SUBCONVEXITY FOR $GL(3) \times GL(2)$ $L$-FUNCTIONS IN $t$-ASPECT

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Abstract. Let $\pi$ be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$ and $f$ be a holomorphic (or Maass) Hecke form for $SL(2, \mathbb{Z})$. In this paper we prove the following subconvex bound

$$L\left(\frac{1}{2} + it, \pi \times f\right) \ll_{\pi, f, \varepsilon} (1 + |t|)^{\frac{3}{2} + \frac{1}{4} + \varepsilon}.$$  

1. Introduction

For $\pi$ a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$, and $f$ a holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$ the associated Rankin-Selberg $L$-series is given by

$$L(s, \pi \times f) = \sum_{n, r=1}^{\infty} \frac{\lambda_{\pi}(n, r) \lambda_{f}(n)}{(nr^2)^s},$$

in the half plane $\sigma > 1$. (Here $\lambda_{\pi}$ and $\lambda_{f}$ are the normalized Fourier coefficients of the forms.) This series extends to an entire function and satisfies a functional equation of the Riemann type $s \mapsto 1 - s$ with a gamma factor of ‘degree six’. This particular $L$-function plays a crucial role in quantum chaos (see [1]), and hence it is important to study its deeper analytic properties. In particular one seeks to understand the size of these functions inside the critical strip. A standard consequence of the functional equation is the easy convexity bound

$$L\left(\frac{1}{2} + it, \pi \times f\right) \ll_{\pi, f, \varepsilon} (1 + |t|)^{\frac{3}{2} + \varepsilon}.$$  

The Lindelöf hypothesis predicts that such a bound holds with any positive exponent in place of $3/2 + \varepsilon$. But even breaking the convexity barrier is hard and has remained open so far. The purpose of this paper is to prove the following subconvex bound.

**Theorem 1.** Let $\pi$ be a Hecke-Maass cusp form for $SL(3, \mathbb{Z})$, and $f$ a holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$. Then we have

$$L\left(\frac{1}{2} + it, \pi \times f\right) \ll_{\pi, f, \varepsilon} (1 + |t|)^{\frac{3}{2} - \frac{1}{12} + \varepsilon}.$$  

Subconvex bounds in the $t$-aspect are known for $L$-functions of degree upto three over the field of rationals (see [12], [3] and [10]). Similar bounds are also known for the Rankin-Selberg $L$-function $L(s, f \times g)$ for two $GL(2)$ forms $f$ and $g$. The $t$-aspect subconvexity for genuine $GL(4)$ $L$-functions remains an important open problem. Our method of proof is similar to the one given in [10] and is based on the separation...
of oscillation technique (as introduced in [8]). The key reason for a similar argument to be effective here is the following observation

\[
\sum_{a \mod q} S(\bar{a}, n; q)e(\bar{a}m/q) \sim qe(-\bar{m}n/q).
\]

In other words, the $GL(3)$, $GL(2)$ Voronoi summations together transform the Ramanujan sums $\sum_{a} e(a(n - m)/q)$ in the delta method to additive characters with respect to the $GL(3)$ variable. As such we save more by applying Poisson summation after Cauchy’s inequality. This is the vital structural input in this paper. The same feature helps us to prove a subconvex bound for these $L$-functions in the $GL(2)$ spectral aspect. This will be taken up in another paper. Let us also note that our argument works for Maass forms $f$, after mild alterations. In fact the argument can be extended to Rankin-Selberg convolutions of a general $GL(3)$ and a general $GL(2)$ automorphic forms over $\mathbb{Q}$.

The main technical heart of [10] was the analysis of the integral transforms. In this paper we give a simpler analysis of these integrals. This is very much desired as the technique of [10] leads to the Weyl bound in the case of $GL(2)$ and $GL(1)$ $L$-functions (see [1]), and now perhaps with this simplification one can go further.

2. The set-up

Let $\lambda_{\pi}(n, m)$ denote the normalised Fourier coefficients of the form $\pi$ (see Chapter 6 of [2]) and let $\lambda_{f}(n)$ denote the normalised Fourier coefficients of the form $f$ (see [4]). Suppose $t > 2$, then by approximate functional equation (see [4]) we have

\[
L \left( \frac{1}{2} + it, \pi \times f \right) \ll t^{\varepsilon} \sup_{N \leq b + \varepsilon} \frac{|S(N)|}{N^{1/2}} + t^{-2018}
\]

where $S(N)$ is a sum of type

\[
S(N) := \sum_{n, r} \lambda_{\pi}(n, r)\lambda_{f}(n)(nr^{2})^{-it}V \left( \frac{nr^{2}}{N} \right)
\]

for some smooth function $V$ supported in $[1, 2]$ and satisfying $V^{(j)}(x) \ll_{j} 1$.

**Remark 1** (Notation). In this paper the notation $\alpha \ll A$ will mean that for any $\varepsilon > 0$, there is a constant $c$ such that $|\alpha| \leq cAt^{\varepsilon}$. The dependence of the constant on $\pi$, $f$ and $\varepsilon$, when occurring, will be ignored.

Using the Ramanujan bound on average, i.e.

\[
\sum_{n_{1}n_{2} \leq x} |D_{2}(n_{1}, n_{2})|^{2} \ll x^{1+\varepsilon},
\]

we further conclude that

\[
L \left( \frac{1}{2} + it, \pi \times f \right) \ll \sup_{r \leq t^{\theta} \frac{3-\theta}{2} \leq N \leq \frac{3+\varepsilon}{r^{2}}} \frac{|S_{r}(N)|}{N^{1/2}} + t^{(3-\theta)/2}
\]

for some smooth function $V$. This will be taken up in another paper.
where

\[ S_r(N) := \sum_{n=1}^{\infty} \lambda_\pi(n, r) \lambda_f(n) n^{-it} V \left( \frac{n}{N} \right) \]

Hence to establish subconvexity we need to show cancellation in the sum \( S_r(N) \) for \( N \) roughly of size \( t^3 \) and \( r \) small. We can and shall further normalize \( V \), for convenience, so that \( \int V(y) dy = 1 \).

2.1. **The delta method.** There are three oscillatory factors contributing to the sum \( S_r(N) \). Our method is based on separating these oscillations using the circle method. In the present situation we will use a version of the delta method of Duke, Friedlander and Iwaniec. More specifically we will use the expansion (20.157) given in Chapter 20 of [4]. Let \( \delta : \mathbb{Z} \to \{0, 1\} \) be defined by

\[ \delta(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise}. \end{cases} \]

We seek a Fourier expansion which matches with \( \delta \) in the range \([-2M, 2M] \). For this we pick \( Q = 2M^{1/2} \). Then we have

\[ \delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sum_{a \text{ mod } q}^* e \left( \frac{na}{q} \right) \int_{\mathbb{R}} g(q, x) e \left( \frac{n\pi}{qQ} x \right) dx \]

for \( n \in \mathbb{Z} \cap [-2M, 2M] \) (and \( e(z) = e^{2\pi iz} \)). The \( * \) on the sum indicates that the sum over \( a \) is restricted by the condition \((a, q) = 1 \). The function \( g \) is the only part in the formula which is not explicitly given. We only need the following two properties (see (20.158) and (20.159) of [4])

\[ g(q, x) = 1 + h(q, x), \quad \text{with} \quad h(q, x) = O \left( \frac{1}{qQ} \left( \frac{q}{Q} + |x| \right)^A \right), \]

\[ g(q, x) \ll |x|^{-A} \]

for any \( A > 1 \). In particular the second property implies that the effective range of the integral in (3) is \([-M^\varepsilon, M^\varepsilon]\).

2.2. **Separation of oscillation.** We apply (3) directly to \( S_r(N) \) as a device to separate the oscillations of \( \lambda(n, r) \) and \( \lambda_f(n)n^{-it} \). This by itself does not suffice, and as in [9] and [10] we need a ‘conductor lowering mechanism’. For this purpose we introduce an extra integral namely

\[ S_r(N) = \frac{1}{K} \int_{\mathbb{R}} V \left( \frac{v}{K} \right) \sum_{n, m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_\pi(n, r) \lambda_f(m) m^{-it} \left( \frac{n}{m} \right)^{iv} V \left( \frac{n}{N} \right) U \left( \frac{m}{N} \right) dv, \]

where \( t^\varepsilon < K < t^{1-\varepsilon} \) is a parameter which will be chosen optimally later, and \( U \) is a smooth function supported in \([1/2, 5/2]\), with \( U(x) = 1 \) for \( x \in [1, 2] \) and \( U^{(j)} \ll_j 1 \).
For $n, m \asymp N$, the integral
\[ \frac{1}{K} \int_{\mathbb{R}} V \left( \frac{v}{K} \right) \left( \frac{n}{m} \right)^i v \, dv \]
is negligibly small (i.e. $O_A(t^{-A})$ for any $A > 0$) if $|n - m| \gg N t^\varepsilon / K$. Hence we can apply (3) with
\[ Q = t^\varepsilon \left( \frac{N}{K} \right)^{1/2} \]
and we get that up to a negligible error term $S_r(N)$ is given by
\[ \frac{1}{QK} \int_{\mathbb{R}} W(x) \int_{\mathbb{R}} V \left( \frac{v}{K} \right) \sum_{1 \leq q \leq Q} \frac{g(q, x)}{q} \sum_{a \bmod q} \lambda_f(m) m^{-i(t+v)} e \left( -\frac{am}{q} \right) e \left( -\frac{mx}{qQ} \right) U \left( \frac{m}{N} \right) dv \, dx, \]
where $W$ is a smooth bump function with support $[-t^\varepsilon, t^\varepsilon]$.

2.3. Sketch of proof. We end this section with a brief sketch of the proof. For simplicity let us focus on the generic case, i.e. $N = t^3$, $r = 1$ and $q \sim Q = t^{3/2} / K^{1/2}$, so that the main object of study is given by
\[ \int \sum_{v \sim K} \sum_{q \sim Q} \sum_{a \bmod q} \lambda_f(m) m^{-i(t+v)} e \left( -\frac{am}{q} \right) e \left( -\frac{mx}{qQ} \right) U \left( \frac{m}{N} \right) \, dv \, dx, \]
where $W$ is a smooth bump function with support $[-t^\varepsilon, t^\varepsilon]$.

Our aim is to save $N$ plus a ‘little more’. First we apply the Voronoi summation formulae to both the $m$ and $n$ sums. In the $GL(2)$ (resp. $GL(3)$) Voronoi we save $(NK)^{1/2} / t$ (resp. $N^{1/4} / K^{3/4}$) and the dual length becomes $m^* \sim t^2 / K$ (resp. $n^* \sim K^{3/2} N^{1/2}$). Also we save $\sqrt{Q}$ in the $a$ sum and $\sqrt{K}$ in the $v$ integral. Hence in total we have saved $N/t$, and it remains to save $t$ plus a little extra in a sum of the form
\[ \sum_{q \sim Q} \sum_{n \sim K^{3/2} N^{1/2}} \lambda_f(1, n) \sum_{m \sim t^2 / K} \lambda_f(m) \mathcal{C} \mathcal{J} \]
where $\mathcal{I}$ is an integral transform which oscillates like $n^{iK}$ with respect to $n$, and the character sum is given by
\[ \mathcal{C} = \sum_{a \bmod q} S(a, n; q) e \left( \frac{\bar{a}m}{q} \right) \sim q e \left( -\frac{\bar{m}n}{q} \right). \]
Next applying the Cauchy inequality we arrive at
\[ \sum_{n \sim K^{3/2} N^{1/2}} \left| \sum_{q \sim Q} \sum_{m \sim t^2 / K} \lambda_f(m) e \left( -\frac{\bar{m}n}{q} \right) \mathcal{C} \mathcal{J} \right|^2 \]
where we seek to save $t^2$ plus extra. Opening the absolute value square we apply the Poisson summation formula on the sum over $n$. We save enough in the zero frequency (diagonal contribution) if $t^2Q/K > t^2$ i.e. if $K < t$. On the other hand we save enough in the non-zero frequencies if $K^{3/2}N^{1/2}/K^{1/2} > t^2$ which boils down to $K > t^{1/2}$.

**Remark 2.** Notice that since the character sum boils down to an additive character we are saving more than the usual. In the usual case we would have saved $K^{3/2}N^{1/2}/QK^{1/2}$, which would be larger than $t^2$ only if we had $K > t^{4/3}$. This would contradict the upper bound $K < t$.

### 3. Voronoi summation formulae

#### 3.1. GL(2) Voronoi

Consider the sum over $m$ in (6). Applying the Voronoi summation formula this transforms into

$$
\frac{N^{1-i(t-v)}}{q} \sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \int_0^\infty U(y) y^{-i(t+\nu)} e\left(-\frac{Nxy}{q}Q\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) dy
$$

where $k$ is the weight of the form $f$. Extracting the oscillation of the Bessel function we see that the above sum is essentially given by a sum of two terms of the form

$$
\frac{N^{3/4-i(t-v)}}{q^{1/2}} \sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m^{1/4}} e\left(\frac{\bar{a}m}{q}\right) \int_0^\infty U(y) y^{-i(t+\nu)} e\left(-\frac{Nxy}{q}Q \pm \frac{2\sqrt{mNy}}{q}\right) dy.
$$

By repeated integration by parts it follows that the integral is negligibly small if $m \gg t^\varepsilon \max\{K, t^2q^2/N\} =: M_0$. In the complementary range the size of the integral is given by the second derivative bound. However we need a more precise analysis of the integral based on the stationary phase expansion. In particular we note that when $Nx/Q < t^{1-\varepsilon}$ then $m \asymp (qt)^2/N$, otherwise the integral is negligibly small.

#### 3.2. GL(3) Voronoi

Next we apply the GL(3) Voronoi summation to the sum over $n$ in (6). A similar sum occurred in [10]. The only difference is that there we had $r = 1$, but here $r$ is allowed to take small values $r \ll t^\theta$. This only introduces certain cosmetic complications. Let $\{\alpha_i : i = 1, 2, 3\}$ be the Langlands parameters for $\pi$. Let $g$ be a compactly supported smooth function on $(0, \infty)$. We define for $\ell = 0, 1$

$$
\gamma_\ell(s) := \pi^{-3s-\frac{3}{2}} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right)}{\Gamma\left(-s-\alpha_i+\ell\right)},
$$

set $\gamma_0(s) = \gamma_1(s)$ and let

$$
G_\pm(y) = \frac{1}{2\pi i} \int_{(\sigma)} y^{-s} \gamma_\pm(s) \tilde{g}(-s) ds,
$$
where $\sigma > -1 + \max\{-\text{Re}(\alpha_1), -\text{Re}(\alpha_2), -\text{Re}(\alpha_3)\}$. The $GL(3)$ Voronoi summation formula (see [6]) is given by

$$\sum_{n=1}^{\infty} \lambda_\pi(n, r) e\left(\frac{an}{q}\right) g(n) = q \sum_{n_1 | qr} \sum_{n_2 = 1}^{\infty} \lambda_\pi(n_1, n_2) S(r\bar{a}, \pm n_2; qr/n_1) G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right).$$

In the present case we have $g(n) = e(nx/qQ) n^iv V(n/N)$. Extracting the oscillation of the integral transform (see e.g. Lemma 2.1 of [5]), as in the case of $GL(2)$ above, we essentially arrive at the following expression

$$\sum_{n_1 \text{ or } qr} \sum_{n_2 = 1}^{\infty} \lambda_\pi(n_1, n_2) S(r\bar{a}, \pm n_2; qr/n_1)$$

$$\times \int_0^{\infty} V(z) z^iv e\left(\frac{Nz}{qQ} \pm \frac{3(Nn_1^2 n_2 z)^{1/3}}{qr^{1/3}}\right) dz. \quad (8)$$

By repeated integration by parts we see that the integral is negligibly small if $n_1^2 n_2 \gg t^e((qK)^3 r/N + K^{3/2} N^{1/2} r x^3) =: N_0$. We now substitute (7) in place of the third line and (8) in place of the second line of (6), to get the object of focus.

4. Reduction of integrals

4.1. Simplifying the integrals. We have transformed the sum in (6) into a new object with four integrals, which we need to simplify. Consider the integral over $x$ which boils down to

$$\int_{\mathbb{R}} W(x) g(q, x) e\left(\frac{N(xz - y)}{qQ}\right) dx.$$

Using (4) this splits as the sum of two integrals

$$\int_{\mathbb{R}} W(x) e\left(\frac{N(xz - y)}{qQ}\right) dx + \int_{\mathbb{R}} W(x) h(q, x) e\left(\frac{N(xz - y)}{qQ}\right) dx,$$

where in the second integral the weight function $h$ has smaller size. In the first integral by repeated integration by parts we see that it is negligibly small unless $|z - y| \ll t^e q/QK$. (We will continue our analysis with the first integral. For the second integral, apart from the fact that the weight function $h$ is of size $1/qQ$, we are able to get a weaker restriction $|z - y| \ll t^e/K$ by considering the $v$ integral. As such we obtain much better final bound in this case.) Writing $z = y + u$ with $|u| \ll t^e q/QK$ we arrive at the $y$ integral

$$I(m, n_1^2 n_2, q) := \int_0^{\infty} U(y) y^{-it} e\left(\frac{\pm 2 \sqrt{mNy}}{q} \pm \frac{3(Nn_1^2 n_2 (y + u))^{1/3}}{qr^{1/3}}\right) dy. \quad (9)$$
4.2. **Size of the integral** $I(\ldots)$. Suppose $K = t^{1-\eta}$ for some $\eta > 0$, then we claim that we essentially have $I(\ldots) \ll t^{-1/2}$. We will prove that the bound holds in $L^2$ sense.

**Lemma 1.** Let

$$L = \int W(w)|I(m, N_0 w^3, q)|^2 dw$$

where $W$ is a bump function. Then we have $L \ll 1/t$.

**Proof.** To prove this assertion we make a change of variable $z = y^{1/2}$, so that the phase function in (9) reduces to

$$P = -\frac{t}{\pi} \log z \pm \frac{2\sqrt{mNz}}{q} \pm \frac{3(NN_0(z^2 + u))^{1/3}w}{q \pi r^{1/3}}.$$

Then

$$P'' = \frac{t}{\pi z^2} \pm \frac{2(NN_0)^{1/3}w}{3rq^{1/3}z^{4/3}} + \text{smaller order terms}.$$

For this to be smaller than $t$ in magnitude one at least needs a negative sign in the second term and $3(NN_0)^{1/3}w/qr^{1/3} \asymp t$. Except this case we have $I(\ldots) \ll t^{-1/2}$ by the second derivative bound. In the special situation we have $N_0 \asymp (tq)^3r/N$. Opening the absolute value square we arrive at

$$L \ll \int \int |U(y_1)U(y_2)| \int W(w) e\left(\frac{3w(NN_0)^{1/3}}{qr^{1/3}}((y_1 + u)^{1/3} - (y_2 + u)^{1/3})\right) dw \leq t^{-2018} \ll 1/t.$$

The lemma follows. \(\square\)

5. **Cauchy and Poisson**

5.1. **Cauchy inequality.** The expression in (6) has essentially reduced to

$$\frac{N^{5/12}}{r^{2/3}} \sum_{1 \leq q \leq Q} \frac{1}{q^{3/2}} \sum_{\substack{a \mod q}}^* \times \sum_{\pm n_1/q} \sum_{n_2 \ll N_0/n_1^2} \frac{\lambda_\pi(n_1, n_2)}{n_2^{1/3}} S(r\vec{a}, \pm n_2; qr/n_1) \times \sum_{m \leq M_0} \frac{\lambda_I(m)}{m^{1/4}} e\left(\frac{\vec{m}q}{q}\right) I(m, n_1^2 n_2, q).$$
Splitting $q$ in dyadic blocks $q \sim C$, and writing $q = q_1 q_2$ with $q_1 \parallel (n_1 r)^\infty$, $(n_1 r, q_2) = 1$, we see that the contribution of the $C$-block to the above sum is dominated by

$$\frac{N^{5/12}}{r^{2/3}C^{3/2}} \sum_{n_1 \ll C r} n_1^{1/3} \sum_{m_1 \parallel q_1 (n_1 r)^\infty} \sum_{n_2 \ll N_0/n_1^2} |\lambda_x(n_1, n_2)| n_2^{1/3}$$

$$\times \left| \sum_{q_2 \sim C/q_1} \lambda_f(m) m^{1/4} C(\ldots) I(m, n_1^2 n_2, q) \right|,$$

where the character sum $C(\ldots)$ is given by

$$\sum_a^{\star} S(r\tilde{a}, \pm n_2; qr/n_1) e\left(\frac{am}{q}\right) = \sum_d d\mu\left(\frac{q}{d}\right) \sum_{\alpha \equiv -m \mod d}^* e\left(\pm \frac{\tilde{a} n_2}{qr/n_1}\right).$$

To analyse the sum in (10) further we break the sum over $m$ into dyadic blocks. Then applying Cauchy’s inequality and using the Ramanujan bound on average we see that the expression in (10) is dominated by

$$\sup_{M_1 \ll M_0} \frac{N^{5/12} N_0^{1/6}}{r^{2/3}C^{3/2}} \sum_{n_1 \ll C r} n_1^{1/3} \sum_{m_1 \parallel q_1 (n_1 r)^\infty} \Omega^{1/2}$$

where

$$\Omega = \sum_{n_2 \ll N_0/n_1^2} \left| \sum_{q_2 \sim C/q_1} \lambda_f(m) m^{1/4} C(\ldots) I(m, n_1^2 n_2, q) \right|^2,$$

and $M_1 \ll M_0 = K + C^2 t^2 / N$, $N_0 = (CK)^3 r / N + K^{3/2} N^{1/2} r$.

5.2. Poisson summation. Smoothing out the outer sum in (12), opening the absolute value square and applying the Poisson summation formula we arrive at

$$\Omega \ll \frac{N_0}{n_1^2 M_1^{1/2}} \sum_{q_2, q_2' \sim C/q_1} \sum_{m, m' \sim M_1} \sum_{n_2 \in \mathbb{Z}} |\mathcal{C}| |\mathcal{J}|,$$

where

$$\mathcal{C} = \sum_{d | q} \sum_{d' | q'} d d' \mu\left(\frac{q}{d}\right) \mu\left(\frac{q'}{d'}\right) \sum_{\alpha \equiv -m \mod d}^* \sum_{\alpha' \equiv -m' \mod d'}^* 1,$$

and

$$\mathcal{J} = \int W(w) I(m, N_0 w, q) I(m', N_0 w, q') e\left(-\frac{N_0 n_1 n_2 w}{q_2 q_2' q_1 r}/n_1\right) dw.$$
Moreover from our analysis in Subsection 4.2 it follows that $\mathcal{I} \ll t^{-1}$.

5.3. The zero frequency. The zero frequency $n_2 = 0$ has to be treated differently. Let $\Omega_0$ denote the contribution of the zero frequency to $\Omega$, and let $\Sigma_0$ be its contribution to (11).

**Lemma 2.** We have

$$
\Omega_0 \ll \frac{N_0 M_1^{1/2} C^2 r}{n_1^2 q_1 t} \left(C + M_1\right),
$$

and

$$
\Sigma_0 \ll r^{1/3} N^{1/2} t^{3/2} \left(t^{-1/2} + \eta t^{-3/2} \right).
$$

**Proof.** In the case $n_2 = 0$ it follows from the congruence conditions that $q_2 = q_2'$ and $\alpha = \alpha'$. So the character sum is bounded as

$$
\mathcal{C} \ll \sum_{d,d'|q_1} \sum_{\alpha \mod q_1 \atop n_1 \alpha \equiv -m \mod d} \sum_{\alpha \equiv -m' \mod d'} 1 \ll \sum_{d,d'|q_1} \sum_{(d,d')(m-m')} q r \left[\frac{d^*}{d,d'}\right],
$$

and hence we get

$$
\Omega_0 \ll \frac{N_0}{n_1^2 M_1^{1/2} t} \sum_{q_2 \sim C/q_1} q r \sum_{(d,d') \atop m,m' \sim M_1} \sum_{(d,d')(m-m')} 1
$$

$$
\ll \frac{N_0}{n_1^2 M_1^{1/2} t} \sum_{q_2 \sim C/q_1} q r \sum_{(d,d') \atop m,m' \sim M_1} \left(M_1(d,d') + M_1^2\right).
$$

Trivially executing the remaining sums we get the first part of the lemma.

This bound when substituted in place of $\Omega$ in (11) yields the bound

$$
N^{3/4} K t^{1/3} \left(1 + \frac{K^{1/2}}{C^{1/2}} + \frac{C^{1/2} t}{N^{1/2}}\right) \left(K^{1/4} + \frac{(C t)^{1/2}}{N^{1/4}}\right)
$$

(14)

Here if we substitute $\sqrt{N/K}$ in place of $C$ and use the fact that $K = t^{1-\eta}$, then we get $O(t^{1/3} N^{1/2} t^{1+\eta/2})$ as the final bound to (11). This takes care of all the terms in (14) except the single term which has $C^{1/2}$ in the denominator. This occurs only when $M_1 \sim K$, which is possible only if $N|x|/CQ \sim t$ (as otherwise the integral in (11) is negligibly small). In this case we get

$$
\frac{N^{3/4} K t^{1/3}}{t^{1/2}} \frac{K^{3/4}}{C^{1/2}} \ll \frac{N^{3/4} K^{7/4} t^{1/3}}{t^{1/2}} \frac{(Q t)^{1/2}}{(N|x|)^{1/2}}.
$$

The integral over $x$ takes care of the $x^{1/2}$ in the denominator, and we see that the total contribution of this term to (11) is dominated by $O(t^{1/3} N^{1/2} t^{3/2-3\eta/2})$. The lemma follows. $\Box$
6. Analysis of non-zero frequencies

6.1. The character sum. Our next lemma gives a bound for \( \mathfrak{C} \).

**Lemma 3.** We have

\[
\mathfrak{C} \ll \frac{q_1^3 r}{n_1} \sum \sum_{d_2 | (q_2, q'_2, n_1 + m n_2) \atop d'_2 | (q_2, q'_2, n_1 + m' n_2)} d_2 d'_2.
\]

**Proof.** The ‘character sum’ \( \mathfrak{C} \) can be dominated by a product of two sums \( \mathfrak{C} \ll \mathfrak{C}_1 \mathfrak{C}_2 \) where

\[
\mathfrak{C}_1 = \sum \sum_{d_1, d'_1 | q_1 \atop \alpha \equiv \alpha' \mod{q_1} \frac{r}{n_1} \atop \frac{n_1}{n_1} \equiv -m \mod{d_1} \atop \frac{n_1 \alpha \equiv -m' \mod{d_1'}}}{1},
\]

and

\[
\mathfrak{C}_2 = \sum \sum_{d_2 | q_2 \atop d'_2 | q'_2 \atop \alpha \equiv \alpha' \mod{q_2} \frac{r}{n_1} \atop \frac{n_1}{n_1} \equiv -m \mod{d_2} \atop \frac{n_1 \alpha \equiv -m' \mod{d_2'}}}{1}.
\]

In the second sum since \((n_1, q_2 q'_2) = 1\), we get \( \alpha \equiv -m \bar{n}_1 \mod{d_2} \) and \( \alpha' \equiv -m' \bar{n}_1 \mod{d'_2} \). Then using the congruence modulo \( q_2 q'_2 \) we are able to conclude that

\[
\mathfrak{C}_2 \ll \sum \sum_{d_2 | (q_2, q'_2, n_1 + m n_2) \atop d'_2 | (q_2, q'_2, n_1 + m' n_2)} d_2 d'_2.
\]

In the first sum \( \mathfrak{C}_1 \) the congruence condition determines \( \alpha' \) uniquely in terms of \( \alpha \), and hence

\[
\mathfrak{C}_1 \ll \sum \sum_{d_1, d'_1 | q_1 \atop \alpha \equiv \alpha' \mod{q_1} \frac{r}{n_1} \atop \frac{n_1}{n_1} \equiv -m \mod{d_1}} 1 \ll \frac{q_1^3 r}{n_1}.
\]

This completes the proof of the lemma. \( \square \)

We now substitute these bounds in \((13)\). Writing \( q_2 d_2 \) in place of \( q_2 \) and \( q'_2 d'_2 \) in place of \( q'_2 \) we get that the contribution of the non-zero frequencies to \( \Omega \) is

\[
\Omega \not= 0 \ll \frac{N_0 q_1 r}{n_1^3 M_1^{1/2}} \sum \sum_{d_1, d'_1 | q_1 \atop \alpha \equiv \alpha' \mod{q_1} \frac{r}{n_1} \atop \frac{n_1}{n_1} \equiv -m \mod{d_1}} \frac{1}{n_1^3 M_1^{1/2}} \sum \sum_{m, m' \sim M_1 \atop n_2 \equiv \alpha \mod{d_1}} \sum \sum_{q_2, q'_2 \sim C \atop q_2 q'_2 \not= 0} \sum \sum_{q_2, q'_2 \sim C \atop q_2 q'_2 \not= 0} \sum \sum_{q_2^2 q_2 q'_2 \not= 0 \mod{d_2} \atop q_2^2 q_2 q'_2 \not= 0 \mod{d_2'}} |\mathfrak{I}|.
\]

We denote by \( \Sigma \not= 0 \) the term we get by substituting this for \( \Omega \) in \((11)\).
6.2. The case of small modulus. In this section we will consider the case where 
$q \sim C \ll t^{1+\epsilon}$. Recall that we have $J \ll 1/t$ and $n_2 \neq 0$.

Lemma 4. The contribution of $q \sim C \ll t^{1+\epsilon}$, and $n_2 \neq 0$ to (11) is bounded by

$$\Sigma_{\neq 0, \text{small}} \ll r^{1/2} t^{3/2} N^{1/2} \left( \frac{t^{3-\eta}}{N} + \frac{t^{3/2-\eta/2}}{N^{1/2}} \right).$$

Proof. We use the congruences to count the number of $(m, m')$ in (15). This comes out to be dominated by

$$O((d_2, q_2' d_2' n_1)(d_2', n_2)(1 + M_1/d_2)(1 + M_1/d_2')).$$

It follows that the contribution of this case to $\Omega \neq 0$ is dominated by

$$\frac{N_0 q_1^3 r}{n_1^3 M_1^{1/2} t} \sum_{d_2, d_2'} \sum_{d_2, d_2'} d_2 d_2' \sum_{q_2 \sim C/q_1} \sum_{q_2' \sim C/q_1'} \sum_{1 \leq n_2 < N_2} (d_2, q_2' d_2' n_1) (d_2', n_2) \left( 1 + \frac{M_1}{d_2} \right) \left( 1 + \frac{M_1}{d_2'} \right).$$

Summing over $n_2$ and $q_2$ we arrive at

$$\frac{N_0 q_1^3 r C N_2}{n_1^3 M_1^{1/2} t} \sum_{d_2} \sum_{d_2'} d_2' \sum_{q_2 \sim C/q_1} \left( d_2, q_2' d_2' n_1 \right) \left( 1 + \frac{M_1}{d_2} \right) \left( 1 + \frac{M_1}{d_2'} \right).$$

Next summing over $d_2$ we get

$$\frac{N_0 q_1^3 r C N_2}{n_1^3 M_1^{1/2} t} \sum_{d_2} d_2' \sum_{q_2 \sim C/q_1} \left( C/q_1 + M_1 \right) \left( 1 + \frac{M_1}{d_2} \right).$$

Executing the remaining sums we get

$$\frac{N_0 q_1^3 r C^2 N_2}{n_1^3 M_1^{1/2} t} \left( \frac{C}{q_1} + M_1 \right)^2 \ll \frac{q_1 r}{n_1^3} \left( \frac{N_0 N_2 C^4}{M_1^{1/2} t q_1^2} + \frac{N_0 N_2 C^2 M_1^{3/2}}{t} \right).$$

(16) \( N_0 q_1^3 r C^2 N_2 \left( \frac{C}{q_1} + M_1 \right)^2 \ll \frac{q_1 r}{n_1^3} \left( \frac{N_0 N_2 C^4}{M_1^{1/2} t q_1^2} + \frac{N_0 N_2 C^2 M_1^{3/2}}{t} \right). \)

Suppose $M_1 \asymp (tC)^2/N$ or $M_1 \gg C/q_1$, then when the above bound is substituted for $\Omega$ in (11) we get the bound

$$r^{1/2} t^{3/2} N^{1/2} \left( \frac{t^{3-\eta}}{N} + \frac{t^{3/2-\eta/2}}{N^{1/2}} \right)$$

for $C \ll t^{1+\epsilon}$. In the complementary range when $M_1 \ll C/q_1$ and $M_1$ is not of size $(tC)^2/N$, then $N_0 \asymp (Ct)^3 r/N$. In this case we adopt a different strategy for counting. (Let $d_2 \sim D \ll t^2$. In this case $q_2 d_2 n_1 + m' n_2 \ll C n_1/q_1 + M_1 N_2 \ll C n_1/q_1 + N/n_1 q_1^2 t^2$). Writing $q_2 d_2 n_1 + m' n_2 = -d_2' h$ we see that $h \ll C n_1/q_1 D' + N/n_1 q_1^2 t^2 D' := H$. With this we transform (15) to

$$\frac{N_0 q_1^3 r}{n_1^3 M_1^{1/2}} \sum_{d_2, d_2'} \sum_{d_2, d_2'} d_2 d_2' \sum_{h \ll H} \sum_{q_2 \sim C/q_1} \sum_{q_2' \sim C/q_1'} \sum_{m, m' \sim M_1} \sum_{n_2 \in \mathbb{Z} - \{0\}} \sum_{d_2' d_2 n_1 + m n_2 \equiv 0 \mod d_2} \sum_{d_2' + m' n_2 \equiv 0 \mod d_2} |\mathcal{E}|.$$
Using the second congruence we count the number of $d'_2$ which comes out to be $O((d_2, m'n_2)D'/D)$. The first congruence gives us the number of $m$ which comes out to be $O((n_2, d_2)(1 + M_1/D))$. It follows that (17) is dominated by

$$\frac{N_0 q_1^2 r}{n_1^2 M_1^{1/2} t} \sum_{d_2 \sim D} D'^2 \sum_{h \ll H} \sum_{m' \sim M_1} \sum_{0 < n_2 \ll N_2} (m'n_2, d_2)(n_2, d_2) \left(1 + \frac{M_1}{D}\right).$$

Then summing over $n_2$, $m'$ and $d_2$ we arrive at

$$\frac{N_0 q_1^3 r}{n_1^3 M_1^{1/2} t} M_1 N_2 D D'^2 \sum_{h \ll H} \sum_{q_2' \sim C/q_1 D'} \left(1 + \frac{M_1}{D}\right),$$

which is dominated by

$$(18) \quad \frac{N_0 q_1^3 r}{n_1^3 M_1^{1/2} t} M_1 N_2 C \left(C n_1 + \frac{N}{n_1 q_1 t^2}\right) (D + M_1).$$

Now we substitute $D \ll C/q_1$, $M_1 \ll C/q_1$ and $C \ll t^{1+\varepsilon}$. When the above bound is substituted in place of $\Omega$ in (11) we get the bound

$$r^{1/2} t^{3/2} N^{1/2} \left(\frac{t^{3/2-\eta}}{N^{1/2} + t^{-\eta/2}}\right).$$

This is dominated by the previous bound. The lemma follows. $\square$

### 6.3. The generic case.

It now remains to tackle the case where $C \gg t^{1+\varepsilon}$ and $n_2 \neq 0$.

**Lemma 5.** The contribution of $q \sim C \gg t^{1+\varepsilon}$, and $n_2 \neq 0$ to (11) is bounded by

$$\Sigma_{\neq 0, \text{generic}} \ll r^{1/2} t^{3/2} N^{1/2} \frac{t^{3n/4} N^{1/4}}{t^{1/12}} \ll N^{1/2} t^{3/2-1/6+3\eta/4+\theta/2}.$$ 

**Proof.** In this case we need a better bound for $I$. To this end we seek to apply stationary phase analysis to the integral $I(\ldots)$ in (9), namely

$$\int_0^\infty U(y)e\left(-\frac{t}{2\pi} \log y \pm A\sqrt{y} \pm B(y + u)\right)^{1/3} dy,$$

where $A = 2\sqrt{mN}/q$ and $B = 3(Nn_1^2 n_2)^{1/3}/qr^{1/3}$. Since $C \gg t^{1+\varepsilon}$, from (7) we conclude that we have plus sign with $A$ and that $A \simeq t$. From (8) we conclude that $B \ll t^{1-\eta/2}$. (Otherwise the integrals in (7) and (8) are negligibly small.) As such the stationary point can be written as $y_0 + y_1 + y_2 + \ldots$ with $y_i \ll (B/t)^i$. Explicit calculation yields

$$y_0 = \left(\frac{t}{\pi A}\right)^2, \quad y_1 = \pm \frac{4\pi B}{3t} \left(\frac{t}{\pi A}\right)^{8/3},$$

and so forth. The lemma follows. $\square$
and in general \( y_k = f_k(t, A)(B/t)^k \) for some function \( f_k \). It follows that \( I(m, n_1^2 n_2, q) \) is essentially given by

\[
\frac{1}{t^{1/2}} y_0^{-u} e \left( Bg_1(A) + B^2 g_2(A) + O \left( \frac{B^3}{t^2} \right) \right)
\]

where \( g_1(A) = \mp t^{2/3}/3(\pi A)^{2/3} \ll 1 \) and \( g_2(A) \ll 1/t \). Also note that \( B \asymp (NN_0)^{1/3}/qr^{1/3} \).

It follows that the integral \( J \) is given by

\[
\frac{1}{t} \int W(y) e \left( (Bg_1(A) - B'g_1(A')) + (B^2 g_2(A) - B'^2 g_2(A')) + O \left( \frac{NN_0}{C^2 r^2} \right) \right)
\]

\[
\times e \left( -\frac{N_0 n_1 n_2 y}{q_2 q_2' q_1 r} \right) dy
\]

where in \( B, B' \) we replace \( n_1^2 n_2 \) by \( N_0 y \). Since \( n_2 \neq 0 \) we get

\[
\frac{N_0 n_1 n_2 y}{q_2 q_2' q_1 r} \gg \frac{N_0 n_1}{C' r} \gg \frac{t^\varepsilon NN_0}{C^3 r t^2}
\]

as \( C \gg t^{1+\varepsilon} \) and \( N \ll t^{3+\varepsilon} \). Making a change of variable \( y = z^3 \) and using the third derivative bound for the exponential integral we get

\[
J \ll \frac{1}{t} \left( \frac{q_2 q_2' q_1 r}{N_0 n_1 n_2} \right)^{1/3} \ll \frac{C r^{1/3} t^{2/3}}{t(NN_0)^{1/3}}.
\]

In our bounds for \( \Omega \) (see (16) and (18)), we had the factor \( N_0 N_2 \) which boils down to \( C(NN_0)^{1/3} r^{2/3} / n_1 q_1 \) by substituting the value of \( N_2 \). Now when we incorporate the new bound for the integral, this factor is replaced by \( C^2 r / n_1 q_1 t^{1/3} \). Making this replacement in the proof of Lemma \([4]\) we get Lemma \([5]\). \( \square \)

We now pull together the bounds from Lemma \([2]\), Lemma \([4]\) and Lemma \([5]\) to get that

\[
\frac{S_r(N)}{N^{1/2} t^{3/2}} \ll t^{-1/3} \left( t^{-1/2 + \eta/2} + t^{-3n/2} \right) + t^{1/2} \left( \frac{t^{3-\eta}}{N} + \frac{t^{3-2-\eta/2}}{N^{1/2}} \right) + t^{-1/2} \frac{t^{3n/4} N^{1/4}}{t^{11/12}},
\]

where \( t^{3-\theta}/r^2 \ll N < t^3/r^2 \). It follows that

\[
\frac{S_r(N)}{N^{1/2} t^{3/2}} \ll t^{-1/2 + \eta/2 + \theta/3} + t^{-3n/2 + \theta/3} + t^{7\theta/2 - \eta} + t^{2\theta - \eta/2} + t^{-1/6 + 3n/4},
\]

for \( r \ll t^{\theta} \). Hence we need \( \eta > 7\theta/2 \), and consequently the third term dominates the second and the fourth terms. Also we see that the last term dominates the first. Hence the above bound reduces to

\[
\frac{S_r(N)}{N^{1/2} t^{3/2}} \ll t^{7\theta/2 - \eta} + t^{-1/6 + 3n/4}.
\]

The optimal choice for \( \eta \) is given by \( \eta = 2\theta + 2/21 \). Plugging this in (2) we get that

\[
L(1/2 + it, \pi \times f) \ll t^{3/2 + 3\theta/2 - 2/21} + t^{3/2 - \theta/2},
\]

and with the optimal choice \( \theta = 1/21 \) we obtain the bound given in Theorem \([4]\).
References

[1] K. Aggarwal; S.K. Singh: t-aspect subconvexity for GL(2) L-functions. (arxiv)
[2] D. Goldfeld: Automorphic Forms and L-Functions for the Group GL(n, ℝ). Cambridge Univ. Press, (2006), vol. 99, Cambridge.
[3] A. Good: The square mean of Dirichlet series associated with cusp forms. Mathematika 29 (1982), 278–295.
[4] H. Iwaniec; E. Kowalski: Analytic Number Theory. Amer. Math. Soc. Coll. Publ. 53, American Mathematical Society, Providence, RI, 2004.
[5] X. Li, Bounds for GL(3) × GL(2) L-functions and GL(3) L-functions. Annals of Math. 173 (2011), 301–336.
[6] S. D. Miller; W. Schmid: Automorphic distributions, L-functions, and Voronoi summation for GL(3). Annals of Math. 164 (2006), 423–488.
[7] R. Munshi: Bounds for twisted symmetric square L-functions - III. Adv. in Math. 235 (2013), 74–91.
[8] R. Munshi: The circle method and bounds for L-functions - I. Math. Annalen 358 (2014), 389–401.
[9] R. Munshi: The circle method and bounds for L-functions - II. Subconvexity for twists of GL(3) L-functions. American J. Math., 137 (2015), 791–812.
[10] R. Munshi: The circle method and bounds for L-functions - III. t-aspect subconvexity for GL(3) L-functions. J. Amer. Math. Soc. 28 (2015), 913–938.
[11] P. Sarnak: Recent progress on the quantum unique ergodicity conjecture. Bull. A.M.S. 48 (2011), 211–228.
[12] H. Weyl: Zur Abschätzung von ζ(1 + ti). Math. Z. 10 (1921), 88–101.

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