Renormalization of Massive Lattice Fermions

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The renormalization of a general action for massive lattice fermions is discussed. The analysis applies for all $m,q$. Preliminary results for the self energy at one loop in perturbation theory are presented.

1. INTRODUCTION

This paper is a progress report of efforts to calculate the renormalization of fermion masses and bilinear currents in one-loop perturbation theory [1]. When these calculations are finished, they will permit a determination of heavy-quark masses, they will give one-loop predictions for the tuning of improvement parameters in the action, and they will give a one-loop guide for extrapolating the matrix elements to the continuum.

After specifying a general action in sect. 2, sect. 3 sketches an all-orders derivation for mass and wavefunction renormalization in terms of the fermion self energy. We have the self energy to one-loop for the simplest action, and present results with and without tadpole improvement. Sect. 4 discusses current renormalization.

2. THE ACTION

Consider the action $S = S_0 + S_B + S_E + \cdots$,

$$S_0 = m_0 \int \tilde{\psi}(x) \psi(x) - \frac{i}{2} a r \zeta \int \tilde{\psi}(x) \Delta(3) \psi(x)$$

$$+ \zeta \int \tilde{\psi}(x) \gamma^j D_j \psi(x)$$

$$+ \frac{1}{2} \int \tilde{\psi}(x) [(1 + \gamma_0) D_0^+ - (1 - \gamma_0) D_0^-] \psi(x),$$

$$S_B = -\frac{1}{2} a c_B \zeta \int \tilde{\psi}(x) \Sigma \cdot B(x) \psi(x),$$

$$S_E = -\frac{1}{2} a c_B \zeta \int \tilde{\psi}(x) \alpha \cdot E(x) \psi(x).$$

Special cases are the Wilson action [2], which sets $r = \zeta = 1$, $c_B = c_E = 0$, and the Sheikholeslami-Wohlert action [3], which sets $r = \zeta = 1$, $c_B = c_E = c_E$.

To remove lattice artifacts in general the parameters $m_0, r, \zeta, c_B$ and $c_E$ must all be adjusted [4]. In a non-relativistic setting, however, it is enough to adjust $m_0, c_B$, and $c_E$ [4].

3. THE SELF ENERGY

The self energy $\Sigma(p)$ is related to the momentum-space propagator by

$$S^{-1}(p) = S_0^{-1}(p) - \Sigma(p),$$

where $S_0(p)$ is the free propagator. In perturbation theory $\Sigma(p)$ is the sum of all one-particle irreducible graphs. The $p_0$-Fourier transform $C(t, p) = (2\pi)^{-1} \int dp_0 e^{ip_0 t} S(p)$ obeys

$$C(t, p) = Z_2(p) e^{-E_p \mid Q + \cdots,}$$

where $E_p$ is the energy of a one fermion state with momentum $p$, and $Q$ is a Dirac matrix satisfying $(Q \gamma_0)^2 = Q \gamma_0$. The $\cdots$ denote multi-particle states, which are irrelevant here.

The self energy has the decomposition

$$\Sigma(p) = i \sum_\mu \gamma_\mu \sin p_\mu A_\mu(p) + C(p)$$

in Dirac matrices. For a Euclidean invariant cutoff $C$ and $A_\mu = A \forall \mu$ are functions of $p^2$ only. With the lattice cutoff, however, they are constrained only by (hyper)cubic symmetry. For emphasis it is convenient to write, say, $C(p_0, p)$.

To obtain an expressions for $E_p$ and $Z_2$ one carries out the $p_0$ integration with the residue theorem. For arbitrary $p$ the energy $E_p$ is the
solution of the implicit equation
\[ 1 + m_0 a + \frac{1}{2} r \zeta p^2 a^2 = C = \cosh E a + (1 - A_0) \sqrt{1 - p^2} \sinh E a, \] (7)
for \( E \). The abbreviation
\[ p^2 = \sum_j \left( \zeta - A_j \right)^2 \sin^2 p_j a \over (1 - A_0)^2 \sin^2 E a. \] (8)
Here the self-energy functions \( A_\mu (p_0, p) \) and \( C(p_0, p) \) are evaluated at \( p_0 = i E \). The solution of eq. (7), \( E = E_p \), defines the (lattice-distorted) mass shell of the fermion. The residue is
\[ Z_2^{-1}(p) = (1 - A_0) \cosh E_p a - \tilde{C} \sqrt{1 - p^2} \\
+ \sum_j \tilde{A}_j (\zeta - A_j) \sin^2 p_j a \\
+ \sum_j \tilde{A}_j (\zeta - A_j) \sin^2 E_p a \] (9)
The notation \( \tilde{f} = (i a)^{-1} (df/dp_0) \). In eq. (9) the self-energy functions \( A_\mu (p_0, p) \) and \( C(p_0, p) \) are evaluated on shell, i.e. \( p_0 = i E_p \).

The \( p \) dependence of \( Z_2(p) \) is an artifact of the lattice cutoff. An acceptable definition of the wavefunction renormalization constant is
\[ Z_2^{-1} = \epsilon^M a_1 - A_0 \cosh M_1 a + \tilde{A} \sinh M_1 a - \tilde{C} \] (10)
at \( p = 0 \), where \( M_1 \equiv E_0 \) is the (all-orders) rest mass of the fermion.

Eq. (7) at \( p = 0 \) determines the rest mass via
\[ \epsilon^{M_1 a} = 1 + m_0 a + A_0 \sinh M_1 a - C \] (11)
and the dynamic mass \( M_2 = (d^2 E_p / dp_1^2)_{p=0}^{-1} \) via
\[ \epsilon^{M_2 a} = r^2 + \frac{(\zeta - A_1)^2}{(1 - A_0) \sinh M_1 a} \sum_j \tilde{A}_j (\zeta - A_j) \sinh E_p a \] (12)
A total \( p_1 \)-derivative includes an explicit part and an implicit part through the \( E_p \) dependence. In eqs. (11) and (12) the self-energy functions and derivatives are evaluated at \( p_0 = i M_1 \) and \( p = 0 \).

For a massless fermion, \( M_1 = M_2 = 0 \). The bare mass that induces \( M_1 = 0 \) obeys
\[ m_{0e} a = C(0, 0; m_{0e} a). \] (13)

The third argument of \( C \) denotes the parametric dependence. It is useful to take care of this term once and for all, and write
\[ \epsilon^{M_1 a} = 1 + M_0 a + A_0 \sinh M_1 a - \tilde{C} \] (14)
where \( M_0 a = m_0 a - m_{0e} a = (2 \kappa)^{-1} - (2 \kappa_e)^{-1} \), and \( \tilde{C}(i M_1 a, 0; m_{0e} a) = C(i M_1 a, 0; m_{0e} a) - m_{0e} a \).

We turn now to one-loop results for \( r = \zeta = 1 \), \( c_B = c_F = 0 \), with and without tadpole improvement. In perturbation theory the rest mass has an expansion
\[ M_1 a = \log(1 + M_0 a) + \sum_{i=1}^\infty \tilde{g}_i a^i M_1^i a. \] (15)

In the tadpole improved version \( \tilde{M}_0 a = M_0 a/\tilde{u}_0 \), where \( \tilde{u}_0 \) is a suitable (gauge invariant) average link. In applications both \( u_0 \) and \( \kappa_e \) would be taken from Monte Carlo calculations. Below we choose \( u_0 = (8 \kappa_e)^{-1} \). Figure 1 shows the one-loop correction to the rest mass \( M_1 \). As expected, \( \tilde{M}_1^{[1]} \) is significantly smaller than \( M_1^{[1]} \).

For the dynamic mass it is better to define a renormalization factor via \( M_2 = m_2 Z_M \), where
\[ m_2 a = \frac{M_0 a (1 + M_0 a) (2 + M_0 a)}{2 \zeta^2 (1 + M_0 a) + r \zeta M_0 a (2 + M_0 a)} \] (16)
is the tree-level expression for the dynamic mass, except that the linear mass divergence is absorbed, order by order, into \( M_0 a \). The tadpole improvement is \( M_2 = \tilde{m}_2 Z_M \), where \( \tilde{m}_2 \) is

\[ \text{Figure 1. Plot comparing } M_1^{[1]} a \text{ vs. } M_b a \text{ (solid curve) and } \tilde{M}_1^{[1]} a \text{ vs. } M_b a \text{ (dashed curve). The static point is indicated by the box [5].} \]
Figure 2. Plot comparing $Z^{[1]}_M$ (solid curve) and $\tilde{Z}^{[1]}_M$ vs. $M_2a$ (dashed curve).

Figure 3. Plot comparing $z^{[1]}_2$ vs. $M_0a$ (solid curve) and $\tilde{z}^{[1]}_2$ vs. $M_0a$ (dashed curve). The static point is indicated by the box [5].

given by the right-hand side of eq. (16), but with $M_0a \rightarrow \tilde{M}_0a$. The factors $Z_M$ have series

$$Z^{(e)}_M = 1 + \sum_{l=1}^{\infty} g_0^{[l]} Z^{[l]}_M, \quad (17)$$

Figure 2 shows the one-loop renormalization of the dynamic mass. Again, $[Z^{[l]}_M]$ is significantly smaller than $[Z^{[l]}_M]$.

To define the perturbative coefficients for the wavefunction renormalization constant, factor out $e^{M_0a}$. The tadpole improved constant is $Z_2 = u_0 Z_2$. The perturbative series are

$$e^{M_1a} Z_2 = 1 + \sum_{l=1}^{\infty} g_0^{[l]} Z^{[l]}_2. \quad \text{(18)}$$

The one-loop coefficient has an infrared divergence, which can be regulated with a gluon mass $\lambda$. Figure 3 plots the IR-finite $\tilde{Z}^{[1]}_2 = \frac{1}{Z^{[1]}_2} \log(\lambda^2 a^2)/(6\pi^2)$. Once again, $\tilde{z}^{[1]}_2$ is significantly smaller than $z^{[1]}_2$.

4. VERTEX CORRECTIONS

A full vertex function takes the form

$$V(p|q) = S(p) \Gamma(p|q) S(q). \quad \text{(19)}$$

In perturbation theory $\Gamma$ is given by the sum of all truncated three-point diagrams. To put the external lines on shell, one Fourier transforms in $p_0$ and $q_0$. Poles arise precisely as in the self-energy derivation, so the on-shell truncated vertex function is $\Gamma(iE_p, p|E_q, q)$. Consequently, when normalization conditions introduce a factor $1 + m_0a$ at tree level, the all-orders generalization is $e^{M_0a}$, just as with $Z^{-1}$.

ACKNOWLEDGEMENTS

This work is being carried out in collaboration with Aida El-Khadra and Paul Mackenzie [1].

B.P.M. is supported in part by the U.S. Department of Energy under Grant No. DE-FG02-90ER40560. Fermilab is operated by Universities Research Association, Inc., under contract DE-AC02-76CH03000 with the U.S. Department of Energy.

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