A REMARK ON RATIONAL CHEREDNIK ALGEBRAS AND DIFFERENTIAL OPERATORS ON THE CYCLIC QUIVER

Abstract. We show that the spherical subalgebra $U_{k,c}$ of the rational Cherednik algebra associated to $S_n \wr C_\ell$, the wreath product of the symmetric group and the cyclic group of order $\ell$, is isomorphic to a quotient of the ring of invariant differential operators on a space of representations of the cyclic quiver of size $\ell$. This confirms a version of [EG, Conjecture 11.22] in the case of cyclic groups. The proof is a straightforward application of work of Oblomkov, [O], on the deformed Harish–Chandra homomorphism, and of Crawley–Boevey, [CB1] and [CB2], and Gan and Ginzburg, [GG], on preprojective algebras.

1. Introduction

1.1. The representation theory of symplectic reflection algebras has links with a number of subjects including algebraic combinatorics, resolutions of singularities, Lie theory and integrable systems. There is a family of symplectic reflection algebras associated to any symplectic vector space $V$ and finite subgroup $\Gamma \leq Sp(V)$, but a simple reduction allows one to study those subgroups $\Gamma$ which are generated by symplectic reflections (i.e. by elements whose set of fixed points is of codimension two in $V$). This essentially focuses attention on two cases:

(1) $\Gamma = W$, a finite complex reflection group, acting on $V = \mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is a reflection representation of $W$;

(2) $\Gamma = S_n \wr G$, where $G$ is a finite subgroup of $SL_2(\mathbb{C})$, acting naturally on $(\mathbb{C}^2)^n$.

The representation theory in the first case is mysterious at the moment: several important results are known but there is no general theory yet. On the other hand a geometric point of view on the representation theory in the second case is beginning to emerge. A key fact is that in this case the singular space $V/\Gamma$ admits a crepant resolution of singularities: the representation theory of the symplectic reflection algebra is then expected to be closely related to the resolution. In the case $\Gamma = S_n$ (i.e. $G$ is trivial) there are two approaches to this: the first is via noncommutative algebraic geometry, [GS], the second via sheaves of differential operators, [GG]. In this paper we extend the second approach to the groups $\Gamma = \Gamma_n = S_n \wr C_\ell$.

1.2. To state the result here we need to introduce a little notation. Let $Q$ be the cyclic quiver with $\ell$ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) $0$. Then let $Q_\infty$ be the quiver obtained by adding one vertex named $\infty$ to $Q$ that is joined to $0$ by a single arrow.

We will consider representation spaces of these quivers. Let $\delta = (1,1,\ldots,1)$ be the affine dimension vector of $Q$, and set $\epsilon = e_\infty + n\delta$, a dimension vector for $Q_\infty$. Let $\text{Rep}(Q, n\delta)$ and $\text{Rep}(Q_\infty, \epsilon)$ be the representation spaces of these quivers with the given dimension vectors. There is an action of $G = \prod_{r=0}^{\ell-1} GL_n(\mathbb{C})$ on both.
these spaces. In fact, the action of the scalar matrices in $G$ is trivial on $\text{Rep}(Q, n\delta)$ (but not on $\text{Rep}(Q, \epsilon)$) so in this case the action descends to an action of $PG = G/\mathbb{C}^*$. Let $\mathfrak{X} = \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}$. There is an action of $PG$ on $\mathfrak{X}$.

1.3. Let $D(\text{Rep}(Q, \epsilon))$ denote the ring of differential operators on the affine space $\text{Rep}(Q, \epsilon)$, $D_X(nk)$ the sheaf of twisted differential operators on $\mathfrak{X}$ and $D(\mathfrak{X}, nk)$ its algebra of global sections. The group action of $G$ (respectively $PG$) on $\text{Rep}(Q, \epsilon)$ (respectively $\mathfrak{X}$) differentiates to an action of $g = \text{Lie}(G)$ (respectively $pg = \text{Lie}(PG)$) by differential operators. This gives mappings

\[ \hat{\tau} : g \rightarrow D(\text{Rep}(Q\infty, \epsilon)), \quad \tau : pg \rightarrow D_X(nk). \]

1.4. Let $U_{k,c}$ be the spherical subalgebra of type $S_n \wr C_\ell$ (this is defined in Section 3.4).

**Theorem.** For all $(k, c)$ there are isomorphisms of algebras

\[ \left( \frac{D(\text{Rep}(Q\infty, \epsilon))}{I_{k,c}} \right)^G \cong \left( \frac{D(\mathfrak{X}, nk)}{I_c} \right)^{PG} \cong U_{k,c}, \]

where $I_{k,c}$ is the left ideal of $D(\text{Rep}(Q\infty, \epsilon))$ generated by $(\hat{\tau} - \chi_{k,c})(g)$ and $I_c$ is the left ideal of $D(\mathfrak{X}, nk)$ generated by $(\tau - \chi_c)(pg)$ for suitable characters $\chi_{k,c} \in g^*$ and $\chi_c \in pg^*$ (which are defined in Section 4).

Note that it is a standard fact that the left hand side is an algebra. The proof of the theorem has two parts. One part constructs a filtered homomorphism from the left hand side to the right hand side using as its main input the work of Oblomkov, [O]. The other part proves that the associated graded homomorphism is an isomorphism and is a simple application of results of Crawley-Boevey, [CB1] and [CB2], and of Gan–Ginzburg, [GG].

1.5. We give an application of this result in Section 4.

1.6. While writing this down, we were informed that the general version of [EG, Conjecture 11.22] has been proved in [EGGO]. That result is more general than the work presented here and requires a new approach and ideas to overcome problems that simply do not arise for the case $\Gamma = S_n \wr C_\ell$.

### 2. Quivers

2.1. Once and for all fix integers $\ell$ and $n$. We assume that both are greater than 1. Set $\eta = \exp(2\pi i / \ell)$.

2.2. Let $Q$ be the cyclic quiver with $\ell$ vertices and cyclic orientation. Choose an extending vertex (in this case any vertex) $0$. Then let $Q\infty$ be the quiver obtained by adding one vertex named $\infty$ to $Q$ that is joined to 0 by a single arrow. Let $\underleftarrow{Q}$ and $\underrightarrow{Q\infty}$ denote the double quivers of $Q$ and $Q\infty$ respectively.

We will consider representation spaces of these quivers. Let $\delta = (1, 1, \ldots, 1)$ be the affine dimension vector of $Q$, and set $\epsilon = e_\infty + n\delta$, a dimension vector for $Q\infty$. Recall that

\[ \text{Rep}(Q, n\delta) = \bigoplus_{r=0}^{\ell-1} \text{Mat}_n(\mathbb{C}) = \{ (X_0, X_1, \ldots, X_{\ell-1}) \} = \{ (X) \} \]
and
\[ \text{Rep}(Q_\infty, \epsilon) = \bigoplus_{r=0}^{\ell-1} \text{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n = \{(X_0, X_1, \ldots, X_{\ell-1}, i) \} = \{(X, i)\}. \]

Let \( G = \prod_{r=1}^{\ell-1} GL_n(\mathbb{C}) \) be the base change group. If \( g = (g_0, \ldots, g_{\ell-1}) \) then \( g \) acts on \( \text{Rep}(Q, n\delta) \) by
\[ g \cdot (X_0, X_1, \ldots, X_{\ell-1}) = (g_0X_0g_1^{-1}, g_1X_1g_2^{-1}, \ldots, g_{\ell-1}X_{\ell-1}g_0^{-1}) \]
and on \( \text{Rep}(Q_\infty, \epsilon) \) by
\[ g \cdot (X_0, X_1, \ldots, X_{\ell-1}, i) = (g_0X_0g_1^{-1}, g_1X_1g_2^{-1}, \ldots, g_{\ell-1}X_{\ell-1}g_0^{-1}, goi). \]

The action of the scalar subgroup \( \mathbb{C}^* \) is trivial in the first action (but not the second), so we can consider the first action as a \( PG \)-action where \( PG = G/\mathbb{C}^* \). Let \( \mathfrak{g} \) and \( \mathfrak{pg} \) be the Lie algebras of \( G \) and \( PG \) respectively.

2.3. Let \( \mathfrak{h}^{\text{reg}} \subset \mathbb{C}^n \) be the affine open subvariety consisting of points \( x = (x_1, \ldots, x_n) \) such that
(i) if \( i \neq j \) then \( x_i \neq \eta^m x_j \) for all \( m \in \mathbb{Z} \),
(ii) for each \( 1 \leq i \leq n \) \( x_i \neq 0 \).

This is the subset of \( \mathbb{C}^n \) on which \( \Gamma_n = S_n \rtimes C_\ell \) acts freely.

2.4. We can embed \( \mathfrak{h}^{\text{reg}} \) into \( \text{Rep}(Q, n\delta) \) by first considering a point \( x = (x_1, \ldots, x_n) \in \mathfrak{h}^{\text{reg}} \) as a diagonal matrix \( X = \text{diag}(x_1, \ldots, x_n) \) and then sending this to \( X = (X, X, \ldots, X) \). We denote the image of \( \mathfrak{h}^{\text{reg}} \) in \( \text{Rep}(Q, n\delta) \) by \( \mathcal{S} \).

Let \( T_\Delta \) be the subgroup of \( G \) with elements \((T, T, \ldots, T)\) where \( T \) is a diagonal matrix in \( GL_n(\mathbb{C}) \). Then \( T_\Delta \) is the stabiliser of \( \mathcal{S} \). So consider the mapping
\[ \pi : G/T_\Delta \times \mathfrak{h}^{\text{reg}} \to \text{Rep}(Q, n\delta) \]
given by \( \pi(gT_\Delta, x) = g \cdot X \). If we let \( G \) act on \( G/T_\Delta \times \mathfrak{h}^{\text{reg}} \) by left multiplication then \( \pi \) is a \( G \)-equivariant mapping.

**Lemma.** \( \pi \) is an étale mapping with covering group \( \Gamma_n \). In fact its image \( \text{Rep}(Q, n\delta)^{\text{reg}} \) is open in \( \text{Rep}(Q, n\delta) \) and we have an isomorphism
\[ \omega : G/T_\Delta \times_{\Gamma_n} \mathfrak{h}^{\text{reg}} \to \text{Rep}(Q, n\delta)^{\text{reg}}. \]

**Proof.** Let \( \mathcal{S} = \{X : x \in \mathfrak{h}^{\text{reg}}\} \) and set \( N_G(\mathcal{S}) = \{g \in G : g \cdot \mathcal{S} = \mathcal{S}\} \) and \( Z_G(\mathcal{S}) = \{g \in G : g \cdot X = X \) for all \( X \in \mathcal{S}\}. \)

Suppose \( g \cdot X = Y \) for some \( X, Y \in \mathcal{S} \). This implies that for each \( 0 \leq i \leq \ell - 1 \)
\[ g_i \text{diag}(x)^{\ell} g_i^{-1} = \text{diag}(y)^{\ell}. \]

The hypotheses on \( \mathfrak{h}^{\text{reg}} \) imply that both \( \text{diag}(x)^{\ell} \) and \( \text{diag}(y)^{\ell} \) are regular semisimple in \( \mathbb{C}^n \). Two such elements are conjugate if and only if \( g_i \in N_{GL_n(\mathbb{C})}(T) = T \rtimes S_n \) where \( T \) is the diagonal subgroup of \( GL_n(\mathbb{C}) \).

So there exists \( \sigma \in S_n \) such that for all \( i \) we have \( g_i = t_i \sigma \) for some \( t_i \in T \), and for all \( 1 \leq r \leq n \) we
have that $x'_{σ (r)} = y'_r$. Hence $x_{σ (r)} = η ^{m_r} y_r$ for some \( m_r \in \mathbb{Z} \). Now we find that $Y = g \cdot X$ implies that $\text{diag}(y_r) = t_i t_{i+1}^{-1} \text{diag}(η ^{m_r} y_r)$. Since $y_r \neq 0$ this shows that $t_{i+1} = \text{diag}(η ^{m_r}) t_i$ for each $i$. Hence we find that $gT_Δ = (σ, \text{diag}(η ^{m_r}) σ, \ldots, \text{diag}(η ^{m_r}) t_{\ell - 1} σ) T_Δ$.

In particular, if $X = Y$ we see from above that each $m_r = 0$, so that $Z_G(S) = T_Δ$. Thus the group $Γ_n$ is isomorphic to $N_G(S)/Z_G(S)$ via the homomorphism that sends $(η ^{m_1}, \ldots, η ^{m_r}) σ$ to $(σ, \text{diag}(η ^{m_r}) σ, \ldots, \text{diag}(η ^{m_r}) t_{\ell - 1} σ) T_Δ$.

Now suppose that $π(gT_Δ, x) = π(hT_Δ, y)$. Then $(h^{-1} g) \cdot X = Y$ and so we see that $h^{-1} g \in N_G(S)$. This shows that $π$ is the composition

$$G/T_Δ \times h\text{reg} \longrightarrow G/T_Δ \times Γ_n \times h\text{reg} \longrightarrow \text{Rep}(Q, nδ)^{\text{reg}}.$$  

The first mapping factors out the action of $Γ_n$, and since $Γ_n$ acts freely on $h\text{reg}$ this is an étale mapping. Hence, to finish the lemma, it suffices to show that $\text{Rep}(Q, nδ)^{\text{reg}}$ is open in $\text{Rep}(Q, nδ)$.

We claim first that $\text{Rep}(Q, nδ)^{\text{reg}}$ is the set $O$ of representations of $Q$ which decompose into $n$ simple modules of dimension $δ$ and whose endomorphism ring is $n$-dimensional. To prove this observe that any element of $\text{Rep}(Q, nδ)^{\text{reg}}$ is isomorphic to a representation of the form $X$ and so it decomposes into the $n$ indecomposable modules $X_1, \ldots, X_n$ of dimension $δ$ where $X_i = (x_1, x_2, \ldots, x_δ)$ (the condition $x_i \neq 0$ implies simplicity). Now the representation $X_i$ is isomorphic to the representation $(1, 1, \ldots, 1, x_i)$. By hypothesis $x_i \neq x_j$ so we deduce that the representations $X_i$ are pairwise non-isomorphic which ensures that the endomorphism ring of $X$ is $n$-dimensional. This proves the inclusion $\text{Rep}(Q, nδ)^{\text{reg}} \subseteq O$. On the other hand, if $V$ belongs to $O$ then $V = V_1 \oplus \ldots \oplus V_n$ where each $V_i$ is isomorphic to a representation $(1, 1, \ldots, 1, ν_i)$ for some non-zero scalars $ν_i$. Moreover, since $\dim \text{End}(V) = n$ the $ν_i$ must be pairwise distinct. Now, let $η_i$ be an $ℓ$-th root of $ν_i$. Then $V_i$ is isomorphic to $(η_1, \ldots, η_i)$. Therefore $V$ is isomorphic to the representation $X$ where $x = (η_1, \ldots, η_n)$.

Now we must show that $O$ is open in $\text{Rep}(Q, nδ)$. We use first the fact that the canonical decomposition of the vector $n δ$ is $δ + δ + \cdots + δ$, [Scho, Theorem 3.6]. This means that the representations of $\text{Rep}(Q, nδ)$ whose indecomposable components all have dimension $δ$ form an open set. Now, consider the morphism $f$ from $\text{Rep}(Q, δ)$ to $C$ which sends the representation $(λ_1, \ldots, λ_ℓ)$ to the product $λ_1 \ldots λ_ℓ$. The open set $f^{-1}(C^*)$ consists of the simple representations of dimension vector $δ$. Therefore the subset of $\text{Rep}(Q, nδ)$ consisting of representations which decompose as the sum of $n$ simple representations of dimension vector $δ$ is open. On the other hand, the function from $\text{Rep}(Q, nδ)$ to $N$ which sends a representation $V$ to $\dim \text{End}(V)$ is upper semi-continuous. Thus $\{ V : \dim \text{End}(ν) \leq n \}$ is an open set in $\text{Rep}(Q, nδ)$. Intersecting these two sets shows that $O$ is open, as required.

\[ \square \]

2.5. Now we’re going to move from $Q$ to $Q_∞$. So let’s start with the following

$$\{(gT_Δ, x, i) : g_0^{-1} i \text{ is a cyclic vector for } \text{diag}(x)\} \subset (G/T_Δ \times Γ_n \times h^{\text{reg}}) \times \mathbb{C}^n.$$
By applying $\omega^{-1} \times \text{id}_{\mathbb{C}^n}$ this corresponds to an open subset of $\text{Rep}(Q, n\delta) \times \mathbb{C}^n = \text{Rep}(Q_\infty, \epsilon)$. Call that set $U_\infty$. This is a $G$–invariant open set since the $G$–action on triples is given by

$$h \cdot ([gT_\Delta, x], i) = ([hgT_\Delta, x], h_0i)$$

so $g_0^{-1}i$ is cyclic for diag$(x)$ if and only if $(h_0g_0)^{-1}h_0i$ is cyclic for diag$(x)$. Observe too that $U_\infty$ is an affine variety. Indeed it is defined by the non–vanishing of the morphism

$$s : (G/T_\Delta \times \Gamma_n, h^{reg}) \times \mathbb{C}^n \rightarrow \mathbb{C}$$

which sends $([gT_\Delta, x], i)$ to $(g_0^{-1}i) \wedge \text{diag}(x) \cdot (g_0^{-1}i) \wedge \cdots \wedge \text{diag}(x)^{n-1} \cdot (g_0^{-1}i)$.

**Lemma.** The $G$–action on $U_\infty$ is free and projection onto the second component

$$\pi_2 : U_\infty \rightarrow h^{reg}/\Gamma_n$$

is a principal $G$–bundle.

**Proof.** Suppose that $h \cdot ([gT_\Delta, x], i) = ([gT_\Delta, x], i)$. Then $[g^{-1}hgT_\Delta, x] = [T_\Delta, x]$, so by Lemma 2.4 $g^{-1}hg \in T_\Delta$.

We have that $h_0i = i$. Setting $i' = g_0^{-1}i$ implies that $g_0^{-1}h_0g_0i' = i'$. By hypothesis $i'$ is a cyclic vector for diag$(x)$. So in the standard basis $i'$ decomposes as $\sum \lambda_j e_j$ where each $\lambda_j$ is non–zero. Therefore the only diagonal matrix that fixes $i'$ is the identity element. In other words $g_0^{-1}h_0g_0 = I_n$. Since $g^{-1}hg \in T_\Delta$ this implies that $g^{-1}hg = \text{id}$. Thus $h = \text{id}$ and this proves that the action is free.

It remains to prove that each fibre of $\pi_2$ is a $G$–orbit. So take $([gT_\Delta, x], i) \in \pi_2^{-1}([x])$. This equals $g \cdot ([T_\Delta, x], g_0^{-1}i)$. Now $g_0^{-1}i$ is a cyclic vector for diag$(x)$ so it has the form $\sum \lambda_j e_j$ with each $\lambda_j$ non–zero. Let $t = \text{diag} (\lambda_1, \ldots, \lambda_n)$ and consider $t = (t, \ldots, t) \in T_\Delta$. We have

$$([gT_\Delta, x], i) = gt^{-1}([T_\Delta, x], g_0^{-1}i) = gt([T_\Delta, x], \sum_{j=1}^n e_j).$$

This proves that each fibre of $\pi_2$ is indeed a $G$–orbit. \hfill $\square$

2.6. Let $\text{Rep}(Q_\infty, \epsilon)$ be the representation space for the doubled quiver $Q_\infty$: we can naturally identify it with $T^* \text{Rep}(Q_\infty, \epsilon)$. The group $G$ acts on the base and hence on the total space of the cotangent bundle. The resulting moment map

$$\mu : \text{Rep}(Q_\infty, \epsilon) \rightarrow \mathfrak{g}^* \cong \mathfrak{g}$$

is given by

$$\mu(X, Y, i, j) = [X, Y] + ij.$$

**Theorem** (Gan–Ginzburg, Crawley–Boevey). Let $\mu^{-1}(0)$ denote the scheme–theoretic fibre of $\mu$.

1. $\mu^{-1}(0)$ is reduced, equidimensional and a complete intersection.

2. The moment map $\mu$ is flat.

3. $\mathbb{C}[\mu^{-1}(0)]^G \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/\Gamma_n$. 

5
Proof. (i) This is [GG, Theorem 3.2.3].

(ii) This follows from [CB1, Theorem 1.1] and the dimension formula in [GG, Theorem 3.2.3(iii)].

(iii) This is [CB2, Theorem 1.1]

2.7. Let \( X = \{(X, i) \in \text{Rep}(Q, n\delta) \times \mathbb{P}^{n-1}\} \). This space is the quotient of the (quasi–affine) open subvariety

\[ U = \{(X, i) : i \neq 0\} \subset \text{Rep}(Q_{\infty}, \epsilon) \]

by the scalar group \( \mathbb{C}^* \). Thus there is an action of \( P G \) on \( X \).

Since \( T^* \mathbb{P}^{n-1} = \{(i, j) : i \neq 0, ji = 0\}/\mathbb{C}^* \)
we have

\[ T^* X = \{(X, Y, i, j) \in \text{Rep}(Q_{\infty}, \epsilon) : i \neq 0, ji = 0\}/\mathbb{C}^* \]

The \( P G \) action on \( X \) gives rise to a moment map

\[ \mu_X : T^* X \to \mathfrak{p}g^* \cong \mathfrak{pg}. \]

Let

\[ \mu_X^{-1}(0) = \{(X, Y, i, j) \in \text{Rep}(Q_{\infty}, \epsilon) : i \neq 0, ji = 0, [X, Y] + ij = 0\}/\mathbb{C}^* \]

denote the scheme theoretic fibre of 0.

Proposition. There is an isomorphism \( \mathbb{C}[\mu_X^{-1}(0)]^{P G} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{G_{\mu}} \).

Proof. Consider the \( G \)–equivariant open subvariety of \( \mu^{-1}(0) \) given by the non–vanishing of \( i \). The variety \( \mu^{-1}(0) \) is determined by the conditions \( [X, Y] + ij = 0 \), so if we take the trace of this equation then we see that \( 0 = Tr(ij) = Tr(ji) = ji \). Thus we see that \( \{(X, Y, i, j) \in \text{Rep}(Q_{\infty}, \epsilon) : i \neq 0, ji = 0\} \cap \mu^{-1}(0) \) is an open subvariety of \( \mu^{-1}(0) \) so in particular reduced by Theorem 2.6(1). Hence factoring out by the action of \( \mathbb{C}^* \leq G \) shows that \( \mu_X^{-1}(0) \) is reduced and that there is a \( P G \)–equivariant morphism

\[ \mu_X^{-1}(0) \to \mu^{-1}(0)/\mathbb{C}^*. \]

This induces an algebra map

\[ \alpha : \mathbb{C}[\mu^{-1}(0)]^G \to \mathbb{C}[\mu_X^{-1}(0)]^{P G}. \]

We now follow some of the proof of [GG, Lemma 6.3.2]. Write \( O_1 \) for the conjugacy class of rank one nilpotent matrices in \( \mathfrak{gl}(n) \), and let \( \overline{O}_1 \) denote the closure of \( O_1 \) in \( \mathfrak{gl}(n) \). The moment map \( v : T^* \mathbb{P}^{n-1} \to \mathfrak{gl}(n)^* \cong \mathfrak{gl}(n) \) that sends \( (i, j) \) to \( ij \) gives a birational isomorphism \( T^* \mathbb{P}^{n-1} \to \overline{O}_1 \). Let \( J \subset \mathbb{C}[\mathfrak{gl}(n)] = \mathbb{C}[Z] \) be the ideal generated by all \( 2 \times 2 \) minors of the matrix \( Z \) and also by the trace function. Then \( J \) is a prime ideal whose zero scheme is \( \overline{O}_1 \) and the pullback morphism \( v^* : \mathbb{C}[\mathfrak{gl}(n)]/J \to \mathbb{C}[T^* \mathbb{P}^{n-1}] \) is a graded isomorphism.
where the first mapping is $\text{id} \times \nu$ and the second mapping $\theta$ sends $(X, Y, Z)$ to $[X, Y] + Z_0$ where $Z_0$ indicates that we place the matrix $Z$ on the copy of $\mathfrak{gl}(n)$ associated to vertex 0. We have a graded algebra isomorphism
\[ \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[\mathfrak{gl}(n)] / J \rightarrow \mathbb{C}[T^* X]. \]

Now write $\mathbb{C}[X, Y, Z] = \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \times \mathbb{C}[\mathfrak{gl}(n)]$, and let $\mathbb{C}[X, Y, Z]/([X, Y] + Z_0)$ denote the ideal in $\mathbb{C}[X, Y, Z]$ generated by all matrix entries of the $\ell$ matrices $[X, Y] + Z_0$. Let $I$ denote the ideal $\mathbb{C}[X, Y, Z]/([X, Y] + Z_0) + \mathbb{C}[X, Y] \otimes J \subset \mathbb{C}[X, Y, Z]$. From the above we have

\[ \mathbb{C}[\mu_X^{-1}(0)] \cong \mathbb{C}[T^* \text{Rep}(Q, n\delta)] / \mathbb{C}[T^* \text{Rep}(Q, n\delta) \times \mathbb{C}[\mathfrak{gl}(n)]] \theta^* (\mathfrak{gl}(n)) = \mathbb{C}[X, Y, Z] / I. \]

Define an algebra homomorphism $r : \mathbb{C}[X, Y, Z] \rightarrow \mathbb{C}[X, Y]$ by sending $P \in \mathbb{C}[X, Y, Z]$ to the function $(X, Y) \mapsto P(X, Y, -[X, Y]_0)$. Obviously $r$ induces an isomorphism $\mathbb{C}[X, Y, Z] / \mathbb{C}[X, Y, Z]/([X, Y] + Z_0) \cong \mathbb{C}[X, Y] / I_1$ where $I_1$ is the ideal of $\mathbb{C}[\text{Rep}(Q, n\delta)] = \mathbb{C}[X, Y]$ generated by the elements
\[ \sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a \]
for all $i$ not equal to zero. Observe that the linear function $P : (X, Y, Z) \mapsto Tr Z = Tr([X, Y] + Z_0)$ belongs to the ideal $\mathbb{C}[X, Y, Z]/([X, Y] + Z_0)$. We deduce that the mapping $r$ sends $\mathbb{C}[X, Y] \otimes J$ to the ideal generated by
\[ \text{rank}( \sum_{h(a)=0} X_a X_{a^*} - \sum_{t(a)=0} X_{a^*} X_a ) \leq 1. \]
Thus we obtain algebra isomorphisms
\[ \mathbb{C}[\mu_X^{-1}(0)] \cong \mathbb{C}[X, Y, Z] / I \cong \mathbb{C}[T^* \text{Rep}(Q, n\delta)] / I_2 \]
where $I_2$ is ideal generated by the elements
\[ \sum_{h(a)=i} X_a X_{a^*} - \sum_{t(a)=i} X_{a^*} X_a \]
for all $1 \leq i \leq \ell - 1$, and
\[ \text{rank}( \sum_{h(a)=0} X_a X_{a^*} - \sum_{t(a)=0} X_{a^*} X_a ) \leq 1. \]

By [LP, Theorem 1] the $G$–invariant (respectively $PG$–invariant) elements of $\mathbb{C}[\text{Rep}(Q_\infty, \epsilon)]$ (respectively $\mathbb{C}[\text{Rep}(Q, n\delta)]$) are generated by traces along oriented cycles. Since all oriented cycles in $Q$ are oriented cycles in $Q_\infty$ we have a surjective composition of algebra homomorphisms
\[ (2.7.1) \quad \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^\ell_n \cong \mathbb{C}[\mu^{-1}(0)]^G \rightarrow \mathbb{C}[\mu_X^{-1}(0)]^{PG} \rightarrow \left( \mathbb{C}[\text{Rep}(Q, n\delta)] / I_2 \right)^{PG}, \]
where the first isomorphism is Theorem 2.6(3). The left hand side is a domain of dimension $2 \dim \mathfrak{h}$, so to see that the mapping is an isomorphism it suffices to prove that the right hand side also has dimension $2 \dim \mathfrak{h}$.
Let $I_3$ be the ideal of $\mathbb{C}[\text{Rep}(Q, n\delta)]$ generated by the elements
\[
\sum_{h(a) = i} X_a X_{a^*} - \sum_{t(a) = i} X_{a^*} X_a
\]
for all $i$. This is the ideal of the zero fibre of the moment map for the $PG$–action on $\text{Rep}(Q, n\delta)$. This ideal contains $I_2$ since the rank condition on the matrices is implied by the commutator condition. So there is a surjective mapping
\[
\frac{\mathbb{C}[\text{Rep}(Q, n\delta)]^{PG}}{I_2^{PG}} \to \frac{\mathbb{C}[\text{Rep}(Q, n\delta)]^{PG}}{I_3^{PG}}.
\]
We do not know yet whether the right hand side is reduced or not, but by [CB2, Theorem 1.1] the reduced quotient of the right hand side is the ring of functions of the variety $(\mathfrak{h} \oplus \mathfrak{h}^*)/\Gamma_n$. As this variety has dimension $2 \dim \mathfrak{h}$ we deduce that the composition in (2.7.1) is an isomorphism, and hence that
\[
\mathbb{C}[\mu_X^{-1}(0)]^{PG} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]/\Gamma_n.
\]

\[\square\]

Remark. In passing let us note that the commutativity of the following diagram
\[
\begin{array}{ccc}
\mathbb{C}[T^* \text{Rep}(Q, n\delta)] & \xrightarrow{\iota} & \mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes \mathbb{C}[T^* \mathbb{P}^{n-1}] \\
& \downarrow{\nu^*} & \downarrow{\iota} \\
\mathbb{C}[T^* \text{Rep}(Q, n\delta)] \otimes C[\Omega_1] & \xrightarrow{\rho} & \mathbb{C}[T^* \text{Rep}(q, n\delta)]/I_2 \\
\end{array}
\]
where $\iota(f) = f \otimes 1$, shows that $\text{im } \iota$ maps surjectively onto $\mathbb{C}[\mu_X^{-1}(0)]$.

3. Differential operators

3.1. Symplectic reflection algebras. Let $C_\ell$ be the cyclic subgroup of $SL_2(\mathbb{C})$ generated by $\sigma = \text{diag}(\eta, \eta^{-1})$.

The vector space $V = (\mathbb{C}^2)^n$ admits an action of $S_n \wr C_\ell = S_n \times (C_\ell)^n$: $(C_\ell)^n$ acts by extending the natural action of $C_\ell$ on $\mathbb{C}^2$, whilst $S_n$ acts by permuting the $n$ copies of $\mathbb{C}^2$. For an element $\gamma \in C_\ell$ and an integer $1 \leq i \leq n$ we write $\gamma_i$ to indicate the element $(1, \ldots, \gamma, \ldots, 1) \in C_\ell^n$ which is non–trivial in the $i$–th factor.

3.2. The elements $S_n \wr C_\ell$ whose fixed points are a subspace of codimension two in $V$ are called symplectic reflections. In this case their conjugacy classes are of two types:

(S) the elements $s_{ij} \gamma_i \gamma_j^{-1}$ where $1 \leq i, j \leq n$, $s_{ij} \in S_n$ is the transposition that swaps $i$ and $j$, and $\gamma \in C_\ell$.

(C_\ell) the elements $\gamma_i$ for $1 \leq i \leq n$ and $\gamma \in C_\ell \setminus \{1\}$.

There is a unique conjugacy class of type $(S)$ and $\ell - 1$ of type $(C_\ell)$ (depending on the non–trivial element we choose from $C_\ell$). We will consider a conjugation invariant function from the set of symplectic reflections
to \( \mathbb{C} \). We can identify it with a pair \((k, c)\) where \( k \in \mathbb{C} \) and \( c \) is an \( \ell - 1 \)–tuple of complex numbers: the function sends elements from \((S)\) to \( k \) and the elements \( (\sigma^m)_i \) to \( c_m \).

3.3. There is a symplectic form on \( V \) which is induced from \( n \) copies of the standard symplectic form \( \omega \) on \( \mathbb{C}^2 \). If we pick a basis \( \{x, y\} \) for \( \mathbb{C}^2 \) such that \( \omega(x, y) = 1 \) then we can extend this naturally to a basis \( \{x_i, y_i : 1 \leq i \leq n\} \) of \( V \) such that the \( x \)'s and the \( y \)'s form Lagrangian subspaces and \( \omega(x_i, y_j) = \delta_{ij} \). We let \( TV \) denote the tensor algebra on \( V \): with our choice of basis this is just the free algebra on generators \( x_i, y_i \) for \( 1 \leq i \leq n \). The symplectic reflection algebra \( H_{k, c} \) associated to \( S_n \wr C_\ell \) is the quotient of \( TV * (S_n \wr C_\ell) \) by the following relations:

\[
\begin{align*}
&x_i x_j = x_j x_i, & y_i y_j = y_j y_i & \text{for all } 1 \leq i, j \leq n \\
y_i x_i - x_i y_i = 1 + \frac{k}{2} \sum_{j \neq i} \sum_{\gamma \in C_\ell} \gamma \gamma_j^{-1} + \sum_{\gamma \in C_\ell \setminus \{1\}} C\gamma \gamma_i & \text{for } 1 \leq i \leq n \\
y_i x_j - x_j y_i = -\frac{k}{2} \sum_{m=0}^{\ell-1} \eta^m s_{ij}(\sigma^m)_i(\sigma^m)_j^{-1} & \text{for } i \neq j.
\end{align*}
\]

(NB: my \( k \) is \(-k\) for Oblomkov.)

3.4. **The spherical algebra.** The symmetrising idempotent of the group algebra \( C(S_n \wr C_\ell) \) is

\[
e = \frac{1}{|S_n \wr C_\ell|} \sum_{w \in S_n \wr C_\ell} w.
\]

The subalgebra \( eH_{k, c}e \) is denoted by \( U_{k, c} \) and called the **spherical algebra.** It will be our main object of study.

3.5. **Rings of differential operators.** Recall the definition of \( \mathfrak{X} \) from 2.7. Let \( D_\mathfrak{X}(nk) \) denote the sheaf of twisted differential operators on \( \mathfrak{X} \) and let \( D(\mathfrak{X}, nk) \) be its algebra of global sections. This is simply the tensor product \( D(\text{Rep}(Q, nk)) \otimes D_{\mathbb{P}^{n-1}}(nk) \). (The twisted differential operators on \( \mathbb{P}^{n-1} \) can be defined as follows. Let \( A_n = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] \) be the \( n \)–th Weyl algebra. This is a graded algebra with \( \text{deg}(x_i) = 1 \) and \( \text{deg}(\partial_i) = -1 \). The degree zero component is the subring generated by the operators \( x_i \partial_j \) which, under the commutator, generate the Lie algebra \( \mathfrak{gl}(n) \). Call this subring \( R \). Let \( E = \sum_{i=1}^n x_i \partial_i \in R \) be the Euler operator. Then \( D(\mathbb{P}^{n-1}, nk) \) is the quotient of \( R \) by the two–sided ideal generated by \( E - nk \).

The group action of \( PG \) on \( \mathfrak{X} \) differentiates to an action of \( \mathfrak{pg} \) on \( \mathfrak{X} \) by differential operators. This gives a mapping

\[(3.5.1) \quad \tau : \mathfrak{pg} \rightarrow D_\mathfrak{X}(nk).\]

(One way to understand this is to start back with \( U \subset \text{Rep}(Q_{\infty}, \epsilon) \) and look at the \( G \) action on \( U \). Differentiating the \( G \)–action gives an action of \( \mathfrak{g} \) by differential operators on \( U \), \( \hat{\tau} : \mathfrak{g} \rightarrow D_U \). Since \( \mathbb{C}^* \) acts trivially on \( \text{Rep}(Q, n\delta) \) and by scaling on \( i \in \text{Rep}(Q_{\infty}, \epsilon) \) we find that \( \hat{\tau}(\text{id}) = 1 \otimes E \) where \( \text{id} = (I_n, I_n, \ldots, I_n) \in \mathbb{C} \subset \mathfrak{g} \). Thus we get an action of \( \mathfrak{pg} \) on \((D_U / D_U(1 \otimes E - nk))^\mathbb{C}^* = D_\mathfrak{X}(nk)\).
3.6. Recall the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ and its quotient $\mathfrak{pg} = \text{Lie}(PG)$ which is simply $\mathfrak{g}/\mathbb{C} \cdot \text{id}$ where $\text{id} = (I_n, \ldots, I_n) \in \mathfrak{g}$. Let $\chi_c : \mathfrak{g} \to \mathbb{C}$ send an element $(X) = (X_0, \ldots, X_{\ell-1}) \in \mathfrak{g}$ to

$$
\chi_c(X) = \sum_{r=0}^{\ell-1} C_r \text{Tr}(X_r)
$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{m \ell} c_m)$ for $1 \leq r \leq \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. Observe that

$$
\chi_c(\text{id}) = \text{Tr}(I_n) \sum_{r=0}^{\ell-1} C_r = n \sum_{r=0}^{\ell-1} \sum_{m=0}^{r-1} -\eta^{rm} c_m = 0.
$$

In particular $\chi_c$ is actually a character of $\mathfrak{pg}$.

Let $\chi_k : \mathfrak{g} \to \mathbb{C}$ send an element $(X) = (X_0, \ldots, X_{\ell-1})$ to $\chi_k(c) = k \text{Tr}(X_0)$.

We will be regularly using the character $\chi_{k,c} \in \mathfrak{g}^*$ defined by $\chi_{k,c} = \chi_c + \chi_k$.

3.7. Let us recall Oblomkov’s deformed Harish–Chandra homomorphism, [O]. By Lemma 2.4 $\omega(\mathfrak{h}^{\text{reg}}/\Gamma_n)$ is a subset of $\text{Rep}(Q, n\delta)^{\text{reg}}$ which is a slice for the $PG$–action on $\text{Rep}(Q, n\delta)$. Let

$$
W'_k = (y_1 \ldots y_n)^{-k} C(0)[y_1^{\pm 1}, \ldots, y_n^{\pm 1}],
$$

a space of multivalued functions on $(\mathbb{C}^*)^n$. The Lie algebra $\mathfrak{g}$ acts on $W'_k$ by projection onto its 0–th summand $\mathfrak{gl}(n)$, and then by the natural action of $\mathfrak{gl}(n)$ on polynomials (so $E_{ij}$ acts as $y_i \partial / \partial y_j$). With this action the identity matrix in $\mathfrak{gl}(n)$ becomes the Euler operator $E$ which acts by multiplication by $-nk$. Thus we can make $W'_k$ a $\mathfrak{pg}$-module by twisting $W'_k$ by the character $\chi_k$ since then id acts trivially. If we call this module $W_k$ then $W_k = W'_k \otimes \chi_k$. Now define $\text{Fun}'$ to be the space of functions on $\text{Rep}(Q, n\delta)$ of the form

$$
f = \hat{f} \prod_{i=0}^{\ell-1} \det(X_i)^{r_i}
$$

where $\hat{f}$ is a rational function on $\text{Rep}(Q, n\delta)^{\text{reg}}$ regular on $\mathcal{S}$, $r_i = \sum_{j=0}^{i} C_j + \sigma$ and $\sigma = \ell^{-1} \sum_{r=0}^{\ell-1} sC_r$. Then $(\text{Fun}' \otimes W_k)^{\mathfrak{pg}}$ is a space of $(\mathfrak{pg}, \chi_c)$–semiinvariant functions defined on a neighbourhood of $\mathcal{S}$ which take values in $W_k$. This space is a free $\mathbb{C}[\mathfrak{h}^{\text{reg}}/\Gamma_n]$–module of rank 1, the isomorphism being given by restriction to $\mathcal{S}$. (Note that the determinant of an element of the form $(X, \ldots, X)$ is $\det(X) \sum r_i = 1$ as $\sum r_i = 0$.) Any $\mathfrak{pg}$–invariant differential operator, $D$, acts on such a function, $f$. Oblomkov defines his homomorphism to be the restriction of $D(f)$ to $\mathcal{S}$.

3.8. We can view the above procedure in terms of $\text{Rep}(Q_\infty, \epsilon)$. Thanks to Lemma 2.5 we use $\mathcal{S}_\infty = \mathcal{S} \times (1, \ldots, 1) \in U_\infty$ as a slice for the $G$–action. The space $\mathcal{S} \times (\mathbb{C}^*)^n$ is a closed subset of $U_\infty$ since the condition that $i$ be cyclic for $\text{diag}(x_1, \ldots, x_n)$ is equivalent to $i \in (\mathbb{C}^*)^n$. Thus functions on a neighbourhood of $\mathcal{S}_\infty$ in $U_\infty$ can be identified with functions from a neighbourhood of $\mathcal{S}$ taking values in functions on $(\mathbb{C}^*)^n$. In particular, we can consider elements on $(\text{Fun}' \otimes W_k)^{\mathfrak{pg}}$ first as $(\mathfrak{g}, \chi_{k,c})$–semiinvariant functions from a neighbourhood of $\mathcal{S}$ taking values in $W'_k$ and hence as $(\mathfrak{g}, \chi_{k,c})$–semiinvariant functions on an open set in a
neighbourhood of \( S_{\infty} \). We can apply any element of \( D \in D(U_{\infty})^g \) to these \((g, \chi_{k,c})\)-semiinvariant functions and then restrict to \( S_{\infty} \) to get a homomorphism

\[
\tilde{\vartheta}_{k,c} : D(U_{\infty})^g \to D(\mathfrak{h}_{\text{reg}}/\Gamma_n).
\]

3.9. Since \( \text{Rep}(Q_{\infty}, \epsilon) = \text{Rep}(Q, n\delta) \times \mathbb{C}^n \) there is a mapping

\[
\mathcal{G} : D(\text{Rep}(Q, n\delta))^p_{\mathbb{R}} \to D(U_{\infty})^g
\]

which sends \( D \in D(\text{Rep}(Q, n\delta))^p_{\mathbb{R}} \) to \((D \otimes 1)\). Oblomkov’s homomorphism is \( \tilde{\vartheta}_{k,c} \circ \mathcal{G} \).

3.10. Differentiating the \( G \)-action on \( U_{\infty} \) gives a Lie algebra homomorphism \( \hat{\tau} : \mathfrak{g} \to \text{Vect}(U_{\infty}) \) which we extend to an algebra map

\[
\hat{\tau} : U(\mathfrak{g}) \to D(U_{\infty}).
\]

By Lemma 2.5 \( U_{\infty} \) is a principle \( G \)-bundle over \( \mathfrak{h}_{\text{reg}}/\Gamma_n \), so (a generalisation of) [Schw, Corollary 4.5] shows that the kernel of \( \tilde{\vartheta}_{k,c} \) is \( (D(U_{\infty})((\hat{\tau} - \chi_{k,c})(\mathfrak{g})))^g \). Moreover, since the finite group \( \Gamma_n \) acts freely on \( \mathfrak{h}_{\text{reg}} \) we can identify \( D(\mathfrak{h}_{\text{reg}}/\Gamma_n) \) with \( D(\mathfrak{h}_{\text{reg}}) \Gamma_n \).

3.11. Recall that

\[
D_X(nk) \cong \left( \frac{D_U}{D_U(\hat{\tau} - \chi_k)(\mathbb{C} \cdot \text{id})} \right)^C.
\]

Hence we have

\[
(D^G_X(nk)) \cong \left( \frac{D_X(nk)}{D_X(nk)(\tau - \chi_c)(pg)} \right)^{PG},
\]

where \( U = \{(X, i) : i \neq 0\} \subset \text{Rep}(Q_{\infty}, n\delta) \) as in 2.7. Now we consider the restriction mapping \( D_U \to D(U_{\infty}) \). Composing the global sections of the above isomorphism with this restriction and the homomorphism \( \tilde{\vartheta}_{k,c} \) gives

\[
\mathcal{R}_{k,c} : \left( \frac{D(X, nk)}{D(X, nk)(\tau - \chi_c)(pg)} \right)^{PG} \to D(\mathfrak{h}_{\text{reg}}) \Gamma_n.
\]

3.12. Let

\[
\delta_{k,c}(x) = \delta^{-k-1} \delta^\sigma_{\Gamma}
\]

where \( \delta = \prod_{1 \leq i < j \leq n} (x_i^f x_j^f) \) and \( \delta_T = \prod_{i=1}^n x_i \). Define a twisted version of \( \mathcal{R}_{k,c} \) above

\[
\mathcal{R}_{k,c}(D) = \delta_{k,c}^{-1} \circ \mathcal{R}_{k,c}'(D) \circ \delta_{k,c}
\]

for any differential operator \( D \).
3.13. Our main result is the following.

**Theorem.** For all values of \( k \) and \( c \), the homomorphism \( \mathcal{R}_{k,c} \) has image \( \im \theta_{k,c} \). In particular we have an isomorphism

\[
\theta_{k,c}^{-1} \circ \mathcal{R}_{k,c} : \left( \frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}} \sim \rightarrow U_{k,c}.
\]

**Proof.** Let us abuse notation by writing \( U_{k,c} \) for the image of \( \mathcal{U}_{k,c} \) in \( D(\mathfrak{h}_{\text{reg}})^{\Gamma_n} \) under \( \theta_{k,c} \).

Since \( \mathfrak{X} = \text{Rep}(Q,n\delta) \times \mathbb{P}^{n-1} \) there is a mapping

\[
D(\text{Rep}(Q,n\delta))^{\mathfrak{p} \mathfrak{g}} \rightarrow D(\mathfrak{X},nk)^{\mathfrak{p} \mathfrak{g}} \rightarrow D(\mathfrak{h}_{\text{reg}})^{\Gamma_n}
\]

which sends \( D \in D(\text{Rep}(Q,n\delta))^{\mathfrak{p} \mathfrak{g}} \) to \( \mathcal{R}_{k,c}(D \otimes 1) \). Recall \( \tau \) from (3.5.1). Since \( \text{gr} \tau = \mu^*_\lambda \) we have an inclusion \( \text{gr}(D(\mathfrak{X},nk))\mu^*_\lambda(\mathfrak{p} \mathfrak{g}) \subseteq \text{gr}(D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g})) \). This gives a graded surjection

\[
p : \left( \frac{D(\mathfrak{X},nk)}{\text{gr}(D(\mathfrak{X},nk))\mu^*_\lambda(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}} \rightarrow \text{gr} \left( \frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}}.
\]

By Remark 2.7 the composition

\[
\text{gr} D(\text{Rep}(Q,n\delta))^{\mathfrak{p} \mathfrak{g}} \rightarrow \text{gr} D(\mathfrak{X},nk)^{\mathfrak{p} \mathfrak{g}} \rightarrow \left( \frac{D(\mathfrak{X},nk)}{\text{gr}(D(\mathfrak{X},nk))\mu^*_\lambda(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}} \rightarrow \text{gr} \left( \frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}}
\]

is surjective. Thus the homomorphism

\[
D(\text{Rep}(Q,n\delta))^{\mathfrak{p} \mathfrak{g}} \rightarrow \left( \frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}}
\]

is also surjective. In particular, by 3.9 this implies that the image of \( \mathcal{R}_{k,c} \) equals the image of Oblomkov’s Harish–Chandra homomorphism, which, by [O, Theorem 2.5], is \( U_{k,c} \).

Thus we have a filtered surjective homomorphism

\[
\mathcal{R}_{k,c} : \left( \frac{D(\mathfrak{X},nk)}{D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}} \rightarrow U_{k,c}.
\]

Thus the dimension of the left hand side is at least \( 2 \dim \mathfrak{h} = \dim U_{k,c} \). By Proposition 2.7

\[
\left( \frac{D(\mathfrak{X},nk)}{\text{gr}(D(\mathfrak{X},nk))\mu^*_\lambda(\mathfrak{p} \mathfrak{g})} \right)^{\mathfrak{p} \mathfrak{g}} \cong \mathbb{C}[\mu^{-1}_\lambda(0)]^{\mathfrak{p} \mathfrak{g}} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\Gamma_n}.
\]

Hence \( p \) is a surjection from a domain of dimension \( 2 \dim \mathfrak{h} \) onto an algebra of dimension at least \( 2 \dim \mathfrak{h} \) and is hence an isomorphism. Thus \( (D(\mathfrak{X},nk)/D(\mathfrak{X},nk)(\tau - \chi_c)(\mathfrak{p} \mathfrak{g}))^{\mathfrak{p} \mathfrak{g}} \) is a domain of dimension \( 2 \dim \mathfrak{h} \). This implies that \( \mathcal{R}_{k,c} \) is an isomorphism. \( \square \)
4. Application: Shift functors

4.1. The Holland-Schwarz Lemma. We want to understand the space

\[ D(\text{Rep}(Q_\infty, \epsilon)) \]

As we observed in the proof of Theorem 3.13 there is a natural surjective homomorphism

\[ \frac{\text{gr} D(\text{Rep}(Q_\infty, \epsilon))}{\text{gr} D(\text{Rep}(Q_\infty, \epsilon)) \mu^*(\mathfrak{g})} \rightarrow \text{gr} \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right). \]

It turns out that this is an isomorphism.

**Lemma (Schwarz, Holland).** The homomorphism (4.1.1) is an isomorphism of \( \mathbb{C}[T^* \text{Rep}(Q_\infty, \epsilon)] \)-modules.

**Proof.** This is [H, Lemma 2.2] since, by Theorem 2.6(2), the moment map \( \mu \) is flat. \( \square \)

4.2. This lets us prove the second part of the isomorphism in the statement of Theorem 1.4.

**Lemma.** There is an algebra isomorphism

\[ \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \rightarrow \left( \frac{D(X, nk)}{D(X, nk)(\hat{\tau} - \chi_{k,c})(\text{pg})} \right)^{PG}. \]

**Proof.** We have a natural \( \text{pg} \)-equivariant mapping

\[ D(\text{Rep}(Q_\infty, \epsilon))^{G^*} \rightarrow D_{U}^{G^*} \rightarrow D_X(nk) \]

which induces a homomorphism

\[ D(\text{Rep}(Q_\infty, \epsilon))^G \rightarrow \left( \frac{D(X, nk)}{D(X, nk)(\hat{\tau} - \chi_{k,c})(\text{pg})} \right)^{PG}. \]

This is surjective since, as we observed in the proof of Theorem 3.13, the image of \( D(\text{Rep}(Q, n\delta))^{PG} \subset D(\text{Rep}(Q_\infty, \epsilon)^G \) spans the right hand side. By (3.11.1) the kernel of this homomorphism includes the ideal \( (D(\text{Rep}(Q, \infty), \epsilon)(\hat{\tau} - \chi_{k,c})(\mathfrak{g})^G. \) Hence we have a surjective homomorphism

\[ \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G \rightarrow \left( \frac{D(X, nk)}{D(X, nk)(\hat{\tau} - \chi_{k,c})(\text{pg})} \right)^{PG}. \]

By Lemma 4.1 and Proposition 2.7 there is an isomorphism

\[ \left( \frac{\text{gr} D(\text{Rep}(Q_\infty, \epsilon))}{\text{gr} D(\text{Rep}(Q_\infty, \epsilon))(\hat{\tau} - \chi_{k,c})(\mathfrak{g})} \right)^G = \left( \frac{\text{gr} D(\text{Rep}(Q_\infty, \epsilon))}{\text{gr} D(\text{Rep}(Q_\infty, \epsilon)) \mu^*(\mathfrak{g})} \right)^G = \mathbb{C}[\mu^{-1}(0)]^G = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^\Gamma. \]

This shows that the algebra on the left is a domain of dimension of \( 2 \dim \mathfrak{h} \) and so (4.2.1) is also injective, as required. \( \square \)
4.3. **Shifting.** The previous two lemmas provide us with an interesting series of bimodules. Given a character $\Lambda$ of $G$ we define

$$B^\Lambda_{k,c} = \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\tau - \chi_{k,c})(g)} \right)^\Lambda$$

to be the set of $(G, \Lambda)$–semiinvariants. Thanks to Lemma 4.2 and Theorem 3.13 this is a right $U_{k,c}$–module.

Now observe that if $x \in g$ and $D \in D(\text{Rep}(Q_\infty, \epsilon))^\Lambda$ then

$$[\tau(x), D] = \lambda(x)D$$

where $\lambda = d\Lambda$. It follows that $B^\Lambda_{k,c}$ is also a left $(D(\text{Rep}(Q_\infty, \epsilon))/D(\text{Rep}(Q_\infty, \epsilon))(\tau - \chi_{k,c} - \lambda)(g))^G$–module. So tensoring sets up a shift functor

$$S^\Lambda_{k,c} : \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\tau - \chi_{k,c})(g)} \right)^G \longrightarrow \left( \frac{D(\text{Rep}(Q_\infty, \epsilon))}{D(\text{Rep}(Q_\infty, \epsilon))(\tau - \chi_{k,c} - \lambda)(g)} \right)^G.$$

4.4. The character group of $G$ is isomorphic to $\mathbb{Z}^\ell$ via

$$(i_0, \ldots, i_{\ell-1}) \mapsto ((g_0, \ldots, g_{\ell-1}) \mapsto \prod_{r=0}^{\ell-1} \det(g_r)^{i_r}).$$

Corresponding to the standard basis element $\epsilon_i$ is the character $\chi_i$ of $g$ which sends $X \in g$ to $\text{Tr}(X_i)$.

**Lemma.** The bimodule corresponding to $\chi_i$ is a $(U_{k,c}, U_{k',c'})$–bimodule where $k' = k + 1$ and $c' = c + (1 - \eta^{-i}, 1 - \eta^{-2i}, \ldots, 1 - \eta^{-(\ell-1)i})$.

**Proof.** Recall that $(k, c)$ corresponds to the character of $g$ we called $\chi_{k,c}$ which is defined as

$$\chi_{k,c}(X) = (C_0 + k) \text{Tr}(X_0) + \sum_{j=1}^{\ell-1} C_j \text{Tr}(X_j),$$

where $C_r = \ell^{-1}(1 - \sum_{m=1}^{\ell-1} \eta^{mr} c_m)$ for $1 \leq r \leq \ell - 1$ and $C_0 = \ell^{-1}(1 - \ell - \sum_{m=1}^{\ell-1} c_m)$. We need to calculate $(k', c')$ so that $\chi_{k,c} + \chi_i = \chi_{k',c'}$. So we have

$$(\chi_{nk,c} + \chi_i)(X) = (C_0 + k) \text{Tr}(X_0) + \text{Tr}(X_i) + \sum_{j=1}^{\ell-1} C_j \text{Tr}(X_j) = (C_0' + k') \text{Tr}(X_0) + \sum_{j=1}^{\ell-1} C'_j \text{Tr}(X_j).$$

Calculation shows that $k' = k + 1$ and that if $i = 0$ then $C'_j = C_j$ and otherwise

$$C'_j = C_j + \begin{cases} -1 & \text{if } j = 0 \\ 1 & \text{if } j = i \\ 0 & \text{otherwise}. \end{cases}$$

These unpack to give $c'_m = c_m + 1 - \eta^{-mi}$. \qed
4.5. **Question.** Thus for each $0 \leq i \leq \ell - 1$ we have a *shift functor*

$$S_i : U_{k,c}-\text{mod} \to U_{k+1,c'}-\text{mod}$$

where $c'$ is as above. When is this an equivalence of categories?

**Remarks.**  
1. *We have been able to prove this is an equivalence when $(k,c)$ can be reached from $(0,0)$ by shifting.*
2. *Shift functors are also constructed in [BC] and [V]. Hopefully they agree with the functors here.*

**References**

[BC] Y. Berest and O. Chalykh, Quasi–invariants of complex reflection groups, *in preparation.*

[BEG] Y. Berest, P. Etingof and V. Ginzburg, Cherednik algebras and differential operators on quasi–invariants, *Duke Math. J.* **118**, 279–337.

[CB1] W. Crawley–Boevey, Geometry of the moment map for representations of quivers, *Compositio Math.*, **126**, (2001), 257–293.

[CB2] W. Crawley–Boevey, Decomposition of Marsden–Weinstein reductions for representations of quivers, *Compositio Math.* **130** (2002), 225–239.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism, *Invent. Math.*, **147** (2002), 243–348.

[EGGO] P. Etingof, W.L. Gan, V. Ginzburg and A. Oblomkov, private communication.

[GG] W.L. Gan and V. Ginzburg, Almost commuting variety, $D$–modules, and Cherednik algebras, *RT:0409262*, March 2005.

[GS] I. Gordon and J.T. Stafford, Rational Cherednik algebras and Hilbert schemes I and II, *to appear in Adv.Math. and Duke Math. Jour.*

[H] M. Holland, Quantization of the Marsden–Weinstein reduction for extended Dynkin quivers, *Ann. scient. Éc. Norm. Sup.*, (1999), 813–834.

[LP] L. Le Bruyn and C. Procesi, Semisimple representations of quivers, *Trans. Amer. Math. Soc.*, **317**, (1990), 585–598.

[LS] T. Levasseur and J.T. Stafford, The kernel of a homomorphism of Harish–Chandra, *Ann. scient. Éc. Norm. Sup.*, **29**, (1996), 385–397.

[O] A. Oblomkov, Deformed Harish–Chandra homomorphism for the cyclic quiver, *RT:0504395*, April 2005.

[Scho] A. Schofield, General representations of quivers, *Proc. London Math. Soc* **65** (1992), 46–64.

[Schw] G.W. Schwarz, Lifting differential operators from orbit spaces, *Ann. scient. Éc. Norm. Sup.*, **28**, (1995), 253–306.

[V] R. Vale, Diagonal coinvariants for $Z_m \wr S_n$, *RT:0505416*, May 2005.