CONGRUENCE PAIRS OF PRINCIPAL MS-ALGEBRAS AND PERFECT EXTENSIONS

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Abstract. The notion of a congruence pair for principal MS-algebras, simpler than the one given by Beazer for K₂-algebras [6], is introduced. It is proved that the congruences of the principal MS-algebras \( L \) correspond to the MS-congruence pairs on simpler substructures \( L^{\circ} \) and \( D(L) \) of \( L \) that were associated to \( L \) in [4].

An analogy of a well-known Grätzer’s problem [11, Problem 57] formulated for distributive p-algebras, which asks for a characterization of the congruence lattices in terms of the congruence pairs, is presented here for the principal MS-algebras (Problem 1). Unlike a recent solution to such a problem for the principal p-algebras in [2], it is demonstrated here on the class of principal MS-algebras, that a possible solution to the problem, though not very descriptive, can be simple and elegant.

As a step to a more descriptive solution of Problem 1, a special case is then considered when a principal MS-algebra \( L \) is a perfect extension of its greatest Stone subalgebra \( L_S \). It is shown that this is exactly when de Morgan subalgebra \( L^{\circ} \) of \( L \) is a perfect extension of the Boolean algebra \( B(L) \). Two examples illustrating when this special case happens and when it does not are presented.

1. Introduction

Blyth and Varlet introduced MS-algebras abstracting de Morgan and Stone algebras in [7] and [8]. In [4] a class of principal MS-algebras was introduced and a simple triple construction of the principal MS-algebras was presented. This was motivated by the second author constructions in [13] and [14]. A one-to-one correspondence between the class of principle MS-algebras and the class of principle MS-triples was proved in [4]. For recent studies of (principle) MS-algebras see also [1] and [3].

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*See Remark 1.3.
In Section 2 of this paper we present some properties of the principal MS-algebras by means of principal MS-triples. Then in Section 3 we introduce the notion of congruence pairs in the principal MS-algebras which is simpler than the notion of congruence pairs given by Beazer in [6] for MS-algebras from the subvariety $K_2$. We prove that every congruence $\theta$ on a principal MS-algebra $L$ determines so-called MS-congruence pair on a triple associated to $L$ and conversely, each MS-congruence pair $(\theta_1, \theta_2)$ of a principle MS-triple uniquely determines a congruence $\theta$ on the principle MS-algebra $L$ associated to it.

Our work in this paper has mainly been motivated by a well-known Grätzer’s problem [11, Problem 57], which was formulated for distributive p-algebras as follows:

Let $B$ be a Boolean algebra, let $D$ be a distributive lattice with unit, and let $A$ be a sublattice of $\text{Con}(B) \times \text{Con}(D)$. Under what conditions does there exist a distributive lattice with pseudocomplementation $L$ such that $B(L) \cong B$, $D(L) \cong D$, and $A$ consists of all congruence pairs of $L$?

In [15] Corollary 1 Katriňák solved this problem and he in fact characterized, in terms of congruence pairs, the congruence lattice of any modular p-algebra (cf. [15] Theorem 1). This characterization has been then generalised to quasi-modular p-algebras by El-Assar [10]. An analogy of the Grätzer’s problem for the principal p-algebras has been considered by the first author in [2].

In Section 4 we firstly present an analogy of the Grätzer’s problem for the principal MS-algebras (Problem 1). Compared to the solution for the principal p-algebras in [2] which aims in a special case (with bounded lattice $D$) to mimic Katriňák’s solution in the general case (with unbounded $D$), our solution here for the principal MS-algebras is much simpler and a short and elegant proof is given.

Then we consider the analogy of the Grätzer’s problem for the principal MS-algebras in a special case when a principal MS-algebra $L$ is a perfect extension of its greatest Stone subalgebra $L_S$. This case is firstly reduced to the property that a de Morgan subalgebra $L^{\vee}$ of $L$ is a perfect extension of its Boolean subalgebra $B(L^{\vee}) = B(L)$. In Section 5 we illustrate on two examples when this special case happens and when it does not.

2. Preliminaries

An Ockham algebra is an algebra $(L; \lor, \land, f, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \lor, \land, 0, 1)$ is a bounded distributive lattice and $f$ is a unary operation such that $f(0) = 1$, $f(1) = 0$ and for all $x, y \in L$,

$$f(x \land y) = f(x) \lor f(y) \quad \text{and} \quad f(x \lor y) = f(x) \land f(y).$$

For an Ockham algebra $(L; \lor, \land, f, 0, 1)$, the subset

$$S(L) := \{f(x) \mid x \in L\}$$
is a subalgebra of $L$ which is called the skeleton of $L$.

By a de Morgan-Stone algebra or briefly an MS-algebra is meant an algebra $(L; \lor, \land, ^*, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \lor, \land, 0, 1)$ is a bounded distributive lattice and $^*$ is a unary operation such that for all $x, y \in L$

\begin{align*}
\text{(MS1)} & \quad x \leq x^{**}; \\
\text{(MS2)} & \quad (x \land y)^* = x^* \lor y^*; \\
\text{(MS3)} & \quad 1^* = 0.
\end{align*}

The classes of all Ockham algebras and of all MS-algebras are equational, i.e. form varieties of algebras. Their relationship is described in [9, Theorem 1.4] as follows: Every MS-algebra is an Ockham algebra with a de Morgan skeleton. An Ockham algebra $(L; \lor, \land, f, 0, 1)$ is an MS-algebra if and only if $x \leq f^2(x)$ for all $x \in L$.

Among important subvarieties of MS-algebras is the class of de Morgan algebras that satisfy the additional identity

\[ x = x^{**}. \]

Another important subvariety is that of Kleene algebras which are de Morgan algebras satisfying the identity

\[ (x \land x^* \lor y \lor y^*) = y \lor y^*. \]

The subvariety of Stone algebras is characterized by the identity $x \land x^* = 0$. The subvariety of $K_2$-algebras is characterized by the identity (MS5) together with the identity

\[ x \land x^* = x^{**} \land x^*. \]

Finally, the subvariety of MS-algebras characterized by the identity $x \lor x^* = 1$ is the well-known class of Boolean algebras.

For an MS-algebra $L$, important subsets playing a role in its investigation are:

(i) the subalgebra $L^{**} = \{ x \in L \mid x = x^{**} \}$ of closed elements of $L$ which is a de Morgan algebra (this is the skeleton $S(L)$ of $L$ when $L$ is considered as an Ockham algebra);

(ii) the filter $D(L) = \{ x \in L \mid x^* = 0 \}$ of dense elements of $L$;

(iii) the Boolean subalgebra $B(L) = \{ x \in L \mid x \lor x^* = 1 \}$ of complemented elements of $L$;

(iv) the generalisation of $B(L)$, the subalgebra $L_S = \{ x \in L \mid x^* \lor x^{**} = 1 \}$ of the elements of $L$ satisfying the Stone identity which is obviously the greatest Stone subalgebra of $L$.

It is easy to see that $B(L) = B(L^{**}) = B(L_S) = L_S^{**}$ and $D(L) = D(L_S)$. If $L$ is a Stone algebra itself, then clearly $B(L) = L^{**}$ and $L_S = L$. It is also obvious that if $L$ is an MS-algebra with a smallest dense element $d_L$, then the map $\varphi(L) : L^{**} \to D(L)$ defined by $\varphi(L)(a) = a \lor d_L$ is a $(0, 1)$-lattice homomorphism.
We recall the following definitions from [4] motivated by the work in [14].

**Definition 2.1.** An MS-algebra \((L; \lor, \land, ^*, 0, 1)\) is called a principal MS-algebra if it satisfies the following conditions:

(i) The filter \(D(L)\) is principal, i.e. there exists an element \(d_L \in L\) such that \(D(L) = [d_L]\);

(ii) \(x = x^* \land (x \lor d_L)\) for any \(x \in L\).

**Definition 2.2.** An (abstract) principal MS-triple is \((M, D, \varphi)\), where

(i) \(M\) is a de Morgan algebra;

(ii) \(D\) is a bounded distributive lattice;

(iii) \(\varphi\) is a \((0, 1)\)-lattice homomorphism from \(M\) into \(D\).

In [4] we proved the following result.

**Theorem 2.3.** Let \((M, D, \varphi)\) be a principal MS-triple. Then

\[ L = \{(x, y) \mid x \in M, y \in D, y \leq \varphi(x)\} \]

is a principal MS-algebra, if one defines

\[
\begin{align*}
(x_1, y_1) \lor (x_2, y_2) &= (x_1 \lor x_2, y_1 \lor y_2) \\
(x_1, y_1) \land (x_2, y_2) &= (x_1 \land x_2, y_1 \land y_2) \\
(x, y)^* &= (x^*, \varphi(x^*)) \\
1_L &= (1, 1) \\
0_L &= (0, 0).
\end{align*}
\]

Also \(L^* = \{(x, \varphi(x)) \mid x \in M\} \cong M\) and \(D(L) = \{(1_M, y) \mid y \in D\} \cong D\).

In [13] the second author introduced a concept of so-called \(K_2\)-triple. By a principal \(K_2\)-triple is meant a triple \((K, D, \varphi)\) in which \(K\) is a Kleene algebra and \(\varphi(K^\land) = \{0_D\}\). In [13 Corollary 1] a restriction of Theorem 2.3 was proven for the principal \(K_2\)-triples which says that if the principal MS-triple \((M, D, \varphi)\) is a principal \(K_2\)-triple, then the MS-algebra \(L\) from Theorem 2.3 is a principal \(K_2\)-algebra, that is, a \(K_2\)-algebra with a principal filter \(D(L)\). We now make a small addition to this result.

While Theorem 2.3 shows that the filter \(D(L)\) of the principal MS-algebra \(L\) associated with a principal MS-triple \((M, D, \varphi)\) is isomorphic to \(D\), we can consider and describe the larger filter \(L^\lor = \{x \lor x^* \mid x \in L\}\) of the principal \(K_2\)-algebra \(L\) associated with a principal \(K_2\)-triple \((K, D, \varphi)\). It is evident that \(L^\lor \supseteq D(L)\). And dually, we can describe the ideal \(L^\land = \{x \land x^* \mid x \in L\}\) of the principal \(K_2\)-algebra \(L\).
Proposition 2.4. Let \((K, D, \varphi)\) be a principal \(K_2\)-triple. Then for the principal \(K_2\)-algebra \(L\) obtained from it by the construction given in Theorem 2.3 we have the following descriptions of its filter \(L^\vee\) and the ideal \(L^\wedge\):

\[
L^\vee = \{(x, y) \in L \mid x \in K^\vee\},
\]
\[
L^\wedge = \{(x, y) \in L \mid x \in K^\wedge\}.
\]

Proof. Let \((K, D, \varphi)\) be a principal \(K_2\)-triple. Using the construction in Theorem 2.3 we obtain a principal \(K_2\)-algebra

\[
L = \{(x, y) \mid x \in K, y \in D, y \leq \varphi(x)\}.
\]

Now

\[
L^\vee = \{(x, y) \vee (x, y)^* \mid (x, y) \in L\}
\]
\[
= \{(x \vee x^*, y \vee \varphi(x^*)) \mid x \in K, y \in D, y \leq \varphi(x)\}
\]
\[
= \{(x \vee x^*, y') \mid x \in K, y' \in D, y' \leq \varphi(x \vee x^*)\}
\]
\[
= \{(x', y') \in L \mid x' \in K^\vee\}.
\]

Dually one can derive the description of \(L^\wedge\). \(\square\)

By a principal \(S\)-triple we mean a principal \(K_2\)-triple \((B, D, \varphi)\) where \(B\) is a Boolean algebra.

Corollary 2.5. Let \((B, D, \varphi)\) be a principal \(S\)-triple. Then the principal \(K_2\)-algebra \(L\) from Theorem 2.3 is a principal Stone algebra with \(L^\vee = D(L)\) and \(L^\wedge = \{0_L\}\).

Proof. Let \((x, y) \in L\). Then we have

\[
(x, y)^* \vee (x, y)^* = (x, \varphi(x)) \vee (x^*, \varphi(x^*))
\]
\[
= (x \vee x^*, \varphi(x \vee x^*))
\]
\[
= (1_M, 1_D),
\]
as \(x \vee x^* = 1\). Hence \(L\) is a Stone algebra. By Proposition 2.4 we obtain

\[
L^\vee = \{(x, y) \in L \mid x \in B^\vee\}
\]
\[
= \{(1_B, y) \in L \mid y \in D\}
\]
\[
= D(L).
\]

and

\[
L^\wedge = \{(x, y) \in L \mid x \in B^\wedge\}
\]
\[
= \{(0_B, y) \mid y \in D, y \leq \varphi(0_B)\}
\]
\[
= \{(0_B, 0_D)\} = \{0_L\}.
\]

\(\square\)
3. Congruence pairs of principal MS-algebras

Let $L$ be a principal MS-algebra with a smallest dense element $d_L$. For a congruence relation $\theta$ of $L$, let $\theta_{L^{\infty}}$ and $\theta_{D(L)}$ denote the restrictions of $\theta$ to $L^{\infty}$ and $D(L) = [d_L]$, respectively. Let $\text{Con}(L^{\infty})$ and $\text{Con}(D(L))$ denote the congruence lattices of the de Morgan algebra $L^{\infty}$ and of the distributive lattice $D(L)$, respectively. Obviously, $(\theta_{L^{\infty}}, \theta_{D(L)}) \in \text{Con}(L^{\infty}) \times \text{Con}(D(L))$.

**Definition 3.1.** Let $L$ be a principal MS-algebra with a smallest dense element $d_L$. A pair of congruences $(\theta_1, \theta_2) \in \text{Con}(L^{\infty}) \times \text{Con}(D(L))$ will be called an MS-congruence pair if

$$(CP)\quad (a, b) \in \theta_1 \text{ implies } (a \lor d_L, b \lor d_L) \in \theta_2.$$

We notice that in [6] R. Beazer defined the notion of a congruence pair for the MS-algebras from the subvariety $K_2$ by two conditions. More precisely, for a $K_2$-algebra $L$ he called a pair of congruences $(\theta_1, \theta_2) \in \text{Con}(L^\infty) \times \text{Con}(L^\prime)$ a $K_2$-congruence pair if

$$(CP_1)\quad (c, d) \in \theta_2 \text{ implies } (c^*, d^*) \in \theta_1;$$

$$(CP_2)\quad (a, b) \in \theta_1 \text{ and } c \in L^\prime \text{ imply } (a \lor c, b \lor d_L) \in \theta_2.$$

One can show that for our MS-congruence pairs $(\theta_1, \theta_2)$, the Beazer first condition $(CP_1)$ is trivially satisfied and so it can be removed from our definition: the reason is that we consider the congruence $\theta_2$ on a smaller filter $D(L) \subseteq L^\prime$. It is also clear that for the principal MS-algebras, the Beazer second condition $(CP_2)$ restricted to the filter $D(L)$ implies our condition $(CP)$ since his condition is quantified for all elements $c$ of the filter. Yet our simpler condition $(CP)$ is equivalent to $(CP_2)$ restricted to $D(L)$ because if $(\theta_1, \theta_2) \in \text{Con}(L^{\infty}) \times \text{Con}(D(L))$ is an MS-congruence pair, then $(a, b) \in \theta_1$ implies $(a \lor d_L, b \lor d_L) \in \theta_2$, whence obviously $(a \lor c, b \lor c) = (a \lor d_L \lor c, b \lor d_L \lor c) \in \theta_2$ for all $c \in D(L)$.

Now we apply our definition of the MS-congruence pair to show that the congruences of principal MS-algebras correspond to the MS-congruence pairs.

**Theorem 3.2.** Let $L$ be a principal MS-algebra with a smallest dense element $d_L$. For every congruence $\theta$ on $L$, the restrictions of $\theta$ to $L^{\infty}$ and $D(L) = [d_L]$ determine the MS-congruence pair $(\theta_{L^{\infty}}, \theta_{D(L)})$.

Conversely, every MS-congruence pair $(\theta_1, \theta_2)$ uniquely determines a congruence $\theta$ on $L$ satisfying $\theta_{L^{\infty}} = \theta_1$ and $\theta_{D(L)} = \theta_2$. This congruence can be defined by the rule

$$(x, y) \in \theta \quad \text{if} \quad (x^*, y^*) \in \theta_1 \text{ and } (x \lor d_L, y \lor d_L) \in \theta_2.$$

**Proof.** For every $\theta \in \text{Con}(L)$, the restrictions of $\theta$ to $L^{\infty}$ and $[d_L]$ determine the MS-congruence pair $(\theta_{L^{\infty}}, \theta_{D(L)})$ because $(a, b) \in \theta_{L^{\infty}}$ gives $(a \lor d_L, b \lor d_L) \in \theta$, thus $(a \lor d_L, b \lor d_L) \in \theta_{D(L)}$. 


Now let \((\theta_1, \theta_2) \in \Con(L^\circ) \times \Con(D(L))\) be an MS-congruence pair and \(\theta\) be the relation on \(L\) defined by the above rule. Clearly \(\theta\) is an equivalence. To show that \(\theta\) is a congruence of \(L\), let \((a, b) \in \theta\) and \((c, d) \in \theta\). Then by the definition of \(\theta\), \((a^\circ, b^\circ), (c^\circ, d^\circ) \in \theta_1\) and \((a \lor d_L, b \lor d_L), (c \lor d_L, d \lor d_L) \in \theta_2\). As \(\theta_1 \in \Con(L^\circ)\), we get \((a \lor c^\circ, b \lor d^\circ) \in \theta_1\), so \[((a \land c)^\circ, (b \land d)^\circ) \in \theta_1\]. Using the distributivity of \(L\) we have \(((a \land c) \lor d_L, (b \land d) \lor d_L) \in \theta_2\). This shows that \((a \land c, b \land d) \in \theta\), so \(\theta\) is preserved by the meet operation. To show that \(\theta\) is preserved by the join operation is immediate as the congruence \(\theta_1\) of \(L^\circ\) contains with the pairs \((a^\circ, b^\circ), (c^\circ, d^\circ)\) also the pair \((a^\circ \land c^\circ, b^\circ \land d^\circ)\), so \(((a \lor c)^\circ, (b \lor d)^\circ) \in \theta_1\). And clearly \(((a \lor c) \lor d_L, (b \lor d) \lor d_L) \in \theta_2\). Hence \((a \lor c, b \lor d) \in \theta_2\).

Now let \((a, b) \in \theta\). Then \((a^\circ, b^\circ) \in \theta_1\) and \((a^\circ, b^\circ) \in \theta_2\). From \((a^\circ, b^\circ) \in \theta_1\) we have \((a^\circ \land d_L, b^\circ \land d_L) \in \theta_2\) by Definition 3.1. Hence \((a^\circ, b^\circ) \in \theta\).

Next we will show that \(\theta_{L^\circ} = \theta_1\) and \(\theta_{D(L)} = \theta_2\). To show \(\theta_1 \leq \theta_{L^\circ}\), let \((a, b) \in \theta_{L^\circ}\) and \((a, b) \in \theta_1\). Then \((a^\circ, b^\circ) \in \theta_1\) so \((a \lor d_L, b \lor d_L) \in \theta_2\) as \((\theta_1, \theta_2)\) is an MS-congruence pair. Hence \((a, b) \in \theta\) and \((a, b) \in \theta_{L^\circ}\). Conversely, let \((a, b) \in \theta_{L^\circ}\), so \((a^\circ, b^\circ) \in \theta_1\). Then \((a^\circ, b^\circ) \in \theta_1\), whence \((a, b) \in \theta_1\).

Now let \((c, d) \in \theta_{D(L)}\) and \((c, d) \in \theta_2\). Then \((c \lor d_L, d \lor d_L) \in \theta_2\) and using the fact that \(c^\circ = d^\circ = 0\) we get \((c, d) \in \theta\), which means \((c, d) \in \theta_{D(L)}\). Conversely, let \((c, d) \in \theta_{D(L)}\). Then \((c, d) \in \theta\) and since \(c = c \lor d_L, d = d \lor d_L\), we obtain \((c, d) \in \theta_2\). Hence \(\theta_{D(L)} = \theta_2\).

To show the uniqueness of \(\theta\), let \(\theta\) and \(\theta'\) be congruences on \(L\) such that \(\theta_{L^\circ} = \theta_{L^\circ}' = \theta_1\) and \(\theta_{D(L)} = \theta_{D(L)}' = \theta_2\). Then \((x, y) \in \theta\) gives \((x^\circ, y^\circ) \in \theta_{L^\circ}\) and \((x \lor d_L, y \lor d_L) \in \theta_{D(L)}\) and also \((x^\circ, y^\circ) \in \theta'_{L^\circ}\) and \((x \lor d_L, y \lor d_L) \in \theta'_{D(L)}\). Therefore \((x^\circ \land (x \lor d_L), y^\circ \land (y \lor d_L)) \in \theta'\). As \(L\) is a principal MS-algebra with a smallest dense element \(d_L\), we get \((x, y) \in \theta'\). The inclusion \(\theta' \subseteq \theta\) can be shown analogously. Hence \(\theta = \theta'\).

The set of all MS-congruence pairs of \(L\) will be denoted by \(A(L)\) which is a traditional notation for the congruence pairs of algebras in the literature.

Corollary 3.3. Let \(L\) be a principal MS-algebra with a smallest dense element \(d_L\). The set \(A(L)\) of MS-congruence pairs of \(L\) forms a sublattice of \(\Con(L^\circ) \times \Con(D(L))\) and \(\theta \mapsto (\theta_{L^\circ}, \theta_{D(L)})\) is an isomorphism between \(\Con(L)\) and \(A(L)\).

Proof. Take \((\phi_1, \psi_1), (\phi_2, \psi_2) \in A(L)\). It is evident that \((\phi_1 \land \phi_2, \psi_1 \land \psi_2) \in A(L)\). To prove that \((\theta_1 \lor \psi_1, \theta_2 \lor \psi_2) \in A(L)\), let \((a, b) \in \theta_1 \lor \psi_1\). Then there exists a finite sequence \(a = a_0, a_1, \ldots, a_n = b\) in \(L^\circ\) with \((a_{i-1}, a_i) \in \theta_1 \lor \psi_1\) for \(i \in \{1, \ldots, n\}\). By Definition 3.1 we have \((a_{i-1} \lor d_L, a_i \lor d_L) \in \theta_2 \lor \psi_2\). Hence the sequence \((a \lor d_L = a_0 \lor d_L, a_1 \lor d_L, \ldots, a_n \lor d_L = b \lor d_L)\)
in $D(L)$ is witnessing $(a \lor d_L, b \lor d_L) \in \theta_2 \lor \psi_2$. Thus $(\theta_1 \lor \psi_1, \theta_2 \lor \psi_2) \in A(L)$ showing that $A(L)$ is a sublattice of $\text{Con}(L^{\text{m}^*}) \times \text{Con}(D(L))$. From Theorem 3.2 it follows that the map $\theta \mapsto (\theta_L^{*\text{m}}, \theta_D(L))$ is an isomorphism. \hfill \Box

4. Grätzer’s Problem and Perfect Extensions

In this section we compare our study with that for (distributive) $p$-algebras. For the following basic facts about $p$-algebras see [16]. We recall that a $p$-algebra is an algebra $(L, \lor, \land, *, 0, 1)$ with a bounded lattice reduct $(L, \lor, \land, 0, 1)$ such that the unary operation of pseudocomplementation $*$ is defined on $L$ by $x \land a = 0$ if and only if $x \leq a^*$.

The first proof that the class of all $p$-algebras is equational is due to P. Ribenboim [17]. A $p$-algebra $L$ is said to be distributive (modular) if the underlying lattice $(L, \lor, \land, 0, 1)$ is distributive (modular). Further, a $p$-algebra $L$ is called quasi-modular if it satisfies the identity $((x \land y) \lor z^{**}) \land x = (x \land y) \lor (z^{**} \land x)$. It follows that it also satisfies the identity $x = x^{**} \land (x \lor x^*)$. The class of all quasi-modular $p$-algebras contains the class of all modular $p$-algebras. Finally, if in a $p$-algebra $L$ the Stone identity $x^* \lor x^{**} = 1$ is satisfied, then it is said to be an $S$-algebra. In the distributive case the $S$-algebras coincide with the Stone algebras.

For a $p$-algebra $L$ the subset $B(L) = \{a \in L \mid a = a^{**}\}$ of closed elements is a Boolean algebra $(B(L), \sqcup, \land, 0, 1)$ with $a \sqcup b = (a^* \land b^*)^*$ and the subset $D(L) = \{a \in L \mid d = 0\}$ of dense elements of $L$ is a filter of $L$.

Grätzer’s problem from his book [11] (Problem 57) for distributive $p$-algebras was formulated at the end of Section 4.

We present an analogy of Grätzer’s problem for the principal MS-algebras here. However, unlike Grätzer’s formulation of [11] Problem 57 for distributive $p$-algebras and its solution by Katriňák in [15], we do not apply the traditional approach of identifying isomorphic objects: in Grätzer’s problem and Katriňák’s solution these were $B(L)$ and $B, D(L)$ and $D$, and the sets $A$ and $A(L)$ of congruence pairs. In our approach below we take into account the isomorphisms between them.

For a homomorphism of algebras $\tau : A \to B$ and a congruence $\theta \in \text{Con}(A)$, the congruence on $B$ generated by all pairs $(\tau(x), \tau(y))$ with $(x, y) \in \theta$ will be denoted by $\text{Con}(\tau)(\theta)$. It is well known (and easy to show) that the mapping $\text{Con}(\tau) : \text{Con}(A) \to \text{Con}(B)$ preserves arbitrary joins.

Now let $M$ be a de Morgan algebra, let $D$ be a bounded distributive lattice and let $A$ be a subset of $\text{Con}(M) \times \text{Con}(D)$. To represent $A$ as the set $A(L)$ of the congruence pairs of some principal $MS$-algebra $L$, we seek for $L$ and isomorphisms $\tau_1 : M \to L^{*\text{m}}$ and $\tau_2 : D \to D(L)$ such that

$$A = \{(\theta_1, \theta_2) \mid (\text{Con}(\tau_1)(\theta_1), \text{Con}(\tau_2)(\theta_2)) \in A(L)\}. \quad (*)$$
Then $A \subseteq \text{Con}(M) \times \text{Con}(D)$ is called representable.

Now we present an analogy of Grätzer’s problem for the principal MS-algebras.

**Problem 1.** Let $M$ be a de Morgan algebra, let $D$ be a bounded distributive lattice, and let $A$ be a sublattice of $\text{Con}(M) \times \text{Con}(D)$. Under what conditions does there exist a principal MS-algebra $L$ and isomorphisms $\tau_1 : M \to L^{\alpha\omega}$ and $\tau_2 : D \to D(L)$ such that $A$ is representable?

A basic answer is the following.

**Theorem 4.1.** Let $M$ be a de Morgan algebra, let $D$ be a bounded distributive lattice, and let $A$ be a sublattice of $\text{Con}(M) \times \text{Con}(D)$. Then $A$ is representable if and only if there exists a lattice $(0, 1)$-homomorphism $\varphi : M \to D$ such that for every $\theta_1 \in \text{Con}(M)$ and every $\theta_2 \in \text{Con}(D)$

$$(\theta_1, \theta_2) \in A \iff \text{Con}(\varphi)(\theta_1) \subseteq \theta_2.$$ (**)

**Proof.** First, let $A$ be representable. So, there exists a principal MS-algebra $L$ and isomorphisms $\tau_1 : M \to L^{\alpha\omega}$ and $\tau_2 : D \to D(L)$ such that (*) is satisfied. We can assume that the MS-algebra $L$ is given by the triple $(M, D, \psi)$, so $L^{\alpha\omega} = \{(x, \psi(x)) \mid x \in M\}$, $D(L) = \{(1_M, y) \mid y \in D\}$, $d_L = (1_M, 0_D)$. The isomorphisms $\tau_1, \tau_2$ must have the form

$$\tau_1(x) = (\sigma_1(x), \psi(\sigma_1(x))),$$

$$\tau_2(y) = (1_M, \sigma_2(y)),$$

for some automorphisms $\sigma_1$ and $\sigma_2$ on $M$ and $D$, respectively. We set $\varphi = \sigma_2^{-1}\psi\sigma_1$. It is not difficult to check (**).

Conversely, let $\varphi : M \to D$ satisfy (**). Let $L$ be the MS-algebra determined by the triple $(M, D, \varphi)$. We define the isomorphisms $\tau_1$ by $\tau_1(x) = (x, \varphi(x))$, $\tau_2(y) = (1_M, y)$. Then it is not difficult to check that (*) holds. $\Box$

Now we present a modification of the above theorem, where only principal congruences on $M$ are needed.

**Theorem 4.2.** Let $M$ be a de Morgan algebra, let $D$ be a bounded distributive lattice, and let $A$ be a sublattice of $\text{Con}(M) \times \text{Con}(D)$. Then $A$ is representable if and only if the following conditions are satisfied:

1. $A$ is join-closed;
2. $A$ is down-closed in first coordinate, that is, $(\theta_1, \theta_2) \in A$ and $\alpha \leq \theta_1$ imply $(\alpha, \theta_2) \in A$;
3. there exists a lattice $(0, 1)$-homomorphism $\varphi : M \to D$ such that for every principal congruence $\theta_1 \in \text{Con}(M)$ and every $\theta_2 \in \text{Con}(D)$

$$(\theta_1, \theta_2) \in A \iff \text{Con}(\varphi)(\theta_1) \subseteq \theta_2.$$
Proof. The necessity of the conditions (1)-(3) for representability of A is clear. For the converse assume that the conditions (1)-(3) are satisfied. Every congruence \( \theta_1 \in \text{Con}(M) \) is a join of principal congruences \( \alpha_i \in \text{Con}(M) \) where \( i \in I \) for some set \( I \). Since the mapping \( \text{Con}(\varphi) \) preserves joins, we obtain

\[
(\theta_1, \theta_2) \in A \iff (\forall i \in I) \ (\alpha_i, \theta_2) \in A \\
\iff (\forall i \in I) \ \text{Con}(\varphi)(\alpha_i) \subseteq \theta_2 \\
\iff \text{Con}(\varphi)(\theta_1) \subseteq \theta_2.
\]

Hence \( A \) is representable. \( \square \)

As a step to achieve a more descriptive solution of Problem 1, for the rest of this section we consider a special case when a principal MS-algebra \( L \) is a so-called perfect extension of its greatest Stone subalgebra \( L_S \). In our first theorem below we are able to reduce such a condition to the usual simpler substructures \( L^\circ \) and \( D(L) \) of \( L \).

It is appropriate to recall here some concepts. An algebra \( A \) satisfies the Congruence Extension Property (briefly \((\text{CEP})\)) if for every subalgebra \( B \) of \( A \) and every congruence \( \theta \) of \( B \), \( \theta \) extends to a congruence of \( A \). An algebra \( A \) is said to be a perfect extension of its subalgebra \( B \), if every congruence of \( B \) has a unique extension to \( A \) (see \([9, \text{page 30}]\)). We notice that in the literature such \( A \) is sometimes called a congruence-preserving extension of \( B \) (we e.g. refer to \([12]\) by Grätzer and Wehrung where this concept is used in case of lattices). Of course, if \( A \) is a perfect extension of \( B \), then \( \text{Con}(A) \cong \text{Con}(B) \).

The classes of distributive lattices and of MS-algebras are known to satisfy the \((\text{CEP})\) \([11], \ [9]\). However, very little seems to be known when in these or other classes of algebras, an algebra \( A \) is a perfect extension of its subalgebra \( B \). In \([12]\), Grätzer and Wehrung proved that every lattice with more than one element has a proper congruence-preserving (i.e. perfect) extension. In \([9, \text{Theorem 2.13}]\) it is shown that for any Ockham algebra \( O \) with a fixed point \( a \), \( O \) is a perfect extension of its subalgebra \( C_a \) which is the cone generated by \( a \):

\[
C_a := \downarrow a \cup \uparrow a = \{x \in O \mid x \leq a\} \cup \{x \in O \mid x \geq a\}.
\]

Consequently, \( \text{Con}(A) \cong \text{Con}(C_a) \). For an Ockham algebra \( O \) its subalgebra \( C_a \) always satisfies the axiom \((\text{MS5})\) and the size of \( C_a \) can in general be much smaller than that of \( O \) (see \([9, \text{page 30}]\)).

Remark 4.3. We notice that in \([19]\) it was shown that in a modular lattice \( L \), the cone \( C_a \) generated by \( a \) (where \( a \in L \) is an arbitrary element) is a sublattice with the property that every congruence of \( C_a \) has at most one extension to a congruence of \( L \).

The theory of modular lattices was the first research topic of Beloslav Riečan to whose memory this paper is dedicated. Under the guidance of his young professor Milan Kolibiar, in 1957 as a student Belo proved that the axiomatic of
Theorem 4.4. Let \( L' \) be a subalgebra of a principal MS-algebra \( L \). Then \( L \) is a perfect extension of \( L' \) if and only if

1. \( D(L) \) is a perfect extension of \( D(L') \) and
2. \( L^\infty \) is a perfect extension of \( (L')^\infty \).

Proof. Let \( L \) be a perfect extension of \( L' \). Let \( \psi \in \text{Con}(D(L')) \). As the (CEP) holds for the class of distributive lattices, we only have to verify that \( \psi \) has a unique extension to a congruence of \( D(L) \). Let \( \psi_1, \psi_2 \in \text{Con}(D(L)) \) such that \( \psi_1 \vDash D(L') = \psi_2 \vDash D(L') = \psi \). Clearly \( (\Delta_{L^\infty}, \psi_1), (\Delta_{L^\infty}, \psi_2) \in A(L) \). Using Theorem 3.2 there exist \( \theta_1 \) and \( \theta_2 \) of \( \text{Con}(L) \) corresponding to \( (\Delta_{L^\infty}, \psi_1) \) and \( (\Delta_{L^\infty}, \psi_2) \), respectively. Since \( (\Delta_{(L')^\infty}, \psi_1 \vDash D(L')), (\Delta_{(L')^\infty}, \psi_2 \vDash D(L')) \) in \( A(L') \) corresponding to the congruences \( \theta_1 \vDash L', \theta_2 \vDash L' \in \text{Con}(L') \) are the same, we obtain \( \theta_1 \vDash L' = \theta_2 \vDash L' \). As \( L \) is a perfect extension of \( L' \), it follows \( \theta_1 = \theta_2 \). Hence \( \psi_1 = \psi_2 \) proving (1). Now we show that \( L^\infty \) is a perfect extension of \( (L')^\infty \). Let \( \phi \in \text{Con}((L')^\infty) \). Then \( \phi \) has an extension to a congruence of \( L^\infty \) by the (CEP). To show that this extension is unique, let \( \phi_1, \phi_2 \in \text{Con}(L^\infty) \) with \( \phi_1 \vDash (L')^\infty = \phi_2 \vDash (L')^\infty = \phi \). Clearly \( (\phi_1, \nabla_{D(L)}), (\phi_2, \nabla_{D(L)}) \in A(L) \). As above, by Theorem 3.2 there exist \( \theta_1 \) and \( \theta_2 \) of \( \text{Con}(L) \) corresponding to \( (\phi_1, \nabla_{D(L)} \vDash L^\infty) \) and \( (\phi_2, \nabla_{D(L)} \vDash L^\infty) \), respectively. Since the pairs \( (\phi_1 \vDash (L')^\infty, \nabla_{D(L)}), (\phi_2 \vDash (L')^\infty, \nabla_{D(L)}) \) corresponding to the congruences \( \theta_1 \vDash L', \theta_2 \vDash L' \) of \( L' \) are the same, we again obtain \( \theta_1 \vDash L' = \theta_2 \vDash L' \). Because \( L \) is a perfect extension of \( L' \), we get \( \theta_1 = \theta_2 \), whence \( \phi_1 = \phi_2 \) proving that \( L^\infty \) is a perfect extension of \( (L')^\infty \).

Conversely, let the conditions (1) and (2) hold and let \( \theta' \in \text{Con}(L') \). By the (CEP) for the class of MS-algebras, \( \theta' \) has an extension to a congruence of \( L \). To show that this extension is unique, assume that \( \theta_1 \) and \( \theta_2 \) are extensions of \( \theta' \) in \( \text{Con}(L) \). By Theorem 3.2 the congruences \( \theta_1, \theta_2 \) can be represented by some congruence pairs \( (\phi_1, \psi_1), (\phi_2, \psi_2) \in A(L) \). It is easy to see that it follows \( (\phi_1 \vDash (L')^\infty, \psi_1 \vDash D(L')), (\phi_2 \vDash (L')^\infty, \psi_2 \vDash D(L')) \in A(L) \) and these pairs both represent the congruence \( \theta' \in \text{Con}(L') \). Hence we have \( \phi_1 \vDash (L')^\infty = \phi_2 \vDash (L')^\infty \) and \( \psi_1 \vDash D(L') = \psi_2 \vDash D(L') \). By the conditions (1) and (2), we obtain \( \phi_1 = \phi_2 \) and \( \psi_1 = \psi_2 \). Therefore \( \theta_1 = \theta_2 \) as required. \( \square \)

The following result is now an immediate consequence of the above theorem and the equalities \( B(L) = L_S^\infty \) and \( D(L) = D(L_S) \).

Corollary 4.5. A principal MS-algebra \( L \) is a perfect extension of its greatest Stone subalgebra \( L_S \) if and only if \( L^\infty \) is a perfect extension of \( B(L) \).
We notice that the lattice of MS-congruence pairs of the greatest Stone subalgebra \( L_S \) of an MS-algebra \( L \) is \( A(L_S) = \{ (\phi_{L_S}, \psi) \in \text{Con}(B(L)) \times \text{Con}(D(L)) \mid (\phi, \psi) \in A(L) \} \), where \( \phi_{L_S} \) is the restriction of \( \phi \) to \( B(L) \).

**Corollary 4.6.** Let a principal MS-algebra \( L \) be a perfect extension of its greatest Stone subalgebra \( L_S \). Then \( \text{Con}(L) \cong \text{Con}(L_S) \).

Let \( L \) be a principal MS-algebra associated to a principal MS-triple \((M, D, \varphi)\). Then obviously \( L_S = \{(a, x) \in L \mid a \in B(M)\} \) is the greatest Stone subalgebra of \( L \) such that \( L_S = B(L) \cong B(M) \) and \( D(L_S) = D(L) \cong D \). Hence our final corollary is stated in terms of principal MS-triples.

**Corollary 4.7.** Let \( L \) be a principal MS-algebra associated with a principal MS-triple \((M, D, \varphi)\) and let \( L_S \) be the greatest Stone subalgebra of \( L \). Then \( L \) is a perfect extension of \( L_S \) if and only if \( M \) is a perfect extension of \( B(M) \).

From the above results it follows that in the considered special case when a principal MS-algebra \( L \) is a perfect extension of its greatest Stone subalgebra \( L_S \), our solution in Theorem 4.2 to Problem 1 can slightly be simplified by not considering all principal congruences \( \alpha \in \text{Con}(M) \) but only special principal congruences \( \alpha \in \text{Con}(M) \) of the form \( \alpha = \theta(0, a) = \theta(a', 1) \).

Our reasoning here is as follows. If \( M \) is a perfect extension of \( B(M) \), then every congruence \( \theta_1 \) on \( M \) is generated by some congruence on \( B(M) \), which is a join of principal congruences on \( B(M) \). Since \( B(M) \) is Boolean, every principal congruence on \( B(M) \) is of form \( \theta(0, a) = \theta(a', 1) \). Hence, \( \theta_1 \) is a join of such congruences and by using the same argument as in the proof of Theorem 4.2 we obtain:

**Corollary 4.8.** Let \( M \) be a de Morgan algebra that is a perfect extension of \( B(M) \), let \( D \) be a bounded distributive lattice, and let \( A \) be a sublattice of \( \text{Con}(M) \times \text{Con}(D) \).

Then \( A \) is representable if and only if

1. \( A \) is join-closed and
2. \( A \) is down-closed in first coordinate, that is, \((\theta_1, \theta_2) \in A \) and \( \alpha \leq \theta_1 \) imply \((\alpha, \theta_2) \in A \);
3. there exists a lattice \((0,1)\)-homomorphism \( \varphi : M \rightarrow D \) such that for every principal congruence \( \alpha \in \text{Con}(M) \) of the form \( \alpha = \theta(0, a) = \theta(a', 1) \) and every \( \theta_2 \in \text{Con}(D) \)
   \[(\theta_1, \theta_2) \in A \iff \text{Con}(\varphi)(\theta_1) \subseteq \theta_2.\]

5. **Examples**

In the second part of the previous section we considered Problem 1 in the special case when a principal MS-algebra \( L \) is a perfect extension of its greatest
Stone subalgebra $L_S$. When we associate the principal MS-algebra $L$ with a principal MS-triple $(M, D, \varphi)$, we know by Corollary 4.7 that this special case occurs when the de Morgan algebra $M$ is a perfect extension of its Boolean subalgebra $B(M)$. In this section we illustrate in examples when this condition is satisfied as well as when it is not satisfied.

In Figure 1 we see two small examples of de Morgan algebras: $M_1$ is the four-element subdirectly irreducible de Morgan algebra and $M_2$ is the de Morgan algebra $1 \oplus 2^3 \oplus 1$. For both $i = 1, 2$ the Boolean subalgebra $B(M_i) = \{0, 1\}$ is simple. (The elements of $B(M_i)$ are in all figures shaded.)

![Figure 1. The de Morgan algebras $M_1$ and $M_2$.](image)

Since the four-element de Morgan algebra $M_1$ is simple, too, it is automatically a perfect extension of $B(M_1)$. So the MS-algebra $L_1$ taken from Example 3.3 and depicted in Figure 2 is a perfect extension of its greatest Stone subalgebra $S := (L_1)_S$ which is the three-element chain. Here $L_1$ is created from the principal MS-triple $(M_1, D, \varphi_1)$ where $D = \{0, 1\}$ is the two-element lattice and $\varphi_1 : M_1 \to D$ is given by $\varphi_1(0) = \varphi_1(a) = 0, \varphi_1(b) = \varphi_1(1) = 1$. It is easy to see that $\theta = \Delta_{L_1} \cup D(L_1)^2 \cup \{(b, 0), (b, 1)\}$ is the only non-trivial congruence of $L_1$ which is the unique extension of the congruence $\theta_S = \Delta_S \cup D(S)^2$.

On the other hand, the ten-element de Morgan algebra $M_2$ is not simple and so it is not a perfect extension of $B(M_2)$. A non-trivial congruence $\theta'$ of $M_2$ is depicted in Figure 1.
Figure 2. The MS algebra $L_1$ and its Stone subalgebra $S$.

Figure 3. The MS algebra $L_2$

For the principal MS-triple $(M_2, D, \varphi_2)$ where $D$ is the same two-element lattice as above and $\varphi : M_2 \to D$ is given by

\[
\begin{align*}
\varphi_2(0) &= \varphi_2(a) = \varphi_2(b) = \varphi_2(c) = \varphi_2(b^\circ) = 0, \\
\varphi_2(d) &= \varphi_2(c^\circ) = \varphi_2(d^\circ) = \varphi_2(a^\circ) = \varphi_2(1) = 1
\end{align*}
\]
we obtain the principal MS-algebra \( L_2 \) in Figure 3. By Corollary 4.3, \( L_2 \) is not a perfect extension of its greatest Stone subalgebra \( (L_2)_S \) which is again the three-element chain \( S = \{(0, 0), (1, 0), (1, 1)\} \). It is easy to check that the only non-trivial congruence \( \theta_S = \Delta_S \cup D(S)^2 \) of \( S \) has at least two different extensions in \( \text{Con}(L_2) \), namely the congruences

\[
\theta_1 = \Delta_{L_2} \cup D(L_2)^2 \cup \{(d, 0), (d, 1)\}^2 \cup \{(d^0, 0), (d^0, 1)\}^2 \cup \{(c^0, 0), (c^0, 1)\}^2 \\
\cup \{(a^0, 0), (a^0, 1)\}^2,
\]

\[
\theta_2 = \Delta_{L_2} \cup D(L_2)^2 \cup \{(d, 0), (d, 1), (d^0, 0), (d^0, 1)\}^2 \cup \{(a, 0), (c, 0)\}^2 \\
\cup \{(b, 0), (b^0, 0)\}^2 \cup \{(c^0, 0), (c^0, 1), (a^0, 0), (a^0, 1)\}^2.
\]

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