Free decomposition spaces

Hackney, Philip; Kock, Joachim

Publication date: 2022

Citation for published version (APA): Hackney, P., & Kock, J. (2022). Free decomposition spaces.
Free decomposition spaces

PHILIP HACKNEY* and JOACHIM KOCK**

*University of Louisiana at Lafayette
**Universitat Autònoma de Barcelona and Centre de Recerca Matemàtica;
currently at the University of Copenhagen

Abstract

We introduce the notion of free decomposition spaces: they are simplicial spaces freely generated by their inert maps. We show that left Kan extension along the inclusion \( j: \delta_{\text{inert}} \to \Delta \) takes general objects to Möbius decomposition spaces and general maps to CULF maps. We establish an equivalence of \( \infty \)-categories \( \PrSh(\Delta_{\text{inert}}) \simeq \text{Decomp}_{/\text{BN}} \). Although free decomposition spaces are rather simple objects, they abound in combinatorics: it seems that all comultiplications of deconcatenation type arise from free decomposition spaces. We give an extensive list of examples, including quasi-symmetric functions. We show that the Aguiar–Bergeron–Sottile map to the decomposition space of quasi-symmetric functions, from any Möbius decomposition space \( X \), factors through the free decomposition space of nondegenerate simplices of \( X \), and offer a conceptual explanation of the zeta function featured in the universal property of \( \text{QSym} \).
# Contents

| Section                                      | Page |
|----------------------------------------------|------|
| Introduction                                 | 2    |
| 1 Preliminaries                              |      |
| 1.1 Active and inert maps                    | 6    |
| 1.2 Decomposition spaces and incidence coalgebras | 8    |
| 2 Free decomposition spaces                  |     |
| 2.1 Left Kan extension along \( j \)          | 8    |
| 2.2 Two identifications                      | 10   |
| 2.3 \( j \)! gives decomposition spaces and CULF maps | 11   |
| 3 CULF-graded decomposition spaces           |     |
| 3.1 Untwisting theorem                        | 12   |
| 3.2 Equivalence between \( \Delta^{op}_{\text{inert}} \)-diagrams and CULF-graded decomposition spaces | 14   |
| 4 Miscellaneous results                      |     |
| 4.1 Remarks about sheaves                    | 15   |
| 4.2 Restriction \( L \)-species              | 16   |
| 5 Examples in combinatorics                  |     |
| 5.1 Paths and words                          | 17   |
| 5.2 Further examples of deconcatenations     | 21   |
| 5.3 Aguiar–Bergeron–Sottile map              | 23   |
| 5.4 Zeta functions                           | 25   |

**Introduction**

**Background**

**Decomposition spaces.** A lot of recent activity in the interface between algebraic combinatorics and simplicial homotopy theory has resulted from the discovery that objects more general than posets admit the construction of incidence algebras and Möbius inversion: these are called *decomposition spaces* by Gálvez, Kock, and Tonks [21], [22], and they are the same thing as the 2-Segal spaces of Dyckerhoff and Kapranov [17] who were motivated mainly by representation theory and homological algebra. (The equivalence between the two notions was completed only recently by Feller et al. [18].) A decomposition space is a simplicial \( \infty \)-groupoid satisfying the exactness condition (weaker than the Segal condition) stating that active-inert pushouts in the simplex category \( \Delta \) are sent to homotopy pullbacks. The active-inert factorization system was already an important ingredient in the combinatorics of higher algebra, as in Lurie's book [40], where the terminology originates.

In the present work, we explore further the fundamental relationship between decomposition spaces and the active-inert factorization system, and single out a new class of decomposition space, the *free decomposition spaces*, being simplicial
$\infty$-groupoids freely generated by their inert maps. We show that most comultiplications in algebraic combinatorics of deconcatenation type arise from free decomposition spaces.

**Active and inert maps for categories and higher categories.** The factorization system $\Delta = (\Delta_{\text{active}}, \Delta_{\text{inert}})$ was first introduced (by Leinster and Berger [5]) to express the interplay between algebra and geometry in the notion of category: for the nerve of a category, the active maps parametrize the algebraic operations of composition and identity arrows, while the inert maps express the bookkeeping that these operations are subject to, namely source and target. The geometric nature of this background fabric is manifest in the fact that the category $\Delta_{\text{inert}}$ has a natural Grothendieck topology, through which the gluing conditions are expressed: a simplicial set $X$ is the nerve of a category if and only if $j^*(X)$, the restriction of $X$ along $j : \Delta_{\text{inert}} \to \Delta$, is a sheaf. In particular, the question whether a simplicial set is the nerve of a category depends only on the inert part. This viewpoint is the starting point for the Segal–Rezk approach to $\infty$-categories, defined by replacing simplicial sets by simplicial spaces, and considering the sheaf condition up to homotopy. Many other developments exploit the active-inert machinery to obtain nerve theorems in fancier contexts, including Weber’s extensive theory of local right adjoint monads and monads with arities, with abstract nerve theorems [56], [57], [6], as well as special nerve theorems for specific operad-like structures (polynomial monads in terms of trees [37], [25], properads in terms of directed graphs [38], modular operads and wheeled properads as well as infinity versions [25], [20], and so on); see [26] for a survey. Recently Chu and Haugseng [15] have even developed a general Segal approach to operad-like structures in terms of algebraic patterns, where the notion of active-inert factorization system is taken as a primitive notion.

**Active and inert maps for decomposition spaces.** For decomposition spaces, the active-inert interplay is more subtle, and the exactness condition that characterizes them can no longer be measured on the inert maps alone. It is now about *decomposition* of ‘arrows’ rather than composition. Roughly, the active maps encode all possible ways to decompose arrows, and the inert maps then separate out the pieces of the decomposition. The exactness condition characterizing decomposition spaces combines active and inert maps. It can be interpreted as a locality condition, stating roughly that the possible decompositions of a local region are not affected by anything outside the region [24], [20].

**Active and inert maps for morphisms.** Turning to morphisms, the situation is more complex for decomposition spaces than for categories. For categories, the nerve functor is fully faithful, meaning that all simplicial maps are relevant: the simplicial identities for simplicial maps simply say that source and target, composition and identities are preserved. For decomposition spaces, this is no longer the case, as there are different ways in which a simplicial map could be said to preserve decompositions. The most well-behaved class of simplicial maps in this respect are the CULF maps [21] (standing for ‘conservative’ and
‘unique lifting of factorizations’), a class of maps well studied in category theory, originating with Lawvere’s work on dynamical systems [42], and exploited in computer science in the algebraic semantics of processes [13], [32], [12]. From the viewpoint of combinatorics the interest in CULF maps is that they preserve decompositions in a way such as to induce coalgebra homomorphisms between incidence coalgebras [44], [43], [21]. The formal characterization of CULF maps is that when interpreted as natural transformations, they are cartesian on active maps. Independently of the coalgebra interpretation, this pullback condition interacts very well with the exactness condition characterizing decomposition spaces. (There is another class of maps that induce coalgebra homomorphisms, but this time contravariantly [21]: they are the IKEO maps (standing for ‘inner Kan and equivalence on objects’), characterized by a pullback condition instead on inert maps. This notion will surface near the end of the paper.)

Contributions of the present paper

In the present work, the focus is not so much on restriction along \( j: \Delta_{\text{inert}} \to \Delta \) as in the Segal case, but rather on its left adjoint \( j_! \vdash j^* \), left Kan extension along \( j \).

It is rarely the case that \( j_!(A) \) is Segal. We show instead that \( j_!(A) \) is always a decomposition space:

**Theorem.** (Cf. Proposition [2.3.2] and Corollary [2.3.3]) For any \( A: \Delta_{\text{op}}^{\text{inert}} \to S \), the left Kan extension \( j_!(A): \Delta_{\text{op}} \to S \) is a M"obius decomposition space, and for any map \( A' \to A \) of \( \Delta_{\text{inert}} \)-diagrams, we have that \( j_!(A') \to j_!(A) \) is CULF.

The decomposition spaces that arise with \( j_! \) we call free decomposition spaces.

The functor \( j_! \) does not seem to be full (but see Remark [3.0.1] below). In order to obtain an equivalence, we take into account the terminal object in \( \PrSh(\Delta_{\text{inert}}) \). The following result is just a calculation:

**Lemma 2.3.1.** We have \( j_!(1) \simeq BN \), the classifying space of the natural numbers.

Now we can characterize free decomposition spaces:

**Proposition.** (Cf. Corollary [3.2.2]) A decomposition space is free if and only if it admits a CULF map to \( BN \).

In fact we derive this from the following more precise result.

**Theorem 3.2.1** The \( j_! \) functor induces an equivalence of \( \infty \)-categories

\[ \PrSh(\Delta_{\text{inert}}) \simeq \text{Decomp}_{/BN}. \]
Here $\text{Decomp}$ is the $\infty$-category of decomposition spaces and CULF maps, and $\text{Decomp}_{/BN}$ its slice over $BN$.

The proof of this result turned out to be quite involved, and ended up developing into a proof of the following very general result, which is an $\infty$-version of a theorem of Kock and Spivak [41]:

**Theorem 3.1.2.** For $D$ a decomposition space, there is a natural equivalence of $\infty$-categories

$$\text{Decomp}_{/D} \simeq R\text{fib}(\text{tw} D).$$

Here $\text{tw}(D)$ denotes the edgewise subdivision of the simplicial space $D$ — when $D$ is a decomposition space, this is a Segal space, called the twisted arrow category of $D$. The right-hand side $R\text{fib}(\text{tw} D)$ is the $\infty$-category of right fibrations over $\text{tw}(D)$, which is equivalent to $\text{PrSh}(\text{tw} D)$ (at least when $D$ is Rezk complete [27]).

In order to apply the general theorem, take $D = BN$, and note the following:

**Lemma 2.3.1.** There is a natural equivalence of categories

$$\Delta_{\text{inert}} \simeq \text{tw}(BN).$$

Theorem 3.2.1 follows essentially from this observation and the general theorem, but there is still some work to do to show that in this special case, the untwisting of the general theorem can actually be identified with left Kan extension along $j$, surprisingly.

Since the general theorem is of independent interest, and since the proof is very long, we have separated it out into a paper on its own [27].

Theorem 3.2.1 readily implies the following classical and more special result due to Street [54]: a category admits a CULF functor to $BN$ if and only if it is free on a directed graph.

We also characterize a large class of free decomposition spaces in terms of a class of species called *restriction* $\mathbb{L}$-species (Proposition 4.2.3).

Although free decomposition spaces are rather simple, they abound in combinatorics. Generally it seems that all comultiplications of deconcatenation type arise from free decomposition spaces. We illustrate and substantiate this principle by giving a long list of examples of deconcatenation-type comultiplications and the free decomposition spaces they are incidence coalgebras of. This includes many variation on paths and words, including parking functions, Dyck paths, noncrossing partitions, as well as processes of transition systems, Petri nets, and rewrite systems. In particular, the comultiplication of the Hopf algebra of quasi-symmetric functions $\text{QSym}$ is shown to be the incidence coalgebra of a free decomposition space $Q$, namely that of words in the alphabet $\mathbb{N}_+$. We analyze this further by showing that the Aguiar–Bergeron–Sottile map [11] expressing
the universal property of QSym, actually factors through a free map. Precisely, for any Möbius decomposition space $X$, there is a free decomposition space of nondegenerate simplices, denoted $J(X)$. For the decomposition space $BN$, we get $J(BN) = Q$. If $X$ is (not just filtered but actually) graded, so that there is a simplicial map $X \to BN$, then the Aguiar–Bergeron–Sottile map is given by

\[
X \xleftarrow{\text{IKEO}} J(X) \xrightarrow{\text{CULF}} J(BN) = Q.
\]

We also show that the relevant special zeta function on QSym (in terms of which the universal property is formulated [1]) has a nice interpretation in terms of free decomposition spaces: it is the only zeta function compatible with the $J$-construction, in a precise sense.

(The decomposition space for QSym as well as the decomposition-space interpretation of the Aguiar–Bergeron–Sottile map, are due to Gálvez–Kock–Tonks (unpublished) [19]. The description in terms of free decomposition spaces is new.)

1 Preliminaries

We run through some standard material just to set up notation.

1.1 Active and inert maps

As usual, $\Delta$ denotes the category of finite nonempty ordinals

\[ [n] = \{0 < 1 < \cdots < n \}. \]

1.1.1. The active-inert factorization system. The category $\Delta$ has an active-inert factorization system: the active maps, written $g: [k] \to [n]$, are those that preserve end-points, $g(0) = 0$ and $g(k) = n$; the inert maps, written $f: [m] \to [n]$, are those that are distance preserving, $f(i+1) = f(i) + 1$ for $0 \leq i \leq m - 1$. The active maps are generated by the codegeneracy maps and the inner coface maps; the inert maps are generated by the outer coface maps $d^\perp$ and $d^\top$. (This orthogonal factorization system is an instance of the important general notion of generic-free factorization system of Weber [57], who referred to the two classes as generic and free. The active-inert terminology is due to Lurie [46].)

1.1.2. Active maps vs. $k$-tuples. For fixed $k \in \mathbb{N}$, write $\text{Act}(k)$ for the set of active maps out of $[k]$. For $i = 1, \ldots, k$, write $\rho_i : [1] \to [k]$ for the inert map that picks out the $i$th principal edge. For an active map $\alpha : [k] \to [n]$, write $[n_i]$ for the ordinal appearing in the active-inert factorization of $\alpha \circ \rho_i$:

\[
\begin{array}{ccc}
[1] & \to & [n_i] \\
\rho_i & \downarrow & \gamma_i^c \\
[k] & \alpha & [n]
\end{array}
\]
This defines a $k$-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$.

Conversely, given a $k$-tuple $(n_1, \ldots, n_k) \in \mathbb{N}^k$, define an active map $\alpha : [k] \to [n]$ (with $n := \sum_{1 \leq i \leq k} n_i$) by sending $j \in [k]$ to $\sum_{1 \leq i \leq j} n_i \in [n]$. Clearly $\alpha(0) = 0$ and $\alpha(k) = n$, so $\alpha$ is indeed active.

These assignments are inverse to each other so as to define a bijection

$$\text{Act}(k) \simeq \mathbb{N}^k.$$ 

(Below (in 2.2.3) we shall vary $k$ and fit these bijections into an isomorphism of categories.)

### 1.2 Decomposition spaces and incidence coalgebras

#### 1.2.1. Decomposition spaces.**

Active and inert maps in $\Delta$ admit pushouts along each other, and the resulting maps are again active and inert. A decomposition space [21] is a simplicial $\infty$-groupoid $X : \Delta^{op} \to S$ that takes all such active-inert pushouts to pullbacks:

$$X \left( \begin{array}{ccc}
[n'] & \leftarrow & [n] \\
\uparrow & \swarrow & \uparrow \\
(m') & \leftarrow & [m]
\end{array} \right) = \begin{array}{ccc}
X_{n'} & \longrightarrow & X_n \\
\downarrow & \swarrow & \downarrow \\
X_{m'} & \longrightarrow & X_m.
\end{array}$$

Every Segal space is also a decomposition space [21], [17]. In particular, posets and categories are decomposition spaces, via the nerve construction.

#### 1.2.2. Incidence coalgebras.**

The motivation for the notion of decomposition space is that they admit the construction of coassociative coalgebras [21], [22], generalizing the classical theory of incidence coalgebras of posets developed by Rota [50], [33] in the 1960s. Just as incidence coalgebras of posets are spanned linearly by the poset intervals, the incidence coalgebra of a decomposition space $X$ is spanned linearly by $X_1$. The comultiplication (which generalizes the case of posets) is given by (for $f \in X_1$)

$$\Delta(f) = \sum_{\sigma \in X_2} d_2(\sigma) \otimes d_0(\sigma)$$

which verbalizes into 'sum over all 2-simplices with long edge $f$ and return the two short edges.'

#### 1.2.3. CULF maps.**

A simplicial map $F : Y \to X$ between simplicial spaces is called CULF [21] (standing for conservative and having unique lifting of factorizations) when it is cartesian on active maps (i.e. the naturality squares on

\[\text{In this paper we work with } \infty\text{-categories model independently, as is often done in the theory of decomposition spaces [21], [22], [23]. For specificity, the reader can take the model of quasi-categories of Joyal [35] and Lurie [45], where in particular, } \infty\text{-groupoid means Kan complex.}\]
active maps are pullbacks). If \( X \) is a decomposition space (e.g. a Segal space) and \( F: Y \to X \) is CULF, then also \( Y \) is a decomposition space (but not in general Segal). We denote by \( \text{Decomp} \) the \( \infty \)-category of decomposition spaces and CULF maps.

From the viewpoint of incidence coalgebras, the interest in CULF maps is that the incidence coalgebra construction is functorial (covariantly) in CULF maps \([21]\).

### 1.2.4. Finiteness conditions and Möbius decomposition spaces.

The theory of incidence coalgebras of decomposition spaces is natively ‘objective,’ meaning that the constructions deal directly with the combinatorial objects rather than with the vector spaces they span \([21]\). At this level the theory does not require any finiteness conditions. However, in order to be able to take (homotopy) cardinality to arrive at coalgebras in vector spaces as in classical combinatorics, it is necessary to impose certain finiteness conditions \([22]\). A decomposition space is locally finite when the maps \( X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2 \) are (homotopy) finite \([22]\). Often they are also discrete, in which case \( X \) is called locally discrete. The first condition ensures that the general incidence-coalgebra construction admits a (homotopy) cardinality. The discreteness condition ensures that the sum formula for comultiplication is free from denominators. An important class of decomposition spaces, called Möbius decomposition spaces, are characterized by (a completeness condition and) yet another finiteness condition, namely that for every 1-simplex \( f \in X_1 \) there is only finitely many higher nondegenerate simplices with long edge \( f \). As suggested by the terminology, these admit a Möbius inversion principle \([22]\).

We shall not need to verify any of these conditions directly. We shall only need the fact \([22]\) that if \( Y \to X \) is CULF and if \( X \) is locally finite, locally discrete, or Möbius, then so is \( Y \). (And then we shall invoke the fact that \( B\mathbb{N} \) has all three properties.)

## 2 Free decomposition spaces

### 2.1 Left Kan extension along \( j \)

Throughout, \( j: \Delta_{\text{inert}} \to \Delta \) denotes the inclusion functor.

**Lemma 2.1.1.** For a presheaf \( A: \Delta_{\text{inert}}^{\text{op}} \to \mathcal{S} \) with corresponding right fibration \( \tilde{A} \to \Delta_{\text{inert}} \), the left Kan extension along \( j: \Delta_{\text{inert}} \to \Delta \) corresponds to the left-hand composite in the pullback diagram

\[
\begin{array}{ccc}
\text{Arr}_{\Delta}(\tilde{A}) & \xrightarrow{\text{cart}} & \tilde{A} \\
\downarrow \text{dom} & & \downarrow \text{codom} \\
\Delta & \xrightarrow{j_!} & \Delta_{\text{inert}}
\end{array}
\]
Here $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}}$ is the category whose objects are the active maps in $\Delta$ and whose arrows from $\alpha'$ to $\alpha$ are commutative squares

$$
\begin{array}{ccc}
[k'] & \longrightarrow & [k] \\
\downarrow & & \downarrow \alpha \\
[n'] & \longrightarrow & [n],
\end{array}
$$

as will be explained further in the following discussion.

The lemma gives the following explicit sum-over-active-maps formula for $j_!(A)$:

**Corollary 2.1.2.** The simplicial space $X = j_!(A)$ has

$$X_k = \sum_{[k] \to [n]} A_n.$$  

**Proof.** In the diagram in 2.1.1, the fiber over $[k] \in \Delta$ is computed by first expanding the fiber in $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}}$ over $[k]$, which is the set $\{[k] \to [n]\}$ (for varying $n$). For each element in this set, the fiber is clearly $A_n$. \qed

The lemma is a special case of a general result about left Kan extension from the right class of a factorization system. Since the proof is the same in the general case, we give the general proof. First, a few recollections on factorization systems, which will also explain the notation $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}}$ already employed.

**2.1.3. Factorization systems and cartesian fibrations.** Let $\mathcal{C}$ be an $\infty$-category with a factorization system $(\mathcal{E}, \mathcal{F})$. Let $\text{Arr}(\mathcal{C}) = \text{Fun}(\Delta^1, \mathcal{C})$ be the $\infty$-category of arrows, with the cartesian fibration $\text{dom}: \text{Arr}(\mathcal{C}) \to \mathcal{C}$, denoted $\text{Arr}^{\mathcal{E}}(\mathcal{C})$. This is again a cartesian fibration, and the cartesian arrows are those for which the codomain component is in $\mathcal{F}$. If we take only those, we thus get a right fibration $\text{Arr}^{\mathcal{E}}(\mathcal{C})^{\text{cart}} \to \mathcal{C}$.

**Lemma 2.1.4.** Suppose that $(\mathcal{E}, \mathcal{F})$ is a factorization system on $\mathcal{C}$. For a presheaf $A: \mathcal{F}^{\text{op}} \to \mathcal{S}$ with corresponding right fibration $\bar{A} \to \mathcal{F}$, the left Kan extension along $j: \mathcal{F} \to \mathcal{C}$ corresponds to the left-hand composite in the pullback diagram

$$
\begin{array}{ccc}
j_!(\bar{A}) & \longrightarrow & \bar{A} \\
\downarrow & & \downarrow \\
\text{Arr}^{\mathcal{E}}(\mathcal{C})^{\text{cart}} & \longrightarrow & \mathcal{F} \\
\downarrow \text{dom} & & \downarrow \text{codom} \\
\mathcal{C} & & \mathcal{C}
\end{array}
$$
Proof. The codomain functor codom admits a right adjoint \( s \) (which is also a right inverse): \( s \) sends an object \( x \) to the identity arrow on \( x \). Since codom is a left adjoint of \( s \), we have \( \text{codom}^* = s_! \) as functors \( \text{Rfib}(\mathcal{F}) \to \text{Rfib}(\mathcal{C}^\text{cart}) \). Further, \( \text{dom} \circ s = j \), hence \( j_! = \text{dom}_! \circ s_! = \text{dom}_! \circ \text{codom}^* \).

2.2 Two identifications

2.2.1. Twisted arrow categories. Recall that for \( \mathcal{C} \) a small category (we shall only need the construction for \( 1 \)-categories), the twisted arrow category \( \text{tw}(\mathcal{C}) \) is the category of elements of the Hom functor \( \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Set} \). It thus has the arrows of \( \mathcal{C} \) as objects, and trapezoidal commutative diagrams

\[
\begin{array}{ccc}
  f' & \rightarrow & f \\
  \downarrow & \searrow & \downarrow \\
  \downarrow & & \downarrow \\
  m & \leftarrow & a + m + b \\
  a & \leftarrow & \\
\end{array}
\]

as morphisms from \( f' \) to \( f \). (Fancier viewpoints in the \( \infty \)-setting play a key role in [27], but in this paper only the naive viewpoint is needed.)

Lemma 2.2.2. There is a natural equivalence of categories

\[
\Delta_{\text{inert}} \simeq \text{tw}(BN).
\]

Proof. The category \( BN \) has only one object, and its set of arrows is \( \mathbb{N} \). Therefore, the object set of \( \text{tw}(BN) \) is \( \mathbb{N} \), just as for \( \Delta_{\text{inert}} \). A general map in \( \text{tw}(BN) \) is of the form

\[
\begin{array}{ccc}
  b & \rightarrow & a + m + b \\
  m & \leftarrow & a \\
\end{array}
\]

and it corresponds precisely to

\[
(a^T)^b \circ (a^\perp)^a : [m] \rightarrow [a + m + b]
\]

in \( \Delta_{\text{inert}} \).

This result should be well known, but we are not aware of a reference for it. It should also be mentioned that it is also closely related to a recent fancier result (Hoang [31] and Burkin [14]) stating that \( \Delta \) itself is the twisted arrow category (in a certain generalized sense) of the operad for unital associative algebras.

Lemma 2.2.3. There is a canonical identification

\[
\text{Arr}^{\text{act}}(\Delta)^{\text{cart}} \simeq \text{el}(BN)
\]

\[
\begin{array}{ccc}
  \text{dom} & \rightarrow & \Delta \\
  \downarrow & & \downarrow \\
  \Delta \leftarrow & & \\
\end{array}
\]
of right fibrations over $\Delta$.

**Proof.** The objects of $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}}$ in the fiber over $[k] \in \Delta$ are the active maps $[k] \to [n]$, whereas the objects in $\text{el}(BN)$ over $[k]$ are $k$-tuples $(n_1, \ldots, n_k)$ of natural numbers. The bijection between these sets was already described in Proposition 1.1.2.

Functoriality amounts to matching up cartesian lifts for the two fibrations. For active maps in $\Delta$, the lifts in $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}}$ are given by composition, and the lifts in $\text{el}(BN)$ are given by addition of natural numbers. For inert maps in $\Delta$, the lifts in $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}}$ are given by active-inert factorization, and the cartesian lifts in $\text{el}(BN)$ are given by projections. The checks are routine. \hfill $\square$

### 2.3 $j_!$ gives decomposition spaces and CULF maps

**Lemma 2.3.1.** For the terminal $\Delta_{\text{inert}}^{\text{op}}$-diagram 1 we have

$$j_!(1) = BN$$

*(nerve of the one-object category $\mathbb{N}$).*

**Proof.** At the level of right fibrations, Lemma 2.1.4 gives us that $j_!(1)$ is $\text{Arr}^{\text{act}}(\Delta)^{\text{cart}} \to \Delta$, and by Lemma 2.2.3 this is equivalent to $\text{el}(BN) \to \Delta$, as asserted. The result can also be proved by a direct calculation using Lemma 2.1.2.

In simplicial degree $k$ we have

$$j_!(1)_k = \sum_{n \in \mathbb{N}} \text{Hom}_{\Delta_{\text{act}}}( [k], [n] ) \simeq \sum_{n \in \mathbb{N}} \text{Hom}_{\Delta}( n, k ) \simeq (\Delta \downarrow k)^{\text{iso}} \simeq \mathbb{N}^k.$$  

It then remains to describe the face and degeneracy maps. \hfill $\square$

**Proposition 2.3.2.** If $A \to B$ is a map of $\Delta_{\text{inert}}^{\text{op}}$-diagrams, then $j_!(A) \to j_!(B)$ is CULF.

**Proof.** By standard arguments (see [21, Lemma 4.1]) it is enough to show that for any $k \in \mathbb{N}$, the naturality square

$$
\begin{array}{ccc}
\text{j}_!(A)_k & \to & \text{j}_!(A)_1 \\
\downarrow & & \downarrow \\
\text{j}_!(B)_k & \to & \text{j}_!(B)_1
\end{array}
$$

is a pullback. We establish this by showing that the fibers of the two horizontal maps are equivalent (for every point $x \in \text{j}_!(A)_1$). Let us compute the fiber of the top map. By the explicit formula in Corollary 2.1.2, this map is

$$
\sum_{[k] \to [n]} A_n \to \sum_{[1] \to [n]} A_n
$$

given on the indexing sets by precomposition with the (unique) active map $[1] \to [k]$ and on the summands by the identity map $A_n \to A_n$. The fiber is thus
discrete, given by the finite set \( \text{Hom}_{\Delta_{\text{act}}}([k], [n]) \) of active maps from \([k]\) to \([n]\).
In particular, the fiber does not depend on the point \(x \in A_n\), and indeed does not even depend on \(A\), only on \(n\). It is therefore the same for \(B\).

Lemma 2.3.1 states that \(j!(1) = BN\). Since \(BN\) is a Möbius decomposition space (in fact even a Möbius category in the sense of Leroux [44]), and since anything CULF over a Möbius decomposition space is again a Möbius decomposition space [22, Prop. 6.5], it follows from Proposition 2.3.2 that:

Corollary 2.3.3. For any \(A: \Delta_{\text{inert}}^{\text{op}} \rightarrow S\), the left Kan extension \(j!(A): \Delta^{\text{op}} \rightarrow S\) is a Möbius decomposition space.

3 CULF-graded decomposition spaces

Since \(j!(1) = BN\), it follows that every free decomposition space admits a CULF map to \(BN\). A decomposition space equipped with such a CULF map is called CULF-graded. We just showed that free decomposition spaces are Möbius. In this section, we establish an equivalence between \(\Delta_{\text{inert}}^{\text{op}}\)-diagrams and CULF-graded decomposition spaces.

3.0.1. Remark. The Möbius condition amounts to the existence of a length filtration [22]. In many cases of interest, the length filtration is actually a grading, which amounts to a simplicial map to \(BN\), and it makes sense to ask if this grading map is CULF. On the other hand, since a CULF-graded decomposition space is Möbius it has a length filtration. We do not know if in this case the two must always coincide, or if the CULF grading could be different. At the moment we have to regard CULF-grading as an extra structure, although it might be a property, so that the category of CULF-graded decomposition spaces would actually be equivalent to the full subcategory of \(\text{Decomp}\) spanned by the Möbius categories for which the length filtration happens to be a CULF grading.

3.1 Untwisting theorem

We briefly reproduce the main result of [27].

3.1.1. Edgewise subdivision and twisted arrow categories. The edgewise subdivision \(sd(X)\) of a simplicial space \(X: \Delta^{\text{op}} \rightarrow S\) is given by precomposing with the functor \(Q: \Delta \rightarrow \Delta, [n] \mapsto [2n + 1]\); see [27]. In particular, \(sd(X)_0 = X_1\) and \(sd(X)_1 = X_3\). When \(X\) is a decomposition space then \(sd(X)\) is in fact a Segal space (and conversely [10]); it is then denoted \(\text{tw}(X)\). Furthermore, \(sd\) takes CULF maps to right fibrations. When \(X\) is the nerve of a category (or Segal space), then \(sd(X)\) is the nerve of the twisted arrow category.

There is a natural transformation \(\lambda: \text{el} \Rightarrow \text{tw}\) from the category of elements to the twisted arrow category [27], given on objects by sending \(\Delta^n \rightarrow X\) (an object in \(\text{el}(X)\)) to the composite \(\Delta^1 \rightarrow \Delta^n \rightarrow X\) (an object in \(\text{tw}(X)\)).

\(^2\)The natural transformation \(\lambda\) goes back to Thomason’s Notebook 85 [55], where it is described in the special case of the nerve of a 1-category.
**Theorem 3.1.2** ([27]). For $D$ a Rezk-complete decomposition space, there is a natural equivalence of ∞-categories

$$
\text{Decomp}_{/D} \simeq \text{Rfib}(\text{tw } D).
$$

In the forward direction, it takes a CULF map $X \to D$ to $\text{tw}(X) \to \text{tw}(D)$. In the backward direction it is given essentially (modulo some translations involving nerves and elements) by pullback along $\lambda$.

In more detail, given a right fibration $X \to \text{tw}(D)$, in the diagram

$$
\begin{array}{ccc}
\lambda^*(X) & \xrightarrow{j} & X \\
\downarrow f & & \downarrow j \\
\text{el}(D) & \xrightarrow{\lambda} & \text{tw}(D) \\
\downarrow & & \downarrow \\
\Delta & & \\
\end{array}
$$

the map $f$ is shown to be CULF.

We shall need the theorem only in the very special case where $D$ is $BN$, and give an explicit description of $\lambda$ in this case.

### 3.2 Equivalence between $\Delta_{\text{inert}}^{\text{op}}$-diagrams and CULF-graded decomposition spaces

**Theorem 3.2.1.** The $j_!$ construction induces an equivalence of ∞-categories

$$
\text{PrSh}(\Delta_{\text{inert}}^{\text{op}}) \simeq \text{Decomp}_{/BN}.
$$

**Corollary 3.2.2.** A decomposition space is free if and only if it admits a CULF map to $BN$.

**Proof of Theorem 3.2.1.** Theorem 3.1.2 and Lemma 2.2.2 give us equivalences

$$
\text{Decomp}_{/BN} \simeq \text{Rfib}(\text{tw } BN) \simeq \text{PrSh}(\Delta_{\text{inert}}).
$$

It only remains to see that the inverse equivalence is actually given by $j_!$.

To this end we need to match up the diagrams

$$
\begin{array}{ccc}
\lambda^*(X) & \xrightarrow{j} & X \\
\downarrow & & \downarrow \\
\text{el}(BN) & \xrightarrow{\lambda} & \text{tw}(BN) \\
\downarrow & & \\
\Delta & & \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\tilde{A} & \xrightarrow{j} & \tilde{A} \\
\downarrow & & \downarrow \\
\text{Arr}^{\text{act}}(\Delta)_{\text{cart}} & \xrightarrow{\text{dom}} & \Delta_{\text{inert}} \text{dom} \\
\downarrow & & \downarrow \\
\Delta & & \\
\end{array}
$$

because the first diagram computes the inverse equivalence according to Theorem 3.1.2 whereas the second diagram computes $j_!$ (sliced over $BN$) according to 2.1.1. But this is the content of the following lemma. □
Lemma 3.2.3. The identifications $\text{Arr}_{\text{cart}}^{\text{act}}(\Delta) \simeq \text{el}(BN)$ (Lemma 2.2.3) and $\Delta_{\text{inert}} \simeq \text{tw}(BN)$ (Lemma 2.2.2) are compatible with the maps $\lambda$ and $\text{codom}$:

$$
\begin{array}{ccc}
\text{el}(BN) & \xrightarrow{\lambda} & \text{tw}(BN) \\
\simeq & & \simeq \\
\text{Arr}_{\text{cart}}^{\text{act}}(\Delta) & \xrightarrow{\text{codom}} & \Delta_{\text{inert}}.
\end{array}
$$

Proof. On objects, start with an element $(n_1, \ldots, n_k)$ in $\text{el}(BN)$ (lying over $[k] \in \Delta$). The component of $\lambda$ takes this to the ‘long edge,’ which is just the sum $n := \sum_{i=1}^{k} n_i$, which is an object of $\text{tw}(BN)$. Under the identification in Lemma 2.2.2, this corresponds to the object $[n] \in \Delta_{\text{inert}}$. The other way around: via the identification of Lemma 2.2.3, $(n_1, \ldots, n_k) \in \text{el}(BN)$ corresponds to $[k] \to [n]$, whose codomain is $[n] \in \Delta_{\text{inert}}$.

To see that the diagram commutes also on morphisms is a bit more involved. A morphism in $\text{el}(BN)$ is the data of $\Delta^h \xrightarrow{\phi} \Delta^k (n_1, \ldots, n_k) \to BN$. To compute $\lambda$ (see [27] for details), we first need to write down the diagram

$$
\begin{array}{ccc}
\Delta^1 & \xrightarrow{d^+} & \Delta^3 \\
\downarrow & & \downarrow \\
\Delta^h & \xrightarrow{\phi} & \Delta^k
\end{array}
$$

The value of $\lambda$ is now the $3$-simplex $\Delta^3 \to BN$ given by composition, namely the $3$-tuple

$$
\left( \sum_{i=1}^{\phi(0)} n_i, \sum_{i=\phi(0)+1}^{\phi(h)} n_i, \sum_{i=\phi(h)+1}^{k} n_i \right) =: (a, m, b).
$$

Under the identification in Lemma 2.2.2, this is the inert map $[m] \xrightarrow{(d^+)^a (d^-)^b} [n]$.

The other way around: starting again with the morphism $\Delta^h \xrightarrow{\phi} \Delta^k (n_1, \ldots, n_k) \to BN$ in $\text{el}(BN)$, under the identification of Lemma 2.2.3 the corresponding morphism in $\text{Arr}_{\text{cart}}^{\text{act}}(\Delta)$ is the commutative square

$$
\begin{array}{ccc}
[h] & \xrightarrow{} & [k] \\
\downarrow & & \downarrow \\
[m] & \xrightarrow{} & [n]
\end{array}
$$

given by active-inert factorization of the composite $[h] \to [k] \to [n]$, so we have

$$
m = \sum_{i=\phi(0)+1}^{\phi(h)} n_i,
$$

and the inclusion is given by writing $n = a + m + b$ where $a = \sum_{i=1}^{\phi(0)} n_i$ and $b = \sum_{i=\phi(h)+1}^{k} n_i$. This is the same inert map as we found above. \qed
4 Miscellaneous results

4.1 Remarks about sheaves

We have shown that \( j! : \Delta \to S \) is always a decomposition space. In this subsection we analyze under what conditions it is actually a category.

It is well known (see e.g. [5] or [37]) that \( \Delta_{\text{inert}} \) (interpreted as the category of nonempty linear graphs) has a Grothendieck topology for which a family of arrows constitute a covering when it is jointly surjective on vertices and edges. The elementary graphs are \([0]\) (the vertex) and \([1]\) (the edge), and every linear graph is canonically covered by its elementary subgraphs. We get in this way a canonical equivalence

\[
\text{PrSh}(\Delta_{\text{el}}) \simeq \text{Sh}(\Delta_{\text{inert}})
\]
given by left Kan extension along the (full) inclusion \( \Delta_{\text{el}} \subset \Delta_{\text{inert}} \) of the category of elementary graphs.

**Proposition 4.1.1.** The simplicial space \( j! : \Delta^{\text{op}} \to S \) is Segal if and only if \( A : \Delta^{\text{op}}_{\text{inert}} \to S \) is a sheaf.

**Proof.** The Segal map for \( X := j! \) is (for each \( k \)) the map

\[
X_k \to X_1 \times X_0 \cdots \times X_0 X_1.
\]

Lemma 2.1.2 gives

\[
X_k = \sum_{\alpha : [k] \to [n]} A_n
\]
on the left, whereas the right-hand side unpacks to

\[
(\sum_{n_1} A_{n_1}) \times A_0 \times \cdots \times A_0 (\sum_{n_k} A_{n_k}) = \sum_{(n_1, \ldots, n_k)} (A_{n_1} \times A_0 \times \cdots \times A_0 A_{n_k}).
\]

The Segal map is the specific combination of inert maps

\[
\sum_{\alpha : [k] \to [n]} A_n \to \sum_{(n_1, \ldots, n_k)} (A_{n_1} \times A_0 \times \cdots \times A_0 A_{n_k})
\]
given by sending the \( \alpha \)-summand to the \((n_1, \ldots, n_k)\)-summand via the map

\[
(\gamma_1^\alpha, \ldots, \gamma_k^\alpha) : A_n \to A_{n_1} \times A_0 \times \cdots \times A_0 A_{n_k}.
\]

Recall from 1.1.2 that the maps \( \gamma_i^\alpha \) are the maps appearing in the active-inert factorization of \( \alpha \circ \rho_i \):

\[
\begin{array}{ccc}
[1] & \cdots & [n_i] \\
\rho_i & \downarrow & \gamma_i^\alpha \\
[k] & \alpha \nearrow & [n]
\end{array}
\]

The families \( \{\gamma_i^\alpha \mid i \in [k]\} \) are precisely the reduced coverings of \([n]\), so if \( A \) is a sheaf, all the maps \( (\gamma_1^\alpha, \ldots, \gamma_k^\alpha)^* \) are thus equivalences, and therefore the Segal
map, which is the sum of them all, is an equivalence, which is to say that \( X \) is Segal. Conversely, if \( X \) is Segal, all these maps \( (\gamma_1^\alpha, \ldots, \gamma_k^\alpha) \) are equivalences, ensuring that \( A \) is a sheaf.

This result is only slightly more precise than the following classical result:

**Corollary 4.1.2** (Street [54]). A category admits a CULF functor to \( BN \) if and only if it is the free category on a directed graph.

The following corollary may be surprising at first sight:

**Corollary 4.1.3.** \( A : \Delta^{op}_{\text{inert}} \to S \) is a sheaf if and only if \( j^* j_!(A) : \Delta^{op}_{\text{inert}} \to S \) is a sheaf.

*Proof.* It is well known that a simplicial space \( X : \Delta^{op} \to S \) is Segal if and only if \( j^* X : \Delta^{op}_{\text{inert}} \to S \) is a sheaf. The result now follows from Proposition 4.1.1. \Box

**4.1.4. Remark.** The results fit together by considering the string of inclusions

\[ \Delta_{\text{el}} \to \Delta_{\text{inert}} \to \Delta_{\text{inj}} \to \Delta. \]

Here \( \Delta_{\text{el}} \) is the truncation of \( \Delta_{\text{inert}} \) in dimension 1, so that presheaves on \( \Delta_{\text{el}} \) are directed graphs. Now we have that left Kan extension along \( \Delta_{\text{el}} \to \Delta \) gives categories (free categories) or more generally Segal spaces (note that left Kan extension along \( \Delta_{\text{inert}} \to \Delta \) gives precisely the sheaves); left Kan extension along \( \Delta_{\text{inert}} \to \Delta \) gives decomposition spaces, by Corollary 2.3.3; and finally, left Kan extension along \( \Delta_{\text{inj}} \to \Delta \) gives split simplicial spaces, by a theorem of Gálvez, Kock, and Tonks [22, Prop. 5.16].

### 4.2 Restriction \( L \)-species

We briefly comment on the relationship between free decomposition spaces and certain species.

**4.2.1. Restriction species.** Where an ordinary species [34], [7], [2] is a functor \( F : B \to \text{Set} \) (where \( B \) is the groupoid of finite sets and bijections), in order to get a comultiplication of the set of \( F \)-structures, \( F \) should furthermore be contravariantly functorial in injections: Schmitt [51] thus defined a restriction species to be a functor \( R : \Pi^{op} \to \text{Set} \). For \( G \in R[S] \) an \( R \)-structure on a finite set \( S \), the comultiplication is then given by

[2]

\[ \Delta(G) = \sum_{A+B=S} G|A \otimes G|B \]

where the sum is over all splittings of the underlying set into two disjoint subsets, and where \( G|A \) denotes the restriction of \( G \) along \( A \subset S \).

Schmitt’s construction was subsumed in decomposition space theory in [24], where the following more general notion was also introduced, covering many interesting examples. Let \( C \) denote the category of finite posets and convex
monotone injections. A *directed restriction species* is a functor $\mathcal{C}^{\text{op}} \to \text{Set}$. The resulting comultiplication formula is very similar to (2), except that the sum is now over splittings where $A$ is required to be a lowerset and $B$ an upperset of the poset $S$. Ordinary restriction species are the special case of directed restriction species supported on discrete posets.

4.2.2. *L*-species and restriction *L*-species. There is another special case, which covers many of the examples of free decomposition spaces listed below, namely those directed restriction species that are supported on linear orders. Classically [7, Ch. 5], *L*-species are functors $\mathbb{L}^{\text{iso}} \to \text{Set}$, where $\mathbb{L}^{\text{iso}}$ is the groupoid of linear orders and monotone bijections (this groupoid is of course discrete). We arrive at the notion of *restriction L*-species, defined to be functors $\mathbb{L}^{\text{op}} \to \text{Set}$,

where now $\mathbb{L}$ denotes the category of linear orders and convex monotone injections.

The point is that the category $\mathbb{L}$ has a presentation like this:

$$
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow \\
d_\top & d_\bot & d_\top & d_\bot \\
\end{array}
\quad d_\top d_\bot = d_\bot d_\top.
$$

The following is now clear:

**Proposition 4.2.3.** A restriction $\mathbb{L}$-species is the same thing as a functor $A : \mathbb{L}^{\text{op}}_{\text{inert}} \to \text{Set}$ for which the two face maps $A_0 \xrightarrow{d_\top} A_1$ coincide.

Note in particular that this happens automatically whenever $A_0 = \ast$, as will be the case in many of the examples below.

5 Examples in combinatorics

The following selection of examples serves to illustrate the nature of free decomposition spaces as a common origin of comultiplications of deconcatenation type[6]. Note however that in many of the examples, the deconcatenation does not arise from a concatenation! More precisely, in these cases the free decomposition space is not Segal.

All the simplicial spaces in this section are actually simplicial sets.

5.1 Paths and words

5.1.1. Graphs and paths. For any directed graph (quiver) $A_0 \xrightarrow{A_1}$, put

$$
A_n = A_1 \times_{A_0} \cdots \times_{A_0} A_1, \quad n \geq 2.
$$

[6]Since these comultiplications are often the very simplest aspect of the combinatorial structure in question, we do not pretend to have made any contribution to the various application areas.
Then X was considered already by Dyckerhoff and Kapranov \cite{17}. This is the length-\( n \) set of all paths of length \( n \geq 1 \). Although words in an alphabet can be seen as special cases of \( \mathit{d} \gamma \)-paths. The face map \( \mathit{d} \) deletes all steps before the marked edge \( e \) and keeps \( e \) and all following steps. The face map \( \mathit{d}_2 \) deletes everything after \( e \), and keeps everything up to and including \( e \). (This \( X := j_i(A') \) is actually (the nerve of) a preorder on the set of edges: we can say that \( f \leq g \) if \( d_0(f) = d_1(g) \).)

Similarly the path \( \Delta^\text{op}_{\text{inert}} \)-diagram of a quiver can be shifted down \( d \) places. Then \( X_1 \) is the set of all paths of length \( \geq d \), and \( X_2 \) is the set of all paths of length \( \geq d \) with a marked \( d \)-path somewhere in the middle. The resulting comultiplication will then comultiply a path by selecting a \( d \)-path \( \gamma \) somewhere inside it, and then returning the prefix (including \( \gamma \)) on the left, and returning the postfix (also including \( \gamma \)) on the right. The group-like elements are thus the \( d \)-paths.

5.1.3. Words. Let \( S \) be an alphabet, whose elements we call \textit{symbols}, reserving the term \textit{letter} for the entries of a word. (Thus \textit{cbabb} is a 5-letter word comprising three symbols.) Although words in an alphabet can be seen as special cases...
of paths in a graph, namely in graphs with only one vertex \( S \), this case deserves special mention for its importance in combinatorics. We define \( W(S): A^\text{op}_{\text{inert}} \to \text{Set} \) by letting \( W(S)_n \) be the set of words of length \( n \); the face maps delete the first or last letter. In the resulting free decomposition space \( X := j_!(W S) \), we have \( X_1 \) the set of all words, and \( X_2 \) the set of all words with a splitting, say \( s_1s_2s_3|s_4s_5 \), and the outer face maps return those prefix and postfix words. The comultiplication of the incidence coalgebra is the standard deconcatenation, splitting a word in all possible ways into two parts.

This example is actually Segal. But we can disturb it by the same techniques: truncate (say, to allow only words of length between 4 and 13); that is, put the words of length 4 in degree 0, etc. The comultiplication in the incidence coalgebra will now sum over all ways to select a (convex) 4-letter subword (so that’s a middle part), and then include that middle subword in both tensor factors of the comultiplication.

An interesting special case of this is nonempty words. This is to put the length-\( n \) words in degree \( n - 1 \). Here the resulting comultiplication splits a word at a letter (not between two letters), and that letter then forms part of both sides of the split. (This coalgebra came up recently in connection with the Baez–Dolan construction [39].) This decomposition space can easily be interpreted as a category: the objects are the symbols, and a word from \( x \) to \( y \) is any word whose first letter is \( x \) and whose last letter is \( y \). Such arrows are composed by gluing words along their one-letter overlap.

5.1.4. Quasi-symmetric functions, I. When the alphabet is \( S = \mathbb{N}_+ \), the set of positive integers, the decomposition space

\[
Q := j_!(W \mathbb{N}_+)
\]

is the decomposition space whose incidence coalgebra is the coalgebra of quasi-symmetric functions \( \text{QSym} \) in the \( M \)-basis [19]. Indeed, the elements of \( Q_1 \) are words in \( \mathbb{N}_+ \), and to a word such as (23114) corresponds the quasi-symmetric function \( M_{23114} \). The comultiplication is simply word splitting (deconcatenation). (This is not the most interesting aspect of the Hopf algebra of quasi-symmetric functions, but it is the aspect that arises from a free decomposition space. It is also the one of relevance for the universal property that we come to in 5.3.)

5.1.5. \( \text{WQSym} \) and \( \text{FQSym} \). (The decomposition spaces here are still from [19], but the \( j_! \) interpretation is new.) We continue with the ordered alphabet \( S = \mathbb{N}_+ \). A word is packed if whenever a symbol occurs in it, then all smaller symbols occur too. If the first or last letter of a packed word is omitted, and if that letter was the only occurrence of a symbol, then the result is not again a packed word, but there is a canonical way to ‘pack’ it, by shifting down all bigger symbols. With this prescription the sets \( A_n \) of packed words of length \( n \) assemble into a \( A^\text{op}_{\text{inert}} \)-diagram, and we get a free decomposition space \( X := j_!(A) \), were \( X_1 \) is the set of all packed words, and \( X_2 \) is the set of all packed words with a word splitting. The resulting comultiplication is that of the \( F \)-basis in the Hopf algebra of word quasi-symmetric functions \( \text{WQSym} \) (cf. [8], [30]).
The same arguments apply to packed words without repetition of symbols. A packed word without repetition is the same thing as a permutation. We obtain the comultiplication of the Hopf algebra \( FQSym \) (free quasi-symmetric functions, also called the Malvenuto–Reutenauer Hopf algebra, after [47]), in the \( F \)-basis.

There is a different way to assemble packed words into a free decomposition space: let \( A_n \) denote the set of packed words on \( n \) symbols (any length), and let \( d^\top \) delete all occurrences of the largest symbol, and let \( d_\bot \) delete all occurrences of the smallest symbol (decrementing all larger symbols, for the word to remain packed). The resulting free decomposition space \( X := j_i(A) \) has \( X_1 \) the set of all packed words, and \( X_2 \) the set of all packed words with a linear splitting of the set of employed symbols into what we can call lower and upper symbols. Now \( d_2 : X_2 \to X_1 \) deletes from a word all upper symbols, while \( d_0 \) deletes all lower symbols (decrementing the remaining symbols until the word is packed again).

5.1.6. Parking functions. A parking function (see [53]) is a word \( w \) in the alphabet \( \mathbb{N}_+ \) such that if reordered to form a monotone word \( m \), then we have \( m_i \leq i, \forall i \). Let \( A_n \) denote the set of parking functions of length \( n \). The face maps are defined by deleting the first or last letter. This may violate the parking condition, but there is a canonical way to ‘parkify,’ by shifting down all bigger symbols (see [48] for details). With this prescription the sets \( A_n \) assemble into a \( \Delta_{\text{inert}}^{\text{op}} \)-diagram, and we get a free decomposition space \( X := j_i(A) \). Here \( X_1 \) is the set of all parking functions, and \( X_2 \) is the set of all parking function with a chosen breakpoint. The resulting comultiplication is that of the \( G \)-basis in the Hopf algebra of parking quasi-symmetric functions \( PQSym \) (see [48] again).

There is another way to assemble parking functions into a \( \Delta_{\text{inert}}^{\text{op}} \)-diagram. A breakpoint of a length-\( \ell \) parking function is an \( i \in \{0, \ldots, \ell\} \) such that there are exactly \( b \) occurrences of symbols smaller or equal to \( i \). For example the breakpoints of the parking function 162436166 are 0, 5, 9. Let \( A_n \) denote the set of parking functions (of any length) with precisely \( n + 1 \) breakpoints. There are face maps \( A_{n-1} \xrightarrow{d^\top} A_n \) given as follows: if the breakpoints of \( w \) are \( (0 = b_0, \ldots, b_n = \ell) \), then \( d^\top \) deletes all occurrences of symbols \( > b_{\ell-1} \), and \( d_\bot \) deletes all occurrences of symbols \( \leq b_1 \); this involves parkification. The resulting \( X := j_i(A) \) has \( X_1 \) the set of all parking functions, and \( X_2 \) the set of all parking functions with a chosen breakpoint. The resulting comultiplication is that of the \( G \)-basis in the Hopf algebra of parking quasi-symmetric functions \( PQSym \) (see [48] again).

5.1.7. Functorialities. All the constructions are functorial: given a homomorphism of graphs \( G \to H \), there is induced a CULF map \( X_G \to X_H \), and this works for all the variations mentioned.

Similarly, given a map of alphabets \( S \to T \), there is induced a CULF functor for the associated free decomposition spaces \( j_i(WS) \to j_i(WT) \). In particular, it is an interesting case when the alphabet \( S \) is positively graded, meaning that it has a map to \( \mathbb{N}_+ \). This gives a CULF map \( j_i(WS) \to Q \), the decomposition space for \( QSym \), which we shall come back to below.

For a graph \( V \xrightarrow{f} E \), consider words on the set of edges \( E \). Then there is a morphism of \( \Delta_{\text{inert}}^{\text{op}} \)-diagrams from paths to words. This can be seen as
coming from the graph homomorphism from $V \xleftarrow{\ } E$ to $1 \xleftarrow{\ } E$ (collapsing all vertices to a single vertex).

5.2 Further examples of deconcatenations

5.2.1. Noncrossing partitions. Let $A_n$ denote the set of noncrossing partitions of $n = \{1, 2, \ldots, n\}$ (see [53] for definitions). Given a noncrossing partition, one can obtain a new one by deleting the element $1 \in n$ or the element $n \in n$.

These assignments define the face maps of a $\Delta_{\text{inert}}$-diagram, and we get thus a free decomposition space $X := j_!(A)$, where $X_1$ is the set of all noncrossing partitions, and $X_2$ is the set of all noncrossing partitions with a marked gap. The resulting comultiplication is given by (for $\pi \in \text{NC}_n$)

$$\Delta(\pi) = \sum_{a+b=n} \pi|_a \otimes \pi|_b.$$

Here the sum is over the splitting of $n$ into an initial segment and a final segment, and $\pi|_a$ and $\pi|_b$ denote the restriction of $\pi$ to these two subsets.

5.2.2. Dyck paths. A Dyck path is a lattice path in $\mathbb{N} \times \mathbb{N}$ starting at $(0, 0)$ and ending at $(2\ell, 0)$ (for some $\ell \in \mathbb{N}$), taking only steps of type $(1, 1)$ and $(1, -1)$.

The height of a Dyck path is the maximal second coordinate. Let $A_n$ be the set of Dyck paths (varying $\ell$) of height $n$. The face maps $A_n \xleftarrow{\ } A_{n+1}$ are given by clipping the path, either at the top or at the bottom, and sliding the disconnected pieces left (and down) until they meet up again. Then $X := j_!(A)$ has $X_1$ the set of all Dyck paths (all lengths and all heights), $X_2$ is the set of all Dyck paths with a marked level, and more generally $X_k$ is the set of all Dyck paths with $k - 1$ marked levels (which may coincide). The inner face maps delete a level marking (without affecting the path), whereas the outer face maps clip the path outside the outermost level. For example, the outer face maps involved in the formula for comultiplication in the incidence coalgebra, namely $X_1 \xleftarrow{\ } X_2$, are exemplified here:

There is another way to assemble Dyck paths into a free decomposition space: A baseline point of a Dyck path is one with second coordinate 0. Let $A_n$ be the set of Dyck paths (any length and height) with $n - 1$ baseline points (and $A_0$ consists of the trivial Dyck path). Then $X := j_!(A)$ has in degree 1 the set of all Dyck paths, and in degree 2 the set of all Dyck paths with a chosen baseline point. The inner face maps forget baseline points, and the outer face maps delete the portion before the first or after the last chosen baseline point. For example
This second example is actually just an instance of the general word example, namely where the alphabet is the set of irreducible Dyck paths (meaning Dyck paths whose only baseline points are the start and the finish). The first example is not of this form, as can be seen by the fact that an element in $X_2$ contains more information than its two layers.

5.2.3. Heap orders. A heap order on a poset $P$ is a bijective monotone map $\phi : P \to n$ to a total order, that is, a numbering of the elements such that $x \leq y \Rightarrow \phi(x) \leq \phi(y)$. (See also Stanley [52].) (In computer science, this is also called a ‘topological sort’ [36].) Let $A_n$ denote the set of heap ordered posets of size $n$, and let the face maps delete the first or the last element. In the resulting free decomposition space $X := j! (A)$, we have $X_1$ the set of heap-ordered posets and $X_2$ the set of heap-ordered posets with a ‘cut’ compatible with the ordering. Heap orderings occur in computer science in connection with sorting and efficient data structures (most notably heap-ordered planar binary trees). A similar idea occurs in scheduling problems, as illustrated next.

5.2.4. Sequential processes of transition systems, Petri nets, or rewrite systems. Labelled transition systems. — For a labelled transition system (see for example [58] or [11]) with states $Q$, input alphabet $\Sigma$, and transition relation $R \subset Q \times \Sigma \times Q$, write $q \xrightarrow{\sigma} q'$ to express $(q, \sigma, q') \in R$. A run is a sequence

$$q_0 \xrightarrow{\sigma_1} q_1 \to \cdots \xrightarrow{\sigma_n} q_n$$

Let $A$ be the $\Delta^{\text{op}}_{\text{inert}}$-diagram with $A_n$ the set of runs of length $n$, with the obvious face maps. With $X := j! (A)$ we get $X_1$ the set of all runs, and $X_2$ the set of runs with a marked intermediate state.

Each run defines a word in $\Sigma$, and altogether a morphism of $\Delta^{\text{op}}_{\text{inert}}$-diagrams to $W(\Sigma)$, and hence a CULF map from $X$ to $j! (W(\Sigma))$. (Note that generally not all words arise like this, and a given word may arise from distinct runs. The set of words arising from the labelled transition system constitutes a language, but we should warn that these languages are not the regular languages usually studied in this context (see for example [59]), since that notion depends on an initial state and a set of terminal states. With the presence of initial and terminal states, one cannot define the face maps $A_n \leftrightarrow A_{n+1}$, as the initial or terminal state would be thrown away. At the level of languages, a standard example of a regular language is the condition ‘an even number of occurrences of a given symbol $\sigma$.’ Clearly this condition is not stable under removing the first or last letter of a word, so it does not fit the present class of examples.)

Petri nets. — More generally one can consider runs (firing sequences) of a given Petri net (see for example [19]). The markings then play the role of states, and the firings play the role of transitions. We shall not give the details, since the overall ideas are the same as for labelled transition systems.
In the algebraic semantics for Petri nets, one is often concerned with concatenation of firing sequences, and faces the trouble that automorphisms of markings prevent unique concatenation (see for example [3]). The decomposition space \( X := j_!(A) \) resulting from the general construction sketched above shows that while unique concatenations may be problematic, unique deconcatenations are easily obtained, a viewpoint advocated in [40] as a first step towards concatenation up to homotopy.

**Rewrite systems.** — More generally still, one can consider linear derivation sequences of a rewrite system (such as for example double-pushout rewrite systems of graphs or other adhesive categories, see for example [16]). The graphs now play the role of states, and specific single applications of a rewrite rule play the role of transitions. Again, we shall not go into technical details here, but wish to mention that the freeness of the resulting decomposition space actually plays a role in a recent Hopf-algebraic approach to double-pushout rewriting in the context of tracelets [4].

### 5.3 Aguiar–Bergeron–Sottile map

#### 5.3.1. Decomposition space of simplices. For any simplicial space \( X : \Delta^{op} \to S \), one can restrict to \( \Delta_{\text{inert}} \) and then left Kan extend back to get the free decomposition space \( j_! j^*(X) \), the decomposition space of all simplices of \( X \). If \( X \) is the nerve of a category, this construction gives the free category on the underlying directed graph of \( X \). The following subtle variation is much richer, though.

#### 5.3.2. The decomposition space of nondegenerate simplices. Instead of considering all simplices, we restrict to nondegenerate simplices. For this to make sense, it is necessary first to demand the simplicial space \( X \) to be complete [22] (necessary to make sense of nondegenerate simplices, denoted \( \vec{X}_n \subset X_n \)), and second to impose the condition on \( X \) that outer faces of nondegenerate simplices are again nondegenerate. The simplicial spaces for which this is true are called stiff in [22], where they are characterized in terms of exactness conditions (certain pushouts in \( \Delta \) being sent to pullbacks). Stiff simplicial spaces include all Möbius decomposition spaces [22], which are the ones of most interest in combinatorics.

For any stiff simplicial space \( X \), one can form the decomposition space \( J(X) \) of nondegenerate simplices [19]. It has the following pleasant description in terms of \( j_! \). Form first the \( \Delta_{\text{inert}}^{op} \)-diagram

\[
\vec{X}_0 \leftarrow \vec{X}_1 \leftarrow \vec{X}_2 \leftarrow \cdots
\]

Now put \( J(X) := j_!(\vec{X}) \). We have

\[
J(X)_0 = X_0, \quad J(X)_1 = \sum_{n \in \mathbb{N}} \vec{X}_n,
\]

the space of all nondegenerate simplices in all dimensions. In higher simplicial dimension, it has subdivided nondegenerate simplices. Precisely, an \( k \)-simplex of
\[ J(X) \text{ is a diagram} \]
\[ \Delta^k \to \Delta^n \overset{\text{nondeg}}{\longrightarrow} X \]
where the second map determines the nondegenerate \( n \)-simplex in \( X \) and the active map \( \Delta^k \to \Delta^n \) gives the subdivision, whereby the \( n \)-simplex is subdivided into \( k \) ‘stages.’ (Note that if \( X \) is Segal, then \( J(X) \) will be Segal again.)

The decomposition space of nondegenerate simplices seems to be interesting in general. (See [4] for a recent use in rewriting theory.) One reason for its importance is the general link with quasi-symmetric functions explained in the remainder of this subsection.

5.3.3. Quasi-symmetric functions, II. In the special case where the decomposition space \( X \) is the nerve of the monoid \( (\mathbb{N},+) \), then the \( \Delta^{\text{op inert}} \)-diagram of nondegenerate simplices has \( \mathbb{N}^n_+ \) in degree \( n \), and the free decomposition space is
\[ J(B\mathbb{N}) = Q \]
the decomposition space of words in \( \mathbb{N}^+ \), whose incidence coalgebra is \( \text{QSym} \), the coalgebra of quasi-symmetric functions already visited in 5.1.4 (cf. [19]).

5.3.4. Functoriality in conservative maps. A convenient class of simplicial maps between stiff simplicial spaces are the conservative maps, meaning cartesian on degeneracy maps. Conservative maps \( X \to Y \) preserve nondegenerate simplices, so as to induce a map of \( \Delta^{\text{op inert}} \)-diagrams \( \vec{X} \to \vec{Y} \), and therefore a CULF map between decomposition spaces \( J(X) \to J(Y) \).

5.3.5. IKEO map back to \( X \). There is a canonical simplicial map
\[ X \leftarrow J(X) \]
given in dimension 1 by sending a nondegenerate simplex to its long edge. In dimension \( k \), it sends a \( k \)-subdivided \( n \)-simplex to its \( k \)-simplex of stages. This map is IKEO (which stands for inner Kan and equivalence on objects), which is the class of simplicial maps that induce coalgebra homomorphisms contravariantly [21, 8.5–8.7].

5.3.6. Graded \( X \). The IKEO map \( X \leftarrow J(X) \) exists for all stiff simplicial spaces, and it defines a coalgebra homomorphism at the objective level whenever \( X \) is a decomposition space. However, in order for it to admit a homotopy cardinality so as to define also a coalgebra homomorphism at the level of vector spaces, it is necessary to demand the map to be finite. This means that for every 1-simplex \( f \), there is only a finite number of nondegenerate higher-dimensional simplices with long edge \( f \). This happens for Möbius decomposition spaces (see [22] for precise definition). The Möbius condition can be interpreted as saying that every 1-simplex has a finite length, which is the maximal dimension of such a nondegenerate simplex. This notion of length defines a filtration of \( X \), so as to make its incidence coalgebra filtered.

When \( X \) is (not just filtered but actually) graded, the assignment sending a 1-simplex to its length assembles into a simplicial map \( X \to B\mathbb{N} \), which is
generally neither CULF nor IKEO [22]. (In fact we have seen in 3.2.1 that having a CULF map to $B\mathbb{N}$ can happen only for $X$ free.) It is however conservative: this is to say that only degenerate 1-simplices have length 0. This is also the condition that allows for inducing a map of $\Delta^{\text{op}}_{\text{inert}}$-diagrams, in this case from $\tilde{X}$ to the $\Delta^{\text{op}}_{\text{inert}}$-diagram $W(N_+)$ of words in the alphabet $N_+$.

In conclusion, the construction $J$ is functorial in conservative maps, and we get in particular

$$J(X) \to J(B\mathbb{N}) = Q,$$

which always becomes CULF, by 2.3.2, and hence defines a coalgebra homomorphism (covariantly) from the incidence coalgebra to QSym.

5.3.7. Aguiar–Bergeron–Sottile map. Altogether, for $X$ a graded Möbius decomposition space, the maps

$$X \overset{\text{IKEO}}{\leftarrow} J(X) \overset{\text{CULF}}{\to} J(B\mathbb{N})$$

define the coalgebra homomorphism from the incidence coalgebra of $X$ to QSym, which is the universal map of combinatorial coalgebras discovered by Aguiar–Bergeron–Sottile [1]. (This map is described at the level of decomposition spaces in the forthcoming paper [19] but without knowing that the second part of it is free.)

5.4 Zeta functions

5.4.1. Standard zeta functions. The Möbius function (which exists for any Möbius decomposition space [22]) is the convolution inverse of the standard zeta function $\zeta$, namely the linear form that takes value 1 on every basis element. Both the Möbius function and the zeta function are linear forms on the incidence coalgebra (forming altogether the incidence algebra, which is the convolution algebra of the incidence coalgebra). At the objective level, they are linear functors rather than linear functions, which means that they are represented by spans of the form

$$X_1 \leftarrow F \to 1.$$

The standard zeta functor [22] is given by the span

$$X_1 \leftarrow X_1 \to 1.$$

(The Möbius functor cannot be represented by a single span, as it involves minus signs; instead it is a formal difference of certain spans [22].) Coalgebra homomorphisms act on linear functions by precomposition. At the objective level this looks different for CULF and for IKEO maps. If $Y \to X$ is a CULF map, then it takes a linear functor $X_1 \leftarrow F \to 1$ to the pullback $Y_1 \leftarrow Y_1 \times_X F \to 1$. In particular, the standard zeta functor $X_1 \leftarrow X_1 \to 1$ is sent to the standard zeta functor $Y_1 \leftarrow Y_1 \to 1$. For IKEO maps it is instead about postcomposition: if $Y \to X$ is IKEO, then it takes a linear functor $Y_1 \leftarrow F \to 1$ to the composite $X_1 \leftarrow Y_1 \leftarrow F \to 1$. 25
5.4.2. Special zeta function of a free decomposition space. For $X := j! (A)$ the free decomposition space on $A: \Delta_{\text{inert}}^{\text{op}} \to \mathcal{S}$, we define the special zeta functor by the span

$$X_1 \leftarrow A_0 + A_1 \rightarrow 1.$$ 

Recall that $X_1 = \sum_n A_n$, so the leftward map in the span is just the inclusion of the first two summands.

Note that the special zeta functions are preserved under free maps: indeed if $A \to B$ is a map of $\Delta_{\text{inert}}^{\text{op}}$-diagrams, then the CULF map $j! (A) \to j! (B)$ has in degree 1 the left vertical map:

$$\sum_n A_n \leftarrow A_0 + A_1 \quad \Downarrow \quad \Downarrow$$

$$\sum_n B_n \leftarrow B_0 + B_1,$$

and the fact that the square is a pullback shows that the special zeta functions are preserved.

The definition of special zeta function may look ad hoc, but it is motivated by the following.

5.4.3. Special zeta function of $J(X)$. Let $X$ be a graded Möbius decomposition space and consider the free decomposition space $J(X)$ of nondegenerate simplices of $X$. Recall that $J(X)_1 = \sum_n \vec{X}_n$. Its special zeta functor is thus

$$\sum_n \vec{X}_n \leftarrow \vec{X}_0 + \vec{X}_1 \rightarrow 1.$$ 

Since the composite

$$X_1 \leftarrow \sum_n \vec{X}_n \leftarrow \vec{X}_0 + \vec{X}_1$$

is clearly the identity (expressing the decomposition of $X_1$ into degenerate and nondegenerate simplices), we see that the induced coalgebra homomorphism sends the special zeta function of $J(X)$ to the standard zeta function of $X$. Furthermore, the definition of special zeta function is the only possible definition to achieve this property.

5.4.4. Special zeta function of $Q = J(BN)$. The special zeta function of QSym is thus given by the span

$$Q_1 \leftarrow \{\emptyset\} + N_+ \rightarrow 1.$$ 

Recall that $Q_1$ is the set of all words in the alphabet $N_+$; the set in the middle of the span consists of the empty word and all the one-letter words, expressing the fact that the zeta function of QSym considered by Aguiar–Bergeron–Sottile \[1\] sends the quasi-symmetric monomials $M_{\{\}}$ and $M_n$ to 1 and all other basis elements to 0. The universal property of QSym refers to this choice of zeta function.

26
function, stating that \((Q\text{Sym}, \zeta_{\text{special}})\) is the terminal object in the category of graded coalgebras equipped with a linear functional.

The description of the Aguiar–Bergeron–Sottile map in terms of the \(J\)-construction does not establish the universal property, but it does show that the standard zeta function on any graded Möbius decomposition space is induced from the special zeta function on \(Q\). Indeed the IKEO map \(X \leftarrow J(X)\) induces the standard zeta function on \(X\) from the special zeta functor on \(J(X)\) (as noted in 5.4.3), and the CULF map \(J(X) \to J(BN) = Q\) induces the special zeta functor on \(J(X)\) from the special zeta functor on \(Q\) (as noted in 5.4.2).

Acknowledgments. This work was supported by a grant from the Simons Foundation (#850849, PH). JK gratefully acknowledges support from grants MTM2016-80439-P and PID2020-116481GB-I00 (AEI/FEDER, UE) of Spain and 2017-SGR-1725 of Catalonia, and was also supported through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D grant number CEX2020-001084-M.

References

[1] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. Combinatorial Hopf algebras and generalized Dehn–Sommerville relations. Compos. Math. 142 (2006), 1–30. doi:10.1112/S0010437X0500165X.

[2] Marcelo Aguiar and Swapneel Mahajan. Monoidal functors, species and Hopf algebras, vol. 29 of CRM Monograph Series. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal. doi:10.1090/crmm/029.

[3] Paolo Baldan, Andrea Corradini, and Ugo Montanari. Concatenable graph processes: relating processes and derivation traces. In Kim Guldstrand Larsen, Sven Skyum, and Glynn Winskel, editors, Automata, Languages and Programming, 25th International Colloquium, ICALP’98, Aalborg, Denmark, July 13-17, 1998, Proceedings, vol. 1443 of Lecture Notes in Computer Science, pp. 283–295. Springer, 1998. doi:10.1007/BFb0055061.

[4] Nicolas Behr and Joachim Kock. Tracelet Hopf algebras and decomposition spaces. In Proceedings of the Fourth International Conference on Applied Category Theory ACT2021 (Cambridge, 2021), Electr. Proc. Theoret. Comput. Sci., 2022. arXiv:2105.06186.

[5] Clemens Berger. A cellular nerve for higher categories. Adv. Math. 169 (2002), 118–175. doi:10.1006/aima.2001.2056.

[6] Clemens Berger, Paul-André Melliès, and Mark Weber. Monads with arities and their associated theories. J. Pure Appl. Algebra 216 (2012), 2029–2048. doi:10.1016/j.jpaa.2012.02.039, arXiv:1101.3064.
[7] François Bergeron, Gilbert Labelle, and Pierre Leroux. Combinatorial species and tree-like structures, vol. 67 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1998. Translated from the 1994 French original by Margaret Readdy, With a foreword by Gian-Carlo Rota. doi:10.1017/CBO9781107325913.

[8] Nantel Bergeron and Mike Zabrocki. The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and co-free. J. Algebra Appl. 8 (2009), 581–600. doi:10.1142/S0219498809003485, arXiv:math/0509265.

[9] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer. 2-Segal sets and the Waldhausen construction. Topology Appl. 235 (2018), 445–484. doi:10.1016/j.topol.2017.12.009, arXiv:1609.02853.

[10] Julia E. Bergner, Angélica M. Osorno, Viktoriya Ozornova, Martina Rovelli, and Claudia I. Scheimbauer. The edgewise subdivision criterion for 2-Segal objects. Proc. Amer. Math. Soc. 148 (2020), 71–82. doi:10.1090/proc/14679, arXiv:1807.05069.

[11] J. A. Bergstra, A. Ponse, and S. A. Smolka, editors. Handbook of process algebra. North-Holland Publishing Co., Amsterdam, 2001. doi:10.1016/B978-0-444-82830-9.X5017-6.

[12] Marta Bunge and Marcelo Fiore. Unique factorisation lifting functors and categories of linearly-controlled processes. Math. Struct. Comput. Sci. 10 (2000), 137–163. doi:10.1017/S0960129599003023.

[13] Marta Bunge and Susan Niefield. Exponentiability and single universes. J. Pure Appl. Algebra 148 (2000), 217–250. doi:10.1016/S0022-4049(98)00172-8.

[14] Sergei Burkin. Twisted arrow categories, operads and Segal conditions. Theory Appl. Categ. 38 (2022), Paper No. 16, 595–660.

[15] Hongyi Chu and Rune Haugseng. Homotopy-coherent algebra via Segal conditions. Adv. Math. 385 (2021), 107733. doi:10.1016/j.aim.2021.107733, arXiv:1907.03977.

[16] Andrea Corradini, Ugo Montanari, Francesca Rossi, Hartmut Ehrig, Reiko Heckel, and Michael Löwe. Algebraic approaches to graph transformation - Part I: basic concepts and double pushout approach. In Grzegorz Rozenberg, editor, Handbook of Graph Grammars and Computing by Graph Transformations, Volume 1: Foundations, pp. 163–246. World Scientific, 1997. doi:10.1142/9789812384720_0003.

[17] Tobias Dyckerhoff and Mikhail Kapranov. Higher Segal spaces, vol. 2244 of Lecture Notes in Mathematics. Springer, Cham, 2019. doi:10.1007/978-3-030-27124-4, arXiv:1212.3563.
[18] Matthew Feller, Richard Garner, Joachim Kock, May U. Proulx, and Mark Weber. Every 2-Segal space is unital. Commun. Contemp. Math. 23 (2021), 2050055, 6. doi:10.1142/S0219199720500558, arXiv:1905.09580.

[19] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces of quasi-symmetric functions. Unpublished/in preparation.

[20] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces in combinatorics. Preprint, arXiv:1612.09225.

[21] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory. Adv. Math. 331 (2018), 952–1015. doi:10.1016/j.aim.2018.03.016, arXiv:1512.07573.

[22] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces, incidence algebras and Möbius inversion II: Completeness, length filtration, and finiteness. Adv. Math. 333 (2018), 1242–1292. doi:10.1016/j.aim.2018.03.017, arXiv:1512.07577.

[23] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces, incidence algebras and Möbius inversion III: The decomposition space of Möbius intervals. Adv. Math. 334 (2018), 544–584. doi:10.1016/j.aim.2018.03.018, arXiv:1512.07580.

[24] Imma Gálvez-Carrillo, Joachim Kock, and Andrew Tonks. Decomposition spaces and restriction species. Int. Math. Res. Notices 2020 (2020), 7558–7616. doi:10.1093/imrn/rny089, arXiv:1708.02570.

[25] David Gepner, Rune Haugseng, and Joachim Kock. ∞-Operads as analytic monads. Int. Math. Res. Notices 2022 (2022), 12516–12624. doi:10.1093/imrn/rnaa332, arXiv:1712.06469.

[26] Philip Hackney. Segal conditions for generalized operads. Preprint, arXiv:2208.13852.

[27] Philip Hackney and Joachim Kock. Cuf maps and edgewise subdivision. With an appendix coauthored with Jan Steinebrunner.

[28] Philip Hackney, Marcy Robertson, and Donald Yau. Infinity properads and infinity wheeled properads, vol. 2147 of Lecture Notes in Mathematics. Springer, Cham, 2015. doi:10.1007/978-3-319-20547-2, arXiv:1410.6716.

[29] Philip Hackney, Marcy Robertson, and Donald Yau. Modular operads and the nerve theorem. Adv. Math. 370 (2020), 107206. doi:10.1016/j.aim.2020.107206, arXiv:1906.01144.

[30] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. Commutative combinatorial Hopf algebras. J. Algebraic Combin. 28 (2008), 65–95. doi:10.1007/s10801-007-0077-0, arXiv:0605262.
[31] Truong Hoang. *Quillen cohomology of enriched operads*. Preprint, arXiv:2005.01198.

[32] Peter Johnstone. *A note on discrete Conduché fibrations*. Theory Appl. Categ. 5 (1999), No. 1, 1–11.

[33] Saj-nicole A. Joni and Gian-Carlo Rota. *Coalgebras and bialgebras in combinatorics*. Stud. Appl. Math. 61 (1979), 93–139. doi:10.1002/sapm197961293.

[34] André Joyal. *Une théorie combinatoire des séries formelles*. Adv. Math. 42 (1981), 1–82. doi:10.1016/0001-8708(81)90052-9.

[35] André Joyal. *The theory of quasi-categories and its applications*. No. 45 in Quaderns. CRM, Barcelona, 2008. Available at http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf.

[36] Donald E. Knuth. *The Art of Computer Programming*, Vol. 1: Fundamental Algorithms. Addison-Wesley, Reading, Mass., third edition, 1997.

[37] Joachim Kock. *Polynomial functors and trees*. Int. Math. Res. Notices 2011 (2011), 609–673. doi:10.1093/imrn/rnq068, arXiv:0807.2874.

[38] Joachim Kock. *Graphs, hypergraphs, and properads*. Collect. Math. 67 (2016), 155–190. doi:10.1007/s13348-015-0160-0, arXiv:1407.3744.

[39] Joachim Kock. *The incidence comodule bialgebra of the Baez–Dolan construction*. Adv. Math. 383 (2021), Paper No. 107693, 79. doi:10.1016/j.aim.2021.107693, arXiv:1912.11320.

[40] Joachim Kock. *Whole-grain Petri nets and processes*. J. ACM. (2022). doi:10.1145/3559103, arXiv:2005.05108.

[41] Joachim Kock and David I. Spivak. *Decomposition-space slices are toposes*. Proc. Amer. Math. Soc. 148 (2020), 2317–2329. doi:10.1090/proc/14834, arXiv:1807.06000.

[42] F. William Lawvere. *State categories and response functors. Dedicated to Walter Noll*. Preprint (May 1986).

[43] F. William Lawvere and Matías Menni. *The Hopf algebra of Möbius intervals*. Theory Appl. Categ. 24 (2010), 221–265.

[44] Pierre Leroux. *Les catégories de Möbius*. Cahiers Topol. Géom. Diff. 16 (1976), 280–282.

[45] Jacob Lurie. *Higher topos theory*, vol. 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009. doi:10.1515/9781400830558.
[46] JACOB LURIE. Higher algebra. Available from http://www.math.harvard.edu/~lurie/ 2013.

[47] CLAUDIA MALVENUTO and CHRISTOPHE REUTENAUER. Duality between quasi-symmetric functions and the Solomon descent algebra. J. Algebra 177 (1995), 967–982. doi:10.1006/jabr.1995.1336.

[48] JEAN-CHRISTOPHE NOVELLI and JEAN-YVES THIBON. Hopf algebras and dendriform structures arising from parking functions. Fund. Math. 193 (2007), 189–241. doi:10.4064/fm193-3-1, arXiv:0511200.

[49] WOLFGANG REISIG. Petri Nets: An Introduction, vol. 4 of EATCS Monographs on Theoretical Computer Science. Springer, 1985. doi:10.1007/978-3-642-69968-9.

[50] GIAN-CARLO ROTA. On the foundations of combinatorial theory. I. Theory of Möbius functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 2 (1964), 340–368. doi:10.1007/BF00531932.

[51] WILLIAM R. SCHMITT. Hopf algebras of combinatorial structures. Canad. J. Math. 45 (1993), 412–428. doi:10.4153/CJM-1993-021-5.

[52] RICHARD STANLEY. Ordered structures and partitions. Memoirs of the American Mathematical Society, no. 119. American Mathematical Society, Providence, 1972.

[53] RICHARD P. STANLEY. Parking functions and noncrossing partitions. Electron. J. Combin. 4 (1997), Research Paper 20, 1–14. The Wilf Festschrift (Philadelphia, PA, 1996). doi:10.37236/1335.

[54] ROSS STREET. Categorical structures. In Handbook of algebra, Vol. 1, pp. 529–577. North-Holland, Amsterdam, 1996. doi:10.1016/S1570-7954(96)80019-2.

[55] ROBERT THOMASON. Notebook 85, 1995. https://www.math-info-paris.cnrs.fr/bibli/digitization-of-robert-wayne-thomasons-notebooks/.

[56] MARK WEBER. Generic morphisms, parametric representations and weakly Cartesian monads. Theory Appl. Categ. 13 (2004), 191–234.

[57] MARK WEBER. Familial 2-functors and parametric right adjoints. Theory Appl. Categ. 18 (2007), 665–732.

[58] GLYNN WINSKEL and MOGENS NIELSEN. Models for concurrency. In Handbook of logic in computer science, vol. 4, pp. 1–148. Oxford Univ. Press, New York, 1995.

[59] SHENG YU. Regular languages. In Handbook of formal languages, Vol. 1, pp. 41–110. Springer, Berlin, 1997.