although it not clear to us if they give rise to the exact same family of relaxations as ours, see [5, 6, 10]. Our relaxations may also be connected to the following collection of conjectures studied by Farmer in [14]: for \( \Lambda \) and each of its derivatives, all local maxima are positive and all local minima are negative. Again, these conjectures encompass 1-HRRH, but it is not clear to us how the higher-order conjectures interact with our order-\( N \) relaxations.

We can apply our main results (Theorems 1.1, 1.2, 1.4) to \( \Lambda \) if \( \Lambda \) is in \( N \)-LP (i.e. if \( N \)-HRRH holds) and if \( \Lambda \) is spaced (or strongly spaced). Under those assumptions, our main results imply that for each \( N \), \( \Lambda \) possesses a specific zero-free region near the real line, and as \( N \) goes to infinity, \( \Lambda \) has no zeros off of the real line. Of course, the specific zero-free regions will depend on the spacing constant for \( \Lambda \), as defined in (5). Unfortunately, explicit bounds on the spacing of the zeta function’s zeros are generally not known and it is an open question as to whether the zeros are even all simple, see [32] and the references therein for recent results. Still, using the numerical evidence in the Odlyzko tables [26], we are motivated to make the following spacing conjecture.

**Conjecture 2.3** (Spacing conjecture) The spacing constant from (5) for \( \Lambda \) is positive and equal to \( \lambda_6 - \lambda_5 > 159.045 \).

This conjecture might seem surprising, so it is worth noting that the spacing (of the real parts) of the zeros of \( \Lambda \) is connected to the spacing of the zeros of the Riemann zeta function via the following equation:
\[
\Re(\lambda_{j+1} - \lambda_j) = (\Im(\rho_{j+1})^2 - \Im(\rho_j)^2) + \left( (\frac{1}{2} - \Re(\rho_j))^2 - (\frac{1}{2} - \Re(\rho_{j+1}))^2 \right).
\]

Because of this quadratic relationship, one expects the zeros of \( \Lambda \) to spread out as \( j \) increases and indeed, looking at the first two million zeros, it appears that the smallest gap happens to occur between the fifth and sixth zeros of \( \Lambda \).

Heuristically, this spacing conjecture says that the gap between the real parts of two adjacent zeros of the zeta function is at least on the order of \( 1/\log(x) \) where \( x \) is the modulus of one of the zeros. For comparison, the average gap is of order \( 1/\log(x) \). It is possible and even likely that isolated small gaps would give rise to Lehmer zeros or other small gaps, although we do not see an exact connection in general. Still, it is worth noting that work of Simonić in [31] related to Lehmer zeros shows that the linear gap \( \Im(\rho_{j+1}) - \Im(\rho_j) \) can be quite small.

Assuming this spacing conjecture, our results show that our hierarchy of relaxations for the Riemann hypothesis gives rise to zero-free regions for the zeta function near the critical line. For example, the \( N = 1 \) case paired with Theorem 1.1 gives
Fig. 1  Allowable regions for zeros \((\sigma + it)\) of the zeta function. The heavy dashed line is \(\sigma = 1\) and the light dashed line is \(\sigma = \frac{1}{2}\).

Corollary 2.4  If 1-HRRH and the spacing conjecture are true and if some zero \(\rho_k\) of the zeta function is not on the critical line, then

\[
|\Im(\rho_k)| \left| \frac{1}{2} - \Re(\rho_k) \right| \geq \frac{159\sqrt{3}}{2\pi}.
\]

This lower bound is not sharp and Remark 4.3 gives one argument improving this estimate. Similarly, if \(N \geq 5\), Theorem 1.2 implies

Corollary 2.5  There is a constant \(M\) independent of \(N\) (and \(\Lambda\)) such that if \(N\)-HRRH and the spacing conjecture are true and if some zero \(\rho_k\) of the zeta function is not on the critical line, then

\[
|\Im(\rho_k)| \left| \frac{1}{2} - \Re(\rho_k) \right| \geq \frac{159M}{2}N^{1/9}.
\]

To prove these corollaries, assume some \(\rho_k\) is not on the critical line and without loss of generality, assume \(\Im(\rho_k) > 0\). Then \(\lambda_k := \eta_k^2\) (defined in (7)) is a zero of \(\Lambda\) with \(\Im(\lambda_k) = 2\Im(\rho_k)\left(\frac{1}{2} - \Re(\rho_k)\right) \neq 0\). As \(\kappa = b/c\) where \(c = 159\) and \(b\) is defined in (6), we have

\[
\frac{2\Im(\rho_k)}{159} \left| \frac{1}{2} - \Re(\rho_k) \right| = \frac{|\Im(\lambda_k)|}{159} \geq \frac{b}{159} = \kappa,
\]

and the results follow from the \(\kappa\) inequalities in Theorems 1.1 and 1.2.

These results are rather odd; zero free regions, as developed classically by de la Vallée Poussin, Littlewood, Chudakov [3, 9, 21], and in more modern terms by Cheng, Ford, Mossinghoff, Trudgian, and other authors [2, 15, 23], usually say that the zeros of the Riemann zeta function have to be away from the \(s = 1\) line. Instead, ours say that there are no zeros close to the critical line \(s = \frac{1}{2}\), see Fig. 1. So, if \(N\)-HRRH holds to some order \(N\) but then suddenly fails, then the Riemann hypothesis has to fail badly— the zeros need to be, in some sense, far away from the critical line. These findings appear to align with qualitative
conclusions suggested by [14]; indeed, Farmer’s work implies that if a function satisfies a family of conditions including 1-HRRH, then its non-real zeros should in some sense be far away from the real line.

It is worth noting that our approach generally fits into the Pólya-Jensen approach [28, 29] to the Riemann hypothesis, which is based upon the belief that there is some naturally occurring self-adjoint operator with eigenvalues equal to the zeros of $\Lambda$. Other Laguerre–Pólya based approaches to the Riemann hypothesis include deBranges’ approach [8], Li’s criterion [20], and Rodgers and Tao’s establishment that the de Bruijn-Newman constant is non-negative [30].

Related exciting results on the asymptotic hyperbolicity of certain Jensen polynomials appeared in recent work by Griffen et al. in [16, 17]. Their ideas uses power series coefficients derived from $\Lambda$ at a single point, but as discussed in the critique [13], the results do not appear to have direct implications for the Riemann hypothesis. Our methods in some sense resolve the apparent disharmony between the Griffen et al. and Farmer perspectives; we use power series coefficients derived from $\Lambda$ at a number of different points simultaneously and under various conjectures, are able to draw conclusions about zero-free regions.

2.2 Alternate spacing regimes

In this section, we mention two alternate (non-constant) spacing regimes for $\Lambda$, one stronger than constant spacing and one weaker than constant spacing. These could (in theory) be used in place of Conjecture 2.3 to apply our main theorems to $\Lambda$ and hence, draw conclusions about the zeros of the zeta function.

2.2.1 Strong spacing

Under the assumption of the Riemann hypothesis, standard conjectures informed by random matrix theory [22, 25] imply that the zeros of the zeta function display quadratic repulsion in the sense that the probability distribution function of the distance from one zero to the next zero apparently vanishes to order 2 at 0. This implies the standard conjecture that $|\rho_{j+1} - \rho_j| < |\rho_j|^{-\alpha}$ at most finitely many times for $\alpha > 1/3$. If one converts this to a statement about the zeros of $\Lambda$, it says: there is a constant $C$ (depending on $\gamma$) such that

$$\lambda_{j+1} - \lambda_j \geq C\lambda_j^\gamma \tag{8}$$

for $\gamma \in [0, 1/3]$.

If we do not assume the Riemann hypothesis (and hence, allow the $\lambda_k$ to lie off of the real line), one way to generalize (8) is the following family of conjectures:

**Conjecture 2.6** (Strong spacing conjecture) *Let $\gamma \in [0, 1/3]$. There exists a $C > 0$ depending on $\gamma$ such that

$$\Re(\lambda_{j+1} - \lambda_j) \geq C\Re(\lambda_j)^\gamma$$

for all $j$ with $\lambda_j \neq \bar{\lambda}_{j+1}$ when $\Im(\lambda_j) \neq 0$.*

It is not clear to us whether this spacing conjecture should be true, especially for $\gamma$ near $1/3$. Indeed, while it seems plausible that the zeros of $\Lambda$ should still repel each other in some sense, it is not clear whether that repulsion would be concentrated in the real direction. Still, it seems worth investigating whether this conjecture might be supported by known paradigms, at least for $\gamma$ in some subinterval of $[0, 1/3)$.
If Conjecture 2.6 holds for some $\gamma > 0$ and one assumes 1-HRRH, Theorem 1.4 applies to $\Lambda$ and gives an intricate condition on any non-real zeros of $\Lambda$. We leave it to future work (or an interested reader) to further investigate the zero constraints implied by $N$-HRRH (for $N \geq 1$) paired with Conjecture 2.6 for different values of $\gamma$.

2.2.2 Weak spacing

We say $\Lambda$ is weakly spaced if there is a $\gamma < 0$ and a $C > 0$ depending on $\gamma$ such that

$$\Re(\lambda_{j+1} - \lambda_j) \geq C\Re(\lambda_j)^\gamma$$

for all $j$ with $\lambda_j \neq \bar{\lambda}_{j+1}$ when $\Im(\lambda_j) \neq 0$. Here, we note that one may be able to apply our results from Sects. 4 and 5 to $\Lambda$ even if $\Lambda$ is only weakly spaced, via the following recipe.

Let $\Lambda_0 = \Lambda$ and define a sequence of entire functions iteratively via the following relation

$$\Lambda_n(z) = \Lambda_{n-1}(\sqrt{z})\Lambda_{n-1}(-\sqrt{z})$$

and observe that the zeros of $\Lambda_n$ are given by $\lambda_j^{2n}$. Suppose that for some $\gamma \in \mathbb{R}$ and all $j$ with $\lambda_j \neq \bar{\lambda}_{j+1}$ when $\Im(\lambda_j) \neq 0$,

$$\Re(\lambda_{j+1} - \lambda_j) \geq C\Re(\lambda_j)^\gamma.$$

Then

$$\Re(\lambda_j^{2n-1}) \cdot \Re(\lambda_{j+1} - \lambda_j) \geq C\Re(\lambda_j)^{\gamma + 2n - 1}.$$

If the real parts of the $\lambda_j$ sufficiently dominate the imaginary parts of the $\lambda_j$ (via a relation that would depend on $C$ and $n$), then this should yield a related constant $\hat{C}$ with

$$\Re(\lambda_{j+1} - \lambda_j) \geq \hat{C}|\Re(\lambda_j^{2n})|^{\gamma + 2n - 1}/2n,$$

as long as $\lambda_{j+1} \neq \bar{\lambda}_j$ when $\Im(\lambda_j) \neq 0$. Then for sufficiently large $n$, the zeros of $\Lambda_n$ would exhibit strong spacing and so, one could apply our results from Sects. 4 and 5 to $\Lambda_n$.

3 Hankel matrices associated to Hadamard products

Before proving our main results, we need to collect some preliminary information about the Hankel matrices defined in (3). To that end, let $f$ be defined via Hadamard product as in (1) with $f(0) \neq 0$. As mentioned earlier, we say $f$ is in the $N$-th order Laguerre–Pólya class ($N$-LP) if $d_2 \in \mathbb{R}$ and (3) holds for all $x \in \mathbb{R} \setminus \{\lambda_j\}_j=1^\infty$. Let us quickly derive the formulas for the entries of each $A_N(x)$. First, one can easily compute

$$g(z) := -\frac{d}{dz} (\log (f(z))) = -\frac{f'(z)}{f(z)} = -d_2 + \sum_{j=1}^{\infty} \frac{z}{\lambda_j(\lambda_j - z)}.$$

For $z$ near each $x \in \mathbb{R} \setminus \{\lambda_j\}$, i.e. if $|z - x| < |\lambda_j - x|$, we can then write

$$\frac{z}{\lambda_j(\lambda_j - x)} = \frac{x}{\lambda_j(\lambda_j - x)} + \sum_{k=1}^{\infty} \left( \frac{1}{\lambda_j - x} \right)^{k+1} (z - x)^k.$$
For each $x \in \mathbb{R} \setminus \{\lambda_j\}_{j=1}^{\infty}$, we can expand $g$ in a power series $g(z) = \sum_{k} a_k(x)(z - x)^k$ around any $x$ as follows:

$$g(z) = -d_2 + \sum_{j=1}^{\infty} \frac{x}{\lambda_j(\lambda_j - x)} + \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} \left( \frac{1}{\lambda_j - x} \right)^{k+1} \right) (z - x)^k,$$

where our assumption that $\sum_{j} \frac{1}{|\lambda_j|^2} < \infty$ implies that the equality holds for all $z$ in some nontrivial interval centered around $x$.

The $N$-LP property implies the following polynomial condition:

Lemma 3.1 Fix $x \in \mathbb{R} \setminus \{\lambda_j\}_{j=1}^{\infty}$ and $N \in \mathbb{N}$ and assume (3) holds. Then for all $t \in \mathbb{R}$,

$$\sum_{j=1}^{\infty} q \left( \frac{t}{\lambda_j - x} \right)^2 \geq 0,$$

for all polynomials $q$ with real coefficients such that $q(0) = 0$ and $\deg q \leq N$.

Proof Let $q(z) = \sum_{n=1}^{N} q_n z^n$ with each $q_n \in \mathbb{R}$. Fix $\tilde{a} = [d_1 \ldots d_N]^T \in \mathbb{R}^N$ with each $d_n = t^n q_n$. Then by our assumptions and the definition of $a_k(x) = \sum_{j} \frac{1}{(\lambda_j - x)^{k+1}}$, we have

$$0 \leq \begin{bmatrix} a_1(x) & a_2(x) & \cdots & a_N(x) \\ a_2(x) & a_3(x) & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_N(x) & \cdots & a_{2N-1}(x) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix} = \sum_{j=1}^{\infty} \left( \frac{1}{\lambda_j - x} \right)^{m+n}.$$  \tag{11}

where $A^j$ denotes the $N \times N$ matrix whose $mn$-th entry is $(\frac{1}{\lambda_j - x})^{m+n}$. Then

$$\left\langle A^j \tilde{a}, \tilde{a} \right\rangle = \sum_{m,n=1}^{N} d_m d_n \left( \frac{1}{\lambda_j - x} \right)^{m+n} = \left( \sum_{n=1}^{N} d_n \left( \frac{1}{\lambda_j - x} \right)^n \right)^2$$

$$= \left( \sum_{n=1}^{N} q_n \left( \frac{t}{\lambda_j - x} \right)^n \right)^2 = q \left( \frac{t}{\lambda_j - x} \right)^2.$$  \tag{12}

Combining (11) and (12) immediately gives (10). \hfill \Box

4 Spaced functions

Let $f$ be a spaced function defined via a Hadamard product as in (1) with $f(0) \neq 0$ and let $c, b, \kappa$ denote the respective spacing constant, height, and aperture of $f$, see (5) and (6). Write each zero $\lambda_j = \alpha_j + i\beta_j$ for $\alpha_j, \beta_j \in \mathbb{R}$.

The following result will be used implicitly in many of our estimates.

Lemma 4.1 Let $f$ be in the first order Laguerre–Pólya class (1-LP). If $\lambda_j$ is a zero of $f$, then $\bar{\lambda}_j$ is also a zero of $f$.

Proof Recall that $g(z) = -\frac{f'(z)}{f(z)}$. Looking at the power series expansion of $f$ near each zero, it is easy to see that

$$g'(z) = \frac{f'(z)^2 - f''(z) f(z)}{f(z)^2}$$
has poles (of order two) exactly at the zeros of \( f \). As \( f \) is in 1-LP, (3) with \( N = 1 \) implies that

\[
g'(x) \in \mathbb{R} \text{ for all } x \in \mathbb{R} \setminus \{\lambda_j\}_{j=1}^\infty.
\]

Thus, we can apply the Schwarz reflection principle on any interval in \( \mathbb{R} \setminus \{\lambda_j\}_{j=1}^\infty \) to conclude that (except at the zeros of \( f \)):

\[
g'(z) = g'(\overline{z}).
\]

Then this implies \( g' \) has a pole at \( w \in \mathbb{C} \) if and only if it has a pole at \( \overline{w} \). Observations show that: \( \lambda_j \) is zero of \( f \) if and only if \( \lambda_j \) is a pole of \( g' \) if and only if \( \lambda_j \) is a pole of \( f \) if and only if \( \lambda \) is a zero of \( f \), which establishes the claim.

**Theorem 4.2** Let \( f \) be in the first order Laguerre–Pólya class (1-LP) with \( f(0) \neq 0 \). Then \( \kappa \geq \frac{\sqrt{3}}{\pi} \).

**Proof** If \( \kappa = \infty \), then we are done. Thus, without loss of generality, assume \( f \) has at least one zero in \( \mathbb{C} \setminus \mathbb{R} \). Fix \( \epsilon > 0 \) and choose \( \lambda_k \) so that \( 0 < |\beta_k| < b + \epsilon \). By reordering, assume this zero is \( \lambda_1 \) and its complex conjugate is \( \lambda_2 \). Then setting \( \alpha_1 = x \) in (3) with \( N = 1 \) gives

\[
0 \leq \Re(a_1(\alpha_1)) \leq \sum_{j \geq 3} \frac{|1}{\lambda_j - \alpha_1}^2 \leq \frac{2\pi^2}{3c^2} - \frac{2}{|b + \epsilon|^2}.
\]

Rearranging and letting \( \epsilon \to 0 \) implies \( \kappa = \frac{b}{c} \geq \frac{\sqrt{3}}{\pi} \).

As detailed in the following remark, the estimate in Theorem 4.2 can be improved.

**Remark 4.3** Define \( \lambda_1, \lambda_2 \) as in the proof of Theorem 4.2 and consider the zeros of \( f \) whose real parts are closest to \( \Re(\lambda_1) = \alpha_1 \). The worst case scenarios (in terms of the estimates) are either simple real zeros at \( \alpha_1 \pm c \) or complex zeros in conjugate pairs at some \( \pm i \beta + (\alpha_1 \pm c) \) where \( b \leq |\beta| \). Then (handling the closest zeros to \( \lambda_1, \lambda_2 \) separately), our estimates become

\[
0 \leq \max \left\{ \frac{4}{c^2 + b^2}, \frac{2}{c^2} \right\} + \left( \frac{2\pi^2}{3c^2} - \frac{4}{c^2} \right) - \frac{2}{|b + \epsilon|^2}.
\]

Applying that refined argument to all of the zeros of \( f \) (excepting \( \lambda_1, \lambda_2 \)) yields the estimate

\[
0 \leq \sum_{n=1}^\infty \max \left\{ \frac{4}{c^2n^2 + b^2}, \frac{2}{c^2n^2} \right\} - \frac{2}{|b + \epsilon|^2}.
\]

Rearranging terms and letting \( \epsilon \to 0 \) gives

\[
1 \leq \sum_{n=1}^\infty \max \left\{ \frac{2\kappa^2}{\kappa^2 + n^2}, \frac{\kappa^2}{n^2} \right\} \leq \sum_{n=1}^\infty \frac{2\kappa^2}{\kappa^2 + n^2},
\]

where the second inequality holds if \( \kappa \leq 1 \) and the right-hand term is increasing in \( \kappa \). Thus to obtain a bound on \( \kappa \), we need only solve

\[
1 = \sum_{n=1}^\infty \frac{2\kappa^2}{\kappa^2 + n^2} = \kappa \pi \coth(\kappa \pi) - 1
\]

for \( \kappa \). This yields \( \kappa \approx 0.609566 \) and so, if \( f \) is in the first order Laguerre–Pólya class with \( f(0) \neq 0 \), then \( \kappa \geq 0.60956 > 0.551329 > \frac{\sqrt{3}}{\pi} \), the original bound from Theorem 4.2.
Now, we prove our estimate for general $N$.

**Theorem 4.4** Let $f$ be an entire function of genus one such that $f(0) \neq 0$. If $f$ is in the $N$-th order Laguerre–Pólya class ($N$-LP) then

$$N \leq \frac{\ln \left( \frac{4\pi^2}{3} + 4 \right)}{\ln 2} + \frac{\pi^3}{\ln 2} \left( \kappa + \sqrt{1 + \kappa^2} \right)^6 (1 + \kappa^2)^{3/2} \approx \kappa^9.$$  

**Proof** Note that the $N$-th order Laguerre–Pólya classes are nested, namely if $f$ is in $N$-LP and $M < N$, then $f$ is also in $M$-LP. Thus, we can assume $f$ is in 1-LP, since otherwise the theorem statement is trivial.

We actually prove the contrapositive of the theorem, i.e. if

$$N \geq \frac{\ln \left( \frac{4\pi^2}{3} + 4 \right)}{\ln 2} + \frac{\pi^3}{\ln 2} \left( \kappa + \sqrt{1 + \kappa^2} \right)^6 (1 + \kappa^2)^{3/2} \approx \kappa^9,$$  

then $f$ is not in $N$-LP.

First if $\kappa = \infty$, then the result is trivially true. Thus, we can assume $\kappa < \infty$. Set $d = \sqrt{b^2 + c^2} = c \sqrt{1 + \kappa^2}$ and $\epsilon = -b + d > 0$. Choose $\lambda_k$ such that $b \leq |\beta_k| < b + \epsilon/2$. After reordering, we can assume $k = 1$ and $\lambda_2 = \tilde{\lambda}_1$. For ease of notation, define $\beta := \beta_1$ and $\alpha := \alpha_1$. Set

$$\tilde{t} = \pi^2 c \left( 1 + \kappa^2 \right)^{3/2} \left( \sqrt{1 + \kappa^2} + \kappa \right)^6.$$  

Let $S_N$ denote the $N$-th degree Taylor polynomial of $\sin(z)$ centered at 0. In this proof, we will show that if

$$N \geq \frac{\ln \left( \frac{4\pi^2}{3} + 4 \right)}{\ln 2} + \frac{3\tilde{t}}{\ln(2)c} \max \{1, \frac{1}{\kappa}\}$$  

then

$$\sum_{j=1}^{\infty} \Re \left( S_N \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right)^2 \right) < 0.$$  

By Theorem 4.2, we know $\kappa \geq \sqrt{3}/\pi$. It is worth noting that we use this lower bound, rather than the stronger one from Remark 4.3, because it implies simpler formulas without affecting the main (9-th order) relationship between $N$ and $\kappa$ that we establish in this proof. Using that estimate in (15) gives the bound in (13) and then the conclusion follows from Lemma 3.1, since $S_N$ has real coefficients with $S_N(0) = 0$ and $\deg S_N \leq N$. Thus, we just need to establish (16).

**Step 1:** Show $\sum_{j=1}^{\infty} \Re \left( \sin \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right)^2 \right) < -1$.

Recall that $\sin(x + iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$. First assume $j \geq 3$ and $t \in \mathbb{R}^+$. Then

$$\Re \left( \sin^2 \left( \frac{t}{\lambda_j - \alpha} \right) \right) \leq \sin^2 \left( \Re \left( \frac{t}{\lambda_j - \alpha} \right) \right) \leq \frac{t^2}{|\lambda_j - \alpha|^2} e^{2|\beta_j| \frac{t}{|\lambda_j - \alpha|^2}}.$$  

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If $|\beta_j| > b + \epsilon$, then

$$\frac{|\beta_j|}{|\lambda_j - \alpha|^2} = \frac{|\beta_j|}{\beta_j^2 + (\alpha - \alpha_j)^2} \leq \frac{1}{|\beta_j|} < \frac{1}{b + \epsilon} = \frac{1}{d}. $$

If $|\beta_j| \leq b + \epsilon$, then

$$\frac{|\beta_j|}{|\lambda_j - \alpha|^2} \leq \frac{b + \epsilon}{b^2 + c^2} = \frac{\sqrt{b^2 + c^2}}{b^2 + c^2} = \frac{1}{d}. $$

This implies that

$$\sum_{j \geq 3} \Re \left( \sin^2 \left( \frac{t}{\lambda_j - \alpha} \right) \right) \leq \sum_{j \geq 3} \frac{t^2}{|\alpha_j - \alpha|^2} e^{2t/d} \leq \frac{2\pi^2 t^2}{3c^2} e^{2t/d}. $$

Similarly, if $j = 1$ or $j = 2$, then

$$\Re \left( \sin^2 \left( \frac{t}{\lambda_j - \alpha} \right) \right) = -\sinh^2 \left( \frac{t}{\beta_j} \right) \leq -\frac{1}{4} e^{2t/|\beta_j|} + \frac{1}{2} \leq -\frac{1}{4} e^{\frac{2t}{b + \epsilon}} + \frac{1}{2} = -\frac{1}{4} e^{\frac{4t}{b + d}} + \frac{1}{2}. $$

Thus, we can conclude that

$$\sum_{j=1}^{\infty} \Re \left( \sin^2 \left( \frac{t}{\lambda_j - \alpha} \right) \right) \leq \frac{2\pi^2 t^2}{3c^2} e^{\frac{2t}{3c^2}} + \frac{1}{2} - \frac{1}{4} e^{\frac{4t}{3c^2}} - 1. $$

(17)

Now, we just need to show that if $t = \tilde{t}$, then the right hand side of (17) is less than $-1$. That inequality is equivalent to

$$e^{\frac{2t}{3c^2}} \left( e^{Bt} - \frac{4\pi^2}{3c^2} t^2 \right) > 4,$$

where

$$B = \frac{4}{b + d} - \frac{2}{d} = \frac{2}{d(b + d)} \frac{d - b}{d} = 2 \frac{\sqrt{1 + \kappa^2} - \kappa}{c \sqrt{1 + \kappa^2} (\sqrt{1 + \kappa^2} + \kappa)} = \frac{2}{c \sqrt{1 + \kappa^2} (\sqrt{1 + \kappa^2} + \kappa)^2}. $$

Observe that $e^{\frac{2t}{3c^2}} > 4$ occurs if and only if

$$t > \frac{d \ln(4)}{2} = \frac{c \sqrt{1 + \kappa^2} \ln(4)}{2}. $$

(18)

Similarly, by expanding $e^{Bt}$ as a power series centered at 0 and cancelling 1 from both sides, we can see

$$e^{Bt} - \frac{4\pi^2}{3c^2} t^2 \geq 1$$

occurs if $\frac{B}{6} t^3 \geq \frac{4\pi^2}{3c^2} t^2$, or equivalently, if

$$t \geq \frac{8\pi^2}{c^2 B^3} = \pi^2 c(1 + \kappa^2)^{3/2} (\sqrt{1 + \kappa^2} + \kappa)^6.$$

(19)
This value is larger than the value in (18) and so we only need to choose $t$ that satisfies this inequality to guarantee that the right hand side of (17) is less than $-1$. The formula for $\tilde{t}$ in (14) clearly satisfies (19), which completes the proof of Step 1.

**Step 2:** Establish (16) for the specified values of $\tilde{t}$ and $N$.

First, the Cauchy integral formula expanded as a power series around 0 implies the following estimate: if $R > 0$ and $|z| \leq R$, then for any $N \in \mathbb{N}$,

$$|\sin(z) - S_N(z)| \leq \frac{2|z|^{N+1}}{(2R)^{N+1}} \sup_{|w|=2R} |\sin(w)| \leq \frac{2|z|^{N+1}e^{2R}}{(2R)^{N+1}}.$$ 

Observe that

$$\left|\frac{\tilde{t}}{\lambda_j - \alpha}\right| \leq \max \left\{ \frac{\tilde{t}}{c}, \frac{\tilde{t}}{b} \right\} := R > 0,$$

since $\sqrt{3}/\pi \leq \kappa < \infty$ implies $b, c \neq 0$. Then

$$\sum_{j \geq 3} \left| \sin^2 \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) - S_N^2 \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right|$$

$$\leq \sum_{j \geq 3} \left| \sin \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) + S_N \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right| \left| \sin \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) - S_N \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right|$$

$$\leq \sum_{j \geq 3} 2e^R \frac{2e^{2R}}{(2R)^{N+1}} \left| \frac{\tilde{t}}{\lambda_j - \alpha} \right|^{N+1}$$

$$\leq \frac{4e^{3R}}{2^{N+1}} \frac{c^{N+1}}{(\tilde{t})^{N+1}} \sum_{j \geq 3} \frac{(\tilde{t})^{N+1}}{|\lambda_j - \alpha|^{N+1}} \leq \frac{2e^{3R}}{2^N} \frac{2\pi^2}{3}.$$ 

Similarly, if $j = 1, 2$, we obtain

$$\sum_{j < 3} \left| \sin^2 \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) - S_N^2 \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right|$$

$$= \left| \sin^2 \left( \frac{\tilde{t}}{\tilde{t}} \right) - S_N^2 \left( \frac{\tilde{t}}{\tilde{t}} \right) \right| + \left| \sin^2 \left( \frac{\tilde{t}}{-\tilde{t}} \right) - S_N^2 \left( \frac{\tilde{t}}{-\tilde{t}} \right) \right|$$

$$\leq 2e^R \frac{1}{b} \frac{(\tilde{t})^{N+1}}{(2R)^{N+1}} \leq \frac{4e^{3R}}{2^N}.$$ 

Setting $\frac{4e^{3R}}{2^N} \left( \frac{\pi^2}{3} + 1 \right) < 1$ and solving for $N$ yields

$$N > \frac{\ln \left( \frac{4\pi^2}{3} + 4 \right)}{\ln 2} + \frac{3R}{\ln 2} = \frac{\ln \left( \frac{4\pi^2}{3} + 4 \right)}{\ln 2} + \frac{3\tilde{t}}{c\ln 2} \max \left\{ 1, \frac{1}{\kappa} \right\},$$

the earlier condition on $N$, (15). From this, we can immediately conclude that

$$\sum_{j=1}^\infty \Re \left( S_N^2 \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right)$$

$$\leq \sum_{j=1}^\infty \Re \left( \sin^2 \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right) + \sum_{j=1}^\infty \Re \left( \sin \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) - S_N \left( \frac{\tilde{t}}{\lambda_j - \alpha} \right) \right)$$

$$< -1 + 1 = 0,$$
which is what we needed to show. □

The proof of Theorem 4.4 encodes the following result, which may be of independent interest:

**Lemma 4.5 (Sine recovery lemma)** Let \( f \) be an entire function of genus one such that \( f(0) \neq 0 \). If \( \sum_j \sin(t/(\lambda_j - x))^2 \geq 0 \) for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R} \setminus \{\lambda_j\} \), then \( \kappa = \infty \).

**Proof** The proof of Theorem 4.4 assumed that \( f \) was an entire function of genus one with \( f(0) \neq 0 \) and that \( \kappa < \infty \). Given those assumptions, the proof produced numbers \( \tilde{t} \in \mathbb{R} \) and \( x \in \mathbb{R} \setminus \{\lambda_j\} \) such that

\[
\sum_{j=1}^{\infty} \Re \left( \sin \left( \frac{t}{\lambda_j - x} \right)^2 \right) < -1. \tag{20}
\]

Note that this contradicts the summation condition in the statement of the lemma, thus \( \kappa = \infty \). □

It is worth noting that the proof of Theorem 4.4 also assumed that \( f \) was in 1-LP which allowed us to use (from Theorem 4.2) that \( \kappa \geq \sqrt{3}\pi \). However that assumption is not used in the part of the proof giving (20) and so it need not be an assumption of Lemma 4.5.

### 5 Strongly spaced functions

Let \( f \) be an entire function defined with a Hadamard product as in (1). Further, assume that all zeros of \( f \) are simple with positive real part and any complex zeros appear in complex conjugate pairs. If \( f \) is in 1-LP, Lemma 4.1 says that the conjugate pair condition is immediate.

Recall that \( f \) is **strongly spaced** if there is some exponent \( \gamma > 0 \) and spacing constant \( C \) such that if the \( \lambda_j \) are ordered with increasing real part, then

\[
\Re(\lambda_{j+1} - \lambda_j) \geq C|\Re(\lambda_j)|^\gamma, \tag{21}
\]

for all \( j \) where \( \lambda_{j+1} \neq \bar{\lambda}_j \). This condition implies the following:

**Theorem 5.1** Let \( f \) be in the first order Laguerre–Pólya class (1-LP) with \( f(0) \neq 0 \) and strongly spaced with exponent \( \gamma > 0 \) and spacing constant \( C \). If \( \lambda_k \) is a zero of \( f \) with \( \Im(\lambda_k) \neq 0 \), \( \lambda_{k+1} = \bar{\lambda}_k \), and \( k > 1 \), then

\[
|\Im(\lambda_k)| \geq C \frac{\sqrt{3}}{\pi} |\Re(\lambda_{\lfloor k/2 \rfloor})|^\gamma, \tag{22}
\]

where \( \lfloor k/2 \rfloor \) denotes the greatest integer less than or equal to \( k/2 \). Similarly, if \( k = 1 \), then

\[
|\Im(\lambda_1)| \geq C \frac{\sqrt{3}}{\pi} |\Re(\lambda_1)|^\gamma. \tag{23}
\]

**Proof** Write each \( \lambda_j = \alpha_j + i\beta_j \). In what follows, we will use the assumption that the \( \lambda_j \) are distinct and the \( \alpha_j \) are positive and increasing in \( j \). Furthermore, by the strong spacing assumption (21), for any \( \alpha > 0 \) there can be most two \( \lambda_j \) with real part equal to \( \alpha \). Thus, for any \( j \), at most one of

\[
|\alpha_{j+2} - \alpha_{j+1}| \text{ and } |\alpha_{j+1} - \alpha_j| \tag{24}
\]

can be zero.
Assume $\lambda_k$ is a zero of $f$ with $\Im(\lambda_k) \neq 0$ and for now, assume $k > 1$. Since $f$ is in the first order Laguerre–Pólya class, we can substitute $x = \alpha_k = \alpha_{k+1}$ into Lemma 3.1 with $q(z) = z$ to conclude

$$0 \leq \sum_{j=1}^{\infty} \Im \left( \left( \frac{1}{\lambda_j - \alpha_k} \right)^2 \right) \leq \sum_{j \neq k, k+1} \frac{1}{|\alpha_j - \alpha_k|^2} - \frac{2}{|\beta_k|^2}. \quad (25)$$

Now, write

$$\sum_{j \neq k, k+1} \frac{1}{|\alpha_j - \alpha_k|^2} = \sum_{j=1}^{[k/2]} \frac{1}{|\alpha_j - \alpha_k|^2} + \sum_{j=[k/2]+1}^{k-1} \frac{1}{|\alpha_j - \alpha_k|^2} + \sum_{j=k+1}^\infty \frac{1}{|\alpha_j - \alpha_k|^2} = S_1 + S_2 + S_3.$$

To handle $S_3$, observe that if $j > k + 1$, then since the $\alpha_j$ are positive and increasing in $j$, we have

$$|\alpha_j - \alpha_k| = |\alpha_j - \alpha_{j-1}| + |\alpha_{j-1} - \alpha_{j-2}| + \cdots + |\alpha_{k+2} - \alpha_{k+1}| \geq \frac{j-(k+1)}{2} C|\alpha_{k+1}|^\gamma.$$

For the above inequality, we used the fact that at most half of the terms in the sum can be zero. This follows from (24) and the fact that since $\alpha_{k+1} = \alpha_k$, we must have $|\alpha_{k+2} - \alpha_{k+1}| \neq 0$. Then since nonzero terms can be bounded below using (21), they can also be bounded below by $C|\alpha_{k+1}|^\gamma$. Then

$$S_3 \leq \sum_{j>k+1} \frac{4}{C^2(j-(k+1))^2|\alpha_{k+1}|^2\gamma} = \frac{2\pi^2}{3C^2} \frac{1}{|\alpha_k|^{2\gamma}} \leq \frac{2\pi^2}{3C^2} \frac{1}{|\alpha_{[k/2]}|^{2\gamma}}. \quad (26)$$

Similarly, for $S_2$, if $j \geq [k/2]$, then

$$|\alpha_k - \alpha_j| = |\alpha_k - \alpha_{k-1}| + |\alpha_{k-1} - \alpha_{k-2}| + \cdots + |\alpha_{j+1} - \alpha_j| \geq \frac{k-j}{2} C|\alpha_j|^\gamma,$$

and so,

$$S_2 \leq \sum_{j=[k/2]+1}^{k-1} \frac{4}{C^2(k-j)^2|\alpha_{[k/2]}|^2\gamma} \leq \frac{2\pi^2}{3C^2} \frac{1}{|\alpha_{[k/2]}|^{2\gamma}}.$$

Lastly, for $S_1$ observe that if $j \leq [k/2]$, then

$$|\alpha_k - \alpha_j| \geq |\alpha_k - \alpha_{[k/2]}| \geq \frac{(k-[k/2])}{2} C|\alpha_{[k/2]}|^\gamma \geq \frac{k}{4} C|\alpha_{[k/2]}|^\gamma,$$

and so,

$$S_1 \leq \sum_{j=1}^{[k/2]} \frac{16}{k^2C^2|\alpha_{[k/2]}|^{2\gamma}} \leq \frac{8}{kC^2|\alpha_{[k/2]}|^{2\gamma}}.$$

Combining our estimates for $S_1$, $S_2$, $S_3$ with (25) give

$$0 \leq \frac{8}{kC^2|\alpha_{[k/2]}|^{2\gamma}} + \frac{4\pi^2}{3C^2} \frac{1}{|\alpha_{[k/2]}|^{2\gamma}} - \frac{2}{|\beta_k|^2},$$
or equivalently

\[ |\beta_k| \geq C |\alpha_{\lfloor k/2 \rfloor}|^\gamma \frac{1}{\sqrt{\frac{4}{k} + \frac{2\pi^2}{3}}} \geq C |\alpha_{\lfloor k/2 \rfloor}|^\gamma \frac{1}{\sqrt{\frac{2\pi^2}{3} + \frac{2\pi^2}{3}}}, \]

which implies the desired inequality (22).

Lastly, if \( k = 1 \), then both \( S_1 \) and \( S_2 \) are trivial and the second-to-last inequality in (26) gives

\[ S_3 \leq \frac{2\pi^2}{3C^2} \frac{1}{|\alpha_1|^{2\gamma}}. \]

and substituting this into (25) gives the desired inequality (23).

\[ \square \]

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