The first order partial differential equations resolved with any derivatives

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Abstract. In this paper we discuss the first order partial differential equations resolved with any derivatives. At first, we transform the first order partial differential equation resolved with respect to a time derivative into a system of linear equations. Secondly, we convert it into a system of the first order linear partial differential equations with constant coefficients and nonlinear algebraic equations. Thirdly, we solve them by the Fourier transform and convert them into the equivalent integral equations. At last, we extend to discuss the first order partial differential equations resolved with respect to time derivatives and the general scenario resolved with any derivatives.

Keywords. linear algebra, Fourier transform, integral equations.

1 Introduction

The goal of this paper is to transform the first order partial differential equations resolved with any derivatives as follows into the integral equations,

\begin{equation}
  v_j = f_j(v_{m+1}, v_{m+2}, \cdots, v_{5m}, x, y, z, t), \quad 1 \leq j \leq m,
\end{equation}

where \( t \in [0, T] \), \( (x, y, z)^T \in \Omega \subset \mathbb{R}^3 \), and \( \Omega \) is bounded, \( \partial \Omega \) is smooth or piecewise smooth,

\begin{equation}
  u = (u_1(x, y, z, t), \cdots, u_m(x, y, z, t))^T \in C^1(\Omega \times (0, T))
\end{equation}

the initial conditions and boundary conditions are

\begin{equation}
  u|_{t=0} \in L^2(\Omega), \quad u|_{\partial \Omega \times (0, T)} \in L^2(\partial \Omega \times (0, T)),
\end{equation}

\( f_j, \quad 1 \leq j \leq m \), are continuous functions, \( v_1, v_2, \cdots, v_{5m} \) is a permutation of the components of \( u_t, u, u_x, u_y, u_z \),

After we did it, most of the Mathematical-Physics equation in [1] as follows can be transformed into the integral equations.

1. Eikonal equation

\begin{equation}
  |Du| = 1.
\end{equation}

2. Nonlinear Poisson equation

\begin{equation}
  -\Delta u = f(u).
\end{equation}

3. p-Laplacian equation

\begin{equation}
  div(|Du|^{p-2}Du) = 0.
\end{equation}
4. Minimal surface equation
\[ \text{div} \left( \frac{Du}{(1 + |Du|^2)^{1/2}} \right) = 0. \] (1.7)

5. Monge-Ampère equation
\[ \det(D^2 u) = f. \] (1.8)

6. Hamilton-Jacobi equation
\[ u_t + H(Du, x) = 0. \] (1.9)

7. Scalar conservation law
\[ u_t + \text{div} F(u) = 0. \] (1.10)

8. Inviscid Burgers’ equation
\[ u_t + uu_x = 0. \] (1.11)

9. Scalar reaction-diffusion equation
\[ u_t - \triangle u = f(u). \] (1.12)

10. Porous medium equation
\[ u_t - \triangle (u^γ) = 0. \] (1.13)

11. Nonlinear wave equations
\[ u_{tt} - \triangle u = f(u), \ u_{tt} - \text{div} a(Du) = 0. \] (1.14)

12. Korteweg-de Vries (KdV) equation
\[ u_t + uu_x + u_{xxx} = 0. \] (1.15)

13. System of conservation law
\[ u_t + \text{div} F(u) = 0. \] (1.16)

14. Reaction-diffusion system
\[ u_t - \triangle u = f(u). \] (1.17)

From \( f_j, \ 1 \leq j \leq m, \) are continuous functions, we can’t use the Cauchy-Kovalevskaya theorem and the linear methods such as elliptic, parabola and hyperbola. In this paper, we will gradually transform Eq.(1.1) into the equivalent integral equations. In order to read easily, we discuss the first order partial differential equation resolved with respect to a time derivative at first.

2 One equation

In this section, we discuss the first order partial differential equation resolved with respect to a time derivative as follows,
\[ u_t = f(u, u_x, u_y, u_z, x, y, z, t), \] (2.1)
where $t \in [0, T]$, $f$ is a continuous function, $(x, y, z)^T \in \Omega \subset R^3$, and $\Omega$ is bounded, $\partial \Omega$ is smooth or piecewise smooth. For simplicity, we assume $u = u(x, y, z, t) \in C^1(\Omega \times (0, T))$. The initial condition and boundary condition are

$$u|_{t=0} \in L^2(\Omega), \ u|_{\partial\Omega \times (0, T)} \in L^2(\partial\Omega \times (0, T)). \quad (2.2)$$

At first we transform Eq.(2.1) into the linear equations on unknown functions as we have done in [2],

$$u_t - au - bu_x - cu_y - du_z - v = 0, \quad (2.3)$$

where $a, b, c, d$ are real constants to be determined,

$$v = f(u, u_x, u_y, u_z, x, y, z, t) - au - bu_x - cu_y - du_z. \quad (2.4)$$

Let's introduce

$$X = (x_1, x_2, x_3, x_4, x_5, x_6)^T, \text{ where}$$

$$x_1 = u_t, \ x_2 = u, \ x_3 = u_x, \ x_4 = u_y, \ x_5 = u_z, \ x_6 = v. \quad (2.5)$$

Then Eq.(2.1) is equivalent to

$$\alpha^T X = 0, \quad (2.6)$$

where $\alpha = (1, -a, -b, -c, -d, -1)^T$.

After we solve Eq.(2.6) by linear algebra, we obtain $\exists$ independent variable vector $Z$, such that

$$X = \begin{pmatrix} \beta^T \\ E \end{pmatrix} Z, \text{ where } \beta = (a, b, c, d, 1)^T. \quad (2.7)$$

We should discuss the independent variable vector $Z$ as follows,

$$u_t = \beta^T Z, \ u = e_1^T Z, \ u_x = e_2^T Z, \ u_y = e_3^T Z, \ u_z = e_4^T Z, \ v = e_5^T Z. \quad (2.8)$$

And we obtain Eq.(2.1) is equivalent to the following system respect to $Z$,

$$\frac{\partial e_1^T Z}{\partial t} = \beta^T Z, \quad (2.9)$$

$$\frac{\partial e_2^T Z}{\partial x} = e_2^T Z, \quad (2.10)$$

$$\frac{\partial e_3^T Z}{\partial y} = e_3^T Z, \quad (2.11)$$

$$\frac{\partial e_4^T Z}{\partial z} = e_4^T Z, \quad (2.12)$$

$$f(e_1^T Z, e_2^T Z, e_3^T Z, e_4^T Z, x, y, z, t) - ae_1^T Z - be_2^T Z - ce_3^T Z - de_4^T Z = e_5^T Z. \quad (2.13)$$

**Remark 2.1** In fact, we should assume

$$X = XI_{\Omega \times (0, T)}(x, y, z, t), \text{ or } Z = ZI_{\Omega \times (0, T)}(x, y, z, t), \quad (2.14)$$

if Eq.(2.1) is only satisfied on $\Omega \times (0, T)$. 

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We will solve Eq.(2.9) to Eq.(2.12) by the Fourier transform as follows. Then Eq.(2.13) is the goal.

\[\int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \frac{\partial e^T Z}{\partial t} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \beta^T Z dx dy dz, \tag{2.15}\]

\[\int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \frac{\partial e^T Z}{\partial x} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} e_2^T Z dx dy dz, \tag{2.16}\]

\[\int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \frac{\partial e^T Z}{\partial y} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} e_3^T Z dx dy dz, \tag{2.17}\]

\[\int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \frac{\partial e^T Z}{\partial z} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} e_4^T Z dx dy dz. \tag{2.18}\]

To denote easily, we define the Fourier transform on \(\Omega \times (0, T)\) as follows.

**Definition 2.1** \(\forall f(x, y, z, t) \in L^2(\Omega \times (0, T)),\)

\[FI(f(x, y, z, t)) = \int_0^T \int_\Omega f(x, y, z, t) e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dx dy dz dt = F(f(x, y, z, t)) I_{\Omega \times (0, T)}(x, y, z, t),\]

where \(F\) means the Fourier transform and \(I_{\Omega \times (0, T)}(x, y, z, t)\) is the characteristic function. In the following, we write \(I_{\Omega \times (0, T)}(x, y, z, t)\) into \(I_{\Omega \times (0, T)}\).

Then we obtain

\[FI\left(\frac{\partial e^T Z}{\partial t}\right) = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \left(\frac{\partial e^T Z}{\partial t}\right) dx dy dz = f_0 + i\xi_0 FI(e^T Z), \tag{2.21}\]

\[FI\left(\frac{\partial e^T Z}{\partial x}\right) = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \left(\frac{\partial e^T Z}{\partial x}\right) dx dy dz = f_1 + i\xi_1 FI(e^T Z), \tag{2.22}\]

\[FI\left(\frac{\partial e^T Z}{\partial y}\right) = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \left(\frac{\partial e^T Z}{\partial y}\right) dx dy dz = f_2 + i\xi_2 FI(e^T Z), \tag{2.23}\]

\[FI\left(\frac{\partial e^T Z}{\partial z}\right) = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix_1 - iy_2 - iz_3} \left(\frac{\partial e^T Z}{\partial z}\right) dx dy dz = f_3 + i\xi_3 FI(e^T Z). \tag{2.24}\]
If we assume then we obtain where

\[
FI(\frac{\partial e_i^T Z}{\partial z}) = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3}(\frac{\partial e_i^T Z}{\partial z}) dx dy dz
\]  
(2.33)

\[
= \int_0^T dt \int_\partial\Omega (e_i^T Z)n_3 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} dx dy dz +
\]
(2.34)

\[
i\xi_3 \int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} e_i^T Z) dx dy dz
\]  
(2.35)

\[
= f_3 + i\xi_3 FI(e_i^T Z),
\]  
(2.36)

where

\[
f_0 = \int_\Omega (A_2 e^{-iT\xi_0} - A_1) e^{-ix\xi_1 - iy\xi_2 - iz\xi_3} dx dy dz,
\]  
(2.37)

\[
f_1 = \int_0^T dt \int_\partial\Omega A_3 n_1 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} dS,
\]  
(2.38)

\[
f_2 = \int_0^T dt \int_\partial\Omega A_3 n_2 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} dS,
\]  
(2.39)

\[
f_3 = \int_0^T dt \int_\partial\Omega A_3 n_3 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} dS.
\]  
(2.40)

\[
A_1 = u|_{t=0}, \ A_2 = u|_{t=T}, \ A_3 = u|_{\partial\Omega \times (0, T)},
\]  
(2.41)

\[n_k\] is the \(k\)th component of the normal vector to \(\partial\Omega\), \(k = 1, 2, 3\). We only need

\[A_1, \ A_2 \in L^2(\Omega), \ A_3 \in L^2(\partial\Omega \times (0, T)).\]

Now we transformed Eq.(2.9) to Eq.(2.12) into the following.

\[BFI(Z) = \beta_1,\]

(2.43)

where

\[B = \begin{pmatrix}
i\xi_0 e_1^T - \beta^T \\
i\xi_1 e_1^T - e_2^T \\
i\xi_2 e_1^T - e_3^T \\
i\xi_3 e_1^T - e_4^T
\end{pmatrix}_{4 \times 5} = (B_1, -B_2),\]

(2.44)

\[B_1 = \begin{pmatrix}
i\xi_0-a, & -b, & -c, & -d \\
i\xi_1, & -1, & 0, & 0 \\
i\xi_2, & 0, & -1, & 0 \\
i\xi_3, & 0, & 0, & -1
\end{pmatrix}_{4 \times 4},\]

(2.45)

\[B_2 = (1, 0, 0, 0)^T, \ \beta_1 = (-f_0, -f_1, -f_2, -f_3)^T.\]

(2.46)

If we assume

\[Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \text{ where } Z_1 \text{ is the first 4 components of } Z,\]

(2.47)

then we obtain

\[B_1 FI(Z_1) = \beta_1 + B_2 FI(Z_2).\]

(2.48)
It is not very difficult to work out

\[
det(B_1) = a + bi\xi_1 + ci\xi_2 + di\xi_3 - i\xi_0 = a_1.
\]

(2.49)

\[
B_1^{-1} = \begin{pmatrix}
    a_1^{-1}, & a_1^{-1}b, & a_1^{-1}c, & a_1^{-1}d \\
    i\xi_1a_1^{-1}, & 1 + i\xi_1a_1^{-1}b, & i\xi_1a_1^{-1}c, & i\xi_1a_1^{-1}d \\
    i\xi_2a_1^{-1}, & i\xi_2a_1^{-1}b, & 1 + i\xi_2a_1^{-1}c, & i\xi_2a_1^{-1}d \\
    i\xi_3a_1^{-1}, & i\xi_3a_1^{-1}b, & i\xi_3a_1^{-1}c, & 1 + i\xi_3a_1^{-1}d
\end{pmatrix},
B_1^{-1}B_2 = -\begin{pmatrix}
    a_1^{-1} \\
    i\xi_1a_1^{-1} \\
    i\xi_2a_1^{-1} \\
    i\xi_3a_1^{-1}
\end{pmatrix}.
\]

(2.50)

If we assume \( C = \{\xi'|a_1 = 0\} \), where \( \xi' = (\xi_0, \xi_1, \xi_2, \xi_3)^T \), then the measure of \( C \) is 0. And we obtain

\[
FI(Z_1)(1 - I_C(\xi')) = B_1^{-1}\beta_1(1 - I_C(\xi')) + B_1^{-1}B_2FI(Z_2)(1 - I_C(\xi')).
\]

(2.51)

We need some lemmas.

**Lemma 2.1** (Plancherel Theorem) If \( f(x, y, z, t) \in L^2(R^4) \), then \( F(f(x, y, z, t)) \) exists, moreover

1. \( \|F(f(x, y, z, t))\|_{L_2} = \|f(x, y, z, t)\|_{L_2} \),
2. \( F^{-1}[F(f(x, y, z, t))] = f(x, y, z, t) \).

(2.52)

**Proof of lemma 2.2.** From the lemma 2.1, we know \( F(f(x, y, z, t)) \in L^2(R^4) \). Therefore,

\[
\int_C F(f(x, y, z, t))e^{it\xi_0 + ix\xi_1 + iy\xi_2 + iz\xi_3}d\xi_0d\xi_1d\xi_2d\xi_3 = 0.
\]

(2.53)

And we obtain

\[
F^{-1}([F(f(x, y, z, t))](1 - I_C(\xi'))) = F^{-1}[F(f(x, y, z, t))] = f(x, y, z, t).
\]

(2.54)

From these two lemmas, we obtain

\[
F^{-1}[FI(Z_1)(1 - I_C(\xi'))] = Z_1I_{\Omega \times (0, T)},
F^{-1}[FI(Z_2)(1 - I_C(\xi'))] = Z_2I_{\Omega \times (0, T)}.
\]

(2.55)

Now we determine the parameters. We choose \( a, b, c, d \), such that \( F^{-1}(a_1^{-1}) \) exists. Then \( F^{-1}(B_1^{-1}B_2) \) exists. And \( F^{-1}(B_1^{-1}\beta_1(1 - I_C(\xi'))) \) exists.

If we assume

\[
w_1(x, y, z, t) = F^{-1}[B_1^{-1}\beta_1(1 - I_C(\xi'))],
w_2(x, y, z, t) = F^{-1}(B_1^{-1}B_2),
\]

(2.56)


then we obtain

\[
Z_1I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) \ast (Z_2I_{\Omega \times (0, T)}),
\]

(2.57)

where

\[
Z_1 = (e_1^T Z, e_2^T Z, e_3^T Z, e_4^T Z)^T = (u, u_x, u_y, u_z)^T,
\]

(2.58)
\[ Z_2 = e_5^T Z = f(e_1^T Z, e_2^T Z, e_3^T Z, e_4^T Z, x, y, z, t) - ae_1^T Z - be_2^T Z - ce_3^T Z - de_4^T Z = v. \] (2.59)

It is obvious \( \exists \psi \), such that \( Z_2 = \psi(Z_1) \). Therefore, we obtain
\[ Z_1 I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) * (\psi(Z_1) I_{\Omega \times (0, T)}). \] (2.60)

If \( Z_1 \) satisfied Eq.(2.60), then we let \( Z_2 = \psi(Z_1) \). We obtain \( BFI(Z) = \beta_1 \), \( \alpha^TX = 0 \), on \( \Omega \times (0, T) \). Therefore, \( e_1^T Z \) is the solution of Eq.(2.1) on \( \Omega \times (0, T) \). Hence we arrive at

**Theorem 2.1** \( w_1, w_2, \psi, \) as we described, then Eq.(2.1) is equivalent to Eq.(2.60).

Maybe you will say we should know \( A_2 = u|_{t=T} \). In fact, we shouldn’t. If we choose the parameter \( a < 0 \), then in \( F^{-1}(a_1^{-1}f_0) \), we obtain
\[
\int_{-\infty}^{+\infty} \frac{e^{-iT\xi_0}a_3}{\xi_0 + a_2} e^{it\xi_0} d\xi_0 = e^{-a_2(t-T)}a_3 I_{\{t\geq T\}},
\] (2.61)

where
\[ a_2 = -(a + bi\xi_1 + ci\xi_2 + di\xi_3), \quad a_3 = \int_{\Omega} A_2 e^{-i\xi_1-y\xi_2-iz\xi_3} dxdydz. \] (2.62)

\( I_{\{t\geq T\}} \) means \( A_2 \) doesn’t work on \( \Omega \times (0, T) \). If we only discuss \( u \) on \( \Omega \times (0, T) \), then we only need to know \( A_1 = u|_{t=0}, A_3 = u|_{\partial\Omega \times (0, T)} \) in Eq.(2.60). Hence we take \( f_0 \) as
\[ f_0 = \int_{\Omega} (-A_1) e^{-i\xi_1-iy\xi_2-iz\xi_3} dxdydz. \] (2.63)

Maybe you will also say Eq.(2.60) is the kind of Hammerstain. We take it at beginning, but we change the mind after we obtain \( w_2(x, y, z, t) \) with \( I_{\{t\geq 0\}} \).

**3 The equations**

In this section, we discuss the first order partial differential equations resolved with respect to time derivatives as follows,
\[ u_t = f(u, u_x, u_y, u_z, x, y, z, t), \] (3.1)

where \( t \in [0, T] \), \( f = (f_1, \ldots, f_m)^T \) is continuous, \( (x, y, z)^T \in \Omega \subset \mathbb{R}^3 \), and \( \Omega \) is bounded, \( \partial \Omega \) is smooth or piecewise smooth. For simplicity, we assume
\[ u = (u_1(x, y, z, t), \ldots, u_m(x, y, z, t))^T \in C^1(\Omega \times (0, T)). \] (3.2)

The initial conditions and boundary conditions are
\[ u|_{t=0} \in L^2(\Omega), \quad u|_{\partial\Omega \times (0, T)} \in L^2(\partial\Omega \times (0, T)). \] (3.3)

We transform Eq.(3.1) into the linear equations on unknown functions as follows,
\[ u_t - Au - Bu_x - Cu_y - Du_z - v = 0, \] (3.4)
where $A$, $B$, $C$, $D$ are real constants $m \times m$ matrices to be determined,

$$v = f(u, u_x, u_y, u_z, x, y, z, t) - Au - Bu_x - Cu_y - Du_z. \quad (3.5)$$

Let’s introduce

$$X = (x_1^T, x_2^T, x_3^T, x_4^T, x_5^T, x_6^T)^T,$$

$$x_1 = u_t, \ x_2 = u, \ x_3 = u_x, \ x_4 = u_y, \ x_5 = u_z, \ x_6 = v. \quad (3.6)$$

Then Eq.(3.14) is equivalent to

$$\alpha^T X = 0, \quad (3.7)$$

where $\alpha^T = (E, -A, -B, -C, -D, -E)$.

After we solve Eq.(3.7) by linear algebra, we obtain $\exists$ independent variable vector $Z$, such that

$$X = \left( \begin{array}{c} \beta^T \\ E_{5m} \end{array} \right) Z,$$

$$\beta^T = (A, B, C, D, E). \quad (3.8)$$

We should discuss the independent variable vector $Z$ as follows,

$$u_t = \beta^T Z, \ u = E_1^T Z, \ u_x = E_2^T Z, \ u_y = E_3^T Z, \ u_z = E_4^T Z, \ v = E_5^T Z, \quad (3.9)$$

where $E_j = (e_{m(j-1)+1}, e_{m(j-1)+2}, \ldots, e_{mj}), \ 1 \leq j \leq 5. \quad (3.10)$

And we obtain Eq.(3.4) is equivalent to the following system respect to $Z$,

$$\frac{\partial E_1^T Z}{\partial t} = \beta^T Z, \quad (3.11)$$

$$\frac{\partial E_1^T Z}{\partial x} = E_2^T Z, \quad (3.12)$$

$$\frac{\partial E_1^T Z}{\partial y} = E_3^T Z, \quad (3.13)$$

$$\frac{\partial E_1^T Z}{\partial z} = E_4^T Z, \quad (3.14)$$

$$f(E_1^T Z, E_2^T Z, E_3^T Z, E_4^T Z, x, y, z, t) - AE_1^T Z - BE_2^T Z - CE_3^T Z - DE_4^T Z = E_5^T Z. \quad (3.15)$$

We will solve Eq.(3.11) to Eq.(3.14) by the Fourier transform as follows. Then Eq.(3.15) is the goal.

$$\int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial E_1^T Z}{\partial t} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \beta^T Z dx dy dz, \quad (3.16)$$

$$\int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial E_1^T Z}{\partial x} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} E_2^T Z dx dy dz, \quad (3.17)$$

$$\int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial E_1^T Z}{\partial y} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} E_3^T Z dx dy dz, \quad (3.18)$$

$$\int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial E_1^T Z}{\partial z} dx dy dz = \int_0^T dt \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} E_4^T Z dx dy dz. \quad (3.19)$$
By using the definition 2.1, we obtain

\[
FI(\frac{\partial E^T Z}{\partial t}) = \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (\frac{\partial E^T Z}{\partial t}) dxdydz \tag{3.20}
\]

\[
= \int_{\Omega} (E^T Z)e^{-it\xi_0}_{|t=0} e^{-ix_1 - iy_2 - iz_3} dxdydz + \tag{3.21}
\]

\[
i\xi_0 \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (E^T Z) dxdydz \tag{3.22}
\]

\[
= f_0 + i\xi_0 FI(E^T Z), \tag{3.23}
\]

\[
FI(\frac{\partial E^T Z}{\partial x}) = \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (\frac{\partial E^T Z}{\partial x}) dxdydz \tag{3.24}
\]

\[
= \int_{\Omega} \int_0^T (E^T Z)n_1 e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dxdydz + \tag{3.25}
\]

\[
i\xi_1 \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (E^T Z) dxdydz \tag{3.26}
\]

\[
= f_1 + i\xi_1 FI(E^T Z), \tag{3.27}
\]

\[
FI(\frac{\partial E^T Z}{\partial y}) = \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (\frac{\partial E^T Z}{\partial y}) dxdydz \tag{3.28}
\]

\[
= \int_{\Omega} \int_0^T (E^T Z)n_2 e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dxdydz + \tag{3.29}
\]

\[
i\xi_2 \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (E^T Z) dxdydz \tag{3.30}
\]

\[
= f_2 + i\xi_2 FI(E^T Z), \tag{3.31}
\]

\[
FI(\frac{\partial E^T Z}{\partial z}) = \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (\frac{\partial E^T Z}{\partial z}) dxdydz \tag{3.32}
\]

\[
= \int_{\Omega} \int_0^T (E^T Z)n_3 e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dxdydz + \tag{3.33}
\]

\[
i\xi_3 \int_0^T dt \int_{\Omega} e^{-it\xi_0 - ix_1 - iy_2 - iz_3} (E^T Z) dxdydz \tag{3.34}
\]

\[
= f_3 + i\xi_3 FI(E^T Z), \tag{3.35}
\]

where

\[
f_0 = \int_{\Omega} (A_2 e^{-iT\xi_0} - A_1) e^{-ix_1 - iy_2 - iz_3} dxdydz, \tag{3.36}
\]

\[
f_1 = \int_0^T dt \int_{\partial\Omega} A_3 n_1 e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dS, \tag{3.37}
\]

\[
f_2 = \int_0^T dt \int_{\partial\Omega} A_3 n_2 e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dS, \tag{3.38}
\]

\[
f_3 = \int_0^T dt \int_{\partial\Omega} A_3 n_3 e^{-it\xi_0 - ix_1 - iy_2 - iz_3} dS, \tag{3.39}
\]

\[
A_1 = u|_{t=0}, A_2 = u|_{t=T}, A_3 = u|_{\partial\Omega \times (0, T)}, \tag{3.40}
\]
$n_k$ is the $k$th component of the normal vector to $\partial \Omega$, $k = 1, 2, 3$. We only need

$$A_1, A_2 \in L^2(\Omega), A_3 \in L^2(\partial \Omega \times (0, T)).$$

(3.41)

Now we transformed the equations Eq.(3.11) to Eq.(3.14) into the following.

$$BFI(Z) = \beta_1,$$

(3.42)

where

$$B = \begin{pmatrix} i \xi_0 E^T - \beta^T \\ i \xi_1 E^T - E_1^T \\ i \xi_2 E^T - E_2^T \\ i \xi_3 E^T - E_3^T \end{pmatrix} = (B_1, -B_2),$$

(3.43)

$$B_1 = \begin{pmatrix} i \xi_0 E - A, & -B, & -C, & -D \\ i \xi_1 E, & -E, & 0, & 0 \\ i \xi_2 E, & 0, & -E, & 0 \\ i \xi_3 E, & 0, & 0, & -E \end{pmatrix}_{4m \times 4m},$$

(3.44)

$$B_2 = (E, 0, 0, 0)^T, \beta_1 = (-f_0^T, -f_1^T, -f_2^T, -f_3^T)^T.$$  

(3.45)

If we assume

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where $Z_1$ is the first $4m$ components of $Z$,

(3.46)

then we obtain

$$B_1FI(Z_1) = \beta_1 + B_2FI(Z_2).$$

(3.47)

It is not very difficult to work out

$$det(B_1) = det(A + i \xi_1 B + i \xi_2 C + i \xi_3 D - i \xi_0 E) = det(B_0),$$

(3.48)

where

$$B_0 = A + i \xi_1 B + i \xi_2 C + i \xi_3 D - i \xi_0 E.$$  

(3.49)

If we assume $C_0 = \{\xi'|det(B_0) = 0\}$, where $\xi' = (\xi_0, \xi_1, \xi_2, \xi_3)^T$, then the measure of $C_0$ is 0. And we obtain

$$FI(Z_1)(1 - I_{C_0}(\xi')) = B_1^{-1}\beta_1(1 - I_{C_0}(\xi')) + B_1^{-1}B_2FI(Z_2)(1 - I_{C_0}(\xi')).$$

(3.51)

From two lemmas in last section, we also obtain

$$F^{-1}[FI(Z_1)(1 - I_{C_0}(\xi'))] = Z_1I_{\Omega \times (0, T)}, F^{-1}[FI(Z_2)(1 - I_{C_0}(\xi'))] = Z_2I_{\Omega \times (0, T)}.$$ 

(3.52)
Now we determine the parameters. We choose $A$, $B$, $C$, $D$, such that $F^{-1}(B_0^{-1})$ exists. Then $F^{-1}(B_1^{-1}B_2)$ exists. And $F^{-1}[B_1^{-1}\beta_1(1 - I_{G_0}(\xi))]$ exists.

If we assume

$$w_1(x, y, z, t) = F^{-1}[B_1^{-1}\beta_1(1 - I_{G_0}(\xi))], \quad w_2(x, y, z, t) = F^{-1}(B_1^{-1}B_2),$$

then we obtain

$$Z_1I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) \ast (Z_2I_{\Omega \times (0, T)}),$$

where

$$Z_1 = (E_1^T Z, E_2^T Z, E_3^T Z, E_4^T Z)^T = (u^T, u_2^T, u_3^T, u_4^T)^T,$$

$$Z_2 = E_3^T Z = f(E_1^T Z, E_2^T Z, E_3^T Z, E_4^T Z, x, y, z, t) - AE_1^T Z - BE_2^T Z - CE_3^T Z - DE_4^T Z = v.$$  

It is obvious $\exists \psi$, such that $Z_2 = \psi(Z_1)$. Therefore, we obtain

$$Z_1I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) \ast (\psi(Z_1)I_{\Omega \times (0, T)}).$$

If $Z_1$ satisfied Eq.(3.58), then we let $Z_2 = \psi(Z_1)$. We obtain $BFI(Z) = \beta_1$, $a^T X = 0$, on $\Omega \times (0, T)$. Therefore, $E_1^T Z$ is the solution of Eq.(3.1) on $\Omega \times (0, T)$. Hence we arrive at

**Theorem 3.1** $w_1$, $w_2$, $\psi$, as we described, then Eq.(3.1) is equivalent to Eq.(3.58).

Maybe you will say this time we should know $A_2 = u|_{t=T}$. In fact, we also shouldn’t. If we choose the parameter matrix $A = aE$, $a < 0$, $B = bE$, $C = cE$, $D = dE$, $b$, $c$, $d \in R$, then in $F^{-1}(B_0^{-1}I)$, we obtain

$$\int_{-\infty}^{+\infty} (i\xi_0 E + a_2)^{-1} e^{-i\xi_0 a_3} e^{it\xi_0} d\xi_0 = e^{-a_2(t-T)} a_3 I_{(t \geq T)},$$

where

$$a_2 = -(A + i\xi_1 B + i\xi_2 C + i\xi_3 D), \quad a_3 = \int_{\Omega} A_2 e^{-ix_1 i\xi_1 - iy_2 i\xi_2 - iz_3 i\xi_3} dxdydz.$$  

$I_{(t \geq T)}$ means $A_2$ doesn’t work on $\Omega \times (0, T)$. If we only discuss $u$ on $\Omega \times (0, T)$, then we only need to know $A_1 = u|_{t=0}$, $A_3 = u|_{\beta \Omega \times (0, T)}$ in Eq.(3.58). Hence we take $f_0$ as

$$f_0 = \int_{\Omega} (-A_1 e^{-ix_1 i\xi_1 - iy_2 i\xi_2 - iz_3 i\xi_3} dxdydz.$$  

**Remark 3.1** Actually, the choice of the parameter matrix $A = aE$, $a < 0$, $B = bE$, $C = cE$, $D = dE$, $b$, $c$, $d \in R$, is not unique. We only need to choose $A$, $B$, $C$, $D$ which satisfy $Re(\lambda) > 0$, where $\lambda$ is the characteristic value of matrix $a_2$. We assume the following,

$$a_2 = PJP^{-1}, \quad J = \begin{pmatrix} J_1 & \cdots & \\
\cdots & \\
J_\sigma & \cdots & \end{pmatrix},\quad J_k = \begin{pmatrix} \lambda_k & 1 & \cdots & \\
\cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & 1 \\
\cdots & \cdots & \cdots & \lambda_k \end{pmatrix}_{m_k \times m_k}, \quad 1 \leq k \leq \sigma.$$  

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where $P$ is not related to $i\xi_0$, $\text{Re}(\lambda_k) > 0$, $1 \leq k \leq \sigma$. Then we obtain

$$
(i\xi_0 E + a_2)^{-1} = P(i\xi_0 E + J)^{-1} P^{-1},
$$

(3.63)

$$
(i\xi_0 E + J)^{-1} = \begin{pmatrix}
(i\xi_0 E_1 + J_1)^{-1} & \cdots & \\
\vdots & \ddots & \\
& & (i\xi_0 E_\sigma + J_\sigma)^{-1}
\end{pmatrix},
$$

(3.64)

$$
(i\xi_0 E_k + J_k)^{-1} = 
\begin{pmatrix}
(i\xi_0 + \lambda_k)^{-1}, & -(i\xi_0 + \lambda_k)^{-2}, & (i\xi_0 + \lambda_k)^{-3}, & \cdots, & (-1)^{m_k-1}(i\xi_0 + \lambda_k)^{-m_k}, \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
& & & & & (i\xi_0 + \lambda_k)^{-3} \\
& & & & & (i\xi_0 + \lambda_k)^{-2} \\
& & & & & (i\xi_0 + \lambda_k)^{-1}
\end{pmatrix}_{m_k \times m_k},
$$

(3.65)

(3.66)

(3.67)

Therefore, we obtain

$$
\int_{-\infty}^{+\infty} (i\xi_0 + \lambda_k)^{-1} e^{it\xi_0} d\xi_0 = e^{-\lambda_k t} I_{\{t \geq 0\}}, \quad \int_{-\infty}^{+\infty} (i\xi_0 + \lambda_k)^{-n} e^{it\xi_0} d\xi_0 = \frac{e^{-\lambda_k t} t^{n-1}}{(n-1)!} I_{\{t \geq 0\}}, \quad n \geq 1.
$$

(3.68)

Hence (3.59) stands.

4 General scenario

In this section, we will consider the general first order partial differential equations. But what the general scenario should be? Is it the scenario as follows,

$$
f_j(u_t, u, u_x, u_y, u_z, x, y, z, t) = 0, \quad 1 \leq j \leq m,
$$

(4.1)

where $t \in [0, T]$, $(x, y, z)^T \in \Omega \subset \mathbb{R}^3$, and $\Omega$ is bounded, $\partial \Omega$ is smooth or piecewise smooth,

$$
u = (u_1(x, y, z, t), \cdots, u_m(x, y, z, t))^T \in C^1(\Omega \times (0, T)),
$$

(4.2)

the initial conditions and boundary conditions are

$$
u|_{t=0} \in L^2(\Omega), \quad \nu|_{\partial \Omega \times (0, T)} \in L^2(\partial \Omega \times (0, T)),
$$

(4.3)

$f_j$, $1 \leq j \leq m$, are continuous functions? Of course it isn’t. We should solve Eq.(4.1) for $m$ components of $u_t$, $u$, $u_x$, $u_y$, $u_z$.

Let’s consider an example of the second order equation at first,

$$
u_t = f(u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}, u_{yy}, u_{yz}, u_{zz}, x, y, z, t).
$$

(4.4)
We should transform it into the first order equations as follows,

\[ u_t = f(u, u_x, u_y, u_z, v_x, v_y, v_z, w_x, w_y, w_z, r, x, y, z, t), \quad (4.5) \]
\[ v = u_x, \quad (4.6) \]
\[ w = u_y, \quad (4.7) \]
\[ r = u_z. \quad (4.8) \]

We notice that it is not always the derivatives in the left hand sides. It should be resolved with any derivatives in the left hand side, including the components of \( u \). Hence the general first order partial differential equations should be like the following,

\[ v_j = f_j(v_{m+1}, v_{m+2}, \cdots, v_{5m}, x, y, z, t), \quad 1 \leq j \leq m, \quad (4.9) \]

where \( t \in [0, T] \), \((x, y, z)^T \in \Omega \subset \mathbb{R}^3 \), and \( \Omega \) is bounded, \( \partial\Omega \) is smooth or piecewise smooth,

\[ u = (u_1(x, y, z, t), \cdots, u_m(x, y, z, t))^T \in C^1(\Omega \times (0, T)). \quad (4.10) \]

The initial conditions and boundary conditions are

\[ u|_{t=0} \in L^2(\Omega), \quad u|_{\partial\Omega \times (0, T)} \in L^2(\partial\Omega \times (0, T)). \quad (4.11) \]

\( f_j, \ 1 \leq j \leq m, \) are continuous functions, \( v_1, v_2, \cdots, v_{5m} \) is a permutation of the components of \( u_t, u, u_x, u_y, u_z \).

We assume there are \( r \) components of \( u \) in the beginning of the right hand sides of the Eq.(4.9)

\[ v_{m+1}, v_{m+2}, \cdots, v_{m+r}, \ 0 \leq r \leq m. \quad (4.12) \]

And there are \( 4m - r \) derivatives of \( u \) in the right hand sides of the Eq.(4.9). This point is very important in the following.

We transform Eq.(4.9) into the linear equations on unknown functions as follows,

\[ v_j - \sum_{k=1}^{4m} c_{jk} v_{m+k} - s_j = 0, \ 1 \leq j \leq m, \quad (4.13) \]

where \( c_{jk}, \ 1 \leq j \leq m, \ 1 \leq k \leq 4m, \) are all real constants to be determined,

\[ s_j = f_j(v_{m+1}, v_{m+2}, \cdots, v_{5m}, x, y, z, t) - \sum_{k=1}^{4m} c_{jk} v_{m+k}, \ 1 \leq j \leq m. \quad (4.14) \]

Let’s introduce \( X = (V_1^T, V_2^T, V_3^T, V_4^T, V_5^T, S^T)^T \), where

\[ V_i = (v_{m(i-1)+1}, v_{m(i-1)+2}, \cdots, v_{mi})^T, \ 1 \leq i \leq 5, \quad S = (s_1, s_2, \cdots, s_5)^T. \quad (4.15) \]

Then Eq.(4.13) is equivalent to

\[ \alpha^T X = 0, \quad (4.16) \]

where

\[ \alpha^T = (E, -C_1, -C_2, -C_3, -C_4, -E), \quad C_i = (c_{j,(m(i-1)+k)})_{m \times m}, \ 1 \leq i \leq 4. \quad (4.17) \]
After we solve Eq.(4.16) by linear algebra, we obtain $\exists$ independent variable vector $Z$, such that

$$X = \left( \begin{array}{c} \beta^T \\ E_{5m} \end{array} \right) Z, \text{ where } \beta^T = (C_1, C_2, C_3, C_4, E). \tag{4.18}$$

We should discuss the independent variable vector $Z$ as follows,

$$u_t = A_{00}Z, \ u = A_0Z, \ u_x = A_{01}Z, \ u_y = A_{02}Z, \ u_z = A_{03}Z, \ S = E_5^T Z, \tag{4.19}$$

where $E_j = (e_{m(j-1)+1}, e_{m(j-1)+2}, \ldots, e_{mj}), \ 1 \leq j \leq 5$, and the rows of $A_{00}, A_0, A_{01}, A_{02}, A_{03}$ are a permutation of the rows of $\beta^T, E_1^T, E_2^T, E_3^T, E_4^T$.

And we obtain Eq.(4.9) is equivalent to the following system respect to $Z$,

$$\frac{\partial A_0 Z}{\partial t} = A_{00} Z, \tag{4.20}$$
$$\frac{\partial A_0 Z}{\partial x} = A_{01} Z, \tag{4.21}$$
$$\frac{\partial A_0 Z}{\partial y} = A_{02} Z, \tag{4.22}$$
$$\frac{\partial A_0 Z}{\partial z} = A_{03} Z, \tag{4.23}$$

$$f(E_1^TZ, E_2^T Z, E_3^T Z, E_4^T Z, x, y, z, t) - C_1 E_1^T Z - C_2 E_2^T Z - C_3 E_3^T Z - C_4 E_4^T Z = E_5^T Z. \tag{4.24}$$

We will also solve Eq.(4.20) to Eq.(4.23) by the Fourier transform as follows. Then Eq.(4.24) is the goal.

$$\int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial A_0 Z}{\partial t} dxdydz = \int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} A_{00} Z dxdydz, \tag{4.25}$$
$$\int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial A_0 Z}{\partial x} dxdydz = \int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} A_{01} Z dxdydz, \tag{4.26}$$
$$\int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial A_0 Z}{\partial y} dxdydz = \int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} A_{02} Z dxdydz, \tag{4.27}$$
$$\int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \frac{\partial A_0 Z}{\partial z} dxdydz = \int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} A_{03} Z dxdydz. \tag{4.28}$$

By using the definition 2.1, we obtain

$$FI\left( \frac{\partial A_0 Z}{\partial t} \right) = \int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \left( \frac{\partial A_0 Z}{\partial t} \right) dxdydz \tag{4.29}$$
$$= \int (A_0 Z) e^{-it\xi_0} \bigg|_{t=0}^T e^{-ix\xi_1 - iy\xi_2 - iz\xi_3} dxdydz + \int i\xi_0 \int_0^T \int_\Omega e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} A_0 Z dxdydz \tag{4.30}$$
$$= g_0 + i\xi_0 FI(A_0 Z), \tag{4.31}$$

$$= g_0 + i\xi_0 FI(A_0 Z), \tag{4.32}$$

\[4\]
where

\[ g_0 = \int_{\Omega} (A_2 e^{-it\xi_0} - A_1) e^{-ix\xi_1 - iy\xi_2 - iz\xi_3} \, dx dy dz, \]  
(4.45)

\[ g_1 = \int_{\partial\Omega} dt \int_{\partial\Omega} A_3 n_1 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \, dS, \]  
(4.46)

\[ g_2 = \int_{\partial\Omega} dt \int_{\partial\Omega} A_3 n_2 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \, dS, \]  
(4.47)

\[ g_3 = \int_{\partial\Omega} dt \int_{\partial\Omega} A_3 n_3 e^{-it\xi_0 - ix\xi_1 - iy\xi_2 - iz\xi_3} \, dS. \]  
(4.48)

\[ A_1 = u|_{t=0}, \quad A_2 = u|_{t=T}, \quad A_3 = u|_{\partial\Omega \times (0, T)}, \]  
(4.49)

\[ n_k \text{ is the } k\text{th component of the normal vector to } \partial\Omega, \quad k = 1, 2, 3. \]  

We only need

\[ A_1, \ A_2 \in L^2(\Omega), \ A_3 \in L^2(\partial\Omega \times (0, T)). \]  
(4.50)

Now we transformed the equations Eq.(4.20) to Eq.(4.23) into the following.

\[ BFI(Z) = \beta_1, \]  
(4.51)

where

\[ B = \begin{pmatrix} i\xi_0 A_0 - A_{00} \\ i\xi_1 A_0 - A_{01} \\ i\xi_2 A_0 - A_{02} \\ i\xi_3 A_0 - A_{03} \end{pmatrix}_{4 \times 5} = (B_1, -B_2), \quad \beta_1 = (-g_0^T, -g_1^T, -g_2^T, -g_3^T)^T, \]  
(4.52)
If we assume then we obtain

\[ v_{m+1}, v_{m+2}, \ldots, v_{m+r}, \ 0 \leq r \leq m. \]  

(4.53)

There are \(4m - r\) derivatives of \(u\) in the right hand sides of the Eq.(4.9). Hence there exists \(P\) is a permutation matrix, such that

\[
P \begin{pmatrix} A_{00} \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix}_{B_1} = \begin{pmatrix} C_{r1} & C_{r2} \\ 0 & E_{4m-r} \end{pmatrix}, \text{ where } \begin{pmatrix} A_{00} \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix}_{B_1} \text{ is the first } 4m \text{ columns of } \begin{pmatrix} A_{00} \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix}.\]

(4.54)

Therefore, we obtain

\[
P B_1 = \begin{pmatrix} A_{r1} - C_{r1} & A_{r2} - C_{r2} \\ A_{r3} & A_{r4} - E_{4m-r} \end{pmatrix},\]

where \(A_{r1}, A_{r2}, A_{r3}, A_{r4}\) are related with \(i\xi_0, i\xi_1, i\xi_2, i\xi_3\), the elements in them are not constants except 0. If \(i\xi_0 = i\xi_1 = i\xi_2 = i\xi_3 = 0\), then \(A_{r4} = 0\). Hence \(\det(A_{r4} - E_{4m-r})\) is not always 0. And \(A_{r4} - E_{4m-r}\) is convertible. We obtain

\[
\det(PB_1) = \det(A_{r1} - C_{r1} - (A_{r2} - C_{r2})(A_{r4} - E_{4m-r})^{-1}A_{r3})\det(A_{r4} - E_{4m-r}).\]

(4.56)

Now we determine the parameters. We choose \(C_1, C_2, C_3, C_4\), such that \(\det(PB_1) \neq 0\). Hence \(\det(B_1) \neq 0\), \(F^{-1}(B_1^{-1})\) and \(F^{-1}(B_1^{-1}B_2)\) exists.

Now we assume

\[
Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \text{ where } Z_1 \text{ is the first } 4m \text{ components of } Z,\]

(4.57)

then we obtain

\[
B_1 FI(Z_1) = \beta_1 + B_2 FI(Z_2).\]

(4.58)

If we assume \(C_0 = \{\xi' | \det(B_1) = 0\}\), where \(\xi' = (\xi_0, \xi_1, \xi_2, \xi_3)^T\), then the measure of \(C_0\) is 0. And we obtain

\[
FI(Z_1)(1 - I_{C_0} (\xi')) = B_1^{-1} \beta_1 (1 - I_{C_0} (\xi')) + B_1^{-1} B_2 FI(Z_2)(1 - I_{C_0} (\xi')).\]

(4.59)

From two lemmas in second section, we also obtain

\[
F^{-1}[FI(Z_1)(1 - I_{C_0}(\xi'))] = Z_1 I_{\Omega \times (0, T)}, \quad F^{-1}[FI(Z_2)(1 - I_{C_0}(\xi'))] = Z_2 I_{\Omega \times (0, T)}.
\]

(4.60)

If we assume

\[
w_1(x, y, z, t) = F^{-1}[B_1^{-1} \beta_1 (1 - I_{C_0}(\xi'))], \quad w_2(x, y, z, t) = F^{-1}(B_1^{-1} B_2),\]

(4.61)

then we obtain

\[
Z_1 I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) * (Z_2 I_{\Omega \times (0, T)}),\]

(4.62)
where

\[
\begin{align*}
Z_1 &= (E_1^T Z, E_2^T Z, E_3^T Z, E_4^T Z)^T = (V_1^T, V_2^T, V_3^T, V_4^T)^T, \\
Z_2 &= E_5^T Z = f(E_1^T Z, E_2^T Z, E_3^T Z, E_4^T Z, x, y, z, t) - \\
& \quad C_1 E_1^T Z - C_2 E_2^T Z - C_3 E_3^T Z - C_4 E_4^T Z = S.
\end{align*}
\] (4.63, 4.64, 4.65)

It is obvious \( \exists \psi \), such that \( Z_2 = \psi(Z_1) \). Therefore, we obtain

\[
Z_1 I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) * (\psi(Z_1)) I_{\Omega \times (0, T)}.
\] (4.66)

If \( Z_1 \) satisfied Eq.(4.66), then we let \( Z_2 = \psi(Z_1) \). We obtain \( BFI(Z) = \beta_1 \), \( \alpha^T X = 0 \), on \( \Omega \times (0, T) \). Therefore, \( A_0 Z \) is the solution of Eq.(4.9) on \( \Omega \times (0, T) \). Hence we arrive at

**Theorem 4.1** \( w_1, w_2, \psi \), as we described, then Eq.(4.9) is equivalent to Eq.(4.66).

At this time, we will also show that we shouldn’t know \( A_2 = u|_{t=T} \). We can get it by

\[
\int_{-\infty}^{+\infty} e^{-it\xi_0} e^{it\xi_0} dt = \delta(t-T),
\] (4.67)
\[
F^{-1}(B_1^{-1} e^{-it\xi_0}) = (\varphi(x, y, z, t) I_{\{t\geq 0\}}) * \delta(t-T) = \varphi(x, y, z, t-T) I_{\{t\geq T\}}.
\] (4.68)

We only need to prove

\[
F^{-1}(B_1^{-1}) = F^{-1}(B_1^{-1}) I_{\{t\geq 0\}}.
\] (4.69)

We can transform \( B_1 \) into the following by the primary row transformations on the rows from \((m+1)th\) to \((4m)th\) and some transpositions of the columns,

\[
\left(\begin{array}{cccc}
i\xi_0 A_{11} - B_{11}, & i\xi_0 A_{12} - B_{12}, & i\xi_0 A_{13} - B_{13}, & i\xi_0 A_{14} - B_{14}, \\
C_{21} & E & 0 & 0 \\
C_{31} & 0 & E & 0 \\
C_{41} & 0 & 0 & E
\end{array}\right),
\] (4.71)

where \( C_{21}, C_{31}, C_{41} \) are not related to \( i\xi_0 \), \( A_{1j}, B_{1j}, 1 \leq j \leq 4 \), are all constant matrices. Hence we obtain

\[
det(B_1) = det(i\xi_0 A_{05} - B_{05}) \phi(i\xi_1, i\xi_2, i\xi_3),
\] (4.72)

where \( A_{05} = A_{11} - \sum_{j=2}^{4} A_{1j} C_{j1}, B_{05} = B_{11} - \sum_{j=2}^{4} B_{1j} C_{j1} \). (4.73)

\( A_{05} \) and \( B_{05} \) are not related to \( i\xi_0 \), neither. Then we choose the parameter matrices \( C_1, C_2, C_3, C_4 \) which satisfy \( A_{05} \) is convertible and \( Re(\lambda) < 0 \), where \( \lambda \) is the characteristic value of matrix \( A_{05}^{-1} B_{05} \).
This is available because there are \( 4m^2 \) variables in the parameter matrices \( C_1, C_2, C_3, C_4 \) and \( A_{05}, B_{05} \) are \( m \times m \). From the remark 3.1, we obtain

\[
\det(B_1) = \det(A_{05}) \det(i\xi_0 E - A_{05}^{-1} B_{05}) \phi(i\xi_1, i\xi_2, i\xi_3),
\]

\[
\int_{-\infty}^{+\infty} (\det(i\xi_0 E - A_{05}^{-1} B_{05}))^{-1} e^{it\xi_0} d\xi_0 = \int_{-\infty}^{+\infty} (\det(i\xi_0 E - A_{05}^{-1} B_{05}))^{-1} e^{it\xi_0} d\xi_0 I_{\{t \geq 0\}},
\]

\[
\int_{-\infty}^{+\infty} (\det(B_1))^{-1} e^{it\xi_0} d\xi_0 = \int_{-\infty}^{+\infty} (\det(B_1))^{-1} e^{it\xi_0} d\xi_0 I_{\{t \geq 0\}}.
\]

Hence (4.69) stands. We also take \( g_0 \) as

\[
g_0 = \int_{\Omega} (-A_1)e^{-ix_1 i\xi_1 - iy_2 i\xi_2 - iz_3 i\xi_3} dx_1 dy_1 dz.
\]

Now we have transformed the first order partial differential equations resolved with any derivatives into the equivalent integral equations as Eq.(4.66).

5 Classical solution and generalized solution

We have transformed the first order partial differential equations resolved with any derivatives into the equivalent integral equations as follows,

\[
Z_1 I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) * (\psi(Z_1) I_{\Omega \times (0, T)}),
\]

where

\[
Z_1 = (Z_{1j})_{p \times 1}, \ w_1 = (w_{1j})_{p \times 1}, \ w_2 = (w_{2j})_{p \times q}, \ \psi(Z_1) = (\psi_j(Z_1))_{q \times 1},
\]

\( p, q \) are natural numbers. We notice that

\[
w_1(x, y, z, t) \neq w_1(x, y, z, t) I_{\Omega \times (0, T)}, \ w_2(x, y, z, t) \neq w_2(x, y, z, t) I_{\Omega \times (0, T)}.
\]

Then (5.1) is equivalent to

\[
Z_1 I_{\Omega \times (0, T)} = w_1(x, y, z, t) + w_2(x, y, z, t) * (\psi(Z_1) I_{\Omega \times (0, T)}), \ \forall (x, y, z, t) \in \Omega_0,
\]

\[
0 = w_1(x, y, z, t) + w_2(x, y, z, t) * (\psi(Z_1) I_{\Omega \times (0, T)}), \ \text{otherwise},
\]

where \( \Omega_0 = \Omega \times (0, T) \).

If there exists \( Z_1 I_{\Omega \times (0, T)} \in C(\Omega \times (0, T)) \) satisfies Eq.(5.4) and Eq.(5.5) both, then we say the classical solution of the first order partial differential equations resolved with any derivatives exists.

If there exists \( Z_1 I_{\Omega \times (0, T)} \in L^2(\Omega \times (0, T)) \) only satisfies Eq.(5.4), then we say the generalized solution of the first order partial differential equations resolved with any derivatives exists. Maybe Eq.(5.5) can explain why sometimes the classical solution doesn’t exist. The generalized solution is always locally exist and unique.
6 Acknowledgements

I give my best thanks to my supervisor Prof. Mark Edelman, for his guidance when I am a Scholar Visitor in Yeshiva University. I sincerely thank Prof. Caisheng Chen in Hohai University, Prof. Junxiang Xu in Southeast University and Prof. Zuodong Yang in Nanjing Normal University for their recommendation and other helps. The financial support of Chinese ministry of education is gratefully acknowledged.

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