Quasi-two-body decays $B \to \eta_c(1S, 2S) [\rho(770), \rho(1450), \rho(1700)] \to \pi\pi$ in the perturbative QCD approach

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In this paper, we calculated the branching ratios of the quasi-two-body decays $B \to \eta_c(1S, 2S) [\rho(770), \rho(1450), \rho(1700)] \to \pi\pi$ by employing the perturbative QCD (PQCD) approach. The contributions from the $P$-wave resonances $\rho(770)$, $\rho(1450)$ and $\rho(1700)$ were taken into account. The two-pion distribution amplitude $\Phi_{\pi\pi}^P$ is parameterized by the vector current time-like form factor $F_v$ to study the considered decay modes. We found that (a) the PQCD predictions for the branching ratios of the considered quasi-two-body decays are in the order of $10^{-7} \sim 10^{-6}$, while the two-body decay rates $B(B \to \eta_c(1S, 2S)(\rho(1450), \rho(1700)))$ are extracted from those for the corresponding quasi-two-body decays; (b) the whole pattern of the pion form factor-squared $|F_v|^2$ measured by the BABAR Collaboration could be understood based on our theoretical results; (c) the general expectation based on the similarity between $B \to \eta_c\pi\pi$ and $B \to J/\psi\pi\pi$ decays are confirmed: $R_2(\eta_c) \approx 0.45$ is consistent with the measured $R_2(J/\psi) \approx 0.56 \pm 0.09$ within errors; and (d) new ratios $R_3(\eta_c(1S))$ and $R_4(\eta_c(2S))$ among the branching ratios of the considered decay modes are defined and could be tested by future experiments.

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I. INTRODUCTION

In recent years, due to the great progress in the theoretical studies and experimental measurements, the three-body hadronic $B$ meson decays become much more attractive than ever before, and begin to play an important role in testing the standard model (SM) and in searching for the signal of the possible new physics beyond the SM.

In the experiment side, the measurements for the branching ratios and $CP$ violating asymmetries for $B \to K\pi\pi$ and other decay modes have been reported by the BABAR [1–5], Belle [6–9] and LHCB Collaboration [10–16]. These three-body decays are known experimentally to be dominated by the low energy resonances on $\pi\pi$, $KK$ and $K\pi$ channels on the Dalitz plots [17, 18], analysed by employing the isobar model [19, 20] in terms of the usual Breit-Wigner model [21] or D.V. Bugg model [22] plus a background. Obviously, such decay modes do receive the resonant and nonresonant contributions, as well as the possible final-state interactions (FSIs) [23–25], but the relative strength of these contributions from different sources are varying significantly from channel to channel.

In the theory side, the three-body hadronic decays of the heavy $B$ meson are clearly much more complicated to be described theoretically than those two-body decays. We firstly can not separate the nonresonant contributions from the resonant ones clearly, and secondly do not know how to calculate or estimate the nonresonant and FSI contributions reliably [26]. As a first step, however, we can restrict ourselves to specific kinematical configurations, in which two energetic final state mesons almost collimating to each other, the three-body interactions for such topologies are expected to be suppressed strongly. Then it seems reasonable to assume the validity of factorization for these quasi-two-body $B$ decays. In the “quasi-two-body” mechanism, the two-body scattering and all possible interactions between the two involved particles are included but the interactions between the third particle and the pair of mesons are neglected.

During the past two decades, several different theoretical frameworks have been developed for the study of the three-body hadronic $B$ meson decays: the one based on the QCD-improved factorization (QCDF) [26–33], the method with the symmetry

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principles [34–41] and the framework relying on the perturbative QCD (PQCD) approach [42–52]. In PQCD factorization approach, for example, we study the three-body hadronic decays of B meson by introducing the two-hadron distribution amplitude (DA) $\Phi_{h_1, h_2}$ [53–59] to describe the system of the two collimating energetic final state mesons. Our estimation proceeds via the idea of the quasi-two-body decays involving resonant and nonresonant contributions, which can be absorbed in the time-like form factors to parameterize these two-hadron DAs.

As discussed in Ref. [42], we here assume that the hard $b$-quark decay kernels containing two virtual gluons at leading order is not important due to the power-suppression. The contributions from the dynamical region, where there is at least one pair of the final state light mesons having an invariant mass below $O(\Lambda m_B)$ [42], $\Lambda = m_B - m_b$ being the $B$ meson and $b$ quark mass difference, is dominant. It’s reasonable that the dynamics associated with the pair of mesons can be factorized into a two-meson distribution amplitude $\Phi_{h_1, h_2}$. In the PQCD approach, one can write down the decay amplitude for a $B \to h_1 h_2 h_3$ decay symbolically in the following form [42]

$$A = \Phi_B \otimes H \otimes \Phi_{h_1, h_2} \otimes \Phi_{h_3},$$

where the hard kernel $H$ describes the dynamics of the strong and electroweak interactions in three-body hadronic decays in a similar way as the one for the two-body $B \to h_1 h_2$ decays, the function $\Phi_B$ and $\Phi_{h_3}$ are the wave functions for the $B$ meson and the final-state $h_3$ meson.

Up to now, the decays of $B$ mesons to the charmonium state plus a pion pair, such as the decay modes $B^0 \to J/\psi \pi^+ \pi^-$ [60–63], $B^0_s \to J/\psi \pi^+ \pi^-$ [64, 65], $B^0_{(s)} \to \psi(2S)\pi^+ \pi^-$ [66] and $B^0_s \to \eta_c \pi^+ \pi^-$ [16], have been measured by BESIII and LHCb Collaboration. For $B^0 \to J/\psi \pi^+ \pi^-$ decay [62], six interfering $\pi^+ \pi^-$ states, $\rho(770), f_0(500), f_2(1270), \rho'(1450), \omega(782)$ and $\rho''(1700)$, are required to give a good description of invariant mass spectra and decay angular distributions. Along with the rapid progress of the LHCb experiment, more information of the $B$ meson three-body decays involving various charmonium states ($\eta_c(1S, 2S)$ etc.) will become available. To improve the description of the invariant mass spectra, more resonant structures should be taken into account. Very recently, based on the PQCD factorization approach, we studied the $S$-wave resonance contributions to the decays $B^0_{(s)} \to \eta_c(1S, 2S)\pi^+ \pi^-$ [47, 51] and $B^0_s \to \psi(2S)\pi^+ \pi^-$ [50], as well as the $P$-wave contributions (i.e. $\rho(770), \rho'(1450)$ and $\rho''(1700)$) to the decays $B \to P_B \to P \pi \pi$ [49, 52].

In this paper, we will extend our previous analysis to the cases for the $\bar{P}$-wave resonance ($\rho, \rho'$ and $\rho''$) contributions to the three-body decays $B \to \eta_c(1S, 2S)\pi^+ \pi$. For the quasi-two-body decays $B \to \eta_c(1S, 2S)(\rho, \rho'), \eta_c(1S, 2S)\pi^+ \pi$, the relevant Feynman diagrams are illustrated in Fig. 1. The vector current time-like form factor $F_\rho$ [67] will be adopted to describe the strong interactions between the $\bar{P}$-wave resonant state $(\rho, \rho', \rho'')$ and the final-state pion pair in our work. In Sec. II, we give a brief introduction for the theoretical framework. The numerical values, some discussions and the conclusions will be given in last two sections. The explicit PQCD factorization formulas for all the decay amplitudes are collected in the Appendix.

II. FRAMEWORK

For the quasi-two-body $B \to \eta_c(1S, 2S)(\rho, \rho', \rho'') \to \eta_c(1S, 2S)\pi^+ \pi$ decays, the $B$ meson momentum $p_B$, the total momentum of the pion pair $p = p_1 + p_2$ and the final-state $\eta_c$ momentum $p_3$, can be expressed in the light-cone coordinates as the following form:

$$p_B = \frac{m_B}{\sqrt{2}}(1, 1, 0_T), \quad p = \frac{m_B}{\sqrt{2}}(1 - r^2, \eta, 0_T), \quad p_3 = \frac{m_B}{\sqrt{2}}(r^2, 1 - \eta, 0_T),$$

where $0_T = \frac{\sqrt{2m_B}}{2}$.

1 For the sake of simplicity, we generally use the abbreviation $\rho = \rho(770)$, $\rho' = \rho(1450)$, $\rho'' = \rho(1700)$ in the following sections.
where \( m_B \) is the mass of \( B \) meson, \( \eta = \frac{q^2}{(1 - r^2)m_B^2} \) with \( r = m_{\pi^+}/m_B \) and the invariant mass squared \( \omega^2 = p^2 \). In the same way, we also define the momentum \( k_B \) of the spectator quark in the \( B \) meson, the momentum \( k = zp^+ \) and \( k_3 = x_3p_3 \) for the quark in the resonant state \((\rho, \rho', \rho'')\) and in the final state \( \eta_3 \) in the following form:

\[
k_B = \left( 0, x_B \frac{m_B}{\sqrt{2}}, k_{BT} \right), \quad k = \left( z(1 - r^2) \frac{m_B}{\sqrt{2}}, 0, k_T \right), \quad k_3 = \left( r^2 x_3 \frac{m_B}{\sqrt{2}} (1 - \eta)x_3 \frac{m_B}{\sqrt{2}}, k_{3T} \right),
\]

where the parameter \( x_B, z, x_3 \) denotes the momentum fraction of the quark in each meson and runs from zero to unity. If we define \( \zeta = p_1^- / p^+ \) as one of the pion pair’s momentum fraction, other kinematic variables of the two pions can be chosen as

\[
p_1^- = (1 - \zeta) \frac{m_B}{\sqrt{2}}, \quad p_2^+ = (1 - \zeta)(1 - r^2) \frac{m_B}{\sqrt{2}}, \quad p_2^- = \zeta \frac{m_B}{\sqrt{2}}.
\]

We assume that the \( B \to \eta_c(\rho, \rho', \rho'') \to \pi \pi \) decays can proceed mainly via quasi-two-body channels, which contain a \( P \)-wave resonant state by introducing the two-pion DAs \( \Phi^P_{\pi \pi} \). As done in Ref. [46], we should introduce the time-like form factor \( F_\pi(s) \), which involves the strong interactions between the \( P \)-wave resonance and two pions, as well as elastic rescattering of pion pair to parameterize the \( P \)-wave two-pion distribution amplitudes \( \Phi^P_{\pi \pi} \). We adopt the same \( F_\pi(s) \) in this work as the one in Ref. [46], the approximate relations \( F_{\pi i}(s) \approx (f_{\rho T}^I / f_{\rho}^I)F_\pi(s) \) [46] will also be used in the following section. By taking the \( \rho - \omega \) interference and the excited states into account, the form factor \( F_\pi(s) \) can be written in the form of

\[
F_\pi(s) = \left[ \frac{\text{GS}_\rho(s, \rho_\pi, \Gamma_\rho)}{1 + c_\omega \cdot \text{BW}_\omega(s, m_\omega, \Gamma_\omega)} + \sum_i c_i \cdot \text{GS}_i(s, m_i, \Gamma_i) \right] \left[ 1 + \sum_i c_i \right]^{-1}
\]

where \( s = m^2(\pi \pi) \) is the two-pion invariant mass squared, \( i = (\rho(1450), \rho(1700), \rho(2254)) \), \( \Gamma_i \) is the decay width for the relevant resonance, \( m_{\rho, \omega, i} \) are the masses of the corresponding mesons, respectively. The function \( \text{GS}_\rho(s, m_\rho, \Gamma_\rho) \) has been parameterized in the Gounaris-Sakurai (GS) model [67] based on the Breit-Wigner (BW) model [21].

\[
\text{GS}_\rho(s, m_\rho, \Gamma_\rho) = \frac{m_\rho^2 [1 + d(m_\rho)\Gamma_\rho / m_\rho]}{m_\rho^2 - s + f(s, m_\rho, \Gamma_\rho) - im_\rho\Gamma(s, m_\rho, \Gamma_\rho)}.
\]

The explicit expressions of the resonant state function \( \text{GS}_\rho, \text{GS}_i \) and \( \text{BW}_\omega \) and the values of the involved parameters can be found for example in Ref. [68].

We here adopt the same two-pion distribution amplitude as the one being used in Ref. [46],

\[
\Phi^P_{\pi \pi} = \frac{1}{\sqrt{2N_c}} \left[ \Phi^P_{\nu \nu = -}^{I = 1}(z, \zeta) + \omega \Phi^P_{\nu \nu = -}^{I = 1}(z, \zeta) + \frac{\beta_1 \beta_2 - \beta_3 \beta_4}{w(2\zeta - 1)} \Phi^P_{\nu \nu = +}^{I = 1}(z, \zeta) \right],
\]

with

\[
\Phi^P_{\nu \nu = -}^{I = 1} = \frac{3 F_\pi(s)}{\sqrt{2N_c}} x(1 - z) \left[ 1 + a_{2p}^0 \cdot \frac{3}{2} \right] 5(1 - 2z)^2 - 1 \right] P_1(2\zeta - 1),
\]

\[
\Phi^P_{\nu \nu = -}^{I = 1} = \frac{3 F_\pi(s)}{\sqrt{2N_c}} x(1 - 2z) \left[ 1 + a_{2p}^0 \cdot \left( 10z^2 - 10z + 1 \right) \right] P_1(2\zeta - 1),
\]

\[
\Phi^P_{\nu \nu = +}^{I = 1} = \frac{3 F_\pi(s)}{\sqrt{2N_c}} (1 - 2z)^2 \left[ 1 + a_{2p}^t \cdot \frac{3}{2} \right] 5(1 - 2z)^2 - 1 \right] P_1(2\zeta - 1),
\]

where the Legendre polynomial \( P_1(2\zeta - 1) = 2\zeta - 1 \). In the numerical calculations, we will use the same set of Gegenbauer moments \( a_{2p}^0, a_{2p}^s, a_{2p}^t \) in the two-pion distribution amplitude \( \Phi^P_{\pi \pi} \), as those used in Refs. [49, 52].

\[
a_{2p}^0 = 0.30 \pm 0.05, \quad a_{2p}^s = 0.70 \pm 0.20, \quad a_{2p}^t = -0.40 \pm 0.10.
\]

### III. NUMERICAL RESULTS AND DISCUSSIONS

The following input parameters (in units of GeV) will be adopted [69] for numerical calculations,

\[
\begin{align*}
\Lambda^4_{MS} &= 0.25, \quad m_B^{\pm, 0} = 5.28, \quad m_b = 4.8, \quad m_c = 1.275 \pm 0.025, \quad m_\rho = 0.775, \quad m_{\eta_3} = 0.140, \\
m_{\pi^0} &= 0.35, \quad m_{\pi^+} = 0.135, \quad m_{\eta_3(1S)} = 2.9834, \quad m_{\eta_3(2S)} = 3.6392, \quad f_B = 0.19 \pm 0.02.
\end{align*}
\]

(12)
The values of the Wolfenstein parameters are the same as given in Ref. [69]: $\lambda = 0.22506 \pm 0.00050$, $\bar{\rho} = 0.1243 \pm 0.0198$, $\bar{\eta} = 0.356 \pm 0.011$.

For the decay $B \to \eta_c(\rho \to \pi \pi)$, the differential decay rate is written as

$$\frac{d\mathcal{B}}{ds} = \tau_B \frac{|\mathcal{A}|^2}{32\pi^3 m_B^3},$$

with the kinematic variables $|\vec{p}_1|$ and $|\vec{p}_3|$,

$$|\vec{p}_1| = \frac{1}{2\sqrt{s-4m_c^2}}, \quad |\vec{p}_3| = \frac{1}{2\sqrt{[(m_B^2 - m_c^2)^2 - 2(m_B^2 + m_c^2)s + s^2]/s}},$$

where $\tau_{B\pm} = 1.638\, \text{ps}$, $\tau_{B^0} = 1.520\, \text{ps}$ is the mean lifetime of $B^\pm$ and $B^0$ meson.

By using the differential decay rate as defined in Eq. (13) and the relevant decay amplitudes as given in the Appendix, we make the PQCD predictions for the branching ratios $\mathcal{B}(B \to \eta_c(1S, 2S)(\rho, \rho', \rho'' \to \pi \pi \pi)) \eta \pi \pi$ and find the following numerical results (in units of $10^{-6}$)

$$\mathcal{B}(B^+ \to \eta_c(1S)(\rho^+ \to \pi^+ \pi^0)) = 8.55^{+3.92}_{-2.35}(\omega_B)_{-1.18}^{+1.85}(a_{2p}^0)_{-0.29}^{+0.57} a_{2p}^0, \quad B(B^+ \to \eta_c(1S)(\rho'' - \to \pi^+ \pi^0)) = 0.15 \pm 0.03(\omega_B)_{-0.01}^{+0.02}(a_{2p}^0)_{-0.02}^{+0.01}(a_{2p}^0)_{0.02}^{+0.01}(c_{\rho''}), \quad B(B^0 \to \eta_c(1S)(\rho^+ \to \pi^+ \pi^-)) = 3.95^{+1.36}_{-1.00}(\omega_B)_{-0.35}^{+0.69}(a_{2p}^0)_{-0.25}^{+0.15}(a_{2p}^0)_{0.15}^{+0.27}(a_{2p}^0), \quad B(B^0 \to \eta_c(1S)(\rho^+ \to \pi^+ \pi^-)) = 0.11^{+0.02}_{-0.01}(\omega_B)_{-0.02}^{+0.02}(a_{2p}^0)_{0.02}^{+0.02}(a_{2p}^0)_{0.02}^{+0.02}(c_{\rho''}).$$

For the decays $B \to \eta_c(1S)(\rho \to \pi \pi)$, the first error of the PQCD predictions comes from the uncertainty of $\omega_B = (0.40 \pm 0.04)$ GeV, the following three errors are due to $a_{2p}^0 = -0.40 \pm 0.10$, $a_{2p}^0 = 0.20 \pm 0.20$, and $a_{2p}^0 = 0.30 \pm 0.05$ respectively. For the decay modes involving $\rho'$ and $\rho''$ resonant states, the fifth error results from the uncertainty of the form factor $F_\rho(s)$ as given in Eq. (5): the one induced by the uncertainties of the coefficients $c_{\rho'} = (0.158 \pm 0.018) \cdot \exp[i(3.76 \pm 0.10)]$ and $c_{\rho''} = (0.068 \pm 0.009) \cdot \exp[i(1.39 \pm 0.20)]$ [68]. One can see from the PQCD predictions as given in Eqs. (15.16) that the major error in our approach comes from the parameter $\omega_B$ in $B$ meson wave function, which can reach 30–50%. The error from the coefficient $c_{\rho''}$ is around 20–30% for the relevant decay modes. The possible errors due to the uncertainties of $m_c$ and CKM matrix elements are very small and can be neglected safely.

For the considered decay modes $B \to \eta_c(1S, 2S)(\pi \pi \pi)$, the dynamical limit on the value of invariant mass $\omega$ is $2m_c \leq \omega \leq (m_B - m_{\eta_c(1S, 2S)})$. For $B \to \eta_c(2S)(\pi \pi \pi)$ decays, since $m(\rho'') > \omega_{max} = (m_B - m_{\eta_c(2S)})$, the resonant $\rho''$ cannot contribute to this decay. We therefore have the following PQCD predictions for the branching ratios (in units of $10^{-6}$):

$$\mathcal{B}(B^+ \to \eta_c(2S)(\rho^+ \to \pi^+ \pi^0)) = 3.82^{+1.45}_{-1.00}(\omega_B)_{-0.49}^{+0.49}(a_{2p}^0)_{-0.58}^{+0.58}(a_{2p}^0)_{0.14}^{+0.14}(a_{2p}^0), \quad B(B^+ \to \eta_c(2S)(\rho'' - \to \pi^+ \pi^0)) = 0.13 \pm 0.03(\omega_B)_{-0.01}^{+0.01}(a_{2p}^0)_{0.02}^{+0.02}(a_{2p}^0)_{0.02}^{+0.02}(a_{2p}^0)_{0.02}^{+0.02}(c_{\rho''}).$$

The errors in above equations have the same meaning as those in Eqs. (15.16).

For the phenomenological study of the two-body decays $B \to \eta_c\rho'$ and $B \to \eta_c\rho''$, we currently still lack the distribution amplitudes of the states $\rho'$ and $\rho''$. But one can extract out the branching fractions for the two-body decays $B \to \eta_c\rho'(\rho'')$ from those PQCD predictions for the quasi-two-body processes $B \to \eta_c\rho'(\rho'') \to \eta_c \pi \pi$ with the input of $\Gamma_{\rho' \to \pi \pi}/\Gamma_{\rho'}$ and $\Gamma_{\rho'' \to \pi \pi}/\Gamma_{\rho''}$. We know that there is a relation of the decay rates between the quasi-two-body and the corresponding two-body decay modes

$$\mathcal{B}(B \to \eta_c(\rho' \to \pi \pi)) = \mathcal{B}(B \to \eta_c(\rho'' \to \pi \pi)) \cdot \mathcal{B}(\rho' \to \pi \pi), \quad \mathcal{B}(B \to \eta_c(\rho'' \to \pi \pi)) \to \eta_c \pi \pi.$$
FIG. 2: (a) The PQCD prediction for the differential decay rate of $B^+ \to \eta_c(1S)(\rho^+ \to \pi^+ \pi^0)$ decay with the inclusion of all contributions from $\rho(770)$, $\rho(1450)$ and $\rho(1700)$. (b) The differential decay rate of $B^+ \to \eta_c(2S)(\rho^+ \to \pi^+ \pi^0)$ decay when the possible contributions from both $\rho(770)$ and $\rho(1450)$ are included.

Here the individual errors from different sources have been added in quadrature.

In Fig. 2(a) and 2(b), we show the $\omega$-dependence of the differential decay rate $dB(B^+ \to \eta_c(1S)\pi^+\pi^0)/d\omega$ and $dB(B^+ \to \eta_c(2S)\pi^+\pi^0)/d\omega$ after the inclusion of the possible contributions from the resonant states. For $B^+ \to \eta_c(1S)\pi^+\pi^0$ decay, the dynamical limit is $0.28\text{GeV} \leq \omega \leq 2.28\text{GeV}$: all three resonant states ($\rho, \rho', \rho''$) can contribute. For $B^+ \to \eta_c(2S)\pi^+\pi^0$ decay, however, the limit is $0.28\text{GeV} \leq \omega \leq 1.60\text{GeV}$: which means that the heavier $\rho''$ can not contribute to this decay mode.

In Fig. 3(a) and 3(b), we show the $\omega$-dependence of the differential decay rate of the five considered decay modes. In Fig. 3(a), we show the PQCD prediction for $dB/d\omega$ for $B^+ \to \eta_c(1S)(\rho(770) \to \pi^+\pi^0)$ decay (the solid curve) and $B^+ \to \eta_c(2S)(\rho(770) \to \pi^+\pi^0)$ decay (the dotted curve), respectively. In Fig. 3(b), similarly, we show the PQCD prediction for $dB/d\omega$ for $B^+ \to \eta_c(1S)(\rho(1450) \to \pi^+\pi^0)$ decay (the solid curve), $B^+ \to \eta_c(1S)(\rho(1700) \to \pi^+\pi^0)$ decay (the short-dashed curve) and $B^+ \to \eta_c(2S)(\rho(1450) \to \pi^+\pi^0)$ decay (the dotted curve), respectively.

From the curves as illustrated in Fig. 2 and Fig. 3 and the PQCD predictions for the decay rates as given in Eqs. (15-21), we have the following observations:

(1) According to the full Dalitz-plot analysis to the $B \to J/\psi \pi^+\pi^-$ decay by the LHCb experiment [62], the dominant contributions come from the $P$-wave resonance $\rho(770)$ and $S$-wave resonance $f_0(500)$. The relative rate between the two contributions was measured to be in the range

$$1.4 \leq R_{J/\psi} \approx \frac{B(B^0 \to J/\psi(\rho(770) \to \pi^+\pi^-))}{B(B^0 \to J/\psi(f_0(500) \to \pi^+\pi^-))} \leq 1.9,$$

(22)
here only the fraction of the helicity $\lambda = 0$ component of the $P$-wave resonance has been taken to account. Because of the analogous properties of the $\eta_c$ and $J/\psi$ meson, it is reasonable for us to expect a similar invariant mass distribution for $B \to \eta_c \pi^+\pi^-$ decay when compared with that of the $B \to J/\psi \pi^+\pi^-$ decay.

In a previous work [47], we calculated the $S$-wave resonance contributions to $B^0 \to \eta_c(1S)\pi^+\pi^-$ decay, and confirmed that the largest contribution is from the $f_0(500)$. The PQCD predictions for the branching ratios are

$$B(B^0 \to \eta_c(1S)(f_0(500) \to \pi^+\pi^-)\approx \frac{1.53^{+0.76}_{-0.35} \times 10^{-6}}{2.31^{+0.96}_{-0.48} \times 10^{-6}}.$$  

(23)

By using the PQCD prediction as given in Eq. (16), one can define the relative ratio of the $P$-wave and $S$-wave contribution as the following form:

$$R_1 = \frac{B(B^0 \to \eta_c(1S)(\rho(770) \to \pi^+\pi^-)}{B(B^0 \to \eta_c(1S)(f_0(500) \to \pi^+\pi^-)} \approx \frac{2.6}{1.7}.$$  

(24)

The ratio $R_1$ agrees well with the ratio $R_{J/\psi}$ for the case of $B \to J/\psi \pi^+\pi^-$ decay and will be tested by the future LHCb and Belle-II experiment.

(2) From Fig. 2(a), one can see one prominent $\rho$ peak, a shoulder around the $\rho(1450)$ and a deep dip near $\omega \approx 1.6$ GeV, followed by an enhancement (the second lower but wider peak) in the $\rho(1700)$ region. Because the differential decay rate $dB/d\omega$ depends on the values of $|F_\pi|^2$, the position of the first peak and deep dip, as well as the pattern of the whole curve do agree well with the curve in Fig. 45 of the Ref. [68], where the pion form factor-squared $|F_\pi|^2$ measured by $BABAR$ are illustrated as a function of $\sqrt{s'}$ (i.e.$m(\pi\pi)$) in the region from 0.3 to 3 GeV.

The first dip around $\omega \approx 1.6$ GeV is in fact caused by the strong destructive interference between the resonant state $\rho(1450)$ and $\rho(1700)$. Taking $B^+ \to \eta_c(1S)\rho(1450) \to \eta_c(1S)\pi^+\pi^0$ and $B^+ \to \eta_c(1S)\rho(1700) \to \eta_c(1S)\pi^+\pi^0$ decay as an example, we calculated the interference terms between $\rho(1450)$ and $\rho(1700)$ amplitudes and found the large negative contribution to the total branching ratio. Numerically, the PQCD predictions for the individual decay rate and the interference term are:

$$B(B^+ \to \eta_c(1S)(\rho(1450) \to \pi^+\pi^0) \approx 9.31 \times 10^{-7},$$

$$B(B^+ \to \eta_c(1S)(\rho(1700) \to \pi^+\pi^0) \approx 2.41 \times 10^{-7},$$

interference term $\approx -6.45 \times 10^{-7}$.  

(25)

By comparing with other two individual contributions, we find that the interference term is indeed large and negative, which leads to the first deep dip in the region around $\omega \approx 1.6$ GeV, as illustrated in Fig. 2(a).

(3) From Fig. 3(a), one can see easily that the differential decay rate $dB/d\omega$ for $B^+ \to \eta_c(2S)(\rho(770) \to \pi^+\pi^0)$ decay is always smaller than that for $B^+ \to \eta_c(1S)(\rho(770) \to \pi^+\pi^0)$ decay, mainly due to the difference between the distribution amplitudes of the $\eta_c(1S)$ and $\eta_c(2S)$: the tighter phase space and the smaller decay constant of the $\eta_c(2S)$ state result in the suppression as shown in Fig. 3(a).

From the numerical results as given in Eqs. (15-18), we obtain the relative ratio $R_2$ between the branching ratios of $B$ meson decays involving $\eta_c(2S)$ and $\eta_c(1S)$ respectively,

$$R_2(\eta_c) = \frac{B(B^+ \to \eta_c(2S)(\rho(770) \to \pi^+\pi^0)}{B(B^+ \to \eta_c(1S)(\rho(770) \to \pi^+\pi^0)} \approx 0.45.$$  

(26)

Owing to the same quark structures between $\eta_c$ and $J/\psi$ mesons, one generally expect that the $B \to \eta_c \pi\pi$ decays should be similar in nature with the decays $B \to J/\psi \pi\pi$: i.e. $R_2(\eta_c) \approx R_2(\eta_c)$). This general expectation, in fact, agrees well with the LHCb measurement [66]:

$$R_2(J/\psi)_{LHCb} = \frac{B(B^0 \to \psi(2S)\pi^+\pi^-)}{B(B^0 \to J/\psi\pi^+\pi^-)} = 0.56 \pm 0.09.$$  

(27)

Here the main contribution also come from $B^0 \to J/\psi\rho(770) \to J/\psi\pi^+\pi^-$.  

(4) From Fig. 3(b), one can see easily that the differential decay rate $dB/d\omega$ for $B^+ \to \eta_c(2S)(\rho(1450) \to \pi^+\pi^0)$ and $B^+ \to \eta_c(1S)(\rho(1700) \to \pi^+\pi^0)$ decay are much smaller than that for $B^+ \to \eta_c(1S)(\rho(1450) \to \pi^+\pi^0)$ decay. From
the numerical results as given in Eqs. (15-18), we find the following relative ratios

\[
R_3(\eta_c(1S)) = \frac{\mathcal{B}(B^+ \rightarrow \eta_c(1S)|[\rho(1700) \rightarrow \pi^+ \pi^0])}{\mathcal{B}(B^+ \rightarrow \eta_c(1S)|[\rho(1450) \rightarrow \pi^+ \pi^0])} \approx 0.26 \wedge \tag{28}
\]

\[
R_4(\eta_c(2S)) = \frac{\mathcal{B}(B^+ \rightarrow \eta_c(2S)|[\rho(1450) \rightarrow \pi^+ \pi^0])}{\mathcal{B}(B^+ \rightarrow \eta_c(1S)|[\rho(1450) \rightarrow \pi^+ \pi^0])} \approx 0.16 \wedge \tag{29}
\]

The ratio \(R_3(\eta_c(1S))\) is mainly governed by the difference between the parameters \((c_{\rho'}, c_{\rho''})\) and the functions \(G_{\rho', \rho''}\), while the ratio \(R_4(\eta_c(2S))\) has a strong dependence on the distribution amplitudes of the \(\eta_c(1S)\) and \(\eta_c(2S)\).

Based on the similarity between \((\eta_c(1S), \eta_c(2S))\) and \((J/\psi, \psi(2S))\) mesons, furthermore, it also be reasonable for us to expect similar \(R_3\) and \(R_4\) ratios for the cases of \(B \rightarrow J/\psi \pi \pi\) and \(B \rightarrow \psi(2S) \pi \pi\) decays. Fortunately, the ratio \(R_3(J/\psi)\) analogous to \(R_3(\eta_c(1S))\) has been measured by LHCb Collaboration recently [62]. If we take only the contributions from the longitudinal component \(\rho(1450)_0\) and \(\rho(1700)_0\) into account, we can obtain the value of the ratio \(R_3(J/\psi)\) from the “Fit fractions of contributing components” as listed in Table VI of Ref. [62]:

\[
R_3(J/\psi) = \frac{\mathcal{B}(B^0 \rightarrow J/\psi|[\rho(1700)_0 \rightarrow \pi^+ \pi^-])}{\mathcal{B}(B^0 \rightarrow J/\psi|[\rho(1450)_0 \rightarrow \pi^+ \pi^-])} \approx 0.29 \pm 0.16 \wedge \text{in Best - Model,} \tag{30}
\]

which indeed agrees very well with \(R_3(\eta_c(1S))\) \(\approx 0.26\). Other predictions will be tested by the forthcoming LHCb and Belle-II experimental measurements.

(5) For \(B^+ \rightarrow \eta_c(1S)|[\rho(770) \rightarrow \pi^+ \pi^0]\) decay, the main portion of the branching ratios lies in the region around the pole mass of \(\rho(770)\) meson, as can be seen clearly in Fig. 3(a). The central values of the branching ratio \(B\) are \(4.6 \times 10^{-6}\) and \(6.4 \times 10^{-6}\) when the integration over \(\omega\) is limited in the range of \(\omega = [m_\rho - 0.5 \Gamma_\rho, m_\rho + 0.5 \Gamma_\rho]\) or \(\omega = [m_\rho - \Gamma_\rho, m_\rho + \Gamma_\rho]\) respectively, which amount to 54% and 75% of the total branching ratio \(B = 8.6 \times 10^{-6}\) as listed in Eq. (15).

**IV. CONCLUSION**

In this work, we studied the contributions from the \(P\)-wave resonance \(\rho(770), \rho(1450)\) and \(\rho(1700)\) to the \(B \rightarrow \eta_c(1S, 2S) \pi \pi\) decays in the PQCD framework. We calculated the branching ratios of the quasi-two-body decays \(B \rightarrow \eta_c(1S, 2S)|[\rho(770), \rho(1450), \rho(1700)]\) \(\pi \pi\) by utilizing the vector current time-like form factor \(F_{\omega}(s)\) with the inclusion of the final state interactions between the pion pair in the resonant regions.

From the analytical analysis and the numerical results, we found the following points:

1. The PQCD predictions for the branching ratios of the considered quasi-two-body decays are generally in the order of \(10^{-7}\) to \(10^{-6}\). We obtained the theoretical predictions for the branching ratios of the two-body decays \(B \rightarrow \eta_c(1S, 2S)|[\rho(1450), \rho(1700)]\) out of the PQCD predictions for the corresponding quasi-two-body decay modes, which will be tested by future LHCb and Belle II experiments.

2. The whole pattern of the \(\omega\)-dependence of the pion form factor-squared \(|F_{\omega}|^2\) measured by the BABAR Collaboration could be understood based on our studies, as illustrated in Fig. 2(a). The dominant contribution comes from the \(\rho(770)\) resonance, while the deep dip around \(\omega \approx 1.6\) GeV is induced by the strong destructive interference between the contribution from \(\rho(1450)\) and \(\rho(1700)\).

3. The general expectation based on the similarity between \(B \rightarrow \eta_c \pi \pi\) and \(B \rightarrow J/\psi \pi \pi\) decays are confirmed: the value of newly defined ratio \(R_2(\eta_c) \approx 0.45\) agrees well with the measured value \(R_2(J/\psi) = 0.56 \pm 0.09\) as reported by LHCb experiments.

4. The new ratios \(R_3(\eta_c(1S))\) and \(R_4(\eta_c(2S))\) among the branching ratios of the considered decay modes are defined, and the PQCD predictions for their values will be tested by future experiments.

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Appendix A: Decay amplitudes

The widely used wave function of $B$ meson is adopted as the one being used in Refs. [70–76],

\[ \Phi_B = \frac{i}{\sqrt{2N_c}}(\bar{p}_B + m_B)\gamma_5\phi_B(k_B), \] (A1)

with the distribution amplitude

\[ \phi_B(x, b) = N_B x^2 (1 - x)^2 \exp \left[ -\frac{M_B^2 x^2}{2\omega_B} - \frac{1}{2}(\omega_B b)^2 \right], \] (A2)

where $N_B$ is the normalization factor defined through the normalization relation $\int_0^1 dx \phi_B(x, b = 0) = f_B/(2\sqrt{6})$. We also set $\omega_B = 0.40 \pm 0.04$ GeV in the numerical calculations.

For the final-state $\eta_c(1S, 2S)$, its wave function can be written as

\[ \Psi_{\eta_c} = \frac{1}{\sqrt{2N_c}} \gamma_5 [\bar{p}_B \psi_v + m_{\eta_c} \psi_s], \] (A3)

where the twist-2 and twist-3 distribution amplitudes $\psi_v$ and $\psi_s$ for the $\eta_c(1S, 2S)$ meson are parameterized as [77, 78]

\[ \psi_v(x, b) = \frac{f_{\eta_c}}{2\sqrt{6}} N_v x \bar{x} T(x) \cdot \exp \left[ -x \bar{x} \frac{m_c}{\omega} [\omega^2 b^2 + (x - \bar{x})^2] \right], \]

\[ \psi_s(x, b) = \frac{f_{\eta_c}}{2\sqrt{6}} N_s T(x) \cdot \exp \left[ -x \bar{x} \frac{m_c}{\omega} [\omega^2 b^2 + (x - \bar{x})^2] \right], \] (A4)

with the function $T(x) = 1$ for the meson $\eta_c(1S)$, and $T(x) = 1 - 4B^2 m_c \omega x \bar{x} + m_c (x - \bar{x})^2 / (\omega x \bar{x})$ for the meson $\eta_c(2S)$. The normalization constants $N_v$ and $N_s$ can be determined by the relation $\int_0^1 dx \psi_i(x, b = 0) dx = f_{\eta_c}/(2\sqrt{6})$. The decay constant $f_{\eta_c(1S)} = 0.42 \pm 0.05$ GeV and $\omega = 0.6 \pm 0.1$ GeV are adopted for $\eta_c(1S)$ meson, while $f_{\eta_c(2S)} = 0.243^{+0.079}_{-0.111}$ GeV and $\omega = 0.2 \pm 0.1$ GeV for $\eta_c(2S)$ meson.

The total decay amplitudes for the considered decay modes $B \to \eta_c(1S, 2S)\pi \pi$ in this work are given as follows:

\[ A(B^+ \to \eta_c(1S, 2S)\pi^+ \pi^-) = \frac{G_F}{\sqrt{2}} \left\{ V_{cb}^* V_{td} \left[ F^{LL} + M^{LL} \right] \right. \] 

\[ - V_{tb}^* V_{td} \left[ F^{LL} + F^{LR} + M^{LL} + M^{SP} \right] \right\}, \] (A5)

\[ A(B^0 \to \eta_c(1S, 2S)\pi^+ \pi^-) = \frac{1}{\sqrt{2}} A(B^+ \to \eta_c(1S, 2S)(\pi^+ \pi^-)) , \] (A6)

where $G_F = 1.16639 \times 10^{-5}$ GeV$^{-2}$ is the Fermi coupling constant and $V_{ij}$'s are the Cabibbo-Kobayashi-Maskawa matrix elements. The functions ($F^{LL}, F^{LL}, F^{LR}, M^{LL}, M^{SP}$) appeared in above equations are the individual decay amplitudes corresponding to different currents, the relevant Wilson coefficients have been included in $F^{LL}$ and other functions, and their explicit expressions can also be found in Ref. [47].

[1] B. Aubert et al. (BABAR Collaboration), Measurements of the branching fractions of charged B decays to $K^\pm \pi^\mp \pi^\mp$ final states, Phys. Rev. D 70, 092001 (2004).
[2] B. Aubert et al. (BABAR Collaboration), Dalitz-plot analysis of the decays $B^\pm \to K^\pm \pi^\mp \pi^\mp$, Phys. Rev. D 72, 072003 (2005).
[3] B. Aubert et al. (BABAR Collaboration), Time-dependent amplitude analysis of $B^0 \to K^0 \pi^+ \pi^-$, Phys. Rev. D 80, 112001 (2009).
[4] B. Aubert et al. (BABAR Collaboration), Dalitz plot analysis of $B^\pm \to \pi^\pm \pi^\mp \pi^\mp$ decays, Phys. Rev. D 79, 072006 (2009).
[5] B. Aubert et al. (BABAR Collaboration), Dalitz plot analysis of the decay $B^0(B^\pm) \to K^\pm \pi^\mp \pi^0$, Phys. Rev. D 78, 052005 (2008).
[6] A. Garmash et al. (Belle Collaboration), Dalitz analysis of three-body charmless $B^{\pm} \to K^{0} \pi^\mp \pi^\mp$ decay, Phys. Rev. D 75, 012006 (2007).
[7] A. Garmash et al. (Belle Collaboration), Evidence for Large Direct CP Violation in $B^\pm \to \rho(770)^0 K^\pm$ from Analysis of Three-Body Charmless $B^\pm \to K^{\pm} \pi^\mp \pi^\mp$ Decays, Phys. Rev. Lett. 96, 251803 (2006).
[8] A. Garmash et al. (Belle Collaboration), Dalitz analysis of the three-body charmless decays $B^+ \to K^+ \pi^0 \pi^-$ and $B^+ \to K^+ K^+ K^-$, Phys. Rev. D 71, 092003 (2005).
[9] J. Dalseno et al. (Belle Collaboration), Time-dependent Dalitz plot measurement of CP parameters in $B^0 \rightarrow K_S^0 \pi^+ \pi^-$ decays, Phys. Rev. D 79, 072004 (2009).

[10] R. Aaij et al. (LHCb Collaboration), Implications of LHCb measurements and future prospects, Eur. Phys. J. C 73, 2373 (2013).

[11] R. Aaij et al. (LHCb Collaboration), Measurement of CP Violation in the Phase Space of $B^{\pm} \rightarrow K^{\pm} \pi^+ \pi^-$ and $B^{\pm} \rightarrow K^{\pm} K^{+} K^{-}$ decays, Phys. Rev. Lett. 111, 101801 (2013).

[12] R. Aaij et al. (LHCb Collaboration), Study of $B_S^{(s)} \rightarrow K_S^0 h^+ h^-$ decays with first observation of $B_S^{0} \rightarrow K_S^0 K^{+} K^{-}$ and $B_S^{0} \rightarrow K_S^0 \pi^+ \pi^-$. JHEP 10, 143 (2013).

[13] R. Aaij et al. (LHCb Collaboration), Measurements of CP violation in the three-body phase space of charmless $B^{\pm}$ decays, Phys. Rev. D 90, 112004 (2014).

[14] R. Aaij et al. (LHCb Collaboration), Measurement of CP Violation in the Phase Space of $B^{\pm} \rightarrow K^{\pm} K^{0} \pi^\mp$ and $B^{\pm} \rightarrow \pi^\pm \pi^\mp \pi^\mp$ decays, Phys. Rev. Lett. 112, 011801 (2014).

[15] R. Aaij et al. (LHCb Collaboration), Observation of the decay $B^0 \rightarrow \phi \pi^+ \pi^-$ and evidence for $B^0 \rightarrow \phi \pi^+ \pi^-$. Phys. Rev. D 95, 012006 (2017).

[16] R. Aaij et al. (LHCb Collaboration), Observation of the decay $B^0 \rightarrow \eta_\ell \phi$ and evidence for $B^0 \rightarrow \eta_\ell \pi^+ \pi^-$. JHEP 07, 021 (2017).

[17] R. H. Dalitz, On the analysis of $\tau$-meson data and the nature of the $\tau$-meson, Phil. Mag. 44, 1068 (1953).

[18] R. H. Dalitz, Decay of $\tau$ Mesons of Known Charge, Phys. Rev. 94, 1046 (1954).

[19] R. M. Sternheimer and S. J. Lindenbaum, Extension of the Isobaric Nucleon Model for Pion Production in Pion-Nucleon, Nucleon-Nucleon, and Antinucleon-Nucleon Interactions, Phys. Rev. 123, 333 (1961).

[20] D. Herndon, P. Soding and R. J. Cashmore, Generalized isobar model formalism, Phys. Rev. D 11, 3165 (1975).

[21] G. Breit and E. Wigner, Capture of Slow Neutrons, Phys. Rev. 49, 519 (1936).

[22] D. V. Bugg, The mass of the $\sigma$ pole, J. Phys. G 34, 151 (2007).

[23] I. Bediaga, T. Frederico and O. Lourenço, CP violation and CPT invariance in $B^{\pm}$ decays with final state interactions, Phys. Rev. D 89, 094013 (2014).

[24] I. Bediaga and P. C. Magalhães, Final state interaction on $B^{\pm} \rightarrow \pi^- \pi^+ \pi^+$, arXiv:1512.09284 [hep-ph].

[25] X. W. Kang, B. Kubis, C. Hanhart and U. G. Meißen, $B_{s}^{0}$ decays and the extraction of $|V_{ub}|$, Phys. Rev. D 89, 053015 (2014).

[26] S. Kräntl, T. Mannel and J. Virto, Three-body non-leptonic $B$ decays and QCD factorization, Nucl. Phys. B 899, 247 (2015).

[27] A. Furman, R. Kamiński, L. Leśniak and B. Loiseau, Long-distance effects and final state interactions in $B \rightarrow \pi\pi K$ and $B \rightarrow K K\bar{K}$ decays, Phys. Lett. B 622, 207 (2005).

[28] B. El-Bennich, A. Furman, R. Kamiński, L. Leśniak and B. Loiseau, Interference between $f_0(980)$ and $\rho(770)$ resonances in $B \rightarrow \pi^+ \pi^- K$ decays, Phys. Rev. D 74, 114009 (2006).

[29] B. El-Bennich, A. Furman, R. Kamiński, L. Leśniak, B. Loiseau, and B. Moussallam, CP violation and kaon-pion interactions in $B \rightarrow K^-\pi^+\pi^-$ decays, Phys. Rev. D 79, 094005 (2009).

[30] H. Y. Cheng and K. C. Yang, Nonresonant three-body decays of $D$ and $B$ mesons, Phys. Rev. D 66, 054015 (2002).

[31] H. Y. Cheng, C. K. Chua and A. Soni, Charmless three-body decays of $B$ mesons, Phys. Rev. D 76, 094006 (2007).

[32] H. Y. Cheng, C. K. Chua and Z. Q. Zhang, Direct CP violation in charmless three-body decays of $B$ mesons, Phys. Rev. D 94, 094015 (2016).

[33] Z. H. Zhang, X. H. Guo and Y. D. Yang, CP violation in $B^{\pm} \rightarrow \pi^\pm \pi^\mp \pi^-$ in the region with low invariant mass of one $\pi^+ \pi^-\pi^-$ pair, Phys. Rev. D 87, 076007 (2013).

[34] M. Gronau and J. L. Rosner, Symmetry relations in charmless $B \rightarrow PPP$ decays, Phys. Rev. D 72, 094031 (2005).

[35] M. Gronau, $U$-spin breaking in $CP$ asymmetries in $B$ decays, Phys. Lett. B 727, 136 (2013).

[36] G. Engelhard, Y. Nir and G. Raz, $SU(3)$ relations and the $CP$ asymmetry in $B \rightarrow K_S K_S K_S$, Phys. Rev. D 72, 075013 (2005).

[37] M. Imbeault and D. London, $SU(3)$ breaking in charmless $B$ decays, Phys. Rev. D 84, 056002 (2011).

[38] B. Bhattacharya, M. Gronau and J. L. Rosner, CP asymmetries in three-body $B^{\pm}$ decays to charged pions and kaons, Phys. Lett. B 726, 337 (2013).

[39] D. Xu, G. N. Li and X. G. He, $U$-spin analysis of CP violation in $B^-$ decays into three charged light pseudoscalar mesons, Phys. Lett. B 728, 579 (2014).

[40] D. Xu, G. N. Li and X. G. He, Large $SU(3)$ breaking effects and CP violation in $B^+$ decays into three charged octet pseudoscalar mesons, Int. J. Mod. Phys. A 29, 1450011 (2014).

[41] X. G. He, G. N. Li and D. Xu, $SU(3)$ and isospin breaking effects on $B \rightarrow PPP$ amplitudes, Phys. Rev. D 91, 014029 (2015).

[42] C. H. Chen and H. N. Li, Three-body nonleptonic $B$ decays in perturbative QCD, Phys. Lett. B 561, 258 (2003).

[43] C. H. Chen and H. N. Li, Vector-pseudoscalar two-meson distribution amplitudes in three-body $B$ meson decays, Phys. Rev. D 70, 054006 (2004).

[44] W. F. Wang, H. C. Hu, H. N. Li and C. D. Lü, Direct $CP$ asymmetries of three-body $B$ decays in perturbative QCD, Phys. Rev. D 89, 074031 (2014).

[45] W. F. Wang, H. N. Li, W. Wang and C. D. Lü, S-wave resonance contributions to the $B^{0}_{s}(s) \rightarrow J/\psi \pi^+ \pi^-$ and $B_s \rightarrow \pi^+ \pi^- \mu^+ \mu^-$ decays, Phys. Rev. D 91, 094024 (2015).

[46] W. F. Wang and H. N. Li, Quasi-two-body decays $B \rightarrow K_{p} \rightarrow K_{p} \pi$ in perturbative QCD approach, Phys. Lett. B 763, 29 (2016).

[47] Y. Li, A. J. Ma, W. F. Wang and Z. J. Xiao, The S-wave resonance contributions to the three-body decays $B^{0}_{s}(s) \rightarrow \eta_{c} f_{0}(X) \rightarrow \eta_{c} \pi^+ \pi^-$ in perturbative QCD approach, Eur. Phys. J. C 76, 675 (2016).

[48] A. J. Ma, Y. Li, W. F. Wang and Z. J. Xiao, The quasi-two-body decays $B_{s}(s) \rightarrow (D(s), \bar{D}(s)) \rho \rightarrow (D(s), \bar{D}(s)) \pi \pi$ in the perturbative QCD factorization approach, Nucl. Phys. B 923, 54 (2017).

[49] Y. Li, A. J. Ma, W. F. Wang and Z. J. Xiao, Quasi-two-body decays $B_{s}(s) \rightarrow P \rho \rightarrow P \pi \pi$ in the perturbative QCD approach, Phys. Rev. D 95, 056008 (2017).
[50] R. Zhou, Y. Li and W. F. Wang, The S-wave resonance contributions in the $B^0 \to \psi(2S, 3S)$ plus pion pair, Eur. Phys. J. C 77, 199 (2017).

[51] A. J. Ma, Y. Li, W. F. Wang and Z. J. Xiao, S-wave resonance contributions to the $B^0_{(s)} \to \eta_c(2S)\pi^+\pi^-$ in the perturbative QCD factorization approach, Chin. Phys. C 41, 083105 (2017).

[52] Y. Li, A. J. Ma, W. F. Wang and Z. J. Xiao, Quasi-two-body decays $B_{(s)} \to P\rho(1450), P\rho^*(1700) \to P\pi\pi$ in the perturbative QCD approach, Phys. Rev. D 96, 036014 (2017).

[53] D. Müller, D. Robaschik, B. Geyer, F. -M. Dittes and J. Hořejší, Wave Functions, Evolution Equations and Evolution Kernels from Light-Ray Operators of QCD, Fortschr. Physik. 42, 101 (1994).

[54] M. Diehl, T. Gousset, B. Pire and O. Teryaev, Probing Partonic Structure in $\gamma^*\gamma \to \pi\pi$ near Threshold, Phys. Rev. Lett. 81, 1782 (1998).

[55] M. Diehl, T. Gousset and B. Pire, Exclusive production of pion pairs in $\gamma^*\gamma$ collisions at large $Q^2$, Phys. Rev. D 62, 073014 (2000).

[56] Ph. Hägele, B. Pire, L. Szymanowski and O. V. Teryaev, Pomeron-Odderon interference effects in electroproduction of two pions, Eur. Phys. J. C 26, 261 (2002).

[57] M. V. Polyakov, Hard exclusive electroproduction of two pions and their resonances, Nucl. Phys. B 555, 231 (1999).

[58] A. G. Grozin, On Wave Functions Of Mesonic Pairs And Mesonic Resonances, Sov. J. Nucl. Phys. 38, 289-292 (1983).

[59] A. G. Grozin, One- and two-particle wave functions of multihadron systems, Theor. Math. Phys. 69, 1109-1121 (1986).

[60] B. Aubert et al. (BABAR collaboration), Measurement of the $B^0 \to J/\psi\pi^+\pi^-$ Branching Fraction, Phys. Rev. Lett. 90, 091801(2003).

[61] R. Aaij et al. (LHCb Collaboration), Analysis of the resonant components in $B^0 \to J/\psi\pi^+\pi^-$, Phys. Rev. D 87, 052001 (2013).

[62] R. Aaij et al. (LHCb Collaboration), Measurement of the resonant and $CP$ components in $B^0 \to J/\psi\pi^+\pi^-$ decays, Phys. Rev. D 90, 012003 (2014).

[63] R. Aaij et al. (LHCb Collaboration), Measurement of the $CP$-violating phase $\beta$ in $B^0 \to J/\psi\pi^+\pi^-$ decays and limits on penguin effects, Phys. Lett. B 742, 38 (2015).

[64] R. Aaij et al. (LHCb Collaboration), Analysis of the resonant components in $\bar{B}^0 \to J/\psi\pi^+\pi^-$, Phys. Rev. D 86, 052006 (2012).

[65] R. Aaij et al. (LHCb Collaboration), Measurement of resonant and $CP$ components in $\bar{B}^0 \to J/\psi\pi^+\pi^-$ decays, Phys. Rev. D 89, 092006 (2014).

[66] R. Aaij et al. (LHCb Collaboration), Observations of $B^0 \to \psi(2S)\eta$ and $B^0_{(s)} \to \psi(2S)\pi^+\pi^-$, Nucl. Phys. B 871, 403 (2013).

[67] G. J. Gounaris and J. J. Sakurai, Finite-Width Corrections to the Vector-Meson-Dominance Prediction for $\rho \to e^+e^-$, Phys. Rev. Lett. 21, 244 (1968).

[68] J. P. Lees et al. (BABAR Collaboration), Precise measurement of the $e^+e^- \to \pi^+\pi^- (\gamma)$ cross section with the initial-state radiation method at BABAR, Phys. Rev. D 86, 032013 (2012).

[69] C. Patrignani et al. (Particle Data Group), Review of Particle Physics, Chin. Phys. C 40, 100001 (2016).

[70] Y. Y. Keum, H. N. Li and A. I. Sanda, Penguin enhancement and $B \to K\pi$ decays in perturbative QCD, Phys. Rev. D 63, 054008 (2001).

[71] T. Kurimoto, H. N. Li and A. I. Sanda, Leading-power contributions to $B \to \pi\rho$ transition form factors, Phys. Rev. D 65, 014007 (2001).

[72] C. D. Lü and M. Z. Yang, $B$ to light meson transition form-factors calculated in perturbative QCD approach, Eur. Phys. J. C 28, 515 (2003).

[73] H. N. Li, QCD aspects of exclusive $B$ meson decays, Prog.Part. & Nucl. Phys. 51, 85 (2003) and references therein.

[74] Z. J. Xiao, W. F. Wang and Y. Y. Fan, Revisiting the pure annihilation decays $B_s \to \pi^+\pi^-$ and $B^0 \to K^+K^-$: The data and the perturbative QCD predictions, Phys. Rev. D 85, 094003 (2012).

[75] Y. Y. Keum, H. N. Li and A. I. Sanda, Fat penguins and imaginary penguins in perturbative QCD, Phys. Lett. B 504, 6 (2001).

[76] C. D. Lü, K. Ukai and M. Z. Yang, Branching ratio and $CP$ violation of $B \to \pi\pi$ decays in the perturbative QCD approach, Phys. Rev. D 63, 074009 (2001).

[77] J. F. Sun, D. S. Du and Y. L. Yang, Study of $B_c \to J/\psi\pi, \eta_c\pi$ decays with perturbative QCD approach, Eur. Phys. J. C 60, 107 (2009).

[78] R. Zhou, W. F. Wang, G. X. Wang, L. H. Song and C. D. Lü, The $B_c \to \psi(2S)\pi, \eta_c(2S)\pi$ decays in the perturbative QCD approach, Eur. Phys. J. C 75, 293 (2015).