HOW TO (SYMPLECTICALLY) THREAD THE EYE OF A 
(LAGRANGIAN) NEEDLE

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Abstract. We show that there exists no Lagrangian embeddings of the Klein bottle into $\mathbb{C}P^2$. Using the same techniques we also give a new proof that any Lagrangian torus in $\mathbb{C}P^2$ is smoothly isotopic to the Clifford torus.

1. LAGRANGIAN EMBEDDINGS IN $\mathbb{C}^2$

The topology of closed Lagrangian embeddings into $\mathbb{C}^n$ (see [1]) is still an elusive problem in symplectic topology. Before Gromov invented the techniques of pseudo–holomorphic curves it was almost intractable and the only known obstructions came from the fact that such a submanifold has to be totally real. Then in [9] he showed that for any such closed, compact, embedded Lagrangian there exists a holomorphic disk with boundary on it. Hence the integral of a primitive over the boundary is different from zero and the first Betti number of the Lagrangian submanifold cannot vanish, excluding the possibility that a three–sphere can be embedded into $\mathbb{C}^3$ as a Lagrangian. A further analysis of these techniques led to more obstructions for the topology of such embeddings in [21] and [23].

For $\mathbb{C}^2$ the classical obstructions restrict the classes of possible closed, compact surfaces which admit Lagrangian embeddings into $\mathbb{C}^2$ to the torus and connected sums of the Klein bottle with oriented surfaces of even genus. There are obvious Lagrangian embeddings of the torus (e.g. the Clifford torus $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$) and not so obvious ones for the connected sums except for the Klein bottle (see [2]). One may further ask which topological types of embeddings may be realized as a Lagrangian embedding. There has been a partial answer to that in [15] and an announcement of a proof by Hofer and Luttinger of the following

Theorem 1 (Hofer/Luttinger). Any Lagrangian embedding of the torus into $\mathbb{C}P^2$ is smoothly isotopic to the Clifford torus $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$.

Here we show that the same circle of ideas, namely the study of holomorphic curves with respect to a certain singular almost complex structure, solves the question for the Klein bottle:

Theorem 2. There is no Lagrangian embedding of the Klein bottle into $\mathbb{C}P^2$.

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Both questions were contributed to Kirby’s problem list [13] by Eliashberg. There have been several attempts to attack this problem ([2, 14, 19] et.al.). Independently, Nemirovski proved in [20] a more general version of Theorem 2 using completely different methods from complex analysis.

The constructions described in this paper will yield the following result:

**Theorem 3.** Let $L \subset \mathbb{C}P^2$ be a closed Lagrangian embedding of a flat surface, i.e. either the Klein bottle or the 2–torus. Then there is an almost complex structure $J$ compatible to $\omega$ with the following property: There are two smooth mappings $D, E : \Delta \rightarrow \mathbb{C}P^2$ of the unit disk with $D(\partial \Delta), E(\partial \Delta) \subset L$ and three embedded $J$–holomorphic spheres $F, G, H : \mathbb{C}P^1 \rightarrow \mathbb{C}P^2 \setminus L$ whose fundamental classes represent the same generator of $H_2(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}$ with the following algebraic intersection numbers:

$$F \cdot D = 1, \quad F \cdot E = 0$$
$$G \cdot D = 0, \quad G \cdot E = 1$$
$$H \cdot D = 0, \quad H \cdot E = 0.$$ 

Let us first see how the main result follows and how we obtain a new proof of the unknottedness of Lagrangian tori.

**Proof of Theorem 3** We claim that the span of $[\partial D], [\partial E] \in H_1(L; \mathbb{Q})$ is two–dimensional. Assume for some pair of integers $(k, l) \neq (0, 0)$

$$(1) \quad k[\partial D] = l[\partial E]$$

as integer homology classes. Hence we can find a 2–chain $C$ in $L$ such that $kD + C + (-lE)$ forms a 2–cycle in $\mathbb{C}P^2$. We test it against $F, G$ and $H$ which yields the same number since these represent the same homology class. By the assumption, $F \cdot C = G \cdot C = H \cdot C = 0$ and therefore

$$(kD + C + (-lE)) \cdot F = k$$
$$(kD + C + (-lE)) \cdot G = -l$$
$$(kD + C + (-lE)) \cdot H = 0$$

Thus $k = l = 0$, verifying the statement. Hence, $b_1(L) \geq 2$, which excludes that $L$ was the Klein bottle. \hfill \Box

**Proof of Theorem 1** By a result of Gromov in [9] there exists a symplectic isotopy of $(\mathbb{C}P^2, \omega)$ which maps the $J$–holomorphic spheres $F, G, H$ to projective lines. Hence the image of $L$ under this map, which we denote by abuse of notation by $L$ again, is a Lagrangian in $\mathbb{C}^2$ in the complement of 2 complex lines, without loss of generality, $0 \times \mathbb{C}, \mathbb{C} \times 0 \subset \mathbb{C}^2$. The lines correspond to $F$ and $G$, respectively. Hence the intersection numbers with the symplectic disks show that the induced homomorphism

$H_1(L) \rightarrow H_1(\mathbb{C}^2 \setminus (0 \times \mathbb{C} \cup \mathbb{C} \times 0))$
is an isomorphism. But \( \mathbb{C}^2 \setminus (0 \times \mathbb{C} \cup \mathbb{C} \times 0) \cong T^*T^2 \) symplectomorphically. The standard Clifford torus \( SS^1 \times SS^1 \subset \mathbb{C}^2 \) is thereby mapped to the zero section of the cotangent bundle \( O_{T^2} \subset T^*T^2 \). Moreover, the Lagrangian embedding \( L \subset T^*T^2 \) induces an isomorphism \( H_1(L) \cong H_1(T^2) \). By a result of Eliashberg and Polterovich \[8\] it follows that \( L \subset T^*T^2 \) is smoothly isotopic to \( O_{T^2} \) in \( T^*T^2 \) and hence the original \( L \subset \mathbb{C}^2 \) is smoothly isotopic to the Clifford torus. \( \square \)

**Remark 4.**

(1) The original idea of the proof goes back to earlier work \[19\], where we observed that there is no Lagrangian embedding of the Klein bottle into the complement of a complex line in \( \mathbb{C}^2 \) nontrivially linking it. The proof presented here would show that an embedding into \( \mathbb{C}^2 \) is symplectically isotopic to one in the complement of a complex line. The nontrivial linking is measured by the disk \( D \). Thus the existence of a Lagrangian embedding into \( \mathbb{CP}^2 \) would contradict the previous result. This idea gave the title for this exposition.

(2) Notice that such a result is bound to dimension 4 not only for the lack of homological intersection of pairs of holomorphic curves: In dimension 6 and higher there are Lagrangian embeddings of manifolds with first Betti number 1 (see \[19\]). Furthermore, the proof of the result does not seem to apply directly to the cases of Lagrangian embeddings of (non–orientable) surfaces of higher genus. Thus one could wonder if it is true at all in these cases.

(3) The argument in \[8\] fails to provide an isotopy through Lagrangian embeddings. Hence this stronger question remains open in both situations, \( L \subset \mathbb{C}^2 \) as well as \( L \subset T^*T^2 \).

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### 2. Geometric set–up and data

Let us first introduce and discuss the geometric data which will be used to define holomorphic curves.

**2.1. Topology and Dynamics of the Unit Cotangent Bundles.** Throughout let \( L \) be either the Klein bottle \( K^2 \) or the 2–torus \( T^2 \) with a flat Riemannian metric \( g \) fixed. Denote by
θ ∈ Ω¹(T*L) the canonical, or Liouville form. Remember that its differential dθ is a symplectic form on T*L. The restriction to the co-sphere bundle λ := θ|U*L, U*L := {α ∈ T*L | |α|g = 1} is a contact form. We denote by ξ := ker λ the corresponding contact distribution. The flow of its Reeb vector field Rλ, given by

(3)

\[ d\alpha(R_\lambda) = 0 \]
\[ \alpha(R_\lambda) = 1, \]

coinsides with the geodesic flow of the chosen metric. Hence, closed orbits of the Reeb flow, in short Reeb orbits, correspond exactly to closed oriented geodesics. Notice that the Reeb flow preserves λ, and therefore induces symplectic isomorphisms of the symplectic vector spaces (ξ, dλ). We will not distinguish (oriented) geodesics and their corresponding closed Reeb orbits by different notation. From the context it will be clear whether we denote by γ either of them. It will not necessarily be simple. By −γ we denote the geodesic with the opposite orientation or its corresponding Reeb orbit. Notice that the latter is geometrically different (actually disjoint) from γ.

The complement of the zero section 0_L in T*L is symplectomorphic to the symplectization of U*L:

(4)

\[ (T*L \setminus 0_L, d\theta) \cong (\mathbb{R} \times U*Ld(e^r\lambda)), \]
\[ v \mapsto (v, \log(|v|)), \]

where r ∈ ℝ denotes the parameter of the second factor. An almost complex structure J on (\mathbb{R} × U*L, d(e^r\lambda)) is called compatible (to λ) if

\[ J(\partial/\partial y) = R_\alpha \]
\[ J(\xi) = \xi \]
\[ d\lambda(., J.) > 0 \]
\[ \text{and symmetric} \]
\[ J \text{ invariant under translation.} \]

Therefore, J is compatible to the symplectic structure: d(e^rα)(., J.) is a Riemannian structure (a symmetric and positive definite bilinear form). It is a cone over U*L. Most importantly, cylinders over Reeb orbits are J–holomorphic. Finally, we call a complex structure J on T*L compatible (to dθ and λ) if dθ(., J.) defines a Riemannian metric and if J coincides with a λ-compatible almost complex structure on the complement T*L \ D^*_ρL of D^*_ρL := {α ∈ T*L | |α|g ≥ ρ} under the above identification \[ T^*L \setminus D^*_\rho L \cong (\log \rho, \infty) \times U*L. \]

On the other hand the metric g on L defines an almost complex structure J_g on T*L, which is compatible to dθ in the following way: we define a complex structure on T*L ⊕ TL via

\[ J_g(\alpha, v) := (-g(v, .), g^{-1} \alpha). \]

Then we use the Levi-Civita connection to globally split T(T*L) ≅ T*L ⊕ TL into horizontal and vertical part with respect to the fibration T*L → L. J_g is not translational invariant, and
hence not compatible to $\lambda$. Instead, we may use it to define a $\lambda$-compatible almost complex structure $J$: We set $J(x) := J_{g}(x)$ for $x \in U^*L$ and extend it to $\mathbb{R} \times U^*L$ via translation.

Given a geodesic $\gamma = \gamma(t)$ in $L$, the map $f_{\gamma} : \mathbb{C} \to T^*L$ given by

$$f_{\gamma}(s + it) := sg(\dot{\gamma}(t), \cdot)$$

is $J_{g}$–holomorphic. Since the image in the complement of $0_L$ consists of two cylinders over the closed Reeb orbits, $\mathbb{R} \times \pm \gamma$, these are also $J$–complex curves. But they will become holomorphic only for a different parameterization.

Now we use a cut–off function $\varphi = \varphi(|\alpha|)$ to interpolate between $J_{g}$ and $J$ and obtain a $d\theta$-compatible almost complex structure $J^-$ on $T^*L$ which coincides with $J_{g}$ in a neighborhood of the zero section $0L$ and with $J$ on $[-1, \infty) \times T^*_1L$ with respect to (4). Under these assumptions, the image of $f_{\gamma}$ is still a complex curve. By abuse of notation we will denote by $f_{\gamma}$ its $J^-$–holomorphic parameterization.

Closed geodesics on $L$ typically arise in 1–dimensional families. However the corresponding Reeb orbits are non–degenerate in the sense of Morse–Bott: any eigenvector of eigenvalue 1 of the linearized Poincaré return map restricted to $\xi$ (which is transversal to it) arises as the deformation of the closed Reeb orbit through a family of such (see [10]). Hence we may use the results of [10], or [4, 5] in our argument. The advantage over perturbing $\lambda$ such that there are only non–degenerate closed Reeb orbits in the strong sense is a clearer structure of the proof.

If $L$ is the flat Klein bottle there are also isolated geodesics, two simple ones and their odd multiples which are parallel (but not homotopic). They are all non-degenerate in the stronger sense. But notice that their even multiple covers are elements of 1–dimensional families of geodesics as before. Thus let us call the isolated geodesics odd and all those which arise in families even.

Now there are three indices one can assign to a geodesic $\gamma$, which is non-degenerate in the sense of Morse–Bott. First of all its Morse index $\text{index}(\gamma)$: it is the maximal dimension of a linear subspace in the space of vector fields at $L$ along $\gamma$ on which the Hessian of the length functional on loops in $L$ is negative definite.

For the other two indices we fix a trivialization of the symplectic vector bundle $(T(T^*L), d\theta)$. With that choice any loop $\gamma : SS^1 \to L$ defines a loop of Legendrian subspaces $\Gamma(x) = T_{\gamma(x)}L \subset \mathbb{C}^2$ and we may assign to the geodesic $\gamma$ the Maslov index $\mu(\gamma)$. It depends only on the homology class $[\gamma]$ of a loop in $L$ and hence defines a homomorphism

$$\mu : H_1(L; \mathbb{Z}) \longrightarrow \mathbb{Z}.$$ 

Recall that $\mu(\gamma)$ is even if and only if $\gamma^*TL$ is orientable. Notice that for $L = T^2$ we may choose the trivialization induced by a trivialization of $TL \to L$. In this case all Maslov–indices vanish.

On the other hand this trivialization induces one of the symplectic bundle $(\xi, d\lambda)$ via

$$\xi \oplus \mathbb{R}R_{\lambda} \oplus \mathbb{R} \frac{\partial}{\partial r} \cong T(T^*L)$$
on $U^*L$. Since the linearization of the Reeb flow induces symplectic isomorphisms of $(\xi, d\lambda)$ any closed Reeb orbit defines a path of symplectic $2 \times 2$–matrices, starting at the identity. Thus we can assign the Conley–Zehnder index $CZ(\gamma)$ to the closed Reeb orbit $\gamma$, which, in our situation, could be a half–integer (see [22]).

Viterbo found a striking relation between them:

Lemma 5 ([23]).

\[(6) \quad CZ(\gamma) = \mu(\gamma) + \text{index}(\gamma) + \frac{1}{2} \dim(\gamma).\]

Here $\dim(\gamma)$ is the dimension of the family of geodesics in which $\gamma$ arises.

Notice that Viterbo gave this formula in the non-degenerate case in the strong sense, but this identity follows from his result and the result of [22].

2.2. Almost complex structures with cylindrical ends. Let $L \subset \mathbb{CP}^2$ be a Lagrangian embedding. There is a (pseudo-convex) neighborhood $U$ of $L \subset \mathbb{CP}^2$ which is symplectomorphic to the unit disk bundle $D^*_1L \subset T^*L$ of $L$ after rescaling the flat metric on $L$. Fix an almost complex structure $J_0$ on $\mathbb{CP}^2$ which is compatible with the symplectic structure $\omega$, coincides with $J^-$ on $U$ using identification (4). These data define an $\omega$–compatible almost complex structure $J^+$ on $\mathbb{CP}^2 \setminus L$ which coincides with $J_0$ on $\mathbb{CP}^2 \setminus U$ and with $J$ on $U \setminus L$ again using (4). $\mathbb{CP}^2 \setminus L$ is said to have a concave end, and $J^+$ is compatible to $\omega$ and $\lambda$. Notice that the Riemannian metric defined by $\omega(., J^+)\omega$ has a singularity which is a cone over the unit cotangent bundle $U^*L$ equipped with the induced metric.

Now we deform $J_0$ into a family $J_\tau$ of $\omega$–compatible almost complex structures to ”approximate” $J^+$. For a convenient description we define spaces $X_\tau$ via

$$X_\tau := U \cup_{\partial U \times [-\tau, 0]} U^*L \times [0, \tau] \times U \cup_{\{0\} \times U^*L} \partial(\mathbb{CP}^2 \setminus U) \mathbb{CP}^2 \setminus U.$$

Then $J_\tau$ coincides with $J^+$ on $\mathbb{CP}^2 \setminus U$, with $J^-$ on $U$ and with $J$ on the cylinder. It is compatible to a symplectic structure $\omega_\tau$ which is given by $\omega$ on $\mathbb{CP}^2 \setminus U$, by $d(e^\tau \alpha)$ on $[-\tau, 0] \times U^*L$, and by the rescaled structure $e^{-\tau}d\theta$ on $U$. Notice that there is a diffeomorphism $\Phi_\tau : \mathbb{CP}^2 \cong X_\tau$ simply given by squeezing the long neck to its original size. With respect to this identification the cohomology classes of $[\omega_\tau]$ are constant $[\Phi_\tau^*\omega_\tau] = [\omega] \in H^2(X_\tau) \cong H^2(\mathbb{CP}^2)$. By Moser’s argument they are all symplectomorphic by diffeomorphisms which are supported in $U$. Hence $J_\tau$ can all be considered as almost complex structures on $\mathbb{CP}^2$ tamed by $\omega$. We remark that this is not crucially important for the result presented here but added for the convenience of the reader.

3. Holomorphic Curves

In this section we collect the parts of the theory necessary for the proof of our main result. This is a version of what is known about punctured holomorphic curves in symplectic manifolds with concave and convex ends (see [7] Chapters 1.1.–1.6, [3, 4, 6]) reduced to what is actually
needed in the argument. The geometric set–up, the compactness result and the index formula
are specialized to the situation at hand.

3.1. Punctured holomorphic spheres. Families of compact \( J_\tau \)–holomorphic curves in symplectic manifolds with ends split into \( J^\pm \) and \( J \)–holomorphic curves as \( \tau \to \infty \) (see below). As part of the limit appear so–called punctured pseudo-holomorphic curves. We may restrict our considerations to spheres:

**Definition 6.** A punctured holomorphic sphere in \( X = \mathbb{CP}^2 \setminus L, T^*L \) or \( \mathbb{R} \times U^*L \) is a punctured sphere \( (\mathbb{CP}^1, \overline{z}_1, \ldots, \overline{z}_p, \overline{z}_1, \ldots, \overline{z}_p) \) together with a map \( f \) holomorphic with respect to \( J^+, J^- \) or \( J \)

\[ f : \tilde{\Sigma} := \mathbb{CP}^1 \setminus \{ \overline{z}_1, \ldots, \overline{z}_p, \overline{z}_1, \ldots, \overline{z}_p \} \longrightarrow X. \]

For each of the punctures \( \overline{z}_i \) (\( \overline{z}_j \)) there exists a closed Reeb orbit \( \gamma(t) = \gamma_i, \gamma_j \), \( \dot{\gamma}(t) = R(\gamma(t)) \),

with the following property: In local complex coordinates \( (s, t) \in \mathbb{R}^+ \times S^1 \mapsto z_i + e^{-(s+it)} \)

\( ((s, t) \in \mathbb{R}^+ \times S^1 \mapsto z_i + e^{(s+it)}) \) around \( z_i \), the following limit is uniform with respect to \( t \)

\[ \lim_{s \to \pm \infty} f(s, t) = \gamma(\frac{T}{2\pi} t), \]

where \( T \) is the period (or action) of \( \gamma \) (see [10, 11]).

In the aforementioned papers the authors show that punctured \( J \)–holomorphic curves are asymptotic to cylinders over the corresponding closed Reeb orbits near their punctures with exponential decays with respect to \( s \).

The asymptotic behavior allows us to compactify \( u \) to a smooth map

\[ \overline{f} : \tilde{\Sigma} \longrightarrow \mathbb{CP}^2 \setminus L \]

or \( \overline{T^*L}, \mathbb{R} \times U^*L \), respectively, where \( \tilde{\Sigma} \) is the compactification of \( \Sigma \) by circles (one for each puncture) and \( \mathbb{CP}^2 \setminus L \) (\( T^*L, \mathbb{R} \times U^*L \)) are the compactifications by copies of \( U^*L \) each time using the cylindrical structures near the ends. The topology of these compactifications and the asymptotics mentioned above are such that the map \( \pi : \mathbb{CP}^2 \setminus L \to \mathbb{CP}^2 \) induced by the projection \( \pi : \tilde{\Sigma} \longrightarrow L \) is smooth. Hence, finally, we obtain a smooth map

\[ \hat{f} : \tilde{\Sigma} \longrightarrow \mathbb{CP}^2 \]

\[ \hat{f} = \pi \circ \overline{f} \]

with \( \hat{f}(\partial \tilde{\Sigma}) \subset L \).

Notice that the \( J^- \)–holomorphic curves \( f_\gamma \) in \( T^*L \) associated to a geodesic \( \gamma \) are examples of two–punctured holomorphic curves in the cotangent bundle. The next lemma summarizes all facts we will need here.

**Lemma 7.** (1) There are no one–punctured \( J^- \)–holomorphic spheres in \( T^*L \), and no one–punctured \( J \)–holomorphic spheres in \( \mathbb{R} \times U^*L \).
Any holomorphic sphere with two punctures in $T^*L$ which intersects the zero section, is of the form $f_\gamma$ for a closed geodesic $\gamma$ as introduced above.

The homological intersection index with $\mathbb{Z}_2$–coefficients between $f_\gamma$ and $0_L$ is 1 if $\gamma$ is an odd closed geodesic.

For each odd geodesic $\gamma$ there is exactly one two–punctured $J^–$–holomorphic sphere in $T^*L$ which is asymptotic to $+\gamma (−\gamma)$ at one of its punctures, namely $f_\gamma$.

Remark 8. The picture is somewhat incomplete, though it suffices for our purposes. We expect that for each rational slope the two–punctured spheres with asymptotics parallel to that slope foliate $T^*L$ similar to the $J_g$–holomorphic shifts of $f_\gamma$ by harmonic 1–forms. However, the shifts are not $J^–$–complex anymore which prevents this from being an elementary fact.

Proof. The first statement is obvious since there are no contractible closed geodesics for a flat structure on $L$.

Let $f$ be a two–punctured holomorphic curve with asymptotics $\gamma', \gamma''$ which intersects $0_L$. Since geodesics parallel to $\gamma'$ and $\gamma''$ foliate $L$ there is a $f_\gamma$ with $\gamma$ parallel to them which intersects $f$ in $f \cap 0_L$. Unless $f$ coincides with $f_\gamma$ (in which case we are done) they intersect in a discrete set of points with a finite positive (algebraic) intersection index in each of them (see [16] [18]. Pick one of these points then it will persist if we change $\gamma$ slightly. Thus we may assume that $±\gamma \neq \gamma', \gamma''$.

There is an involution on $L$ which preserves all geodesics parallel to $\gamma$ but reverses their orientations. That gives rise to a canonical involution on $T^*L$ which preserves $J^–$. Thus we obtain a curve $\tau(f)$ with asymptotics $−\gamma', −\gamma''$. Since $±\gamma \neq \gamma', \gamma''$ we obtain a 2–cycle in $T^*L$ via $\tau(f) + \hat{f} - f_{\gamma'} - f_{\gamma''}$ which non-trivially intersects $f_{\gamma}$. This is impossible, since the latter can be moved away from the zero section through shifting it by a nowhere vanishing 1–form and $H_2(T^*L; \mathbb{Z}_2) \cong H_2(L; \mathbb{Z}_2)$ is, of course, generated by $[0_L]$.

For the third statement we perform an explicit deformation of $0_L$ which intersects $f_\gamma$ once transversally. There is a one form $\beta \in \Omega^1(L)$ which vanishes on the geodesics parallel to $\gamma$ and intersects with $0_L$ in one geodesic $\gamma'$ perpendicular to $\gamma$. Thus image($\beta$) and $f_\gamma$ intersect in $\gamma \cap \gamma'$, and this intersection can be made easily transversal.

To prove the last assertion notice first of all that the two simple odd closed geodesics of the flat Klein bottle are not homotopic. Hence, if a two–punctured holomorphic curve $f$ is asymptotic to an odd geodesic $\gamma$ at one puncture it has to be asymptotic to $−\gamma$ at the other. Therefore, $\hat{f} - \hat{f}_\gamma$ forms a 2–cycle in $T^*L$ which intersects $0_L$ homological trivially. Thus, due to the previous fact the intersection index of $f$ and $0_L$ with $\mathbb{Z}_2$–coefficients is non-trivial. From the second assertion we conclude that $f = f_\gamma$.

3.2. The moduli of punctured $J$–holomorphic spheres. $J$–holomorphic curves described in the last section typically arise in families neglecting reparameterizations. Let us study deformations of such a $J$–holomorphic curve $f$. Notice that the Reeb orbits to which the curves are asymptotic to at their punctures may also vary within the family of Reeb orbits in which
they arise. Since all closed Reeb orbits (or geodesics) are non-degenerate in the sense of Morse–Bott the curves are asymptotic to cylinders over closed Reeb orbits near their punctures with exponential decays with respect to \( s \) (see [10, 11]). Hence tangencies to deformations are elements in the kernel of a Fredholm operator \( \partial_f \), which is defined using the linearization of the Cauchy–Riemann equation acting on sections of the pull-back, \( f^* T\mathbb{CP}^2 \) with similar asymptotics additionally taking into account the variation of the Reeb orbits (see [4], Paragraph 2.4 for details).

If \( \partial_f \) is surjective the virtual dimension actually coincides with the dimension of all possible deformations modulo reparameterizations. Such a punctured holomorphic curve is called regular. In this case the dimension is equal to the index of \( \partial_f \) which is given by the following formula (see [4], Proposition 2.7):

\[
\text{v–dim}(f) := \text{index}(\partial_f) = -(2 - \bar{p} - p) + \sum_{i=1}^{\bar{p}} (CZ(\gamma_i) + \frac{\dim(\gamma_i)}{2}) - \sum_{i=1}^{p} (CZ(\gamma_i) - \frac{\dim(\gamma_i)}{2}) + 2c_1(T\mathbb{CP}^2)[f].
\]

The number \( c_1(T\mathbb{CP}^2)[u] \) is the Chern number of the bundle extended into the punctures using the fixed trivialization of it in the neighborhood \( U \) of \( L \) given by that of \( T(T^*L) \to T^*L \). Here we use the symplectic identification of the neighborhood \( U \) of \( L \) with a neighborhood of the zero section \( V \supset 0_L \). The Conley–Zehnder index \( CZ \) is given with respect to the same trivialization.

Notice that all geodesics on the flat surface are minimizing. Thus their Morse indices \( \text{index}(\gamma) \) vanish altogether. Hence the virtual dimension of a holomorphic sphere \( f \) with \( p \) (negative) punctures in \( \mathbb{CP}^2 \setminus L \) is given by

\[
\text{v–dim}(f) = p - 2 - \sum_{i=1}^{p} \mu(\gamma_i) + 2c_1(T\mathbb{CP}^2)[f].
\]

3.3. Limits of smooth holomorphic spheres. In this section we describe the behavior of sequences of \( J_\tau \)-holomorphic curves as the parameter \( \tau \) increases. We need to describe the objects which will be the limits of sequences of \( J_N \)-holomorphic curves in \( \mathbb{CP}^2 \).

**Definition 9** (Broken holomorphic curves). A broken \( J_\infty \)-holomorphic sphere is a finite collection of punctured holomorphic curves \( F := (F^{(1)}, ..., F^{(N)}) \) where

\[
F^{(1)} : \dot{S}^{(1)} \to X^+ \\
F^{(k)} : \dot{S}^{(k)} \to \mathbb{R} \times M \quad \text{for} \ k = 2, ..., N - 1 \\
F^{(N)} : \dot{S}^{(N)} \to X^-
\]
and for each level $S^{(k)}$ denotes the disjoint union of a finite number of $\mathbb{C}P^1$'s with finite sets of (pairwise disjoint) positive and negative punctures $z^{(k)}_1, ..., z^{(k)}_p$ and $\bar{z}^{(k)}_1, ..., \bar{z}^{(k)}_q$ given on them. As before, we denote by $\hat{S}^{(k)}$ the corresponding punctured curves.

They satisfy the following conditions

1. $p^{(k)} = p^{(k+1)}$ with the understanding that the first level has no positive and the last no negative punctures: $p^{(1)} = p^{(N)} = 0$.
2. The closed Reeb orbits in $M$ which describe the asymptotics at the punctures agree correspondingly: $\gamma^{(k)}_j = \gamma^{(k+1)}_j$.
3. Each level is stable, i.e. $F^{(k)}$ contains a component which is not a cylinder over a closed Reeb orbit without marking.
4. We compactify $\hat{S}^{(k)}$ as described in the definition of punctured holomorphic spheres to obtain oriented surfaces with boundaries, $\tilde{S}^{(k)}$ which are spheres with holes.
5. A diffeomorphism used for gluing of the boundary components of $\tilde{S}^{(k)}$ and $\tilde{S}^{(k+1)}$ corresponding to $z^{(k)}_j$ and $\bar{z}^{(k)}_j$, is chosen in such a way that it commutes with the restrictions of $\tilde{F}^{(k)}$ and $\tilde{F}^{(k+1)}$ to them. There are $m(\gamma^{(k)}_j)$ possible choices. In the present work we do not care which we choose to formally glue the levels.
6. The surface obtained by gluing all boundary components and denoted by $\tilde{S}$ is diffeomorphic to the sphere. If we assign a graph to $F$ whose nodes correspond to the connected components of the $F^{(k)}$ and whose edges to the punctures of two adjacent levels which are identified, we therefore obtain a tree.

Notice that from the conditions on the asymptotics follows that $F$ induces a continuous map

$$F : \tilde{S} \longrightarrow X := \bar{X}^+ \cup_M \mathbb{R} \times M \cup_M \ldots \cup_M \bar{X}^-$$

given by the union of the closures of the $F^{(k)}$. Of course, the two spaces $\bar{X} \cong X$ are diffeomorphic. Hence the fundamental class associated to $F$, is naturally an element in the homology of $X$: $[F] \in H_2(X; \mathbb{Z})$.

Finally, notice that the univalent nodes (the branch tips) of the tree correspond to one–punctured spheres. Since there are no one–punctured spheres in $T^*L$ or $\mathbb{R} \times U^*L$ (see Lemma 7) these correspond to components of $F^{(1)}$ in $\mathbb{C}P^2 \setminus L$.

We are now in the position to formulate the splitting theorem for holomorphic curves in the course of splitting the underlying almost complex structures $J_\tau$ into two almost complex structures $J^\pm$ as described in Section refneck. Let $J := (J^+, J_M, J^-)$ be the triple of almost complex structures.

**Proposition 10 (K, G).** Let $f_n : \mathbb{C}P^1 \rightarrow X_{\tau_n} \cong \mathbb{C}P^2$ be a sequence of $J_{\tau_n}$–holomorphic curves, $\tau_n \rightarrow \infty$, with fixed fundamental class $[f_n] = A \in H_2(X; \mathbb{Z}) \cong \mathbb{Z}$, equal to the generator. Then there is a subsequence which converges to a broken $J$–holomorphic curve $F = (F^{(1)}, \ldots, F^{(N)})$ in the following sense:

There exists diffeomorphisms $\varphi_n : \tilde{S} \rightarrow \mathbb{C}P^1$ such that:
(1) \( \lim_{n \to \infty} \varphi_n^* j|_{\dot{S}(k)} = j^{(k)} \) in \( C^\infty_{\text{loc}}(\dot{S}(k)) \), \( j^{(k)} \) being the conformal structure of the punctured holomorphic curve \( \dot{S}(k) \) for all \( k \).

(2) \( \lim_{n \to \infty} f_n \circ \varphi_n|_{\dot{S}(k)} = F^{(k)} \) in \( C^\infty_{\text{loc}}(\dot{S}(k)) \).

Note that it follows that \( [F] = A \in H_2(X; \mathbb{Z}) \).

**Remark 11.** Usually there are two types of possible phenomena for such sequences of holomorphic curves: non-trivial "pinching" of closed curves (in special situation also known under a different name, "bubbling") as present in Gromov compactness and "breaking" apart into different levels of punctured holomorphic curves similar to the breaking of gradient trajectories in Floer theory. But if \( [f_n] \in H^2(\mathbb{C}P^2; \mathbb{Z}) \) are all equal to the generator there is no pinching. This can be seen as follows. Assume there is a curve \( \gamma \) pinching non-trivially. If we cut \( \mathbb{C}P^1 \) open along \( \gamma \) we will obtain two holomorphic curves of area greater than some threshold \( a_0 \). Assuming \( \gamma \) is very small with respect to the metric we may fill either of the parts by some disk with small symplectic area and obtain a nontrivial splitting of \( A \in H^2(\mathbb{C}P^2; \mathbb{Z}) \) into homology classes which both have positive symplectic area, which is impossible. Notice that the metric \( \omega(., J_\tau) \) degenerates as \( \tau \to \infty \). In the case that the pinching approaches the singular locus we have to make use of the fact, that any holomorphic curve with boundary \( \gamma \) either intersects \( \mathbb{C}P^2 \setminus U \) or the function \( |.|_g \) attains its maximum on the boundary. If the latter occurs for one of the two pieces of \( \mathbb{C}P^1 \setminus \gamma \) it turns out that in the limit there was no non-trivial pinching. Otherwise we obtain a contradiction as indicated above.

We need some more facts about the limit for the proof of the main result:

**Lemma 12.**

(1) With respect to any trivialization of \( T(T^*L) \) the Chern number of \( F^{(1)} \) satisfies

\[
C_1(F^{(1)}) = f_n^* c_1[\mathbb{C}P^1] = 3.
\]

(2) The virtual dimension of \( F^{(1)} \) turns out to be

\[
v\text{-dim}(F^{(1)}) = 4.
\]

Equality holds if and only if \( N = 2 \) and \( F^{(2)} \) consists of a disjoint union of 2–punctured \( J^- \)–holomorphic spheres in \( T^*L \).

(3) The multiplicity of any point \( x \in F^{(1)} \) or \( x \in F^{(N)} \) is

\[
m_x(F) = 1.
\]

**Proof.** There is a compact set \( K \subset \dot{S}^{(1)} \) such that \( (f_n \circ \varphi_n)^{-1}(\mathbb{C}P^2 \setminus U) \subset K \) for all \( n \) sufficiently large. Since \( f_n \circ \varphi_n|_K \) converge in \( C^\infty(K) \), the bundle \( (f_n \circ \varphi_n|_K)^*T\mathbb{C}P^2 \) does. On the other hand \( (f_n \circ \varphi_n)^*T\mathbb{C}P^2 \) is trivialized over \( (f_n \circ \varphi_n)^{-1}(U) \) containing all \( \dot{S}(k) \) for \( k > 1 \). By the choice of \( K \) we may define Chern numbers for \( (f_n \circ \varphi_n|_K)^*T\mathbb{C}P^2 \), which are equal to the Chern number of \( f_n \), \( f_n^* c_1(T\mathbb{C}P^2)|[\mathbb{C}P^1] = 3 \) by that remark.
The second statement is a corollary of the first. We find that

\[ v\text{-dim}(F^{(1)}) = -(2\iota^{(1)} - p^{(1)}) - \sum_{j=1}^{\ell^{(1)}} (C_{\mathbb{Z}}(\gamma^{(1)}_j) - \frac{\dim(\gamma^{(1)}_j)}{2}) + 2c_1(F^{(1)}) \]

\[ = 4 - \chi(F \setminus F^{(1)}) - \sum_{j} \mu(\gamma^{(1)}_j) \]

\[ = 4 - \chi(F \setminus F^{(1)}) \leq 4 \]

Here \( \iota^{(1)} \) denotes the number of connected components of the first level \( S^{(1)} \). The second line is due to Lemma 5 and the fact, that \( S \cong S^2 \), the third line uses that the projection to \( L \), \( \pi \circ (F \setminus F^{(1)}) \) provides a 2–chain in \( L \) whose boundary is

\[ \partial(\pi \circ (F \setminus F^{(1)})) = \sum_j [\gamma^{(1)}_j] \]

therefore providing the cancellation of the Maslov indices in the formula. Finally the last inequality uses the fact that there are no contractible geodesics in \( L \). Therefore the Euler characteristic of each connected component of \( F \setminus F^{(1)} \) is non–positive. Equality holds therefore if all components of \( F \setminus F^{(1)} \) consist of annuli. But notice that there the only holomorphic cylinders in \( \mathbb{R} \times U^*L \) (punctured holomorphic spheres with one positive and one negative puncture), due to the simple fact that closed geodesic (Reeb orbits) which are homotopic have the same length (action). Therefore, in this case \( F \setminus F^{(1)} = F^{(2)} \) consist of 2–punctured holomorphic spheres in \( T^*L \).

The last statement follows from the general principle we will use in the proof of the main result. Fix a non–singular point \( x \) on the top level, and a complex line \( \nu \subset T_x \mathbb{C}P^2 \). For any \( \tau \) there is a (unique) \( J_\tau \)–holomorphic sphere \( g_\tau \) through \( x \) and tangent to \( \nu \) representing the generator of \( H_2(\mathbb{C}P^2; \mathbb{Z}) \). For a subsequence of \( \tau_n \) of the compactness statement, we have convergence of the \( g_n := g_{\tau_n} \) in that sense, to a broken holomorphic sphere \( G \) passing through \( x \). Moreover, either the image is tangent to \( \nu \) or it has a cusp singularity at \( x \). In both cases the images of the two punctured holomorphic spheres do not coincide. Hence there is a neighborhood \( W \) of \( x \) such that \( F \cap G \cap W = \{x\} \) and they intersect at \( x \) with a finite algebraic intersection index (see \[ \text{[16],[18]} \]), which is equal to the product \( \iota_x(F,G) = m_x(G)m_x(F) \) of the multiplicities of the curves at \( x \). For \( n \) large enough the sum of intersection indices of points in \( f_n \cap g_n \cap W \) is equal to that same number. Since each further point of \( f_n \cap g_n \) would contribute a positive number to the total sum of intersection indices and the algebraic intersection index of \( [f_n] = [g_n] = [H] \) is \( f_n \cdot g_n = 1 \) we conclude that

\[ \iota_x(F,G) = m_x(F)m_x(G) = 1. \]

\[ \square \]

We choose the compatible almost complex structure \( J^+ \) on \( \mathbb{C}P^2 \setminus L \) such that all \textit{simple} punctured \( J^+ \)–holomorphic curves are regular. Much less than that is needed to achieve that
HOW TO (SYMPLECTICALLY) THREAD THE EYE OF A (LAGRANGIAN) NEEDLE

In particular, the linearization of the $\bar{\partial}$-equation at each punctured holomorphic sphere in $\mathbb{CP}^2$ which appears in the limits we consider, is surjective. That means by the implicit function theorem that the number of true deformation parameters of it is exactly given by the virtual dimension. Notice that we do not have to change the structure of $J^+$ on the ends since there is no non-constant punctured $J^+$–holomorphic curve in $U \setminus L \subset \mathbb{CP}^2 \setminus L$ by the usual argument using maximum principle (see [12]).

4. Proof of Theorem 3

We will successively construct two broken holomorphic spheres with relations to $L$ and to each other which will provide the disks and spheres we are searching for.

4.1. Construction of the disks. Choose a point $x \in L$ which, in the case that $L \cong K$ is the Klein bottle, lies on one of the odd geodesics. Pick a complex line $\nu \subset T_x \mathbb{CP}^2$ which is not tangent to $L$: $\nu \cap T_x L = \{0\}$. Now, for each parameter $\tau$ there is exactly one $J_\tau$–holomorphic sphere homologous to the generator of $H_2(\mathbb{CP}^2; \mathbb{Z})$ passing through $x$ and tangent to $\nu$. Let $f = (f^{(1)}, \ldots, f^{(N)})$ denote the broken holomorphic sphere which is the limit of a subsequence $f_{\tau_n}$ due to Proposition 10. Since all $f_\tau$ pass through $x \in L$ the lowest level $f^{(N)}$ contains a punctured holomorphic sphere which passes through $x$ and either has a cusp singularity at $x$ or is tangent to $\nu$. Due to Lemma 7, (1) and (2) it has to have at least three (positive) punctures. Pick any one of these punctures and remove the corresponding edge of the graph corresponding to $f$. Consider the connected component of the resulting graph which does not contain the node associated to the punctured holomorphic sphere through $x$. Pick one of the univalent nodes lying in that part. It corresponds to a one–punctured holomorphic sphere. Due to Lemma 7 (1), this will be a one-punctured $J^+$–holomorphic sphere in $\mathbb{CP}^2 \setminus L$. We compactify it to obtain one of the desired disks. Since the punctured holomorphic curve through $x$ has at least three punctures and due to Lemma 12 (3), we obtain three disjoint disks $C, D$ and $E$ with boundary on $L$.

4.2. Threading the eyes. We pick a point $y \in C \setminus L$, i.e. on the image of the corresponding one–punctured $J^+$–holomorphic sphere. Suppose that $y$ is a smooth point. Next choose a complex line $\eta \subset T_y \mathbb{CP}^2$. Assume that there is no punctured $J^+$–holomorphic sphere in $\mathbb{CP}^2 \setminus L$ which appears as a limit of $J_\tau$–holomorphic spheres homologous to the primitive class, has less than 4 deformation parameters and passes through $y$ and is tangent to $\eta$. Notice that the tangency makes a priori sense, since $y$ cannot be a singular point according to the proof of Lemma 12 (3). Notice also that this gadget can be achieved by choosing $(y, \eta)$ in general position, since this is a 4–dimensional family of parameters to choose from. As in the same proof we consider the family $g_\tau$ of unique $J_\tau$–holomorphic spheres passing through $y$ while being tangent to $\eta$. A subsequence of these converges to a broken holomorphic sphere $g$ with the same property. We claim that it contains only one level $g = (g^{(1)})$. Therefore, it will be a smooth $J^+$–holomorphic
spheretext{i.e. a mapping}
\[ g : \mathbb{CP}^1 \rightarrow \mathbb{CP}^2 \setminus L. \]
Assume on the contrary, that \( g = (g^{(1)}, \ldots, g^{(M)}) \) with \( M > 1 \). The component \( g' \) of \( g^{(1)} \) passing through \( y \) has at least 4 real deformation parameters. Since all components are simple and by the choice of \( J^+ \) thus regular, the virtual dimensions of each component of \( g^{(1)} \) is non-negative. By Lemma 12 (2), this is only possible if
\[ v\text{-dim}(g') = 4, \]
\( M = 2 \), and \( g^{(2)} \) consists of a disjoint union of two–punctured \( J^– \)-holomorphic spheres in \( T^*L \).
Notice that since the index of a component of \( g^{(1)} \) is either 0 or 4, it is always even. Hence, due to \([10]\) the Reeb orbit to which the one–punctured spheres are asymptotic to must all be odd. It is not difficult to see that therefore all components of \( g^{(2)} \) are of the form \( f_{\gamma} \), for an odd geodesic \( \gamma \). Since all components are simple (Lemma 12 (3)), the geodesics have to be simple. For the same reason each such \( f_{\gamma} \) occurs at most once. On the other hand, the number of such components of \( g \) has to be even. Indeed, due to Lemma 7 (3), the homological intersection of \( f_{\gamma} \) with \( 0_L \) with \( \mathbb{Z}_2 \) coefficients is 1. But \( L \) could be homotoped away from itself (since there are nowhere vanishing 1–forms on \( L \)). Since \( H_2(\mathbb{CP}^2; \mathbb{Z}_2) = \mathbb{Z}_2[H] \), \( H \) being the projective line, and \( H \cdot H = 1 \), we learn that \( [L] = 0 \in H_2(\mathbb{CP}^2; \mathbb{Z}_2) \) (that fact was brought to our attention by \([20]\)). Therefore, \( [g_\tau] \cdot L = 0 \) for all \( \tau \) and the above claim follows since \( f \) is the limit of a subsequence of these \( g_\tau \). Hence both possible, \( f_{\gamma} \) and \( f_{\gamma'} \), appear in \( g^{(2)} \). But then \( g \) intersects \( f \) also in \( x \) with a finite positive algebraic intersection index. Since \( g_\tau \cdot f_\tau = 1 \) we obtain a contradiction along the lines of the proof of Lemma 12 (3). Therefore, our assumption on \( M \) was wrong and we verified the claim. Notice that \( g \) may not intersect neither \( D \) nor \( E \), again due to the same reasoning using intersection indices. We call the so-obtained \( J^+ \)-holomorphic sphere without punctures \( H \).

The same procedure applied to \( D \) and \( E \) yields \( J^+ \)-holomorphic spheres without punctures \( F \) and \( G \) with the desired properties. Notice that since \( F, G \) and \( H \) are compact they are \( J_– \)-holomorphic for sufficiently large \( \tau \).

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