Form Factor of a Quantum Graph in a Weak Magnetic Field

Taro Nagao and Keiji Saito†

Institut für Theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Germany (Permanent Address: Department of Physics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan)

†Department of Applied Physics, Graduate School of Engineering, University of Tokyo, Bunkyo-ku, Tokyo 113-8656, Japan

Abstract

Using periodic orbit theory, we evaluate the form factor of a quantum graph to which a very weak magnetic field is applied. The first correction to the diagonal approximation describing the transition between the universality classes is shown to be in agreement with Pandey and Mehta’s formula of parametric random matrix theory.

PACS: 05.45.Mt; 05.40.-a

KEYWORDS: quantum graph; periodic orbit theory; random matrix
1 Introduction

Since more than a decade, it has been known that the energy level statistics of classically chaotic systems universally follows the prediction of random matrix theory. As the semiclassical theory of chaotic systems is described in terms of classical periodic orbits, it is preferable if we can also understand the universal level statistics in terms of them. However, until recently, such an understanding had been limited within the framework of Berry’s diagonal approximation for double sums over periodic orbits[1].

A recent breakthrough was brought about by Sieber and Richter[2, 3]. They identified the pairs of periodic orbits giving the first off-diagonal correction and showed that such pairs indeed exist in chaotic systems. The first off-diagonal correction was then found to be in agreement with random matrix theory. Berkolaiko, Schanz and Whitney rederived the same correction term for quantum graphs[4] and further succeeded in evaluating the second correction[5].

In this paper, we investigate a quantum graph and extend the calculation of the first off-diagonal correction to the case intermediate between the absence and presence of a magnetic field. In random matrix theory, this intermediate case corresponds to Pandey and Mehta’s two matrix model[6, 7]. Chaotic systems with time reversal symmetry (without magnetic field) are described by GOE(Gaussian Orthogonal Ensemble) of random matrices. On the other hand, when time reversal symmetry is broken by the application of a magnetic field, GUE(Gaussian Unitary Ensemble) becomes suitable. Pandey and Mehta formulated an intermediate random matrix ensemble (two matrix model) between GOE and GUE and were able to derive the correlation functions among the eigenvalues. Within the diagonal approximation, Bohigas et al.[8] already showed that the periodic orbit theory was in accordance with the intermediate random matrix ensemble. We will show that, by applying Berkolaiko, Schanz and Whitney’s method, the first off-diagonal correction is also in agreement with Pandey and Mehta’s formula.

Let us explain the model. We consider quantum mechanics on a graph[9, 10, 11, 12]. A graph consists of vertices connected by bonds. For example, in a globally coupled graph, every vertex is connected to all vertices, while in a star graph, one particular central vertex is connected to all of the others.

On the bonds a particle behaves like a free particle and on the vertices it is scattered according to a given scattering matrix. On the bond \((j, l)\) connecting the \(j\)-th vertex and \(l\)-th vertex, the Schroedinger equation

\[
-\left( -i \frac{d}{dx_{jl}} - A_{jl} \right) \Psi(x_{jl}) = k^2 \Psi(x_{jl})
\]  

(1.1)

holds, where \(A_{jl}\) is a magnetic vector potential satisfying \(A_{jl} = -A_{lj}\). Therefore, on the bond \((j, l)\), the wave function \(\Psi(x_{jl})\) is proportional to

\[
\exp(ikx_{jl} + iA_{jl}x_{jl}).
\]

(1.2)

The relative amplitude of this wave function is decided by the scattering matrices on the vertices. Let us suppose that, if a wave function has the amplitude 1 on the bond \((j, l)\), then the amplitude of the wave function on the bond \((l, m)\) is \(\sigma_{j,m}^{l}\). Then the scattering
matrix connecting the bond \((j, l')\) and \((l, m)\) is
\[
S_{jl',lm} = \sigma^{(l)}_{jm} \exp (ikL_{lm} + iA_{lm}L_{lm}) \delta_{l,l'},
\]
where \(L_{jl}\) is the length of the bond \((j, l)\) satisfying \(L_{jl} = L_{lj}\). The total number of the vertices and directed bonds are denoted by \(N\) and \(B\), respectively. In this paper, we consider only a globally coupled graph (every vertex is connected to all vertices), so that \(B\) is equal to \(N^2\). We further assume that the scattering matrix is given by the discrete Fourier transform
\[
\sigma^{(l)}_{mn} = \frac{1}{\sqrt{N}} e^{2\pi imn/N}.
\]

Chaotic dynamics in the classical limit is characterized by an approach to an equidistribution over all bonds. Noting that an analogue of the classical Frobenius-Perron operator is given by \(M_{m'l'lm} \equiv |S_{m'l'lm}|^2\), we can write it as
\[
\lim_{t \to \infty} M_{m'l'lm} = 1/B.
\]
For the scattering matrix \((1.4), (1.3)\) holds even for finite \(t\). Berkolaiko, Schanz and Whitney[4] argued that, if the convergence of \((1.5)\) in the limit \(t \to \infty\) is sufficiently fast, the form factor is in agreement with random matrix theory. We expect that a similar condition holds in the intermediate case. However it is not discussed here and left for future works.

## 2 Periodic Orbit Theory

Let us define \(P\) and \(Q\) as sequences of vertices \([p_1, p_2, \ldots, p_l]\) and \([q_1, q_2, \ldots, q_l]\), respectively. Each of \(p_j\) and \(q_j\) takes the values \(1, 2, \ldots, N\). According to Berkolaiko, Schanz and Whitney[4], the form factor \(K(\tau)\) can be written in terms of the scattering matrices as
\[
K(\tau) = \frac{1}{B} \langle |\mathrm{Tr}S|^2 \rangle
= \frac{1}{B} \lim_{\kappa \to \infty} \kappa^{-1} \sum_{P,Q} \int_0^\kappa dk \sigma^{(p_1)}_{p_{l-1}p_l} e^{iL_{p_{l-1}p_l} (k+A_{p_{l-1}p_l})} \sigma^{(p_2)}_{p_{l-1}p_l} e^{iL_{p_{l-1}p_l} (k+A_{p_{l-1}p_l})} \ldots
\]
\[
\sigma^{(p_l)}_{p_{l-1}p_l} e^{iL_{p_{l-1}p_l} (k+A_{p_{l-1}p_l})} \sigma^{(q_1)}_{q_{l-1}q_l} e^{-iL_{q_{l-1}q_l} (k+A_{q_{l-1}q_l})} \ldots \sigma^{(q_l)}_{q_{l-1}q_l} e^{-iL_{q_{l-1}q_l} (k+A_{q_{l-1}q_l})}
\]
\[
= \frac{1}{B} \sum_{P,Q} A_P A_Q \sigma^{(p_1)}_{p_{l-1}p_l} \ldots \sigma^{(p_l)}_{p_{l-1}p_l} e^{i(L_{p_{l-1}p_l} A_{p_{l-1}p_l} + \ldots + L_{p_{l-1}p_l} A_{p_{l-1}p_l} - L_{q_{l-1}q_l} A_{q_{l-1}q_l})} \delta_{L_P, L_Q},
\]
where
\[
A_P = \sigma^{(p_1)}_{p_{l-1}p_l} \ldots \sigma^{(p_l)}_{p_{l-1}p_l}, \quad L_P = L_{p_{l-1}p_l} + \ldots + L_{p_{l-1}p_l},
\]
\[
A_Q = \sigma^{(q_1)}_{q_{l-1}q_l} \ldots \sigma^{(q_l)}_{q_{l-1}q_l}, \quad L_Q = L_{q_{l-1}q_l} + \ldots + L_{q_{l-1}q_l}
\]
and
\[
\tau = t/B.
\]
We are interested in the scaling limit $t \to \infty$, $B \to \infty$ with $\tau$ fixed. Then we expect that the diagonal term and its corrections give the expansion around $\tau = 0$.

The diagonal term comes from the cases in which circular permutations of $P$ and $Q$ (or the reversal of $Q$) coincide. It is explicitly written as

$$K^{\text{diag}}(\tau) = \frac{t}{B} \sum_P |A_P|^2 \left[ \epsilon^{2i} \sum_j L_{p_j p_{j+1}} A_{p_j p_{j+1}} + 1 \right],$$

where $p_{t+1} = p_1$. On the other hand, the first off-diagonal correction $K^{\text{off}}(\tau)$ comes from the pairs of self-intersecting orbits differing in the orientation of a single loop. We suppose that the orbits are self-intersecting at a vertex $\alpha \equiv p_1 = p_\tau$. Then, in the first off-diagonal correction, the orbit sum is taken over the pair $P = [\alpha, l_1, \alpha, l_2]$ and $Q = [\alpha, l_1, \alpha, \bar{l}_2]$. Here $l_1 = [p_2, p_3, \cdots, p_{t-1}]$, $l_2 = [p_{\tau+1}, p_{\tau+2}, \cdots, p_t]$, respectively, and $\bar{l}_2 = [p_t, p_{t-1}, \cdots, p_{\tau+1}]$ is the reversal of the sequence $l_2$. That is, $K^{\text{off}}(\tau)$ can be written as

$$K^{\text{off}}(\tau) = \frac{\tau^2}{B} \sum_{t'=1}^{t-2} \sum_P (1 - \delta_{\alpha \alpha}) |\sigma^{(a)}|^2 \cdots |\sigma^{(b)}_{p_{\tau'-2} \alpha}|^2 |\sigma^{(c)}_{p_{\tau'} \alpha}|^2 \cdots |\sigma^{(d)}_{p_\tau \alpha}|^2$$

$$\times \sigma^{(a)}_{d a} \sigma^{(a)}_{b c} \sigma^{(a)}_{c a} \sigma^{(a)}_{b d} \exp \left[ 2i \left( L_{d a} A_{d a} + L_{p_{t-1} p_t} A_{p_{t-1} p_t} + \cdots + L_{p_{\tau} p_{\tau+1}} A_{p_{\tau} p_{\tau+1}} \right) \right].$$

Here $a, b, c, d$ are identified with $p_2, p_{\tau-1}, p_{\tau+1}, p_t$, respectively. As indicated by the prime, the sum over $P$ obeys the restriction that $l_1$ is not identical to the reversal of itself($l_1 \neq \bar{l}_1$).

We are now in a position to calculate the diagonal contribution and the first off-diagonal correction. For simplicity, we set $L_{jl} = 1/2$ from now on.

(1) the diagonal contribution

$$K^{\text{diag}}(\tau) = \frac{t}{B} \sum_P |A_P|^2 \left[ \epsilon^{2i} \sum_j L_{p_j p_{j+1}} A_{p_j p_{j+1}} + 1 \right]$$

$$= \frac{t}{B} \sum_P \frac{1}{N^t} \left[ \epsilon^{2i} \sum_j L_{p_j p_{j+1}} A_{p_j p_{j+1}} + 1 \right]$$

$$= \frac{t}{B N^t} \left[ N^t + \text{Tr}g^t \right].$$

(2.7)

Here the elements $g_{jl}$ of an $N \times N$ transfer matrix $g$ is defined as

$$g_{jl} = \begin{cases} 1, & j = l, \\ \epsilon^{iA_{jl}}, & j \neq l. \end{cases}$$

(2.8)

Since we are interested in the long time limit $t \to \infty$, we only need to know the behavior of the largest eigenvalue $\Lambda$ of $g$. For small $A_{jl}$, the largest eigenvalue $\Lambda$ can be evaluated by the perturbation method. Let us decompose $g$ as

$$g = g_0 + g',$$

where $g_0$ is an $N \times N$ matrix with all the elements equal to 1. Then the largest eigenvalue of $g_0$ is $N$ and the corresponding normalized eigenvector $\mathbf{x}_1$ is $(1/\sqrt{N}, 1/\sqrt{N}, \cdots, 1/\sqrt{N})^T$.
(Here a superscript $T$ means a transpose). All the other eigenvalues of $g_0$ are 0 and let us denote the corresponding orthonormalized eigenvectors by $x_2, x_3, \ldots, x_N$. Then we regard $g'$ as a small perturbation and apply the standard technique of perturbation theory to evaluate the largest eigenvalue. The second order perturbation gives

$$\Lambda \sim N + x_1^T g' x_1 + \frac{1}{N} \sum_{n=2}^{N} |x_n^T g' x_1|^2,$$

which yields

$$\Lambda \sim N - \frac{1}{N} \sum_{j<l} A_{jl}^2 + \frac{1}{N} \sum_{n=2}^{N} |x_n^T a x_1|^2,$$

where $a$ is an $N \times N$ matrix with the elements

$$a_{jl} = \begin{cases} 0, & j = l, \\ iA_{jl}, & j \neq l. \end{cases}$$

In the limit $N \to \infty$, the sum $C$ of the contributions from the second and third terms is normally $O(N)$ (for fixed $A_{jl}$). We then generally write $C \sim -Nb$ with $b$ proportional to the magnetic field applied to the graph. For example, if $\sum_{j \neq l} A_{jl} = 0$ for all $j, l$ (this is possible when $N$ is odd), the third term vanishes and the second term gives $C = -(N-1)A^2/2 \sim -NA^2/2$, so that $b = A^2/2$. The trace of the multiples of the matrix $g$ can be estimated as

$$\text{Tr} g^t \sim \Lambda^t \sim (N - Nb)^t \sim N^t e^{-bt}$$

with $t \to \infty$, $b \to 0$ and $bt$ fixed. Putting this, we arrive at

$$K^{\text{diag}}(\tau) \sim \frac{t}{B} (1 + e^{-bt}).$$

This is the diagonal contribution to the form factor in a weak magnetic field, as we shall see below, in agreement with random matrix theory. We now proceed to the first off-diagonal correction.

(2) the first off-diagonal correction

As noted before, the contribution $K^{\text{off}}(\tau)$ from self-retracing orbits with $l_1 = \bar{l}_1$ is removed from the periodic orbit sum (2.4) for $K^{\text{off}}(\tau)$, since it is already included in the diagonal contribution. In spite of that, it is convenient to first consider the sum with no such restriction. The periodic sum including the self-retracing orbits can be readily evaluated as

$$K^{\text{off}}(\tau) + K^{\text{off}}(\tau) = \frac{t^2}{B} \sum_{t'=4}^{t-2} N^{t'-2} \sum_{abcd} (1 - \delta_{cd}) g_{ac} \left[ g^{t'-1} \right]_{c,d} g_{da} \sigma^{(a)}_{da} \sigma^{(a)}_{bc} \sigma^{(a)*}_{ca} \sigma^{(a)*}_{bd}$$

$$= \frac{t^2}{B} \sum_{t'=4}^{t-2} N^{t'-2} \sum_{abcd} \delta_{cd} (1 - \delta_{cd}) g_{ac} \left[ g^{t'-1} \right]_{c,d} g_{da} = 0.$$

Here we utilized the unitarity of the matrices $\sigma^{(a)}$. As to the contribution from the self-retracing orbits, if we neglect completely self-retracing paths which are exponentially few,
a diagrammatic cancellation leaves only the terms with $t' = 4$ and $t' = 5$, in which the factors $1 - \delta_{cd}$ are removed. Therefore we find

$$K_{\text{off}}(\tau) = -K_{\text{off}}(\tau) = \frac{-t^2}{B} \left[ N^{2-t} \sum_{aa'dd} \sigma_{a'd}^{(a)} \sigma_{ac}^{(a)} \sigma_{ca}^{(a)} g_{ac} \left[ g_{t=5}^{t-5} \right]_{c,d} g_{da} \right] + N^{3-t} \sum_{aa'dd} \sigma_{a'd}^{(a)} \sigma_{ac}^{(a)} \sigma_{ad}^{(a)} g_{ac} \left[ g_{t=6}^{t-6} \right]_{c,d} g_{da} \right]$$

$$= \frac{-t^2}{B} \left[ N^{1-t} \text{Tr} \left( g_{t=3}^{t-3} \right) + N^{2-t} \text{Tr} \left( g_{t=4}^{t-4} \right) \right]. \quad (2.16)$$

Substitution of (2.13) into (2.16) leads to

$$K_{\text{off}}(\tau) \sim -2 \frac{t^2}{B} e^{-bt}. \quad (2.17)$$

As a result, the diagonal and the first off-diagonal terms are summed up to yield

$$K_{\text{diag}}(\tau) + K_{\text{off}}(\tau) \sim \tau + \tau e^{-bt} - 2 \frac{t^2}{B} e^{-bt}. \quad (2.18)$$

3 Random Matrix Result: Pandey-Mehta Formula

In this section we calculate the Fourier transform of random matrix result (Pandey-Mehta formula) to derive a prediction of the form factor $K(\tau)$. Let us define the Fourier transform of the two energy level correlation function $Y(r; \rho)$ as $Y(k; \rho)$. The form factor $K(k)$ can be then written as

$$K(k) = 1 - Y(k; \rho). \quad (3.1)$$

Pandey and Mehta’s formula [6] is

$$Y(r; \rho) = \left( \frac{\sin \frac{\pi r}{\rho}}{\frac{\pi r}{\rho}} \right)^2 - \frac{1}{\pi^2} \int_0^\pi dk_1 \int_{-\infty}^{\infty} dk_2 \left( \frac{k_1}{k_2} \right) \sin(k_1 r) \sin(k_2 r) e^{2\rho^2(k_1-k_2)(k_1+k_2)}. \quad (3.2)$$

Here $r$ is the distance between the two energy levels and $\rho$ is a parameter corresponding to a weak magnetic field. In the limit $\rho \rightarrow 0$, $Y(r; \rho)$ becomes the two level correlation function of GOE, while, in the limit $\rho \rightarrow \infty$, $Y(r; \rho)$ approaches that of GUE.

The Fourier transform $Y(k; \rho)$ can be evaluated as

$$Y(k; \rho) = 1 - k - \frac{1}{2\pi} \int_{-\pi - \bar{k}}^{\pi - \bar{k}} dk_1 \frac{k_1}{k_1 + k} e^{-2\rho^2(2k_1 + k)\bar{k}}, \quad 0 \leq \bar{k} \leq \pi, \quad (3.3)$$

where $\bar{k} = 2\pi k$. Let us define

$$F(\bar{k}) = \frac{1}{2\pi} \int_{-\pi - \bar{k}}^{\pi - \bar{k}} dk_1 \frac{k_1}{k_1 + k} e^{-2c(2k_1 + k)\bar{k}} \quad (3.4)$$

with $c = \rho^2 \bar{k}$ fixed. Then, for small $\bar{k}$, $F(\bar{k})$ can be expanded as

$$F(\bar{k}) = F(0) + F'(0)\bar{k} + \frac{1}{2} F''(0)\bar{k}^2 + \cdots, \quad (3.5)$$
where \( F'(\bar{k}) \) and \( F''(\bar{k}) \) are the first and second derivatives of \( F(\bar{k}) \), respectively. The derivatives are readily evaluated as
\[
F'(0) = \frac{1}{2\pi} e^{-4\pi c}, \\
F''(0) = -\frac{1}{\pi^2} e^{-4\pi c},
\]
which lead to (for small \( k \))
\[
Y(k; \rho) \sim 1 - k - F'(0)\bar{k} - \frac{1}{2} F''(0)\bar{k}^2 \\
= 1 - k - ke^{-4\pi c} + 2k^2 e^{-4\pi c}.
\]
Therefore we arrive at
\[
K(k) \sim k + ke^{-4\pi c} - 2k^2 e^{-4\pi c},
\]
which is in agreement with periodic orbit theory with an identification
\[
\tau = k, \quad bt = 4\pi c = 8\pi^2 \rho^2 k.
\]

In summary, using the periodic orbit theory we evaluated the first off-diagonal correction to the form factor of a quantum graph in a very weak magnetic field. It was found that the result was consistently in agreement with Pandey-Mehta formula of random matrix theory. The extension to the second off-diagonal correction seems promising, since it was already worked out in the absence of a magnetic field [5].

**References**

[1] M.V. Berry, Proc. R. Soc. London **A400** (1985) 229.

[2] M. Sieber and K. Richter, Physica Scripta, **T90** (2001) 128.

[3] K. Richter and M. Sieber, [cond-mat/0205158](http://arxiv.org/abs/cond-mat/0205158).

[4] G. Berkolaiko, H. Schanz, and R. S. Whitney, Phys. Rev. Lett. **88** (2002) 104101-1.

[5] G. Berkolaiko, H. Schanz, and R. S. Whitney, [nlin.CD/0205014](http://arxiv.org/abs/nlin.CD/0205014).

[6] A. Pandey and M.L. Mehta, Commun. Math. Phys. **87** (1983) 449.

[7] M.L. Mehta and A. Pandey, J. Phys. **A16** (1983) 2655.

[8] O. Bohigas, M. -J. Giannoni, A. M. Ozorio de Almeida, and C. Schmit, Nonlinearity **8** (1995) 203.

[9] T. Kottos and U. Smilansky, Phys. Rev. Lett. **79** (1997) 4794.

[10] H. Schanz and U. Smilansky, Phil. Mag. **B80** (2000) 1999.

[11] H. Schanz and U. Smilansky, Phys. Rev. Lett. **84** (2000) 1427.

[12] G. Berkolaiko and J. P. Keating, J. Phys. A: Math. Gen. **32** (1999) 7827.