A SEMIDEFINITE PROGRAM FOR LEAST DISTORTION EMBEDDINGS OF FLAT TORI INTO HILBERT SPACES

ARNE HEIMENDAHL, MORITZ LÜCKE, FRANK VALLENTIN, AND MARC CHRISTIAN ZIMMERMANN

Abstract. We derive and analyze an infinite-dimensional semidefinite program which computes least distortion embeddings of flat tori $\mathbb{R}^n/L$, where $L$ is an $n$-dimensional lattice, into Hilbert spaces.

This enables us to provide a constant factor improvement over the previously best lower bound on the minimal distortion of an embedding of an $n$-dimensional flat torus.

As further applications we prove that every $n$-dimensional flat torus has a finite dimensional least distortion embedding, that the standard embedding of the standard tours is optimal, and we determine least distortion embeddings of all 2-dimensional flat tori.

1. Introduction

Lattices (discrete subgroups of $n$-dimensional Euclidean spaces) are central to the geometry of integer programming. One good way to describe geometric properties of a given lattice $L$ are fundamental domains of $\mathbb{R}^n$ with respect to translations by $L$. The quotient $\mathbb{R}^n/L$ is itself a metric space; a flat torus.

Approximating metric spaces by Euclidean spaces to design efficient approximation algorithms has been a central theme in theoretical computer science in the last two decades. There the starting point is computing a least distortion embedding of a “difficult” metric spaces into an “easy” normed space.

Least distortion embeddings of flat tori into Hilbert spaces were first studied by Khot and Naor [12] in 2006. One motivation is that studying the Euclidean distortion of flat tori might have applications to the complexity of lattice problems, like the closest vector problem, and might also lead to more efficient algorithms for lattice problems through the use of least distortion embeddings. Another motivation comes from comparing the Riemannian setting to the bi-Lipschitz setting we are discussing here. On the one hand, by the Nash embedding theorem, flat tori can be embedded isometrically as Riemannian submanifolds into Euclidean space; we refer to [4] for spectacular visualizations of such an isometric embedding in the case of the two-dimensional square flat torus. On the other hand, Khot and Naor showed that flat tori can be highly non-Euclidean in the bi-Lipschitz setting.

1.1. Notation and review of the relevant literature. We review the relevant results of the literature which appeared since the pioneering work of Khot and Naor. At the same time we set the notation for this paper.
A flat torus is the metric space given by the quotient \( \mathbb{R}^n / L \) with some \( n \)-dimensional lattice \( L \subseteq \mathbb{R}^n \) and with metric
\[
d_{\mathbb{R}^n/L}(x, y) = \min_{v \in L} |x - y - v|.
\]

An \( n \)-dimensional lattice is a discrete subgroup of \((\mathbb{R}^n, +)\) consisting of all integral linear combinations of a basis of \( \mathbb{R}^n \). Furthermore, \( | \cdot | \) denotes the standard norm of \( \mathbb{R}^n \) given by \( |x| = \sqrt{x^* x} \).

A Euclidean embedding of \( \mathbb{R}^n / L \) is an injective function \( \varphi: \mathbb{R}^n / L \to H \) mapping the flat torus \( \mathbb{R}^n / L \) into some (complex) Hilbert space \( H \). The distortion of \( \varphi \) is
\[
\text{dist}(\varphi) = \sup_{x, y \in \mathbb{R}^n / L, \ x \neq y} \frac{\|\varphi(x) - \varphi(y)\|}{d_{\mathbb{R}^n/L}(x, y)} \sup_{x, y \in \mathbb{R}^n / L} \frac{d_{\mathbb{R}^n/L}(x, y)}{\|\varphi(x) - \varphi(y)\|}
\]
where \( \| \cdot \| \) is the norm of the Hilbert space \( H \). Here the first supremum is called the expansion of \( \varphi \) and the second supremum is the contraction of \( \varphi \). When we minimize the distortion of \( \varphi \) over all possible embeddings of \( \mathbb{R}^n / L \) into Hilbert spaces we speak of the least (Euclidean) distortion of the flat torus; it is denoted by
\[
c_2(\mathbb{R}^n / L) = \inf \{ \text{dist}(\varphi) : \varphi: \mathbb{R}^n / L \to H \text{ for some Hilbert space } H, \varphi \text{ injective} \}.
\]

Similarly one can define \( c_1(\mathbb{R}^n / L) \) by replacing the Hilbert space by some \( L_1 \) space.

Khot and Naor showed (see [12, Corollary 4]) that flat tori can be highly non-Euclidean in the sense that there is a family of flat tori \( \mathbb{R}^n / L_n \) with
\[
c_2(\mathbb{R}^n / L_n) = \Omega(\sqrt{n}).
\]
On the other hand, they noticed (see [12, Remark 5]) that the standard embedding of the standard flat torus \( \mathbb{R}^n / \mathbb{Z}^n \) embeds into \( \mathbb{R}^{2n} \) with distortion \( O(1) \).

The standard embedding is given by
\[
\varphi(x_1, \ldots, x_n) = (\cos 2\pi x_1, \sin 2\pi x_1, \ldots, \cos 2\pi x_n, \sin 2\pi x_n).
\]

In fact, Khot and Naor are mainly concerned with bounding \( c_1(\mathbb{R}^n / L) \), which immediately provides bounds for \( c_2(\mathbb{R}^n / L) \) because \( c_1(\mathbb{R}^n / L) \leq c_2(\mathbb{R}^n / L) \). To state their main result, leading to (1), we make use of the Voronoi cell of \( L \), which is an \( n \)-dimensional polytope defined as
\[
V(L) = \{ x \in \mathbb{R}^n : |x| \leq |x - v| \text{ for all } v \in L \}.
\]
The Voronoi cell is a fundamental domain of \( \mathbb{R}^n / L \) under the action of \( L \). We denote the volume of \( V(L) \) by \( \text{vol} L \). Clearly, \( |x| = d_{\mathbb{R}^n/L}(x, 0) \) for all \( x \in V(L) \). The covering radius of \( L \) is \( \mu(L) = \max\{|x| : x \in V(L)\} \), which is the circumradius of \( V(L) \). The length of a shortest vector of \( L \) is \( \lambda(L) = \min\{|v| : v \in L \setminus \{0\}\} \) that is two times the inradius of \( V(L) \). Now the main result (see [12, Theorem 5]) is
\[
c_1(\mathbb{R}^n / L) = \Omega \left( \frac{\lambda(L^*) \sqrt{n}}{\mu(L^*)} \right).
\]
Here, as usual, \( L^* = \{ u \in \mathbb{R}^n : u^* v \in \mathbb{Z} \text{ for all } v \in L \} \) denotes the dual lattice of \( L \). They also give an alternative proof of their main result for \( c_2(\mathbb{R}^n / L) \) (see [12, Lemma 11]). The main result leads to the lower bound (1) when plugging in duals of lattices which simultaneously provide dense packings and economical coverings. Such a family of lattices exist by a theorem of Butler [5].

\[\text{In fact, we have } \text{dist}(\varphi) = \pi/2 \text{ and } \varphi \text{ is an optimal embedding, see Theorem 6.1.}\]
Using Korkine-Zolotarev reduction Khot and Naor determine an embedding of $\mathbb{R}^n / L$ into $\mathbb{R}^{2n}$ with distortion $O(n^{3n/2})$ (see [12, Theorem 6]).

Haviv and Regev [11, Theorem 1.3] found an improved embedding that yields $c_2(\mathbb{R}^n / L) = O(n\sqrt{\log n})$. They also improved on (3) and showed in [11, Theorem 1.5] that for any $n$-dimensional lattice $L$ we have
\begin{equation}
    c_2(\mathbb{R}^n / L) \geq \frac{\lambda(L^*)\mu(L)}{4\sqrt{n}},
\end{equation}
which improves on (3) because $\mu(L)\mu(L^*) \geq \Omega(n)$ holds for every $n$-dimensional lattice. This follows from a simple volume argument giving $\mu(L) = \Omega(\sqrt{n}(\text{vol } L)^{1/n})$ and $\text{vol } L = (\text{vol } L)^{-1}$.

Recently, Agarwal, Regev, Tang [2] constructed excellent embeddings of flat tori having low distortion and showed that the lower bound (1) is nearly tight: For every lattice $L \subseteq \mathbb{R}^n$ there exists an embedding of $\mathbb{R}^n / L$ into Hilbert space with distortion $O(\sqrt{n \log n})$.

1.2. Aim and method. In this paper we want to add a semidefinite optimization perspective to this story.

For finite metric spaces it is known that one can compute least distortion Euclidean embeddings via a semidefinite program (SDP), which is linear optimization over the cone of positive semidefinite matrices. We want to extend this result from finite metric spaces to flat tori. This will yield, via semidefinite programming duality, an algorithmic method for proving nonembeddability results. In particular, this leads to a new, simple proof of (4). In fact we even get a constant factor improvement that is tight in the case of the standard torus.

First we recall the semidefinite program for finding least Euclidean distortion embeddings of finite metric spaces. Suppose we consider a finite metric space $(X, d)$ with distance function $d$. Then, as first observed by Linial, London, Rabinovich [16], we can find a least distortion embedding of $(X, d)$ into a Hilbert space algorithmically by solving the following semidefinite program
\begin{equation}
    \min \{ C : C \in \mathbb{R}_+, Q \in S^X_+, \quad d(x, y)^2 \leq Q_{xx} - 2Q_{xy} + Q_{yy} \leq Cd(x, y)^2 \quad \text{for all } x, y \in X \},
\end{equation}
where $S^X_+$ denotes the convex cone of positive semidefinite matrices whose rows and columns are indexed by the elements of $X$. The optimal solution $C$ of this semidefinite program equals $c_2(X, d)^2$ and if $Q$ attains the optimal solution, then we can determine a least distortion embedding $\varphi : X \to \mathbb{R}^X$ with the property $\varphi(x) \cdot \varphi(y) = Q_{xy}$ by considering a Cholesky decomposition of $Q$.

This shows how to compute (in fact in polynomial time) an optimal Euclidean embedding of a finite metric space. Another benefit of this formulation is that we can apply duality theory of semidefinite programs. Then the dual maximization problem will play a key role to determine lower bounds for $c_2(X, d)$. By using strong duality we arrive at the following result: The least distortion of a finite metric space $(X, d)$, with $X = \{x_1, \ldots, x_n\}$, into Euclidean space is given by
\begin{equation}
    c_2(X, d)^2 = \max \left\{ \sum_{i,j=1}^n Y_{ij}d(x_i, x_j)^2 : Y \in S^X_+, Ye = 0 \right\}.
\end{equation}
The condition $Ye = 0$ says that the all-ones vector $e$ lies in the kernel of $Y$. A proof of this result is detailed in Matoušek [19] or in Laurent, Vallentin [15].
This lower bound has been extensively used to determine the least distortion Euclidean embeddings of the shortest path metric of several graph classes. Linial, Magen [17] computed least distortion embeddings of products of cycles and of expander graphs. Least distortion Euclidean embeddings of strongly regular graphs and of more general distance regular graphs were first considered by Vallentin [22]. This was further extended by Kobayashi, Kondo [13], Cioabă, Gupta, Ihringer, Kurihara [6]. Linial, Magen, Naor [18] considered graphs of high girth using this approach.

To apply the bound (6) one has to construct a matrix $Y$, which sometimes appears to come out of the blue. By complementary slackness, which is the same as analyzing the case of equality in the proof of weak duality, we get hints where to search for an appropriate matrix $Y$: If $Y$ is an optimal solution of the maximization problem (6), then $Y_{ij} > 0$ only for the most contracted pairs. These are pairs $(x_i, x_j)$ for which $\frac{d(x_i, x_j)}{\|f(x_i) - f(x_j)\|}$ is maximized. Similarly, then $Y_{ij} < 0$ only for the most expanded pairs, maximizing $\frac{\|f(x_i) - f(x_j)\|}{d(x_i, x_j)}$.

Linial, Magen [17] realized that for graphs most expanded pairs are simply adjacent vertices. However, most contracted pairs are more mysterious and there is no characterization known. The first intuition that the largest contraction occurs at pairs at maximum distance is wrong in general.

1.3. Contribution and structure of the paper. In Section 2 of this paper we derive a new infinite-dimensional semidefinite program for determining a least distortion embedding of flat tori into Hilbert spaces which is analogous to (5). It is given in Theorem 2.1 where we additionally apply symmetry reduction techniques in the spirit of [3] to reduce the original infinite-dimensional SDP into an infinite-dimensional linear program that involves Fourier analysis. Then we realize that in a Euclidean embedding of a flat torus there are no most expanded pairs: The expansion is only attained in the limit by pairs whose distance tends to zero. This is in perfect analogy to the graph case where the most expanded pairs are also attained at minimal distance. This insight has the advantage that in the infinite dimensional linear program some of the infinitely many constraints can be replaced by only one finite-dimensional semidefinite constraint. This is the content of Theorem 2.4. Its dual program is derived in Theorem 2.5 which is analogous to (6).

In Section 3 we further investigate the properties of the optimization problems given in Theorem 2.4 and Theorem 2.5. These properties will be used in the next sections.

In the last sections we apply our new methodology. In Section 4 we prove that an $n$-dimensional flat torus always admits a finite dimensional least distortion embedding, a space of (complex) dimension $2^n - 1$ suffices. Section 5 contains a new and simple proof of our constant factor improvement of the lower bound given in (4). In Section 6 we show that the standard embedding (2) of the standard torus is indeed optimal and has distortion $\pi/2$. We give an optimal embedding of the lattices $D^*_n$ in Section 7. In Section 8 we determine least distortion embeddings of all two-dimensional flat tori. Open questions are discussed in Section 9.

2. An infinite-dimensional SDP

Starting from (5) we want to derive a similar, but now infinite-dimensional, semidefinite program which can be used to determine $c_2(\mathbb{R}^n/L)$. 
2.1. Primal program. The first step is to apply a classical theorem of Moore [20] which enables us to optimize over all embeddings \( \varphi : \mathbb{R}^n/L \rightarrow H \) into some Hilbert space \( H \). In our situation Moore’s theorem says that there exists a (complex) Hilbert space \( H \) and a map \( \varphi : \mathbb{R}^n/L \rightarrow H \) if and only if there is a positive definite kernel

\[
Q : \mathbb{R}^n/L \times \mathbb{R}^n/L \rightarrow \mathbb{C}
\]

such that \( Q(x, y) = (\varphi(x), \varphi(y)) \) for all \( x, y \in \mathbb{R}^n/L \), where \((\cdot, \cdot)\) denotes the inner product of \( H \). Therefore we get

\[
c_2(\mathbb{R}^n/L)^2 = \inf\{ C : C \in \mathbb{R}_+, Q \text{ positive definite},
\]

\[
d_{\mathbb{R}^n/L}(x, y)^2 \leq Q(x, x) - 2\Re(Q(x, y)) + Q(y, y) \leq Cd_{\mathbb{R}^n/L}(x, y)^2 \text{ for all } x, y \in \mathbb{R}^n/L.\]

Here we scaled the embedding \( \varphi \) which is defined through \( Q \) so that the contraction of \( \varphi \) equals 1. The real part \( \Re(Q) \) of a positive definite kernel is positive definite again and we can restrict to real-valued positive definite kernels for determining \( c_2(\mathbb{R}^n/L) \).

For the second step we apply a standard group averaging argument. If \( Q \) is a feasible solution for the minimization problem above, so is its group average

\[
\overline{Q}(x, y) = \frac{1}{\text{vol}(\mathbb{R}^n/L)} \int_{\mathbb{R}^n/L} Q(x - z, y - z) \, dz.
\]

By this averaging the kernel \( \overline{Q} \) becomes continuous and only depends on the difference \( x - y \). Thus, instead of minimizing over positive definite kernels \( Q \) it suffices to minimize over continuous, real functions \( f : \mathbb{R}^n/L \rightarrow \mathbb{R} \) which are of positive type, i.e. the kernel \((x, y) \mapsto f(x - y)\) is positive definite; see also the proof of Theorem 3.1 in the paper [1] by Aharoni, Maurey, Mityagin.

For the convenience of the reader we provide the argument why the positive type function \( f(x) = \overline{Q}(x, 0) \) is continuous: For every \( x, y \) the matrix

\[
\begin{pmatrix}
   f(0) & f(x) & f(x + y) \\
   f(x) & f(0) & f(y) \\
   f(x + y) & f(y) & f(0)
\end{pmatrix}
\]

is positive semidefinite and it is congruent (simultaneously subtract the second row/column of the third row/column) to the positive semidefinite matrix

\[
\begin{pmatrix}
   f(0) & f(x) & f(x + y) - f(x) \\
   f(x) & f(0) & f(y) - f(0) \\
   f(x + y) - f(x) & f(y) - f(0) & 2f(0) - 2f(y)
\end{pmatrix}
\]

Taking the minor of the first and third row/column gives

\[
2f(0)(f(0) - f(y)) \geq (f(x + y) - f(x))^2.
\]

This inequality implies that \( f \) is continuous at every \( x \) if and only if \( f \) is continuous at 0. Then \( f \) is continuous at 0 because it satisfies the constraint

\[
d_{\mathbb{R}^n/L}(0, y)^2 \leq \overline{Q}(0, 0) - 2\overline{Q}(0, y) + \overline{Q}(y, y) = 2(f(0) - f(y)) \leq Cd_{\mathbb{R}^n/L}(0, y)^2
\]

for every \( y \).

---

2 A kernel \( Q \) is called positive definite if and only if for all \( N \in \mathbb{N} \) and for all \( x_1, \ldots, x_N \in \mathbb{R}^n/L \) the matrix \((Q(x_i, x_j))_{1 \leq i, j \leq N} \in \mathbb{C}^{N \times N}\) is Hermitian and positive semidefinite. This naming convention is unfortunate but for historical reasons unavoidable.
Note that also \((x, y) \mapsto d_{\mathbb{R}^n/L}(x, y)^2\) only depends on the difference \(x - y\). So we can replace \((x, y)\) by \((x - y, 0)\) and we can move \(x - y\) by a lattice vector translation into the Voronoi cell \(V(L)\). Hence,

\[
c_2(\mathbb{R}^n/L)^2 = \inf \left\{ C : C \in \mathbb{R}_+, f : \mathbb{R}^n/L \to \mathbb{R} \text{ continuous and of positive type}, \right. \\
\left. |x|^2 \leq 2(f(0) - f(x)) \leq C|x|^2 \text{ for all } x \in V(L) \right\}.
\]

In the third step we parametrize continuous positive type functions by the Fourier coefficients using Bochner’s theorem, cf. Folland [10, (4.18)], which says that a continuous function \(f : \mathbb{R}^n/L \to \mathbb{C}\) is of positive type if and only if all its Fourier coefficients

\[
\hat{f}(u) = \int_{\mathbb{R}^n/L} f(x)e^{-2\pi i u^\top x} \, dx,
\]

with \(u \in L^*\) are nonnegative and \(\hat{f}\) lies in

\[
\ell^1(L^*) = \left\{ z : L^* \to \mathbb{C} : \sum_{u \in L^*} |z(u)| < \infty \right\}.
\]

Then if \(f\) is real, continuous and of positive type we have the representation

\[
f(x) = \sum_{u \in L^*} \hat{f}(u)e^{2\pi i u^\top x},
\]

where the convergence is absolute and uniform, with \(\hat{f} \in \ell^1(L^*)\), \(\hat{f}(u) \geq 0\) and \(\hat{f}(u) = \hat{f}(-u)\) for all \(u \in L^*\). Thus,

\[
f(x) = \sum_{u \in L^*} \hat{f}(u) \cos(2\pi u^\top x).
\]

Writing \(f\) in this form, one can express \(c_2(\mathbb{R}^n/L)^2\) as an infinite-dimensional linear program:

**Theorem 2.1.** The least distortion Euclidean embedding of a flat torus \(\mathbb{R}^n/L\) is given by

\[
c_2(\mathbb{R}^n/L)^2 = \inf \left\{ C : C \in \mathbb{R}_+, z \in \ell^1(L^*), z(u) = z(-u) \geq 0 \text{ for all } u \in L^*, \right. \\
\left. |x|^2 \leq 2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^\top x)) \leq C|x|^2 \right. \\
\left. \text{ for all } x \in V(L) \right\}.
\]

A feasible solution of the above minimization problem \((C, z)\) determines a Euclidean embedding \(\varphi\) of \(\mathbb{R}^n/L\) with distortion \(\text{dist}(\varphi) \leq \sqrt{C}\) by

\[
\varphi : \mathbb{R}^n/L \to \ell^2(L^*), \quad x \mapsto \left(\sqrt{z(u)}e^{2\pi i u^\top x}\right)_{u \in L^*},
\]

with complex Hilbert space

\[
\ell^2(L^*) = \left\{ z : L^* \to \mathbb{C} : \left( \sum_{u \in L^*} |z(u)|^2 \right)^{1/2} < \infty \right\}.
\]

**Remark 2.2.** The \(\inf\) in (7) is in fact a min because the set of bounded continuous functions of positive type is weak* compact due to the Banach-Alaoglu theorem; see for example Folland [10, Chapter 3.3].
It is worth to mention that the embedding \( \varphi \) of Theorem 2.1 embeds the flat torus \( \mathbb{R}^n/L \) into a direct product of circles
\[
\prod_{u \in L^*} \sqrt{z(u)} S^1 \quad \text{with} \quad \|\varphi(x)\|^2 = \sum_{u \in L^*} z(u) \text{ for all } x \in L.
\]
The support of \( z \) contains a basis of \( L^* \) since the embedding is injective. Using the fact \( z(u) = z(-u) \) we could also use the real embedding \( \varphi' \) with
\[
[\varphi'(x)]_u = \sqrt{z(u)}(\cos 2\pi u^\top x, \sin 2\pi u^\top x),
\]
where \( u \) runs through \( L^*/\{\pm 1\} \) and which has the same distortion as \( \varphi \).

On the other hand, the constraint \( z(u) = z(-u) \) is clearly redundant in the minimization problem of Theorem 2.1.

Now we want to simplify the infinitely many inequalities
\[
2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^\top x)) \leq C|x|^2 \text{ for all } x \in V(L),
\]
which occur in (7), by only one finite-dimensional semidefinite condition. For this we observe that in any embedding there are no most expanded pairs: the corresponding supremum \( \sup_{\|\varphi(x)-\varphi(y)\|} \{ \|\varphi(x)-\varphi(y)\| : x, y \in \mathbb{R}^n/L, x \neq y \} \) is only attained by a limit of pairs whose distance tends to 0.

**Lemma 2.3.** Let \( L \subseteq \mathbb{R}^n \) be an \( n \)-dimensional lattice. Let \( (C, z) \) be as in (7). Inequality (9) is satisfied if and only if
\[
4\pi^2 \sum_{u \in L^*} z(u)(u^\top x)^2 \leq C|x|^2 \text{ for all } x \in \mathbb{R}^n.
\]
Note that (9) holds for all \( x \in \mathbb{R}^n \).

**Proof.** By the cosine double angle formula \( 1 - \cos(\alpha) = 2 \sin(\alpha/2)^2 \) and by the inequality \( |\sin(\alpha)| \leq |\alpha| \) we have
\[
2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^\top x)) \leq 4\pi^2 \sum_{u \in L^*} z(u)(u^\top x)^2.
\]
Thus, (10) implies (9).

Conversely, assume that (10) is not satisfied. There exists \( x^* \in \mathbb{R}^n \) with
\[
4\pi^2 \sum_{u \in L^*} z(u)(u^\top x^*)^2 > C|x^*|^2.
\]
For \( r \geq 0 \) define the function
\[
f(r) = 2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^\top (rx^*))) - C|rx^*|^2
\]
and consider its Taylor expansion
\[
f(r) = \left(4\pi^2 \sum_{u \in L^*} z(u)(u^\top x^*)^2 - C|x^*|^2\right) r^2 + \text{h.o.t. (in } r)
\]
Writing \( f \) this way and using the assumption, \( f(r) \) is positive for sufficiently small \( r \). Thus, (9) is not satisfied. \( \square \)
Inequality (10) can also be rewritten as an inequality of the largest eigenvalue $\lambda_{\text{max}}$ of a corresponding matrix

$$\lambda_{\text{max}} \left( 4\pi^2 \sum_{u \in L^*} z(u) uu^T \right) \leq C$$

or equivalently as a semidefinite condition

$$CI - 4\pi^2 \sum_{u \in L^*} z(u) uu^T \in S_n^+,$$

where $I$ denotes the identity matrix. With this lemma we arrive at the following simplification of (7).

**Theorem 2.4.** The least distortion Euclidean embedding of a flat torus $\mathbb{R}^n/L$ is given by

$$c_2(\mathbb{R}^n/L)^2 = \inf \left\{ C : C \in \mathbb{R}_+, z \in \ell^1(L^*), z(u) = z(-u) \geq 0 \text{ for all } u \in L^*, \right.$$  

$$|x|^2 \leq 2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^T x)) \text{ for all } x \in V(L),$$

$$CI - 4\pi^2 \sum_{u \in L^*} z(u) uu^T \in S_n^+ \right\}. \tag{11}$$

2.2. Dual program. We derive the dual of (11) to systematically find lower bounds for $c_2(\mathbb{R}^n/L)$.

**Theorem 2.5.** Suppose that $(C, z)$ is feasible for (11), then

$$C \geq c_2(\mathbb{R}^n/L)^2 \geq \sup \left\{ 2\pi^2 \int_{V(L)} |x|^2 \, d\nu(x) : \right.$$  

$$\nu \in \mathcal{M}_+(V(L)), Y \in S_n^+, \text{Tr}(Y) = 1,$$

$$\int_{V(L)} (1 - \cos(2\pi u^T x)) \, d\nu(x) \leq u^T Y u$$

$$\text{for all } u \in L^* \right\}, \tag{12}$$

where $\mathcal{M}_+(V(L))$ is the cone of Borel measures supported on $V(L)$. In (12) equality holds for a feasible $(\nu, Y)$ if and only if

$$\left( CI - 4\pi^2 \sum_{u \in L^*} z(u) uu^T \right) Y = 0,$$

and the measure $\nu$ is only supported on vectors $x \in V(L)$ for which equality

$$|x|^2 = 2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^T x))$$

holds, and for all vectors $u \in L^*$ with $z(u) \neq 0$ we have

$$\int_{V(L)} (1 - \cos(2\pi u^T x)) \, d\nu(x) = u^T Y u.$$
Proof. For two symmetric matrices $A, B$ we define $\langle A, B \rangle = \text{Tr}(AB)$. Using the feasibility of $(C, z)$ and $(\nu, Y)$ we get

$$C - 2\pi^2 \int_{V(L)} |x|^2 d\nu(x) \geq 4\pi^2 \sum_{u \in L^*} z(u) \nu(u^T, Y) - 4\pi^2 \int_{V(L)} \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^T x)) d\nu(x)$$

$$= 4\pi^2 \sum_{u \in L^*} z(u) \left( \langle uu^T, Y \rangle - \int_{V(L)} (1 - \cos(2\pi u^T x)) d\nu(x) \right)$$

$$\geq 0.$$

When analyzing the case of equality we find the three conditions of the theorem. □

Remark 2.6. As a side note we would like to mention that in (12) we even have equality $c_2(\mathbb{R}^n/L)^2 = \sup$. This follows again by the weak* compactness of the set of bounded, continuous functions of positive type together with the Hahn-Banach (strict) separation theorem.

3. Properties and observations

We collect some results that are consequences of the primal and dual formulation of the preceding section, including some auxiliary results used in later sections.

3.1. Subquadratic inequality. First, we show that the functions of the form

$$f(x) = 2 \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^T x)) \text{ with } z(u) \geq 0$$

are subquadratic, this auxiliary result is going to be used a number of times. Note that we have

$$f(x - y) = \|\varphi(x) - \varphi(y)\|^2$$

for the embedding $\varphi$ in (8). Suppose for a moment that $\varphi$ was an isometry, then $f$ would satisfy the parallelogram law

$$f(x - y) + f(x + y) = 2f(x) + 2f(y)$$

and it would be a homogeneous quadratic form

$$f(\lambda x) = \lambda^2 f(x).$$

However, $\varphi$ cannot be a Hilbert space isometry, but the next lemma shows that we have at least two inequalities.

Lemma 3.1. The function $f$ defined in (13) is subquadratic, i.e. it satisfies

$$f(x + y) + f(x - y) \leq 2f(x) + 2f(y) \text{ for all } x, y \in \mathbb{R}^n.$$  \hspace{1cm} (14)

Furthermore,

$$f(\lambda x) \leq \lambda^2 f(x) \text{ for all } \lambda \in \mathbb{N}, x \in \mathbb{R}.$$  \hspace{1cm} (15)

If $f$ defines an embedding, we have equality in (14) and (15) if and only if $x$ or $y$ lie in $L$.

A proof for (15) can also be found in [14]; we provide it here for the convenience of the reader.
Proof. To show that $f$ is subquadratic it suffices to prove the inequality
\[
1 - \cos(\alpha + \beta) + 1 - \cos(\alpha - \beta) \leq 2(1 - \cos(\alpha)) + 2(1 - \cos(\beta))
\]
for all $\alpha, \beta \in \mathbb{R}$. This is elementary by the cosine addition formula:
\[
1 - \cos(\alpha + \beta) + 1 - \cos(\alpha - \beta) = 2 - 2 \cos(\alpha) \cos(\beta)
\]
\[
= 2 \cos(\beta)(1 - \cos(\alpha)) + 2(1 - \cos(\beta)) \leq 2(1 - \cos(\alpha)) + 2(1 - \cos(\beta))
\]
where equality holds if and only if $\alpha$ or $\beta$ is an integral multiple of $2\pi$.

Now consider $f$ as in (13) and assume $f$ defines an embedding. Then the claim about equality comes from the fact that $\alpha = 2\pi u^T x$ or $\beta = 2\pi u^T y$ is an integral multiple of $2\pi$ for all $u \in \text{supp}(z)$ if and only if $x \in L$ or $y \in L$, since $\text{supp}(z)$ contains a basis of $L^\ast$.

For even $\lambda$ we directly use (14)
\[
f(\lambda x) = f\left(\frac{\lambda}{2}x + \frac{\lambda}{2}x\right) + f\left(\frac{\lambda}{2}x - \frac{\lambda}{2}x\right)
\]
\[
\leq 4 \left(\frac{\lambda}{2}\right)^2 f(x) = \lambda^2 f(x)
\]
since $f(0) = 0$. For odd $\lambda \geq 3$ we use (14) and proceed by induction
\[
f(\lambda x) + f(x) = f\left(\left(\frac{\lambda - 1}{2} + 1\right)x + \frac{\lambda - 1}{2}x\right) + f\left(\left(\frac{\lambda - 1}{2} + 1\right)x - \frac{\lambda - 1}{2}x\right)
\]
\[
\leq 2f\left(\left(\frac{\lambda - 1}{2} + 1\right)x\right) + 2f\left(\frac{\lambda - 1}{2}x\right)
\]
\[
\leq 2 \left(\frac{\lambda - 1}{2} + 1\right)^2 f(x) + \left(\frac{\lambda - 1}{2}\right)^2 f(x)
\]
\[
= \lambda^2 f(x) + f(x).
\]

\[\blacksquare\]

3.2. Dual feasibility. In general the dual program (12) has infinitely many conditions of the form
\[
\int_{V(L)} \left(1 - \cos(2\pi v^T x)\right) dv(x) \leq \text{Tr}(v v^T Y), \quad v \in L^\ast.
\]
We will now show that sometimes already finitely many constraints are sufficient to imply all conditions (16). The first observation is the following:

Lemma 3.2. Let $q_a(x) = 1 - \cos(2\pi a^T x)$. The (in-)equalities
\[
\int_{V(L)} q_a(x) dv(x) \leq \text{Tr}(aa^T Y), \quad \int_{V(L)} q_b(x) dv(x) \leq \text{Tr}(bb^T Y),
\]
\[
\int_{V(L)} q_{a-b}(x) dv(x) = \text{Tr}((a - b)(a - b)^T Y)
\]

imply
\[
\int_{V(L)} q_{a+b}(x) dv(x) \leq \text{Tr}((a + b)(a + b)^T Y).
\]
The corresponding result also holds when $q_{a-b}$ and $q_{a+b}$ are interchanged.
Proof. As shown in the proof of Lemma 3.1, the function \( q_a \) is subquadratic and therefore
\[
\int_{V(L)} q_{a+b}(x) \, d\nu(x) + \int_{V(L)} q_{a-b}(x) \, d\nu(x) \leq 2 \int_{V(L)} q_a(x) + q_b(x) \, d\nu(x) \leq 2 \text{Tr}(aa^T Y) + 2 \text{Tr}(bb^T Y),
\]
which by (18) is equivalent to
\[
\int_{V(L)} q_{a+b}(x) \, d\nu(x) \leq 2 \text{Tr}(aa^T Y) + 2 \text{Tr}(bb^T Y) - \text{Tr}((a-b)(a-b)^T Y) = \text{Tr}((a+b)(a+b)^T Y) .
\]

The above lemma can be used to replace the infinitely many constraints (16) by finitely many using the shortest vectors in cosets of the form \( v + 2L^* \) for \( v \in L^* \).

The proof of the lemma relies on a characterization of Voronoi vectors. These are lattice vectors \( v \in L \setminus \{0\} \) such that the set \( F_v := V(L) \cap \{ x : v^T x \leq \frac{1}{2} v^T v \} \) defines a non-empty face of \( V(L) \). Moreover, \( v \in L \) is called Voronoi relevant if \( F_v \) is a facet of \( V(L) \), i.e. an \((n-1)\)-dimensional face of \( V(L) \).

An element \( v \in L \setminus \{0\} \) is a Voronoi vector of \( V(L) \) if and only if \( \pm v \) are shortest vectors in the coset \( v + 2L \) and \( \pm v \) are Voronoi relevant if and only if they are the only shortest vectors in \( v + 2L \). For a proof see [7, Chapter 21, Theorem 10] and [8, Theorem 2].

Lemma 3.3. If (16) is tight for at least one shortest vector in each coset of the form \( v + 2L^* \), \( v \in L^* \), then (16) holds for all \( v \in L^* \).

Proof. Assume that (16) is tight for at least one shortest vector in each coset \( v + 2L^* \). We will first prove by induction that (16) also holds for all Voronoi vectors.

The statement holds by assumption for all Voronoi relevant vectors \( v \) because in this case the normal vectors \( v \) and \( -v \) are the only shortest vectors in \( v + 2L \) due to the above characterization.

So assume that the statement holds for all vectors \( u \in L^* \) such that \( F_u \) is a face of dimension smaller than \( n-k \). Now consider \( v \in L^* \) such that \( v \) is the normal vector of some \((n-(k+1))\)-dimensional face of \( V(L^*) \) with \( k \geq 2 \). Further, assume that \( u \in v + 2L^* \) with \( u \neq v \) is such that (16) is tight for \( u \) (otherwise there is nothing to show for \( v \)).

Now \( \frac{1}{2}(u \pm v) \in L^* \) and the inequality \( v^T x \leq \frac{1}{2} v^T v \) for all \( x \in V(L^*) \) is implied by
\[
\frac{1}{2}(u + v)^T x + \frac{1}{2}(u - v)^T x \leq \frac{1}{4}(u + v)^T (u + v) + \frac{1}{4}(u - v)^T (u - v) \leq \frac{1}{4} u^T u + \frac{1}{4} v^T v \leq \frac{1}{2} v^T v .
\]

Due to \( u \neq v \) and the above inequality, the sets \( F_{\frac{1}{2}(u+v)}, F_{\frac{1}{2}(u-v)} \) define non-empty faces of \( V(L^*) \) of dimension strictly larger than \( n-(k+1) \). Hence, (16) holds for \( \frac{1}{2}(u + v), \frac{1}{2}(u - v) \) by the induction hypothesis. Applying Lemma 3.2 (with \( a = \frac{1}{2}(u + v), b = \frac{1}{2}(u - v), a + b = u \)) shows that (16) also holds for \( v \).

Now assume that \( v \) is a not a Voronoi vector. Then there exists a a shortest vector \( u \in v + 2L^* \) for which (16) is tight and \( |u| < |v| \).
Then, again $\frac{1}{2}(u \pm v) \in L^*$ and as

$$|\frac{1}{2}(u \pm v)| \leq \frac{1}{2}(|u| + |v|) < |v|,$$

we can argue by an analogous inductive argument (based on the norm) as before that (16) holds for $\frac{1}{2}(u \pm v)$. Finally, we can use Lemma 3.2 to infer that (16) is valid for $v$ as well. \hfill \Box

4. LEAST EUCLIDEAN DISTORTION EMBEDDINGS ALWAYS HAVE FINITE DIMENSION

The goal of this section is to prove that for every $n$-dimensional lattice, there always exists a least distortion embedding of $\mathbb{R}^n/L$ that is finite-dimensional. In the sense of Theorem 2.1, this means that there is always an optimal solution $(C, z)$ for (11) such that the support of $z$ is finite.

Additionally, our arguments will reveal that the constructed optimal solution with finite support has only support on at most one vector per coset $v + 2L^*$ of $L^*/2L^*$ and that supp($z$) only contains primitive lattice vectors. An element $v \in L$ is called primitive for $L$ if $\alpha v \in L$ with $\alpha \in \mathbb{Z}$ implies $\alpha = \pm 1$.

The first step towards proving that there is always a finite-dimensional least Euclidean distortion embedding is the following observation.

Lemma 4.1. Assume that $(C, z)$ is a solution for (11).

1. If there are $u, v \in \text{supp}(z)$, $u \neq v$ with $u \pm v \in 2L^*$ and $z(v) \leq z(u)$, then $(C, \tilde{z})$ with

$$\tilde{z}(t) = \begin{cases} 
    z(u) - z(v), & \text{if } t = \pm u, \\
    0, & \text{if } t = \pm v, \\
    2z(v) + z(t), & \text{if } t \in \{\pm u \pm v\}, \\
    z(t), & \text{otherwise},
\end{cases}$$

is a solution for (11).

2. If there is $u \in \text{supp}(z)$ and $u = kv$ for some integer $k \geq 2$, then $(C, \tilde{z})$ with

$$\tilde{z}(t) = \begin{cases} 
    0, & \text{if } t = \pm u, \\
    z(v) + k^2z(u), & \text{if } t = \pm v, \\
    z(t), & \text{otherwise},
\end{cases}$$

is a solution for (11).

In both cases, $\tilde{z}$ satisfies

$$\sum_{t \in L^*} z(t)t^T = \sum_{t \in L^*} \tilde{z}(t)t^T \quad \text{and} \quad \sum_{t \in L^*} z(t) < \sum_{t \in L^*} \tilde{z}(t).$$

Proof. (1) By construction, we have

$$\sum_{t \in L^*} z(t) < \sum_{t \in L^*} z(t) + 4z(v) = \sum_{t \in L^*} \tilde{z}(t).$$
Computing
\[ \begin{align*}
    z(u)uu^\top + z(v)vv^\top &= (z(u) - z(v))uu^\top + z(v)(uu^\top + vv^\top) \\
    &= (z(u) - z(v))uu^\top + 2z(v)
    \left( \left( \frac{u + v}{2} \right)^\top \right)
    \left( \frac{u + v}{2} \right)
    + \left( \frac{u - v}{2} \right)
    \left( \frac{u - v}{2} \right)^\top 
\end{align*} \]
(and analogously for the pair \(-u, -v\) we obtain \(\sum_{\ell \in L^*} z(t)tt^\top = \sum_{\ell \in L^*} \tilde{z}(t)tt^\top\) and \(CI - 4\pi^2 \sum_{\ell \in L^*} \tilde{z}(t)tt^\top \subseteq S_n^+\).

Moreover, by the subquadratic inequality,
\[ 1 - \cos(2\pi u^\top x) + 1 - \cos(2\pi v^\top x) \]
\[ = 1 - \cos \left( 2\pi \left( \frac{u + v}{2} + \frac{u - v}{2} \right)^\top x \right)
    + 1 - \cos \left( 2\pi \left( \frac{u + v}{2} - \frac{u - v}{2} \right)^\top x \right) \]
\[ \leq 2 \left( 1 - \cos \left( 2\pi \left( \frac{u + v}{2} \right)^\top x \right) \right)
    + 2 \left( 1 - \cos \left( 2\pi \left( \frac{u - v}{2} \right)^\top x \right) \right). \]

Thus, for every \(x \in V(L)\)
\[ |x|^2 \leq 2 \sum_{\ell \in L^*} z(t)(1 - \cos(2\pi t^\top x)) \leq 2 \sum_{\ell \in L^*} \tilde{z}(t)(1 - \cos(2\pi t^\top x)), \]
implying that \((C, \tilde{z})\) is feasible for (11) with the desired properties.

(2) The proof is analogous to (1). \(\square\)

The lemma gives rise to an algorithmic way to transform a feasible solution \((C, z)\)
towards a solution \((C, \tilde{z})\) such that \(\tilde{z}\) has only support on at most one primitive lattice element per coset \(u + 2L^*\). Roughly speaking, start with any solution and apply the above lemma “as long as possible”, i.e. as long as there are pairs of vectors that satisfy (1) or (2) of the above lemma.

**Theorem 4.2.** For any \(n\)-dimensional lattice \(L\), the torus \(\mathbb{R}^n/L\) has a finite-dimensional least Euclidean distortion embedding. In particular, the program (11) has an optimal solution \((C, z)\) such that

1. \(|\text{supp}(\tilde{z}) \cap (v + 2L^*)| \leq 1 \text{ for every coset } v + 2L^* \text{ of } L^*/2L^*\).

2. Every \(u \in \text{supp}(z)\) is primitive in \(L^*\).

Note that claim (1) shows that there are at most \(2^n - 1\) non-zero elements in the support of \(z\), therefore we obtain an embedding into a space of dimension at most \(2^n - 1\).

**Proof.** As a consequence of Remark 2.2, there is an optimal solution \((C, z_0)\) with \(z_0 \in \ell_1(L^*)\) for (11). Our goal is to construct a sequence of solutions \((C, z_m)\) for (11) that converges to a solution that satisfies (1) and (2). Let
\[ A_z = \{ \{u, v\} : u \neq -v, u \pm v \in 2L^*, u, v \in \text{supp}(z) \} \]
and let \((z_m)_m\) be a sequence where \(z_m\) is obtained from \(z_{m-1}\) by applying transformation (1) of Lemma 4.1 to an arbitrary pair \(\{u, v\} \in A_z\) (the actual choice of the pair does not matter). Due to Lemma 4.1, the pair \((C, z_m)\) is feasible for (11) and we have
\[ \sum_{u \in L^*} z_m(u)uu^\top = \sum_{u \in L^*} z_{m+1}(u)uu^\top \text{ and } Z_m < Z_{m+1} \text{ for all } m \in \mathbb{N}, \]
where $Z_m := \sum_{u \in L^*} z_m(u)$. The sequence $Z_m$ is monotonically increasing but bounded since $CI - 4\pi^2 \sum_{u \in L^*} z_m(u)uu^\top \in S_n^+$ enforces that
\[
0 \leq \text{Tr}\left(CI - 4\pi^2 \sum_{u \in L^*} z_m(u)uu^\top\right) = Cn - 4\pi^2 \sum_{u \in L^*} z_m(u)|u|^2 \leq Cn - 4\pi^2 \lambda(L^*) \sum_{u \in L^*} z_m(u).
\]
Hence, by monotone convergence, the sequence $Z_m$ converges.

Now we claim that $\lim_{m \to \infty} z_m(u)$ exists for all $u \in L^*$. Therefore, assume that $\{u_m, v_m\} \in A_{z_{m-1}}$ is chosen in the iteration from $z_{m-1}$ to $z_m$. Assume that $z_{m-1}(u_m) \geq z_{m-1}(v_m)$. Then, using Lemma 4.1, we obtain
\[
\sum_{u \in L^*} |z_m(u) - z_{m-1}(u)| = 3 \cdot 4 z_{m-1}(v_m) = 3(Z_m - Z_{m-1}).
\]
The right hand side converges to zero, therefore the sequence $z_m$ converges pointwise, i.e. there is $z \in \ell_1(L^*)$ such that
\[
\lim_{m \to \infty} z_m(u) = z(u) \quad \text{for all } u \in L^*.
\]
Next, we will show that $z$ satisfies (1), which is equivalent to $A_z = \emptyset$. But this simply follows by construction: For every pair $\{u, v\} \in A_{z_m}$ we have
\[
\lim_{m \to \infty} \min\{z_m(u), z_m(v)\} = 0.
\]
This holds because if there was $\{u, v\} \in A_z$ and $\varepsilon > 0$ such that for all $M$ there was $m \geq M$ with $\min\{z_m(u), z_m(v)\} \geq \varepsilon$, then according to the construction of $z_m$ there would also be $m' \geq m$ with
\[
Z_{m'} - Z_m \geq 4 \min\{z_m(u), z_m(v)\} \geq 4\varepsilon.
\]
This would be a contradiction to the convergence of the sequence $Z_m$.

Now, if there is $u \in \text{supp}(z)$ with $u = kv$ for some $k \geq 2$, we may apply (2) of Lemma 4.1 to obtain a new feasible solution $\tilde{z}$ with $v \in \text{supp}(\tilde{z})$ and $z(u) = 0$. This solution may contain a pair $(u, v) \in A_{z_{\tilde{z}}}$. But in this case, we may again apply (1) of Lemma 4.1.

By continuing like this, we will finally end up with a solution that has only support on primitive vectors and on one vector per coset, thus satisfying properties (1) and (2). \hfill \square

Unfortunately, the proof does not give a bound on $\max\{|u| : u \in \text{supp}(\tilde{z})\}$ for $\tilde{z}$ constructed in Theorem 4.2.

5. IMPROVED LOWER BOUND

In this section we apply Theorem 2.5 to get a constant factor improvement over (4), basically without any effort.

**Theorem 5.1.** Let $L$ be an $n$-dimensional lattice, then
\[
c_2(\mathbb{R}^n / L) \geq \frac{\pi \lambda(L^*) \mu(L)}{\sqrt{n}}.
\]
A semidefinite program for least distortion embeddings of flat tori into Hilbert spaces

Proof. Let $y$ be a vertex of the Voronoi cell $V(L)$ which realizes the covering radius, that is $|y| = \mu(L)$ and so $y$ is a “deep hole” of $L$. Choose $\nu = \frac{\lambda(L^n)^2 \delta_y}{2n}$ to be a point measure supported at $y$ and set $Y = \frac{1}{n} I$. Then $(\nu, Y)$ is feasible for (12) because

$$Z \left( V(L) \right) \left( 1 - \cos(2\pi u^T x) \right) d\nu(x) = \left( 1 - \cos(2\pi u^T y) \right) \frac{\lambda(L^n)^2}{2n} \leq \frac{2n}{n} = u^T Y u$$

for every $u \in L^* \setminus \{0\}$. Hence, by Theorem 2.5,

$$c_2(\mathbb{R}^n/L)^2 \geq 2\pi^2 \int_{V(L)} |x|^2 d\nu(x) = \frac{\pi^2 \lambda(L^n)^2 \mu(L)}{n}.$$

6. Least distortion embeddings of $\mathbb{R}^n/\mathbb{Z}^n$ and of orthogonal decompositions

As our second application of Theorem 2.5, through Theorem 5.1, we prove that the standard embedding (2) of the standard torus is indeed a least distortion embedding. It is somewhat surprising that this result is new. We also note that one can easily use the same argument to capture the case of flat tori whose lattices have an orthogonal basis.

Theorem 6.1. The standard embedding $\varphi : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^2$ of the standard torus $\mathbb{R}^n/\mathbb{Z}^n$ given by

$$\varphi(x_1, \ldots, x_n) = (\cos 2\pi x_1, \sin 2\pi x_1, \ldots, \cos 2\pi x_n, \sin 2\pi x_n)$$

is a least distortion embedding with distortion $c_2(\mathbb{R}^n/\mathbb{Z}^n) = \pi/2$.

Proof. We have $\lambda(\mathbb{Z}^n) = 1$ and $\mu(\mathbb{Z}^n) = \sqrt{n}/4$, so $c_2(\mathbb{R}^n/\mathbb{Z}^n) \geq \pi/2$ by Theorem 5.1.

To show the corresponding upper bound we show that the embedding

$$\phi(x_1, \ldots, x_n) = \frac{1}{\sqrt{32}} (e^{-2\pi i x_1}, \ldots, e^{-2\pi i x_n})$$

has contraction 1 and expansion $\pi/2$. Then turning $\phi$ into a real embedding and rescaling does not change the distortion and gives the standard embedding $\varphi$.

To show that $\phi$ has contraction 1 and expansion $\pi/2$ it suffices to prove that $(\pi^2, z)$ with

$$z(u) = \begin{cases} \frac{1}{32} & \text{if } u = \pm e_i, \\ 0 & \text{otherwise}, \end{cases}$$

is a feasible solution for (11).

The expansion equals $\pi/2$ because

$$CI - 4\pi^2 \sum_{u \in \mathbb{Z}^n} z(u) uu^T = \frac{\pi^2}{4} I - 4\pi^2 \frac{\delta}{32} I = 0.$$ 

Moreover, to show that the contraction equals 1 we need to verify the inequality

$$\sum_{i=1}^n x_i^2 \leq \frac{4}{32} \sum_{i=1}^n (1 - \cos(2\pi x_i)) \quad \text{for all } x \in V(\mathbb{Z}^n) = [-1/2, 1/2]^n,$$
which we check summand by summand, that is \( x_i^2 \leq \frac{1}{8}(1 - \cos(2\pi x_i)) \), where we have equality if \( x_i = \pm 1/2 \). To do so we show that the logarithm of the quotient

\[
x_i \mapsto \log \left( \frac{x_i^2}{1 - \cos(2\pi x_i)} \right)
\]

is convex on \((-1/2, 1/2)\) which follows by taking the second derivative: For \( x_i \in (0, 1/2) \) we have

\[
\frac{\partial^2}{\partial x_i^2} \log \left( \frac{x_i^2}{1 - \cos(2\pi x_i)} \right) = -\frac{2}{x_i^2} + \frac{4\pi^2}{1 - \cos(2\pi x_i)} \geq 0,
\]

where we used the inequality \( 1 - \cos(2\pi x_i) \leq 2\pi^2 x_i^2 \).

Here it is interesting to note that even for the rather trivial standard embedding of the standard torus the structure of the most contracted pairs is rich. Every center of every face of the Voronoi cell \( V(Z^n) \) gives a most contracted pair, see Figure 1a.

Recapitulating the above proof, one recognizes that at its heart is the verification of inequality (20). Here one reduces the situation from \( Z^n \) to \( Z \). This works because \( Z^n \) can be orthogonally decomposed as the direct sum of \( n \) copies of \( Z \) and this can be done in generality as the following theorem demonstrates.

**Theorem 6.2.** Let \( L \subseteq \mathbb{R}^n \) be a lattice such that \( L \) decomposes as the orthogonal direct sum of lattices \( L_1, \ldots, L_m \), i.e.

\[
L = L_1 \perp L_2 \perp \ldots \perp L_m.
\]

Then

\[
c_2(\mathbb{R}^n/L) = \max\{c_2(\mathbb{R}^{n_j}/L_j) : j = 1, \ldots, m\},
\]

where \( \mathbb{R}^{n_j} \) is (isometric) to the Euclidean space spanned by \( L_j \).

**Proof.** Any Euclidean embedding of \( \mathbb{R}^n/L \) gives a Euclidean embedding of \( \mathbb{R}^{n_j}/L_j \) which immediately gives the inequality \( c_2(\mathbb{R}^n/L) \geq \max\{c_2(\mathbb{R}^{n_j}/L_j) : j = 1, \ldots, m\} \).

Also the reverse inequality is easy to see. Let \( \varphi: \mathbb{R}^{n_j}/L_j \to H_j \) be a Euclidean embedding of \( \mathbb{R}^{n_j}/L_j \) with distortion \( C_j \) scaled so that the contraction is 1 and the expansion is \( C_j \). We identify \( \mathbb{R}^n/L \cong \mathbb{R}^{n_1}/L_1 \perp \ldots \perp \mathbb{R}^{n_m}/L_m \), write \( x \in \mathbb{R}^n/L \) as \( x = (x_1, \ldots, x_m) \) with \( x_j \in \mathbb{R}^{n_j}/L_j \) so that \( d_{\mathbb{R}^n/L}(x, y)^2 = \sum_{j=1}^m d_{\mathbb{R}^{n_j}/L_j}(x_j, y_j)^2 \).

Then

\[
\varphi: \mathbb{R}^n/L \to H := H_1 \perp \ldots \perp H_m, \quad (x_1, \ldots, x_m) + L \mapsto (\varphi_1(x_1), \ldots, \varphi_m(x_m))
\]

is a Euclidean embedding of \( \mathbb{R}^n/L \) into the Hilbert space \( H \). Its distortion is at most \( \max\{C_j : j = 1, \ldots, m\} \) because for every pair \( x, y \in \mathbb{R}^n/L \)

\[
|\varphi(x) - \varphi(y)|^2 = \sum_{j=1}^m |\varphi_j(x_j) - \varphi_j(y_j)|^2 \leq \sum_{j=1}^m C_j^2 d_{\mathbb{R}^{n_j}/L_j}(x_j, y_j)^2
\]

\[
\leq \max_{j=1,\ldots,m} C_j^2 \sum_{j=1}^m d_{\mathbb{R}^{n_j}/L_j}(x_j, y_j)^2 = \max_{j=1,\ldots,m} C_j^2 d_{\mathbb{R}^n/L}(x, y)^2,
\]

showing that the expansion of \( \varphi \) at most \( \max\{C_j : j = 1, \ldots, m\} \) and, in exactly the same way, one shows that the contraction of \( \varphi \) is at most 1. \( \square \)
7. Least Euclidean Distortion Embedding of the Lattice $D_n^*$

In this section, we will construct a least Euclidean distortion embedding for the lattice $D_n^*$. To construct this embedding we have to assign weights $z(u)$ for $u \in (D_n^*)^* = D_n$, a root lattice, which is defined as

$$ D_n = \left\{ u \in \mathbb{Z}^n : \sum_{i=1}^n u_i \text{ even} \right\}. $$

The shortest vectors of $D_n$ are precisely its roots, that is, the vectors of squared length 2. They are given by $R(D_n) = \{ \pm (e_i \pm e_j) : i \neq j \}$.

To prove optimality of the embedding for $D_n^*$ we will make use of the lattice’ symmetries. Generally, these symmetries can be exploited to construct a symmetrized solution for (11) as explained below.

Let $G_{L^*}$ be the orthogonal group of the lattice $L^*$, which is the group of orthogonal matrices $S$ satisfying $SL^* = L^*$. Note, that by definition of $L^*$, the group $G_L$ equals $G_{L^*}$.

Now if $(C, z)$ is a solution of (11), then this solution can be symmetrized to a solution $(\bar{C}, \bar{z})$ by taking the group average over $G_L$:

$$ \bar{z}(u) = \frac{1}{|G_L|} \sum_{S \in G_L} z(Su). $$

The tuple $(\bar{C}, \bar{z})$ is feasible because $\bar{z} \geq 0$ and

$$ CI - 4\pi^2 \sum_{u \in L^*} \bar{z}(u) uu^T = \frac{1}{|G_L|} \sum_{S \in G_L} S^T(CI - 4\pi^2 \sum_{u \in L^*} z(u) uu^T)S \in S_n^+. $$

Moreover, for all $x \in V(L)$:

$$ 2 \sum_{u \in L^*} \bar{z}(u)(1 - \cos(2\pi u^T x)) = \frac{2}{|G_L|} \sum_{S \in G_L} \sum_{u \in L^*} z(Su)(1 - \cos(2\pi u^T x)) $$

$$ = \frac{2}{|G_L|} \sum_{S \in G_L} \sum_{u \in L^*} z(u)(1 - \cos(2\pi (S^{-1}u)^T x)) $$

$$ = \frac{2}{|G_L|} \sum_{S \in G_L} \sum_{u \in L^*} z(u)(1 - \cos(2\pi u^T (Sx))) $$

$$ \geq \frac{1}{|G_L|} \sum_{S \in G_L} |Sx|^2 $$

$$ = |x|^2. $$

By construction, $\bar{z}$ is constant on orbits under the action of $G_L$:

$$ 21 \quad \text{If } Su = w \text{ for some } S \in G_L, \text{ then } \bar{z}(u) = \bar{z}(v) $$

Moreover, we have

$$ \sum_{u \in L^*} z(u) = \sum_{u \in L^*} \bar{z}(u) $$

Back to $D_n^*$. The matrix group generated by the reflections $(I - xx^T)$ for $x \in \pm (e_i \pm e_j)$, for $i \neq j$, is called the Weyl group of $D_n$ and preserves the lattice. It acts transitively on the roots; see, for examples, [9, Chapter 1] for details.
Theorem 7.1. An optimal solution for (11) for $D_n^*$ is given by $(C, z)$ where

$$z(u) = \begin{cases} \alpha, & \text{if } u \in R(D_n), \\ 0, & \text{otherwise} \end{cases}$$

and

$$C = \max \left\{ \frac{|x|^2}{2\alpha \sum_{u \in R(D_n)} (1 - \cos(2\pi u^T x))} : x \in V(L) \setminus \{0\} \right\}$$

with $\alpha > 0$ chosen such that $CI = 4\pi^2 \alpha \sum_{u \in R(D_n)} uu^T$.

Proof. Let $(C, z)$ be an optimal solution of (11). By Theorem 4.2, we may assume that every element in supp$(z)$ is primitive and there is maximally one pair $\pm u \in \text{supp}(z) \cap v + 2D_n$ for every coset $v + 2D_n$. Now suppose that there is $u \in \text{supp}(z)$ such that $|u|^2 > 2$. Assume that $u = \sum_{i=1}^n u_i e_i \in \text{supp}(z)$ with $u_i \in \mathbb{Z}$ and $\sum_{i=1}^n u_i$ even. Furthermore, assume without loss of generality that $u_1 + u_2$ is even (such a pair exists, as $u \in D_n$).

Now define

$$v = (I - (e_1 + e_2)(e_1 + e_2)^T)u = -u_2 e_1 - u_1 e_2 + \sum_{i=3}^n u_i e_i.$$ 

As supp$(z)$ contains only primitive vectors and $|u|^2 > 2$, it follows that $v \neq -u$.

Using (21), we can symmetrize the solution $(C, z)$ to a solution $(C, \tilde{z})$ which satisfies $\tilde{z}(u) = \tilde{z}(v)$. Since $u_1 + u_2$ is even, it follows that $\frac{u_1 + u_2}{2} \in D_n$ and we can apply Lemma 4.1 to obtain a new solution $(C, \tilde{z})$.

Iterating this process and arguing as in the proof of Theorem 4.2, we will obtain a sequence of solutions which will finally converge to a solution which has only support on the points of the form $\pm (e_i \pm e_j)$ for $i \neq j$. As the Weyl group acts transitively on the roots, it follows, that there is an optimal solution $(C, \hat{z})$ of (11) which has uniform support on the roots.

Finally, we have that the matrix $\sum_{u \in R(D_n)} uu^T$ is a positive multiple of the identity because the roots of $D_n$ form a spherical 2-design (see [23] for details). \qed

Remark 7.2. The argument cannot be straightforwardly carried over to the lattice $A_n^*$. However, we conjecture that there is also an optimal solution for (11) which assigns uniform weights to the roots of $A_n$. This has been proved by Moustrou and Vallentin [21] for $n = 2$.

8. LEAST DISTORTION EMBEDDINGS OF TWO-DIMENSIONAL FLAT TORI

In this section we will construct least Euclidean distortion embeddings of flat tori in dimension 2.

First, as a simple corollary of Theorem 2.4, we will give a recipe to construct (possibly non-optimal) embeddings of flat tori of arbitrary dimension provided that they satisfy the following assumption:

(23) There exist $u_1, \ldots, u_k \in L^*, z_1, \ldots, z_k \geq 0$ such that $4\pi^2 \sum_{i=1}^k z_i u_i u_i^T = I$.

As we will prove in Lemma 8.2, condition (23) can be realized for every 2-dimensional lattice, Another example for lattices that satisfy condition (23) are
duals of root lattices. If \( L^* \) is a root lattice, then assumption (23) is satisfied. In this case the root system \( R \subseteq L^* \) of \( L^* \) forms a spherical 2-design, implying that
\[
4\pi^2 \alpha \sum_{u \in R} uu^T = I
\]
for some positive constant \( \alpha \). We refer to the monograph by Venkov [23] for more information on root lattices and spherical designs.

**Corollary 8.1.** Let \( L \subseteq \mathbb{R}^n \) be a lattice that satisfies (23). Then
\[
\varphi : \mathbb{R}^n / L \to \mathbb{C}^k, \quad \varphi(x) = (\sqrt{Dz_1} e^{2\pi i x_1}, \ldots, \sqrt{Dz_k} e^{2\pi i u_x})
\]
with
\[
D = \max \left\{ \frac{|x|^2}{2 \sum_{i=1}^k z_i (1 - \cos(2\pi u_i^T x))} : x \in V(L) \setminus \{0\} \right\}
\]
is a Euclidean embedding of \( \mathbb{R}^n / L \) with distortion \( \sqrt{D} \). In particular,
\[
c_2(\mathbb{R}^n / L)^2 \leq D.
\]

**Proof.** The pair \( ((Dz_i)_{1 \leq i \leq k}, D) \) is a feasible solution for the primal optimization problem (10).

Except for the easiest case of the standard torus we do not know how to determine \( D \) explicitly. Unfortunately, it seems to be difficult to compute most contracted pairs \( (0, x) \), i.e. vectors \( x \in V(L) \) that are maximizers of the right hand side of (24).

Next, we show that Corollary 8.1 can be applied to every 2-dimensional lattice. For this we will use the concept of an obtuse superbasis. An obtuse superbasis of an \( n \)-dimensional lattice \( L \) is a basis \( u_1, \ldots, u_n \) of \( L \) enlarged by the vector \( u_0 = -u_1 - \cdots - u_n \) so that these \( n + 1 \) vectors pairwise form non-acute angles, i.e.
\[
u_i^T u_j \leq 0 \quad \text{for all } 0 \leq i < j \leq n.
\]

It is known that up to dimension 3 all lattices have an obtuse superbasis, but from dimension 4 on this is no longer the case, see for instance [8].

**Lemma 8.2.** If \( L \) is a two-dimensional lattice, then its dual lattice \( L^* \) satisfies (23).

**Proof.** Let \( u_0, u_1, u_2 \) be an obtuse superbasis of \( L^* \). We will show that there are non-negative coefficients \( z_0, z_1, z_2 \) such that
\[
I = z_0 u_0 u_0^T + z_1 u_1 u_1^T + z_2 u_2 u_2^T,
\]
and therefore condition (23) holds.

We may assume that \( |u_1| \geq |u_0| = 1 \), by scaling and renumbering. Then, by Gram-Schmidt orthogonalization, \( u_0 \) and \( u_1 := u_1 - (u_0^T u_1) u_0 \) are orthogonal and so
\[
I = u_0 u_0^T + \frac{1}{|u_1|^2} u_1 u_1^T
= \left( 1 + \frac{(u_0^T u_1)^2}{|u_1|^2} \right) u_0 u_0^T + \frac{1}{|u_1|^2} |u_1|^2 u_1 u_1^T - \frac{u_0^T u_1}{|u_1|^2} (u_0 u_1^T + u_1 u_0^T).
\]

Using
\[
u_2 u_2^T = (-u_0 - u_1)(-u_0 - u_1)^T = u_0 u_0^T + u_1 u_0^T + u_0 u_1^T + u_1 u_1^T,
\]

and
\[
u_1 u_1^T = \left( 1 + \frac{(u_0^T u_1)^2}{|u_1|^2} \right) u_0 u_0^T + \frac{1}{|u_1|^2} |u_1|^2 u_1 u_1^T - \frac{u_0^T u_1}{|u_1|^2} (u_0 u_1^T + u_1 u_0^T).
\]

Here \( u_0, u_1, u_2 \) are normalized so that \( |u_0|^2 = |u_1|^2 = |u_2|^2 = 1 \).
yields
\[ I = \left( 1 + \frac{(u_0^T u_1)^2 + u_0^T u_1}{|u_1|^2} \right) u_0 u_1^T + \frac{1 + u_0^T u_1}{|u_1|^2} u_1 u_1^T - \frac{u_0^T u_1}{|u_1|^2} u_2 u_2^T. \]

To prove that the three coefficients in the above sum are non-negative, observe for the third coefficient that \( u_0^T u_1 \leq 0 \). For the second coefficient
\[
0 \leq -u_0^T u_2 = u_0^T u_0 + u_0^T u_1 = 1 + u_0^T u_1.
\]

For the first coefficient we compute the squared norm \(|u_1|^2 = |u_1|^2 - (u_0^T u_1)^2\) and see that the first coefficient is nonnegative if and only if \(|u_1|^2 \geq -u_0^T u_1\), which is true because \(|u_1| \geq |u_0| = 1\) by assumption.

Now, to verify that the embedding of Corollary 8.1 is indeed a least Euclidean distortion embedding for two-dimensional flat tori, we construct a dual solution for (12) that shows that the upper bound (25) is sharp.

**Theorem 8.3.** Let \( L \subseteq \mathbb{R}^2 \) be a 2-dimensional lattice, then \( c_2(\mathbb{R}^2/L)^2 = D \), where \( D \) is defined in (24).

**Proof.** Let \( u_0, u_1, u_2 \) be an obtuse superbasis of \( L^\ast \). We may assume, see for example [8], that this superbasis is chosen in a way such that \( u_i \) is a shortest vector in its coset \( u_i + 2L^\ast \), with \( i = 0, 1, 2 \). By Lemma 8.2 we can determine coefficients \( z_0, z_1, z_2 \geq 0 \) such that \( 4\pi^2 \sum_{i=0}^2 z_i u_i u_i^T = I \).

Furthermore, let \( \bar{x} \in V(L) \) be a vector such that \((0, \bar{x})\) is a most contracted pair for the embedding \( \varphi \) of Corollary 8.1, that is, \( \bar{x} \) is a maximizer for (24).

We define the pair \((Y, \nu)\) as follows: Set \( \beta = \frac{D}{2\pi^2|\bar{x}|^2} \), define \( Y \) via
\[ \text{Tr}(u_i u_i^T Y) = \beta(1 - \cos(2\pi u_i^T \bar{x})), \quad i \in \{0, 1, 2\}, \]
and let \( \nu = \beta \delta_{\bar{x}} \) be a point measure supported only on \( \bar{x} \). We now verify that this pair is a feasible dual solution with objective value \( D \).

We have
\[ \text{Tr}(Y) = \text{Tr}(YI) = 4\pi^2 \sum_i z_i \text{Tr}(u_i u_i^T Y) = 4\pi^2 \beta \sum_i z_i (1 - \cos(2\pi u_i^T \bar{x})) = 1. \]
Equation (27) together with Lemma 3.3 implies
\[ \text{Tr}(u_i u_i^T Y) \geq \beta(1 - \cos(2\pi u_i^T \bar{x})) \quad \text{for all } u \in L^\ast. \]

Finally, it remains to show that \( Y \) is positive semidefinite. For this we compute its Gram matrix \( B \) with respect to \( u_0, u_1 \), that is
\[
B_{ij} = u_i^T Y u_j = \frac{1}{2} \text{Tr}((u_i u_j^T + u_j u_i^T) Y), \quad 0 \leq i, j \leq 1.
\]
Then
\[
B_{ii} = \text{Tr}(u_i u_i^T Y) = 2\beta \sin^2(\pi u_i^T \bar{x}).
\]
Since
\[
u_0 u_1^T + u_1 u_0^T = u_2 u_2^T - u_0 u_0^T - u_1 u_1^T,
\]
we get
\[
B_{01} = \frac{\beta}{2} \left( 2 \sin^2(\pi u_0^T \bar{x}) - 2 \sin^2(\pi u_1^T \bar{x}) - 2 \sin^2(\pi u_2^T \bar{x}) \right) = 2\beta \sin(\pi u_0^T \bar{x}) \sin(\pi u_1^T \bar{x}) \cos(\pi (u_0 + u_1)^T \bar{x}).
\]
From this we see that matrix $B$ is the Schur-Hadamard (entry-wise) product of the positive semidefinite rank-one matrix $xx^T$ with $x_i = \sqrt{2}/\beta \sin(\pi u_i^T \bar{x})$ and the symmetric matrix $M \in \mathbb{R}^{2 \times 2}$ defined by

$$M_{ij} = \begin{cases} 1 & \text{if } i = j \\ \cos(\pi (u_0 + u_1)^T \bar{x}) & \text{if } (i, j) \in \{(0, 1), (1, 0)\}. \end{cases}$$

The matrix $M$ is positive semidefinite because $M_{ii} \geq 0$ and

$$\det(M) = 1 - \cos^2(\pi (u_0 + u_1)^T \bar{x}) \geq 0.$$  

Thus $B$, and therefore also $Y$, is positive semidefinite, which finishes the proof. □

To conclude the discussion of 2-dimensional lattices Figure 1 collects an illustration of the behavior of the distortion function defined in (24) and the most contracted pairs, applying the above results, for a selection of 2-dimensional lattices.

9. Discussion and open questions

In this paper we derived an infinite-dimensional semidefinite program to determine the least distortion Euclidean embedding of a flat torus. It would be very interesting to show that this infinite-dimensional semidefinite program can in fact be turned into a finite-dimensional semidefinite program. Then one could, similarly to the case of finite metric spaces, algorithmically determine least distortion Euclidean embeddings of flat tori; at least up to any desired precision.

For this a characterization of the most contracted pairs is needed. We believe that the most contracted pairs are always of the form $(0, y)$ and $y$ is a center of a face of the Voronoi cell. However, we do not know whether such a $y$ can only lie on the Voronoi cell’s boundary. We do not even know whether there are only finitely many most contracted pairs.

Moustrou and Vallentin [21] computed the most contracted pairs for the lattices $A_2^*$ and $E_8$ using a modification of the linear programming method for spherical designs.

We also do not know how to restrict the variable $z \in \ell_1(L^*)$ to finite dimension, even though Theorem 4.2 shows that we can always find a finite-dimensional least distortion embedding. Obtaining a bound on the maximally needed length of a support vector in the cosets $L^*/2L^*$ would solve this problem.

Another interesting problem is to determine $n$-dimensional lattices which maximize the distortion among all $n$-dimensional lattices.

Acknowledgements

This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie agreement No 764759. F.V. is partially supported by the SFB/TRR 191 “Symplectic Structures in Geometry, Algebra and Dynamics”, F.V. and M.C.Z. are partially supported “Spectral bounds in extremal discrete geometry” (project number 414898050), both funded by the DFG. A.H. is partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – Cluster of Excellence Matter and Light for Quantum Computing (ML4Q) EXC 2004/1 – 390534769.
Figure 1. Let $L_\varphi$ be the lattice spanned by $v_1 = e_1$, $v_2 = R_\varphi e_1$, where $R_\varphi$ is the rotation by $\varphi$ degrees (counter-clockwise).

The Voronoi cell of any lattice in $\mathbb{R}^2$ is either a rectangle or a hexagon. We plot contour lines of the distortion function for a selection of lattices that illustrate how the distortion function and the most contracted pairs vary with the shape of the Voronoi cell. $L_{93^\circ}$ shows what happens for almost degenerated hexagons, i.e. lattices close to the standard lattice. A zoom into the behavior around the short edge of the hexagon is included to illustrate that the most contracted points are all vertices of the Voronoi-cell.
A semidefinite program for least distortion embeddings of flat tori into Hilbert spaces

References

[1] I. Aharoni, B. Maurey, B.S. Mityagin, Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces, Israel J. Math. 52 (1985), 251–265.

[2] I. Agarwal, O. Regev, Y. Tang, Nearly optimal embeddings of flat tori, APPROX/RANDOM 2020.

[3] C. Bachoc, D.C. Gijswijt, A. Schrijver, F. Vallentin, Invariant semidefinite programs, pp. 219–269 in: Handbook on semidefinite, conic, and polynomial optimization (M.F. Anjos, J.B. Lasserre, eds.), Springer, 2012.

[4] V. Borrelli, S. Jabrane, F. Lazarus, B. Thibert, Flat tori in three-dimensional space and convex integration, Proceedings of the National Academy of Sciences (PNAS) 109 (2012), 7218–7223.

[5] G.J. Butler, Simultaneous packing and covering in Euclidean space, Proc. London Math. Soc. 25 (1972) 721–735.

[6] S.M. Cioab˘ a, H. Gupta, F. Ihringer, H. Kurihara, The least Euclidean distortion constant of a distance-regular graph, Discrete Appl. Math. 325 (2023), 212–225.

[7] J.H. Conway, N.J.A. Sloane, Sphere packings, lattices, and groups, Springer, 1988.

[8] J.H. Conway, N.J.A. Sloane, Low-Dimensional Lattices VI: Voronoi Reduction of Three-Dimensional Lattices, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 436 (1992), 55–68.

[9] W. Ebeling, Lattices and codes, Friedr. Vieweg & Sohn, 1994.

[10] G.B. Folland, A course in abstract harmonic analysis, CRC Press, 1995.

[11] I. Haviv, O. Regev, The Euclidean distortion of flat tori, J. Topol. Anal. 5 (2013), 205–223 (extended abstract appeared in APPROX/RANDOM 2010).

[12] S. Khot, A. Naor, Nonembeddability theorems via Fourier analysis, Math. Ann. 334 (2006), 821–852 (extended abstract appeared in FOCS 2005)

[13] T. Kobayashi, T. Kondo, The Euclidean distortion of generalized polygons, Adv. Geom. 15 (2015), 499–506.

[14] Z. Kominek, K. Troczka-Pawelec, Some remarks on subquadratic functions, Demonstratio Math. 39 (2006), 751–758.

[15] M. Laurent, F. Vallentin, A course on semidefinite optimization, Cambridge University Press, in preparation.

[16] N. Linial, E. London, Y. Rabinovich, The geometry of graphs and some of its algorithmic applications, Combinatorica 15 (1995), 215–246.

[17] N. Linial, A. Magen, Least-distortion Euclidean embeddings of graphs: products of cycles and expanders, J. Combin. Theory Ser. B 79 (2000), 157–171.

[18] N. Linial, A. Magen, A. Naor, Girth and Euclidean distortion, Geom. Funct. Anal. 12 (2002), 380–394.

[19] J. Matouˇ sek, Lectures on discrete geometry, Springer, 2002.

[20] E.H. Moore, On properly positive Hermitian matrices, Bull. Amer. Math. Soc. 23 (1916), 66–67.

[21] P. Moustrou, F. Vallentin, Least distortion Euclidean embeddings of flat tori. Proceedings of the International Symposium on Symbolic & Algebraic Computation (ISSAC 2023, July 24–27, 2023, Tromso, Norway), 13–23, ACM, 2023.

[22] F. Vallentin, Optimal distortion embeddings of distance regular graphs into Euclidean spaces, J. Combin. Theory Ser. B 98 (2008), 95–104.

[23] B.B. Venkov, Réseaux et designs sphériques, pages 10–86 in: Réseaux euclidiens, designs sphériques et formes modulaires, Monogr. Enseign. Math., 37, 2001.

A. Heimendahl, M. Lücke, F. Vallentin, M.C. Zimmermann, Department Mathematik/Informatik, Abteilung Mathematik, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany

Email address: arne.heimendahl@uni-koeln.de

Email address: moritz.luecke1997@gmail.com

Email address: frank.vallentin@uni-koeln.de

Email address: marc.christian.zimmermann@gmail.com