ASYMPTOTIC BEHAVIOUR OF AN INFINITELY-MANY-ALLELES DIFFUSION WITH SYMMETRIC OVERDOMINANCE

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Abstract. This paper considers the limiting distribution of \( \pi_{\lambda,\theta} \), the stationary distribution of the infinitely-many-alleles diffusion with symmetric overdominance [Ethier and Kurtz, 1998]. In [Feng, 2009] the large deviation principle for \( \pi_{\lambda,\theta} \) indicates that there are countably many phase transitions for the limiting distribution of \( \pi_{\lambda,\theta} \), and the critical points are \( \lambda = k(k + 1), k \geq 1 \). The asymptotic behaviours at those critical points, however, are unclear. This article provides a definite description of the critical cases.

1. Introduction

The infinitely many alleles model is an extensively studied model in population genetics. In this model, mutations always generate completely new allele types, and \( x = (x_1, x_2, \cdots) \), where \( x_i \)'s are arranged decreasingly and \( \sum_{i=1}^{\infty} x_i = 1 \), is usually used to represent the allele frequency. The infinitely-many-neutral-alleles diffusion [Ethier and Kurtz, 1981] is the associated diffusion process characterized by generator

\[
G = \frac{1}{2} \sum_{i,j=1}^{\infty} x_i (\delta_{ij} - x_j) \frac{\partial^2}{\partial x_i \partial x_j} - \theta \sum_{i=1}^{\infty} x_i \frac{\partial}{\partial x_i}, x \in \mathcal{V}_\infty,
\]

where \( \mathcal{V}_\infty = \{(x_1, x_2, \cdots) | x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \} \). The Poisson-Dirichlet distribution [Kingman et al., 1975], hereafter denoted as \( \text{PD}(\theta) \), is its stationary distribution. If symmetric overdominant selection is considered, we will end up with the infinitely-many-alleles diffusion with symmetric overdominance [Ethier and Kurtz, 1998], characterized by

\[
G_\sigma = G + \sigma \sum_{i=1}^{\infty} x_i (x_i - \varphi_2(x)) \frac{\partial}{\partial x_i}, x \in \mathcal{V}_\infty,
\]

where \( \varphi_2(x) = \sum_{i=1}^{\infty} x_i^2 \). In population genetics, the homozygosity, denoted by \( H_2 \), is defined to be \( \varphi_2(x) \) for a given allele frequency \( x \). \( H_2 \) is a random variable for allele frequency is random.

The infinitely-many-alleles diffusion with symmetric overdominance has stationary distribution \( \pi_\sigma \) defined as

\[
\pi_\sigma(dx) = C_\sigma \exp\{\sigma \varphi_2(x)\} \text{PD}(\theta)(dx), x \in \mathcal{V}_\infty.
\]
where $C_\sigma$ is a normalized constant. If $\sigma > 0$, then the selection is underdominant; if $\sigma < 0$, then the selection is overdominant. In this article, only overdominant selection is considered.

For the infinitely-many-alleles diffusion with symmetric overdominance, random sampling, mutations and selection are all the evolutionary forces involved. It is commonly accepted that mutations contribute substantially to the genetic variability. Even when the very low mutation rate is presented, overdominant selection, however, can maintain certain amount of biological diversity, please refer to [Maruyama and Nei, 1981] and references therein. Random sampling is the evolutionary force constantly deleting some types. The interactions of those forces determine the unique configuration of the whole population. When mutations and overdominant selection are both large, the effect of overdominant selection is hardly pronounced. This intuitive statement is in fact verified by the numerical results in [Gillespie, 1999]. Also some numerical results in [Gillespie, 1999] are confirmed theoretically by Joyce, Krone and Kurtz [Joyce et al., 2003], Dawson and Feng [Dawson and Feng, 2006]. On the contrary, when mutation is small and overdominant selection is large, the effect of overdominant selection can be observable. Similarly, this statement is also shown in [Feng, 2009] through large deviation principle (hereafter LDP, for detailed introduction of LDP, please refer to [Dembo and Zeitouni, 2010]) for PD($\theta$) and $\pi_\sigma$ with small mutation and large selection. For PD($\theta$), the speed is $\log 1/\theta$ and the LDP rate function is

$$J(x) = \begin{cases} 0, & x \in L_1 \\ n - 1, & x \in L_n, x_n > 0, n \geq 2 \\ +\infty, & x \notin L_n \end{cases}$$

where $L_n = \{(x_1, \ldots, x_n, 0, \ldots) \in \mathbb{N}_\infty \mid \sum_{i=1}^n x_i = 1\}$, $L = \bigcup_{n=1}^\infty L_n$. Here $J(x)$ exhibits some properties similar to energy ladder structure. In order to understand the interaction of small mutation and large selection, the selection intensity $\sigma$ is regarded as $\sigma(\theta)$, and the LDP for $\pi_\sigma(\theta)$ was also considered. Especially, when $\sigma = \lambda \log \theta (\lambda > 0, 0 < \theta < 1)$, $\pi_\sigma(\theta)$ is denoted as $\pi_{\lambda, \theta}$. The rate function of the LDP for $\pi_{\lambda, \theta}$ is

$$S_\lambda(x) = J(x) + \lambda \varphi_2(x) - \inf \left\{ \frac{\lambda}{n} + n - 1 \mid n \geq 1 \right\}.$$ 

Thus, the effects of overdominant selection are pronounced because the LDP for PD($\theta$) and the LDP for $\pi_{\lambda, \theta}$ have different rate function. It was observed in [Feng, 2009] that when $\lambda \in (k(k-1), k(k+1))$, $k \geq 1$, the limiting distribution of $\pi_{\lambda, \theta}$ is $\delta_{(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots)}$; but for the critical case $\lambda = k(k+1)$, $k \geq 1$, the LDP rate function $S(x)$ has two zero points. Therefore the law of large numbers of $\pi_{\lambda, \theta}$ at critical points remains open.

The main result of this paper confirms that the asymptotic distribution at critical values $\lambda = k(k+1)$ is also

$$\delta_{(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots)}.$$ 

In general, as $\theta \to 0$, the limiting distribution of $\pi_{\lambda, \theta}$ can be written as

$$\sum_{k=1}^\infty I_{(k(k-1), k(k+1))}(\lambda) \delta_{(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots)}.$$
Therefore, for different selection intensity, the asymptotic distribution varies, and there are countably many phase transitions. This is indeed a bit of surprising.

The possible explanation of the phase transition can be two fold. On one hand, mathematically, $\varphi_2(x) = \sum_{i=1}^{\infty} x_i^2$ can be regarded as the potential function of the infinitely-many-alleles diffusion with symmetric overdominance. Obviuously, $\varphi_2(x)$ has a minimum point $(\frac{1}{k}, \cdots, \frac{1}{k}, 0, \cdots)$ in each domain $L_k, k \geq 1$. The graph of $\varphi_2(x)$ thus indicates a “multi-valley energy landscape”. For a given selection intensity, the system is trapped in a specific valley with bottom point say $(\frac{1}{k}, \cdots, \frac{1}{k}, 0, \cdots)$.

On the other hand, biologically, random sampling constantly deletes a great amount of gene types. Therefore, the system is often likely to stay in $L_k, k \geq 1$. Once the mutation is present, it will move the system upward along the energy ladder, i.e. the system will gradually move to $L_{k+1}$ from $L_k, k \geq 1$. The symmetry of the overdominant selection guarantees that the existed types are evenly distributed. Hence, the three evolutionary forces will balance out a single state such as $(\frac{1}{k}, \cdots, \frac{1}{k}, 0, \cdots)$.

Furthermore, the limiting distribution of homozygosity under $\pi_{\lambda, \theta}$ is also obtained and it is

$$\sum_{k=1}^{\infty} I_{(k(k-1), k(k+1))}(\lambda)\delta_{\frac{1}{k}}.$$

All these results are concerned with the infinitely many alleles model. For finitely many alleles model, say two alleles model, the asymptotic distribution of homozygosity is quite similar to the infinitely many alleles model and reads as

$$I_{(0, 2)}(\lambda)\delta_{\frac{1}{2}} + I_{(2, \infty)}(\lambda)\delta_1.$$ 

But the limiting distribution of the stationary distribution for the two alleles model with symmetric overdominance is different since it is a labelled model. Please refer to [Zhou, 2010] for the derivation of the limiting distribution for the two alleles model with symmetric overdominance. Presumably, for other finitely many alleles model, the asymptotic homozygosity should behave similarly; but the proof is missing.

The whole paper is organized as follows. In section 2, we will present the main theorem on limiting distribution of homozygosity and its proof. In section 3, proofs of all lemmas will be shown in detail.

2. Main results

In this section, we are going to derive the limiting distribution of $\pi_{\lambda, \theta}$ at the critical points $\lambda = k(k+1), k \geq 1$. Due to the LDP estimation of $\pi_{\lambda, \theta}$, we can conclude that the limiting distribution of $\pi_{\lambda, \theta}$, if any, should concentrate on two points $(\frac{1}{k}, \cdots, \frac{1}{k}, 0, \cdots)$ and $(\frac{1}{k+1}, \cdots, \frac{1}{k+1}, 0, \cdots)$. But we can not determine the probability weights of these two points using only LDP estimation of $\pi_{\lambda, \theta}$.

To this end, we are going to find the limits of homozygosity $H_2$ first. Then, making use of the LDP estimation of $\pi_{\lambda, \theta}$, the limits of $\pi_{\lambda, \theta}$ are finally obtained. To obtain the limit of $H_2$, we need to estimate its asymptotic moment generating function. Hence, we expand terms such as the normalized constant $C_\sigma$, which is usually called partition function in statistical physics. Since $C_\sigma$ is a function of $\theta$, we will expand it near 0. Thanks to a ratio limit theorem, i.e. Lemma 4, the limiting moment generating function of $H_2$ will be determined by the leading term in $C_\sigma$’s expansion.
Theorem 2.1. For \( \lambda > 0 \), the limiting distributions of homozygosity, \( H_2 \), under the distribution \( \pi_{\theta, \lambda} \) is

\[
\sum_{k=1}^{+\infty} I_{(k(k-1), k(k+1))}(\lambda) \delta_x(dx).
\]

Before we present the proof, we need the following lemmas, the proofs of which are postponed.

Lemma 1. The moment of the heterozygosity \( m_k = E(1 - H_2)^k \) has the form

\[
m_k = \sum_{l=1}^{k} A_{k,l}(\theta) \theta^l,
\]

where

\[
A_{k,1}(\theta) = \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + \theta)}{\Gamma(2k + 1 + \theta)}
\]

and

\[
A_{k,p}(\theta) = \sum_{l=p-1}^{k-1} \frac{2k + l}{2l!} \frac{2^k l! \Gamma(l + \theta)}{\Gamma(2k + 1 + \theta)} A_{l,p-1}(\theta), \quad p \geq 2.
\]

Let us define \( A_{k,p} = A_{k,p}(0) \); then

\[
A_{k,1} = \frac{2^k k!(k-1)!}{(2k)!},
\]

and

\[
A_{k,p} = \sum_{l=p-1}^{k-1} \frac{2^k l! \Gamma(l + \theta)}{2l! \Gamma(2k + 1)} A_{l,p-1}, \quad p \geq 2.
\]

Thus \( A_{k,p} \) does not depend on \( \theta \) anymore; but it is an appropriate approximation of \( A_{k,p}(\theta) \), as can be seen in the proof of Lemma 7.

Lemma 2. If we fix integer \( p \geq 1 \), then, \( \forall \theta \in [0, 1] \), we have

\[
\frac{1}{2p} A_{k,p} \leq A_{k,p}(\theta) \leq A_{k,p} \leq \frac{1}{2p-2}, \quad \forall k \geq p \geq 1;
\]

\[
|A_{k,p}(\theta) - A_{k,p}| \leq \theta p A_{k,p}, \quad 1 \leq p \leq k.
\]

Lemma 3. For \( \lambda > 0 \), we have

\[
\lim_{\theta \to 0} \sum_{l=|\lambda|+1}^{\infty} \theta^l \sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,l}(\theta) = 0.
\]

and

\[
\lim_{\theta \to 0} \sum_{l=|\lambda|+1}^{\infty} \theta^l \sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k+n,l}(\theta) = 0.
\]

Lemma 4. Let \( a_n, b_n \) be two positive sequences. Suppose that \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \) and \( \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \) and \( \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \) are both convergent in \( \mathbb{R} \). Then \( \lim_{x \to +\infty} \frac{\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}}{\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}} = c. \)


**Proof.** Let us use $\phi$ and $\psi$ to denote the moment generating function of the homozygosity $H_2$ under $\pi_{\lambda, \theta}$. Thus,

$$\phi_H(t) = \frac{E \exp\{tH_2\} \cdot \exp\{\lambda \log \theta H_2\}}{E \exp\{\lambda \log \theta H_2\}}.$$ 

For some technical reason, we need to multiply the numerator and denominator in the above equation by the common term $\lambda \log \frac{1}{\theta}$; then

$$\phi_H(t) = e^{t E \exp\{-t(1 - H_2)\} \cdot \exp\{\lambda \log \theta (1 - H_2)\}} = e^{t \left( 1 + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} E(1 - H_2)^n \exp\{\lambda \log \frac{1}{\theta} (1 - H_2)\} \right)} = e^{t \left( 1 + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \sum_{m=0}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^m}{m!} \sum_{l=1}^{\infty} \frac{A_{k+l} \theta^l}{l!} \right)}.$$ 

In the above expansion, all terms are positive, which greatly facilitates our calculations. If we denote the limit of $\phi_H(t)$, as $\theta \to 0$, by $\psi_H(t)$, then we have

$$\psi_H(t) = e^{t \left[ 1 + \lim_{\theta \to 0} \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \left( \sum_{k=1}^{\infty} \frac{A_{l} \theta^l}{l!} \right) + m + \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} \sum_{l=1}^{\infty} \frac{A_{k+l} \theta^l}{l!} \right]}.$$ 

**Lemma 5.** For any fixed integer $p \geq 1$, we have, as $k \to +\infty$,

$$A_{k,p} \sim C_p \left( \frac{p}{p+1} \right)^k,$$

where $C_1 = \sqrt{\pi}$, and $C_{p+1} = C_p \sqrt{\frac{p+2}{p}}$.

**Lemma 6.** Define $C_{k,l} = \sum_{s=0}^{k-l} \left( \frac{k-l}{s} \right) A_{k-s,l}$. Then, as $k \to +\infty$,

$$C_{k,l} \sim C_l \left( 1 + \frac{\lambda - l}{\lambda} \right)^{\frac{l}{\lambda}} \frac{1}{k^\frac{l}{\lambda}} \left( \frac{\lambda - l}{\lambda} + \frac{l}{1 + l} \right)^k.$$ 

**Lemma 7.** For $\lambda > 2$, define

$$K^\lambda_n(\theta) = \sum_{l=1}^{[\lambda]} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k+n,l}(\theta)$$

and

$$\tilde{K}^\lambda_n(\theta) = \sum_{l=1}^{[\lambda]} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k+n,l}.$$ 

When $u(u - 1) < \lambda \leq u(u + 1)$, $u \geq 2$, we have

$$\lim_{\theta \to 0} K^\lambda_n(\theta) = \lim_{\theta \to 0} \tilde{K}^\lambda_n(\theta) = \left( \frac{u - 1}{u} \right)^n.$$ 

**[PROOF OF THEOREM 2.1]:**
By the Lebesgue dominant convergence theorem, we can switch the order of summation and limit. Thus,

\begin{align*}
(1) \quad \psi_H(t) &= e^t \left[ 1 + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \left( \lim_{\theta \to 0} \frac{\sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k}{\sum_{l=1}^{\infty} A_{k+l}(\theta) \theta^l} + \lim_{\theta \to 0} m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=1}^{k+n} A_{k+l}(\theta) \theta^l \right) \right].
\end{align*}

Now we claim that

\begin{align*}
(2) \quad \lim_{\theta \to 0} m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=1}^{k+n} A_{k+l}(\theta) \theta^l = 0.
\end{align*}

Indeed, we have

\begin{align*}
0 \leq \frac{m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=1}^{k+n} A_{k+l}(\theta) \theta^l}{1 + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=1}^{k} A_{k+l}(\theta) \theta^l} \leq m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=1}^{k+n} A_{k+l}(\theta) \theta^l.
\end{align*}

By Lemma 2 we have

\begin{align*}
m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=1}^{k+n} A_{k+l}(\theta) \theta^l &\leq m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \sum_{l=k+n}^{k+n} \theta^l \\
&= m_n + \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k \left(1 - \frac{\theta}{2}\right)^n (e^{\lambda \log \frac{\theta}{k}} - 1) \to 0, \text{ as } \theta \to 0.
\end{align*}

We have used the fact that \( m_n \to 0 \), as \( \theta \to 0 \), which is due to \( PD(\theta)(dx) \to \delta_{\{1,0,\ldots\}}(dx) \) as \( \theta \to 0 \). Thus, claim (2) is true. Therefore, by switching the summation order in (1), we have

\begin{align*}
\psi_H(t) = e^t \left[ 1 + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \lim_{\theta \to 0} \frac{\sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k A_{k+l}(\theta)}{1 + \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k A_{k+l}(\theta)} \right].
\end{align*}

Now what we need to show is, for \( u(u-1) < \lambda \leq u(u+1), u \geq 1, \)

\begin{align*}
(3) \quad \lim_{\theta \to 0} \frac{\sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k A_{k+l}(\theta)}{1 + \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} (\lambda \log \frac{\theta}{k})^k A_{k+l}(\theta)} = \left(\frac{u-1}{u}\right)^n.
\end{align*}

Once we have got the above equation, then

\begin{align*}
\psi_H(t) &= e^t \left[ 1 + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \left(\frac{u-1}{u}\right)^n \right] \\
&= e^t \left[ 1 + \sum_{n=1}^{\infty} \frac{-t^n (u/u-1)^n}{n!} \right] = e^t e^{-\frac{t(u-1)}{u}} = e^{\frac{t}{u}}.
\end{align*}

Thus, \( \psi_H(t) = \sum_{n=1}^{\infty} I_{\{u(u-1),u(u+1)\}}(\lambda)e^{\frac{t}{u}}. \) Therefore, the limiting distribution of \( H_2 \) under \( \pi_{\lambda,\theta} \) is \( \sum_{u=1}^{\infty} I_{\{u(u-1),u(u+1)\}}(\lambda)\delta_{\frac{t}{u}}. \)
Now we are going to verify the claim \( (3) \). Firstly, when \( 0 < \lambda \leq 2 \), we have

\[
0 \leq \lim_{\theta \to 0} \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta) \leq \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta)
\]

\[
\leq \theta \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,1}(\theta) + \theta^2 \sum_{k=2}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,2}(\theta)
\]

\[
+ \sum_{l=3}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta).
\]

We can actually show that the above three terms approach 0 as \( \theta \to 0 \). Indeed, by Lemma\(\text{[2]}\) we have

\[
0 \leq \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta) \leq \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,1}
\]

\[
\leq \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,1} \to 0, \quad \text{as} \quad \theta \to 0.
\]

The above limit is due to Lemma\(\text{[5]}\) and Lemma\(\text{[4]}\). Similarly,

\[
0 \leq \theta^2 \sum_{k=2}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,2}(\theta) \leq \sum_{k=2}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,2}
\]

\[
\leq \sum_{k=2}^{\infty} \frac{(2 \log \frac{1}{\pi})^k}{k!} A_{k+n,2} \to 0, \quad \text{as} \quad \theta \to 0.
\]

Thus, we have for \( 0 < \lambda \leq 2 \),

\[
\lim_{\theta \to 0} \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta) = \lim_{\theta \to 0} \left( 1 - \frac{1}{1} \right)^n = 0.
\]

Secondly, for \( u(u-1) < \lambda \leq u(u+1), \) \( u \geq 2 \), then \( \lambda > 2 \). We can show that

\[
\lim_{\theta \to 0} \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta) = \lim_{\theta \to 0} K_n^\lambda(\theta);
\]

then by Lemma\(\text{[7]}\) we have proved claim \( (3) \). Now we only need to verify \( (4) \). To this end, we rewrite

\[
\lim_{\theta \to 0} \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k+n,l}(\theta)
\]

\[
1 + \sum_{l=1}^{\infty} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\pi})^k}{k!} A_{k,n}(\theta).
\]
as
\[
\sum_{i=1}^{[\lambda]} \theta^{i} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k+n,i}(\theta) + \sum_{i=1}^{[\lambda]} \theta^{i} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,i}(\theta) \leq \frac{1}{1 + \sum_{i=1}^{[\lambda]} \theta^{i} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,i}(\theta) + \sum_{i=1}^{[\lambda]} \theta^{i} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,i}(\theta)}.
\]

By Lemma 3, we know, as \(\theta \to 0\)
\[
\sum_{i=1}^{[\lambda]} \theta^{i} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k+n,i}(\theta) \to 0,
\]
and
\[
\sum_{i=1}^{[\lambda]} \theta^{i} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,i}(\theta) \to 0.
\]

Therefore, claim (4) is proved. Theorem 2.1 is thus proved!  \(\square\)

Now we are ready to show the weak law of large numbers for \(\pi_{\lambda,\theta}\). Define \(d(\cdot,\cdot)\) as
\[
d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}.
\]
Under \(d(\cdot,\cdot)\), \((\mathcal{V}_\infty,d)\) is a compact space, and the LDP for \(\pi_{\lambda,\theta}\) was originally proved under the metric \(d(\cdot,\cdot)\) in [Feng, 2009]. Denote \(C(\mathcal{V}_\infty)\) to be the set of all continuous functions in \((\mathcal{V}_\infty,d)\).

**Theorem 2.2.** For a given \(\lambda > 0\), \(\pi_{\lambda,\theta}\) converges weakly to the following
\[
\sum_{k=1}^{\infty} I(k(k-1),k(k+1))(\lambda)\delta((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots)).
\]

**Proof.** For integer \(k \geq 1\), and \((k-1)k < \lambda < k(k+1)\), the limit of \(\pi_{\lambda,\theta}\) has already been verified in [Feng, 2009]. Therefore, we only need to consider the critical case \(\lambda = (k+1)k, k \geq 1\). For a fixed \(k \geq 1\), and \(\lambda = (k+1)k\), we need to prove that \(\pi_{\lambda,\theta}\) converges weakly to \(\delta((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))\).

Define \(B_\delta((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))\) to be a ball with center \((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots)\) and radius \(\delta\). For a given \(f \in C(\mathcal{V}_\infty)\), one can conclude that, \(\forall \epsilon > 0, \exists \delta < \frac{1}{k(k+1)+1}\), such that \(\forall x \in B_\delta((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))\),
\[
|f(x) - f((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))| < \epsilon.
\]
Thus,
\[
\int_{\mathcal{V}_\infty} |f(x)\pi_{\lambda,\theta}(dx) - f((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))| \pi_{\lambda,\theta}(dx)
\leq \int_{\mathcal{V}_\infty} |f(x) - f((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))| \pi_{\lambda,\theta}(dx)
\leq \int_{\mathcal{V}_\infty} |f(x)| \pi_{\lambda,\theta}(dx)
\]

Thus,
\[
\int_{\mathcal{V}_\infty} |f(x)\pi_{\lambda,\theta}(dx) - f((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))| \pi_{\lambda,\theta}(dx)
\leq \int_{\mathcal{V}_\infty} |f(x) - f((\frac{1}{k},\cdots,\frac{1}{k},0,\cdots))| \pi_{\lambda,\theta}(dx)
\leq \int_{\mathcal{V}_\infty} |f(x)| \pi_{\lambda,\theta}(dx)
\]
Moreover, we claim that
\[ \lim_{\theta \to 0} \pi_{\lambda, \theta}(S_\lambda \geq \delta) = \lim_{\theta \to 0} \pi_{\lambda, \theta}\left(|\varphi_2 - \frac{1}{k}| \geq \delta\right) = 0. \]

Moreover, we claim that
\[ (S_\lambda < \delta) \cap \left(|\varphi_2 - \frac{1}{k}| < \delta\right) \subset B_\delta\left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots\right). \]

Then we have
\[ \lim_{\delta \to 0} \sup_{\theta} \int_{\varphi_2} f(x)\pi_{\lambda, \theta}(dx) - f\left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots\right) \leq \epsilon. \]

Letting \( \epsilon \to 0 \), we have
\[ \lim_{\theta \to 0} \int_{\varphi_2} f(x)\pi_{\lambda, \theta}(dx) = f\left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots\right). \]

Therefore, \( \pi_{\lambda, \theta} \) converges weakly to \( \delta_{\left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots\right)} \). Now we need to show the claim \([5]\). For \( \lambda = (k + 1)k \), since
\[ S_\lambda(x) = J(x) + (k + 1)k\varphi_2(x) - \inf_{n \geq 1} \left\{ \frac{(k + 1)k}{n} + n - 1 \right\}, \]
clearly
\[ S_\lambda|_{L_n}(x) = n - 1 + k(k + 1)\varphi_2|_{L_n}(x) - 2k. \]

Because \( \varphi_2|_{L_n}(x) \) has a unique minimum point \( \left(\frac{1}{n}, \ldots, \frac{1}{n}, 0, \ldots\right) \); then
\[ S_\lambda|_{L_n}(x) \geq n - 1 + \frac{k(k + 1)}{n} - 2k. \]

By the monotonicity of the righthand function in \( n \), we know it attains its minimum at \( k \) and \( k + 1 \). Since \( \delta < \frac{1}{k(k + 1) + 1} < \frac{2}{k + 2} \), one can see, \( \forall n \neq k, k + 1 \),
\[ S_\lambda|_{L_n}(x) \geq \min \left\{ k - 1 + \frac{k(k + 1)}{k - 1} - 2k, k + 2 - 1 + \frac{k(k + 1)}{k + 2} - 2k \right\} = \frac{2}{k + 2} > \delta. \]

Then \( (S_\lambda < \delta) = (S_\lambda < \delta) \cap (L_k \cup L_{k+1}) \). Thus,
\[ (S_\lambda < \delta) \cap \left(|\varphi_2 - \frac{1}{k}| < \delta\right) \]
\[ = \left[(S_\lambda < \delta) \cap \left(|\varphi_2 - \frac{1}{k}| < \delta\right) \cap L_k\right] \cup \left[(S_\lambda < \delta) \cap \left(|\varphi_2 - \frac{1}{k}| < \delta\right) \cap L_{k+1}\right]. \]
But $S_{\lambda} \mid L_{k+1} = k(k+1)(\varphi_2 \mid L_{k+1} - \frac{1}{k+1})$, then $(S_{\lambda} < \delta) \cap L_{k+1} = (\varphi_2 < \frac{\delta}{k(k+1)} + \frac{1}{k+1}) \cap L_{k+1}$. Since $\delta < \frac{1}{k(k+1)+1}$, then $\frac{\delta}{k(k+1)} + \frac{1}{k+1} < \frac{1}{k}$, thus $\forall x \in L_{k+1} \cap (|\varphi_2 - \frac{1}{k}| < \delta) \cap (S_{\lambda} < \delta)$, we have
\[ \varphi_2(x) < \frac{\delta}{k(k+1)} + \frac{1}{k+1} < \frac{1}{k} - \delta < \varphi_2(x). \]
Therefore, $(S_{\lambda} < \delta) \cap (|\varphi_2 - \frac{1}{k}| < \delta) \cap L_{k+1} = \emptyset$, hence,
\[ (S_{\lambda} < \delta) \cap \left( |\varphi_2 - \frac{1}{k}| < \delta \right) = (S_{\lambda} < \delta) \cap \left( |\varphi_2 - \frac{1}{k}| < \delta \right) \cap L_k. \]
Since $\forall x \in L_k \cap (|\varphi_2(x) - \frac{1}{k}| < \delta)$, we have $\frac{1}{k} - \delta < \sum_{i=1}^{k} x_i^2 < \frac{1}{k} + \delta$, and
\[ \frac{1}{k^2} - \delta \leq \frac{\sum_{i=1}^{k} x_i^2}{k} < \frac{1}{k^2} + \delta. \]
Therefore, $\sqrt{\frac{1}{k^2} - \frac{\delta}{k}} < \min_{1 \leq i \leq k} x_i \leq \max_{1 \leq i \leq k} x_i \leq \sqrt{\frac{1}{k^2} + \frac{\delta}{k}}$. Then
\[ d \left( x, \left( \frac{1}{k}, \cdots, \frac{1}{k}, 0, \cdots \right) \right) = \sum_{i=1}^{k} \frac{|x_i - 1/k|}{2^i} \leq \sum_{i=1}^{k} \frac{1}{2^i} \max_{1 \leq i \leq k} |x_i - 1/k| = \left( 1 - \frac{1}{2^k} \right) \max_{1 \leq i \leq k} |x_i - 1/k| \leq \max \left\{ \left| \sqrt{\frac{1}{k^2} - \frac{\delta}{k}} - \frac{1}{k} \right|, \left| \sqrt{\frac{1}{k^2} + \frac{\delta}{k}} - \frac{1}{k} \right| \right\} \leq \max \left\{ \frac{\frac{\delta}{k}}{\sqrt{\frac{1}{k^2} - \frac{\delta}{k}} + \frac{1}{k}}, \frac{\frac{\delta}{k}}{\sqrt{\frac{1}{k^2} + \frac{\delta}{k}} + \frac{1}{k}} \right\} \leq \delta. \]
Therefore $(S_{\lambda} < \delta) \cap (|\varphi_2 - \frac{1}{k}| < \delta) \subset B_\delta(\frac{1}{k}, \cdots, \frac{1}{k}, 0, \cdots)$, the claim (5) is thus proved. \hfill \square

3. Proof of Lemmas

The LDP estimations of binomial distribution and negative binomial distribution will be needed in this section. Hence, example 1 is presented.

Example 1. Let $X_{\alpha}^k = \sum_{i=1}^{k} Y_{\alpha}^i$, and $U_{\alpha}^k = \sum_{i=1}^{k} V_{\alpha}^i$, where $\{Y_{\alpha}^i, 1 \leq l \leq k\}$ and $\{V_{\alpha}^i, 1 \leq l \leq k\}$ are i.i.d. geometric random variables and Bernoulli random variables respectively; i.e.
\[ P(Y_{\alpha}^i = u) = (1 - \alpha)^u \alpha, u \geq 0; \quad P(V_{\alpha}^i = v) = \alpha^v (1 - \alpha)^{1-v}, v = 0, 1. \]
Then the distributions of $X_{\alpha}^k$ and $U_{\alpha}^k$, denoted by $\mu_k$ and $\nu_k$, satisfy LDPs with speed $k$ and rate function $I_1(x)$ and $I_2(x)$ respectively, where
\[ I_1(x) = x \log x - (x + 1) \log(1 + x) - [x \log(1 - \alpha) + \log \alpha] \]
and
\[ I_2(x) = x \log \frac{x}{\alpha} + (1 - x) \log \frac{1 - x}{1 - \alpha}. \]
Proof. This can be shown by the Cramér theorem. Please refer to [Dembo and Zeitouni, 2010]. □

Next, we will embark on a long journey to prove the previous lemmas.

[PROOF OF LEMMA 1]:

Proof. Define $V_1 = U_1, V_i = (1 - U_1) \cdots (1 - U_{i-1})U_i, i \geq 2$, where $\{U_i, i \geq 1\}$ are i.i.d. Beta$(1, \theta)$. Then $(V_1, V_2, \cdots)$ follows the GEM distribution. Denote $(V_1, V_2, \cdots)$ to be the descending statistics of $(V_1, V_2, \cdots)$; then $(V_1, V_2, \cdots)$ follows the Poisson-Dirichlet distribution PD$(\theta)$. Since $H_2 = \sum_{i=1}^{\infty} V_i = \sum_{i=1}^{\infty} V_i^2$, one can observe that $1 - H_2 = (1 - U_1^2) - (1 - U_1)^2 + (1 - U_1)^2(1 - H_2) = 2U_1(1 - U_1) + (1 - U_1)^2(1 - H_2)$, where $H_2 = \sum_{i=1}^{\infty} V_i^2$, and $\tilde{V}_i = U_2, \tilde{V}_i = (1 - U_2) \cdots (1 - U_i)U_{i+1}, i \geq 2$. We can see that $(\tilde{V}_1, \tilde{V}_2, \cdots)$ follows the GEM distribution as well and is independent of $U_1$. Thus, $E(1 - H_2)^k = E(1 - \tilde{H}_2)^k$, and

$$m_k = E(1 - H_2)^k = E \left(2U_1(1 - U_1) + (1 - U_1)^2(1 - \tilde{H}_2)\right)^k$$

$$= E \sum_{l=0}^{k} \binom{k}{l} (2U_1(1 - U_1))^{k-l} (1 - \tilde{H}_2)^l$$

$$= \sum_{l=0}^{k} \binom{k}{l} E(2U_1(1 - U_1))^{k-l} E(1 - \tilde{H}_2)^l$$

$$= \sum_{l=0}^{k} \binom{k}{l} 2^{k-l-1} \Gamma(k - l + 1) \Gamma(k + l + \theta) \theta m_l \frac{\Gamma(2k + 1 + \theta)}{\Gamma(2k + 1 + \theta)} m_l.$$

If we isolate $m_k$, we have

$$m_k = \theta \sum_{l=1}^{k-1} \frac{2k + \theta}{2k} \frac{2k!}{2^l l!} \frac{2^l l!}{\Gamma(2k + 1 + \theta)} m_l + \frac{2k + \theta}{2k} \frac{2^l l!}{\Gamma(2k + 1 + \theta)} \frac{2^k k! \Gamma(k + \theta)}{\Gamma(2k + 1 + \theta)},$$

where $k \geq 2$, and $m_1 = \frac{\theta}{1 + \theta}$. We claim that $m_k$ has the following expansion

$$m_k = \sum_{l=1}^{k} A_{k,l}(\theta) \theta^l,$$

where

$$A_{k,l}(\theta) = \sum_{u=1}^{k-1} \frac{2k + \theta}{2k} \frac{2^l l!}{\Gamma(2k + 1 + \theta)} A_{u,l-1}(\theta), l \geq 2;$$

$$A_{k,1}(\theta) = \frac{2k + \theta}{2k} \frac{2k! \Gamma(k + \theta)}{\Gamma(2k + 1 + \theta)}.$$
Thus, we are going to show

\[ A_{k,1}(\theta) = \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + \theta)}{\Gamma(2k + 1)} \]

\[ A_{k,p}(\theta) = \sum_{l=p}^{k-1} \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + l + \theta)}{2^l! \Gamma(2k + 1 + \theta)} A_{l,p-1}(\theta), \quad p \geq 2. \]

\[ \square \]

**Proof.** We use mathematical induction with respect to \( p \) to show these conclusions.

**Step 1:** We are going to show \( \frac{1}{2^p} A_{k,p+1} \leq A_{k,p+1}(\theta) \leq A_{k,p+1} \).

When \( p = 1 \), we have, \( \forall \theta \in [0, 1] \),

\[ A_{k,1}(\theta) = \frac{2k + \theta}{2k} \frac{2^k k!}{(2k + 1)!} = \frac{2^k k!}{(2k + 1)!} A_{k,1} \]

and

\[ A_{k,1}(\theta) = \frac{2^k k!}{(2k + 1)!} \frac{(2k + 1 + \theta)!}{\Gamma(2k + 1 + \theta)} A_{k,1} \]

Moreover, \( A_{k,1} = \frac{2^k k!}{(2k)!} \leq 1 < \frac{1}{2^k} = 2 \); therefore,

\[ \frac{1}{2} A_{k,1} \leq A_{k,1}(\theta) \leq A_{k,1} \leq \frac{1}{2^k}. \]

Now we assume that \( A_{k,p}(\theta) \) satisfies the inequality

\[ \frac{1}{2p} A_{k,p} \leq A_{k,p}(\theta) \leq A_{k,p} \leq \frac{1}{2p-2}, \quad k \geq p \geq 1. \]

Since \( A_{k,p+1}(\theta) = \sum_{l=p}^{k-1} \frac{2^k k!}{2^l!} \frac{A_{l,p}(\theta)}{(2l + 1)! (k + l + 1)!} \), by assumption \( \text{[1]} \), we have

\[ \frac{1}{2p} \sum_{l=p}^{k-1} \frac{2^k k!}{2^l!} \frac{A_{l,p}}{(2l + 1)! (k + l + 1)!} \leq A_{k,p+1}(\theta) \]

\[ \leq \sum_{l=p}^{k-1} \frac{2^k k!}{2^l!} \frac{A_{l,p}}{(2l + 1)! (k + l + 1)!}. \]

Thus,

\[ \frac{1}{2p} \sum_{l=p}^{k-1} \frac{k + l}{2k} \frac{2^k k!}{2^l!} \frac{\Gamma(k + l)}{\Gamma(2k + 1)} A_{l,p} \leq A_{k,p+1}(\theta) \leq \sum_{l=p}^{k-1} \frac{2^k k!}{2^l!} \frac{\Gamma(k + l)}{\Gamma(2k + 1)} A_{l,p} = A_{k,p+1}. \]

But \( \frac{k + l}{2k} > \frac{k}{2k} = \frac{1}{2} \), then

\[ \frac{1}{2p} \sum_{l=p}^{k-1} \frac{k + l}{2k} \frac{2^k k!}{2^l!} \frac{\Gamma(k + l)}{\Gamma(2k + 1)} A_{l,p} \geq \frac{1}{2p+1} \sum_{l=p}^{k-1} \frac{2^k k!}{2^l!} \frac{\Gamma(k + l)}{\Gamma(2k + 1)} A_{l,p} = \frac{1}{2p+1} A_{k,p+1}. \]

Hence,

\[ \frac{1}{2p+1} A_{k,p+1} \leq A_{k,p+1}(\theta) \leq A_{k,p+1}. \]
Step 2: We are going to show

\[ A_{k,p} \leq \frac{1}{2^{p-2}}, \forall k \geq p \geq 1. \]

Since \( A_{k,1} \leq \frac{1}{2^{k-2}} \), we can assume that \( A_{k,p} \leq \frac{1}{2^{p-2}}, k \geq p \). So

\[
A_{k,p+1} = \sum_{l=p}^{k-1} \frac{2^k k^l l! \Gamma((k + l)/2) \Gamma(2k + 1)}{2^l l! \Gamma(2k + 1)} A_{l,p} \leq \frac{1}{2^{p-2}} \sum_{l=p}^{k-1} \frac{2^k k^l \Gamma((k + l)/2) \Gamma(2k + 1)}{2^l l! \Gamma(2k + 1)},
\]

where we will show that \( \sum_{l=p}^{k-1} \frac{2^k k^l \Gamma((k + l)/2)}{2^l l! \Gamma(2k + 1)} < \frac{1}{2} \). Therefore, \( A_{k,p+1} \leq \frac{1}{2^{p-2}} \), and (8) is thus proved. Next, to show \( \sum_{l=p}^{k-1} \frac{2^k k^l \Gamma((k + l)/2)}{2^l l! \Gamma(2k + 1)} < \frac{1}{2} \), let us define \( B_{k,l} = \frac{2^k k^l \Gamma((k + l)/2)}{2^l l! \Gamma(2k + 1)} \). Since \( k - 1 \leq l \leq p, k \geq 2 \), we can assume that \( B_{k,l} \) is increasing in \( l \) only if \( l < k - 2 \). Because when \( l < k - 2 \)

\[
B_{k,l+1} = \frac{2^l l! \Gamma((k + l)/2) \Gamma(2k + 1)}{2^{l+1}(l+1)! \Gamma(2k + 1)} = \frac{k + l + 1}{2(l + 1)} > 1.
\]

Therefore, \( B_{k,k-2} \) or \( B_{k,k-1} \) should be the maximum term. Since

\[
B_{k,k-2} = \frac{2^k k! \Gamma(2k - 2)}{2^{k-2} (k - 2)! \Gamma(2k + 1)} = \frac{1}{2k - 1},
\]

\[
B_{k,k-1} = \frac{2^k k! \Gamma(2k - 1)}{2^{k-1} (k - 1)! \Gamma(2k + 1)} = \frac{1}{2k - 1},
\]

we obtain that

\[
\sum_{l=p}^{k-1} \frac{2^k k^l \Gamma((k + l)/2)}{2^l l! \Gamma(2k + 1)} \leq k - p \leq \frac{k - p}{2(k - p) + 2p - 1} < \frac{1}{2}.
\]

Step 3: We are going to show

\[
|A_{k,p}(\theta) - A_{k,p}| \leq \theta p A_{k,p}.
\]

When \( p = 1 \), \( |A_{k,1}(\theta) - A_{k,1}| \leq \theta A_{k,1} \), for

\[
|A_{k,1}(\theta) - A_{k,1}| = \left| \frac{2^k k! \Gamma(k + \theta)}{2^k k! \Gamma(2k + \theta)} - \frac{2^k k! \Gamma(k)}{2^k k! \Gamma(2k + 1)} \right| \leq \theta A_{k,1} \left| \frac{2^k k! \Gamma(k)}{2^k k! \Gamma(2k + 1)} \right| \leq \theta A_{k,1} \left| \frac{2^k k! \Gamma(k)}{2^k k! \Gamma(2k + 1)} \right|.
\]
\[
\leq \theta A_{k,1} \sum_{u=1}^{k} \sum_{k \leq l_1 < \cdots < l_u \leq 2k-1} \frac{1}{l_1 \cdots l_u} \\
= \theta A_{k,1} \left( (1 + \frac{1}{2k-1}) \cdots (1 + \frac{1}{k}) - 1 \right) \\
= \theta A_{k,1} \left( \frac{2k}{2k-1} \cdots \frac{k+1}{k} - 1 \right) \geq \theta A_{k,1} |2 - 1| = \theta A_{k,1}.
\]

Therefore we assume that

\[
(9) \quad \left| A_{k,p}(\theta) - A_{k,p} \right| \leq \theta p A_{k,p},
\]

then, for \( |A_{k,p+1}(\theta) - A_{k,p+1}| \), we have

\[
|A_{k,p+1}(\theta) - A_{k,p+1}| = \left| \sum_{l=p}^{k-1} \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + l + \theta)} A_{l,p} \right| - \left| \sum_{l=p}^{k-1} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + l + \theta)} A_{l,p} \right|
\]

\[
\leq \left| \sum_{l=p}^{k-1} \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + l + \theta)} (A_{l,p} - A_{l,p}) \right|
\]

\[
+ \left| \sum_{l=p}^{k-1} \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + l + \theta)} A_{l,p} \right| - \left| \sum_{l=p}^{k-1} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + l + \theta)} A_{l,p} \right|
\]

\[
\leq \sum_{l=p}^{k-1} \left| \frac{2k + \theta}{2k} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + l + \theta)} A_{l,p} \right| | \Gamma(k + l + \theta) \Gamma(2k) - \Gamma(2k + \theta) - 1 |.
\]

By the assumption \( (9) \), we have

\[
|A_{k,p+1}(\theta) - A_{k,p+1}| \leq \theta p \sum_{l=p}^{k-1} \frac{2^k k! \Gamma(k + l + \theta)}{2^l l! \Gamma(2k + \theta)} A_{l,p}
\]

\[
+ \sum_{l=p}^{k-1} \frac{2^k k! \Gamma(k + l)}{2^l l! \Gamma(2k + l + \theta)} A_{l,p} | \Gamma(k + l + \theta) \Gamma(2k) - \Gamma(2k + \theta) | - 1 |
\]

where

\[
| \Gamma(k + l + \theta) \Gamma(2k) \Gamma(2k + \theta) | - 1 = | \frac{(2k - 1) \cdots (k + l)}{(2k + \theta - 1) \cdots (k + l + \theta)} - 1 |
\]

\[
= \left| 1 - \frac{\theta}{2k - 1 + \theta} \right| \cdots \left| 1 - \frac{\theta}{k + l + \theta} \right| - 1 |
\]

\[
= \sum_{u=1}^{k-l} (-1)^u \sum_{k+l \leq l_1 < \cdots < l_u \leq 2k-1} \left| \frac{\theta^u}{(l_1 + \theta) \cdots (l_u + \theta)} \right|
\]

\[
\leq \sum_{u=1}^{k-l} \sum_{k+l \leq l_1 < \cdots < l_u \leq 2k-1} \left| \frac{\theta^u}{(l_1 + \theta) \cdots (l_u + \theta)} \right|
\]
\[ \leq \theta \sum_{u=1}^{k-l} \sum_{k+1 \leq t_1 < \ldots < t_u \leq 2k-1} \frac{1}{t_1 \cdots t_u} \]
\[ = \theta \left| (1 + \frac{1}{2k-1}) \cdots (1 + \frac{1}{k+l}) - 1 \right| \]
\[ = \frac{k-l}{k+l} < \theta, \]
and
\[ \frac{\Gamma(k+l+\theta)}{2k\Gamma(2k+\theta)} \leq \frac{1}{2k(2k-1)\cdots(k+l+\theta)} = \frac{\Gamma(k+l)}{\Gamma(2k+1)}. \]

Therefore,
\[ \left| A_{k,p+1}(\theta) - A_{k,p+1} \right| \leq \theta \sum_{l=p}^{k-1} \frac{2^kk!}{2l!} \frac{\Gamma(k+l+\theta)}{\Gamma(2k+1)} A_{l,p} + \theta \sum_{l=p}^{k-1} \frac{2^kk!}{2l!} \frac{\Gamma(k+l)}{\Gamma(2k+1)} A_{l,p} \]
\[ = \theta(p+1)A_{k,p+1}. \]

Thus, we have proved the lemma. \( \square \)

[PROOF OF LEMMA 3]:

Proof. By Lemma 2, we have
\[ \sum_{l=0}^{\lambda+1} \theta^l \sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,l}(\theta) \leq \sum_{l=0}^{\lambda+1} \theta^l \sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} \frac{1}{2^{l-2}} \]
\[ = 4 \sum_{l=0}^{\lambda+1} \frac{\theta^l}{2^{l}} \sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} \]
\[ \leq 4 \sum_{l=0}^{\lambda+1} \frac{(\theta/2)^l e^{\lambda \log \frac{1}{\theta}}}{2^{l}} = 4 \sum_{l=0}^{\lambda+1} \frac{(\theta/2)^l e^{\lambda \log \frac{1}{\theta}}}{2^{l}} = 4 \frac{(\theta/2)^{\lambda+1}}{2\lambda+1} \frac{2}{2-\theta} \]
\[ \to 0, \text{ as } \theta \to 0, \text{ due to } [\lambda]+1 > \lambda. \]

Similarly, we can also show
\[ \lim_{\theta \to 0} \sum_{l=0}^{\lambda+1} \theta^l \sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k+n,l}(\theta) = 0. \]

[PROOF OF LEMMA 5]:

Proof. By mathematical induction with respect to \( p \), we can prove this lemma. For \( p = 1 \), by Stirling’s formula,
\[ \Gamma(z) \sim \sqrt{\frac{2\pi}{x}} \left( \frac{z}{e} \right)^z, \]

(10)
we have $A_{k,1} \sim \sqrt{\frac{\pi}{2} \left(\frac{1}{2}\right)^k}$, as $k \to +\infty$. We can therefore assume that, as $k \to +\infty$,

$$\tag{11} A_{k,p} \sim C_p \frac{1}{k^2} \left(\frac{p}{p+1}\right)^k, \quad C_p = C_{p-1} \sqrt{\frac{\pi}{2}} \left(\frac{p+1}{p-1}\right)^{\frac{p-1}{p+1}}.$$ 

For $A_{k,p+1}$, we have

$$A_{k,p+1} = \sum_{l=p}^{k-1} \frac{2^{k-l} \Gamma(k+l)}{2^l \Gamma(2k+1)} A_{l,p}.$$ 

By the assumption (11), $\forall \epsilon > 0$, $\exists M > 0$, such that $\forall k > M$,

$$1 - \epsilon < \frac{A_{k,p}}{C_p \frac{1}{k^2} \left(\frac{p}{p+1}\right)^k} < 1 + \epsilon;$$

then we rewrite $A_{k,p+1}$ as $X + Y$, where

$$X = \sum_{l=p}^{M} \frac{2^{k-l} \Gamma(k+l)}{2^l \Gamma(2k+1)} A_{l,p}, \quad Y = \sum_{l=M+1}^{k-1} \frac{2^{k-l} \Gamma(k+l)}{2^l \Gamma(2k+1)} A_{l,p},$$

Define $a_k(l) = \frac{2^{k-l} \Gamma(k+l)}{2^l \Gamma(2k+1)} \left(\frac{p}{p+1}\right)^l$, and $\Sigma_1 = \sum_{l=p}^{k-1} C_p \frac{1}{k^2} a_k(l)$. Now we are going to show lim$_{k \to +\infty}$ $\frac{X}{\Sigma_1} = 0$, and lim$_{k \to +\infty}$ $\frac{Y}{\Sigma_1} = 1$.

Indeed, since

$$0 \leq X \leq \frac{\max_{p \leq l \leq M} \{A_{l,p}\}(M-p+1)}{2^p} \frac{2^{k-l} \Gamma(k+M)}{\Gamma(2k+1)}$$

and

$$(1 - \epsilon) \sum_{l=M+1}^{k-1} C_p \frac{1}{l^2} a_k(l) \leq Y \leq (1 + \epsilon) \sum_{l=M+1}^{k-1} C_p \frac{1}{l^2} a_k(l),$$

we have

$$0 \leq \frac{X}{\Sigma_1} \leq \frac{X}{\sum_{l=p}^{M} \{A_{l,p}\}(M-p+1)} \frac{2^{k-l} \Gamma(k+M)}{\Gamma(2k+1)} \leq \max_{p \leq l \leq M} \frac{\{A_{l,p}\}(M-p+1)}{2^p} \frac{2^{k-l} \Gamma(k+M)}{\Gamma(2k+1)} a_k(k-1).$$

By Stirling’s formula (10), we have, as $k \to +\infty$,

$$\tag{12} (k-1)^{\frac{1}{2}} \left(\frac{2(p+1)}{p}\right)^{k-1} \frac{\Gamma(k+M)}{\Gamma(2k-1)} \sim k^{M+\frac{1}{2}} (\frac{p+1}{2p})^{k-1} \to 0.$$ 

Thus, lim$_{k \to +\infty}$ $\frac{X}{\Sigma_1} = 0$. Similarly,

$$0 \leq \sum_{l=p}^{M} C_p \frac{1}{l^2} a_k(l) \leq \frac{C_p(M-p+1)}{p^2 2^p} \frac{2^{k-l} \Gamma(k+M)}{\Gamma(2k+1)}.$$ 

then

$$0 \leq \frac{\sum_{l=p}^{M} C_p \frac{1}{l^2} a_k(l)}{\Sigma_1} \leq \frac{(M-p+1)(\frac{p}{p+1})^p}{p^2 2^p} \frac{2^{k-l} \Gamma(k+M)}{\Gamma(2k+1)}.$$

By Stirling’s formula (10), we have, as $k \to +\infty$,
Thus, \( \frac{\sum_{i=p}^{M} C_p a_k(l)}{\Sigma_1} \to 0 \) due to 12. Therefore, for the following inequality,

\[
(1 - \epsilon) \left( 1 - \frac{\sum_{i=p}^{M} C_p a_k(l)}{\Sigma_1} \right) \leq \frac{Y}{\Sigma_1} \leq (1 + \epsilon) \left( 1 - \frac{\sum_{i=p}^{M} C_p a_k(l)}{\Sigma_1} \right),
\]

if we let \( k \to +\infty \), and \( \epsilon \to 0 \), then we have

\[
\lim_{k \to \infty} \frac{Y}{\Sigma_1} = 1;
\]

hence \( A_{k,p+1} \sim \Sigma_1 \). Moreover,

\[
a_k(l) = \frac{2^k k!(k - 1)!}{\Gamma(2k + 1)} \left( \frac{2(p + 1)}{p + 2} \right)^k \Gamma(k + l) \left( 1 - \frac{p + 2}{2(p + 1)} \right)^l \left( \frac{p + 2}{2(p + 1)} \right)^k.
\]

Let \( X_{\alpha}^k \) be negative binomial, \( NB(k, \alpha) \), where \( \alpha = \frac{p + 1}{2(p + 1)} \), then

\[
a_k(l) = \frac{2^k k!(k - 1)!}{\Gamma(2k + 1)} \left( \frac{2(p + 1)}{p + 2} \right)^k P(X_{\alpha}^k = l),
\]

and

\[
\Sigma_1 = C_p \frac{2^k k!(k - 1)!}{\Gamma(2k + 1)} \left( \frac{2(p + 1)}{p + 2} \right)^k \sum_{l=p}^{k-1} \frac{1}{l^k} P(X_{\alpha}^k = l).
\]

We claim that, as \( k \to +\infty \),

\[
\sum_{l=p}^{k-1} \frac{1}{l^k} P(X_{\alpha}^k = l) \sim \frac{1}{l_0^k} \sum_{l=p}^{k-1} P(X_{\alpha}^k = l) \sim \frac{1}{l_0^k},
\]

where \( l_0 = \frac{(1-\alpha)k}{\alpha} \). Therefore,

\[
\Sigma_1 \sim \frac{1}{l_0^k} C_p \frac{4^k k!(k - 1)!}{\Gamma(2k + 1)} \left( \frac{p + 1}{p + 2} \right)^k.
\]

Then, by Stirling’s formula 10, we know

\[
A_{k,p+1} \sim \Sigma_1 \sim C_p \sqrt{\pi} \left( \frac{p + 2}{p} \right)^{p/2} \frac{1}{k^{p/2}} \left( \frac{p + 1}{p + 2} \right)^k.
\]

This lemma is thus proved. Now we only need to show claim 13. Indeed,

\[
\sum_{l=p}^{k-1} \frac{1}{l^k} P(X_{\alpha}^k = l) = \sum_{l=p}^{k-1} \left( \frac{l_0}{T} \right)^p P(X_{\alpha}^k = l).
\]

\( \forall \epsilon > 0 \), we have

\[
\sum_{l=p}^{k-1} \left( \sqrt{\frac{l_0}{T}} \right)^p P(X_{\alpha}^k = l)
\]

\[
= \sum_{p \leq l \leq l_0(1-\epsilon)} \left( \sqrt{\frac{l_0}{T}} \right)^p P(X_{\alpha}^k = l) + \sum_{l_0(1-\epsilon) \leq l \leq l_0(1+\epsilon)} \left( \sqrt{\frac{l_0}{T}} \right)^p P(X_{\alpha}^k = l)
\]

\[
+ \sum_{l_0(1+\epsilon) \leq l \leq k-1} \left( \sqrt{\frac{l_0}{T}} \right)^p P(X_{\alpha}^k = l),
\]
where
\[
0 \leq \sum_{p \leq l \leq l_0(1-\epsilon)} \left( \sqrt{\frac{l_0}{p}} \right)^p P(X^k_\alpha = l) \leq \left( \frac{l_0}{p} \right)^p \sum_{p \leq l \leq l_0(1-\epsilon)} P(X^k_\alpha = l) = \left( \frac{l_0}{p} \right)^p P(X^k_\alpha \leq l_0(1-\epsilon)),
\]

\[
0 \leq \sum_{l_0(1+\epsilon) \leq l \leq k-1} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \leq \left( \frac{1}{1+\epsilon} \right)^p \sum_{l_0(1+\epsilon) \leq l \leq k-1} P(X^k_\alpha = l) \leq \left( \frac{1}{1+\epsilon} \right)^p P(X^k_\alpha \geq l_0(1+\epsilon)),
\]

and
\[
\left( \frac{l_0}{l_0(1+\epsilon)} \right)^p P(l_0(1-\epsilon) \leq X^k_\alpha \leq l_0(1+\epsilon)) \leq \sum_{l_0(1-\epsilon) \leq l \leq l_0(1+\epsilon)} \left( \frac{l_0}{l} \right)^p P(X^k_\alpha = l) \leq \left( \frac{l_0}{l_0(1-\epsilon)} \right)^p P(l_0(1-\epsilon) \leq X^k_\alpha \leq l_0(1+\epsilon)).
\]

By LDP for \(NB(k, \alpha)\) in example 1, we have
\[
P(X^k_\alpha \leq l_0(1-\epsilon)) \sim e^{-k \left[ \inf_{x < -\epsilon + 1} \frac{\epsilon}{\alpha} I_1(x) \right]},
\]

and
\[
P(X^k_\alpha \geq l_0(1+\epsilon)) \sim e^{-k \left[ \inf_{x > \epsilon + 1} \frac{1}{\alpha} I_1(x) \right]},
\]

Therefore, as \(k \rightarrow +\infty\),
\[
\sum_{p \leq l \leq l_0(1-\epsilon)} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \rightarrow 0,
\]

and
\[
\sum_{l_0(1+\epsilon) \leq l \leq k-1} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \rightarrow 0.
\]

By the central limit theorem of \(X^k_\alpha\), if we let \(k \rightarrow +\infty, \epsilon \rightarrow 0\), we have
\[
\sum_{l_0(1-\epsilon) \leq l \leq l_0(1+\epsilon)} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \rightarrow 1.
\]

The claim (13) is thus proved. \(\square\)

[PROOF OF LEMMA 6:]

\[
\begin{align*}
0 & \leq \sum_{p \leq l \leq l_0(1-\epsilon)} \left( \sqrt{\frac{l_0}{p}} \right)^p P(X^k_\alpha = l) \leq \left( \frac{l_0}{p} \right)^p \sum_{p \leq l \leq l_0(1-\epsilon)} P(X^k_\alpha = l) \\
& = \left( \frac{l_0}{p} \right)^p P(X^k_\alpha \leq l_0(1-\epsilon)),
\end{align*}
\]

\[
0 \leq \sum_{l_0(1+\epsilon) \leq l \leq k-1} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \leq \left( \frac{1}{1+\epsilon} \right)^p \sum_{l_0(1+\epsilon) \leq l \leq k-1} P(X^k_\alpha = l) \leq \left( \frac{1}{1+\epsilon} \right)^p P(X^k_\alpha \geq l_0(1+\epsilon)),
\]

and
\[
\left( \frac{l_0}{l_0(1+\epsilon)} \right)^p P(l_0(1-\epsilon) \leq X^k_\alpha \leq l_0(1+\epsilon)) \leq \sum_{l_0(1-\epsilon) \leq l \leq l_0(1+\epsilon)} \left( \frac{l_0}{l} \right)^p P(X^k_\alpha = l) \leq \left( \frac{l_0}{l_0(1-\epsilon)} \right)^p P(l_0(1-\epsilon) \leq X^k_\alpha \leq l_0(1+\epsilon)).
\]

By LDP for \(NB(k, \alpha)\) in example 1, we have
\[
P(X^k_\alpha \leq l_0(1-\epsilon)) \sim e^{-k \left[ \inf_{x < -\epsilon + 1} \frac{\epsilon}{\alpha} I_1(x) \right]},
\]

and
\[
P(X^k_\alpha \geq l_0(1+\epsilon)) \sim e^{-k \left[ \inf_{x > \epsilon + 1} \frac{1}{\alpha} I_1(x) \right]},
\]

Therefore, as \(k \rightarrow +\infty\),
\[
\sum_{p \leq l \leq l_0(1-\epsilon)} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \rightarrow 0,
\]

and
\[
\sum_{l_0(1+\epsilon) \leq l \leq k-1} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \rightarrow 0.
\]

By the central limit theorem of \(X^k_\alpha\), if we let \(k \rightarrow +\infty, \epsilon \rightarrow 0\), we have
\[
\sum_{l_0(1-\epsilon) \leq l \leq l_0(1+\epsilon)} \left( \sqrt{\frac{l_0}{l}} \right)^p P(X^k_\alpha = l) \rightarrow 1.
\]

The claim (13) is thus proved. \(\square\)
Proof. Define
\[ \Sigma_2 = \sum_{s=0}^{k-l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k-s)^2} \left( \frac{l}{l+1} \right)^{k-s}. \]
Then
\[ \Sigma_2 = C_l \left( \frac{\lambda - l + l}{\lambda} \right) \sum_{s=0}^{k-l} \frac{1}{(k-s)^2} P(X_{\beta}^k = s), \]
where \( X_{\beta}^k \) follows binomial distribution \( B(k, \beta) \), with \( \beta = \frac{k_{\lambda}}{\lambda^2 + k_{\lambda}} \).

Next, we show, as \( k \to +\infty \),
\[ \sum_{s=0}^{k-l} \frac{1}{(k-s)^2} P(X_{\beta}^k = s) \sim \left( \frac{\beta}{(1-\beta)s_0} \right)^{\frac{1}{2}}, \]
where \( s_0 = \beta k \). To this end, \( \forall \epsilon > 0 \), let us consider
\[ \sum_{s=0}^{k-l} \frac{1}{(k-s)^2} P(X_{\beta}^k = s) = \sum_{s=0}^{k-l} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s). \]
\( \forall \epsilon > 0 \), we have,
\[ \sum_{s=0}^{k-l} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) \]
\[ = \sum_{0 \leq s \leq s_0(1-\epsilon)} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) + \sum_{(1-\epsilon)s_0 \leq s \leq (1+\epsilon)s_0} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) \]
\[ + \sum_{(1+\epsilon)s_0 \leq s \leq k-1} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s), \]
where
\[ 0 \leq \sum_{0 \leq s \leq s_0(1-\epsilon)} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) \leq \left( \sqrt{s_0 \frac{1}{k-s_0(1-\epsilon)}} \right)^{l} P(X_{\beta}^k \leq s_0(1-\epsilon)) \]
and
\[ 0 \leq \sum_{(1+\epsilon)s_0 \leq s \leq k-1} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) \leq \left( \sqrt{s_0 \frac{1}{k-(1+\epsilon)s_0}} \right)^{l} P(X_{\beta}^k \geq s_0(1+\epsilon)). \]

Then by the LDP for \( B(k, \beta) \) in example 1, we have
\[ \lim_{k \to +\infty} \sum_{0 \leq s \leq s_0(1-\epsilon)} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) = 0, \]
and
\[ \lim_{k \to +\infty} \sum_{k-l \geq s \geq s_0(1+\epsilon)} \left( \sqrt{s_0 \frac{1}{k-s}} \right)^{l} P(X_{\beta}^k = s) = 0. \]
Moreover,
\[ \left( \sqrt{s_0 \frac{1}{k-(1+\epsilon)s_0}} \right)^{l} P((1-\epsilon)s_0 \leq X_{\beta}^k \leq (1+\epsilon)s_0) \]
\[ \to \frac{\beta}{(1-\beta)s_0} \]
\[
\begin{align*}
\sum_{(1-\epsilon)s_0 \leq s \leq (1+\epsilon)s_0} \left( \sqrt{\frac{s_0}{k-s}} \right)^l P(X_\beta^k = s) \\
\leq \left( \sqrt{\frac{s_0}{k-(1-\epsilon)s_0}} \right)^l P((1-\epsilon)s_0 \leq X_\beta^k \leq (1+\epsilon)s_0).
\end{align*}
\]

Analogously, if we let \( k \to \infty \) and \( \epsilon \to 0 \), then we have
\[
\sum_{(1-\epsilon)s_0 \leq s \leq (1+\epsilon)s_0} \left( \sqrt{\frac{s_0}{k-s}} \right)^l P(X_\beta^k = s) \to \left( \frac{\beta}{1-\beta} \right)^\frac{1}{2},
\]
due to the central limit theorem of binomial distributions. Therefore,
\[
\Sigma_2 \sim C_l \left( \frac{1}{1-\beta} \right)^\frac{1}{2} \frac{1}{k^{l/2}} \left( \frac{\lambda-l}{\lambda} + \frac{l}{l+1} \right)^k.
\]

Next we are going to show \( C_{k,l} \sim \Sigma_2 \). Because of Lemma 5, \( \forall \epsilon > 0, \exists M > 1, \) such that \( \forall k-s > M, \)
\[
1 - \epsilon \leq \frac{A_{k-s,l}}{C_l (k-s)^{l/2} (l+1)} \leq 1 + \epsilon.
\]
Then we rewrite \( C_{k,l} \) as \( A + B \), where
\[
A = \sum_{s=0}^{k-M-1} \binom{k}{s} \left( \frac{\lambda-l}{\lambda} \right)^s A_{k-s,l}
\]
and
\[
B = \sum_{k-M \leq s \leq k-l} \binom{k}{s} \left( \frac{\lambda-l}{\lambda} \right)^s A_{k-s,l}.
\]
Since
\[
0 \leq B \leq \max_{k-M \leq s \leq k-l} \left\{ A_{k-s,l} \right\} \binom{k}{k-M} \left( \frac{\lambda-l}{\lambda} \right)^{k-M},
\]
and
\[
0 \leq \frac{B}{\Sigma_2} \leq \frac{\max_{k-M \leq s \leq k-l} \left\{ A_{k-s,l} \right\} \binom{k}{k-M} \left( \frac{\lambda-l}{\lambda} \right)^{k-M}}{C_l M!} \times \frac{1}{(k-s)^{l/2} (l+1)^2} \left( \frac{\lambda-l}{\lambda} \right)^M \left( 1 - \beta \right)^{l/2} k^{l/2} + M \left( \frac{\lambda-l}{\lambda} + \frac{l}{l+1} \right)^k
\]
\[
\to 0.
\]
we can easily see \( \lim_{k \to +\infty} \frac{B}{\Sigma_2} = 0 \). Moreover,
\[
(1-\epsilon) \sum_{0 \leq s \leq k-M-1} \binom{k}{s} \left( \frac{\lambda-l}{\lambda} \right)^s C_l \frac{1}{(k-s)^{l/2}} \left( \frac{l}{l+1} \right)^{k-s} \leq A
\]
\[
\leq (1+\epsilon) \sum_{0 \leq s \leq k-M-1} \binom{k}{s} \left( \frac{\lambda-l}{\lambda} \right)^s C_l \frac{1}{(k-s)^{l/2}} \left( \frac{l}{l+1} \right)^{k-s}.
\]
and also
\[
(1 - \epsilon) \left( \sum_{k - M \leq s \leq k - l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k - s) \lambda^s} \left( \frac{l}{l + 1} \right)^k \right)
\]
\[
\leq A \leq (1 + \epsilon) \left( \sum_{k - M \leq s \leq k - l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k - s) \lambda^s} \left( \frac{l}{l + 1} \right)^k \right).
\]

Since
\[
0 \leq \sum_{k - M \leq s \leq k - l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k - s) \lambda^s} \left( \frac{l}{l + 1} \right)^k \leq C_l \frac{1}{l^l} \binom{k}{k - M} \left( \frac{\lambda - l}{\lambda} \right)^{k - M},
\]
we have
\[
0 \leq \frac{\sum_{k - M \leq s \leq k - l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k - s) \lambda^s} \left( \frac{l}{l + 1} \right)^k}{\sum_{k - M \leq s \leq k - l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k - s) \lambda^s} \left( \frac{l}{l + 1} \right)^k} \leq \frac{C_l \frac{1}{l^l} \binom{k}{k - M} \left( \frac{\lambda - l}{\lambda} \right)^{k - M}}{\sum_{k - M \leq s \leq k - l} \binom{k}{s} \left( \frac{\lambda - l}{\lambda} \right)^s C_l \frac{1}{(k - s) \lambda^s} \left( \frac{l}{l + 1} \right)^k} \to 0 \text{ as } k \to \infty.
\]

Hence, letting \( k \to \infty, \epsilon \to 0 \), we have \( A \sim \Sigma_2 \). Thus, \( C_{k,l} \sim \Sigma_2 \). Then,
\[
C_{k,l} \sim \Sigma_2 \sim C_l \left( \frac{1}{1 - \beta} \right)^{\frac{1}{k^{\beta}}} \frac{1}{l^l} \left( \frac{\lambda - l}{\lambda} + \frac{l}{l + 1} \right)^k
\]
\[
= C_l \left( 1 + \frac{(\lambda - l)(l + 1)}{\lambda} \right)^{\frac{1}{k^{\beta}}} \frac{1}{l^l} \left( \frac{\lambda - l}{\lambda} + \frac{l}{l + 1} \right)^k.
\]

\[
\square
\]

**Proof.** Let us assume that
\[
\lim_{\theta \to 0} \tilde{K}_n^\lambda(\theta) = \left( \frac{u}{u - 1} \right)^n \text{ for } u(u - 1) < \lambda \leq u(u + 1), u > 2;
\]
then we are going to show \( \lim_{\theta \to 0} K_n^\lambda(\theta) = \lim_{\theta \to 0} \tilde{K}_n^\lambda(\theta) \). Note that \( K_n^\lambda(\theta) \) can be rewritten as
\[
K_n^\lambda(\theta) = \frac{\tilde{K}_n^\lambda(\theta) + F_n^\lambda(\theta)}{1 + G^\lambda(\theta)},
\]
where
\[
F_n^\lambda(\theta) = \sum_{l=1}^{[\lambda]} l^l \sum_{k=1}^{[\lambda]} \sum_{k=1}^{[\lambda]} \frac{\lambda k l^l}{k!} \left( A_{k+n,l}(\theta) - A_{k+n,l} \right),
\]
and
\[
G^\lambda(\theta) = \sum_{l=1}^{[\lambda]} l^l \sum_{k=1}^{[\lambda]} \sum_{k=1}^{[\lambda]} \frac{\lambda k l^l}{k!} \left( A_{k,l}(\theta) - A_{k,l} \right).
\]
We claim that \( \lim_{\theta \to 0} F_n^\lambda(\theta) = \lim_{\theta \to 0} G^\lambda(\theta) = 0 \). Then
\[
\lim_{\theta \to 0} K_n^\lambda(\theta) = \lim_{\theta \to 0} \tilde{K}_n^\lambda(\theta)
\]
follows. Indeed, by Lemma 2 we have

\[ 0 \leq |F_n^\lambda(\theta)| \leq \sum_{l=1}^{[\lambda]} \theta^l \sum_{k=1}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} |A_{k+n,l}(\theta) - A_{k,n,l}| \]

Similarly, by Lemma 2, we also have

\[ \sum_{l=1}^{[\lambda]} \theta^l \sum_{k=1}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k+n,l} \]

Then, we need to show

\[ \lim_{\theta \to 0} \lambda \theta^l \sum_{k=1}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k+n,l} = [\lambda] \theta K_n^\lambda(\theta) = 0. \]

Now we are going to show the assumption (15). We can rewrite \( K_n(\theta) \) as

\[ \sum_{v=1}^{[\lambda]} \sum_{k=1}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k+n,v} \]

Then,

\[ \lim_{\theta \to 0} K_n^\lambda(\theta) = \sum_{v=1}^{[\lambda]} \lim_{\theta \to 0} \frac{\theta^v}{\theta^l} \sum_{k=1}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k+n,v}. \]

By Lemma 4 and Lemma 5 we know

\[ \lim_{\theta \to 0} \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k+n,v} = \frac{v}{v+1}. \]

Then we need to show

\[ \lim_{\theta \to 0} \theta^v \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} = \delta(v_1)(v). \]

Once we have obtained this, then \( \lim_{\theta \to 0} K_n^\lambda(\theta) = \sum_{v=1}^{[\lambda]} \delta(v_1)(v) \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k+n,v} = (\frac{v}{v+1})^n. \)

To this end, both the numerator and the denominator of

\[ \frac{\theta^v}{\theta^l} \sum_{k=1}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} \sum_{k=l}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,n,l} \]

are divided by \( \theta^\lambda \). Thus, we need consider

\[ \frac{\theta^{v-\lambda}}{\theta^{l-\lambda}} \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} \sum_{k=l}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,n,l}. \]

Since \( 1 \leq v \leq [\lambda] \), it is not difficult to see that

\[ \theta^{v-\lambda} \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} = \left( \frac{1}{\theta^\lambda} \right)^{v-\lambda} \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} \]

\[ \left( \sum_{s=0}^\infty \frac{(\log \frac{1}{\theta})^s}{s!} (\lambda - v)^s \right) \left( \sum_{k=v}^\infty \frac{(\frac{\log \frac{1}{\theta}}{k})^k}{k!} A_{k,v} \right) \]
Thus, by Lemma 4 and Lemma 6, we have

$$\lambda \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} C_{k,v},$$

where

$$C_{k,v} = \sum_{s=0}^{k-v} \binom{k}{s} \left( \frac{\lambda - v}{\lambda} \right)^s A_{k-s,v}.$$

Then

$$\sum_{l=1}^{\lfloor \lambda \rfloor} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} A_{k,l} = \sum_{l=1}^{\lfloor \lambda \rfloor} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} C_{k,l}.$$

Thus, to figure out the limit (16), we must find the leading term among

$$\sum_{k=l}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} C_{k,l}, 1 \leq l \leq \lfloor \lambda \rfloor.$$

By Lemma 4 and Lemma 6 we have

$$\lim_{\theta \to 0} \sum_{l=1}^{\lfloor \lambda \rfloor} \sum_{k=1}^{\infty} \frac{(\lambda \log \frac{1}{\theta})^k}{k!} C_{k,l} = \lim_{k \to +\infty} \frac{C_{k,v}}{C_{k,l}}$$

$$= \lim_{k \to +\infty} \frac{(1 + \frac{(\lambda - v)(v+1)}{\lambda})^\frac{1}{2}}{(1 + \frac{(\lambda - l)(l+1)}{\lambda})^\frac{1}{2}} \left( \frac{\lambda - v}{\lambda} + \frac{u}{l+1} \right)^k.$$

To find the leading term, we need to figure out the maximum term among \(\frac{\lambda - l}{\lambda} + \frac{1}{l+1}\), \(1 \leq l \leq \lfloor \lambda \rfloor\). Consider \(f(x) = \frac{\lambda - x}{\lambda} + \frac{x}{x+1} = 2 - (\frac{1}{x} + \frac{1}{x+1})\); then

$$f'(x) = \frac{1}{(x+1)^2} - \frac{1}{\lambda}.$$  

We know \(f'(x)\) is

\[
\begin{cases} 
  \geq 0, & \text{if } x \leq \sqrt{\lambda} - 1 \\
  < 0, & \text{if } x > \sqrt{\lambda} - 1.
\end{cases}
\]

Therefore, \(\frac{\lambda - l}{\lambda} + \frac{1}{l+1}\) attains its maximum at \([\sqrt{\lambda}] - 1\) or \([\sqrt{\lambda}]\).

**Case 1:** For \((u - 1)u < \lambda < u^2\), \([\sqrt{\lambda}] = u - 1\), since

$$f(u - 2) = 2 - \left( \frac{u - 2}{\lambda} + \frac{1}{u - 1} \right),$$

and

$$f(u - 1) = 2 - \left( \frac{u - 1}{\lambda} + \frac{1}{u} \right),$$

$$f(u - 2) - f(u - 1) = \frac{1}{\lambda} - \frac{1}{u(u-1)} < 0.$$  

So \(f(u - 1)\) is the maximum term.

$$\frac{\lambda - u + 1}{\lambda} + \frac{u - 1}{u} < 1, \forall 1 \leq l \leq \lfloor \lambda \rfloor, l \neq u - 1.$$

Thus

$$\lim_{k \to +\infty} \frac{C_{k,l}}{C_{k,u-1}} = \delta_{u-1}(l), \forall 1 \leq l \leq \lfloor \lambda \rfloor.$$
**Case 2:** For \( u^2 \leq \lambda < u(u + 1) \), \( \sqrt{\lambda} = u \); then the maximum term of \( \frac{\lambda - l}{\lambda} + \frac{l}{l+1} \) should be \( f(u - 1) \) or \( f(u) \). Since

\[
\frac{\lambda - l}{\lambda} + \frac{l}{l+1} < 1, \forall 1 \leq l \leq \lfloor \lambda \rfloor, l \neq u - 1.
\]

we can see \( f(u - 1) \) is the maximum term; and

\[
\frac{\lambda - u + 1}{\lambda} + \frac{u - 1}{u} < 1, \forall 1 \leq l \leq \lfloor \lambda \rfloor.
\]

Thus,

\[
\lim_{k \to +\infty} \frac{C_{k,l}}{C_{k,u-1}} = \delta_{u-1}(l), \forall 1 \leq l \leq \lfloor \lambda \rfloor.
\]

**Case 3:** For \( \lambda = u(u + 1) \), \( \sqrt{\lambda} = u \), then the maximum term of \( \frac{\lambda - l}{\lambda} + \frac{l}{l+1} \) is \( f(u - 1) \) or \( f(u) \). Since

\[
\frac{\lambda - l}{\lambda} + \frac{l}{l+1} < 1, \forall 1 \leq l \leq \lfloor \lambda \rfloor, l \neq u - 1,
\]

one can have

\[
\frac{\lambda - u + 1}{\lambda} + \frac{u - 1}{u} < 1, \forall 1 \leq l \leq \lfloor \lambda \rfloor, l \neq u - 1,
\]

and

\[
\frac{\lambda - u + 1}{\lambda} + \frac{u - 1}{u} = 1.
\]

But

\[
\lim_{k \to +\infty} \frac{C_{k,u}}{C_{k,u-1}} = \lim_{k \to +\infty} \frac{(1 + (\lambda - u + 1)v) \frac{1}{k^v}}{(1 + (\lambda - u + 1)v) \frac{1}{k^v}} = 0;
\]

therefore, \( C_{k,u-1} \) is the leading term among \( C_{k,l}, 1 \leq l \leq \lfloor \lambda \rfloor \). We have

\[
\lim_{k \to +\infty} \frac{C_{k,l}}{C_{k,u-1}} = \delta_{u-1}(l), 1 \leq l \leq \lfloor \lambda \rfloor.
\]

Thus,

\[
\lim_{\theta \to u} \sum_{l=1}^{\lfloor \lambda \rfloor} \theta^l \sum_{k=1}^{\infty} \frac{(\lambda \frac{1}{k})^k}{k!} A_{k,l} = \delta_{u-1}(v), 1 \leq v \leq \lfloor \lambda \rfloor.
\]

\[
\Box
\]

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