Towards optimized suppression of dephasing in systems subject to pulse timing constraints

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We investigate the effectiveness of different dynamical decoupling protocols for storage of a single qubit in the presence of a purely dephasing bosonic bath, with emphasis on comparing quantum coherence preservation under uniform vs. non-uniform delay times between pulses. In the limit of instantaneous bit-flip pulses, this is accomplished by establishing a new representation of the controlled qubit evolution, where the resulting decoherence behaviour is directly expressed in terms of the free evolution. Simple analytical expressions are given to approximate the long- and short-term coherence behaviour for both ohmic and supra-ohmic environments. We focus on systems with physical constraints on achievable time delays, with emphasis on pure dephasing of exciton qubits in quantum dots. Our analysis shows that little advantage of high-level decoupling schemes based on concatenated or optimal design is to be expected if operational constraints prevent pulses to be applied sufficiently fast. In such constrained scenarios, we demonstrate how simple modifications of repeated periodic echo protocols can offer significantly improved coherence preservation in realistic parameter regimes.

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I. INTRODUCTION

The ability to effectively counteract decoherence processes in physical quantum information processing (QIP) devices is a fundamental prerequisite for taking advantage of the added power promised by quantum computation and quantum simulation as compared to purely classical methods. Dynamical decoupling (DD) techniques for realistic qubit devices has spurred an intense theoretical and experimental effort, taking DD well beyond the original nuclear magnetic resonance (NMR) setting. While earlier DD schemes relied on the simple periodic repetition of instantaneous pulses (so-called ‘bang-bang’ periodic DD, PDD), and its closely-related time-symmetrized version, so-called Carr-Purcell DD, CPDD[24], recent theoretical investigations have explored the benefits of more sophisticated control design in a number of ways. In particular, this has led to devising recursive and randomized pulse sequences for generic decoherence models on finite-dimensional quantum systems – so-called ‘concatenated’ DD (CDD[6]) and ‘randomized’ DD[25]; to identifying ‘optimal’ protocols for a single qubit undergoing pure dephasing – most notably, the so-called Uhrig DD (UDD)[6,10,11,12,13,14], and its extension to ‘locally optimized’[25,16] DD sequences tailored to specific noise environments; and, most recently, to combining the advantages of concatenation and optimization for a single qubit exposed to arbitrary decoherence[17,18,19]. As a key common feature, these investigations highlight the sensitivity of DD performance to the details of the applied control path, and point to the importance of carefully tuning the relative pulse delays in order to boost the efficiency of the achievable decoherence suppression[20].

In view of the above rich scenario, assessing the performance of different DD protocols in specific qubit devices and/or in the presence of specific control constraints becomes especially important. Recently, the effectiveness of traditional multi-pulse spin-echo sequences based on PDD and CPDD, as compared to ‘high-level’ protocols based on CDD and UDD, has been scrutinized in several control settings. In particular, a number of theoretical studies have addressed suppression of pure dephasing associated to spectral diffusion[26] and hyperfine-induced decoherence[27] from a quantum spin-bath for an electron-spin qubit, as well as suppression of classical $1/f$ phase noise in a superconducting qubit[25,26]. Experimentally, the performance of CDD protocols has been characterized for an NMR spin qubit[27], while optimal UDD implementations have been reported for both a trapped ion qubit exposed to engineered classical phase noise[15,16,28] and, most recently, for electron spin qubits undergoing spin-bath decoherence in a malonic acid crystal[29]. These studies have demonstrated, in particular, how UDD can significantly outperform low-level DD schemes provided that the noise spectrum has a sharp high-frequency cutoff and sufficiently high pulse repetition rates may be afforded.

Amongst prospective solid-state QIP platforms, exciton qubits in self-assembled quantum dots (QDs) have likewise received vast attention in recent years[30,31]; due to the coupling to photons, excitons can be driven all-optically on sub-picosecond time scales[32]. Excitonic implementations also allow the flexibility of designing hybrid solid state-flying qubit schemes[33]. Pure dephasing turns out to be the dominant factor limiting the coherence lifetime in such qubit devices, where strong coupling with phonon modes of the host crystal result in typical decoherence ($T_2$) time scales of a few picoseconds[34]. We have previously shown in Ref. [35] that, remarkably, PDD allows for substantial exciton coherence recovery in experimentally relevant parameter regimes (up to 90% recovery over ~ 10 ps at room temperature), the control performance being especially enhanced for QD shapes and bias fields optimized for quantum computing architectures. Our goal in this paper is to quantitatively assess to what extent
more elaborated DD schemes – in particular, sequences employing non-uniform pulse timings – can improve beyond the simplest PDD setting when a lower bound on the achievable control time scale (minimum pulse separation) is present.

We find that in the presence of such a timing limitation, simple protocols such as PDD or CPDD may outperform high-level sequences based on CDD/UDD. Interestingly, on the one hand this reinforces similar conclusions drawn in Ref. 26 for classical dephasing in superconducting qubits. On the other hand, we additionally show how it is possible to engineer a suitable ‘preparatory’ sequence that enhances the performance of a subsequent PDD pulse train. In the process, we take advantage of the exact solvability of a purely dephasing model in the presence of instantaneous pulses to obtain an exact representation of the controlled dynamics in terms of the free evolution. This allows rigorous results on the long-time asymptotic decoherence behaviour to be established for generic noise spectral densities, by allowing in particular a comparison between ohmic and supra-ohmic environments. Furthermore, our work provides a first explicit analysis of CDD performance in the presence of a quantum bosonic bath. From a practical standpoint, our results suggest that simple DD protocols may remain a method of choice if significant timing constraints are in place, and that incorporating such constraints from the outset is necessary before further optimization can show its benefits. While our numerical results are tailored to excitons in QDs, we expect the above conclusions to be relevant for other constrained qubit devices.

II. SINGLE-QUBIT DEPHASING DYNAMICS

We consider the pure dephasing dynamics of a single qubit coupled to a non-interacting bath of harmonic oscillators. The Hamiltonian of such a system may be written in the form

$$H = \frac{E}{2} \sigma_z + \hbar \sum_j \omega_j b_j^\dagger b_j + \hbar \sum_j (g_j^0 b_j^\dagger + g_j b_j)(1 - \alpha)\sigma_0 + \alpha \sigma_z$$

(1)

$$\equiv H_0 + \hbar \sum_j (g_j^0 b_j^\dagger + g_j b_j)(1 - \alpha)\sigma_0 + \alpha \sigma_z,$$

(2)

where $E$ gives the energy difference between the qubit’s levels, $b_j^\dagger$ and $b_j$ are canonical creation and annihilation operators of the oscillator mode $j$, and $g_j$ describes the coupling between the qubit and the $j$-th bath mode. In the above expression for $H$, the parameter $\alpha$ accounts for the possibility that either both or only one of the spin (or pseudo-spin) qubit computational levels effectively couple to the bath: $\alpha = 1$ corresponds to the standard purely-dephasing spin-boson model, whereas if $\alpha = 1/2$, only the $\sigma_z = +1$ eigenstate couples to the bath. Specifically, for an excitonic qubit, the logical states are represented by the presence or absence a single (ground-state) exciton in the QD, and $E$ is the energy relative to the crystal ground state.

As time evolves, the qubit becomes entangled with the environment and the off-diagonal elements of the qubit density matrix evaluated at time $t$ in the interaction picture with respect to $H_0$ read

$$\rho_{01}(t) = \rho_{01}^r(t) = \rho_{01}(t = 0)e^{-\Gamma(t)},$$

(3)

$$\Gamma(t) = \Gamma_0(t)$$

$$= (2\alpha)^2 \int_0^\infty d\omega I(\omega) \frac{\omega^2}{2} \coth \left( \frac{\hbar \omega}{2k_BT} \right) [1 - \cos(\omega t)],$$

(4)

where $T$ is the temperature, $k_B$ the Boltzmann’s constant, and

$$I(\omega) = \sum_j |\delta(\omega - \omega_j)| g_j^2$$

(5)

is the spectral density function characterizing the interaction of the qubit with the oscillator bath. For a supra-ohmic environment, $I(\omega) \sim \omega^3$, as opposed, for instance, to an ohmic reservoir where $I(\omega) \sim \omega$. Likewise, the high-frequency behaviour is characterized by a frequency cut-off $\omega_c$, for instance, for excitons one can assume that $I(\omega) \sim e^{-\omega_c^2/\omega^2}$.

As it turns out, the decoherence of the qubit in the presence of an arbitrary sequence of bang-bang pulses, each effecting an instantaneous $\pi$ rotation, can still be exactly described by Eq. (3), provided a modified decoherence function is used. Consider an arbitrary storage time $t$, during which a total number $s$ of pulses is applied, at instants $\{t_1, \ldots, t_n, \ldots, t_s\}$, with $0 < t_1 < t_2 < \ldots < t_s < t$. By using the theory developed by Uhrig in Refs. 10,11, we can define a controlled coherence function $\Gamma(t)$ in the following way:

$$\Gamma(t) = \left\{ \begin{array}{ll}
\Gamma_0(t) & t \leq t_1, \\
\Gamma_n(t) & t_n < t \leq t_{n+1}, 0 < n < s, \\
\Gamma_s(t) & t_s < t.
\end{array} \right.$$

(6)

Here, $\Gamma_0(t)$ is given in Eq. (4) whereas for $1 \leq n \leq s$ we let

$$\Gamma_n(t) = (2\alpha)^2 \int_0^\infty \frac{I(\omega)}{2\omega^2} \coth \left( \frac{\hbar \omega}{2k_BT} \right) |y_n(\omega t)|^2 d\omega,$$

$$y_n(z) = 1 + (-1)^n e^{iz} + 2 \sum_{m=1}^n (-1)^m e^{iz\delta_m},$$

(7)

with the $n$-th pulse being understood to occur at time $t_n = \delta_n t$, and $0 < \delta_1 < \ldots < \delta_n < \ldots < \delta_s < 1$. While the instantaneous pulse assumption must be handled with care in general, we have discussed in Ref. 35 how it translates into reasonable physical constraints for an excitonic qubit coupled to a phononic bath.

We now proceed to directly relate $\Gamma_n(t)$ to $\Gamma_0(t)$ for arbitrary $n$. Let us first rewrite the above coherence function $\Gamma_n(t)$ in a compact way as

$$\Gamma_n(t) = \int_0^\infty \eta(\omega) |y_n(\omega t)|^2 d\omega, \quad n \geq 0,$$

(8)
where we have defined
\[ |y_0(\omega t)|^2 \equiv |1 - e^{i\omega t}|^2, \tag{8} \]
and
\[ \eta(\omega) = (2\alpha)^2 I(\omega) \frac{\hbar \omega}{2\omega^2} \coth \left( \frac{\hbar \omega}{2k_B T} \right). \tag{9} \]

By relating \(|y_1(\omega t)|^2\) to \(|y_0(\omega t)|^2\) we can write
\[ \Gamma_1(t) = -\Gamma_0(t) + 2\Gamma_0(t_1) + 2\Gamma_0(t - t_1). \tag{10} \]

Upon continuing this iteration we find
\[
\begin{align*}
\Gamma_2(t) &= -\Gamma_1(t) + 2\Gamma_1(t_2) + 2\Gamma_0(t - t_1), \\
\vdots \\
\Gamma_n(t) &= -\Gamma_{n-1}(t) + 2\Gamma_{n-1}(t_n) + 2\Gamma_0(t - t_n). \tag{11}
\end{align*}
\]

Furthermore, by expressing \(|y_n(\omega t)|^2\) as a function of \(|y_0(\omega t)|^2\), we are able to write the entire evolution in the presence of an arbitrary pulse sequence only in terms of the uncontrolled evolution. Explicitly, we find:
\[
\Gamma_n(t) = 2 \sum_{m=1}^{n} (-1)^{m+1}\Gamma_0(t_m) + 4 \sum_{m=2}^{n} \sum_{j=m}^{m-1} \Gamma_0(t_m - t_j)(-1)^{m-1+j} + 2 \sum_{m=1}^{n} (-1)^{n+m}\Gamma_0(t - t_m) + (-1)^n\Gamma_0(t). \tag{12}
\]

The above equation is one of the main results of this paper. By using Eq. (12), it is, in particular, straightforward to see that
\[ \Gamma_{n-1}(t_n) = \lim_{t \to t_n} \Gamma_n(t) = \Gamma_n(t_n). \tag{13} \]

This confirms that the function \(\Gamma(t)\) as defined in Eqs. (6) is continuous at the (instantaneous) pulse timings, as expected on physical grounds.

As a first example of the usefulness of this representation, we consider how two pulses may be used to increase the asymptotic coherence of a super-ohmic system, in which the free dephasing dynamics saturates in the long-time limit to a finite value, \(\Gamma_0(\infty) > 0\). Taking the \(t \to \infty\) limit in Eq. (10) or, equivalently, letting \(n = 1\) in Eq. (12), yields \(\Gamma_1(\infty) = 2\Gamma_0(t_1) + \Gamma_0(\infty)\). Since \(\Gamma_0(t) \geq 0\) for all \(t\), this shows how a single pulse cannot decrease the asymptotic decoherence level. However, after two pulses we have
\[
\begin{align*}
\Gamma_2(t) &= \Gamma_0(t) - 2\Gamma_0(t - t_1) - 2\Gamma_0(t_2) \\
&\quad + 2\Gamma_0(t_1) + 4\Gamma_0(t_2 - t_1) + 2\Gamma_0(t - t_2). \tag{14}
\end{align*}
\]

Therefore,
\[ \Gamma_2(\infty) = \Gamma_0(\infty) - 2\Gamma_0(t_2) + 2\Gamma_0(t_1) + 4\Gamma_0(t_2 - t_1) + 2\Gamma_0(t - t_2), \tag{15} \]
and \(t_1, t_2\) can be chosen to decrease the asymptotic decoherence provided that
\[ 2\Gamma_0(t_2) - 2\Gamma_0(t_1) > 4\Gamma_0(t_2 - t_1). \tag{16} \]

In Fig. 1 we plot \(\Gamma(t)\) for an exciton qubit coupled to phonon modes and subject to two control pulses at \(t_1 = 0.2\) ps and \(t_2 = 0.31\) ps. For comparison, we also plot the evolution under a single control pulse at \(t_1 = 0.2\) ps and the free evolution \(\Gamma_0(t)\). As one can see, Eq. (16) can indeed be satisfied. Numerical results showing how a few pulses can increase the asymptotic coherence have been reported for excitonic dephasing in Ref. [11].

For the case of Fig. 1, as well as for all the numerical examples in this paper, we consider (unless otherwise stated) an exciton qubit tightly confined within a GaAs QD at 77 K. The QD potentials are modeled as parabolic in all three dimensions, with confinement energies in the \(z\)-direction of \(\hbar \omega_c = 505\) meV and \(\hbar \omega_h = 100\) meV, while \(\hbar \omega_e = 30\) meV and \(\hbar \omega_h = 24\) meV in the in-plane directions [10,11]. The subscript \(e/h\) indicates electron/hole, respectively. For this exciton, in the absence of control most of the coherence is lost after a few picoseconds [11]. Having this specific system in mind, we shall set \(\alpha = 1/2\) henceforth in our numerical calculations, and plot the quantity \(\exp(-\Gamma_0(t))^2\), which is directly proportional to the square modulus of the measured optical polarization \(P(t)\).

As discussed in detail in Ref. [35], the spectral density of this system is given by
\[ I(\omega) = I_e(\omega) + I_h(\omega) + I_{eh}(\omega), \tag{17} \]
where the indices \(e/h/eh\) correspond to single particle spectral densities of the electron and the hole, and to the electron-hole interference term respectively, and
\[ I_{e/h/eh}(\omega) = \sum_i F_{i}^{e/h/eh}(\omega) \exp\left(-\frac{\omega^2}{\omega_{e/h/eh}^2}\right). \tag{18} \]
Here, \(i\) labels different phonon modes, whereas \(F_{i}^{e/h/eh}(\omega)\)
is a mode-dependent function for which \( F_s^{c/\hbar/2h} (\omega) \leq \omega^3 \). The spectral density may be further approximated as

\[
I(\omega) \approx F \omega^3 \exp \left( -\frac{\omega^2}{\omega_c^2} \right),
\]

where the parameters \( F \) and \( \omega_c \) are determined by fitting a curve of the form Eq. (19) to the actual exciton spectral density. For the particular exciton parameters listed above, this yields \( F = 1.14 \times 10^{-20} \text{s} \) and \( \hbar \omega_c = 2 \text{meV} \).

III. PERIODIC DD: PERFORMANCE AND EXACT ASYMPTOTIC PROPERTIES

For a Hamiltonian as in Eq. (1), a DD cycle consisting of two uniformly spaced rotations by \( \pi \) about the \( x \) axis,

\[
X \Delta t X \Delta t,
\]

where time ordering is understood from right to left, removes the interaction between the qubit and the boson bath\(^2\) to the lowest (perturbative) order in \( \omega_c T_c \), with \( T_c \geq 2 \Delta t \). The simplest DD protocol, PDD, is obtained by iterating the above control cycle in time.

Fig. 2 compares the free evolution with the PDD-controlled dephasing for the exciton qubit under examination, computed from the exact expressions given in Sec II. Sequences with three different pulse delays are shown, \( \Delta t = 0.1 \text{ ps}, \Delta t = 0.2 \text{ ps}, \) and \( \Delta t = 0.3 \text{ ps}, \) respectively. For the exciton qubit, two conditions determine a suitable range of \( \Delta t \) for effective PDD: i) On the one hand, it is necessary that the control time scale \( T_c \) be sufficiently short with respect to the (shortest) correlation time of the decoherence dynamics, which means in this case \( 2 \Delta t \lesssim \tau_c = 2 \pi/\omega_c \). Physically, this can also be interpreted by requiring that the characteristic frequency introduced by the periodic control,

\[
\omega_{res} = \frac{\pi}{\Delta t},
\]

be significantly higher than the spectral cut-off frequency itself, \( \omega_{res} \gtrsim \omega_c \), in such a way that the DD-renormalized spectral density function, \( I(\omega) \tan^2(\omega \Delta t/2) \), is effectively ‘up-shifted’ beyond the bath cutoff\(^2\). ii) On the other hand, the existence of a lower bound on the pulse duration implies a lower bound on the separation \( \Delta t \) in order for the instantaneous-pulse description to be accurate. As discussed in Ref. \(^2\), this means \( \Delta t \gtrsim 0.1 \text{ ps} \) for semiconductor self-assembled QDs of interest for QIP.

The values of \( \Delta t \) used in Fig. 2 are consistent with both these conditions. It can be seen that coherence decays until the first bit-flip occurs, after which it rises, reaches a local maximum before decohering once again – with this pattern repeating between every two bit-flips. It can also be seen that DD recovers most of the dephasing, that is, \( \exp(-\Gamma(t)) \) is much closer to unity than in the uncontrolled evolution, which falls rapidly before saturating to \( \exp(-\Gamma_0(\infty)) \). After the first few initial pulses, the dephasing enters a phase in which the behavio after the \((n+1)\)-th pulse is approximately the same as the one after the \(n\)-th pulse. For \( \Delta t = 0.1 \text{ ps} \) and \( \Delta t = 0.2 \text{ ps} \), the average dephasing \( \text{over each cycle} \) in this ‘steady-state’ phase is very small, leading to a practical ‘freezing’ of the average decoherence over a period much longer than the estimated (sub-picosecond) gating times\(^3\). For \( \Delta t = 0.3 \text{ ps} \), however, the increase of decoherence due to this average dephasing with time is more noticeable, leading to worse DD performance overall. It can also be seen that, to minimize the effects of dephasing, any readout on the qubit should be made half-way between two control pulses. As it is well known in NMR, this motivates a proper choice of the observation window, which underlies the Carr-Purcell (CP) sequence and is also discussed in Ref. \(^4\) in the spin-boson context.

A. Long-time dynamics: Ohmic versus supraohmic behaviour

A main advantage of the exact representation established in Eq. (12) is that it allows detailed quantitative insight on the controlled dephasing behaviour to be gained. In particular, we focus on long-time coherence properties, which have also received recent attention in view of control-dependent ‘saturation’ effects observed in the context of spin-bath decoherence\(^4\) (see also Ref. \(^4\)). We start by quantifying how the decoherence function in the presence of \( n \) pulses differs between two consecutive control times. Let

\[
\Delta \Gamma_n = \Gamma_n(t_{n+1}) - \Gamma_n(t_n).
\]

By using Eq. (12) we obtain:

\[
\Delta \Gamma_n \sim (-1)^n [\Gamma_0(t_{n+1}) - \Gamma_0(t_n)] + 2 \sum_{j=1}^{n-1} \Gamma_0(t_{n+1} - t_j)(-1)^{n+j} - 2 \sum_{j=1}^{n-1} \Gamma_0(t_n - t_j)(-1)^{j+n}.
\]
Let now $\Delta \Gamma_n^{\text{PDD}}$ denote the above ‘differential dephasing function,’ Eq. (21), specialized to a PDD protocol. Then, as showed in Appendix 1, the following asymptotic result holds for an arbitrary dephasing environment:

$$\Delta \Gamma_{\infty} \equiv \lim_{n \to \infty} \Delta \Gamma_n^{\text{PDD}} = 8 \omega_{\text{res}} \eta(\omega_{\text{res}}).$$

Interestingly, Eq. (23) can be used to describe how the dephasing function changes between any two instants separated by $\Delta t$, for large enough $t$. That is, consider

$$\Delta \Gamma_n^{\text{PDD}}(\tilde{t}) \equiv \Gamma_{n+1}(\tilde{t} + t_{n+1}) - \Gamma_n(\tilde{t} + t_n),$$

where $0 \leq \tilde{t} \leq \Delta t$, $t_n = n \Delta t$. By using Eq. (13) we can verify that $\Delta \Gamma_n^{\text{PDD}}(0) = \Delta \Gamma_n^{\text{PDD}}$ sat. Then one may also prove (see Appendix 1 for detail) that

$$\Delta \Gamma_n^{\text{PDD}}(\tilde{t}) \overset{n > n_{\text{sat}}}{\approx} \Delta \Gamma_{\infty},$$

where $n_{\text{sat}} \equiv t_{\text{sat}}/\Delta t$ is a sufficiently large integer defined in the same Appendix. Eq. (25) shows that the dephasing increment becomes independent of $n$ and $\tilde{t}$ for $t > t_{\text{sat}}$, that is, dephasing asymptotically enters a periodic oscillation ‘in phase’ with the PDD sequence. Thus, $\Delta \Gamma_{\infty}$ in Eq. (23) may be used to describe the difference in dephasing between any two times separated by $\Delta t$—in particular, between consecutive coherence maxima which for $t > t_{\text{sat}}$ occur at $\tilde{t} \approx \Delta t/2$.

For a supra-ohmic environment as in the exciton qubit, the convergence of $\Delta \Gamma_n^{\text{PDD}}$ to $\Delta \Gamma_{\infty}$, Eq. (23), is very fast. This is illustrated in Fig. 3 for two representative values of $\Delta t$.

Because $\Delta \Gamma_{\infty}$ in Eq. (23) is non-zero as long as $\Delta t$ is finite, we can infer that $\Gamma_n$ diverges for fixed $\Delta t$ as $n \to \infty$. While this in principle implies a decay of $\exp(-\Gamma(t))$ to zero under the PDD, details of the spectral density function (including the nature of the coupling spectrum and the form of spectral cutoff) become essential to characterize different dynamical regimes of interest. In what follows, we illustrate these features by contrasting ohmic and supra-ohmic dephasing environments, and by considering stroboscopic sampling, $t_n = 2n\Delta t$, in which case explicit analytic expressions for the PDD ‘filter function’ $|y_{2n}(2n\omega\Delta t)|^2$ are available. Specifically, upon combining Eq. (11b) of Ref. 10 with Eq. (13) we recovers the well-known result $11,12$:

$$\Gamma_{\text{sat}}(2n\Delta t) = \int_0^\infty 4\eta(\omega) \sin^2(\omega n\Delta t) \tan^2\left(\frac{\omega \Delta t}{2}\right) d\omega.$$

(26)

In general, we expect two dominant contributions to the above integral: the one from small values of $\omega$, where $\eta(\omega)$ is not small, and the one from the region of the resonance, $\omega \approx \omega_{\text{res}}$, where $|y_{2n}(\omega)|^2$ may be large. First, note that for both a ohmic and supra-ohmic spectral density, the contributions from the small-$\omega$ region saturate to a finite value with time. For the ohmic case, this is true irrespective of the fact that the free dephasing dynamics does not exhibit a similar long-time saturation. This behavior is due to the control term $\tan^2(\omega \Delta t/2)$, which increases the rate at which the integrand goes to zero as $\omega \to 0$. Second, the contribution from the $\omega \approx \omega_{\text{res}}$ region is more or less relevant depending on the form of the spectral cutoff. Clearly, such ‘resonating’ contributions do not pose a problem in the limiting situation of an arbitrarily ‘hard’ spectral cutoff of the form $\Theta(\omega - \omega_n) \Theta(\omega_n)$ (denoting the step function), since, as remarked earlier, $\omega_{\text{res}} > \omega_n$ in a good DD limit. For a smooth (‘soft’) spectral cutoff, the resonating contribution increases with time and will ultimately be responsible for the divergence of $\Gamma_{\text{sat}}(2n\Delta t)$ as $n \to \infty$.

As shown by Eq. (25), we can approximate $\Delta \Gamma_n \approx \Delta \Gamma_{\infty}$ for $t > t_{\text{sat}}$ since at such long times, the contributions to Eq. (26) from small $\omega$ have saturated, dephasing is indeed dominated from the region around $\omega_{\text{res}}$. Thus, for both ohmic and supra-ohmic systems under PDD, the coherence will eventually decay to zero for large enough times and soft cutoffs.

The above considerations are illustrated in Fig. 4, where we plot exact results calculated from Eq. (26) for a representative ohmic spectral density with an exponential cutoff:

$$I_\omega(\omega) = F \omega \exp\left(-\frac{\omega}{\omega_c}\right),$$

(27)

In order to highlight the different contributions to the overall dephasing function, we also explicitly compute and plot the following quantities: (i) (dotted line)

$$\Gamma_{\text{sat}}(2n\Delta t) = \int_0^{\omega_{\text{res}}/2} \eta(\omega) |y_{2n}(\omega 2n \Delta t)|^2 d\omega,$$

(28)

FIG. 3: Differential dephasing function, $\Delta \Gamma_n^{\text{PDD}}$, for the exciton qubit under examination in the presence of PDD with $\Delta t = 0.3$ ps (top) and $\Delta t = 0.25$ ps (bottom), calculated from Eq. (27). The dotted lines show, in each case, the limiting value $\Delta \Gamma_{\infty}$ given by Eq. (23). Notice that for $n < n_{\text{sat}}$, where $n_{\text{sat}} \sim 15$, the sign of $\Delta \Gamma_n^{\text{PDD}}$ oscillates, in agreement with Eqs. (25) and (27).
which isolates the small-$\omega$ contributions, and (ii) (dashed line)

$$\Gamma_{\text{res}}(2n\Delta t) = \int_{\omega_{\text{res}}}^{3\omega_{\text{res}}/2} \eta(\omega)|y_{2n}(\omega 2n\Delta t)|^2 d\omega,$$

which isolates the contributions from the $\omega \approx \omega_{\text{res}}$ region. Three distinct regions may be identified: an initial drop in coherence due to the low-frequency modes, until saturation of Eq. (28) occurs at about $t = \tau_c$; a plateau region where the contributions from Eq. (29) are not important enough to cause further decoherence; and a final decay of coherence to zero caused by increasing contributions from the $\omega \approx \omega_{\text{res}}$ region. Fig. 4 also compares (bottom panel) the resonanting contributions calculated from Eq. (29) with the asymptotic prediction $\exp(-\Delta \Gamma_\infty t/\Delta t)$ (solid line), with $\Delta \Gamma_\infty = 4.507 \times 10^{-7}$. The data confirm that $\Delta \Gamma_\infty$ does indeed arise from the resonanting contributions as expected, and that as long as the low-frequency contributions have saturated, $\Delta \Gamma_\infty$ may be used to accurately describe dephasing under PDD in the long time limit, that is, $\Delta \Gamma_n \approx \Delta \Gamma_\infty$, for $t > t_{\text{sat}}$.

Additional insight may be gained by examining how the above different regimes (initial decay, plateaux, final coherence decay) are affected by the harder or softer spectral cutoff function. Beside the ohmic spectral density of Eq. (27), consider the following supra-ohmic spectral densities:

$$I_b(\omega) = F\omega^3 \exp\left(-\frac{\omega}{\omega_c}\right),$$

$$I_c(\omega) = F\omega^3 \exp\left(-\frac{\omega^2}{\omega_c^2}\right),$$

where, in particular, $I_c(\omega)$ has a Gaussian tail, similar to the excitonic qubit case. When comparing $I_b(\omega)$ and $I_c(\omega)$ (see Fig. 5), the harder cut-off due to the Gaussian tail strongly reduces the value of $\eta(\omega_{\text{res}})$, and hence greatly increases the duration of the plateau regime. In fact, for the set of parameter chosen, our numerics loose the necessary precision well before the third regime sets on for $I_c(\omega)$. The harder cut-off of the Gaussian case also decreases $\Gamma_{\text{sm}}$ and, in turn, decreases the decoherence that occurs before the plateau.

B. Short time dynamics

In the previous section we analyzed the dephasing dynamics in the presence of PDD for $t > t_{\text{sat}}$. Here, we focus on $t < t_{\text{sat}}$. The long time regime is entered when $\Gamma_{n+1} = \Gamma_n + \Delta \Gamma_\infty$, and for this to occur the coherence must oscillate in phase with the DD pulses. However, the natural response of the coherence after the first PDD pulse is instead to oscillate with a period of $2\Delta t$ (twice that of PDD pulses, recall Fig. 4). This follows from the fact that the first bit-flip occurs an in-
terval $\Delta t$ after a maximum, $\Gamma_0(0)$, and for sufficiently small $\Delta t$, the dephasing function is roughly symmetrical about the control pulse, so the coherence maximum following the first pulse occurs at $t \approx 2\Delta t$. The PDD sequence quickly drives the coherence into phase with it, (see Fig. 3), but the first few even bit-flips in PDD occur near the coherence maxima, and this worsens the performance of the control sequence. This may be seen by considering Eq. (11) at time $t = t_n + \Delta t$, with $0 < \Delta t \leq \Delta t$. By expanding the first and last terms to first order in $\Delta t$, and considering that $\Gamma_0(0)$ is a maximum, Eq. (11) rewrites as

$$\Gamma_n(t) \approx -\frac{d\Gamma_{n-1}(t_n)}{dt} t + \Gamma_{n-1}(t_n).$$

The second term in the above equation is a constant, hence there can be a coherence peak after the $n$-th control pulses only if the derivative $d\Gamma_{n-1}(t_n)/dt > 0$, as also pointed out in Ref. 42. In particular, again using Eq. (11), we can calculate

$$\frac{d\Gamma_n(t_n)}{dt} \approx \frac{\Gamma_n(t_n + \Delta t) - \Gamma_n(t_n)}{\Delta t} = -\frac{d\Gamma_{n-1}(t_n)}{dt},$$

which shows that the larger the gradient of $\Gamma_{n-1}(t_n)$, the faster the coherence is retrieved immediately following the $n$th pulse. In particular, if $\Gamma_{n-1}(t)$ is locally flat at the time of the $n$-th pulse, no coherence gain can occur after that pulse.

We can see from Fig. 6 that as PDD drives the coherence oscillations into phase with it, $\Delta \Gamma_n$ has alternating sign for odd and even $n$ (cf. Eqs. (B5) and (B7)). $\Delta \Gamma_n$ is initially negative for odd $n$ and positive for even $n$, while its magnitude decreases until a time $t_{av}$ after which $\Delta \Gamma_n$ becomes positive for odd $n$ and negative for even $n$, before saturating to $\Delta \Gamma_n = \Delta \Gamma_{\infty}$. We see numerically that $t_{av}$ is independent of $\Delta t$, with $t_{av} \approx 0.5$ ps in our case. Furthermore, we can show from Eq. (11) that if we consider the times at which the control pulses occur ($t = 0$), then

$$\Delta \Gamma_{PDD}^n(0) = \Delta \Gamma_{PDD}^{n-1}(0) + (-1)^n \Delta t^2 \Gamma_0''(n \Delta t),$$

where

$$\frac{d^2 \Gamma_0(n \Delta t)}{dt^2} = \frac{\Gamma_0((n - 1) \Delta t) - 2\Gamma_0(n \Delta t) + \Gamma_0((n + 1) \Delta t)}{\Delta t^2}.$$  

From this expression we can understand the behaviour of the dephasing for PDD as the coherence oscillations are driven into phase with the PDD pulses. As $n$ increases, the sign of the last term in Eq. (33) alternates, and its magnitude decreases as $d^2 \Gamma_0(n \Delta t)/dt^2$ reaches a maximum before decreasing and tending to zero (recall the behavior of $\Gamma_0(t)$ in Fig. 1). Thus, we can now rigorously define $t_{av}$ by the condition $d^2 \Gamma_0(t_{av})/dt^2 = 0$, that is, when the gradient of $\Gamma_0(t)$ is maximum.

C. Practical considerations

Even if the qubit coherence eventually decays to zero under PDD in our excitonic system, for practical purposes we only need to suppress the dephasing for the qubit lifetime, $T_1$. From the above discussion, we can estimate more precisely how short $\Delta t$ must be, in order for this to happen. For $t = n \Delta t + \Delta t/2 > t_{sat}$, we can approximate the off-diagonal density matrix element at the maxima of coherence (where a measurement would be made) as

$$\rho_{01}[(n + 1/2)\Delta t] \approx \rho_{01}(0)e^{-\Gamma_n t_{av}[(n_{sat} + 1/2)\Delta t] - (n - n_{sat})\Delta \Gamma_{\infty}}.$$  

(34)

Considering the long-time limit, if $\Delta t$ is sufficiently small and $n \gg n_{sat}$, we may neglect the coherence that is lost whilst $t < t_{sat}$, and further approximate the dephasing as

$$\rho_{01}[(n + 1/2)\Delta t] \approx \rho_{01}(0)e^{-n \Delta \Gamma_{\infty}} \approx \rho_{01}(0)e^{-\Delta t / \Delta \Gamma_{\infty}}.$$  

(35)

Thus, in the long-time limit, we effectively have $1/T_2^eff = \Delta \Gamma_{\infty} / \Delta t$. A sufficient condition for the dephasing to be suppressed for the entire qubit lifetime is then

$$T_2^eff = \frac{\Delta t}{\Delta \Gamma_{\infty}} \gtrsim T_1.$$  

(36)

Fig. 7 shows $\Delta \Gamma_{\infty} / \Delta t$ as a function of $\Delta t$ for the exciton qubit under consideration, for which $1/T_1 = 1$ ns$^{-1}$. It can be seen that for $\Delta t \lesssim 0.2$ ps, PDD effectively suppresses dephasing for the entire lifetime. This is in excellent agreement with our previous results in Ref. 35, where we found numerically that $\Delta t = 0.2$ ps leads to efficient PDD, but, in comparison, $\Delta t = 0.3$ ps could only suppress the dephasing for relatively short times.

IV. COMPARISON OF PDD WITH NON-UNIFORM DD SCHEMES

Having characterized the performance of the simplest DD scheme, where the control involves a single time scale $\Delta t$, we proceed to examine some of the high-level protocols mentioned in the introduction, which involve non-uniform pulse delays to a lesser or greater extent. While CPDD is both, his-
Given $T_{\text{storage}}$ and the presence of a physical constraint on $\Delta t$, a first way to exploit CDD is to identify a minimum concatenation level, $\ell^*$, for which the length of the corresponding sequence, $T_{\ell^*} = 2^\ell \Delta t$, exceeds $T_{\text{storage}}$. For a given $\Delta t$, increasing $\ell$ beyond this point would not modify the results because the pulse timings over $T_{\text{storage}}$ would be unchanged. (see Table I). Figure 8 compares CDD and PDD for storage of an exciton qubit for different $\Delta t$. As expected from the general analysis of Ref. 3, the efficiency of CDD increases with decreasing $\Delta t$. However, in the range of values under exploration, and with readout effected at the maxima of the coherence curve, CDD is found to be more efficient than PDD only if $\Delta t \lesssim 0.036$ ps. The latter time scale is substantially smaller than physically allowed in our system.

We can understand the possible advantage of CDD by comparing it with the long- and short-time behaviour of PDD (Secs. IIIA and IIIB respectively). Eq. (24) shows that the long-time performance of the protocol depends on $\Delta t$, $\Delta t$, and $\Gamma_{\text{sat}}(1/2\Delta t)$.

1. Single CDD cycle

Instead of repeating the basic control cycle given in Eq. (20), CDD recursively concatenates it within itself. Let $S_\ell$ denote the sequence corresponding to the $\ell$-th level of concatenation, as given in Table II.

A. Concatenated decoupling

For a qubit undergoing arbitrary decoherence, CDD with a ‘universal decoupling’ cycle given, for instance, by $\Delta t X \Delta t Z \Delta t X \Delta t Z$, has been shown to significantly outperform PDD in the limit $\Delta t \to 0$. However, for purely dephasing systems for which $\Delta t$ has a finite lower limit, and for single-axis protocols constructed out of the basic cycle in Eq. (20), the advantages of CDD are largely lost, and PDD may be more efficient. While different ways for comparing different DD protocols can be considered, we shall focus here on comparing the efficiency of PDD and CDD at ensuring dephasing-protected storage of the exciton qubit for a fixed time $T_{\text{storage}}$. In particular, for our calculations we choose $T_{\text{storage}} = 10$ ps. This time is appropriate given the typical gating time for exciton-based QIP, which is of the order of $1$ ps.
cycle’, constructed from Table I. For instance, truncation at $\ell = 2$ results in our purely dephasing case in a cycle of length $4\Delta t$, which is identical in structure to a CP cycle (see Sec. IV C), and whose periodic repetition we term PCDD$_2$. For a single qubit undergoing arbitrary decoherence, the corresponding PCD$_2$ protocol (constructed from a 16-pulse base cycle) has been shown to be the best performer in suppressing the effects of a quantum spin bath$^{24,41,43}$.

Figure 8 shows a comparison of PDD and PCDD$_2$ protocols for $\ell = 2, 3$, for the shortest pulse separation compatible with the exciton qubit constraint, $\Delta t = 0.1$ ps. One can infer that, for the $\Delta t$ and $T_{\text{storage}}$ values considered, PCDD$_2$ performs better (that is, displays higher coherence maxima) than PDD for $\ell = 2$, but worse for $\ell = 3$. The difference between PCDD$_2$ and PCDD$_3$ may be understood as a consequence of the fact that in terms of a Magnus expansion$^{21}$, concatenated cycles with even $\ell$ are time-symmetric, thus cancel the interaction with the phonon bath up to (at least) the second order. Over the time period shown, PCDD$_2$ also outperforms standard PDD (see Fig. 9 upper panel). However, the coherence oscillations for PCDD$_2$ occur over a period of $2\Delta t$ since, after the initial pulse, the sequence is equivalent to PDD with a base time interval of $2\Delta t$. Therefore, we expect PDD to be more efficient for long storage times, as PCDD$_2$ will yield a larger $\Delta \Gamma_{\infty}$ than a PDD sequence characterized by $\Delta t$, hence worse asymptotic performance.

### B. Uhrig decoupling

We now assess the limitations of the optimal sequence proposed by Uhrig$^{10}$ when significant restrictions on $\Delta t$ are in place. In UDD, consecutive pulses are spaced according to

$$\delta_j = \sin^2 \left( \frac{\pi j}{2n+2} \right), \quad (37)$$

which implies, in particular, closely spaced pulses at the beginning and the end of the evolution period. Such a control sequence strongly suppresses the dephasing for a storage time
that such a sequence with first pulse, and after the last pulse. From Eq. (37) we see that the efficiency of UDD is rapidly lost. The shortest interval may not be allowed by the physical constraints we are considering, this study will pave the way to being developed in the next section.

For our QD system, however, the main physical limitation is on the time delay between pulses. The shortest interval between control pulses in UDD, $\Delta t_{\text{UDD}}^{\text{min}}$, is before the first pulse, and after the last pulse. From Eq. (37) we see that such a sequence with $n = 100$ pulses over a period of $T_{\text{storage}} = 10$ ps corresponds to $\Delta t_{\text{UDD}}^{\text{min}} = 2.4 \times 10^{-3}$ ps, which is roughly two orders of magnitude less than that allowed by the physical constraints for the exciton qubit in question. Even for a sequence consisting of $n = 40$ pulses only (for which the efficiency is already poor as shown in Fig. 10 curve c), $\Delta t_{\text{UDD}}^{\text{min}} = 1.5 \times 10^{-2}$ ps, which is still an order of magnitude shorter than allowed.

To respect the physical constraints, one may estimate that allowed UDD sequences should have a number of pulses $n \lesssim 14$ within the intended $T_{\text{storage}} = 10$ ps. Such a sequence corresponds to curve e) in Fig. 10. It is then clear that any UDD sequence compatible with our physical constraints is outperformed by the best allowed PDD sequence which would preserve a coherence close to 1 for the same time window (see Fig. 2). Fig. 10 also shows that any constrained UDD sequence performing like curve d) or worse would increase the dephasing compared with the free evolution, that is, would result in decoherence acceleration. The reason for the shortfalls of UDD in our setting stems from the large spread of the control intervals $(t_i - t_{i-1})$. If we impose a lower bound on the minimum time interval, other intervals must take up a considerable proportion of the total evolution time. This places a relatively large restriction on how many pulses may be used within a given storage period, and eventually results in large amounts of dephasing during the long time delays in which no pulses occur.

C. Carr-Purcell decoupling

We now focus on analyzing more closely CPDD, which results from the periodic repetition of a CP cycle of the form

$$\Delta t_{\text{CP}} X 2 \Delta t_{\text{CP}} X \Delta t_{\text{CP}}.$$  (39)

This also corresponds, as noted, to PCDD$_2$ with $\Delta t_{\text{CP}} = \Delta t$ (cf. Table I). Specifically, we are interested in comparing a PDD sequence with a CPDD having the same cycle time, $T_c = 2 \Delta t$, thus $\Delta t_{\text{CP}} = \Delta t/2$; though the corresponding pulse time interval may not be allowed by the physical constraints we are considering, this study will pave the way to be the analysis to be developed in the next section.

Basically, CPDD may be viewed as a PDD protocol where pulses are uniformly spaced by $2 \Delta t_{\text{CP}}$, except that the sequence is displaced forward by $t_1 = \Delta t/2$, the time at which the first pulse is applied. As a consequence of the symmetry of the control propagator in Eq. (39) with respect to the cycle mid-point, it is well known that CPDD is a second-order protocol as compared to standard (asymmetric) PDD, with leading corrections of order $T_c^3$. Using the exact representation established in Eq. (12), we will now assess the extent to which CPDD improves over PDD for a purely dephasing system, and gain insight into asymptotic properties.

We begin by determining the dephasing half-way between consecutive control pulses for the case of PDD. Using Eq. (12), we find

$$\Gamma_{\text{PDD}}^{\text{t}} = \sum_{m=1}^{n} (-1)^n \Gamma_0 [(n + 1/2 - m) \Delta t]$$

with

$$\Gamma_{\text{PDD}}^{\text{t}} = \sum_{m=1}^{n} (-1)^n \Gamma_0 [(n + 1/2 - m) \Delta t]$$

(40)
which we may rewrite as
\[
\Gamma_n^{\text{PDD}} \left[ (n + 1/2) \Delta t \right] = 2 \sum_{m=1}^{n} (-1)^{m+1} \Gamma_0 (m \Delta t) \\
+ 4 \sum_{m=2}^{n} \sum_{j < m} \Gamma_0 ((m-j) \Delta t) (-1)^{m-1-j} \\
+ 2 \sum_{k=1}^{n} (-1)^{k+1} \Gamma_0 \left[ (k - 1/2) \Delta t \right] \\
+ (-1)^n \Gamma_0 \left[ (n + 1/2) \Delta t \right],
\]
where \( k = n - m + 1 \). Similarly, by using Eq. (12), we may also determine the dephasing for CPDD with \( \Delta \tau^{\text{CP}} = \Delta t/2 \) and \( t_i = (i - 1/2) \Delta t \), that is,
\[
\Gamma_n^{\text{CPDD}} \left[ (n + 1/2) \Delta t \right] = \Gamma_n^{\text{CPDD}} \left[ (n + 1/2) \Delta t \right].
\]
This exact result is illustrated in Fig. 11 where we plot the dephasing behaviour under PDD and CPDD for the exciton qubit with \( \Delta t = 0.1 \) ps. As predicted by Eq. (12), the coherence in the presence of each sequence is equal at times \( t = (n + 1/2) \Delta t \). Interestingly, for \( t > t_{\text{sat}} \), \( \Gamma_n^{\text{PDD}} \left[ (n + 1/2) \Delta t \right] \) are local maxima of coherence whereas \( \Gamma_n^{\text{CPDD}} \left[ (n + 1/2) \Delta t \right] \) are local minima, proving CPDD to be much more efficient than PDD provided that the time of the first pulse is allowed to be \( t_1 = 0.05 \) ps.

\[\text{FIG. 11: Comparison of CPDD (dotted line) with } \Delta \tau^{\text{CP}} = 0.05 \text{ ps and PDD (solid line) with } \Delta t = 0.1 \text{ ps for the exciton system.}\]

V. TOWARDS OPTIMIZED SEQUENCES IN THE PRESENCE OF PULSE TIMING CONSTRAINTS

Building on the understanding gained from the comparison between different protocols in Sec. IV, we now specifically aim to optimize DD performance for a bosonic dephasing environment when pulses are subject to a minimum pulse-delay constraint.

The basic observation is to note that if after an initial arbitrary pulse sequence, PDD is turned on at time \( t_{\text{PDD}} \), then for \( t > t_{\text{PDD}} + t_{\text{sat}} \), we have \( \Gamma_{n+1}(t + \Delta t) - \Gamma_n(t) = \Delta \Gamma_{\infty} \) (recall Eq. (25)). This naturally suggests an interpolated DD approach, where an initial sequence is chosen to minimize \( \Gamma(t_{\text{max}}) \), whilst transforming the oscillations of coherence into phase with a PDD sequence to be turned on immediately afterwards. Interestingly, a similar philosophy has been invoked to optimally merge deterministic and randomized DD methods to enhance performance over the entire storage times.

In our case, CPDD is indeed the simplest example of this interpolation: as already noted, CPDD can be thought of as a PDD sequence applied at \( t_{\text{PDD}} = \Delta t/2 + \Delta t \), following a preparatory sequence consisting of a single pulse at \( t = \Delta t/2 \).

Unfortunately, standard CPDD is not allowed in our system due to the physical constraint: the time interval between pulses in the initial sequence is smaller than the minimum allowed \( \Delta t \) which characterizes the subsequent PDD sequence. Simply using a CPDD sequence which does not break the time constraint is clearly not optimal. If the smallest allowed pulse interval is \( \Delta t_{\text{min}} \), then the best CPDD sequence consists of periodic repetitions of a CPDD cycle with \( \Delta \tau^{\text{CP}} = \Delta t_{\text{min}} \), and the most efficient allowed PDD sequence is repetitions of \( X \Delta t_{\text{min}} X \Delta t_{\text{min}} \). Since CPDD cancels the terms in the Magnus expansion up to to the second order, over the first few repetitions it performs much better than PDD, which only cancels them up to the first order. However, for longer times the effects due to the higher-order Magnus corrections accumulate, and they turn out to do so more favorably for PDD. This manifests itself in a smaller \( \Delta \Gamma_{\infty} \) for PDD than for the best allowed CPDD protocol. As shown by Eqs. (B3) and (B6), the coherence oscillations are independent of the timing of any pulses applied before \( t - t_{\text{sat}} \). Therefore, CPDD can be treated as a PDD sequence with \( \Delta t = 2 \Delta \tau^{\text{DP}} \) for \( t > t_{\text{sat}} \). This justifies defining a \( \Delta \Gamma_{\infty} \) for a CPDD sequence.

Physically, what is needed is a different initial sequence that efficiently ‘engineers’ the transition of the coherence oscillations – from the natural response frequency determined by the first bit-flip to the frequency of the following PDD sequence. To accomplish this, we propose to use \( \text{CP cycles with varying } \Delta \tau^{\text{CP}} \). That is, we define such an interpolated sequence by letting the \( i \)th cycle to be characterized by a pulse delay \( \Delta \tau_i^{\text{CP}} \), and begin immediately after the previous cycle at \( t_i = t_{i-1} + 4 \Delta \tau_{i-1} \). The analysis of the resulting averaging properties may be carried out by adapting the derivation of Ref. [5] to the pure dephasing bosonic setting of Eq. (1).

While the detail of the calculations are included in Appendix C, the result is that, similar to standard CP, the proposed DD sequence still cancels the terms in the Magnus expansion up to the second order. Therefore, the interpolated scheme does not only perform well for small \( t \), but also quickly results in pulses uniformly separated by \( \Delta t_{\text{min}} \) – resulting in a small \( \Delta \Gamma_{\infty} \), hence high performance for long storage times.

The simplest way to generate a good interpolated DD sequence is to apply a CP cycle with \( \Delta \tau^{\text{CP}} = \Delta t_{\text{min}} \), followed by periodic repetitions of one with \( \Delta \tau^{\text{CP}} = \Delta t_{\text{min}}/2 \). The sequence is then given by
\[
\begin{align*}
  t_1 & = \Delta t_{\text{min}}, \\
  t_2 & = 3 \Delta t_{\text{min}}, \\
  t_3 & = 3 \Delta t_{\text{min}} + \frac{3}{2} \Delta t_{\text{min}}, \\
  t_i & = t_{i-1} + \Delta t_{\text{min}}, \quad i > 3.
\end{align*}
\]
We compare this sequence with standard PDD with $\Delta t = \Delta t_{\text{min}}$ in Fig. 12 upper panel. One clearly sees that the sequence in Eq. (43) is more efficient.

By construction, the first two CP cycles in the above sequence play the role of modifying the frequency of the dephasing oscillations in such a way that they are brought in phase with the following repeated cycles. We can perform this process more smoothly by gradually reducing $\Delta t_{\text{CP}}$ from $\Delta t_{\text{min}}$ to $\Delta t_{\text{min}}/2$ over more than a control cycle. Though for very small $\Delta t_{\text{min}}$ the two cases would be equivalent, for systems such as the exciton qubit where the time restrictions are relatively severe, the smoother transition sequence may decouple the qubit more efficiently. Such a modified sequence may be implemented by applying CP cycles with decreasing $\Delta t_{\text{CP}}$, that is,

$$\Delta t_{\text{CP}} = \begin{cases} 
\Delta t_{\text{min}}, & i = 1 \\
\Delta t_{\text{min}} - (i - 1)\Delta_2, & 1 < i \leq \text{i}_{\text{PDD}}, \\
\Delta t_{\text{min}}/2, & i > \text{i}_{\text{PDD}},
\end{cases} \tag{44}$$

where $\text{i}_{\text{PDD}} = \Delta t_{\text{min}}/(2\Delta_2)$ and $\Delta_2$ is an arbitrary time defined such that $\text{i}_{\text{PDD}}$ is an integer. The greater $\text{i}_{\text{PDD}}$, the longer the time over which the decreasing length cycles are applied. Alternatively, we may describe the above sequence in terms of the pulse times:

$$\begin{align*}
t_i &= \Delta t_{\text{min}} + (i - 1)2\Delta t_{\text{min}} - \frac{(i - 1)(i - 2)}{2}\Delta_2, & i < \frac{\Delta t_{\text{min}}}{\Delta_2} - 2, \\
t_i &= t_{i-1} + \Delta t_{\text{min}}, & i \geq \frac{\Delta t_{\text{min}}}{\Delta_2} - 2. \tag{45}
\end{align*}$$

The above modified sequence is compared to PDD with $\Delta t = \Delta t_{\text{min}}$ in Fig. 12 lower panel. As before, we see that DD with varying CP cycles outperform PDD. Furthermore, it can be seen that there is an improvement over the more abrupt sequence described by Eq. (43).

In Sec. [IV A 2] we showed that a constrained CPDD sequence could outperform PDD over time scales of the order of 10 ps (see top panel of Fig. 11 for PCDD$_2$) even though, for longer times, the smaller $\Delta \Gamma_{\infty}$ for PDD would eventually make it more efficient than CPDD. In Fig. 11 we further compare CPDD with the interpolated sequence given in Eq. (44). The latter is found to be slightly more efficient than the best allowed CPDD sequence over short time scales. Furthermore, because of the smaller $\Delta \Gamma_{\infty}$, as time progresses it will also outperform CPDD asymptotically. A main advantage of the sequence given in Eq. (44), however, is that it not only leads to higher maxima than CPDD, but also, after the first few pulses, to a much smaller coherence oscillation amplitude. This reflects the fact that the oscillation period has been tuned to the minimum allowed time interval $\Delta t_{\text{min}}$. In this respect, the performance of the sequence in Eq. (44) is more robust against the precise readout times, or, equivalently, a readout offset error relative to the coherence maxima would not significantly affect the coherence recovered using this sequence.

VI. CONCLUSION AND OUTLOOK

We have investigated the ability of DD to inhibit decoherence of a single qubit coupled to a purely dephasing bosonic environment, by comparing the performance of low-level periodic DD schemes based on uniform pulse separations to higher-level non-uniform DD schemes.

For arbitrary spectral density functions characterizing different dephasing environments, we have derived an exact representation of the controlled dynamics available for instantaneous pulses, and exact relationships for the asymptotic dynamics in the case of DD. Building on these results, we have shown that a main weakness of PDD is due to the oscillation of coherence following the first bit-flip being out of phase with the rest of the sequence. This has naturally suggested the application of a suitably engineered preparatory sequence as a strategy to enhance DD efficiency, by bringing the coherence oscillations into phase with a subsequent DD sequence. The resulting ‘interpolated’ DD protocols are found to be especially efficient for physical systems where the minimum time interval between control pulses is strongly constrained. For such systems, DD protocols like concatenated or Uhrig DD, which are designed to achieve peak performance when the asymptotic regime of arbitrarily small pulse separations is
fully accessible, tend to largely lose their advantages.

For the excitonic dephasing environment of interest, in particular, we have shown how a sequence of Carr-Purcell cycles with suitably chosen (analytically generated) time delays provides a very efficient DD protocol for realistic QD parameters and qubit storage times. Our process of constructing a DD sequence under which the coherence oscillates asymptotically with the minimum period allowed by the physical contraints offers, as by-product, the advantage of a significantly smaller coherence oscillation amplitude relative to constrained PCDD or UDD sequences. This makes the proposed interpolated sequence more robust against readout.

While our analytically-designed interpolated DD protocol might be compelling in its simplicity, identifying DD schemes that are guaranteed to yield optimal performance subject to non-trivial timing constraints appears as an interesting control-theoretic problem for further investigation. Revisiting the local numerical optimization approach recently proposed in Ref. 12 in a constrained minimization perspective might offer a concrete starting point in this respect. Likewise, the investigation of dynamical error-control schemes based on bounded-strength ‘Eulerian’ DD, along with the recently proposed extension to decoherence-protected quantum gates27,44, might prove especially fruitful for exciton qubits, in view of the reduced control overheads associated with purely dephasing environments. Lastly, an interesting general question is to what extent exact representations of the controlled coherence dynamics in terms of the uncontrolled one may exist for an arbitrary purely dephasing error model, allowing, for instance, exact insight into the long-time controlled dynamics to be gained.

**FIG. 13**: (Color online) Comparison of the best allowed CPDD sequence (light line) with a sequence of CP cycles of decreasing length (dark line) as described in Eq. (44) with $\Delta_t = 0.01$ ps and for $\Delta t_{\text{min}} = 0.1$ ps. Inset: Zoom over the initial part of the time window with timings of the pulse sequences explicitly indicated (squares for the modified sequence and crosses for CPDD, respectively).

**APPENDIX A: DERIVATION OF $\Delta \Gamma_\infty$**

For PDD, $t_n = n \Delta t$, using this expression and recasting the sums in Eq. (22) in terms of $k = n - j$, we can write

$$\Delta \Gamma_{\text{PDD}}^{n} = (-1)^{n} \Gamma_{0}(n+1)\Delta t - 3 \Gamma_{0}(n\Delta t)(-1)^{n} + 4 \sum_{k=1}^{n-1} \Gamma_{0}(k\Delta t)(-1)^{k}. \quad (A1)$$

Using Eq. (7) for $n = 0$ and extending the sum to include $k = 0$, Eq. (A1) becomes

$$\Delta \Gamma_{\text{PDD}}^{n} = -4 \int_{0}^{\infty} \eta(\omega) d\omega + (-1)^{n} \int_{0}^{\infty} 6 \eta(\omega) \cos(\omega n \Delta t) d\omega$$

$$- (-1)^{n} \int_{0}^{\infty} 2 \eta(\omega) \cos(\omega(n+1) \Delta t) d\omega$$

$$+ 8 \int_{0}^{\infty} \eta(\omega) \sum_{k=0}^{n-1} \cos(\omega k \Delta t)(-1)^{k} d\omega. \quad (A2)$$

By using the relationship

$$\sum_{k=1}^{n} (-1)^{k} \cos(kx) = -\frac{1}{2} + \frac{(-1)^{n} \cos\left(\frac{2n+1}{2}x\right)}{2 \cos\left(\frac{x}{2}\right)}, \quad (A3)$$

we can rewrite the above equation as

$$\Delta \Gamma_{\text{PDD}}^{n} = 2 \int_{0}^{\infty} \eta(\omega) \left\{ (-1)^{n+1} \cos((n+1)\omega n \Delta t) \right.$$

$$+ (n+1) \omega n \Delta t) + 2(-1)^{n+1} \cos\left(\frac{2n+1}{2} \Delta t \omega \right) \cos\left(\frac{\Delta t \omega}{2}\right) \right\} d\omega$$

$$= 4 \int_{0}^{\infty} \eta(\omega) \left\{ (-1)^{n+1} \cos\left(\frac{2n+1}{2} \Delta t \omega \right) \cos\left(\frac{\Delta t \omega}{2}\right) \right.\right. + (n+1) \omega n \Delta t) + 2(-1)^{n+1} \cos\left(\frac{2n+1}{2} \Delta t \omega \right) \cos\left(\frac{\Delta t \omega}{2}\right) \right\} d\omega. \quad (A4)$$

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This finally can be rearranged as
\[
\Delta \Gamma_n^{\text{PDD}} = 4 \int_0^\infty \eta(\omega) \sin^2 \left(\frac{\omega \Delta t}{2}\right) \times (-1)^n \cos \left(\omega \left(n + \frac{1}{2}\right) \Delta t\right) \frac{1}{\cos \left(\frac{\omega}{2}\right)} d\omega. \tag{A5}
\]
We now take the limit of \( n \to \infty \), and note that
\[
\lim_{n \to \infty} (-1)^n \cos \left(\omega \left(n + \frac{1}{2}\right) \Delta t\right) \frac{1}{2\pi} = \frac{1}{\Delta t} \sum_{l=0}^\infty \delta \left(\omega - (\omega_{\text{res}} + 2l\omega_{\text{res}})\right). \tag{A6}
\]
If we assume that \( \eta(\omega_{\text{res}}) \gg \eta(\omega_{\text{res}} + 2l\omega_{\text{res}}) \) for \( l > 0 \), which is true for sufficiently small \( \Delta t \) (that is, \( \omega_{\text{res}} \gg \omega_c \)), we can neglect contributions from \( l > 0 \) and define
\[
\Delta \Gamma_\infty = \lim_{n \to \infty} \Delta \Gamma_n^{\text{PDD}} \tag{A7}
= \int_0^\infty 8d\omega \eta(\omega - \omega_{\text{res}}) \eta(\omega) \omega_{\text{res}} \sin^2 \left(\frac{\omega \Delta t}{2}\right)
= 8\eta(\omega_{\text{res}})\omega_{\text{res}}.
\]

APPENDIX B: LONG-TIME LIMIT OF \( \Delta \Gamma_n^{\text{PDD}}(\hat{t}) \)

By using Eq. (B2) and straightforward manipulations, we can separate \( \Delta \Gamma_n^{\text{PDD}}(\hat{t}) \) from Eq. (24) into two parts,
\[
\Delta \Gamma_n^{\text{PDD}}(\hat{t}) = \Delta \Gamma_n^{\text{T1}} + \Delta \Gamma_n^{\text{TD}}(\hat{t}), \tag{B1}
\]
with
\[
\Delta \Gamma_n^{\text{T1}}(\hat{t}) = (-1)^n \left[ \Gamma_0(t_n + \hat{t}) - \Gamma_0(t_{n+1} + \hat{t}) \right], \tag{B2}
\]
and the second term
\[
\Delta \Gamma_n^{\text{TD}}(\hat{t}) = 2(-1)^n \left[ \Gamma_0(t_{n+1}) + 2 \sum_{j=1}^{n} (-1)^j \Gamma_0(t_{n+1} - t_j) \right], \tag{B3}
\]
independent of \( \hat{t} \). By using
\[
\Gamma_0(t) = 2 \int_0^\infty \eta(\omega) [1 - \cos(\omega t)] d\omega, \tag{B4}
\]
we can rewrite \( \Delta \Gamma_n^{\text{TD}}(\hat{t}) \) as
\[
\Delta \Gamma_n^{\text{TD}}(\hat{t}) = -4(-1)^n \int_0^\infty \eta(\omega) \sin \left(\frac{\Delta t \omega}{2}\right) \sin \left\{\left(\frac{n + 1}{2}\right) \Delta t + \hat{t} \right\} d\omega. \tag{B5}
\]
The last term in the integrand above is fast oscillating for large \( n \), so we will have that for \( n > n_{\text{sat}} \),
\[
|\Delta \Gamma_n^{\text{TD}}(\hat{t})| < \epsilon, \tag{B6}
\]
where \( \epsilon \) can be made arbitrarily small.

Let us now consider \( \Delta \Gamma_n^{\text{T1}} \). By using Eq. (B4), the relation Eq. (A3) and some tedious but straightforward manipulations, we can rewrite Eq. (B3) as
\[
\Delta \Gamma_n^{\text{T1}}(\hat{t}) = -(-1)^n 4 \int_0^\infty \eta(\omega) \cos[\omega(n + 1) \Delta t] d\omega
+ 4 \int_0^\infty \eta(\omega) \cos \left(\frac{2n+1}{2} \Delta t \omega\right) d\omega. \tag{B7}
\]
Again, the integrand in the first term of the above equation is fast oscillating for large \( n \), while the second term tends to \( \Delta \Gamma_\infty \) for \( n \to \infty \) (see Eq. (A6). We can then write that for \( n > n_{\text{sat}} \),
\[
|\Delta \Gamma_n^{\text{T1}}(\hat{t}) - \Delta \Gamma_\infty| < \epsilon, \tag{B8}
\]
By combining Eqs. (B6) and (B8), we finally obtain that for any \( \hat{t} \) and \( n > n_{\text{sat}} \),
\[
\Delta \Gamma_n^{\text{PDD}}(\hat{t}) \approx \Delta \Gamma_\infty. \tag{B9}
\]
We note that in the case of supraohmic environment, by using that \( \Gamma_0(\infty) \equiv \lim_{n \to \infty} \Gamma_0(t_{n+1}) = 2 \int_0^\infty \eta(\omega) d\omega \) is finite, and using Eqs. (B3) and (B7), we can recast the conditions Eqs. (B9) and (B9) as
\[
|\Gamma_0(t > t_{\text{sat}}) - \Gamma_0(\infty)| < \epsilon, \tag{B10}
\]
where \( t_{\text{sat}} = n_{\text{sat}} \Delta t \). This emphasizes that condition (B9) applies for times at which the natural evolution saturates to its long-term behavior.

APPENDIX C: AVERAGING PROPERTIES OF INTERPOLATED DD SCHEME

We begin by casting the QD Hamiltonian Eq. (1) (with \( \alpha = 1/2 \)) in the following form:
\[
H_1 = \sigma_z \otimes B_z + \sigma_0 \otimes B_0, \tag{C1}
\]
where \( B_z \) and \( B_0 \) are operators acting on the phonon bath, and \( \sigma_0 \) and \( \sigma_z \) denote the identity and the Pauli matrix acting on the exciton qubit, respectively. This allows us to express the evolution in the presence of the \( i \)-th CP cycle by the propagator
\[
U_i^{\text{CP}}(4\Delta t_i^{\text{CP}}) = U_f(\Delta t_i^{\text{CP}}) X U_f(2\Delta t_i^{\text{CP}}) X U_f(\Delta t_i^{\text{CP}}), \tag{C2}
\]
where \( U_f(t) = \exp(-tH_1) \) represents free evolution for a time \( t \). If we define
\[
H_2 \equiv -\sigma_z \otimes B_z + \sigma_0 \otimes B_0 = X H_1 X, \tag{C2}
\]
we can write the entire sequence propagator as a Magnus series expansion\(^2\),

\[
U(t) = \exp \left( \sum_{i=1}^{\infty} A_i(t) \right)
\]

for which, in the limit of sufficiently fast control, we can only consider the first two lowest-order terms in \( \Delta t \). Specifically (in units where \( \hbar = 1 \)):

\[
A_1 = -i \int_0^t dt_1 H(t_1), \tag{C4}
\]

\[
A_2 = -\frac{1}{2} \int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)], \tag{C5}
\]

\[
A_2 = \int_{t_0}^{t_{n+4\Delta t_C}} dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)] = \sum_{i=1}^{n} \int_{t_i}^{t_{i+4\Delta t_C}} dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)]
\]

\[
= \sum_{i=1}^{n} \left\{ \left( \int_{t_i}^{t_{i+4\Delta t_C}} dt_1 \int_0^{t_1} dt_2 \right) + \left( \int_{t_i}^{t_{i+3\Delta t_C}} dt_1 \int_0^{t_1} dt_2 \right) + \left( \int_{t_i}^{t_{i+3\Delta t_C}} dt_1 \int_0^{t_1} dt_2 \right) \right\}
\]

\[
= \sum_{i=1}^{n} \left\{ 2\Delta t_C [H_1, H_2] + \left( \int_{t_i}^{t_{i+4\Delta t_C}} dt_1 \int_0^{t_1} dt_2 \right) \right\}
\]

\[
= 0,
\]

again up to irrelevant pure-bath terms. This confirms the second-order cancellation claimed in the main text.

\[H(t) = U^\dagger(t) H U(t)\] is the time-dependent (piecewise constant for instantaneous pulses) effective Hamiltonian that describes the evolution under the control propagator \( U(t) \) resulting from the applied pulse\(^2\).\(^3\).\(^3\).

For the sequence of different CP cycles described in Sec. \(\Box\), \(A_1\) is proportional to the identity operator, and hence does not contribute to dephasing. This is a simple consequence of the qubit spending equal amounts of time in each of the computational basis states. More interestingly, we find

\[
{1\over \Delta t} \sum_{i, j=1}^{n} 2\Delta t_C [H_1, H_2] + 2\Delta t_C \sum_{i, j=1}^{n} 2\Delta t_C [H_2, H_1] + \Delta t_C \sum_{i, j=1}^{n} 2\Delta t_C [H_1, H_2] = 0,
\]

\[
\text{where } H(t) = U^\dagger(t) H U(t) \text{ is the time-dependent (piecewise constant for instantaneous pulses) effective Hamiltonian that describes the evolution under the control propagator } U(t) \text{ resulting from the applied pulse.}
\]

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