Disorder Induced Transitions in Layered Coulomb Gases and Superconductors

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A 3D layered system of charges with logarithmic interaction parallel to the layers and random dipoles is studied via a novel variational method and an energy rationale which reproduce the known phase diagram for a single layer. Increasing interlayer coupling leads to successive transitions in which charge rods correlated in \( N > 1 \) neighboring layers are nucleated by weaker disorder. For layered superconductors in the limit of only magnetic interlayer coupling, the method predicts and locates a disorder-induced defect-unbinding transition in the flux lattice. While \( N = 1 \) disorder induced defect rods are predicted for multi-layer superconductors.

In this Letter we develop a theory for a 3D defect-unbinding transition in the FL tilt modulus. Furthermore, for the other scenario \([4]\) is based on a disorder-induced deformation of the lattice resulting in random unbinding transition. Indeed, as we argue, possibility is that a similar, but now disorder-induced, vacancy-interstitial unbinding transition can be demonstrated in 3D layered superconductors, relevant to many layered and multilayer materials.\([8]\)

Topological phase transitions induced by quenched disorder are relevant for numerous physical systems. Such transitions are likely to shape the phase diagram of type II superconductors. It was proposed \([3]\) that the flux lattice (FL) remains a topologically ordered Bragg glass at low field, unstable to the proliferation of dislocations above a threshold disorder or field, providing one scenario for the controversial "second peak" line \([3]\). Another scenario \([4]\) is based on a disorder-induced decoupling transition (DT) responsible for a sharp drop in the FL tilt modulus. Furthermore, for the pure system, it was shown recently \([3]\) that in the absence of Josephson coupling, point "pancake" vortices, i.e vacancies and interstitials in the FL, are nucleated at a temperature \( T_{\text{def}} \), distinct from melting above some field. It is believed that this pure system topological transition merges with the thermal DT \([3]\) once the Josephson coupling is finite, being two anisotropic limits of the same transition \([3]\) (at which superconducting order is destroyed while FL positional correlations are maintained). Thus an interesting possibility is that a similar, but now disorder-induced, vacancy-interstitial unbinding transition can be demonstrated in 3D layered superconductors, relevant to many layered and multilayer materials.\([8]\)

In 2D recent progress was made to describe disorder induced topological transitions, in terms of Coulomb gases of charges with logarithmic long range interactions. It was shown \([14,15]\) that quenched random dipoles lead to a transition, via defect proliferation, at a finite threshold disorder, even at \( T = 0 \).

In this Letter we develop a theory for a 3D defect-unbinding transition in presence of disorder. It is achieved for systems which can be mapped onto a layered Coulomb gas with quenched random dipoles, in which the interaction energy between two charges on layers \( n \) and \( n' \) is \( 2J_{n-n'} \ln r \) with \( r \) the charge separation parallel to the layers. One physical realization is the FL in layered superconductors \([2,4]\) with only magnetic coupling, for which we predict and locate the vacancy-interstitial unbinding transition. Indeed, as we argue, disorder induced deformations of the lattice result in random dipoles as seen by the defects. To study this problem we develop an efficient variational method which allows for fugacity distributions, known \([13]\) to be important in 2D as they become broad at low \( T \). We test the method on a single layer and reproduce the phase diagram, known from renormalization group (RG) with a \( T = 0 \) disorder threshold \( \sigma_{\text{cr}} = 1/8 \) \([13]\). For the 2-layer system we find that above a critical anisotropy \( \eta \equiv -J_1/J_0 = \eta_c = 1 - \frac{1}{T_\text{def}} \) the single layer type transition is preempted by a transition induced by bound states of two pancake vortices on the two layers with \( \sigma_{\text{cr}} < 1/8 \). We develop a \( T = 0 \) energy rationale by an approximate mapping to a Cayley tree problem and find that it reproduces the 2-layer result. Extension to many layers with only nearest layer coupling shows a cascade of transitions in which the number of correlated charges on \( N \) neighboring layers increases, while the critical disorder decreases with \( \eta \), with \( N \to \infty \), \( \sigma_{\text{cr}} \to 0 \) as \( \eta \to 1/2 \). Finally we consider arbitrary range \( n_0 \) for \( J_n \) with the constraint \( \sum_n J_n = 0 \), as appropriate for layered superconductors. For \( N > n_0^2 \) states with \( \sigma_{\text{cr}} \sim n_0^2/N \to 0 \) are possible but only at exponentially large length scales for \( n_0 \gg 1 \). Thus for layered superconductors we expect that the \( N=1 \) state dominates and find its phase diagram. Varying the system parameters by forming multilayers reduces \( n_0 \) and allows for realization of the new \( N > 1 \) phases.

We study the Hamiltonian:

\[
\mathcal{H} = -\frac{1}{2} \sum_{\mathbf{r} \neq \mathbf{r}'} \sum_{n,n'} 2J_{n-n'} s_n(\mathbf{r}) \ln(|\mathbf{r} - \mathbf{r}'|) s_{n'}(\mathbf{r}') \\
- \sum_{\mathbf{r},n} V_n(\mathbf{r}) s_n(\mathbf{r})
\]  

where \( s_n(\mathbf{r}) = \pm 1,0 \) define the positions \( \mathbf{r} \) of charges on the \( n \)-th layer, \( V_n(\mathbf{r}) \) is a disorder potential with long range correlations \( V_n(\mathbf{q})V_{n'}(-\mathbf{q}) = 4\pi \sigma J_0^2 \Delta_{n-n'}/|\mathbf{q}|^2 \) with \( \Delta_0 = 1 \) (the short distance cutoff being set to unity). For simplicity we start with uncorrelated disorder from layer to layer \( \Delta_{n-n'} = \delta_{n,n'} \) with

\[
|V_n(\mathbf{r}) - V_{n'}(\mathbf{r}')|^2 = 4\pi \sigma J_0^2 \ln |\mathbf{r} - \mathbf{r}'|
\]

representing quenched dipoles on each layer. At \( T = 0 \) the problem amounts to find minimal energy configurations of charges in a logarithmically correlated random
potential. For a single layer it was studied either using a “random energy model” (REM) approximation or, more accurately using a representation in terms of directed polymers on a Cayley tree (DPCT) shown to emerge (as a continuum branching process) from the Coulomb gas RG of the single layer problem. Schematically, the tree has independent random potentials (Fig. 1) \( v_i \) on each bond with variance \( v_i^2 = 2\sigma J_i^2 \). After \( l \) generations one has \( \sim 2\eta^l \) sites which are mapped onto a 2D layer, i.e. two points separated by \( r \sim \eta^l \) have a common ancestor at the previous \( l \) in \( r \) generation. Each point \( r \) has a unique path on the tree (DP) with \( v_i, \ldots, v_l \) potentials and is assigned a potential \( V(r) = v_1 + \ldots + v_l \). Since all bonds previous to the common ancestor are identiﬁcal \( [V(r) - V(r')]^2 = 2 \sum_i v_i^2 \) reproducing (3) on each layer. Exact solution of the DPCT (12) yields the energy gained from disorder \( V_{\text{min}} = \min_r V(r) \approx -\sqrt{8\sigma J_0} \ln L \) for a volume \( L^2 \), with only \( O(1) \) ﬂuctuations (13), i.e. \(-\sqrt{8\sigma J_0} \) per generation \( l = \ln L \).

![FIG. 1. Critical disorder values with only nearest neighbor coupling \( \eta \) vs. the anisotropy \( \eta = -J_1/J_0 \). Transitions between different \( N \) phases are marked with arrows. Inset: the Cayley tree representation (for \( N = 3 \) neighboring layers) with + (at the tree endpoints) separated by \( L' \) along the layers, and separated by \( L \) from the \( N = 3 \) - charges.](image)

Optimal energy conﬁgurations for \( M \) coupled layers are constructed considering \( N \) neighboring layers with a +,− pair on each layer and no charges on the other layers. We can take \( J_0 > 0 \) and \( J_{n>0} \leq 0 \) so that equal charges on different layers attract. The DPCT representation now involves, on a single tree, \( N \) + polymers (each seeing different disorder) and \( N \) − polymers (each seeing opposite disorder −\( v_i \) to their + partner). A plausible conﬁguration is that the + charges bind within a scale \( L' \) (0 ≤ \( \epsilon \) ≤ 1), so do the − charges, while the + to − charge separations deﬁne the scale \( L \). Its tree representation (Fig. 1) has \( 2N \) branches with \( \ln L \) generations, i.e. an optimal energy of \(-2N\sqrt{8\sigma}J_0 \ln L \). On the scale between \( L' \) and \( L \) the + charges act as a single charge with a potential \( \sum_{i=1}^{N} V_i(r) \) (the \( N \) polymers share the same branch) of variance \( \text{Var}N \) hence the optimal energy is \(-2\sqrt{8\sigma}N\sqrt{8\sigma}J_0(1 - \epsilon) \ln L \). The total disorder energy is (3):

\[ E_{\text{dis}} \approx -2J_0\sqrt{8\sigma}[\epsilon N + (1 - \epsilon)\sqrt{N}] \ln L. \]  

(4)

The competing interaction energy \( E_{\text{int}} \) is the sum of the one for the ++ pairs, \([2J_0 N + 4 \sum_{n=1}^{N} J_n (N - n)] \ln L \) and for the ++ / −− pairs, \(-4 \sum_{n=1}^{N} J_n (N - n) \ln L \). The total energy \( E_{\text{tot}} = E_{\text{dis}} + E_{\text{int}} \) being linear in \( \epsilon \), its minimum is at either \( \epsilon = 1 \) or \( \epsilon = 0 \). Since \( \epsilon = 1 \) implies that the + charges unbind, it is sufﬁcient to consider \( \epsilon = 0 \) with all \( N \geq 1 \), i.e. a rod with \( N \) correlated charges has energy (with \( \eta = -J_n/J_0 \)):

\[ E_{\text{tot}} = 2J_0 N[1 - \sum_{n=1}^{N} \eta_n (1 - \frac{n}{N}) - \sqrt{8\sigma}J_0 \ln L. \]  

(5)

Disorder induces the \( N \) vortex state at the critical value:

\[ \sigma_{\text{cr}} = \frac{N}{8}[1 - \sum_{n=1}^{N} \eta_n (1 - \frac{n}{N})^2]. \]  

(6)

(i.e. \( E_{\text{tot}} = 0 \)). Consider ﬁrst only nearest neighbor coupling \( \eta_i = \eta_0 \delta_i \). Then \( \sigma_{\text{cr}} \) is minimal at \( N = 1 \) with \( \sigma_{\text{cr}} = 1/8 \) if \( \eta_1 < 1 - 1/\sqrt{2} \). For larger anisotropies successive \( N \) states form at \( 1/(1 - 2\eta_1) = 1 + \sqrt{N}N/(N - 1) \sim N \) with diverging \( N \) as \( \eta_1 \to \frac{1}{2} \) (Fig 1) (21).

Consider now \( J_0 \) of range \( n_0 \) constrained by \( \sum_{n=1}^{N} J_n = 0 \) as for the superconductor, e.g. \( \eta_\infty = \eta_0 e^{-n(n-1)/n_0} \) for which \( \sigma_{\text{cr}} \approx (1 - e^{-N/n_0})/\sqrt{8\sigma}N(1 - e^{-1/n_0}) \). For \( n_0 \gg 1 \), each \( \eta_{n=0} \) is small: for \( N \lesssim n_0 \) the lowest \( \sigma_{\text{cr}} \) is at \( N = 1 \). However, the combined strength of \( N \approx n_0 \) vortices being signiﬁcant \( \sigma_{\text{cr}} \) has a maximum and decreases back to zero for \( N > n_0 \) as \( \sigma_{\text{cr}} \approx n_0^2/8\sigma \). Hence \( \sigma_{\text{cr}} \to 0 \) as \( N \to \infty \) and any small disorder seems to nucleate such vortices. This is because the perfect screening of the zero mode \( \sum_n J_n = 0 \) implies that an inﬁnite charge rod has a vanishing \( \ln r \) interaction; hence a logarithmically correlated disorder is always dominant.

The realization of the large \( N \) rods depends, however, on the type of thermodynamic limit. Adding to (6) the core energy \( E_c, N \) and minimizing yields a \( N \)-vortex scale

\[ L \approx \exp\{E_c N/\sqrt{[2J_0(\sqrt{8\sigma} - \sqrt{8\sigma_{\text{cr}}})]} \}. \]  

(7)

Hence as \( \sigma \to 0 \) such states are only achievable when \( L/N \) diverges exponentially. Using \( \sigma_{\text{cr}} \approx n_0^2/8\sigma \), for \( N > n_0^2/8\sigma \) the lowest scale \( L \) in this range is achieved at \( N = n_0^2/2\sigma \) and leads to a lower bound \( L_{\text{min}} \approx \exp[E_c n_0/4J_0 \sigma] \) for observing large \( N \) states with a given \( \sigma < \frac{1}{2} \). For layered superconductors \( E_c/J_0 \gg 1 \) (22) and \( n_0 \gg 1 \) and this large \( N \) instability occurs at unattainable scales, thus \( N = 1 \) dominates. One needs \( n_0 \approx 2 - 3 \), as in multilayers, to realize the \( N > 1 \) states.

To substantiate these results we develop a variational method for \( M \) layers which allows for fugacity distributions, an essential feature in the one-layer problem. Disorder averaging (2) in Fourier using replicas yields:

\[ \beta H_r = \frac{1}{2\pi \sigma} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} s_a(q,k)(G_0)_{ab}(q,k)s_b^*(q,k) + \beta E_c \sum_{r,n} s_{na}^2(r) \]  

(8)
where \( (G_0)_{ab}(q,k) = (4\pi/q^2)(g(k)\delta_{ab} - \sigma J_0^2\beta^2\Delta(k)) \),
g(k) = \beta J(k) = \beta d\sum_a J_a \exp(ikdn), \) d the interlayer spacing \(d\) (for uncorrelated layers \(\Delta(k) = d\) ), \(a,b = 1,...,m\) are replica indices and \(m \to 0\) is to be carefully taken. In transforming to a sine-Gordon Hamiltonian \( \mathcal{H}_{SG} \) it is crucial to keep all charge fugacities \( \beta \), which yields:

\[
\beta \mathcal{H}_{SG} = \frac{1}{2} \int_{kq} \chi_a(q,k)(G_0)^{-1}_{ab}\chi_b(q,k) - \sum_{\mathbf{r}} \sum_{\mathbf{s} \neq 0} Y[\mathbf{s}] \exp(i\mathbf{s} \cdot \chi(\mathbf{r})) .
\]

From now on \( s = \{ s_{na} \}_{n=1,M, a=1..m} \) is an integer vector both in layer label and replica space (i.e. of length \( m \) M) of entries 0,±1 and the summation is over all such non null vectors (also \( \chi(\mathbf{r}) \equiv \{ \chi_n(\mathbf{r}) \} \), \( \mathbf{s} \cdot \chi = \sum_{n,a} s_{na} \chi_{na} \)). We now look for the best gaussian approximation of \( (9) \) with propagator \( G_{ab}(q,k) = (G_0)_{ab}^{-1}(q,k) + \sigma_a(\mathbf{k})\delta_{ab} + \sigma_0(k) \). The bare fugacity being \( Y[\mathbf{s}] = \exp(-\beta E_c \sum_n s_{na}^2) \) the naive approach would be to restrict to charges \( \mathbf{s} \) with a single non zero entry, leading to a uniform fugacity term \(-y \sum_{n,a} \cos(\chi_{na}(\mathbf{r})) \) and a diagonal \( k \)-independent replica mass term. Instead we keep all composite charges \( \mathbf{s} \), which allow for variational solutions with off diagonal and \( k \)-dependent replica mass terms. This corresponds respectively to fluctuations of fugacity and \( N > 1 \) charge rods being generated and becoming relevant as also seen from RG. The variational free energy is \( F_{var} = F_0 + \langle \mathcal{H}_{SG} - \mathcal{H}_0 \rangle_0 \) where \( \langle ... \rangle \) is an average using \( \beta \mathcal{H}_0 = \frac{1}{2} \int_{q,k} \chi_a(q,k)G_{ab}(q,k)\chi_b(q,k) \) and \( \beta F_0 = -\frac{1}{2}{\rm tr} \ln G \). The Gaussian average \( F[\mathbf{s}] = \langle Y[\mathbf{s}] \rangle \langle \exp i \mathbf{s} \cdot \chi(\mathbf{r}) \rangle \) yields:

\[
F[\mathbf{s}] = \exp\{-\frac{1}{2\beta^2} \int_{k} (\tilde{G}_c(k)\delta_{ab} - A(k))s_a(k)s_b^*(k) \}
\]

\[
G_c(k) = g(k)\ln[A/(4\pi g(k)\sigma_c(k))] \]

\[
A(k) = \sigma^2 J_0^2\Delta(k)/g(k) + g(k)\sigma_0(k)/\sigma_c(k)
\]

where \( s_a(k) = d\sum_n s_{na}\delta^{+k_{nd}} \), \( \tilde{G}_c(k) = G_c(k) + 2\beta E_c d \), \( \Lambda \) the UV cutoff on \( q^2 \), \( F_{var} \) is minimized by \( \sigma_c(\mathbf{k})\delta_{ab} + \sigma_0(k) = \Lambda d^{-2} \sum_a s_a(k)s_b^*(k) \). Writing the \( A(k) \) term as an average over \( M \) random gaussian fugacities \( w_k \):

\[
\exp\{\frac{1}{2} \sum a s_a(k)^2 A(k) \} = \langle \exp w_k \sum a s_a(k) \rangle_w \]

where \( (\ldots)_w = \prod_k \int_{-e^{-|w_k|^2/2A(k)}}^{e^{-|w_k|^2/2A(k)}} \) allows to perform the exact sum on replicas yielding \( \sum a F[\mathbf{s}] = \langle Z^m \rangle \) with \( Z = \sum_{(s_{na}=0,\pm 1)} \exp(-\frac{1}{2\beta^2} \int_{k} \tilde{G}_c(k)|s(k)|^2 ) \)

\[
\sum_{s} F[\mathbf{s}] = \langle (Z_m)^w \rangle \]

The variational equations for \( m \to 0 \) become \( \mathcal{H}_{var}^{\mathbf{m}} = \sum_{\mathbf{r},p,n,a} s_{na}(\mathbf{r})G_p(\mathbf{r},\mathbf{u}_p - \mathbf{r}, n' - n) \) where, in Fourier space \( \sum_{\mathbf{r},p,n,a} s_{na}(\mathbf{r})G_p(\mathbf{r},\mathbf{u}_p) = (\phi_0^2/4\pi\lambda_0^2)\langle 1 + \mathbf{f}(\mathbf{q}, \mathbf{k}) \rangle \)

solved for the critical line where \( \sigma_c(0) \to 0 \). The phase diagram shown in Fig. 2 (full line) reproduces precisely recent RG results. The variational scheme, allowing for all replica charges \( \mathbf{s} \), therefore treated disorder correctly. For two layers \( kd = 0, \pi \) we need two fugacity distributions \( w_0, w_\pi \) and \( Z \) is a ”nominal”, i.e. \( Z = 1 + 8 \) eight exponentials involving \( G_c(0), G_c(\pi) \). Focusing on the low \( T \) boundary, where \( \sigma_c(\pi) \approx \sigma_c(0) \) as \( \eta_k \to 1 \), \( \beta \) representing decoupled layers, or (ii) \( \beta \to \infty \) for \( \eta_k \to 0 \), representing a + + bound states on the two layers. The \( T = 0 \) energy rational is therefore reproduced. The phase diagram for two layers with \( \eta_k < \eta < 1/2 \) is shown in Fig. 2.
function $f(q,k) = (d/4L_{2D}^2 q) \sin(qd/\sin^2(qd/2) + \sin^2(kd/2))$; 
$\phi_0 = B a^2$ is the flux quantum, $a$ the FL spacing, $\lambda_{ab}$ 
the penetration length along the layers. To 0-th order in $u_n^0$ 
the defects feel a periodic potential fixing their position 
in a unit cell, hence $s(q,k)$ involve only $|q| < 1/a$.

In the limit $q \rightarrow 0$ the longitudinal modes, to 
which defects couple, have for (tilt) elastic energy

$$H_{el} = \frac{\phi_0^2}{2} \sum_{q \neq 0} G_v(q,k)^2 D(k)$$

with $D(k) = \frac{1}{2} \sum_{\neq 0} G_v(q,k) G_v(q,k) = \lim_{n \rightarrow 0} G_v(q,k)^2 = \frac{\phi_0^2}{4} k^2 / (4\pi (1 + \lambda_{ab}^2 k^2))$ and $k_z = (2/d) \sin(kd/2)$. The sum on $q$ is due to the high 
momentum components of the magnetic field and is responsible 
for a single contributions of the defect interaction 
and to a finite $T_{def}$. Minimizing $H_{vac} + H_{el}$ yields 
$u_v(q,k) = i q s(q,k) G_v(q,k)a^2 / D(k)q^2$ and 

$$g(k) = \beta G_v(q,k)(1 - G_v(q,k)/D(k)) / 4\pi$$

The long range interaction is $\sim \ln r$ and its 
coefficient determines $T_{def} = 2 J_0$ (via $g(k)$ is 2). Since 
$g(k)$ is the correlation in the un-
perturbed Bragg glass, 

$$\sigma^2 = \frac{\phi_0^2}{4} a^2 / 4\pi$$

for $k < a$. We thus propose that FL 
exists only if $a > \lambda_{ab}$ (similarly thermal decoupling occurs at 
$T_{dec} = 2\lambda_{ab}$ in 2D).

Remarkably 

$$D(k) = \beta G_v(q,k)(1 - G_v(q,k)/D(k))$$

for $k > a$. The present methods may be useful for other 2D 
disordered systems, such as quantum Hall.

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With $s_n(q,k) = d \sum_{j \neq n} s_{n,j}(r) e^{i qr + i k d}$ and $\beta = 1/T$. 

In layered superconductors $a/d \approx 10 - 100$ and the 
The $N = 1$ transition at $\sigma_{cr} = 1/8$ dominates for realistic 

sizes. The disorder-induced decoupling transition, 

neglecting defects, predicted if $\sigma_{dec} = 2$ is thus above 
the defect transition (with $B \sim \sigma$) in the $B - \sigma$ plane 
(similarly thermal decoupling occurs at $T_{dec} = 2\lambda_{ab}$ for 
$d \ll a \ll \lambda$). A natural scenario is again of a single transition 
at $\sigma$ varying from 2 to 1/8 as the bare Josephson 

coupling is reduced, e.g. by increasing $d$ in multilayers.

In conclusion, we developed a variational method 
and a Cayley tree rationale to study layered Coulomb gas. 
The results are relevant to flux lattices where we find 
the phase boundaries and propose new $N > 1$ phases for 
$d \gg a$. The present methods may be useful for other 2D 
disordered systems, such as quantum Hall.