Thermo-magneto coupling in a dipole plasma

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On a dipole plasma, we observe the generation of magnetic moment, as the movement of the levitating magnet-plasma compound, in response to electron-cyclotron heating and the increase of $\beta$ (magnetically-confined thermal energy). We formulate a thermodynamic model with interpreting heating as injection of microscopic magnetic moment; the corresponding chemical potential is the ambient magnetic field.

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The RT-1 device confines a high-temperature (electron temperature $T_e \sim 10$ keV) plasma in a dipole magnetic field that is generated by a levitating superconducting magnet [1-4]; see Fig. 1. When a high-beta (local $\beta \sim 0.7$) plasma is produced, we observe an appreciable amplitude of vertical motion of the levitating superconducting magnet. Interpreting this phenomenon form thermodynamic view point, we will delineate an interesting property of magnetized plasmas.

Let us start by analyzing mechanics. We denote by $z$ the vertical displacement of the magnet from the equilibrium position (we use $r$-$\theta$-$z$ cylindrical coordinates). From the time-series data of the coil position and the controlled current ($I_L$) in the lifting coil, we can estimate the change of forces acting on the levitating magnet-plasma compound: denoting by $M$ (= 112 kg) the mass of the magnet (the mass of the plasma is ignorable), the equation of motion in the vertical direction can be written as

$$M\frac{d^2z}{dt^2} = F_m - Mg. \quad (1)$$

On the right-hand side, $Mg$ is the gravity and $F_m$ is the magnetic force in the vertical direction:

$$F_m := \int e_z \cdot (J \times B_L) d^3x \approx -2\pi RI B_{L,r}. \quad (2)$$

where $B_L$ is the magnetic field applied by the lifting magnet ($B_{L,r}$ is its radial component), $J$ is the current density in the magnet-plasma compound, and $I$ is the total current in $\theta$-direction. Here we have approximated $J$ by a ring current of radius $R$. Invoking the conventional magnetic moment $M = \pi R^2 I$, we may write (using $\nabla \cdot B = r^{-1} \partial (r B_r) / \partial r + \partial B_z / \partial z = 0$ and $\partial B_r / \partial r \approx B_r / r$)

$$F_m = -2M \frac{B_{L,r}}{r} = M \frac{\partial B_{L,z}}{\partial z}. \quad (3)$$

We define $M$ to be positive and $B_{L,r}$ to be negative ($B_{L,z}$ and $\partial B_{L,z} / \partial z$ to be positive), and then, $F_m$ is positive (upward). We denote

$$\mathcal{G} := \frac{\partial B_{L,z}}{\partial z} \quad (4)$$

to write $F_m = M \mathcal{G}$. At the equilibrium point ($z = 0$), we define $M = M_0$ and $\mathcal{G} = \mathcal{G}_0$. The equilibrium condition reads as $M_0 \mathcal{G}_0 = Mg$.

While (3) is derived for the conventional magnetic moment of a loop current, we may use it to “define” the total magnetic moment of the magnet-plasma system. In what follows, we evaluate $\mathcal{G}$ at the barycenter of the levitating magnet, and define $M := F_m / \mathcal{G}$ by the total magnetic force $F_m$ on the levitating magnet-plasma compound. Linearizing (1) in the neighborhood of the equilibrium point ($z = 0$, $\mathcal{G} = \mathcal{G}_0$ and $M = M_0$), we obtain

$$M \frac{d^2z}{dt^2} = M_0 \frac{\partial \mathcal{G}}{\partial z} z + M_0 \frac{\partial \mathcal{G}}{\partial I_L} dI_L + \mathcal{G}_0 dM, \quad (5)$$

where $(\partial \mathcal{G} / \partial I_L) dI_L$ represents the variation of $\mathcal{G}$ due to a perturbation $dI_L$ in the lifting magnet (we define the sign of $I_L$ so that $dI_L > 0$ increases $B_{L,z}$ and $\mathcal{G}$) [5]. The inertial force on the left-hand-side of (5) can be estimated by the time-series data of the coil position. Evaluating the first and second

![FIG. 1: Schematic drawing of the RT-1 device. A dipole magnetic field is produced by the levitating superconducting magnet. The field strength in the confinement region varies from 0.5 T to 0.01 T. Plasma is produced and heated by ECH (8.25 GHz, 25 kW, and 2.45GHz, 20 kW systems).]
FIG. 2: Typical waveforms of (a) the vertical plasma position $z$ (measured by laser position sensors), (b) the lifting-magnet current $I_L$ (feedback controlled in response to the position signal), (c) (1) the inertial force, (2) magnetic force by perturbed $dI_L$, and (3) the remaining term of $\delta$ corresponding to the magnetic force by the perturbed magnetic moment $dM$, (d) the perturbed magnetic moment normalized by the magnetic moment $M_0$ of the superconducting magnet, and (e) the normalized plasma energy ($\beta$ averaged over the plasma volume) estimated by diamagnetic signals. ECH is injected for $0.5 < t < 1.5$ s.

The main theme of this brief communication, however, is not the practical application of the magnetic moment for measurement. Examining this phenomenon from a thermodynamic view point, we notice an interesting implication, and that is the subject of present practice. Injecting electron cyclotron heating (ECH) power, we increase the internal (thermal) energy of the plasma (electrons). In the language of thermodynamics, giving a heat $\delta Q$ causes a change $\delta U$ of the internal energy $U$ (here we denote a general variation by $\delta X$, while a variation of a state variable $Y$ is written as $dY$); the energy $U$ is the combination of the thermal, mechanical, gravitational and electromagnetic energies, and $\delta Q$, in general, may cause variations in every component of the energy, resulting in changes in macroscopic quantities including those mechanical (vertical velocity), gravitational (vertical position), and electromagnetic (magnetic moment). Writing the first law as

$$dU = \delta W + \delta Q,$$

the term $\delta W$ represents whole such contributions from macroscopic quantities to the energy balance. In textbook thermodynamics, we often assume that $\delta W = -PdV$ with a pressure $P$ and volume $V$, and then, the coupling of the thermodynamic energy and the macroscopic mechanical energy is only through compressible motion of fluid. Needless to say, possible processes are much more rich in a plasma.

As mentioned above, we observe that heating $\delta Q$ causes a change in the magnetic moment $M$ and subsequent changes in the vertical position and (feedback controlled) lifting-magnet current. To describe the “thermodynamics” of this system, we have to formulate the relations among $\delta Q$, $dM$, $dI_L$, and $dz$. Here we proffer a “grand-canonical model” to understand this thermo-magneto coupling. We do not intend to challenge the aforementioned elementary understanding in terms of the diamagnetic current. Instead, our new perspective will delineate an interesting property of a magnetized plasma in a more succinct picture.

The energy of a magnetic moment $M$ is at the core of the first law connecting the plasma, the magnet, the heating system, and the lifting system. When an external magnetic field...
\( B_L \) is applied, the magnetic moment \( \mathcal{M} \) has a mechanical potential energy [7]

\[
V_m = -MB_L,
\]

(7)

where \( B_L \) is the average of \( e_z \cdot B_L \) over the levitating magnet-plasma compound. As well known [8], the “total energy” of a magnetic moment, including the electric energies of the levitating and lifting currents, is \(-V_m\), but we must use \( V_m \) to derive mechanical forces and corresponding works. Combining \( V_m \) with the gravitational energy \( U_g = \mathcal{M}g_z \) and the kinetic energy \( U_k = p^2/(2M) \) \( (p_z = Mdz/dt \) is the momentum), we obtain a Hamiltonian

\[
H := U_k + U_g + V_m.
\]

(8)

The corresponding Hamilton’s equation of motion reproduces [5]. The explicit dependences of \( V_m \) on the parameters \( I_L \) and \( \mathcal{M} \) yield changes of \( H \):

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial V_m}{\partial I_L} \frac{dI_L}{dt} + \frac{\partial V_m}{\partial \mathcal{M}} \frac{d\mathcal{M}}{dt}.
\]

(9)

As already remarked, \( V_m \) is not the right energy to be inserted into the first law; we have to add the electric energies in the levitating and lifting systems, which amounts \( 2MB_L = -2V_m \). Hence, the magnetic moment acquires an energy, when put in an external magnetic field \( B_L \),

\[
U_m = V_m + 2MB_L = MB_L = -V_m.
\]

(10)

Notice the flip of the sign of energy. In addition to this mutual energy, the magnetic field of the total system (consisting of \( B_D \) that is produced by the dipole magnet-plasma compound and \( B_L \) that is produced by the lifting magnet) has also the self-energy \( U_s \), which may be written as [6]

\[
U_s = \frac{1}{2\mu_0} \int (B_D^2 + B_L^2) \, d^3x = \frac{1}{2} B_D \mathcal{M} + \frac{1}{2\mu_0} \int B_L^2 \, d^3x,
\]

(11)

where \( B_D \) is an average of \( e_z \cdot B_D \).

Including the thermal energy \( U_t \) of the plasma, the total energy of the system is

\[
U = U_k + U_g + U_m + U_s + U_t.
\]

(12)

The first law, combined with the “mechanical law” [9], reads as

\[
dU = \mathcal{M}dB_L + B_Ld\mathcal{M} + du_s + du_t.
\]

(13)

We find that \( \mathcal{M}dB_L = \mathcal{M}(dB_L/dI_L)dI_L \) contributes to the mechanical work \( dH \) (on the magnet-plasma subsystem by \( (\partial V_m/\partial I_L)dI_L = -(\mathcal{M}(\partial^2 B_L/\partial I_L \partial z)dI_L)dz \) (notice the flip of the sign). On the other hand, \( d\mathcal{M} \) is “caused” by heating \( \delta Q \), thus we may relate the term \( B_Ld\mathcal{M} \) with \( \delta Q \) (the latter also includes energy loss).

To delineate the relation between \( d\mathcal{M} \) and \( \delta Q \), we invoke the microscopic magnetic moment \( \mu = (m_e e_z^2)/(2B) \), where \( B \) is the local magnetic field in the plasma region, \( m_e \) is the mass of an electron, and \( v_e \) is the velocity of cyclotron motion. The power of ECH, first of all, increases \( v_e^2 \) (and then, excitates macroscopic processes). The perpendicular thermal energy \( U_{t, \perp} \) is the sum of \( B\mu_j \) over all particles (labeled by \( j = 1, 2, \cdots \)). With an average magnetic field \( B \), we write

\[
U_{t, \perp} := \sum_j B\mu_j = B \sum_j \mu_j.
\]

(14)

In view of (14), we may rephrase “heating” as injection of microscopic magnetic moments \( \mu_j \), and then, \( B \) is an effective chemical potential.

To relate the microscopic magnetic moments \( \mu_j \) with the macroscopic one \( \mathcal{M}_p \) (we denote by \( \mathcal{M}_p \) the plasma’s contribution to \( \mathcal{M} \)), we put

\[
\mathcal{M}_p = D \sum_j \mu_j,
\]

(15)

with a geometric factor \( D \), which we can estimate as follows. By the levitating magnets’s current \( I_0 \) and the length scale \( \ell \) of poloidal magnetic field lines \( (\ell \sim 2\pi a \) with a minor radius \( a \)), we estimate \( B = \mu_0 I_0/\ell \). Normalizing by \( \mathcal{M}_0 = \pi R^2 I_0 \), we obtain

\[
\frac{\mathcal{M}_p}{\mathcal{M}_0} = \frac{D}{\pi R^2 \ell B^2/\mu_0} = \frac{V}{\pi R^2 \ell \beta_{\perp}},
\]

(16)

where \( V \) is the volume of the plasma and \( \beta_{\perp} := U_{t, \perp}/(V B^2/2\mu_0) \) is the average beta ratio of the perpendicular plasma pressure. On the other hand, we estimate \( \mathcal{M}_p = \pi R^2 I_p \), where \( I_p \) is the diamagnetic current induced by the perpendicular pressure \( P_{\perp} \). Estimating \( I_p = \ell P_{\perp}/B \), we obtain

\[
\frac{\mathcal{M}_p}{\mathcal{M}_0} = \frac{\beta_{\perp}}{2}.
\]

(17)

Figure 3 shows a reasonable agreement. Comparing (16) and (17), we estimate \( D = \pi R^2 \ell/V \sim R/a \). Since the change of the superconductor’s current in response to \( dz, dI_L \) or \( d\mathcal{M} \) is of second order, we may assume \( d\mathcal{M} = d\mathcal{M}_p \).

Now we have a more explicit representation of the thermomagnet coupling processes included in the first law (13): denoting by \( U_{t, \parallel} \) the remaining parallel component of the thermal energy \( U_t \) and by \( \mathcal{M}_L \) the coefficient such that \( \int B_L^2 d^3x/\mu_0 = \mathcal{M}_L B_L \),

\[
dU = \left( \mathcal{M} + \frac{\mathcal{M}_L}{2} \right) \frac{dB_L}{dI_L} dI_L
\]

\[
+ \left( B_L + \frac{B_D}{2} \right) d\mathcal{M}
\]

\[
+ du_{t, \parallel} + \delta Q'.
\]

(18)

The first term on the right-hand side (induced by \( dI_L \)) is the process connected to the lifting magnet system. The second term (induced by \( d\mathcal{M} \)) is the “ECH heating” \( \delta Q_{ECH} \) (or, in our language, injection of magnetic moments); the component \( (B/D)d\mathcal{M} \) goes to the thermal energy \( U_{t, \perp} \), while the other
components change macroscopic magnetic energies $U_m$ and $U_s$, as well as mechanical energies $U_k$ and $U_g$ (through the mechanical potential energy $V_m = -U_m$), which we observe as the change of $z$. The remaining abstract terms $dU_{s,\parallel}$ (parallel energy change) and $\delta Q^\prime$ (heat processes including thermal conduction, energy loss with particle transport, etc.) are not the direct subject of the present analysis.

We have made an attempt to understand and interpret the observed macroscopic thermo-magneto coupling in a dipole plasma produced on the RT-1 magnetospheric device. The most abstract thermodynamic first law has been given a more concrete and dissected form that elucidates the internal and external thermo-magneto processes; the conventional expression of ECH as heating $\delta Q$ has been rewritten an injection of magnetic moment $dM$, and its partition into different terms of energy has been specified.

What is rather nontrivial is that a magnetic moment $\mathbf{m}$ is an axial vector (or, a pseudo-vector) having an odd parity; the $z$-component of $\mathbf{m}$ (i.e. $\mathbf{m} = \mathbf{Me}_z$), which can be regarded as a pseudo-scalar. Multiplying $\mathbf{m} (\mathcal{M})$ by the other axial vector $\mathbf{B}$ (a pseudo-scalar $\mathcal{B}$), we obtain a scalar that can be related to an energy or some thermodynamic potential.

In this paper, we have made an attempt to understand and interpret the direct subject of the present analysis. We have made an attempt to understand and interpret the direct subject of the present analysis.

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[6] Because the levitating magneto-plasma system consists of a superconductor and high-temperature plasma, the magnetic flux is conserved. Under the flux-conserving condition, the magnetic energy is better evaluated in terms of magnetic moment. The most general expression of the magnetic energy is (ignoring the displacement current) $U_M = \int \mathbf{B}^2 d^3x/(2\mu_0) = \int \mathbf{A} \cdot \mathbf{J} d^3x/2$. In an axisymmetric dipole configuration, $\mathbf{J} = J_0 e_\theta$, and then, $U_M = \pi \int \psi J_0 r dr dz$, where $\psi := r A_\theta = \int B_z r dr$ is the magnetic flux. With an average value $\bar{\psi} = B_z S/(2\pi)$ (at $r = R$, $\psi = \bar{\psi}$; $S := 2\pi r f r dr$ is the area of the disk of radius $R$, and $B_z$ is the average of $B_z$ on the disk), and the current $I = \int J_\theta dr dz$, we may write $U_M = \bar{B}_z S l/2 = \bar{B}_z M/2$. Decomposing $\mathbf{B} = B_D + B_L$ ($B_D$ is the dipole magnetic field; $\nabla \times B_D = \mu_0 J$ produces $\mathcal{M}$), we may write $U_M = \int \mathbf{A}_D \cdot \mathbf{J} d^3x/2 + \int \mathbf{A}_L \cdot \mathbf{J} d^3x + \int B_L^2 d^3x/(2\mu_0) = \bar{B}_z \pi S M/2 + \bar{B}_z l M + \int B_L^2 d^3x/(2\mu_0)$. The second term is the “mutual” energy $U_m$.

[7] In general, a magnetic moment is an axial vector $\mathbf{m}$, and $\nabla \times \mathbf{m} = -\mathbf{J} \cdot \mathbf{B}$ Here, $\mathcal{M}$ is the $z$-component (i.e. $\mathbf{m} = \mathbf{Me}_z$). The axisymmetric geometry of the present system allows us to omit the torque on the magnetic moment.

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