On Blow-up of a Semilinear Heat Equation with Nonlinear Boundary Conditions

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Abstract

This paper deals with the blow-up properties of the solutions of the semilinear heat equation

\[ u_t = \Delta u + \lambda e^{pu} \] in \( B_R \times (0, T) \) with the nonlinear boundary conditions \( \frac{\partial u}{\partial \eta} = e^{qu} \) on \( \partial B_R \times (0, T) \), where \( B_R \) is a ball in \( \mathbb{R}^n \), \( \eta \) is the outward normal, \( p > 0, q > 0, \lambda > 0 \). The upper and lower blow-up rate estimates are established. It is also proved under some restricted assumptions, that the blow-up occurs only on the boundary.

1 Introduction

In this paper, we consider the initial-boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + \lambda e^{pu}, & (x, t) &\in B_R \times (0, T), \\
\frac{\partial u}{\partial \eta} &= e^{pu}, & (x, t) &\in \partial B_R \times (0, T), \\
u(x, 0) &= u_0(x), & x &\in B_R,
\end{align*}
\] (1.1)

where \( p > 0, q > 0, \lambda > 0, B_R \) is a ball in \( \mathbb{R}^n \), \( \eta \) is the outward normal, \( u_0 \) is nonnegative, radially symmetric, nondecreasing, smooth function satisfies the conditions

\[
\begin{align*}
\frac{\partial u_0}{\partial \eta} &= e^{pu_0}, & x &\in \partial \Omega, \\
\Delta u_0 + \lambda e^{pu_0} &\geq 0, & u_0(|x|) &\geq 0, & x &\in \overline{\Omega}_R.
\end{align*}
\] (1.2) (1.3)

The problem of the semilinear heat equation with nonlinear boundary conditions:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + \lambda f(u), & (x, t) &\in \Omega \times (0, T), \\
\frac{\partial u}{\partial \eta} &= g(u), & (x, t) &\in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), & x &\in \Omega,
\end{align*}
\] (1.4)
has been studied by many authors (see for example [1, 10, 6]). The crucial point of these works was the question whether the reaction term in the semilinear equation can prevent (affect) blow-up. For instance, in [1] it has been studied the blow-up solutions of problem (1.4), where \( \lambda < 0 \) and
\[
f(u) = u^p, \quad g(u) = u^q, \quad p, q > 1, \tag{1.5}
\]
for \( n = 1 \) or \( \Omega = B_R \). Particularly, it was shown that the exponent \( p = 2q - 1 \) is critical for blow-up in the following sense:

(i) If \( p < 2q - 1 \) (or \( p = 2q - 1 \) and \( -\lambda < q \)), then there exist solutions, which blow up in finite time and the blow-up occurs only on the boundary.

(ii) If \( p > 2q - 1 \) (or \( p = 2q - 1 \) and \( -\lambda > q \)), then all solutions exist globally and are globally bounded.

In [9] J. D. Rossi has proved for the case (i), where \( n = 1, \Omega = [0, 1] \), that there exist positive constants \( C, c \) such that the upper (lower) blow-up rate estimate take the following forms
\[
c \leq \max_{[0,1]} u(\cdot, t)(T - t)^{1/(q - 1)} \leq C, \quad 0 < t < T.
\]

In [6] it has been studied another special case of problem (1.4), where \( \lambda = 1, f, g \) as in (1.6), \( \Omega = [0, 1] \) or it is a bounded domain with \( C^2 \) boundary, it was proved that the solutions of (1.4) exist globally if and only if \( \max\{p, q\} \leq 1 \), otherwise, every solution has to blow up in finite time. Moreover, the blow-up occurs only on the boundary. The blow-up rate estimate for this case has been studied in [6, 9], for \( n = 1, \Omega = [0, 1] \), it has been shown that there exist positive constants \( c, C \) such that
\[
c \leq \max_{[0,1]} u(\cdot, t)(T - t)\alpha \leq C, \quad 0 < t < T,
\]
where \( \alpha = 1/(p - 1) \) if \( p \geq 2q - 1 \), and \( \alpha = 1/[2(q - 1)] \) if \( p < 2q - 1 \).

We observe that if \( p < 2q - 1 \), then the nonlinear term at the boundary determines and gives the blow-up rate while, if \( p > 2q - 1 \), then the reaction term in the semilinear equation dominates and gives the blow-up rate.

Later, in [10] it was considered a second special case of (1.4), where \( \lambda = -a, a > 0 \), \( f, g \) are of exponential forms, namely
\[
\begin{align*}
&u_t = \Delta u - ae^{pu}, \quad (x, t) \in \Omega \times (0, T), \\
&\frac{\partial u}{\partial \eta} = e^{qu}, \quad (x, t) \in \partial \Omega \times (0, T), \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*}
\]
where \( p, q > 0 \), \( u_0 \) satisfies (1.2), (1.3).

It has been shown that in case of \( \Omega \) is a bounded domain with smooth boundary, the critical exponent can be given as follows
(i) If $2q < p$, the solutions of problem (1.6) are globally bounded.

(ii) If $2q > p$, the solutions of problem (1.6) blow up in finite time for large initial data.

(iii) If $2q = p$, the solutions may blow up in finite time for large initial data. Moreover, in case $\Omega = B_R$, the blow-up occurs only on the boundary and there exist positive constants $c, C$ such that the upper (lower) blow-up rate estimate take the following form

$$
\log C_1 - \frac{1}{2q} \log(T - t) \leq \max_{\overline{B}_R} u(\cdot, t) \leq \log C_2 - \frac{1}{2q} \log(T - t), \quad 0 < t < T.
$$

Therefore, the blow-up properties (blow-up location and bounds) of problem (1.6) are the same as that of problem (1.4), where $a = 0$, which has been considered in [2].

In this paper, we study the blow-up solutions of problem (1.1). The upper (lower) blow-up rate estimates is obtained. Moreover, under some restricted assumptions, we prove that blow-up occurs only on the boundary.

2 Preliminaries

Since $f(u) = \lambda e^{pu}$, $g(u) = e^{qu}$ are smooth functions, and problem (1.1) is uniformly parabolic, also $u_0$ satisfies the compatibility condition (1.2), it follows that the existence and uniqueness of local classical solutions to problem (1.1) are known by the standard theory [5]. On the other hand, the nontrivial solutions of this problem blow up in finite time and the blow-up set contains $\partial B_R$, and that due to comparison principle, [7], and the known blow-up result of problem (1.1), where $\lambda = 0$ (see [2]).

In this paper, we denote for simplicity $u(x, t) = u(r, t)$. The following lemma shows some properties of the classical solutions to problem (1.1).

Lemma 2.1. Let $u$ be a classical solution to problem (1.1), where $u_0$ satisfies the assumptions (1.2), (1.3). Then

(i) $u > 0$, radial in $B_R \times (0, T)$.

(ii) $u_r \geq 0$, in $[0, R] \times [0, T)$.

(iii) $u_t > 0$ in $B_R \times (0, T)$.

3 Blow-up Rate Estimates

Since $u_r \geq 0$, in $[0, R] \times (0, T)$, it follows that

$$
\max_{\overline{B}_R} u(\cdot, t) = u(R, t), \quad 0 < t < T.
$$

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Therefore, it is sufficient to derive the upper (lower) bounds of blow-up rate for $u(R, t)$.

**Theorem 3.1.** Let $u$ be a solution to problem \([1.1]\), where $u_0$ satisfies the assumptions \([1.2], [1.3]\), $T$ is the blow-up time. Then there is a positive constant $c$ such that

$$
\log c - \frac{1}{2\alpha} \log(T - t) \leq u(R, t), \quad t \in (0, T),
$$

where $\alpha = \max\{p, q\}$.

**Proof.** Define

$$
M(t) = \max_{B_R} u(\cdot, t) = u(R, t), \quad \text{for} \quad t \in [0, T).
$$

Clearly, $M(t)$ is increasing in $(0, T)$ (due to $u_t > 0$, for $t \in (0, T), \ x \in \overline{B}_R$). As in \([10]\), for $0 < z < t < T, x \in B_R$, the integral equation of problem \([1.1]\) with respect to $u$, can be written as follows

$$
u(x, t) = \int_{B_R} \Gamma(x - y, t - z)u(y, z)dy + \lambda \int_{z}^{t} \int_{B_R} \Gamma(x - y, t - \tau)e^{pu(y, \tau)}dyd\tau
+ \int_{z}^{t} \int_{S_R} \Gamma(x - y, t - \tau)e^{qu(y, \tau)}ds_yd\tau
- \int_{z}^{t} \int_{S_R} u(y, \tau)\frac{\partial \Gamma}{\partial \eta_y}(x - y, t - \tau)ds_yd\tau,
$$

(3.1)

where $\Gamma$ is the fundamental solution of the heat equation, namely

$$
\Gamma(x, t) = \frac{1}{(4\pi t)^{(n/2)}} \exp[-\frac{|x|^2}{4t}].
$$

(3.2)

Since $u(y, t) \leq u(R, t)$ for $y \in \overline{B}_R$, so, the last equation becomes

$$
u(x, t) \leq u(R, z) \int_{B_R} \Gamma(x - y, t - z)dy + \lambda \int_{z}^{t} \int_{B_R} e^{pu(R, \tau)} \int_{B_R} \Gamma(x - y, t - \tau)dyd\tau.
+ \int_{z}^{t} \int_{S_R} e^{qu(R, \tau)} \int_{S_R} \Gamma(x - y, t - \tau)ds_yd\tau
+ \int_{z}^{t} u(R, \tau) \int_{S_R} |\frac{\partial \Gamma}{\partial \eta_y}(x - y, t - \tau)|ds_yd\tau.
$$
Since \( u \) is a continuous function on \( B_R \), the last inequality leads to

\[
M(t) \leq M(z) \int_{B_R} \Gamma(x - y, t - z)dy + \lambda e^{\alpha M(t)} \int_{t}^{\tau} \int_{B_R} \Gamma(x - y, t - \tau)dyd\tau \\
+ e^{\beta M(t)} \int_{S_R} \Gamma(x - y, t - \tau)ds_yd\tau \\
+ M(t) \int_{S_R} |\partial_{\eta y}(x - y, t - \tau)| ds_yd\tau.
\]

(3.3)

It is known from [3, 7] that for \( 0 < t_1 < t_2, \ x, y \in R^n \), \( \Gamma \) satisfies

\[
\int_{B_R} \Gamma(x - y, t_2 - t_1)dy \leq 1.
\]

Moreover, there exist positive constants \( k_1, k_2 \) such that

\[
\Gamma(x - y, t_2 - t_1) \leq \frac{k_1}{(t_2 - t_1)^{\mu_0}} \cdot \frac{1}{|x - y|^{n-2+\mu_0}}, \quad 0 < \mu_0 < 1,
\]

\[
|\partial_{\eta y}(x - y, t_2 - t_1)| \leq \frac{k_2}{(t_2 - t_1)^{\mu}} \cdot \frac{1}{|x - y|^{n+1-2\mu-\sigma}}, \quad \sigma \in (0, 1), \ \mu \in (1 - \frac{\sigma}{2}, 1).
\]

If we choose \( \mu_0 = 1/2 \), then from [3], there exist positive constants \( d_1, d_2 \) such that

\[
\int_{S_R} \frac{ds_y}{|x - y|^{n-2+\mu_0}} \leq d_1, \quad \int_{S_R} \frac{ds_y}{|x - y|^{n+1-2\mu-\sigma}} \leq d_2.
\]

From above it follows that there exist \( C_1, C_2 > 0 \) such that, the inequality (3.3) becomes

\[
M(t) \leq M(z) + \lambda e^{\alpha M(t)}(t - z) + C_1 e^{\alpha M(t)} \sqrt{T - z} + C_2 M(t)(t - z)^{1-\mu}.
\]

Since \( t - z \leq T - z \), it follows that

\[
M(t) \leq M(z) + \lambda e^{\alpha M(t)} \sqrt{T - z} + C_1 e^{\alpha M(t)} \sqrt{T - z} + C_2 M(t)(T - z)^{1-\mu}, \quad (3.4)
\]

provided \( (T - z) \leq 1 \).

Clearly,

\[
\frac{M(t)}{e^{\alpha M(t)}} \to 0, \quad \text{when} \quad t \to T.
\]

Thus

\[
\frac{M(t)}{e^{\alpha M(t)}} \leq (T - z)^{\frac{1}{2}-(1-\mu)}, \quad \text{for} \ t \ \text{close to} \ T.
\]

Therefore, the inequality (3.4) becomes

\[
M(t) \leq M(z) + \lambda e^{\alpha M(t)} \sqrt{T - z} + C_1 e^{\alpha M(t)} \sqrt{T - z} + C_2 e^{\alpha M(t)} \sqrt{T - z},
\]
thus there is a constant $C^*$ such that
\[ M(t) \leq M(z) + C^* e^{\alpha M(t)} \sqrt{T - z}, \quad z < t < T, \; t \text{ close to } T. \]

For any $z$ close to $T$, we can choose $z < t < T$ such that
\[ M(t) - M(z) = C_0 > 0, \]
which implies
\[ C_0 \leq C^* e^{\alpha M(z) + \alpha C_0} \sqrt{T - z}. \]

Thus
\[ \frac{C_0}{C^* e^{(\alpha C_0) \sqrt{T - z}}} \leq e^{\alpha u(R, z)}. \]

Therefore, there exist a positive constant $c$ such that
\[ \log c - \frac{1}{2\alpha} \log(T - t) \leq u(R, t), \quad t \in (0, T). \]

The next theorem shows similar results to Theorem 3.1 with adding more restricted assumptions on $q$ and $u_0$. The proof relies on the maximum principle rather than the integral equation.

**Theorem 3.2.** Let $u$ be a solution to problem (1.1), where $q \geq 1$, $T$ is the blow-up time, $u_0$ satisfies the assumptions (1.2), (1.3), moreover, it satisfies the following condition
\[ u_0(r) - \frac{r}{R} e^{u_0(r)} \geq 0, \quad r \in [0, R]. \] (3.5)

Then there is a positive constant $c$ such that
\[ \log c - \frac{1}{2\alpha} \log(T - t) \leq u(R, t), \quad t \in (0, T), \]
where $\alpha = \max\{p, q\}$.

**Proof.** Define the functions $J$ as follows:
\[ J(x, t) = u_r(r, t) - \frac{r}{R} e^{u(r, t)}, \quad x \in B_R \times (0, T). \]

A direct calculation shows
\[ J_t = u_{rt} - \frac{r}{R} e^u [u_{rr} + \frac{n - 1}{r} u_r + \lambda p e^{pu}], \]
\[ J_r = u_{rr} - \frac{r}{R} e^u u_r - \frac{1}{R} e^u, \]
\[ J_{rr} = \left[ u_{rr} - \frac{n - 1}{r} u_{rr} + \frac{n - 1}{r^2} u_r - \lambda p e^{pu} u_r \right] - \frac{r}{R} \left( e^u u_{rr} + e^u u_r^2 \right) - \frac{2}{R} e^u u_r. \]
From above it follows that
\[
J_t - J_{rr} - \frac{n-1}{r} J_r = - \frac{n-1}{r^2} [u_r - \frac{r}{R} e^u] + \lambda pe^{pu} [u_r - \frac{r}{R} e^u] + \frac{r}{R} e^u u_r^2 + \frac{2}{R} e^u u_r.
\]
Thus
\[
J_t - \Delta J - bJ = \frac{r}{R} e^u u_r^2 + \frac{2}{R} e^u u_r \geq 0,
\]
for \((x, t) \in B_R \times (0, T) \cap \{r > 0\}, \) where \(b = [\lambda pe^{pu} - \frac{n-1}{r^2}].\)

Clearly, from (3.5), it follows that
\[
J(x, 0) \geq 0, \quad x \in B_R,
\]
and
\[
J(0, t) = u_r(0, t) \geq 0, \quad J(R, t) = 0 \quad t \in (0, T).
\]
Since
\[
\sup_{(0, R) \times [0, t]} b < \infty, \quad \text{for} \quad t < T,
\]
from above and maximum principle [8], it follows that
\[
J \geq 0, \quad (x, t) \in B_R \times (0, T).
\]
Moreover,
\[
\frac{\partial J}{\partial \eta}|_{\partial B_R} \leq 0.
\]
This means
\[
(u_{rr} - \frac{r}{R} e^u u_r - \frac{1}{R} e^u)|_{\partial B_R} \leq 0.
\]
Thus
\[
u_t \leq (\frac{n-1}{r} u_r + \lambda pe^{pu} + e^u u_r + \frac{1}{R} e^u)|_{\partial B_R},
\]
which implies that
\[
u_t(R, t) \leq \frac{n-1}{R} e^{qu(R,t)} + \lambda pe^{pu(R,t)} + e^{(1+q)u(R,t)} + \frac{2}{R} e^{u(R,t)}, \quad t \in (0, T).
\]

Thus, there exist a constant \(C\) such that
\[
u_t(R, t) \leq Ce^{2\alpha u(R,t)}, \quad t \in (0, T).
\]
Integrate this inequality from \(t\) to \(T\) and since \(u\) blows up at \(R\), it follows
\[
\frac{c}{(T-t)^{\frac{1}{2}}} \leq e^{\alpha u(R,t)}, \quad t \in (0, T)
\]
or
\[
\log c - \frac{1}{2\alpha} \log(T-t) \leq u(R, t), \quad t \in (0, T).
\]
Remark 3.3. From Theorems 3.1 and 3.2 we conclude that, when $q > p$ the boundary term plays the dominating role and the lower blow-up rate takes the form:

$$\log c - \frac{1}{2q} \log(T - t) \leq u(R,t), \quad t \in (0,T),$$

moreover, this estimate is coincident with lower blow-up rate estimate of problem (1.1), where $\lambda = 0$, which has been considered in [2], while when $p > q$ the reaction term is dominated and gives the lower blow-up rate as follows

$$\log c - \frac{1}{2p} \log(T - t) \leq u(R,t), \quad t \in (0,T).$$

We next consider the upper bound

**Theorem 3.4.** Let $u$ be a solution of problem (1.1), where $T$ is the blow-up time, $u_0$ satisfies the assumptions (1.2), (1.3) moreover, assume that

$$\Delta u_0 + f(u_0) \geq a > 0, \quad \text{in } \overline{B_R}. \quad (3.6)$$

Then there is a positive constant $C$ such that

$$u(R,t) \leq \log C - \frac{1}{q} \log(T - t), \quad t \in (0,T). \quad (3.7)$$

**Proof.** Define the function $J$ as follows

$$J(x,t) = u_t(r,t) - \varepsilon u_r(r,t), \quad (x,t) \in B_R \times (0,T).$$

Since $u_0(r)$ is bounded in $B_R$, and by (3.6), for some $\varepsilon > 0$, we have

$$J(x,0) = \Delta u_0(r) + f(u_0(r)) - \varepsilon u_0(r) \geq 0, \quad x \in \overline{B_R}.$$

A simple computation shows

$$J_t = u_{rrt} + \frac{n-1}{r} u_{rt} + \lambda p \varepsilon p u u_t - \varepsilon u_t,$$

$$J_r = u_{tr} - \varepsilon u_{rr},$$

$$J_{rr} = u_{trr} - \varepsilon u_{tr} + \varepsilon \frac{n-1}{r} u_{rr} - \varepsilon \frac{(n-1)}{r^2} u_r + \varepsilon \lambda_ p \varepsilon p u u_r.$$

From above, it follows that

$$J_t - J_{rr} - \frac{n-1}{r} J_r - \lambda p \varepsilon p u J = \varepsilon \frac{(n-1)}{r^2} u_r \geq 0,$$

i.e.

$$J_t - \Delta J - \lambda p \varepsilon p u J \geq 0, \quad (x,t) \in B_R \times (0,T).$$
Moreover,

\[
\frac{\partial J}{\partial \eta} |_{x \in \partial B_R} = u_{rt}(R, t) - \varepsilon u_{rr}(R, t)
\]

\[
= q e^{qu(R,t)} u_t - \varepsilon [u_t(R, t) - \frac{n-1}{r} u_r(R, t) - \lambda e^{pu(R,t)}]
\]

\[
\geq [q e^{qu(R,t)} - \varepsilon] u_t(R, t)
\]

Since, \(u_t > 0\) in \(\overline{B}_R \times (0, T)\), if follows that

\[
\frac{\partial J}{\partial \eta} \geq 0, \quad \text{on} \quad \partial B_R \times (0, T),
\]

provided \(\varepsilon \leq q e^{\{qu_0(R)\}}\).

Since \(e^{pu}\) is bounded on \(B_R \times (0, t]\) for \(t < T\), from maximum principle \[7\] and above, we have

\[J \geq 0, \quad (x, t) \in \overline{B}_R \times (0, T).\]

In particular, \(J(x, t) \geq 0\) for \(x \in \partial B_R\), that is

\[u_t(R, t) \geq \varepsilon u_r(R, t) = \varepsilon e^{qu(R,t)}, \quad t \in (0, T).\]

Upon integration the above inequality from \(t\) to \(T\) and since \(u\) blows up at \(R\), it follows that

\[e^{qu(R,t)} \leq \frac{1}{q \varepsilon (T - t)}, \quad t \in (0, T),\]

or

\[u(R, t) \leq \log C - \frac{1}{q} \log(T - t), \quad t \in (0, T).\]

\[\square\]

**Remark 3.5.** The upper blow-up rate estimate for problem (1.1), which has been derived in Theorem 3.4, is governed by the boundary term even in case \(p > q\). On the other hand, it is known that the upper blow-up bound of problem (1.1), where \(\lambda = 0\) (see [2]) takes the form:

\[u(R, t) \leq \log \frac{C}{(T - t)^{\frac{1}{2q}}}.\]

Therefore, we conclude that the presence of the reaction term has an important effect on the upper blow-up rate estimate.
4 Blow-up Set

We shall prove in this section that the blow-up to problem (1.1) occurs only on the boundary, restricting ourselves to the special case $p = q = 1$ with some restriction assumption on $\lambda$.

**Theorem 4.1.** Suppose that the function $u(x, t)$ is $C^{2,1}(\overline{B}_R \times [0, T])$, and satisfies

$$
\begin{align*}
&u_t = \Delta u + \lambda e^u, \quad (x, t) \in B_R \times (0, T), \\
&u(x, t) \leq \log \frac{C}{(T-t)}, \quad (x, t) \in \overline{B}_R \times (0, T), \\
&u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*}
$$

where

$$
\lambda [4R^2(n + 1) + 1] \leq \min \left\{ \frac{1}{C}, \frac{4(n + 1)}{[R^2 + 4(n + 1)T]} e^{-\|u_0\|_\infty} \right\},
$$

(4.1)

$C < \infty$. Then for any $0 \leq a < R$, there exist a positive constant $A$ such that

$$
u(x, t) \leq \log \left[ \frac{1}{A(R^2 - r^2)^2} \right] < \infty \quad \text{for} \quad 0 \leq |x| \leq a < R, 0 < t < T.
$$

Proof. Let

$$
v(x) = A(R^2 - r^2)^2, \quad r = |x|, \quad 0 \leq r \leq R,
$$

$$
z(x, t) = z(r, t) = \log \left[ \frac{1}{v(x) + B(T - t)} \right], \quad \text{in} \ \overline{B}_R \times (0, T),
$$

where $B > 0, A \geq \lambda$.

A direct calculation shows that

$$
\begin{align*}
z_t &= \frac{B}{[v(x) + B(T - t)]}, \\
z_r &= \frac{4rA(R^2 - r^2)}{[v(x) + B(T - t)]}, \\
z_{rr} &= \frac{[v(x) + B(T - t)][4A(R^2 - 3r^2)] + 16A^2 r^2 (R^2 - r^2)^2}{[v(x) + B(T - t)]^2}.
\end{align*}
$$

Thus

$$
\begin{align*}
z_t - z_{rr} - \frac{n-1}{r} z_r - \lambda e^z &= \frac{[B - 4A(n - 1)(R^2 - r^2) - \lambda][v(x) + B(T - t)]}{[v(x) + B(T - t)]^2} \\
&- \frac{[4A(R^2 - 3r^2)][v(x) + B(T - t)] + 16Ar^2 v(x)}{[v(x) + B(T - t)]^2} \\
&\geq \frac{[B - 4A(n - 1)(R^2 - r^2) - \lambda - 4A(R^2 - 3r^2) - 16Ar^2 v(x)]}{[v(x) + B(T - t)]^2} \\
&\geq \frac{[B - 4AR^2 n - 4AR^2 - \lambda]v(x)}{[v(x) + B(T - t)]^2} \\
&\geq \frac{[B - 4AR^2 n - 4AR^2 - A]v(x)}{[v(x) + B(T - t)]^2} \geq 0
\end{align*}
$$

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provided

\[ B \geq A[4R^2(n + 1) + 1]. \]

i.e.

\[ z_t - \Delta z - \lambda e^z \geq 0, \quad \text{in} \quad B_R \times (0, T) \]

Moreover,

\[
\begin{align*}
z(x, 0) &= \log \frac{1}{[v(x)]^{+BT}} \geq \log \frac{1}{[AR^4+BT]} \geq u(x, 0), \quad x \in B_R, \\
z(R, t) &= \log \frac{C}{B(T-t)} \geq \log \frac{C}{(T-t)} \geq u(R, t), \quad t \in (0, T)
\end{align*}
\]

provided

\[ B \leq \min \left\{ \frac{1}{C}, \frac{4(n+1)}{R^2 + 4(n+1)T} e^{-\|u_0\|_\infty} \right\}. \]

From above, and the comparison principle \cite{7}, we obtain

\[ z(x, t) \geq u(x, t) \quad \text{in} \quad B_R \times (0, T). \]

Thus

\[ u(x, t) \leq \log \left[ \frac{1}{A(R^2 - r^2)^2} \right] < \infty \quad \text{for} \quad 0 \leq |x| \leq a < R, 0 < t < T. \]

\[ \square \]

Remark 4.2. From Theorem 4.1 and the upper blow-up rate estimate \cite{3.7}, it follows that, for the special case of problem \cite{11} \((p = q = 1 \text{ and } \lambda \text{ satisfies } \cite{4.1})\), the blow-up occurs only on the boundary. Therefore, we conclude that, the blow-up set of \cite{1.1}, where \(\lambda\) is small enough, is the same that of \cite{1.1}, where \(\lambda = 0\) \(\text{ see } \cite{2}\).}

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