On Boolean Control Networks with Maximal Topological Entropy

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Abstract

Boolean control networks (BCNs) are discrete-time dynamical systems with Boolean state-variables and inputs that are interconnected via Boolean functions. BCNs are recently attracting considerable interest as computational models for genetic and cellular networks with exogenous inputs.

The topological entropy of a BCN with \( m \) inputs is a nonnegative real number in the interval \([0, m \log 2]\). Roughly speaking, a larger topological entropy means that asymptotically the control is “more powerful”. We derive a necessary and sufficient condition for a BCN to have the maximal possible topological entropy. Our condition is stated in the framework of Cheng’s algebraic state-space representation of BCNs. This means that verifying this condition incurs an exponential time-complexity. We also show that the problem of determining whether a BCN with \( n \) state variables and \( m = n \) inputs has a maximum topological entropy is NP-hard, suggesting that this problem cannot be solved in general using a polynomial-time algorithm.

Key words: Boolean control networks, algebraic state-space representation, topological entropy, symbolic dynamics, computational complexity, Perron-Frobenius theory.

1 Introduction

Boolean networks (BNs) are useful modeling tools for dynamical systems whose state-variables can attain two possible values. Examples range from artificial neural networks with ON/OFF type neurons (see, e.g. Hassoun\textsuperscript{(1995)}), to models for the emergence of social consensus between simple agents that can either agree or disagree with a certain opinion (see, e.g. Green \textit{et al.}\textsuperscript{(2007)}).

There is a growing interest in modeling biological systems using BNs and, in particular, genetic regulation networks, where each gene can be either expressed (ON) or not expressed (OFF) (Chaos \textit{et al.}\textsuperscript{(2006)}; Kauffman \textit{et al.}\textsuperscript{(2003)}; Li \textit{et al.}\textsuperscript{(2004)}). Although being highly abstract, BNs seem to capture the real behavior of gene-regulatory processes well (Bornholdt\textsuperscript{(2008)}; Hopfensitz \textit{et al.}\textsuperscript{(2012)}).

Kauffman\textsuperscript{(1969)} has studied the order and stability of large, randomly constructed nets of such binary genes. He also related the behavior of these random nets to various cellular control processes, including cell differentiation, by associating every possible cell type with a stable attractor of the BN. This work has stimulated the analysis of large-scale BNs using tools from the theory of complex systems and statistical physics (see, e.g. Albert and Barabasi\textsuperscript{(2000)}; Aldana\textsuperscript{(2003)}; Drossel \textit{et al.}\textsuperscript{(2003)}; Kauffman\textsuperscript{(1993)}).

BNs have also been used to model various cellular processes including the complex cellular signaling network controlling stomatal closure in plants (Li \textit{et al.}\textsuperscript{(2006)}), the molecular pathway between two neurotransmitter systems, the dopamine and glutamate receptors (Gupta \textit{et al.}\textsuperscript{(2007)}), carcinogenesis, and the effects of therapeutic intervention (Szallasi and Liang\textsuperscript{(1998)}).

BNs with (Boolean) inputs are referred to as Boolean control networks (BCNs). BCNs have been used to model biological systems with exogenous inputs. For example, Faure \textit{et al.}\textsuperscript{(2006)} (see also Faure and Thieffry\textsuperscript{(2009)}) have developed a BCN model for the core network regulating the mammalian cell cycle. Here the nine
The remainder of this note is organized as follows. Section 2 reviews BNs, BCNs, and some definitions and tools from SD. Section 3 includes our main results. Section 4 concludes and describes some possible directions for further research.

2 Preliminaries

We begin by reviewing BCNs and their ASSRs. Let $\mathcal{S} := \{0,1\}$. A BCN is a discrete-time logical dynamical system

$$X_1(k+1) = f_1(X_1(k), \ldots, X_n(k), U_1(k), \ldots, U_m(k)),$$

$$\vdots$$

$$X_n(k+1) = f_n(X_1(k), \ldots, X_n(k), U_1(k), \ldots, U_m(k)),$$

where $X_i, U_i \in \mathcal{S}$, and each $f_i$ is a Boolean function, i.e. $f_i : \mathcal{S}^{n+m} \rightarrow \mathcal{S}$. It is useful to write this in vector form as

$$X(k+1) = f(X(k), U(k)).$$

(2)

A BN is a BCN without inputs, i.e.

$$X(k+1) = f(X(k)).$$

(3)

Cheng et al. (2011) have developed an algebraic state-space representation (ASSR) of BCNs (and, in particular, of BNs). This representation has proved useful for studying control-theoretic questions, as they reduce a BCN to a positive linear switched system whose input, state and output variables are canonical vectors. Topics that have been analyzed using the ASSR include optimal control (Zhao et al. 2011), controllability and observability (Laschov and Margaliot 2011, 2013), disturbance decoupling (Cheng and Qi 2009; Fornasini and Valcher 2013), identification (Cheng and Zhao 2011), and more.

The ASSR of a BN with $n$ state-variables and $m$ inputs includes a $2^n \times 2^{n+m}$ matrix. Thus, any algorithm based on the ASSR has an exponential time complexity. A natural question is whether better algorithms exist. Zhao (2005) has shown that determining whether a BN has a fixed point is NP-complete. Akutsu et al. (2007) have shown that several control problems for BCNs are NP-hard. Laschov and Margaliot (2013) have shown that the observability problem for BCNs is also NP-hard. Thus, unless $P = NP$, these analysis problems for BCNs cannot be solved in polynomial time.

Hochma et al. (2013) noted the connection between BCNs and symbolic dynamics (SD). The main object of study in SD is shift spaces (Lind and Marcus 1995). The set of all possible trajectories of a BCN is a shift space, so many results and analysis tools from SD are immediately applicable to BCNs. In particular, Hochma et al. (2013) noted that an important notion from SD called topological entropy can be defined for BCNs, and computed using the Perron root of a certain non-negative matrix that appears in the ASSR of a BCN. The topological entropy of a BCN with $n$ state-variables and $m$ inputs (we always assume that $m \leq n$) is a number in the range $[0, m \log 2]$ that indicates how “rich” the control is.

In this paper, we derive a necessary and sufficient condition for a BCN to have a maximal topological entropy. This condition is stated in terms of the ASSR. We also show that for a BCN with $n$ state variables and $m = n$ inputs the problem of determining whether the BCN has maximal topological entropy is NP-hard. This implies that unless $P = NP$, there does not exist an algorithm with polynomial time complexity that solves this problem.

The remainder of this note is organized as follows. Section 2 reviews BNs, BCNs, and some definitions and
where \( x(k) \in \mathcal{L}^{2^n} \) and \( L \in \mathcal{L}^{2^n \times 2^n} \).

The fact that a BN may be represented in a linear form using the vector of minterms has been known for a long time (see, e.g., [Cull (1971, 1973)]), but the ASSR provides an explicit algebraic form that is particularly suitable for control-theoretic analysis.

Given the ASSR (5) of a BN, we can associate with it a directed graph \( G = G(V, E) \), where \( V = \{ e_2^n, \ldots, e_2^n \} \), and there is a directed edge from vertex \( e_2^n \rightarrow e_2^n \) if and only if \( [L]_{ij} = 1 \). In other words, there is a directed edge from vertex \( e_2^n \) to vertex \( e_2^n \) if and only if \( x(k) = e_2^n \) implies that \( x(k+1) = e_2^n \).

We now briefly review some results from [Hochma et al (2013)] by relating BCNs and symbolic dynamics (SD) ([Lind and Marcus (1995)])). SD has evolved from analyzing general dynamical systems by discretizing the state-space into finitely many pieces, each labeled by a different symbol. An orbit of the dynamical system is then transformed into a symbolic orbit composed of the sequence of symbols corresponding to the successive pieces visited by the orbit. The original evolution is transformed into a symbolic dynamics given by a shift operator \( \sigma \). The main object of study in SD is shift spaces.

Given the BCN (2), define its set of state-trajectories of length \( j \) by

\[
\mathcal{A}_S^j := \{ X(0)X(1) \ldots X(j-1) : X(k+1) = f(X(k), U(k)), U(k) \in \mathcal{S}^m, X(0) \in \mathcal{S}^n \},
\]

i.e., the state trajectories of length \( j \) over all possible controls and initial conditions. Note that for a BN this becomes

\[
\{ X(0) \ldots X(j-1) : X(k+1) = f(X(k)), X(0) \in \mathcal{S}^n \}.
\]

The topological entropy of a BCN is

\[
h_S := \lim_{j \to \infty} \frac{1}{j} \log |\mathcal{A}_S^j|.
\]

In other words, \( h_S \) is the asymptotic “growth rate” of the number of state-sequences of a given length. A higher \( h_S \) corresponds to a “richer” control in the sense that asymptotically more state-sequences can be produced.

**Example 1.** Consider the BCN:

\[
\begin{align*}
X_1(k+1) &= U_1(k), \\
X_2(k+1) &= U_2(k), \\
& \quad \vdots \\
X_m(k+1) &= U_m(k), \\
X_{m+1}(k+1) &= f_1(X_1(k), \ldots, X_n(k)), \\
X_{m+2}(k+1) &= f_2(X_1(k), \ldots, X_n(k)), \\
& \quad \vdots \\
X_n(k+1) &= f_{m-1}(X_1(k), \ldots, X_n(k)).
\end{align*}
\]

It is straightforward to see that here \( |\mathcal{A}_S^j| = 2^{n((j-1)m)} \), so (6) yields

\[
h_S = \lim_{j \to \infty} \frac{1}{j} ((n + (j-1)m) \log 2) = m \log 2.
\]

Intuitively speaking, each of the \( m \) control inputs in (7) contributes \( \log 2 \) to the topological entropy. ■

[Hochma et al (2013)] have shown that in the ASSR, the set of state trajectories of a BCN is a shift space (more precisely, a 1-step shift space of finite type) over the alphabet \( \{ e_2^n, \ldots, e_2^n \} \). Combining this with known results from SD yields the following.

**Theorem 1.** ([Hochma et al (2013)]) Consider a BCN in the ASSR (4). Let \( L_i := L \times e_2^n, i = 1, \ldots, 2^m \), where \( L \) is the transition matrix of the BCN, and let

\[
M := L_1 \lor L_2 \lor \ldots \lor L_{2^m}.
\]

Then the topological entropy of the BCN is

\[
h_S = \log \lambda_M.
\]

where \( \lambda_M \) is the Perron root of the non-negative matrix \( M \).

**Remark 1.** Note that \( L_i \in \mathcal{L}^{2^n \times 2^n} \) and thus \( M \in \mathcal{S}^{2^n \times 2^n} \).

**Example 2.** Consider the BCN defined by

\[
X(k+1) = U(k) \lor \bar{X}(k).
\]

The ASSR is given by (4) with \( n = m = 1 \), and \( L = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \). Thus, \( L_1 = L \times e_2^n = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, L_2 = L \times e_2^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \) and \( M = L_1 \lor L_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \) The eigenvalues of \( M \) are \( (1 \pm \sqrt{5})/2 \), so (10) yields \( h_S = \log ((1 + \sqrt{5})/2). \) ■

For easy reference, we recall the following result from the Perron-Frobenius theory of non-negative matrices.
Theorem 2. (Horn and Johnson, 1983, Ch. 8) Suppose that $A \in \mathbb{R}^{n \times n}_+$ and let $\lambda_A$ denote its Perron root. Then
\[ \min_{1 \leq j \leq n} \sum_{i=1}^{n} A_{ij} \leq \lambda_A \leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} A_{ij}. \] (11)

Furthermore, there exists $w \in \mathbb{R}_+^n \setminus \{0\}$ such that $Aw = \lambda_A w$.

3 Main results

Let $\mathcal{B}_n^m$ denote the set of all BCNs with $n$ state-variables and $m$ inputs (with $m \leq n$). Let $h_{\text{max}}$ be the maximum of the topological entropy over the BCNs in $\mathcal{B}_n^m$. Let $\mathcal{B}_n^m \subset \mathcal{B}_n^m$ denote the subset of BCNs with topological entropy equal to $h_{\text{max}}$.

Our first result shows in particular that the BCN in (7) is in $\mathcal{B}_n^m$.

Proposition 3. The maximal topological entropy of a BCN in $\mathcal{B}_n^m$ is $h_{\text{max}} = m \log 2$.

Proof. Fix a BCN in $\mathcal{B}_n^m$, and consider its ASSR. Since $M = \sqrt{\sum_{i=1}^{n} L_i}$, and every $L_i$ has a single one entry in every column, every column of $M$ has no more than $2^m$ one entries. By (11), $\lambda_M \leq 2^m$ so $h_S \leq m \log 2$. The BCN (7) attains this bound and this completes the proof. $\square$

Combining Theorems 1 and 2 suggests that we can relate the topological entropy of a BCN with the maximum of the column (or row) sums of the matrix $M$. The next result shows that this is indeed so. Let $\alpha_{k,k}$ denote the $k \times k$ matrix with all entries equal to $\alpha$. We use $\alpha_k$ as a shorthand for $\alpha_{k,1}$.

Proposition 4. Consider a BCN in the ASSR (4). Let
\[ v := \max_{1 \leq i \leq 2^m} \sum_{j=1}^{2^m} M_{ij}, \] (12)
where $M$ is the matrix defined in (9). Then the following two conditions are equivalent.

(a) $h_S = \log v$.

(b) There exist a permutation matrix $P \in \{0, 1\}^{2^m \times 2^m}$ and $r \geq v$ such that
\[ PMP' = \begin{bmatrix} B & C \\ 0^{n-r} & D \end{bmatrix}, \] (13)
where $B \in S^{r \times r}$, each column of $B$ has exactly $v$ non zero elements, $D \in S^{(2^n-r) \times (2^n-r)}$, and $C \in S^{r \times (2^n-r)}$.

Proof. Assume that condition (b) holds. Let $w \in \mathbb{R}_+^n$ denote an eigenvector of $B$ corresponding to its Perron root $\lambda_B$. Let $\bar{w} := \begin{bmatrix} w \\ 0^{2^n-r} \end{bmatrix}$. Then
\[ PMP' \bar{w} = \begin{bmatrix} B & C \\ 0^{n-r} & D \end{bmatrix} \begin{bmatrix} w \\ 0^{2^n-r} \end{bmatrix} = \lambda_B \bar{w}. \]

This implies that $P' \bar{w}$ is an eigenvector of $M$ corresponding to the eigenvalue $\lambda_B$. Since every column of $B$ has exactly $v$ one entries, Theorem 2 implies that $\lambda_B = v$. Combining this with (12) and Theorem 2 implies that $\lambda_M = v$, so $h_S = \log \lambda_M = \log v$. This shows that condition (b) implies condition (a).

To prove the converse implication, assume that $h_S = \log v$. Then $\lambda_M = v$. By Theorem 2, there exists a vector $w \in \mathbb{R}_+^n \setminus \{0\}$ such that $Mw = vw$. Let $r \geq 1$ be the number of entries in $w$ that are strictly positive, and let $P$ be a permutation matrix such that
\[ \bar{w} := Pw = \begin{bmatrix} w_1 & \ldots & w_r & 0 & \ldots & 0 \end{bmatrix}. \]

(note that if $r = 2^n$ then this vector includes no zeros). Then
\[ \tilde{M} \bar{w} = vw, \]
where $\tilde{M} := PMP'$. Multiplying this on the left by $1_{2^n}$ yields
\[ \tilde{s}_1 \bar{w}_1 + \ldots + \tilde{s}_r \bar{w}_r = v(\bar{w}_1 + \ldots + \bar{w}_r), \]
where $\tilde{s}_i$ denotes the sum of the elements in column $i$ of $M$. By (12), $\tilde{s}_i \leq v$ for all $i$, so (16) implies that
\[ \tilde{s}_i = v \quad \text{for all } i \in \{1, \ldots, r\}. \]
(17)

Let $\tilde{M} = \begin{bmatrix} M_1 & \tilde{M}_2 \\ M_3 & \tilde{M}_4 \end{bmatrix}$, where $\tilde{M}_1 \in S^{r \times r}$. Then (15) becomes
\[ \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 \\ \tilde{M}_3 & \tilde{M}_4 \end{bmatrix} \begin{bmatrix} \bar{w}_1 & \ldots & \bar{w}_r & 0 & \ldots & 0 \end{bmatrix}' = v \begin{bmatrix} \bar{w}_1 & \ldots & \bar{w}_r & 0 & \ldots & 0 \end{bmatrix}'. \]
Since the $\bar{w}_i$s are strictly positive, we conclude that $\tilde{M}_3 = 0^{2^n-r}$. Thus, (17) implies that every column of $\tilde{M}_1$ has exactly $v$ one entries, so condition (b) holds. $\square$
Remark 2. We can provide an intuitive explanation of (13) as follows. For a state $a \in \{e_2^1, \ldots, e_2^n\}$, let
\[
R(a) := \{L \times e_2^m \times a, \ldots, L \times e_2^m \times a\},
\]
i.e., the reachable set from $a$ in one time step. By the definition of $M$, $|R(e_2^m)|$ is equal to the number of one entries in column $j$ of $M$. Thus, $v$ is the maximal cardinality of the one step reachable sets. Proposition 4 asserts that the topological entropy is equal to $\log v$ if and only if there exists a set $Y$ containing $r \geq v$ states such that $|R(a)| = v$ for all $a \in Y$, and any transition from a state in $Y$ is to a state in $Y$. 

Example 3. Consider the two-state, one-input BCN:
\[
\begin{align*}
X_1(k+1) = X_1(k), \\
X_2(k+1) = [U(k) \land X_1(k) \land X_2(k)] \\
&\quad \lor [U(k) \land X_1(k) \land X_2(k)].
\end{align*}
\] (18)

Fig. 1 depicts the state-space transition graph of this BCN, i.e., a directed arrow from state $a$ to state $b$ means that $b$ belongs to the one step reachable set of $a$. It is easy to see from Fig. 1 that $v = 2$ and that $Y := \{e_1^1, e_1^2\}$ satisfies the properties described in Remark 2. By Proposition 4, the topological entropy of (18) is $h_S = \log 2$.

Example 4. Consider again the BCN in Example 1. To analyze its topological entropy using Proposition 4 we first derive an expression for the matrix $M$.

Let $A \in \mathbb{L}^{2^n-m \times 2^n}$ denote the transition matrix in the ASSR with $s := n - m$ state-variables and $m$ control inputs BCN given by:
\[
\begin{align*}
Y_1(k+1) &= f_1(W_1(k), \ldots, W_m(k), Y_1(k), \ldots, Y_s(k)), \\
&\vdots \\
Y_s(k+1) &= f_s(W_1(k), \ldots, W_m(k), Y_1(k), \ldots, Y_s(k)).
\end{align*}
\]

Pick $i \in \{1, \ldots, 2^m\}$. Consider the dynamics of (7) for $u(k) = e_2^m$. By (7), $x_1(k+1) \times x_2(k+1) \times \cdots \times x_m(k+1) = e_2^m$. Thus,
\[
x(k+1) = x_1(k+1) \times \cdots \times x_m(k+1) \\
= e_2^m \times x_m(k+1) \times \cdots \times x_1(k+1) \\
= e_2^m \times Ax(k) \\
= (e_2^m \otimes I_{2^n-2^m}) Ax(k) \\
= \begin{bmatrix} O_{(i-1)2^n-2^m} & \cdots & 0 \end{bmatrix} Ax(k).
\]

On the other hand, for $u(k) = e_2^m$, $x(k+1) = L_i x(k)$, so we conclude that
\[
L_i = \begin{bmatrix} A \\
0 \end{bmatrix}.
\]

This with (9) implies that
\[
M = \begin{bmatrix} A \\
A \end{bmatrix} \in S^{2^n \times 2^n}. \quad (19)
\]

Since every column of $A$ is canonical vector, $M$ is a Boolean matrix and every column of $M$ has exactly $2^m$ ones. Thus $M$ has the form (13) with $r = 2^n$, $v = 2^m$. Proposition 4 implies that $h_S = m \log 2$, and this agrees with (8).

One may perhaps expect that (7) is a “canonical form” of a BCN in $E_2^n$, i.e., that for every BCN in this set there exists an invertible logical transformation of the state-variables taking it to the form (7). However, the next example shows that this is not so.

Example 5. Consider again the two-state, one-input BCN in Example 3. Its ASSR is given by $n = 2$, $m = 1$, and
\[
L = \begin{bmatrix} e_1^1 e_1^2 e_2^1 e_2^2 \\
e_1^1 e_1^2 e_1^3 e_1^4 e_1^5 e_1^6 e_1^7 e_1^8 \end{bmatrix}.
\]

Thus, $L_1 = L \times e_1^1 = \begin{bmatrix} e_1^1 e_1^2 e_1^3 e_1^4 \end{bmatrix}$, $L_2 = L \times e_2^2 = \begin{bmatrix} e_2^1 e_1^1 e_1^3 e_1^4 \end{bmatrix}$, and $M = L_1 \lor L_2 = \begin{bmatrix} e_1^1 e_2^2 e_1^3 e_1^4 e_1^5 e_1^6 e_1^7 e_1^8 \end{bmatrix}$. The eigenvalues of $M$ are $\{2, 1, 0, 0\}$, so $\lambda_M = 2$ and $h_S = \log 2$. Since $2^m = 2$, Proposition 3 and Theorem 1 imply that $h_S = h_{\max}$. Since $M$ has a unique zero row, $P^T M P$ will also have a unique zero row, for any permutation matrix $P$. Therefore $P^T M P$ cannot have the form (19) for any permutation matrix $P$.

The next two results follow immediately from Proposition 4.
Corollary 1. A BCN is in $\mathcal{B}_n^m$ if and only if condition (b) in Proposition 4 holds and each column in the matrix $B$ has $2^m$ non zero elements.

Corollary 2. A BCN is in $\mathcal{B}_n^m$ if and only if

$$M = 1_{2^n,2^n}.$$  \hspace{1cm} (20)

Remark 3. Recall that a BCN is called fixed-time controllable if for any $a, b \in \{e_1^n, \ldots, e_2^n\}$ there exists a control that steers the BCN from $x(0) = a$ to $x(k) = b$ (see Laschow and Margolius (2013)). Eq. (20) means that any state can be reached from any state in one time step. Thus, the BCN is 1 fixed-time controllable.

3.1 Computational complexity

Consider the following problem.

Problem 1. Given a BCN in $\mathcal{B}_n^m$ determine whether its topological entropy is $h_S = h_{\max}$. 

Proposition 5. Problem 1 is NP-hard.

This implies that there does not exist an algorithm with polynomial complexity that solves Problem 1, unless $P = NP$.

Proof of Proposition 5. The proof is based on a polynomial-time reduction of the famous SAT problem (see e.g. [Garvey and Johnson (1990)]) to Problem 1.

Consider a set of Boolean variables $z_1, \ldots, z_n$ taking values in $S$. A formula $g : S^n \rightarrow S$ is a rooted tree. The leaves include either a variable or its negation. Each internal node includes the operator $\land$ or $\lor$. The root of the tree then computes a formula in a natural way. The length of the formula is the number of leaves in the tree. Formulas are often written as strings (e.g., $g(z_1, z_2) = (z_1 \land z_2) \lor \neg z_1$), obtained by an inorder traversal of the rooted tree.

A formula is called satisfiable if there exists an assignment of its variables for which it attains the value 1. For example, $g(z_1, z_2) = z_1 \land z_1 \land z_2$ is not satisfiable.

Problem 2. (SAT) Given a Boolean formula $g : S^n \rightarrow S$, determine whether it is satisfiable.

Given a formula $g : S^n \rightarrow S$, consider the BCN in $\mathcal{B}_n^m$ defined by

$$X_1(k+1) = U_1(k) \land (1 - g(X_1(k), \ldots, X_n(k))),$$

$$\vdots$$

$$X_n(k+1) = U_n(k) \land (1 - g(X_1(k), \ldots, X_n(k))).$$

It is clear that if $g$ is not satisfiable then this BCN is in $\mathcal{B}_n^m$. On the other hand, if $g$ is satisfiable then there is at least one state that is mapped to $0_n$ for any control. This implies that in the ASSR, at least one column of $M$ is the vector $e_{2^n}^n$. Then Corollary 2 implies that the BCN is not in $\mathcal{B}_n^m$. Summarizing, this provides a polynomial reduction from the SAT problem to Problem 1. Since SAT is NP-complete even if the length of $g$ is polynomial in $n$, this completes the proof. □

4 Conclusions

BNs and BCNs are recently attracting considerable interest as computational models in systems biology.

The topological entropy of a BCN is a measure of how rich the control is. A natural question is what is the structure of BCNs that have the maximal possible topological entropy. In this paper, we derived a necessary and sufficient condition for a BCN to have this property, stated in terms of the ASSR.

Since the ASSR of a BCN with $n$ state variables and $m$ inputs includes a matrix $L \in \mathcal{L}^{2^n \times 2^{n+m}}$, verifying this conditions incurs an exponential time complexity. We also showed that the problem of determining whether a BCN has a maximal topological entropy is NP-hard. Thus, there does not exist an algorithm with polynomial time complexity that solves this problem, unless $P = NP$.

Further research is needed in order to clarify the biophysical meaning of the topological entropy in BCNs that model biological systems. Another interesting topic for further research is to characterize all the possible values $h$ such that there exists a BCN in $\mathcal{B}_n^m$ with topological entropy $h$.

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