Article

Stability analysis of a single-species Markovian jumping ecosystem

Ruofeng Rao 1,2

1 Department of Mathematics, Chengdu Normal University, Chengdu 61130, China; ruofengrao@163.com or ruofengrao@cdnu.edu.cn
2 Institute of Financial Mathematics, Chengdu Normal University, Chengdu 61130, China

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Abstract: In this paper, impulsive control on a single-species Markovian jumping ecosystem leads to a stability criterion, and the newly-obtained theorems improve the related existing results. Numerical examples illuminate the effectiveness.

Keywords: a single-species ecosystem; variational methods; global stability; reaction-diffusion; Sobolev spaces

1. Introduction

As pointed out in [1], the following Logistic system has been widely concerned and studied due to its importance in the development of ecology:

\[ \frac{d\mathcal{Z}}{dt} = R\mathcal{Z}(t)(1 - \frac{\mathcal{Z}(t)}{K}), \quad (1) \]

where \(\mathcal{Z}(t)\) represents the density or quantity of population \(\mathcal{Z}\) at time \(t\), \(R > 0\) and \(K\) represent the intrinsic growth rate of population and environmental capacity, respectively. Because all solutions of nonlinear ecosystem are difficult to be given accurately, people pay more attention to the long-term dynamic trend of ecosystem, i.e., the long-term trend of population density (see, e.g. [1-5]). People especially want to know whether the population will tend to a positive constant after a long time, which is related to the final long-term existence of the population. For example, the authors of [2] investigated the long time behavior of the following stochastic ecosystem for a single-species:

\[ d\mathcal{Z} = 3[a - b\mathcal{Z}]dt + \gamma \mathcal{Z}dB(t). \quad (2) \]

Animal populations will inevitably spread because of climate, foraging and random walking. And hence the reaction-diffusion ecological models well simulate the real ecosystem, and ([7-21]). Particularly, reaction-diffusion ecosystem were studied in [11-21]. For example, in [12], a single-species Markovian jumping ecosystem with diffusion and delayed feedback under Dirichlet boundary value was investigated:

\[ \begin{cases} \frac{\partial u(t,x)}{\partial t} = q\Delta u(t,x) + u(t,x)[a - bu(t,x)] + c(r(t))[u(t,x) - u(t - \tau(t),x)], & t \geq 0, \ x \in \Omega, \\ u(t,x) = 0, & x \in \partial\Omega, \ t \geq 0. \end{cases} \quad (3) \]

Markov systems often occurred in various engineering technologies (see, e.g. [24-26]). Particularly, Markovian jumping delayed feedback model reflects well the influence of stochastic factors on time delays of the changes of populations, such as weather, temperature, humidity, ventilation status, and...
so on. But the case of Neumann boundary value on a single-species ecosystem is seldom researched. In fact, Neumann zero boundary value model well simulates the biosphere boundary without population migration. For example, freshwater fish do not enter the sea through rivers. Inspired by some ideas or methods of the related literature [11-28, 30], the author is to investigate the stability of a single-species Markovian jumping ecosystem with diffusion and delayed feedback under Neumann zero boundary value.

For convenience, throughout of this paper, \( \Omega \subset R^N(1 \leq N \leq 3) \) is denoted as a bounded domain with a smooth boundary \( \partial \Omega, \overline{\Omega} = \Omega \cup \partial \Omega \). Denote by \( \|u\|_H = \sqrt{\int_{\Omega} |\nabla u(x)|^2 dx} \) the norm of \( H_0^1(\Omega) \), and by \( \lambda_1 \) the first positive eigenvalue of Laplace operator \( -\Delta \) in \( H_0^1(\Omega) \). Besides, we denote \( [v] = ([v_1], [v_2])^T \) for \( v = (v_1, v_2)^T \in \mathbb{R}^2 \), and \( |C| = (|c_{ij}|)_{2 \times 2} \) for matrix \( C = (c_{ij})_{2 \times 2} \). \( \mathbb{N} \) represents the set of natural numbers \( \{1, 2, \cdots \} \). Denote by \( L^2(\Omega) \) the space of all real-valued square integrable functions with the inner product \( \langle \chi, \eta \rangle = \int_{\Omega} \chi(x)\eta(x)dx \), for \( \chi, \eta \in L^2(\Omega) \) which derives the norm \( \|\chi\| = (\int_{\Omega} \chi^2(x)dx)^{1/2} \) for \( \chi \in L^2(\Omega) \). Denote by \( \Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2} \) the Laplace operator, with domain \( \mathcal{D}(\Delta) = W_0^{1,2}(\Omega) \cap W_0^{2,2}(\Omega) \), which generates a strongly continuous semigroup \( e^{\tau \Delta} \), where \( W_0^{1,2}(\Omega) \) and \( W_0^{2,2}(\Omega) \) are the Sobolev spaces with compactly supported sets. Next, the definition of the mild solution of the system (9) will be introduced. For convenience, the author takes \( U = \{U(t,.)\}_{[0,T]} \) for any given \( T > 0 \) such that \( U(t) \) is a \( L^2(\Omega) \)-valued function.

2. System descriptions

Denote by \( (Y, \mathcal{F}, P) \) the complete probability space with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \). Let \( S = \{1, 2, \cdots, n_0 \} \) and the random form process \( \{r(t) : [0, +\infty) \to S \} \) be a homogeneous, finite-state Markovian process with right continuous trajectories with generator \( \Pi = (\gamma_{ij})_{n_0 \times n_0} \) and transition probability from mode \( i \) at time \( t \) to mode \( j \) at time \( t + \delta \), \( i, j \in S \),

\[
\mathbb{P}(r(t + \delta) = j \mid r(t) = i) = \begin{cases} 
\gamma_{ij}\delta + o(\delta), & j \neq i \\
1 + \gamma_{ii}\delta + o(\delta), & j = i,
\end{cases}
\]

where \( \gamma_{ij} \geq 0 \) is transition probability rate from \( i \) to \( j(j \neq i) \) and \( \gamma_{ii} = -\sum_{j=1,j\neq i}^{n_0} \gamma_{ij}, \delta > 0 \) and \( \lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0 \).

Consider the following ecosystem with diffusion and delayed feedback

\[
\begin{cases}
\frac{\partial u(t,x)}{\partial t} = q\Delta u(t,x) + u(t,x)[a - bu(t,x)] - c(r(t))[u(t,x) - u(t - \tau(t),x)], & t \geq 0, x \in \Omega, \\
\frac{\partial u(t,x)}{\partial x} = 0, & x \in \partial \Omega, t \geq 0, \\
u(s,x) = \Gamma(s,x), & (s,x) \in [-\tau,0] \times \Omega,
\end{cases}
\]

where \( a > 0 \) and \( b > 0 \) describe the growth rate and the intra-specific competition. Besides, the initial value function \( \Gamma(s,x) \) is bounded on \( [-\tau,0] \times \Omega \). For convenience, \( c(r(t)) \) is denoted simply by \( c_r \) for \( r \in S \).

In addition, due to the limited resources of nature, the population density should have an upper limit. At the same time, the population density should also have a lower limit. For example, if the population density of whales is less than a certain degree, the population will become extinct, because male whales and female whales cannot meet in the sea. So the following assumption is suitable:

(H1) There exist two positive constants \( N_1 \) and \( N_2 \) with \( N_1 \leq \frac{a}{b} \leq N_2 \) such that

\[
0 < N_1 \leq u(t,x) \leq N_2, \quad \forall x \in \overline{\Omega}, \quad t \geq -\tau.
\]

Definition 1. Set \( u_s(t,x) \equiv u_s(x), \forall (t,x) \in [-\tau, +\infty) \times \overline{\Omega} \), then \( u_s(x) \) is said to be a stationary solution of the ecosystem system (5) if \( u_s(x) \) satisfies the boundedness assumption (H1), and
integral equations hold for any $U$ below. Furthermore, applying an impulse control on the natural ecosystem (5) or (8) results in the ecosystem (5). And hence, the stability of the above-mentioned zero solution will be investigated.\[\text{Definition 2}\]

\begin{align*}
\text{Definition 2. For an arbitrarily given } u &
\qquad \subset \mathbb{R}^+ \text{, then the system (5) is translated into} \\
&
\frac{\partial U(t,x)}{\partial t} = q\Delta U(t,x) - aU(t,x) - bU^2(t,x) - c_r[U(t,x) - U(t - \tau(t),x)], \quad t \geq 0, t \neq t_k, x \in \Omega, \\
&
U(t^+,x) = M_k U(t^-,x), \quad k \in \mathbb{N}, \\
&
\frac{\partial U(t,x)}{\partial x} = 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
&
U(s,x) = \Gamma(s,x) - \frac{a}{b}u, \quad (s,x) \in [-\tau,0] \times \Omega,
\end{align*}

where the zero solution of the system (8) is corresponding to the positive stationary solution $u_* \equiv \frac{a}{b}$ of the ecosystem (5). And hence, the stability of the above-mentioned zero solution will be investigated below. Furthermore, applying an impulse control on the natural ecosystem (5) or (8) results in

\begin{align*}
\text{Definition 2. For an arbitrarily given } U = \{U(t)\}_{t \in [0,T]} \text{ is called a mild solution of (9) if } U(t,x) \in \mathcal{C}([0,T]; L^2(\Omega)) \text{ satisfies } \int_0^T \|U(s)\|^p ds < \infty, \quad i = 1,2, \quad \text{and the following integral equations hold for any } t \in [0,T] \text{ and } x \in \Omega, \\
U(t,x) = e^{a(t-s)}\xi(0,x) + \int_0^t e^{a(t-s)}\left(-aU(s,x) - bU^2(s,x) - c_r[U(s,x) - U(s - \tau(s),x)]\right)ds \\
+ e^{a(t-s)} \sum_{0<t_k<s} e^{-a(t-t_k)}(M_k-1)U(t_k,x), \quad t \geq 0,
\end{align*}

and

\begin{align*}
\frac{\partial U}{\partial s} = 0, \quad x \in \partial \Omega, \quad t \geq 0, \\
U(s,x) = \xi(s,x), \quad s \in [-\tau,0], \quad x \in \Omega.
\end{align*}

\text{Remark 1. Definition 2 is well defined in view of [22] and [23].}

In this paper, the following condition is also required:
which contradicts the inequality (13).

Suppose (H1) holds. For all $\nu \in \mathbb{R}$, the condition (12), Definition 2 and the continuity of $u$ make the following equations hold:

$$q\Delta u_s + u_s[a - bu_s] = 0, \quad t \geq 0, \quad x \in \Omega,$$

and

$$\frac{\partial u_s}{\partial x} = 0, \quad x \in \partial\Omega, \quad t \geq 0.$$

Thus, Definition 1 yields that $u_s > 0$ defined in Theorem 1 is the unique stationary solution of the system (5).

Below, the author claims that $u_s$ is the unique stationary solution of the ecosystem (5). Indeed, if $u_s$ and $v_s(x)$ are two different stationary solutions of the system (5), then Poincare inequality and the boundary condition yield

$$a \int_{\Omega} (u_s - v_s(x))^2 dx - b \int_{\Omega} (u_s - v_s(x))^2 (u_s + v_s(x)) dx \geq \lambda_1 q \int_{\Omega} |u_s - v_s(x)|^2 dx. \quad (13)$$

The condition (12), Definition 2 and the continuity of $u_s$ and $v_s$ lead to

$$a \int_{\Omega} (u_s - v_s(x))^2 dx - b \int_{\Omega} (u_s - v_s(x))^2 (u_s + v_s(x)) dx < \lambda_1 \int_{\Omega} |u_s - v_s(x)|^2 dx,$$

which contradicts the inequality (13).

This completes the proof. \qed

**Proof.** Obviously, for $(t, x) \in [-\tau, 0] \times \overline{\Omega}$, $u_s \equiv \frac{q}{b}$ makes the following equations hold:

$$a \int_{\Omega} (u_s - v_s(x))^2 dx - b \int_{\Omega} (u_s - v_s(x))^2 (u_s + v_s(x)) dx \geq \lambda_1 q \int_{\Omega} |u_s - v_s(x)|^2 dx. \quad (13)$$

The condition (12), Definition 2 and the continuity of $u_s$ and $v_s$ lead to

$$a \int_{\Omega} (u_s - v_s(x))^2 dx - b \int_{\Omega} (u_s - v_s(x))^2 (u_s + v_s(x)) dx < \lambda_1 \int_{\Omega} |u_s - v_s(x)|^2 dx,$$

which contradicts the inequality (13).

This completes the proof. \qed
Remark 2. As far as I am concerned, Theorem 1 is the first theorem to give the unique existence of the equilibrium point of a single-species ecosystem under Neumann boundary value.

Next, the global stability of the stationary solution \( u_* \equiv \frac{a}{b} \) should be investigated.

**Theorem 2.** Set \( p \geq 1 \). Suppose all the conditions of Theorem 1 hold. Assume, in addition, the condition (H2) holds, and

\[
0 < \omega < 1,
\]

then zero solution of the system (9) is globally exponential stability in the \( p \)th moment, equivalently, \( u_* \equiv \frac{a}{b} \) of the system (10) is globally exponential stability in the \( p \)th moment. where \( N_0 = \max\{|N_1 - \frac{a}{b}|, N_2 - \frac{a}{b}\}, \mu = \inf_{k \in \mathbb{N}}(t_{k+1} - t_k) > 0, \)

\[
\omega = \left[ 4^{p-1} \left( (a + c_r) \left( \frac{M}{q\gamma} \right)^p + b \left( \frac{2M N_0}{q\gamma} \right)^p + c_r \left( \frac{M}{q\gamma} \right)^p + M^2 \left( \max_k |M_k - 1| \right)^p \left( 1 + \frac{1}{q\gamma \mu} \right)^p \right) \right]^{\frac{1}{2}}.
\]

**Proof.** Banach contraction mapping principle will play a important role in this proof, so the author formulates a contraction mapping on a suitable complete metric space firstly.

Let \( \mathbb{I} \) be the normed space consisting of functions \( g(t, x) : [-\tau, +\infty) \times \Omega \rightarrow [N_1 - \frac{a}{b}, N_2 - \frac{a}{b}] \), where \( g \) satisfies:

(A1) \( g \) is \( p \)th moment continuous at \( t \geq 0 \) with \( t \neq t_k (k \in \mathbb{N}) \);

(A2) for any given \( x \in \Omega \), \( \lim_{t \rightarrow t_k^-} g(t, x) \) and \( \lim_{t \rightarrow t_k^+} g(t, x) \) exist, and \( \lim_{t \rightarrow t_k} g(t, x) = g(t_k, x) \);

(A3) \( g(s, x) = \tilde{g}(s, x) \), \( \forall s \in [-\tau, 0], x \in \Omega \);

(A4) \( \lim_{t \rightarrow +\infty} \|g(t)\|^p = 0 \) as \( t \rightarrow +\infty \), where \( a \) is a positive scalar with \( 0 < a < q\gamma \).

It is not difficult to verify that the normed space \( \mathbb{I} \) is a complete metric space if it is equipped with the following metric:

\[
dist(U, V) = \left( \sup_{t \geq -\tau} \|U(t, x) - V(t, x)\|^p \right)^{\frac{1}{p}}, \quad \forall U, V \in \mathbb{I}.
\]

Construct an operator \( \Theta \) such that for any given \( U \in \mathbb{I} \),

\[
\begin{cases}
\Theta(U)(t) = e^{\rho \Delta} \tilde{g}(0, x) + \int_0^t e^{\rho \Delta(t-s)} \left( -aU(s, x) - bU^2(s, x) - c_r[U(s, x) - U(s - \tau(s), x)] \right) ds \\
+ e^{\rho \Delta} \sum_{0 < \xi_1 < t} e^{-\rho \xi_1 \Delta}(M_k - 1)U(t_\xi, x), \quad t \geq 0,
\end{cases}
\]

\[
\frac{\partial \Theta(U)}{\partial \nu} = 0, \quad x \in \partial \Omega, t \geq 0,
\]

\[
\Theta(U)(s, x) = \tilde{g}(s, x), \quad s \in [-\tau, 0], x \in \Omega.
\]

Below, it is necessary to show that \( \Theta : \mathbb{I} \rightarrow \mathbb{I} \), which may require four steps to achieve the goal.

**Step 1.** The author claim that for \( U \in \mathbb{I} \), \( \Theta(U) \) must be \( p \)th moment continuous at \( t \geq 0 \) with \( t \neq t_k (k \in \mathbb{N}) \).

Indeed, \( U \in [N_1 - \frac{a}{b}, N_2 - \frac{a}{b}] \) means the boundedness of \( U \), and let \( \delta \) be small enough scalar, a routine proof yields that if \( \delta \rightarrow 0 \), for \( t \in [0, +\infty) \setminus \{ t_k \}_{k=1}^{\infty} \),

\[
\|\Theta(U)(t + \delta, x) - \Theta(U)(t, x)\|^p \leq 4^{p-1} \|e^{\rho(t+\delta)\Delta}(0, x) - e^{\rho t \Delta}(0, x)\|^p
\]

\[
+ 4^{p-1} \left\| \int_0^{t+\delta} e^{\rho(t+\delta-s)\Delta} \left[ -a(U(s, x) - bU^2(s, x)) - c_r[U(s, x) - U(s - \tau(s), x)] \right] ds \right\|^p
\]

\[
+ 4^{p-1} \int_0^{t+\delta} e^{\rho(t+\delta-s)\Delta} \left[ c_r[U(s, x) - U(s - \tau(s), x)] \right] ds = \int_0^{t+\delta} e^{\rho(t+\delta-s)\Delta} [c_r(U(s, x) - U(s - \tau(s), x))] ds \left\| U(t_\xi, x) \right\|^p
\]

\[
+ 4^{p-1} \int_0^{t+\delta} e^{\rho(t+\delta-s)\Delta} \sum_{0 < \xi_1 < t+\delta} e^{-\rho \xi_1 \Delta}(M_k - 1)U(t_\xi, x) - e^{\rho \Delta} \sum_{0 < \xi_1 < t} e^{-\rho \xi_1 \Delta}(M_k - 1)U(t_\xi, x) \right\|^p \rightarrow 0.
\]
which proves the claim. And then (A1) is verified.

**Step 2.** Verifying $\Theta(U)$ satisfies (A2), where $U \in \mathcal{I}$.

In fact, for $U \in \mathcal{I}$, people can easily see it from (17) that $\lim_{t \to \tau_k^+} \Theta(U)(t, x)$ and $\lim_{t \to \tau_k^+} \Theta(U)(t, x)$ exist, and $\lim_{t \to \tau_k^+} \Theta(U)(t, x) = \Theta(U)(t, x)$, which verifies (A2).

**Step 3.** Verifying $\Theta(U)$ satisfies (A3), where $U \in \mathcal{I}$. Indeed, the third equation of (17) verifies (A3) directly.

**Step 4.** Verifying (A4), i.e., for $U \in \mathcal{I}$, verifying

$$e^{\alpha t} \left\| \Theta(U)(t) \right\|^p \to 0, \text{ if } t \to +\infty. \tag{19}$$

Indeed,

$$e^{\alpha t} \left\| \Theta(U)(t, x) \right\|^p = e^{\alpha t} \left\| e^{\rho\Delta} \xi(0, x) + \int_0^t e^{\rho(t-s)\Delta} \left( -a U(s, x) - b U^2(s, x) - c_r [U(s, x) - U(s - \tau(s, x))] \right) ds \right\|^p.$$

Moreover,

$$e^{\alpha t} \left\| e^{\rho\Delta} \xi(0, x) \right\|^p \leq M^p e^{\alpha t} \left\| e^{-\gamma q t} \left\| \xi(0, x) \right\|\right\|^p \to 0, \text{ if } t \to +\infty. \tag{21}$$

$U \in \mathcal{I}$ means $U \in [N_1, N_2]$, and

$$U^2 \leq N_0 |U|, \text{ where } N_0 = \max\{|N_1 - \frac{a}{b}|, N_2 - \frac{a}{b}\}. \tag{22}$$

and Holder inequality yield

$$e^{\alpha t} \left\| \int_0^t e^{\rho(t-s)\Delta} \left( -a U(s, x) - b U^2(s, x) \right) ds \right\|^p \leq 2^{p-1} M^p \left[ a^p \left( \frac{1}{q} \right)^{p-1} \int_0^t e^{-(q \gamma - a)(t-s)} e^{\alpha s} \left\| U(s, x) \right\|^p ds + b^p N_0^p \left( \frac{1}{q q} \right)^{p-1} \int_0^t e^{-q \gamma (t-s)} \left\| U \right\|^p ds \right]. \tag{23}$$

On the other hand, $e^{\alpha t} \left\| U(t, t) \right\|^p \to 0$ means that for any $\varepsilon > 0$, there exists $t^* > 0$ such that all $e^{\alpha t} \left\| U(t, t) \right\|^p < \varepsilon$. And so,

$$\int_0^t e^{-q \gamma - a}(t-s) e^{\alpha s} \left\| U(s, x) \right\|^p ds \leq \max_{s \in [0, t^*]} \left\{ e^{\alpha s} \left\| U(s, x) \right\|^p \right\} e^{-(q \gamma - a) t} \frac{1}{q q - a} e^{q \gamma - a} t^* + \varepsilon \frac{1}{q q - a},$$

which together with the arbitrariness of $\varepsilon$ implies that

$$\int_0^t e^{-q \gamma - a}(t-s) e^{\alpha s} \left\| U(s, x) \right\|^p ds \to 0, \text{ as } t \to +\infty. \tag{24}$$
Similarly as the proof of (23), people can prove

\[
e^\Delta t \left\| \int_0^t e^{\Delta(t-s)} \left[ -aU(s,x) - bU^2(s,x) \right] ds \right\|^p, \quad t \to +\infty.
\]

(25)

\[
e^\Delta t \left\| \int_0^t e^{\Delta(t-s)} \right\|^p, \quad t \to +\infty.
\]

(26)

Due to \( U(s,x) = \xi(s,x) \) is bounded on \([ -\tau, 0] \times \Omega \), it is not difficult to prove similarly

\[
e^\Delta t \left\| \int_0^t e^{\Delta(t-s)} \right\|^p, \quad t \to +\infty.
\]

(27)

Next, using the definition of Riemann integral \( \int b^a e^x dx \) results in

\[
e^\Delta t \left\| e^{\Delta t} \right\| \sum_{0 < t_k < t} e^{-\alpha t_k} (M_k - 1) |U(t_k, x)|^p
\]

\[
\leq 2^{p-1} \max_k |M_k - 1| \left[ e^{-\alpha \gamma x} \left( \sum_{0 < t_k < t} e^{\alpha t_k} \|U(t_k, x)\|^p \right) + \varepsilon \left( \frac{1}{\gamma^b q - 1} \right) \right] \to 0.
\]

(28)

Combining (20)-(28) yields (19).

It follows from the above four steps that

\[
\Theta(\mathcal{I}) \subset \mathcal{I}.
\]

(29)

Finally, the author claims that \( \Theta \) is a contractive mapping on \( \mathcal{I} \). Indeed, for any \( U, V \in \mathcal{I} \), Hölder inequality and (H2) yield

\[
\left\| \int_0^t e^{\Delta(t-s)} \right\| \leq \left( M \int_0^t e^{-\gamma(t-s)} \|U(s,x) - V(s,x)\| ds \right)
\]

\[
\leq M^p \left( \frac{1}{\gamma^b q} \right) \left( \frac{1}{\gamma} \right) \left( \sup_{t_0 \leq t} \|U(t, x) - V(t, x)\| \right)^p
\]

\[
\leq \left( \frac{M}{\gamma^b q} \right)^p [\text{dist}(U, V)]^p.
\]

(30)

Similarly,

\[
\left\| \int_0^t e^{\Delta(t-s)} \right\| \leq \left( 2MN_0 \int_0^t e^{-\gamma(t-s)} \|U(s,x) - V(s,x)\| ds \right)
\]

\[
\leq \left( \frac{2MN_0}{\gamma^b q} \right)^p [\text{dist}(U, V)]^p.
\]

(31)

and

\[
\left\| \int_0^t e^{\Delta(t-s)} \right\| \leq \left( \frac{M}{\gamma^b q} \right)^p [\text{dist}(U, V)]^p
\]

(32)
Suppose $t_{j-1} < t \leq t_j$, then the definition of Riemann integral $\int_0^t e^s ds$ yields
\[
\|e^{t \Delta} \sum_{0 < t_k < t} e^{-t_k \Delta} (M_k - 1)[U(t_k, x) - V(t_k, x)]\| \leq \sum_{0 < t_k < t} e^{-t_k \Delta} (t_{k+1} - t_k) \cdot \text{dist}(U, V)
\]
which derives
\[
\text{dist}(\Theta(U), \Theta(V)) \leq (\max_{r \leq t} \omega_r) \cdot \text{dist}(U, V), \quad \forall U, V \in \mathbb{L}.
\]

It follows from (30)-(33) that
\[
\|\Theta(U) - \Theta(V)\| \leq 4p^{-1} (a + c_1) \int_0^t e^{(t-s) \Delta} [U(s, x) - V(s, x)] ds + 4p^{-1} b \int_0^t e^{(t-s) \Delta} [U^2(s, x) - V^2(s, x)] ds \|
\]
where $\omega_r$ satisfies $0 < \omega_r < 1$. This shows that $\Theta : \mathbb{L} \to \mathbb{L}$ is a contraction mapping such that there exists a fixed point $U$ of $\Theta$ in $\mathbb{L}$, which implies that $U$ is the solution of the system (9), satisfying $e^{t \Delta} U \to 0$, $t \to +\infty$ so that $e^{t \Delta} [U - U_*] \to 0$, $t \to +\infty$. Therefore, zero solution of the system (9) is globally exponential stability in the $p$th moment, equivalently, $U_* \equiv \frac{d}{N}$ of the system (10) is globally exponential stability in the $p$th moment. $\square$

**Remark 3.** As far as I am concerned, it is the first paper to apply the Laplacian semigroup theory to deal with the stability of a single-species ecosystem with Markovian jumping and delayed feedback.

4. Numerical example

**Example 1.** Set $S = \{1, 2\}$, $c_1 = 0.1, c_2 = 0.06, q = 0.2, a = 0.5, b = 0.2, N_1 = 2, N_2 = 3$, then $N_0 = 0.5, u_* = 2.5$. Suppose, in addition, $\Omega = (0, \pi)$, then by computing the eigenfunctions of $-\Delta$, one can get $\|e^{t \Delta}\| \leq e^{-\pi^2 t}$, $t \geq 0$, and so $\gamma = \pi^2 = \lambda_1, M = 1$. Direct computation yields
\[
0.5 = a < 2.7739 = \lambda_1 q + 2 b N_1
\]
which implies that the condition (12) is satisfied. Theorem 1 yields that the positive solution $u_* \equiv 2.5$ is the unique stationary solution of the system (5).

**Example 2.** Suppose all the data of Example are applied to this example. Assume, in addition, $p = 1.5, M_k \equiv 1.02, \mu = 5$, Obviously the condition (H2) holds in Example 1, and direct calculations yield:
\[
\omega_1 = \left[4p^{-1} \left(\frac{M}{q^2} + b \left(\frac{2MN_0}{q^2}\right) + c_1 \left(\frac{M}{q^2}\right)^p + M^2 p \left(\max_k |M_k - 1|\right) p \left(1 + \frac{1}{q^2 \mu}\right)\right) \right]^{\frac{1}{p}}
\]
and
\[
\omega_2 = \left[4p^{-1} \left(\frac{M}{q^2} + b \left(\frac{2MN_0}{q^2}\right) + c_2 \left(\frac{M}{q^2}\right)^p + M^2 p \left(\max_k |M_k - 1|\right) p \left(1 + \frac{1}{q^2 \mu}\right)\right) \right]^{\frac{1}{p}}
\]
And hence the condition (14) is satisfied.

Thereby, Theorem 2 yields that zero solution of the system (9) is globally exponential stability in the \( p \)th moment, equivalently, \( u_1 \equiv 2.5 \) of the system (10) is globally exponential stability in the \( p \)th moment.

5. Conclusions and further considerations

In this paper, there are some improvements on mathematical methods, for it is the first paper to employ fixed point theory, Laplacian semigroup theory and variational methods to solve the unique existence of the globally stable positive equilibrium point of a single-species ecosystem with Markovian jumping and delayed feedback. Numerical examples are given to show the feasibility of artificial management of nature by way of impulse control.

As pointed out in [28, Definition 1], the author originally proposed a class of global asymptotical stability in the meaning of switching in [27, Definition 3]. Particularly, in the case of one subsystem, the switched system becomes non-switched common system, the global asymptotical stability in the meaning of switching becomes that in the classical meaning. The author actually gave both of the mentioned two classes of global stability for the unique (positive) stationary solution in [27]. Now the author wants to know whether the two classes of global stability can be applied to ecosystem, and what meanings about the stability in the meaning of switching for an ecosystem? This is an interesting problem. Besides, in [27, Statement 2], the author originally design an example to show that under the influence of diffusion, the unique equilibrium point of ordinary differential system with Lipschitz continuous activation function becomes at least three equilibrium points of its corresponding partial differential system. Now the author wants to know whether the unique equilibrium point is globally stable. Furthermore, the author in [27, Section 5] originally proposed four problems, particularly [27, Problem 1] and [27, Problem 4] can also be suitable for the case of Neumann boundary value in this paper. Such problems are also interesting. Moreover, how to consider the case of infinite delays in [10, Problem 6] to ecosystem? It is also an interesting problem. To sum up, the following problems are more interesting:

**Problem 1.** Is the zero solution of the ordinary differential equation in [21, Theorem 3] or [27, Statement 2] global stable?

**Problem 2.** How to design another example somewhat similar to [21, Theorem 3] or [27, Statement 2] with Lipschitz continuous activation function, where the ordinary differential equation possesses a globally stable equilibrium point, but its corresponding partial differential equation owns at least two stationary solutions.

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