MULTIVARIATE LÉVY-TYPE DRIFT CHANGE DETECTION WITH UNKNOWN POST-CHANGE PARAMETER AND MORTALITY MODELING

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ABSTRACT. In this paper, using Bayesian approach, we solve the quickest drift change detection problem for a multidimensional Lévy process consisting of both a continuous gaussian part and a jump component. We allow a general a priori distribution of the change point as well as random post-change drift parameter. Classically, our optimality criterion is based on a probability of false alarm and an expected delay of the detection. The main technique uses the optimal stopping theory and is based on solving a certain free-boundary value problem. The paper is supplemented by an extensive numerical analysis related with the construction of the Generalized Shiryaev-Roberts statistic applied to analysis of Polish life tables (after proper calibration) and predicting the drift change in the correlated force of mortality of men and women jointly.

KEYWORDS. Lévy process • quickest detection • longevity • optimal stopping • force of mortality • life tables • change of measure • multidimensional jump-diffusion

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1. INTRODUCTION

Quickest detection problems, often called also disorder problems, arise in various fields of applications of mathematics, such as finance, engineering or economy. All of them address a question how to detect some changes of observed system using statistical methods. One of the main methods was based on the drift change detection using Bayesian approach; see e.g. Shiryaev [23, 24], where Brownian motion with linear drift was considered and the drift has been changing according to an exponential distribution. The original problem was reformulated in terms of a free-boundary problem and solved using optimal stopping methods. All details of this analysis are also given in a surveys [29, 31] (see also references therein). Later, the minimax approach have also been used. This method is based on identifying the optimal detection time based on so-called cumulative sums (CUSUM) strategy; see e.g. Beibel [3], Shiryaev [26] or Moustakides [13] in the Wiener case, or El Karoui et al. [9] in the Poisson case. Many of these quickest detection problems and used methods are gathered in the book of Poor and Hadjiliadis [20]. In this paper we choose the first approach.

Our first main goal is to perform this analysis for multivariate process to take into account the dependence between components. We allow a general a priori distribution of the change point and that a post-change parameter could be random as well. The novelty of this article lies in allowing observed process to have, apart of diffusion ingredient, jumps as well. This is very important in many applications appearing in actuarial science, finance, etc.

Most of works on the detection problems has been devoted to the one-dimensional diffusion processes, hence to the one-dimensional processes with continuous trajectories; see e.g. Beibel [2] or Shiryaev [24], [30] Chap. 4 and references therein. Only some particular cases of jump models without diffusion component have been already analysed, e.g. by Galchuk and Rozovskii [7], Peskir and Shiryaev [17] or Bayraktar et al. [4] for the Poisson process, by Gapeev [8] for the compound Poisson process with the exponential jumps or by Dayanik and Sezer [5] for more general compound Poisson problem and by Krawiec et al. [10] for a certain Lévy process. All of these results concern one-dimensional case only.

Finally, we assume that a drift change point θ has a general prior distribution G. In most works it has been assumed that θ can have only exponential distribution (conditioned that it is strictly positive). Furthermore, similarly like in Dayanik et al. [6], we assume that the direction of disorder is a random vector ζ with known prior multivariate distribution function H(r). That is, after the change of the drift, this new drift is chosen according to the law H. In the case when H is a one-point distribution we end up with classical question where the after-change-drift is fixed and different than zero. In the case when H is supported on two points in Rd we know that after-change-drift may take two possible values with known weights. Even for d = 1 this additional feature of our model gives much more freedom and has not been analysed in detail yet.

The methodology used in this paper is based on transferring the detection problem to a certain boundary value problem. More formally, in this paper we consider the process \(X = (X_t)_{t \geq 0}\) with

\[
X_t = \begin{cases} 
X_t^\infty, & t < \theta, \\
X_\theta^\infty + X_{t-\theta}^{(0,r)}, & t \geq \theta,
\end{cases}
\]

where \(X_t^\infty\) and \(X_{t}^{(0,r)}\) are both independent jump-diffusion processes with values in \(\mathbb{R}^d\). We assume that \(X_t^\infty\) and \(X_{t}^{(0,r)}\) are related with each other via the exponential change of measure described e.g. in Palmowski and Rolski [15]. This change of measure can be seen as a form of the drift change
between $X_t^\infty$ and $X_t^{(0,r)}$. The parameter $r$ corresponds to the rate (direction) of disorder that can be observed after time $\theta$.

Let $\theta$ has an additional atom at zero with mass $x > 0$. We choose the optimality criterion based on both probability of a false alarm and a mean delay time. That is, in this paper, we are going to find an optimal detection rule $\tau^* \in T$ for which the following infimum is attained

$$V^*(x) = \inf_{\tau \in T} \left\{ \mathbb{P}^{G,H}_x (\tau < \theta) + cx \mathbb{E}^{G,H}_x [ (\tau - \theta)^+] \right\},$$

where $T$ is the family of stopping times and the subscript $x$ associated with $\mathbb{E}^{G,H}$ indicates the starting position equal to $x$. Measure $\mathbb{P}^{G,H}$ will be formally introduced later. Firstly, we transfer above detection problem into the following optimal stopping problem

$$V^*(x) = \inf_{\tau \in T} \mathbb{E}^{G,H}_x \left[ 1 - \pi_x + c \int_0^\tau \pi_s ds \right],$$

for the a posteriori probability process $\pi = (\pi_t)_{t \geq 0}$ that also will be formally introduced later. In the next step, using the change of measure technique and stochastic calculus, we can identify the infinitesimal generator $A$ of the Markov process $\pi$. Finally, we formulate the free-boundary value problem

$$Af(x) = -cx, \quad 0 \leq x < A^*,
\quad f(x) = 1 - x, \quad A^* \leq x \leq 1,$$

with the boundary conditions

$$f(A^{*-}) = 1 - A^* \quad \text{(continuous fit)},
\quad f'(A^{*-}) = -1 \quad \text{(smooth fit)},
\quad f'(0^+) = 0 \quad \text{(normal entrance)}$$

for some optimal level $A^*$ which allows to identify the threshold optimal alarm rule as

$$\tau^* = \inf\{t \geq 0 : \pi_t \geq A^*\}.$$

We solve above boundary problem for two basic models: two-dimensional Brownian motion with two possible post-change drifts and two-dimensional Brownian motion with downward exponential jumps.

Our second main goal is to apply the solution of above detection problem to the analysis of correlated change of drift in force of mortality of men and women. The life expectancies of men and women are widely recognized as dependent on each other. For example, married people live statistically longer than single ones. Since many insurance products are engineered for marriages or couples it is crucial to detect the change of mortality rate of marriages. Indeed, the observed improvements of longevity produce challenges related with the capital requirements that has to be constituted to face this long-term risk and with creating new ways to cross-hedge or to transfer part of the longevity risk to reinsurers or to financial markets. To do this we need to perform accurate longevity projections and hence to predict the change of the drift observed in prospective life tables (national or the specific ones used in insurance companies). In this paper we focus on an extensive numerical analysis of the (Polish) life tables for both men and women. We proceed as follows. We take logarithm of force of mortality of men and women creating two-dimensional process modelled by jump-diffusion process. This process consists of observed two-dimensional drift that can be calibrated from the historical data and random mean zero Lévy-type perturbation. Based on previous theoretical work we construct a
statistical and numerical procedure based on the generalized version of the Shiryaev-Roberts statistics introduced by Shiryaev [23, 24] and Roberts [21], see also Polunchenko and Tartakovsky [18], Shiryaev [27], Pollak and Tartakovsky [19] and Moustakides et al. [14]. Precisely, we start from a continuous statistics derived from the solution of the optimal detection problem in continuous time. Then we take discrete moments $0 < t_1 < t_2 < \ldots < t_N$, construct an auxiliary statistic and raise the alarm when it exceeds certain threshold $A^*$ identified in the first part of the paper.

The paper is organized as follows. In Section 2 we describe basic setting of the problem, introduce main definitions and notation. In this section we also formulate main theoretical results of the paper. Section 3 is devoted to the construction of the Generalized Shiryaev-Roberts statistic. To apply it, we first need to find some density processes related to the processes $X$ prior and post the drift change. This is done in Section 3 as well. Particular examples are analyzed in Section 4. Next, in Section 5, we provide extensive numerical analysis based on a real life tables data. We finish our paper with some technical proofs given in Section 6.

2. Model description and main results

The main observable process is a regime-switching $d$-dimensional process $X = (X_t)_{t \geq 0}$. It changes its behavior at a random moment $\theta$ in the following way:

(2) \[ X_t = \begin{cases} X_{t}^\infty, & t < \theta, \\ X_{\theta}^\infty + X_{t-\theta}^{(0,r)}, & t \geq \theta, \end{cases} \]

where $X_{t}^\infty$ and $X_{t}^{(0,r)}$ are two different independent Lévy processes related with each other via exponential change of measure specified later. The parameter $r$ describes the drift after change $\theta$. However, we will assume that $r$ is driven by a random vector $\zeta$ with a given distribution $H(r)$ which we will formally introduce later.

More precisely, we assume the following set-up for the case when the post-change drift equals $r$. The process that we can observe after the change of the drift is a $d$-dimensional processes $X_{t}^{(0,r)} = (X_{t,1}^{(0,r)}, \ldots, X_{t,d}^{(0,r)})$ defined as

(3) \[ X_{t}^{(0,r)} := \sigma W_{t}^{r} + rt + \sum_{k=1}^{N_{t}^{(0,r)}} J_{k}^{(0,r)} - \mu^{r} m^{r} t, \]

where

- $W_{t}^{r} = (W_{t,1}^{r}, \ldots, W_{t,d}^{r})^T$ is a vector of standard independent Brownian motions,
- $\sigma = (\sigma_{i,j})_{i,j=1,\ldots,d}$ is a matrix of real numbers, responsible for the correlation of the diffusion components of $X_{t,1}^{(0,r)}, \ldots, X_{t,d}^{(0,r)}$, we assume that $\sigma_{ii} > 0$,
- $r = (r_1, \ldots, r_d)^T$ is a vector of an additional drift,
- $N_{t}^{(0,r)}$ is a Poisson process with intensity $\mu^{r}$,
- $(J_{k}^{(0,r)})_{k \geq 1}$ is a sequence of i.i.d. random vectors responsible for jump sizes; we denote each coordinate of $J_{k}^{(0,r)}$ by $J_{k,i}^{(0,r)}$ for $i = 1, \ldots, d$ and its distribution by $F_{t}^{r}$ with mean $m_{t}^{r}$; we also denote by $F^{r}$ a joint distribution of vector $J_{k}^{(0,r)}$ and by $m^{r} = (m_{1}^{r}, \ldots, m_{d}^{r})^T$ its mean.

We assume that all components of $X_{t}^{(0,r)}$ are stochastically independent, i.e. $W_{t}^{(0,r)}$, $N_{t}^{(0,r)}$ and the sequence $(J_{k}^{(0,r)})_{k=1,2,\ldots}$ are independent.
Similarly, we assume that the process that we observe prior the drift change is a $d$-dimensional process
\[ X_t^\infty = (X_t^\infty_1, \ldots, X_t^\infty_d) \]
defined as
\[ X_t^\infty := \sigma W_t^\infty + \sum_{k=1}^{N_t^\infty} J_k^\infty - \mu^\infty m^\infty t, \]
where
\[ W_t^\infty \] is again a vector of standard independent Brownian motions,
\[ \sigma \] is the same as for the process $X_t^{(0,r)}$,
\[ N_t^\infty \] is a Poisson process with intensity $\mu^\infty$,
\[ (J_k^\infty)_{k \geq 1} \] is a sequence of i.i.d. random vectors, where each coordinate $J_k^\infty$ of $J_k^\infty$ has distribution $F_t^\infty$ with mean $m_k^\infty$, we also denote by $F_t^\infty$ a joint distribution of vector $J_k^\infty$ and by $m^\infty = (m_1^\infty, \ldots, m_d^\infty)^T$ its mean.

To do it formally, following Zhitlukhin and Shiryaev [35] we consider a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ with a right-continuous filtration $\{\mathcal{F}_t\}_{t \geq 0}$ on which we define a stochastic system with disorder as follows. First, on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ for $r \in \mathbb{R}^d$ we introduce probability measures $\mathbb{P}^\infty$ and $\mathbb{P}^{(0,r)}$. Later, added subscript $t$ to a measure denotes its restriction to $\mathcal{F}_t$, that is, $\mathbb{P}_t^\infty := \mathbb{P}^\infty|_{\mathcal{F}_t}$ and $\mathbb{P}_t^{(0,r)} := \mathbb{P}^{(0,r)}|_{\mathcal{F}_t}$. We assume that for each $t \geq 0$ the restrictions $\mathbb{P}_t^\infty$ and $\mathbb{P}_t^{(0,r)}$ are equivalent. The measure $\mathbb{P}^\infty$ corresponds to the case when there is no drift change in the system at all and $\mathbb{P}^{(0,r)}$ describes the measure under which there is a drift present from the beginning (i.e. from $t = 0$). Both measures correspond to laws of the processes $X_t^\infty$ and $X_t^{(0,r)}$ described above, respectively. We also introduce a probability measure $\mathbb{P}$ that dominates $\mathbb{P}^\infty$ and $\mathbb{P}^{(0,r)}$ for each $r \in \mathbb{R}^d$ and which restriction $\mathbb{P}_t$ is equivalent to $\mathbb{P}_t^\infty$ and $\mathbb{P}_t^{(0,r)}$ for each $t \geq 0$.

We define the Radon-Nikodym derivatives
\[ L_t^{(0,r)} := \frac{d\mathbb{P}_t^{(0,r)}}{d\mathbb{P}_t^\infty}, \quad L_t^\infty := \frac{d\mathbb{P}_t^\infty}{d\mathbb{P}_t}. \]
Furthermore, for $s \in (0, \infty)$ we define
\[ L_t^{(s,r)} := L_t^\infty I(t < s) + \frac{L_t^\infty}{L_s^\infty} L_s^{(0,r)} I(t \geq s). \]

Finally, for any fixed $s \in (0, \infty)$ and $r \in \mathbb{R}^d$, by the Kolmogorov’s existence theorem, we can define measures $\mathbb{P}^{(s,r)}$ such that $\mathbb{P}^{(s,r)}|_{\mathcal{F}_t} = \mathbb{P}_t^{(s,r)}$ for $(\mathbb{P}_t^{(s,r)})_{t \geq 0}$ defined via
\[ \frac{d\mathbb{P}_t^{(s,r)}}{d\mathbb{P}_t} = L_t^{(s,r)}. \]
Note that for $t < s$ and all $r \in \mathbb{R}^d$ the following equality holds
\[ \mathbb{P}_t^\infty = \mathbb{P}_t^{(s,r)}, \]
since disorder after time $t$ does not affect the behavior of the system before time $t$.

We consider Bayesian framework, that is, we assume that the moment of disorder is a random variable $\theta$ with a given distribution function denoted by $G(s)$ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. We assume that $G(s)$ is continuous for $s > 0$ with right derivative $G'(0) > 0$. Similarly, we assume that the rate (direction) of disorder is a random vector $\zeta$ with multivariate distribution function $H(r)$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Hence, to catch this additional randomness we have to introduce an extended filtered probability
space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^{G,H})\) such that:
\[(7) \quad \Omega := \Omega \times \mathbb{R}_+ \times \mathbb{R}^d, \quad \mathcal{F} := \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d), \quad \mathcal{F}_t := \mathcal{F}_t \otimes \{\emptyset, \mathbb{R}_+\} \otimes \{\emptyset, \mathbb{R}^d\}.
\]

For \(A \in \mathcal{F}, B \in \mathcal{B}(\mathbb{R}_+)\) and \(C \in \mathcal{B}(\mathbb{R}^d)\) we define probability measure
\[
\mathbb{P}^{G,H}(A \times B \times C) := \int_C \int_B \mathbb{P}^{G,H}(s,r)(A)dG(s)dH(r).
\]

On this extended space random variables \(\theta\) and \(\zeta\) are defined by \(\theta(\omega, s, r) := s\) and \(\zeta(\omega, s, r) := r\) with
\[
\mathbb{P}^{G,H}(\theta \leq s) = G(s) \quad \text{and} \quad \mathbb{P}^{G,H}(\zeta_1 \leq r_1, \ldots, \zeta_d \leq r_d) = H(r) \quad \text{for} \quad \zeta = (\zeta_1, \ldots, \zeta_d) \quad \text{and} \quad r = (r_1, \ldots, r_d).
\]

Observe that measure \(\mathbb{P}^{G,H}\) describes formally the process \(X\) defined in (2).

In the problem of the quickest detection we are looking for an optimal stopping time \(\tau^*\) that minimizes certain optimality criterion. This criterion incorporates both the probability of false alarm and the mean delay time. Let \(T\) denotes the class of all stopping times with respect to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). Our problem can be stated as follows:

**Problem 1.** For each \(c > 0\) calculate the optimal value function
\[(8) \quad V^*(x) = \inf_{\tau \in T} \left\{ \mathbb{E}^{G,H}_x (\tau < \theta) + c\mathbb{E}^{G,H}_x [\pi^*_r] \right\}
\]
and find the optimal stopping time \(\tau^*\) for which above infimum is attained.

Above \(\mathbb{E}^{G,H}_x\) means the expectation with respect to \(\mathbb{P}^{G,H}\).

The key role in solving this problem plays a posterior probability process \(\pi := (\pi_t)_{t \geq 0}\) defined as
\[(9) \quad \pi_t := \int_{\mathbb{R}^d} \pi_t^r dH(r) \quad \text{where} \quad \pi_t^r := \mathbb{P}^{G,H}_t (\theta \leq t | \mathcal{F}_t, \zeta = r).
\]

Note that \(\pi_0 = G(0) = x\). Using this posterior probability, one can reformulate criterion (8) in the equivalent form.

**Problem 2.** For a given \(c > 0\) find the optimal value function
\[
V^*(x) = \inf_{\tau \in T} \mathbb{E}^{G,H}_x \left[ 1 - \pi^* + c \int_0^\tau \pi_s ds \right]
\]
and the optimal stopping time \(\tau^*\) such that
\[
V^*(x) = \mathbb{E}^{G,H}_x \left[ 1 - \pi^* + c \int_0^{\tau^*} \pi_s ds \right].
\]

That is, formally, the following result holds true.

**Lemma 1.** The criterion given in Problem 1 is equivalent to the criterion given in Problem 2.

Although the proof follows classical arguments, we added it in Section 3 for completeness.

Value function \(V^*(x)\) from Problem 2 is equivalent to a Mayer-Lagrange optimal stopping problem that, using general optimal stopping theory, could be transferred into the following free-boundary value problem (called Stefan problem as well):

Find a function \(f : \mathbb{R} \to \mathbb{R}\) and \(A^* \in [0, 1]\) satisfying the system
\[(10) \quad Af(x) = -cx, \quad 0 \leq x < A^*, \quad f(x) = 1 - x, \quad A^* \leq x \leq 1,
\]
with the boundary conditions
\[(11) \quad f(A^*) = 1 - A^* \quad \text{(continuous fit)},
\]
\[ f'(A^*) = -1 \quad \text{(smooth fit)}, \]
\[ f'(0^+) = 0 \quad \text{(normal entrance)}, \]
where \( A \) is the infinitesimal generator of the process \( \pi = (\pi_t)_{t \geq 0} \); see Peskir and Shiryaev [16, Chap. VI. 22], Krylov [11, p. 41], Strulovici and Szydlowski [34, Thm. 4] and [1] for details. This equivalence is formally presented in the following theorem. We will prove it in Section 6.

**Theorem 1.** Let \( A \) be a generator of the Markov process \( \pi = (\pi_t)_{t \geq 0} \) given by (15). Then the optimal value function \( V^*(x) \) from the Problem 2 solves the free-boundary problem formulated in (10) – (13) for the unique point \( A^* \in (0, 1] \). Furthermore, the optimal stopping time for the Problem 2 is given by
\[
\tau^* = \inf \{ t \geq 0 : \pi_t \geq A^* \}.
\]
To formulate properly above free-boundary problem we have to identify this infinitesimal generator \( A \) which is given in next theorem.

**Theorem 2.** The generator of the process \( \pi = (\pi_t)_{t \geq 0} \) defined in (9) for functions \( f \in C^2 \) is given by
\[ Af(x) = \int_{\mathbb{R}^d} \left\{ f'(x) \left( G'(0) + x(1-x)(\mu^\infty - \mu^r) \right) \right. \]
\[ + \frac{1}{2} f''(x) x^2 (1-x)^2 \sum_{i=1}^d \sum_{j=1}^d z_{r,i} z_{r,j} (\sigma \sigma^T)_{ij} \]
\[ + \int_{\mathbb{R}^d} \left[ f \left( \frac{x \exp \left\{ \sum_{i=1}^d z_{r,i} u \right\}}{x \exp \left( \sum_{i=1}^d z_{r,i} u \right) - 1} + 1 \right) - f(x) \right] \]
\[ \cdot \left[ (1-x) \mu^\infty dF^\infty(u) + x \mu^r dF^{(0,r)}(u) \right] \right\} dH(r). \]

We will prove this theorem later in Section 6. In the final step we solve (10)–(13) for some specific choice of model parameters, then find the optimal threshold \( A^* \) and hence the optimal alarm time. This allows us to construct a Generalized Shiryaev-Roberts statistic in this general set-up. Later we apply it to detect the changes of drift in joint (correlated) mortality of men and women based on life tables.

### 3. Generalized Shiryaev-Roberts statistic

According to Zhitlukhin and Shiryaev [35] and Shiryaev [25, II.7], by the generalized Bayes theorem the process \( \pi \) defined in (9) equals
\[
\pi_t = \int_{\mathbb{R}^d} \frac{\int_0^t L^r(s,r) \, dG(s)}{\int_0^\infty L^r(s,r) \, dG(s)} \, dH(r).
\]
We will give another representation of the process \( \pi \) in terms of the process \( L^r = (L^r_t)_{t \geq 0} \) defined by
\[
L^r_t := \frac{L^{(0,r)}_t}{L^r_t} = \frac{d\mathbb{P}^{(0,r)}}{d\mathbb{P}^r}.
\]
To find above Radon-Nikodym derivative such that process \( X \) defined in (2) indeed admits representation (5) under the measure \( \mathbb{P}^{(0,r)} \) and (4) under \( \mathbb{P}^\infty \), we assume that for given \( r = (r_1, \ldots, r_d) \in \mathbb{R}^d \)
the following relation holds
\[ \forall x \in \mathbb{R}^d \quad \mu^r F^r(dy) = \frac{h_r(x + y)}{h_r(x)} \mu^{\infty} F^{\infty}(dy), \]
where for \( x = (x_1, \ldots, x_d) \) the function \( h_r(x) : \mathbb{R}^d \to \mathbb{R} \) is given by
\[ h_r(x) = \exp \left\{ \sum_{j=1}^d z_{r,j} x_j \right\}. \]

Above the coefficients \( z_{r,1} \ldots z_{r,d} \) solve the following system of equations:
\[ \begin{cases} r_1 - \mu^r m_1^r + \mu^{\infty} m_1^{\infty} = \sum_{j=1}^d z_{r,j} (\sigma \sigma^T)_{1,j}, \\ \vdots \\ r_d - \mu^r m_d^r + \mu^{\infty} m_d^{\infty} = \sum_{j=1}^d z_{r,j} (\sigma \sigma^T)_{d,j}. \end{cases} \]

**Theorem 3.** Under the assumption (18) for given \( r \in \mathbb{R}^d \), if the Radon-Nikodym derivative \( L^r = (L_t^r)_{t \geq 0} \)
defined in (17) is given by
\[ L_t^r = \exp \left\{ \sum_{i=1}^d z_{r,i}(X_{t,i} - X_{0,i}) - K_r t \right\}, \]
where
\[ K_r = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d z_{r,i} z_{r,j} (\sigma \sigma^T)_{i,j} - \sum_{i=1}^d z_{r,i} \mu^{\infty} m_i^{\infty} + \mu^r - \mu^{\infty}, \]
then the process \( X \) defined in (2) admits representation (3) under the measure \( \mathbb{P}^{(0,r)} \) and (4) under \( \mathbb{P}^{\infty} \).

The proof will be given in Section 6.

Having the density process \( L^r \) defined in (17) identified in above theorem, we introduce an auxiliary process
\[ \psi^r_t := \int_0^t L_s^r \frac{dG(s)}{L_s^r}. \]
Then by (16) the following representation of \( \pi_t \) holds true
\[ \pi_t = \int_{\mathbb{R}^d} \frac{\psi^r_t}{\psi^r_t + \int_t^{\infty} \frac{L_{s}^{(r)}}{L_s^r} dG(s)} dH(r) = \int_{\mathbb{R}^d} \frac{\psi^r_t}{\psi^r_t + 1 - G(t)} dH(r), \]
where the last equality follows from the definition of \( L_{s}^{(r)} \) in (6) for \( t < s \). By the Itô’s formula applied to (23) we obtain that \( \psi^r_t \) solves the following SDE
\[ d\psi^r_t = dG(t) + \frac{\psi^r_t}{L_t^r} dL_t^r. \]

The construction of the classical Shiryaev-Roberts statistic (SR) is in detail described and analyzed e.g. by Shiryaev [27], Pollak and Tartakovsky [19] and Moustakides et al. [14]. In this paper we consider Generalized Shiryaev-Roberts statistic (GSR). We start the whole construction from taking the discrete-time data \( X_{t_n} \in \mathbb{R}^d \) observed in moments \( 0 = t_0 < t_1 < \ldots < t_n \), where \( n \) is a fixed integer. Without lost of generality we can assume that \( t_i - t_{i-1} = 1 \) for \( i = 1, \ldots n \). Let \( x_k := X_{t_k} - X_{t_{k-1}} \) for \( k = 1, \ldots, n \). Since \( X \) is a \( d \)-dimensional process, \( x_k \) is a \( d \)-dimensional vector \( x_k = (x_{k,1}, \ldots, x_{k,d}) \).
Considering a discrete analogue of (23) we define the following statistic

$$\tilde{\psi}_n := L_n^r G(0) + \sum_{j=0}^{n-1} \frac{L_j^r}{L_j} G'(j) = L_n^r G(0) + \sum_{j=0}^{n-1} \prod_{k=j+1}^{n} \exp \left\{ \sum_{i=1}^{d} z_{r,i} x_{k,i} - K_r \right\} G'(j),$$

where from equation (21) we take

$$L_n^r := \exp \left\{ \sum_{i=1}^{d} z_{r,i} \sum_{k=1}^{n} x_{k,i} - K_r n \right\} = \prod_{k=1}^{n} \exp \left\{ \sum_{i=1}^{d} z_{r,i} x_{k,i} - K_r \right\}$$

for $n > 0$ and $L_0^r = 1$. Above $G(0) = x$ corresponds to an atom at 0.

For convenience it can be also calculated recursively as follows:

$$\tilde{\psi}_{n+1} = (\tilde{\psi}_n + G'(n)) \cdot \exp \left\{ \sum_{i=1}^{d} z_{r,i} x_{n+1,i} - K_r \right\}, \quad \tilde{\psi}_0 = x.$$

Recall from Theorem 1 that the optimal stopping time is given by

$$\tau^* = \inf \{ t \geq 0 : \pi_t \geq A^* \}$$

for some optimal level $A^*$. Therefore from identity (24) we can introduce the following Generalized Shiryaev-Roberts statistic

$$\bar{\pi}_n = \int_{\mathbb{R}^d} \frac{\tilde{\psi}_n^r}{\tilde{\psi}_n^r + 1 - G(n)} dH(r)$$

and raise the alarm of the drift change at the optimal time of the form

$$\bar{\tau}^* := \inf \{ n \geq 0 : \bar{\pi}_n \geq A^* \}.$$

Note that formally, we first choose a direction of new drift $r$ according to distribution $H$. Then we apply statistic $\tilde{\psi}_n^r$ to identify the GSR statistic by $\bar{\pi}_n = \frac{\tilde{\psi}_n^r}{\tilde{\psi}_n^r + 1 - G(n)}$ and hence to raise the alarm at the optimal level $A^*$.

We emphasize that the GSR statistic is more appropriate in longevity modeling analyzed in this article than the standard one (i.e. SR). Indeed, the classical statistic is a particular case when $\theta$ has an exponential distribution with parameter $\lambda$ tending to 0. The latter case corresponds to passing with mean value of the change point $\theta$ to $\infty$ and hence it becomes conditionally uniform, see e.g. Shiryaev [27]. Still, in longevity modeling it is more likely that life tables will need to be revised more often and therefore keeping dependence on $\lambda > 0$ in our statistic seems to be much more appropriate. For the similar reasons we also prefer to fix average moment of drift change $\theta$ instead of fixing the expected moment of the revision time $\tau$.

To apply above strategy we have to identify the optimal alarm level $A^*$ in the first step and hence we have to solve the free-boundary value problem (10) – (13). We analyze two particular examples in the next section.

4. Examples

4.1. Two-dimensional Brownian motion with two possible post-change drifts. Consider the process $X$ without jumps (i.e. with jump intensities $\mu^\infty = \mu^r = 0$). In terms of processes $X^{(0,r)}$ and $X^\infty$ given in (3) and (4) it means that

$$X^{(0,r)}_t = \sigma W^r_t + rt \quad \text{and} \quad X^\infty_t = \sigma W^\infty_t.$$
Assume that
\[
\sigma = \begin{pmatrix}
\sigma_1 & 0 \\
\sigma_2 \rho & \sigma_2 \sqrt{1 - \rho^2}
\end{pmatrix}.
\]
Then the first coordinate \(X^{(0,r)}_{t,1}\) is a Brownian motion with drift and with variance \(\sigma_1^2\) and the second coordinate \(X^{(0,r)}_{t,2}\) is also a Brownian motion with drift and with variance \(\sigma_2^2\). The correlation of the Brownian motions on both coordinates is equal to \(\rho\). Process \(X^\infty\) has similar characteristics but without any drift.

Next, assume that, conditioned on \(\theta > 0\), \(\theta\) is exponentially distributed with parameter \(\lambda > 0\), i.e.
\[
\mathcal{T}^{G,H}(\theta \leq t) = G(t) = x + (1 - x)(1 - e^{-\lambda t}), \quad t \geq 0.
\]
Moreover, assume that after the disorder moment \(\theta\), the drift of process \(X\) attains one of the two possible values in \(\mathbb{R}^2\): either \(r_1\) or \(r_2\). That is, \(\zeta\) equals \(r_1\) or \(r_2\) with probabilities \(\xi_1 > 0\) and \(\xi_2 > 0\) respectively, such that \(\xi_1 + \xi_2 = 1\). More formally,
\[
dH(r) = \xi_1 \delta_{r_1} + \xi_2 \delta_{r_2}.
\]
Then the generator of process \(\pi\) according to (15) is equal to
\[
Af(x) = f'(x)\lambda(1 - x) + \frac{1}{2}f''(x)x^2(1 - x)^2 \sum_{k=1}^{2} \xi_k \left[ z_{r_k,1}^2 \sigma_1^2 + z_{r_k,2}^2 \sigma_2^2 + 2z_{r_k,1} z_{r_k,2} \sigma_1 \sigma_2 \rho \right],
\]
where \(z_{r_k,1}\) and \(z_{r_k,2}\) for \(k \in \{1, 2\}\) solve the following system
\[
\begin{cases}
  r_{k,1} &= \sum_{j=1}^{2} z_{r_k,j} (\sigma \sigma^T)_{1,j}, \\
  r_{k,2} &= \sum_{j=1}^{2} z_{r_k,j} (\sigma \sigma^T)_{2,j}.
\end{cases}
\]
Our goal is to solve the boundary value problem (10) – (13) where generator \(A\) is given by (27). Note that the system (10) takes now the following form
\[
f'(x)\lambda(1 - x) + \frac{1}{2}f''(x)x^2(1 - x)^2 \cdot B = -cx, \quad 0 \leq x < A^*,
\]
\[
f(x) = 1 - x, \quad A^* \leq x \leq 1,
\]
where
\[
B := \sum_{k=1}^{2} \xi_k \left[ z_{r_k,1}^2 \sigma_1^2 + z_{r_k,2}^2 \sigma_2^2 + 2z_{r_k,1} z_{r_k,2} \sigma_1 \sigma_2 \rho \right].
\]
From Shiryaev [23, 29] it follows that solution of above equation is given by
\[
V^*(x) = \begin{cases}
  1 - A^* - \int_x^{A^*} y(s)ds, & x \in [0, A^*) \\
  1 - x, & x \in [A^*, 1],
\end{cases}
\]
where
\[
y(s) = -\frac{2c}{B} \int_0^s e^{-\frac{4c}{B} |Z(u) - Z(s)|} \frac{1}{u(1 - u)^2} du
\]
for
\[
Z(u) = \log \frac{u}{1 - u} - \frac{1}{u}.
\]
The exact values of function \(y(x)\) can be found numerically, while the threshold \(A^*\) can be found from the equation \(y(A^*) = -1\), which is the boundary condition (12).
4.2. Two-dimensional Brownian motion with one-sided jumps. The second example concerns similar 2-dimensional Brownian motion model as in the previous example, but with additional exponential jumps. Assume that $\mu^\infty, \mu^r > 0$ and

\begin{equation}
F^\infty(dy) = \prod_{j=1}^{2} F_j^\infty(dy) = \prod_{j=1}^{2} \frac{1}{w_j} e^{-y_j/w_j} I(y_j \geq 0) dy.
\end{equation}

In other words, jump sizes on each coordinate $j \in \{1, 2\}$ of process $X^\infty$ are independent of each other and distributed exponentially with mean $w_j > 0$. Additionally, we assume as in the previous example that

$$P^{G,H}(\theta \leq t) = G(t) = x + (1-x)(1-e^{-\lambda t}), \quad t \geq 0$$

and that there is only one possible post-change drift $r_0 \in \mathbb{R}^2$, that is

$$dH(r) = \delta_{r_0}.$$ 

Jump distribution given by (28) together with Theorem 3 allows us to formulate the following lemma.

**Lemma 2.** Assume that jump distribution $F^\infty$ of the process $X^\infty$ is given by (28) and jump intensity is equal to $\mu^\infty$. Assume also that there exists a vector $z_{r_0} = (z_{r_0,1}, \ldots, z_{r_0,d})$ satisfying the system (20) such that $(\forall 1 \leq j \leq 2) (|w_j z_{r_0,j}| < 1)$. Then the following distribution function $F^{r_0}$ and intensity $\mu^{r_0}$ satisfy the condition (28):

\begin{equation}
F^{r_0}(dy) = \frac{1}{w_j} e^{-y_j/(w_j z_{r_0,j})} I(y_j \geq 0) dy,
\end{equation}

\begin{equation}
\mu^{r_0} = \mu^\infty \prod_{j=1}^{2} \frac{1}{1 - w_j z_{r_0,j}}.
\end{equation}

**Proof.** By the combination of (18) and (19) we obtain that

$$\mu^{r_0} F^{r_0}(dy) = e^{\sum_{j=1}^{2} z_{r_0,j} y_j} \mu^\infty F^\infty(dy) = \mu^\infty \prod_{j=1}^{2} \frac{1}{w_j} e^{-y_j/(w_j z_{r_0,j})} I(y_j \geq 0) dy,$$

which can be rearranged to

$$\mu^{r_0} = \mu^\infty \prod_{j=1}^{2} \frac{1}{1 - w_j z_{r_0,j}} \prod_{j=1}^{2} \frac{1}{w_j} e^{-y_j/(w_j z_{r_0,j})} I(y_j \geq 0) dy.$$ 

Now it is sufficient to observe that above formula is equal to the product $\mu^{r_0} F^{r_0}$ given by (29) and that $F^{r_0}$ is indeed a proper distribution by the assumption that $(\forall 1 \leq j \leq 2) (|w_j z_{r_0,j}| < 1)$. \qed

**Remark 1.** Considering the jump distributions $F^\infty$ and $F^{r_0}$ given by (28) and (29), the system (20) consists of equations

$$r_{0,k} + \mu^\infty m_k^\infty - \mu^{r_0} m_k^{r_0} - \sum_{j=1}^{2} z_{r_0,j} (\sigma \sigma^T)_{k,j} = 0, \quad k = 1, \ldots, 2,$$

where

$$\mu^\infty m_k^\infty = \mu^\infty w_k$$

and

$$\mu^{r_0} m_k^{r_0} = \mu^\infty \frac{w_k}{1 - w_k z_{r_0,k}} \prod_{j=1}^{2} \frac{1}{1 - w_j z_{r_0,j}}.$$
Remark 2. Distribution $F_{\epsilon_0}$ given by (29) has similar characteristics to $F^\infty$. More precisely: jumps on both coordinates $X_{t,1}^{(0,\epsilon_0)}$ and $X_{t,2}^{(0,\epsilon_0)}$ are independent, exponentially distributed with means $\frac{w_1}{1-w_2 \epsilon_{r_0,1}}$ and $\frac{w_2}{1-w_2 \epsilon_{r_0,2}}$, respectively.

The generator $A$ given by (15) for jump distributions specified above can be expressed as

$$A f(x) = f'(x) (\lambda (1 - x) + x (1 - x) (\mu^\infty - \mu^\circ)) + \frac{1}{2} f''(x) x^2 (1 - x)^2 \left[ \epsilon_{r_0,1} \sigma_1^2 + \epsilon_{r_0,2} \sigma_2^2 + 2 \epsilon_{r_0,1} \epsilon_{r_0,2} \sigma_1 \sigma_2 \rho \right] - f(x)$$

$$+ \int_{[0,\infty)^2} f \left( \frac{x \exp \{ \sum_{i=1}^2 \epsilon_{r_0,i} y_i \} } {x \exp \{ \sum_{i=1}^2 \epsilon_{r_0,i} y_i \} - 1 + 1} \right)$$

$$\cdot \left[ (1 - x) \mu^\infty \sum_{j=1}^2 \frac{1}{w_j} e^{-y_j/w_j} + x \mu^\circ \sum_{j=1}^2 \frac{1 - \epsilon_{r_0,j} e_{r_0,j} } {w_j} e^{-y_j/w_j} \right] \, dy.$$  

The integral part of $A$ can be further simplified. For $\alpha_1, \alpha_2 > 0$ we define the following integrals

$$I_+^\alpha(x) := \int_{[0,\infty)^2} f \left( \frac{x \exp \{ \sum_{i=1}^2 \epsilon_{r_1,i} y_i^1 \} } {x \exp \{ \sum_{i=1}^2 \epsilon_{r_1,i} y_i^1 \} - 1 + 1} \right) \prod_{j=1}^2 \alpha_j e^{-\alpha_j y_j} \, dy$$

and

$$I_-^\alpha(x) := \int_{(-\infty,0)^2} f \left( \frac{x \exp \{ \sum_{i=1}^2 \epsilon_{r_2,i} y_i^2 \} } {x \exp \{ \sum_{i=1}^2 \epsilon_{r_2,i} y_i^2 \} - 1 + 1} \right) \prod_{j=1}^2 \alpha_j e^{\alpha_j y_j} \, dy.$$  

Lemma 3. Assume that $\alpha_1, \alpha_2, \epsilon_{r_1,1}, \epsilon_{r_1,2} > 0$ and $\frac{\alpha_1}{\epsilon_{r_1,1}} \neq \frac{\alpha_2}{\epsilon_{r_1,2}}$. Then for $x \in (0,1]$,

$$I_+^\alpha(x) = f(x) - \frac{\beta_1}{\beta_2 - \beta_1} \left( \frac{1 - x}{x} \right)^{-\beta_2} \int_x^1 f'(v) \left( \frac{v}{1 - v} \right)^{-\beta_2} \, dv$$

$$+ \frac{\beta_2}{\beta_2 - \beta_1} \left( \frac{1 - x}{x} \right)^{-\beta_1} \int_x^1 f'(v) \left( \frac{v}{1 - v} \right)^{-\beta_1} \, dv$$

for $\beta_1 = \frac{\alpha_1}{\epsilon_{r_1,1}}$ and $\beta_2 = \frac{\alpha_2}{\epsilon_{r_1,2}}$ and

$$I_-^\alpha(x) = f(x) + \frac{\beta_1}{\beta_2 - \beta_1} \left( \frac{1 - x}{x} \right)^{\beta_2} \int_0^x f'(v) \left( \frac{v}{1 - v} \right)^{\beta_2} \, dv$$

$$- \frac{\beta_2}{\beta_2 - \beta_1} \left( \frac{1 - x}{x} \right)^{\beta_1} \int_0^x f'(v) \left( \frac{v}{1 - v} \right)^{\beta_1} \, dv.$$  

Remark 3. In Lemma 5 we restrict calculations to the case $\beta_1 \neq \beta_2$ since this will be the actual case in further applications. Nevertheless, similar transformations of the integral may be made for the case $\beta_1 = \beta_2$ as well. The difference will appear in the distribution of random variable $S^\alpha$ being the sum of two exponential random variables.
The representation of the force of mortality process is given by the distribution jumps. However, the whole analysis can be also conducted for negative exponential jumps, i.e. for analogous to the one described in Krawiec et. al [10].

\[ \text{Equation } A f(x) = -cx \text{ in the free-boundary value problem can be further simplified to get rid of the integrals and then solved numerically to find the threshold } A^*. \text{ This numerical method of solution is analogous to the one described in Krawiec et. al [10].} \]

**Remark 4.** The results of above example are derived under the assumption of positive exponential jumps. However, the whole analysis can be also conducted for negative exponential jumps, i.e. for the distribution

\[ \mathcal{F}^\infty(dy) = \prod_{j=1}^2 \frac{1}{w_j} \alpha(y_j/w_j) I(y_j \leq 0)dy. \]

Then we can use part of Lemma 5 concerning \( I^- \) to derive the generator \( A \) given by

\[ A f(x) = f(x) [(1 - x) \mu^\infty + x \mu^{t_0} - 1] + f'(x) (\lambda(1 - x) + x(1 - x)(\mu^\infty - \mu^{t_0})) \]

\[ + \frac{1}{2} f''(x)x^2 (1 - x)^2 \left[ \tau^2 r_{1,1} \sigma_1^2 + \tau^2 r_{1,2} \sigma_2^2 + 2 \tau r_{1,1} \tau r_{1,2} \sigma_1 \sigma_2 \rho \right] \]

\[ + (1 - x)^{-\gamma_2 + 1} x^{-\gamma_2} \int_0^x f'(v) \left[ \mu^{\infty} \frac{\gamma_1}{\gamma_2 - \gamma_1} \left( \frac{v}{1 - v} \right)^{-\gamma_2} + \mu^{t_0} \frac{\gamma_1}{\gamma_2 - \gamma_1} \left( \frac{v}{1 - v} \right)^{-\gamma_2} \right] dv \]

\[ - (1 - x)^{\gamma_1 + 1} x^{\gamma_1} \int_0^x f'(v) \left[ \mu^{\infty} \frac{\gamma_2}{\gamma_2 - \gamma_1} \left( \frac{v}{1 - v} \right)^{\gamma_1} + \mu^{t_0} \frac{\gamma_2 + 1}{\gamma_2 - \gamma_1} \left( \frac{v}{1 - v} \right)^{\gamma_1} \right] dv. \]

### 5. Application to the Force of Mortality

Now we are going to give an important example of applications, which concerns modeling of the force of mortality process. We will analyze the joint force of mortality for both men and women. We observe this process over the past decades and check if and when there have been significant changes of drift.

To achieve this goal, we introduce two-dimensional process of the force of mortality \( \mu := (\mu_t)_{t \geq 0} = (\mu^1_t, \mu^2_t)_{t \geq 0} \). We interpret this process as follows:

- the first coordinate \( \mu^1_t \) represents force of mortality of men, while the second one \( \mu^2_t \) represents force of mortality of women (of course they are correlated),
- the time \( t \) runs through consecutive years of life tables, e.g. if \( t = 0 \) corresponds to the year 1990, then \( t = 10 \) corresponds to the year 2000,
- the age of people is fixed for a given process \( \mu \), i.e. if \( \mu_0 \) concerns 50-year old men and women, then \( \mu_{10} \) also concerns 50-year old men and women, but in the other year.

The representation of the force of mortality process is given by

\[ \log \mu_t = \log \mu_0 + \mu X_t. \]
where \( \log \bar{\mu}_t := (\log \mu_{1,t}, \log \mu_{2,t}) \) is a deterministic part equal to
\[
\log \bar{\mu}_t = a_0 + a_1 t.
\]

Above \( a_0 = (a_0^1, a_0^2) \) is a known initial force of mortality vector of men and women and \( a_1 = (a_1^1, a_1^2) \) is a vector of a historical drift per one year. It is worth to mention here that our model is similar to the Lee-Carter model (for fixed age \( \omega \), cf. \[12\]):
\[
\log \mu_{\omega,t} = a_\omega + b_\omega k_t + \epsilon_{\omega,t},
\]
where \( a_\omega \) is a chosen number, \( k_t \) is certain univariate time series and \( \epsilon_{\omega,t} \) is a random error. However, Lee-Carter method focuses on modelling the deterministic part of the force of mortality, while our detection procedure concerns controlling the random perturbation in time, precisely the moment when it substantially changes. This model is univariate as well in contrast to our two-dimensional mortality process.

In our numerical analysis the stochastic part \( X_t \) will be modeled by the two-dimensional Brownian motion with two possible post-change drifts analyzed in Example \[4.1\]. We apply this model to the life tables downloaded from the Statistics Poland website \[33\].

The first step concerns the model calibration. We start with some historical values of the force of mortality \( \bar{\mu}_0, \ldots, \bar{\mu}_n \), where each \( \bar{\mu}_i = (\bar{\mu}_{i,1}, \bar{\mu}_{i,2}) \) is a two-dimensional vector (one coordinate for women and one for men). We estimate \( a_1 \) as a mean value of log-increments of \( \bar{\mu}_0, \ldots, \bar{\mu}_n \). Precisely,
\[
\hat{a}_1 := \frac{1}{n} \sum_{i=1}^{n} y_i,
\]
where
\[
y_i := \log \bar{\mu}_i - \log \bar{\mu}_{i-1}, \quad i = 1, \ldots, n.
\]

A little more attention is needed to calibrate the stochastic part \( X \), which includes correlation. Denote
\[
\hat{X}_i = \log \bar{\mu}_i - a_0 - \hat{a}_1 i, \quad i = 0, \ldots, n
\]
and the increments
\[
x_i := \hat{X}_{i+1} - \hat{X}_i, \quad i = 1, \ldots, n.
\]

We estimate \( \sigma_1 \) as a standard deviation of the vector \( (x_{1,1}, x_{2,1}, \ldots, x_{n,1}) \). Similarly, \( \sigma_2 \) is calculated as a standard deviation of a vector \( (x_{1,2}, x_{2,2}, \ldots, x_{n,2}) \). Finally, we calculate \( \rho \) as the sample Pearson correlation coefficient of vectors \( (x_{1,1}, x_{2,1}, \ldots, x_{n,1}) \) and \( (x_{1,2}, x_{2,2}, \ldots, x_{n,2}) \).

There are still some model parameters that have to be chosen a priori. In particular, we have to declare the incoming drifts \( r_1, r_2 \), the probabilities \( \xi_1 = P^{G,H}(\zeta = r_1) = 1 - P^{G,H}(\zeta = r_2) = 1 - \xi_2 \), the probability \( x = \pi_0 = P^{G,H}(\theta = 0) \) that the drift change occurs immediately, the parameter \( \lambda > 0 \) of the exponential distribution of \( \theta \) and parameter \( c \) present in criterion stated in the Problem \[2\].

We assume their values at the following level:

- \( \lambda = 0, 2 \). It is the reciprocal of the mean value of \( \theta \) distribution conditioned to be strictly positive. Such choice reflects the expectation that the drift will change in 5 years on average.
- \( x(= P^{G,H}(\theta = 0)) = 0, 1 \). This parameter should be rather small (unless we expect the change of drift very quickly).
- \( c = 0, 2 \). It is the weight of the mean delay time inside the optimality criterion stated in the Problem \[1\]. It reflects how large delay we can accept comparing to the risk of false alarm. We have chosen rather small value and connected it to \( \lambda \) by choosing \( c = \lambda \).
• Drift incoming after the moment $\theta$ – we have connected the two possible values of $r$ to $\sigma$: $r_1 = 2(\sigma_1, \sigma_2)$ and $r_2 = (\sigma_1, \sigma_2)$ with probabilities $\xi_1 = 0,3$ and $\xi_2 = 0,7$, respectively. In practice we suggest to adjust the choice of $r$ to the analysis of sensitivity of e.g. price of an insurance contract.

In Table 1 we sum up all parameters that were used (both calibrated and arbitrary chosen ones) in the numerical analysis. The calibration interval was set to years 1990 – 2000.

| calibrated | arbitrary chosen |
|------------|------------------|
| $\sigma_1$ | 0,03             |
| $\sigma_2$ | 0,02             |
| $\rho$     | 0,33             |
| $c$        | 0,2              |
| $\lambda$  | 0,1              |
| $x$        | (2$\sigma_1$, 2$\sigma_2$) |
| $r_1$      | (2$\sigma_1$, 2$\sigma_2$) |
| $r_2$      | $\xi_1$, $\xi_2$ |
| $\xi_1$    | 0,3              |
| $\xi_2$    | 0,7              |

Table 1. Parameters used to drift change detection

In the Figure 1 we present exemplary plot of the force of mortality for women at age 60 through years 1990 – 2017. Most of the time it is decreasing, but we can observe a stabilization period around years 2002 – 2009. According to (31) we first take logarithm of the force of mortality, separate deterministic linear part and then model the remaining part by the process $X$ given by (26). Figure 2 presents historical observations of this remaining part for the same data as in the Figure 1.

Figure 1. Force of mortality of women aged 60 in 1990-2017

The results of the detection algorithm for the force of mortality of 60-year old men and women jointly are presented in the Figure 3. The change of drift for given parameters was detected in year 2006 (red vertical line in the first two plots). The threshold $A^*$ for the optimal stopping time is here equal to 0,85, which is indicated by the red horizontal line in the third plot presenting values of
π = (π_t)_{t \in \{1990, \ldots, 2017\}}. In year 2005 π_t almost crossed A*, but it actually crosses the threshold in year 2006. Note that calibration of parameters (including historical drift) has been done for interval 1990 – 2000, when the force of mortality was mostly decreasing. After year 2002 it stayed at a stable level for several years, which was detected as a change of drift. This change of behavior is even more evident in the Figure 2, where we can observe that process X is mostly increasing through the years 2002 – 2009.

An important note need to be given at the end. This procedure is strongly dependent on parameters chosen to the model. Post-change drift vectors r_1 and r_2 was chosen depending on σ_1 and σ_2, to give appropriate order of magnitude. However, this method of detection would not be efficient if we choose r_1 and r_2 of the opposite signs.

6. PROOFS

Proof of Lemma 1. Note that

\begin{equation}
\mathbb{P}^{G,H}(\tau < \theta) = \mathbb{E}_x^{G,H}[I(\tau < \theta)|\mathcal{F}_t] = \mathbb{E}_x^{G,H}[1 - \mathbb{P}^{G,H}(\theta \leq \tau|\mathcal{F}_t)] = \mathbb{E}_x^{G,H}[1 - \pi_\tau].
\end{equation}

Moreover, observe that by Tonelli’s theorem we have:

\begin{equation}
\mathbb{E}_x^{G,H}[(\tau - \theta)^+] = \int_{\mathbb{R}_+} \mathbb{E}_x^{G,H}[(t - \theta)^+]d\mathbb{P}^{G,H}(\tau \in dt) = \int_{\mathbb{R}_+} \mathbb{E}_x^{G,H} \left[ \int_0^t I(\theta \leq s)ds \right]d\mathbb{P}^{G,H}(\tau \in dt).
\end{equation}

Putting together (34) and (35) completes the proof. □
Proof of Theorem 1. We start from the observation that the optimal value function $V^*(x)$ is concave, which follows from [10, Lem. 3] and the assumption that distribution function $G(t)$ of $\theta$ is continuous for $t > 0$. Let

$$C = \{ x : V^*(x) > 1 - x \}$$

be an open continuation set and $D = C^c$ be a stopping set. From Lemmas 4 and 5 of [10] we know that $C = [0, \lambda^*)$ and that the stopping rule given by

$$\tau^* = \inf \{ t \geq 0 : \pi_t \in D \}.$$

is optimal for Problem 2. Moreover, we have

$$\mathbb{P}_x(\tau^* < \infty) = 1$$

and the optimal value function $V^*(x)$ satisfies the system

$$\begin{cases} A V^*(x) = -c x, & x \in C, \\ V^*(x) = 1 - x, & x \in D, \end{cases}$$

where $A$ is the infinitesimal generator of process $\pi$ given by (15). Finally, from [10, Lem. 6] and from the form of this infinitesimal generator we can conclude the normal entrance condition (13) as $G'(0) > 0$. Using the same arguments like in the proof of [10, Lem. 7] we can prove the smooth fit condition (12). This completes the proof. $\square$

Proof of Theorem 2. First, we will find the SDE which is satisfied by process $\pi_t^r$. By the definition of the process $X$ for each $i = 1, \ldots, d$, we get

$$dX_{t,i} = \sum_{j=1}^{d} \sigma_{ij} dW_{t,j} + \Delta X_{t,i} + r_i I(t \geq \theta) dt - (\mu^\infty m_\infty^i I(t < \theta) + \mu^r m_r^i I(t \geq \theta)) dt.$$

Denote the continuous part of the process by an additional upper index $c$. Then

$$d \langle X_{t,i}^c, X_{t,k}^c \rangle = \sum_{j=1}^{d} \sigma_{ij} \sigma_{kj} dt = (\sigma \sigma^T)_{ik} dt.$$

For the process $L^r$ given in (21), by the Itô’s formula we obtain

$$dL_t^r = \left\{ \mu^\infty - \mu^r + \sum_{i=1}^{d} z_{r,i}(r_i + \mu^\infty m_\infty^i - \mu^r m_r^i) I(\theta \leq t) \right\} L_t^r dt$$

$$+ \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} L_t^r dW_{t,j} + \Delta L_t^r,$$

where

$$\Delta L_t^r = L_{t^-}^r \left( \frac{L_t^r}{L_{t^-}^r} - 1 \right) = L_{t^-}^r \left( e^{\sum_{i=1}^{d} z_{r,i} \Delta X_{t,i}^-} - 1 \right).$$

By (25) we conclude that

$$d\psi_t^r = dG(t) + \left\{ \mu^\infty - \mu^r + \sum_{i=1}^{d} z_{r,i}(r_i + \mu^\infty m_\infty^i - \mu^r m_r^i) I(\theta \leq t) \right\} \psi_t^r dt$$

$$+ \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{ij} \psi_t^r dW_{t,j} + \psi_t^r \left( e^{\sum_{i=1}^{d} z_{r,i} \Delta X_{t,i}^-} - 1 \right).$$
Recall that by (24) we have
\[ \pi^r_t = \frac{\psi^r_t}{\psi^r_t + 1 - G(t)}. \]
Then, using Itô’s formula once again we obtain
\[
d\pi^r_t = \frac{\pi^r_t(1 - \pi^r_t)}{1 - G(t)} dG(t) + \frac{(1 - \pi^r_t)^2}{1 - G(t)} d\psi^r_t - \frac{(1 - \pi^r_t)^3}{(1 - G(t))^2} d\langle \psi^r, \psi^r \rangle_t + \Delta \pi_t.
\]
Moreover,
\[
d\langle \psi^r, \psi^r \rangle_t = \sum_{i=1}^{d} \sum_{j=1}^{d} z_{r,i} z_{r,j} (\sigma \sigma^T)_{i,j} (\psi^r_t)^2 dt.
\]
Together with the system of equations (20) it produces
\[
d\pi^r_t = \frac{1 - \pi^r_t}{1 - G(t)} dG(t) + \frac{(1 - \pi^r_t)^2}{1 - G(t)} (\mu^\infty - \mu^r) dt
\begin{align*}
&+ \frac{(1 - \pi^r_t)^2}{1 - G(t)} \sum_{i=1}^{d} z_{r,i} \sum_{j=1}^{d} \sigma_{i,j} \psi^r_t dW_{t,j} \\
&+ \frac{(1 - \pi^r_t)^2}{1 - G(t)} \sum_{i=1}^{d} \sum_{j=1}^{d} z_{r,i} z_{r,j} (\sigma \sigma^T)_{i,j} I(\theta \leq t) \psi^r_t dt \\
&- \frac{(1 - \pi^r_t)^3}{(1 - G(t))^2} \sum_{i=1}^{d} \sum_{j=1}^{d} z_{r,i} z_{r,j} (\sigma \sigma^T)_{i,j} (\psi^r_t)^2 dt + \Delta \pi^r_t.
\end{align*}
\]
Jump part of \( \pi_t \) is equal to \( \int_{\mathbb{R}^d} \Delta \pi^r_t dH(r) \), where
\[
\Delta \pi^r_t = \pi^r_{t-} \left( \frac{\psi^r_t}{\psi^r_{t-}} + 1 - G(t) \right) - \pi^r_{t-} \left( \frac{\exp\{\sum_{i=1}^{d} z_{r,i} \Delta X_{t,i}\} - 1}{\psi^r_{t-} \exp\{\sum_{i=1}^{d} z_{r,i} \Delta X_{t,i}\} + 1 - G(t)} \right).
\]
Using the Itô’s formula one more time completes the proof. \( \square \)

**Proof of Theorem**

The proof is based on the technique of exponential change of measure described in [15].

Firstly, we will prove that the process \( (L^r_t)_{t \geq 0} \) satisfies the following representation
\[
(38) \quad L^r_t = \frac{h(X_t)}{h(X_0)} \exp \left( - \int_{0}^{t} \frac{(A^\infty h)(X_s)}{h(X_s)} ds \right)
\]
for the good function \( h(x) := h_c(x) \) given in (19), where \( A^\infty \) is an extended generator of the process \( X \) under \( \mathbb{P}^\infty \) and \( h \) is in its domain since it is twice continuously differentiable. Then from Theorem 4.2 by Palmowski and Rolski [15] it follows that the generator of \( X \) under \( \mathbb{P}^{(0,r)} \) is related with \( A^\infty \) by
\[
(39) \quad A^r f = \frac{1}{h} \left[ A^\infty (fh) - f A^\infty h \right].
\]
On the other hand, from the definition of the infinitesimal generator or using the Theorem 31.5 in Sato [22] it follows that for twice continuously differentiable function \( f(x_1, \ldots, x_d) : \mathbb{R}^d \to \mathbb{R} \) generators \( A^\infty \) and \( A^r \) are given by
\[
(40) \quad A^\infty f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)(\sigma \sigma^T)_{i,j} - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x) \mu^\infty m^\infty_i
\begin{align*}
&+ \int_{\mathbb{R}^d} (f(x + y) - f(x)) \mu^\infty F^\infty(dy),
\end{align*}
\]
\[ \mathcal{A}^r f(x) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) (\sigma^T)_{i,j} - \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x) (\mu^T m^r_i - r_i) \]

\[ + \int_{\mathbb{R}^d} (f(x + y) - f(x)) \mu^T F^r(dy). \]

Finally, using the system of equations (20) completes the proof.

For \( h_r(x) \) given by (19) we obtain

\[ \frac{h_r(X_t)}{h_r(X_0)} = \exp \left\{ \sum_{j=1}^{d} z_{r,j} (X_{t,j} - X_{0,j}) \right\}. \]

Further, since

\[ \frac{\partial h_r}{\partial x_i} = z_{r,i} h_r \]

and

\[ \int_{\mathbb{R}^d} \frac{h(X_x + y) - h(X_x)}{h(X_x)} \mu^\infty F^\infty(dy) = \int_{\mathbb{R}^d} \mu^T F^r(dy) - \int_{\mathbb{R}^d} \mu^\infty F^\infty(dy) = \mu^r - \mu^\infty, \]

then

\[ \frac{(A^\infty h_r)(X_s)}{h(X_s)} = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} z_{r,i} z_{r,j} (\sigma^T)_{i,j} - \sum_{i=1}^{d} z_{r,i} \mu^\infty m^r_i + \mu^r - \mu^\infty = K_r. \]

Hence, we obtain

\[ L^r_t = \exp \left\{ \sum_{j=1}^{d} z_{r,j} (X_{t,j} - X_{0,j}) \right\} \cdot \exp \left\{ - \int_{0}^{t} K_r ds \right\} \]

\[ = \exp \left\{ \sum_{j=1}^{d} z_{r,j} (X_{t,j} - X_{0,j}) - K_r t \right\} \]

and thus \( L^r_t \) given in (21) indeed satisfies the representation (38) for function \( h_r(x) \) given by (19). To finish the proof it is sufficient to show that the generator \( A^r \) given by (41) indeed coincides with the generator given in (39) for \( h(x) = h_r(x) \). First, by (22) we get

\[ \frac{1}{h} [A^\infty (fh) - f A^\infty h] = \frac{A^\infty (fh)}{h} - \frac{f A^\infty h}{h} = \frac{A^\infty (fh)}{h} - f K_r. \]

Second, (40) produces

\[ \frac{A^\infty (fh)}{h} = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_i} z_{r,j} + \frac{\partial f}{\partial x_j} z_{r,i} + f z_{r,j} z_{r,i} \right) (\sigma^T)_{i,j} \]

\[ - \sum_{i=1}^{d} \left( \frac{\partial f}{\partial x_i} + f z_{r,i} \right) \mu^\infty m^r_i + \int_{\mathbb{R}^d} f(x + y) h(x + y) - f(x) h(x) \mu^\infty F^\infty(dy). \]

Hence

\[ \frac{A^\infty (fh)}{h} - f K_r = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} (\sigma^T)_{i,j} \]

\[ + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} \sum_{j=1}^{d} (z_{r,j} (\sigma^T)_{i,j} - \mu^\infty m^r_i) + \int_{\mathbb{R}^d} (f(x + y) - f(x)) \mu^T F^r(dy). \]

Finally, using the system of equations (20) completes the proof. \( \square \)
Proof of Lemma 3. First observe that $I_2^r(x)$ is equal to the expectation

$$
E \left[ f \left( \frac{x \exp \left( \sum_{i=1}^{2} z_{r,i} T_i \right)}{x \exp \left( \sum_{i=1}^{2} z_{r,i} T_i \right) - 1} + 1 \right) \right],
$$

where $T_1$ and $T_2$ are two independent random variables with exponential distributions $\text{Exp}(\alpha_1)$ and $\text{Exp}(\alpha_2)$, respectively. Then $z_{r,1} T_1 \sim \text{Exp} \left( \frac{\alpha_1}{z_{r,1}} \right)$, $z_{r,2} T_2 \sim \text{Exp} \left( \frac{\alpha_2}{z_{r,2}} \right)$ and the density of $S^r := \sum_{i=1}^{2} z_{r,i} T_i$ is given by

$$f_{S^r}(y) = \frac{\beta_1 \beta_2}{\beta_1 - \beta_2} \left( e^{-\beta_1 y} - e^{-\beta_2 y} \right) I(y \geq 0) dy.$$

Hence, the expectation (42) equals

$$E \left[ f \left( \frac{xe^{S^r}}{x(e^{S^r} - 1) + 1} \right) \right] = \int_0^\infty f \left( \frac{xe^y}{(x(e^y - 1) + 1) \beta_1 \beta_2} \left( e^{-\beta_1 y} - e^{-\beta_2 y} \right) \right) dy.$$

Next, we can integrate above integral by parts to obtain

$$f(x) - \int_0^\infty f' \left( \frac{xe^y}{(x(e^y - 1) + 1) \beta_1 \beta_2} \left( e^{-\beta_1 y} - e^{-\beta_2 y} \right) \right) dy \frac{xe^y}{x(e^y - 1) + 1} \beta_1 \beta_2 \left( e^{-\beta_1 y} - e^{-\beta_2 y} \right) dy,$$

and by substitution $v := \frac{xe^y}{x(e^y - 1) + 1}$ (hence $y = \ln \left( \frac{v(1-x)}{x(1-v)} \right)$) we derive

$$f(x) - \frac{\beta_1}{\beta_2 - \beta_1} \left( \frac{1-x}{x} \right)^{-\beta_2} \int_0^1 f'(v) \left( \frac{v}{1-v} \right)^{-\beta_2} dv + \frac{\beta_2}{\beta_2 - \beta_1} \left( \frac{1-x}{x} \right)^{-\beta_1} \int_0^1 f'(v) \left( \frac{v}{1-v} \right)^{-\beta_1} dv,$$

which completes the first part of the proof.

The formula for $I_1^r(x)$ can be derived by substitution $u := -y$ to get

$$I_1^r(x) = \int_{(0,\infty)^2} f \left( \frac{x \exp \left( - \sum_{i=1}^{2} z_{r,i} u_i^r \right)}{x \exp \left( - \sum_{i=1}^{2} z_{r,i} u_i^r \right) - 1} + 1 \right) \prod_{j=1}^{2} \alpha_j e^{-\alpha_j u_i^r} dy,$$

which, by the same arguments as for $I_2^r(x)$, is equal to

$$E \left[ f \left( \frac{xe^{-S^r}}{x(e^{-S^r} - 1) + 1} \right) \right] = \int_0^\infty f \left( \frac{xe^y}{x(e^y - 1) + 1} \beta_1 \beta_2 \left( e^{-\beta_1 y} - e^{-\beta_2 y} \right) \right) dy.$$

Integration by parts together with substitution of $v := \frac{xe^{-y}}{x(e^{-y} - 1) + 1}$ (hence $y = -\ln \left( \frac{v(1-x)}{x(1-v)} \right)$) gives

$$f(x) + \frac{\beta_1}{\beta_2 - \beta_1} \left( \frac{1-x}{x} \right)^{\beta_2} \int_0^x f'(v) \left( \frac{v}{1-v} \right)^{\beta_2} dv - \frac{\beta_2}{\beta_2 - \beta_1} \left( \frac{1-x}{x} \right)^{\beta_1} \int_0^x f'(v) \left( \frac{v}{1-v} \right)^{\beta_1} dv,$$

which completes the second part of the proof. \hfill \square

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