The interval intersection method for FE model updating

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Abstract. In this work a method for finite element (FE) model updating based on interval analysis is presented. The interval analysis inclusion theorem is exploited to achieve a solution of the model updating problem. The solution algorithm is implemented in the automatic procedure named interval intersection method (INTIM). A basic and a refined version of the method and their implementations are presented. Solutions of a numerical example are provided to show the method capabilities compared with a classical sensitivity method.

1. Introduction
Damage detection can be achieved by updating finite element models parameters [1-3], when modal properties of a system are made available by previous experiments. Experimentally identified modal properties describe the actual linear state of a structure; hence model updating interprets damage as a decreasing of structure elastic properties (e.g. stiffness). In this frame modelling and experimental uncertainties play a key role in the problem conditioning. Standard model updating methods are based on sensitivity calculation and they could lead to solutions without physical meaning, or they could have a lack of solution uniqueness depending on the given initial conditions. To cope with the problem of solution uniqueness and to achieve a final physical bound of the updated parameters interval based techniques can be used on purpose [4,5]. A new model updating method is hence presented, called the Interval Intersection Method (INTIM) [6]. INTIM embodies the possibility to use uncertain quantities defined as interval numbers. The core of the method exploits the inclusion theorem of the interval analysis to apply the concept of interval admissibility to structural FE models. The basic sensitivity approach is recalled in section 2 and is then used as a comparison solution. The global optimisation capabilities of interval methods are shown in section 3 whereas in section 4 the method is presented as well as the implemented algorithm. Both the sensitivity and the interval methods are finally used to update a selected classic literature example in section 5.

2. Sensitivity based model updating
The reference model updating technique is the sensitivity approach [3], where defining \( \mathbf{z}_s \) as the vector of experimental measures \((\mathbf{F}_s, \mathbf{U}_s)\), eigenfrequencies \(\mathbf{F}_s\) and eigenvectors \(\mathbf{U}_s\), respectively, an equivalent vector of numerical measures \(\mathbf{z}_0 = (\mathbf{F}_0, \mathbf{U}_0)\) can be defined, which is calculated from an FEM of the structure.
From the modelling point of view a model class \(M(\mathbf{x})\) can be defined, which depends on the constitutive parameters vector \(\mathbf{x}\) from which it can be chosen an a priori model \(M(\mathbf{x}_0)\) that gives the...
defined modal response $Z_0 = z(x_0)$. The distance between the a priori model and the experimental one is defined by the error $e = z_s - z_0 = z_s - z(x_0)$.

This error can be minimized to find the optimum set of model parameters $x$ that better match the experimental results. For this purpose it can be defined an objective function to be minimized, which is given by

$$f(x) = \frac{1}{2} e^T \cdot e = \frac{1}{2} \left[ z_s - z(x) \right]^T \cdot \left[ z_s - z(x) \right]$$

(1)

In general the model function $z(x)$ is nonlinear in $x$, so a series expansion can be performed in order to linearize it

$$z(x) \approx z(x_0) + S_0 \cdot (x - x_0) + o(x^2)$$

(2)

where $S_0$ is the sensitivity matrix, whose components represent the measured ($z$) variations with respect to the constitutive model parameters.

By substituting the expansion (2) in the objective function expression (1), equation (3) is obtained.

$$f(x) = \frac{1}{2} e^T \cdot e = \frac{1}{2} \left[ z_s - z(x_0) - S_0 \cdot (x - x_0) \right]^T \cdot \left[ z_s - z(x_0) - S_0 \cdot (x - x_0) \right]$$

(3)

By applying then the minimum stationary condition (4) to equation (3), the expression (5) of the updated parameters $x$ is obtained.

$$\min_{x \in \mathbb{R}^n} \left[ f(x) \right] \Rightarrow \nabla f(x) = 0$$

$$x = x_0 + \left[ S_0^T \cdot S_0 \right]^{-1} \cdot S_0^T \left[ z_s - z(x_0) \right]$$

(5)

Equation (6) represents the first step of the iterative updating procedure and for the $h$-th iteration step the updating expression is:

$$x_{h+1} = x_h + \left[ S_h^T \cdot S_h \right]^{-1} \cdot S_h^T \left[ z_s - z(x_h) \right]$$

(6)

The presented sensitivity procedure is here applied to validate the alternative interval approach. This procedure is implemented in a self developed tool named FEMUP and developed according to the work referenced in [7].

3. Optimisation of interval functions

In interval analysis uncertain quantities are replaced by intervals. To distinguish between certain quantities and intervals, lower case letters are reserved for the former, whereas upper case letters are used for the latter. Italic or bold characters denote scalar or vector quantities as usual.

An interval $X$ can be alternatively denoted by its infimum $X_{\inf}$ and supremum limits $X_{\sup}$ or by the central notation:
\[ X = \left[ x_{\text{inf}}, x_{\text{sup}} \right] = x_c + \Delta x \cdot e \]  
(7)

where \( x_c = (x_{\text{inf}} + x_{\text{sup}})/2 \) is the central value or mid point, \( \Delta x = (x_{\text{inf}} - x_{\text{sup}})/2 \) is the interval radius and \( e = [ -1, 1 ] \) the unit interval. In the central notation the uncertainty is expressed by the radius \( \Delta x \). Intervals can be regarded as special kind of sets. Sets operations are hence admissible for intervals as well as algebraic.

### Table 1. Basic optimisation interval algorithm pseudo code

| Variables |
|-----------|
| \( X_0 \) - initial search domain |
| \( X_n \) - candidate solution subinterval |
| \( Q \) - list of generated subintervals \( X_n \subset X_0 \) |
| \( Q_s \) - list of solution subintervals |
| \( f_s \) - actual minimum value |
| \( \Delta_{\text{tol}} \) - uncertainty radius stopping tolerance |

**Initialize**

\[ Q := \{ X_0 \}; Q_s := \{ \} \text{ (null list)}; f_s := +\infty; \]

**Start**

while \( Q \neq \{ \} \) do

1. \( X_0 := Q \{ 1 \}; \) push the first element of \( Q \)
2. \( Q := Q - Q \{ 1 \}; \) remove \( Q \{ 1 \} \) from \( Q \)
3. subdivide \( X_n = X_1 \cup X_2 \); according to a bisection rule
4. calculate \( F(X_1) \) and \( F(X_2) \);
5. \( \hat{f} := \min (f_{1,\text{sup}}, f_{2,\text{sup}}); \)
6. if \( \hat{f} < f_s \) then
   - \( f_s := \hat{f}; \) update the actual minimum
7. for \( i = 1, 2 \) do
   - if \( f_{1,\text{inf}} \leq f_s \) then
     - if \( \Delta x_i \leq \Delta_{\text{tol}} \) then
       - \( Q_s := \{ Q_s, X_i \} \)
     - else
       - \( Q := \{ Q, X_i \} \)
   end
end
end

### 3.1. Interval functions

Let \( f(\mathbf{x}) \) be a real valued function that depends on the crisp vector \( \mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \). There exist different ways to define its interval extension \( F(\mathbf{X}) \) [4]. The interval functions considered in the paper are called natural extensions and are obtained by replacing every single occurrence \( x_i \) in the expression of \( f \) with the correspondent interval \( X_i \) in \( F \). \( F(\mathbf{X}) \) maps the vector \( \mathbf{X} = (X_1, \ldots, X_i) \) into the interval space and converges to \( f \), i.e. \( F(\mathbf{x}) = f(\mathbf{x}) \), whenever \( \mathbf{X} \) shrinks to \( \mathbf{x} \). For natural extensions, the inclusion property is settled by the inclusion theorem [5] which ensures that \( F(\mathbf{X}) \) bounds all minima and maxima of \( f(\mathbf{x}) \) when \( \mathbf{x} \in \mathbf{X} \). The strict form of the theorem is based on natural extensions that are inclusion monotonic:
\[ F(X) \subseteq F(Y) \quad \text{if} \quad \{ X \subset Y \mid X_i \subset Y_i, \forall i \} \] (8)

3.2. Interval basic algorithm

The interval optimisation algorithm is presented in its basic version, as originally developed in [5] by Hansen. It is framed in the so-called Branch and Bound (B&B) family of algorithms, in which an initial search domain \( X_0 \) is iteratively subdivided (branched) in subdomains \( X \), that in our case are subintervals. For every generated subinterval a selected criterion is applied to verify if a minimum could be contained in it. If this is proven then the evaluated subinterval is considered as a bounding of the solution, otherwise it is discarded from the search.

When this concept are applied to interval functions, then the bounding criterion is naturally given by the inclusion properties of the interval calculus. For this reason it is very easy to define a robust discarding criterion.

The main steps of the interval algorithm are reported in the pseudo code of Table 1. With reference to the symbols in Table 1 and in the hypothesis of the validity of the inclusion theorem for interval functions, one can suppose that at certain iteration of the optimisation procedure \( f_i \) is the actual evaluated minimum (i.e. the smaller non-interval function value estimated at that iteration). Then for every generated subinterval, if the evaluated lower bound of the function \( f_{\text{inf}} \) is greater than \( f_s \), this means that \( f_s \) is not included in the interval \( F(X) = [f_{\text{inf}}, f_{\text{sup}}] \) and hence \( X \) can be discarded. The survived subintervals are subdivided until a radius tolerance \( \Delta_{\text{tol}} \) is reached, that defines the acceptable uncertainty in the solution subintervals \( X \).

The main interesting properties of an interval B&B optimisation algorithm is that it allows to reach global optima as a solution and it is able to find more than one global minimum if occurred.

The main drawback is that, in its basic form it is not very efficient from a computational point of view, in fact it requires a lot of function evaluations being a derivative free algorithm.

3.3. Example – Minimisation of the Branin function

The Branin function is a test function generally used in global optimisation problems [8] as it has the peculiarity of having three global minima (see Fig. 1).

![Contour representation of the Branin function](image_url)
Expression of the Branin function: \( f(x_1, x_2) = (x_2 - \frac{5.1}{4\pi^2} x_1^2 + \frac{5}{\pi} x_1 - 6)^2 + 10 \left(1 - \frac{1}{8\pi}\right) \cos(x_1) + 10 \)

Initial search domain: \( X_0 = [-5, 10] \times [0, 15] \)

Tolerance radius: \( \Delta_{tol} = 0.375 \)

True global solutions in \( X_0 \):
(1) \((-\pi, 12.275)\); (2) \((\pi, 2.275)\); (3) \((3\pi, 2.475)\)

Interval obtained bounds for the global solutions:
(1) \([-3.36, -2.89], [11.48, 13.12]\); (2) \([2.97, 3.20], [1.88, 2.81]\); (3) \([9.30, 9.53], [1.88, 3.05]\)

True value of the global minimum: \( f^* = 0.397887 \)

Interval inclusion of the global minimum: \( F^* = [0.398, 1.61] \);

Global minimum value estimated by the interval algorithm: \( f_s = 0.3985 \).

In Figures 1 and 2 it is depicted the Branin function shape and the interval solution, respectively.

![Figure 2. Interval bounding of the Branin function's global minima](image)

4. The Interval Intersection Method

The INTIM method has been developed by the author and referenced in [6]. It develops from the basic B&B method presented in the previous section. The concise rationale of the method is as follows. It is supposed to start from a search interval \( K_0 \) such that \( K = \{K_1, \ldots, K_n\} \in K_0 \). The branching step is performed according to the previous optimisation case and a bisection rule is iteratively applied. The repeated application of the branching step produces progressively smaller subintervals, \( K \subset K_0 \). The bounding step is properly formulated to decide whether \( F(K) = F \) includes the experimental data \( F_s \) or not and consequently if \( K \) should be retained or rejected. To check whether the solution is included in a particular subinterval \( K \) the intersection operation is adopted:

\[
\begin{align*}
F_s \cap F &= F_s \\
F_s \cap F &= \emptyset
\end{align*}
\]
The first row of relation (9) is an acceptance criterion and states that a subinterval $K$ survives only if $F$ includes totally $F_s$. By reverting the previous statement one obtains the discarding criterion, which is given by the second row of equation (9).

The main steps of the algorithm are contained in the pseudo code of Table 2.

**Table 2. INTIM algorithm pseudo code**

| Variables |
|-----------|
| $K_0$ - initial search domain in the parameters space |
| $K_n$ - candidate solution subinterval |
| $Q$ - list of generated subintervals $K_n \subset D_0$ |
| $Q_s$ - list of solution subintervals |
| $\Delta_{tol}$ - uncertainty radius stopping tolerance |

**Initialize**

$Q := \{K_0\}; Q_s := \{\}$ (null list);

**Avvio**

while $Q \neq \{\}$

$K_n := Q \{1\}$; - push the first element of $Q$

$Q := Q - Q \{1\}$; - remove $Q \{1\}$ from $Q$

subdivide $K_n = K_1 \cup K_2$; - according to a bisection rule

calculate $F(K_1)$ and $F(K_2)$;

for $i = 1, 2$

if $F_s \cap F(K_i) = F_s$ then

if $\Delta k_i \leq \Delta_{tol}$ then

$Q_s := \{Q_s, K_i\}$;

else

$Q := \{Q, K_i\}$;

end

elseif $F_s \cap F(K_i) = \emptyset$ then

discard $K_i$

end

end

As can be noted by comparison of Table 1 with Table 2, the INTIM algorithm does not make use of the definition of an objective function (as $f(x)$ of Eq. (1)) to be minimize. It is in fact constructed by thinking the existence of a comparison solution, that is $F_s$. The intersection defined in (9) practically substitutes the objective function evaluation, as in this case $F(K)$ is the evaluation of the interval numerical measurements, and in the hypothesis that the inclusion theorem holds $F(K)$ have to include the experimental counterpart $F_s$, the parameters subinterval $K$ to be considered as a possible solution.

In INTIM the solution is again iteratively found. As $K_0$ is consecutively branched in subintervals $K$ then the procedure stops when for some $K$ the inclusion criterion holds together with the radius tolerance is reached. The selected criterion to stop the iterations, in the $n$-dimensional case, $K: K_1 \times \ldots \times K_n$, is as following

$$\min \{\Delta k_1, \ldots, \Delta k_n\} \leq \Delta_{tol}$$

(10)

that is when the radius $\Delta k$ of at least one of the $n$ parameters is smaller than an user selectable tolerance $\Delta_{tol}$, a discussion about how $\Delta_{tol}$ affect the result could be found in [6].

The INTIM procedure has been coded and implemented in the Matlab® environment through the INTLAB toolbox [9].
4.1. Refined version of INTIM

As in the original formulation [6] INTIM does not allow the automatic use of measured eigenmodes, in this paper a refined version of the acceptance criterion is also proposed by adding a comparison on calculated interval eigenvectors $U = [u_{\text{inf}}, u_{\text{sup}}]$. By defining the interval $MAC$ index, such that

$$MAC(u, u_s) = \frac{(u^T \cdot u_s)^2}{(u^T \cdot u)(u_s^T \cdot u_s)}$$

with $u_{\text{inf}}$, $u_{\text{sup}}$ being the lower and upper bound of the interval calculated eigenvectors respectively. Then the mean of the mid points $MAC$ are subject to the following restriction,

$$\frac{\sum_{i=1}^{m} \text{mid}([MAC]_i)}{m} < MAC_{\text{tol}}$$

with $m$ being the number of considered modes, $\text{mid}$ represent the mid-point value of the calculated $i$-th $MAC$ and $MAC_{\text{tol}}$ being a tolerance on the acceptable mean $MAC$ value. The $MAC_{\text{tol}}$ value depends on the quality of the available experimental data and a high level of acceptance could be fixed as $MAC_{\text{tol}} = 0.95$.

**Figure 3.** Selected mechanical example

| Mode | Experimental Frequency [Hz] | Initial [Hz] | Updated [Hz] | Initial (% error) | Updated (% error) |
|------|-----------------------------|--------------|--------------|-------------------|-------------------|
| 1    | 37.46                       | 36.78        | 37.44        | 1.82%             | 0.05%             |
| 2    | 236.8                       | 234.0        | 236.8        | 1.18%             | 0.00%             |
| 3    | 657.3                       | 650.1        | 657.3        | 1.10%             | 0.00%             |
| 4    | 1255                        | 1238         | 1255         | 1.35%             | 0.00%             |
5. Numerical example
The selected example is taken from the classical model updating book of Mottershead and Friswell [1], and it is the example 8.1. This to provide a validation of the INTIM approach by comparison with a well known literature mechanical example. The solution given in the book is a sensitivity based solution as that given in section 2. In the example the uncertain updating parameters are the springs stiffnesses $K_t$ and $K_r$ and the flexural stiffness $EI$. The set of pseudo-experimental data is given by the first four eigenfrequencies that are generated by taking as solution parameters the following values:

$k_t = 4.0 \times 10^4$ kN/m, $k_r = 100$ kNm/rad, $ei = 4.56$ kNm$^2$

5.1. Sensitivity solution
Initial parameters  $K_t = 2.0 \times 10^4$ kN/m, $K_r = 50$ kNm/rad, $EI = 4.5$ kNm$^2$

Updated parameters  $K_t = 3.767 \times 10^4$ kN/m, $K_r = 96.9$ kNm/rad, $EI = 4.592$ kNm$^2$

Even if already published in [1], the frequency updating comparison is reported and differently presented in Table 3 for completeness.

5.2. Interval solution obtained by INTIM
The initial search interval is taken as:

$K_{t0} = [3.0, 5.12] \times 10^4$ kN/m, $K_{r0} = [50, 160]$ kNm/rad, $EI_0 = [4.43, 4.75]$ kNm$^2$
The radius stopping tolerance $\Delta_{\text{rel}}$ is taken one thousand smaller than the max relative radius of the initial search interval box (it is called relative the radius scaled by the interval central value).

The obtained solution interval is given by the interval set depicted in Figures 4, where the three dimensional solution in the parameters space of the upper left corner, is then represented in orthographic projections. In Fig. 4 the parameter $EI$ variation is considered by fixing the inertia moment $I$ value and by updating the Young modulus $E$. The obtained solution intervals are:

$K_t = [3.911, 4.226] \times 10^4$ kN/m, $K_r = [98.99, 103.1]$ kNm/rad, $EI = [4.564, 4.615]$ kNm$^2$

having with the crisp central values:

$k_{tc} = 4.07 \times 10^4$ kN/m, $k_{rc} = 101.1$ kNm/rad, $e_{ic} = 4.590$ kNm$^2$

| Mode | Experimental Frequency [Hz] | Interval solution [Hz] | Model Mid value [Hz] | Mid (% error) | Interval (% error) |
|------|-----------------------------|------------------------|----------------------|---------------|-------------------|
| 1    | 37.46                       | [37.30, 37.63]         | 37.46                | 0.00%         | [-0.45, 0.43]%    |
| 2    | 236.8                       | [235.7, 237.8]         | 236.7                | 0.04%         | [-0.42, 0.46]%    |
| 3    | 657.3                       | [654.3, 660.5]         | 657.4                | 0.02%         | [-0.48, 0.46]%    |
| 4    | 1255                        | [1248.6, 1262.8]       | 1256                 | 0.08%         | [0.49, 0.51]%     |

In Table 4 it is reported the INTIM solution in terms of eigenfrequencies, whereas in the Fig. 5 a graph of the min and max radii, scaled by the central values of the parameters (relative uncertainty).

In this example it is not used the refined version of the algorithm as the mode shapes were not used in the original example of the book [1].
It can be noted that the sensitivity solution (Section 5.1) is a very good one, in terms of eigenfrequency result, as the relative error of the updating model is zero, in practice. We can also note that in real the starting point is not very far from the solution (see Table 3) but against that the obtained solution in terms of parameters does not match the true parameter values, in particular k, and kr. This means that the sensitivity solution jump to a minimum different from that correspond to the true solution, and this minimum could be a local one or a different global minimum, as we obtain a zero for the eigenfrequency error.

The obtained INTIM solution is comparable with the sensitivity solution. The experimental frequencies are included with a very small uncertainty radius, and the mid point values has very small error, very near to zero (Table 4). If we look at the solution in terms of model parameters, it can be noted that the parameter EI is practically updated as in the sensitivity case if the central value is considered. On the contrary the parameters k, and kr are very well updated, with mid point values that are very close to the true solution, and the true solutions are included in the updated intervals.

It is good to remember that in the case of INTIM, the obtained solution says that no other solution could be found outside the calculated intervals, due to the inclusion theorem. And as the sensitivity solution is not completely included in the interval solution, it should be revised.

On the drawbacks it is worth to note from Fig. 5 that the INTIM convergence process is very slow even in this simple case, and when the number of parameters increase it could become not practicable. Improvements in that sense could be proposed by adding interval gradients calculations in the optimisation algorithm. This will possibly accelerate the algorithm convergence (see reference [5] for interval methods using gradients), but no trial in updating problems has been still made at author knowledge.

6. Conclusions
In this work a new interval model updating procedure, called Interval Intersection Method (INTIM) is presented in its algorithmic form and discussed. A proper selected example from the classic literature is solved and the INTIM solution is compared with a standard sensitivity approach. The sensitivity result validates the INTIM solution as they are comparable, and according to an inclusion criterion a revision of the former could be needed.

It is also put in evidence that, against the INTIM robustness in finding global solutions, the implemented algorithm is not very efficient from a computational point of view and needs to be ameliorated.

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