GLOBAL WELL-POSEDNESS, SCATTERING AND BLOW-UP FOR THE ENERGY CRITICAL FOCUSING NON-LINEAR WAVE EQUATION

CARLOS E. KENIG AND FRANK MERLE

1. Introduction

In this paper we consider the energy critical non-linear wave equation

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u &= \pm |u|^{\frac{4}{N-2}} u \\
|u|_{t=0} &= u_0 \in H^1(\mathbb{R}^N) \\
\partial_t u |_{t=0} &= u_1 \in L^2(\mathbb{R}^N)
\end{align*}
\]

Here the sign corresponds to the defocusing problem, while the + sign corresponds to the focusing problem. The theory of the local Cauchy problem (CP) for this equation was developed in many papers, see for instance [26], [9], [21], [29], [30], [31], [15] etc. In particular, one can show that if \( \|(u_0, u_1)\|_{\dot{H}^{1/2} \times L^2} \leq \delta, \delta \) small, there exists a unique solution with \( (u, \partial_t u) \in C(\mathbb{R}; \dot{H}^{1}(\mathbb{R}^N) \times L^2(\mathbb{R}^N)) \) with the norm

\[
\|u\|_{L^{\frac{2(N+1)}{N-2}}} < \infty
\]

(i.e., the solution scatters in \( \dot{H}^{1}(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \)). See section 2 of this paper for a review and an update of the results.

In the defocusing case, Struwe [34] in the radial case, when \( N = 3 \), Grillakis [11] in the general case when \( N = 3 \), and then Grillakis [12], Shatah-Struwe [28], [29], [30] (and others [15]) in higher dimensions, proved that this also holds for any \( (u_0, u_1) \) with \( \|(u_0, u_1)\|_{\dot{H}^{1/2} \times L^2} < \infty \) and that, (for \( 3 \leq N \leq 5 \)) for more regular \( (u_0, u_1) \) the solution preserves the smoothness for all time. This topic has been the subject of intense investigation. See the recent work of Tao [36] for a recent installment in it and further references.

The first author was supported in part by NSF and the second one in part by CNRS. Part of this research was carried out during visits of the second author to the University of Chicago and I.H.E.S. and of the first author to Paris XIII.
In the focusing case, these results do not hold. In fact, the classical identity
\begin{equation}
\frac{d^2}{dt^2} \int |u(x,t)|^2 = 2 \left( \int (\partial_t u)^2 - |\nabla u|^2 - |u(t)|^{\frac{2N}{N-2}} \right)
\end{equation}
(see the work of H. Levine [20] and also sections 3 and 5) was used by Levine [20] to show that if \((u_0, u_1) \in H^1 \times L^2\) is such that
\begin{equation}
E((u_0, u_1)) = \int \left( \frac{1}{2} |\nabla u_0|^2 + \frac{1}{2} |u_1|^2 - \frac{(N-2)}{2N} |u_0|^{\frac{2N}{N-2}} \right) < 0,
\end{equation}
the solution must break down in finite time. Moreover,
\begin{equation}
W(x) = W(x,t) = \frac{1}{\left(1 + \frac{|x|^2}{N(N-2)}\right)^{(\frac{N-2}{2})}}
\end{equation}
is in \(\dot{H}^1(\mathbb{R}^N)\) and solves the elliptic equation
\begin{equation}
\Delta W + |W|^{\frac{4N}{N-2}} W = 0,
\end{equation}
so that scattering cannot always occur even for global (in time) solutions.

In this paper we initiate the detailed study of the focusing case (see also [19] for an interesting recent work in this direction). We show:

**Theorem 1.1.** Let \((u_0, u_1) \in \dot{H}^1 \times L^2, 3 \leq N \leq 5\). Assume that
\begin{equation}
E((u_0, u_1)) < E((W, 0)).
\end{equation}
Let \(u\) be the corresponding solution of the Cauchy problem, with maximal interval of existence
\begin{equation}
I = (-T_-(u_0, u_1), T_+(u_0, u_1)).
\end{equation}
(See Definition 2.13.) Then:

i) If \(\int |\nabla u_0|^2 < \int |\nabla W|^2\), then
\begin{equation}
I = (-\infty, +\infty) \quad \text{and} \quad \|u\|_{L^{\frac{2(N+1)}{N-2}}(\mathbb{R}^N_x)} < \infty.
\end{equation}

ii) If \(\int |\nabla u_0|^2 > \int |\nabla W|^2\), then
\begin{equation}
T_+(u_0, u_1) < +\infty, \quad T_-(u_0, u_1) < +\infty.
\end{equation}

Our proof follows the new point of view into these problems that we introduced in [16], where we obtained the corresponding result for the energy critical non-linear Schrödinger equation for radial data. In section 3 we prove some elementary variational estimates which yield the necessary coercivity for our arguments and which follows from arguments in [16]. In section 4, using the work of Bahouri-Gerard [4] and the concentration compactness argument from [16] we produce a “critical element” for which scattering fails and which enjoys a compactness property because of its criticality. (Propositions 4.1 and 4.2.)
At this point, we show a crucial orthogonality property of “critical elements” related to a second conservation law in the energy space (Proposition 4.10 and Proposition 4.11), which exploits the finite speed of propagation for the wave equation and its Lorentz invariance. This is the extra ingredient that allows us to go beyond the radial case as in [16]. In sections 5 and 6 we prove a rigidity theorem (Theorem 5.1), which allows us to conclude the argument. The first case of the rigidity theorem deals with infinite time of existence. This uses localized conservation laws of the type (1.1) and related ones, very much in the spirit of the corresponding localized virial identity used in [16]. The second case of the rigidity theorem deals with finite time of existence. This case is dealt with in [16] through the use of the $L^2$ conservation law, which is absent for the wave equation. We proceed in two stages. First we show that the solution must have self-similar behavior (Proposition 5.7). Then, in section 6, following Merle-Zaag ([23]) and earlier work on non-linear heat equations by Giga-Kohn ([8]), we introduce self-similar variables and the new resulting equation, which has a monotonic energy. We then show that there exists a non-trivial asymptotic solution $w^*$, which solves a (degenerate) elliptic non-linear equation. Finally, using the estimates we proved on $w^*$ and the unique continuation principle, we show that $w^*$ must be zero, a contradiction which gives our rigidity theorem. In section 7 we prove our main theorem as a consequence of the rigidity theorem.

Finally, we would like to point out that we expect that our arguments will extend to $N \geq 6$, using arguments similar to those in the work of Tao-Visan [37] for the local solvability in time of the equation and the corresponding extension of the work of Bahouri-Gerard [4] (the rest of argument is independent of the dimension).

2. A review of linear estimates and the Cauchy problem

In this section we will review the theory of the Cauchy problem

\[
\begin{align*}
\partial_t^2 u - \Delta u &= |u|^\frac{4}{N-2} u \\
|u|_{t=0} &= u_0 \in \dot{H}^1(\mathbb{R}^N) \\
\partial_t u |_{t=0} &= u_1 \in L^2(\mathbb{R}^N)
\end{align*}
\]

i.e. the $\dot{H}^1$ critical, focusing Cauchy problem for NLW, and some of the associated linear theory. We start out with some preliminary notation
and linear estimates. Consider thus

\[
\begin{aligned}
\text{(LCP)} & \quad \begin{cases}
\partial_t^2 w - \Delta w = h \\ w_{t=0} = w_0 \in \dot{H}^1(\mathbb{R}^N) \\ \partial_t w_{t=0} = w_1 \in L^2(\mathbb{R}^N)
\end{cases} \\
\end{aligned}
\]

the associated linear problem. The solution operator to (LCP) is given by:

\[
w(x, t) = \cos(t\sqrt{-\Delta})w_0 + (-\Delta)^{1/2}\sin(t\sqrt{-\Delta})w_1 + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} h(s) ds
\]

\[
= S(t)(w_0, w_1) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} h(s) ds.
\]

**Lemma 2.1** (Strichartz estimates [21, 10]). There is a constant $C$, independent of $T$, such that

\[
\begin{aligned}
\left\| \partial_t w \right\|_{L^2_t(W^{N+1}_t \dot{H}^{N+1}_x)} + \left\| w \right\|_{L^2_t(W^{N+1}_t \dot{H}^{N+1}_x)} &
\leq C \left[ \left\| w_0 \right\|_{\dot{H}^1(\mathbb{R}^N)} + \left\| w_1 \right\|_{L^2(\mathbb{R}^N)} + \left\| h \right\|_{L^2_t(\dot{H}^{N+3}_x)\dot{L}^{2(N+3)}_x} \right].
\end{aligned}
\]

**Lemma 2.2** (Trace Theorem). Let $w_0, w_1, h, w$ be as in Lemma 2.1. Then, for $|d| \leq 1/4$,

\[
\begin{aligned}
\sup_t \left\| \nabla_x w \left( \frac{x_1 - dt}{\sqrt{1 - d^2}}, x', \frac{t - dx_1}{\sqrt{1 - d^2}} \right) \right\|_{L^2(dx_1 dx')}
&+ \sup_t \left\| \partial_t w \left( \frac{x_1 - dt}{\sqrt{1 - d^2}}, x', \frac{t - dx_1}{\sqrt{1 - d^2}} \right) \right\|_{L^2(dx_1 dx')}
\leq C \left\{ \left\| w_0 \right\|_{\dot{H}^1(\mathbb{R}^N)} + \left\| w_1 \right\|_{L^2(\mathbb{R}^N)} + \left\| h \right\|_{L^1_t L^2_x} \right\}.
\end{aligned}
\]

**Proof.** Let $v(x, t) = U(t)\hat{f}$ be given by $\hat{v}(\xi, t) = e^{it\xi} \hat{f}(\xi)$, with $f \in L^2$.

We will show that

\[
\sup_t \left\| v \left( \frac{x_1 - dt}{\sqrt{1 - d^2}}, x', \frac{t - dx_1}{\sqrt{1 - d^2}} \right) \right\|_{L^2(dx_1 dx')} \leq C \left\| f \right\|_{L^2}.
\]
which easily implies the desired estimate. But
\[ v(x, t) = \int e^{ix \xi} e^{it |\xi|} \hat{f}(\xi) d\xi = \int e^{i x_1 \xi_1} e^{it |\xi|} e^{ix' \xi'} \hat{f}(\xi) d\xi_1 d\xi', \]
so that
\[ v \left( \frac{x_1 - dt}{\sqrt{1 - d^2}}, x', \frac{t - dx_1}{\sqrt{1 - d^2}} \right) = \int e^{i(x_1 - dt) \xi_1 / \sqrt{1 - d^2}} e^{i(t - dx_1) \xi_1 / \sqrt{1 - d^2}} e^{ix' \xi'} \hat{f}(\xi) d\xi_1 d\xi', \]
\[ = \int e^{i x_1 (\xi_1 - d|\xi|) / \sqrt{1 - d^2}} e^{it \xi_1 / \sqrt{1 - d^2}} e^{it |\xi|} e^{ix' \xi} e^{-idt \xi_1 / \sqrt{1 - d^2}} e^{it |\xi|} e^{ix' \xi} \hat{f}(\xi) d\xi_1 d\xi', \]
\[ = \int e^{ix_1 (\xi_1 - d|\xi|) / \sqrt{1 - d^2}} e^{ix' \xi} \hat{g}_t(\xi) d\xi_1 d\xi', \]
where \( \hat{g}_t(\xi) = e^{-idt \xi_1 / \sqrt{1 - d^2}} e^{it |\xi| / \sqrt{1 - d^2}} \hat{f}(\xi) \). We now define \( \eta_1 = \frac{\xi_1 - d|\xi|}{\sqrt{1 - d^2}} \),
\( \eta' = \xi' \) and compute
\[ \left| \frac{d\eta}{d\xi} \right| = \det \left( \begin{array}{cccc} \frac{-d \xi_1 / |\xi|}{\sqrt{1 - d^2}} & \frac{-d \xi_2 / |\xi|}{\sqrt{1 - d^2}} & \cdots & \frac{-d \xi_N / |\xi|}{\sqrt{1 - d^2}} \\ \frac{1}{\sqrt{1 - d^2}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{1 - d^2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right) \]
\[ = \left( \frac{1 - d \xi_1 / |\xi|}{\sqrt{1 - d^2}} \right) \approx 1 \text{ for } |d| \leq 1/4. \]
The result now follows from Plancherel’s Theorem. \( \square \)

Remark 2.3. A density argument in fact shows that
\[ t \mapsto w \left( \frac{x_1 - dt}{\sqrt{1 - d^2}}, x', \frac{t - dx_1}{\sqrt{1 - d^2}} \right) \in C \left( \mathbb{R}; \dot{H}^1 \left( \mathbb{R}^N, dx_1, dx' \right) \right), \]
and similarly for \( \partial_t w \).

Remark 2.4. Let \( F(u) = |u|^\frac{4}{N-2} u \), then clearly for \( 3 \leq N \leq 6 \),
\[ |F(u)| \leq |u|^\frac{N+2}{N-2}, \quad |(\nabla F)(u)| \leq C |u|^\frac{4}{N-2}, \]
\[ |(\nabla F)(u) - (\nabla F)(v)| \leq C |u - v| \left\{ |u|^\frac{6-N}{N-2} + |v|^\frac{6-N}{N-2} \right\}, \]
\[ |\nabla_x (F(u(x))) - \nabla_x (F(v(x)))| \leq C |u(x)|^\frac{4}{N-2} |\nabla u(x) - \nabla v(x)| \]
\[ + C |\nabla v(x)| \left\{ |u|^\frac{6-N}{N-2} + |v|^\frac{6-N}{N-2} \right\} |u - v|. \]
We will need also a version of the chain rule for fractional derivatives (see \[6, 17, 32, 38\]).

**Lemma 2.5.** Assume \( F(0) = F'(0) = 0 \) and that for all \( a, b \)
\[
|F'(a + b)| \leq C \{ |F'(a)| + |F'(b)| \}, \quad |F''(a + b)| \leq C \{ |F''(a)| + |F''(b)| \}.
\]

We then have, for \( 0 < \alpha < 1 \)
\[
\|D^\alpha F(u)\|_{L^p} \leq C \|F'(u)\|_{L^p} \|D^\alpha u\|_{L^p},
\]
where \( \frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}, \quad 1 < p < \infty \) and
\[
\|D^\alpha (F(u) - F(v))\|_{L^p} \leq C \{ \|F'(u)\|_{L^{p_1}} + \|F'(v)\|_{L^{p_2}} \} \|D^\alpha (u - v)\|_{L^p}
\]
\[+ C \{ \|F''(u)\|_{L^{r_1}} + \|F''(v)\|_{L^{r_2}} \} \{ \|D^\alpha u\|_{L^p} + \|D^\alpha v\|_{L^p} \} \|u - v\|_{L^p},
\]
where \( \frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \quad 1 < r_i < \infty, \quad 1 < p < \infty. \]

**Remark 2.6.** In our application of Lemma 2.5, we will have
\[
F(u) = |u|^{\frac{4}{N-2}} u, \quad 3 \leq N \leq 5, \quad \text{and} \quad F'(u) = C_N |u|^{\frac{4}{N-2}} \cdot
\]
\[
F''(u) = \tilde{C}_N \text{sign}(u) \cdot |u|^{\frac{4}{N-2} - 1} = \tilde{C}_N \text{sign}(u) |u|^\frac{6-N}{N-2}.
\]

We will choose \( p = \frac{2(N+1)}{N+3}, \quad \frac{1}{p_1} = \frac{1}{p} - \frac{1}{p_2} = \frac{2}{N+1}; \)
\( r_2 = \frac{2(N+1)}{N-1}, \quad \frac{1}{r_2} = \frac{1}{p} - \frac{1}{r_3} = \frac{6-N}{2(N+1)}. \)

Let us now define the \( S(I) \), \( W(I) \) norm for an interval \( I \) by
\[
\|v\|_{S(I)} = \|v\|_{L^\frac{2(N+1)}{N-2}} \quad \text{and} \quad \|v\|_{W(I)} = \|v\|_{L^\frac{2(N+1)}{N-2}}.
\]

**Theorem 2.7** (See \[26, 29, 29\]). Assume \( (u_0, u_1) \in \dot{H}^1 \times L^2 \), \( 0 \in I \)
an interval and \( \|u_0 \times u_1\|_{\dot{H}^1 \times L^2} \leq A \). Then, (for \( 3 \leq N \leq 5 \)) there exists \( \delta = \delta(A) \) such that if
\[
\|S(t) ((u_0, u_1))\|_{S(I)} < \delta,
\]
there exists a unique solution \( u \) to (CP) in \( \mathbb{R}^N \times I \), with \( (u, \partial_t u) \in C(I; \dot{H}^1 \times L^2) \), \( \|D_t^{1/2} u\|_{W(I)} + \|\partial_t D_t^{1/2} u\|_{W(I)} < +\infty \), \( \|u\|_{S(I)} \leq 2\delta. \)

Moreover, if \( (u_0, u_1) \to (u_0, u_1) \) as \( k \to +\infty \) in \( \dot{H}^1 \times L^2 \) (so that, for \( k \) large \( \|S(t) ((u_0, u_1))\|_{S(I)} < \delta \), the corresponding solutions as \( k \to +\infty \) \( (u_k, \partial_t u_k) \to (u, \partial_t u) \) in \( C(I; \dot{H}^1 \times L^2) \).

**Sketch of the proof.** (CP) is equivalent to the integral equation
\[
u(t) = S(t) ((u_0, u_1)) + \int_0^t \frac{\sin ((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u)(s) ds,
\]
where \( F(u) = |u|^\frac{4}{N-2} u \). We let

\[
B_{a,b} = \left\{ v \text{ on } \mathbb{R}^N \times I : \|v\|_{S(I)} \leq a \text{ and } \|D_x^{1/2}v\|_{W(I)} \leq b \right\},
\]

\[
\Phi_{(u_0,u_1)}(v) = S(t)(u_0, u_1) + \int_0^t \sin \left( \frac{(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} \right) F(v)(s)\,ds.
\]

We will next choose \( \delta, a, b \) so that \( \Phi_{(u_0,u_1)} : B_{a,b} \to B_{a,b} \) and is a contraction there. Note that, by Lemma 2.1,

\[
\|D_x^{1/2} \Phi_{(u_0,u_1)}(v)\|_{W(I)} \leq CA + C\|F(v)\|_{L_x^{2(N+1)}W_x^{\frac{1}{2}(N+3)}}.
\]

But, by Lemma 2.6, \( \|D_x^{1/2} F(v)\|_{L_x^{\frac{2(N+1)}{N+3} \frac{2(N+1)}{N+3}}} \) is bounded by

\[
C\|F'(v)\|_{L_x^{\frac{N-2}{2(N+1)}}} \|D_x^{1/2} v\|_{L_x^{\frac{2(N+1)}{N+3}}} \leq C\|v\|_{L_x^{\frac{N-2}{2(N+1)}}} \|D_x^{1/2} v\|_{L_x^{\frac{2(N+1)}{N+3}}}.
\]

so that

\[
\|D_x^{1/2} F(v)\|_{L_x^{\frac{2(N+1)}{N+3} \frac{2(N+1)}{N+3}}} \leq C\|v\|_{S(I)} \|D_x^{1/2} v\|_{W(I)}.
\]

Hence, for \( v \in B_{a,b} \),

\[
\|D_x^{1/2} \Phi_{(u_0,u_1)}(v)\|_{W(I)} \leq CA + Ca^{\frac{4}{N-2}}b.
\]

Similarly, using Lemma 2.1 for the second term in \( \Phi_{(u_0,u_1)} \), and the argument above, together with our assumption on \((u_0, u_1)\) for the first term, we obtain:

\[
\|\Phi_{(u_0,u_1)}\|_{S(I)} \leq \delta + Ca^{\frac{4}{N-2}}b.
\]

Next, choose \( b = 2AC \), \( a \) so that \( Ca^{\frac{4}{N-2}} \leq \frac{1}{2} \). Then,

\[
\|D_x^{1/2} \Phi_{(u_0,u_1)}(v)\|_{W(I)} \leq b.
\]

If \( \delta = a/2 \) and \( Ca^{\left(\frac{4}{N-2}-1\right)}b \leq 1/2 \) (possible if \( N < 6 \)) we obtain \( \|\Phi_{(u_0,u_1)}(v)\|_{S(I)} \leq a \), so that \( \Phi_{(u_0,u_1)} : B_{a,b} \to B_{a,b} \). Next, for the
contraction, we again use Lemma 2.1 and Lemma 2.5 to see that:

\[
\|D_x^{1/2}(\Phi(u_0,u_1)(v) - \Phi(u_0,u_1)(v'))\|_{W(I)} + \|\Phi(u_0,u_1)(v) - \Phi(u_0,u_1)(v')\|_{S(I)} \\
\leq C \left\{ \left\| v \right\|_{L_1^{N+2} L_x^{2(N+1)}}^{4/2} \right\} + \left\{ \left\| v' \right\|_{L_1^{N+2} L_x^{2(N+1)}}^{4/2} \right\} \\
+ \left\{ \left\| D_x^{1/2}v \right\|_{L_1^{N+2} L_x^{2(N+1)}}^{4/2} \right\} \\
+ \left\{ \left\| D_x^{1/2}v' \right\|_{L_1^{N+2} L_x^{2(N+1)}}^{4/2} \right\} \\
\leq 2Ca^{4N/2} \| D_x^{1/2}(v - v') \|_{W(I)} + 2Ca^{4N/2} 2b \| v - v' \|_{S(I)}
\]

and the contraction property follows for \( N < 6 \). We then find \( u \in B_{a,b} \) solving \( \Phi(u_0,u_1)(u) = u \). To show that \( (u,\partial_t u) \in C(I; \tilde{H}^1 \times L^2) \) we use Lemma 2.1 together with the fact that \( D_x^{1/2}F(u) \in L_1^{2(N+3)} L_x^{2(N+3)} \). This also shows that \( \partial_tD_x^{-1/2}u \in W(I) \). The continuity statement at the end is an easy consequence of the fixed point argument, so that the proof is complete.

Remark 2.8. \( u \in L_1^{N+2} L_x^{2(N+1)} \), because of Lemma 2.1 and the fact that \( D_x^{1/2}F(u) \in L_1^{2(N+3)} L_x^{2(N+3)} \). Note that because of this and the integral equation, the conclusion of Lemma 2.2 holds for \( u \), provided the integrations on the left hand side are restricted to \( (x,x',t) \in \mathbb{R}^N \) so that \( \left( \frac{x_1-d}{\sqrt{1-d}}, \frac{x_2}{\sqrt{1-d}}, \frac{t-\rho(x', x)}{\sqrt{1-d}} \right) \in \mathbb{R}^N \times I \).

Remark 2.9 (Higher regularity of solutions, see for example [9]). If \((u_0, u_1) \in (\tilde{H}^1 \cap \tilde{H}^{1+\mu}, \tilde{H}^\mu), 0 \leq \mu \leq 1 \), and \((u_0, u_1)\) verifies the conditions in Theorem 2.7, then \((u, \partial_t u) \in C(I; \tilde{H}^{1+\mu} \times \tilde{H}^\mu) \) and

\[
\| D_x^{1/2+\mu} u \|_{W(I)} + \| D_x^{1/2} u \|_{W(I)} + \| \partial_t D_x^{-1/2} u \|_{W(I)} + \| \partial_t D_x^{1/2} u \|_{W(I)} < \infty,
\]

\( \| u \|_{S(I)} \leq 2\delta \). (In this result we also need to use the assumption \( 3 \leq N \leq 5 \)).
Remark 2.10. There exists \( \tilde{\delta} \) such that if \( \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq \tilde{\delta} \), the conclusion of Theorem 2.7 applies to any interval \( I \). In fact, by Lemma 2.1 \( \|S(t)(u_0, u_1)\|_{S((\infty, +\infty))} \leq C\tilde{\delta} \) and the claim follows.

Remark 2.11. Given \( (u_0, u_1) \in \dot{H}^1 \times L^2 \), there exists \( (0 \in I) \) such that the hypothesis of Theorem 2.7 is verified on \( I \). This is clear because, by Lemma 2.1 \( \|S(t)(u_0, u_1)\|_{S(I)} < +\infty \).

Remark 2.12 (Finite speed of propagation, see for instance [30]). Let \( R \) denote the fundamental solution of the Cauchy problem, i.e. \( R \) solves

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\partial_t^2 - \Delta_x)u = 0 & (x, t) \in \mathbb{R}^N \times \mathbb{R} \\
u|_{t=0} = 0 \\
\partial_t u |_{t=0} = \delta(x),
\end{array} \right.
\]

(2.1)

where \( \delta(x) \) is the Dirac mass at 0. Then, we can write the solution of (LCP) in the form

\[
w(t) = \partial_t R(t) \ast w_0 + R(t) \ast w_1 - \int_0^t R(t-s) \ast h(s) ds,
\]

where \( \ast \) denotes convolution in the spatial variable. As is well known, \( \text{supp } R(-t) \subset B(0, t) \) and \( \text{supp } \partial_t R(-t) \subset B(0, t) \). Thus, if \( \text{supp } u_0 \subset c \overline{B(x_0, a)} \), \( \text{supp } u_1 \subset c \overline{B(x_0, a)} \) and \( \text{supp } h \subset c \left( \bigcup_{0 \leq t \leq a} \big[ \overline{B(x_0, a-t)} \times (a-t) \big] \right) \), we have

\[
w \equiv 0 \quad \text{on} \quad \bigcup_{0 \leq t \leq a} \big[ B(x_0, a-t) \times (a-t) \big].
\]

These remarks have immediate consequences for the solutions of (CP) given in Theorem 2.7. In fact, suppose that \( (u_0, u_1), (u_0', u_1') \) are data verifying the conditions of Theorem 2.7 and such that \( (u_0, u_1) = (u_0', u_1') \) in \( B(x_0, a) \). Then, the corresponding solutions \( u, u' \) agree on \( \bigcup_{0 \leq t \leq a} \big[ B(x_0, (a-t)) \times (a-t) \big] \cap \{ \mathbb{R}^N \times I \} \). To see this, for \( n \in \mathbb{N} \), define \( u^{(n+1)}(x, t) = S(t) ((u_0, u_1)) + \int_0^t \frac{1}{\sqrt{-\Delta}} S(t-s) \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} F(u^{(n)}(s)) ds \) (for \( n = 0 \), we set \( u^{(0)}(x, t) = S(t) ((u_0, u_1)) \)). We define correspondingly \( u^{(n+1)}(x, t) \). The proof of Theorem 2.7 gives us \( u = \lim_n u^{(n)} \) and \( u' = \lim_n u^{(n)} \). The previous remarks allow us to show inductively that \( u^{(n+1)} = u^{(n+1)} \) on \( \bigcup_{0 \leq t \leq a} \big[ B(x_0, (a-t)) \times (a-t) \big] \cap \{ \mathbb{R}^N \times I \} \), which establishes the claim. Typical applications of this remark are the following:

a) If \( \text{supp}(u_0) \subset B(0, b) \), \( \text{supp}(u_1) \subset B(0, b) \) and \( (u_0, u_1) \) verifies the hypothesis of Theorem 2.7 then \( u(x, t) \equiv 0 \) on \( \{(x, t) : |x| > b + t, t \geq 0, t \in I \} \).
b) We can approximate solutions $u$ in $\mathbb{R} \times I'$, $I' \subset I$ by means of regular, compactly supported solutions, combining a), Remark 2.9 and the last statement in Theorem 2.7.

Similar statements hold for $t < 0$, for instance if $(u_0, u_1) = (u_0', u_1')$ in $\bar{B}(x_0, a)$ then $u, u'$ agree on $\bigcup_{-a \leq t \leq 0} [B(x_0, (a + t)) \times (a + t)] \cap \mathbb{R}^N \times I$.

**Definition 2.13.** Let $t_0 \in I$. We say that $u$ is a solution of (CP) in $I$ if $(u, \partial_t u) \in C(I; \dot{H}^1(\mathbb{R}^N) \times L^2)$, $D_x^{1/2} u \in W(I)$, $u \in S(I)$, $(u, \partial_t u)_{t=t_0} = (u_0, u_1)$ and the integral equation

$$u(t) = S(t)((u_0, u_1)) + \int_{t_0}^{t} \sin \left(\frac{(t - s)\sqrt{-\Delta}}{\sqrt{-\Delta}}\right) F(u(s)) \, ds$$

holds, with $F(u) = |u|^{\frac{4}{N-2}} u$, for $x \in \mathbb{R}^N$, $t \in I$.

Note that if $u^{(1)}$, $u^{(2)}$ are solutions of (CP) on $I$, and

$$(u^{(1)}(t_0), \partial_t u^{(1)}(t_0)) = (u^{(2)}(t_0), \partial_t u^{(2)}(t_0)),$$

then $u^{(1)} \equiv u^{(2)}$ on $\mathbb{R}^N \times I$. (See the argument in [16], Definition 2.10). This allows one to define a maximal interval

$$I((u_0, u_1)) = (t_0 - T_-(u_0, u_1), t_0 + T_+((u_0, u_1)))$$

with $T_\pm((u_0, u_1)) > 0$ where the solution is defined. If $T_1 > t_0 - T_-(u_0, u_1)$ and $T_2 < t_0 + T_+(u_0)$, $t_0 \in (T_1, T_2)$, then $u$ solves (CP) in $\mathbb{R}^N \times [T_1, T_2]$, so that

$$(u, \partial_t u) \in C([T_1, T_2]; \dot{H}^1 \times L^2)), \quad D_x^{1/2} u \in W([T_1, T_2]), \quad u \in S([T_1, T_2]),$$

$$u \in L^{\frac{N+2}{N-2}}([T_1, T_2]; L_x^{2\left(\frac{N+2}{N-2}\right)}), \quad \partial_t D_x^{-1/2} u \in W([T_1, T_2]).$$

**Remark 2.14.** If $u$ is such that $(u, \partial_t u) \in C(I; \dot{H}^1 \times L^2)$, $\|u\|_{S(I)} \leq B$ and there exist $u_j$ with $(u_j, \partial_t (u_j)) \in C(I; \dot{H}^1 \times L^2)$, $(u_j, \partial_t (u_j)) \to (u, \partial_t u)$ in $C(I; \dot{H}^1 \times L^2)$, with $u_j$ a solution of (CP) in $I$ together with $\|u_j\|_{S(I)} \leq B$, then $\|D_x^{1/2} u_j\|_{W(I)} < +\infty$ and $u$ is a solution of (CP) in $I$. This follows by showing that $\|D_x^{1/2} u_j\|_{W(I)} \leq B'$, where $B'$ is independent of $j$. To show this, first find $A$ so that $\sup_{t \in I} \|(u_j, \partial_t (u_j))\|_{\dot{H}^1 \times L^2} \leq A$, for all $j$. Next, partition $I = \bigcup_{k=1}^{M} I_k$, where $I_k$ is such that $\|u_j\|_{S(I_k)} \leq \delta$, where $\delta = \delta(A)$ is to be chosen. Note that $M = M(B, \delta)$. We then use the integral equation for $u_j$ and the estimate


\[ \left\| D_x^{1/2} F(u_j) \right\|_{L_t^\infty L_x^{2N+3j}} \leq C \delta^{\frac{1}{N-2}} \left\| D_x^{1/2} u_j \right\|_{W(I_k)} \]

(see the proof of Theorem 2.7), so that

\[ \left\| D_x^{1/2} u_j \right\|_{W(I_k)} \leq CA + C \delta^{\frac{1}{N-2}} \left\| D_x^{1/2} u_j \right\|_{W(I_k)}. \]

Thus, for \( \delta \) small we obtain \( \left\| D_x^{1/2} u_j \right\|_{W(I_k)} \leq 2CA \) and adding in \( k \) we obtain the desired bound.

**Lemma 2.15** (Standard finite blow-up criterion). If \( T_+ ((u_0, u_1)) < \infty \), then

\[ \| u \|_{S([t_0, t_0 + T_+ (u_0, u_1)])} = +\infty. \]

A corresponding result holds for \( T_- ((u_0, u_1)) \).

The proof is similar to the one in Lemma 2.11 of [16].

**Remark 2.16** (Energy and moment identities). Let \( (u_0, u_1) \in \dot{H}^1 \times L^2 \) and let \( 0 \in I \) be the maximal interval of existence. Then, for \( t \in I \), with \( \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N} (2^* = \frac{2N}{N-2}) \), we have

\[ E(u(t), \partial_t u(t)) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\partial_t u(x, t)|^2 + \frac{1}{2} |\nabla_x u(x, t)|^2 - \frac{1}{2^*} |u(x, t)|^{2^*} \right) dx \]

\[ = E((u_0, u_1)), \]

and

\[ \int \nabla_x u(x, t) \partial_t u(x, t) dx = \int \nabla u_0 u_1. \]

**Proof.** Let \( e(u)(x, t) = \frac{1}{2} (\partial_t u)^2(x, t) + \frac{1}{2} |\nabla_x u(x, t)|^2 - \frac{1}{2^*} |u(x, t)|^{2^*}. \) Then, for sufficiently smooth solutions of (CP) we have

\[ \partial_t e(u)(x, t) = \sum_{j=1}^N \partial_{x_j} \left( \partial_{x_j} u(x, t) \partial_t u(x, t) \right), \]

as is readily seen. Now, fix any \( I' \subset I \), so that \( \| u \|_{S(I')} < +\infty \). By dividing \( I' = \bigcup_{k=1}^M I_k \), with \( \| u \|_{S(I_k)} \leq \delta(A) \), where

\[ A = \sup_{t \in I'} \| (u(t), \partial_t u(t)) \|_{\dot{H}^1 \times L^2}, \]

we can use Theorem 2.7 to approximate \( u \) by compactly supported solutions in \( \mathbb{R}^N \times I_k \) (see Remarks 2.9, 2.12). We then apply (2.3) and integrate by parts, and then pass to the limit, for \( t \in I_k \). The proof of second equality is similar.
Lemma 2.17. Let \((u_0, u_1) \in \dot{H}^1 \times L^2, \| (u_0, u_1) \|_{\dot{H}^1 \times L^2} \leq A\) with maximal interval of existence \(I = \dot{-T_-} (u_0, u_1), \dot{T_+} (u_0, u_1))\). There exists \(\epsilon_0 > 0\) so that, if for some \(M > 0\) and \(0 < \epsilon < \epsilon_0\), we have \(\int_{|x| \geq M} |\nabla_x u_0|^2 + \frac{|u_0|^2}{|x|^2} + |u_1|^2 \leq \epsilon\), then for \(t \in I_+ = [0, \infty) \cap I\), we have

\[
\int_{|x| \geq \frac{3}{2} M + t} |\nabla_x u(x, t)|^2 + |\partial_t u(x, t)|^2 \, dx \leq C \epsilon.
\]

Proof. Choose \(\Psi_M \equiv 1\) for \(|x| \geq \frac{3}{2} M\), \(\Psi_M \equiv 0\) for \(|x| \leq M\), \(|\nabla_x \Psi_M| \leq C/M\). Define \(u_{0,M} = \Psi_M u_0, u_{1,M} = \Psi_M u_1\). Because of our assumption, we have \(\|(u_{0,M}, u_{1,M})\|_{\dot{H}^1 \times L^2} \leq C \epsilon\). Choose now \(\epsilon_0\) so small that \(C \epsilon_0 \leq \delta\), where \(\delta\) is as in Remark 2.10. Then, there exists \(u_M\) solving the (CP) in \(I = (-\infty, +\infty), \) with \((u_M(0), \partial_t u_M(0)) = (u_{0,M}, u_{1,M})\) and such that \(\sup_{t \in (-\infty, +\infty)} \|(u_M(t), \partial_t u_M(t))\|_{\dot{H}^1 \times L^2} \leq 2C \epsilon\). But, by Remark 2.12, \(u_M(x, t) = u(x, t)\) for \(|x| \geq \frac{3M}{2} + t, t \in I_+\). The Lemma follows. \(\Box\)

Definition 2.18. Let \((v_0, v_1) \in \dot{H}^1 \times L^2, v(x, t) = S(t)((v_0, v_1))\) and let \(\{t_n\}\) be a sequence, with \(\lim_{n \to -\infty} t_n = \bar{t} \in [-\infty, +\infty]\). We say that \(u(x, t)\) is a non-linear profile associated with \(((v_0, v_1), \{t_n\})\) if there exists an interval \(I\), with \(\bar{t} \in \dot{I}\) (if \(\bar{t} = \pm \infty, I = [a, +\infty)\) or \((-\infty, a]\)) such that \(u\) is a solution of (CP) in \(I\) and

\[
\lim_{n \to -\infty} \|(u(t_n) - v(t_n), \partial_t u(t_n) - \partial_t v(t_n))\|_{\dot{H}^1 \times L^2} = 0.
\]

Remark 2.19. There always exists a non-linear profile associated to \(((v_0, v_1), \{t_n\})\). The proof is similar to the one in [16], Remark 2.13, once we use the proof of Theorem 2.7 and the linear estimates, (with \(w(x, t) = \int_t^\infty \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}} h(s)ds, I = (a, +\infty), a > 0)\)

\[
\sup_{t \in I} \|(w(t), \partial_t w(t))\|_{\dot{H}^1 \times L^2} + \|D_x^{1/2} w\|_{W(I)} + \|w\|_{S(I)} 
\leq \frac{C \|h\|_{L^{2(N+1)} W_x^{1/2} S^{2(N+1)}(N+1)}},
\]

which follow from [16], Proposition 3.1, (2) and (3). Also, as in [16], Remark 2.13, we have uniqueness of the non-linear profile and a maximal interval of existence of the non-linear profile associated to \(((v_0, v_1), \{t_n\})\).

Theorem 2.20 (Long time perturbation theory, see also [37], [16].) Let \(I \subset \mathbb{R}\) be a time interval. Let \(t_0 \in I, (u_0, u_1) \in \dot{H}^1 \times L^2\) and some constants \(M, A, A' > 0\). Let \(\bar{u}\) be defined on \(\mathbb{R}^N \times I, 3 \leq N \leq 5\), and satisfy \(\sup_{t \in I} \|(\bar{u}(t), \partial_t \bar{u}(t))\|_{\dot{H}^1 \times L^2} \leq A, \|\bar{u}(t)\|_{S(I)} \leq M\) and
Then there exists $\varepsilon_0 = \varepsilon_0(M, A, A')$ such that there exists a solution of $(CP)$ in $I$ with $(u(t_0), \partial_t u(t_0)) = (u_0, u_1)$, for $0 < \varepsilon < \varepsilon_0$, with $\|u\|_{S(I)} \leq C(M, A, A')$ and $\forall t \in I$,

$$\|(u(t), \partial_t u(t)) - (\bar{u}(t), \partial_t \bar{u}(t))\|_{H^1 \times L^2} \leq C(A, A', M)(A' + \varepsilon).$$

The proof is analogous to the one given in [16], Theorem 2.14, using the ideas in the proof of Theorem 2.7.

**Remark 2.20.** Theorem 2.20 yields the following continuity fact, which will be used later. Let $(\bar{u}_0, \bar{u}_1) \in \dot{H}^1 \times L^2$, $\|(\bar{u}_0, \bar{u}_1)\|_{H^1 \times L^2} \leq A$, let $\bar{u}$ be the solution of $(CP)$, with maximal interval of existence

$$(-T_-((\bar{u}_0, \bar{u}_1)), T_+((\bar{u}_0, \bar{u}_1))).$$

Let $\left(\begin{smallmatrix} u_0^{(n)} \\ u_1^{(n)} \end{smallmatrix}\right) \to (\bar{u}_0, \bar{u}_1)$ in $\dot{H}^1 \times L^2$ and let $u^{(n)}$ be the corresponding solution of $(CP)$, with maximal interval of existence

$$(-T_-((u_0^{(n)}, u_1^{(n)})), T_+((u_0^{(n)}, u_1^{(n)}))).$$

Then $T_-((\bar{u}_0, \bar{u}_1)) \leq \lim T_-((u_0^{(n)}, u_1^{(n)}))$, $T_+((\bar{u}_0, \bar{u}_1)) \leq \lim T_+((u_0^{(n)}, u_1^{(n)}))$ and for each $t \in (-T_-((\bar{u}_0, \bar{u}_1)), T_+((\bar{u}_0, \bar{u}_1)))$ we have

$$(u^{(n)}(t), \partial_t u^{(n)}(t)) \to (\bar{u}(t), \partial_t \bar{u}(t))$$

in $\dot{H}^1 \times L^2$.

Indeed, let $I \subset (-T_-((\bar{u}_0, \bar{u}_1)), T_+((\bar{u}_0, \bar{u}_1)))$, so that

$$\sup_{t \in I} \|(\bar{u}(t), \partial_t \bar{u}(t))\|_{\dot{H}^1 \times L^2} \leq \bar{A}, \quad \|\bar{u}\|_{S(I)} \leq M < +\infty.$$

We will show that, for $n$ large, $u^{(n)}$ exists on $I$, and that

$$\sup_{t \in I} \|((u^{(n)}(t), \partial_t u^{(n)}(t)) - (\bar{u}(t), \partial_t \bar{u}(t))\|_{\dot{H}^1 \times L^2} \leq C(M, \bar{A}) \|((u_0^{(n)}, u_1^{(n)}) - (\bar{u}_0, \bar{u}_1))\|_{\dot{H}^1 \times L^2},$$
and additionally, \( \|u^{(n)}\|_{S(I)} \leq \tilde{M}(\tilde{A}, M) \). To show this, apply Theorem 2.20 with \( u = u^{(n)} \), \( (u_0, u_1) = (u_0^{(n)}, u_1^{(n)}) \), \( \epsilon = 0 \). If \( \epsilon_0 = \epsilon_0(M, \tilde{A}, 2\tilde{A}) \) and \( n \) is large enough that
\[
\left\| S(t)((\tilde{u}_0 - u_0^{(n)}, \tilde{u}_1 - u_1^{(n)})) \right\|_{S(I)} \leq \epsilon, \quad \left\| (\tilde{u}_0 - \tilde{u}_0^{(n)}, \tilde{u}_1 - \tilde{u}_1^{(n)}) \right\|_{H^1 \times L^2} \leq 2\tilde{A},
\]
the desired conclusions follow from Theorem 2.20. Note also that if we choose \( u_0^{(n)}, u_1^{(n)} \) in \( C_0^\infty(\mathbb{R}^N) \), the approximating solutions \( u^{(n)} \) will be regular in view of Remark 2.19 and for \( t \in I \) will have compact support in \( x \), in view of Remark 2.12 and will verify \( \|u^{(n)}\|_{S(I)} \leq M \).

Remark 2.22. If \( u \) is a solution of (CP) in \( \mathbb{R}^N \times I \), for each \( I' \subset I \), \( I = [a, +\infty) \) (or \( I = (-\infty, a] \)), such that \( \|u\|_{S(I)} < \infty \), there exists \( (u_0^+, u_1^+) \in H^1 \times L^2 \) such that
\[
\lim_{t \uparrow +\infty} \left\| (\partial_t u(t), S(t)(u_0^+, u_1^+), \partial_t S(t)(u_0^+, u_1^+)) \right\|_{H^1 \times L^2} = 0.
\]
See Remark 2.15 in [16, 4] for a similar proof, based on the fact, in our case, that \( \left\| D_x^{1/2} F(u) \right\|_{L_{L^2(N+3)}^2} < \infty \) and the inequality used in the proof of Remark 2.19.

Remark 2.23. We recall that, since we are working in the focusing case, from the work of Levine [20, 33] we have that if \( (u_0, u_1) \in H^1 \times L^2 \) is such that \( E((u_0, u_1)) < 0 \), then the maximal interval of existence is finite. We will return to the issue of break-down in finite time (blow-up), in the next section and at the end of the paper.

3. Variational estimates

Let \( W(x) = W(x, t) = \frac{1}{(1 + \frac{|x|^2}{\lambda_0^2})^{N+2}} \) be a stationary solution of (CP). That is \( W \) solves the non-linear elliptic equation
\[
\Delta W + |W|^\frac{N-2}{N-4} W = 0.
\]
Moreover, \( W \geq 0 \) and it is radially symmetric and decreasing. Note that \( W \in H^1 \), but \( W \) need not belong to \( L^2 \), depending on the dimension. By invariances of the equation [3.1], for \( \theta_0 \in [-\pi, \pi] \), \( \lambda_0 > 0 \), \( x_0 \in \mathbb{R}^N \), \( W_{\theta_0, x_0, \lambda_0}(x) = e^{i\theta_0} \lambda_0^{-\frac{N-2}{2}} W(\lambda_0(x - x_0)) \) is still a solution of [3.1]. By the work of Aubin [3], Talenti [33] we have the following characterization of \( W \):
\[
\forall u \in H^1, \quad \|u\|_{L^2} \leq C_N \|\nabla u\|_{L^2};
\]
moreover,
(3.3)

If \( \| u \|_{L^2^*} = C_N \| \nabla u \|_{L^2} \neq 0 \), then \( \exists (\theta_0, \lambda_0, x_0) : u = W_{\theta_0, x_0, \lambda_0} \), where \( C_N \) is the best constant of the Sobolev inequality (3.2) in dimension \( N \).

Remark that 
\[ \int |\nabla W|^2 = \frac{1}{C_N} \quad \text{and} \quad \mathcal{E}(W) = \frac{1}{N} \frac{1}{C_N}, \]
where
\[ \mathcal{E}(u) = \int \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right). \]

Indeed, the equation (3.1) gives 
\[ \int |\nabla W|^2 = \frac{1}{C_N} \int |W|^{2^*}. \]
Also, (3.3) yields 
\[ C_N^2 \int |\nabla W|^2 = \left( \int |W|^{2^*} \right)^\frac{N-2}{2}, \]
so that 
\[ C_N^2 \int |\nabla W|^2 = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int |\nabla W|^2 = \frac{1}{N C_N^2}. \]

**Lemma 3.1.** Let \( u \in \dot{H}^1(\mathbb{R}^N) \) be such that for \( \delta_0 > 0 \),
\[ |\nabla u|_{L^2}^2 < \| \nabla W \|_{L^2}^2 \quad \text{and} \quad \mathcal{E}(u) \leq (1 - \delta_0) \mathcal{E}(W). \]
Then there exists \( \bar{\delta} = \bar{\delta}(\delta_0) > 0 \) such that
\[ |\nabla u|_{L^2}^2 \leq (1 - \bar{\delta}) \| \nabla W \|_{L^2}^2 \quad \text{and} \quad \mathcal{E}(u) \geq 0. \]

**Proof.** It is contained in Lemma 3.4 of [16]. \( \square \)

**Corollary 3.2.** If \( u \) is as in Lemma 3.1, then there exists \( C_{\bar{\delta}} > 0 \) so that
\[ \int |\nabla u|^2 - |u|^{2^*} \geq C_{\bar{\delta}} \int |\nabla u|^2. \]

**Proof.** Note that (3.2) implies that
\[
\int |\nabla u|^2 - |u|^{2^*} \geq \int |\nabla u|^2 - C_N^{2^*} \left( \int |u|^2 \right)^\frac{2^*}{N-2^*}
\geq \int |\nabla u|^2 \left[ 1 - C_N^{2^*} \left( \int |\nabla u|^2 \right)^\frac{1}{N-2} \right]
\geq \int |\nabla u|^2 \left[ 1 - C_N^{2^*} (1 - \bar{\delta}) \frac{1}{N-2} \left( \int |\nabla W|^2 \right)^\frac{2}{N-2} \right],
\]
by Lemma 3.1. But \( \left( \int |\nabla W|^2 \right)^\frac{1}{N-2} = \frac{1}{C_N^{\frac{N-2}{2}}} = \frac{1}{C_N^2} \), so that the corollary follows with \( C_{\bar{\delta}} = [1 - (1 - \bar{\delta})^{\frac{1}{N-2}}]. \) \( \square \)

**Corollary 3.3.** Let \( u \in \dot{H}^1, \| \nabla u \|_{L^2} < \| \nabla W \|_{L^2}. \) Then \( \mathcal{E}(u) \geq 0. \)
Proof. If $\mathcal{E}(u) < \mathcal{E}(W) = \frac{1}{N} \frac{1}{C_N}$, the claim follows from Lemma 3.1. If, on the other hand $\mathcal{E}(u) \geq \mathcal{E}(W)$, the statement is obvious. \qed

Remark 3.4. Let $u \in \dot{H}^1(\mathbb{R}^N)$ be such that $\mathcal{E}(u) \leq (1 - \delta_0)\mathcal{E}(W)$. Assume that $\|\nabla u\|_{L^2}^2 > \|\nabla W\|_{L^2}^2$. Then there exists $\delta = \bar{\delta}(\delta_0, N)$ such that

$$\|\nabla u\|_{L^2}^2 \geq (1 + \delta) \|\nabla W\|_{L^2}^2.$$  

The proof of this is similar to the one of Lemma 3.1. See Remark 3.14 in [16].

Theorem 3.5 (Energy trapping). Let $u$ be a solution of (CP), with $(u, \partial_t u)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2$ and maximal interval of existence $I$.

Assume that, for $\delta_0 > 0$,

$$E((u_0, u_1)) \leq (1 - \delta_0)E((W, 0)) \quad \text{and} \quad \|\nabla u_0\|_{L^2}^2 < \|\nabla W\|_{L^2}^2.$$  

Then, there exists $\bar{\delta} = \bar{\delta}(\delta_0)$ such that, for $t \in I$, we have

$$\|\nabla u(t)\|_{L^2}^2 \leq (1 - \delta) \|\nabla W\|_{L^2}^2$$  

(3.4)  

$$\int |\nabla_x u(t)|^2 - |u(t)|^2^* \geq C_\delta \int |\nabla_x u(t)|^2$$  

(3.5)  

$$\mathcal{E}(u(t)) \geq 0 \quad \text{(and hence } E((u(t), \partial_t u(t))) \geq 0).$$  

(3.6)

Proof. By Remark 2.16, $E((u(t), \partial_t u(t))) = E((u_0, u_1))$, $t \in I$. Also, $\mathcal{E}(u(t)) \leq E((u(t), \partial_t u(t)))$. Thus, the Theorem follows from Lemma 3.1 Corollary 3.2 Corollary 3.3 and a continuity argument. \qed

Corollary 3.6. Let $u$ be as in Theorem 3.5. Then for all $t \in I$ we have

$E((u(t), \partial_t u(t))) \simeq \|(u(t), \partial_t u(t))\|_{H^1 \times L^2}^2 \simeq \|(u_0, u_1)\|_{H^1 \times L^2}^2$, with comparability constants which depend only on $\delta_0$.

Proof. For $t \in I$, $E((u(t), \partial_t u(t))) \leq \|(u(t), \partial_t u(t))\|_{H^1 \times L^2}^2$. Also,

$$E((u(t), \partial_t u(t))) = \frac{1}{2} \int (\partial_t u(t))^2 + \mathcal{E}(u(t))$$  

$$= \frac{1}{2} \int (\partial_t u(t))^2 + \frac{1}{2} \left[ \int |\nabla_x u(t)|^2 - |u(t)|^2^* \right] + \left( \frac{1}{2} - \frac{1}{2^*} \right) \int |u(t)|^2$$  

$$\geq \frac{1}{2} \int (\partial_t u(t))^2 + C_\delta \int |\nabla_x u(t)|^2.$$  

Finally, $E((u(t), \partial_t u(t))) = E((u_0, u_1)) \simeq \|(u_0, u_1)\|_{H^1 \times L^2}^2$. \qed
Theorem 3.7 (Finite time blow-up, see also Remark 2.23). Assume that \((u_0, u_1) \in H^1 \times L^2, u_0 \in L^2\) and that \(u\) is the solution of (CP) with maximal interval of existence \(I\). Assume that \(E((u_0, u_1)) < E((W, 0))\) and \(\int |\nabla u_0|^2 > \int |\nabla W|^2\). Then \(I\) must be a finite interval.

Proof. Fix \(\delta_0\) positive so that \(E((u_0, u_1)) \leq (1 - \delta_0)E((W, 0))\). Define

\[ y(t) = \int |u(x, t)|^2 \, dx. \]

We then have

\[ y'(t) = 2 \int u \partial_t u \quad \text{and} \quad y''(t) = 2 \left[ \int (\partial_t u)^2 - |\nabla_x u|^2 + |u|^2 \right]. \]

(To check these identities, we proceed as in Remark 2.16 starting with data in \(C^\infty_0\) and using a limiting argument.) Let \(\delta_0 = \delta_0 E((W, 0))\), so that \(E((W, 0)) \geq E((u(t), \partial_t u(t))) + \delta_0\) and hence \(\frac{1}{2} \int |u(t)|^2 \geq \frac{1}{2} \int \left( (\partial_t u(t))^2 + |\nabla_x u(t)|^2 \right) - E((W, 0)) + \delta_0\) so that

\[ \int |u(t)|^2 \geq \frac{N}{N-2} \int \left( (\partial_t u(t))^2 + |\nabla_x u(t)|^2 \right) - 2^* E((W, 0)) + 2^* \tilde{\delta}_0. \]

But then, (with \(\tilde{\delta}_0 = 22^* \tilde{\delta}_0\)) we have

\[ y''(t) \geq 2 \int (\partial_t u(t))^2 + \frac{2N}{N-2} \int (\partial_t u(t))^2 - 22^* E((W, 0)) \]

\[ + \frac{2N}{N-2} \int |\nabla_x u(t)|^2 - 2 \int |\nabla_x u(t)|^2 + \tilde{\delta}_0 \]

\[ = 4 \frac{(N-1)}{N-2} \int (\partial_t u(t))^2 + \frac{4}{N-2} \int |\nabla_x u|^2 - \frac{4}{N-2} \int |\nabla W|^2 + \tilde{\delta}_0 \]

\[ \geq 4 \frac{(N-1)}{N-2} \int (\partial_t u(t))^2 + \tilde{\delta}_0 \]

(by Remark 3.4 and a continuity argument.) Assume now that \(I \cap [0, \infty) = [0, \infty)\). Then, by our lower bound on \(y''(t)\), there exists \(t_0 > 0\) such that \(y'(t_0) > 0\), and hence \(y'(t) > 0\) for \(t > t_0\). Hence, for \(t > t_0\),

\[ y''(t)y(t) \geq 4 \left[ \frac{N-1}{N-2} \right] \left( \int (\partial_t u)^2(t) \right) \left( \int u(t)^2 \right) \geq \frac{N-1}{N-2} y'(t)^2, \]

so that, for \(t > t_0\),

\[ \frac{y''(t)}{y'(t)} \geq \frac{N-1}{N-2} \frac{y'(t)}{y(t)} \quad \text{or} \quad (\log y'(t))' \geq \frac{N-1}{N-2} (\log y(t))'. \]
Hence for \( t > t_0 \),
\[
\log y' \geq \frac{N-1}{N-2} \log y - C_0 \quad \text{or} \quad y'(t) \geq \tilde{C}_0 y^{\frac{N-1}{N-2}},
\]
which leads to finite time blow-up of \( y \), because \( \frac{N-1}{N-2} > 1 \). This is a contradiction which gives the result. \( \square \)

An extension of Theorem 3.7 will be given in Section 7.

4. Existence and compactness of a critical element; further properties of critical elements

Let us consider the statement:

\( (SC) \) For all \( (u_0, u_1) \in H^1 \times L^2 \), with \( \int |\nabla u_0|^2 < \int |\nabla W|^2 \) and \( E((u_0, u_1)) < E((W, 0)) \), if \( u \) is the corresponding solution of (CP) with maximal interval of existence \( I \) (see Definition 2.13) then \( I = (-\infty, +\infty) \) and \( \| \nabla u \|_{C((-\infty, +\infty))} < \infty \).

In addition, for a fixed \( (u_0, u_1) \in \dot{H}^1 \times L^2 \), with \( \int |\nabla u_0|^2 < \int |\nabla W|^2 \) and \( E((u_0, u_1)) < E((W, 0)) \), we say that \( SC((u_0, u_1)) \) holds if, for \( u \) the corresponding solution of (CP), with maximal interval of existence \( I \), we have \( I = (-\infty, +\infty) \) and \( \| \nabla u \|_{C((-\infty, +\infty))} < \infty \).

Note that, because of Remark 2.10 if \( \| (u_0, u_1) \|_{\dot{H}^1 \times L^2} \leq \tilde{\delta} \), then \( SC((u_0, u_1)) \) holds. Thus, in light of Corollary 3.6 there exists \( \eta_0 > 0 \) such that if \( (u_0, u_1) \) is as in \( (SC) \), and \( E((u_0, u_1)) \leq \eta_0 \), then \( SC((u_0, u_1)) \) holds. Moreover, for any \( (u_0, u_1) \) as in \( (SC) \), (3.6) shows that
\[
E((u_0, u_1)) \geq 0.
\]

Thus, there exists a number \( E_C, \eta_0 \leq E_C \leq E((W, 0)) \) such that, if \( (u_0, u_1) \) is as in \( (SC) \) and \( E((u_0, u_1)) < E_C \), then \( SC((u_0, u_1)) \) holds and \( E_C \) is optimal with this property. For the rest of this section we will assume that \( E_C < E((W, 0)) \). Using concentration compactness ideas, following the argument in [16], Section 4, we prove that there exists a critical element \( (u_{0,C}, u_{1,C}) \) at the critical level of energy \( E_C \), so that \( SC((u_{0,C}, u_{1,C})) \) does not hold and from the minimality, this element has a compactness property up to the symmetries of the equation (which will give rigidity in the problem). We then use the finite speed of propagation and Lorentz transformations to establish support and orthogonality properties of critical elements, which are essential to treat the nonradial case.

**Proposition 4.1.** There exists \( (u_{0,C}, u_{1,C}) \in \dot{H}^1 \times L^2 \), with
\[
E((u_{0,C}, u_{1,C})) = E_C < E((W, 0)), \quad \int |\nabla u_{0,C}|^2 < \int |\nabla W|^2
\]
such that if \( u_C \) is the solution of \((CP)\) with data \((u_{0,C}, u_{1,C})\) and with maximal interval of existence \( I, 0 \in I \), then \( \|u_C\|_{S(I)} = +\infty \).

**Proposition 4.2.** Assume that \( u_C \) is as in Proposition 4.1 and that (say) \( \|u_C\|_{S(I_+)} = +\infty \), where \( I_+ = [0, \infty) \cap I \). Then there exists \( x(t) \in \mathbb{R}^N, \lambda(t) \in \mathbb{R}^+ \), for \( t \in I_+ \), such that \( K = \{ \tilde{v}(x,t), t \in I_+ \} \) has the property that \( K \) is compact in \( H^1 \times L^2 \), where

\[
\tilde{v}(x,t) = \left( \frac{1}{\lambda(t)^{\frac{N+2}{2}}} u_C \left( \frac{x-x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{\frac{2}{2}}} \partial_t u_C \left( \frac{x-x(t)}{\lambda(t)}, t \right) \right).
\]

A corresponding conclusion is reached if \( \|u_C\|_{S(I_-)} = +\infty \), where \( I_- = (-\infty, 0) \cap I \).

The proofs of Propositions 4.1 and 4.2 are identical to the corresponding ones in [16], using Lemma 4.3 below and the results of Section 2.2. We will therefore omit them.

**Lemma 4.3 (Concentration compactness).** Let \( \{(v_{0,n}, v_{1,n})\} \in \dot{H}^1 \times L^2 \), \( \|(v_{0,n}, v_{1,n})\|_{\dot{H}^1 \times L^2} \leq A \). Assume that

\[
\|S(t)((v_{0,n}, v_{1,n}))\|_{S(-\infty, +\infty)} \geq \delta > 0,
\]

where \( \delta = \delta(A) \) is as in Theorem 2.7. Then there exists a sequence \( \{(V_{0,j}, V_{1,j})\} \) in \( \dot{H}^1 \times L^2 \), a subsequence of \( \{(v_{0,n}, v_{1,n})\} \) (which we still call \( \{(v_{0,n}, v_{1,n})\} \)) and a triple \( (\lambda_{j,n}; x_{j,n}; t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R} \) with

\[
\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{t_{j,n} - t_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to \infty
\]

as \( n \to \infty \), for \( j \neq j' \) (we say that \( (\lambda_{j,n}; x_{j,n}; t_{j,n}) \) is orthogonal if this property is verified) such that

\[
(1.1) \quad \|(V_{0,1}, V_{1,1})\|_{\dot{H}^1 \times L^2} > \alpha_0(A) > 0.
\]

If \( V_{j}(x,t) = S(t)((V_{0,j}, V_{1,j})) \), then given \( \epsilon > 0 \), there exists \( J = J(\epsilon) \) and \( \{(w_{0,n}, w_{1,n})\} \in \dot{H}^1 \times L^2 \), so that

\[
(2.1) \quad v_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^2} V_{j} \left( \frac{x-x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}} \right) + w_{0,n},
\]

\[
(2.2) \quad v_{1,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^2} \partial_t V_{j} \left( \frac{x-x_{j,n}}{\lambda_{j,n}}, -\frac{t_{j,n}}{\lambda_{j,n}} \right) + w_{1,n},
\]

with \( \|S(t)((w_{0,n}, w_{1,n}))\|_{S((-\infty, +\infty))} \leq \epsilon_0 \), for \( n \) large.
\[
\int \frac{1}{2} |\nabla_x v_{0,n}|^2 + \frac{1}{2} |v_{1,n}|^2 = \sum_{j=1}^{J} \int \frac{1}{2} |\nabla_x V_{0,j}|^2 + \frac{1}{2} |V_{1,j}|^2 \\
+ \int \frac{1}{2} |\nabla_x w_{0,n}|^2 + \frac{1}{2} |w_{1,n}|^2 + o(1) \text{ as } n \to \infty
\]

\[
E((v_{0,n}v_{1,n})) = \sum_{j=1}^{J} E \left( V_{j} \left( \frac{-t_{j,n}}{\lambda_{j,n}} \right), \partial_t V_{j} \left( \frac{-t_{j,n}}{\lambda_{j,n}} \right) \right) + \\
E((w_{0,n}, w_{1,n})) + o(1) \text{ as } n \to \infty.
\]

Remark 4.4. Lemma 4.3 is due to Bahouri-Gérard [4]. There it is proved for \( N = 3 \), but the proof extends to all \( N \geq 3 \). Also, the norm \( \| \cdot \|_{S(\mathbb{R}^N)} \) is replaced by \( \| \cdot \|_{\dot{L}_2^{N+2} L_{L}^{N+2}} \) in [4], but as is mentioned in page 136 of [4], it works equally well for \( \| \cdot \|_{S(\mathbb{R}^N)} \). See the Remark on page 159 of [4] to eliminate their condition (1.6). (See also the work of Keraani [18], where the corresponding result is proved for NLS and where the analogue of (4.1) is shown.) See also Remark 4.8 in [16].

Corollary 4.5. There exists a decreasing function \( g : (0, E_C] \to [0, \infty) \) such that for every \( (u_0, u_1) \) as in \( (SC) \), with \( E((u_0, u_1)) = E_C - \eta \), we have

\[
\|u\|_{S(\mathbb{R}^N)} \leq g(\eta).
\]

For a proof of Corollary 4.5, see Corollary 2 in [4] and Corollary 1.14 in [18].

We next turn our attention to further properties of critical elements.

Lemma 4.6. Assume that \( u \) is a solution of \( (CP) \), with maximal interval of existence \( I \). Assume that for \( t \in I^+ = I \cap [0, \infty) \), there exist \( x(t) \in \mathbb{R}^N, \lambda(t) \in \mathbb{R}^+ \) so that \( K = \{ v(x,t), t \in I^+ \} \) has the property that \( K \) is compact in \( \dot{H}^1 \times L^2 \), where

\[
v(x,t) = \left( \frac{1}{\lambda(t)^{\frac{N-2}{2}}} u \left( \frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{\frac{N}{2}}} \partial_t u \left( \frac{x - x(t)}{\lambda(t)}, t \right) \right).
\]

Then we can choose \( \tilde{\lambda}(t), \tilde{x}(t) \), continuous in \( I^+ \), so that the corresponding \( \tilde{K} \) has compact closure in \( \dot{H}^1 \times L^2 \).

Proof. The proof given in Remark 5.4 of [16] applies verbatim. \( \square \)

From now on, we always use the \( \tilde{\lambda}(t), \tilde{x}(t) \) provided by Lemma 4.6.
Lemma 4.7. Let $u$ be as in Lemma 4.6 and assume that $I_+$ is a finite interval. After scaling, we can assume then that $I_+ = [0,1)$. Then,

$$0 < \frac{C_0(K)}{1-t} \leq \lambda(t).$$

Proof. Consider $0 < t_j \to 1$. (Because of Lemma 4.6, this suffices.) Let $(v_{0,j}, v_{1,j}) = \left( \frac{1}{\lambda(t_j)^{\frac{N-2}{4}}} u \left( \frac{x - x(t_j)}{\lambda(t_j)}, t_j \right), \frac{1}{\lambda(t_j)^{\frac{2}{N}}} \partial_t u \left( \frac{x - x(t_j)}{\lambda(t_j)}, t_j \right) \right).$

Since $(v_{0,j}, v_{1,j}) \in K$, $(\bar{K}$ is compact in $\dot{H}^1 \times L^2$, there exists $C_0 = C_0(K) > 0$ independent of $j$, so that $T_+((v_{0,j}, v_{1,j})) \geq C_0$. (Here we are using the notation in Definition 2.13.) Let $v_j(t)$ be the corresponding solution of (CP). Note that $\lambda(t_j)^{\frac{(N-2)}{4}} v_{0,j}(\lambda(t_j)y + x(t_j)) = u(y, t_j), \lambda(t_j)^{\frac{N}{2}} v_{1,j}(\lambda(t_j)y + x(t_j)) = \partial_t u(y, t_j).$ Hence, by uniqueness in (CP) (see the argument in Definition 2.13) for $t$ such that $t_j + t \leq T_+((u_0, u_1)) = 1$ we have

$$\lambda(t_j)^{\frac{(N-2)}{4}} v_{j}(\lambda(t_j)y + x(t_j), \lambda(t_j)t) = u(y, t_j + t).$$

Thus, we have $t_j + t \leq 1$, for all $0 < \lambda(t_j)t \leq C_0$. But then, choose $t = C_0/\lambda(t_j)$, so that $\lambda(t_j) \geq C_0/(1 - t_j)$, as desired. \hfill \Box

Lemma 4.8. Let $u$ be as in Lemma 4.7. Then $\exists \bar{x} \in \mathbb{R}^N$ such that

$$\text{supp } u \subset B(\bar{x}, (1 - t)), \text{supp } \partial_t u \subset B(\bar{x}, (1 - t)).$$

Proof. We first start by showing that for $t \in [0,1)$, there is a ball $B_{(1-t)}$ of radius $(1-t)$ so that $\text{supp } \nabla u, \text{supp } \partial_t u$ are contained in $B_{(1-t)}$. If not, for a fixed $t$, there exist $\epsilon_0 > 0, \eta_0 > 0$ such that, for all $x_0 \in \mathbb{R}^N$ we have

$$\int_{|x-x_0| \geq (1+\eta_0)(1-t)} |\nabla_x u(t)|^2 + |\partial_t u(t)|^2 \geq \epsilon_0.$$

Choose a sequence $t_n \uparrow 1$. Recall from Lemma 4.7 that $\lambda(t_n) \geq \frac{C_0}{1-t_n}$. We claim that, given $R_0 > 0$, $M > 0$, for $n$ large we have

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq R_0} \left| \nabla_x u(x, t_n) \right|^2 + \left| \partial_t u(x, t_n) \right|^2 + \frac{\left| u(x, t_n) \right|^2}{|x|^2} \leq \frac{\epsilon_0}{M}.$$

Indeed, if $\bar{v}(x, t) = \frac{1}{\lambda(t)^\frac{2}{N}} \left( \nabla u \left( \frac{x - x(t)}{\lambda(t)}, t \right), \partial_t u \left( \frac{x - x(t)}{\lambda(t)}, t \right) \right),$

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq R_0} \left| \nabla_x u(x, t) \right|^2 + \left| \partial_t u(x, t) \right|^2 dx = \int_{|y| \geq \lambda(t)R_0} |\bar{v}(y, t)|^2 dy$$
and our claim follows from the compactness of $\overline{K}$ and the fact that $\lambda(t_n) \uparrow +\infty$. Using this estimate, we apply Lemma 2.17 backward in time, to conclude that for $n$ large,

$$\forall t \in [0, t_n], \int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq 2R_0 + (t_n-t)} |\nabla_x u(x,t)|^2 + |\partial_t u(x,t)|^2 \leq \epsilon_0.$$ 

But, if $(1 + \eta_0)(1 - t) \geq 2R_0 + (t_n-t)$, we reach a contradiction. But, for $0 \leq t < 1$, fixed, we can always choose $n$ large and $R_0$ small so that this is the case.

The next step is to show that $\frac{x(t)}{\lambda(t)} \leq M$, for $0 \leq t < 1$. Assume not, so that we can find (in light of Lemma 4.6 we can assume $\frac{x(t)}{\lambda(t)} \leq M_T$, $0 \leq t \leq T < 1$) $t_n \uparrow 1$ so that $\frac{x(t_n)}{\lambda(t_n)} \rightarrow +\infty$. Fix a ball $B = B(x_0, 1)$, such that $\operatorname{supp} \nabla u_0, \operatorname{supp} u_1 \subset B$. But, for fixed $R_0 > 0$, $\epsilon_0 > 0$ given, our previous argument shows that for $n$ large,

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq 2R_0 + t_n} |\nabla_x u_0|^2 + |u_1|^2 \leq \epsilon_0.$$ 

But, if $\frac{x(t_n)}{\lambda(t_n)} \rightarrow +\infty$, $B(x_0, 1) \subset \left\{ x + \frac{x(t_n)}{\lambda(t_n)} \geq 2R_0 + t_n \right\}$, so that $\nabla u_0, u_1$ are identically 0, contradicting $I_+ = [0,1)$. Let now $t_n \uparrow 1$, and choose a subsequence so that $-\frac{x(t_n)}{\lambda(t_n)} \rightarrow \bar{x}$. Arguing as before, for $0 \leq t < t_n$, we see that, for $n$ large,

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq R_0} |\nabla_x u(t_n)|^2 + |\partial_t u(t_n)|^2 \leq \epsilon_0/M,$$

for $R_0, M, \epsilon_0$ given and hence, by our previous argument,

$$\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq 2R_0 + (t_n-t)} |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, dx \leq \epsilon_0,$$

for $n$ large. Letting $n \rightarrow \infty$, we obtain for all $R_0 > 0$ and $\epsilon_0 > 0$ small,

$$\int_{|x-x'| \geq 2R_0 + (1-t)} |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \, dx \leq \epsilon_0,$$

so that $\operatorname{supp} \nabla u(-,t), \operatorname{supp} \partial_t u(-,t) \subset B(\bar{x}, 1-t)$. Assume now that $-\frac{x(t_n)}{\lambda(t_n)} \rightarrow \bar{x}, -\frac{x(t'_n)}{\lambda(t'_n)} \rightarrow \bar{x}'$ for two different sequences $t_n, t'_n \rightarrow 1$. If $\bar{x} \neq \bar{x}'$ and $(1-t)$ is so small that $1-t < |\bar{x} - \bar{x}'|$, we must have $\nabla u(-,t), \partial_t u(-,t) \equiv 0$, a contradiction to $I_+ = [0,1)$. \hfill \Box

Remark 4.9. After a translation we can assume $\bar{x} = 0$. Also, since $u(-,t) \in L^2$ for each $t$, the conditions $\operatorname{supp} u \subset B(0, 1-t)$ and $\operatorname{supp} \nabla_x u \subset B(0, 1-t)$ are equivalent.
We turn now to the next important property of \( u_C \) (at least in the nonradial situation): the second invariant of the equation for \( u_C \) is zero. We consider the cases \( I_+ \) is a finite interval and then an infinite interval.

**Proposition 4.10.** Assume that \( u_C \) is as in Proposition 4.2 and \( I_+ \) is a finite interval. Then,

\[
\int \nabla u_{0,C} \cdot u_{1,C} = 0.
\]

**Proof.** By scaling, we can assume that \( I_+ = [0,1) \). By Lemma 4.8, \( \text{supp } u_C \subset B(0,1-t) \). Note also that for any \( u \) a solution of (CP) in \( I \), the maximal interval of existence, and \( t \in I \), we have from \( (2.2) \),

\[
\int \nabla_x u(t) \cdot \partial_t u(t) \, dx = \int \nabla u_0 \cdot u_1.
\]

Assume now that, (without loss of generality)

\[
\gamma = \int \partial_{x_1}(u_{0,C}) \cdot u_{1,C} > 0.
\]

We will reach a contradiction, by considering (for convenience) \( u(x,t) = u_C(x,1+t), -1 \leq t < 0 \). Clearly, for \(-1 \leq t < 0 \),

\[
E((u(t), \partial_t u(t))) = E_C, \quad \int |\nabla u(t)|^2 \leq (1-\delta) \|\nabla W\|_{L^2}^2, \quad \gamma = \int \partial_{x_1} u(t) \cdot \partial_t u(t),
\]

by Theorem 3.5 and our assumption above. We will consider the action of Lorentz transformations on \( u \). (Now, \( \text{supp } u(-,t) \subset B(0,-t) \), \( \text{supp } \partial_t u(-,t) \subset B(0,-t) \), \(-1 \leq t < 0 \).) Thus, for \( 0 < d < 1/4 \), consider

\[
(4.5) \quad z_d(x_1, \bar{x}, t) = u \left( \frac{x_1 - dt}{\sqrt{1-d^2}}, \frac{\bar{x}}{\sqrt{1-d^2}} \right),
\]

where \( x = (x_1, \bar{x}) \in \mathbb{R}^N, t \in \mathbb{R} \) and \( s = \frac{t-dx_1}{\sqrt{1-d^2}} \) is such that \(-1 \leq s < 0 \).

Note that, for this range of \( s \) and \( y = (y_1, \bar{y}) \) such that \((y,s) \in \text{supp } u \), we have \(|y| \leq |s| \). Thus, if \( y_1 = \frac{x_1 - dt}{\sqrt{1-d^2}}, \bar{y} = \bar{x} \), we obtain \( x_1^2 + |\bar{x}|^2 \leq t^2 \) in support of \( z_d, \partial_t z_d \). Fix now \(-\frac{1}{2} \leq t < 0 \) and \( x_1^2 + |\bar{x}|^2 \leq t^2 \). Then, \( \frac{t-dx_1}{\sqrt{1-d^2}} \geq \frac{(1+dt)}{\sqrt{1-d^2}} \geq -\frac{1}{2} + \frac{1-d^2}{\sqrt{1-d^2}} \geq -1 \), while \( \frac{t-dx_1}{\sqrt{1-d^2}} \leq \frac{(1-dt)}{\sqrt{1-d^2}} < 0 \). Thus, for such \((x,t) \), \( z_d \) is defined and \( \partial_t z_d(x,t) = 0 \), \( \nabla_x z_d(x,t) = 0 \), \( \partial_t z_d(x,t) = 0 \) for \( x^2 + |\bar{x}|^2 = t^2 \). We extend \( z_d(-,t) \) to be zero for \(|x| \geq |t|, -\frac{1}{2} \leq t < 0 \). An elementary calculation shows that if \( u \) is a regular solution (by regular solution we will mean one as in Remark 2.9 with \( \mu = 1 \)) of

\[
\partial_t^2 u - \Delta u = |u|^\frac{4}{N-2} u \quad \text{in } \mathbb{R}^N \times [-1,0)
\]

the resulting \( z_d \) is a solution of the (CP) for this equation in \(-\frac{1}{2} \leq t < 0 \), \( x \in \mathbb{R}^N \).
We will now show that the $z_d$ we defined in (4.5) is a solution of (CP) in $\mathbb{R}^N \times [-1/2, 0]$. To this end, fix $\epsilon_0 > 0$ and consider $-1/2 \leq t \leq -\epsilon_0$, $x \in \mathbb{R}^N$. Note that in this range we have, on supp $z_d$, that $-1 \leq s \leq -\frac{3}{\sqrt{15}}\epsilon_0$. Note that since $S\left([-1, -\frac{3}{\sqrt{15}}\epsilon_0]\right)$ norm of $u$ is finite, and $u \in L^2_{[-1, -\epsilon_0]} L^2_x$ (see Definition 2.13), in light of Remark 2.14 we have that $(z_d, \partial_t(z_d)) \in C([-1/2, -\epsilon_0]; \dot{H}^1 \times L^2)$. Also, if we let $J = \left|\det \frac{\partial(y,s)}{\partial(x,t)}\right|$, then $J \equiv 1$ and hence, if $D_{\epsilon_0} = \mathbb{R}^N \times [-1/2, -\epsilon_0]$, $\tilde{D}_{\epsilon_0} = \Phi(D_{\epsilon_0})$, where $\Phi(x, t) = (y, s)$, then

$$
\int_{D_{\epsilon_0}} |z_d(x, t)|^{2(n+1)/(n-1)} \, dxdt = \int_{\tilde{D}_{\epsilon_0}} |u(y, s)|^{2(n+1)/(n-1)} \, dyds
$$

$$
\leq \int_{-1 \leq s \leq -\frac{3}{\sqrt{15}}\epsilon_0} |u(y, s)|^{2(n+1)/(n-1)} \, dyds \leq C_{\epsilon_0}.
$$

Moreover, pick $u_{0,j} \in C_0^\infty(B(0, \frac{3}{\sqrt{15}}\epsilon_0))$, $u_{1,j} \in C_0^\infty(B(0, \frac{3}{\sqrt{15}}\epsilon_0))$ with $(u_{0,j}, u_{1,j}) \to (u(-\frac{3}{\sqrt{15}}\epsilon_0), \partial_s u(-\frac{3}{\sqrt{15}}\epsilon_0))$ in $H^1 \times L^2$. Let $u_j$ be the solution of (CP), defined for $s < -\left(\frac{3}{\sqrt{15}}\right)\epsilon_0$. Note that, because of Remark 2.21 we know that, for $j$ large, $u_j$ is a solution of (CP) for $-1 \leq s < -\left(\frac{3}{\sqrt{15}}\right)\epsilon_0$,

$$
\|\left(u_j, \partial_s(u_j)\right)\|_{C\left([-1, -\frac{3}{\sqrt{15}}\epsilon_0]; \dot{H}^1 \times L^2\right)} \leq C
$$

and

$$
\|u_j\|_{S\left([-1, -\frac{3}{\sqrt{15}}\epsilon_0]\right)} + \|u_j\|_{L^2_{[-1, -\frac{3}{\sqrt{15}}\epsilon_0]} L^2_x} \leq \tilde{C}_{\epsilon_0}.
$$

Also, by virtue of Remark 2.9 $u_j$ is regular for $s \in \left[-1, -\frac{3}{\sqrt{15}}\epsilon_0\right]$ and for $-1 \leq s \leq -\frac{3}{\sqrt{15}}\epsilon_0$, we have supp $u_j(-, s) \subset B(0, |s|)$, by Remark 2.12. If we now consider $z_{j,d}$ given by (4.5) with $u$ replaced by $u_j$, the $z_{j,d}$ are solutions of (CP) in $-\frac{1}{2} \leq t \leq -\epsilon_0$. Moreover, from the proof of Remark 2.21 and the proof that $(z_d, \partial_t(z_d)) \in C\left([-1/2, -\epsilon_0]; \dot{H}^1 \times L^2\right)$ we can conclude that $(z_{j,d}, \partial_t(z_{j,d})) \to (z_d, \partial_t z_d)$ in $C([-1/2, -\epsilon_0]; \dot{H}^1 \times L^2)$ and similarly that $\|z_{j,d}\|_{S((-1/2, -\epsilon_0)]} \leq C_{\epsilon_0}$. From Remark 2.14 it now follows that $z_d$ is a solution of (CP) for $t \in [-1/2, -\epsilon_0]$. Since $\epsilon_0 > 0$ is arbitrary, we conclude that $T_\epsilon((z_d(-1/2), \partial_t z_d(-1/2))) \geq 0$. But, since for each $t \in [-1/2, 0]$, supp $z_d, \partial_t z_d \subset \{|x| \leq |t|\}$, either $T_\epsilon((z_d(-1/2), \partial_t z_d(-1/2))) = 0$, or $z_d \equiv 0$. We will soon see that $z_d \not\equiv 0$. 
We have, by Remark 2.16, that

\[\int_{-1/2}^{-1/4} E((z_d(t), \partial_t z_d(t)))dt = \frac{1}{4} E((z_d(-1/2), \partial_t z_d(-1/2))).\]

We are now going to estimate the left-hand side. Note that

\[
\partial_{x_1} z_d = \frac{d}{\sqrt{1-d^2}} \partial_s u + \frac{1}{\sqrt{1-d^2}} \partial_{y_1} u, \quad \partial_{y_1} z_d = \partial_y u,
\]

\[
\partial_t z_d = \frac{1}{\sqrt{1-d^2}} \partial_s u - \frac{d}{\sqrt{1-d^2}} \partial_{y_1} u.
\]

Thus, the left-hand side of (4.6) equals \(I_1 + I_2\), where

\[
I_1 = \int_{-1/2}^{-1/4} \int \frac{1}{2} \left\{ \left( \frac{1 + d^2}{1-d^2} \right) ((\partial_s u)^2 + (\partial_{y_1} u)^2) + |\nabla_y u|^2 \right\} - \frac{1}{2^*} |u|^{2^*} d x_1 d \bar{x} dt
\]

\[
I_2 = -\frac{2d}{(1-d^2)} \int_{-1/2}^{-1/4} \int \partial_{y_1} u \partial_s u d x_1 d \bar{x} dt.
\]

We now have

\[\lim_{d \to 0} \frac{I_2}{d} = -2 \left( \frac{1}{4} \right) \int \partial_{y_1} u (-\frac{1}{2}) \partial_t u (-\frac{1}{2}) = -\frac{1}{2} \gamma.\]

To see this, we consider the change of variables \(\Phi(x,t) = (y,s)\), introduced before. Let \(D_{-1/4} = \mathbb{R}^N \times [-1/2, -1/4], \tilde{D}_{-1/4} = \Phi(D_{-1/4}).\)

Since \(\left| \det \frac{\partial (y,s)}{\partial (x,t)} \right| = 1\), we have,

\[
\lim_{d \to 0} \frac{I_2}{d} = \lim_{d \to 0} -2 \int_{D_{-1/4}} \partial_{y_1} u \partial_s u d x_1 d \bar{x} dt
\]

\[
= \lim_{d \to 0} -2 \int_{\tilde{D}_{-1/4}} \partial_{y_1} u \partial_s y_1 d y_1 d s.
\]

Since \(y_1 = \frac{x_1 - dt}{\sqrt{1-d^2}}, s = \frac{t - dx_1}{\sqrt{1-d^2}}\), we have that \(t \sqrt{1-d^2} = dy_1 + s\), so that

\(\tilde{D}_{-1/4} = \{ (y,s) : |y| \leq |s| \text{ and } -\frac{1}{2} \sqrt{1-d^2} \leq dy_1 + s \leq -\frac{1}{4} \sqrt{1-d^2} \}\).

Note that the restriction \(|y| \leq |s|\) comes from the support of \(\partial_y u\),
\frac{\partial_s u}{s}. Thus,
\frac{\int \int_{\tilde{D}_{\gamma/4}} \partial_{y_1} u \partial_s u dy ds}{2} = \int_{|y| \leq 2} \int \sqrt{1-d^2 - dy_1} \frac{\partial_{y_1} u \partial_s u dy ds}{2} \\
= \int_{|y| \leq 2} \int \sqrt{1-d^2 - dy_1} \frac{\partial_{y_1} u \partial_s u dy ds}{2} - \int_{|y| \leq 2} \int \sqrt{1-d^2 - dy_1} \frac{\partial_{y_1} u \partial_s u dy ds}{2}.

Consider, for instance, the second term. There, \(-\frac{1}{2} \leq s \leq -\frac{1}{2} \sqrt{1-d^2 + 2d}, so that, it is bounded by
\left| \int_{|y| \leq 2} \int \sqrt{1-d^2 + 2d} \left| \partial_{y_1} u \right| \left| \partial_s u \right| dy ds \right| \leq C d \sup_{-\frac{1}{2} \leq s \leq -\frac{1}{2} \sqrt{1-d^2 + 2d}} \|(u, \partial_s u)\|_{L^2_{\gamma} L^2}.

Thus, the second term goes to 0 as \(d \to 0\), and the third one can be treated similarly. (Recall that, by compactness, we have \(\sup_{-1 \leq s \leq 0} \|(u(s), \partial_s u(s))\|_{L^2} \leq A\).) But, using now (2.2), we obtain (4.7).

To study \(I_1\), we introduce
\[ I_3 = \int_{-\frac{1}{2} \sqrt{1-d^2}} \sqrt{1-d^2} \int_{|y| \leq 2} \frac{1}{2} \left\{ \frac{1 + d^2}{1 - d^2} ((\partial_s u)^2 + (\partial_y u)^2) + |\nabla_y u|^2 \right\} - \frac{|u|^2}{2} dy ds. \]

Then, in light of the fact that \(\sup_{-1 \leq s \leq 0} \|(u(s), \partial_s u(s))\|_{L^2} \leq A\), the identity in Remark [2.16] and the support properties of \(u, \partial_s u\), we have
\[ I_3 = \frac{1}{4} E(u(-1/2), \partial_s u(-1/2)) + O(d^2) = \frac{1}{4} E \gamma + O(d^2). \]

We next claim that
\[ \lim_{d \to 0} \frac{I_1 - I_3}{d} = - \int y_1 e(u)(-1/4) + \int y_1 e(u)(-1/2). \]
(Recall the definition of \(e(u)\) from the proof of Remark [2.16].) Let us assume (4.8) temporarily. Recall that (2.8), the support properties of \(u\) and integration by parts, yield
\[ \partial_s \int y_1 e(u(s)) dy = - \int \partial_y u(s) \partial_s u(s) dy, \]
so that in light of (2.2), \(- \int y_1 e(u)(-1/4) + \int y_1 e(u)(-1/2) = \frac{1}{4} \gamma \) and hence \(\lim_{d \to 0} \frac{1}{d} \frac{1}{4} E \gamma = \frac{1}{4} \gamma \) and so, using (4.7), we obtain from (4.6),
\[ \lim_{d \to 0} \frac{1}{d} \frac{1}{4} E \left( z_d(-1/2), \partial_t z_d(-1/2) \right) - \frac{1}{4} E \gamma = \frac{1}{4} \gamma - \frac{1}{2} \gamma = - \frac{1}{4} \gamma. \]
(Note that, for $d$ small this already implies that $z_d$ cannot by identically 0.) (4.9) implies that, for $d > 0$ small, $E (z_d(-1/2), \partial_t z_d(-1/2)) < E_C$, since $\gamma > 0$.

We now turn to the verification of (4.8). Note that, for $d < \delta > 0$ in

$$I_3 = \int_{|y| \leq 2} \left\{ \frac{1}{2} \int_{|y| \leq 2} \frac{1}{d} \int_{|y| \leq 2} \int_{-\sqrt{1-d^2} \leq y, \sqrt{1-d^2} \leq y} e(u)(y, \bar{y}, s)dsdy - \frac{1}{d} \int_{|y| \leq 2} \int_{-\sqrt{1-d^2} \leq y, \sqrt{1-d^2} \leq y} e(u)(y, \bar{y}, s)dsdy \right\}$$

By using the change of variables used in the proof of (4.7) we see that

$$\lim_{d \downarrow 0} \frac{I_1 - I_3}{d} = \lim_{d \downarrow 0} \left\{ \frac{1}{d} \int_{|y| \leq 2} \int_{-\sqrt{1-d^2} \leq y, \sqrt{1-d^2} \leq y} e(u)(y, \bar{y}, s)dsdy - \frac{1}{d} \int_{|y| \leq 2} \int_{-\sqrt{1-d^2} \leq y, \sqrt{1-d^2} \leq y} e(u)(y, \bar{y}, s)dsdy \right\}$$

$$A(d) = \frac{1}{d} \int_{|y| \leq 2} \int_{-\sqrt{1-d^2} \leq y, \sqrt{1-d^2} \leq y} e(u)(y, \bar{y}, s)dsdy$$

where we have made the change of variables $h = \sqrt{\frac{-1/4}{d} - t}$. Since $(\nabla u, \partial_t u) \in C([-1, -\epsilon_0]; L^2(\mathbb{R}^N))$, for every $\epsilon_0 > 0$, we see that, for $d$ small, we have, for $|h| \leq 2$, that

$$\int_{|y| \leq 2} [e(u)(y, \sqrt{1-d^2}(-1/4) - hd) - e(u)(y, -1/4)]dy = o(1)$$

as $d \to 0$, uniformly in $h$. Hence,

$$A(d) = - \int_{|y| \leq 2} \int_{0}^{y_1} e(u)(y, -1/4)dydy + o(1)$$

Similarly, $B(d) = - \int y_1 e(u)(y, -1/2)dy + o(1)$ and hence (4.8) follows.

Finally, since $E_C < E((W, 0))$, $\|\nabla u(-1)\|_{L^2}^2 < \|\nabla W\|_{L^2}^2$, because of Theorem 3.5, we have that, for $-1 \leq s \leq 0$, $\|\nabla_x u(-s)\|_{L^2}^2 \leq (1 - \delta) \|\nabla W\|_{L^2}^2$, $\delta > 0$. We now consider $\int_{-1/2}^{-1/4} \int |\nabla_x z_d(x, t)|^2 dxdt$. The
argument that we used in the estimate for $\frac{L}{d}$ above, (together with the calculation of $\partial_z z, \partial_x z$) show that
\[
\lim_{d \to 0} \int_{-1/2}^{1/4} \int \left| \nabla_v z_d(x, z) \right|^2 dx dt = \int_{-1/2}^{1/4} \int \left| \nabla_y u(y, s) \right|^2 dy ds \\
\leq \frac{1}{4} (1 - \delta) \| \nabla W \|_{L^2}^2.
\]
But then, for $d$ small, $\int_{-1/2}^{1/4} \int \left| \nabla_v z_d(x, t) \right|^2 dx dt \leq \frac{1}{4} (1 - \delta)/2 \| \nabla W \|_{L^2}^2$. Thus, there exists $t_0 = t_0(d) \in (-1/2, -1/4)$ such that, for $d$ small,
\[
\int \left| \nabla_v z_d(x, t_0) \right|^2 dx < \| \nabla W \|_{L^2}^2, \quad E(z_d(-1/2), \partial_z z_d(-1/2)) < E((W, 0)).
\]
By Theorem 3.5 we have, for all $d$ such that for all $s \in [0, \infty)$, $E((W, 0)) < 0$, we have:
\[
\int \left| \nabla_v z_d(x, t_0) \right|^2 dx < \| \nabla W \|_{L^2}^2, \quad E(z_d(-1/2), \partial_z z_d(-1/2)) < E((W, 0)).
\]
Since the interval of existence of $z_d$ is finite, this contradicts the definition of $E_C$ taking $d > 0$ small, and thus $\gamma = 0$. \hfill \Box

**Proposition 4.11.** Assume that $u_C$ is as in Proposition 4.2 and $I_+ = [0, +\infty)$. Assume in addition that for $t > 0$, $\lambda(t) > A_0 > 0$. Then,
\[
\int \nabla u_{0,C} u_{1,C} = 0.
\]

**Proof.** Because of Proposition 4.10 we can assume that $T_-(u_0, u_1) = +\infty$. To abbreviate the notation, let us denote $u(x, t) = u_C(x, t)$. Again, without loss of generality, if the conclusion does not hold, we can assume that $\gamma = \int \partial_y u_0 u_1 > 0$ and hence, by (2.23), for all $s \in \mathbb{R}$ we have
\[
\int \partial_y u(s) \partial_s u(s) = \gamma > 0.
\]
We will see that this assumption leads to a contradiction. We first start out by showing: given $\epsilon > 0$,
\[
(4.10) \quad \text{there exists } R_0(\epsilon) > 0 \text{ such that, for all } s \geq 0, \text{ we have}
\int \left[ y + \frac{\epsilon}{\lambda(s)} \right] \left| \partial_s u \right|^2 + \left| \nabla_y u \right|^2 + \frac{|u|^2}{|y|^2} + |u^*|^2 \leq \epsilon.
\]
In fact, by compactness of $K$, given $\epsilon > 0$, $\exists \bar{R}_0 = \bar{R}_0(\epsilon) > 0$ such that,
\[
\forall s \in [0, \epsilon), \int \left[ y + \frac{\epsilon(s)}{\lambda(s)} \right] \left| \partial_s u \right|^2 + \left| \nabla_y u \right|^2 + \frac{|u|^2}{|y|^2} + |u^*|^2 \leq \epsilon.
\]
Since $\lambda(s) \geq A_0$, $R_0(\epsilon) = \bar{R}_0(\epsilon)/A_0$ does the job.
Next, we show that, as a consequence of (4.10) we have good bounds
ENERGY-CRITICAL FOCUSING WAVE 29

(4.11) For $M > 0$, we have, for all $s \in [0, \infty)$, \[ \left| \frac{y(s)}{\lambda(s)} \right| \leq s + M. \]

To verify (4.11), recall that, since $E((u_0, u_1)) = E_C > 0$, $(u_0, u_1)$ is not identically 0 and we have, because of Corollary 3.6

\[ \inf_{s \geq 0} \int |\nabla_y u(y, s)|^2 + |\partial_s u(y, s)|^2 \, dy \geq C \|(u_0, u_1)\|_{H^1 \times L^2}^2 = B_0 > 0. \]

Then, use (4.10) to choose $M_0 > 0$ so that

\[ \int_{|y + \frac{y(s)}{\lambda(s)}| \geq M_0} |\nabla u| + |\partial_s u| \leq B_0/2; \quad s \in [0, \infty), \]

to conclude that

\[ \int_{|y + \frac{y(s)}{\lambda(s)}| \leq M_0} |\nabla u|^2 + |\partial_s u|^2 \geq B_0/2; \quad s \in [0, \infty). \]

Recall now from Lemma 2.17, that there exists $\epsilon_0 > 0$ so that if for some $M_1 > 0$, we have

(4.12) \[ \int_{|x| > M_1} |\nabla_y u_0|^2 + \frac{|u_0|^2}{|y|^2} + |u_1|^2 \leq \epsilon \]

then

\[ \int_{|x| \geq 2M_1 + s} |\nabla_y u(y, s)|^2 + |\partial_s u(y, s)|^2 \leq C\epsilon, \]

wherever $0 < \epsilon < \epsilon_0$ and $s \geq 0$. Since we can assume, without loss of generality, that $y(0) = 0, \lambda(0) = 1$, in light of (4.10) we can always achieve (4.12). We will show that we can choose $\epsilon$ so small that $y \lambda(s) \leq s + 3 \max(M_0, M_1)$. If not, $\left| \frac{y(s)}{\lambda(s)} \right| \geq s + 3 \max(M_0, M_1)$, and if $y + \frac{y(s)}{\lambda(s)} \leq M_0, |y| \geq s + 3 \max(M_0, M_1) - M_0 \geq s + 2 \max(M_0, M_1) \geq s + 2M_1$. But then,

\[ \frac{B_0}{2} \leq \int_{|y + \frac{y(s)}{\lambda(s)}| \leq M_0} |\nabla_y u|^2 + |\partial_s u|^2 \leq \int_{|y| \geq s + 2M_1} |\nabla_y u|^2 + |\partial_s u|^2 \leq C\epsilon, \]

by (4.12). If $C\epsilon < B_0/2$, we reach a contradiction, which establishes (4.11).

Having (4.10), (4.11) at our disposal, we now define for $R > 0, d > 0,$

(4.13) \[ z_{d,R}(x_1, \bar{x}, t) = u_R \left( \frac{x_1 - dt}{\sqrt{1 - d^2}}, \frac{\bar{x}}{\sqrt{1 - d^2}}, \frac{t - dx_1}{\sqrt{1 - d^2}} \right), \]
Moreover, we will use the fact that, when \((x,t) \in R^1\), and define \(u\) be shown by approximating \(u\) by compactly supported regular solutions and making the observation that the corresponding \(z\) solutions of \((CP)\) on finite time intervals.

We also have \(\sup_{s < \bar{\eta}} (u, \partial_s u)_{L^1} \leq A\) and \(\|u_R\|_{L^\infty(S_{(0,\infty)})} = +\infty\). Moreover, we will use the fact that, when \((x,t)\) are in a compact set, the identity \(\partial_t e(z_{d,R})(x,t) = \sum_{j=1}^{N} A_j (\partial_{x_j} z_{d,R} \partial_t z_{d,R})\) holds, which can be shown by approximating \(u\) by compactly supported regular solutions and making the observation that the corresponding \(z_{d,R}\) are then solutions of \((CP)\) on finite time intervals.

We now have:

There exists \(d_0 > 0\) so that, for \(0 < d < d_0\)

\[
\int_1^2 \int_{3 \leq |x| \leq 8} |\nabla_x z_{d,R}|^2 + |\partial_t z_{d,R}|^2 + |z_{d,R}|^2 \leq \eta_1(R, d),
\]

where \(\eta_1(R, d) \to 0\), uniformly in \(d < d_0\).

To establish (4.14), we use the change of variables \(\Phi(x,t) = (y,s)\) where \(y_1 = \frac{x_1 - d}{\sqrt{1 - d^2}}\), \(\bar{y} = \bar{x}\), \(s = \frac{d_1}{\sqrt{1 - d^2}}\). Then, for \(d\) small, we have, after changing variables, that the left-hand side of (4.14) is bounded by

\[
\int_{1 - 1/8}^{2 + 1/8} \int_{3 - 1/8 \leq |y| \leq 8 + 1/8} |\nabla_y u_R|^2 + |\partial_s u_R|^2 + |u_R|^2 dy ds,
\]

which after rescaling, becomes

\[
\frac{1}{R} \int_{1 - 1/8R}^{2 + 1/8R} \int_{3 - 1/8R \leq |y| \leq 8 + 1/8R} |\nabla_y u|^2 + |\partial_s u|^2 + |u|^2 dy ds.
\]

But, by (4.11), \(\frac{|y(s)|}{\lambda(s)} \leq (2 + 1/8)R + M\), for \(0 \leq s \leq (2 + 1/8)R\), so that, for \(R\) large, \(\{y : |y| \geq (3 - 1/8)R\} \subset \{y : \frac{|y + y(s)|}{\lambda(s)} \geq R/2\} \) and our claim then follows from (4.10).

We now pick \(\theta_1 = \theta_1(x) \in C_0^\infty(|x| < 5), \theta_1 \equiv 1\) on \(|x| < 4, 0 \leq \theta_1 \leq 1\), and define \(\theta(x) = \theta_1(x_1) \theta_1(|\bar{x}|)\). Note that \(\theta(x) \equiv 1\) on \(|x| \leq 4\) and supp \(\theta \subset \{|x| \leq \sqrt{25 + 25} = \sqrt{50}\}\). Our next task is to study

\[
\int_1^2 \int \theta^2 e(z_{d,R})(x_1, \bar{x}, t) dx_1 d\bar{x} dt.
\]

For this, we first change variables \(\Phi(x,t) = (y,s)\), as before. Our integral then becomes, (for \(d_0\) small), recalling that \(t \sqrt{1 - d^2} = dy_1 + s\),
\[(4.15) \int_{\sqrt{1-d^2} \leq dy_1 + s \leq 2\sqrt{1-d^2}} \int \theta^2 \left( \Phi^{-1}(y, s) \right) e(z_{d,R}) \left( \Phi^{-1}(y, s) \right) dy_1 dy ds.\]

Recall that \(\theta^2(x) = \theta^2_1(x_1)\theta^2_1(|\vec{x}|)\), \(\vec{y} = \vec{x}\), \(y_1 = \frac{x_1 - dt}{\sqrt{1-d^2}}\), \(s = \frac{t - dx_1}{\sqrt{1-d^2}}\), so that \(x_1 = \frac{y_1 + ds}{\sqrt{1-d^2}}\) and \(\theta^2 \left( \Phi^{-1}(y, s) \right) = \theta^2_1 \left( \frac{y_1 + ds}{\sqrt{1-d^2}} \right) \theta^2_1 (\vec{y})\). Note that

\[
\left| e(z_{d,R}) \left( \Phi^{-1}(y, s) \right) \right| \leq C \left\{ |\nabla_y u_R(y, s)|^2 + |\partial_s u_R(y, s)|^2 + |u^2_R(y, s)| \right\},
\]

for \(0 < d < d_0\), and that

\[
\theta^2_1 \left( \frac{y_1 + ds}{\sqrt{1-d^2}} \right) = \theta^2_1 \left( \sqrt{1-d^2}y_1 + \frac{d}{\sqrt{1-d^2}} (s + dy_1) \right).
\]

Thus, since in our domain of integration we have \(\sqrt{1-d^2} \leq dy_1 + s \leq 2\sqrt{1-d^2}\), for \(0 \leq d < d_0\), \(d_0\) small, we have

\[
\theta^2_1 \left( \sqrt{1-d^2}y_1 + \frac{d}{\sqrt{1-d^2}} (s + dy_1) \right) - \theta^2_1 (\sqrt{1-d^2}y_1) = O(d)(\theta^2_1)' \left( \sqrt{1-d^2}y_1 + \eta O(d) \right),
\]

where \(|\eta| \leq 1\). Note that \(\text{supp}(\theta^2_1)'(\alpha) \subset \{4 \leq |\alpha| \leq 5\}\), so that, for \(d_0\) small, this can only be nonzero for \(3 + \frac{1}{4} \leq |y_1| \leq 5 + \frac{1}{4}\). Using a similar argument for \(\theta^2_1 \left( \sqrt{1-d^2}y_1 \right) - \theta^2_1 (y_1)\), and the argument used in the proof of (4.14), we see that the integral in (4.15) equals

\[(4.16) \int \int \theta^2(y)e(z_{d,R})(\Phi^{-1}(y, s))dy ds + d\eta_2(R, d),\]

where \(|\eta_2(R, d)| \to 0\), uniformly in \(d < d_0\).

Now, using the formulas after (4.6), we see that the integral in (4.16) equals \(J_1 + J_2\), where

\[
J_1 = \int \int_{\frac{y_1 + s}{\sqrt{1-d^2}} \leq 2} \theta^2(y) \left\{ \frac{1}{2} \left[ \frac{1 + d^2}{1 - d^2} (\partial_s u_R)^2 + \frac{1 + d^2}{1 - d^2} (\partial_{y_1} u_R)^2 + |\nabla_y u_R|^2 \right] \\
- \frac{1}{2^*} |u_R|^{2^*} \right\} dy ds,
\]

\[
J_2 = -\frac{2d}{(1-d^2)} \int \int_{\frac{y_1 + s}{\sqrt{1-d^2}} \leq 2} \theta^2(y) \partial_{y_1} u_R \partial_s u_R dy ds.
\]
Let us first analyze $\frac{J_2}{d}$. We clearly have

$$\frac{J_2}{d} = -2 \int \int \theta^2(y) \partial_{y_1} u_R \partial_s u_R dy ds + O(d^2),$$

where $O(d^2)$ is uniform in $R$. Consider

$$\tilde{J}_2 = -2 \int \int \theta^2(y) \partial_{y_1} u_R \partial_s u_R dy ds$$

$$= -2 \int \int \partial_{y_1} u_R \partial_s u_R dy ds - 2 \int \int [\theta^2(y) - 1] \partial_{y_1} u_R \partial_s u_R$$

$$= \tilde{J}_{21} + \tilde{J}_{22}.$$  

Note that supp[$\theta^2 - 1$] $\subset \{|y| \geq 4\}$, so that, with the argument in the proof of (4.14) we see that $\tilde{J}_{22} = \eta_3(R, d)$, with $\eta_3 \to 0$, uniformly in $d$. Moreover, (2.12) and scaling show that $\tilde{J}_{21} = -2 \gamma \sqrt{1 - d^2}$, so that

$$\frac{J_2}{d} = -2 \int \int \theta^2(y) \partial_{y_1} u_R \partial_s u_R dy ds + 2 \int \int \theta^2(y) \partial_{y_1} u_R \partial_s u_R dy ds$$

$$- 2 \gamma \sqrt{1 - d^2} + O(d^2) + \eta_3(R, d).$$

We turn to the difference of the two integrals on the right hand side. It is dominated by

$$2 \int \int_{y_1 > 0} \int_{2 \sqrt{1 - d^2} - dy_1}^{2 \sqrt{1 - d^2}} \theta^2(y) |\partial_{y_1} u_R| |\partial_s u_R| ds dy$$

$$+ 2 \int \int_{y_1 < 0} \int_{2 \sqrt{1 - d^2} - dy_1}^{2 \sqrt{1 - d^2}} \theta^2(y) |\partial_{y_1} u_R| |\partial_s u_R| ds dy$$

$$+ 2 \int \int_{y_1 > 0} \int_{\sqrt{1 - d^2}}^{\sqrt{1 - d^2} - dy_1} \theta^2(y) |\partial_{y_1} u_R| |\partial_s u_R| ds dy$$

$$+ 2 \int \int_{y_1 < 0} \int_{\sqrt{1 - d^2}}^{\sqrt{1 - d^2} - dy_1} \theta^2(y) |\partial_{y_1} u_R| |\partial_s u_R| ds dy = \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D}.$$  

We will estimate $\tilde{A}$, the other terms being similar. In our region of integration, we have $|y_1| \leq 5$. We make the change of variables in the $s$ integral $h = \frac{2 \sqrt{1 - d^2} - s}{d}$. We then have, in our region of integration,
$0 \leq h \leq y_1$. Thus,

$$\tilde{A} \leq 2d \int \int_{y_1 > 0} \int_{0}^{y_1} \theta^2(y) \left| \partial_{y_1} u_R(y, 2\sqrt{1 - d^2} - dh) \right| dh dy$$

$$\leq 2d \int_{0}^{5} \int \theta^2(y) \left| \partial_{y_1} u_R(y, 2\sqrt{1 - d^2} - dh) \right| dh dy$$

$$\leq 2d \int \int \theta^2(y) \left| \partial_{s} u_R(y, 2\sqrt{1 - d^2} - dh) \right| dy dh$$

$$\leq 2Ad.$$

We thus have

\begin{equation}
J_2 = -2\gamma + O(d) + \eta_3(R, d),
\end{equation}

where $O(d)$ is uniform in $R$ and $\eta_3(R, d) \to 0$ uniformly for $0 \leq d < d_0$.

Next, $\frac{J_1}{d} = \frac{1}{d} \int \int \theta^2(y)e(u_R)(y, s)dy ds + O(d)$, where $O(d)$ is uniform in $R$ large. We now consider $\tilde{J}_1 = \int \int \theta^2(y)e(u_R)(y, s)ds$.

Note that, arguing as in the case of $\tilde{J}_2$, it is easy to see that

\begin{equation}
\tilde{J}_1 = \sqrt{1 - d^2}EC + \eta_4(R, d), \quad \text{with } \eta_4(R, d) \to 0,
\end{equation}

uniformly for $0 \leq d \leq d_0$.

We next turn to $\frac{h - \tilde{J}_1}{d}$, which equals after rescaling to

$$\frac{1}{dR} \left\{ \int \int_{R \leq \sqrt{1 - d^2} \leq 2R} \theta^2(y/R)e(u)(y, s)dy ds - \int \int_{R \leq \sqrt{1 - d^2} \leq 2R} \theta^2(y/R)e(u)(y, s)dy ds \right\}.$$
The difference then equals

\[
\frac{1}{dR} \left\{- \int \int_{y_1 > 0} \int_{2R\sqrt{1-d^2}}^{2R} \theta^2(y/R)e(u)(y, s) dy ds \right. \\
+ \int \int_{y_1 < 0} \int_{2R\sqrt{1-d^2}}^{2R} \theta^2(y/R)e(u)(y, s) dy ds \\
+ \frac{1}{dR} \left\{ \int \int_{y_1 > 0} \int_{R\sqrt{1-d^2}}^{R} \theta^2(y/R)e(u)(y, s) dy ds \right. \\
- \int \int_{y_1 < 0} \int_{R\sqrt{1-d^2}}^{R} \theta^2(y/R)e(u)(y, s) dy ds \\
- \int \int \int_{y_1 > 0} \int_{2R\sqrt{1-d^2}}^{2R} \theta^2(y/R)e(u)(y, s) dy ds \\
+ \int \int \int_{y_1 < 0} \int_{2R\sqrt{1-d^2}}^{2R} \theta^2(y/R)e(u)(y, s) dy ds \\
- \int \int \int_{y_1 > 0} \int_{R\sqrt{1-d^2}}^{R} \theta^2(y/R)e(u)(y, s) dy ds \\
- \int \int \int_{y_1 < 0} \int_{R\sqrt{1-d^2}}^{R} \theta^2(y/R)e(u)(y, s) dy ds \\
\left. \right\} = \frac{1}{dR} (L_1 + L_2). 
\]

We will first study \( \frac{1}{dR} L_1 \). In the first of its integrals, we interchange

the order of integration, to obtain, in supp \( \theta^2 \),

\[
- \int_{2R\sqrt{1-d^2}}^{2R} \int_{2R\sqrt{1-d^2}}^{5R} \int_{2R\sqrt{1-d^2}}^{5R} \theta^2(y/R)e(u)(y, s) dy ds. 
\]

We then perform the change of variables \( s = 2R\sqrt{1-d^2} - 5\alpha dR \), so that the integral equals

\[
-5dR \int_0^1 \int_{5\alpha R}^{5R} \int_{2R\sqrt{1-d^2}}^{2R\sqrt{1-d^2}} \theta^2(y/R)e(u)(y, 2R\sqrt{1-d^2} - 5\alpha dR) dy dy_1 d\alpha. 
\]

Similarly, the second integral equals

\[
5dR \int_0^1 \int_{5\alpha R}^{5R} \theta^2(y/R)e(u)(y, R\sqrt{1-d^2} - 5\alpha dR) dy dy_1 d\alpha, 
\]
Therefore, the argument after (4.14) shows that the second term, for $1 \leq \nu_0 \leq 1$, by virtue of (2.3). Integrating by parts, we obtain that

\[ \frac{1}{dR} L_1 = \begin{cases} - \int_0^1 \int_{5\alpha R}^{5R} \int \theta^2(y/R)e(u)(y, 2R\sqrt{1 - d^2} - 5\alpha dR)dyd\alpha \\ + \int_0^1 \int_{5\alpha R}^{5R} \int \theta^2(y/R)e(u)(y, R\sqrt{1 - d^2} - 5\alpha dR)dyd\alpha \end{cases} \]

by virtue of (2.3). Integrating by parts, we obtain that $\frac{1}{dR} L_1$ equals

\[ \frac{1}{dR} L_1 = \begin{cases} \frac{1}{dR} L_1 = \begin{cases} -5 \int_0^1 \int_{5\alpha R}^{5R} \int_{R\sqrt{1 - d^2}}^{2R\sqrt{1 - d^2}} \theta^2(y/R)\partial_{\nu_0}e(u)(y, s - 5\alpha dR)dsdyd\alpha \\ -5 \int_0^1 \int_{5\alpha R}^{5R} \int_{R\sqrt{1 - d^2}}^{2R\sqrt{1 - d^2}} \theta^2(y/R)\sum_{j=1}^N \partial_{\nu_j}(\partial_{\nu_j}u(y, s - 5\alpha dR))dsdyd\alpha \end{cases} \\ -5 \int_0^1 \int_{5\alpha R}^{5R} \int_{R\sqrt{1 - d^2}}^{2R\sqrt{1 - d^2}} \theta^2(y/R)\partial_{\nu_0}e(u)(y, s - 5\alpha dR)dsdyd\alpha \end{cases} \]

For the second term, note that

\[ |\partial_{\nu_j}(\theta^2(y/R))| \leq C/R \text{ and } \text{supp} \partial_{\nu_j}(\theta^2(y/R)) \subset \{4 \leq |y| \leq 8\}. \]

Also, $R(\sqrt{1 - d^2} - 5\alpha d) \leq s - 5\alpha dR \leq 2R(\sqrt{1 - d^2} - 5\alpha d), 0 \leq \alpha \leq 1$. Therefore, the argument after (1.14) shows that the second term, for $0 \leq d \leq d_0$ is of the form $\eta_5(R, d)$, with $\eta_5(R, d) \rightarrow 0$, uniformly in $0 \leq d \leq d_0$. A similar argument shows that

\[ \frac{1}{dR} L_2 = 5 \int_{-1}^{0} \int_{R\sqrt{1 - d^2}}^{2R\sqrt{1 - d^2}} \theta^2(5\alpha\theta_1^2(|\tilde{y}| / R)\partial_{\nu_0}u(5\alpha R, \tilde{y}, s - 5\alpha dR)dsdyd\alpha \]

with $\eta_6$ behaving like $\eta_5$. Hence,

\[ \frac{\bar{J}_1 - \bar{J}_1}{d} = 5 \int_{-1}^{1} \int_{R\sqrt{1 - d^2}}^{2R\sqrt{1 - d^2}} \theta^2(5\alpha\theta_1^2(|\tilde{y}| / R)\partial_{\nu_0}u(5\alpha R, \tilde{y}, s - 5\alpha dR)dsdyd\alpha + \eta_5(R, d) + \eta_6(R, d). \]
Making the change of variables $y_1 = 5\alpha R, \zeta = s - 5\alpha dR$, the integral on the right hand side gets transformed and $\frac{4-J_1}{d}$ equals

$$\frac{1}{R} \int \int \int \frac{1}{\sqrt{1+4d^2}}(y_1/R)\theta^2(|y|/R)\partial_{y_1} u(y_1, y) \partial_s u(y_1, y) d\zeta dy_1 dy$$

$$+ \eta_5(R, d) + \eta_6(R, d).$$

The calculation of $J_2$ above now yields that the integral equals $\gamma + O(d) + \eta_3(R, d)$, so that

$$(4.19) \quad \frac{J_1 - \tilde{J}_1}{d} = \gamma + O(d) + \eta_3(R, d) + \eta_5(R, d) + \eta_6(R, d),$$

where $\eta_i(R, d) \to 0$, uniformly for $0 \leq d < d_0$ and $O(d)$ is uniform in $R$ large.

Next, we recall that for fixed $R, u_R \in L^2_t L^{\frac{N+2}{N-2}}_x \varepsilon^2$, for any compact time interval. From this and Lemma 2.2 we can see that $\theta(x)z_{d,R}(x, t)$ is in $C([1, 2]; H^1 \times L^2)$. Fix now $t_0 \in [1, 2]$ and recall, from the beginning of the proof that $\partial_t e(z_{d,R})(x, t) = \sum \partial_{x_j} (\partial_{x_j} z_{d,R} \partial_t z_{d,R})$. Hence,

$$\int \theta^2(x) e(z_{d,R})(x, t_0) dx = \int_1^2 \int \theta^2(x) e(z_{d,R})(x, t) dx dt$$

$$+ \int_1^2 \int \theta^2(x) \int_0^{t_0} \sum_{j=1}^N \partial_{x_j} (\partial_{x_j} z_{d,R} \partial_t z_{d,R}) d\alpha dx dt$$

$$= \int_1^2 \int \theta^2(x) e(z_{d,R})(x, t) dx dt$$

$$- \sum_{j=1}^N \int_1^2 \int_0^{t_0} \partial_{x_j} (\theta^2(x)) \partial_{x_j} z_{d,R}(x, \alpha) \partial_t z_{d,R}(x, \alpha) d\alpha dx dt.$$ 

Because of (11.14), the second term equals $\eta_7(R, d, t_0)$, with $\eta_7(R, d, t_0) \to 0$, uniformly in $t_0 \in [1, 2], 0 \leq d \leq d_0$. Thus, if

$$E(t_0, d, R) = \int \theta^2(x) e(z_{d,R})(x, t_0) dx,$$

we have (using our previous estimates):

$$(4.20) E(t_0, d, R) = \eta_7(R, d, t_0) + d\eta_2(R, d) - 2\gamma d + O(d^2) + d\eta_3(R, d)$$

$$+ E_C + \eta_4(R, d) + \gamma d + d\eta_3(R, d) + d\eta_5(R, d) + d\eta_6(R, d)$$

$$= E_C - \gamma d + d \{ \eta_2(R, d) + \eta_3(R, d) + \eta_5(R, d) + \eta_6(R, d) \}$$

$$+ \eta_4(R, d) + \eta_7(R, d, t_0) + O(d^2).$$
We now need to consider
\[ \int_1^2 \int \theta^2(x) |\nabla_x z_{d,R}(x,t)|^2 \, dx \, dt = \int_1^2 \int \theta^2(x) \left\{ \frac{1}{1 - d^2} |\partial_y u_R|^2 + |\nabla_y u_R|^2 \right\} \, dx \, dt. \]

The arguments used to establish (4.20) easily yield that the right hand side equals \( \int \int_1^{\sqrt{1-d^2}} \theta^2(y) |\nabla_y u_R|^2 + O(d) \), where \( O(d) \) is uniform in \( R \), i.e.,
\[
(4.21) \quad \int_1^2 \int \theta^2(x) |\nabla_x z_{d,R}(x,t)|^2 \, dx \, dt = \int_1^{\sqrt{1-d^2}} \theta^2(y) |\nabla_y u_R|^2 + O(d).
\]

Define now \( h_{d,R}(x,t) = \theta(x) z_{d,R}(x,t) \). Then,
\[
|\nabla_x h_{d,R}(x,t)|^2 = |\nabla_x z_{d,R}|^2 + |\nabla \theta|^2 |z_{d,R}|^2 + 2\theta \nabla \theta \cdot \nabla z_{d,R} \overline{z_{d,R}}
\]
and note that the last two terms are supported in \( 3 \leq |x| \leq 8 \). Also, \( |h_{d,R}|^2 = \theta^2(x) |z_{d,R}|^2 + \left( |\theta|^2 - |\theta|^2 \right) |z_{d,R}|^2 \) and the last term is supported in \( 3 \leq |x| \leq 8 \).

We are now able to conclude the proof. Choose \( d_0 \) so that for \( 0 < d < d_0 \), uniformly in \( R \), we have
\[
\int_0^1 \int \theta^2 |\nabla_x z_{d,R}|^2 \leq (1 - \delta/2) \int |\nabla W|^2,
\]
which we can do because of (4.21). Let \( 1 + \bar{\delta} = \frac{1-\delta}{1-\delta/2} \). Consider
\[
S_1 = S_1(d,R) = \left\{ t \in [1,2] : \int \theta^2(x) |\nabla_x z_{d,R}|^2 (x,t) \, dx \right\}
\]
\[
\leq (1 + \bar{\delta})(1 - \delta/2) \int |\nabla W|^2 = (1 - \bar{\delta}/4) \int |\nabla W|^2 \right\}.
\]

Then \( |S_1| \geq \frac{\bar{\delta}}{1 + \bar{\delta}} \), for all \( 0 < d < d_0 \), \( R > 0 \). Next, choose \( d_1 \) so small and \( R > R_0(d_1) \) so that, for all \( t_0 \in [1,2] \), \( E(t_0,d,R) \leq E_C - \frac{\delta}{2} d_1 \). In addition, we can choose \( d_1 \leq d_0 \). This is possible in view of (4.20).

Now, for an \( \epsilon > 0 \) to be chosen, find \( R_1(\epsilon) \) so large that for \( R \geq R_1(\epsilon) \), we have \( \eta_1(R,d_1) \leq \epsilon \), where \( \eta_1 \) is as in (1.14).

Consider next the set
\[
S_2 = S_2(R,d_1,\epsilon,M) = \left\{ t \in [1,2] : \int_{3 \leq |x| \leq 8} |\nabla_x z_{d,R}|^2 + |\partial_t z_{d,R}|^2 + |z_{d,R}|^{2*} \leq M \epsilon \right\}.
\]
Because of \( |S_2| \geq (1 - 1/M) \), and if we choose \( M = M_\delta \) so large that \((1 - 1/M_\delta) + \bar{\delta}/(1 + \bar{\delta}) > 1\), we can find \( t_0 = t_0(R, \epsilon) \in S_1 \cap S_2 \). We then have:

\[
(4.22) \quad \int |\nabla_x h_{d,R}(t_0)|^2 \leq \int \theta^2 |\nabla z_{d,R}(t_0)|^2 + CM\epsilon \\
\leq (1 - \bar{\delta}/4) \int |\nabla W|^2 + CM\epsilon \leq (1 - \bar{\delta}/8) \int |\nabla W|^2 ,
\]

if we choose \( CM\epsilon \leq \bar{\delta}/8 \int |\nabla W|^2 \), \( R \geq R_1(\epsilon) \).

\[
(4.23) \quad \int e(h_{d,R}(t_0)) \leq \int \theta^2 e(z_{d,R}(t_0)) + C\epsilon M \leq E_C - \frac{\gamma d_1}{2} + C\epsilon M,
\]

for \( R \geq R_0(d_1) \), \( R \geq R_1(\epsilon) \). If we now choose \( C\epsilon M \leq \frac{\gamma}{4}d_1 \), we have then

\[
(4.24) \quad \int e(h_{d,R}(t_0)) \leq E_C - \frac{\gamma d_1}{4},
\]

for all \( R > \text{Max}(R_0(d_1), R_1(\epsilon)) \). Let us now consider \( w_R(x,t) \) to be the solution of (CP) with data at \( t = t_0 \), \( (h_{d_1,R}(t_0), \partial_t h_{d_1,R}(t_0)) \). In light of the definition of \( E_C \), \( w_R(x,t) \) exists for all time and verifies, in view of Corollary 4.5,

\[
(4.25) \quad \int \int |w_R(x,t)|^{\frac{2(N+1)}{N-2}} dxdt \leq C_{d_1,\gamma},
\]

uniformly for all \( R > \text{Max}(R_0(d_1), R_1(\epsilon)) \).

Next, observe that, by finite speed of propagation (Remark 2.12), \( w_R(x,t) = z_{d,R}(x,t) \) on \( \cup_{-2 \leq t \leq 1}(B(0, 2 + t) \times t) \). To justify the application of Remark 2.12, we approximate \((u_0, u_1)\) and hence \((u_{0,R}, u_{1,R})\) by \((u_{0,R}^{(j)}, u_{1,R}^{(j)})\) which are in \( C_0^\infty \times C_0^\infty \). The resulting \( w_{R}^{(j)} \) exists on any finite time interval, for \( j \) large by Remark 2.21 and the corresponding \( z_{d,R}^{(j)} \) are now solutions of (CP) on each finite time interval. We then have, for \( j \) large, \( w_{R}^{(j)} = z_{d,R}^{(j)} \) on the required set, and a passage to the limit, (since \( x \) and \( t \) are in fixed bounded sets, we can apply Lemma 2.22), gives the required identity. But then,

\[
\int \int |z_{d_1,R}|^{\frac{2(N+1)}{N-2}} dxdt \leq C_{d_1,\gamma}.
\]

We now use our change of variables \((y,s) = \Phi(x,t)\), and observe that, (for \( d_1 \) small enough), \( \Phi(\cup_{-2 \leq t \leq 1}(B(0, 2 + t) \times t)) \supset \{(y,s): 0 \leq s \leq 1/4, |y| \leq 1/4\} \). But then, we obtain \( \int \int |u_R|^{\frac{2(N+1)}{N-2}} dyds \leq C_{d_1,\gamma} \), for
all \( R \geq \text{Max}(R_0(d_1), R_1(\epsilon)) \). If we now rescale the above interval, we find that for all \( R \geq \text{Max}(R_0(d_1), R_1(\epsilon)) \),

\[
\int \int_{0 \leq s \leq R/4 \atop |y| \leq R/4} |u|^{rac{2(N+1)}{N-2}} \, dy \, ds \leq C_{d_1, \gamma}.
\]

But, since we have \( \int \int_{s \geq 0} |u|^2(\frac{N+1}{N-2})dy \, ds = +\infty \), we reach a contradiction, which establishes the proposition. \( \square \)

5. RIGIDITY THEOREM. PART 1: INFINITE TIME INTERVAL AND SELF-SIMILARITY FOR FINITE TIME INTERVALS

In this and the following section we will prove the following:

**Theorem 5.1.** Assume that \((u_0, u_1) \in \dot{H}^1 \times L^2\) is such that

\[
E((u_0, u_1)) < E((W, 0)), \quad \int |\nabla u_0|^2 < \int |\nabla W|^2, \quad \int \nabla u_0, u_1 = 0.
\]

Let \( u \) be the solution of (CP) with \((u(0), \partial_t u(0)) = (u_0, u_1)\), with maximal interval of existence \((-T_-(u_0, u_1), T_+(u_0, u_1))\). Assume that there exist \( \lambda(t) > 0, x(t) \in \mathbb{R}^N \), for \( t \in [0, T_+(u_0, u_1)) \), with the property that

\[
K = \left\{ \vec{v}(x, t) = \left( \frac{1}{\lambda(t)^{\frac{(N-2)}{2}}} u \left( \frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{\frac{N}{2}}} \partial_t u \left( \frac{x - x(t)}{\lambda(t)}, t \right) \right), \right. \\
\left. t \in [0, T_+(u_0, u_1)) \right\}
\]

has the property that \( K \) is compact in \( \dot{H}^1 \times L^2 \).

Then, \( T_+(u_0, u_1) < \infty \) is impossible.

Moreover, if \( T_+(u_0, u_1) = +\infty \) and we assume that \( \lambda(t) \geq A_0 > 0 \), for \( t \in [0, \infty) \), we must have \( u \equiv 0 \).

**Remark 5.2.** This Theorem shows the rigidity of (CP) for optimal small data (consider the solution of (CP), \( u(x, t) = W(x) \)). The momentum condition is the ingredient which allows us to treat the nonradial situation and is always true for a radial solution. Lemma 4.6 implies that we can choose \( x(t), \lambda(t) \) continuous in \( [0, T_+(u_0, u_1)) \). Its proof also shows that we can preserve the property \( \lambda(t) \geq A_0 > 0 \).

We next turn to the proof of Theorem 5.1 in the case when

\( T_+(u_0, u_1) = +\infty, \lambda(t) \geq A_0 \).
Assume that \((u_0, u_1) \neq (0, 0)\). Because of Corollary 3.6, we have 
\[ E((u_0, u_1)) = E > 0 \]  
and 
\[ \sup_{t>0} \| (\nabla u, \partial_t u) \|_{L^2} \leq CE \]  
as well as, from Theorem 3.5,

\begin{equation}
\int |\nabla_x u(t)|^2 - |u(t)|^2^* \geq C_7 \int |\nabla_x u(t)|^2 \tag{5.1}
\end{equation}

and

\begin{equation}
\alpha \int (\partial_t u)^2 + (1 - \alpha) \left( \int |\nabla_x u(t)|^2 - |u(t)|^2^* \right) \geq C_\alpha E, \tag{5.2}
\end{equation}

for \(0 < \alpha < 1\).

We will also be applying (4.10), which gives the following:

Given \(\epsilon > 0\), there exists \(R_0(\epsilon) > 0\) such that, for all \(t \geq 0\)

\begin{equation}
\int_{|x + \frac{x(t)}{\lambda(t)}| \geq R_0(\epsilon)} |\partial_t u|^2 + |\nabla_x u|^2 + \frac{|u|^2}{|x|^2} + |u|^{2^*} \leq \epsilon E. \tag{5.3}
\end{equation}

(Here we use the assumptions \(\lambda(t) \geq A_0 > 0, E > 0\).)

We will next summarize some algebraic properties that will be needed in the sequel. Let us fix \(\phi \in C_0^\infty(\mathbb{R}^N), \phi \equiv 1\) for \(|x| \leq 1\), \(\phi \equiv 0\) for \(|x| \geq 2\), and also define, for \(R > 0\),

\[ \phi_R(x) = \phi(x/R), \quad \psi_R(x) = x\phi(x/R). \]

We will set

\[ r(R) = \int_{|x| \geq R} \frac{|u|^2}{|x|^2} + |u|^{2^*} + |\nabla u|^2 + |\partial_t u|^2 \, dx. \]

**Lemma 5.3.** The following identities hold: for all \(t \geq 0\)

i) \( \partial_t \left( \int \frac{1}{2}(\partial_t u)^2 + \frac{1}{2} |\nabla_x u|^2 - \frac{1}{2^*} |u|^{2^*} \right) = 0 \)

ii) \( \partial_t \int \nabla u \cdot \partial_t u = 0 \)

iii) \( \partial_t \left( \int \psi_R(x) \cdot \nabla u \partial_t u \right) = -\frac{N}{2} \int (\partial_t u)^2 + \frac{(N-2)}{2} \left[ \int |\nabla_x u|^2 - |u|^{2^*} \right] + O(r(R)) \)

iv) \( \partial_t \left( \int \phi_R uu_t \right) = \int (\partial_t u)^2 - \frac{1}{2} |\nabla u|^2 + \frac{1}{2^*} |u|^{2^*} + O(r(R)) \)

v) \( \partial_t \left( \int \psi_R \left\{ \frac{1}{2}(\partial_t u)^2 + \frac{1}{2} |\nabla_x u|^2 - \frac{1}{2^*} |u|^{2^*} \right\} \right) = -\int \nabla u \partial_t u + O(r(R)). \)

Note that i) is Remark 2.16, ii) is (2.2), v) follows from (2.3), iv) follows from the arguments in the proof of Theorem 3.7 and iii) follows by an integration by parts (and a limiting argument).

We now will prove the lemmas crucial for our purpose. Recall that we can assume \(x(0) = 0\).
Lemma 5.4. There exist \( \epsilon_1 > 0, C > 0 \), such that, if \( \epsilon \in (0, \epsilon_1) \), there exists \( R_0(\epsilon) \) so that if \( R > 2R_0(\epsilon) \), then there exists \( t_0 = t_0(R, \epsilon) \), \( 0 \leq t_0 \leq CR \), with the property that for all \( 0 < t < t_0 \) we have

\[
\left| \frac{x(t)}{\lambda(t)} \right| < R - R_0(\epsilon) \quad \text{and} \quad \left| \frac{x(t_0)}{\lambda(t_0)} \right| = R - R_0(\epsilon).
\]

Proof. Since \( x(0) = 0, \lambda(t) \geq A_0 > 0 \), if not, for all \( 0 < t < CR \) (where \( C \) is large), we have

\[
\left| \frac{x(t)}{\lambda(t)} \right| < R - R_0(\epsilon).
\]

Let

\[
z_R(t) = \int \psi_R(x) \nabla_x uu_t + \left( \frac{N}{2} - \alpha \right) \int \phi_R uu_t, \quad 0 < \alpha < 1.
\]

Then, by Lemma 5.3 and by (5.2),

\[
z'_R(t) = -\frac{N}{2} \int (\partial_t u)^2 + \left( \frac{N - 2}{2} \right) \left( \int |\nabla u|^2 - |u|^{2^*} \right) + O(r(R)) + \left( \frac{N}{2} - \alpha \right) \left[ \int (\partial_t u)^2 - \int |\nabla u|^2 + \int |u|^{2^*} \right] O(r(R))
\]

\[
= -\alpha \int (\partial_t u)^2 - (1 - \alpha) \left[ \int |\nabla u|^2 - |u|^{2^*} \right] + O(r(R))
\]

\[
\leq -C\alpha E + O(r(R)).
\]

But, for \( |x| \geq R \), \( \left| x + \frac{x(t)}{\lambda(t)} \right| \geq R_0(\epsilon) \), by our assumption, so that, by (5.3), \( |r(R)| \leq \tilde{C}\epsilon E \). Now, choose \( \epsilon \), so small that \( z'_R(t) \leq -\frac{C\alpha E}{2} \). Note that \( |z_R(t)| \leq \tilde{C}_1 RE \), so that, integrating in \( t \), between 0 and \( CR \),

\[
CR \frac{C\alpha E}{2} \leq 2\tilde{C}_1 RE.
\]

This is a contradiction for \( C \) large.

Note that in the radial case, we have \( x(t) = 0 \) (see [16]) and a contradiction follows from Lemma 5.4. This proof, using the momentum, is the algebraic counterpart of virial identity used in [16] for the NLS equation.

Lemma 5.5. There exist \( \epsilon_2 > 0, R_1(\epsilon) > 0, C_0 > 0 \) such that if \( R > R_1(\epsilon) \), \( t_0 = t_0(R, \epsilon) \) is as in Lemma 5.4, then for \( 0 < \epsilon < \epsilon_2 \),

\[
t_0(R, \epsilon) \geq \frac{C_0 R}{\epsilon}.
\]

Proof. Let for \( t \in [0, t_0] \),

\[
y_R(t) = \int \psi_R(x) e(u)(x, t) dx.
\]
Since $\int \nabla u_0 u_1 = 0$, because of ii) in Lemma 5.3 and v) in Lemma 5.3, we have $|y'_R(t)| = O(r(R))$. Since for $0 < t < t_0$, if $|x| \geq R$, then
\[
| x + \frac{x(t)}{\lambda(t)} | \geq R - (R - R_0(\epsilon)) = R_0(\epsilon),
\]
we have, integrating in $t$,
\[
|y_R(t_0) - y_R(0)| \leq \tilde{C} \epsilon E t_0.
\]

On the one hand, by (5.3)
\[
|y_R(0)| \leq \tilde{C} R_0(\epsilon) E + O(R r(R_0(\epsilon))) \leq \tilde{C} E \{ R_0(\epsilon) + \epsilon R \}.
\]

On the other hand,
\[
|y_R(t_0)| \geq \left| \int_{x + \frac{x(t_0)}{\lambda(t_0)} \leq R_0(\epsilon)} \psi_R e(u)(t_0) \right| - \left| \int_{x + \frac{x(t_0)}{\lambda(t_0)} \geq R_0(\epsilon)} \psi_R e(u)(t_0) \right|.
\]

In the first integral, $|x| \leq | x + \frac{x(t_0)}{\lambda(t_0)} | + \frac{x(t_0)}{\lambda(t_0)} \leq R$, so that $\psi_R(x) = x$.

Note also that the second integral is bounded by $M R \epsilon E$. Hence,
\[
|y_R(t_0)| \geq \left| \int_{x + \frac{x(t_0)}{\lambda(t_0)} \leq R_0(\epsilon)} x e(u)(t_0) \right| - M R \epsilon E.
\]

But, $\int_{x + \frac{x(t_0)}{\lambda(t_0)} \leq R_0(\epsilon)} x e(u)(t_0)$ equals
\[
- \frac{x(t_0)}{\lambda(t_0)} \int_{x + \frac{x(t_0)}{\lambda(t_0)} \leq R_0(\epsilon)} e(u)(t_0) + \int_{x + \frac{x(t_0)}{\lambda(t_0)} \leq R_0(\epsilon)} \left[ x + \frac{x(t_0)}{\lambda(t_0)} \right] e(u)(t_0)
\]
\[
= - \frac{x(t_0)}{\lambda(t_0)} \int e(u)(t_0) + \frac{x(t_0)}{\lambda(t_0)} \int_{x + \frac{x(t_0)}{\lambda(t_0)} \geq R_0(\epsilon)} e(u)(t_0)
\]
\[
+ \int_{x + \frac{x(t_0)}{\lambda(t_0)} \leq R_0(\epsilon)} \left[ x + \frac{x(t_0)}{\lambda(t_0)} \right] e(u)(t_0).
\]

The first term is, in absolute value $|R - R_0(\epsilon)| E$, while the last two are bounded in absolute value by $\tilde{C}(R - R_0(\epsilon)) E + \tilde{C} R_0(\epsilon) E$. We then find:
\[
|y_R(t_0)| \geq (R - R_0(\epsilon)) E (1 - \tilde{C} \epsilon) - M R \epsilon E - \tilde{C} R_0(\epsilon) E.
\]

The quantity on the right exceeds $\frac{R}{4} E$, if, for $0 < \epsilon < \epsilon_2$ we have $(1 - \tilde{C} \epsilon - M \epsilon) \geq 1/2$ and for $R > R_1(\epsilon)$ we have $R/4 \geq (1 + \tilde{C}) R_0(\epsilon)$.

Thus,
\[
\frac{R}{4} E - \tilde{C} E \{ R_0(\epsilon) + \epsilon R \} \leq \tilde{C} \epsilon E t_0,
\]
which yields the result for $0 < \epsilon < \epsilon_2'$ and $R > R_1'(\epsilon)$. \qed
Proof of Theorem 5.1, in the case when \( T_+((u_0, u_1)) = +\infty \). By Lemma 5.3, for \( 0 < \epsilon < \epsilon_1 \), \( R > 2R_0(\epsilon) \) we have \( t_0(R, \epsilon) \leq CR \), while by Lemma 5.5, for \( 0 < \epsilon < \epsilon_2 \), \( R > R_1(\epsilon) \), \( t_0(R, \epsilon) \geq C_0 R/\epsilon \). Hence, for \( R > \max(2R_0(\epsilon), R_1(\epsilon)) \), \( \epsilon < \min(\epsilon_1, \epsilon_2) \), \( C_0 R/\epsilon \leq CR \), which is a contradiction for \( \epsilon \) small. □

We now turn to the start of the analysis of the case \( T_+((u_0, u_1)) < +\infty \). By scaling we can assume, without loss of generality, that \( T_+((u_0, u_1)) = 1 \).

Recall, from Lemma 4.7 that

\[
\lambda(t) \geq \frac{C_0(K)}{1-t}
\]

and, from Lemma 4.8, that (after translation in \( x \)),

\[
\text{supp } u(-, t) \subset B(0, 1-t) \quad \text{and} \quad \text{supp } \partial_t u(-, t) \subset B(0, 1-t).
\]

**Lemma 5.6.** Let \( u \) be as above. Then, there is \( C_1(K) > 0 \) such that

\[
\frac{C_1(K)}{1-t} \geq \lambda(t).
\]

**Proof.** Assume not. In light of Lemma 4.6, there exist \( t_n \uparrow 1 \), such that \( \lambda(t_n)(1-t_n) \uparrow +\infty \). Consider now

\[
z(t) = \int x \nabla u \partial_t u + \left( \frac{N}{4} - \alpha \right) \int u \partial_t u, \quad 0 < \alpha < 1,
\]

which is defined for \( 0 \leq t < 1 \) (recall (5.5)).

In view of Lemma 5.3, iii), iv) we have

\[
z'(t) = -\alpha \int (\partial_t u)^2 - (1-\alpha) \left[ \int |\nabla_x u|^2 - |u|^2 \right].
\]

Because of Corollary 3.6 (\( u \neq 0 \) since \( T_+((u_0, u_1)) = 1 \)) we have \( E((u_0, u_1)) = E > 0 \), \( \sup_{0<t<1} \| (\nabla u, \partial_t u) \|_{L^2} \leq CE \) and

\[
\alpha \int (\partial_t u)^2 + (1-\alpha) \left[ \int |\nabla_x u|^2 - |u|^2 \right] \geq C_\alpha E.
\]

Then, we have

\[
z'(t) \leq -C_\alpha E, \quad 0 < t < 1.
\]

Moreover, (5.5) and Hardy’s inequality give that \( z(t) \to 0 \). Also, the assumption \( \int \nabla u_0 \cdot u_1 = 0 \) and ii) in Lemma 5.3 give that \( \int \nabla u \cdot \partial_t u = 0 \), \( 0 \leq t < 1 \).
Note that, integrating in $t$, $z(t) \geq C_\alpha E(1 - t)$. We have:

$$
\frac{z(t_n)}{(1 - t_n)} = \int \left( x + \frac{x(t_n)}{\lambda(t_n)} \right) \nabla u \partial_t u \left( 1 - t_n \right) + \left( \frac{N}{2} - \alpha \right) \frac{\int u \partial_t u}{(1 - t_n)} \geq C_\alpha E.
$$

We will show that

$$
(5.6) \quad \frac{z(t_n)}{(1 - t_n)} \to 0,
$$

yielding a contradiction. In fact, for $\epsilon > 0$ given,

$$
\frac{1}{(1 - t_n)} \int_{\left| x + \frac{x(t_n)}{\lambda(t_n)} \right| \leq \epsilon(1 - t_n)} \left| x + \frac{x(t_n)}{\lambda(t_n)} \right| |\nabla u(t_n)| |\partial_t u(t_n)| \leq C\epsilon E.
$$

Next, note that

$$
(5.7) \quad \frac{|x(t_n)|}{\lambda(t_n)} \leq 2(1 - t_n).
$$

If not, $B\left( -\frac{x(t_n)}{\lambda(t_n)} , (1 - t_n) \right) \cap B(0, 1 - t_n) = \emptyset$, so that

$$
\int_{B\left( -\frac{x(t_n)}{\lambda(t_n)} , (1 - t_n) \right)} |\nabla u(x, t_n)|^2 dx = 0,
$$

while

$$
\int_{\left| x + \frac{x(t_n)}{\lambda(t_n)} \right| \geq (1 - t_n)} |\nabla u| = \int_{\left| \lambda(t_n) x + x(t_n) \right| \geq \lambda(t_n)(1 - t_n)} |\nabla u|,
$$

by compactness of $K$, since $\lambda(t_n)(1 - t_n) \to +\infty$. But then,

$$
E\left( (u(x, t_n), \partial_t u(x, t_n)) \right) \to 0
$$

(arguing for $\partial_t u$ in a similar way) which is a contradiction to $E > 0$, and thus establishing (5.7). But then,

$$
\frac{1}{(1 - t_n)} \int_{\left| x + \frac{x(t_n)}{\lambda(t_n)} \right| \geq \epsilon(1 - t_n)} \left| x + \frac{x(t_n)}{\lambda(t_n)} \right| |\nabla u(x, t_n)| |\partial_t u(x, t_n)| dx
$$

$$
\leq 3 \int_{\left| x + \frac{x(t_n)}{\lambda(t_n)} \right| \geq \epsilon(1 - t_n)} |\nabla u(x, t_n)| |\partial_t u(x, t_n)| dx
$$

$$
\leq \frac{3}{\lambda(t_n)^{N}} \int_{|y| \geq \epsilon(1 - t_n)\lambda(t_n)} \left| \nabla u\left( \frac{y - x(t_n)}{\lambda(t_n)} , t_n \right) \right| \left| \partial_t u\left( \frac{y - x(t_n)}{\lambda(t_n)} , t_n \right) \right| dy
$$

$$
\longrightarrow 0,
$$
by compactness of $\overline{K}$, and the assumption that $\lambda(t_n)(1 - t_n) \uparrow +\infty$. This shows \cite{5.6} for the first term in $\frac{z(t_n)}{(1-t_n)}$. The second one gives the same result, using the same argument and the fact that

$$\frac{1}{(1-t_n)} \int u(t_n) |\partial_t u(t_n)| \leq \frac{1}{(1-t_n)} \int |x + \frac{x(t_n)}{\lambda(t_n)}| |u(x, t_n)| |\partial_t u(x, t_n)|,$$

and Hardy’s inequality. \hfill $\Box$

**Proposition 5.7.** Assume that $(u_0, u_1)$ is as in Theorem \ref{5.1} with $T_+(u_0, u_1) = 1$. Then supp $\nabla u, \partial_t u \subset B(0, 1 - t)$ and

$$\overline{K} = \left\{(1 - t)^\frac{N}{2} \left(\nabla u((1 - t)x(t), t), \partial_t u((1 - t)x(t), t)\right)\right\}$$

has compact closure in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

**Proof.** We first claim that

$$(1 - t)^\frac{N}{2} \left(\nabla u((1 - t)(x - x(t)), t), \partial_t u((1 - t)(x - x(t)), t)\right)$$

has compact closure in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. This is because $C_0(K) \leq (1 - t)\lambda(t) \leq C_1(K)$ and if $\overline{K}$ is compact,

$$K_1 = \left\{\lambda^\frac{N}{2} \tilde{\sigma}(\lambda x) : \tilde{v} \in \overline{K}, c_0 \leq \lambda \leq c_1\right\}$$

also has the property that $\overline{K}_1$ is compact. Next, let

$$\tilde{v}(x, t) = (1 - t)^\frac{N}{2} \left(\nabla u((1 - t)x(t), t), \partial_t u((1 - t)x(t), t)\right),$$

so that $\tilde{v}(x, t) = \tilde{v}(x + x(t), t)$, where

$$\tilde{v}(x, t) = (1 - t)^\frac{N}{2} \left(\nabla u((1 - t)(x - x(t)), t), \partial_t u((1 - t)(x - x(t)), t)\right).$$

Note that, by \cite{5.3}, supp $\tilde{v}(-t) \subset \{x : |x - x(t)| \leq 1\}$. The fact that $E > 0$ and the compactness of $\tilde{v}(x, t)$ and preservation of energy now imply that $|x(t)| \leq C$. But, if

$$K_2 = \left\{\tilde{v}(x + x_0) : \tilde{v} \in K_1, |x_0| \leq C\right\},$$

$\overline{K}_2$ is also compact and hence the Proposition follows. \hfill $\Box$

6. **Rigidity Theorem. Part 2: Self-similar variables and conclusion of the proof of the rigidity theorem**

In this section our point of departure is Proposition \ref{5.1}

For this case, in \cite{16}, we proved an extra decay estimate which allowed us to use the $L^2$ invariance and get a contradiction.

Following Merle and Zaag \cite{23}, see also \cite{11}, we will introduce self-similar variables to show that a solution as in Proposition \ref{5.1} cannot exist. Merle and Zaag considered the case of power non-linearities
\[|u|^{p-1} u \] which have \( p \leq 1 + \frac{4}{N-2} \), while here we consider the energy critical case \( p = 1 + \frac{4}{N-2} \). Nevertheless, many of the calculations in [23] also apply to our case and one can use an extra Liapunov function. We remark that a similar structure exists in the case of nonlinear heat equations, as has been used by Giga and Kohn [8] and others ([24]). Again here, we obtain some extra decay estimates which allow us to reduce to an elliptic problem with no solution.

We now set,

\[ y = x / (1 - t), \quad s = - \log(1 - t), \quad 0 \leq t < 1 \]

and define

(6.1) \[ w(y, s, 0) = (1 - t)^{\frac{N-2}{2}} u(x, t) = e^{-s} (1 - t)^{\frac{N-2}{2}} u(e^{-s} y, 1 - e^{-s}). \]

Note that \( w(y, s, 0) \) is defined for \( 0 \leq s < +\infty \), and that \( \text{supp} w(\cdot, s, 0) \subset \{|y| \leq 1\} \). We also consider, for \( \delta > 0 \), small,

\[ y = \frac{x}{1 + \delta - t}, \quad s = - \log(1 + \delta - t), \]

(6.2) \[ w(y, s, \delta) = (1 + \delta - t)^{\frac{N-2}{2}} u(x, t) = e^{-s} (1 + \delta - t)^{\frac{N-2}{2}} u(e^{-s} y, 1 + \delta - e^{-s}) \]

Note that \( w(y, s, \delta) \) is defined for \( 0 \leq s < -\log \delta \), and that

\[ \text{supp} w(-, \delta) \subset \left\{ |y| \leq \frac{e^{-s} - \delta}{e^{-s}} = \frac{(1 - t)}{(1 + \delta - t)} \leq 1 - \delta \right\} . \]

The \( w \) solve, in their domain of definition, the equation (see [23]):

\[
\partial_s^2 w = \frac{1}{\rho} \text{div} (\rho \nabla w - \rho (y, \nabla w)y) - \frac{N(N-2)}{4} w \\
+ |w|^\frac{4}{N-2} w - 2y \nabla \partial_s w - (N-1) \partial_s w,
\]

where \( \rho = (1 - |y|^2)^{-\frac{1}{2}} \).

**Lemma 6.1.** For \( \delta > 0 \) fixed, the following hold: For \( s \in [0, -\log \delta) \),

i) \( \text{supp} w(-, s, \delta) \subset \left\{ |y| \leq \frac{e^{-s} - \delta}{e^{-s}} \leq 1 - \delta \right\} \)

\( \text{supp} \partial_s w(-, s, \delta) \subset \left\{ |y| \leq \frac{e^{-s} - \delta}{e^{-s}} \leq 1 - \delta \right\} \)

ii) \( w(-, s, \delta) \in H_0^1(B_1) \) and

\[
\int |w|^2 \, dy \leq C, \quad \int |\nabla y w|^2 < \int |\nabla W|^2, \\
\int |w|^2 + \frac{|w|^2}{(1 - |y|^2)^2} \leq C, \quad \int |\partial_s w|^2 \leq C.
\]
iii) \[
\int \left( |\nabla w|^2 + |\partial_s w|^2 + |w|^2 + |w|^2 \right) \log \left( \frac{1}{1 - |y|^2} \right) dy \leq C \log \left( \frac{1}{\delta} \right).
\]

iv) \[
\int \left( |\nabla w|^2 + |\partial_s w|^2 + |w|^2 + |w|^2 \right) (1 - |y|^2)^{-1/2} dy \leq \frac{C}{\delta^{1/2}}.
\]

Proof. The first part of i) was pointed out after (6.2). For the second part, we have, using the notation in (6.2),
\[
\partial_s w(y, s, \delta) = -\frac{(N-2)}{2} e^{-s(N-2)/2} u(e^{-s}y, 1 + \delta - e^{-s})
\]
\[
+ e^{-s} e^{-s(N-2)/2} \partial_y u(e^{-s}y, 1 + \delta - e^{-s})
\]
\[
- e^{-s} e^{-s(N-2)/2} y \cdot \nabla u(e^{-s}y, 1 + \delta - e^{-s})
\]
and i) follows from (5.3).

ii) follows from the support property of \(w\), which gives \(w(-, s, \delta) \in H_0^2(B_1)\), a change of variables in \(y\) and (3.4), Sobolev embedding and Corollary 3.6 the Hardy inequality (5), for example and (6.4).

For iii), iv), note that on \(\text{supp} \ w\), \(\text{supp} \ \partial_s w\), we have \((1 - |y|^2) \geq 1 - (1 - \delta e^s)^2 = 2\delta e^s - \delta^2 e^{2s} \geq \delta\), for \(\delta\) small, \(0 \leq s < -\log \delta\).

For \(w(y, s, \delta)\), \(\delta > 0\) as above, we now define (see 23)
\[
\tilde{E}(w(s)) = \int_{B_1} \frac{1}{2} \left[ (\partial_s w)^2 + |\nabla w|^2 - (y, \nabla w)^2 \right] \]  
\[
+ \frac{N(N-2)}{8} w^2 - \frac{(N-2)}{2N} |w|^2 \}
\] \[
y \int (1 - |y|^2)^{1/2}.
\]

Proposition 6.2. Let \(w = w(y, s, \delta)\), \(\delta > 0\) be as above. Then, for \(0 < r_1 < r_2 < \log(1/\delta)\), the following identities hold:

i) \(\tilde{E}(w(s_2)) - \tilde{E}(w(s_1)) = \int_{s_1}^{s_2} \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} \] \[
\quad \frac{dy}{(1 - |y|^2)^{1/2}}.
\]

ii) \(\frac{1}{2} \left[ \int_{B_1} \left( \frac{1}{2} + N_2 \right) w^2 \right] \frac{dy}{(1 - |y|^2)^{1/2}} \] \[
\left. \right|_{s_1}^{s_2}
\]
\[
= - \int_{s_1}^{s_2} \tilde{E}(w(s)) ds + \frac{1}{N} \int_{s_1}^{s_2} \int_{B_1} \frac{|w|^2}{(1 - |y|^2)^{1/2}} \] \[
+ \int_{s_1}^{s_2} \int_{B_1} \left\{ (\partial_s w)^2 + \partial_s w, \nabla w + \frac{\partial_s w}{(1 - |y|^2)^{1/2}} \right\} \] \[
\frac{dy}{(1 - |y|^2)^{1/2}}.
\]
iii) \[ \lim_{s \to \log(\frac{1}{2})} \tilde{E}(w(s)) \leq E = E(u_0, u_1). \]

Proof. For i) see the proof of Lemma 2.1 in [23]. For ii), see the proof of (11) in [23]. We turn to the proof of iii). We analyze term by term, using the notation in (6.2).

\[
\int_{B_1} \frac{w^2}{(1 - |y|^2)^{1/2}} dy = \int_{|y|<(1-t)/(1+\delta-t)} (1 + \delta - t)^N |u((1 + \delta - t)y, t)|^2 \frac{dy}{(1 - |y|^2)^{1/2}} \leq C \int_{|x|<1-t} (1 + \delta - t)^{-2} |u(x, t)|^2 \frac{dx}{\delta^{1/2}} \leq \frac{C}{\delta^{1/2}(1 + \delta - t)^2} \left( \int_{|x|<(1-t)} |u(x, t)|^{2^*} dx \right)^{2/2^*} (1 - t)^{2/N} \to 0. 
\]

Recall that \[ |y|^2 = \frac{|x|^2}{(1+\delta-t)^2}, \] and assume that \[ 1 - \epsilon \delta \leq t \leq 1. \] Then, we have \[ \frac{1}{(1+\epsilon)^2} \leq (1 - |y|^2)^{1/2} \leq 1, \] since \[ |x| \leq (1 - t). \] Thus,

\[
\int_{B_1} \frac{|w|^{2^*}}{(1 - |y|^2)^{1/2}} dy \geq \int_{|x|<1-t} |u(x, t)|^{2^*} dx,
\]

and a similar computation gives that

\[
\int_{B_1} \frac{|
abla w|^2}{(1 - |y|^2)^{1/2}} dy \leq \frac{1}{(1 + \epsilon)^{1/2}} \int_{|x|\leq1-t} |
abla u|^2 dx.
\]

Also,

\[
\int_{B_1} (y \cdot \nabla w)^2 \frac{dy}{(1 - |y|^2)^{1/2}} = \int_{|x|\leq(1-t)} |x \cdot \nabla_x u(x, t)|^2 \frac{dx}{(1 + \delta - t)^2} \cdot \frac{dx}{(1 - |y|^2)^{1/2}} \leq \frac{1}{(1 + \epsilon)} \int_{|x|\leq(1-t)} |
abla_x u(x, t)|^2 dx \frac{|1 - t|^2}{(1 + \delta - t)^2} \to 0.
\]
With these computations and (6.4) we see that
\[
\lim_{t \to 1} \frac{1}{2} \int |\partial_t w|^2 \frac{dy}{(1 - |y|^2)^{1/2}} = \frac{1}{2} \int |\partial_t u|^2 dx,
\]
which combined with the previous calculations yields iii). \[\square\]

**Corollary 6.3.** For \( s \in [0, \log(\frac{1}{\delta})] \), we have
\[
-C/\delta^{3/2} \leq \tilde{E}(w(s)) \leq E.
\]

**Proof.** The first statement follows from Proposition 6.2 i), iii), while the second one follows from Lemma 6.1, iv) and (6.5). \[\square\]

Using space-time estimates, we now obtain our first improvement of the space decay of \( w \).

**Lemma 6.4.** For \( \delta > 0 \), we have
\[
\int_0^1 \int \frac{|\partial_s w|^2}{(1 - |y|^2)} dy ds \leq C \log(\frac{1}{\delta}).
\]

**Proof.** We start out with the readily verified identity
\[
\frac{d}{ds} \left\{ \int \left[ \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} (|\nabla w|^2 - (y, \nabla w)^2) \right] \right\} - \frac{(N - 2)N}{2N} |w|^2 \left( -\log(1 - |y|^2) \right) dy
\]
\[
+ \int \left[ \log(1 - |y|^2) + 2 \right] y, \nabla w \partial_s w - \int \log(1 - |y|^2) (\partial_s w)^2 - 2 \int (\partial_s w)^2
\]
\[
= -2 \int \frac{(\partial_s w)^2}{(1 - |y|^2)}.
\]

We now integrate between 0 and 1, change signs. In the estimate of the left hand side, we can drop the term \( \int \log(1 - |y|^2) (\partial_s w)^2 \) since it is negative. The \( \frac{d}{ds} \) term, and the \( \int_0^1 (\partial_s w)^2 \) term are controlled by Lemma 6.1 (using \( -\log(1 - |y|^2) \leq C \log(\frac{1}{\delta}) \)). It remains to bound
\[
\left| \int_0^1 \int \left[ \log(1 - |y|^2) + 2 \right] y, \nabla w \partial_s w dy ds \right| \leq \left( \int_0^1 \int \frac{|\partial_s w|^2}{(1 - |y|^2)} \right)^{1/2}
\]
\[
\times \left( \int_0^1 \int (1 - |y|^2) \left| \log(1 - |y|^2) + 2 \right|^2 |\nabla w|^2 dy ds \right)^{1/2}.
\]

The second factor is bounded because of Lemma 6.1 ii). The proof is concluded by using \( ab \leq \epsilon a^2 + \frac{1}{\epsilon} b^2 \). \[\square\]

**Lemma 6.5.** For \( \delta > 0 \), we have
i) \( \int_0^1 \int_{B_1} \frac{|w|^2}{(1-|y|^2)^{1/2}} dy ds \leq C \left( \log \left( \frac{1}{\delta} \right) \right)^{1/2}, \)

ii) \( \tilde{E}(w(1)) \geq -C \left| \log \left( \frac{1}{\delta} \right) \right|^{1/2} \)

**Proof.** We will use Proposition 6.2 ii) to handle i). We have

\[
\frac{1}{N} \int_0^1 \int_{B_1} \frac{|w|^2}{(1-|y|^2)^{1/2}} dy ds = \frac{1}{2} \left[ \int_{B_1} (\partial_s w w - \frac{(1+N)w^2}{2} ) \frac{dy}{(1-|y|^2)^{1/2}} \right]_0^1 + \int_0^1 \tilde{E}(w(s)) ds - \int_0^1 \int_{B_1} \left\{ (\partial_s w)^2 + \partial_s wy \nabla w + \partial_s w \frac{w |y|^2}{(1-|y|^2)^2} \right\} \frac{dy ds}{(1-|y|^2)^{1/2}}.
\]

By Proposition 6.2 i) and iii), the second term on the right hand side is bounded by \( E \). The first term on the right hand side is bounded using Lemma 6.1 ii) and Cauchy-Schwarz. For the third term, because of the sign, we only need to consider the last two summands, which are bounded in absolute value by

\[
\left| \int_0^1 \int_{B_1} \frac{\partial_s w}{(1-|y|^2)^{1/2}} \left( \frac{|w|}{(1-|y|^2)} + |\nabla w| \right) dy ds \right| 
\leq 2 \left( \int_0^1 \int_{B_1} \frac{|\partial_s w|^2}{(1-|y|^2)^2} dy ds \right)^{1/2} \left( \int_0^1 \int_{B_1} \frac{w^2}{(1-|y|^2)^2} + |\nabla w|^2 dy ds \right)^{1/2} 
\leq C \left( \log \left( \frac{1}{\delta} \right) \right)^{1/2},
\]

because of Lemma 6.1 ii) and Lemma 6.4. This establishes i).

To prove ii), we first consider \( \int_0^1 \tilde{E}(w(s)) ds \), which is bounded from below by \( -C \left( \log \left( \frac{1}{\delta} \right) \right)^{1/2} \), by i). The monotonicity of \( \tilde{E} \) (i) in Proposition 6.2 concludes the proof of ii). \( \square \)

We now obtain our second improvement of decay on \( w \).

**Lemma 6.6.** For \( \delta > 0 \), we have

\[
\int_1^\left( \log \left( \frac{1}{\delta} \right) \right)^{3/4} \int_{B_1} (\partial_s w)^2 \frac{dy ds}{(1-|y|^2)^{3/2}} \leq C \left( \log \left( \frac{1}{\delta} \right) \right)^{1/2}.
\]
Proof. Because of i) in Proposition 6.2 we have:
\[
\int_{1}^{\left(\log\left(\frac{1}{\delta}\right)\right)^{3/4}} \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} dy \, ds = \mathcal{E}(w) \leq \mathcal{E}(1) + C \left(\log\left(\frac{1}{\delta}\right)\right)^{1/2},
\]
where we have used Corollary 6.3 and Lemma 6.5 ii). \qed

**Corollary 6.7.** For each \( \delta > 0 \), there exists \( \bar{\delta}_\delta \in (1, (\log(\frac{1}{\delta}))^{3/4}) \) such that
\[
\int_{\bar{\delta}_\delta}^{(\log(\frac{1}{\delta}))^{1/8}} \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} dy \, ds \leq \frac{2C}{\left(\log\left(\frac{1}{\delta}\right)\right)^{1/8}}.
\]

**Proof.** Split the interval \((1, (\log(\frac{1}{\delta}))^{3/4})\) into disjoint intervals of length \((\log(\frac{1}{\delta}))^{-1/8}\). The number of such intervals is of the order of \((\log(1/\delta))^{-5/8}\).
For at least one such interval \((\bar{\delta}_\delta, \bar{\delta}_\delta + (\log(\frac{1}{\delta}))^{-1/8})\), with \(\bar{\delta}_\delta \in (1, (\log(\frac{1}{\delta}))^{3/4})\), we must have
\[
\int_{\bar{\delta}_\delta}^{\bar{\delta}_\delta + (\log(\frac{1}{\delta}))^{1/8}} \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} dy \, ds \leq \frac{2C}{(\log(\frac{1}{\delta}))^{1/8}},
\]
where \(C\) is the constant in Lemma 6.6 which proves the Corollary. \qed

**Remark 6.8.** Let \( \bar{\delta}_\delta = -\log(1 + \delta - \bar{\delta}_\delta) \). Note that
\[
\left| \frac{(1 - \bar{\delta}_\delta)}{1 + \delta - \bar{\delta}_\delta} - 1 \right| = \frac{\delta}{(1 + \delta - \bar{\delta}_\delta)} = \frac{\delta e^{-\bar{\delta}_\delta}}{\delta^{1/4}} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.
\]

Let us now reduce the time evolution problem to a stationary problem in the \( w \) variable (i.e. self-similar solutions). Pick \( \delta_j \downarrow 0 \), so that
\[
\left( (1 - \bar{\delta}_j) \frac{\nabla u}{(1 - \bar{\delta}_j)y, \bar{\delta}_j) \right), (1 - \bar{\delta}_j) \frac{\partial_t u}{(1 - \bar{\delta}_j)y, \bar{\delta}_j) \right) \rightarrow (\nabla u_0^*, u_1^*)
\]
in \( L^2 \). This is possible by Proposition 5.7. Note that, because of Remark 6.8 and the compact closure of \( \mathcal{K}^2 \) in Proposition 5.7, we also have that
\[
\left( (1 + \delta_j - \bar{\delta}_j) \frac{\nabla u}{(1 + \delta_j - \bar{\delta}_j)y, \bar{\delta}_j) \right), (1 + \delta_j - \bar{\delta}_j) \frac{\partial_t u}{(1 + \delta_j - \bar{\delta}_j)y, \bar{\delta}_j) \right) \rightarrow (\nabla u_0^*, u_1^*) \text{ in } L^2.
\]

Let now \( u_j^*, u^* \) be solutions of (CP) with data
\[
\left( (1 + \delta_j - \bar{\delta}_j) \frac{(N-2)}{2} u((1 + \delta_j - \bar{\delta}_j)y, \bar{\delta}_j) \right), (1 + \delta_j - \bar{\delta}_j) \frac{\nabla u}{(1 + \delta_j - \bar{\delta}_j)y, \bar{\delta}_j) \right)
\]
and \((u^*_j, u^*_1)\) respectively, in a time interval \([0, T^*]\), independent of \(j\), which we take to have \(T^* < 1\). By uniqueness in the (CP), we have

\[
u^*_j(y, \tau) = (1 + \delta_j - \overline{\tau}_\delta_j)^{\frac{(N-2)}{2}} u((1 + \delta_j - \overline{\tau}_\delta_j)y, \overline{\tau}_\delta_j + (1 + \delta_j - \overline{\tau}_\delta_j)\tau).
\]

Note that, \(\text{supp} u^*_j(-, \tau) \subset \{(1 + \delta_j - \overline{\tau}_\delta_j)y \leq 1 - \overline{\tau}_\delta_j - (1+\delta_j-\overline{\tau}_\delta_j)\tau\}\) and hence \(|y| \leq \frac{1-\overline{\tau}_\delta_j}{1+\delta_j-\overline{\tau}_\delta_j} - \tau < 1 - \tau\) on the support of \(u^*_j(-, \tau)\). Similarly,

\[
\text{supp } \partial_\tau u^*_j(-, \tau) \subset \left\{y : |y| \leq \frac{(1-\overline{\tau}_\delta_j)}{(1+\delta_j-\overline{\tau}_\delta_j)} - \tau < 1 - \tau\right\}.
\]

Let us compare the solutions in the self-similar variables. Recall from (6.2), that if \(s = -\log(1 + \delta_j - t)\), then

\[
w(y, s, \delta_j) = (1 + \delta_j - t)^{\frac{(N-2)}{2}} u((1 + \delta_j - t)y, t).
\]

Define now \(\tau\) by \(t = \overline{\tau}_\delta_j + (1 + \delta_j - \overline{\tau}_\delta_j)\tau\), so that \((1 + \delta_j - t) = (1 + \delta_j - \overline{\tau}_\delta_j)(1 - \tau)\). Define also \(s = -\log((1 + \delta_j - \overline{\tau}_\delta_j)(1 - \tau))\). We then have

\[
w(y, s, \delta_j) = [(1 + \delta_j - \overline{\tau}_\delta_j)(1 - \tau)]^{\frac{(N-2)}{2}} u((1 + \delta_j - \overline{\tau}_\delta_j)(1 - \tau)y, \overline{\tau}_\delta_j + (1 + \delta_j - \overline{\tau}_\delta_j)\tau).
\]

If we now set

\[
s' = -\log(1 - \tau), y' = \frac{y}{(1 - \tau)} \quad \text{and} \quad w^*_j(y', s') = (1-\tau)^{\frac{(N-2)}{2}} u^*_j(y, \tau),
\]

then \(w^*_j\) is a solution of (6.3), for \(0 < \tau < T^*\). But, because of (6.6), (6.7),

\[
w^*_j(y', s') = (1 - \tau)^{\frac{(N-2)}{2}} (1 + \delta_j - \overline{\tau}_\delta_j)^{\frac{(N-2)}{2}} u((1 + \delta_j - \overline{\tau}_\delta_j)y, \overline{\tau}_\delta_j + (1 + \delta_j - \overline{\tau}_\delta_j)\tau)
\]

\[
= w(y', s, \delta_j),
\]

where \(s = -\log(1 + \delta_j - t) = -\log((1 + \delta_j - \overline{\tau}_\delta_j)(1 - \tau)) = -\log(1 + \delta_j - \overline{\tau}_\delta_j) - \log(1 - \tau) = \overline{\tau}_\delta_j + s', \ i.e.,
\]

\[
w^*_j(y', s') = w(y', \overline{\tau}_\delta_j + s', \delta_j).
\]

Consider also,

\[
w^*(y', s') = (1 - \tau)^{\frac{(N-2)}{2}} u^*(y, \tau).
\]

We clearly have \(\text{supp } u^*(-, \tau) \subset \{|y| \leq (1 - \tau)\}\) and \(w^*\) solves (6.3) for \(0 < \tau < T^*\). Also, recall that \((u^*_j(-, \tau), \partial_\tau u^*_j(-, \tau)) \to (u^*(-, \tau), \partial_\tau u^*(-, \tau))\)
in $\dot{H}^1 \times L^2$, uniformly for $t \in [0, T^*]$, by continuity in (CP). But then if $0 \leq t \leq T^*/2 = \bar{T}$ and $0 \leq s' \leq -\log(1 - \bar{T})$, we have that

$$\left( w^*_{s'}(-, s'), \partial_{s'} w^*_{s'}(-, s') \right) \xrightarrow{j \to \infty} \left( w^*(-, s'), \partial_{s'} w^*(-, s') \right) \in \dot{H}^1 \times L^2$$

uniformly for $0 \leq s' \leq -\log(1 - \bar{T})$. But, by (6.8), we have:

$$(6.9) \quad \left( w(y', \overline{s}_j + s', \delta_j), \partial_{s'} w(y', \overline{s}_j + s', \delta_j) \right) \xrightarrow{j \to \infty} \left( w^*(-, s'), \partial_{s'} w^*(-, s') \right),$$

in $\dot{H}^1 \times L^2$, uniformly in $0 \leq s' \leq -\log(1 - \bar{T})$ and $w^*$ is a solution of (6.3) and $\text{supp}(w^*(-, s'), \partial_{s'} w^*(-, s')) \subset \{|y| \leq 1\}$.

**Lemma 6.9.** Let $w^*$ be as above. Then,

$$w^*(y', s') = w^*(y') \text{ and } w^* \not= 0.$$

**Proof.** Let $S = -\log(1 - \bar{T})$ and choose $j$ large. Then

$$\int_0^S \int_{B_1} \frac{(\partial_{s'} w^*(y', s'))^2}{(1 - |y'|^2)^{3/2}} dy'ds' \leq \lim_{j \to \infty} \int_0^S \int_{B_1} \frac{(\partial_{s'} w(y', \overline{s}_j + s', \delta_j))^2}{(1 - |y'|^2)^{3/2}} dy'ds'$$

by (6.9). The right hand side is bounded by

$$\lim_{j \to \infty} \int_{B_1} \frac{\left( \partial_{s'} w(y', s', \delta_j) \right)^2}{(1 - |y'|^2)^{3/2}} dy'ds' \leq 2C \lim_{j \to \infty} 1/(\log(1/\delta_j))^{1/8} = 0,$$

by Corollary 6.4. This shows that $w^*(y', s') = w^*(y')$.

To show that $w^* \not= 0$, assume $w^* \equiv 0$. Then, by (6.8) and (6.9), we would have $\nabla_{y'} w(y', \overline{s}_j, \delta_j) \rightarrow 0$ in $L^2(\mathbb{R}^N)$, so that $\left( 1 + \delta_j - \overline{t}_{\delta_j} \right) \frac{N}{2} \nabla_y u((1 + \delta_j - \overline{t}_{\delta_j})y) \rightarrow 0$ in $L^2(\mathbb{R}^N)$. Because of Corollary 3.8, we have, for $0 < t < 1$, $\int_{B_1} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 dx \geq CE > 0$.

But,

$$\int_{B_1} |\nabla u(x, \overline{t}_{\delta_j})|^2 dx = \int |(1 + \delta_j - \overline{t}_{\delta_j}) \frac{N}{2} \nabla_y u((1 + \delta_j - \overline{t}_{\delta_j})y, \overline{t}_{\delta_j})|^2 dy \rightarrow 0,$$

so for $j$ large we obtain

$$\int_{B_1} |\partial_t u(x, \overline{t}_{\delta_j})|^2 dx \geq CE/2.$$
gives
\[
\partial_s w(y', \sigma_{\delta_j}, \delta_j) = -\frac{(N-2)}{2}(1+\delta_j-\tau_{\delta_j})^{(N-2)/2} u((1+\delta_j-\tau_{\delta_j})y', \tau_{\delta_j}) + \\
+ (1+\delta_j-\tau_{\delta_j})^2 \partial_t u((1+\delta_j-\tau_{\delta_j})y', \tau_{\delta_j}) - \\
- (1+\delta_j-\tau_{\delta_j})^2 \partial_y \nabla u((1+\delta_j-\tau_{\delta_j})y', \tau_{\delta_j}).
\]

From our assumption, we see that, since \(|y'| \leq 1\), the \(L^2\) norm of the last term goes to 0. The same can be said for the \(L^2\) norm of the first term, by Sobolev embedding. But this contradicts (6.10), so that \(w^* \not\equiv 0\). □

**Proposition 6.10.** Let \(w^*\) be as above. Then, \(w^* \in H^1_0(B_1)\),
\[
\int_{B_1} \frac{|w^*(y)|^2}{(1-|y|^2)^{1/2}} < \infty \quad \text{and} \quad w^* \text{ solves the (degenerate) elliptic equation}
\]
(6.11) \[
\frac{1}{\rho} \text{div}(\rho \nabla w^* - \rho(y, \nabla w^*)y) - \frac{N(N-2)}{4} w^* + |w^*|^{\frac{4}{N-2}} w^* = 0,
\]
where \(\rho(y) = (1-|y|^2)^{-1/2}\).
Moreover, \(w^* \not\equiv 0\) and
(6.12) \[
\int \frac{|w^*(y)|^2}{(1-|y|^2)^{1/2}} dy + \int \frac{[|\nabla w^*(y)|^2 - (y, \nabla w^*(y))^2]}{(1-|y|^2)^{1/2}} dy < +\infty,
\]

**Remark 6.11.** We will see that (6.12) are the critical estimates which allow us to conclude the proof.

**Proof.** It only remains to prove (6.12). Because of (6.9) and Lemma 6.5 to bound the first term in (6.12) it suffices to show that
\[
\int_{\tau_{\delta_j}}^{\sigma_{\delta_j} + S} \int_{B_1} \frac{|w(y', s', \delta_j)|^2}{(1-|y'|^2)^{1/2}} dy' ds' \leq C,
\]
where \(C\) is independent of \(j\). In order to show this, we use ii) in Proposition 6.2 so that
\[
\frac{1}{N} \int_{\tau_{\delta_j}}^{\sigma_{\delta_j} + S} \int_{B_1} \frac{|w(y', s', \delta_j)|^2}{(1-|y'|^2)^{1/2}} dy' ds' = \int_{\tau_{\delta_j}}^{\sigma_{\delta_j} + S} \tilde{E}(w(s')) ds'
\]
\[
+ \frac{1}{2} \left[ \int_{B_1} \left( \partial_s w w - \frac{1+N}{2} w^2 \right) (1-|y'|^2)^{1/2} \right]_{\tau_{\delta_j}}^{\sigma_{\delta_j} + S}
\]
\[
- \int_{\tau_{\delta_j}}^{\sigma_{\delta_j} + S} \int_{B_1} \left\{ (\partial_s w)^2 + \partial_s w y' \nabla w + \frac{w \partial_s w |y'|^2}{(1-|y'|^2)} \right\} (1-|y'|^2)^{1/2} dy'.
\]
The first term of the right hand side is bounded by Corollary 6.3, the second one by Lemma 6.1 ii). To bound the last one we only need to
estimate the last two summands. To bound the last summand, we use Cauchy-Schwarz to bound it by
\[
\left(\int_{\delta_j}^{s_j+S} \int_{B_1} \frac{w^2}{1-|y'|^2} dy' ds' \right)^{1/2} \left(\int_{\delta_j}^{s_j+S} \int_{B_1} \frac{|\partial_s w|^2}{1-|y'|^2} dy' ds' \right)^{1/2}
\leq C (\log(1/\delta_j))^{-1/16},
\]
by Lemma 6.1 ii) and Corollary 6.7.

To bound the second summand we use Cauchy-Schwarz to bound it by
\[
\left(\int_{\delta_j}^{s_j+S} \int_{B_1} |\nabla w|^2 dy' ds' \right)^{1/2} \left(\int_{\delta_j}^{s_j+S} \int_{B_1} \frac{|\partial_s w|^2}{1-|y'|^2} dy' ds' \right)^{1/2},
\]
which can be estimated similarly. This proves the first estimate. To prove the gradient estimate, use Corollary 6.3 and the previous proof to conclude that
\[
\int_{\delta_j}^{s_j+S} \int_{B_1} \left\{ |\nabla w(y', s', \delta_j)|^2 - (y'.\nabla w(y', s', \delta_j))^2 \right\} \frac{dy'}{(1-|y'|^2)^{1/2}} \leq C.
\]
Using (6.9) and Lemma 6.9 this leads to
\[
\int_{B_1} \left\{ |\nabla w^*|^2 - (y'.\nabla w^*)^2 \right\} \frac{dy'}{(1-|y'|^2)^{1/2}} \leq C,
\]
which concludes the proof. \(\square\)

The contradiction which finishes the proof of Theorem 5.1 is then provided by the following elliptic result:

**Proposition 6.12.** Assume that \(w \in H_0^1(B_1)\), is such that
\[\]
i) \(\int \frac{|w(y)|^2}{(1-|y|^2)^2} dy < \infty\) (a consequence of \(w \in H_0^1(B_1)\))
\[\]
ii) \(\int \frac{|w(y)|^2}{(1-|y|^2)^{1/2}} dy + \int \frac{|\nabla w(y)|^2 - (y.\nabla w(y))^2}{(1-|y|^2)^{1/2}} dy < \infty\)
\[\]
iii) \(w\) verifies the (degenerate) elliptic equation (6.11).

Then, \(w \equiv 0\).

**Proof.** We write again the equation (6.11), with \(\rho = (1-|y|^2)^{-1/2}\):
\[
(6.13) \quad \frac{1}{\rho} \text{div}(\rho \nabla w - \rho(y.\nabla w)y) - \frac{N(N-2)}{4} w + |w|^{d-2} w = 0.
\]
Consider first the linear part
\[
Lw = \frac{1}{\rho} \text{div}(\rho \nabla w - \rho(y.\nabla w)y) = \frac{1}{\rho} \text{div}(\rho(I - y \otimes y) \nabla w).
\]
For \( |y| < 1 - \delta, \delta > 0 \), \( L \) is a second order elliptic operator with smooth coefficients. Thus, the well-known argument of Trudinger \cite{trudinger1} shows that \( w \in L^\infty(B_{1-\delta}) \) and hence \( w \in C^2(B_{1-\delta}) \), where \( B_{1-\delta} = \{ y : |y| < 1 - \delta \} \), for each \( \delta > 0 \). From this and the classical unique continuation theorem of Aronszajn, Krzywicki and Szarski (see \cite{aronszajn} and \cite{kryzwicki}, Section 17.2) we see that if \( w \equiv 0 \) on \( 1 - \delta < |y| < 1 \), we will have \( w \equiv 0 \).

In order to establish this for \( 1 - \delta < |y| < 1 \), it is convenient to write our equation in polar coordinates \((r, \theta), 0 < r < \infty, \theta \in S^{N-1}. \) In these coordinates, \((6.13)\) becomes:

\[
(1 - r^2)^{1/2} \frac{\partial}{\partial r} (1 - r^2)^{1/2} \frac{\partial w}{\partial r} + \frac{1}{r^2} \Delta_\theta w + \frac{(N-1)}{r} (1 - r^2) \frac{\partial w}{\partial r} = \frac{N(N-2)}{4} w - |w|^{\frac{4}{N-2}} w,
\]

(6.14)

where \( \Delta_\theta \) denotes the spherical Laplacian on \( S^{N-1}. \)

For \( 1 - \delta < r < 1 \), we perform the change of variables \( v(s, \theta) = w(r(s), \theta), \) with \( r(s) = 1 - \frac{(1-s)^2}{4}. \) For suitable \( \delta, \) we have \( 1 - \tilde{\delta} \leq s \leq 1, \)

when \( 1 - \delta \leq r \leq 1. \) Also, \( r'(s) = \frac{1-\delta}{2}, \) \( \frac{r'(s)}{(1-r(s))^{1/2}} = 1. \) Since

\[
(1 + r(s))^{1/2} \frac{\partial}{\partial s} v(s, \theta) = (1 - r^2(s))^{1/2} \frac{\partial w}{\partial r} (r(s), \theta),
\]

\( v \) verifies the equation

\[
\frac{\partial}{\partial s} (1 + r(s))^{1/2} \frac{\partial v}{\partial s} + \frac{1}{(1 + r(s))^{1/2} r(s)^2} \Delta_\theta v
\]

\[
+ \frac{(N-1)}{r(s)} (1 - r(s)^2)^{1/2} \frac{\partial v}{\partial s} = \frac{N(N-2)}{4 (1 + r(s))^{1/2}} \frac{4}{1 + r(s)^4} v - \frac{|v|^{\frac{4}{N-2}} v}{(1 + r(s))^{1/2}}.
\]

(6.15)

The advantage of (6.15) is that it is elliptic, not degenerate elliptic, near \( s = 1 \) (or \( r = 1 \)). Moreover, since \( (1 + r(s)) \) is bounded above and below and smooth, the coefficients in (6.15) are smooth. We now turn to some estimates for \( v, \) for \( 1 - \tilde{\delta} \leq s \leq 1, \theta \in S^{N-1}. \)

We first claim that

\[
\int_{1-\delta < s < 1} \int_{S^{N-1}} |v(s, \theta)|^2 s d\theta < \infty.
\]

(6.16)

In fact, the integral in (6.16) equals \( \int_{1 - \delta < r < 1} \int_{S^{N-1}} |w(s, \theta)|^2 r(1-r)^{1/2} d\theta d\theta, \) which is finite by virtue of ii).

Next, we notice that, for \( 1 - \delta \leq |y| \leq 1, \)

\[
|\nabla_\theta w(y)| \simeq \left| \nabla w - \left( \frac{y}{|y|} \cdot \nabla w \right) \frac{y}{|y|} \right|
\]
\[
|\nabla w|^2 - (y, \nabla w)^2 = \left( \frac{1}{|y|^2} - 1 \right) (y, \nabla w)^2 + \left| \nabla w - \frac{y}{|y|} \nabla w \frac{y}{|y|} \right|^2 = (1 - |y|^2) \left( \frac{y}{|y|}, \nabla w \right)^2 + \left| \nabla w - \frac{y}{|y|} \nabla w \frac{y}{|y|} \right|^2.
\]

Thus, since \( w \in H_0^1(B_1) \), ii) holds, we see that
\[
\int_{1-\delta<r<1} \int_{S^{N-1}} |\nabla_\theta w(s, \theta)|^2 \frac{dr \, d\theta}{(1-r)^{1/2}} < \infty
\]
and hence
\[(6.17) \quad \int_{1-\delta<s<1} \int_{S^{N-1}} |\nabla_\theta v(s, \theta)|^2 \, ds \, d\theta < \infty.
\]

Next, we show that
\[(6.18) \quad \int_{1-\tilde{\delta}<s<1} \int_{S^{N-1}} \left| \frac{\partial v}{\partial s}(s, \theta) \right|^2 \frac{ds \, d\theta}{(1-s)} < \infty.
\]

This estimate, combined with \( v \in H_0^{1,2}(B_1) \) is the one that forces \( v \) to vanish, since it means that the Cauchy data for the solution \( v \) of (6.15) vanishes. This is a consequence of the fact that \( w \in H_0^1(B_1) \) and the degeneracy of (6.13). On the other hand, (6.16) and (6.17) show that we are dealing with a “standard solution” to (6.15). To obtain (6.18), change variables. The integral equals
\[
\int_{1-\delta<s<1} \int_{S^{N-1}} \left| \frac{\partial w}{\partial r}(r(s), \theta) \right|^2 \frac{|r'(s)|^2}{(1-s)} \, ds \, d\theta
\]
\[
= \int_{1-\delta<s<1} \int_{S^{N-1}} \left| \frac{\partial w}{\partial r}(r(s), \theta) \right|^2 \frac{|r'(s)|^2}{2} \, ds \, d\theta
\]
\[
= \int_{1-\delta<r<1} \int \left| \frac{\partial w}{\partial r}(r, \theta) \right|^2 \frac{dr}{2} \, d\theta \leq C \int_{1-\delta<r<1} \int \left| \frac{\partial w}{\partial r}(r, \theta) \right|^2 \, dr \, d\theta.
\]

Finally, a similar argument, using i) shows that
\[(6.19) \quad \int_{1-\tilde{\delta}<s<1} \int \left| v(s, \theta) \right|^2 \frac{ds \, d\theta}{(1-s)^3} < \infty.
\]

Once we have the estimates (6.16), (6.17), (6.18) and (6.19), we define
\[(6.20) \quad \tilde{v}(s, \theta) = \begin{cases} 
v(s, \theta) & 1 - \tilde{\delta} < s < 1, \\
0 & 1 < s < 2. \end{cases}
\]
Since $v(s,\theta) \in H^1(ds\,d\theta)$, for $1 - \tilde{\delta} < s < 1$, in light of (6.17), (6.18) and (6.19), $\tilde{v} \in H^1(ds\,d\theta)$, $1 - \tilde{\delta} < s < 2$, $\theta \in S^{N-1}$. We claim that $\tilde{v}$ solves (6.15) for $1 - \tilde{\delta} < s < 2$: to show this, let $\eta(s,\theta)$ be a test function. Let $\mu_\epsilon(s)$ be a smooth approximation of the characteristic function of $s < 1$. We have to show that,

$$\int \int (1 + r(s))^{1/2} \frac{\partial \tilde{v}}{\partial s} \frac{\partial \eta}{\partial s} = \lim_{\epsilon \to 0} \int \int (1 + r(s))^{1/2} \frac{\partial v}{\partial s} \frac{\partial}{\partial s} (\eta \mu_\epsilon).$$

But, this reduces to showing that

$$\lim_{\epsilon \to 0} \left| \int \int \eta (1 + r(s))^{1/2} \frac{\partial v}{\partial s} \frac{\partial}{\partial s} \mu_\epsilon \right| \leq C \int_{1-2\epsilon < s < 1-\epsilon} |\eta| (1 + r(s))^{1/2} \left| \frac{\partial v}{\partial s} \right| ds d\theta \to 0,$$

because of (6.18). We can now apply Trudinger’s argument in the critical case [39] to $\tilde{v}$, to show that $\tilde{v} \in C^2((1 - \tilde{\delta} < s < 2) \times S^{N-1})$. Once we have this, $\tilde{v} \equiv 0$ on $1 - \tilde{\delta} < s < 2$, because of the fact that $\tilde{v} \equiv 0$ for $1 < s < 2$ and the unique continuation theorem of [1]. (See also [13], Section 17.2.) From this, we conclude that $w \equiv 0$, as desired. \[\Box\]

Remark 6.13. One can skip the use of Trudinger’s argument in [39] and use directly the more delicate unique continuation theorem of [14], or rather its variable coefficient version, due to T. Wolff ([40]).

Remark 6.14. For this part of the argument, no size or energy conditions are needed. In addition, in the radial case, Lemma 6.1 and one dimensional Sobolev inequalities give that $\tilde{E}(w(0))$ is bounded in absolute value, which allows us to reduce directly to the elliptic problem.

The results in this section yield the contradiction which completes the proof of Theorem 5.1.

7. Main Theorem

In this section we establish our main result (see [25] and [27] for the subcritical case, where energy controls yield the result).

Theorem 7.1. Let $(u_0, u_1) \in \dot{H}^1 \times L^2$, $3 \leq N \leq 5$. Assume that $E((u_0, u_1)) < E((W, 0))$. Let $u$ be the corresponding solution of the Cauchy problem, with maximal interval of existence $I = (-T_-(u_0, u_1), T_+(u_0, u_1))$. (See Definition 6.1.) Then:
If \( \int |\nabla u_0|^2 < \int |\nabla W|^2 \), then

\[
I = (-\infty, +\infty) \text{ and } \|u\|_{L_{q(t)}^{2(N+1)\over N-2}} < \infty.
\]

\( \text{Remark 7.2. } \int |\nabla u_0|^2 = \int |\nabla W|^2 \) is incompatible with the energy condition from (3.2). (Indeed, in this case \( E((u_0, u_1)) \geq E((W, 0)) \)).

To establish i) we argue by contradiction. If not, \( E_C \), defined in Section 4, must satisfy \( \eta_0 \leq E_C < E((W, 0)) \). Let \( u_C \) be as in Proposition 4.2 and assume that \( I_+ \) is finite. Then, by Proposition 4.10

\[
\int \nabla u_{0,C}u_{1,C} = 0.
\]

But then we reach a contradiction from Theorem 5.1. If \( I_+ \) is infinite, and \( \lambda(t) \geq A_0 > 0 \), Proposition 4.11 shows that \( \int \nabla u_{0,C}u_{1,C} = 0 \) and Theorem 5.1 gives \( u_C \equiv 0 \), a contradiction because \( E((u_C, \partial_t u_C)) = E_C \geq \eta_0 \).

To conclude the proof, we need to reduce to case \( \lambda(t) > A_0 > 0 \), for \( t \geq 0 \), using the argument in the proof of Theorem 5.1 of [16] (also see [22] for a similar proof). Recall that \( E((u_C, \partial_t u_C)) = E_C \geq \eta_0 > 0 \). Because of Lemma 4.6 we can assume that there exist \( t_n \uparrow +\infty \) so that \( \lambda(t_n) \to 0 \). After possibly redefining \( \{t_n\}_{n=1}^\infty \), we can assume that

\[
\lambda(t_n) \leq \inf_{t \in [0, t_n]} \lambda(t).
\]

From Proposition 4.2

\[
(w_{0,n}(x), w_{1,n}(x)) = \left( {1 \over \lambda(t_n)^{N \over 2}} u_C \left( x - x(t_n) \over \lambda(t_n) \right), t_n \right),
\]

\[
\int \nabla u_C \left( x - x(t_n) \over \lambda(t_n), t_n \right) \to (w_0, w_1) \text{ in } H^1 \times L^2.
\]

Note that \( E((w_0, w_1)) = E_C \). Moreover, \( \int |\nabla w_0|^2 < \int |\nabla W|^2 \), by the corresponding properties of \( u_C \) and Theorem 3.5. Let \( w_0(x, \tau), \tau \in (-T_-(w_0, w_1), 0) \) be the corresponding solution of (CP). If \( T_- (w_0, w_1) < \infty \), then Proposition 4.2 and Proposition 4.10 yield \( \int \nabla w_0w_1 = 0 \), and Theorem 5.1 and Proposition 4.2 give a contradiction. Hence \( T_-(w_0, w_1) = +\infty \). Let \( w_n(x, \tau) \) be the solution of (CP), with data \( (w_{0,n}(x), w_{1,n}(x)) \), \( \tau \in (-T_-(w_{0,n}, w_{1,n}), 0] \). Because of Remark 2.21

\[
\lim T_- (w_{0,n}, w_{1,n}) = +\infty, \text{ and for any } \tau \in (-\infty, 0],
\]

\[
(w_n(x, \tau), \partial_\tau w_n(x, \tau)) \to (w_0(x, \tau), \partial_\tau w_0(x, \tau))
\]
in \( \dot{H}^1 \times L^2 \). Note that, by uniqueness in (CP), for \( 0 \leq t_n + \frac{\tau}{\lambda(t_n)} \),
\[
(7.1) \quad w_n(x, \tau) = \frac{1}{\lambda(t_n)} \frac{(N-2)}{2} u_C \left( \frac{x - x(t_n)}{\lambda(t_n)}, t_n + \frac{\tau}{\lambda(t_n)} \right).
\]

Note that,
\[
\lim_n ( - \tau_n ) = \lim_n (t_n \lambda(t_n)) \geq T_- (w_0, w_1) = +\infty,
\]
so that for all \( \tau \in (-\infty, 0] \), for \( n \) large, \( 0 \leq t_n + \frac{\tau}{\lambda(t_n)} \leq t_n \). In fact, if
\[
- \tau_n \to - \tau_0 < \infty,
\]
then
\[
w_n(x, - \tau_n) = \frac{1}{\lambda(t_n)} \frac{(N-2)}{2} u_C \left( \frac{x - x(t_n)}{\lambda(t_n)}, 0 \right),
\]
\[
\partial_\tau w_n(x, - \tau_n) = \frac{1}{\lambda(t_n)} \frac{(N-1)}{2} \partial_\tau u_C \left( \frac{x - x(t_n)}{\lambda(t_n)}, 0 \right),
\]
would converge to \((w_0(x, - \tau_0), \partial_\tau w_0(x, - \tau_0))\) in \( \dot{H}^1 \times L^2 \), with \( \lambda(t_n) \to 0 \), which is a contradiction from \( (u_0, C, u_1, C) \neq (0, 0), (w_0, w_1) \neq (0, 0) \).

Next, note that we must have \( \|w_0\|_{S(-\infty, 0)} = +\infty \). Otherwise, by Theorem 2.20 for \( n \) large, \( T_-(w_{0,n}, w_{1,n}) = \infty \) and \( \|w_n\|_{S(-\infty, 0)} \leq M \), uniformly in \( n \), which, in view of (7.1), contradicts \( u_C \|_{S(0,+\infty)} = +\infty \).

Fix now \( \tau \in (-\infty, 0] \). For \( n \) sufficiently large, \( t_n + \frac{\tau}{\lambda(t_n)} \geq 0 \) and \( \lambda(t_n + \frac{\tau}{\lambda(t_n)}) \) is defined. Let
\[
\left( \frac{1}{\lambda(t_n + \frac{\tau}{\lambda(t_n)})} \frac{(N-2)}{2} u_C \left( \frac{x - x(t_n + \frac{\tau}{\lambda(t_n)})}{\lambda(t_n + \frac{\tau}{\lambda(t_n)})}, t_n + \frac{\tau}{\lambda(t_n)} \right),
\right.
\]
\[
\left. \frac{1}{\lambda(t_n + \frac{\tau}{\lambda(t_n)})} \frac{(N-1)}{2} \partial_\tau u_C \left( \frac{x - x(t_n + \frac{\tau}{\lambda(t_n)})}{\lambda(t_n + \frac{\tau}{\lambda(t_n)})}, t_n + \frac{\tau}{\lambda(t_n)} \right) \right)
\]
\[
= \left( \frac{1}{\lambda_n(\tau)} \frac{(N-2)}{2} w_n \left( \frac{x - x_n(\tau)}{\lambda_n(\tau)}, \tau \right), \frac{1}{\lambda_n(\tau)} \frac{(N-1)}{2} \partial_\tau w_n \left( \frac{x - x_n(\tau)}{\lambda_n(\tau)}, \tau \right) \right) \in K,
\]
with
\[
(7.2) \quad \tilde{\lambda}_n(\tau) = \frac{\lambda(t_n + \frac{\tau}{\lambda(t_n)})}{\lambda(t_n)} \geq 1, \quad \tilde{x}_n(\tau) = x(t_n + \frac{\tau}{\lambda(t_n)}) - \frac{x(t_n)}{\lambda(t_n)}.
\]
Now, since \( \frac{1}{\lambda_n^2} \tilde{v} \left( \frac{\tau - x_n}{\lambda_n} \right) \nrightarrow \tilde{v} \) in \( L^2 \), with either \( \lambda_n \to 0 \) or \( +\infty \), or \(|x_n| \to \infty \) implies that \( \tilde{v} \equiv 0 \), we see that (since \( E_C > 0 \) we can
assume, after passing to a subsequence, that \( \tilde{\lambda}_n(\tau) \to \tilde{\lambda}(\tau), 1 \leq \tilde{\lambda}(\tau) < \infty \) and \( \tilde{x}_n(\tau) \to \tilde{x}(\tau) \in \mathbb{R}^N \). But then

\[
\left( \frac{1}{\tilde{\lambda}_n(\tau)^{\frac{N-2}{2}}} w_0 \left( \frac{x - \tilde{x}_n(\tau)}{\tilde{\lambda}_n(\tau)}, \tau \right), \frac{1}{\tilde{\lambda}_n(\tau)^{\frac{N-2}{2}}} \partial_\tau w_0 \left( \frac{x - \tilde{x}_n(\tau)}{\tilde{\lambda}_n(\tau)}, \tau \right) \right) \in K.
\]

But then, by Proposition 4.11 and Theorem 5.1 \((w_0, w_1) = (0, 0)\), contradicting \( E_C = E((w_0, w_1)) \). This proves i).

For ii) note that if \( u_0 \in L^2 \), this is the result in Theorem 3.7. The proof of the general case is a modification of the one of Theorem 3.7. Let \( A = \| (u_0, u_1) \|_{H^1 \times L^2} > 0 \). Recall that (from Lemma 2.17 and its proof) there exists \( \varepsilon_0 > 0 \) such that, for \( 0 < \varepsilon < \varepsilon_0 \), there exists \( M_0 = M_0(\varepsilon) \), with

\[
\int_{|x| \geq M_0 + t} |\nabla_x u(x, t)|^2 + |\partial_t u(x, t)|^2 + |u(x, t)|^{2^*} + \frac{|u(x, t)|^2}{|x|^2} \, dx \leq \varepsilon,
\]

for \( t \in [0, T_+(u_0, u_1)) \). Assume that \( T_+(u_0, u_1) = +\infty \) to reach a contradiction.

Let \( f(\tau) \) be a solution to the differential inequality \((f \geq 0)\)

\[
(7.3) \quad f'(\tau) \geq B f(\tau)^{\frac{N-2}{2}}, \quad f(0) = 1.
\]

Then, the time of blow-up for \( f \) is \( \tau_* \), with \( \tau_* \leq K_N B^{-1} \).

Consider now, for \( R \) large, \( \phi \in C_0^\infty(B_2), \phi \equiv 1 \) on \( |x| < 1, 0 \leq \phi \leq 1, \)

\[
y_R(t) = \int u^2(x, t) \phi(x/R) \, dx.
\]

Then, \( y'_R(t) = 2 \int u \partial_t u \phi(x/R) \, dx \), and, using the notation in Lemma 5.3, we have that

\[
y''_R(t) = 2 \left[ \int (\partial_t u)^2 - |\nabla_x u|^2 + |u|^{2^*} \, dx \right] + O(r(R)).
\]

Arguing as in the proof of Theorem 5.7 we find that

\[
y''_R(t) \geq 2 \left[ 1 + \frac{N}{N-2} \right] \int (\partial_t u)^2 \phi(x/R) + \tilde{\delta}_0 + O(r(R)).
\]

Choose now \( \varepsilon_1 \) so small, and \( M_0 = M_0(\varepsilon_1) \), as above, so that, for \( R > 2M_0, O(r(R)) \leq \varepsilon_1, \varepsilon_1 \leq \tilde{\delta}_0/2 \). We then have, for \( 0 < t < R/2, \)

\[
y''_R(t) \geq \tilde{\delta}_0/2,
\]

\[
y''_R(t) \geq 2 \left[ 1 + \frac{N}{N-2} \right] \int (\partial_t u)^2 \phi(x/R).
\]
Note also that
\begin{equation}
(7.5) \quad y_R(0) \leq CM_0^2 A^2 + \epsilon_1 R^2, \quad |y_R'(0)| \leq CM_0 A^2 + \epsilon_1 R.
\end{equation}

Let \( T = \frac{4CM_0^2A^2 + 2\epsilon_1 R + 2R/\sqrt{\epsilon_1}}{\delta_0} \). Then, (if \( T < R/2 \))
\begin{align*}
y_R'(T) & \geq \frac{\delta_0}{2} + y_R'(0) \geq 2CM_0 A^2 + \epsilon_1 R + R\sqrt{\epsilon_1} - CM_0 A^2 - \epsilon_1 R \\
&= CM_0 A^2 + R\sqrt{\epsilon_1}.
\end{align*}

Thus, there exists \( 0 < t_1 < T \) such that \( y_R'(t_1) = CM_0 A^2 + R\sqrt{\epsilon_1} \), and for \( 0 < t < t_1 \), we have \( y_R'(t) < CM_0 A^2 + R\sqrt{\epsilon_1} \). Note that, in light of \( (7.4) \), \( y_R'(t) > y_R'(t_1) > 0 \), \( t > t_1 \) \( (t < R/2) \) and also
\begin{align*}
y_R(t_1) & \leq y_R(0) + \int_0^{t_1} y_R' \leq y_R(0) + t_1(CM_0 A^2 + R\sqrt{\epsilon_1}) = y_R(0) + t_1 y_R'(t_1).
\end{align*}

We next estimate \( T = \frac{4CM_0^2A^2 + 2\epsilon_1 R + 2R/\sqrt{\epsilon_1}}{\delta_0} \). We first choose \( \epsilon_1 \) so small that \( \frac{2\epsilon_1}{\delta_0} + \frac{2\sqrt{\epsilon_1} R}{\delta_0} \leq \frac{1}{32K_N} \), where \( K_N \) is the constant defined at the beginning of the proof, and \( R \) so large that \( \frac{4CM_0^2A^2}{\delta_0} \leq \frac{1}{16K_N} R \). We then have \( T \leq \frac{1}{8K_N} R \). We can also ensure \( T \leq \frac{R}{8} \). Thus,
\[ y_R(t_1) \leq CM_0^2 A^2 + \epsilon_1 R^2 + \frac{R}{8K_N} y_R'(t_1). \]

If we now use the argument in the proof of Theorem 3.7, for the function \( \tilde{y}_R(\tau) = y_R(t_1 + \tau), 0 \leq \tau \leq R/4 \), in light of \( (7.4) \), we see that, for \( 0 < \tau < R/4 \), we have \( \log(\tilde{y}_R'(\tau)) \geq \frac{(N-1)}{(N-2)} \log(\tilde{y}_R(\tau))' \), so that, by integration,
\[ \frac{\tilde{y}_R'(\tau)}{\tilde{y}_R'(0)} \geq \left[ \frac{\tilde{y}_R(\tau)}{\tilde{y}_R(0)} \right]^{\frac{(N-1)}{(N-2)}} \quad \text{for } 0 \leq \tau \leq R/4. \]

Thus, if \( f(\tau) = \frac{\tilde{y}_R'(\tau)}{\tilde{y}_R'(0)} \) and \( B = \frac{\tilde{y}_R'(0)}{\tilde{y}_R'(0)} = \frac{y_R'(t_1)}{y_R'(0)} \), we have that \( f \) is a solution of \( (7.3) \) for \( 0 \leq \tau \leq R/4 \). Thus, we must have
\[ \frac{R}{4} \leq y_R(t_1) \leq K_N(CM_0^2 A^2 + \epsilon_1 R^2) \frac{R}{8K_N} y_R'(t_1), \]
or
\begin{align*}
1 \leq \frac{K_N(CM_0^2 A^2 + \epsilon_1 R^2)}{CM_0 A^2 R + \sqrt{\epsilon_1} R^2} & = \frac{K_N(CM_0^2 A^2/R^2 + \epsilon_1)}{[CM_0 A^2/R + \sqrt{\epsilon_1}]^2} \\
& \leq K_N M_0/R + K_N \sqrt{\epsilon_1}.
\end{align*}
By taking $K_N \sqrt{\epsilon_1} < \frac{1}{32}$, and $\frac{K_N M_0}{R} < \frac{1}{32}$ we reach a contradiction, which gives the proof of ii). \hfill \Box

To conclude, let us give some Corollaries of our main results similarly to the NLS case (We will refer to [10] for the proofs, which are identical).

**Corollary 7.3.** Let $(u_0, u_1) \in \dot{H}^1 \times L^2$, $3 \leq N \leq 5$. Assume that $E((u_0, u_1)) < E((W,0))$ and $\int |\nabla u_0|^2 < \int |\nabla W|^2$. Then the solution $u$ of the Cauchy problem (CP) with data $(u_0, u_1)$ at $t = 0$ has time interval of existence $I = (-\infty, +\infty)$, and there exists $(u_{0, \pm}, u_{1, \pm})$ in $\dot{H}^1 \times L^2$ such that if we denote by $v_{\pm}(t)$ the solutions of (LCP) corresponding to these initial data, we have

$$\lim_{t \to \pm \infty} \|(u(t), \partial_t u(t)) - (v_{\pm}(t), \partial_t v_{\pm}(t))\|_{\dot{H}^1 \times L^2} = 0.$$ 

Moreover, if we define $\delta_0$ so that $E(u_0, u_1) \leq (1 - \delta_0)E((W,0))$, there exists a function $M(\delta_0)$ so that $\|u\|_{S((-\infty, +\infty))} \leq M(\delta_0)$.

Let us give now a different version of the main result.

**Corollary 7.4.** Let $(u_0, u_1)$ in $\dot{H}^1 \times L^2$ and assume that for all $t \in (-T_-(u_0), T_+(u_0))$ we have $\int |\nabla u(t)|^2 + |\partial_t u(t)|^2 \leq \int |\nabla W|^2 - \delta_0$, for $\delta_0 > 0$. Then the solution $u$ of the Cauchy problem (CP) with data $u_0$ at $t = 0$ has time interval of existence $I = (-\infty, +\infty)$, $\|u\|_{S((-\infty, +\infty))} < +\infty$.

**Corollary 7.5.** Let $3 \leq N \leq 5$, $(u_0, u_1) \in \dot{H}^1 \times L^2$ (no size restrictions) be such that $T_+(u_0, u_1) < +\infty$ and $\forall t \in [0, T_+(u_0, u_1)], \int |\nabla u(t)|^2 + |\partial_t u(t)|^2 \leq C_0$. Then, we have for $x_1(t)$ and $x_2(t)$, and for all $R > 0$,

$$\lim_{t \to T_+(u_0)} \int_{|x-x_1(t)| \leq R} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \geq \frac{2}{N} \int |\nabla W|^2$$

$$\overline{\lim}_{t \to T_+(u_0)} \int_{|x-x_2(t)| \leq R} |\nabla u(t)|^2 + |\partial_t u(t)|^2 \geq \int |\nabla W|^2.$$ 

**References**

[1] C. Antonini and F. Merle, Optimal bounds on positive blow-up solutions for a semilinear wave equation, Internat. Math. Res. Notices 21 (2001), 1141–1167.
[2] N. Aronszajn, A. Krzywicki and J. Szarski, A unique continuation theorem for exterior differential forms on Riemannian manifolds, Ark. Mat. 4 (1962), 417–453.
[3] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant le courbure scalaire, J. Math. Pures Appl. 55, 1976, 3, 269–296.
[4] H. Bahour and P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math 121 (1999), 131–175.
[5] H. Brézis and M. Marcus, *Hardy’s inequalities revisited*, Ann. Scuola Norm. Pisa **25** (1997), 217–237.

[6] M. Christ and M. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg–de Vries equation*, J. Funct. Anal. **100** (1991), 87–109.

[7] P. Gérard, *Description du défaut de compacité de l’injection de Sobolev*, ESAIM Control Optim. Calc. Var. **3** (1998), 213–233.

[8] Y. Giga and R. Kohn, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math. **42** (1989), 223–241.

[9] J. Ginibre, A. Soffer and G. Velo, *The global Cauchy problem for the critical nonlinear wave equation*, J. Funct. Anal. **110** (1992), 96–130.

[10] J. Ginibre and G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Funct. Anal. **133** (1995), 50–68.

[11] M. Grillakis, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. **132** (1990), 485–509.

[12] M. Grillakis, *Regularity for the wave equation with a critical nonlinearity*, Comm. Pure Appl. Math. **45** (1992), 749–774.

[13] L. Hörmander, “The analysis of linear partial differential operators III”, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[14] D. Jerison and C. E. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math. **121** (1985), 463–494.

[15] L. Kapitanski, *Global and unique weak solutions of nonlinear wave equations*, Math. Res. Lett., **1** (1994), no. 2, 211–223.

[16] C. E. Kenig and F. Merle, *Global well-posedness, scattering and blow-up for the energy critical, focusing, non-linear Schrödinger equation in the radial case*, to appear, Invent. Math.

[17] C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math. **46** (1993), 527–620.

[18] S. Keraani, *On the defect of compactness for the Strichartz estimates of the Schrödinger equations*, J. Differential Equations **175** (2001), 353–392.

[19] J. Krieger and W. Schlag, *On the focusing critical semi-linear wave equation*, to appear, Amer. J. of Math.

[20] H. Levine, *Instability and nonexistence of global solutions to nonlinear wave equations of the form Pt = −Au + F(u)*, Trans. Amer. Math. Soc. **192** (1974), 1–21.

[21] H. Lindblad and C. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. **130** (1995), 357–426.

[22] F. Merle, *Existence of blow-up solutions in the energy space for the critical generalized KdV equation*, J. Amer. Math. Soc. **14** (2001), 555–578.

[23] F. Merle and H. Zaag, *Determination of the blow-up rate for the semilinear wave equation*, Amer. J. of Math. **125** (2003), 1147–1164.

[24] F. Merle and H. Zaag, *A Liouville theorem for vector-valued nonlinear heat equations and applications*, Math. Ann. **316** (2000), no. 1, 103–137.

[25] L.E. Payne and D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math., **22**, (1975), 273–303.

[26] H. Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equation*, Math. Z. **185** (1984), 261–270.
[27] D.H. Sattinger, *On global solutions of nonlinear hyperbolic equations*, Arch. Rational Mech. Anal., 30, (1968), 148–172.
[28] J. Shatah and M. Struwe, *Regularity results for nonlinear wave equations*, Ann. of Math. 138 (1993), 503–518.
[29] J. Shatah and M. Struwe, *Well-posedness in the energy space for semilinear wave equations with critical growth*, Internat. Math. Res. Notices 7 (1994), 303–309.
[30] J. Shatah and M. Struwe, “Geometric wave equations,” Courant Lecture Notes in Mathematics, 2 (1998).
[31] C. Sogge, “Lectures on nonlinear wave equations,” Monographs in Analysis II, International Press, 1995.
[32] G. Staffilani, *On the generalized Korteweg-de Vries-type equations*, Differential Integral Equations 10 (1997), 777–796.
[33] W. Strauss, “Nonlinear wave equations,” CBMS Regional Conference Series in Mathematics, 73, American Math. Soc., Providence, RI, 1989.
[34] M. Struwe, *Globally regular solutions to the u^5 Klein-Gordon equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 15 (1988), 495–513.
[35] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) 110 (1976), 353–372.
[36] T. Tao, *Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions*, preprint, [http://arxiv.org/abs/math.AP/0601164](http://arxiv.org/abs/math.AP/0601164).
[37] T. Tao and M. Visan, *Stability of energy-critical nonlinear Schrödinger equations in high dimensions*, Electron. J. Differential Equations 118 (2005), 28 pp. (electronic).
[38] M. Taylor, “Tools for PDE. Pseudodifferential operators, paradifferential operators and layer potentials,” Math. Surveys and Monographs 81, AMS, Providence RI 2000.
[39] N. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa 22 (1968), 265–274.
[40] T. Wolff, *Recent work on sharp estimates in second-order elliptic unique continuation problems*, J. Geom. Anal. 3 (1993), 621–650.

CARLOS E. KENIG:
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF CHICAGO,
CHICAGO, IL 60637
USA

FRANK MERLE:
DÉPARTEMENT DE MATHÉMATIQUES,
UNIVERSITÉ DE CERGY-PONTOISE,
PONTOISE,
95302 CERGY-PONTOISE,
FRANCE