A New Nonconvex Strategy to Affine Matrix Rank Minimization Problem

Angang Cui, Jigen Peng, Haiyang Li, Junxiong Jia, and Meng Wen

Abstract

The affine matrix rank minimization (AMRM) problem is to find a matrix of minimum rank that satisfies a given linear system constraint. It has many applications in some important areas such as control, recommender systems, matrix completion and network localization. However, the problem (AMRM) is NP-hard in general due to the combinational nature of the matrix rank function. There are many alternative functions have been proposed to substitute the matrix rank function, which lead to many corresponding alternative minimization problems solved efficiently by some popular convex or nonconvex optimization algorithms. In this paper, we propose a new nonconvex function, namely, $T_L^{\alpha, \epsilon}$ function (with $0 \leq \alpha < 1$ and $\epsilon > 0$), to approximate the rank function, and translate the NP-hard problem (AMRM) into the $T_L^{\alpha, \epsilon}$ function affine matrix rank minimization (TLAMRM) problem. Firstly, we study the equivalence of problem (AMRM) and (TLAMRM), and proved that the uniqueness of global minimizer of the problem (TLAMRM) also solves the NP-hard problem (AMRM) if the linear map $A$ satisfies a restricted isometry property (RIP). Secondly, an iterative thresholding algorithm is proposed to solve the regularization problem (RTLAMRM) for all $0 \leq \alpha < 1$ and $\epsilon > 0$. At last, some numerical results on low-rank matrix completion problems illustrated that our algorithm is able to recover a low-rank matrix, and the extensive numerical on image inpainting problems shown that our algorithm performs the best in finding a low-rank image compared with some state-of-art methods.

Index Terms

Affine matrix rank minimization problem, $T_L^{\alpha, \epsilon}$ function, Equivalence, Iterative thresholding algorithm.

I. INTRODUCTION

The problem of recovering a low-rank matrix from a given linear system constraint, namely, affine matrix rank minimization (AMRM) problem, has been actively studied in different fields such as control [1], [2], recommender systems [3], [4], matrix completion [5], [6], [7], [8], [9] and network localization [10]. This rank minimization problem can be described as follows

\[
\text{(AMRM)} \quad \min_{X \in \mathbb{R}^{m \times n}} \text{rank}(X) \quad \text{s.t.} \quad A(X) = b,
\]
where \( b \in \mathbb{R}^d \) is a given vector, and \( \mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^d \) is a linear map determined by \( d \) matrices \( A_1, A_2, \cdots, A_d \in \mathbb{R}^{m \times n} \), i.e.,
\[
\mathcal{A}(X) := (\langle A_1, X \rangle, \langle A_2, X \rangle, \cdots, \langle A_d, X \rangle)^\top \in \mathbb{R}^d
\]
with \( \langle A_i, X \rangle = \text{trace}(A_i^\top X) \), \( i = 1, 2, \cdots, d \). Without loss of generality, we assume that \( m \leq n \) throughout this paper. An important special case of the problem (AMRM) is the matrix completion (MC) problem:
\[
\text{(MC)} \quad \min_{X \in \mathbb{R}^{m \times n}} \text{rank}(X) \quad \text{s.t.} \quad X_{i,j} = M_{i,j}, \quad (i,j) \in \Omega,
\]
where \( X, M \in \mathbb{R}^{m \times n} \) are both \( m \times n \) real matrices, \( \Omega \) is the set of indices of samples and the subset \( \{M_{i,j} | (i,j) \in \Omega\} \) of the entries is known. This problem has been widely applied in signal and image processing [7], [11], machine learning [12], computer vision [13] and the famous Netflix problem [14]. Unfortunately, problem (1) is NP-hard [5], [7] for which all known finite time algorithms have at least doubly exponential running times in both theory and practice. To overcome such a difficulty, Recht [5], Fazel [7] and other researchers (e.g., [3], [6], [15]) introduced the convex envelope of \( \text{rank}(X) \) on the set \( \{X \in \mathbb{R}^{m \times n} : \|X\|_2 \leq 1\} \), namely, nuclear-norm \( \|X\|_* \) of \( X \), to relax the rank of \( X \). It leads to the nuclear-norm affine matrix rank minimization (NAMRM) problem
\[
\text{(NAMRM)} \quad \min_{X \in \mathbb{R}^{m \times n}} \|X\|_* \quad \text{s.t.} \quad \mathcal{A}(X) = b
\]
for the constrained problem and
\[
\text{(RNAMRM)} \quad \min_{X \in \mathbb{R}^{m \times n}} \left\{ \|\mathcal{A}(X) - b\|_2^2 + \lambda \|X\|_* \right\}
\]
for the regularized unconstrained problem, where \( \lambda > 0 \) is the regularization parameter and \( \|X\|_* = \sum_{i=1}^m \sigma_i(X) \) is defined as the sum of the nonzero singular values of \( X \in \mathbb{R}^{m \times n} \).

Recht et al. [5] have shown that if a certain restricted isometry property holds for the linear map \( \mathcal{A} \), the minimum rank solution can be recovered by solving the problem (NAMRM), and the sharp results can be seen in [16], [17]. Many algorithms for solving the problems (NAMRM) and (RNAMRM) have been proposed. These include semidefinite programming and interior point SDP solver [5], [18], singular value thresholding (SVT) algorithm [15], accelerated proximal gradient (APG) algorithm [19], inexact proximal point algorithms [20], fixed point and Bregman iterative algorithms [21], [22]. However, the problem (RNAMRM) may yield a matrix with much higher rank and need more observations to recover a real low-rank matrix [3], and it may tend to lead to biased estimation by shrinking all the singular values toward zero simultaneously [15].

On the other hand, with recent development of non-convex relaxation approaches in sparse signal recovery problems, a large number of non-convex surrogate functions have been proposed to approximate the \( l_0 \)-norm, including \( l_p \)-norm \( (0 < p < 1) \) [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], MCP (Mini-max Concave Plus) [35], SCAD (Smoothly Clipped Absolute Deviation) [36], Laplace [37], [38], Logarithm [39], capped \( l_1 \)-norm [40], smoothed \( l_0 \)-norm [41]. Inspired by the good performance of the non-convex surrogate functions in sparse signal recovery problems, these popular nonconvex surrogate functions have been extended on the singular values to better approximate the rank function (e.g., [42], [43], [44], [45], [46], [47], [48], [49]). Some empirical evidence has also shown that the corresponding non-convex algorithms can really make a better recovery in some
matrix rank minimization problems. Different from previous studies, in this paper, a new continuous promoting low-rank function
\[
\sum_{i=1}^{m} \varphi_{\alpha}^{\epsilon}(\sigma_i(X)) = \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{\left(\sigma_i(X) + \epsilon\right)^{1/2-\alpha}}
\]
\(5\)
in terms of the singular values of matrix \(X\) is considered to approximate the rank function, where the continuous function
\[
\varphi_{\alpha}^{\epsilon}(|t|) = \frac{|t|^{1/2}}{(|t| + \epsilon)^{1/2-\alpha}}
\]
\(6\) is the \(TL_{\alpha}^{\epsilon}\) function for all \(0 \leq \alpha < 1\) and \(\epsilon > 0\). It is easy to verify that the \(TL_{\alpha}^{\epsilon}\) function \(\varphi_{\alpha}^{\epsilon}\) is concave for any \(\alpha \in (0, 1/2]\). Moreover, with the change of parameters \(\alpha\) and \(\epsilon\), we have
\[
\lim_{\alpha \to 0^+} \lim_{\epsilon \to 0^+} \varphi_{\alpha}^{\epsilon}(|t|) = \begin{cases} 0, & \text{if } t = 0; \\ 1, & \text{if } t \neq 0,
\end{cases}
\]
and therefore the function (5) interpolates the rank of matrix \(X\):
\[
\lim_{\alpha \to 0^+} \lim_{\epsilon \to 0^+} \sum_{i=1}^{m} \varphi_{\alpha}^{\epsilon}(\sigma_i(X)) = \lim_{\alpha \to 0^+} \lim_{\epsilon \to 0^+} \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{\left(\sigma_i(X) + \epsilon\right)^{1/2-\alpha}} = \text{rank}(X).
\]
\(7\) Then, by this transformation, we propose the new approximation optimization problem of the problem (AMRM) which has the following form
\[
(\text{TLAMRM}) \quad \min_{X \in \mathbb{R}^{m \times n}} \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{\left(\sigma_i(X) + \epsilon\right)^{1/2-\alpha}} \quad \text{s.t. } A(X) = b
\]
\(8\) for the constrained problem and
\[
(\text{RTLAMRM}) \quad \min_{X \in \mathbb{R}^{m \times n}} \left\{ \|A(X) - b\|_2^2 + \lambda \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{\left(\sigma_i(X) + \epsilon\right)^{1/2-\alpha}} \right\}
\]
\(9\) for the regularization problem.

This paper is organized as follows. Section II presents some useful notions and crucial preliminary results that are used in this paper. Section III presents the equivalence between minimization problems (AMRM) and (TLAMRM). Section IV presents an iterative thresholding algorithm to solve the problem (RTLAMRM) for all \(0 \leq \alpha < 1\) and \(\epsilon > 0\). The experimental results are presented in Section V. Finally, some conclusion remarks are presented in Section VI.

II. NOTIONS AND PRELIMINARY RESULTS

In this section, we present some useful notions and crucial preliminary results that are used in this paper.

A. Notions

The space of \(m \times n\) real matrices is denoted by \(\mathbb{R}^{m \times n}\). Given any \(X \in \mathbb{R}^{m \times n}\), the Frobenius norm of \(X\) is denoted by \(\|X\|_F\), namely, \(\|X\|_F = \sqrt{\text{tr}(X^T X)}\), where \(\text{tr}(\cdot)\) denotes the trace of a matrix. Given any matrices \(X, Y \in \mathbb{R}^{m \times n}\), the standard inner product of matrices \(X\) and \(Y\) is denoted by \(\langle X, Y \rangle\), and \(\langle X, Y \rangle = \text{Tr}(Y^T X)\).

The linear map \(A : \mathbb{R}^{m \times n} \to \mathbb{R}^d\) determined by \(d\) matrices \(A_1, A_2, \ldots, A_d \in \mathbb{R}^{m \times n}\) is given by \(A(X) = (\langle A_1, X \rangle, \langle A_2, X \rangle, \ldots, \langle A_d, X \rangle)^T \in \mathbb{R}^d\). Define \(A = (\text{vec}(A_1), \text{vec}(A_2), \ldots, \text{vec}(A_d))^T \in \mathbb{R}^{d \times mn}\) and \(x = (\text{vec}(A_1), \text{vec}(A_2), \ldots, \text{vec}(A_d))^T \in \mathbb{R}^{d \times mn}\) and \(x =\)
In order to get equation (12), it suffices to impose Lemma 3.

Since the function \( \sum_{i=1}^{d} y_i A_i \). The singular value decomposition of matrix \( X \in \mathbb{R}^{m \times n} \) is \( X = U_X [\text{Diag}(\sigma(X)), 0_{m,n-m}] V_X^T \), where \( U_X \) is an \( m \times m \) unitary matrix, \( V_Y \) is an \( n \times n \) unitary matrix, \([\text{Diag}(\sigma(X)), 0] \in \mathbb{R}^{m \times n} \), \( 0_{m,n-m} \in \mathbb{R}^{m \times n-m} \) is a \( m \times (n-m) \) zero matrix, and the vector \( \sigma(X) : \sigma_1(X) \geq \sigma_2(X) \geq \cdots \geq \sigma_r(X) \geq \sigma_{r+1}(X) = \cdots = \sigma_m(X) = 0 \), arranged in descending order, denotes the singular value vector of matrix \( X \).

**B. Preliminary results**

**Lemma 1.** (see [5]) Let \( M, N \in \mathbb{R}^{m \times n} \). Then there exist matrices \( N_1, N_2 \in \mathbb{R}^{m \times n} \) such that, for any \( \eta \in \mathbb{R}^{m \times n} \), such that, for any \( \eta \in \mathbb{R}^{m \times n} \),

1. \( N = N_1 + N_2 \);
2. \( \text{rank}(N_1) \leq 2 \text{rank}(M) \);
3. \( MN_2 = 0_{m,m} \) and \( M^T N_2 = 0_{n,n} \);
4. \( \langle N_1, N_2 \rangle = 0 \).

**Lemma 2.** Let \( M, N \in \mathbb{R}^{m \times n} \). If \( MN^T = 0_{m,m} \) and \( M^T N = 0_{n,n} \), then

\[
\sum_{i=1}^{m} \frac{\sigma_i(M+N)}{(\sigma_i(M)+\epsilon)^{1-2\alpha}} = \sum_{i=1}^{m} \frac{\sigma_i(M)}{(\sigma_i(M)+\epsilon)^{1-2\alpha}} + \sum_{i=1}^{m} \frac{\sigma_i(N)}{(\sigma_i(N)+\epsilon)^{1-2\alpha}}.
\]

**Proof.** Consider the singular value decompositions of matrices \( M \) and \( N \):

\[
M = U_M [\text{Diag}(\sigma(M)), 0_{m,n-m}] V_M^T, \quad N = U_N [\text{Diag}(\sigma(N)), 0_{m,n-m}] V_N^T.
\]

Since the unitary matrices \( U_M, U_N \in \mathbb{R}^{m \times m} \) are invertible, the condition \( MN^T = 0_{m,m} \) implies that \( V_M^T V_N = 0_{n,n} \). Similarly, \( M^T N = 0_{n,n} \) implies that \( U_M^T U_N = 0_{m,m} \). Thus, the following is a valid SVD for \( M+N \),

\[
M + N = \begin{bmatrix} U_M & U_N \end{bmatrix} \begin{bmatrix} \text{Diag}(\sigma(M)) & 0_{m,n-m} & 0_{m,m} & 0_{m,n-m} \\ 0_{m,m} & 0_{m,n-m} & \text{Diag}(\sigma(N)) & 0_{m,n-m} \end{bmatrix} \begin{bmatrix} V_M \\ V_N \end{bmatrix}^T.
\]

This shows that the singular values of \( M+N \) are equal to the union (with repetition) of the singular values of \( M \) and \( N \). Hence, we get the equation (10).

**Lemma 3.** Let \( X = U_X \text{Diag}(\sigma(X)) V_X^T \) be the singular value decomposition of matrix \( X \in \mathbb{R}^{m \times n} \), and \( \text{rank}(X) = r \). For any \( \alpha \in [0, 1/2] \) and \( \epsilon \in (0, 1/3] \), there exists

\[
\eta_1 = \frac{r \sigma_1(X)}{2^{2\alpha-1/3-2\alpha \epsilon} 1^{1-2\alpha}}
\]

such that, for any \( \eta \geq \eta_1 \),

\[
\sum_{i=1}^{m} \frac{\sigma_i(\eta^{-1}X)}{(\sigma_i(\eta^{-1}X)+\epsilon)^{1-2\alpha}} \leq 2^{2\alpha-1/3-2\alpha}.
\]

**Proof.** Since the function \( t/(t+\epsilon)^{1-2\alpha} \) is increasing in \( t \in [0, +\infty) \), we have

\[
\sum_{i=1}^{m} \frac{\sigma_i(\eta^{-1}X)}{(\sigma_i(\eta^{-1}X)+\epsilon)^{1-2\alpha}} \leq \frac{r \sigma_1(\eta^{-1}X)}{(\eta \epsilon)^{1-2\alpha}} \leq \frac{r \sigma_1(X)}{\eta \epsilon^{1-2\alpha}}.
\]

In order to get equation (12), it suffices to impose

\[
\frac{r \sigma_1(X)}{\eta \epsilon^{1-2\alpha}} \leq 2^{2\alpha-1/3-2\alpha},
\]

(14)
equivalently,
\[ \eta \geq \frac{r\sigma_1(X)}{2^{2\alpha - 1/3} - 2\alpha - 1 - 2\alpha}. \]

This completes the proof. \[\Box\]

**Lemma 4.** ([24]) For any fixed \( \lambda > 0 \) and \( y_i \in \mathbb{R} \), let

\[ h_\lambda(y_i) := \arg \min_{x_i \geq 0} \left\{ (x_i - y_i)^2 + \lambda x_i^{1/2} \right\}, \quad (15) \]

then the half thresholding function \( h_\lambda \) can be analytically expressed by

\[ h_\lambda(y_i) = \begin{cases} h_{\lambda, 1/2}(y_i), & \text{if } y_i > \frac{3\pi^2}{4} \lambda^{2/3}; \\ 0, & \text{if } y_i \leq \frac{3\pi^2}{4} \lambda^{2/3}; \end{cases} \quad (16) \]

where

\[ h_{\lambda, 1/2}(y_i) = \frac{2}{3} \left( 1 + \cos \left( \frac{2\pi}{3} - \frac{2}{3} \phi_\lambda(y_i) \right) \right) \quad (17) \]

with

\[ \phi_\lambda(y_i) = \arccos \left( \frac{\lambda}{8} \left( \frac{|y_i|}{3} \right)^{-3/2} \right). \quad (18) \]

**Definition 1.** ([24]) For any \( \lambda > 0 \) and \( y = (y_1, y_2, \ldots, y_m)^T \in \mathbb{R}^m \), the vector half thresholding operator \( H_\lambda \) is defined as

\[ H_\lambda(y) = (h_\lambda(y_1), h_\lambda(y_2), \ldots, h_\lambda(y_m))^T, \quad (19) \]

where \( h_\lambda \) is defined in Lemma 3.

**Lemma 5.** ([24]) For any \( y_i > \frac{3\pi^2}{4} \lambda^{2/3} \), the half thresholding function \( h_\lambda(y_i) \) defined in (15) is strict increasing.

**Definition 2.** Suppose matrix \( Y \in \mathbb{R}^{m \times n} \) admits a singular value decomposition as \( Y = UV \), where \( U \) and \( V \) are the singular value vectors of \( Y \), and effec-ctively shrinks the singular values towards zero. If \( Y \) is defined as

\[ Y = [\Diag(Y)], \quad (20) \]

where \( H_\lambda \) is defined in Definition 1.

The matrix half thresholding operator \( H_\lambda \) simply applies the vector half thresholding operator \( H_\lambda \) defined in Definition 1 to the singular value vector of a matrix, and effectively shrinks the singular values towards zero. If there are some nonzero singular values of matrix \( X \) are below the threshold value \( \frac{\pi^2}{4} \lambda^{2/3} \), we can immediately get that the rank of \( H_\lambda(Y) \) lower than the rank of matrix \( Y \).

Combing Lemma 5 and Definition 2, we can get the following crucial Lemma.

**Lemma 6.** Let \( Y = UV \), where \( U \) and \( V \) are the singular value vectors of \( Y \), and

\[ H_\lambda(Y) = U \Diag(H_\lambda(Y)), \quad (21) \]

Then

\[ \mathcal{H}_\lambda(Y) = \arg \min_{X \in \mathbb{R}^{m \times n}} \left\{ \|X - Y\|_F^2 + \lambda \|X\|_1^{1/2} \right\}. \]

**Proof.** Similar argument as used in the proof of ([50], Theorem 2.1). \[\Box\]
Theorem 1. Let \( X^* \) and \( X_0 \) be the minimizers to the problem (TLAMRM) and (AMRM) respectively. For any \( \alpha \in [0, 1/2] \) and \( \epsilon \in (0, 1/3] \), if there is a number \( k > 2\beta \), such that

\[
e^{1-2p}(2\beta)^{-3/2} \sqrt{1 - \delta_{2\beta+k}(A)} - \frac{1 + \delta_k(A)}{k} > 0,
\]

then the unique minimizer \( X^* \) of the problem (TLAMRM) is exactly \( X_0 \), where \( \beta = \text{rank}(X_0) \).
\textbf{Proof.} Let \( E = X^* - X_0 \). Applying Lemma 1 to the matrices \( X_0 \) and \( E \), there exist matrices \( E_0 \) and \( E_c \) such that \( E = E_0 + E_c \), \( \text{rank}(E_0) \leq 2\text{rank}(X_0), X_0 E_0^\top = 0_{m,m}, X_0^\top E_c = 0_{n,n} \) and \( \langle E_0, E_c \rangle = 0 \). Then

\[
\sum_{i=1}^{m} \frac{\sigma_i(X_0)}{(\sigma_i(X_0) + \epsilon)^{1-2\alpha}} \geq \sum_{i=1}^{m} \frac{\sigma_i(X^*)}{(\sigma_i(X^*) + \epsilon)^{1-2\alpha}} = \sum_{i=1}^{m} \frac{\sigma_i(X_0 + E)}{(\sigma_i(X_0 + E) + \epsilon_1)^{1-2\alpha}} \geq \sum_{i=1}^{m} \frac{\sigma_i(X_0 + E_c)}{(\sigma_i(X_0 + E_c) + \epsilon)^{1-2\alpha}} - \sum_{i=1}^{m} \frac{\sigma_i(E_0)}{(\sigma_i(E_0) + \epsilon_1)^{1-2\alpha}} = \sum_{i=1}^{m} \frac{\sigma_i(E_0)}{(\sigma_i(E_0) + \epsilon)^{1-2\alpha}} - \sum_{i=1}^{m} \frac{\sigma_i(E_0)}{(\sigma_i(E_0) + \epsilon)^{1-2\alpha}},
\]

where the first inequality follows from the optimality of \( X^* \), third assertion follows the triangle inequality and the last one follows Lemma 2. Rearranging (26), we can conclude that

\[
\sum_{i=1}^{m} \frac{\sigma_i(E_0)}{(\sigma_i(E_0) + \epsilon)^{1-2\alpha}} \geq \sum_{i=1}^{m} \frac{\sigma_i(E_c)}{(\sigma_i(E_c) + \epsilon)^{1-2\alpha}}.
\]

We partition \( E_c \) into a sum of matrices \( E_{1j}, E_{2j}, \ldots \), each of rank at most \( k \). Let \( E_{c} = U_{E_c} \{ \text{Diag}(\sigma(E_{c})) \}, 0 \} V_{E_c}^\top \) be the singular value decomposition of matrix \( E_c \). For each \( j \geq 1 \), define the index set \( I_j = \{ k(j-1) + 1, \ldots, kj \} \), and let \( E_j = U_{E_{1j}} \{ \text{Diag}(\sigma(E_{j})) \}, 0 \} V_{E_{1j}}^\top \) (notice that \( \langle E_k, E_l \rangle = 0 \text{ if } k \neq l \)). For each \( \nu \in I_j \), by Lemma 3, there exist \( \gamma_1 = \frac{\kappa_\nu(E_{1j})}{2\alpha-1-2\alpha} \), for any \( \gamma > \gamma_1, \alpha \in [0, 1/2] \) and \( \epsilon \in (0, 1/3] \), we have

\[
\frac{\sigma_\nu(\gamma^{-1} E_j)}{(\sigma_\nu(\gamma^{-1} E_j) + \epsilon)^{1-2\alpha}} \leq \sum_{\nu \in I_j} \frac{\sigma_\nu(\gamma^{-1} E_j)}{(\sigma_\nu(\gamma^{-1} E_j) + \epsilon)^{1-2\alpha}} \leq 2^{2\alpha-1-3-2\alpha}.\]

Also since

\[
\frac{\sigma_\nu(\gamma^{-1} E_j)}{(\sigma_\nu(\gamma^{-1} E_j) + \epsilon)^{1-2\alpha}} \leq 2^{2\alpha-1-3-2\alpha} \Leftrightarrow \sigma_\nu(\gamma^{-1} E_j) \leq 1/3,
\]

we have

\[
\sigma_\nu(\gamma^{-1} E_j) \leq \frac{\sigma_\nu(\gamma^{-1} E_j)}{(\sigma_\nu(\gamma^{-1} E_j) + \epsilon)^{1-2\alpha}}, \quad \forall \nu \in I_j.
\]

Moreover, by the construction of matrices \( E_{ij} \)s, we can get that

\[
\sigma_\mu(\gamma^{-1} E_{j+1}) \leq \sum_{\nu \in I_j} \sigma_\nu(\gamma^{-1} E_j), \quad \forall \mu \in I_{j+1}.
\]

It follows that

\[
\| \gamma^{-1} E_{j+1} \|_F \leq \frac{1}{\sqrt{k}} \sum_{\nu \in I_j} \frac{\sigma_\nu(\gamma^{-1} E_j)}{(\sigma_\nu(\gamma^{-1} E_j) + \epsilon)^{1-2\alpha}}
\]

and

\[
\sum_{j \geq 2} \| \gamma^{-1} E_{j+1} \|_F \leq \frac{1}{\sqrt{k}} \sum_{j \geq 1} \frac{\sigma_\nu(\gamma^{-1} E_j)}{(\sigma_\nu(\gamma^{-1} E_j) + \epsilon)^{1-2\alpha}} \leq \frac{1}{\sqrt{k}} \sum_{\nu \in I_j} \sigma_\nu(\gamma^{-1} E_0) \leq \frac{1}{\sqrt{k}} \left( \sum_{i=1}^{m} \frac{\sigma_i(\gamma^{-1} E_0)}{\sigma_i(\gamma^{-1} E_0) + \epsilon} \right)^{1/2} \leq \left( \sum_{i=1}^{m} \frac{\sigma_i(\gamma^{-1} E_0)}{\sigma_i(\gamma^{-1} E_0) + \epsilon} \right)^{1/2} \leq \left( \sum_{i=1}^{m} \frac{\sigma_i(\gamma^{-1} E_0)}{\sigma_i(\gamma^{-1} E_0) + \epsilon} \right)^{1/2},
\]

where the second inequality follows from (27).
At the next step, we will derive two inequalities between the Frobenius norm and function \( \sum_{i=1}^{m} |t_i|^{1/2} / (|t_i| + \epsilon)^{1/2-\alpha} \). Since
\[
\frac{(\sigma_i(\gamma^{-1}E_0))^{1/2}}{(\sigma_i(\gamma^{-1}E_0) + \epsilon)^{1/2-\alpha}} \leq \frac{(\sigma_i(\gamma^{-1}E_0))^{1/2}}{\epsilon^{1/2-\alpha}}
\]
we have
\[
\sum_{i=1}^{m} \frac{(\sigma_i(\gamma^{-1}E_0))^{1/2}}{(\sigma_i(\gamma^{-1}E_0) + \epsilon)^{1/2-\alpha}} \leq \sum_{i=1}^{m} \frac{(\sigma_i(\gamma^{-1}E_0))^{1/2}}{\epsilon^{1/2-\alpha}} \leq \|\gamma^{-1}E_0\|_F^1/2(2\beta)^{3/4}/\epsilon^{1/2-\alpha}
\]
where the second inequality follows from the Hölder’s inequality.

Finally, we put all these together as
\[
\|A(\gamma^{-1}E)\|_2 = \|A(\gamma^{-1}(E_0 + E_c))\|_2
\]
\[
= \|A(\gamma^{-1}(E_0 + E_1)) + \sum_{j \geq 2} A(\gamma^{-1}E_j)\|_2
\]
\[
\geq \|A(\gamma^{-1}(E_0 + E_1))\|_2 - \|\sum_{j \geq 2} A(\gamma^{-1}E_j)\|_2
\]
\[
\geq \|A(\gamma^{-1}(E_0 + E_1))\|_2 - \|\sum_{j \geq 2} A(\gamma^{-1}E_j)\|_2 - \|1 - \delta_{2\beta + k}(A)\|F \|\gamma^{-1}(E_0 + E_1)\|_F
\]
\[
= \left(\epsilon^{1-2\alpha}(2\beta)^{-3/2} \sqrt{1 - \delta_{2\beta + k}(A)} - \frac{1 + \delta_k(A)}{k} \left(\sum_{i=1}^{m} \frac{(\sigma_i(\gamma^{-1}E_0))^{1/2}}{\sigma_i(\gamma^{-1}E_0) + \epsilon)^{1/2-\alpha}}\right)^2 \right).
\]

Since \( A(E) = A(X^* - X_0) = 0 \), and the factor
\[
\epsilon^{1-2\alpha}(2\beta)^{-3/2} \sqrt{1 - \delta_{2\beta + k}(A)} - \frac{1 + \delta_k(A)}{k}
\]
is strictly positive, we can get that \( E_0 = 0 \). Furthermore, according to (27), we have \( E_c = 0 \). Therefore, \( X^* = X_0 \).

**IV. Iterative thresholding algorithm for solving the problem (RTLAMRM)**

In this section, we propose an \( TL_{\epsilon}^\alpha \) iterative half thresholding (TLIHT) algorithm to solve the problem (RTLAMRM) for all \( 0 \leq \alpha < 1 \) and \( \epsilon > 0 \).

For any \( \lambda, \mu, \epsilon \in (0, +\infty) \), \( \alpha \in [0, 1) \) and \( Z \in \mathbb{R}^{m \times n} \), let
\[
C_{\lambda}(X) = \|A(X) - b\|_2^2 + \lambda \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{\sigma_i(X) + \epsilon)^{1/2-\alpha}},
\]
\[
C_{\lambda,\mu}(X, Z) = \mu \|A(X) - b\|_2^2 + \lambda \mu \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{\sigma_i(Z) + \epsilon)^{1/2-\alpha}} - \mu \|A(X) - A(Z)\|_F^2 + \|X - Z\|_F^2
\]
and
\[
B_\mu(X) = X + \mu A^*(b - A(X)).
\]

Note that by (33) and (34), we have
\[
C_{\lambda,\mu}(X, X) = \mu C_{\lambda}(X).
\]
Lemma 7. For any fixed $\lambda > 0$, $\mu > 0$, $\epsilon > 0$, $\alpha \in [0, 1)$ and $Z \in \mathbb{R}^{m \times n}$, if $X^* \in \mathbb{R}^{m \times n}$ is a global minimizer of $C_{\lambda, \mu}(X, Z)$, then

$$X^* = \mathcal{H}_{\lambda \mu/(\sigma(Z)+\epsilon)^{1/2-\alpha}}(B_{\mu}(Z)),$$

(37)

where $\mathcal{H}_{\lambda \mu/(\sigma(Z)+\epsilon)^{1/2-\alpha}}$ is obtained by replacing $\lambda/((\sigma(Z)+\epsilon)^{1/2-\alpha}$ with $\lambda\mu/((\sigma(Z)+\epsilon)^{1/2-\alpha}$ in $H_{\lambda/\sigma(Z)+\epsilon^{1/2-\alpha}}$.

Proof. By definition, $C_{\mu}(X, Z)$ can be rewritten as

$$C_{\lambda, \mu}(X, Z) = ||X - (Z - \mu A^*A(Z) + \mu A^*(b))||_F^2 + \lambda \mu \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{(\sigma_i(Z) + \epsilon)^{1/2-\alpha}} + \mu b||2 - \mu \|A(Z)||_2$$

$$= ||X - B_{\mu}(Z)||_F^2 + \lambda \mu \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{(\sigma_i(Z) + \epsilon)^{1/2-\alpha}} + \mu \|b||2 - \mu \|A(Z)||_2 + \mu \|Z||_F^2 - \lambda \|A(Z)||_2$$

which implies that minimizing $C_{\lambda, \mu}(X, Z)$ for any fixed $\lambda > 0$, $\mu > 0$, $\epsilon > 0$, $\alpha \in [0, 1)$ and matrix $Z \in \mathbb{R}^{m \times n}$ is equivalent to

$$\min_{X \in \mathbb{R}^{m \times n}} \left\{ ||X - B_{\mu}(Z)||_F^2 + \lambda \mu \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{(\sigma_i(Z) + \epsilon)^{1/2-\alpha}} \right\}.$$

By Lemma 6, the expression (37) immediately follows.

Lemma 8. For any fixed $\lambda > 0$, $\epsilon > 0$ and $0 < \mu < \frac{1}{\|A\|_2}$. If $X^*$ is a global minimizer of $C_{\lambda}(X)$, then $X^*$ is also a global minimizer of $C_{\lambda, \mu}(X, X^*)$, that is

$$C_{\lambda, \mu}(X^*, X^*) \leq C_{\lambda, \mu}(X, X^*)$$

(38)

for all $X \in \mathbb{R}^{m \times n}$.

Proof. The condition $0 < \mu < \frac{1}{\|A\|_2}$ implies that

$$||X - X^*||_F^2 - \mu \|A(X) - A(X^*)\|_2^2 \geq (1 - \mu \|A\|_2^2)||X - X^*||_F^2 \geq 0.$$

Therefore, for any $X \in \mathbb{R}^{m \times n}$, we have

$$C_{\lambda, \mu}(X, X^*) = \mu \|A(X) - b\|_2^2 + \lambda \mu \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{(\sigma_i(X^*) + \epsilon)^{1/2-\alpha}} - \mu \|A(X) - A(X^*)\|_2^2 + ||X - X^*||_F^2$$

$$\geq \mu \left[ \|A(X) - b\|_2^2 + \lambda \sum_{i=1}^{m} \frac{(\sigma_i(X))^{1/2}}{(\sigma_i(X^*) + \epsilon)^{1/2-\alpha}} \right]$$

$$\geq \mu C_{\lambda}(X^*)$$

$$= C_{\lambda, \mu}(X^*, X^*).$$

This completes the proof.

Lemma 8 show us that, if $X^*$ is a global minimizer of $C_{\lambda}(X)$, it is also a global minimizer of $C_{\lambda, \mu}(X, Z)$ with $Z = X^*$. Combing with Lemma 6, we now derive the following important alternative theorem, which underlies the algorithm to be proposed.
Theorem 2. For any fixed $\lambda > 0$ and $0 < \mu < \frac{1}{\|A\|_2}$, let $X^* \in \mathbb{R}^{m \times n}$ be a global solution of the problem (RTLAMRM) and $B_\mu(X^*) = X^* + \mu A^*(b - A(X^*))$ admit the following singular value decomposition

$$B_\mu(X^*) = U^*[\text{Diag}(\sigma_i(B_\mu(X^*))), 0_{m, n-m}](V^*)^T.$$ (39)

Then $X^*$ satisfies the following fixed point inclusion

$$X^* = \mathcal{H}_{\lambda \mu/\sigma_i(X^*)+\epsilon}^{1/2-\alpha}(B_\mu(X^*)) = U^*[\text{Diag}(H_{\lambda \mu/\sigma_i(X^*)+\epsilon}^{1/2-\alpha}(\sigma(B_\mu(X^*))))], 0_{m, n-m}](V^*)^T,$$ (40)

where $\mathcal{H}_{\lambda \mu/\sigma_i(X^*)+\epsilon}^{1/2-\alpha}$ and $H_{\lambda \mu/\sigma_i(X^*)+\epsilon}^{1/2-\alpha}$ are obtained by replacing $\lambda$ with $\lambda \mu/\sigma_i(X^*) + \epsilon$ in $\mathcal{H}_\lambda$ and $H_\lambda$, which means that, per iteration, the singular values of matrix $X^{k+1}$ satisfy

$$
\sigma_i(X^{k+1}) = \begin{cases} 
\frac{h_{\lambda \mu/\sigma_i(X^*)+\epsilon}^{1/2-\alpha}(\sigma_i(B_\mu(X^k)))}{\sigma_i(B_\mu(X^k))} & \text{if } \sigma_i(B_\mu(X^k)) > t^*; \\
0 & \text{if } \sigma_i(B_\mu(X^k)) \leq t^*.
\end{cases}
$$ (41)

for $i = 1, \cdots, m$, where the threshold function $t^*$ is defined as

$$t^* = \frac{\sqrt{54}}{4}(\lambda \mu/\sigma_i(X^*) + \epsilon)^{1/2-\alpha/2}.$$ (42)

With the representation (40), the TLIHT algorithm for solving the problem (RTLAMRM) can be naturally given by

$$X^{k+1} = \mathcal{H}_{\lambda \mu/\sigma_i(X^*)+\epsilon}^{1/2-\alpha}(X^k + \mu A^*(b - A(X^k)))$$ (43)

for all $0 \leq \alpha < 1$ and $\epsilon > 0$.

Next, we analyze the convergence of the above TLIHT algorithm, and the convergence of the TLIHT algorithm is very important in guaranteeing that the algorithm can be successfully applied.

Theorem 3. Given $\lambda > 0$, let $\{X^k\}$ be the sequence generated by the TLIHT algorithm with the step size $\mu$ satisfying $0 < \mu < \frac{1}{\|A\|_2}$, then

i) The sequence $\{X^k\}$ is a minimization sequence, and the sequence $\{C_\lambda(X^k)\}$ is decreasing and converges to $C_\lambda(X^*)$, where $X^*$ is any accumulation point of the sequence $\{X^k\}$.

ii) The sequence $\{X^k\}$ is asymptotically regular, i.e., $\lim_{k \to \infty} \|X^{k+1} - X^k\|_F^2 = 0$.

iii) Any accumulation point of the sequence $\{X^k\}$ is a stationary point of the problem (RTLAMRM).

Proof. Its proof follows from the fact that the step size $\mu$ satisfying $0 < \mu < \frac{1}{\|A\|_2}$, and a similar argument as used in the proof of ([24], Theorem 3). \qed

As we all know, the quality of the solution to a regularization problem depends seriously on the setting of the regularization parameter $\lambda > 0$. However, the selection of proper parameter is a very hard problem and there is no optimal rule in general. In this paper, we suppose that the matrix $X^*$ of rank $r$ is the optimal solution of the regularization problem (RTLAMRM), and set

$$\lambda = \frac{\sqrt{96}(\sigma_{r+1}(B_\mu(X^k)))^{3/2}(\sigma_{r+1}(X^k) + \epsilon)^{1/2-p}}{9 \mu}$$ (44)

in each iteration. That is, (44) can be used to adjust the value of the regularization parameter $\lambda$ during iteration, and the TLIHT algorithm will be adaptive and free from the choice of regularization parameter $\lambda$. Moreover, we
also find that the quantity of the solution of the MIHT algorithm also depends seriously on the setting of the parameter \( \epsilon \). In TLIHT algorithm, a proper choice for the value of \( \epsilon \) at \( k \)-th iteration is given by

\[
\epsilon = \max\{\sigma_{r+1}(X^k), 10^{-3}\}. \tag{45}
\]

**Algorithm 1 : TLIHT algorithm**

- **Input**: \( \mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^d \), \( b \in \mathbb{R}^d \)
- **Initialize**: \( X^0 \in \mathbb{R}^{m \times n} \), \( \mu = \frac{1 - \eta}{\|\mathcal{A}\|_2} (\eta \in (0, 1)) \), \( \alpha \in [0, 1) \) and \( \epsilon > 0 \);
- \( k = 0 \);
- while not converged do
  - \( B_\mu(X^k) = X^k - \mu \mathcal{A}^* \mathcal{A}(X^k) + \mu \mathcal{A}^*(b) \);
  - Compute the SVD of \( B_\mu(X^k) \) as \( B_\mu(X^k) = U^k [\text{Diag}(\sigma_i(B_\mu(X^k))), 0_{m,n-m}](V^k)^\top \);
  - \( \lambda = \sqrt{\frac{\max_i \sigma_{r+1}(B_\mu(z^k))^{3/2}([z^k]_{r+1})^{1/2-\alpha}}{\mu}} \), \( t^* = \frac{5}{4} \left( \lambda \mu/\sigma_i(B_\mu(z^k)) + \epsilon \right)^{1/2-\alpha} \);
    - for \( i = 1 : m \)
      - 1. \( \sigma_i(B_\mu(X^k)) > t^* \), then \( \sigma_i(B_\mu(X^{k+1})) = h_{\lambda \mu/\sigma_i(B_\mu(z^k))^{1/2-\alpha}}(\sigma_i(B_\mu(X^k))) \);
      - 2. \( \sigma_i(B_\mu(X^k)) < t^* \), then \( \sigma_i(B_\mu(X^{k+1})) = 0 \);
    - end
  - \( X^{k+1} = U^k [\text{Diag}(\sigma_i(B_\mu(X^{k+1}))), 0_{m,n-m}](V^k)^\top \);
  - \( k \rightarrow k + 1 \);
- end while
- return: \( X^{k+1} \);

V. Numerical experiments

In this section, we present a series of numerical experiments to test the performance of the TLIHT algorithm for some matrix completion problems and compare it with some state-of-art methods (SVT algorithm [15] and SVP algorithm [51]) in some grayscale image inpainting problems.

A. Completion of random matrices

In this subsection, we present a series of numerical experiments to test the performance of the TLIHT algorithm for some random low rank matrix completion problems. In our experiments, we aim to recover a random matrix \( X \in \mathbb{R}^{m \times n} \) with rank \( r \) from a subset of observe entries, \( \{M_{i,j} | (i, j) \in \Omega \} \). We generate random matrices \( M_1 \in \mathbb{R}^{n \times r} \) and \( M_2 \in \mathbb{R}^{r \times n} \) with independent identically distributed Gaussian entries. Let \( M = M_1 M_2 \), and the matrix \( M \) has rank at most \( r \). We sample the observe set \( \Omega \) with the sampling ratio \( sr = s/mn \), where \( s \) is the cardinality of the set \( \Omega \). One quantity helps to quantify the difficulty of a recovery problem is the freedom ratio \( fr = s/r(m + n - r) \), which is the freedom of rank \( r \) matrix divided by the number of measurement. If \( fr < 1 \),
There is always an infinite number of matrices with rank $r$ with the given entries, so we cannot hope to recover the matrix in this situation [21]. The stopping criterion used in our algorithm is defined as follows:

$$\|X^k - X^{k-1}\|_F \leq \text{Tol},$$

where $X^k$ is the matrix at iteration $k$. This criterion ensures that the algorithm stops when the difference between successive iterates is sufficiently small. The stopping criterion is used in conjunction with the iterative algorithm to ensure convergence to a solution that satisfies the given constraints.
where $Tol$ is a given small number. In our numerical experiments, we set $Tol = 10^{-8}$. Given an approximate recovery $X^*$ for $M$, the relative error is defined as

$$RE = \frac{\|X^* - M\|_F}{\|M\|_F}.$$

In order to implement our algorithm, we need to determine the parameter $\alpha$, which influences the behaviour of penalty function $\sum_{i=1}^{m} \left( \frac{\sigma_i(X)}{\sigma(X)+\epsilon_i} \right)^{1-\alpha}$. In the numerical tests, we test our algorithm on a series of low rank matrix completion problems with different $\alpha$ values, and set $\alpha = 0, 0.1, 0.3, 0.4, 0.5, 0.6, 0.7, 0.9$, respectively. We only take $m = n = 100$, and the results are shown in Tables I, II. Comparing the performances of TLIHT algorithm for matrix completion problems with different rank $r$, parameter $\alpha$ and $fr$, we can find that the parameters $\alpha = 0, 0.1, 0.3, 0.4, 0.5$ seem to be the optimal strategy for our algorithm when $fr$ is closed to one.

![Original Peppers image](image1)
![Approximated Peppers image with rank 30](image2)

Fig. 1. Original $256 \times 256$ Peppers image and its approximated image with rank 30

![Original Cameraman image](image3)
![Approximated Cameraman image with rank 30](image4)

Fig. 2. Original $256 \times 256$ Cameraman image and its approximated image with rank 30
B. Application for image inpainting

In this subsection, we demonstrate the performance of the TLIHT algorithm on some image inpainting problems. The TLIHT algorithm is tested on two standard $256 \times 256$ grace images (Peppers and Cameraman). We first use the singular value decomposition to obtain their approximated images with rank $r = 30$. Original images and their corresponding approximated images are displayed in Figs 1 and 2. We take $sr = 0.40$ and $sr = 0.30$ for the two low rank images. We only take $\alpha = 0.1, 0.5$ in the TLIHT algorithm. Numerical results of the three algorithms for image inpainting problems are reported in Table III. We display the recovered Peppers and Cameraman images via the three algorithms in Figs 3, 4 respectively. We can see that the TLIHT algorithm with $\alpha = 0.1$ performs the best in image inpainting problems compared with SVT algorithm and SVP algorithm.

| Image      | TLIHT, $\alpha = 0.1$ | TLIHT, $\alpha = 0.5$ | SVT  | SVP  |
|------------|----------------------|----------------------|------|------|
| (Peppers, 30, 1.8129) | 3.66e-07, 5.08 | 4.70e-07, 6.12 | 1.43e-02, 12.05 | 3.66e-07, 1.79 |
| (Cameraman, 30, 1.8129) | 1.08e-06, 11.85 | 1.12e-06, 12.52 | 7.99e-02, 7.84 | 7.59e-01, 2.35 |

VI. Conclusion

In this paper, we proposed a nonconvex function to approximate the rank function in the NP-hard problem (AMRM), and studied the transformed minimization problem in terms of theory, algorithm and computation. We discussed the equivalence of problem (AMRM) and (TLAMRM), and the uniqueness of global minimizer of the problem (TLAMRM) also solves the NP-hard problem (AMRM) if the linear map $A$ satisfies a restricted isometry property (RIP). In addition, an iterative thresholding algorithm is proposed to solve the regularization problem (RTLAMRM). Numerical results on low-rank matrix completion problems illustrated that our algorithm is able to recover a low-rank matrix, and the extensive numerical on image inpainting problems shown that our algorithm performs the best in finding a low-rank image compared with some state-of-art methods.

Acknowledgment

The work was supported by the National Natural Science Foundations of China (11771347, 11131006, 41390450, 11761003, 11271297) and the Science Foundations of Shaanxi Province of China (2016JQ1029, 2015JM1012).

References

[1] M. Fazel, H. Hindi and S. Boyd, A rank minimization heuristic with application to minimum order system approximation. In proceedings of American Control Conference, Arlington, VA, 6, 4734–4739 (2001)
[2] M. Fazel, H. Hindi and S. Boyd, Log-det heuristic for matrix minimization with applications to Hankel and Euclidean distance matrices. In Proceedings of American Control Conference, Denever, Colorado, 3, 2156–2162 (2003)

[3] E. J. Candès, B. Recht, Exact matrix completion via convex optimization. Foundations of Computational Mathematics, 9, 717–772 (2009)

[4] D. Jannach, M. Zanker, A. Felfernig and G. Friedrich, Recommender System: An Introduction. Cambridge university press, New York (2012)

[5] B. Recht, M. Fazel and P. A. Parrilo, Guaranteed minimum-rank solution of linear matrix equations via nuclear norm minimization. SIAM Review, 52, 471–501 (2010)

[6] E. J. Candès, T. Tao, The power of convex relaxation: Near-optimal matrix completion. IEEE Transactions on Information Theory, 56, 2053–2080 (2010)

[7] M. Fazel, Matrix rank minimization with applications. PhD thesis, Stanford University (2002)

[8] E. J. Candè s, Y. Plan, Matrix completion with noise. Proceedings of the IEEE, 98, 925–936 (2010)

[9] A. Cui, J. Peng, H. Li, C. Zhang, and Y. Yu, Affine matrix rank minimization problem via non-convex fraction function penalty. Journal of Computational and Applied Mathematics, 336: 353–374, 2018.

[10] S. Ji, K. F. Sze and Z. Zhou, Beyond Convex Relaxation: A polynomial-time nonconvex optimization approach to network localization. INFOCOM, 2013 Proceedings IEEE, 12, 2499–2507 (2013)

[11] A. Singer, A remark on global positioning from local distances. Proceedings of the National Academy of Sciences of the United States of America, 105(28), 9507–9511 (2008)

[12] N. Srebro, Learning with matrix factorizations. Ph.D. thesis, Massachusetts Institute of Technology (2004)

[13] Y. Hu, D. Zhang, J. Ye, X. Li, and X. He, Fast and accurate matrix completion via truncated nuclear norm regularization. IEEE Transactions on Pattern Analysis and Machine Intelligence, 35(9), 2117–2130 (2013)

[14] Netflix prize website. https://www.netflixprize.com/

[15] J. Cai, E. J. candès and Z. W. Shen, A singular value thresholding algorithm for matrix completion. SIAM Journal on Optimization, 20, 1956–1982 (2010)

[16] T. T. Cai, A. Zhang, Sparse representation of a polytope and recovery of sparse signals and Low-Rank matrices. IEEE Transactions on information theory, 60(1), 122–132 (2014)

[17] T. T. Cai, A. Zhang, Sharp RIP bound for sparse signal and low-rank matrix recovery. Applied and Computational Harmonic Analysis, 35, 74–93 (2013)

[18] R. H. TüttüncüK, C. TohM, J. Todd, Solving semidefinite-quadratic-linear programs using SDPT3. Mathematical Programming, Ser. B 95: 189–217 (2003)

[19] K. C. Toh, S. Yun, An accelerated proximal gradient algorithm for nuclear norm regularized linear least squares problems. Pacific Journal of Optimization, 6(3), 615–640 (2010)

[20] Y. Liu, D. Sun K.-C. Toh, An implementable proximal point algorithmic framework for nuclear norm minimization. Mathematical Programming, Ser. A 133: 399C-43695 (2012)

[21] S. Ma, D. Goldfarb and L. Chen, Fixed point and Bregman iterative methods for matrix rank minimization. Mathematical Programming, Ser. A 128, 321–353 (2011)

[22] D. Goldfarb, S. Ma, Convergence of Fixed-Point continuation algorithms for matrix rank minimization. Foundations of computational mathematics, 11: 183–210 (2011)

[23] R. Charttrand, Exact reconstruction of sparse signals via nonconvex minimization. IEEE Signal Processing Letters, 14,707–710 (2007)

[24] Z. Xu, X. Chang, F. Xu, H. Zhang, L1/2 regularization: A thresholding representation theory and a fast solver, IEEE Transactions on Neural Networks and Learning Systems 24(7), 1013–1027 (2012)

[25] S. Foucart, M. Lai, Sparest solutions of underdetermined linear systems via $\ell_q$ minimization for $0 < q \leq 1$. Applied and Computational Harmonic Analysis, 26, 395–407 (2009)

[26] M. Lai, J. Wang, An unconstrained $\ell_q$ minimization with $0 < q \leq 1$ for sparse solution of underdetermined linear systems. SIAM Journal on Optimization, 21, 82–101 (2011)

[27] X. Chen, F. Xu and Y. Ye, Lower bound theory of nonzero entries in solutions of $\ell_2$-$\ell_p$ minimization. SIAM Journal on Scientific Computing, 32, 2832–2852 (2010)

[28] I. Daubechies, R. Devore, M. Fornasier and C. S. Gunturk, Iteratively reweighted least squares minimization for sparse recovery. Communications on Pure and Applied Mathematics, 63, 1–38 (2010)
[29] N. Mourad, J. P. Reilly, Minimizing nonconvex functions for sparse vector reconstruction. IEEE Transactions on Signal Processing, 58, 3485-3496 (2010)

[30] Q. Sun, Recovery of sparsest signals via $\ell_q$ minimization. Applied and Computational Harmonic Analysis, 32, 329–341 (2010)

[31] R. Chartrand, V. Staneva, Restricted isometry properties and nonconvex compressive sensing. Inverse Problems, 24(3), 657–682 (2008)

[32] J. Peng, S. Yue and H. Li, NP/MP Equivalence: A phenomenon hidden among sparsity models $\ell_0$ minimization and $\ell_p$ minimization for information processing. IEEE Transaction on Information Theory, 61, 4028–4033 (2015)

[33] Z. Xu, H. Zhang, Y. Wang, X. Chang and Y. Liang, L1/2 regularization. Science China Information Sciences, 53, 1159–1169 (2010)

[34] W. Cao, J. Sun, Z. Xu, Fast image deconvolution using closed-form thresholding formulas of $\ell_q$ ($q = \frac{1}{3}, \frac{2}{3}$) regularization, Journal of Visual Communication and Image Representation 24(1), 31–41 (2013)

[35] C. Zhang, Nearly unbiased variable selection under minimax concave penalty. The Annals of Statistics, 38(2), 894–942 (2010)

[36] J. Fan, R. Li, Variable selection via nonconcave penalized likelihood and its oracle properties. Journal of the American Statistical Association, 96(456), 1348C-1360 (2001)

[37] J. Weston, Elisseeff, A. B. Scholkopf and M. Tipping, Use of the zero-norm with linear models and kernel methods. Journal of Machine Learning Research, 3, 1439C-1461 (2003)

[38] J. Trzasko, A. Manduca, Highly undersampled magnetic resonance image reconstruction via homotopic $\ell_0$-minimization. IEEE Transactions on Medical Imaging, 28(1), 106–121 (2009)

[39] H. A. Le Thi, T. Pham Dinh, H. M. Le and X. T. Vo, DC approximation approaches for sparse optimization. European Journal of Operational Research, 244, 26–46 (2015)

[40] T. Zhang, Analysis of multi-stage convex relaxation for sparse regularization. Journal of Machine Learning Research, 11, 1081–1107 (2010)

[41] H. Mohimani, M. Babaie-Zadeh, and C. Jutten, A fast approach for overcomplete sparse decomposition based on smoothed $\ell_0$-norm. IEEE Transactions on Signal Processing, 57(1), 289–301 (2009)

[42] C. Lu, J. Tang, S. Yan, Z. Lin, Nonconvex nonsmooth low rank minimization via iteratively reweighted nuclear norm. IEEE Transactions on Image Processing, 25(2), 829–839 (2016)

[43] Z. Lu, Y. Zhang, J. Lu, $\ell_p$ Regularized low-rank approximation via iterative reweighted singular value minimization. Computational Optimization and Applications, 68(3), 619–642 (2017)

[44] S. Li, K. Li, Y. Fu, Self-Taught Low-Rank coding for visual learning. IEEE Transaction on Networks and Learning Systems, 29(3), 645–656 (2018)

[45] Y. Zhang, G. Cai, J. Sun, Y. Wang, J. Chen, A new sparse Low-rank matrix decomposition method and its application on train passenger abnormal action identification. Neural Network World, 25(6), 657-668 (2015)

[46] Y. Chen, Y. Guo, Y. Wang, D. Wang, C. Peng, G. He, Denoising of hyperspectral images using nonconvex low rank matrix approximation. IEEE Transactions on Geoscience and Remote Sensing, 55(9), 5366–5380 (2017)

[47] M. Malek-Mohammadi, Babaie-Zadeh, A. Amini, C. Jutten, Recovery of Low-Rank matrices under affine constraints via a smoothed rank function. IEEE Transactions on Signal Processing, 62(4), 981–992 (2014)

[48] M. Lai, Y. Xu, W. Yin, Improved iteratively reweighted least squares for unconstrained smoothed $\ell_q$ minimization. SIAM Journal on Numerical Analysis, 51 (2), 927-957 (2013)

[49] K. Mohan, M. Fazel, Iterative reweighted algorithms for matrix rank minimization. Journal of Machine Learning Research, 13, 3441-3473 (2012)

[50] Y. Yu, J. Peng, S. Yue, A new nonconvex approach to low-rank matrix completion with application to image inpainting. Multidimensional Systems and Signal Processing (2018). https://doi.org/10.1007/s11045-018-0549-5

[51] R. Meka, P. Jain, I. Dhillon, Guaranteed rank minimization via singular value projection, in: Proceeding of the Neural Information Processing Systems Conference (NIPS), 937-945 (2010)
Fig. 3. Comparisons of TLIHT, SVT and SVP algorithms for recovering the approximated low-rank Peppers image with \( sr = 0.30 \).
Fig. 4. Comparisons of TLIHT, SVT and SVP algorithms for recovering the approximated low-rank Cameraman image with \( sr = 0.30 \).