Conformally flat slices of asymptotically flat spacetimes

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Abstract
For mathematical convenience initial data sets in numerical relativity are often taken to be conformally flat. Employing the dual-foliation formalism, we investigate the physical consequences of this assumption. Working within a large class of asymptotically flat spacetimes we show that the ADM linear momentum is governed by the leading Lorentz part of a boost even in the presence of supertranslation-like terms. Following up, we find that in spacetimes that are asymptotically flat, and admit spatial slices with vanishing linear momentum that are sufficiently close to conformal flatness, any boosted slice can not be conformally flat. Consequently there are no conformally flat boosted slices of the Schwarzschild spacetime. This confirms the previously anticipated explanation for the presence of junk-radiation in Brandt–Brügmann puncture data.

Keywords: conformal, flatness, asymptotic

(Some figures may appear in colour only in the online journal)

1. Introduction

The construction of initial data for the Cauchy problem in general relativity (GR) relies on the one hand upon a suitable formulation of the constraints, and on the other on a suitable choice for the given data within this formulation. The former serves to provide a theoretical framework in which data can be constructed, while the latter encodes modeling choices of the physics under consideration. Ideally this framework will allow for a straightforward proof that data of physical interest exists, and for the required given data to be easily interpreted. In numerical relativity the most popular approach to solve the constraints is to make use of the
York–Lichnerowicz conformal transverse traceless decomposition [1–3] plus developments of the approach culminating in the extended conformal thin-sandwich equations [4–9]. In this setting the constraints become a coupled nonlinear elliptic system. A particularly popular choice is the puncture data [10]. Other strategies include gluing [11, 12] and the reformulation of the constraints as a hyperbolic evolution system [13].

An important aspect in formulating the constraints is the study of exact solutions or those with special properties as this helps to understand the physical nature of the constructed data. This is the case for puncture data, which is in some sense inspired by the form of Schwarzschild in isotropic coordinates. In fact, in all approaches employing a free conformal metric, it often simplifies matters to make that metric flat. Simplifying choices may however have unfortunate physical consequences on the data being constructed. It is known, for example, that the Kerr spacetime admits no spatial slice which is conformally flat [14, 15]. Therefore the use of this restriction, even in the construction of a single spinning black hole, must result in data which corresponds not to Kerr, but to some physical deformation thereof. This deviation often appears as high-frequency gravitational wave content, and is therefore referred to as junk-radiation. Similar radiation is also observed in evolutions of conformally flat initial data in which the black holes have linear momentum. This feature becomes the crucial stumbling block for highly boosted data [16, 17]. With the expectation that the restriction to conformal flatness was the cause of this problem, several practical [18] and more sophisticated [19, 20] cures have been implemented. Another strategy is to try and account for the physical effect of the junk-radiation. In the recent paper [21], for example, a fitting method is used to do so.

The relationship between linear momentum, conformal flatness and junk-radiation has notably been studied in the literature by York [22–24], but usually with a fixed background spacetime and a Taylor expansion in the boost. Here, to avoid those simplifications, we employ the dual-foliation (DF) formalism [25–30] and consider spacetimes which are asymptotically flat at spatial-infinity. Properties of asymptotic charges, in particular of the ADM 4-momentum are then examined under our definitions. We then show that if there is a spatial slice with vanishing ADM-momentum which is in some sense close to conformal flatness, then no slice asymptotically related to the first by a boost near spatial-infinity can be conformally flat. Morally this result can be summarized by saying that no slice of Schwarzschild with linear momentum is conformally flat and therefore, in concordance with the expectation mentioned above, conformal flatness is a cause of junk-radiation in single black hole spacetimes with linear momentum. This is presumably also true in a more general context.

We begin in section 2 with an overview of the DF formalism and the various definitions and asymptotics that are assumed afterward. We demonstrate that the ADM 4-momentum is governed by the leading Lorentz part of a boost even when supertranslation terms are present. Section 3 contains the main argument that, under refined assumptions on the asymptotics, boosted slices can not be conformally flat. As a corollary we show that axisymmetric slices of Kerr can not be conformally flat. We conclude in section 4. Geometric units are used throughout.

2. Asymptotic flatness and the ADM 4-momentum

In this section we describe the DF formalism before giving a relevant formulation of asymptotic-flatness at spatial-infinity. We then define a change of coordinates that preserves this notion of asymptotic flatness, and end by discussing the transformation of the ADM energy-momentum under changes of coordinates that asymptote to Poincaré transformations plus a supertranslation term near spatial-infinity.
2.1. DF formalism overview

Given two families of observers, one associated with upper case coordinates \( X^\mu \), the other with the lower case \( x^\mu \), spacetime will be described in two different but related ways. The DF formalism [26] provides a means to relate these two worldviews from a \( 3 + 1 \) perspective. Throughout the paper, Latin indices \( a, b, c, d, e \) will be abstract, underlined Greek indices denote the components of tensors in the upper case coordinate tensor basis, whereas plain Greek indices are used for the lower case basis. Underlined and plain Latin indices \( i, j, k, l \) stand for the spatial components in the upper case and lower case bases respectively. The two time coordinates \( T \) and \( t \) provide, in general, two distinct foliations of the spacetime, thus creating different spatial tensors, spatial metrics, extrinsic curvatures and so on. We denote with \((N)^{\gamma} \gamma^{ab}\) the upper case spatial metric, and \( \gamma^{ab} \) the lower case metric. The future pointing unit normal vectors \( N^a \) and \( n^a \) of upper case and lower case foliations are related by,

\[
N^a = W(n^a + v^a),
\]

where we have defined the Lorentz factor \( W \) and lower case boost vector \( v_a \).

\[
W = -(N^a n_a), \quad v_a = \frac{1}{W} \overline{J}^b_a n_b.
\]

Here \( \overline{J}^b_a \) is the projection operator on to the lower case slice. Since the normal vectors have unit magnitude the Lorentz factor and boost vector satisfy,

\[
W = \frac{1}{\sqrt{1 - \gamma^2}}, \quad W \geq 1 > \gamma^i v^i \equiv v^2,
\]

where \( \gamma^i \) is the inverse lower case metric. Tensors orthogonal on every slot to \( N^a \) and \( n^a \) are called upper case and lower case respectively. The \( 3 + 1 \) form of the spacetime metric \( g_{ab} \) can be written as

\[
ds^2 = (-\alpha^2 + \beta^i \beta^j) dT^2 + 2\beta^i dT dx^i + (N)^{\gamma} \gamma^{ij} dx^i dx^j = (-\alpha^2 + \beta^i \beta^j) dT^2 + 2\beta^i dT dx^i + \gamma^{ij} dx^i dx^j,
\]

with standard definitions for the lapse and shift variables. Subsequent definitions, such as that for the extrinsic curvature of each foliation \((N)^{\gamma} K_{ab} \) and \( K_{ab} \), follow the standard lines. Their explicit relationship is given in [26].

The two tensor bases are of course related by the Jacobian \( J^a_
u \equiv \partial X^a / \partial x^\mu \), which can we represented as,

\[
J = \begin{pmatrix} A^{-1} W(\alpha - \beta^i v_i) & \alpha \pi^a + \beta^i \phi^a \\ -A^{-1} W v_i & \phi^a \end{pmatrix},
\]

where \( \phi^a \equiv J^a_i \).

The projected upper case induced metric defined by \( \partial_{ab} = \gamma^c_a \gamma^d_b (N)^{\gamma} \gamma^{cd} \) is,

\[
g_{ij}^{(N)} = \gamma_{ij} + W^2 v_i v_j.
\]

This object can be considered a metric on the lower case foliation and it is called boost metric, with covariant derivative \( \partial \) and connection \( \Gamma \). The boost metric has inverse,

\[
(g^{(N)}\gamma)_{ij} = \gamma_{ij} - v^i v^j.
\]

For more details of the formalism we direct the reader to [26].
2.2. Asymptotic flatness

Physically speaking, an asymptotically flat spacetime is characterized by the requirement that the metric asymptotes to the Minkowski metric sufficiently fast at large distances. A point well made in [31] is that no absolute preferred definition of asymptotic flatness can be given or expected. Rather there is an interplay between the field equations, the physics under consideration, and the rate at which the metric becomes flat. Therefore several distinct precise formulations of the concept have arisen. A key development in these definitions has been the use of conformal compactification [32], which was used [14, 15] in the demonstration that there is no conformally flat slice of the Kerr spacetime. We instead work with a more pedestrian definition, which is motivated and stated in the following.

Basic notion of asymptotic flatness: consider a globally hyperbolic spacetime foliated by a family of spacelike Cauchy hypersurfaces $\Sigma T$ and a boost-type domain $\Omega$ defined as,

$$\Omega := \{ R > R_0, |T| < qR + T_0 \},$$  

(8)

where $R$ is a radial coordinate on that foliation, defined in the standard way in terms of $X^\alpha$, to be introduced momentarily, and $R_0$, $q > 0$ and $T_0$ are constants. The spacetime is said to be asymptotically flat if there exists a preferred coordinate system $X_{\mu} = (T, X, Y, Z)$, which will in general be highly nonunique, with $X^\mu$ on $\Sigma_T$, in which the metric $g_{\alpha\beta}$ satisfies the following condition within $\Omega$:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + O_p(R^{-1}),$$  

(9)

where $\eta_{\alpha\beta}$ is the Minkowski metric, $p \geq 1$ and $O_p(R^{-m})$ means that its $\partial_\alpha$ partial derivatives of order $n$ decay as $R^{-m-n}$ for all $n = 0, \ldots, p$. Following a hint given in [33], we note that by combining the boost theorem of [34] with the improved Sobolev embedding of [35], the present definition of asymptotic flatness can be propagated from its natural restriction to initial data, in the vacuum setting, inside a boost-type domain for any $p \geq 1$ provided that suitable data, belonging to sufficiently high order weighted Sobolev spaces, is given. Details can be found in appendix A.

Permissible coordinate changes: if we are to restrict ourselves to the study of asymptotically flat spacetimes, it is helpful to know which class of coordinate transformations preserves the asymptotic form of the metric. In the following we determine the form of these coordinate changes, which we dub permissible coordinate changes. Intuitively, slices that we can obtain from $\Sigma_T$ through a permissible coordinate change are called permissible slices. Let us assume momentarily that there are two coordinate systems $X^\alpha$ and $x^\alpha$ in which the metric, close to spatial infinity, takes the form (9). Then we have,

$$g_{\alpha\beta} = g_{\alpha\beta} J_{\alpha}^\alpha J_{\beta}^\beta \Rightarrow \eta_{\alpha\beta} + O_p(R^{-1}) = \eta_{\alpha\beta} J_{\alpha}^\alpha J_{\beta}^\beta + O_p(R^{-1}).$$  

(10)

In order to retrieve any information about the Jacobian, we have to know how the two error terms relate to one another. For that we assume that the upper case coordinate system can be expanded in powers of $r^{-1}$ in the following way:

$$X^\alpha = X^\alpha_0(t, \theta, \phi)r + X^\alpha_1(t, \theta, \phi) + O_{p+1}(r^{-1}),$$  

(11)

where $t := x^0$ and $\theta$ and $\phi$ are the standard polar and azimuthal angles associated with $x^\alpha$. We can then write the upper case radial coordinate near spatial infinity as,
\[ R^2 := \delta_{ij}X^i_X^j = \delta_{i,j}X^\alpha_X^\beta r^2 + O_{p+1}(r), \]  
(12)

which implies an equivalence of orders,
\[ O_p(r^{-9}) = O_p(R^{-9}), \]
and we can conclude from (10) that the Jacobian must have a leading Lorentz term,
\[ J^\alpha = \Lambda^\alpha + O_p(r^{-1}), \]
(14)

where \( \Lambda^\alpha \) is the standard Lorentz matrix. Using the fact that \( J^\alpha := \partial_\alpha X^\beta \), we can differentiate (11) with respect to \( x^\alpha \) and use (14) to get four equations, one for each derivative of \( X^\alpha \). The \( \partial_\alpha X^\beta \) equations give
\[ X^\alpha \partial_\beta = \Lambda^\alpha + O_p(r^{-1}), \]
(15)

and the \( \partial_\beta X^\alpha \) equation yields,
\[ \partial_\beta X^\alpha = \Lambda^\alpha \Rightarrow \partial_\beta X^\alpha = \Lambda^\alpha t + c^\alpha(\theta, \phi), \]
(16)

where \( c^\alpha \) are arbitrary functions of lower case angles. Plugging (15) and (16) in (11), we get
\[ X^\alpha = \Lambda^\alpha x^0 + c^\alpha(\theta, \phi) + O_{p+1}(r^{-1}). \]
(17)

We conclude that any coordinate transformation that preserves the asymptotic form of the metric must have this form, and this is the class that we will use throughout this work. Note that the Poincaré transformations are precisely the subset of this large class with constant \( c^\alpha \) and vanishing error terms. For completeness we write here the explicit form of the Jacobian in our notation, as well as that of its inverse, which will be useful throughout this work:

\[
J = \begin{pmatrix}
W & -W\tilde{v}_i\delta^i_j \\
-\tilde{v}_j & \delta^j_i + \frac{W^2v_j\tilde{v}_i}{W+1}
\end{pmatrix} + \partial c(\theta, \phi) + O_p(r^{-2}),
\]

and

\[
J^{-1} = \begin{pmatrix}
W & Wv^i_j \\
W\tilde{v}_j & \delta^j_i + \frac{Wv_j\tilde{v}_i}{W+1}
\end{pmatrix} + \partial C(\Theta, \Phi) + O_p(R^{-2}),
\]

(18)

with \( W = (1 - \tilde{v}_i\tilde{v}^i)^{-1/2}, \tilde{v}_j \) constant and \( \tilde{v}^i \equiv \tilde{v}_i\delta^i_j. \) Now we need to check that all transformations in the class (17) preserve the asymptotic form of the metric. For that we assume the metric to behave like (9) and the coordinate transformation to be of the form (17), and compute the behavior of \( g_{\alpha\beta} \) close to spatial infinity,

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + O_p(r^{-1})
\]

and

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + O_p(r^{-1}) + O_p(R^{-1}) = \eta_{\alpha\beta} + O_p(r^{-1}),
\]

where the last equality comes from (13). This concludes the proof that the largest class of coordinate transformations that preserve the asymptotic behavior required by our definition of asymptotic flatness (9) is given by (17). We can then write the following result:

**Proposition 1.** Let \((M, g)\) be an asymptotically flat spacetime with preferred coordinates \( X^\alpha \). A coordinate transformation is permissible if and only if it is of the form,

\[ X^\alpha = \Lambda^\alpha x^0 + c^\alpha(\theta, \phi) + O_{p+1}(r^{-1}). \]

(19)
It is straightforward to see that the inverse transformation takes the analogous form,
\begin{equation}
\xi^\alpha = (\Lambda^{-1})^\alpha_\mu X^\mu + C^\alpha(\theta, \phi) + O_{p+1}(R^{-1}),
\end{equation}
with the new quantities and \( O \) defined in the obvious manner, and \( C^\alpha = -(\Lambda^{-1})^\alpha_\mu c^\mu \).
Additionally, the angular coordinates can be seen to satisfy,
\begin{align}
\Theta &= \frac{\Lambda^Z x^i}{R} + O_p + 1(R^{-1}), \\
\Phi &= \frac{\Lambda^Y x^i}{\Lambda^X x^j} + O_p + 1(R^{-1}).
\end{align}
(21)
Note that the leading order terms in \( \Theta \) and \( \Phi \) depend only on \( \theta \) and \( \phi \), which is why we can define \( C^\alpha \) in terms of \( \theta \) and \( \phi \) in (20).

The ADM 4-momentum: the ADM 4-momentum is defined as,
\begin{equation}
P_{\text{ADM}}^\alpha := (-m, P_X, P_Y, P_Z),
\end{equation}
where \( m \) is the ADM mass and \( P^i \) are the components of the ADM linear momentum, given in terms of the intrinsic metric and extrinsic curvatures by,
\begin{align}
P_{\text{ADM}}^\alpha &= \lim_{R \to \infty} \frac{-1}{16\pi} \int_{S_R} (\partial_i \gamma_{ij} - \partial_j \gamma_{ij}) dS^i_R, \\
P_{\text{ADM}}^i &= \lim_{R \to \infty} \frac{1}{8\pi} \int_{S_R} (\gamma_{ij} - \gamma_{ij}) dS^i_R.
\end{align}
(24)
Here, \( S_R \) is a coordinate 2-sphere of radius \( R \). Our definition of asymptotic flatness is sufficient for the ADM four-momentum to be well defined. \( P_{\text{ADM}}^\alpha \) behaves as a four dimensional linear form under coordinate change (17) [36]. In fact, \( \bar{\text{ADM}} \) showed [37] that a Poincaré transformation transforms the 4-momentum according to,
\begin{equation}
P_{\text{ADM}}^\alpha = \Lambda^\alpha_\beta P_{\text{ADM}}^\beta,
\end{equation}
(25)
where \( \Lambda^\alpha_\beta = \Lambda^\alpha_\beta \eta^{\beta\gamma} \eta^{\gamma\delta} \). For an introduction of the ADM conserved quantities at spatial infinity based on differentiability requirements for the Hamiltonian see [38]. Here a slightly more restrictive definition of asymptotic flatness is used.

2.3. Supertranslations do not affect the transformation of the ADM 4-momentum

We work in this subsection along the lines of the discussion given in [39] (section 1.2.3). The Einstein–Hilbert action contains second derivatives of the metric, but one can remove a total divergence from it, leaving the action with only first derivatives of the metric. As the term removed is a total divergence, the field equations are unchanged. This can be achieved by introducing a background metric.

Let \( S_0 \) be the spacelike hypersurface \( \{ T = 0 \} \cap \Omega \), where \( \Omega \) is a boost-type domain. On \( S_0 \), we define the upper case background metric \( B_{ab} \) by requiring that \( B_{ij} = \eta_{ij} \). Note that, due to the Poincaré-invariance of the Minkowski metric, if we took \( c^\mu \) in (17) constant, there would be no difference between \( B_{ab} \) and the analogously defined lower case background metric \( b_{ab} \).

In our case, that difference takes the form,
\[ B_{\alpha\beta} - b_{\alpha\beta} = 2 \partial_{\alpha} [\Lambda^{\alpha\beta} \partial^\gamma \eta_{\gamma\beta}] + O_p(R^{-2}) =: 2 \partial_{\alpha} \hat{c}_{\beta} + O_p(R^{-2}). \] (26)

We define the following tensor,
\[ \hat{g}^{ab} := \frac{1}{16\pi} \sqrt{-g} \hat{g}^{ab}, \] (27)
where \( g \) and \( B \) are the determinants of \( g_{ab} \) and \( B_{ab} \), respectively. As in [39] (section 1.2.3), given a vector field \( X \), the Hamiltonian generating the flow of \( X \) can be written as,
\[ H(X, S_0) := \int_{S_0} \left( p^{bc}_a \mathcal{L}_X \hat{g}^{bc} - X^a L \right) dS_a = \int_{S_0} \hat{\nabla}^b \hat{U}^{ab} dS_a, \]
where \( \hat{\nabla} \) is the covariant derivative associated with \( \hat{g}^{ab} \), \( L \) is the Lagrangian and \( p^{bc}_a \) is the momentum canonically conjugate to \( \hat{\nabla} \hat{g}^{ab} \) defined as,
\[ p^{bc}_a := \frac{\partial L}{\partial (\hat{\nabla} \hat{g}^{bc})} \] (29)

The ADM 4-momentum on \( S_0 \) is then,
\[ P^{\text{ADM}}_{\alpha} := H(\partial_{\alpha}, S_0). \] (30)

By Chruściel’s proposition 1.2.1 in [39], if the metric \( g_{ab} \) satisfies our notion of asymptotic flatness (9), then the integral \( H(\partial_{\alpha}, S_0) \) converges. Now let \( S \) be the spacelike hypersurface \( \{t = 0\} \cap \Omega \) which is related to \( S_0 \) by a change of coordinates of the form (17). As we have shown that the boundary conditions are preserved under (17), we immediately get convergence of the integral \( H(\partial_{\alpha}, S) \). In order to show that (25) is unchanged by supertranslations, we define the following three dimensional region of spacetime,
\[ T = \{ R = R_0, T > 0, t < 0 \} \cup \{ R = R_0, T < 0, t > 0 \}, \]
so that its boundary \( \partial T \) consists of two 2-spheres of radius \( R_0 \), \( S_0 \cap \{ R = R_0 \} \) and \( S \cap \{ R = R_0 \} \) respectively, as shown on figure 1. We integrate \( \hat{\nabla} \hat{U}^{ab} \) over \( T \) and use Gauss’s theorem to write,
\[ \int_{S \cap \{ R = R_0 \}} \hat{U}^{ab} dS_{ab} = \int_{T} \hat{\nabla} \hat{U}^{ab} dS_a + \int_{S_0 \cap \{ R = R_0 \}} \hat{U}^{ab} dS_{ab}. \] (31)

From [39] we can see that the integrand in the first term on the right-hand side can be written as,
\[ 16\pi \hat{\nabla} \hat{U}^{ab} = \sqrt{-\hat{B}} (\hat{\nabla}^{a} X^{b} + Q^{ab} X + Q^{ab} \hat{\nabla} X^b). \]
where $T^{ab}_{\circ}$ is called canonical stress and is defined as,

$$T^{ab}_{\circ} := \frac{1}{8\pi} \sqrt{-g} \left( R^{ab}_{\circ} - \frac{1}{2} R g^{ab}_{\circ} \right),$$

(32)

$Q^a_{\circ b}$ is, to leading order, quadratic in $\nabla^\circ g_{ab}$ and $Q^{ab}_{\circ c}$ is bilinear in $\nabla^\circ g_{ab}$ and $g_{ab} - B_{ab}$ with bounded coefficients. At this point we have to make the additional assumption that the space-time satisfies Einstein’s equations with a stress-energy tensor decaying as $O_0(R^{-4})$. This implies that $T^{ab}_{\circ}$ decays near spatial infinity at least as,

$$T^{ab}_{\circ} = O_0(R^{-4}).$$

(33)

This requirement is necessary because the integral on $T$ involves integrating over two angular coordinates, yielding $R^2_0$ in the volume element, and one time coordinate, yielding $R_0$ as the time interval grows with $R_0$. Then, as $R_0 \to \infty$, the first term on the right-hand side of (32) is,

$$\int_T \nabla^\circ b \nabla^\circ g_{ab} dS_a = O_0(R_0^{-1}),$$

(34)

and the remaining two terms give,

$$\lim_{R_0 \to \infty} \int_{\partial S \cap \{ R = R_0 \}} \nabla^\circ g_{ab} dS_{ab} = H(\partial_{\circ} \cdot S_0).$$

(35)

We would like the left-hand side to reduce to $H(\partial_{\circ} \cdot S)$. For that, we have to rewrite the integrand in terms of the following tensor,

$$g^{-1}_{ab} := \frac{1}{16\pi} \sqrt{-B} \frac{\sqrt{-g}}{\sqrt{-B}} g_{ab} = \frac{\sqrt{-B}}{\sqrt{-B}} g^{-1}_{ab},$$

(36)

where $b$ is the determinant of the lower case background metric, and replace the covariant derivative $\nabla^\circ a$ with the one associated to $b_{ab}$, $\nabla^\circ a$, by making use of the tensor,

$$C^a_{b\cdot c} := \Gamma_b^a_{c} - \Gamma_b^a_{\circ c} = \frac{1}{2} (B^{-1})^{ad} (\nabla^\circ_b B_{cd} + \nabla^\circ_c B_{bd} - \nabla^\circ_d B_{bc}),$$

(37)
where $\Gamma^a_{bc}$ and $\Gamma^a_{bc}$ are Levi-Civita connections of $B_{ab}$ and $b_{ab}$, respectively. In order to compute the ADM 4-momentum, the vector $X$ is chosen to be $\partial_{\alpha}$ (see (30)) and the second term on the definition of $\tilde{U}^{ab}$ (28) vanishes, so let us write,

$$\tilde{U}^{ab}_c = 2 \frac{\sqrt{B}}{\sqrt{-g}} \left[ \frac{1}{2} \Gamma^a \delta^{[b} g^{c]} \right]_{\alpha} + \frac{B}{b} \delta^{[b} g^{c]} \nabla_a \frac{b}{B} + \frac{B}{b} \delta^{[b} g^{c]} \nabla_a \frac{b}{B} c_{\alpha} + \frac{B}{b} \delta^{[b} g^{c]} \nabla_a \frac{b}{B} n_{\alpha} + C_{e f g} \delta^{[b} g^{c]} \nabla_a \frac{b}{B} n_{\alpha} \right] .$$

(38)

The quotient of the determinants of the background metrics can be computed using (26):

$$\frac{b}{B} = 1 - 2(b^{-1})^{ab} \nabla_a n_{\alpha} + O_p(R^{-2}).$$

(39)

We must now define the future-pointing vectors normal to $S_0$ and $S$, normalized with respect to the background metrics. Respectively,

$$\hat{n}_a := -\alpha \nabla_a t, \quad \alpha := -B^{-1} \nabla_a \nabla_a t,$$

$$\hat{s}_a := \nabla_a R.$$ 

(40)

We must also define the outward-pointing vectors normal to $S_0 \cap \{ R = R_0 \}$ and $S \cap \{ R = R_0 \}$,

$$\tilde{n}_a := \nabla_a \tilde{R}, \quad \tilde{s}_a := \tilde{\Gamma}_a b_c = \delta^{b}_{a} + \hat{n}_a n_{\alpha} (B^{-1})^{bc},$$

(41)

respectively, in order to build the integrand on the left-hand side of (37):

$$\tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c = \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c + 2 \nabla_a \nabla_a t \left[ g^{ab} g^{cd} - g^{bd} g^{ad} \right] n_{\alpha} \phi X^a + O_p(R^{-3}).$$

(42)

In order to understand the last non-error term in (42), we have to perform the following simple computation,

$$\tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c = \nabla_a \nabla_a t \left( n_{\alpha} \phi X^a + O_p(R^{-3}) \right).$$

(43)

The third term on the right-hand side of (43) is zero to leading order because $\tilde{c}_a$ only depends on angular coordinates (17). Note that, to this order, whether the dependence is on lower case angles or upper case ones is irrelevant because of (21). Also, from (41) we have $\nabla_a \phi = \nabla a \phi$, so we get,

$$\nabla_a \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c = \nabla_a \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c + O_p(R^{-3}).$$

(44)

Plugging this into (44) yields,

$$\tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c = \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c + \nabla_a \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c + \nabla a \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c + \nabla a \tilde{U}^{ab}_c \hat{n}_a \hat{s}_b X^c + O_p(R^{-3}).$$

(45)

The vector $\partial_{\alpha}$ is a Killing vector with respect to the upper case background metric $B_{ab}$, so we can write,

$$\mathcal{L}_a B_{ab} = 2 \nabla_a [B_{ab}] X^c = 0.$$ 

(46)
Naturally, we have that,
\[
(\partial_T)^a = -(B^{-1})^{ab}\nabla_b T,
\]
\[
(\partial_Z)^a = (B^{-1})^{ab}\nabla_b X^2,
\]
which, together with (46), gives
\[
\nabla_a X^b = 0.
\] (48)

Note that this result implies \(\nabla_a X^b = O_p(R^{-2})\), because the difference between the background metrics has a fall-off (26), and this allows us to push \(X\) through the covariant derivative in (45) while only getting higher order additional terms. From (26) and (40), we can easily see that,
\[
\hat{\alpha} = 1 + O_p(R^{-1}),
\] (49)

and hence that,
\[
\nabla_a \hat{n}_b = O_p(R^{-2}),
\] (50)

Now, using (48) and (50) in the integrand (45), we find,
\[
\nabla^a \hat{n} b a X^c = \nabla^a \hat{n} b a X^c + \nabla^a [\hat{n} b a X^c] + \nabla^a [\nabla^a \hat{n} b a X^c] + O_p(R^{-3}).
\] (51)

In the first term on the right-hand side we have replaced \(\nabla^a \hat{n} b a\) with \(\nabla^a \hat{n} b a\) because, to leading order, they are equal (49). Moreover, we have replaced \(\nabla^a \hat{n} b a\) with \(\nabla^a \hat{n} b a\) because the antisymmetry of the first two indices of \(\nabla^a \hat{n} b a\) guarantees that whichever component of \(\nabla^a \hat{n} b a\) that is not orthogonal to \(\nabla^a \hat{n} b a\) vanishes. The last two terms are total divergences on the sphere and thus integrate to zero. Finally, we get the result,
\[
H(X, S) = H(X, S_0).
\] (52)

Let us now use this in order to find how the ADM momentum transforms under (17):
\[
P_{\alpha}^{ADM} := H(\partial^a S_0) = H(\partial^a S) = \Lambda_{a}^{\alpha} H(\partial^a S) = \Lambda_{a}^{\alpha} P_{\alpha}^{ADM},
\] (53)

where in the second equality we used (52) and in the third we used the fact that \(\Lambda_{a}^{\alpha}\) are constants while the rest of the terms in the Jacobian are of order \(O(R^{-1})\), so that they cannot contribute to the integral. This is the result that we wanted (25) and we state it concisely in the following theorem:

**Theorem 1.** Let \((M, g)\) be an asymptotically flat spacetime and a solution of Einstein’s equations with stress-energy tensor components decaying as \(O_0(R^{-4})\). Then, any permissible coordinate change transforms the ADM 4-momentum as
\[
P_{\alpha}^{ADM} = \Lambda_{a}^{\alpha} P_{\alpha}^{ADM}.
\] (54)
Discussion: note that related results can be found in [39, 40], but for our main result, theorem 2, this specific statement is needed. The definition of asymptotic flatness at spatial infinity given above makes no assumption about the linear momentum, and nor should such a definition in general. By the result on the ADM 4-momentum above (53) however, assuming that the spacetime extends long enough near spatial infinity, we can transform to an asymptotic rest-frame, or just rest-frame for short, which we define as a slice in which the linear momentum vanishes. If we wish, we can then refine the definition of asymptotic flatness within this preferred slice. In view of the boost theorem [34], we expect that given suitable initial data, with appropriate care, our requirements on the asymptotics, to be stated momentarily, can be propagated long enough in time to apply our results. We required that Einstein’s equations are satisfied due to the fact that \( P_{\text{ADM}} \) was defined according to the corresponding action. That said, we expect that different actions would yield similar results, but with different definitions of the canonical stress tensor (33).

3. Conformal flatness of boosted slices

In this section, the Cotton–York tensor of the lower case spatial metric is computed assuming the upper case slice to have zero ADM linear momentum and the coordinate change to be given by (17). For that a stronger definition of asymptotic flatness is needed, namely, assumptions have to be made on the first order terms in \( R^{-1} \) of the metric components. It turns out that a crucial component of the Cotton–York tensor is given by the boost vector itself. In the presence of linear momentum this component gives the leading obstruction to conformal flatness in the lower case foliation. Throughout this section we shall be concerned with coordinate transformations of the form,

\[
X^\alpha = \Lambda^\alpha_\mu x^\mu + c^\alpha(\theta, \phi) + O_4(R^{-1}),
\]

where we can take the error term in terms of the upper case radial coordinate because of the equivalence of orders implied by (13).

3.1. Strong asymptotic flatness

We call a globally hyperbolic asymptotically flat spacetime with \( p = 3 \) strongly asymptotically flat of order \( O_3(R^{-2}) \) at spatial infinity if there exist coordinates \( X^\mu = (T, X, Y, Z) \) defining a rest-frame in which, in a neighborhood of spatial infinity, the spatial metric takes the form,

\[
^{(5)}g_{ij} = \psi^4 \left( \delta_{ij} + h_{ij} \right),
\]

where \( h_{ij} = O_3(R^{-2}) \), and we fix the ambiguity in this decomposition by taking \( \psi = 1 + \frac{m}{R} \), whilst the lapse and shift satisfy,

\[
A = 1 - \frac{m}{R} + O_3(R^{-2}) \quad B^i = O_3(R^{-2}).
\]

To highlight the differences between the notion of strong asymptotic flatness and its weaker version given in section 2.2, the former can be written in a more concise way:

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + \frac{2m}{R} \delta_{\alpha\beta} + O_3(R^{-2}).
\]

Ultimately this amounts to requiring that the spacetime is asymptotically flat with \( p = 3 \) and the coefficient of the \( R^{-1} \) term is \( 2m\delta_{\alpha\beta} \). We are not aware of a general theorem guaranteeing...
that such fall-off will be propagated from initial data, but this definition is satisfied by the Kerr–Newman metric, and by the Schwarzschild metric with vanishing error terms, and so is not absolutely prohibitive. It is similar in spirit but not identical to the notion of strong asymptotic flatness employed in [41], but we expect that we could adjust our definition to match the conventions therein.

3.2. Definition of conformal flatness

It is well known that in three dimensions conformal flatness is characterized by the vanishing of the Cotton, or equivalently Cotton–York, tensor \( C_{abc} \). Working in the lower case foliation, the Cotton tensor and the Cotton–York tensor associated with \( \gamma_{ij} \) are given by,

\[
C_{abc} := D_c \left( R_{ab} - \frac{1}{4} R \gamma_{ab} \right) - D_b \left( R_{ac} - \frac{1}{4} R \gamma_{ac} \right)
\]

\[
C^{ab} := -\frac{1}{2} \varepsilon^{acde} C_{ced} \gamma^{eb} = \varepsilon^{cd} D_c R^{eb} \quad \text{(59)}
\]

respectively, where the last equality makes use of the fact that \( C^{ab} \) is a symmetric tensor. Here, \( \varepsilon^{abcd} := n_a \varepsilon^{abcd} \) is the Levi-Civita totally antisymmetric tensor with indices raised with the metric \( g_{ab} \). The definitions for the upper case foliation are analogous. An important point to make here is that if a metric is conformally flat, then there is a coordinate system \( X^\alpha \) in which we can write that locally,

\[
(N) \gamma_{ij} = \Omega^4 \delta_{ij} \quad \text{(60)}
\]

We then say that \( (N) \gamma_{ab} \) is explicitly conformally flat in coordinates \( X^\alpha \). Taking the spacetime to be strongly asymptotically flat of order \( O(R^{-2}) \), our primary assumption, the upper case Cotton–York tensor is easily seen to be at worst,

\[
(N) C^{ij} = O(R^{-5}) \quad \text{(61)}
\]

We call such a non vanishing Cotton–York tensor an upper case obstruction to conformal flatness. Any such obstruction must be, in some sense, generated by the traceless part of \( h_{ij} \).

3.3. Conformal flatness and the boost metric

Let us consider a spacetime which is strongly asymptotically flat of order \( O(R^{-2}) \). We want to show the result that the boost metric components (6) have the same type of fall-off near spatial infinity as the upper case spatial metric components in an appropriate set of spatial coordinates. This observation will be helpful when computing the lower case Cotton–York tensor. The lower case metric can be written as,

\[
\gamma_{ij} = g_{\alpha\beta} \mathbf{J}_\alpha^i \mathbf{J}_\beta^j = (N) \gamma_{ij} - W^2 \phi_i \phi_j + A^{-2} B_i B_j W^2 v_i v_j - A^{-1} B_i \phi^j \left( W v_j \right), \quad \text{(62)}
\]

where, in the second equality, we have used (5). Then the boost metric is exactly,

\[
g_{ij} = (N) \gamma_{ij} - W^2 \phi_i \phi_j + A^{-2} B_i B_j W^2 v_i v_j - A^{-1} B_i \phi^j \left( W v_j \right). \quad \text{(63)}
\]
Strong asymptotic flatness on our metric $g_{ab}$ gives,

$$g_{ij} = (N)\gamma_{ij} + O_3(R^{-2}), \quad (64)$$

which does not depend on $J^T$, so there must be a set of spatial coordinates $x^i$ that allows us to write

$$\gamma_{ij} = \psi^4(\delta_{ij} + h_{ij}), \quad (65)$$

with $h_{ij} = O_3(R^{-2})$. Notice that having made no assumption on the form of the boost, $g_{ij}$ inherits the asymptotic form of the upper case metric $(56)$. In fact, these coordinates are easily seen to be given by $x^i = X^i$, so that the full composite transformation is given by

$$\tilde{t} = t = \tilde{W}(T + \tilde{v}^i\delta^i_j X^j) + C(\Theta, \Phi) + O_4(R^{-1}),$$

$$x^i = X^i, \quad (66)$$

which renders the spatial part of the Jacobian $\phi_j = \delta_j i$. Note that this coordinate transformation does not give a Lorentz transformation at leading order, and hence must be treated carefully when evaluating asymptotic charges. Although the slice is boosted, the time derivative associated with these coordinates still coincides with $\partial_T$, which means that the solution still appears time independent at order $O(R^{-2})$ in the transformed tensor basis.

It is interesting to note also that in the static case, taking the upper case coordinates to have vanishing shift, the error term in $(64)$ vanishes and $g_{ij}$ is conformally flat whenever the upper case spatial metric is. Moreover, Einstein’s equations were not used to reach this result, meaning that it is fair to say that the following fact is purely geometrical: in a static space-time with a foliation with vanishing shift in which the spatial metric is conformally flat, the boost metric relative to that foliation is conformally flat with the same conformal factor. More generally, since the boost metric is conformally related to $\delta_{ij} + h_{ij}$, we can say that the obstruction to conformal flatness in the boost Cotton–York tensor is at worst $O(R^{-5})$. Naturally, we recover the precise obstruction of the upper case Cotton–York tensor continuously as $v \to 0$. In other words in strongly asymptotically flat spacetimes of order $O_3(R^{-2})$, boost metrics have the same obstruction to conformal flatness as the spatial metric in the preferred rest-frame. On this basis one would therefore expect that the spatial metric in such a boosted slice would pick up an obstruction to conformal flatness at lower order in $R^{-1}$. This we examine in the following.

### 3.4. The lower case Cotton–York tensor

From $(53)$ we can see that if we assume the upper case slice to have zero ADM linear momentum, then any slice that we get by changing coordinates according to $(55)$ has non-vanishing linear momentum if and only if $\tilde{v}_i \neq 0$. In the last section we saw that the boost metric of Schwarzschild spacetime is conformally flat. Then, looking at $(6)$, we expect that $\gamma_{ij}$ is not. In this section we compute the lower case Cotton–York tensor using $(62)$ to show that our expectations are correct for a large class of spacetimes. We begin by assuming that our spacetime is strongly asymptotically flat of order $O_3(R^{-2})$. While it is possible to do this computation directly, it proves more efficient to use the conformal invariance of the Cotton tensor and compute it for a metric that is conformal to the lower case spatial metric. For that, let us expand $W_{ij}$ under $(55)$,
This implies that, if we assume our metric to have a ‘rest-frame’ (and only if \( \bar{\gamma}_{ij} \)) and plug it in to (62) to get,
\[
\bar{\gamma}_{ij} := \psi^{-4} \gamma_{ij} = \delta_{ij} + \frac{4m}{R} \bar{W}^{k} \bar{v}_{i} \bar{v}_{j} + 2 \bar{W}^{l} \bar{v}_{i} \partial_{l} \bar{c} + 2 \phi_{l} \partial_{l} \bar{c} + O_{2}(R^{-2}),
\]
where the second equality is obtained from equations (18) and (67) and \( \phi_{l} := \delta_{ij} \phi_{j}^{l} \). The Levi-Civita connection associated with \( \bar{\gamma}_{ij} \) is,
\[
\bar{\Gamma}^{k}_{ij} = \frac{2m}{R^{2}} \bar{W}(\bar{v}_{i} \bar{v}_{j} \bar{s}^{k} + \bar{v}_{i} \bar{s} \delta_{j}^{k} - \bar{v}_{j} \bar{s} \delta_{i}^{k}) + \bar{W} \bar{v}^{l} \partial_{l} \bar{c} \bar{c}^{T} + \phi_{l}^{k} \partial_{l} \bar{c} \bar{c}^{T} + O_{2}(R^{-2}),
\]
where \( \phi_{l}^{k} : = \delta^{l} \phi_{j}^{k} \) and \( s_{i} \) defined as,
\[
s_{i} := L \partial_{i} R, \quad L^{-2} : = (g^{-1})_{ij} \partial_{i} R \partial_{j} R.
\]
Here we raise and lower the indices on \( s_{i} \) and \( \bar{v}_{i} \) with \( g^{ij} \). In order to compute the Ricci tensor of the conformal metric, we will need to take one derivative of \( s_{i} \). From (12), we get,
\[
\partial_{i} s_{i} = \frac{1}{R} (g_{ij} - s_{i} s_{j}) + O_{2}(R^{-2}),
\]
and hence,
\[
\bar{R}_{ij} = \partial_{i} \bar{\Gamma}^{k}_{ij} - \partial_{j} \bar{\Gamma}^{k}_{ik} + \bar{\Gamma}^{l}_{ij} \bar{\Gamma}^{k}_{lk} - \bar{\Gamma}^{l}_{ik} \bar{\Gamma}^{k}_{lj}
= - \frac{2m}{R^{3}} \bar{W}^{2} (12 \bar{v}_{i} s_{k} \bar{v}_{j} s_{l} \delta^{k}_{l} - \bar{v}_{i} \bar{v}_{j} - \bar{v}_{k} \bar{v}^{k} \delta_{ij}) + O_{1}(R^{-4}).
\]
Note that, to leading order, the Ricci tensor of the conformal metric does not depend on super-translations. We can finally compute the Cotton–York tensor of the conformal metric using (59),
\[
\bar{C}^{ij} = \frac{30 \bar{W}^{2} m}{R^{4}} \bar{v}_{k} \bar{v}^{k} \left( \frac{1}{5} (\bar{v}^{j} + \bar{v}^{j}) \left[ \bar{v}_{m} s^{m} \right]^{2} - s^{j} \bar{v}_{m} s^{m} \right) + O_{0}(R^{-5}).
\]
In order to obtain the lower case Cotton–York tensor from the conformal one, we use the conformal invariance of the Cotton tensor and verify that, to leading order, the Cotton–York tensors must agree,
\[
C_{ijk} = \bar{C}_{ijkl} \Leftrightarrow C^{ij} = \bar{C}^{ij} + O(R^{-5}).
\]
Notice that \( \bar{v}_{i} \) and \( s_{i} \) cannot be parallel because, in Cartesian coordinates, \( \bar{v}_{i} \) is constant. To leading order, all the five independent components of \( C^{ij} \) (symmetric and trace-free) vanish if and only if \( \bar{v} = 0 \) or \( m = 0 \), except \( C^{ij} s_{i} s_{j} \) which is zero regardless of the values of the constants. This implies that, if we assume our metric to have a ‘rest-frame’ \( P_{\text{ADM}} = 0 \) that is close to conformal flatness in the asymptotic sense of (56) and (57), no slice with \( \bar{v}_{i} \neq 0 \) can be conformally flat. It is interesting to note that this fact is purely geometrical, in the sense that it does not assume GR to hold. It is only when we talk about linear momentum that this ceases too be true, because its definition and transformation law (25) rely on GR. However we do expect that similar results can be obtained for different theories. For clarity we state this result in the following theorem:

**Theorem 2.** Let \( (M, g) \) be a strongly asymptotically flat spacetime of order \( O_{3}(R^{-2}) \) at spatial infinity and a solution of Einstein’s equations with non-trivial \( m \) and stress-energy tensor
components decaying as $O_0(R^{-4})$. Then, there is no permissible slice with non-zero ADM linear momentum which is conformally flat.

3.5. The Kerr case

It is straightforward to see that the Kerr spacetime satisfies the hypotheses of both theorems 1 and 2. Therefore there can be no conformally flat boosted slice of Kerr. In fact it is already known [14, 15, 42] that there is no such slice with vanishing linear momentum in Kerr either. Presently, as a corollary of theorem 2, we recover the latter result in the special case that the slice is axially symmetric. Details of the calculations of this section can be found the mathematica notebook that accompanies the paper [43].

We start with Boyer–Lindquist coordinates and adjust the radial coordinate $R_{BL}$ as,

$$R_{BL} = \psi^2 R,$$  \hspace{1cm} (74)

with $\psi$ defined as before. Constructing Cartesian coordinates in the standard way from $(T, R, \Theta, \Phi)$ brings the metric into the form (56) employed in theorem 2. Computing the Cotton–York tensor, one readily finds an obstruction to conformal flatness or order $O(R^{-7})$ at large radius. Therefore our aim would be to adjust the slice so that this obstruction is somehow absorbed. We consider only axisymmetric slices, and so make the ansatz,

$$t = W(T + \bar{v}_i \delta^i_j X^j) + C_t(0) + R^{-1} C_t(1),$$ \hspace{1cm} (75)

with $C_t(0)$ and $C_t(1)$ functions of $\Theta$ to be determined. Working with axisymmetric slices means that we end up with only a simple ODE analysis to perform. Generalizing this would instead require treating a PDE problem. Adding higher order terms to this ansatz will not affect the calculations to the order at which we work. Presently we do not alter the spatial coordinates, since doing so will only complicate the computation, and can not help to impose conformal flatness on the adjusted spatial slice, which is determined solely by the choice of $t$. By theorem 2 we must furthermore choose $v_2$ trivial, otherwise there will be an obstruction to conformal flatness of order $O(R^{-4})$ on the adjusted slice. Computing the Cotton tensor of the lower case spatial metric in powers of $R^{-1}$ reveals that there is an obstruction to conformal flatness at order $O(R^{-5})$ unless,

$$C_t(0) = c_1 + c_2 \cos^2 \Theta,$$ \hspace{1cm} (76)

with $c_1$ and $c_2$ arbitrary real constants. There is furthermore an obstruction of order $O(R^{-6})$ unless $c_2 = 0$; in other words the supertranslation term must belong to the Poincaré class. Using these conditions and computing one order further we find that there is no choice of $C_t(1)$ that removes the $O(R^{-7})$ obstruction. In particular, we must have,

$$\sin \Theta \partial_\Theta C_t(1) + 3 \cos \Theta \partial_\Theta C_t(1) = 0,$$ \hspace{1cm} (77)

but that even when this condition is satisfied there remains an obstruction at the same order. The obstruction is proportional to the dimensionless spin parameter. Thus the Kerr spacetime admits no axially symmetric conformally flat spatial slice.

4. Conclusions

Working with asymptotically flat spacetimes and using the DF formalism we have made a number of interesting findings. Starting from a set of coordinates in which the metric has good
asymptotic behavior and performing a boost that preserves this fall-off of the metric near spatial infinity, we first found that the ADM 4-momentum is governed solely by the leading Lorentz transformation of the boost even in the presence of supertranslation terms, generalizing the result beyond the Poincaré group.

We then restricted our notion of asymptotic flatness in order to study conformal flatness of boosted frames. The special property of our class is that there exist rest-frames, slices with vanishing linear momentum, in which the spatial metric is close to conformal flatness. Working with spatial slices that can be boosted with respect to such a rest-frame we showed that the boost metric inherits properties from its unprojected counterpart. Using this fact and restricting our attention to boosted slices with nonvanishing linear momentum, from which it follows that the ADM mass and asymptotic boost must be nontrivial by our first result, we found that the Cotton tensor in the boosted slice picks up an $O(R^{-4})$ term. Linear momentum therefore serves as an obstruction to conformal flatness in these space-times. A consequence is that no boosted slice of Schwarzschild can be conformally flat, and therefore our results help explain problems in the ultra-boost regime with Brand–Brügmann data.

Turning our attention to the Kerr spacetime we recovered a special case of the result [14, 15, 42] that axisymmetric slices in this spacetime can not be conformally flat. More generally it is clear that even in strongly asymptotically flat space-times of order $O_3(R^{-2})$, adjustment of slices can only annihilate an obstruction to conformal flatness if that obstruction has a very special structure. A complete characterization of that structure is still lacking, however.

From a practical point of view, for applications in numerical relativity, our findings suggest that it may be natural to adopt a conformally flat boost metric as an ingredient in the construction of initial data. For that one could employ a method similar to the standard conformal-transverse-traceless decomposition of the constraints. Such a construction would then proceed in the spirit of [19, 20]. Likewise a natural suggestion for the extrinsic curvature, which still needs to be properly formalized, would be to make it ‘essentially’ a Lie-derivative of the 3-metric along the boost vector. In the case of a single black hole, such data would reduce to a boosted slice of Schwarzschild. Therefore we expect that data so constructed would contain less junk-radiation as compared with the present moving-puncture approach. These physically motivated choices do not obviously lead to a mathematically simple formulation of the constraints, so we postpone further discussion for future work.

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Appendix A. Propagation of asymptotic flatness from initial data

It has been shown that requiring initial data for vacuum GR to be asymptotically flat, for some definition of the term, gives a time development that preserves the asymptotic fall off [34]. As our definition of asymptotic flatness is a set of conditions on the whole of a boost region, it is
interesting to check whether this definition is a consequence of the initial data requirements of [34]. If that is true, then we need only to impose conditions on an initial slice that are sufficient to guarantee that they are preserved in a boost region. Let $U$ be any open set in $\mathbb{R}^3$ and let $\sigma$ be the function,

$$\sigma(R) = \left(1 + R^2\right)^{1/2}. \quad (A1)$$

The weighted Sobolev space $H^k(U)$, with $s \in \mathbb{N}$ and $\delta \in \mathbb{R}$, is the class of all functions $u$ on $U$ with values in some finite dimensional vector space $V$, defined by the norm:

$$\|u\|_{H^k(U)} := \sum_{j=0}^k \|\sigma^{\delta+j} D^j u\|_{L^2(U)}. \quad (A2)$$

The first statement of the boost theorem, as stated in [34], is the following: Let $(N)\gamma$ be a Riemannian metric and $(N)K$ a 2-covariant symmetric tensor field on $\Sigma_t$. If, $(N)\gamma - e \in H^{k+1}_{\delta+\frac{1}{2}}(\Sigma_t)$, $(N)K \in H^{k-1}_{\delta+\frac{1}{2}}(\Sigma_t)$, (A3)

where $k \geq 4$, $\delta > -2$ and $e$ is the three dimensional flat metric, then there exists a metric $g$ solution of Einstein’s equations in a boost-type domain $\Omega$, such that $g - \eta \in H^k(\Omega)$ and $(N)\gamma, (N)K$ are respectively the first and second fundamental forms of $g$ associated with $\Sigma_t$. On $\Omega$ we now define the function:

$$\tau(t, R) := \frac{t}{\sigma(R)}, \quad (A4)$$

whose level surfaces define a foliation,

$$\Omega = \bigcup_{\tau \in I_\theta} \Sigma_{\tau}, \quad I_\theta = (-\theta, \theta). \quad (A5)$$

Then lemma 2.4 in [34] states that, for each $\tau \in I_\theta$, the following inclusion holds and is continuous:

$$H^k_{\delta}(\Omega) \subset H^{k-1}_{\delta+\frac{1}{2}}(\Sigma_{\tau}, \Omega), \quad (A6)$$

where the space $H^k_{\delta}(\Sigma_{\tau}, \Omega)$ is defined by the norm,

$$\|u\|_{H^k_{\delta}(\Sigma_{\tau}, \Omega)} := \sum_{j=0}^k \|D^j u|_{\Sigma_{\tau}}\|_{H^{k-1}_{\delta+j}(\mathbb{R}^n)}. \quad (A7)$$

(A6) then gives that, for each $j \leq k$,

$$D_j(g - \eta)|_{\Sigma_{\tau}} \in H^{k-1-i-j}_{\delta+\frac{1}{2}+j}(\mathbb{R}^n), \quad (A8)$$

where $D_j$ is a time derivative. By definition of the weighted Sobolev norm we know that if we take a spatial derivative $D_i$, we get,

$$D_i D_j(g - \eta)|_{\Sigma_{\tau}} \in H^{k-1-i-j}_{\delta+\frac{1}{2}+i+j}(\mathbb{R}^n), \quad (A9)$$

where $i + j$ is the number of derivatives taken in all directions. We introduce the weighted Sobolev norms defined by,
\[
\|u\|_{W^{k,\infty}_{\delta}(\Omega)} := \sum_{j=0}^{k} \text{ess sup}_{U} |\sigma^{i+j}D^{i+j}u|,
\]
(A10)

\[
\|u\|_{W^{k,\infty}_{\delta}(\Sigma, \Omega)} := \sum_{j=0}^{k} \|D^{i+j}u|_{\Sigma}\|_{W^{k-i,\infty}_{\delta+j}(\mathbb{R}^{n})},
\]
(A11)

where (A11) can be written in a more convenient way as,

\[
\|u\|_{W^{k,\infty}_{\delta}(\Sigma, \Omega)} := \sum_{j=0}^{k} \sum_{i=0}^{k-j} \text{ess sup}_{\mathbb{R}^{n}} |\sigma^{i+j}D^{i+j}D^{i+j}u|_{\Sigma}|
= \sum_{j=0}^{k} \sum_{i=0}^{k-j} \|D^{i+j}D^{i+j}u|_{\Sigma}\|_{W^{k-i,\infty}_{\delta+j}(\mathbb{R}^{n})},
\]
(A12)

In [35], equation (1.9) shows the Sobolev embedding result that we need,

\[
H_{\delta+\frac{3}{2}}^{k}(\mathbb{R}^{n}) \subset W_{\delta}^{0,\infty}(\mathbb{R}^{n}),
\]
(A13)

for any \(k \geq 2\). Note that this result seems different from the one in [35] because our definitions for the weighted Sobolev norms are more in line with [34], where \(\delta\) is defined differently. From (A9) and (A13) we find,

\[
D^{i+j}D^{i+j}(g - \eta)|_{\Sigma} \in W^{0,\infty}_{\delta-1+i+j}(\mathbb{R}^{n}),
\]
(A14)

with \(k \geq i + j + 3\). Then, if we want that \(p\) derivatives in any directions improve the fall off of the metric, we must choose \(k \geq p + 3\). This, together with (A12) implies that,

\[
g - \eta \in W^{0,\infty}_{\delta}(\Sigma, \Omega),
\]
(A15)

which in turn implies our definition of asymptotic flatness,

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + O(R^{-1}),
\]
(A16)

with \(p \geq 1\), given that we choose \(\delta = 2\). Note that the possible choices of \(k\) and \(\delta\) that give the desired asymptotic conditions trivially satisfy the requirements of the boost theorem. It is thus shown that our definition of asymptotic flatness holds if we require our initial data to have the asymptotic behavior of the boost theorem.

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