From Short-Range to Mean-Field Models in Quantum Lattices

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Abstract

Realistic effective interparticle interactions of quantum many-body systems are widely seen as being short-range. However, the rigorous mathematical analysis of this type of model turns out to be extremely difficult, in general, with many important fundamental questions remaining open still nowadays. By contrast, mean-field models come from different approximations or Ansätze, and are thus less realistic, in a sense, but are technically advantageous, by allowing explicit computations while capturing surprisingly well many real physical phenomena. Here, we establish a precise mathematical relation between mean-field and short-range models, by using the long-range limit that is known in the literature as the Kac, or van der Waals, limit. If both attractive and repulsive long-range forces are present then it turns out that the limit mean-field model is not necessarily what one traditionally guesses. One important innovation of our study, in contrast with previous works on the subject, is the fact that we are able to show the convergence of equilibrium states, i.e., of all correlation functions. This paves the way for studying phase transitions, or at least important fingerprints of them like strong correlations at long distances, for models having interactions whose ranges are finite, but very large. It also sheds a new light on mean-field models. Even on the level of pressures, our results go considerably further than previous ones, by allowing, for instance, a continuum of long-range interaction components, as well as very general short-range Hamiltonians for the “free” part of the model. The present results were made possible by the variational approach of [1] for equilibrium states of mean-field models, as well as the game theoretical characterization of these states. Our results are obtained in an abstract, model-independent, way.

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\section{Introduction}

In theoretical physics, realistic effective interparticle interactions of quantum many-body systems are widely seen as being short-range, like in the celebrated Hubbard model or models with Yukawa-type potentials. However, the rigorous mathematical analysis of this type of model turns out to be extremely difficult, in general, with many important fundamental questions (referring to the structure of phase diagrams, transport properties, etc.) remaining open still nowadays. In contrast with short-range models, mean-field models come from different approximations or Ansätze, and are thus less realistic, in a sense, but handy, by allowing explicit computations while capturing surprisingly well many important real physical phenomena, like for instance the superconductivity. They are thus essential in condensed matter physics. Note that a mean-field interaction is, at least in a formal sense, an extreme form of interaction that is weak, but acts across very long distances. The aim of this paper is thus to establish a precise mathematical relation between mean-field and short-range models, by using the long-range limit known in the literature as the Kac limit, or the \textit{van der Waals limit}. This is done in an abstract, model-independent, way.

To understand our motivations and results, it is instructive to consider the following example:

\textbf{Notation.} $\mathcal{L} \cong \mathbb{Z}^d$ represents a cubic crystal of dimension $d \in \mathbb{N}$. In order to define the thermodynamic limit, we use the cubic boxes $\Lambda_L \cong \{ \mathbb{Z} \cap [-L, L] \}^d$ of volume $|\Lambda_L|$, where $L \in \mathbb{N}_0$. Let $S$ be some finite set representing an orthonormal basis of spin modes. For instance, $S = \{\uparrow, \downarrow\}$, referring to fermions with spin $\uparrow$ and $\downarrow$, like electrons. Fermionic annihilation operators are denoted by $a_{x,s}$ for $x \in \mathcal{L}$ and $s \in S$. The parameter $\beta \in \mathbb{R}^+$ fixes the inverse temperature.

\textbf{Short-range model.} On the one hand, given two functions $h, f : \mathbb{R}^d \to \mathbb{R}$ and a parameter $\gamma \in (0, 1)$, consider a translation-invariant lattice fermion system via box Hamiltonians defined by

$$H_{L \mathbb{Q}} = \sum_{x,y \in \Lambda_L, s \in S} h(x - y) a_{x,s}^* a_{y,s} + \sum_{x,y \in \Lambda_L, s,t \in S} \gamma^d f(\gamma(x - y)) a_{y,t}^* a_{y,t} a_{x,s}^* a_{x,s}, \quad L \in \mathbb{N}_0.$$

The function $h$ represents the hopping term\footnote{For instance, take $h(x - y) = -\langle \mathcal{e}_x, \mathcal{D}_d \mathcal{e}_y \rangle_{\mathcal{E}(\mathcal{L})}$ for $x,y \in \mathcal{L}$, where $\mathcal{D}_d \in \mathcal{B}(\ell^2(\mathcal{L}))$ is the usual $d$-dimensional discrete Laplacian and with $\{ \mathcal{e}_x \}_{x \in \mathcal{E}(\mathcal{L})}$ being the canonical orthonormal basis of $\ell^2(\mathcal{L})$.}, or kinetic part, of the lattice fermion system and is assumed to decay fast at large distances, while $f$ is a pair potential characterizing the interparticle interaction, whose range is tuned by the parameter $\gamma \in (0, 1)$. As is usual in theoretical physics, $f$ is assumed to be fast decaying\footnote{Take for instance a compactly supported function $f$.}, reflection-symmetric\footnote{I.e., $f(x) = f(-x)$. Usually, $f(x) = v(|x|)$ for some function $v : \mathbb{R}^d_0 \to \mathbb{R}$.} and positive definite (i.e., the Fourier transform $\hat{f}$ of $f$ is a positive function on $\mathbb{R}^d$). This choice for $f$ is reminiscent of a superstability condition, which is essential in the bosonic case [2 Section 2.2 and Appendix G].

For any inverse temperature $\beta \in \mathbb{R}^+$, the infinite volume pressure

$$P_{\text{SR}}(\gamma) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \ln \text{Trace}(e^{-\beta H_{L \mathbb{Q}}}), \quad \gamma \in (0, 1),$$

is well-defined and given by a variational problem for translation-invariant states. See, e.g., [1 Theorem 2.12]. Equilibrium states can thus be naturally defined as being the solutions (minimizers) to this variational problem. However, a mathematically rigorous computation of the pressure and equilibrium states to show possible phase transitions is elusive, beyond perturbative arguments, even after decades of mathematical studies.

\textbf{Mean-Field model.} On the other hand, instead of the above model, one may study a mean-field version of it, defined by the box Hamiltonians

$$H_{L \mathbb{Q}} = \sum_{x,y \in \Lambda_L, s \in S} h(x - y) a_{x,s}^* a_{y,s} + \frac{\eta}{|\Lambda_L|} \sum_{x,y \in \Lambda_L, s,t \in S} a_{y,t}^* a_{y,t} a_{x,s}^* a_{x,s}, \quad L \in \mathbb{N}_0.$$

\footnotesize
1\footnotesize{For instance, take $h(x - y) = -\langle \mathcal{e}_x, \mathcal{D}_d \mathcal{e}_y \rangle_{\mathcal{E}(\mathcal{L})}$ for $x,y \in \mathcal{L}$, where $\mathcal{D}_d \in \mathcal{B}(\ell^2(\mathcal{L}))$ is the usual $d$-dimensional discrete Laplacian and with $\{ \mathcal{e}_x \}_{x \in \mathcal{E}(\mathcal{L})}$ being the canonical orthonormal basis of $\ell^2(\mathcal{L})$.}

2\footnotesize{Take for instance a compactly supported function $f$.}

3\footnotesize{I.e., $f(x) = f(-x)$. Usually, $f(x) = v(|x|)$ for some function $v : \mathbb{R}^d_0 \to \mathbb{R}$.}
for some positive parameter \( \eta \in \mathbb{R}_0^+ \). For any \( \beta \in \mathbb{R}_0^+ \), the infinite volume pressure

\[
P_{\text{MF}}(\eta) = \lim_{L \to \infty} \frac{1}{|A_L|} \ln \text{Trace}(e^{-\beta H_{\text{MF}}^c}) , \quad \eta \in \mathbb{R}_0^+ ,
\]
is well-defined and given by a variational problem for translation-invariant states, thanks to [1, Theorem 2.12]. A general notion of (generalized) equilibrium states can again be given via this variational problem. What is more, they can be explicitly computed through an alternative version of the approximating Hamiltonian method [3–8] (see [1, Sections 2.10 and 10.2] for more details), named *thermodynamic game* (Section 4.4.3), introduced in [1, Section 2.7]. In this case, [1, Theorem 2.36] shows that

\[
P_{\text{MF}}(\eta) = \inf_{c \in \mathbb{C}} \{ |c|^2 + P(c, \eta) \} , \quad \eta \in \mathbb{R}_0^+ ,
\]
where \( P : \mathbb{C} \times \mathbb{R}_0^+ \to \mathbb{R} \) is the infinite volume pressure defined by

\[
P(c, \eta) = \lim_{L \to \infty} \frac{1}{|A_L|} \ln \text{Trace}(e^{-\beta H_L(c)}) , \quad c \in \mathbb{C}, \eta \in \mathbb{R}_0^+ ,
\]
and

\[
H_L(c) = \sum_{x,y \in A_L, a \in S} h(x-y) a^*_x a_y + 2\eta \Re\{c\} \sum_{x \in A_L, a \in S} a^*_x a_x .
\]

This Hamiltonian is quadratic in terms of creation and annihilations operators and can thus be explicitly diagonalized, as is well-known. It means that the function (4) can be computed and the variational problem (3) studied by analytic or rigorous numerical methods. Last but not least, [1, Theorems 2.21 and 2.39] also show that (generalized) equilibrium states can be obtained by using self-consistency conditions, which refer, in a sense, to Euler-Lagrange equations for the variational problem (3). See Section 4.5.3 for a concise explanation of this point.

**Kac Limit.** The Kac, or long-range, limit refers here to the limit \( \gamma \to 0^+ \) of short-range models that are already in the thermodynamic limit. For small parameters \( \gamma \ll 1 \), the short-range model defined in finite volume by [1] has an interparticle interaction with very large range \( (O(\gamma^{-1})) \), but the interaction strength is small as \( \gamma^d \), in such a way that the first Born approximation is the scattering length of the interparticle potential remains constant, as is usual. One therefore expects to have some effective long-range model in the limit \( \gamma \to 0^+ \). This is explicitly what we prove here by showing, among other things, that

\[
\lim_{\gamma \to 0^+} P_{\text{SR}}(\gamma) = P_{\text{MF}}(\hat{f}(0)) ,
\]

where \( \hat{f} \) is the (positive) Fourier transform of the two-body interaction potential \( f \). See Equation (116). Explicit computations of the rate of convergence can also be done by using our estimates, see Proposition 5.6 as well as the proof of Theorem 5.7. More importantly, the equilibrium states of the short-range model can also be approximated by generalized equilibrium states of the corresponding mean-field model, as shown by Theorem 5.7. Recall that a precise study of phase transitions for short-range models is notoriously difficult, but we show here that it can be done in the Kac limit via a mean-field model for which efficient mathematical methods (for instance, what we call the thermodynamic game) can be used.

\[ ^4 \text{I.e., } \int_{\mathbb{R}^d} \gamma^d f(\gamma x) \, dx = \int_{\mathbb{R}^d} f(x) \, dx = \hat{f}(0). \]

\[ ^5 \text{The existence of a phase transition, like a first-order one, in the limiting mean-field model does not necessarily imply a phase transition in the corresponding short-range model for small, but nonzero } \gamma \ll 1. \text{ However, this convergence can imply important properties on correlation functions of the short-range model for small } \gamma. \text{ This is similar to what occurs in the thermodynamic limit: Recall that the equilibrium states are always unique at finite volume, whereas first order phase transitions do take place at infinite volume. See discussions in [1, Section 2.6].} \]
The results referring to the above examples are already highly non-trivial and use in an essential way various fundamental outcomes of [1]. Nevertheless, our present results go far beyond this example, by showing, for instance, that the kinetic part (represented by the function $h$) can be replaced with a very general short-range fermionic Hamiltonian. See Theorem 5.7 which is in turn generalized by Theorem 5.9 to include the case of possibly infinitely many long-range repulsions. The main limitation in this example is the fact that the function $f$ has to be a positive definite function, which leads to a purely repulsive mean-field model (Section 4.2), in the long-range (Kac) limit. This situation excludes some important models of physics like BCS-type models (e.g., the reduced BCS Hamiltonian or the strong coupling BCS Hamiltonian), which have purely attractive mean-field (interparticle) interactions, see discussions after Theorem 5.11.

In the present paper, we provide results that do not depend upon the positive definiteness of the pair potential $f$. In particular, general purely attractive interparticle interactions, like the BCS model (see (91) and (122)), are included. See Theorems 5.11 and 5.13. The general case, i.e., the case of long-range interaction forces which are neither purely attractive nor purely repulsive, refers to Proposition 5.14, Theorems 5.15, 5.17 and 5.19 as well as Corollaries 5.16 and 5.18. See also Section 5.6.2. As expected, any Kac limit leads to mean-field pressures and equilibrium states. However, the limit mean-field model is not necessarily what one traditionally guesses. In fact, it strongly depends upon the hierarchy of ranges between attractive and repulsive interparticle forces. For instance, if the range of repulsive forces is much larger than the range of the attractive ones, then in the Kac limit for these forces one may get a mean-field model that is non-conventional. See, e.g., Theorems 5.17 and Sections 4.4.2, 4.5.2. However, we also show in Theorems 5.9 and 5.13 that, in the case of purely attractive or purely repulsive long-range forces, the formally expected mean-field model is in fact the correct effective long-range model for the Kac limit. See also Theorem 5.22.

Our paper follows a rather old sequence of studies on the Kac limit, basically starting from 1959 with Kac’s work on classical one-dimensional spin systems. The first important result [9] in this period was provided by Penrose and Lebowitz in 1966, who proved the convergence of the free energy of a classical system towards the one of the van der Waals theory. Shortly after, the results of this seminal paper were extended to quantum systems (Boltzmann, Bose, or Fermi statistics) by Lieb in [10]. In 1971, Penrose and Lebowitz went considerably further than [9] with the paper [11]. See also [12] for a review of all these results of classical statistical mechanics. These outcomes form the mainstays of the subsequent results on the Kac limit and we recommend the book [13], published in 2009, for a more recent review on the subject in classical statistical mechanics, including the so-called Lebowitz-Penrose theorem and a more exhaustive list of references.

Studies on the Kac limit are still performed nowadays in classical statistical mechanics, see, e.g., [14–16]. By contrast, to our knowledge, not much was done on the subject for quantum systems after the first main important results [10] in 1966 (which refer to quantum particles in the continuum, but may certainly be extended to lattice systems). The second important results [17] concern the Kac limit (i.e., the van der Waals limit) of a Bose gas in presence of two-body interactions in the grand-canonical ensemble and in the continuum, in the spirit of [10]. [17] is liminal because, by using a scaled external field, the authors show for the first time that the Kac limit can lead to another mean-field-type model than the expected mean-field gas at high densities. See discussions at the end of [17] for more details. More than three decades later, we recover this general observation by combining instead repulsive and attractive interactions, without any scaled-external-field perturbation method as used in [17]. Since the 2000’s we are only aware of a few papers in relation to the Kac limit for quantum systems: In 2003 [18, 19], the Kac limit is used within a diagrammatic approach and in 2005 [20] the same authors study corrections to the Kac limit for the Bose system used in [17], but without scaled external fields. In 2019, the Kac limit is also used again for the Bose gas in the Hartree-Fock approximation to analyze (again) the Bose–Einstein condensation [21]. This shows that

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6Such models were introduced for the first time in [1].
The research activity on the subject is still nowadays quite restricted.

The main innovation of our present study is the fact that we are able to show convergence in the Kac limit not only for pressure-like quantities (for instance, the thermodynamic limit of the logarithm of canonical or grand-canonical partition functions), as in previous works, but also for equilibrium states, i.e., for all correlation functions. Our results on states were made possible by the variational approach of [1] for equilibrium states of mean-field models. Additionally, also in contrast with previous results on Kac limits, our method allows for coexistence of both attractive and repulsive long-range forces. This important extension is related to the game theoretical characterization of equilibrium states of mean-field models (cf. thermodynamic game) also introduced in [1]. Our study thus paves the way for studying phase transitions, or at least important fingerprints of them like strong correlations at long distances, for models having interactions whose ranges are finite, but very large. It also sheds a new light on mean-field models by connecting them with short-range ones, in a mathematically precise manner (like in [17]). Even on the level of the partition functions (here the grand-canonical pressure), our results go considerably further than previous ones, by allowing for instance a continuum of mean-field components (cf. Definitions 5.1 and 5.2), as well as very general short-range Hamiltonians for the “free” part of the model.

To conclude, similar to the seminal paper [22] reviving in 1998 an old problem on dilute Bose gases studied by Dyson in 1957, our paper sheds new light on old questions about long-range limits of short-range interactions. We think that such studies will be important for future theoretical developments in many-body theory, since long-range interactions are expected to imply effective, classical background fields, in the spirit of the Higgs mechanism of quantum field theory. This is shown in [23–25] for mean-field models.

The paper is organized as follows: Section 2 presents the general mathematical framework. Observe that we focus on lattice fermion systems. See Remark 1.1. In Section 3 we give a brief account on the theory of fermion systems on the lattice with short-range interactions, which is used as a springboard to introduce the theory of mean-field models in Section 4. Note that the mathematical setting used in the current paper is – up to minor modifications – the one of [1, 23], including the notation. We thus provide it in a concise way. Section 5 establishes precise mathematical relations between short-range and mean-field models via Kac limits, in which concerns both the pressure and equilibrium states. This section includes our main results, which are Theorems 5.15, 5.17, 5.19 and 5.22, Propositions 5.14 and 5.21 as well as Corollaries 5.16 and 5.18. Since the results are general and abstract, for convenience, in Section 6 we illustrate them on models of the form (5), to which an attractive long-range interaction term is incorporated (so that a competition between attractive and repulsive long-range forces takes place). In fact, this section was designed to be read almost independently of the other ones, allowing the (possibly non-expert) reader to catch more easily the main results of the paper. Section 7 is an appendix that gathers some useful technical results.

Remark 1.1 (Quantum spin systems)
Our study focuses on lattice fermion systems, which are, from a technical point of view, slightly more difficult than quantum spin systems, because of a non-commutativity issue at different lattice sites. However, all the results presented here hold true for quantum spin systems, via obvious modifications.

Remark 1.2 (Periodic quantum lattice systems)
Our study focuses on lattice fermion systems that are translation-invariant (in space). However, all

Mean-field repulsions have generally a geometrical effect by possibly breaking the face structure of the set of (generalized) equilibrium states (see [1, Lemma 9.8]). When this appears, we have phase transitions with long-range order. See [1, Section 2.9].

See all studies done during the last two decades in relation with the Gross-Pitaevskii theory and mean-field systems for indistinguishable particles (bosons).
the results presented here hold true for (space-)periodic lattice fermion systems, by appropriately redefining the spin set

A similar argument holds true for quantum spin systems.

\section{Algebraic Formulation of Lattice Fermion Systems}

\subsection{CAR Algebra for Lattice Fermions}

\subsubsection{Background Lattice}

Fix once and for all the dimension \( d \in \mathbb{N} \) of the (cubic) lattice. Let \( \mathcal{L} = \mathbb{Z}^d \) and \( \mathcal{P}_L \subseteq 2^\mathcal{L} \) be the set of all non-empty finite subsets of the lattice \( \mathcal{L} \). In order to define the thermodynamic limit, we use the cubic boxes

\[ \Lambda_L = \{ (x_1, \ldots, x_d) \in \mathcal{L} : |x_1|, \ldots, |x_d| \leq L \} \in \mathcal{P}_L, \quad L \in \mathbb{N}_0, \]

as a so-called van Hove sequence.

\subsubsection{The CAR \( C^* \)-Algebra}

For any nonempty subset \( \Lambda \subseteq \mathcal{L} \), \( \mathcal{U}_\Lambda \) denotes the universal unital \( C^* \)-algebra generated by elements \( \{a_{x,s}\}_{x \in \Lambda, s \in S} \) satisfying the canonical anti-commutation relations (CAR):

\[
\begin{align*}
    a_{x,s}a_{y,t} + a_{y,t}a_{x,s}^* &= 0, \\
    a_{x,s}a_{y,t}^* + a_{y,t}a_{x,s} &= \delta_{x,y}\delta_{s,t}\mathbf{1},
\end{align*}
\]

where \( \mathbf{1} \) stands for the unit of the algebra, \( \delta_{x,y} \) is the Kronecker delta and \( S \) is some finite set (representing an orthonormal basis of spin modes), which is fixed once and for all. If \( \Lambda = \emptyset \) then \( \mathcal{U}_\Lambda = \mathbb{C} \).

We use the notation

\[ |A|^2 = A^*A, \quad A \in \mathcal{U}_\Lambda, \quad \Lambda \subseteq \mathcal{L}, \]

(8)

to shorten mathematical expressions, in particular in the context of mean-field models.

By identifying the units and the generators \( \{a_{x,s}\}_{x \in \Lambda \setminus \Lambda', s \in S} \) in any two \( C^* \)-algebras \( \mathcal{U}_\Lambda \) and \( \mathcal{U}_{\Lambda'} \), \( \{\mathcal{U}_\Lambda\}_{\Lambda \in \mathcal{P}_L} \) canonically forms a net of unital \( C^* \)-algebras with respect to inclusion: For all subsets \( \Lambda, \Lambda' \subseteq \mathcal{L} \) so that \( \Lambda \subseteq \Lambda' \), one has \( \mathcal{U}_\Lambda \subseteq \mathcal{U}_{\Lambda'} \). For \( \Lambda = \mathcal{L} \) we use the notation \( \mathcal{U} \equiv \mathcal{U}_\mathcal{L} \).

Observe additionally that the subspace

\[ \mathcal{U}_0 = \bigcup_{\Lambda \in \mathcal{P}_L} \mathcal{U}_\Lambda \subseteq \mathcal{U} \equiv \mathcal{U}_\mathcal{L}, \]

(9)

is a dense *-algebra of the CAR \( C^* \)-algebra \( \mathcal{U} \) of the infinite lattice. In particular, \( \mathcal{U} \) is separable, because \( \mathcal{U}_\Lambda \) has finite dimension for all (finite subsets) \( \Lambda \in \mathcal{P}_L \) and the collection \( \mathcal{P}_L \) of sets is countable. Elements of \( \mathcal{U}_0 \) are called local elements of \( \mathcal{U} \). The (real) Banach subspace of all self-adjoint elements of \( \mathcal{U} \) is denoted by \( \mathcal{U}^{\text{sa}} \subseteq \mathcal{U} \).

The local causality of quantum field theory is broken in CAR algebras and physical quantities are therefore defined from the subalgebra of even elements, which are defined as follows: Given a fixed parameter \( \theta \in \mathbb{R}/(2\pi\mathbb{Z}) \), the condition

\[ g_\theta(a_{x,s}) = e^{-i\theta}a_{x,s}, \quad x \in \mathbb{Z}^d, \quad s \in S, \]

(10)

defines a unique *-automorphism \( g_\theta \) of the \( C^* \)-algebra \( \mathcal{U} \). Note that, for any \( \Lambda \subseteq \mathcal{L} \), \( g_\theta(\mathcal{U}_\Lambda) \subseteq \mathcal{U}_\Lambda \) and thus \( g_\theta \) canonically defines a *-automorphism of the subalgebra \( \mathcal{U}_\Lambda \). A special role is played by \( g_\pi \). Elements \( A, B \in \mathcal{U}_\Lambda, \Lambda \subseteq \mathcal{L} \), satisfying \( g_\pi(A) = A \) and \( g_\pi(B) = -B \) are respectively called

\footnote{In fact, one can see the lattice points in a (space) period as a single point in an equivalent lattice on which particles have an enlarged spin set.}
even and odd. Every element of the algebra can be decomposed into a sum of even and odd terms. (Elements \( A \in \mathcal{U}_\Lambda \) satisfying \( g_{\pi}(A) = A \) for any \( \theta \in \mathbb{R}/(2\pi\mathbb{Z}) \) are called gauge invariant.) The space of even elements of \( \mathcal{U} \) is denoted by
\[
\mathcal{U}^+ = \{ A \in \mathcal{U} : A = g_{\pi}(A) \} \subseteq \mathcal{U} .
\] (11)

It is a unital \( C^* \)-subalgebra of the \( C^* \)-algebra \( \mathcal{U} \). In physics, \( \mathcal{U}^+ \) is seen as more fundamental than \( \mathcal{U} \), because of the local causality in quantum field theory, which holds in the first \( C^* \)-algebra, but not in the second one. See, e.g., discussions in [23], Section 2.3.

### 2.2 States of Lattice Fermion Systems

#### 2.2.1 Even States

States on the \( C^* \)-algebra \( \mathcal{U} \) are, by definition, linear functionals \( \rho : \mathcal{U} \to \mathbb{C} \) which are positive, i.e., for all elements \( A \in \mathcal{U} \), \( \rho(|A|^2) \geq 0 \), and normalized, i.e., \( \rho(1) = 1 \). Equivalently, the linear functional \( \rho \) is a state iff \( \rho(1) = 1 \) and \( \|\rho\|_{\mathcal{U}^*} = 1 \). The set of all states on \( \mathcal{U} \) is denoted by
\[
E = \bigcap_{A \in \mathcal{U}} \{ \rho \in \mathcal{U}^* : \rho(1) = 1, \ \rho(|A|^2) \geq 0 \} .
\] (12)

This convex set is metrizable and compact with respect to the weak* topology. Mutatis mutandis, for every \( \Lambda \subseteq \mathcal{L} \), we define the set \( E_\Lambda \) of all states on the sub-algebra \( \mathcal{U}_\Lambda \subseteq \mathcal{U} \). For any \( \Lambda \subseteq \mathcal{L} \), the symbol \( \rho_\Lambda \) denotes the restriction of any \( \rho \in E \) to the sub-algebra \( \mathcal{U}_\Lambda \). This restriction is clearly a state on \( \mathcal{U}_\Lambda \).

Even states on \( \mathcal{U}_\Lambda \), \( \Lambda \subseteq \mathcal{L} \), are, by definition, the states \( \rho \in E_\Lambda \) satisfying \( \rho \circ g_\pi = \rho \). In other words, the even states on \( \mathcal{U}_\Lambda \) are exactly those vanishing on all odd elements of \( \mathcal{U}_\Lambda \). The set of even states on \( \mathcal{U} \) can be canonically identified with the set of states on the \( C^* \)-subalgebra \( \mathcal{U}^+ \) of even elements, by [23], Proof of Proposition 2.1. As a consequence, physically relevant states on \( \mathcal{U} \) are even.

#### 2.2.2 Translation-Invariant States

Lattice translations refer to the group homomorphism \( x \mapsto \alpha_x \) from \((\mathbb{Z}^d, +)\) to the group of \(*\)-automorphisms of the CAR \( C^* \)-algebra \( \mathcal{U} \) of the (infinite) lattice \( \mathcal{L} \), which is uniquely defined by the condition
\[
\alpha_x(a_{y,s}) = a_{y+x,s}, \quad y \in \mathcal{L}, \ s \in S .
\] (13)

Via this group homomorphism we define the translation invariance of states and interactions of lattice fermion systems.

The state \( \rho \in E \) is said to be translation-invariant iff it satisfies \( \rho \circ \alpha_x = \rho \) for all \( x \in \mathbb{Z}^d \). The space of translation-invariant states on \( \mathcal{U} \) is the convex set
\[
E_1 = \bigcap_{x \in \mathbb{Z}^d} \{ \rho \in \mathcal{U}^* : \rho(1) = 1, \ \rho(|A|^2) \geq 0, \ \rho = \rho \circ \alpha_x \} ,
\] (14)

which is again metrizable and compact with respect to the weak* topology. Any translation-invariant state is even. See, for instance, [1], Lemma 1.8]. Thanks to the Krein-Milman theorem [26], Theorem 3.23], \( E_1 \) is the weak*‐closure of the convex hull of the (non-empty) set \( \mathcal{E}(E_1) \) of its extreme points, which turns out to be a weak*‐dense \((G_{\delta})\) subset [1], Corollary 4.6]:
\[
E_1 = \overline{\mathcal{E}(E_1)} = \overline{\mathcal{E}(E_1)} ,
\] (15)
where $\overline{\omega}(K)$ denotes the weak$^*$-closed convex hull of a set $K$. This fact is well-known and is also true for quantum spin systems on lattices [27, Example 4.3.26 and discussions p. 464].

Since $E_1$ is metrizable (because of the separability of $\mathcal{U}$), the Choquet theorem applies: By [1, Theorem 1.9], for any $\rho \in E_1$, there is a unique probability measure $\mu_\rho$ on $E_1$ such that
\[
\mu_\rho(E(E_1)) = 1
\]
and
\[
\rho(A) = \int_{E_1} \hat{\rho}(A) \, \mu_\rho(d\hat{\rho}) , \quad A \in \mathcal{U} .
\]
(16)

In particular, $E_1$ is a Choquet simplex. In fact, up to an affine homeomorphism, $E_1$ is the so-called Poulsen simplex [1, Theorem 1.12]. (Note in passing that $\mu_\rho$ is an orthogonal measure, thanks to [24, Theorem 5.1].)

The unique decomposition of a translation-invariant state $\rho \in E_1$ in terms of extreme translation-invariant states $\hat{\rho} \in E(E_1)$ is also called the ergodic decomposition of $\rho$ because of the following fact: Define the space-averages of any element $A \in \mathcal{U}$ by
\[
A_L = \frac{1}{|A_L|} \sum_{x \in A_L} \alpha_x(A) , \quad L \in \mathbb{N}_0 .
\]
(17)

Then, by definition, a translation-invariant state $\hat{\rho} \in E_1$ is said to be ergodic if
\[
\lim_{L \to \infty} \hat{\rho}(|A_L|^2) = |\hat{\rho}(A)|^2 , \quad A \in \mathcal{U} .
\]
(18)

(Recall Equation (13).) By [1, Theorem 1.16], any extreme translation-invariant state is ergodic and vice-versa. In other words, the set of extreme translation-invariant states is equal to
\[
E(E_1) = \{ \hat{\rho} \in E_1 : \hat{\rho} \text{ is ergodic} \} = \bigcap_{A \in \mathcal{U}} \left\{ \hat{\rho} \in E_1 : \lim_{L \to \infty} \hat{\rho}(|A_L|^2) = |\hat{\rho}(A)|^2 \right\} .
\]
(19)

\section{3 Short-Range Models}

\subsection{3.1 The Banach Space of Interactions}

In the algebraic setting, a (complex) interaction is, by definition, any mapping $\Phi : \mathcal{P}_1 \to \mathcal{U}^+$ from the set $\mathcal{P}_1 \subseteq 2^\mathcal{L}$ of all finite subsets of $\mathcal{L}$ to the $C^*$-subalgebra $\mathcal{U}^+$ of even elements such that $\Phi_\Lambda \in \mathcal{U}_\Lambda$ for all $\Lambda \in \mathcal{P}_1$. The set $\mathcal{V}$ of all interactions can be naturally endowed with the structure of a complex vector space (via the usual point-wise vector space operations), as well as with the antilinear involution
\[
\Phi \mapsto \Phi^* = (\Phi_\Lambda^*)_{\Lambda \in \mathcal{P}_1} .
\]
(20)

An interaction $\Phi$ is said to be self-adjoint iff $\Phi = \Phi^*$. The set $\mathcal{V}^R$ of all self-adjoint interactions forms a real subspace of the space of all (complex) interactions.

By definition, the interaction $\Phi \in \mathcal{V}$ is translation-invariant if
\[
\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda) , \quad x \in \mathbb{Z}^d , \quad \Lambda \in \mathcal{P}_1 ,
\]
(21)
where $\{\alpha_x\}_{x \in \mathbb{Z}^d}$ is the set of (translation) $*$-automorphisms of $\mathcal{U}$ defined by (13), while
\[
\Lambda + x = \{y + x : y \in \Lambda\} , \quad x \in \mathbb{Z}^d , \quad \Lambda \in \mathcal{P}_1 .
\]
(22)

\footnote{For $E$ is a metrizable compact space, any finite Borel measure is regular and tight. Thus, here, probability measures are just the same as normalized Borel measures.}
The space of translation-invariant interactions is denoted by

\[ \mathcal{V}_1 \doteq \bigcap_{x \in \mathbb{Z}^d, \Lambda \in \mathcal{P}_f} \{ \Phi \in \mathcal{V} : \Phi_{\Lambda+x} = \alpha_x (\Phi_{\Lambda}) \} \subsetneq \mathcal{V}. \]

Using the norm

\[ \| \Phi \|_{\mathcal{W}_1} \doteq \sum_{\Lambda \in \mathcal{P}_f, \Lambda \supseteq \{0\}} |\Lambda|^{-1} \| \Phi_{\Lambda} \|_{\mathcal{U}}, \quad \Phi \in \mathcal{V}_1, \]  

we obtain a separable Banach space

\[ \mathcal{W}_1 \doteq \{ \Phi \in \mathcal{V}_1 : \| \Phi \|_{\mathcal{W}_1} < \infty \} \]

of translation-invariant interactions. The (real) Banach subspace of interactions that are simultaneously self-adjoint and translation-invariant, is denoted by

\[ \mathcal{W}_1^{\mathbb{R}} \doteq \mathcal{V}^{\mathbb{R}} \cap \mathcal{W}_1, \]  

similar to \( \mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U} \) and \( \mathcal{V}^{\mathbb{R}} \subsetneq \mathcal{V} \).

Additionally, for any \( \Lambda \in \mathcal{P}_f \) with \( \Lambda \ni 0 \), we define the closed subspace\(^{11}\)

\[ \mathcal{W}_{\Lambda} \doteq \{ \Phi \in \mathcal{W}_1 : \Phi_{\mathcal{Z}} = 0 \text{ whenever } \mathcal{Z} \not\subseteq \Lambda, \mathcal{Z} \ni 0 \} \]

of interactions that are simultaneously finite-range and translation-invariant. Similar to \( \mathcal{U} \)\(^9\),

\[ \mathcal{W}_0 \doteq \bigcup_{L \in \mathbb{N}_0} \mathcal{W}_{\Lambda_L} \subseteq \mathcal{W}_1 \]

is a dense subspace of \( \mathcal{W}_1 \). A translation-invariant interaction \( \Phi \in \mathcal{W}_1 \) is said to be \textit{finite-range} if it lies in \( \mathcal{W}_0 \). As usual, \( \mathcal{W}_0^{\mathbb{R}} \doteq \mathcal{V}^{\mathbb{R}} \cap \mathcal{W}_0 \).

To conclude, note that the norm defined by \( \| \Phi \|_{\mathcal{W}_1} \) is rather weak. In fact, the Banach space \( \mathcal{W}_1 \) is quite large and includes interactions having a long-range character in which concerns dynamics, in the sense that usual Lieb-Robinson bounds may not hold true for them. Models associated with \( \mathcal{W}_1 \) can however be properly named (quasi-)short-range, in contrast with the mean-field models of Sections 4, which refer to an idealization of interactions that are weak, but act across extremely long distances.

### 3.2 Energy Density Functionals on Translation-Invariant States

Local energy elements associated with a given complex interaction \( \Phi \in \mathcal{V} \) correspond to the following sequence within the \( C^\ast \)-subalgebra \( \mathcal{U}^\ast \)\(^{11}\) of even elements:

\[ U^\Phi_L \doteq \sum_{\Lambda \subseteq \Lambda_L} \Phi_{\Lambda} \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^\ast, \quad L \in \mathbb{N}_0, \]

where we recall that \( \Lambda_L, L \in \mathbb{N}_0 \), are the cubic boxes\(^6\) used to define the thermodynamic limit.

The energy density of a state \( \rho \in \mathcal{E} \) with respect to a given interaction \( \Phi \in \mathcal{V} \) is defined by

\[ e_\Phi (\rho) \doteq \limsup_{L \to \infty} \frac{\text{Re}\{\rho (U^\Phi_L)\}}{|\Lambda_L|} + i \limsup_{L \to \infty} \frac{\text{Im}\{\rho (U^\Phi_L)\}}{|\Lambda_L|} \in [-\infty, \infty] + i[\infty, \infty]. \]

If \( \Phi \in \mathcal{V}^{\mathbb{R}} \) is self-adjoint then \( (U^\Phi_L)_{L \in \mathbb{N}_0} \) is a sequence (of local Hamiltonians) in \( \mathcal{U}^{\mathbb{R}} \) and thus, \( e_\Phi (\rho) \) belongs to \( [-\infty, \infty] \) for all states \( \rho \in \mathcal{E} \).

\(^{11}\)\( \mathcal{W}_{\Lambda} \) is a closed subspace of \( \mathcal{W}_1 \) because of the continuity and linearity of the mappings \( \Phi \mapsto \Phi_{\mathcal{Z}} \) for all \( \mathcal{Z} \in \mathcal{P}_f \).
By [23, Proposition 3.2], for any translation-invariant state \( \rho \in E_1 \) (14) and each translation-invariant interaction \( \Phi \in \mathcal{W}_1 \),

\[
e_\Phi (\rho) = \lim_{L \to \infty} \frac{\rho(U^\Phi_L)}{|\Lambda_L|} = \rho(e_\Phi),
\]

where \( e_\phi : \mathcal{W}_1 \to \mathcal{U}^+ \) is the continuous mapping from the Banach space \( \mathcal{W}_1 \) to the \( C^* \)-algebra \( \mathcal{U}^+ \subseteq \mathcal{U} \), defined by

\[
e_\Phi = \sum_{Z \in \mathcal{P}, \, Z \ni 0} e_Z \Phi Z |Z| \in \mathcal{U}^+, \quad \Phi \in \mathcal{W}_1.
\]

In particular, for any fixed \( \Phi \in \mathcal{W}_1 \), the mapping \( \rho \mapsto e_\Phi (\rho) \) from \( E_1 \) to \( \mathbb{C} \) is a weak*-continuous affine functional. Last but not least, by straightforward estimates using Equation (23), note that, for all translation-invariant interactions \( \Phi, \Psi \in \mathcal{W}_1 \),

\[
\|U^\Phi_L - U^\Psi_L\|_{\mathcal{U}} \leq |\Lambda_L| \|\Phi - \Psi\|_{\mathcal{W}_1}, \quad L \in \mathbb{N}_0,
\]

and, for all translation-invariant states \( \rho \in E_1 \),

\[
|e_\Phi (\rho) - e_\Psi (\rho)| \leq \|\Phi - \Psi\|_{\mathcal{W}_1}, \quad \Phi, \Psi \in \mathcal{W}_1.
\]

In particular, given any fixed translation-invariant state \( \rho \in E_1 \), the linear mapping \( \Phi \mapsto e_\Phi (\rho) \) from \( \mathcal{W}_1 \) to \( \mathbb{C} \) is Lipschitz continuous.

### 3.3 Entropy Density Functional on Translation-Invariant States

The entropy density functional \( s : E_1 \to \mathbb{R}_+^0 \) maps any translation-invariant state to its von Neumann entropy per unit volume in the thermodynamic limit, that is,

\[
s(\rho) = -\lim_{L \to \infty} \left\{ \frac{1}{|\Lambda_L|} \text{Trace} \left( d_{\rho_{\Lambda_L}} \ln d_{\rho_{\Lambda_L}} \right) \right\}, \quad \rho \in E_1,
\]

where we recall that \( \rho_{\Lambda_L} \) is the restriction of the translation-invariant state \( \rho \in E_1 \) to the finite-dimensional CAR \( C^* \)-algebra \( \mathcal{U}_{\Lambda_L} \) of the cubic box \( \Lambda_L \) defined by (6). Here, \( d_{\rho_{\Lambda_L}} \in \mathcal{U}_{\Lambda_L} \) is the (uniquely defined) density matrix representing the state \( \rho_{\Lambda_L} \) via a trace:

\[
\rho_{\Lambda_L} (\cdot) = \text{Trace} \left( \cdot \, d_{\rho_{\Lambda_L}} \right).
\]

By [11, Lemma 4.15], the functional \( s \) is well-defined on the set \( E_1 \) of translation-invariant states. See also [28, Section 10.2]. By [11, Lemma 1.29], the entropy density functional \( s \) is a weak*-upper semicontinuous affine functional.

### 3.4 Equilibrium States as Minimizers of the Free Energy Density

Equilibrium states of lattice fermion systems are always defined from a fixed self-adjoint interaction, which determines the energy density of states as well as the microscopic dynamics. Here, we define equilibrium states as minimizers of the free energy density functional, in direct relation with the notion of (grand-canonical) pressure: For a fixed \( \beta \in \mathbb{R}_+^0 \), the infinite volume pressure \( P \) is the

\[\text{For } \Lambda \in \mathcal{P}_f, \text{ the trace on the finite-dimensional } C^* \text{-algebra } \mathcal{U}_\Lambda \text{ refers to the usual trace on the fermionic Fock space representation.}\]
real-valued function on the real Banach subspace $\mathcal{W}_1^\mathbb{R}$ (25) of interactions that are self-adjoint and translation-invariant, defined by

$$\Phi \mapsto P_\Phi \doteq \lim_{L \to \infty} \frac{1}{\beta |\Lambda_L|} \ln \text{Trace}(e^{-\beta U_L^\Phi}) .$$

(34)

Recall that the parameter $\beta \in \mathbb{R}^+$ is the inverse temperature of the system. It is fixed once and for all and, therefore, it is often omitted in our discussions or notation. By [11, Theorem 2.12], the above pressure is well-defined and, for any $\Phi \in \mathcal{W}_1^\mathbb{R}$,

$$P_\Phi = -\inf f_\Phi(E_1) < \infty ,$$

(35)

where the mapping $f_\Phi : E_1 \to \mathbb{R}$ is the free energy density functional defined on the set $E_1$ of translation-invariant states by

$$f_\Phi \doteq e_\Phi - \beta^{-1} s .$$

(36)

Recall that $e_\Phi : E_1 \to \mathbb{R}$ is the energy density functional defined in Section 3.2 for any $\Phi \in \mathcal{W}_1$, while $s : E_1 \to \mathbb{R}_+^n$ is the entropy density functional presented in Section 3.3.

As explained in Sections 3.2–3.3, the functionals $e_\Phi, F \in \mathcal{W}_1^\mathbb{R}$, and $-\beta^{-1} s, \beta \in \mathbb{R}^+$, are weak*-lower semicontinuous and affine (with $e_\Phi$ being actually weak*-continuous). In particular, the functional $f_\Phi$ is weak*-lower semicontinuous and affine. Therefore, for any $\Phi \in \mathcal{W}_1^\mathbb{R}$, this functional has minimizers in the weak*-compact set $E_1$ of translation-invariant states. Similarly to what is done for translation-invariant quantum spin systems (see, e.g., [29][30]), for any $\Phi \in \mathcal{W}_1^\mathbb{R}$, the set $M_\Phi$ of translation-invariant equilibrium states of fermions on the lattice is, by definition, the (non-empty) set

$$M_\Phi \doteq \{ \omega \in E_1 : f_\Phi(\omega) = \inf f_\Phi(E_1) = -P_\Phi \}$$

(37)

of all minimizers of the free energy density functional $f_\Phi$ over the set $E_1$. By affineness and weak*-lower semicontinuity of $f_\Phi$, $M_\Phi$ is a (non-empty) weak*-closed face of $E_1$ for any $\Phi \in \mathcal{W}_1^\mathbb{R}$.

Recall that this is not the only reasonable way of defining equilibrium states. For fixed interactions, they can also be defined as tangent functionals to the corresponding pressure functional or via other conditions like the local stability, the Gibbs condition or the Kubo-Martin-Schwinger (KMS) condition. All these definitions are generally not completely equivalent to each other. For more details, we recommend the paper [28].

4 Mean-Field Models

4.1 The Banach Space of Mean-Field Models

Let

$$\mathbb{S} \doteq \{ \Phi \in \mathcal{W}_1 : \|\Phi\|_\mathcal{U} = 1 \}$$

(38)

be the unit sphere of the Banach space $\mathcal{W}_1$ (24) of translation-invariant short-range interactions. Let $\mathcal{S}_1$ denote the space of signed Borel measures of bounded variation on $\mathbb{S}$, which is a real Banach space whose norm is the total variation of measures

$$\|a\|_{\mathcal{S}_1} \doteq |a| (\mathbb{S}) , \quad a \in \mathcal{S}_1 .$$

The space of translation-invariant mean-field, or long-range, models is the separable (real) Banach space

$$\mathcal{M}_1 \doteq \{ m \in \mathcal{W}_1^\mathbb{R} \times \mathcal{S}_1 : \|m\|_{\mathcal{M}_1} < \infty \} ,$$

(39)

Recall that a face $F$ of a convex set $K$ is a subset of $K$ with the property that, if $\rho = \lambda_1 \rho_1 + \cdots + \lambda_n \rho_n \in F$ with $\rho_1, \ldots, \rho_n \in K, \lambda_1, \ldots, \lambda_n \in (0, 1)$ and $\lambda_1 + \cdots + \lambda_n = 1$, then $\rho_1, \ldots, \rho_n \in F$. 

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whose norm is

\[ \|m\|_{\mathcal{M}_1} \doteq \|\Phi\|_{\mathcal{W}_1} + \|a\|_{\mathcal{S}_1}, \quad m \doteq (\Phi, a) \in \mathcal{M}_1. \]  

(40)

The spaces \( \mathcal{W}_1^\mathbb{R} \) and \( \mathcal{S}_1 \) are seen as subspaces of \( \mathcal{M}_1 \), i.e.,

\[ \mathcal{W}_1^\mathbb{R} \subseteq \mathcal{M}_1 \quad \text{and} \quad \mathcal{S}_1 \subseteq \mathcal{M}_1, \]  

(41)

with the canonical identifications \( \Phi \equiv (\Phi, 0) \in \mathcal{M}_1 \) for \( \Phi \in \mathcal{W}_1^\mathbb{R} \) and \( a \equiv (0, a) \in \mathcal{M}_1 \) for \( a \in \mathcal{S}_1 \).

The local Hamiltonians associated with any mean-field model \( m \doteq (\Phi, a) \in \mathcal{M}_1 \) are the (well-defined) self-adjoint elements

\[ U^m_L \doteq U^\Phi_L + \frac{1}{|\Lambda_L|} \int_{\mathcal{S}} |U^\Phi_L|^2 a \, (d\Psi), \quad L \in \mathbb{N}_0, \]  

(42)

where we recall that \( |A|^2 \doteq A^* A \) for any \( A \in \mathcal{U} \), see (8). Note that \( U^{(\Phi, 0)}_L = U^\Phi_L \) for any self-adjoint interaction \( \Phi \in \mathcal{W}_1^\mathbb{R} \) (cf. (41)) and straightforward estimates yield the bound

\[ \|U^m_L\|_{\mathcal{U}} \leq |\Lambda_L| \|m\|_{\mathcal{M}_1}, \quad L \in \mathbb{N}_0, \; m \in \mathcal{M}_1, \]  

(43)

by Equations (31) and (40).

Remark 4.1 (Equivalent definitions of translation-invariant mean-field models)

[1] and [23, 24] have different definitions of mean-field models. We use here the formalism introduced in [23,24]. Compare (39) with [1, Definition 2.1]. In fact, [23, Section 8] shows that the results of [1] apply equally to all translation-invariant mean-field models \( m \in \mathcal{M}_1 \).

### 4.2 Purely Repulsive and Purely Attractive Mean-Field Models

By the Hahn decomposition theorem, any signed measure \( a \) of bounded variation on the unit sphere \( S \) of the Banach space \( \mathcal{W}_1 \) has a unique decomposition

\[ a = a_+ - a_-, \]  

(44)

\( a_\pm \) being two positive finite measures vanishing on disjoint Borel sets, respectively denoted by \( S_\mp \subseteq S \). Recall that such a decomposition is called the Jordan decomposition of the measure of bounded variation \( a \) and \( |a| = a_+ + a_- \). Mean-field attractions are represented by the measure \( a_- \), whereas \( a_+ \) refers to mean-field repulsions. A mean-field model \( m \doteq (\Phi, a) \in \mathcal{M}_1 \) is said to be purely attractive iff \( a_- = 0 \), while it is purely repulsive iff \( a_+ = 0 \).

Distinguishing between these two special types of models is important because the effects of mean-field attractions and repulsions on the structure of the corresponding sets of (generalized) equilibrium states can be very different: By [1, Theorem 2.25], mean-field attractions have no particular effect on the structure of the set of (generalized, translation-invariant) equilibrium states, which is still a (non-empty) weak*-closed face of the set \( E_1 \) of translation-invariant states, like for interactions in \( \mathcal{W}_1^\mathbb{R} \). By contrast, mean-field repulsions have generally a geometrical effect by possibly breaking the face structure of the set of (generalized) equilibrium states (see [1, Lemma 9.8]).

### 4.3 The Space-Averaging Functional on Translation-Invariant States

In addition to the energy density and entropy density functionals, defined respectively in Sections 3.2–3.3, we need the so-called space-averaging functional in order to study the thermodynamic properties
of long-range models. This new density functional is defined on the set \(E_1\) of translation-invariant states as follows: For any \(A \in \mathcal{U}\), the mapping \(\Delta_A : E_1 \to \mathbb{R}\) is (well-)defined by
\[
\rho \mapsto \Delta_A (\rho) \doteq \lim_{L \to \infty} \rho \left( |A_L|^2 \right) \in \left[ |\rho(A)|^2, \|A\|^2 \right],
\] where \( |A_L|^2 \doteq A_L^* A_L\) (see (8)) and \(A_L\) is defined by (17) for any \(L \in \mathbb{N}_0\). Compare this last definition with Equation (18). See also [1, Section 1.3]. By [1, Theorem 1.18], the functional \(\Delta_A\) is affine and weak*-upper semicontinuous. Thanks again to [1, Theorem 1.18], note additionally that, at any fixed translation-invariant state \(\rho \in E_1\),
\[
|\Delta_A (\rho) - \Delta_B (\rho)| \leq (\|\rho\| + \|B\|) \|A - B\|, \quad A, B \in \mathcal{U}.
\]
In particular, at fixed \(\rho \in E_1\), the mapping \(A \mapsto \Delta_A (\rho)\) from \(\mathcal{U}\) to \(\mathbb{R}\) is locally Lipschitz continuous.

For any \(A \in \mathcal{U}\), [1, Theorem 1.19] also proves, among other things, that \(\Delta_A\) can be decomposed in terms of an integral over the set \(\mathcal{E}_1\): For all \(\rho \in E_1\) with ergodic (or extreme) decomposition given by the probability measure \(\mu_\rho\) of Equation (16),
\[
\Delta_A (\rho) = \int_{\mathcal{E}_1} |\hat{\rho} (A)|^2 \, d\mu_\rho (\hat{\rho}) .
\]
Its \(\Gamma\)-regularization \(\Gamma (\Delta_A)\) on \(E_1\) is the weak*-continuous convex mapping \(\rho \mapsto |\rho (A)|^2\). Recall that the so-called \(\Gamma\)-regularization of a functional \(h\) on \(E_1\), denoted here by \(\Gamma (h)\), is defined by
\[
\Gamma (h) (\rho) \doteq \sup \left\{ m(\rho) : m \in \Lambda (\mathcal{U}^*) \text{ and } m|_{E_1} \leq h|_{E_1} \right\}, \quad \rho \in E_1,
\] with \(\Lambda (\mathcal{U}^*)\) being the set of all affine and weak*-continuous functionals on the dual space \(\mathcal{U}^*\) of the \(C^*\)-algebra \(\mathcal{U}\). The \(\Gamma\)-regularization \(\Gamma (h)\) of a functional \(h\) on \(E_1\) equals its twofold Legendre-Fenchel transform, also called the biconjugate (functional) of \(h\). Indeed, \(\Gamma (h)\) is the largest lower weak*-semicontinuous and convex minorant of \(h\). Note that the \(\Gamma\)-regularization \(\Gamma (h_1 + h_2)\) of the sum of two functionals \(h_1\) and \(h_2\) is generally not equal to the sum \(\Gamma (h_1) + \Gamma (h_2)\). For more details on \(\Gamma\)-regularizations, we recommend [1, Section 10.5] and, in particular, [1, Corollary 10.30].

For any signed Borel measure \(\alpha \in \mathcal{S}_1\) of bounded variation on \(\mathbb{S}\), we define the space-averaging functional \(\Delta_\alpha : E_1 \to \mathbb{R}\) on translation-invariant states by
\[
\Delta_\alpha (\rho) \doteq \int_{\mathbb{S}} \Delta_{\xi_\Psi} (\rho) \, d\alpha (\Psi), \quad \rho \in E_1.
\]
Recall that the continuous mapping \(\xi (\cdot) : \mathcal{W}_1 \to \mathcal{U}\) is defined by Equation (30). By [1, Theorem 1.18], \(\Delta_\alpha\) is a well-defined, affine and weak*-upper semicontinuous functional on the set \(E_1\) of translation-invariant states. By [1, Theorem 1.19], for any positive measure \(\alpha \in \mathcal{S}_1\), i.e., \(\alpha = \alpha_+ - \alpha_- = \alpha_+\), its \(\Gamma\)-regularization equals the functional
\[
\Gamma (\Delta_\alpha) (\rho) = \int_{\mathbb{S}} |\rho (\xi_\Psi)|^2 \, d\alpha_+ (\Psi), \quad \rho \in E_1.
\]

### 4.4 Infinite Volume Pressures

#### 4.4.1 Conventional Pressures

As is usual in quantum statistical mechanics, at any given inverse temperature \(\beta \in \mathbb{R}^+\), the conventional infinite volume pressure for translation-invariant mean-field models is, by definition, the real-valued function \(P^\beta\) on the Banach space \(\mathcal{M}_1\) (39) of translation-invariant mean-field models, defined by
\[
m \mapsto P^\beta_m \doteq \lim_{L \to \infty} \frac{1}{\beta |A_L|} \ln \text{Trace}(e^{-\beta U^m_L}).
\]
By [1 Theorem 2.12], this mapping is well-defined and, for any \( m = (\Phi, a) \in \mathcal{M}_1 \),

\[
P^\Phi_m = - \inf f^\Phi_m (E_1) \in \mathbb{R} ,
\]
where \( f^\Phi_m : E_1 \to \mathbb{R} \) is the free energy density functional defined by

\[
f^\Phi_m \doteq \Delta_a + f_\Phi = \Delta_a + e_\Phi - \beta^{-1} s .
\]

See Equation (36), defining the free energy density functional \( f_\Phi \) for any self-adjoint translation-invariant and short-range interaction \( \Phi \in \mathcal{W}_1^{\mathbb{R}} \). Observe that Equation (50) is an extension of (35) — which refers to the space \( \mathcal{W}_1^{\mathbb{R}} \) of (quasi-)short-range models only — to the space \( \mathcal{M}_1 \supseteq \mathcal{W}_1^{\mathbb{R}} \) of mean-field models.

### 4.4.2 Non-conventional Pressures

In [1 Section 2.5], we introduce a new free energy density functional, as well as the corresponding pressure associated with mean-field models via a variational problem. More precisely, we refer to [1 Equations (2.16) and (2.18)]. They were used to study the structure of generalized equilibrium states for mean-field models.

The new free energy density functional on \( E_1 \) is defined, for any fixed \( m = (\Phi, a) \in \mathcal{M}_1 \), by

\[
f^m (\rho) \doteq \int_{\mathbb{R}} |\rho| \big((e_\Phi)^2 a_+ + (d\Psi)\big) - \Delta_{a_-} (\rho) + e_\Phi (\rho) - \beta^{-1} s (\rho) , \quad \rho \in E_1 , \tag{52}
\]

where \( a_\pm \in \mathcal{S}_1 \) are the two positive finite measures resulting from the Jordan decomposition of the signed measure \( a \in \mathcal{S}_1 \). Here, \( \Delta_{a_-} \) is the space-averaging functional defined by (47) for \( a_- \in \mathcal{S}_1 \).

By Equation (48), note that

\[
\Delta_{a_+} (\rho) \geq \Gamma (\Delta_{a_+}) (\rho) = \int_{\mathbb{R}} |\rho| \big((e_\Phi)^2 a_+ + (d\Psi)\big) , \quad \rho \in E_1 , \tag{53}
\]

while \( \Delta_a = \Delta_{a_+} - \Delta_{a_-} \), by Equation (44) and linearity of the mapping \( a \mapsto \Delta_a \). It follows from (51) and (52) that

\[
f^{\Phi}_m \leq f^m , \quad m \in \mathcal{M}_1 , \tag{54}
\]

inspiring the notation \( b \) (“lower in pitch”) and \( \sharp \) (“higher in pitch”). In fact, we replace the functional \( \Delta_{a_+} \) in \( f^m \) by its \( \Gamma \)-regularization (48) to define the new free energy density functional \( f^m \).

Similar to (50), the non-conventional pressure associated with a mean-field model \( m \in \mathcal{M}_1 \) is then defined to be the real-valued function

\[
m \mapsto P^\Phi_m \doteq - \inf f^\Phi_m (E_1) \in \mathbb{R} \tag{55}
\]
on the Banach space \( \mathcal{M}_1 \) of translation-invariant mean-field models. Note from (55), (50) and (54) that \( P^\Phi_m = P^\Phi_m = P_\Phi \) for any \( \Phi \in \mathcal{W}_1 \subseteq \mathcal{M}_1 \). Moreover, for any purely repulsive or purely attractive mean-field model \( m \in \mathcal{M}_1 \), as defined in Section 4.2, \( P^\Phi_m = P^\Phi_m \doteq P_m \), thanks to [1 Theorem 2.25]. However, for mean-field models with non-trivial attractive and repulsive mean-field interactions, one only has

\[
P^\Phi_m \leq P^\Phi_m , \quad m \in \mathcal{M}_1 ,
\]
by virtue of Equation (53). In other words, the non-conventional pressure is only an upper bound of the conventional one, in general. In fact, in [1 Section 2.7] we give examples of models \( m \in \mathcal{M}_1 \) for which \( P^\Phi_m < P^\Phi_m \).

For general mean-field models, the non-conventional pressure is not equal to the thermodynamic limit of lattice fermion systems at finite-volume, in contrast with the conventional one (cf. (49)–(50)). However, in the present paper, we show that the non-conventional pressure introduced in [1] is not only an interesting mathematical object, but also physically relevant in the context of the Kac limit studied in Section 5.
4.4.3 Thermodynamic Games

As stated above, the conventional and non-conventional pressures are given via infima over translation-invariant states, see Equations (50) and (54). They are pivotal in [11] for the study of infinite volume equilibrium states. Nonetheless, it is a priori not clear how useful these variational formulae are to study phase transitions. To tackle this question, it is convenient to consider the so-called Bogoliubov approximations of mean-field models, which refer to the approximating Hamiltonian method used in the past to compute the conventional pressure associated with particular mean-field models, as explained in [1, Section 2.10]. In [11], we generalize this method to all models of the Banach space $\mathcal{M}_1$ and all corresponding equilibrium states. We use the viewpoint of game theory by interpreting the mean-field attractions $a$ and all corresponding equilibrium states. As stated above, the conventional and non-conventional pressures are given via infima over translation-invariant states, see Equations (50) and (54). They are pivotal in [11] for the study of infinite volume equilibrium states. Nonetheless, it is a priori not clear how useful these variational formulae are to study phase transitions. To tackle this question, it is convenient to consider the so-called Bogoliubov approximations of mean-field models, which refer to the approximating Hamiltonian method used in the past to compute the conventional pressure associated with particular mean-field models, as explained in [1, Section 2.10]. In [11], we generalize this method to all models of the Banach space $\mathcal{M}_1$ and all corresponding equilibrium states. We use the viewpoint of game theory by interpreting the mean-field attractions $a$ and all corresponding equilibrium states.

For any mean-field model $m = (\Phi, a) \in \mathcal{M}_1$ and every function $c = (c_\Psi)_{\Psi \in \mathcal{S}} \in L^2(\mathcal{S}; \mathbb{C}; |a|)$, we define a so-called approximating (self-adjoint, short-range) interaction by

$$\Phi_m(c) = \Phi + 2 \int_{\mathcal{S}} \Re \{ c_\Psi \} a(\Psi) \in \mathcal{W}^R_1. \tag{56}$$

The integral in the last definition, which refers to a self-adjoint interaction, i.e., an element of the space $\mathcal{W}^R_1$, has to be understood as follows:

$$\left( \int_{\mathcal{S}} \Re \{ c_\Psi \} a(\Psi) \right)_\Lambda = \int_{\mathcal{S}} \Re \{ c_\Psi \} a(\Psi), \quad \Lambda \in \mathcal{P}_1. \tag{57}$$

Note that the integral in the definition is well-defined because, for each $\Lambda \in \mathcal{P}_1$, the integrand is an absolutely integrable (measurable) function taking values in a finite-dimensional normed space, which is $U_\Lambda$. By Equation (28), the energy observables associated with $\Phi_m(c)$ equal

$$U^\Phi_L(c) = U^\Phi_L + \int_{\mathcal{S}} 2\Re \{ c_\Psi \} a(\Psi), \quad L \in \mathbb{N}_0. \tag{58}$$

One then deduces from Equations (55)–(56) that

$$P_{\Phi_m(c)} = -\inf f_{\Phi_m(c)}(E_1), \quad c \in L^2(\mathcal{S}; \mathbb{C}; |a|), \tag{59}$$

where, for any translation-invariant state $\rho \in E_1$,

$$f_{\Phi_m(c)}(\rho) = \int_{\mathcal{S}} 2\Re \{ c_\Psi \} a(\Psi) + c_\Psi(\rho) - \beta^{-1}s(\rho).$$

As compared to the pressure $P^d_m$ for translation-invariant mean-field models $m \in \mathcal{M}_1$, $P_{\Phi_m(c)}$ is, in principle, easier to analyze, because it comes from a purely short-range interaction $\Phi_m(c) \in \mathcal{W}^R_1$. In many cases that are important for condensed matter, $P_{\Phi_m(c)}$ is even explicitly known or given by converging perturbative expansions around some simple (unperturbed) object.

Recall the Jordan decomposition of the measure $a$ stated in Equation (44): $a = a_+ - a_-$ with $a_\pm$ being two positive finite measures vanishing on disjoint Borel subsets $S_\mp \subseteq S$, respectively, referring to the Hahn decomposition theorem. Then, we define two Hilbert spaces corresponding respectively to the repulsive and attractive components, $a_+$ and $a_-$, of any mean-field model $m \in \mathcal{M}_1$:

$$L^2_\pm(\mathcal{S}; \mathbb{C}) \doteq L^2(\mathcal{S}; \mathbb{C}; a_\pm). \tag{60}$$

Note that we canonically have the identification

$$L^2(\mathcal{S}; \mathbb{C}; |a|) = L^2_+(\mathcal{S}; \mathbb{C}) \oplus L^2_-(\mathcal{S}; \mathbb{C}).$$
The approximating free energy density functional

\[ f_m : L^2_-(S; \mathbb{C}) \times L^2_+(S; \mathbb{C}) \to \mathbb{R} \]

is then defined by

\[ f_m (c_-, c_+) = -\|c_+\|^2_2 + \|c_-\|^2_2 - P_{\Phi_m (c_- + c_+)} , \quad c_{\pm} \in L^2_{\pm} (S; \mathbb{C}) . \]  

(61)

The \textit{thermodynamic game} is the two-person zero-sum game defined from \( f_m \), with its conservative values being equal to \(-P_m^+\) and \(-P_m^-\). By [1, Theorem 2.36], the conventional and non-conventional pressures associated with any mean-field model \( m \in M_1 \) are equal to

\[ P_m^+ = -\inf_{c_- \in L^2_-(S; \mathbb{C})} \sup_{c_+ \in L^2_+(S; \mathbb{C})} f_m (c_- , c_+) \quad \text{and} \quad P_m^- = -\sup_{c_+ \in L^2_+(S; \mathbb{C})} \inf_{c_- \in L^2_-(S; \mathbb{C})} f_m (c_- , c_+) . \]  

(62)

Compare these equalities with Equations (50)-(52) and (54).

Note that the \( \sup \) and the \( \inf \) in (62) do not commute in general. See [1, p. 42]. A sufficient condition for \( \sup \) and \( \inf \) to commute is given through Sion’s minimax theorem [31] as follows:

**Lemma 4.2 (Sufficient condition for \( P_m^+ = P_m^- \))**

\textit{Let} \( m \in M_1 \) \textit{be any mean-field model. If, for any fixed} \( c_+ \in L^2_+(S; \mathbb{C}) \), \textit{the function} \( f_m (\cdot, c_+) \) \textit{on} \( L^2_-(S; \mathbb{C}) \) \textit{is quasi-convex, i.e., for all} \( r \in \mathbb{R} \), \textit{the level set}

\[ \{ c_- \in L^2_-(S; \mathbb{C}) : f_m (c_- , c_+) \leq r \} \]

\textit{is convex, then} \( P_m^+ = P_m^- \).

**Proof.** Take \( m \in M_1 \). By [1, Lemma 8.1], for any \( c_+ \in L^2_+(S; \mathbb{C}) \), the function \( f_m (\cdot, c_+) \) on \( L^2_-(S; \mathbb{C}) \) is weakly lower semicontinuous, while, for any \( c_- \in L^2_-(S; \mathbb{C}) \), the function \( f_m (c_-, \cdot) \) on \( L^2_+(S; \mathbb{C}) \) is weakly upper semicontinuous and concave. By [1, Lemmata 8.3-8.4], there is a closed ball \( B_R (0) \subseteq L^2_-(S; \mathbb{C}) \) of radius \( R < \infty \) such that

\[ \inf_{c_- \in L^2_-(S; \mathbb{C})} \sup_{c_+ \in L^2_+(S; \mathbb{C})} f_m (c_- , c_+) = \inf_{c_- \in B_R (0)} \sup_{c_+ \in L^2_+(S; \mathbb{C})} f_m (c_- , c_+) , \]

\[ \sup_{c_+ \in L^2_+(S; \mathbb{C})} \inf_{c_- \in L^2_-(S; \mathbb{C})} f_m (c_- , c_+) = \sup_{c_+ \in B_R (0)} \inf_{c_- \in L^2_-(S; \mathbb{C})} f_m (c_- , c_+) . \]

Note that any closed ball of \( L^2_-(S; \mathbb{C}) \) is convex and weakly compact, by the Banach-Alaoglu theorem and the reflexivity of Hilbert spaces\footnote{The weak and the weak* topologies are the same in this case.}. Therefore, if we additionally assume the quasi-convexity of the function \( f_m (\cdot, c_+) \) at fixed \( c_+ \in L^2_+(S; \mathbb{C}) \), then the lemma follows from (62) and Sion’s minimax theorem [31]. ■

### 4.5 Generalized Equilibrium States

#### 4.5.1 Conventional Equilibrium States

We give here the extension of the notion of equilibrium states of Section 3.4 to mean-field models, by using the variational principle associated with the conventional pressure, as is usual in quantum statistical mechanics at equilibrium. An important issue appears in this situation, because of the lack of weak*-lower semicontinuity of the free energy density functional in presence of mean-field repulsions (Section 4.2).
In fact, similar to \([37]\), for any translation-invariant mean-field model \(m \in \mathcal{M}_1\), one might define the set of equilibrium states by
\[
M^*_m \doteq \left\{ \omega \in E_1 : f^*_m(\omega) = \inf_{E_1} f^*_m(E_1) = -P^*_m \right\}.
\] (63)

Note however that the free energy density functional \(f^*_m\) is in general not weak*-lower semicontinuous on \(E_1\) and it is thus a priori not clear whether \(M^*_m\) is empty or not. In fact, by Equation (44), for any translation-invariant mean-field model \(m = (\Phi, a) \in \mathcal{M}_1\),
\[
f^*_m = \frac{\Delta_{a_+}}{\text{weak*-upper semic.}} + \frac{(-\Delta_{a_-} + e_a - \beta^{-1}s)}{\text{weak*-lower semic.}}.
\] (64)

See Sections 3.2–3.3 and 4.3. Therefore, instead of considering \(M^*_m\), we define
\[
\Omega^*_m = \left\{ \omega \in E_1 : \exists \{\rho_n\}_{n=1}^\infty \subseteq E_1 \text{ weak*-converging to } \omega \text{ such that } \lim_{n\to\infty} f^*_m(\rho_n) = \inf_{E_1} f^*_m(E_1) \right\}
\] (65)
as being the (conventional) set of generalized equilibrium states of any fixed translation-invariant mean-field model \(m \in \mathcal{M}_1\) (at inverse temperature \(\beta \in \mathbb{R}^+\)). Observe for instance that, under periodic boundary conditions, the weak* accumulation points of (finite-volume) Gibbs states associated with any mean-field model \(m \in \mathcal{M}_1\) and \(\beta \in \mathbb{R}^+\) always belong to \(\Omega^*_m\), but not necessarily to \(M^*_m\), by [11] Theorem 3.13.

Obviously, by weak*-compactness of \(E_1\), the set \(\Omega^*_m\) is non-empty and \(\Omega^*_m \supseteq M^*_m\). This definition can be expressed in terms of the graph of \(f^*_m\):
\[
\Omega^*_m \times \left\{ \inf_{E_1} f^*_m(E_1) \right\} = (E_1 \times \left\{ \inf_{E_1} f^*_m(E_1) \right\}) \cap \text{Graph}(f^*_m),
\]
where the closure of the graph of \(f^*_m\) refers to the product topology of the weak* topology on \(E_1\) and the usual topology on \(\mathbb{R}\). It follows that \(\Omega^*_m\) is weak*-closed and convex, by affineness of \(f^*_m\). Thus, \(\Omega^*_m\) is a (non-empty) weak*-compact convex subset of \(E_1\). See [11] Lemma 2.16. If \(a_+ = 0\) then \(\Omega^*_m = M^*_m\) is a (non-empty) weak*-closed face of the Poulsen simplex \(E_1\). By contrast, as already mentioned above, a mean-field repulsion \(a_+\) has generally a geometrical effect on the set \(\Omega^*_m\), by possibly breaking its face structure in \(E_1\). This effect can lead to long-range order of generalized equilibrium states. See [11] Section 2.9.

### 4.5.2 Non-conventional Equilibrium States

In Section 4.4.2 we introduce a non-conventional pressure by means of a new free energy density functional. See Equations (52) and (54). We show in Section 5 that these mathematical objects are physically relevant in the context of the Kac limit. Therefore, at inverse temperature \(\beta \in \mathbb{R}^+\) and for any mean-field model \(m \in \mathcal{M}_1\), similar to the set \(\Omega^*_m\) of generalized equilibrium states defined by (65), we define a set \(\Omega^d_m\) of non-conventional equilibrium states by
\[
\Omega^d_m = \left\{ \omega \in E_1 : f^d_m(\omega) = \inf_{E_1} f^d_m(E_1) = -P^d_m \right\}
\] (66)
with \(f^d_m\) being the free energy density functional (52). In contrast with the affine functional \(f^*_m\) (see (64)), note that \(f^d_m\) is weak*-lower semicontinuous but only convex (and not affine). To prove these properties, use Equation (52) and Sections 3.2–3.3 and 4.3 together with the obvious inequality
\[
\|\epsilon_\Phi\|_U \leq \|\Phi\|_{\mathcal{W}_1}, \quad \Phi \in \mathcal{W}_1,
\] (67)
and Lebesgue’s dominated convergence theorem to deduce the weak*-continuity of the convex functional
\[
\rho \mapsto \int_\mathcal{S} |\rho(\epsilon_\Psi)|^2 a_+ \text{ (d}\Psi),
\]
on $E_1$. Hence, $\Omega^b_m$ is a nonempty weak*-compact convex subset of $E_1$.

If $m = (\Phi, a = a_+ - a_-) \in M_1$, with $a_+ = 0$ then $\Omega^b_m = \Omega^b_1 = M_m$ is a (nonempty) weak*-closed face of the Poulsen simplex $E_1$. In fact, for any purely repulsive or attractive mean-field model $m \in M_1$, as defined in Section 4.4.2 the equality $\Omega^b_m = \Omega^b_1$ always holds true, thanks to [1] Theorem 2.25. However, for mean-field models $m \in M_1$ with non-trivial attractive and repulsive mean-field interactions, the sets $\Omega^b_m$ and $\Omega^b_1$ of conventional and non-conventional equilibrium states are not equal to each other, in general.

4.5.3 Self-Consistency of Generalized Equilibrium States

In Section 4.4.3 we introduce the thermodynamic game, which provides an efficient mathematical method to study phase transitions induced by mean-field interactions. It refers to a two-person zero-sum game defined from the approximating free energy density functional (61). By Equation (62), the conventional and non-conventional pressures associated with any mean-field model are, up to a minus sign, the conservative values of this game. The thermodynamic game also provides a complete characterization of the sets (65) and (66) of (translation-invariant) conventional and non-conventional equilibrium states, as follows:

The sup and the inf in the variational problems (62) are attained, i.e., they are respectively a max and a min. For any mean-field model $m \in M_1$, the sets

$$C^b_m \doteq \left\{ d_- \in L^2_-(S; \mathbb{C}) : \max_{c_+ \in L^2_+(S; \mathbb{C})} f_m(d_-, c_+) = -P^b_m \right\} \quad (68)$$

$$C^b_m \doteq \left\{ d_+ \in L^2_+(S; \mathbb{C}) : \min_{c_- \in L^2_-(S; \mathbb{C})} f_m(c_-, d_+) = -P^b_m \right\} \quad (69)$$

of conservative strategies of the repulsive and attractive players, respectively, are non-empty. By [1] Lemma 8.4, $C^b_m$ is norm-bounded and weakly compact, while the set $C^b_m$ has exactly one element $d_+$ when $m = (\Phi, a = a_+ - a_-) \in M_1$ with $a_+ \neq 0$. In the particular case of purely repulsive mean-field models, i.e., when $a_- = 0$, $C^b_m = \{0\} = L^2_-(S; \mathbb{C})$, as $f_m$ is independent of $c_-$. Similarly, if $a_+ = 0$ then $C^b_m = \{0\} = L^2_+(S; \mathbb{C})$.

In [1] Lemma 8.3 (2) it is proven that, when $a_+ \neq 0$, for all functions $c_- \in L^2_-(S; \mathbb{C})$, the set

$$\left\{ d_+ \in L^2_+(S; \mathbb{C}) : \max_{c_+ \in L^2_+(S; \mathbb{C})} f_m(c_-, c_+) = f_m(c_-, d_+) \right\} \quad (70)$$

has exactly one element, which is denoted by $r_+(c_-)$. By [1] Lemma 8.8, if $a_+ \neq 0$ then the mapping

$$r_+ : c_- \mapsto r_+(c_-) \quad (71)$$

defines a continuous functional from $L^2_-(S; \mathbb{C})$ to $L^2_+(S; \mathbb{C})$, where $L^2_-(S; \mathbb{C})$ and $L^2_+(S; \mathbb{C})$ are endowed with the weak and norm topologies, respectively. This mapping is called (by us) the thermodynamic decision rule of the mean-field model $m \in M_1$. In the particular case of purely attractive mean-field models, i.e., when $a_+ = 0$, $f_m$ is independent of $c_+$ and one trivially has $r_+ = 0$, since $L^2_+(S; \mathbb{C}) = \{0\}$ in this case.

For any mean-field model $m = (\Phi, a = a_+ - a_-) \in M_1$, it is convenient to introduce a family of approximating, purely attractive mean-field models by

$$m(c_+) \doteq (\Phi + \Phi_m(c_+), -a_-) \in M_1, \quad c_+ \in L^2_+(S; \mathbb{C}). \quad (72)$$

Then, for every pair of functions $c_\pm \in L^2_\pm(S; \mathbb{C})$, we define the (possibly empty) sets

$$\Omega_m(c_-, c_+) \doteq \{ \omega \in M_{\Phi_m(c_+ + c_-)} : e_\omega(\omega) = c_+ + c_- \ | |a| - a_- e_\omega \} \subseteq E_1 \quad (73)$$
and
\[
\Omega_m (c_+) \triangleq \left\{ \omega \in \Omega^2_{m(c_+)} : e_{(\cdot)} (\omega) = c_+ \quad a_+\text{-a.e.} \right\} \subseteq E_1 ,
\] (74)
where, for any fixed translation-invariant state \( \rho \in E_1 \), the continuous and bounded mapping \( e_{(\cdot)} (\rho) : S \to \mathbb{C} \) is defined from (29)–(30) by
\[
e_{\Psi} (\rho) = \rho (e_{\Psi}) , \quad \Psi \in S ,
\] (75)
while \( M_{\Phi_m(c)} \) and \( \Omega^2_{m(c_+)} \) are the sets respectively defined by (37) and (65). Note that \( M_{\Phi_m(c)} \) and \( \Omega^2_{m(c_+)} \) are (non-empty) weak\(^*\)-closed faces of \( E_1 \), since \( m (c_+) \) is a purely attractive mean-field model. Then, we obtain a (static) self-consistency condition for (generalized) equilibrium states, which refers, in a sense, to Euler-Lagrange equations for the variational problem defining the thermodynamic game. More precisely, we have the following statements:

**Theorem 4.3 (Self-consistency of equilibrium states)**

Let \( m \in \mathcal{M}_1 \) be any translation-invariant mean-field model.

(i) \[
\Omega^2_m = \overline{\text{co}} \left( \bigcup_{d_+ \in C^0_m} \Omega_m (d_+, r_+(d_-)) \right) .
\]

(ii) The set \( \mathcal{E} (\Omega^2_m) \) of extreme points of the weak\(^*\)-compact convex set \( \Omega^2_m \) is included in the union of the sets
\[
\mathcal{E} (\Omega_m (d_+, r_+(d_-))) , \quad d_- \in C^0_m ,
\]
of all extreme points of \( \Omega_m (d_+, r_+(d_-)) \), \( d_- \in C^0_m \), which are non-empty, convex, mutually disjoint, weak\(^*\)-closed subsets of \( E_1 \).

(b) \[
C^0_m = \{ d_+ \} \quad \text{and} \quad \Omega^2_m = \Omega_m (d_+) .
\]

**Proof.** Assertion (i) results from \([1] \) Theorem 2.21 (i)] and \([1] \) Theorem 2.39 (i)], while (ii) corresponds to \([1] \) Theorem 2.39 (ii)]. It remains to prove Assertion (b). As already mentioned, the fact that \( C^0_m = \{ d_+ \} \) refers to \([1] \) Lemma 8.4]. However, the identity \( \Omega^2_m = \Omega_m (d_+) \) was not considered in \([1] \), but its proof is similar to the one of \([1] \) Lemma 9.2]: Fix \( m \in \mathcal{M}_1 \). The set \( \Omega_m (d_+) \cap \Omega^2_m \) is non-empty, by \([1] \) Lemma 8.5]. So, take some \( \omega \in \Omega_m (d_+) \cap \Omega^2_m \). In particular, the equilibrium state \( \omega \) satisfies the self-consistency condition
\[
e_{(\cdot)} (\omega) = \omega (e_{(\cdot)}) = d_+ \quad a_+\text{-a.e.}
\] (76)
We now observe that, for any \( \rho \in E_1 \),
\[
2 \int_S \text{Re} \left\{ \overline{d_+} \rho (e_{\Psi}) \right\} a_+ (d \Psi) = \int_S |\rho (e_{\Psi})|^2 a_+ (d \Psi) + ||d_+||^2 - \int_S |\rho (e_{\Psi}) - d_+|^2 a_+ (d \Psi)
\] (77)
and since \( \omega \in \Omega_m (d_+) \cap \Omega^2_m \) satisfies (76), we conclude that
\[
\inf_{\rho \in E_1} \left\{ 2 \int_S \text{Re} \left\{ \overline{d_+} \rho (e_{\Psi}) \right\} a_+ (d \Psi) - \Delta_{a_-} (\rho) + e_{}\Phi (\rho) - \beta^{-1} s (\rho) \right\}
\] (78)
\[
= \int_S |\omega (e_{\Psi})|^2 a_+ (d \Psi) - \Delta_{a_-} (\omega) + e_{}\Phi (\omega) - \beta^{-1} s (\omega) + ||d_+||^2 
\]
\[
= f^0_m (\omega) + ||d_+||^2, 
\]
\[
= \inf f^0_m (E_1) + ||d_+||^2.
\] (79)
Going backwards from (79) to (78) and using then (77), for any non-conventional equilibrium state \( \omega \in \Omega_m^0 \), we obtain the inequality

\[
f_m^0(\omega) + \|d_+\|_2^2 \leq f_m^0(\omega) - \int_{\Sigma} |\omega(\epsilon \phi) - d_+ \phi|^2 a_+ (d\Psi) + \|d_+\|_2^2 ,
\]
i.e.,

\[
\int_{\Sigma} |\omega(\epsilon \phi) - d_+ \phi|^2 a_+ (d\Psi) \leq 0 .
\]

As a consequence, any non-conventional equilibrium state \( \omega \in \Omega_m^0 \) satisfies the self-consistency condition (76) with \( d_+ \in C_m^0 \). Combining this with (77)–(79), it follows that \( \Omega_m^0 \subseteq \Omega_m(d_+) \). Conversely, for any \( \omega \in \Omega_m(d_+) \) with \( d_{n,+} \in C_m^0 \). Such a state \( \omega \in \Omega_m(d_+) \) is a solution to the variational problem (78) and we easily deduce that \( \omega \in \Omega_m^0 \), again from (78)–(79).

Theorem 4.3 implies in particular that, for any extreme state \( \hat{\omega} \in \mathcal{E}(\Omega_m^0) \) of \( \Omega_m^0 \), there is a unique \( d_- \in C_m^0 \) such that

\[
d \doteq d_- + r_+(d_-) = e(\omega) .
\]

In the physics literature on superconductors, the above equality refers to the so-called gap equations. Conversely, for any \( d_- \in C_m^0 \), there is some generalized equilibrium state \( \omega \) satisfying the condition above, but \( \omega \) is not necessarily an extreme point of \( \Omega_m^0 \).

To conclude, note that Theorem 4.3 yields the equality \( \Omega_m^0 = \Omega_m^0 \) for any purely repulsive or purely attractive mean-field model \( m \in \mathcal{M}_1 \), as defined in Section 4.2. However, for mean-field models \( m \in \mathcal{M}_1 \) with both non-trivial attractive and repulsive mean-field interactions, there is no reason to have this equality, in general.

### 5 From Short-Range to Mean-Field Models

#### 5.1 Banach Spaces of Reflection-Symmetric Functions

Recall that the dimension of the (cubic) lattice \( \mathcal{L} = \mathbb{Z}^d \) is denoted by \( d \in \mathbb{N} \) and is always fixed. In [10, Equation (2.8)], the function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is reflection-symmetric and has to decay like \( |x|^{-(d+\epsilon)} \) in the limit \( |x| \rightarrow \infty \), for some strictly positive parameter \( \epsilon \in \mathbb{R}^+ \), being bounded and continuous at \( x = 0 \), cf. [10, Conditions (\( \varphi 1 \)) and (\( \varphi 2 \))]. In the sequel, we use a similar, albeit stronger, condition. In fact, we do not only impose the decay of \( f \) itself, but also of its derivatives up to order \( 2d \), at least.

Given the parameters \( \epsilon \in \mathbb{R}^+ \) and \( \kappa \in \{2d, 2d + 1, \ldots, \infty\} \), we consider the real space

\[
\mathcal{D}_{\epsilon,\kappa} \doteq \left\{ f \in C^\kappa(\mathbb{R}^d, \mathbb{R}) : \forall \ell \in \mathbb{N}_0^d, |\ell| \leq \kappa, \lim_{|x| \rightarrow \infty} |x|^{d+\epsilon+|\ell|} \partial^\ell_x f (x) = 0, \right. \\
\left. \text{and } \forall x \in \mathbb{R}^d, f (-x) = f (x) \right\}
\]  (81)

of continuously \( \kappa \)-differentiable, reflection-symmetric, real-valued functions on \( \mathbb{R}^d \) that decay faster than \( |x|^{-(d+\epsilon)} \) as \( |x| \rightarrow \infty \). We use above the standard multi-index notation\(^{15}\),

\[
\ell! \doteq \ell_1! \ldots \ell_d! , \quad |\ell| \doteq \ell_1 + \cdots + \ell_d , \quad \partial^\ell_x \doteq \partial^\ell_{x_1} \cdots \partial^\ell_{x_d} ,
\]

for any \( \ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}_0^d \). Using the norm

\[
\|f\|_{\mathcal{D}_{\epsilon,\kappa}} \doteq \sum_{\ell \in \mathbb{N}_0^d : |\ell| \leq \kappa} \frac{1}{\ell!} \sup_{x \in \mathbb{R}^d} \left( 1 + |x|^{d+\epsilon+|\ell|} \right) \partial^\ell_x f (x) ,
\]

\(^{15}\partial^0_{x_l} \doteq 1 \) for \( l = 1, \ldots, d \).
the space $D_{\varepsilon, \kappa}$ becomes a separable real Banach space: The fact that the norm $\| \cdot \|_{D_{\varepsilon, \kappa}}$ is complete follows from the closedness of differential operators. In order to prove the separability of the Banach space $D_{\varepsilon, \kappa}$, take a positive, compactly supported and smooth real-valued function $g$ on $\mathbb{R}^d$ with $\|g\|_1 = 1$, and define a second function $h$ of this type by

$$h(x_1, \ldots, x_d) = \int_{[-1/2,1/2]^d} g(x_1 - y_1, \ldots, x_d - y_d) \, d^d y, \quad x_1, \ldots, x_d \in \mathbb{R}.$$ 

For any $n \in \mathbb{N}$, $z \in \mathbb{Z}^d$ and all multi-indices $\ell \in \mathbb{N}_0^d$, we then define the function

$$h_{n,z,\ell}(x) \equiv (z + x)^\ell h(2^n (z + x)) + (z - x)^\ell h(2^n (z - x)), \quad x \in \mathbb{R}^d.$$ 

The set of all rational linear combinations of the subset $\{h_{n,z,\ell}\}_{n \in \mathbb{N}, z \in \mathbb{Z}^d, \ell \in \mathbb{N}_0^d}$ is dense in the Banach space $D_{\varepsilon, \kappa}$. Here, for any $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ and $\ell \in \mathbb{N}_0^d$, as is usual, $x^\ell$ stands for the product $x_1^{\ell_1} \cdots x_d^{\ell_d}$.

Unless it is not explicitly mentioned, from now on, we omit in the notation the parameter $\kappa$, which is taken by default as $\kappa = 2d$. I.e., $D_{\varepsilon} = D_{\varepsilon, 2d}$ if the parameter $\kappa$ is not explicitly specified. In fact, a parameter $\kappa > 2d$ is only relevant in Section 5.6.2. For the moment, note only that $\kappa = 2d$ corresponds to the biggest space in the scale $(D_{\varepsilon, \kappa})_{\kappa = 2d, 2d+1, \ldots}$ of vector spaces: For any $\kappa_1, \kappa_2 = 2d, 2d+1, \ldots$ with $\kappa_2 \geq \kappa_1$, $D_{\varepsilon, \kappa_1}$ is a vector subspace of $D_{\varepsilon, \kappa_2}$ and

$$\|f\|_{D_{\varepsilon, \kappa_1}} \leq \|f\|_{D_{\varepsilon, \kappa_2}}, \quad f \in D_{\varepsilon, \kappa_1},$$

by Equation (83).

A closed convex cone in $D_{\varepsilon} \equiv D_{\varepsilon, 2d}$ for any $\varepsilon \in \mathbb{R}^+$ is given by the subset $D_{\varepsilon, +} \subseteq D_{\varepsilon}$ of all positive definite functions, i.e., of all those functions whose Fourier transforms are everywhere non-negative. Note that any function $f \in D_{\varepsilon}$ has a well-defined Fourier transform, which is a function $\mathbb{R}^d \to \mathbb{R}$ denoted by

$$F(f)(k) \equiv \hat{f}(k) \equiv \int_{\mathbb{R}^d} f(x) \, e^{-ik \cdot x} \, d^d x, \quad k \in \mathbb{R}^d. \quad (84)$$

The notation $F(f)$ for the Fourier transform of $f$ is only used to make some expressions simpler, $\hat{f}$ being the default notation. Recall that the Fourier transform of any reflection-symmetric real-valued function is again a reflection-symmetric real-valued function.

In order to study approximations of mean-field attractions via Kac interactions, we also need the following property of functions: We say that a function $g : \mathbb{R}^d \to \mathbb{R}$ is scaling-monotone if

$$g(\gamma^{-1} k) \leq g(k), \quad k \in \mathbb{R}^d, \quad \gamma \in (0, 1). \quad (85)$$

In fact, if $g$ is differentiable, then this condition is equivalent to

$$\nabla g(k) \cdot k \leq 0, \quad k \in \mathbb{R}^d.$$ 

Using this definition, for any $\varepsilon \in \mathbb{R}^+$, we introduce the closed convex cones

$$\mathcal{C}_{\varepsilon, +} \equiv \left\{ f \in D_{\varepsilon, +} : \hat{f} \text{ is scaling-monotone} \right\}, \quad \varepsilon \in \mathbb{R}^+.$$ 

Examples of functions in the above cone are given as follows:

- The Yukawa-type potential $f_1$ defined by its Fourier transform

  $$\hat{f}_1(k) \equiv \frac{c_0}{|k|^2 + c_1} e^{-c_2 |k|^2}, \quad k \in \mathbb{R}^d,$$

  for constants $c_0, c_1, c_2 \in \mathbb{R}^+$, belongs to $\mathcal{C}_{\varepsilon, +}$ for all $\varepsilon \in \mathbb{R}^+$. Note that $c_2 = 0$ corresponds to the usual Yukawa potential, whereas the case $c_2 \in \mathbb{R}^+$ refers to a regularization at short distances.
• Take any finite positive Borel measure \( \mu \) on \((\mathbb{R}^+)^d\) such that, for some \( \eta \in \mathbb{R}^+\),

\[
\mu(\{(s_1, \ldots, s_d) : \exists j \in \{1, \ldots, d\}, \ s_j < \eta\}) = 0
\]

and define the function \( f_2 \) by

\[
f_2(x) = \int_{(\mathbb{R}^+)^d} e^{-(s_1 x_1^2 + \cdots + s_d x_d^2)} \mu(\text{d}s), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.
\]

Then, \( f_2 \in \mathcal{C}_{\varepsilon, +} \) for all strictly positive parameters \( \varepsilon \in \mathbb{R}^+ \).

In the sequel, we use the notation

\[
\mathcal{D}_0 \doteq \bigcup_{\varepsilon \in \mathbb{R}^+} \mathcal{D}_\varepsilon, \quad \mathcal{D}_{0,+} \doteq \bigcup_{\varepsilon \in \mathbb{R}^+} \mathcal{D}_{\varepsilon,+} \quad \text{and} \quad \mathcal{C}_{0,+} \doteq \bigcup_{\varepsilon \in \mathbb{R}^+} \mathcal{C}_{\varepsilon,+}.
\]  

(86)

5.2 Kac Interactions

5.2.1 Simple-Field Case

The link between interactions (in the sense of Section 5.1) and mean-field models is explicitly established in the sequel by using the so-called Kac limit. Like, for instance, in [10], a Kac model is a short-range model that depends upon parameters that fix the range of some of its interaction terms. The Kac limit then refers to taking these ranges to infinity, after the thermodynamic limit. By contrast, in the usual mean-field setting described in Section 4, both the range of interaction terms and the size of the system are taken simultaneously to infinity, in the thermodynamic limit.

The study presented here uses arguments that are of abstract nature and, hence, not model-dependent. We define a general notion of Kac interactions as follows:

**Definition 5.1 (Kac interactions – simple-field case)**

For any parameter \( \gamma \in (0, 1) \), we define \( \gamma \)-the Kac function \( K_\gamma \) to be the mapping from \( \mathcal{W}_1 \times \mathcal{D}_0 \) to \( \mathcal{W}_1^\mathbb{R} \) defined, for any \( \Phi \in \mathcal{W}_1 \) and \( f \in \mathcal{D}_0 \), by

\[
K_\gamma(\Phi, f)_\Lambda = \sum_{Z_1, Z_2 \in \mathcal{P}_f : Z_1 \cup Z_2 = \Lambda} \frac{|Z_1 \cup Z_2|}{|Z_1| + |Z_2|} \Phi_{Z_1}^* \Phi_{Z_2} \sum_{x \in Z_1} \sum_{y \in Z_2} \frac{\gamma^d f(\gamma (x - y))}{|Z_1||Z_2|}, \quad \Lambda \in \mathcal{P}_f.
\]

This interaction is named the Kac interaction associated with \( \Phi \) and \( f \).

The fact that \( K_\gamma \) maps elements of \( \mathcal{W}_1 \times \mathcal{D}_0 \) to the real Banach space \( \mathcal{W}_1^\mathbb{R} \) is proven in Lemma 7.7 (i), which additionally shows that it is a locally Lipschitz continuous function. Definition 5.1 can be extended to all interactions \( \Phi \in \mathcal{V} \supseteq \mathcal{W}_1 \), real-valued functions \( f \) on \( \mathbb{R}^d \) and \( \gamma \in \mathbb{R} \). This generalization is not considered here since we focus on the limit \( \gamma \to 0^+ \) for Kac interactions in the Banach space \( \mathcal{W}_1 \).

Definition 5.1 includes, mutatis mutandis, much more general models than [10], where, translated to the lattice fermion case, only the following particular example is considered: Using the finite-range interaction \( N \in \mathcal{W}_0^\mathbb{R} \subseteq \mathcal{W}_1^\mathbb{R} \) defined by

\[
N_\Lambda = \begin{cases} 
\sum_{s \in S} a^*_x a_x & \text{if } \Lambda = \{x\} \text{ for } x \in \mathcal{L}, \\
0 & \text{otherwise}
\end{cases}, \quad \Lambda \in \mathcal{P}_f,
\]

(87)

In the definition of Kac interactions and functions, we have a factor of 2 as compared to Lieb’s and Penrose’s papers: Their sum is basically over pairs of enumerated particles \( i, j \) such that \( i < j \), while our definition applied to their special case would refer to a full sum over \( i, j \).
for any $f \in \mathcal{D}_0$ and $\gamma \in (0, 1)$, the associated Kac interaction is defined\(^{[7]}\) by

\[
K_{\gamma} (N, f)_\Lambda \doteq 2 \sum_{s,t \in S} \left(1 - \frac{3}{4} \delta_{x,y}\right) \gamma^d f(\gamma (x - y)) a^*_s a_y a_x a^*_t a_x a_y,
\]

whenever $\Lambda = \{x, y\}$ for $x, y \in \mathcal{L}$, and $K_{\gamma} (N, f)_\Lambda \doteq 0$ otherwise. Therefore, for any $\Phi \in \mathcal{V}^R$, the finite-volume Hamiltonian \(^{[28]}\) associated with the interaction $\Phi + K_{\gamma} (N, f)$ is equal in this case to

\[
U_{L}^{\Phi + K_{\gamma} (N,f)} \doteq U_{L}^{\Phi} + \frac{\gamma^d f (0)}{2} \sum_{x \in \Lambda_L, y \in S} a^*_x a_y a_x a^*_y + \sum_{x \in \Lambda_L, y \in S} \gamma^d f(\gamma (x - y)) a^*_y a_y a_x a_y a_x,
\]

where $L \in \mathbb{N}_0$.

In other words, the function $f \in \mathcal{D}_0$ is interpreted as a pair potential whose range\(^{[8]}\) is tuned by the parameter $\gamma \in (0, 1)$.

Another interesting example is given by BCS-type models of superconductivity. They refer to the following Kac interactions: Let the spin set $S = \{\uparrow, \downarrow\}$. Consider the finite-range interaction $C \in \mathcal{W}_0^R \subset \mathcal{W}_1^R$ defined by

\[
C_\Lambda \doteq \begin{cases} a_{x,\uparrow} a_{x,\downarrow} & \text{if } \Lambda = \{x\} \text{ for } x \in \mathcal{L} \\ 0 & \text{otherwise} \end{cases}, \quad \Lambda \in \mathcal{P}_f.
\]

For any $f \in \mathcal{D}_0$ and $\gamma \in (0, 1)$, the associated Kac interaction is defined by

\[
K_{\gamma} (C, -f)_\Lambda \doteq \gamma^d f(\gamma (x - y)) \left(\frac{3}{4} \delta_{x,y} - 1\right) \left(a^*_{y,\uparrow} a^*_{y,\downarrow} a_{x,\uparrow} a_{x,\downarrow} + a^*_{x,\uparrow} a^*_{x,\downarrow} a_{y,\uparrow} a_{y,\downarrow}\right)
\]

whenever $\Lambda = \{x, y\}$ for $x, y \in \mathcal{L}$, and $K_{\gamma} (C, -f)_\Lambda \doteq 0$ otherwise. Therefore, for any $\Phi \in \mathcal{V}^R$, the finite-volume Hamiltonian \(^{[28]}\) associated with the interaction $\Phi + K_{\gamma} (C, -f)$ is equal in this case to

\[
U_{L}^{\Phi + K_{\gamma} (C,-f)} \doteq U_{L}^{\Phi} + \frac{\gamma^d f (0)}{2} \sum_{x \in \Lambda_L} a^*_x a_{x,\uparrow} a_{x,\downarrow} a_x - \sum_{x \in \Lambda_L} \gamma^d f(\gamma (x - y)) a^*_x a_y a_{x,\uparrow} a_{x,\downarrow}, \quad L \in \mathbb{N}_0.
\]

The function $f \in \mathcal{D}_0$ encodes the hopping strength of Cooper pairs. This model thus implements a BCS interaction whose range is tuned by the parameter $\gamma \in (0, 1)$.

**Remark 5.1**

*The second term of the right-hand side of Equation \(^{[89]}\) can be absorbed in the short-range component $\Phi$. Additionally, it tends to zero like $\gamma^d$ in the sense of $\mathcal{W}_1$, as $\gamma \rightarrow 0^+$, and it is therefore irrelevant for the pressure in the Kac limit. The same is true for the second term of the right-hand side of Equation \(^{[97]}\).*

\[\text{[7]}\text{Here, } \delta_{x,y} \text{ denote the Kronecker delta.}\]

\[\text{[8]}\text{Take for instance a compactly supported function } f \in \mathcal{D}_0. \text{ See, more generally, Equation \(^{[83]}\).}\]

5.2.2 Multiple-Field Case

In order to approximate models with possibly infinitely many long-range components, we generalize Definition 5.1. To this end, recall that $S$ is the unit sphere of the Banach space $\mathcal{V}_1$ \(^{[24]}\) of translation-invariant short-range interactions, see Equation \(^{[38]}\). $S_1$ is the (real) Banach space of signed Borel measures of bounded variation on $S$. Similar to this construction, for any $\varepsilon \in \mathbb{R}^+$, we introduce the function sets

\[
\mathbb{D}_\varepsilon \doteq \{ f \in \mathcal{D}_\varepsilon : \| f \|_{\mathcal{D}_\varepsilon} = 1 \}, \quad \mathbb{D}_{\varepsilon,+} \doteq \mathbb{D}_\varepsilon \cap \mathbb{D}_{\varepsilon,+} \quad \text{and} \quad \mathbb{C}_{\varepsilon,+} \doteq \mathbb{D}_\varepsilon \cap \mathbb{C}_{\varepsilon,+}
\]
and consider the (real) Banach space of Borel measures of bounded variation on \( S \times D_\varepsilon, S \times D_\varepsilon^+, \) and \( S \times C_\varepsilon, +, \) denoted by \( D_\varepsilon, D_\varepsilon^+, \subseteq D_\varepsilon \) and \( C_\varepsilon^+, \subseteq D_\varepsilon, \) respectively. Here, we identify each element of \( D_\varepsilon^+ \) and \( C_\varepsilon^+ \) with an element of \( D_\varepsilon \) whose support lies in \( S \times D_\varepsilon^+ \) and \( S \times C_\varepsilon^+ \), respectively. Similar to (96), let
\[
D_0 \triangleq \bigcup_{\varepsilon \in \mathbb{R}^+} D_\varepsilon, \quad D_{0,+} \triangleq \bigcup_{\varepsilon \in \mathbb{R}^+} D_{\varepsilon,+} \quad \text{and} \quad C_{0,+} \triangleq \bigcup_{\varepsilon \in \mathbb{R}^+} C_{\varepsilon,+}. \tag{92}
\]

Recall that \( D_\varepsilon \equiv D_{\varepsilon,\kappa} \) still depends upon the parameter \( \kappa = 2d, 2d + 1, \ldots, \) which is by default set to \( 2d. \) See Section 5.1. Hence, all the objects \( D_{\varepsilon}, D_{\varepsilon,+}, C_{\varepsilon,+}, D_{\varepsilon}, D_{\varepsilon,+}, C_{\varepsilon,+}, D_0, D_{0,+} \) and \( C_{0,+} \) depend upon \( \kappa. \) As explained in Section 5.1 this parameter is omitted in the notation, unless we have to refer to it explicitly. For any \( \varepsilon \in \mathbb{R}^+ \) and \( \kappa_1, \kappa_2 = 2d, 2d + 1, \ldots \) with \( \kappa_2 \geq \kappa_1, \) we can canonically identify \( D_{\varepsilon,\kappa_2} \) with a subspace of \( D_{\varepsilon,\kappa_1}. \) Let \( \Xi \) denote the continuous mapping
\[
(\Psi, f) \mapsto (\Psi, \|f\|_{D_{\varepsilon,\kappa_1}}^{-1} f)
\]
from \( S \times D_{\varepsilon,\kappa_2} \) to \( S \times D_{\varepsilon,\kappa_1} \) and let the continuous, bounded, positive valued function \( d : S \times D_{\varepsilon,\kappa_1} \rightarrow \mathbb{R}_0^+ \) be defined by \( d(\Psi, f) \triangleq \|f\|_{D_{\varepsilon,\kappa_1}}. \) Then any \( b \in D_{\varepsilon,\kappa_2} \) is identified with the pushforward
\[
\Xi_*(\mathbb{b}) \in D_{\varepsilon,\kappa_1}
\]
of \( \mathbb{b} \) under the (Borel measurable) mapping \( \Xi, \) where \( \mathbb{b} \in D_{\varepsilon,\kappa_2} \) is the Borel measure of finite variation defined by
\[
\mathbb{b}(A) \triangleq \int_A d(\Psi, f) b(d(\Psi, f))
\]
for any Borel set \( A \subseteq S \times D_{\varepsilon,\kappa_2}. \) Canonical vector space inclusions \( D_{0,\kappa_2} \subseteq D_{0,\kappa_1}, D_{0,\kappa_2,+} \subseteq D_{0,\kappa_1,+}, \) and \( C_{0,\kappa_2,+} \subseteq C_{0,\kappa_1,+} \) are defined in a similar manner.

Recall that the Kac function \( K_\gamma \) is the mapping from \( \mathcal{W}_1 \times D_0 \rightarrow \mathcal{W}_1^R \) of Definition 5.1 for any fixed \( \gamma \in (0, 1). \) By Lemma 7.7(i), it is locally Lipschitz continuous. Therefore, we can extend the definition of Kac interactions by replacing simple interactions with Borel measures in \( D_0: \)

**Definition 5.2 (Kac interactions – multiple-field case)**

*For any \( \varepsilon \in \mathbb{R}^+, b \in D_{\varepsilon} \) and \( \gamma \in (0, 1), \) we define the corresponding Kac interaction to be
\[
\Phi^{b,\gamma} \triangleq \int_{S \times D_{\varepsilon}} K_\gamma(\Psi, f) b\left(d\left(\Psi, f\right)\right) \in \mathcal{W}_1^R.
\]

Thanks to Lemma 7.7(i), the above integral is a Bochner integral because the measure \( b \) has bounded variation and \( \mathcal{W}_1 \) and \( D_{\varepsilon} \) are both separable Banach spaces. See, e.g., [32, Chapter III, Theorems 1.1 and 1.2].

Note that the mappings \( b \mapsto \Phi^{b,\gamma}, \gamma \in (0, 1), \) from \( D_\varepsilon \) to \( \mathcal{W}_1^R, \) are (real) linear. With the canonically identification of \( D_{\varepsilon,\kappa_2} \) with a subspace of \( D_{\varepsilon,\kappa_1} \) for \( \kappa_2 \geq \kappa_1, \) as explained above, observe also that for all \( b \in D_{\varepsilon,\kappa_2} \) and \( \gamma \in (0, 1), \) one has
\[
\int_{S \times D_{\varepsilon,\kappa_1}} K_\gamma(\Psi, f) b\left(d\left(\Psi, f\right)\right) = \int_{S \times D_{\varepsilon,\kappa_2}} K_\gamma(\Psi, f) b\left(d\left(\Psi, f\right)\right),
\]
i.e., in the above definition of \( \Phi^{b,\gamma} \in \mathcal{W}_1^R \) it does not matter on which space \( S \times D_{\varepsilon,\kappa}, \kappa \in 2d, 2d + 1, \ldots, \) one sees \( b \) as a measure.

Recall also that the subset \( \mathcal{W}_0 \) of finite-range interactions defined by (27) is dense in the Banach space \( \mathcal{W}_1. \) We have a similar property for Kac interactions, in the following sense:
Lemma 5.2 (Approximation by finite-range interactions)

For any \( \varepsilon \in \mathbb{R}^+ \), \( b \in D_\varepsilon \) and \( \eta \in \mathbb{R}^+ \), there is a finite sequence \( (\Psi_1, f_1), \ldots, (\Psi_n, f_n) \) in \( S \times D_\varepsilon \) such that, for any \( \gamma \in (0, 1) \),

\[
\{\Psi_1, \ldots, \Psi_n\} \subseteq W_0 \quad \text{and} \quad \|\Phi^{b,\gamma} - \Phi^{b,\eta,\gamma}\|_{W_1} \leq \eta,
\]

where \( b_\eta \in D_\varepsilon \) is the finite sum of Dirac measures \( \delta_{(\Psi, f)} \) on \( S \times D_\varepsilon \):

\[
b_\eta = \sum_{j=1}^n \delta_{(\Psi_j, f_j)}.
\]

If \( b \in C_\varepsilon \) then the sequence can be chosen such that \( f_1, \ldots, f_n \in C_\varepsilon \).

Proof. Fix \( \varepsilon \in \mathbb{R}^+ \), \( b \in D_\varepsilon \) and \( \eta \in \mathbb{R}^+ \). Again, \( W_1 \) and \( D_\varepsilon \) are both separable Banach spaces. Then, for any \( \vartheta \in \mathbb{R}^+ \), there is a step function \( 1_\vartheta \) from \( S \times D_\varepsilon \) to itself such that

\[
\int_{S \times D_\varepsilon} \|1_\vartheta (\Psi, f) - (\Psi, f)\|_{W_1 \times D_\varepsilon} b (d (\Psi, f)) \leq \vartheta,
\]

thanks to [32, Chapter III, Theorems 1.1 and 1.2]. Because of the density of \( W_0 \subseteq W_1 \), we can assume that

\[
1_\vartheta (S \times D_\varepsilon) \subseteq W_0 \times D_\varepsilon.
\]

We then define \( b_\eta \) to be the pushforward of \( b \) through \( 1_\vartheta \). By Lemma [77](i),

\[
\|\Phi^{b,\gamma} - \Phi^{b_\eta,\gamma}\|_{W_1} \leq \eta, \quad \gamma \in (0, 1),
\]

for sufficiently small \( \vartheta \in \mathbb{R}^+ \). The proof for the special case \( b \in C_\varepsilon \) only needs an obvious adaptation of the above arguments.

5.3 Kac Limit of Energy Densities

5.3.1 Simple-Field Case

Recall that the energy density functional on translation-invariant states is the mapping \( e_\Psi : E_1 \to \mathbb{C} \), defined by (29) for each translation-invariant interaction \( \Psi \in W_1 \), that is,

\[
e_\Psi (\rho) = \rho (e_\Psi) \quad \text{with} \quad e_\Psi \doteq \sum_{Z \in P_t, Z \geq 0} \frac{\Psi_Z}{|Z|} \in U
\]

for any \( \Psi \in W_1 \) and \( \rho \in E_1 \). Note that any translation-invariant interaction \( \Phi \in W_1 \) yields a self-adjoint and translation-invariant Kac interaction \( \mathcal{K}_\gamma (\Phi, f) \in W_R^1 \) for each \( f \in D_0 \) and \( \gamma \in (0, 1) \), see Definition [51]. In particular, Equation (92) with \( \Psi = \mathcal{K}_\gamma (\Phi, f) \) gives the energy density of the Kac interaction in the simple-field case. In the Kac limit \( \gamma \to 0^+ \), we obtain a multiple of the space-averaging functional \( \Delta_{e_\Phi} : E_1 \to \mathbb{R} \) defined on translation-invariant states by (45) for \( A = e_\Phi \):

Theorem 5.3 (Energy density in the Kac limit)

For any interaction \( \Phi \in W_1 \) and function \( f \in D_0 \), the family \( (e_{\mathcal{K}_\gamma (\Phi, f)})_{\gamma \in (0, 1)} \) of energy density functionals converges pointwise to the functional \( \hat{f}(0) \Delta_{e_\Phi} \) in the Kac limit, i.e.,

\[
\lim_{\gamma \to 0^+} e_{\mathcal{K}_\gamma (\Phi, f)} (\rho) = \hat{f}(0) \Delta_{e_\Phi} (\rho), \quad \rho \in E_1,
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \).
Proof. Fix \( \Phi \in \mathcal{W}_1 \) and \( f \in \mathcal{D}_0 \). The proof is done in several steps:

Step 1: The element \( c_{\gamma} \) given by Equation (93) for the Kac interaction \( \Psi = K_{\gamma} (\Phi, f) \in \mathcal{W}_1^{\mathbb{R}} \) of Definition 5.1 satisfies the equality

\[
e_{\mathcal{K}_{\gamma} (\Phi, f)} + R_{\gamma}^{f} = \sum_{z_1, z_2 \in \mathbb{P}_1, z_1 \cap z_2 \geq \{0\}} \left( \frac{\Phi_{z_1}^2 \Phi_{z_2}}{|z_1| + |z_2|} \sum_{x \in E} \alpha_x (\Phi_{z_2}) \sum_{y \in E} \gamma_d f (\gamma (x - y) - \gamma z) \right) \frac{|z_1| |z_2|}{|z_1| + |z_2|} + \sum_{x \in \mathbb{Z}, y \in \mathbb{Z}} \alpha_x (\Phi_{z_1}^2) \Phi_{z_2} \sum_{y \in E} \gamma_d f (\gamma (x - y) + \gamma z) \frac{|z_1| |z_2|}{|z_1| + |z_2|}, \tag{94}\]

where \( \{\alpha_x\}_{x \in \mathbb{Z}^d} \) is the set of (translation) \(*\)-automorphisms of \( \mathcal{U} \) defined by (13) and

\[
R_{\gamma}^{f} = \sum_{z_1, z_2 \in \mathbb{P}_1, z_1 \cap z_2 \geq \{0\}} \frac{\Phi_{z_1}^2 \Phi_{z_2}}{|z_1| + |z_2|} \sum_{x \in \mathbb{Z}, y \in \mathbb{Z}} \gamma_d f (\gamma (x - y)) \frac{|z_1| |z_2|}{|z_1| + |z_2|}. \tag{95}\]

Equation (94) results from arguments that are very similar to the ones used to get (166). See also (21). Note from Lemma (7.8) and the triangle inequality that

\[
||R_{\gamma}^{f}||_{\mathcal{U}} \leq \mathfrak{F}_{\gamma} (\Phi, f) \quad \text{and thus,} \quad \lim_{\gamma \to +0} ||R_{\gamma}^{f}||_{\mathcal{U}} = 0. \tag{96}\]

We therefore only need to study the right-hand side of (94) more precisely. This is done by using a cyclic representation of translation-invariant states to have the spectral theorem at our disposal.

Step 2: Any state \( \rho \in E \) induces a cyclic representation of the \( C^* \)-algebra \( \mathcal{U} \) [27, Theorem 2.3.16]:

For any \( \rho \in E \), there exist a Hilbert space \( \mathcal{H}_\rho \), a representation \( \pi_\rho : \mathcal{U} \to \mathcal{B} (\mathcal{H}_\rho) \) of \( \mathcal{U} \) on \( \mathcal{H}_\rho \), and a norm-one vector \( \Omega_\rho \in \mathcal{H}_\rho \), which is cyclic with respect to \( \pi_\rho (\mathcal{U}) \), such that

\[
\rho (A) = \langle \Omega_\rho, \pi_\rho (A) \Omega_\rho \rangle_{\mathcal{H}_\rho}, \quad A \in \mathcal{U}. \]

Recall that any state \( \rho \) is translation-invariant, i.e., \( \rho \in \mathbb{E}_1 \), iff \( \rho \circ \alpha_x = \rho \) for any \( x \in \mathbb{Z}^d \). See Equation (14). Since the mapping \( x \mapsto \alpha_x \) from \( \mathbb{Z}^d \) into the group of \(*\)-automorphisms of \( \mathcal{U} \) is a group homomorphism, if \( \rho \) is translation-invariant, then there is a uniquely defined family \( \{U_x\}_{x \in \mathbb{Z}^d} \) of unitary operators in \( \mathcal{B} (\mathcal{H}_\rho) \) with invariant vector \( \Omega_\rho \), i.e., \( \Omega_\rho = U_x \Omega_\rho \) for any \( x \in \mathbb{Z}^d \), and such that

\[
\pi_\rho (\alpha_x (A)) = U_x \pi_\rho (A) U_x^*, \quad A \in \mathcal{U}, \quad x \in \mathbb{Z}^d. \]

In particular, \( U_{x+y} = U_x U_y \) for any \( x, y \in \mathbb{Z}^d \) and \( U_x^* = U_{-x} \). See for instance [27, Corollary 2.3.17]. Since \( (\mathbb{Z}^d, +) \) is abelian, the normal operators

\[
U_{(1,0,0,\ldots)} U_{(0,1,0,\ldots)} \cdots U_{(0,0,1,\ldots,1)} \in \mathcal{B} (\mathcal{H}_\rho) \]

commute with each other and their joint spectrum is contained in the \( d \)-dimensional torus

\[
\mathbb{T}_d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : |z_i| = 1, \ i = 1, \ldots, d\}. \]

The spectral theorem\(^{19}\) ensures the existence of a projection-valued measure \( P \) on the Borel \( \sigma \)-algebra of \( \mathbb{T}_d \) (with respect to the usual Euclidean metric for \( \mathbb{C}^d \)), such that

\[
U_z = \int_{\mathbb{T}_d} x^2 \cdots x_d^2 dP (x) = \int_{\mathbb{T}_d} e^{i \theta \cdot x} d (P_\theta (x)) = \int_{\mathbb{T}_d} e^{i \theta \cdot x} d (P_\theta (x)) \tag{97}, \quad z \in \mathbb{Z}^d,
\]

\(^{19}\)For any family \( \{A_j, A_j^*\}_{j \in J} \) of commuting bounded operators on some Hilbert space, the real and imaginary parts of these operators commute, i.e., \( \{\text{Re} (A_j), \text{Im} (A_j)\}_{j \in J} \) is also a commuting family. The spectral theorem for such general commuting families \( \{A_j, A_j^*\}_{j \in J} \) of (normal) operators directly follows from the one for (commuting) self-adjoint operators, as in [33, Chap. 6, Theorem 2 and compare with Sect. 5].
where \( \Theta_d \equiv [-\pi, \pi]^d \) and \( F_\gamma P \) is the pushforward of \( P \) under the (Borel measurable) mapping \( \Gamma : T_d \to \Theta_d \) defined by

\[
F(x_1, \ldots, x_d) \doteq (\arg(x_1), \ldots, \arg(x_d)) , \quad (x_1, \ldots, x_d) \in T_d .
\] (98)

By the measurable functional calculus, the mapping

\[
f \mapsto \int_{\Theta_d} f(\theta) d(F_* P)(\theta)
\]

is a \(*\)-homomorphism from the (commutative) \( C^* \)-algebra of bounded Borel-measurable functions on \( \Theta_d \) to \( B(H_\rho) \). In particular, for any pair \( f, g \) of bounded Borel-measurable functions on \( \Theta_d \), one has

\[
\int_{\Theta_d} f(\theta) g(\theta) d(F_* P)(\theta) = \int_{\Theta_d} f(\theta) d(F_* P)(\theta) \int_{\Theta_d} g(\theta) d(F_* P)(\theta) .
\] (99)

See, e.g., [33, Section 3 of Chapter 5, in particular Equation (5) and Theorem 1].

**Step 3:** Using the reflection-symmetry of \( f \) and a cyclic representation (Step 2), for any translation-invariant state \( \rho \in E_1 \), we infer from Equations (94)–(95) that

\[
\rho \left( e^{\mathcal{K}_i(\Phi, f)} \right) + \rho (\mathcal{R}_\Phi f) = \sum_{Z_1, Z_2 \in \mathcal{P}; Z_1 \cap Z_2 \not\subset \{0\}} \left( \frac{\pi_\rho (\Phi_{Z_1})}{|Z_1|} \right) \sum_{x \in \mathcal{Z}_1, y \in \mathcal{Z}_2} \frac{B_{\gamma} (y - x) \pi_\rho (\Phi_{Z_2})}{|Z_1| |Z_2|} \Omega_{Z_1} Z_2 \right) \in H_\rho .
\] (100)

where, for any \( a \in \mathcal{L} \) and \( \gamma \in (0, 1) \), we define the operator

\[
B_{\gamma} (a) \doteq \sum_{z \in \mathcal{L}} \gamma^d f(\gamma a + \gamma z) U_z \in B(H_\rho) .
\] (101)

Note that \( B_{\gamma}(a) \) is well-defined because of (161) and the triangle inequality. The next step is a study of the operator (101), since we already know that, as \( \gamma \to 0^+ \), \( \rho (\mathcal{R}_\Phi f) \to 0 \) uniformly in \( \rho \in E \), thanks to Equation (95).

**Step 4:** Using Equations (97)–(98), the operator (101) is equal for \( a \in \mathcal{L} \) and \( \gamma \in (0, 1) \) to

\[
B_{\gamma} (a) = \int_{\Theta_d} \sum_{z \in \mathcal{L}} \gamma^d f(\gamma a + \gamma z) e^{i\theta z} d(F_* P)(\theta) .
\] (102)

Defining the function \( g \) on the lattice \( \mathcal{L} \) by

\[
g(y) \doteq f(y) e^{\gamma^{-1} i\theta y} e^{-i\theta a} , \quad y \in \mathcal{L} ,
\]

for any \( a \in \mathcal{L}, \gamma \in (0, 1) \) and \( \theta \in \Theta_d \), we obviously have the equalities

\[
g(\gamma a + \gamma z) = f(\gamma a + \gamma z) e^{i\theta z} , \quad z \in \mathcal{L} ,
\]

as well as

\[
g(\gamma a + \gamma z) = f(k - \gamma^{-1} \theta) e^{-i\theta a} , \quad k \in \mathbb{R}^d .
\]

Since \( f \in \mathcal{D}_0 \), we can thus infer from Equation (102), combined with the Poisson summation formula applied to \( g \) (Proposition [7, 3]), that

\[
B_{\gamma} (a) = \int_{\Theta_d} \hat{f} (-\gamma^{-1} \theta) e^{-i\theta a} d(F_* P)(\theta) + R_{f,a,\gamma} ,
\] (103)
for any \( a \in \mathcal{L} \) and \( \gamma \in (0, 1) \), where

\[
R_{f,a,\gamma} := \int_{\Theta_d} \sum_{\varepsilon \in \mathcal{L} \setminus \{0\}} \hat{f} \left( \gamma^{-1} (2\pi z - \theta) \right) e^{i(2\pi z - \theta) a} d(F_sP)(\theta) .
\]  

(104)

By Lemma [7.5] observe that, for any \( \varepsilon \in \mathbb{R}^+ \), there is a constant \( M_\varepsilon \in \mathbb{R}^+ \) such that, for any \( \gamma \in (0, 1), \theta \in (-\pi, \pi)^d \) and \( f \in \mathcal{D}_\varepsilon \),

\[
\sum_{k \in \mathcal{L} \setminus \{0\}} \left| \hat{f} \left( \gamma^{-1} (2\pi z - \theta) \right) \right| \leq \gamma^2 M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} .
\]

It then directly follows from (104) that

\[
\|R_{f,a,\gamma}\|_{\mathcal{B}(\mathcal{H}_0)} \leq \gamma^2 M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} , \quad f \in \mathcal{D}_\varepsilon, \; \varepsilon \in \mathbb{R}^+ .
\]  

(105)

We are now in a position to study the Kac limit of the energy density given by Equations (100)–(101).

Step 5: The function \( f \in \mathcal{D}_{0,+} \) being reflection-symmetric, \( \hat{f} \) is reflection-symmetric. It follows from Equations (100) and (103) that

\[
\rho \left( \mathcal{L}_\varepsilon(\phi_0, f) \right) + \rho(\mathcal{L}_\varepsilon(\phi_\gamma)) + \zeta(\gamma)
= \sum_{z_1, z_2 \in \mathcal{P}_{1}; z_1 \neq z_2 \geq 0} \left( \pi_\rho(\Phi_{z_1}) |Z_1| \Omega, \int_{\Theta_d} \hat{f} \left( \gamma^{-1} \theta \right) \sum_{x \in \mathbb{Z}_1, y \in \mathbb{Z}_2} \frac{e^{i\theta \cdot (x-y)}}{|Z_1| |Z_2|} d(F_sP)(\theta) \pi_\rho(\Phi_{z_2}) |Z_2| \Omega \right)_{\mathcal{H}_\rho}
\]

for some complex-valued function \( \zeta \) satisfying

\[
|\zeta(\gamma)| \leq \gamma^2 M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi\|^2_{\mathcal{W}_2} , \quad f \in \mathcal{D}_\varepsilon, \; \varepsilon \in \mathbb{R}^+ ,
\]  

(107)

thanks to Equations (23) and (105). On the other hand, we infer from [1] Equation (4.17)] that

\[
\Delta_A(\rho) = \|P_\rho \pi_\rho(A) \Omega\|_{\mathcal{H}_\rho}^2, \quad \rho \in E_1 ,
\]

(108)

where \( P_\rho \) is the orthogonal projection on the subspace of elements of \( \mathcal{H}_\rho \) that are invariant with respect to all unitary operators \( U_x \) for \( x \in \mathbb{Z}^d \). In other words, by Equation (97) and other well-known properties of the measurable spectral calculus, this projection can be written as

\[
P_\rho = \int_{\Theta_d} h(\theta) d(F_sP)(\theta) ,
\]

(109)

where \( h: \Theta_d \to \mathbb{R} \) is the function defined by \( h(0) \equiv 1 \) and \( h(\theta) \equiv 0 \) for all \( \theta \in \Theta_d \setminus \{0\} \). Equality (108) is in fact a direct consequence of (17) and (43) together with the von Neumann ergodic theorem [1] Theorem 4.2]. Now, for every (nonempty) finite subset \( Z \in \mathcal{P}_1 \) with \( 0 \in Z \), observe that

\[
\lim_{\gamma \to 0^+} \hat{f} \left( \gamma^{-1} \theta \right) \sum_{x \in \mathbb{Z}_1, y \in \mathbb{Z}_2} \frac{e^{i\theta \cdot (x-y)}}{|Z_1| |Z_2|} = \hat{f}(0) h(\theta) , \quad \theta \in \Theta_d .
\]

By Lebesgue’s dominated convergence theorem, it follows that the operators

\[
\int_{\Theta_d} \hat{f} \left( \gamma^{-1} \theta \right) \sum_{x \in \mathbb{Z}_1, y \in \mathbb{Z}_2} \frac{e^{i\theta \cdot (x-y)}}{|Z_1| |Z_2|} d(F_sP)(\theta) \in \mathcal{B}(\mathcal{H}_\rho) , \quad \gamma \in (0, 1) ,
\]

converge strongly to the operator

\[
\int_{\Theta_d} \hat{f}(0) h(\theta) d(F_sP)(\theta) = \hat{f}(0) P_\rho ,
\]

as \( \gamma \to 0^+ \). Together with Equations (93), (96) and (106)–(108), the theorem then follows. □
5.3.2 Multiple-Field Case

Recall that Kac interactions of Definition 5.1 are generalized by Definition 5.2 to accommodate possibly infinitely many mean-field components in the Kac limit. This generalization uses Borel measures of bounded variation on either $S \times D_\epsilon, S \times D_{\epsilon,+}$ or $S \times C_{\epsilon,+}$, corresponding to the spaces $D_\epsilon, D_{\epsilon,+} \subseteq D_\epsilon$ and $C_{\epsilon,+} \subseteq D_\epsilon$, respectively, for any fixed $\epsilon \in \mathbb{R}^+$. See also Equation (92). In the same way we obtain the energy density of the Kac interaction in the limit $\gamma \to 0^+$ for the simple-field case (Theorem 5.3), we get an analogous result for the general (multiple-field) case. To this end, we need a standard result of measure theory, referring to the disintegration of measures:

**Theorem 5.4 (Disintegration of measures)**

Let $X, Y$ be complete separable metric spaces. Take $\mu$ a finite (positive) Borel measure on $X \times Y$ and let $\pi^* \mu$ be the pushforward of $\mu$ through the projection $\pi : X \times Y \to X$ defined by

$$\pi(x, y) = x, \quad x \in X, \ y \in Y.$$  

Then, there exists a $\pi^* \mu$-a.e. uniquely defined family $\{\mu_x\}_{x \in X}$ of probability measures on $Y$, such that, for any Borel set $A \subseteq Y$, the mapping $x \mapsto \mu_x(A)$ is Borel measurable and, for any measurable function $g : X \times Y \to [0, \infty]$,  

$$\int_{X \times Y} g(x, y) \, \mu(dx, dy) = \int_X \left( \int_Y g(x, y) \, \mu_x(dy) \right) (\pi^* \mu)(dx).$$  

**Proof.** This theorem is a straightforward application of [34, Theorem 5.3.1].

This disintegration theorem is useful in the sequel by allowing us to extend Theorem 5.3 to Kac interactions of Definition 5.2.

To explain how we use this result, note first that, for any $\epsilon \in \mathbb{R}^+$, the set $S \times D_\epsilon$ is a complete and separable metric space with respect to the restriction of the norm of $W_1 \times D_\epsilon$, which is a separable Banach space. Then, for any $\epsilon \in \mathbb{R}^+$ and any finite (positive) measure $\mu \in D_\epsilon$, we use Theorem 5.4 to define the Borel measure $a_\mu \in S_1$ by

$$a_\mu(A) = \int_A \left( \int_{D_\epsilon} \hat{f}(0) \, \mu_{\Psi}(df) \right) (\pi^* \mu)(d\Psi)$$  

for all Borel sets $A \subseteq S$, with $\pi^* \mu$ being the pushforward of $\mu$ under the projection mapping $\pi : S \times D_\epsilon \to S$ defined by

$$\pi(\Psi, f) = \Psi, \quad \Psi \in S, \ f \in D_\epsilon.$$  

Note that this measure is well-defined because the mapping $f \mapsto \hat{f}(0)$ from $D_\epsilon$ to $\mathbb{R}_0^+$ is continuous. Observe also that the mapping $\mu \mapsto a_\mu$ from $D_\epsilon$ to $S_1$ is (real) linear.

We then obtain the following theorem:

**Theorem 5.5 (Energy density in the Kac limit)**

For any $\mu \in D_0$, the family $(e_{\Phi, \gamma}^\mu)_{\gamma \in (0,1)}$ of energy density functionals converges pointwise to the functional $\Delta a_\mu$ in the Kac limit, i.e.,

$$\lim_{\gamma \to 0^+} e_{\Phi, \gamma}^\mu(\rho) = \Delta a_\mu(\rho), \quad \rho \in E_1.$$  

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Proof. Take a finite measure \( b \in D_0 \) and observe from Equation (93) and Definition 5.2, along with the continuity of states and basic properties of the Bochner integral, that

\[
e_{\Phi, b, \gamma} (\rho) = \int_{\mathbb{S} \times D_\varepsilon} e_{\mathcal{K}_\gamma (\Psi, f)} (\rho) \, b \left( d (\Psi, f) \right), \quad \rho \in E_1.
\] (112)

See also Lemma 7.7 (i). So, since \( b \) is a finite measure, it suffices to invoke Equation (32), Theorem 5.3, and Lemma 7.7 (i), as well as Lebesgue’s dominated convergence theorem to arrive at

\[
\lim_{\gamma \to 0^+} e_{\Phi, b, \gamma} (\rho) = \int_{\mathbb{S} \times D_\varepsilon} \hat{f} (0) \, \Delta_{\epsilon_\gamma} (\rho) \, b \left( d (\Psi, f) \right), \quad \rho \in E_1.
\]

Combined with Theorem 5.4, this last equality in turn implies the pointwise convergence, as \( \gamma \to 0^+ \), of the family \( (e_{\Phi, b, \gamma})_{\gamma \in (0, 1)} \) to the functional \( \Delta_{a_b} \), because, for any translation-invariant state \( \rho \in E_1 \),

\[
\int_{\mathbb{S} \times D_\varepsilon} \hat{f} (0) \, \Delta_{\epsilon_\gamma} (\rho) \, b \left( d (\Psi, f) \right) = \int_{\mathbb{S}} \Delta_{\epsilon_\gamma} (\rho) \left( \int_{D_\varepsilon} \hat{f} (0) \, \mu_\Phi (df) \right) \left( \pi \ast b \right) (d\Psi)
\]

\[
= \int_{\mathbb{S}} \Delta_{\epsilon_\gamma} (\rho) \, a_b (d\Psi) \doteq \Delta_{a_b} (\rho),
\] (113)

by Equations (47) and (110).

5.4 Thermodynamics in the Kac Limit – Repulsive Case

In this section we study the convergence of Fermi systems associated with Kac interactions towards purely repulsive mean-field models in the Kac limit, after taking the thermodynamic limit. As explained in Section 4.2, purely repulsive mean-field models refer here to translation-invariant mean-field models \( m = (\Phi, a) \in M_1 \), with \( a = a_+ \) being a finite positive measure on \( \mathbb{S} \). The attractive case is studied in the next section. Similar to the study of the Kac limit of energy densities done in Section 5.3, we divide this study in two steps, by starting with the pedagogical case of models with a single mean-field interaction.

5.4.1 Simple-Field Case

In order to study the convergence of Fermi systems associated with Kac interactions towards purely repulsive mean-field models, we use an estimate on energy densities of Kac interactions with respect to space-averaging functionals on translation-invariant states (Section 4.3). It is obtained from a slight modification of the proof of Theorem 5.3 and corresponds to the following assertion:

Proposition 5.6 (Energy densities of Kac interactions and space-averaging functionals)

For any parameter \( \varepsilon \in \mathbb{R}^+ \), there is a constant \( M_\varepsilon > 0 \) such that, for any \( \gamma \in (0, 1) \), each function \( f \in D_{\varepsilon, +} \), any interaction \( \Phi \in \mathcal{W}_1 \) and \( \rho \in E_1 \),

\[
\hat{f} (0) \, \Delta_{\epsilon_\gamma} (\rho) \leq e_{\mathcal{K}_\gamma (\Phi, f)} (\rho) + \mathcal{F}_\gamma (\Phi, f) + \gamma^2 M_\varepsilon \| f \|_{D_\varepsilon} \| \Phi \|_{\mathcal{W}_1}^2,
\]

where \( \hat{f} \) denotes the Fourier transform of \( f \) and \( \mathcal{F}_\gamma \) is the mapping from \( \mathcal{W}_1 \times \mathbb{D}_0 \) to \( \mathbb{R}^+ \) defined by Equation (167).

Proof. Using all mathematical objects described in the proof of Theorem 5.3, the identity

\[
h(\theta)g(\theta) = h(\theta)g(0), \quad \theta \in \Theta_d,
\]
for any function \( g : \Theta_d \rightarrow \mathbb{C} \), as well as Equation (99), we obtain from (93) and (108)–(109) that
\[
\hat{f}(0) \Delta_{\epsilon \phi} (\rho) = \left\| \int_{\Theta_d} \sqrt{\hat{f}(0)} h(\theta) d(F_{*}P)(\theta) \sum_{z \in \Pi : |z| \geq 0} \frac{\pi_{\rho}(\Phi z)}{|z|} \Omega_{\rho} \right\|_{\mathcal{H}_{\rho}}^2
\]
\[= \left\| \sum_{z \in \Pi : |z| \geq 0} \int_{\Theta_d} h(\theta) \sqrt{\hat{f}(\gamma^{-1} \theta)} \sum_{x \in Z} e^{i \theta \cdot x} d(F_{*}P)(\theta) \frac{\pi_{\rho}(\Phi z)}{|z|} \Omega_{\rho} \right\|_{\mathcal{H}_{\rho}}^2
\]
\[= \rho \sum_{z \in \Pi : |z| \geq 0} \int_{\Theta_d} \sqrt{\hat{f}(\gamma^{-1} \theta)} \sum_{x \in Z} e^{i \theta \cdot x} d(F_{*}P)(\theta) \frac{\pi_{\rho}(\Phi z)}{|z|} \Omega_{\rho} \],
which, together with Equation (106), directly implies the upper bound
\[
\hat{f}(0) \Delta_{\epsilon \phi} (\rho) \leq \rho (\epsilon_{\Phi,\gamma}(f,\phi)) + \rho(\mathcal{R}_{\gamma \phi}) + \zeta(\gamma)
\]
for any \( \gamma \in (0, 1) \). Now, it suffices to invoke Equations (96) and (107) to get the assertion. \( \blacksquare \)

Proposition 5.6 gives in particular an upper bound for the space-averaging functional on translation-invariant states via an energy density functional of a short range interaction in \( \mathcal{W}_{1} \). This upper bound is \textit{uniform} with respect to translation-invariant states, by (67) and Lemma 7.8. This is a strong property which directly implies that the infinite volume pressure \( P_{\kappa_{\gamma}(\Phi, f)} \) and the equilibrium states of \( M_{\kappa_{\gamma}(\Phi, f)} \) associated with some Kac interaction \( \kappa_{\gamma}(\Phi, f) \), as respectively defined by (34) and (37), converge in the Kac limit \( \gamma \to 0^+ \) to those of the expected purely repulsive mean-field model \( m \in \mathcal{M}_{1} \). See (49) and (63) for the definitions of the infinite volume pressure and (generalized) equilibrium states of any mean-field model. More precisely, one gets the following result:

**Theorem 5.7 (From short-range interactions to repulsive mean-field models)**

Take interactions \( \Phi \in \mathcal{W}_{1}^{\mathbb{R}}, \Psi \in \mathbb{S} \) (see (38)) and a positive definite function \( f \in \mathcal{D}_{0,+} \), whose Fourier transform is denoted by \( \hat{f} \). Let \( \delta_{\psi} \in S_{1} \) be the Dirac measure on \( \Psi \in \mathbb{S} \).

(i) Convergence of infinite-volume pressures:
\[
\lim_{\gamma \to 0^+} P_{\Phi+\kappa_{\gamma}(\Psi, f)} = P_{\Phi, f}(0, \delta_{\psi}) = P_{\Phi, f}(0, \delta_{\psi}) \cdot \rho_{\Phi, f}(0, \delta_{\psi})
\]

(ii) Convergence of equilibrium states: Weak* accumulation points of any net of equilibrium states \( \omega_{\gamma} \in M_{\Phi+\kappa_{\gamma}(\Psi, f)} \) as \( \gamma \to 0^+ \) are generalized equilibrium states of the purely repulsive mean-field model \( \Phi, \hat{f}(0) \delta_{\psi} \), i.e., they belong to the weak* compact convex set \( \Omega_{\Phi, f}(0, \delta_{\psi}) = \Omega_{\Phi, f}(0, \delta_{\psi}) \).

**Proof.** Before starting, recall that \( P_{\Phi_{\sigma}} = P_{\Phi} \) and \( \Omega_{\Phi_{\sigma}} = \Omega_{\Phi} \) for any purely repulsive mean-field model \( m \in \mathcal{M}_{1} \), thanks to [1] Theorem 2.25. By Equations (35) and (36), for any \( \Phi \in \mathcal{W}_{1}^{\mathbb{R}}, \Psi \in \mathbb{S} \) and \( f \in \mathcal{D}_{0} \),
\[
P_{\Phi+\kappa_{\gamma}(\Psi, f)} = - \inf_{\rho \in E_{1}} \{ f_{\Phi}(\rho) + e_{\kappa_{\gamma}(\Psi, f)}(\rho) \}, \quad \gamma \in (0, 1),
\]
while from (50)–(51),
\[
P_{\Phi, f}(0, \delta_{\psi}) = - \inf_{\rho \in E_{1}} \{ f_{\Phi}(\rho) + \hat{f}(0) \Delta_{\epsilon \phi}(\rho) \}. \quad (115)
\]

Assertion (i) is therefore a direct consequence of (114)–(115) combined with Theorem 5.3, Proposition 5.6 and Lemma 7.8. The proof of Assertion (ii) is similar: For any net of equilibrium states \( \omega_{\gamma} \in M_{\Phi+\kappa_{\gamma}(\Psi, f)} \), we combine (114) with Proposition 5.6 to deduce that, for any \( \gamma \in (0, 1) \),
\[
f_{\Phi}(\omega_{\gamma}) + \Delta_{\epsilon \phi}(\omega_{\gamma}) \leq \inf_{\rho \in E_{1}} \{ f_{\Phi}(\rho) + e_{\kappa_{\gamma}(\Psi, f)}(\rho) \} + \mathcal{F}_{\gamma}(\Phi, f) + \gamma^{2} M_{\epsilon} \| f \|_{D_{\varepsilon}} \| \Phi \|_{W_{1}}^{\frac{\gamma}{2}},
\]

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which, combined with Equations (65) and (115), Lemma 7.8 and Assertion (i), implies the second statement of the theorem.

If one considers the simple example given by (87)–(89), then Theorem 5.7 and Remark 5.4 imply that, for any translation-invariant interaction $\Phi \in \mathcal{W}_1^\mathbb{R}$ and positive definite function $f \in \mathcal{D}_{0,+}$,

$$
\lim_{\gamma \to 0^+} \lim_{L \to \infty} \frac{1}{|A_L|} \ln \text{Trace} \left( \exp \left\{ -\beta \left( U_L^\Phi + \sum_{x,y \in A_L, s,t \in S} \gamma^d f (x-y) a^*_{y,t} a_{y,t} a^*_{x,s} a_{x,s} \right) \right\} \right)
$$

while the equilibrium states of this Kac interaction can be approximated by generalized equilibrium states of the corresponding mean-field model. Note that the thermodynamics of this repulsive mean-field model can be explicitly computed when $\Phi$ defines a quasi-free fermion system, by using the thermodynamic game (cf. Sections 4.4.3 and 4.5.3).

This application of Theorem 5.7 is nontrivial, albeit simple, keeping in mind that the results hold true for all translation-invariant interactions $\Phi \in \mathcal{W}_1^\mathbb{R}$. In particular, $\mathcal{W}_1^\mathbb{R}$ encodes – by far – all short-range lattice fermion models used in condensed matter physics, the norm on $\mathcal{W}_1^\mathbb{R}$ being quite weak. E.g.,

$$
\| \Phi \|_{\mathcal{W}_1} := \sum_{\Lambda \in \mathcal{P}_1, \Lambda \geq \{0\}} |\Lambda|^{-1} \| \Phi_\Lambda \|_{\mathcal{U}} \leq \sum_{\Lambda \in \mathcal{P}_1, \Lambda \geq \{0\}} \| \Phi_\Lambda \|_{\mathcal{U}}.
$$

See Equation (23).

Regarding the simple example given by (87)–(89), $f$ is a pair potential characterizing an inter-particle interaction whose range is tuned by the parameter $\gamma \in (0, 1)$. The main limitation at this point concerning the choice of $f$ is the fact that this function has to be positive definite. Conditions of this type are relaxed, from Section 5.5 on, and very general results are proven in Section 5.6. Remark, however, that it is current in theoretical physics to use a positive definite function for such two-body interaction potential. This property of $f$ is reminiscent of a superstability condition, which is important in the bosonic case [2 Section 2.2 and Appendix G].

In the next section we extend our results to approximate – via (Kac) short-range interactions of $\mathcal{W}_1$ – models of $\mathcal{M}_1$ with possibly infinitely many repulsive mean-field components.

5.4.2 Multiple-Field Case

In order to approximate a model with possibly infinitely many mean-field repulsions, we use the Kac interactions $\Phi^{b,\gamma} \in \mathcal{W}_1^\mathbb{R}$ of Definition 5.2 for positive measures $b \in \mathcal{D}_{0,+}$ and $\gamma \in (0, 1)$. In this context, we need to adapt Proposition 5.6. For any positive measure $b \in \mathcal{D}_0$, recall that we use Theorem 5.4 to define the positive measure $a_b \in \mathcal{S}_1$ by (110). Then, we have the following estimate:

**Proposition 5.8 (Energy densities of Kac interactions and space-averaging functionals)**

For any parameter $\varepsilon \in \mathbb{R}^+$, there is a constant $M_\varepsilon > 0$ such that, for any $\gamma \in (0, 1)$, positive measure $b \in \mathcal{D}_{\varepsilon,+}$, interaction $\Phi \in \mathcal{W}_1$ and $\rho \in E_1$,

$$
\Delta_{ab} (\rho) \leq e_{\Phi^{b,\gamma}} (\rho) + \int_{\mathcal{S} \times \mathcal{D}_{\varepsilon,+}} \mathfrak{F}_\gamma (\Phi, f) b (d (\Psi, f)) + \gamma^2 M_\varepsilon b (\mathbb{S} \times \mathcal{D}_{\varepsilon,+})
$$

with $\mathfrak{F}_\gamma$, being the mapping from $\mathcal{W}_1 \times \mathcal{D}_0$ to $\mathbb{R}_0^+$ defined by Equation (167) for any fixed $\gamma \in (0, 1)$.

**Proof.** The assertion is a consequence of Proposition 5.6 along with Equations (112) and (113). Note that the above integral is well-defined. See for instance Lemma 7.8.

We can now generalize Theorem 5.7 to models with possibly infinitely many long-range repulsions, by replacing Proposition 5.6 with Proposition 5.8 and by using basically the same arguments:
Theorem 5.9 (From short-range interactions to repulsive mean-field models)

Take an interaction $\Phi \in \mathcal{W}_1^\mathbb{R}$ and a positive measure $b \in \mathcal{D}_{0,+}$.

(i) Convergence of infinite-volume pressures:

$$\lim_{\gamma \to 0^+} P_{\Phi+\phi,\gamma} = P_{\Phi,ab}^\Phi = P_{(\Phi,ab)}^\Phi.$$ 

(ii) Convergence of equilibrium states: Weak* accumulation points of any net of equilibrium states $\omega_\gamma \in M_{\Phi+\phi,\gamma}$ as $\gamma \to 0^+$ are generalized equilibrium states of the purely repulsive model $(\Phi,ab)$, i.e., they belong to the weak*-compact convex set $\Omega^\phi_{(\Phi,ab)} = \Omega^\phi_{(\Phi,ab)}$.

Proof. Recall again that $P_{m}^\phi = P\Phi_m$ and $\Omega^\phi_{m} = \Omega^\phi_m$ for any purely repulsive model $m \in \mathcal{M}_1$, thanks to [1] Theorem 2.25. By Equations (35) and (36), for any $\Phi \in \mathcal{W}_1^\mathbb{R}$ and positive measure $b \in \mathcal{D}_0$,

$$P_{\Phi+\phi,\gamma} = - \inf_{\rho \in E_1} \left\{ f_\phi(\rho) + \epsilon_{\Phi,\gamma}(\rho) \right\}, \quad (117)$$

while from (50)–(51),

$$P_{\phi,ab} = - \inf_{\rho \in E_1} \left\{ f_\phi(\rho) + \Delta_{ab}(\rho) \right\}. \quad (118)$$

Assertion (i) thus results from Equations (117)–(118), together with Proposition 5.8, Theorem 5.5 and Lemma 7.8, while the proof of (ii) requires the additional use of Equations (65), all combined with Lebesgue’s dominated convergence theorem. We omit the details.

5.5 Thermodynamics in the Kac Limit – Attractive Case

In the present section we study the convergence of Fermi systems associated with Kac interactions towards purely attractive mean-field models in the Kac limit, after taking the thermodynamic limit. As explained in Section 4.2, purely attractive mean-field models refer to translation-invariant mean-field models $m = (\Phi,a) \in \mathcal{M}_1$, with $a_- = -a$ being a finite positive measure on $\mathbb{S}$. Similar to the repulsive case (Section 5.4), we split this study in two steps, by starting with the pedagogical case of models with a single mean-field interaction.

5.5.1 Simple-Field Case

The upper bound given by Proposition 5.4 for the purely repulsive case does not hold true anymore in the attractive case. In fact, Theorem 5.3 and Equations (114)–(115) only lead to the inequality

$$P_{\phi,f(0)\delta_\phi} = P_{\phi,f(0)\delta_\phi} \leq \inf_{\gamma \in (0,1)} P_{\phi+K_{\gamma},\Psi}^\phi, \quad \Psi \in \mathcal{S}, \Phi \in \mathcal{W}_1, f \in \mathcal{D}_{0,+}, \quad (119)$$

with $\delta_\phi \in \mathcal{S}_1$ being the Dirac measure on $\Psi$. In the present case, Proposition 5.6 is replaced by the monotonicity of energy densities of Kac interactions with respect to $\gamma$, allowing us to use Lemma 7.6 which is reminiscent of Dini’s theorem. To this end, we need to restrict the choice of functions $f \in \mathcal{D}_{0,+}$ to the subcone $\mathcal{C}_{0,+} \subseteq \mathcal{D}_{0,+}$, which is defined by (86). More precisely, to tackle the attractive case, Proposition 5.6 is replaced by the following result:

Proposition 5.10 (Monotonicity of energy density functionals)

For $\Psi \in \mathcal{W}_1$, $f \in \mathcal{C}_{0,+}$ and $\eta \in \mathbb{R}^+$, there exists $\gamma_0 \in (0,1)$ such that, for all $\gamma_1, \gamma_2 \in (0,\gamma_0)$ with $\gamma_1 \geq \gamma_2$, $e_{\kappa_{\gamma_2},\Psi,f} \geq e_{\kappa_{\gamma_1},\Psi,f} - \eta$.
Proof. By Lemma 7.6 (i), Equation (32) and the density of $\mathcal{W}_0$ in $\mathcal{W}_1$, it suffices to consider the special case $\Psi \in \mathcal{W}_0$ to prove the proposition. Using the characteristic function $\chi$, of the ball of radius $r \in \mathbb{R}^+$ centered at zero in the torus $\Theta_d = [-\pi, \pi]^d$, we observe from (106) that, for any $\Psi \in \mathcal{W}_0$, $r \in \mathbb{R}^+$, $\gamma \in (0,1)$ and $f \in \mathcal{C}_{0,+}$,

$$
\rho\left(C_{\gamma}(\Psi,f)\right) + \rho(\mathcal{R}_{f}^{\Psi,\gamma}) + \zeta(\gamma) + \chi_{f}^{\gamma}(r, \rho) + \mathcal{Q}_{f}^{\gamma}(r, \rho)
\begin{aligned}
= & \sum_{z_1, z_2 \in \mathcal{P}_{1}: z_1 \in 2z_2 \{0\}} \left\langle \frac{\pi_{\rho}(\Psi_{z_1})}{|z_1| \rho_{\theta}}, \int_{\Theta_{d}} \hat{f}(\gamma^{-1}\theta) \chi_{r}(\theta) d(F_{\rho}^{\Psi})(\theta) \frac{\pi_{\rho}(\Psi_{z_2})}{|z_2| \rho_{\theta}} \right\rangle_{\mathcal{M}_{\rho}},
\end{aligned}
$$

where

$$
\chi_{f}^{\gamma}(r, \rho) \equiv \sum_{z_1, z_2 \in \mathcal{P}_{1}: z_1 \in 2z_2 \{0\}} \left\langle \frac{\pi_{\rho}(\Psi_{z_1})}{|z_1| \rho_{\theta}}, \int_{\Theta_{d}} \hat{f}(\gamma^{-1}\theta) \chi_{r}(\theta) \right\rangle_{\mathcal{M}_{\rho}}
$$

and

$$
\mathcal{Q}_{f}^{\gamma}(r, \rho) \equiv \sum_{z_1, z_2 \in \mathcal{P}_{1}: z_1 \in 2z_2 \{0\}} \left\langle \frac{\pi_{\rho}(\Psi_{z_1})}{|z_1| \rho_{\theta}}, \int_{\Theta_{d}} \hat{f}(\gamma^{-1}\theta) (\chi_{r}(\theta) - 1) \right\rangle_{\mathcal{M}_{\rho}} + \sum_{x \in \mathcal{Z}_{1}, y \in \mathcal{Z}_{2}} \frac{1 - e^{i\theta(x-y)}}{|z_1| |z_2|} d(F_{\rho}^{\Psi})(\theta) \frac{\pi_{\rho}(\Psi_{z_2})}{|z_2| \rho_{\theta}} \right\rangle_{\mathcal{M}_{\rho}}.
$$

For any $\varepsilon \in \mathbb{R}^+$ and finite-range interaction $\Psi \in \mathcal{W}_0$, there is a constant $C_{\varepsilon, \Psi} \in \mathbb{R}^+$ such that, for any $f \in \mathcal{C}_{\varepsilon, +}$ and $\gamma \in (0,1)$,

$$
\left|\chi_{f}^{\gamma}(r, \rho)\right| \leq C_{\varepsilon, \Psi} r \|f\|_{\mathcal{D}_{\varepsilon}}, \quad \left|\mathcal{Q}_{f}^{\gamma}(r, \rho)\right| \leq C_{\varepsilon, \Psi} \left(\frac{\gamma}{r}\right)^{2d} \|f\|_{\mathcal{D}_{\varepsilon}}.
$$

Observe meanwhile that, for any function $f \in \mathcal{C}_{0,+}$, the family of functions $(g_{\gamma})_{\gamma \in (0,1)}$ defined on the torus $\Theta_d$ by

$$
g_{\gamma}(\theta) = \hat{f}(\gamma^{-1}\theta) \chi_{r}(\theta), \quad \theta \in \Theta_{d},
$$

is monotonically increasing with respect to $\gamma \in (0,1)$. It suffices now to use Equations (106), (107) and (120)-(121) to get the assertion. 

Proposition 5.10 combined with Lemma 7.6 implies that the infinite volume pressure $P_{K_{\gamma}(\Phi,-f)}$ and the equilibrium states of $M_{K_{\gamma}(\Phi,-f)}$ associated with some Kac interaction $K_{\gamma}(\Phi,-f)$, as respectively defined by (34) and (37), converges in the Kac limit $\gamma \to 0^+$ to those of some purely attractive mean-field model $m \in \mathcal{M}_1$. See (49) and (65) for the definitions of the infinite volume pressure and equilibrium states of any mean-field model. More precisely, one gets the following result:

**Theorem 5.11 (From short-range interactions to attractive mean-field models)**

Take interactions $\Phi \in \mathcal{W}_{1}^{\mathbb{R}}, \Psi \in \mathbb{S} \cap \mathcal{W}_{0}$ and a positive definite function $f \in \mathcal{C}_{0,+}$, whose Fourier transform is denoted by $\hat{f}$. Let $\delta_{\Phi} \in \mathcal{S}_{1}$ denote the Dirac measure on $\Psi \in \mathbb{S}$.

(i) Convergence of infinite-volume pressures:

$$
\lim_{\gamma \to 0^+} P_{K_{\gamma}(\Phi,-f)} = P_{\Phi, -\hat{f}(0)\delta_{\Phi}} = P_{\Phi,-\hat{f}(0)\delta_{\Phi}}.
$$

(ii) Convergence of equilibrium states: Weak* accumulation points of any net of equilibrium states $\omega_{\gamma} \in M_{K_{\gamma}(\Phi,-f)}$ as $\gamma \to 0^+$ are equilibrium states of $(\Phi, -\hat{f}(0)\delta_{\Phi})$, i.e., they belong to the weak*-compact convex set $\Omega_{(\Phi,-\hat{f}(0)\delta_{\Phi})}^{\Phi}$.
Proof. Assertion (i) is a direct consequence of Proposition 5.10, Equations (114)–(115) and Theorem 5.3 together with Lemma 7.6. To prove the second assertion, fix \( \Phi \in \mathcal{W}^1 \), \( \Psi \in \mathcal{S} \cap \mathcal{W}_0 \) and a positive definite function \( f \in \mathcal{C}_{0,\infty} \). For any \( \gamma \in (0,1) \), pick a minimizer \( \omega_\gamma \in M_{\Phi+\kappa_\gamma(\Psi,-f)} \) of the variational problem

\[
\inf_{\rho \in E_1} \left\{ f(\rho) - e_{\kappa_\gamma(\Psi,-f)}(\rho) \right\} = -P_{\Phi+\kappa_\gamma(\Psi,-f)}.
\]

Observe that, for any \( \gamma \in (0,1) \), \( \omega_\gamma \) can be equivalently seen as a tangent functional at \( \Phi \in \mathcal{W}^1 \) to the convex continuous function \( \Phi \mapsto P_{\Phi+\kappa_\gamma(\Psi,-f)} \) from the real Banach space \( \mathcal{W}^1 \) to \( \mathbb{R} \). See [1] Section 2.6 for more details. In particular, for any \( \Phi_1 \in \mathcal{W}_1^1 \),

\[
P_{\Phi+\kappa_\gamma(\Psi,-f)} - P_{\Phi+\kappa_\gamma(\Psi,-f)} \geq -\omega_\gamma(e_{\Phi_1}).
\]

By weak* compactness of \( E_1 \), the net \( (\omega_\gamma)_{\gamma \in (0,1)} \) has weak* converging subnets and any weak* accumulation point of the net is the limit of such a subnet. By Assertion (i), each weak* accumulation point can be seen as a tangent functional at \( \Phi \in \mathcal{W}^1 \) to the convex continuous function \( \Phi \mapsto P^d_{(\Phi,-f(0)\delta_\phi)} \) from the real Banach space \( \mathcal{W}^1 \) to \( \mathbb{R} \). By [1] Theorem 2.28, Assertion (ii) then follows. 

If one considers the simple example given by (90)–(91), then Theorem 5.11 and Remark 5.1 imply that, for any translation-invariant interaction \( \Phi \in \mathcal{W}^1_1 \) and positive definite function \( f \in \mathcal{C}_{0,\infty} \),

\[
\lim_{\gamma \to 0^+} \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \ln \text{Trace} \left( \exp \left\{ -\beta \left( U^\Phi_L - \sum_{x,y \in \Lambda_L} \gamma^d f(\gamma(x-y)) \right) \right\} \right) = \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \ln \text{Trace} \left( \exp \left\{ -\beta \left( U^\Phi_L - \sum_{x,y \in \Lambda_L} \gamma^d f(0) \right) \right\} \right),
\]

(122)

while the equilibrium states of this Kac interaction can be approximated by equilibrium states of the corresponding mean-field model. Note that the thermodynamics of this attractive mean-field model can again be explicitly computed when \( \Phi \) defines a quasi-free fermion system, by using the thermodynamic game (cf. Sections 4.4.3 and 4.5.3).

This application of Theorem 5.11 is again nontrivial, albeit simple, keeping in mind that the results hold true for all translation-invariant interactions \( \Phi \in \mathcal{W}_1^1 \). Regarding the simple example given by (90)–(91), \( f \) encodes the hopping strength of Cooper pairs, characterizing the BCS interaction whose range is tuned by the parameter \( \gamma \in (0,1) \). The limit mean-field models are also important in theoretical physics. For instance, the reduced BCS Hamiltonian or the strong coupling BCS Hamiltonian, which are mean-field models, qualitatively display most of the basic properties of real conventional type I superconductors. See, e.g., [3] Chapter VII, Section 4. Concerning the choice of \( f \), the main limitation at this point is the fact that this function has to be positive definite with scaling-monotone Fourier transform, see [5]. This kind of condition is relaxed in Section 5.6 where very general results are proven. Observe, however, that functions of this type are used in theoretical physics, like in the original BCS theory of superconductivity [36].

In the next section we extend our results to approximate – via (Kac) short-range interactions of \( \mathcal{W}_1 \) – models of \( M_1 \) with possibly infinitely many attractive mean-field components.

### 5.5.2 Multiple-Field Case

In order to approximate a model with possibly infinitely many mean-field attractions, we use the Kac interactions \( \Phi^{-b,\gamma} \in \mathcal{W}_1^1 \) of Definition 5.2 for positive measures \( b \in \mathcal{C}_{0,\infty} \) and \( \gamma \in (0,1) \). In this context, we need to adapt Proposition 5.10. For any positive measure \( b \in \mathcal{D}_{0,\infty} \), recall that we use Theorem 5.4 to define the positive measure \( a_b \in \mathcal{S}_1 \) by (110). Then, we have the following property of energy densities of Kac interactions:
Proposition 5.12 (Monotonicity of energy density functionals)

For any positive measure \( b \in C_{0,+} \) and \( \eta \in \mathbb{R}^+ \), there exists \( \gamma_0 \in (0, 1) \) such that, for all \( \gamma_1, \gamma_2 \in (0, \gamma_0) \) with \( \gamma_1 \geq \gamma_2 \),

\[
e_{\Phi - b, \gamma_2} \geq e_{\Phi - b, \gamma_1} - \eta.
\]

Proof. By Lemma 5.2 and Equation (32), we can assume without loss of generality that

\[
b = \sum_{j=1}^{n} \delta_{(\Psi_j, f_j)}
\]

for some finite sequence \( (\Psi_1, f_1), \ldots, (\Psi_n, f_n) \in \mathcal{S} \times (\mathbb{D}^n \cap C_{0,+}) \). Then, by Proposition 5.10 and the linearity of the mapping \( \Phi \mapsto e_{\Phi} (\rho) \) from \( \mathcal{W}_1 \) to \( \mathbb{C} \) at fixed \( \rho \in E_1 \), the assertion follows.

We can now generalize Theorem 5.11 to models with possibly infinitely many mean-field attractions by replacing Proposition 5.10 with Proposition 5.12 and by using basically the same arguments as in the simple-field case:

Theorem 5.13 (From short-range interactions to attractive mean-field models)

Take an interaction \( \Phi \in \mathcal{W}_1^{\mathbb{R}} \) and a positive measure \( b \in C_{0,+} \).

(i) Convergence of infinite-volume pressures:

\[
\lim_{\gamma \to 0^+} P_{\Phi + \Phi - b, \gamma} = P_{\Phi, -a_b}^b = P_{\Phi, -a_b}^b.
\]

(ii) Convergence of equilibrium states: Weak* accumulation points of any net of equilibrium states \( \omega_\gamma \in M_{\Phi + \Phi - b, \gamma} \), as \( \gamma \to 0^+ \), are equilibrium states of \( (\Phi, -a_b) \), i.e., they belong to the weak*-compact convex set \( \Omega_{\Phi, -a_b}^b = \Omega_{\Phi, -a_b}^b \).

Proof. Assertion (i) is a direct consequence of Proposition 5.12, Equations (117)–(118) and Theorem 5.11(ii). We omit the details.

It is easy to check that all purely attractive mean-field models can be approximated in the sense of Theorem 5.13 via a Kac interaction of Definition 5.2 for positive measures \( b \in C_{0,+} \).

5.6 Thermodynamics in the Kac Limit – General Case

5.6.1 Mixed Case

In this section we generalize the results on purely attractive or repulsive mean-field models to all mean-field models in \( M_1 \), with possibly infinitely many mean-field attractions and repulsions. We first generalize Inequality (119), which relates the conventional and non-conventional pressures \( P_\gamma^\sharp \) and \( P_\gamma^\flat \) of mean-field models with the pressures of the corresponding Kac interactions. Recall that \( P_\gamma^\sharp \) and \( P_\gamma^\flat \) are defined by (49)–(51) and (52)–(54), respectively. In fact, in the general case, we have the following inequalities:

Proposition 5.14 (Bounds on Kac pressures)

Take an interaction \( \Phi \in \mathcal{W}_1^{\mathbb{R}} \) and two positive measures \( b_+ \in D_{0,+} \) and \( b_- \in C_{0,+} \). Then, for any sequences \( (\gamma_+, n)_{n \in \mathbb{N}} \) and \( (\gamma_-, n)_{n \in \mathbb{N}} \) of real numbers in the interval \((0, 1)\) converging to zero,

\[
P_{\Phi, a_{b_+} - a_{b_-}}^\gamma \leq \liminf_{n \to \infty} P_{\Phi + \Phi - b_+ + \gamma, n + \Phi - b_- - \gamma, n} \leq \limsup_{n \to \infty} P_{\Phi + \Phi - b_+ + \gamma, n + \Phi - b_- - \gamma, n} \leq P_{\Phi, a_{b_+} - a_{b_-}}^\gamma,
\]

where, for any measure \( b \in D_0 \), the measure \( a_b \in S_1 \) is defined by Equation (110).
Proof. Fix once and for all $\Phi \in \mathcal{W}_{b,1}^\mathbb{R}$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in \mathcal{C}_{0,+}$. On the one hand, using Proposition 5.3 and Equation (117), for any parameters $\gamma_-, \gamma_+ \in (0, 1)$,

$$P_{\Phi+\Phi^b_{\gamma+}+\Phi^{-\gamma-}} \leq -\inf h_{\gamma_+} (E_1) + \int_{\mathbb{S} \times \mathbb{D}_{\varepsilon,+}} \mathfrak{S}_{\gamma+} (\Phi, f) \ b \ (d \ (\Psi, f)) + \gamma^2_+ M_\varepsilon \ b \ (\mathbb{S} \times \mathbb{D}_{\varepsilon,+})$$

(123)

with $h_{\gamma_+}$ being the functional defined on the space $E_1$ of translation-invariant states by

$$h_{\gamma_+} (\rho) \doteq f_{\Phi+\Phi^b_{\gamma+}+\Phi^{-\gamma-}} (\rho) + \Delta_{\Phi b_+} (\rho) , \quad \rho \in E_1 .$$

(124)

Note that this functional is generally not lower semicontinuous in the weak$^*$ topology, because $\Delta_{\Phi b_+}$ is only upper semicontinuous for non-trivial measures $a_{\Phi b_+} \neq 0$. By [37] Theorem 1.4, observe that

$$\inf h_{\gamma_+} (E_1) = \inf \Gamma (h_{\gamma_+}) (E_1) ,$$

(125)

where $\Gamma (h)$ denotes the so-called $\Gamma$-regularization of a functional $h$ on $E_1$, defined by (46). In contrast with $h_{\gamma_+}$, the functional $\Gamma (h_{\gamma_+})$ is weak$^*$-lower semicontinuous and, moreover, it can be explicitly computed, as done in the proof of [1] Theorem 2.21:

$$\Gamma (h_{\gamma_+}) (\rho) = f_{\Phi+\Phi^{-\gamma-}} (\rho) + \int_{\mathbb{S}} |\rho (\varepsilon \phi)|^2 \ a_{\Phi b_+} (d\varepsilon \phi) , \quad \rho \in E_1 .$$

(126)

We now invoke Proposition 5.12, Theorem 5.3 and Lemma 7.6 to deduce from (125)–(126) that

$$\lim_{\gamma_+ \to 0^+} \inf h_{\gamma_+} (E_1) = \inf f^{\gamma}_{\Phi b_{\Phi b_+ - a_{\Phi b_-}}_+} (E_1) = -P^{\Phi b_{\Phi b_+ - a_{\Phi b_-}}}_1 (E_1) .$$

(127)

By Inequality (123), Lemma 7.8 and Lebesgue’s dominated convergence theorem, it follows that

$$\limsup_{n \to \infty} P_{\Phi+\Phi^b_{\gamma_+,n}+\Phi^{-\gamma_-,n}} \leq P^{\Phi b_{\Phi b_+ - a_{\Phi b_-}}}_1$$

for any sequences $(\gamma_+,n) \in \mathbb{N}$ and $(\gamma_-,n) \in \mathbb{N}$ of real numbers in the interval $(0, 1)$ converging to zero.

On the other hand, using Equation (117), for any $\gamma_-, \gamma_+ \in (0, 1)$,

$$P_{\Phi+\Phi^b_{\gamma_+,n}+\Phi^{-\gamma_-,n}} \geq -f^{\gamma}_{\Phi+\Phi^b_{\gamma_+,n}+\Phi^{-\gamma_-,n}} (\rho) , \quad \rho \in E_1 .$$

By Theorem 5.5, it follows that, for any $\rho \in E_1$ and all sequences $\gamma_+,n) \in \mathbb{N}$ and $(\gamma_-,n) \in \mathbb{N}$ of real numbers in the interval $(0, 1)$ converging to zero,

$$\liminf_{n \to \infty} P_{\Phi+\Phi^b_{\gamma_+,n}+\Phi^{-\gamma_-,n}} \geq \liminf_{n \to \infty} f^{\gamma}_{\Phi+\Phi^b_{\gamma_+,n}+\Phi^{-\gamma_-,n}} (\rho) = -f^{\gamma}_{\Phi b_{\Phi b_+ - a_{\Phi b_-}}} (\rho) ,$$

which, combined with Equation (60), in turn implies that

$$\liminf_{n \to \infty} P_{\Phi+\Phi^b_{\gamma_+,n}+\Phi^{-\gamma_-,n}} \geq P^{\Phi b_{\Phi b_+ - a_{\Phi b_-}}} .$$

In other words, Proposition 5.14 tells us that the limit of Kac pressures or, at least, its accumulation points belong to the interval $[P^b_{\Phi b_+, \Phi b_-}]$ for some mean-field model $m \in \mathcal{M}_1$. It also means that the pressures associated with attractive and repulsive Kac interactions may not converge to its natural mean-field approximation, in contrast with the intuitive guess. Below, we show that the extremes of the interval $[P^b_{\Phi b_+, \Phi b_-}]$ can be attained by a convenient choice of the sequences $(\gamma_+,n) \in \mathbb{N}$ and $(\gamma_-,n) \in \mathbb{N}$. We start with the lower boundary of the interval:

---

20Such kind of discrepancy between quantum models in the Kac limit and their expected mean-field approximations already appears in a different context for the Kac limit of a Bose gas in presence of two-body interactions, see [17]. However, in this previous case [17], there is no competition between repulsive and attractive long-range forces, but instead a perturbation with a scaled external field. See also Section 6.4 for more details.
Theorem 5.15 (From short-range interactions to conventional mean-field models)

Take an interaction $\Phi \in \mathcal{W}_1^R$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in C_{0,+}$.

(i) Convergence of infinite-volume pressures:

$$\lim_{\gamma_- \to 0^+} \lim_{\gamma_+ \to 0^+} P_{\Phi + \Phi^b + \gamma + \Phi^b - \gamma} = P^b_{(\Phi, a_+ - a_-)}.$$ 

(ii) Convergence of equilibrium states: For all $\gamma_+ \in (0,1)$, take any weak$^*$ accumulation point $\omega_{\gamma_+}$ of any net $(\omega_{\gamma_+})_{\gamma_+ \in (0,1)} \subseteq M_{\Phi + \Phi^b + \gamma + \Phi^b - \gamma}$ as $\gamma_- \to 0^+$. Pick any weak$^*$ accumulation point $\omega$ of the net $(\omega_{\gamma_+})_{\gamma_+ \in (0,1)}$ as $\gamma_+ \to 0^+$. Then, $\omega \in \Omega^R_{(\Phi, a_+ - a_-)}$.

Proof. Pick $\Phi \in \mathcal{W}_1^R$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in C_{0,+}$. Fix $\gamma_+ \in (0,1)$. Then, by Theorem 5.13

$$\lim_{\gamma_- \to 0^+} \lim_{\gamma_+ \to 0^+} P_{\Phi + \Phi^b + \gamma + \Phi^b - \gamma} = P^b_{(\Phi, a_+ - a_-)} = - \inf_{\rho \in E_1} \left\{ f_\Phi (\rho) + c_{\Phi^b + \gamma} (\rho) - \Delta_{a_+} (\rho) \right\}.$$ 

(128)

In the same way we prove Theorem 5.9 we use Proposition 5.8, Theorem 5.5, Lemma 7.8 and Lebesgue’s dominated convergence theorem as well as Equations (118) and (128) to get Assertion (i). Finally, for any $\gamma_+ \in (0,1)$, take any weak$^*$ accumulation point $\omega_{\gamma_+}$ of any net $(\omega_{\gamma_+})_{\gamma_+ \in (0,1)} \subseteq M_{\Phi + \Phi^b + \gamma + \Phi^b - \gamma}$ as $\gamma_- \to 0^+$. Then, pick any weak$^*$ accumulation point $\omega$ of the net $(\omega_{\gamma_+})_{\gamma_+ \in (0,1)}$, as $\gamma_+ \to 0^+$. Similar to the proof of Theorem 5.11(ii), one shows that $\omega$ can be seen as a tangent functional at $\Phi \in \mathcal{W}_1^R$ to the convex continuous function $\Phi \mapsto P^b_{(\Phi, a_+ - a_-)}$ from the real Banach space $\mathcal{W}_1^R$ to $\mathbb{R}$. Then, by [1], Theorem 2.28, the assertion follows. ■

Corollary 5.16 (From short-range interactions to conventional mean-field models)

Take an interaction $\Phi \in \mathcal{W}_1^R$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in C_{0,+}$. There exist two sequences $(\gamma_+, n)_{n \in \mathbb{N}}$ and $(\gamma_-, n)_{n \in \mathbb{N}}$ of real numbers in the interval $(0,1)$ converging to zero and a sequence $(\omega_n)_{n \in \mathbb{N}}$ of equilibrium states $\omega_n \in M_{\Phi + \Phi^b + \gamma_+ + n + \Phi^b - \gamma_-}$ such that

$$\lim_{n \to \infty} P_{\Phi + \Phi^b + \gamma_+ + n + \Phi^b - \gamma_-} = P^b_{(\Phi, a_+ - a_-)}$$

and $(\omega_n)_{n \in \mathbb{N}}$ weak$^*$ converges to a generalized equilibrium state of $\Omega^R_{(\Phi, a_+ - a_-)}$.

Proof. It suffices to combine Theorem 5.15 with [38, Chapter 2, 4 Theorem]. Note that the weak$^*$-topology is metrizable on the weak$^*$-compact convex set of all states on $\mathcal{U}$, because $\mathcal{U}$ is separable. See, e.g., [1, Theorem 10.10]. ■

We show now that the upper bound of Proposition 5.14 is also attained by a convenient choice of the sequences $(\gamma_+, n)_{n \in \mathbb{N}}$ and $(\gamma_-, n)_{n \in \mathbb{N}}$:

Theorem 5.17 (From short-range interactions to non-conventional mean-field models)

Take an interaction $\Phi \in \mathcal{W}_1^R$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in C_{0,+}$.

(i) Convergence of infinite-volume pressures:

$$\lim_{\gamma_- \to 0^+} \lim_{\gamma_+ \to 0^+} P_{\Phi + \Phi^b + \gamma + \Phi^b - \gamma} = P^b_{(\Phi, a_+ - a_-)}.$$ 

(ii) Convergence of equilibrium states: For all $\gamma_- \in (0,1)$, take any weak$^*$ accumulation point $\omega_{\gamma_-}$ of any net $(\omega_{\gamma_-})_{\gamma_- \in (0,1)} \subseteq M_{\Phi + \Phi^b + \gamma + \Phi^b - \gamma}$ as $\gamma_- \to 0^+$. Pick any weak$^*$ accumulation point $\omega$ of the net $(\omega_{\gamma_-})_{\gamma_- \in (0,1)}$, as $\gamma_- \to 0^+$. Then, $\omega \in \Omega^b_{(\Phi, a_+ - a_-)}$. 

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Proof. Pick $\Phi \in \mathcal{W}_1^\mathbb{R}$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in \mathcal{C}_{0,+}$. Fix $\gamma_- \in (0, 1)$. By Theorem 5.9,
\[
\lim_{\gamma_- \to 0^+} P_{\Phi + \Phi^b + \gamma_+ + \Phi^b - \gamma_-} = P_{\Phi + \Phi^b - \gamma_- - \delta b_+} \tag{129}
\]
and weak* accumulation points of any net of equilibrium states $\omega_{\gamma_-} \in M_{\Phi + \Phi^b + \gamma_+ + \Phi^b - \gamma_-}$ as $\gamma_+ \to 0^+$. Let $\gamma_- \to 0^+$. Then, pick any weak* accumulation point $\omega$ of the net $\omega_{\gamma_-} \in (0, 1)$, as $\gamma_- \to 0^+$. Similar to the proof of Theorem 5.13(ii), one shows that $\omega$ can be seen as a tangent functional at $\Phi \in \mathcal{W}_1^\mathbb{R}$ to the convex continuous function $\Phi \mapsto P_{\Phi^b - \gamma_- - \delta b_-}$ from the real Banach space $\mathcal{W}_1^\mathbb{R}$ to $\mathbb{R}$. Then, by [37, Theorems 1.4 and 1.8], the assertion follows, since $\Omega_{\Phi^b - \gamma_- - \delta b_-}$ is a weak*-compact convex set.

Corollary 5.18 (From short-range interactions to non-conventional mean-field models)
Take an interaction $\Phi \in \mathcal{W}_1^\mathbb{R}$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in \mathcal{C}_{0,+}$. There exist two sequences $(\gamma_{+n})_{n \in \mathbb{N}}$ and $(\gamma_{-n})_{n \in \mathbb{N}}$ of real numbers in the interval $(0, 1)$ converging to zero and a sequence $(\omega_n)_{n \in \mathbb{N}}$ of equilibrium states $\omega_n \in M_{\Phi + \Phi^b + \gamma_{+n} + \Phi^b - \gamma_{-n}}$ such that
\[
\lim_{n \to \infty} P_{\Phi + \Phi^b + \gamma_{+n} + \Phi^b - \gamma_{-n}} = P_{\Phi^b - \gamma_- - \delta b_-}
\]
and $(\omega_n)_{n \in \mathbb{N}}$ weak* converges to a non-conventional equilibrium state of $\Omega_{\Phi^b - \gamma_- - \delta b_-}$.

Proof. Exactly like for Corollary 5.16 one combines Theorem 5.17 with [38, Chapter 2, 4 Theorem].

We conclude by showing that the limit of Kac pressures can attain all the values of the interval $[P^*_m, P^b_m]$ for the corresponding mean-field model $m \in \mathcal{M}_1$.

Theorem 5.19 (Reachability of all pressures in $[P^*_m, P^b_m]$)
Take an interaction $\Phi \in \mathcal{W}_1^\mathbb{R}$ and two positive measures $b_+ \in \mathcal{D}_{0,+}$ and $b_- \in \mathcal{C}_{0,+}$. Let $m = (\Phi, a b_+ - a b_-) \in \mathcal{M}_1$. For any $p \in [P^*_m, P^b_m]$, there are two sequences $(\gamma_{+n})_{n \in \mathbb{N}}$ and $(\gamma_{-n})_{n \in \mathbb{N}}$ of real numbers in the interval $(0, 1)$ converging to zero, such that
\[
\lim_{n \to \infty} P_{\Phi + \Phi^b + \gamma_{+n} + \Phi^b - \gamma_{-n}} = p.
\]

Proof. Fix all parameters of the theorem. By Corollaries 5.16 and 5.18 without loss of generality, we can take $p \in (P^*_m, P^b_m)$ and, for any $n \in \mathbb{N}$, there exist real numbers $\gamma_{+n}^p, \gamma_{-n}^p, \gamma_{+n}^b, \gamma_{-n}^b$ in the interval $(0, 1)$, converging to zero as $n \to \infty$, such that
\[
P_{\Phi + \Phi^b + \gamma_{+n}^p + \Phi^b - \gamma_{-n}^p} < p < P_{\Phi + \Phi^b + \gamma_{+n}^b + \Phi^b - \gamma_{-n}^b}, \quad n \in \mathbb{N}.
\]
Note from Lemma [77, (ii)] and Lebesgue’s dominated convergence theorem that the mappings $\gamma \mapsto P_{\Phi^b + \gamma}$ from $(0, 1)$ to $\mathcal{W}_1$ are continuous. Hence, by continuity of the pressure $\Phi \mapsto P_{\Phi}$ on $\mathcal{W}_1$ and Inequality (130), there are sequences $(\gamma_{+n})_{n \in \mathbb{N}}$ and $(\gamma_{-n})_{n \in \mathbb{N}}$ of real numbers in the interval $(0, 1)$ converging to zero such that
\[
P_{\Phi + \Phi^b + \gamma_{+n} + \Phi^b - \gamma_{-n}} = p, \quad n \in \mathbb{N}.
\]
5.6.2 Relaxing Positive-Definiteness and Scaling-Monotonicity

The general results of Section 5.6.1 use two functions \( f_+ \in \mathcal{D}_{0,+} \) and \( f_- \in C_{0,+} \) in the simple field case, or two positive measures \( b_+ \in \mathcal{D}_{0,+} \) and \( b_- \in C_{0,+} \) in the multiple-field case. The objects \( f_+ \) and \( b_+ \) encode repulsions, while \( f_- \) and \( b_- \) represent attractions of fermions on the lattice \( \mathcal{L} \). Therefore, up to this point, the repulsions and attractions are defined from positive definite, 2\( d \) times continuously differentiable, reflection-symmetric, real-valued functions on \( \mathbb{R}^d \), whose derivatives of order \( l \leq 2d \) decay faster than \( |x|^{-(d+\varepsilon)} \), for some \( \varepsilon > 0 \), as \( |x| \to \infty \). Note additionally that the attractive part also requires the scaling-monotone property of Fourier transforms, as defined by (85). In the present section we show that, by only imposing sufficient regularity on general functions \( f \in \mathcal{D}_0 \), the case of Section 5.6.1 is recovered and we arrive at a fully general result for sufficiently regular interactions.

Heuristically, the main part of the argument is as follows: A (not necessarily positive definite) function \( f \in \mathcal{D}_0 \) can be written as a difference \( f = f_+ - f_- \), where \( f_+ \in \mathcal{D}_{0,+} \) and \( f_- \in C_{0,+} \). In fact, one may add and subtract a positive definite and scaling-monotone function \( \xi_f \in C_{0,+} \) to \( f \) in such a way that \( f_+ = f + \xi_f \in \mathcal{D}_{0,+} \) and \( f_- = \xi_f \in C_{0,+} \). A similar fact is true in the multiple-field case. This argument can be rigorously carried out by adding stronger regularity and asymptotic properties to functions defining Kac interactions. In fact, it suffices to take \( \kappa = 3d+1 \) in the definitions (81) and (83) of \( D_{e,\kappa} \) and its norm. Recall that \( D_{e,3d+1} \) is a vector subspace of \( D_e \equiv D_e,2d \) and

\[
\| f \|_{D_{e,2d}} \leq \| f \|_{D_{e,3d+1}}, \quad f \in D_{e,3d+1}.
\]

For any parameter \( \varepsilon \in \mathbb{R}^+ \) and function \( f \in D_{e,3d+1} \),

\[
\hat{f}(k) \leq \kappa! \| f \|_{D_{e,\kappa}} \min \left\{ 1, \min_{j \in \{1, \ldots, d\}} |k_j|^{-\kappa} \right\} \int_{\mathbb{R}^d} (1+|x|)^{-(d+\varepsilon)} \, dx \leq C_\varepsilon \| f \|_{D_{e,\kappa}} \hat{\xi}(k) \quad (131)
\]

for some constant \( C_\varepsilon \in \mathbb{R}^+ \) depending only upon \( \varepsilon \), where \( \kappa = 3d+1 \) and

\[
\hat{\xi}(k) \equiv (1+|k|^2)^{-\frac{3d+1}{2}}, \quad k \in \mathbb{R}^d.
\]

Now, we define \( \xi \in \bigcap_{\varepsilon \in \mathbb{R}^+} D_{e,2d} \) to be the function whose Fourier transform is \( \hat{\xi} \). Note that \( \xi \in C_{0,+} \), see Inequality (85). As a consequence, we infer from (131) that, for any \( \varepsilon \in \mathbb{R}^+ \) and \( f \in D_{e,3d+1} \),

\[
f_+ \doteq f + C_\varepsilon \| f \|_{D_{e,3d+1}} \xi \in D_{0,+2d} \quad \text{and} \quad f_- \doteq C_\varepsilon \| f \|_{D_{e,3d+1}} \xi \in C_{0,+2d} \quad (132)
\]

for some constant \( C_\varepsilon \in \mathbb{R}^+ \) depending only upon \( \varepsilon \). We now extend this argument to the multiple field case and arrive at the following lemma:

**Lemma 5.20 (Positive decompositions of measures)**

For any fixed \( \varepsilon \in \mathbb{R}^+ \) and \( \mathbf{b} \in D_{e,3d+1} \),

\[
\mathcal{J}_{\mathbf{b}} \doteq \left\{ (\mathbf{b}_+, \mathbf{b}_-) \in D_{e,+,2d} \times C_{e,+,2d} : \mathbf{a}_{b_+} - \mathbf{a}_{b_-} = \mathbf{a}_{\mathbf{b}}, \ \Phi^{b,\gamma} = \Phi^{b_+,\gamma} \Phi^{b_-,\gamma} \ for \ \gamma \in (0,1) \right\} \neq \emptyset.
\]

**Proof.** Fix the parameter \( \varepsilon \in \mathbb{R}^+ \). We consider first the case of a positive measure \( \mathbf{b} \in D_{e,3d+1} \). Let \( \delta : S \times D_{e,3d+1} \to \mathbb{R}_0^+ \) be the (continuous, bounded) positive valued function defined by

\[
\delta(\Psi, f) \doteq \| f + 2C_\varepsilon \xi \|_{D_{e,2d}}, \quad \Psi \in S, \ f \in D_{e,3d+1},
\]

with the constant \( C_\varepsilon \in \mathbb{R}^+ \) defined by (131). This function is used to define the (positive) Borel measure \( \delta \mathbf{b} \) of finite variation by

\[
\delta \mathbf{b}(A) \doteq \int_A \delta(\Psi, f) \mathbf{b}(\delta(\Psi, f))
\]

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for any Borel set $\mathcal{A} \subseteq \mathbb{S} \times \mathbb{D}_{e,3d+1}$. Let $\tilde{b}_+ \doteq \xi^+_b(\mathcal{A})$ be the pushforward of $\mathcal{A}$ through the (continuous) mapping $\xi^+_b : \mathbb{S} \times \mathbb{D}_{e,3d+1} \to \mathbb{S} \times \mathbb{D}_{e,2d}$ defined by

$$
\xi^+_b(\Psi, f) = \left( \Psi, \|f + 2C_e \xi\|^{-1}_{\mathbb{D}_{e,2d}} (f + 2C_e \xi) \right), \quad \Psi \in \mathbb{S}, \ f \in \mathbb{D}_{e,3d+1}.
$$

Remark that $f + 2C_e \xi \neq 0$ for every $f \in \mathbb{D}_{e,3d+1}$. Define further $\tilde{b}_- \in \mathcal{C}_{e,+2d}$ by

$$
\tilde{b}_- \doteq 2C_e \|\xi\|_{\mathbb{D}_{e,2d}} \xi^+ \mathcal{B},
$$

where $\xi^+_b \mathcal{B}$ is the pushforward of $\mathcal{B}$ through the (continuous) mapping $\xi^- : \mathbb{S} \times \mathbb{D}_{e,3d+1} \to \mathbb{S} \times \mathbb{D}_{e,2d}$ defined by

$$
\xi^- (\Psi, f) = \left( \Psi, \|\xi\|^{-1}_{\mathbb{D}_{e,2d}} \xi \right), \quad \Psi \in \mathbb{S}, \ f \in \mathbb{D}_{e,3d+1}.
$$

It is easy to check that $(\tilde{b}_+, \tilde{b}_-) \in \mathfrak{B}_b$. To prove the general case, remark that, for any $b \in \mathcal{D}_{e,3d+1}$, one has $a_b = -a_b$ and $\Phi^{b,\gamma} = -\Phi^{b,\gamma}$ for any $\gamma \in (0, 1)$, where $b$ is the pushforward of $\mathcal{B}$ under the continuous mapping $(\Psi, f) \mapsto (\Psi, f)$ from $\mathbb{S} \times \mathbb{D}_{e,3d}$ to itself. Hence, for a general (not necessarily positive) $b \in \mathcal{D}_{e,3d+1}$, one may consider any decomposition $b = b_+ - b_- \in D_{e,+3d+1}$.

(Take, for instance, the Jordan decomposition of $b$.) Then, replace $b$ with the positive measure $b_+ + b_-$. This change in the measure $b$ does not affect the corresponding objects $a_b$ and $\Phi^{b,\gamma}$ for any $\gamma \in (0, 1)$.

Recall that the mappings $b \mapsto a_b$ and $b \mapsto \Phi^{b,\gamma}$ at fixed $\gamma \in (0, 1)$ are (real) linear.

By means of Lemma 5.20, one can apply all results of Section 5.6.1 to Kac and mean-field models associated with general elements $b \in \mathcal{D}_{0,3d+1}$ and get the following result:

**Proposition 5.21 (From short-range interactions to mean-field models - I)**

Let $\Phi \in \mathcal{W}_1^\mathbb{R}$ and $b \in \mathcal{D}_{0,3d+1}$. Take any decomposition $(\tilde{b}_+, \tilde{b}_-) \in \mathfrak{B}_b \neq \emptyset$ of $b$. (See Lemma 5.20.) Then, Proposition 5.14 Theorems 5.15 5.17 and 5.19 as well as Corollaries 5.16 and 5.18 hold true with $b_\pm = b_+ + b_-$ and mean-field interaction $a_{b_+} - a_{b_-} = a_b$.

**Proof.** The proposition directly follows from Lemma 5.20 along with Proposition 5.14, Theorems 5.15 5.17 and 5.19 as well as Corollaries 5.16 and 5.18 from Section 5.6.1. ■

Remark that, by Proposition 5.21 if the conventional and non-conventional pressures of the mean-field model $m = (\Phi, a_b)$ are the same for some $\Phi \in \mathcal{W}_1^\mathbb{R}$ and $b \in \mathcal{D}_{0,3d+1}$, then we obtain the convergence of the Kac pressure $P_{\Phi^{b,\gamma}}$ to the pressure associated with the corresponding mean-field model. Mutatis mutandis for the convergence of equilibrium states. More precisely, we have the following:

**Theorem 5.22 (From short-range interactions to mean-field models - II)**

Take any interaction $\Phi \in \mathcal{W}_1^\mathbb{R}$ and a positive measure $b \in \mathcal{D}_{0,3d+1}$ such that $P^\mathbb{D}_{\Phi, a_b} = P^\mathbb{D}_{\Phi, a_b}$.

(i) Convergence of infinite-volume pressures:

$$
\lim_{\gamma \to 0^+} P_{\Phi^{b,\gamma}} = P^\mathbb{D}_{\Phi, a_b} = P^\mathbb{D}_{\Phi, a_b}.
$$

(ii) Convergence of equilibrium states: If $P^\mathbb{D}_{\Phi, a_b} = P^\mathbb{D}_{\Phi, a_b}$ for all $\Psi \in \mathcal{W}_1^\mathbb{R}$, then weak accumulation points of any net of equilibrium states $\omega_\gamma \in M_{\Phi^{b,\gamma}}$ as $\gamma \to 0^+$ are generalized equilibrium states of the mean-field model $(\Phi, a_b)$, i.e., they belong to the weak* compact convex set $\Omega^\mathbb{D}_{\Phi, a_b} = \Omega^\mathbb{D}_{\Phi, a_b}$.

**Proof.** The first part of the theorem is a direct consequence of the part of Theorem 5.21 referring to Proposition 5.14 with $\gamma_+ = \gamma_- = \gamma$. The second part follows from the first one, via a simple adaptation of the proof of the second part of Theorem 5.14. ■
Recall that $P^d$ and $P^b$ are defined by (49)–(51) and (52)–(54), respectively. As already explained above, $P^d(\psi, a_b) = P^b(\psi, a_b)$ whenever the mean-field field interaction $a_b \in S_1$ is purely attractive, or purely repulsive. Thus, Theorem 5.22 is a stronger version of Theorems 5.9 and 5.13 because $b_\pm \in D_0$ are not restricted anymore to be in $D_{0,\pm}$, nor in $C_{0,\pm}$, i.e., the requirement that functions have to be of positive type and have scaling monotone Fourier transforms can be relaxed. Recall also that a sufficient condition on mean-field models that have non-trivial attractive and repulsive components for $P^d = P^b$ to hold true is given by Lemma 4.2.

6 Illustration of the General Results: Two-Body Potentials

For the reader’s convenience, we illustrate our general abstract results in a particular example, which however generalizes the model (1), presented in the introduction for $S = \{\uparrow, \downarrow\}$, in the sense that it also includes long-range attractions, as described in Sections 5.5 and 5.6. This is an important generalization, because attractive interactions have always been an issue in the Kac limit, and, what is more, the competition between repulsions and attractions can lead to unconventional mean-field effective models in the Kac limit (Section 5.6.1). Note that this section is based on the short paper [39], which we summarize below.

6.1 Short-Range Model with Two-Body Interaction Potentials

Using the cubic boxes $\Lambda_L = \{Z \cap [-L, L]\}^d$ of volume $|\Lambda_L|$, we define the translation-invariant local Hamiltonians

$$H_{\Lambda_L} (\gamma_-, \gamma_+) \doteq T_{\Lambda_L} - H_{\Lambda_L, -} + H_{\Lambda_L, +} ,$$

for two parameters $\gamma_-, \gamma_+ \in (0, 1)$ and length $L \in N_0$, where

$$T_{\Lambda_L} \doteq \sum_{x, y \in \Lambda_L, s \in \{\uparrow, \downarrow\}} h (x - y) a^*_x a_y ,$$

$$H_{\Lambda_L, -} \doteq \sum_{x, y \in \Lambda_L} \gamma_-^d f_- (\gamma_- (x - y)) a^*_y a^*_x a_x a_{x, \uparrow} ,$$

$$H_{\Lambda_L, +} \doteq \sum_{x, y \in \Lambda_L, s \in \{\uparrow, \downarrow\}} \gamma_+^d f_+ (\gamma_+ (x - y)) a^*_y a_y a^*_x a_{x, \uparrow} .$$

For $\gamma_- = 0$, the Hamiltonian is nothing else than the Hamiltonian $H_{\Lambda_L}^{SR}$ defined by (1), presented in the introduction for $S = \{\uparrow, \downarrow\}$. Compare also (133) with (89) and (91). Here, $h$ is some reflection-symmetric real-valued function on $Z^d$, i.e.,

$$h(-z) = h(z) \in R , \quad z \in Z^d .$$

It represents the kinetic part of the model. Usually, $h(x) = v(|x|)$ for some function $v : R^+_0 \to R$. Instead, one could have used the condition

$$h(-z) = \overline{h(z)} \in C , \quad z \in Z^d ,$$

on the one-particle hopping strength with minor changes. This slightly more general situation allows, for instance, for an external magnetic potential in the model. For simplicity we stick to the real case and assume additionally that $h$ is finitely supported. This encompasses the case of the discrete Laplacian, which corresponds to the choice

$$h(z) = \begin{cases} 0 & \text{for } |z| > 1 \\ -1 & \text{for } |z| = 1 \\ 2d & \text{for } z = 0 \end{cases} , \quad z \in Z^d .$$
As explained in all the paper, the reflection-symmetric real-valued function \( f_+ \) is a (non-zero) pair potential encoding interparticle forces, whose range is tuned by the parameter \( \gamma_+ \in (0, 1) \). The (non-zero) reflection-symmetric real-valued function \( f_- \) encodes the hopping strength of Cooper pairs. This model thus implements a BCS interaction whose range is tuned by the parameter \( \gamma_- \in (0, 1) \). Similar to the one-particle hopping strength \( h \), with minor modifications in order to include external magnetic forces, one could have used a complex-valued function \( f_- \) satisfying

\[
f_-(-z) = \hat{f}_-(z) \in \mathbb{C}, \quad z \in \mathbb{Z}^d.
\]

Again for simplicity we stick to the real case. As is usual in theoretical physics, \( f_- \) and \( f_+ \) are assumed to be fast decaying and positive definite, i.e., the Fourier transforms \( \hat{f}_-, \hat{f}_+ \) of \( f_- \), \( f_+ \) are positive functions on \( \mathbb{R}^d \). This choice for \( f_+ \) is reminiscent of a superstability condition, which is essential in the bosonic case [2, Section 2.2 and Appendix G]. For simplicity, we assume that \( f_- \), \( f_+ \in C_0^{\mathbb{R}^d} (\mathbb{R}^d, \mathbb{R}) \), that is, they are both compactly supported and sufficiently regular. For technical reasons, we also assume that the Fourier transform of the Cooper pair hopping strength is scaling monotone, in the sense of (85), that is,

\[
\hat{f}_-(\gamma^{-1} k) \leq \hat{f}_-(k), \quad k \in \mathbb{R}^d, \gamma (0, 1).
\]

All these properties are already explained in Section 5.1, see in particular Equations (81) and (86) as well as the explicit examples given there (like the Yukawa-type potential).

By (35)–(36), the pressure of the model can be represented in the thermodynamic limit as a variational problem over translation-invariant states: First, for \( \gamma_- \), \( \gamma_+ \in (0, 1) \), the energy density functional

\[
\epsilon(\gamma_-, \gamma_+): E_1 \rightarrow \mathbb{R}
\]

is defined by

\[
\epsilon(\gamma_-, \gamma_+) (\rho) \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \rho \left(H_{\Lambda_L} (\gamma_-, \gamma_+)\right) < \infty
\]

for any translation-invariant state \( \rho \in E_1 \). By (29)–(30), this limit exists and it can be split into three parts:

\[
\epsilon(\gamma_-, \gamma_+) = \underbrace{\epsilon_+}_{\text{repulsive interaction term (+)}} - \underbrace{\epsilon_-}_{\text{attractive interaction term (-)}} + \underbrace{\epsilon_0}_{\text{kinetic term}}, \quad (134)
\]

where, for any translation-invariant state \( \rho \in E_1 \),

\[
\epsilon_{\pm} (\rho) \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \rho \left(H_{\Lambda_L, \pm}\right) < \infty \quad \text{and} \quad \epsilon_0 (\rho) \doteq \lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \rho \left(T_{\Lambda_L}\right) < \infty. \quad (135)
\]

Then, by (36), using the specific notation of this section, for any inverse temperature \( \beta \in (0, \infty) \) and \( \gamma_- \), \( \gamma_+ \in (0, 1) \), the free energy density functional \( f_\beta(\gamma_-, \gamma_+): E_1 \rightarrow \mathbb{R} \) is defined by

\[
f_\beta(\gamma_-, \gamma_+) \doteq \epsilon(\gamma_-, \gamma_+) - \beta^{-1} s, \quad (136)
\]

where \( s : E_1 \rightarrow \mathbb{R}_+^\ast \) is the entropy density functional defined as the thermodynamic limit (33) of the von Neumann entropy per unit volume. With these definitions, the thermodynamic limit of the (grand-canonical) pressure equals

\[
P_\beta (\gamma_- \gamma_+) \doteq \lim_{L \rightarrow \infty} \frac{1}{\beta |\Lambda_L|} \ln \text{Tr}(e^{-\beta H_{\Lambda_L}(\gamma_- \gamma_+)}) = - \inf f_\beta(\gamma_-, \gamma_+) (E_1) < \infty \quad (137)
\]

for any inverse temperature \( \beta \in (0, \infty) \) and parameters \( \gamma_- \), \( \gamma_+ \in (0, 1) \). See Equation (35)–(36). The free energy density functional \( f_\beta(\gamma_-, \gamma_+) \) is weak*-lower semicontinuous and the (space homogeneous) infinite volume equilibrium states of the short-range model are defined as being the minimizers of this functional. They form thus the set \( \Omega_{\beta} (\gamma_-, \gamma_+) \doteq \{ \omega \in E_1 : f_\beta(\gamma_-, \gamma_+) (\omega) = -P_\beta (\gamma_- \gamma_+) \} \).
for any $\beta \in (0, \infty)$ and $\gamma_- , \gamma_+ \in (0, 1)$.

As explained in Section 4.5, $\Omega_\beta (\gamma_- , \gamma_+ )$ is a (non-empty) convex weak*-compact subset of the space $E_1$ of translation-invariant states. It can be directly related to the limit of Gibbs states associated with the local Hamiltonians $H_{A_L} (\gamma_- , \gamma_+ )$, $L \in \mathbb{N}_0$: The set of all states on $\mathcal{X}$ being weak*-compact, the sequence of Gibbs states of the local Hamiltonians, seen as periodic states on $\mathcal{X}$, has weak*-convergent subsequences. However, it is not clear that such limits always belong to the set $E_1$ of translation-invariant states. In fact, if a weak*-convergent sequence of Gibbs states has a translation-invariant state $\omega$ as its limit, then the state $\omega$ must belong to $\Omega_\beta (\gamma_- , \gamma_+ )$. This condition can be ensured by imposing periodic boundary conditions, as explained in [11] Chapter 3. In particular, in this case, the weak*-accumulation points of Gibbs states are equilibrium states, i.e., minimizers of the free energy density functional. See [11] Theorem 3.13.

## 6.2 Mean-Field Approximations

Recall now that the Kac, or van der Waals, limit refers to the limits $\gamma_\pm \to 0^+$ of the short-range model that is already in infinite volume, i.e., $\gamma_\pm \to 0^+$ after the thermodynamic limit $L \to \infty$. For small parameters $\gamma_\pm \ll 1$, the short-range model defined in finite volume by (133) has interparticle (+) and BCS (−) interactions whose ranges ($O(\gamma_\pm^{-1})$) are very large as compared to lattice constant (here, $O(1)$), but the interaction strength is small as $\gamma_\pm^2$, in such a way that the first Born approximation\(^2\) to the scattering lengths of the interparticle and BCS potentials remains constant, as is usual. One therefore expects to have some effective mean-field, or long-range, model in the limits $\gamma_\pm \to 0^+$.

The effective local Hamiltonians for the long-range limit of the above short-range model should be

$$
H^L_{A_L} (\eta_- , \eta_+) \doteq T_{A_L} + \frac{\eta_+}{|A_L|} \sum \limits_{x,y \in \Lambda_L, s \in \{\uparrow, \downarrow\}} a^*_x a_{y,s} a^*_y a_{x,s} - \frac{\eta_-}{|A_L|} \sum \limits_{x,y \in \Lambda_L} a^*_x a_{y,\downarrow} a_{x,\uparrow} a_{y,\uparrow}
$$

(138)

for all $L \in \mathbb{N}_0$ and some positive parameters $\eta_- , \eta_+ \in \mathbb{R}^+_0$. Compare these local Hamiltonians with (2) and (133). They refer to a mean-field model, as defined in Section 4. See, e.g., (116) and (122).

As a consequence, we can define from (45) the space-averaging functionals associated with the mean-field repulsions (+) and attractions (−), which are respectively denoted by $\Delta_\pm : E_1 \to \mathbb{R}$ and equals

$$
\Delta_\pm (\rho) \doteq \lim \frac{1}{|A_L|^2} \sum \limits_{x,y \in \Lambda_L} \rho \left( \alpha_y ( A^*_x A_\pm ) : A_\pm ( A^*_x ) \right) \in \left[ |\rho(A_\pm)|^2 , \|A_\pm\|_{L^2}^2 \right],
$$

(139)

for any translation-invariant state $\rho \in E_1$, where

$$
A_\downarrow \doteq a_{0,\downarrow} a_{0,\uparrow} \quad \text{and} \quad A_\uparrow \doteq a^*_{0,\uparrow} a_{0,\uparrow} + a^*_{0,\downarrow} a_{0,\downarrow}.
$$

(140)

Compare this functional with Equations (17)—(19) and recall the ergodic property of extreme points of the convex weak*-compact space $E_1$ of translation-invariant states.

For any inverse temperature $\beta \in (0, \infty)$ and $\eta_- , \eta_+ \in \mathbb{R}^+_0$, the (conventional) free energy density functional of the mean-field model, as defined by (51), is well-defined and denoted here by

$$
\mathcal{F}_\beta (\eta_- , \eta_+) \doteq \eta_+ \Delta_+ - \eta_- \Delta_- + \epsilon_0 - \beta^{-1} s.
$$

(141)

Compare this definition with Equations (134)—(136), the corresponding one for the short-range model. By (49)—(51), the thermodynamic limit of the (grand-canonical) pressure of the mean-field model

\(^{21}\text{i.e., } \int_{\mathbb{R}^d} \gamma_\pm^d f_\pm (\gamma_\pm x) \, dx = \int_{\mathbb{R}^d} f_\pm (x) \, dx \doteq \hat{f}_\pm (0).\)
equals
\[ P^\beta_\beta (\eta_-, \eta_+) = \lim_{L \to \infty} \frac{1}{\beta |\Lambda|} \ln \text{Tr}(e^{-\beta H^\beta_\beta (\eta_-, \eta_+)} = -\inf \beta^\beta_\beta (\eta_-, \eta_+) (E_1) < \infty \] (142)
for any inverse temperature \( \beta \in (0, \infty) \) and parameters \( \eta_-, \eta_+ \in \mathbb{R}_0^+ \).

As explained in Section 4.5.1 the free energy density functional \( P^\beta_\beta (\eta_-, \eta_+) \) is generally not weak*-lower semicontinuous on the convex weak*-compact space \( E_1 \) of translation-invariant states because of repulsive mean-field interactions. See Equation (64), that is here,
\[ P^\beta_\beta (\eta_-, \eta_+) = \eta_+ \Delta_+ + \left(-\eta_- \Delta_- + c_0 - \beta^{-1} s\right) \]  

upper semicont. lower semicont.

In particular, in contrast with the short-range model, this functional does not necessarily have minimizers and this leads to the definition of (conventional) equilibrium states given via (65), that is in this special case,
\[ \Omega^\beta_\beta (\eta_-, \eta_+) = \left\{ \omega \in E_1 : \exists (\rho_n)_{n \in \mathbb{N}} \subseteq E_1 \text{ weak*-converging to } \omega \right\} \]
so that \( \lim_{n \to \infty} P^\beta_\beta (\eta_-, \eta_+) (\rho_n) = -P^\beta_\beta (\eta_-, \eta_+) \).

Again, this last set can be directly related to the limit of Gibbs states associated with the local Hamiltonians \( H^\beta_\beta (\eta_-, \eta_+) \), \( L \in \mathbb{N}_0 \): If a weak*-convergent subsequence of Gibbs states has a translation-invariant state \( \omega \) as its limit, then the state \( \omega \) must belong to \( \Omega^\beta_\beta (\eta_-, \eta_+) \). Exactly as in the short-range case, this condition can be ensured by imposing periodic boundary conditions, as explained in [1] Chapter 3. In particular, in this case, the weak*-accumulation points of Gibbs states are conventional equilibrium states, i.e., minimizers of the conventional free energy density functional defined by (141). See again [1] Theorem 3.13.

The set \( \Omega^\beta_\beta (\eta_-, \eta_+) \) gathers all conventional equilibrium states, being associated with approximating minimizers of the free energy density functional (141). Non-conventional equilibrium states are instead minimizers of the non-conventional free energy density functional (52), which is in the present case denoted by
\[ P^\beta_\beta (\eta_-, \eta_+) (\rho) = \eta_+ (a^*_0 a_0 + a^*_0 a_0) + \left(-\eta_- \Delta_- (\rho) + c_0 (\rho) - \beta^{-1} s (\rho)\right) \]  

convex cont. affine lower semicont.

for any translation-invariant state \( \rho \in E_1 \). In particular, this leads to the set (66) of non-conventional equilibrium states, that is in this case,
\[ \Omega^\beta_\beta (\eta_-, \eta_+) = \left\{ \omega \in E_1 : \beta^\beta_\beta (\eta_-, \eta_+) (\omega) = \inf \beta^\beta_\beta (\eta_- \eta_+) (E_1) = -P^\beta_\beta (\eta_-, \eta_+) \right\} \]

where \( P^\beta_\beta (\eta_-, \eta_+) \) is nothing else than the non-conventional pressure defined in the general case by Equation (54).

### 6.3 Thermodynamic Game

The thermodynamic game explained in Section 4.4.3 can be explicitly given in this case. Applying the concept of approximating (self-adjoint, short-range) interactions (56) and Hamiltonians (58) to the present example, we introduce so-called approximating local short-range Hamiltonians for the mean-field model:
\[ \hat{H}_{\Lambda} \left( \eta_-, \eta_+, c_-, c_+ \right) = T_{\Lambda} + \eta_{\hat{\beta}}^{1/2} \left( \bar{c}_+ + c_+ \right) \sum_{x \in \Lambda, s \in \{\uparrow, \downarrow\}} a^*_{x,s} a_{x,s} \]
\[ -\eta^{1/2} \sum_{x \in \Lambda} \left( \bar{c}_- a^*_{x,\uparrow} a_{x,\downarrow} + c_- a_{x,\downarrow} a_{x,\uparrow} \right) \] (143)
for \( c_- , c_+ \in \mathbb{C}, L \in \mathbb{N}_0 \) and \( \eta_-, \eta_+ \in \mathbb{R}_0^+ \). Given an inverse temperature \( \beta \in (0, \infty) \), we define the function \( \tilde{P}_\beta : \mathbb{C}^2 \times (\mathbb{R}_0^+)^2 \to \mathbb{R} \) by the infinite volume pressure

\[
\tilde{P}_\beta (c_- , c_+, \eta_-, \eta_-) \doteq \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \ln \text{Tr}(e^{-\beta \hat{H}_{\Lambda_L} (\eta_-, \eta_+ , c_- , c_+ )}) < \infty , \quad c_- , c_+ \in \mathbb{C}, \ \eta_-, \eta_+ \in \mathbb{R}_0^+ ,
\]

which exists, thanks to (59). Then, for fixed \( \beta \in (0, \infty) \) and \( \eta_-, \eta_+ \in \mathbb{R}_0^+ \), the approximating free energy density, as defined by (61), is equal to

\[
\mathcal{h}_\beta (c_- , c_+) \doteq - |c_+|^2 + |c_-|^2 - \tilde{P}_\beta (c_- , c_+, \eta_-, \eta_-) , \quad c_- , c_+ \in \mathbb{C} .
\]

Given an inverse temperature \( \beta \in (0, \infty) \) and fixed parameters \( \eta_-, \eta_+ \in \mathbb{R}_0^+ \), this function is seen as the payoff function of a two-person zero-sum game, the thermodynamic game associated with the mean-field model. See Section 4.4.3 or [39] for more pedagogical explanations. Note from (62) that

\[
P^\dagger_\beta (\eta_-, \eta_+) = - \inf_{c_- \in \mathbb{C}} \sup_{c_+ \in \mathbb{C}} \mathcal{h}_\beta (c_- , c_+) \quad \text{and} \quad P^\circ_\beta (\eta_-, \eta_+) = - \sup_{c_+ \in \mathbb{C}} \inf_{c_- \in \mathbb{C}} \mathcal{h}_\beta (c_- , c_+) .
\]

The sup and the inf of this last equation do not commute in general, but a sufficient condition for the sup and inf to commute is given by Lemma 4.2.

Notice that the approximating Hamiltonians (143) are quadratic in the annihilation and creation operators. It can thus be diagonalized by a so-called Bogoliubov transformation and the pressures \( P^\dagger_\beta \) and \( P^\circ_\beta \), as well as the payoff function \( \mathcal{h}_\beta \) of the thermodynamic game, can be analytically and numerically studied. Additionally, the sets \( \Omega^\dagger_\beta (\eta_-, \eta_+) \) and \( \Omega^\circ_\beta (\eta_-, \eta_+) \) of all equilibrium states can be explicitly determined, thanks to Theorem 4.3.

### 6.4 The Kac Limit and Mean-Field Approximations

By Theorem 5.3, the energy densities (135) associated with the short-range attractions and repulsions converge pointwise to a mean-field one associated with the elements \( A_\pm \) (140):

\[
\lim_{\gamma_\pm \to 0^+} \epsilon_\pm (\rho) \doteq \lim_{\gamma_\pm \to 0^+} \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \rho (H_{\Lambda_L, \pm}) = \hat{f}_{\pm} (0) \Delta_{\pm} (\rho) \doteq \lim_{L \to \infty} \frac{\hat{f}_{\pm} (0)}{|\Lambda_L|^2} \sum_{x, y \in \Lambda_L} \rho (\alpha_y (A_{\pm}^x ) \alpha_x (A_{\pm}) )
\]

for any translation-invariant state \( \rho \in E_\delta \), where we have from (84) that

\[
\hat{f}_{\pm} (0) \doteq \int_{\mathbb{R}^d} f (x) \, dx \geq 0 .
\]

Recall that \( f_- , f_+ \) are assumed to be positive definite, i.e., the Fourier transforms \( \hat{f}_-, \hat{f}_+ \) of \( f_- , f_+ \), respectively, are positive functions on \( \mathbb{R}^d \). Comparing (134)–(137) and (141)–(142) in light of (145), it demonstrates that the parameters \( \eta_- , \eta_+ \in \mathbb{R}_0^+ \) of the mean-field model to be taken in the limit \( \gamma_\pm \to 0^+ \) of the short-range one are the first Born approximation to the scattering length of the interparticle potentials

\[
\eta_\pm = \hat{f}_{\pm} (0) \in \mathbb{R}_0^+ ,
\]

as expected of course. This is rigorously proven by Theorem 5.15 which, in the example presented here, refers to the following statement:

**Theorem 6.1 (Conventional mean-field models)**

*Fix an arbitrary reflection-symmetric finitely supported real-valued function \( \varepsilon \) on \( \mathbb{Z}^d \). Let \( f_- , f_+ \in C_0^\text{ref} (\mathbb{R}^d, \mathbb{R}) \) be reflection-symmetric, positive definite functions on \( \mathbb{R}^d \) with \( \hat{f}_-(\gamma^{-1} k) \leq f_- (k) \) for \( k \in \mathbb{R}^d \). Fix an inverse temperature \( \beta \in (0, \infty) \).*
i.) Convergence of infinite volume pressures:

\[
\lim_{\gamma_+ \to 0^+} \lim_{\gamma_- \to 0^+} P_\beta \left( \gamma_-, \gamma_+ \right) = P_\beta^2 \left( \hat{f}_-(0), \hat{f}_+(0) \right).
\]

ii.) Convergence of equilibrium states: For any \( \gamma_+ \in (0, 1) \), take any weak* accumulation point \( \omega_{\gamma_+} \) of any net \( (\omega_{\gamma_-, \gamma_+})_{\gamma_- \in (0,1)} \subseteq \Omega_\beta(\gamma_-, \gamma_+) \) as \( \gamma_- \to 0^+ \). Pick any weak* accumulation point \( \omega \) of the net \( (\omega_{\gamma_+})_{\gamma_+ \in (0,1)} \), as \( \gamma_+ \to 0^+ \). Then,

\[
\omega_{\gamma_- \to 0^+, \gamma_+} \xrightarrow{\text{weak}*} \omega \xrightarrow{\text{weak}*} \Omega_\beta^2(\hat{f}_-(0), \hat{f}_+(0)) \subset \Omega_\beta^2(f_-(0), f_+(0))
\]

Theorem 6.1 uses the particular order for the limits: first \( \gamma_- \to 0^+ \) and then \( \gamma_+ \to 0^+ \), meaning that the attractive range has to be much larger than the repulsive one. In the opposite case, Theorem 5.17 shows that one gets non-conventional pressures:

**Theorem 6.2 (Non-conventional mean-field models)**

Fix an arbitrary reflection-symmetric finitely supported real-valued function \( \varepsilon \in \mathbb{Z}^d \). Let \( f_-, f_+ \in C_0^d(\mathbb{R}^d, \mathbb{R}) \) be reflection-symmetric, positive definite functions on \( \mathbb{R}^d \) with \( f_-(\gamma^{-1}k) \leq f_-(k) \) for \( k \in \mathbb{R}^d \). Fix an inverse temperature \( \beta \in (0, \infty) \).

i.) Convergence of infinite volume pressures:

\[
\lim_{\gamma_- \to 0^+} \lim_{\gamma_+ \to 0^+} P_\beta \left( \gamma_-, \gamma_+ \right) = P_\beta^2 \left( \hat{f}_-(0), \hat{f}_+(0) \right).
\]

ii.) Convergence of equilibrium states: For any \( \gamma_- \in (0, 1) \), take any weak* accumulation point \( \omega_{\gamma_-} \) of any net \( (\omega_{\gamma_-, \gamma_+})_{\gamma_+ \in (0,1)} \subseteq \Omega_\beta(\gamma_-, \gamma_+) \) as \( \gamma_+ \to 0^+ \). Pick any weak* accumulation point \( \omega \) of the net \( (\omega_{\gamma_-})_{\gamma_- \in (0,1)} \), as \( \gamma_- \to 0^+ \). Then,

\[
\omega_{\gamma_- \to 0^+, \gamma_+} \xrightarrow{\text{weak}*} \omega \xrightarrow{\text{weak}*} \Omega_\beta^2(\hat{f}_-(0), \hat{f}_+(0))
\]

Recall that the sup and the inf in Equation (144) do not commute in general and we generally have

\[
P_\beta^k(\eta_-, \eta_+) \neq P_\beta^0(\eta_-, \eta_+).
\]

Hence, depending upon how the double limit \( \gamma_+ \to 0^+ \) of the short-range model is taken, one can get an effective long-range system that is different from the one described by the conventional mean-field model, which is the thermodynamic limit of the finite-volume system described from local Hamiltonians (138). See also Theorem 5.19 showing that the Kac limit of pressures can attain all the values of the interval

\[
\left[ P_\beta^2(\hat{f}_-(0), \hat{f}_+(0)), P_\beta^2(\hat{f}_-(0), \hat{f}_+(0)) \right]
\]

by a convenient choice of the sequences \( (\gamma_+, n)_{n \in \mathbb{N}} \) and \( (\gamma_-, n)_{n \in \mathbb{N}} \).

As explained in Section 5.6.1 in the general case, the results referring to the above examples are highly non-trivial: As expected, any Kac or van der Waals limit leads to mean-field pressures and equilibrium states. However, the limit mean-field model is not necessarily what one traditionally guesses when one mixes repulsive and attractive long-range components. In fact, it strongly depends upon the hierarchy of ranges between attractive and repulsive interparticle forces. For instance, if the range of repulsive forces is much larger than the range of the attractive ones, then in the Kac limit for these forces one may get a mean-field model that is unconventional. See Theorem 6.2.
The observation that models in the Kac (or van der Waals) limit are not equivalent to their expected mean-field approximation has already been observed in the past, but in a different context. In 1987, de Smedt and Zagrebnov studied a Bose gas in the continuum and in presence of two-body interactions with positive Fourier transforms and whose range is tuned by the parameter $\gamma = \gamma_+ \in (0, 1)$, exactly as in (1) or as in (133) for $\gamma = \gamma_+$ and $\gamma_- = 0$. To this model, they added a rescaled external potential $V_L$ defined by $V_L(x) \doteq V(x/L)$, $V \in C^\infty (\mathbb{R}^d, \mathbb{R})$, where $L \in \mathbb{R}^+$ parametrizes the $d$-dimensional cube $[0, L]^d$ where the Bose gas is enclosed. Indeed, the Bose-Einstein condensation of the perfect Bose gas (i.e., without any Kac limit) was previously shown by van den Berg and Lewis to be very sensitive to scaled external fields, even in low dimensions and even if the rescaled external potentials. Using this setting, de Smedt and Zagrebnov [17] show that a discrepancy marking the appearance of a Bose-Einstein condensation) In this case [17], there is no combination of repulsive and attractive interactions, but instead a perturbation with a (weak) scaled external field. The latter refers to the scaled-external-field perturbation method (used in [17] to the Kac limit), which has been used in many situations.

7 Appendix

7.1 Elementary Technical Results

In this section, we give a sequence of useful technical results, mainly on summations of real-valued functions of the $d$-dimensional vector space $\mathbb{R}^d$, as in the first lemma:

**Lemma 7.1 (Integral test for series estimates – I)**

Let $f$ be an arbitrary function $\mathbb{R}^d \to \mathbb{R}$ and $g : [0, \infty) \to [0, \infty)$ a monotonically decreasing function such that, for any $x \in \mathbb{R}^d$,

$$|f(x)| \leq g(|x|) \quad \text{and} \quad \int_0^\infty g(r) r^{d-1} dr < \infty.$$  \hspace{1cm} (146)

Then, it follows that

$$\sum_{z \in \mathbb{Z}^d} |f(z)| \leq g(0) + \sum_{n=1}^d \binom{d}{n} \frac{2\pi^n}{\Gamma(\frac{d}{2})} \int_0^\infty g(r) r^{n-1} dr < \infty.$$  \hspace{1cm} (147)

**Proof.** Note first that monotonic functions $[0, \infty) \to [0, \infty)$ are Borel measurable and thus $\int_0^\infty g(r) r^{d-1} dr$ is well-defined as the integral of a positive measurable function, with respect to the Lebesgue integral. Fix all parameters of the lemma. For any subset $I \in 2^{\{1, \ldots, d\}}$ with $|I| = n$ and $I = \{j_1, \ldots, j_n\}$ so that $j_k < j_l$ for $k < l$, we define the mapping $\xi_I : \mathbb{Z}^n \to \mathbb{Z}^d$ by

$$\xi_I (z) \doteq (y_1, \ldots, y_d), \quad z = (z_1, \ldots, z_n) \in \mathbb{Z}^n,$$  \hspace{1cm} (148)

with $y_k = z_k$ for any $k \in \{1, \ldots, n\}$ and $y_j = 0$ otherwise. Then, the following equality holds true:

$$\sum_{z \in \mathbb{Z}^d} |f(z)| = |f(0)| + \sum_{n=1}^d \sum_{I \in 2^{\{1, \ldots, d\}} : |I| = n} \sum_{z \in \mathbb{Z}^n \setminus \{0\}} |f \circ \xi_I (z)|.$$  \hspace{1cm} (149)

For any $n \in \{1, \ldots, d\}$, observe from (146) that, obviously,

$$|f \circ \xi_I (z)| \leq g(|z|), \quad z \in \mathbb{Z}^n \setminus \{0\},$$

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and, since $g$ is monotonically decreasing, one obtains that

$$
|f \circ \xi(z)| \leq g(|z|) \leq g(|x|), \quad z \in \mathbb{Z}^n \setminus \{0\}, \ x \in \mathcal{A}_z \subseteq \mathbb{R}^n,
$$

by defining the subset

$$
\mathcal{A}_{(z_1, \ldots, z_d)} = \mathcal{A}_{z_1} \times \cdots \times \mathcal{A}_{z_d}
$$

with

$$
\mathcal{A}_{z_i} = \begin{cases} 
(z_i - 1, z_i] & \text{if } z_i > 0, \\
[z_i, z_i + 1) & \text{if } z_i < 0.
\end{cases}
$$

By integrating (149) over $\mathcal{A}_z$, we deduce the inequality

$$
|f \circ \xi(z)| \leq \int_{\mathcal{A}_z} g(|x|) \, dx_1 \cdots dx_n, \quad z \in \mathbb{Z}^n \setminus \{0\}.
$$

We incorporate this upper estimate in (148) to arrive at

$$
\sum_{z \in \mathbb{Z}^d} |f(z)| \leq g(0) + \sum_{n=1}^{d} \sum_{I \in \mathcal{A}_{z_i} : |I| = n} \sum_{z \in \mathbb{Z}^n \setminus \{0\}} \int_{\mathcal{A}_z} g(|x|) \, dx_1 \cdots dx_n
$$

from which the assertion follows.

**Corollary 7.2 (Integral test for series estimates – II)**

*Under the conditions of Lemma 7.1, for any $\gamma \in \mathbb{R}^+$ and $a \in \mathbb{Z}^d$, the series*

$$
\sum_{z \in \mathbb{Z}^d} \gamma^d f(\gamma z + a)
$$

*is absolutely convergent. If $\gamma < 1$ then*

$$
\sum_{z \in \mathbb{Z}^d} \left| \gamma^d f(\gamma z + a) \right| \leq M_g,
$$

*where $M_g$ is a constant that only depends upon $g$ and is additive and homogeneous with respect to $g$.*

**Proof.** Fix all parameters of the corollary. First, observe that, for any $\gamma \in \mathbb{R}^+$ and $a \in \mathbb{Z}^d$, there is $b^{(\alpha, \gamma)} \in \mathbb{R}^d$ such that $|b^{(\alpha, \gamma)}| \leq \sqrt{d}/2$ and

$$
\sum_{z \in \mathbb{Z}^d} \gamma^d f(\gamma z + a) = \sum_{z \in \mathbb{Z}^d} \gamma^d f(\gamma(z + b^{(\alpha, \gamma)})).
$$

Therefore, fix now $\gamma \in \mathbb{R}^+$ and $b \in \mathbb{R}^d$ such that $|b| \leq \sqrt{d}/2$. Define the functions

$$
h(x) = \gamma^d f(\gamma(x + b)) , \quad x \in \mathbb{R}^d, \quad \text{and} \quad m(r) = \begin{cases} 
\gamma^d g(0) & \text{if } 0 \leq r \leq |b|, \\
\gamma^d g(\gamma |r - |b||) & \text{if } r > |b|.
\end{cases}
$$

Using the reverse triangle inequality and the monotonicity of the function $g$, we deduce from (146) that

$$
|h(x)| \leq \gamma^d g(\gamma |x + b|) \leq m(|x|) \leq \gamma^d g(0), \quad x \in \mathbb{R}^d,
$$

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as well as, for any \( n \in \{1, \ldots, d\} \),

\[
\int_0^\infty m(r) r^{n-1} dr \leq \gamma^d \left( |b|^n g(0) + \int_{|b|}^\infty g(\gamma (r - |b|)) r^{n-1} dr \right)
\]

\[
= \gamma^d \left( |b|^n g(0) + \gamma^{-1} \int_0^\infty g(u) \left( \frac{u}{\gamma} + |b| \right)^{n-1} du \right)
\]

\[
= \gamma^d |b|^n g(0) + \sum_{k=0}^{n-1} \binom{n-1}{k} |b|^{n-(k+1)} \gamma^{d-(k+1)} \int_0^\infty g(u) u^k du
\]

\[
\leq \gamma^d \left( \frac{\sqrt{d}}{2} \right)^n g(0) + \sum_{k=0}^{n-1} \binom{n-1}{k} \gamma^{d-(k+1)} \left( \frac{\sqrt{d}}{2} \right)^{n-(k+1)} \int_0^\infty g(u) u^k du < \infty .
\]

(152)

We can thus invoke Lemma 7.1 with the functions given by (151) to deduce that, for any \( \gamma \in \mathbb{R}^+ \) and \( b \in \mathbb{R}^d \) so that \( |b| \leq \sqrt{d}/2 \),

\[
\sum_{z \in \mathbb{Z}^d} |\gamma^d f(\gamma (z + b))| \leq \gamma^d g(0) + \sum_{n=1}^d \binom{d}{n} \frac{2\pi^n}{\Gamma(n/2)} \int_0^\infty m(r) r^{n-1} dr < \infty .
\]

If \( \gamma \in (0, 1) \) then one checks from (152) that, for any \( b \in \mathbb{R}^d \) so that \( |b| \leq \sqrt{d}/2 \),

\[
\sum_{z \in \mathbb{Z}^d} |\gamma^d f(\gamma (z + b))| \leq M_g
\]

with

\[
M_g = g(0) + \sum_{n=1}^d \binom{d}{n} \frac{2\pi^n}{\Gamma(n/2)} \left( \frac{\sqrt{d}}{2} \right)^n g(0) + \sum_{k=0}^{n-1} \binom{n-1}{k} \gamma^{d-(k+1)} \left( \frac{\sqrt{d}}{2} \right)^{n-(k+1)} \int_0^\infty g(u) u^k du .
\]

(153)

By Equation (150), the corollary then follows. ■

We recall now the so-called Poisson summation formula. It is a well-known identity in Fourier analysis. Its precise formulation, as we need it here, is given in the next proposition and relates the summation of a rescaled, continuously differentiable and integrable function \( f \) to a summation of the Fourier transform \( \hat{f} \) of this function.

**Proposition 7.3 (Poisson summation formula)**

Let \( f \in C^1(\mathbb{R}^d, \mathbb{C}) \) be a continuously differentiable function satisfying

\[
\sup_{k \in \{0, \ldots, d\}} \sup_{x \in \mathbb{R}^d} \left| (1 + |x|)^{d+\varepsilon} \partial_{x_k} f(x) \right| < \infty
\]

(153)

for some \( \varepsilon \in \mathbb{R}^+ \), with the convention \( \partial_{x_0} f = f \). Then, for any \( \gamma \in \mathbb{R}^+ \),

\[
\sum_{z \in \mathbb{Z}^d} \gamma^d f(\gamma a + \gamma z) = \sum_{z \in \mathbb{Z}^d} \hat{f}(2\pi \gamma^{-1} z) e^{2\pi i z \cdot a} .
\]

**Proof.** For the reader’s convenience, we shortly give the proof of this well-known identity. Fix all parameters of the proposition. By Inequality (153) and Lemma 7.1 one checks that all the series

\[
\sum_{z \in \mathbb{Z}^d} \partial_{x_k} f(x + \gamma z) , \quad k \in \{0, \ldots, d\} ,
\]

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converge uniformly with respect to \( x \in \mathbb{R}^d \). So, we can define a continuously differentiable function \( g \in C^1(\mathbb{R}^d, \mathbb{C}) \) by the series

\[
g(x) = \sum_{z \in \mathbb{Z}^d} \gamma^d f(x + \gamma z), \quad x \in \mathbb{R}^d.
\]  

(154)

As previously used in (150), this function is \( \gamma \)-periodic in the sense that, for any vector of the form \( y = \gamma z \in \mathbb{R}^d \) for some \( z \in \mathbb{Z}^d \), one has \( g(x + y) = g(x) \) for any \( x \in \mathbb{R}^d \). Thus, this function can be written as a Fourier series:

\[
g(x) = \sum_{z \in \mathbb{Z}^d} u_z e^{2\pi i \gamma z \cdot x}, \quad x \in \mathbb{R}^d,
\]  

(155)

where, for any \( z \in \mathbb{Z}^d \),

\[
u_z = \gamma^{-d} \int_{\Gamma} g(x) e^{-2\pi i \gamma z \cdot x} dx \quad \text{with} \quad \Gamma = \left[ -\frac{\gamma}{2}, \frac{\gamma}{2} \right]^d.
\]  

(156)

Additionally, the Fourier coefficients \( u_z, z \in \mathbb{Z}^d \), are absolutely summable, for \( g \) is continuously differentiable. Now, from (154) and (156) we compute that, for any \( z \in \mathbb{Z}^d \),

\[
u_z = \sum_{y \in \mathbb{Z}^d} \int_{\Gamma} f(x + y) e^{-2\pi i \gamma z \cdot x} dx = \int_{\mathbb{R}^d} f(u) e^{-2\pi i \gamma z \cdot u} du = \hat{f}(2\pi \gamma^{-1} z),
\]

which together with (154)–(155) for \( x = \gamma a \) yields the proposition.

We conclude the series of elementary results by giving two estimates on summations of Fourier transforms of integrable functions.

Lemma 7.4 (Summations of Fourier transforms of functions – I)

For any \( \varepsilon \in \mathbb{R}^+ \), there is a constant \( M_\varepsilon \in \mathbb{R}^+ \) such that, for any \( f \in \mathcal{D}_e \),

\[
\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{f}(\gamma^{-1} k)| \leq \gamma^2 M_\varepsilon \|f\|_{\mathcal{D}_e}, \quad \gamma \in (0, 1).
\]

Proof. Fix all parameters of the lemma. For any \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \) such that \( k_{j_1}, \ldots, k_{j_n} \neq 0 \) for some subset \( J = \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, d\} \) with \( |J| = n \), one has that

\[
F(f)(k) = \hat{f}(k) = \int_{\mathbb{R}^d} f(x) e^{-ik \cdot x} dx = \frac{(-1)^n}{k_{j_1}^2 \cdots k_{j_n}^2} \frac{\partial^{2n} f}{\partial x_{j_1}^2 \cdots \partial x_{j_n}^2}(k),
\]

(157)

where the Fourier transform

\[
F \left( \frac{\partial^{2n} f}{\partial x_{j_1}^2 \cdots \partial x_{j_n}^2} \right)
\]

is well-defined and bounded, because all derivatives of \( f \) of order up to \( 2d \) belong to \( L^1(\mathbb{R}^d) \), by assumption. Define

\[
C_f = (2d)! \|f\|_{\mathcal{D}_e} \int_{\mathbb{R}^d} (1 + |x|)^{-(d+\varepsilon)} d^d x < \infty
\]

(158)

and note that

\[
\max_{n \in \{1, \ldots, d\}} \max_{(j_1, \ldots, j_n) \in \mathbb{Z}^d} \left\| \frac{\partial^{2n} f}{\partial x_{j_1}^2 \cdots \partial x_{j_n}^2} \right\|_{\infty} \leq C_f.
\]
Similar to (148), observe that
\[ \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{f}(\gamma^{-1}k)| = \sum_{n=1}^{d} \sum_{I \in 2^{\{1,\ldots,d\}: |I| = n}} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |\hat{f} \circ \xi_I(\gamma^{-1}k)|, \]
where \( \xi_I : \mathbb{Z}^n \to \mathbb{Z}^d \) is the mapping defined by (147) for any subset \( I \in 2^{\{1,\ldots,d\}} \). Then, using (157) and (158), we bound the last equality to get that, for any \( \gamma \in \mathbb{R}^+ \),
\[ \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{f}(\gamma^{-1}k)| \leq C_f \sum_{n=1}^{d} \sum_{I \in 2^{\{1,\ldots,d\}: |I| = n}} \gamma^{2n} \sum_{k = (k_1,\ldots,k_n) \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{k_1^2 \cdots k_n^2} = C_f \sum_{n=1}^{d} \binom{d}{n} \left( \gamma^2 \frac{\pi^2}{3} \right)^n. \]
In particular, for any \( \gamma \in (0, 1) \), we obtain that
\[ \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\hat{f}(\gamma^{-1}k)| \leq \gamma^2 M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon}, \]
where \( M_\varepsilon \in \mathbb{R}^+ \) is the constant defined by
\[ M_\varepsilon = (2d)! \int_{\mathbb{R}^d} (1 + |x|)^{-(d+\varepsilon)} \, dx \sum_{n=1}^{d} \binom{d}{n} \left( \gamma^2 \frac{\pi^2}{3} \right)^n. \]

\[ \text{Proof.} \] The proof is very similar to the one of Lemma 7.4. For instance, in the same way one gets (159), one checks that, for any \( \gamma \in (0, 1) \), \( \theta \in (-\pi, \pi]^d \).
\[ \lim_{n \to \infty} \inf_{X} f_n(X) = \inf_{X} f(X) > -\infty. \]

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**Proof.** Note that \( \inf f_n (\mathcal{X}) > -\infty \) because \( f_n \) is lower semicontinuous on a compact set. Additionally, for any \( \eta \in \mathbb{R}^+ \), there is \( N \in \mathbb{N} \) such that, for any natural number \( n \geq N, f \geq f_n - \eta \), which in turn implies that

\[
\inf f (\mathcal{X}) \geq \inf f_n (\mathcal{X}) - \eta \geq \inf f_N (\mathcal{X}) - 2\eta > -\infty. \tag{160}
\]

Thus the sequence \( (\inf f_n (\mathcal{X}))_{n \in \mathbb{N}} \) of real numbers is bounded and it suffices to prove that it has only one accumulation point to prove its convergence. Take any subsequence \( (f_{n_k})_{k \in \mathbb{N}} \) such that \( (\inf f_{n_k} (\mathcal{X}))_{k \in \mathbb{N}} \) converges to some accumulation point of the whole sequence. Given \( \eta \in \mathbb{R}^+ \), by taking again a subsequence of \( (f_{n_k})_{k \in \mathbb{N}} \), we can assume without loss of generality that \( f_{n_l} \geq f_{n_k} - \frac{\eta}{2^k} \) for all natural numbers \( l \geq k \). For any \( l \in \{2, 3, \ldots, \infty\} \), define the preimage

\[
E_l = f_{n_l}^{-1}(I_l)
\]

of the open interval

\[
I_l = \left( \inf f (\mathcal{X}) - \sum_{j=1}^{l-1} \frac{\eta}{2^j}, \infty \right).
\]

Then, it follows that \( E_l \subseteq E_{l+1} \), and since \( (f_n)_{n \in \mathbb{N}} \) converges pointwise to \( f \), one has

\[
\mathcal{X} = \bigcup_{j=1}^{\infty} E_j.
\]

Since \( f_n \) is lower semicontinuous for any \( n \in \mathbb{N} \), \( (E_j)_{j \in \mathbb{N}} \) is an open cover for \( \mathcal{X} \), and by the compactness of \( \mathcal{X} \), there exists \( q \in \mathbb{N} \) such that

\[
\mathcal{X} = \bigcup_{j=1}^{q} E_j = E_q.
\]

Hence, for any \( \eta \in \mathbb{R}^+ \), there is some \( q \in \mathbb{N} \) so that, for any natural number \( j \geq q \),

\[
\inf f_{n_j} (\mathcal{X}) \geq \inf f (\mathcal{X}) - \eta.
\]

Combined with (160), we deduce that

\[
\inf f (\mathcal{X}) + \eta \geq \lim_{j \to \infty} \inf f_{n_j} (\mathcal{X}) \geq \inf f (\mathcal{X}) - \eta,
\]

for all subsequences \( (f_{n_k})_{k \in \mathbb{N}} \) such that \( (\inf f_{n_k} (\mathcal{X}))_{k \in \mathbb{N}} \) converges to some accumulation point. Since \( \eta \in \mathbb{R}^+ \) is arbitrary, the assertion follows. 

### 7.2 Estimates on Kac Interactions

In this section we give important estimates on the Kac interactions of Definition 5.1. We start by showing in the next lemma that they are locally Lipschitz continuous:

**Lemma 7.7 (Locally Lipschitz continuity of the Kac function)**

(i) For any fixed \( \gamma \in (0,1) \), the range of the Kac function \( K_\gamma \) of Definition 5.1 is a subspace of \( \mathcal{W}_1^\mathbb{R} \) and, for every \( \varepsilon \in \mathbb{R}^+ \), there is a constant \( D_\varepsilon \in \mathbb{R}^+ \) such that

\[
\|K_\gamma (\Phi, f) - K_\gamma (\Psi, g)\|_{\mathcal{W}_1} \leq D_\varepsilon (\|f\|_{\mathcal{D}_1} \|\Phi - \Psi\|_{\mathcal{W}_1} (\|\Phi\|_{\mathcal{W}_1} + \|\Psi\|_{\mathcal{W}_1}) + \|f - g\|_{\mathcal{D}_1} \|\Psi\|^2_{\mathcal{W}_1}).
\]


for any $\Phi, \Psi \in \mathcal{W}_1$ and $f, g \in \mathcal{D}_\varepsilon$. In particular, the mapping $\mathcal{K}_\gamma : \mathcal{W}_1 \times \mathcal{D}_0 \to \mathcal{W}_1^\mathbb{R}$ is locally Lipschitz continuous.

(ii) For any fixed $\varepsilon \in \mathbb{R}^+$, there is a constant $\tilde{D}_\varepsilon \in \mathbb{R}^+$ such that, for any $f \in \mathcal{D}_\varepsilon$ and $\Phi \in \mathcal{W}_1$,

$$
\|\mathcal{K}_{\gamma_1} (\Phi, f) - \mathcal{K}_{\gamma_2} (\Phi, f)\|_{\mathcal{W}_1} \leq \tilde{D}_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi\|^2_{\mathcal{W}_1} \left| \frac{\gamma_2}{\gamma_1} \right|, \quad \gamma_1, \gamma_2 \in (0, 1) .
$$

**Proof.** (i): In view of Corollary 7.2, for any $\varepsilon \in \mathbb{R}^+$, there is a constant $M_\varepsilon$ such that, for any $f \in \mathcal{D}_\varepsilon$, $\gamma \in (0, 1)$ and $a \in \mathcal{L}$,

$$
\sum_{z \in \mathcal{L}} \gamma^d |f (\gamma z + a)| \leq M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} .
$$

(161)

In fact, choose $g(r) = \|f\|_{\mathcal{D}_\varepsilon} (1 + r)^{-(d+\varepsilon)}$ in Corollary 7.2 observing from (83) that, for any $\varepsilon \in \mathbb{R}^+$ and $f \in \mathcal{D}_\varepsilon$,

$$
\sup_{x \in \mathbb{R}^d} \left| (1 + |x|)^{d+\varepsilon} f (x) \right| \leq \|f\|_{\mathcal{D}_\varepsilon} < \infty .
$$

Take now an interaction $\Phi \in \mathcal{W}_1$ and a function $f \in \mathcal{D}_\varepsilon$ for a fixed $\varepsilon \in \mathbb{R}^+$. Then, using the norm (23), the fact that $f$ is reflection-symmetric and (161), we compute from Definition 5.1 that

$$
\|\mathcal{K}_\gamma (\Phi, f)\|_{\mathcal{W}_1} \leq 2 \sum_{Z_1, Z_2 \in \mathcal{P}_1: Z_1 \cup Z_2 \geq \{0\}} \|\Phi_{Z_1}\|_{\mathcal{U}} \|\Phi_{Z_2}\|_{\mathcal{U}} \frac{\sum_{x \in Z_1, y \in Z_2} \gamma^d \|f (\gamma \langle x - y \rangle)\|}{|Z_1| + |Z_2|} \leq 2 \sum_{z \in \mathcal{L}} \gamma^d |f (\gamma z)| \leq 2M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi\|^2_{\mathcal{W}_1} .
$$

(162)

(163)

(164)

In particular, the Kac function $\mathcal{K}_\gamma$ maps $\mathcal{W}_1 \times \mathcal{D}_0$ to $\mathcal{W}_1^\mathbb{R}$. Similarly, for any $\Phi, \Psi \in \mathcal{W}_1$ and $f \in \mathcal{D}_0$, from the inequality

$$
\|\mathcal{K}_\gamma (\Phi, f) - \mathcal{K}_\gamma (\Psi, f)\|_{\mathcal{W}_1} \leq 2 \sum_{Z_1, Z_2 \in \mathcal{P}_1: Z_1 \cup Z_2 \geq \{0\}} \|\Phi_{Z_1} - \Psi_{Z_1}\|_{\mathcal{U}} \|\Phi_{Z_2}\|_{\mathcal{U}} \frac{\sum_{x \in Z_1, y \in Z_2} \gamma^d \|f (\gamma \langle x - y \rangle)\|}{|Z_1| + |Z_2|} \leq 2 \sum_{z \in \mathcal{L}} \gamma^d |f (\gamma z)| \leq 2M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi - \Psi\|_{\mathcal{W}_1} \left( \|\Phi\|_{\mathcal{W}_1} + \|\Psi\|_{\mathcal{W}_1} \right) .
$$

(165)

Since the mapping $f \mapsto \mathcal{K}_\gamma (\Phi, f)$ is linear for any fixed $\gamma \in (0, 1)$ and $\Phi \in \mathcal{W}_1$, the first bound stated in the lemma then follows. This is done by using (165) and the identity

$$
\mathcal{K}_\gamma (\Phi, f) - \mathcal{K}_\gamma (\Psi, g) = \mathcal{K}_\gamma (\Phi, f) - \mathcal{K}_\gamma (\Psi, f) + \mathcal{K}_\gamma (\Psi, f) - \mathcal{K}_\gamma (\Psi, g)
$$

for any $\Phi, \Psi \in \mathcal{W}_1$, $\varepsilon \in \mathbb{R}^+$ and $f, g \in \mathcal{D}_\varepsilon$. 

55
(ii): By Definition 5.1, for any $\varepsilon \in \mathbb{R}^+$, there is a constant $\tilde{D}_\varepsilon \in \mathbb{R}^+$ such that, for any $f \in \mathcal{D}_\varepsilon$, $\Phi \in \mathcal{W}_1$ and $\gamma_1, \gamma_2 \in (0, 1),$

\[
\|K_{\gamma_1} (\Phi, f) - K_{\gamma_2} (\Phi, f)\|_{\mathcal{W}_1} \leq 2 \sum_{z_1 \in P_1 : z_1 \geq 0} \frac{\|\Phi_{z_1}\|_{U}}{|Z_1|} \sum_{z_2 \in P_1 : z_2 \geq 0} \frac{2 \|\Phi_{z_2}\|_{U}}{|Z_2|} \frac{1}{|Z_2|} \times \sum_{x \in Z_1, y \in Z_2} \int_{\gamma_1}^{\gamma_2} \frac{\gamma^d |f (\gamma (x - y)) - f (\gamma (x - y))|}{\gamma^1} d\gamma 
\]

\[
\leq \tilde{D}_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi\|^2_{\mathcal{W}_1} \ln \frac{\gamma_2}{\gamma_1},
\]

similar to Equations (162)–(163), observing that

\[
|f (x) d + \nabla f (x) \cdot x| \leq d \|f\|_{\mathcal{D}_\varepsilon} (1 + |x|)^{- (d + \varepsilon)}, \quad x \in \mathbb{R}^d.
\]

Note that the bound (164) on the norm of the Kac interaction $K_{\gamma} (\Phi, f)$, referring to Lemma 7.7 (i) can be refined, yielding the inequality

\[
\|K_{\gamma} (\Phi, f)\|_{\mathcal{W}_1} + \|f\|_{\mathcal{D}_\varepsilon} \|\Phi\|^2_{\mathcal{W}_1} \leq 2 M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi\|^2_{\mathcal{W}_1},
\]

where, for any $\gamma \in (0, 1)$, $\mathcal{F}_\gamma$ is the mapping from $\mathcal{W}_1 \times \mathcal{D}_0$ to $\mathbb{R}^+$ defined by

\[
\mathcal{F}_\gamma (\Phi, f) = \sum_{z_1, z_2 \in P_1 : z_1 \geq 0} \frac{\|\Phi_{z_1}\|_{U} \|\Phi_{z_2}\|_{U}}{|Z_1| |Z_2|} \sum_{x \in Z_1, y \in Z_2} \gamma^d |f (\gamma (x - y))| \frac{1}{|Z_1| + |Z_2|}, \quad \Phi \in \mathcal{W}_1, f \in \mathcal{D}_\varepsilon.
\]

(167)

To prove (166), it suffices to refine Inequality (162) in an obvious way. However, such a correction to the bound (164) pointwise vanishes in the Kac limit $\gamma \to 0^+$:

**Lemma 7.8 (Vanishing of $\mathcal{F}_\gamma$ in the Kac limit)**

(i) For any $\varepsilon \in \mathbb{R}^+$, there is a constant $M_\varepsilon \in \mathbb{R}^+$ such that, for any $\gamma \in (0, 1)$, $\Phi, \Psi \in \mathcal{W}_1$, and $f, g \in \mathcal{D}_\varepsilon$,

\[
|\mathcal{F}_\gamma (\Phi, f) - \mathcal{F}_\gamma (\Psi, g)| \leq M_\varepsilon \|\Phi - \Psi\|_{\mathcal{W}_1} \|f\|_{\mathcal{D}_\varepsilon} (\|f\|_{\mathcal{W}_1} + 2 \max \{\|\Psi\|_{\mathcal{W}_1}, \|\Phi\|_{\mathcal{W}_1}\})
\]

\[
+ 3 M_\varepsilon \|f - g\|_{\mathcal{D}_\varepsilon} \|\Psi\|^2_{\mathcal{W}_1}.
\]

(ii) For any $\Phi \in \mathcal{W}_1$ and $f \in \mathcal{D}_0$,

\[
\lim_{\gamma \to 0^+} \mathcal{F}_\gamma (\Phi, f) = 0.
\]

**Proof.** Let $\varepsilon \in \mathbb{R}^+$ and $f \in \mathcal{D}_\varepsilon$. For any $\gamma \in (0, 1)$ and $\Phi, \Psi \in \mathcal{W}_1$, from (161), (167) and the triangle inequality, we arrive at

\[
|\mathcal{F}_\gamma (\Phi, f) - \mathcal{F}_\gamma (\Psi, g)| \leq 2 \sum_{z_1, z_2 \in P_1 : z_1 \geq 0} \frac{\|\Psi_{z_1}\|_{U} \|\Phi_{z_1} - \Psi_{z_1}\|_{U}}{|Z_1| |Z_2|} \sum_{x \in Z_1, y \in Z_2} \frac{\gamma^d |f (\gamma (x - y))|}{|Z_1| + |Z_2|}
\]

\[
+ \sum_{z_1, z_2 \in P_1 : z_1 \geq 0} \frac{\|\Phi_{z_1} - \Psi_{z_1}\|_{U} \|\Phi_{z_2} - \Psi_{z_2}\|_{U}}{|Z_1| |Z_2|} \sum_{x \in Z_1, y \in Z_2} \frac{\gamma^d |f (\gamma (x - y))|}{|Z_1| + |Z_2|}
\]

\[
\leq M_\varepsilon \|f\|_{\mathcal{D}_\varepsilon} \|\Phi - \Psi\|_{\mathcal{W}_1} (\|\Phi - \Psi\|_{\mathcal{W}_1} + 2 \max \{\|\Psi\|_{\mathcal{W}_1}, \|\Phi\|_{\mathcal{W}_1}\}) + 3 M_\varepsilon \|f - g\|_{\mathcal{D}_\varepsilon} \|\Psi\|^2_{\mathcal{W}_1}.
\]

(168)
Using the reverse triangle inequality, observe that
\[ |\mathfrak{F}_\gamma (\Phi, f) - \mathfrak{F}_\gamma (\Psi, g)| \leq |\mathfrak{F}_\gamma (\Phi, f) - \mathfrak{F}_\gamma (\Psi, f)| + |\mathfrak{F}_\gamma (\Psi, f - g)| \]
for any $\Phi, \Psi \in \mathcal{W}_1$, $\varepsilon \in \mathbb{R}^+$ and $f, g \in \mathcal{D}_\varepsilon$. By combining this estimate with (168), we obtain Assertion (i). Finally, note that, for finite-range interactions (see (27)), we clearly have
\[ \lim_{\gamma \to 0^+} \mathfrak{F}_\gamma (\Phi, f) = 0, \quad \Phi \in \mathcal{W}_0 \subseteq \mathcal{W}_1, \]
for any $\Phi, \Psi \in \mathcal{W}_1$, $\varepsilon \in \mathbb{R}^+$ and $f, g \in \mathcal{D}_\varepsilon$. By combining this estimate with (168), we obtain Assertion (i). Finally, note that, for finite-range interactions (see (27)), we clearly have
\[ \lim_{\gamma \to 0^+} \mathfrak{F}_\gamma (\Phi, f) = 0, \quad \Phi \in \mathcal{W}_0 \subseteq \mathcal{W}_1, \]
since in this case the sum in (167) is finite. Therefore, by density of $\mathcal{W}_0 \subseteq \mathcal{W}_1$ and the equicontinuity of $(\mathfrak{F}_\gamma (\cdot, f))_{\gamma \in (0,1)}$ at fixed $f \in \mathcal{D}_0$ (Assertion (i)), (169) extends to all interactions $\Phi \in \mathcal{W}_1$.

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References

[1] J.-B. Bru and W. de Siqueira Pedra, Non-cooperative Equilibria of Fermi Systems With Long Range Interactions. Memoirs of the AMS 224, no. 1052 (2013).

[2] V.A. Zagrebnov and J.-B. Bru, The Bogoliubov Model of Weakly Imperfect Bose Gas. Phys. Rep. 350 (2001) 291-434.

[3] N.N. Bogoliubov Jr., On model dynamical systems in statistical mechanics. Physica 32 (1966) 933.

[4] J.G. Brankov, N.S. Tonchev and V.A. Zagrebnov, A nonpolynomial generalization of exactly soluble models in statistical mechanics, Ann. Phys. (N. Y.) 107(1-2) (1977) 82-94.

[5] J.G. Brankov, N.S. Tonchev and V.A. Zagrebnov, On a class of exactly soluble statistical mechanical models with nonpolynomial interactions, J. Stat. Phys. 20(3) (1979) 317-330.

[6] N.N. Bogoliubov Jr., J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, Metod approksimiruyushchego gamil’toniana v statisticheskoi fizike. Sofia: Izdat. Bulgar. Akad. Nauk, 1981

[7] N.N. Bogoliubov Jr., J.G. Brankov, V.A. Zagrebnov, A.M. Kurbatov and N.S. Tonchev, Some classes of exactly soluble models of problems in Quantum Statistical Mechanics: the method of the approximating Hamiltonian. Russ. Math. Surv. 39, 1-50 (1984)

[8] J.G. Brankov, D.M. Danchev and N.S. Tonchev, Theory of Critical Phenomena in Finite–size Systems: Scaling and Quantum Effects. Singapore–New Jersey–London–Hong Kong: Word Scientific, 2000

[9] J. Lebowitz and O. Penrose, A Rigorous Treatment of the Van der Waals-Maxwell Theory of the Vapor-Liquid Transition, J. Math. Phys. 7 (1966) 98.

---

22 The Approximating Hamiltonian Method in Statistical Physics.
23 Publ. House Bulg. Acad. Sci.
[10] E. Lieb, Quantum-mechanical extension of the Lebowitz-Penrose theorem on the Van Der Waals theory, *J. Math. Phys.* **7**(6) (1966) 1016-1024.

[11] O. Penrose I and J. L. Lebowitz, Rigorous Treatment of Metastable States in the van der Waals-Maxwell Theory, *J. Stat. Phys.* **3**(2) (1971) 211-236.

[12] P. C. Hemmer and J. L. Lebowitz, Systems with Weak Long-Range Potentials, pp 107-203 in *Phase Transitions and Critical Phenomena (Volume 5b)*, by C. Domb, and M.S. Green (eds), Academic Press Inc, 1976.

[13] E. Presutti, *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*, Berlin, Heidelberg: Springer, 2009.

[14] S. Franz and F.L. Toninelli, Kac Limit for Finite-Range Spin Glasses, *Phys. Rev. Lett.* **92** (2004) 030602 (3 pages).

[15] S. Franz and F.L. Toninelli, *Finite-range spin glasses in the Kac limit: free energy and local observables*, *J. Phys. A: Math. Gen.* **37** (2004) 7433-7446.

[16] S. Franz, Spin glass models with Kac interactions, *Eur. Phys. J. B* **64** (2008) 557-561.

[17] P. de Smedt and V. A. Zagrebnov, van der Waals limit of an interacting Bose gas in a weak external field, *Phys. Rev. A* **35**(11), (1987) 4763-4769.

[18] P.A. Martin and J. Piasecki, Self-consistent equation for an interacting Bose gas, *Phys. Rev. E* **68**(12) (2003) 161131-1611314.

[19] P.A. Martin, Quantum Mayer Graphs: application to Bose and Coulomb Gases, *34**(7) (2003) 3629.

[20] P.A. Martin and J. Piasecki, Bose gas beyond mean field, *Phys. Rev. E* **71** (2005) 016109 (9pp)

[21] A. Alastuey, J. Piasecki and P. Szymczak, Hartree-Fock analysis of the effects of long-range interactions on the Bose-Einstein condensation, *J. Stat. Mech.* **2019** (2019) 033101.

[22] E.H. Lieb and J. Yngvason, Ground State Energy of the Low Density Bose Gas, *Phys. Rev. Lett.* **80**(12) (1998) 2504-2507.

[23] J.-B. Bru and W. de Siqueira Pedra, Classical Dynamics Generated by Long-Range Interactions for Lattice Fermions and Quantum Spins, *J. Math. Anal. Appl.* **493**(1) (2021) 124434 (pp 61).

[24] J.-B. Bru and W. de Siqueira Pedra, Quantum Dynamics Generated by Long-Range Interactions for Lattice-Fermion and Quantum Spins, *J. Math. Anal. Appl.* **493**(1) (2021) 124517 (pp 65).

[25] J.-B. Bru and W. de Siqueira Pedra, Entanglement of Classical and Quantum Short-Range Dynamics in Mean-Field Systems, *Annals of Physics* **434** (2021) 168643 (pp 31).

[26] W. Rudin, *Functional Analysis*. McGraw-Hill Science, 1991.

[27] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. I, 2nd ed.* New York: Springer-Verlag, 1987.

[28] H. Araki and H. Moriya, Equilibrium Statistical Mechanics of Fermion Lattice Systems. *Rev. Math. Phys.* **15** (2003) 93-198.
[29] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. II, 2nd ed.* New York: Springer-Verlag, 1997.

[30] G.L. Sewell, *Quantum Theory of Collective Phenomena*, Oxford: Clarendon Press, 1986.

[31] H. Komiya, Elementary Proof For Sion’s minimax theorem, *Kodai Math. J.* **11**(1) (1988) 5-7.

[32] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Mathematical Surveys and Monographs, Volume 49. Providence, American Mathematical Society, 1997.

[33] M. Sh. Birman and M.Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Mathematics and its Applications, Springer Netherlands, D. Reidel Publishing Company, Dordrecht, Holland, 1987.

[34] L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Birkhäuser Verlag: Basel–Boston–Berlin, 2005.

[35] D. J. Thouless, *The Quantum Mechanics of Many-Body Systems*, Second Edition, Academic Press: New York, 1972.

[36] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Theory of Superconductivity, *Phys. Rev.* **108** (1957) 1175-1204.

[37] J.-B. Bru and W. de Siqueira Pedra, Remarks on the $\Gamma$–regularization of Non-convex and Non-Semi-Continuous Functionals on Topological Vector Spaces, *J. Convex Analysis* **19**(2) (2012) 467-483.

[38] J.L. Kelley, *General Topology*. Graduate Texts in Mathematics, vol. 27, Springer-Verlag New York, 1975.

[39] J.-B. Bru, W. de Siqueira Pedra and K. Rodrigues Alves, Thermodynamic Game and The Kac Limit in Quantum Lattices, in Quantum Mathematics II, Eds. M. Correggi and M. Falconi, Springer Nature Singapore Pte Ltd. (2023).

[40] M. van den Berg and J. T. Lewis, On the free boson gas in a weak external potential, *Commun. Math. Phys.* **81** (1981) 475-494.

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