On identity theorem for real functions

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Abstract

Identity theorem for analytic complex functions says that a function is uniquely defined by its values on a set that contains a density point. The paper presents sufficient conditions for classes of real analytic functions that ensures similar property.

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1 Introduction

Identity theorem for complex functions states that an analytic in a domain function is uniquely defined by its values on a subdomain. It is well known that the real functions do not have this feature. For instance, let $x_1(t) = 0$ for all $t \in \mathbb{R}$ and let $x_2(t)$ be the so-called Sobolev kernel defined as $x_2(t) = \exp(t^2(1-t^2)^{-1})$, $t \in (-1, 1)$ and $x_2(t) = 0$ if $t \in \mathbb{R} \setminus (-1, 1)$. Both functions are infinitely differentiable, and $x_1(t) = x_2(t)$ for $t \leq -1$. This example shows that the identity property does not hold for infinitely differentiable real functions. On the other hand, there are some classes of real functions that are uniquely defined by their values on a part of the real axis (for instance, periodic functions and band-limited functions). It is interesting to find wider classes of real functions where identity theorem holds. It may be useful for the extrapolation problems and forecasts.

The paper presents sufficient conditions for some classes of real analytic functions that ensures identity if the functions are identical on a semi-infinite interval. These conditions are expressed in the term of boundaries for growth for $L_2$-norms of derivatives as well as some integrability condition for the Fourier transforms.
2 Problem setting and main result

For $C > 0$, consider a class $\mathcal{M}(C)$ of infinitely differentiable functions $x(t) : \mathbb{R} \to \mathbb{R}$ such that there exists $M = M(x(\cdot)) > 0$ such that
\[
\left\| \frac{d^k x}{dt^k}(\cdot) \right\|_{L_2(\mathbb{R})}^2 \leq C^k M, \quad k = 0, 2, 4, \ldots.
\]

For $C > 0$, consider a class $\mathcal{N}(C)$ of functions $x(t) : \mathbb{R} \to \mathbb{R}$ such that there exists $M = M(x(\cdot)) > 0$ such that
\[
\frac{1}{2} \left( \left\| \frac{d^{k-1} x}{dt^{k-1}}(\cdot) \right\|_{L_2(\mathbb{R})}^2 + \left\| \frac{d^{k+1} x}{dt^{k+1}}(\cdot) \right\|_{L_2(\mathbb{R})}^2 \right) \leq k! C^{-k} M, \quad k = 1, 3, 5, \ldots. \tag{2.1}
\]

For $q \in \{1, 2\}$, let $\mathcal{X}(q) = \mathcal{X}(q, T)$ be the set of processes $x(\cdot) \in L_2(\mathbb{R}) \cup L_1(\mathbb{R})$ such that
\[
\int_{-\infty}^{+\infty} e^{qT|\omega|} |X(i\omega)|^q d\omega < +\infty,
\]
where $X(i\omega) = \mathcal{F}x$ is the Fourier transform of $x(\cdot)$.

Let $\mathcal{M} \triangleq \cup_{C > 0} \mathcal{M}(C)$, $\mathcal{N} \triangleq \cup_{C > 0} \mathcal{N}(C)$, and $\mathcal{X}(q) \triangleq \cup_{T > 0} \mathcal{X}(q, T)$.

**Theorem 1** Let $x(\cdot) \in \mathcal{M} \cup \mathcal{N} \cup \mathcal{X}(2)$ be such that $x(t) = 0$ for $t < 0$. Then $x(t) \equiv 0$.

**Proof of Theorem 1**. By Propositions 1 and 2, $\mathcal{M} \subset \mathcal{X}(2)$ and $\mathcal{N} \subset \mathcal{X}(2)$. (In this proposition, the set of infinitely differentiable functions is denoted as $C^\infty(\mathbb{R})$). Therefore, it suffices to prove the theorem for $x(\cdot) \in \mathcal{X}(2)$.

Let $T > 0$ be such that $x(\cdot) \in \mathcal{X}(2, T)$.

The following lemma is a special case of Theorem 1.

**Lemma 1** The process $x(\cdot) \in \mathcal{X}(2, T)$ is weakly predictable in the following sense: for any $\varepsilon > 0$ and any kernel $k \in L_\infty(0, T)$, there exists a kernel $\tilde{k}(\cdot) \in L_2(0, +\infty) \cap L_\infty(0, +\infty)$ such that
\[
\|y - \tilde{y}\|_{L_2(\mathbb{R})} \leq \varepsilon,
\]
where
\[
y(t) \triangleq \int_0^{t+T} k(t-s)x(s)ds, \quad \tilde{y}(t) \triangleq \int_0^t \tilde{k}(t-s)x(s)ds.\]

We will use this lemma to prove Theorem 1. First, let us observe that
\[
\tilde{y}(t) = 0 \quad \forall t < 0. \tag{2.2}
\]
Further, let us show that \( x(t) = 0 \) for \( t \in [0, T] \). Let \( \{k_i(\cdot)\}_{i=1}^{+\infty} \) be a basis in \( L_2(-T, 0) \), with continuous bounded functions \( k_i \). Let \( y_i(t) = \int_t^{t+T} k_i(t-s)x(s)ds \). By Lemma \( \[ \] \), it follows from (2.2) that \( y_i(\cdot) |_{t \leq 0} = 0 \) as an element of \( L_2(-\infty, 0) \). Since \( y_i(t) \) is a continuous function, it follows that \( y_i(t) = 0 \) for \( t \leq 0 \). It follows that \( x(\cdot)|_{[0,T]} = 0 \) as an element of \( L_2(0,T) \). By the properties of the class \( \mathcal{X}(2) \), it follows that \( x(t) \) is continuous. Hence \( x(t) = 0 \) for \( t \leq T \).

Further, let us apply the proof given above to the function \( x_1(t) = x(t+T) \). Clearly, \( x_1(\cdot) \in \mathcal{X}(2) \). We found that \( x_1(t) = 0 \) for \( t < 0 \). Similarly, we obtain that \( x_1(t) = 0 \) for all \( t \leq T \), i.e., \( x(t) = 0 \) for all \( t < 2T \). Repeating this procedure \( n \) times, we obtain that i.e., \( x(t) = 0 \) for all \( t < nT \) for all \( n \geq 1 \). This completes the proof of Theorem \( \[ \] \) \[ \]

3 Concluding remarks

We established identity theorem for the functions with exponential rate of decay for the spectrum. We don’t claim that this rate of decay is a necessary condition for identity theorem. However, it can be observed that the polynomial rate of decay for spectrum is insufficient for identity property. For instance, consider a process \( x(\cdot) \in L_2(\mathbb{R}) \) with Fourier transform \( X(i\omega) = (1 + i\omega)^{-N-1} \), where \( N > 0 \) is an arbitrarily large integer. Then \( x(t) = 0 \) for all \( t \leq 0 \) and

\[
\int_{-\infty}^{+\infty} |\omega|^{2N} |X(i\omega)|^2 d\omega < +\infty.
\]

Further, Lemma \( \[ \] \) claims predicability functions from \( \mathcal{X}(T) \) on the finite horizon \( T \) only. At the same time, Theorem \( \[ \] \) states that are uniquely defined on \( \mathbb{R} \) by their values at a semi-infinite interval. It does not mean that Lemma \( \[ \] \) is strengthened here: prediction on a horizon larger than \( T \) would be a ill-posed problem that yet has a unique solution.

References

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