Coboson formalism for Cooper pairs used to derive Richardson’s equations

Monique Combescot\textsuperscript{1,2} and Guojun Zhu\textsuperscript{2}

\textsuperscript{1} Institut des NanoSciences de Paris, Université Pierre et Marie Curie, CNRS, Campus Boucicaut, 140 rue de Lourmel, 75015 Paris, France

\textsuperscript{2} Department of Physics, University of Illinois at Urbana-Champaign, 1110 W Green St, Urbana, IL, 61801

July 22, 2010

Abstract. We propose a many-body formalism for Cooper pairs which has similarities to the one we recently developed for composite boson excitons (coboson in short). Its Shiva diagram representation evidences that \( N \) Cooper pairs differ from \( N \) single pairs through electron exchange only: no direct coupling exists due to the very peculiar form of the BCS potential. As a first application, we here use this formalism to derive Richardson’s equations for the exact eigenstates of \( N \) Cooper pairs. This gives hints on why the \( N(N - 1) \) dependence of the \( N \)-pair ground state energy we recently obtained by solving Richardson’s equations analytically in the low density limit, stays valid up to the dense regime, no higher order dependence exists even under large overlap, a surprising result hard to accept at first. We also briefly question the BCS wave function ansatz compared to Richardson’s exact form, in the light of our understanding of coboson many-body effects.

PACS. XX.XX.XX No PACS code given

1 Introduction

It is commonly accepted that the Pauli exclusion principle plays a key role in superconductivity. None the less, the precise way Pauli blocking transforms a collection of single Cooper pairs into a superconducting condensate, still is an open problem. This precise understanding goes through the study of Cooper pairs not within the grand canonical ensemble as done in the standard theory proposed by Bardeen, Cooper and Schrieffer (BCS), but within the canonical ensemble. To handle the Pauli exclusion principle between a fixed number of interacting fermions is however known to be quite difficult when these fermions are paired. Turning to the grand canonical ensemble makes the task far easier. This is why superconductivity has been tackled this way, the two procedures being equally valid in the thermodynamical limit. Yet, adding fermion pairs one by one constitutes the one and only way to fully control the increasing effect of Pauli blocking from the dilute to the dense regime of pairs.

Five years after the BCS milestone paper\textsuperscript{[1]} on superconductivity, Richardson succeeded to solve this \( N \)-body problem formally\textsuperscript{[2,3,4,5]} (see also reference \textsuperscript{[6,7]}). He showed that the exact eigenstates of the Schrödinger equation for an arbitrary number \( N \) of pairs can be expressed in terms of \( N \) parameters, \( R_1, ..., R_N \) which are solutions of \( N \) coupled non-linear equations, the energy of these \( N \) pairs reading as \( E_N = R_1 + ... + R_N \). Although this exact form is definitely very smart, to use it in practice is not that easy: Except in the infinite \( N \) limit for which the BCS energy has been recovered \textsuperscript{[5]}, the equations giving \( R_1, ..., R_N \) have not been, up to now, analytically solved for arbitrary \( N \) and arbitrary interaction strength. The only approaches are numerical\textsuperscript{[8,9,10]}. This probably is why Richardson’s solution has not had so far the attention it deserves. Nowadays, these equations are commonly addressed numerically to study superconducting granules having small number of pairs\textsuperscript{[5,11,12,13]}.

Last year, we decided to tackle again these Richardson’s equations because we wanted to reveal the deep connection which has to exist between two well-known problems, namely the one-pair problem solved by Cooper and the many-pair problem considered by Bardeen, Cooper and Schrieffer. These two problems have intrinsic similarities: In both cases, there is a “frozen” core of non-interacting electrons. Above this core, there is a potential layer with attraction between up and down spin electrons having opposite momenta. In the one-pair problem, this layer contains one electron pair only, while in the standard BCS configuration, the potential layer is half-filled - a symmetrical potential on both sides of the Fermi level just corresponds to fill half the layer. It is clear that, by adding more and more pairs to the frozen Fermi sea, we must go from the one-pair problem studied by Cooper to the dense regime studied by Bardeen, Cooper and Schrieffer.
Although, at the present time, such a continuous pair increase does not seem easy to experimentally achieve, this increase can at least be seen as a gedanken experiment to study the evolution of the energy spectrum when the filling of the potential layer is changed, in order to understand the exact role of the Pauli exclusion principle in superconductivity. This procedure can also be seen as a simple but well-defined toy model to tackle the BEC-BCS crossover since, by changing the number of pairs, we change their overlap. An overlap change has already been considered by Eagles [14], and also by Leggett [15], through the change of the interaction strength between pairs. In their approach, the number of pairs is fixed, so that Pauli blocking does not change when the overlap changes while it increases when the pair number increases. Consequently, the two types of overlap change do not involve the same physics. This is why to change the overlap by changing the pair number at constant potential is a fully relevant problem, complementary to the one studied in the past by Eagles and by Leggett. We wish to mention that Ortiz and Dukelsky have applied Richardson’s approach to the BEC-BCS crossover problem with a different perspective, the overlap being varied by changing the interaction strength[16], along Eagles’ and Leggett’s idea. Some interesting comparisons between the BCS ansatz and Richardson’s solution also follow from their work.

Since the Richardson’s procedure allows one to fix the pair number and vary this number at will from one to half filling, we seriously reconsidered solving these equations analytically in order to better understand how superconductivity develops from a collection of single pairs through the $N$ dependence of their ground state energy. By turning to the dimensionless form of Richardson’s equations, we succeeded to solve these equations analytically at lowest order of density in the dilute limit on the single pair scale. Indeed, these equations do have a small dimensionless parameter which is the inverse of the number of pairs $N_c$ from which overlap between noninteracting single pairs would start. This allowed us to demonstrate that, for $N$ arbitrary large but $N/ N_c$ still small, the energy of $N$ Cooper pairs reads, at lowest order in $1/ N_c$, as

$$E_N = N \left[ \left( 2 \epsilon_{F_0} + \frac{N - 1}{\rho_0} \right) - \epsilon_c \left( 1 - \frac{N - 1}{N \Omega} \right) \right]$$

$\epsilon_{F_0}$ is the Fermi energy of the frozen sea.

$\epsilon_c \approx 2 \Omega \exp (-2/ \rho_0 V)$ is the single pair binding energy for a small potential amplitude $V$ (weak-coupling limit).

$\rho_0$ is the density of states within the potential layer, taken as constant. It linearly increases with sample size. $N_c = \rho_0 \epsilon_c$ while $N \Omega = \rho_0 \Omega$ is the number of free pair states in the potential layer, $\Omega$ being the layer extension.

A $N(N - 1)$ dependence in the energy of $N$ pairs suggest interaction treated at lowest order in density. In spite of it, the above result fully agrees with the textbook energy obtained in the dense BCS configuration. In other words, all higher order terms in the $N$ dependence of the energy cancel exactly, even under strong overlap. Indeed, the first term of Eq. (1) is the exact energy of $N$ pairs in the normal state whatever this pair is. For a constant density of states $\rho_0$, the kinetic energy of $N$ free pairs above the frozen Fermi sea, is given by

$$E_N^{(normal)} = 2 \left[ \epsilon_{F_0} + (\epsilon_{F_0} + 1/\rho_0) + \ldots + (\epsilon_{F_0} + (N - 1)/\rho_0) \right]$$

which is exactly equal to the first term of Eq. (1).

If we now turn to the condensation energy in the BCS configuration, obtained, for a number of pairs corresponding to fill half the potential layer, we find, according to Eq. (1)

$$E_N^{(normal)} - E_N = \frac{N \Omega \epsilon_c}{2} = \frac{1}{2} \rho_0 \Omega^2 \epsilon_c^2 e^{-2/ \rho_0 V}$$

This result exactly matches the energy $\rho_0 \Delta^2/2$ obtained by Bardeen, Cooper, and Schrieffer within the grand canonical ensemble using their wave function ansatz, since the gap $\Delta$ reads as $2 \omega_c \exp (-1/ \rho_0 V)$ where $2 \omega_c$ is just the potential layer extension $\Omega$.

The validity of a $N(N - 1)$ interaction term over the whole density range seems to indicate that either Cooper pairs are not involved in many-body effects higher than 2x2, or some magic cancellation takes place even in the dilute limit on the single pair scale. This strongly indicates that some unrevealed physics must hide behind such a surprising $N$-dependence, hard to accept at first.

As the Pauli exclusion principle is said to play a key role in superconductivity while Pauli blocking between $N$ paired fermions is commonly known to be difficult to handle properly, it can appear of interest to approach many-body effects with Cooper pairs through a composite boson formalism similar to the one we have successfully developed for the many-body physics of excitons in semiconductors[15].

The main purpose of the present work is to settle such a formalism. The physics being fully determined by the Hamiltonian, the major difference between an exciton gas and a set of Cooper pairs of course lies in the potential. Excitons interact via the Coulomb potential between its carriers. For Cooper pairs, we here take the usual “reduced” BCS potential without questioning it. While its relevance has been proved in many physical effects, its main advantage surely is its simplicity which makes it “solvable” - even more than originally thought by Bardeen, Cooper and Schrieffer, as seen from Richardson’s works and the quite recent analytical solution we found to his equations.

Coulomb potential is long-range, the BCS potential, taken as separable, is short range. This can be a reason for Cooper pairs to stay bound at large densities while excitons break through a Mott transition when the density increases. However, to our opinion, the crucial difference between excitons and Cooper pairs lies in the fact that usual excitons are made of fermion pairs having two degrees of freedom while the electrons which interact by the BCS potential have opposite momenta, so that electron pairs have one degree of freedom only. Actually, there also are excitons made of pairs with one degree of freedom: Frenkel excitons. Those exist in organic materials while Wannier excitons are found in inorganic materials.
The latter are made from a free electron and a free hole, attracted by intraband Coulomb processes. By contrast, Frenkel excitons are made of atomic excitations on ion sites which are delocalized into exciton by Coulomb processes between atomic levels. It is worth noting that we have found the same $N(N - 1)$ dependence for the hamiltonian mean value taken between $N$ ground state Frenkel excitons [19], while Wannier excitons have been shown to have terms in $N(N - 1)(N - 2)$ and higher [20].

Another important difference between excitons and Cooper pairs is that Cooper pairs are said to stay bound in the dense regime, i.e., under strong overlap, while excitons dissociate through a Mott transition. As a result, excitons when they exist, always are in the dilute limit while the relevant regime for superconductivity is the dense regime. Due to this, creation operators for single exciton eigenstates are relevant operators to tackle the exciton many-body physics while the single Cooper pair operator is probably not a relevant operator in the dense regime. We can nevertheless develop a composite boson many-body formalism not for correlated pairs but for the free pairs out of which the BCS condensate is made. This is what we here do.

As a first interesting outcome of this formalism, we clearly see that, due to the very peculiar form of the reduced BCS potential, two pairs of free electrons with opposite spins and opposite momenta have an interaction scattering which is a succession of a fermion exchange between pairs followed by a fermion interaction inside one pair. Since fermion exchange physically comes from the Pauli exclusion principle, this formalism evidences that two electron pairs interact, within the BCS potential, due to Pauli blocking only. However, as the Pauli exclusion principle acts between any number of pairs, exchange interaction scatterings - which originate from this Pauli exclusion - should be a priori exist between more than two pairs. This strongly questions the $N(N - 1)$ dependence of the $N$-pair energy we found: Why Pauli-induced $N \times N$ exchanges do not show up through higher order terms in the energy?

The coboson formalism is capable of exactly handling Pauli blocking between an arbitrary number of composite bosons. Since the exact eigenstates of $N$ pairs have been shown to follow from Richardson’s equations, a relevant first application of this formalism is to address the eigenenergies of $N$ pairs in order to show how Richardson’s equations follow from this formalism, and to possibly understand the origin of the eigenenergy $N$ dependency we found. In doing so, we see that $N - 2$ pairs stay unchanged when the Hamiltonian $H$ acts on $N$ pairs. Since these pairs have one degree of freedom only, they cannot exchange their fermions in order to generate higher order exchange Coulomb scattering as in the case of Wannier excitons. This can be the physical reason for the ground state energy of $N$ pairs to depend on $N$ as $N(N - 1)$ only, with no higher order term, whatever $N$, a result hard to accept, especially when pairs strongly overlap.

The paper is organized as follow:

In section 2, we present the coboson formalism appropriate to many-body effects between the free electron pairs on which Cooper pairs are constructed. We derive the Pauli and interaction scatterings for these free pairs.

In section 3, we use this formalism to rederive Richardson’s form of the exact eigenstates for $N = 1, 2, 3, \ldots$ pairs interacting through the reduced BCS potential, in order to see how the solution for general $N$ develops. We then derive this general $N$ solution explicitly.

In section III, we physically analyze the role of the Pauli exclusion principle in a collection of Cooper pairs. We, in particular, show that the $1/(R_i - R_j)$ terms in Richardson’s equations readily follow from Pauli scatterings for fermion exchanges between electron pairs. The Richardson’s parameters $R_i$ do have $N$ different values just because of Pauli blocking between Cooper pair components. As a direct consequence, the $N$-pair ground state must be fundamentally different from the BCS ansatz. We then question this ansatz in the light of our general understanding of the many-body physics of composite bosons. In this section, we also briefly discuss the major difference between Cooper pairs and Wannier excitons which could possibly explain why the energy of $N$ Wannier excitons has terms in $N(N - 1)(N - 2)$ and higher while they do not exist for Cooper pairs.

In the last section, we conclude.

2 Commutation technique for free fermion pairs making Cooper pairs

In our recent works on the many-body physics of composite bosons - essentially concentrated on semiconductor excitons - we have proposed a “commutation technique” which allows an exact treatment of Pauli blocking between the fermionic components of these composite bosons (cobosons in short). They appear through dimensionless “Pauli scatterings” which describe fermion exchanges in the absence of fermion interaction. These dimensionless scatterings, when mixed with energy-like scatterings coming from interactions between the coboson fermionic components, allow us to deal with fermion exchanges between any number of composite particles in an exact way. For a review on this formalism and its applications to the many-body physics of semiconductor excitons, see Refs. [18].

We here construct a similar formalism for the free electron pairs on which Cooper pairs are made.

2.1 Exchange scattering

We consider free fermion pairs with zero total momentum

$$\beta_k^\dagger = a_k^\dagger b_k^- \quad (4)$$

In the case of Cooper pairs, $a_k^\dagger$ creates a up-spin electron with momentum $k$ while $b_k^- \dagger$ creates a down-spin electron with momentum $-k$. These $\beta_k^\dagger$ pairs have one degree of freedom only, namely $k$. This has to be contrasted to the
most general fermion pairs \( a_k^\dagger b_k^\dagger \), such as Wannier exciton pairs, which have two, \( \beta_k \) pairs actually have some similarity to Frenkel exciton pairs \([10]\), the index \( \mathbf{k} \) being then replaced by the excited ion site \( n \).

It is straightforward to show that the creation operators of these free fermion pairs commute

\[
[\beta_{k'}, \beta_k^\dagger] = 0
\]

These free pairs thus are boson-like particles. It however is worth noting that while \( \langle a_k^\dagger a_k \rangle^2 = 0 \) simply follows from the anticommutation of \( a_k^\dagger \) operators, the cancellation of \( \langle \beta_k \rangle^2 \) does not follow from Eq.\((5)\), but from the fact that \( \langle \beta_k \rangle^2 \) contains \( \langle a_k^\dagger \rangle^2 \). The \( \langle \beta_k \rangle^2 \) cancellation which comes from Pauli blocking, may appear to be lost when working with pair operators instead of single fermion operators. We will see that this Pauli blocking is yet preserved in the commutation algebra for free fermion pairs we develop.

If we now turn to creation and annihilation operators, their commutator reads

\[
[\beta_{k'}, \beta_k^\dagger] = \delta_{k'k} - D_{k'k}
\]

where the “deviation-from-boson operator” of two zero-momentum free fermion pairs \( D_{k'k} \) reduces to

\[
D_{k'k} = \delta_{k'k} (a_k^\dagger a_k + b_k^\dagger b_{-k})
\]

This operator which would be zero for fermion pairs taken as elementary bosons, allows us to generate the dimensionless Pauli scatterings for fermion exchanges between composite bosons in the absence of fermion interaction. Following our works on excitons\([13]\), these are formally defined through

\[
D_{k'k',k,k_2} = \sum_{k_2} \left\{ \lambda \left( k_2, k_1' \right) + (k_1' \leftrightarrow k_2') \right\} \beta_{k_2}^\dagger
\]

By noting that

\[
[\beta_{k_1}'^\dagger \beta_{k_2}^\dagger] = \delta_{k'k} \lambda (a_k^\dagger a_k + b_k^\dagger b_{-k})
\]

it is then easy to show that

\[
D_{k'k,k,k_2} = 2\beta_{k_2}^\dagger \delta_{k,k_2}^\dagger \delta_{k_1,k_2}
\]

This leads us to identify the Pauli scattering of two zero-momentum free fermion pairs appearing in Eq.\((5)\), with a product of Kronecker symbols

\[
\lambda \left( k_2, k_1' \right) = \delta_{k_1,k_2} \delta_{k_2,k_1} \delta_{k_1,k_2}
\]

Such a simple expression results from the fact that these pairs are made of two free bosons, but also from the fact that they have one degree of freedom only. Actually, this Pauli scattering is just the one we expect for fermion exchanges between \((k_1, k_2)\) pairs in the absence of fermion interaction, as visualized by the Shiva diagram of Fig.\((1a)\).

2.2 Interaction scattering

We now turn to the interaction scatterings resulting from fermion-fermion interaction. For a free fermion Hamiltonian

\[
H_0 = \sum_k \epsilon_k \left( a_k^\dagger a_k + b_k^\dagger b_k \right)
\]

Eq.\((12)\), readily gives

\[
[H_0, \beta^\dagger] = 2\epsilon_p \beta^\dagger
\]

In standard BCS superconductivity, these fermion pairs interact through the reduced potential

\[
V_{BCS} = \sum_{k} v_{k'k} \beta_k^\dagger \beta_{k'}^\dagger
\]

We will show below that this potential must be taken as separable \( v_{k'k} = -V \epsilon_{k'} w_k \), with moreover \( w_k^2 = w_k^2 \).
order to possibly find the $N$-pair eigenstates of $H_0 + V_{BCS}$ analytically.

It is of importance to note that this potential fundamentally is a (1x1) potential in the fermion pair subspace since fermion $k$ interacts with one fermion only of the other species, namely fermion ($-k$)(see Fig.(1b)). As a crucial consequence, this prevents direct interaction between two zero-momentum pairs. The only way these pairs feel each other, i.e., interact in the most general sense, is through the Pauli exclusion principle.

For this (1x1) potential, we do have

$$[V_{BCS}, \beta_p^\dagger] = \gamma_p^\dagger + V_p^\dagger$$  \hspace{1cm} (15)

where $\gamma_p^\dagger = \sum_k \beta_{kp}^\dagger \delta_{kp}$ while $V_p^\dagger$, that we will call “creation potential” of the free fermion pair, is given by

$$V_p^\dagger = -\gamma_p^\dagger \left( a_p^\dagger b_p + b_p^\dagger a_p \right)$$  \hspace{1cm} (16)

The general property of creation potentials is that they give zero when acting on vacuum. As now shown, this operator allows us to generate the interactions of the $p$ pair with the rest of the system.

While the $\gamma_p^\dagger$ part of Eq.(15) commutes with $\beta_p^\dagger$, this is not so for the creation potential $V_p^\dagger$. Using Eq. (9), its commutator precisely reads

$$[V_{p_1}^\dagger, \beta_{p_2}^\dagger] = -2\delta_{p_1 p_2} \gamma_{p_1}^\dagger \beta_{p_2}^\dagger$$

$$= -2\delta_{p_1 p_2} \sum_k \beta_{kp_2}^\dagger \beta_{p_1 k}^\dagger v_{kp_2}$$  \hspace{1cm} (17)

This allows us to identify the interaction scattering for zero-momentum free pairs, formally defined as [15]

$$[V_{p_1}^\dagger, \beta_{p_2}^\dagger] = \sum_k \chi \left( p_k^\dagger p_2^\dagger p_1^\dagger p_k^\dagger \right) \beta_{p_k}^\dagger \beta_{p_1}^\dagger$$  \hspace{1cm} (18)

with a sequence of one (2x2) fermion exchange between two pairs and one (1x1) fermion interaction inside one pair. Indeed, this sequence leads to

$$\chi (p_k^\dagger p_2^\dagger p_1^\dagger p_k^\dagger) = -\sum_k \left\{ v_{kp_1 k}^\dagger \lambda \left( p_k^\dagger p_2^\dagger p_1^\dagger p_k^\dagger \right) \right\}$$

$$= -\left(v_{kp_1 p_k}^\dagger \delta_{p_1 p_2} + v_{kp_2 p_1}^\dagger \delta_{p_2 p_1} \right) \lambda$$  \hspace{1cm} (19)

When inserted into Eq. (18), this readily gives Eq. (17).

This interaction scattering is visualized by the diagram of Fig.(1c): the free pairs $p_1$ and $p_2$ first exchange an electron. As for any exchange, this brings a minus sign. In a second step, the electrons of one of the two pairs, $p_1$ or $p_2$, interact via the BCS potential. It is clear that, since the BCS potential has a (1x1) structure within the pair subspace, the scattering between two pairs can only result, as already said, from electron exchange between pairs, i.e., Pauli blocking. This diagram evidences it.

It is worth noting that electron exchange and electron interaction do not play a symmetrical role in this interaction scattering. Indeed, process in which the interaction takes place before the exchange - instead of after as in Fig.1c - would lead to

$$- \sum_k \lambda \left( p_k^\dagger p_2^\dagger p_k^\dagger \right) v_{kp_1}^\dagger = -\delta_{p_1 p_2} \delta_{p_2 p_1} v_{p_2 p_1}$$  \hspace{1cm} (20)

which is definitely different from the first term of Eq.(19).

In the next section, we use this commutation formalism to rederive the equations that Richardson has obtained for the eigenstates of $N$ Cooper pairs through a totally different route. The new derivation we have proposed, through its diagrammatic support, enlightens some important physical aspects of this exactly solvable problem.

3 Richardson’s equations for N Cooper pairs

In order to better grasp how these equations develop, we are going to increase the number of pairs in the potential layer one by one, starting from a single pair.

3.1 One pair

We first consider a state in which one free pair $(k, -k)$ is added to a “frozen” Fermi sea $|F_0\rangle$, i.e., a sea which does not feel the BCS potential. This means that the $v_{kp}$ prefactors in Eq.(14) cancel for all $k$ belonging to $|F_0\rangle$ in order to have $V_{BCS}|F_0\rangle = 0$.

Note that this “one-pair” state actually contains $N_0+1$ electron pairs, $N_0$ being the number of pairs in the frozen sea, so that this state is a many-body state already, but in the most simple sense since the Fermi sea $|F_0\rangle$ is just there to block states by the Pauli exclusion principle. This Fermi sea mainly brings a finite density of state for all states above it, a crucial point to have a bound state in 3D whatever the weakness of the attracting BCS potential.

By choosing the zero energy such that $H_0|F_0\rangle = 0$, Eqs.(13,15) gives the hamiltonian $H = H_0 + V_{BCS}$ acting on a one-pair-free state as

$$H|\beta_k^\dagger |F_0\rangle = \left[ H, \beta_k^\dagger \right] |F_0\rangle = \left( 2\epsilon_k \beta_k^\dagger + V_k^\dagger \right) |F_0\rangle$$  \hspace{1cm} (21)

We then note that

$$V_k^\dagger |F_0\rangle = 0$$  \hspace{1cm} (22)

since the $v_{kp}$ factor included in the $\gamma_k^\dagger$ part of $V_k^\dagger$, brings $v_{kp}\delta_{p_1 p_2} |F_0\rangle = v_{kp}b_{p_2}^\dagger b_{p_1} |F_0\rangle = 0$.

Next, we subtract $E_1 \beta_k^\dagger |F_0\rangle$ to the two sides of the above equation, with $E_1$ yet undefined, but assumed to be different from any $2\epsilon_k$. We then divide the resulting equation by $(2\epsilon_k - E_1)$. This gives

$$(H - E_1) \frac{1}{2\epsilon_k - E_1} \beta_k^\dagger |F_0\rangle = \beta_k^\dagger |F_0\rangle + \frac{1}{2\epsilon_k - E_1} \gamma_k^\dagger |F_0\rangle$$  \hspace{1cm} (23)

To go further and possibly obtain the one-pair eigenstate of the hamiltonian $H$ in a compact analytical form,
it is necessary to approximate the BCS potential by a separable potential \( v_{kp} = -V w_k w_p \). The operator \( \gamma_k \) in Eq. (15) then reduces to

\[
\gamma_k = -V w_k \beta^\dagger
\]

where \( \beta^\dagger \) is given by

\[
\beta^\dagger = \sum_p w_p \beta_p^\dagger
\]

(24)

(25)

If we now multiply Eq. (23) by \( w_k \) and sum over \( k \), we end with

\[
(H - E_1) B_k^1(E_1) |F_0\rangle = \left[ 1 - V \sum_k \frac{w_k^2}{2k - E_1} \right] \beta^\dagger |F_0\rangle
\]

(26)

where the operator \( B_k^1(E) \) is defined as

\[
B_k^1(E) = \sum_k B_k^1(E) B_k^1(E) = \frac{w_k}{2k - E} \beta^\dagger
\]

(27)

Eq. (26) readily shows that \( B_k^1(E_1) |F_0\rangle \) is one-pair eigenstate of the Hamiltonian \( H \) with energy \( E_1 \), provided that the bracket in the RHS is zero, i.e., \( E_1 \) fulfills

\[
1 = V \sum_k \frac{w_k^2}{2k - E_1}
\]

(28)

This is just the well-known equation for the single pair energy derived by Cooper.

### 3.2 Two pairs

We now add two pairs to the frozen sea \( |F_0\rangle \). Eqs. (13,15) yield

\[
H \gamma_k^\dagger \beta_k^\dagger |F_0\rangle = \left( \begin{array}{cc} H, \beta_k^\dagger \beta_k^\dagger \end{array} \right) |F_0\rangle
\]

\[
= (2\epsilon_k + \epsilon_{k_2}) \beta_k^\dagger \beta_k^\dagger |F_0\rangle
\]

\[
+ |V_{k_1,k_2}\rangle + |W_{k_1,k_2}\rangle
\]

The last two terms come from interactions between the four electrons of the two pairs. The first term of Eq. (15) readily gives \( |V_{k_1,k_2}\rangle \) as

\[
|V_{k_1,k_2}\rangle = \left( \begin{array}{cc} \gamma_k^\dagger \beta_k^\dagger + \gamma_{k_2}^\dagger \beta_{k_2}^\dagger \end{array} \right) |F_0\rangle
\]

\[
= -V \left( w_{k_1} \beta_{k_1}^\dagger + \omega_{k_2} \beta_{k_2}^\dagger \right) \beta^\dagger |F_0\rangle
\]

(30)

The second term of Eq. (15) yields

\[
|W_{k_1,k_2}\rangle = \left( \begin{array}{cc} \gamma_{k_1}^\dagger \beta_{k_1}^\dagger + \gamma_{k_2}^\dagger \beta_{k_2}^\dagger \end{array} \right) |F_0\rangle
\]

(31)

To calculate it, we again use commutators. Since \( V_k^\dagger \) acting on the frozen sea \( |F_0\rangle \) gives zero (see Eq. (22)), we find from Eqs. (11,19)

\[
|W_{k_1,k_2}\rangle = \sum_{p_1,p_2} \chi_{p_1,p_2} \beta_{p_1}^\dagger \beta_{p_2}^\dagger |F_0\rangle
\]

\[
= 2V \delta_{k_1,k_2} \omega_{k_1} \beta_{k_1}^\dagger \beta^\dagger |F_0\rangle
\]

(32)

The interaction part of \( H \) acting on two free pairs is visualized by the diagram of Fig. 2. In \( |V_{k_1,k_2}\rangle \) and \( |W_{k_1,k_2}\rangle \) of the Hamiltonian \( H \) acting on two free pairs, as given in Eqs.(30) and (32)

\[
(H - E_2) B_{k_1}^1(R_1) B_{k_2}^1(R_2) |F_0\rangle = \left\{ B_{k_1}^1(R_1) \left[ w_{k_1} \beta_{k_1}^\dagger - V \omega_{k_2} \beta_{k_2}^\dagger \right] + (1 \leftrightarrow 2) \right\} |F_0\rangle
\]

\[
+ 2V \left[ \delta_{k_1,k_2} \left( V_{k_1,k_2} + V_{k_2,k_1} \right) \right] \beta^\dagger |F_0\rangle
\]

(33)

the last term coming from the exchange interaction term \( |W_{k_1,k_2}\rangle \).

As a last step, we sum over \( (k_1,k_2) \). The sum over \( (k_2) \) in the first bracket readily gives

\[
\left( 1 - V \sum_k \frac{w_k^2}{2k - E_R} \right) \beta^\dagger
\]

(34)

To calculate the sum over \( (k_1,k_2) \) in the second bracket, we first note that

\[
\frac{1}{(2\epsilon_{k_1} - R_1)(2\epsilon_{k_1} - R_2)} = \left[ \frac{1}{(2\epsilon_{k_1} - R_1)} - \frac{1}{(2\epsilon_{k_1} - R_2)} \right] \frac{1}{(R_1 - R_2)}
\]

(35)

which is valid, provided that \( R_1 \neq R_2 \), a condition that we can always enforce since the unique requirement is to have \( R_1 + R_2 = E_2 \). For \( w_k^2 = w_k \), we then find

\[
\sum_{k_1,k_2} \delta_{k_1,k_2} \frac{w_{k_1}^2}{(2\epsilon_{k_1} - R_1)(2\epsilon_{k_1} - R_2)} \beta_{k_1}^\dagger
\]

\[
= \frac{1}{(R_1 - R_2)} \left[ B^1(R_1) - B^1(R_2) \right]
\]

(36)
Summation over \((k_1, k_2)\) of Eq. 33 then yields

\[
(H - E_2) B^\dagger (R_1) B^\dagger (R_2) |F_0\rangle = \\
\left\{ B^\dagger (R_1) \left[ 1 - V \sum \frac{w_{k}^2}{2\epsilon_k - R_1} + \frac{2V}{R_1 - R_2} \right] + (1 \leftrightarrow 2) \right\} \beta^\dagger |F_0\rangle \tag{37}
\]

The above equation evidences that \(B^\dagger (R_1) B^\dagger (R_2) |F_0\rangle\) is a two-pair eigenstate of the hamiltonian \(H\) with energy \(E_2 = R_1 + R_2\) provided that the bracket in the above equations is zero, i.e., \((R_1, R_2)\) fulfill two equations, known as Richardson’s equations for two pairs

\[
1 = V \sum \frac{w_{k}^2}{2\epsilon_k - R_1} + \frac{2V}{R_1 - R_2} = (1 \leftrightarrow 2) \tag{38}
\]

### 3.3 Three pairs

We now turn to three pairs in order to see how these equations develop for an increasing number of pairs. Two usually is not generic, while three most often is. We will here see that when \(H\) acts on three pairs, one at least among the three pairs stays unchanged. This is a step toward understanding why the \(N\) dependence of the \(N\)-pair ground state energy is in \(N(N - 1)\) only, with no term in \(N(N - 1)(N - 2)\) and higher, as the validity of our low density result extrapolated to the high density BCS regime, seems to indicate.

We start with

\[
H_{\beta^\dagger_{k_1} \beta^\dagger_{k_2} \beta^\dagger_{k_3} |F_0\rangle} = \left\{ \left[ H_{\beta^\dagger_{k_1}} \beta^\dagger_{k_2} \beta^\dagger_{k_3} + \beta^\dagger_{k_1} \left[ H_{\beta^\dagger_{k_2}} \beta^\dagger_{k_3} \right] + \beta^\dagger_{k_1} \beta^\dagger_{k_2} \left[ H_{\beta^\dagger_{k_3}} \right] \right] \right\} |F_0\rangle \tag{39}
\]

Using Eqs. 15 and 15, we again split the above equation into a kinetic part and two interaction parts

\[
H_{\beta^\dagger_{k_1} \beta^\dagger_{k_2} \beta^\dagger_{k_3} |F_0\rangle} = \left\{ 2(\epsilon_{k_1} + \epsilon_{k_2} + \epsilon_{k_3}) \beta^\dagger_{k_1} \beta^\dagger_{k_2} \beta^\dagger_{k_3} |F_0\rangle + |V_{k_1, k_2, k_3}\rangle + |W_{k_1, k_2, k_3}\rangle \right\} \tag{40}
\]

As in the case of two pairs, the BCS potential generates direct processes which are given by

\[
|V_{k_1, k_2, k_3}\rangle = \left\{ \gamma^\dagger_{k_1, k_2, k_3} \beta^\dagger_{k_1, k_2, k_3} + \gamma^\dagger_{k_1, k_2, k_3} \beta^\dagger_{k_1, k_2, k_3} + \gamma^\dagger_{k_1, k_2, k_3} \beta^\dagger_{k_1, k_2, k_3} \right\} |F_0\rangle \tag{41}
\]

since \(\gamma^\dagger_{k}\) and \(\beta^\dagger_{k'}\) commute. It also generates exchange processes, which appear as

\[
|W_{k_1, k_2, k_3}\rangle = \left\{ \delta^\dagger_{k_1} \beta^\dagger_{k_2} \beta^\dagger_{k_3} + \beta^\dagger_{k_1} \delta^\dagger_{k_2} \beta^\dagger_{k_3} + \beta^\dagger_{k_1} \beta^\dagger_{k_2} \delta^\dagger_{k_3} \right\} |F_0\rangle \tag{42}
\]

To calculate them, we again use commutators and the fact that \(\gamma^\dagger_{k} |F_0\rangle = 0\). Eq. 15 allows us to rewrite \(|W_{k_1, k_2, k_3}\rangle\) in a more symmetrical form as

\[
\left\{ \delta^\dagger_{k_1} \beta^\dagger_{k_2} \beta^\dagger_{k_3} + \beta^\dagger_{k_1} \delta^\dagger_{k_2} \beta^\dagger_{k_3} + \beta^\dagger_{k_1} \beta^\dagger_{k_2} \delta^\dagger_{k_3} \right\} |F_0\rangle = \\
\sum_{k_1, k_2, k_3} \delta^\dagger_{k_1} \beta^\dagger_{k_2} \beta^\dagger_{k_3} \tag{43}
\]

The interaction part of Eq. 40, namely \(|V_{k_1, k_2, k_3}\rangle\) and \(|W_{k_1, k_2, k_3}\rangle\), is represented by the diagrams of Fig. 3. These diagrams show that \(|V_{k_1, k_2, k_3}\rangle\) has interactions inside a single pair, two pairs staying unchanged, \(|W_{k_1, k_2, k_3}\rangle\) contains processes in which the pair suffering the potential exchanges one of its electrons with a second pair, the third pair staying unchanged. There are three similar contributions, obtained by circular permutations.

![Fig. 3: Shiva diagram for the interaction part](image)
We then sum over \((k_1, k_2, k_3)\). By calculating the sums of the two brackets as for two pairs, Eqs. (45) and (39) then yield

\[
(H - E_3)B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3) | F_0 \rangle = \\
\left[ 1 - V \sum \frac{w^2_{k_i}}{2\epsilon_{k_i} - R_i} - \frac{2V}{R_i - R_j} + \frac{2V}{R_k - R_l} \right] \\
+ 2 \text{ perm.} \beta^\dagger | F_0 \rangle
\]

(46)

This leads us to conclude that the three-pair state \(B^\dagger(R_1)B^\dagger(R_2)B^\dagger(R_3) | F_0 \rangle\) is an eigenstate of the Hamiltonian \(H\) with the energy \(E_3 = R_1 + R_2 + R_3\), provided that \((R_1, R_2, R_3)\) fulfill the three equations,

\[
1 = V \sum \frac{w^2_{k_i}}{2\epsilon_{k_i} - R_i} + \frac{2V}{R_i - R_j} + \frac{2V}{R_k - R_l} \\
1 = V \sum \frac{w^2_{k_i}}{2\epsilon_{k_i} - R_i} + \frac{2V}{R_i - R_j} + \frac{2V}{R_i - R_k} \\
1 = V \sum \frac{w^2_{k_i}}{2\epsilon_{k_i} - R_i} + \frac{2V}{R_i - R_j} + \frac{2V}{R_i - R_k}
\]

(47)

### 3.4 \(N\) pairs

The above commutation technique can be easily extended to \(N\) pairs. As nicely visualized by the diagrams of Figs. 2 and 3, the effect of the BCS potential on these \(N\) pairs splits into direct and exchange processes: In the direct set, one pair is affected by the \((1\times1)\) scattering while the other \((N-1)\) pairs stay unchanged. In the exchange set, this pair, before interaction, also exchanges one of its electrons with another pair, the remaining \((N-2)\) pairs staying unchanged. This understanding shows that an increase of pair number above two, does not really change the structure of the equations since \((N-2)\) pairs stay unchanged, the pair which exchanges its fermions with the pair suffering the interaction being just one among \((N-1)\) pairs.

The procedure is rather straightforward once we have understood that either \((N-1)\) or \((N-2)\) pairs stay unaffected in the BCS interaction process. The general form of the \(N\)-pair eigenstates ultimately appears as

\[
(H - E_N)B^\dagger(R_1) \cdots B^\dagger(R_N) | F_0 \rangle = 0
\]

with \(E_N = R_1 + \cdots + R_N\), these \(R_N\)'s being solutions of \(N\) coupled equations

\[
1 = V \sum \frac{w^2_{k_i}}{2\epsilon_{k_i} - R_i} + \sum_{j \neq i} \frac{2V}{R_i - R_j} \quad \text{for} \quad i = (1, \ldots, N)
\]

(49)

Let us explicitly derive these \(N\) equations following the procedure we have used for three pairs.

The Hamiltonian acting on \(N\) pairs can be written in terms of commutators with any of these \(N\) pairs as

\[
H | \beta^\dagger_{k_1} \cdots \beta^\dagger_{k_N} | F_0 \rangle = H | \beta^\dagger_{k_1} \cdots \beta^\dagger_{k_N} | F_0 \rangle
\]

(50)

We then use Eqs. (43) to replace \(H | \beta^\dagger_{k_i} \rangle\) by \(2\epsilon_{k_i} \beta^\dagger_{k_i} + \gamma_{k_i} + V_{k_i} \beta^\dagger_{k_i}\). The first two contributions commute with the other \(\beta^\dagger_{k_i}\)'s, so that, when inserted into the above equation, they yield

\[
2 (\epsilon_{k_1} + \cdots + \epsilon_{k_N}) \prod_{i=1}^{N} \beta^\dagger_{k_i} | F_0 \rangle + | V_{k_1} \cdots k_N \rangle
\]

(51)

where the direct interaction part is given by

\[
| V_{k_1} \cdots k_N \rangle = \sum_{i=1}^{N} \gamma_{k_i} \prod_{m \neq i} \beta^\dagger_{k_m} | F_0 \rangle
\]

(52)

which is similar to Eq. (41).

The part with the creation potential \(V^\dagger_{k_i}\) is much cumbersome. We again calculate it through commutators. Let us consider one term. We start as

\[
V^\dagger_{k_i} \beta^\dagger_{k_{i+1}} \cdots \beta^\dagger_{k_N} | F_0 \rangle = \\
\left[ V^\dagger_{k_i}, \beta^\dagger_{k_{i+1}} \cdots \beta^\dagger_{k_N} | F_0 \rangle \\
+ \cdots + \beta^\dagger_{k_{i+1}} \cdots \beta^\dagger_{k_{j-1}} | V^\dagger_{k_i}, \beta^\dagger_{k_i} | F_0 \rangle \beta^\dagger_{k_{j+1}} \cdots \beta^\dagger_{k_N} | F_0 \rangle \\
+ \cdots + \beta^\dagger_{k_{i+1}} \cdots \beta^\dagger_{k_{N-1}} | V^\dagger_{k_i}, \beta^\dagger_{k_i} | F_0 \rangle \right]
\]

(53)

\[
\left[ V^\dagger_{k_i}, \beta^\dagger_{k_j} \right]
\]

makes appear the exchange interaction scattering between the \(k_i\) and \(k_j\) pairs, so that the above term generate the exchange interaction scatterings between the \(k_i\) pair and all the \(k_j\) pairs with \(i < j \leq N\). When inserted into Eq. (50), we find all possible exchange interactions between fermion pairs, so that the set of terms with creation potential ultimately gives

\[
| W_{k_1} \cdots k_N \rangle = \sum_{p_1, p_2} \beta^\dagger_{p_1} \beta^\dagger_{p_2} \sum_{i < j} \left( \frac{p_{i+1}}{p_i} \frac{k_{i+1}}{k_i} \right) \prod_{m \neq (i, j)} \beta^\dagger_{k_m} | F_0 \rangle
\]

(54)

which is similar to Eq. (42). If we now use Eq. (52) for the sum over \((p_1, p_2)\), we end with

\[
| W_{k_1} \cdots k_N \rangle = 2V \beta^\dagger \left[ \sum_{i < j} \delta_{k_i k_j} w_{k_i} / \beta^\dagger_{k_i} \prod_{m \neq (i, j)} \beta^\dagger_{k_m} \right] | F_0 \rangle
\]

(55)

which also reads in a compact form as

\[
| W_{k_1} \cdots k_N \rangle = 2V \beta^\dagger \left[ \prod_{m \neq (i, j)} \beta^\dagger_{k_m} \right] | F_0 \rangle
\]

(56)
All this leads for \( E_N \) written as \( R_1 + \ldots + R_N \) to

\[
(H - E_N) \beta_{k_1} \beta_{k_2} \cdots \beta_{k_N} |F_0\rangle \\
= \left[ (2 \epsilon_{k_1} - R_1) + \cdots + (2 \epsilon_{k_N} - R_N) \right] \beta_{k_1} \beta_{k_2} \cdots \beta_{k_N} |F_0\rangle \\
- V \beta_{k_1} \sum_{i \neq m} w_{k_i} \beta_{k_i} \prod_{m \neq i} \beta_{k_m} |F_0\rangle \\
+ 2V \beta_{k_i} \sum_{i < j} \delta_{k_i,k_j} w_{k_i} \beta_{k_i} \prod_{m \neq (i,j)} \beta_{k_m} |F_0\rangle \\
\]

(57)

To go further, we do as before: we multiply both sides of the equation by \( w_{k_1} \cdots w_{k_N}/(2 \epsilon_{k_1} - R_1) \cdots (2 \epsilon_{k_N} - R_N) \) and we sum over \((k_1, \ldots, k_N)\). The LHS readily gives

\[
(H - E_N) B^1(R_1) \cdots B^1(R_N) |F_0\rangle \\
(58)
\]

The first term in the RHS, which comes from the free pair kinetic energy, yields

\[
\left[ (\sum_{k_i} w_{k_i} \beta_{k_i}) B^1(R_2) \cdots B^1(R_N) + \cdots \right] |F_0\rangle \\
= \beta_{k_1} \sum_{i=1}^N \prod_{m \neq i} B^1(R_m) |F_0\rangle \\
(59)
\]

The first interaction term, induced by direct processes within one pair, readily leads to

\[
\beta_{k_1} \sum_{i=1}^N \left(-V \sum_k \frac{w_{k_i}^2}{2 \epsilon_{k_i} - R_i} \right) \prod_{m \neq i} B^1(R_m) |F_0\rangle \\
(60)
\]

while contributions coming from exchange interaction processes appear as

\[
2V \beta_{k_1} \left[ \sum_{k_1,k_2} \delta_{k_1,k_2} \frac{w_{k_1}^3}{(2 \epsilon_{k_1} - R_1)(2 \epsilon_{k_2} - R_2)} \beta_{k_1} \right] \\
B^1(R_3) \cdots B^1(R_N) |F_0\rangle \\
(61)
\]

By using Eq. 59 for the above bracket, we end with

\[
(H - E_N) B^1(R_1) \cdots B^1(R_N) |F_0\rangle = \\
\beta_{k_1} \sum_{i=1}^N \left[ 1 - V \sum_k \frac{w_{k_i}^2}{2 \epsilon_k - R_i} - \sum_{j \neq i} \frac{2V}{R_i - R_j} \right] \\
\sum_{m \neq i} B^1(R_m) |F_0\rangle \\
(62)
\]

This evidences that \( B^1(R_1) \cdots B^1(R_N) |F_0\rangle \) is \( N \)-pair eigenstate of the hamiltonian \( H \), with energy \( E_N = R_1 + \cdots + R_N \) provided that all the brackets in the above equation cancel. These just are the \( N \) Richardson’s equations written in Eq.(49).

### 4 Physical understanding

#### 4.1 Richardson equations and the Pauli exclusion principle

The above derivation of Richardson’s equations makes crystal clear the parts in these equations which are directly linked to the Pauli exclusion principle between fermion pairs through electron exchanges.

From a mathematical point of view, the link is rather obvious: In the absence of terms in \( V/(R_i - R_j) \), the \( N \) equations for \( R_i \) would reduce to the same equation (25), so that the solution would be \( R_i^{(0)} = E_1 \) for all \( i \). The fact that the energy of \( N \) pairs differs from \( N \) times the single pair energy \( E_1 \) entirely comes from the set of \( (R_i - R_j) \)’s different from zero.

Physically, the fact that \( E_N \) differs from \( NE_1 \) comes from interactions between Cooper pairs. Due to the (1x1) structure of the BCS potential within the pair subspace, interaction between pairs can only be mediated by fermion exchanges as evidenced from the diagram of Fig. 1C. Consequently, interactions between pairs are solely the result of the Pauli exclusion principle. This Pauli blocking mathematically appears through the various \( \delta_{p'p} \) factors in the Pauli scatterings \( \langle \beta_{p_1} \beta_{p_1} | R_{i_1} \cdots R_{i_j} \rangle \). These \( \delta \) factors are the ones of the \( |W_{k_1} \cdots k_N \rangle \) term. They ultimately lead to the various \( (R_i - R_j) \) differences in the Richardson’s equations, as easy to follow from our procedure.

In short, the Kronecker symbols in the Pauli scatterings of fermion pairs take care of states which are excluded by the Pauli exclusion principle. They induce the \( 2V/(R_i - R_j) \) terms of the Richardson’s equations which ultimately makes the energy of \( N \) pairs different from the energy of \( N \) independent pairs.

#### 4.2 Excitons versus Cooper pairs

An important feature of the ground state energy \( E_N \) for \( N \) pairs that this derivation possibly explains, is the fact that the part of \( E_N \) coming from interaction, namely \( E_N - NE_1 \) depends on \( N \) as \( N(N-1) \) only, with no higher order dependence. Indeed, Eq.(1) also reads

\[
E_N = NE_1 + N(N-1) \left( \frac{1}{\rho_0} + \frac{\epsilon_0}{N\Omega} \right) \\
(63)
\]

In order to have terms in the energy in \( N(N-1)(N-2) \), we need topologically connected diagrams between 3 pairs. The diagram of Fig 3 shows that, when \( H \) acts on three pairs, one out of them do not participate to the scattering, so that the three pairs are not connected. In the case of \( N \) pairs, this is \( (N-2) \) out of the \( N \) pairs which are not connected. Connections thus seem to exist between two pairs only.

This however is not enough to explain that higher order terms do not exist in the energy because the energy of \( N \) Wannier excitons has terms in \( N(N-1)(N-2) \) and higher \( \Omega \) while the hamiltonian acting on \( N \) Wannier excitons also leaves \( N - 2 \) excitons unchanged. Indeed, let
$B^\dagger_i$ be the creation operator of the $i$ exciton with energy $E_i$, i.e., $(H - E_i)B^\dagger_i |0\rangle = 0$. We do have, as in Eq. (60),

$$
H B^\dagger_i B^\dagger_j B^\dagger_i |0\rangle = \left\{ [H, B^\dagger_i] B^\dagger_j B^\dagger_i + B^\dagger_i [H, B^\dagger_j] B^\dagger_i + B^\dagger_i B^\dagger_j [H, B^\dagger_i] B^\dagger_i \right\} |0\rangle
$$

(64)

To calculate it, we introduce the $i$ exciton creation operator $V^\dagger_i$ defined as

$$
[H, B^\dagger_i] = E_i B^\dagger_i + V^\dagger_i
$$

(65)

which has similarity with Eqs. (12) and (13). This operator is such that $V^\dagger_i |0\rangle = 0$, as readily seen from the above equation acting on vacuum. From it, we construct the interaction scatterings of two excitons through

$$
[V^\dagger_i, B^\dagger_j] = \sum \xi \left( \frac{n}{m} \frac{i}{j} \right) B^\dagger_m B^\dagger_n
$$

(66)

which is similar to Eq. (18). When used into Eq. (64), this yields

$$
(H - E_{i_1} - E_{i_2} - E_{i_3}) B^\dagger_{i_1} B^\dagger_{i_2} B^\dagger_{i_3} |0\rangle
$$

= \sum B^\dagger_m B^\dagger_n \left\{ \xi \left( \frac{n}{m} \frac{i_1}{i_3} \right) B^\dagger_{i_3} + \xi \left( \frac{n}{m} \frac{i_1}{i_2} \right) \right\} |0\rangle
$$

(67)

As in Eq. (13), one out of the three excitons seems to stay outside in the scattering process.

Here comes the crucial difference between Wannier excitons and Cooper pairs. Wannier exciton, made of $a^\dagger_{k_1} b^\dagger_{k_2}$ pairs, have two degrees of freedom. As a direct consequence, two electrons and two holes can be associated in two different ways to form two excitons. It is possible to show that

$$
B^\dagger_i B^\dagger_j = -\sum \lambda \left( \frac{n}{m} \frac{i}{j} \right) B^\dagger_m B^\dagger_n
$$

(68)

where $\lambda \left( \frac{n}{m} \frac{i}{j} \right)$ is the Pauli scattering of two excitons, defined, as for $\beta^\dagger_{k_3}$ pairs, through

$$
B^\dagger_m B^\dagger_i = \delta_{m_i} - D_{m_i}
$$

$$
D_{m_i} B^\dagger_i = \sum \lambda \left( \frac{n}{m} \frac{j}{i} \right) + \lambda \left( \frac{m}{n} \frac{i}{j} \right) B^\dagger_i
$$

(69)

The fact that, in the first term of the RHS of Eq.(67), the $B^\dagger_i$ exciton does not participate to the scattering, is in fact somewhat artificial because, using Eq. (68), we could as well write this first term as

$$
\sum B^\dagger_m B^\dagger_p B^\dagger_q \left\{ \lambda \left( \frac{n}{m} \frac{i_1}{i_3} \right) + \lambda \left( \frac{p}{n} \frac{n}{i_3} \right) \right\} \xi \left( \frac{n}{m} \frac{i_1}{i_3} \right)
$$

(70)

This shows that the $(i_1, i_2, i_3)$ excitons can actually be involved in $3 \times 3$ connected diagram, as shown in Fig. 4.

Let us return to the $\beta^\dagger_{k_3}$ pairs making Cooper pairs. We see that, since the $k$ electron with up spin is associated to the $-k$ electron with down spin only, an equation similar to Eq. (68) does not exist. As a result the $k_3$ pair in Eq. (43) cannot be mixed with the $k_3'$ pair as in Eq. (70) to generate $3 \times 3$ connected diagram, as the one of Fig. 4.

4.3 Richardson’s exact eigenstate versus BCS ansatz

The Richardson’s procedure we have here redired, gives the exact form of the $H_0 + V_{BCS}$ eigenstates as

$$
B^\dagger(R_1) \cdots B^\dagger(R_N) |F_0\rangle
$$

(71)

with $B^\dagger(R)$ given by Eq. (27). The fact that, by construction, all the $R_i$’s are different in order for the $1/(R_i - R_j)$ factors in the Richardson’s equation not to diverge, strongly questions the standard BCS ansatz for the $N$-pair wave function since this ansatz reduces to $(B^\dagger)^N |F_0\rangle$ when projected into the $N$-pair subspace. In this ansatz, all the pairs are taken as condensed into the same state. This is physically hard to accept for composite bosons due to Pauli blocking between pairs which makes each added pair necessarily different from the previous ones, due to the fact that more and more states are occupied already.

There were in past several discussions about differences and similarities between BCS ansatz and Richardson’s exact solution, or more generally a Bethe ansatz like

$$
\prod B^\dagger_i |0\rangle
$$

from various perspectives and physical situation. However, the discussions essentially focus on recovering the correct energy or some other physical quantities, not the wave function itself, more difficult to experimentally evidence. This wave function actually is attached to the picture people commonly have of superconductivity. This is why a correct wave function is important for physical understanding, at least.

To discuss this problem, let us again start with two pairs. In a previous work, we have shown that the Richardson’s parameters for two pairs read as $R_1 = R + iR'$ and $R_2 = R - iR'$ with $R$ and $R'$ real. In the large sample limit, i.e. for $1/\rho_0$ small, the dominant terms of $R$ and $R'$ are given by $R \approx \epsilon_c + 1/\rho_0 + \epsilon_c/\rho_0 \Omega$ and $R' \approx \sqrt{2\epsilon_c/\rho_0}$. By writing $B^\dagger(R)$ as $[B^\dagger(R) + B^\dagger(R') - B^\dagger(R)]$ and sim-
ilarly for $B^\dagger(R_2)$, we get, from Eq. (72),

$$B^\dagger(R_1)B^\dagger(R_2) - [B^\dagger(R_1)]^2 = R^2 \left\{ C^\dagger_\pm C^\dagger_- - 2B^\dagger(R)D^\dagger \right\}$$

where we have set

$$C^\dagger_\pm = \sum_{\nu_k} \frac{\nu_k}{(2\epsilon_k - R)(2\epsilon_k - R \pm iR')} \beta^\dagger_k$$

$$D^\dagger = \sum_{\nu_k} \frac{\nu_k}{(2\epsilon_k - R)^2 + R'^2} \beta^\dagger_k$$

Eq. (72) shows that, in order to possibly replace $B^\dagger(R_1)B^\dagger(R_2)$ by $[B^\dagger(R_1)]^2$ as in the BCS ansatz, we must neglect terms in $R'^2$, i.e., in $1/\rho_0$. However, these $1/\rho_0$ terms are precisely those which make $E_1$ different from $E_2/2 \approx E_1 + 1/\rho_0 + \epsilon_c/\rho_0$; so that the replacement of $B^\dagger(R_1)B^\dagger(R_2)$ by a “condensed two-pair state” $(B^\dagger(E_2/2))^2$ with $E_2$ different from $E_1$ is fully inconsistent because, in this two-pair operator, we would keep contributions in $1/\rho_0$ which are as large as the ones we drop by neglecting the RHS of Eq. (72); two pairs do not condense into the same state.

Actually, it is claimed that the BCS ansatz is valid in the thermodynamical limit when $N$ is very large. It is possible to show that, for $N$ large but still in the dilute regime on the single pair scale, the $R_i$’s stay two by two complex conjugate, the imaginary part of $R_i$’s getting larger and larger as $\sqrt{N}\epsilon_c/\rho_0$ when $N$ increases. By using a similar procedure as the one we used for $N = 2$, we hardly see how, starting from the exact form of the $N$-pair eigenstate $B^\dagger(R_1) \cdots B^\dagger(R_N) |F_0\rangle$, we can possibly recover the BCS ansatz with the same creation operator for all pairs when $N$ ventures outside the dilute limit because nothing special happens in the behavior of the $R_i$’s when $N$ crosses $N_c$.

In a recent work, Ortiz and Dukelsky [16] have also considered Richardson’s equations in the thermodynamical limit. While they do recover the energy obtained from the ansatz, they conclude, like us, that Richardson’s exact wave function is substantially different from the BCS ansatz in many ways.

We wish to stress that, to the best of our knowledge, derivations of the validity of the BCS ansatz for the ground state of $N$ pairs mainly concentrate on the energy it provides (see, e.g., [24] and references therein). We of course agree that the BCS ansatz gives the correct ground state energy for $N$ pairs because the energy obtained using this ansatz is just the one we derived from the exact Richardson’s procedure, extrapolated outside the dilute limit. However, agreement on the energy by no means proves agreement on the wave function. Many examples have been given in the past with wave functions very different from the exact one, although giving correct energy. Direct experiments supporting the form of the ground state wave function however seems to be even harder to achieve than the ones possibly checking the $N$ dependence of the ground state energy given in Eq. (1). Nevertheless, it seems to us highly desirable to carefully reconsider “agreement with experiments” in the light of the exact Richardson’s wave function. It is still a rather intriguing question to understand why the minimization of the hamiltonian mean value calculated with this ansatz, leads to exactly the same energy as the one we derived by analytically solving Richardson’s equations in the dilute limit.

It is worth noting that the reduced potential used in standard BCS superconductivity has the great advantage to allow an analytical resolution of the $N$-body Schrödinger equation - which is quite inefrquent. It however is clear that this potential is highly simplified. A certain amount of corrections are necessary to make this potential more realistic. These are going to destroy the possibility to get the eigenstates analytically. However, since the BCS ansatz for the wave function with all the pairs condensed into the same state - which is commonly considered as one of the essential features of superconductivity - has been worked out within this reduced BCS potential, a precise comparison between this conventional ansatz and the exact solution of the model in the canonical ensemble, is definitely quite relevant to better understand the deep physics hidden in this ansatz.

Finally, we wish to stress that the possible replacement of $B^\dagger(R_1) \cdots B^\dagger(R_N) |F_0\rangle$ by $(B^\dagger)^N |F_0\rangle$ is crucial to support the overall picture of superconductivity commonly in mind, with all the pairs in the same state, “as an army of little soldiers, all walking similarly”. This picture actually seems to be a rather naive extrapolation to composite bosons, of the standard Bose-Einstein condensation demonstrated in the case of elementary bosons. It is hard for us to accept that, in the case of composite bosons, Pauli blocking between fermionic components is not going to destroy nice harmony in this “army”. More work on the validity of the BCS ansatz in the thermodynamical limit in the context of the coboson nature of Cooper pairs, seems a necessity to more deeply understand some unrevealed aspects of basic superconductivity as the ones at the origin of Eq. (1) for the $N$-pair ground state energy. The coboson many-body formalism we have here constructed, should appear as quite valuable because it gives a fresh view to this famous field, its Shiva diagram representation helping to support physical understanding.

5 Conclusion

We have constructed a coboson formalism for the electron pairs on which Cooper pairs are made. It has similarity with the one we have constructed for composite boson excitons. This formalism evidences that the scatterings of two zero-momentum electron pairs in the BCS potential, are mediated by the Pauli exclusion principle. No direct process exists

As a first application, we here rederive Richardson’s equations for the exact eigenstates of $N$ Cooper pairs. This derivation allows us to trace back the physical origin of the various terms. In particular, we clearly see that $N$ pairs differ from $N$ independent pairs, due to Pauli blocking only. This Pauli blocking enforces the $R_i$ parameters of Richardson’s equations to be all different. As a direct
consequence, the exact wave function for \( N \) interacting pairs is definitely different from the BCS ansatz, although the \( N \)-pair energy this ansatz provides, is the correct one in the large sample limit.

The diagrammatic representation of our derivation also shows that, because electron pairs with zero total momentum have one degree of freedom instead of two, they scatter within the BCS potential through \((2 \times 2)\) exchange interaction scatterings only. This possibly explains why the \( N \)-pair ground state energy that we have recently found in the dilute limit, has interaction terms in \( N(N - 1) \) but not in \( N(N - 1)(N - 2) \) and higher, as expected for \( N \)-body problems.

One of us (M.C.) wishes to thank the University of Illinois at Urbana-Champaign, and Tony Leggett in particular, for enlightening discussions during her invitation at the Institute for Condensed Matter Physics where most of the present work has been performed. We also wish to thank Walter Pogosov for his constructive comments on the manuscript.

References

1. J. Bardeen, L.N. Cooper, J.R. Schrieffer, Physical Review 106, 162 (1957)
2. R.W. Richardson, physics letters 3(6), 277 (1963)
3. R.W. Richardson, N. Sherman, Nucl. Phys. 52(6), 221 (1964)
4. R.W. Richardson, Journal of Mathematical Physics 9, 1327 (1968)
5. R.W. Richardson, Journal of Mathematical Physics 18, 1802 (1977)
6. M. Gaudin, États Propres et Valeurs Propres de l’Hamiltonien d’Appariement (Les Éditions de Physique, France, 1995)
7. A.G. Ushveridze, Quasi-exactly solvable models in quantum mechanics (Taylor & Francis Group LLC, 1994), ISBN 0730302666
8. J. Dukelsky, S. Pittel, G. Sierra, Rev. Mod. Phys. 76(3), 643 (2004)
9. G. Ortiz, R. Somma, J. Dukelsky, S. Rombouts, Nuclear Physics B 707(3), 421 (2005), ISSN 0550-3213
10. F. Braun, J. von Delft, Phys. Rev. Lett. 81(21), 4712 (1998)
11. J. von Delft, D.C. Ralph, Physics Reports 345(2-3), 61 (2001), ISSN 0370-1573
12. G. Sierra, J. Dukelsky, G.G. Dussel, J. von Delft, F. Braun, Phys. Rev. B 61(18), R11890 (2000)
13. M. Schechter, Y. Imry, Y. Levinson, J.v. Delft, Phys. Rev. B 63(21), 214518 (2001)
14. D.M. Eagles, Physical Review 186(2), 456 (1969)
15. A.J. Leggett, Modern Trends in the Theory of Condensed Matter, in Proceedings of the XVth Karpacz Winter School of Theoretical Physics, Karpacz, Poland (Springer-Verlag, 1980), pp. 13-27
16. G. Ortiz, J. Dukelsky, Phys. Rev. A 72(4), 043611 (2005)
17. W.V. Pogosov, M. Combescot, "moth-eaten effect" driven by pauli blocking, revealed for cooper pairs (2009), 0911.0849v2. http://arxiv.org/abs/0911.0849v2
18. M. Combescot, O. Betbeder-Matibet, F. Dubin, Physics Reports 463, 215 (2008)
19. M. Combescot, W.V. Pogosov, Eur.Phys.J.B, 68(2), 161 (2009)
20. M. Combescot, O. Betbeder-Matibet, Eur.Phys.J.B, 31(4), 517 (2003)
21. J. Bang, J. Krumlinde, Nuclear Physics A 141(1), 18 (1970), ISSN 0375-9474
22. M. Hasegawa, S. Tazaki, Phys. Rev. C 35(4), 1508 (1987)
23. W.V. Pogosov, M. Combescot, M. Crouzeix, Phys. Rev. B 81(17), 174514 (2010)
24. J.R. Schrieffer, Theory Of Superconductivity, revised edition edn. (Perseus Books, 1999), ISBN 0738201200