A recursive construction of units in a class of rings

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Abstract

Let $R$ be an associative ring with identity and let $N$ be a nil ideal of $R$. It is shown that units of $R/N$ can be lifted to units in $R$. Under some mild conditions on the ring, a procedure is given to determine those lifted units in a recursive way. As an application, the units of several classes of rings are determined, including: matrix rings, chain rings, and group rings where the ring is a chain ring. Numerical examples are given illustrating the main results.

Keywords unit, nil ideal, matrix ring, chain ring, group ring.

1 Introduction

The problem of determining the structure of the group of units of a ring has been the subject of considerable research over time. In the literature several cases of information about the group of units of a ring can be found. For instance, if the ring is a finite field $\mathbb{F}$ its group of units is almost the entire ring: $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. If the ring is local commutative or finite commutative, an answer is provided in [10, Chapter XVIII pag. 353 and Chapter XXI pag. 398], respectively. In [3] is determined all nonisomorphic rings whose group of units is isomorphic to a finite group of $n$ elements, where $n$ os a power of a prime or any product of prime powers, not divisible by 4. In [6] the authors determine the structure of the group of units of a finite commutative chain ring, and in [7] the units of the matrix rings $M_2(\mathbb{Z}_b[x])$ for $b = 2, 3$ are determined. Other descriptions of the units of a ring can also be found in the literature, although in general it is an unresolved question.

Of course the determination of the group of units of a ring is closely related to Fuchs’ problem on determining which groups can be the groups of units of a ring. Answers to
this question have been obtained in some cases. For instance, in [1] a class of \( p \)-groups is considered. In [4] the authors provide an answer to Fuchs’ problem when the ring is torsion-free, and when the ring has characteristic zero. However Fuchs’ problem is also far from being solved in general.

The knowledge of a unit and its inverse in a ring is important in several instances, for example, in matrix theory. The group of units of a ring is also related with other areas of mathematics as well as physics, chemistry, economics. For example in [8],[2] a relation with number theory and geometry, particularly with Dirichlet’s Unit Theorem is treated.

There are several approaches to obtain the units and their inverse of a ring, one of which is to determine the units by lifting units from another ring, usually a quotient ring. For instance in [12] the question of lifting units to a ring \( R \) from the ring \( R/I \) for the class of separative exchange ideals \( I \) and a relation with \( C^* \)-algebras is treated. In [11] the following question is studied: If \( R \) is a right self-injective ring and \( I \) an ideal of \( R \), when can a unit of \( R/I \) be lifted to units of \( R \)? In [13] lifting units in clean rings are considered.

In this manuscript the units of a ring \( R \) with certain properties will be determined by lifting units from a quotient ring \( R/I \). This process gives a precise expression for determining the inverse of a unit of the ring \( R \). Several consequences are obtained from this result and the units are determined in cases which include the following: matrix rings with entries in modular integers; commutative rings containing a nilpotent ideal; group rings \( RG \) not necessarily commutative; commutative group rings \( RG \) where the ring \( R \) is a chain ring; and the commutative group ring \( \mathbb{Z}_m G \) where \( \mathbb{Z}_m \) is the ring of integers modulo \( m \).

The manuscript is organized as follows: in Section 2 basic facts on units of a ring and a quotient ring are discussed. Section 3 is devoted to the recursive construction of the inverse of a unit of a ring with a certain property (see Definition 3.2), which is lifted from a unit of a quotient ring; and in Section 4, by using the main result of the previous Section, the units of the rings mentioned above are determined. Numerical examples are provided illustrating the discussed results.

### 2 Basic facts

Recall that an element \( x \) of a ring \( R \) is called *invertible* if there exists \( y \in R \) such that \( xy = yx = 1 \). The element \( y \) is unique and it is denoted by \( y = x^{-1} \) (the inverse of \( x \)). In addition, the set of units denoted by \( R^* \) of a ring \( R \), form a group under the multiplication of the ring.

The starting point for the results presented in this manuscript is the following one on the construction of inverse elements on a ring \( R \) from those of a quotient ring \( R/N \), where \( N \) is a nil ideal. This is a particular version of the result established in [3, Lemma 2.1], where \( N \) is the Jacobson radical. For the sake of completeness, the proof is presented in here.

**Proposition 2.1.** Let \( R \) be a ring, \( N \) a nil ideal of \( R \) and \( - : R \to R/N \) the canonical homomorphism from \( R \) to the quotient ring \( R/N \). Then, \( \bar{f} = f + N \in (R/N)^* \), if and only if, \( f + N \subset R^* \).
Proof: First it is shown that if \( \bar{f} \) is an invertible element in \( R/N \), then for all \( x \in f + N \), \( x \) is an invertible element in \( R \). Since \( \bar{x} = \bar{f} \) is an invertible element in the quotient ring \( R/N \), there exists \( \bar{g} \in R/N \) such that \( xg - 1 \in N \), \( gx - 1 \in N \) and, since \( N \) is a nil ideal, \((xg - 1)^{2n+1} = (gx - 1)^{2m+1} = 0\) for some integers \( n, m > 0 \). It is easy to see that:

\[
x e_1 = 1, \quad \text{with} \quad e_1 = g \left[ \sum_{i=0}^{2n} \binom{2n+1}{i} (xg)^{2n-i}(-1)^i \right],
\]

and,

\[
e_2 x = 1, \quad \text{with} \quad e_2 = \left[ \sum_{i=0}^{2m} \binom{2m+1}{i} (gx)^{2m-i}(-1)^i \right] g.
\]

Since \( R \) is associative, it follows that \( e_1 = e_2 \). Hence, if \( x^{-1} = e_1 \), \( xx^{-1} = x^{-1}x = 1 \), i.e., \( x \) is an invertible element in \( R \). The proof of the other implication of the statement is straightforward.

Given an invertible element \( \bar{f} \in R/N \), each element \( x \in f + N \subset R \) will be called a lifted invertible element associated to \( \bar{f} \). The previous result implies that in order to compute the inverse \( x^{-1} \) of a lifted invertible element \( x \in \bar{f} \), it is necessary to compute the inverse \( \bar{g} \) of \( \bar{x} \) in \( R/N \) and then, calculate any of the expressions for \( e_1 = x^{-1} \) or \( e_2 = x^{-1} \) given in the proof of Proposition 2.1. One of the goals of this manuscript is to give a better (a simplified) expression to calculate \( e_1 = e_2 = x^{-1} \). Another, fundamental implication of the proposition is about the relation between the cardinality of the group \( R^* \) and the cardinality of the group \((R/N)^*\), this is the content of the following,

Remark 2.2. The following observations are easy consequences of the previous result.

1. If \( R \) is ring and \( N \) is a nil ideal of \( R \), the group \( R^* \) of invertible elements of \( R \) can be described as a disjoint union of equivalence classes of invertible elements of the quotient ring \( R/N \). That is

\[
R^* = \bigcup_{\bar{f} \in (R/N)^*} (f + N).
\]  

Indeed, if \( y \in R^* \), then \( \bar{y} \in (R/N)^* \). Since \( y \in \bar{y} \), it follows that \( R^* \subset \bigcup_{\bar{f} \in (R/N)^*} \bar{f} \). The other inclusion is a consequence of Proposition 2.1.

2. With the same assumptions as in the previous observation, from relation (1) and the Lagrange theorem:

\[
|R^*| = \sum_{f \in (R/N)^*} (f + N) = |N| \sum_{f \in (R/N)^*} 1 = |N||(R/N)^*|.
\]

Finally, it is to be noticed that Proposition 2.1 is not true if the hypothesis of \( N \) being a nil ideal is removed. For instance, considering \( R = \mathbb{Z}_{12} \) and \( N = 2\mathbb{Z}_{12} \), \( 1 + N \) is an invertible element in \( R/N \), but \( 3 \in 1 + N \) is a non-invertible element in \( R \).
3 The recursive construction

In this section, under certain conditions on the ring $R$, it is provided a simple way to compute the inverse $x^{-1}$ of any lifted invertible element $x \in \bar{f} \in (R/N)^*$ given in Proposition 2.1. More precisely, if a collection $\{N_1, \ldots, N_k\}$ of ideals of a ring $R$ satisfies the CNC-condition (Definition 3.2), and $x + N_1 = f + N_1$ is an invertible element in the ring $R/N_1$, then $x^{-1}$ can be computed as a product of $g$ and a power of $xg$ in the ring $R$, where $g$ is any element in the class $\bar{f}^{-1}$ (Theorem 3.3). For this purpose we need the following,

**Proposition 3.1.** Let $R$ be a ring and $N$ a nilpotent ideal of index $t \geq 2$ in $R$. Then the following statements holds:

1. For any prime number $p$ such that $p \geq t$ and for all $n \in N$, there exists $r \in R$ such that
   $$(1 + n)^p = 1 + pnr.$$ 

2. Let $\bar{f}$ be an invertible element in the quotient ring $R/N$. If there exists a natural number $s > 1$, such that $sN = 0$, and all the prime factors of the number $s$ are greater than or equal to the nilpotency index $t$ of the ideal $N$, then $f$ is an invertible element in $R$ and
   $$f(g(fg)^{s-1}) = (fg)^s = 1,$$
   where $g \in R$ is such that $\bar{f}g = \bar{1}$. Thus, the inverse of the invertible element $f$ is given by
   $$f^{-1} = g(fg)^{s-1}.$$ 

**Proof:**

1. Since $n^t = 0$,
   $$\sum_{j=0}^{p} \binom{p}{j}n^j = 1 + \sum_{j=1}^{t-1} \binom{p}{j}n^j.$$ 
   Since $p$ is a prime number, $p$ divides $\binom{p}{j}$ for all $1 \leq j \leq p - 1$. Also, since $t \leq p$,
   $$(1 + n)^p = 1 + pn(k_1 + k_2n + \cdots + k_{t-1}n^{t-2})$$
   where $k_i = \binom{p}{i}/p$. Therefore,
   $$\sum_{j=0}^{p} \binom{p}{j}n^j = 1 + pnr,$$
   with $r = k_1 + k_2n + \cdots + k_{t-1}n^{t-2} \in R$.

2. Since $N$ is a nilpotent ideal of $R$ and $f + N$ is an invertible element in $R/N$, Proposition 2.1 implies that $f$ is an invertible element in $R$. Let $p_1, p_2, p_3, \ldots, p_m$ be the prime numbers, not necessarily different, appearing in the prime factor decomposition of the
integer \( s \), with \( p_i \geq t \), for \( i = 1, 2, 3, \ldots, m \). Since \( \bar{f} \bar{g} = 1 \), there exists \( n \in N \) such that \( \bar{f} = 1 + n \). Since \( p_1 \geq t \), from item 1 there exists \( r_1 \in R \) such that

\[
(fg)^{p_1} = (1 + n)^{p_1} = 1 + p_1nr_1.
\]

Similarly, since \( p_2 \geq t \) and \( p_1nr_1 \in N \), item 1 implies that there exists \( r_2 \in R \) such that

\[
(fg)^{p_1p_2} = ((fg)^{p_1})^{p_2} = (1 + p_1nr_1)^{p_2} = 1 + p_2(p_1nr_1)r_2.
\]

In the same way, it is possible to verify that \( r_3, r_4, \ldots, r_m \in R \) exists such that

\[
(fg)^s = 1 + sn(r_1r_2 \cdots r_m).
\]

In other words,

\[
(fg)^s = 1 + sh,
\]

where \( h = nr_1r_2 \cdots r_s \in N \). Finally, since \( h \in N \) and \( sN = 0 \), \( 1 = (fg)^s \) and

\[
f^{-1} = g(fg)^{s-1}.
\]

It is evident that the hypothesis in item 2 of Proposition 3.1, which requires that all prime factors of \( s \) be greater or equal than the nilpotency index \( t \) of the ideal \( N \), restricts enormously the number of applications of that proposition. For instance, if we consider \( 2 \leq t \), \( R = \mathbb{Z}_{2t} \) and \( N = 2\mathbb{Z}_{2t} \), it is clear that \( N^t = \{0\} \) and \( 2^{t-1}N = \{0\} \). Thus, according to the item 2 of Proposition 3.1 in order to describe the invertible elements of the ring \( R \) in terms of the invertible elements of the quotient ring \( R/N \cong \mathbb{Z}_2 \), it is necessary that \( t \leq 2 \), thus \( t = 2 \). Hence, we can only describe the invertible elements of \( \mathbb{Z}_4 \) in terms of the invertible elements of \( \mathbb{Z}_2 \), which is easily done by hand. In the following lines, we show how to overcome such restrictions.

**Definition 3.2.** [9, Definition 3.2] We say that a collection \( \{N_1, \ldots, N_k\} \) of ideals of a ring \( R \) satisfies the CNC-condition if the following properties hold:

1. **Chain condition:** \( \{0\} = N_k \subset N_{k-1} \subset \cdots \subset N_2 \subset N_1 \subset R \).

2. **Nilpotency condition:** for \( i = 1, 2, 3, \ldots, k-1 \), there exists \( t_i \geq 2 \) such that \( N_i^{t_i} \subset N_{i+1} \).

3. **Characteristic condition:** for \( i = 1, 2, 3, \ldots, k-1 \), there exists \( s_i \geq 1 \) such that \( s_iN_i \subset N_{i+1} \). In addition, the prime factors of \( s_i \) are greater than or equal to \( t_i \).

The minimum number \( t_i \) satisfying the nilpotency condition will be called the nilpotency index of the ideal \( N_i \) in the ideal \( N_{i+1} \). Similarly, the minimum number \( s_i \) satisfying the characteristic condition will be called the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \).
The nilpotency condition and the characteristic condition of the previous definition can be stated as follows:

a. The nilpotency condition is equivalent to the following condition: for \( i = 1, 2, \ldots, k-1 \), \( N_i/N_{i+1} \) is a nilpotent ideal of index \( t_i \) in the ring \( R/N_{i+1} \), (for details see [9 Definition 3.2]).

b. The characteristic condition is equivalent to the following condition: for \( i = 1, 2, \ldots, k-1 \), there exists a natural number \( s_i \geq 1 \) such that \( s_i(N_i/N_{i+1}) = 0 \) in the ring \( R/N_{i+1} \), (for details see [9 Definition 3.2]).

**Theorem 3.3.** Let \( R \) be a ring, \( \{N_1, N_2, \ldots, N_k\} \) a collection of ideals of \( R \) satisfying the CNC-condition and \( s_i \) the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \). Then, \( f + N_1 \) is an invertible element in the ring \( (R/N_1) \), if and only if, \( f + N_1 \subset R^* \). Furthermore for each \( x \in f + N_1 \),

\[
x^{-1} = g(xy)^{s_1s_2\cdots s_{k-1}-1},
\]

where \( g \in R \) is such that \( (f + N_1)(g + N_1) = 1 + N_1 \). Moreover,

\[
|R^*| = |(R/N_1)^*||N_1|.
\]  

**Proof:** Since \( (f + N_1)(g + N_1) = 1 + N_1 \), for any \( x \in f + N_1 \), \( (x + N_1)(g + N_1) = 1 + N_1 \).

The isomorphism

\[
R/N_1 \cong \frac{(R/N_2)}{(N_1/N_2)},
\]  

implying that \( (x + N_2 + N_1/N_2)(g + N_2 + N_1/N_2) = 1 + N_2 + N_1/N_2 \), so \( x + N_2 + N_1/N_2 \) is an invertible element in the ring \( \frac{(R/N_2)}{N_1/N_2} \). Since \( N_1/N_2 \) is a nilpotent ideal of index \( t_1 \) in the ring \( R/N_2 \) and \( s_1 \) satisfies the hypothesis of claim 2 of Proposition 3.1, it follows that \( x + N_2 \) is invertible in \( R/N_2 \) and

\[
(x + N_2)(g(xy)^{s_1-1} + N_2) = (xg)^{s_1} + N_2 = 1 + N_2.
\]

From the isomorphism

\[
R/N_2 \cong \frac{(R/N_3)}{(N_2/N_3)},
\]  

it follows that \( (x + N_2 + N_2/N_3)(g(fg)^{s_1-1} + N_2 + N_2/N_3) = 1 + N_3 + N_2/N_3 \), so \( x + N_3 + N_2/N_3 \) is an invertible element in the ring \( \frac{(R/N_3)}{N_2/N_3} \). Since \( N_2/N_3 \) is a nilpotent ideal of index \( t_2 \) in the ring \( R/N_3 \) and \( s_2 \) satisfies the hypothesis of claim 2 of Proposition 3.1, it follows that \( x + N_3 \) is invertible in \( R/N_3 \) and

\[
(x + N_3)(g(xy)^{s_1s_2-1} + N_3) = (xg)^{s_1s_2} + N_3 = 1 + N_3.
\]

Since

\[
R/N_i \cong \frac{(R/N_{i+1})}{(N_i/N_{i+1})},
\]  

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\(x + N_{i+1}\) is invertible in \(R/N_{i+1}\) and
\[(x + N_{i+1})(g(xg)^{s_1s_2\ldots s_i-1} + N_{i+1}) = (xg)^{s_1s_2\ldots s_i} + N_{i+1} = 1 + N_{i+1}.
\]
Finally, in the last step of the chain of ideals, \(x\) is invertible in \(R\) and
\[x(g(xg)^{s_1s_2\ldots s_{k-1}-1}) = (xg)^{s_1s_2\ldots s_{k-1}-1} = 1,
\]
which implies that \(f + N_1 \subset R^*\) and \(x^{-1} = g(xg)^{s_1s_2\ldots s_{k-1}-1}\). The proof of relation (2) follows. From the isomorphism in (3) and the identity in (1),
\[|(R/N_{i+1})^*| = |(R/N_i)^*|N_i/N_{i+1}|
\]
for \(i = 1, 2, \ldots, k - 1\). Applying the previous relation \((k - 1)\) times,
\[|R^*| = |(R/N_k)^*| = |(R/N_1)^*|N_1/N_2|N_2/N_3|\cdots|N_{k-2}/N_{k-1}|N_{k-1}/N_k|.
\]
Finally, from the Lagrange theorem,
\[|R^*| = |(R/N_1)^*|N_1/|N_k| = |(R/N_1)^*|N_1|
\]
proving the result.

Remark 3.4. Observe that if \(\{N_1, N_2, \ldots, N_k\}\) is a collection of ideals of the ring \(R\) satisfying the CNC-condition, any invertible element \(x + N_1\) of the ring \(R/N_1\) is lifted up to the invertible element \(x + N_2\) of the ring \(R/N_2\). This new invertible element is lifted up to the invertible element \(x + N_3\) of the ring \(R/N_3\), and so on. At the end of this process, \(x\) is an invertible element in \(R\) with \(x^{-1} = g(xg)^{s_1s_2\ldots s_{k-1}-1}\). The following chain of ring homomorphisms,
\[R \xrightarrow{\phi_{k-1}} \frac{R}{N_{k-1}} \xrightarrow{\phi_{k-2}} \cdots \xrightarrow{\phi_3} \frac{R}{N_3} \xrightarrow{\phi_2} \frac{R}{N_2} \xrightarrow{\phi_1} \frac{R}{N_1},
\]
appears naturally in the lifting process of the invertible element \(x + N_1 \in R/N_1\).

Remark 3.5. If in Theorem 3.3 it is assumed that the ring \(R\) is commutative, the inverse \(x^{-1}\) of \(x\) in the ring \(R\) can be obtained as
\[x^{-1} = g^{s_1s_2\ldots s_{k-1}}x^{s_1s_2\ldots s_{k-1}-1},
\]
where \((x + N_1)(g + N_1) = (1 + N_1). If \((x + N_1)(g + N_1) = (1 + N_1)\) and \(g + N_1 = h + N_1\), then
\[x^{-1} = h^{s_1s_2\ldots s_{k-1}}x^{s_1s_2\ldots s_{k-1}-1}.
\]
Since \(x^{-1}\) is unique, \(g^{s_1s_2\ldots s_{k-1}} = h^{s_1s_2\ldots s_{k-1}}\). Consequently, the function
\[\psi : (R/N_1)^* \to R^*, \quad g + N_1 \to \psi(g + N_1) = g^{s_1s_2\ldots s_{k-1}},
\]
defines a group homomorphism. Furthermore, the classical isomorphism \(I : R^* \to R^*,\) that maps \(x \in R^*\) to its inverse \(x^{-1}\), takes the following explicit expression:
\[I(x) = \psi \left((x + N_1)^{-1}\right) x^{s_1s_2\ldots s_{k-1}-1}.
\]
Hence, the inverse of an invertible element \(x\) can be computed as a multiplication of a power of \(x\) by a power of an element in the equivalence class of the inverse of \(x + N_1 \in R/N_1\).
4 Applications of the main result

In this section Theorem 3.3 will be applied in order to determine the group of units of several rings which include: rings containing a nilpotent ideal; matrix ring $M_n(R)$ where $R$ is a commutative ring containing a collection of ideals satisfying the CNC-condition; group ring $RG$ where the ring $R$ contains a nilpotent ideal; group rings $RG$ where $R$ is a chain ring; the group ring $\mathbb{Z}_mG$ where $\mathbb{Z}_m$ is the ring of integers modulo $m$. Examples are given illustrating the results.

4.1 Rings containing a nilpotent ideal

In the following result, for a ring $R$ containing a nilpotent ideal $N$, starting from the set of invertible elements of the quotient ring $R/N$, the set of invertible elements of $R$ is determined.

Proposition 4.1. Let $R$ be a ring and $N$ a nilpotent ideal of nilpotency index $k \geq 2$ in $R$. Let $s > 1$ be the characteristic of the quotient ring $R/N$. Then, $f + N$ is an invertible element in $R/N$, if and only if, $f + N \subset R^*$. Furthermore for each $x \in f + N$,

$$x^{-1} = g(xg)^{(s^{k-1}-1)}$$

where $g \in R$ is such that $\bar{fg} = \bar{1}$. Moreover, $|R^*| = |(R/N)^*||N|$.

Proof: The proof of this proposition is a consequence of Theorem 3.3. First it is shown that the collection $B = \{N, N^2, \ldots, N^k\}$ of ideals of the ring $R$ satisfies the CNC-condition with nilpotency index and characteristic of the ideal $N^i$ in the ideal $N^{i+1}$ being $t_i = 2$ and $s_i = s$ for all $i = 1, 2, 3, \ldots, k - 1$. Thus,

1. It is clear that the collection $B$ satisfies the chain condition.

2. Since $(N^i)^2 = N^{2i}$ and $i + 1 \leq 2i$ for all $i = 1, 2, 3, \ldots, k - 1$, it follows that $(N^i)^2 \subset N^{i+1}$. Hence, the collection $B$ satisfies the nilpotency condition.

3. Since the ring $R/N$ has characteristic $s$, there exists $n \in N$ such that $\sum_{i=1}^{s} 1_R = n$. Since

$$sN^i = (1_R + \cdots + 1_R)N^i = nN^i \subset N^{i+1},$$

it follows that $sN^i \subset N^{i+1}$ for all $i = 1, 2, 3, \ldots, k - 1$. In addition, all prime factors of $s_i = s$ are greater or equal to the nilpotency index $t_i = 2$, proving that the collection $B$ satisfies the characteristic condition.

Theorem 3.3 implies that $f + N \subset R^*$ and the inverse of an invertible element $x \in f + N$ is $x^{-1} = g(xg)^{(s^{k-1}-1)}$. □

An application of the previous result is the following:
Corollary 4.2. Let $R$ be a commutative ring, $a \in R$ be a nilpotent element of index $k$ and $s > 1$ be the characteristic of the quotient ring $R/(a)$. Then, $f + \langle a \rangle$ is an invertible element in $R/(a)$, if and only if, $f + \langle a \rangle \subset R^*$. Furthermore for each $x \in f + \langle a \rangle$,

$$x^{-1} = g(xg)^{(s^k-1)}$$

where $g \in R$ is such that $\bar{f}g = \bar{1}$. Moreover, $|R^*| = |(R/\langle a \rangle)^*|\langle a \rangle|$.

Proof: Since $R$ is a commutative ring, $\langle a \rangle$ is a nilpotent ideal of nilpotency index $k$ in $R$, and the result follows immediately from Proposition 4.1

Remark 4.3. If in Proposition 4.1 it is assumed that the ring $R$ is commutative, the inverse $f^{-1}$ of $f \in R$ can be written as

$$f^{-1} = g^{s^k-1} f^{s^k-1-1},$$

where $g \in R$ is such that $\bar{f}g = \bar{1}$.

Example 4.4. Let

$$\mathbb{Z}_{p^k}[i] = \{a + bi : a, b \in \mathbb{Z}_{p^k}, i^2 = -1\},$$

where $p > 2$ is a prime and $k > 1$ a natural number. It is easy to see that $a = p$ is a nilpotent element of index $k$ in the ring $\mathbb{Z}_{p^k}[i]$. Since

$$\frac{\mathbb{Z}_{p^k}[i]}{\langle p \rangle} \cong \mathbb{Z}_p[i],$$

and the ring $\mathbb{Z}_p[i]$ has characteristic $s = p$, if $f + \langle p \rangle$ is a unit in $\frac{\mathbb{Z}_{p^k}[i]}{\langle p \rangle}$, Corollary 4.2 and Remark 4.3 imply that if $x \in f + \langle p \rangle$ then

$$x^{-1} = g^{s^k-1} x^{s^k-1-1},$$

and $|\mathbb{Z}_{p^k}[i]^*| = |(\mathbb{Z}_{p^k}[i]/\langle p \rangle)^*|\langle p \rangle|$. It is easy to see that for any even prime $\mathbb{Z}_p[i]$ is a finite field with $p^2$ elements. Hence $(\mathbb{Z}_p[i])^*$ is a cyclic group with $(p^2 - 1)$ elements. Therefore,

$$(\mathbb{Z}_{p^k}[i])^* = \bigcup_{u \in (\mathbb{Z}_p[i])^*} (u + \langle p \rangle),$$

with a total of $(p^2 - 1)p^k$ elements.

For example, if $p = 3$ and $k = 2$, it is not difficult to see that $(\mathbb{Z}_3[i])^* = \{1, 2, i, 2i, 1 + i, 2 + i, 1 + 2i, 2 + 2i\}$. Hence,

$$(\mathbb{Z}_{3^2}[i])^* = \bigcup_{u \in (\mathbb{Z}_3[i])^*} (u + \langle 3 \rangle),$$

with a total of $(3^2 - 1) \times 3^2 = 72$ elements.
4.2 Matrix ring

The following result describes the group of units of the matrix ring $M_n(R)$ obtained from the units of the matrix ring $M_n(R/N_1)$, where $R$ is a commutative ring containing a collection of ideals $\{N_1, \ldots, N_k\}$ satisfying the CNC-condition.

**Proposition 4.5.** Let $R$ be a commutative ring and let $M_n(R)$ be the ring of $n \times n$ matrices with entries from $R$. Let $\{N_1, N_2, \ldots, N_k\}$ be a collection of ideals of $R$ satisfying the CNC-condition, and let $s_i$ be the characteristic of the ideal $N_i$ in the ideal $N_{i+1}$. Let $f = [f_{ij}] \in M_n(R)$, then $f = [f_{ij} + N_1]$ is an invertible element in the matrix ring $M_n(R/N_1)$, if and only if, $f + M_n(N_1) \subset (M_n(R))^*$. Furthermore for each $x \in f + M_n(N_1)$,

$$x^{-1} = g(xy)^{s_1s_2 \cdots s_{k-1}-1},$$

where $g = [g_{ij}] \in M_n(R)$ is such that $f g = I$, with $I$ being the identity matrix. Moreover,

$$|(M_n(R))^*| = |(M_n(R/N_1))^*||N_1|^n.$$  \hspace{1cm} (8)

**Proof:** The proof of this proposition is a consequence of Theorem 3.3. First, we claim that the collection $B = \{M_n(N_1), M_n(N_2), \ldots, M_n(N_k)\}$ of ideals of the ring $M_n(R)$ satisfies the CNC-condition with nilpotency index and characteristic of the ideal $M_n(N_i)$ in the ideal $M_n(N_{i+1})$ being exactly the same nilpotency index and characteristic of the ideal $N_i$ in the ideal $N_{i+1}$. Indeed,

1. It is obvious that the collection $B$ satisfies the chain condition.

2. If $t_i$ denotes the nilpotency index of the ideal $N_i$ in the ideal $N_{i+1}$, then $N_i^{t_i} \subset N_{i+1}$. It is easy to see that $(M_n(N_i))^{t_i} \subset M_n(N_{i+1})$. Hence,

$$(M_n(N_i))^{t_i} \subset (M_n(N_{i+1})), $$

proving that the collection $B$ satisfies the nilpotency condition.

3. If $s_i$ is the characteristic of the ideal $N_i$ in the ideal $N_{i+1}$, then $s_i N_i \subset N_{i+1}$. It is clear that $s_i (M_n(N_i)) = M_n(s_i N_i)$. Hence,

$$s_i (M_n(N_i)) \subset M_n(N_{i+1}).$$

Now, since the collection $\{N_1, N_2, \ldots, N_k\}$ satisfies the CNC-condition, all prime factors of the characteristic $s_i$ are greater than or equal to the nilpotency index $t_i$ for all $i = 1, 2, 3, \ldots, k-1$. Thus the collection $B$ satisfies the characteristic condition.

From the isomorphism

$$\frac{M_n(R)}{M_n(N_1)} \cong M_n(R/N_1),$$  \hspace{1cm} (9)
it follows that \( \overline{f} = [f_{ij} + N_1] \) is an invertible element in the matrix ring \( M_n(R/N) \) if and only if \( f + M_n(N_1) \) is an invertible element in the ring \( M_n(R)/M_n(N_1) \). From Theorem 3.3 it follows that \( f + M_n(N_1) \subset (M_n(R))^* \) and for each \( x \in f + M_n(N_1) \),

\[
x^{-1} = g(xg)^{n_1s_2\cdots s_{k-1}1},
\]

where \( g = [g_{ij}] \in M_n(R) \) is such that \( \overline{g} = \overline{T} \). Finally, from relation (2) in Theorem 3.3 and the isomorphism given in (9), we conclude that

\[
\|(M_n(R))^*\| = \|(M_n(R/N_1))^*\|\|M_n(N_1)\|,
\]

and the proposition is proved.

The following results describe the set of invertible elements of the matrix ring \( M_n(R/N) \), where \( R \) is a commutative ring containing a nilpotent ideal \( N \).

**Corollary 4.6.** Let \( R \) be a commutative ring and let \( M_n(R) \) be the ring of \( n \times n \) matrices with entries from \( R \). Let \( N \) be a nilpotent ideal of \( R \) of index \( k \) in \( R \), and let \( s \) be the characteristic of the quotient ring \( R/N \). Let \( f = [f_{ij}] \in M_n(R) \), then \( \overline{f} = [f_{ij} + N] \) is an invertible element of the matrix ring \( M_n(R/N) \), if and only if, \( f + M_n(N) \subset (M_n(R))^* \). Furthermore for each \( x \in f + M_n(N) \),

\[
x^{-1} = g(xg)^{s_{k-1}1},
\]

where \( g = [g_{ij}] \in M_n(R) \) is such that \( \overline{g} = \overline{I} \), with \( I \) being the identity matrix. Moreover,

\[
\|(M_n(R))^*\| = \|(M_n(R/N))^*\|\|N\|^{n^2}.
\]

**Proof:** It is obvious that the collection \( \{N, N^2, \ldots, N^k\} \) of ideals of the ring \( R \) satisfies the CNC-condition with constant characteristic \( s_i = s \), for all \( i = 1, 2, 3, \ldots, k - 1 \). Thus, the result follows from Proposition 4.5.

**Corollary 4.7.** Let \( R \) be a commutative ring and \( M_n(R) \) be the ring of \( n \times n \) matrices with entries from \( R \). Let \( a \) be a nilpotent element of index \( k \) in \( R \) and \( s \) be the characteristic of the quotient ring \( R/\langle a \rangle \). Let \( f = [f_{ij}] \in M_n(R) \), then \( \overline{f} = [f_{ij} + \langle a \rangle] \) is an invertible element of the matrix ring \( M_n(R/\langle a \rangle) \), if and only if, \( f + M_n(\langle a \rangle) \subset (M_n(R))^* \). Furthermore for each \( x \in f + M_n(\langle a \rangle) \)

\[
x^{-1} = g(xg)^{s_{k-1}1},
\]

where \( g = [g_{ij}] \in M_n(R) \) is such that \( \overline{g} = \overline{I} \), with \( I \) being the identity matrix. Moreover,

\[
\|(M_n(R))^*\| = \|(M_n(R/\langle a \rangle))^*\|\|\langle a \rangle\|^{n^2}.
\]
**Proof:** It is enough to observe that the ideal $N = \langle a \rangle$ is nilpotent of index $k$ in the ring $R$. The result follows from Corollary 4.6.

In the following lines the above results will be illustrated in the context of the ring of matrices with entries in the ring $R = \mathbb{Z}_{p^k}$. First, recall that a matrix $f \in M_n(\mathbb{Z}_{p^k})$ is invertible if and only if $\det(f)$ is an invertible element of the ring $\mathbb{Z}_{p^k}$, where $\det(f)$ denotes the determinant of the matrix $f$. In this case the inverse of $f$ is given by the relation

$$f^{-1} = \det(f)^{-1} \text{adj}(f),$$

where $\text{adj}(f)$ is the adjoint matrix of $f$, that is, the transpose of its cofactor matrix. Since $p$ is a nilpotent element of index $k$ in the ring $\mathbb{Z}_{p^k}$ and $\mathbb{Z}_{p^k}/\langle p \rangle \cong \mathbb{Z}_p$ has characteristic $p$, Corollary 4.7 can be used to determine the inverse of the matrix $f$, if it exists. The following steps define an algorithm to compute the inverse of a matrix $f = [f_{ij}] \in M_n(\mathbb{Z}_{p^k})$ by using this Corollary:

1. Check if $f$ is invertible by recalling that $f = [f_{ij}]$ is invertible in $M_n(\mathbb{Z}_{p^k})$ if and only if $\overline{f} = [f_{ij} \mod (p)]$ is invertible in $M_n(\mathbb{Z}_p)$, which is equivalent to $\det(\overline{f}) \neq 0$ in $\mathbb{Z}_p$.
2. If $\overline{f}$ is invertible in $M_n(\mathbb{Z}_p)$ proceed to compute its inverse $\overline{g}$, that is, $\overline{f} \overline{g} = \overline{I} = I$, by using relation (12) or any other known procedure for this purpose.
3. Finally, compute the inverse of $f \in M_n(\mathbb{Z}_{p^k})$ by using the relation:

$$f^{-1} = g(\overline{f})^{p^k - 1 - 1}.$$  

(13)

There are some advantages to using relation (13) for computing the inverse of a matrix $f \in M_n(\mathbb{Z}_{p^k})$. If $x \in M_n(\mathbb{Z}_{p^k})$ is such that $\overline{x} = f$ and $\overline{f}$ is invertible, then $x$ is invertible and the inverse of $x$ can be computed as $x^{-1} = g(x \overline{g})^{p^k - 1 - 1}$, where $\overline{g}$ denotes the inverse of $\overline{f}$. Hence, when $\overline{x} = \overline{f}$, $f$ is invertible and the inverse $\overline{g}$ of $\overline{f}$ has been computed, in order to compute $x^{-1}$, so steps 1 and 2 in the algorithm described above can be avoided.

As mentioned above, the invertibility of a matrix $f \in M_n(\mathbb{Z}_{p^k})$ can be analyzed directly by computing $\det(f)$ in the ring $\mathbb{Z}_{p^k}$, or indirectly by computing $\det(\overline{f})$ in the field $\mathbb{Z}_p$. Since the number of entries equal to zero in the matrix $\overline{f}$ could be greater than the number of entries equal to zero of the matrix $f$, the computation of $\det(f)$ could be easier than the computation of $\det(\overline{f})$.

**Example 4.8.** Let $f \in M_3(\mathbb{Z}_{3^3})$ be given by

$$f = \begin{bmatrix} 19 & 12 & 22 \\ 6 & 5 & 24 \\ 0 & 16 & 11 \end{bmatrix} \quad \text{then,} \quad \overline{f} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$  

(14)
It is easy to see that \( \det(f) = 1 \), so \( f \) is invertible in \( M_3(\mathbb{Z}_3) \) and likewise, it is easy to check that \( g \), the inverse of \( f \) is given by:

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 2 & 0 \\
0 & 2 & 2 \\
\end{pmatrix}.
\]  

(15)

In order to determine \( f^{-1} \), using relation (13) with \( p = 3 \), \( k = 3 \), \( f \) given in (14) and \( g = \bar{g} \) given in (15), we find that

\[
f^{-1} = g(fg)^8 = g \begin{bmatrix} 19 & 6 & 9 \end{bmatrix}^8 = g \begin{bmatrix} 19 & 21 & 18 \end{bmatrix} = \begin{bmatrix} 13 & 22 & 7 \\ 15 & 2 & 0 \end{bmatrix},
\]

this finish our example.

Next, it is discussed how to compute the inverse of an invertible matrix in \( M_n(\mathbb{Z}_m) \). Let \( m = p_1^{k_1}p_2^{k_2}\ldots p_j^{k_j} \) be a positive integer with its prime factorization. It is obvious that the function

\[
\Psi : M_n(\mathbb{Z}_m) \rightarrow M_n(\mathbb{Z}_{p_1^{k_1}}) \times \cdots \times M_n(\mathbb{Z}_{p_j^{k_j}}),
\]

\[
\Psi(f) = (f_1, \ldots, f_j), \quad f_i = f \mod (p_i^{k_i})
\]

induced by the function given by the Chinese Remainder Theorem (CRT) on \( \mathbb{Z}_m \), is a ring isomorphism. For \( m_i = m/p_i^{k_i} \) and \( s_i \) such that \( s_im_i = 1 \mod (p_i^{k_i}) \), the inverse of \( \Psi \) is given by

\[
\Psi^{-1}(h_1, \ldots, h_j) = s_1m_1h_1 + \cdots + s_jm_jh_j.
\]

Moreover, a matrix \( f \) is invertible in \( M_n(\mathbb{Z}_m) \) if and only if each of the coordinates \( f_i \) of \( f \) is invertible in \( M_n(\mathbb{Z}_{p_i^{k_i}}) \). Finally, if \( f \) is invertible, the inverse of \( f \) can be computed by using the relation

\[
f^{-1} = s_1m_1f_1^{-1} + \cdots + s_jm_jf_j^{-1},
\]

where \( f_i^{-1} = g_i(f_if_i)_{(p_i^{k_i-1})} \) and \( \bar{g}_i \) is the inverse of the matrix \( f_i \) in the ring \( M_n(\mathbb{Z}_{p_i}) \).

### 4.3 Group rings

If \( R \) is a ring containing a collection of ideals satisfying the CNC-condition and \( G \) is a group, by invoking Theorem 3.3, the inverse of the units of the group ring \( RG \) are determined as follows:

**Proposition 4.9.** Let \( R \) be a ring and \( G \) a group. Let \( \{N_1, N_2, \ldots, N_k\} \) be a collection of ideals of \( R \) satisfying the CNC-condition with \( s_i \) being the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \). Then, \( f + N_1G \) is an invertible element in the group ring \( (R/N_1)G \), if and only if, \( f + N_1G \subset (RG)^* \). Furthermore for \( x \in f + N_1G \),

\[
x^{-1} = g(xg)^{s_1s_2\ldots s_{k-1}-1},
\]

13
where \( g \in RG \) is such that \((f + N_1G)(g + N_1G) = 1 + N_1G\). Moreover, \(|(RG)^*| = |N_1|^{[G]}|((R/N_1)G)^*|\).

**Proof:** We claim that the collection \( B = \{N_1G, N_2G, \ldots, N_kG\} \) of ideals of the group ring \( RG \) satisfies the CNC-condition with nilpotency index and characteristic of the ideal \( N_iG \) in the ideal \( N_{i+1}G \) being the same nilpotency index and characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \). Indeed,

1. It is clear that the collection \( B \) satisfies the chain condition.
2. If \( t_i \) denotes the nilpotency index of the ideal \( N_i \) in the ideal \( N_{i+1} \), then \( N_i^{t_i} \subset N_{i+1} \).
   It is not difficult to see that \((N_iG)^{t_i} = N_i^{t_i}G\). Hence,
   \[(N_iG)^{t_i} = N_i^{t_i}G \subset N_{i+1}G,\]
   proving that the collection \( B \) satisfies the nilpotency condition.
3. If \( s_i \) is the characteristic of the ideal \( N_i \) in the ideal \( N_{i+1} \), then \( s_iN_i \subset N_{i+1} \).
   It is clear that \( s_i(N_iG) = (s_iN_i)G \). Hence,
   \[s_i(N_iG) = (s_iN_i)G \subset N_{i+1}G.\]
   Since the collection \( \{N_1, N_2, \ldots, N_k\} \) satisfies the CNC-condition, it is obvious that all prime factors of the characteristic \( s_i \) are greater than or equal to the nilpotency index \( t_i \) for all \( i = 1, 2, 3, \ldots, k - 1 \). Thus the collection \( B \) satisfies the characteristic condition.

From Theorem 3.3 and the following isomorphism,
\[
\frac{RG}{N_1G} \cong \left( \frac{R}{N_1} \right) G,
\]
it follows that \( f + N_1G \subset (RG)^* \), the inverse of an invertible element \( x \in f + N_1G \) is \( x^{-1} = g(xg)^{s_1s_2\cdots s_k-1} \), and \(|(RG)^*| = |N_1|^{[G]}|((R/N_1)G)^*|\). \( \blacksquare \)

**Remark 4.10.** If it is assumed in Proposition 4.9 that the ring \( R \) and the group \( G \) are commutative, the inverse \( f^{-1} \) of \( f \in RG \) can be expressed as
\[
f^{-1} = g^{s_1s_2\cdots s_k-1}f^{(s_1s_2\cdots s_k-1)-1},
\]
where \( g \in RG \) is such that \((f + N_1G)(g + N_1G) = 1 + N_1G\).

**Corollary 4.11.** Let \( R \) be a ring, \( N \) a nilpotent ideal of index \( k \) in \( R \), \( G \) a group and \( s \) the characteristic of the quotient ring \( R/N \). Then, \( f + NG \) is an invertible element in the group ring \( (R/N)G \), if and only if, \( f + NG \subset (RG)^* \). Furthermore for each \( x \in f + NG \),
\[x^{-1} = g(xg)^{s^{-1}-1},\]
where \( g \in RG \) is such that \((f+NG)(g+NG) = 1+NG\). Moreover, \(|(RG)^*| = |N|^{[G]}|((R/N)G)^*|\).
**Proof:** It readily seen that the collection \( \{N, N^2, \ldots, N^k\} \) of ideals of the ring \( R \) satisfies the CNC-condition with constant characteristic \( s_i = s \) for all \( i = 1, 2, 3, \ldots, k - 1 \). The result follows from Proposition 4.9.

**Example 4.12.** Considering the group ring \( M_n(\mathbb{Z}_9)S_3 \), where \( S_3 \) is the symmetric group in three symbols. Observe that \( N = \langle 3 \rangle \) is a nilpotent ideal of index 2 in the ring \( \mathbb{Z}_9 \), so the collection \( \{M_n(N)S_3, M_n(N^2)S_3\} \) of ideals of the ring \( M_n(\mathbb{Z}_9)S_3 \) satisfies the CNC-condition with constant characteristic \( s = 3 \). Now, for \( \sigma = (123) \in S_3 \) and \( I \) the identity matrix in \( M_n(\mathbb{Z}_9) \), the element \( f = 2I + 8I\sigma \) is an invertible element in the group ring \( M_n(\mathbb{Z}_9)S_3 \) and \( \bar{g} = I + 2I\sigma + I\sigma^2 \in M_n(\mathbb{Z}_9/N)S_3 \) is such that \( \bar{f}\bar{g} = 1 \) in \( M_n(\mathbb{Z}_9/N)S_3 \), by Corollary 4.11, it follows that \( f + M_n(N)S_3 \subset (M_n(\mathbb{Z}_9)S_3)^* \) and \( x^{-1} = g(xg)^2 \), where \( x \) denotes any element in \( f + M_n(N)S_3 \). For instance, since \( f \in f + M_n(N)S_3 \), the inverse of \( f \) in the group ring \( M_n(\mathbb{Z}_9)S_3 \) is given by

\[
 f^{-1} = g(fg)^2 = 7I + 8I\sigma + 4I\sigma^2.
\]

In the same fashion, since \( x = 2I + 2I\sigma \in f + M_n(N)S_3 \), then

\[
 x^{-1} = g(xg)^2 = 4I + I\sigma + 4I\sigma^2,
\]

is the inverse of \( x \) in the group ring \( M_n(\mathbb{Z}_9)S_3 \).

**Corollary 4.13.** Let \( R \) be a commutative ring, let \( a \) be a nilpotent element of index \( k \) in \( R \), \( G \) a group and \( s \) the characteristic of the quotient ring \( R/(a) \). Then, \( f + (a)G \) is an invertible element in the group ring \( (R/(a))G \), if and only if, \( f + (a)G \subset ((R/(a))G)^* \). Furthermore for each \( x \in f + (a)G \),

\[
 x^{-1} = g(xg)^{s^{-k}-1},
\]

where \( g \in RG \) is such that \( (f + (a)G)(g + (a)G) = 1 + (a)G \). Moreover, \( |(RG)^*| = |(a)| |G| |((R/(a))G)^*| \).

**Proof:** Since \( R \) is a commutative ring, the ideal \( N = \langle a \rangle \) is a nilpotent ideal of index \( k \) in \( R \), and the result follows from Corollary 4.11.

**Remark 4.14.** If it is assumed in Corollaries 4.11 and 4.13 that the ring \( R \) and the group \( G \) are both commutative, then the inverse \( f^{-1} \) of \( f \in RG \) can be expressed as:

\[
 f^{-1} = g^{s^{-k}} f^{s^{-k}-1},
\]

where \( g \in RG \) is such that \( \bar{f}\bar{g} = \bar{1} \).

### 4.4 Commutative group rings \( RG \) with \( R \) a chain ring

Let \( R \) be a commutative chain ring and \( G \) a commutative group. It is well known that \( R \) contains a unique maximal nilpotent ideal \( N = \langle a \rangle \) for some \( a \in R \). If \( k \) denotes the
nilpotency index of \(a\), and \(p\) denotes the characteristic of the residue field \(\mathbb{F} = R/(a)\), from

Corollary 4.13 it follows that

\[
(RG)^* = \left\{ g^r f^{r-1} : r = p^{k-1}, \bar{f}, \bar{g} \in (\mathbb{F}G)^* \text{ and } \bar{f}\bar{g} = \bar{1} \right\},
\]

and

\[
|(RG)^*| = |(\mathbb{F}G)^*| |(a)|^{[G]}.
\]

Examples of finite commutative chain rings include the ring of modular integers \(R = \mathbb{Z}_{p^k}\), where \(p\) is a prime number and \(k > 1\) is an integer. In this example the maximal nilpotent ideal is \(N = \langle p \rangle\), with nilpotency index equal to \(k\) in \(\mathbb{Z}_{p^k}\), and \(\mathbb{F} \cong \mathbb{Z}_p\). Thus,

\[
(Z_{p^k}G)^* = \left\{ g^r f^{r-1} : r = p^{k-1}, \bar{f}, \bar{g} \in (\mathbb{Z}_pG)^* \text{ and } \bar{f}\bar{g} = \bar{1} \right\}.
\]

If the group \(G\) is cyclic of order \(n\), it is known that \(\mathbb{Z}_{p^k}G \cong \mathbb{Z}_{p^k}[x]/\langle x^n - 1 \rangle\), then (19) can be rewritten in the following equivalent form,

\[
(\mathbb{Z}_{p^k}[x]/\langle x^n - 1 \rangle)^* = \left\{ g^r f^{r-1} : r = p^{k-1}, \bar{f}, \bar{g} \in (\mathbb{Z}_p[x]/\langle x^n - 1 \rangle)^* \text{ and } \bar{f}\bar{g} = \bar{1} \right\}.
\]

Another interesting example of chain rings is represented by Galois rings, \(R = \mathbb{Z}_{p^k}[x]/\langle q(x) \rangle\) where \(p\) is a prime number, \(k > 1\) and \(q(x)\) is a monic polynomial of degree \(r\) whose image in \(\mathbb{Z}_p[x]\) is irreducible. In this example the maximal nilpotent ideal is \(N = pR\), with nilpotency index equal to \(k\) in \(R\), and \(\mathbb{F} = R/N \cong \mathbb{F}_p^r\). Thus,

\[
((\mathbb{Z}_{p^k}[x]/\langle q(x) \rangle)G)^* = \left\{ g^r f^{r-1} : r = p^{k-1}, \bar{f}, \bar{g} \in (\mathbb{F}_p^rG)^* \right\}.
\]

**Example 4.15.** Consider the group ring \(\mathbb{Z}_{25}C_5\), where \(C_5\) is the cyclic group of order five generated by \(a\). Observe that \(N = \langle 5 \rangle\) is a nilpotent ideal of index \(2\) in the ring \(\mathbb{Z}_{25}\), so the collection \(\{NC_5, N^2C_5\}\) of ideals of the ring \(\mathbb{Z}_{25}C_5\) satisfies the CNC-condition with constant characteristic \(s = 5\). Now, since the element \(f = 2 - a = 2 + 4a\), is an invertible element in the group ring \(\mathbb{Z}_{25}C_5\) and \(g = 1 + 3a + 4a^2 + 2a^3 + a^4 \in (\mathbb{Z}_{25}/N)C_5\) is such that \(\bar{f}\bar{g} = 1\) in \((\mathbb{Z}_{25}/N)C_5\), from Corollary 4.17 it follows that \(f + NC_5 \subset (\mathbb{Z}_{25}C_5)^*\) and \(f^{-1} = (fg)^4\). Thus, \(fg = 1 + 5a + 5a^2\), \((fg)^4 = 1 + 20a + 20a^2\) and finally

\[
f^{-1} = g(fg)^4 = 11 + 18a + 9a^2 + 17a^3 + 21a^4
\]

is the inverse of \(f\) in the group ring \(\mathbb{Z}_{25}C_5\).

### 4.5 Commutative group rings \(\mathbb{Z}_mG\) with \(G\) a commutative group

The previously discussed results for \(\mathbb{Z}_{p^k}G\) can be extended to the group ring \(\mathbb{Z}_mG\), where \(m > 1\) and \(G\) is a finite commutative group.

**Theorem 4.16.** Let \(m = p_1^{r_1}p_2^{r_2}\cdots p_j^{r_j}\) be the prime factorization of the integer \(m \geq 2\). Set \(m_i = m/p_i^{r_i}\) and let \(s_i\) be a natural number such that \(s_im_i = 1 \mod (p_i^{r_i})\) for \(i = 1, 2, 3, \ldots, j\). If \(\bar{f}_i\) is an invertible element of the group ring \(\mathbb{Z}_{p_i}G\) for \(i = 1, 2, 3, \ldots, j\), then

\[
f = s_1m_1f_1 + s_2m_2f_2 + \cdots + s_jm_jf_j,
\]
is an invertible element in the group ring $\mathbb{Z}_m G$, with
\[
f^{-1} = s_1 m_1 g_1 f_1^{\alpha_1 - 1} + s_2 m_2 g_2 f_2^{\alpha_2 - 1} + \cdots + s_j m_j g_j f_j^{\alpha_j - 1},
\] (22)
where $\alpha_i = p_i^{r_i - 1}$ and $\bar{g}_i \in \mathbb{Z}_{p_i} G$ is such that $\bar{f}_i \bar{g}_i = 1$. Moreover,
\[
|\langle \mathbb{Z}_m G \rangle^*| = (m/(p_1 p_2 \cdots p_j))^G |\langle \mathbb{Z}_{p_1} G \rangle^*| |\langle \mathbb{Z}_{p_2} G \rangle^*| \cdots |\langle \mathbb{Z}_{p_j} G \rangle^*|.
\] (23)

**Proof:** From the Chinese Remainder Theorem (CRT), $\mathbb{Z}_m \cong \mathbb{Z}_{p_1^r_1} \times \mathbb{Z}_{p_2^r_2} \times \cdots \times \mathbb{Z}_{p_r^r_j}$, and
\[
\mathbb{Z}_m G \cong \mathbb{Z}_{p_1^r_1} G \times \mathbb{Z}_{p_2^r_2} G \times \cdots \times \mathbb{Z}_{p_r^r_j} G.
\]
Therefore,
\[
(\mathbb{Z}_m G)^* \cong \mathbb{Z}_{p_1^r_1} G^* \times \mathbb{Z}_{p_2^r_2} G^* \times \cdots \times \mathbb{Z}_{p_r^r_j} G^*.
\] (24)

If $\bar{f}_i$ is an invertible element in $\mathbb{Z}_{p_i} G$, from relation (19), $f_i$ is an invertible element in the group ring $\mathbb{Z}_{p_i^r_i} G$ with $f_i^{-1} = g_i^{\alpha_i} f_i^{\alpha_i - 1}$, where $\bar{f}_i \bar{g}_i = 1$ for each $i = 1, 2, 3, \cdots, j$. Thus, $h = (f_1, f_2, \cdots, f_j)$ is an invertible element in the product group given in (24), with
\[
h^{-1} = (g_1^{\alpha_1} f_1^{\alpha_1 - 1}, g_2^{\alpha_2} f_2^{\alpha_2 - 1}, \cdots, g_j^{\alpha_j} f_j^{\alpha_j - 1}).
\]
Consequently, $f = \phi^{-1}(h)$ is an invertible element in the group ring $\mathbb{Z}_m G$, with $f^{-1} = \phi^{-1}(h^{-1})$. Finally, from the CRT, $f$ and $f^{-1}$ can be expressed in the form (21) and (22) respectively. The equality in (23) follows from (18) and (24).

The following result provides an alternative way to compute the invertible elements in $\mathbb{Z}_m G$.

**Theorem 4.17.** Let $m = p_1^{r_1} p_2^{r_2} \cdots p_j^{r_j}$ be the prime factorization of the integer $m \geq 2$. Set $k = \max\{r_1, r_2, \ldots, r_j\}$, $c_i = (p_1 p_2 \cdots p_j)/p_i$, and let $t_i$ be a natural number such that $t_i c_i = 1 \mod (p_i)$ for $i = 1, 2, 3, \cdots, j$. If $\bar{f}_i$ is an invertible element of the group ring $\mathbb{Z}_{p_i} G$ for $i = 1, 2, 3, \ldots, j$, then
\[
f = t_1 c_1 f_1 + t_2 c_2 f_2 + \cdots + t_j c_j f_j,
\] (25)
is an invertible element in the group ring $\mathbb{Z}_m G$, with
\[
f^{-1} = (t_1 c_1 g_1 + t_2 c_2 g_2 + \cdots + t_j c_j g_j)^{(p_1 p_2 \cdots p_j)^k - 1} (t_1 c_1 f_1 + t_2 c_2 f_2 + \cdots + t_j c_j f_j)^{(p_1 p_2 \cdots p_j)^{k-1} - 1},
\] (26)
where $\bar{g}_i \in \mathbb{Z}_{p_i} G$ is such that $\bar{f}_i \bar{g}_i = 1$. Moreover,
\[
|\langle \mathbb{Z}_m G \rangle^*| = (m/(p_1 p_2 \cdots p_j))^G |\langle \mathbb{Z}_{p_1 p_2 \cdots p_j} G \rangle^*|.
\] (27)

**Proof:** If $\bar{f}_i$ is an invertible element in the group ring $\mathbb{Z}_{p_i} G$ for $i = 1, 2, 3, \cdots, j$, from Theorem 4.16 it follows that
\[
f = t_1 c_1 f_1 + t_2 c_2 f_2 + \cdots + t_j c_j f_j
\] (28)
is an invertible element in the group ring $\mathbb{Z}_{p_1p_2 \cdots p_j} G$, with

$$w := f^{-1} = t_1c_1g_1 + t_2c_2g_2 + \cdots + t_jc_jg_j,$$  

(29)

where $\tilde{g}_i \in \mathbb{Z}_{p_i} G$ is such that $\tilde{f}_i \tilde{g}_i = \tilde{1}$. By observing that $a = p_1p_2 \cdots p_j$ has nilpotency index $k$ in the ring $\mathbb{Z}_m$ and,

$$\frac{\mathbb{Z}_m}{a\mathbb{Z}_m} \cong \mathbb{Z}_a,$$  

(30)

has characteristic $a = p_1p_2 \cdots p_j$, from Corollary 4.13 it follows that, $f$ is an invertible element in $\mathbb{Z}_m G$, with

$$f^{-1} = w(p_1p_2 \cdots p_j)^{k-1} f(p_1p_2 \cdots p_j)^{k-1}-1.$$ 

Relation (27) follows from Corollary 4.13 and (30).

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