LOG INTERMEDIATE JACOBIANS

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§0. INTRODUCTION

Let $S = \Delta = \{ q \in \mathbb{C} | |q| < 1 \}$, and let

$$ f : X \to S $$

be a projective morphism which is smooth over $S^* = \Delta^* = \Delta \setminus \{0\}$ and which is semi-stable at $q = 0$.

Put $X^* = f^{-1}(S^*)$. Fix a polarization of $X^*/S^*$ and fix $r \geq 1$. Then we have the variation of polarized Hodge structure $H^{2r-1}(X^*/S^*)$ over $S^*$ (cf. [Gri70]):

$$ H_{Z}^{2r-1}(X^*/S^*) = R^{2r-1}f_*\mathbb{Z}/(\text{torsion}), $$

$$ H_{\mathcal{O}}^{2r-1}(X^*/S^*) = R^{2r-1}f_*(\Omega^*_{X^*/S^*}) \simeq \mathcal{O}_{S^*} \otimes_{\mathbb{Z}} H_{Z}^{2r-1}(X^*/S^*), $$

$$ F^p H_{\mathcal{O}}^{2r-1}(X^*/S^*) = R^{2r-1}f_*(\Omega^{2p}_{X^*/S^*}). $$

Let us twist it into $H^{2r-1}(X^*/S^*)(r)$ whose weight is $-1$. Fix a point $s \in S^*$.

Put

$$ H^* = H_{Z}^{2r-1}(X^*/S^*)(r)_s, \quad H = H^* \oplus \mathbb{Z}. $$
Let $W$ be an increasing filtration on $H_Q$ such that
\[ \text{gr}^W_0(H_Q) = Q, \quad \text{gr}^W_{-1}(H_Q) = H'_Q, \quad \text{gr}^W_k(H_Q) = 0 \quad \text{for } k \neq 0, -1. \]

Let
\[ \langle , \rangle_0 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}; \quad (a, b) \mapsto ab, \]
\[ \langle , \rangle' = \langle , \rangle_{-1} : H' \times H' \to \mathbb{Q} \quad \text{the pairing defined by the polarization of } X^*/S^*. \]

Let $(h^{p,q}_k)$ be the evident Hodge numbers.

Let $D'_r = D'_r(H', (h^{p,q}_r), (, )')$ be the classifying space of polarized Hodge structures of weight $-1$, and let $D = D(H, W, (h^{p,q}_r), (, )_k))$ be the classifying space of gradedly polarized mixed Hodge structures. Let $\gamma' : H' \to H'$ be the local monodromy and let $\Gamma'$ be the group generated by $\gamma'$. Let $\Gamma$ be the subgroup of $G_Z = \text{Aut}(H, W, (, )_k))$ consisting of all elements whose restrictions to $H'$ are contained in $\Gamma'$.

Then, the intermediate Jacobian $J^{\ast}_{X/\ast}/S^*$ introduced by Griffiths (cf. [GS69]) is
\[
J^{\ast}_{X/\ast}/S^* = (H^{2r-1}_Q(X^*/S^*/(F^r H^{2r-1}_Q(X^*/S^*) + H^{2r-1}_Z(X^*/S^*)))(r).
\]

There is an isomorphism of functors (cf. [Car80])
\[
\text{Mor}(?, J^{\ast}_{X/\ast}/S^*) \simeq \mathcal{E}xt_{\text{MHS}}^1(\mathbb{Z}, H^{2r-1}(X^*/S^*)/(r))
\]
from the category of complex spaces over $S^*$ to the category of sets. Here $\mathcal{E}xt^1$ is taken in the category (MHS) of mixed Hodge structures with polarized graded quotients (= Variation of gradedly polarized mixed Hodge structure without assuming Griffiths transversality). This is also expressed as the fiber product (cf. [Usu84])
\[
J^{\ast}_{X/\ast}/S^* \longrightarrow \Gamma \backslash D
\]
\[
\downarrow \quad \downarrow \text{gr}^W_{-1}
\]
\[
S^* \quad \varphi \quad \Gamma' \backslash D',
\]
where $\varphi$ is the period map associated to the VPHS $H^{2r-1}(X^*/S^*)/(r)$.

The purpose of this article is to extend $J^{\ast}_{X/\ast}/S^*$ in expression (0.3) to a “log intermediate Jacobian $J^{\ast}_{X/S, \Sigma}$” over $S$ by using log geometry ($\S7$). Our “$J^{\ast}_{X/S, \Sigma}$” is related to works by Zucker [Zuc76], Clemens [Cle83], and Saito [Sai96] ($\S8$).

$\S\S1–10$ in this article are based on our note on which the third author gave a talk at the JAMI conference at Johns Hopkins university in March, 2005.
§1. Situation

Let \( S = \Delta = \{ q \in \mathbb{C} \mid |q| < 1 \} \), and let

\[
    f : X \to S
\]

be a projective morphism which is smooth over \( S^* = \Delta \setminus \{ 0 \} \) and which is semi-stable at \( q = 0 \). Endow \( X \) and \( S \) with the standard log structures ([Kak89]):

\[
    M_S := \{ g \in \mathcal{O}_S \mid g \text{ is invertible outside the point } 0 \in S \} \hookrightarrow \mathcal{O}_S,
\]

\[
    M_X := \{ g \in \mathcal{O}_X \mid g \text{ is invertible outside the fiber } f^{-1}(0) \} \hookrightarrow \mathcal{O}_X.
\]

Associated with these, [KN99] introduced ringed spaces \((S_{\log}, \mathcal{O}_{S}^{\log})\), \((X_{\log}, \mathcal{O}_{X}^{\log})\), a morphism \( f_{\log} : X_{\log} \to S_{\log} \), and the following commutative diagram:

\[
\begin{array}{ccc}
    X & \xrightarrow{\tau_X} & X_{\log} \\
    \downarrow f & & \downarrow f_{\log} \\
    S & \leftarrow \tau_S & S_{\log}
\end{array}
\]

An intrinsic definition of them is as follows. As a set, \( X_{\log} \) is defined to be the set of all pairs \((x, h)\) consisting of a point \( x \in X \) and an argument function \( h \) which is a homomorphism \( M_{X,x}^{\log} \to S^1 \) whose restriction to \( \mathcal{O}_{X,x} \) is \( u \mapsto u(x)/|u(x)| \). Here \( S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \} \). For definition of the topology of \( X_{\log} \), we work locally on \( X \). Take a chart \( S \to M_X \), then we have an injective map

\[
    X_{\log} \hookrightarrow X \times \text{Hom}(S^{\text{gp}}, S^1), \quad (x, h) \mapsto (x, h_S),
\]

where \( h_S \) denotes the composite map \( S^{\text{gp}} \to M_{X,x}^{\text{gp}} \xrightarrow{h} S^1 \). We endow \( X_{\log} \) with the topology as a subset of \( X \times \text{Hom}(S^{\text{gp}}, S^1) \). This topology is independent of the choice of a chart, and hence is globally well-defined. The canonical map

\[
    \tau_X : X_{\log} \to X, \quad (x, h) \mapsto x,
\]

is surjective, continuous and proper. For \( x \in X \), the inverse image \( \tau^{-1}(x) \) is homeomorphic to \( (S^1)^r \), where \( r := \text{rank}_\mathbb{Z}(M_{X,x}^{\text{gp}}/\mathcal{O}_{X,x}^{\text{adm}}) \). The continuous map \( \tau_S : S_{\log} \to S \) is defined similarly, and the continuous map \( f_{\log} : X_{\log} \to S_{\log} \) is defined to be \((x, h) \mapsto (f(x), M_{S,f(x)} h_{M_{X,x}} \to S^1)\).

The sheaf of rings \( \mathcal{O}_{X_{\log}}^{\log} \) on \( X_{\log} \) is defined as follows. We will define first the sheaf of logarithms \( \mathcal{L} \) of \( M_{X,x}^{\text{gp}} \) on \( X_{\log} \), and will then define \( \mathcal{O}_{X_{\log}}^{\log} \) as a sheaf of \( \tau^{-1}(\mathcal{O}_{X}) \)-algebras generated by \( \mathcal{L} \), where \( \tau = \tau_X \). Let \( \mathcal{L} \) be the fiber product of

\[
    \tau^{-1}(M_{X,x}^{\text{gp}}) \xrightarrow{\exp} \text{Cont(}, i\mathbb{R}) \xrightarrow{\text{exp}} \text{Cont(}, S^1),
\]

where \( \text{Cont(}, \cdot \) is the complex analytic completion.
where $\text{Cont}(-, T)$, for a topological space $T$, denotes the sheaf on $X^{\log}$ of continuous maps to $T$, and $\tau^{-1}(M_{X}^{\gp}) \to \text{Cont}(-, S^1)$ comes from the definition of $X^{\log}$. We define

$$\mathcal{O}_{X}^{\log} := (\tau^{-1}(\mathcal{O}_{X}) \otimes_{\mathbb{Z}} \text{Sym}_{\mathbb{Z}}(\mathcal{L}))/\mathfrak{a},$$

where $\text{Sym}_{\mathbb{Z}}(\mathcal{L})$ denotes the symmetric algebra of $\mathcal{L}$ over $\mathbb{Z}$, and $\mathfrak{a}$ is the ideal of $\tau^{-1}(\mathcal{O}_{X}) \otimes_{\mathbb{Z}} \text{Sym}_{\mathbb{Z}}(\mathcal{L})$ generated by the image of

$$\tau^{-1}(\mathcal{O}_{X}) \to \tau^{-1}(\mathcal{O}_{X}) \otimes_{\mathbb{Z}} \text{Sym}_{\mathbb{Z}}(\mathcal{L}), \quad f \mapsto f \otimes 1 - 1 \otimes \iota(f).$$

Here the map $\iota : \tau^{-1}(\mathcal{O}_{X}) \to \mathcal{L}$ is the one induced by

$$\begin{array}{c}
\tau^{-1}(\mathcal{O}_{X}) \\
\exp
\end{array} \tau^{-1}(\mathcal{O}_{X}^{\times}) \subset \tau^{-1}(M_{X}^{\gp}).$$

In the above, $\bar{}$ means the complex conjugation. We denote the projection $\mathcal{L} \to \tau^{-1}(M_{X}^{\gp})$ by exp, and the inverse $\tau^{-1}(M_{X}^{\gp}) \to \mathcal{L}/(2\pi i \mathbb{Z})$ by log. Then we have a commutative diagram with exact rows:

$$\begin{array}{cccc}
0 & \to & \mathbb{Z} & \xrightarrow{2\pi i} & \tau^{-1}(\mathcal{O}_{X}) & \xrightarrow{\exp} & \tau^{-1}(\mathcal{O}_{X}^{\times}) & \to & 1 \\
\| & & \iota \downarrow \cap & & \| \downarrow \cap & & \| \downarrow \cap \\
0 & \to & \mathbb{Z} & \xrightarrow{2\pi i} & \mathcal{L} & \xrightarrow{\exp} & \tau^{-1}(M_{X}^{\gp}) & \to & 1.
\end{array}$$

The morphism $f : X \to S$ induces a morphism

$$f^{\log} : (X^{\log}, \mathcal{O}_{X}^{\log}) \to (S^{\log}, \mathcal{O}_{S}^{\log})$$

of ringed spaces over $\mathbb{C}$ in the evident way.

[Usu01] showed that $f^{\log} : X^{\log} \to S^{\log}$ is topologically locally trivial over the base.

For $y \in X^{\log}$, the stalk $\mathcal{O}_{X,y}^{\log}$ is described as follows. Put $x = \tau(y) \in X$ and $r = \text{rank}_{\mathbb{Z}}(M_{X, x}^{\gp}/\mathcal{O}_{X, x}^{\times})$. Let $(\ell_{j})_{1 \leq j \leq r}$ be a family of elements of $\mathcal{L}_{y}$ whose images in $(M_{X, x}^{\gp}/\mathcal{O}_{X, x}^{\times})$ form a system of free generators. Then we have an isomorphism of $\mathcal{O}_{X, x}$-algebras

$$\tau^{-1}(\mathcal{O}_{X, x})[T_{1}, \ldots, T_{r}] \cong \mathcal{O}_{X,y}^{\log}, \quad T_{j} \mapsto \ell_{j}.$$  

Note that this is a polynomial algebra of $r$-variables over $\tau^{-1}(\mathcal{O}_{X, x})$, which is not a local ring if $r \geq 1$.

§2. Associated log Hodge structure

For each $m \geq 0$, we have the polarizable log Hodge structure $H^{m}(X/S)$ of weight $m$:

$$H_{\mathbb{Z}}^{m}(X/S) = R^{m}f_{\ast}^{\log}(\mathbb{Z}/(\text{torsion})), \quad H_{\mathcal{O}}^{m}(X/S) = R^{m}f_{\ast}(\Omega_{X/S}^{\ast}(\log)),$$

where $\Omega_{X/S}^{\ast}(\log)$ is the de Rham complex with log poles, with the Hodge filtration

$$F^{p}H_{\mathcal{O}}^{m}(X/S) = R^{m}f_{\ast}(\Omega_{X/S}^{\geq p}(\log)).$$
Theorem. There is a fan $g$ \textsuperscript{(1)} tent cones, which satisfies the following

Let $(H^m_{\mathcal{O}}(X/S))$ be the evident Hodge numbers.

Let $(h_k^{p,q})$ be the evident Hodge numbers.

§4. MAIN THEOREM

Let $\gamma' : H' \to H'$ be the local monodromy, i.e., the action of the standard generator of $\pi_1(S^*)$, and let $\Gamma' \subset \text{Aut}(H', \langle , \rangle')$ be the group generated by $\gamma'$. Let $\Gamma$ be the subgroup of $G_{\mathbb{Z}} = \text{Aut}(H, W, \langle , \rangle_k)$ consisting of all elements whose restrictions to $H'$ are contained in $\Gamma'$.

Let $N' = \text{log}(\gamma') : H'_Q \to H'_Q$ (note that $\gamma'$ is unipotent). Let $\Sigma'$ be the fan in $g' := \text{End}(H'R, \langle , \rangle') = \{ h : H'R \to H'R \mid \langle h(x), y \rangle' + \langle x, h(y) \rangle' = 0 \}$ defined by

\[ \Sigma' = \{ (R_{\geq 0})N', \{0\} \}. \]

Theorem. There is a fan $\Sigma$ in $g := \text{End}(H_R, W, \langle , \rangle_k)$, consisting of rational nilpotent cones, which satisfies the following (1) and (2).

(1) For any $\sigma \in \Sigma$, $\sigma$ is admissible for $W$ and the restriction of any element of $\sigma$ to $H_R'$ is contained in $(R_{\geq 0})N'$. Furthermore, $\Sigma$ is strongly compatible with $\Gamma$.

(2) (Relative completeness.) Let $\sigma$ be any rational nilpotent cone in $g$ which is admissible for $W$ such that the restriction of any element of $\sigma$ to $H_R'$ is contained in
(R_{\geq 0})N'. Then there exists a finite subdivision \{\sigma_j\} of \sigma such that each \sigma_j is contained in some element of \Sigma.

We explain some terminology in the above theorem. A nilpotent cone \sigma in g is a cone over \mathbb{R}_{\geq 0} in g generated by a finite number of mutually commutative nilpotent elements of g. We also assume that \sigma is sharp, i.e., \sigma \cap (-\sigma) = \{0\}. A nilpotent cone \sigma is rational if its generators as an \mathbb{R}_{\geq 0}-cone can be taken in gQ. A nilpotent cone \sigma is admissible for W if, for any element \sigma of \sigma, there exists the W-relative \sigma-filtration M = M(\sigma, W), i.e., the filtration M on \text{H}_R satisfying

\[ NM_k \subset M_{k-2} \quad \text{for any } k, \quad \text{and} \]
\[ N^l : \text{gr}_{k+l}^M \text{gr}_k^W \cong \text{gr}_{k-l}^M \text{gr}_k^W \quad \text{for any } k, l \geq 0. \]

Furthermore, this filtration M depends only on the smallest face of \sigma containing \sigma.

See [SZ85], [Kas85], [Kas86] for the details on admissibility.

See [KU09] for the definition of the strong compatibility.

**Definition.** We call a fan \Sigma in g satisfying the conditions (1) and (2) of Theorem as a relatively complete fan.

§5. EXAMPLE OF A RELATIVELY COMPLETE FAN

We prove the main theorem by giving an explicit example of a relatively complete fan. Let

\[ P := \text{Im}(N' : H'_Q \to H'_Q), \quad Q := \text{Ker}(N' : H'_Q \to H'_Q) \cap P. \]

Take a finitely generated \mathbb{Z}-submodule L of \text{H}_Q containing \text{H}'' + N'(\text{H}'). Take a section s : (P \cap L)/(Q \cap L) \to P \cap L of \mathbb{Z}-modules to the surjection P \cap L \to (P \cap L)/(Q \cap L).

Write as s also for the induced Q-linear map P/Q \to P by abuse of notation. For an element \sigma of P/Q, let \sigma(x) be the order of the image of \sigma in P/(P \cap L) + Q.

Fix a \mathbb{Z}-basis \{(e_j)_{1 \leq j \leq m}\} of Q \cap L, and fix an element \sigma \in H inducing 1 \in \mathbb{Z} = \text{gr}_W^W (H).

For \sigma \in P/Q and \sigma = (n_{ij})_{j \in \mathbb{Z}^m}, let \sigma(x, n) be the \text{H}_R of \sigma generated by all elements \sigma = \text{gr}_W^W (H).

The restriction of \sigma to \text{H}_R coincides with \text{H}_Q, and \sigma(N') is an element of \text{H}_R of the form \sigma(x) + (1/\sigma(x)) \sum_{1 \leq j \leq m} c \sigma e_j with \sigma \leq c \leq \sigma + 1 for all j.

Let \Sigma be the set of all faces of \sigma(x, n) for all \sigma \in P/Q and \sigma \in \mathbb{Z}^m.

It is easy to see that this \Sigma satisfies the desired conditions (1) and (2), and also readers can find a proof in §11 in a more general setting.

§6. LMHS AND THEIR MODULI

In this section, we review the notion of LMHS and their moduli in [KNU08], [KNU09], [KNU0.p].

A log mixed Hodge structure with polarized graded quotients over an fs log analytic space S is \((H_S, H_O, F, W, (\langle , \rangle_k))\), consisting of a locally constant sheaf of free \mathbb{Z}-modules \text{H}_S[S] on S, with \text{O}_S^{\log} \otimes S \text{H}_S \simeq \text{O}_S^{\log} \otimes S \text{H}_O, a decreasing filtration F on \text{H}_O such that \text{gr}_P^P is a locally free \text{O}_S-module for any p, a
Q-rational increasing filtration $W$ of the sheaf $H_{\mathbf{R}}$, and a family of polarizations $\langle \cdot, \cdot \rangle_k$ on each $\text{gr}^W_k$, which satisfies the following conditions (1)–(3) (See [KU09] 2.6, [KKN08a] 2.3, 2.5 for the details):

1. **Admissibility** for any $s \in S$: the action of $\pi_1(s^{\log})$ (= the local monodromy action at $s$) is admissible with respect to $W$.

2. **Small Griffiths transversality** for any $\sigma$. (This is a weaker version of Griffiths transversality, and is imposed only at the place of degeneration.)

3. **Graded positivity**: $(W_k H_\mathbf{Z}/W_{k-1} H_\mathbf{Z}, F(\text{gr}_k^W), \langle \cdot, \cdot \rangle_k)$ is a polarized log Hodge structure of weight $k$ for each $k$.

Let $D = D(H, W, (h_{k}^{p,q}), (\langle \cdot, \cdot \rangle_k))$ be the classifying space of log mixed Hodge structures with polarized graded quotients, and let $\tilde{D} = \tilde{D}(H, W, (h_{k}^{p,q}), (\langle \cdot, \cdot \rangle_k))$ be its “compact dual”. They and their toroidal partial enlargements are defined roughly as follows. See [KU99], [KU09], [KNU08], [KNU09], [KNU.p] for the details.

\[ \mathcal{F} := \{ F \text{ decreasing filtration on } H_{\mathbf{C}} \mid \dim_C \text{gr}^p_F \text{gr}_k^W = h_{k}^{p,w-p} \} \]
\[ \supset \tilde{D} := \{ F \in \mathcal{F} \mid \langle F^p(\text{gr}_k^W), F^q(\text{gr}_k^W) \rangle_k = 0 \text{ for } p + q > k \} \]
\[ \supset D := \{ F \in \tilde{D} \mid i^{p,q}(x, \tau)_k > 0 \text{ for } x \in H_{p,q}(\text{gr}_k^W) - \{0\}, p + q = k \}, \]

where $H_{p,q}(\text{gr}_k^W) = F^p(\text{gr}_k^W) \cap F^q(\text{gr}_k^W)$ for $p + q = k$.

Let $\Sigma$ be a fan satisfying (1) in Theorem.

$D_\Sigma = D_\Sigma(H, W, (h_{k}^{p,q}), (\langle \cdot, \cdot \rangle_k), \Sigma)$ be the space of nilpotent orbits.

$\tilde{D}' = \tilde{D}'(H', (h_{k}^{p,q}), (\langle \cdot, \cdot \rangle_k)) \supset D' = D'(H', (h_{k}^{p,q}), (\langle \cdot, \cdot \rangle_k))$

$D_{\Sigma'} = D_{\Sigma'}(H', (h_{k}^{p,q}), (\langle \cdot, \cdot \rangle_k), \Sigma')$ be the space of nilpotent orbits.

Then $D' = \text{gr}_W^1(D), \tilde{D}' = \text{gr}_W^1(\tilde{D}), D_{\Sigma'} = \text{gr}_W^1(D_\Sigma)$.

The polarized log Hodge structure $H^{2r-1}(X/S)(r)$ defines a canonical morphism (period map)

\[ S \rightarrow \Gamma' \backslash D_{\Sigma'}. \]

§7. **Log intermediate Jacobian**

Let $\Sigma$ be a fan satisfying (1) in Theorem. Then

**Proposition.** The space $\Gamma' \backslash D_{\Sigma}$ is a log manifold and Hausdorff.

This is a main result of [KNU.p], which is a mixed version of [KU09], and proved similarly to the pure case in loc. cit. As mentioned in [KNU08] 0.8, in the proof, the SL(2)-orbit theorem 0.5 in loc. cit. plays a key role, that is, to guarantee the continuity of the CKS-map (cf. [KU09]), exactly as the SL(2)-orbit theorem of Cattani-Kaplan-Schmid [CKS86] did so in the pure case [KU09].

Define $J_{X/S, \Sigma}^r$ to be the fiber product

\[ J_{X/S, \Sigma}^r \quad \longrightarrow \quad \Gamma' \backslash D_{\Sigma} \]
\[ \downarrow \quad \quad \quad \downarrow \text{gr}_W^1 \]
\[ S \quad \underbrace{\text{period map}} \quad \Gamma' \backslash D_{\Sigma'} \]

in the category $B(\log)$ ([KU09] 3.2.4).
Corollary. The space $J^r_{X/S,\Sigma}$ is a log manifold and Hausdorff.

If $T$ also satisfies (1) in Theorem and if $T \subset \Sigma$, then $J^r_{X/S,T}$ is an open subset of $J^r_{X/S,\Sigma}$.

For $\Sigma$ which also satisfies (2) in Theorem, i.e., for a relatively complete fan $\Sigma$, we call $J^r_{X/S,\Sigma}$ the $r$-th log intermediate Jacobian associated to $X/S$ and $\Sigma$.

§8. Relationship with other works

What we have just constructed is closely related to some works of Zucker [Zuc76] (cf. also [EZ84]), Clemens [Cle83], and Saito [Sai96]. What they considered are essentially $J^r_{X/S,\Sigma}$ for the following fans $\Sigma = \Sigma_j$ ($j = 0, 1, 2$). We denote Zucker’s (resp. Clemens’, Saito’s) fan by $\Sigma_0$ (resp. $\Sigma_1$, $\Sigma_2$). We also denote our relatively complete fan in §5 by $\Sigma_3$. $\Sigma_j$ for $j = 1, 2$ are considered under the condition:

$$ (8.1) \quad (N')^2 = 0 \quad \text{and} \quad gr^{W'}_0 \text{ has Hodge type } (0, 0). $$

Here $W' = W(N')[1]$, that is, the $N'$-filtration on $H'$.

Now we define $\Sigma_j$ ($j = 0, 1, 2$):

$$ \Sigma_0 = \{(R_{\geq 0})N \mid N \in g_{\mathbb{Q}}, N|_{H'} = N', N(e) \in N'(H')\} \cup \{0\}. $$

$$ \Sigma_1 = \{(R_{\geq 0})N \mid N \in g_{\mathbb{Q}}, N|_{H'} = N', N(e) \in Q \cap H'\} \cup \{0\}. $$

$$ \Sigma_2 = \{\text{face of } \sigma_n \mid n \in \mathbb{Z}^m\}, $$

where for a fixed $\mathbb{Z}$-basis $e'_1, \ldots, e'_m$ of $N'(H')$, $\sigma_n$ is the $(R_{\geq 0})$-cone generated by

$$ \{N \in g_{\mathbb{Q}} \mid N|_{H'} = N', N(e) = \sum_{1 \leq j \leq m} c_j e'_j \text{ with } n_j \leq c_j \leq n_j + 1 \text{ for any } j\}. $$

Under the condition (8.1), we see that $P = Q$ so that $s$ must be 0, and for any $L$, we have the following relationship among the four fans:

$$ \Sigma_2 \subset \Sigma_3 \cup \Sigma_1, \quad \Sigma_1 \text{ and } \Sigma_2 \text{ are not necessarily contained in each other.} $$

Here $A < B$ means $B$ is a subdivision of $A$.

Precisely speaking, as Saito pointed out in [Sai96, (3.5) (iv)], Zucker’s space is not Hausdorff. This is because Zucker did not put the Griffiths transversality. On the other hand, our space $J^r_{X/S,\Sigma_0}$ or $J^r_{X/S,\Sigma_3}$ is always Hausdorff thanks to slits coming from the Griffiths transversality (see the next section). In the case (8.1), the Griffiths transversality is automatically satisfied and hence slits do not appear, and Clemens and Saito considered exactly $J^r_{X/S,\Sigma_j}$ ($j = 1, 2$, respectively) which is Hausdorff. Clemens’ is “Néron model” which is not necessarily proper over $S$, whereas Saito’s is proper over $S$.

Proposition. Let $\Sigma$ be a fan satisfying (1) in Theorem. Assume (8.1). Then, $J^r_{X/S,\Sigma} \to S$ is proper if and only if $\Sigma$ is relatively complete.

This is because slits do not appear under (8.1).
\section{Example of slit}

Let \( Y = \mathbb{C}/\mathbb{Z}[i] \) be the elliptic curve with period \( i \), and let \( g : E \to \Delta \) be the standard degeneration of elliptic curves, i.e., \( g^{-1}(q) = \mathbb{C}^\times /q\mathbb{Z} \) for \( q \neq 0 \) and \( g^{-1}(0) \) is a rational curve with one node, and consider the family

\[ f := g \circ \text{pr}_2 : X = Y^2 \times E \to S = \Delta. \]

Then we see

\[ (\text{Zucker's space for } \Sigma_0) = Y^4 \times \big( ((\mathbb{C}^\times)^4 \times \mathbb{C}^2 \times \Delta)/ \sim \big). \]

Here \( ((t_j), (a_k), q) \sim ((t'_j), (a'_k), q') \) if and only if

\[
\begin{aligned}
\text{when } q \neq 0; & \quad q' = q, t'_j/t_j \in q^\mathbb{Z} \text{ for any } j, \text{there exists } b \in \mathbb{Z}[i] \\
& \quad \text{such that } a'_2 - a_2 - b = 0 \text{ and } a'_1 - a_1 - b \cdot \log(q)/(2\pi i) \in \mathbb{Z}[i]; \\
\text{when } q = 0; & \quad q' = q = 0, t'_j = t_j \text{ for any } j, \quad a'_2 - a_2 = 0, \quad a'_1 - a_1 \in \mathbb{Z}[i].
\end{aligned}
\]

This is not Hausdorff. In fact, let \( c \in \mathbb{C} \) and \( t \in (\mathbb{C}^\times)^4 \), and for \( n \gg 0 \), let \( q_n = \exp(2\pi i(c+n i)). \) Then \( (t, (c, 1), q_n) \sim (t, (0, 0), q_n), \) and \( (t, (c, 1), q_n) \) (resp. \( (t, (0, 0), q_n) \)) converges to \( (t, (c, 1), 0) \) (resp. \( (t, (0, 0), 0) \)) as \( n \to \infty. \) But we see \( (t, (c, 1), 0) \not\sim (t, (0, 0), 0). \)

On the other hand, our space for the fan \( \Sigma_0 \) is

\[ J^2_{X/S, \Sigma_0} = Y^4 \times \{ \text{class of } (t, a, q) \mid q = 0 \Rightarrow a_2 = 0 \}. \]

Here the condition “\( q = 0 \Rightarrow a_2 = 0 \)” comes from the Griffiths transversality, produces a slit and makes the space \( J^2_{X/S, \Sigma_0} \) Hausdorff for the usual topology and hence for the strong topology.

\section{Comments on relative completeness}

We have an embedding

\[ \text{Mor}(?, J^r_{X/S, \Sigma}) \subset \text{Ext}_{LMHS}^1(\mathbb{Z}, H^{2r-1}(X/S)(r)) \]

of functors from the category \( (\text{fs}/S) \) of fs log analytic spaces over \( S \) to the category of sets. Here, \( \text{Ext}^1 \) is taken in the category \( (\text{LMHS}) \) of log mixed Hodge structures with polarized graded quotients \( (\text{fs}) \).

**Proposition.** Let \( \Sigma \) be a relatively complete fan (§4). Then, for any fs log analytic space \( S' \) over \( S \) and any \( a \in \text{Ext}_{LMHS}^1(\mathbb{Z}, H^{2r-1}(X/S)(r)) \), locally on \( S' \), there is a log modification \( S'' \to S' \) ([KU09], 3.6) and a subdivision \( \Sigma' \) of \( \Sigma \) satisfying (1) in Theorem in §4 such that the image of \( a \) in \( \text{Ext}_{LMHS}^1(\mathbb{Z}, H^{2r-1}(X/S)(r)) \) belongs to \( \text{Mor}(S'', J^r_{X/S, \Sigma'}) \).

The proof of the above fact is similar to the pure case [KU09] 4.3, where the extensions of period maps are explained.

In particular, we have the following. For any fs log analytic space \( S' \) over \( S \) which is log smooth over \( C \) ([KU09], 2.1.11), let \( U \) be the open subspace of \( S' \) where the
log structure is trivial. Let $a \in \text{Ext}_1^r(\mathbb{Z}, H^{2r-1}(X/S))(r)$ be an extension of graded polarized variation of MHS, regarded as a morphism $a : U \rightarrow J_{X_U/U}^r$ to the usual intermediate Jacobian. Assume that $a$ is admissible with respect to $S'$. Then, locally on $S'$, there is a log modification $S'' \rightarrow S'$ with $U \subset S''$ and a subdivision $\Sigma'$ of $\Sigma$ satisfying (1) in Theorem such that $U \rightarrow J_{X_U/U}^r \rightarrow J_{X/S,\Sigma'}^r$ extends to a morphism $S'' \rightarrow J_{X/S,\Sigma'}^r$.

More specifically, assume $S' = S = \Delta$. Then, $\text{Ext}_1^r(\mathbb{Z}, H^{2r-1}(X/S))(r)$ is nothing but the space of admissible normal functions ([Sai96]), and the above fact says that any admissible normal function extends to some log intermediate Jacobian because, in this case, there is no non-trivial log modification, that is, $S'' = S'$.

Since a cycle on $X$ gives an admissible normal function by a theory of Saito ([Sai90], [Sai96]), we also have the Abel-Jacobi map into the log intermediate Jacobian.

§11. Case of Extensions with Negative Weights

Here we give a construction of a relatively complete fan for an extension $0 \rightarrow (H')/(H')$, $H \rightarrow \mathbb{Z} \rightarrow 0$ with $k < 0$, which generalizes the case $k = -1$ in §5. Here the base $S$ is the disc $\Delta$.

11.1. Let $W'$ be the filtration $W(N')[-k]$ on $H'_Q$. Let

$$P = \text{Im}(N' : H'_Q \rightarrow H'_Q) + W'_{-2}, \quad Q = \text{Ker}(N' : H'_Q \rightarrow H'_Q) \cap W'_{-2}.$$

Let $L$ be a finitely generated $\mathbb{Z}$-submodule of $H'_Q$ which contains $H' + N'(H')$. Fix a homomorphism $s : (P \cap L)/(Q \cap L) \rightarrow P \cap L$ of $\mathbb{Z}$-modules such that the composition $(P \cap L)/(Q \cap L) \xrightarrow{s} P \cap L \rightarrow (P \cap L)/(Q \cap L)$ is the identity map of $(P \cap L)/(Q \cap L)$, and denote the $\mathbb{Q}$-linear map $P/Q \rightarrow P$ induced by $s$ by the same letter $s$. For an element $x$ of $P/Q$, let $a(x)$ be the smallest integer $a$ such that $ax \in (P \cap L)/(Q \cap L)$.

Fix $e \in H'$ whose image in $H'/H'$ is $\mathbb{Z}$ is 1. Fix a $\mathbb{Z}$-basis $(e_j)_{1 \leq j \leq r}$ of $Q \cap L$, where $r = \dim_{\mathbb{Q}}(Q)$. For $x \in P/Q$ and $n \in \mathbb{Z}^r$, let $\sigma(x, n)$ be the cone generated by all $N \in \mathfrak{g}$ such that the restriction of $N$ to $H'_R$ coincides with $N'$ and such that $N(e) = s(x) + a(x)^{-1} \sum_{j=1}^r t_j e_j$ with $n_j \leq t_j \leq n_j + 1$ for $1 \leq j \leq r$.

Let $\Sigma$ be the set of all faces of $\sigma(x, n)$ for all $x \in P/Q$ and $n \in \mathbb{Z}^r$.

Proposition 11.2. Let $\Gamma$ be as in §4. Then $\Sigma$ is strongly compatible with $\Gamma$.

Proof. It is enough to show that $\text{Ad}(\gamma) \sigma \in \Sigma$ for any $\gamma \in \Gamma$ and $\sigma \in \Sigma$. It is sufficient to prove $\text{Ad}(\gamma) \sigma(x, n) \in \Sigma$ for any $x \in P/Q$ and $n \in \mathbb{Z}^r$. Write $\gamma^{-1} e = e + h$ with $h \in H'$, and write $\gamma s(x) + N'(\gamma h) = s(y) + q$ with $y \in P/Q$ and $q \in Q$. We have $a(y) = a(x)$ and $a(x) q \in Q \cap L$. Write $a(x) q = \sum_{j=1}^r m_j e_j$ with $m_j \in \mathbb{Z}$, and let $m = (m_j) \in \mathbb{Z}^r$. We prove

$$\text{Ad}(\gamma) \sigma(x, n) = \sigma(y, n + m).$$

In fact, let $N \in \sigma(x, n)$ and assume that the restriction of $N$ to $H'_R$ is $N'$ and $N(e) = s(x) + a(x)^{-1} \sum_{j=1}^r t_j e_j$ with $n_j \leq t_j \leq n_j + 1$. Then, since $\gamma$ and $N'$ commute, we have

$$\gamma N^{-1}(e) = \gamma N(e + h) = \gamma s(x) + N'(\gamma h) + a(x)^{-1} \sum_{j=1}^r t_j e_j.$$
\[ = s(y) + a(y)^{-1} \sum_{j=1}^{r} (t_j + m_j)e_j. \]

Hence \( \gamma N \gamma^{-1} \) belongs to \( \sigma(y, n + m) \). \( \square \)

**Proposition 11.3.** \( \Sigma \) is a relatively complete fan.

This is deduced from the following two facts:

**Fact 1.** Let \( N : H_R \to H_R \) be a homomorphism such that \( N(H_R) \subset H'_R \) and such that the restriction of \( N \) to \( H'_R \) coincides with \( N' \). Then the relative monodromy filtration \( M(N, W) \) exists if and only if \( N(e) \in \text{Im}(N' : H_R \to H_R) + W'_{-2,R} \).

**Fact 2.** Let \( N_j : H_R \to H_R \) (\( j = 1, 2 \)) be homomorphisms such that \( N_j(H_R) \subset H'_R \) and such that the restrictions of \( N_j \) to \( H'_R \) coincide with \( N' \). Then \( N_1N_2 = N_2N_1 \) if and only if \( N_1(e) \equiv N_2(e) \mod \text{Ker}(N' : H'_R \to H'_R) \).

## §12. Remarks on Néron models

Here the base \( S \) is any fs log analytic space unless otherwise stated. Let \( H' \) be a polarized log Hodge structure of weight \(-1\) over \( S \).

**12.1.** (The case where \( F^1 = 0 \) of this subparagraph is in [KKN08c] §5.)

From the viewpoint of the theory of log intermediate Jacobian, it is fundamental to consider the exact sequences

\[ 0 \to H' \to (O^\log \otimes H')/F^0 \to H'/(O^\log \otimes H')/F^0 \to 0 \]

of abelian sheaves on \((fs/S)^{\log}\) (see [KKN08a] for the definition of \((fs/S)^{\log}\)) and the induced

\[ 0 \to \tau_*H' \to (H'_{\text{hor}}/F^0)_{\text{hor}} \to (\tau_*(H'/(O^\log \otimes H')/F^0))_{\text{hor}} \to R^1\tau_*H' \]

of abelian sheaves on \((fs/S)\), where “hor” means the horizontal parts, i.e., the parts consisting of sections corresponding to pre-log mixed Hodge structures that satisfy the small Griffiths transversality.

There are several important subgroups of \((\tau_*(H'/(O^\log \otimes H')/F^0))_{\text{hor}}\) which are, respectively, the inverse images of some subgroups of the monodromy group \( R^1\tau_*H' \) under the connecting homomorphism \( \partial \) in the last exact sequence (*). The sheaf \( \mathcal{E}xt^1_{\text{LMHS}}(Z, H') \) is one of them, which is the inverse image of the “admissible part” of the monodromy group. From the viewpoint of log geometry, it is this sheaf that should be called the “log intermediate Jacobian”, and what have been called log intermediate Jacobians so far in this article should be called “models of the log intermediate Jacobian.”

Note that this sheaf \( \mathcal{E}xt^1_{\text{LMHS}}(Z, H') \) is a group object, and, in a sense, log smooth (even when \( H' \) corresponds to some geometric object which degenerate in the usual sense), as so are log complex tori introduced in [KKN08a]. The authors expect that it would be possible to generalize the theory of log complex tori and the theory of their proper models developed in [KKN08a] and [KKN08c] to the log intermediate Jacobians.

**12.2.** Let the situation be as in §1. Let \( \Sigma_1 \) be the fan consisting of \( \{0\} \) and the cones \((R_{\geq 0})N\) for \( N \in \mathfrak{g}_Q \) satisfying \( N|_{H'} = N' \), \( N(e) = N'(a) \) for some \( a \in H'_Q \) such that \( \gamma a - a \in H' \). We define the Néron model as \( J_{X/S, \Sigma_1} \). This is a log manifold
whose log structure is the inverse image of that of the base, and "represents" (in some
suitable senses\(^1\)) the subgroup of \((\tau_*(H'\setminus (\mathcal{O}^{\text{log}} \otimes H')/F^0))\)\(_{\text{hor}}\) which is the inverse image of \(\psi^{-1}(\mathcal{R}_N^1\tau_*H'_{\text{tor}}) \subset \mathcal{R}_N^1\tau_*H'\) by \(\partial\) in (\(\ast\)). Here \(\tau_S : S^{\text{log}} \to S\) and \(\iota : (\mathfrak{f}/S) \to S\) are the natural morphisms. Note that this \(J^r_{X/S,\Sigma_1}\) generalizes Clemens' model constructed in \(\S 8\) under the condition (8.1).

It is easy to see that for a sufficiently large \(L\) (and for any \(s\)), our fan \(\Sigma\) in \(\S 5\) contains \(\Sigma_1\) as a subfan. Hence our log intermediate Jacobian associated to \(\Sigma\) contains the Néron model as an open subspace. In this sense, our construction gives a kind of compactification of the Néron model. See \(\S 8\) for the special case of this fact under the condition (8.1).

By the proof of [KU09] 4.3.1 (i), which works also in this mixed Hodge theoretic situation, any admissible normal function extends to the Néron model.

The relationship does not seem to be known between this \(J^r_{X/S,\Sigma_1}\) and the Néron model constructed by Green-Griffiths-Kerr [GGK.p].

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\end{thebibliography}

\footnote{\(1\)Precisely, we can say as follows. In general, \(J^r_{X/S,\Sigma}\) is defined as an object of \(B(\text{log})\) ([KU09] 3.2.4) (even if \(S\) is not \(\Delta\) but any \(\mathfrak{fs}\) log analytic space). If we do 12.1 over \(B(\text{log})/S\) instead of \((\mathfrak{fs}/S)\), we can say \(J^r_{X/S,\Sigma_1}\) represents the indicated subgroup in the usual sense. Alternatively, in general, if the base \(S\) is log smooth ([KU09] 2.1.11), \(J^r_{X/S,\Sigma}\) is a log manifold and so is regarded as an object of the category \((\mathfrak{fs}/S)^\sim\) of sheaves on \((\mathfrak{fs}/S)\). Then, \(J^r_{X/S,\Sigma_1}\) is naturally isomorphic to the indicated subgroup as an object of \((\mathfrak{fs}/S)^\sim\).}
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