Some approach to key exchange protocol based on non-commutative groups

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Abstract — We consider non-commutative generalization of CDH problem [1,2] on base of metacyclic group $G$ of type Millera Moreno (minimal non-abelian group). We show that conjugacy problem in this group are intractable. The algorithm of generating (designing) of common key in non-commutative group with 2 mutually commuting subgroups are constructed by us.

Keywords — CDH and CCP problem, non-commutative cryptography, Millera Moreno group, subdirect product, generalization of CDH problem.

I. INTRODUCTION

In this investigation effective method of key exchange which based on non-commutative group $G$ is proposed. The results of Ko K, Lee S, is improved and generalized [1,2,3]. Public key cryptographic schemes based on the new systems are established. One of them is most notable due to Anshel and Goldfeld [9], and another due to Ko Lee etc. As we know if CSP problem is tractable in group $G$ then problem of finding $w^a$ by given $w$, $w^a=a^{-1}wa$, $w^b=b^{-1}wb$ is tractable too for arbitrary fixed $w \in G$ such that is not from center of $G$, where $w^a$ is the common key that Alice and Bob have to generate.

As well known if CCP problem is tractable in $G$ then problem of finding $w^a$ by given $w$, $w^a=a^{-1}wa$, $w^b=b^{-1}wb$ is tractable too for arbitrary fixed $w \in G$ such that is not from center of $G$. Note that $w^a$ is the common key that Alice and Bob have to generate.

We denote by $w^*$ the conjugated element $u=x^{-1}wx$. We show that no efficient algorithm exists that can distinguish between the two probability distributions of $(w^a, w^b, w^a)$ and $(w^a, w^b, w^a)$. Also no efficient algorithm exists to recover $w^a$ from $w, w^a$ and $w^b$. Metacyclic Millera Moreno group has representation $G=\langle a,b \mid a^{p^m} = e, b^{p+1} = e, b^{-1}ab = a^{i+1}, m \geq 2, n \geq 1 \rangle$, where is $p$ prime. As a generators $a, b$ can be chosen two arbitrary non commuting elements [4, 5, 6].

II. PROOF THAT CONJUGACY PROBLEM IS NP-HARD IN $G$

A. Size of conjugacy class

We need to have an effective algorithm for computation of conjugated elements, if we want to design a key exchange algorithm based on non-commutative DH problem [3]. Due to the relation in metacyclic group, which define the homomorphism $\varphi : \langle b \rangle \to \text{Aut}(\langle a \rangle)$ to the automorphism group of $A=\langle a \rangle$, we obtain a formula for finding a conjugated element. This formula give us possibility to efficiently calculate the conjugated to $a$ element by using the raising to the $1 + p^{m-1}$ -th power, where $m > 1$. Also due to cyclic structure of groups $A=\langle a \rangle$ and $B=\langle b \rangle$ in this group $G$ exists effectively method of checking of equality of elements.

Indeed the reducing by finite modulo $n$ give us an effective method of checking the equality of elements in the additive group $\mathbb{Z}_n$.

The goal of this investigation is effective method of key exchange which based on non-commutative group $G$. The results of Ko K, Lee S, is improved and generalized.

We consider non-commutative generalization of CDH problem [1,2] on base of metacyclic group $G$ of Miller’s Moreno type (minimal non-abelian group). We show that conjugacy problem in this group is intractable. Effectivity of computation is provided due to using groups of residues by modulo $n$. The algorithm of generating (designing) common key in non-commutative group with 2 mutually commuting subgroups is constructed by us.

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To problem of DL or equivalently problem of conjugacy in non-commutative group $G$ be NP-hard it have to be enough long orbit of the given base element $w \in G$.

Let elements of $G$ acts by conjugation on $w \in G$, where $w \in Z(G)$.

**Proof of NP-hardness of the conjugacy problem in $G$.**

Size of a conjugacy class.

Let elements of $G$ acts by conjugation on $w \in G$, where $w \in Z(G)$. To problem of DL or equivalent problem of conjugacy in non-commutative group $G$ be NP-hard, the orbit of the given base element $w \in G$ must be enough long if we want to have stability of DL problem or equally problem of conjugacy in non-commutative group $G$ like NP-hard problem.

**Theorem 1.** The length of conjugacy class of non-central element $w$ is equal to $p$.

**Proof.** Recall the inner automorphism in $G$ is determined by formula $b^{-1}ab = a^{b(a^{-1})}$. Let us recall the structure of minimal non-abelian Metacyclic group:

$$G = B \circ A,$$

where $A = (a)$ and $B = (b)$ are finite cyclic groups. Therefore formula $b^{-1}ab = a^{b(a^{-1})}$ define a homomorphism $\varphi$ in the subgroup of inner automorphisms $\text{Aut}(G)$. As well-known each finite cyclic group is isomorphic to the correspondent additive cyclic group modulo $n$ residue $\mathbb{Z}_n$ (or $\mathbb{Z}_p$). In this group equality of elements can be checked effectively due to reducing the elements of the module group.

Consider the orbit of element $w$ under action by conjugation. The length of such orbit can be found from equality $w^{(b(a^{-1}))^r} = w$ as minimal power $s$ for which this equality will be true. We apply Newton binomial formula to the expression $(1 + p^{m-1})^r = 1 (\text{mod } p^n)$ and take account the relation $a^r = e$. We obtain

$$1 + C_s 1^{p^{m-1}} + C_s 1^{2p^{m-1}} + \ldots + C_s 1^{s(p^{m-1})} = 1 (\text{mod } p^n),$$

where $l < m$ because of $1 + C_s 1^{p^{m-1}} = 1 + sp^{m-1} = 1 (\text{mod } p^l)$ if $s < p$. It means that minimal $s$ when this congruence start to holds is equal to $p$. The prime number $p$ can be chosen as big as we need [7]. The proof is fully completed.

Consider non-metacyclic group of Millera Moreno.

**Theorem 2.** The length of conjugacy class of non-central element $w$ is equal to $p$ in non-metacyclic group of Millera Moreno.

This group has representation

$$G = \left\{ a, b \left| \begin{array}{l} a, b \in P, |a| = p^m, |b| = p^n, m, n \geq 1, b^{-1}ab = ac, \\
 b^{-1}cb = c. \end{array} \right. \right\}$$

To find a length of orbit of action by conjugation by $b$ we consider the class of conjugacy of elements of form $a^ib^jc$. This class has length $p$ because of action $b^{-1}a^ib^jc = a^{b(a^{-1})}b^jc$, as well as $b^{-1}a^ib^jc = a^{b(a^{-1})+1}b^jc$ increase the power of $c$ on 1. Thus, the first repetition of initial power $j$ in $a^ib^jc$ occurs through $n$ conjugations of this word by $b$, where $1 \leq j \leq p$. Therefore, the length of the orbit is $p$.

**B. Key exchange protocol**

Let $S_1, S_2$ are subsets from $G$ consisting of mutually commutative elements. We consider subgroups $H_1 = \{ S_1 \}$ and $H_2 = \{ S_2 \}$. Due to mutually commutative generating sets this subgroups are mutually commutative too.

**Consider base steps of protocol.**

Input: Elements $w, w'$ and $w''$.

Alice chose random element $x$ from the subgroup $H_1$ and compute $w'$. After she sends it to Bob.

Bob chose random element $y$ from the subgroup $H_2$ and compute $w''$. After he sends it to Alice.

Bob computes $(w')^y = w''$ and Alice computes $(w'')^y = w''$. Taking into consideration that $H_1$ and $H_2$ are mutually commutative groups we obtain that $xy = yx$.

Therefore we have $w'' = w''$. Thus, common key [6] $w''$ was successfully generated.

**Resistance to a cryptanalysis.** But if we will use for cryptoanalysys solving of conjugacy search problem in group using method of reduction to solving of decomposition problem [8] then it leads us to solving of discrete logarithm problem in the group that have structure of semidirect product of multiplicative group $\mathbb{Z}_p \times \mathbb{Z}_p$. This problem is NP-hard even in multiplicative group $\mathbb{Z}_p \times \mathbb{Z}_p$ for enough big $p$ or for essentially big $m$.

If one try to solve conjugacy search problem in $G$ using the method of Barrett [9, 10] then complexity of the method Barret [9, 10] uses $2\log_2(2p^{n-1}p + 1)$ multiplications. Also the complexity of such multiplication by modulo $p^n$ is $\log_2 p^n = m \log_2 p$.

**C. Conclusion**

We can chose mutually commutative $H_1, H_2$ as a subgroups of $Z(G)$. As we said above, $x, y$ as components of key a chosen from $H_1, H_2$. According to [4] $Z(G) = p^{n-1}$, so size of key-space is $O(p^{n-1})$. Note that size of key-space can be chosen as arbitrary big number by choice of parameters $p, n, m$. 

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