Remarks on the Aharonov–Bohm Green function

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Some elementary algebraic points regarding the Green function for a localised flux tube are developed. A calculation of the effective action density is included.

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1. Introduction

This is a continuation of an investigation into quantities associated with a free conformal scalar field propagating in the presence of a simple, localised flux tube (or monodromy defect) in flat space.

In order to deal with the infinities arising on taking a coincidence limit a subtracted Green function, $G_{\text{sub}}$, was derived in [1,2] which automatically takes care of the divergences and allows any coincidence limit to be found in terms of Beta functions. In the present brief note I show how equivalent results are obtained directly from the complete Green function, $G$, again without encountering any infinities. For rapidity, only the coincidence limit of the Green function/correlator will be considered leading to the simplest one point function, $\langle \phi \bar{\phi}(x) \rangle$.

Although the system is somewhat basic (just free fields) there are still areas that can be subject to some refinement.

2. The Green function

An explicit formula for the Green function was derived in [3], [4], and can be written, conveniently, in terms of the Appell $F_1$ function. In a modified form, the result is,

$$G(x_1, x_2) = N \left\{ \frac{\sin \pi \Delta}{\pi} \frac{\Gamma(1 - \Delta) \Gamma(\delta + \Delta)}{\Gamma(\delta + 1)} \left( \frac{z \bar{z}}{\Delta} \right)^{\Delta/2} \frac{\Delta}{\Gamma(\delta + 1)} F_1(\delta + \Delta; 1, \Delta, \delta + 1; z, z \bar{z}) + \right.$$  

$$+ \left. \left( \frac{z}{r_1 r_2} \right)^{\Delta} F_1(\delta + \Delta; 1, \Delta, \delta + 1; 1, 1) \right\} \; (1)$$

The notation is that $z$ and $\bar{z}$ are light cone coordinates and $r_1$ and $r_2$ radial coordinates in the space orthogonal to the defect. $N$ is a convention-dependent normalisation.

The complete coincidence limit is $z = 1$, $\bar{z} = 1$ and $r_1 = r_2 \equiv r$ so that, in this limit, (1) becomes, in the spirit of dimensional regularisation,

$$G(\Delta, \delta, x, x) = \frac{N}{r^{2\Delta}} \frac{\sin \pi \Delta}{\pi} \frac{\Gamma(1 - \Delta) \Gamma(\delta + \Delta)}{\Gamma(\delta + 1)} F_1(\delta + \Delta; 1, \Delta, \delta + 1; 1, 1) +$$  

$$+ \left( \frac{z}{r_1 r_2} \right)^{\Delta} F_1(\delta + \Delta; 1, \Delta, \delta + 1; 1, 1) \right\} \; (2)$$

$$= \frac{N}{r^{2\Delta}} \sec \pi \Delta \left[ \frac{\Gamma(\delta + \Delta)}{\Gamma(\delta - \Delta)} + \frac{\Gamma(1 - \delta + \Delta)}{\Gamma(1 - \delta - \Delta)} \right]$$

$$= \frac{N}{r^{2\Delta}} \frac{\sin \pi \delta}{\pi} \frac{\Gamma(\delta + \Delta) \Gamma(1 - \delta + \Delta)}{\Gamma(2\Delta + 1)},$$
after some elementary Gamma combinations. $\delta$ is the monodromy flux and $\Delta$ the conformal dimension equal to $d/2 - 1$ for conventional fields.

In deriving this result use has been made of the Gauss–like equality, [5],

$$F_1(a, b, b', c; 1, 1) = \frac{\Gamma(c)\Gamma(c-a-b-b')}{\Gamma(c-a)\Gamma(c-b-b')},$$

valid for $\text{Re} (c-a-b-b') > 0$. For the parameters in (1) this condition is just $\Delta < 0$.

The dimension, $\Delta$, is to be continued to a physical value, e.g. to an integer or a half–integer. The resulting values of $G$ are finite without needing any explicit renormalisation or subtraction which is a consequence of the manifold being flat. In fact, $[G]$ decreases monotonically as $\Delta$ tends to infinity but there are now singularities for negative $\Delta$.

3. Using the subtracted Green function

The Green function has the structure,

$$G(\Delta, \delta, x, x') = G(\Delta, 0, x, x') + G_{sub}(\Delta, \delta, x, x'),$$

where the coincidence limit infinities (for physical $\Delta$) reside in the flux–free Green function. In [2], via a simple contour manipulation, a closed form was derived for $G_{sub}$ which can be expressed neatly in terms of an Appell $F_3$ function,

$$G_{sub}(\Delta, \delta, x, x') = \frac{N}{(r_1 r_2)^\Delta} C(\Delta, \delta) \sqrt{z\bar{z}}^\Delta F_3(\Delta+\delta, \Delta+1-\delta, ; \Delta, \Delta, 2\Delta+1; 1-\bar{z}, 1-z),$$

where the constant, $C$, is,

$$C(\Delta, \delta) = \frac{\sin \pi \delta}{\pi} \text{B}(\Delta + \delta, \Delta + 1 - \delta),$$

in terms of the Beta function. The regularised coincidence limit of $G$ is thus, for any $\Delta$,

$$G_{sub}(\Delta, \delta, x, x) = \frac{N}{r^{2\Delta}} C(\Delta, \delta).$$

Actually, it is not necessary to introduce $F_3$ to obtain this simple coincidence limit. It follows quickly, at an earlier stage, from the originating contour integral. Indeed, classically, formula (3) follows from the integral form of $F_1$, [5].

Comments on these results are made in section 6.
5. The local effective action

A standard expression for the effective action density, $W(x)$, is the proper time integral,

$$W(x) = -\frac{1}{2} \int_{0}^{\infty} \frac{d\tau}{\tau} \langle x | K(\tau) | x \rangle,$$

in terms of the diagonal element (i.e. coincidence limit) of the heat–kernel, $K$. The integral generally diverges at the lower limit and some sort of UV regularisation, such as a cutoff, is usually necessary. However, it is possible to proceed, somewhat formally, by noting, as is well known, that the integral in (6) without the $1/\tau$ factor, is, to $-1/2$, the Green function.

Hence, as above, for sufficiently negative $\Delta$, the coincidence limit is finite, and the effect of the $1/\tau$ in (6) is just to replace $\Delta$ in (1) by $\Delta + 1$, up to a factor of $4\pi$ which can be absorbed into $N$ and (6) then takes the explicit form,

$$W(\Delta, \delta, x) = -\frac{1}{2} G(\Delta + 1, \delta, x, x),$$

and so the effective action density is obtained with no extra work, in this particular case.

There are no infinities as $\Delta$ is continued to a physical value, which is a consequence of the lack of curvature in the manifold. For example, there is no local trace anomaly, in any dimension.$^2$

It is also possible to construct a local $\zeta$–function but there seems to be no technical advantage in doing so.

As a mathematical aside, I remark that the derivative of $W$ with respect to $\delta$, which should be related in some way to a current one point function, vanishes correctly at $\delta = 1/2$ but not at $\delta = 0$ or at $\delta = 1$ where the values are equal, opposite and, for even dimensions, proportional to the inverse of the Apery number, $\Delta \left( \frac{2\Delta}{\Delta} \right)$. For odd dimensions the values at the ends of the unit cell are again equal and opposite but this time proportional to $\pi 2^{-4\Delta+1} \left( \frac{2\Delta-1}{\Delta-1/2} \right).$

$^2$ There might be an integrated, or global, anomaly. The relation between local and global is non–trivial.
5. Partial coincidence limits

Equation (3) is a particular case of the following standard reduction formulae, [5] pp.22,23,

\[ F_1(a, b, b', c; x, 1) = \frac{\Gamma(c)\Gamma(c - a - b')}{\Gamma(c - a)\Gamma(c - b')} \, 2F_1(a, b, c - b'; x), \]

for \( \text{Re} \, (c - a - b') > 0 \),

or,

\[ F_1(a, b, b', c; x, x) = \, 2F_1(a, b + b', c; x) . \]

(7) (8)

Applied to (1), these would give \( G \) at say, \( z = 1 \), as a specific power series in \( \bar{z} \). The hypergeometric functions can be subject to further transformation.

6. Conclusion

Not unexpectedly, the result (5) agrees with (1). The reason is rather trivial (and probably routinely employed already, cf [6]). It is that, in dimensional regularisation, the coincidence limit of the flux–free Green function in flat space (the standard Green function) vanishes and so, for the appropriate range of \( \Delta \), one has the equality,

\[ G(\Delta, \delta, x, x) = G_{\text{sub}}(\Delta, \delta, x, x) = \langle \phi \bar{\phi}(x) \rangle . \]

The local average of the energy–momentum tensor was computed in [7] and more recently in [8] and [6]. The last reference seems to use dimensional continuation implicitly while in [8] the infinities are removed by hand.

I note that the Gamma function combinations that produce the result in (2) must automatically be contained in the contour manipulations that turn \( G \) into \( G_{\text{sub}} \).

Using \( G \), (1), for higher coincidence limits (which are essentially obtained by differentiation) is not algebraically efficient. These limits have been discussed more generally in [1,2].
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