Tuning Parameter Selection for Penalized Estimation via $R^2$

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Abstract

The tuning parameter selection strategy for penalized estimation is crucial to identify a model that is both interpretable and predictive. However, popular strategies (e.g., minimizing average squared prediction error via cross-validation) tend to select models with more predictors than necessary. This paper proposes a simple, yet powerful cross-validation strategy based on maximizing squared correlations between the observed and predicted values, rather than minimizing squared error loss. The strategy can be applied to all penalized least-squares estimators and we show that, under certain conditions, the metric implicitly performs a bias adjustment. Specific attention is given to the lasso estimator, in which our strategy is closely related to the relaxed lasso estimator. We demonstrate our approach on a functional variable selection problem to identify optimal placement of surface electromyogram sensors to control a robotic hand prosthesis.

Keywords: Cross-Validation; Model/Variable Selection; Functional Data; Relaxed Lasso.

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1 Introduction

Many statistical problems aim to build a predictive model from a large set of potential predictor variables. Variable selection is often performed to select a predictive model that depends on as few predictor variables as possible. For example, Stallrich et al. (2020) recently discussed an important functional variable selection problem to develop a prosthesis controller (PC) for a robotic hand. Electromyogram (EMG) signals from surface sensors placed on the residual muscles were input into a PC and translated into movement of the robotic hand. For able-bodied subjects, it is know that contracting few muscles cause certain movements, implying a predictive PC requires a few strategically-placed EMG sensors.

This paper concerns problems that are well approximated by a sparse linear model:

\[ y_{n\times 1} = X_{n\times p} \beta^*_{p\times 1} + \epsilon_{n\times 1} \]  

where \( E(\epsilon) = 0, \ V(\epsilon) = \Sigma, \ beta^T = (\beta_1^*, ..., \beta_p^*, 0, ..., 0), \) and \( p^* \) is the number of important variables. Without loss of generality, assume \( y \) and predictor variables, \( X \), are centered and the diagonals of \( X^T X \) equal \( n \). Let \( M^* = \{ j : \beta_j^* \neq 0 \} \) denote the support of \( \beta^* \). A predictive model’s estimate for \( \beta^* \), denoted \( \hat{\beta} \), will ideally also have support \( M^* \) and will be close to \( \beta^* \) in some other sense, such as \( ||\hat{\beta} - \beta^*||_2^2 = \sum_j (\hat{\beta}_j - \beta_j^*)^2 \).

For high-dimensional data like that in Stallrich et al. (2020), simultaneous support recovery and parameter estimation is performed via penalized estimation. A penalized estimator is represented generally by \( \hat{\beta}_\lambda = \arg \min L(\beta) + P_\lambda(\beta) \) where \( L(\beta) \) is a loss function comparing \( y \) to its predicted values \( \hat{y} = X\hat{\beta} \), and \( P_\lambda(\beta) \) is a penalty function that depends on tuning parameter(s) \( \lambda \geq 0 \). Henceforth we let \( L(\beta) = (2n)^{-1} ||y - \hat{y}_\lambda||_2^2 \). Penalty functions can take myriad forms but we are interested in those that increase as \( \beta \) moves away from \( 0 \). The Lasso (Tibshirani 1996) penalty, \( P_\lambda(\beta) = \lambda ||\beta||_1 = \lambda \sum_{j=1}^p |\beta_j| \), is one such penalty that can force estimates to equal 0, thereby performing simultaneous
variable selection and estimation. For such sparsity-inducing estimators, we are interested
in comparing the estimated support, \( M_{\lambda} = \{ j \mid \hat{\beta}_{\lambda,j} \neq 0 \} \), to \( M^* \).

The value of \( \lambda \) balances the importance of minimizing \( P_{\lambda}(\beta) \) relative to \( L(\beta) \), so it is
recommended to explore the tuning parameter space to identify an “optimal” value. Potential
criteria for an optimal value include identifying a \( \hat{\beta}_\lambda \) that minimizes \( ||\hat{\beta}_\lambda - \beta^*||^2 \),
minimizes \( ||X\hat{\beta}_\lambda - X\beta^*||^2 \), or has \( M_\lambda = M^* \). The latter criterion is referred to as support
recovery and is the primary focus of this paper. Even if a \( \lambda \) exists where \( M_\lambda = M^* \), there is
no guarantee that we will be able to correctly identify it. Popular approaches, such as mini-
mizing information criteria, like AIC (Akaike 1974) and BIC (Schwarz 1978), or minimizing
squared prediction error from \( K \)-fold Cross Validation often choose a \( \lambda \) that overselects the
number of important variables (Feng and Yu 2013; Hastie et al. 2017), i.e., \( M^* \subset M_\lambda \).
Post-selection inference techniques (e.g., Covariance Test for the Lasso (Lockhart et al.
2014) and the Debiased Lasso (Zhang and Zhang 2014)) and multi-stage modifications
(e.g., Adaptive (Zou 2006) and Relaxed Lasso (Meinshausen 2007)) can correct for this
overselection, albeit with added computations and assumptions.

This paper proposes a new \( K \)-fold Cross Validation strategy that assesses the predic-
tive quality of \( \hat{\beta}_\lambda \) via squared prediction correlation rather than squared prediction error.
The invariance of correlation to the scale of \( \hat{\beta}_\lambda \) reduces the impact of the potential shrink-
age when estimating large \( \beta_j^* \) that would otherwise burgeon squared prediction error. To
demonstrate, we simulated a dataset \( \{ y, X \} \) with \( n = p = 100, p^* = 5 \), and \( \sigma^2 = 1 \), and gen-
erated the entire solution path of the Lasso. \( X \) were independently sampled from a standard
Normal distribution and the active coefficients in \( \beta^* \) were \( \{ 2.13, 1.81, -2.46, -1.89, -2.51 \} \).
Many \( \lambda \) values had \( M_\lambda = M^* \), so a wide range were optimal for variable selection. Figure
1 plots \( y \) against their in-sample predictions, \( \hat{y}_\lambda \), under four different \( \lambda \) values, and sum-
marizes the models’ number of false positives (FP), average prediction error (APE) from
10-fold CV, and in-sample squared prediction correlation (\( R^2 \)). Figure 1(a) corresponds to
the $\lambda$ determined by a Cross Validation one-standard-error rule (CV 1SE) and (b) corresponds to the $\lambda$ that minimizes $\|X\beta^* - X\hat{\beta}_\lambda\|_2$ (Min PB). Among the $\lambda$ having $\mathcal{M}_\lambda = \mathcal{M}^*$, we consider the largest and smallest in Figures 1(c) and 1(d) respectively.

(a) CV 1SE  
(b) Min PB  
(c) Max $\lambda$, $\mathcal{M}_\lambda = \mathcal{M}^*$  
(d) Min $\lambda$, $\mathcal{M}_\lambda = \mathcal{M}^*$

Figure 1: In-sample observations ($y_i$) versus predictions ($\hat{y}_i$) for Lasso estimator at four values of $\lambda$ for a data set with $n = 100$, $p = 100$, and $p^* = 5$. We also report average prediction error (APE) under 10-fold CV, in-sample squared prediction correlation ($R^2$), and the number of false positives (FP). The minimum APE plus-or-minus one standard error is $0.2844 \pm 0.0262$ and all estimates have $\mathcal{M}^* \subseteq \mathcal{M}_\lambda$.

All models shown in Figure 1 include the 5 important variables. Both the CV 1SE and Min PB models have small APE and large $R^2$, but many false positives. The model in Figure 1(c) has no false positives, but has relatively large APE. The model under Figure 1(d) also has no false positives and its in-sample $R^2$ approximately equals that for the Min PB estimate. This motivates a tuning parameter selection strategy based on an $R^2$ metric rather than APE to compromise between prediction and variable selection.

After justifying the $R^2$ metric, we highlight and investigate a surprising equivalence between the metric and a multiplicative adjustment on $\hat{\beta}_\lambda$, referred to here as the $\alpha$-Modification. We argue that for $\hat{\beta}_\lambda$ with certain statistical properties, the adjustment can reduce the bias of $\hat{\beta}_\lambda$ thereby improving its predictive potential. We go on to study the $\alpha$-Modification for the Lasso, highlighting its similarities to the Nonnegative Garrote (Breiman 1995) and Relaxed Lasso. However, unlike these two methods, the $\alpha$-Modification can be applied to any penalized least-squares estimator, including non-convex penalties, without additional computational complexity.
This paper is organized as follows. In Section 2 we review classes of penalized estimators, popular tuning parameter selection strategies, and post-selection inference methods. Section 3 justifies the value in the $R^2$ metric and provides statistical properties for a general class of penalized estimators. Finite-sample properties are then derived for the $\alpha$-Modification for the Lasso in Section 4. Section 5 presents a simulation study of the new approaches and Section 6 applies our new methods to the EMG data of Stallrich et al. (2020). Section 7 provides a discussion on the implications of our new framework for evaluating model fit and propose avenues of future research. Proofs of all results may be found in the Appendix, with additional details given in the Supplementary Materials.

2 Background

2.1 Classes of Penalized Estimators

Consider the class of penalties $P_\lambda(\beta) = ||\beta||_q^q = \sum_j |\beta_j|^q$, $q > 0$, corresponding to the so-called bridge estimators (Frank and Friedman 1993). Knight and Fu (2000) show that for $q \leq 1$ and large enough $\lambda$, the penalized estimate will have some $\hat{\beta}_{\lambda,j} = 0$, yielding a continuous approach to the intractable exploration of all submodels. They went on to show that under appropriate regularity conditions, the limiting distributions of such $\hat{\beta}_{\lambda,j}$ whose corresponding $\beta_j^* = 0$ can have positive probability mass at 0 when $p$ is fixed, meaning that they are capable of support recovery. Huang et al. (2008) extended this work to prove the same result as $p$ and $n$ grow to infinity, under certain growth rate conditions. However, the large $\lambda$ necessary for this to occur may cause $|\hat{\beta}_{\lambda,j}| << |\beta_j^*|$ for $j \in M^*$, thereby inflating criteria commonly used for tuning parameter selection. That is, a $\lambda$ yielding support recovery may be ignored by popular tuning parameter selection strategies.

The Smoothly Clipped Absolute Deviation (SCAD) penalty (Fan and Li 2001) is a quadratic spline function with knots at two tuning parameters and the Minimax Concave
penalty (MCP) gives a continuous, nearly unbiased method of penalized estimation (Zhang 2010). Fan and Li (2001) proved that SCAD is support recovery consistent (i.e., $P(M_{\lambda_n} = M^*) \to 1$ as $n \to \infty$) for appropriately chosen $\lambda$. MCP shares a similar result but again under certain conditions on $\lambda$ (Zhang 2010). These penalties are less likely to have $|\hat{\beta}_{\lambda,j}| << |\beta^*_j|$ but suffer from having to explore a multidimensional tuning parameter space.

Another approach to prevent $|\hat{\beta}_{\lambda,j}| << |\beta^*_j|$ is to adjust bridge penalties. The Ridge penalty ($q = 2$) cannot shrink any coefficient estimates to exactly zero, so Wu (2021) recently proposed $P_{\lambda}(\beta) = \sum_{j=1}^{p} \lambda_j |\beta_j|^2$, to allow for this behavior. The Adaptive Lasso (Zou 2006) is a similar adjustment but for the Lasso penalty. It follows a two-stage process: first $\hat{\beta}$—a consistent estimator for $\beta^*$ such as Ordinary Least Squares (OLS)—is calculated. Then, with $\gamma > 0$, Adaptive Lasso estimates are found using the penalty $P_{\lambda}(\beta) = \lambda \sum_{j=1}^{p} |\hat{\beta}_j|^{-\gamma} |\beta_j|$. When $\gamma = 1$ and $\hat{\beta} = \hat{\beta}_{OLS}$, the objective function reduces to the Nonnegative Garotte (Breiman 1995). This approach is support recovery consistent when $\lambda_n/\sqrt{n} \to 0$ and $\lambda_n n^{(\gamma-1)/2} \to \infty$ as $n \to \infty$. The Adaptive Lasso is easily generalized to non-Lasso penalties, however the selection of $\gamma$ and consistent estimation of $\beta^*$ for high dimensional data can be difficult to establish.

The Relaxed Lasso (Meinshausen 2007) minimizes the objective function:

$$\frac{1}{2n} ||y - X(\beta \circ 1_{M_{\lambda}})||_2^2 + \lambda \phi ||\beta||_1 . \quad (2)$$

where $\phi \in (0, 1]$ and $\beta \circ 1_{M_{\lambda}}$ is the Hadamard product of $\beta$ with the support vector under $M_{\lambda}$. In simulations, the Relaxed Lasso returns sparse models with low bias. Meinshausen (2007) showed that the expected value of the loss function of the Relaxed Lasso converges faster than the Lasso when $p$ increases quickly relative to $n$, meaning that Relaxed Lasso estimates are likelier than Lasso estimates to be close to $\beta^*$ for smaller $n$. This result is again achieved assuming that $\lambda$ is sufficiently large. The Relaxed Lasso is computationally effi-
cient to calculate but its extensions to more complicated penalties can be computationally intensive, and to our knowledge has not been well-studied.

The Group Lasso \cite{YuanLin} penalty assumes \( E(y_i) = \sum_{j=1}^{p} x_{ij}^T \beta_j \) where each \( x_{ij} \) is a \( d_j \times 1 \) vector corresponding with the \( i \)th observation of the \( j \)th covariate group and shrinks coefficients at the group level. This class of models includes general additive models (GAMs) and functional linear models. GAMs \cite{HastieTibshirani} have the form \( y_i = \sum_{j=1}^{p} f_j(x_{ij}) + \epsilon_i \) where the \( f_j \)'s are functions of one or more covariates. Each \( f(\cdot) \) is commonly approximated by a pre-specified basis expansion, such as B-splines, so that estimation of \( f(\cdot) \) is equivalent to estimating the corresponding group of basis coefficients. The functional linear model \cite{RamsaySilverman}, an example of which may be found in Section 6, has functional covariates \( x_{ij}(t) \) on domain \( T \) and models \( y_i = \sum_{i=1}^{p} \int x_{ij}(t) \beta_j(t) dt + \epsilon_i \). The model is often approximated by imposing a basis expansion of the \( \beta_j(\cdot) \). The Group Lasso penalty for such models is \( P(\lambda) = \sum_{j=1}^{p} \lambda ||\beta_j||_{K_j} \) where \( ||z||_K = (z^T K z)^{1/2} \) and \( K_1, \ldots, K_J \) are known positive definite matrices. The group of coefficient estimates, \( \hat{\beta}_{\lambda,j} \), are then either all zero or all nonzero. For the linear model, and under some regularity conditions, \cite{NardiRinaldo} proved that the Group Lasso is support recovery consistent as long as \( \sqrt{n} \lambda_j \rightarrow \infty \) for all \( j \notin M^* \).

2.2 Tuning Parameter Selection Strategies

Desirable statistical properties of penalized estimators require certain tuning parameter values, making tuning parameter selection a pivotal step in the analysis. One popular approach is to choose \( \lambda \) that minimizes an information criterion \( IC(\lambda) = -2 \log L(\hat{\beta}_\lambda) + h(k_\lambda) \) where \( L(\hat{\beta}_\lambda) \) is the likelihood of the data under \( \hat{\beta}_\lambda \) and \( h(\cdot) \) is a penalty to prevent overselection based on the \( k_\lambda = |M_\lambda| \). Two well-known criteria are AIC, where \( h(k_\lambda) = 2k_\lambda \) \cite{Akaike}, and BIC, where \( h(k_\lambda) = k_\lambda \log(n) \) \cite{Schwarz}. AIC often overselects, particularly for small sample sizes, so a corrected AIC (AICc), where \( h(k_\lambda) = 2k_\lambda + \frac{2k_\lambda^2-2k_\lambda}{n-k_\lambda-1} \)
is recommended (Hurvich and Tsai 1989). BIC, unlike AIC, is support recovery consistent when \( \epsilon_i \overset{\text{iid}}{\sim} N(0, \sigma^2) \) (Nishii 1984). For Gaussian errors, \( L(\hat{\beta}_\lambda) \) involves \( \sigma^2 \) and for unknown \( \sigma^2 \), \(-2 \log L(\hat{\beta}_\lambda) \propto n \log(\hat{\sigma}_\lambda^2) \) where \( \hat{\sigma}_\lambda^2 \) is the estimated model variance at \( \lambda \). Buhlmann and van de Geer (2011) point out this can lead to overselection when \( \hat{\sigma}_\lambda^2 < 1 \). When possible, \( \sigma^2 \) is substituted with \( \hat{\sigma}^2 \) from a presumed low bias model (Hastie et al. 2017), but this may be challenging to identify for high-dimensional data. Hui et al. (2015) proposed the Extended Regularized Information Criterion (ERIC) specifically for tuning parameter selection of penalized estimators, where \( h(k_\lambda) = 2\nu k_\lambda \log(\hat{\sigma}_\lambda^2 / \lambda) \) and \( \nu > 0 \). ERIC outperformed popular tuning parameter selection approaches in their simulations for the Adaptive Lasso, but the choice of \( \nu \) is subjective and determines the balance between fit and sparsity.

Cross Validation (CV), is the process of splitting data into training and validation sets, in which the models are fitted on the training sets and overfitting is assessed by predicting observations in the validation sets. CV takes many forms, but \( K \)-fold CV (Geisser 1975; Allen 1974; Stone 1974) is arguably the most common. In \( K \)-fold CV, the data are partitioned into \( K \) sets, or folds, of size \( n_k \) each. For each \( \lambda \), \( K \) sets of estimates are generated using \( K - 1 \) of the \( K \) folds and predictions are generated for the remaining fold, denoted \( \hat{y}_{\lambda,k} \). Prediction error is calculated for each \( \lambda \) and fold as \( \frac{1}{n_k} ||y_k - \hat{y}_{\lambda,k}||_2^2 \) and is averaged across the \( K \) folds to give the average prediction error (APE) for each \( \lambda \). The \( \lambda \) with the minimum APE is selected, or a 1SE rule—which chooses the largest \( \lambda \) within one standard error of the minimum APE—is implemented. The use of a 1SE rule is most common for a one-dimensional \( \lambda \), but Stallrich et al. (2020) proposed a multidimensional extension where the \( \hat{\beta}_\lambda \) chosen lies withing one standard error of the minimum and also minimizes some penalty function. \( K \)-fold CV still has a tendency towards overselection, even with a 1SE rule (Krstajic et al. 2014).

Generalized Cross Validation (GCV) is an efficient alternative to \( n \)-fold Cross Validation (Craven and Wahba 1978; Golub et al. 1979). GCV is appropriate when the estimation
procedure admits linear predictions $\hat{y} = Sy$ for some matrix $S$ (Hastie et al. 2017). For example, in Ridge Regression, $S = X((X^TX)^{-1} + \lambda I)X^T$. GCV minimizes a function that divides $L(\hat{\beta}_\lambda) = ||y - \hat{y}_\lambda||_2^2$ by a function of $k_\lambda$, the effective degrees of freedom. However, just like with other information criteria, GCV has been shown to lead to overselection (Homrighausen and McDonald 2018).

2.3 Post-selection Inference

Post-selection inference techniques carry out further variable selection after a tuning parameter value has been selected (van de Geer et al. 2014; Javanmard and Montanari 2013; Taylor and Tibshirani 2015; Lee et al. 2016; Shi et al. 2020). The Debiased Lasso estimator is $\frac{1}{n} \Theta X^T (y - X\hat{\beta}_\lambda)$, where $\Theta$ is an estimate of $(X^TX)^{-1}$, to $\hat{\beta}_\lambda$ (Zhang and Zhang 2014). This method focuses on estimation bias rather than direct variable selection, so the use of an additive bias correction is coherent; estimates for zero effect coefficients are small and hence their bias is small. The correction leads to an approximate Normal distribution of $\hat{\beta}_\lambda$ so one may perform hypothesis testing and construct confidence intervals.

The Covariance Test (Lockhart et al. 2014) takes advantage of the LARS algorithm, which constructs the Lasso solution path by adding variables one at a time (Efron et al. 2004). It is distinctive in that it assesses model fit using covariance rather than squared error loss. Suppose the knots of the LARS algorithm are $\lambda_1 > \lambda_2 > ... \lambda_L$ and $\epsilon_i \overset{iid}{\sim} N(0, \sigma^2)$. To test the significance of the predictor which first enters the model at $\lambda_l$, we refit the Lasso, setting $\lambda = \lambda_{l+1}$ and using $M_{\lambda_{l-1}}$ as predictors. This gives coefficient estimates that lead to predictions for the response, $\hat{y}^*$. The test statistic is:

$$F = \frac{1}{\hat{\sigma}^2} \left( y^T \hat{y}_{\lambda_{l+1}} - y^T \hat{y}^* \right)$$

where $\hat{\sigma}^2$ is the mean squared error of a model including all parameters. P-values are found
using the $F$-distribution with degrees of freedom 2 and $n - p$. The test requires estimation of $\sigma^2$ and its extension to non-Lasso penalties is not well studied, but it provides precedence for the use of correlation in the evaluation of model fit for variable selection.

In general, when the tuning parameter selection event can be written as $\{Ay \leq b\}$ for some matrix $A$ and vector $b$, there exists a general scheme for post-selection inference that gives exact confidence intervals and p-values for Gaussian errors. Choose $\eta$ such that inference about $\eta^T E[y]$ is of interest. Lee et al. (2016) and Lockhart et al. (2014) use the polyhedral lemma for Gaussian errors to represent this event in terms of $\eta^T y$ to perform conditional inference. This allows for inference upon multiple $\lambda$ or a fixed $\lambda$. When used for successive steps of LARS, it is known as the Spacing Test (Tibshirani et al. 2014) and is a non-asymptotic version of the Covariance Test.

3 Methodology

3.1 Why Correlation Over Average Prediction Error?

Distinguish the magnitude of $\beta^*$, denoted $\alpha^* = ||\beta^*||_2$, from its direction, $\xi^* = \beta^*/\alpha^*$. Then $\xi^*$ contains all information about $\mathcal{M}^*$. The same summaries may be computed from a penalized estimate, $\hat{\beta}_\lambda$, denoted by $\hat{\alpha}_\lambda$ and $\hat{\xi}_\lambda$. Comparing $\mathcal{M}_\lambda$ to $\mathcal{M}^*$ is equivalent to comparing the supports of $\hat{\xi}_\lambda$ and $\xi^*$. A tuning parameter selection strategy based on squared prediction error, however, concerns both magnitude and direction. Let $\{y, X\}$ denote a holdout sample where $y$ and $X$ have been centered. The squared prediction error for $y_i = x_i^T \beta^* + \epsilon_i$ is

$$\sum_i (y_i - x_i^T \hat{\beta}_\lambda)^2 = \sum_i (\epsilon_i + x_i^T (\alpha \xi^* - \hat{\alpha}_\lambda \hat{\xi}_\lambda))^2.$$  (4)
In the ideal situation where $\hat{\xi}_\lambda = \xi^*$, (4) will be inflated when $\hat{\alpha}_\lambda \neq \alpha^*$. Indeed, for penalized estimators typically $\hat{\alpha}_\lambda < \alpha^*$. Therefore, it is possible for an estimate having $\hat{\xi}_\lambda = \xi^*$ to have a large APE and so would be unlikely to be chosen by an APE-based tuning parameter selection strategy.

Consider now the correlation between $y$ and its prediction $\hat{y}_\lambda = X\hat{\beta}_\lambda$. After some simplification, we get the expression

$$\text{Corr}(y, \hat{y}_\lambda) = \frac{(X\xi^* + \tilde{e})^TX\hat{\xi}_\lambda}{||X\xi^* + \tilde{e}||_2 ||X\xi^*||_2}$$

(5)

where $\tilde{e} = e/\alpha^*$ is a scaled error vector that does not depend on $\hat{\beta}_\lambda$. The $\hat{\alpha}_\lambda$ has no influence over this summary so this measure better compares the $\xi^*$ and $\hat{\xi}_\lambda$, and hence better assesses support recovery than squared prediction error. This is especially so for penalized estimators that struggle with estimating $\alpha^*$. Our proposed tuning parameter selection strategy, called AR2 CV, employs $K$-fold CV with folds $\{y_k, X_k\}$ but replaces APE with

$$\text{AR2} = \frac{1}{K} \sum_{k=1}^{K} 1 - \text{Corr}(y_k, \hat{y}_{\lambda,k})^2.$$  

(6)

The optimal $\lambda$ may be chosen as the one that minimizes AR2, but we have found significant improvements in support recovery under an analogous one-standard-error rule.

Applying AR2 CV with a 1SE rule to the toy example in Section 1, the optimal $\lambda$ is that given in Figure 1(d). The AR2 CV estimator was then able to compromise between support recovery and prediction error, while the APE-based CV prioritized prediction error. The coefficient estimates for the $j \in M^*$ under the APE CV model exhibit less bias than those of the AR2 CV model. A potential drawback then of AR2 CV is that its ignorance to $\hat{\alpha}_\lambda$ may lead to a $\hat{\beta}_\lambda$ that exhibits more shrinkage than is desired. A potential remedy is to proceed selection of $M_\lambda$ with unpenalized estimation for only predictors in $M_\lambda$. We next
discuss an alternative strategy that is related to AR2 CV that adjusts the shrinkage of $\hat{\beta}_\lambda$.

### 3.2 The $\alpha$-Modification

Consider now the in-sample $y$ and their predictions, $\hat{y}_\lambda$. If $\hat{y}_\lambda \neq 0$, calculate the least-squares estimate $\hat{\alpha}_\lambda = \arg\min_{\alpha}||y - \alpha \hat{y}_\lambda||_2^2 = (\hat{y}_\lambda^T \hat{y}_\lambda)^{-1} \hat{y}_\lambda^T y$. Note this $\hat{\alpha}_\lambda$ likely differs from $||\hat{\beta}_\lambda||_2$. By definition, the predictions $\hat{\alpha}_\lambda \hat{y}_\lambda$ will be as close or closer to $y$ as $\hat{y}_\lambda$. Note these $\hat{\alpha}_\lambda \hat{y}_\lambda$ are equivalent to prediction under the adjusted penalized estimate, $\hat{\alpha}_\lambda \hat{\beta}_\lambda$, and this estimate results from minimizing the biconvex objective function

$$\frac{1}{2n} ||y - \alpha X \xi||_2^2 + \lambda P(\xi)$$

by first fixing $\alpha = 1$ to estimate $\hat{\xi} = \hat{\beta}_\lambda$ and then minimizing the function for $\alpha$ given $\hat{\xi}$.

Note that as $\lambda \to 0$, $\hat{\alpha}_\lambda \to 1$ because the objective function focuses most of its attention on the loss function. Thus the impact of the $\alpha$-Modification is more noticeable for larger values of $\lambda$. For such $\lambda$, the hope is that the $\alpha$-Modified estimate will correct the bias of $\hat{\beta}_\lambda$ due to shrinkage. We now study the properties of these $\alpha$-Modified estimates.

The $\alpha$-Modification is similar to many other existing methods. First, one can view the modification as reversing the process of calculating the Nonnegative Garrote estimator, which starts with OLS estimates and then performs penalization. Zou and Hastie (2005) also recommended a multiplicative adjustment to the Elastic Net estimator, although the adjustment only involves one of the tuning parameters. For penalties that satisfy $P(\alpha \xi) = \alpha P(\xi)$ for $\alpha > 0$, we may rewrite the penalty in (7) as $\lambda \alpha^{-1} P(\alpha \xi)$ which resembles the Relaxed Lasso penalty, so long as $\alpha^{-1} \in (0, 1]$. Finally, the Debiased Lasso tries to reduce bias through an additive adjustment, but this will may some $\hat{\beta}_{\lambda,j} = 0$ to become nonzero. The multiplicative adjustment we propose does not change the support of $\hat{\beta}_\lambda$.

Penalized estimates are typically shrunk towards zero so the $\alpha$-Modification will correct
this type of bias only if $\hat{\alpha}_\lambda \geq 1$. This property is guaranteed for common penalties:

**Theorem 1.** Suppose $P_\lambda(\beta) = \sum_{\ell=1}^{L} \lambda g_\ell(\beta)$ where $g_\ell(\beta)$ is convex and minimized at $0_p$. Then $\hat{\alpha}_\lambda \geq 1$ when $\hat{\beta}_\lambda \neq 0$.

Amplifying penalized estimates does not necessarily decrease bias. To evaluate the $\alpha$-Modification as a bias-reduction tool, we have the following result.

**Lemma 1.** If there exists a $\lambda$ where $\hat{\xi}_\lambda = \xi^*$ as defined in Section 3.1, then $E(\hat{\alpha}_\lambda \hat{\beta}_\lambda) = \beta^*$.

Lemma 1 conditions on an event that may have probability 0. The following lemma considers a broader condition, whereby the penalized estimate recovers the direction of an OLS estimate under the submodel $\mathcal{M}_\lambda$, denoted by $\hat{\beta}_\lambda^{\mathcal{M}_\lambda}$. Here $\hat{\beta}_\lambda^{\mathcal{M}_\lambda}_{OLS,j} = \hat{\beta}_{OLS,j}$ when $j \in \mathcal{M}_\lambda$ and zero otherwise.

**Lemma 2.** If $\hat{\beta}_\lambda \propto \hat{\beta}_\lambda^{\mathcal{M}_\lambda}_{OLS}$ then $\hat{\alpha}_\lambda \hat{\beta}_\lambda = \hat{\beta}_\lambda^{\mathcal{M}_\lambda}$. There are multiple examples that satisfy the condition of Lemma 1. The OLS estimator itself qualifies as a scaled OLS estimator, where $\hat{\alpha}_\lambda = 1$. Ridge estimates when $X^T X = nI_p$ also take this form, having $\hat{\beta}_\lambda = \frac{1}{1+\lambda} \hat{\beta}_{OLS}$. Lemma 2 also applies whenever $\hat{\beta}_\lambda$ contains exactly one non-zero entry. Finally, note the Relaxed Lasso always includes $\hat{\beta}_{\lambda,\phi} = \hat{\beta}_\lambda^{\mathcal{M}_\lambda}_{OLS}$ among its solutions by setting $\phi = 0$. Lemma 2 shows that this is can sometimes occur for the $\alpha$-Modified estimates as well.

The $\alpha$-Modification serves to improve predictions under a given $\hat{\beta}_\lambda$ through a positive, multiplicative adjustment and so its ability to improve estimation depends on the properties of $\hat{\beta}_\lambda$. While this paper is mainly concerned with tuning parameter selection for finite sample sizes, properties of $\hat{\beta}_\lambda$ are easier to study as $n \to \infty$ and the same is true for $\alpha$-Modified estimators. Concerning model selection consistency, since the support of $\hat{\alpha}_\lambda \hat{\beta}_\lambda$ equals the support of $\hat{\beta}_\lambda$, the $\alpha$-Modified estimator is model selection consistent if and only if $\hat{\beta}_\lambda$ is model selection consistent. The following theorem establishes estimation consistency of $\alpha$-Modified estimators.
**Theorem 2.** Fix $p$ and suppose there exists a positive definite matrix $C$ where $\frac{1}{n}X_n^T X_n = \frac{1}{n} C_n \to C$ as $n \to \infty$. For a $P_\lambda(\cdot)$, if there exists a $\lambda_n$ where $\hat{\beta}_{\lambda_n} \to_p c\xi^*$ for some $c > 0$, then $\hat{\alpha}_{\lambda_n} \hat{\beta}_{\lambda_n} \to_p \beta^*$.

We hypothesize that the rate of convergence for the $\alpha$-Modified estimators is generally faster than the unmodified versions, but leave the proof for future investigation.

### 3.3 $\alpha$-Modified Cross Validation

In addition to AR2 CV, we propose $\alpha$-Modified CV based on average prediction error under the $\alpha$-Modified estimates. Returning to the simulated data example from Section [1] the $\lambda$ chosen through $\alpha$-Modified CV with a 1SE Rule returns a model with $\mathcal{M}_\lambda = \mathcal{M}^*$. Its APE is 0.3821, which is marginally larger than that of AR2 CV. This demonstrates the possibility that the two CV strategies may point to different optimal $\lambda$.

One benefit of $\alpha$-Modified CV over traditional CV surprisingly derives from the potential drawbacks of the modification. As the modification unshrinks the $\hat{\beta}_\lambda$ without changing its direction, the modification may exacerbate an estimate of poor quality, causing its corresponding APE to be greater than APE based on $\hat{\beta}_\lambda$. Similarly, when $\hat{\beta}_\lambda$ recovers $\xi^*$ but has small magnitude, the modified APE will decrease significantly over the traditional APE. Theorem [8] gives a theoretical justification for the latter situation.

**Theorem 3.** Suppose that $\hat{\beta}_\lambda$ recovers the direction of $\beta^*$ and assume all $\epsilon_i$ are independent with constant variance $\sigma^2$. Then the expected value of the $\alpha$-Modified APE is less than the expected value of the traditional APE when:

$$\frac{\alpha^2}{E[\|\hat{\beta}_\lambda\|_2 - \alpha]^2]} \leq \frac{\beta^{*T} X^T X \beta^*}{\sigma^2}.$$  

(8)

The $\alpha$-Modified APE is then expected to be smaller than the traditional APE as long as the signal-to-noise ratio, represented by $\beta^{*T} X^T X \beta^*/\sigma^2$, is sufficiently large.
4 The α-Modified Lasso

The methods in Section 3 generalize to many types of penalties, but to better understand their properties we must focus on a specific penalty. Due to its popularity, we explore the properties for the Lasso penalty and finite $n$. First, note that the α-Modified Lasso may be thought of as a reverse Non-negative Garotte in that it starts with shrunken estimates and uses a least squares modification to readjust and “un-shrink” them. The use of correlation to assess model quality is also consistent with the premise of the Covariance Test. The α-Modified Lasso is most similar to the Relaxed Lasso. The α-Modification penalty is $\lambda \alpha^{-1}||\alpha \xi||_1$ and the Relaxed Lasso penalty is $\lambda \phi ||\beta||_1$ for $\phi \in (0, 1]$. Theorem 6 establishes $\alpha^{-1} \in (0, 1]$ but α is estimated while the Relaxed Lasso treats $\phi$ as a tuning parameter.

While estimating $\hat{\alpha}_\lambda$ reduces computations, the possible estimators are less flexible than the Relaxed Lasso. Specifically, the α-Modification cannot change the direction of $\hat{\beta}_\lambda$.

Figure 2 illustrates the distinction between the α-Modified and Relaxed Lasso estimator for an orthogonal $X$ with $n = 100$, $p = 50$ and $p^* = 2$ where $(\beta_1^*, \beta_2^*) = (-6, 6)$. The α-Modified Lasso and Relaxed Lasso solutions were generated for a set of 100 $\lambda$ and 20 $\phi$ values. Figure 2(a) simultaneously shows the original Lasso estimates and the α-Modified estimates for $(\beta_1^*, \beta_2^*)$ across all $\lambda$ values. Figure 2(b) shows the Relaxed Lasso estimates for $(\beta_1^*, \beta_2^*)$ at a fixed $\lambda$ and across the $\phi$ values. Note how the α-Modified estimates amplify the Lasso estimates in the same direction, while the Relaxed Lasso is able to change the direction of the original Lasso estimate. However, the α-Modified estimates do seem to approximate the Relaxed Lasso estimates as $\lambda$ is varied.

We now derive properties of the α-Modified estimator for orthogonal $X$. The Lasso estimator for such $X$ is $\hat{\beta}_{\lambda,j} = \text{sign}(\hat{\beta}_{OLS,j})(|\hat{\beta}_{OLS,j}| - \lambda)_+$, where $\hat{\beta}_{OLS,j}$ is the OLS estimate of the $\beta_j^*$ and $(z)_+ = \max(0, z)$. If the Lasso estimate for a $j^*$ is non-zero, the closed form expression for $\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*}$ may be derived.
Figure 2: Lasso, Modified Lasso, and Relaxed Lasso estimates for two active coefficients in simulated data. The dot indicates the true values of the coefficients, and estimates at \( \lambda = 0.429, \phi = 1 \) drawn in grey. Shorter vectors correspond to larger tuning parameters.

Lemma 3. When \( X \) is orthogonal, the \( \alpha \)-Modified Lasso estimator is:

\[
\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} = w_1 \hat{\beta}_{\text{OLS},j^*} + (1 - w_1) \hat{\beta}_{j^*,\lambda} + w_2 \hat{\beta}_{j^*,\lambda}
\]

where \( w_1 = \frac{d_{j^*}^2}{\sum_{j \neq j^*} d_j^2} \), \( w_2 = \frac{\lambda \sum_{j \neq j^*} d_j}{\sum_{j = 1}^{p} d_j^2} \), and \( d_j = (|\hat{\beta}_{\text{OLS},j}| - \lambda)_+ \).

If \( d_{j^*} = 0 \), then \( \hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} = \hat{\beta}_{\lambda,j^*} = 0 \). If \( \hat{\alpha}_\lambda \hat{\beta}_{\lambda,j} = 0 \) for all \( j \neq j^* \) and \( d_{j^*} > 0 \) then \( \hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} = \hat{\beta}_{\text{OLS},j} \) which is consistent with Lemma 3. When \( |M_\lambda| > 1 \), (9) involves a convex combination of the OLS and Lasso estimates, as well as an additive term, \( w_2 \hat{\beta}_{j^*,\lambda} \), that may overcorrect the \( \alpha \)-Modified Lasso beyond the OLS estimate. This behavior is illustrated for a simple example in Figure 3. In that example, for \( \lambda < 10 \) there are potential values of \( \hat{\beta}_{\text{OLS},j^*} \) where the \( \alpha \)-Modified Lasso exceeds \( \hat{\beta}_{\text{OLS},j^*} \). To better understand this behavior, Theorem 4 provides an upper bound for \( |\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{\text{OLS},j^*}| \).

Theorem 4. Suppose \( X \) is orthogonal and consider a given predictor, \( j^* \), and \( \lambda \) where at least one \( j \neq j^* \) has \( |\hat{\beta}_{\text{OLS},j}| > \lambda \). Then across all possible \( \hat{\beta}_{\text{OLS},j^*} \),

\[
|\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{\text{OLS},j^*}| \leq \lambda \times \max \left( 1, \frac{\sqrt{u^2 v + v^2} - v}{2v} \right)
\]

where \( u = \sum_{j \neq j^*} d_j \) and \( v = \sum_{j \neq j^*} d_j^2 \). Moreover, \( \lim_{|\hat{\beta}_{\text{OLS},j^*}| \to \infty} |\hat{\alpha}_\lambda \hat{\beta}_{j^*,\lambda} - \hat{\beta}_{\text{OLS},j^*}| \to 0 \).
Figure 3: \(\alpha\)-modified estimates of \(\alpha\beta_1\) as \(\hat{\beta}_{OLS,1}\) changes, all other parameters being fixed. \(\hat{\beta}_{OLS,j} = [-8, 5, -3, 1]\) for \(j = [2, ..., 5]\). The dotted line gives the Lasso estimates and the solid line gives the \(\alpha\)-modified estimates.

Based on Theorem 9, we see that when \((\sqrt{u^2v + u^2} - v)/2v > 1\) there exists some \(\hat{\beta}_{OLS,j}^*\) in which the \(\alpha\)-Modified estimate blows up \(\hat{\beta}_{\lambda,j}^*\), thereby introducing bias in the opposite direction. This occurs as \(u^2/v\) increases, meaning multiple \(d_j\) are nonzero and are close to 0. This will likely occur for relatively large \(\lambda\) values and certain patterns of \(\beta^*\).

To generalize Theorem 9 to an arbitrary \(X\) we condition on the event that the \(\lambda\) recovers the sign vector of \(\beta^*\). Theorem 10 shows that the absolute difference between the \(\alpha\)-Modified Lasso estimate and the OLS estimate generally do not approach 0 as \(|\hat{\beta}_{OLS,j}^*| \to \infty\). Rather, they approach a non-zero constant that is always smaller than \(|\hat{\beta}_{\lambda,j}^* - \hat{\beta}_{OLS,j}^*|\), indicating the \(\alpha\)-Modified estimate improves \(\hat{\beta}_{\lambda,j}^*\).

**Theorem 5.** Suppose the Lasso recovers the correct sign vector of \(\beta^*\). Let \(s_1\) be the sign vector of active coefficients, and let \(X_1\), the active subset of \(X\), have full column rank. Let \(G = n\lambda (s_j^* - ((X_1^T X_1)^{-1})_{j, s_1})\). As \(|\hat{\beta}_{OLS,j}^*| \to \infty\) for \(\hat{\beta}_{\lambda,j}^* \neq 0\):

\[
|\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j}^* - \hat{\beta}_{OLS,j}^*| \to G < |\hat{\beta}_{\lambda,j}^* - \hat{\beta}_{OLS,j}^*|.
\] (11)

Theorems 9 and 10 confirm that the \(\alpha\)-Modification cannot generally guarantee bias reduction. However, a poor bias adjustment may be detected by CV. To illustrate, we performed a simulation study for the \(X\) used in Figure 1 and considered two \(\beta^*\) vectors. For each \(\beta^*\), we generated 500 \(\epsilon \sim N(0, I)\) and averaged the CV results, including APE, \(\alpha\)-Modified APE, and AR2. The results are presented in Figure 4.
\[
\begin{align*}
(a) \quad & \beta^* = (4.3, 2.9, -5.8, -3.3, -6.1, 0, \ldots, 0)^T. \\
(b) \quad & \beta^* = (50, 50, 50, 50, 0, \ldots, 0)^T.
\end{align*}
\]

**Figure 4:** Average 10-fold Cross Validation results across 500 replications of linear data. \(X\) was taken from the example in Section 1. Average values of \(\lambda\) where the correct submodel was selected are represented by the grey box, the dashed line gives the APE, the solid line gives \(\alpha\)-Modified APE, and the dotted line gives Average \(R^2\). Standard errors from these simulations were too small to be depicted.

In Figure 4(a) the \(\alpha\)-Modified APE is less than or equal to APE for all \(\lambda\), and especially so for larger \(\lambda\) that recover \(M^*\). The optimal \(\lambda\) according to the minimum for both \(R^2\) and \(\alpha\)-Modified APE result in sparser models than the minimum APE. With a 1SE rule, traditional CV recovered the support in only 24.4\% of the replications, whereas \(\alpha\)-Modified and \(R^2\) CV did so in 91.6\% and 93.6\% of the replications, respectively.

Figure 4(b) corresponds to \(\beta^*\) with larger and equal magnitude coefficients. Generally the \(\alpha\)-Modified APE is equal to or smaller than regular APE except for \(\log(\lambda) \approx 4\). These models exhibit false negatives and \(\hat{\alpha}_\lambda\) tends to overamplify the few active coefficients in the wrong direction, leading to poor prediction. Fortunately, the \(\alpha\)-Modified APE highlights this issue and will not choose such \(\lambda\). The support recovery percentages were 19.8\% for APE CV, 98.2\% for \(\alpha\)-Modified CV, and 98.8\% for \(R^2\) CV. Surprisingly, the support recovery percentage decreased for APE CV despite a higher signal to noise ratio.

### 5 Numerical Results

To evaluate the new methods for the Lasso, we conducted a simulation study consisting of 100 replications of data. Following Meinshausen (2007), the rows of \(X\) were drawn from a multivariate Normal distribution with mean \(0\) and covariance \(\Sigma_X\). For each \(X\),
nonzero elements of \( \beta^* \) were drawn from a Gamma distribution with shape 10 and scale 0.25 with negative and positive signs randomly assigned with equal probability. For each \( X \) and \( \beta^* \), two independent error vectors were generated from \( N(0, \sigma^2) \) distributions, with \( \sigma \) determined by a fixed signal-to-noise ratio \( \text{SNR} = \beta^* \Sigma \beta^* / \sigma^2 \).

Each simulated data set was analyzed using 10-fold CV with and without a 1SE rule under traditional APE (CV Min, CV 1SE), Average \( R^2 \) (AR2 Min, AR2 1SE), \( \alpha \)-Modified APE (Mod Min, Mod 1SE), and the Relaxed Lasso. Meinshausen (2007) recommended choosing tuning parameters for the Relaxed Lasso based on the minimum APE. We implemented this as well as a 1SE Rule based on Stallrich et al. (2020), wherein the optimal combination of \( \phi \) and \( \lambda \) is the model with the smallest \( ||\hat{\beta}_{\lambda,\phi}||_1 \) among all models within one standard error of the minimum. Relaxed Lasso estimates were found using a coordinate descent algorithm, capable of admitting more than \( n - 1 \) covariates into models, unlike LARS which was used by Meinshausen (2007). Simulations are evaluated using false discovery rate (FDR), average number of false positives, and average prediction bias, where prediction bias is \( ||X\beta^* - X\hat{\beta}_\lambda||_2 \) or \( ||X\beta^* - X\hat{\alpha}\hat{\beta}_\lambda||_2 \) for the \( \alpha \)-Modification.

5.1 Simulation Study for Independent Predictors

We first consider the case of \( \Sigma_X = I_p \). Tables 1, 2, and 3 give the average number of active coefficients for the traditional APE CV technique with a 1SE Rule, AR2 CV with a 1SE rule, and the Relaxed Lasso with a minimum APE CV approach. AR2 CV and Mod APE CV perform fairly similarly; the latter results may be found in the supplementary materials. For all methods, false negatives were close to or exactly zero, so it is reasonable to evaluate variable selection properties using model size. In general, AR2 CV returns sparser models than APE CV and, for small \( p^* \), the Relaxed Lasso.
Table 1: Average number of selected variables 100 replications of CV 1SE. Standard errors given in parentheses.

| n   | p 50 | 100 | 200 | 400 | 800 | p 50 | 100 | 200 | 400 | 800 |
|-----|------|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|
| 50  | 6.4  | 10.6| 9.7 | 11.5| 11.2| 11   | 12.7| 13.9| 17.9| 18.6|
|     | (0.6)| (1.2)| (1.3)| (1.4)| (1.6)| (0.8)| (0.8)| (0.9)| (1.2)| (1.5)|
| 100 | 6.8  | 8.7 | 9.9 | 10.3| 12  | 7.8  | 9.8 | 11.6| 14.5| 15.7|
|     | (0.3)| (0.7)| (1) | (1.3)| (1.5)| (0.4)| (0.6)| (0.7)| (1.1)| (1.4)|
| 200 | 6.5  | 6.8 | 8   | 9   | 9.8 | 6.6  | 7.6 | 8.6 | 10.3| 11.3|
|     | (0.2)| (0.2)| (0.4)| (0.5)| (0.6)| (0.2)| (0.3)| (0.4)| (0.9)| (0.9)|

Table 2: Average number of selected variables 100 replications of AR2 CV 1SE. Standard errors given in parentheses.

| n   | p 50 | 100 | 200 | 400 | 800 | p 50 | 100 | 200 | 400 | 800 |
|-----|------|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|
| 50  | 7.8  | 9   | 6.1 | 5.8 | 5.8 | 22.2 | 14.8| 10.7| 10  | 8.4 |
|     | (1.1)| (1.3)| (1.3)| (1.3)| (1.2)| (1.4)| (1.5)| (1.5)| (1.5)| (1.5)|
| 100 | 19.2 | 11.6| 8.6 | 6.8 | 7   | 43.6 | 44.3| 28.4| 20  | 12.6|
|     | (1.2)| (1.6)| (1.6)| (1.6)| (1.7)| (0.5)| (1.9)| (2.4)| (2.5)| (2.2)|
| 200 | 39.1 | 37.8| 33.7| 20.6| 15  | 49.2 | 68.8| 82.8| 80.9| 58.1|
|     | (0.6)| (1.4)| (2.2)| (2.2)| (2.6)| (0.1)| (0.6)| (1.4)| (2.4)| (3) |

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Table 3: Average number of selected variables 100 replications of Relaxed Lasso. Standard errors given in parentheses.

| n   | $p^* = 5; SNR = 1.25; \bar{\sigma} = 5.23$ | $p^* = 5; SNR = 5; \bar{\sigma} = 2.61$ |
|-----|------------------------------------------|------------------------------------------|
| 50  | 9.1 (0.8) 22.4 (2.2) 30.4 (2.7) 31.8 (2.7) 36.5 (2.9) | 12.2 (1.1) 19.5 (2) 27.5 (2.6) 39.9 (2.7) 43 |
|     | 7.8 (0.6) 10.5 (1.3) 11.5 (1.6) 18.6 (3.2) 30.2 (4.4) | 6 (0.2) 7.2 (0.6) 8.2 (1) 13.7 (2.8) 13.4 |
| 100 | 6.5 (0.3) 6.5 (0.4) 7.2 (0.9) 8.3 (1.1) 8.7 (1.7) | 5.6 (0.2) 5.7 (0.2) 6.8 (1) 5.4 (2.8) 5.6 |
| 200 | 16.1 (1.3) 27.4 (2.5) 20 (2.4) 25.8 (2.7) 26.8 (2.8) | 27.9 (1.4) 35.7 (2.5) 43.1 (2.6) 36.4 (2.7) 28.1 |
|     | 31 (1.1) 26.2 (2.1) 37.5 (4.4) 41.6 (4.9) 38.8 (4.8) | 46.7 (0.5) 57.4 (1.8) 81.5 (4.5) 77.6 (5.6) 61 |
| 200 | 45.1 (0.4) 56.8 (1.5) 60.5 (2.7) 61.7 (5.3) 62 (7.8) | 49.4 (0.1) 71 (1.1) 92 (2.3) 159.9 (8.5) 205.7 |
|     |                                           |                                           |

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We considered more granular results for a wider set of methods under the $n = 100$ and $p = 100$ setting in Figure 5. More methods and results may be found in the supplementary materials. In the high SNR case, the two new methods almost always have fewer false positives than CV 1SE, and Mod CV maintains this pattern for low SNR as well. Mod CV consistently performs similarly to Relaxed 1SE and out performs the CV 1SE in terms of variable selection.

![Graphs showing average false positives and prediction error plots](image)

**Figure 5:** Average false positives and prediction error plots from 100 replications of simulated data with $n = 100$ and $p = 100$ for both SNR = 1.25 and SNR = 5. Thin dotted lines represent the mean ± one standard error.

Unlike the Relaxed Lasso, the $\alpha$-Modification is straightforward to apply to other penalties. We also compared $\alpha$-Modified Lasso with the non-convex penalties SCAD and MCP, both traditionally and with an $\alpha$-Modification. As expected, the $\alpha$-Modification had a minimal effect on the predictive models. MCP and SCAD estimates were prone to underselection whereas the $\alpha$-Modified Lasso estimates were more prone to overselection. Details may be found in the supplementary materials.
5.2 Simulation Study for Correlated Predictors

Here the $\Sigma_X$ satisfied $\Sigma_{ii} = 1$ and $\Sigma_{ij} = 0.75$ whenever $i \neq j$. Figure 6 gives the average false positives and prediction error for 100 replications with $n = p = 100$, but further results can be found in the supplementary material. The two new methods of CV are highly competitive with the Relaxed 1SE, whereas Relaxed Min has a tendency to overselect as $p^*$ increases. For lower SNR, Mod 1SE has even fewer False Positives than Relaxed 1SE and almost identical average prediction bias. When SNR is larger and signals are more easily distinguishable, the differences between Relaxed 1SE, Mod 1SE, and AR2 1SE are negligible, in both Average False Positives and Average Prediction Bias. These results suggest that, for correlated $X$, the $\alpha$-Modified CV approach is the best alternative.

6 Optimal EMG Placement for a Robotic PC

For the EMG application introduced in Section 1, we want to identify as few EMG sensors as needed to reliably predict hand movement. The data were collected from an able-bodied
subject and consist of concurrent measurements of the subject’s finger position and 16 EMG signals as predictors. A full description of the data, some of its challenges, and SAFE can be found in Stallrich et al. (2020). Six data sets were analyzed, corresponding to three consistent finger movement patterns (FC1, FC2, FC3) and three random patterns (FR1, FR2, FR3). As data were collected from an able-bodied subject, it was known that three of the 16 sensors, denoted $X_5$, $X_7$, $X_{12}$, targeted muscles known to fully explain finger movement. Sensors $X_5$ and $X_7$ collected information from the same muscle, however, and so only one of the pair is necessary to predict finger position. An ideal model would thus include $X_{12}$ and either $X_5$ or $X_7$, but recovery of all three sensors is also acceptable.

Due to known biomechanical features of hand movement, Stallrich et al. (2020) use finger velocity as the response and treat the recent past EMG signals as functional covariates. The model is:

$$y_i = \sum_{j=1}^{16} \int_{-\delta}^{0} X_{ij}(t) \gamma_j(t, z_i) dt + \epsilon_i$$

where $y_i$ is the velocity, $X_{ij}(t)$ represents past EMG signals, $t \in [-\delta, 0]$ is the recent past time, and $z_i$ is the recent finger position. Following Gertheiss et al. (2013), Stallrich et al. (2020) proposed a penalized estimation procedure where the penalty accounted for both sparsity and smoothness of the $\gamma_j(\cdot, \cdot)$:

$$P_\lambda(\gamma_j) = \lambda f_j ||\gamma_j||^2_2 + g_j \lambda_t ||\gamma_j''||^2_2 + h_j \lambda_z ||\gamma_j''||^2_2$$

where $||\gamma_j||^2_2 = \int \int \gamma_j(t, z_i)^2 dtdz$ and $\gamma_j'' = \partial^2 \gamma_j(t, z_i)/\partial t^2$. There are three tuning parameters, $\lambda, \lambda_t, \lambda_z$, and adaptive weights $f_j, g_j, h_j$. To facilitate estimation, the $\gamma_j(\cdot, \cdot)$ were written using a tensor product basis expansion, leading to a Group lasso-type penalty; more details may be found in Stallrich et al. (2020) and the supplementary materials.

To perform variable selection, Stallrich et al. (2020) proposed Sequential Adaptive Functional Estimation (SAFE) that performs penalized estimation in stages. The first stage set
$f_j = g_j = h_j = 1$ and chose optimal tuning parameters based on an APE 1SE rule following 10-fold CV. Let $\mathcal{M}_{\lambda,1}$ denote the support of this estimator. Adaptive weights were updated based on the estimates of the nonzero effects and penalized estimation was performed again using these weights and only those $j \in \mathcal{M}_{\lambda,1}$. The process was repeated for up to 5 stages. While effective in identifying the correct submodel, the analysis can be very time consuming. We modified their method based on AR2 CV and modified APE for this application in hopes to reduce the number of stages required to perform variable selection.

Table 4 gives the variable selection results for the three CV methods based on the initial stage; results from subsequent stages of SAFE can be found in the supplementary materials. AR2 CV and Mod APE generally give smaller models than traditional APE CV, with one exception: the Average $R^2$ method has a large model size for FC3. On average, however, both new methods have fewer false positives and smaller model sizes. Mod APE gives very similar results to APE, suggesting that the $\alpha$-Modified CV approach requires further study for the Group Lasso. Although AR2 CV is not perfect in this application, in general it reduces model size and decreases false positives at no additional computational cost.

Table 4: Variable selection results for EMG finger movements without adaptive weighting. FP indicates the total number of false positives in the model, and Size is the total number of EMG signals contained in the model.

|     | FC1 | FC2 | FC3 | FR1 | FR2 | FR3 | Mean |
|-----|-----|-----|-----|-----|-----|-----|------|
| APE | FP  | 2   | 1   | 2   | 2   | 1   | 1.5  |
|     | Size| 4   | 3   | 4   | 4   | 4   | 4    |
| AR2 | FP  | 0   | 1   | 3   | 0   | 1   | 1    |
|     | Size| 2   | 3   | 6   | 2   | 4   | 3.33 |
| Mod | FP  | 1   | 1   | 2   | 1   | 1   | 1.33 |
|     | Size| 3   | 3   | 4   | 5   | 4   | 3.83 |

7 Discussion

In this paper, we proposed AR2 CV to choose tuning parameters to balance support recovery and prediction performance. This led to the $\alpha$-Modification, a multiplicative adjustment
to predictions from penalized estimates which can also be used for $\alpha$-Modified CV. The $\alpha$-Modification is simple and efficient, making it an attractive option when less flexible approaches are unavailable. A simulation study on the capabilities of AR2 and $\alpha$-Modified CV found that their variable selection results were highly competitive with—or, in some cases, better than—the Relaxed Lasso. In order to ensure fair comparison, we introduced a 1SE Rule for the Relaxed Lasso. Finally, we applied the approaches to a functional data problem in a demonstration of their flexibility.

The $\alpha$-Modification and the tuning parameter selection methods proposed here inspire several research questions. First, further theoretical analysis of the methods is of interest. Because the two new methods of tuning parameter selection are CV-based approaches, the theoretical properties of CV are central. Theoretical justifications for the use of CV for penalized estimators are relatively new and still evolving. [Chetverikov et al. (2021)] may be extended to show that $\alpha$-Modified and AR2 CV lead to estimates that are low bias and appropriately sparse. A second area of future research is the theoretical properties of $\alpha$-Modified estimates themselves. In Theorem 7, it was shown that any consistent estimator will still be consistent with the $\alpha$-Modification. We posit that the rates of convergence for $\alpha$-Modified estimators are faster than unmodified, but proof of this conjecture is the work of future research. Similarly, many of the results from this paper assume finite sample sizes. Further study is needed to determine more of the asymptotic properties of $\alpha$-Modified estimators and to adapt the specific results given for the Lasso penalty to other penalties.

There are also a few extensions and adaptations of the $\alpha$-Modification that may prove fruitful. We are currently expanding the $\alpha$-Modification to Generalized Linear Models (GLMs). This extension will require an iterative algorithm to find estimates of $\alpha$ because closed form solutions do not exist and an accommodation for the inclusion of an intercept term will be necessary. Additionally, as noted in some of the theoretical results in this paper, the $\alpha$-Modification does not always reduce bias. We are interested in exploring a further
penalty on $\alpha$ itself to ensure bias reduction. Finally, the calculation of $\hat{\alpha}_\lambda$ described here uses in-sample predictions and observations. As overspecification is a particular concern, the question of whether out-of-sample data can be used to find estimates of $\alpha$ is another subject of further research.

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**Appendix**

**Proof of Theorem 6.** Let \( \hat{\beta}_\lambda \neq 0 \) and \( g(\beta) = \sum_{l=1}^{L} \lambda_l g_l(\beta) \). Denote the subgradient vector of \( g(\beta) \) by \( \nabla g^* \). Then the KKT conditions give \( \frac{1}{n} X^T (y - X \hat{\beta}_\lambda) = \nabla g^* \) and there exists a generalized inverse \( (X^T X)^{-1} \) where \( \hat{\beta}_\lambda = (X^T X)^{-1} (X^T y - n\nabla g^*) \). It is easy to show \( \hat{y}_\lambda^T \hat{y}_\lambda = \hat{y}_\lambda^T y - n\nabla g^T \hat{\beta}_\lambda \). Therefore:

\[
\hat{\alpha}_\lambda = 1 + \frac{n\nabla g^T \hat{\beta}_\lambda}{\hat{y}_\lambda^T \hat{y}_\lambda}.
\] (14)

Because \( \hat{y}_\lambda^T \hat{y}_\lambda, n, \) and \( \lambda \) are greater than zero, we must show \( \nabla g^T \hat{\beta}_\lambda \geq 0 \). As \( \nabla g^* \) is the subgradient of a convex function at \( \hat{\beta}_\lambda \), it satisfies

\[
\nabla g^T \hat{\beta}_\lambda \geq g(\hat{\beta}_\lambda) - g(\beta) + \nabla g^T \beta
\] (15)

for all \( \beta \). For \( \beta = 0 \), \( g(\hat{\beta}_\lambda) - g(\beta) + \nabla g^T \beta = g(\hat{\beta}_\lambda) - g(0) \geq 0 \). Hence, \( \nabla g^T \hat{\beta}_\lambda \geq 0 \). \( \square \)
Proof of Lemma 4. When $\boldsymbol{\xi}^*$ is recovered, we have $\hat{\beta}_\lambda = \|\hat{\beta}_\lambda\|_2 \boldsymbol{\xi}^*$. The response $\mathbf{y} = \alpha^* \mathbf{X} \boldsymbol{\xi}^* + \epsilon$, so:

$$\hat{\alpha}_\lambda \hat{\beta}_\lambda = \alpha^* \boldsymbol{\xi}^* + \left( \frac{\xi^T X^T \epsilon}{\xi^T X^T \xi} \right) \boldsymbol{\xi}^*. \quad (16)$$

Thus $E[\hat{\alpha}_\lambda \hat{\beta}_\lambda] = \alpha^* \boldsymbol{\xi}^* = \beta^*$. □

Proof of Lemma 5. Fix $\lambda$ and let $\hat{\beta}_\lambda = a \hat{\beta}_{OLS}^{M_\lambda}$ where $\hat{\beta}_{OLS}^{M_\lambda}$ is the OLS estimate for submodel $M_\lambda$. Let $X_{M_\lambda}$ be the submatrix of $X$ containing only the columns indexed by $M_\lambda$. Then $\hat{\beta}_{OLS}^{M_\lambda} = (X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T y$. Now:

$$\hat{\alpha}_\lambda = \frac{a y^T X_{M_\lambda} (X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T y}{a^2 y^T X_{M_\lambda} (X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T X_{M_\lambda} (X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T y} = \frac{1}{a}. \quad (17)$$

Therefore:

$$\hat{\alpha}_\lambda \hat{\beta}_\lambda = \frac{1}{a} \times a \hat{\beta}_{OLS}^{M_\lambda} = \hat{\beta}_{OLS}^{M_\lambda}. \quad (18)$$

□

Proof of Theorem 7. Consider $\hat{\alpha}_n$:

$$\hat{\alpha}_n = \frac{\frac{1}{n} \hat{\beta}_n^T X_n^T y}{\frac{1}{n} \hat{\beta}_n^T X_n^T X_n \hat{\beta}_n}. \quad (19)$$

By Continuous Mapping Theorem, the denominator of $(19)$ converges in probability to $c^2 \xi^T C \xi^*$. Expanding $\mathbf{y} = \alpha^* \mathbf{X} \boldsymbol{\xi}^* + \epsilon$ establishes the numerator converges to $\alpha c \xi^T C \xi^*$. Then $\hat{\alpha}_n \rightarrow_p \frac{a}{c}$ and $\hat{\alpha}_n \hat{\beta}_n \rightarrow \frac{a}{c} \alpha \boldsymbol{\xi}^* = \alpha \boldsymbol{\xi}^* = \beta^*$. □

Proof of Theorem 8. It is sufficient to show that $E[\text{ModPE}] < E[PE]$ where $PE = \sum_i (y_{k_i} - x_{k_i}^T \hat{\beta}_\lambda)^2$ and $ModPE = \sum_i (y_{k_i} - \hat{\alpha}_\lambda x_{k_i}^T \hat{\beta}_\lambda)^2$. We can write $\hat{\beta}_\lambda = \|\hat{\beta}_\lambda\|_2 \boldsymbol{\xi}^*$. Expand $y_{k_i} = \alpha^* x_{k_i}^T \boldsymbol{\xi}^* + \epsilon_i$ to find:

$$PE = \sum_i \left( \epsilon_i - x_{k_i}^T \xi^* (\|\hat{\beta}_\lambda\|_2 - \alpha^*) \right)^2. \quad (20)$$
Similarly,
\[
ModPE = \sum_i \left( \epsilon_i - x_k^T \xi \left( \frac{\xi^T X^T \epsilon}{\xi^T X^T \xi} \right) \right)^2. \tag{21}
\]

For \( V(\epsilon) = \sigma^2 I \)
\[
E[PE] = n_k \sigma^2 + \sum_i (x_k^T \xi)^2 E[\|\hat{\beta}_x\|_2 - \alpha^2] \tag{22}
\]
\[
E[ModPE] = E \left[ \sum_i \left( \epsilon_i - x_k^T \xi \left( \frac{\xi^T X^T \epsilon}{\xi^T X^T \xi} \right) \right)^2 \right] \tag{23}
\]
\[
= n_k \sigma^2 + \sum_i (x_k^T \xi)^2 E \left[ \epsilon^T X^T \xi^T X^T \epsilon \right] \left( \frac{\xi^T X^T \xi}{(\xi^T X^T \xi)^2} \right) \tag{24}
\]
\[
= n_k \sigma^2 + \frac{\sigma^2}{\xi^T X^T X \xi^T} \sum_i (x_k^T \xi)^2. \tag{25}
\]

For \( \beta^* = \alpha \xi^* \), \( E[PE] \geq E[ModPE] \) whenever
\[
E[\|\hat{\beta}_x\|_2 - \alpha^2] \geq \frac{\sigma^2}{\xi^T X^T X \xi^T} \iff \frac{\alpha^2}{E[\|\hat{\beta}_x\|_2 - \alpha^2]} \leq \frac{\beta^T X^T X \beta^*}{\sigma^2}. \tag{26}
\]

\(\Box\)

**Proof of Lemma 6.** When \( X \) is orthonormal,
\[
\hat{\alpha}_x \hat{\beta}_{\lambda,j}^* = \left( \frac{\sum_{j=1}^p |\hat{\beta}_{OLS,j}| (|\hat{\beta}_{OLS,j}| - \lambda)_+}{\sum_{j=1}^p (|\hat{\beta}_{OLS,j}| - \lambda)^2_+} \right) \times \text{sign}(\hat{\beta}_{OLS,j}^*) \times (|\hat{\beta}_{OLS,j}^*| - \lambda)_+ \tag{27}
\]
Letting \( d_j = (|\hat{\beta}_{OLS,j}| - \lambda)_+ \) and \( s_j^* = \text{sign}(\hat{\beta}_{OLS,j}^*) \), we can rewrite this expression as
\[
\hat{\alpha}_x \hat{\beta}_{\lambda,j}^* = w_1 \hat{\beta}_{OLS,j}^* + \hat{\beta}_{\lambda,j}^* \frac{\sum_{j \neq j^*} (d_j + \lambda) d_j}{\sum_{j=1}^p d_j^2}, \tag{28}
\]
which uses \( s_j^* |\hat{\beta}_{OLS,j}^*| = \hat{\beta}_{OLS,j}^*, \ w_1 = d_j^2 / \sum_j d_j^2, \ \hat{\beta}_{\lambda,j}^* = s_j^* d_j^*, \) and \( |\hat{\beta}_{OLS,j}| = d_j + \lambda \) whenever \( d_j > 0 \). Hence we have the expression
\[
\hat{\alpha}_x \hat{\beta}_{\lambda,j}^* = w_1 \hat{\beta}_{OLS,j}^* + (1-w_1) \hat{\beta}_{\lambda,j}^* + w_2 \hat{\beta}_{\lambda,j}^*, \tag{29}
\]
33
Proof of Theorem 9. For a given \( \lambda \) we are allowing \( \hat{\beta}_{OLS,j^*} \) to vary. For \( |\hat{\alpha}_j \hat{\beta}_{OLS,j^*} - \hat{\beta}_{OLS,j^*}| \leq \lambda \), we have \( \hat{\alpha}_j \hat{\beta}_{OLS,j^*} = 0 \) so \( |\hat{\alpha}_j \hat{\beta}_{OLS,j^*} - \hat{\beta}_{OLS,j^*}| = |\hat{\beta}_{OLS,j^*}| \leq \lambda \) so the theorem is satisfied for this range of values. Next we consider \( \hat{\beta}_{OLS,j^*} > \lambda \) and by Lemma 6, and some simplification, we have

\[
\hat{\alpha}_j \hat{\beta}_{OLS,j^*} - \hat{\beta}_{OLS,j^*} = \frac{\lambda (\sum_{j \neq j^*} d_j) \hat{\beta}_{OLS,j^*} - \lambda (\sum_{j \neq j^*} d_j^2 + \lambda \sum_{j \neq j^*} d_j)}{(\hat{\beta}_{OLS,j^*} - \lambda)^2 + \sum_{j \neq j^*} d_j^2}
\]

(30)

Let \( u = \sum_{j \neq j^*} d_j \), \( v = \sum_{j \neq j^*} d_j^2 \), and \( x = \hat{\beta}_{OLS,j^*} \). We can express (30) as \( f(x) = \frac{\lambda ux - \lambda (v + \lambda u)}{[(x - \lambda)^2 + v]} \), a differentiable function for \( x \geq \lambda \). Between \( x = \lambda \) and \( x = (u\lambda + v + \sqrt{u^2v + v^2})/u \equiv x^* \), \( f(x) \) is an increasing function bounded below by \(-\lambda\) and bounded above by

\[
\frac{\lambda}{2} \left( \sqrt{\frac{u^2}{v}} + 1 - 1 \right).
\]

(31)

For \( x > x^* \), \( f(x) \) is decreasing so we consider \( \lim_{x \to \infty} f(x) \), which is easily shown to equal \( 0 > -\lambda \). Therefore,

\[
|f(x)| \leq \lambda \times \max \left( 1, \frac{1}{2} \left( \sqrt{\frac{u^2}{v}} + 1 - 1 \right) \right),
\]

(32)

and \( \lim_{x \to \infty} |f(x)| = 0 \). For \( \hat{\beta}_{OLS,j^*} < -\lambda \), note that \( \hat{\alpha}_j \hat{\beta}_{OLS,j^*} - \hat{\beta}_{OLS,j^*} = -(\hat{\alpha}_j |\hat{\beta}_{OLS,j^*}| - |\hat{\beta}_{OLS,j^*}|) \) and so analogous arguments to the case of \( \hat{\beta}_{OLS,j^*} \) also apply. This completes the proof. \( \square \)

Proof of Theorem 10. When the sign vector of the true model, \( s \), is known there exists a closed form expression for the non-zero Lasso estimates. Let \( s_1 \) be the sign of the non-zero coefficient estimates and \( \hat{\beta}_{\lambda,1} \) be the lasso estimates for non-zero coefficients. Then \( \hat{\beta}_{\lambda,1} = (\hat{\beta}_{OLS,1} - n\lambda(X_1^T X_1)^{-1} s) \) where \( X_1 \) is the covariate matrix including all variables.
s_1 and \( \hat{\beta}_{OLS,1} = (X_1^T X_1)^{-1} X_1^T y \). Note that:

\[
\hat{y}_{\lambda}^T y - \hat{y}_{\lambda}^T \hat{y}_{\lambda} = n \lambda s_j, \hat{\beta}_{OLS,j*} + n \lambda \sum_{j \neq j*} |\hat{\beta}_{OLS,j}| - n^2 \lambda^2 s_1^T (X_1^T X_1)^{-1} s_1. \tag{33}
\]

Hence:

\[
\hat{\alpha}_{\lambda} \hat{\beta}_{\lambda,j*} - \hat{\beta}_{OLS,j*} = \frac{(\hat{y}_{\lambda}^T y - \hat{y}_{\lambda}^T \hat{y}_{\lambda} \hat{\beta}_{OLS,j*} - n \lambda \hat{y}_{\lambda}^T y [(X_1^T X_1)^{-1} s_1]_{j*}}{\hat{y}_{\lambda}^T \hat{y}_{\lambda}}. \tag{34}
\]

Using the expression for \( \hat{y}_{\lambda}^T y - \hat{y}_{\lambda}^T \hat{y}_{\lambda} \) derived above, it can be shown that as \( |\hat{\beta}_{OLS,j*}| \to \infty \):

\[
\hat{\alpha}_{\lambda} \hat{\beta}_{\lambda,j*} - \hat{\beta}_{OLS,j*} \to n \lambda (s_{j*} - [(X_1^T X_1)^{-1} s_1]) = G, \tag{35}
\]

where \( [(X_1^T X_1)^{-1}]_{j*} \) denotes the \( j^* \)-th row of \( (X_1^T X_1)^{-1} \). It is simple to show by contradiction that \( |G| \leq |\hat{\beta}_{\lambda,j*} - \hat{\beta}_{OLS,j*}| \). \( \square \)
SUPPLEMENTARY MATERIAL of “Tuning Parameter Selection for Penalized Estimation via $R^2$”

The supplementary material in the following document includes three sections. Section A gives more details of the proofs for the theorems and lemmas presented in the paper. Section B provides tables and figures that more fully explore the simulation studies discussed in the Numerical Results. This section also evaluates results from a simulation study of the non-convex SCAD and MC+ penalties. Finally, Section C describes the approximations used to achieve the estimates from the EMG data application and includes results from multiple stages of adaptive estimation.

A  Proofs for Theoretical Results

Theorem 6. The value $\hat{\alpha}_\lambda$, calculated as $(\hat{y}_\lambda^T \hat{y}_\lambda)^{-1} \hat{y}_\lambda^T y$, will always be greater than or equal to 1 for any penalized estimator minimizing an objective function of the form:

\[
\frac{1}{2n} ||y - X\beta||_2^2 + \sum_{\ell=1}^L \lambda_\ell g_\ell(\beta)
\]  \hspace{1cm} (36)

where each $g_\ell(\beta)$ for $\ell = \{1, ..., L\}$ is convex and minimized at $0$.

Proof of Theorem 6. Suppose $\hat{\beta}_\lambda$ minimizes the objective function in Equation (36) and not all elements of $\hat{\beta}_\lambda$ are equal to 0. Let $\lambda g(\beta) = \sum_{\ell=1}^L \lambda_\ell g_\ell(\beta)$ and $\nabla g^*$ be the subgradient vector of $g(\beta)$. Then we know the following is true:

\[
-\frac{1}{n} X^T(y - X\hat{\beta}_\lambda) + \nabla g^* = 0
\]  \hspace{1cm} (37)

under KKT conditions. Further, we can say that there exists some generalized inverse of
Given this definition, $\hat{\alpha}_\lambda$ may be written:

$$\hat{\alpha}_\lambda = \hat{\beta}_\lambda^T \hat{\beta}_\lambda + n \nabla g^* T \hat{\beta}_\lambda \hat{y}_\lambda^T \hat{y}_\lambda \lambda (\hat{y}_\lambda^T \hat{y}_\lambda).$$

(39)

Therefore:

$$\hat{y}_\lambda^T \hat{y}_\lambda = \hat{y}_\lambda^T \hat{y}_\lambda + \lambda \nabla g^* T \hat{\beta}_\lambda \hat{y}_\lambda^T \hat{y}_\lambda + n \nabla g^* T \hat{\beta}_\lambda \hat{y}_\lambda^T \hat{y}_\lambda \lambda (\hat{y}_\lambda^T \hat{y}_\lambda).$$

(40)

$$\hat{y}_\lambda^T \hat{y}_\lambda = 1 + n \nabla g^* T \hat{\beta}_\lambda \hat{y}_\lambda^T \hat{y}_\lambda.$$ 

(41)

Because the denominator of the second term is greater than 0, as is $n$, this expression is greater than or equal to 1 whenever $\nabla g^* T \hat{\beta}_\lambda$ is greater than or equal to zero. The vector $\nabla g^*$ the subgradient of a convex function at $\hat{\beta}_\lambda$, therefore for any $\beta$:

$$g(\beta) \geq g(\hat{\beta}_\lambda) + \nabla g^* T (\beta - \hat{\beta}_\lambda) \geq 0.$$ 

(42)

Because this is true for all $\beta$ vectors, it must be the case when $\beta = 0$. Hence, we have:

$$\nabla g^* T \hat{\beta}_\lambda \geq g(\hat{\beta}_\lambda) - g(0).$$ 

(43)

Thus $\hat{\alpha}_\lambda \geq 1$. □

**Lemma 4.** If there exists a $\lambda$ where $\hat{\xi}_\lambda = \xi^*$ as defined in Section 3.1, then $E(\hat{\alpha}_\lambda \hat{\beta}_\lambda) = \beta^*$.

**Proof of Lemma 4.** When $\|\hat{\beta}_\lambda\|_2^{-1} \hat{\beta}_\lambda = \xi^*$, $\hat{\beta}_\lambda = \|\hat{\beta}_\lambda\|_2 \xi^*$. Expand he response $y = \alpha X \xi^* + \epsilon$, to define:

$$\hat{\alpha}_\lambda = \frac{\alpha \xi^T X^T X \xi^* + \xi^* X^T \epsilon}{\|\hat{\beta}_\lambda\|_2 \xi^T X \xi^*}.$$ 

(45)
Therefore:
\[
\hat{\alpha}_\lambda \hat{\beta}_\lambda = \alpha \| \hat{\beta}_\lambda \|_2 \xi^*
\]  
\[
= \frac{1}{\| \hat{\beta}_\lambda \|_2} \left( \alpha + \frac{\xi^T X^T \epsilon}{\xi^T X^T \xi^*} \right) \| \hat{\beta}_\lambda \|_2 \xi^*
\]  
\[
= \alpha \xi^* + \left( \frac{\xi^T X^T \epsilon}{\xi^T X^T \xi^*} \right) \xi^*.
\]

Thus \( E[\hat{\alpha}_\lambda \hat{\beta}_\lambda] = \alpha \xi^* = \beta^* \) and the estimator is unbiased.

\[\square\]

Lemma 5. If \( \hat{\beta}_\lambda \propto \hat{\beta}_{\text{OLS}}^{M_\lambda} \) then \( \hat{\alpha}_\lambda \hat{\beta}_\lambda = \hat{\beta}_{\text{OLS}}^{M_\lambda} \).

Proof of Lemma 5. Fix \( \lambda \) and let \( \hat{\beta}_\lambda = a \hat{\beta}_{\text{OLS}}^{M_\lambda} \) where \( \hat{\beta}_{\text{OLS}}^{M_\lambda} \) is the OLS estimate for submodel \( M_\lambda \). Let \( X_{M_\lambda} \) be the submatrix of \( X \) containing only the columns indexed by \( M_\lambda \). Then \( \hat{\beta}_{\text{OLS}}^{M_\lambda} = (X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T y \). Now:
\[
\hat{\alpha}_\lambda = \frac{ay^T X_{M_\lambda}(X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T y}{a^2 y^T X_{M_\lambda}(X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T X_{M_\lambda}(X_{M_\lambda}^T X_{M_\lambda})^{-1} X_{M_\lambda}^T y} = \frac{1}{a}.
\]

Therefore:
\[
\hat{\alpha}_\lambda \hat{\beta}_\lambda = \frac{1}{a} \times a \hat{\beta}_{\text{OLS}}^{M_\lambda} = \hat{\beta}_{\text{OLS}}^{M_\lambda}.
\]  
\[\square\]

Theorem 7. Fix \( p \) and suppose there exists a non-singular matrix \( C \) such that: \( \frac{1}{n} X_n^T X_n = \frac{1}{n} C_n \rightarrow C \) as \( n \rightarrow \infty \). Then for any \( P_\lambda(\cdot) \), \( \hat{\alpha}_\lambda \hat{\beta}_\lambda_n \rightarrow_p \beta^* \) whenever \( \hat{\beta}_\lambda_n \rightarrow_p c \xi^* \) for some non-zero constant, \( c \).

Proof of Theorem 7. Consider \( \hat{\alpha}_\lambda_n \) :
\[
\hat{\alpha}_\lambda_n = \frac{\frac{1}{n} \hat{\beta}_\lambda_n^T X_n^T y}{\frac{1}{n} \hat{\beta}_\lambda_n^T X_n X_n \beta_\lambda_n}.
\]

By Continuous Mapping Theorem, the denominator of this expression converges in proba-
bility to $c^2\xi^T C\xi^*$. The numerator is equal to:

$$ \frac{1}{n} \hat{\beta}_{\lambda_n}^T X_n^T X_n \beta + \hat{\beta}_{\lambda_n}^T X_n^T e \to_p \alpha c \xi^T C \xi^*. $$

(52)

Then $\hat{\alpha}_{\lambda_n} \to_p \frac{\alpha}{\epsilon}$ and, once again by Continuous Mapping Theorem, $\hat{\alpha}_{\lambda_n} \hat{\beta}_{\lambda_n} \to_p \beta^*$. $\square$

For a fixed $p$, $\lambda_n$ showed that, when $\epsilon_i$ is iid with $E[\epsilon_i] = 0$, $E|\epsilon_i| < \infty$, and $\frac{\lambda_n}{n} \to 0$, the Lasso estimate converges in probability to $\beta^*$. Hence, by Theorem 7, $\hat{\alpha}_{\lambda_n} \hat{\beta}_{\lambda_n} \to_p \beta^*$ where $\hat{\beta}_{\lambda_n}$ is the Lasso estimate.

**Theorem 8.** Suppose that $\hat{\beta}_{\lambda}$ recovers the direction of $\beta^*$ and assume all $\epsilon_i$ are independent with constant variance $\sigma^2$. Then the expected value of $\alpha$-Modified APE is less than the expected value of unmodified APE when:

$$ \frac{\alpha^2}{E[\|\|\beta_{\lambda}\|_2 - \alpha]\|^2]} \leq \frac{\beta^T X^T X \beta^*}{\sigma^2} $$

(53)

**Proof of Theorem 8.** Average Prediction Error for $K$-fold CV is calculated as:

$$ \frac{1}{K} \sum_k \frac{1}{n_k} \sum_i (y_{ki} - \hat{\beta}_{\lambda})^2. $$

(54)

To show that the expected value of $\alpha$-Modified APE is less than that of regular APE, it is sufficient to show that $E[ModPE] < E[PE]$ where $PE = \sum_i (y_{ki} - x_{ki}^T \hat{\beta}_{\lambda})^2$ and $ModPE = \sum_i (y_{ki} - \hat{\alpha}_{\lambda} x_{ki}^T \hat{\beta}_{\lambda})^2$. When the direction of $\beta^*$ has been captured by $\hat{\beta}_{\lambda}$, we can write the estimate as $\|\hat{\beta}_{\lambda}\|_2 \xi^*$. Expand $y_{ki} = \alpha x_{ki}^T \xi^* + \epsilon_i$ to find:

$$ PE = \sum_i \left( \epsilon_i - x_{ki}^T \xi^* (\|\hat{\beta}_{\lambda}\|_2 - \alpha) \right)^2. $$

(55)
Similarly,

\[ \text{ModPE} = \sum_i \left( \varepsilon_i - x_k^T \xi^* \left( \frac{\xi^* X^T \varepsilon}{\xi^* X^T X \xi^*} \right) \right)^2. \]  

(56)

Let \( \Sigma \) be the covariance matrix for \( \varepsilon \). Therefore, \( \Sigma = \sigma^2 I \)

\[
E[PE] = n_k \sigma^2 + \sum_i (x_k^T \xi^*)^2 E[\|\hat{\beta}_\lambda\|_2 - \alpha]^2 
\]

(57)

\[
E[\text{ModPE}] = E \left[ \sum_i \left( \varepsilon_i - x_k^T \xi^* \left( \frac{\xi^* X^T \varepsilon}{\xi^* X^T X \xi^*} \right) \right)^2 \right]
\]

(58)

\[
= n_k \sigma^2 + \sum_i (x_k^T \xi^*)^2 E[\varepsilon^T X \xi^* \varepsilon^T X^T \varepsilon] \left( \frac{\xi^* X^T X \xi^*}{\xi^* X^T X \xi^*} \right)^2 
\]

(59)

\[
= n_k \sigma^2 + \sum_i (x_k^T \xi^*)^2 (tr(\xi^* X^T X \xi^*) + E[\varepsilon^T X \xi^* \varepsilon^T X^T \varepsilon]) \left( \frac{\xi^* X^T X \xi^*}{\xi^* X^T X \xi^*} \right)^2 
\]

(60)

\[
= n_k \sigma^2 + \sum_i (x_k^T \xi^*)^2 \sigma^2 \left( \frac{\xi^* X^T X \xi^*}{\xi^* X^T X \xi^*} \right)^2 
\]

(61)

\[
= n_k \sigma^2 + \frac{\sigma^2}{\xi^* X^T X \xi^*} \sum_i (x_k^T \xi^*)^2 
\]

(62)

Therefore \( E[PE] \geq E[\text{ModPE}] \) whenever

\[
E[\|\hat{\beta}_\lambda\|_2 - \alpha]^2 \geq \frac{\sigma^2}{\xi^* X^T X \xi^*} 
\]

(63)

\[
\frac{\alpha^2}{E[\|\hat{\beta}_\lambda\|_2 - \alpha]^2} \leq \frac{\beta^* X^T X \beta^*}{\sigma^2} 
\]

(64)

because \( \beta^* = \alpha \xi^* \). Therefore we have the result. \( \Box \)

**Lemma 6.** When \( X \) is orthonormal, the modified Lasso estimate for coefficient \( j^* \) can be expressed as:

\[
\hat{\alpha}_\lambda \hat{\beta}_{\lambda, j^*} = w_1 \hat{\beta}_{\text{OLS}, j^*} + (1 - w_1) \hat{\beta}_{j^*, \lambda} + w_2 \hat{\beta}_{j^*, \lambda} 
\]

(65)

where \( w_1 = \frac{d_{j^*}}{\sum_{j=1}^p d_j} \), \( w_2 = \frac{\lambda \sum_{j \neq j^*} d_j}{\sum_{j=1}^p d_j} \), and \( d_j = (|\hat{\beta}_{\text{OLS}, j}| - \lambda)_+ \).
Proof of Lemma 6. When $X$ is orthonormal,

$$
\hat{\alpha}_\lambda = \frac{\hat{\beta}_\lambda^T \hat{\beta}_{OLS}}{\hat{\beta}_\lambda^T \hat{\beta}_\lambda} = \frac{\sum_{j=1}^{p} |\hat{\beta}_{OLS,j}|(|\hat{\beta}_{OLS,j}| - \lambda)_+}{\sum_{j=1}^{p} (|\hat{\beta}_{OLS,j}| - \lambda)_+^2}.
$$

(66)

So following expression arises:

$$
\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} = \left(\frac{\sum_{j=1}^{p} |\hat{\beta}_{OLS,j}|(|\hat{\beta}_{OLS,j}| - \lambda)_+}{\sum_{j=1}^{p} (|\hat{\beta}_{OLS,j}| - \lambda)_+^2}\right) \times \text{sign}(\hat{\beta}_{OLS,j^*}) \times (|\hat{\beta}_{OLS,j^*}| - \lambda)_+.
$$

(67)

Letting $d_j = (|\hat{\beta}_{OLS,j}| - \lambda)_+$ and $s_j = \text{sign}(\hat{\beta}_{OLS,j^*})$, the expression becomes:

$$
\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} = \frac{s_j|\hat{\beta}_{OLS,j^*}|}{\sum_{j=1}^{p} d_j^2 + s_j d_j^*} \sum_{j=1}^{p} \frac{|\hat{\beta}_{OLS,j}| d_j^*}{d_j} + \frac{s_j d_j^* \sum_{j=1}^{p} |\hat{\beta}_{OLS,j}| d_j^*}{d_j} + \frac{s_j d_j^* \sum_{j=1}^{p} |\hat{\beta}_{OLS,j}| d_j^*}{d_j}
$$

(68)

which uses $s_j|\hat{\beta}_{OLS,j^*}| = \hat{\beta}_{OLS,j^*}$, $w_1 = d_j^2 / \sum_{j=1}^{p} d_j^2$, $\hat{\beta}_{\lambda,j^*} = s_j d_j^*$, and $|\hat{\beta}_{OLS,j}| = (|\hat{\beta}_{OLS,j}| - \lambda)_+ + \lambda$ whenever $d_j > 0$. Therefore:

$$
\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} = w_1 \hat{\beta}_{OLS,j^*} + \hat{\beta}_{\lambda,j^*} \left(\frac{\sum_{j=1}^{p} d_j^2}{\sum_{j=1}^{p} d_j^2} + \frac{\lambda \sum_{j=1}^{p} d_j^2}{\sum_{j=1}^{p} d_j^2}\right)
$$

(70)

$$
= w_1 \hat{\beta}_{OLS,j^*} + (1 - w_1) \hat{\beta}_{\lambda,j^*} + w_2 \hat{\beta}_{\lambda,j^*},
$$

(71)

where $w_2 = \lambda \sum_{j=1}^{p} d_j^2$. Hence we have the result. \(\square\)

Theorem 9. Suppose $X$ is orthogonal and consider a given predictor, $j^*$, and $\lambda$ where at least one $j \neq j^*$ has $|\hat{\beta}_{OLS,j}| > \lambda$. Then across all possible $\hat{\beta}_{OLS,j^*}$,

$$
|\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}| \leq \lambda \times \max\left(1, \frac{\sqrt{u^2v + v^2} - v}{2v}\right)
$$

(72)

where $u = \sum_{j \neq j^*} d_j$ and $v = \sum_{j \neq j^*} d_j^2$. Moreover, $\lim_{|\hat{\beta}_{OLS,j^*}| \to \infty} |\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}| \to 0$ .

Proof of Theorem 9. For a given $\lambda$ we are allowing $\hat{\beta}_{OLS,j^*}$ to vary. For $|\hat{\beta}_{OLS,j^*}| \leq \lambda$,
we have $\hat{\alpha}_l \hat{\beta}_{\lambda,j^*} = 0$ so $|\hat{\alpha}_l \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}| = |\hat{\beta}_{OLS,j^*}| \leq \lambda$ so the theorem is satisfied for this range of values. Next we consider $\hat{\beta}_{OLS,j^*} > \lambda$. By Lemma 6.

$$\hat{\alpha}_l \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*} = (w_1 - 1)\hat{\beta}_{OLS,j^*} + (1 - w_1 + w_2)\hat{\beta}_{\lambda,j^*}$$

$$= w_2 \hat{\beta}_{OLS,j^*} + (1 - w_1 + w_2)(-\lambda)$$

$$= \frac{\lambda \left(\sum_{j \neq j^*} d_j\right) \hat{\beta}_{OLS,j^*} - \lambda \left(\sum_{j \neq j^*} d_j^2 + \lambda \sum_{j \neq j^*} d_j\right)}{\left(\hat{\beta}_{OLS,j^*} - \lambda\right)^2 + \sum_{j \neq j^*} d_j^2}$$

Let $u = \sum_{j \neq j^*} d_j$, $v = \sum_{j \neq j^*} d_j^2$, and $x = \hat{\beta}_{OLS,j^*}$. This a function, $f(x)$, that is differentiable for $x \geq \lambda$:

$$f(x) = \frac{\lambda ux - \lambda(v + \lambda u)}{(x - \lambda)^2 + v}$$

$$\frac{\partial f(x)}{\partial x} = \frac{u \lambda}{(x - \lambda)^2 + v} - \frac{2(x - \lambda)(u \lambda x - \lambda(u \lambda + v))}{((x - \lambda)^2 + v)^2}$$.

The derivative $\frac{\partial f(x)}{\partial x}$ is equal to zero whenever $x = x^*$, where $x^*$ is defined:

$$x^* = \frac{u \lambda + v \pm \sqrt{u^2 v + v^2}}{u}$$.

Note that $x^* < \lambda$ when the negative part of the square root is taken, therefore the root is restricted to $x^* = (u \lambda + v + \sqrt{u^2 v + v^2})/u$. The second derivative of $f(x)$ is:

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{2\lambda[-3v(x - \lambda)(u + x - \lambda) + u(x - \lambda)^3 + v^2]}{(v + (x - \lambda)^2)^3}$$.

When $x = x^*$, the second derivative equals:

$$-\frac{u^4 \lambda(u^2 + v)(\sqrt{u^2 v + v^2} + v)}{2v^2(\sqrt{u^2 v + v^2} + u^2 + v)^3} < 0$$

meaning that $x^*$ gives the maximum of $f(x)$ when $x > \lambda$. $f(x)$ is an increasing function.
bounded below by $-\lambda$ and bounded above by

$$
\frac{\lambda}{2} \left( \sqrt{\frac{u^2}{v} + 1 - 1} \right).
$$

(81)

For $x > x^*$, $f(x)$ is decreasing so we consider $\lim_{x \to \infty} f(x)$, which is easily shown to equal $0 > -\lambda$. Therefore,

$$
|f(x)| \leq \lambda \times \max \left( 1, \frac{1}{2} \left( \sqrt{\frac{u^2}{v} + 1 - 1} \right) \right),
$$

(82)

and $\lim_{x \to \infty} |f(x)| = 0$. For $\hat{\beta}_{OLS,j^*} < -\lambda$, note that $\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*} = -(\hat{\alpha}_\lambda |\hat{\beta}_{\lambda,j^*}| - |\hat{\beta}_{OLS,j^*}|)$ and so analogous arguments to the case of $\hat{\beta}_{OLS,j^*}$ also apply. This completes the proof. □

**Theorem 10.** Suppose the Lasso recovers the correct sign vector of $\beta^*$. Let $s_1$ be the sign vector of active coefficients, and let $X_1$, the active subset of $X$, be invertible. Let $G = n\lambda \left( s_{j^*} - [(X_1^T X_1)^{-1}]_{j^*} s_1 \right)$. For $\hat{\beta}_{\lambda,j^*} \neq 0$:

$$
|\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}| \to G < |\hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}|
$$

(83)

as $|\hat{\beta}_{OLS,j^*}| \to \infty$.

**Proof of Theorem 10.** When the sign vector of the true model, $s_1$, is known there exists a closed form expression for the Lasso solutions: $\hat{\beta}_\lambda = (\hat{\beta}_{OLS} - n\lambda (X_1^T X_1)^{-1}s_1)$ where $X_1$ is the covariate matrix including all variables in the support of the true model. The following is also known:

$$
\hat{\alpha}_\lambda \hat{\beta}_\lambda = \left( \frac{\hat{\beta}_\lambda X_1^T y}{\hat{\beta}_\lambda X_1^T X_1 \hat{\beta}_\lambda} \right) \hat{\beta}_\lambda.
$$

(84)
Hence:

\[ \hat{y}_\lambda^T y = \hat{\beta}_\lambda^T X_1^T y \]  
\[ = \hat{\beta}_{OLS,1}^T X_1^T X_1 \hat{\beta}_{OLS,1} - n \lambda s_1^T \hat{\beta}_{OLS,1} \]  
\[ = \left( \sum_{i=1}^{n} x_{ij}^2 \right) \hat{\beta}_{OLS,j}^* + \left( 2 \sum_{i=1}^{n} x_{ij} \left( \sum_{j \neq j^*} x_{ij} \hat{\beta}_{OLS,j} \right) \right) - n \lambda s_j^* \hat{\beta}_{OLS,j}^* \]  
\[ - n \lambda \sum_{j \neq j^*} |\hat{\beta}_{OLS,j}| + \sum_{i=1}^{n} \left( \sum_{j \neq j^*} \hat{\beta}_{OLS,j} x_{ij} \right)^2 \]  
\[ \hat{y}_\lambda^T \hat{y}_\lambda = \hat{\beta}_{OLS,1}^T X_1^T X_1 \hat{\beta}_{OLS,1} - 2n \lambda s_1^T \hat{\beta}_{OLS,1} + n^2 \lambda^2 s_1^T \left( X_1^T X_1 \right)^{-1} s_1 \]  
\[ = \left( \sum_{i=1}^{n} x_{ij}^2 \right) \hat{\beta}_{OLS,j}^* + \left( 2 \sum_{i=1}^{n} x_{ij} \left( \sum_{j \neq j^*} x_{ij} \hat{\beta}_{OLS,j} \right) - 2n \lambda s_j^* \hat{\beta}_{OLS,j}^* \right) \]  
\[ - 2n \lambda \sum_{j \neq j^*} |\hat{\beta}_{OLS,j}| + \sum_{i=1}^{n} \left( \sum_{j \neq j^*} \hat{\beta}_{OLS,j} x_{ij} \right)^2 + n^2 \lambda^2 s_1^T \left( X_1^T X_1 \right)^{-1} s_1. \]

Here, \( x_{ij} \) is the \( i \)th element of the \( j \)th column of \( X_1 \). Note that:

\[ \hat{y}_\lambda^T y - \hat{y}_\lambda^T \hat{y}_\lambda = n \lambda s_j^* \hat{\beta}_{OLS,j}^* + n \lambda \sum_{j \neq j^*} |\hat{\beta}_{OLS,j}| - n^2 \lambda^2 s_1^T \left( X_1^T X_1 \right)^{-1} s_1. \]

Hence:

\[ \hat{\lambda}_\lambda \hat{\beta}_{OLS,j}^* - \hat{\beta}_{OLS,j}^* = \hat{\lambda}_\lambda \left( \hat{\beta}_{OLS,j}^* - n \lambda \left[ \left( X_1^T X_1 \right)^{-1} s_1 \right]_{j^*} \right) - \hat{\beta}_{OLS,j}^* \]
\[ = (\hat{\lambda}_\lambda - 1) \hat{\beta}_{OLS,j}^* - \hat{\lambda}_\lambda n \lambda \left[ \left( X_1^T X_1 \right)^{-1} s_1 \right]_{j^*} \]
\[ = \hat{y}_\lambda^T \hat{y}_\lambda \left( \hat{\beta}_{OLS,j}^* - n \lambda \left[ \left( X_1^T X_1 \right)^{-1} s_1 \right]_{j^*} \right) \]
\[ = \left( \hat{y}_\lambda^T y - \hat{y}_\lambda^T \hat{y}_\lambda \right) \hat{\beta}_{OLS,j}^* - n \lambda \hat{y}_\lambda^T \left[ \left( X_1^T X_1 \right)^{-1} s_1 \right]_{j^*} \]
\[ \hat{y}_\lambda^T \hat{y}_\lambda \]

Now, using the expression for \( \hat{y}_\lambda^T y - \hat{y}_\lambda^T \hat{y}_\lambda \) derived above, it can be shown that as \( |\hat{\beta}_{OLS,j^*}| \rightarrow \)
\[
\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*} \rightarrow \frac{n\lambda s_j^* \hat{\beta}_{OLS,j^*}^2 - n\lambda [(X_i^T X_j)^{-1} s_1]_{j^*} (\sum_{i=1}^n x_{ij}^2) \hat{\beta}_{OLS,j^*}^2}{(\sum_{i=1}^n x_{ij}^2) \hat{\beta}_{OLS,j^*}^2}
\]

(97)

\[
\rightarrow \frac{n\lambda s_j^* - n\lambda [(X_i^T X_j)^{-1} s_1]_{j^*} (\sum_{i=1}^n x_{ij}^2)}{(\sum_{i=1}^n x_{ij}^2)}
\]

(98)

\[
= n\lambda \left( \frac{s_j^*}{\sum_{i=1}^n x_{ij}^2} - [(X_i^T X_j)^{-1} s_1]_{j^*} \right). 
\]

(99)

If we assume that the columns of \( X \) have been centered and scaled, we can say:

\[
\hat{\alpha}_\lambda \hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*} \rightarrow n\lambda \left( s_j^* - [(X_i^T X_j)^{-1} s_1]_{j^*} \right) = G.
\]

(100)

By the definition of the Lasso estimate:

\[
\hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*} = -n\lambda [(X_i^T X_j)^{-1} s_1],
\]

(101)

Because accurate sign recovery has been assumed, when \( s_j^* > 0 \), \(-n\lambda [(X_i^T X_j)^{-1} s_1] < 0\) and when \( s_j^* < 0 \), \(-n\lambda [(X_i^T X_j)^{-1} s_1] > 0\). Now for proof by contradiction, assume \(|G| > |\hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}|\). The absolute value inequality can be broken into two cases, first where \( s_j^* > 0 \) and then where \( s_j^* < 0 \). Begin with \( s_j^* > 0 \):

\[
G < -n\lambda [(X_i^T X_j)^{-1} s_1]
\]

(102)

\[
n\lambda \left( s_j^* - [(X_i^T X_j)^{-1} s_1] \right) < -n\lambda [(X_i^T X_j)^{-1} s_1]
\]

(103)

\[
n\lambda s_j^* < 0.
\]

(104)
This is a contradiction. Now consider $s_{j^*} < 0$:

\[
G > -n\lambda[(X_1^T X_1)^{-1}]_{j^*} s_1 \tag{105}
\]

\[
n\lambda(s_{j^*} - [(X_1^T X_1)^{-1}]_{j^*} s_1) > -n\lambda[(X_1^T X_1)^{-1}]_{j^*} s_1 \tag{106}
\]

\[
n\lambda s_{j^*} > 0 , \tag{107}
\]

giving us another contradiction. Note that $G = |\hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}|$ if and only if $n\lambda s_{j^*} = 0$, which is another contradiction. Therefore, $|G| < |\hat{\beta}_{\lambda,j^*} - \hat{\beta}_{OLS,j^*}|$. Thus the absolute value of the difference between the $\alpha$-Modified Lasso estimate and the OLS estimate approaches a constant that is always less than the absolute value of the difference between the unmodified Lasso estimate and the OLS estimate as the OLS estimate approaches infinity.

\[\square\]

## B Further Numerical Results

### B.1 Simulation Study for Independent Predictors

In this section we give additional results from the simulation studies conducted in the main document. Tables ?? through ?? give the average model size, average False Discovery Rates, average false negatives, and average prediction bias for the four methods considered in this part of the study: APE CV, AR2 CV, Mod CV, and the Relaxed Lasso for an additional $p^*$, $p^* = 15$. Mod CV was left out of the main document because of its similarity to AR2 CV. We can see from these results that all False negatives are close to zero. For smaller $p^*$, AR2 and Mod CV generally produce lower False Discovery Rates than APE CV. Average Prediction Bias, however generally appears lower for the two new approaches than APE CV, but higher than the Relaxed Lasso on average.
Table 5: Average number of selected variables 100 replications of CV 1SE. Standard errors given in parentheses.

| p  | 50   | 100  | 200  | 400  | 800   | 50   | 100  | 200  | 400  | 800   |
|----|------|------|------|------|-------|------|------|------|------|-------|
| n  |      |      |      |      |       |      |      |      |      |       |
| 50 | 6.4  | 10.6 | 9.7  | 11.5 | 11.2  | 11   | 12.7 | 13.9 | 17.9 | 18.6  |
|    | (0.6)| (1.2)| (1.3)| (1.4)| (1.6) | (0.8)| (0.8)| (0.9)| (1.2)| (1.5) |
| 100| 6.8  | 8.7  | 9.9  | 10.3 | 12    | 7.8  | 9.8  | 11.6 | 14.5 | 15.7  |
|    | (0.3)| (0.7)| (1)  | (1.3)| (1.5) | (0.4)| (0.6)| (0.7)| (1.1)| (1.4) |
| 200| 6.5  | 6.8  | 8    | 9    | 9.8   | 6.6  | 7.6  | 8.6  | 10.3 | 11.3  |
|    | (0.2)| (0.2)| (0.4)| (0.5)| (0.6) | (0.2)| (0.3)| (0.4)| (0.9)| (0.9) |
| n  |      |      |      |      |       |      |      |      |      |       |
| 50 | 8.9  | 8    | 8.5  | 5.6  | 6.6   | 18.9 | 20.6 | 15.1 | 13.7 | 13.1  |
|    | (1)  | (1.2)| (1.3)| (1.2)| (1.3) | (1)  | (1.5)| (1.4)| (1.6)| (1.6) |
| 100| 15.9 | 15.2 | 13.5 | 11   | 11.5  | 22.8 | 28.8 | 33.7 | 35   | 29.9  |
|    | (0.6)| (1.1)| (1.3)| (1.6)| (1.9) | (0.4)| (0.9)| (1.4)| (1.7)| (1.8) |
| 200| 19.4 | 21.7 | 22.8 | 23.5 | 21.4  | 21.1 | 25.3 | 31.9 | 38   | 43.5  |
|    | (0.4)| (0.6)| (0.9)| (1.1)| (1.2) | (0.3)| (0.5)| (0.9)| (1.6)| (1.4) |
| n  |      |      |      |      |       |      |      |      |      |       |
| 50 | 7.8  | 9    | 6.1  | 5.8  | 5.8   | 22.2 | 14.8 | 10.7 | 10   | 8.4   |
|    | (1.1)| (1.3)| (1.3)| (1.3)| (1.2) | (1.4)| (1.5)| (1.5)| (1.5)| (1.5) |
| 100| 19.2 | 11.6 | 8.6  | 6.8  | 7     | 43.6 | 44.3 | 28.4 | 20   | 12.6  |
|    | (1.2)| (1.6)| (1.6)| (1.6)| (1.7) | (0.5)| (1.9)| (2.4)| (2.5)| (2.2) |
| 200| 39.1 | 37.8 | 33.7 | 20.6 | 15    | 49.2 | 68.8 | 82.8 | 80.9 | 58.1  |
|    | (0.6)| (1.4)| (2.2)| (2.2)| (2.6) | (0.1)| (0.6)| (1.4)| (2.4)| (3)   |
Table 6: Average number of selected variables 100 replications of AR2 CV 1SE. Standard errors given in parentheses.

| n  | 50  | 100 | 200  | 400  | 800  | 50  | 100 | 200  | 400  | 800  |
|----|-----|-----|------|------|------|-----|-----|------|------|------|
|    | \( p^* = 5; SNR = 1.25; \sigma = 5.23 \) | \( p^* = 5; SNR = 5; \sigma = 2.61 \) | \( p^* = 15; SNR = 1.25; \sigma = 9.04 \) | \( p^* = 15; SNR = 5; \sigma = 4.52 \) | \( p^* = 50; SNR = 1.25; \sigma = 16.52 \) | \( p^* = 50; SNR = 5; \sigma = 8.26 \) |
| 50 | 6.7 | 7.3 | 6    | 6    | 5.4  | 7.4 | 7.9 | 6.8  | 7.5  | 7.8  |
|    | (0.9)| (1) | (0.9)| (0.8)| (0.9)| (0.7)| (0.7)| (0.6)| (0.6)| (0.8)|
| 100| 4.8 | 5.2 | 5.6  | 6.2  | 6.4  | 5.3 | 5.8 | 6.3  | 6.6  | 6.7  |
|    | (0.2)| (0.4)| (0.7)| (1.1)| (0.8)| (0.1)| (0.2)| (0.4)| (0.3)| (0.4)|
| 200| 4.9 | 5.1 | 5.2  | 5.3  | 5.2  | 5.1 | 5.2 | 5.2  | 5.2  | 5.2  |
|    | (0.1)| (0.2)| (0.2)| (0.3)| (0.3)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)|
| 50 | 10.8| 10.3| 9.1  | 5.9  | 9.2  | 16.3| 13.5| 10.8 | 8.4  | 9.2  |
|    | (1.3)| (1.2)| (1.2)| (0.8)| (1.2)| (1) | (1.2)| (1.2)| (1.1)| (1.2)|
| 100| 15.6| 15   | 15.1 | 9.8  | 12.8 | 18.3| 22.9| 24.1 | 21.9 | 22.9 |
|    | (0.7)| (1.4)| (1.8)| (1.3)| (1.7)| (0.4)| (0.7)| (1.1)| (1.2)| (1.6)|
| 200| 16.8| 17   | 17.9 | 17.3 | 15.9 | 16.7| 17.6| 20.8 | 22.9 | 25.9 |
|    | (0.5)| (0.6)| (0.8)| (1)  | (1)  | (0.2)| (0.3)| (0.5)| (0.7)| (1.1)|
| 50 | 14.4| 10   | 8.3  | 6.6  | 7.2  | 22.5| 11.7| 9.9  | 7.3  | 6.3  |
|    | (1.3)| (1.3)| (1.2)| (1)  | (1)  | (1.4)| (1.3)| (1.2)| (0.9)| (0.9)|
| 100| 26.4| 23   | 16.4 | 14.6 | 13.3 | 44  | 46.6| 36.1 | 24.2 | 15.6 |
|    | (1.3)| (2.3)| (2.2)| (2.1)| (1.9)| (0.6)| (1.7)| (2.3)| (2.4)| (2)  |
| 200| 42.4| 48.3 | 48.8 | 43.8 | 42   | 49  | 64.8| 76.5 | 73.1 | 55.7 |
|    | (0.4)| (1.5)| (2.6)| (3.6)| (4.7)| (0.1)| (0.7)| (1.5)| (2.7)| (3.1)|
Table 7: Average number of selected variables 100 replications of α-modified CV 1SE. Standard errors given in parentheses.

|      | p 50 | 100 | 200 | 400 | 800 | p 50 | 100 | 200 | 400 | 800 |
|------|------|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|
|      |      |     |     |     |     |      |     |     |     |     |     |
| n    |      |     |     |     |     |      |     |     |     |     |     |
| 50   | 4.5  | 8.7 | 8.4 | 10  | 8.9 | 7.7  | 9.3 | 8.8 | 11.1| 11.6|
|      | (0.6)| (1.3)| (1.3)| (1.4)| (1.4)| (0.7)| (0.7)| (0.8)| (1) | (1.2)|
| 100  | 4.7  | 5   | 6.2 | 6.1 | 8.1 | 5.4  | 6.2 | 6.6 | 7   | 7.6 |
|      | (0.2)| (0.3)| (0.8)| (1.1)| (1.3)| (0.1)| (0.3)| (0.4)| (0.3)| (0.5)|
| 200  | 5    | 5.1 | 4.9 | 5.4 | 5.1 | 5.1  | 5.3 | 5.6 | 5.6 | 5.8 |
|      | (0.1)| (0.2)| (0.1)| (0.3)| (0.2)| (0)  | (0.1)| (0.1)| (0.3)| (0.3)|
|      |      |     |     |     |     |      |     |     |     |     |     |
| n    |      |     |     |     |     |      |     |     |     |     |     |
| 50   | 6.7  | 6.7 | 8.8 | 5.3 | 5.8 | 17.3 | 18.7| 13.1| 12.2| 13.3|
|      | (1.1)| (1.3)| (1.5)| (1.2)| (1.3)| (1.1)| (1.5)| (1.4)| (1.6)| (1.7)|
| 100  | 11.8 | 10.8| 9.3 | 8.4 | 8.6 | 19.2 | 23.5| 27  | 27  | 24.3|
|      | (0.8)| (1) | (1.4)| (1.8)| (1.8)| (0.4)| (0.7)| (1.4)| (1.7)| (1.6)|
| 200  | 15.8 | 15.4| 14.6| 14.8| 11.8| 17.1 | 18.5| 21.9| 24.1| 27.5|
|      | (0.5)| (0.5)| (0.7)| (1) | (1) | (0.2)| (0.3)| (0.6)| (0.7)| (1.1)|
|      |      |     |     |     |     |      |     |     |     |     |     |
| n    |      |     |     |     |     |      |     |     |     |     |     |
| 50   | 6.3  | 8.8 | 6.3 | 6.8 | 5.5 | 20.8 | 15  | 10.7| 10.2| 7.8 |
|      | (1) | (1.3)| (1.4)| (1.4)| (1.2)| (1.5)| (1.6)| (1.6)| (1.6)| (1.5)|
| 100  | 15.4 | 7.3 | 7.1 | 6.5 | 7.2 | 43   | 41.4| 29.5| 22.6| 12.9|
|      | (1.5)| (1.5)| (1.9)| (1.7)| (2) | (0.7)| (2.1)| (2.7)| (2.9)| (2.4)|
| 200  | 39.9 | 29.7| 23.4| 12.3| 11.1| 49.1 | 65  | 76.3| 73.4| 51.5|
|      | (0.7)| (1.9)| (2.5)| (2.3)| (3.3)| (0.1)| (0.7)| (1.6)| (2.8)| (3.8)|
Table 8: Average number of selected variables 100 replications of Relaxed Lasso. Standard errors given in parentheses.

| n   | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| p 50 | 9.1 (0.8) | 22.4 (1.3) | 30.4 (1.6) | 31.8 (1.6) | 36.5 (4.4) | 12.2 (0.2) | 19.5 (0.3) | 27.5 (0.6) | 39.9 (1) | 43 |
| (SNR = 1.25; $\sigma = 5.23$) | | | | | | 6 | 7.2 (0.2) | 8.2 | 13.7 | 13.4 |
| p 100 | 7.8 (0.6) | 10.5 (1.3) | 11.5 (1.6) | 18.6 (3.2) | 30.2 (4.4) | 6 | 7.2 | 8.2 | 13.7 | 13.4 |
| (SNR = 5; $\sigma = 2.61$) | | | | | | (0.2) (0.2) | (1) | (2.8) | (2.7) |
| p 200 | 6.5 (0.3) | 6.5 (0.4) | 7.2 (0.9) | 8.3 (1.1) | 8.7 (1.7) | 5.6 (0.2) | 5.7 | 6.8 | 5.4 | 5.6 |
| (SNR = 1.25; $\sigma = 9.04$) | | | | | | (0.2) (0.2) | (1) | (2.8) | (2.7) |
| p 50 | 15.2 (1.3) | 26.4 (2.4) | 31 (2.7) | 26.6 (2.7) | 30.8 (2.9) | 23.8 (0.7) | 47.1 (1.3) | 40 | 44.8 | 39.4 |
| (SNR = 1.25; $\sigma = 9.04$) | | | | | | (1.1) (2.2) | (2.6) | (2.6) | (2.7) |
| p 100 | 21.1 (0.8) | 23.6 (1.6) | 40.1 (3.9) | 35.2 (4.3) | 46.2 (5) | 21.6 (0.7) | 25.9 (1) | 49.8 | 78.6 | 91.5 |
| (SNR = 5; $\sigma = 4.52$) | | | | | | (1.1) (1.2) | (4.2) | (5.3) | (5.1) |
| p 200 | 21.2 (0.7) | 25.3 (1.2) | 29.9 (1.8) | 30 (2.6) | 34.1 (4.2) | 17.1 (0.4) | 18.5 | 21.8 | 23.4 | 25.8 |
| (SNR = 1.25; $\sigma = 16.52$) | | | | | | (0.4) (0.7) | (1.3) | (1.5) | (2.4) |
| p 50 | 16.1 (1.3) | 27.4 (2.5) | 20 (2.4) | 25.8 (2.7) | 26.8 (2.8) | 27.9 (1.4) | 35.7 (2.5) | 43.1 | 36.4 | 28.1 |
| (SNR = 1.25; $\sigma = 16.52$) | | | | | | (1.4) (2.5) | (2.6) | (2.7) | (2.7) |
| p 100 | 31 (1.1) | 26.2 (2.1) | 37.5 (4.4) | 41.6 (4.9) | 38.8 (4.8) | 46.7 (0.5) | 57.4 (1.8) | 81.5 | 77.6 | 61 |
| (SNR = 5; $\sigma = 8.26$) | | | | | | (1.8) (4.5) | (5.6) | (5.7) |
| p 200 | 45.1 (0.4) | 56.8 (1.5) | 60.5 (2.7) | 61.7 (5.3) | 62 (7.8) | 49.4 (0.1) | 71 (1.1) | 92 | 159.9 | 205.7 |
| (SNR = 1.25; $\sigma = 16.52$) | | | | | | (1.1) (2.3) | (8.5) | (9) | |
Table 9: Average false discovery rate from 100 replications of CV 1SE. Standard errors given in parentheses.

| n   | p 50 | 100 | 200 | 400 | 800 | p 50 | 100 | 200 | 400 | 800 |
|-----|------|-----|-----|-----|-----|------|-----|-----|-----|-----|-----|
|     | p^* = 5; SNR = 1.25; \( \bar{\sigma} \) = 5.23 |  |  |  |  | p^* = 5; SNR = 5; \( \bar{\sigma} \) = 2.61 |  |  |  |  |  |
| 50  | 0.3  | 0.4 | 0.4 | 0.5 | 0.4 | 0.4  | 0.5 | 0.6 | 0.6 | 0.6 | 0.6 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
| 100 | 0.3  | 0.4 | 0.4 | 0.4 | 0.4 | 0.3  | 0.4 | 0.5 | 0.5 | 0.5 | 0.5 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
| 200 | 0.2  | 0.2 | 0.3 | 0.3 | 0.4 | 0.2  | 0.3 | 0.3 | 0.4 | 0.4 | 0.4 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
|     | p^* = 15; SNR = 1.25; \( \bar{\sigma} \) = 9.04 |  |  |  |  | p^* = 15; SNR = 5; \( \bar{\sigma} \) = 4.52 |  |  |  |  |  |
| 50  | 0.2  | 0.3 | 0.3 | 0.2 | 0.3 | 0.4  | 0.4 | 0.4 | 0.4 | 0.4 | 0.4 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
| 100 | 0.3  | 0.3 | 0.4 | 0.3 | 0.3 | 0.3  | 0.5 | 0.6 | 0.6 | 0.6 | 0.6 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
| 200 | 0.3  | 0.4 | 0.4 | 0.5 | 0.5 | 0.3  | 0.4 | 0.5 | 0.6 | 0.6 | 0.6 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
|     | p^* = 50; SNR = 1.25; \( \bar{\sigma} \) = 16.52 |  |  |  |  | p^* = 50; SNR = 5; \( \bar{\sigma} \) = 8.26 |  |  |  |  |  |
| 50  | 0.0  | 0.2 | 0.2 | 0.2 | 0.2 | 0.0  | 0.2 | 0.3 | 0.3 | 0.3 | 0.3 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
| 100 | 0.0  | 0.1 | 0.2 | 0.2 | 0.2 | 0.0  | 0.3 | 0.3 | 0.3 | 0.3 | 0.2 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
| 200 | 0.0  | 0.2 | 0.3 | 0.3 | 0.3 | 0.0  | 0.3 | 0.5 | 0.5 | 0.5 | 0.5 |
|     | (0)  | (0) | (0) | (0) | (0) | (0)  | (0) | (0) | (0) | (0) | (0) |
Table 10: Average false discovery rate for 100 replications of AR2 CV 1SE. Standard errors given in parentheses.

|    | 50 | 100 | 200 | 400 | 800 | 50 | 100 | 200 | 400 | 800 |
|----|----|-----|-----|-----|-----|----|-----|-----|-----|-----|
| n  | p* = 5; SNR = 1.25; \( \bar{\sigma} = 5.23 \) | p* = 5; SNR = 5; \( \bar{\sigma} = 2.61 \) |
| 50 | 0.3 0.3 0.4 0.4 0.4 | 0.2 0.3 0.3 0.3 0.4 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
| 100| 0.1 0.1 0.2 0.2 0.2 | 0.1 0.1 0.1 0.2 0.2 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
| 200| 0 (0) 0.1 0.1 0.1 | 0 (0) 0 (0) 0 (0) 0 (0) |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |

| n  | p* = 15; SNR = 1.25; \( \bar{\sigma} = 9.04 \) | p* = 15; SNR = 5; \( \bar{\sigma} = 4.52 \) |
|----|----|-----|-----|-----|-----|----|-----|-----|-----|-----|
| 50 | 0.3 0.4 0.5 0.6 0.7 | 0.3 0.3 0.4 0.5 0.5 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
| 100| 0.3 0.3 0.4 0.3 0.4 | 0.2 0.4 0.4 0.5 0.5 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
| 200| 0.2 0.2 0.3 0.3 0.4 | 0.1 0.2 0.3 0.3 0.4 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |

| n  | p* = 50; SNR = 1.25; \( \bar{\sigma} = 16.52 \) | p* = 50; SNR = 5; \( \bar{\sigma} = 8.26 \) |
|----|----|-----|-----|-----|-----|----|-----|-----|-----|-----|
| 50 | 0 (0) 0.2 0.5 0.6 0.7 | 0 (0) 0.2 0.4 0.5 0.7 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
| 100| 0 (0) 0.2 0.4 0.5 0.6 | 0 (0) 0.3 0.4 0.4 0.5 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
| 200| 0 (0) 0.3 0.4 0.5 0.5 | 0 (0) 0.3 0.4 0.5 0.5 |
|    | (0) (0) (0) (0) (0) | (0) (0) (0) (0) (0) |
Table 11: Average false discovery rate of 100 replications for \( \alpha \)-modified CV 1SE. Standard errors given in parentheses.

| \( p \) | \( 50 \) | \( 100 \) | \( 200 \) | \( 400 \) | \( 800 \) | \( 50 \) | \( 100 \) | \( 200 \) | \( 400 \) | \( 800 \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( p^* = 5; SNR = 1.25; \bar{\sigma} = 5.23 \) | \( 0.2 \) | \( 0.3 \) | \( 0.3 \) | \( 0.4 \) | \( 0.3 \) | \( 0.2 \) | \( 0.3 \) | \( 0.3 \) | \( 0.4 \) | \( 0.4 \) |
| 50 | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) |
| \( p^* = 15; SNR = 1.25; \bar{\sigma} = 9.04 \) | \( 0.2 \) | \( 0.2 \) | \( 0.3 \) | \( 0.2 \) | \( 0.2 \) | \( 0.3 \) | \( 0.4 \) | \( 0.3 \) | \( 0.4 \) | \( 0.4 \) |
| 100 | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) |
| \( p^* = 50; SNR = 1.25; \bar{\sigma} = 16.52 \) | \( 0.2 \) | \( 0.2 \) | \( 0.3 \) | \( 0.2 \) | \( 0.2 \) | \( 0.1 \) | \( 0.2 \) | \( 0.2 \) | \( 0.3 \) | \( 0.3 \) |
| 200 | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) |
| \( p^* = 50; SNR = 5; \bar{\sigma} = 2.61 \) | \( 0.2 \) | \( 0.3 \) | \( 0.3 \) | \( 0.4 \) | \( 0.4 \) | \( 0.1 \) | \( 0.1 \) | \( 0.1 \) | \( 0.1 \) | \( 0.1 \) |
| 50 | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) |
| \( p^* = 15; SNR = 5; \bar{\sigma} = 4.52 \) | \( 0.2 \) | \( 0.2 \) | \( 0.3 \) | \( 0.2 \) | \( 0.2 \) | \( 0.2 \) | \( 0.1 \) | \( 0.2 \) | \( 0.2 \) | \( 0.2 \) |
| 100 | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) | \( (0) \) |
| \( p^* = 50; SNR = 5; \bar{\sigma} = 8.26 \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) |
| 200 | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) | \( 0 (0) \) |
Table 12: Average false discovery rate for 100 replications of Relaxed Lasso. Standard errors given in parentheses.

| p  | 50   | 100  | 200  | 400  | 800  | 50   | 100  | 200  | 400  | 800  |
|----|------|------|------|------|------|------|------|------|------|------|
| n  |      |      |      |      |      |      |      |      |      |      |
| 50 | 0.4  | 0.6  | 0.6  | 0.7  | 0.7  | 0.4  | 0.5  | 0.5  | 0.7  | 0.7  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 100| 0.3  | 0.3  | 0.3  | 0.4  | 0.5  | 0.1  | 0.1  | 0.2  | 0.2  | 0.2  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 200| 0.1  | 0.1  | 0.1  | 0.2  | 0.2  | 0.1  | 0.1  | 0.1  | 0   | 0   |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 50 | p* = 5; SNR = 1.25; σ = 5.23 | 0.4  | 0.5  | 0.6  | 0.7  | 0.7  | 0.7  | 0.7  | 0.7  | 0.7  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 100| p* = 15; SNR = 1.25; σ = 9.04 | 0.4  | 0.5  | 0.6  | 0.5  | 0.6  | 0.3  | 0.4  | 0.6  | 0.7  | 0.8  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 200| p* = 50; SNR = 1.25; σ = 16.52 | 0.3  | 0.4  | 0.5  | 0.5  | 0.5  | 0.1  | 0.2  | 0.2  | 0.3  | 0.3  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 50 | p* = 15; SNR = 5; σ = 2.61 | 0.4  | 0.5  | 0.6  | 0.7  | 0.7  | 0.7  | 0.7  | 0.7  | 0.7  | 0.7  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
| 100| p* = 50; SNR = 5; σ = 8.26 | 0.3  | 0.4  | 0.5  | 0.5  | 0.5  | 0.3  | 0.5  | 0.5  | 0.7  | 0.8  |
|    | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  | (0)  |
Table 13: Average number of false negatives from 100 replications of CV 1SE. Standard errors given in parentheses.

|         | p = 5; SNR = 1.25; $\bar{\sigma}$ = 5.23 |         | p = 5; SNR = 5; $\bar{\sigma}$ = 2.61 |
|---------|------------------------------------------|---------|----------------------------------------|
|         | 50                                       | 100     | 200                                    | 400 | 800 | 50 | 100 | 200 | 400 | 800 |
| n       | 1.9 (0.2) 2.2 (0.2) 2.9 (0.2) 2.9 (0.2) 3.5 (0.2) | 0.2 (0.1) 0.3 (0.1) 0.6 (0.1) 0.6 (0.1) 1.1 (0.1) | 0.1 (0.1) 0.1 (0.1) 0.1 (0.1) 0.1 (0.1) 0.1 (0.1) |
|         | 0.6 (0.1) 0.6 (0.1) 0.9 (0.1) 1.2 (0.1) 1.4 (0.1) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
|         | 0.2 (0.1) 0.2 (0.1) 0.2 (0.1) 0.2 (0.1) 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
| n       | 10.1 (0.4) 12.2 (0.3) 12.8 (0.3) 14 (0.2) 14.3 (0.2) | 4.2 (0.4) 6.5 (0.5) 9.7 (0.4) 11.7 (0.3) 12.7 (0.2) | 0.4 (0.1) 0.8 (0.1) 1.6 (0.1) 3.2 (0.1) 5.5 (0.1) |
|         | 4.5 (0.3) 6.9 (0.4) 9.1 (0.4) 11.4 (0.3) 12.3 (0.3) | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) |
|         | 1 (0) 2 (0) 3 (0) 4.2 (0.2) 5.7 (0.2) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) | 0 (0) |
| n       | 42.2 (1.1) 44.4 (0.8) 47.9 (0.4) 48.7 (0.3) 49.2 (0.3) | 27.8 (1.4) 40.4 (1.0) 45.3 (0.9) 47.3 (0.9) 48.6 (0.9) | 0.5 (0.5) 1.1 (0.6) 1.1 (0.6) 0.9 (0.6) 1.1 (0.6) |
|         | 30.8 (1.2) 41.5 (1.1) 45.8 (0.7) 47.6 (0.5) 48.5 (0.5) | 6.3 (1.2) 19.3 (1.1) 34.5 (1.1) 42 (1.0) 46.3 (1.0) | (0.1) | (0.2) | (0.3) | (0.6) | (0.8) | (0.8) |
|         | 10.9 (0.6) 21.6 (0.9) 29.8 (1.1) 40 (0.9) 44.4 (0.7) | 0.8 (0.1) 2.1 (0.2) 5.8 (0.3) 13.6 (0.3) 26.3 (0.6) | (0.1) | (0.2) | (0.3) | (0.6) | (0.8) | (0.8) |
Table 14: Average number of false negatives for 100 replications of AR2 CV 1SE. Standard errors given in parentheses.

|      | p 50 | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|      | \(p^* = 5; SNR = 1.25; \bar{\sigma} = 5.23\) | \(p^* = 5; SNR = 5; \bar{\sigma} = 2.61\) |
| n 50 | 2.1  | 2.4 | 2.9 | 3   | 3.6 | 0.6 | 0.8 | 1.4 | 1.4 | 1.9 |
|      | (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)|
| n 100| 1.1  | 1.2 | 1.6 | 1.8 | 1.9 | 0.1 | 0.1 | 0.1 | 0.2 | 0.3 |
|      | (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0) | (0) | (0) | (0.1)| (0.1)|
| n 200| 0.4  | 0.4 | 0.5 | 0.6 | 0.7 | 0 (0)| 0 (0)| 0 (0)| 0.1 | 0 (0)|
|      | (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0) | (0) | (0) | (0) | (0) |
|      | \(p^* = 15; SNR = 1.25; \bar{\sigma} = 9.04\) | \(p^* = 15; SNR = 5; \bar{\sigma} = 4.52\) |
| n 50 | 9.5  | 11.1| 12.3| 13.6| 13.7| 5.2 | 8.4 | 10.5| 12.4| 13.1|
|      | (0.4)| (0.3)| (0.3)| (0.1)| (0.1)| (0.4)| (0.5)| (0.3)| (0.2)| (0.2)|
| n 100| 4.9  | 7.2 | 9.1 | 11.3| 11.6| 1   | 1.4 | 2.7 | 4.8 | 6.5 |
|      | (0.3)| (0.4)| (0.4)| (0.3)| (0.3)| (0.1)| (0.2)| (0.2)| (0.3)| (0.4)|
| n 200| 1.9  | 2.9 | 4   | 5.2 | 6.8 | 0.2 | 0.4 | 0.4 | 0.7 | 0.9 |
|      | (0.2)| (0.2)| (0.2)| (0.3)| (0.3)| (0) | (0.1)| (0.1)| (0.1)| (0.1)|
|      | \(p^* = 50; SNR = 1.25; \bar{\sigma} = 16.52\) | \(p^* = 50; SNR = 5; \bar{\sigma} = 8.26\) |
| n 50 | 35.6 | 43.7| 46.8| 48.2| 49  | 27.5| 41.8| 45.2| 47.5| 48.7|
|      | (1.3)| (0.7)| (0.4)| (0.3)| (0.1)| (1.4)| (0.8)| (0.5)| (0.3)| (0.2)|
| n 100| 23.6 | 34.7| 42.2| 45.1| 46.7| 6   | 18.1| 31  | 40.1| 45.1|
|      | (1.3)| (1.3)| (0.9)| (0.6)| (0.4)| (0.6)| (1)  | (1) | (0.8)| (0.5)|
| n 200| 7.7  | 16.1| 24.2| 32.9| 38.4| 1   | 3   | 7   | 15.4| 26.9|
|      | (0.4)| (0.8)| (0.9)| (1)  | (0.8)| (0.1)| (0.2)| (0.4)| (0.7)| (0.8)|
Table 15: Average number of false negatives of 100 replications for \( \alpha \)-modified CV 1SE. Standard errors given in parentheses.

|     | \( p^* = 5; SNR = 1.25; \bar{\sigma} = 5.23 \) | \( p^* = 5; SNR = 5; \bar{\sigma} = 2.61 \) | \( p^* = 15; SNR = 1.25; \bar{\sigma} = 9.04 \) | \( p^* = 15; SNR = 5; \bar{\sigma} = 4.52 \) | \( p^* = 50; SNR = 1.25; \bar{\sigma} = 16.52 \) | \( p^* = 50; SNR = 5; \bar{\sigma} = 8.26 \) |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
|     | \( n \)                        | \( n \)                        | \( n \)                        | \( n \)                        | \( n \)                        | \( n \)                        |
| 50  | 2.6 (0.2)                      | 0.4 (0.1)                      | 4.9 (0.4)                      | 0 (0)                          | 1.1 (0.1)                      | 29.2 (1.5)                     |
|     | 2.7 (0.2)                      | 0.5 (0.1)                      | 7.3 (0.5)                      | 0 (0)                          | 1.1 (0.1)                      | 40.4 (1.5)                     |
|     | 3.2 (0.2)                      | 0.5 (0.1)                      | 10.3 (0.6)                     | 0 (0)                          | 1.4 (0.1)                      | 45.4 (1.6)                     |
|     | 3 (0.1)                        | 0.5 (0.1)                      | 12.1 (0.4)                     | 0 (0)                          | 2.4 (0.1)                      | 48.8 (1.4)                     |
|     | 3.6 (0.1)                      | 0.7 (0.1)                      | 12.8 (0.3)                     | 0 (0)                          | 4.2 (0.1)                      | 50 (1.1)                       |
| 100 | 1.1 (0.1)                      | 0 (0)                          | 6.3 (0.2)                      | 0 (0)                          | 0.7 (0.1)                      | 7 (0.1)                        |
|     | 1.1 (0.1)                      | 0.7 (0.1)                      | 8.2 (0.3)                      | 0 (0)                          | 2.4 (0.1)                      | 10 (0.1)                       |
|     | 1.4 (0.1)                      | 0.7 (0.1)                      | 8.2 (0.3)                      | 0 (0)                          | 4.2 (0.1)                      | 12 (0.1)                       |
|     | 1.9 (0.1)                      | 0.7 (0.1)                      | 8.2 (0.3)                      | 0 (0)                          | 6 (0.1)                        | 14 (0.1)                       |
|     | 2 (0.1)                        | 0.7 (0.1)                      | 8.2 (0.3)                      | 0 (0)                          | 8 (0.1)                        | 16 (0.1)                       |
| 200 | 0.4 (0.1)                      | 0 (0)                          | 0.2 (0.1)                      | 0 (0)                          | 0.2 (0.1)                      | 0 (0)                          |
|     | 0.4 (0.1)                      | 0 (0)                          | 0.3 (0.1)                      | 0 (0)                          | 0.3 (0.1)                      | 0 (0)                          |
|     | 0.5 (0.1)                      | 0 (0)                          | 0.4 (0.1)                      | 0 (0)                          | 0.4 (0.1)                      | 0 (0)                          |
|     | 0.7 (0.1)                      | 0 (0)                          | 0.6 (0.1)                      | 0 (0)                          | 0.6 (0.1)                      | 0 (0)                          |
|     | 0.7 (0.1)                      | 0 (0)                          | 0.8 (0.1)                      | 0 (0)                          | 0.8 (0.1)                      | 0 (0)                          |
Table 16: Average number of false negatives for 100 replications of Relaxed Lasso. Standard errors given in parentheses.

| n   | p  | 50 | 100 | 200 | 400 | 800 | 50 | 100 | 200 | 400 | 800 |
|-----|----|----|-----|-----|-----|-----|----|-----|-----|-----|-----|
|     | p* | 50 | 1.4 | 1.4 | 2   | 2   | 2.5| 0.2 | 0.3 | 0.6 | 0.4 | 0.7 |
|     |    | (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)|
| 50  | p* | 100| 0.6 | 0.7 | 0.8 | 1.1 | 1.2| 0   | 0   | 0.1 | 0.1 | 0.1 |
|     |    | (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)|
|     | p* | 200| 0.2 | 0.1 | 0.3 | 0.4 | 0.6| 0   | 0   | 0   | 0   | 0   |
|     |    | (0)| (0.1)| (0.1)| (0.1)| (0.1)| (0.1)| (0)| (0)| (0)| (0)| (0)|
|     | p* | 50 | 7.8 | 8.4 | 10  | 12  | 12.8| 2.8 | 3.2 | 6.5 | 8.3 | 10.6|
|     |    | (0.4)| (0.4)| (0.4)| (0.3)| (0.2)| (0.3)| (0.3)| (0.4)| (0.3)| (0.3)|
| 100 | p* | 100| 3.2 | 5   | 5.7 | 8.5 | 9.5| 0.6 | 1.2 | 1.8 | 2.4 | 3.6 |
|     |    | (0.2)| (0.3)| (0.3)| (0.4)| (0.3)| (0.1)| (0.1)| (0.2)| (0.2)| (0.3)|
|     | p* | 200| 0.9 | 1.9 | 2.7 | 4.1 | 5.3| 0.2 | 0.4 | 0.6 | 0.9 | 1.1 |
|     |    | (0.1)| (0.2)| (0.2)| (0.3)| (0.3)| (0)| (0.1)| (0.1)| (0.1)| (0.1)|
|     | p* | 50 | 33.9| 34.6| 43.2| 44.9| 47.2| 22.1| 29.5| 34.9| 41.4| 46.1|
|     |    | (1.3)| (1.3)| (0.8)| (0.5)| (0.3)| (1.4)| (1.3)| (0.9)| (0.6)| (0.4)|
| 100 | p* | 100| 19  | 32.9| 35.9| 40.1| 43.9| 3.3 | 13.4| 19.8| 30.7| 38.7|
|     |    | (1.1)| (1.2)| (1.3)| (1)| (0.6)| (0.5)| (0.9)| (1.1)| (1.1)| (0.9)|
|     | p* | 200| 4.9 | 12.6| 20.7| 29.8| 37  | 0.7 | 2.2 | 5   | 8.8 | 15.2|
|     |    | (0.4)| (0.8)| (0.9)| (1)| (1)| (0.1)| (0.3)| (0.4)| (0.6)| (0.8)|
Table 17: Average prediction bias from 100 replications of CV 1SE. Standard errors given in parentheses.

| p    | 50       | 100      | 200      | 400      | 800      | p* = 5; SNR = 1.25; σ = 5.23 | 50       | 100      | 200      | 400      | 800      | p* = 5; SNR = 5; σ = 2.61 |
|------|----------|----------|----------|----------|----------|----------------------------|----------|----------|----------|----------|----------|----------------------------|
| n    |          |          |          |          |          | 28.8 (0.9) 31.4 (0.7)       | 14.2 (0.4) 16.5 (0.5)       |
|      | 50       | 28.8     | 31.4     | 28.5     | 32.4     | (0.8)         | (0.8)     | (0.7)     | (0.8)     | (0.8)     | (0.8)     | (0.8)         | (0.8)     | (0.7)     | (0.8)     | (0.8)     | (0.8)     |
|      | 100      | 28.2     | 32.1     | 29.6     | 35.8     | (0.8)         | (0.9)     | (0.9)     | (0.9)     | (0.9)     | (0.9)     | (0.8)         | (0.8)     | (0.8)     | (0.8)     | (0.8)     | (0.8)     |
|      | 200      | 29.9     | 33.3     | 32.2     | 36.4     | (0.8)         | (0.9)     | (0.8)     | (0.8)     | (0.8)     | (0.8)     | (0.8)         | (0.8)     | (0.8)     | (0.8)     | (0.8)     | (0.8)     |
|      |          |          |          |          |          | n = 15; SNR = 1.25; σ = 9.04 | p* = 15; SNR = 5; σ = 4.52 |
|      | 50       | 56.4 (1.4) | 61.4 (1.1) | 61.7 (1.1) | 65.2 (1) | (1)         | (1.4)     | (1.4)     | (1.5)     | (1.7)     | (1.7)     | (1.1)         | (1.4)     | (1.5)     | (1.7)     | (1.7)     | (1.7)     |
|      | 100      | 60 (1.2) | 69.4 (1.4) | 77.5 (1.3) | 86.4 (1.5) | (1.5)     | (0.6)     | (0.6)     | (0.9)     | (1.2)     | (1.5)     | (0.5)         | (0.6)     | (0.9)     | (1.2)     | (1.5)     | (1.5)     |
|      | 200      | 63.9 (1.2) | 71.9 (1.3) | 79.9 (1.2) | 87.3 (1.5) | (1.5)     | (0.5)     | (0.6)     | (0.6)     | (0.7)     | (0.7)     | (0.5)         | (0.6)     | (0.7)     | (0.7)     | (0.7)     | (0.7)     |
|      |          |          |          |          |          | n = 50; SNR = 1.25; σ = 16.52 | p* = 50; SNR = 5; σ = 8.26 |
|      | 50       | 113.4 (2) | 113.2 (2.1) | 122.6 (1.8) | 123.8 (2) | (2)         | (2.8)     | (3)       | (3.1)     | (3.4)     | (3)       | (2.8)         | (3)       | (3.1)     | (3.4)     | (3)       | (3)       |
|      | 100      | 133.6 (2.8) | 158.6 (2.8) | 167.7 (2.5) | 174.8 (2.2) | (2.2)     | (1)       | (2.6)     | (3.4)     | (3.9)     | (3.5)     | (2.8)         | (2.6)     | (3.4)     | (3.9)     | (3.5)     | (3.5)     |
|      | 200      | 133.9 (2.1) | 169.7 (2.6) | 192.2 (3.2) | 221.6 (3.3) | (3)       | (0.8)     | (0.9)     | (1.1)     | (2)       | (3.5)     | (2.6)         | (3.3)     | (1.1)     | (2)       | (3.5)     | (3.5)     |
Table 18: Average prediction bias for 100 replications of AR2 CV 1SE. Standard errors given in parentheses.

| n  | 50     | 100    | 200    | 400    | 800    | 50     | 100    | 200    | 400    | 800    |
|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|    | \(p^* = 5; SNR = 1.25; \bar{\sigma} = 5.23\) | \(p^* = 5; SNR = 5; \bar{\sigma} = 2.61\) |
| 50 | 27.4(0.8) | 28.6(0.6) | 29.5(0.6) | 31.4(0.7) | 33(0.6) | 15.3(0.6) | 15.4(0.5) | 18.3(0.6) | 19.4(0.7) | 21(0.7) |
| 100| 27.5(0.8) | 28.3(0.9) | 29.5(0.9) | 32.9(0.8) | 34.9(0.9) | 13.7(0.4) | 14.8(0.4) | 14.3(0.4) | 15.9(0.5) | 16.8(0.6) |
| 200| 27.6(0.8) | 28.8(0.9) | 28.9(0.9) | 30.9(0.9) | 32.1(0.9) | 14(0.4) | 15(0.4) | 14.2(0.4) | 15.9(0.5) | 15.8(0.6) |

| n  | 50     | 100    | 200    | 400    | 800    | 50     | 100    | 200    | 400    | 800    |
|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|    | \(p^* = 15; SNR = 1.25; \bar{\sigma} = 9.04\) | \(p^* = 15; SNR = 5; \bar{\sigma} = 4.52\) |
| 50 | 57.5(1.1) | 60.6(0.9) | 61.6(0.9) | 63.7(0.9) | 66.1(0.8) | 35.7(1.2) | 43.1(1.2) | 46.2(1.2) | 50.7(1.2) | 52.3(1.2) |
| 100| 61.7(1.1) | 71.1(1.2) | 78.4(1.1) | 83.3(1.1) | 85.1(1.1) | 32.2(0.7) | 34.6(0.6) | 41.5(0.6) | 47.8(0.6) | 53.8(0.6) |
| 200| 62.2(1.3) | 70.6(1.4) | 78.3(1.2) | 85.1(1.3) | 93.6(1.5) | 32(0.6) | 34.1(0.6) | 37.1(0.6) | 40.2(0.6) | 43.9(0.6) |

| n  | 50     | 100    | 200    | 400    | 800    | 50     | 100    | 200    | 400    | 800    |
|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|    | \(p^* = 50; SNR = 1.25; \bar{\sigma} = 16.52\) | \(p^* = 50; SNR = 5; \bar{\sigma} = 8.26\) |
| 50 | 112.3(1.7) | 117.1(1.7) | 122.1(1.4) | 121.6(1.4) | 123.3(1.5) | 73(2.4) | 90.9(2.4) | 97.8(2.3) | 103.7(2.5) | 103.5(2.1) |
| 100| 131.1(1.9) | 151.9(1.7) | 162.1(1.7) | 168.8(1.8) | 170.5(1.8) | 67(1.7) | 86.1(1.7) | 105.8(2.5) | 126.2(3.1) | 141.9(3.2) |
| 200| 134.5(1.8) | 167.6(1.7) | 188.1(2.1) | 210.1(1.9) | 223(2) | 70.2(1.7) | 84.9(1.7) | 99.3(1.7) | 119(1.7) | 142.1(3) |
Table 19: Average number of prediction bias of 100 replications for \( \alpha \)-modified CV 1SE. Standard errors given in parentheses.

| \( n \) | \( p^* = 5; SNR = 1.25; \bar{\sigma} = 5.23 \) | \( p^* = 5; SNR = 5; \bar{\sigma} = 2.61 \) |
|-------|--------------------------|--------------------------|
| 50    | 29 (0.9) 30.4 (0.7) 32.1 (0.8) 32.7 (0.7) 34.9 (0.7) | 13.5 (0.4) 14.3 (0.5) 16.5 (0.5) 17.8 (0.5) 19.6 (0.6) |
| 100   | 26.9 (0.9) 27.5 (0.9) 29.5 (0.9) 33.7 (0.9) 35.8 (0.9) | 12.7 (0.4) 13.5 (0.4) 13.2 (0.4) 14.9 (0.4) 15.8 (0.5) |
| 200   | 26.9 (0.9) 28.4 (0.9) 29 (0.9) 31 (0.9) 32.1 (0.9) | 13.3 (0.4) 13.9 (0.4) 14.6 (0.4) 14.5 (0.4) |
|       | (0.9) (0.8) (0.9) (0.9) (0.9) | (0.4) (0.4) (0.4) (0.4) (0.4) |
| 50    | 62.1 (1.3) 66 (1) 64.9 (1) 66.9 (0.9) 68.4 (0.8) | 35 (1.1) 40.8 (1.4) 46.3 (1.5) 51.5 (1.6) 53.4 (1.7) |
| 100   | 66 (1.4) 74.7 (1.5) 83.9 (1.4) 89.8 (1.3) 90.9 (1.4) | 30.9 (0.6) 34 (0.6) 40.3 (0.9) 46.1 (1.2) 52.5 (1.5) |
| 200   | 64.3 (1.4) 73 (1.5) 82.4 (1.5) 89.5 (1.6) 100.4 (1.9) | 30.6 (0.5) 32.7 (0.6) 36 (0.6) 39.2 (0.7) 42.5 (0.7) |
|       | (1.4) (1.5) (1.5) (1.6) (1.9) | (0.5) (0.6) (0.6) (0.7) (0.7) |
| 50    | 121.5 (1.6) 120.2 (1.7) 125.6 (1.5) 126.5 (1.6) 125.3 (1.4) | 78.6 (2.8) 90.5 (3) 104.8 (3.1) 108.5 (3.4) 111 (2.9) |
| 100   | 150.1 (2.9) 170.5 (2.2) 176.8 (1.9) 179.5 (1.8) 181.6 (1.8) | 68 (1.1) 93.9 (2.8) 119.4 (3.5) 140.2 (4.3) 158.2 (3.8) |
| 200   | 140 (2.2) 192.1 (3.5) 216.4 (3.4) 240.5 (2.9) 247.7 (2.4) | 68.8 (0.9) 84.7 (1) 99.2 (1.1) 119.2 (1.8) 152.7 (4.3) |
|       | (2.2) (3.5) (3.4) (2.9) (2.4) | (0.9) (1) (1.1) (1.8) (4.3) |

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Table 20: Average prediction bias for 100 replications of Relaxed Lasso. Standard errors given in parentheses.

| n   | \( p^* = 5; SNR = 1.25; \sigma = 5.23 \) | \( p^* = 5; SNR = 5; \sigma = 2.61 \) | \( p^* = 15; SNR = 1.25; \sigma = 9.04 \) | \( p^* = 15; SNR = 5; \sigma = 4.52 \) | \( p^* = 50; SNR = 1.25; \sigma = 16.52 \) | \( p^* = 50; SNR = 5; \sigma = 8.26 \) |
|-----|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|-----------------------------------------|
| 50  | 24.5 (0.7) 27.8 (0.8) 31.8 (0.8) 32.4 (0.7) 34.1          | 12.3 (0.5) 13.4 (0.5) 15.6 (0.5) 17 (0.5) 17.8 | 8.5 (0.3) 9.8 (0.4) 9.8 (0.4) 11.6 (0.6) 12.3 | 7.7 (0.3) 8.3 (0.4) 7.8 (0.4) 8.1 (0.4) 7.9 | 28.3 (0.5) 32.7 (0.5) 34.8 (0.9) 36.9 (0.9) 39.9 | 25.9 (0.6) 31.6 (0.7) 38 (0.7) 42.9 (0.7) 47 |
| 100 | 21.4 (0.7) 23.2 (0.9) 24.4 (0.8) 28.9 (1.1) 34.5 (1.1) | 8.5 (0.3) 9.8 (0.4) 9.8 (0.4) 11.6 (0.6) 12.3 | 7.7 (0.3) 8.3 (0.4) 7.8 (0.4) 8.1 (0.4) 7.9 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 |
| 200 | 18.2 (0.8) 17.4 (0.8) 18.9 (1) 22.7 (1) 23.7 (1)       | 7.7 (0.3) 8.3 (0.4) 7.8 (0.4) 8.1 (0.4) 7.9 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 28.3 (0.5) 32.7 (0.5) 34.8 (0.9) 36.9 (0.9) 39.9 | 25.9 (0.6) 31.6 (0.7) 38 (0.7) 42.9 (0.7) 47 |
|     | 50.5 (1) 58.1 (1.1) 59.6 (1) 61.8 (1) 63.7 (1)           | 28.3 (0.5) 32.7 (0.5) 34.8 (0.9) 36.9 (0.9) 39.9 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 |
|     | 52 (0.9) 61.1 (1) 68.7 (1.3) 75.4 (1.3) 80.8 (1.4)      | 25.9 (0.6) 31.6 (0.7) 38 (0.7) 42.9 (0.7) 47 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 28.3 (0.5) 32.7 (0.5) 34.8 (0.9) 36.9 (0.9) 39.9 | 25.9 (0.6) 31.6 (0.7) 38 (0.7) 42.9 (0.7) 47 |
|     | 50.8 (1) 60.9 (1) 68.2 (1.1) 75.7 (1.2) 83.2 (1.5)     | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 | 23.2 (0.5) 25.7 (0.7) 29.6 (0.7) 33.1 (0.7) 36.6 |
|     | 102.7 (1.9) 109.4 (1.5) 114.8 (1.4) 117.3 (1.4) 119.6 (1.5) | 63.9 (1.6) 71.4 (2.1) 73.5 (2.4) 80.5 (3.1) 86 (2.8) | 60.2 (1.6) 75.3 (2.1) 89 (3.1) 101.9 (3.1) 116.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 |
|     | 113.4 (1.7) 135.8 (2.5) 150.8 (2.2) 158.2 (2.3) 161.4 (2.3) | 60.2 (1.6) 75.3 (2.1) 89 (3.1) 101.9 (3.1) 116.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 |
|     | 117.2 (1.4) 144.7 (1.7) 163.3 (2) 186.8 (2.6) 206.3 (3.2) | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 | 61.4 (1.7) 77 (2.6) 90 (3.1) 108.1 (3.1) 119.9 |
We also evaluated the results for more limited data settings—fixing $n$ and $p$, but varying $p^*$ and SNR across more methods of tuning parameter selection, including AIC, BIC, ERIC with $\nu = 0.5$, and GCV. For AIC and BIC, fixed $\hat{\sigma}^2$ was calculated as the MSE of the model chosen by CV 1SE whereas ERIC was calculated assuming unknown variance for the likelihood. An additional alternative we consider is a simple Cutoff Method. Without splitting the data, $R^2$ is calculated for all $\lambda$ and the largest $\lambda$ that achieves some threshold is chosen as the optimal. For the following results, a simple cutoff of 0.95 was used in the R2 Cut method. Table 21 gives the average false negatives from the setting considered in the main document: $n = p = 100$. Figure 7 gives the average false positives and prediction bias for data with $n = 100$ and $p = 800$. Table 8 gives average false positives and prediction bias for $n = 1000$ and $p = 100$. Tables 22, 23, and 24 give the false discovery rates, prediction bias, and false negatives for this last setting.
Table 21: Average false negatives for 100 replications of simulated data with \( n = 100 \). Standard errors given in parentheses.

|               | \( p = 100 \) |               |               | \( p = 800 \) |               |               |
|---------------|----------------|---------------|----------------|----------------|---------------|----------------|
|               | \( p^* = 5 \) | \( p^* = 33 \) | \( p^* = 5 \) | \( p^* = 33 \) | \( p^* = 5 \) | \( p^* = 33 \) |
| CV Min        | 0.21           | 14.69         | 0.01           | 3.86           | 0.65          | 27.88         |
|               | (0.05)         | (0.66)        | (0.01)         | (0.25)         | (0.07)        | (0.47)        |
| CV 1SE        | 0.66           | 23.84         | 0.02           | 7.32           | 1.36          | 31.36         |
|               | (0.09)         | (0.72)        | (0.01)         | (0.48)         | (0.12)        | (0.32)        |
| Mod Min       | 0.43           | 19.07         | 0.01           | 4.31           | 1.03          | 29.33         |
|               | (0.07)         | (0.97)        | (0.01)         | (0.31)         | (0.11)        | (0.54)        |
| Mod 1SE       | 1.15           | 26.38         | 0.08           | 8.35           | 1.98          | 31.49         |
|               | (0.11)         | (0.76)        | (0.03)         | (0.59)         | (0.13)        | (0.32)        |
| AR2 Min       | 0.44           | 11.7          | 0.02           | 4.04           | 0.95          | 26.77         |
|               | (0.07)         | (0.69)        | (0.01)         | (0.3)          | (0.09)        | (0.49)        |
| AR2 1SE       | 1.19           | 19.71         | 0.13           | 8.17           | 1.90          | 29.92         |
|               | (0.1)          | (0.78)        | (0.04)         | (0.54)         | (0.13)        | (0.32)        |
| R2 Cut        | 0.07           | 4.36          | 0.01           | 2.90           | 0.47          | 23.25         |
|               | (0.03)         | (0.18)        | (0.01)         | (0.15)         | (0.06)        | (0.24)        |
| AIC           | 0.09           | 0.49          | 0.01           | 0.75           | 0.5           | 22.01         |
|               | (0.03)         | (0.09)        | (0.01)         | (0.1)          | (0.06)        | (0.28)        |
| BIC           | 0.06           | 2.20          | 0.01           | 1.67           | 0.52          | 22.39         |
|               | (0.02)         | (0.2)         | (0.01)         | (0.12)         | (0.06)        | (0.27)        |
| ERIC          | 1.40           | 32.99         | 0.06           | 31.91          | 0.44          | 20.18         |
|               | (0.15)         | (0.01)        | (0.03)         | (0.57)         | (0.06)        | (0.28)        |
| GCV           | 0.19           | 12.02         | 0.01           | 3.48           | 0.44          | 20.18         |
|               | (0.04)         | (0.62)        | (0.01)         | (0.25)         | (0.06)        | (0.28)        |
| Relaxed Min   | 0.74           | 17.5          | 0.10           | 5.59           | 1.41          | 27.82         |
|               | (0.09)         | (0.77)        | (0.03)         | (0.43)         | (0.11)        | (0.54)        |
| Relaxed 1SE   | 1.15           | 23.55         | 0.14           | 8.71           | 1.98          | 31.12         |
|               | (0.1)          | (0.76)        | (0.04)         | (0.63)         | (0.13)        | (0.36)        |

Figure 7: Average false positives and prediction error plots from 100 replications of simulated data with \( n = 100 \) and \( p = 800 \) for both \( \text{SNR} = 0.5 \) and \( \text{SNR} = 2 \). Thin dotted lines represent the mean ± one standard error.
Figure 8: Average false positives and prediction error plots from 100 replications of simulated data with \( n = 1000 \) and \( p = 100 \) for both \( \text{SNR} = 0.5 \) and \( \text{SNR} = 2 \). Thin dotted lines represent the mean \( \pm \) one standard error.

B.2 Simulation Study for Non-Convex Penalties

This set of simulations examines the effect of the modification on the non-convex penalties, SCAD (Fan and Li 2001) and MC+ (Zhang 2010). Both penalties are most commonly expressed in terms of its derivative which, for SCAD at a single point, \( \beta_j \), is:

\[
P'_{\lambda}(\beta_j) = \lambda_1 I(\beta_j \leq \lambda_1) + \frac{(\lambda_2 \beta_j - \beta_j)}{(\lambda_2 - 1) \beta_j} I(\beta_j > \lambda_1)\]

(108)

The derivative of the MC+ penalty for a single coefficient, \( \beta_j \), is:

\[
P'_{\lambda}(\beta_j) = \begin{cases} 
    \text{sign}(\beta_j) \left( \lambda_1 - \frac{|\beta_j|}{\lambda_2} \right) & \text{if } |\beta_j| \leq \lambda_1 \lambda_2 \\
    0 & \text{otherwise}
\end{cases}
\]

(109)

We investigated SCAD with \( \lambda_2 = 3.7 \) and MC+ with \( \lambda_2 = 3 \), both with and without the \( \alpha \)-Modification using 10-fold CV with a 1SE rule. We generated data from the following
Table 22: Average false discovery rate for 100 replications of simulated data with \( n = 1000 \). Standard errors given in parentheses.

|                | SNR = 1.25 | SNR = 5 |
|----------------|------------|---------|
|                | \( p^* = 5 \) | \( p^* = 33 \) | \( p^* = 5 \) | \( p^* = 33 \) |
| CV Min         | 0.66 (0.02) | 0.49 (0.01) | 0.7 (0.01) | 0.51 (0) |
| CV 1SE         | 0.05 (0.01) | 0.23 (0.01) | 0.06 (0.01) | 0.24 (0.01) |
| Mod Min        | 0.32 (0.03) | 0.36 (0.01) | 0.29 (0.03) | 0.32 (0.01) |
| Mod 1SE        | 0 (0)       | 0.08 (0.01) | 0 (0)       | 0.05 (0)  |
| AR2 Min        | 0.32 (0.03) | 0.38 (0.01) | 0.28 (0.03) | 0.32 (0.01) |
| AR2 1SE        | 0 (0)       | 0.08 (0.01) | 0 (0)       | 0.03 (0)  |
| R2 Cut         | 0 (0)       | 0.07 (0.01) | 0 (0)       | 0.06 (0.01) |
| AIC            | 0.94 (0)    | 0.66 (0)   | 0.92 (0)    | 0.65 (0)  |
| BIC            | 0.85 (0.01) | 0.63 (0)   | 0.54 (0.02) | 0.6 (0)   |
| ERIC           | 0.05 (0.01) | 0.09 (0)   | 0.11 (0.01) | 0.12 (0.01) |
| GCV            | 0.67 (0.02) | 0.5 (0.01) | 0.69 (0.02) | 0.5 (0.01) |
| Relaxed Min    | 0.06 (0.02) | 0.11 (0.01) | 0.07 (0.01) | 0.03 (0.01) |
| Relaxed 1SE    | 0 (0)       | 0.04 (0)   | 0 (0)       | 0.01 (0)  |

model:

\[
y_{n \times 1} = \beta^*_{0} + X_{n \times p} \beta^*_{p \times 1} + \epsilon_{n \times 1} \tag{110}\]

where \( X \) was drawn from a Standard Normal distribution, \( \beta^*_{0} \) and \( \beta^* \) were drawn from the same distribution as the Lasso simulations above, and SNR = \{1.25, 5\}. We include an intercept here because the R package ncvreg does not have include a no-intercept option. The optimization to find \( \hat{\alpha}_\lambda \) is slightly more complicated with an intercept term. We now optimize the likelihood function over both \( \alpha \) and \( \beta_0 \), the intercept. Figures 9 and 10 include the average False Positives, False Negatives, and Prediction Bias across increasing values of \( p^* \) when \( n = 100 \) and \( p = 100 \) over 500 replications. Note the similarity between the two approaches.

Although average false positives for the non-convex penalties are lower than or equal to that of the \( \alpha \)-Modified Lasso, their average false negatives tend to be higher, particularly as \( p^* \) increases. Prediction bias is slightly worse for \( \alpha \)-Modified SCAD and MC+ than unmodified, but similar to the \( \alpha \)-Modified Lasso for small \( p^* \) and similar to unmodified for larger \( p^* \). This is a fairly limited simulation study, capturing only one \( n \) and \( p \). Furthermore, for non-convex penalties, we no longer have a guarantee that \( \hat{\alpha}_\lambda \) will always be greater than
Table 23: Average prediction bias for 100 replications of simulated data with \( n = 1000 \). Standard errors given in parentheses.

|                | SNR = 1.25 |              | SNR = 5  |              |
|----------------|------------|-------------|----------|-------------|
|                | \( p^* = 5 \) | \( p^* = 33 \) | \( p^* = 5 \) | \( p^* = 33 \) |
| CV Min         | 23.91 (0.63) | 111.74 (1.28) | 12.25 (0.3) | 57.55 (0.64) |
| CV 1SE         | 38.88 (0.92) | 140.13 (1.62) | 19.3 (0.44) | 70.78 (0.91) |
| Mod Min        | 17.44 (0.63) | 111.09 (1.39) | 8.55 (0.29) | 52.7 (0.68)  |
| Mod 1SE        | 33.79 (0.89) | 137.34 (1.79) | 17.16 (0.44) | 66.12 (0.93) |
| AR2 Min        | 17.55 (0.66) | 111.28 (1.41) | 8.52 (0.28) | 52.75 (0.68) |
| AR2 1SE        | 34.6 (0.91)  | 136.49 (1.71) | 19.07 (0.49) | 69.65 (0.87) |
| R2 Cut         | 80.99 (2.05) | 193.08 (2.39) | 51.8 (1.63) | 110.93 (2.11) |
| AIC            | 45.52 (0.89) | 132.32 (1.34) | 19.59 (0.49) | 66.35 (0.67) |
| BIC            | 28.16 (0.84) | 125.86 (1.36) | 12.49 (0.3) | 61.21 (0.72) |
|ERIC           | 31.45 (0.78) | 168.25 (2.4)  | 15.55 (0.37) | 81.95 (0.89) |
|GCV             | 24.54 (0.65) | 112.68 (1.32) | 12.58 (0.3) | 58.01 (0.64) |
|Relaxed Min     | 14.07 (0.59) | 101.53 (1.87) | 7.63 (0.36) | 44.14 (0.79) |
|Relaxed 1SE     | 33.99 (0.92) | 132.93 (1.91) | 17.04 (0.41) | 60.95 (0.88) |

Figure 9: Average false positives, prediction error, and false negatives from 500 replications of simulated data with \( n = 100 \) and \( p = 100 \) for SNR = 5. Thin dotted lines represent the mean ± one standard error.

1, so the bias-reduction aim of the modification is in question. Allowing the \( \alpha \)-Modification to compete with and enhance non-convex penalties may be the subject of future research.
Table 24: Average false negatives for 100 replications of simulated data with $n = 1000$. Standard errors given in parentheses.

|                | SNR = 1.25 |            |            | SNR = 5 |            |            |
|----------------|------------|------------|------------|---------|------------|------------|
|                | $p^* = 5$  | $p^* = 33$ | $p^* = 5$  | $p^* = 33$ |            |            |
| CV Min         | 0 (0)      | 0.04 (0.02)| 0 (0)      | 0 (0)    |            |            |
| CV 1SE         | 0 (0)      | 0.26 (0.05)| 0 (0)      | 0 (0)    |            |            |
| Mod Min        | 0 (0)      | 0.11 (0.03)| 0 (0)      | 0 (0)    |            |            |
| Mod 1SE        | 0.02 (0.01)| 0.97 (0.1) | 0 (0)      | 0.06 (0.02)|            |            |
| AR2 Min        | 0 (0)      | 0.1 (0.03) | 0 (0)      | 0 (0)    |            |            |
| AR2 1SE        | 0.03 (0.02)| 0.97 (0.1) | 0 (0)      | 0.09 (0.03)|            |            |
| R2 Cut         | 0.02 (0.01)| 1.23 (0.11)| 0 (0)      | 0.07 (0.03)|            |            |
| AIC            | 0 (0)      | 0 (0)      | 0 (0)      | 0 (0)    |            |            |
| BIC            | 0 (0)      | 0 (0)      | 0 (0)      | 0 (0)    |            |            |
| ERIC           | 0 (0)      | 0.97 (0.13)| 0 (0)      | 0.03 (0.02)|            |            |
| GCV            | 0 (0)      | 0.04 (0.02)| 0 (0)      | 0 (0)    |            |            |
| Relaxed Min    | 0 (0)      | 1.01 (0.12)| 0 (0)      | 0.13 (0.03)|            |            |
| Relaxed 1SE    | 0 (0)      | 1.94 (0.18)| 0 (0)      | 0.22 (0.04)|            |            |

Figure 10: Average false positives, prediction error, and false negatives from 500 replications of simulated data with $n = 100$ and $p = 100$ for SNR = 5. Thin dotted lines represent the mean ± one standard error.
B.3 Simulation Study for Correlated Predictors

Additional results from the simulation study described in the main document for correlated predictors can be found in Tables 29 through 40. These results include average model size, average FDR, average false negatives, and average prediction bias for traditional CV, \( \alpha \)-modified and Average \( R^2 \) CV, and the Relaxed Lasso using a minimum APE rule.

**Table 25:** Average number of selected variables from 100 replications of CV 1SE with correlated predictors. Standard errors given in parentheses.

| p  | 50  | 100 | 200 | 400 | 800  | 50  | 100 | 200 | 400 | 800  |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| n  | p* = 5; SNR = 1.25; \( \sigma = 4.86 \) | p* = 5; SNR = 5; \( \sigma = 2.43 \) | p* = 5; SNR = 1.25; \( \sigma = 4.86 \) | p* = 5; SNR = 5; \( \sigma = 2.43 \) |
| 50 | 5.2 (0.5) | 6.1 (0.3) | 6.9 (0.2) | 9.4 (0.6) |
|    | 9.7 (1.2) | 7.6 (0.6) | 7.5 (0.3) | 11.8 (0.3) |
|    | 9.2 (1.1) | 9.6 (0.9) | 8.8 (0.4) | 10.6 (0.4) |
|    | 11.6 (1.2) | 10.2 (1.1) | 10.5 (0.6) | 12.4 (0.5) |
|    | 9.6 (1.2) | 11.1 (1.2) | 11.7 (0.8) | 15.6 (0.8) |
|    | 50 | 100 | 200 | 400 | 800  | 50  | 100 | 200 | 400 | 800  |
| 50 | 5.6 (0.5) | 10.2 (0.6) | 13.6 (0.6) | 14.3 (0.6) |
|    | 7 (0.9) | 10.1 (0.8) | 15 (0.8) | 14.8 (0.6) |
|    | 9.8 (1.4) | 10.3 (1.0) | 15.3 (1.0) | 13.9 (1.2) |
|    | 7.2 (1) | 9.7 (1.0) | 16.2 (1.5) | 13.9 (1.2) |
|    | 9 (1.3) | 13.1 (1.8) | 15 (1.3) | 15.3 (1.4) |
|    | 14.3 (1.3) | 18.6 (1.8) | 19 (1.3) | 18.3 (1.4) |
|    | 14.8 (1.2) | 21 (1.3) | 23.1 (1.8) | 12.7 (1.1) |
|    | 13.9 (1.2) | 25.1 (2.1) | 27.4 (2.1) | 15.5 (1.4) |
|    | 13.6 (1.1) | 27.2 (2.1) | 29.9 (2.1) | 13.9 (1.3) |
|    | 15.3 (1.4) | 25.7 (2.2) | 34.4 (2.1) | 11.1 (1.5) |
| 100| 69 |
Table 26: Average number of selected variables from 100 replications of AR2 CV 1SE with correlated predictors. Standard errors given in parentheses.

| n   | $p^* = 5; SNR = 1.25; \bar{\sigma} = 4.86$ | $p^* = 5; SNR = 5; \bar{\sigma} = 2.43$ | $p^* = 50; SNR = 1.25; \bar{\sigma} = 14.71$ | $p^* = 50; SNR = 5; \bar{\sigma} = 7.36$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 50  | p = 5; SNR = 1/2; $\bar{\sigma} = 4.86$ | p = 5; SNR = 5; $\bar{\sigma} = 2.43$ | p = 50; SNR = 1/2; $\bar{\sigma} = 14.71$ | p = 50; SNR = 5; $\bar{\sigma} = 7.36$ |
| 100 | p = 5; SNR = 1/2; $\bar{\sigma} = 4.86$ | p = 5; SNR = 5; $\bar{\sigma} = 2.43$ | p = 50; SNR = 1/2; $\bar{\sigma} = 14.71$ | p = 50; SNR = 5; $\bar{\sigma} = 7.36$ |
| 200 | p = 5; SNR = 1/2; $\bar{\sigma} = 4.86$ | p = 5; SNR = 5; $\bar{\sigma} = 2.43$ | p = 50; SNR = 1/2; $\bar{\sigma} = 14.71$ | p = 50; SNR = 5; $\bar{\sigma} = 7.36$ |

| p   | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 50  | 3.2 | 5.8 | 7.4 | 6.8 | 4   | 6.2 | 7.5 | 6.5 | 6.7 | 5.7 | 6.2 | 7.5 | 6.5 | 6.7 | 5.7 |
| 100 | 3.9 | 4.1 | 6.5 | 6.5 | 7.2 | 6   | 7   | 7.9 | 8.6 | 10.1| 6   | 6.1 | 6.6 | 7   | 7.8 |
| 200 | 4.7 | 4.5 | 5.4 | 6   | 5.6 | 6   | 6.1 | 6.6 | 7   | 7.8 | 6   | 6.1 | 6.6 | 7   | 7.8 |

| p   | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 50  | 5.4 | 5.5 | 4.4 | 4.2 | 4   | 11.9| 7.3 | 6.6 | 5.7 | 6.9 | 6.2 | 7.5 | 6.5 | 6.7 | 5.7 |
| 100 | 8.5 | 9.2 | 7.8 | 6.8 | 8.6 | 15.6| 17.1| 18.1| 16.3| 13.7| 11.9| 7.3 | 6.6 | 5.7 | 6.9 |
| 200 | 10.9| 11.6| 11.2| 11.7| 10.8| 16.1| 18.6| 21  | 21.4| 26  | 16.1| 18.6| 21  | 21.4| 26  |

| p   | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 50  | 6.1 | 7.3 | 6.7 | 5.2 | 5   | 10.9| 9.1 | 7.9 | 5.8 | 5.7 | 10.9| 9.1 | 7.9 | 5.8 | 5.7 |
| 100 | 12.2| 10.1| 6.7 | 7   | 5   | 29.5| 27.5| 17.6| 12.4| 8.9 | 29.5| 27.5| 17.6| 12.4| 8.9 |
| 200 | 22.1| 20.1| 16.2| 19.5| 13.1| 40.9| 43.8| 45.1| 44.7| 34.1| 40.9| 43.8| 45.1| 44.7| 34.1|
Table 27: Average number of selected variables from 100 replications of α-modified CV 1SE with correlated predictors. Standard errors given in parentheses.

| n  | \(p^* = 5; SNR = 1.25; \sigma = 4.86\) | \(p^* = 5; SNR = 5; \sigma = 2.43\) |
|----|---------------------------------------|--------------------------------------|
|    | 50                                    | 100                                  |
|    | 100                                   | 200                                  |
|    | 200                                   | 400                                  |
|    | 400                                   | 800                                  |
|    | 50                                    | 100                                  |
|    | 100                                   | 200                                  |
|    | 200                                   | 400                                  |
|    | 400                                   | 800                                  |
|    | \(n\) \(p^* = 5; SNR = 1.25; \sigma = 4.86\) | \(p^* = 5; SNR = 5; \sigma = 2.43\) |
|    | \(n\) \(p^* = 15; SNR = 1.25; \sigma = 8.22\) | \(p^* = 15; SNR = 5; \sigma = 4.11\) |
|    | \(n\) \(p^* = 50; SNR = 1.25; \sigma = 14.71\) | \(p^* = 50; SNR = 5; \sigma = 7.36\) |

| n  | 50 | 100 | 200 | 400 | 800 | 50 | 100 | 200 | 400 | 800 |
|----|----|-----|-----|-----|-----|----|-----|-----|-----|-----|
| p  | 3.5 | 7.7 | 8.5 | 10.1 | 8.2 | 7.1 | 9.8 | 10 | 12.8 | 11.9 |
|    | (0.5) | (1.2) | (1.3) | (1.4) | (1.3) | (0.7) | (1) | (1) | (1.3) | (1.4) |
| p  | 3.8 | 4.1 | 6.2 | 8 | 7.5 | 6.8 | 8 | 8.8 | 10.9 | 12.6 |
|    | (0.3) | (0.3) | (0.9) | (1.3) | (1.3) | (0.3) | (0.4) | (0.7) | (0.8) | (1.4) |
| p  | 4.9 | 4.6 | 5.1 | 5.8 | 5.6 | 6.3 | 6.7 | 7.3 | 7.8 | 9.4 |
|    | (0.3) | (0.3) | (0.3) | (0.5) | (0.6) | (0.2) | (0.3) | (0.4) | (0.4) | (0.7) |
| p  | 4 | 5.5 | 9.9 | 7.7 | 8.6 | 12.4 | 12.8 | 12.2 | 12.8 | 13.3 |
|    | (0.5) | (1) | (1.5) | (1.3) | (1.4) | (1.1) | (1.3) | (1.4) | (1.5) | (1.6) |
| p  | 7.2 | 7 | 6.8 | 6.9 | 11.1 | 16.9 | 17.8 | 21.2 | 24.9 | 20 |
|    | (0.7) | (0.9) | (1.2) | (1.7) | (2.2) | (0.6) | (1.1) | (1.5) | (2.1) | (2.1) |
| p  | 10.8 | 10.3 | 9.8 | 9.8 | 7.8 | 17.1 | 19.2 | 22.9 | 23 | 28 |
|    | (0.7) | (0.8) | (0.8) | (1) | (1.3) | (0.5) | (0.6) | (0.9) | (1.2) | (1.6) |
| p  | 4 | 8.5 | 9 | 8.8 | 6.9 | 11.4 | 14.5 | 11.6 | 11.2 | 10.9 |
|    | (0.7) | (1.4) | (1.5) | (1.3) | (1.3) | (1.2) | (1.6) | (1.3) | (1.5) | (1.4) |
| p  | 8.2 | 5 | 5 (1) | 5.8 | 4.7 | 29.5 | 27.7 | 18.4 | 14 | 13.2 |
|    | (1) | (1.1) | (1.4) | (1.1) | (1.5) | (2.2) | (2.2) | (1.9) | (1.9) |
| p  | 19.6 | 11.8 | 8.6 | 7.8 | 12.4 | 42.1 | 45.7 | 47.1 | 45.5 | 32.4 |
|    | (1.6) | (1.5) | (1.5) | (1.7) | (3.2) | (0.9) | (2.1) | (3.1) | (3.5) | (3.7) |
Table 28: Average number of selected variables from 100 replications of Relaxed Lasso with correlated predictors. Standard errors given in parentheses.

| n  | 50      | 100     | 200     | 400     | 800     | 50      | 100     | 200     | 400     | 800     |
|----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
|    | p = 5; SNR = 1.25; σ = 4.86 | p = 5; SNR = 5; σ = 2.43 |                      |                      |                      |                      |                      |                      |                      |                      |
| 50 | 7.9     | 24.1    | 39.6    | 46.6    | 44.6    | 13.1    | 24.8    | 31.4    | 44.1    | 45      |
|    | (0.6)   | (2.7)   | (3.5)   | (3.5)   | (3.4)   | (1)     | (2.7)   | (3.1)   | (3.4)   | (3.4)   |
| 100| 7.9     | 11.5    | 16.4    | 24.9    | 29.1    | 6.6     | 10.8    | 12      | 28.9    | 29.6    |
|    | (0.5)   | (1)     | (2.1)   | (4)     | (4.3)   | (0.6)   | (0.9)   | (1)     | (4.1)   | (4.5)   |
| 200| 8.3     | 8.8     | 10.6    | 16.4    | 14      | 7.2     | 8.1     | 9.4     | 11.6    | 12.7    |
|    | (0.4)   | (0.6)   | (1.1)   | (2.5)   | (1.5)   | (0.4)   | (0.5)   | (0.8)   | (1.3)   | (1.9)   |

| n  | p = 15; SNR = 1.25; σ = 8.22 | p = 15; SNR = 5; σ = 4.11 |                      |                      |                      |                      |                      |                      |                      |                      |
|----|-----------------------------|---------------------------|                      |                      |                      |                      |                      |                      |                      |                      |
| 50 | 10                          | 26.9                      | 36.2                | 40                  | 37.2                | 18.3                | 36.5                | 48.6                | 51.3                | 50.8                |
|    | (0.8)                       | (2.8)                     | (3.4)               | (3.6)               | (3.4)               | (1.1)               | (2.8)               | (3.4)               | (3.3)               | (3.3)               |
| 100| 17.3                        | 19.5                      | 21.7                | 30                  | 45.5                | 21.5                | 24                  | 34.9                | 59.5                | 66.8                |
|    | (1)                         | (1.5)                     | (2.8)               | (4.5)               | (5.8)               | (0.9)               | (1.3)               | (3)                 | (5.1)               | (6)                 |
| 200| 17.3                        | 22                        | 24.4                | 29                  | 27.7                | 19.2                | 23.7                | 29.9                | 31.1                | 38.5                |
|    | (0.8)                       | (1.3)                     | (1.6)               | (2.3)               | (2.8)               | (0.5)               | (1.1)               | (2)                 | (2.2)               | (3.4)               |

| n  | p = 50; SNR = 1.25; σ = 14.71 | p = 50; SNR = 5; σ = 7.36 |                      |                      |                      |                      |                      |                      |                      |                      |
|----|--------------------------------|---------------------------|                      |                      |                      |                      |                      |                      |                      |                      |
| 50 | 10.8                          | 29.5                      | 33                  | 39.1                | 43.3                | 18.9                | 39                  | 47.1                | 45.5                | 49.8                |
|    | (1)                           | (2.8)                     | (3.5)               | (3.5)               | (3.6)               | (1.2)               | (2.9)               | (3.1)               | (3.5)               | (3.4)               |
| 100| 18.5                          | 18.6                      | 22.1                | 35.9                | 32.8                | 36.9                | 41.3                | 49.4                | 58.2                | 66.8                |
|    | (1.4)                         | (1.8)                     | (3.5)               | (4.6)               | (4.9)               | (1.3)               | (2.3)               | (4.9)               | (5.8)               | (6.7)               |
| 200| 32.2                          | 35.3                      | 34.7                | 34.5                | 42.8                | 46.2                | 60.3                | 69.8                | 79.2                | 90.5                |
|    | (1.4)                         | (2.3)                     | (2.7)               | (3.3)               | (5.8)               | (0.6)               | (1.9)               | (3.4)               | (5.3)               | (8.5)               |
Table 29: Average false discovery rate from 100 replications of CV 1SE with correlated predictors. Standard errors given in parentheses.

| n   | p 50 | 100 | 200 | 400 | 800 | 50 | 100 | 200 | 400 | 800 |
|-----|------|-----|-----|-----|-----|----|-----|-----|-----|-----|
| 50  | p^* = 5; SNR = 1.25; \( \bar{\sigma} = 4.86 \) | p^* = 5; SNR = 1.25; \( \bar{\sigma} = 4.86 \) |
| 100 | (0) 0.6 0.6 0.7 0.7 | (0) 0.6 0.7 0.7 0.8 |
| 200 | (0) 0.4 0.6 0.7 0.7 | (0) 0.4 0.5 0.6 0.6 |

| n   | p^* = 15; SNR = 1.25; \( \bar{\sigma} = 8.22 \) | p^* = 15; SNR = 1.25; \( \bar{\sigma} = 8.22 \) |
|-----|------|------|-----|-----|-----|----|-----|-----|-----|
| 50  | 0.4 0.5 0.6 0.6 0.6 | 0.4 0.5 0.6 0.7 0.7 |
| 100 | (0) 0.4 0.5 0.6 0.6 | (0) 0.5 0.6 0.6 0.7 |
| 200 | (0) 0.3 0.4 0.5 0.6 0.6 | (0) 0.3 0.4 0.5 0.6 0.7 |

| n   | p^* = 50; SNR = 1.25; \( \bar{\sigma} = 14.71 \) | p^* = 50; SNR = 1.25; \( \bar{\sigma} = 14.71 \) |
|-----|------|------|-----|-----|-----|----|-----|-----|-----|
| 50  | 0 (0) 0.3 0.5 0.6 0.6 | 0 (0) 0.3 0.5 0.6 0.7 |
| 100 | (0) 0.2 0.4 0.6 0.7 | (0) 0.3 0.4 0.5 0.6 |
| 200 | (0) 0.2 0.4 0.5 0.6 | (0) 0.3 0.4 0.5 0.6 |

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Table 30: Average false discovery rate for 100 replications of AR2 CV 1SE with correlated predictors. Standard errors given in parentheses.

| n   | 50  | 100 | 200 | 400 | 800  | 50  | 100 | 200 | 400 | 800  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |     |     |     |
| 50  | 0.4 | 0.5 | 0.6 | 0.7 | 0.7 | 0.3 | 0.4 | 0.4 | 0.5 | 0.5  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
| 100 | 0.3 | 0.3 | 0.4 | 0.5 | 0.6 | 0.2 | 0.3 | 0.3 | 0.4 | 0.5  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
| 200 | 0.2 | 0.2 | 0.3 | 0.3 | 0.3 | 0.1 | 0.2 | 0.2 | 0.2 | 0.3  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |

| n   | 50  | 100 | 200 | 400 | 800  | 50  | 100 | 200 | 400 | 800  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |     |     |     |
| 50  | 0.4 | 0.6 | 0.7 | 0.8 | 0.8 | 0.4 | 0.4 | 0.5 | 0.7 | 0.7  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
| 100 | 0.3 | 0.4 | 0.5 | 0.5 | 0.7 | 0.3 | 0.4 | 0.5 | 0.5 | 0.5  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
| 200 | 0.2 | 0.3 | 0.4 | 0.4 | 0.5 | 0.2 | 0.3 | 0.4 | 0.5 | 0.5  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |

| n   | 50  | 100 | 200 | 400 | 800  | 50  | 100 | 200 | 400 | 800  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     |     |     |     |     |     |     |     |     |     |     |
| 50  | 0 (0) | 0.3 | 0.6 | 0.8 | 0.8 | 0 (0) | 0.3 | 0.5 | 0.6 | 0.8  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
| 100 | 0 (0) | 0.2 | 0.5 | 0.6 | 0.7 | 0 (0) | 0.3 | 0.4 | 0.5 | 0.6  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
| 200 | 0 (0) | 0.2 | 0.4 | 0.5 | 0.6 | 0 (0) | 0.2 | 0.4 | 0.5 | 0.5  |
|     | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0) | (0)  |
Table 31: Average false discovery rate of 100 replications for α-modified CV 1SE with correlated predictors. Standard errors given in parentheses.

| n  | p = 50; SNR = 1.25; $\bar{\sigma}$ = 4.86 | p = 5; SNR = 5; $\bar{\sigma}$ = 2.43 |
|----|---------------------------------------------|----------------------------------------|
| 50 | 0.4 0.5 0.6 0.7 0.6                      | 0.4 0.4 0.5 0.6 0.5                    |
|    | (0) (0) (0) (0) (0)                      | (0) (0) (0) (0) (0)                    |
| 100| 0.3 0.3 0.4 0.4 0.6                      | 0.3 0.4 0.4 0.5 0.5                    |
|    | (0) (0) (0) (0) (0)                      | (0) (0) (0) (0) (0)                    |
| 200| 0.2 0.2 0.3 0.3 0.3                      | 0.2 0.2 0.3 0.3 0.4                    |
|    | (0) (0) (0) (0) (0)                      | (0) (0) (0) (0) (0)                    |
| 50 | 0.3 0.5 0.6 0.5 0.7 0.5 0.7 0.5 0.6       | 0.3 0.4 0.5 0.7 0.6 0.7 0.7 0.6 0.7    |
|    | (0) (0) (0) (0) (0) (0) (0) (0) (0)     | (0) (0) (0) (0) (0) (0) (0) (0) (0)   |
| 100| 0.3 0.4 0.4 0.4 0.4 0.5 0.6 0.6 0.6       | 0.3 0.4 0.5 0.6 0.6 0.6 0.6 0.6 0.6    |
|    | (0) (0) (0) (0) (0) (0) (0) (0) (0)     | (0) (0) (0) (0) (0) (0) (0) (0) (0)   |
| 200| 0.2 0.3 0.3 0.4 0.4 0.2 0.3 0.3 0.4      | 0.2 0.3 0.4 0.5 0.6                    |
|    | (0) (0) (0) (0) (0) (0) (0) (0) (0)     | (0) (0) (0) (0) (0) (0) (0) (0) (0)   |
| 50 | 0 (0) 0 (0) 0 (0) 0 (0) 0 (0) 0 (0) 0 (0) | 0 (0) 0 (0) 0 (0) 0 (0) 0 (0) 0 (0) 0 (0) |
|    | (0) (0) (0) (0) (0) (0) (0) (0) (0)     | (0) (0) (0) (0) (0) (0) (0) (0) (0)   |
| 100| 0 (0) 0.2 0.4 0.5 0.6 0 (0) 0.3 0.4 0.5   | 0 (0) 0.3 0.4 0.5 0.6                   |
|    | (0) (0) (0) (0) (0) (0) (0) (0) (0)     | (0) (0) (0) (0) (0) (0) (0) (0) (0)   |
| 200| 0 (0) 0.2 0.3 0.4 0.5 0 (0) 0.2 0.4 0.5   | 0 (0) 0.2 0.4 0.5 0.5                   |
|    | (0) (0) (0) (0) (0) (0) (0) (0) (0)     | (0) (0) (0) (0) (0) (0) (0) (0) (0)   |
Table 32: Average false discovery rate for 100 replications of Relaxed Lasso with correlated predictors. Standard errors given in parentheses.

| n   | p  | p* = 5; SNR = 1.25; \sigma = 4.86 | p* = 5; SNR = 5; \sigma = 2.43 |
|-----|----|----------------------------------|----------------------------------|
|     | 50 | 0.6 0.7 0.8 0.9 0.9             | 0.5 0.6 0.7 0.8 0.8              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
|     | 100| 0.5 0.5 0.6 0.6 0.8             | 0.3 0.4 0.4 0.6 0.6              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
|     | 200| 0.4 0.4 0.4 0.5 0.5             | 0.2 0.3 0.3 0.3 0.3              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |

| n   | p  | p* = 15; SNR = 1.25; \sigma = 8.22 | p* = 15; SNR = 5; \sigma = 4.11 |
|-----|----|----------------------------------|----------------------------------|
|     | 50 | 0.5 0.7 0.8 0.8 0.9             | 0.5 0.7 0.8 0.9 0.9              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
|     | 100| 0.4 0.6 0.6 0.7 0.8             | 0.4 0.5 0.6 0.7 0.8              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
|     | 200| 0.4 0.5 0.6 0.7 0.7             | 0.2 0.3 0.4 0.5 0.6              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |

| n   | p  | p* = 50; SNR = 1.25; \sigma = 14.71 | p* = 50; SNR = 5; \sigma = 7.36 |
|-----|----|----------------------------------|----------------------------------|
|     | 50 | 0 (0) 0.4 0.6 0.8 0.8             | 0 (0) 0.4 0.6 0.7 0.8              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
|     | 100| 0 (0) 0.3 0.5 0.7 0.8             | 0 (0) 0.3 0.5 0.7 0.7              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
|     | 200| 0 (0) 0.3 0.5 0.6 0.7             | 0 (0) 0.3 0.5 0.6 0.7              |
|     | (0) (0) (0) (0) (0)             | (0) (0) (0) (0) (0)              |
Table 33: Average number of false negatives from 100 replications of CV 1SE with correlated predictors. Standard errors given in parentheses.

|     | 50       | 100      | 200      | 400      | 800      | 50       | 100      | 200      | 400      | 800      |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| n   |          |          |          |          |          |          |          |          |          |          |
| 50  | 3        | 3        | 3.6      | 3.7      | 4.1      | 1.2      | 1.3      | 1.8      | 2        | 2.5      |
|     | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    |
| 100 | 1.9      | 2        | 2.1      | 2.4      | 2.9      | 0.6      | 0.4      | 0.6      | 0.8      | 1        |
|     | (0.1)    | (0.1)    | (0.1)    | (0.2)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    |
| 200 | 0.8      | 1.1      | 1.3      | 1.5      | 1.7      | 0.1      | 0.2      | 0.2      | 0.3      | 0.4      |
|     | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    | (0.1)    |
|     |          |          |          |          |          |          |          |          |          |          |
| 50  | 11.9     | 12.8     | 13.2     | 14.1     | 14.3     | 7.2      | 9.3      | 11.1     | 12.4     | 12.8     |
|     | (0.3)    | (0.2)    | (0.2)    | (0.1)    | (0.1)    | (0.4)    | (0.3)    | (0.3)    | (0.2)    | (0.2)    |
| 100 | 8.7      | 10.1     | 11.4     | 12.6     | 13.1     | 2.9      | 4.7      | 5.5      | 6.9      | 8.6      |
|     | (0.4)    | (0.4)    | (0.3)    | (0.3)    | (0.2)    | (0.3)    | (0.4)    | (0.4)    | (0.4)    | (0.3)    |
| 200 | 5.6      | 6.4      | 8.2      | 9        | 10.4     | 1.7      | 1.6      | 2.5      | 3.6      | 3.9      |
|     | (0.4)    | (0.4)    | (0.4)    | (0.4)    | (0.3)    | (0.2)    | (0.2)    | (0.3)    | (0.4)    | (0.4)    |
|     |          |          |          |          |          |          |          |          |          |          |
| 50  | 44.4     | 44.4     | 47       | 48       | 49.3     | 36.2     | 40.2     | 44.9     | 46.7     | 48.1     |
|     | (0.7)    | (0.8)    | (0.4)    | (0.3)    | (0.2)    | (1.1)    | (0.8)    | (0.5)    | (0.3)    | (0.2)    |
| 100 | 38.6     | 44.3     | 46.7     | 47.4     | 48.6     | 19.4     | 29.3     | 39.3     | 43.4     | 46       |
|     | (1)      | (0.7)    | (0.4)    | (0.3)    | (0.2)    | (1.3)    | (1.2)    | (0.8)    | (0.5)    | (0.3)    |
| 200 | 28.1     | 35.8     | 41       | 44.4     | 45.8     | 7.7      | 14.9     | 21.6     | 27.1     | 35.9     |
|     | (1.3)    | (1)      | (0.7)    | (0.6)    | (0.4)    | (0.9)    | (1.3)    | (1.4)    | (1.1)    | (0.9)    |
Table 34: Average number of false negatives for 100 replications of AR2 CV 1SE with correlated predictors. Standard errors given in parentheses.

| n   | p  | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|-----|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|     | p* = 5; SNR = 1.25; σ = 4.86 |     |     |     |     |     | p* = 5; SNR = 5; σ = 2.43 |     |     |     |     |     |
| 50  | 3.5 | 3.5 | 3.7 | 3.8 | 4.3 |     | 2.1 | 1.9 | 2.5 | 2.8 | 3.3 |
|     | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) |     | (0.2) | (0.2) | (0.1) | (0.2) | (0.1) |
| 100 | 2.7 | 2.5 | 2.6 | 2.8 | 3.2 |     | 0.9 | 0.7 | 1.3 | 1.4 |     |
|     | (0.2) | (0.2) | (0.2) | (0.2) | (0.2) |     | (0.1) | (0.1) | (0.1) | (0.2) | (0.1) |
| 200 | 1.4 | 1.9 | 1.8 | 2.1 | 2.2 |     | 0.2 | 0.5 | 0.6 | 0.7 |     |
|     | (0.1) | (0.2) | (0.2) | (0.2) | (0.2) |     | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) |
|     | p* = 15; SNR = 1.25; σ = 8.22 |     |     |     |     |     | p* = 15; SNR = 5; σ = 4.11 |     |     |     |     |     |
| 50  | 12.1 | 13 | 13.7 | 14.3 | 14.6 |     | 8.5 | 11.5 | 12.5 | 13.5 | 13.6 |
|     | (0.3) | (0.2) | (0.2) | (0.1) | (0.1) |     | (0.5) | (0.3) | (0.2) | (0.2) | (0.1) |
| 100 | 9.6 | 10.9 | 12.1 | 12.9 | 13.4 |     | 3.9 | 5.8 | 7.1 | 8.4 | 10.3 |
|     | (0.4) | (0.4) | (0.3) | (0.3) | (0.2) |     | (0.4) | (0.5) | (0.5) | (0.4) | (0.4) |
| 200 | 6.8 | 7.7 | 9.1 | 9.9 | 11.2 |     | 2.5 | 2.1 | 3.1 | 4.7 | 4.9 |
|     | (0.5) | (0.5) | (0.5) | (0.4) | (0.4) |     | (0.4) | (0.3) | (0.4) | (0.4) | (0.4) |
|     | p* = 50; SNR = 1.25; σ = 14.71 |     |     |     |     |     | p* = 50; SNR = 5; σ = 7.36 |     |     |     |     |     |
| 50  | 43.9 | 45.6 | 47.6 | 48.8 | 49.4 |     | 39.1 | 44 | 46.4 | 48.2 | 49 |
|     | (0.9) | (0.6) | (0.3) | (0.2) | (0.1) |     | (1.2) | (0.7) | (0.4) | (0.2) | (0.1) |
| 100 | 37.8 | 43.1 | 46.8 | 47.7 | 48.9 |     | 20.5 | 30.9 | 40.9 | 45 | 47.4 |
|     | (1.2) | (1) | (0.5) | (0.3) | (0.2) |     | (1.6) | (1.4) | (0.9) | (0.5) | (0.3) |
| 200 | 27.9 | 35.6 | 41.3 | 43 | 46.1 |     | 9.1 | 17.4 | 24.4 | 28.6 | 36.8 |
|     | (1.6) | (1.4) | (1) | (0.9) | (0.5) |     | (1.1) | (1.5) | (1.6) | (1.3) | (1.1) |
Table 35: Average number of false negatives of 100 replications for α-modified CV 1SE with correlated predictors. Standard errors given in parentheses.

| p    | 50    | 100   | 200   | 400   | 800   | 50    | 100   | 200   | 400   | 800   |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| n    |       |       |       |       |       |       |       |       |       |       |
|      | p* = 5; SNR = 1.25; σ = 4.86 |       |       |       |       |       | p* = 5; SNR = 5; σ = 2.43 |
| 50   | 3.4   | 3.8   | 3.8   | 4.3   | 1.9   | 1.7   | 2.2   | 2.4   | 2.9   |
|      | (0.1) | (0.1) | (0.1) | (0.1) | (0.2) | (0.2) | (0.2) | (0.2) | (0.2) |
| 100  | 2.7   | 2.8   | 2.8   | 3.2   | 0.8   | 0.7   | 0.9   | 1     | 1.3   |
|      | (0.2) | (0.2) | (0.2) | (0.2) | (0.1) | (0.1) | (0.1) | (0.1) | (0.1) |
| 200  | 1.3   | 1.8   | 2.1   | 2.2   | 0.2   | 0.4   | 0.4   | 0.5   | 0.6   |
|      | (0.1) | (0.2) | (0.2) | (0.2) | (0)   | (0.1) | (0.1) | (0.1) | (0.1) |
|      | p* = 15; SNR = 1.25; σ = 8.22 |       |       |       |       |       | p* = 15; SNR = 5; σ = 4.11 |
| 50   | 12.8  | 13.3  | 13.3  | 14.1  | 14.4  | 8.1   | 10    | 11.5  | 12.7  | 13.1  |
|      | (0.2) | (0.2) | (0.3) | (0.2) | (0.1) | (0.5) | (0.4) | (0.3) | (0.3) | (0.2) |
| 100  | 10.3  | 12.5  | 13.4  | 13.4  | 3.4   | 5.5   | 6.3   | 7.4   | 9.7   |
|      | (0.4) | (0.4) | (0.3) | (0.2) | (0.2) | (0.4) | (0.4) | (0.5) | (0.4) | (0.4) |
| 200  | 7     | 8     | 9.5   | 10.3  | 11.9  | 2.1   | 2     | 2.8   | 4.4   | 4.6   |
|      | (0.5) | (0.5) | (0.4) | (0.4) | (0.3) | (0.3) | (0.3) | (0.4) | (0.4) | (0.4) |
|      | p* = 50; SNR = 1.25; σ = 14.71 |       |       |       |       |       | p* = 50; SNR = 5; σ = 7.36 |
| 50   | 46    | 45.1  | 47.1  | 48.3  | 49.3  | 38.6  | 41    | 45.2  | 47    | 48.4  |
|      | (0.7) | (0.8) | (0.5) | (0.3) | (0.2) | (1.2) | (0.9) | (0.6) | (0.4) | (0.2) |
| 100  | 41.8  | 46.5  | 47.7  | 48.4  | 49.1  | 20.5  | 30.9  | 40.9  | 45    | 46.9  |
|      | (1)   | (0.7) | (0.4) | (0.4) | (0.1) | (1.5) | (1.4) | (1)   | (0.6) | (0.4) |
| 200  | 30.4  | 41.1  | 44.9  | 46.9  | 47.2  | 7.9   | 16.4  | 23.5  | 29    | 37.8  |
|      | (1.6) | (1.1) | (0.7) | (0.6) | (0.5) | (0.9) | (1.4) | (1.6) | (1.3) | (1.1) |
Table 36: Average number of false negatives for 100 replications of Relaxed Lasso with correlated predictors. Standard errors given in parentheses.

| n   | p = 50; SNR = 1.25; $\bar{\sigma}$ = 4.86 | p = 50; SNR = 5; $\bar{\sigma}$ = 2.43 |
|-----|-------------------------------------------|----------------------------------------|
|     | 50                                        | 100                                    |
|     | 50                                        | 100                                    |
|     | 2.6                                       | 2.3                                    |
|     | 2.5                                       | 2.8                                    |
|     | 2.8                                       | 3.3                                    |
|     | (0.1)                                     | (0.2)                                  |
|     | (0.1)                                     | (0.1)                                  |
|     | (0.1)                                     | (0.1)                                  |
|     | 50                                        | 50                                    |
|     | 1.7                                       | 1.4                                    |
|     | 1.8                                       | 2.1                                    |
|     | 2.5                                       | 2.5                                    |
|     | (0.1)                                     | (0.1)                                  |
|     | (0.1)                                     | (0.1)                                  |
|     | (0.1)                                     | (0.1)                                  |
|     | 100                                       | 100                                    |
|     | 0.5                                       | 1                                     |
|     | 1.2                                       | 1.4                                    |
|     | 1.4                                       | 1.5                                    |
|     | (0.1)                                     | (0.1)                                  |
|     | (0.1)                                     | (0.1)                                  |
|     | (0.1)                                     | (0.1)                                  |
|     | 200                                       | 200                                    |
|     | 9.1                                       | 9.4                                    |
|     | 10.6                                      | 12                                    |
|     | 13.2                                      | 13.2                                  |
|     | (0.4)                                     | (0.4)                                  |
|     | (0.4)                                     | (0.4)                                  |
|     | (0.3)                                     | (0.2)                                  |
|     | 50                                        | 50                                    |
|     | 6.1                                       | 6.4                                    |
|     | 10.6                                      | 10.8                                  |
|     | 11.4                                      | 11.4                                  |
|     | (0.4)                                     | (0.4)                                  |
|     | (0.4)                                     | (0.4)                                  |
|     | (0.3)                                     | (0.3)                                  |
|     | 100                                       | 100                                    |
|     | 4.5                                       | 5.1                                    |
|     | 6.7                                       | 7.4                                    |
|     | 9.2                                       | 9.2                                    |
|     | (0.4)                                     | (0.4)                                  |
|     | (0.4)                                     | (0.4)                                  |
|     | (0.3)                                     | (0.3)                                  |
|     | 200                                       | 200                                    |
|     | 39.2                                      | 34.2                                  |
|     | 40.3                                      | 43.5                                  |
|     | 46.2                                      | 46.2                                  |
|     | (1)                                       | (1.4)                                 |
|     | (1)                                       | (0.6)                                 |
|     | (0.4)                                     | (0.4)                                  |
|     | 50                                        | 50                                    |
|     | 31.5                                      | 38.1                                  |
|     | 42.2                                      | 43                                    |
|     | 45.9                                      | 45.9                                  |
|     | (1.4)                                     | (1)                                   |
|     | (1)                                       | (0.7)                                 |
|     | (0.5)                                     | (0.5)                                  |
|     | 100                                       | 100                                    |
|     | 17.8                                      | 26.8                                  |
|     | 33.8                                      | 38.5                                  |
|     | 41.5                                      | 41.5                                  |
|     | (1.4)                                     | (1)                                   |
|     | (1.1)                                     | (0.9)                                 |
|     | (0.7)                                     | (0.7)                                  |

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Table 37: Average prediction bias from 100 replications of CV 1SE with correlated predictors. Standard errors given in parentheses.

| n   | p  | 50  | 100 | 200 | 400  | 800  | 50  | 100 | 200 | 400  | 800  |
|-----|----|-----|-----|-----|------|------|-----|-----|-----|------|------|
| 50  | p∗ | 5; SNR = 1.25; σ = 4.86 |      |      |      |      | p∗ | 5; SNR = 5; σ = 2.43 |      |      |      |      |
|     |    | 25.4 | 22.4 | 24.2 | 23.8 | 25.1 | 14.7 | 13.7 | 14.6 | 15.2 | 16.3 |
|     |    | (1.2) | (0.8) | (1) | (0.9) | (1) | (0.7) | (0.6) | (0.6) | (0.6) | (0.6) |
| 100 |    | 26.8 | 25.6 | 26.6 | 26.3 | 30.1 | 14.1 | 13.4 | 15   | 15.2 | 17.3 |
|     |    | (1.2) | (1.1) | (1.2) | (0.9) | (1.3) | (0.7) | (0.7) | (0.7) | (0.6) | (0.8) |
| 200 |    | 25.4 | 29.6 | 29.7 | 32.2 | 33.5 | 13.1 | 15.6 | 15.1 | 16.7 | 17.9 |
|     |    | (1.2) | (1.4) | (1.4) | (1.4) | (1.5) | (0.6) | (0.8) | (0.8) | (0.7) | (0.9) |
| n   | p∗ | 15; SNR = 1.25; σ = 8.22 |      |      |      |      | p∗ | 15; SNR = 5; σ = 4.11 |      |      |      |      |
| 50  |    | 40.6 | 41.7 | 44.9 | 44.4 | 44.7 | 26.2 | 28.1 | 31.1 | 31.3 | 32.7 |
|     |    | (1.3) | (1.2) | (1.9) | (1.7) | (1.6) | (1.1) | (0.8) | (0.9) | (0.9) | (0.9) |
| 100 |    | 46   | 53.1 | 55.2 | 54.1 | 55.8 | 25.6 | 32.4 | 33.4 | 34.4 | 38.8 |
|     |    | (1.7) | (2) | (1.6) | (1.3) | (1.3) | (1.1) | (1.4) | (1.2) | (1.2) | (1.1) |
| 200 |    | 56   | 57.8 | 63.6 | 66.4 | 69.3 | 30.3 | 31   | 34.8 | 39.7 | 42.3 |
|     |    | (2.1) | (2.4) | (2.4) | (2) | (1.8) | (1.4) | (1.5) | (1.5) | (1.6) | (1.7) |
| n   | p∗ | 50; SNR = 1.25; σ = 14.71 |      |      |      |      | p∗ | 50; SNR = 5; σ = 7.36 |      |      |      |      |
| 50  |    | 80.1 | 78.5 | 78.8 | 81.5 | 79.3 | 55.1 | 55.3 | 60.3 | 62.9 | 60.5 |
|     |    | (2.5) | (2.8) | (2.5) | (2.6) | (2.5) | (1.5) | (2.1) | (1.8) | (1.7) | (1.4) |
| 100 |    | 90.3 | 101.9 | 111.4 | 104 | 108.1 | 54.4 | 64.7 | 79.6 | 84.1 | 87.5 |
|     |    | (2.8) | (2.8) | (3) | (2.1) | (2.3) | (2.1) | (2.3) | (1.9) | (1.6) | (1.5) |
| 200 |    | 105.4 | 125 | 137 | 136.3 | 140 | 57.8 | 72.2 | 86   | 90.4 | 104.3 |
|     |    | (3.1) | (3.6) | (3.8) | (2.7) | (3.1) | (2.3) | (2.8) | (3.3) | (2.8) | (2.5) |
Table 38: Average prediction bias for 100 replications of AR2 CV 1SE with correlated predictors. Standard errors given in parentheses.

| p  | 50  | 100 | 200 | 400 | 800 | 50  | 100 | 200 | 400 | 800 |
|----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| n | \(p^* = 5; SNR = 1.25; \sigma = 4.86\) | \(p^* = 5; SNR = 5; \sigma = 2.43\) |
|    |     |     |     |     |     |     |     |     |     |     |
| 50 | 21.6 | 20.1 | 21.9 | 21.7 | 20.9 | 14.9 | 13.1 | 15  | 15.9 | 16.3 |
|    | (0.8) | (0.6) | (0.8) | (0.7) | (0.7) | (0.7) | (0.5) | (0.6) | (0.6) | (0.6) |
| 100| 25.3 | 23.8 | 23.7 | 25.3 | 28.2 | 14.2 | 13.9 | 14.7 | 15.9 | 17.1 |
|    | (1.1) | (0.9) | (0.9) | (1.1) | (1.2) | (0.7) | (0.7) | (0.7) | (0.7) | (0.7) |
| 200| 25.2 | 29.1 | 27.3 | 30.5 | 33  | 13.8 | 15.9 | 15.5 | 17.1 | 18  |
|    | (1.3) | (1.5) | (1.4) | (1.5) | (1.9) | (0.7) | (0.8) | (0.9) | (0.9) | (1)  |
|    |     |     |     |     |     |     |     |     |     |     |
| n | \(p^* = 15; SNR = 1.25; \sigma = 8.22\) | \(p^* = 15; SNR = 5; \sigma = 4.11\) |
|    |     |     |     |     |     |     |     |     |     |     |
| 50 | 37  | 38.5 | 40.1 | 40  | 40.2 | 26.7 | 30.5 | 31.5 | 32.2 | 32.1 |
|    | (1.1) | (1.1) | (1.3) | (1.2) | (1.2) | (1.2) | (0.8) | (0.8) | (0.9) | (0.8) |
| 100| 44.1 | 51.5 | 51.3 | 50  | 54.4 | 27.4 | 33.6 | 34.7 | 36.9 | 40.2 |
|    | (1.4) | (1.8) | (1.3) | (1.1) | (1.4) | (1.2) | (1.5) | (1.2) | (1.4) | (1.2) |
| 200| 55.3 | 57.8 | 62.1 | 65.5 | 68.8 | 32.4 | 31.6 | 35.9 | 40.7 | 42.8 |
|    | (2.3) | (2.4) | (2.2) | (2.6) | (2.2) | (1.6) | (1.6) | (1.7) | (1.7) | (1.7) |
|    |     |     |     |     |     |     |     |     |     |     |
| n | \(p^* = 50; SNR = 1.25; \sigma = 14.71\) | \(p^* = 50; SNR = 5; \sigma = 7.36\) |
|    |     |     |     |     |     |     |     |     |     |     |
| 50 | 75.8 | 72.1 | 75  | 73.5 | 71.6 | 56.9 | 58.2 | 59.4 | 61.6 | 59.5 |
|    | (2.7) | (2.4) | (2.4) | (2.1) | (1.8) | (1.6) | (1.9) | (1.7) | (1.4) | (1.5) |
| 100| 84.4 | 94.7 | 99.8 | 102.6 | 99.7 | 56.7 | 66.4 | 78.5 | 82  | 85.5 |
|    | (2.1) | (2.5) | (2.6) | (2.4) | (2.3) | (2.3) | (2.4) | (2.1) | (1.7) | (1.5) |
| 200| 107 | 119.5 | 135.4 | 132 | 133.5 | 60.8 | 76.9 | 88.6 | 89.9 | 101.4 |
|    | (3.5) | (2.9) | (4) | (2.9) | (2.6) | (2.4) | (3.4) | (3.4) | (2.7) | (2.6) |
Table 39: Average prediction bias of 100 replications for $\alpha$-modified CV 1SE with correlated predictors. Standard errors given in parentheses.

| p   | 50   | 100  | 200  | 400  | 800  | 50   | 100  | 200  | 400  | 800  |
|-----|------|------|------|------|------|------|------|------|------|------|
| n   | p\textsuperscript{*} = 5; SNR = 1.25; $\bar{\sigma}$ = 4.86 | p\textsuperscript{*} = 5; SNR = 5; $\bar{\sigma}$ = 2.43 |
| 50  | 21.7 (0.8) | 21 (1) | 23.7 (1) | 23.7 (0.8) | 23.2 (0.6) | 14.3 (0.5) | 12.8 (0.6) | 14.3 (0.6) | 14.7 (0.6) | 15.6 (0.6) |
| 100 | 25.2 (1.1) | 23.4 (0.9) | 24.1 (0.9) | 25.7 (1.1) | 28.3 (1.2) | 13.2 (0.7) | 13.3 (0.7) | 14.2 (0.7) | 14.8 (0.7) | 16.5 (0.7) |
| 200 | 24.9 (1.3) | 28.7 (1.4) | 27.4 (1.4) | 30.9 (1.4) | 33.2 (1.8) | 13 (0.6) | 14.8 (0.8) | 14.6 (0.8) | 16.1 (0.8) | 17.1 (0.8) |
| n   | p\textsuperscript{*} = 15; SNR = 1.25; $\bar{\sigma}$ = 8.22 | p\textsuperscript{*} = 15; SNR = 5; $\bar{\sigma}$ = 4.11 |
| 50  | 38 (1.1) | 40.7 (1.1) | 43.2 (1.6) | 44.6 (1.7) | 44 (1.4) | 26.3 (1.1) | 28.2 (0.8) | 30.5 (0.9) | 30.3 (0.9) | 30.8 (0.9) |
| 100 | 45.6 (1.4) | 52 (1.8) | 52.8 (1.3) | 51.9 (1.1) | 56 (1.4) | 25.9 (1.1) | 32.5 (1.3) | 33.4 (1.3) | 34.5 (1.3) | 38.8 (1.3) |
| 200 | 56.4 (2.4) | 58.9 (2.5) | 63.4 (2.3) | 66.2 (2.5) | 70.8 (2.1) | 30.4 (1.4) | 30.9 (1.6) | 34.5 (1.6) | 39.6 (1.6) | 41.8 (1.6) |
| n   | p\textsuperscript{*} = 50; SNR = 1.25; $\bar{\sigma}$ = 14.71 | p\textsuperscript{*} = 50; SNR = 5; $\bar{\sigma}$ = 7.36 |
| 50  | 79 (2.7) | 78.2 (2.5) | 79.6 (2.6) | 82 (2.9) | 78.2 (2.3) | 56.3 (1.4) | 54.3 (1.8) | 58.4 (1.8) | 59.3 (1.6) | 57.7 (1.5) |
| 100 | 88.5 (1.9) | 99.2 (2.3) | 102.5 (2.3) | 106.9 (2.4) | 102.1 (2.2) | 56.2 (1.4) | 66.1 (1.8) | 77.5 (1.8) | 81.8 (1.6) | 83.6 (1.5) |
| 200 | 110.4 (3.4) | 126.6 (2.6) | 140.9 (2.6) | 137.8 (2.4) | 138.2 (2.2) | 59 (2.3) | 74.7 (3.2) | 87 (3.3) | 90.4 (2.7) | 102.5 (2.7) |
Table 40: Average prediction bias for 100 replications of Relaxed Lasso with correlated predictors. Standard errors given in parentheses.

| p  | 50     | 100    | 200  | 400  | 800  | 50     | 100    | 200  | 400  | 800  |
|----|--------|--------|------|------|------|--------|--------|------|------|------|
| n  | \(p^* = 5; SNR = 1.25; \hat{\sigma} = 4.86\) | \(p^* = 5; SNR = 5; \hat{\sigma} = 2.43\) | \(p^* = 5; SNR = 5; \hat{\sigma} = 4.11\) |
| 50 | 19.4   | 21.8   | 26   | 26.9 | 26   | 12.7   | 12.6   | 14.4 | 14.9 | 15.6 |
|    | (0.7)  | (1)    | (1.4)| (1.2)| (1.3)| (0.6)  | (0.6)  | (0.7)| (0.7)| (0.8)|
| 100| 21.9   | 20.1   | 21.6 | 24.4 | 27.1 | 10.6   | 11.1   | 11.6| 14.2 | 15.9 |
|    | (1)    | (0.9)  | (0.9)| (1.3)| (1.3)| (0.6)  | (0.7)  | (0.6)| (0.9)| (1)  |
| 200| 17.7   | 21.7   | 21.8 | 25.5 | 25.5 | 8.3    | 10.4   | 10.6| 11.7 | 13.3 |
|    | (1)    | (1.1)  | (1.3)| (1.4)| (1.4)| (0.5)  | (0.6)  | (0.7)| (0.8)| (1)  |
| 50 | \(p^* = 50; SNR = 1.25; \hat{\sigma} = 14.71\) | \(p^* = 50; SNR = 5; \hat{\sigma} = 7.36\) |
| 50 | 69.4   | 75.5   | 76.4 | 89.3 | 83.1 | 47.4   | 49.3   | 50.6| 55.3 | 51.2 |
|    | (2.4)  | (3.5)  | (2.8)| (4.8)| (3.7)| (1.3)  | (2)    | (2.2)| (1.9)| (1.9)|
| 100| 75.6   | 84.8   | 94   | 97.8 | 95.6 | 47.6   | 56.5   | 64.5| 69   | 73.3 |
|    | (1.7)  | (2.1)  | (3.2)| (2.8)| (2.3)| (1.6)  | (1.8)  | (1.8)| (1.7)| (2.1)|
| 200| 90.8   | 103.9  | 114.4| 116.7| 124.2| 52.1   | 64.8   | 75.2| 75.4 | 84.9 |
|    | (2.8)  | (2.8)  | (3)  | (2.8)| (3.2)| (1.9)  | (2.6)  | (2.8)| (2.3)| (2.5)|
Finally, we evaluate the methods for limited data settings, letting \( n = 100, \ p = \{100, 800\} \), \( \text{SNR} = \{1.25, 5\} \), and varying \( p^* \). Once again, \( \mathbf{X} \) contained correlated columns and more methods are evaluated in Figure 11 and Tables 41, 42, and 43. Figure 12 and Tables 44 through 46 give results from the case where \( n = 1000 \). Average false negatives are higher for all methods when predictors are correlated. Particularly for large \( p^* \) all methods struggle to capture the true model, though the Relaxed Lasso seemingly performs the best. For the \( n = 1000 \) case, Mod 1SE, AR2 1SE, and Relaxed 1SE all have the smallest average FDRs so it seems that the new methods are most competitive with the Relaxed Lasso when sample sizes are large.

**Figure 11:** Average false positives and prediction error plots from 100 replications of simulated data with \( n = 100 \) and \( p = 800 \) for both SNR = 0.5 and SNR = 2 with correlated predictors. Thin dotted lines represent the mean ± one standard error.
Table 41: Average false discovery rate for 100 replications of simulated data with correlated predictors and \( n = 100 \). Standard errors given in parentheses.

|               | \( p = 100 \) | \( p = 800 \) |
|---------------|---------------|---------------|
|               | \( p^* = 5 \) | \( p^* = 33 \) | \( p^* = 5 \) | \( p^* = 33 \) |
| SNR = 1.25    | 0.73          | 0.51          | 0.62          | 0.33          |
|               | (0.01)        | (0.03)        | (0.03)        | (0.03)        |
| CV Min        | 0.74          | 0.51          | 0.66          | 0.33          |
|               | (0.01)        | (0.02)        | (0.02)        | (0.02)        |
| SNR = 5       | 0.49          | 0.39          | 0.48          | 0.35          |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| CV 1SE        | 0.49          | 0.39          | 0.48          | 0.35          |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| Mod Min       | 0.67          | 0.38          | 0.33          | 0.33          |
|               | (0.02)        | (0.03)        | (0.03)        | (0.03)        |
| SNR = 1.25    | 0.70          | 0.44          | 0.39          | 0.33          |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| Mod 1SE       | 0.70          | 0.44          | 0.39          | 0.33          |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| AR2 Min       | 0.94          | 0.66 (0)      | 0.67 (0)      | 0.67 (0)      |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| AR2 1SE       | 0.94          | 0.66 (0)      | 0.67 (0)      | 0.67 (0)      |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| R2 Cut        | 0.94          | 0.66 (0)      | 0.67 (0)      | 0.67 (0)      |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| AIC           | 0.94          | 0.67 (0)      | 0.67 (0)      | 0.67 (0)      |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| BIC           | 0.85          | 0.66 (0)      | 0.69 (0)      | 0.69 (0)      |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| ERIC          | 0.37          | 0.18          | 0.42          | 0.42          |
|               | (0.03)        | (0.02)        | (0.02)        | (0.02)        |
| GCV           | 0.77          | 0.55          | 0.74          | 0.74          |
|               | (0.01)        | (0.01)        | (0.01)        | (0.01)        |
| Relaxed Min   | 0.55          | 0.44          | 0.39          | 0.45          |
|               | (0.03)        | (0.02)        | (0.02)        | (0.02)        |
| Relaxed 1SE   | 0.44          | 0.25          | 0.25          | 0.36          |
|               | (0.03)        | (0.03)        | (0.03)        | (0.03)        |

**Figure 12:** Average false positives and prediction error plots from 100 replications of simulated data with \( n = 1000 \) and \( p = 100 \) for both SNR = 0.5 and SNR = 2 with correlated predictors. Thin dotted lines represent the mean ± one standard error.
Table 42: Average prediction bias for 100 replications of simulated data with correlated predictors and $n = 100$. Standard errors given in parentheses.

|                  | $p = 100$ |                  | $p = 800$ |
|------------------|-----------|------------------|-----------|
|                  | SNR = 1.25 | SNR = 5         | SNR = 1.25 | SNR = 5 |
|                  | $p^* = 5$  | $p^* = 33$      | $p^* = 5$  | $p^* = 33$ |
| CV Min           | 20.1       | 63.85           | 11.83      | 43.46    |
|                  | (0.76)     | (1.42)          | (0.46)     | (1.39)   |
| CV 1SE           | 26.36      | 79.79           | 14.34      | 50.96    |
|                  | (1.16)     | (1.94)          | (0.65)     | (1.72)   |
| Mod Min          | 21.24      | 72.94           | 11.96      | 45.45    |
|                  | (0.83)     | (1.57)          | (0.5)      | (1.44)   |
| Mod 1SE          | 23.95      | 81.54           | 13.92      | 52.59    |
|                  | (0.88)     | (1.9)           | (0.66)     | (1.85)   |
| AR2 Min          | 21.65      | 73.11           | 11.92      | 45.59    |
|                  | (0.89)     | (1.75)          | (0.52)     | (1.43)   |
| AR2 1SE          | 23.96      | 78              | 14.99      | 52.87    |
|                  | (0.91)     | (2.07)          | (0.73)     | (1.83)   |
| R2 Cut           | 41.31      | 105.19          | 17.04      | 45.12    |
|                  | (1.83)     | (4.59)          | (0.74)     | (1.77)   |
| AIC              | 47.95      | 140.64          | 19.45      | 62.39    |
|                  | (2.83)     | (7.01)          | (1.32)     | (3.19)   |
| BIC              | 33.97      | 126.16          | 13.17      | 54.1     |
|                  | (2.44)     | (7.45)          | (0.76)     | (2.87)   |
| ERIC             | 29.44      | 98.14           | 15.06      | 76.35    |
|                  | (1.56)     | (3.53)          | (0.7)      | (1.76)   |
| GCV              | 22.53      | 67.48           | 12.43      | 44.43    |
|                  | (0.99)     | (1.81)          | (0.49)     | (1.61)   |
| Relaxed Min      | 20.9       | 66.67           | 11.37      | 44.59    |
|                  | (0.88)     | (1.62)          | (0.57)     | (1.49)   |
| Relaxed 1SE      | 25.38      | 77.77           | 13.27      | 50.73    |
|                  | (1.14)     | (1.83)          | (0.69)     | (1.76)   |

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Table 43: Average false negatives for 100 replications of simulated data with correlated predictors and $n = 100$. Standard errors given in parentheses.

|                  | $p = 100$ |                | $p = 800$ |                |
|------------------|-----------|----------------|-----------|----------------|
|                  | SNR = 1.25| SNR = 5        | SNR = 1.25| SNR = 5        |
|                  | $p^* = 5$ | $p^* = 33$     | $p^* = 5$ | $p^* = 33$     |
| CV Min           | 1.35 (0.14) | 0.25 (0.05) | 11.17     | 2.02 (0.14) | 0.47 (0.07) | 25.14     |
| CV 1SE           | 2.18 (0.13) | 0.57 (0.08) | 16.68     | 2.94 (0.14) | 0.69 (0.08) | 29.15     |
| Mod Min          | 1.58 (0.15) | 0.42 (0.08) | 11.93     | 2.45 (0.16) | 0.56 (0.09) | 25.61     |
| Mod 1SE          | 2.72 (0.15) | 0.89 (0.08) | 18.82     | 3.28 (0.16) | 0.97 (0.09) | 30.14     |
| AR2 Min          | 1.5 (0.16)  | 0.43 (0.08) | 11.31     | 2.27 (0.15) | 0.71 (0.11) | 26.9      |
| AR2 1SE          | 2.64 (0.15) | 1.04 (0.08) | 18.9      | 3.3 (0.15)  | 1.17 (0.14) | 30.49     |
| R2 Cut           | 0.35 (0.06) | 0.07 (0.03) | 6.2       | 1.56 (0.13) | 0.39 (0.07) | 22.5      |
| AIC              | 0.2 (0.04)  | 0.07 (0.03) | 1.49      | 1.58 (0.13) | 0.39 (0.07) | 21.18     |
| BIC              | 0.47 (0.06) | 0.19 (0.05) | 4.28      | 1.76 (0.12) | 0.46 (0.08) | 22.21     |
| ERIC             | 2.95 (0.12) | 0.78 (0.09) | 29.26     | 1.99 (0.11) | 0.26 (0.06) | 16.1      |
| GCV              | 1.19 (0.13) | 0.24 (0.05) | 10.45     | 1.19 (0.11) | 0.26 (0.06) | 16.1      |
| Relaxed Min      | 1.7 (0.14)  | 0.44 (0.08) | 13.4      | 2.38 (0.14) | 0.63 (0.09) | 24.62     |
| Relaxed 1SE      | 2.42 (0.14) | 0.85 (0.1)  | 18.19     | 3.09 (0.14) | 0.99 (0.13) | 28.83     |

Table 44: Average false discovery rate for 100 replications of simulated data with correlated predictors and $n = 1000$. Standard errors given in parentheses.

|                  | SNR = 1.25 |                | SNR = 5   |                |
|------------------|-----------|----------------|-----------|----------------|
|                  | $p^* = 5$ | $p^* = 33$     | $p^* = 5$ | $p^* = 33$     |
| CV Min           | 0.69 (0.01)| 0.5 (0.01)     | 0.7 (0.01)| 0.5 (0.01)     |
| CV 1SE           | 0.23 (0.02)| 0.25 (0.01)    | 0.18 (0.02)| 0.25 (0.01)    |
| Mod Min          | 0.57 (0.02)| 0.46 (0.01)    | 0.56 (0.02)| 0.44 (0.01)    |
| Mod 1SE          | 0.08 (0.01)| 0.18 (0.01)    | 0.08 (0.02)| 0.18 (0.01)    |
| AR2 Min          | 0.58 (0.02)| 0.47 (0.01)    | 0.57 (0.02)| 0.44 (0.01)    |
| AR2 1SE          | 0.08 (0.02)| 0.19 (0.01)    | 0.04 (0.01)| 0.16 (0.01)    |
| R2 Cut           | 0.11 (0.02)| 0.19 (0.01)    | 0.09 (0.02)| 0.19 (0.01)    |
| AIC              | 0.94 (0)   | 0.66 (0)       | 0.88 (0.01)| 0.64 (0)       |
| BIC              | 0.75 (0.02)| 0.63 (0)       | 0.51 (0.02)| 0.56 (0.01)    |
| ERIC             | 0.22 (0.02)| 0.1 (0.01)     | 0.21 (0.02)| 0.13 (0.01)    |
| GCV              | 0.71 (0.01)| 0.5 (0.01)     | 0.72 (0.01)| 0.49 (0.01)    |
| Relaxed Min      | 0.22 (0.03)| 0.27 (0.02)    | 0.11 (0.02)| 0.11 (0.01)    |
| Relaxed 1SE      | 0.05 (0.01)| 0.13 (0.01)    | 0.03 (0.01)| 0.07 (0.01)    |
Table 45: Average prediction bias for 100 replications of simulated data with correlated predictors and $n = 1000$. Standard errors given in parentheses.

|               | SNR = 1.25 |               | SNR = 5       |
|---------------|------------|---------------|---------------|
|               | $p^* = 5$  | $p^* = 33$    | $p^* = 5$    |
| CV Min        | 21.43 (1.09) | 94.15 (3.68) | 11.57 (0.58) |
| CV 1SE        | 35.04 (1.82) | 120.22 (5.14) | 18.32 (0.95) |
| Mod Min       | 20.4 (1.13) | 96.17 (3.79) | 10.86 (0.59) |
| Mod 1SE       | 33.86 (1.87) | 120.82 (5.01) | 17.98 (0.93) |
| AR2 Min       | 20.49 (1.15) | 96 (3.76) | 10.88 (0.6) |
| AR2 1SE       | 32.98 (1.66) | 118.94 (5.01) | 19.53 (0.95) |
| R2 Cut        | 66.13 (6.79) | 166.62 (15.68) | 34.89 (4.76) |
| AIC           | 41.1 (2.19) | 115.3 (5.02) | 17.87 (1.16) |
| BIC           | 26.46 (1.92) | 109 (4.92) | 12.62 (0.68) |
| ERIC          | 29.81 (1.52) | 163.59 (5.88) | 15.94 (0.9) |
| GCV           | 21.7 (1.08) | 94.73 (3.66) | 11.96 (0.63) |
| Relaxed Min   | 17.02 (1.19) | 93.33 (4) | 8.38 (0.61) |
| Relaxed 1SE   | 32.64 (1.91) | 119.98 (5.42) | 16.68 (0.91) |

Table 46: Average false negatives for 100 replications of simulated data with correlated predictors and $n = 1000$. Standard errors given in parentheses.

|               | SNR = 1.25 |               | SNR = 5       |
|---------------|------------|---------------|---------------|
|               | $p^* = 5$  | $p^* = 33$    | $p^* = 5$    |
| CV Min        | 0.04 (0.02) | 1.95 (0.35) | 0.01 (0.01) |
| CV 1SE        | 0.25 (0.05) | 5.26 (0.67) | 0.07 (0.03) |
| Mod Min       | 0.06 (0.02) | 2.3 (0.41) | 0.01 (0.01) |
| Mod 1SE       | 0.45 (0.08) | 6.3 (0.8) | 0.1 (0.03) |
| AR2 Min       | 0.05 (0.02) | 2 (0.35) | 0.01 (0.01) |
| AR2 1SE       | 0.38 (0.07) | 6.31 (0.81) | 0.14 (0.04) |
| R2 Cut        | 0.39 (0.07) | 6.12 (0.82) | 0.1 (0.03) |
| AIC           | 0 (0) | 0.11 (0.04) | 0 (0) |
| BIC           | 0 (0) | 0.33 (0.07) | 0 (0) |
| ERIC          | 0.24 (0.04) | 13.53 (0.98) | 0.07 (0.03) |
| GCV           | 0.02 (0.01) | 1.87 (0.33) | 0 (0) |
| Relaxed Min   | 0.1 (0.03) | 3.69 (0.48) | 0.01 (0.01) |
| Relaxed 1SE   | 0.35 (0.06) | 7.47 (0.71) | 0.08 (0.03) |
C EMG Application Approximations and Results

The model given in the main document does not involve raw data, so some manipulations are required in order to perform estimation. The response, velocity \((y_i)\), must be estimated from hand position, \(z_i\) and covariate functions in the model must be created from discrete observations. These tasks are performed exactly as described in Stallrich et al. (2020).

Furthermore, the model also requires approximations for \(\gamma_j\). We represent all \(\gamma_j\) using:

\[
\gamma_j(t, z) \approx \sum_{l=1}^{L} \sum_{m=1}^{M} \omega_l(t)\tau_m(z)\beta_{jlm}
\]

where \(\omega_l(t)\) and \(\tau_m(z)\) are the same univariate basis functions for all \(\gamma_j\), and the \(\beta_{jlm}\)'s are basis coefficients that must be estimated. In matrix form, we have

\[
\gamma_j(t, z) \approx \omega^T(t)B_j\tau(z) \quad \text{where} \quad \omega^T(t) = (\omega_1(t), ..., \omega_L(t)), \quad \tau^T(z) = (\tau_1(z), ..., \tau_M(z)), \quad \text{and} \quad B_j = (\beta_{jlm}) \text{ is the } L \times M \text{ coefficient matrix.}
\]

The integral in the model is approximated:

\[
\int_{-\delta}^{0} X_{ij}(t) \gamma_j(t, z_i) dt \approx \left( \sum_{k=-\delta}^{0} x_{(i+k)j} \omega(k)^T \right) B_j \tau(z_i) = X_{ij}^T \omega B_j \tau(z_i).
\]

As noted in the main document, we use the penalized least squares approach to functional estimation proposed in Gertheiss et al. (2013) and adapted in Stallrich et al. (2020).

Tables 47 through 50 show the results from multiple stages of adaptive weighting, the SAFE procedure. After only one round, the differences in tuning parameter selection approaches disappear, giving the same results for APE, AR2, and Mod CV. The differences reappear again after three stages, but only for AR2 CV, which still overselects by a single coefficient for FR2, even after a fourth stage.
### Table 47: Variable selection results for EMG finger movements after one stage of adaptive weighting.
FP indicates the total number of false positives in the model and Size is the total number of EMG signals contained in the model.

|       | FC1 | FC2 | FC3 | FR1 | FR2 | FR3 | Mean |
|-------|-----|-----|-----|-----|-----|-----|------|
| APE   | FP  | 0   | 0   | 0   | 1   | 1   | 0.33 |
| Size  |     | 2   | 2   | 2   | 4   | 3   | 2.5  |
| AR2   | FP  | 0   | 0   | 1   | 0   | 1   | 0.33 |
| Size  |     | 2   | 2   | 3   | 2   | 4   | 2.5  |
| Mod   | FP  | 0   | 0   | 0   | 0   | 1   | 0.33 |
| Size  |     | 2   | 2   | 2   | 4   | 3   | 2.5  |

### Table 48: Variable selection results for EMG finger movements after two stages of adaptive weighting.
FP indicates the total number of false positives in the model and Size is the total number of EMG signals contained in the model.

|       | FC1 | FC2 | FC3 | FR1 | FR2 | FR3 | Mean |
|-------|-----|-----|-----|-----|-----|-----|------|
| APE   | FP  | 0   | 0   | 0   | 0   | 1   | 0.17 |
| Size  |     | 2   | 2   | 2   | 4   | 2   | 2.33 |
| AR2   | FP  | 0   | 0   | 0   | 0   | 1   | 0.17 |
| Size  |     | 2   | 2   | 2   | 4   | 2   | 2.33 |
| Mod   | FP  | 0   | 0   | 0   | 0   | 1   | 0.17 |
| Size  |     | 2   | 2   | 2   | 4   | 2   | 2.33 |

### Table 49: Variable selection results for EMG finger movements after three stages of adaptive weighting.
FP indicates the total number of false positives in the model and Size is the total number of EMG signals contained in the model.

|       | FC1 | FC2 | FC3 | FR1 | FR2 | FR3 | Mean |
|-------|-----|-----|-----|-----|-----|-----|------|
| APE   | FP  | 0   | 0   | 0   | 0   | 0   | 0    |
| Size  |     | 2   | 2   | 2   | 3   | 2   | 2.17 |
| AR2   | FP  | 0   | 0   | 0   | 0   | 1   | 0.17 |
| Size  |     | 2   | 2   | 2   | 4   | 2   | 2.33 |
| Mod   | FP  | 0   | 0   | 0   | 0   | 0   | 0    |
| Size  |     | 2   | 2   | 2   | 3   | 2   | 2.17 |

### Table 50: Variable selection results for EMG finger movements after the four stages of adaptive weighting recommended by Stallrich et al. (2020). FP indicates the total number of false positives in the model and Size is the total number of EMG signals contained in the model.

|       | FC1 | FC2 | FC3 | FR1 | FR2 | FR3 | Mean |
|-------|-----|-----|-----|-----|-----|-----|------|
| APE   | FP  | 0   | 0   | 0   | 0   | 0   | 0    |
| Size  |     | 2   | 2   | 2   | 3   | 2   | 2.17 |
| AR2   | FP  | 0   | 0   | 0   | 0   | 1   | 0.17 |
| Size  |     | 2   | 2   | 2   | 4   | 2   | 2.33 |
| Mod   | FP  | 0   | 0   | 0   | 0   | 0   | 0    |
| Size  |     | 2   | 2   | 2   | 3   | 2   | 2.17 |
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