Real AlphaBeta-Geometries

Abstract
By a real $\alpha\beta$-geometry we mean a four-dimensional manifold $M$ equipped with a neutral metric $h$ such that $(M, h)$ admits both an integrable distribution of $\alpha$-planes and an integrable distribution of $\beta$-planes. We obtain a local characterization of the metric when at least one of the distributions is parallel (i.e., is a Walker geometry) and the three-dimensional distribution spanned by the $\alpha$- and $\beta$-distributions is integrable. The case when both distributions are parallel, which has been called two-sided Walker geometry, is obtained as a special case. We also consider real $\alpha\beta$-geometries for which the corresponding spinors are both multiple Weyl principal spinors.

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1. Introduction

By a neutral geometry \((M,h)\) we shall mean a 2n-dimensional manifold \(M\) equipped with a metric \(h\) of neutral signature. A neutral geometry admitting a parallel distribution of totally null \(n\)-planes is called a Walker geometry, see Walker (1950). In this paper we restrict attention to the case of four dimensions. In Law & Matsushita (2008a), the authors provided a spinor approach to the study of four-dimensional Walker geometry and we refer to that paper for a full account of notation, conventions, and details. In particular, a Walker geometry has a canonical orientation, Law & Matsushita (2008a) §1, with respect to which the parallel distribution is a distribution of self-dual (SD) two-planes, i.e., a distribution of \(\alpha\)-planes, see Law & Matsushita (2008a) §2, which we subsequently call an \(\alpha\)-distribution. Locally, and globally when \((M,g)\) is \(\text{SO}^\pm\)-orientable, the \(\alpha\)-distribution is equivalent to a projective spinor field \([\pi^\alpha]\): the \(\alpha\)-plane at \(m \in M\) is \(\{ \mu^A \nu^A : \mu^A \in S_m \}\), where \(S_m\) is the space of unprimed spinors at \(m\) (and \(S'_m\) that of primed spinors), and \(\nu^A\) belongs to the projective class \([\pi^\alpha]\). We therefore denote a Walker geometry by \((M,g,[\pi^\alpha])\) and the \(\alpha\)-distribution by \(Z_{[\pi]}\). For any (local) projective spinor field \([\pi^A]\), a (local) spinor field \(\nu^A\) whose (pointwise) projectivization equals \([\pi^A]\) is called a local scaled representative (LSR) of \([\pi^A]\).

By a two-sided Walker geometry (to be distinguished from a double Walker geometry, see Law & Matsushita (2008a) §4) is meant a Walker geometry \((M,g,[\pi^A])\) which also admits a parallel distribution of \(\beta\) (i.e., anti-self-dual)-planes. This \(\beta\)-distribution is determined, at least locally, by a projective spinor field \([\lambda_A]\), and so denoted \(W_{[\lambda]}\). Recently, Chudecki and Przanowski (2008b) exploited the hyperheavenly formalism, for which see e.g., Finley and Plebański (1976) and Boyer et al. (1980), to provide a local characterization of two-sided Walker geometry. In this paper, we provide a simple and transparent derivation of this characterization and, in the process, generalize the result by formulating it in a broader context. See Law (2008) for the formalism of null tetrads and spin frames in the context of neutral geometry; in particular, spin coefficients and their application to describing null geometry.

2. Real AlphaBeta-Geometries

A (four-dimensional) neutral geometry \((M,h)\) admitting an integrable \(\alpha\)-distribution will be called a real \(\alpha\)-geometry and denoted \((M,h,[\pi^A])\), see Law & Matsushita (2008b). The condition for integrability of the \(\alpha\)-distribution in terms of \([\pi^A]\) is

\[
\pi_A^B \pi^B \nabla_{BB'} \pi^A' = 0, \tag{2.1}
\]

where \(\pi^A'\) is any LSR of \([\pi^A]\). By Law (2008) (6.2.9), any solution \([\pi^A]\) of (2.1) is a Weyl principal spinor (WPS), i.e., a principal spinor (PS) of the Weyl curvature spinor \(\Psi_{A'B'C'D'}\) (see Law 2006). A Walker geometry \((M,g,[\pi^A])\) is a real \(\alpha\)-geometry for which \([\pi^A]\) satisfies, in place of (2.1), the stronger condition

\[
\pi_A^B \nabla_{BB'} \pi^A' = 0. \tag{2.2}
\]

Equation (2.1) may admit nontrivial complex solutions, i.e., solutions \(\pi^A' \in \text{CS}'\) which are not just complex scalar multiples of elements of \(S'\) (i.e., \(\pi^A' \pi^\alpha = 0\), where \(\pi^\alpha\) denotes the complex conjugate of \(\pi^A\)). Such solutions induce an interesting, but different, geometry on \((M,h)\) which we call complex \(\alpha\)-geometries and discuss elsewhere. In this paper, we restrict attention to real \(\alpha\)-geometries and may, for convenience, omit the qualifier ‘real’.

2.1 Definitions

An \(\alpha\beta\)-geometry \((M,h,[\pi^A],[\lambda^A])\) is a neutral geometry admitting both an integrable \(\alpha\)-distribution \(Z_{[\pi]}\) and an integrable \(\beta\)-distribution \(W_{[\lambda]}\). By \(\lambda^A\) we denote any LSR of \([\lambda^A]\), and any such LSR satisfies

\[
\lambda_A \lambda_B \nabla_{BB'} \lambda^A = 0. \tag{2.3}
\]

Defining the null distributions \(\mathcal{D} := \langle \lambda^A \pi^A \rangle_R\) and \(\mathcal{H} := \mathcal{D}^\perp\), then \((M,h,[\pi^A],[\lambda^A])\) contains the nested null distributions (of types I, II, and III, respectively, in the terminology of Law 2008)

\[
\mathcal{D} \leq Z_{[\pi]} \leq \mathcal{H} \quad \mathcal{D} \leq W_{[\lambda]} \leq \mathcal{H}. \tag{2.4}
\]
Note that $\mathcal{H} = (Z_{[\pi]}, W_{[\lambda]})^R$. Equation (2.1) is equivalent to each of:

$$ S_b := \pi_A \nabla_b \pi^{A'} = \omega_B \pi^{B'}; \quad \pi^{B'} \nabla_{B'} \pi^{A'} =: \eta_B \pi^{A'}; \quad (2.5a) $$

while (2.3) is equivalent to each of

$$ \hat{S}_b := \lambda_A \nabla_b \lambda^A = \lambda_B \kappa_{B'}; \quad \lambda^B \nabla_{B'} \lambda^A =: \zeta_{B'} \lambda^A; \quad (2.5b) $$

for some spinors $\omega_A$, $\eta_A$, $\kappa_A$, and $\zeta_{A'}$. The significance of the spinors $\omega_A$ and $\eta_A$ was studied in Law (2008), §6.2.

A Walker geometry $(M, g, [\pi^{A'}])$ together with an integrable $\beta$-distribution $W_{[\lambda]}$, i.e., an $\alpha\beta$-geometry for which $[\pi^{A'}]$ satisfies (2.2), will be called a sesquiWalker $\alpha\beta$-geometry (here the order of $\alpha\beta$ is meant to indicate which distribution is Walker and which only integrable). A sesquiWalker $\alpha\beta$-geometry for which $[\lambda^A]$ satisfies the analogue of (2.2):

$$ \lambda_A \nabla_{B'} \lambda^A = 0, \quad (2.6) $$

is of course a two-sided Walker geometry.

### 2.2 Lemma

For an $\alpha\beta$-geometry $(M, h, [\pi^{A'}], [\lambda^A])$, the distribution $\mathcal{D}$ is auto-parallel in the sense of Law (2008) (6.1.7). In a two-sided Walker geometry, both $\mathcal{D}$ and $\mathcal{H}$ are parallel.

**Proof.** By (2.5), $\lambda^B \pi^{B'} \nabla_{B'} \lambda^A \pi^{A'} \propto \lambda^A \pi^{A'}$, i.e., $\mathcal{D}$ is auto-parallel. In two-sided Walker geometry, both $Z_{[\pi]}$ and $W_{[\lambda]}$ are parallel. As $\mathcal{D} = Z_{[\pi]} \cap W_{[\lambda]}$, it is therefore parallel, as is $\mathcal{H} = \mathcal{D} \perp = (Z_{[\pi]}, W_{[\lambda]})^R$.

For an $\alpha\beta$-geometry, $Z_{[\pi]}$ and $W_{[\lambda]}$ are each integrable by assumption, and $\mathcal{D}$ is integrable being one dimensional. To check the integrability of $\mathcal{H}$, it suffices to check whether $[\lambda^B \nu^{B'}, \mu^B \pi^{B'}] \in \mathcal{H}$, for arbitrary $\nu^{B'}$ and $\mu^B$, i.e., whether that expression is orthogonal to $\mathcal{D}$. As

$$ \lambda_A \pi^{A'} [\lambda^B \nu^{B'}, \mu^B \pi^{B'}]^{AA'} = \lambda_A \pi^{A'} (\lambda^B \nu^{B'} \nabla_{B'} \mu^A - \mu^B \pi^{B'} \nabla_{B'} \lambda^A) $$

$$ = (\mu^B \lambda_D) (\nu^{D'} \pi_D) (\lambda^B \omega_B - \pi^{B'} \kappa_{B'}), $$

$\mathcal{H}$ is integrable iff

$$ \lambda^B \omega_B = \pi^{B'} \kappa_{B'}. \quad (2.7) $$

One does not, therefore, expect $\mathcal{H}$ to be integrable in a general $\alpha\beta$-geometry. In fact, in any (four-dimensional) neutral geometry, the condition for a null distribution $\mathcal{H}$ of type III, i.e., $\mathcal{H} \perp = \mathcal{D} := (\lambda^A \pi^{A'})^R$, to be integrable is

$$ \pi^{A'} \lambda_B \lambda^A \nabla_{AB'} \pi_{A'} = \lambda^A \pi^B \pi^{A'} \nabla_{BA'} \lambda_A, $$

see the proof of Law (2008) (6.3.2) where this condition was shown to be equivalent to: $\mathcal{D}$ is auto-parallel together with an equation involving spin coefficients. In the context of an $\alpha\beta$-geometry, $\mathcal{D}$ is automatically auto-parallel by 2.2, and the previous equation reduces, by (2.5), to (2.7).

### 2.3 Lemma

For a Walker geometry $(M, g, [\pi^{A'}])$ with a $\beta$-distribution $W_{[\lambda]}$, the single condition $\hat{S}_b = \lambda_A \nabla_b \lambda^A \propto \lambda_B \pi^{B'}$ is the necessary and sufficient condition for both $W_{[\lambda]}$ and $\mathcal{H}$ to be integrable.

For a sesquiWalker $\alpha\beta$-geometry $(M, g, [\pi^{A'}], [\lambda^A])$, the distribution $\mathcal{H}$ is integrable iff $\pi^{B'} \kappa_{B'} = 0$, i.e., $\kappa_{A'}$ is an LSR for $[\pi^{A'}]$. For a given LSR $\lambda^A$ of $[\lambda^A]$, there is then an LSR $\pi^{A'}$ of $[\pi^{A'}]$ such that $\hat{S}_b = \lambda_B \pi^{B'}$, and, for a given LSR $\pi^{A'}$ of $[\pi^{A'}]$, an LSR $\lambda^A$ of $[\lambda^A]$ such that $\hat{S}_b = \lambda_B \pi^{B'}$. Moreover, $\mathcal{H}$ is integrable iff auto-parallel.

We will call a (sesquiWalker) $\alpha\beta$-geometry for which $\mathcal{H}$ is integrable, an integrable (sesquiWalker) $\alpha\beta$-geometry. Clearly, two-sided Walker geometry is a special case of integrable sesquiWalker $\alpha\beta$-geometry.
Proof. The first assertion is clear from (2.5b) and (2.7). With $Z_{[\pi]}$ parallel, $\mathcal{H}$ is auto-parallel iff, for any spinor $\nu^A$, the covariant derivative of $\lambda^A \nu^A$, where $\lambda^A$ is any LSR of $[\lambda^A]$, along $\mathcal{H}$ lies in $\mathcal{H}$, i.e., is orthogonal to $\mathcal{D}$, i.e., $\lambda_A \pi_A \lambda^B \nabla_{B'} \lambda^A \nu^A = 0$ and $\lambda_A \pi_A \pi^B \nabla_{B'} \lambda^A \nu^A = 0$. The first equality is equivalent to (2.3) and the second to $\pi^B \kappa_{B'} = 0$. (Of course, any auto-parallel distribution is necessarily integrable, see Law 2008 (6.1.7).)

The interest in integrability of $\mathcal{H}$ is not just passing curiosity.

2.4 Construction
Beginning with an integrable $\alpha \beta$-geometry $(M, h, [\pi^A], [\lambda^A])$, choose Frobenius coordinates $(p, q, x, y)$ for the first of the nested distributions in (2.4), i.e., so that $\mathcal{D} = (\partial_p) \mathcal{R}$, $Z_{[\pi]} = (\partial_p, \partial_q) \mathcal{R}$, and $\mathcal{H} = (\partial_p, \partial_q, \partial_x, \partial_y) \mathcal{R}$. Since $y$ is constant on the integral manifolds of each of the distributions, $dy = \lambda_A \pi^A$ for some LSRs $\lambda_A$ of $[\lambda_A]$ and $\pi_A$ of $[\pi_A]$; as $x$ is constant on the integral surfaces (called $\alpha$-surfaces) of $Z_{[\pi]}$, then $dx = \mu_A \pi^A$, for some spinor $\mu_A$ satisfying $\lambda^A \mu_A \neq 0$. At this stage, one can proceed to construct new coordinates $(u, v, x, y)$ with respect to which the metric takes a coordinate form generalizing the Walker coordinate form, see Law & Matsushita (2008b) (3.32). The construction of these coordinates is essentially the first step in the hyperheavenly formalism. Such coordinates are Frobenius for $Z_{[\pi]}$ but will not, in general, respect the nesting of $\mathcal{D}$ within $Z_{[\pi]}$.

We therefore specialize the context to that of an integrable sesquiWalker $\alpha \beta$-geometry $(M, g, [\pi^A], [\lambda^A])$. The assumption that $\mathcal{H}$ is integrable is thus characterized by lemma 2.3. With $dy = \lambda_A \pi^A$, and $dx = \mu_A \pi^A$ as in the previous paragraph, one can apply the construction of Law & Matsushita (2008a), 2.3, to obtain Walker coordinates for $(M, g, [\pi^A])$, i.e., one obtains coordinates $(u, v, x, y)$ which are Frobenius coordinates for $Z_{[\pi]}$, yield Walker’s canonical coordinate form for the metric

$$(g_{ab}) = \begin{pmatrix} 0 & 1_2 \\ 1_2 & W \end{pmatrix}, \quad \text{where} \quad W = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \tag{2.8}$$

and $a$, $b$, and $c$ are arbitrary functions of the coordinates, and moreover

$$\partial_u = \mu^A \pi^A, \quad \partial_v = \lambda^A \pi^A, \quad dx = \mu_A \pi^A, \quad dy = \lambda_A \pi^A. \tag{2.9}$$

As in Law & Matsushita (2008a), 2.4, it proves convenient to specialize the choice of Walker coordinates to oriented Walker coordinates. First, by rescaling the LSRs as follows: $\pi^A \mapsto \gamma \pi^A$; $\lambda^A \mapsto \lambda^A / \gamma$; $\mu^A \mapsto \mu^A / \gamma$, one preserves (2.9) but, by appropriate choice of $\gamma$, can ensure $\lambda^A \mu_A = \pm 1$. The LSRs are now fixed up to a common sign. If $\lambda^A \mu_A = 1$, write $\mu_A$ as $\alpha_A$, so that (2.9) becomes

$$\partial_u = \alpha^A \pi^A, \quad \partial_v = \lambda^A \pi^A, \quad dx = \alpha_A \pi^A, \quad dy = \lambda_A \pi^A. \tag{2.10a}$$

In this case, $(v, u, x, y)$ are Frobenius coordinates respecting the nested distributions (note, in particular, that the coordinate tangent vectors of $x$ with respect to the two coordinate systems $(p, q, x, y)$ and $(u, v, x, y)$ differ by an element of $Z_{[\pi]}$; whence, as that with respect to $(p, q, x, y)$ lies in $\mathcal{H}$, so does that with respect to $(u, v, x, y)$) and $(u, v, x, y)$ are oriented Walker coordinates (i.e., satisfying Law & Matsushita 2008a, (2.8)).

If, however, $\lambda^A \mu_A = -1$, then, as in Law & Matsushita (2008a), 2.4, to achieve oriented Walker coordinates one can resort to Law & Matsushita (2008a), (A1.7), interchanging $u$ with $v$ and $x$ with $y$. After relabelling the coordinates and writing $\mu_A$ as $\beta_A$, one obtains oriented Walker coordinates $(u, v, x, y)$, with (2.9) now taking the form

$$\partial_u = \lambda^A \pi^A, \quad \partial_v = \beta^A \pi^A, \quad dx = \lambda_A \pi^A, \quad dy = \beta_A \pi^A, \tag{2.10b}$$

and where now $(u, v, y, x)$ are Frobenius coordinates respecting the nested distributions.

Alternatively, when $\lambda^A \mu_A = -1$, and noting that the Walker Lagrangian in Law & Matsushita (2008a), A1.2, is also invariant under $x \mapsto -x$, $u \mapsto -u$ and $c \mapsto -c$, one can replace $x$ by $-x$ and $u$ by $-u$ (which

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in effect replaces $\mu_A$ by $-\mu_A$). After relabelling the coordinates one obtains oriented Walker coordinates $(u, v, x, y)$, with (2.9) now taking the same form as (2.10a) but with $\alpha_A = -\mu_A$, and where $(v, u, x, y)$ are again Frobenius coordinates for the nested distributions.

Thus, by an appropriate tactic, one can always find oriented Walker coordinates $(u, v, x, y)$ satisfying (2.10a), where $\lambda^A \alpha_A = 1$, and with $(v, u, x, y)$ Frobenius coordinates respecting the nested distributions. From Law & Matsushita (2008a), (2.11), the Walker spin frames associated to these oriented Walker coordinates are $\{\alpha^A, \lambda^A\}$ and $\{\pi^A, \xi^A\}$, i.e., $\beta^A = \lambda^A$ and in terms of the associated Walker null tetrad, one has

\[
D = (\partial_v)^R = (\ell^a)^R \\
W_\lambda = (\ell^a, n^a)^R = (\partial_v - \frac{\alpha}{2} \partial_u - \frac{\beta}{2} \partial_x)^R
\]

For the form (2.10b), one has instead $\lambda^A = \alpha^A$ in the Walker spin frames, whence

\[
D = (\partial_u)^R = (\ell^a)^R \\
W_\lambda = (\ell^a, \tilde{m}^a)^R = (\partial_v - \frac{\alpha}{2} \partial_u - \frac{\beta}{2} \partial_x)^R
\]

Note that it is the assumption of integrability of $\mathcal{H}$ which allows one to write $dy = \lambda_A \pi_A$ and ultimately achieve the oriented Walker coordinates satisfying (2.10a) (or (2.10b)). Without that assumption, all one can do is construct oriented Walker coordinates without any relationship to $\mathcal{D}$ or $\mathcal{H}$ (the integrability of $\mathcal{D}$ is automatic and does not facilitate matters).

### 2.5 Proposition

Let $(M, g, [\pi^A], [\lambda^A])$ be an integrable sesquiWalker $\alpha\beta$-geometry. Then, for oriented Walker coordinates satisfying the form (2.10a), the metric components in (2.8) satisfy $a_v = 0$, i.e., the component $g_{22}$ is constant along the integral curves of $\mathcal{D}$. Note that this coordinate condition is in fact geometric in nature in that: $\mathcal{D}$ is determined by the sesquiWalker $\alpha\beta$-geometry: $\mathcal{H} = (Z_\tau, \partial_\tau)^R$ and $a = g(\partial_x, \partial_z)$.

For the form (2.10b), one obtains instead that $b = g(\partial_y, \partial_z)$ is constant along the integral curves of $\mathcal{D}$, i.e., $b_y = 0$, which has an analogous geometric interpretation.

If $(M, g, [\pi^A], [\lambda^A])$ is a 2-sided Walker geometry, then $\mathcal{H}$ is integrable and for oriented Walker coordinates satisfying the form (2.10a), $a_v = c_v = 0$, i.e., the metric components $a$ and $c$ are constant along integral curves of $\mathcal{D}$. For oriented Walker coordinates satisfying the form (2.10b), $a_v = c_v = 0$, i.e., the metric components $b$ and $c$ are constant along the integral curves of $\mathcal{D}$. Chudecki & Przanowski (2008b), §5, gave one version of this result for 2-sided Walker geometry.

**Proof.** For any spinor $\kappa^A$ and null tetrad,

\[
\nabla_b \kappa^A = n_b D\kappa^A + \ell_b D' \kappa^A - \tilde{m}_b \delta \kappa^A - m_b \Delta \kappa^A.
\]

If $(M, g, [\pi^A], [\lambda^A])$ is an integrable sesquiWalker $\alpha\beta$-geometry, then $[\lambda^A]$ satisfies (2.3). As $\mathcal{H}$ is integrable, one can exploit oriented Walker coordinates satisfying (2.10). For form (2.10a), $\lambda_A = \beta_A$ in the Walker spin frame, so, exploiting (2.12),

\[
0 = \beta_A \beta^B \nabla_b \beta^A = \beta^B \ell_b \beta^A D' \beta^A - \delta^B m_b \lambda_A = \pi_B (\beta_A D' \beta^A) - \xi_B (\beta_A \Delta \beta^A).
\]

By Law (2008) (5.8), $\beta_A D' \beta^A = -(a_v / 2)$ while $\beta_A \Delta \beta^A = 0$, which proves the relevant assertion. For the form (2.10b), $\lambda^A = \alpha^A$, and the analogous computation yields

\[
0 = \alpha_A \alpha^B \nabla_b \alpha^A = -\xi_B \alpha_A D \alpha^A + \pi_B \alpha_A \delta \alpha^A = -\frac{b_y}{2} \pi_B.
\]

In passing, note that, by 2.3, integrability of $\mathcal{H}$ can be stated as $\lambda_A \pi^B \nabla_b \kappa^A = 0$. Exploiting the form (2.10a) with $\lambda^A = \beta^A$, one finds from (2.12), $\beta_A \pi^B \nabla_b \beta^A = \beta_A \pi^B n_b D \beta^A - \beta_A \pi^B m_b \Delta \beta^A = 0$, since the
Walker spin frames are parallel with respect to $D$ and $\Delta$, i.e., on $\alpha$-surfaces of $Z_{[\pi]}$ (Law 2008 (5.8)). This computation confirms that, apart from providing the construction of oriented Walker coordinates satisfying (2.10), the assumption of integrability of $\mathcal{H}$ involves no further conditions on the metric components.

Now suppose $(M, g, [\pi^A], [\lambda^A])$ is 2-sided Walker; in particular $[\pi^A]$ satisfies (2.2) and $[\lambda^A]$. Hence, in (2.5), $\omega^A$ and $\kappa^A$ are zero and (2.7) is trivially satisfied, i.e., $\mathcal{H}$ is integrable. Thus, one can construct oriented Walker coordinates satisfying (2.10). For form (2.10a), with $\lambda^A = \beta^A$, one obtains, using (2.12) and formulae for the action of $D, D', \delta$ and $\Delta$ on the elements of the Walker spin frames given in Law (2008) (5.8),

$$0 = \beta_A \nabla_b \beta^A = \beta_A \left[ \ell_b \left( \frac{a_v}{2} \alpha^A + \frac{c_v - a_v}{4} \beta^A \right) + \hat{m}_b \left( -\frac{c_v}{2} \alpha^A + \frac{c_v - b_v}{4} \beta^A \right) \right] = -\frac{a_v}{2} \ell_b + \frac{c_v}{2} \hat{m}_b,$$

whence $a_v = c_v = 0$ as claimed. The analogous computation for form (2.10b) with $\lambda^A = \alpha^A$ is

$$0 = \alpha_A \nabla_b \alpha^A = \alpha_A \left[ \ell_b \left( \frac{a_v - c_v}{4} \alpha^A + \frac{c_v}{2} \beta^A \right) + \hat{m}_b \left( \frac{b_v - c_v}{4} \alpha^A - \frac{b_v}{2} \beta^A \right) \right] = \frac{c_v}{2} \ell_b + \frac{b_v}{2} \hat{m}_b.$$

For an $\alpha\beta$-geometry $(M, h, [\pi^A], [\lambda^A])$, by Law (2008) (6.2.9), $[\pi^A]$ is a WPS of $\Psi_{ABCD}$ and $[\lambda^A]$ a WPS of $\Psi_{ABCD}$. For a sesquiWalker $\alpha\beta$-geometry, by Law & Matsushita (2008a), 2.5, $[\pi^A]$ is in fact a multiple WPS and also a PS of $\Phi_{ABCD}$, i.e., a Ricci Principal Spinor (RPS). Hence, for a 2-sided Walker geometry, $[\lambda^A]$ is also a multiple WPS and a RPS. These facts are readily seen to be consistent with Proposition 2.5 by consulting the expressions for the components of the curvature spinors with respect to the Walker spin frames given in Law & Matsushita (2008a) §2. For example, for form (2.10a), $a_v = c_v = 0$ substituted into Law & Matsushita (2008a) (2.25) yields $\Psi_3 = \Psi_4 = 0$, i.e., $[\beta^A]$ is indeed a multiple WPS. Substituting $a_v = c_v = 0$ into Law & Matsushita (2008a) A1.8, shows that $\nu = 0$, and into A1.7 that $\zeta = 0$ (these quantities are defined at those locations in Law & Matsushita 2008a), whence, in Law & Matsushita (2008a) (2.32–33), one sees that $A_{AB} \beta^A \beta^B = B_{AB} \beta^A \beta^B = 0$, hence $\Phi_{ABCD} \beta^A \beta^B = 0$. In an integrable sesquiWalker $\alpha\beta$-geometry, $a_v = 0$ yields only $\Psi_4 = 0$ in Law & Matsushita (2008a) (2.25), as expected. For form (2.10b) in the two-sided Walker case, one obtains instead that $\Psi_0 = \Psi_1 = 0$ from Law & Matsushita (2008a) (2.25) (i.e., $[\alpha^A]$ is a multiple WPS) and $\mu = 0$ from Law & Matsushita (2008a) A1.8 and $\nu = 0$ from Law & Matsushita (2008a) A1.7, which entail in Law & Matsushita (2008a) (2.32–33) that $B_{AB} \alpha^A \alpha^B = A_{AB} \alpha^A \alpha^B = 0$.

The covariant derivatives of the Walker null tetrad and spin frames in a Walker geometry were given in Law (2008) (5.5) and (5.8) respectively. For an integrable sesquiWalker $\alpha\beta$-geometry $(M, h, [\pi^A], [\lambda^A])$, using coordinates of either form of (2.10), one readily computes from (2.12) and Law (2008), (5.8), that

$$\lambda^B \nabla_{BB'} \lambda^A \propto \pi_{BB'} \lambda^A; \quad \text{whence} \quad \lambda^B \nabla_{BB'} \lambda^A \propto \pi_{BB'} \lambda^A \lambda^A; \quad \text{consequently, from (2.5b),} \quad [\zeta_{BB'}] = [\pi_{BB'}] = [\kappa_{BB'}].$$

For two-sided Walker geometries one has in addition

$$D' \lambda^A \propto \lambda^A, \quad \delta \lambda^A \propto \lambda^A, \quad \nabla_{BB'} \lambda^A \propto \lambda_{BB'} \lambda^A \lambda^A; \quad \text{consequently, from (2.5b),} \quad [\zeta_{BB'}] = [\pi_{BB'}] = [\kappa_{BB'}].$$

From Law & Matsushita (2008a) (A1.8), in integrable sesquiWalker geometry, the Walker coordinate parametrising integral curves of $D$ ($v$ for form (2.10a), $u$ for form (2.10b)) is an affine parameter for such curves as null geodesics. These geodesics are the intersections of $\alpha$- and $\beta$-surfaces, and the tangent vector ($\partial_v = \lambda^A \pi^A$ for (2.10a), $\partial_u = \lambda^A \pi^A$ for (2.10b)) along such a null geodesic is in fact parallel over the $\alpha$-surface in which the null geodesic lies, and, in the two-sided Walker case, parallel over both the $\alpha$- and $\beta$-surface whose intersection is that null geodesic. The final equation in (2.14) implies that $D$ is parallel, as it must be in two-sided Walker geometry because both $Z_{[\pi]}$ and $W_{[\lambda]}$ are parallel, and of course $\mathcal{H} = D^\perp$ is also parallel.

The null distributions $\partial_u |_R$ and $\partial_v |_R$ for Walker coordinates were studied in Law (2008) (6.1.33) and (6.1.44) and those results therefore provide a local description of the distribution $D$ here. That discussion confirms that $D$ is auto-parallel in an integrable sesquiWalker $\alpha\beta$-geometry, and parallel in the two-sided
Walker case. Note that the treatment of $\langle \partial \rangle_R$ given in Law (2008) (6.1.33) and (6.1.44) is, in regard to the assumptions concerning coordinates and spin frames employed there, consistent with the assumptions for (2.10b) (whereas the coordinates and spin frames employed in Law 2008 (6.1.33) for the treatment of $\langle \partial \rangle_R$ are not the same as in form (2.10a)), so we will restrict attention here to that form, though the geometric results will be valid generally. The computation of $\alpha^A \alpha^B \nabla_{BB} \alpha^A = 0$ in the proof of proposition 2.5 above could have been stated as the analogue of Law (2008) (6.2.4): i.e., integrability of $W_{[\lambda]}$ is equivalent (for form (2.10b)) to $\kappa = \sigma = 0$. For oriented Walker coordinates, Law (2008) (5.6) gives $\kappa = 0$ and $\sigma = -b_1/2$, confirming proposition 2.5 in this case. Conditions on the spin coefficients for $\mathcal{D}$ to be parallel are stated in Law (2008) (6.1.12) which, together with Law (2008) (5.6), confirm proposition 2.5 for two-sided Walker geometry. The particularly simple forms one obtains in Law (2008) (6.1.33a–c) with $b_u = c_u = 0$ (for form (2.10b)) are consistent with the fact that $\mathcal{D}$ is parallel in two-sided Walker geometry (in particular, connecting vector fields between the null geodesics of $\mathcal{D}$ within $\mathcal{H}$, when expressed in terms of the null tetrad, have constant components of $m^a$ and $\tilde{m}^a$).

The assumptions underlying form (2.10b) are also consistent with the assumptions employed in the description of null distributions of type III in Law (2008) §3. In particular, one confirms that Law (2008) (6.3.9b) is consistent with the curvature results for integrable sesquiWalker $\alpha\beta$-geometries and (6.3.9c) with the curvature results for two-sided Walker geometries. Note that the assertion there that $\Psi_2 = S/12 = \Psi_2$, where $S$ is the Ricci scalar curvature, is confirmed, for form (2.10b), by Law & Matsushita (2008a) (2.20) and (2.25) ($S = a_{uu} + b_{uv} + 2c_u = a_{uu} + b_{uv}$, when $c_u = 0$). Note that the results in the case of form (2.10b) are entirely consistent with Law (2008) (6.3.13).

In an integrable sesquiWalker $\alpha\beta$-geometry which is not two-sided Walker, Lemma 2.3 entails that $\mathcal{D} = (\tilde{S}^n)_{\mathcal{R}}$ (the latter is of course not defined in two-sided Walker geometry). The analogue of $(\tilde{S}^n)_{\mathcal{R}}$, i.e., $(\tilde{S}^n)_{\mathcal{R}}$ in an $\alpha$-geometry which is not Walker, was studied in Law (2008). This coincidence does not appear to be illuminating, however, as the results obtained in Law (2008) would mainly concern integrability conditions for $(\tilde{S}^n)_{\mathcal{R}} = \mathcal{H}$, and we have already assumed integrability for $\mathcal{H}$ to obtain the coincidence.

We now turn to curvature restrictions in the context of $\alpha\beta$-geometries. Law (2008) (6.2.45) showed that Ricci-null Walker geometries $(M, g, [\pi^A], [\lambda^A])$, i.e., a Walker geometry for which $[\pi^A]$ is a multiple RPS, take a special form, viz., for Walker coordinates $(u, v, x, y)$ there are functions $\vartheta(u, v, x, y)$, $F(u, x, y)$ and $G(v, x, y)$ satisfying only $F_{uu} = G_{uv} =: h(x, y)$ in terms of which $W$ in (2.8) takes the form

$$ W = -2 \begin{pmatrix} \vartheta_{uv} & -\vartheta_{uu} \\ -\vartheta_{uv} & \vartheta_{uu} \end{pmatrix} + \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix}. $$

Law (2008) (6.2.47–57) expresses the curvature in terms of $\vartheta$, $F$, and $G$. In particular, the Ricci scalar curvature $\tilde{S} = 2h(x, y)$.

2.6 Proposition

Let $(M, g, [\pi^A], [\lambda^A])$ be a sesquiWalker $\alpha\beta$-geometry for which $[\pi^A]$ is a multiple RPS. Then, $[\lambda^A]$ is a multiple WPS. Moreover, for any LSR $\lambda^A$ of $[\lambda^A]$, $\phi_{\lambda'\lambda'} := \Phi_{\lambda'\lambda'\lambda\lambda} \lambda^A \lambda^B$ satisfies

$$ \lambda^A \nabla_{\lambda'} \phi_{\lambda'\lambda'} = 2\zeta_{\lambda'} \phi_{\lambda'\lambda'}, $$

where $\zeta_{\lambda'}$ is defined in (2.5b), whence there is an LSR $\lambda^A$ of $[\lambda^A]$ (with the freedom to scale by functions constant on $\beta$-surfaces) such that $\lambda^A \nabla_{\lambda'} \phi_{\lambda'\lambda'} = 0$.

Proof. For specificity, choose oriented Walker coordinates $(u, v, x, y)$ satisfying (2.10a). By 2.5, $a_v = 0$. From Law (2008) (6.2.45d), $a_v = 0 \Leftrightarrow \vartheta_{evv} = 0$, whence, by Law (2008) (6.2.53), $\Psi_1 = \Psi_4 = 0$, i.e., $[\lambda^A]$ is a multiple WPS. Since $[\lambda^A]$ also satisfies (2.3), by the Generalized Goldberg-Sachs Theorem (GGST) (see e.g., Law 2008, (6.2.17))

$$ 0 = \lambda^A \lambda^B \lambda^C \nabla_{\lambda'} \phi_{\lambda'\lambda'\lambda'\lambda'}, $$

where the second equality follows by a spinor Bianchi identity. Hence

$$ \lambda^A \nabla_{\lambda'} \phi_{\lambda'\lambda'} = \lambda^A \lambda^B \lambda^C \nabla_{\lambda'} \phi_{\lambda'\lambda'\lambda'} + (\lambda^B \lambda^C \nabla_{\lambda'} \lambda^A \lambda^B) \Phi_{\lambda'\lambda'\lambda'}, $$

$$ = 2\phi_{\lambda'\lambda'} \zeta'_{\lambda'}, $$

7
Since $[\lambda^A]$ is a solution of (2.3) and a multiple WPS, by Law (2008) (6.2.32), there is an LSR $\chi^A$ of $[\lambda^A]$ for which $\zeta^A = 0$.

The condition $\vartheta_{uuvv} = 0$, i.e., $\vartheta$ quadratic in $v$ with coefficients functions of $u$, $x$, and $y$, can be exploited to generate Ricci-null Walker geometries on the chart $(u, v, x, y)$ for which the Walker spin frame element $\beta^A$ defines via $[\beta^A]$ an integrable $\beta$-distribution, i.e., a sesquiWalker $\alpha\beta$-geometry satisfying Proposition 2.6.

One obtains a further specialization of 2.6 if $(M, g, [\pi^A], [\lambda^A])$ is two-sided Walker. The condition $c_v = 0$ is equivalent, by Law (2008) (6.2.45d), to $\vartheta_{uuvv} = 0$, i.e., the leading coefficient in the quadratic expression of $\vartheta$ as a function of $v$ is independent of $u$. One then notes from Law (2008) (6.2.53) that $\Psi_2 = h/6 = S/12 = \tilde{\Psi}_2$ and from Law (2008) (6.2.57) that $A_{11} = 0$, i.e., $[\lambda^A]$ is a RPS, these results confirming those obtained above for any two-sided Walker geometry.

In a self-dual (SD) Walker geometry $(M, g, [\pi^A])$, the absence of ASD Weyl curvature means that at each point $p$ and for each $\beta$-plane $W \subseteq T_p M$, there exists a $\beta$-surface $W$ such that $T_p W = W$ (see, for example, LeBrun & Mason 2007, §3). In fact, one can construct, locally, integrable $\beta$-distributions. Let $[\lambda^A]$ be a (local) spinor field defining an integrable $\beta$-distribution so that $(M, g, [\pi^A], [\lambda^A])$ is sesquiWalker. Díaz-Ramos et al. (2006) and Davidov & Muskarov (2006) provided a local characterization of SD Walker metrics. With respect to Walker coordinates $(u, v, x, y)$, a Walker metric is SD iff the metric components $a$, $b$, and $c$ take the form:

\[
\begin{align*}
    a &= A u^3 + B u^2 v + C u v^2 + 2 D u v^3 + E u v^4 + F v^5 + G; \\
    b &= B v^3 + A u v^2 + K u v^2 + 2 L u v + M u + N v + H; \\
    c &= A u^2 v + B u v^2 + L u^2 v + D v^2 + C^2 + K u v + P u + Q v + T;
\end{align*}
\]

(2.15)

where the coefficients are arbitrary functions of $x$ and $y$. Assuming the geometry is in fact integrable sesquiWalker, choosing Walker oriented coordinates satisfying (2.10a), then $a_v = 0$, i.e., $B = D = F = 0$. If one further supposes the geometry is two-sided Walker, then $c_v = 0$ too, i.e., altogether one has $A = B = D = F = Q = 0$ and $C = -K$, resulting in the simpler equations:

\[
\begin{align*}
    a &= C u^2 + E u + G; \\
    b &= -C v^3 + 2 L u v + M u + N v + H; \\
    c &= L u^2 + P u + T.
\end{align*}
\]

(2.16)

Now suppose $(M, g, [\pi^A])$ is a SD Ricci-null Walker geometry. Once again, due to the self-duality, one can introduce an integrable $\beta$-distribution and consider Ricci-null, SD integrable sesquiWalker $(M, g, [\pi^A], [\lambda^A])$ using the description in Law (2008), (6.2.45). The SD condition, by Law (2008) (6.2.53) is equivalent to

\[
\vartheta_{uuvv} = \vartheta_{uuvv} = \vartheta_{uuvu} = \vartheta_{uuvu} = 0 \text{ and } \vartheta_{uuvv} = \frac{h}{6}. 
\]

(2.17)

Supposing the oriented Walker coordinates $(u, v, x, y)$ have been chosen to satisfy (2.10a), then $a_v = 0$, i.e., $\vartheta_{uuv} = 0$, which is compatible with (2.17). A simplification is obtained by assuming the Ricci scalar curvature vanishes ($S = 0$), which is equivalent to $h = 0$. Then, all fourth order partial derivatives in $u$ and $v$ of $\vartheta$ vanish and $\vartheta$ is a cubic polynomial in $u$ and $v$, with coefficients functions of $x$ and $y$. The condition $\vartheta_{uuvv} = 0$ means that there is no term in $v^3$.

If one considers a SD, two-sided Walker geometry $(M, g, [\pi^A], [\lambda^A])$ for which $[\pi^A]$ is a multiple RPS, then as noted previously for two-sided Walker geometries, $\Psi_2 = S/12 = \tilde{\Psi}_2$. Hence, self duality entails $0 = \Psi_2 = S/12 = \tilde{\Psi}_2$, and $[\pi^A]$ is a WPS of multiplicity at least three. If the oriented Walker coordinates $(u, v, x, y)$ are chosen to satisfy (2.10a), then $a_v = c_v = 0$, i.e., $\vartheta_{uuvv} = \vartheta_{uuvv} = 0$. Together with the vanishing of all the fourth order partial derivatives in $u$ and $v$, $\vartheta$ must therefore take the form

\[
\vartheta = K_2 u^2 v + K_4 u^3 + K_5 u^2 v + K_6 u v + K_7 v^2 + K_8 u + K_9 v + K_{10},
\]

(2.18)

where the coefficients are functions of $x$ and $y$. One special case is a left-flat Walker geometry $(M, g, [\pi^A])$, i.e., for which the SD Weyl curvature is the only nontrivial curvature. Such curvature permits the construction, locally, of parallel unprimed spin frames, whence, locally, of parallel $\beta$-distributions, so left-flat
Walker geometries are, locally, automatically two-sided Walker, and Ricci null with respect to \([\pi^A]\). As \(S = 0\), from Law (2008) (6.2.58), \(F(u, x, y) = uf(x, y)\) and \(G(v, x, y) = vg(x, y)\), for some functions \(f\) and \(g\). Hence, locally, such geometries are characterized by the expression (2.18) satisfying Law (2008) (6.2.63). Substitution of (2.18) into Law (2008) (6.2.63) yields two constraints:

\[
(8K_2 - g)_x = f_y \quad (3K_4)_x + (K_2)_y - 4(K_2)^2 + \frac{g}{2}K_2 - \frac{f}{2}(3K_4) = 0.
\]

The first implies the existence of a function \(X(x, y)\) satisfying \(X_x = f\) and \(X_y = 8K_2 - g\). Upon substituting these expressions into the second constraint, one obtains

\[
(3K_4)_x + (K_2)_y = \frac{X_x}{2}(3K_4) + \frac{X_y}{2}K_2,
\]

which is of the form of Chudecki & Przanowski (2008b) (5.35) and therefore has solution \(3K_4 = Y_x\exp(X/2)\) and \(K_2 = -Y_x\exp(X/2)\), for some function \(Y(x, y)\). Substituting into Law (2008) (6.2.59) yields

\[
a = uX_x - 4K_7 \quad c = -4uY_x\exp(X/2) + 2K_6 \quad b = -4(uY_y + vY_x)\exp(X/2) - vX_y - 4K_5,
\]

which is essentially Chudecki & Przanowski (2008b) (5.37).

3. Algebraically Special AlphaBeta-Geometries

Let \((M, h, [\pi^A], [\lambda^A])\) be an \(\alpha\beta\)-geometry. As noted in §2, \([\pi^A]\) and \([\lambda^A]\) are each WPSs. In this section we assume that \([\pi^A]\) is a multiple WPS and say that \((M, h, [\pi^A])\) is a real algebraically special (AS)\(\alpha\)-geometry. By Law & Matsushita (2008b), (3.27), such is locally conformal to a Walker geometry, i.e., each point \(p \in M\) has a neighbourhood \(U\) such that on \(U\), \(h = \Omega^2 g\), for some smooth function \(\Omega : U \to \mathbb{R}^+\), and \((U, g, [\pi^A])\) is Walker. Since integrability of distributions is a differential-topological condition, \((U, g, [\pi^A], [\lambda^A])\) is a sesquiWalker \(\alpha\beta\)-geometry. Moreover, the integrability of \(\mathcal{H}\) is also unaffected by the conformal rescaling, whence one can evaluate that condition on \(U\) with respect to \((U, g, [\pi^A], [\lambda^A])\) using lemma 2.3. Granted integrability of \(\mathcal{H}\), one can then deduce properties of the local geometry of \((M, h, [\pi^A], [\lambda^A])\) using conformal rescaling formulae as in Law & Matsushita (2008b) and the known local geometric properties of integrable sesquiWalker \(\alpha\beta\)-geometry.

Now suppose that \([\lambda^A]\) is a multiple WPS too. As this property is conformally invariant, it remains valid for \((U, g, [\pi^A], [\lambda^A])\), whence \((M, g, [\lambda^A])\) is a real AS\(\beta\)-geometry and it is locally conformal to a geometry which is Walker for the \(\beta\)-distribution, i.e., each point \(p \in U\) has a neighbourhood \(V\) on which \(g = \chi^2 k\), for some smooth \(\chi : V \to \mathbb{R}^+\), where \((V, k, [\lambda^A])\) is Walker. We will refer to \((V, k, [\pi^A], [\lambda^A])\) as a sesquiWalker \(\beta\alpha\)-geometry.

A natural question to ask is when \((V, k, [\pi^A], [\lambda^A])\) can be constructed so as to be two-sided Walker, i.e., in effect, when is \((M, h, [\pi^A], [\lambda^A])\) locally conformal to two-sided Walker geometry? Note that a necessary condition is that \(\mathcal{H}\) must be integrable as it is for two-sided Walker geometry and is a differential topological condition.

If one begins with an \(\alpha\beta\)-geometry \((M, h, [\pi^A], [\lambda^A])\), and conformally rescales, one requires

\[
\pi_A\nabla_b\pi^A = 0 \quad \lambda_A\nabla_b\lambda^A = 0,
\]

i.e.,

\[
0 = S_b + \pi_B\Upsilon_{BX}\pi^X \quad 0 = \tilde{S}_b + \lambda_B\Upsilon_{XB}\lambda^X,
\]

i.e., one must solve

\[
\pi^{B'}\nabla_{BB'}f = \omega_B \quad \lambda^{B'}\nabla_{BB'}f = \kappa_{B'},
\]

for some \(f\). Taking components gives four equations but two are equivalent under the assumption that \(\mathcal{H}\) is integrable.
In fact, from Law & Matsushita (2008b), a necessary and sufficient condition to solve the first equation of (3.1) is that $[\pi^A]$ is a multiple WPS, and to solve the second equation is that $[\lambda^A]$ is a WPS. Moreover, one obviously has the necessary condition that

$$\pi^A \kappa^A = \lambda^A \pi^A \nabla_{AA'} f = \lambda^A \omega_A,$$

which is (2.7), i.e., integrability of $\mathcal{H}$. A natural question is whether the three necessary conditions: integrability of $\mathcal{H}$; each of $[\pi^A]$ and $[\lambda^A]$ multiple WPSs; are also sufficient to simultaneously solve the pair of equations in (3.1)?

To investigate this question, note that if $(U, \Omega^{-2}h, [\pi^A], [\lambda^A])$ is two-sided Walker for some smooth function $\Omega : U \to \mathbb{R}^+$, then, by Law & Matsushita (2008b) (3.1), for any smooth function $\chi : U \to \mathbb{R}^+$ which is constant on $\alpha$-surfaces, $(U, \chi^2 \Omega^{-2}h, [\pi^A])$ is still Walker. Hence, if it is possible to locally conformally rescale $h$ to be two-sided Walker, then it is possible to first rescale $h$ to be Walker for $[\pi^A]$ and then to locally rescale that Walker metric to be two-sided Walker. We take this approach to study our question so as to utilize the Walker coordinates introduced in §2.

Since $(M, h, [\pi^A], [\lambda^A])$ is an AS0-geometry, as in the opening paragraph of this section each point $p \in M$ has a neighbourhood $U$ such that $(U, \Omega^{-2}h, [\pi^A], [\lambda^A])$ is a sesquiWalker $\alpha\beta$-geometry for some $\Omega : U \to \mathbb{R}^+$. Can we choose, on a possibly smaller neighbourhood $V$ of $p$, a smooth $\chi : V \to \mathbb{R}^+$, constant on $\alpha$-surfaces and such that $(V, \chi^2 \Omega^{-2}h, [\pi^A], [\lambda^A])$ is two-sided Walker? Thus, one requires that $\chi$ solve $\pi^B \nabla_{BB'} \chi = 0$ and $\lambda^B \nabla_{BB'} [\ln(\chi^{-1})] = \kappa_{BB'}$, where the latter equation is the appropriate analogue of Law & Matsushita (2008b) (3.27.1), i.e., the second equation of (3.1) above. For simplicity, we may write these equations as

$$\pi^B \nabla_{BB'} f = 0, \quad \lambda^B \nabla_{BB'} f = \kappa_{BB'}, \quad f := \ln(\chi^{-1}). \quad (3.2)$$

Thus, as must be the case, $\kappa_{BB'} = 0$ iff $c_\nu = 0$, i.e., iff $(U, \Omega^2h, [\pi^A], [\lambda])$ is already two-sided Walker.

The equations (3.2) are a system of PDEs on the integral surfaces of $\mathcal{H}$. The coordinates $(v, u, x, y)$ are Frobenius coordinates for the nested distributions $\mathcal{P} \leq \mathcal{H} \leq \mathcal{C}$. Taking components, by (2.10a–11a), (3.2) are equivalent to

$$X_1 f := \partial_v f = 0, \quad X_2 f := \partial_u f = 0, \quad X_3 f := -\lambda^B \xi^B \nabla_{BB'} f = -\xi^B \kappa_{BB'} = \frac{c_\nu}{2}.$$

By Law & Matsushita (2008a), (2.11),

$$X_3 f = -n_b \nabla_b f = \left(\frac{a}{2} \partial_u + \frac{c}{2} \partial_v - \partial_x\right) f = \frac{c_\nu}{2}.$$

Writing $[X_i, X_j] =: \Phi_{ij}^k X_k$, one computes: $[X_1, X_2] = 0$, $[X_1, X_3] = (a_\nu/2)X_2 + (c_\nu/2)X_1 = (c_\nu/2)X_1$, $[X_2, X_3] = (a_\nu/2)X_2 + (c_\nu/2)X_1$. Hence, the only nonzero $\Phi_{ij}^k$ are

$$\Phi_{13}^1 = \frac{c_\nu}{2}, \quad \Phi_{23}^1 = \frac{c_\nu}{2}, \quad \Phi_{23}^2 = \frac{a_\nu}{2}.$$

With $\phi_1 = \phi_2 = 0$ and $\phi_3 = c_\nu/2$, the integrability conditions for $X_i f = \phi_i$, $i = 1–3$, are $X_i \phi_j - X_j \phi_i = \Phi_{ij}^k \phi_k$. The non vacuous conditions are: $X_1 \phi_3 - X_3 \phi_1 = \Phi_{13}^k \phi_k$, i.e., $(c_\nu/2) = 0$; $X_2 \phi_3 - X_3 \phi_2 = \Phi_{23}^k \phi_k$, i.e., $(c_\nu/2) = 0$. Thus, the integrability conditions for (3.2), in these Walker coordinates, are

$$c_{12} = c_{22} = 0.$$
As \( a_v = 0 \) already, from Law & Matsushita (2008a), (2.25), \([\lambda^A] = [\beta^A]\) is a multiple WPS iff \( c_{uv} = 0\). Thus, granted our assumptions, the remaining obstruction to solving (3.2) in terms of the Walker coordinates satisfying (2.10a) is the single condition

\[
c_{uv} = 0.
\] (3.4)

Exactly the same result is obtained utilizing the coordinate form (2.10b). The condition (3.4) indicates that the multiplicity of each \([\pi^A]\) and \([\lambda^A]\) as WPSs together with the integrability of \( \mathcal{H} \) are necessary but not sufficient conditions for \((M, h, [\pi^A], [\lambda^A])\) to be locally conformally two-sided Walker. A geometric characterization of this obstruction is desirable. From Law & Matsushita (2008a) (2.25), (3.4) is equivalent to

\[
\Psi_2 = \frac{S}{12}.
\] (3.5)

for the Walker geometry \((U, g, [\pi^A])\). Given that \([\beta^A]\) is here a multiple WPS \((\Psi_4 = \Psi_3 = 0)\) and that the freedom in the spin frame \(\{\alpha^A, \beta^A\}\) is \(\beta^A \mapsto \lambda \beta^A, \alpha^A \mapsto \chi^{-1} \alpha^A + \mu \beta^A\), then \(\Psi_2 = \Psi_{ABCD} A^A B^C D^D\) is in fact a geometric quantity. Thus, under the stated constraints, (3.5) is a geometric condition, albeit expressed in terms of the Walker geometry \((U, g, [\pi^A])\) rather than the \(\alpha\beta\)-geometry \((M, h, [\pi^A], [\lambda^A])\). (Note that while (3.4) is equivalent to (3.5) for arbitrary Walker coordinates, \(\Psi_2\) is only geometric in nature in the circumstances stated above.)

Under any conformal rescaling \(g \mapsto \chi^2 g\), exploiting the spin frames of Law & Matsushita (2008b) (3.15) for the rescaled metric, one has

\[
\hat{\Psi}_2 = \chi^{-2} \Psi_2 \quad \hat{S} = \chi^{-2} [S - 6 \chi^{-1} \Box \chi]
\] (3.6)

(see, for example, Law & Matsushita 2008b, (2.8)). In Walker geometry,

\[
\Box \chi = -a \chi_{uu} - 2 c \chi_{uv} - b \chi_{vv} + 2 \chi_{ux} + 2 \chi_{vy} - (a_u + c_v) \chi_u - (b_v + c_u) \chi_v,
\] (3.7)

Law & Matsushita (2008a), (3.9). Note that if \(\chi\) is constant on \(\alpha\)-surfaces, i.e., is a function of \((x, y)\) only, then \(\Box \chi = 0\) and under such conformal rescalings (3.5) is invariant. Thus, choosing a \(\chi\) to solve the first equation of (3.2), the obstruction to solving the second equation of (3.2) is again (3.5) within the rescaled geometry. As noted previously, (3.5) follows when \(\mathcal{H}\) is parallel and thus is a necessary condition if \((V, \chi^2 \Omega^{-2} h, [\pi^A], [\lambda^A])\) is to be two-sided Walker for some choice of \(\chi\).

3.1 Proposition

An \(\alpha\beta\)-geometry \((M, h, [\pi^A], [\lambda^A])\) is locally conformally two-sided Walker iff \(\mathcal{H}\) is integrable, each of \([\pi^A]\) and \([\lambda^A]\) are multiple WPSs, and (3.5) holds in every locally conformally-related Walker geometry \((U, \Omega^{-2} h, [\pi^A])\).

We end with some observations on an explicit example presented by Chudecki & Przanowski (2008a), (4.23), which we write in the form:

\[
(h_{ab}) = v^{-2} (g_{ab})
= v^{-2} \left[ 2 (du^2 + dv^2) + \left( \frac{e^4 u^4}{3 v^2} + 4 u F_x \right) dx^2 + 2 \left( \frac{2 e^4 u^3}{3 v} + 2 u F_y \right) dy^2 \right],
\] (3.8)

where \(F\) is a function of \((x, y)\) only. It is clear that the metric \((g_{ab})\) is of the form (2.8), i.e., is Walker, with \((u, v, x, y)\) Walker coordinates. The metric \((h_{ab})\) is obviously conformally Walker. Chudecki & Przanowski (2008a) constructed this metric as a solution of the hyperheavenly equation and thus \(h\) is Einstein; moreover, they constructed it so that each Weyl spinor is of type \(\{4\}\) (i.e., null). We shall denote the multiple WPSs by \([\pi^A]\) and \([\lambda^A]\); since they are the only WPSs, \([\pi^A]\) must be the projective spinor field defining the integrable \(\alpha\)-distribution. It follows that \(h\) is not itself Walker for this \(\alpha\)-distribution \(Z_{\chi}\) since the conformal factor \(\chi := v^{-1}\) is not constant on \(\alpha\)-surfaces; there are no other integrable \(\alpha\)-distributions as there is only one WPS.
for $\Psi_{ABCD}$. There is also at most one integrable $\beta$-distribution. Indeed, since the metric $h$ is Einstein, a spinor Bianchi identity gives $\nabla^a \Psi_{ABCD} = 0$, whence the GGST (Law 2008 (6.2.19)) ensures that $[\lambda^A]$ is in fact a solution of (2.3) and does define an integrable $\beta$-distribution $W_{[\lambda]}$.

Since $[\pi^A]$ is of multiplicity four, the Ricci scalar curvature $S$ of the Walker metric $g$ must vanish (Law & Matsushita (2008a), 2.5 or 2.6). The Ricci scalar curvature $\tilde{S}$ of the metric $h$ is then

$$\tilde{S} = \chi^{-2}[S - 6\chi^{-1} \Box \chi] = -6\chi^{-3} \Box \chi.$$

From (3.7) one computes that $\Box \chi = 0$, whence $\tilde{S} = 0$. From Law & Matsushita (2008a) A1.8 and (2.33), however, one easily checks that $[\pi^A]$ is not a multiple RPS for the Walker metric $g$ (the $\theta$, $\mu$, and $\nu$ in (2.33) are each nonzero); in particular, the Walker metric $g$ is not Einstein.

Using the given Walker coordinates $(u, v, x, y)$ for $g$, and their associated Walker spin frames, one can readily calculate the SD Weyl curvature spinor from Law & Matsushita (2008a) (2.25–26), obtaining:

$$\Psi_{ABCD} = e^{AF} \left( \frac{u^4}{v^3} \alpha_A \alpha_B \alpha_C \alpha_D + 4 \frac{u^3}{v^2} \alpha_A \alpha_B \beta_C \beta_D + 6 \frac{u^2}{v} \alpha_A \alpha_B \beta_C \beta_D + 4 \frac{u}{v} \alpha_A \beta_B \beta_C \beta_D + \beta_A \beta_B \beta_C \beta_D \right).$$

To determine the WPS of $\Psi_{ABCD}$, one need only equate the last expression with $\lambda_A \alpha_B \lambda_C \lambda_D$. One may take $[\lambda^A] = [u\alpha^A + v\beta^A]$. Using (2.12) and the covariant derivatives of the Walker spin frames (Law 2008, (5.8)) one readily computes:

$$\lambda_A \nabla_b \lambda^A = \left( \frac{4F}{6v} u^4 \right) \ell_b + \left( \frac{e^{4F} u^3}{6v} \right) \kappa_b - v\pi_b - u\xi_b = \lambda_B \left[ \frac{e^{4F} u^3}{6v} \pi_{B'} - \xi_{B'} \right] \neq 0 \quad \lambda_A \lambda_B \nabla_{B'} \lambda^A = 0.$$

Hence the Walker metric $g$ is indeed sesquiWalker, but not two-sided Walker. Moreover, from (2.5b), $\kappa_{B'} = (e^{4F} u^3/6v)\pi_{B'} - \xi_{B'}$, so $\kappa_{B'} \pi_{B'} = -1$ whence, by lemma 2.3, $\mathcal{H}$ is not integrable. Thus, the metric $h$, though an $\alpha\beta$-metric which is algebraically special for both WPSs, is not locally conformally two-sided Walker. Note, however, that the condition (3.5) is valid for the metric $g$ (both sides of the equation vanish), so the non-integrability of $\mathcal{H}$ is the only obstruction to $h$ being locally conformally two-sided Walker.

References

Boyer, C. P., Finley III, J. D. & Plebański, J. F. 1980 Complex General Relativity, $\mathcal{H}$ and $\mathcal{HH}$ Spaces-A Survey of One Approach, in General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein, Vol. 2, A. Held (ed.), Plenum Press, New York, NY, 241–281.

Chudecki, A. and Przanowski, M. 2008a A simple example of type-$[N]$ $\otimes [N]$$\mathcal{H}$-spaces admitting twisting null geodesic congruence. Classical Quantum Gravity, 25, 055010 (13pp).

Chudecki, A. and Przanowski, M. 2008b From hyperheavenly spaces to Walker spaces and Osserman spaces. I. Classical Quantum Gravity, 25 (145010) (18 pp).

Davidov, J. & Muškarov, O. 2006 Self-dual Walker metrics with two-step nilpotent Ricci operator. J. Geom. Phys. 57, 157–165.

Díaz-Ramos, J. C., García-Río, E. & Vázquez-Lorenzo, R. 2006 New Examples of Osserman metrics with non-diagonalizable Jacobi operators. Differential Geometry and Its Applications 24, 433–442.

Finley III, J. D. & Plebański, J. F. 1976 The intrinsic spinorial structure of hyperheavens. Journal of Mathematical Physics 17, 2207–2214.

Law, P. R. 2006 Classification of the Weyl curvature spinors of neutral metrics in four dimensions. J. Geo. Math. Phys. 56, 2093–2108.

Law, P. R. 2008 Spin Coefficients for Four-Dimensional Neutral Metrics, and Null Geometry. [arXiv:0802.1761v1] [math.DG] (13 Feb 2008).

Law, P. R. & Matsushita, Y. 2008a A Spinor Approach to Walker Geometry. Communications in Mathematical Physics, 282, 577-623. [arXiv:math/0612804v3 [math.DG] 17 Aug 2008.]

Law, P. R. & Matsushita, Y. 2008b Algebraically Special, Real Alpha-Geometries. [arXiv:0808.2082v1 [math.DG] (15 Aug 2008).]
LeBrun, C. and Mason, L. J. 2007 Nonlinear Gravitons, Null Geodesics, and Holomorphic Disks. *Duke Mathematics Journal* **136**, 205–273.

Walker, A. G. 1950 Canonical form for a Riemannian space with a parallel field of null planes. *Quart. J. Math. Oxford(2)* **1**, 69–79.