On Special Semigroups Derived From an Arbitrary Semigroup

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Abstract

Let $S$ be a semigroup, $\Lambda$ a non-empty set and $P$ a mapping of $\Lambda$ into $S$. The set $S \times \Lambda$ together with the operation $\circ_P$ defined by $(s, \lambda) \circ_P (t, \mu) = (sP(\lambda)t, \mu)$ form a semigroup which is denoted by $(S, \Lambda, \circ_P)$. Using this construction, we prove a common connection between the semigroups $S$, $S/\theta$ and $S/\theta^* = (S/\theta)/(\theta^*/\theta)$, where $\theta$ and $\theta^*/\theta$ are the kernels of the right regular representations of $S$ and $S/\theta$, respectively. We also prove an embedding theorem for the semigroup $(S, S/\theta, \circ_P)$, where $S$ is a left equalizer simple semigroup without idempotents, and $P$ maps every $\theta$-class of $S$ into itself.

1 Introduction

Let $S$ be an arbitrary semigroup. It is known that the relation $\theta$ on $S$ defined by $(a, b) \in \theta$ if and only if $xa = xb$ for all $x \in S$ is a congruence on $S$. This congruence is the kernel of the right regular representation $\varphi : a \mapsto \varphi_a$ $(a \in S)$ of $S$: $\varphi_a : s \mapsto sa$ $(s \in S)$ is the inner right translations of $S$ defined by $a$. For convenience, the semigroup $\varphi(S) = S/\theta$ is also called the right regular representation of $S$. The $\theta$-class of $S$ containing an element $s \in S$ will be denoted by $[s]_{\theta}$.

Let $\theta^*$ denote the congruence on the semigroup $S$ for which $\theta \subseteq \theta^*$ and $\theta^*/\theta$ is the kernel of the right regular representation on $S/\theta$, where $\theta^*/\theta$ is defined by $([s]_{\theta}, [t]_{\theta}) \in \theta^*/\theta$ if and only if $(s, t) \in \theta^*$ (see Theorem 5.6 of [5]). It is easy to see that $(a, b) \in \theta^*$ if and only if $(xa, xb) \in \theta$ for all $x \in S$, that is, $sa = sb$ for all $s \in S^2$ (see also [7] and [8]). The $\theta^*$-class of $S$ containing an element $s \in S$ will be denoted by $[s]_{\theta^*}$.

The right regular representation of semigroups plays an important role in the examination of semigroups. Here we cite some results of papers [1], [2] and [9], in which special types of semigroup are characterized by the help of their right regular representation.

A semigroup satisfying the identity $ab = a$ (resp. $ab = b$) is called a left zero (resp. right zero) semigroup. A semigroup is called a left group (resp. right group...
group) if it is a direct product of a group and a left zero (resp. right zero) semigroup.

**Lemma 1** ([1]) A semigroup $S$ is a left group if and only if the right regular representation $S/\theta$ of $S$ is a group.

A semigroup $S$ is called an $M$-inversive semigroup ([10]) if, for each $a \in S$, there are elements $x, y \in S$ such that $ax$ and $ya$ are middle units of $S$, that is, $caxd = cd$ and $cyad = cd$ is satisfied for all $c, d \in S$.

**Lemma 2** ([2]) A semigroup $S$ is $M$-inversive if and only if the right regular representation $S/\theta$ of $S$ is a right group.

In [9], a semigroup $S$ is called a left equalizer simple semigroup if, for arbitrary elements $a, b \in S$, the assumption $x_0a = x_0b$ for some $x_0 \in S$ implies $xa = xb$ for all $x \in S$.

**Lemma 3** ([9]) A semigroup $S$ is left equalizer simple if and only if the right regular representation $S/\theta$ of $S$ is left cancellative.

The previous lemmas show connections between $S$ and $S/\theta$, in special cases. In this paper we would like to find a common connection between the semigroups $S$, $S/\theta$ and $S/\theta^* = (S/\theta)/(\theta^*/\theta)$ in a general case. In our examination, the following construction plays an important role.

Let $S$ be a semigroup, $\Lambda$ an arbitrary set, and $P$ is a mapping of $\Lambda$ into $S$. It is easy to see that the set $S \times \Lambda$ together with the operation $(s, \lambda) \circ P (t, \mu) = (sP(\lambda)t, \mu)$ is a semigroup. This semigroup is the dual of the semigroup constructed in Exercise 6 for §8.2 of [4]. This semigroup will be denoted by $(S, \Lambda; \circ_P)$.

In Section 2, we deal with the semigroups $(S, \Lambda; \circ_P)$. We show that the left cancellativity and the right simplicity of a semigroup $S$ are inherited from $S$ to the semigroup $(S, \Lambda; \circ_P)$.

In Section 3, the semigroups $(S, S/\theta; \circ_P)$ are in the focus of our examination, where $S$ is an arbitrary semigroup and $P$ is an arbitrary mapping of the factor semigroup $S/\theta$ into $S$ with condition $P([s]_\theta) \in [s]_\theta$. We show that, for an arbitrary semigroup $S$, the right regular representation of the semigroup $(S, S/\theta; \circ_P)$ is isomorphic to the semigroup $(S/\theta^*, S/\theta; \circ_P')$, where $P'$ is the mapping of $S/\theta$ into $S/\theta^*$ defined by $P'(\[s]_\theta) = \[s]_{\theta^*}$.

In Section 4, we prove an embedding theorem for the semigroups $(S, S/\theta; \circ_P)$, where $S$ is an idempotent-free left equalizer simple semigroup. We prove that if $S$ is a left equalizer simple semigroup without idempotent then the semigroup $(S, S/\theta; \circ_P)$ can be embedded into a simple semigroup $(S'', S/\theta; \circ_{P''})$ containing a minimal left ideal.

For notations and notions not defined here, we refer to [3], [4] and [6].
2 Hereditary properties

In this section we show that the left cancellativity and the right simplicity of a semigroup $S$ are inherited from $S$ to the semigroup $(S, \Lambda; \circ_P)$.

**Lemma 4** If $S$ is a left cancellative semigroup then the semigroup $(S, \Lambda; \circ_P)$ is also left cancellative for any set $\Lambda$ and any mapping $P : \Lambda \mapsto S$.

**Proof.** Assume

$$(s, \lambda) \circ_P (t, \mu) = (s, \lambda) \circ_P (r, \tau)$$

for some $s, t, r \in S$ and $\lambda, \mu, \tau \in S$. Then

$$(sP(\lambda)t, \mu) = (sP(\lambda)r, \tau)$$

from which it follows that

$$sP(\lambda)t = sP(\lambda)r \quad \text{and} \quad \mu = \tau$$

As $S$ is left cancellative, we get $t = r$. Hence

$$(t, \mu) = (r, \tau).$$

Thus the semigroup $(S, \Lambda; \circ_P)$ is left cancellative. \qed

**Lemma 5** If $S$ is a right simple semigroup then the semigroup $(S; \Lambda; \circ_P)$ is also right simple for any set $\Lambda$ and any mapping $P : \Lambda \mapsto S$.

**Proof.** Let $(s, \lambda), (t, \mu) \in (S, \Lambda; \circ_P)$ be arbitrary elements. As $S$ is right simple,

$$sP(\lambda)S = S.$$

Then there is an element $x \in S$ such that

$$sP(\lambda)x = t,$$

and so

$$(s, \lambda) \circ_P (x, \mu) = (sP(\lambda)x, \mu) = (t, \mu).$$

From this it follows that the semigroup $(S, \Lambda; \circ_P)$ is right simple. \qed

**Corollary 1** If $S$ is a right group then the semigroup $(S; \Lambda; \circ_P)$ is also a right group for any semigroup $S$ and any mapping $P : \Lambda \mapsto S$.

**Proof.** As a semigroup is a right group if and only if it is right simple and left cancellative, our assertion follows from Lemma 4 and Lemma 5. \qed
3 The right regular representation

In this section we deal with the right regular representation of semigroups $(S, \Lambda, \circ_P)$ in that case when $S$ is an arbitrary semigroup, $\Lambda$ is the factor semigroup $S/\theta$ and $P$ is an arbitrary mapping of $S/\theta$ into $S$ with condition that $P([a]_\theta) \in [a]_\theta$ for every $s \in S$. We note that the product $\circ_P$ in the semigroup $(S, S/\theta, \circ_P)$ does not depend on choosing $P$, because $(s, [a]_\theta) \circ_P (t, [b]_\theta) = (sP([a]_\theta)t, [b]_\theta)$ for every $s, t, a, b \in S$, and $sa' = sa''$ for every $a', a'' \in [a]_\theta$.

**Theorem 1** Let $S$ be an arbitrary semigroup. Let $P$ be a mapping of $S/\theta$ into $S$ with condition $P([a]_\theta) \in [a]_\theta$ for every $[a]_\theta \in S/\theta$. Let $P'$ denote the mapping of $S/\theta$ onto $S/\theta^*$ defined by $P'([a]_\theta) = [a]_{\theta^*}$. Then the right regular representation of the semigroup $(S, S/\theta; \circ_P)$ is isomorphic to the semigroup $(S/\theta^*, S/\theta; \circ_{P'})$.

**Proof.** Let $\theta^*$ denote the kernel of the right regular representation of the semigroup $(S, S/\theta, \circ_P)$. Let $\phi$ be the mapping of the factor semigroup $(S, S/\theta, \circ_P)/\theta^*$ onto the semigroup $(S/\theta^*, S/\theta, \circ_{P'})$ defined by

$$\phi(([a, [b]_\theta])_{\theta^*}) = ([a]_{\theta^*}, [b]_\theta),$$

where $([a, [b]_\theta])_{\theta^*}$ denotes the $\theta^*$-class of $(S, S/\theta, \circ_P)$ containing the element $(a, [b]_\theta)$ of $(S, S/\theta, \circ_P)$. To show that $\phi$ is injective, assume

$$\phi(([a, [b]_\theta])_{\theta^*}) = \phi(([c, [d]_\theta])_{\theta^*})$$

for some $([a, [b]_\theta])_{\theta^*}, ([c, [d]_\theta])_{\theta^*} \in (S, S/\theta, \circ_P)/\theta^*$. Then

$$([a]_{\theta^*}, [b]_\theta) = ([c]_{\theta^*}, [d]_\theta)$$

and so

$$[a]_{\theta^*} = [c]_{\theta^*} \quad \text{and} \quad [b]_\theta = [d]_\theta.$$
be arbitrary. Then
\[
\phi((a, [b]_{\theta})\theta \cdot ([c, [d]_{\theta}]_{\theta \cdot}) = \phi(((a, [c, [d]_{\theta}]_{\theta \cdot})) =
\]
\[
= (\{abc\}_{\theta \cdot}, [d]_{\theta}) = (\{a, [b]_{\theta \cdot} [c]_{\theta \cdot}, [d]_{\theta}) =
\]
\[
= ([a]_{\theta \cdot}, [b]_{\theta \cdot} \circ P (\{c]_{\theta \cdot}, [d]_{\theta}) = \phi((a, [b]_{\theta \cdot})_{\theta \cdot} \circ P (\{((a, [b]_{\theta \cdot})_{\theta \cdot}).
\]
Hence \( \phi \) is a homomorphism, and so it is an isomorphism of the right regular representation of the semigroup \((S, S/\theta, \circ_P)\) onto the semigroup \((S/\theta^*, S/\theta, \circ_{P'})\).

\( \square \)

**Corollary 2** If \( S \) is a left group then the semigroup \((S, S/\theta, \circ_P)\) is M-inversive.

**Proof.** If \( S \) is a left group then \( S/\theta \) is a group by Lemma 1. As \( S^2 = S \), we have \( \theta = \theta^* \). Thus \( S/\theta^* \) is a group. By Corollary 1, the semigroup \((S/\theta^*, S/\theta, \circ_{P'})\) is a right group and so, by Theorem 1 and Lemma 2, \((S, S/\theta, \circ_P)\) is an M-inversive semigroup.

\( \square \)

**Corollary 3** If \( S \) is an M-inversive semigroup then the semigroup \((S, S/\theta, \circ_P)\) is M-inversive.

**Proof.** If \( S \) is M-inversive then \( S/\theta \) is a right group by Lemma 2. As the kernel of the right regular representation of a right group is the identity relation, \( \theta^*/\theta \) is the identity relation on \( S/\theta \). Then \( \theta^* = \theta \) and so \( S/\theta^* \) is a right group. By Corollary 1, \((S/\theta^* \times S/\theta, \circ_{P'})\) is a right group. By Theorem 1 and Lemma 2, \((S, S/\theta, \circ_P)\) is an M-inversive semigroup.

\( \square \)

**Corollary 4** If \( S \) is a left equalizer simple semigroup then the semigroup \((S, S/\theta, \circ_P)\) is left equalizer simple.

**Proof.** Let \( S \) be a left equalizer simple semigroup. Then, by Lemma 3, \( S/\theta \) is a left cancellative semigroup. As the right regular representation of a left cancellative semigroup is the identity relation, \( \theta^*/\theta \) is the identity relation on \( S/\theta \). From this it follows that \( \theta^* = \theta \) and so \( S/\theta^* \) is a left cancellative semigroup. By Lemma 4, \((S/\theta^*, S/\theta, \circ_{P'})\) is a left cancellative semigroup. By Theorem 1 and by Lemma 3, the semigroup \((S, S/\theta, \circ_P)\) is M-inversive.

\( \square \)

### 4 An embedding theorem

In this section we deal with such semigroups \((S, S/\theta, \circ_P)\) in which \( S \) is an idempotent-free left equalizer simple semigroup.

**Theorem 2** If \( S \) is a left equalizer simple semigroup without idempotents then the semigroup \((S, S/\theta, \circ_P)\) can be embedded into a simple semigroup \((S'', S/\theta, \circ_{P''})\) containing a minimal left ideal.
Proof. Let $S$ be a left equalizer simple semigroup without idempotent. Then, by Theorem 8.19 of [4], there is an embedding $\tau$ of $S$ into a left simple semigroup $S''$ without idempotents. Consider the semigroup $(S'', S/\theta \circ P')$, where the mapping $P'' : S/\theta \rightarrow S''$ is defined by $P''([s]_\theta) = \tau(P([s]_\theta))$. By the dual of Exercise 6 for §8.2 of [4], $(S'', S/\theta \circ P')$ is a simple semigroup containing a minimal left ideal. We show that

$$\Phi : (a, [b]_\theta) \mapsto (\tau(a), [b]_\theta)$$

is an embedding of the semigroup $(S, S/\theta \circ P)$ into the semigroup $(S'', S/\theta \circ P')$.

First we show that $\Phi$ is injective. Assume $\Phi((a, [b]_\theta)) = \Phi((c, [d]_\theta))$ for some $a, b, c, d \in S$. Then

$$\tau(a) = \tau(c) \quad \text{and} \quad [b]_\theta = [d]_\theta.$$ 

As $\tau$ is injective, we get

$$(a, [b]_\theta) = (c, [d]_\theta).$$

Next we show that $\Phi$ is a homomorphism. Let

$$(a, [b]_\theta), (c, [d]_\theta) \in (S, S/\theta \circ P)$$

be arbitrary elements. Then

$$\Phi((a, [b]_\theta) \circ_P (c, [d]_\theta)) = \Phi((a P([b]_\theta)c, [d]_\theta)) = (\tau(a P([b]_\theta)c), [d]_\theta) =$$

$$(\tau(a)\tau(P([b]_\theta))\tau(c), [d]_\theta) = (\tau(a)P''([b]_\theta)c, [d]_\theta) =$$

$$= (\tau(a), [b]_\theta) \circ_{P'} (\tau(c), [d]_\theta) = \Phi((a, [b]_\theta)) \circ_{P'} \Phi((c, [d]_\theta))$$

and so $\Phi$ is a homomorphism. Consequently $\Phi$ is an embedding of the semigroup $(S, S/\theta \circ P)$ into the simple semigroup $(S'', S/\theta \circ P')$ containing a minimal left ideal. \[\square\]

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