Testing for unobserved heterogeneous treatment effects

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ABSTRACT

Unobserved heterogeneous treatment effects have been emphasized in recent policy evaluation literature. In this paper, we extend Lu and White (2014)’s testing method for unobserved heterogeneous treatment effects by developing nonparametric tests under the standard exogenous instrumental variable assumption and allowing for endogenous treatment. Specifically, we propose Kolmogorov–Smirnov–type statistics that are consistent and simple to implement. To illustrate, we apply the proposed test method with two empirical applications: treatment effects of job training program on earnings as well as the impact of fertility on family income. The null hypotheses, i.e., lack of unobserved heterogeneous treatment effects, cannot be rejected at a 10% significance level in the former case, but should be rejected at all usual significance levels in the latter.

Keywords: Specification test, nonseparability, unobserved heterogeneous treatment effects

JEL codes: C12, C14, C31

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1 Introduction

Unobserved heterogeneous treatment effects have been emphasized in recent policy evaluation literature. See e.g. Heckman, Smith, and Clements (1997); Matzkin (2003); Chesher (2003, 2005); Chernozhukov and Hansen (2005); Imbens and Newey (2009), Heckman and Vytlacil (2001, 2005). Recently, Lu and White (2014) and Su, Tu, and Ullah (2014) develop nonparametric tests for unobserved heterogenous treatment effects via testing for additive separability of the error term in the structural relationship. A key assumption in their approaches is to assume that treatment is (conditional) independent of the error term. Motivated by Lu and White (2014), in this paper we propose nonparametric tests under the standard exogenous instrumental variable assumption and allowing for endogenous treatment.

In this paper, we consider the following structure model

\[ Y = g(D, X, \epsilon) \]

where \( Y \) is the outcome variable of interest, \( X \) is a vector of observed covariates, \( D \) denotes the binary treatment status, and \( \epsilon \) is an unobserved error term of general form. In particular, \( \epsilon \) represents the unobserved individual heterogeneity and we allow for the correlation between \( \epsilon \) and \( D \). Such a structural relationship is nonseparable in \( \epsilon \), which implies treatment effect from \( D \) on \( Y \) varies across individuals, even after we control for observed heterogeneity \( X \). When there is no unobserved heterogeneous treatment effects, we show that the structural model can be represented by

\[ Y = m(D, X) + \nu(X, \epsilon) \]

for some measurable functions \( m \) and \( \nu \). With additive separability of the error term, treatment effects are the same across individuals with the same covariates. Formally, we test such an additive separability of the structural model.

A key feature of our approach is to allow for the presence of treatment endogeneity. Due to the sample selection issue highlighted in Heckman (1979), the treatment status \( D \) and error term \( \epsilon \) are statistically dependent on each other given \( X \). With additive
separability, identification and estimation of average treatment effects directly obtained by Imbens and Angrist (1994)'s “Local Average Treatment Effects” and Heckman and Vytlacil (1999)'s Marginal Treatment Effect (MTE). For instance, Angrist and Krueger (1991) use a two-stage least square approach to estimate treatment effects in a linear specification. As is pointed out by Heckman, Smith, and Clements (1997), however, the conventional assumption of identical treatment effects across individuals, while convenient, is implausible. If so, then the usual linear parametric specification with additive errors is not only for feasibility and simplification of estimation, but also essential for identification as well as interpretations of treatment effects.

Though unobserved heterogeneity is crucially important, there are only a handful of papers studying testing for the existence of. In a framework without endogeneity, Heckman, Smith, and Clements (1997) focus on observed heterogeneous treatment effects. In particular, they provide tests for whether treatment effects depend on observed exogenous covariates via testing zero-variance of $g(1,X,\epsilon) - g(0,X,\epsilon)$. Recently, Hoderlein and Mammen (2009) briefly discuss specification tests for unobserved heterogeneity in structured non-separable models. This paper is intrinsically motivated by Lu and White (2014) and Su, Tu, and Ullah (2014), who study nonparametric testing for unobserved heterogeneous treatment effects under the unconfoundedness assumption. In particular, Lu and White (2014) test such hypotheses via testing an equivalent independence condition on observables under additional weak assumptions. Our paper extends Lu and White (2014) by representing the existence of unobserved heterogeneous treatment effects by a similar conditional independence restriction on observables, allowing for the endogeneity of treatment variable.

Another closely related paper is Heckman, Schmierer, and Urzua (2010) who study testing for the absence of selection on the gain to treatment in the generalized Roy model framework, allowing for unobserved heterogeneous treatment effect. This paper complements to Heckman, Schmierer, and Urzua (2010) in the sense that presence of both unobserved heterogeneity and selection into treatment is the so-called “essential heterogeneity” in Heckman, Urzua, and Vytlacil (2006).

The proposed testing approach distinguishes the cases where exogenous covariates
contains or not a continuously distributed element. We propose Kolmogorov–Smirnov type statistics that are consistent and simple to implement. Motivated by Stinchcombe and White (1998), when there is a continuous covariate, we modify the classic Kolmogorov–Smirnov test statistics by using primitive functions of CDF to represent probability distributions of a (nonparametrically) generated variable. Such a modification is novel and plays a key role for developing our test statistics. Moreover, we establish the asymptotic properties of the proposed test statistics under the null and alternative hypothesis.

The paper is organized as follows. In Section 2, we introduce the framework and motivate our testing idea. Section 3 discusses our test statistics and main asymptotic results. We distinguish the cases whether covariates include continuous variables. Section 4 presents Monte Carlo experiments to study finite sample performance of our test statistics. Section 5 applies our testing approach to two empirical applications: treatment effects of a job training program and treatment effects of fertility on earnings. All proofs are collected in the Appendix.

2 Model and Testable Restrictions

We consider the following nonparametric nonseparable model:

$$Y = g(D, X, \epsilon)$$  \hspace{1cm} (1)

where $Y \in \mathbb{R}$, $D \in \{0, 1\}$ and $X \in \mathbb{R}^{d_X}$ are observables, $\epsilon \in \mathbb{R}^{d_\epsilon}$ is an unobserved random disturbance of general form, and $g$ is an unknown but smooth function defined on $\{0, 1\} \times \mathcal{S}_{X, \epsilon}$. In particular, $D$ is an endogenous treatment variable that is correlated with $\epsilon$ due to the selection issue. See e.g. Heckman, Smith, and Clements (1997). To deal with the endogeneity issue, we follow the literature by introducing a binary instrumental variable $Z \in \{0, 1\}$. Throughout the paper, we use upper case letters to denote random variables, and their corresponding lower case letters to stand for the realizations. Moreover, we use $\mathcal{S}_A$ for the support of a vector of generic random variables $A$.

Note that the non-additivity of the structural relationship $g$ in $\epsilon$ captures the idea that individual treatment effect, i.e., $g(1, X, \epsilon) - g(0, X, \epsilon)$, depends on the unobserved indi-
individual heterogeneity \( \epsilon \), even after one controls for \( X \) covariates. Following Lu and White (2014), the null hypothesis for testing heterogeneous individual treatment effects is equivalent to testing the following null hypothesis:

\[
H_0 : g(D, X, \epsilon) = m(D, X) + \nu(X, \epsilon),
\]

where \( m : \mathcal{S}_{DX} \mapsto \mathbb{R} \) and \( \nu : \mathcal{S}_{X\epsilon} \mapsto \mathbb{R} \). Such an equivalence is summarized formally in the following proposition.

**Proposition 2.1.** Suppose (1) holds, then \( H_0 \) holds if and only if

\[
g(1, X, \cdot) - g(0, X, \cdot) = \delta(X),
\]

(2) holds for some measurable function \( \delta(\cdot) : \mathcal{S}_X \mapsto \mathbb{R} \).

Note that under \( H_0 \), \( \delta(x) = m(1, x) - m(0, x) \), which represents homogenous individual treatment effects across individuals with the same value of covariates. Proposition 2.1 shows that the additive separability of (1) is equivalent to homogenous individual treatment effects. A similar result can be found in Lu and White (2014).

Another key insight from Lu and White (2014) is the equivalence between the additive separability hypotheses and a conditional independence restriction on observables under the unconfoundedness assumption. Following Lu and White (2014), we derive a similar set of model restrictions in the presence of endogeneity. For each \( x \in \mathcal{S}_X \) and \( z = 0, 1 \), let \( p(x, z) = \mathbb{P}(D = 1 | X = x, Z = z) \) be the propensity score.

**Assumption A.** Suppose \( Z \perp \epsilon \mid X \) and \( p(x, 0) \neq p(x, 1) \) for all \( x \in \mathcal{S}_X \). Without loss of generality, let \( p(x, 0) < p(x, 1) \) for all \( x \in \mathcal{S}_X \).

**Assumption B (Single-index error term).** There exist measurable functions \( \tilde{g} : \mathcal{S}_{DX} \times \mathbb{R} \mapsto \mathbb{R} \) and \( \nu : \mathcal{S}_{X\epsilon} \mapsto \mathbb{R} \) such that

\[
g(D, X, \epsilon) = \tilde{g}(D, X, \nu(X, \epsilon)).
\]

Moreover, \( \tilde{g}(d, x, \cdot) \) is strictly increasing in the scalar–valued index \( \nu \) for \( d = 0, 1 \) and all \( x \in \mathcal{S}_X \).

Assumption A requires the instrumental variable \( Z \) to be (conditionally) exogenous and relevant, which is standard in the literature. See e.g., Imbens and Angrist (1994), Chernozhukov and Hansen
(2005), Vuong and Xu (2017), and references therein. Assumption B imposes monotonicity on the structural relationship, which has also been widely assumed in the literature of nonseparable models. For instance, Matzkin (2003) Chesher (2003) and Chernozhukov and Hansen (2005) assume the structural function \( g \) is strictly increasing in the scalar–valued error term \( \epsilon \). It is worth pointing out that Assumption B holds under \( \mathcal{H}_0 \).

Under Assumption A, \( \mathcal{H}_0 \) implies that \( \mathbb{E}(Y|X, Z = z) = \mathbb{E}[g(0, X, \epsilon)|X] + \delta(X) \times p(X, z) \) for \( z = 0, 1 \). Thus, for each \( x \in \mathcal{X} \), \( \delta(x) \) can be identified by:

\[
\delta(x) \equiv \frac{\mu(x, 1) - \mu(x, 0)}{p(x, 1) - p(x, 0)},
\]

(3)

where \( \mu(x, z) = \mathbb{E}(Y|X = x, Z = z) \) for \( z = 0, 1 \). Note that \( \delta(x) \) takes the conditional version of LATE in Imbens and Angrist (1994).

Let \( W \equiv Y + (1 - D) \times \delta(X) \). Given that Assumptions A and B hold, by Proposition 2.1, \( \mathcal{H}_0 \) implies that \( W = g(1, X, \epsilon) \). We can also obtain \( g(0, X, \epsilon) \) by a similar argument. Therefore, by Assumption A, \( W \) is conditionally independent of \( Z \) given \( X \). The next lemma summarizes above discussion.

**Lemma 2.1.** Suppose (1), and Assumptions A and B hold. Then, \( \mathcal{H}_0 \) implies that \( W \perp\!\!\!\perp Z \mid X \).

Lemma 2.1 derives a testable model restriction under \( \mathcal{H}_0 \), i.e., \( W \perp\!\!\!\perp Z \mid X \). Note that \( W \) can be consistently estimated by \( Y + (1 - D) \times \hat{\delta}(X) \), where \( \hat{\delta}(X) \) is a consistent nonparametric estimator of \( \delta(X) \). To provide a consistent test, we next provide weak conditions under which the conditional independence restriction is also sufficient for testing \( \mathcal{H}_0 \).

**Assumption C** (Monotone selection). The selection to the treatment is given by

\[
D = \mathbb{1}[\theta(X, Z) - \eta \geq 0],
\]

(4)

where \( \theta \) is an unknown smooth function and \( \eta \in \mathbb{R} \) is an unobserved error term.

Imbens and Angrist (1994) first introduce the assumption that the selection to the treatment is monotone which implies the “no defier” condition. Vytlačil (2002) shows that such a monotonicity condition is observationally equivalent to the weak monotonicity of (4) in the error term \( \eta \). Furthermore, Vuong and Xu (2017) show that Assumption C can
be relaxed to the strict monotonicity of \( P(Y \leq y; D = 1|X, Z = 1) - P(Y \leq y; D = 1|X, Z = 0) \) in \( y \in \mathcal{Y}_{Y|X,D=1} \).

Under Assumption A, note that \( \theta(x,0) < \theta(x,1) \) for all \( x \in \mathcal{X} \). Let \( \mathcal{C}_x \equiv \{ \eta \in \mathbb{R} : \theta(x,0) < \eta < \theta(x,1) \} \) be the “complier group” given \( X = x \) (see Imbens and Angrist, 1994).

**Assumption D.** The support of \( g(d,x,\epsilon) \) given \( X = x \) and the complier group \( \mathcal{C}_x \) equals to the support of \( g(d,x,\epsilon) \) given \( X = x \), i.e.,

\[
\mathcal{S}_{g(d,x,\epsilon)|X=x,\eta \in \mathcal{C}_x} = \mathcal{S}_{g(d,x,\epsilon)|X=x}. 
\]

Assumption D is a support condition. This assumption is testable, since the distribution of \( g(d,x,\epsilon) \) given \( X = x \) and \( \eta \in \mathcal{C}_x \) can be identified, see, e.g., Imbens and Rubin (1997). Specifically, for all \( t \in \mathbb{R} \),

\[
F_{g(d,x,\epsilon)|X=x,\eta \in \mathcal{C}_x}(t) = \frac{P(Y \leq t, D = d|X = x, Z = 1) - P(Y \leq t, D = d|X = x, Z = 0)}{P(D = d|X = x, Z = 1) - P(D = d|X = x, Z = 0)},
\]

from which we can identify the support \( \mathcal{S}_{g(d,x,\epsilon)|X=x,\eta \in \mathcal{C}_x} \).

**Theorem 2.1.** Suppose (1), and Assumptions A to D hold. Then \( H_0 \) holds if and only if \( W \perp \perp Z|X \).

Throughout, we maintain Assumptions A to D. By Theorem 2.1, \( H_0 \) can be tested by testing the conditional independence condition in the theorem. As a matter of fact, Theorem 2.1 is the basis of our approach to test for (unobserved) heterogeneity in treatment effects.

### 2.1 Discussion: Testing for Full Additive Separability

One might be also interested in testing for the (full) additive separability of the error term in the outcome equation (1), which has been widely used in the empirical treatment effect literature. Lu and White (2014) and Su, Tu, and Ullah (2014) consider testing additive error structure under the unconfoundedness assumption. Specifically, their null hypothesis are given by

\[ H_0^\dagger : g(D,X,\epsilon) = m^\dagger(D,X) + \epsilon \]
for some measurable function $m^\dagger$. Clearly, $H_0^\dagger$ is more restrictive than our null hypotheses $H_0$. In particular, $H_0^\dagger$ rules out both observed and unobserved heterogeneity, which has been widely discussed in the treatment effect literature. See e.g. Heckman and Vylacil (2007) and Imbens and Wooldridge (2009).

Following Lu and White (2014) and Theorem 2.1, we derive a similar set of conditional independence restrictions that are equivalent to $H_0^\dagger$.

**Theorem 2.2.** Suppose Assumptions A to D hold. Suppose in addition $X$ and $\epsilon$ is independent, i.e., $X \perp \epsilon$. Then $H_0^\dagger$ holds if and only if

$$W - E(W|X) \perp (X, Z).$$

(5)

It is worth noting that (5) is equivalent to (i) $W \perp Z \mid X$; and (ii) $W - E(W|X) \perp X$ (see e.g. Dawid, 1979). Condition (i) is the same as that in Theorem 2.1 and (ii) has also been derived in Lu and White (2014) for testing $H_0^\dagger$ under the unconfoundedness assumption.

### 3 Consistent Test for Unobserved Treatment Effect Heterogeneity

In this section, we propose tests for unobserved treatment effect heterogeneity via testing the conditional independence restriction, i.e., $W \perp Z \mid X$. Because $Z$ is binary, the conditional independence becomes $F_{W|X,Z}(w|x,0) = F_{W|X,Z}(w|x,1)$ for all $x \in \mathcal{X}$. Difficulties arise due to the fact that $W = Y + (1 - D) \times \delta(X)$ needs to be nonparametrically estimated from the data, in particular when $X$ includes continuous variables.

In the following discussion, we distinguish the cases whether covariates $X$ include continuous variables. For expositional simplicity, throughout we assume $X \in \mathbb{R}$ be a scalar-valued random variable. It is straightforward to generalize our result to vector-valued covariates. For the discrete case, our test adopts the classic two-sample Kolmogorov–Smirnov test. When $X$ is continuous, we propose a modified Kolmogorov–Smirnov test that also converges to a limiting distribution at the $\sqrt{n}$-rate.
3.1 Discrete Covariates

Let \( \{(Y_i, D_i, X_i, Z_i) : i = 1, \ldots, n\} \) be an i.i.d. random sample of \((Y, D, X, Z)\), where \(X\) is distributed on a finite support. By Theorem 2.1, we can test unobserved treatment effect heterogeneity by testing the model restriction

\[
F_W(-|X,Z = 0) = F_W(-|X,Z = 1).
\]

For each \(x \in \mathcal{S}_X\), let

\[
\delta(x) = \frac{\hat{\mu}(x, 1) - \hat{\mu}(x, 0)}{\hat{\rho}(x, 1) - \hat{\rho}(x, 0)},
\]

where, for \(z = 0, 1,\)

\[
\hat{\mu}(x, z) = \frac{\sum_{i=1}^n Y_i \mathbb{1}(X_i = x, Z_i = z)}{\sum_{i=1}^n \mathbb{1}(X_i = x, Z_i = z)} \quad \text{and} \quad \hat{\rho}(x, z) = \frac{\sum_{i=1}^n D_i \mathbb{1}(X_i = x, Z_i = z)}{\sum_{i=1}^n \mathbb{1}(X_i = x, Z_i = z)}.
\]

It is straightforward that \(\delta(x)\) converges to \(\delta(x)\) at the \(\sqrt{n}\)-rate under additional regularity conditions. Let \(\hat{W}_i = Y_i + (1 - D_i)\delta(X_i)\). By definition, \(\hat{W}_i - W_i = (1 - D_i)\left[\delta(X_i) - \delta(X_i)\right]\) is the first-stage estimation error.

We are now ready to define our test statistic as

\[
\hat{f}_n = \sup_{w \in \mathcal{W}; x \in \mathcal{S}_X} \sqrt{n} \left| \hat{F}_{\hat{W}|XZ}(w|x, 0) - \hat{F}_{\hat{W}|XZ}(w|x, 1) \right|
\]

where \(\hat{F}_{\hat{W}|XZ}(w|x, z)\) is the empirical CDF of \(\hat{W}\) conditional on \((X, Z) = (x, z)\), i.e.,

\[
\hat{F}_{\hat{W}|XZ}(w|x, z) = \frac{\sum_{i=1}^n \mathbb{1}(\hat{W}_i \leq w) \mathbb{1}(X_i = x, Z_i = z)}{\sum_{i=1}^n \mathbb{1}(X_i = x, Z_i = z)}.
\]

Next, we establish the limiting distribution of the proposed test statistic. For notational simplicity, let \(\mathbb{1}_{XZ}(x, z) = \mathbb{1}(X = x, Z = z)\) and \(f_{WD|XZ}(w, d|x, z) = f_{W|DXZ}(w|d, x, z) \times \mathbb{P}(D = d|X = x, Z = z)\). Let

\[
\kappa(w, x) = -\frac{f_{WD|XZ}(w, 0|x, 1) - f_{WD|XZ}(w, 0|x, 0)}{p(x, 1) - p(x, 0)}.
\]

It is worth noting that \(\kappa(w, x) \geq 0\) can be interpreted as the p.d.f. of the potential outcome
Moreover, under Hypothesis $H_0$,

$$g(1, X, \epsilon)$$
given the complier group under Assumptions A and C. Moreover, let

$$\psi_{wx} = \left[ \mathbb{1}(W \leq w) - F_{W|X}(w|x) \right] \times \left[ \frac{\mathbb{1}_{XZ}(x, 0)}{\mathbb{P}(X = x, Z = 0)} - \frac{\mathbb{1}_{XZ}(x, 1)}{\mathbb{P}(X = x, Z = 1)} \right];$$

(6)

$$\phi_{wx} = \kappa(w, x) \times [W - \mathbb{E}(W|X)] \times \left[ \frac{\mathbb{1}_{XZ}(x, 0)}{\mathbb{P}(X = x, Z = 0)} - \frac{\mathbb{1}_{XZ}(x, 1)}{\mathbb{P}(X = x, Z = 1)} \right].$$

(7)

By definition, $\psi_{wx}$ and $\phi_{wx}$ and are random objects indexed by $(w, x)$. In particular, $\mathbb{E}(\psi_{wx}|X, Z) = \mathbb{E}(\phi_{wx}|X, Z) = 0$ under $H_0$.

**Assumption E.** Let $X$ be a discrete random variable. Moreover, the probability distribution of $Y$ given $(D, X, Z)$ admits a uniformly continuous density function $f_{Y|DXZ}$.

**Theorem 3.1.** Suppose Assumptions A to E hold. Then, under $H_0$,

$$\hat{\mathcal{T}}_n \xrightarrow{d} \sup_{w \in \mathbb{R}; x \in \mathcal{X}} |\mathcal{Z}(w, x)|$$

where $\mathcal{Z}(\cdot, x)$ is a mean–zero Gaussian process with covariance kernel: for $(w, x), (w', x') \in \mathbb{R} \times \mathcal{X}$,

$$\text{Cov}[\mathcal{Z}(w, x), \mathcal{Z}(w', x')] = \mathbb{E}[(\psi_{wx} + \phi_{wx})(\psi_{w'x'\prime} + \phi_{w'x'\prime})].$$

Moreover, under $H_1$, we have

$$n^{-\frac{1}{2}} \hat{\mathcal{T}}_n \xrightarrow{p} \sup_{w \in \mathbb{R}; x \in \mathcal{X}} |F_{W|XZ}(w|x, 0) - F_{W|XZ}(w|x, 1)|.$$

Theorem 3.1 forms the basis for the following one-sided test against any alternative to $H_0$: reject $H_0$ significance level $\alpha$ if and only if $\hat{\mathcal{T}}_n \geq c_\alpha$. Regarding the limiting distribution $\mathcal{Z}(\cdot, x)$, $\phi_{wx}$ in the covariance kernel appears due to the first–stage estimation of $\delta(x)$.

Because the asymptotic distribution of $\hat{\mathcal{T}}_n$ under $H_0$ is complicated and it is computationally difficult to derive the limiting distribution for the critical value, then we apply the multiplier bootstrap method in van der Vaart and Wellner (1996), Barrett and Donald (2003), and Hsu (2016) to approximate the entire process and then to approximate critical values. Specifically, we simulate a sequence of i.i.d. pseudo random variables $\{U_i : i = 1, \ldots, n\}$ with $\mathbb{E}(U) = 0$, $\mathbb{E}(U^2) = 1$, and $\mathbb{E}(|U|^4) < +\infty$. Moreover, the simulated sample $\{U_i : i = 1, \ldots, n\}$ is independent of the random sample $\{(Y_i, X_i, D_i, Z_i) : i = 1, \ldots, n\}$. Then, we obtain the following simulated empirical process:

$$\hat{\mathcal{Z}}^u(w, x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \times (\hat{\psi}_{wx,i} + \hat{\phi}_{wx,i}),$$
where $\hat{\psi}_{wx} + \hat{\psi}_{wx}$ is the estimated influence function such that

$$
\hat{\psi}_{wx} = [\mathbb{1}(\hat{W} \leq w) - \hat{f}_{WD|XZ}(w|x)] \times \left[ \frac{\mathbb{1}_{XZ}(x, 0)}{\hat{p}(X = x, Z = 0)} - \frac{\mathbb{1}_{XZ}(x, 1)}{\hat{p}(X = x, Z = 1)} \right];
$$

$$
\hat{\psi}_{wx} = \hat{r}(w, x) \times \left[ \hat{W} - \frac{\sum_{i=1}^{n} \mathbb{1}_{Z}(X_i = x)}{\hat{p}(x, 1)} \right] \times \left[ \frac{\mathbb{1}_{XZ}(x, 0)}{\hat{p}(X = x, Z = 0)} - \frac{\mathbb{1}_{XZ}(x, 1)}{\hat{p}(X = x, Z = 1)} \right]
$$

where

$$
\hat{p}(X = x, Z = z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(X_i = x, Z_i = z), \quad z = 0, 1,
$$

and

$$
\hat{r}(w, x) = \frac{\hat{f}_{WD|XZ}(w, 0|x, 1) - \hat{f}_{WD|XZ}(w, 0|x, 0)}{\hat{\rho}(x, 1) - \hat{\rho}(x, 0)},
$$

in which $\hat{f}_{WD|XZ}(w, 0|x, z)$ and $\hat{\rho}(x, z)$ are nonparametric estimators of $f_{WD|XZ}(w, 0|x, z)$ and $p(x, z)$, respectively. See e.g. Hsu (2016) for more details. By a similar argument to Barrett and Donald (2003) and Hsu (2016), $\hat{Z}^u(\cdot, x)$ converges to the same limiting process $Z(\cdot, x)$. Next, to derive the critical values, we first let $\Pi_U$ be the multiplier probability measure. Then, for a given significant level $\alpha$, the simulated critical value $\hat{c}_{n}(\alpha)$ is defined as

$$
\hat{c}_{n}(\alpha) = \sup \left\{ q : \Pi_U \left( \sup_{w \in \mathbb{R}, \ x \notin X} \left| \hat{Z}^u(w, x) \right| \leq q \right) \leq 1 - \alpha \right\}.
$$

By definition, $\hat{c}_{n}(\alpha)$ is the $(1 - \alpha)$ quantile of the simulated distribution. With the simulated critical value, we reject $H_0$ if and only if $\hat{T}_n > \hat{c}_{n}(\alpha)$.

### 3.2 Continuous Covariates

When $X$ contains continuous covariates, the generated regressor $\hat{W}$ involves estimating a nonparametric function $\delta(\cdot)$. Because $\hat{W}$ appears in the indicator function of Kolmogorov–Smirnov–type test statistics, then the empirical process argument in the proof of Theorem 3.1 does not apply to the continuous covariates case. Therefore, we propose a modified Kolmogorov–Smirnov test that we can derive its limiting distribution.

Let $\lambda(t) = -t \times \mathbb{1}(t \leq 0)$ and $\Pi(w|x, z) = \mathbb{E}[\lambda(W - w)|X = x, Z = z]$. As a matter of fact, $\Pi(\cdot|x, z)$ is the primitive function of the $F_{W|XZ}(\cdot|x, z)$, i.e., $\frac{\partial}{\partial w} \Pi(w|x, z) = F_{W|XZ}(w|x, z)$. Then $\Pi(\cdot|X, Z)$ characterizes the (conditional) distribution of $W$ given $X$ and $Z$ the same
as the c.d.f. $F_{W|XZ}$. Hence, $W \perp Z \mid X$ is equivalent to the following:

$$\mathcal{H}_0^\pi: \Pi(\cdot|x,0) = \Pi(\cdot|x,1), \quad \forall x \in \mathcal{S}_X$$

Note that $\lambda(\cdot)$ is a differentiable function. W.l.o.g., we assume $\mathcal{S}_W$ is bounded.\footnote{When $\mathcal{S}_W$ is unbounded, we need to modify $\Pi(x|z)$ by $\hat{\Pi}(w|x,z) = \frac{2(w^2 + C)}{C} \Pi(x|z)$ where $C \geq \mathbb{E}(W^2)$. The modification ensures $\hat{\Pi}(\cdot|x,z)$ is uniformly bounded above. Then all our arguments remain valid.}

Let $G(w,x;z) = \mathbb{E}\left[\mathbb{1}_{XZ}(x,z) q(X,Z') \lambda(W - w)\right]$, where $z' = 1 - z$, $q(x,z) = f_{X|Z}(x|z) \Pi(Z = z)$ and $\mathbb{1}_{XZ}(x,z) = \mathbb{1}(X \leq x; Z = z)$. Following Stinchcombe and White (1998), we rewrite the conditional moments by the following unconditional restrictions:

$$\mathcal{H}_0^G: G(w,x;0) - G(w,x;1) = 0, \quad \forall (w,x) \in \mathbb{R} \times \mathcal{S}_X.$$

To see the equivalence between $\mathcal{H}_0^\pi$ and $\mathcal{H}_0^G$, note that

$$\frac{\partial}{\partial x} \mathbb{E}[\mathbb{1}(X \leq x) \lambda(W - w) f_{X|Z}(X|1 - z)|Z = z] = \Pi(w|x,z) q(x,0) q(x,1).$$

Thus, our test statistic is constructed based on $\mathcal{H}_0^G$.

Let $K$ and $h$ be a bounded kernel function and a smoothing bandwidth, respectively. By eq. (3), we nonparametrically estimate $\delta(X_i)$ by

$$\hat{\delta}(X_i) = \frac{\hat{\mu}(X_i,1) - \hat{\mu}(X_i,0)}{\hat{\rho}(X_i,1) - \hat{\rho}(X_i,0)}.$$

where, for $z = 0,1$,

$$\hat{\mu}(X_i,z) = \frac{\sum_{j \neq i} Y_j K\left(\frac{X_j - X_i}{h}\right) \mathbb{1}(Z_j = z)}{\sum_{j \neq i} K\left(\frac{X_j - X_i}{h}\right) \mathbb{1}(Z_j = z)} \quad \text{and} \quad \hat{\rho}(X_i,z) = \frac{\sum_{j \neq i} D_j K\left(\frac{X_j - X_i}{h}\right) \mathbb{1}(Z_j = z)}{\sum_{j \neq i} K\left(\frac{X_j - X_i}{h}\right) \mathbb{1}(Z_j = z)}.$$

Let $\hat{q}(X_i,z) = \frac{1}{(n-1)h} \sum_{j \neq i} K\left(\frac{X_j - X_i}{h}\right) \mathbb{1}(Z_j = z)$ be the estimator of $q(X_i,z)$. Moreover, define

$$\hat{T}_n^c = \sup_{(w,x) \in \mathcal{S}_{WX}} \sqrt{n} |\hat{G}(w,x;0) - \hat{G}(w,x;1)|$$

where

$$\hat{G}(w,x;z) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{X_iZ_i}(x,z) \times \hat{q}(X_i,1 - z) \times \lambda(\hat{W}_i - w).$$

In above definition, the support $\mathcal{S}_{WX}$ is assumed to be known for simplicity. In practice, this assumption can be relaxed by using a consistent set estimator $\hat{\mathcal{S}}_{WX}$ of $\mathcal{S}_{WX}$.\footnote{When $\mathcal{S}_W$ is unbounded, we need to modify $\Pi(w|x,z)$ by $\hat{\Pi}(w|x,z) = \frac{2(w^2 + C)}{C} \Pi(w|x,z)$ where $C \geq \mathbb{E}(W^2)$. The modification ensures $\hat{\Pi}(\cdot|x,z)$ is uniformly bounded above. Then all our arguments remain valid.}
As is shown below, the proposed test statistics $\tilde{T}_n$ converges in distribution to a limit at the regular $\sqrt{n}$ rate. The proofs proceed in two steps: we first show that $\hat{G}(w, x; z)$ can be approximate by

$$\hat{G}(w, x; z) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{X_i, Z_i}(x, z) \times \hat{q}(X_i, 1 - z) \times (w - \hat{W}_i) \times \mathbb{I}(W_i \leq w).$$

It is worth noting that $\tilde{G}(w, x; z)$ has no nonparametric estimator in the indicator function, which allows us to apply $U$-processes theorem to establish its limiting distribution in the second step.

The first step of our proof is to show that for $z = 0, 1$

$$\sup_{(w,x) \in \mathcal{S}_W X \mathcal{Z}} |\hat{G}(w, x; z) - \tilde{G}(w, x; z)| = o_p \left( n^{-1/2} \right).$$

A key condition for above result is that the nonparametric estimator $\hat{\delta}(\cdot)$ converges to $\delta(\cdot)$ uniformly at a rate faster than $n^{-\iota}$ for some $\iota > \frac{1}{4}$. To show the approximation, we assume the following regularity assumptions.

**Assumption F.** $\mathcal{S}_W \subseteq \mathbb{R}$ is a compact subset. Let $\sup_{(x,z) \in \mathcal{S}_X \mathcal{Z}} f(x|z) \leq \mathcal{F}$ for some $\mathcal{F} < +\infty$. Moreover, $\inf_{x \in \mathcal{S}_X} |q(x, 1) - q(x, 0)| > 0$.

**Assumption G.** For some $\iota > \frac{1}{4}$, $h \to 0$ and $n^t \frac{1}{\sqrt{nh}} \to 0$ as $n \to \infty$.

**Assumption H.** The first stage nonparametric estimators satisfy:

$$\sup_{(x,z) \in \mathcal{S}_X \mathcal{Z}} \left| \mathbb{E} \left[ \frac{1}{nh} \sum_{j=1}^{n} Y_j K \left( \frac{X_j - x}{h} \right) \mathbb{I}(Z_j = z) \right] - q(x, z) \right| = O_p(n^{-\iota}),$$

$$\sup_{(x,z) \in \mathcal{S}_X \mathcal{Z}} \left| \mathbb{E} \left[ \frac{1}{nh} \sum_{j=1}^{n} D_j K \left( \frac{X_j - x}{h} \right) \mathbb{I}(Z_j = z) \right] - p(x, z)q(x, z) \right| = O_p(n^{-\iota}),$$

$$\sup_{(x,z) \in \mathcal{S}_X \mathcal{Z}} \left| \mathbb{E} \left[ \frac{1}{nh} \sum_{j=1}^{n} Y_j K \left( \frac{X_j - x}{h} \right) \mathbb{I}(Z_j = z) \right] - \mathbb{E}(Y|X = x, Z = z)q(x, z) \right| = O_p(n^{-\iota}),$$

Assumption F is weak and standard in the literature. Assumptions G and H require the standard deviation and bias term of nonparametric estimation converge to zero uniformly at a rate no slower than $n^{-\iota}$, respectively. In particular, Assumption H is a high-level assumption that can be derived by primitive conditions on $K$ and $h$. 

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**Lemma 3.1.** Suppose Assumptions F to H hold. Then, for \( z = 0, 1 \), we have

\[
\sup_{(w, x) \in \mathcal{W}_x} |\hat{G}(w; x; z) - \hat{G}(w; x; z)| = o_p(n^{-1/2}).
\]

By Lemma 3.1, it suffices to establish the limiting distribution of \( \hat{G}(w; x; 1) - \hat{G}(w; x; 0) \) for the asymptotic properties of our test statistics.

Note that

\[
\hat{G}(w, x, z) = \frac{1}{n} \sum_{i=1}^{n} \left[ \mathbb{1}_{X_i, Z_i}(x, z) \hat{q}(X_i, z') (W_i - \hat{W}_i) \mathbb{1}(W_i \leq w) \right] + \frac{1}{n} \sum_{i=1}^{n} \left[ \mathbb{1}_{X_i, Z_i}(x, z) \hat{q}(X_i, z') (w - W_i) \mathbb{1}(W_i \leq w) \right] \equiv U_1(w, x; z) + U_2(w, x; z).
\]

where \( z' = 1 - z \). Moreover, we have

\[
U_1(w, x; z) = \frac{1}{n} \sum_{i=1}^{n} \left[ \mathbb{1}_{X_i, Z_i}(x, z) q(X_i, z') (1 - D_i) \mathbb{1}(W_i \leq w) [\delta(X_i) - \hat{\delta}(X_i)] \right] + o_p(n^{-1/2})
\]

provided that \( \sup_{x \in \mathcal{X}} |[\hat{q}(x, z) - q(x, z)] \times [\hat{\delta}(x) - \delta(x)]| = o_p(n^{-1/2}) \) holds. Therefore, the leading terms in \( U_1(w, x; z) \) and \( U_2(w, x; z) \) are essentially two \( \mathcal{U} \)-processes indexed by \( w \) and \( x \). Following Nolan and Pollard (1988, Theorem 5) and Powell, Stock, and Stoker (1989, Theorem 3.1), \( \sqrt{n}[\hat{G}(\cdot; z) - G(\cdot; z)] \) converges in distribution to a mean–zero Gaussian process with bounded and nonzero covariance kernel.

We denote \( F_{WD|XZ}(w, d| X, z) = F_{W|DXZ}(w|d, X, z) \times p(D = d| X, Z = z) \). Let

\[
\kappa^\epsilon(w, x) = \frac{F_{WD|XZ}(w, 0| x, 1) - F_{WD|XZ}(w, 0| x, 0)}{p(X, 1) - p(x, 0)};
\]

\[
\psi_{\epsilon wx} = \left\{ \lambda(w - W) - \mathbb{E}[\lambda(w - W)|X] \right\} \left[ \frac{\mathbb{1}_{XZ}(x, 0)}{q(X, 0)} - \frac{\mathbb{1}_{XZ}(x, 1)}{q(X, 1)} \right] q(X, 0)q(X, 1);
\]

\[
\phi_{\epsilon wx} = \kappa^\epsilon(w, X) \times [W - \mathbb{E}(W|X)] \times \left[ \frac{\mathbb{1}_{XZ}(x, 0)}{q(X, 0)} - \frac{\mathbb{1}_{XZ}(x, 1)}{q(X, 1)} \right] q(X, 0)q(X, 1).
\]

By definition, \( \psi_{\epsilon wx} \) and \( \phi_{\epsilon wx} \) are random processes indexed by \( (w, x) \). Moreover, we have \( \mathbb{E}(\psi_{\epsilon wx}^\epsilon|X, Z) = \mathbb{E}(\phi_{\epsilon wx}^\epsilon|X, Z) = 0 \) under \( \mathcal{H}_0^\epsilon \).

To establish the weak convergence, we make the following assumptions.

**Assumption I.** \( f_{X|Z}(x|z), \delta(x), p(x, z) \) and \( \mathbb{E}(Y|X = x, Z = z) \) are continuously differentiable in \( x \) for \( z = 0, 1 \).
Assumption J. Let $nh_q^3 \to \infty$ and $nh^3 \to \infty$ as $n \to \infty$. Moreover, the support of $K$ (resp. $K_q$) is a convex (possibly unbounded) subset of $\mathbb{R}$ with nonempty interior, with the origin as an interior point. $K(\cdot)$ (resp. $K_q(\cdot)$) is a bounded differentiable function such that $\int K(u) = 1$, $\int uK(u) = 0$, and $K(u) = K(-u)$ holds for all $u$ in the support.

Assumption K. $\sup_{x \in \mathcal{X}} |\mathbb{E}[\hat{\delta}(x)] - \delta(x)| = o_p(n^{-\frac{1}{2}})$ and $\sup_{x \in \mathcal{X}} |\mathbb{E}[\hat{q}(x,z)] - q(x,z)| = o_p(n^{-\frac{1}{2}})$.

Assumption F is a smoothness condition that can be further relaxed by the Lipschitz condition. Assumption J is standard in the kernel regression literature. In particular, the first part strengthens the conditions for bandwidth choice in Assumption G. Assumption K strengthens Assumption H by requiring the bias term in the first-stage nonparametric estimation to be smaller than $o_p(n^{-1/2})$, which can be established under high order kernels (see e.g. Powell, Stock, and Stoker, 1989).

The limiting distribution of our test statistic is summarized in the following theorem.

Theorem 3.2. Suppose the assumptions in Lemma 3.1 and in addition Assumptions I to K hold. Then, under $\mathcal{H}_0^c$,

$$
\hat{T}_n^c \overset{d}{\to} \sup_{w \in \mathbb{R}; x \in \mathcal{X}} |Z^c(w,x)|
$$

where $Z^c(w,x)$ is a mean-zero Gaussian process with the following covariance kernel

$$
\text{Cov}[Z^c(w,x), Z^c(w',x')] = \mathbb{E}
\left[
(\psi_{wx}^c + \phi_{wx}^c)(\psi_{wx'}^c + \phi_{wx'}^c)
\right], \forall w, w' \in \mathbb{R}.
$$

Moreover, under $\mathcal{H}_1^c$, we have

$$
n^{-\frac{1}{2}}\hat{T}_n^c \overset{P}{\to} \sup_{w \in \mathbb{R}; x \in \mathcal{X}} |G(w,x;0) - G(w,x;1)|.
$$

It is worth pointing out that our test is one-sided against any alternative to $\mathcal{H}_0$: reject $\mathcal{H}_0$ significance level $\alpha$ if and only if $\hat{T}_n^c \geq c_\alpha$.

Similar to the discrete–covariates case, we apply the multiplier bootstrap method to approximate the entire process and therefore to approximate critical values. The estimates
of the pointwise influence function can be estimated by
\[ \hat{\psi}_{wx}^c = \{\lambda(w - \hat{W}) - \hat{E}[\lambda(w - \hat{W})|X]\} \left[ \frac{1_{XZ}(x, 0)}{\hat{q}(X, 0)} - \frac{1_{XZ}(x, 1)}{\hat{q}(X, 1)} \right] \hat{q}(X, 0)\hat{q}(X, 1), \]
\[ \hat{\phi}_{wx}^c = -\hat{\kappa}^c(w, X) \times [\hat{W} - \hat{E}(W|X)] \times \left[ \frac{1_{XZ}(x, 0)}{\hat{q}(X, 0)} - \frac{1_{XZ}(x, 1)}{\hat{q}(X, 1)} \right] \hat{q}(X, 0)\hat{q}(X, 1), \]
where
\[ \hat{E}[\lambda(w - \hat{W})|X] = \frac{\sum_{j=1}^{n} \lambda(w - \hat{W}_j)K(\frac{X_j - X}{h})}{\sum_{j=1}^{n} K(\frac{X_j - X}{h})} \]
and
\[ \hat{E}(W|X) = \frac{\sum_{j=1}^{n} \hat{W}_jK(\frac{X_j - X}{h})}{\sum_{j=1}^{n} K(\frac{X_j - X}{h})} \]
are estimators of \( E[\lambda(w - \hat{W})|X] \) and \( E(W|X) \), respectively, and
\[ \hat{\kappa}^c(w, x) = -\frac{\hat{F}_{WD|XZ}(w, 0|x, 1) - \hat{F}_{WD|XZ}(w, 0|x, 0)}{\hat{p}(X, 1) - \hat{p}(x, 0)} \]
in which
\[ \hat{p}(X, z) = \frac{\sum_{j=1}^{n} D_jK(\frac{X_j - X}{h})\mathbb{1}(Z_j = z)}{\sum_{j=1}^{n} K(\frac{X_j - X}{h})\mathbb{1}(Z_j = z)}, \quad z = 0, 1 \]
and
\[ \hat{F}_{WD|XZ}(w, 0|x, z) = \frac{\sum_{j=1}^{n} \mathbb{1}(\hat{W}_j \leq w; D_j = 0)K(\frac{X_j - X}{h})\mathbb{1}(Z_j = z)}{\sum_{j=1}^{n} K(\frac{X_j - X}{h})\mathbb{1}(Z_j = z)}. \]
Note that we can simulate the limiting process by the following result:
\[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i \times (\hat{\psi}_{wx,i}^c + \hat{\phi}_{wx,i}^c) \Rightarrow Z^c(w, x). \]
The test can be constructed similar to the discrete case and we omit the details for brevity.

4 Monte Carlo Simulations

We investigate the finite sample performance of our tests using Monte Carlo methods. First, we examine both size and power by using some simple data generating processes (DGPs). Two DGPs are considered.

DGP 1: \( Y = DX + X\varepsilon; \ D = \mathbb{1}(Z - \eta > 0); \)

DGP 2: \( Y = DX + (1 + \gamma D)X\varepsilon; \ D = \mathbb{1}(Z - \eta > 0), \)
where $X$ is uniformly distributed on $\{1, 2, 3, 4, 5\}$, $\epsilon \in \mathbb{R}$ is uniformly distributed on $(0, 1)$, and $\eta = \rho \epsilon + \sqrt{1-\rho^2}u$, where $u$ is uniformly distributed on $(0, 1)$ and independent of $\epsilon$, and $\rho = 0.5, 0.7$, or $0.9$, respectively. The value of $\rho$ represents level of the endogeneity. The instrumental variable $Z \in \{0, 1\}$, independent of $X$, follows Bernoulli distribution with the $\mathbb{P}(Z = 1) = 0.3, 0.5,$ or $0.7$, respectively. Moreover, $\gamma = 0.1, 0.3$ and $0.5$ represents the “degree” of nonseparability in DGP 2. By design, $\mathbb{H}_0$ holds in DGP 1, but not in DGP 2.

We choose the sample size $n = 500, 1000, \text{and } 2000$, and the rejection rate is approximated by 1000 repetitions. To simulate the stochastic processes in the limiting distribution for deriving the critical values, we follow the the method of multiplier bootstrap stated in Section 3 with 1000 bootstrap repetition. Moreover, we use 100 grids on the support of $(\min(\hat{W}), \max(\hat{W}))$ for the suprema of simulated stochastic processes.

Table 1 reports size performances under DGP 1 with different values of $\rho$, $\alpha$ and $n$. In all cases, our tests have reasonable size. Tables 2 to 4 report power performances under DGP 2 with different values of $\gamma$. In particular, the rejection rates increase rapidly with the sample size, which verifies the consistency of our test. Moreover, the rejection rate increases with $\gamma$: Less separable of the error term, more likely to be rejected.

Next, we investigate the case with continuously distributed covariates $X$. DGP 3 and DGP 4 are the same as DGP 1 and DGP 2, respectively, except that $X$ is a continuous random variable uniformly distributed on $[0, 1]$. We choose sample size $n = 1000, 2000, \text{and } 4000$, and the rejection rate is based on 1000 repetitions. For each repetition, the $p$-value is approximated by 500 simulations. To compute the suprema of the simulated stochastic processes, we use 100 grids on the support of $(\min(\hat{W}), \max(\hat{W}))$ and 100 grids on $[0, 1]$ for $w$ and $x$ respectively. We choose $\gamma = 0.5$ and $0.7$.

In this continuous covariates case, the performance of the proposed test behaves similarly to the discrete covariates case. For simplicity, we only present the results for $\mathbb{P}(Z = 1) = 0.5$ and $\rho = 0.7$. The results for other settings exhibit similar patterns. Table 5 reports empirical levels at various nominal levels. The level of our test is fairly well behaved.
and it converges to the nominal level as the sample size increases. Table 6 presents the empirical power of our test. Clearly, our test is consistent.

5 Empirical Applications

5.1 The Effect of Job Training Program on Earnings

In this section we apply our testing approach to the effect of job training program on earnings. The National Job Training Partnership Act (JTPA), commissioned by the Department of Labor, began funding training from 1983 to late 1990’s to increase employment and earnings for participants. The major component of JTPA aims to support training for the economically disadvantaged.

Our sample consists of 11,204 observations from the JTPA, a survey dataset from over 20,000 adults and out-of-school youths who applied for JTPA in 16 local areas across the country between 1987 and 1989. Each participant were assigned randomly to either a program group or a control group (1 out of 3 on average). Program group are eligible to participate JTPA services, including classroom training, on-the-job training or job search assistance, and other services, while members of control group were not eligible for JTPA services for 18 months.

Following the literature, we use the program eligibility as an instrumental variable for the endogenous individual’s participation decision. The outcome variable is individual earnings, measured by the sum of earnings in the 30-month period following the offer. The effects of JTPA training programs on earnings has also been studied by Abadie, Angrist, and Imbens (2002) under a general framework allowing for unobserved heterogeneous treatment effects.

The observed covariates include a set of dummies for races, for high-school graduates, and for marriage, whether the applicant worked at least 12 weeks in the 12 months.

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2 JTPA services are provided at 649 sites, which might not be randomly chosen. For a given site, the applicants were randomly selected for the JTPA dataset.

3 The data is publicly available at http://upjohn.org/services/resources/employment-research-data-center/national-jtpa-study.
preceding random assignment, and also 5 age-group dummies (22-24, 25-29, 30-35, 36-44, and 45-54), among others. See Table 7 for descriptive statistics. For simplicity, we group all applicants into 3 age categories (22-29, 30-35, and 36 and above), and pool all non-White applicants as minority applicants.

To implement, we use the Gaussian kernel with Silverman bandwidth selection. For the critical value, we use 10,000 bootstrapped simulations and search for the suprema from 5,000 grids. The $p$-value of our test is 0.1204. Therefore, $H_0$, i.e. there is no unobserved heterogenous treatment effects, cannot be rejected at a 10% significance level. It is worth noting that our results are robust to the number of simulations and the number of grids.

5.2 The Impact of Fertility on Family Income

The second empirical illustration example is to explore the impact of children on parents’ labor supply and income. The endogeneity issue arises due to the fertility variable. Rosenzweig and Wolpin (1980) suggest the usage of the twin births as an instrumental variable. The “twin-strategy” IV has been widely used in the literature. See eg. Angrist and Evans (1998) and Vere (2011). The impact of fertility on family income has also been studied by Frölich and Melly (2013) under a general framework allowing for unobserved heterogeneous treatment effects.

Our sample come from 1990 and 2000 censuses, consisting of 602,767 and 573,437 observations, respectively. Similar to Frölich and Melly (2013), we use the 1% and 5% Public Use Microdata Sample (PUMS) from the 1990 and 2000 censuses. Moreover, our sample is restricted to 21–35 years old married mothers with at least one child. The outcome variable of interest is the family’s annual labor income. The treatment variable is the dummy for a mother has two or more children. The instrument variable is the dummy for the first birth is a twin. The covariates includes mother’s and father’s age, race, educational level, and working status. Table 8 provides descriptive statistics. Some covariates, e.g., age, wages, salary, armed forces pay, commissions, tips, piece-rate payments, cash bonuses earned before deductions were made for taxes, bonds, pensions, union dues, etc. See Frölich and Melly (2013) for more details.

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4The data is publicly available at https://www.census.gov/main/www/pums.html.
5It includes wages, salary, armed forces pay, commissions, tips, piece-rate payments, cash bonuses earned before deductions were made for taxes, bonds, pensions, union dues, etc. See Frölich and Melly (2013) for more details.
years in education, and working hours per week, are treated as continuous variables.

For the critical value, we use 5000 bootstrapped simulations and search for the suprema from 1000 grids. The $p$-values of our tests are 0.0031 and 0.0004 for the 1990 and 2000 censused, respectively. These results suggest that the null hypothesis, i.e., homogeneous treatment effects, should be rejected at all usual significance levels.
APPENDIX

A  PROOFS OF LEMMAS AND THEOREMS

A.1  PROOF OF PROPOSITION 2.1

Proof. For the “only if” part, under $H_0$, we have

$$g(1, x, \epsilon) - h(0, x, \epsilon) = m(1, x) - m(0, x) \equiv \delta(x), \quad \forall x \in \mathcal{X}.$$

For the “if part”, (2) implies

$$g(d, x, \epsilon) = d \times [g(1, x, \epsilon) - g(0, x, \epsilon)] + g(0, x, \epsilon) = d \times \delta(x) + g(0, x, \epsilon).$$

Therefore, $H_0$ holds in the sense $m(d, x) = d \times \delta(x)$ and $\nu(x, \epsilon) = g(0, x, \epsilon)$.  Q.E.D.

A.2  PROOF OF THEOREM 2.1

Proof. Given Proposition 2.1, it suffices to show the if part. Suppose $W \perp Z \mid X$ holds. Recall that $W = Y + (1 - D) \times \delta(X)$. It follows that

$$\mathbb{P}(Y \leq y, D = 1 \mid X, Z = 1) + \mathbb{P}(Y + \delta(X) \leq y, D = 0 \mid X, Z = 1) = \mathbb{P}(Y \leq y, D = 1 \mid X, Z = 0) + \mathbb{P}(Y + \delta(X) \leq y, D = 0 \mid X, Z = 0), \quad \forall y \in \mathbb{R}.$$  

Equivalently, we have

$$\mathbb{P}(Y \leq y, D = 1 \mid X, Z = 1) - \mathbb{P}(Y \leq t, D = 1 \mid X, Z = 0) = \mathbb{P}(Y \leq y - \delta(X), D = 0 \mid X, Z = 1) - \mathbb{P}(Y \leq y - \delta(X), D = 0 \mid X, Z = 0). \quad (A.1)$$

Let $V \equiv \nu(X, \epsilon)$ and define

$$\Delta_0(\tau, x) \equiv \mathbb{P}(V \leq \tau, D = 0 \mid X = x, Z = 1) - \mathbb{P}(V \leq \tau, D = 0 \mid X = x, Z = 0);$$

$$\Delta_1(\tau, x) \equiv \mathbb{P}(V \leq \tau, D = 1 \mid X = x, Z = 0) - \mathbb{P}(V \leq \tau, D = 1 \mid X = x, Z = 1).$$

By Assumptions A and C, we have

$$\Delta_0(\tau, x) = \mathbb{P}(V \leq \tau, \eta \in \mathcal{C}_x \mid X = x) = \Delta_1(\tau, x)$$
which is strictly monotone in $\tau \in \mathcal{J}_{V|X=x, \eta \in \mathcal{C}}$ and, by Assumptions B and D,

$$\mathcal{J}_{V|X=x, \eta \in \mathcal{C}_x} = \mathcal{J}_{V|X=x}.$$ 

Therefore, we have

$$P(Y \leq y, D = 1|X = x, Z = 0) - P(Y \leq y, D = 1|X = x, Z = 1) = \Delta_1(\check{g}^{-1}(1, x, y), x) = \Delta_0(\check{g}^{-1}(1, x, y), x)$$

$$= P(Y \leq \check{g}(0, x, \check{g}^{-1}(1, x, y)), D = 0|X = x, Z = 1) - P(Y \leq \check{g}(0, x, \check{g}^{-1}(1, x, y)), D = 0|X = x, Z = 0),$$

where $\check{g}^{-1}(1, x, \cdot)$ is the inverse function of $\check{g}(1, x, \cdot)$. Note that both sides are strictly monotone in $y \in \mathcal{J}_{\check{g}(1,X,V)|X=x}$ since $\Delta_1(\cdot, x)$ is strictly monotone on $\mathcal{J}_{V|X=x}$.

Combining the above result with (A.1), we have

$$\check{g}(0, x, \check{g}^{-1}(1, x, y)) = y - \delta(x), \quad \forall x \in \mathcal{J}_x, \ y \in \mathcal{J}_{\check{g}(1,X,V)|X=x}.$$

Let $y = \check{g}(1, x, \tau)$ for $\tau \in \mathcal{J}_{V|X=x}$. It follows that

$$\check{g}(0, x, \tau) = \check{g}(1, x, \tau) - \delta(x),$$

which gives us the result by Proposition 2.1. \hfill Q.E.D.

### A.3 Proof of Theorem 2.2

**Proof.** For notational simplicity, let $W^* = W - E(W|X)$. For the “only if” part, under $\mathbb{H}_0^t$, since $W^* = h(1, X, \epsilon) - E[h(1, X, \epsilon)|X]$ is a function of $\epsilon$, we have $W^* \perp (X, Z)$.

For the “if” part, first note that $W^* \perp Z|X$ is equivalent to $W \perp Z|X$. Then, by Theorem 2.1, $\check{g}(D, X, \epsilon) = m(D, X) + v(X, \epsilon)$ and

$$W^* = m(1, X) + v(X, \epsilon) - E[m(1, X) + v(X, \epsilon)|X] = v(X, \epsilon) - E[v(X, \epsilon)|X] = \overline{v}(X, \epsilon)$$

is a function of $X$ and $\epsilon$. Note that $\overline{v}(X, \epsilon)$ is strictly increasing in $\epsilon$. We now show that there exists a measurable function $v_1 : \mathbb{R} \to \mathbb{R}$ such that $v(x, \epsilon) = v_1(\epsilon)$ almost everywhere. To see this, first note that $\epsilon \perp X$ implies $P(\epsilon > \epsilon|X = x) = P(\epsilon > \epsilon)$ for almost all $x$. We also
have

\[ P(\nu(x,e) > \varepsilon|X = x) = P(\varepsilon > \nu^{-1}(x,\varepsilon)|X = x) = P(\varepsilon > \nu^{-1}(x,\varepsilon)) \]

where \( \nu^{-1} \) is the inverse function of \( \nu \) with respect to its second argument. Since the c.d.f. of \( \varepsilon \) is strictly increasing and \( P(\nu(x,e) > \varepsilon|X = x) \) does not depend \( x \), \( \nu^{-1}(x,\varepsilon) \) must be constant in \( x \) for almost all \( (x,\varepsilon) \). Thus, \( \nu(x,e) \) is constant in \( x \) almost everywhere and this completes the proof.

Q.E.D.

A.4 Proof of Theorem 3.1

Proof. For discrete \( X \), we have \( \hat{\delta}^*(\cdot) = \delta^*(\cdot) + O_p(n^{-1/2}) \) by the central limit theorem. The \( O_p(n^{-1/2}) \) term is uniformly over the finite support \( \mathcal{X} \). Moreover, let \( \mathbb{1}_{WXZ}(w,x,z;\delta) = \mathbb{1}(W \leq w) \cdot \mathbb{1}_{XZ}(x,z) \). By definition, we have

\[ F_{W|XZ}(w|x,z) = \frac{\mathbb{E}[\mathbb{1}_{WXZ}(w,x,z;\delta)]}{\mathbb{E}[\mathbb{1}_{XZ}(x,z)]} \quad \text{and} \quad \hat{F}_{W|XZ}(w|x,z) = \frac{\mathbb{E}_n[\mathbb{1}_{WXZ}(w,x,z;\hat{\delta})]}{\mathbb{E}_n[\mathbb{1}_{XZ}(x,z)]}. \]

Note that

\[
\mathbb{E}_n[\mathbb{1}_{WXZ}(\cdot,x,z;\hat{\delta})] = \mathbb{E}_n[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] - \mathbb{E}[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] \\
+ \left\{ \mathbb{E}_n[\mathbb{1}_{WXZ}(\cdot,x,z;\hat{\delta})] - \mathbb{E}[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] - \mathbb{E}_n[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] + \mathbb{E}[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] \right\} \\
+ \mathbb{E}[\mathbb{1}_{WXZ}(\cdot,x,z;\hat{\delta})].
\]

Since \( \hat{\delta} \) is a consistent estimator of \( \delta \), by the empirical process theory (see e.g. van der Vaart and Wellner, 2007), we have

\[ \mathbb{E}_n[\mathbb{1}_{WXZ}(\cdot,x,z;\hat{\delta})] - \mathbb{E}[\mathbb{1}_{WXZ}(\cdot,x,z;\hat{\delta})] - \mathbb{E}_n[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] + \mathbb{E}[\mathbb{1}_{WXZ}(\cdot,x,z;\delta)] = o_p(n^{-1/2}). \]

Moreover, by Taylor expansion,

\[ \sqrt{n} \mathbb{E}[g(\cdot,x,z;\delta)] = \sqrt{n} P(W \leq w, X = x, Z = z) + \frac{\partial \mathbb{E}[\mathbb{1}_{WXZ}(w,x,z;\delta)]}{\partial \delta} \times \sqrt{n}(\delta - \delta) + o_p(1) \]

where

\[ \frac{\partial \mathbb{E}[\mathbb{1}_{WXZ}(w,x,z;\delta)]}{\partial \delta(x')} = 0 \text{ for all } x' \neq x . \]
and

\[
\frac{\partial \mathbb{E}[\mathbb{I}_{WXZ}(w, x, z; \delta)]}{\partial \delta(x)} = -f_{Y|DXZ}(w - \delta(x)|0, x, z) \times \mathbb{P}(D = 0, X = x, Z = z)
\]

\[
= -f_{W|DXZ}(w|0, x, z) \times \mathbb{P}(D = 0, X = x, Z = z).
\]

It follows that

\[
\sqrt{n} \mathbb{E}_n[\mathbb{I}_{WXZ}(\cdot, x, z; \delta)] = \sqrt{n} \left\{ \mathbb{E}_n[\mathbb{I}_{WXZ}(\cdot, x, z; \delta)] - \mathbb{E}[\mathbb{I}_{WXZ}(\cdot, x, z; \delta)] \right\}
\]

\[
+ \sqrt{n} \mathbb{P}(W \leq w, X = x, Z = z) + \frac{\partial \mathbb{E}[\mathbb{I}_{WXZ}(w, x, z; \delta)]}{\partial \delta(x)} \times \sqrt{n}(\delta(x) - \delta(x)) + o_p(1).
\]

Moreover, by the central limit theorem, we have

\[
\mathbb{E}_n[\mathbb{I}_{XZ}(x, z)] = \mathbb{P}(X = x, Z = z) + O_p(n^{-1/2}).
\]

Thus,

\[
\sqrt{n} \left[ \hat{f}_{W|XZ}(w|x, 1) - \hat{f}_{W|XZ}(w|x, 0) \right]
\]

\[
= \frac{\sqrt{n} \left\{ \mathbb{E}_n[\mathbb{I}_{WXZ}(w, x, 1; \delta)] - \mathbb{E}[\mathbb{I}_{WXZ}(w, x, 1; \delta)] \right\} + \frac{\partial \mathbb{E}[\mathbb{I}_{WXZ}(w, x, 1; \delta)]}{\partial \delta(x)} \times \sqrt{n}(\delta(x) - \delta(x))}{\mathbb{P}(X = x, Z = 1)}
\]

\[
+ \frac{\sqrt{n} \mathbb{P}(W \leq w, X = x, Z = 1)}{\mathbb{E}_n[\mathbb{I}_{XZ}(x, 1)]} + \frac{\mathbb{P}(W \leq w, X = x, Z = 0)}{\mathbb{E}_n[\mathbb{I}_{XZ}(x, 0)]} + o_p(1)
\]

Moreover, we apply Taylor expansion and have

\[
\sqrt{n} \mathbb{P}(W \leq w, X = x, Z = z)
\]

\[
= \frac{\sqrt{n} \mathbb{P}(W \leq w, X = x, Z = z)}{\mathbb{E}_n[\mathbb{I}_{XZ}(x, z)]} - \frac{\mathbb{P}(W \leq w, X = x, Z = z)}{\mathbb{E}^2[\mathbb{I}_{XZ}(x, z)]} \times \sqrt{n} \left[ \mathbb{E}_n[\mathbb{I}_{XZ}(x, z)] - \mathbb{E}[\mathbb{I}_{XZ}(x, z)] \right] + o_p(1)
\]

\[
= \sqrt{n} F_{W|X}(w|x) - F_{W|X}(w|x) \times \frac{\sqrt{n} \left[ \mathbb{E}_n[\mathbb{I}_{XZ}(x, z)] - \mathbb{E}[\mathbb{I}_{XZ}(x, z)] \right]}{\mathbb{P}(X = x, Z = z)} + o_p(1)
\]

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Thus, we have
\[ F_{W|X}(w|x) \times \mathbb{I}_{XZ}(x,z) = P(W \leq w, X = x, Z = z) = \mathbb{E}[\mathbb{I}_{WZ}(w,x,z;\delta)]. \]

Thus, we have
\[
\sqrt{n} \mathbb{P}(W \leq w, X = x, Z = z) \\
\mathbb{E}_n \mathbb{I}_{XZ}(x,z) \\
= \sqrt{n} F_{W|X}(w|x) - \mathbb{E}_n \mathbb{I}_{XZ}(x,z) - \mathbb{E}[\mathbb{I}_{WZ}(w,x,z;\delta)].
\]

Summarizing above steps, we now have
\[
\sqrt{n} \left[ \hat{F}_{W|XZ}(w|x,1) - \hat{F}_{W|XZ}(w|x,0) \right] \\
= \sqrt{n} \mathbb{E}_n \left\{ [\mathbb{I}(W \leq w) - F_{W|X}(w|x)] \times \left[ \frac{\mathbb{I}_{XZ}(x,1)}{\mathbb{P}(X = x, Z = 1)} - \frac{\mathbb{I}_{XZ}(x,0)}{\mathbb{P}(X = x, Z = 0)} \right] \right\} \\
+ \left[ f_{WD|XZ}(w,0|x,0) - f_{WD|XZ}(w,0|x,1) \right] \times \sqrt{n} [\delta(x) - \delta(x)] + o_p(1).
\]

To compute the covariance kernel, we first derive \( \hat{\delta} - \delta \). Fix \( X = x \). For expositional simplicity, we suppress \( x \) in the following notation. Note that
\[
\delta = \frac{A_n(1)C_n(0) - A_n(0)C_n(1)}{B_n(1)C_n(0) - B_n(0)C_n(1)} \quad \text{and} \quad \delta = \frac{A(1)C(0) - A(0)C(1)}{B(1)C(0) - B(0)C(1)},
\]

where \( A_n(z) = \mathbb{E}_n[Y \cdot \mathbb{I}(X = x, Z = z)] \), \( B_n(z) = \mathbb{E}_n[D \cdot \mathbb{I}(X = x, Z = z)] \), \( C_n(z) = \mathbb{E}_n \mathbb{I}(X = x, Z = z) \), \( A(z) = \mathbb{E}[Y \cdot \mathbb{I}(X = x, Z = z)] \), \( B(z) = \mathbb{E}[D \cdot \mathbb{I}(X = x, Z = z)] \), and \( C(z) = \mathbb{P}(X = x, Z = z) \). Therefore,
\[
\delta - \delta = \frac{A_n(1)C_n(0) - A_n(0)C_n(1) - [A(1)C(0) - A(0)C(1)]}{B_n(1)C_n(0) - B_n(0)C_n(1)} \\
+ \left\{ \frac{A(1)C(0) - A(0)C(1)}{B(1)C(0) - B(0)C(1)} - \frac{A(1)C(0) - A(0)C(1)}{B(1)C(0) - B(0)C(1)} \right\} \equiv \mathbb{I} + \mathbb{II}.
\]

Let \( p_c(x) = \mathbb{P}(X = x, Z = 1) - \mathbb{P}(X = x, Z = 0) \), which is strictly positive under Assumption A.
Note that

\[
I = \frac{[A_n(1) - A(1)] \cdot C_n(0) + A(1) \cdot [C_n(0) - C(0)]}{B_n(1)C_n(0) - B_n(0)C_n(1)} - \frac{[A_n(0) - A(0)] \cdot C_n(1) + A(0) \cdot [C_n(1) - C(1)]}{B_n(1)C_n(0) - B_n(0)C_n(1)} \\
= \frac{[A_n(1) - A(1)] \cdot C(0) + A(1) \cdot [C_n(0) - C(0)]}{B(1)C(0) - B(0)C(1)} - \frac{[A_n(0) - A(0)] \cdot C(1) + A(0) \cdot [C_n(1) - C(1)]}{B(1)C(0) - B(0)C(1)} + o_p(n^{-1/2})
\]

where the last step comes from the fact: \( A_n(z) = A(z) + O_p(n^{-1/2}) \), \( B_n(z) = B(z) + O_p(n^{-1/2}) \) and \( C_n(z) = C(z) + O_p(n^{-1/2}) \). Therefore,

\[
I = \frac{\mathbb{E}_n \left[ \left( Y - \mathbb{E}(Y|X = x, Z = 0) \right) \cdot \mathbbm{1}_{XZ}(x, 1) \right] \cdot \mathbb{P}(X = x, Z = 0)}{B(1)C(0) - B(0)C(x, 1)} - \frac{\mathbb{E}_n \left[ \left( Y - \mathbb{E}(Y|X = x, Z = 1) \right) \cdot \mathbbm{1}_{XZ}(x, 0) \right] \cdot \mathbb{P}(X = x, Z = 1)}{B(1)C(0) - B(0)C(1)} \\
+ 2 \frac{[A(0) \cdot C(1) - A(1) \cdot C(0)]}{B(1)C(0) - B(0)C(1)} + o_p(n^{-1/2}) \\
= -\frac{1}{p_c(x)} \times \mathbb{E}_n \left\{ \left[ Y - \mathbb{E}(Y|X = x, Z = 0) \right] \cdot \frac{\mathbbm{1}_{XZ}(x, 1)}{\mathbb{P}(X = x, Z = 1)} \right\} \\
+ \frac{1}{p_c(x)} \times \mathbb{E}_n \left\{ \left[ Y - \mathbb{E}(Y|X = x, Z = 1) \right] \cdot \frac{\mathbbm{1}_{XZ}(x, 0)}{\mathbb{P}(X = x, Z = 0)} \right\} \\
- 2\delta(x) + o_p(n^{-1/2})
\]

where the last step comes from

\[
B(1)C(0) - B(0)C(1) = -p_c(x) \times \mathbb{P}(X = x, Z = 1) \times \mathbb{P}(X = x, Z = 0).
\]

Similarly, by Taylor expansion,

\[
\mathbb{I} = -\delta(x) \times \frac{[B_n(1) - B(1)] \cdot C(0) + B(1) \cdot [C_n(0) - C(0)]}{B(1)C(0) - B(0)C(1)} \\
+ \delta(x) \times \frac{[B_n(0) - B(0)] \cdot C(1) + B(0) \cdot [C_n(1) - C(1)]}{B(1)C(0) - B(0)C(1)} + o_p(n^{-1/2}) \\
= \frac{1}{p_c(x)} \times \mathbb{E}_n \left\{ [D - p(x, 0)] \cdot \delta(x) \times \frac{\mathbbm{1}_{XZ}(x, 1)}{\mathbb{P}(X = x, Z = 1)} \right\} \\
- \frac{1}{p_c(x)} \times \mathbb{E}_n \left\{ [D - p(x, 1)] \cdot \delta(x) \times \frac{\mathbbm{1}_{XZ}(x, 0)}{\mathbb{P}(X = x, Z = 0)} \right\} \\
+ 2\delta(x) + o_p(n^{-1/2}).
\]
Note that under the null hypothesis $H_0$:

$$\mathbb{E}(Y|X = x, Z = 1) - \mathbb{E}(D|X = x, Z = 1) \times \delta(x) = \mathbb{E}[W - \delta(X)|X = x, Z = 1]$$

$$= \mathbb{E}[W - \delta(X)|X = x, Z = 0] = \mathbb{E}(Y|X = x, Z = 0) - \mathbb{E}(D|X = x, Z = 0) \times \delta(x).$$

Moreover, we have

$$Y - D \cdot \delta(X) - \mathbb{E}(Y|X, Z) + \mathbb{E}(D|X, Z) \cdot \delta(X) = W - \mathbb{E}(W|X, Z) = W - \mathbb{E}(W|X).$$

Thus, we have

$$\sqrt{n} [\hat{\delta}(x) - \delta(x)] = -\frac{1}{p_c(x)} \times \sqrt{n} \mathbb{E}_n \left\{ \hat{W} \times \left[ \frac{1_{XZ}(x, 1)}{\mathbb{P}(X = x, Z = 1)} - \frac{1_{XZ}(x, 0)}{\mathbb{P}(X = x, Z = 0)} \right] \right\} + o_p(1). \quad (A.2)$$

It follows that

$$\sqrt{n} \left[ \hat{P}_{W|XZ}(w|x, 1) - \hat{P}_{W|XZ}(w|x, 0) \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\psi_{wx} + \phi_{wx}).$$

where $\psi_{wx}$ and $\phi_{wx}$ are defined by (6) and (7).

To conclude, the empirical process $\sqrt{n} \left[ \hat{P}_{W|XZ}(\cdot|x, 1) - \hat{P}_{W|XZ}(\cdot|x, 0) \right]$ converges to a zero–mean Gaussian process $\mathcal{Z}(\cdot, x)$ with the given covariance kernel. Moreover, following e.g., Kim and Pollard (1990), we have $\hat{P}_n \overset{d}{\to} \sup_{w \in \mathbb{R}; x \in \mathcal{X}} |\mathcal{Z}(w, x)|$. Q.E.D.

### A.5 Proof of Lemma 3.1

**Proof.** Fix $X = x$ and w.l.o.g., let $z = 1$. Note that

$$\hat{G}(w, x, 1) - \tilde{G}(w, x, 1)$$

$$= \mathbb{E}_n \left\{ 1_{XZ}(x, 1) \hat{q}(X, 0)(w - \hat{W}) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \right\}$$

$$= \mathbb{E}_n \left\{ 1_{XZ}(x, 1) \hat{q}(X, 0)(w - \hat{W}) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \times 1(|W - w| \leq n^{-r}) \right\}$$

$$+ \mathbb{E}_n \left\{ 1_{XZ}(x, 1) \hat{q}(X, 0)(w - \hat{W}) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \times 1(|W - w| > n^{-r}) \right\}$$

$$\equiv T_1 + T_2$$

where $r \in \left( \frac{1}{2}, 1 \right)$. It suffices to show both $T_1$ and $T_2$ are $o_p(n^{-\frac{1}{2}})$. 

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For term $T_1$, note that
\[
T_1 = E_n \left\{ 1^*_X(x, 1) \hat{q}(X, 0)(w - W) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \times 1(|W - w| \leq n^{-r}) \right\}
+ E_n \left\{ 1^*_X(x, 1) \hat{q}(X, 0)(W - \hat{W}) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \times 1(|W - w| \leq n^{-r}) \right\}.
\]
Because
\[
E \left| 1^*_X(x, 1) \hat{q}(X, 0)(w - W) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \times 1(|W - w| \leq n^{-r}) \right| 
\leq E \left| \hat{q}(X_1, z_2) \times (w - W) \times 1(|W - w| \leq n^{-r}) \right| = O(1) \times O(n^{-2r}) = o(n^{-\frac{1}{2}}),
\]
where last step holds because $r > \frac{1}{2}$. Moreover,
\[
E \left| 1^*_X(x, 1) \hat{q}(X, 0)(W - \hat{W}) \left[ 1(\hat{W} \leq w) - 1(W \leq w) \right] \times 1(|W - w| \leq n^{-r}) \right| 
\leq E \left| \hat{q}(X_1, z_2)(W - \hat{W}) \times 1(|W - w| \leq n^{-r}) \right| = O(1) \times O(n^{-r}) \times O(n^{-r}) = o(n^{-\frac{1}{2}}).
\]
Then, we have $T_1 = o_p(n^{-\frac{1}{2}})$.

Next, for term $T_2$, note that
\[
E|T_2| \leq \frac{K}{h} \times E \left[ |w - \hat{W}| \times 1(|\hat{W} - W| > n^{-r}) \right] 
\leq \frac{K}{h} \times \sqrt{E(w - \hat{W})^2} \times \sqrt{P(|\hat{W} - W| > n^{-r})}
\leq \frac{K}{h} \times \sqrt{E\hat{W}^2 - 2w \cdot E(\hat{W}) + w^2} \times \sqrt{P(|\hat{\delta}(X) - \delta(X)| > n^{-r})},
\]
where $K$ is the upper bound of $K(\cdot)$. Because $W$ is a bounded random variable and $w$ belongs to a compact set, then $\sqrt{E\hat{W}^2 - 2w \cdot E(\hat{W}) + w^2} = O(1)$. Moreover, by Lemma B.1, $E|T_2| \leq o(n^{-k})$ for any $k > 0$. Hence, $T_2 = o_p(n^{-\frac{1}{2}})$. Q.E.D.

A.6 Proof of Theorem 3.2

Proof. By Lemma 3.1, we have
\[
\hat{\tau}_n^* = \sqrt{n} \left| \tilde{G}(w, x; 1) - \tilde{G}(w, x; 0) \right| + o_p(1).
\]
Let $1^*_{WXZ}(w, x, z) = 1(W \leq w, X \leq x, Z = z)$. Note that
\[
\tilde{G}(w, x, z_\ell) = U_1(w, x; z_\ell) + U_2(w, x; z_\ell) + o_p(n^{-1/2})
\]
where \( U_1(w, x; z) = \frac{1}{n} \sum_{i=1}^{n} [\mathbb{1}_{W_i, X_i, Z_i}(w, x, z_i) \times \hat{q}(X_i, z_i) \times (W_i - \hat{W}_i)] \) and \( U_2(w, x; z) = \frac{1}{n} \sum_{i=1}^{n} [\mathbb{1}_{W_i, X_i, Z_i}(w, x, z_i) \times \hat{q}(X_i, z_i) \times (w - W_i)]. \) Therefore,

\[
\sqrt{n} \left[ \hat{G}(w, x; 1) - \hat{G}(w, x; 0) \right]
\]

\[
= \sqrt{n} \left[ U_1(w, x; 1) - U_1(w, x; 0) \right] + \sqrt{n} \left[ U_2(w, x; 1) - U_2(w, x; 0) \right]
\]

\[
= \sqrt{n} \left\{ U_1(w, x; 1) - U_1(w, x; 0) - [EU_1(w, x; 1) - EU_1(w, x; 0)] \right\}
\]

\[
+ \sqrt{n} \left\{ U_2(w, x; 1) - U_2(w, x; 0) - [EU_2(w, x; 1) - EU_2(w, x; 0)] \right\}
\]

\[
+ \sqrt{n} \left[ EU_1(w, x; 1) - EU_1(w, x; 0) \right] + \sqrt{n} \left[ EU_2(w, x; 1) - EU_2(w, x; 0) \right]
\]

We first look at those terms with \( U_2. \) By definition,

\[
U_2(w, x; z)
\]

\[
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \{1_{X_i, Z_i}(x, z) \lambda(W_i - w) \times \frac{1}{h_q} K_q(\lambda \frac{X_j - X_i}{h_q}) \mathbb{1}(Z_j = 1 - z) \}
\]

\[
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \zeta_{n, ij}(w, w, z_i)
\]

where \( \zeta_{n, ij}(w, x, z) = 1_{X_i, Z_i}(x, z) \lambda(W_i - w) \times \frac{1}{h_q} K_q(\lambda \frac{X_j - X_i}{h_q}) \mathbb{1}(Z_j = 1 - z). \)

Let \( \zeta_{n, ij}^*(w, x, z) = \frac{1}{2} [\zeta_{n, ij}(w, x, z) + \zeta_{n, ji}(w, x, z)]. \) Then, \( \zeta_{n, ij}^* \) is symmetric in indices \( i \) and \( j. \) Therefore,

\[
U_2(w, x; z) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \zeta_{n, ij}^*(w, x, z),
\]

which is a \( \mathcal{U} \)-process indexed by \( (w, x, z_i). \) By Nolan and Pollard (1988, Theorem 5) and Powell, Stock, and Stoker (1989, Lemma 3.1),

\[
U_2(w, x; z) - EU_2(w, x; z)
\]

\[
= \frac{2}{n} \sum_{i=1}^{n} \left\{ EU[\zeta_{n, ij}^*(w, x, z)|Y_i, D_i, X_i, Z_i] - EU[\zeta_{n, ij}^*(w, x, z)] \right\] + o_p(n^{-1/2}).
\]

where the \( o_p(n^{-1/2}) \) applies uniformly over \( (w, x). \) Note that

\[
EU[\zeta_{n, ij}^*(w, x, z)|Y_i, D_i, X_i, Z_i]
\]

\[
= \frac{1}{2} \left\{ 1_{XZ}(x, z_i) q(X, 1 - z) \lambda(W - w) + 1_{XZ}(x, 1 - z) q(X, z) \Pi(w|X, z) \right\} + o_p(1).
\]
We now derive $\mathbb{E}[\zeta_{n,ij}^*(w, x, z)]$. Let $\mu_1(w, x, z) = \mathbb{E}[\mathbb{I}_{X(Z)(x, z)}q(X, 1 - z)\delta(W - w)]$ and $\mu_2(w, x, z) = \mathbb{E}[\mathbb{I}_{X(Z)(x, 1 - z)}q(X, z)\Pi(w|X, z)]$. Note that

$$
\mu_1(w, x, z) = \mu_2(w, x, z) = \int \mathbb{I}(X \leq x)\Pi(w|X)f_{X|Z}(X|1)f_{X|Z}(X|0)dX \times \mathbb{P}(Z = 1)\mathbb{P}(Z = 0)
$$

under the $H^G_0$, which are invariant with $z$. Therefore, $\mathbb{E}[\zeta_{n,ij}^*(w, x, z)] = \frac{1}{2}\left[\mu_1(w, x, z) + \mu_2(w, x, z)\right]$ is also invariant with $z$. Let $\mu^*(w, x) = \mathbb{E}[\zeta_{n,ij}^*(w, x, z)]$.

By Powell, Stock, and Stoker (1989, Theorem 3.1),

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \mathbb{E}[\zeta_{n,ij}^*(w, x, z)|Y_i, D_i, X_i] - \mathbb{E}[\zeta_{n,ij}^*(w, x, z)] \right\} = \mathbb{E}_n \left\{ \mathbb{I}_{X(Z)(x, z)}q(X, z)\delta(W - w) - \mu^*(w, x) \right\} + \mathbb{E}_n \left\{ \mathbb{I}_{X(Z)(x, z)}q(X, z)\Pi(w|X, z) - \mu^*(w, x) \right\} + o_p(n^{-\frac{1}{2}}),
$$

where the $o_p(n^{-1/2})$ holds uniformly over $(w, x)$. It follows that

$$
U_2(w, x; 1) - U_2(w, x; 0) - [\mathbb{E}U_2(w, x; 1) - \mathbb{E}U_2(w, x; 0)] = \mathbb{E}_n \psi_{\alpha x} + o_p(n^{-\frac{1}{2}}).
$$

We now turn to $U_1(w, x, z)$. Note that

$$
U_1(w, x; z) = -\frac{1}{n} \sum_{i=1}^{n} \left\{ \mathbb{I}_{W_i, X_i, Z_i}^*(w, x, z)q(X_i, 1 - z)(1 - D_i)[\delta(X_i) - \delta(X_i)] \right\} + o_p(n^{-\frac{1}{2}}),
$$

provided that $\sup_{x \in \mathcal{X}} \left[ |\delta(x, z) - q(x, z)| \times |\delta(x) - \delta(x)| \right] = o_p(n^{-\frac{1}{2}})$ holds. By a similar decomposition argument on $\delta(X) - \delta(X)$ in Lemma B.1, we have

$$
U_1(w, x; z) = -\frac{1}{n(n - 1)} \sum_{i=1}^{n} \sum_{j \neq i} \left\{ \mathbb{I}_{W_i, X_i, Z_i}^* (w, x, z)q(X_i, 1 - z)(1 - D_i) \right\} \times \frac{\mathbb{W}_{ji,n}^{1/2} K(X_i - X_j)}{p(X_i, 1) - p(X_i, 0)} \left[ \mathbb{I}(Z_j = 1) - \mathbb{I}(Z_j = 0) \right] + o_p(n^{-1/2})
$$

$$
= -\frac{1}{n(n - 1)} \sum_{i=1}^{n} \sum_{j \neq i} \zeta_{n,ij}(w, x, z) + o_p(n^{-1/2})
$$

where $\mathbb{W}_{ji} = W_j - \mathbb{E}(W_j X_i)$ and

$$
\zeta_{n,ij}(w, x, z) = \mathbb{I}_{W_i, X_i, Z_i}^* (w, x, z)q(X_i, 1 - z)(1 - D_i) \times \frac{\mathbb{W}_{ji,n}^{1/2} K(X_i - X_j)}{p(X_i, 1) - p(X_i, 0)} \left[ \mathbb{I}(Z_j = 1) - \mathbb{I}(Z_j = 0) \right].
$$
Moreover, Let \( \hat{\xi}_{n, ij}(w, x, z) = \frac{1}{2}[\xi_{n, ij}(w, x, z) + \xi_{n, ji}(w, x, z)] \). By a similar argument as that for \( \mathbb{U}_2 \), we have

\[
\mathbb{U}_1(w, x, z) - \mathbb{E}\mathbb{U}_1(w, x, z) = -\frac{2}{n} \sum_{i=1}^{n} \left\{ \mathbb{E}[\hat{\xi}_{n, ij}^*(w, x, z)|Y_i, D_i, X_i, Z_i] - \mathbb{E}[\hat{\xi}_{n, ij}^*(w, x, z)] \right\} + o_p(n^{-1/2}).
\]

Because

\[
\mathbb{E}[\hat{\xi}_{n, ij}^*(w, x, z)|Y_i, D_i, X_i, Z_i] = \mathbb{E} \left\{ \mathbb{E}[\hat{\xi}_{n, ij}(w, x, z)|X_j, Z_j, Y_i, D_i, X_i, Z_i]|Y_i, D_i, X_i, Z_i \right\} = \mathbb{E} \left\{ 1_{X_i, Z_j}(x, z) q(X_j, 1 - z) \mathbb{P}(W \leq w; D = 0|X_j, Z_j)[W_i - \mathbb{E}(W|X_j)] \times \frac{\frac{1}{n} K(\frac{X_i - X_j}{p(X_j, 1) - p(X_j, 0)}) [1_{(Z_i = 1)} \mathbb{E}(W|X_j) - 1_{(Z_i = 0)} q(X_j, 1)]}{q(X_j, 1) - q(X_j, 0)} \right\} + o_p(1)
\]

where the last step comes from the Bochner’s Lemma and uses the fact the integrant equals zero if \( Z_j = 1 - z \), and the expectation of the first term in the above equation equals to zero.

Thus, we have

\[
\mathbb{U}_1(w, x, z) - \mathbb{E}\mathbb{U}_1(w, x, z) = -\mathbb{E} \left\{ \frac{F_{WD|XZ}(w, 0|X, z)}{p(X, 1) - p(X, 0)} \times \left[ \frac{1_{XZ}(x, 1)}{q(X, 1) - q(X, 0)} \right] - \frac{1_{WXZ}(x, 0)}{q(X, 0)} \right\} + o_p(n^{-1/2}),
\]

where the \( o_p(n^{-1/2}) \) holds uniformly over \((w, x)\). Moreover,

\[
\mathbb{U}_1(w, x, 1) - \mathbb{E}\mathbb{U}_1(w, x, 1) - [\mathbb{U}_1(w, x, 0) - \mathbb{E}\mathbb{U}_1(w, x, 0)] = \mathbb{E}_{n}\phi_{wx} + o_p(n^{-1/2}).
\]

Finally, by Assumption K,

\[
\mathbb{E}\mathbb{U}_2(w, x; z) = \int_{-\infty}^{x} q(X, 1) q(X, 0) \mathbb{P}(w|X, Z = z) dX + o_p(n^{-1/2})
\]
which is invariant with \( z \) under \( H_0^G \), and
\[
\mathbb{E}U_1(w, x; z) = o_p(n^{-\frac{1}{2}}) .
\]

Therefore,
\[
\sqrt{n} [\mathbb{E}U_1(w, x; 1) - \mathbb{E}U_1(w, x; 0)] + \sqrt{n} [\mathbb{E}U_2(w, x; 1) - \mathbb{E}U_2(w, x; 0)] = o_p(n^{-\frac{1}{2}}).
\]

To conclude, the empirical process \( \sqrt{n} [\hat{G}(w, x; 1) - \hat{G}(w, x; 0)] \) is asymptotically equivalent to the following process
\[
\sqrt{n} \times \mathbb{E}_n(\psi_cwx + \phi_cwx).
\]

which converges to a zero-mean Gaussian process \( Z(\cdot, x) \) with the given covariance kernel. \( \quad Q.E.D. \)

**B  ** TECHNICAL LEMMAS

**Lemma B.1.** Suppose Assumptions F to H hold. Then for any \( k > 0 \) and \( r \in (\frac{1}{4}, \iota) \),
\[
\sup_{x \in \mathcal{S}_X} n^k \times \mathbb{P} [ | \hat{\delta}(x) - \delta(x) | > n^{-r} ] \rightarrow 0.
\]

**Proof.** First, by a similar decomposition of \( \hat{\delta}(x) - \delta(x) \) as that in the proof of Theorem 3.1, it suffices to show
\[
\sup_{x} n^k \times \mathbb{P} \{|a_n(x, z) - a(x, z)| > \lambda_a \times n^{-r}\} \rightarrow 0;
\]
\[
\sup_{x} n^k \times \mathbb{P} \{|b_n(x, z) - b(x, z)| > \lambda_b \times n^{-r}\} \rightarrow 0;
\]
\[
\sup_{x} n^k \times \mathbb{P} \{|q_n(x, z) - q(x, z)| > \lambda_q \times n^{-r}\} \rightarrow 0,
\]

where \( \lambda_a, \lambda_b \) and \( \lambda_q \) are strictly positive constants, and
\[
a_n(x, z) = \frac{1}{nh} \sum_{j=1}^{n} Y_j K(\frac{X_j - x}{h}) \mathbb{1}(Z_j = z), \quad a(x, z) = \mathbb{E}(Y | X = x, Z = z) \times q(x, z);
\]
\[
b_n(x, z) = \frac{1}{nh} \sum_{j=1}^{n} D_j K(\frac{X_j - x}{h}) \mathbb{1}(Z_j = z), \quad b(x, z) = \mathbb{E}(D | X = x, Z = z) \times q(x, z);
\]
\[
q_n(x, z) = \frac{1}{nh} \sum_{j=1}^{n} K(\frac{X_j - x}{h}) \mathbb{1}(Z_j = z).
\]
For expositional simplicity, we only show the first result. It is straightforward that the rest follow a similar argument.

Let $T_{nxz} = Y_j K(\frac{X_j - x}{h}) 1(Z_j = z)$ and $\tau_{nxz} = h \times [\lambda_a n^{-r} - |Ea_n(x, z) - a(x, z)|]$. Note that

$$\mathbb{P} \left( |a_n(x, z) - a(x, z)| > \lambda_a \times n^{-r} \right)$$

$$\leq \mathbb{P} \left( |a_n(x, z) - Ea_n(x, z)| + |Ea_n(x, z) - a(x, z)| > \lambda_a \times n^{-r} \right)$$

$$= \mathbb{P} \left\{ \frac{1}{n} \sum_{j=1}^{n} \left( T_{nxz} - \mathbb{E} T_{nxz} \right) > \tau_{nxz} \right\}.$$

Moreover, by Bernstein’s tail inequality,

$$\mathbb{P} \left\{ \frac{1}{n} \left| \sum_{j=1}^{n} (T_{nxz} - \mathbb{E} T_{nxz}) \right| > \tau_{nxz} \right\} \leq 2 \exp \left( -\frac{n \times \tau_{nxz}^2}{2 \text{Var}(T_{nxz}) + \frac{3}{4} \mathcal{K} \times \tau_{nxz}} \right),$$

where $\mathcal{K}$ is the upper bound of kernel $K$.

By Assumption H, $|Ea_n(x, z) - a(x, z)| = O(n^{-i}) = o(n^{-r})$. Then, for sufficient large $n$, there is $0.5 \lambda_a n^{-r} h \leq \tau_n(x, z) \leq \lambda_a n^{-r} h$. Moreover,

$$\text{Var}(T_{nxz}) \leq \mathbb{E} T_{nxz}^2 \leq \mathbb{E} \left[ \mathbb{E}(Y^2|X) K^2(\frac{X - x}{h}) \right] \leq Ch,$$

where $C = \sup_x \mathbb{E}[Y^2|X = x] \times \sup_x f_X(x) \times \mathcal{K} \times \int |K(u)| du < \infty$. It follows that

$$\mathbb{P} \left\{ \frac{1}{n} \sum_{j=1}^{n} \left| T_{nxz} - \mathbb{E} T_{nxz} \right| > \tau_{nxz} \right\} \leq 2 \exp \left( -\frac{\lambda_a n h n^{-2r}}{2C + \frac{2}{3} \mathcal{K} \lambda_a n^{-r}} \right).$$

For sufficiently large $n$, we have $\frac{2}{3} \mathcal{K} \lambda_a n^{-r} \leq 1$. Therefore, for sufficiently large $n,$

$$\mathbb{P} \left\{ \frac{1}{n} \sum_{j=1}^{n} \left| T_{nxz} - \mathbb{E} T_{nxz} \right| > \tau_{nxz} \right\} \leq 2 \exp \left( -\frac{n^{2-2r}}{2C + 1} \right) = o(n^{-k})$$

where the inequality comes from Assumption G. Note that the upper bound does not depend on $x$ or $z.$ Therefore,

$$\sup_{x, z} \mathbb{P} \left[ |a_n(x, z) - a(x, z)| > \lambda_a \times n^{-r} \right] = o(n^{-k}).$$

Q.E.D.
REFERENCES

ABADIE, A., J. ANGRIST, AND G. IMBENS (2002): “Instrumental variables estimates of the effect of subsidized training on the quantiles of trainee earnings,” *Econometrica*, 70(1), 91–117.

ANGRIST, J. D., AND W. N. EVANS (1998): “Children and Their Parents’ Labor Supply: Evidence from Exogenous Variation in Family Size,” *American Economic Review*, pp. 450–477.

ANGRIST, J. D., AND A. B. KRUEGER (1991): “Does Compulsory School Attendance Affect Schooling and Earnings?,” *The Quarterly Journal of Economics*, 106(4), 979–1014.

BARRETT, G. F., AND S. G. DONALD (2003): “Consistent Tests for Stochastic Dominance,” *Econometrica*, 71, 71–104.

CHERNOZHUKOV, V., AND C. HANSEN (2005): “An IV Model of Quantile Treatment Effects,” *Econometrica*, 73(1), 245–261.

CHESHER, A. (2003): “Identification in Nonseparable Models,” *Econometrica*, 71(5), 1405–1441.

——— (2005): “Nonparametric Identification under Discrete Variation,” *Econometrica*, 73(5), 1525–1550.

DAVID, A. P. (1979): “Conditional independence in statistical theory,” *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 1–31.

FRÖLICH, M., AND B. MELLY (2013): “Unconditional quantile treatment effects under endogeneity,” *Journal of Business & Economic Statistics*, 31(3), 346–357.

HECKMAN, J. J. (1979): “Sample selection bias as a specification error,” *Econometrica: Journal of the econometric society*, pp. 153–161.

HECKMAN, J. J., D. SCHMIERER, AND S. URZUA (2010): “Testing the Correlated Random Coefficient Model,” *Journal of econometrics*, 158(2), 177–203.
HECKMAN, J. J., J. SMITH, AND N. CLEMENTS (1997): “Making the most out of programme evaluations and social experiments: Accounting for heterogeneity in programme impacts,” *The Review of Economic Studies*, 64(4), 487–535.

HECKMAN, J. J., S. URZUA, AND E. VYTLACIL (2006): “Understanding instrumental variables in models with essential heterogeneity,” *The Review of Economics and Statistics*, 88(3), 389–432.

HECKMAN, J. J., AND E. VYTLACIL (2001): “Policy-relevant treatment effects,” *The American Economic Review*, 91(2), 107–111.

——— (2005): “Structural equations, treatment effects, and econometric policy evaluation1,” *Econometrica*, 73(3), 669–738.

HECKMAN, J. J., AND E. J. VYTLACIL (1999): “Local instrumental variables and latent variable models for identifying and bounding treatment effects,” *Proceedings of the national Academy of Sciences*, 96(8), 4730–4734.

——— (2007): “Econometric evaluation of social programs, part II: Using the marginal treatment effect to organize alternative econometric estimators to evaluate social programs, and to forecast their effects in new environments,” *Handbook of econometrics*, 6, 4875–5143.

HODERLEIN, S., AND E. MAMMEN (2009): “On the role of the propensity score in efficient semiparametric estimation of average treatment effects,” *Econometrics Journal*, pp. 1–25.

HSU, Y.-C. (2016): “Multiplier Bootstrap for Empirical Processes,” *Working Paper*.

IMBENS, G. W., AND J. D. ANGRIST (1994): “Identification and estimation of local average treatment effects,” *Econometrica*, 62(2), 467–475.

IMBENS, G. W., AND W. K. NEWEY (2009): “Identification and estimation of triangular simultaneous equations models without additivity,” *Econometrica*, 77(5), 1481–1512.

IMBENS, G. W., AND D. B. RUBIN (1997): “Estimating Distributions for Outcome Compliers Models in Instrumental Variables,” *Review of Economic Studies*, 64(4), 555–574.
IMBENS, G. W., AND J. M. WOOLDRIDGE (2009): “Recent developments in the econometrics of program evaluation,” Journal of economic literature, 47(1), 5–86.

KIM, J., AND D. POLLARD (1990): “Cube root asymptotics,” The Annals of Statistics, pp. 191–219.

LU, X., AND H. WHITE (2014): “Testing for Separability in Structural Equations,” Journal of Econometrics, 182(1), 14–26.

MATZKIN, R. L. (2003): “Nonparametric estimation of nonadditive random functions,” Econometrica, 71(5), 1339–1375.

NOLAN, D., AND D. POLLARD (1988): “Functional limit theorems for U-processes,” The Annals of Probability, pp. 1291–1298.

POWELL, J. L., J. H. STOCK, AND T. M. STOKER (1989): “Semiparametric estimation of index coefficients,” Econometrica: Journal of the Econometric Society, pp. 1403–1430.

ROSENZWEIG, M. R., AND K. I. WOLPIN (1980): “Testing the quantity-quality fertility model: The use of twins as a natural experiment,” Econometrica: journal of the Econometric Society, pp. 227–240.

STINCHCOMBE, M. B., AND H. WHITE (1998): “Consistent specification testing with nuisance parameters present only under the alternative,” Econometric theory, 14(03), 295–325.

SU, L., Y. TU, AND A. ULLAH (2014): “Testing Additive Separability of Error Term in Nonparametric Structural Models,” Econometric Reviews, (October 2014).

VAN DER VAART, A. W., AND J. A. WELLNER (1996): “Weak Convergence,” in Weak Convergence and Empirical Processes, pp. 16–28. Springer.

——— (2007): “Empirical processes indexed by estimated functions,” Lecture Notes-Monograph Series, pp. 234–252.

VERE, J. P. (2011): “Fertility and parents labour supply: new evidence from US census data Winner of the OEP prize for best paper on Women and Work,” Oxford Economic Papers, 63(2), 211–231.
Vuong, Q., and H. Xu (2017): “Counterfactual mapping and individual treatment effects in nonseparable models with discrete endogeneity,” *Quantitative Economics*, forthcoming.

Vytlacil, E. (2002): “Independence, Monotonicity, and Latent Index Models: An Equivalence Result,” *Econometrica*, 70(1), 331—341.
### Table 1: Empirical level for DGP 1

| N  | $p \setminus \rho$ | $\alpha$ | 0.01 | 0.05 | 0.10 |
|----|-----------------|--------|------|------|------|
|    |                 | 0.5    | 0.7  | 0.9  | 0.5  | 0.7  | 0.9  | 0.5  | 0.7  | 0.9  |
| 500| 0.3             | 0.0010 | 0.0010| 0.0010| 0.0190| 0.0210| 0.0210| 0.0590| 0.0640| 0.0640|
|    | 0.5             | 0.0050 | 0.0040| 0.0020| 0.0270| 0.0290| 0.0230| 0.0570| 0.0610| 0.0590|
|    | 0.7             | 0.0010 | 0.0020| 0.0010| 0.0200| 0.0220| 0.0190| 0.0560| 0.0590| 0.0530|
| 1000| 0.3          | 0.0040 | 0.0080| 0.0060| 0.0450| 0.0390| 0.0360| 0.0810| 0.0730| 0.0710|
|    | 0.5           | 0.0030 | 0.0030| 0.0050| 0.0370| 0.0360| 0.0290| 0.0740| 0.0700| 0.0750|
|    | 0.7           | 0.0050 | 0.0040| 0.0020| 0.0430| 0.0340| 0.0320| 0.0780| 0.0740| 0.0750|
| 2000| 0.3         | 0.0050 | 0.0030| 0.0030| 0.0390| 0.0340| 0.0360| 0.0970| 0.0970| 0.0990|
|    | 0.5         | 0.0060 | 0.0070| 0.0080| 0.0480| 0.0470| 0.0470| 0.0930| 0.0950| 0.0930|
|    | 0.7         | 0.0080 | 0.0100| 0.0100| 0.0450| 0.0460| 0.0430| 0.0810| 0.0800| 0.0780|

### Table 2: Empirical power for DGP 2: $\gamma = 0.1$

| N  | $p \setminus \rho$ | $\alpha$ | 0.01 | 0.05 | 0.10 |
|----|-----------------|--------|------|------|------|
|    |                 | 0.5    | 0.7  | 0.9  | 0.5  | 0.7  | 0.9  | 0.5  | 0.7  | 0.9  |
| 500| 0.3             | 0.0040 | 0.0030| 0.0070| 0.0360| 0.0330| 0.0300| 0.0880| 0.0820| 0.0860|
|    | 0.5             | 0.0100 | 0.0090| 0.0070| 0.0380| 0.0450| 0.0370| 0.1050| 0.0950| 0.0900|
|    | 0.7             | 0.0060 | 0.0050| 0.0020| 0.0300| 0.0380| 0.0330| 0.0900| 0.0940| 0.0800|
| 1000| 0.3          | 0.0170 | 0.0170| 0.0160| 0.0930| 0.0900| 0.0840| 0.1670| 0.1580| 0.1640|
|    | 0.5           | 0.0210 | 0.0210| 0.0210| 0.1050| 0.0950| 0.0950| 0.1650| 0.1590| 0.1410|
|    | 0.7           | 0.0160 | 0.0130| 0.0120| 0.0800| 0.0780| 0.0730| 0.1650| 0.1590| 0.1410|
| 2000| 0.3         | 0.0630 | 0.0650| 0.0630| 0.2260| 0.1900| 0.1940| 0.3500| 0.3160| 0.3220|
|    | 0.5         | 0.0820 | 0.0690| 0.0670| 0.2500| 0.2330| 0.2340| 0.3910| 0.3710| 0.3750|
|    | 0.7         | 0.0440 | 0.0450| 0.0400| 0.1830| 0.1730| 0.1740| 0.3190| 0.3020| 0.2970|
Table 3: Empirical power for DGP 2: $\gamma = 0.3$

| $N$  | $p \backslash \rho$ | 0.01 | 0.05 | 0.1 |
|------|-----------------|------|------|-----|
|      | 0.5             | 0.7  | 0.9  |     |
| 500  | 0.3             | 0.0440 | 0.0370 | 0.0350 | 0.2070 | 0.1750 | 0.1820 | 0.3340 | 0.3120 | 0.3130 |
| 1000 | 0.5             | 0.0520 | 0.0590 | 0.0550 | 0.2380 | 0.2220 | 0.2340 | 0.4190 | 0.3870 | 0.3900 |
| 2000 | 0.7             | 0.0270 | 0.0230 | 0.0200 | 0.1630 | 0.1470 | 0.1390 | 0.3300 | 0.2980 | 0.2950 |

Table 4: Empirical power for DGP 2: $\gamma = 0.5$

| $N$  | $p \backslash \rho$ | 0.01 | 0.05 | 0.1 |
|------|-----------------|------|------|-----|
|      | 0.5             | 0.7  | 0.9  |     |
| 500  | 0.3             | 0.1800 | 0.1310 | 0.1420 | 0.5120 | 0.4550 | 0.4640 | 0.7050 | 0.6540 | 0.6630 |
| 1000 | 0.5             | 0.2350 | 0.1870 | 0.2150 | 0.6840 | 0.6200 | 0.6180 | 0.8540 | 0.8170 | 0.8150 |
| 2000 | 0.7             | 0.1190 | 0.0860 | 0.0920 | 0.4630 | 0.3950 | 0.4390 | 0.6920 | 0.6440 | 0.6850 |

Table 5: Empirical size for DGP 3: $(p, \rho) = (0.5, 0.7)$

| $N \backslash \alpha$ | 0.01 | 0.05 | 0.1 |
|----------------------|------|------|-----|
| 1000                 | 0.0040 | 0.0300 | 0.0480 |
| 2000                 | 0.0080 | 0.0500 | 0.0880 |
| 4000                 | 0.0080 | 0.0360 | 0.0780 |
Table 6: Empirical power for DGP 4: \((p, \rho) = (0.5, 0.7)\)

| \(N \setminus \alpha\) | 0.01  | 0.05  | 0.1   |
|-------------------------|-------|-------|-------|
| \(\gamma = 0.5\)       |       |       |       |
| 1000                    | 0.0040| 0.0280| 0.0520|
| 2000                    | 0.0080| 0.0580| 0.1100|
| 4000                    | 0.1060| 0.3350| 0.5180|
| \(\gamma = 0.7\)       |       |       |       |
| 1000                    | 0.0040| 0.0340| 0.0620|
| 2000                    | 0.0044| 0.1880| 0.3260|
| 4000                    | 0.7240| 0.9400| 0.9840|

Table 7: Descriptive Statistics for the National JTPA Study

|                   | All   | \(Z = 1\) (eligible) | \(Z = 0\) (not eligible) |
|-------------------|-------|-----------------------|---------------------------|
| **Men**           |       |                       |                           |
| Number of observations | 5,102 | 3,399                 | 1,703                     |
| Training \((D = 1)\) | 41.87%| 62.28%                | 1.12%                     |
| High school or GED | 69.32%| 69.26%                | 69.43%                    |
| Married            | 35.26%| 36.01%                | 33.75%                    |
| Minorities         | 38.38%| 38.69%                | 37.76%                    |
| Work less than 13 weeks in the past year | 40.02%| 40.28%                | 39.05%                    |
| 30 months earnings | 19,147| 19,520                | 18,404                    |
| **Women**         |       |                       |                           |
| Number of observations | 6,102 | 4,088                 | 2,014                     |
| Training \((D = 1)\) | 44.61%| 65.73%                | 1.74%                     |
| High school or GED | 72.06%| 72.85%                | 70.45%                    |
| Married            | 21.93%| 22.48%                | 20.82%                    |
| Minorities         | 40.41%| 40.58%                | 51.86%                    |
| Work less than 13 weeks in the past year | 51.79%| 51.75%                | 51.86%                    |
| 30 months earnings | 13,029| 13,439                | 12,197                    |

Note: Means are reported in this table for the National JTPA study 30-month earnings sample.
Table 8: Descriptive Statistics for the 1999 and 2000 Censuses

|                           | 1990          | 1990 (twin birth) | 1990 (no twin birth) | 2000          | 2000 (twin birth) | 2000 (no twin birth) |
|---------------------------|---------------|-------------------|----------------------|---------------|-------------------|----------------------|
| Observations              | 602,767       | 6,524             | 596,243              | 573,437       | 8,569             | 564,868              |
| Number of children        | 1.9276        | 2.5318            | 1.9209               | 1.8833        | 2.5196            | 1.8734               |
| At least two children \(D = 1\) | 0.6500        | 1.0000            | 0.6461               | 0.6163        | 1.0000            | 0.6104               |

**Mother**
- Age in years: 29.7894, 29.9530, 29.7867, 30.0562, 30.3943, 30.0510
- Years of education: 12.9196, 12.9623, 12.9191, 13.1131, 13.2615, 13.1108
- Black: 0.0637, 0.0757, 0.0636, 0.0724, 0.0816, 0.07228
- Asian: 0.0326, 0.0321, 0.0326, 0.0447, 0.0335, 0.0448
- Other Races: 0.0537, 0.0592, 0.0536, 0.0912, 0.0806, 0.0914
- Currently at work: 0.5781, 0.5444, 0.5785, 0.5629, 0.5132, 0.5637
- Usual hours per work: 24.5660, 23.3537, 24.5795, 25.1400, 23.0491, 25.1723
- Wage or salary income last year: 8942, 8593, 8946, 14200, 13757, 14206

**Father**
- Age in years: 32.5358, 32.7534, 32.5333, 32.9291, 33.3102, 32.9232
- Years of education: 13.0436, 13.0748, 13.0432, 13.0331, 13.1806, 13.0308
- Black: 0.0671, 0.0796, 0.0670, 0.0800, 0.0945, 0.0798
- Asian: 0.0291, 0.0263, 0.0292, 0.0402, 0.0318, 0.0403
- Other Races: 0.0488, 0.0529, 0.0488, 0.0919, 0.0802, 0.0921
- Currently at work: 0.8973, 0.8922, 0.8974, 0.8512, 0.8584, 0.8511
- Usual hours per work: 42.7636, 42.7704, 42.7635, 43.8805, 43.8789, 43.8805
- Wage or salary income last year: 27020, 28039, 27010, 38041, 41584, 37987

**Parents**
- Wages or salary income last year: 35,963, 36,632, 35,956, 52,241, 55,342, 52,193

Note: Data from the 1% and 5% PUMS in 1990 and 2000. Own calculations using the PUMS sample weights. The sample consists of married mother between 21 and 35 years of age with at least one child.