Invariants of Third Type Almost Geodesic Mappings of Generalized Riemannian Space

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Abstract

We studied rules of transformations of Christoffel symbols under third type almost geodesic mappings in this paper. From this research, we obtained some new invariants of these mappings. These invariants are analogies of Thomas projective parameter and Weyl projective tensor.

Key words: almost geodesic mapping, difference, invariant

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1 Introduction

Following the Eisenhart’s work [6–8], it is started the research about Riemannian spaces endowed with non-symmetric metrics [26,27]. An N-dimensional manifold endowed with metric tensor \( g_{ij} \) non-symmetric by indices \( i \) and \( j \) is the generalized Riemannian space \( \mathbb{GR}_N \). Affine connection coefficients of the space \( \mathbb{GR}_N \) are Christoffel symbols of the second kind \( \Gamma_{jk}^i \) with respect to the connection of the metric \( g_{ij} \). Because \( \Gamma_{jk}^i \neq \Gamma_{kj}^i \), the symmetric and antisymmetric parts of the Christoffel symbol \( \Gamma_{jk}^i \) are defined as

\[
\Gamma_{jk}^i = \frac{1}{2}(\Gamma_{jk}^i + \Gamma_{kj}^i) \quad \text{and} \quad \Gamma_{\nu}^i = \frac{1}{2}(\Gamma_{jk}^i - \Gamma_{kj}^i). \tag{1.1}
\]

The anti-symmetric part \( \Gamma_{\nu}^i \) of the coefficient \( \Gamma_{jk}^i \) is called the torsion tensor of the space \( \mathbb{GR}_N \). The Riemannian space \( \mathbb{R}_N \), endowed with affine connection coefficients \( \Gamma_{jk}^i \) is the associated space of the space \( \mathbb{GR}_N \) [2][23][25][28][29].

With regard to the affine connection of Riemannian space, it is defined one kind of covariant derivation

\[
a_{j;k}^i = a_{j,k}^i + \Gamma_{\alpha}^i a_{\alpha}^j - \Gamma_{\alpha}^j a_{\alpha}^i, \tag{1.2}
\]

for a tensor \( a_j^i \) of the type \((1,1)\) and partial derivation denoted by comma. Curvature tensor \( R_{jmn}^i \) of the associated space \( \mathbb{R}_N \) is

\[
R_{jmn}^i = \Gamma_{jm,n}^i - \Gamma_{jn,m}^i + \Gamma_{\alpha}^j \Gamma_{\alpha}^m - \Gamma_{\alpha}^m \Gamma_{\alpha}^j. \tag{1.3}
\]

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Four kinds of covariant differentiation with regard to affine connection of the space $\mathbb{GR}_N$ are [15]:

1. $a_{jk}^i = a_{j,k}^i + \Gamma_{ak}^i a_j^a - \Gamma_{jk}^i a_k^a$,  
2. $a_{jk}^i = a_{j,k}^i + \Gamma_{ak}^i a_j^a - \Gamma_{kj}^i a_k^a$,  
3. $a_{jk}^i = a_{j,k}^i + \Gamma_{ak}^i a_j^a - \Gamma_{kj}^i a_k^a$,  
4. $a_{jk}^i = a_{j,k}^i + \Gamma_{ak}^i a_j^a - \Gamma_{kj}^i a_k^a$.

(1.4)

There are twelve curvature tensors of the space $\mathbb{GR}_N$ [15]. They are elements of the family

$K_{jmn}^i = R_{jmn}^i + u_{\gamma n}^i \Gamma_{\gamma m}^i + v_{\gamma m}^i \Gamma_{\gamma n}^i + w_{\gamma n}^i \Gamma_{\gamma m}^i$,  

(1.6)

for real constants $u, u', v, v', w$. Five of these curvature tensors are linearly independent [16].

1.1 Mappings of generalized Riemannian spaces

Riemannian and generalized Riemannian spaces are special affine connected spaces. We are going to pay attention on mappings between generalized Riemannian spaces in this paper.

Different mappings of Riemannian and generalized Riemannian spaces as well as their invariants have been investigated (see [1,2,6,9–25,28,29,31]). N. S. Sinyukov started the research about almost geodesic mappings [19]. His work has been continued by by J. Mikeš and his research group (see [12–14]).

Sinyukov [19] generalized the term of geodesics. He involved the terms of almost geodesic lines and almost geodesic mappings of symmetric affine connection spaces. He founded that are three types $\pi_1, \pi_2, \pi_3$ of almost geodesic of a symmetric affine connection space.

The terms of almost geodesic lines and almost geodesic mappings are generalized in [20–22]. It is founded that there are three types and two kinds of almost geodesic mappings of a non-symmetric affine connection space. Because generalized Riemannian spaces are special non-symmetric affine connection spaces, the basic equations of an almost geodesic mapping $f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N$ of the third type and $s$-th kind, $s = 1, 2$, are:

$$\pi_3^s : \begin{cases} 
\Gamma_j^i = \Gamma_j^i + \psi_j \delta^i_k + \psi_k \delta^i_j + 2\sigma_{jk} \varphi^i + \xi_{jk}^i, \\
\varphi_j^i + (-1)^{s-1} \xi_{ij}^i \varphi^a = \nu_j \varphi^i + \mu \delta_j^i,
\end{cases}$$

(1.7)

for tensor $\xi_{jk}^i$ anti-symmetric by indices $j$ and $k$, covariant vectors $\psi_j, \nu_j$, contravariant vector $\varphi^i$ and scalar function $\mu$. Third type almost geodesic mapping $f$ has the property of reciprocity if its inverse mapping is third type almost geodesic mapping.

In this paper, we will pay attention to invariants of equitortion third type almost geodesic mappings which satisfy the property of reciprocity.

1.2 Motivation

A. Einstein was the first scientist who applied the non-symmetric affine connection in the theory of gravitation [3–5]. In his theory, Weyl projective tensor is related to gravity.

In this paper, we wish to generalize Weyl projective tensor as an invariant of equitortion third type almost geodesic mappings. It will be obtained transformation rules of covariantly...
differentiated torsion tensor $\Gamma_{jk}^i$ under equitorsion third type almost geodesic mappings which have the property of reciprocity. In the next, we will find families of invariants of these mappings with regard to the changes of curvature tensors $K_{jmn}^i$ given by the equation (1.6).

2 Main results

Let $f: GR_N \rightarrow GR_N$ be an equitorsion almost geodesic mapping of the third type and $s$-th kind $s = 1, 2$, which satisfies the property of reciprocity. Basic equations of this mapping are

$$\pi_s: \begin{cases} \Gamma_{jk}^i = \Gamma_{jk}^i + \psi_j \delta_k^i + \psi_k \delta_j^i + 2\sigma_{jk} \varphi^i, \\ \varphi^i_j = \nu_j \varphi^i + \mu \delta_j^i, \end{cases} \quad (2.1)$$

for scalar function $\mu$, covariant vectors $\psi_j, \nu_j$, contravariant vector $\varphi^i$ and tensor $\sigma_{jk}$ of the type (0,2) symmetric by indices $j$ and $k$. It is obtained in [30] that the geometrical objects

$$\begin{aligned} W_{(1)jmn}^i &= R_{jmn}^i + \delta_j^i \eta_{(1)[mn]} - \frac{1}{N+1} \delta_j^i (\Gamma_{j\alpha;m}^\alpha - (N+1)(\eta_{jm} + \mu \sigma_{jm})) \\
&+ \frac{1}{N+1} \delta_j^i (\Gamma_{j\alpha;m}^\alpha - (N+1)(\eta_{jm} + \mu \sigma_{jm})) \\
&- (\sigma_{jm;m} - \sigma_{jm;m} - (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \varphi^\alpha) \varphi^i + \sigma_{jm} \Gamma_{\alpha\gamma}^i \varphi^\alpha - \sigma_{jn} \Gamma_{\gamma\alpha}^i \varphi^\alpha; \\
W_{(2)jmn}^i &= R_{jmn}^i + \delta_j^i \eta_{(2)[mn]} - \frac{1}{N+1} \delta_j^i (\Gamma_{j\alpha;m}^\alpha - (N+1)(\eta_{jm} + \mu \sigma_{jm})) \\
&+ \frac{1}{N+1} \delta_j^i (\Gamma_{j\alpha;m}^\alpha - (N+1)(\eta_{jm} + \mu \sigma_{jm})) \\
&- (\sigma_{jm;m} - \sigma_{jm;m} - (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \varphi^\alpha) \varphi^i - \sigma_{jm} \Gamma_{\alpha\gamma}^i \varphi^\alpha + \sigma_{jn} \Gamma_{\gamma\alpha}^i \varphi^\alpha; 
\end{aligned} \quad (2.2)$$

for $\eta_{ij}^{(1)}$ and $\eta_{ij}^{(2)}$ given by the following equations:

$$\eta_{j}^{(1)} = \frac{1}{(N+1)^2} ((N+1) \varphi^\alpha \sigma_{jk}(\Gamma_{\alpha\beta}^\beta + \sigma_{\alpha\beta} \varphi^\beta) - (\Gamma_{\alpha j}^\alpha + \sigma_{ja} \varphi^\alpha)(\Gamma_{k\beta}^\beta + \sigma_{k\beta} \varphi^\beta))$$

$$\eta_{j}^{(2)} = \frac{1}{(N+1)^2} ((N+1) \varphi^\alpha \sigma_{jk}(\Gamma_{\alpha\beta}^\beta + \sigma_{\alpha\beta} \varphi^\beta) - (\Gamma_{\alpha j}^\alpha + \sigma_{ja} \varphi^\alpha)(\Gamma_{k\beta}^\beta + \sigma_{k\beta} \varphi^\beta))$$

(2.4)

(2.5)

scalar function $\mu$ and antisymmetrization without division denoted by square brackets, are invariants of the mapping $f$. 

3
2.1 Transformations of covariant derivative of torsion tensor

Let \( f : \mathbb{G}_{RN} \rightarrow \mathbb{G}_{RN} \) be an equitorsion third type almost geodesic mapping of an \( s \)-th kind, \( s = 1, 2 \), which has the property of reciprocity. Based on the equation (1.2) and the invariance \( \Gamma^i_{jk} = \Gamma^i_{jk} \), we obtain that

\[
\Gamma^i_{jm\gamma} - \Gamma^i_{jm;\gamma} = \Gamma^i_{an} \Gamma^a_{jm} - \Gamma^i_{am} \Gamma^a_{jm} - \Gamma^i_{mn} \Gamma^m_{jm} + \Gamma^a_{jm} \Gamma^i_{an} + \Gamma^m_{jm} \Gamma^i_{an} - \Gamma^a_{jm} \Gamma^m_{an} - \Gamma^m_{jm} \Gamma^a_{an} \tag{2.6}
\]

Because the mapping \( f \) has the property of reciprocity, it is got in [30] that is

\[
\tilde{\psi}_j = -\psi_j \quad \text{and} \quad \tilde{\sigma}_{jk\varphi}^i = -\sigma_{jk\varphi}, \tag{2.7}
\]
i.e.

\[
\Gamma^i_{jk} - \Gamma^i_{jk} = \frac{1}{N+1} \left( \left( \Gamma^a_{ja} + \sigma_{ja\varphi^a} \right) \delta_k^i + \left( \Gamma^a_{ka} + \sigma_{ka\varphi^a} \right) \delta_j^i \right) - \sigma_{jk\varphi} \tag{2.8}
\]

With regard to the expressions (2.6, 2.8), we obtain that it holds

\[
\Gamma^i_{jm\gamma} - \Gamma^i_{jm;\gamma} = \tilde{\sigma}_{(p)jm\gamma}^i - \sigma_{(p)jm\gamma}^i, \tag{2.9}
\]

for \( p = 1, \ldots, 8 \), for

\[
\sigma_{(1)jm\gamma}^i = \Gamma^a_{jm} \Gamma^i_{an} - \Gamma^i_{am} \Gamma^a_{jm} - \Gamma^i_{ja} \Gamma^a_{mn}, \tag{2.10}
\]

\[
\sigma_{(2)jm\gamma}^i = \Gamma^a_{jm} \Gamma^i_{an} + \Gamma^i_{am} \Gamma^a_{jm} + \Gamma^i_{jm} \Gamma^a_{\sigma mn} - \frac{1}{N+1} \left( 2 \Gamma^i_{jm} \Gamma^a_{\sigma an} + \Gamma^i_{jm} \Gamma^a_{\sigma mn} - \Gamma^i_{jm} \Gamma^a_{\sigma an} \right), \tag{2.11}
\]

\[
\sigma_{(3)jm\gamma}^i = \Gamma^a_{jm} \Gamma^i_{an} - \Gamma^i_{am} \Gamma^a_{jm} + \Gamma^i_{jm} \Gamma^a_{\sigma mn}, \tag{2.12}
\]

\[
\sigma_{(4)jm\gamma}^i = \Gamma^a_{jm} \Gamma^i_{an} - \Gamma^i_{ja} \Gamma^a_{mn} + \Gamma^i_{am} \Gamma^a_{\sigma jm}, \tag{2.13}
\]

\[
\sigma_{(5)jm\gamma}^i = -\Gamma^a_{jm} \Gamma^i_{\sigma an} - \Gamma^i_{am} \Gamma^a_{jm} - \Gamma^i_{jm} \Gamma^a_{\sigma an}, \tag{2.14}
\]
\[ \sigma^i_{(6)jmn} = -\Gamma^i_{jm} \varphi^i \sigma_{an} + \Gamma^i_{am} \varphi^a \sigma_{jn} + \Gamma^i_{ja} \varphi^a \sigma_{mn} \]
\[ + \frac{1}{N+1} \left( \delta^i_n \Gamma^a_{jm} \Gamma^\beta_{a\beta} - \Gamma^i_{jm} \Gamma^a_{am} - \Gamma^i_{jn} \Gamma^a_{ma} + \Gamma^i_{mn} \Gamma^a_{ja} \right) \]
\[ + \frac{1}{N+1} \left( \delta^i_n \Gamma^a_{jm} \varphi^\beta \sigma_{a\beta} - \Gamma^i_{jm} \varphi^a \sigma_{am} + \Gamma^i_{mn} \varphi^a \sigma_{aj} \right), \]  
\[ \sigma^i_{(7)jmn} = -\Gamma^i_{jm} \varphi^i \sigma_{an} - \Gamma^i_{am} \Gamma^\alpha \sigma_{jn} + \Gamma^i_{ja} \varphi^a \sigma_{mn} \]
\[ + \frac{1}{N+1} \left( \delta^i_n \Gamma^a_{jm} \Gamma^\beta \sigma_{a\beta} - \Gamma^i_{jm} \Gamma^a_{ma} + \Gamma^i_{mn} \varphi^a \sigma_{aj} \right), \]  
\[ \sigma^i_{(8)jmn} = -\Gamma^i_{jm} \varphi^i \sigma_{an} - \Gamma^i_{am} \Gamma^\alpha \sigma_{jn} + \Gamma^i_{ja} \varphi^a \sigma_{mn} \]
\[ + \frac{1}{N+1} \left( \delta^i_n \Gamma^a_{jm} \Gamma^\beta \sigma_{a\beta} + \Gamma^i_{mn} \varphi^a \sigma_{aj} \right), \]  
and the corresponding \( \overline{\sigma}^i_{(p)jmn} \).

Let be
\begin{align*}
U_1 &= \Gamma^a_{jm} \varphi^i, & U_2 &= \Gamma^a_{jm} \varphi^i, & U_3 &= \Gamma^a_{jm} \varphi^i, & U_4 &= \Gamma^a_{jm} \varphi^i, & U_5 &= \Gamma^a_{jm} \varphi^i, \\
U_6 &= \Gamma^a_{jm} \varphi^i, & U_7 &= \Gamma^a_{jm} \varphi^i, & U_8 &= \Gamma^a_{jm} \varphi^i, & U_9 &= \Gamma^a_{jm} \varphi^i, & U_{10} &= \Gamma^a_{jm} \varphi^i, \\
U_{11} &= \Gamma^a_{jm} \varphi^i, & U_{12} &= \Gamma^a_{jm} \varphi^i, & U_{13} &= \Gamma^a_{jm} \varphi^i, & U_{14} &= \Gamma^a_{jm} \varphi^i, & U_{15} &= \Gamma^a_{jm} \varphi^i, & U_{16} &= \Gamma^a_{jm} \varphi^i, & U_{17} &= \Gamma^a_{jm} \varphi^i, & U_{18} &= \Gamma^a_{jm} \varphi^i, & U_{19} &= \Gamma^a_{jm} \varphi^i, & U_{20} &= \Gamma^a_{jm} \varphi^i.
\end{align*}

It holds the following lemma:

**Lemma 1.** Let \( f : \mathbb{GR}_N \rightarrow \mathbb{GR}_N \) be an equitor spoil third type almost geodesic mapping which has the property of reciprocity.

a) Covariant derivatives \( \Gamma^i_{jm} \) and \( \Gamma^i_{jm} \) of the torsion tensor of the spaces \( \mathbb{GR}_N \) and \( \mathbb{GR}_N \) satisfy the equations
\[ \Gamma^i_{jm} = \Gamma^i_{jm} + \overline{\sigma}^i_{(p)jmn} - \sigma^i_{(p)jmn} = \Gamma^i_{jm} + \sum_{p=1}^{8} \sum_{\theta=1}^{20} u^\rho_{\theta} (U - U), \]  
\[ p = 1, \ldots, 8, \text{ for the corresponding real constants } u^\rho_{\theta}, \text{ geometrical objects } \sigma^i_{(p)jmn}, U \text{ given by the equations (2.10) (2.18).} \]

b) The rank of matrix \([u^\rho_{\theta}]_{8 \times 20}, \rho = 1, \ldots, 8, \text{ is 4, i.e. there are four linearly independent transformations of the transformations from (2.19).} \]
Corollary 1. Geometrical objects

\[ \rho_j^{i} \frac{\mathcal{T}_{j,m,n}}{\mathcal{V} - \mathcal{U}_{j,m,n}} = \Gamma_{j,m,n}^{i} - \sum_{\theta=1}^{20} u_{\theta}^{i} U_{j,m,n} \]  \hspace{1cm} (2.20)

\( \rho \in \{1, \ldots, 8\} \), for the corresponding real constants \( u_{\theta}^{i} \), are invariants of an equitorsion almost geodesic mapping \( f : \mathcal{G}_{\mathcal{R}} \rightarrow \mathcal{G}_{\mathcal{R}} \) which has the property of reciprocity. Four of these invariants are linearly independent.

2.2 Transformations of curvature tensors under almost geodesic mappings

Let \( f : \mathcal{G}_{\mathcal{R}} \rightarrow \mathcal{G}_{\mathcal{R}} \) be an equitorsion almost geodesic mapping of the third type and \( s \)-th kind, \( s = 1,2 \), which has the property of reciprocity. From the invariance of the geometrical objects \( \mathcal{W}^{i}_{(1)j,m,n} \) and \( \mathcal{W}^{i}_{(2)j,m,n} \) given by the equations (2.2, 2.3), we directly obtain that is

\[ \mathcal{R}^{i}_{j,m,n} = R^{i}_{j,m,n} - \delta_{j}^{i} \eta^{[mn]} + \frac{1}{N + 1} \delta_{j}^{i} (\Gamma_{j,m,n}^{\alpha} - (N + 1)(\eta_{j,n}^{(1)} + \mu \sigma_{j}) \mathcal{W}^{i}_{(1)m,n}) \]

\[ - \frac{1}{N + 1} \delta_{j}^{i} (\Gamma_{j,m,n}^{\alpha} - (N + 1)(\eta_{j,m}^{(1)} + \mu \sigma_{j}) \mathcal{W}^{i}_{(1)n,m}) \]

\[ + (\sigma_{j,m}^{\alpha} - \sigma_{j,n}^{\alpha} - (\sigma_{j,m}^{\alpha} - \sigma_{j,n}^{\alpha}) \varphi^{\alpha}) \varphi^{i} - \sigma_{j,m}^{\alpha} \mathcal{W}^{i}_{(2)m,n} + \sigma_{j,n}^{\alpha} \mathcal{W}^{i}_{(2)n,m} \]

\[ + \delta_{j}^{i} \eta^{[mn]} - \frac{1}{N + 1} \delta_{j}^{i} (\Gamma_{j,m,n}^{\alpha} - (N + 1)(\eta_{j,n}^{(2)} + \mu \sigma_{j}) \mathcal{W}^{i}_{(2)m,n}) \]

\[ \mathcal{R}^{i}_{j,m,n} = R^{i}_{j,m,n} - \delta_{j}^{i} \eta^{[mn]} + \frac{1}{N + 1} \delta_{j}^{i} (\Gamma_{j,m,n}^{\alpha} - (N + 1)(\eta_{j,m}^{(2)} + \mu \sigma_{j}) \mathcal{W}^{i}_{(2)n,m}) \]

\[ - \frac{1}{N + 1} \delta_{j}^{i} (\Gamma_{j,m,n}^{\alpha} - (N + 1)(\eta_{j,n}^{(2)} + \mu \sigma_{j}) \mathcal{W}^{i}_{(2)n,m}) \]

\[ + (\sigma_{j,m}^{\alpha} - \sigma_{j,n}^{\alpha} - (\sigma_{j,m}^{\alpha} - \sigma_{j,n}^{\alpha}) \varphi^{\alpha}) \varphi^{i} + \sigma_{j,m}^{\alpha} \mathcal{W}^{i}_{(2)m,n} - \sigma_{j,n}^{\alpha} \mathcal{W}^{i}_{(2)n,m} \]  \hspace{1cm} (3.1.1)

Hence, based on these transformations and the equation (2.19) we establish the following equations:

\[ \]
\[
\overline{\mathcal{K}}_{jmn} = K_{jmn} - \delta^i_j \hat{\eta}_{[mn]} + \frac{1}{N+1} \delta^i_m (\Gamma^\alpha_{j\alpha,m} - (N+1)(\hat{\eta}_{jn} + \mu \sigma_{jn}))
\]
\[
- \frac{1}{N+1} \delta^i_n (\Gamma^\alpha_{j\alpha,m} - (N+1)(\hat{\eta}_{jm} + \mu \sigma_{jm}))
\]
\[
+ (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha} - (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha}) \varphi^{\alpha}) \varphi_i - \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu} + \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu}
\]
\[
+ \delta^j_{[mn]} - \frac{1}{N+1} \delta^i_m (\Gamma^\alpha_{j\alpha,n} - (N+1)(\hat{\eta}_{jn} + \mu \sigma_{jn}))
\]
\[
+ \frac{1}{N+1} \delta^i_n (\Gamma^\alpha_{j\alpha,n} - (N+1)(\hat{\eta}_{jm} + \mu \sigma_{jm}))
\]
\[
- (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha} - (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha}) \varphi^{\alpha}) \varphi_i + \sigma_{jm} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu} - \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu}
\]
\[
+ u \varphi_i^{(p)\eta_{jmn}} + u' \varphi_i^{(q)\sigma_{jmn}} - u \sigma_i^{(p)\eta_{jmn}} - u' \sigma_i^{(q)\sigma_{jmn}}
\]
\[
= K_{jmn} - \delta^i_j \hat{\eta}_{[mn]} + \frac{1}{N+1} \delta^i_m (\Gamma^\alpha_{j\alpha,m} - (N+1)(\hat{\eta}_{jn} + \mu \sigma_{jn}))
\]
\[
- \frac{1}{N+1} \delta^i_n (\Gamma^\alpha_{j\alpha,m} - (N+1)(\hat{\eta}_{jm} + \mu \sigma_{jm}))
\]
\[
+ (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha} - (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha}) \varphi^{\alpha}) \varphi_i - \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu} + \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu}
\]
\[
+ \delta^j_{[mn]} - \frac{1}{N+1} \delta^i_m (\Gamma^\alpha_{j\alpha,n} - (N+1)(\hat{\eta}_{jn} + \mu \sigma_{jn}))
\]
\[
+ \frac{1}{N+1} \delta^i_n (\Gamma^\alpha_{j\alpha,n} - (N+1)(\hat{\eta}_{jm} + \mu \sigma_{jm}))
\]
\[
- (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha} - (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha}) \varphi^{\alpha}) \varphi_i + \sigma_{jm} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu} - \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu}
\]
\[
+ u \varphi_i^{(p)\eta_{jmn}} + u' \varphi_i^{(q)\sigma_{jmn}} - u \sigma_i^{(p)\eta_{jmn}} - u' \sigma_i^{(q)\sigma_{jmn}}
\]

Based on these transformations, we obtain that is

\[
\hat{\mathcal{W}}^{i}_{(p,q)\eta_{jmn}} = \hat{\mathcal{W}}^{i}_{(p,q)\eta_{jmn}} \quad \text{and} \quad \hat{\mathcal{W}}^{i}_{(p,q)\sigma_{jmn}} = \hat{\mathcal{W}}^{i}_{(p,q)\sigma_{jmn}},
\]

for \((p, q) \in \{1, \ldots, 8\}^2\) and

\[
\hat{\mathcal{W}}^{i}_{(p,q)\eta_{jmn}} = K^{i}_{jmn} + \delta^j_{[mn]} - \frac{1}{N+1} \delta^i_m (\Gamma^\alpha_{j\alpha,n} - (N+1)(\hat{\eta}_{jn} + \mu \sigma_{jn}))
\]
\[
+ \frac{1}{N+1} \delta^i_n (\Gamma^\alpha_{j\alpha,n} - (N+1)(\hat{\eta}_{jm} + \mu \sigma_{jm}))
\]
\[
- (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha} - (\sigma_{jm\sigma\alpha} - \sigma_{jn\sigma\alpha}) \varphi^{\alpha}) \varphi_i + \sigma_{jm} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu} - \sigma_{jn} \Gamma^{\alpha \nu}_{\alpha \nu} \varphi^{\nu}
\]
\[
- u \sigma_i^{(p)\eta_{jmn}} - u' \sigma_i^{(q)\sigma_{jmn}}
\]

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\[
\begin{align*}
\mathcal{W}_{(p,q)}^{i,jmn} &= K_{jmn}^{i} + \delta_{(2)}^{i} \eta_{jmn}^{m} - \frac{1}{N+1} \delta_{n}^{i} (\Gamma_{jm;n}^{\alpha} - (N+1)(\eta_{jm}^{n} + \mu \sigma_{jm})) \\
&+ \frac{1}{N+1} \delta_{n}^{i} (\Gamma_{jm;n}^{\alpha} - (N+1)(\eta_{jm}^{n} + \mu \sigma_{jm})) \\
&- (\sigma_{jm;n} - \sigma_{jm;m} - (\sigma_{jm}^{m} - \sigma_{jm;m}^{m})\varphi^{\alpha}) \phi^{i} - \sigma_{jm}^{i} \Gamma_{jm,n}^{\alpha} \varphi^{\alpha} + \sigma_{jm}^{i} \Gamma_{jm,n}^{\alpha} \varphi^{\alpha} \\
&- \mu \sigma_{jm}^{i} - \mu' \sigma_{jm}^{i} - N^{1}(\mu) \Gamma_{jm,n}^{\alpha} \varphi^{\alpha} + \mu' \Gamma_{jm,n}^{\alpha} \varphi^{\alpha},
\end{align*}
\] (2.24)

**Theorem 1.** Let \( f : \mathcal{G}R_{N} \to \mathcal{G}R_{N} \) be an equitorsion almost geodesic mapping of the third type and \( s \)-th kind, \( s = 1, 2 \), which has the property of reciprocity. Families \( \mathcal{W}_{(p,q)}^{i,jmn} \) and \( \mathcal{W}_{(p,q)}^{i,jmn} \) are families of invariants of mapping of the corresponding kind.

**Corollary 2.** The families \( \mathcal{W}_{(p,q)}^{i,jmn} \), \( s = 1, 2 \), of invariants of an equitorsion almost geodesic mapping \( f : \mathcal{G}R_{N} \to \mathcal{G}R_{N} \) which has the property of reciprocity and the invariants \( \mathcal{W}_{(s)}^{i,jmn} \) given by the equations (2.23) and (2.24) satisfy the equations

\[
\begin{align*}
\mathcal{W}_{(p,q)}^{i,jmn} &= \mathcal{W}_{(s)}^{i,jmn} - \mu \sigma_{jm}^{i} - \mu' \sigma_{jm}^{i} - N^{1}(\mu) \Gamma_{jm,n}^{\alpha} \varphi^{\alpha} + \mu' \Gamma_{jm,n}^{\alpha} \varphi^{\alpha},
\end{align*}
\] (2.25)

for \( (p,q) \in \{1, \ldots, 8\}^{2} \).

**Corollary 3.** The rank of matrix

\[
\mathcal{W} = \begin{bmatrix}
1 & -u'_{1} & \ldots & -u'_{8} & u & u' & v & v' & w
\end{bmatrix}
\] (2.26)

of the type \( 64 \times 26 \) is equal 6, i.e. there are six linearly independent families \( \mathcal{W}_{(p,q)}^{i,jmn}, s \in \{1, 2\} \), \( (p,q) \in \{1, \ldots, 8\}^{2} \), of invariants given by the equations (2.23) and (2.24).

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