Estimates of Constants in Error Estimates for $H^2$ Conforming Finite Elements for Regularized Nonlinear Elliptic Geometric Evolution Equations and Question of Optimality

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Abstract. Geometric evolution equations in level set form are usually singular and a well-known regularization procedure generates a family of approximating non-singular equations (e.g. useful for analytical or numerical aspects). In the previous work [13] upper bounds for constants which appear in the standard finite element error estimates for elliptic regularized geometric evolution equations in dependence on the regularization parameter have been addressed and an exponential relation in the inverse regularization parameter has been observed. In this paper the aim is twofold: First, we extend the results from [13] to $H^2(\Omega)$ conforming approaches which are of interest in the special case of higher regularity of the solution in order to detect level sets, and second, we present a strong indication that the previously mentioned exponential bound which carries over to the higher conforming case is optimal (independent from the degree of conformity). This is in accordance with practical experience in own previous work and suggests at least in the special case of mean curvature flow that a parabolic approach is preferable (in order to get better FE error estimates).

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1. Introduction

This paper extends the finite element error analysis for regularized geometric evolution equations considered in [13]. We are concerned with a finite element approximation formulated in Feng, Neilan, and Prohl [6] for the following family of equations

\[
\begin{aligned}
\text{div} \left( \frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}} \right) - \eta \left( \sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} \right) &= 0 \quad \text{in } \Omega, \\
u^\varepsilon &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

with

\[
\eta(r) := \sigma r^\alpha
\]

which was introduced in Huisken and Ilmanen [9] and Schulze [14] in order to approximate weak solutions of the inverse mean curvature flow and the flow by powers of the mean curvature, respectively. Before we formulate the issue under consideration in detail, we describe the equations more precisely. The parameter \( \varepsilon \) is a positive regularization parameter which prevents the equations from becoming singular. We assume that \( \Omega \subset \mathbb{R}^3 \) is an open, bounded domain with a smooth boundary of positive mean curvature. The equation in case \( \varepsilon = 0 \) models (assuming sufficient regularity) level set motion (in the sense that the evolving surface at time \( t \geq 0 \) is given as the \( t \)-level set of \( u^0 \)) with normal speed given by a power of the mean curvature, so that we have in the case \( \sigma = 1 \) and \( \alpha = 1 \) inverse mean curvature flow and in the case \( \sigma = -1 \) and \( \alpha = -1/k, \ k \geq 1 \), a contracting flow by a power \( k \) of the mean curvature (this includes mean curvature flow for \( k = 1 \)). Motivated by these applications for the rest of the paper we will always stay in these two cases for the regimes of \( \sigma \) and \( \alpha \). In the case \( \sigma = -1 \) and \( \alpha = -1/k, \ k \geq 1 \), equation (1.1) has smooth solutions for each sufficiently small \( \varepsilon > 0 \), cf. [14]. Assuming that \( u^\varepsilon \) converges to a continuous \( u \) as \( \varepsilon \to 0 \) which satisfies equation (1.1) with \( \varepsilon = 0 \) in a viscosity sense (such a convergence is available in the case of the flow by a power of the mean curvature [14]), then, when aiming to solve equation (1.1) computationally with \( \varepsilon = 0 \), it is tempting and reasonable to circumvent the possible singularity of the equation by approximating it with (1.1) for \( \varepsilon > 0 \) small and fixed. We remark that in the case \( \sigma = \alpha = 1 \) the model equation (1.1) is a simplified case of the approximating problems for a weak formulation of inverse mean curvature flow in [9]. Therein the weak solutions of the inverse mean curvature flow are approximated by problems of type (1.1) in which in addition to \( \varepsilon \) also the domain and the boundary values serve as a parameter in the approximation process and a certain minimization property is required in order to define weak solutions leading to a more complicated setting. Numerical error bounds which take these general approximating problems into account are not content of this paper, instead we focus on the simplified (not realistic in the case of the parameters chosen within the inverse mean curvature flow regime) case given by (1.1) in combination with \( \varepsilon \) as the only parameter at first. In [6] from where we take the numerical scheme also numerical analysis only for this case is addressed. Therein the authors report about relevance of analyzing constants in FE error estimates (especially in the most general inverse mean curvature flow setting) and about some numerical experiences in special cases concerning the constants in error estimates. While the authors therein expect a polynomial growth of the constants in the inverse regularization.
parameter, our own experimental work [11] suggests a stronger growth which is in accordance with the exponential bound in the inverse regularization parameter from [13] and the present paper. Furthermore, in the final section of the present paper we give strong indication that our bound is optimal.

Let us explain the background and contribution of our paper. The work [6] shows $W^{1,p}$- and $L^p$-error bounds being of type of a product of a constant which depends in an unknown way on $\varepsilon$ and the usually expected powers of the discretization parameter. In previous work [13] we were able to derive an estimate of these constants in form of an upper bound $\varepsilon^{P(1/\varepsilon)}$ provided the finite elements are piecewise cubic, globally $H^1$ conform, the space dimension is $n + 1 = 3$ and by using an appropriate boundary approximation. The contribution of this paper is twofold. First, we extend the results from [13] to an $H^2$–conforming approach and second, we present a strong indication that one can not expect a better dependence of the constants in the numerical estimates on the inverse regularization parameter $\varepsilon$ than exponential. Note that $H^2$–conformity is, at least in the two-dimensional case, a well-established approach; cf. the Argyris element or the Clough-Tocher element for the biharmonic equation; we remark, nevertheless also nonconforming approaches have shown their relevance in numerical computations. The resulting error estimates in correspondingly higher order Sobolev norms which in turn lead to $C^1$ error estimates via embedding theorems are very reasonable from the viewpoint of modeling evolving level sets. Evolving level sets cannot be detected from lower order error estimates. Explicit dependencies of the constants on the regularization parameter are needed when putting discretization error estimates together with pure regularization error estimates to a full error estimate. Estimates for the regularization errors can be found in [12] for the flow by powers of the mean curvature and in [11] in a crucially simplified rotationally symmetric case for the inverse mean curvature flow, see also [13] for a full error estimate in the $C^0$ norm in certain cases. Note that $C^0$ is from the viewpoint of viscosity solutions a natural norm and that our higher order FE error estimates contribute usefully only to a full error estimate if the regularization error estimate (and the regularity of the solution) is of sufficiently high order; but they are in any case of own interest.

The quantitative convergence results for approximations of geometric evolution equations (with respect to the result itself and rather less concerning the methods and the technical level) developed in this paper as well as previous papers [11, 12, 13] of the authors are inspired by the works Crandall and Lions [3] and Deckelnick [4], the paper [3] shows convergence of a difference scheme to parabolic level set mean curvature flow, and [4] shows bounds for constants in error estimates (and in the sequel full error estimates) for a scheme based on [3] which are polynomial in the inverse regularization parameter. While in both references finite difference schemes for a parabolic problem are used, here finite element approximations for an elliptic problem are considered. Clearly, it is of interest to have also here convergence with rates available at which roughly said (note that for the precise statement the norms

\begin{equation}
\text{P}(1/\varepsilon) \leq \frac{c}{\varepsilon^m}
\end{equation}

for small $\varepsilon$ where $c, m > 0$ are fixed constants which do not depend on $\varepsilon$ and which may vary from line to line where the expression $P(1/\varepsilon)$ appears.

\[1\] Here and in the sequel $P$ applied to a list of positive arguments $l_1, \ldots, l_m, m \in \mathbb{N}$, stands for an expression which depends at most polynomial on $\max\{l_i, 1/l_i\}, i = 1, \ldots, m$, hence to illustrate the notation in this example case we especially have

\[P(1/\varepsilon) \leq \frac{c}{\varepsilon^m}\]
and limit processes have to be specified in details) our own contributions solely
address the rates aspect while the convergence was already known before. The
elliptic formulation has the advantage that even for nonlinear speeds in the mean
curvature the equation is in divergence form and the nonlinearity arising from the
velocity appears only in lower order terms. On the other hand parabolic problems
are sometimes more convenient for numerical methods. An effect of this kind seems
to appear here in the sense that for our special elliptic problem the dependence of
constants on the regularization parameter (compared to the analogous issue in [4])
is of more implicit type, which we handle with a different method in comparison
to [4], namely, by using techniques from [13]. See the parabolic versus elliptic
discussion in Section 5.

Our approach uses \( L^\infty \)-estimates due to the proof of the Alexandrov weak max-
imum principle [8, Thm. 9.1] which are available for the linearization of equation
(1.1) in case of \( H^2 \)-conforming finite elements. While in that cited theorem explicit
dependencies (between assumed bounds on the coefficients and the constant in the
a priori estimate) are not formulated, its proof allows to work out these more ex-
plicitly yielding an exponential relation. For additional references about numerical
analysis for geometric evolution equations we refer to the variety of references in
[11, 10, 7] as well as to these papers themselves.

In this final section we discuss some practical experiences concerning the issue of
polynomial and exponential rates. Numerical examples for a rotationally symmetric
setting (the symmetry avoids the singularity of the equation in the computational
domain) using \( H^1 \)-conforming finite elements are presented in [6]. Thereby the
authors apply a moderate coupling of the regularization and discretization param-
eter \( \varepsilon \) and \( h \) of type \( h \approx \varepsilon^2 \). Under this polynomial coupling the numerical scheme
seems to work well in the considered numerical examples; for further details, see
[6]. For a more general case (without symmetry and with singularity in the com-
putational domain) the observation from [11] is that numerical computations with
small \( \varepsilon \) are difficult at all which makes it at least plausible that the bound could
be worse than of moderate polynomial type in the case of \( H^1 \)-conforming finite
elements, e.g. even exponential.

Note that the method presented here generalizes to several other situations and
equations which inherit certain parameters for which a certain asymptotic is as-
sumed.

2. Main result

We adopt the setting from [13]. Throughout the paper we use for a bounded
domain \( \Omega \subset \mathbb{R}^{n+1} \) the usual notation for Lebesgue spaces \( L^p(\Omega) \), \( 1 \leq p \leq \infty \),
and Sobolev spaces \( W^{m,p}(\Omega) \), \( 0 \leq m < \infty \), \( 1 \leq p \leq \infty \). For \( p = 2 \) we write
\( H^m(\Omega) = W^{m,2}(\Omega) \). By \( H^1_0(\Omega) \) we denote the closure of the smooth functions
being compactly supported in \( \Omega \) in the \( H^1 \)-norm. We denote generic constants in
estimates usually by \( c \) and use the summation convention that we sum over repeated
indices. By \( | \cdot | \) we denote the Euclidean norm, and by \( \| \cdot \|_1 \) the 1-norm. The domain
of the level set function is considered to be \( n+1 = 3 \) dimensional, so that level sets
are in case of sufficient regularity \( n = 2 \) dimensional surfaces. Let

\[
\{ T_h : 0 < h < h_0 \}
\]
be a family of shape-regular and uniform triangulations of $\Omega$, $h$ the mesh size of $\mathcal{T}_h$ and $h_0 = h_0(\Omega) > 0$ small, where we allow boundary elements to be curved and define
\begin{equation}
\Omega^h := \bigcup_{T \in \mathcal{T}_h} T;
\end{equation}

since $\Omega$ might lack convexity, we have not in general $\Omega^h \subset \bar{\Omega}$. We will specify the triangulation concerning their boundary approximation in the following two assumptions and will already here stipulate that we will consider piecewise polynomial functions with polynomial degree at most $\deg \geq 3$, $\deg \in \mathbb{N}$.

**Assumption 2.1.** For $h > 0$ we assume the existence of the following $H^2$-conforming finite element space:
\begin{equation}
V_h := \left\{ w \in C^1(\bar{\Omega}^h) \mid \begin{array}{l}
\text{for all } T \in \mathcal{T}_h : \quad w|_T \text{ polynomial of degree } \leq \deg \\
\text{up to the transformation in case } T \text{ is a boundary cell}, \\
w|_{\partial \Omega^h} = 0.
\end{array} \right\}.
\end{equation}

Hereby, we allow for the finite elements restricted to the boundary cells to be accordingly transformed polynomials, see Appendix A.

The corresponding space when dropping the requirement $w|_{\partial \Omega^h} = 0$ in (2.3) is denoted by $V_h$.

**Assumption 2.2.** For $0 < h \leq h_0$ we assume that there exists an interpolation operator $I_h : H^{\deg}(\Omega^h) \to V_h$ such that for $1 \leq p \leq \infty$
\begin{equation}
\|u - I_h u\|_{W^{m,p}(\Omega^h)} \leq c h^{\deg - m} \|u\|_{W^{\deg,p}(\bar{\Omega}^h)} \quad \forall u \in H^{\deg}(\Omega^h), \quad m = 1, 2.
\end{equation}

Furthermore, there is a constant $0 < \hat{c} := \hat{c}(\Omega)$ so that
\begin{equation}
\partial \Omega^h \subset \Omega_{\hat{c} \hat{\deg}} \setminus \Omega_{-\hat{c} \hat{\deg}}.
\end{equation}

where we use the notation (2.6) and assume that $\hat{\deg} \leq \frac{\deg}{2}$, $\hat{\deg} \in \mathbb{N}$.

Note that the $\hat{\deg} \leq \deg/2$, $\hat{\deg} \in \mathbb{N}$, inequality is assumed for simplification and is to avoid a later consideration of two different cases and that the excluded range for $\hat{\deg}$ could be treated analogously but is not that interesting since it corresponds to a rather high boundary approximation.

We denote the set of nodes of $\mathcal{T}_h$ by $N_h$. We recall that a continuous piecewise polynomial function whose derivatives are continuous along faces is an element in $H^2(\bar{\Omega})$, cf. [1, Thm. 5.2]. Since the curved elements at the boundary can be treated in the weak formulation of the discrete problem analogously as if they were exact tetrahedra we will refer in the following in our notation to these elements as to the usual tetrahedra. Let $d : \mathbb{R}^3 \to \mathbb{R}$ be the signed distance function of $\partial \Omega$ where the sign convention is so that $d$ is negative inside $\Omega$ and nonnegative outside $\Omega$. Let $\delta_0 = \delta_0(\Omega) > 0$ be small and define for $0 < \delta < \delta_0$ that
\begin{equation}
\Omega_\delta := \{ x \in \mathbb{R}^3 \mid d(x) < \delta \}
\end{equation}
then we have
\begin{equation}
\partial \Omega_\delta \in C^\infty, \quad \|\partial \Omega_\delta\|_{C^2} \leq c(\Omega)\|\partial \Omega\|_{C^2}.
\end{equation}

Let $h_0 > 0$ be chosen such that
\begin{equation}
\Omega^h \subset \Omega_{\delta_0} \quad \text{for all } 0 < h \leq h_0.
\end{equation}
We extend \( u^\varepsilon \) to a function in \( C^m(\Omega_0) \) with \( m \in \mathbb{N} \) sufficiently large, denote the extension again \( u^\varepsilon \), and assume that
\[
\|u^\varepsilon\|_{C^m(\Omega_0)} \leq c\|u^\varepsilon\|_{C^m(\overline{\Omega})}.
\]
Furthermore, when (tacitly) extending functions \( v_h \in V_h \) to \( \mathbb{R}^{n+1} \) by zero we denote the extended function again by \( v_h \).

We formulate our main result.

**Theorem 2.3.** Let \( \mu \geq 2 \) and \( \nu := \frac{3}{\mu} - \frac{7}{2} + \frac{4}{3}\tilde{\deg} \). Then choosing \( \delta \) such that
\[
\begin{cases}
\nu > \delta > 2, & \text{if } \mu > 3, \\
\nu > \delta > \frac{3}{2} - \frac{1}{2}, & \text{if } 2 \leq \mu \leq 3
\end{cases}
\]
and setting
\[
\rho = e^{\varepsilon - \gamma} h^\delta
\]
with \( \gamma > 0 \) suitable constant with can be calculated explicitly the fol lowing holds:
For every \( 0 < \varepsilon < \varepsilon_0 \) and \( 0 < h \leq h_0 \) the equation
\[
\int_{\Omega_h} \frac{(Du^\varepsilon_h \cdot D\varphi_h)}{\sqrt{\varepsilon^2 + |Du^\varepsilon_h|^2}} = -\int_{\Omega_h} \eta \left( \sqrt{\varepsilon^2 + |Du^\varepsilon_h|^2} \right) \varphi_h \quad \forall \varphi_h \in V_h,
\]
has a unique solution \( u^\varepsilon_h \) in
\[
\tilde{B}_\rho^h := \{ w_h \in V_h : \|w_h - u^\varepsilon\|_{H^2(\Omega_h)} \leq \rho \}.
\]

**Remark 2.4.** Condition (2.10) implies a lower bound for \( \tilde{\deg} \), namely \( \tilde{\deg} > \frac{3}{4}(11/2 - \frac{3}{\mu}) \) for \( \mu > 3 \) and \( \tilde{\deg} > \frac{9}{4} \) for \( 2 \leq \mu \leq 3 \). Furthermore, we have \( \tilde{\deg} \leq \frac{4}{3}\tilde{\deg} \), see the discussion below Assumption 2.2.

Throughout the paper we assume \( \mu \geq 2 \) which will be used in (4.22).

**Example 2.5.** For \( \tilde{\deg} = 3 \), \( \deg = 9 \) (see for a realization Section A), and \( \mu = 2 \) the condition for \( \delta \) reads as follows:
\[
2 > \delta > 1.
\]

In comparison with our previous result [13, Thm. 1.2] there are two main differences: In [13]
(i) we used \( H^1 \) instead of \( H^2 \)--conforming finite elements,
(ii) we estimated \( W^{1,p} \)-- instead of \( W^{2,\mu} \)--norms.

Furthermore, note that the validity of the assumptions in Theorem 2.3—exactly seen—do not imply that the assumptions of the paper [13] are satisfied because we fixed therein the concrete space of piecewise polynomial functions (while here we keep the space more abstract), nevertheless, clearly, an easy observation shows that the former conclusion holds for the present case as well.

The remaining part of the paper deals with the proof of Theorem 2.3.

### 3. Discrete \( W^{2,\mu} \)-estimates with explicit constants

Here, we formulate some auxiliary estimates for a class of linear equations which includes in particular the linearization of (1.1); we proceed analogously to [13].

The linearized operator \( L_\varepsilon := L_\varepsilon(u^\varepsilon) : H^1_0(\Omega) \to H^{-1}(\Omega) \) for (1.1) in a solution \( u^\varepsilon \) is given by
\[
Lu := D_i(a^{ij} D_j u) + c^i D_i u + du
\]
in $\Omega$ with coefficients given as follows (such a calculation can be found in [13]): We define for $\varepsilon > 0$ and $z \in \mathbb{R}^{n+1}$

\begin{equation}
|z|_\varepsilon := f_\varepsilon(z) := \sqrt{|z|^2 + \varepsilon^2}
\end{equation}

and denote derivatives of $f_\varepsilon$ with respect to $z^i$ by $D_z f_\varepsilon$, i.e. there holds

\begin{equation}
D_z f_\varepsilon(z) = \frac{z_i}{|z|_\varepsilon}, \quad D_z D_z f_\varepsilon(z) = \frac{\delta_{ij} - \frac{z_i z_j}{|z|_\varepsilon^2}}{|z|_\varepsilon^2};
\end{equation}

with these notations we set

\begin{equation}
a^{ij} := -D_z D_z f_\varepsilon(D u^\varepsilon), \quad c^i := \eta_i(\|Du^\varepsilon\|)D_z f_\varepsilon(D u^\varepsilon) \text{ and } d := 0.
\end{equation}

Furthermore, we introduce the equation

\begin{equation}
Lu = g + D_i f^i.
\end{equation}

with $f^i \in W^{1,p}(\Omega)$, $1 \leq i \leq n+1$, and $g$ in $L^p(\Omega)$, $p \geq 1$. The special formal structure of the right-hand side is chosen as in [13] where when working with $H^1$ conforming elements one only has $f^i \in L^p(\Omega)$.

For convenience we recall the following well-known inverse estimate [2, Sec. 4.5] which will be used without mentioning it each time.

**Lemma 3.1.** For $1 \leq p, q \leq \infty$ and $0 \leq m \leq l \leq 2$ there exists a constant $c > 0$ such that

\begin{equation}
\|v_h\|_{W^{l,p}(\Omega^h)} \leq c h^{m-l+\min\left(0, \frac{1}{q} - \frac{2}{p}\right)} \|v_h\|_{W^{m,q}(\Omega)}
\end{equation}

for all $v_h \in V_h$.

Using standard elliptic regularity theory it is shown in [13, Sec. 2] that the solution $u^\varepsilon$ of (1.1) is smooth and satisfies the estimate

\begin{equation}
\|u^\varepsilon\|_{H^m(\Omega)} = P(1/\varepsilon)
\end{equation}

for all $m \in \mathbb{N}$.

**Lemma 3.2.** Let $\delta_0$ be given as in Section 2 and $h_0 > 0$ such that (2.8) is satisfied. We allow unless specified concretely that

\begin{equation}
\begin{cases}
\Omega = \Omega^h \text{ with } 0 < h < h_0 \\
\tilde{\Omega} = \Omega_{\delta_0}
\end{cases}
\end{equation}

and assume $D_i f^i + g \in L^{n+1}(\tilde{\Omega})$.

(i) There exists a unique solution $u \in H^2(\tilde{\Omega}) \cap H^1_0(\tilde{\Omega})$ of (3.5) with $L = L_\varepsilon$ satisfying

\begin{equation}
\|u\|_{H^2(\tilde{\Omega})} \leq \frac{\varepsilon}{P(1/\varepsilon)} \|D_i f^i + g\|_{L^{n+1}(\tilde{\Omega})}.
\end{equation}

(ii) Furthermore, assuming $\tilde{\Omega} = \Omega^h$ in (i) we have the best-approximation property

\begin{equation}
\|u - u_h\|_{H^1(\Omega^h)} \leq \varepsilon P(1/\varepsilon) \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega^h)}.
\end{equation}

**Proof.** (i) As an intermediate step we prove

\begin{equation}
\|Du\|_{L^2(\tilde{\Omega})} \leq \frac{\varepsilon}{P(1/\varepsilon)} \|D_i f^i + g\|_{L^{n+1}(\tilde{\Omega})}.
\end{equation}

This estimate follows from [13, Lem. 7.1] replacing [13, Thm. 6.2] therein by the $L^\infty$-estimate in Theorem B.1. Thereby, $D_i f^i + g$ from our setting plays the role of
g in [13, Lem. 7.1], note that the two situations differ since we assume now higher regularity for \( f^i \). Estimate (3.9) is a straightforward calculation by combining the standard proof for higher regularity with the estimate (3.11).

(ii) The proof follows exactly the lines in [13, estimate (7.12)] using (3.9) instead of [13, (7.10)]: we remark that in the latter reference cubic elements are used, however, since here we have ansatz functions of higher polynomial degree and a boundary approximation as given in (2.5), the situation here is even at least as convenient as before and allows the application of the former arguments. \( \square \)

**Theorem 3.3.** Let \( u \) be the unique solution of (3.5) in \( \Omega^h \) where \( L = L_\varepsilon \). Then, there is \( h_0 > 0 \) so that for

\[
0 < h \leq h_0
\]

there exists a unique finite element solution \( u_h \in V_h \) of (2.12) in \( \Omega^h \) satisfying

\[
\| u_h \|_{H^2(\Omega^h)} \leq e^{P(1/\varepsilon)} \| D_i f^i + g \|_{L^{n+1}(\Omega^h)}.
\]

**Proof.** We have

\[
\| u - u_h \|_{H^2(\Omega^h)} \leq \| u - I_h u \|_{H^2(\Omega^h)} + \| u_h - I_h u \|_{H^2(\Omega^h)}
\]

\[
\leq \| u - I_h u \|_{H^2(\Omega^h)} + Ch^{-1} \| u_h - I_h u \|_{H^1(\Omega^h)}
\]

\[
\leq \| u - I_h u \|_{H^2(\Omega^h)} + Ch^{-1} \left( \| u - I_h u \|_{H^1(\Omega^h)} + \| u - u_h \|_{H^1(\Omega^h)} \right)
\]

\[
\leq \| u - I_h u \|_{H^2(\Omega^h)} + C \| u \|_{H^2(\Omega^h)} + Ch^{-1} \| u - u_h \|_{H^1(\Omega^h)};
\]

here, \( I_h \) is the interpolation operator introduced in Assumption 2.2. Since by (3.10)

\[
\| u - u_h \|_{H^1(\Omega^h)} \leq C \| u - I_h u \|_{H^1(\Omega^h)},
\]

we have by interpolation estimate (2.4) that

\[
\| u - u_h \|_{H^2(\Omega^h)} \leq e^{P(1/\varepsilon)} \| D_i f^i + g \|_{L^{n+1}(\Omega^h)}
\]

and the claim follows from the triangle inequality and stability estimate (3.9).

\( \square \)

Note that we will apply Banach’s fixed point theorem in \( W^{2,\mu} (\Omega^h) \)-balls but go via \( H^2(\Omega^h) \)-norms for the reason to work out the constants explicitly; this is analogous to [13] where we apply Banach’s fixed point theorem in \( W^{2,\mu} (\Omega^h) \)-balls.

4. **Banach’s fixed point theorem in \( W^{2,\mu} \) balls with radii given explicitly in terms of \( h \) and \( \varepsilon \)**

We prove Theorem 2.3.

(i) We present the general outline of the proof which follows the proofs in [6, Sec. 2] and [13, Thm. 4.2] with the following differences. The novelties which we implement here in the proof are the higher order Sobolev norms and explicit constants. Both is new compared with [6] and the first (and hence the combination of both) also compared with [13]. The higher regularity in the present paper allows to differentiate second order derivative expressions explicitly which makes the estimates different from the latter reference. Existence and uniqueness of a solution \( u_h^\varepsilon \) of (2.12) will be shown by identification of this solution as the unique fixed point of a map \( T : V_h \to \overline{B}_R^h \), cf. (2.13), which will be defined in (4.4). Uniqueness and existence of the fixed point follows by Banach’s fixed point theorem. Therefore, we
will check the standard assumptions of the fixed point theorem in a quantitative way with respect to constants. We use the following selection of three sufficient conditions:

\begin{equation}
\bar{B}^h_\rho \neq \emptyset,
\end{equation}

recalling \(2 \leq \mu \leq 3\) condition

\begin{equation}
\|T w_h - T v_h\|_{W^{2,\mu} (\Omega_h)} \leq c h^\eta \|w_h - v_h\|_{W^{2,\mu} (\Omega_h)} \quad \forall w_h, v_h \in \bar{B}^h_\rho
\end{equation}

with some \(\eta > 0\), and

\begin{equation}
T(\bar{B}^h_\rho) \subset \bar{B}^h_\rho.
\end{equation}

Hereby, we define \(T : V_h \rightarrow V_h\) by

\begin{equation}
L_\varepsilon (w_h - T w_h) = \Phi_\varepsilon (w_h), \quad w_h \in V_h
\end{equation}

with the operator \(\Phi_\varepsilon\) given by

\begin{equation}
\Phi_\varepsilon : H^1_0 (\Omega_h) \rightarrow H^{-1} (\Omega_h), \quad \Phi_\varepsilon (\nu) := -D_i \left( \frac{D_i \nu}{|D \nu|_\varepsilon} \right) + \eta (|D \nu|_\varepsilon).
\end{equation}

(ii) Condition (4.1) follows directly from (2.4).

(iii) Here, an estimate for \(\|T(v_h - w_h)\|_{W^{2,\mu} (\Omega_h)}\) with \(\mu \geq 2\) is shown. Let \(v_h\) and \(w_h\) be in \(\bar{B}^h_\rho\), \(\xi_h = v_h - w_h\), \(\alpha(t) = w_h + t \xi_h\), \(0 \leq t \leq 1\). In view of (4.4) we have

\begin{equation}
L_\varepsilon (T v_h - T w_h) = L_\varepsilon \xi_h + \Phi_\varepsilon (w_h) - \Phi_\varepsilon (v_h).
\end{equation}

Recalling the convention that when \(\eta\) and \(f_\varepsilon\) have no arguments it is meant \(\eta = \eta(|D u^\varepsilon|_\varepsilon)\) and \(f_\varepsilon = f_\varepsilon (u^\varepsilon)\) the right-hand side of (4.6) is of the form \(D_i f^t + g \in L^{n+1}\) with (see [13])

\begin{equation}
f^t = D_2 f_\varepsilon (D v_h) - D_2 f_\varepsilon (D w_h) - D_2 D_2 f_\varepsilon D_2 \xi_h
\end{equation}

\begin{equation}
= \int_0^1 (D_2 D_2 f_\varepsilon (D \alpha(t))) - D_2 D_2 f_\varepsilon D_2 \xi_h
\end{equation}

and (using the convention that \(\eta'\) denotes the derivative of simply the function \(r \mapsto \eta(r)\) and that the argument if omitted is understood to be \(|D u^\varepsilon|_\varepsilon\)) we have

\begin{equation}
g = \eta' D_2 f_\varepsilon D_2 \xi_h + \eta (f_\varepsilon (D w_h)) - \eta (f_\varepsilon (D v_h))
\end{equation}

\begin{equation}
= \int_0^1 (\eta' D_2 f_\varepsilon - \eta' (f_\varepsilon (D \alpha(t)))) D_2 f_\varepsilon D_2 \xi_h.
\end{equation}

Since the finite element space is \(H^2\)-conforming we may rewrite \(D_i f^t\) by performing the differentiation and get by using the abbreviation

\begin{equation}
G(t) = D_2 f_\varepsilon (D \alpha(t)) D_1 D_1 \alpha(t)
\end{equation}

that

\begin{equation}
D_i f^t = D_2 D_2 f_\varepsilon (D \alpha(0)) D_1 D_1 \alpha(0) - D_2 D_2 f_\varepsilon (D \alpha(t)) D_1 D_1 \alpha(0)
\end{equation}

\begin{equation}
- D_2 D_2 f_\varepsilon D_1 D_1 D_1 u^\varepsilon D_2 \xi_h - D_2 D_2 f_\varepsilon D_1 D_1 \xi_h
\end{equation}

\begin{equation}
= \int_0^1 \frac{d}{dt} G(t) - D_2 D_2 f_\varepsilon D_1 D_1 D_1 u^\varepsilon D_2 \xi_h - D_2 D_2 f_\varepsilon D_1 D_1 \xi_h.
\end{equation}
There holds
\begin{equation}
\frac{d}{dt} G(t) = D_{z^m} D_{z^r} f_{\varepsilon}(D\alpha(t)) D_m \xi_h D_t \alpha(t) \\
+ D_{z^r} D_{z^r} f_{\varepsilon}(D\alpha(t)) D_t D_r \xi_h.
\end{equation}

Now we rewrite \( \frac{d}{dt} G(t) \) by using the identity
\begin{equation}
ABC = (A - a)(B - b)C + (A - a)bC + a(B - b)C + abC
\end{equation}
for real numbers \( A, B, C, a, b \), i.e. we have
\begin{equation}
\frac{d}{dt} G(t) = (D_{z^m} D_{z^r} f_{\varepsilon}(D\alpha(t)) - D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon)) D_m \xi_h (D_t D_r \alpha(t) - D_t D_r u^\varepsilon) \\
+ (D_{z^m} D_{z^r} f_{\varepsilon}(D\alpha(t)) - D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon)) D_m \xi_h D_t D_r u^\varepsilon \\
+ D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon) D_m \xi_h (D_t D_r \alpha(t) - D_t D_r u^\varepsilon) \\
+ D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon) D_m \xi_h D_t D_r u^\varepsilon \\
+ (D_{z^r} D_{z^r} f_{\varepsilon}(D\alpha(t)) - D_{z^r} D_{z^r} f_{\varepsilon}(Du^\varepsilon)) D_t D_r \xi_h + D_{z^r} D_{z^r} f_{\varepsilon}(Du^\varepsilon) D_t D_r \xi_h.
\end{equation}

Due to cancellations we can write
\begin{equation}
\frac{d}{dt} f^t = \int_0^t \left[ (D_{z^m} D_{z^r} f_{\varepsilon}(D\alpha(t)) - D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon)) D_m \xi_h (D_t D_r \alpha(t) - D_t D_r u^\varepsilon) \\
+ (D_{z^m} D_{z^r} f_{\varepsilon}(D\alpha(t)) - D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon)) D_m \xi_h D_t D_r u^\varepsilon \\
+ D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon) D_m \xi_h (D_t D_r \alpha(t) - D_t D_r u^\varepsilon) \\
+ D_{z^m} D_{z^r} f_{\varepsilon}(Du^\varepsilon) D_m \xi_h D_t D_r u^\varepsilon \\
+ (D_{z^r} D_{z^r} f_{\varepsilon}(D\alpha(t)) - D_{z^r} D_{z^r} f_{\varepsilon}(Du^\varepsilon)) D_t D_r \xi_h + D_{z^r} D_{z^r} f_{\varepsilon}(Du^\varepsilon) D_t D_r \xi_h \right] dt.
\end{equation}

We will apply Theorem 3.3 with the right-hand side given by (4.6), or more explicitly expressed by using (4.14) and (4.8) and the estimates below. The needed \( L^{n+1}(\Omega^h) \) norm of that right-hand side will be related for that purpose to available norms of the expressions on the right-hand side by using the interpolation estimate (2.4) and switching between discrete norms by applying inverse estimates. We then arrive at
\begin{equation}
\|Dw_h - Du^\varepsilon\|_{L^\infty(\Omega^h)} \leq \|Dw_h - D\mathcal{I}_h u^\varepsilon\|_{L^\infty(\Omega^h)} + \|D\mathcal{I}_h u^\varepsilon - Du^\varepsilon\|_{L^\infty(\Omega^h)} \\
\leq c h^{1 - \frac{n+1}{p}} (\|Dw_h - Du^\varepsilon\|_{W^{1,p}(\Omega^h)} + c h \|Du^\varepsilon - D\mathcal{I}_h u^\varepsilon\|_{W^{1,p}(\Omega^h)}) + c h^{\deg - 1} \|u^\varepsilon\|_{C^{\deg}(\tilde{\Omega}^h)} + c h^{\deg - 1} \|u^\varepsilon\|_{C^{\deg}(\tilde{\Omega}^h)}.
\end{equation}

Analogously, we have
\begin{equation}
\|D^2 w_h - D^2 u^\varepsilon\|_{L^\infty(\Omega^h)} \leq \|D^2 w_h - D^2 \mathcal{I}_h u^\varepsilon\|_{L^\infty(\Omega^h)} + \|D^2 \mathcal{I}_h u^\varepsilon - D^2 u^\varepsilon\|_{L^\infty(\Omega^h)} \\
\leq c h^{\frac{n}{2}} (\|D^2 w_h - D^2 u^\varepsilon\|_{L^p(\Omega^h)} + \|D^2 u^\varepsilon - D^2 \mathcal{I}_h u^\varepsilon\|_{L^p(\Omega^h)}) + c h^{\deg - 2} \|u^\varepsilon\|_{C^{\deg}(\tilde{\Omega}^h)} + c h^{\deg - 2} \|u^\varepsilon\|_{C^{\deg}(\tilde{\Omega}^h)}.
\end{equation}
Clearly, since \( \rho \) plays the role of the radius of the ball \( \bar{B}_\rho \) in which we confirm the assumptions of Banach’s fixed point theorem (and we are interested in the asymptotic regime of the parameter values) it will naturally have a small value so that we may assume \( \rho \), \( h \), its boundedness by a moderate constant, e.g. \( \rho < 1 \), which will be used for the following tacitly (and will vanish in generic constants).

We estimate by using (4.14), (4.15) and (4.16) as follows

\[
\|D_1 f\|_{L^{n+1}(\Omega^h)} \leq e^{P(1/\varepsilon)}(\|D w_h - Du \|_{L^\infty(\Omega^h)} + \|D v_h - Du \|_{L^\infty(\Omega^h)})
\]

\[
\|D \xi_h\|_{L^{n+1}(\Omega^h)}(\|D^2 w_h - D^2 u \|_{L^\infty(\Omega^h)} + \|D^2 v_h - D^2 u \|_{L^\infty(\Omega^h)})
\]

\[
e^{P(1/\varepsilon)}(\|D w_h - Du \|_{L^\infty(\Omega^h)} + \|D v_h - Du \|_{L^\infty(\Omega^h)})\|D \xi_h\|_{L^{n+1}(\Omega^h)}
\]

Furthermore, we have

\[
\|g\|_{L^{n+1}(\Omega^h)} \leq e^{P(1/\varepsilon)}(\|D w_h - Du \|_{L^\infty(\Omega^h)} + \|D u \|_{L^\infty(\Omega^h)})\|D \xi_h\|_{L^{n+1}(\Omega^h)}
\]

We estimate the right-hand sides of (4.17) and (4.18) further from above in terms of \( \rho, h \) and \( \varepsilon \) which will we done by relating the appearing \( L^{n+1}(\Omega^h) \) norms to \( L^\mu \) norms and by using the estimate

\[
\|D^m w_h - D^m u \|_{L^\infty(\Omega^h)} \leq c h^{2m-\frac{n+1}{\nu}} \rho + c h^{deg-\mu} \|u\|_{C^\mu(\Omega^h)},
\]

for \( m = 1, 2 \). Proceeding in this way we obtain as upper bound for \( \|D_1 f\|_{L^{n+1}(\Omega^h)} \) the expression

\[
e^{P(1/\varepsilon)}(c h^{1-\frac{n+1}{\nu}} \rho + c h^{deg-1} \|u\|_{C^\mu(\Omega^h)})(c h^{1-\frac{n+1}{\nu}} \rho + c h^{deg-2} \|u\|_{C^\mu(\Omega^h)} + 1 + h^{-1})
\]

\[
h^{1+\min(0,1-\frac{n+1}{\nu})} + e^{P(1/\varepsilon)}(c h^{1-\frac{n+1}{\nu}} \rho + c h^{deg-2} \|u\|_{C^\mu(\Omega^h)})h^{1+\min(0,1-\frac{n+1}{\nu})}
\]

\[
=: A\rho
\]

\[
e^{P(1/\varepsilon)}(I_1 + I_2)h^{-1}(I_1 + I_2 + h + 1)I_3\rho + e^{P(1/\varepsilon)}h^{-1}(I_1 + I_2)I_3\rho
\]

where we abbreviated quantities in an obvious way. For \( \|g\|_{L^{n+1}(\Omega^h)} \) we obtain the upper bound

\[
B\rho := e^{P(1/\varepsilon)}(c h^{1-\frac{n+1}{\nu}} \rho + c h^{deg-1} \|u\|_{C^\mu(\Omega^h)})h^{1+\min(0,1-\frac{n+1}{\nu})}
\]

\[
e^{P(1/\varepsilon)}(I_1 + I_2)I_3\rho.
\]

Theorem 3.3 and an inverse estimate lead to

\[
\|T v_h - T w_h\|_{W^2(\Omega^h)} \leq h^{\frac{n+1}{\nu} - \frac{n+1}{2}}\|T v_h - T w_h\|_{H^2(\Omega^h)}
\]

\[
e^{P(1/\varepsilon)}h^{\frac{n+1}{\nu} - \frac{n+1}{2}}(A\rho + B\rho)
\]

where we used that \( \mu \geq 2 \). Sufficient for \( T \) to satisfy the contraction property of Banach’s fixed point theorem is that the right-hand side of (4.22) is a multiple less than 1 of \( \rho \), i.e.

\[
e^{P(1/\varepsilon)}h^{\frac{n+1}{\nu} - \frac{n+1}{2}}(A + B) < 1.
\]
In the following we derive sufficient conditions for (4.23) and expand for it in details the relevant quantity. An easy check shows that the quantities in $B$ can be subsumed under those of $A$. The terms from $A$ are as follows

\begin{align*}
(4.24) & \quad h^{-1}I_1^2I_3, \quad h^{-1}I_1I_2I_3, \quad h^{-1}I_1I_3, \quad h^{-1}I_2^2I_3, \quad h^{-1}I_2I_3
\end{align*}

from which we extract the relevant overall power of $h$ by multiplying the factor $h^{\frac{n+1}{\mu} - 1 + \frac{1}{2}}$ to each of the terms and making the ansatz $\rho = h^\delta$ with some $\delta > 0$. We obtain in the corresponding order as in (4.24) the following list of powers of $h$

\begin{align*}
(4.25a) & \quad 2(\delta + 1) - \frac{n+1}{\mu} + \min \left( 0, 1 - \frac{n+1}{\mu} \right) - \frac{n+1}{2}, \\
(4.25b) & \quad \delta + \text{deg} + \min \left( 0, 1 - \frac{n+1}{\mu} \right) - \frac{n+1}{2}, \\
(4.25c) & \quad \delta + 1 + \min \left( 0, 1 - \frac{n+1}{\mu} \right) - \frac{n+1}{2}, \\
(4.25d) & \quad 2(\text{deg} - 1) + \min \left( 0, 1 - \frac{n+1}{\mu} \right) + \frac{n+1}{\mu} - \frac{n+1}{2}, \\
(4.25e) & \quad \text{deg} - 1 + \min \left( 0, 1 - \frac{n+1}{\mu} \right) + \frac{n+1}{\mu} - \frac{n+1}{2}.
\end{align*}

As mentioned above we have $n = 2$ and $\text{deg} \geq 4$. Comparing the five terms in (4.25) we see immediately that (4.25a), (4.25c) as well as (4.25e) are the relevant (in the sense of smallness) terms from which we obtain the following sufficient conditions for positivity of all five terms. In the case $\mu > 3$ we require

\begin{align*}
(4.26) & \quad \delta > 2 \\
(4.27) & \quad \delta > \frac{3}{\mu} - \frac{1}{2}.
\end{align*}

(iv) Proof of conditions (4.3) and (4.2). In order to fulfill (4.3) we choose $z_h \in \tilde{V}_h$ by requiring that

\begin{align*}
(4.28) & \quad z_h = \begin{cases} 
I_h u^\varepsilon & \text{in } \partial \Omega^h, \\
0 & \text{in } N_h \setminus \partial \Omega^h
\end{cases}
\end{align*}

and set

\begin{align*}
(4.29) & \quad \tilde{u}^\varepsilon := I_h u^\varepsilon - z_h.
\end{align*}

Then $\tilde{u}^\varepsilon \in V_h$ and for all $1 \leq q \leq \infty$

\begin{align*}
(4.30) & \quad \| \tilde{u}^\varepsilon - u^\varepsilon \|^q_{W^{2,q}(\Omega^h)} \leq ch^{\text{deg} - 2 + \frac{2}{q}} \| u^\varepsilon \|^q_{C^{\text{deg}+1}(\tilde{\Omega}^h)}
\end{align*}

which follows from the standard interpolation error estimate and the consideration at the boundary by using

\begin{align*}
(4.31) & \quad \| z_h \|_{C^0(\tilde{\Omega}^h)} \leq ch^{\text{deg}}, \quad \| Dz_h \|_{L^\infty(\Omega^h)} \leq ch^{\text{deg}-1}, \quad \| D^2z_h \|_{L^\infty(\Omega^h)} \leq ch^{\text{deg}-2}
\end{align*}

and that the support of $z_h$ lies in a boundary strip of width $\leq ch^{\text{deg}}$.
We conclude $\tilde{u}^\varepsilon \in \tilde{B}_\rho^h$ provided $h_0 e^{P(1/\varepsilon)} < 1$ and
\begin{equation}
(4.32) \quad \tilde{d}e g - 2 + \frac{\tilde{d}e g}{\mu} > \delta.
\end{equation}
By using the triangle inequality we estimate
\begin{equation}
(4.33) \quad \|T w_h - u^\varepsilon\|_{W^2,\nu(\Omega^h)} \leq \|T w_h - T \tilde{u}^\varepsilon\|_{W^2,\nu(\Omega^h)} + \|T \tilde{u}^\varepsilon - \tilde{u}^\varepsilon\|_{W^2,\nu(\Omega^h)}
\end{equation}
and in the following we continue this estimate by considering all three terms separately. The third term is estimated according to (4.30). Using the contraction property (4.2) we deduce that
\begin{equation}
(4.34) \quad \|T w_h - T \tilde{u}^\varepsilon\|_{W^2,\nu(\Omega^h)} \leq ch^\eta \|w_h - \tilde{u}^\varepsilon\|_{W^2,\nu(\Omega^h)}
\end{equation}
Let $\xi = u^\varepsilon - \tilde{u}^\varepsilon$, $\alpha(t) = \tilde{u}^\varepsilon + t \xi$, $0 \leq t \leq 1$. We have in $\Omega^h$
\begin{equation}
L_\xi (\tilde{u}^\varepsilon - T(\tilde{u}^\varepsilon)) = \Phi_\varepsilon (\tilde{u}^\varepsilon)
\end{equation}
and the right-hand side of this equation is of the form $D_i f^i + g \in L^{n+1}$ with
\begin{equation}
(4.36) \quad f^i = -D_{z^i} f_\varepsilon (D \tilde{u}^\varepsilon) + D_{z^i} f_\varepsilon (Du^\varepsilon)
\end{equation}
and
\begin{equation}
(4.37) \quad g = \eta (f_\varepsilon (D \tilde{u}^\varepsilon)) - \eta (f_\varepsilon (Du^\varepsilon)).
\end{equation}
Since the finite element space is $H^2$-conforming we may rewrite $D_i f^i$ by performing the differentiation and get by using the abbreviation
\begin{equation}
(4.38) \quad G(t) = D_{z^i} D_{z^i} f_\varepsilon (D\alpha(t)) D_i D_r \alpha(t)
\end{equation}
that
\begin{equation}
(4.39) \quad D_{z^i} f^i = D_{z^i} D_{z^i} f_\varepsilon (D\alpha(1)) D_i D_r \alpha(1) - D_{z^i} D_{z^i} f_\varepsilon (D\alpha(0)) D_i D_r \alpha(0)
\end{equation}
\begin{equation}
\quad = \int_0^1 \frac{d}{dt} G(t).
\end{equation}
There holds
\begin{equation}
(4.40) \quad \frac{d}{dt} G(t) = D_{z^m} D_{z^r} D_{z^i} f_\varepsilon (D\alpha(t)) D_m \xi_h D_i D_r \alpha(t)
\end{equation}
\begin{equation}
\quad + D_{z^i} D_{z^r} f_\varepsilon (D\alpha(t)) D_i D_r \xi_h
\end{equation}
with expanded version (4.13) exactly as before concerning the appearing symbols but with different meanings ($\tilde{u}^\varepsilon$ instead of $w_h$, $u^\varepsilon$ instead of $v_h$). Estimation of that expression gives
\begin{equation}
(4.41) \quad \|D_1 f^i + g\|_{L^{n+1}(\Omega^h)} \leq e^{P(1/\varepsilon)} \|u^\varepsilon - \tilde{u}^\varepsilon\|_{W^{2,n+1}(\Omega^h)}
\end{equation}
and hence
\begin{equation}
(4.42) \quad \|\tilde{u}^\varepsilon - T(\tilde{u}^\varepsilon)\|_{W^2,\nu(\Omega^h)} \leq e^{P(1/\varepsilon)} h^{\frac{n+1}{\rho} - \frac{n+1}{\nu}} \|D_1 f^i + g\|_{L^{n+1}(\Omega^h)}
\end{equation}
\begin{equation}
\quad \leq e^{P(1/\varepsilon)} h^{\frac{n+1}{\rho} - \frac{n+1}{\nu}} \|u^\varepsilon - \tilde{u}^\varepsilon\|_{W^{2,n+1}(\Omega^h)}
\end{equation}
\begin{equation}
\quad \leq e^{P(1/\varepsilon)} h^{\frac{n+1}{\rho} - \frac{n+1}{\nu} + \tilde{d}e g - 2 + \frac{\tilde{d}e g}{\mu}} \|u^\varepsilon\|_{C^{4,\theta}(\Omega^h)}
\end{equation}
from which we can immediately obtain an estimate of the $W^{2,\mu}(\Omega^h)$ norm of
\[(4.43)\]
$$u^\varepsilon - T(\tilde{u}^\varepsilon) = u^\varepsilon - \tilde{u}^\varepsilon + \tilde{u}^\varepsilon - T(\tilde{u}^\varepsilon).$$
From this we find the sufficient conditions in order to satisfy (4.3) that
\[(4.44)\]
$$\frac{n+1}{\mu} - \frac{n+1}{2} + \tilde{\deg} - 2 + \frac{\tilde{\deg}}{n+1} > \delta$$
\[(4.45)\]
$$\tilde{\deg} - 2 + \frac{\tilde{\deg}}{\mu} > \delta.$$  

Conditions (4.26), (4.27), (4.44), and (4.45) lead to the following sufficient conditions when using $n=2, \tilde{\deg} \geq 4$: For $\mu > 3$ we have
\[(4.46)\]
$$\frac{3}{\mu} - \frac{7}{2} + \frac{4}{3}\tilde{\deg} > \delta > 2$$
while for $2 \leq \mu \leq 3$ we have
\[(4.47)\]
$$\frac{3}{\mu} - \frac{7}{2} + \frac{4}{3}\tilde{\deg} > \delta > \frac{3}{\mu} - \frac{1}{2}.$$
Thereby we complete the proof of Theorem 2.3.

5. Discussion of the optimality of the bound

Having fixed the equations and the approximation scheme according to the previous sections we discuss the question if the upper bound $e^{\mathcal{O}(\varepsilon)}$ for the constant in the error estimate according to Theorem 2.3 is optimal. It is quite plausible that the quality of the bound is prescribed by the ‘quality’ of the constant in the a priori estimate for the linearized equation (3.5). This linearized equation evaluated in $u^\varepsilon \in C^\infty$ with stationary points (or being stationary even on open subsets) has lower order (e.g. first order) coefficients locally of size $\approx 1/\varepsilon m, m \in \mathbb{N}$, and arbitrary sign. A simple prototype equation of that type for a function $u = u(t)$,
\[(5.1)\]
$$-u'' + \frac{1}{\varepsilon}u' = 0$$
in $(0, 1)$, already shows that $u' \approx ce^{\frac{t}{\varepsilon}}$ and $u \approx \varepsilon e^{\frac{t}{\varepsilon}}$. This is a strong indication for optimality of our exponential bound in Theorem 2.3.

We draw the following conclusions.

First this indicates that the experimentally motivated polynomial coupling between $\varepsilon$ and $h$ (even of moderate polynomial size $h = \varepsilon^2$) in [6, p.114] which is interpreted therein also as a suggestion for being sufficient in the general case for simulations is rather unlikely to work in general.

Second an above-polynomial optimal bound is supported by the limitations of the scheme for small values of $\varepsilon$ which became experimentally visible in [11].

Third from our overall analysis we come to the conclusion that the parabolic method [3], [4] has advantages concerning the error estimates in the mean curvature flow case compared with the scheme of our paper, originally proposed in [6] for the inverse mean curvature flow case.

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Appendix A. On the realization of Assumptions 2.1 and 2.2

This section is devoted to the explanation that Assumptions 2.1 and 2.2 can be indeed realized. For this purpose we arrange for convenience here some known arguments and results from the literature. We recall first a statement on a polyhedral domain in Lemma A.1 using the realization of $H^2$-conforming finite elements form [15] and general interpolation estimates from [5]. In a second step we extend this result to smooth domains as introduced in Section 2. For this purpose we vary the definition of $\Omega$ within this Section A. At first we recall some results from [15] for which we need the following notation:

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply or multiply connected domain with boundary $\partial \Omega$ consisting of a finite number of polyhedrons $\Gamma_i$ ($i = 0, \ldots, s$) with $\Gamma_1, \ldots, \Gamma_r$ lying inside of $\Gamma_0$ and having no intersection. Let $\mathcal{M}$ be a set of a finite number of closed tetrahedrons having the following properties: (1) the union of all tetrahedrons is $\Omega$; (2) two arbitrary tetrahedrons are either disjoint or have a common vertex or a common edge or a common face. Let $N_i$, $N_v$, and $N_f$ be the total numbers of the tetrahedrons, of the vertices and of the triangular faces in the division $\mathcal{M}$, respectively. Let the tetrahedrons of $\mathcal{M}$ be denoted by $U_i$ ($i = 1, \ldots, N_i$), the vertices by $P_i$ ($i = l, \ldots, N_v$), and the triangular faces by $T_i$ ($i = l, \ldots, N_f$). Let $Q_i$ denote the center of gravity of $T_i$, and $P_i^{(0)}$ the center of gravity of $U_i$. $Q_j^{(r,s)}$ denote the points dividing the segment $[P_j, P_k]$ into $s + 1$ equal parts. The normal to the triangular face whose center of gravity is $Q_i$ is denoted by $n_i$. We orientate $n_i$ according to the right-hand screw rule with respect to the increasing indices $j < k < l$ of the vertices $P_j$, $P_k$, $P_l$ of the face.

We set $D^\alpha f := \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$, and for $\beta = (\beta_1, \beta_2)$ and $|\beta|_1 = \beta_1 + \beta_2$ we define $D^\beta_j f := \partial |\beta|_1/\partial x_j^{\beta_1_j} \partial x_j^{\beta_2_j}$ with $\partial f/\partial s_i$, $\partial f/\partial t_i$, $\partial f/\partial s_{jk}$, and $\partial f/\partial t_{jk}$ denote the derivatives of the function $f$ in the directions $s_i$, $t_i$, $s_{jk}$, and $t_{jk}$. Let there be prescribed at each point $P_i$ thirty-five values $D^\alpha f(P_i)$ ($|\alpha|_1 \leq 4$), at each point $Q^{(1)}_{jk}$ two values $D^\beta_j f(Q^{(1)}_{jk})$ ($|\beta|_1 = 1$), at each point $Q^{(2)}_{jk}$ three values $D^\beta_j f(Q^{(2)}_{jk})$ ($|\beta|_1 = 2$), at each point $P^{(i)}_0$ four values $D^\alpha f(P^{(i)}_0)$ ($|\alpha|_1 \leq 1$) and at each point $Q_i$ one value $f(Q_i)$ and six values $D^\alpha f(Q_i)/\partial n_i$ ($|\beta|_1 \leq 2$). Then on each tetrahedron $D_i$ this determines a unique polynomial of the ninth degree $p_i(x, y, z)$. The function

$$g(x, y, z) = p_i(x, y, z), \quad (x, y, z) \in \tilde{U}_i, \quad i = 1, \ldots, N$$

is once continuously differentiable on the domain $\tilde{\Omega}$, see [15, Thm. 2]. Denoting all functions satisfying (A.1) by $W_h$, we have

$$\dim W_h = 35N_v + 7N_f + 4N_t + 8N_e$$

with $N_v$ number of vertices, $N_f$ number of triangular faces, $N_t$ number of tetrahedrons, and $N_e$ number of edges, and $h$ the mesh parameter.

We further remark, that the simplest polynomial $p(x, y, z)$ on a tetrahedron which leads to piecewise polynomial functions which are globally continuously differentiable is expected to be of order nine [15].
We introduce the following set of conditions for functions \( w \in C^1(\bar{\Omega}) \) being cellwise polynomial

\[
\begin{align*}
D^\alpha w(P_i) &= 0, \quad |\alpha|_1 \leq 4; \\
D^{\beta}_{jk} w(Q^{(r,s)}_{jk}) &= 0, \quad |\beta|_1 = s, \quad r = 1, \ldots, s, \quad s = 1, 2; \\
w(Q_0) &= 0; \\
D^\beta_i \left( \frac{\partial w(Q_i)}{\partial n_i} \right) &= 0, \quad |\beta|_1 \leq 2; \\
D^\alpha w(P_0) &= 0, |\alpha|_1 \leq 1
\end{align*}
\]

(A.3)

where \( i = 1, \ldots, 4, \quad j = 1, 2, 3, \quad k = 2, 3, 4 \) \((j < k)\). This allows to relax Assumptions 2.1 and 2.2.

**Lemma A.1.** (i) For \( h > 0 \) the ansatz space

\[
V_h := \left\{ w \in C^1(\bar{\Omega}) \mid \text{for all } T \in T_h : w|_T \text{ polyom of degree 9 satisfying (A.3), } w|_{\partial \Omega^h} = 0 \right\}
\]

is \( H^2 \)-conform.

(ii) For \( 0 < h \leq h_0 \) there exists a function \( \hat{\varphi} := \varphi \in V_h \) having the same values at the points \( P_i, \ Q^{(r,s)}_{jk}, \ P_0^{(i)}, \ Q_i \) as the exact solution \( u \). Then, we have for \( 1 \leq p \leq \infty \) that

\[
\|u - \hat{\varphi}\|_{W^{m,\infty}(\Omega)} \leq ch^{\deg - m}\|u\|_{W^{m+\infty,\infty}(\Omega)} \quad \forall \ u \in H^{\deg}(\Omega), \quad m = 1, 2.
\]

Proof. (i) By the consideration above, cf. [15], the space \( V_h \) is well-defined, the conformity follows by classical result, cf. Ženšík [1, Thm. 5.2].

(ii) Consequence of Ern and Guermond [5, Thm. 1.109] setting in this reference \( l \) equal to the \( m \) from above, and \( p = \infty \). \( \square \)

In the following we explain for convenience how Assumptions 2.1 and 2.2 can be realized by using Lemma A.1. On the one hand this can be found in standard text books (e.g. in the \( H^1 \) case and for convex and smooth domains) under the terminus ‘curved boundary elements’ or ‘boundary approximation’, on the other hand we do not have a concrete reference for our specific scenario, so we present the following argument for convenience. Let \( \Omega \) be again as in Section 2 and \( \hat{\Omega}^h \) a triangulation where we assume that all faces of the boundary elements are flat (and not curved). Let \( \hat{V}_h \) be the finite element space according to Lemma A.1 on the polygonal domain \( \hat{\Omega}^h \). We will now construct \( V_h \) and \( \hat{\Omega}^h \) which satisfy Assumptions 2.1 and 2.2 on the basis of the previously mentioned spaces. As already explained \( d \) denotes the signed distance function with respect to \( \partial \Omega \) such that

\[
d_j(\Omega) < 0, \quad d_j((\mathbb{R}^n + 1) \setminus \Omega) > 0, \quad d_j(\partial \Omega) = 0.
\]

Let \( \varepsilon > 0 \) and consider

\[
\begin{align*}
\Omega_\varepsilon &= \{ d < \varepsilon \}, \\
\Omega_{-\varepsilon} &= \{ d < -\varepsilon \}, \\
U_\varepsilon &= \Omega_\varepsilon \setminus \overline{\Omega_{-\varepsilon}} = \{ |d| < \varepsilon \}.
\end{align*}
\]

Let \( N_h \) be the set of nodes of the triangulation of \( \hat{\Omega}^h \). We may assume \( h = o(\varepsilon) \) so that

\[
\hat{\partial} \hat{\Omega}^h \subset U_\varepsilon.
\]

We would like to define a diffeomorphism

\[
\Phi : \Omega_\varepsilon \rightarrow \hat{\Omega}, \quad \Phi|_{\Omega_{-2\varepsilon}} = \text{id}, \quad \Phi(N_h \cap \partial \hat{\Omega}^h) \subset \partial \Omega
\]

(A.9)
where \( \tilde{\Omega} \) is an open auxiliary set containing \( \Omega_{-\varepsilon} \) and so that

\[
|D\Phi| + |D^2\Phi| \leq c
\]

uniformly in \( h \) and roughly spoken so that the triangulation using the nodes \( N_h \) carries over via \( \Phi \) to a triangulation of \( \Phi(N_h) \), meaning that the distortion of the nodes is not too large. Without making this formulation precise here we will see that we are far away from such a kind of criticality.

Then we can define

\[
\Omega^h = \Phi(\tilde{\Omega}^h), \tag{A.11}
\]

\[
V_h = \{ \varphi \circ \Phi^{-1} : \varphi \in \hat{V} \} \tag{A.12}
\]

and

\[
I_h u = \tilde{I}_h(u \circ \Phi) \tag{A.13}
\]

for \( u \in H_{deg}(\Omega^h) \). Introducing coordinates \( p = (\hat{x}, x_{n+1}) \in U_\varepsilon, \hat{x} \in \partial \Omega, \text{dist}(\hat{x}, p) = x_{n+1} \), we can write \( N_h \cap \partial \hat{\Omega}^h \) as graph over a suitable discrete set \( D \subset \partial \Omega \), i.e.

\[
N_h \cap \partial \hat{\Omega}^h = \{ (\hat{x}, u(\hat{x})) : \hat{x} \in D \}. \tag{A.14}
\]

Clearly, we can extend \( u \) to \( \partial \Omega \) as smooth function such that

\[
\frac{|u|}{h^2} + \frac{|Du|}{h} + |D^2u| \leq c \tag{A.15}
\]

where \( |\cdot| \) refers to the Euclidean norm of \( Du \) and \( D^2u \) with respect to a fixed selection of finitely many local coordinate systems. (Note that one basically has here the choice of more specific extensions, e.g. piecewise polynomial and sufficiently regular, depending on the concrete situation.) Let \( \rho \in C^\infty(\mathbb{R}) \) (note that on the \( C^2 \) level instead anything like this can be achieved by using polynomial functions as well) such that

\[
\rho(t) = \begin{cases}
0, & t \leq -2\varepsilon \\
1, & t \geq -\varepsilon
\end{cases} \tag{A.16}
\]

and

\[
0 \leq \rho, \quad 0 \leq \rho' \leq \frac{c}{\varepsilon}, \quad |\rho''| \leq \frac{c}{\varepsilon^2}. \tag{A.17}
\]

Then \( \Phi \) is now defined as follows

\[
\Phi(y) = y, \quad y \in \Omega_{-2\varepsilon} \tag{A.18}
\]

and

\[
\Phi : U_{2\varepsilon} \cap \Omega_{\varepsilon} \ni (\hat{x}, x_{n+1}) \mapsto (\hat{x}, x_{n+1} + u(\hat{x})\rho(x_{n+1})) \in \mathbb{R}^{n+1} \setminus \Omega_{-2\varepsilon}. \tag{A.19}
\]

A calculation shows that \( \Phi \) is as desired provided \( h = h(\varepsilon) \), e.g. \( h = \varepsilon^2 \), is sufficiently small: The Jacobian of \( \Phi \) is given by

\[
\begin{pmatrix}
I & 0 \\
-\rho(x_{n+1})Du(\hat{x}) & 1 - u(\hat{x})\rho'(x_{n+1})
\end{pmatrix} \tag{A.20}
\]

and in view of (A.16), (A.17) and (A.15) invertible. And concerning injectivity of \( \Phi \) we have that

\[
(\hat{x}, x_{n+1} + u(\hat{x})\rho(x_{n+1})) = (\hat{y}, y_{n+1} - u(\hat{y})\rho(y_{n+1})) \tag{A.21}
\]
implies \( \hat{x} = \hat{y} \) and in consequence that
\[
(A.22) \quad x_{n+1} - y_{n+1} = u(\hat{x})(\rho(x_{n+1}) - \rho(y_{n+1})) = u(\hat{x})\rho'(\xi)(x_{n+1} - y_{n+1})
\]
with some \( \xi \) which implies \( x_{n+1} = y_{n+1} \) since \( u(\hat{x})\rho'(\xi) \) is small. Note that the so piecewisely defined \( \Phi \) defines a diffeomorphism from \( \Omega_\varepsilon \) onto a set \( \tilde{\Omega} \) as desired.

**Appendix B. \( L^\infty \)-estimate with exponential asymptotics in the constant with respect to the bounds of the coefficients**

In this section we derive an \( L^\infty \)-estimate for strong solutions for linear equations which is obtained from reworking the proof of the Alexandrov weak maximum principle for strong solutions from [8, Thm. 9.1]. In that reference the supremum is estimated by a product consisting of a first factor being a constant which depends on the differential operator and the domain and a second factor being the \( L^{n+1} \) norm of the right hand side (assuming an \( (n + 1) \)-dimensional domain). For our purposes we need an explicit dependence of the constant in this product on the assumed bounds for the coefficients of the differential operator. The formulation of [8, Thm. 9.1] itself does not provide such an explicit dependence. When following the proof of the latter reference one can straightforward extract an explicit relation in addition to the statement of the theorem. According to this explicit relation the constant grows exponentially in the bounds for the coefficients. We recall the well-known proof from that text book here for convenience in details.

In this section we consider the differential operator
\[
(B.1) \quad Lu := a^{ij}D_iD_ju + b^iD_iu + cu
\]
with measurable, bounded coefficients \( a^{ij}, b^i \) and \( c \) on \( \Omega \) satisfying the bounds
\[
(B.2) \quad 0 < \lambda \leq a^{ij} \leq \Lambda \quad \text{(in the sense of quadratic forms)}, \quad |b^i| \leq c_1, \quad |c| \leq c_2, \quad c \leq 0
\]
in \( \Omega \) where \( \lambda, \Lambda, c_1 \) and \( c_2 \) are some positive constants.

**Theorem B.1.** ([8, Thm. 9.1]) Let \( f \in L^{n+1}(\Omega) \), \( c \leq 0 \) and \( u \in C^0(\bar{\Omega}) \cap W_{loc}^{2,n+1}(\Omega) \) with
\[
(B.3) \quad Lu \geq f.
\]
Then
\[
(B.4) \quad \sup_{\Omega} u \leq e^{P(\lambda, \Lambda, c_1, c_2)}(\text{diam } \Omega)\|f\|_{L^{n+1}(\Omega)}.
\]

Recall our convention for \( P \) in (1.3).

**Proof.** Note that the constant \( P(\lambda, \Lambda, c_1, c_2) \) in Theorem B.1 does not depend on \( \Omega \) and that the functions \( \text{diam}(\cdot) \) and \( \|f\|_{L^{n+1}(\cdot)} \) are monotone with respect to the partial ordering of sets by inclusion, hence we may—by considering connected components of \( \{x \in \Omega : u(x) > 0\} \)—assume for the proof that \( u \geq 0 \) in \( \Omega \). We use the following lemma.

**Lemma B.2.** ([8, Lemma. 9.4]) Let \( g \) be a nonnegative, locally integrable function on \( \mathbb{R}^{n+1} \). Then, for any \( u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \), we have
\[
(B.5) \quad \int_{B_{\delta \lambda}(0)} g \leq \frac{1}{((n + 1)\lambda)^{n+1}} \int_{\Gamma^+} g(Du) \left( -a^{ij}D_iD_ju \right)^{n+1}
\]
where

\[ \tilde{M} := (\sup_{\Omega} u - \sup_{\partial\Omega} u)/d, \quad d := \text{diam } \Omega \]

and

\[ \Gamma^+ := \left\{ y \in \Omega \left| u(x) \leq u(y) + p(x - y) \text{ for all } x \in \Omega \right. \text{ and some } p = p(y) \in \mathbb{R}^{n+1} \right\}. \]

Note that \( D^2 u \) is nonpositive in \( \Gamma^+ \). First, we show Theorem B.1 under the assumption that \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) and second, we deduce the general case \( u \in C_0^0(\overline{\Omega}) \cap W^{2,n+1}_{\text{loc}}(\Omega) \) by an approximation argument. So let us assume this higher regularity for \( u \) and we will use Lemma B.2 with

\[ g(p) := \left( |p|^{\frac{n+1}{n}} + \mu^{\frac{n+1}{n}} \right)^{-n}, \quad p \in \mathbb{R}^{n+1}, \]

where \( \mu > 0 \) is a parameter which will later on be set to \( \|f\|_{L^{n+1}(\Omega)} \) provided this norm does not vanish. Now we estimate in \( \Gamma^+ \)

\[ g(Du) \left( -a^{ij} D_i D_j u \right)^{n+1} \leq \frac{(|b| |Du| + |f|)^{n+1}}{|Du|^{\frac{n+1}{n}} + \mu^{\frac{n+1}{n}}} \]

\[ \leq c \frac{|b|^{n+1} |Du|^{n+1} + |f|^{n+1}}{|Du|^{\frac{n+1}{n}} + \mu^{\frac{n+1}{n}}} \]

\[ \leq c \left( |b|^{n+1} + \frac{|f|^{n+1}}{\mu^{n+1}} \right). \]

From below we estimate

\[ g \geq 2^{(1-n)} \left( |p|^{n+1} + \mu^{n+1} \right)^{-1}. \]

Integration of the lower bound yields

\[ \int_{B_{\tilde{M}}(0)} \left( |p|^{n+1} + \mu^{n+1} \right)^{-1} = c \int_0^{\tilde{M}} \frac{r^{n}}{(r^{n+1} + \mu^{n+1})} dr \]

\[ = c \log \left( r^{n+1} + \mu^{n+1} \right) \bigg|_0^{\tilde{M}} \]

\[ = c \log \left( \tilde{M}^{n+1} + \mu^{n+1} \right) - c \log(\mu^{n+1}). \]

Together we obtain

\[ \tilde{M} \leq \left[ \exp \left( P(\lambda, \Lambda, c_1, c_2) \int_\Omega \left( |b|^{n+1} + \frac{|f|^{n+1}}{\mu^{n+1}} \right) dx \right) \mu^{n+1} \right]^{\frac{1}{n+1}} \]

where we assumed for the last two equations that \( \|f\|_{L^{n+1}(\Omega)} \neq 0 \) and set \( \mu = \|f\|_{L^{n+1}(\Omega)} \). The \( f = 0 \) case is clear. The general case concerning the assumed regularity for \( u \) follows by using an approximation argument as (and even simpler in view of the available uniform ellipticity of \( L \) in \( \Omega \)) than in the proof of [8, Lem. 9.4]. \( \square \)
References

1. Dietrich Braess, *Finite Elemente - Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie*, Springer, Berlin [u.a.], 1992.
2. Susanne C. Brenner and L. Ridgway Scott, *The mathematical theory of finite element methods*, Texts in Applied Mathematics, vol. 15, Springer-Verlag, New York, 1994. MR 1278258
3. Michael G. Crandall and Pierre-Louis Lions, *Convergent difference schemes for nonlinear parabolic equations and mean curvature motion*, Numer. Math. 75 (1996), no. 1, 17–41. MR 1417861
4. Klaus Deckelnick, *Error bounds for a difference scheme approximating viscosity solutions of mean curvature flow*, Interfaces Free Bound. 2 (2000), no. 2, 117–142. MR 1760409
5. Alexandre Ern and Jean-Luc Guermond, *Theory and practice of finite elements*, Applied Mathematical Sciences, vol. 159, Springer-Verlag, New York, 2004. MR 2050138
6. Xiaobing Feng, Michael Neilan, and Andreas Prohl, *Error analysis of finite element approximations of the inverse mean curvature flow arising from the general relativity*, Numer. Math. 108 (2007), no. 1, 93–119. MR 2350186
7. Xiaobing Feng, Markus von Oehsen, and Andreas Prohl, *Rate of convergence of regularization procedures and finite element approximations for the total variation flow*, Numer. Math. 100 (2005), no. 3, 441–456. MR 2194526
8. David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983. MR 737190
9. Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. 59 (2001), no. 3, 353–437. MR 1916951
10. Balázs Kovács, Buyang Li, and Christian Lubich, *A convergent evolving finite element algorithm for mean curvature flow of closed surfaces*, Numer. Math. 143 (2019), no. 4, 797–853. MR 4026373
11. Axel Kröner, Eva Kröner, and Heiko Kröner, *Finite element approximation of level set motion by powers of the mean curvature*, SIAM J. Sci. Comput. 40 (2018), no. 6, A4158–A4183. MR 3892432
12. Heiko Kröner, *Approximation rates for regularized level set power mean curvature flow*, Port. Math. 74 (2017), no. 2, 115–126. MR 3734408
13. Heiko Kröner, *Analysis of constants in error estimates for the finite element approximation of regularized nonlinear geometric evolution equations*, SIAM J. Numer. Anal. 57 (2019), no. 5, 2413–2435 (English).
14. Felix Schulze, *Nonlinear evolution by mean curvature and isoperimetric inequalities*, J. Differential Geom. 79 (2008), no. 2, 197–241. MR 2420018
15. Alexander Ženíšek, *Polynomial approximation on tetrahedrons in the finite element method*, J. Approximation Theory 7 (1973), 334–351. MR 350260