Polynomial Bottleneck Congestion Games with Optimal Price of Anarchy

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Abstract—We study bottleneck congestion games where the social cost is determined by the worst congestion of any resource. These games directly relate to network routing problems and also job-shop scheduling problems. In typical bottleneck congestion games, the utility costs of the players are determined by the worst congested resources that they use. However, the resulting Nash equilibria are inefficient, since the price of anarchy is proportional on the number of resources which can be high. Here we show that we can get smaller price of anarchy with the bottleneck social cost metric. We introduce the polynomial bottleneck games where the utility costs of the players are polynomial functions of the congestion of the resources that they use. In particular, the delay function for any resource \( r \) is \( C_{r}^{\alpha_{r}} \), where \( C_{r} \) is the congestion measured as the number of players that use \( r \), and \( M \geq 1 \) is an integer constant that defines the degree of the polynomial. The utility cost of a player is the sum of the individual delays of the resources that it uses. The social cost of the game remains the same, namely, it is the worst bottleneck resource congestion: \( \max_{r} C_{r} \). We show that polynomial bottleneck games are very efficient and give price of anarchy \( O(|R|^{1/(M+1)}) \), where \( R \) is the set of resources. This price of anarchy is tight, since we demonstrate a game with price of anarchy \( \Omega(|R|^{1/(M+1)}) \), for any \( M \geq 1 \). We obtain our tight bounds by using two proof techniques: transformation, which we use to convert arbitrary games to simpler games, and expansion, which we use to bound the price of anarchy in a simpler game.

I. INTRODUCTION

We consider non-cooperative congestion games with \( n \) players, where each player has a pure strategy profile from which it selfishly selects a strategy that minimizes the player’s utility cost function (such games are also known as atomic or unsplittable-flow games). We focus on bottleneck congestion games where the objective for the social outcome is to minimize \( C \), the maximum congestion on any resource. Typically, the congestion on a resource is a non-decreasing function on the number of players that use the resource; here, we consider the congestion to be simply the number of players that use the resource.

Bottleneck congestion games have been studied in the literature [1], [3], [2] in the context of routing games, where each player’s utility cost is the worst resource congestion on its strategy. For any resource \( r \), we denote by \( C_{r} \) the number of users that use \( r \) in their strategies. In typical bottleneck congestion games, each player \( i \) has utility cost function \( C_{i} = \max_{r \in S_{i}} C_{r} \), where \( S_{i} \) is the strategy of the player. The social cost is worst congested resource: \( C = \max_{i} C_{i} = \max_{r} C_{r} \).

In [1] the authors observe that bottleneck games are important in networks for various practical reasons. In networks, each resource corresponds to a network link, each player corresponds to a packet, and a strategy represents a path for the packet. In wireless networks, the maximum congested link is related to the lifetime of the network since the nodes adjacent to high congestion links transmit large number of packets which results to higher energy utilization. High congestion links also result to congestion hot-spots which may slow-down the network throughput. Hot spots also increase the vulnerability of the network to malicious attacks which aim to increase the congestion of links in the hope to bring down the network. Thus, minimizing the maximum congested edge results to hot-spot avoidance and more load-balanced and secure networks.

In networks, bottleneck games are also important from a theoretical point of view since the maximum resource congestion is immediately related to the optimal packet scheduling. In a seminal result, Leighton et al. [13] showed that there exist packet scheduling algorithms that can deliver the packets along their chosen paths in time very close to \( C + D \), where \( D \) is the maximum chosen path length. When \( C \gg D \), the congestion becomes the dominant factor in the packet scheduling performance. Thus, smaller \( C \) immediately implies faster packet delivery time.

A natural problem that arises in games concerns the effect of the players’ selfishness on the welfare of the whole system measured with the social cost \( C \). We examine the consequence of the selfish behavior in pure Nash equilibria which are stable states of the game in which no player can unilaterally improve her situation. We quantify the effect of selfishness with the price of anarchy (PoA) [12], [18], which expresses how much larger is the worst social cost in a Nash equilibrium compared to the social cost in the optimal coordinated solution. The price of anarchy provides a measure for estimating how closely do Nash equilibria of bottleneck congestion games approximate the optimal \( C^{*} \) of the respective coordinated optimization problem.

Ideally, the price of anarchy should be small. However, the current literature results have only provided weak bounds...
A price of anarchy bound is a significant improvement over previous bounds on the price of anarchy in congestion games, which model the behavior of self-interested agents sharing a resource. In wireless networks, the social cost reflects the total delivery delay. In wireless networks, the overall delay of anarchy but also because they represent interesting and important real-life problems. In networks, the overall delay of anarchy but also because they represent interesting and important real-life problems.

## A. Contributions

The lower bound in [3] suggests that in order to obtain better price of anarchy in bottleneck congestion games (where the social cost is the bottleneck resource $C$), we need to consider alternative player utility cost functions. Towards this goal, we introduce polynomial bottleneck games where the player cost functions are polynomial expressions of the congestions along the resources. In particular, the player cost function for player $i$ is: $c_i = \sum_{r \in S_i} c_i^r$, for some integer constant $M \geq 1$. Note that the new utility cost is a sum of polynomial terms on the congestion of the resources in the chosen strategy (instead of the max that we described earlier). The social cost remains the maximum bottleneck congestion $C$, the same as in typical congestion games.

The new player utility costs have significant benefits in improving both the upper and lower bounds on the price of anarchy. For the bottleneck social cost $C$ we prove that the price of anarchy of polynomial games is:

$$\text{PoA} = O(|R|^{1/(M+1)}),$$

for any constant $M \geq 1$. We show that this bound is asymptotically tight by providing an instance of a polynomial bottleneck game with $\text{PoA} = \Omega(|R|^{1/(M+1)})$, for any constant $M \geq 1$. Our price of anarchy bound is a significant improvement over the price of anarchy from the typical bottleneck games described above.

Polynomial congestion games are interesting variations of bottleneck games not only because they provide good price of anarchy but also because they represent interesting and important real-life problems. In networks, the overall delay that a packet experiences is directly related with the link congestions along the path and hence the polynomial utility cost function reflects the total delivery delay. In wireless networks, the polynomial player utilities correspond to the total energy that a packet consumes while it traverses the network, and the social cost reflects to the worst energy utilization in any node in the network. Similar benefits from polynomial congestion games appear in the context of job-shop scheduling, where computational tasks require resources to execute. In this context, the social bottleneck cost function $C$ represents the task load-balancing efficiency of the resources, and the player utility costs relate to the makespan of the task schedule. In all the above problems, the polynomial degree $M$ is chosen appropriately to model precisely the involved costs of the resource utilization in each computational environment.

In our analysis, we obtain the price of anarchy upper bound by using two techniques: transformation and expansion. Consider a game $G$ with a Nash equilibrium $S$ and congestion $C$. We identify two kinds of players in $S$: type-A players which use only one resource in their strategies, and type-B players which use two or more resources. In our first technique, transformation, we convert $G$ to a simpler game $\tilde{G}$, having a Nash equilibrium $\tilde{S}$ with congestion $\tilde{C}$, such that $\tilde{C} = O(C)$, and all players in $\tilde{S}$ with congestion above a threshold $\tau$ are of type-A; that is, we transform type-B players to type-A players. Having type-A players is easier to bound the price of anarchy. Then, we use a second technique, expansion, which is used to give an upper bound on the price of anarchy of game $\tilde{G}$, which implies an upper bound on the price of anarchy of the original game $G$.

In [10], we have derived upper bounds for the price of anarchy of games with exponential utility cost functions using similar techniques (transformation and expansion). While exponential cost games have a unique substructure which makes the analysis of Price of Anarchy much simpler, we believe these techniques are general enough to adapt in a non-trivial manner for a large class of utility cost functions. For the case of exponential cost games, we obtained logarithmic price of anarchy upper bounds, which was related to the problem structure. Here we obtain tight (optimal) price of anarchy bounds for polynomial bottleneck games using a non-trivial application of the general transformation and expansion techniques.

## B. Related Work

Congestion games were introduced and studied in [17], [19]. In [19], Rosenthal proves that congestion games have always pure Nash equilibria. Koutsoupias and Papadimitriou [12] introduced the notion of price of anarchy in the specific parallel link networks model in which they provide the bound $\text{PoA} = 3/2$. Roughgarden and Tardos [22] provided the first result for splittable flows in general networks in which they showed that $\text{PoA} \leq 4/3$ for a player cost which reflects to the sum of congestions of the resources of a path. Pure equilibria with atomic flow have been studied in [3], [4], [14], [24] (our work fits into this category), and with splittable flow in [20], [21], [23], [24]. Mixed equilibria with atomic flow have been studied in [6], [8], [11], [12], [15], [16], [18], and with splittable flow in [5], [7].

Most of the work in the literature uses a cost metric related to the sum of congestions of all the resources of the player’s path [4], [21], [23], [24]. In terms of our notation, the player cost functions are polynomials of degree $M = 1$. However, the social cost in those games is different than ours since it is an aggregate function of the player flows and congestion of all the resources. On the other hand, the social cost in our
case corresponds to the bottleneck congestion which is a metric that reduces to a single resource. The vast majority of the work on congestion games has been performed for parallel link networks, with only a few exceptions on network topologies. Our work immediately applies to network topologies.

In [1], the authors consider bottleneck routing games in networks with player cost $C_i$ and social cost $C$. They prove that the price of stability is 1 (the price of stability measures the ratio of the best Nash equilibrium social cost versus the coordinated optimal solution). They show that the price of anarchy is bounded by $O(L + \log |V|)$, where $L$ is the maximum allowed path length, and $V$ is the set of nodes. They also prove that $\kappa \leq \text{PoA} \leq c(\kappa^2 + \log^2|V|)$, where $\kappa$ is the size of the largest resource-simple cycle in the graph and $c$ is a constant. That work was extended in [2] to the $C + D$ routing problem. Bottleneck routing games have also been studied in [3], where the authors consider the maximum congestion metric in general networks with splittable and atomic flow (but without considering path lengths). They prove the existence and non-uniqueness of equilibria in both the splittable and atomic flow models. They show that finding the best Nash equilibrium that minimizes the social cost is an NP-hard problem. Further, they show that the price of anarchy may be unbounded for specific resource congestion functions. In [9], the authors prove the existence of strong Nash equilibria (which concern coalitions of players) for games with the lexicographic improvement property; such games include Bottleneck congestion games and our polynomial games.

Outline of Paper

In Section II we give basic definitions. In Section III we convert games with type-B players to games with type-A players. In Section IV we give a bound on the price of anarchy. We finish with providing a lower bound in Section V.

II. Definitions

A congestion game is a strategic game $G = (\Pi_G, R, S, (d_r)_{r \in R, (pc_\pi)_{\pi \in \Pi_G}})$ where:

- $\Pi_G = \{\pi_1, \ldots, \pi_n\}$ is a non-empty and finite set of players.
- $R = \{r_1, \ldots, r_z\}$ is a non-empty and finite set of resources.
- $S = S_{\pi_1} \times S_{\pi_2} \times \cdots \times S_{\pi_n}$, where $S_{\pi_i}$ is a strategy set for player $\pi_i$, such that $S_{\pi_i} \subseteq \text{powerset}(R)$; namely, each strategy $S_{\pi_i} \in S_{\pi_i}$ is pure, and it is a collection of resources. A game state (or pure strategy profile) is any $S \in S$. We consider finite games which have finite $S$ (finite number of states).
- In any game state $S$, each resource $r \in R$ has a delay cost denoted $d_r(S)$.
- In any game state $S$, each player $\pi \in \Pi_G$ has a player cost $pc_\pi(S) = \sum_{r \in S_{\pi_i}} d_r(S)$.

Consider a game $G$ with a state $S = (S_{\pi_1}, \ldots, S_{\pi_n})$. The (congestion) of a resource $r$ is defined as $C_r(S) = |\{\pi_i : r \in S_{\pi_i}\}|$, which is the number of players that use $r$ in state $S$. The (bottleneck) congestion of a set of resources $Q \subseteq R$ is defined as $C_Q(S) = \max_{r \in Q} C_r(S)$, which is the maximum congestion over all resources in $Q$. The (bottleneck) congestion of state $S$ is denoted $C(S) = C_R(S)$, which is the maximum congestion over all resources in $R$. The length of state $S$ is defined to be $L(S) = \max_{\pi} |S_{\pi_i}|$, namely, the maximum number of resources used in any player. When the context is clear, we will drop the dependence on $S$. We examine polynomial congestion games:

- **Polynomial games**: The delay cost function for any resource $r$ is $d_r = C^M_r$, for some integer constant $M \geq 1$.

For any state $S$, we use the standard notation $S = (S_{\pi_1}, S_{-\pi_1})$ to emphasize the dependence on player $\pi_1$. Player $\pi_1$ is locally optimal (or stable) in state $S$ if $pc_{\pi_1}(S) \leq pc_{\pi_1}((S'_{\pi_1}, S_{-\pi_1}))$ for all strategies $S'_{\pi_1} \in S_{\pi_1}$. A greedy move by a player $\pi_i$ is any change of its strategy from $S'_{\pi_i}$ to $S_{\pi_i}$ which improves the player’s cost, that is, $pc_{\pi_1}((S_{\pi_i}, S_{-\pi_i})) < pc_{\pi_1}((S'_{\pi_i}, S_{-\pi_i}))$. Best response dynamics are sequences of greedy moves by players. A state $S$ is in a Nash Equilibrium if every player is locally optimal. Nash Equilibria quantify the notion of a stable selfish outcome. In the games that we study there could exist multiple Nash Equilibria.

For any game $G$ and state $S$, we will consider a social cost (or global cost) which is simply the bottleneck congestion $C(S)$. A state $S^*$ is called optimal if it has minimum attainable social cost: for any other state $S$, $C(S^*) \leq C(S)$. We will denote $C^* = C(S^*)$. We quantify the quality of the states which are Nash Equilibria with the price of anarchy (PoA) (sometimes referred to as the coordination ratio). Let $P$ denote the set of distinct Nash Equilibria. Then the price of anarchy of game $G$ is:

$$\text{PoA}(G) = \sup_{S \in P} \frac{C(S)}{C^*}.$$  

We continue with some more special definitions that we use in the proofs. Consider a game $G$ with a socially optimal state $S^* = (S^*_{\pi_1}, \ldots, S^*_{\pi_n})$, and let $S = (S_{\pi_1}, \ldots, S_{\pi_n})$ denote the equilibrium state. We consider two special kinds of players with respect to states $S$ and $S^*$:

- **Type-A players**: any player $\pi_i$ with $|S_{\pi_i}| = 1$.
- **Type-B players**: any player $\pi_i$ with $|S_{\pi_i}| \geq 2$.

For any resource $r \in R$, we will let $\Pi_r$ and $\Pi_r^*$ denote the set of players with $r$ in their equilibrium and socially optimal strategies respectively, i.e $\Pi_r = \{\pi_i \in \Pi_G | r \in S^*_{\pi_i}\}$ and $\Pi_r^* = \{\pi_i \in \Pi_G | r \in S^*_{\pi_i}\}$.

Let $G = (\Pi_G, R, S, (d_r)_{r \in R, (pc_\pi)_{\pi \in \Pi_G}})$ and $\tilde{G} = (\Pi_{\tilde{G}}, \tilde{R}, \tilde{S}, (\tilde{d}_r)_{r \in \Pi_{\tilde{G}}})$ be two games. We say that $\tilde{G}$ dominates $G$ if the following conditions hold between them for the highest cost Nash equilibrium and optimal states and:

$$|\tilde{R}| \leq |R|, \quad d = \tilde{d}, \quad C = C, \quad C^* \leq \tilde{C}^* \leq \beta C^*, \quad \text{where } \beta > 1$$

is a constant and $C, C^*$ and $\tilde{C}, \tilde{C}^*$ represent the bottleneck congestion.
congestions in the highest cost Nash equilibrium and optimal states of G and \(G'\), respectively.

**Corollary 2.1:** \(\text{PoA}(G) \leq \beta \cdot \text{PoA}(G')\) for an arbitrary game G and dominated game \(G'\).

In the next section, we will describe how an arbitrary game G in Nash equilibrium state S can be transformed into a dominated game \(G'\) containing type A players of arbitrary cost and type B players restricted to costs below a given threshold.

### III. TYPE-B TO TYPE-A GAME TRANSFORMATION

We first state our main results in this section.

**Theorem 3.1:** Every game G with highest-cost Nash equilibrium state S can be transformed into a game \(G'\) with Nash equilibrium state \(\tilde{S}\) in which all resources \(r\) with congestion \(C_r > \psi = \max(2M, 3C^*)\) are occupied exclusively by type-A players.

**Theorem 3.2:** \(\tilde{G}\) is dominated by G, i.e. the bottleneck congestion in optimal states \(S^*\) and \(\tilde{S}^*\) of G and \(G\) satisfies \(C^* \leq \tilde{C}^* \leq 7C^*\).

We prove Theorem 3.1 by constructing \(\tilde{G}\) via the transformation algorithm below and defer the proof of the domination of \(G\) for later. We first describe some needed preliminaries.

**Preliminaries:** We initialize \(\tilde{G}\), the input to our transformation algorithm as a restricted version of game G with exactly two strategies per player: \(\tilde{S}_\pi = S_\pi\) and \(\tilde{S}^*_\pi = S^*_\pi\). We iteratively transform \(G\) by converting type-B players of cost at least \(T = \psi^M + 1\) into type-A players, one at a time in decreasing order of player costs until all type-B players remaining either fall below the threshold cost function T or no type-B players exist. We add and delete players/resources from \(\tilde{G}\) iteratively and have a working set of players. However \(\tilde{G}\) will always remain in equilibrium state \(\tilde{S}\) at every step of the transformation process. When we add a new player \(\pi_k\) to \(\tilde{G}\) we will assign two strategy sets to \(\pi_k\): an ‘equilibrium’ strategy \(\tilde{S}_\pi_k\) and an optimal strategy \(\tilde{S}^*_\pi_k\). Thus \(\tilde{S} = \tilde{S}_\pi \cup \tilde{S}^*_\pi\) and \(\tilde{S}^* = S^*_\pi \cup S^*_\pi_k\).

We then convert \(\tilde{G}\) into a ‘clean’ version in which every type-B player \(\pi \in \tilde{G}\) has distinct resources in its equilibrium and optimal strategies i.e \(\tilde{S}_\pi \cap \tilde{S}^*_\pi = \emptyset\). If not already true, this can be achieved by creating \(\tilde{S}_\pi \cup \tilde{S}^*_\pi\) new type-A players with identical and one type-B player with disjoint equilibrium and optimal strategies for each original player \(\pi\). The new type-B player has \(\tilde{S}_\pi - \tilde{S}^*_\pi\) and \(\tilde{S}^*_\pi - \tilde{S}_\pi\) as its equilibrium and optimal strategy respectively while the new type-A players each use one resource from \(\tilde{S}_\pi \cup \tilde{S}^*_\pi\) as their identical equilibrium and optimal strategies. Note that the new players are also in equilibrium in \(\tilde{S}\). We also assume throughout that type-A players in \(\tilde{G}\) have no redundant resources in their optimal strategies, i.e if a resource can be removed from \(\tilde{S}^*_\pi\) without affecting \(pc_{\tilde{G}}(\tilde{S}_\pi) \leq pc_{\tilde{G}}(\tilde{S}^*_\pi)\), then it is removed.

Let \(\pi_i\) be an arbitrary type-B player using \(k\) resources \(r_1, r_2, \ldots, r_k\) in its equilibrium strategy \(\tilde{S}_\pi_i\) and distinct from the \(m\) resources \(r'_1, \ldots, r'_m\) in its optimal strategy \(\tilde{S}^*_\pi_i\). Let \(C_{r_j}, C_{r'_j}\) denote the congestion on these resources in Nash equilibrium state \(\tilde{S}\). Define procedure PMS—\(\text{Partition}(\pi_i)\) as follows:

**Procedure 1:** Partition \(\tilde{S}, \pi_i\) into t pairs \((L_1, L_1), (L_2, L_2), \ldots, (L_t, L_t)\) where

1. The \(L_j\)'s form a disjoint resource partition of \(\tilde{S}\).
2. \(L_j \subseteq S^*_\pi_i\) and \(|L_j \cap L_k| \leq 1\) for \(1 \leq j, k \leq t\).
3. \[\sum_{r \in L_j}(C_r + 1)^M \geq \sum_{r \in L_j} C_r^M, \quad 1 \leq j \leq t\] (1)

Without loss of generality, assume the resources in \(\tilde{S}, \pi_i\) have been sorted in decreasing order of congestion and vice versa for resources in \(S^*_\pi_i\), i.e \(C_{r_1} \geq C_{r_2} \geq \cdots \geq C_{r_m}\) and \(C_{r_j} \leq C_{r_j} \leq \cdots \leq C_{r_{m_j}}\). Then we have the following:

**Lemma 3.3:** There exists an implementation of PMS—\(\text{Partition}(\pi_i)\) in which

1. The \(L_j\)'s, \(1 \leq j \leq t\), form a linear partition of \(S^*_\pi_i\) into contiguous resources with \(|L_j \cap L_{j+1}| \leq 1\). If \(|L_j \cap L_{j+1}| = 1\) then the last resource in \(L_j\) is the first resource in \(L_{j+1}\).
2. \(\forall j: 1 \leq j \leq t\), either \(|L_j| = 1\) or \(|L_j| = 1\) or both. If \(|L_j| > 1\) and \(|L_j| = 1\) with \(L_j = \{r_p\}\) we have \(C_{r_p} \geq \max(C_r|r \in L_j)\).

**Proof:** We provide a simple proof sketch due to space limitations. Start with \(L_1 = \{r_1\}\). We add resources \(r'_1, \ldots, r'_q\) to \(L_1\) where \(r'_q\) is the first resource such that \(\sum_{j=1}^q (C_{r'_j} + 1)^M \geq C_{r'_q}^M\). Then we proceed with \(L_2 = \{r_2\}\) and start forming \(L_3\) with \(r'_q\). As we continue this process, due to the fact that

\[\sum_{j=1}^m (C_{r'_j} + 1)^M \geq \sum_{i=1}^k C_{r_i}^M,\] (2)

eventually resources in L will have smaller congestion than the resources in \(L^*\). At this point the \(L^*\) partitions will contain single resources while the corresponding L partition will contain multiple resources. At each step, we maintain the invariant in Equation 1 which implies condition 2 in the lemma. No resource in \(L^*_j\) need be used more than twice during this process, which is assured because of Eq. 2. We skip the remaining technical details of the proof which ensure that as many resources as possible from \(S^*_\pi_i\) are used in the partition-pairs.

Procedure PMS—\(\text{Partition}(\pi_i)\) is used to create new players and forms the basic step in our transformation algorithm. We ensure the equilibrium of these new players in \(G\) using the key constructs of exact matching sets and potential matching sets.

A set of resources \(\tilde{R}\) in \(\tilde{G}\) forms an exact matching set for a newly created player \(\pi_k\) with newly assigned equilibrium strategy \(\tilde{S}_{\pi_k}\) if \(\sum_{r \in \tilde{R}} (C_r + 1)^M \geq pc_{\tilde{G}}(\tilde{S}_{\pi_k}, \tilde{S}_{\pi_k}) = \sum_{r \in \tilde{S}_{\pi_k}} C_r^M\). Clearly, \(\tilde{R}\) can be assigned as the new optimal strategy \(\tilde{S}^*_{\pi_k}\) in game \(\tilde{G}\) without violating the equilibrium of \(\tilde{S}_{\pi_k}\).
Potential matching sets are defined for newly created type-$B$ players. A potential matching set $R$ is an exact matching set that can ‘potentially’ be added to the optimal set of resources $\tilde{S}_{\pi^*}$ of a type-$B$ player $\pi_k \in G$ without increasing the optimal bottleneck congestion in $G$ from original game $G$ by a constant factor i.e $C^* \leq \beta C^*$, where $\beta > 1$ is a constant.

Now consider a type-$B$ player $\pi_i$ to be transformed. We partition the resources in its equilibrium and optimal strategies $\tilde{S}_{\pi_i}$ and $S^*_{\pi_i}$ according to $\text{PMS-Partition}(\pi_i)$ and remove it from $\tilde{G}$, i.e $\tilde{S} = \tilde{S} - \tilde{S}_{\pi_i}$ and $S^* = S^* - S^*_{\pi_i}$.

Consider those partition-pairs $(L_j, L_j^*)$ with $|L_j| = 1$. We can create a new type-$A$ player $\pi_k$ and add it to to $\tilde{G}$ with an equilibrium strategy $\tilde{S}_{\pi_k}$ that is the singleton resource in $L_j$. Due to the condition in Eq. [1] the set of resources in $L_j^*$ forms an exact matching set for $\pi_k$ and can therefore be assigned to $\tilde{S}_{\pi_k}$. $\pi_k$ is in equilibrium in $\tilde{G}$ and the equilibrium and optimal congestion on resources in $\tilde{S}_{\pi_k}$ and $S^*_{\pi_k}$ are now the same as before. This forms the ‘easy’ part of the transformation process.

Consider however, those partitions $(L_j, L_j^*)$ with $1 < |L_j| \leq |R|$ and $L_j^* = \{r_j\}$. Similar to the above, we can create $|L_j|$ new type-$A$ players and assign a distinct resource in $L_j$ to each such players equilibrium strategy. However if, as above, we assign $r_j$, the single resource in $L_j^*$, to each players optimal strategy, we might increase the socially optimal congestion $C^*$ of $\tilde{G}$ to as much as $C^* + |R|$, thereby violating the domination of $G$ over $\tilde{G}$. Thus we need to find an appropriate potential matching set from among existing resources and assign them to these players, without increasing the optimal congestion beyond $O(C^*)$. Finding such a set is the ‘hard’ part of the transformation process and forms the core of our algorithm below.

We define a subroutine (PARTITION-TRANSFORM) that executes procedure $\text{PMS-Partition}(\pi_i)$ for a type-$B$ player $\pi_i$ and creates several new type-$A$ and type-$B$ players. Specifically, we first obtain $\text{PMS-Partition}(\pi_i) = \{(L_1, L_1^*), \ldots, (L_l, L_l^*)\}$. We then delete the strategies of $\pi_i$ from $\tilde{G}$ and transform $\pi_i$ into $l$ type-$A$ and type-$B$ sub-players as follows: for each partition member $L_q$, we create a new player $\pi_{L_q}$ which is either a type-$A$ player if $|L_q| = 1$, or a type-$B$ player if $|L_q| > 1$. $\pi_{L_q}$ is created with two strategy sets: equilibrium strategy $\tilde{S}_{\pi_{L_q}} = L_q$ and socially optimal strategy $S^*_{\pi_{L_q}} = L_q$.

The following lemma is a direct consequence of lemma [3] and is needed for later analysis.

**Lemma 3.4:** For a given type-$B$ player $\pi_i$, every new type-$B$ player $\pi_{L_q}$ created after an execution of subroutine PARTITION-TRANSFORM($\pi_i$) is in equilibrium in $\tilde{G}$ with $pc_{\pi_{L_q}} \leq (C_p + 1)^M \leq pc_{\pi_i}$, where $C_p = \max_{r \in S^*_{\pi_i}} r$. Every new type-$A$ player $\pi_{L_q}$ is also in equilibrium with $pc_{\pi_{L_q}} \leq pc_{\pi_i}$.

We are now ready to prove our main result.

**Proof of Theorem 3.7** We describe the transformation via an iterative algorithm for which the pseudocode is attached below.

The main challenge is to find potential matching sets for newly created type-$B$ players without increasing the optimal congestion in the game beyond a constant factor. To achieve this, we will transform type-$B$ players in distinct phases corresponding to decreasing ranges of player costs. As a preprocessing step in the algorithm we call PARTITION-TRANSFORM($\pi$) for all type-$B$ players $\pi$ with $pc_{\pi} > (C + 1)^M$. By lemma [3] we are now left only with players with cost $\leq (C + 1)^M$ in $\tilde{G}$.

**Algorithm 1** TRANSFORMATION ALGORITHM

1. **Preprocessing:**
2. $\forall$ Type-$B$ players $\pi$ with $pc_{\pi} > (C + 1)^M$
3. Execute PARTITION-TRANSFORM($\pi$)
4. **Main Procedure**
5. for $i = 1$ to $C + 1 - \psi$
6. Phase Index $\tilde{C} = C + 1 - i$
7. $\Pi_{\tilde{C}} \leftarrow \{\text{type-}B\pi_j \in \Pi | \tilde{C}^M < pc_{\pi_j}(\tilde{S}_{\pi_j}) \leq (\tilde{C} + 1)^M\}$
8. for all $\pi_j \in \Pi_{\tilde{C}}$
9. PARTITION-TRANSFORM($\pi_j$)
10. end for
11. $\Pi_{C^*} \leftarrow \Pi_{\tilde{C}} \cup \{\text{type-}A\ \text{players with cost} = \tilde{C}^M\}$
12. for all $\pi_i \in \Pi_{C^*}$
13. ELIMINATE-HIGH-CONGESTION-RESOURCES($\pi_i$)
14. end for
15. $D \leftarrow \{\pi_i\}$ where $1|\tilde{S}_{\pi_i}^*| > 1, 2) \max_{r \in S_{\pi_i}^*} \{C_r\} \leq \tilde{C} - 1$
16. and 3) $\sum_{r \in S_{\pi_i}^*} (C_r + 1)^M \geq (\tilde{C} + 1)^M$
17. for all $\pi_i \in D$
18. PARTITION-TRANSFORM($\pi_i$)
19. end for
20. $E \leftarrow \{\pi_i\}$ where $|\tilde{S}_{\pi_i}^*| = 1$ and $C_{\tilde{S}_{\pi_i}} = \tilde{C}$
21. $X \leftarrow \{r \in R|C_{r} = \tilde{C}\}$
22. while $|E| > 0$
23. Choose any $\pi_i \in E$ and select UNMARKED
24. type-$A$ player $\pi_j \in \Pi_X$ in round-robin fashion.
25. Update $\tilde{S}_{\pi_i} \leftarrow \tilde{S}_{\pi_i} \cup \tilde{S}_{\pi_j}$
26. Update $\tilde{S}_{\pi_i}^* \leftarrow \tilde{S}_{\pi_i}^* \cup \tilde{S}_{\pi_j}$ and MARK $\pi_j$
27. PARTITION-TRANSFORM($\pi_i$) and add the resultant new player to $E$ if qualified
28. end while
29. end for

Let $\tilde{C} = C + 1 - i$ denote a (decreasing) phase index. During the $i^{th}$ phase, $1 \leq i \leq C + 1 - \psi$, we transform all type-$B$ players with player costs $\tilde{C}^M < pc_{\pi} \leq (\tilde{C} + 1)^M$ into either type-$A$ players or type-$B$ players of cost $\leq \tilde{C}^M$. We will use the fact that in phase $i$, all resources with congestion $\geq \tilde{C} + 1$ in equilibrium state $\tilde{S}$ are occupied only by type-$A$ players (since any type-$B$ player using a resource $r$ with $C_r \geq \tilde{C} + 1$ would
have a player cost strictly \(>(\bar{C}+1)^M\), a contradiction).

In phase \(i\), let \(\Pi_C\) denote the set of type-\(B\) players \(\pi_i\) whose player costs are in the range \(\bar{C}^M < pc_{\pi_i} \leq (\bar{C}+1)^M\). We first call on PARTITION-TRANSFORM for all players in \(\Pi_C\). This results in a new set of type-\(A\) and type-\(B\) players with the same or lower player costs. New type-\(B\) players with cost \(\leq \bar{C}^M\), are dealt with in subsequent phases while we form \(\Pi_C\) again with the remaining type-\(B\) players. At this point, every type-\(B\) player \(\pi_i \in \Pi_C\) has exactly one resource in its optimal strategy (by definition of lemma \(3.3\)). Moreover, this resource must have congestion \(C_r \geq \bar{C}\) in equilibrium state \(\bar{S}\), since \(pc_{\pi_i}(\bar{S}) > \bar{C}^M\).

As discussed before, the key challenge is to find a larger potential matching set for such type-\(B\) players without increasing \(\bar{C}\) significantly. For technical reasons, we first eliminate all high-congested resources from consideration as potential matching sets. In particular, resources with equilibrium congestion \(C_r \geq \bar{C} + 1\) are occupied only by type-\(A\) players. By eliminating these resources (using subroutine ELIMINATE-HIGH-CONGESTION-RESOURCES(\(\pi_i\)) described below), we ensure that the optimal congestion \(\bar{C}_r\) on any resource \(r\) with equilibrium congestion \(C_r\) remains unchanged during all phases with phase index \(\bar{C} < \bar{C}_r\).

Let \(\pi_i\) denote a generic player from the set of type-\(B\) players in \(\Pi_C\) and the set of type-\(A\) players in \(\bar{G}\) with player cost exactly \(\bar{C}^M\) and a single resource in their optimal strategy sets. Let \(\hat{S}_{\pi_i} = \{x\}\). We check to see if \(C_x \geq \bar{C} + 1\). If so, the type-\(A\) player \(\pi_q \in \Pi_x\) (recall that \(\Pi_x\) is the set of players using \(x\) in equilibrium) with the largest socially optimal strategy set \(\hat{S}_{\pi_q}\). Let \(F = \arg\min_{C_r \geq \bar{C}}\{r \in \hat{S}_{\pi_q}\}\), i.e the resource in \(\hat{S}_{\pi_q}\) with the smallest congestion \(\geq \bar{C}\). If \(F\) above does not exist, then set \(F = \hat{S}_{\pi_q}\). We now change the socially optimal strategy of \(\pi_1\) to \(F\) instead of \(x\). Since \(\sum_{y \in F}(C_y + 1)^M \geq (\bar{C} + 1)^M \geq pc_{\pi_i}\), we are assured that \(\pi_1\) will remain in equilibrium in \(\bar{S}\) after this. Simultaneously, change the optimal strategy of \(\pi_q\) to its equilibrium strategy i.e \(\hat{S}_{\pi_q} = \hat{S}_{\pi_i} = x\). Note that after this step, the equilibrium and optimal congestion on all resources involved, i.e \(x\) and \(F\), remain unchanged in \(\bar{G}\). If the resource \(x\) above has congestion \(C_x \geq \bar{C}\), we repeat the steps for player \(\pi_i\). We execute the subroutine for all such qualified players.

It must now be the case that the set of players \(\{\pi_i\}\) can be divided into two subsets \(D\) and \(E\), where \(D\) contains all the players with \(|\hat{S}_{\pi_i}| = 1\), \(\max_{r \in \hat{S}_{\pi_i}}\{C_r\} \leq \bar{C} - 1\) and \(\sum_{r \in \hat{S}_{\pi_i}}(C_r + 1)^M \geq (\bar{C} + 1)^M\) while \(E\) contains players with \(|\hat{S}_{\pi_i}| = 1\) and \(C_y = \bar{C}\), where \(\hat{S}_{\pi_i} = \{y\}\). We now execute PARTITION-TRANSFORM on all type-\(B\) players in the set \(D\). By lemma \(3.3\) the cost of a newly created type-\(B\) player after PARTITION-TRANSFORM on the set \(D\) can be at most \(\bar{C}^M\). These players will be further transformed in subsequent phases.

It now only remains to transform type-\(B\) players from the set \(E\) in this phase. Let \(X = \{r \in R|C_r = \bar{C}\}\) denote the set of resources with congestion exactly \(\bar{C}\) in equilibrium. Let \(\Pi_X\) denote the set of players using resources from \(X\) in equilibrium. From the above discussions, we note the following:

1) Every type-\(B\) and type-\(A\) player \(\pi_l \in E\) is using a resource from \(X\) in its optimal strategy, i.e \(\hat{S}_{\pi_l} = x\) for some \(x \in X\).
2) Every type-\(A\) player in \(D \cup E\) is using a resource from \(X\) in its equilibrium strategy.

Essentially every untransformed type-\(B\) player is using a resource in \(X\) in its optimal strategy. Let \(X^B \subseteq E\) denote the set of type-\(B\) players in \(\Pi_X\). Then \(X^A = \Pi_X \setminus X^B\), the remaining set of players in \(\Pi_X\), must be of type-\(A\). We can focus on the set \(X^A\) to obtain larger potential matching sets for type-\(B\) players in \(E\).

Order the resources in \(X\) as \(L = r_1, r_2, \ldots, r_k\), in increasing order of number of type-\(A\) players using them in equilibrium, where \(k \leq |X|\). For each type-\(B\) player \(\pi_l \in E\), we select an unmarked type-\(A\) player \(\pi_j\) using a resource, say \(r_m\) in \(L\), i.e \(\hat{S}_{\pi_l} = r_m\) where \(\hat{S}_{\pi_l} \cap \hat{S}_{\pi_j} = \emptyset\). Now define \(PM(S_{\pi_l}) = S^*_\pi_l \cup S^*_\pi_j\) as the new potential matching set for \(\pi_l\). Update the optimal strategy of \(\pi_l\) to \(S^*_\pi_l = PMS(\pi_l)\) and execute PARTITION-TRANSFORM(\(\pi_l\)). We claim that at least two new players are created, at most one of which could be a type-\(B\) player with cost \(> \bar{C}^M\). This is because \(\sum_{r \in S^*_\pi}(C_r + 1)^M \geq \bar{C}^M\) while every resource in \(S^*_\pi\) has congestion \(\leq \bar{C}\). Thus resources from \(S^*_\pi\) will be in at least one partition pair of \(PMS – Partition(\pi_l)\) while the (existing) resource in \(S^*_\pi\) will be in another partition pair. If after \(PMS – Partition(\pi_l)\) there is still a type-\(B\) player with cost \(> \bar{C}^M\) add this player to the set \(E\). Simultaneously update the optimal strategy of \(\pi_l\) to \(S^*_\pi_l = r_m\) and mark \(\pi_j\). (Note that this increases the optimal congestion on \(r_m\) by one. We will bound the total increase in optimal congestion later). Also note that all players are in equilibrium in \(\bar{S}\). We repeat the process as long as there exist type-\(B\) players in set \(E\).

In order to show that this is a valid transformation, we need to show that there exist a sufficient number of unmarked type-\(A\) players in \(X\). We do this by a simple counting argument. Let \(\alpha = \lfloor e^{\bar{M}} \rfloor\). First note that each type-\(B\) player in \(X^B\) can be using at most \(|(\bar{C} + 1)^M/\bar{C}^M| \leq \alpha\) resources in \(X\) in equilibrium state \(\bar{S}\) and hence \(|X^A| \geq |X| - (\bar{C} - \alpha)|X^B|\). Secondly, since each resource in \(X\) can be in the optimal strategy of at most \(\alpha\) type-\(A\) players and every type-\(B\) player in \(X^A\) at the start of the transformation process has its optimal strategy in \(X\), we must have \(|X^B| \leq \alpha^{|X|}\). Finally, noting that each type-\(B\) player in \(E\) at the start of the transformation process can make at most \(2\alpha\) calls for marking type-\(A\) players before it is completely transformed we get the total number of type-\(A\) players in \(X\) required for marking as \(\leq 2\alpha\alpha^{|X|}\). Using the given threshold value \(\psi = \max(2M,3\alpha)\), we
obtain the number of type-A players in $X$ as
\[ |X^A| \geq |X| \cdot \tilde{C} - |X^B| \geq |X|((\tilde{C} - C^*) \geq 2C^*|X|) \] (3)
for $\tilde{C} > \psi \geq 3C^*$ which proves the result.

**Proof of Theorem 3.2** First note that the optimal congestion $\tilde{C}_r^*$ on a resource $r$ does not change in any phase with phase index $\tilde{C} < C_r$. There are only two occasions when $C_r^*$ increases:

1) During the phase with index $\tilde{C} = C_r$, $\tilde{C}_r^*$ increases by one whenever a type-A player on $r$ is marked. The number of resources in $X$ that contain type-A players to be marked is $|R^A| \geq |X^A|/\tilde{C} \geq |X|((\tilde{C} - C^*)/\tilde{C})$. In order to bound the increase in $\tilde{C}_r^*$, we will select (and mark) type-A players from $R^A$ during each step of the transformation in cyclic round-robin fashion, i.e. after marking $r_m$, we select a type-A player from $r_{m+1}$ etc. Since at most $2C^*|X|$ players are required to be marked, the maximum number of marked players on any resource in $X$ is $\leq 2C^*|X|/|R^A|$ which is bounded by
\[ \frac{2\tilde{C}C^*}{C - C^*} \leq \frac{2\psi C^*}{\psi - C^*} \leq 3C^* \]
Hence for any resource $r$ optimal congestion $\tilde{C}_r^*$ increases by at most $3C^*$ in phase index $C_r$ due to marked players.

2) At the beginning of the transformation process, $r$ can be in the optimal strategy set, and consequently partition pairs of up to $C^*$ players. In the discussion below, focus on a particular player in this set and consider the increase in $\tilde{C}_r^*$ throughout the transformation process only due to $r$’s presence in some partition pair $L^*_k$ due to this particular player. $r$ via $L^*_k$ can be involved in multiple calls to PMS-Partition() in multiple phases with phase index $> C_r$. In any such call to PMS-Partition(), $\tilde{C}_r$ can increase by 1 only if $r$ is either the first or last resource in a new partition pair $L^*_{k'} \subset L^*_k$. Once $r$ is the first or last resource in a partition $L^*_k$, its $\tilde{C}_r^*$ can increase by at most one when $r$ become part of a new singleton partition pair (i.e $L^*_k = \{r\}$). From this point onwards, $\tilde{C}_r^*$ cannot increase due to further PMS-Partition() calls. Thus the total increase in $\tilde{C}_r^*$ is bounded by 3. Given that $r$ can be in up to $C^*$ optimal strategy sets at the start of the first phase, the total increase in $\tilde{C}_r^*$ due to calls to PMS-Partition() is bounded by $3C^*$.

Putting the two facts above together, consequently, we get $\tilde{C}^* \leq C^* + 3C^* + 3C^* = 7C^*$ and hence $G$ is dominated by $G$. \[ \Box \]

**IV. Price of Anarchy**

**A. Price of Anarchy for Type-A Player Games**

We now consider equilibria where highly congested resources are occupied only by type-A players and use this to bound the price of anarchy of games with polynomial cost functions. Consider a game with optimal solution $S^* = (S^*_1, \ldots, S^*_\pi)$ and congestion $C^*$. Let $\psi = \max(2M, 3C^*)$ be a threshold value. Let $S = (S^*_1, \ldots, S^*_\pi)$ denote the Nash equilibrium state which has the highest congestion $C$ among all Nash equilibria states, and further all players on resources $r$ with $C_r > \psi$ are of type-A. We will obtain a price of anarchy result by bounding the ratio $C/C^*$.

We first define a resource graph $N$ for state $S$. There are $V = V_1 \cup V_2$ nodes in $N$. Each resource $r \in R$ with $C_r > \psi$ ($C_r \leq \psi$, resp.) corresponds to the equivalent node $r \in V_1$ ($r \in V_2$). Henceforth we will use the term resource and node interchangeably. For every player $\pi$ using a resource $x \in V_1$ in equilibrium, there is a directed edge $(x, y)$ between node $x$ and all nodes $y \in V$, where $y \neq x$ is in the optimal strategy set of $\pi$, i.e. $S_\pi = x$ and $y \in S_2$. The set of nodes in $\bigcup_{\pi | \pi \succ \psi} S^*_\pi$ are called the children of node $x$. We use the notation $\text{Ch}(x)$ to denote this set. Note that there could be multiple links directed at $x$ from the same node, however $x$ can be the child of at most $C^*$ nodes and $x$ cannot be its own child. Also note that nodes in $V_2$ are terminal nodes that have no outgoing links.

We first observe the following about nodes in $V_1$:

**Lemma 4.1:** For any node $x \in N$ with $C_x > \psi$, it holds that
\[ \sum_{y \in \text{Ch}(x) \cap V_1} C^*_y + \sum_{y \in \text{Ch}(x) \cap V_2} \psi^M \geq \frac{C_x - C^*}{2C^*} C^*_x \]

**Proof:** Let $\Pi$ be the set of players such that $\forall \pi \in \Pi : S^*_\pi = x$ and $x \notin S^*_\pi$. We must have $C_x - C^* \leq |\Pi| \leq C_x$ since up to $C^*$ players could be using resource $x$ simultaneously in their optimal as well as equilibrium strategies. Since $\pi$ is in equilibrium state, we must have $\sum_{y \in S^*_\pi} (C_y + 1)^M \geq C^*_x$. Let $Z_\pi = S^*_\pi \cap V_1$ and $W_\pi = S^*_\pi \cap V_2$. Using the fact that $\forall z \in Z_\pi : C_z > \psi$ and $C^* \geq 1$, we get that $((C_z + 1)/C^*_z)^M \leq (1 + \frac{1}{C^*}) \leq \sqrt{e} < 2$. Also $\forall w \in W : C^*_w + 1 \leq \psi + 1 < 2\psi$ and hence $\sum_{z \in Z_\pi} 2(C^*_z)^M + \sum_{w \in W} \psi^M \geq C^*_x/2$. Now consider the sum
\[ \sum_{\pi \in \Pi} \left( \sum_{z \in Z_\pi} C^*_z + \sum_{w \in W} \psi^M \right) \]
Since $|\Pi| \geq C_x - C^*$ and each resource $z$ and $w$ in the inside term above can be in the sets $Z_\pi$ and $W_\pi$ for up to $C^*$ players from $\Pi$, we get
\[ \sum_{y \in \text{Ch}(x) \cap V_1} C^*_y + \sum_{y \in \text{Ch}(x) \cap V_2} \psi^M \geq \frac{C_x - C^*}{2C^*} C^*_x \] (4)
as desired. \[ \Box \]

To get a bound on the price of anarchy we need to relate the number of resources $|R|$ with the parameters $C$ and $C^*$. Note that the second term on the LHS of Eq. 4 is bounded by $|R|\psi^M$. Unfortunately, since $N$ has cycles, we cannot apply the lemma recursively to nodes from the first term on the LHS and their children in $N$ (to eventually replace these nodes with
nodes from $V_2$). We will therefore modify $\mathcal{N}$ to eliminate cycles and construct an expansion Directed Acyclic Graph (DAG) $T$ (without increasing the size of $\mathcal{N}$), which will help us obtain our price of anarchy bound. This is stated in the form of the lemma below (We omit the proof due to space considerations).

**Lemma 4.2:** Resource graph $\mathcal{N}$ can be transformed into expansion DAG $T$ without affecting the equilibrium state $\mathcal{S}$ and optimal congestion $C^*$, where $|T| \leq |\mathcal{N}|$.

Since $T$ is a DAG we know that it has sink nodes (with outdegree 0). Every node in $V_1$ is an internal node (with non-zero indegree and outdegree) since it has congestion $> C^*$ and hence the sink nodes in $T$ are nodes from $V_2$. Consider the DAG starting at the root node with congestion $C$. By applying lemma 4.1 recursively to the root and its descendants in $V_1$, we count the number of nodes until we reach terminating sinks in $V_2$. Noting as before that each resource in $T$ is counted at most $C^*$ times in the lemma, we get the result

**Lemma 4.3:** For DAG $T$ with root node $r$ and congestion $C_r = C$, it holds that

$$\sum_{y \in \text{Descendants}(r) \cap V_2} \psi^M \geq \frac{C - C^*}{2C^*} C^M$$

**Theorem 4.4 (Price of Anarchy for Type-A players):** The upper bound on the price of anarchy is $\text{PoA} = O(|R|^{M+1})$.

**Proof:** The number of descendants of $r$ in $T$ is at most $|R| - 1$. Using the fact that $\psi \geq 3C^*$ and substituting in lemma 4.3 we get that

$$|R| - 1 \geq (\text{PoA} - 1) \cdot \frac{\text{PoA}^M}{2\cdot 3^M} > \frac{\text{PoA}^M}{4\cdot 3^M}$$

for any $\text{PoA} > 2$, and hence for the given constant $M$, we get the desired result $\text{PoA} = O(|R|^{M+1})$.

**B. Price of Anarchy for Arbitrary Games**

By Theorem 4.1 we only need to consider games in equilibrium with type-A players occupying resources with congestion $> \psi$. By combining Theorem 4.1 Theorem 4.4 and Corollary 2.1 we obtain the main result for price of anarchy:

**Theorem 4.5 (PoA for Arbitrary Polynomial Cost Games):** The upper bound on the price of anarchy for polynomial cost games is $O(|R|^{M+1})$.

**V. LOWER BOUND**

We show that the upper bound of $O(|R|^{1/(M+1)})$ in the price of anarchy is tight by demonstrating a congestion game with a lower bound on the price of anarchy of $\Omega(|R|^{1/(M+1)})$. We construct a game instance represented as a graph in the figure below, such that each edge in the graph corresponds to a resource, and each player $\pi_i$ has two strategies available: either the path from $u$ to $v$ through the direct edge $e = (u, v)$, or an alternative path $p_i = (u, x_1, \ldots, y_1, v)$ (note that different player paths are edge-disjoint). Each path has length $|p_i| = |R|^{M/(M+1)}$ edges and the number of players is $n = |R|^{1/(M+1)}$. (For simplicity assume that the values of $|p_i|$ and $n$ are integers, since if they are not we can always round to the nearest ceiling.) The edge $e$ is actually part of one of the paths; for ease of presentation, in the figure below the edge is depicted as separate from the paths.

Let $S$ be the state depicted on the left part of the figure, where each player chooses the first strategy, and let $S^*$ be the state on the right part of the figure, where each player chooses the alternative path. We have that $C(S^*) = 1$, which is the smallest congestion possible. Thus, $S^*$ represents a socially optimal solution. For state $S$ we have that $C(S) = n$, since all players use edge $(u, v)$. Note that $S$ is a Nash Equilibrium, since each player $\pi_i$ has cost $p_{\pi_i}(S) = nM = |R|^{M/(M+1)}$, and the cost of switching to path $p_i$ would be $1M \cdot |p_i| = |R|^{M/(M+1)}$, which is the same at the cost of using edge $(u, v)$. Consequently, a lower bound on the price of anarchy is $C(S)/C(S^*) = n/1 = |R|^{1/(M+1)}$. Therefore, $\text{PoA} = \Omega(|R|^{1/(M+1)})$, as needed.

**VI. CONCLUSIONS**

We have considered bottleneck congestion games with polynomial cost functions and shown that the Price of Anarchy is bounded by $O(|R|^{M+1})$. This Price of Anarchy result is optimal, as demonstrated by a game with this exact $\text{PoA}$. We also demonstrate two novel techniques, $B$ to $A$ player conversion and expansion which help us obtain this result. These techniques which enable us to simplify games for analysis are sufficiently general. In future work, we plan to use these techniques to analyze the PoA of games with arbitrary player cost functions.

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