A $G_2$ Unification of the Deformed and Resolved Conifolds

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ABSTRACT

We find general first-order equations for $G_2$ metrics of cohomogeneity one with $S^3 \times S^3$ principal orbits. These reduce in two special cases to previously-known systems of first-order equations that describe regular asymptotically locally conical (ALC) metrics $\mathbb{B}_7$ and $\mathbb{D}_7$, which have weak-coupling limits that are $S^1$ times the deformed conifold and the resolved conifold respectively. Our more general first-order equations provide a supersymmetric unification of the two Calabi-Yau manifolds, since the metrics $\mathbb{B}_7$ and $\mathbb{D}_7$ arise as solutions of the same system of first-order equations, with different values of certain integration constants. Additionally, we find a new class of ALC $G_2$ solutions to these first-order equations, which we denote by $\tilde{C}_7$, whose topology is an $\mathbb{R}^2$ bundle over $T^{1,1}$. There are two non-trivial parameters characterising the homogeneous squashing of the $T^{1,1}$ bolt. Like the previous examples of the $\mathbb{B}_7$ and $\mathbb{D}_7$ ALC metrics, here too there is a $U(1)$ isometry for which the circle has everywhere finite and non-zero length. The weak-coupling limit of the $\tilde{C}_7$ metrics gives $S^1$ times a family of Calabi-Yau metrics on a complex line bundle over $S^2 \times S^2$, with an adjustable parameter characterising the relative sizes of the two $S^2$ factors.

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1 Introduction

Seven-dimensional manifolds of $G_2$ holonomy are of considerable interest in string theory and M-theory, since they are closely related to the six-dimensional Calabi-Yau manifolds that have formed the basis of most attempts to extract phenomenologically realistic four-dimensional physics from string theory. Although the principal phenomenological focus would be on compact internal manifolds, implying a discrete spectrum of massive and massless four-dimensional fields, it is often useful to study non-compact manifolds, since these can form the “building blocks” that describe the regions of localised curvature near an orbifold limit of a smooth compact space.

If a non-compact $G_2$ metric has an asymptotically locally conical (ALC) geometry, meaning that at large distance it approaches a twisted product of a circle and a six-dimensional asymptotically conical (AC) metric, then this circle may be used in a Kaluza-Klein reduction that allows a reinterpretation of an M-theory solution as a type IIA string solution. If there is a non-trivial parameter in the $G_2$ metric that allows one to adjust the asymptotic radius of the circle, while holding the scale-size of the interior “core” fixed, then one can take a limit of vanishingly-small radius for the circle that therefore corresponds to weak coupling in the type IIA string picture. In this weak-coupling limit, known mathematically as the “Gromov-Hausdorff limit,” the remaining six-dimensional metric becomes exactly a Ricci-flat Kähler metric; i.e. a Calabi-Yau metric, with $SU(3)$ holonomy.

The non-compact Calabi-Yau metrics that have received the most attention are those associated with the singular conifold, and its smoothed-out versions. The conifold itself is the Ricci-flat metric on the cone over the homogeneous Einstein metric on $T^{1,1} = (S^3 \times S^3)/S^1$, which becomes singular at the origin where the radius of the $T^{1,1}$ collapses to zero. It can be smoothed out in two different ways, with the apex of the cone either becoming a smooth 2-sphere or a smooth 3-sphere. The former case is called the resolved conifold and the latter is called the deformed conifold.

An ansatz for a class of cohomogeneity one $G_2$ metrics with $S^3 \times S^3$ principal orbits was introduced. The ansatz has four metric functions, and a system of four first-order equations was derived by requiring that the metrics have $G_2$ holonomy. These equations admit solutions describing ALC metrics with a degenerate $S^3$ orbit (an $S^3$ bolt) at short distance. A specific solution was obtained, and it was argued that this was but one member of a one-parameter family of ALC metrics. In , the general regular short-distance Taylor expansions for solutions with an $S^3$ bolt were obtained, which indeed contained one non-trivial adjustable parameter. By numerical integration of the first-order
equations governing the $G_2$ holonomy, it was shown that within a certain range for this non-trivial parameter, the metrics are ALC and regular also at large distance. The adjustable parameter in this family of solutions, which were denoted by $B_7$ in [1], characterises the asymptotic radius of the circle factor, while the radius of the $S^3$ bolt is held fixed. At the upper end of the parameter range the radius becomes infinite, and the $G_2$ metric becomes the original AC metric found in [7, 8]. At the lower end of the parameter range, the asymptotic radius of the circle goes to zero, and the metric approaches $S^1$ times the Ricci-flat Kähler metric on the deformed conifold. An important feature of the one-parameter family of $B_7$ metrics is that the $S^3$ bolt is always “round” (with its Einstein metric), regardless of the value of the parameter.

A more general class of metric ansatz, again with cohomogeneity one and $S^3 \times S^3$ principal orbits, was introduced in [6]. There are nine functions in the metric ansatz, and the isometry group is $SU(2) \times SU(2)$ acting by left translations on the orbits. In order to be able to make a Kaluza-Klein relation to Calabi-Yau metrics, a specialisation of this to a six-function ansatz was made recently in [3]. The restriction from nine to six functions, the metric ansatz appearing in [1] below, is chosen so as to give an additional right-acting diagonal $U(1)$ isometry. This $U(1)$ isometry plays an important role in the subsequent Kaluza-Klein reduction. By then writing down a natural choice of associative 3-form, and imposing closure and co-closure (the necessary and sufficient conditions for $G_2$ holonomy), a new system of first-order equations for $G_2$ holonomy was obtained in [1]. This system comprised four independent first-order equations, together with two algebraic expressions for the remaining two metric functions. (The previous system of first-order equations found in [3] can also be embedded within the more general class considered in [1], as an inequivalent set of four independent first-order equations with two algebraic expressions for the remaining two metric functions.) The new system of four first-order equations obtained in [1] was shown to admit short-distance Taylor expansions describing regular metrics with an $S^3$ bolt. This time, however, there is a non-trivial adjustable parameter that characterises the degree of homogeneous “squashing” of the $S^3$ bolt. It was then shown in [3], by using numerical integration methods, that for a suitable range of the parameter one gets ALC metrics, denoted by $D_7$, that are regular also at large distance. The upper end of the parameter range, as the radius of the stabilising circle goes to infinity, corresponds again to the case of the AC metric found in [7, 8]. At the lower end of the parameter range, i.e. in the Gromov-Hausdorff limit, the $S^3$ bolt becomes infinitely squashed to $S^2$, and one obtains

\footnote{An apparently equivalent system, but with a second-order equation, has since appeared in [10].}
$S^1$ times the resolved conifold [9].

With these two examples, the $B_7$ and the $D_7$ metrics, one has the ability to obtain both the deformed conifold and the resolved conifold as weak-coupling limits of families of seven-dimensional $G_2$ metrics. In fact since both $B_7$ and $D_7$ have a coincident AC limit at the upper ends of their parameter ranges, one could say that deformed and the resolved conifold solutions in weakly-coupled string theory are related via a strong-coupling regime and the eleven-dimensional M-theory [9].

The two systems of four first-order equations that give rise to the $B_7$ and $D_7$ families of $G_2$ metrics both, of course, imply that the conditions of Ricci-flatness are satisfied. Thus the $B_7$ and $D_7$ metrics come from a common six-function metric ansatz and they satisfy the same system of Ricci-flat equations, with certain specific (consistent) truncations implied by the pair of algebraic constraints that were imposed in the two cases. Thus the statement in [9] about the strong-coupling relation between the deformed and the resolved conifolds was at the level of solutions of the M-theory field equations. It could not be argued in [9] that there was a connecting path within the stronger criterion of $G_2$ holonomy, since the two systems of first-order equations for the two cases were ostensibly disconnected.

In order to be able to link the deformed and the resolved conifolds via a $G_2$ (and hence supersymmetric) path, we need to find a larger system of first-order equations, within the framework of the six-function metric ansatz, which not only ensures $G_2$ holonomy but also encompasses both of the previous four-equation first-order systems. The search for such an enlarged $G_2$ system forms the subject of this paper. We shall show that the most general system of equations determining $G_2$ holonomy, starting from our six-function metric ansatz, is a system comprising five independent first-order equations, together with one algebraic expression for the sixth metric function. Both of the previous systems of four first-order equations are then contained as (consistently truncated) special cases of our more general equations. Thus, in particular, we now have a supersymmetric path of $G_2$ metrics relating the deformed and the resolved conifolds.

An additional outcome from our new $G_2$ equations is that we can also fit some further six-dimensional Calabi-Yau metrics into the picture. Long ago, a general construction of Ricci-flat Kähler metrics on line bundles over Einstein-Kähler bases spaces was found [11, 12]. A particular example is the six-dimensional case of the line bundle over $S^2 \times S^2$. This is a metric of cohomogeneity one, with principal orbits that are $T^{1,1}/Z_2$. In fact this particular metric was shown in [13] to arise as the Gromov-Hausdorff limit of another class of regular ALC $G_2$ metrics that were obtained there. They are solutions of the four-function
The general $G_2$ metrics with $S^3 \times S^3$ principal orbits

2.1 The metric ansatz, and first-order equations

Our starting point is an ansatz for seven-dimensional metrics of cohomogeneity one, and $S^3 \times S^3$ principal orbits, that was used in [3]:

$$ds_7^2 = dt^2 + a^2 [(\Sigma_1 + g \sigma_1)^2 + (\Sigma_2 + g \sigma_2)^2] + b^2 (\sigma_1^2 + \sigma_2^2) + c^2 (\Sigma_3 + g_3 \sigma_3)^2 + f^2 \sigma_3^2, \quad (1)$$

In fact, it turns out that the subsequent equations for $G_2$ holonomy are greatly simplified by working with a different set of variables ($\tilde{c}, \tilde{f}, \tilde{g}_3$), in terms of which (1) becomes

$$ds_7^2 = dt^2 + a^2 ((\Sigma_1 + g \sigma_1)^2 + (\Sigma_2 + g \sigma_2)^2) + b^2 (\sigma_1^2 + \sigma_2^2) + c^2 (\Sigma_3 - \sigma_3)^2 + f^2 (\Sigma_3 + \tilde{g}_3 \sigma_3)^2, \quad (2)$$

where $a$, $b$, $\tilde{c}$, $\tilde{f}$, $g$ and $\tilde{g}_3$ are functions only of the radial variable $t$, and $\sigma_i$ and $\Sigma_i$ are left-invariant 1-forms of $SU(2) \times SU(2)$. They can be expressed in terms of Euler angles as

$$\sigma_1 + i \sigma_2 = e^{-i\psi} (d\theta + i \sin \theta \, d\phi), \quad \sigma_3 = d\psi + \cos \theta \, d\phi,$$

$$\Sigma_1 + i \Sigma_2 = e^{-i\tilde{\psi}} (d\tilde{\theta} + i \sin \tilde{\theta} \, d\tilde{\phi}), \quad \Sigma_3 = d\tilde{\psi} + \cos \tilde{\theta} \, d\tilde{\phi}. \quad (3)$$

The metric is a specialisation of a nine-function ansatz introduced in [6], in which all three directions $\sigma_i$ and $\Sigma_i$ were given distinct metric functions. The nine-function ansatz has the
left-acting \( SU(2) \times SU(2) \) group as isometries. As discussed in \([2]\), by setting the \( i = 1 \) and \( i = 2 \) directions equal as in \((2)\), we gain an additional diagonal \( U(1) \) of right-acting isometries, generated by the Killing vector

\[
K = \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \tilde{\psi}}. \tag{4}
\]

In \([8]\), we obtained first-order equations for \( G_2 \) holonomy, by writing down a specific ansatz for an associative 3-form. Here, we shall follow a slightly different strategy, and instead demand that there exist a covariantly-constant (i.e. parallel) spinor \( \eta \). Having a covariantly-constant spinor is equivalent to having a closed and co-closed associative 3-form; they both imply \( G_2 \) holonomy. However, our result that we shall get from using the requirement of a covariantly-constant spinor is more general than the result we obtained in \([8]\), since the associative 3-form in \([8]\) was not the most general that is allowed by the isometries of the metric, whilst we make no analogous restrictive assumption for the covariantly-constant spinor. In consequence we shall obtain the most general possible set of first-order equations giving \( G_2 \) holonomy for the metric ansatz \((2)\).

After imposing the requirement on \((3)\) that there exist a covariant-constant spinor \( \eta \), \( D \eta \equiv d \eta + \frac{1}{4} \omega_{ab} \Gamma^{ab} \eta = 0 \), we obtain an algebraic expression for \( \tilde{g}_3 \),

\[
\tilde{g}_3 = \frac{a^3 \tilde{c} g + (b \tilde{f} - a \tilde{c} g) W^2}{a^2 b \tilde{f}}, \tag{5}
\]
together with first-order equations for the five remaining metric functions,

\[
\begin{align*}
a' &= \frac{a^4 g^2 - \tilde{c}^2 W^2}{2 a b \tilde{c} W}, \\
b' &= \frac{a^5 g^4 - [a \tilde{c}^2 + 2b \tilde{c} \tilde{f} g + a (a^2 - \tilde{c}^2) g^2] W^2}{2 a b^2 \tilde{c} W}, \\
\tilde{c}' &= \frac{a^2 \tilde{c}^2 + (\tilde{c}^2 - 2a^2) W^2}{2 a^2 b W}, \\
\tilde{f}' &= \frac{a^2 \tilde{c} g (a \tilde{f} g - 2b \tilde{c}) - g [a \tilde{c} \tilde{f} g - 2b (\tilde{c}^2 + \tilde{f}^2)]] W^2}{2 a b^3 W}, \\
g' &= \frac{a^3 b g + \tilde{c} \tilde{f} W^2}{a^3 \tilde{c} W},
\end{align*}
\tag{6}
\]

where

\[
W \equiv \sqrt{b^2 + a^2 g^2}, \tag{7}
\]

and a prime denotes a derivative with respect to \( t \). One can straightforwardly verify, using the results for the Ricci tensor obtained in \([3]\), that the metric \((2)\) is Ricci-flat if \((3)\) and \((3)\) are satisfied.
2.2 Parallel spinor and calibrating 3-form

It is quite easy to give an explicit expression for the covariantly-constant spinor that we found in the analysis in section 2.1. We take the vielbein for the metric (2) to be

\[
e^0 = dt, \quad e^1 = a (\Sigma_1 + g \sigma_1), \quad e^2 = a (\Sigma_2 + g \sigma_2), \quad e^3 = c (\Sigma_3 - \sigma_3),
\]
\[
e^4 = b \sigma_1, \quad e^5 = b \sigma_2, \quad e^6 = f (\Sigma_3 + \tilde{g}_3 \sigma_3).
\]

(8)

Using the natural choice of spinor frame, we then find that the covariantly-constant spinor \( \eta \) is given by

\[
\eta = W^{-\frac{1}{2}} \left( \sqrt{b + i a g} \epsilon_1 + \sqrt{b - i a g} \epsilon_2 \right),
\]

(9)

where \( \epsilon_1 \) and \( \epsilon_2 \) are spinors with constant components, defined uniquely up to overall scale by

\[
\Gamma_{14} \epsilon_1 = \Gamma_{25} \epsilon_1 = \Gamma_{36} \epsilon_1 = i \epsilon_1, \quad \epsilon_2 = \Gamma_{123} \epsilon_1.
\]

(10)

If we normalise \( \epsilon_1 \), and hence \( \epsilon_2 \), to unit length, \( \epsilon_1^\dagger \epsilon_1 = \epsilon_2^\dagger \epsilon_2 = 1 \), then we shall also have \( \eta^\dagger \eta = 1 \).

From \( \eta \), we can construct the closed and co-closed associative 3-form \( \Phi \), whose tangent-space components are given by \( \Phi_{abc} = i \eta^\dagger \Gamma_{abc} \eta \). This turns out to be

\[
\Phi = e^0 \wedge (e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6) + (e^1 \wedge e^2 - e^4 \wedge e^5) \wedge (\alpha e^3 + \beta e^6)
\]
\[
+ (e^1 \wedge e^5 - e^2 \wedge e^4) \wedge (\beta e^3 - \alpha e^6),
\]

(11)

where

\[
\alpha \equiv \frac{a g}{\sqrt{b^2 + a^2 g^2}}, \quad \beta \equiv \frac{b}{\sqrt{b^2 + a^2 g^2}}.
\]

(12)

Note that \( \alpha^2 + \beta^2 = 1 \), and so the explicitly-appearing radial dependence in (11) is an \( SO(2) \) rotation in the \( (e^3, e^6) \) plane. One can easily verify that \( \Phi \) given in (11) is indeed a closed and co-closed associative 3-form.

2.3 Two constants of the motion

Note that two combinations of the five first-order equations can be integrated. Specifically, if we define

\[
p \equiv -(a \tilde{c} g + b \tilde{f} \tilde{g}_3) W, \quad q \equiv \frac{a^2 (a \tilde{c} g + b \tilde{f})}{W},
\]

(13)

where \( \tilde{g}_3 \) is given by (3) and \( W \) is given by (7), then we shall have \( dp/dt = 0 \) and \( dq/dt = 0 \). This can be seen by substituting the first-order equations into these expressions, but a
simpler way to see the result is by noting that \( p \) and \( q \) are nothing but the coefficients of the volume forms of the two 3-spheres in the expression (11) for the associative 3-form \( \Phi \);

\[
\Phi = p \sigma_1 \wedge \sigma_2 \wedge \sigma_3 + q \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 + \cdots .
\] (14)

It is now evident, as observed in [10], that the closure condition \( d\Phi = 0 \) implies that \( p \) and \( q \) must be constants. If it happens that \( p = -q \), then the metric has a \( \mathbb{Z}_2 \) symmetry under which the two \( S^3 \) factors in the \( S^3 \times S^3 \) principal orbits are interchanged [10]. When \( p \neq -q \), the metric is not \( \mathbb{Z}_2 \) symmetric.

The two expressions (13) could be used in order to reduce the system of five first-order equations to a system with three first-order equations and the two constants \( p \) and \( q \). (A formulation apparently equivalent to this, starting from a \( G_2 \)-invariant ansatz for \( \Phi \) involving \( p \) and \( q \) as given constants, and reducing to a second-order differential equation, was obtained in [10]. We understand that \( G_2 \) metrics with constants equivalent to \( p \) and \( q \) have also been considered by S. Gukov, K. Saraikin and A. Volovich [15].)

It is very important, however, that in our complete system of five first-order equations the quantities \( p \) and \( q \) arise as constants of integration, rather than being fixed, given constants in an ansatz with fewer independent metric functions. In particular, it means that solutions with different values of \( p \) and \( q \) can all be viewed as solutions of the same system of five first-order equations that imply \( G_2 \) holonomy.

2.4 An alternative form for the metric

For some purposes, it is useful to make a further reorganisation of the metric on the orbit space, replacing (2) by the more symmetrical ansatz

\[
\begin{align*}
\text{ds}_7^2 &= dt^2 + a^2 \left[ (\Sigma_1 + \tilde{g} \sigma_1)^2 + (\Sigma_2 + \tilde{g} \sigma_2)^2 \right] + b^2 \left[ (\Sigma_1 - \tilde{g} \sigma_1)^2 + (\Sigma_2 - \tilde{g} \sigma_2)^2 \right] \\
&\quad + c^2 \left( \Sigma_3 - \sigma_3 \right)^2 + f^2 \left( \Sigma_3 + \tilde{g} \sigma_3 \right)^2.
\end{align*}
\] (15)

After doing this, we find that the algebraic relation (6) becomes

\[
\tilde{g}_3 = \tilde{g}^2 - \frac{\tilde{c} (a^2 - b^2)(1 - \tilde{g}^2)}{2a \, b \, \tilde{f}},
\] (16)

while the first-order equations (7) for the five remaining metric functions become

\[
\begin{align*}
\tilde{a}' &= \frac{\tilde{c}^2 (\tilde{a}^2 - \tilde{b}^2) + 4\tilde{a}^2 (\tilde{a}^2 - \tilde{b}^2) - \tilde{c}^2 (5\tilde{a}^2 - \tilde{b}^2) - 4\tilde{a} \, \tilde{b} \, \tilde{c} \, \tilde{f} \, \tilde{g}^2}{16\tilde{a}^2 \, \tilde{b} \, \tilde{c} \, \tilde{g}^2}, \\
\tilde{b}' &= -\frac{\tilde{c}^2 (\tilde{a}^2 - \tilde{b}^2) + 4\tilde{b}^2 (\tilde{a}^2 - \tilde{b}^2) + \tilde{c}^2 (5\tilde{b}^2 - \tilde{a}^2) - 4\tilde{a} \, \tilde{b} \, \tilde{c} \, \tilde{f} \, \tilde{g}^2}{16\tilde{a} \, \tilde{b} \, \tilde{c} \, \tilde{g}^2},
\end{align*}
\]
\[ \dot{c} = \frac{c^2 + (c^2 - 2a^2 - 2b^2) \hat{g}^2}{4\hat{a} \hat{b} \hat{g}^2}, \]  
\[ \dot{f} = -\frac{(\hat{a}^2 - \hat{b}^2) [4\hat{a} \hat{b} \hat{f}^2 \hat{g}^2 - \hat{c} (4\hat{a} \hat{b} \hat{c} + \hat{a}^2 \hat{f} - \hat{b}^2 \hat{f}) (1 - \hat{g}^2)]}{16\hat{a}^3 \hat{b}^3 \hat{g}^2}, \]  
\[ \dot{g} = \frac{-\hat{c} (1 - \hat{g}^2)}{4\hat{a} \hat{b} \hat{g}}. \]

Note that unlike (11), the equations in terms of these variables no longer involve any square roots. The conserved quantities \( p \) and \( q \) defined in (14) are given in terms of the entirely tilded variables by

\[ p = [(\hat{a}^2 - \hat{b}^2) \hat{c} - 2\hat{a} \hat{b} \hat{f} \hat{g} \hat{s} \hat{g}] \hat{g}^2, \quad q = -(\hat{a}^2 - \hat{b}^2) \hat{c} + 2\hat{a} \hat{b} \hat{f}. \]  

In the parameterisation of (15), we find that, with respect to the vielbein basis

\[ e^0 = dt, \quad \dot{e}^1 = \hat{a} (\Sigma_1 + \hat{g} \sigma_1), \quad e^2 = \hat{a} (\Sigma_2 + \hat{g} \sigma_2), \quad \dot{e}^3 = \hat{c} (\Sigma_3 - \sigma_3), \]
\[ \dot{e}^4 = \hat{b} (\Sigma_1 - \hat{g} \sigma_1), \quad \dot{e}^5 = \hat{b} (\Sigma_2 - \hat{g} \sigma_2), \quad \dot{e}^6 = \hat{f} (\Sigma_3 + \hat{g} \sigma_3), \]  

which we denote by \( \tilde{e}^a \) to distinguish it from the basis \( e^a \) in (8)), and the natural choice of spin frame, the covariantly-constant spinor has purely constant components;

\[ \eta = \frac{\epsilon_1 + i \epsilon_2}{\sqrt{2}}, \]  

where \( \epsilon_1 \) and \( \epsilon_2 \) are the two constant-component spinors defined in (14). The calibrating 3-form is now given simply by

\[ \Phi = \tilde{e}^0 \wedge (\tilde{e}^1 \wedge \tilde{e}^4 + \tilde{e}^2 \wedge \tilde{e}^5 + \tilde{e}^3 \wedge \tilde{e}^6) - (\tilde{e}^1 \wedge \tilde{e}^2 \wedge \tilde{e}^5 - \tilde{e}^4 \wedge \tilde{e}^7 \wedge \tilde{e}^8) \wedge \tilde{e}^3 \]
\[ + (\tilde{e}^1 \wedge \tilde{e}^3 \wedge \tilde{e}^2 \wedge \tilde{e}^4) \wedge \tilde{e}^5, \]  

Of course the first-order equations (17) are implied by \( d\Phi = 0 \) and \( d^\ast \Phi = 0 \). It is interesting that in the metric parameterisation (15) with the vielbein (19), \( \Phi \) takes a “canonical” form with constant tangent-space components.

### 3 Deformed and resolved conifolds as weak-coupling limits

The system of five first-order equations that we have obtained here encompasses both the four-function first-order system first found in [3], and also the inequivalent system of four first-order equations plus two constraints that we obtained in [4]. To see the reduction of (3) and (5) to give this latter case, it is merely necessary to impose the following further consistent algebraic constraint

\[ g = -\frac{b \hat{f}}{\hat{a} \hat{c}}, \]  

\[ (22) \]
The metric functions then satisfy a reduced system of first-order equations, which is most conveniently expressed in terms of the untilded variables in the form (1) for the metric ansatz:

\[
\dot{a} = -\frac{c}{2a} + \frac{a^5 f^2}{8b^4 c^3}, \quad \dot{b} = -\frac{c}{2b} + \frac{a^2 (a^2 - 3c^2) f^2}{8b^4 c^3},
\]

\[
\dot{c} = -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - \frac{3a^2 f^2}{8b^4 c^3}, \quad \dot{f} = -\frac{a^4 f^3}{4b^4 c^3}.
\]

These are the first-order equations that were obtained in \([9]\). The conserved quantities \(p\) and \(q\) in (13) are given by

\[
p = f \left( a^4 f^2 + 4b^4 c^2 - 4a^2 b^2 c^2 \right) / (4b^2 c^2) \quad \text{and} \quad q = 0.
\]

The first-order equations for the resolved conifold emerge from (23) as the Gromov-Hausdorff limit, by setting \(f \to 0\).

To see how the first-order equations (6) and (5) can instead be reduced to give the four-function system in \([5]\), one has to impose a different additional algebraic constraint. This can be read off by comparing the six-function metric ansatz (2) with the four-function ansatz of \([5]\), which is

\[
ds^2 = dt^2 + a_1^2 \left[ (\Sigma_1 - \sigma_1)^2 + (\Sigma_2 - \sigma_2)^2 \right] + a_3^2 \left[ (\Sigma_3 - \sigma_3)^2 + b_1^2 \left[ (\Sigma_1 + \sigma_1)^2 + (\Sigma_2 + \sigma_2)^2 \right] + b_3^2 (\Sigma_3 + \sigma_3)^2 \right].
\]

Thus one has

\[
a_1^2 = \frac{1}{2} a^2 (1 - g), \quad b_1^2 = \frac{1}{2} a^2 (1 + g), \quad a_3 = \tilde{c}, \quad b_3 = \tilde{f}, \quad \tilde{g}_3 = 1.
\]

The first-order system (23) then consistently reduces to that of \([5]\), which is mostly simply expressed after changing to the variables \((a_1, b_1, a_3, b_3)\):

\[
\dot{a}_1 = -\frac{a_1^2}{4a_3 b_1} + a_3 \frac{b_1}{4a_3} + b_3 \frac{a_1}{4a_3} + \frac{b_3}{2a_1} \frac{b_1}{2a_1}, \quad \dot{a}_3 = -\frac{a_3^2}{2a_1 b_1} + a_1 \frac{b_1}{2a_1} + \frac{b_1}{2a_1},
\]

\[
\dot{b}_1 = -\frac{b_1^2}{4a_1 a_3} + a_1 \frac{a_3}{4a_3} + a_3 \frac{b_3}{4a_1} - \frac{b_3}{4a_1} \frac{b_1}{4a_1}, \quad \dot{b}_3 = -\frac{b_3^2}{4a_1} + \frac{b_3^2}{4b_1^2}.
\]

The conserved quantities \(p\) and \(q\) defined in (13) are given by

\[
p = -q = a_3 (b_1^2 - a_1^2) + 2a_1 b_1 b_3.
\]

By taking the limit \(b_3 \to 0\) in (26), the equations reduce to those that describe the deformed conifold in \(D = 6\).

Note that the more symmetrical version of the metric (13) is especially suited to taking the Gromov-Hausdorff limit to the deformed conifold. By comparing (13) and (24), we see that the constraint that reduces (13) to (24) is now simply \(\tilde{g} = -1\).

The first-order system (26) gives rise to the \(\mathbb{B}_7\) solutions, whose Gromov-Hausdorff limit is \(S^1\) times the deformed conifold, and to the \(\mathbb{C}_7\) solutions, whose Gromov-Hausdorff limit
is $S^1$ times the Kähler metric on the complex line bundle over $S^2 \times S^2$ obtained in [11, 12]. These both have $p = -q$, with $p = -a_0^3$ for $\mathbb{B}_7$, where $a_0$ is the radius of the $S^3$ bolt, and $p = a_0^2 c_0$ for $\mathbb{C}_7$, where $a_0$ and $c_0$ are the radii of $S^2$ in the $S^2 \times S^2$ base and the $U(1)$ fibre of the $T^{1,1}$ bolt. By contrast, the first-order system obtained in [3] gives rise to the $\mathbb{D}_7$ solutions, with $q = 0$ and $p$ proportional to the volume of the squashed $S^3$ bolt. The Gromov-Hausdorff limit of the $\mathbb{D}_7$ metrics is the resolved conifold.

4 New $G_2$ metrics $\tilde{\mathbb{C}}_7$ with $T^{1,1}$ bolt

We find that as well as yielding all the above $\mathbb{B}_7$, $\mathbb{C}_7$ and $\mathbb{D}_7$ solutions, the full system with five first-order equations that we have obtained in this paper gives rise to another new class of complete non-singular $G_2$ metrics, which we shall denote by $\tilde{\mathbb{C}}_7$, with two non-trivial parameters. At an upper boundary of the parameter range the metrics are AC, whilst away from this boundary they are ALC. At a lower boundary of the parameter range the Gromov-Hausdorff limit is reached, where the $\tilde{\mathbb{C}}_7$ metrics approach $S^1$ times the Ricci-flat Kähler 6-metrics obtained in [14]. These 6-metrics, which we shall denote by $\tilde{K}_6$, have the topology of a complex line bundle over $S^2 \times S^2$, and they are generalisations of the example found in [11, 12]. Specifically, the radii of the two 2-spheres in the $S^2 \times S^2$ bolt in the metrics of [14] can be chosen arbitrarily, with the metric in [11, 12], which we shall denote by $K_6$, being the special case where the two radii are equal. Thus the $\tilde{\mathbb{C}}_7$ metrics with the radii of the two $S^2$ factors chosen to be equal are precisely the $\mathbb{C}_7$ metrics found in [13].

To obtain these new solutions $\tilde{\mathbb{C}}_7$, we begin by constructing regular short-distance expansions in the form of Taylor series. Substituting into (6) and (5), we find that the metric functions at short distance take the form

$$
a = a_0 + \frac{(4a_0^2 - \tilde{f}_0^3) t^2}{16a_0^3} + \cdots,
$$

$$
b = b_0 + \frac{(4a_0^4 - 3b_0^2 \tilde{f}_0^3) t^2}{16b_0 a_0^4} + \cdots,
$$

$$
c = -t + \frac{(4a_0^2 b_0^2 + 4a_0^4 - b_0^2 \tilde{f}_0^3) t^2}{24a_0^4} + \cdots,
$$

$$
\tilde{f} = \tilde{f}_0 + \frac{\tilde{f}_0^3 t^2}{4a_0^4} + \cdots,
$$

$$
g = \frac{b_0 \tilde{f}_0 t}{2a_0^3} + \frac{\tilde{f}_0 (4a_0^4 - 20a_0^2 b_0^2 + 11b_0^2 \tilde{f}_0^3) t^3}{96a_0^6 b_0} + \cdots,
$$

$$
\tilde{g}_3 = \frac{b_0^2}{a_0^2} + \frac{(4a_0^4 - b_0^4) t^4}{8a_0^6 b_0^2} + \cdots,
$$

(27)
where $a_0$, $b_0$ and $\tilde{f}_0$ are free parameters. One of the three parameters here can be viewed as being trivial, since it is just associated with the overall scale of the solution. Two out of the three, for example $\tilde{f}_0/a_0$ and $\tilde{f}_0/b_0$, constitute non-trivial parameters in the solutions. They characterise the homogeneous squashings of the $T^{1,1}$ bolt at $t = 0$.

By using the Taylor expansions to set initial data just outside the bolt, and then integrating the first-order equations numerically, we find that the solutions are regular also at large distances, provided that the parameters $a_0$, $b_0$ and $\tilde{f}_0$ lie in appropriate ranges. There is a two-dimensional non-trivial parameter space of regular solutions $\tilde{C}_7$. For generic points in this modulus space, the metric is ALC, with a $T^{1,1}$ bolt whose squashing is characterised by $a_0$, $b_0$ and $c_0$. At large distance, the metric approaches a twisted product of $S^1$ and an AC six-metric, namely the Ricci-flat Kähler metric on the $\mathbb{R}^2$ bundle over $S^2 \times S^2$ with, generically, unequal $S^2$ radii. There is a boundary of the modulus space corresponding to the situation where the radius of this circle goes to infinity, and the seven-metric becomes AC. As in the special case $C_7$ when $a_0 = b_0$ [13], the magnitude of the Killing vector $K$ given in (1) is everywhere finite and non-zero in the ALC $\tilde{C}_7$ metrics. It is given by $|K|^2 = \tilde{f}^2 (1 + \tilde{g}_3)^2$, and this runs from a minimum value at $t = 0$, where from (27) we have $|K|_{t=0} = \tilde{f}_0 (1 + b_0^2/a_0^2) = \tilde{f}_0 (1 + \sqrt{-p/q})$, to a final value $|K|_{t=\infty} = 2 \tilde{f}_\infty$ (since $\tilde{g}_3$ goes to 1 at infinity), which stabilises at infinity. In the Gromov-Hausdorff limit the $\tilde{C}_7$ metrics become $S^1$ times the Ricci-flat Kähler metrics found in [14].

It is straightforward to see, by substituting the Taylor expansions (27) into the definitions of the first integrals $p$ and $q$ given in (13), that we have

$$ p = -\frac{b_0^4 \tilde{f}_0}{a_0^6}, \quad q = a_0^2 \tilde{f}_0. \tag{28} $$

Thus the conserved quantities $p$ and $q$ characterise the relative sizes of the two $S^2$ factors in the $S^2 \times S^2$ base space of the $T^{1,1}$ bolt. Note, in particular, that if we set the two radii equal, $a_0 = b_0$, then we shall have $p = -q = -a_0^2 \tilde{f}_0$. This special case, which corresponds to the $C_7$ metrics found in [13], has the $Z_2$ symmetry $p = -q$ under the interchange of the two $S^3$ factors in the $S^3 \times S^3$ principal orbits. The $\tilde{C}_7$ metrics that we have found here have $p \neq -q$, and hence they are not $Z_2$ symmetric. Note that it is because of the geometry of the $\tilde{C}_7$ manifolds, with the $T^{1,1}$ bolt as opposed to the $S^3$ bolt of the $\mathbb{B}_7$ and $\mathbb{D}_7$ manifolds, that we can obtain smooth $G_2$ metrics with $p$ and $q$ both non-zero and $p \neq -q$. In fact this is the only possible geometry for smooth metrics with such $p$ and $q$ values.
5 Discussion

The $G_2$ metrics can be used in order to construct supersymmetric Ricci-flat solutions of M-theory, by simply writing $ds_{11}^2 = dx^\mu dx_\mu + ds_7^2$. There is a $U(1)$ isometry, and so we can reduce this solution to $D = 10$, using the standard Kaluza-Klein formula

$$ds_{11}^2 = e^{-\frac{1}{3} \phi} ds_{10}^2 + e^{\frac{4}{3} \phi} (dz + A)^2. \quad (29)$$

Since the $U(1)$ Killing vector is given by (3), we shall have $z = \frac{1}{2}(\psi + \tilde{\psi})$ as the fibre coordinate, together with $y = \frac{1}{2}(\psi - \tilde{\psi})$ as a coordinate in $ds_{10}^2$. The $G_2$ metric (2), or (15), which has the form $ds_7^2 = ds_6^2 + \tilde{f}^2 (\Sigma_3 + \tilde{g}_3 \sigma_3)^2$, can now be written as

$$ds_7^2 = ds_6^2 + \tilde{f}^2 (1 + \tilde{g}_3)^2 (dz + A)^2, \quad (30)$$

where the Kaluza-Klein vector is given by

$$A = \frac{1}{1 + \tilde{g}_3} \left[ \tilde{g}_3 (dy + \cos \theta d\phi) - (dy - \cos \tilde{\theta} d\tilde{\phi}) \right]. \quad (31)$$

The $D = 10$ configuration therefore carries non-trivial Ramond-Ramond 2-form flux. The functions $\tilde{f}$ and $(1 + \tilde{g}_3)$ are everywhere finite and non-vanishing in the $\mathbb{C}_7$ and $\mathbb{D}_7$ solutions, and also in the $\tilde{\mathbb{C}}_7$ solutions that we have found in this paper. This implies that in the reduction (29) to the type IIA theory, the string coupling $g_{\text{string}} = e^{\phi} = \tilde{f}^{3/2} (1 + \tilde{g}_3)^{3/2}$ is everywhere finite and non-zero. This may provide a good supergravity dual for $\mathcal{N} = 1$ field theory in four dimensions.

It is useful to summarise the results for known classes of $G_2$ metrics with $S^3 \times S^3$ principal orbits in a Table.

| $G_2$ Metric | Gromov-Hausdorff Limit | Bolt | $(p, q)$ |
|--------------|------------------------|------|----------|
| $\mathbb{B}_7$ | Deformed conifold | $S^3_1$ | $p = -q$ |
| $\mathbb{C}_7$ | $\mathbb{K}_6$ | $T^{1,1}_{\lambda}$ | $p = -q$ |
| $\tilde{\mathbb{C}}_7$ | $\tilde{\mathbb{K}}_6$ | $T^{1,1}_{\lambda,\mu}$ | $p \neq -q$ |
| $\mathbb{D}_7$ | Resolved conifold | $S^3_3$ | $q = 0$ |
| $\text{AC}_7$ | – | $S^3_1$ | $(p, q) = (p, 0), (p, -p), (0, q)$ |

Table 1: The $G_2$ metrics with $S^3 \times S^3$ principal orbits

The space denoted by $\text{AC}_7$ is the original AC $G_2$ metric found in [2, 3]. The spaces denoted by $\mathbb{K}_6$ and $\tilde{\mathbb{K}}_6$ are the Ricci-flat Kähler metrics on the complex line bundle over
the Einstein $S^2 \times S^2$ base \cite{11, 12}, and its generalisation found in \cite{14}, respectively. The subscripts $\lambda$ and $\mu$ on the spaces $S^3_\lambda, T^{1,1}_\lambda$ and $T^{1,1}_{\lambda,\mu}$ denote the adjustable parameters that characterise the homogeneous squashings of the bolt. There are three possible choices for the relation between $p$ and $q$ in the original AC metric AC$_7$, since there is a triality symmetry in this case \cite{3, 10, 15}.

The most general solution of our system of five first-order equations for $G_2$ holonomy would have five modulus parameters, corresponding to the constants of integration of the equations. Two of these are trivial, corresponding to the freedom to redefine the radial coordinate by the addition and multiplication by constants. (The latter changes the overall scale of the metric.) Three non-trivial parameters remain, but this solution would in general be singular. All the regular solutions, listed in Table 1, arise in special regions of the modulus space. The original AC$_7$ metric of \cite{7, 8} has no non-trivial parameter; the $\mathbb{B}_7$ and $\mathbb{D}_7$ metrics have one non-trivial parameter; and the new $\tilde{\mathbb{C}}_7$ metrics have two non-trivial parameters ($\mathbb{C}_7$ is a special case of these with one non-trivial parameter). Thus all of the metrics in Table 1, and their weak-coupling limits, are unified within the general $G_2$ metrics described by the five first-order equations, provided one includes the singular metrics that interconnect them.

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