A scenario for the dynamics in the small entropy production limit

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We present a scenario for the nonequilibrium dynamics in the limit of small entropy production. We discuss (i) the appearance of different time-scales, (ii) the modification of the fluctuation-dissipation theorem and its relation to effective temperatures and partial equilibrations and (iii) the validity of Onsager reciprocity relations. We distinguish these properties by their reaction to infinitesimal perturbations. We recall that one can easily change the time dependence of observables by applying an infinitesimal force while time-reparametrization invariant features remain unchanged under the same perturbations. With the aim of better understanding these properties, we consider the effect of several baths with different temperatures and time-scales on the dynamics. This is done in two ways: numerically, by using a especially developed Monte Carlo algorithm that mimics the coupling to multiple baths; analytically, by computing the time-dependent probability density of simple systems in contact with multiple baths. We finally argue that these features are related to supersymmetry, the reparametrization invariance of the slow dynamics and its spontaneous breaking. This scenario is consistent within any perturbative resummation scheme. A brief version of this article appeared in Ref. 1.

KEYWORDS: nonequilibrium dynamics, effective temperatures

§1. Introduction

In many dynamical systems, there is a quantity that can be naturally interpreted as the entropy production. It depends on the time-dependent probability distribution and vanishes only when the system reaches the Gibbs-Boltzmann (GB) measure. The limit of small, though non-vanishing, entropy production (SEP) is achieved in a relaxing system at long times, and in stationary systems driven by external forces (or by different thermal baths) when the external power input tends to zero.

A finite system in which all forces derive from a potential approaches, for any reasonable dynamics and in the long-time limit, the GB distribution. In contrast, an extended system might stay far from equilibrium even at very long times, producing entropy at a small rate. In these cases the systems are explicitly kept far from equilibrium by the ‘non-potential’ forces.

If the SEP limit is not the GB measure, is there any generic, model independent statement we can make about it? The purpose of this paper is to discuss several aspects of a scenario with partial equilibrations at separate time-scales inspired by ‘mean-field’ glass theory. Although we cannot show that this scenario is the most general one, we can show that it is consistent in any dimension. It leads to definite predictions, some of which have already been tested. and some new ones that, to a limited extent, we test here.

The discussion will be aimed at extended systems satisfying reasonable spatial homogeneity properties. We may easily construct counterexamples of the present scenario in models with disconnected or almost disconnected parts, in which microscopic couplings scale with the system size, etc. Throughout this paper whenever a quantity is declared to be ‘small’, it must be understood with the zero entropy production limit taken first, and this itself taken after the infinite size limit.

§2. Formalism

Consider a system described by the variables \( \phi_i, i = 1, \ldots, N \), that we encode in a vector \( \vec{\phi} \), with energy \( E(\vec{\phi}) \). A nonconservative force \( \vec{f} \), scaled with a parameter \( D \), acts on \( \vec{\phi} \). If the system is set in contact with a white...
thermal bath, the uncorrelated Gaussian noises \( \eta_i \) have zero mean and variance \( 2T_0 \delta \). \( \Gamma_i \) is the friction coefficient. The equations of motion are

\[
-m_i \ddot{\phi}_i - \frac{\partial E(\phi)}{\partial \phi_i} + D f_i(\phi) = \Gamma_i \dot{\phi}_i - \eta_i \quad (2.1)
\]

and the masses \( m_i \) can be zero in particular.

If the system is coupled to a coloured Gaussian bath,\(^7\) the noise and friction terms are

\[
\int_t^s ds \nu(t,s) \dot{\phi}_i(s) - \rho_i \quad (2.2)
\]

with \( \langle \rho_i(t) \rho_j(t_w) \rangle = T^* \delta_{ij} \nu(t,t_w) \) and \( T^* \) the temperature of the bath. The presence of the same kernel \( \nu(t,s) \) in the noise-noise correlation and in the friction term is a consequence of having taken a bath in equilibrium.

We define the correlation functions and the linear response of the variable \( \phi_i \) to a kick applied to \( \phi_j \) at \( t_w \):

\[
C_{ij}(t, t_w) = \langle \phi_i(t) \phi_j(t_w) \rangle , \quad R_{ij}(t, t_w) = \frac{\delta \langle \phi_i(t) \rangle}{\delta h_j(t)} \bigg|_{h=0} .
\]

Using a standard diagrammatic approach,\(^21\) one can derive a set of two coupled dynamic equations for \( C_{ij}(t, t_w) \) and \( R_{ij}(t, t_w) \) for any model. In general though, one does not succeed in performing the sum of diagrams involved. There are two main strategies to obtain a closed set of equations. The first consists in using simple models, i.e., simple expressions for \( E(\phi) \). This is the choice made when working with \( O(N) \)-type models for ferromagnetism,\(^3\) fully-connected spin-glass models,\(^10, 22, 23\) models in infinite dimensional embedding spaces,\(^24, 25\) etc. The second consists in choosing a recipe to select a subset of the infinite set of diagrams in such a way that one can sum them all and express this sum as an explicit functional of \( C_{ij} \) and \( R_{ij} \). This is a route commonly followed in field theory and it is used, for example, to derive the mode-coupling theory for super-cooled liquids.\(^8, 10, 26\)

In fully-connected models all \( n \)-point functions can be expressed in terms of two-point functions, which then provide a complete description of the dynamics. In finite dimensional models this cannot be done in closed, exact form, and the description in terms of two-point functions is not complete.

For all models, the equations for two-time functions have the structure of Schwinger-Dyson equations. Our claim is that in the SEP limit their solutions share the aspects discussed in Sect. 3. This is a consequence of the spontaneous breaking of a large reparametrization invariance appearing in the long waiting-time limit.\(^1\)

We wish to stress that neither the methods used to obtain the dynamic equations nor the scenario discussed in this paper rely on the presence of quenched disorder. This is unnecessary in a dynamic treatment.

\section{The dynamic scenario}

\subsection{Fast and slow dynamics. Infinite sensitivity of the slow dynamics}

Two-time functions have different behaviours in two time-regimes separated by a model-dependent characteristic time \( T(t_w) \) that diverges in the SEP limit.\(^27\) The relaxation is fast for \( t-t_w \) small with respect to \( T(t_w) \) and it is fast for \( t-t_w \) large with respect to \( T(t_w) \), \( t-t_w \geq 0 \). The correlation and response are then

\[
C_{ij}(t, t_w) = C_{ij}^F(t-t_w) + \tilde{C}_{ij}(t, t_w) ,
\]

\[
R_{ij}(t, t_w) = R_{ij}^F(t-t_w) + \tilde{R}_{ij}(t, t_w) ,
\]

where \( C_{ij}^F \) and \( R_{ij}^F \) are nonnegligible only for small time-differences, and \( \tilde{C}_{ij} \) and \( \tilde{R}_{ij} \) vary very slowly. This separation becomes sharper and sharper in the SEP limit. In this sense our treatment is asymptotic.

A very general feature of systems with slow dynamics is that the time-dependence of \( \tilde{C}_{ij} \) and \( \tilde{R}_{ij} \) is sensitive to vanishingly small changes in the equations of motion. The most familiar example is ferromagnetic coarsening in which an arbitrary small random field changes the growth law\(^28\) from power to log, making the slow part of the autocorrelation change from being a function of \( t_w/t \) to being a function of \( \ln(t_w/\tau_0)/\ln(t/\tau_0) \). Another extreme example is the case of mean-field glasses, in which weak nonconservative forces completely destroy aging, rendering the problem stationary.\(^13\)

The “infinite sensitivity” is due to the presence of flat directions in the free-energy landscape\(^27, 29, 30\) and this is also the very reason why there is slow dynamics. This physical fact has a mathematical counterpart in the invariances of the equations.\(^1\)

The fact that one can change from aging to stationary slow dynamics by applying infinitesimal perturbations is indeed the justification for treating both situations on the same footing.
temperatures take a single value

pairs of observables and for each time-scale the effective

ing of a degree of freedom

temperature measured by a thermometer we proceed as in

M

Fig. 1.

A

observable

Equations (3.2) and (3.5) do not necessarily coincide.

In or-App. A we show that the equation of motion for

M

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eratures in the glassy phase. On the experimental side, Grigera and Israeloff gave evidence for the existence of non-trivial effective temperatures in glycerol by showing that Nyquist FDT is violated below $T_g$. In the context of spin-glasses, an indirect – and not exact – method also suggests that non-trivial effective temperatures appear in the glassy phase.

On the numerical side, the simulations of many groups gave evidence for the existence of non-trivial FDT violations (and, consequently, effective temperatures) in a large variety of models. Among them we can mention models undergoing domain growth, spin-glasses in finite dimensions, and Lenard-Jones systems.

3.3 Response to slow, auxiliary baths. Thermalization of subsystems.

If glassy systems have natural effective temperatures associated with the slow degrees of freedom, it is then natural to ask how would they react to a small coupling to an additional slow auxiliary bath of temperature $T^*$.

The effect of the additional bath is taken into account by adding a term like (2.2) to the Langevin equation. The simplest choice is

\[ \dot{\nu}(t) = v(t) \] (3.9)

$C_{AA}$ is the auto-correlation of the observable $A$ at two different times and $R_{AA}$ is the response of the same observable with respect to an infinitesimal perturbation $h$ that acts at time $t_w$ modifying the Hamiltonian according to $H \rightarrow H - A(t_w)h$. If $R_{AA}$ and $C_{AA}$ are linked by a time-scale dependent effective temperature that satisfies (3.7), we can decompose them as a sum of terms, each evolving in a different time-scale. Hence, in the weak coupling limit $k^2 \rightarrow 0$, Eq. (3.9) is a Langevin equation with several ‘baths’ of the form (2.2) acting on widely separated time-scales and each with its own temperature. If $x$ is tuned to respond to a single time-scale, it will measure only its temperature.

A simple way to view the relation of $\beta^{\text{eff}}$ to the zeroth law is to connect alternatively $x$ to an observable $A$ and an observable $B$ (simultaneously for an ensemble of $M$ systems, as in the thermometer case). One shows that the average heat flow goes from the observable having a lower to the one having a higher $\beta^{\text{eff}}$.

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On the numerical side, the simulations of many groups gave evidence for the existence of non-trivial FDT violations (and, consequently, effective temperatures) in a large variety of models. Among them we can mention models undergoing domain growth, spin-glasses in finite dimensions, and Lenard-Jones systems.

In order to show that $\beta^{\text{eff}}(\omega_1, t_w)$ is indeed a temperature measured by a thermometer we proceed as in Ref. We measure, with a ‘thermometer’ consisting of a degree of freedom $x$ with a potential $V(x)$ (see Fig. 1), the temperature of an observable $A(\hat{\phi})$. In order to do so, we couple the thermometer linearly to the observable $A$ of an ensemble of many ($M$) independent copies of the system, subject to the same history. In App. A we show that the equation of motion for $x$ becomes, for large $M$,

\[ m \ddot{x}(t) = -\frac{\partial V(x)}{\partial x(t)} + k^2 \int_{t_0}^{t} ds R_{AA}(t, s)x(s) + \rho(t) \] (3.8)

with $\rho$ a Gaussian noise with zero mean and correlation

\[ \langle \rho(t)\rho(t_w) \rangle = k^2 C_{AA}(t, t_w) \] . (3.9)

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but will still have aging for time-scales with $T^* < T_{	ext{eff}}$.

We hence have argued,\textsuperscript{1} very much in the spirit of Refs.\textsuperscript{45–48} that the coupling to slow auxiliary baths are perturbations ‘conjugate to the natural temperatures’. The arguments put forward in the previous two paragraphs are supported by simulations of spin-glasses both mean-field and 3D (the latter very encouraging).\textsuperscript{1} In App. C we describe the algorithm used to simulate a spin system in contact with a multiple bath.

Another relevant question is mutual thermalisation between two subsystems having different natural effective temperatures, but sharing the same bath.\textsuperscript{19} This is related to the response to an auxiliary slow bath, since, to a certain extent, each system acts as an auxiliary bath for the other when placed in mutual contact.

Within the present scenario two situations are possible:\textsuperscript{1} strong coupling, in which the effective temperatures equalise; and weak coupling, in which the combined system preserves essentially the temperatures of its constituents, but rearranges them in widely separated time-scales. In the latter case, the combined system has three temperatures, the bath temperature for the fast processes, the lowest and highest original effective temperatures for the intermediate and slowest timescales, respectively.

### 3.4 Reciprocity relations.

An unexpected feature that appears in this scenario are the reciprocity (Onsager) relations:\textsuperscript{1}

$$\langle A(t)B(t_w) \rangle = \langle B(t)A(t_w) \rangle . \quad (3.10)$$

If this equality holds for the observables $A$ and $B$, Eqs. (3.6) and (3.7) imply a similar relation for the responses. There is no reason a priori why the reciprocity relations should hold in a situation in which FDT is strongly violated. This is all the more surprising in an aging case in which the system is not even stationary. The interest of these relations is that they are relatively easy to measure in a simulation (results for 3DEA appeared in Ref.\textsuperscript{1}) or in an experiment. The reciprocity relations are intimately related to the partial equilibria.

### §4. Multiply thermalized systems

A standard procedure in the dynamics of extended systems is to keep some macroscopic variables of interest and integrate away the others. The eliminated variables become part of a ‘thermal’ bath.

In equilibrium, the validity of FDT and of time-translational invariance (TTI) ensure that the friction and noise terms, coming from the projected sector, satisfy FDT (of the second kind). The whole procedure is self-consistent since the variables evolving in contact with a bath satisfying FDT will eventually satisfy FDT, now called of the first kind.\textsuperscript{2}

In an out of equilibrium situation FDT (and sometimes TTI) do not hold. How can we then picture the effect of the projected variables on our chosen ones?

We treat a system in contact with a multiple bath with different temperatures and time-scales that is based on subsequent adiabatic approximations of the slower baths. This approach has points in common with the ‘pinning field’ approach of Kirkpatrick and Thirumalai\textsuperscript{46} and Monasson\textsuperscript{47} and with recent developments of Allahverdyan et al.\textsuperscript{48} and Franz and Virasoro\textsuperscript{49} We claim that, in the SEP limit, correlation and response functions of the selected macroscopic variables behave as if they were subject to such a multiple bath.

By studying in detail the evolution of a harmonic oscillator in contact with a multiple bath, we show how, even in this very simple model, different effective temperatures are induced by the coupling to several baths.\textsuperscript{50}

#### 4.1 The construction

Consider a particle of coordinate $x$ moving in a potential $V(x)$ under the influence of two thermal baths: a fast white bath of temperature $T$ and friction coefficient $\Gamma_0$, and a slow bath of temperature $T^*$, characteristic time-scale $\tau^*$ and strength $\Gamma^*_1$. Note that such a system has a non-Gibbsian stationary measure. We are interested in the evolution of the particle taking $\tau^*$ large with respect to any other characteristic time. (The following argument can be easily extended to any number of variables $x_i, i = 1, \ldots, N$.)

This problem, and its solution, is very similar to the one studied by Allahverdyan and Nieuwenhuizen in the second article of.\textsuperscript{48} These authors analysed two interacting variables that are respectively coupled to two baths with different temperatures and very different time-scales.

Neglecting the effect of inertia one has

$$\Gamma_0 \ddot{x}(t) + \int_{-\infty}^{t} ds \, \hat{\nu}(t-s) \hat{x}(s) = - \frac{\partial V}{\partial x(t)} + \eta(t) + \rho(t) .$$

where $\eta$ and $\rho$ are the Gaussian thermal noises of the fast and slow baths, respectively, both with zero mean and variances $\langle \eta(t)\eta(t_w) \rangle = 2T\Gamma_0 \delta(t-t_w)$ and $\langle \rho(t)\rho(t_w) \rangle = T^* \nu(t-t_w)/\tau^*$. We assume that $\nu(0) \equiv \Gamma_1$ and $\nu(0^+) = 0$. We have set the system in contact with the slow bath at the initial time $t_0 = -\infty$. If we integrate by parts,

$$\Gamma_0 \ddot{x}(t) = - \frac{\partial V}{\partial x(t)} - \Gamma_1 x(t) + \eta(t) + h(t) \quad (4.1)$$

$$h(t) \equiv - \int_{-\infty}^{t} ds \, \hat{\nu}(t-s)x(s) + \rho(t) . \quad (4.2)$$

In the adiabatic limit, the slow bath generates a quasi-static field $h(t)$. (Indeed, using $\hat{\nu}(0^+) = 0$ and $\tau^* \gg 1$, one shows $h = O(1/\tau^*) \ll 1$.) Hence, $x$ has a fast evolution given by Eq. (4.1) with $h$ fixed and it achieves a distribution

$$P(x/h) = \frac{e^{-\beta(V(x)+\Gamma_1 x - h x)}}{\int dx \, e^{-\beta(V(x)+\Gamma_1 x - h x)}} . \quad (4.3)$$

The denominator defines $Z(h)$ and $F(h) \equiv - \beta^{-1} \ln Z(h)$. Henceforth we denote $P(a/b)$ the conditional probability of $a$ given $b$ at stationarity.

The approximate evolution of $h$ is given by Eq. (4.2) with the replacement of $x$ in the friction term by its
average with respect to the fast evolution:
\[ h(t) = \int_{-\infty}^{t} ds \nu(t-s) \frac{\partial F(h)}{\partial h}(s) + \rho(t) \]  
(4.4)
This equation is a non-Markovian and it has all the properties of a system coupled to a (slow) bath of temperature \( T^* \). In particular, the stationary distribution is
\[ \dot{P}(h) = \frac{e^{-\beta h}(F(h)+\frac{\delta^2}{\delta h^2})}{\int dh e^{-\beta (F(h)+\frac{\delta^2}{\delta h^2})}}. \]  
(4.5)
Reciprocity and FDT for functions of \( h \) also hold at stationarity. The stationary distribution of \( x \) now reads
\[ P(x) = \int dh P(x/h) \dot{P}(h). \]  
(4.6)
We prove these properties in App. B. In particular, if \( T = T^* \) we recover the usual GB distribution for \( P(x) \) regardless of the details of the slow bath.

Note that the distribution function \( P(x) \) contains the superposition of different time-scales and is not a very eloquent quantity.

A case of great interest is the one of a slow bath that is not itself stationary, but ages. Suppose that we have
\[ \nu(t, s) = \nu(L(s)/L(t)) \quad t \geq s \]  
(4.7)
for some monotonically increasing function \( L \). For the fast motion, at large enough times, we recover Eq. (4.3) while Eq. (4.2) follows from the change of variables
\[ T = \ln L(t), \quad T' = \ln L(s). \]  
(4.8)
In these new time-like variables, the rest of the derivation can be carried through identically. ‘Stationary’ means in this case invariant with respect to \( L \to \Delta \times L \) that leads to \( T \to T + \ln \Delta \) in time-like variables (\( \Delta \) is a parameter).

### 4.2 Reciprocity and FDT

In its strongest form, reciprocity means that the joint probability of having \( x \) at time \( t \) and \( x' \) at \( t_w \) is equal at stationarity\(^{531} \) to the probability of having \( x' \) at time \( t \) and \( x \) at \( t_w \). To prove that this indeed happens, we use the separation of time-scales and consider two cases:

- **i.** If \( t - t_w \) is small, \( h \) is approximately fixed and we have an equilibrium problem at temperature \( T \), with a fixed field \( h \). Reciprocity holds in the usual way.
- **ii.** If \( t - t_w \) is such that \( h(t) - h(t_w) \neq 0 \), we have
\[ P(x, t; x', t_w) = \int dh dh' P(x/h) P(h, t/h', t_w) P(h'/x') P(x') \]
Since \( P(h, t; h', t_w) = P(h', t; h, t_w) \) (see App. B),
\[ P(x, t; x', t_w) = P(x', t; x, t_w). \]  
(4.9)
In a similar way one shows that FDT holds separately for each timescale with its own temperature.

- **i.** If \( t - t_w \) is small FDT holds for each \( h \). Denoting \( \delta P(x, t)/\delta H_A(t_w) \) the variation of the distribution at time \( t \) due to a field conjugate to \( A \) that has been on from \( -\infty \) to \( t_w \), we have
\[ \frac{\delta P(x, t)}{\delta H_A(t_w)} = \beta \int dx' P(x, t/x', t_w) A(x') P(x'). \]  
(4.10)
The integrated form of FDT for two observables \( A \) and \( B \) can be obtained by multiplying by \( B(x) \) and integrating over \( x \).

- **ii.** If \( t - t_w \) is such that \( h(t) - h(t_w) \neq 0 \), while the field conjugate to \( A \) is on, \( h \) evolves with Eq. (4.4) and
\[ F_A(h) = -\frac{1}{\beta} \int dx' e^{-\beta (V(x') - H_A(x') + \Gamma_1 \frac{x'^2}{2} - h'x')} \]
\[ = F(h) - H_A A(h), \]  
(4.11)
with \( A(h) = \int dx' P(x'/h) A(x') \). If \( H_A \) is turned off at \( t_w \), \( P(x/h) \), is first modified in a short time-scale with respect to that of \( h \). Then, \( h \) continues to evolve but with a modified effective potential \( F(h) \). Now, FDT holds for the evolution of \( h \) at temperature \( T^* \).

### 4.3 The effect of a weak coloured bath

The simplest example where to check the ideas described in Sect. 4.1 and 4.2 is a harmonic oscillator in contact with a white and a coloured bath with exponential correlation. This problem can be tackled using a variety of techniques.

From the exact asymptotic solution we extract the behaviour of the integrated response \( \chi(\tau) \equiv \int_0^\tau dx' R(x') \) vs. \( C(\tau) \) and read \( T_{\text{eff}}^{\text{asy}} \) from it. In Fig. 2 we show \( \chi(C) \) for \( T = 0.5, \Gamma_0 = 1, T^* = 1 \). In the plot above, \( (\omega + \Gamma_1)\tau^* = 2000 \gg \Gamma_0 = 1 \), with \( \omega \) the frequency of the oscillator. The evolution takes place in two timescales (or correlation-scales) characterised by temperatures \( T \) and \( T^* \). The straight lines have slopes \(-1/T\) and \(-1/T^*\) showing that there are two temperatures associated to the motion of the particle: a fast motion for \( q/C < q_d \) controlled by \( T \) and a slow motion for \( 0 < C < q \) that is instead controlled by \( T^* \). In the plot below \((\omega + \Gamma_1)\tau^* = 2 = O(\Gamma_0) = 1 \), the timescales (or correlation-scales) are not well separated, and \( \chi(C) \) continuously interpolates between a region of slope of \(-1/T\) and a region of slope of \(-1/T^*\).

This problem can be alternatively analysed using the technique described in Sect. 4.1. One recovers, as expected, the exact results of the previous paragraphs.

### 4.4 Many scales

The construction of Sect. 4.1 can be generalised to many nested scales. Let us see how this is done for three
before, we write \( \ast \) with \( \rho \) mean, \( \nu \) with \( h \) and \( \hat{P} \).

Fig. 2. The \( \chi(C) \) plot for a harmonic oscillator subject to a white bath of temperature \( T = 0.5 \) and a coloured noise of temperature \( T^* = 1 \). Above: \( (\omega + \Gamma^\ast) \tau^\ast = 2000 \gg \Gamma_0 = 1 \). The straight lines (dots) have inverse slopes \(-1/T = -2 \) and \(-1/T^* = -1 \). Below: \( (\omega + \Gamma^\ast) \tau^\ast = 2 = O(\Gamma_0 = 1) \).

baths; the equation of motion is
\[
\Gamma_0 \dot{x}(t) + \int_0^t ds (\nu_1(t-s) + \nu_2(t-s)) \dot{x}(s) = -\frac{\partial V(x)}{\partial x(t)} + \eta(t) + \rho_1(t) + \rho_2(t) \tag{4.12}
\]
with \( \rho_1 \) and \( \rho_2 \) two Gaussian thermal noises with zero mean, \( \langle \rho_1(t) \rho_1(t_w) \rangle = T_1^\ast \nu_1 \frac{1}{\tau_1^\ast} (t-t_w) / \tau_1^\ast \), \( \langle \rho_2(t) \rho_2(t_w) \rangle = T_2^\ast \nu_2 \frac{1}{\tau_2^\ast} (t-t_w) / \tau_2^\ast \) and \( 1 \ll \tau_i^\ast \ll \tau_i^\ast \). Proceeding as before, we write
\[
\Gamma_0 \dot{x}(t) = -\frac{\partial V(x)}{\partial x(t)} + (\Gamma_1 + \Gamma_2) x(t) + \eta(t) + h_1(t) + h_2(t) \tag{4.13}
\]
with \( h_i(t) \equiv -\int_0^t ds \dot{\nu}_i(t-s) x(s) + \rho_i(t), i = 1, 2, \) and \( \nu_i(0) = 0, i = 1, 2 \).

In the fastest evolution \( x \) achieves distribution,
\[
P(x/h_1, h_2) = \frac{e^{-\beta (V(x) + (\Gamma_1 + \Gamma_2) x^2 - (h_1 + h_2) x) / \beta}}{Z_{h_1, h_2}}, \tag{4.14}
\]
with \( Z_{h_1, h_2} \) ensuring \( \int dx P(x/h_1, h_2) = 1 \). The evolution of \( h_1 \) is obtained by taking \( h_2 \) to be adiabatic
\[
h_1(t) = \int_{-\infty}^t ds \dot{\nu}_1(t-s) \frac{\partial F_1(h_1 + h_2)}{\partial h_1}(s) + \rho_1(t) \tag{4.15}
\]
with \( e^{-\beta F_1(h_1 + h_2)} = \int dx e^{-\beta (V(x) + (\Gamma_1 + \Gamma_2) x^2 - (h_1 + h_2) x) / \beta} \) yielding a stationary distribution for \( h_1 \)
\[
P(h_1/h_2) \propto \exp \left[ -\beta_1^\ast \left( F_1(h_1 + h_2) + \frac{h_1^2}{2 \beta_1^\ast} \right) \right]. \tag{4.16}
\]
Finally, \( h_2 \) evolves at inverse temperature \( \beta_2^\ast \) and has a stationary distribution:
\[
\hat{P}(h_2) \propto \exp \left[ -\beta_2^\ast \left( F_2(h_2) + \frac{h_2^2}{2 \beta_2^\ast} \right) \right]. \tag{4.17}
\]
with \( e^{-\beta_2^\ast F_2(h_2)} \equiv \int dh_1 e^{-\beta_1^\ast \left( F_1(h_1 + h_2) + \frac{h_1^2}{2 \beta_1^\ast} \right)} \). One can also consider two slow baths that are non-stationary. In order to have two well separated time-scales, we need two (increasing) scaling functions \( L_1(t) \), \( L_2(t) \) with
\[
\lim_{t \to \infty} \left( d_t \ln L_2(t) / d_t \ln L_1(t) \right) = 0 \tag{4.18}
\]
and, for example,
\[
\nu_1(t, s) = \tilde{\nu} \left( L_1(s) / L_1(t) \right), \nu_2(t, s) = \tilde{\nu} \left( L_2(s) / L_2(t) \right),
\]
t \geq s. After performing two transformations like the ones in Eq. (4.8), the rest of the derivation can be repeated.

For this example one concludes that the ‘probability cloud’ of each scale acts as a configuration for the slower scale — moving in an effective potential.

\section{Conclusions}

In the last years great progress has been made in understanding the asymptotic nonequilibrium evolution of systems with slow dynamics. Much of these developments came from the study of simplified models that include classical glassy disordered and non-disordered models, quantum disordered systems and, importantly enough, systems that are constantly driven out of equilibrium by external forces. The results collected from the solution of several of these cases made apparent the existence of various general features. In this article (as well as in Ref.\cite{1}) we stressed are the features that we believe build a scenario for the slow dynamics in the limit of small entropy production. Properties \( i-iv \) discussed in Sect. 3, can be quite surprising, in particular the latter. In addition, in view of the infinite sensitivity of the slow time dependence of the correlation and response functions, it is a non-trivial fact that properties \( ii-iv \) are preserved under small perturbations. In Ref.\cite{1} we justified this fact by studying the invariances (and breakdown of) the dynamic equations in the SEP limit.\cite{54}

In short, the image we have in mind is better grasped by comparing the general case to the simple problem studied in Sect. 4. The Langevin equation for a variable \( x \) including a non-local friction kernel and a correlated noise, Eq. (4.1), is reminiscent of the equation for a single effective variable that one obtains in fully connected models, large dimension approximations, etc.\cite{25,52} Indeed, in the large \( N \) limit, one can derive such an equation by performing a saddle-point approximation of the dynamic generating functional. For any dynamic variable \( \phi_i \), the ‘single-variable equation’ reads
\[
\Gamma_0 \dot{\phi}_i(t) = -\mu_0(t) \phi_i(t) + \int_{t_0}^t ds \Sigma(t, s) \phi_i(s) + \rho_i(t) + \eta_i(t) \tag{5.1}
\]
There are two noise sources in this equation: \( \eta_i(t) \) is the original white noise while \( \rho_i(t) \) is an effective (Gaussian) noise with zero mean and correlations self-consistently given by \( \langle \rho_i(t) \rho_j(t_w) \rangle \).

vertex $D(t, t_w)$ plays the rôle of the coloured noise correlation in a usual Langevin equation. The self-energy $\Sigma(t, t_w)$ appears here as an ‘integrated friction’.

In the case of a system in contact with an external coloured bath, the correlation of the noise $T^\ast \nu(t, t_w)$ is simply related to the retarded friction $\nu(t, t_w)$. This is the statement of FDT applies for the bath. In the case of the single-variable equation for a more complicated model, the friction kernel $\Sigma(t, t_w)$ and the correlation of the coloured noise $T^D(t, t_w)$ are properties of the system and are not necessarily related in a simple manner. In the SEP limit, they get related by a modification of FDT with time-scale dependent effective temperature $T_{\mathrm{eff}}(t, t_w)$.

One then concludes that the structure of these two problems is indeed very similar:

- If one weakly couples a simple system to a slow bath of temperature $T^\ast$, at sufficiently slow time scales the system acquires the temperature $T^\ast$.
- Glassy systems arrange their internal degrees of freedom in such a way that slow degrees of freedom select their own effective temperatures.

Implicit in the construction presented in Section 4.1 is the fact that all observables have the same effective temperatures at the same two-times. Also implicit are the reciprocity relations. These are important in that they are potentially measurable, and quite unexpected out of equilibrium.

### Appendix A: Measurements of natural effective temperatures

We here recall the definition of a time-scale dependent effective temperature, for systems out of equilibrium, given in Ref.\textsuperscript{19} The presentation follows closely this reference but it makes some points of the derivation more precise.

Let us consider $M$ non-interacting copies of the system,\textsuperscript{37} and couple each of them to a simple system that acts as a thermometer in the manner sketched in Fig. 1. For simplicity, we describe this thermometer with a single variable $x$ and each system with a variable $\phi_{\alpha}$, $\alpha = 1, \ldots, M$. The total energy of the complex is

$$E_{\mathrm{TOT}} = \frac{\dot{x}^2}{2} + V(x) + \sum_{\alpha=1}^{M} E(\phi_{\alpha}) - \frac{k}{M^{1/2}} x \sum_{\alpha=1}^{M} A(\phi_{\alpha}) ,$$

where $V(x)$ and $E(\phi_{\alpha})$ are the potential energies of the isolated thermometer and each isolated system. For each copy the last term yields an infinitesimal (for $M$ large) perturbation corresponding to a field $kx/\sqrt{M}$ conjugate to the observable $A(\phi_{\alpha})$. The equation of motion for $x$ is

$$m\ddot{x}(t) = -\frac{\partial V(x)}{\partial x(t)} - \frac{k}{M^{1/2}} \sum_{\alpha=1}^{M} A(\phi_{\alpha})(t) . \quad (A.1)$$

For simplicity we choose an operator $A$ such that $\langle A(\phi_{\alpha}) \rangle_{k=0} = 0$ where $\langle \bullet \rangle$ represents either an average over different histories of the same system or an average over different systems, e.g. $\langle f \rangle = 1/M \sum_{\alpha=1}^{M} f_{\alpha}$. (If the average of the operator $A$ does not vanish, one has to work with the difference between the operator and its average.) The subindex $k = 0$ indicates that the average is taken in the absence of the external field $kx/M^{1/2}$. Whenever we take the average in the presence of the field we shall denote it $\langle \bullet \rangle_{k}$.

Assuming that linear response holds for each system, the total variation of the average $\langle A(\phi_{\alpha}) \rangle_{k}$ caused by a field that has been applied from $t = 0$ upto $t$ reads

$$\delta \langle A(\phi_{\alpha}) \rangle_{k}(t) = \langle A(\phi_{\alpha}) \rangle_{k}(t) - \langle A(\phi_{\alpha}) \rangle_{k=0}(t) = \int_{0}^{t} ds \ R_{\alpha, \alpha}(t, s) \ \frac{k}{\sqrt{M}} \ x(s) , \quad (A.2)$$

where $R_{\alpha, \alpha}(t, s)$ is the linear response of the observable $A(\phi_{\alpha})$ at time $t$ to a change in energy $-k/\sqrt{M} x A(\phi_{\alpha})$ at time $s$. These thermal history-averaged responses are equal for all systems, we henceforth denote them $R(t, s)$ for all $\alpha$. (We also simplify the notation by eliminating the subindex $A$ that identifies the observable.) Adding the last equation over $\alpha$ and multiplying by $k/\sqrt{M}$ we have

$$\frac{k}{\sqrt{M}} \sum_{\alpha=1}^{M} \langle A(\phi_{\alpha}) \rangle_{k}(t) = k^{2} \int_{0}^{t} ds \ R(t, s) x(s) . \quad (A.3)$$

Adding and subtracting from the rhs of Eq. (A.1) the average (A.3), we recast it as

$$m\ddot{x}(t) = -\frac{\partial V(x)}{\partial x(t)} + k^{2} \int_{0}^{t} ds \ R(t, s) x(s) + \rho(t) \quad (A.4)$$

with

$$\rho(t) = \frac{k}{\sqrt{M}} \sum_{\alpha=1}^{M} \left( \langle A(\phi_{\alpha}) \rangle(t) - \langle A(\phi_{\alpha}) \rangle_{k}(t) \right) . \quad (A.5)$$

The ‘force’ $\rho(t)$ is the sum of $M$ independent identically distributed random variables. For large $M$ it becomes a Gaussian variable with zero mean and variance:

$$\langle \rho(t) \rho(t_{w}) \rangle_{k} = \frac{k^{2}}{M} \sum_{\alpha, \beta=1}^{M} C_{\alpha \beta}^{\mathrm{CONN}}(t, t_{w}) \bigl|_{k} . \quad (A.6)$$

with $C_{\alpha \beta}^{\mathrm{CONN}}(t, t_{w})_{k} = \langle (A(\phi_{\alpha})_{k})(t) - \langle A(\phi_{\alpha})_{k} \rangle(t) \rangle \times (A(\phi_{\beta})_{k}(t_{w}) - \langle A(\phi_{\beta})_{k}(t_{w}) \rangle_{k}) $ the connected correlation in the presence of the field. Up to leading order in $k$

$$C_{\alpha \beta}^{\mathrm{CONN}}(t, t_{w})_{k} \sim C_{\alpha \beta}^{\mathrm{CONN}}(t, t_{w})_{k=0} . \quad (A.7)$$

The connected correlation can also be substituted by the usual correlation since $A(\phi_{\alpha})_{k}(t) = O(1)$ implies $\langle A(\phi_{\alpha})_{k}(t) \rangle_{k} = O(1/\sqrt{M})$. Thus

$$C_{\alpha \beta}^{\mathrm{CONN}}(t, t_{w})_{k=0} \sim C_{\alpha \beta}(t, t_{w}) \sim \delta_{\alpha \beta} C(t, t_{w}) \quad (A.8)$$

since the bare systems are completely independent. Finally,

$$\langle \rho(t) \rho(t_{w}) \rangle_{k} = k^{2} C(t, t_{w}) . \quad (A.9)$$

Thus the dynamic equation governing the evolution of the thermometer reads

$$m\ddot{x}(t) = -\frac{\partial V(x)}{\partial x(t)} + \rho(t) + k^{2} \int_{0}^{t} ds \ R(t, s) x(s) , \quad (A.10)$$

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\[ \langle \rho(t) \rho(t_w) \rangle_k = k^2 C(t, t_w). \]  
\[ (A-10) \]

If the response and the correlation of the system are related through the FDT relation
\[ R(t, s) = \frac{1}{T} \frac{\partial C(t, s)}{\partial s} \Theta(t - s). \]  
\[ (A-11) \]

one integrates by parts the last term on the rhs of Eq. (A-10) and obtains
\[ m \ddot{x}(t) = -\frac{\partial V}{\partial x(t)} + \rho(t) - \frac{k^2}{T} \int_0^t ds \ C(t, s) \dot{x}(s) 
+ \frac{k^2}{T} (C(t, t)x(t) - C(t, 0)x(0)). \]  
\[ (A-12) \]

that in the limit of weak coupling, \( k^2 \to 0 \), becomes a Langevin equation since the last term can be neglected.

From this equation we can conclude that the thermometer, being a system with fast dynamics, will eventually equilibrate at the temperature that relates response and correlation of the system, Eq. (A-11). The system acts as a thermal bath on the thermometer.

We can also derive this result by choosing a harmonic oscillator as a thermometer.\(^{19)}\) Using \( V(x) = m \omega_o^2 x^2 / 2 \), one easily shows that the averaged potential energy of the thermometer over a time-window around the measuring time \( t_w \) is
\[ \frac{1}{2} \langle \rho_{osc}(t_w) \rangle = \frac{1}{2} \omega_o^2 \langle x^2 \rangle_{t_w}, \]  
\[ (A-13) \]

with \( \omega_o \) is the probing frequency of the oscillator and \( m = 1 \). In the large \( t_w \) limit, it reaches the following limit
\[ \langle \rho_{osc}(t_w) \rangle = \frac{\omega_o \hat{C}(\omega_o, t_w)}{\chi''(\omega_o, t_w)} \]  
\[ (A-14) \]

with \( \hat{C} \) and \( \chi'' \) defined in Eqs. (3.3) and (3.4). If equipartition holds for the oscillator, one has
\[ T^{eff}(\omega_o, t_w) = \frac{\omega_o \hat{C}(\omega_o, t_w)}{\chi''(\omega_o, t_w)}, \]  
\[ (A-15) \]

i.e. the definition given in Eq. (3.5).

Under the assumption (3.6) we can decompose the correlations and responses as follows:
\[ C(t, t_w) = C^F(t, t_w) + C^1(t, t_w) + C^2(t, t_w) + ... \]
\[ R(t, t_w) = R^F(t, t_w) + R^1(t, t_w) + R^2(t, t_w) + ... \]

with
\[ R^i(t, t_w) = \beta^i \frac{\partial C^i(t, t_w)}{\partial t_w} \Theta(t - t_w), \]  
\[ (A-16) \]

\( i = 0, 1, \ldots \), \( i = 0 \) corresponding to the FDT scale. Then, Eqs. (3.9) and (A-12) corresponds to a system coupled to a series of baths of inverse temperatures \( \beta^F, \beta^1, \beta^2 \) acting on widely separated scales. The probing frequency of the thermometer can be selected to measure any of the values of the effective temperature.\(^{38}\)

Actually, the effective temperature \( T^{eff} \) in Eq. (A-15) might depend on the observable \( A \) considered. The picture in this paper describes the way in which we think these dependences arrange in a physical system.

### Appendix B: Stationary distribution of the quasi-static field

We show that Eq. (4.4) leads to Eq. (4.5) and that at stationarity reciprocity and FDT hold. The Fourier transform of Eq. (4.4) reads
\[ h(\omega) = r(\omega) \frac{\partial F}{\partial \omega}(\omega) + \rho(\omega), \]  
\[ (B-1) \]

\[ \langle \rho(\omega) \rho(\omega') \rangle = T^* \nu(\omega)2\pi\delta(\omega + \omega'), \]  
\[ (B-2) \]

where \( r(\omega) = \int_{-\infty}^{\infty} d\tau \exp(i\omega \tau) \nu(r) \Theta(r) \) is the slow bath’s response function. It is then analytical in the upper half complex plane and satisfies, for real \( \omega \),\(^{53} \) \( r^*(\omega) = r(-\omega) \). FDT holds for the bath: \( r(\omega) - r(-\omega) = -i\omega \nu(\omega) \). In order to prove Eq. (4.5), we introduce a set of auxiliary variables \( y_j \) satisfying the ordinary Langevin equation:
\[ m_j \ddot{x}_j + \Gamma_j \dot{x}_j + \Omega_j \dot{x}_j = y_j = \xi_j(t) - \frac{\partial F}{\partial y_j} \langle \sum_j A_j y_j \rangle \]  
\[ (B-3) \]

with \( \xi_j(t) = 2T^* \Gamma_j \delta_j \delta(t - t_w) \). The aim is to show that \( \sum_j A_j y_j \) satisfies the same equations as \( h \), Eqs. (B-1) and (B-2).\(^{55} \) The introduction of \( y_j \) allows us to convert the original non-Markovian problem into a Markovian one. If \( F \) is such that Eq. (B-3) drives the ensemble \( y_j \) to equilibrium,
\[ P(\bar{y}) \propto e^{-\beta \left( \frac{1}{2} F \sum_j A_j y_j \right)} \]  
\[ (B-4) \]

In Fourier space, Eq. (B-3) reads
\[ -m_j(\omega - \omega_j^\pm)(\omega - \omega_j^-) + \frac{\partial F}{\partial y_j} \langle \sum_j A_j y_j \rangle y_j(\omega) = \xi_j(\omega) \]  
\[ \text{where} \quad \omega_j^\pm = \text{the roots of} \quad -m_j \omega^2 \pm 2 - i\omega \Gamma_j + \Omega_j = 0. \]

Defining \( h(\omega) = \sum_j A_j y_j(\omega) \) one has
\[ h(\omega) = -\sum_j \frac{A_j \xi_j(\omega) + A_j^2 \partial h F(h)}{m_j(\omega - \omega_j^\pm)(\omega - \omega_j^-)} \]  
\[ (B-5) \]

Since \( \text{Im}(\omega_j^\pm) \leq 0 \), we can choose the variables \( y_j \) and the parameters \( m_j, \Gamma_j, \Omega_j \) in such a way to identify
\[ r(\omega) = \sum_j \frac{A_j^2}{m_j(\omega - \omega_j^\pm)(\omega - \omega_j^-)} \]  
\[ (B-6) \]

Furthermore, we associate the first term in the rhs of Eq. (B-5) to the coloured noise \( \rho(\omega) \) and we use Eq. (B-3) to obtain the noise-noise correlation:
\[ \langle \rho(\omega) \rho(\omega') \rangle = T^* \delta(\omega + \omega') \times \sum_j \frac{\Gamma_j A_j^2}{m_j(\omega - \omega_j^\pm)(\omega - \omega_j^-)(\omega + \omega_j^-)} \]  
\[ (B-7) \]

The properties of the poles \( \omega_j^\pm \) are \( \omega_j^\pm = -\omega_j^\mp \) and \( m_j(\omega_j^+ + \omega_j^-) = -i\Gamma_j y_j \) yield FDT \( \nu(\omega) = -\frac{1}{\pi} \left[ r(\omega) - r(-\omega) \right] \).

We need an expression for \( \Gamma_1 \equiv \nu(t = 0) = \int_{-\infty}^{\infty} d\omega / (2\pi) \nu(\omega) \). Completing the integration over the upper or the lower half plane, we have
\[ \Gamma_1 = \sum_j \frac{A_j^2}{\Omega_j} \]  
\[ (B-8) \]
We have shown that Eq. (4.4) is equivalent to an ordinary Markovian equation for \( \bar{y} \), provided one identifies \( h = \sum A_j y_j \). The stationary distribution for the \( y_j \) is then given by Eq. (B.4) \( \bar{P}(h) \) is calculated by introducing a delta-function:

\[
\bar{P}(h) \propto \int d\lambda \left( \sum \frac{A^2}{h^2} - \beta h + F(h) \right) \exp\left( -\beta \lambda \sum A_j y_j - h \right)
\]

\[
\propto \int d\lambda e^{-\beta \left( \sum \frac{A^2}{h^2} - i\lambda h + F(h) \right)} \approx e^{-\beta \left( \frac{A^2}{h^2} + F(h) \right)}.
\]

(B.9)

The reciprocity property is immediate from the fact that, for a Markovian Langevin process in equilibrium,

\[
P(\bar{y}^{1}, t; \bar{y}^{\prime}, t_{\omega}) = P(\bar{y}^{\prime}, t; \bar{y}^{1}, t_{\omega}).
\]

Similarly FDT follows from the fact that it holds for any two functions of the \( y_j \), it holds in particular for any two functions of \( h \).

Appendix C: A Monte Carlo algorithm with several baths

We describe the algorithm used to simulate the evolution of a system interacting with several thermal baths of different type. Let us consider a dynamic spin \( s = \pm 1 \) with energy \( E(s) \) that is in contact with a ‘fast’ bath of inverse temperature \( \beta \) and a ‘slow’ bath of inverse temperature \( \beta^* \) and time-scale \( \tau^* \). The generalization to several variables and many baths is straightforward.

The algorithm is as follows.

i With probability \( 1/\tau^* \) we generate a Gaussian random variable \( h \) with \( s \)-dependent mean and variance:

\[
\langle h \rangle = \beta^* \Gamma^2 s, \quad \langle h^2 \rangle - \langle h \rangle^2 = \Gamma^2.
\]

(C.1)

\( \Gamma_1 \) is the strength of the bath.

ii With probability \( 1 - 1/\tau^* \) we make ordinary Montecarlo updates of \( s \) at temperature \( 1/\beta \), with an energy \( E_{\text{tot}} = E(s) - hs \).

We justify this algorithm as follows. We model the slow bath by a system of \( N \) non-interacting spins \( \sigma_a = \pm 1, a = 1, \ldots, N \), weakly coupled to \( s \). The total energy of system and bath is given by

\[
E_{\text{tot}} = E(s) - \frac{\Gamma_1}{\sqrt{N}} \sum_a \sigma_a = E(s) - hs,
\]

(C.2)

with \( E(s) \) the energy of the free spin, \( \Gamma_1 \) the coupling constant and the ‘field’ \( h \) defined as \( h \equiv \Gamma_1/\sqrt{N} \sum \sigma_a \).

We propose the following dynamics to the coupled system:

i Frequent updates of \( s \) keeping \( \sigma_a \) fixed. The spin is then in presence of a constant external field \( h \). These updates are done with any Montecarlo procedure, at inverse temperature \( \beta \).

ii Infrequent updates of all \( \sigma_a \), i.e. of the field \( h \), keeping \( s \) fixed. This evolution is controlled by an inverse temperature \( \beta^* \).

If the sequence is long, one can assume that each \( \sigma_a \) is in thermal equilibrium, at temperature \( 1/\beta^* \) with an energy \( E(\sigma_a) \sim -\Gamma_1/\sqrt{N} s \sigma_a \). Each configuration \( \sigma_a = \pm 1 \) has probabilities \( p_+ \) and \( p_- \) given by:

\[
p_{\pm} = \frac{\exp(\pm \beta^* \frac{\Gamma_1}{\sqrt{N}} s \sigma_a)}{2 \cosh(\beta^* \frac{\Gamma_1}{\sqrt{N}} s)}.
\]

(C.3)

The \( \sigma_a \) are then identically distributed independent random variables. It follows that, in the limit of large \( N \), \( h \) is a Gaussian random variable. Its mean and variance are given by:

\[
\langle h \rangle = \frac{\Gamma_1}{\sqrt{N}} \sum_a \langle \sigma_a \rangle = \beta^* \Gamma_2^2 s + O \left( \frac{1}{\sqrt{N}} \right)
\]

\( \langle h^2 \rangle - \langle h \rangle^2 = \frac{\Gamma_1^2}{N} \sum_a (\langle \sigma_a^2 \rangle - \langle \sigma_a \rangle^2) = \Gamma_1^2 + O \left( \frac{1}{\sqrt{N}} \right) \)

As a check of the consistency of this procedure we examine if detailed balance holds, as expected, in the particular case of a variable coupled to two baths with equal temperatures \( \beta = \beta^* \). The transition probability per unit time to go from the state \( s_A \) to the state \( s_B \) is

\[
P(A \rightarrow B) = \frac{\Gamma_1}{\sqrt{2\pi}} \int dh e^{-\frac{(h - \beta^* \Gamma_2 A)^2}{2\Gamma_1^2}} P_h(A \rightarrow B)
\]

(C.4)

where \( P_h(A \rightarrow B) \) is the transition probability given \( h \). We assume that \( P_h(A \rightarrow B) \) satisfies detailed balance

\[
P_h(A \rightarrow B) \frac{P_h(B \rightarrow A)}{P_h(B \rightarrow A)} = e^{-\beta(V(s_B) - V(s_A) - h(s_B - s_A))}
\]

(C.5)

and we then prove that this implies that the transition probability over a time interval \( \tau \), \( P(A \rightarrow B, \tau) \), does too. Inserting Eq. (C.5) into Eq. (C.4) we have:

\[
P(A \rightarrow B) = \frac{\Gamma_1}{\sqrt{2\pi}} \int dh e^{-\frac{(h - \beta^* \Gamma_2 A)^2}{2\Gamma_1^2}} P_h(B \rightarrow A)
\]

\[
\times e^{-\beta(V(s_B) - V(s_A) - h(s_B - s_A))}
\]

(C.6)

\[= e^{-\beta(V(s_B) - V(s_A))} P(B \rightarrow A)\]

i.e. detailed balance.

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