Square of Planar Graphs of Max Degree Four without Five Cycles

Eric Culver  
Department of Mathematical and Statistical Sciences  
University of Colorado Denver  
eric.culver@ucdenver.edu

Stephen G. Hartke∗  
Department of Mathematical and Statistical Sciences  
University of Colorado Denver  
stephen.hartke@ucdenver.edu

May 2022

Abstract

We show that the choosability of the square of planar graphs of max degree 4 without five cycles is at most 12.

Keywords: planar graph, choosability  
AMS Mathematics Subject Classification: 05C15

1 Introduction

In 1977, Wegner conjectured the following upper bounds on the chromatic numbers of squares of planar graphs:

\textbf{Conjecture 1} (Wegner [13]). \textit{Let }\textit{G be a planar graph with maximum degree }\Delta. \textit{Then}

\[
\chi(G^2) \leq \begin{cases}
7 & \Delta \leq 3 \\
\Delta + 5 & 4 \leq \Delta \leq 7 \\
\left\lfloor \frac{3\Delta}{2} \right\rfloor + 1 & \Delta \geq 8
\end{cases}
\]

In this paper, we will be focusing on the \(\Delta = 4\) case.

\textbf{Conjecture 2}. \textit{Let }\textit{G be a planar graph with maximum degree }4, \textit{then }\chi(G^2) \leq 9.

We can see that this conjecture, if true, would be sharp, from the graph in Figure 1, which is a planar graph of maximum degree four on nine vertices. We can see that the square of this graph is the complete graph on nine vertices. Also, since this graph contains vertices of degree three, there is an infinite number of examples of planar graphs of maximum degree four which contain this graph as a subgraph, and therefore would also require nine colors in their squares.

Not much progress was made on the \(\Delta = 4\) case of this conjecture until 2002, when Borodin et al. [6] showed that for planar graphs \(G\) with maximum degree \(\Delta \leq 20\) that \(\chi(G^2) \leq 59\). This

∗Supported in part by a Collaboration Grant from the Simons Foundation (#316262 to Stephen G. Hartke).
was improved by Zhu and Bu [14] in 2018 who showed that planar graphs $G$ with maximum degree $\Delta \leq 5$ have $\chi(G^2) \leq 20$.

In this paper, we will be looking at the more general case of list coloring, which was first introduced independently by Vizing in 1976 [12] and by Erdős, Rubin, and Taylor in 1979 [8]. A list assignment for a graph $G$ is a function $L$ that assigns to each vertex a list of colors. An $L$-coloring of $G$ is a coloring of $G$ such that for each vertex, the color assigned to it is picked from its list. A graph $G$ is $k$-choosable if there exists an $L$-coloring of $G$ for every assignment $L$ of lists of size $k$ to the vertices of $G$.

Every $k$-choosable graph is $k$-colorable. However, the converse is known to not be true in general. For example, the graph in Figure 2 is 2-colorable but not 2-choosable.

The choosability, $\chi_\ell(G)$, of $G$ is the minimum $k$ such that $G$ is $k$-choosable. Then, this relationship is expressed as:

$$\chi(G) \leq \chi_\ell(G)$$

for all graphs $G$.

Where the inequality is sometimes strict.

In this paper, we will be proving Theorem 2.

**Theorem 2.** Let $G$ be a planar graph with no 5-cycles such that $\Delta(G) \leq 4$. Then $\chi_\ell(G^2) \leq 12$.

Our technique for proving this statement will be the discharging method. The method of discharging was developed by Birkoff [4] and Heesch [9, 10], and was ultimately used by Appel, Haken, and Koch [1, 3, 2] to prove the Four Color Theorem. It has since been used to great effect for many results in graph theory.

For a survey of results proven by discharging, I refer the reader to [5, 11]. For a more thorough explanation of the discharging method than this paper will provide, and a further survey of results, see [7].

All discharging arguments follow the same steps:
a. We suppose our graph \( G \) is a minimal counterexample to the statement we want to prove.

b. We argue that \( G \) cannot contain certain configurations, called reducible configurations. If \( G \) did contain a reducible configuration, then we can construct a smaller graph \( G' \) for which the statement holds by the minimality of \( G \). We then show that the statement (usually a coloring) can be extended from \( G' \) to \( G \). This then shows the statement holds for \( G \), which is a contradiction. Therefore, \( G \) cannot contain any of these reducible configurations.

c. We then use the technique of discharging to show that \( G \) must contain one of the reducible configurations. We then call these configurations unavoidable.

d. This is a contradiction. Therefore, the statement is true.

Our proof will follow this same basic outline. We will handle Step b. in Section 2, where our chief tool will be Lemma 1. We will handle Step c. in Section 3.

In order to show this, we will need some definitions.

Let \([k] = \{0, 1, \cdots, k-1\}\).

**Definition 1.** In a graph \( G \), a \( k \)-vertex is a vertex of degree \( k \). Similarly, a \( k^+ \)-vertex is a vertex of degree at least \( k \), and a \( k^- \)-vertex is a vertex of degree at most \( k \).

A \( k \)-face, \( k^+ \)-face, and \( k^- \)-face are defined similarly, referring to the length of the face instead of the degree of the vertex.

**Definition 2.** Given a set \( S \subseteq V(G) \) of vertices of \( G \), the neighborhood \( N(S) \) of \( S \) is the set of vertices in \( G \setminus S \) which are adjacent to at least one vertex in \( S \).

The closed neighborhood \( \overline{N}(S) \) of \( S \), i.e., \( \overline{N}(S) = N(S) \cup S \).

If the set \( S \) is small, we often omit the curly braces, so: \( N(x) = N\{x\} \) and \( N(x, y) = N\{x, y\} \).

**Definition 3.** Given a plane graph \( G \), the set of faces of \( G \) will be notated \( F(G) \).

## 2 Reducibility

The intuitive idea behind the following lemma is that \( X \) is the set of vertices and \( Y \) the set of edges that are removed from \( G \) to produce \( H \). The vertices in \( R \) are being recolored. The first condition of the lemma simply says that if we are removing a vertex, we also must remove all the edges incident to that vertex. The second condition ensures that the coloring on \( H^2 \) is a valid coloring of the vertices of \( P \) in \( G^2 \). The third condition then checks that we can extend the coloring on the vertices in \( P \) to the vertices in \( X \) and \( R \).

**Lemma 1.** Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( X, R, P \subseteq V(G) \) be three disjoint subsets of the vertex set of \( G \) whose union is \( V(G) \). And let \( Y, Q \subseteq E(G) \) be two disjoint subsets of the edge set of \( G \) whose union is \( E(G) \). Let \( H \) be the subgraph of \( G \) on vertex set \( R \cup P \) and edge set \( Q \). If \( X, R, P, Y, Q, H, \) and \( G \) satisfy:

1. Any edge of \( G \) incident to a vertex in \( X \) must be in \( Y \).
2. Any edges in \( G^2 \) not in \( H^2 \) must be incident to vertices in \( X \cup R \).
3. Define \( f : X \cup R \rightarrow \mathbb{N} \) by \( f(v) = 12 - |N_{G^2}(v) \cap P| \). Then the subgraph of \( G^2 \) induced by \( X \cup R \) is \( f \)-choosable.
4. $\chi_\ell(H^2) \leq 12$

Then $\chi_\ell(G^2) \leq 12$.

Proof. Let $L$ be a list assignment for $G$, mapping each vertex to a list of 12 colors. Since $H$ is a subgraph of $G$, this is also a list assignment for $H$. Since $\chi_\ell(H^2) \leq 12$, $H$ is $L$-square-colorable. Since the only edges of $G^2$ not in $H^2$ are incident to vertices in $X \cup R$, if we erase the color on vertices in $R$, then extend the coloring to the vertices in $X \cup R$, we will have square colored $G$. The condition that the subgraph of $G^2$ on vertex set $X \cup R$ is $f$-choosable for those specific $f$-values is sufficient to extend this coloring. Therefore, $G$ is $L$-square-colorable. Since $L$ was arbitrarily chosen, this shows $\chi_\ell(G^2) \leq 12$. \hfill $\square$

![Figure 3: Guide for Lemma 1](image)

Note that in all of our applications of this lemma except one, we show that all the elements of $X \cup R$ must be distinct, and the subgraph of $G^2$ on $X \cup R$ is complete. This means we need only consider the configurations in the most general way possible, since any amount of overlap in the configuration will only increase the $f$-values, and it cannot add any edges into the subgraph on $X \cup R$, since that is already complete.

**Theorem 2.** Let $G$ be a planar graph with no 5-cycles such that $\Delta(G) \leq 4$. Then $\chi_\ell(G^2) \leq 12$.

Let $\mathcal{C}$ be the family of all planar graphs with no 5-cycles such that $\Delta(G) \leq 4$. For the rest of this paper, let $G$ be a minimal counterexample to Theorem 2, that is, $G \in \mathcal{C}$ and $\chi_\ell(G^2) > 12$ and for any graph $H \in \mathcal{C}$ with a smaller number of edges than $G$ or a smaller number of vertices than $G$, $\chi_\ell(H^2) \leq 12$.

From these assumptions, we can prove certain lemmas about $G$. All of these lemmas will take the form: “The graph $G$ cannot contain structure $X$”, and most will be proven by assuming that $G$ does have structure $X$ and finding $X, R, P, Y, Q, H$ which satisfy the conditions of Lemma 1. Since this lemma concludes that $\chi_\ell(G^2) \leq 12$, while by assumption $\chi_\ell(G^2) > 12$, this leads to a contradiction, showing that $G$ cannot have structure $X$. Note that to derive the condition that
We will use that \( H \in \mathcal{C} \) and that \( H \) is a smaller graph than \( G \), meaning that at least one of \( X, Y \) must be nonempty. These are further conditions we will need to check.

Note 1. In the following figures, we will mark the removed vertices (elements of \( X \)) by filling them in with black, the removed edges (elements of \( Y \)) by dashing them, and recolored vertices (element of \( R \)) by filling them in with gray. We need only show the unremoved edges and precolored vertices of \( G \) that are within distance two of an element of \( X, Y, \) or \( R \), as those are the only ones that contribute to the count of the \( f \)-values. We will be showing those in the most general way possible, as if nothing overlapped, and every vertex was of maximum degree. To facilitate the checking of Lemma 1 by the reader, the subgraph of \( G^2 \) on the vertex set \( X \cup R \) with the \( f \) values is also given, after an arrow.

**Lemma 3.** The graph \( G \) is connected.

**Proof.** Suppose \( G \) did have multiple connected components, and let \( G_1 \) be one of them, while \( G_2 \) is the rest of the graph. By the minimality of \( G \), we can assume that \( \chi_\ell(G_1^2) \leq 12 \) and \( \chi_\ell(G_2^2) \leq 12 \). Since they are disconnected, given any assignment of lists of 12 colors to the vertices of \( G \), we can color \( G_1 \), without affecting \( G_2 \), and then color \( G_2 \), without affecting \( G_1 \). In this way, we can always color \( G \). Therefore, \( G^2 \) can be colored from any lists of size 12, which contradicts the assumption that \( \chi_\ell(G^2) > 12 \). Therefore, \( G \) is connected. \( \square \)

**Lemma 4.** The graph \( G \) cannot have a 1-vertex.

**Proof.** Suppose it did have a 1-vertex \( v \). Let \( X = \{ v \} \), let \( Y \) be the edge incident to \( v \), and let \( R \) be empty. (See Figure 4) We can see that the conditions of Lemma 1 are satisfied. Therefore, \( G^2 \) can be colored from any lists of size 12, which contradicts the assumption that \( \chi_\ell(G^2) > 12 \). Therefore, \( G \) cannot have a 1-vertex. \( \square \)

![Figure 4: A 1-vertex](image)

**Lemma 5.** The graph \( G \) cannot have a 2-vertex incident to a 3-face.

**Proof.** Suppose it did have a 2-vertex \( v \) incident to a 3-face. Let \( X = \{ v \} \), let \( Y \) be the two edges incident to \( v \), and let \( R \) be empty. (See Figure 5) We can see that the conditions of Lemma 1 are satisfied. Therefore, \( G^2 \) can be colored from any lists of size 12, which contradicts the assumption that \( \chi_\ell(G^2) > 12 \). Therefore, \( G \) cannot have a 2-vertex incident to a 3-face. \( \square \)

**Lemma 6.** The graph \( G \) cannot have a 2-vertex incident to a 4-face.

**Proof.** Suppose it did have a 2-vertex \( v \) incident to a 4-face. Let \( X = \{ v \} \), let \( Y \) be the two edges incident to \( v \), and let \( R \) be empty. (See Figure 6) We can see that the conditions of Lemma 1 are satisfied. Therefore, \( G^2 \) can be colored from any lists of size 12, which contradicts the assumption that \( \chi_\ell(G^2) > 12 \). Therefore, \( G \) cannot have a 2-vertex incident to a 4-face. \( \square \)
Lemma 7. The graph $G$ cannot have adjacent 2-vertices.

Proof. Suppose it did have vertices $u, v$ of degree 2 which are adjacent. Let $X = \{u, v\}$, let $Y$ be the three edges incident to $u$ and $v$, and let $R$ be empty. (See Figure 7) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have adjacent 2-vertices. □

Lemma 8. The graph $G$ cannot have a 2-vertex adjacent to a 3-vertex.

Proof. Suppose it did have vertex $u$ of degree 2 and $v$ of degree 3 which are adjacent. Let $X$ be empty, let $Y = \{uv\}$, and let $R = \{u, v\}$. (See Figure 8) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 2-vertex adjacent to a 2-vertex. □

Lemma 9. The graph $G$ cannot have a 3-vertex adjacent to a 3-vertex.
Figure 8: A 2-vertex adjacent to a 3-vertex

Proof. Suppose it did have vertex $u$ of degree 3 and $v$ of degree 3 which are adjacent. Let $X$ be empty, let $Y = \{uv\}$, and let $R = \{u, v\}$. We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 3-vertex adjacent to a 3-vertex.

Figure 9: Adjacent 3-vertices

Lemma 10. The graph $G$ cannot have a 2-vertex distance at most two away from another 2-vertex.

Proof. Suppose it did have vertices $u, v, w$ such that $u, w$ are 2-vertices, and $uvw$ is a path. Note that by Lemma 8, $v$ must be a 4-vertex. Let $X = \{u, w\}$, let $Y$ be the four edges incident to $u, w$, and let $R = \{v\}$. (See Figure 10) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 2-vertex distance two away from another 2-vertex.

Figure 10: 2-vertices at distance at most two
Lemma 11. The graph $G$ cannot have a 2-vertex distance at most two away from a 3-vertex.

Proof. Suppose it did have vertices $u, v, w$ such that $u$ is a 2-vertex, $w$ is a 3-vertex, and $uvw$ is a path. Note that by Lemma 8, $v$ must be a 4-vertex. Let $X = \{u\}$, let $Y$ be the two edges incident to $u$, and let $R = \{v, w\}$. (See Figure 11) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 2-vertex distance two away from a 3-vertex.

![Figure 11: A 2-vertex at distance at most two from a 3-vertex](image)

Lemma 12. The graph $G$ cannot have a 2-vertex adjacent to one of the vertices of a 3-face.

Proof. Suppose it did have vertices $u, v$ such that $u$ is a 2-vertex, $v$ is incident to a 3-face $f$ and $u, v$ are adjacent. Note that by Lemmas 7, 8, 10, and 11, $v$ and all the vertices incident to the 3-face must be 4-vertices. Let $X = \{u\}$, let $Y$ be the two edges incident to $u$, and let $R = \{v\}$. (See Figure 12) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 2-vertex adjacent to one of the vertices of a 3-face.

![Figure 12: A 2-vertex adjacent to a vertex of a 3-face](image)

Lemma 13. The graph $G$ can have no 3-vertex incident to two 3-faces.

Proof. Suppose it did have a 3-vertex $v$ which is incident to two 3-faces $f, g$, then those 3-faces must share an edge $vu$. Let $X$ be empty, $Y = \{vu\}$ and $R = \{v\}$. (See Figure 13) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 3-vertex incident to two 3-faces.

![Figure 13: A 2-vertex adjacent to a vertex of a 3-face](image)
**Lemma 14.** The graph $G$ can have no 3-face sharing two edges with 3-faces.

*Proof.* If the graph $G$ contains a 3-face $f$ which shares two edges $uv, vw$ with the same 3-face $g$, then the vertex $v$ must be a 2-vertex incident to a 3-face, which by Lemma 5 cannot happen.

If the graph $G$ contains a 3-face which shares two edges with two distinct 3-faces which do not share an edge themselves, then the outside cycle around the three faces is a 5-cycle, which is forbidden. Therefore, this cannot happen.

Suppose the graph $G$ contains a 3-face which shares two edges $uv, vw$ with two distinct 3-faces which also share an edge $vx$. Then $v$ is a 3-vertex which is incident to three 3-faces, which by Lemma 13 cannot happen.

Since none of these possibilities can happen, $G$ cannot have a 3-face sharing two edges with 3-faces. \(\square\)

**Lemma 15.** The graph $G$ can have no 3-face sharing an edge with a 4-face.

*Proof.* If the graph $G$ contains a 3-face which shares two edges $uv, vw$ with the same 4-face, then the vertex $v$ must be a 2-vertex incident to a 3-face, which by Lemma 5 cannot happen.

If the graph $G$ contains a 3-face which shares a single edge with a 4-face, then the outside cycle around the two faces is a 5-cycle, which is forbidden. Therefore, this cannot happen. \(\square\)

**Lemma 16.** The graph $G$ can have no 3-vertex incident to two 4-faces.

*Proof.* If the graph $G$ contains a 3-vertex $v$ incident to two 4-faces, then those two 4-faces must share an edge $vu$. Let $X$ be empty, $Y = \{vu\}$, and let $R = \{v\}$. We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ cannot have a 3-vertex incident to two 4-faces. \(\square\)
Lemma 17. The graph $G$ can have no 3-vertex incident to a 3-face which shares an edge with another 3-face.

Proof. Suppose it did have a 3-vertex $v$ incident to a 3-face $f$ which shares an edge with another 3-face $g$. By Lemma 13, $v$ cannot be incident to the 3-face $g$ also, therefore, the shared edge between the faces $f, g$ must not be incident to $v$. Called the shared edge $uw$. Therefore, $f$ is incident to the vertices $v, u, w$, and $g$ is incident to $u, w$ and a third vertex. Let $X$ be empty, $Y = \{uw\}$ and let $R = \{u, w\}$. (See Figure 15) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ can have no 3-vertex incident to a 3-face which shares an edge with another 3-face. □

Figure 15: A 3-vertex incident a 3-face sharing an edge with another 3-face

Lemma 18. The graph $G$ can have no 3-vertex incident to a 3-face which shares a vertex with another 3-face.

Proof. Suppose it does have a 3-vertex $v$ incident to a 3-face $f$ which shares a vertex with another 3-face $g$. By Lemma 13, $v$ cannot also be incident to the 3-face $g$. Therefore, the vertex shared between faces $f$ and $g$ must be some other vertex we will call $u$. Let $X$ be empty, let $Y = \{uv\}$, and let $R = \{u, v\}$. (See Figure 16) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ can have no 3-vertex incident to a 3-face which shares a vertex with another 3-face. □

Lemma 19. The graph $G$ can have no 3-vertex adjacent to a vertex incident to a 3-face which is incident to another 3-vertex.

Proof. Suppose it does have a 3-vertex $v$ which is adjacent to a vertex $u$ incident to a 3-face $f$ that is incident to another 3-vertex. By Lemma 9, $u$ cannot be a 3-vertex, therefore, the 3-vertex incident to $f$ must be some other vertex $w$, and the vertex $u$ must be a 4-vertex. Let $X$ be empty, $Y = \{uw\}$, and let $R = \{u, w\}$. (See Figure 17) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_\ell(G^2) > 12$. Therefore, $G$ can have no 3-vertex adjacent to a vertex incident to a 3-face which is incident to another 3-vertex. □
Figure 16: A 3-vertex incident a 3-face sharing a vertex with another 3-face

Figure 17: A 3-vertex adjacent to a vertex incident to a 3-face incident to another 3-vertex

Lemma 20. The graph $G$ can have no $3^-$-vertex adjacent to one of the endpoints of the shared edge between two 3-faces.

Proof. Suppose it does have a $3^-$-vertex $v$ which is adjacent to a vertex $u$ such that the edge $uw$ is the shared edge between two 3-faces $f, g$. We know that $w \neq v$ by Lemma 13 if $v$ is a 3-vertex, or by Lemma 5 if $v$ is a 2-vertex. Without loss of generality, we can now assume that $v$ is a 3-vertex. Let $X$ be empty, let $Y = \{uw\}$, and let $R = \{u, w\}$. (See Figure 18) We can see that the conditions of Lemma 1 are satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_l(G^2) > 12$. Therefore, $G$ can have no 3-vertex adjacent to one of the endpoints of the shared edge between two 3-faces. \qed

Lemma 21. The graph $G$ can have no 2-vertex distance at most two away from the midpoint of one of the endpoints of the shared edge between two 3-faces.

Proof. Suppose it does have a path $vuwx$ such that $v$ is of degree 2, and $wx$ is the shared edge between two 3-faces. By Lemma 20, $u$ must be of degree 4. Let $X$ be empty, let $Y = \{wx\}$, and let $R = \{v, u, w, x\}$. (See Figure 19) We can see that the conditions of Lemma 1 are satisfied. The subgraph of $G^2$ induced by $X \cup R$ in this general instance is not complete, it is missing one edge.
However, even if that edge were present and the $f$-values remained the same, the resulting graph would still be $f$-choosable, and so the conditions of Lemma 1 are still satisfied. Therefore, $G^2$ can be colored from any lists of size 12, which contradicts the assumption that $\chi_l(G^2) > 12$. Therefore, $G$ can have no 2-vertex distance at most two away from the midpoint of one of the endpoints of the shared edge between two 3-faces.

3 Discharging

In this proof, we will use the technique of intermediate discharging. This is a technique where the main discharging is completed by a series of smaller steps within the proof. Usually this is done in order to clarify the proof.

Proof. (Theorem 2) To each vertex of $G$, give a charge equal to its degree minus four, and to each face of $G$ give a charge equal to its length minus four. This is commonly known as balanced charging. Let $c$ be this charging function. By using the handshaking lemma and Euler’s formula
(which we can use by Lemma 3), we can derive the total charge on \( G \):

\[
\sum_{v \in V(G)} c(v) + \sum_{f \in F(G)} c(f) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (l(f) - 4)
\]

\[
= \left( \sum_{v \in V(G)} d(v) \right) - 4|V(G)| + \left( \sum_{f \in F(G)} l(f) \right) - 4|F(G)|
\]

\[
= 2|E(G)| - 4|V(G)| + 2|E(G)| - 4|F(G)|
\]

\[
= -4(|V(G)| - |E(G)| + |F(G)|)
\]

\[
= -4 \cdot 2
\]

\[
= -8
\]

Thus, the total initial charge is negative.

We then redistribute the charge by the following discharging rules:

R1: A 2-vertex takes 1 charge from each incident 6\(^+\)-face.

R2: A 3-vertex takes \( \frac{1}{2} \) charge from each incident 6\(^+\)-face.

R3: A 3-face incident to at least one 3-face takes \( \frac{1}{2} \) charge from each incident 6\(^+\)-face.

R4: A 3-face not incident to any 3-faces takes \( \frac{1}{3} \) charge from each incident 6\(^+\)-face.

In the above rules, if a vertex is incident to a face in multiple ways, and it would take charge from that face according to the above rules, we want it to take charge from that face multiple times. Similarly, we want the same to apply for faces. For example, a 2-vertex \( v \) is generally incident to two faces, \( f, g \), but if \( f = g \), we still want \( v \) to take charge from \( f = g \) twice if \( f \) is a 6\(^+\)-face.

Let \( c^* \) be the resulting charge function after this redistribution. We will show that for all \( v \in V(G) \), \( c^*(v) \geq 0 \) and for all \( f \in F(G) \), \( c^*(f) \geq 0 \).

**2-vertex** Let \( v \in V(G) \) be a 2-vertex. Note \( c(v) = -2 \). Let \( f, g \) be the faces incident to \( v \). By Lemmas 5 and 6, both \( f \) and \( g \) are 6\(^+\)-faces. Therefore, \( c^*(v) = c(v) + 2 = -2 + 2 = 0 \geq 0 \).

**3-vertex** Let \( v \in V(G) \) be a 3-vertex. Note \( c(v) = -1 \). Let \( f, g, h \) be the faces incident to \( v \). By Lemmas 13, 16, and 15, at most one of \( f, g, h \) can be a 4\(^-\)-face, and therefore the remaining two are 6\(^+\)-faces. Therefore, \( c^*(v) = c(v) + 2 \cdot \frac{1}{2} = -1 + 1 = 0 \geq 0 \).

**4-vertex** Let \( v \in V(G) \) be a 4-vertex. Then \( c^*(v) = c(v) = 0 \geq 0 \).

**3-face** Let \( f \in F(G) \) be a 3-face. Note \( c(f) = -1 \). Let \( g, h, k \) be the faces incident to \( f \). By Lemmas 14 and 15, we know that none of \( g, h, k \) can be 4\(^-\)-faces and at most one can be a 3-face. Therefore, at least two of \( g, h, k \) are 6\(^+\)-faces, and so \( c^*(f) = c(f) + 2 \cdot \frac{1}{2} = -1 + 1 = 0 \geq 0 \).

**4-face** Let \( f \in F(G) \) be a 4-face. Note \( f \) gives no charge and takes no charge and so remains unchanged. Then \( c^*(f) = c(f) = 0 \geq 0 \).

**6\(^+\)-face** Let \( f \in F(G) \) be a 6\(^+\)-face. Let \( l \) be the length of the face. Note \( c(f) = l - 4 \). We will use intermediate discharging to implement the above rules. First, we will give each edge of the face \( \frac{1}{3} \) charge. After this step, the final charge left in \( f \) is:

\[
l - 4 - \frac{l}{3} = \frac{2l}{3} - 4
\]
This is nonnegative since $l \geq 6$. We will now distribute the charges from the edges around $f$ to the 2-vertices, 3-vertices, and 3-faces incident to $f$ that according to the above rules should be pulling charge from $f$, and each of these receives the charge that they should receive.

**SubR1** A 3-face takes $\frac{1}{3}$ charge from the shared edge with $f$. (See Figure 20)

**SubR2** A 3-face adjacent to at least one 3-face additionally takes $\frac{1}{6}$ charge from the edge on the same side as the incident 3-face. (See Figure 21)

**SubR3** A 3-vertex incident to a 3-face takes $\frac{1}{3}$ charge from the other edge incident to it, and it takes $\frac{1}{6}$ charge from the edge on the other side of the 3-face along $f$. (See Figure 22)

**SubR4** A 3-vertex not incident to a 3-face takes $\frac{1}{4}$ charge from each edge incident to it. (See Figure 23)

**SubR5** A 2-vertex takes $\frac{1}{3}$ charge from the two edges incident to it, and takes $\frac{1}{6}$ charge from the edges distance two away from it along $f$. (See Figure 24)

We can see that these rules provide enough charge to satisfy the 2-vertices, 3-vertices, and 3-faces around $f$.

Note that charges of $\frac{1}{3}$ and $\frac{1}{4}$ travel a “short” distance, from an edge to a vertex or face incident to it, while charges of $\frac{1}{6}$ travel a “long” distance, from an edge to a vertex or face incident to an adjacent edge. Also note that since we are moving charge around a particular face $f$, there are only two directions to move it, restricting options significantly.

If an edge $e$ has two short distance charges leaving, or a short and a long distance charge leaving, or if it has two long distance charges leaving in the same direction, then it could end up with negative charge. We will show that this cannot happen. Lemmas 5, 7, 8, 9, and 13 ensure
Figure 22: A 3-vertex incident to a 3-face takes $\frac{1}{3}$ charge from the other edge incident to it, and it takes $\frac{1}{6}$ charge from the edge on the other side of the 3-face along $f$.

Figure 23: A 3-vertex not incident to a 3-face takes $\frac{1}{4}$ charge from each edge incident to it.

that $e$ cannot have two short distance charges leaving. Lemmas 11, 14, 19, and 20 show reducible the configurations which would occur if $e$ had a short distance charge and a long distance charge leaving, necessarily in different directions. Lemmas 5 and 20 show that the if $e$ has two long distance charges leaving in the same direction going to a vertex and a face, then there is a reducible configuration on that side. Since there cannot be any reducible configurations in $G$, we cannot have this happen.

Therefore, we see that every edge ends up with nonnegative charge. Since every edge of the face $f$ ends up with nonnegative charge, and the face $f$, as shown above, ends up with nonnegative charge, the total is also nonnegative.

We have shown that the final charge after redistribution is nonnegative for all vertices, edges, and faces of $G$. However, the initial charge was negative, and we only moved charge around. This is a contradiction. Therefore, the assumption that a minimal counterexample to our statement $G$ exists is wrong. And so the theorem is proved.

\[\square\]

4 Future Work

Future work will look towards extending Theorem 2 to the more general statement for all planar graphs with max degree 4:

Conjecture 3. Let $G$ be a planar graph such that $\Delta(G) \leq 4$. Then $\chi_e(G^2) \leq 12$.

References

[1] K. Appel and W. Haken. Every planar map is four colorable Part I: Discharging. Illinois Journal of Mathematics, 21(3):429–490, 1977.
Figure 24: A 2-vertex takes $\frac{1}{3}$ charge from the two edges incident to it, and takes $\frac{1}{6}$ charge from the edges distance two away from it along $f$.

[2] K. Appel and W. Haken. *Every Planar Map Is Four Colorable (Contemporary Mathematics)*. American Mathematical Society, 1989.

[3] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable Part II: Reducibility. *Illinois Journal of Mathematics*, 21(3):491–567, 1977.

[4] G. D. Birkhoff. The reducibility of maps. *American Journal of Mathematics*, 35(2):115, Apr 1913.

[5] O. V. Borodin. Colorings of plane graphs: A survey. *Discrete Mathematics*, 313(4):517–539, Feb 2013.

[6] O. V. Borodin, H. J. Broersma, A. Glebov, and J. van den Heuvel. Stars and bunches in planar graphs. part II: General planar graphs and colourings. Technical Report 0169-2690, University of Twente, Department of Applied Mathematics, 2002.

[7] D. W. Cranston and D. B. West. An introduction to the discharging method via graph coloring. *Discrete Mathematics*, 340(4):766–793, 2017.

[8] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. *Congressus Numerantium*, 26:125–127, 1979.

[9] H. Heesch. *Untersuchungen zum Vierfarbenproblem*. B.I.-Hochschulskripten, 810/810a/810b. Mannheim, Bibliographisches Institut, 1969.

[10] H. Heesch. Chromatic reduction of the triangulations $T_e$, $e = e_5 + e_7$. *Journal of Combinatorial Theory, Series B*, 13(1):46–55, Aug 1972.

[11] S. Jendrol’ and H. J. Voss. Light subgraphs of graphs embedded in the plane - A survey. *Discrete Mathematics*, 313(4):406–421, 2013.

[12] V. Vizing. Vertex coloring of a graph with assigned colors (in russian). *Metody Diskret. Analiz. (Novosibirsk)*, 29:3–10, 1976.

[13] G. Wegner. Graphs with given diameter and a colouring problem. Preprint, 1977.

[14] J. Zhu and Y. Bu. Minimum 2-distance coloring of planar graphs and channel assignment. *Journal of Combinatorial Optimization*, 36(1):55–64, Jul 2018.