SCATTERING STATES AND SYMMETRIES IN THE MATRIX MODEL AND TWO DIMENSIONAL STRING THEORY

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ABSTRACT

We study the correspondence between the linear matrix model and the interacting nonlinear string theory. Starting from the simple matrix harmonic oscillator states, we derive in a direct way scattering amplitudes of 2-dimensional strings, exhibiting the nonlinear equation generating arbitrary N-point tree amplitudes. An even closer connection between the matrix model and the conformal string theory is seen in studies of the symmetry algebra of the system.
1. Introduction

Matrix models provide not only a novel formulation of low dimensional string theory but one which is integrable and exactly solvable. They lead to exact string equations in $D < 1$ and a wealth of results for the free energy and correlation functions [1]. The largest model understood so far is the two dimensional string theory, described by a simple dynamics of a (matrix) harmonic oscillator [2-8].

While the numerical results follow straightforwardly, the physical picture encoded in the matrix model is however not seen directly. It is exhibited once appropriate physical observables (collective fields) [3] are identified. For the tachyon one has the bosonic collective field defining perturbative states. While the matrix model is linear, the collective field exhibits a nonlinear interaction which leads to nontrivial physical scattering processes [4]. A fermi liquid description can be used to give a semiclassical picture of the scattering [5]. The field theory is integrable: it exhibits an infinite sequence of conserved charges and an even larger symmetry of $W_{\infty}$ generators [6].

String theory is however most naturally described in terms of the world sheet string coordinates and associated conformal vertex operators [9]. These indeed exhibit similar symmetries [10] and can be seen to give the same correlation functions. Except for the coincidence of various results a closer connection between the matrix model description and the string language is still lacking.

It is the purpose of this paper to address this problem and give a more direct relationship between linear states of the matrix model and nonlinear scattering states of string theory. One has the matrix harmonic oscillator

\begin{equation}
L = \frac{1}{2} \text{Tr}(\dot{M}^2 + M(t)^2),
\end{equation}

with
\[ A_{\pm} \equiv P \pm M = \dot{M} \pm M, \]
\[ A_{\pm}(t) = A_{\pm}(0) e^{\pm t} \]

being standard creation-annihilation operators. In terms of these one easily writes down the eigenstates of the hamiltonian

\[ H = \frac{1}{2} \text{Tr} \left( (P + M)(P - M) \right) = \frac{1}{2} \text{Tr} A_+ A_. \]

For example, the one-parameter set

\[ A_{\pm}^n = \text{Tr} (P \pm M)^n \]

gives imaginary eigenvalues with energies \( \epsilon_n = \mp i n \). Real energy states are obtained by analytic continuation \( n = i k \):

\[ B_k^{(\pm)} = \text{Tr} (P \pm M)^{ik}. \quad (1.2) \]

The question is then how this simple set of exact matrix model states translates into nontrivial string scattering states. Continuing on the constructions begun in [6], we shall explain a correspondence in section 2 and describe a simple derivation of general string scattering amplitudes using the integrable states. As such we exhibit how the nonlinear string dynamics follows from the linear and integrable matrix dynamics.

In section 3 we discuss the symmetry algebra of the theory. We demonstrate there a close connection between the matrix \( W_\infty \) generators and those of the conformal string theory. In particular we shall see that the collective (tachyon) field representation of these operators is nothing but the representation defined in the conformal approach by Klebanov in [11].
2. From States to Scattering

Strings in two dimensions are described by the coordinates $X^\mu \equiv (X, \phi)$, where $X$ is (usually) taken as spacelike and $\phi$ is the nontrivial Liouville coordinate [9]. One has translation invariance in the $X$ direction (this is the time coordinate of the matrix model, \textit{i.e.} $X = it$) and only asymptotic translation invariance in $\phi$ due to an exponential wall $\mu e^{-\sqrt{2}\phi}$. The vertex operators of the lowest string modes (massless tachyons) are

$$V_\pm = e^{ipX + \beta_\pm \phi},$$

$$\beta_\pm = -\sqrt{2} \pm |k|.$$  \hfill (2.1)

Only the $+$ branch describes physical scattering states. The $-$ operators grow at $\phi \to -\infty$ and are termed “wrongly dressed”. For scattering one has left movers (as initial states) and right movers (as final states) respectively denoted by

$$T_k^{(\pm)} = e^{ikX + (-\sqrt{2} + |k|)\phi},$$  \hfill (2.2)

where $\pm = \text{sign } k$.

States of the matrix model can be seen to be in close correspondence. In particular, of (1.2) half of the states have a scattering interpretation as

$$B_{-k}^{(-)} |0\rangle = \text{Tr} (P - M)^{-ik} |k; \text{in}\rangle,$$

$$B_k^{(+)} |0\rangle = \text{Tr} (P + M)^{ik} |k; \text{out}\rangle.$$  \hfill (2.3)

This physical interpretation will arise once the spatial (Liouville) coordinate is identified. This was understood to be related to the eigenvalue index of the matrix variable. The physical world is the positive real axis with a barrier at the origin, and so one only considers an in state that is left moving and an out state that is right moving.
The identification of physical states and of the extra Liouville momentum is seen in a transition to the collective field theory language [3]. This transition can be summarized [3-6] by the following set of replacement rules:

\[ M \rightarrow x, \]
\[ P \rightarrow \alpha(x,t), \]
\[ \text{Tr} \rightarrow \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha. \]  

The matrix hamiltonian then becomes

\[ H = \frac{1}{6} \int \frac{dx}{2\pi} \left( \alpha_+^3 - \alpha_-^3 \right) - \frac{1}{2} \int \frac{dx}{2\pi} x^2 \left( \alpha_+ - \alpha_- \right), \]  

(2.5)

describing a scalar field \( \phi(x,t) \) and its conjugate \( \Pi(x,t) \), with \( \alpha_{\pm} = \partial_x \Pi \pm \pi \phi \).

The collective representation exhibits in addition to the time \( t \) a spatial dimension \( x \). One has a classical background field \( \pi \phi_0 = \sqrt{x^2 - 2\mu} \), which induces a reparametrization of the new spatial coordinate to \( \tau = \int \frac{dx}{\pi \phi_0(x)} \), or

\[ x(\tau) = \sqrt{2\mu} \cosh(\tau), \]
\[ \pi \phi_0(\tau) = \sqrt{2\mu} \sinh(\tau). \]  

(2.6)

Asymptotic translations in \( \tau \) are scale transformations of \( x \) since

\[ x(\tau) \sim \sqrt{\frac{\mu}{2}} e^{\tau}. \]  

(2.7)

Indeed, in addition to time translation the collective Lagrangian transforms covariantly under scale transformations
\[ x \rightarrow \lambda x, \]
\[ \alpha(x,t) \rightarrow \frac{1}{\lambda} \alpha(\lambda x,t), \]
\[ H \rightarrow \frac{1}{\lambda^4} H. \] (2.8)

This symmetry is the origin of a second (spatial momentum) quantum number \( p_\tau \).

In linearized approximation with

\[ \phi(x) = \phi_0(x) + \partial_x \psi(x), \quad p(x) = -\partial_x \Pi(x), \]
\[ \psi(\tau) = \psi(x), \quad p(\tau) = \pi \phi_0 p(x), \]
\[ \alpha_\pm(x) = \pm \pi \phi_0 + \frac{1}{\pi \phi_0} \tilde{\alpha}_\pm(\tau). \] (2.9)

one has right-left moving massless modes (tachyons)

\[ \tilde{\alpha}_\pm(\tau, t) = f(t \mp \tau) = \pm \int_{-\infty}^{\infty} dk \alpha_k^\pm e^{-ik(t \mp \tau)}, \] (2.10)

satisfying

\[ (\partial_t \mp \partial_\tau) \tilde{\alpha}_\pm = 0. \] (2.11)

with the energy momentum values

\[ \alpha_{\pm k}^\pm : \quad p_0 = k, \quad p_\tau = \pm k. \] (2.12)

The exact states of the matrix model are directly translated into the field theoretic representation. We have as exact tachyon eigenstates
\[ T_n^{(\pm)} = \int \frac{dx}{2\pi} \int d\alpha (\alpha \pm x)^n = \int \frac{dx}{2\pi} \frac{(\alpha \pm x)^{n+1}}{n+1}, \quad (2.13) \]

introduced by Avan and one of the authors in [6]. Using the Poisson brackets \( \{\alpha(x), \alpha(y)\} = 2\pi \delta'(x - y) \) one easily shows

\[ \{H, T_n^{(\pm)}\} = \pm n T_n^{(\pm)} \quad (2.14) \]

and one has eigenstates with \( ip_0 = \pm n \). Defining

\[ p_\tau = \text{scale dimension} - 4, \quad (2.15) \]

one has

\[ p_\tau = -2 + n. \]

These states stand in comparison with the vertex operators of conformal field theory

\[ T_p^{(\pm)} \equiv e^{ip X + (-\sqrt{2}+|p|)\varphi} \leftrightarrow \int \frac{dx}{2\pi} \frac{(\alpha \pm x)^{n+1}}{n+1} \leftrightarrow \text{Tr} (P \pm M)^n. \quad (2.16) \]

The tachyon vertex operators with opposite (Liouville) dressing correspond to singular operators in the matrix model
\[ e^{ipX+(2-|p|)\varphi} \leftrightarrow \int \frac{dx}{2\pi} \frac{(\alpha \pm x)^{1-n}}{1-n} \leftrightarrow \text{Tr} (P \pm M)^{-n}. \] (2.17)

We have now described a one to one correspondence between the matrix model states and string states. Scattering amplitudes can be derived immediately once this correspondence is understood.

We note that the collective field theory seemingly introduces a degeneracy. For each state of the matrix model one can define two states in collective field theory since we can replace \( P \rightarrow \alpha \pm (x, t) \). Each of the separate fields \( \alpha_+ \) or \( \alpha_- \) can be used to define states with the above quantum numbers. In particular

\[ \int \frac{dx}{2\pi} \frac{(\alpha_+ \pm x)^{1\pm ik}}{1 \pm ik} \]

and

\[ \int \frac{dx}{2\pi} \frac{(\alpha_- \pm x)^{1\pm ik}}{1 \pm ik} \]

both have the same quantum numbers

\[ p_0 = k, \quad p_\tau = -2 \pm ik. \]

These have to be identified, up to a phase factor. It can be shown (below) that boundary conditions fix the phase factor to be \(-1\). So one has

\[ \int \frac{dx}{2\pi} \frac{(\alpha_+ \pm x)^{1\pm ik}}{1 \mp ik} = -\int \frac{dx}{2\pi} \frac{(\alpha_\pm \pm x)^{1\pm ik}}{1 \mp ik}, \] (2.18)

implying a nonlinear relation between left and right movers. This equation, which
follows from simple kinematical reasoning, determines the complete tree level scattering amplitude. Expanding
\[
\alpha_{\pm}(x) = \pm x \mp \frac{1}{x} (\mu \mp \hat{\alpha}_{\pm}(\tau)) + \frac{1}{x^2} \text{ terms,} \tag{2.19}
\]
we shall find the relation
\[
\int_{-\infty}^{\infty} d\tau e^{\pm ik\tau} \frac{\hat{\alpha}_{\pm}}{\mu} = -\int_{-\infty}^{\infty} \frac{d\tau}{ik \pm 1} e^{\mp ik\tau} \left[ \left[ 1 + \frac{\hat{\alpha}_{\mp}}{\mu} \right]^{ik\pm 1} - 1 \right]. \tag{2.20}
\]
This functional equation relating left and right moving waves of the collective field was shown to represent a solution to the scattering problem in [8]. Here we exhibited how this nonlinear scattering equation emerges directly from the exact oscillator states. The fact that the left and the right hand side of the equation are interpreted as eigenstates of collective field theory implies also the following: a complete quantization procedure was given [4] for the field theory Hamiltonian, involving normal ordering and the subtraction of counterterms. The same procedure can be applied to the states and will lead to a fully quantum version of the scattering equation.

The main ingredient in obtaining the scattering equation are the proper boundary conditions. Let us now elaborate on this question. The issue of boundary conditions is of paramount importance in a correct treatment of the spectrum within the collective approach. In QCD-like unitary matrix models, it is well known that as the system moves from a strong coupling regime to a weak coupling regime where the classical density of states $\phi_0$ has only finite support, Dirichlet boundary conditions must be imposed on the shifted field $\psi(\tau)$. This is essentially due to the fact that $\phi_0(\tau = 0) = \phi_0(\tau = L \to \infty) = 0$, and in this way the time independence of the original constraint condition $\int dx \phi = N$ is preserved [3]. For $c = 1$ strings, this “constraint” equation determines the value of the cosmological constant. Therefore, apart from problems of consistency, a choice other than Dirichlet
boundary conditions would result in a time dependent cosmological constant. Notice that this implies that in a density variable description of “wall” scattering, the “wall” at $\tau = 0$ is rigid. A creation-annihilation basis that automatically enforces Dirichlet boundary conditions on the scalar field $\psi$ is defined by the expansion

$$\tilde{\alpha}_\pm(\tau) = \pm \int_{-\infty}^{\infty} \frac{dk}{\sqrt{|k|}} e^{\pm i k \tau} a_k, \quad a_{-k} \equiv a_k^\dagger,$$

$$(2.21)$$

We could equally well have chosen the “left-right” basis (2.10). Once one expresses a scalar theory with fields satisfying boundary conditions in a left-right basis, there is a standard problem, also present in the critical open string: the functions $e^{i k \tau}$ are not orthogonal over the half line, and therefore the computation of Fourier coefficients require some modification. To this standard problem there is a standard solution [12]: one notices that the definitions of all the fields in (2.10) naturally extend to negative values of $\tau$. Therefore we extend the definition of the fields from $0 \leq \tau < \infty$ to $-\infty < \tau < \infty$ by requiring

$$\psi(-\tau) = -\psi(\tau),$$

$$\tilde{\alpha}_\pm(-\tau) = -\tilde{\alpha}_\mp(\tau).$$

$$(2.22)$$

In other words, the fields of interest to us are the fields defined on the full line which are odd (in coordinate free form) under the involution $\tau \to -\tau$. This point of view has been extensively used in works relating critical open string amplitudes to those of the closed string [13]. One can then compute Fourier coefficients of $\tilde{\alpha}_+$, say:

$$\int_{0}^{\infty} \frac{d\tau}{2\pi} e^{\pm i k \tau} \tilde{\alpha}_\pm(\tau) - \int_{0}^{\infty} \frac{d\tau}{2\pi} e^{\pm i k \tau} \tilde{\alpha}_\mp(\tau) = \pm \alpha_k.$$ 

$$(2.23)$$

We can now reformulate the problem as follows: suppose we introduce the arbitrary
left-right expansion (2.10). Equation (2.22) is then equivalent to

\[ \alpha_k^- = -\alpha_k^+. \quad (2.24) \]

Physically, this simply means that in order to preserve the boundary conditions of the system, if a right mover is created then a left mover must also be created with amplitude minus one, and similarly for annihilation operators. This means that the Dirichlet boundary conditions cause the left and right movers to combine into standing waves, which are perturbative tachyon states in the matrix model.

Now, in terms of the matrix variables (2.21) described above, this condition is immediately built into the expansion of the fields. However, for asymptotic incoming and outgoing states, which are naturally defined on the full line, the analogue of condition (2.24), imposed on the the exact states of the system, leads to the nonlinear scattering matrix.

We now concentrate on \( T_{ik}^{(+)} \) and introduce the following notation to represent the two degenerate states described previously

\[
T_{ik}^{(+)} = \int \frac{dx}{2\pi} \int_{\alpha^-}^{\alpha^+} d\alpha (\alpha + x)^{ik} \equiv T_{ik}^{+} - T_{ik}^{(-)}
\]

\[
= \int \frac{dx}{2\pi} \frac{\alpha_{\pm} + x}{ik + 1}^{ik+1} - \int \frac{dx}{2\pi} \frac{\alpha_{\pm} + x}{ik + 1}^{ik+1}.
\]

The equation relating left and right moving fields reads
The range of integration has been extended as described above equation (2.23). This is a restatement of equation (2.18). Since the c-number contributions $C_{\pm}^{(+)}$ to the above operators are the same, we rewrite this condition as

\[
T_{ik,+}^{(+)} = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \, \frac{\pi \phi_0}{2 \pi i k + 1} \left[ \left( \pi \phi_0 + x \right) + \frac{\alpha_-(\tau)}{\pi \phi_0} \right]^{ik+1} - x^{ik+1} + 1
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} d\tau \, \frac{\pi \phi_0}{2 \pi i k + 1} \left[ \left( -\pi \phi_0 + x \right) + \frac{\alpha_-(\tau)}{\pi \phi_0} \right]^{ik+1} - x^{ik+1}
\]

\[
= -T_{ik,-}^{(+)}. \tag{2.26}
\]

This equality is a necessary consequence of Dirichlet boundary conditions. To linear order, it is straightforward to show that equation (2.27) is equivalent to equation (2.24).

As $x \gg \sqrt{2\mu}$ we wish to express this condition in terms of the asymptotic fields $\hat{\alpha}_\pm$ defined in equation (2.19), using the asymptotic behaviour (2.7). We remind ourselves that $\hat{\alpha}_-(t + \tau)$ is an incoming left-moving wave and $\hat{\alpha}_+(t - \tau)$ is the outgoing, right-moving wall scattered wave. In the asymptotic description the fact that, from the collective field theory point of view, the “wall” at $\tau = 0$ is rigid, is not immediately built into the definition of the fields. This condition has to be imposed on the exact states of the system, i.e., equation (2.27) must be satisfied. Expressing the exact states in terms of the variables (2.19) we get

\[
T_{ik,+}^{(+)} - C_+^{(+)} = - \left( T_{ik,-}^{(+)} - C_-^{(+)} \right). \tag{2.27}
\]
\[ T_{ik,+}^{(+)} - C_+^{(+)} = \frac{1}{ik + 1} \int \frac{dx}{2\pi} (\alpha_+ + x)^{ik+1} - (\pi\phi_0 + x)^{ik+1} \]
\[ = \frac{\sqrt{2\mu}}{ik + 1} \int_{-\infty}^{\infty} \frac{d\tau}{8\pi} e^{ik\tau} e^{2\tau} \]
\[ \left[ \sum_{j=0}^\infty \frac{\Gamma(ik + 2)}{\Gamma(ik + 2 - j) j!} (-)^j e^{-2j\tau} \left\{ \left( 1 - \frac{\hat{\alpha}_+}{\mu} \right)^j - 1 \right\} \right], \tag{2.28} \]
\[ - (T_{ik,-}^{(+)} - C_-^{(+)} ) = - \frac{1}{ik + 1} \int \frac{dx}{2\pi} (\alpha_- + x)^{ik+1} - (-\pi\phi_0 + x)^{ik+1} \]
\[ = - \frac{\sqrt{2\mu}}{ik + 1} \int_{-\infty}^{\infty} \frac{d\tau}{8\pi} e^{-ik\tau} \sum_{p=1}^\infty \frac{\Gamma(ik + 2)}{\Gamma(ik + 2 - p) p!} \left( \frac{\hat{\alpha}_-}{\mu} \right)^p. \tag{2.29} \]

Equating these expressions as required by the condition (2.27), and applying partial integrations and a Fourier transform, we obtain

\[ \sum_{j=0}^\infty e^{-2(j-1)\tau} \frac{\Gamma(-\partial + 2)}{\Gamma(-\partial + 2 - j) j!} (-\tau)^j e^{-2j\tau} \left\{ \left( 1 - \frac{\hat{\alpha}_+}{\mu} \right)^j - 1 \right\} \]
\[ = - \sum_{p=1}^\infty \frac{\Gamma(\partial + 2)}{\Gamma(\partial + 2 - p) p!} \left( \frac{\hat{\alpha}_-(-\tau)}{\mu} \right)^p. \tag{2.30} \]

We can now extract the asymptotic limit by letting \( \tau \to \infty \). We find that on the left hand side only the \( j = 1 \) term contributes (lower order terms would correspond, on the right hand side, to terms dropped in the asymptotic definition (2.7)). As \( \tau \to \infty \)

\[ (-\partial + 1) \frac{\hat{\alpha}_+}{\mu} \]
\[ = - \sum_{p=1}^\infty \frac{\Gamma(-\partial + 2)}{\Gamma(-\partial + 2 - p) p!} \left( \frac{\hat{\alpha}_-}{\mu} \right)^p, \tag{2.31} \]
where \( \bar{\alpha}_-(\tau) \equiv \hat{\alpha}_-(-\tau) \). It follows that

\[
\alpha_+(\tau) = -\sum_{p=1}^{\infty} \frac{\Gamma(-\partial + 1)}{\Gamma(-\partial + 2 - p) p!} \left( \frac{1}{\mu} \right)^{p-1} \bar{\alpha}_-(\tau)^p. \tag{2.32}
\]

This relation expressing left moving fields in terms of right moving ones is the result (2.20) for the scattering problem [8].

3. Symmetries

The spacetime field theory given by the collective field exhibits a large \( (W_\infty) \) spacetime symmetry of 2-dimensional string theory [6]. The generators of this symmetry can be directly found or simply induced from the matrix model. There one has the invariant operators

\[
\text{Tr} \left( P^r M^s \right), \tag{3.1}
\]

which are closed under commutation. The field theory operators read

\[
H_m^m = \int \frac{dx}{2\pi} \frac{\alpha_+^{m-n}}{m-n} x^{m-1} \tag{3.2}
\]

and can be shown to satisfy the \( w_\infty \) algebra

\[
[H_{m_1}^{n_1}, H_{m_2}^{n_2}] = i \left[ (m_2 - 1)n_1 - (m_1 - 1)n_2 \right] H_{m_1+m_2-2}^{n_1+n_2}. \tag{3.3}
\]

Of particular relevance to us are the spectrum generating operators

\[
O_{JM} \equiv \text{Tr} \left( P + M \right)^{J-M} \left( P - M \right)^{J+M}, \tag{3.4}
\]

which become, in the collective field theory representation
\[
O_{JM} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \, (\alpha + x)^{J+M+1} (\alpha - x)^{J-M-1}.
\] (3.5)

One sees that these are linear combinations of the basic \(w_\infty\) operators

\[
O_{JM} = H_1^{-2J-2} + 2M H_2^{-2j} + (2M^2 - J - 1) H_3^{2J+2} + \ldots
\] (3.6)

and it follows that the spectrum generating algebra is precisely a \(w_\infty\), i.e.,

\[
[O_{J_1,M_1}, O_{J_2,M_2}] = i [(M_2 - 1) J_1 - (M_1 - 1) J_2] O_{J_1+J_2,M_1+M_2}.
\] (3.7)

This can also be shown directly from (3.5) by doing partial integrations [6].

There is a close connection between these \(w_\infty\) operators and the operators describing exact tachyon states of the field theory. Recall the one parameter family of operators

\[
T_n^{(\pm)} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \, (\alpha \pm x)^n.
\] (3.8)

Their commutators give the generators of the \(w_\infty\) algebra, i.e., one can show that

\[
O_{JM} = \frac{1}{2i (J - M + 2) (J + M + 2)} [T_{J+M+2}^+, T_{-J-M+2}^-].
\] (3.9)

The symmetry generators were also written down in the conformal field theory approach [10]. There is a close parallel with all of the matrix model relationships and the commutators are simply replaced by operator products. The above implies
for example that the $w_\infty$ generators are obtained as operator products of basic tachyon vertex operators. A closer correspondence is seen by comparing the above field theory forms with the representations deduced for the action of the symmetry generators on the tachyon module [11].

3.1. Fourier Expansion

We now consider the spectrum generating operators $O_{JM}$ of (3.5) in more detail. We shall see that the correspondence with the conformal field theory results of [11] will then follow. To expand in terms of creation-annihilation operators we make the substitutions

$$\alpha_+ = x + \bar{\alpha}_+, \quad \alpha_- = -x + \bar{\alpha}_- \quad (3.10)$$

in the spectrum generating operators (3.5). Applying partial integration to (3.5) and inserting the limits (3.10), one finds

$$O_{JM} = \int \frac{dx}{2\pi} \sum_{k=0}^{J+M+1} (-)^k \frac{(J-M+1)! (J+M+1)!}{(J-M+2+k)! (J+M+1-k)!} \times$$

$$\times \{ \bar{\alpha}_+^{J-M+2+k} (\bar{\alpha}_+ + 2x)^{J+M+1-k}$$

$$- (\bar{\alpha}_- - 2x)^{J-M+1+k} \bar{\alpha}_-^{J+M+2-k} \} \quad (3.11)$$

The leading term in $\bar{\alpha}_+$ is of order $\bar{\alpha}_+^{J-M+2}$. The leading term in $\bar{\alpha}_-$ seems to be linear in $\bar{\alpha}_-$. However, this is not true, as careful consideration shows that there are two terms linear in $\bar{\alpha}_-$ which cancel. One might expect that in general something similar happens also for higher order terms in $\bar{\alpha}_-$. This indeed turns out to be the case. The easiest way to see this, is to do the partial integration of (3.5) in the other “direction”. One finds
The terms in $\bar{\alpha}_+$ and $\bar{\alpha}_-$ in (3.11) and (3.12) must separately be equal, up to c-number terms of the form $\int \frac{dx}{2\pi} x^{2J+3}$. Substituting the change of variables (2.7), this becomes $\sim \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{(2J+3)\tau}$, which can in general be argued to vanish after an analytic continuation $\tau \to i\tau$ (see below). It therefore follows that we can write the expansion

\begin{equation}
O_{JM} = \int \frac{dx}{2\pi} \frac{J-M+1}{2\pi} \sum_{k=0}^{J-M+1} (-)^{k} \frac{(J + M + 1)! (J - M + 1)!}{(J - M + 2 + k)! (J - M + 1 - k)!} \times \left\{ \bar{\alpha}_+^{J-M+1-k} (\bar{\alpha}_+ + 2x)^{J+M+2+k} \right. \\
- \left. -(\bar{\alpha}_- - 2x)^{J-M+1-k} \bar{\alpha}_-^{J+M+2+k} \right\}.
\end{equation}

\section{3.13}

Thus to lowest order in the fields, one finds

\begin{equation}
O_{JM} = \frac{1}{J-M+2} \int \frac{dx}{2\pi} (2x)^{J+M+1} \bar{\alpha}_+^{J-M+2} \\
- \frac{1}{J+M+2} \int \frac{dx}{2\pi} (2x)^{J+M+1} \bar{\alpha}_-^{J+M+2}.
\end{equation}

\section{3.14}

Now, applying the change of variables (2.7), \textit{i.e.,}
\[
x = \sqrt{\frac{\mu}{2}} e^{\tau},
\]
\[
\bar{\alpha}_\pm \rightarrow \frac{d\tau}{dx} \bar{\alpha}_\pm,
\]

one finds that the leading order expression for the charges is given by

\[
O_{JM} = \frac{2^{J+1} \mu^M}{J-M+2} \int \frac{d\tau}{2\pi} e^{2M\tau} \bar{\alpha}_+^{J-M+2} - (-)^{J-M+1} \frac{2^{J+1} \mu^{-M}}{J+M+2} \int \frac{d\tau}{2\pi} e^{-2M\tau} \bar{\alpha}_-^{J+M+2}.
\] (3.16)

Expanding in right and left moving modes

\[
\bar{\alpha}_+ = \int dk \bar{\alpha}(k) e^{-ik(t-\tau)},
\]
\[
\bar{\alpha}_- = \int dk \bar{\beta}(k) e^{-ik(t+\tau)}
\] (3.17)

and applying the rotation \( \tau \rightarrow i\tau, k \rightarrow -ik \), we find that in terms of the analyti-

\[
\alpha(k) \equiv \bar{\alpha}(-ik),
\]
\[
\beta(k) \equiv \bar{\beta}(-ik)
\] (3.18)

the charges have the form

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\[ O_{JM} = \frac{2^{J+1} \mu^M}{J - M + 2} i \int dk_1 \ldots dk_{J-M+2} \times \]
\[ \times \alpha(k_1) \ldots \alpha(k_{J-M+2}) \delta \left( \sum k_i + 2M \right) \]
\[ - (-)^{J-M+1} \frac{2^{J+1} \mu^{-M}}{J + M + 2} i \int dp_1 \ldots dp_{J+M+2} \times \]
\[ \times \beta(p_1) \ldots \beta(p_{J+M+2}) \delta \left( \sum p_i + 2M \right). \]

(3.19)

We emphasize that this is the expression for the charges to lowest order in the fields, which corresponds to the leading order in \( \mu \). The full expression (3.13) has corrections in \( 1/\mu \) that are higher order polynomials in the fields. In the remainder of the discussion we do not consider these corrections.

Defining

\[ a(k) \equiv \alpha(k), \quad b(p) \equiv \beta(p), \]
\[ a^\dagger(k) \equiv \alpha(-k)/k, \quad b^\dagger(p) \equiv \beta(-p)/p \]

satisfying \([a(k), a^\dagger(k')] = \delta(k-k')\), \([b(p), b^\dagger(p')] = \delta(p-p')\), we have the expressions of [11] (up to an inessential difference in normalization), plus additional contributions. To see these, note that in addition to the term

\[ 2^{J+1} \mu^M \int_0^\infty dk \int_0^\infty dk_1 \ldots dk_{J-M+1} \times \]
\[ \times k a^\dagger(k) a(k_1) \ldots a(k_{J-M+1}) \delta \left( \sum k_i - k + 2M \right) \]

(3.21)

found in [11], we in general also have terms of higher order in the creation operators. The next term would be, for example
$$2^{J+1} \mu^M (J - M + 1) i \int_0^\infty dk \int_0^\infty dk' \int_0^\infty dk_1 \ldots dk_{J-M} \times$$

$$\times kk' a^\dagger(k) a^\dagger(k') a(k_1) \ldots a(k_{J-M}) \delta \left( \sum k_i - k - k' + 2M \right). \quad (3.22)$$

If $M < 0$, we also get an additional contribution of the form

$$\frac{2^{J+1} \mu^M}{J - M + 2} i \int_0^\infty dk_1 \ldots dk_{J-M+2} \times$$

$$\times a(k_1) \ldots a(k_{J-M+2}) \delta \left( \sum k_i + 2M \right) . \quad (3.23)$$

These additional contributions have to be included in order to obtain a representation of the algebra (3.7). The reason for this is that terms of the type (3.22), commuted with terms of the type (3.23), give additional contributions of the type (3.21), which are needed to again obtain a member of the algebra on the right hand side. This effect cannot be produced by only using terms of the type (3.21). To see where the representation (3.21) fails, one has to take careful account of the regions of momentum integration. For example, if one were to use only terms of the type (3.21) one would find for the commutator

$$\left[ O_{MM}, O_{\frac{J}{2}, -\frac{J}{2}} \right] = \int_0^\infty dk_1 dk_2 (-2k_1 - 2k_2 - 4M) (k_1 + k_2 + M - \frac{1}{2}) \times$$

$$\times a^\dagger(k_1 + k_2 + M - \frac{1}{2}) a(k_1) a(k_2)$$

$$+ \int_{0, k_1 + k_2 > \frac{1}{2}}^\infty dk_1 dk_2 (2k_1 + 2k_2 - 1) (k_1 + k_2 + M - \frac{1}{2}) \times$$

$$\times a^\dagger(k_1 + k_2 + M - \frac{1}{2}) a(k_1) a(k_2). \quad (3.24)$$
which would give $(-4M - 1) O_{M + \frac{1}{2}, M - \frac{1}{2}}$, were it not for the fact that the regions of integration do not match. It is now not difficult to see how to fix the representation (3.21). Simply remove the restrictions on the ranges of integration, i.e., take them to be $\int_{-\infty}^{\infty} dk$ instead of $\int_{0}^{\infty} dk$. This solves the problem on a formal level, and imposing the reality conditions $a_{-n} = n a_{n}^\dagger \equiv \alpha_{-n}$, we recover our full representation (3.19).

One can now ask whether the Ward identities derived in [11] for the tachyon scattering amplitudes will be affected by these corrections. As we will show in the next section, they will not be affected.

### 3.2. Ward Identities

One can now identify the spectrum generating operators as we did for the tachyons by comparing quantum numbers as in (2.18), or alternatively, by imposing Dirichlet boundary conditions as in (2.26). One simply requires

$$O_{JM,+} = -O_{JM,-},$$

which implies that to leading order

$$O_{JM} = 2i \frac{2^{J+1} \mu^M}{J - M + 2} \int dk_1 \cdots dk_{J-M+2} \times$$
$$\times \alpha(k_1) \cdots \alpha(k_{J-M+2}) \delta \left( \sum k_i + 2M \right)$$
$$= 2i (-)^{J-M+1} \frac{2^{J+1} \mu^{-M}}{J + M + 2} \int dp_1 \cdots dp_{J+M+2} \times$$
$$\times \beta(p_1) \cdots \beta(p_{J+M+2}) \delta \left( \sum p_i + 2M \right).$$

This identification will, in practice, be very useful in explicit calculations of Ward identities, as will be seen below.
The “bulk” scattering amplitudes only involve fixed, discrete values of the outgoing momenta. This can be interpreted in our formalism as follows: Imposing the above identification of quantum numbers, one has

\[
T^{(-)}_{2M} \equiv O_{M-1,M} = 2i (-)^{2M+1} \left( \frac{2}{\mu} \right)^M \beta(2M) \\
= 2i (2\mu)^M \int dk_1 \ldots dk_{2M+1} \alpha(k_1) \ldots \alpha(k_{2M+1}) \delta\left( \sum k_i - 2M \right).
\]

(3.27)

Thus, in terms of the oscillators \( \alpha(k) \) and \( \beta(p) \) defined in (3.10) and (3.18), we find an S-matrix that is different from the one we previously calculated in terms of the “asymptotic” variables (2.7). In particular, to leading order an out state \( \langle 0 | \beta(2M) \rangle \) is \( (2M+1) \)-linear in \( \alpha \), so that a correlation function \( \langle \beta(p) \alpha(k_1) \ldots \alpha(k_N) \rangle \) can only be nonvanishing to this order if

\[
p = N - 1.
\]

(3.28)

Except for an overall factor of \( \frac{1}{2} \), due to different normalization of the momentum, this agrees with the “sum rule” stated in [11].

Now, to see how the Ward identities can be derived in our formalism, note that if one has an operator \( O \) that annihilates the vacuum from the left and the right, i.e.,

\[
\langle 0 | O = 0 = O | 0 \rangle,
\]

(3.29)

then, starting from the expectation value

\[
\langle \beta(p) O \alpha(k_1) \ldots \alpha(k_N) \rangle
\]

(3.30)

one can write, commuting \( O \) respectively to the left and to the right.
\[ \langle [\beta(p), O] \alpha(k_1) \ldots \alpha(k_N) \rangle = \langle \beta(p) [O, \alpha(k_1) \ldots \alpha(k_N)] \rangle. \] (3.31)

This equation expresses the Ward identities, and for suitable choices of the charges \(O\), can be used to derive recursion relations relating scattering amplitudes.

In general, however, our representation (3.19) of the charges \(O_{JM}\) have terms of the type \(aa\ldots a\) or \(a\dagger a\dagger \ldots a\dagger\), and therefore would fail to annihilate the vacuum from either the left or the right. Also, in addition to the terms (3.21), which were considered in the analysis of [11], one might expect corrections to the Ward identities derived in [11] due to our extra terms such as (3.22) and (3.23). However, counting numbers of creation and annihilation operators, one sees that indeed only terms of the type (3.21) contribute to the Ward identities. For example, consider the Ward identity relating \(N + 1 \rightarrow 1\) amplitudes to \(N \rightarrow 1\) amplitudes. The relevant charge is \(O_{\frac{N}{2}, -\frac{N}{2}} \sim aaa + a\dagger aa + a\dagger a\dagger a\), and we should take the momentum of the out state to be \(p = N - 1\). The out state is then, from our previous discussion, of order

\[ \langle 0 | \beta(p) \sim \langle 0|a^N, \] (3.32)

so that one can write (3.30) as

\[ \langle \beta(p) O_{\frac{N}{2}, -\frac{N}{2}} \alpha(k_1) \ldots \alpha(k_{N+1}) \rangle \sim \langle a^N (aaa + a\dagger aa + a\dagger a\dagger a) (a\dagger)^{N+1} \rangle. \] (3.33)

It immediately follows that only the term linear in \(a\dagger\) contributes. This argument generalizes to the identity for expressing \(N \rightarrow 1\) amplitudes directly in terms of \(2 \rightarrow 1\) amplitudes, where the relevant charge is \(Q_{N/2-1, -N/2-1}\). The conclusion is therefore that for the purpose of deriving these Ward identities, it is sufficient to consider only the terms (3.21) linear in the creation operators, as was done in [11].
Finally, as an example, we calculate the Ward identity relating $3 \to 1$ amplitudes to $2 \to 1$ amplitudes. Using the identification (3.26), we have

$$Q_{\frac{3}{2}, -\frac{1}{2}} = \frac{4\sqrt{2}i}{\sqrt{\mu}} \int_0^\infty dk_1 dk_2 dk_3 \ k_1 a^\dagger(k_1) a(k_2) a(k_3) \delta(-k_1 + k_2 + k_3 - 1)$$

$$= 4\sqrt{2\mu}i \int_0^\infty dp_1 dp_2 \ p_1 b^\dagger(p_1) b(p_2) \delta(-p_1 + p_2 - 1).$$

(3.34)

up to terms that we have argued to be irrelevant. Inserting this into the general formula (3.31), for $p = 1$ and $k_1 + k_2 + k_3 = 2$, we obtain the Ward identity

$$\langle b(2) a^\dagger(k_1) a^\dagger(k_2) a^\dagger(k_3) \rangle = \frac{1}{\mu} (k_1 + k_2 - 1) \langle b(1) a^\dagger(k_1 + k_2 - 1) a^\dagger(k_3) \rangle + \text{cyclic.}$$

It is also possible to derive recursion relations in our formalism using methods similar to those used in [11]. The argument roughly goes as follows: The operators

$$O_{NN} \equiv \int dx \int d\alpha (\alpha + x)^{2N+1}(\alpha - x)$$

have quantum numbers $p_x = N = p_\tau$, while $T_k$ has $p_x = k$, $p_\tau = -1 + k$. Adding these, it follows that the commutator $[O_{NN}, T_k]$ has quantum numbers $p_x = N+k$, $p_\tau = -1 + (n + k)$, and should therefore be identified with $T_{k+N}$, i.e.,

$$[O_{NN}, T_k] \sim T_{k+n}. \quad (3.35)$$

Similarly, the charge $O_{N+M,M}$ has quantum numbers $p_x = M$, $p_\tau = N + M$. Thus it follows that $[O_{N+M,M}, T_{k_1}, \ldots T_{k_{N+1}}]$ has quantum numbers $p_x = M + \sum k_i$, $p_\tau = -1 + (m + \sum k_i)$, so that we have to identify
\[
\left[ O_{N+M,M}, T_{k_1} \ldots T_{k_{N+1}} \right] \sim T_{M+\sum_{i=1}^{N+1} k_i}.
\] (3.36)

In conclusion, we stress that the matrix model \( w_\infty \) generators, when expanded, give the conformal field theory expressions of [11] and the associated bulk Ward identities. But in addition they also contain higher corrections in \( 1/\mu \), which could be used to derive improved Ward identities which would give the full amplitudes.
REFERENCES

1. D. J. Gross and A. A. Migdal, Phys. Rev. Lett. 64 (1990) 127; M. R. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635; E. Brézin and V. Kazakov, Phys. Lett. B236 (1990) 144.

2. D. J. Gross and N. Miljković, Phys. Lett. B238 (1990) 217; E. Brézin, V. A. Kazakov and A. B. Zamolodchikov, Nucl.Phys. B338 (1990) 673; P. Ginsparg and J. Zinn-Justin, Phys. Lett. B240 (1990) 333; D. J. Gross and I. R. Klebanov, Nucl. Phys. B344 (1990) 475.

3. S. R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639; A. Jevicki and B. Sakita, Nucl.Phys. B165 (1980) 511.

4. K. Demeterfi, A. Jevicki and J. P. Rodrigues, Nucl.Phys. B362 (1991) 173; B365 (1991) 499; Mod. Phys. Lett. A35 (1991) 3199.

5. J. Polchinski, Nucl.Phys. B362 (1991) 25.

6. J. Avan and A. Jevicki, Phys. Lett. B266 (1991) 35; B272 (1992) 17; D. Minic, J. Polchinski and Z. Yang, Nucl. Phys. B369 (1992) 324; M. Awada, S. J. Sin; UFIFT (Florida)-HEP 90-33 and 91-03; G. Moore, N. Seiberg; Int. J. Mod. Phys. A7 (1992) 2601; S. R. Das, A. Dhar, G. Mandal and S. Wadia; Mod. Phys. Lett. A7 (1992) 71; U. H. Danielsson, Princeton Preprint PUPT-1199 (1992).

7. D. Gross and I. Klebanov, Nucl. Phys. B352 (1991) 671; A. M. Sengupta and S. Wadia, Int. J. Mod. Phys. A6 (1991) 1961; G. Moore, Nucl. Phys. B368 (1992) 557.

8. G. Moore and R. Plesser, “Classical Scattering in 1+1 Dimensional String Theory”, Yale preprint YCTP-P7-92, March 1992.

9. A. M. Polyakov, Mod. Phys. Lett. A6 (1991) 635; Preprint PUPT (Princeton) -1289 (Lectures given at 1991 Jerusalem Winter School); D. Kutasov, “Some Properties of (non) Critical Strings”, PUPT-1272, 1991.
10. E. Witten, *Nucl. Phys.* **B373** (1992) 187; I. Klebanov and A. M. Polyakov, *Mod. Phys. Lett.* **A6** (1991) 3273; N. Sakai and Y. Taniì, *Prog. Theor. Phys.* **86** (1991) 547; Y. Matsumura, N. Sakai and Y. Taniì, TIT (Tokyo) -HEEP 127, 186 (1992).

11. I. R. Klebanov, “Ward Identities in Two-Dimensional String Theory”, PUPT-1302 (1991); D. Kutasov, E. Martinec and N. Seiberg, PUPT-1293, RU-31-43.

12. M. Green, J. Schwarz and E. Witten, “Superstring Theory”, Volume 1, Cambridge University Press, (1987), p.72.

13. A. Morozov and A. Rosly, *Phys. Lett.* **B195** (1987) 554; J. P. Rodrigues, *Phys. Lett.* **B202** (1988) 227; S. K. Blau et al., *Nucl. Phys.* **B301** (1988) 285; W. de Beer, A. J. van Tonder and J. P. Rodrigues, *Phys. Lett.* **B248** (1990) 67.