1. Introduction and Foundations

Let \( p \) be an odd prime number, \( D \geq 2 \) a \( p \)-th power free integer, and \( \zeta_p \) be a primitive \( p \)-th root of unity. The coarse cohomology types of pure metacyclic fields \( N = \mathbb{Q}(\zeta_p, \sqrt[D]{D}) \), which are cyclic Kummer extensions of the cyclotomic field \( K = \mathbb{Q}(\zeta_p) \) with relative automorphism group \( G = \text{Gal}(N/K) = \langle \sigma \rangle \), are based on the Galois cohomology of the unit group \( U \) viewed as a \( G \)-module. The primary invariant is the group \( H^0(G, U_N) \simeq U_K/N_{N/K}(U_N) \) of order \( p^U \) with \( 0 \leq U \leq \frac{p-1}{2} \) which is related to the group \( H^1(G, U_N) \simeq (U_N \cap \ker(N_{N/K}))/U_N^{1-\sigma} \) of order \( p^P \) by the Takagi/Hasse/Iwasawa Theorem on the Herbrand quotient of \( U_N \):

\[
\#H^1(G, U_N) = \#H^0(G, U_N) \cdot [N : K], \quad \text{respectively } P = U + 1.
\]

The secondary invariant is a natural decomposition of the group \( H^1(G, U_N) \simeq \mathcal{P}_{N/K}/\mathcal{P}_K \) of primitive ambiguous principal ideals of \( N/K \), which can be viewed as principal ideals dividing the relative different \( \mathcal{D}_{N/K} \) and are therefore called differential principal factors (DPF) of \( N/K \):

\[
\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \mathcal{P}_{L/Q}/\mathcal{P}_Q \times ((\mathcal{P}_{N/K}/\mathcal{P}_K) \cap \ker(N_{N/L})) \text{, respectively } U + 1 = A + R,
\]

where \( L = \mathbb{Q}(\sqrt[D]{D}) \) denotes the real non-Galois pure subfield of degree \( p \) of \( N \). The subgroup of absolute DPF, \( \mathcal{P}_{L/Q}/\mathcal{P}_Q \), is of order \( p^A \), and the subgroup of relative DPF (the norm kernel), \( (\mathcal{P}_{N/K}/\mathcal{P}_K) \cap \ker(N_{N/L}) \), is of order \( p^R \). We present \( p = 7 \) in comparison to \( p \in \{3, 5\} \).

2. Pure Septic Fields

For pure septic fields \( L = \mathbb{Q}(\sqrt[7]{D}) \) and their Galois closure \( N = \mathbb{Q}(\zeta_7, \sqrt[7]{D}) \), that is the case \( p = 7 \), the coarse classification of \( N \) according to the invariants \( U \) and \( A \) alone is illustrated in Fig. 1. The coarse types are \( \alpha, \beta, \gamma, \delta \) with \( U = 3 \), \( \varepsilon, \zeta, \eta \) with \( U = 2 \), \( \vartheta, \iota \) with \( U = 1 \), and \( \kappa \) with \( U = 0 \). The possibility that the primitive seventh root of unity \( \zeta_7 \) occurs as the relative norm \( N_{N/K}(Z) \) of a unit \( Z \in U_N \) will cause a splitting of all types with \( 1 \leq U \leq 2 \), similar to the splitting into \( \delta/\zeta \) and \( \varepsilon/\eta \) in the pure quintic case of §3. Due to the existence of radicals in the pure septic field, the \( \mathbb{F}_7 \)-dimension \( A \) of the vector space of absolute DPF is at least one: \( 1 \leq A \leq 4 \).

![Figure 1. Classification of pure septic fields](image-url)
3. Pure Quintic Fields

For pure quintic fields \( L = \mathbb{Q}(\sqrt[5]{D}) \) and their Galois closure \( N = \mathbb{Q}(\zeta_5, \sqrt[5]{D}) \), that is the case \( p = 5 \), the coarse classification of \( N \) according to the invariants \( U \) and \( A \) alone is closely related to the classification of totally real dihedral fields by Nicole Moser [13, Thm. III.5, p. 62], as illustrated in Figure 2. The coarse types \( \alpha, \beta, \gamma, \delta, \varepsilon \) are completely analogous in both cases. Additional types \( \zeta, \eta, \vartheta \) are required for pure quintic fields, because there arises the possibility that the primitive fifth root of unity \( \zeta_5 \) occurs as the relative norm \( N_{N/K}(Z) \) of a unit \( Z \in U_N \). Due to the existence of radicals in the pure quintic case, the \( \mathbb{F}_p \)-dimension \( A \) of the vector space of absolute DPF exceeds the corresponding dimension for totally real dihedral fields by one [8, 10, 11].

![Figure 2. Classification of totally real dihedral and pure quintic fields](image)

4. Pure Cubic Fields

For pure cubic fields \( L = \mathbb{Q}(\sqrt[3]{D}) \) and their Galois closure \( N = \mathbb{Q}(\zeta_3, \sqrt[3]{D}) \), that is the case \( p = 3 \), the coarse classification of \( N \) according to the invariants \( U \) and \( A \) alone is closely related to the classification of simply real dihedral fields by Nicole Moser [13, Dfn. III.1 and Prop. III.3, p. 61], as illustrated in Figure 3. The coarse types \( \alpha \) and \( \beta \) are completely analogous in both cases. The additional type \( \gamma \) is required for pure cubic fields, because there arises the possibility that the primitive cube root of unity \( \zeta_3 \) occurs as the relative norm \( N_{N/K}(Z) \) of a unit \( Z \in U_N \). Due to the existence of radicals in the pure cubic case, the \( \mathbb{F}_p \)-dimension \( A \) of the vector space of absolute DPF exceeds the corresponding dimension for simply real dihedral fields by one [11, 24, 7].

![Figure 3. Classification of simply real dihedral and pure cubic fields](image)

5. Common Features

In all sections, §§2, 3, and 4 the symbol \( \bullet \) indicates a fine structure splitting of the remaining \( \mathbb{F}_p \)-dimension \( R = U + 1 - A \) into relative DPF, and either capitulation or intermediate DPF.
6. Conclusion

The purpose of this brief note was to present the fundamental ideas for a classification of pure septic fields $L = \mathbb{Q}(\sqrt[7]{D})$ and their Galois closures $N = \mathbb{Q}(\zeta_7, \sqrt[7]{D})$. Figure 1 illustrates the increase of complexity in comparison with the pure quintic situation in Figure 2: we shall have 10 coarse DPF types instead of only 6 (neglecting the quintic splitting of $\delta/\zeta$ and $\zeta/\eta$).

Our theory of fine DPF types, as developed in \[3, 9\] for $p = 5$, showed the crucial impact of splitting prime divisors of the conductor $f$ of $N/K$ on the possibility of DPF types with non-maximal extent of absolute principal factorizations $A < U + 1$, which will appear in aggravated form for $p \geq 7$.

On the other hand, splitting prime divisors of $f$ have been proved to enforce non-trivial $p$-class numbers of $L$ and $N$: according to Ishida \[3\], a prime divisor $\ell \equiv +1 \pmod{p}$ of $f$ implies $p \mid h_L$ and $p \mid h_N$, for any $p \geq 3$. Such a prime divisor $\ell$ splits completely in the cyclotomic field $K$, that is, into $p - 1$ prime ideals. More recently, Kobayashi \[4, 5\] has proved that a prime divisor $\ell \equiv -1 \pmod{5}$ of $f$ implies $5 \mid h_L$ and $5 \mid h_N$ and he conjectures the truth of this behavior for $p \geq 7$. Such a prime divisor $\ell$ splits into prime ideals of $K$. Therefore, we were surprised that other splitting prime divisors $\ell$ of $f$, whose occurrence starts with $p = 7$, do not exert such severe constraints on class numbers, and we conclude with the following interesting proven phenomenon.

**Theorem 6.1.** Let $L = \mathbb{Q}(\sqrt[7]{D})$ be a pure septic field with 7-th power free radicand $D > 1$ and Galois closure $N = \mathbb{Q}(\zeta_7, \sqrt[7]{D})$. If $D = \ell \equiv 2, 4 \pmod{7}$ is a prime radicand, then it splits into $\frac{7 - 1}{2} = 2$ prime ideals of $K = \mathbb{Q}(\zeta_7)$. If the radicand belongs to the range $2 \leq D < 200$, then $\ell$ causes relative principal factorizations in the norm kernel $(P_{N/K}/P_K) \cap \ker(N_{N/L})$, but $h_L$ and $h_N$ are not divisible by $7$.

**Proof.** By direct investigation with the aid of the computer algebra system Magma \[6\]. Explicitly, the radicands are $D \in \{2, 11, 23, 37, 53, 67, 79, 107, 109, 137, 149, 151, 163, 179, 191, 193\}$. \hfill \Box

7. Acknowledgements

We gratefully acknowledge that our research was supported by the Austrian Science Fund (FWF): projects J 0497-PHY and P 26008-N25.

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