Applications of Asymptotic Riesz Representation Theorem

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Abstract. We review the relation between compact asymptotic spectral measures and certain positive asymptotic morphism on locally compact spaces via asymptotic Riesz representation theorem, as introduced by Martinez and Trout [3]. Applications to this theorem shall be discus.

1 Introduction

In [3], Martinez and Trout introduced the positive asymptotic morphism, defined as:

A positive asymptotic morphism from a C*-algebra $A$ to a C*-algebra $B$ is a family of maps $\{Q_h\}_{h \in (0, 1]} : A \to B$, parameterized by $h \in (0, 1]$, such that the following hold:

1. Each $Q_h$ is a positive linear map;
2. The map $h \mapsto Q_h(f) : (0, 1] \to B(H)$ is continuous for each $f \in A$;
3. For all $f, g \in A$ we have
   \[ \lim_{h \to 0} \|Q_h(fg) - Q_h(f)Q_h(g)\| = 0. \]

Also, Martinez and Trout introduced the concept of an asymptotic spectral measure $\{A_h\}_{h \in (0, 1]} : \Sigma \to B(H)$ associated to a measurable space $(X, \Sigma)$ (Definition 3.1). Roughly, an asymptotic spectral measure $\{A_h\}_{h \in (0, 1]} : \Sigma \to B(H)$ is a continuous family of positive operator-valued measures which has the property:

\[ \lim_{h \to 0} \left\| A_h(\Delta_1 \cap \Delta_2) - A_h(\Delta_1)A_h(\Delta_2) \right\| = 0, \]

for each $\Delta_1, \Delta_2 \in \Sigma$.

Let $X$ be a locally compact Hausdorff topological space with Borel C*-algebra $\Sigma_X$ and let $C_X \subset \Sigma_X$ denote the collection of all pre-compact open subsets of $X$. Let $C_0(X)$ denote the C* - algebra of all continuous functions which vanish at infinity on $X$. Define $B_0(X)$ to be the C* - subalgebra of $B(X)$ (C* - algebra of

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2010 Mathematics Subject Classification: 46-01;46G10

Keywords: local spectrum; local resolvent set; asymptotic equivalence; asymptotic quasinilpotent equivalence
all bounded Borel functions on $X$) generated by $\{\chi_U | U \in C_X\}$, where $\chi_U$ denotes the characteristic function of $U \subseteq X$.

Let $H$ be a separable Hilbert space, $B(H)$ be the C*-algebra of all bounded linear operators on $H$, and $B$ denote a hereditary C*-algebra of $B(H)$.

The asymptotic Riesz representation theorem, formulated by Martinez and Trout [3], gives a bijective correspondence between positive asymptotic morphisms $\{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H)$, having property $Q_h(C_0(X)) \subset B$ for any $h \in (0,1]$, and asymptotic spectral measures $\{A_h\}_{h \in (0,1]} : X \to B(H)$, having property $A_h(C_X) \subset B$ for any $h \in (0,1]$. This correspondence is given by

$$Q_h(f) = \int_X f(x)dA_h(x),$$

for any $f \in B_0(X)$. (Teorema 4.2. [3])

In this paper, we study some applications of this theorem.

## 2 Positive Asymptotic Morphisms

In this section we review the basic definitions and properties of positive asymptotic morphisms.

Let $A$ and $B$ be two C*-algebras. An linear operator $Q : A \to B$ is positive if $Q(f) \geq 0$ for any $f \geq 0$.

**Definition 1.** A positive asymptotic morphism from a C*-algebra $A$ to a C*-algebra $B$ is a family of maps $\{Q_h\}_{h \in (0,1]} : A \to B$, parameterized by $h \in (0,1]$, such that the following hold:

1. Each $Q_h$ is a positive linear map;
2. The map $h \mapsto Q_h(f) : (0,1] \to B(H)$ is continuous for each $f \in A$;
3. For all $f, g \in A$ we have

$$\lim_{h \to 0} \|Q_h(fg) - Q_h(f)Q_h(g)\| = 0.$$

(Definition 2.1. [3])

**Definition 2.** Two positive asymptotic morphisms $\{Q_h\}, \{P_h\}_{h \in (0,1]} : A \to B$ are called asymptotically equivalent if for all $f \in A$ we have that

$$\lim_{h \to 0} \|Q_h(f) - P_h(f)\| = 0.$$

(Definition 2.2. [3])
Remark 3. The asymptotic equivalence relation of two positive asymptotic morphisms is symmetric, reflexive and transitive.

Let $H$ be a separable Hilbert space and $B(H)$ be the C*-algebra of all bounded linear operators on $H$. Let $X$ be a set equipped with a C*-algebra $\Sigma_X$ of measurable sets and let $C_X \subset \Sigma_X$ denote the collection of all pre-compact open subsets of $X$. Define $B_0(X)$ to be the C*-subalgebra of $B_b(X)$ (C*-algebra of all bounded Borel functions on $X$) generated by $\{\chi_U | U \in C_X\}$, where $\chi_U$ denotes the characteristic function of $U \subseteq X$. If $X$ is also $\sigma$-compact, then let $C_0(X)$ be the set of all continuous functions which vanish at infinity on $X$.

We call support of a morphism $Q : C_0(X) \to B(H)$ the set

$$supp(Q) = \cap \{F \subset X | F \text{ closed and } Q(f) = 0, \forall f \in supp(f) \subset X \setminus F\}.$$ 

A morphism $Q : C_0(X) \to B(H)$ will be said to have compact support if there is a compact subset $K$ of $X$ such that $supp(Q) \subset K$.

Definition 4. A positive asymptotic morphism $\{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H)$ will be said to have compact support if there is a compact subset $K$ of $X$ such that $supp(Q_h) \subset K, \forall h \in (0,1]$.

Definition 5. Let $\{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H)$ be a positive asymptotic morphism. The support of $\{Q_h\}$ is defined as the set

$$supp(\{Q_h\}) = \cap \left\{F \text{ closed } \mid \lim_{h \to 0} \|Q_h(f)\| = 0, \forall f \in B_0(X) \text{ with } supp(f) \bigcap F = \emptyset \right\}.$$ 

Remark 6. 1. $supp(\{Q_h\}) \subseteq \bigcup_{h \in (0,1]} supp(Q_h)$.

2. If $\{Q_h\}$ has compact support $supp(\{Q_h\})$ is a compact set.

Theorem 7. Let $\{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H)$ be a positive asymptotic morphism such that $\lim_{h \to 0} \|Q_h(1) - I\| = 0$. Then

$$Sp(\{Q_h(f)\}) \subseteq f(supp(\{Q_h\})),$$

for any $f \in B_0(X)$.

Proof. Let $f \in B_0(X)$ and $\lambda \notin f(supp(\{Q_h\}))$. Then $\lambda - f(x) \neq 0, \forall x \in supp(\{Q_h\})$. Thus there is an open set $G \supset supp(\{Q_h\})$ such that $\lambda - f(x) \neq 0, \forall x \in G$. Therefore, the application $x \mapsto 1/(f(x) - \lambda) \in C(G)$.

Let $g \in B_0(X)$ such that $g(x) = 1/(f(x) - \lambda), \forall x \in G$. Then
\[ g(x)(f(x) - \lambda) = (f(x) - \lambda)g(x) = 1, \]
\[ \forall x \in G. \] Taking into account the above relation and since \( \lim_{h \to 0} \| Q_h(1) - I \| = 0, \)
it follows that
\[
\lim_{h \to 0} \| Q_h(g)(\lambda - Q_h(f)) - I \| = \\
\lim_{h \to 0} \| Q_h(g)(\lambda - Q_h(f)) - Q_h(g)Q_h(\lambda - f) + Q_h(g)Q_h(\lambda - f) - Q_h(g(\lambda - f)) + \\
Q_h(g(\lambda - f)) - Q_h(1) + Q_h(1) - I \| \leq \\
\leq \lim_{h \to 0} \| Q_h(g)(\lambda - Q_h(f)) - Q_h(g)(\lambda - f) \| + \\
+ \lim_{h \to 0} \| Q_h(g)Q_h(\lambda - f) - Q_h(g(\lambda - f)) \| + \\
+ \lim_{h \to 0} \| Q_h(g(\lambda - f)) - Q_h(1) \| + \lim_{h \to 0} \| Q_h(1) - I \| = 0.
\]

Analogously \( \lim_{h \to 0} \| (\lambda - Q_h(f))Q_h(g) - I \| = 0. \) Therefore \( \lambda \in r(\{Q_h(f)\}). \)

We have showed that
\[ \text{Sp}(\{Q_h(f)\}) \subseteq f(\text{supp}(\{Q_h\}), \forall f \in B_0(X). \]

\[ \square \]

**Corollary 8.** Let \( \{Q_h\}_{h \in (0,1]} : B_0(\mathbb{C}) \to B(H) \) be a positive asymptotic morphism such that \( \lim_{h \to 0} \| Q_h(1) - I \| = 0. \) Then
\[ \text{Sp}(\{Q_h(z)\}) \subseteq \text{supp}(\{Q_h\}). \]

**Proof.** We take in Theorem 7 \( f = z, \) where \( z \) represents the identity application.

\[ \square \]

### 3 Asymptotic Spectral Measures

Let \((X, \Sigma)\) be a measurable space and \( H \) a separable Hilbert space. Let \( \varepsilon \subset \Sigma \) be a fixed collection of measurable sets.

**Definition 9.** A positive operator-valued measure on the measurable space \((X, \Sigma)\) is a mapping \( A : \Sigma \to B(H) \) which satisfies the following properties:

1. \( A(\emptyset) = 0; \)
2. \( A(\Delta) \geq 0, \forall \Delta \in \Sigma; \)
3. \( A(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} A(\Delta_n), \) for disjoint measurable sets \( (\Delta_n)_{n=1}^{\infty} \subset \Sigma, \) where the series converges in weak operator topology.
Definition 10. An asymptotic spectral measure on \((X, \Sigma, \varepsilon)\) is a family of maps \(\{A_h\}_{h \in (0, 1]} : \Sigma \to B(H)\), parameterized by \(h \in (0, 1]\), such that the following hold:

1) Each \(A_h\) is a positive operator-valued measure;
2) \(\lim_{h \to 0} \|A_h(X)\| \leq 1\);
3) The map \(h \mapsto A_h(\Delta) : (0, 1] \to B(H)\) is continuous, for any \(\Delta \in \varepsilon\);
4) For each \(\Delta_1, \Delta_2 \in \varepsilon\) we have

\[
\lim_{h \to 0} \left\| A_h \left( \Delta_1 \cap \Delta_2 \right) - A_h(\Delta_1)A_h(\Delta_2) \right\| = 0.
\]

The triple \((X, \Sigma, \varepsilon)\) will be called asymptotic measure space.

If \(\varepsilon = \Sigma\), then \(\{A_h\}\) will be called total (full) asymptotic spectral measure on \((X, \Sigma)\).

If each \(A_h\) is normalized, i.e. \(A_h(X) = I_H\), then \(\{A_h\}\) will be called normalized. (Definition 3.1. [3])

If \(\lim_{h \to 0} \|A_h(X) - I_H\| = 0\), then \(\{A_h\}\) will be called asymptotically normalized.

Definition 11. Two asymptotic spectral measures \(\{A_h\}, \{B_h\}_{h \in (0, 1]} : \Sigma \to B(H)\) on \((X, \Sigma)\) are said to be (asymptotically) equivalent if for each measurable set \(\Delta \in \varepsilon\), we have

\[
\lim_{h \to 0} \|A_h(\Delta) - B_h(\Delta)\| = 0.
\]

(Definition 2.2 [3])

Let \(X\) denote a locally compact Hausdorff topological space with Borel \(\sigma\)-algebra \(\Sigma\).

Definition 12. Let \(A\) be a Borel positive operator-valued measure on \(X\). The cospectrum of \(A\) is defined as the set

\[
cospec(A) = \bigcup \{ U \subset X | U \text{ is open and } A(U) = 0 \}.
\]

The spectrum of \(A\) is the complement, i.e.

\[
spec(A) = X \setminus \cospec(A).
\]

Definition 13. Let \(A\) be a Borel positive operator-valued measure on \(X\). \(A\) is said to be compact if \(spec(A)\) is a compact subset of \(X\).

Theorem 14. Let \(A\) be a Borel positive operator-valued measure on \(X\). Then

\[
A(spec(A)) = A(X).
\]

(Theorem 23 [5])
Definition 15. An asymptotic spectral measure \( \{A_h\} \) on \( X \) will have compact support if there is a compact subset \( K \) of \( X \) such that \( \text{spec}(A_h) \subset K, \forall h \in (0,1] \). (Definition 3.4 [3])

Remark 16. i) If \( \{A_h\} \) has compact, then \( A_h \) has compact support, \( \forall h \in (0,1] \); ii) If \( \{A_h\} \) has compact support, then

\[
A_h(\text{spec}(A_h)) = A_h(K) = A_h(X), \forall h \in (0,1].
\]

Definition 17. Let \( \{A_h\}_{h \in (0,1]} : \Sigma_X \to B(H) \) be an asymptotic spectral measure. The cospectrum of \( \{A_h\} \) is defined as the set

\[
\text{cospec}(\{A_h\}) = \bigcup \left\{ a \subset X \mid a \text{ open and } \lim_{h \to 0} \|A_h(a)\| = 0 \right\}.
\]

The spectrum of \( \{A_h\} \) is the complement of \( \text{cospec}(\{A_h\}) \), i.e.

\[
\text{spec}(\{A_h\}) = X \setminus \text{cospec}(\{A_h\}).
\]

Remark 18. i) \( \text{spec}(\{A_h\}) \subseteq \bigcup_{h \in (0,1]} \text{spec}(A_h) \) and \( \bigcap_{h \in (0,1]} \text{cospec}(A_h) \subseteq \text{cospec}(\{A_h\}) \). ii) If \( \{A_h\} \) has compact support, then \( \text{spec}(\{A_h\}) \) is also a compact set.

Proof. i) Let \( a \subset \bigcap_{h \in (0,1]} \text{cospec}(A_h) \) be an open set. Thus \( A_h(a) = 0, \forall h \in (0,1] \) and

\[
\lim_{h \to 0} \|A_h(a)\| = 0.
\]

Therefore

\[
a \subset \text{cospec}(\{A_h\}), \forall a \subset \bigcap_{h \in (0,1]} \text{cospec}(A_h).
\]

It results

\[
\bigcap_{h \in (0,1]} \text{cospec}(A_h) \subseteq \text{cospec}(\{A_h\})
\]

and, taking the complement we have

\[
\text{spec}(\{A_h\}) \subseteq \bigcup_{h \in (0,1]} \text{spec}(A_h).
\]

ii) Let \( \{A_h\} \) be a compact set. Thus there is a compact subset \( K \) of \( X \) such that \( \text{spec}(A_h) \subset K, \forall h \in (0,1] \).

By i), it follows
\[ \text{spec} \left( \{ A_h \} \right) \subseteq \bigcup_{h \in (0,1]} \text{spec} \left( A_h \right) \subset K. \]

**Lemma 19.** Let \( \{ A_h \}_{h \in (0,1]} : \Sigma X \rightarrow B(H) \) be an asymptotic spectral measure. Then

\[ \lim_{h \to 0} \| A_h(K) \| = 0, \]

for each compact subset \( K \) of \( \cospec(\{ A_h \}) \).

**Proof.** Let \( K \) be a compact subset of \( \cospec(\{ A_h \}) \). Thus each element of \( K \) belongs to an open set \( a \) having property \( \lim_{h \to 0} \| A_h(a) \| = 0 \). Since \( K \) is a compact set, hence there is a family of open set \( (a_i)^n \subset X \) such that \( K \subset a_1 \cup \cdots \cup a_n \). Therefore

\[ A(K) \leq A(a_1) + \cdots + A(a_n) = 0 \]

and

\[ \lim_{h \to 0} \| A_h(K) \| \leq \lim_{h \to 0} \| A_h(a_1) \| + \cdots + \lim_{h \to 0} \| A_h(a_n) \| = 0. \]

**Proposition 20.** Let \( \{ A_h \}_{h \in (0,1]} : \Sigma X \rightarrow B(H) \) be an asymptotic spectral measure. Then

\[ \lim_{h \to 0} \| A_h(X) \| = \lim_{h \to 0} \| A_h(\text{spec}(\{ A_h \})) \|. \]

**Proof.** We show that

\[ \lim_{h \to 0} \| A_h(\cospec(\{ A_h \})) \| = 0. \]

Let \( K \) be a compact subset of \( \cospec(\{ A_h \}) \). By Lemma 19, it follows that

\[ \lim_{h \to 0} \| A_h(K) \| = 0. \]

Since \( A_h \) is regular, for any \( h \in (0,1) \), by above relation, it results that

\[ \lim_{h \to 0} \| A_h(\cospec(\{ A_h \})) \| = 0. \]

Since

\[ \text{spec}(\{ A_h \}) = X \setminus \cospec(\{ A_h \}), \]

we have that

\[ \lim_{h \to 0} \| A_h(X) \| = \lim_{h \to 0} \| A_h(\text{spec}(\{ A_h \})) + A_h(\cospec(\{ A_h \})) \| \leq \]
\[ \lim_{h \to 0} \| A_h(\text{spec}(\{A_h\})) \| + \lim_{h \to 0} \| A_h(\cospec(\{A_h\})) \| = \lim_{h \to 0} \| A_h(\text{spec}(\{A_h\})) \|. \]

In addition, we have that
\[ \lim_{h \to 0} \| A_h(\text{spec}(\{A_h\})) \| = \lim_{h \to 0} \| A_h(\cospec(\{A_h\})) + A_h(\text{spec}(\{A_h\})) - A_h(\cospec(\{A_h\})) \| \leq \lim_{h \to 0} \| A_h(\cospec(\{A_h\})) \| + \lim_{h \to 0} \| A_h(\text{spec}(\{A_h\})) - A_h(\cospec(\{A_h\})) \| \leq \lim_{h \to 0} \| A_h(X) \| . \]

From two preceding relations, it follows that
\[ \lim_{h \to 0} \| A_h(X) \| = \lim_{h \to 0} \| A_h(\text{spec}(\{A_h\})) \|. \]

**Theorem 21.** Let \( \{A_h\}, \{B_h\}_{h \in [0,1]} : \Sigma_X \to B(H) \) be two asymptotic spectral measures. If \( \{A_h\}, \{B_h\} \) are asymptotically equivalent, then
\[ \text{spec}(\{A_h\}) = \text{spec}(\{B_h\}). \]

**Proof.** Let be an open set \( a \subset \cospec(\{A_h\}) \). Thus
\[ \lim_{h \to 0} \| A_h(a) \| = 0. \]

Since \( \{A_h\}, \{B_h\} \) are asymptotically equivalent, it results that
\[ \lim_{h \to 0} \| A_h(a) - B_h(a) \| = 0. \]

By two preceding relations, we have that
\[ \lim_{h \to 0} \| B_h(a) \| = \lim_{h \to 0} \| A_h(a) \| = 0. \]

Thus \( a \subset \cospec(\{B_h\}), \forall a \subset \cospec(\{A_h\}) \) open. Therefore
\[ \text{spec}(\{B_h\}) \subset \text{spec}(\{A_h\}). \]

Reciprocal: Analog.

**Remark 22.** Let \( \{A_h\}, \{B_h\} \) be two asymptotic spectral measures on \((X, \Sigma)\). If \( \{A_h\}, \{B_h\} \) are asymptotically equivalent, then \( \{A_h\} \) has compact support if and only if \( \{B_h\} \) has compact support.
Proof. By preceding Proposition, for each compact subset $K$ of $X$, we have $\text{spec} \{A_h\} \subset K$ if and only if $\text{spec} \{B_h\} \subset K$.

\[ \square \]

**Proposition 23.** Let $\{A_h\}$ be a full asymptotic spectral measure on $(X, B)$, where $B$ is the $\sigma$ – algebra of Borel subsets of $X$, and $a \in B$. Then $\{A^a_h\} : B \rightarrow B(H)$, parameterized by $h \in (0, 1]$, given by

$A^a_h (b) = A_h (a \cap b), \forall b \in B$ and $\forall h \in (0, 1]$,

is an asymptotic spectral measure.

Proof. By definition of $\{A^a_h\}$, we have

$A^a_h (\emptyset) = A_h (a \cap \emptyset) = A_h (\emptyset) = 0, \forall h \in (0, 1]$

and

\[ \lim_{h \to 0} \|A^a_h (X)\| = \lim_{h \to 0} \|A_h (a \cap X)\| = \lim_{h \to 0} \|A_h (a)\| \leq \lim_{h \to 0} \|A_h (X)\| \leq 1. \]

Let $(b_n)_{n \in \mathbb{N}} \subset B$ be a family of disjoint sets. Thus $(a \cap b_n)_{n \in \mathbb{N}} \subset B$ is also a family of disjoint sets. Since $A_h$ is numerable additive, $\forall h \in (0, 1]$, it results

$A^a_h \left( \bigcup_{n \in \mathbb{N}} b_n \right) = A_h \left( \bigcup_{n \in \mathbb{N}} (a \cap b_n) \right) = \sum_{n \in \mathbb{N}} A_h (a \cap b_n) = \sum_{n \in \mathbb{N}} A^a_h (b_n), \forall h \in (0, 1].$

As the map $(0, 1] \rightarrow B(H) : h \rightarrow A_h (a \cap b)$ is continuous $\forall b \in B$, then the map is also continuous $\forall b \in B$.

Let $b_1, b_2 \in B$. Thus

\[ \lim_{h \to 0} \left\| A^a_h \left( b_1 \cap b_2 \right) - A^a_h (b_1)A^a_h (b_2) \right\| = \lim_{h \to 0} \left\| A_h \left( a \cap b_1 \cap b_2 \right) - A_h (a \cap b_1)A_h (a \cap b_2) \right\| = \lim_{h \to 0} \left\| A_h \left( (a \cap b_1) \cap (a \cap b_2) \right) - A_h (a \cap b_1)A_h (a \cap b_2) \right\| = 0. \]

Therefore, $\{A^a_h\} : B \rightarrow B(H)$ is a full asymptotic spectral measure.

\[ \square \]
Proposition 24. Let \{A_h\} be a full asymptotic spectral measure on \((X, B)\) and \(\{A^a_h\} : B \to B(H)\), parameterized by \(h \in (0, 1]\), given by \(A^a_h(b) = A_h(a \cap b), \forall b \in B\) and \(\forall h \in (0, 1]\). Then

\[
\text{spec}(A^a_h) \subseteq \overline{\alpha} \cap \text{spec}(A_h), \forall h \in (0, 1].
\]

Proof. Let \(b\) be a compact set such that \(b \subset C \setminus \overline{\alpha}\). Thus \(a \cap b = \emptyset\). By this relation we have

\[
A^a_h(b) = A_h(a \cap b) = A_h(\emptyset) = 0, \forall h \in (0, 1],
\]

hence

\[
b \subset \text{cospec}(A^a_h) \Rightarrow C \setminus \overline{\alpha} \subset \text{cospec}(A^a_h), \forall h \in (0, 1]\]

(by regularity property of measures \(A_h\) - Teorema 23 [5]). Therefore

\[
\text{spec}(A^a_h) \subseteq \overline{\alpha}, \forall h \in (0, 1].
\]

Let \(b\) be a compact set such that \(b \subset C \setminus \text{spec}(A_h)\). Thus there is a family of open sets \((b_i)_{i=1}^n\) such that

\[
b \subset \bigcup_{i=1}^n b_i, \quad b_i \subset C \setminus \text{spec}(A_h) \Rightarrow A_h(b_i) = 0.
\]

Since each \(A_h\) is additive, we have

\[
A_h(b) \leq A_h\left(\bigcup_{i=1}^n b_i\right) = \sum_{i=1}^n A_h(b_i) = 0.
\]

Taking into account the following relation

\[
A^a_h(b) = A_h(a \cap b) \leq \sum_{i=1}^n A_h(a \cap b_i) \leq \sum_{i=1}^n A_h(b_i) = 0
\]

it results

\[
b \subset C \setminus \text{spec}(A^a_h),
\]

for any compact set \(b\) such that \(b \subset C \setminus \text{spec}(A_h)\). Since each \(A_h\) is regular, it follows

\[
C \setminus \text{spec}(A_h) \subseteq C \setminus \text{spec}(A_h), \forall h \in (0, 1].
\]

Therefore

\[
\text{spec}(A^a_h) \subseteq \text{spec}(A_h), \forall h \in (0, 1].
\]

\(\square\)
Remark 25. If \( \{ A_h \} \) is an asymptotic spectral measure having compact support, then \( \{ A_h^a \} \) is an asymptotic spectral measure having compact support, \( \forall a \in B \).

Proposition 26. Two full asymptotic spectral measures on \((X, B)\) \( \{ A_h \} \), \( \{ B_h \} \) are asymptotically equivalent if and only if \( \{ A_h^a \} \), \( \{ B_h^a \} : B \to B(H) \), given by \( A_h^a (b) = A_h (a \cap b) \) and \( B_h^a (b) = B_h (a \cap b) \), \( \forall b \in B \), \( \forall h \in (0, 1] \), are asymptotically equivalent \( \forall a \in B \).

Proof. Let \( a \in B \) be fixed. Since \( \{ A_h \} \), \( \{ B_h \} \) are asymptotically equivalent, thus

\[
\lim_{h \to 0} \| A_h (a \cap b) - B_h (a \cap b) \| = 0, \forall b \in B.
\]

It follows that

\[
\lim_{h \to 0} \| A_h^a (b) - B_h^a (b) \| = 0, \forall b \in B.
\]

Reciprocal. Since \( \{ A_h^a \} \), \( \{ B_h^a \} \) are asymptotically equivalent \( \forall a \in B \) and

\[
A_h^a (a) = A_h (a), B_h^a (a) = B_h (a),
\]

it results

\[
\lim_{h \to 0} \| A_h (a) - B_h (a) \| =
\]

\[
= \lim_{h \to 0} \| A_h^a (a) - B_h^a (a) \| = 0, \forall a \in B.
\]

Therefore, \( \{ A_h \} \), \( \{ B_h \} \) are asymptotically equivalent.

\[ \square \]

4 Asymptotic Riesz Representation Theorem

Let \( X \) be a locally compact Hausdorff topological space with Borel \( \sigma \)-algebra \( \Sigma_X \) and let \( C_X \subset \Sigma_X \) denote the collection of all pre-compact open subsets of \( X \).

Let \( H \) be a separable Hilbert space, \( B(H) \) be the \( C^* \) - algebra of all bounded linear operators on \( H \) and \( B \) denote a hereditary \( C^* \) - algebra of \( B(H) \).

Lemma 27. There is a bijective correspondence between Borel positive operator-valued measure \( A : \Sigma_X \to B(H) \), having property \( A(C_X) \subset B \), and positive morphism \( Q : C_0(X) \to B \). This correspondence is given by

\[
Q(f) = \int_X f(x) dA(x).
\]
(Lemma 4.1. [3])

Let $C_0(X)$ denote the C*-algebra of all continuous functions which vanish at infinity on $X$. Define $B_0(X)$ to be the C* - subalgebra of $B_b(X)$ (C* - algebra of all bounded Borel functions on $X$) generated by $\{\chi_U | U \in \mathcal{C}_X\}$, where $\chi_U$ denotes the characteristic function of $U \subseteq X$.

**Proposition 28.** If $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$ is a compact asymptotic spectral measure, then $\{A_h\}$ verifies property $A_h(C_X) \subset B$, $\forall h \in (0,1]$, where $B$ is the hereditary subalgebra generated by $\{A_h (\text{spec}(A_h))\}_{h \in (0,1]}$.

**Proof.** By Theorem 14 we have

$$A_h (\cospec (A_h)) = 0.$$  

Let $a \in C_X$. Then

$$0 \leq A_h (a \cap \cospec (A_h)) \leq A_h (\cospec (A_h)) = 0$$

and thus

$$0 \leq A_h (a \cap \text{spec}(A_h)) \leq A_h (\text{spec}(A_h)).$$

Since $B$ is the hereditary subalgebra generated by $\{A_h (\text{spec}(A_h))\}_{h \in (0,1]}$, then $A_h (a) \in B$.

\[ \square \]

**Theorem 29.** (Asymptotic Riesz Representation Theorem): There is a bijective correspondence between positive asymptotic morphisms $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$, having property $Q_h (C_0(X)) \subset B$, $\forall h \in (0,1]$, and asymptotic spectral measures $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$, having property $A_h(C_X) \subset B$, $\forall h \in (0,1]$, given by

$$Q_h(f) = \int_X f(x) dA_h(x), \ \forall f \in B_0(X).$$

(Theorem 4.2. [3])

### 5 Application of Asymptotic Riesz Representation Theorem

**Proposition 30.** Let $\{A_h\}_{h \in (0,1]} : \Sigma_X \rightarrow B(H)$ be an asymptotic spectral measure, as in asymptotic Riesz representation theorem, and $\{Q_h\}_{h \in (0,1]} : B_0(X) \rightarrow B(H)$ the corresponding positive asymptotic morphism. Then the following assertions hold:

1. $\{Q_h\}$ is unitary if and only if $\{A_h\}$ is normalized;
2. $\lim_{h \to 0} \| A_h(X) - I_H \| = 0$ if and only if $\lim_{h \to 0} \| Q_h(1) - I_H \| = 0$.

3. Let $\{ T_h \} \subset B(H)$. Then $\lim_{h \to 0} \| T_h Q_h(f) - Q_h(f) T_h \| = 0$, $\forall f \in B_0(X)$ if and only if $\lim_{h \to 0} \| T_h A_h(\Delta) - A_h(\Delta) T_h \| = 0$, $\forall \Delta \in \Sigma_X$.

Proof. i) $\{ Q_h \}_{h \in (0,1]} : B_0(X) \to B(H)$ is unitary if $Q_h(1) = I_H$, $\forall h \in (0,1]$. Since $A_h(X) = Q_h(\chi_X) = \int_X \chi_X(x) dA_h(x) = \int_X dA_h(x) = Q_h(1) = I_H, \forall h \in (0,1]$, it follows that $\{ A_h \}_{h \in (0,1]}$ is normalized.

Reciprocal. Since $\{ A_h \}_{h \in (0,1]} : \Sigma_X \to B(H)$ is normalized, i.e. $A_h(X) = I_H$, $\forall h \in (0,1]$, then taking $f = 1$ we have $Q_h(1) = \int_X dA_h(x) = A_h(X) = I_H, \forall h \in (0,1]$.

ii) It results from $A_h(X) = Q_h(1)$, $\forall h \in (0,1]$.

iii) Since $\lim_{h \to 0} \| T_h Q_h(f) - Q_h(f) T_h \| = 0$, $\forall f \in S_0(X) \subset B_0(X)$, taking $f = \chi_\Delta$ it follows $\lim_{h \to 0} \| T_h A_h(\Delta) - A_h(\Delta) T_h \| = 0$, $\forall \Delta \in \Sigma_X$.

Reciprocal. Since $\lim_{h \to 0} \| T_h A_h(\Delta) - A_h(\Delta) T_h \| = 0$, $\forall \Delta \in \Sigma_X$, and having in view $A_h(\Delta) = Q_h(\chi_\Delta)$ it results $\lim_{h \to 0} \| T_h Q_h(\chi_\Delta) - Q_h(\chi_\Delta) T_h \| = 0$, $\forall \Delta \in \Sigma_X$.

Let $f \in S_0(X)$. Then there are disjoint sets $(\Delta_i)_{i=1,n}$ such that $f = \sum_{i=1}^n \alpha_i \chi_\Delta_i$.

By above relation, we have

$$\lim_{h \to 0} \| T_h Q_h(f) - Q_h(f) T_h \| = \lim_{h \to 0} \left\| \sum_{i=1}^n \alpha_i (T_h Q_h(\chi_\Delta_i) - Q_h(\chi_\Delta_i) T_h) \right\| \leq \sum_{i=1}^n \lim_{h \to 0} |\alpha_i| \| T_h Q_h(\chi_\Delta_i) - Q_h(\chi_\Delta_i) T_h \| \leq$$
\[ \leq \sum_{i=1}^{n} |\alpha_i| \lim_{h \to 0} \| T_h Q_h (\chi_{\Delta_i}) - Q_h (\chi_{\Delta_i}) T_h \| = 0. \]

Let \( f \in B_0(X) \). Then there are functions \( (f_n)_{n \in \mathbb{N}} \subset S_0(X) \) such that \( f \to f_n \). By preceding relation, we have
\[ \lim_{h \to 0} \| T_h Q_h (f) - Q_h (f) T_h \| = 0, \forall f \in B_0(X). \]

\[ \Box \]

**Proposition 31.** Two asymptotic spectral measures having compact support
\[ \{ A_h \}, \{ B_h \}_{h \in (0,1]} : \Sigma_X \to B(H) \] are asymptotically commutative, i.e. \( \lim_{h \to 0} \| A_h (\Delta_1) B_h (\Delta_2) - B_h (\Delta_2) A_h (\Delta_1) \| = 0, \forall \Delta_1, \Delta_2 \in \Sigma_X, \) if and only if the corresponding positive asymptotic morphisms \( \{ Q_h \}, \{ P_h \}_{h \in (0,1]} : B_0(X) \to B(H) \) are asymptotically commutative, i.e. \( \lim_{h \to 0} \| Q_h (f) P_h (g) - P_h (g) Q_h (f) \| = 0, \forall f, g \in B_0(X). \)

**Proof.** Since \( A_h (\Delta_1) = Q_h (\chi_{\Delta_1}) \in B(H) \) and taking into account Proposition 30 iii), we have
\[ \lim_{h \to 0} \| Q_h (\chi_{\Delta_1}) P_h (g) - P_h (g) Q_h (\chi_{\Delta_1}) \| = 0, \forall \Delta_1 \in \Sigma_X \text{ and } \forall g \in B_0(X). \]

Let \( f \in S_0(X) \). Then there are disjoint sets \( (\Delta_i)_{i=1,n} \) such that \( f = \sum_{i=1}^{n} \alpha_i \chi_{\Delta_i}. \)

By above relation, we have
\[ \lim_{h \to 0} \| \sum_{i=1}^{n} \alpha_i (Q_h (\chi_{\Delta_i}) P_h (g) - P_h (g) Q_h (\chi_{\Delta_i})) \| \leq \]
\[ \leq \sum_{i=1}^{n} \lim_{h \to 0} \| \alpha_i (Q_h (\chi_{\Delta_i}) P_h (g) - P_h (g) Q_h (\chi_{\Delta_i})) \| = \]
\[ \sum_{i=1}^{n} |\alpha_i| \lim_{h \to 0} \| Q_h (\chi_{\Delta_i}) P_h (g) - P_h (g) Q_h (\chi_{\Delta_i}) \| = 0. \]

Let \( f \in B_0(X) \). Then there are functions \( (f_n)_{n \in \mathbb{N}} \subset S_0(X) \) such that \( f \to f_n \). By preceding relation, we have
\[ \lim_{h \to 0} \| Q_h (f) P_h (g) - P_h (g) Q_h (f) \| = 0, \forall f, g \in B_0(X). \]

Reciprocal. Since
\[ \lim_{h \to 0} \|Q_h(f)P_h(g) - P_h(g)Q_h(f)\| = 0, \quad f, g \in S_0(X) \subset B_0(X), \]

taking \( f = \chi_{\Delta_1} \) and \( g = \chi_{\Delta_2} \) follows
\[ \lim_{h \to 0} \| A_h(\Delta_1)B_h(\Delta_2) - B_h(\Delta_2)A_h(\Delta_1) \| = 0, \quad \forall \Delta_1, \Delta_2 \in \Sigma_X. \]

\[ \textbf{Theorem 32.} \quad \text{Two asymptotic spectral measures having compact support are asymptotically equivalent if and only if the corresponding positive asymptotic morphisms are asymptotically equivalent.} \]

\[ \textbf{Proof.} \quad \text{Let} \{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma_X \to B(H) \text{ be two asymptotic spectral measures having compact support and} \{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \to B(H) \text{ the corresponding positive asymptotic morphisms. We suppose that} \{A_h\}, \{B_h\}_{h \in (0,1]} : \Sigma_X \to B(H) \text{ are asymptotically equivalent, i.e.} \]
\[ \lim_{h \to 0} \| A_h(\Delta) - B_h(\Delta) \| = 0, \forall \Delta \in \Sigma_X. \]

Since \( A_h(\Delta) = Q_h(\chi_\Delta), B_h(\Delta) = P_h(\chi_\Delta), \)

from preceding relation, we obtain
\[ \lim_{h \to 0} \| Q_h(\chi_\Delta) - P_h(\chi_\Delta) \| = 0, \forall \Delta \in \Sigma_X. \]

Let \( f \in S_0(X). \) Then there are disjoint sets \((\Delta_i)_{i=1}^n\) such that \( f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i}. \)

By above relation, we have
\[ \lim_{h \to 0} \| Q_h(f) - P_h(f) \| = \lim_{h \to 0} \| Q_h \left( \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \right) - P_h \left( \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \right) \| = \]
\[ = \lim_{h \to 0} \left\| \sum_{i=1}^n \alpha_i Q_h(\chi_{\Delta_i}) - \sum_{i=1}^n \alpha_i P_h(\chi_{\Delta_i}) \right\| = \]
\[ = \lim_{h \to 0} \left\| \sum_{i=1}^n \alpha_i (Q_h(\chi_{\Delta_i}) - P_h(\chi_{\Delta_i})) \right\| \leq \sum_{i=1}^n |\alpha_i| \left\| \lim_{h \to 0} \| Q_h(\chi_{\Delta_i}) - P_h(\chi_{\Delta_i}) \| \right\| = 0. \]

therefore
\[ \lim_{h \to 0} \| Q_h(f) - P_h(f) \| = 0, \forall f \in S_0(X). \]
Let \( f \in B_0(X) \). Then there are functions \( (f_n)_{n \in \mathbb{N}} \subset S_0(X) \) such that \( f \to f_n \). Since \( Q_h, P_h \in B(H) \) and from above relation, we have

\[
\lim_{h \to 0} \|Q_h(f) - P_h(f)\| = 0, \forall f \in B_0(X).
\]

Reciprocal. We suppose that \( \{Q_h\}, \{P_h\} \subset B_0(X) \) are asymptotically equivalent, i.e. \( \lim_{h \to 0} \|Q_h(f) - P_h(f)\| = 0, \forall f \in B_0(X) \).

Taking \( f = \chi_\Delta \) we have

\[
\lim_{h \to 0} \|A_h(\Delta) - B_h(\Delta)\| = \lim_{h \to 0} \|Q_h(\chi_\Delta) - P_h(\chi_\Delta)\| = 0, \forall \Delta \in \Sigma_X.
\]

\[\square\]

**Proposition 33.** Let \( \{A_h\}_{h \in (0,1]} : \Sigma_X \to B(H) \) be an asymptotic spectral measure having property \( A_h(C_X) \subset B \forall h \in (0,1] \) and \( \{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H) \) the corresponding positive asymptotic morphism. Then

\[
\text{spec}(A_h) = \text{supp}(Q_h), \forall h \in (0,1].
\]

**Proof.** Let \( \Delta \) be an open set such that \( \text{supp}(Q_h) \cap \Delta = \emptyset \). Then

\[
Q_h(f) = 0, \forall f \text{ with } \text{supp}(f) \subset \Delta.
\]

Taking \( f = \chi_\Delta \), we have \( A_h(\Delta) = Q_h(\chi_\Delta) = 0 \). Thus \( \Delta \subset X\setminus \text{spec}(A_h) \), for each open set \( \Delta \) such that \( \text{supp}(Q_h) \cap \Delta = \emptyset \). Therefore

\[
\text{spec}(A_h) \subseteq \text{supp}(Q_h), \forall h \in (0,1].
\]

Reciprocal. Let \( \Delta \) be an open set such that \( \Delta \subset X\setminus \text{spec}(A_h) \). Then \( Q_h(\chi_\Delta) = A_h(\Delta) = 0 \).

Let \( f \in S_0(X) \) such that \( \text{supp}(f) \subset X\setminus \text{spec}(A_h) \). Then there are disjoint sets \( (\Delta_i)_{i=1,n} \) such that \( f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \), and \( \Delta_i \subset X\setminus \text{spec}(A_h) \), \( \forall i \in [1,n] \). From the preceding relation, we have

\[
Q_h(f) = Q_h \left( \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \right) = \sum_{i=1}^n \alpha_i Q_h(\chi_{\Delta_i}) = 0.
\]

Let \( f \in B_0(X) \) such that \( \text{supp}(f) \subset X\setminus \text{spec}(A_h) \). Then there are functions \( (f_n)_{n \in \mathbb{N}} \subset B_0(X) \) such that \( f \to f_n \) and \( \text{supp}(f_n) \subset X\setminus \text{spec}(A_h) \), \( \forall n \in \mathbb{N} \). From the preceding relation, we have \( Q_h(f) = 0, \forall f \in B_0(X) \) such that \( \text{supp}(f) \subset X\setminus \text{spec}(A_h) \). Thus

\[
\text{supp}(Q_h) \subseteq \text{spec}(A_h), \forall h \in (0,1].
\]

\[\square\]
Remark 34. Let \( \{ A_h \}_{h \in [0,1]} : \Sigma_X \to B(H) \) be an asymptotic spectral measure having property \( A_h(C_X) \subset B \forall h \in (0,1] \) and \( \{ Q_h \}_{h \in (0,1]} : B_0(X) \to B(H) \) the corresponding positive asymptotic morphism. Then \( \{ A_h \} \) has compact support if and only if \( \{ Q_h \} \) has compact support.

Proposition 35. Let \( \{ A_h \}_{h \in [0,1]} : \Sigma_X \to B(H) \) be an asymptotic spectral measure having compact support and \( \{ Q_h \}_{h \in [0,1]} : B_0(X) \to B(H) \) the corresponding positive asymptotic morphism. The corresponding positive asymptotic morphism of asymptotic spectral measure \( \{ A_h^a \}_{h \in [0,1]} : \Sigma_X \to B(H) \), given by \( A_h^a(b) = A_h(a \cap b) \) \( \forall b \in B \) and \( \forall h \in (0,1] \), is \( \{ Q_h^a \}_{h \in [0,1]} : B_0 \to B(H) \) given by

\[
Q_h^a(f) = Q_h(\chi_a f), \quad \forall f \in B_0(X) \text{ and } \forall h \in (0,1].
\]

Proof. Since \( \{ A_h \}_{h \in [0,1]} : \Sigma_X \to B(H) \) is an asymptotic spectral measure having compact support, by Proposition 23 follows that \( \{ A_h^a \}_{h \in [0,1]} : \Sigma_X \to B(H) \), given by relation \( A_h^a(b) = A_h(a \cap b) \) \( \forall b \in B \) and \( \forall h \in (0,1] \), is an asymptotic spectral measure. In addition, by Remark of Proposition 24, results that \( \{ A_h^a \} \) has a compact support.

Taking into account the following relation \( \chi_{a \cap b} = \chi_a \chi_b \), we have

\[
A_h^a(b) = A_h(a \cap b) = Q_h(\chi_{a \cap b}) = Q_h(\chi_a \chi_b) = Q_h^a(\chi_b),
\]

\( \forall b \in B \) and \( \forall h \in (0,1] \).

Let \( f \in B_0(C) \). Since \( \{ Q_h \} \) is the corresponding positive asymptotic morphism of asymptotic spectral measure \( \{ A_h \} \) and having in view that

\[
A_h^a(x) = \chi_a(x) A_h(x),
\]

we have that

\[
Q_h^a(f) = \int_X f(x) \chi_a(x) dA_h(x) = \int_a f(x) dA_h(x) = \int_X f(x) d\chi_a(x) A_h(x) = \int_X f(x) dA_h^a(x).
\]

Remark 36. Even if \( \{ Q_h \} \) is a positive asymptotic morphism, \( \{ Q_h^a \} \) is not necessary a positive asymptotic morphism, because \( Q_h^a(1) = Q_h(\chi_a), \forall h \in (0,1] \).

Corollary 37. Let \( \{ A_h \}_{h \in [0,1]} : \Sigma_X \to B(H) \) and \( \{ A_h^a \}_{h \in [0,1]} : \Sigma_X \to B(H) \) as in the preceding Proposition. Then

\[
\lim_{h \to 0} \left\| \int_X f(x) dA_h^a(x) - A_h(a) \int_X f(x) dA_h(x) \right\| = 0.
\]

\( \forall f \in B_0(X) \).
Proof. Let \( \{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H) \) be the corresponding positive asymptotic morphism of \( \{A_h\} \) and \( \{Q_h^a\} \) given by \( Q_h^a(f) = Q_h(\chi_a f), \forall f \in B_0(X) \). Then

\[
\lim_{h \to 0} \left\| \int_X f(x) dA_h(x) - A_h(\chi_{\Delta}) \right\| = \lim_{h \to 0} \left\| Q_h(f) - A_h(\chi_{\Delta}) Q_h(f) \right\| = 0.
\]

\( \forall f \in B_0(X) \).

Corollary 38. Let \( \{Q_h\}, \{P_h\}_{h \in (0,1]} : B_0(X) \to B(H) \) be two positive asymptotic morphisms and \( \{Q_h^a\}, \{P_h^a\}_{h \in (0,1]} : B \to B(H) \) as in Proposition 35. Then \( \{Q_h^a\}, \{P_h^a\}_{h \in (0,1]} \) are asymptotically equivalent if and only if \( \{Q_h\}, \{P_h^a\}_{h \in (0,1]} \) are asymptotically equivalent for any \( a \in B \).

Proof. Applying Proposition 35 and Proposition 26.

Theorem 39. Let \( \{A_h\}_{h \in (0,1]} : \Sigma_X \to B(H) \) be an asymptotic spectral measure having property \( A_h(C_X) \subset B \forall h \in (0,1] \) and \( \{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H) \) the corresponding positive asymptotic morphism. Then

\[
\text{spec}(\{A_h\}) = \text{supp}(\{Q_h\}).
\]

Proof. Let \( \Delta \) be an open set such that \( \text{supp}(\{Q_h\}) \cap \Delta = \emptyset \). Then

\[
\lim_{h \to 0} \|Q_h(f)\| = 0, \forall f \text{ with } \text{supp}(f) \subset \Delta.
\]

Taking \( = \chi_{\Delta} \), we have

\[
\lim_{h \to 0} \|A_h(\chi_{\Delta})\| = \lim_{h \to 0} \|Q_h(\chi_{\Delta})\| = 0.
\]

Thus \( \Delta \subset X \setminus \text{spec}(\{A_h\}) \), for each open set \( \Delta \) such that \( \text{supp}(\{Q_h\}) \cap \Delta = \emptyset \). Therefore

\[
\text{spec}(\{A_h\}) \subseteq \text{supp}(\{Q_h\}).
\]

Reciprocal. Let \( \Delta \) be an open set such that \( \Delta \subset X \setminus \text{spec}(A_h) \). Then

\[
Q_h(\chi_{\Delta}) = A_h(\Delta) = 0.
\]

Let \( f \in S_0(X) \) such that \( \text{supp}(f) \subset X \setminus \text{spec}(\{A_h\}) \). Then there are disjoint sets \( (\Delta_i)_{i=1,n} \) such that \( f = \sum_{i=1}^n \alpha_i \chi_{\Delta_i} \) and \( \Delta_i \subset X \setminus \text{spec}(\{A_h\}) \), \( \forall i = 1, n \). From the previous relation, we have
\[ Q_h(f) = Q_h \left( \sum_{i=1}^{n} \alpha_i \chi_{\Delta_i} \right) = \sum_{i=1}^{n} \alpha_i Q_h(\chi_{\Delta_i}) = 0. \]

When \( h \to 0 \) into the prior relation, it results

\[ \lim_{h \to 0} \| Q_h(f) \| = 0, \]

\( \forall f \in S_0(X) \) such that \( \text{supp}(f) \subset X \setminus \text{spec}({A_h}). \)

Let \( f \in B_0(X) \) such that \( \text{supp}(f) \subset X \setminus \text{spec}(A_h) \). Then there are functions \((f_n)_{n \in \mathbb{N}} \subset S_0(X)\) such that \( f \to f_n \) and \( \text{supp}(f_n) \subset X \setminus \text{spec}(A_h), \forall n \in \mathbb{N}. \) From the previous relation, we have

\[ \lim_{h \to 0} \| Q_h(f) \| = 0, \]

\( \forall f \in B_0(X) \) such that \( \text{supp}(f) \subset X \setminus \text{spec}(A_h). \)

Therefore

\[ \text{supp}({Q_h}) \subseteq \text{spec}({A_h}). \]

**Corollary 40.** Let \{\( Q_h \), \( P_h \}_{h \in (0,1]} : B_0(X) \to B(H)\} be two corresponding positive asymptotic morphisms having property \( Q_h(C_0(X)) \subset B \) and \( P_h(C_0(X)) \subset B \) \( \forall h \in (0,1]. \) Then

\[ \text{supp}({Q_h}) = \text{supp}({P_h}). \]

**Proof.** By Theorem 39 and Theorem 21.

**Corollary 41.** Let \{\( Q_h \), \( P_h \)\}_{h \in (0,1]} : \( B_0(X) \to B(H) \) be two corresponding positive asymptotic morphisms of two compact asymptotic spectral measures. Then \{\( Q_h \)\} has compact support if and only if \{\( P_h \)\} has compact support.

**Proof.** By Theorem 39 and Theorem 21.

**Theorem 42.** Let \{\( A_h \)\}_{h \in [0,1]} : \( \Sigma_X \to B(H) \) be an asymptotic spectral measure having compact support and \{\( A_h^a \)\}_{h \in [0,1]} : \( \Sigma_X \to B(H), \) given by \( A_h^a(b) = A_h(a \cap b) \) \( \forall b \in B \) and \( \forall h \in (0,1]. \) Then

\[ \text{spec}({A_h^a}) = \text{spec}({A_h}), \forall h \in [0,1]. \]
Proof. We show that
\[ \text{spec}\left(\{A_h^a\}\right) \subseteq \pi \cap \text{spec}\left(\{A_h\}\right). \]
By Proposition 24 we have
\[ \text{spec}(A_h^a) \subset \pi \cap \text{spec}(A_h). \]
By Remark 18 i), it follows
\[ \text{spec}\left(\{A_h^a\}\right) \subset \pi. \]
Let \( b \) be a compact set such that \( b \subset \mathbb{C} \setminus \text{spec}(\{A_h^a\}) \). Then there are open sets \((b_i)_{i=1}^n\) such that \( b \subset \bigcup_{i=1}^n b_i \), \( b_i \subset \mathbb{C} \setminus \text{spec}(\{A_h\}) \) and it results
\[ \lim_{h \to 0} \|A_h(b_i)\| = 0. \]
Since each \( A_h \) is additive, we have
\[ A_h(b) \leq A_h\left(\bigcup_{i=1}^n b_i\right) = \sum_{i=1}^n A_h(b_i). \]
When \( h \to 0 \), it follows
\[ \lim_{h \to 0} \|A_h(b)\| \leq \sum_{i=1}^n \lim_{h \to 0} \|A_h(b_i)\| = 0. \]
How
\[ \lim_{h \to 0} \|A_h^a(b)\| = \lim_{h \to 0} \|A_h(a \cap b)\| \leq \sum_{i=1}^n \lim_{h \to 0} \|A_h(a \cap b_i)\| = \sum_{i=1}^n \lim_{h \to 0} \|A_h(a)A_h(b_i)\| \leq \sum_{i=1}^n \lim_{h \to 0} \|A_h(b_i)\| = 0, \]
it results \( b \subset \mathbb{C} \setminus \text{spec}(\{A_h^a\}) \) for each compact set \( b \) such that \( b \subset \mathbb{C} \setminus \text{spec}(\{A_h\}) \).
Therefore
\[ \text{spec}(\{A_h^a\}) \subseteq \text{spec}(\{A_h\}) \]
(by regularity of measures \( A_h \)).
Reciprocal. Let \( \{Q_h\}_{h \in (0,1]} : B_0(X) \to B(H) \) be the corresponding positive asymptotic morphism of \( \{A_h\} \). We show
\[
\text{supp } (\{Q^a_h\}) \supseteq \pi \cap \text{supp } (\{Q_h\}) ,
\]

where \( \{Q^a_h\}_{h \in (0,1]} : B_0 \to B(H) \) is given by

\[
Q^a_h (f) = Q_h (\chi_a f) .
\]

Let \( a \) be a set such that \( \pi \cap \text{supp } (\{Q_h\}) \subset F \). Let \( f \in B_0 (X) \) such that \( \text{supp } (f) \subset X \setminus F \). Then \( \text{supp } (f) \cap F = \emptyset \) and thus \( \lim_{h \to 0} \| Q_h (f) \| = 0 \).

Let \( \{Q^a_h\}_{h \in (0,1]} : B_0 \to B(H) \) given by

\[
Q^a_h (f) = Q_h (\chi_a f), \ \forall f \in B_0 (X) \text{ and } \forall h \in (0,1].
\]

The, form the previous relation and since \( \lim_{h \to 0} \| Q_h (\chi_a) \| \leq 1 \), we have

\[
\lim_{h \to 0} \| Q^a_h (f) \| = \lim_{h \to 0} \| Q_h (\chi_a f) \| \leq \lim_{h \to 0} \| Q_h (\chi_a) Q_h (f) \| \leq \lim_{h \to 0} \| Q_h (\chi_a) \| \lim_{h \to 0} \| Q_h (f) \| \leq \lim_{h \to 0} \| Q_h (f) \| = 0 ,
\]

\( f \forall \in B_0 (X) \) such that \( \text{supp } (f) \subset X \setminus F \). Thus \( c \supp (\{Q^a_h\}) \subset F \).

By Theorem 39 we have

\[
\text{spec } (\{A_h\}) = \text{supp } (\{Q_h\}),
\]

thus

\[
\pi \cap \text{spec } (\{A_h\}) \subseteq \text{spec } (\{A^a_h\}) .
\]

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