STABILITY CONDITIONS ON CY\textsubscript{N} CATEGORIES ASSOCIATED TO A\textsubscript{n}-QUIVERS AND PERIOD MAPS

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Abstract. In this paper, we study the space of stability conditions on a certain \(N\)-Calabi-Yau (CY\textsubscript{N}) category associated to an \(A\textsubscript{n}\)-quiver. Recently, Bridgeland and Smith constructed stability conditions on some CY\textsubscript{3} categories from meromorphic quadratic differentials with simple zeros. Generalizing their results to higher dimensional Calabi-Yau categories, we describe the space of stability conditions as the universal cover of the space of polynomials of degree \(n + 1\) with simple zeros. In particular, central charges of stability conditions on CY\textsubscript{N} categories are constructed as the periods of quadratic differentials with zeros of order \(N - 2\) which are associated to polynomials.

1. Introduction

The notion of a stability condition on a triangulated category \(\mathcal{D}\) was introduced by Bridgeland [Bri07] following Douglas’s work on II-stability [Dou02]. Bridgeland also showed that the space of stability conditions \(\text{Stab}(\mathcal{D})\) has the structure of a complex manifold.

In this paper, we study the space of stability conditions on a certain \(N\)-Calabi-Yau (CY\textsubscript{N}) category \(\mathcal{D}_N\) associated to an \(A\textsubscript{n}\)-quiver [ST01, Tho06]. The category \(\mathcal{D}_N\) is defined by the derived category of finite dimensional dg modules over the Ginzburg CY\textsubscript{N} dg algebra [Gin] of the \(A\textsubscript{n}\)-quiver.

We show that the distinguished connected component \(\text{Stab}^\circ(\mathcal{D}_N) \subset \text{Stab}(\mathcal{D}_N)\) is isomorphic to the universal cover of the space of polynomials \(p_n(z) = z^{n+1} + u_1 z^{n-1} + \cdots + u_n (u_i \in \mathbb{C})\) with simple zeros. The central charges of stability conditions in \(\text{Stab}^\circ(\mathcal{D}_N)\) are constructed as the periods of quadratic differentials of the form \(p_n(z)^{N-2} z^{\otimes 2}\) on the Riemann sphere \(\mathbb{P}^1\).

To prove the result, we use the recent results of Bridgeland and Smith [BS]. Motivated by the work of physicists Gaiotto-Moore-Neitzke [GMN], they constructed stability conditions on a class of CY\textsubscript{3} categories associated to triangulations of marked bordered surfaces from quadratic differentials with simple zeros.

By considering quadratic differentials with zeros of order \(N - 2\) instead of simple zeros, we can generalize their framework in [BS] for CY\textsubscript{3} categories to CY\textsubscript{N} categories. Here we only treat the simplest case that the surface is a disk with some marked points on the boundary. In the future, we expect that their framework is generalized to a class of CY\textsubscript{N} categories associated to \(N\)-angulations of marked bordered surfaces, though the definition of these categories are not yet.

1.1. Background geometry. Let \(N \geq 2\) be an integer. Here we see background geometry of this paper and explain why we consider the period of the differential form \(p_n(z)^{N-2} dz\). Section 2 in [Tho06] is the excellent survey of related topics.
The category $D^N_n$ was first introduced as the derived Fukaya category [KS02, Tho06] of Lagrangian $N$-spheres which form the basis of vanishing cycles in the universal deformation of the $A_n$-singularity
\[ X_p^N := \{ x_1^2 + x_2^2 + \cdots + x_N^2 = p_n(z) \} \subset \mathbb{C}^N \times \mathbb{C} \]
where $p_n(z) = z^{n+1} + u_1 z^{n-1} + \cdots + u_n (u_i \in \mathbb{C})$ with simple zeros. We consider a symplectic structure on $X_p^N$ coming from the standard Kähler form on $\mathbb{C}^N \times \mathbb{C}$, and a nowhere vanishing holomorphic $N$-form $\Omega_p$ on $X_p^N$ given by taking Poincaré residue of the standard form $dx_1 \cdots dx_N dz$ on $\mathbb{C}^{N+1}$.

Consider the projection
\[ \pi: X_p^N \to \mathbb{C}, \quad (x_1, \ldots, x_N, z) \mapsto z. \]
If $z \in \mathbb{C}$ is not a root of $p_n$, then the fiber $\pi^{-1}(z)$ is an affine quadric of complex dimension $N - 1$ with a real slice $S^{N-1} \subset \pi^{-1}(z)$. Thus, each Lagrangian $N$-sphere is constructed by the total space of the $S^{N-1}$-fibration over a path $\gamma: [0, 1] \to \mathbb{C}$ which connects distinct roots of $p_n$ (see [TY02, Section 6] or [Tho06, Section 2]).

As in [Dou02], a central charge “in the physics literature” of a Lagrangian $L \subset X_p^N$ is given by the period
\[ \int_L \Omega_p. \]
Assume that $L$ is a $S^{N-1}$-fibration over the path $\gamma: [0, 1] \to \mathbb{C}$. Then, up to constant multiplication the above integration is reduced to the integration of the differential form $p_n(z)\frac{dz}{z^2}$ along the path $\gamma$ (see [TY02, Section 6]):
\[ \int_{\gamma} p_n(z)\frac{dz}{z^2}. \]
From this observation, we expect that central charges of Bridgeland stability conditions on $D^N_n$ can be constructed by using the period of the differential form $p_n(z)\frac{dz}{z^2}$.

1.2. **Summary of results.** First we explain the space of polynomials and the associated period map (Section 2). Let $M_n$ be the space of polynomials $p_n(z) = z^{n+1} + u_1 z^{n-1} + \cdots + u_n (u_i \in \mathbb{C})$ with $n + 1$ simple zeros. Note that the fundamental group of $M_n$ is the Artin group $B_{n+1}$. The action of $\mathbb{C}^*$ on $M_n$ is given by $(u_1, \ldots, u_n) \mapsto (ku_1, \ldots, k^n u_n)$ for $k \in \mathbb{C}^*$.

We regard $p_n(z)$ as a meromorphic function on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{ \infty \}$. For $p_n(z) \in M_n$, we define the quadratic differential on $\mathbb{P}^1$ by $\phi(z) := p_n(z)^{N-2}dz \otimes dz$ and denote by $Q(N,n)$ the space of such differentials.

To consider the period of $\sqrt{\phi} = p_n(z)\frac{dz}{N-2}$, we need to introduce homology groups $H_+ (\phi)$ and $H_- (\phi)$ which provide appropriate cycles for the integration of $\sqrt{\phi}$. If $N$ is even, the homology group is defined by the relative homology group $H_+ (\phi) = H_1(\mathbb{C}, \text{Zero}(\phi); \mathbb{Z})$ where $\text{Zero}(\phi) \subset \mathbb{C}$ is the set of zeros of $\phi$. If $N$ is odd, the homology group is defined by $H_- (\phi) := H_1(S^\pi^{-1}(\infty); \mathbb{Z})$ where $\pi: S \to \mathbb{P}^1$ is the double cover of the hyperelliptic curve $S = \{ y^2 = p_n(z) \}$.

A linear map $Z_{\phi}: H_{\pm}(\phi) \to \mathbb{C}$, called the period of $\phi$, is defined by
\[ Z_{\phi}(\gamma) := \int_{\gamma} \sqrt{\phi} \]
for $\gamma \in H_{\pm}(\phi)$. 
Let $\Gamma$ be a free abelian group of rank $n$. An isomorphism of abelian groups $\theta: \Gamma \xrightarrow{\sim} H_+(\phi)$ is called a $\Gamma$-framing of $\phi \in Q(N, n)$. Denote by $Q(N, n)^\Gamma$ the set of framed differentials. For a framed differential $(\phi, \theta) \in Q(N, n)^\Gamma$, the composition of $\theta: \Gamma \to H_+^\pm(\phi)$ and $Z_\phi: H_+^\pm(\phi) \to \mathbb{C}$ gives a linear map $Z_\phi \circ \theta: \Gamma \to \mathbb{C}$. We define the period map by

$$W_N : Q(N, n)^\Gamma \to \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}), \quad (\phi, \theta) \mapsto Z_\phi \circ \theta.$$ 

Fix a point $* = (\phi_0, \theta_0) \in Q(N, n)^\Gamma$ and let $Q(N, n)_c^\Gamma \subset Q(N, n)^\Gamma$ be the connected component containing $*$. Then the universal cover of $Q(N, n)_c^\Gamma$ is isomorphic to the universal cover $\widetilde{M}_n$ of $M_n$. Thus the period map $W_N$ is extended on $\widetilde{M}_n$:

$$W_N : \widetilde{M}_n \to \text{Hom}(\Gamma, \mathbb{C}).$$

Next we consider the category $\mathcal{D}_n^N$ (Section 4) and stability conditions on it (Section 5). For a quiver $Q$, we can define a dg algebra $\Gamma_N Q$, called the Ginzburg CY $N$ dg algebra [Gin]. Let $\mathcal{D}_d(\Gamma_N Q)$ be the derived category of dg modules over $\Gamma_N Q$ with finite total dimension. It was proved by Keller [Kel11] that the category $\mathcal{D}_d(\Gamma_N Q)$ is a CY $N$ triangulated category. Let $Q = \xrightarrow{n} A_n$ be the $A_n$-quiver and set $\mathcal{D}_n^N := \mathcal{D}_d(\Gamma_N \xrightarrow{n} A_n)$. An object $S \in \mathcal{D}_n^N$ is called $N$-spherical if

$$\text{Hom}_{\mathcal{D}_n^N}(S, S[i]) = \begin{cases} k & \text{if } i = 0, N \\ 0 & \text{otherwise.} \end{cases}$$

There is a full subcategory $\mathcal{H}_0$, called the standard heart of $\mathcal{D}_n^N$, and there are $N$-spherical objects $S_1, \ldots, S_n \in \mathcal{H}_0$ such that $\mathcal{H}_0$ is generated by these $N$-spherical objects.

From such objects, Seidel and Thomas defined autoequivalences $\Phi_{S_1}, \ldots, \Phi_{S_n} \in \text{Aut}(\mathcal{D}_n^N)$, called spherical twists [ST01]. They also proved that the subgroup $\text{Br}(\mathcal{D}_n^N) \subset \text{Aut}(\mathcal{D}_n^N)$ generated by $\Phi_{S_1}, \ldots, \Phi_{S_n}$ is isomorphic to the braid group $B_{n+1}$.

A stability condition $\sigma = (Z, \mathcal{H})$ on a triangulated category $\mathcal{D}$ consists of a heart $\mathcal{H} \subset \mathcal{D}$ of a bounded t-structure, and a linear map $Z: K(\mathcal{H}) \to \mathbb{C}$ called the central charge, which satisfy some axioms. In [Bri07], Bridgeland showed that the set of stability conditions $\text{Stab}(\mathcal{D})$ has the structure of a complex manifold, and the projection map

$$Z : \text{Stab}(\mathcal{D}) \to \text{Hom}_Z(K(\mathcal{D}), \mathbb{C}), \quad (Z, \mathcal{H}) \mapsto Z$$

is a local isomorphism. There are two group actions on $\text{Stab}(\mathcal{D})$, one is the action of $\mathbb{C}$ and the other is the action of $\text{Aut}(\mathcal{D})$.

Consider the category $\mathcal{D}_n^N$. There is a distinguished connected component $\text{Stab}^\diamond(\mathcal{D}_n^N) \subset \text{Stab}(\mathcal{D}_n^N)$ which contains stability conditions with the standard heart $\mathcal{H}_0$. The action of the group $\text{Br}(\mathcal{D}_n^N)$ preserves the connected component $\text{Stab}^\diamond(\mathcal{D}_n^N)$. Since $\text{Br}(\mathcal{D}_n^N) \cong B_{n+1}$, the space $\text{Stab}^\diamond(\mathcal{D}_n^N)$ has the action of the braid group $B_{n+1}$.

Our main result is the following.
Theorem 1.1 (Theorem 7.14). Assume that \( N \geq 2 \). There is a \( B_{n+1} \)-equivariant and \( \mathbb{C} \)-equivariant isomorphism of complex manifolds \( \tilde{K} \) such that the diagram

\[
\begin{array}{ccc}
\tilde{M}_n & \xrightarrow{\phi} & \tilde{K} \\
\downarrow \cong & & \downarrow \text{Stab}^\circ(\mathcal{D}_n^N) \downarrow \\
W_N & \xleftarrow{\text{Hom}_{\mathbb{T}}(\Gamma, \mathbb{C})} & Z
\end{array}
\]

commutes.

When \( N = 2 \), the result was proved by Thomas [Tho06]. When \( N = 3 \) and \( n = 2 \), the result was proved by Sutherland [Sut]. The result for \( N = 3 \) and arbitrary \( n \) follows from the result of Bridgeland-Smith [BS] with some modifications (see Example 12.1 in [BS]).

The result is compatible with the observation in [Bri06] that if there is a Frobenius structure on the space of stability conditions on a CY \( N \) category, then the central charge corresponds to the twisted period of the Frobenius structure with the twist parameter \( \nu = (N-2)/2 \). The twisted periods are characterized as the horizontal sections of the second structure connection of the Frobenius structure with some parameter \( \nu \) (see [Dub04, Section 3]). By the K. Saito’s theory of primitive forms [Sai83], there is a Frobenius structure on \( M_n \), and the map \( W_N \) naturally arises as the period map associated to the primitive form. As in [Dub04, Section 5], the map \( W_N \) coincides with the twisted period map with the parameter \( \nu = (N-2)/2 \). Therefore, the result supports the Bridgeland’s conjecture in [Bri06].

1.3. Basic ideas. Many arguments of this paper are parallel to those of [BS] with appropriate generalizations. We replace CY \( 3 \) categories with CY \( N \) categories, quivers with \( (N-2) \)-colored quivers, simple zeros with zeros of order \( (N-2) \) and triangulations with \( N \)-angulations.

However as in Section 6, there are some points where parallel arguments cannot be applied. In the case of \( N \)-angulations, it is not known what objects correspond to quivers with potential and their mutation in the case of triangulations. Hence we do not have natural dg algebras associated to \( N \)-angulations of surfaces, and the category equivalence of Keller-Yang [KY11] is not established yet.

Fortunately, King-Qiu’s result [KQ] on the hearts of \( \mathcal{D}_n^N \) together with the geometric realization of the \( (N-2) \)-cluster category of type \( A_n \) by Baur-Marsh [BM08] allows us to apply the framework of [BS] for our cases.

The key idea of constructing stability conditions on \( \mathcal{D}_n^N \) from quadratic differentials is the fact that saddle-free quadratic differentials determine \( N \)-angulations of surfaces as follows. The differential \( \phi \) determines a foliation on \( \mathbb{P}^1 \) by

\[
\text{Im} \int p_n(z) \frac{N-2}{z^2} \, dz = \text{constant}.
\]

A leaf which connects zeros of \( \phi \) is called a saddle trajectory. Near a zero of order \( N-2 \), the foliation behaves as in Figure 2. As a result, the foliation of a saddle-free differential \( \phi \) determines the \( N \)-angulation \( \Delta_\phi \) of the \( ((N-2)(n+1)+2) \)-gon (Lemma 6.10) similar to the WKB triangulations in [BS, GMN]. This is the reason why we consider zeros of order \( N-2 \). Using the result of Baur-Marsh and King-Qiu, we have a natural heart \( \mathcal{H}(\Delta_\phi) \subset \mathcal{D}_n^N \) with simple objects \( S_1, \ldots, S_n \) (Theorem 6.3).
Corresponding these simple objects, there are cycles $\gamma_1, \ldots, \gamma_n \in H_*(\phi)$, called standard saddle classes (Corollary 6.12). The central charge of $S_i$ is defined by the period of the corresponding saddle class $\gamma_i$:

$$Z(S_i) := Z_\phi(\gamma_i) = \int_{\gamma_i} p_n(z) \frac{N-2}{2} dz.$$ 

Thus, we can construct the stability condition $(Z, H(\Delta_\phi))$ on $D_n^N$ (Lemma 7.5).

1.4. **Homotopy type problem.** Ishii-Ueda-Uehara [IUU10] proved that the space Stab$(D_2^n)$ is connected, so Stab$(\overline{\Delta_2^n}) = \overline{\Delta_2^n}$. Together with the result of Thomas in [Tho06], we have Stab$(D_2^n) \cong \tilde{M}_n$. In particular, Stab$(D_2^2)$ is contractible. For $N \geq 3$, Qiu proved that Stab$(D_2^N)$ is simply connected (Corollary 5.5 in [Qiu]). Our Theorem 1.1 implies the following stronger result (Conjecture 5.8 in [Qiu]).

**Corollary 1.2** (Corollary 7.15). *The distinguished connected component Stab$(D_2^N)$ is contractible.*

The remaining problem is the connectedness of the space Stab$(D_2^n)$. As in the case $N = 2$, we naively expect the following.

**Conjecture 1.3.** *The space Stab$(D_2^n)$ is connected.*

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**Notations.** We put $d_n^N := (N - 2)(n + 1) + 2$. We work over the algebraically closed field $k$. We use homological gradings $d: M_0 \to M_{n-1}$ for dg algebras and dg modules. All triangulated categories in this paper are considered to be $k$-linear, and assume that their Grothendieck groups are free of finite rank ($K(D) \cong \mathbb{Z}^n$ for some $n$).

# 2. Quadratic differentials associated to polynomials

In the following, we assume that $N \geq 3$ and $n \geq 1$.

## 2.1. Braid groups and their representations

In this section, we introduce the braid groups and their representations on some lattices associated to $A_n$-quivers.

**Definition 2.1.** *The Artin braid group $B_{n+1}$ is the group generated by generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ with the following relations:

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2.$$*
It is known that the element

$$C := (\sigma_1 \sigma_2 \cdots \sigma_n)^{n+1}$$

generates the center of $B_{n+1}$.

We consider two representations $\rho_+$ and $\rho_-$ of $B_{n+1}$ on some lattices associated to the $A_n$-quivers. Such representations are equivalent to reduced Burau representations with the parameter $\pm 1$ (see [KT08, Section 3]).

An $A_n$-quiver $A_n^\pm$ is a quiver obtained by giving orientations on edges of the Dynkin diagram of type $A_n$. We set vertices of $A_n^\pm$ by $\{1, \ldots, n\}$ and assume that vertices $i$ and $i+1$ are connected by some arrow. Let $q_{ij}$ be the number of arrows from $i$ to $j$.

Let $L$ be a free abelian group of rank $n$ with generators $\alpha_1, \ldots, \alpha_n$:

$$L := \bigoplus_{i=1}^n \mathbb{Z} \alpha_i.$$  

We define a symmetric bilinear form $\langle , \rangle_+$ and a skew-symmetric bilinear form $\langle , \rangle_-$ on $L$ by

$$\langle \alpha_i, \alpha_j \rangle_+ := \delta_{ij} + \delta_{ji} - (q_{ij} + q_{ji})$$
$$\langle \alpha_i, \alpha_j \rangle_- := \delta_{ij} - \delta_{ji} - (q_{ij} - q_{ji})$$

where $\delta_{ij}$ is the Kronecker delta.

Reflections (which depend on the sign $\pm$) $r_1^\pm, \ldots, r_n^\pm : L \to L$ via the bilinear forms $\langle , \rangle_\pm$ is defined by

$$r_i^\pm(\alpha_j) := \alpha_j - \langle \alpha_i, \alpha_j \rangle_\pm \alpha_i.$$  

Let $W_\pm := (r_1^\pm, \ldots, r_n^\pm)$ be the groups generated by these reflections.

Two representations $\rho_\pm : B_{n+1} \to \text{GL}(L, \mathbb{Z})$ are defined by

$$\rho_\pm(\sigma_i) := r_i^\pm.$$  

We put the kernel of $\rho_\pm$ by $P_\pm$. Then we have a short exact sequence

$$1 \to P_\pm \to B_{n+1} \to W_\pm \to 1.$$  

Note that the group $W_\pm$ is a symmetric group of degree $n+1$ and the group $P_\pm$ is a pure braid group.

2.2. Spaces of polynomials. Let $p_n(z)$ be a polynomial of the following form

$$p_n(z) = z^{n+1} + u_1 z^{n-1} + u_2 z^{n-2} + \cdots + u_n, \quad u_1, \ldots, u_n \in \mathbb{C}.$$  

We denote by $\Delta(p_n)$ the discriminant of $p_n$. (If $p_n(z) = \prod_{i=1}^{n+1} (z - a_i)$, then $\Delta(p_n) = \prod_{i<j} (a_i - a_j)^2$.)

Denote by $M_n$ the space of such polynomials with simple zeros:

$$M_n := \{ p_n(z) | \Delta(p_n) \neq 0 \}.$$  

Define the free $\mathbb{C}^*$-action on $M_n$ by

$$(k \cdot p_n)(z) := z^{n+1} + ku_1 z^{n-1} + k^2 u_2 z^{n-2} + \cdots + k^n u_n \quad (k \in \mathbb{C}^*).$$  

Note that for the above action, we have the equation

$$(k \cdot p_n)(kz) = k^{n+1} p_n(z).$$
Consider the description of $M_n$ as the configuration space of $\mathbb{C}$. Let $C_{n+1}(\mathbb{C})$ be the configuration space of $(n+1)$ distinct points in $\mathbb{C}$ and $C_{n+1}^0(\mathbb{C}) \subset C_{n+1}(\mathbb{C})$ be the subspace consists of configurations with the center of mass zero:

$$C_{n+1}(\mathbb{C}) := \{ (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1} \mid a_i \neq a_j \text{ for } i \neq j \}/S_n$$

$$C_{n+1}^0(\mathbb{C}) := \{ (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} a_i = 0, a_i \neq a_j \text{ for } i \neq j \}/S_n.$$

A complex number $c \in \mathbb{C}$ acts on $C_{n+1}(\mathbb{C})$ as a parallel transportation $(a_1, \ldots, a_{n+1}) \mapsto (a_1 + c, \ldots, a_{n+1} + c)$. Hence we have $C_{n+1}^0(\mathbb{C}) = C_{n+1}(\mathbb{C})/C$.

As a result, we have an isomorphism $C_{n+1}^0(\mathbb{C}) \sim M_n$ given by

$$(a_1, \ldots, a_{n+1}) \mapsto \prod_{i=1}^{n+1} (z - a_i).$$

Next we consider the description of a braid group $B_{n+1}$ as the fundamental group of $M_n$. Take a configuration $\{a_1, \ldots, a_{n+1}\} \in C_{n+1}(\mathbb{C})$ and fix some order of $n+1$ points $(a_1, \ldots, aR_{n+1})$. We consider the collection of non-crossing paths $\gamma_1, \ldots, \gamma_n$ in $\mathbb{C}$ such that $\gamma_i$ connects $a_i$ and $a_{i+1}$. We call it the $A_n$-chain of paths. Let $\tau_i$ be a closed path in $C_{n+1}(\mathbb{C})$ which corresponds to the half-twist along $\gamma_i$ in clockwise (see Figure 1), and consider its image in $M_n$ via the projection $C_{n+1}(\mathbb{C}) \rightarrow M_n$ (we also write it by $\tau_i$).

![Figure 1. Half-twists along the $A_n$-chain.](image)

It is well-known that there is an isomorphism of groups

$$\pi_1(M_n) \sim B_{n+1}, \quad [\tau_i] \mapsto \sigma_i.$$

For more details, we refer to [KT08, Section 1.6].

2.3. Quadratic differentials associated to polynomials. Let $\mathbb{P}^1$ be the Riemann sphere, and we fix the coordinate $z \in \mathbb{C} \cup \{ \infty \} \cong \mathbb{P}^1$. For a polynomial $p_n \in M_n$, define the meromorphic quadratic differential $\phi$ on $\mathbb{P}^1$ by

$$\phi(z) := p_n(z)^{N-2}dz^2.$$

$\phi$ has zeros of order $(N - 2)$ at distinct $n + 1$ points on $\mathbb{C} \cap \mathbb{P}^1$, and a unique pole of order $(d_n^N + 2)$ at $\infty \in \mathbb{P}^1$.

We denote by $Q(N, n)$ the space of such quadratic differentials associated to $p_n \in M_n$:

$$Q(N, n) := \{ p_n(z)^{N-2}dz^2 \mid p_n \in M_n \}.$$
Lemma 2.4. Both group in \([\text{BS}]\) which is written as \(\hat{\text{the fiber}} \ \pi \ \text{hyperelliptic curve} \ S \ \text{of} \ C^k \) where for Lemma 2.2. holds.

We introduce the action of \(\mathbb{C}^*\) on \(Q(N, n)\). For \(\phi(z) = p_n(z)^{N-2}dz^\otimes 2\), the action of \(\mathbb{C}^*\) is defined by

\[
(k \cdot \phi)(z) := \{(k \cdot p_n)(z)\}^{N-2}dz^\otimes 2 \quad (k \in \mathbb{C}^*)
\]

where \(k \cdot p_n\) is the action defined in Section 2.2. By using the equation \((k \cdot p_n)(kz) = k^{n+1}p_n(z)\), we have the following formula.

Lemma 2.2. For \(\phi \in Q(N, n)\), the equation

\[
(k \cdot \phi)(kz) = k^{d_n} \phi(z) \quad (k \in \mathbb{C}^*)
\]

holds.

For \(\phi \in Q(N, n)\), denote by Zero(\(\phi\)) \(\subset \mathbb{P}^1\) the set of zeros of \(\phi\) and by Crit(\(\phi\)) \(\subset \mathbb{P}^1\) the set of critical points of \(\phi\). Note that \(\text{Crit}(\phi) = \text{Zero}(\phi) \cup \{\infty\}\).

The action \(z \mapsto kz\) of \(k \in \mathbb{C}^*\) on \(\mathbb{P}^1\) maps the critical points of \(\phi\) to those of \(k \cdot \phi\):

\[
k \cdot \text{Crit}(\phi) = \text{Crit}(k \cdot \phi).
\]

2.4. Homology groups and periods.

Definition 2.3. Depending on the parity of \(N\), we introduce homology groups \(H_+ (\phi)\) and \(H_- (\phi)\) in the following.

- If \(N\) is even, \(H_+ (\phi)\) is defined by the relative homology group
  \[
  H_+ (\phi) := H_1(\mathbb{C}, \text{Zero}(\phi); \mathbb{Z}).
  \]

- If \(N\) is odd, \(H_- (\phi)\) is given as follows. Let \(S\) be a hyperelliptic curve defined by the equation \(y^2 = p_n(z)\) and \(\pi : S \rightarrow \mathbb{P}^1\) be an associated branched covering map of degree 2. Then the homology group \(H_- (\phi)\) is defined by
  \[
  H_- (\phi) := H_1(S \setminus \pi^{-1}(\infty); \mathbb{Z}).
  \]

Note that if \(N = 3\), the homology group \(H_- (\phi)\) corresponds to the hat-homology group in [BS] which is written as \(H(\phi)\).

Lemma 2.4. Both \(H_+ (\phi)\) and \(H_- (\phi)\) are free abelian groups of rank \(n\).

Proof. If \(N\) is even, \(|\text{Zero}(\phi)| = n + 1\) implies rank \(H_+ (\phi) = n\).

If \(N\) is odd, there are two cases, \(n\) is even or odd. If \(n\) is even, the genus of the hyperelliptic curve \(S\) is \(n/2\) and the fiber \(\pi^{-1}(\infty)\) consists of one point. Hence we have rank \(H_- (\phi) = 2(n/2) = n\). When \(n\) is odd, the genus of \(S\) is \((n - 1)/2\) and the fiber \(\pi^{-1}(\infty)\) consists of two points. Hence we have rank \(H_- (\phi) = 2\{(n - 1)/2\} + 1 = n\).

Let \(\psi := \sqrt{\phi} = p_n(z)^{N-2}dz\) be the square root of \(\phi\). (If \(N\) is odd, \(\psi\) is determined up to the sign.) If \(N\) is even, then \(\psi\) is a holomorphic 1-form on \(\mathbb{P}^1 \setminus \{\infty\}\), and if \(N\) is odd, then \(\psi\) is a holomorphic 1-form on \(S \setminus \pi^{-1}(\infty)\).

Therefore, for any cycle \(\alpha \in H_\pm(\phi)\) the integration of 1-form \(\psi\) via the cycle \(\alpha\)

\[
Z_\phi(\alpha) := \int_\alpha \psi \in \mathbb{C}
\]

is well-defined.

Thus we have a group homomorphism \(Z_\phi : H_\pm(\phi) \rightarrow \mathbb{C}\), called the period of \(\phi\).
2.5. **Framings.** Following [BS, Section 2.6], we introduce framings on \( Q(N, n) \). Let \( \Gamma \) be a free abelian group of rank \( n \). A \( \Gamma \)-framing of \( \phi \in Q(N, n) \) is an isomorphism of abelian groups

\[
\theta: \Gamma \xrightarrow{\sim} H_+(\phi).
\]

Denote by \( Q(N, n)^\Gamma \) the space of pairs \((\phi, \theta)\) consisting of a quadratic differential \( \phi \in Q(N, n) \) and a \( \Gamma \)-framing \( \theta \). The projection

\[
Q(N, n)^\Gamma \to Q(N, n), \quad (\phi, \theta) \mapsto \phi
\]

defines a principal \( \text{GL}(n, \mathbb{Z}) \)-bundle on \( Q(N, n) \).

For \((\phi, \theta) \in Q(N, n)^\Gamma\), the composition of a framing \( \theta \) and a period \( Z_\phi \) gives a group homomorphism \( Z_\phi \circ \theta: \Gamma \to \mathbb{C} \). Thus, we have the map

\[
W_N: Q(N, n)^\Gamma \to \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})
\]

defined by \( W_N((\phi, \theta)) := Z_\phi \circ \theta \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) \). We call \( W_N \) a period map.

**Proposition 2.5.** The period map

\[
W_N: Q(N, n)^\Gamma \to \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})
\]

is a local isomorphism of complex manifolds.

**Proof.** Theorem 4.12 in [BS] can be easily extended to our cases. In our cases, the result also follows from the K. Saito’s theory on primitive forms [Sai83, Section 5] since \( W_N \) is interpreted as the period map associated to the primitive form of the \( A_n \)-singularity. \( \square \)

2.6. **Intersection forms.** In this section, we introduce intersection forms on \( H_+(\phi) \).

First we consider the case \( N \) is even. The exact sequence of relative homology groups gives the isomorphism of homology groups

\[
H_+(\phi) = H_1(\mathbb{C}, \text{Zero}(\phi); \mathbb{Z}) \cong \tilde{H}_0(\text{Zero}(\phi); \mathbb{Z})
\]

where \( \tilde{H}_0(\text{Zero}(\phi); \mathbb{Z}) \) is the reduced homology group of \( H_0(\text{Zero}(\phi); \mathbb{Z}) \), and this isomorphism is induced by the boundary map. There is a trivial intersection form on \( H_0(\text{Zero}(\phi); \mathbb{Z}) \), and it induces the intersection form on \( \tilde{H}_0(\text{Zero}(\phi); \mathbb{Z}) \). Through the above isomorphism of homology groups, we have the intersection form

\[
I_+: H_+(\phi) \times H_+(\phi) \to \mathbb{Z}.
\]

Note that \( I_+ \) is symmetric and non-degenerate.

Next consider the case \( N \) is odd. Let \( \pi: S \to \mathbb{P}^1 \) be the hyperelliptic curve introduced in Section 2.4, and \( \iota: S \setminus \pi^{-1}(\infty) \to S \) be the inclusion. This inclusion induces the surjective group homomorphism

\[
\iota_*: H_-(\phi) = H_1(S \setminus \pi^{-1}(\infty); \mathbb{Z}) \to H_1(S; \mathbb{Z}).
\]

Since there is a skew-symmetric non-degenerate intersection form on \( H_1(S; \mathbb{Z}) \), through the inclusion \( \iota_* \), we have the intersection form

\[
I_-: H_-(\phi) \times H_-(\phi) \to \mathbb{Z}.
\]

The map \( \iota_* \) is an isomorphism if \( n \) is even, and has the rank one kernel if \( n \) is odd. Therefore, \( I_- \) is non-degenerate if \( n \) is even, or has the rank one kernel if \( n \) is odd.

For \( \phi \in Q(N, n) \), picking the order of \( \text{Zero}(\phi) \) and consider the \( A_n \)-chain of path \( \gamma_1, \ldots, \gamma_n \). If \( N \) is even, the path \( \gamma_i \) determines the homology class \( \hat{\gamma}_i \in H_+(\phi) \) since
γ_i connects zeros of φ. If N is odd, since the lift \( \pi^{-1}(γ_i) \) on the hyperelliptic curve \( S \) is a closed path in \( S \setminus \pi^{-1}(∞) \), we have the homology class \( \tilde{γ}_i := [\pi^{-1}(γ_i)] \in H_1(φ) \).

The pair of the homology group \( H_±(φ) \) and the intersection form \( I \) gives a geometric realization of the lattice \( (L, ⟨ , ⟩_±) \) which is introduced in Section 2.1.

**Lemma 2.6.** There is an isomorphism \( H_±(φ) \cong L \) given by \( \tilde{γ}_i \mapsto α_i \).

Further, by choosing suitable orientations of \( \tilde{γ}_1, \ldots, \tilde{γ}_n \), this isomorphism takes the intersection form \( I_± \) to the bilinear form \( ⟨ , ⟩_± \).

**Proof.** The result follows from the direct calculation.

### 2.7. Local system

For each \( φ \in Q(N, n) \), by corresponding to the homology group \( H_±(φ) \), we have a local system

\[
\bigcup_{φ \in Q(N, n)} H_±(φ) \rightarrow Q(N, n).
\]

For a continuous path \( c : [0, 1] \rightarrow Q(N, n) \) with \( φ_0 = c(0) \), \( φ_1 = c(1) \), a parallel transportation along \( c \) gives the identification

\[
GM_c : H_±(φ_0) \sim H_±(φ_1),
\]

and this identification determines a flat connection on the local system. We call it the Gauss-Manin connection.

Fix a point \( (φ_0, θ_0) \in Q(N, n)^Γ \) and denote by \( Q(N, n)^Γ \) the connected component containing \( (φ_0, θ_0) \). Any point \( (φ, θ) \in Q(N, n)^Γ \) is contained in \( Q(N, n)^Γ \) if and only if there is a continuous path \( c : [0, 1] \rightarrow Q(N, n) \) connecting \( φ_0 \) and \( φ \) such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{θ} & H_±(φ) \\
\downarrow{θ_0} & & \downarrow{GM_c} \\
H_±(φ_0) & \xrightarrow{GM_c} & H_±(φ)
\end{array}
\]

commutes.

Note that different connected components of \( Q(N, n)^Γ \) are all isomorphic since these are related by the action of \( GL(n, \mathbb{Z}) \).

We introduce the \( \mathbb{C} \)-action on \( Q(N, n)^Γ \), which is compatible with the \( \mathbb{C} \)-action on the space of stability conditions given in Section 5.3.

**Definition 2.7.** For a framed differential \( (φ, θ) \in Q(N, n)^Γ_+ \), the new framed differential \( (φ', θ') := t \cdot (φ, θ) \) for \( t ∈ \mathbb{C} \) is defined by

\[
φ' := e^{-(2πi/d_n)t} \cdot φ, \quad θ' := GM_c \circ θ
\]

where \( GM_c \) is the Gauss-Manin connection along the path \( c(s) := e^{-(2πi/d_n)ts} \cdot φ \) for \( s ∈ [0, 1] \) which connects \( φ \) and \( φ' \).

### 2.8. Monodromy representations

By using the intersection form \( I_± \) on \( H_±(φ) \) defined in Section 2.6, we can compute how do cycles change by the Gauss-Manin connection along a closed path in \( Q(N, n) \). The result is known as the Picard-Lefschetz formula.

Fix a base point \( * = \prod_{i=1}^{n+1} (z - a_i) \) and let \( γ_1, \ldots, γ_n \) be the \( A_n \)-chain of paths. Let \( φ_0 := \prod_{i=1}^{n+1} (z - a_i) \) be the differential corresponding to the base.
point *. Under the identification $M_n \cong Q(N, n)$, consider the half-twists $\tau_1, \ldots, \tau_n$ as in Section 2.2 and take the orientations of $\hat{\gamma}_1, \ldots, \hat{\gamma}_n \in H_\pm(\phi_0)$ as in Lemma 2.6.

**Proposition 2.8.** The monodromy representation of $[\tau_i] \in \pi_1(Q(N, n), \ast)$ on $H_\pm(\phi_0)$ is given by

$$GM_{\tau_i}(\beta) = \beta - I_\pm(\hat{\gamma}_i, \beta) \hat{\gamma}_i \quad (\beta \in H_\pm(\phi_0)).$$

Lemma 2.6 and Proposition 2.8 imply that the monodromy representation of the braid group $\pi_1(M_n, \ast) \cong B_{n+1}$ on $H_\pm(\phi_0)$ gives a geometric realization of two representations $\rho^+$ and $\rho^-$ introduced in Section 2.1. Further recall from Section 2.1 that the subgroup $P_\pm \subset B_{n+1}$ is the kernel of $\rho_\pm$ and the group $W_\pm$ is the image of $\rho_\pm$ on $L \cong H_\pm(\phi_0)$.

Let $\tilde{M}_n$ be a universal cover of $M_n$. Note that $B_{n+1}$ acts on $\tilde{M}_n$ as the group of deck transformations.

**Corollary 2.9.** The connected component $Q(N, n)^\Gamma_\ast$ is isomorphic to the space $\tilde{M}_n/P_\pm$ and the projection map $Q(N, n)^\Gamma_\ast \to Q(N, n)$ gives a principal $W_\pm$-bundle.

**Proof.** It follows from the above remarks. □

3. Trajectories on $\mathbb{P}^1$

In Section 3, in order to fix notations, we collect basic facts and definitions for trajectories on $\mathbb{P}^1$ from [BS, Section 3]. Since our Riemann surface is only $\mathbb{P}^1$, many results are simplified.

3.1. **Foliations.** Let $\phi \in Q(N, n)$. For any neighborhood of a point in $\mathbb{P}^1 \setminus \text{Crit}(\phi)$, the differential $\phi(z) = \varphi(z)dz \otimes 2$ defines the distinguished local coordinate $w$ by

$$w := \int \sqrt{\varphi(z)} \, dz,$$

up to the transformation $w \mapsto \pm w + c (c \in \mathbb{C})$. The coordinate $w$ also characterized by

$$\phi(w) = dw \otimes dw.$$

The coordinate $w$ determines a foliation of the phase $\theta$ on $\mathbb{P}^1 \setminus \text{Crit}(\phi)$ by

$$\text{Im} \,(w/e^{i\pi \theta}) = \text{constant}.$$

A straight arc $\gamma : I \to \mathbb{P}^1 \setminus \text{Crit}(\phi)$ of the phase $\theta$ is an integral curve along the foliation $\text{Im} \,(w/e^{i\pi \theta}) = \text{constant}$, defined on an open interval $I \subset \mathbb{R}$. Note that a maximal straight arc of the phase $\theta$ is a leaf of the foliation of the phase $\theta$.

A foliation of the phase $\theta = 0$ is called a horizontal foliation, and a maximal straight arc of the phase $\theta = 0$ is called a trajectory.

**Lemma 3.1.** Let $\phi \in Q(N, n)$. For $k = r e^{i \pi \theta} \in \mathbb{C}^*$, set $\phi' := k \cdot \phi$. Then the distinguished coordinates $w$ and $w'$ which come from $\phi$ and $\phi'$ have the relation

$$w' = k^{a_n^N} w.$$

In addition, the transformation $z \mapsto kz$ on $\mathbb{P}^1$ maps the horizontal straight arcs for $\phi$ to the straight arcs of the phase $(a_n^N \theta)/2$ for $\phi'$. 
Proof. Let $\gamma: I \to \mathbb{P}^1 \setminus \text{Crit}(\phi)$ be a horizontal straight arc of $\phi$. Then the following computation gives the result:

$$\int^{k\gamma(t)} \phi(z) dz = \int^{k\gamma(t)} (k \cdot \phi)(z) dz = \int^{\gamma(t)} \sqrt{(k \cdot \phi)(kz')} dz' \quad \text{(put } z = kz').$$

$$= k^{\frac{dN}{2}} \int^{\gamma(t)} \phi(z') dz' \quad \text{(by Lemma 2.2).}$$

A trajectory $\gamma$ defined on a finite open interval $(a, b)$ is called saddle trajectory. Since $\mathbb{P}^1$ is compact, $\gamma$ is extended to a continuous path $\gamma: [a, b] \to \mathbb{P}^1$ with $\gamma(a), \gamma(b) \in \text{Zero}(\phi)$.

Similarly, a saddle connection of the phase $\theta$ is defined to be a maximal straight arc of the phase $\theta$ defined on a finite open interval.

3.2. Foliations near critical points. Here we see the local behavior of the horizontal foliation near a critical point. In details, we refer to [BS, Section 3.3]. First we consider the behavior near a zero. Let $\phi \in Q(N, n)$ and $p \in \text{Zero}(\phi)$ be a zero of order $N - 2$. Then there is a local coordinate $t$ of a neighborhood $p \in U \subset \mathbb{P}^1$ ($t = 0$ corresponds to $p$) such that

$$\phi = c^2 t^{N-2} dt^2, \quad c = \frac{1}{2} N.$$ 

Hence on $U \setminus \{p\}$, the distinguished local coordinate $w$ takes the form $w = t^{N/2}$. Figure 2 illustrates the local behavior of the horizontal foliation in the case $N = 3, 4, 5$.

![Figure 2. Trajectories near a zero of order 1, 2, 3.](image)

Note that there are $N$ horizontal straight arcs departing from $p$. This fact is the reason why the differential $\phi$ determines the $N$-angulation.

Next we consider the behavior near a pole. Recall from Section 2.3 that $\phi$ has a unique pole of order $d_n^N + 2$ at $\infty \in \mathbb{P}^1$. Therefore it is sufficient to consider a pole of order $> 2$. In this case, there is a neighborhood $\infty \in U \subset \mathbb{P}^1$ and a collection of $d_n^N$ distinguished tangent directions $v_i (i = 1, \ldots, d_n^N)$ at $\infty$, such that any trajectory entering $U$ tends to $\infty$ and becomes asymptotic to one of the $v_i$. For more details, we refer to [BS, Section 3.3].
3.3. **Classification of trajectories.** In this section, we summarize the global structure of the horizontal foliation on $\mathbb{P}^1$ following [BS, Section 3.4], and [Str84, Section 9-11 and 15].

We first note that $\phi$ is holomorphic on the contractible domain $\mathbb{P}^1 \setminus \{\infty\}$. Then Theorem 14.2.2 in [Str84] implies that there are no closed trajectories and Theorem 15.2 in [Str84] implies that there are no recurrent trajectories. Hence in our setting, every trajectory of $\phi$ is classified to exactly one of the following classes.

1. A saddle trajectory approaches distinct zeros of $\phi$ at both ends.
2. A separating trajectory approaches a zero of $\phi$ at one end and a pole $\infty \in \mathbb{P}^1$ at the other end.
3. A generic trajectory approaches a pole $\infty \in \mathbb{P}^1$ at both ends.

As explained in the previous section, there are only finite horizontal straight arcs departing from zeros of $\phi$. Therefore, the number of saddle trajectories and separating trajectories are finite. For a quadratic differential $\phi$, denote by $s_\phi$ the number of saddle trajectories, and by $t_\phi$ the number of separating trajectories. Then the fact that there are $N$ horizontal straight arcs departing from each zero of $\phi$ implies the equation

\[
2s_\phi + t_\phi = N(n + 1).
\]

By removing these two type trajectories and $\text{Crit}(\phi)$ from $\mathbb{P}^1$, the remaining open surface splits into a disjoint union of connected components. In our setting, each connected component coincides with one of the following two type surfaces.

1. A half-plane is the upper half-plane
   \[
   \{ w \in \mathbb{C} \mid \text{Im } w > 0 \}
   \]
   equipped with the differential $dw^\otimes 2$. It is swept out by generic trajectories with both end points at $\infty \in \mathbb{P}^1$. The boundary $\{\text{Im } w = 0\}$ consists of saddle trajectories and two separating trajectories.
2. A horizontal strip is the strip domain
   \[
   \{ w \in \mathbb{C} \mid a < \text{Im } w < b \}
   \]
   equipped with the differential $dw^\otimes 2$. It is also swept out by generic trajectories with both end points at $\infty \in \mathbb{P}^1$. Each of two boundaries $\{\text{Im } w = a\}$ and $\{\text{Im } w = b\}$ consists of saddle trajectories and two separating trajectories.

Note that every saddle trajectory or separating trajectory appears just two times as a boundary element of these open surfaces.

![Figure 3. Examples of a half plane and a horizontal strip.](image)
Let \( k \) be the number of half planes and \( l \) be the number of horizontal strips appearing in such a decomposition of \( \mathbb{P}^1 \). Since every half plane has 2 separating trajectories on the boundary, and every horizontal strip has 4 separating trajectories on the two boundaries, we have the equation
\[
2k + 4l = 2t_\phi.
\]

We also note that near a pole of order \( m > 2 \), just \( m - 2 \) half planes appear. Since \( \phi \) has a unique pole of order \((d_n^N + 2)\) at \( \infty \in \mathbb{P}^1 \), we have
\[
k = d_n^N = (N - 2)(n + 1) + 2.
\]

**Lemma 3.2.** For \( \phi \in \mathsf{Q}(N,n) \), the number of saddle trajectories \( s_\phi \) and the number of horizontal strips \( l \) satisfy the equation
\[
s_\phi + l = n.
\]

**Proof.** This immediately follows from the equations (3.1), (3.2) and (3.3). \( \square \)

**Definition 3.3.** A differential \( \phi \in \mathsf{Q}(N,n) \) is called saddle-free if \( \phi \) has no saddle trajectories \( (s_\phi = 0) \).

For a saddle-free differential \( \phi \in \mathsf{Q}(N,n) \), Lemma 3.2 implies that exactly \( n \) horizontal strips appear in the decomposition of \( \mathbb{P}^1 \) by \( \phi \). In this situation, any boundary component of a half plane or a horizontal strip consists of two separating trajectories.

### 3.4. Hat-homology classes and standard saddle connections

In this section, we collect some constructions of homology classes from [BS, Section 3.2 and 3.4].

Take \( \phi \in \mathsf{Q}(N,n) \). Recall from Section 2.4 that the square root of \( \phi \) defines a holomorphic 1-form \( \psi = \sqrt{\phi} \) on \( \mathbb{P}^1 \setminus \{\infty\} \) or \( S \setminus \pi^{-1}(\infty) \) depending on the parity of \( N \). We set
\[
\hat{S}^o := \begin{cases} \mathbb{P}^1 \setminus \{\infty\} & \text{if } N \text{ is even} \\ S \setminus \pi^{-1}(\infty) & \text{if } N \text{ is odd} \end{cases}
\]

We first note that the 1-form \( \psi \) defines the distinguished local coordinate \( \hat{w} \) on \( \hat{S}^o \setminus \text{Zero}(\psi) \) by \( \psi = \hat{w}d\hat{w} \), up to the transformation \( \hat{w} \mapsto \hat{w} + c (c \in \mathbb{C}) \). As a result, \( \psi \) determines an oriented horizontal foliation \( \text{Im} \hat{w} = \text{constant} \) on \( \hat{S}^o \setminus \text{Zero}(\psi) \). The orientation of the foliation is determined to make the tangent vectors lie in \( \mathbb{R}_{>0} \).

For a saddle trajectory \( \gamma : [a, b] \to \mathbb{P}^1 \), we can define a homology class \( \hat{\gamma} \in H_\pm(\phi) \), called the hat-homology class of \( \gamma \) by the following.

If \( N \) is even, we orient \( \gamma \) to be compatible with the above orientation. Since \( \gamma(a), \gamma(b) \in \text{Zero}(\phi) \), the oriented curve \( \gamma \) defines a homology class
\[
\hat{\gamma} := [\gamma] \in H_+ (\phi) = H_1 (\mathbb{C}, \text{Zero}(\phi); \mathbb{Z})
\]

. By definition, it satisfies \( Z_\phi (\hat{\gamma}) \in \mathbb{R}_{>0} \).

If \( N \) is odd, since zeros of \( \phi \) are branched points of the double cover \( \pi : S \to \mathbb{P}^1 \), the inverse image \( \pi^{-1}(\gamma) \) becomes a closed curve in \( \hat{S}^o \). We orient \( \pi^{-1}(\gamma) \) to be compatible with the above orientation. Then the oriented closed curve \( \pi^{-1}(\gamma) \) defines a homology class homology class
\[
\hat{\gamma} := [\pi^{-1}(\gamma)] \in H_- (\phi) = H_1 (S \setminus \pi^{-1}(\infty); \mathbb{Z})
\]

As in the case \( N \) is even, it satisfies \( Z_\phi (\hat{\gamma}) \in \mathbb{R}_{>0} \).
A similar construction works for a saddle connection \( \gamma: [a, b] \to \mathbb{P}^1 \) with the phase \( \theta \in (0, \pi) \). In this case, the orientation is determined to satisfy that \( Z_{\phi}(\hat{\gamma}) \in \{ r e^{i \pi \theta} \mid r > 0 \} \subset \mathbb{H} \).

Let \( h = \{ a < \text{Im } w < b \} \) be a horizontal strip appearing in the domain decomposition of \( \mathbb{P}^1 \) by a saddle-free differential \( \phi \). Then each of two boundary components \( \{ \text{Im } w = a \} \) and \( \{ \text{Im } w = b \} \) contains exactly one zero of \( \phi \) and consists of two separating trajectories intersecting at the zero. Hence, there is a unique saddle connection \( l_h \) with the phase \( \theta \in (0, \pi) \) which connects two zeros appearing in two boundary components. We denote by \( \gamma_h := \hat{l}_h \in H_\pm(\phi) \) the hat-homology class associated to the saddle connection \( l_h \).

Since \( \phi \) is saddle-free, Lemma 3.2 implies that there are precisely \( n \) horizontal strips in the decomposition of \( \mathbb{P}^1 \) by \( \phi \). Thus, we obtain \( n \) saddle connections \( l_{h_1}, \ldots, l_{h_n} \) and \( n \) hat-homology classes \( \gamma_{h_1}, \ldots, \gamma_{h_n} \). \( l_h \) is called a standard saddle connection, and \( \gamma_h \) is called a standard saddle class.

**Lemma 3.4** ([BS], Lemma 3.2). Standard saddle classes \( \gamma_{h_1}, \ldots, \gamma_{h_n} \) form a basis of \( H_\pm(\phi) \).

**Proof.** The argument of Lemma 3.2 in [BS] for \( N = 3 \) also works for \( N \geq 3 \). \( \square \)

4. **CY\(_N\) categories associated to acyclic quivers**

For objects \( E, F \) in a triangulated category \( \mathcal{D} \), we set

\[
\text{Hom}^i(E, F) := \text{Hom}_\mathcal{D}(E, F[i]) \\
\text{hom}^i(E, F) := \dim_k \text{Hom}_\mathcal{D}(E, F[i])
\]

where \([i]\) is the \( i \)-th shift functor.

4.1. **Ginzburg dg algebras.** Let \( Q = (Q_0, Q_1) \) be a finite acyclic quiver with vertices \( Q_0 = \{1, \ldots, n\} \) and arrows \( Q_1 \). The Ginzburg dg algebra \( \Gamma_N Q := (k\overline{Q}, d) \) is defined as follows. Define a graded quiver \( \overline{Q} \) with vertices \( \overline{Q}_0 = \{1, \ldots, n\} \) and following arrows:

- an original arrow \( a: i \to j \in Q_1 \) (degree 0);
- an opposite arrow \( a^* : j \to i \) for the original arrow \( a : i \to j \in Q_1 \) (degree \( N - 2 \));
- a loop \( t_i \) for each vertex \( i \in Q_0 \) (degree \( N - 1 \)).

Let \( k\overline{Q} \) be a graded path algebra of \( \overline{Q} \), and define a differential \( d: k\overline{Q} \to k\overline{Q} \) of degree \(-1\) by
\[ da = da^* = 0 \text{ for } a \in Q_1; \]
\[ dt_i = e_i \left( \sum_{a \in Q_1} (aa^* - a^*a) \right) e_i; \]

where \( e_i \) is the idempotent at \( i \in Q_0 \). Thus we have the dg algebra \( \Gamma_N Q = (k\overline{Q}, d) \).

Let \( D(\Gamma_N Q) \) be the derived category of right dg-modules over \( \Gamma_N Q \), and \( D_N^Q \) be the full subcategory of \( D(\Gamma_N Q) \) consists of dg modules \( M \) whose homology is of finite total dimension:

\[ D_N^Q := \{ M \in D(\Gamma_N Q) \mid \sum_{i \in \mathbb{Z}} \dim_k H_i(M) < \infty \}. \]

A triangulated category \( D \) is called \( N \)-Calabi-Yau (CY \( N \)) if for any objects \( E, F \in D \), there is a natural isomorphism of \( k \)-vector spaces

\[ \nu: \text{Hom}(E, F) \xrightarrow{\sim} \text{Hom}^N(F, E)^*. \]

(For a vector space \( V \), we denote by \( V^* \) the dual vector space.)

**Theorem 4.1** ([Kel11], Theorem 6.3). The category \( D_N^Q \) is a CY \( N \) triangulated category.

Let \( K(D_N^Q) \) be the Grothendieck group of \( D_N^Q \). A bilinear form \( \chi: K(D_N^Q) \times K(D_N^Q) \to \mathbb{Z} \), called the Euler form, is defined to be

\[ \chi(E, F) := \sum_{i \in \mathbb{Z}} (-1)^i \text{hom}^i(E, F), \quad \text{for } E, F \in D_N^Q. \]

The CY \( N \) property of \( D_N^Q \) implies that \( \chi \) is symmetric if \( N \) is even, and skew-symmetric if \( N \) is odd.

For a quiver \( A_n \), we set \( D_N^A_n := D_N^{A_n} \).

**4.2. Hearts of bounded t-structures.**

**Definition 4.2** ([BBD82]). A t-structure on \( D \) is a full subcategory \( P \subset D \) satisfying the following conditions:

(a) \( P[1] \subset P \),

(b) define \( P^\perp := \{ G \in D \mid \text{Hom}(F, G) = 0 \text{ for all } F \in P \} \), then for every object \( E \in D \) there is an exact triangle \( F \to E \to G \to F[1] \) in \( D \) with \( F \in P \) and \( G \in P^\perp \).

In addition, the t-structure \( P \subset D \) is said to be bounded if \( P \) satisfies the condition:

\[ D = \bigcup_{i, j \in \mathbb{Z}} P^\perp[i] \cap P[j]. \]

For a t-structure \( P \subset D \), we define its heart \( \mathcal{H} \) by

\[ \mathcal{H} := P^\perp[1] \cap P. \]

It was proved in [BBD82] that \( \mathcal{H} \) becomes an abelian category.

**Remark 4.3.** For a heart \( \mathcal{H} \subset D \) of a bounded t-structure, we have a canonical isomorphism of Grothendieck groups

\[ K(\mathcal{H}) \cong K(D). \]

In the following, we only treat bounded t-structures and their hearts. The word heart always refers to the heart of some bounded t-structure.
Proposition 4.4 ([Ami09], Section 2). There is a canonical bounded t-structure \( \mathcal{P}_0 \subset D_Q^N \) with the heart \( \mathcal{H}_0 \) such that the functor \( H_0 : D_Q^N \to \text{mod-} kQ \) induces an equivalence of abelian categories:

\[
H_0 : \mathcal{H}_0 \xrightarrow{\sim} \text{mod-} kQ.
\]

We call \( \mathcal{H}_0 \) the standard heart of \( D_Q^N \). \( \mathcal{H}_0 \) is generated by simple \( \Gamma_N Q \) modules \( S_1, \ldots, S_n \) which correspond to vertices 1, \ldots, n of \( Q \).

4.3. Spherical twist. We define some autoequivalences of \( D_Q^N \), called spherical twists, introduced by P. Seidel and R. Thomas [ST01].

An object \( S \in D_Q^N \) is called \( N \)-spherical if

\[
\text{Hom}_i(S, S) = \begin{cases} k & \text{if } i = 0, N \\ 0 & \text{otherwise.} \end{cases}
\]

Proposition 4.5 ([ST01], Proposition 2.10). For a spherical object \( S \in D_Q^N \), there is an exact autoequivalence \( \Phi_S \in \text{Aut}(D_Q^N) \) defined by the exact triangle

\[
\text{Hom}^\bullet(S, E) \otimes S \to E \to \Phi_S(E)
\]

for any object \( E \in D_Q^N \). The inverse functor \( \Phi_S^{-1} \in \text{Aut}(D_Q^N) \) is given by

\[
\Phi_S^{-1}(E) \to E \to S \otimes \text{Hom}^\bullet(E, S)^*.
\]

Let \( S_1, \ldots, S_n \in \mathcal{H}_0 \) be simple \( \Gamma_N Q \) modules. Then \( S_1, \ldots, S_n \) are \( N \)-spherical in \( D_Q^N \) (see Lemma 4.4 in [Kel11]). Hence they define spherical twists \( \Phi_{S_1}, \ldots, \Phi_{S_n} \in \text{Aut}(D_Q^N) \). The Seidel-Thomas braid group \( \text{Br}(D_Q^N) \) is defined to be the subgroup of \( \text{Aut}(D_Q^N) \) generated by these spherical twists:

\[
\text{Br}(D_Q^N) := \langle \Phi_{S_1}, \ldots, \Phi_{S_n} \rangle.
\]

In the rest of this section, we assume that \( Q = \mathbb{A}_n^+ \).

Theorem 4.6 ([ST01], Theorem 1.2 and Theorem 1.3). The correspondence of generators \( \sigma_i \mapsto \Phi_{S_i} \) gives an isomorphism of groups

\[
B_{n+1} \cong \text{Br}(D_n^N).
\]

On \( K(D_n^N) \), a spherical twist \( \Phi_{S_i} \) induces the reflection \( [\Phi_{S_i}] : K(D_n^N) \to K(D_n^N) \) given by

\[
[\Phi_{S_i}](E) = [E] - \chi(S_i, E)[S_i].
\]

Recall from Section 2.1 that for a quiver \( \mathbb{A}_n^+ \), we define the lattice \( (L, \langle , \rangle_\pm) \). By the map \( [S_i] \mapsto \alpha_i \), we have an isomorphism of abelian groups

\[
K(D_n^N) \cong L.
\]

Further, this map takes the Euler form \( \chi \) to the bilinear form \( \langle , \rangle_\pm \). Therefore, under the above isomorphism, spherical twists \( \Phi_{S_1}, \ldots, \Phi_{S_n} \) act on \( K(D_n^N) \) as reflections \( r_{1\pm}^+, \ldots, r_{n\pm}^+ \) in Section 2.1:

\[
[\Phi_{S_i}] = r_{1\pm}^+ , \ldots, [\Phi_{S_n}] = r_{n\pm}^+.
\]
4.4. **Simple tilted hearts.** Let \( \mathcal{H} \subset \mathcal{D} \) be a heart in a triangulated category \( \mathcal{D} \). We denote by \( \text{Sim} \mathcal{H} \) the set of simple objects in \( \mathcal{H} \).

**Definition 4.7.** \( \mathcal{H} \) is called finite if \( \text{Sim} \mathcal{H} \) is a finite set and any object of \( \mathcal{H} \) is generated by finitely many extensions of \( \text{Sim} \mathcal{H} \).

For a given heart, one can construct a new heart by using the technique of tilting in [HRS96]. In particular for a finite heart \( \mathcal{H} \subset \mathcal{D} \) and a simple object \( S \in \mathcal{H} \), there is a new heart \( \mathcal{H}^S \) (respectively, \( \mathcal{H}^S \)), called the forward (respectively, backward) simple tilt of \( \mathcal{H} \) by \( S \). For the precise definition of \( \mathcal{H}^S \) (\( \mathcal{H}^S \)), we refer to [KQ, Section 3]. (Our notation is based on [KQ].) In the present paper, we do not use the explicit construction of simple tilted hearts.

We say an object \( S \in \mathcal{H} \) is rigid if \( \text{Hom}^1(S, S) = 0 \).

**Proposition 4.8** ([KQ], Proposition 5.2). Let \( \mathcal{D} \) be a triangulated category and \( \mathcal{H} \subset \mathcal{D} \) be a finite heart. For any rigid simple object \( S \in \mathcal{H} \), the sets of simple objects in the simple tilted hearts \( \mathcal{H}^S \) and \( \mathcal{H}^S \) are given by

\[
\text{Sim} \mathcal{H}^S = \{ S[1] \} \cup \{ \psi^S(X) \mid X \in \text{Sim} \mathcal{H}, X \neq S \}
\]

\[
\text{Sim} \mathcal{H}^S = \{ S[-1] \} \cup \{ \psi^S(X) \mid X \in \text{Sim} \mathcal{H}, X \neq S \}
\]

where

\[
\psi^S(X) = \text{Cone}(X \to S[1] \otimes \text{Hom}^1(X, S)^*)[-1]
\]

\[
\psi^S(X) = \text{Cone}(S[-1] \otimes \text{Hom}^1(S, X) \to X).
\]

In addition, \( \mathcal{H}^S \) and \( \mathcal{H}^S \) are also finite.

4.5. **Exchange graphs.** In this section, we summarize basic properties of finite hearts in \( \mathcal{D}^N_Q \) showed in [KQ].

**Definition 4.9.** An oriented graph \( \text{EG}(\mathcal{D}^N_Q) \), called the exchange graph of \( \mathcal{D}^N_Q \), is defined as follows. The set of vertices in \( \text{EG}(\mathcal{D}^N_Q) \) consists of all finite hearts of \( \mathcal{D}^N_Q \), and two hearts \( \mathcal{H} \) and \( \mathcal{H}' \) are connected by the arrow \( \mathcal{H} \rightarrow \mathcal{H}' \) if there is some rigid simple object \( S \in \mathcal{H} \) such that they are related by the simple forward tilt \( \mathcal{H}^S \) = \( \mathcal{H}' \).

The connected subgraph \( \text{EG}^*(\mathcal{D}^N_Q) \subset \text{EG}(\mathcal{D}^N_Q) \) is defined to be the connected component which contains the standard heart \( \mathcal{H}_0 \).

For \( \mathcal{H} \in \text{EG}^*(\mathcal{D}^N_Q) \) and a rigid simple object \( S \in \mathcal{H} \), we define inductively

\[
\mathcal{H}^S\mathcal{H}^S := (\mathcal{H}^{(m-1)H^S})^{(m-1)H^S} \quad \mathcal{H}^S := (\mathcal{H}^{(m-1)H^S})^{(m-1)H^S}
\]

for \( m \geq 1 \).

**Definition 4.10.** We say that a heart \( \mathcal{H} \) is

- spherical, if all its simples are spherical,
- monochromatic, if for any simples \( S \neq T \in \mathcal{H} \), \( \text{Hom}^*(S, T) \) is concentrated in a single positive degree.

**Proposition 4.11** ([KQ], Corollary 8.3). Every heart in \( \text{EG}^*(\mathcal{D}^N_Q) \) is finite, spherical and monochromatic. Let \( \mathcal{H} \in \text{EG}^*(\mathcal{D}^N_Q) \) a finite heart and consider simple
objects $\text{Sim} \mathcal{H} = \{T_{1}, \ldots, T_{n}\}$. Then spherical twists $\Phi_{T_{1}}, \ldots, \Phi_{T_{n}}$ generate the Seidel-Thomas braid group $\text{Br}(D_{Q}^{N})$. In addition, we have

$$H_{T_{i}}^{(N-1)b} = \Phi_{T_{i}}^{-1}(H), \quad H_{T_{i}}^{(N-1)g} = \Phi_{T_{i}}(H).$$

The result implies that the action of $\text{Br}(D_{Q}^{N})$ on $\text{EG}(D_{Q}^{N})$ preserves the connected component $\text{EG}^{g}(D_{Q}^{N})$ and is free on it.

Let $\mathcal{H}_{i}$ ($i = 1, 2$) be the hearts of bounded t-structures $\mathcal{P}_{i}$ ($i = 1, 2$). We denote by

$$\mathcal{H}_{1} \leq \mathcal{H}_{2}$$

if they satisfy the condition $\mathcal{P}_{2} \subset \mathcal{P}_{1}$.

We define the full subgraph of $\text{EG}^{g}(D_{Q}^{N})$ by

$$\text{EG}^{g}_{N}(D_{Q}^{N}) := \left\{ \mathcal{H} \in \text{EG}^{g}(D_{Q}^{N}) \mid \mathcal{H}_{0} \leq \mathcal{H} \leq \mathcal{H}_{0}[N-2] \right\}.$$

Let $\mathcal{H} \in \text{EG}(D_{Q}^{N})$ be a finite heart and $S \in \mathcal{H}$ be a simple object. Since

$$\Phi(\mathcal{H}^{g}_{S}) = (\Phi(\mathcal{H}))^{g}_{\Phi(S)}$$

for any autoequivalence $\Phi \in \text{Aut}(D_{Q}^{N})$, the element of $\text{Aut}(D_{Q}^{N})$ acts on $\text{EG}(D_{Q}^{N})$ as an automorphism of the oriented graph. Denote by $\text{EG}^{g}(D_{Q}^{N})$ the subgroup consisting of autoequivalences which preserve the connected component $\text{EG}^{g}(D_{Q}^{N})$.

The following implies that $\text{EG}^{g}_{N}(D_{Q}^{N})$ is a fundamental domain of the action of $\text{Br}(D_{Q}^{N})$ on $\text{EG}^{g}(D_{Q}^{N})$.

**Theorem 4.12** ([KQ], Theorem 8.5). There is a map of oriented graphs

$$p_{0} : \text{EG}^{g}_{N}(D_{Q}^{N}) \rightarrow \text{EG}^{g}(D_{Q}^{N}) / \text{Br}(D_{Q}^{N})$$

such that $p_{0}$ is embedding and bijective on vertices.

### 4.6. Colored quivers and mutation

Following [BT09, Section 2], we introduce the notion of a colored quiver and its mutation.

An $(N-2)$-colored quiver $Q$ consists of vertices $\{1, \ldots, n\}$ and colored arrows $i \xrightarrow{c} j$, where $c \in \{0, 1, \ldots, N-2\}$. Write by $q_{ij}^{(c)}$ the number of arrows from $i$ to $j$ of color $(c)$.

In addition, we assume the following conditions:

1. No loops, $q_{ii}^{(c)} = 0$ for all $c$.
2. Monochromaticity, if $q_{ij}^{(c)} \neq 0$, then $q_{ij}^{(c')} = 0$ for $c \neq c'$.
3. Skew-symmetry, $q_{ij}^{(c)} = q_{ji}^{(N-2-c)}$.

Let $Q$ be a $(N-2)$-colored quiver satisfying additional conditions and fix a vertex $v$ in $Q$. Then a new $(N-2)$-colored quiver $\mu_{v}^{+}Q = (\tilde{q}_{ij}^{(c)})$ is defined by

$$\tilde{q}_{ij}^{(c)} = \begin{cases} q_{ij}^{(c)} & \text{if } v = i \\ q_{ij}^{(c) - 1} - \max\{0, q_{ij}^{(c)} - \sum_{t \neq c} q_{ij}^{(t)} + (q_{iv}^{(c)} - q_{iv}^{(c+1)} + q_{ij}^{(c) - 1})\} & \text{if } v = j \\ \max\{0, q_{ij}^{(c)} - \sum_{t \neq c} q_{ij}^{(t)} + (q_{iv}^{(c)} - q_{iv}^{(c+1)} + q_{ij}^{(c) - 1})\} & \text{otherwise.} \end{cases}$$

We call this operation the forward colored quiver mutation of $Q$ at a vertex $v$. The backward mutated colored quiver $\mu_{v}^{-}Q$ can be similarly defined. (Our notation is
different to the notation in [BT09]. The backward mutation $\mu^-_i Q$ corresponds to the colored quiver mutation $\mu_{\iota}(Q)$ in [BT09, Section 2].

Next we consider the colored quiver associated to a finite heart in $\text{EG}^q(D_Q^N)$.

**Definition 4.13.** For a finite heart $H \in \text{EG}^q(D_Q^N)$, we define the colored quiver $Q(H)$ whose vertices $1, \ldots, n$ correspond to simple objects $S_1, \ldots, S_n \in \text{Sim} H$ and the number of arrows $q_{ij}^{(c)}$ from $i$ to $j$ of color $(c)$ is given by $q_{ij}^{(c)} := \text{hom}^{c+1}(S_i, S_j)$.

By definition, $Q(H)$ has no loops. Further, the CY$_N$ property of $D_Q^N$ implies that $Q(H)$ is skew-symmetric, and Proposition 4.11 implies the monochromaticity of $Q(H)$. Therefore, the colored quiver $Q(H)$ satisfies additional conditions.

**Theorem 4.14** ([BT09], Theorem 2.1 and [KQ], Theorem 8.6). Let $H \in \text{EG}^q(D_Q^N)$ be a finite heart and $Q(H)$ be the corresponding colored quiver. Then for any simple object $S_v \in H$, we have

$$\mu^-_v Q(H) = Q(H'_{S_v}), \quad \mu^+_v Q(H) = Q(H''_{S_v}).$$

**Proof.** For hearts $H$ and $H'_{S_v}$, we have the corresponding cluster tilting objects $T$ and $T'$ in $C_{N-2}(Q)$ induced by the exact sequence of triangulated categories

$$0 \to D(\Gamma_N Q) \to \text{per}(\Gamma_N Q) \to C_{N-2}(Q) \to 0$$

where $C_{N-2}(Q)$ is the $(N-2)$-cluster category associated to $Q$ (see [KQ, Section 8]). For cluster tilting objects $T$ and $T'$, we can define certain colored quivers $Q_T$ and $Q_{T'}$ (see [KQ, Section 6] or [BT09, Section 2]). Theorem 8.6 in [KQ] implies that the colored quivers $Q_T$ and $Q_{T'}$ coincide with $Q(H)$ and $Q(H'_{S_v})$ respectively.

On the other hand, Theorem 2.1 in [BT09] implies that

$$\mu^+_v Q_T = Q_{T'}.$$

$\square$

### 4.7. Colored quiver lattices

In this section, we define a lattice associated to a colored quiver.

For a $(N-2)$-colored quiver $Q$ with vertices $\{1, \ldots, n\}$, let $L_Q$ be a free abelian group generated by vertices of $Q$. Denote by $\alpha_1, \ldots, \alpha_n$ generators corresponding to vertices $1, \ldots, n$. Then we have

$$L_Q = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i.$$

Define a bilinear form $\langle \cdot, \cdot \rangle : L_Q \times L_Q \to \mathbb{Z}$ by

$$\langle \alpha_i, \alpha_j \rangle := \delta_{ij} + (-1)^N \delta_{ij} - \sum_{c=0}^{N-2} (-1)^c q_{ij}^{(c)}.$$

Since $q_{ij}^{(c)} = q_{ji}^{(N-2-c)}$, the bilinear form $\langle \cdot, \cdot \rangle$ is symmetric if $N$ is even, and skew-symmetric if $N$ is odd.

Let $Q(H)$ be a colored quiver of a heart $H \in \text{EG}^q(D_Q^N)$ with simple objects $S_1, \ldots, S_n$. We write by $\alpha_i$ the basis of colored quiver lattice $L_{Q(H)}$ which corresponds to a vertex $i$ of $Q(H)$. Then we have the following obvious result.
Lemma 4.15. There is an isomorphism of abelian groups
\[ \lambda : L_{Q(H)} \to K(D^N_{Q}) \]
defined by \( \alpha_i \mapsto [S_i] \). The map \( \lambda \) takes the bilinear form \( \langle , \rangle \) on \( L_{Q(H)} \) to the Euler form \( \chi \) on \( K(D^N_{Q}) \).

Proof. Since
\[ K(D^N_{Q}) \cong K(H) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[S_i], \]
\( \lambda \) is an isomorphism. Recall from Definition 4.13 that \( q_{ij}^{(c)} = \text{hom}^{c+1}(S_i, S_j) \). In addition, since \( H \) is spherical, \( \text{hom}^{0}(S_i, S_j) = \text{hom}^{N}(S_i, S_j) = \delta_{ij} \). Hence the result follows. \( \square \)

Let \( H^\sharp_{S_v} \) be the forward simple tilted heart and \( S'_1, \ldots, S'_1 \) be its simple objects. By Proposition 4.8, we have
\[ S'_j = \begin{cases} S_v[1] & \text{if } j = v \\ \psi^S_{S_v}(S_j) & \text{if } j \neq v \end{cases} \]
where
\[ \psi^S_{S_v}(S_j) = \text{Cone}(S_j \to S_v[1] \otimes \text{Hom}^1(S_v, S_j)^{-})[-1]. \]

Let \( \mu^\sharp_{v} Q(H) \) be the forward colored quiver mutation of \( Q(H) \) at \( v \).
Consider two colored quiver lattices
\[ L_{Q(H)} = \bigoplus_{j=1}^{n} \mathbb{Z} \alpha_j, \quad L_{\mu^\sharp_{v} Q(H)} = \bigoplus_{j=1}^{n} \mathbb{Z} \alpha'_j. \]

We define a linear map \( F_v : L_{\mu^\sharp_{v} Q(H)} \to L_{Q(H)} \) by
\[ F_v(\alpha'_j) := \begin{cases} -\alpha_v & \text{if } j = v \\ \alpha_j + q_{vj}^{(0)} \alpha_v & \text{if } j \neq v. \end{cases} \]

Since \( \mu^\sharp_{v} Q(H) = Q(H^\sharp_{S_v}) \) by Theorem 4.14, we have \( L_{\mu^\sharp_{v} Q(H)} \cong L_{Q(H^\sharp_{S_v})} \). Hence from Lemma 4.15, we have an isomorphism
\[ \lambda' : L_{\mu^\sharp_{v} Q(H)} \to K(D^N_{Q}) \]
defined by \( \alpha'_j \mapsto [S'_j] \).

In the above notations, we have the following diagram.

\[ \begin{array}{ccc} L_{\mu^\sharp_{v} Q(H)} & \xrightarrow{F_v} & L_{Q(H)} \\ \downarrow{\lambda'} & & \downarrow{\lambda} \\ K(D^N_{Q}) & & \end{array} \]

Lemma 4.16. Let \( \lambda : L_{Q(H)} \to K(D^N_{Q}) \) and \( \lambda' : L_{\mu^\sharp_{v} Q(H)} \to K(D^N_{Q}) \) be the isomorphisms given by Lemma 4.15. Then the diagram

\[ \begin{array}{ccc} L_{\mu^\sharp_{v} Q(H)} & \xrightarrow{F_v} & L_{Q(H)} \\ \downarrow{\lambda'} & & \downarrow{\lambda} \\ K(D^N_{Q}) & & \end{array} \]

commutes.
Proof. First note that by the equation (4.1), we have
\[
[S'_j] = \begin{cases} 
- [S_v] & \text{if } j = v \\
[S_j] + \text{hom}^1(S_v, S_j)[S_v] & \text{if } j \neq v
\end{cases}
\]
in \( K(D^b_Q) \). Recall from Definition 4.13 that \( q^{(0)}_{v,j} = \text{hom}^1(S_v, S_j) \). Since \( \lambda(\alpha_j) = [S_j] \) and \( \lambda'(\alpha'_j) = [S'_j] \), we have the result. \( \square \)

5. Bridgeland stability conditions

5.1. Spaces of stability conditions. In this section, we recall the notion of stability conditions on triangulated categories introduced by T. Bridgeland [Bri07] and collect some results for spaces of stability conditions.

Definition 5.1. Let \( D \) be a triangulated category and \( K(D) \) be its \( K \)-group. A stability condition \( \sigma = (Z, P) \) on \( D \) consists of a group homomorphism \( Z : K(D) \to \mathbb{C} \) called central charge and a family of full additive subcategories \( P(\phi) \subset D \) for \( \phi \in \mathbb{R} \) satisfying the following conditions:

(a) if \( 0 \neq E \in P(\phi) \), then \( Z(E) = m(E) \exp(i\pi \phi) \) for some \( m(E) \in \mathbb{R}_{>0} \),

(b) for all \( \phi \in \mathbb{R} \), \( P(\phi + 1) = P(\phi)[1] \),

(c) if \( \phi_1 > \phi_2 \) and \( A_i \in P(\phi_i) \) (\( i = 1, 2 \)), then \( \text{Hom}_D(A_1, A_2) = 0 \),

(d) for \( 0 \neq E \in D \), there is a finite sequence of real numbers \( \phi_1 > \phi_2 > \cdots > \phi_m \) and a collection of exact triangles

\[
0 = E_0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E_m = E
\]

with \( A_i \in P(\phi_i) \) for all \( i \).

It follows from the definition that the subcategory \( P(\phi) \subset D \) is an abelian category (see Lemma 5.2 in [Bri07]). Nonzero objects in \( P(\phi) \) are called semistable of phase \( \phi \) in \( \sigma \), and simple objects in \( P(\phi) \) are called stable of phase \( \phi \) in \( \sigma \).

For a stability condition \( \sigma = (Z, P) \), we introduce the set of semistable classes \( C^\text{ss}(\sigma) \subset K(D) \) by

\[
C^\text{ss}(\sigma) := \{ \alpha \in K(D) \mid \text{there exists a semistable object } E \in D \text{ in } \sigma \text{ such that } [E] = \alpha \}.
\]

We always assume that our stability conditions satisfy the additional assumption, called the support property in [KS].

Definition 5.2. Let \( \| \cdot \| \) be some norm on \( K(D) \otimes \mathbb{R} \). A stability condition \( \sigma = (Z, P) \) has a support property if there is a some constant \( C > 0 \) such that

\[
C \cdot |Z(\alpha)| > \|\alpha\|
\]

for all \( \alpha \in C^\text{ss}(\sigma) \).

Let \( \text{Stab}(D) \) be the set of all stability conditions on \( D \) with the support property.

In [Bri07], Bridgeland introduced a natural topology on the space \( \text{Stab}(D) \) characterized by the following theorem.
Theorem 5.3 ([Bri07], Theorem 1.2). The space \( \text{Stab}(\mathcal{D}) \) has the structure of a complex manifold and the projection map of central charges

\[
\pi: \text{Stab}(\mathcal{D}) \to \text{Hom}_\mathbb{Z}(K(\mathcal{D}), \mathbb{C})
\]

defined by \((Z, \mathcal{P}) \mapsto Z\) is a local isomorphism of complex manifolds onto an open subset of \(\text{Hom}_\mathbb{Z}(K(\mathcal{D}), \mathbb{C})\).

In [Bri07], Bridgeland gave the alternative description of a stability condition on \(\mathcal{D}\) as a pair of a bounded \(t\)-structure and a central charge on its heart.

Definition 5.4. Let \(\mathcal{H}\) be an abelian category and let \(K(\mathcal{H})\) be its Grothendieck group. A central charge on \(\mathcal{H}\) is a group homomorphism \(Z: K(\mathcal{H}) \to \mathbb{C}\) such that for any nonzero object \(0 \neq E \in \mathcal{H}\), the complex number \(Z(E)\) lies in semi-closed upper half-plane \(H = \{ \text{re}^{\pi \phi} \in \mathbb{C} | r \in \mathbb{R}_{>0}, \phi \in (0, 1)\}\).

The real number

\[
\phi(E) := \frac{1}{\pi} \arg Z(E) \in (0, 1)
\]

for \(0 \neq E \in \mathcal{H}\) is called the phase of \(E\).

A nonzero object \(0 \neq E \in \mathcal{H}\) is said to be \(Z\)-(semi)stable if for any nonzero proper subobject \(0 \neq A \subset E\) satisfies \(\phi(A) < (\leq) \phi(E)\).

For a central charge \(Z: K(\mathcal{H}) \to \mathbb{C}\), the set of semistable classes can be defined similarly.

Proposition 5.5 ([Bri07], Proposition 5.3). Let \(\mathcal{D}\) be a triangulated category. To give a stability condition on \(\mathcal{D}\) is equivalent to giving the heart \(\mathcal{H} \subset \mathcal{D}\) of a bounded structure on \(\mathcal{D}\) and a central charge with the Harder-Narasimhan property on \(\mathcal{H}\).

For details of the Harder-Narasimhan property (HN property), we refer to [Bri07, Section 2].

We denote by \(\text{Stab}(\mathcal{H})\) the set of central charges on the heart \(\mathcal{H} \subset \mathcal{D}\) with the HN property and the support property.

5.2. Simple tilting and gluing. Here by using Proposition 5.5, we construct stability conditions on a finite heart.

Let \(\mathcal{H} \subset \mathcal{D}\) be a finite heart with simple objects \(\{S_1, \ldots, S_n\}\). Then \(K(\mathcal{H}) \cong \oplus_{i=1}^n Z[S_i]\). For any point \((z_1, \ldots, z_n) \in H^n\), the central charge \(Z: K(\mathcal{H}) \to \mathbb{C}\) is constructed by \(Z(S_i) := z_i\). Conversely, for a given central charge \(Z: K(\mathcal{H}) \to \mathbb{C}\), the complex number \(Z(S_i)\) lies in \(H\) for all \(i\). Hence \(Z\) determines a point \((z_1, \ldots, z_n) \in H^n\) where \(z_i := Z(S_i)\). As a result, the set of central charges on \(\mathcal{H}\) is isomorphic to \(H^n\).

Lemma 5.6 ([Bri09], Lemma 5.2). Let \(Z: K(\mathcal{H}) \to \mathbb{C}\) be a central charge given by the above construction. Then \(Z\) has a Harder-Narasimhan property. In particular, we have

\[
\text{Stab}(\mathcal{H}) \cong H^n.
\]

Let \(\mathcal{H} \subset \mathcal{D}\) be a finite rigid heart with simple objects \(\{S_1, \ldots, S_n\}\). From the above results, there is a natural inclusion

\[
\text{Stab}(\mathcal{H}) \subset \text{Stab}(\mathcal{D}).
\]

From such a heart, we can construct the another hearts \(\mathcal{H}^\perp_{S_i}\) (or \(\mathcal{H}^\uparrow_{S_i}\)) by simple tilting as in Section 4.4. The relationship between \(\text{Stab}(\mathcal{H})\) and \(\text{Stab}(\mathcal{H}^\perp_{S_i})\) (or \(\text{Stab}(\mathcal{H}^\uparrow_{S_i})\)) in \(\text{Stab}(\mathcal{D})\) is given by the following lemma.
Lemma 5.7 ([BS], Lemma 7.9). Let $\sigma = (Z, P) \in \text{Stab}(D)$ lies on a unique codimension one boundary of the region $\text{Stab}(H)$ so that $\text{Im} Z(S_i) = 0$ for a unique simple object $S_i$. Then there is a neighborhood $\sigma \subset U \subset \text{Stab}(D)$ such that one of the following holds:

1. $Z(S_i) \in \mathbb{R}_{>0}$ and $U \subset \text{Stab}(H) \cup \text{Stab}(H_{S_i}^t)$,
2. $Z(S_i) \in \mathbb{R}_{<0}$ and $U \subset \text{Stab}(H) \cup \text{Stab}(H_{S_i}^s)$.

Consider $D = D_n^N$, which is the derived category of finite total dimensional dg modules over the Ginzburg dg algebra $\Gamma_N \tilde{A}_n$ associated to the $A_n$-quiver (see section 4.1). By Proposition 4.4, there is a special heart $H_0 \subset D_n^N$, called the standard heart. Since $H_0$ is finite, we have $\text{Stab}(H_0) \cong H^n$.

Definition 5.8. The distinguished connected component $\text{Stab}^o(D_n^N) \subset \text{Stab}(D_n^N)$ is defined to be the connected component which contains the connected subset $\text{Stab}(H_0)$.

By using the result in [Woo10], Qiu described the space $\text{Stab}^o(D_n^N)$ as follows.

Proposition 5.9 ([Qiu], Corollary 5.3). The distinguished connected component $\text{Stab}^o(D_n^N)$ is given by

$$\text{Stab}^o(D_n^N) = \bigcup_{H \in \text{EG}^o(D_n^N)} \text{Stab}(H).$$

5.3. Group actions. For the space $\text{Stab}(D)$, we introduce two commuting group actions.

First we consider the action of $\text{Aut}(D)$ on $\text{Stab}(D)$. Let $\Phi \in \text{Aut}(D)$ be an autoequivalence of $D$ and $(Z, P) \in \text{Stab}(D)$ be a stability condition on $D$. Then the element $\Phi \cdot (Z, P) = (Z', P')$ is defined by

$$Z'(E) := Z(\Phi^{-1}(E)), \quad P'(\phi) := \Phi(P(\phi)),$$

where $E \in D$ and $\phi \in \mathbb{R}$.

Second we define the $\mathbb{C}$-action on $\text{Stab}(D)$. For $t \in \mathbb{C}$ and $(Z, P) \in \text{Stab}(D)$, the element $t \cdot (Z, P) = (Z', P')$ is defined by

$$Z'(E) := e^{-i\pi t} \cdot Z(E), \quad P'(\phi) := P(\phi + \text{Re}(t))$$

where $E \in D$ and $\phi \in \mathbb{R}$. Clearly, this action is free.

Consider the category $D = D_n^N$. Recall from Section 4.3 that there is a subgroup $\text{Br}(D_n^N) \subset \text{Aut}(D_n^N)$ consisting of spherical twists. By Proposition 4.11, the action of $\text{Br}(D_n^N)$ on $\text{EG}^o(D_n^N)$ is free. Together with Proposition 5.9 we have the following.

Lemma 5.10. The action of $\text{Br}(D_n^N)$ on $\text{Stab}^o(D_n^N)$ is free and properly discontinuous.

6. $N$-angulations and hearts

6.1. $N$-angulations of polygons. In [BM08], they give a geometric description of the $(N - 2)$-cluster category of type $A_n$. By using their model, we give a geometric realization of the cluster exchange graph of an $(N - 2)$-cluster category of type $A_n$.

Let $\Pi_{d_n^N}$ be a $d_n^N$-gon. An $(N - 2)$-diagonal of $\Pi_{d_n^N}$ is a diagonal which divides $\Pi_{d_n^N}$ into an $((N - 2)i + 2)$-gon and an $((N - 2)(n + 1 - i) + 2)$-gon ($i = 1, \ldots, n$). A collection of $(N - 2)$-diagonals is called non-crossing if they intersect only the boundary of $\Pi_{d_n^N}$. 

Definition 6.1. An $N$-angulation $\Delta$ of $\Pi_{d_N^n}$ is the maximal set of non-crossing $(N - 2)$-diagonals of $\Pi_{d_N^n}$.

Note that the number of $(N - 2)$-diagonals in the $N$-angulation $\Delta$ is just $n$, and they divide $\Pi_{d_N^n}$ into $n + 1$ pieces of $N$-gons.

For any $(N - 2)$-diagonal $\delta \in \Delta$, there are just two $N$-gons whose common edge is $\delta$. By removing $\delta$, we have a $(2N - 2)$-gon. We call $\delta$ a diameter of the $(2N - 2)$-gon.

For an $N$-angulation $\Delta$ and an $(N - 2)$-diagonal $\delta \in \Delta$, we define the operation to make new $N$-angulations $\mu^\sharp_\delta \Delta$ and $\mu^\flat_\delta \Delta$ by the following.

For an $(N - 2)$-diagonal $\delta$, there is a unique $(2N - 2)$-gon which has $\delta$ as a diameter. By rotating two boundary points of $\delta$ one step in clockwise (respectively, counterclockwise), we have a new $(N - 2)$-diagonal $\delta^\sharp$ ($\delta^\flat$). Thus, we obtain a new $N$-angulation $\mu^\sharp_\delta \Delta := (\Delta \setminus \{\delta\}) \cup \{\delta^\sharp\}$

Definition 6.2. Define the graph $A(\Pi_{d_N^n}, N)$ to be an oriented graph whose vertices are all $N$-angulations of $\Pi_{d_N^n}$, and for any two $N$-angulations $\Delta$ and $\Delta'$, the arrow $\Delta \rightarrow \Delta'$ is given if there is some diagonal $\delta \in \Delta$ such that $\Delta' = \mu^\sharp_\delta \Delta$.

6.2. Correspondence between hearts and $N$-angulations. Combining the construction in the previous section with the correspondence between cluster exchange graphs and heart exchange graphs in [KQ], we have the following result.

Theorem 6.3 ([BM08], Theorem 5.6 and [KQ], Theorem 8.5). There is an isomorphism of oriented graphs with labeled arrows

$$A(\Pi_{d_N^n}, N) \sim \rightarrow \text{EG}^\circ(\mathcal{D}_n^N)/\text{Br}(\mathcal{D}_n^N).$$

For $\Delta \in A(\Pi_{d_N^n}, N)$ and the corresponding heart $\mathcal{H}(\Delta) \in \text{EG}^\circ(\mathcal{D}_n^N)$ (which is determined up to modulo $\text{Br}(\mathcal{D}_n^N)$), there is a canonical bijection between $(N - 2)$-diagonals in $\Delta$ and simple objects in $\mathcal{H}(\Delta)$:

$$\Delta = \{\delta_1, \ldots, \delta_n\} \sim \rightarrow \{S_1, \ldots, S_n\} = \text{Sim} \mathcal{H}(\Delta).$$

In addition, we have

$$\mathcal{H}(\mu^\sharp_\delta \Delta) \equiv \mathcal{H}(\Delta^\sharp_{S_\delta}) \mod \text{Br}(\mathcal{D}_n^N)$$

Figure 5. Examples of $N$-angulations and the rotation operation in the case $N = 4$ and $n = 3$. 
where $\delta_v \in \Delta$ is an $(N - 2)$-diagonal and $S_v \in \text{Sim}\mathcal{H}(\Delta)$ is the corresponding simple object.

**Proof.** By Theorem 5.6 in [BM08], the graph $A(\Pi_{dN}, N)$ gives the cluster exchange graph of the $(N - 2)$-cluster category of type $A_n$. Each $N$-angulation corresponds to the cluster tilting object and diagonals correspond to indecomposable direct summands of the cluster tilting object. (For the precise definition of the cluster exchange graph, see Definition 4.4 in [KQ].)

On the other hand, by Theorem 8.5 in [KQ], there is an isomorphism of the cluster exchange graph of type $A_n$ and the heart exchange graph $\text{EG}^o(D_n)/\text{Br}(D_n)$. In addition, each indecomposable direct summand of the cluster tilting object corresponds to the simple object of the corresponding hearts. □

We fix the ambiguity of $\mathcal{H}(\Delta)$ as follows.

**Remark 6.4.** By Theorem 4.12, for an $N$-angulation $\Delta$ we can take the corresponding heart $\mathcal{H}(\Delta)$ uniquely in the fundamental domain $\text{EG}^o_N(D_n)$. In the following, we always assume that $\mathcal{H}(\Delta)$ is in $\text{EG}^o_N(D_n)$.

**Lemma 6.5.** Let $\Delta \in A(\Pi_{dN}, N)$ be an $N$-angulation and $\mathcal{H}(\Delta) \in \text{EG}^o_N(D_n)$ be the corresponding heart. Take a $(N - 2)$-diagonal $\delta_v \in \Delta$ and consider the new $N$-angulation $\mu^\sharp_{\delta_v} \Delta$. Then there is a unique spherical twist $\Phi \in \text{Br}(D_n)$ such that

$$\Phi \cdot \mathcal{H}(\mu^\sharp_{\delta_v} \Delta) = \mathcal{H}(\Delta)^{\sharp}_{S_v}.$$

**Proof.** By Theorem 6.3, such a spherical twist exists. In addition, since the action of $\text{Br}(D_n)$ on $\text{EG}^o(D_n)$ is free, the result follows. □

### 6.3 Colored quivers from $N$-angulations

Following [BT09, Section 11], we associate the colored quiver $Q(\Delta)$ to an $N$-angulation $\Delta$.

**Definition 6.6.** For an $N$-angulation $\Delta \in A(\Pi_{dN}, N)$, define the $(N - 2)$-colored quiver $Q(\Delta)$ whose vertices are $(N - 2)$-diagonals in $\Delta$ and the colored arrow $\delta_i \xrightarrow{(c)} \delta_j$ is given if both $\delta_i$ and $\delta_j$ lie on the same $N$-gon in the $N$-angulated $d_n$-gon and there are just $c$ edges forming the segment of the boundary of the $N$-gon from $\delta_i$ to $\delta_j$ in counterclockwise.

**Proposition 6.7.** Let $\Delta \in A(\Pi_{dN}, N)$ be an $N$-angulation and $\mathcal{H}(\Delta) \in \text{EG}^o(D_n)$ be the corresponding heart. Then there is a canonical isomorphism of colored quivers

$$Q(\Delta) = Q(\mathcal{H}(\Delta)).$$

In addition, the correspondence between vertices of two colored quivers is given by the correspondence between $(N - 2)$-diagonals and simple objects in Theorem 6.3.

**Proof.** By Proposition 11.1 in [BT09], the colored quiver $Q(\Delta)$ is equal to the colored quiver $Q_{T(\Delta)}$ which corresponds to the cluster tilting object $T(\Delta)$. On the other hand, Theorem 8.6 in [KQ] implies that the colored quiver $Q_{T(\Delta)}$ coincides with the colored quiver $Q(\mathcal{H}(\Delta))$. □

The definition of the colored quiver $Q(\Delta)$ is compatible to the mutation and the rotation of a diagonal.
Lemma 6.8. Let $\delta \in \Delta$ be an $N$-angulation. Then we have

$$\mu^\delta Q(\Delta) = Q(\mu^\delta \Delta).$$

Let $\Delta = \{\delta_1, \ldots, \delta_n\}$ be an $N$-angulation and $\mu^\delta \Delta$ be its rotation of $\delta_v$. By Lemma 6.7 and Lemma 6.8, we have canonical isomorphisms of colored quiver lattices

$$L_{Q(\Delta)} \cong L_{Q(H(\Delta))},$$

$$L_{\mu^\delta \Delta} \cong L_{Q(\mu^\delta \Delta)} \cong L_{Q(H(\mu^\delta \Delta))}.$$

Let $S_1, \ldots, S_n$ and $S_1', \ldots, S_n'$ be simple objects in $H(\Delta)$ and $H(\mu^\delta \Delta)$. By using Lemma 4.15, we have isomorphisms of lattices

$$\lambda: L_{Q(\Delta)} \cong K(D^N_n)$$

$$\lambda': L_{\mu^\delta \Delta} \cong K(D^N_n)$$

given by $\lambda(\alpha) := [S_1]$ and $\lambda'(\alpha') := [S_1']$.

Then we have the following diagram.

Lemma 6.9. Let $F_v: L_{\mu^\delta \Delta} \to L_{Q(\Delta)}$ be the linear map defined in Section 4.7 and $\Phi \in Br(D^N_n)$ be a unique spherical twist satisfying $\Phi \cdot H(\mu^\delta \Delta) = H(\Delta)^{S_1}_v$ (see Lemma 6.5). Then the diagram

$$\begin{align*}
L_{\mu^\delta \Delta} & \xrightarrow{F_v} L_{Q(\Delta)} \\
| & \downarrow \lambda \\
K(D^N_n) & \xrightarrow{[\Phi]} K(D^N_n)
\end{align*}$$

commutes.

Proof. As in Section 4.7, define the isomorphism $\lambda': L_{\mu^\delta \Delta} \to K(D^N_n)$ by $\alpha' \mapsto [S_1']$ where $S_1', \ldots, S_n'$ are simple objects in $H(\Delta)^{S_1}_v$. Then since $[\Phi] \cdot [S_1'] = [S_1']$ by Lemma 6.5, we have a commutative diagram

$$\begin{align*}
L_{\mu^\delta \Delta} & \xrightarrow{F_v} L_{Q(\Delta)} \\
| & \downarrow \lambda' \\
K(D^N_n) & \xrightarrow{[\Phi]} K(D^N_n)
\end{align*}$$

Combining the above diagram with the diagram in Lemma 4.16, we have the result.

6.4. $N$-angulations from saddle-free differentials. Let $\phi \in Q(N,n)$ be a quadratic differential. Recall from Section 3.2 that $\phi$ defines $d^N_n$ distinguished tangent directions at $\infty \in \mathbb{P}^1$ since $\phi$ has a pole of order $(d^N_n + 2)$ at $\infty$. Consider the real oriented blow-up at $\infty \in \mathbb{P}^1$ (we replace $\infty$ with $S^1$ which corresponds to tangent directions at $\infty$), then we have a disk $\Pi$. By adding $d^N_n$ points on the boundary of $\Pi$, which correspond to $d^N_n$ distinguished tangent directions, we have a $d^N_n$-gon $\Pi_{d^N_n}$. 

In [GMN, Section 6], they constructed ideal triangulations of marked bordered surfaces from saddle-free differentials with simple zeros (see also [BS, Section 10.1]). Motivated their works, we consider the following construction.

**Lemma 6.10.** Let $\phi \in Q(N,n)$ be a saddle-free differential. By taking one generic trajectory from each horizontal strip in the decomposition of $\mathbb{P}^1$ by $\phi$, we have an $N$-angulation $\Delta_{\phi}$ of $\Pi_{dN}$.

**Proof.** By Lemma 3.2, there are just $n$ horizontal strips in the decomposition, and generic trajectories taken from different horizontal strips are non-crossing. Hence, it is sufficient to prove that these generic trajectories are $(N-2)$-diagonals. Assume that some generic trajectory divides $\Pi_{dN}$ into two disks $\Pi_1$ and $\Pi_2$, and $\Pi_1$ contain $i$ zeros. Then the restriction of $\phi$ on the interior of $\Pi_1$ is also saddle-free with $i$ zeros. Therefore, the foliation in $\Pi_1$ has $(i-1)$ horizontal strips and $Ni$ separating trajectories. By the equation (3.2), the number of half planes in $\Pi_1$ is $((N-2)i + 2)$, and this implies that $\Pi_1$ is $((N-2)i + 2)$-gon. □

![Figure 6. An example of an $N$-angulation from a saddle-free differential in the case $N = 4$ and $n = 3.](image)

We write by $\Delta_{\phi}$ the $N$-angulation determined by a saddle-free differential $\phi \in Q(N,n)$. We also write by $h(\delta)$ the horizontal strip which contains a generic trajectory $\delta$.

Set $\Delta_{\phi} = \{\delta_1, \ldots, \delta_n\}$ and consider the colored quiver lattice

$$L_{Q(\Delta_{\phi})} = \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$$

where $\alpha_1, \ldots, \alpha_n$ are the basis corresponding to $\delta_1, \ldots, \delta_n$. Recall from Section 3.4 that we can associate standard saddle classes $\gamma_1, \ldots, \gamma_n \in H_{\pm}(\phi)$ to horizontal strips $h(\delta_1), \ldots, h(\delta_n)$.

**Lemma 6.11.** There is an isomorphism of abelian groups

$$\mu: L_{Q(\Delta_{\phi})} \sim \rightarrow H_{\pm}(\phi)$$

defined by $\alpha_i \rightarrow \gamma_i$. The map $\mu$ takes the bilinear form $\langle , \rangle$ on $L_{Q(\Delta_{\phi})}$ to the intersection form $I_{\pm}$ on $H_{\pm}(\phi)$.

**Proof.** Since standard saddle classes $\gamma_1, \ldots, \gamma_n$ form the basis of $H_{\pm}(\phi)$ by Lemma 3.4, the map $\mu$ is an isomorphism.
The remaining part is to prove that \( \langle \alpha_i, \alpha_j \rangle = I_\pm(\gamma_i, \gamma_j) \). First note that
\[
\langle \alpha_i, \alpha_i \rangle = I_\pm(\gamma_i, \gamma_i) = 1 + (-1)^N \text{ is clear by the definitions of } \langle \cdot, \cdot \rangle \text{ and } I_\pm.
\]
Next consider the case \( q^{(c)}_{ij} = 1 \) for some color \( c \) and \( i \neq j \). If \( c = 2m \), there are \( 2m + 1 \) separating trajectories from \( \gamma_i \) to \( \gamma_j \) in counterclockwise as in the left of Figure 7. Hence the oriented foliation of the 1-form \( \sqrt{\phi} \) is locally given as in the left of Figure 7. Therefore, we have
\[
\langle \alpha_i, \alpha_j \rangle = (-1)^{2m+1} q_{ij}^{(2m)} = -1 = I_\pm(\gamma_i, \gamma_j).
\]
Similarly, if \( c = 2m - 1 \), the oriented foliation is given as in the right of Figure 7, and we have
\[
\langle \alpha_i, \alpha_j \rangle = (-1)^{2m} q_{ij}^{(2m-1)} = 1 = I_\pm(\gamma_i, \gamma_j).
\]

**Figure 7.** Oriented foliations near a zero.

Let \( \phi \in Q(N, n) \) be a saddle-free differential and \( \Delta_\phi = \{ \delta_1, \ldots, \delta_n \} \) be the \( N \)-angulation associated to \( \phi \). By Theorem 6.3, we have simple objects \( S_1, \ldots, S_n \in \mathcal{H}(\Delta_\phi) \) which correspond to \( \delta_1, \ldots, \delta_n \).

The next result is the analogue of Lemma 10.5 in [BS].

**Corollary 6.12.** There is an isomorphism of abelian groups
\[
\nu: K(D_N^\phi) \xrightarrow{\sim} H_\pm(\phi)
\]
defined by \( [S_i] \mapsto \gamma_i \). The map \( \nu \) takes the Euler form \( \chi \) on \( K(D_N^\phi) \) to the intersection form \( I_\pm \) on \( H_\pm(\phi) \).

**Proof.** First note that by Proposition 6.7, we have a canonical isomorphism \( L_{Q(\Delta_\phi)} \cong L_{Q(\mathcal{H}(\Delta_\phi))} \). Then combining Lemma 4.15 and Lemma 6.11, the result immediately follows. \( \square \)

7. **Proof of main theorem**

7.1. **Stratifications.** Following [BS, Section 5.2], we introduce the stratification on \( Q(N, n) \). Many results of [BS] for the stratification also work in our cases. For \( \phi \in Q(N, n) \), denote by \( s_\phi \) the number of saddle trajectories. Define the subsets of \( Q(N, n) \) by
\[
D_p := \{ \phi \in Q(N, n) \mid s_\phi \leq p \}.
\]
(Note that our notation differs from that of [BS]. Our \( D_p \) corresponds to \( B_{2p} \) in [BS].) Since Lemma 3.2 implies that \( s_\phi \leq n \), we have \( D_n = Q(N, n) \). Note that the stratum \( B_0 \) consists of all saddle-free differentials. These subsets have the following property.
Lemma 7.1 ([BS], Lemma 5.2). The subsets \( D_p \subset Q(N, n) \) form an increasing chain of dense open subsets

\[
D_0 \subset D_1 \subset \cdots \subset D_n = Q(N, n).
\]

Let \( F_0 := D_0 \) and \( F_p := D_p \setminus D_{p-1} \) for \( p \geq 1 \). Then we have a finite stratification

\[
Q(N, n) = \bigcup_{p=0}^{n} F_p
\]

by locally closed subset \( F_p \).

Proposition 7.2 ([BS], Proposition 5.5 and Proposition 5.7). Let \( p \geq 1 \) and assume that \( \phi \in F_p \). Then there is a neighborhood \( \phi \in U \subset D_p \) and a class \( 0 \neq \alpha \in H_\pm(\phi) \) such that

\[
\phi \in U \cap F_p \implies Z_\phi(\alpha) \in \mathbb{R}.
\]

Further, if \( p \geq 2 \) we can take a neighborhood \( \phi \in U \subset D_p \) which satisfies that \( U \cap D_{p-1} \) is connected.

Following in [BS, Section 5.2], we define generic differentials. \( \phi \in Q(N, n) \) is called generic if for any \( \gamma_1, \gamma_2 \in H_\pm(\phi) \),

\[
\mathbb{R}Z_\phi(\gamma_1) = \mathbb{R}Z_\phi(\gamma_2) \implies Z\gamma_1 = Z\gamma_2.
\]

Lemma 7.3. If \( \phi \in Q(N, n) \) is generic, then \( \phi \) has at most one saddle trajectory, i.e. \( \phi \in D_1 \).

Proof. Assume that \( \phi \in Q(N, n) \) is generic and \( \phi \in Q(N, n) \setminus D_1 \). Then \( \phi \) has at least two different saddle trajectories. We write them by \( \gamma_1, \gamma_2 \), and let \( \gamma_1, \gamma_2 \in H_\pm(\phi) \) be the corresponding saddle homology classes. Since different saddle trajectories define linearly independent saddle classes, it follows that \( Z\gamma_1 \neq Z\gamma_2 \). On the other hand, by the definition of the saddle trajectory, we have \( Z_\phi(\gamma_1), Z_\phi(\gamma_2) \in \mathbb{R} \). This contradicts to the definition of a generic differential. \( \square \)

A differential \( \phi \in Q(N, n) \) defines the length \( l_\phi(\gamma) \) of a smooth path \( \gamma : [0, 1] \to P^1 \setminus \{\infty\} \) by

\[
l_\phi(\gamma) := \int_0^1 |\varphi(\gamma(t))|^{1/2} \left| \frac{d\gamma(t)}{dt} \right| dt
\]

where \( \varphi(z) \) is given by \( \phi(z) = \varphi(z)dz^{\otimes 2} \).

Proposition 6.8 in [BS] is easily extended to our cases.

Proposition 7.4 ([BS], Proposition 6.8). Assume that for a sequence of framed differentials

\[
(\phi_k, \theta_k) \in Q(N, n)^*_k, \quad k \geq 1,
\]

the periods \( Z_{\phi_k} \circ \theta_k : \Gamma \to \mathbb{C} \) converge as \( k \to \infty \). Further, there is some constant \( L > 0 \), which is independent of \( k \), such that any saddle connection \( \gamma \) of \( \phi_k \) satisfies \( l_{\phi_k}(\gamma) \geq L \). Then there is some subsequence of \( \{(\phi_k, \theta_k)\}_{k \geq 1} \) which converges in \( Q(N, n)^*_\)
7.2. Stability conditions from saddle-free differentials. Following [BS, Section 10.4], we construct a stability condition on $D_n^N$ from a saddle-free differential $\phi \in Q(N, n)$.

**Lemma 7.5.** \( \phi \in B_0 \subset Q(N, n) \) be a saddle-free differential. Consider the corresponding heart \( H(\Delta_\phi) \in \text{EG}_N(D_n^N) \) and define a linear map
\[
Z : K(D_n^N) \rightarrow \mathbb{C}
\]
by \( Z := Z_\phi \circ \nu \) where \( Z_\phi : \mathbb{H}(\phi) \rightarrow \mathbb{C} \) is a period of \( \phi \) and \( \nu : K(D_n^N) \rightarrow H(\Delta_\phi) \) is an isomorphism given by Corollary 6.12. Then the pair \( (Z, H(\Delta_\phi)) \) determines a stability condition \( \sigma(\phi) = (Z, H(\Delta_\phi)) \in \text{Stab}(D_n^N) \).

**Proof.** By Proposition 5.5, we only need to check that \( Z(E) \in H \) for any nonzero object \( E \in H(\Delta_\phi) \). Let \( h_1, \ldots, h_n \) be horizontal strips determined by \( \phi \) and \( S_1, \ldots, S_n \) be corresponding simple objects in \( H(\Delta_\phi) \). By the definition of \( \nu \) in Corollary 6.12, we have \( \nu(S_i) = \gamma_i \) where \( \gamma_i \) is the standard saddle connection associated to the horizontal strip \( h_i \). As explained in Section 3.4, the period of \( \gamma_i \) satisfies that \( Z_\phi(\gamma_i) \in \mathbb{H} \subset H \). Hence \( Z(S_i) \in H \).

7.3. Wall crossing. Let \( \phi_1 \in F_1 \) be a differential which has just one saddle trajectory. Write by \( \Delta_{\phi_1} \) the set of \((N - 2)\)-diagonals obtained by taking one generic trajectory from each horizontal strip of \( \phi_1 \). By Lemma 3.2, \( n - 1 \) horizontal strips appear in the decomposition of \( \mathbb{P}^1 \) by \( \phi_1 \), so the number of \((N - 2)\)-diagonals in \( \Delta_{\phi_1} \) is \( n - 1 \).

By Proposition 7.2, if we take \( r > 0 \) sufficiently small, then for any \( 0 < t \leq r \), the differentials
\[
\phi(t) := e^{i\pi t} \cdot \phi_1, \quad \phi^\star(t) := e^{-i\pi t} \cdot \phi_1
\]
are saddle-free. Write \( \phi := \phi(r) \) and \( \phi^\star := \phi^\star(r) \).

Let \( \Delta_{\phi} \) and \( \Delta_{\phi^\star} \) be \( N \)-angulations determined by these saddle-free differentials. Write by \( \delta \) a unique \((N - 2)\)-diagonal of \( \Delta_{\phi} \) which is not contained in \( \Delta_{\phi_1} \):
\[
\delta := \Delta_{\phi} \setminus \Delta_{\phi_1}.
\]

We recall from Section 6.1 that a new \( N \)-angulation \( \mu_{\delta}^\star(\Delta_{\phi}) \) is defined by rotating \( \delta \) one step in clockwise.

**Lemma 7.6.** Two \( N \)-angulations \( \Delta_{\phi} \) and \( \Delta_{\phi^\star} \) are related by
\[
\mu_{\delta}^\star(\Delta_{\phi}) = \Delta_{\phi^\star}.
\]

**Proof.** The center of Figure 8 illustrates a unique saddle trajectory of \( \phi_1 \) and two boundaries of half planes or horizontal strips which contain the saddle trajectory.

By Lemma 3.1, the saddle trajectory maps to the saddle connection of phase \(-\theta \) or \(+\theta \) \( (\theta := d_n^N r/2) \) for \( \phi = e^{-i\pi r} \cdot \phi_1 \) or \( \phi^\star = e^{i\pi r} \cdot \phi_1 \) as the left or right of Figure 8. Comparing the left and right of Figure 8, we have the result.

Set \( \Delta_\phi = \{\delta_1, \ldots, \delta_n\} \) and \( \delta^\star = \Delta_{\phi^\star} \setminus \Delta_{\phi_1} \). By Lemma 6.8 and Lemma 7.6, we have
\[
\mu_{\delta}^\star Q(\Delta_{\phi}) = Q(\Delta_{\phi^\star}).
\]

As a result, we have a natural isomorphism of colored quiver lattices \( L_{\mu_{\delta}^\star} Q(\Delta_{\phi}) \cong L_{Q(\Delta_{\phi^\star})} \).
For $\phi$ and $\phi^\sharp$, we fix the natural basis as follows:

$$
L_{Q(\Delta_o)} \cong \bigoplus_{i=1}^n \mathbb{Z}\alpha_i, \quad L_{Q(\Delta_{o1})} \cong \bigoplus_{i=1}^n \mathbb{Z}\alpha'_i, \\
H_\pm(\phi) \cong \bigoplus_{i=1}^n \mathbb{Z}\gamma_i, \quad H_\pm(\phi^\sharp) \cong \bigoplus_{i=1}^n \mathbb{Z}\gamma'_i
$$

where $\gamma_1, \ldots, \gamma_n$ and $\gamma'_1, \ldots, \gamma'_n$ are standard saddle classes.

Let

$$
F_v: L_{Q(\Delta_{o1})} \rightarrow L_{Q(\Delta)}
$$

be the linear isomorphism defined in Section 4.7.

The relationship between $F_v$ and the Gauss-Manin connection

$$
\text{GM}_c: H_\pm(\phi^\sharp) \rightarrow H_\pm(\phi)
$$

along the path $c(t) = e^{i\pi t} \cdot \phi_1 (t \in [-r, r])$ is given by the following diagram.

**Lemma 7.7.** Let $\mu: L_{Q(\Delta_o)} \rightarrow H_\pm(\phi)$ and $\mu^\sharp: L_{Q(\Delta_{o1})} \rightarrow H_\pm(\phi^\sharp)$ be the isomorphisms given by Lemma 6.11. Then the diagram

$$
\begin{array}{ccc}
L_{Q(\Delta_{o1})} & \xrightarrow{F_v} & L_{Q(\Delta)} \\
\mu^\sharp & \downarrow & \mu \\
H_\pm(\phi^\sharp) & \xrightarrow{\text{GM}_c} & H_\pm(\phi)
\end{array}
$$

commutes.

**Proof.** As in Figure 9 if $q_{ij}^{(0)} = 1$, then we have

$$
\text{GM}_c(\gamma'_j) = \gamma_j + \gamma_v, \quad \text{GM}_c(\gamma'_v) = -\gamma_v.
$$

On the other hand if $q_{ij}^{(0)} = 0$, then we can check $\text{GM}_c(\gamma'_j) = \gamma_j$. Since $\mu(\alpha_j) = \gamma_j$ and $\mu^\sharp(\alpha'_j) = \gamma'_j$, the result follows.

For $0 < t \leq r$, we denote by

$$
\sigma(t) := \sigma(\phi(t)), \quad \sigma^\sharp(t) := \sigma(\phi^\sharp(t))
$$
stability conditions in $\text{Stab}^\circ(D^N_n)$ constructed by Lemma 7.5 for saddle-free differentials $\phi(t)$ and $\phi^\sharp(t)$.

The following result is the analogue of Proposition 10.7 in [BS].

**Proposition 7.8.** Let $\nu: K(D^N_n) \to H_\pm(\phi)$ and $\nu^\sharp: K(D^N_n) \to H_\pm(\phi^\sharp)$ be the isomorphisms given by Corollary 6.12. Then there is a unique spherical twist $\Phi \in \text{Br}(D^N_n)$ such that the following (1) and (2) hold.

1. The diagram

$$
\begin{array}{ccc}
K(D^N_n) & \xrightarrow{[\Phi]} & K(D^N_n) \\
\downarrow{\nu^\sharp} & & \downarrow{\nu} \\
H_\pm(\phi^\sharp) & \xrightarrow{\text{GM}_c} & H_\pm(\phi)
\end{array}
$$

commutes.

2. \[
\lim_{t \to +0} \sigma(t) = \lim_{t \to +0} \Phi \cdot \sigma^\sharp(t).
\]

**Proof.** Part (1) immediately follows from Lemma 6.9 and Lemma 7.7. For the proof of part (2), we first note that by Lemma 6.5 and Lemma 7.6, there is a unique spherical twist $\Phi \in \text{Br}(D^N_n)$ such that

$$
\Phi \cdot \mathcal{H}(\Delta_{\phi^\sharp}) = \mathcal{H}(\Delta_{\phi})^\sharp_{S_v}.
$$

By the definition of $\sigma(t)$ and $\sigma^\sharp(t)$ in Lemma 7.5, $\sigma(t)$ is contained in $\text{Stab}(\mathcal{H}(\Delta_{\phi}))$ and $\Phi \cdot \sigma^\sharp(t)$ is contained in $\text{Stab}(\mathcal{H}(\Delta_{\phi})^\sharp_{S_v})$. Further by Part (1), central charges of $\sigma(t)$ and $\Phi \cdot \sigma^\sharp(t)$ approach the same point in $\text{Hom}(K(D^N_n), \mathbb{C})$ as $t \to +0$. Hence by Lemma 5.7, we can take some open subset

$$
U \subset \text{Stab}(\mathcal{H}(\Delta_{\phi})) \cup \text{Stab}(\mathcal{H}(\Delta_{\phi})^\sharp_{S_v})
$$

such that $U$ is mapped isomorphically on $\text{Hom}(K(D^N_n), \mathbb{C})$, and $U$ contains both $\sigma(t)$ and $\Phi \cdot \sigma^\sharp(t)$. This implies the result. \qed
7.4. Construction of isomorphisms.

**Definition 7.9.** Let \((\phi, \theta) \in Q(N, n)^\Gamma\) be a framed saddle-free differential and \(\nu: K(D_n^N) \to H_\pm(\phi)\) be an isomorphism given by Corollary 6.12. We define the isomorphism \(\kappa\) by the composition of \(\theta\) and \(\nu^{-1}\):

\[
\kappa := \nu^{-1} \circ \theta: \Gamma \sim \to K(D_n^N).
\]

Fix a framed saddle-free differential \((\phi_0, \theta_0) \in Q(N, n)^\Gamma\) and let \(Q(N, n)^\Gamma\) be the connected component which contains \((\phi_0, \theta_0)\).

In the following, we identify \(K(D_n^N)\) with \(\Gamma\) by using the isomorphism \(\kappa_0\) determined from \((\phi_0, \theta_0)\) by Definition 7.9.

**Lemma 7.10.** Let \((\phi, \theta) \in Q(N, n)^\Gamma\) be a framed saddle-free differential and \(\kappa: \Gamma \to K(D_n^N)\) be an isomorphism given by Definition 7.9. Then there is a spherical twist \(\Phi \in Br(D_n^N)\) such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\kappa} & K(D_n^N) \\
& | & | \\
& \downarrow{\kappa_0} & \downarrow{[\Phi]} \\
K(D_n^N) & \xrightarrow{[\Phi]} & K(D_n^N) \\
\end{array}
\]

commutes. In addition, if there are two such spherical twists \(\Phi_1\) and \(\Phi_2\), then the induced linear isomorphisms on \(K(D_n^N)\) are the same:

\[
[\Phi_1] = [\Phi_2].
\]

**Proof.** \([\Phi_1] = [\Phi_2]\) immediately follows from

\[
[\Phi_1] \circ \kappa = \kappa_0 = [\Phi_2] \circ \kappa.
\]

Therefore, we show that such a spherical twist exists. Take a path \(c: [0, 1] \to Q(N, n)^\Gamma\) from \((\phi, \theta)\) to \((\phi_0, \theta_0)\). By Proposition 7.2, we can deform the path \(c\) to lie in \(D_1\) and to intersect only finitely many points of \(F_1\), which consists of framed differentials with just one saddle trajectory.

By applying Proposition 7.8 to each of these points, we have a spherical twist \(\Phi \in Br(D_n^N)\) fitting into a diagram

\[
\begin{array}{ccc}
K(D_n^N) & \xrightarrow{[\Phi]} & K(D_n^N) \\
| & \downarrow{\nu} & | \\
H_\pm(\phi) & \xrightarrow{GM_c} & H_\pm(\phi_0) \\
& \downarrow{\nu_0} & \downarrow{[\Phi_0]} \\
\end{array}
\]

where \(\nu\) and \(\nu_0\) are isomorphisms given by Corollary 6.12. Further, recall the diagram

\[
\begin{array}{ccc}
H_\pm(\phi) & \xrightarrow{GM_c} & H_\pm(\phi_0) \\
\downarrow{\theta} & \downarrow{\theta_0} & \downarrow{} \\
\Gamma & \xrightarrow{} & \Gamma \\
\end{array}
\]

in Section 2.7. Combining the above two diagrams, we have the result. □

Let \(Br^0(D_n^N) \subset Br(D_n^N)\) be the subgroup of spherical twists which act as the identity on \(K(D_n^N)\). We recall that \(Br(D_n^N) \cong B_{n+1}\) by Theorem 4.6, and \(Br(D_n^N)\)
acts as the group $W_\pm$ on $K(D_n^N)$ (see Section 4.3). Hence the group $\text{Br}^0(D_n^N)$ is isomorphic to the group $P_\pm$ in Section 2.1.

By completely the same argument of Proposition 11.3 in [BS], we can construct the following equivariant holomorphic map.

**Proposition 7.11.** There is a $W_\pm$-equivariant and $\mathbb{C}$-equivariant holomorphic map of complex manifolds $K$ such that the diagram

$$
\begin{array}{ccc}
Q(N, n)^\Gamma_z & \xrightarrow{K} & \text{Stab}^\sigma(D_n^N)/\text{Br}^0(D_n^N) \\
\downarrow_{\mathcal{W}_N} & & \downarrow_{\mathcal{Z}} \\
\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) & & \end{array}
$$

commutes.

**Proof.** First we see the $\mathbb{C}$-equivariant property of $K$. Define the action of $t \in \mathbb{C}$ on $\text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$ by $Z \mapsto e^{-it^1}Z$ for $Z \in \text{Hom}_\mathbb{Z}(\Gamma, \mathbb{C})$. Then by Definition 2.7 and Lemma 3.1, the map $\mathcal{W}_N$ is $\mathbb{C}$-equivariant. On the other hand, by the definition of $\mathbb{C}$-action on $\text{Stab}^\sigma(D_n^N)$ given in Section 5.3, the map $\mathcal{Z}$ is also $\mathbb{C}$-equivariant. Since both $\mathcal{W}_N$ and $\mathcal{Z}$ are local isomorphisms by Proposition 2.5 and Theorem 2.5, if $K$ exists then $K$ is $\mathbb{C}$-equivariant.

Next we construct the map $K$. Recall from Section 7.1 that there is a stratification

$$D_0 \subset D_1 \subset \cdots \subset D_n = Q(N, n).$$

We extend the stratification of $Q(N, n)$ to $Q(N, n)^\Gamma_z$ by the obvious way (and use the same notation).

Here we give the construction of $K$ only on the stratum $D_0$ which consists of framed saddle-free differentials. By using Proposition 7.2 and Proposition 7.8, we can extend $K$ on the stratum $D_p (p \geq 1)$ by doing the same argument of Proposition 11.3 in [BS].

Let $(\phi, \theta) \in D_0$ be a framed saddle-free differential. By Lemma 7.5, we can construct a stability condition $\sigma(\phi) \in \text{Stab}^\sigma(D_n^N)$. Let $\Phi \in \text{Br}(D_n^N)$ be a spherical twist for $(\phi, \theta)$ determined by Lemma 7.10. We note that $\Phi$ is determined up to the action of $\text{Br}^0(D_n^N) \subset \text{Br}(D_n^N)$. Therefore, if we define the map $K: D_0 \to \text{Stab}^\sigma(D_n^N)/\text{Br}^0(D_n^N)$ by

$$K((\phi, \theta)) := \Phi(\sigma(\phi)) \in \text{Stab}^\sigma(D_n^N)/\text{Br}^0(D_n^N),$$

then, $K$ is well-defined.

Finally we see that the map $K$ is $W_\pm$-equivariant. Note that the action of $W_\pm$ preserves the stratification of $Q(N, n)^\Gamma_z$ and commutes with the action of $\mathbb{C}$. Hence, it is sufficient to prove that $K$ is $W_\pm$-equivariant only on $D_0$. But, it is clear by the construction of $K$ on $D_0$. \qed

Let $(\phi, \theta) \in Q(N, n)^\Gamma_z$ and we set $\sigma = K((\phi, \theta))$. The stability condition $\sigma$ is determined up to the action of $\text{Br}^0(D_n^N) \subset \text{Aut}(D_n^N)$. For a class $0 \neq \alpha \in \Gamma$, let $\mathcal{M}_\alpha(\alpha)$ be the moduli space of $\sigma$-stable objects which have the class $\alpha$ and the phase in $(0, 1]$ (here, we treat $\mathcal{M}_\alpha(\alpha)$ as a set).

**Proposition 7.12** ([BS], Proposition 11.7). Let $\phi$ be a generic differential. Then the moduli space $\mathcal{M}_\alpha(\alpha)$ is empty or one point. In particular, $\mathcal{M}_\alpha(\alpha)$ is one point
if and only if the class $\alpha$ corresponds to the class of a saddle connection of $\phi$ (up to sign).

**Proof.** By Lemma 7.3, $\phi$ has at most one saddle trajectory. Hence the result follows from Proposition 11.7 in [BS].

The same argument of Proposition 11.11 in [BS] gives the following.

**Proposition 7.13.** The map $K$ in Proposition 7.11 is an isomorphism.

**Proof.** Here we recall the argument of Proposition 11.11 in [BS].

We first show that $K$ is injective. Let $(\phi_1, \theta_1), (\phi_2, \theta_2) \in Q(N,n)_r^\ddag$ such that $K((\phi_1, \theta_1)) = K((\phi_2, \theta_2))$. Since the maps $W_N$ and $Z_1$ are local isomorphisms by Proposition 2.5 and Theorem 5.3 and $K$ commutes with these two maps, we can deform $\phi_1$ and $\phi_2$ to be saddle-free differentials.

By the construction of $K$ in Proposition 7.11, we can write $K((\phi_1, \theta_1)) = \Phi_1(\sigma(\phi_1))$ for some $\Phi_1 \in \text{Br}(D^N_n)$ determined by Lemma 7.10. Since $\Phi_1(\sigma(\phi_1))$ and $\Phi_2(\sigma(\phi_2))$ have the same heart (up to the action of $\text{Br}^0(D^N_n)$), we have

$$\Phi_1 \cdot H(\Delta_{\phi_1}) \equiv \Phi_2 \cdot H(\Delta_{\phi_2}) \mod \text{Br}^0(D^N_n).$$

Recall Remark 6.4 that $H(\Delta_{\phi_1})$ and $H(\Delta_{\phi_2})$ are in the fundamental domain $\text{EG}^\ddag_N(D^N_n)$. Hence Theorem 4.12 implies that $H(\Delta_{\phi_1}) = H(\Delta_{\phi_2})$, and $[\Phi_1] = [\Phi_2]$ as linear maps on $K(D^N_n)$. As a result, we have $\Delta_{\phi_1} = \Delta_{\phi_2}$ by Theorem 6.3. Thus $\phi_1$ and $\phi_2$ have the same $N$-angulations and the same periods. Then Proposition 4.9 in [BS] implies that $\phi_1 = \phi_2$. On the other hand, $[\Phi_1] = [\Phi_2]$ together with the diagram of Lemma 7.10 implies that $\theta_1 = \theta_2$.

Next to prove the surjectivity of $K$, we show that the image of $K$ is open and closed. The injectivity of $K$ together with the local isomorphism property of $W_N$ and $Z_1$ implies that the image of $K$ is open.

Set $\sigma_k := K((\phi_k, \theta_k)) \in \text{Stab}^\ddag(D^N_n)/\text{Br}^0(D^N_n)$ for the sequence of framed differentials $\{(\phi_k, \theta_k)\}_{k \geq 1}$ in $Q(N,n)_r^\ddag$, and assume that $\sigma_k$ converge to the point $\sigma_\infty$ as $k \to \infty$. We can assume that $(\phi_k, \theta_k)$ is generic saddle-free since generic saddle-free differentials are dense in $Q(N,n)_r^\ddag$. In this setting, we show that there is a subsequence of $\{(\phi_k, \theta_k)\}_{k \geq 1}$ which converges some point in $Q(N,n)_r^\ddag$. First note that the periods $Z_{\phi_k} \circ \theta_k$ converge as $k \to \infty$ since the central charge $Z_k$ of $\sigma_k$ also converge as $k \to \infty$ by the assumption. Set

$$M_k := \inf\{ |Z_k(E)| \in \mathbb{R}_{\geq 0} \mid [E] \in C^\ddag(\sigma_k) \} \quad \text{for } k = 1, \ldots, \infty$$

where $Z_k$ is the central charge of $\sigma_k$. The support property of $\sigma_k$ implies that $M_k > 0$. Since $M_k \to M_\infty$ as $k \to \infty$, there is some constant $L > 0$ such that $M_k \geq L$ for all $k$. By Proposition 7.12, the length $l_{\phi_k}(\gamma)$ of the saddle connection $\gamma$ for $\phi_k$ coincides with the mass $|Z_k(S_\gamma)|$ of the corresponding stable object $S_\gamma$. Thus we can apply Proposition 7.4 for $\{(\phi_k, \theta_k)\}_{k \geq 1}$, and obtain the convergent subsequence. $\square$
Theorem 7.14. There is a $B_{n+1}$-equivariant and $C$-equivariant isomorphism of complex manifolds $\widetilde{K}$ such that the diagram

$$
\begin{array}{ccc}
\widetilde{M}_n & \xrightarrow{\gamma} & \text{Stab}^\circ(D^N_n) \\
\downarrow W_N & & \downarrow Z \\
\text{Hom}_Z(\Gamma, C) & & \\
\end{array}
$$

commutes.

Proof. First recall from Corollary 2.9 that $\widetilde{M}_n$ is a universal cover of the space $Q(N, n)\Gamma^\circ$ and $\widetilde{M}_n/P\xrightarrow{\sim} Q(N, n)\Gamma^\circ$. Further, by Proposition 7.11 and Proposition 7.13, we have the isomorphism $\widetilde{M}_n/P\xrightarrow{\sim} \text{Stab}^\circ(D^N_n)/\text{Br}^0(D^N_n)$.

Since $\text{Br}^0(D^N_n) \cong P_\pm$ and the quotient map $\text{Stab}^\circ(D^N_n) \rightarrow \text{Stab}^\circ(D^N_n)/\text{Br}^0(D^N_n)$ is a covering map by Lemma 5.10, we can lift the map $K$ to the $B_{n+1}$-equivariant isomorphism $\widetilde{M}_n \xrightarrow{\sim} \text{Stab}^\circ(D^N_n)$.

□

Corollary 7.15. The distinguished connected component $\text{Stab}^\circ(D^N_n)$ is contractible.

Proof. Since $\widetilde{M}_n$ is contractible (for example, see Lemma 1.28 in [KT08]), the result follows.

□

7.5. Autoequivalence groups. In this section, we determine the group of autoequivalences $\text{Aut}^\circ(D^N_n)$ which preserves the distinguished connected component of the exchange graph $\text{EG}^\circ(D^N_n)$.

Lemma 7.16. Let $\Phi \in \text{Aut}^\circ(D^N_n)$ act trivially on $\text{EG}^\circ(D^N_n)$. Then $\Phi = \text{id}$.

Proof. Since any object in $D^N_n$ is generated by finite extensions of elements in some finite heart $H \in \text{EG}^\circ(D^N_n)$ and their shift, it is sufficient to prove that $\Phi \in \text{Aut}^\circ(D^N_n)$ acts trivially on $H$.

Note that any autoequivalence $\Phi \in \text{Aut}(D^N_n)$ induces a canonical isomorphism of colored quivers $Q(H) \xrightarrow{\sim} Q(\Phi(H))$ for any heart $H \in \text{EG}^\circ(D^N_n)$. Hence if $\Phi$ acts trivially on $\text{EG}^\circ(D^N_n)$, then $\Phi$ induces an automorphism of the colored quiver $Q(H)$ for any heart $H \in \text{EG}^\circ(D^N_n)$. However, we can easily see that there is some heart $H$ whose colored quiver $Q(H)$ has no automorphisms. This implies that $\Phi$ acts trivially on $H$ since $\Phi$ fixes simple objects in $H$.

□

Let $\Pi_d^N_n$ be a $d^N_n$-gon as in Section 6.1. The mapping class group of $\Pi_d^N_n$ is given by a cyclic group $\mathbb{Z}/d^N_n\mathbb{Z}$. We assume that the generator $1 \in \mathbb{Z}/d^N_n\mathbb{Z}$ corresponds to one step clockwise rotation.

Theorem 7.17 ([BS], Theorem 9.10). Assume that $n \geq 2$. Then there is a short exact sequence of groups

$$1 \rightarrow \text{Br}(D^N_n) \rightarrow \text{Aut}^\circ(D^N_n) \rightarrow \mathbb{Z}/d^N_n\mathbb{Z} \rightarrow 1.$$
Proof. The same argument of Theorem 9.10 in [BS] gives the following short exact sequence:

\[ 1 \rightarrow \text{Br}(D^N_n) \rightarrow \text{Aut}^\circ(D^N_n)/\text{Nil}^\circ(D^N_n) \rightarrow \mathbb{Z}/d_n^N \mathbb{Z} \rightarrow 1 \]

where Nil\(^\circ\)(D\(^N\)\(_n\)) is the subgroup of Aut\(^\circ\)(D\(^N\)\(_n\)) consisting of autoequivalences which act trivially on EG\(^\circ\)(D\(^N\)\(_n\)). Lemma 7.16 implies that Nil\(^\circ\)(D\(^N\)\(_n\)) = 1.

For an integer \( m \neq 0 \), let \( \mathbb{Z}[m] \subset \text{Aut}^\circ(D^N_n) \) be the infinite cyclic group generated by the degree \( m \) shift functor \( [m] \in \text{Aut}^\circ(D^N_n) \).

Recall from Theorem 4.6 that there is an isomorphism

\[ B_{n+1} \sim \text{Br}(D^N_n), \quad \sigma_i \mapsto \Phi_{S_i}. \]

By Lemma 4.14 in [ST01], the generator of center \( C \in B_{n+1} \) (see Section 2.1) is mapped to the shift functor \([-d_n^N]\) \( \in \text{Br}(D^N_n) \) under the above isomorphism.

Define the diagonal action of the element \( d_n^N \in \mathbb{Z}^+ \) on \( B_{n+1} \times \mathbb{Z} \) by

\[ (\sigma, l) \mapsto (\sigma \cdot C, l + d_n^N) \text{ for } (\sigma, l) \in B_{n+1} \times \mathbb{Z}. \]

Theorem 7.18. Assume that \( n \geq 2 \). Then the group Aut\(^\circ\)(D\(^N\)\(_n\)) is given by

\[ \text{Aut}^\circ(D^N_n) \cong (B_{n+1} \times \mathbb{Z})/d_n^N \mathbb{Z} \]

where 1 \( \in \mathbb{Z} \) acts as the shift functor \( [1] \in \text{Aut}^\circ(D^N_n) \).

Proof. Consider the group homomorphism

\[ \rho: B_{n+1} \times \mathbb{Z} \rightarrow \text{Aut}^\circ(D^N_n) \]

defined by \( (\sigma_i, l) \mapsto \Phi_{S_i, [l]} \).

By Definition 2.7, 1 \( \in \mathbb{C} \) acts as a rotation of the phase \( 2\pi/d_n^N \) in clockwise on \( Q(N, n) \). This implies that single shift functor \( [1] \in \text{Aut}^\circ(D^N_n) \) is mapped to the generator \( 1 \in \mathbb{Z}/d_n^N \mathbb{Z} \) under the map of the short exact sequence in Theorem 7.17.

Hence \( \rho \) is surjective. On the other hand, the kernel of \( \rho \) is given by

\[ \{(C^m, md_n^N) \in B_{n+1} \times \mathbb{Z} | m \in \mathbb{Z}\} \]

since \( \rho(C) = [-d_n^N] \).

Remark 7.19. In the case \( n = 1 \), the direct calculation gives that Br(D\(^1\)\(_n\)) \( \cong \mathbb{Z}[-N + 1] \) and Aut\(^\circ\)(D\(^N\)\(_1\)) \( \cong \mathbb{Z}[1] \).

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