Duflo-Moore Operator for The Square-Integrable Representation of the 2-Dimensional Affine Lie Group

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Abstract. In this paper, we study the quasi-regular and the irreducible unitary representation of affine Lie group Aff⁺(1) of dimension two. First, we prove a sharpening of Fuhr’s work of Fourier transform of quasi-regular representation of Aff⁺(1). The second, in such the representation of affine Lie group Aff⁺(1) is square-integrable then we compute its Duflo-Moore operator instead of using Fourier transform as in Führ’s work.

Keywords: Affine Lie group; Duflo-Moore operator; Square-integrable representation.
1. Introduction

The current research about square-integrable representations of Lie groups can be found, for instance in [1] and [2]. In the previous work, the notion of square-integrable representation of a Lie group associating to wavelet transforms was introduced by Grossmann, Morlet, and Paul (see [3]). Particularly, they investigated the nice examples of a square-integrable representation of $ax + b$-group, known as affine Lie group $\text{Aff}(1)$ as can be seen in [4]. In the other hand, the research about $ax + b$-groups can also be found, for instance in [5] and [6].

It is well known that $\text{Aff}(1)$ is the exponential solvable Lie group which is non unimodular group whose Lie algebra of $\text{Aff}(1)$ is Frobenius. Other examples are parabolic subgroups which are Frobenius as well (see [7] and[8]). But we thought that Grossmann’s work is the best example for young researchers how to understand the square-integrable representations for case nonunimodular groups which is started from the $\text{Aff}(1)$ Lie group. Moreover, other examples of nonunimodular groups are Lie groups whose Lie algebras are 4-dimensional real Frobenius Lie algebras. Kurniadi and Ishi [9] showed that irreducible unitary representations of these Lie groups are square-integrable representations and they wrote the Duflo-Moore operators in the terms of groups Fourier transforms.

Many researchers study affine Lie algebras and the structure of affine for instance we see some results in [10], [11], [12], [13],[14], [15], [16], and [17].

In the other hand, in easier stage we can also study square-integrable representations for unimodular Lie groups case. Heisenberg Lie groups of dimension $2n + 1$ and filiform Lie groups are in these types. In fact, the Duflo-Moore operators for square-integrable representations of unimodular groups are scalar multiple (see [18]). In current work, Kurniadi in [19] proved that irreducible-unitary representation of Lie group of 4-dimensional standard filiform Lie algebra is square-integrable and its Duflo-Moore operator is scalar multiple of identity which is equal to one.

In this work, we shall give another alternative to compute the Duflo-Moore operator for square-integrable representation of $\text{Aff}^+(1)$ by direct computations instead of forming in group Fourier transform which was written in [18].

2. Preliminaries

Let $\text{Aff}^+(1)$ be the 2-dimensional affine Lie group which is expressed as a semidirect product of the set of all real numbers $\mathbb{R}$ and the set of all positive real numbers $\mathbb{R}_+$. Namely, we can write this group as $\text{Aff}^+(1) = \mathbb{R} \rtimes \mathbb{R}_+$. Particularly, in this work we concentrate to $\text{Aff}^+(1)$ which is the exponential solvable nonunimodular Lie group. To make easier in computations we write $\text{Aff}^+(1)$ in matrix terms. Namely, we have

$$\text{Aff}^+(1) \ni \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}, \alpha \in \mathbb{R}_+, \beta \in \mathbb{R}. \quad (1)$$

Regarding this notations, we denote $g(\alpha, \beta) := \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, $\Delta(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$, and $\nabla(\beta) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$. The Lie algebra of $\text{Aff}^+(1)$ is denoted by $\text{aff}(1)$ whose basis is $\{e_1, e_2\}$. The nonzero bracket of $\text{aff}(1)$ is given by $[e_1, e_2] = e_2$. The Lie algebra $\text{aff}(1)$ is a Frobenius Lie algebra which has two open coadjoint orbits as follows (see [20]).

$$\Omega_{\pm} := \{(a, b) ; a, b \in \mathbb{R}, \pm b > 0\}. \quad (2)$$
The representations of the affine Lie group Aff(1) can be realized on the Hilbert space of all square-integrable functions $L^2(\mathbb{R}_+)$. Before doing that, let us mention here some basic notion of representation theory of Lie groups corresponding to our research.

**Definition 1** [21]. Let $\pi$ be a representation of a Lie group $G$ on the carrier space $\mathcal{H}$. $\pi$ is said to be irreducible if $\pi$ has no nontrivial $\pi$-invariant subspace $\mathcal{H}_0$ in $\mathcal{H}$. Moreover, $\pi$ is said to be unitary if for each $f \in \mathcal{H}$ and each $g \in G$

\[
\|\pi(g)f\| = \|f\|. \quad (3)
\]

**Proposition 2** [20]. The irreducible unitary representations of $\text{Aff}^+(1)$ corresponding to open coadjoint orbit $\Omega_+$ in eqs. (2) in the space $L^2(\mathbb{R}_+)$ is of the form

\[
\pi_+(g)f(x) = e^{2\pi i \beta x} f(ax), \quad (4)
\]

where $g := g(\alpha, \beta) \in \text{Aff}^+(1), \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \text{ and } f \in L^2(\mathbb{R}_+)$. Furthermore, the representation of affine Lie group $\text{Aff}^+(1)$ can be realized as a quasi-regular representations (see [18]). It is written in the formula as follows.

\[
\pi(g(\alpha, \beta)) = \alpha^{-\frac{1}{2}} \psi(\frac{x-\beta}{\alpha}), \quad \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \text{ and } \psi \in L^2(\mathbb{R}_+). \quad (5)
\]

We are mainly interested in the square-integrable representation. Let $\pi$ be an irreducible unitary representation of a Lie group $G$ realized on the space $\mathcal{H}$ and $L^2(G)$ be the space of all square-integrable functions on $G$. For vector $f_1 \in \mathcal{H}$, we define the operator on $\mathcal{H}$ given by

\[
\mathcal{E}_{f_1} : \mathcal{H} \ni f_2 \mapsto \mathcal{E}_{f_1}f_2 \in L^2(G). \quad (6)
\]

where $\mathcal{E}_{f_1}f_2(x) = \langle f_1 | \pi(x)f_2 \rangle$.

**Definition 3** [22]. The irreducible unitary representation $\pi$ of locally compact topological group $G$ realized on a space $\mathcal{H}$ is said to be square-integrable if there exist two vectors $f_1, f_2 \in \mathcal{H} - \{0\}$ such that

\[
\|\mathcal{E}_{f_1}f_2\|^2 = \langle f_1 | \pi(x)f_2 \rangle = \int_G f_1(g)\pi(x)f_2(g) \, d\mu(g) < +\infty. \quad (7)
\]

In the other words, $\langle f_1 | \pi(x)f_2 \rangle \in L^2(G, \mu_G)$ where $\mu_G$ is a measure on $G$. Such vectors which satisfied eqs. (7) are called admissible vectors.

Duflo-Moore state their results in the following theorem

**Theorem 4** [23]. If $\pi$ is square-integrable representations of locally compact group $G$ realized on the space $\mathcal{H}$ then there exists a positive selfadjoint operator $C_\pi : \mathcal{H} \to \mathcal{H}$ which is called the **Duflo-Moore operator** such that

a. a vector $\psi \in \mathcal{H} - \{0\}$ is admissible if and only if $\psi$ is an element of domain of $C_\pi$.

b. if $f_1, f_2 \in \mathcal{H}$ and $f_3, f_4 \in \text{Dom}(C_\pi)$ then

\[
\langle \mathcal{E}_{f_1}f_3 | \mathcal{E}_{f_2}f_4 \rangle_{L^2(G, \mu_G)} = \langle f_1 | f_2 \rangle_{\mathcal{H}} \langle C_\pi f_4 | C_\pi f_3 \rangle_{\mathcal{H}}. \quad (8)
\]
2. Methods

In this research we apply the literature reviews method, particularly we focus on results in [18] and [20]. We obtain the quasi-regular representation of $\text{Aff}^+(1)$ in Fuhr’s work and we compute the Fourier transform of its representation to determine the Duflo-Moore operator. On the other hand, we also obtain the irreducible unitary representation of $\text{Aff}^+(1)$ corresponding to open coadjoint orbits and we show that representation is square-integrable. Using direct computations, we obtain the Duflo-Moore operator for that representation.

3. Results and Discussion

Our results and discussion consist of two main part as follows.

3.1 The Duflo-Moore Operator for The Quasi-Regular Representation of $\text{Aff}^+(1)$.

The following statement can be deduced from [18] in page 30--31. However, we give a detail proof for its own interest.

**Lemma 5** [18]. The Fourier transform of quasi-regular representation $\pi$ of $\text{Aff}^+(1)$ as in eqs. (5) is of the form

$$\mathcal{F}(\pi(g(\alpha, \beta)))\psi(\xi) = \frac{1}{\alpha^2} e^{-2\pi i \beta} \mathcal{F}(\alpha \xi).$$  \hspace{1cm} (9)

**Proof.**

By direct computation we obtain

$$\mathcal{F}(\pi(g(\alpha, \beta)))\psi(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} (\pi(g(\alpha, \beta))\psi(x)) \, dx$$

$$= \int_{\mathbb{R}} e^{-2\pi i \xi x} \alpha^{-1/2} \psi \left( \frac{x - \beta}{\alpha} \right) \, dx$$

$$= \int_{\mathbb{R}} e^{-2\pi i \xi (\alpha \eta + \beta)} \alpha^{-1/2} \psi(\eta) \alpha \, d\eta$$

( Substituting $\eta = \frac{x - \beta}{\alpha}$)

$$= \int_{\mathbb{R}} e^{-2\pi i (\alpha \eta)} e^{-2\pi i \beta} \alpha^{1/2} \psi(\eta) \, d\eta$$

$$= \int_{\mathbb{R}} e^{-2\pi i (\alpha \xi)} e^{-2\pi i \beta} \alpha^{1/2} \psi(\eta) \, d\eta$$

$$= e^{-2\pi i \beta} \alpha^{1/2} \int_{\mathbb{R}} e^{-2\pi i (\alpha \xi)} \psi(\eta) \, d\eta$$

$$= e^{-2\pi i \beta} \alpha^{1/2} \mathcal{F}(\alpha \xi).$$

**Proposition 6** [18]. The Duflo-Moore operator for quasi-regular representation $\pi$ of $\text{Aff}^+(1)$ as in eqs. (5) in the term of Fourier transform can be written as follows.
\[ \mathcal{F}(C_\pi \psi)(\xi) = \xi^{-1/2} \mathcal{F}\psi(\xi). \] (10)

**Proof.** Let \( \psi_1 \) and \( \psi_2 \) be elements of continuous functions space of compact support on \( \text{Aff}^+(1) \) denoted by \( C_c(\text{Aff}^+(1)) \). Using Plancherel’s theorem and Fubini’s theorem we obtain

\[
\int_{\text{Aff}^+(1)} |(\psi_1|\pi(g(\alpha, \beta))\psi_2)|^2 \frac{d\alpha}{\alpha^2} d\beta = \int_{\text{Aff}^+(1)} |(\mathcal{F}\psi_1|\mathcal{F}\pi(g(\alpha, \beta))\psi_2)|^2 \frac{d\alpha}{\alpha^2} d\beta
\]

\[
= \int_{\text{Aff}^+(1)} \int_{\mathbb{R}} |\mathcal{F}\psi_1(\xi)e^{-2\pi i \xi \beta} \alpha^{1/2}\mathcal{F}\psi(\alpha \xi)|^2 \frac{d\alpha}{\alpha^2} d\beta
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}^+} |\mathcal{F}\psi_1(\xi)\mathcal{F}\psi(\alpha \xi)|^2 \frac{d\alpha}{\alpha} d\beta
\]

\[
= \int_{\mathbb{R}} |\mathcal{F}\psi_1(\xi)|^2 \left\{ \int_{\mathbb{R}^+} |\mathcal{F}\psi(\alpha \xi)|^2 \frac{d\alpha}{\alpha} \right\} d\xi
\]

\[
= \|\mathcal{F}\psi_1\|^2 \left\{ \int_{\mathbb{R}} \left( \mathcal{F}\psi(\alpha') \right|^2 \frac{d\alpha'}{\alpha'} \right\}
\]

(\( \alpha' := \alpha \xi \)).

Thus, from the latter equation we obtain the Duflo-Moore operator is equal to \( \mathcal{F}(C_\pi \psi)(\xi) = \xi^{-1/2} \mathcal{F}\psi(\xi) \) as desired.

\[ \blacksquare \]

3.2 **The Duflo-Moore Operator for The Irreducible Unitary Representation of \( \text{Aff}^+(1) \)**

This session is the main result. First, we recall that the irreducible unitary representation of group \( \text{Aff}^+(1) \) in Proposition 2 can be written in the following proposition.
Proposition 7. The irreducible unitary representations of \( \text{Aff}^+(1) \) corresponding to open coadjoint orbit \( \Omega_1 \) in eqs. (2) in the space \( L^2(\mathbb{R}_+^+) \) is of the form

\[
\pi_+(\Delta(\alpha))f(x) = f(ax),
\]
\[
\pi_+(\nabla(\beta))f(x) = e^{2\pi i \beta x} f(x),
\]  \hspace{1cm} (11)

where \( \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \) and \( f \in L^2(\mathbb{R}_+) \).

**Proof.** Let \( \text{aff}(1) \) be a Lie algebra of \( \text{Aff}^+(1) \) whose basis is \( \{e_1, e_2\} \). We consider its dual space as \( \text{aff}(1)^* \ni \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \), where \( a, b \in \mathbb{R} \). Moreover, let \( \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \), \( \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \) be an element of group affine \( \text{Aff}^+(1) \). We shall construct the irreducible unitary representation of \( \text{Aff}^+(1) \) corresponding to open coadjoint orbit \( \Omega_1 = \{(a, b) : b > 0\} \). To do that, fix a point \( \tau := e_1^* \in \Omega_1 \subset \text{aff}(1)^* \) as a linear functional. For subalgebra \( \mathcal{N} := \langle e_2 \rangle \) we have \( \mathcal{N} \) has maximal dimension and the value of linear functional \( \tau \) on the commutator \( [\mathcal{N}, \mathcal{N}] \) is given by \( \tau([\mathcal{N}, \mathcal{N}]) = 0 \). Therefore, \( \mathcal{N} \) is a polarization in \( \text{aff}(1) \). Let \( \mathcal{N}^\perp \) be the orthogonal subspace. Furthermore, since \( \tau + \mathcal{N}^\perp \) is contained in \( \Omega_1 \) then \( \mathcal{N} \) satisfies Pukanszky condition.

Now we construct a 1-dimensional representation \( \lambda_{\tau} \) of \( \mathcal{N} := \exp \mathcal{N} \) as follows.

\[
\lambda_{\tau}(\exp e) := e^{2\pi i \tau|e|} = e^{2\pi i \beta} e := \alpha e_1 + \beta e_2, \tau \in \Omega_1. \]  \hspace{1cm} (12)

We identify the coset \( \text{Aff}^+(1)/\mathcal{N} \) by \( \mathbb{R}_+^+ \) and we obtain the section given by

\[
s: \mathbb{R}_+^+ \ni x \mapsto \exp xe_1 \in \text{Aff}^+(1). \]  \hspace{1cm} (13)

To obtain the explicit formula of the representation of \( \text{Aff}^+(1) \) we need to solve what we called the master equation

\[
s(x)g = h_5(x, g)s(xg), \quad (x \in \mathbb{R}_+, g \in \text{Aff}^+(1), h_5(x, g) \in \mathcal{N}). \]  \hspace{1cm} (14)

Using the basis \( \{e_1, e_2\} \) we solve the following master equations with respect to its basis:

a. \[ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}. \]

by solving with respect to \( u \) and \( y \) we obtain \( y = \alpha x \). Therefore, \( \pi_+(\Delta(\alpha))f(x) = f(ax) \). We mention here that we apply a right action of \( \text{Aff}^+(1) \) in space \( L^2(\mathbb{R}_+) \).

b. \[ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}. \]

In this case, we have \( y = x \) and \( u = \beta x \). Therefore, \( \pi_+(\nabla(\beta))f(x) = e^{2\pi i \beta x} f(x) \) as desired.

In the next section, we shall compute the Duflo-Moore operator for the representation of \( \text{Aff}^+(1) \) with respect to its right Haar measure. The result of Duflo-Moore operator for the representation of \( \text{Aff}^+(1) \) with respect left Haar measure can be found in [24] pages 82-85.
Proposition 8. The Duflo-Moore operator for the irreducible unitary representation $\pi_+$ of $\text{Aff}^+(1)$ as written in eqs. (11) is of the form

$$C_{\pi_+} f(\Delta(x)) = x^{-1/2} f(x), \quad (f \in L^2(\mathbb{R}_+), \ x \in \mathbb{R}_+)$$

(15)

Proof. Let $\vartheta_1$ and $\vartheta_2$ be elements in $C_c(\text{Aff}^+(1))$. Using the right Haar measure, we shall compute the integral

$$\int_{\text{Aff}^+(1)} |\langle \vartheta_1 | \pi_+ (\nabla(\beta)) \pi_+ (\Delta(\alpha)) \vartheta_2 \rangle_{L^2(\mathbb{R}_+)}|^2 \frac{d\beta}{\alpha} d\alpha$$

To do that, first we compute the following inner product.

$$\langle \vartheta_1 | \pi_+ (\nabla(\beta)) \pi_+ (\Delta(\alpha)) \vartheta_2 \rangle_{L^2(\mathbb{R}_+)} = \int_{\mathbb{R}_+} \vartheta_1(x) \pi_+ (\nabla(\beta)) \pi_+ (\Delta(\alpha)) \vartheta_2(x) \frac{dx}{x}$$

$$= \int_{\mathbb{R}_+} \vartheta_1(x) \pi_+ (\Delta(\alpha)) e^{2\pi i \beta x} \vartheta_2(x) \frac{dx}{x}$$

$$= \int_{\mathbb{R}_+} e^{-2\pi i \beta x} \vartheta_1(x) \pi_+ (\Delta(\alpha)) \vartheta_2(x) \frac{dx}{x}$$

Using Plancherel’s theorem we have

$$\int_{\mathbb{R}} |\langle \vartheta_1 | \pi_+ (\nabla(\beta)) \pi_+ (\Delta(\alpha)) \vartheta_2 \rangle_{L^2(\mathbb{R}_+)}|^2 d\beta = \int_{\mathbb{R}_+} |e^{-2\pi i \beta x} \vartheta_1(x) \pi_+ (\Delta(\alpha)) \vartheta_2(x)|^2 \frac{dx}{x^2}$$

$$= \int_{\mathbb{R}_+} |\vartheta_1(x) \pi_+ (\Delta(\alpha)) \vartheta_2(x)|^2 \frac{dx}{x^2}$$

$$= \int_{\mathbb{R}_+} |\vartheta_1(x) \pi_+ (\alpha x)|^2 \frac{dx}{x^2}$$

Therefore, using Fubini’s theorem we obtain

$$\int_{\text{Aff}^+(1)} |\langle \vartheta_1 | \pi_+ (\nabla(\beta)) \pi_+ (\Delta(\alpha)) \vartheta_2 \rangle_{L^2(\mathbb{R}_+)}|^2 \frac{d\beta}{\alpha} d\alpha = \int_{\mathbb{R}_+} \bigg\{ \int_{\mathbb{R}_+} |\vartheta_1(x)|^2 \frac{d\alpha}{\alpha} \bigg\} \int_{\mathbb{R}_+} |\vartheta_2(\alpha x)|^2 \frac{d\alpha}{\alpha} \frac{dx}{x^2}$$

$$= \int_{\mathbb{R}_+} |\vartheta_1(x)|^2 \frac{dx}{x^2} \left\{ \int_{\mathbb{R}_+} |\vartheta_2(\alpha')|^2 \frac{d\alpha'}{\alpha'} \right\} (\alpha' := \alpha x)$$

$$= \int_{\mathbb{R}_+} |x^{-1/2} \vartheta_1(x)|^2 \frac{dx}{x} \left\{ \int_{\mathbb{R}_+} |\vartheta_2(\alpha')|^2 \frac{d\alpha'}{\alpha'} \right\}$$

$$= \int_{\mathbb{R}_+} |x^{-1/2} \vartheta_1(x)|^2 \frac{dx}{x} \cdot ||\vartheta_2||^2.$$
Therefore, The Duflo-Moore operator for the irreducible unitary representation of \( \text{Aff}^+(1) \) as written in eqs. (11) is of the form \( C_{\pi_x} f(\Delta(x)) = x^{-1/2} f(x) \) as desired.

4. Conclusions

The Duflo-Moore operator for the representations of \( \text{Aff}^+(1) \) in this paper is considered in two cases. The first case, it is for the quasi-regular representation and written in the term of Fourier transform. Namely, we obtain \( \mathcal{F}(C_{\pi_x} \psi)(\xi) = \xi^{-1/2} \mathcal{F}(\psi)(\xi) \) (see [18]). The second case, the Duflo-Moore operator is considered for irreducible unitary representation with respect to its right Haar measure and we have \( C_{\pi_x} f(\Delta(x)) = x^{-1/2} f(x) \). On the other hand, the Duflo-Moore operator for a square-interagible representation of \( \text{Aff}^+(1) \) with respect to its left Haar measure can be seen in [24] pages 82-85.

It is more interesting to compute the Duflo-Moore operator for the representation of higher dimension of affine Lie groups.

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