A COUNTEREXAMPLE IN QUASI-CATEGORY THEORY

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Abstract. We give an example of a morphism of simplicial sets which is a monomorphism, bijective on 0-simplices, and a weak categorical equivalence, but which is not inner anodyne. This answers an open question of Joyal. Furthermore, we use this morphism to refute a plausible description of the class of fibrations in Joyal’s model structure for quasi-categories.

1. Introduction

A morphism of simplicial sets is said to be inner anodyne (or an inner anodyne extension) if it belongs to the saturated class generated by the inner horn inclusions $\Lambda^n_k \to \Delta^n$, $0 < k < n$, i.e. if it can be expressed as a retract of a countable composite of pushouts of coproducts of inner horn inclusions. It is easily shown that every inner anodyne extension is (i) a monomorphism, (ii) bijective on 0-simplices, and (iii) a weak categorical equivalence (i.e. a weak equivalence in Joyal’s model structure for quasi-categories [J08b, L09]). This statement admits a partial converse: if a morphism of simplicial sets whose codomain is a quasi-category satisfies the properties (i)–(iii), then it is inner anodyne.

The purpose of this note is to show that the complete converse of this statement is false; we give in §2 an example of a morphism of simplicial sets (therein denoted $f: \Delta^1 \to S$, see Definition 1) which satisfies the properties (i)–(iii), but which is not inner anodyne (see Theorem 4). This answers an open question of Joyal (see [J08a, §2.10]).

The existence of such a morphism yields answers to a few related open questions. For example, we show (Corollary 5) that the class of inner anodyne extensions does not have the left cancellation property. Furthermore, we refute a plausible description of the fibrations in Joyal’s model structure for quasi-categories. Recall that a morphism of simplicial sets is said to be an inner fibration if it has the right lifting property with respect to the inner horn inclusions (and hence all inner anodyne extensions). Joyal proved that a morphism of simplicial sets $p: X \to Y$ whose codomain $Y$ is a quasi-category is a fibration in the model structure for quasi-categories if and only if it is an inner fibration and an isofibration on homotopy categories (i.e. the induced functor $\text{ho}(p): \text{ho}(X) \to \text{ho}(Y)$ between homotopy categories is an isofibration). We show (Corollary 6) that these two properties fail to describe the fibrations with arbitrary codomain.

2. The counterexample

Definition 1. Let $S$ denote the simplicial set freely generated by a 2-simplex $\alpha$ whose 2nd face $d_2(\alpha)$ is degenerate, which is defined by the following pushout diagram in the category of simplicial sets.

$$
\begin{array}{ccc}
\Delta^1 & \longrightarrow & \Delta^0 \\
\downarrow_{d^2} & & \downarrow_{\gamma} \\
\Delta^2 & \longrightarrow & S \\
\end{array}
$$

Let $f: \Delta^1 \to S$ denote the composite of the morphisms $d^1: \Delta^1 \to \Delta^2$ and $\alpha: \Delta^2 \to S$, which picks out the 1st face of the 2-simplex $\alpha$ in $S$.

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Stevenson [S18a] proved that the class of inner anodyne extensions does have the right cancellation property.
The simplicial set \( S \) may be displayed as a quotient of the 2-simplex \( \Delta^2 \) as below.

Thus \( S \) has two 0-simplices \( x \) and \( y \) (say), two non-degenerate 1-simplices \( f \) and \( g \) with \( d_1(f) = d_1(g) = x \) and \( d_0(f) = d_0(g) = y \), and one non-degenerate 2-simplex \( \alpha \) with \( d_2(\alpha) = s_0(x) \), \( d_1(\alpha) = f \), and \( d_0(\alpha) = g \).

**Remark 2.** The simplicial set \( S \) of Definition 1 can also be described as the “right suspension” of the simplicial set \( \Delta^1 \), which is denoted variously as \( C_R(\Delta^1) \) in \([\text{DS11}, \S 4.3] \), \( \Delta^2_{12} \) in \([\text{R14}, \S 15.4] \), and \( \Sigma \Delta^1 \) in \([\text{RV18}, \S 7.1] \). Thus a morphism \( u \) from \( S \) to any simplicial set \( X \) amounts to a 1-simplex in the right hom-space \( \text{Hom}^R_X(u(x), u(y)) \) of \( X \) \([\text{L09}, \S 1.2.2] \). Moreover, the morphism \( f: \Delta^1 \to S \) can be described as the right suspension of the morphism \( d^1: \Delta^0 \to \Delta^1 \).

We shall see from the following list of its properties that the morphism \( f: \Delta^1 \to S \) of Definition 1 is a counterexample to the statements considered in \( \S 1 \).

**Lemma 3.** The morphism \( f: \Delta^1 \to S \) is

(i) a monomorphism,
(ii) bijective on 0-simplices,
(iii) a weak categorical equivalence, and
(iv) an inner fibration.

**Proof.** The properties (i) and (ii) are immediate from the definition of \( f \).

To prove (iii), observe that the morphism \( g: \Delta^1 \to S \) (the composite of \( d^0: \Delta^1 \to \Delta^2 \) and \( \alpha: \Delta^2 \to S \)) is a weak categorical equivalence; for it is the pushout of the inner horn inclusion \( \Lambda^1_1 \to \Delta^2 \) along the morphism \( \Lambda^1_1 \to \Delta^1 \) which picks out the \((2,1)\)-horn

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & 1
\end{array}
\]

in \( \Delta^1 \), as displayed on the left below. Since the two morphisms \( f, g: \Delta^1 \to S \) admit a common retraction, namely the unique morphism \( r: S \to \Delta^1 \) that sends \( \alpha \) to the degenerate 2-simplex \( 001 \) in \( \Delta^1 \), it follows by the two-out-of-three property that \( f \) is a weak categorical equivalence.

\[
\begin{array}{ccc}
\Lambda^1_2 \to & \Delta^1 \\
\downarrow & \downarrow & \downarrow \\
\Delta^2 & \to & S \\
\alpha & \alpha & \alpha
\end{array}
\]

To prove (iv), observe that the pullback of \( f: \Delta^1 \to S \) along the epimorphism \( \alpha: \Delta^2 \to S \) is the outer horn inclusion \( \Lambda^2_0 \to \Delta^2 \), which is an inner fibration (since it is a morphism between nerves of categories). It then follows by a simple argument (indicated by the diagram on the right above) that \( f: \Delta^1 \to S \) is an inner fibration.

**Theorem 4.** There exists a morphism of simplicial sets which is a monomorphism, bijective on 0-simplices, and a weak categorical equivalence, but which is not inner anodyne.

**Proof.** By Lemma 3, the morphism \( f: \Delta^1 \to S \) of Definition 1 has the first three listed properties. To prove that \( f \) is not inner anodyne, it suffices to observe that \( f \) is an inner fibration by Lemma 3 and is evidently not an isomorphism; for if a morphism is both inner anodyne and an inner fibration, then it has the right lifting property with respect to itself, and is therefore an isomorphism.
Corollary 5. The class of inner anodyne extensions does not have the left cancellation property; that is, there exists a composable pair of monomorphisms of simplicial sets $i,j$ such that $j$ and $ji$ are inner anodyne, but such that $i$ is not inner anodyne.

Proof. Let $i: A \to B$ be a morphism as described in Theorem 4, and let $j: B \to C$ be an inner anodyne extension with $C$ a quasi-category (as may be constructed by the small object argument). Then their composite $ji: A \to C$ is a monomorphism, bijective on 0-simplices, and a weak categorical equivalence, and hence is inner anodyne, since its codomain is a quasi-category (see for instance [S18b, Lemma 2.19]). But $i$ is not inner anodyne. □

Corollary 6. There exists a morphism of simplicial sets which is an inner fibration and an isofibration on homotopy categories, but which is not a fibration in Joyal’s model structure for quasi-categories.

Proof. This can be proved as a direct corollary of Theorem 4 by a factorisation and retract argument, but we will instead show that the morphism $f: \Delta^1 \to S$ of Definition 1 is itself such a morphism as in the statement. By Lemma 3, $f$ is an inner fibration, and is an isomorphism (and hence an isofibration) on homotopy categories, since it is bijective on 0-simplices and a weak categorical equivalence. But $f$ is a trivial cofibration in the model structure for quasi-categories, since it is both a monomorphism and a weak categorical equivalence, and hence is not a fibration, since it is not an isomorphism. □

Remark 7. In unpublished work of Joyal, there is constructed a model structure on the category of simplicial sets whose cofibrations are the “immersions” (i.e. those morphisms sent to monomorphisms by the reflection to the full subcategory of reduced simplicial sets), whose weak equivalences are the bijective-on-0-simplices weak categorical equivalences, and whose fibrant objects are the quasi-categories. Theorem 4 implies moreover that, while every fibration in this model structure is an inner fibration (and conversely for morphisms to quasi-categories), not every inner fibration is a fibration.

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