Modular Invariants, Graphs and \(\alpha\)-Induction for Nets of Subfactors. III

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Abstract

In this paper we further develop the theory of \(\alpha\)-induction for nets of subfactors, in particular in view of the system of sectors obtained by mixing the two kinds of induction arising from the two choices of braiding. We construct a relative braiding between the irreducible subsectors of the two “chiral” induced systems, providing a proper braiding on their intersection. We also express the principal and dual principal graphs of the local subfactors in terms of the induced sector systems. This extended theory is again applied to conformal or orbifold embeddings of \(SU(n)\) WZW models. A simple formula for the corresponding modular invariant matrix is established in terms of the two inductions, and we show that it holds if and only if the sets of irreducible subsectors of the two chiral induced systems intersect minimally on the set of marked vertices, i.e. on the “physical spectrum” of the embedding theory, or if and only if the canonical endomorphism sector of the conformal or orbifold inclusion subfactor is in the full induced system. We can prove either condition for all simple current extensions of \(SU(n)\) and many conformal inclusions, covering in particular all type I modular invariants of \(SU(2)\) and \(SU(3)\), and we conjecture that it holds also for any other conformal inclusion of \(SU(n)\) as well. As a by-product of our calculations, the dual principal graph for the conformal inclusion \(SU(3)_5 \subset SU(6)_1\) is computed for the first time.
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1 Introduction

In a previous paper [4], motivated by work of Feng Xu [44], we analyzed nets of subfactors \( \mathcal{N} \subset \mathcal{M} \) associated to type I (or block-diagonal) modular invariants through a notion of induction and restriction of sectors between the two nets of factors [3] — a notion introduced by Longo and Rehren in [31]. As the main application we considered type I modular invariants of \( SU(n) \).

Here we take the analysis further to understand the modular invariant matrix \( Z \) in terms of the inductions \( \alpha^+ \) and \( \alpha^- \), which depend on the choice
of braiding and opposite braiding in the $SU(n)_k$ sectors of the smaller net $\mathcal{N}$. In fact we find for our examples (and believe it to be true in general)

$$Z_{\Lambda,\Lambda'} = \langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda'} \rangle_M,$$

where $\langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda'} \rangle_M$ is the dimension of the intertwiner space $\text{Hom}_M(\alpha^+_{\Lambda}, \alpha^-_{\Lambda'})$ and where $\Lambda, \Lambda'$ are weights in the Weyl alcove $\mathcal{A}^{(n,k)}$, labelling the $SU(n)_k$ sectors, and $M$ is a local factor of the enveloping net $\mathcal{M}$.

We recall the story so far. The fusion graph of $[\alpha^+_{\Lambda_{(1)}}]$ (or $[\alpha^-_{\Lambda_{(1)}}]$), where $\Lambda_{(1)}$ is the (first) fundamental weight and corresponds to the generator of the $SU(n)_k$ fusion algebra, is the graph which in the $SU(2)$ case appears in the A-D-E classification of Capelli, Itzykson and Zuber [6] and empirically associated to the $SU(3)$ modular invariants by Di Francesco and Zuber [8, 9]. In general, the non-zero diagonal terms of the modular invariant matrix correspond exactly to the eigenvalues of (the adjacency matrix of) the fusion graph of $[\alpha^+_{\Lambda_{(1)}}]$ (or $[\alpha^-_{\Lambda_{(1)}}]$), as long as the fusion coefficients of the sectors of the extended theory are diagonalized by the corresponding modular S-matrix.

Let us restrict our discussion to the conformal inclusion case for a while. The set $\mathcal{T}$ of the original sectors of the induced net appears in the set $\mathcal{V}^+$ of irreducible subsectors of the induced system $\{[\alpha^+_{\Lambda}] : \Lambda \in \mathcal{A}^{(n,k)}\}$, and similarly in $\mathcal{V}^-$ corresponding to $\{[\alpha^-_{\Lambda}] : \Lambda \in \mathcal{A}^{(n,k)}\}$. Consequently, the “chiral” sets of sectors $\mathcal{V}^+$ and $\mathcal{V}^-$ intersect at least on $\mathcal{T}$, the “marked vertices”. Note that although there is a canonical bijection between $\mathcal{V}^+$ and $\mathcal{V}^-$ (see Subsect. 3.4 below), they rarely coincide as sets of sectors. Indeed it will be shown in Proposition 5.1 that the following conditions are equivalent:

- $Z_{\Lambda,\Lambda'} = \langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda'} \rangle_M$ for all $\Lambda, \Lambda' \in \mathcal{A}^{(n,k)}$,
- $\mathcal{T} = \mathcal{V}^+ \cap \mathcal{V}^-.$

Although it is shown that the matrix $\langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda'} \rangle_M$ is $T$-invariant (see Lemma 3.10 below) we have no direct proof why it is $S$-invariant or why either of the conditions holds in the general framework. However, the above conditions are also shown to be equivalent to either of the following which say that the set $\mathcal{V}$ of irreducible subsectors of the full induced system $\{[\alpha^+_{\Lambda} \circ \alpha^-_{\Lambda'}] : \Lambda, \Lambda' \in \mathcal{A}^{(n,k)}\}$ is complete in a certain sense:

- Each irreducible subsector of the canonical endomorphism sector $[\gamma]$ belongs to $\mathcal{V}$,
- $\sum_{x \in \mathcal{V}} d_x^2 = \sum_{\Lambda \in \mathcal{A}^{(n,k)}} d^2_{\Lambda}.$
Here the $d$’s denote the statistical dimensions of the sectors. In concrete examples, in particular for the conformal embeddings $SU(2)_4 \subset SU(3)_1$, $SU(2)_{10} \subset SO(5)_1$, $SU(2)_{28} \subset (G_2)_1$, $SU(3)_{3}_3 \subset SO(8)_1$, $SU(3)_{5} \subset SU(6)_1$, $SU(3)_9 \subset (E_6)_1$, $SU(3)_{21} \subset (E_7)_1$ and $SU(4)_4 \subset SO(15)_1$, such conditions can be shown to be satisfied, and thus $\mathcal{V}^+$ and $\mathcal{V}^-$ only intersect on the marked vertices or we recover the modular invariant matrix $Z$ from the induced $\langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda'} \rangle_M$.

The completeness of the induced system has another important aspect. If each irreducible subsector of the canonical endomorphism sector $[\gamma]$ belongs to $\mathcal{V}$, then, besides the principal graph, also the dual principal graph of the conformal inclusion subfactor is determined in terms of the induced system. We use this fact to compute the two basic graphical invariants of conformal inclusion subfactors in examples. This includes the computation of the dual principal graph for the conformal embedding $SU(3)_5 \subset SU(6)_1$ which has, to the best of our knowledge, not been obtained before.

We also extend the discussion of $\mathbb{Z}_n$ orbifold inclusions (or “simple current extensions”) in [9] to the $\mathbb{Z}_m$ case, where $m$ is any divisor of $n$ if $n$ is not prime, and this covers all simple current extension modular invariants of $SU(n)$. For these cases we can in fact show that $Z_{\Lambda,\Lambda'} = \langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda'} \rangle_M$ holds (see Theorem 6.9 below), and in consequence that each irreducible subsector of the canonical endomorphism sector $[\gamma]$ belongs to the induced system and that $\sum_{x \in \mathcal{V}} d_x^2 = \sum_{\Lambda \in A(n,k)} d_{\Lambda}^2$. The intersection $\mathcal{V}^+ \cap \mathcal{V}^-$ corresponds to the “localized sectors” or the “physical spectrum” of the extended theory and is expected to coincide with the labelling set of the conjectured extended $S$-matrix in [17]. In fact, we construct a non-degenerate braiding on this intersection (see Theorem 6.12 below), and by Rehren’s methods [35] this provides a representation of the modular group, thus a matrix $S$. Although we have no proof we expect that this is the $S$-matrix of [17].

Together with our conformal embedding examples we obtain completeness of the induced system for all the type I cases of the modular invariants of $SU(2)$ and $SU(3)$ which were classified by Cappelli, Itzykson and Zuber [6] and Gannon [18].

Ocneanu has classified in [33] irreducible bi-unitary connections on the A-D-E Dynkin diagrams. The family of bi-unitary connections as in Fig. [1], where the horizontal graph $G$ is an A-D-E Dynkin diagram and the vertical graphs are arbitrary, form a fusion ring with generators $W$ and $\overline{W}$. He then obtains the graphs of Figs. [2] [3] [8] and [9] below, the vertices describing all such irreducible connections, and the edges arise from the fusion graphs of the generators. The open string bimodule construction (see [1] for details) identifies such connections or vertices with bimodules, arising from
the Goodman-de la Harpe-Jones \([19]\) inclusion \(N \subset M\). If \(Z = (Z_{i,j})_{i,j=0}^k\) is the \(SU(2)_k\) modular invariant matrix associated to the graph \(\mathcal{G}\) by \([1]\) then the sum \(\sum_{j,j'} Z_{j,j'}^2\), coincides with the total number of vertices, and the irreducible \(M-M\) bimodules form a subset of the even vertices, which exhausts all of them in the \(E_6\) and \(E_8\) cases. Each non-zero entry \(Z_{j,j'}\) of the modular invariant matrix is claimed to be identified with an irreducible representation of the full fusion ring (cf. Proposition 5.3 and Conjecture 7.2 below). A relative braiding between the chiral halves is also constructed (cf. Proposition 3.12 below) which yields a braiding on the “ambichiral” intersection (cf. Corollary 3.13 below). The off-diagonal terms in the modular invariant matrix

\[
Z_{j,j'} = \sum_{t \in T} b_{t,j} b_{\omega(t),j'},
\]

is given a subfactor interpretation as \(b_{t,j} \equiv V_{j,1}^t\) can be computed in terms of “essential paths”; here \(V\) is the \(A - \mathcal{G}_{\text{flat}}\) intertwiner matrix introduced in \([9]\) (cf. also \([14, \text{Sect. 5.4}]\)). \(\mathcal{G}_{\text{flat}}\) the “flat part” of the graph \(\mathcal{G}\) (\(\mathcal{G}_{\text{flat}} = D_{10}\) for \(\mathcal{G} = E_7\), \(\mathcal{G}_{\text{flat}} = A_{4\ell-1}\) for \(\mathcal{G} = D_{2\ell+1}\), \(\ell = 2,3,\ldots\), and \(\mathcal{G}_{\text{flat}} = \mathcal{G}\) for the type I cases \(A, D_{\text{even}}, E_6\) and \(E_8\)), \(T\) the set of “marked vertices” of the modular invariant labelled by \(\mathcal{G}_{\text{flat}}\) and \(\omega\) the corresponding fusion rule automorphism of \(T\) (which is trivial in the type I case). The relationship between our net of subfactors approach and Ocneanu’s bimodule-connection approach will be discussed in \([5]\).

## 2 Preliminaries

In this section we recall several mathematical tools we use and the general framework of \([8,14]\).
2.1 Sectors between different factors

For our purposes it turns out to be convenient to make use of the formulation of sectors between different factors, we follow here (up to minor notational changes) Izumi’s presentation [22, 23] based on Longo’s sector theory [24].

Let $A$, $B$ infinite factors. We denote by $\text{Mor}(A, B)$ the set of unital $*$-homomorphisms from $A$ to $B$. For $\rho \in \text{Mor}(A, B)$ we define the statistical dimension $d_\rho = [B : \rho(A)]^{1/2}$, where $[B : \rho(A)]$ is the minimal index [24, 27].

A morphism $\rho \in \text{Mor}(A, B)$ is called irreducible if the subfactor $\rho(A) \subset B$ is irreducible, i.e. if the relative commutant $\rho(A)' \cap B$ consists only of scalar multiples of the identity in $B$. Two morphisms $\rho, \rho' \in \text{Mor}(A, B)$ are called equivalent if there exists a unitary $u \in B$ such that $\rho'(a) = u \rho(a) u^*$ for all $a \in A$. We denote by $\text{Sect}(A, B)$ the quotient of $\text{Mor}(A, B)$ by unitary equivalence, and we call its elements $B$-$A$ sectors. Similar to the case $A = B$, $\text{Sect}(A, B)$ has the natural operations, sums and products: For $\rho_1, \rho_2 \in \text{Mor}(A, B)$ choose generators $t_1, t_2 \in B$ of a Cuntz algebra $\mathcal{O}_2$, i.e. such that $t_1^* t_j = \delta_{i,j} 1$ and $t_1 t_1^* + t_2 t_2^* = 1$. Define $\rho \in \text{Mor}(A, B)$ by putting $\rho(a) = t_1 \rho_1(a) t_1^* + t_2 \rho_2(a) t_2^*$ for all $a \in A$, and then the sum of sectors is defined as $[\rho_1] \oplus [\rho_2] = [\rho]$. The product of sectors comes from the composition of endomorphisms, $[\rho_1] \times [\rho_2] = [\rho_1 \circ \rho_2]$. The statistical dimension is an invariant for sectors (i.e. equivalent morphisms have equal dimension) and is additive and multiplicative with respect to these operations. Moreover, for $[\rho] \in \text{Sect}(A, B)$ there is a unique conjugate sector $\overline{[\rho]} \in \text{Sect}(B, A)$ such that, if $[\rho]$ is irreducible and has finite statistical dimension, $[\rho] \times [\rho]$ contains the identity sector $[\text{id}_A]$ and $[\rho] \times [\rho]$ contains $[\text{id}_B]$ precisely once.

Then we denote a suitable representative endomorphism of $[\rho]$ naturally by $\overline{\rho}$, thus $\overline{[\rho]} = [\rho]$. For conjugates we have $d_{\overline{\rho}} = d_\rho$. As for bimodules one may decorate $B$-$A$ sectors $[\rho]$ with suffixes, $B[\rho]_A$, and then we can multiply $B[\rho]_A \times_A [\sigma]_B$ but not, for instance, $B[\rho]_A$ with itself. For $\rho, \tau \in \text{Mor}(A, B)$ we denote

$$\text{Hom}_{A,B}(\rho, \tau) = \{ t \in B : t \rho(a) = \tau(a)t, \ a \in A \}$$

and

$$\langle \rho, \tau \rangle_{A,B} = \dim \text{Hom}_{A,B}(\rho, \tau).$$

If $[\rho] = [\rho_1] \oplus [\rho_2]$ then

$$\langle \rho, \tau \rangle_{A,B} = \langle \rho_1, \tau \rangle_{A,B} + \langle \rho_2, \tau \rangle_{A,B}.$$

If $A = B$ we just write $\text{Hom}_A(\rho, \tau)$ and $\langle \rho, \tau \rangle_A$ for $\rho, \tau \in \text{Mor}(A, A) \cong \text{End}(A)$, as usual. If $C$ is another factor, $\rho \in \text{Mor}(A, B)$, $\sigma \in \text{Mor}(A, C)$,
\(\tau \in \text{Mor}(B, C)\) are morphisms with finite statistical dimension and \(\varpi \in \text{Mor}(C, B)\), \(\varphi \in \text{Mor}(B, A)\), representative conjugates of \(\rho\) and \(\tau\), respectively, then we have Frobenius reciprocity \([22, 33]\),

\[
\langle \tau \circ \rho, \sigma \rangle_{A,C} = \langle \rho, \tau \circ \sigma \rangle_{A,B} = \langle \tau, \sigma \circ \varphi \rangle_{B,C} .
\]

Now let \(N \subset M\) be an infinite subfactor of finite index. Let \(\gamma \in \text{End}(M)\) be a canonical endomorphism from \(M\) into \(N\) and \(\theta = \gamma|_N \in \text{End}(N)\). By \(\iota \in \text{Mor}(N, M)\) we denote the injection map, \(\iota(n) = n \in M, n \in N\). Then a conjugate \(\tau \in \text{Mor}(M, N)\) is given by \(\tau(m) = \gamma(m) \in N, m \in M\). (These formulae could in fact be used to define the canonical and dual canonical endomorphism.) Note that \(\gamma = \iota \circ \tau\) and \(\theta = \tau \circ \iota\). Denote by \(\mathcal{P}_0 \subset \text{Sect}(N), \mathcal{P}_1 \subset \text{Sect}(M, N), \mathcal{D}_0 \subset \text{Sect}(M)\) and \(\mathcal{D}_1 \subset \text{Sect}(N, M)\) the set of all irreducible subsectors of \([\theta^\ell], [\theta^\ell \circ \tau], [\gamma^\ell]\) and \([\gamma^\ell \circ \iota], \ell = 0, 1, 2, 3...,\) respectively. Note that there is a bijection from \(\mathcal{P}_1\) to \(\mathcal{D}_1\) arising from sector conjugation. The principal graph of the inclusion \(N \subset M\) is obtained as follows. The even vertices are labelled by the elements of \(\mathcal{P}_0\), the odd vertices by those of \(\mathcal{P}_1\), and we connect any even vertex labelled by \([\lambda] \in \mathcal{P}_0\) with any odd vertex labelled by \([\rho] \in \mathcal{P}_1\) by \(\langle \lambda \circ \tau, \rho \rangle_{N,M}\) edges. Similarly we obtain the dual principal graph. We label the even vertices by \(\mathcal{D}_0\) and the odd vertices by \(\mathcal{D}_1\), and then connect even vertices labelled by \([\beta] \in \mathcal{D}_0\) with odd vertices labelled by \([\tau] \in \mathcal{D}_1\) by \(\langle \beta \circ \iota, \tau \rangle_{M,N}\) edges.

### 2.2 Braiding

Let \(A\) be an infinite factor and \(\Delta \subset \text{End}(A)\) a subset such that \(\text{Ad}(u) \in \Delta\) for any unitary \(u \in A\) and \(\lambda \circ \mu \in \Delta\) whenever \(\lambda, \mu \in \Delta\), moreover, if \(t_1, t_2 \in A\) are Cuntz algebra \((O_2)\) generators, i.e. \(t_i^* t_j = \delta_{i,j} 1\) and \(t_1 t_1^* + t_2 t_2^* = 1\), and \(\lambda, \lambda_1, \lambda_2 \in \text{End}(A)\) such that \(\lambda(a) = t_1 \lambda_1(a) t_1^* + t_2 \lambda_2(a) t_2^*\) for all \(a \in A\), then \(\lambda \in \Delta\) whenever \(\lambda_1, \lambda_2 \in \Delta\) and conversely \(\lambda_1, \lambda_2 \in \Delta\) whenever \(\lambda \in \Delta\). In other words, \(\Delta\) is a set of representative endomorphisms of some set of sectors which is closed under products and sums and decomposition. We say that \(\Delta\) is braided if for any pair \(\lambda, \mu \in \Delta\) there is a unitary operator \(\varepsilon(\lambda, \mu) \in \text{Hom}_A(\lambda, \mu)\), called braiding operator, subject to initial conditions

\[
\varepsilon(\text{id}, \mu) = \varepsilon(\lambda, \text{id}) = 1 ,
\]

composition rules \((\nu \in \Delta)\)

\[
\varepsilon(\lambda \circ \mu, \nu) = \varepsilon(\lambda, \nu) \lambda(\varepsilon(\mu, \nu)) , \quad \varepsilon(\lambda, \mu \circ \nu) = \mu(\varepsilon(\lambda, \nu)) \varepsilon(\lambda, \mu) ,
\]

and whenever \(t \in \text{Hom}_A(\lambda, \mu)\) we have the naturality equations \((\rho \in \Delta)\)

\[
\rho(t) \varepsilon(\lambda, \rho) = \varepsilon(\mu, \rho) t , \quad t \varepsilon(\rho, \lambda) = \varepsilon(\rho, \mu) \rho(t) .
\]
Note that from Eqs. (2) and (3) one obtains easily the braiding fusion equations, that is if $s \in \text{Hom}_A(\lambda, \mu \circ \nu)$ we have
\begin{align*}
\rho(s) \varepsilon(\lambda, \rho) &= \varepsilon(\mu, \rho) \mu(\varepsilon(\nu, \rho)) s , \\
\varepsilon\rho(s) \mu(s) &= \mu\varepsilon(\rho, \lambda) \varepsilon(\rho, \mu) \rho(s) .
\end{align*}
(4)

Also note that for a unitary $u \in A$ the braiding operators transform as
\begin{align*}
\varepsilon(\text{Ad}(u) \circ \lambda, \mu) &= \mu(u) \varepsilon(\lambda, \mu) u^* , \\
\varepsilon(\lambda, \text{Ad}(u) \circ \mu) &= u \varepsilon(\lambda, \mu) \lambda(u)^* , \\
\varepsilon(\rho, \lambda) &= \sum_{i=1}^{n} t_i \varepsilon(\rho, \lambda_i) \rho(t_i^*) , \\
\varepsilon(\lambda, \rho) &= \sum_{i=1}^{n} \rho(t_i) \varepsilon(\lambda_i, \rho) t_i^* .
\end{align*}
(5)

Moreover, putting $\varepsilon^-(\lambda, \mu) = (\varepsilon^+(\mu, \lambda))^*$, $\varepsilon(\mu, \lambda) \equiv \varepsilon^+(\mu, \lambda)$ gives another “opposite” braiding, i.e. satisfying the same relations.

Now let $\mathcal{X} \subset \text{Sect}(A)$ be a sector basis. (A sector basis is a finite set of irreducible sectors of finite dimensions containing the trivial sector and being closed under conjugation and irreducible decomposition of sector products.) We obtain a set $\Delta \equiv \Delta(\mathcal{X}) \subset \text{End}(A)$ from $\mathcal{X}$ by taking all representative endomorphisms of all sector products and sums. We say $\mathcal{X}$ is braided if $\Delta$ is braided. Note that, if we take a choice of representatives for all elements of $\mathcal{X}$ and there is a collection of unitaries satisfying the braiding fusion relation Eq. (3) for these representatives, then we obtain a braiding of $\Delta$ by using Eqs. (2), (3) and (4) as definitions. In particular, if $\rho \in \text{End}(A)$ is a representative for $[\rho] \in \mathcal{X}$ (hence irreducible) then $\varepsilon(\text{id}, \rho)$ (and $\varepsilon(\rho, \text{id})$) is a phase, and from Eq. (4) with $\lambda = \mu = \nu = \text{id}, s = 1$, it follows that it is idempotent, hence the initial condition.

For a sector basis $\mathcal{X} \subset \text{Sect}(A)$ we may make a choice of representative endomorphisms, as usual denoted by $\lambda$ for $[\lambda] \in \mathcal{X}$. We call a braiding on $\mathcal{X}$ non-degenerate if for some $[\lambda] \in \mathcal{X}$ trivial monodromy, $\varepsilon(\lambda, \mu) \varepsilon(\mu, \lambda) = 1$ for all $[\nu] \in \mathcal{X}$ implies that $[\lambda]$ is the trivial sector. Note that by Eq. (3) this definition does not depend on the choice of representatives.

### 2.3 Nets of subfactors and $\alpha$-induction

Here we briefly review our basic notation and some results of our previous papers [3, 4]. There we considered certain nets of subfactors $\mathcal{N} \subset \mathcal{M}$...
on the punctured circle, i.e. we were dealing with a family of subfactors $N(I) \subset M(I)$ on a Hilbert space $\mathcal{H}$, indexed by the set $\mathcal{I}_z$ of open intervals $I$ on the unit circle $S^1$ that do neither contain nor touch a distinguished point “at infinity” $z \in S^1$. The defining representation of $\mathcal{N}$ possesses a subrepresentation $\pi_0$ on a distinguished subspace $\mathcal{H}_0$ giving rise to another net $\mathcal{A} = \{A(I) = \pi_0(N(I)), I \in \mathcal{I}_z\}$. We assumed this net to be strongly additive and to satisfy Haag duality, and also locality of the net $\mathcal{M}$. Fixing an interval $I_0 \in \mathcal{I}_z$ we used the crucial observation in [31] that there is an endomorphism $\gamma$ of the $C^*$-algebras $\mathcal{M}$ into $\mathcal{N}$ (the $C^*$-algebras associated to the nets are denoted by the same symbols as the nets itself, as usual) such that it restricts to a canonical endomorphism of $M(I)$ into $N(I)$ whenever $I \supset I_0$. By $\theta$ we denote its restriction to $\mathcal{N}$. We denote by $\Delta_N(I_0)$ the set of transportable endomorphisms localized in $I_0$. It is a result of [31] that $\theta \in \Delta_N(I_0)$. Elements of $\Delta_N(I_0)$ leave $N(I_0)$ invariant and can therefore also be considered as elements of $\text{End}(N(I_0))$. The elements of $\Delta_N(I_0)$ are braided endomorphisms, and the braiding operators $\varepsilon(\lambda, \mu) \equiv \varepsilon^+(\lambda, \mu)$, $\varepsilon^-(\lambda, \mu) = (\varepsilon^+(\mu, \lambda))^*$, $\varepsilon^\pm(\lambda, \mu) \in \text{Hom}_{N(I_0)}(\lambda, \mu)$, for $\lambda, \mu \in \Delta_N(I_0)$ are given by the DHR statistics operators [11, 21]. The $\pm$-sign here is due to the two possibilities of the statistics operators coming from the non-trivial space-time topology of the punctured circle (see [13, 14]). Therefore the two statistics operators, corresponding to braiding and opposite braiding, are in general different but they may coincide for some $\lambda$ and $\mu$. We call $\alpha$-induction the two maps $\Delta_N(I_0) \to \text{End}(\mathcal{M})$, $\lambda \mapsto \alpha^\pm_\lambda$, where

$$\alpha^\pm_\lambda = \gamma^{-1} \circ \text{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \gamma.$$

As endomorphisms in $\Delta_N(I_0)$ leave $N(I_0)$ invariant it makes sense to define the quotient $[\Delta_N(I_0)]$ by inner equivalence in $N(I_0)$. Similarly, the endomorphisms $\alpha^\pm_\lambda$ leave $M(I_0)$ invariant, hence we can consider them also as elements of $\text{End}(M(I_0))$ and form their inner equivalence classes $[\alpha^\pm_\lambda]$ in $M(I_0)$. We derived that in terms of these sectors, $\alpha$-induction preserves sums and products, and we proved

$$\langle \alpha^\pm_\lambda, \alpha^\pm_\mu \rangle_{M(I_0)} = \langle \theta \circ \lambda, \mu \rangle_{N(I_0)}, \quad \lambda, \mu \in \Delta_N(I_0),$$

which is useful to determine the structure of the induced sectors. We also have a map $\text{End}(\mathcal{M}) \to \text{End}(\mathcal{N})$, $\beta \mapsto \sigma_\beta$, where $\sigma_\beta = \gamma \circ \beta|_N$, called $\sigma$-restriction. If $\beta$ is a localized (in $I_0$) and transportable endomorphism of $\mathcal{M}$ which leaves $M(I_0)$ invariant (the invariance follows automatically from the localization in $I_0$ if the net $\mathcal{M}$ is Haag dual — as is the case in all our applications) then $\sigma_\beta \in \Delta_N(I_0)$, and then we have also $\alpha\sigma$-reciprocity, $\langle \alpha, \sigma_\beta \rangle_{M(I_0)} = \langle \lambda, \sigma_\beta \rangle_{N(I_0)}$.
In [4] we have already applied this theory to several conformal and to the 
Zₙ orbifold inclusions of SU(n). If SU(n)_k ⊂ G₁ is a conformal inclusion at level k with G some connected compact simple Lie group then the associated net of subfactors is given in terms of the local inclusions defined by local loop groups,

\[ N(I) = π^0(L_I SU(n))'' \subset π^0(L_I G)' = M(I), \]

where π₀ is the level 1 vacuum representation of LG. In the orbifold case the net of subfactors is obtained by a certain crossed product construction from the net of factors A(I) = π₀(L_I SU(n))'', relative to a simple current; here π₀ denotes the level k vacuum representation of LSU(n). The orbifold inclusions appear only for certain values of the level, and this turns out to be related to the locality condition of the extended net. In any case we apply induction to the set of sectors \[ \Lambda \] which correspond to the positive energy representation πₖ of LSU(n), \( \Lambda \in \mathcal{A}^{(n,k)} \), and obey the Verlinde fusion rules by the results of Wassermann [42].

3 Mixing Two Inductions

From now on let us assume that we are dealing with a given quantum field theoretic net of subfactors \( \mathcal{N} \subset \mathcal{M} \) over the index set \( \mathcal{J} \) as in [3], i.e. we assume that \( \mathcal{N} \) is strongly additive and Haag dual in the vacuum representation and we assume \( \mathcal{M} \) to be local. We also assume the index to be finite. We fix an arbitrary interval \( I_o \in \mathcal{J} \) and take the endomorphism \( γ \in \text{End}(\mathcal{M}) \) of [31] which restricts to a canonical endomorphism from \( M(I_o) \) into \( N(I_o) \), and we denote \( \theta = γ|_\mathcal{N} \). To simplify notation, we will abbreviate \( N = N(I_o) \) and \( M = M(I_o) \) for the rest of this paper.

3.1 Subsectors of \([α_+^\lambda] \) and \([α_-^\lambda] \)

We now consider \( α \)-induction defined by means of the two different braidings simultaneously.

Lemma 3.1 Let \( λ \in \Delta_N(I_o) \) and \( β \in \text{End}(M) \) such that \([ β ]\) is a subsector of \([α_+^\lambda] \). If there is a \( μ \in \Delta_N(I_o) \) such that \([ β ]\) is a subsector of \([α_+^\mu] \) as well, then \([ β ]\) is also subsector of \([α_+^\lambda] \).

Proof. By assumption there is an isometry \( t \in \text{Hom}_M(β, α_+^\lambda) \). Now if \([ β ]\) is also a subsector of \([α_+^\mu] \) for some \( μ \in \Delta_N(I_o) \) then we have an isometry
Lemma 3.2 Let $\beta_i \in \text{End}(M)$ such that $[\beta_i]$ is a subsector of $[\alpha^\pm_\mu]$, $\lambda_i \in \Delta_N(I_\circ)$, $i = 1, 2$. If $u \in M$ fulfills $u \beta_1(n) = \beta_2(n) u$, $n \in N$, then $u \in \text{Hom}_M(\beta_1, \beta_2)$.

Proof. By assumption there are isometries $t_i \in \text{Hom}_M(\beta_i, \alpha^\pm_\mu)$. If $u \in M$ fulfills $u \beta_1(n) = \beta_2(n)u$, then $t_2ut_1^* \lambda_1(n) = \lambda_2(n)t_2ut_1^*$, $n \in N$, hence $t_2ut_1^* \in \text{Hom}_M(\alpha^\pm_{\lambda_1}, \alpha^\pm_{\lambda_2})$ by [3, Lemma 3.5], and thus

$$u \beta_1(m) = t_2^* t_2 u t_1^* \alpha^\pm_{\lambda_1}(m) t_1 = t_2^* \alpha^\pm_{\lambda_2}(m) t_2 ut_1^* t_1 = \beta_2(m) u,$$

we are done. \hfill \Box

Next we present a slight generalization of our “main formula”, [3, Thm. 3.9]. Recall that $v \in M$ and $w \in N$ are the intertwining isometries from the identity of $M$ and $N$ to $\gamma$ and $\theta$, respectively, and satisfying $w^* v = [M : N]^{-1/2} 1$. Also recall that we have pointwise equality $M = Nv$ [3].

Proposition 3.3 Let $\beta \in \text{End}(M)$ such that $[\beta]$ is a subsector of $[\alpha^\pm_\mu]$ for some $\mu \in \Delta_N(I_\circ)$. Then we have

$$\langle \alpha^\pm_\lambda, \beta \rangle_M = \langle \lambda, \sigma_\beta \rangle_N$$

for all $\lambda \in \Delta_N(I_\circ)$.

Proof. First we show “$\leq$”: Assume $s \in \text{Hom}_M(\alpha^\pm_\lambda, \beta)$. Then, by restriction, $s\lambda(n) = \beta(n) s$ for all $n \in N$, hence $\gamma(s) \in \text{Hom}_N(\theta \circ \lambda, \sigma_\beta)$, hence $\gamma(s) w \in \text{Hom}_N(\lambda, \sigma_\beta)$. As the map $s \mapsto \gamma(s) w$ is injective [3, Lemma 3.8], this proves “$\leq$”.

Next we show “$\geq$”: Let $r \in \text{Hom}_N(\lambda, \sigma_\beta)$. Put $s = v^* r$. Then

$$s \lambda(n) = v^* r \lambda(n) = v^* \gamma \circ \beta(n) \cdot r = \beta(n) v^* r = \beta(n) s, \quad n \in N.$$
By Lemma 3.2 it follows \( s \in \text{Hom}_M(\alpha^+_\lambda, \beta) \). Now the map \( r \mapsto s = v^* r \) is injective, proving “\( \geq \)”.

Let \( \lambda \in \Delta_N(I_0) \) and let

\[
\alpha^+_\lambda(m) = \sum_a t_a \beta^+_a(m) t_a^*, \quad m \in M,
\]

be an irreducible decomposition with some set \( \{t_a\}_a \) of Cuntz algebra generators. Here we allow multiplicities, so some of the \( \beta^+_a \)'s may be equivalent. Now

\[
t_a t_a^* \in \alpha^+_\lambda(M)' \cap M = \lambda(N)' \cap M = \alpha^-_\lambda(M)' \cap M,
\]

again by \( \chi \). Lemma 3.5]. Therefore putting

\[
\beta^-_a(m) = t_a^* \alpha^-_\lambda(m) t_a, \quad m \in M,
\]

defines endomorphisms of \( M \) and we have

\[
\alpha^-_\lambda(m) = \sum_a t_a \beta^-_a(m) t_a^*, \quad m \in M.
\]

Clearly, \( \beta^+_a(n) = t_a^* \lambda(n) t_a = \beta^-_a(n) \) for all \( n \in N \). We now easily obtain from Lemma 3.2 the following

**Corollary 3.4** The \( \beta^-_a \)'s are irreducible as well. Moreover, if \( [\beta^+_a] = [\beta^-_b] \) then \( [\beta^+_a] = [\beta^-_a] = [\beta^+_a] = [\beta^-_b] \). We have \( [\beta^+_a] = [\beta^-_b] \) if and only if \( [\beta^-_a] = [\beta^-_b] \).

We also find

**Lemma 3.5** We have \( d_{\beta^+_a} = d_{\beta^-_a} \).

*Proof.* Since \( \sigma_{\beta^+_a} \) is the restriction of \( \gamma \circ \beta^+_a \) to \( N \) we have \( d_{\beta^+_a} = d_{\gamma} d_{\beta^+_a} \), cf. \( \chi \) Subsect. 3.3], but also \( \sigma_{\beta^+_a} = \sigma_{\beta^-_a} \), implying the statement.

3.2 Comparing \([\alpha^+_\lambda]\) and \([\alpha^-_\mu]\)

For \( \mu \in \Delta_N(I_0) \) define

\[
\text{triv}(\mu, \theta) = \{ t \in M : \varepsilon(\mu, \theta) \varepsilon(\theta, \mu) \gamma(t) = \gamma(t) \}.
\]

Recall from \( \chi \) Subsect. 3.1] that we have \( \alpha^\pm_\mu(v) = \varepsilon^\pm(\mu, \theta)^* v \).
Lemma 3.6 For $\lambda, \mu \in \Delta_N(I_o)$ we have

$$\text{Hom}_M(\alpha^\lambda_\lambda, \alpha^-\mu) = \text{Hom}_M(\alpha^\lambda_\lambda, \alpha^-\mu) \cap \text{triv}(\mu, \theta).$$

(8)

Proof. Ad “$\subset$”: Let $t \in \text{Hom}_M(\alpha^\lambda_\lambda, \alpha^-\mu)$. Restriction and [3, Lemma 3.5] clearly implies $t \in \text{Hom}_M(\alpha^\lambda_\lambda, \alpha^-\mu)$. Moreover, $t \in \text{Hom}_M(\alpha^\lambda_\lambda, \alpha^-\mu)$ implies also

$$t \alpha^\lambda\lambda(v) = \alpha^-\mu(v) t = \varepsilon^-(\mu, \theta)^* v t = \varepsilon(\theta, \mu) \gamma(t)v .$$

Whereas $t \in \text{Hom}_M(\alpha^\lambda_\lambda, \alpha^\mu_\mu)$ yields

$$t \alpha^\lambda\lambda(v) = \alpha^\mu\mu(v) t = \varepsilon(\mu, \theta)^* \gamma(t)v ,$$

thus

$$\varepsilon(\mu, \theta) \varepsilon(\theta, \mu) \gamma(t)v = \gamma(t)v .$$

By [3, Lemma 3.8] this implies

$$\varepsilon(\mu, \theta) \varepsilon(\theta, \mu) \gamma(t) = \gamma(t) ,$$

proving “$\subset$”.

Ad “$\supset$”: Let $t \in \text{Hom}_M(\alpha^\lambda_\lambda, \alpha^\mu_\mu) \cap \text{triv}(\mu, \theta)$. As $\alpha^\lambda_\lambda$ and $\alpha^\mu_\mu$ restrict to $\lambda$ and $\mu$ on $N$, respectively, it suffices to show $t \alpha^\lambda\lambda(v) = \alpha^-\mu(v) t$. From $\varepsilon(\mu, \theta) \varepsilon(\theta, \mu) \gamma(t) = \gamma(t)$ we obtain $\varepsilon(\mu, \theta)^* \gamma(t) = \varepsilon^-(\mu, \theta)^* \gamma(t)$, hence

$$t \alpha^\lambda\lambda(v) = \alpha^\mu\mu(v) t = \varepsilon^+(\mu, \theta)^* \gamma(t)v = \varepsilon^-(\mu, \theta)^* vt = \alpha^-\mu(v) t ,$$

proving the lemma.

Trivially, we obtain

Corollary 3.7 We have $\langle \alpha^\lambda_\lambda, \alpha^-\mu \rangle_M \leq \langle \alpha^\lambda_\lambda, \alpha^\mu_\mu \rangle_M$ for all $\lambda, \mu \in \Delta_N(I_o)$.

For a reducible $\mu \in \Delta_N(I_o)$ take an irreducible decomposition

$$\mu(n) = \sum_{i=1}^{s} t_i \mu_i(n) t_i^* , \quad n \in N ,$$

with $\mu_i \in \Delta_N(I_o)$ and a set $\{ t_i : i = 1, 2, \ldots, s \}$ of Cuntz algebra generators in $N$. We allow that some of the irreducible $\mu_i$’s may be equivalent.
Lemma 3.8 For $\lambda \in \Delta_N(I_o)$ and $\mu$ as above we have $\varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda) = 1$ if and only if $\varepsilon(\lambda, \mu_i)\varepsilon(\mu_i, \lambda) = 1$ for all $i = 1, 2, \ldots, s$.

Proof. Since $t_i \in \text{Hom}(\mu_i, \mu)$ we have the naturality equations

$$\lambda(t_i)\varepsilon(\mu_i, \lambda) = \varepsilon(\lambda, \mu) t_i, \quad t_i \varepsilon(\lambda, \mu_i) = \varepsilon(\lambda, \mu) \lambda(t_i).$$

Therefore

$$\varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda) = \sum_{i=1}^{s} \varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda) t_i t_i^* = \sum_{i=1}^{s} t_i \varepsilon(\lambda, \mu_i)\varepsilon(\mu_i, \lambda) t_i^*,$$

hence $\varepsilon(\lambda, \mu_i)\varepsilon(\mu_i, \lambda) = t_i^*\varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda)t_i$.

Now let $[\theta] = \bigoplus_{i=1}^{s} [\theta_i]$ be an irreducible decomposition where $\theta_i \in \Delta_N(I_o)$, $i = 1, 2, \ldots, s$. We recall that the monodromy is diagonalized as follows (see also Sect. 8.2, in particular Figs. 8.30 and 8.31):

$$\varepsilon(\lambda, \mu)\varepsilon(\mu, \nu)\varepsilon(\nu, \mu) t = \frac{\kappa_\lambda}{\kappa_\mu} t, \quad t \in \text{Hom}_N(\lambda, \nu \circ \mu), \quad (9)$$

for irreducible $\lambda, \mu, \nu \in \Delta_N(I_o)$, where the $\kappa$'s are the statistical phases which are invariants of the sectors. Now for any $\lambda \in \Delta_N(I_o)$ write the statistical phase as $\kappa_\lambda = e^{2\pi i h_\lambda}$ with some $h_\lambda \geq 0$. In our applications, $h_\lambda$ will be the conformal dimension of the sector $[\lambda]$, and for the subssectors of $[\theta]$ we will also have $h_{\theta_i} = 0 \mod Z$, i.e. $\kappa_{\theta_i} = 1$ for all $i = 1, 2, \ldots, s$. We then obtain easily from [3] Prop. 3.23 the following

Corollary 3.9 Let $\lambda \in \Delta_N(I_o)$ be irreducible. If there is an $i = 1, 2, \ldots, s$ and a $\mu \in \Delta_N(I_o)$ such that $\kappa_{\theta_i} = 1$, $N_{[\theta_i],[\lambda]}^{[\mu]} \equiv \langle \mu, \theta_i \circ \lambda \rangle_N \neq 0$ and $h_\mu \neq h_\lambda \mod Z$, then $[\alpha_{\lambda}^+] \neq [\alpha_{\lambda}^-]$.

Similarly we have

Lemma 3.10 Assume $\kappa_{\theta_i} = 1$ for all $i = 1, 2, \ldots, s$ and let $\lambda, \mu \in \Delta_N(I_o)$ be irreducible. Then $\langle \alpha_{\lambda}^+, \alpha_{\mu}^- \rangle_M = 0$ if $h_\mu \neq h_\lambda \mod Z$.

Proof. Assume that $\langle \alpha_{\lambda}^+, \alpha_{\mu}^- \rangle_M \neq 0$, i.e. there is a non-zero intertwiner $t \in \text{Hom}_M(\alpha_{\lambda}^+, \alpha_{\mu}^-)$. It follows that $t\lambda(n) = \mu(n)t$ for all $n \in N$, hence $\gamma(t) \in \text{Hom}_N(\theta \circ \lambda, \theta \circ \mu)$, and we have $\varepsilon(\mu, \theta)\varepsilon(\theta, \mu)\gamma(t) = \gamma(t)$ by Lemma 3.6. It follows that

$$\varepsilon(\mu, \theta)\varepsilon(\theta, \mu)\gamma(t)w = \gamma(t)w,$$
hence with isometries $w_i \in \text{Hom}_N(\theta_i, \theta)$ such that $\theta(n) = \bigoplus_{i=1}^s w_i \theta_i(n) w_i^*$ for $n \in N$ (so that one may choose $w_1 = w$) we obtain

$$\varepsilon(\mu, \theta_i) \varepsilon(\theta_i, \mu) w_i^* \gamma(t) w = w_i^* \gamma(t) w$$

for all $i = 1, 2, \ldots, s$. Now $w_i^* \gamma(t) w \in \text{Hom}_N(\lambda, \theta_i \circ \mu)$, hence this is

$$\frac{\kappa_\lambda}{\kappa_\mu} w_i^* \gamma(t) w = w_i^* \gamma(t) w.$$

Multiplying by $w_i$ from the left and summing over $i$ yields

$$\frac{\kappa_\lambda}{\kappa_\mu} \gamma(t) w = \gamma(t) w,$$

and $\gamma(t) w \neq 0$ since $t \neq 0$ by [3, Lemma 3.8], hence $\kappa_\lambda = \kappa_\mu$. \hfill \qed

### 3.3 A relative braiding

Representative endomorphisms of subsectors of $[\alpha^+]$ (or $[\alpha^-]$) will not possess a braiding since they do not even commute as sectors in general. However, we have seen in [3, Prop. 3.26] that if $\beta, \delta \in \text{End}(M)$ are such that $[\beta]$ is a subsector of $[\alpha^+]$ and $[\delta]$ is a subsector of $[\alpha^-]$ for some $\lambda, \mu \in \Delta_N(I_o)$, then $[\beta]$ and $[\delta]$ commute, $[\beta \circ \delta] = [\delta \circ \beta]$, and that a relevant unitary which we will now denote by $\varepsilon_r(\beta, \delta)$ is given by

$$\varepsilon_r(\beta, \delta) = s^* \alpha^- \varepsilon(\lambda, \mu) \alpha^+(s) t \in \text{Hom}_M(\beta \circ \delta, \delta \circ \beta)$$

with isometries $t \in \text{Hom}_M(\beta, \alpha^+)$ and $s \in \text{Hom}_M(\delta, \alpha^-)$.

**Lemma 3.11** The operator $\varepsilon_r(\beta, \delta)$ of Eq. (10) does not depend on $\lambda, \mu \in \Delta_N(I_o)$ and not on the isometries $s, t$, in the sense that, if there are isometries $x \in \text{Hom}_M(\beta, \alpha^+)$ and $y \in \text{Hom}_M(\delta, \alpha^-)$ with some $\nu, \rho \in \Delta_N(I_o)$, then

$$\varepsilon_r(\beta, \delta) = y^* \alpha^- \varepsilon(\nu, \rho) \alpha^+(y)x.$$  \hfill (11)

**Proof.** If $x, y$ are as above then clearly $xt^* \in \text{Hom}_M(\alpha^+, \alpha^-)$ and $sy^* \in \text{Hom}_M(\alpha^-, \alpha^-)$. Hence $\alpha^-(xt^*) \varepsilon(\lambda, \mu) = \varepsilon(\nu, \mu) xt^*$ and also $sy^* \varepsilon(\nu, \rho) = 

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\(\varepsilon(\nu, \mu)\alpha^+_\nu(s\gamma^*)\) by [3, Lemma 3.25], therefore
\[
y^*\alpha^-_\rho(x^*)\varepsilon(\nu, \rho)\alpha^+_\nu(y)x = s^*s^*\alpha^-_\rho(x^*)\varepsilon(\nu, \rho)\alpha^+_\nu(y)xt^*t
\]
\[
s^*\alpha^+_\mu(x^*)\varepsilon(\nu, \rho)\alpha^+_\nu(y)xt^*t
\]
\[
s^*\alpha^-_\mu(x^*)\varepsilon(\nu, \rho)\alpha^+_\nu(s)xt^*t
\]
\[
s^*\alpha^-_\mu(x^*)\varepsilon(\nu, \rho)\alpha^+_\nu(s)t
\]
\[
s^*\alpha^-_\mu(t^*)\varepsilon(\lambda, \mu)\alpha^+_\lambda(s)\tau,
\]
yielding the invariance. \(\square\)

We will now strengthen the relative commutativity statement of [3, Prop. 3.26] to the following

**Proposition 3.12** The system of unitaries of Eq. (10) provides a relative braiding between representative endomorphisms of subsectors of \(\alpha^+_\lambda\) and \(\alpha^-_\mu\) in the sense that, if \(\beta, \delta, \omega, \xi \in \text{End}(M)\) are such that \([\beta], [\delta], [\omega], [\xi]\) are subsectors of \([\alpha^+_\lambda], [\alpha^-_\mu], [\alpha^+_\mu], [\alpha^-_\mu]\), respectively, \(\lambda, \mu, \nu, \rho \in \Delta_N(I_o)\), then we have initial conditions

\[
\varepsilon_\tau(\text{id}_M, \delta) = \varepsilon_\tau(\beta, \text{id}_M) = 1, \quad (12)
\]

composition rules

\[
\varepsilon_\tau(\beta \circ \omega, \delta) = \varepsilon_\tau(\beta, \delta)\beta(\varepsilon_\tau(\omega, \delta)), \quad \varepsilon_\tau(\beta, \delta \circ \xi) = \delta(\varepsilon_\tau(\beta, \xi))\varepsilon_\tau(\beta, \delta), \quad (13)
\]

and whenever \(q_+ \in \text{Hom}_M(\beta, \omega)\) and \(q_- \in \text{Hom}_M(\delta, \xi)\) then

\[
\delta(q_+)\varepsilon_\tau(\beta, \delta) = \varepsilon_\tau(\omega, \delta)q_+, \quad q_-\varepsilon_\tau(\beta, \delta) = \varepsilon_\tau(\beta, \xi)\beta(q_-). \quad (14)
\]

**Proof.** For \(\beta = \text{id}_M\) (\(\delta = \text{id}_M\)) we are free to choose \(\lambda = \text{id}_N\) (\(\mu = \text{id}_N\)) and \(t = 1\) (\(s = 1\)) by Lemma 3.11, and then Eq. (12) is obvious. To show Eq. (13) we first note that, if \(t \in \text{Hom}_M(\beta, \alpha^+_\lambda)\) and \(x \in \text{Hom}_M(\omega, \alpha^+_\mu)\) are isometries then \(\alpha^+_\lambda(x)t \in \text{Hom}_M(\beta \circ \omega, \alpha^+_\lambda)\) is an isometry. With an
isometry $s \in \text{Hom}_M(\delta, \alpha^-_\mu)$ we can therefore write
\[
\varepsilon_t(\beta \circ \omega, \delta) = s^* \alpha^-_\mu(t^* \alpha^+_\lambda(x^*)) \varepsilon(\lambda \circ \nu, \mu) \alpha^+_\lambda(s) \alpha^+_\lambda(x)t
\]
\[
= s^* \alpha^-_\mu(t^*) \alpha^-_\mu \circ \alpha^+_\lambda(x^*) \cdot \varepsilon(\lambda, \mu) \lambda(\varepsilon(\nu, \mu)) \cdot \alpha^+_\lambda \circ \alpha^+_\lambda(s) \cdot \alpha^+_\lambda(x)t
\]
\[
= s^* \alpha^-_\mu(t^*) \varepsilon(\lambda, \mu) \cdot \alpha^+_\lambda \circ \alpha^-_\mu(x^*) \cdot \alpha^+_\lambda(\varepsilon(\nu, \mu)) \cdot \alpha^+_\lambda \circ \alpha^+_\lambda(s) \cdot \alpha^+_\lambda(x)t
\]
\[
= s^* \alpha^-_\mu(t^*) \varepsilon(\lambda, \mu) \cdot \alpha^+_\lambda(\varepsilon(\nu, \mu)) \cdot \alpha^+_\lambda \circ \alpha^+_\lambda(s) \cdot \alpha^+_\lambda(x)t
\]
\[
= s^* \alpha^-_\mu(t^*) \cdot \beta(s^* \alpha^-_\mu(x^*)) \varepsilon(\nu, \mu) \alpha^+_\mu(s)x
\]
\[
= \varepsilon_t(\beta, \delta) \beta(\varepsilon_t(\omega, \delta)),
\]
where we used [3, Lemmata 3.24 and 3.25]. The proof for the second relation in Eq. (3) is analogous. Now let $q_+ \in \text{Hom}_M(\beta, \omega)$. Note that then $xq_+t^* \in \text{Hom}_M(\alpha^+_\lambda, \alpha^+_\mu)$. Hence
\[
\delta(q_+) \varepsilon_t(\beta, \delta) = s^* \alpha^-_\mu(q_+ss^* \alpha^-_\mu(t^*) \varepsilon(\lambda, \mu) \alpha^+_\lambda(s)t
\]
\[
= s^* \alpha^-_\mu(q_+ t^*) \cdot \varepsilon(\lambda, \mu) \alpha^+_\lambda(s)t
\]
\[
= s^* \alpha^-_\mu(t^*) \cdot \varepsilon(\lambda, \mu) \alpha^+_\lambda(\varepsilon(\nu, \mu)) \cdot \alpha^+_\lambda \circ \alpha^+_\lambda(s) \cdot \alpha^+_\lambda(x)t
\]
\[
= s^* \alpha^-_\mu(t^*) \cdot \beta(s^* \alpha^-_\mu(x^*)) \cdot \varepsilon(\nu, \mu) \alpha^+_\mu(s)xq_+
\]
\[
= \varepsilon_t(\omega, \delta) q_+,
\]
where we used [3, Lemma 3.25] again. The proof for the second relation in Eq. (4) is analogous.

Now consider sectors $[\beta]$ that can be obtained by both inductions, i.e. $[\beta]$ is a subsector of $[\alpha^+_\lambda]$ and $[\alpha^-_\lambda]$ for some $\lambda \in \Delta_N(I_0)$, cf. Lemma 3.1. (And for a representative $\beta \in \text{End}(M)$ we can in fact use the same intertwining isometry.) We easily obtain the following

**Corollary 3.13** For the collection of endomorphisms $\beta, \delta \in \text{End}(M)$ of that kind that $[\beta]$ is a subsector of both, $[\alpha^+_\lambda]$ and $[\alpha^-_\lambda]$, and similarly $[\delta]$ is a subsector of $[\alpha^+_\mu]$ and $[\alpha^-_\mu]$ for some (varying) $\lambda, \mu \in \Delta_N(I_0)$, the unitaries $\varepsilon_t(\beta, \delta)$ and $\varepsilon_t(\delta, \beta)$ define a braiding.

Later we will use the following

**Lemma 3.14** Let $\beta \in \text{End}(M)$ such that $[\beta]$ is a subsector of both, $[\alpha^+_\lambda]$ and $[\alpha^-_\lambda]$, for some irreducible $\lambda \in \Delta_N(I_0)$. Let further $\mu \in \Delta_N(I_0)$ such that $[\alpha^+_\mu] = [\alpha^-_\mu]$, and let $\delta_i \in \text{End}(M)$ such that $[\alpha^+_\mu] = \bigoplus_{i=1}^q [\delta_i]$. Then if $\varepsilon_t(\beta, \delta_i) = \varepsilon_t(\delta_i, \beta)^*$ for all $i = 1, 2, \ldots, q$, then $\varepsilon(\lambda, \mu) \varepsilon(\mu, \lambda) = 1$.
Proof. First note that if \( t \in \text{Hom}_M(\beta, \alpha_\pm^i) \) is an isometry then \( \sigma_\beta(n) = \gamma(t)^* \cdot \theta \cdot \lambda(n) \cdot \gamma(t) \) for all \( n \in N \), and by extending this formula to \( n \in N' \) we can consider \( \sigma_\beta \in \Delta_{N'}(I_\circ) \). Note that for our isometry \( v \in \text{Hom}_M(\text{id}_M, \gamma) \) we have \( v \in \text{Hom}_M(\beta, \alpha_{\sigma_\beta}^\pm) \) by, for instance, Lemma 3.3. Let \( s_i \in \text{Hom}_M(\delta_i, \alpha_\pm^i) \), \( i = 1, 2, \ldots, q \), be isometries generating a Cuntz algebra (recall \( \alpha_\pm^i = \alpha_\mu^i \) by \([\textit{[3]}, \text{Prop. 3.23}]\)). Then \( \varepsilon_i(\beta, \delta_i) = \varepsilon_i(\delta_i, \beta)^* \) yields

\[
\varepsilon_i^*(\sigma_\beta, \mu) \alpha_{\sigma_\beta}^i(s_i) v^* \alpha_{\sigma_\beta}^i(s_i)^* \varepsilon_i(\mu, \sigma_\beta) \alpha_{\mu}^i(v) s_i = 1.
\]

Since we can switch the \( \pm \)-signs as \( \alpha_\pm^i = \alpha_\mu^i \) and \( \alpha_{\sigma_\beta}^i(s_i) v = v \beta(s_i) = \alpha_{\sigma_\beta}^i(s_i) v \) we obtain by left multiplication by \( \alpha_\mu^i(v) s_i \) and by use of \([\textit{[3]}, \text{Lemma 3.25}]\)

\[
\varepsilon_i(\sigma_\beta, \mu) \varepsilon(\mu, \sigma_\beta) \alpha_{\mu}^i(v) s_i = \alpha_{\mu}^i(v) s_i,
\]

and we obtain \( \varepsilon_i(\sigma_\beta, \mu) \varepsilon(\mu, \sigma_\beta) \alpha_{\mu}^i(v) = \alpha_{\mu}^i(v) \) by right multiplication by \( s_i^* \) and summation over \( i \). Now recall \( \alpha_{\mu}^i(v) = \varepsilon(\mu, \theta)^* v \), and therefore we obtain \( \varepsilon_i(\sigma_\beta, \mu) \varepsilon(\mu, \sigma_\beta) = 1 \) by \([\textit{[3]}, \text{Lemma 3.8}]\). Now \( \langle \lambda, \sigma_\beta \rangle_N = \langle \alpha_{\lambda, \beta}^\pm, \beta \rangle_M \neq 0 \) by Prop. 3.3, hence \( [\lambda] \) is a subsector of \( [\sigma_\beta] \), and hence \( \varepsilon(\lambda, \mu) \varepsilon(\mu, \lambda) = 1 \) by Lemma 3.8.

For the rest of this subsection we assume that the enveloping net \( M \) is Haag dual. Let \( \beta \in \Delta_M(I_0) \) where \( \Delta_M(I_0) \) denotes the set of localized transportable endomorphisms, localized in \( I_0 \). Then \( \sigma_\beta \in \Delta_{N'}(I_0) \), in particular if \( Q_{\beta, \pm} \in M \) and \( u_{\theta, \pm} \in N' \) are unitaries such that \( \beta_\pm = \text{Ad}(Q_{\beta, \pm}) \circ \beta \in \Delta_M(I_\circ) \) and \( \theta_\pm = \text{Ad}(u_{\theta, \pm}) \circ \theta \in \Delta_{N'}(I_\circ) \) with intervals \( I_+, I_- \in J_\circ \) lying in the right respectively left complement of \( I_\circ \), one checks easily (cf. \([\textit{[3]}, \text{Lemma 3.18}]\))

\[
\sigma_{\beta, \pm} = \text{Ad}(u_{\theta, \pm} \gamma(Q_{\beta, \pm})) \circ \sigma_\beta \in \Delta_{N'}(I_\circ),
\]

so that (cf. \([\textit{[3]}, \text{Lemma 3.19}]\))

\[
\varepsilon(\sigma_\beta, \theta) = \gamma^2(Q_{\beta, \pm})^* \varepsilon(\theta, \theta)^* \gamma(Q_{\beta, \pm}).
\]

Now \( [\beta] \) is a subsector of both, \( [\alpha_{\sigma_\beta}^+] \) and \( [\alpha_{\sigma_\beta}^-] \), in particular we have \( v \in \text{Hom}_M(\beta, \alpha_{\sigma_\beta}^\pm) \). Note that we therefore find

\[
\gamma(Q_{\beta, \pm}^\ast) v^* Q_{\beta, \pm} = \gamma(Q_{\beta, \pm}) v = v \beta(v) = \alpha_{\sigma_\beta}^\pm(v) v = \alpha_{\sigma_\beta}^\pm(v),
\]

since \( v \beta(v) = v Q_{\beta, \pm}^* \beta_\pm(v) Q_{\beta, \pm} = v Q_{\beta, \pm}^* v Q_{\beta, \pm}^* = \gamma(Q_{\beta, \pm}) v Q_{\beta, \pm}. \) Now let also \( \delta \in \Delta_M(I_0) \) and choose unitaries \( Q_{\delta, \pm} \in M \) such that \( \text{Ad}(Q_{\delta, \pm}) \circ \delta \in \Delta_M(I_\circ) \). Putting \( u_{\theta, \pm} = u_{\theta, \pm} \gamma(Q_{\delta, \pm}) \) we can write

\[
\varepsilon(\sigma_\beta, \sigma_\delta) = u_{\theta, \pm}^* \alpha_{\sigma_\delta}(u_{\theta, \pm}) = \gamma(Q_{\delta, \pm}^* u_{\theta, \pm}^* \sigma_\beta(u_{\theta, \pm}) \cdot \gamma \circ \beta \circ \gamma(Q_{\delta, \pm}).
\]

\[
= \gamma(Q_{\delta, \pm}) \varepsilon(\sigma_\beta, \theta) \cdot \gamma \circ \beta \circ \gamma(Q_{\delta, \pm}).
\]
Endomorphisms in $\Delta_M(I_0)$ are clearly braided and the statistics operators are given by

$$\varepsilon^{\pm}(\beta, \delta) = Q_{\delta, \pm}^{*}(Q_{\delta, \pm}) .$$

We now have the following

**Proposition 3.15** Assume that $\mathcal{M}$ is Haag dual. For $\beta, \delta \in \Delta_M(I_0)$ we have

$$\varepsilon_r(\beta, \delta) = \varepsilon^{+}(\beta, \delta), \quad \varepsilon_r(\delta, \beta)^* = \varepsilon^{-}(\beta, \delta).$$

**(15)**

**Proof.** Since $v \in \text{Hom}_M(\beta, \alpha_{\sigma_\delta}^{\pm})$ and $v \in \text{Hom}_M(\delta, \alpha_{\sigma_\delta}^{\pm})$, we can write

$$\varepsilon_r(\beta, \delta) = v^* \alpha_{\sigma_\delta}^{\pm}(v^*) \varepsilon(\sigma_\beta, \sigma_\delta) \alpha_{\sigma_\delta}^{+}(v)v$$

$$= Q_{\delta, +}^{*} v^* v^* \gamma(Q_{\delta, +}) \varepsilon(\sigma_\beta, \sigma_\delta) \beta(v)$$

$$= Q_{\delta, +}^{*} v^* v^* \varepsilon(\sigma_\beta, \theta) \cdot \gamma \circ \beta \circ \gamma(Q_{\delta, +}) \cdot \beta(v)$$

$$= Q_{\delta, +}^{*} v^* v^* \varepsilon(\sigma_\beta, \theta) \beta(v) \beta(Q_{\delta, +})$$

$$= Q_{\delta, +}^{*} Q_{\beta, -}^{*} v^* v^* \varepsilon(\theta, \theta) \gamma(Q_{\beta, -}) \beta(v) \beta(Q_{\beta, +})$$

$$= Q_{\delta, +}^{*} \beta(Q_{\beta, +}) \equiv \varepsilon^{+}(\beta, \delta) ,$$

where we used the locality relation $\varepsilon(\theta, \theta)v^2 = v^2$ from [31] (or [3, Lemma 3.4]). The second relation follows from $\varepsilon^{-}(\beta, \delta) = \varepsilon^{+}(\delta, \beta)^*$. \qed

### 3.4 Subsectors of $[\gamma]$ and $[\alpha_\lambda^+ \circ \alpha_\mu^-]$

Let $\iota \in \text{Mor}(N,M)$ be the injection map from $N$ into $M$ and recall that $\tau \in \text{Mor}(M,N)$ given by $\tau(m) = \gamma(m) \in N$ for $m \in M$ is a conjugate, see Subsect. [27]. We first note a simple fact.

**Lemma 3.16** We have $\langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N$.

**Proof.** This is just because we have $[\gamma] = [\iota \circ \tau]$ as a sector of $M$ and $[\theta] = [\tau \circ \iota]$ as a sector of $N$. Hence

$$\langle \gamma, \gamma \rangle_M = \langle \iota \circ \tau, \iota \circ \tau \rangle_M = \langle \iota, \iota \circ \tau \circ \iota \rangle_{N,M} = \langle \tau \circ \iota, \tau \circ \iota \rangle_N = \langle \theta, \theta \rangle_N$$

by Frobenius reciprocity. \qed

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The extension property of $\alpha$-induction, $\alpha_\lambda^\pm(n) = \lambda(n)$ for all $n \in N$, $\lambda \in \Delta_N(I_o)$, can be written as $\alpha_\lambda^\pm \circ \iota = \iota \circ \lambda$ as morphisms in $\text{Mor}(N, M)$. Now recall $\langle \alpha_\lambda^\pm, \alpha_\mu^\pm \rangle_M = \langle \theta \circ \lambda, \mu \rangle_N$ by [3, Thm. 3.9].

**Lemma 3.17** For any $\lambda, \mu \in \Delta_N(I_o)$ we have
\[
\langle \alpha_\lambda^\pm, \iota \circ \mu \circ \tau \rangle_M = \langle \theta \circ \lambda, \mu \rangle_N = \langle \alpha_\lambda^\pm, \alpha_\mu^\pm \rangle_M.
\] (16)

**Proof.** We compute
\[
\langle \alpha_\lambda^\pm, \iota \circ \mu \circ \tau \rangle_M = \langle \alpha_\lambda^\pm \circ \iota, \iota \circ \mu \rangle_{N,M} = \langle \iota \circ \lambda, \iota \circ \mu \rangle_{N,M} = \langle \tau \circ \iota \circ \lambda, \mu \rangle_N
\]
by Frobenius reciprocity. □

Taking $\mu$ to be trivial we immediately obtain the following

**Corollary 3.18** Only the identity sector $[\text{id}_M]$ can be a common subsector of $[\alpha_\lambda^\pm]$ and $[\gamma]$ for $\lambda \in \Delta_N(I_o)$.

Now assume $d_\mu < \infty$ and let $\tau \in \Delta_N(I_o)$ be a conjugate of $\mu \in \Delta_N(I_o)$. Then $[\alpha_\mu^+] = \overline{[\alpha_\mu^-]}$ by [3, Lemma 3.14], and therefore we find
\[
\langle \alpha_\lambda^+, \tau \circ \mu \circ \gamma \rangle_M = \langle \alpha_\lambda^+, \alpha_\mu^- \circ \gamma \rangle_M = \langle \alpha_\lambda^+, \iota \circ \mu \circ \tau \rangle_M = \langle \alpha_\lambda^+, \alpha_\mu^+ \rangle_M.
\]
Thus we have the following

**Corollary 3.19** For $\lambda, \mu \in \Delta_N(I_o)$, $d_\mu < \infty$, and $\tau \in \Delta_N(I_o)$ a conjugate of $\mu$ we have $\langle \alpha_\lambda^+ \circ \alpha_\mu^-, \gamma \rangle_M = \langle \alpha_\lambda^+, \alpha_\mu^+ \rangle_M$. In particular, if $[\alpha_\mu^+]$ is irreducible then $[\alpha_\mu^+ \circ \alpha_\mu^-]$ has one irreducible subsector in common with $[\gamma]$ which cannot be the identity if $[\alpha_\mu^+] \neq [\alpha_\mu^-]$.

Recall that $[\alpha_\mu^+] \neq [\alpha_\mu^-]$ if and only if the monodromy $\varepsilon(\mu, \theta)\varepsilon(\theta, \mu)$ is non-trivial [3, Prop. 3.23].

We now further investigate subsectors of mixed products $[\alpha_\mu^+ \circ \alpha_\mu^-]$. Recall that subsectors of $[\alpha_\mu^+]$ commute with $[\alpha_\lambda^+]$, $\lambda, \mu \in \Delta_N(I_o)$, by [3, Prop. 3.16]. We will now generalize this result.

**Lemma 3.20** Let $\beta \in \text{End}(M)$ such that $[\beta]$ is a subsector of $[\alpha_\mu^+ \circ \alpha_\mu^-]$ for some $\mu, \nu \in \Delta_N(I_o)$. Then $[\alpha_\lambda^+ \circ \beta] = [\beta \circ \alpha_\lambda^+]$ for any $\lambda \in \Delta_N(I_o)$.
Proof. For any $\lambda, \mu \in \Delta_N(I_o)$ we have by (the plus- and minus-version of) [3, Cor. 3.11]
\[
\varepsilon^\pm(\lambda, \mu) \cdot \alpha_\lambda^\pm \circ \alpha_\mu^\pm(m) = \alpha_\mu^\pm \circ \alpha_\lambda^\pm(m) \cdot \varepsilon^\pm(\lambda, \mu)
\]
and similarly by [3, Lemma 3.24]
\[
\varepsilon^\pm(\lambda, \mu) \cdot \alpha_\lambda^\pm \circ \alpha_\mu^\pm(m) = \alpha_\mu^\pm \circ \alpha_\lambda^\pm(m) \cdot \varepsilon^\pm(\lambda, \nu).
\]
Recall $\varepsilon^\pm(\lambda, \mu \circ \nu) = \mu(\varepsilon^\pm(\lambda, \nu))\varepsilon^\pm(\lambda, \mu)$ for $\lambda, \mu, \nu \in \Delta_N(I_o)$. Therefore
\[
\varepsilon^\pm(\lambda, \mu \circ \nu) \cdot \alpha_\lambda^\pm \circ \alpha_\mu^\pm \circ \alpha_\nu^\pm(m) = \mu(\varepsilon^\pm(\lambda, \nu)) \cdot \alpha_\mu^\pm \circ \alpha_\lambda^\pm \circ \alpha_\nu^\pm(m) \cdot \varepsilon^\pm(\lambda, \nu)
\]
\[
= \alpha_\mu^\pm \circ \alpha_\nu^\pm \circ \alpha_\lambda^\pm(m) \cdot \varepsilon^\pm(\lambda, \mu \circ \nu)
\]
for all $m \in M$. By assumption, there is an isometry $t \in \text{Hom}_M(\beta, \alpha_\mu^\pm \circ \alpha_\nu^-)$. Hence
\[
t^*\varepsilon^\pm(\lambda, \mu \circ \nu)\alpha_\lambda^\pm(t) \cdot \alpha_\mu^\pm \circ \beta(m) =
\]
\[
= t^*\varepsilon^\pm(\lambda, \mu \circ \nu) \cdot \alpha_\lambda^\pm \circ \alpha_\mu^\pm \circ \alpha_\nu^\pm(m) \cdot \alpha_\lambda^\pm(t)
\]
\[
= t^* \cdot \alpha_\mu^\pm \circ \alpha_\nu^- \circ \alpha_\lambda^\pm(m) \cdot \varepsilon^\pm(\lambda, \mu \circ \nu) \alpha_\lambda^\pm(t)
\]
\[
= \beta \circ \alpha_\lambda^\pm(m) \cdot t^*\varepsilon^\pm(\lambda, \mu \circ \nu) \alpha_\lambda^\pm(t)
\]
for all $m \in M$. It remains to be shown that $u = t^*\varepsilon^\pm(\lambda, \mu \circ \nu)\alpha_\lambda^\pm(t)$ is unitary. Note that $tt^* \cdot \mu \circ \nu(n) = \mu \circ \nu(n) \cdot tt^*$ for all $n \in N$, hence
\[
tt^*\varepsilon^\pm(\lambda, \mu \circ \nu) = \varepsilon^\pm(\lambda, \mu \circ \nu)\alpha_\lambda^\pm(tt^*)
\]
by [3, Lemma 3.25]. With this relation one checks easily that $u^*u = uu^* = 1$.
\[\square\]

4 Induction-Restriction Graphs and $\gamma$-Multiplication Graphs

In this section we will relate $\alpha$-induction to basic invariants of the subfactor $N \subset M$ (and hence to each local subfactor $N(I) \subset M(I)$, $I \in \mathcal{J}_z$, since the choice of the interval $I_o$ was arbitrary), namely the principal graph and the dual principal graph. In our applications these results can be used to determine the graphs for several examples.
4.1 $\alpha$-induction and (dual) principal graphs

Choose a sector basis $\mathcal{W} \subset [\Delta]_N(I_0)$ which contains (at least) all the irreducible subsectors of $[\theta]$. Since a sector basis is by definition finite and closed under products (after irreducible decomposition) such a choice is possible if and only if the subfactor $N \subset M$ has finite depth. A representative endomorphism of $\Lambda \in \mathcal{W}$ is denoted by $\lambda_{\Lambda}$, $[\lambda_{\Lambda}] \equiv \Lambda$. We define the chiral induced sector bases $\mathcal{V}^+, \mathcal{V}^- \subset \text{Sect}(M)$ to be the sector bases given by all irreducible subsectors of $[\alpha_\Lambda^+], [\alpha_\Lambda^-], \Lambda \in \mathcal{W}$, respectively, $\alpha_\Lambda^+ \equiv \alpha_{\lambda_{\Lambda}}^\pm$. We denote representative endomorphisms of $a \in \mathcal{V}^+$ by $\beta_a^\pm$, $[\beta_a^\pm] \equiv a$. If for $a \in \mathcal{V}^+$ we have $\beta_a^+(m) = t^*\alpha_a^+(m)t$, $m \in M$, for some $\Lambda \in \mathcal{W}$ and some isometry $t \in M$, we denote by $\beta_a^- \in \text{End}(M)$ the endomorphism given by $\beta_a^-(m) = t^*\alpha_a^+(m)t$, $m \in M$. Note that then $\mathcal{V}^- = \{[\beta_a^-], a \in \mathcal{V}^+\}$ by Corollary 3.4. Also note that $\beta_a^+(n) = \beta_a^-(n)$ for all $n \in N$, thus $\beta_a^+ \circ t = \beta_a^- \circ t$ for the injection map $t$. Now let $\overline{\mathcal{V}} \subset \text{Sect}(N,M)$ be the set of all irreducible subsectors of $M$-$N$ sectors $[t \circ \lambda_{\Lambda}], \Lambda \in \mathcal{W}$.

Lemma 4.1 We have $\overline{\mathcal{V}} = \{[\beta_a^\pm \circ t], a \in \mathcal{V}^+\}$.

Proof. For a representative endomorphism $\beta_a^\pm \in \text{End}(M)$ for $a \in \mathcal{V}^+$ we have, by definition, some $\Lambda \in \mathcal{W}$ and some isometry $t \in \text{Hom}_M(\beta_a^+, \alpha_\Lambda^\pm)$. Put $\tau_a = \beta_a^\pm \circ t \in \text{Mor}(N,M)$. Note that the definition does not depend on the $\pm$-sign. Then $t\tau_a(n) = t\beta_a(n) = \lambda_{\Lambda}(n)t$ for all $n \in N$, hence $[\tau_a]$ is a subsector of $[\lambda_{\Lambda}]$. Moreover, as $\text{Hom}_{N,M}(\tau_a, \tau_b) = \text{Hom}_M(\beta_a^+, \beta_b^\pm)$ for $a, b \in \mathcal{V}^+$ by Lemma 3.2, the $[\tau_a]$’s are irreducible and $[\tau_a] = [\tau_b]$ if and only if $[\beta_a^\pm] = [\beta_b^\pm]$, i.e. $a = b$.

Conversely, let $[\tau] \in \overline{\mathcal{V}}$, i.e. $[\tau]$ is an irreducible $M$-$N$ sector and there is some $\Lambda \in \mathcal{W}$ and an isometry $t \in \text{Hom}_{N,M}(\tau, t \circ \lambda_{\Lambda})$. Hence $tt^* \in \lambda_{\Lambda}(N)' \cap M = \alpha_\Lambda^\pm(M)' \cap M$, and therefore putting $\beta_x^\pm(m) = t^*\alpha_x^\pm(m)t$, $m \in M$, defines $\beta^\pm_x \in \text{End}(M)$ fulfilling $\beta^\pm_x \circ t = \tau$. As $\beta^\pm_x(M)' \cap M = \beta^\pm_x(N)' \cap M = \tau(N)' \cap M = \mathbb{C}1$ by Lemma 3.2 we find that $\beta_x^\pm$ is irreducible, thus $[\beta_x^\pm] \in \mathcal{V}^\pm$. Similarly, $[\beta_x^\pm] = [\beta_x^\pm]$ if and only if $[\tau] = [\tau']$ for $[\tau'] \in \overline{\mathcal{V}}$.

From now on, we use the notation $[\tau_a] = [\overline{\beta}_a^\pm \circ t] \in \overline{\mathcal{V}}$ for $a \in \mathcal{V}^+$. This makes sense since $\mathcal{V}^+$ (and $\mathcal{V}^-$) is closed under conjugation. We also denote $[\rho_a] = [\overline{\tau}_a] = [\tau \circ \beta_a^\pm], a \in \mathcal{V}^+$, and define the set $\mathcal{Y} = \{[\rho_a], a \in \mathcal{V}^+\}$. Next we define $\mathcal{\check{V}} \subset \text{Sect}(M)$ to be the set of all irreducible subsectors of some $[\beta_a^\pm \circ \gamma], a \in \mathcal{V}^+$, and also this definition is obviously independent of the $\pm$-sign. We denote representative endomorphisms of $x \in \mathcal{\check{V}}$ by $\beta_x, [\beta_x] \equiv x$. Clearly $[\text{id}_M] \in \mathcal{\check{V}}$. 

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We can now draw a bipartite graph as follows. We label the even vertices by the elements of \( W \) and the odd vertices by the elements of \( Y \). We connect any even vertex labelled by \( \Lambda \in W \) with any odd vertex labelled by \( [\rho_a] \), \( a \in V^+ \), by \( \langle \lambda, \sigma_{\beta^\pm} \rangle_N \) edges. Due to Prop. 3.3 we call the (possibly disconnected) graph obtained this way the \textit{induction-restriction graph}. We can draw another bipartite graph as follows. We label the even vertices by the elements of \( \tilde{V} \) and the odd vertices by the elements of \( Y \). We connect any even vertex labelled by \( x \in \tilde{V} \) with any odd vertex labelled by \( [\tau_a] \), \( a \in V^+ \), by \( \langle \beta^\pm x, \beta^\pm \gamma \rangle_M \) edges. We call the (possibly disconnected) graph obtained this way the \textit{\( \gamma \)-multiplication graph}.

**Theorem 4.2** The principal graph of the inclusion \( N \subset M \) is given by the connected component of \([\text{id}_N] \in W \) of the induction-restriction graph. The dual principal graph is given by the connected component of \([\text{id}_M] \in \tilde{V} \) of the \( \gamma \)-multiplication graph.

**Proof.** Note that \( P_0 \), defined in Subsect. 2.1, is contained in \( W \) since it is closed under reduction of products and contains the irreducible subsectors of \([\theta]\). As \( \tilde{V} \) is the set of irreducible subsectors of \([\iota \circ \lambda \Lambda]\), it follows that \( Y \) is the set of irreducible subsectors of \([\lambda \Lambda \circ \iota]\), \( \Lambda \in W \). Since \( W \) is closed under conjugation, it follows in particular that \( P_1 \subset Y \). Recall that the elements of \( Y \) are of the form \([\rho_a] = [\iota \circ \beta^\pm]\). Now for \( \Lambda \in W \) and \( a \in V^+ \) we have

\[
\langle \lambda \Lambda \circ \iota, \rho_a \rangle_{M,N} = \langle \lambda \Lambda \circ \iota, \beta^\pm_a \rangle_{M,N} = \langle \lambda \Lambda \circ \iota, \beta^\pm_a \rangle_{N} = \langle \lambda \Lambda, \sigma_{\beta^\pm} \rangle_{N}
\]

by Frobenius reciprocity, therefore the induction-restriction graph has the principal graph as a subgraph. This must be the connected component of \([\text{id}_N] \in P_0 \).

Similarly, as \( P_1 \subset Y \), we have \( D_1 \subset \tilde{V} \). Since any subsector in \( D_0 \) can be obtained by decomposing sectors \([\tau \circ \iota], [\tau] \in D_1 \), we find that also \( D_0 \subset \tilde{V} \). Now for \( x \in \tilde{V} \) and \( a \in V^+ \) we have

\[
\langle \beta^\pm x \circ \iota, \tau_a \rangle_{N,M} = \langle \beta^\pm x \circ \iota, \beta^\pm_a \circ \iota \rangle_{N,M} = \langle \beta^\pm x, \beta^\pm_a \circ \gamma \rangle_M.
\]

Hence the connected component of \([\text{id}_M] \in \tilde{V} \) is the dual principal graph. \( \square \)

Now let \( V^+_0 \subset V^\pm \) be the subset of those sectors \([\beta^\pm_a] \) such that \([\beta^\pm_a \circ \iota] = [\beta^\pm_a \circ \iota] \) appears (as a label of some odd vertex) in the dual principal graph. As \([\gamma] \) possesses the identity sector as a subsector, \([\beta^\pm_a \circ \gamma] = [\beta^\pm_a \circ \gamma] \) contains \([\beta^\pm_a] \) and, if different, also \([\beta^\pm_a] \) as a subsector. Recall that a sector algebra associated to a sector basis is the vector space with the sector basis as a basis endowed with the sector operations as algebraic structure. From Theorem 1.2 we now obtain immediately the following
Corollary 4.3 Elements of $\mathcal{V}_0^+ \cup \mathcal{V}_0^-$ appear as labels of the even vertices of the dual principal graph. Therefore, as sector algebras, the algebra of $M$-$M$ sectors of the dual principal graph possesses two subalgebras corresponding to the sector bases $\mathcal{V}_0^+$ and $\mathcal{V}_0^-$ (which may be identical).

4.2 Global indices

It is known that the $N$-$N$, $N$-$M$, $M$-$N$ and $M$-$M$ bimodules arising from a subfactor $N \subset M$ and labelling the vertices of the principal and dual principal graph have the same global indices. Here we mean by global index the sum over the squares of the Perron-Frobenius weights which correspond to the statistical dimensions in the sector context. We will now show that an analogous statement holds for the sectors labelling the vertices of the (possibly larger) induction-restriction and $\gamma$-multiplication graphs. We denote $d_\Lambda \equiv d_\lambda \Lambda$, $\Lambda \in \mathcal{W}$, $d_a \equiv d_\beta^\pm a$, $a \in \mathcal{V}^+$, $d_x \equiv d_\beta_x x$, $x \in \tilde{\mathcal{V}}$, and define global indices

$$[[\mathcal{W}]] = \sum_{\Lambda \in \mathcal{W}} d_\Lambda^2,$$
$$[[\mathcal{V}^\pm]] = \sum_{a \in \mathcal{V}^+} d_a^2,$$
$$[[\tilde{\mathcal{V}}]] = \sum_{x \in \tilde{\mathcal{V}}} d_x^2.$$

Recall $d_\gamma = [M : N]$.

Lemma 4.4 We have $[[\mathcal{W}]] = d_\gamma [[\mathcal{V}^\pm]]$.

Proof. We define a rectangular matrix $P$ by

$$P_{a,\Lambda} = \langle \sigma_{\beta^\pm a}, \lambda_\Lambda \rangle_N,$$ $a \in \mathcal{V}^+$, $\Lambda \in \mathcal{W}$.

We then have $[\sigma_{\beta^\pm a}] = \bigoplus_{\Lambda \in \mathcal{W}} P_{a,\Lambda}[\lambda_\Lambda]$, hence $d_\gamma d_a = \sum_{\Lambda \in \mathcal{W}} P_{a,\Lambda} d_\Lambda$. As then $P_{a,\Lambda} = \langle \beta^\pm_a \circ \iota, \iota \circ \lambda_\Lambda \rangle_{N,M}$ by Frobenius reciprocity, and all irreducible subsectors of $[\iota \circ \lambda_\Lambda]$ are of the form $[\beta^\pm_a \circ \iota]$, $a \in \mathcal{V}^+$, we have similarly $[\iota \circ \lambda_\Lambda] = \bigoplus_{a \in \mathcal{V}^+} P_{a,\Lambda}[\beta^\pm_a \circ \iota]$, hence $d_\Lambda = \sum_{a \in \mathcal{V}^+} P_{a,\Lambda} d_a$. Therefore

$$[[\mathcal{W}]] = \sum_{\Lambda \in \mathcal{W}} d_\Lambda^2 = \sum_{\Lambda \in \mathcal{W}} \sum_{a \in \mathcal{V}^+} d_\Lambda P_{a,\Lambda} d_a = \sum_{a \in \mathcal{V}^+} d_\gamma d_a^2 = d_\gamma [[\mathcal{V}^\pm]],$$

and so we are done. \qed

Lemma 4.5 We have $[[\mathcal{W}]] = [[\tilde{\mathcal{V}}]]$. 

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Proof. We define a rectangular matrix $D$ by
\[
D_{a,x} = \langle \beta_a^\pm \circ \gamma, \beta_x \rangle_N, \quad a \in V^+, \quad x \in \tilde{V}.
\]
Since then $[\beta_a^\pm \circ \gamma] = \bigoplus_{x \in \tilde{V}} D_{a,x}[\beta_x]$ we find $d_y d_a = \sum_{x \in \tilde{V}} D_{a,x} d_x$. As $[\beta_x] \in \tilde{V}$ is (by definition) a subsector of $\beta_b^\pm \circ \gamma$ for some $b \in V^+$, we find that $[\beta_x \circ \iota]$ is a subsector of
\[
[\beta_b^\pm \circ \gamma \circ \iota] = [\beta_b^\pm \circ \iota \circ \theta] = \bigoplus_{c \in V^+} \langle \theta, \sigma_{\beta_c^\pm} \rangle_N [\beta_b^\pm \circ \beta_c^\pm \circ \iota]
\]
\[
= \bigoplus_{a,c \in V^+} \langle \theta, \sigma_{\beta_a^\pm} \rangle_N [\beta_a^\pm \circ \beta_c^\pm, \beta_a^\pm]_M [\beta_a^\pm \circ \iota],
\]
hence $[\beta_x \circ \iota]$ decomposes only into elements of $\mathcal{Y}$. Therefore $[\beta_x \circ \iota] = \bigoplus_{a \in V^+} D_{a,x}[\beta_a^\pm \circ \iota]$, hence
\[
d_x = \sum_{a \in V^+} d_{a,x} d_a = \sum_{a \in V^+} d_a^2 = d_x [\mathcal{Y}] + \sum_{x \in \tilde{V}} d_x d_a = \sum_{a \in V^+} d_a^2 = d_x [\mathcal{Y}],
\]
and the statement now follows from Lemma 4.4. \hfill \Box

Let $\mathcal{V}$ be the set of irreducible subsectors of $[\alpha_{\Lambda}^+ \circ \alpha_{\Lambda'}^-]$, $\Lambda, \Lambda' \in \mathcal{W}$. As the maps $[\lambda_\Lambda] \mapsto [\alpha_{\Lambda}^\pm]$ are multiplicative, conjugation preserving and $[\alpha_{\Lambda}^\pm]$ and $[\alpha_{\Lambda'}^-]$ commute, $\mathcal{V}$ must be in fact a sector basis and we call it the full induced sector basis.

Lemma 4.6 We have $\mathcal{V} \subset \tilde{\mathcal{V}}$.

Proof. Let $[\beta]$ be an irreducible subsector of $[\alpha_{\Lambda}^+ \circ \alpha_{\Lambda'}^-]$ for some $\Lambda, \Lambda' \in \mathcal{W}$. Then there are sectors $a, b \in \mathcal{V}^+$ such that $[\beta]$ is a subsector of $[\beta_a^\pm \circ \beta_b^-]$, hence of
\[
[\beta_a^\pm \circ \beta_b^- \circ \gamma] = [\beta_a^\pm \circ \beta_b^+ \circ \gamma] = \bigoplus_{c \in \mathcal{V}^+} \langle \beta_a^\pm \circ \beta_b^+, \beta_c^+ \circ \gamma \rangle_M [\beta_c^+ \circ \gamma],
\]
and therefore $[\beta]$ must be a subsector of $[\beta_c^+ \circ \gamma]$ for some $c \in \mathcal{V}^+$. \hfill \Box

Since $\mathcal{V} \supset \mathcal{V}^\pm$ is a sector basis, we have equality $\mathcal{V} = \tilde{\mathcal{V}}$ if and only if each irreducible subsector of $[\gamma]$ is in $\mathcal{V}$. This is not the case in general but we will find this situation in our conformal field theory examples. The point is that then the $\alpha$-induction machinery provides useful methods to compute, besides the principal graphs, the dual principal graphs of conformal inclusion subfactors.
5 Two Inductions and Modular Invariants

Before turning to the concrete examples, we will now discuss the application of \( \alpha \)-induction to certain conformal or orbifold embeddings involving \( SU(n) \), and we relate the \( \pm \)-inductions to the entries of the corresponding modular invariant mass matrix \( Z \).

5.1 Some equivalent conditions

Let us now consider the more specific situation as already treated in [4], namely that the net of subfactors \( \mathcal{N} \subset \mathcal{M} \) arises from a conformal or orbifold inclusion of \( SU(\mathcal{N}) \). We extend the orbifold analysis from the \( \mathbb{Z}_n \) case of [4] to the \( \mathbb{Z}_m \) case, where \( m \) is any divisor of \( n \) since there are also associated type I modular invariants; that these inclusions also lead to suitable nets of subfactors will be shown in Subsect. 6.2. For later reference, we now also include the conformal inclusions \( SU(n)_k \otimes SU(m)_\ell \subset G_1 \), with \( G \) some simple Lie group, in our discussion. Let \( Z_{\Lambda,\Lambda'} \) denote the entries of the mass matrix of the corresponding modular invariant. Here \( \Lambda \) denotes weights in the Weyl alcove \( \mathcal{A}^{(n,k)} \) in the former case, and in the latter case it labels pairs of weights, denoted \( \Lambda = (\hat{\Lambda}, \check{\Lambda}) \), with \( \hat{\Lambda} \in \mathcal{A}^{(n,k)} \) and \( \check{\Lambda} \in \mathcal{A}^{(m,\ell)} \). Therefore we are dealing with a fusion algebra \( \mathcal{W}_\Lambda \), where \( \mathcal{W}_\Lambda = \{ [\lambda_\Lambda] \} \).

We sometimes identify \( \mathcal{W}_\Lambda \) with its labelling set \( \mathcal{A}^{(n,k)} \) or \( \mathcal{A}^{(n,k)} \times \mathcal{A}^{(m,\ell)} \). As usual, we write \( \alpha_\Lambda^\pm \equiv \alpha_\Lambda^\pm \lambda_\Lambda \). We obtain two chiral induced sector bases \( \mathcal{V}^\pm \) given by all irreducible subsectors of \( [\alpha_\Lambda^\pm] \)'s. Further, we obtain the full induced sector basis \( \mathcal{V} \) by taking all the irreducible subsectors of \( [\alpha_\Lambda^\pm \circ \alpha_\Lambda^-] \). Clearly, \( \mathcal{V}^\pm \subset \mathcal{V} \). We denote representative endomorphisms for \( x \in \mathcal{V} \) by \( \beta_x \) so that we may identify \( \beta_x \).

Let us now define a matrix \( \tilde{Z}_{\Lambda,\Lambda'} \) by

\[
\tilde{Z}_{\Lambda,\Lambda'} = \langle \alpha_\Lambda^+, \alpha_\Lambda^- \rangle_M, \quad \Lambda, \Lambda' \in \mathcal{W}.
\]

We remark that Lemma 3.10 states \( T \)-invariance of this matrix. Let \( M_y \) be the sector product matrices \( M_y \) of \( (\mathcal{V}, \mathcal{V}) \), with

\[
(M_y)_{x,z} \equiv M_{x,y}^z = \langle \beta_x \circ \beta_y, \beta_z \rangle_M, \quad x, y, z \in \mathcal{V}.
\]

We define a collection of matrices \( R_{\Lambda,\Lambda'}^N \), \( \Lambda, \Lambda' \in \mathcal{W} \) by

\[
R_{x,y}^\Lambda N' = \langle \beta_x \circ \alpha_\Lambda^+ \circ \alpha_\Lambda^- \circ \beta_y \rangle_M, \quad x, y \in \mathcal{V}.
\]

First note that \( \tilde{Z}_{\Lambda,\Lambda'} = R_{0,0}^{\Lambda,\Lambda'} \). As \( [\alpha_\Lambda^+] \) and \( [\alpha_\Lambda^-] \) commute with \( [\beta_x] \), the matrices \( R_{\Lambda,\Lambda'} \) commute with \( M_x \), \( \Lambda, \Lambda' \in \mathcal{W}, x \in \mathcal{V} \). It follows from the
homomorphism property of $\alpha$-induction that
\[
R_{\Lambda',\Omega'}^{\Lambda,\Omega} = \sum_{\Phi, \Phi' \in \mathcal{W}} N_{\Lambda,\Omega}^{\Phi, \Omega'} N_{\Lambda',\Omega'}^{\Phi', \Omega} \cdot R^{\Phi, \Phi'},
\]
where the $N$’s are the fusion coefficients in $W$. Thus these matrices constitute a representation of the fusion algebra $W \otimes W$ and hence must decompose into its characters $\gamma_{\Phi_1} \otimes \gamma_{\Phi_2}$, where $\gamma_{\Phi_\epsilon}(\Lambda) = S_{\Lambda,\Phi_\epsilon}/S_{0,\Phi_\epsilon}$, $\Lambda, \Phi_\epsilon \in \mathcal{W}$, $\epsilon = 1, 2$. Here $S$ is the S-matrix of the $SU(n)_k$ (or the $SU(n)_k \otimes SU(m)_\ell$) theory, implementing the modular transformations of the conformal characters and diagonalizing the sector fusion rules at the same time by Wassermann’s result [42]. Similar to the procedure in [3, Subsect. 4.2] we conclude that there is an orthonormal basis \[ \{ \xi_i : i = 0, 1, ..., D - 1 \} \]
indexed by $x \in \mathcal{V}$, such that
\[
R_{x,y}^{\Lambda',\Omega'} = \sum_{i=0}^{D-1} S_{\Lambda,\Phi_1(i)} S_{\Lambda',\Phi_2(i)} \cdot \xi_i^*(\xi_y)
\]
with a map $\Phi : i \mapsto (\Phi_1(i), \Phi_2(i)) \in \mathcal{W} \times \mathcal{W}$. We have $S^*ZS = S$ by modular invariance, hence in particular $(S^*ZS)_{0,0} = 1$. By $d_\lambda$ and $d_x$ we denote the statistical (or “quantum”) dimension of $\lambda, \Lambda \in \mathcal{W}$, and $\beta_x$, $x \in \mathcal{V}$, respectively. We have in particular $d_\lambda = S_{\Lambda,0}/S_{0,0}$.

Let us now concentrate on the conformal inclusion case. As in [3], we denote by $(T, \mathcal{T})$ the fusion algebra corresponding to the (level one) positive energy representations of the ambient theory. We know from $\alpha \sigma$-reciprocity that
\[
\mathcal{T} \subset \mathcal{V}^+ \cap \mathcal{V}^-.
\]
Recall that $Z_{\Lambda,\Lambda'} = \sum_{t \in \mathcal{T}} b_{t,\Lambda} b_{t,\Lambda'}$, where $b_{t,\Lambda} = \langle \lambda_t, \sigma_{\beta_t} \rangle_N$ are the restriction coefficients.

**Proposition 5.1** For conformal inclusions we have $Z_{\Lambda,\Lambda'} \leq \langle \alpha_\Lambda^+, \alpha_{\Lambda'}^- \rangle_M$, $\Lambda, \Lambda' \in \mathcal{W}$ and $\sum_{x \in \mathcal{V}} d_x^2 \leq \sum_{\Lambda \in \mathcal{W}} d_\lambda^2$. Moreover, the following conditions are equivalent:

1. $\mathcal{T} = \mathcal{V}^+ \cap \mathcal{V}^-$,

2. $Z_{\Lambda,\Lambda'} = \langle \alpha_\Lambda^+, \alpha_{\Lambda'}^- \rangle_M$, $\Lambda, \Lambda' \in \mathcal{W}$,

3. $\sum_{x \in \mathcal{V}} d_x^2 = \sum_{\Lambda \in \mathcal{W}} d_\lambda^2$,

4. each irreducible subsector of $[\gamma]$ is in $\mathcal{V}$.

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choose some irreducible matrices see e.g. \([19]\)) is seen as follows. For given \(Q\) the sum matrix \(Q\) thus we have

\[
\langle \alpha^+, \alpha^- \rangle = \langle \alpha^+, \alpha^- \rangle_M = \langle \alpha^+, \alpha^- \rangle_M,
\]

and it is clear that we have equality for all \(\Lambda, \Lambda' \in \mathcal{W}\) if and only if \(\mathcal{T} = \mathcal{V}^+ \cap \mathcal{V}^-\).

Next we show the inequality for the dimensions and equivalence 2 \(\iff\) 3: We compute

\[
\langle \alpha^+, \alpha^- \rangle = \langle \alpha^+, \alpha^- \rangle_M = \langle \alpha^+, \alpha^- \rangle_M,
\]

where we used \(\hat{Z}_{\Lambda, \Lambda'} = R_{\Lambda, \Lambda'}^{\mathcal{W}, \mathcal{V}}\) and \(S_{\mathcal{W}, \mathcal{V}}^{\mathcal{V}} = S_{\mathcal{V}, \mathcal{V}}^{\mathcal{V}} = (S^*)_{\mathcal{V}, \mathcal{V}}^{\mathcal{V}}.\) The sector product matrices obey \(\sum_{x \in \mathcal{V}} M_x y d_x = d_x y d_y\), hence \(\xi_0^0 = \|\xi_0^0\|^{-1} d_x,\) \(x \in \mathcal{V},\) realizes a (normalized) eigenvector for each \(M_y\) with eigenvalue \(d_y\) and since \(R_{\Lambda, \Lambda'} = \sum_y \langle \alpha^+ \circ \alpha^-, \beta_y \rangle_M d_y = d_{\alpha^+} d_{\alpha^-} = d_{\Lambda} d_{\Lambda'} = S_{\Lambda, \Lambda'} S_{\Lambda', \Lambda} / S_{00},\) the corresponding eigenvalues are given by

\[
\sum_{y \in \mathcal{V}} \langle \alpha^+ \circ \alpha^-, \beta_y \rangle_M d_y = d_{\alpha^+} d_{\alpha^-} = d_{\Lambda} d_{\Lambda'} = S_{\Lambda, \Lambda'} S_{\Lambda', \Lambda} / S_{00},
\]

thus we have \(S_{\Lambda, \Lambda'} S_{\Lambda', \Lambda} / S_{00} = S_{\Lambda, \mathcal{W}} S_{\mathcal{W}, \mathcal{V}} / S_{00, \mathcal{V}} S_{\mathcal{V}, \mathcal{V}} / S_{00, \mathcal{V}} S_{\mathcal{V}, \mathcal{V}} / S_{00, \mathcal{V}}\) for all \(\Lambda, \Lambda' \in \mathcal{W},\) implying \(\Phi_1(0) = \Phi_2(0) = 0.\) Clearly, \(\xi_0^0\) is also an eigenvector of the sum matrix \(Q = \sum_{\Lambda, \Lambda'} R_{\Lambda, \Lambda'}\) which is irreducible, hence it is in fact a Perron-Frobenius eigenvector. Irreducibility of \(Q\) (for the definition of irreducible matrices see e.g. \([19]\)) is seen as follows. For given \(x, y \in \mathcal{V}\) choose some \(z \equiv [\beta_z] \in \mathcal{V}\) in the irreducible decomposition of \(\text{lin} \beta_x \circ \beta_y\), then \(\langle \beta_x \circ \beta_z, \beta_y \rangle_M \neq 0.\) Since \(z\) is realized as an irreducible subsector
of \([\alpha^\dagger_\Lambda \circ \alpha_{\Lambda'}]\) for some \(\Lambda, \Lambda' \in \mathcal{W}\) it follows that the corresponding matrix element of \(R_{\Lambda, \Lambda'}\) is non-zero, \(R_{x,y} \neq 0\). Hence any matrix element \(Q_{x,y}\) of the sum matrix \(Q\) is strictly positive, implying irreducibility. Therefore its Perron-Frobenius eigenvector is unique and its eigenvalue is non-degenerate, hence there cannot be an \(i \neq 0\) such that \(\Phi_1(i) = \Phi_2(i) = 0\). This means \(\delta_{\Phi_1(i),0} \delta_{\Phi_2(i),0} = \delta_{i,0}\), hence we obtain

\[
(S^\ast \bar{Z}S)_{0,0} = \frac{|\xi_0^0|^2}{S_{0,0}^2}.
\]

On the other hand we compute

\[
(S^\ast \bar{Z}S)_{0,0} = \sum_{\Lambda, \Lambda' \in \mathcal{W}} (S^\ast)_{0,\Lambda} \bar{Z}_{\Lambda, \Lambda'} S_{\Lambda',0} \geq \sum_{\Lambda, \Lambda' \in \mathcal{W}} (S^\ast)_{0,\Lambda} Z_{\Lambda, \Lambda'} S_{\Lambda',0} = Z_{0,0} = 1,
\]

since \((S^\ast)_{0,\Lambda} = S_{\Lambda,0} > 0\) for all \(\Lambda \in \mathcal{W}\), and therefore we have also equality if and only if \(Z_{\Lambda, \Lambda'} = \bar{Z}_{\Lambda, \Lambda'}\) for all \(\Lambda, \Lambda' \in \mathcal{W}\). Hence we have obtained \(|\xi_0^0|^2/S_{0,0}^2 \geq 1\) with equality if and only if \(Z = \bar{Z}\). Now, by normalization, \(|\xi_0^0|^2 = (\sum_{x \in \mathcal{V}} d_x^2)^{-1}\) and \(S_{0,0}^2 = (\sum_{\Lambda \in \mathcal{W}} d_\Lambda^2)^{-1}\), hence

\[
\sum_{x \in \mathcal{V}} d_x^2 \leq \sum_{\Lambda \in \mathcal{W}} d_\Lambda^2,
\]

and we have equality if and only if \(Z = \bar{Z}\).

Finally we show the equivalence 3 \(\iff\) 4: The inequality for sums over the squared dimensions (“global indices”) is also a corollary of Lemmata 4.5 and 4.6, and clearly we have exact equality if and only if \(\mathcal{V} = \bar{\mathcal{V}}\), and this is clearly equivalent to having each irreducible subsector of \([\gamma]\) in \(\mathcal{V}\), the proof is complete.

\[\square\]

5.2 Modular invariants and exponents of graphs revisited

Similar to the analysis in [4, Subsect. 4.2] we now investigate the relation between non-vanishing entries in the mass matrix of the modular invariant and exponents of fusion graphs obtained by \(\alpha\)-induction. We denote

\[
\text{Eig}(\Lambda, \Lambda') = \text{span}\{\xi^i : i \in \Phi^{-1}(\Lambda, \Lambda')\}, \quad \Lambda, \Lambda' \in \mathcal{W}.
\]

Also we put

\[
\|\xi_0\|_{\Lambda, \Lambda'} = \sqrt{\sum_{i \in \Phi^{-1}(\Lambda, \Lambda')} |\xi^i_0|^2}.
\]
Lemma 5.2 If $\text{Eig}(\Lambda, \Lambda') \neq 0$ for some $\Lambda, \Lambda' \in \mathcal{W}$ then $\| \xi_0 \|_{\Lambda, \Lambda'} \neq 0$.

Proof. As the matrices $R^{\Lambda, \Lambda'}$ commute with the matrices $M_x$ we find for $i = 0, 1, \ldots, D - 1$,

$$R^{\Lambda, \Lambda'} M_x \xi^i = M_x R^{\Lambda, \Lambda'} \xi^i = \gamma_{\Phi_1(i)}(\Lambda) \gamma_{\Phi_2(i)}(\Lambda') M_x \xi^i, \quad \Lambda, \Lambda' \in \mathcal{W}, \quad x \in \mathcal{V},$$

i.e. $M_x \xi^i \in \text{Eig}(\Phi_1(i), \Phi_2(i))$. In other words, the matrices $M_x$ are block-diagonal in the basis $\xi^i$. It follows that there are matrices, namely the “blocks” $B_{\Lambda, \Lambda'}(x)$, $B_{\Lambda, \Lambda'}(x)_{i,j} \in \mathbb{C}$, $i,j \in \Phi^{-1}(\Lambda, \Lambda')$ such that $M_x \xi^i = \sum_{j \in \Phi^{-1}(\Lambda, \Lambda')} B_{\Lambda, \Lambda'}(x)_{i,j} \xi^j$, hence in particular for the 0-components

$$(M_x \xi^i)_0 = \sum_{j \in \Phi^{-1}(\Lambda, \Lambda')} B_{\Lambda, \Lambda'}(x)_{i,j} \xi^j_0.$$ 

Since $(M_x \xi^i)_0 = \sum_{y \in \mathcal{Y}} M^b_{0,x} \xi^i_y = \xi^i_x$ we have for any $i \in \Phi^{-1}(\Lambda, \Lambda')$ and any $x \in \mathcal{V}$,

$$\xi^i_x = \sum_{j \in \Phi^{-1}(\Lambda, \Lambda')} B_{\Lambda, \Lambda'}(x)_{i,j} \xi^j_0.$$ 

It follows if $\xi^j_0 = 0$ for all $j \in \Phi^{-1}(\Lambda, \Lambda')$ then $\xi^i_x = 0$ for all $i \in \Phi^{-1}(\Lambda, \Lambda')$ and $x \in \mathcal{V}$, i.e. $\text{Eig}(\Lambda, \Lambda') = 0$. □

Let us denote $\text{Exp} = \text{Im} \Phi$, the set of exponents. Clearly $(\Lambda, \Lambda') \in \text{Exp}$ if and only if $\text{Eig}(\Lambda, \Lambda') \neq 0$.

Proposition 5.3 Provided $Z = \tilde{Z}$ we have $Z_{\Lambda, \Lambda'} \neq 0$ if and only if $(\Lambda, \Lambda') \in \text{Exp}$.

Proof. If $Z = \tilde{Z}$ then

$$Z_{\Lambda, \Lambda'} = (S^* Z S)_{\Lambda, \Lambda'} = (S^* \tilde{Z} S)_{\Lambda, \Lambda'} = \sum_{\Omega, \Omega' \in \mathcal{W}} (S^*)_{\Lambda, \Omega} \tilde{Z}_{\Omega, \Omega'} S_{\Omega', \Lambda'} = \sum_{\Omega, \Omega' \in \mathcal{W}} \sum_{i=0}^{D-1} \frac{D-1}{S_{0, \Phi_1(i)} S_{0, \Phi_2(i)} S_{\Omega', \Lambda'}} |\xi^i_0|^2,$$

the statement follows now by Lemma 5.2. □

Recall that in [4] we considered a set of exponents, which we will now denote by $\text{Exp}^+$, labelling the joint spectrum of matrices $V_{\Lambda}$, where $V_{\Lambda}^b_{a} = \cdots$
\( \langle \beta_a \circ \alpha^{-1}_\Lambda, \beta_b \rangle_M, \ a, b \in \mathcal{V}^+ \) and \( \Lambda \in \mathcal{W} \). We proved that \( Z_{\Lambda, \Lambda} \neq 0 \) if and only if \( \Lambda \in \text{Exp}^+ \), provided that the extended S-matrix diagonalizes the (endomorphism) fusion rules of the marked vertices \( \mathcal{T} \). This condition is not particularly difficult to prove for the conformal inclusions since the set \( \mathcal{T} \) is given in terms of the level 1 positive energy representations of the ambient WZW theory. However, for the orbifold inclusions this seems to be hardly possible without computer aid since the formulae for the extended S-matrices are complicated. On the other hand we prove \( Z = \tilde{Z} \) for all orbifold inclusions. Therefore it is useful to check the relations between \( \text{Exp} \) and \( \text{Exp}^+ \).

**Lemma 5.4** If \( (\Omega, \Omega') \in \text{Exp} \) then \( \Omega \in \text{Exp}^+ \). Conversely, if \( \Omega \in \text{Exp}^+ \) then there is some \( \Omega' \in \mathcal{W} \) such that \( (\Omega, \Omega') \in \text{Exp} \).

**Proof.** Since the subset \( \mathcal{V}^+ \subset \mathcal{V} \) is itself a sector basis, the matrices \( R^{\Lambda,0} \), corresponding to \( [\alpha_\Lambda] \), decompose block-diagonally with respect to the labels in \( \mathcal{V}^+ \) and \( \mathcal{V} \setminus \mathcal{V}^+ \). Thus we can write \( R^{\Lambda,0} = V_\Lambda \oplus \tilde{V}_\Lambda \). Assume \( (\Omega, \Omega') \in \text{Exp} \). Let \( \xi^i \) be a corresponding simultaneous eigenvector of the \( R^{\Lambda,\Lambda'} \)'s, i.e. \( \Phi(i) = (\Omega, \Omega') \) and we have in particular \( R^{\Lambda,0} \xi^i = \gamma_{\Omega}(\Lambda) \xi^i \). We can write \( \xi^i = \psi^i + \tilde{\psi}^i \), then this reads in particular \( V_\Lambda \psi^i = \gamma_{\Omega}(\Lambda) \psi^i \). Therefore \( \Omega \in \text{Exp}^+ \) if \( \psi^i \neq 0 \). However, by the same argument as in [4, Cor. 4.6], the eigenvectors \( \xi^i \) can be chosen such that \( \xi^i_0 > 0 \), thus \( \psi^i \neq 0 \).

Conversely, assume \( \Omega \in \text{Exp}^+ \). This means \( \gamma_{\Omega}(\Lambda) \) belongs to the spectrum of \( V_\Lambda \), therefore it belongs to the spectrum of \( R^{\Lambda,0} = V_\Lambda \oplus \tilde{V}_\Lambda \). As the eigenvalues of the \( R^{\Lambda,\Lambda'} \)'s are all of the form \( \gamma_{\Omega}(\Lambda) \gamma_{\Omega'}(\Lambda') \) and the characters \( \gamma_{\Omega} \) are linearly independent, there must be some \( \Omega' \in \mathcal{W} \) such that \( \gamma_{\Omega}(\Lambda) \gamma_{\Omega'}(\Lambda') \) gives in fact the eigenvalues of the \( R^{\Lambda,\Lambda'} \)'s, i.e. \( (\Omega, \Omega') \in \text{Exp} \).

\( \square \)

Now we can prove the following

**Proposition 5.5** Provided \( Z = \tilde{Z} \) we have \( Z_{\Lambda, \Lambda} \neq 0 \) if and only if \( \Lambda \in \text{Exp}^+ \).

**Proof.** If \( Z_{\Lambda, \Lambda} \neq 0 \) then \( (\Lambda, \Lambda) \in \text{Exp} \) by Proposition 5.3. Hence \( \Lambda \in \text{Exp}^+ \) by Lemma 5.4. Conversely, if \( \Lambda \in \text{Exp}^+ \) then by Lemma 5.4 there is a \( \Lambda' \in \mathcal{W} \) such that \( (\Lambda, \Lambda') \in \text{Exp} \), hence \( Z_{\Lambda, \Lambda'} \neq 0 \) by Proposition 5.3. As \( Z_{\Lambda, \Lambda'} \neq 0 \) implies \( Z_{\Lambda, \Lambda} \neq 0 \) for block-diagonal modular invariants (cf. also Lemma 3.1), the statement follows.

\( \square \)

Analogous statements hold for \( \text{Exp}^- \) corresponding to \( \mathcal{V}^- \). In particular \( Z = \tilde{Z} \) implies \( \text{Exp}^+ = \text{Exp}^- \).

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6 Applications to Embeddings of $SU(n)$: Examples

We will now apply our results to conformal and orbifold inclusions of $SU(n)$. By the spin and statistics theorem \(20\) we have for the statistics phase $\kappa_{\lambda_\Lambda} = e^{2\pi i h_\Lambda}$, where $h_\Lambda$ is the conformal dimension, $\Lambda \in \mathcal{A}^{(n,k)}$. Due to $T$-invariance of the modular mass matrix $Z$ we have $h_\Lambda = 0 \mod \mathbb{Z}$ whenever $\lambda_\Lambda$ is a subsector of $[\theta]$. With this, Corollary \(3.9\) and Lemma \(3.10\) become useful criteria to compare $[\alpha_\Lambda^+]$ and $[\alpha_\Lambda^-]$, and this will lead us to the validity of $Z_{\Lambda,\Lambda'} = \langle \alpha_\Lambda^+, \alpha_{\Lambda'}^- \rangle_M$, $\Lambda, \Lambda' \in \mathcal{A}^{(n,k)}$, for all our examples.

6.1 Conformal embeddings of $SU(2)$ revisited

$E_6$ revisited: $SU(2)_{10} \subset SO(5)_1$. We have derived the algebraic structure of $\mathcal{V}^+$ in \(3\), and clearly the same results are obtained for $\mathcal{V}^-$, hence we have

$$\mathcal{V}^\pm = \{[\alpha_0], [\alpha_1^\pm], [\alpha_2^\pm], [\alpha_3^{(1)}], [\alpha_6^\pm], [\alpha_{10}]\},$$

where we omit here (and similar in the examples discussed below) the $\pm$-index for the marked vertices $[\alpha_0]$, $[\alpha_3^{(1)}]$ and $[\alpha_{10}]$ as we know that always $T \subset \mathcal{V}^+ \cap \mathcal{V}^-$.  

**Lemma 6.1** For the $E_6$ example we have $[\gamma] = [\text{id}_M] \oplus [\alpha_1^+ \circ \alpha_1^-]$.  

**Proof.** Recall that $[\theta] = [\lambda_0] \oplus [\lambda_6]$. We have the fusion rule $N_{6,2}^8 = 1$ but $h_8 - h_2 = 5/3 - 1/6 = 3/2 \notin \mathbb{Z}$. Hence it follows from Corollary \(3.9\) that $[\alpha_2^+] \neq [\alpha_2^-]$. Since $N_{j,j}^2 = 1$ for all $j \neq 0,10$, it follows that $[\alpha_j^+] \neq [\alpha_j^-]$ for all $j \neq 0,10$, because equality of $[\alpha_j^+]$ and $[\alpha_j^-]$ clearly implies equality of their squares, and if $[\alpha_2^-]$ appears in the decomposition of the square of $[\alpha_j^+]$, then $[\alpha_2^-]$ equals some subsector of some $[\alpha_j^+]$, implying equality of $[\alpha_2^-]$ and $[\alpha_2^+]$ by Lemma \(3.1\), a contradiction. In particular we find $[\alpha_1^+] \neq [\alpha_1^-]$. Now

$$\langle \alpha_1^+ \circ \alpha_1^- , \alpha_1^+ \circ \alpha_1^- \rangle_M = \langle \alpha_1^+ \circ \alpha_1^+ , \alpha_1^- \circ \alpha_1^- \rangle_M = 1 + \langle \alpha_2^+, \alpha_2^- \rangle_M = 1,$$

thus $\alpha_1^+ \circ \alpha_1^-$ is irreducible. Moreover, by Corollary \(3.19\) we find that $[\alpha_1^+ \circ \alpha_1^-]$ is a subsector of $[\gamma]$, different from the identity since $[\alpha_1^+] \neq [\alpha_1^-]$. Since $\langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 2$ by Lemma \(3.16\), the statement follows. \(\square\)

One can also check that

$$d_\gamma = 1 + d_1^2 = 1 + d_6 = d_0 = 3 + \sqrt{3}.$$
In fact, with similar arguments as used in the proof of Lemma 6.1, it is not difficult to solve the system completely, i.e. to determine the algebraic structure of $V$. We find

$$V = \{[\alpha_0], [\alpha^+_1], [\alpha^-_1], [\alpha^+_2], [\alpha^-_2], [\alpha^{(1)}_3], [\alpha^+_4], [\alpha^-_9], [\alpha_{10}], [\delta], [\zeta], [\delta']\},$$

where $[\delta] = [\alpha^+_1 \circ \alpha^-_1], [\zeta] = [\alpha^+_1 \circ \alpha^+_2] = [\alpha^+_2 \circ \alpha^-_1]$ and $[\delta'] = [\alpha^+_9 \circ \alpha^-_1] = [\alpha^+_1 \circ \alpha^-_9]$. The fusion graphs of $[\alpha^+_1]$ (straight lines) and $[\alpha^-_1]$ (dashed lines) are given in Figure 2. We have encircled the even vertices by small circles, the marked vertices by larger circles. It is easy to write down the principal graph of the subfactor $N \subset M$ by taking the connected component of $[\lambda_0] \equiv [\text{id}_N]$ of the induction-restriction graph, as already drawn in Figure 4. The correctly labelled graph is given in Figure 3. Having determined the subsectors of $[\gamma]$, we can now similarly determine the dual principal graph by taking the connected component of $[\alpha_0] \equiv [\text{id}_M]$ of the $\gamma$-multiplication graph, presented in Figure 4. It is straightforward to check that the $M$-$M$ sectors, labelling the even vertices in Figure 4, obey in fact the fusion rules determined by Kawahigashi [25] as the correct fusion table of the five possibilities given in Figure 2. Another result of [25], namely that this fusion al-
Figure 3: E$_6$: Principal graph for the conformal inclusion $SU(2)_{10} \subset SO(5)_1$

Figure 4: E$_6$: Dual principal graph for the conformal inclusion $SU(2)_{10} \subset SO(5)_1$

The algebra contains a subalgebra corresponding to the even vertices of E$_6$ turns up quite naturally here as $[\alpha_0]$, $[\alpha_2^\pm]$ and $[\alpha_{10}]$ appear as even vertices of the dual principal graph due to the general fact stated in Corollary 4.3.

E$_8$ revisited: $SU(2)_{28} \subset (G_2)_1$. Recall from [4] that

$V^\pm = \{[\alpha_0], [\alpha_1^\pm], [\alpha_2^\pm], [\alpha_3^\pm], [\alpha_4^\pm], [\alpha_5^{\pm(1)}], [\alpha_5^{\pm(2)}], [\alpha_6^{(1)}]\}$.

**Lemma 6.2** For the E$_8$ example we have $[\gamma] = [id_M] \oplus [\delta] \oplus [\omega] \oplus [\eta]$, where $[\delta] = [\alpha_1^+ \circ \alpha_1^-]$ and $[\omega] = [\alpha_2^+ \circ \alpha_2^-]$ irreducible and $[\alpha_3^+ \circ \alpha_3^-] = [\eta] \oplus [\eta']$ with $[\eta], [\eta']$ irreducible.

**Proof.** Recall $[\theta] = [\lambda_0] \oplus [\lambda_{10}] \oplus [\lambda_{18}] \oplus [\lambda_{28}]$. We have the fusion rule $N^8_{j,j} = 1$, but $h_{28} - h_2 = 2/3 - 1/15 = 3/5 \notin \mathbb{Z}$. Hence $[\alpha_2^+] \neq [\alpha_2^-]$ by Corollary 3.9. Since $N^2_{j,j} = 1$ for all $j \neq 0, 28$, it follows immediately that $[\alpha_3^+] \neq [\alpha_3^-]$ for all $j \neq 0, 28$. Note that $[\alpha_5^+] = [\alpha_{23}^+]$. Since $h_{23} - h_5 = 115/24 - 7/24 = 9/2$ it follows $\langle \alpha_5^+, \alpha_5^- \rangle_M = 0$ by Lemma 3.10. Thus the subsectors of $[\alpha_5^+]$ and $[\alpha_5^-]$ are all disjoint. We have shown $\langle \alpha_j^+, \alpha_j^- \rangle_M = 0$ for $j = 1, 2, 3, 4, 5$, and since $[\alpha_6^+] \neq [\alpha_6^-]$ but $[\alpha_6^{(1)}]$ is a marked vertex we have $\langle \alpha_6^+, \alpha_6^- \rangle_M = 1$. 

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With these relations one checks easily that \([\delta] = [\alpha_1^+ \circ \alpha_1^-], [\omega] = [\alpha_2^+ \circ \alpha_2^-]\) and \([\alpha_3^+ \circ \alpha_3^-]\) are disjoint, e.g.

\[
\langle \alpha_2^+ \circ \alpha_2^-, \alpha_3^+ \circ \alpha_3^- \rangle_M = \langle \alpha_2^+ \circ \alpha_3^+, \alpha_2^- \circ \alpha_3^- \rangle_M = \sum_{j,j'=1,3,5} \langle \alpha_j^+, \alpha_j^- \rangle_M = 0,
\]

and similarly that \([\delta]\) and \([\omega]\) are irreducible, whereas

\[
\langle \alpha_3^+ \circ \alpha_3^-, \alpha_3^+ \circ \alpha_3^- \rangle_M = \langle \alpha_3^+ \circ \alpha_3^+, \alpha_3^- \circ \alpha_3^- \rangle_M = \langle \alpha_0, \alpha_0 \rangle_M + \langle \alpha_6^+, \alpha_6^- \rangle_M = 2.
\]

Hence \([\alpha_3^+ \circ \alpha_3^-] = [\eta] \oplus [\eta']\) with \([\eta], [\eta']\) irreducible. Since (Corollary 3.19)

\[
\langle \alpha_j^+ \circ \alpha_j^-, \gamma \rangle_M = \langle \alpha_j^+, \alpha_j^+ \rangle = 1, \quad j = 1, 2, 3,
\]

it follows that \([\delta], [\omega]\) are subsectors of \([\gamma]\) and \([\alpha_3^+ \circ \alpha_3^-]\) has one common subsector with \([\gamma]\), say \([\eta]\). As \(\langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 4\), the statement follows.

With a little more computation, the full induced sector basis \(\mathcal{V}\) and its algebraic structure (the associated “sector algebra”, see [3] for definitions) can be determined. One finds that \(\mathcal{V}\) has 32 elements. The fusion graphs of \([\alpha_1^+]\) (straight lines) and \([\alpha_1^-]\) (dashed lines) are given in Figure 6. Here we denote

\[
\begin{align*}
[\xi^\pm] &= [\alpha_2^+ \circ \alpha_1^\mp], & [\zeta^\pm] &= [\alpha_3^+ \circ \alpha_1^\mp], \\
[\psi^\pm] &= [\alpha_4^+ \circ \alpha_1^\mp], & [\kappa^\pm] &= [\alpha_5^{\pm(2)} \circ \alpha_1^\mp], \\
[\tau^\pm] &= [\alpha_3^+ \circ \alpha_2^\mp], & [\varsigma^\pm] &= [\alpha_5^{\pm(2)} \circ \alpha_2^\mp], \\
[\chi] &= [\alpha_6^{\pm(3)} \circ \alpha_1^-], & [\varpi] &= [\alpha_4^+ \circ \alpha_2^-].
\end{align*}
\]

By the induction-restriction mechanism it is easy to write down the principal graph of \(N \subset M\), presented in Figure 7. With a bit more calculation we can also determine the \(\gamma\)-multiplication and therefore write down the dual principal graph, presented in Figure 8. Note that the fusion algebra of the \(M-M\) sectors labelling the even vertices of the dual principal graph possesses again a subalgebra (in fact two copies) which corresponds to the even vertices of the \(E_8\) graph, due to Corollary 4.3.

\(D_4\) revisited: \(SU(2)_1 \subset SU(3)_1\). Recall from [1] that

\[
\mathcal{V}^\pm = \{[\alpha_0], [\alpha_1^\pm], [\alpha_2^{(1)}], [\alpha_2^{(2)}]\},
\]

and the marked vertices \([\alpha_0], [\alpha_2^{(1)}]\) and \([\alpha_2^{(2)}]\) obey \(Z_3\) fusion rules.
Lemma 6.3 For the $D_4$ example we have $[\gamma] = [\text{id}_M] \oplus [\epsilon]$, where $[\epsilon]$ is an irreducible subsector of $[\alpha_1^+ \circ \alpha_1^-]$.

Proof. We have $[\alpha_3^+] = [\alpha_1^+]$ but $h_3 - h_1 = 5/8 - 1/8 = 1/2 \notin \mathbb{Z}$, hence $[\alpha_1^+] \neq [\alpha_1^-]$. Since

$$\langle \alpha_1^+ \circ \alpha_1^- , \alpha_1^+ \circ \alpha_1^- \rangle_M = \langle \alpha_0, \alpha_0 \rangle_M + \langle \alpha_2, \alpha_2 \rangle_M = 3$$

and $d_1 = \sqrt{3}$, we find that $[\alpha_1^+ \circ \alpha_1^-]$ decomposes into three different subsectors,

$$[\alpha_1^+ \circ \alpha_1^-] = [\epsilon] \oplus [\eta] \oplus [\eta'] ,$$

Figure 5: $E_8$: Fusion graph of $[\alpha_1^+]$ and $[\alpha_1^-]$
with statistical dimensions \(d_\epsilon = d_\eta = d_{\eta'} = 1\). Since \(\langle \alpha_1^+ \circ \alpha_1^-, \gamma \rangle_M = \langle \alpha_1^+, \alpha_1^- \rangle_M = 1\), \([\alpha_1^+ \circ \alpha_1^-]\) and \([\gamma]\) have one subsector in common, say \([\epsilon]\). Note that \([M : N] = d_\gamma = 2\), hence we are dealing with the unique \(\mathbb{Z}_2\) subfactor and therefore we can choose \(\epsilon \in \text{Aut}(M)\) such that \(\epsilon^2 = \text{id}_M\) and \(N = M^\epsilon\) and \([\gamma] = [\text{id}_M] \oplus [\epsilon]\).

By Proposition 5.1 we find \(\sum_{x \in \mathcal{V}} d_x^2 = \sum_{j=0}^4 d_j^2 = 12\), hence we have already found all the sectors of \(\mathcal{V}\),

\[
\mathcal{V} = \{[\alpha_0], [\alpha_1^+], [\alpha_1^-], [\alpha_2^{(1)}], [\alpha_2^{(2)}], [\epsilon], [\eta], [\eta']\}.
\]

We remark that we will show in a forthcoming joint work with Y. Kawahigashi [5] that the corresponding sector algebra is non-commutative. The simultaneous fusion graph of \([\alpha_1^+]\) and \([\alpha_1^-]\) is given in Figure 8.

### 6.2 \(\mathbb{Z}_m\) orbifold inclusions of \(SU(n)\)

In [4] we discussed the \(\mathbb{Z}_n\) orbifold modular invariants of \(SU(n)\) which correspond to a simple current extension by a simple current of order \(n\). However, for any decomposition \(n = mq\) with \(m, q \in \mathbb{N}\) there appears a series of block-diagonal orbifold modular invariants corresponding to a simple current of order \(m\), see [35, 38]. We would like to extend our analysis to this more general situation. Note that the case \(q = 1\) corresponds to the familiar \(\mathbb{Z}_n\) orbifold situation whereas the other extreme case \(m = 1\) corresponds to the completely diagonal invariant, i.e. there is no extension at all, but if \(n\) is not prime then there are intermediate cases. The \(\mathbb{Z}_m\) invariants appear whenever \(2n\) divides \(k'q^2\) or, equivalently, when \(2m\) divides \(k'q\), where \(k' = k + n\).
if the level \( k \) and \( n \) are both odd and \( k' = k \) otherwise. One can check that this is equivalent to the condition that \( kq \in 2m \mathbb{Z} \) if \( n \) is even and \( kq \in m \mathbb{Z} \) if \( n \) is odd. The corresponding modular invariant mass matrix reads

\[
Z_{\Lambda, \Lambda'} = \delta^m(t(\Lambda)) \sum_{j=0}^{m-1} \delta(\Lambda', \sigma^j(\Lambda)).
\]  

(17)

Here, as usual, \( \sigma \) is the \( \mathbb{Z}_n \) rotation of \( A^{(n,k)} \),

\[
\sigma(\Lambda) = (k - m_1 - \ldots - m_{n-1})\Lambda_{(1)} + m_1\Lambda_{(2)} + m_2\Lambda_{(3)} + \ldots + m_{n-2}\Lambda_{(n-1)},
\]

for \( \Lambda = \sum_{i=1}^{n-1} m_i\Lambda_{(i)} \in A^{(n,k)} \) with fundamental weights \( \Lambda_{(i)} \), \( t(\Lambda) = \sum_{i=1}^{n-1} i m_i \), and \( \delta^y(x) \) equals 1 or 0 dependent whether or not \( x/y \) is an integer, respectively. In terms of sectors, the \( \mathbb{Z}_n \) rotation \( \sigma \) corresponds to \( [\lambda_{k\Lambda_{(1)}}, \mathbb{Z}_n] \) and the \( \mathbb{Z}_m \) rotation \( \sigma^q \) to \( [\lambda_{k\Lambda_{(q)}}, \mathbb{Z}_m] \) which realize the rotations \( \sigma \) respectively \( \sigma^q \) as fusion rules. In fact as the vacuum block of Eq. (17) is easily read off as \( |\chi_0 + \sum_{j=1}^{m-1} \chi_{k\Lambda_{(q)}}|^2 \) one notices that Eq. (17) corresponds to an extension by the simple current \( [\lambda_{k\Lambda_{(q)}}, \mathbb{Z}_m] \). We will denote by \( \sigma_q \) the representative automorphism corresponding to the \( \mathbb{Z}_m \) rotation, \( \sigma_q = \lambda_{k\Lambda_{(q)}} \). We can assume \( \sigma_q \) to be \( m \)-periodic, \( \sigma_q^m = \text{id} \), by Lemma 4.4, since the statistics phase fulfills \( \kappa \sigma_q = 1 \) as \( \kappa \sigma_q = e^{2\pi i h_{\sigma_q}} \), \( h_{\sigma_q} \equiv h_{k\Lambda_{(q)}} = kq(n - q)/2n = kq(m - 1)/2m \in \mathbb{Z} \) exactly at the relevant levels. (Note that we can no longer use our simple argument relying on the fixed point as in [4, Lemma 3.1] since in this more general case there is not necessarily a fixed point.) Therefore we can construct the extension net of

\[
\begin{array}{cccccccccccccccc}
[\alpha_0] & [\delta] & [\omega] & [\eta] & [\alpha_+^\perp] & [\alpha_-^\perp] & [\zeta^+] & [\zeta^-] & [\eta'] & [\alpha_+^\perp] & [\alpha_-^\perp] & [\varphi] & [\kappa^+] & [\kappa^-] & [\chi] & [\alpha_0^{(1)}] \\
\end{array}
\]
subfactors exactly as in [4, Subsect. 3.1], replacing \( n \) by \( m \). Similarly we find that \( \mathcal{M} \) is local and even Haag dual since \( \varepsilon(\sigma_q, \sigma_q) = \kappa_{\sigma_q} = 1 \). By construction we have

\[
[\theta] = \bigoplus_{j=0}^{m-1} [\sigma_q^j].
\]

We have shown

**Proposition 6.4** At levels \( k \) satisfying \( kq \in 2m\mathbb{Z} \) if \( n \) is even and \( kq \in m\mathbb{Z} \) if \( n \) is odd the simple current extension by the simple current \( \sigma_q \) is realized as a quantum field theoretical net of subfactors \( \mathcal{N} \subset \mathcal{M} \), where \( \mathcal{M} \) is Haag dual and as a sector the dual canonical endomorphism decomposes as in Eq. (18).

Therefore we can apply \( \alpha \)-induction. Now \( [\lambda_{k\Lambda(\ell)}] \times [\lambda_{\Lambda}] = [\lambda_{\sigma^\ell(\Lambda)}] \) is irreducible for any \( \Lambda \in \mathcal{A}_{n,k} \) and any \( \ell = 0, 1, 2, ..., n-1 \), hence it follows from Eq. (8),

\[
\varepsilon(\lambda_{\Lambda}, \lambda_{k\Lambda(\ell)})\varepsilon(\lambda_{k\Lambda(\ell)}, \lambda_{\Lambda}) = e^{2\pi i (h_{\sigma^\ell(\Lambda)} - h_{k\Lambda(\ell)} - h_{\Lambda})} 1,
\]
where we also used \( \kappa_\Lambda = e^{2\pi i \Lambda}, \Lambda \in \mathcal{A}^{(n,k)} \).

**Lemma 6.5** For any \( \ell = 1, 2, \ldots, n - 1 \) and any \( \Lambda = \sum_{i=1}^{n-1} m_i \Lambda(i) \in \mathcal{A}^{(n,k)} \), we have

\[
h_{\sigma^\ell(\Lambda)} - h_\Lambda = \frac{\ell}{n} \left( \frac{(n-\ell)k}{2} - t(\Lambda) \right) + \sum_{i=1}^{\ell-1} (\ell - i) m_{n-i}. \tag{19}
\]

**Proof.** By induction. For \( \ell = 1 \) the formula reduces to [26, Lemma 2.7],

\[
h_{\sigma(\Lambda)} - h_\Lambda = \frac{1}{n} \left( \frac{(n-1)k}{2} - t(\Lambda) \right).
\]

Recall \( \sigma(\Lambda) = \sum_{i=1}^{n-1} m_i^\sigma \Lambda(i) \) with \( m_i^\sigma = k - m_1 - \ldots - m_{n-1} \) and \( m_i^\sigma = m_{i-1} \) for \( i \geq 1 \). Hence we have \( t(\sigma(\Lambda)) = t(\Lambda) + k - nm_{n-1} \). The induction from \( \ell - 1 \) to \( \ell \) is now as follows. First

\[
h_{\sigma^\ell(\Lambda)} - h_{\sigma^{\ell-1}(\sigma(\Lambda))} = \frac{\ell-1}{n} \left( \frac{(n+\ell)k}{2} - t(\sigma(\Lambda)) \right) - \sum_{i=1}^{\ell-2} (\ell - i) m_{n-i} \]

\[
= \frac{\ell-1}{n} \left( \frac{(n+\ell)k}{2} - t(\Lambda) - k + nm_{n-1} \right) - \sum_{i=1}^{\ell-2} (\ell - i) m_{n-1-i} \]

\[
= \frac{\ell-1}{n} \left( \frac{(n+\ell-1)k}{2} - t(\Lambda) \right) - \sum_{i=1}^{\ell-1} (\ell - i) m_{n-i}.
\]

Hence

\[
h_{\sigma^\ell(\Lambda)} - h_\Lambda = \frac{\ell-1}{n} \left( \frac{(n+\ell+1)k}{2} - t(\Lambda) \right) + \frac{1}{n} \left( \frac{(n-1)k}{2} - t(\Lambda) \right) \]

\[
= \frac{\ell}{n} \left( \frac{(n+\ell)k}{2} - t(\Lambda) \right) - \sum_{i=1}^{\ell-1} (\ell - i) m_{n-i},
\]

and the induction is complete. \( \square \)

As \( h_{k\Lambda(i)} = k\ell(n-\ell)/2n \) we obtain immediately the following

**Corollary 6.6** For \( \Lambda \in \mathcal{A}^{(n,k)} \) and \( \ell = 0, 1, 2, \ldots, n \) we have \( h_{\sigma^\ell(\Lambda)} - h_{k\Lambda(i)} - h_\Lambda = -t(\Lambda)\ell/n \mod \mathbb{Z} \) and hence

\[
\varepsilon(\lambda_\Lambda, \lambda_{k\Lambda(i)}) \varepsilon(\lambda_{k\Lambda(i)}, \lambda_\Lambda) = e^{-2\pi i t(\Lambda)\ell/n}. \tag{20}
\]
Using this for $\ell = jq$, $j = 0, 1, 2, \ldots, m - 1$, and Lemma 5.8 we finally find

**Corollary 6.7** We have trivial monodromy, $\varepsilon(\lambda_\Lambda, \theta)\varepsilon(\theta, \lambda_\Lambda) = 1$, if and only if $t(\Lambda) = 0 \mod m$, $\Lambda \in \mathcal{A}^{(n,k)}$.

Now we can investigate the $\alpha$-induced endomorphisms.

**Lemma 6.8** For a $\mathbb{Z}_m$ orbifold inclusion of $SU(n)$ we have $[\alpha^+_\Lambda] = [\alpha^-_\Lambda]$ if $t(\Lambda) = 0 \mod m$ and $\langle \alpha^+_\Lambda, \alpha^-_\Lambda \rangle_M = 0$ if $t(\Lambda) \neq 0 \mod m$, $\Lambda \in \mathcal{A}^{(n,k)}$.

**Proof.** The first statement follows from Corollary 6.7 and [3, Prop. 3.23]. Now note that the decomposition $[\theta] = \bigoplus_{j=0}^{m-1} [\sigma_j]$ implies $[\alpha^+_\Lambda, [\alpha^-_\Lambda]] = [\alpha^+_\sigma(\Lambda), [\alpha^-_\sigma(\Lambda)]]$ since then

$$\langle \alpha^+_\Lambda, \alpha^-_\Lambda \rangle_M = \langle \alpha^+_\Lambda, [\alpha^-_\sigma(\Lambda)] \rangle_M = \langle \theta \circ \lambda, \lambda \rangle_N.$$

Now $h_{k\alpha}(q) = kq(m-1)/2m \in \mathbb{Z}$ for the levels $k$ where the $\mathbb{Z}_m$ orbifold inclusions appear, therefore $h_{\alpha(\Lambda)} - h_{\Lambda} \not\in \mathbb{Z}$ if and only if $t(\Lambda) \neq 0 \mod m$. It follows that then $\langle \alpha^+_\Lambda, \alpha^-_\Lambda \rangle_M = 0$ by Lemma 3.10.

Now we are ready to prove the main result of this subsection.

**Theorem 6.9** For all $\mathbb{Z}_m$ orbifold inclusions of $SU(n)$, where $n = mq$, $m, q \in \mathbb{N}$, appearing at levels $k$ such that $kq \in 2m\mathbb{N}$ if $n$ is even and $kq \in m\mathbb{N}$ if $n$ is odd, we have:

1. $Z_{\Lambda,\Lambda'} = \langle \alpha^+_\Lambda, \alpha^-_{\Lambda'} \rangle_M$ for all $\Lambda, \Lambda' \in \mathcal{A}^{(n,k)}$,
2. $\sum_{x \in V} d_x^2 = \sum_{\Lambda \in \mathcal{A}^{(n,k)}} d^2_\Lambda$,
3. each irreducible subsector of $[\gamma]$ is in $V$.

**Proof.** As by Lemma 3.1, $\langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda} \rangle_M = 0$, $\Lambda \in \mathcal{A}^{(n,k)}$, implies $\langle \alpha^+_{\Lambda}, \alpha^-_{\Lambda} \rangle_M = 0$ for all other $\Lambda' \in \mathcal{A}^{(n,k)}$, we can write

$$\langle \alpha^+_\Lambda, \alpha^-_{\Lambda'} \rangle_M = \delta^m(t(\Lambda)) \langle \alpha^+_\Lambda, \alpha^+_{\Lambda'} \rangle_M$$

by Lemma 6.8. Now

$$\langle \alpha^+_\Lambda, \alpha^-_{\Lambda'} \rangle_M = \langle \theta \circ \lambda, \lambda \rangle_N = \sum_{j=0}^{m-1} \langle \sigma^j \circ \lambda, \lambda \rangle_N = \sum_{j=0}^{m-1} \delta(\Lambda', \sigma^j(\Lambda)),$$
\[ \langle \alpha_\Lambda^+, \alpha_{\Lambda'}^- \rangle_M = \delta^m(t(\Lambda)) \sum_{j=0}^{m-1} \delta(\Lambda', \sigma^j(\Lambda)) = Z_{\Lambda, \Lambda'}, \]

proving the first statement. The second statement is derived from the first in exactly the same way as in the proof of Proposition 5.1 for the conformal inclusion case. Finally the third statement follows from the second by Lemmata 4.5 and 4.6.

It is instructive to find the subsectors of \[ \gamma \] in \( \mathcal{V} \) more constructively. Since \[ \langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = m, \] \[ \gamma \] contains at most \( m \) different irreducible subsectors. In fact, analogously to [4, Cor. 3.4] we find that \( M \simeq N \rtimes \sigma_\gamma \mathbb{Z}_m \), hence it follows (see e.g. [30])

\[ \gamma = \bigoplus_{j=0}^{m-1} [\epsilon^j], \]

where \( \epsilon \in \text{Aut}(M) \) is \( m \)-periodic and \( N \) is the fixed point algebra of \( M \) under the action of \( \epsilon \), \( N = M^\epsilon \). Now for \( i, j = 1, 2, ..., m-1 \) we have, by Lemma 6.8,

\[ \langle \alpha_{\Lambda(i)}^+ \circ \alpha_{\Lambda(n-j)}^- \circ \alpha_{\Lambda(j)}^+ \rangle_M = \langle \alpha_{\Lambda(i)}^+ \circ \alpha_{\Lambda(n-j)}^+ \circ \alpha_{\Lambda(j)}^- \rangle_M = 0 \]

if \( i \neq j \), since then \( \alpha_{\Lambda(i)}^+ \circ \alpha_{\Lambda(n-j)}^+ \) decomposes into \( \alpha_{\Lambda}^+ \)'s with \( t(\Lambda) \neq 0 \mod m \). Similarly

\[ \langle \alpha_{\Lambda(j)}^+ \circ \alpha_{\Lambda(n-j)}^- \circ \text{id}_M \rangle_M = \langle \alpha_{\Lambda(j)}^+ \rangle_M = 0, \quad j = 1, 2, ..., m - 1. \]

We conclude that \( [\text{id}_M] \) and \( [\alpha_{\Lambda(i)}^+ \circ \alpha_{\Lambda(n-j)}^-] \), \( j = 1, 2, ..., m-1, \) are all disjoint, and they all have (at least) one subsector in common with \( [\gamma] \) by Corollary 3.19. Therefore all the subsectors \( [\epsilon^j] \) of \( [\gamma] \) are subsectors.

It is straightforward to determine \( \mathcal{V} \) for the orbifold inclusions of \( SU(2) \). The simultaneous fusion graphs of \( [\alpha_1^+] \) and \( [\alpha_1^-] \) for \( D_6 \) and \( D_8 \) are given in Figure 6. For \( D_{2\varrho+2} \) we denote \( [\beta_{2j}] = [\epsilon \circ \alpha_{2j}], 1 \leq j \leq \varrho - 1, \) and \( [\eta] = [\epsilon \circ \alpha_{2\varrho}^1], [\eta'] = [\epsilon \circ \alpha_{2\varrho}^2] \).

### 6.3 Non-degenerate braidings on orbifold graphs

Let \( \mathcal{W}_0 \subset \mathcal{W} \) (recall that \( \mathcal{W} \) is the sector basis in \( [\Delta]_{\mathcal{N}}(I_\varrho) \) corresponding to — and as a set identified with — \( \mathcal{A}^{(n,k)} \)) be a sector sub-basis, i.e. a subset of \( \mathcal{W} \) which is itself a sector basis. The following lemma is from [12] (cf. also
Lemma 6.10 For any sector sub-basis $W_0 \subset W$ we have that if $\Omega \equiv [\lambda_\Omega] \in W_0$ is degenerate in $W_0$ then $S_{\Lambda,\Omega}S_{0,0} = S_{\Lambda,0}S_{\Omega,0}$ for all $\Lambda \equiv [\lambda_\Lambda] \in W_0$.

Proof. As a consequence of the Verlinde formula and the modular relation $(ST)^3 = S^2$ for the modular matrices $S$ and $T$ we obtain with $T_{\Lambda,\Lambda'} = \delta_{\Lambda,\Lambda'}e^{2\pi i h_\Lambda}T_{0,0}$ the following formula (see e.g. [15, Eq. (2.35)]; compare also
for a derivation in the DHR context),
\[
S_{\Lambda, \Omega} = S_{0,0} \sum_{\Phi \in \mathcal{A}^{(n,k)}} N_{\Lambda, \Omega}^\Phi d_\Phi e^{2\pi i (h_\Lambda + h_\Omega - h_\Phi)} , \quad \Lambda, \Omega \in \mathcal{A}^{(n,k)} .
\]

Now assume that \( \Lambda, \Omega \) belong to \( \mathcal{W}_0 \). Since \( \mathcal{W}_0 \) is a sector sub-basis it follows \( N_{\Lambda, \Omega}^\Phi \) can only be non-zero if \( \Phi \) belongs to \( \mathcal{W}_0 \). Moreover, if \( [\lambda_\Omega] \) is degenerate in \( \mathcal{W}_0 \) then we find for the eigenvalues of the monodromy \( e^{2\pi i (h_\Lambda + h_\Omega - h_\Phi)} = 1 \) whenever \( N_{\Lambda, \Omega}^\Phi \neq 0 \) by Eq. (9). Hence
\[
S_{\Lambda, \Omega} = S_{0,0} \sum_{\Phi \in \mathcal{A}^{(n,k)}} N_{\Lambda, \Omega}^\Phi d_\Phi = S_{0,0} d_\Lambda d_\Phi = \frac{S_{\Lambda,0} S_{\Omega,0}}{S_{0,0}}
\]
whenever \( \Lambda \) belongs to \( \mathcal{W}_0 \). \( \square \)

Now we return to the \( Z_m \) orbifold situation, i.e. we write \( n = mq \) and consider levels \( k \) with \( kq \in 2m\mathbb{Z} \) if \( n \) is even and \( kq \in m\mathbb{Z} \) if \( n \) is odd. The following lemma is a slight generalization of \([12, \text{Lemma 3.4}]\), but since there is a mistake in the proof we give a corrected and generalized proof here.

**Lemma 6.11** Let \( \mathcal{W}_0 \) be the sector basis of sectors \( [\lambda_\Lambda] \), \( \Lambda \in \mathcal{A}^{(n,k)} \), with \( t(\Lambda) = 0 \mod m \), and \( n, m, k, q \) as above. Then the degenerate elements within \( \mathcal{W}_0 \) are given by \([\sigma^j_0] \), \( j = 0, 1, 2, \ldots, m - 1 \).

**Proof.** For any \( \Omega \in \mathcal{A}^{(n,k)} \) let \( s^\Omega \) be the column vector of the S-matrix of \( SU(n)_k \) corresponding to the weight \( \Omega \), i.e. in components \((s^\Omega)_\Lambda = S_{\Lambda, \Omega}, \Lambda \in \mathcal{A}^{(n,k)} \). By the Verlinde formula these vectors are eigenvectors of the fusion matrices \( N_\Lambda \) (defined by \((N_\Lambda)_{\Lambda', \Lambda''} = (N_\Lambda'')_{\Lambda', \Lambda''} \)),
\[
N_\Lambda s^\Omega = \gamma_\Omega(\Lambda) s^\Omega , \quad \gamma_\Omega(\Lambda) = \frac{S_{\Lambda, \Omega} S_{0,0}}{S_{0,0}} , \quad \Lambda \in \mathcal{A}^{(n,k)} .
\]

We may split \( s^\Omega \) into \( m \) pieces, \( s^\Omega = t(s^\Omega_0, s^\Omega_1, \ldots, s^\Omega_{m-1}) \), where each vector \( s^\Omega_j \) consists of components \( S_{\Lambda, \Omega} \) with \( t(\Lambda) = j \mod m \). Since we have \( N_{\Lambda', \Lambda''}^\Lambda = 0 \) whenever \( t(\Lambda) + t(\Lambda') \neq t(\Lambda'') \mod n \), hence in particular if \( t(\Lambda) + t(\Lambda') \neq t(\Lambda'') \mod m \), and \( t(\Lambda(1)) = 1 \), \( t(\Lambda(n-1)) = n - 1 \), we can write
\[
N_{\Lambda(1)} s^\Omega_j = \gamma_\Omega(\Lambda(1)) s^\Omega_{j+1} , \quad N_{\Lambda(n-1)} s^\Omega_{j+1} = \gamma_\Omega(\Lambda(n-1)) s^\Omega_j ,
\]
and the index \( j \) can be read mod \( m \). Therefore we find with \( N_{\Lambda(n-1)} = t N_{\Lambda(1)} \),
\[
\gamma_\Omega(\Lambda(1)) \|s^\Omega_{j+1}\|^2 = (N_{\Lambda(1)} s^\Omega_j , s^\Omega_{j+1}) = (s^\Omega_j , N_{\Lambda(n-1)} s^\Omega_j) = \gamma_\Omega(\Lambda(1)) \|s^\Omega_j\|^2 ,
\]
where we used \( \gamma_\Omega(\Lambda_{(n-1)}) = S_{\Lambda_{(n-1)},\Omega}/S_{0,\Omega} = S_{\Lambda_{(1)},\Omega}/S_{0,\Omega} = \gamma_\Omega(\Lambda_{(1)}) \). Since \( \|s^\Omega\| = 1 \) by unitarity of the S-matrix we first conclude that \( \gamma_\Omega(\Lambda_{(1)}) \neq 0 \) and hence \( \|s_j^\Omega\| = m^{-1/2} \) for all \( j = 0, 1, 2, ..., m-1 \) and all \( \Omega \in \mathcal{A}^{(n,k)} \). Now assume that \( [\lambda_\Omega] \) is degenerate in \( \mathcal{W}_0 \). Then \( S_{\Lambda,\Omega} = S_{\Omega,0}S_{\Lambda,0}/S_{0,0} \) by Lemma 6.10 for all \( \Lambda \in \mathcal{A}^{(n,k)} \) with \( t(\Lambda) = 0 \mod m \), and this is \( s_0^\Omega = \gamma_0(\Omega)s_0^0 \). By \( \|s_0^\Omega\| = m^{-1/2} = \|s_0^0\| \) it follows \( \gamma_0(\Omega) = 1 \), i.e. \( S_{\Omega,0} = S_{0,0} \). This means that \( \Omega \) is the weight of a simple current, i.e. either 0 or one of the weights \( k\Lambda(\ell), \ell = 1, 2, 3, ..., n-1 \). Now take, for example, the weight \( m\Lambda(1) \) which has braiding, given by the relative braiding operators. The following \( T \) with \( t \) belongs to \( \mathcal{W} \) yields the expression for all \( \Lambda \in \mathcal{A}^{(n,k)} \) and it plays exactly the role of the set of marked vertices in the conformal inclusion case. One also checks easily that putting \( \varepsilon(\lambda_{m\Lambda_{(1)}}, \lambda_{k\Lambda(\ell)}\varepsilon(\lambda_{k\Lambda(\ell)}, \lambda_{m\Lambda_{(1)}}) = e^{2\pi i/4} \), \( \ell = 0, 1, 2, ..., n-1 \), and therefore \( [\lambda_{k\Lambda(\ell)}] \) can be degenerate in \( \mathcal{W}_0 \) only if \( \ell \) is a multiple of \( q \). On the other hand it follows similarly from Corollary 6.10 that \( [\lambda_{k\Lambda(\ell)}] \) is in fact degenerate in \( \mathcal{W}_0 \) for \( \ell = jq \), \( j = 0, 1, 2, ..., m-1 \). Finally we note that \( t(k\Lambda_{(jq)}) = kjq = 0 \mod m \) at the relevant levels, so in fact \( [\sigma_q^j] = [\lambda_{k\Lambda_{(jq)}}] \) belongs to \( \mathcal{W}_0 \), the proof is complete.

Defining \( \mathcal{T} \) to be the set of all irreducible subsectors of \( [\alpha_\Lambda^+] \), \( \Lambda \in \mathcal{A}^{(n,k)} \) with \( t(\Lambda) = 0 \mod m \), gives a sector basis with \( \mathcal{T} = \mathcal{V}^+ \cap \mathcal{V}^- \) by Lemma 6.8, and it plays exactly the role of the set of marked vertices in the conformal inclusion case. One also checks easily that putting

\[
\beta_t \lambda = (\lambda, \sigma_t, \beta_t)_N = (\alpha_{\Lambda_t}^+, \beta_t)_M, \quad t \in \mathcal{T}, \quad \Lambda \in \mathcal{A}^{(n,k)},
\]

yields the expression \( Z_{\Lambda,\Lambda'} = \sum_{t \in \mathcal{T}} b_{t,\Lambda} b_{t,\Lambda'} \). By Corollary 3.13 we find that \( \mathcal{T} \) has braiding, given by the relative braiding operators. The following theorem nicely reflects Rehren’s conjecture \( \mathcal{E} \) which was proven by Müger \( \mathcal{E} \).

**Theorem 6.12** For any \( \mathcal{Z}_m \) orbifold inclusion of \( SU(n) \) the sector basis \( \mathcal{T} \) as above has a non-degenerate braiding.

**Proof.** First note that \( \mathcal{T} \) is the image of \( \mathcal{W}_0 \) by \( \alpha \)-induction (+ or −). Now let \( \beta_t \in End(M) \) such that \( [\beta_t] \equiv t \) is an irreducible subsector of \( [\alpha_\Lambda^+] \) for some \( \Lambda \equiv [\lambda_\Lambda] \in \mathcal{W}_0 \), i.e. \( t \in \mathcal{T} \), and assume \( \varepsilon_t(\beta_t, \beta_{t'}) = \varepsilon_{t'}(\beta_{t'}, \beta_t)^* \) for all \( t' \in \mathcal{T} \), where, as usual, \( \beta_t \) denotes a representative endomorphism for each \( t' \). Since \( [\alpha_\Lambda^+] = [\alpha_\Lambda^-] \) for all \( \Lambda' \in \mathcal{W}_0 \) we obtain \( \varepsilon(\lambda, \lambda') \varepsilon(\lambda_\Lambda, \lambda_\Lambda) = 1 \) for all \( \Lambda' \in \mathcal{W}_0 \) by Lemma 3.14. By Lemma 6.11 we conclude that \( [\lambda_\Lambda] = [\sigma_q^j] = [\lambda_{k\Lambda_{(jq)}}] \) for some \( j = 0, 1, 2, ..., m-1 \). But \([\alpha_\Lambda^+] = [\text{id}_M] \), hence \( [\beta_t] = [\text{id}_M] \), showing that the braiding is non-degenerate. \( \square \)
For the $SU(2)$ orbifold inclusions, the set $T$ corresponds to the even vertices of the D-graphs, constructed as fusion graphs of either $[\alpha^+]$ or $[\alpha^-]$. So here Theorem 6.12 can be rephrased, roughly speaking, as “there is a non-degenerate braiding on the even vertices of the graphs $D_{even}$”, which is a known result, see [33, 12, 40]. For $SU(3)$ the corresponding statement is that there is a non-degenerate braiding associated to the triality zero vertices of Kostov’s graphs, see [28, Fig. 3b] or [6, Fig. 25], [11, Fig. 8.32]. Analogous statements hold now for a huge variety of orbifold graphs of the graph $A^{(n,k)}$. Let us finally remark that the analogue of Theorem 6.12 for the conformal inclusions is not very interesting since, by Proposition 3.13, we just rediscover the (non-degenerate) braiding of the enveloping WZW level 1 theory (cf. $SO(5)$ and $G_2$ for the $E_6$ and $E_8$ modular invariants in the $SU(2)$ case).

6.4 More conformal inclusions of $SU(n)$

Let us now continue with the treatment of conformal inclusions of $SU(n)$. We first present a useful lemma. Recall that the fusion rules of the simple current $\lambda_{k(\Lambda)}$ correspond to the $\mathbb{Z}_n$-rotation $\sigma$ of the Weyl alcove, i.e. $[\lambda_{k(\Lambda)}] = [\lambda_{\sigma(\Lambda)}]$ for $\Lambda \in A^{(n,k)}$. The map $\tau : A^{(n,k)} \to \mathbb{Z}_n$, $\Lambda \mapsto \tau(\Lambda) = t(\Lambda) \mod n$, is sometimes called colouring or “$n$-ality”, and recall that the fusion coefficients vanish, $N^{\Lambda''}_{\Lambda\Lambda'} = 0$, unless $\tau(\Lambda) + \tau(\Lambda') = \tau(\Lambda'')$.

**Lemma 6.13** For conformal inclusions at levels $k \in 2n\mathbb{N}$ if $n$ is even and $k \in n\mathbb{N}$ if $n$ is odd we have the following: If $[\theta]$ is $\mathbb{Z}_n$-rotation invariant, i.e. $[\lambda_{k(\Lambda)}] \circ \theta = [\theta]$, then $\langle \alpha^+_{\Lambda}, \alpha^-_{\theta(\Lambda)} \rangle_M = 0$ whenever $\tau(\Lambda) \neq 0$, hence in particular $\langle \alpha^+_{\Lambda}, \alpha^-_{\theta(\Lambda)} \rangle_M = 0$ whenever $\tau(\Lambda) \neq \tau(\Lambda')$, $\Lambda, \Lambda' \in A^{(n,k)}$.

**Proof.** As in the proof of Lemma 6.8 we find if $[\theta]$ is rotation invariant then

$$\langle \alpha^+_{\Lambda}, \alpha^-_{\theta(\Lambda)} \rangle_M = \langle \alpha^+_{\Lambda}, \alpha^-_{\sigma(\Lambda)} \rangle_M = \langle \alpha^+_{\sigma(\Lambda)}, \alpha^-_{\sigma(\Lambda)} \rangle_M = \langle \theta \circ \lambda_{\theta(\Lambda)}, \lambda_{\Lambda} \rangle_N,$$

hence $[\alpha^+_{\Lambda}] = [\alpha^+_{\sigma(\Lambda)}]$. Then by Eq. (14) with $q = 1$ we see that at levels $k \in 2n\mathbb{N}$ if $n$ is even and $k \in n\mathbb{N}$ if $n$ is odd then $h_{\sigma(\Lambda)} - h_\Lambda \notin \mathbb{Z}$ if $t(\Lambda) \notin n\mathbb{Z}$, i.e. if $\tau(\Lambda) \neq 0$. Hence $\langle \alpha^+_{\Lambda}, \alpha^-_{\sigma(\Lambda)} \rangle_M = \langle \alpha^+_{\Lambda}, \alpha^-_{\sigma(\Lambda)} \rangle_M$ vanishes if $\tau(\Lambda) \neq 0$ by Lemma 3.10. Then clearly

$$\langle \alpha^+_{\Lambda}, \alpha^-_{\theta(\Lambda)} \rangle_M = \langle \alpha^+_{\Lambda}, \alpha^-_{\theta(\Lambda)} \rangle_M = 0,$$

since $[\alpha^+_{\Lambda}]$ decompose into $[\alpha^\pm_{\Omega}]$’s, $\Omega \in A^{(n,k)}$, with $\tau(\Omega) = \tau(\Lambda) - \tau(\Lambda')$ non-zero if $\tau(\Lambda) \neq \tau(\Lambda')$. \qed

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\[ \mathcal{D}^{(6)} \text{ revisited: } SU(3)_3 \subset SO(8)_1: \] This is the first case of the \( \mathcal{D} \)-series for \( SU(3) \) and it happens to be a conformal embedding at the same time, similar to the \( D_4 \) example for \( SU(2) \). Although the discussion is in principle covered by our treatment of the orbifold inclusions it is instructive to do the calculations for this case. Recall from [4] that

\[ \mathcal{V}^\pm = \{ [\alpha_{(0,0)}], [\alpha_{(1,0)}^\pm], [\alpha_{(1,1)}^\pm], [\alpha_{(2,1)}^{(1)}], [\alpha_{(2,1)}^{(2)}], [\alpha_{(2,1)}^{(3)}] \}, \]

where \([\alpha_{(0,0)}] \) and \([\alpha_{(2,1)}^{(i)}], i = 1, 2, 3, \) are the marked vertices corresponding to the four level 1 representations of \( SO(8) \). Note that Lemma 6.13 directly yields \( \mathcal{V}^+ \cap \mathcal{V}^- = T \) in this case since \([\theta] \) is rotation invariant and \([\alpha_{(0,0)}] = [\alpha_{(3,0)}] = [\alpha_{(3,3)}] \) and \([\alpha_{(2,1)}] \) are the only sectors in \( \mathcal{V}^\pm \) of the form \([\alpha_\Lambda^\pm] \) with \( \Lambda \) of colour zero.

**Proposition 6.14** For the \( \mathcal{D}^{(6)} \) example we have \([\gamma] = [id_M] \oplus [\eta_1] \oplus [\eta_2] \), where \([\eta_1] \) is an irreducible subsector of \([\alpha_{(1,0)}^+ \circ \alpha_{(1,1)}^-] \) and \([\eta_2] \) is an irreducible subsector of \([\alpha_{(1,1)}^+ \circ \alpha_{(1,0)}^-] \). Hence the equivalent conditions of Proposition 5.4 are fulfilled.

**Proof.** Note that \( \langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 3 \) and \( d_\gamma = d_\theta = 3 \) so that \([\gamma] \) must decompose into three sectors of statistical dimension one. As \([\theta] \) is rotation invariant we learn from Lemma 5.13 that

\[ [\alpha_{(0,0)}] \equiv [id_M], \ [\alpha_{(1,0)}^+ \circ \alpha_{(1,1)}^-], \ [\alpha_{(1,1)}^+ \circ \alpha_{(1,0)}^-], \]

are disjoint sectors and as \([\alpha_{(1,0)}^\pm] \) and \([\alpha_{(1,1)}^\pm] \) are irreducible they have all one subsector in common with \([\gamma] \) by Corollary 3.19.

\[ \mathcal{E}^{(8)} \text{ revisited: } SU(3)_5 \subset SU(6)_1: \] First note that \([\theta] \) is not rotation invariant here. However, the treatment of this example is not particularly difficult. Recall from [4] that \( \mathcal{V}^\pm \) consists of six marked vertices \([\alpha_{(0,0)}], [\alpha_{(2,0)}^{(1)}], [\alpha_{(2,2)}^{(1)}], [\alpha_{(5,0)}], [\alpha_{(5,5)}] \) (forming a \( \mathbb{Z}_6 \) fusion subalgebra) and six further sectors \([\alpha_{(1,0)}^\pm], [\alpha_{(1,1)}^\pm], [\alpha_{(4,4)}^\pm], [\alpha_{(5,1)}^\pm], [\alpha_{(5,5)}^\pm] \).

**Proposition 6.15** For the \( \mathcal{E}^{(8)} \) example we have \([\gamma] = [id_M] \oplus [\alpha_{(1,0)}^+ \circ \alpha_{(1,1)}^-] \), hence the equivalent conditions of Proposition 5.4 are fulfilled.

**Proof.** We have \( N_{(4,2),(1,0)}^{(3,1)} = 1 \) but \( h_{(3,1)} - h_{(1,0)} = 2/3 - 1/6 = 1/2 \notin \mathbb{Z} \), hence \([\alpha_{(1,0)}^+] \neq [\alpha_{(1,0)}^-] \) by Corollary 3.13 and thus disjoint since \([\alpha_{(1,0)}^\pm] \) is irreducible. Therefore

\[ \langle \alpha_{(1,0)}^+ \circ \alpha_{(1,1)}^- \rangle_M = \langle \alpha_{(1,0)}^+ \circ \alpha_{(1,0)}^- \rangle_M = 0. \]
\[ \langle \alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}, \gamma \rangle_M = \langle \alpha^+_{(1,0)}, \alpha^+_{(1,0)} \rangle_M = 1 \]

by Corollary 3.19. Now note that \( d_+ = d_\theta = 1 + d_{(4,2)} = 4 + 2\sqrt{2} \), and \( \langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 2 \), therefore \( [\gamma] = [\text{id}_M] \oplus [\delta] \) with \( [\delta] \) irreducible and \( d_\delta = 3 + 2\sqrt{2} \). But we have \( d_{\alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}} = d^2_{(1,0)} = (1 + \sqrt{2})^2 = 3 + 2\sqrt{2} \). Hence we must have \( [\delta] = [\alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}] \).

With the results of [4] it is easy to write down the principal graph of \( N \subset M \), presented in Figure 10. This graph first appeared in [43] and was, as a principal graph, associated to the conformal inclusion \( SU(3)_5 \subset SU(6)_1 \) in [44]. With our machinery, we can now easily determine the dual principal graph of \( N \subset M \). Let us first determine \( \mathcal{V} \). First we check that for \( [\beta^\pm_a], [\beta^\pm_b] \in \mathcal{V}^\pm \) we have

\[ \langle \beta^\pm_a \circ \alpha^+_{(1,0)}, \beta^\pm_b \circ \alpha^+_{(1,0)} \rangle_M = \langle \beta^\pm_a \circ \beta^\pm_b, \alpha^+_{(1,0)} \circ \alpha^+_{(1,1)} \rangle_M = \delta_{a,b}, \]

since \( [\alpha^+_{(1,0)} \circ \alpha^+_{(1,1)}] = [\alpha_{(0,0)}] \oplus [\alpha^+_{(5,1)}] \oplus [\alpha^+_{(5,4)}] \), and the identity is the only marked vertex on the right hand side. Hence, besides \( [\beta^\pm_b] \in \mathcal{V}^\pm \) we have the irreducible sectors \( [\beta^\pm_a \circ \alpha^+_{(1,0)}] \) in \( \mathcal{V} \). But since \( [[\mathcal{V}]] = d_\gamma [[\mathcal{V}^\pm]] \equiv (1 + d^2_{(1,0)})[[\mathcal{V}^\pm]] \) by Lemmata 4.4 and 4.5, it follows that these sectors are already all sectors in \( \mathcal{V} \). Just by looking at the fusion graph of \( [\alpha^+_{(1,0)}] \) given in Figure 10 (and clearly the fusion graph of \( [\alpha^-_{(1,0)}] \) looks the same way) we find that

\[ [\alpha^+_{(2,0)} \circ \alpha^+_{(1,0)}] = [\alpha^+_{(5,4)}] \]

and similarly \( [\alpha_{(5,0)} \circ \alpha^+_{(1,0)}] = [\alpha^+_{(5,1)}] \). We denote \( [\delta] = [\alpha^+_{(4,4)} \circ \alpha^-_{(1,0)}] \). Now using \( [\gamma] = [\text{id}_M] \oplus [\delta] \) with \( [\delta] = [\alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}] = \)}
Here and some useful identities

\[ \alpha \]

With this we can determine the sector bases \( Z \) as in Corollary 4.3 consist obviously of the sectors \( [\alpha(0,0)] \), \( [\alpha^\pm(5,1)] \), \( [\alpha^\pm(3,4)] \) and \( [\alpha^{(1)}(3,0)] \).

Example \( \mathcal{E}^{(12)} \): \( SU(3)_9 \subset (E_6)_1 \): The corresponding modular invariant reads

\[
Z_{\mathcal{E}^{(12)}} = |\lambda(0,0) + \lambda(9,0) + \lambda(9,9) + \lambda(5,1) + \lambda(8,4) + \lambda(5,4)|^2 + 2|\lambda(4,2) + \lambda(7,2) + \lambda(7,7)|^2
\]

and therefore

\[
[\theta] = [\lambda(0,0)] \oplus [\lambda(9,0)] \oplus [\lambda(9,9)] \oplus [\lambda(5,1)] \oplus [\lambda(8,4)] \oplus [\lambda(5,4)].
\]

With this we can determine the sector bases \( V^\pm \). We find

\[
V^\pm = \{ [\alpha(0,0)], [\alpha^\pm(1,1)], [\alpha^\pm(2,0)], [\alpha^\pm(2,1)], [\alpha^\pm(2,2)], [\alpha^\pm(3,1)], [\alpha^\pm(3,2)], [\eta_1], [\eta_2] \}
\]

and some useful identities

\[
[\alpha^\pm(3,1)] = [\alpha^\pm(2,2)] \oplus [\alpha^\pm(3,1)] \oplus [\alpha^\pm(2,1)],
[\alpha^\pm(3,2)] = [\alpha^\pm(2,0)] \oplus [\alpha^\pm(3,1)] \oplus [\alpha^\pm(2,2)] \oplus [\alpha^\pm(3,2)],
[\alpha^\pm(5,2)] = [\alpha^\pm(1,0)] \oplus [\alpha^\pm(2,2)] \oplus [\alpha^\pm(3,1)] \oplus [\alpha^\pm(2,1)],
\]

Here \( [\alpha(0,0)] \), \( [\eta_1] \) and \( [\eta_2] \) are the marked vertices corresponding to the three level 1 representations of \( E_6 \). One checks by matching statistical dimensions
that they are simple sectors and hence are forced to satisfy the $\mathbb{Z}_3$ fusion rules of $(E_6)_1$. From

$$[\alpha^\pm_{(1,0)}] \times [\alpha^\pm_{(4,2)}] = [\alpha^\pm_{(3,1)}] \oplus [\alpha^\pm_{(3,3)}] \oplus [\alpha^\pm_{(5,2)}]$$

and its conjugation we obtain

$$(\eta_1 \oplus \eta_2) \times [\alpha^\pm_{(1,0)}] = [\alpha^\pm_{(3,1)}] \oplus [\alpha^\pm_{(3,3)}],$$

$$(\eta_1 \oplus \eta_2) \times [\alpha^\pm_{(1,1)}] = [\alpha^\pm_{(3,2)}] \oplus [\alpha^\pm_{(3,3)}].$$

We have the freedom to choose the notation such that $[\eta_1] \times [\alpha^\pm_{(1,0)}] = [\alpha^\pm_{(3,1)}]$, and this will actually provide a nice $\mathbb{Z}_3$ symmetry of the fusion graphs of $[\alpha^\pm_{(1,0)}]$. We remark that the homomorphisms $[\alpha^\pm]$ of sector algebras are not surjective as we cannot isolate $[\eta_1]$ and $[\eta_2]$ separately. However, the sector algebras associated to $\mathcal{V}^\pm$ are uniquely determined, and the fusion graph of either $[\alpha^\pm_{(1,0)}]$ in $\mathcal{V}^+$ or $[\alpha^-_{(1,0)}]$ in $\mathcal{V}^-$ is given in Figure 12. To determine the full induced sector basis $\mathcal{V}$ is much more involved, and we do not present the calculations here. However, we briefly show that we also have $Z = \widetilde{Z}$ in this case.

**Proposition 6.16** For the $\mathcal{E}^{(12)}$ example the equivalent conditions of Proposition 5.7 are fulfilled.

**Proof.** We show that each subsector of $[\gamma]$ is in $\mathcal{V}$. Consider the following sectors:

$$[\alpha_{(0,0)}], \quad [\alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}], \quad [\alpha_{(1,1)} + \alpha^-_{(1,0)}],$$

$$[\alpha^+_{(2,0)} \circ \alpha^-_{(2,2)}], \quad [\alpha^+_{(2,2)} \circ \alpha^-_{(2,0)}], \quad [\alpha^+_{(2,1)} \circ \alpha^-_{(2,1)}].$$

They all have a subsector in common with $[\gamma]$ by Corollary 5.10, and we now show that they are all disjoint. By Lemma 6.13 we only need to show

$$\langle \alpha_{(0,0)}, \alpha^+_{(2,1)} \circ \alpha^-_{(2,1)} \rangle_M = 0,$$

$$\langle \alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}; \alpha^+_{(2,2)} \circ \alpha^-_{(2,0)} \rangle_M = 0,$$

$$\langle \alpha^+_{(1,1)} \circ \alpha^-_{(1,0)}; \alpha^+_{(2,0)} \circ \alpha^-_{(2,2)} \rangle_M = 0.$$

First we find $[\alpha^+_{(2,1)}] \neq [\alpha^-_{(2,1)}]$ using Corollary 3.9 since $N^{(3,0)}_{(2,1),(5,1)} = 1$ but $h_{(3,0)} - h_{(2,1)} = 1/2 - 1/4 = 1/4 \notin \mathbb{Z}$. We even have $\langle \alpha^+_{(2,1)}, \alpha^-_{(2,1)} \rangle_M = 0$ as
Figure 12: $\mathcal{E}_1^{(12)}$: Fusion graph of either $[\alpha_{(1,0)}^+]$ in $\mathcal{V}^+$ or $[\alpha_{(1,0)}^-]$ in $\mathcal{V}^-$

$[\alpha_{(2,1)}^\pm]$ is irreducible, and this is the first relation. The other relations follow by use of the sector products

$$[\alpha_{(1,0)}^\pm] \times [\alpha_{(2,0)}^\pm] = [\alpha_{(1,1)}^\pm] \times [\alpha_{(2,2)}^\pm] = 2[\alpha_{(2,1)}^\pm].$$

Since $\langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 6$ we have already identified all subsectors of $[\gamma]$ as subsectors of some products $[\alpha_{(p,q)}^+] \times [\alpha_{(r,s)}^-]$.

Example $\mathcal{E}^{(24)}$: $SU(3)_{21} \subset (E_7)_1$: The corresponding modular invariant reads

$$Z_{\mathcal{E}^{(24)}} = \left| \chi(0,0) + \chi(21,0) + \chi(21,21) + \chi(8,4) + \chi(17,4) + \chi(17,13) + \chi(11,1) + \chi(11,10) + \chi(20,10) + \chi(12,6) + \chi(15,6) + \chi(15,9) \right|^2
+ \left| \chi(6,0) + \chi(21,6) + \chi(15,15) + \chi(15,0) + \chi(21,15) + \chi(6,6) + \chi(11,4) + \chi(17,7) + \chi(14,10) + \chi(11,7) + \chi(14,4) + \chi(17,10) \right|^2,$$

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\[ \theta = [\lambda(0, 0)] \oplus [\lambda(21, 0)] \oplus [\lambda(21, 21)] \oplus [\lambda(8, 4)] \oplus [\lambda(17, 4)] \oplus [\lambda(17, 13)] \oplus [\lambda(11, 1)] \oplus [\lambda(11, 10)] \oplus [\lambda(20, 10)] \oplus [\lambda(12, 6)] \oplus [\lambda(15, 6)] \oplus [\lambda(15, 9)]. \]

With this we can determine the sector bases \( \mathcal{V}^\pm \). We find
\[
\mathcal{V}^\pm = \{ [\alpha(0, 0)], [\alpha(1, 1)], [\alpha(2, 0)], [\alpha(2, 1)], [\alpha(3, 0)], [\alpha(3, 1)], \\
[\alpha(3, 2)], [\alpha(3, 3)], [\alpha(4, 0)], [\alpha(4, 1)], [\alpha(4, 2)], [\alpha(4, 3)], [\alpha(4, 4)], [\alpha(5, 0)], [\alpha(5, 1)], [\alpha(5, 5)], [e] \}. 
\]

We also give some irreducible decompositions,
\[ [\alpha^\pm_{(4, q)}] = [\alpha^{(1)}_{(4, q)}] \oplus [\alpha^{(2)}_{(4, q)}], \quad q = 0, 1, 2, 3, 4, \]
and
\[ [\alpha^\pm_{(5, 0)}] = [\alpha^{(1)}_{(4, 1)}] \oplus [\alpha^{(1)}_{(5, 0)}], \quad [\alpha^\pm_{(5, 1)}] = [\alpha^{(1)}_{(5, 1)}] \oplus [\alpha^{(1)}_{(5, 5)}] \oplus [\alpha^{(1)}_{(4, 4)}], \]
\[ [\alpha^\pm_{(5, 2)}] = [\alpha^{(1)}_{(3, 1)}] \oplus [\alpha^{(1)}_{(5, 1)}] \oplus [\alpha^{(1)}_{(5, 3)}], \quad [\alpha^\pm_{(5, 3)}] = [\alpha^{(1)}_{(3, 2)}] \oplus [\alpha^{(1)}_{(4, 1)}] \oplus [\alpha^{(1)}_{(4, 4)}], \]
\[ [\alpha^\pm_{(5, 4)}] = [\alpha^{(1)}_{(3, 3)}] \oplus [\alpha^{(1)}_{(4, 2)}] \oplus [\alpha^{(1)}_{(5, 1)}], \quad [\alpha^\pm_{(5, 5)}] = [\alpha^{(1)}_{(4, 3)}] \oplus [\alpha^{(1)}_{(5, 5)}], \]
and
\[ [\alpha^\pm_{(6, 0)}] = [\alpha^{(2)}_{(4, 2)}] \oplus [\alpha^{(1)}_{(5, 1)}] \oplus [e], \]
\[ [\alpha^\pm_{(6, 1)}] = [\alpha^{(2)}_{(3, 1)}] \oplus [\alpha^{(2)}_{(4, 0)}] \oplus [\alpha^{(1)}_{(4, 3)}] \oplus [\alpha^{(1)}_{(5, 5)}], \]
\[ [\alpha^\pm_{(6, 2)}] = [\alpha^{(2)}_{(2, 0)}] \oplus [\alpha^{(2)}_{(3, 2)}] \oplus [\alpha^{(1)}_{(4, 1)}] \oplus [\alpha^{(2)}_{(4, 1)}] \oplus [\alpha^{(2)}_{(4, 4)}], \]
\[ [\alpha^\pm_{(6, 3)}] = [\alpha^{(2)}_{(2, 1)}] \oplus [\alpha^{(2)}_{(3, 2)}] \oplus [\alpha^{(1)}_{(4, 1)}] \oplus [\alpha^{(1)}_{(4, 2)}] \oplus [\alpha^{(2)}_{(4, 4)}], \]
\[ [\alpha^\pm_{(6, 4)}] = [\alpha^{(2)}_{(2, 2)}] \oplus [\alpha^{(2)}_{(3, 1)}] \oplus [\alpha^{(2)}_{(4, 0)}] \oplus [\alpha^{(1)}_{(4, 3)}] \oplus [\alpha^{(2)}_{(4, 4)}], \]
\[ [\alpha^\pm_{(6, 5)}] = [\alpha^{(2)}_{(2, 3)}] \oplus [\alpha^{(1)}_{(4, 1)}] \oplus [\alpha^{(2)}_{(4, 4)}] \oplus [\alpha^{(1)}_{(5, 0)}], \]
\[ [\alpha^\pm_{(6, 6)}] = [\alpha^{(1)}_{(4, 2)}] \oplus [\alpha^{(1)}_{(5, 1)}] \oplus [e], \]
and also
\[ [\alpha^\pm_{(7, 0)}] = [\alpha^{(1)}_{(4, 3)}] \oplus [\alpha^{(2)}_{(5, 5)}], \quad [\alpha^\pm_{(7, 7)}] = [\alpha^{(1)}_{(4, 1)}] \oplus [\alpha^{(2)}_{(4, 1)}] \oplus [\alpha^{(1)}_{(5, 0)}]. \]

Here \([\alpha_{(0, 0)}]\) and \([e]\) are the marked vertices corresponding to the two level 1 representations of \(E_7\). These formulae are indeed enough to isolate each irreducible sector, i.e. can be inverted; in fact the homomorphisms \([\alpha^\pm]\) are
Figure 13: $E^{(24)}$: Fusion graph of either $[\alpha_{(1,0)}^+]$ in $\mathcal{V}^+$ or $[\alpha_{(1,0)}^-]$ in $\mathcal{V}^-$ surjective in this case. The fusion graph of either $[\alpha_{(1,0)}^+]$ in $\mathcal{V}^+$ or $[\alpha_{(1,0)}^-]$ in $\mathcal{V}^-$ is given in Figure 13.

To determine the full induced sector basis $\mathcal{V}$ is much more involved, and we do not present the calculations here, but we just show the following

**Proposition 6.17** For the $E^{(24)}$ example the equivalent conditions of Proposition 5.1 are fulfilled.

**Proof.** As all elements of $\mathcal{V}^\pm$ except the marked vertex $[\epsilon]$ appear as subsectors of some $[\alpha_{(p,q)}^\pm]$, $0 \leq q \leq p \leq 5$, it suffices to show $\langle \alpha_{(p,q)}^+, \alpha_{(p,q)}^- \rangle_M = 0$ for all $0 \leq q \leq p \leq 5$, except $(p, q) = (0, 0)$, in order to prove $\mathcal{V}^+ \cap$
\( \mathcal{V}^- = \mathcal{T} = \{ [\alpha(0,0)], [\theta] \}. \) Since \([\theta]\) is rotation invariant we only need to show \( \langle \alpha^{+}_{(p,q)}, \alpha^{-}_{(p,q)} \rangle_M = 0 \) for the colour zero cases, i.e. for \((p,q) = (2,1), (3,0), (3,3), (4,2), (5,1), (5,4)\), by Lemma 6.13. As \([\alpha^\pm_{(3,0)}]\) is irreducible it suffices to show \([\alpha^+_{(2,1)}] \neq [\alpha^-_{(2,1)}]\). It follows from \(N_{(2,1),(3,0)} = 1\) and \(h_{(2,1)} - h_{(3,0)} = 5/8 - 1/8 = 1/2 \neq \mathbb{Z} \). Similarly \(\langle \alpha^{+}_{(3,0)}, \alpha^{-}_{(3,0)} \rangle_M = 0\), since \(N_{(3,0),(3,0)} = 1\) but \(h_{(3,0)} = 1/2 - 1/4 = 1/4 \neq \mathbb{Z} \). Finally \(\langle \alpha^{+}_{(5,1)}, \alpha^{-}_{(5,1)} \rangle_M = 0\) it only remains to show that \([\alpha^+_{(5,1)}] \neq [\alpha^-_{(5,1)}]\), and this follows since \([\alpha^\pm_{(5,1)}]\) is a subsector of \([\alpha^\pm_{(6,0)}]\) but \(h_{(6,0)} - h_{(5,1)} = 3/4 - 1/2 = 1/4 = \mathbb{Z} \). Finally \(\langle \alpha^{+}_{(5,1)}, \alpha^-_{(5,1)} \rangle_M = 0\) follows by conjugation.

In turn one can also show that the following sectors,

\[
[\alpha^+_{(1,0)} \circ \alpha^-_{(1,1)}], \ [\alpha^+_{(2,2)} \circ \alpha^-_{(2,2)}], \ [\alpha^+_{(1,1)} \circ \alpha^-_{(1,0)}], \ [\alpha^+_{(2,0)} \circ \alpha^-_{(2,2)}], \ [\alpha^+_{(2,1)} \circ \alpha^-_{(2,1)}], \ [\alpha^+_{(3,0)} \circ \alpha^-_{(3,3)}], \ [\alpha^+_{(4,2)} \circ \alpha^-_{(4,2)}], \n\]

are all disjoint and have one sector in common with \([\gamma]\), and that the two further disjoint sectors \([\alpha^+_{(4,3)} \circ \alpha^-_{(4,1)}]\) and \([\alpha^+_{(4,1)} \circ \alpha^-_{(4,3)}]\) have two sectors in common with \([\gamma]\). This already yields all the subsectors of \([\gamma]\) since \(\langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 12\).

SU(4)\(_4\) \(\subset SO(15)\) revisited. We first remark that \([\theta]\) is not \(\mathbb{Z}_4\)-rotation invariant here. Recall from [6] that

\[
\mathcal{V}^\pm = \begin{cases} [\alpha(0,0,0)], [\alpha^\pm_{(1,0,0)}], [\alpha^\pm_{(1,1,0)}], [\alpha^\pm_{(1,1,1)}], [\alpha(4,0,0)], [\alpha_{(3,2,1)}], [\alpha^\pm_{(i,1,1)}], [\alpha^\pm_{(2,1,0)}], [\alpha^\pm_{(2,2,0)}], [\alpha^\pm_{(2,2,1)}], & i = 1, 2 \end{cases},
\]

and

\[
\mathcal{T} = \{ [\alpha(0,0,0)], [\alpha(4,0,0)], [\alpha_{(3,2,1)}] \}.
\]

This example is the first one which leads to non-commutative chiral sector algebras, however, it is not an exception in the sense that the following holds.

**Proposition 6.18** For the conformal embedding SU(4)\(_4\) \(\subset SO(15)\) the equivalent conditions of Proposition 6.7 are fulfilled.
Proof. We first claim that \( \langle \alpha_{(2,1,0)}^+, \alpha_{(2,1,0)}^- \rangle_M = 0 \). This follows since 
\[ [\alpha_{(2,1,0)}^\pm] = [\alpha_{(3,2,2)}^\pm] \] but 
\[ h_{(3,2,2)} - h_{(2,1,0)} = 55/64 - 39/64 = 1/4 \notin \mathbb{Z} \]. By conjugation we obtain 
\[ \langle \alpha_{(2,2,1)}^+, \alpha_{(2,2,1)}^- \rangle_M = 0 \). Note that we have the irreducible decompositions
\[
[\alpha_{(2,1,0)}^\pm] = [\alpha_{(1,1,1)}^\pm] \oplus [\alpha_{(2,1,0)}^{\pm(1)}] \oplus [\alpha_{(2,1,0)}^{\pm(2)}],
\]
\[
[\alpha_{(2,2,1)}^\pm] = [\alpha_{(1,0,0)}^\pm] \oplus [\alpha_{(2,2,1)}^{\pm(1)}] \oplus [\alpha_{(2,2,1)}^{\pm(2)}],
\]
therefore we also find \([\alpha_{(1,0,0)}^+ \neq \alpha_{(1,0,0)}^-] \) and \([\alpha_{(1,1,1)}^+ \neq \alpha_{(1,1,1)}^-] \). Further
we recall that \([\alpha_{(2,2,0)}^\pm] = [\alpha_{(2,2,0)}^{\pm(1)}] \oplus [\alpha_{(2,2,0)}^{\pm(2)}] \) is a subsector of \([\alpha_{(2,1,1)}^\pm] \) but
\[ h_{(2,2,0)} - h_{(2,1,1)} = 3/4 - 1/2 = 1/4 \notin \mathbb{Z} \], hence also \( \langle \alpha_{(2,2,0)}^+, \alpha_{(2,2,0)}^- \rangle_M = 0 \).
Now \([\alpha_{(2,2,0)}^\pm] \) appears (twice) in the square of \([\alpha_{(1,1,0)}^\pm] \), hence also \([\alpha_{(1,1,0)}^+ \neq \alpha_{(1,1,0)}^-] \). We have established \( T = \mathcal{V}^+ \cap \mathcal{V}^- \). \( \Box \)

In turn one easily checks that
\[ [\alpha_{(0,0,0)}^+], \ [\alpha_{(1,0,0)}^+ \circ \alpha_{(1,1,1)}^-], \ [\alpha_{(1,1,1)}^+ \circ \alpha_{(1,0,0)}^-], \ [\alpha_{(1,1,0)}^+ \circ \alpha_{(1,1,0)}^-], \]
are disjoint sectors and they all have a sector in common with \([\gamma] \), exhibiting all subsectors of \([\gamma] \) since \( \langle \gamma, \gamma \rangle_M = \langle \theta, \theta \rangle_N = 4 \).

7 Summary and Outlook

We have analyzed the structure of the induced sector systems obtained by mixing the \( \pm \)-inductions for conformal and \( \mathbb{Z}_m \) orbifold embeddings of \( SU(n) \). We proved the formula \( Z_{\Lambda, \Lambda'} = \langle \alpha_{\Lambda}^+, \alpha_{\Lambda'}^- \rangle_M \), \( \Lambda, \Lambda' \in A^{(n,k)} \), for the associated modular invariant mass matrix for all \( \mathbb{Z}_m \) orbifold inclusions and several conformal inclusions. As a consequence, all subsectors of \([\gamma] \) can be obtained by decomposing suitable sectors \([\alpha_{\Lambda}^+ \circ \alpha_{\Lambda'}^-] \), the “global index” of the induced sector basis is maximal, i.e. coincides with the one of the original \( SU(n)_k \) fusion algebra, and we have \( Z_{\Lambda, \Lambda'} \neq 0 \) if and only if \( \Lambda, \Lambda' \in \text{Exp} \) as well as \( Z_{\Lambda, \Lambda} \neq 0 \) if and only if \( \Lambda \in \text{Exp}^+ \). Our results cover in particular all type I modular invariants of \( SU(2) \) and \( SU(3) \). The proof for the conformal inclusions is, unfortunately, case by case and therefore covers only a limited number of examples. However, we believe that it holds for all of them:

**Conjecture 7.1** The equivalent conditions of Proposition 5.4 hold for any conformal inclusion of \( SU(n) \).
Recall from the proof of Lemma 5.2 that the “regular” representation of the induced fusion algebra, given in terms of the sector product matrices \( M_x, x \in \mathcal{V} \), decomposes into representations \( B_{\Lambda, \Lambda'} \) labelled by the set of exponents, namely \( M_x = \bigoplus_{(\Lambda, \Lambda') \in \text{Exp}} B_{\Lambda, \Lambda'}(x), x \in \mathcal{V} \). We believe that this decomposition is minimal in the following sense:

**Conjecture 7.2** For any conformal or orbifold inclusion of SU\((n)\) we have \( B_{\Lambda, \Lambda'} \cong \pi_{(\Lambda, \Lambda')} \otimes 1 Z_{\Lambda, \Lambda'} \), where the \( \pi_{(\Lambda, \Lambda')} \)'s are the irreducible, pairwise inequivalent representations of the full induced sector algebra, and the dimension of \( \pi_{(\Lambda, \Lambda')} \) is \( Z_{\Lambda, \Lambda'} \), \((\Lambda, \Lambda') \in \text{Exp}\). In consequence, \( \dim \text{Eig}(\Lambda, \Lambda') = Z_{\Lambda, \Lambda'}^2 \) for \( \Lambda, \Lambda' \in \mathcal{A}^{(n,k)} \).

Our results provide powerful methods to compute the induced sector bases \( \mathcal{V} \) and their algebraic structure, yet the computations may become more and more involved with increasing rank and level. (For large \( n \) and \( k \) it might not even be possible to determine \( \mathcal{V} \) completely with our results at hand.) However, if \( \mathcal{V} \) and its sector algebra is determined one can easily write down the principal and the dual principal graph of the conformal inclusion subfactors, (this is certainly less interesting for the orbifold inclusions since there the subfactors \( N \subset M \) are just of \( Z_m \) type), and we have illustrated these powerful methods by several examples, including the computation of the dual principal graph for the conformal inclusion \( SU(3)_5 \subset SU(6)_1 \), which has, to the best of our knowledge, not been computed before. Thus our theory can also be used to determine basic invariants of new subfactor examples.

Let us finally remark that there are type I modular invariants which come neither from conformal inclusions nor from simple current extensions as, for instance, the exceptional \( SU(10) \) level 2 modular invariant found in [1] which arises by level-rank duality from the \( E_6 \) modular invariant of \( SU(2) \). It is natural to presume that there will still be an associated net of subfactors such that \( \alpha \)-induction can be applied. If so, the next thing to check is whether the equivalent conditions of Prop. 5.1 even hold for these cases. However, it does not seem reasonable to expect that there are associated nets of subfactors for all type I modular invariants. In fact there are type I modular invariants for which there is no “fixed point resolution” of the S-matrix, see [16, Sect. 4]. For example, there is a type I modular invariant of \( SU(5) \) at level 5 of this kind [39, Eq. (B.3)] which has the same vacuum block as the different (type I) modular invariant [39, Eq. (B.6)] corresponding to the conformal embedding \( SU(5)_5 \subset SO(24)_1 \). So here we expect an associated net of subfactors to exist only for the conformal
embedding invariant. Type I invariants without fixed point resolution appear to be rather rare, however.

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