Two-particle wave function in four dimensional Ising model

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An exploratory study of two-particle wave function is carried out with a four dimensional simple model. The wave functions not only for two-particle ground and first excited states but also for an unstable state are calculated from three- and four-point functions using the diagonalization method suggested by Lüscher and Wolff. The scattering phase shift is evaluated from these wave functions.

1. Introduction

Calculation of the scattering phase shift is important to understand scatterings and decays of hadrons. The phase shift in \( I = 2 \pi \pi \) scattering system \([3]\) was evaluated with the finite volume method \([12]\) derived by Lüscher. In the finite volume method two-particle wave function plays an important role, because the phase shift is extracted by the wave function.

Study of the wave function was carried out by Balog et al. \([4]\). Using two-dimensional statistical model they evaluated the phase shift from the wave function for several two-particle states. Recently Ishizuka et al. and CP-PACS collaboration studied the wave function in \( I = 2 \pi \pi \) scattering system \([5]\). They extracted the scattering length and scattering effective potential from the wave function for the ground state. Here calculation of the wave function for the first excited and an unstable state is attempted with a four dimensional simple Ising model. It is also aimed to evaluate the phase shift from these wave functions.

2. Wave function

Lüscher proved that the wave function \( \phi(\vec{r}) \) satisfies effective Shrödinger equation,

\[
(\nabla^2 + p^2)\phi(\vec{r}) = \int d^3r' U_E(r, r') \phi(r'),
\]

where \( \vec{r} \) is relative coordinate of two particles, \( p \) is the relative momentum, and \( U_E(r, r') \) is the Fourier transform of the modified Bethe-Salpeter kernel introduced in ref. \([1]\). The effective potential \( U_E(r, r') \) depends on energy \( E \) and decays exponentially in \( r \) and \( r' \). The wave function satisfies the Helmholtz equation \((\nabla^2 + p^2)\phi(\vec{r}) = 0 \) in \( r > R \). The \( R \) is the effective range where \( U_E(r, r') \) becomes sufficiently small in exterior region of \( R \). Lüscher found general solution of the Helmholtz equation in a finite volume,

\[
G(\vec{r}) = (1/L^3) \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}}/((\vec{k}^2 - p^2)^2),
\]

where \( \vec{k} = (2\pi/L) \cdot \vec{n} \) and \( \vec{n} \) is an integer vector.

We can extract \( p^2 \) by fitting \( \phi(\vec{r}) \) in \( r > R \) with \( G(\vec{r}) \), and can then evaluate the phase shift with the finite volume method.

3. Methods

A simple model, which is constructed with a lighter mass field \( \pi \) coupled to a heavier mass field \( \sigma \) with three-point coupling, is employed to calculate an unstable state. This model has been successfully used to observe a resonance by Gattringer and Lang \([6]\) in two dimensions, and by Rummukainen and Gottlieb \([7]\) in four dimensions.

Three- and four-point functions, \( F_i(\vec{r}, t) = \langle 0 | W(\vec{r}) | \mathcal{O}_i(t) - \mathcal{O}_i(t + 1) | 0 \rangle \) and \( C_{ij}(t) = \langle 0 | \mathcal{O}_i(0) | \mathcal{O}_j(t) - \mathcal{O}_j(t + 1) | 0 \rangle \) for \( i, j = 0, 1 \) and \( \sigma \), are calculated to obtain the wave function for some states. The operator \( \mathcal{O}_i(t) \) is \( \pi \pi \) operator with \( p = 0, (2\pi/L) \) (\( i = 0 \)) and \( \sigma \) operator (\( i = \sigma \)), and the subtraction \( | \mathcal{O}_i(t) - \mathcal{O}_j(t + 1) | \) is performed to eliminate the vacuum contribution.

The wave function operator \( W(\vec{r}) \) is defined by \( W(\vec{r}) = (1/L^3) \sum_{\vec{X}, \vec{R}} \pi(\vec{R}[\vec{r}] + \vec{X}) \pi(\vec{X}) \), and the \( \vec{R} \) is an element of cubic group, and summation over \( \vec{X} \) and \( \vec{R} \) projects to \( A^+ \) sector. The lattice size is \( L^3 \times T = 20^3 \times 64 \) and the number of the
configurations is $405 \times 10^3$. In this work the energy of the first excited state is larger than that of the $\sigma$ state.

The wave function is defined by $\phi_\alpha(\vec{r}) = \langle 0|\mathcal{W}(\vec{r})|\alpha\rangle$ for the $\alpha = 0, 1$ and $\sigma$ state. Using $\phi_\alpha(\vec{r})$ and the state overlap of the operator $V_{\alpha i} = \langle \alpha|O_i|0\rangle$, one has $F_i(\vec{r}, t) = \sum_\alpha \phi_\alpha(\vec{r}) \Delta_\alpha(t)V_{\alpha i}$, where $\Delta_\alpha(t) = (1 - e^{-E_\alpha})e^{-E_\alpha t} + (1 - e^{E_\alpha})e^{-E_\alpha(t-t)}$. It is also easy to show that $C_{ij}(t) = \sum_\alpha V_{i\alpha}^* \Delta_\alpha(t)V_{\alpha j}$. The assumption of this calculation is that higher energy states than the first excited state do not contribute to $C_{ij}(t)$ and $F_i(\vec{r}, t)$. To obtain the wave function, at first the overlaps are extracted by the diagonalization method with the correlation function matrix $C^{-1/2}(t_0)C(t)C^{-1/2}(t_0)$, where $t_0$ is reference point. The diagonalization also yields the energy $E_\alpha$. Then the wave function is extracted by the projection $\phi_\alpha(\vec{r}) = \left(\sum_i F_i(\vec{r}, t)V_{i\alpha}^{-1}\right)/\left|\sum_i F_i(\vec{r}_0, t)V_{i\alpha}^{-1}\right|$ (3) apart from the normalization, where the normalization point $r_0$ is chosen $r_0 = 1$.

4. Results

Fig. 1 displays the overlaps determined at some $t$. The $i = 0$ operator is almost dominated by the ground state, while other operators has contributions from each state. All the overlaps are stable at small $t$ region, so that $V_{i\alpha}$ is determined at $t = 1$. Using the overlaps it is possible to calculate the wave function not only for the first excited state but also for the $\sigma$ state as shown in Fig. 2. The projection eq. (3) is carried out with $F_i(\vec{r}, t)$ at $t = 3$ for $\alpha = 0, 1$, and at $t = 4$ for $\alpha = \sigma$. The function $F_i(\vec{r}, t)$ normalized at $r_0$ is also displayed in the figure for comparing ones before and after the projection. The difference between $\phi_\alpha(\vec{r})$ and $F_i(\vec{r}, t)$ for the ground state is small, while those for the first excited and $\sigma$ states are large, as expected from the operator overlaps in Fig. 1.

The effective range is required for fitting of $\phi_\alpha(\vec{r})$ in $r > R$ using eq. (2). In Ref. 4 the effective range is estimated from the quantity $(\nabla^2 \phi_\alpha(\vec{r})/\phi_\alpha(\vec{r}))$, which is approximately effective potential. However, the effective range cannot be estimated from the quantity, because the statistical noise of the quantity is very large. Hence the fit of $\phi_\alpha(\vec{r})$ is carried out by assuming the effective range $R = 7.9$ and 8 for $\alpha = 0, 1$ and $\sigma$ states. The effective range depends on the state, because $U_E(r, r')$ in eq. (1) depends on the energy. The fitting parameters are an overall constant of eq. (2) and the relative momentum $p^2$. Fig. 3 shows the fitting result $G_\alpha(r)$ for each state. The fitting results for all states are consistent with the wave functions. This figure illustrates that the wave function in $r > R$ for the
\( \sigma \) state can be described by the general solution of the Helmholtz equation. The consistency of the effective ranges is checked by the deviations of \( \phi_\alpha(r) \) from \( G_\alpha(r) \). As expected, the deviation vanishes in \( r > R \).

As shown in Fig. 4, the phase shift is evaluated using \( p^2 \) determined from the wave function. In the figure the phase shift calculated with \( p^2 \) obtained from the energy \( E_\alpha = 2\sqrt{m^2 + p^2} \) is also presented. All the phase shifts are consistent with the two methods. The results obtained from the wave function for the ground and first excited states have smaller error than those obtained from the energies. This feature is also seen in refs. [4] and [5]. The wave function result for the \( \sigma \) state, however, has larger error. The reason is not well understood at present. In order to understand this feature, more detailed investigation is needed for the wave function of the \( \sigma \) state.

5. Conclusions

Calculation of the two-particle wave function for the first excited and an unstable states is demonstrated with the diagonalization method. It is found that the wave function in \( r > R \) for an unstable state can be described by the general solution of the Helmholtz equation as same as that for two-particle states, and the phase shift is extracted from these wave functions.

The numerical calculations have been carried out on workstations at Center for Computational Sciences, University of Tsukuba.

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