Semiparametric Mixture Regression with Unspecified Error Distributions

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Abstract

In fitting a mixture of linear regression models, normal assumption is traditionally used to model the error and then regression parameters are estimated by the maximum likelihood estimators (MLE). This procedure is not valid if the normal assumption is violated. To relax the normal assumption on the error distribution hence reduce the modeling bias, we propose semiparametric mixture of linear regression models with unspecified error distributions. We establish a more general identifiability result under weaker conditions than existing results, construct a class of new estimators, and establish their asymptotic properties. These asymptotic results also apply to many existing semiparametric mixture regression estimators whose asymptotic properties have remained unknown due to the inherent difficulties in obtaining them. Using simulation studies, we demonstrate the superiority of the proposed estimators over the MLE when the normal error assumption is violated and the comparability when the error is normal. Analysis of a newly collected Equine Infectious Anemia Virus data in 2017 is employed to illustrate the usefulness of the new estimator.

Keywords: EM algorithm; Kernel estimation; Mixture of regressions; Semiparametric

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1. Introduction

Mixtures of regressions provide a flexible tool to investigate the relationship between variables coming from several unknown latent components. It is widely used in many fields, such as engineering, genetics, biology, econometrics and marketing. A typical mixture of regressions model is as follows. Let \( Z \) be a latent class indicator with
\[
\Pr(Z = j | X) = \pi_j \text{ for } j = 1, 2, \ldots, m,
\]
where \( X \) is a \((p + 1)\)-dimensional vector with the first component the constant 1 and the rest random predictors. Given \( Z = j \), the response \( Y \) depends on \( X \) through a linear regression model
\[
Y = X^T \beta_j + \epsilon_j, \tag{1}
\]
where \( \beta_j = (\beta_{0j}, \beta_{1j}, \ldots, \beta_{pj})^T \), and \( \epsilon_j \sim N(0, \sigma_j^2) \) is independent of \( X \). Thus the conditional density of \( Y \) given \( X = x \) can be written as
\[
f_{Y|X}(y, x) = \sum_{j=1}^{m} \pi_j \phi(y; x^T \beta_j, \sigma_j^2), \tag{2}
\]
where \( \phi(\cdot; \mu, \sigma^2) \) is the normal density with mean \( \mu \) and variance \( \sigma^2 \). The unknown parameters in model (2) can be estimated by the maximum likelihood estimator (MLE) using the EM algorithm (Dempster et al., 1977). See, for example, Wedel and Kamakura (2000), Skrondal and Rabe-Hesketh (2004), Jacobs et al. (1991) and Jiang and Tanner (1991) for some applications of model (2).

A major drawback of model (2) is the normal assumption of the error density, which does not always hold in practice. Unfortunately, unlike the equivalence between the MLE and the least squares estimator (LSE) in linear regression, the normal assumption of \( \epsilon \)
in (1) is indispensable for the consistency of MLE. Furthermore, the normal assumption is also critical for the computation of MLE because it is needed when calculating the classification probabilities in the E step of the EM algorithm.

In order to reduce the modeling bias, we relax the normal assumption of the component error distributions and propose a class of flexible semiparametric mixture of linear regression models by replacing the normal error densities in (2) with unspecified component error densities. Specifically, we propose a semiparametric mixture of linear regressions model of the form

\[
f_{Y|X}(y, x, \theta, g) = \sum_{j=1}^{m} \pi_j \tau_j g\{(y - x^T \beta_j) \tau_j\},
\]

where \(\theta = (\pi_1, \ldots, \pi_{m-1}, \beta_1^T, \ldots, \beta_m^T, \tau_1, \ldots, \tau_m)^T\) and \(g\) is an unspecified density function with mean zero and variance one. Note that \(\pi_m = 1 - \sum_{j=1}^{m-1} \pi_j\) and we can view \(\tau_j\) as the scale parameter or precision parameter playing the role of \(\sigma_j^{-1}\) in (2). For a special case of (3) where \(\tau_1 = \tau_2 = \cdots = \tau_m\) and \(g\) is a symmetric function, some existing work on identifiability exists. For example, Bordes et al. (2006) and Hunter et al. (2007) established the model identifiability when \(m \leq 3\) and \(X = 1\), i.e. when the regression model degenerates to a mixture of density functions, while Hunter and Young (2012) allowed any \(m\) and included covariates in \(X\). In this work, we establish the identifiability result for model (3) in a more general setting than the existing literatures, where the identifiability is shown for the arbitrary component densities \(g_j\) with mean 0 without the identical constraint on the \(\tau_j\)’s. We also propose a semiparametric EM algorithm to estimate the regression parameters \(\theta\) and the unspecified function \(g\). We further prove the consistency and the asymptotic properties of the new semiparametric estimator. Our asymptotic results directly apply to many existing semiparametric mixture regression estimators whose asymptotic properties have not been established in the literature. Using
a Monte Carlo simulation study, we demonstrate that our methods perform better than
the traditional MLE when the errors have distributions other than normal and provide
comparable results when the errors are normal. An empirical analysis of a newly collected
Equine Infectious Anemia Virus (EIAV) data set in 2017 is carried out to illustrate the
usefulness of the proposed methodology.

The rest of the paper is organized as follows. Section 2 introduces the new mixture of
regressions model with unspecified error distributions, proposes the new semiparametric
regression estimator, and establishes the asymptotic properties of the proposed estimator.
In Section 3, we use a simulation study to demonstrate the superior performance of the
new method. We illustrate the effectiveness of the new method on an EIAV data set in
Section 4. Some discussions are given in Section 5.

2. Mixture of regressions with nonparametric error densities

2.1. Identifiability results

Before proposing estimation procedures, we first investigate the identifiability of the
model in (3). Let \( X = (1, X^T_s)^T, \beta_j = (\beta_{0j}, \beta_{sj}^T)^T \) with \( \beta_{sj} = (\beta_{1j}, \ldots, \beta_{pj})^T \).

**Theorem 1.** (Identifiability) Assume that \( \pi_j > 0, j = 1, \ldots, m, \) and \( \beta_{sj}, j = 1, \ldots, m, \)
are distinct vectors in \( \mathbb{R}^p \). Assume further that the support of \( X_s \) contains an open set
in \( \mathbb{R}^p \). Then the semiparametric mixture regression model (3) is identifiable up to a
permutation of the \( m \) components.

**Remark 1.** A more general identifiability result is proved in the supplementary document.
More specifically, under the assumptions in Theorem 1, the model

\[
f_{Y|X}(y, \mathbf{x}, \theta, g) = \sum_{j=1}^{m} \pi_j g_j(y - \mathbf{x}^T \beta_j),
\]

(4)
is identifiable, where \( g_j \) has mean 0 and \( g = (g_1, \ldots, g_m) \). Note that model (3) is a special case of (4) when \( g_j(\cdot) \)'s belong to the same distribution family with different precision parameters. Our identifiability result benefits from the information carried in the random covariates \( X \). This allows us to establish the identifiability result for general number of components \( m \) and arbitrary \( g_j(\cdot) \)'s.

2.2. Estimation algorithms

Suppose that \( \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \) are random observations from (3). In this section, we propose a Kernel DeNsity based EM type algorithm (KDEEM) to estimate the parameter \( \theta \) and the nonparametric density function \( g(\cdot) \) in (3):

**Algorithm 1.** Starting from an initial parameter \( \theta^{(0)} \) and initial density function \( g^{(0)}(\cdot) \), at the \((k + 1)\)th step,

**E step:** Calculate the classification probabilities,

\[
p_{ij}^{(k+1)} = \Pr(Z_i = j \mid x_i, y_i) = \frac{\pi_j^{(k)} g^{(k)}(r_{ij}^{(k)}) \tau_j^{(k)}}{\sum_{j=1}^{m} \pi_j^{(k)} g^{(k)}(r_{ij}^{(k)}) \tau_j^{(k)}}, \quad i = 1, \ldots, n, \ j = 1, \ldots, m,
\]

where \( \epsilon_{ij}^{(k)} = y_i - x_i^T \beta_j^{(k)} \) and \( r_{ij}^{(k)} = \epsilon_{ij}^{(k)} \tau_j^{(k)} \).

**M step:** Update \( \theta \) and \( g(\cdot) \), via

1. \( \pi_j^{(k+1)} = n^{-1} \sum_{i=1}^{n} p_{ij}^{(k+1)} \),
2. \( (\beta_j^{(k+1)}, \tau_j^{(k+1)})^T = \arg \max_{\beta_j, \tau_j} \sum_{i=1}^{n} p_{ij}^{(k+1)} \log[g^{(k)}((Y_i - x_i^T \beta_j) \tau_j)], \) for \( j = 1, \ldots, m \).
3. \( g^{(k+1)}(t) = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij}^{(k+1)} K_h(r_{ij}^{(k+1)} - t), \) where \( j = 1, \ldots, m, \ K_h(t) = h^{-1} K(t/h), \) and \( K(t) \) is a kernel function, such as the Epanechnikov kernel.
For the conventional MLE, the normal density for $g(\cdot)$ is used to calculate the classification probabilities in the E step. In the KDEEM, the error density used in the E step is estimated by a weighted kernel density estimator in stage 3 of the M step, with classification probabilities as weights, to avoid the modelling bias of component error densities. Bordes et al. (2007) and Benaglia et al. (2009) have used similar idea of combining kernel density and EM algorithm for the mixture of location shifted densities when there are no predictors involved. Note that the above EM type algorithm cannot guarantee to increase the likelihood at each iteration due to the kernel density estimation in the M step. One could use the maximum smoothed loglikelihood method proposed by Levine et al. (2011) to produce a modified algorithm that does increase smoothed version of the loglikelihood at each iteration but provides similar performance to the KDEEM.

Hunter and Young (2012) considered a special case of Algorithm 1 by assuming homogeneous scales, i.e. $\tau_1 = \tau_2 = \cdots = \tau_m$, denoted by KDEEM.H. For completeness of the presentation, we also present out the EM algorithm for this special case.

**Algorithm 2.** Starting from an initial parameter $\theta^{(0)}$ and initial density function $g^{(0)}(\cdot)$, at the $(k + 1)$th step,

**E step:** Calculate the classification probabilities,

$$p_{ij}^{(k+1)} = P(Z_i = j \mid x_i, y_i) = \frac{\pi_j^{(k)} g_j^{(k)}(\epsilon_{ij}^{(k)})}{\sum_{j=1}^{m} \pi_j^{(k)} g_j^{(k)}(\epsilon_{ij}^{(k)})} \quad i = 1, \ldots, n, j = 1, \ldots, m,$$

where $\epsilon_{ij}^{(k)} = y_i - x_i^T \beta_j^{(k)}$.

**M step:** Update $\theta$ and $g$, via

1. $\pi_j^{(k+1)} = n^{-1} \sum_{i=1}^{n} p_{ij}^{(k+1)},$
2. $\beta_j^{(k+1)} = \arg \max_{\beta_j} \sum_{i=1}^{n} p_{ij}^{(k+1)} \log [g_j^{(k)}(Y_i - x_i^T \beta_j)],$ for $j = 1, \ldots, m.$
where \( j = 1, \ldots, m \).

To simplify the computation, Hunter and Young (2012) also recommended to use the least squares criterion to update \( \beta \) in the M step of Algorithm 2, i.e.,

\[
\beta_j^{(k+1)} = (X^T W_j^{(k+1)} X)^{-1} X^T W_j^{(k+1)} Y,
\]

where \( X = (x_1, \ldots, x_n)^T, Y = (y_1, \ldots, y_n)^T, W_j^{(k+1)} = \text{diag}(p_{1j}^{(k+1)}, \ldots, p_{nj}^{(k+1)}) \). Let \( \tilde{\theta} \) and \( \tilde{g}(\cdot) \) be the resulting estimators, denoted by KDEEM.LSE. Note that \( \tilde{\theta} \) is different from the classic MLE in that the classification probabilities are calculated based on the weighted kernel density estimator (5) instead of the normal density to avoid the misspecification of the component error densities.

2.3. Asymptotic properties

We now establish the asymptotic properties of the estimators presented in Section 2.2. Let \( \tilde{\theta} \) and \( \tilde{g}(\cdot) \) be the resulting estimators of Algorithm 1 and \( \theta_0 \) and \( g_0(\cdot) \) be the corresponding true values. Next, we provide the asymptotic properties of \( \tilde{\theta} \) and \( \tilde{g}(\cdot) \). We make the following mild Assumptions.

A1 The probabilities \( \pi_j \in (0, 1) \) for \( j = 1, \ldots, m, \sum_{j=1}^m \pi_j = 1 \).

A2 The precision values satisfy \( 0 < \tau_j < \infty \) for \( j = 1, \ldots, m \).

A3 The true parameter value is in the interior of an open set \( \Theta \subset \mathcal{R}^d \) where \( d = \dim(\theta) \).

A4 The pdf \( g(\cdot) \) has a compact support and is bounded away from zero on its support.

In addition, \( g(\cdot) \) is continuous and has continuously bounded second derivative with mean 0 and variance 1.
The kernel function $K(\cdot)$ is symmetric, bounded, and twice differentiable with bounded second derivative, compact support and finite second moment.

The bandwidth $h$ satisfies $nh^2 \to \infty$ and $nh^4 \to 0$ when $n \to \infty$.

In the neighborhood of the true parameter values $(\theta_0, g_0)$, there is a unique value $(\hat{\theta}, \hat{g})$ where the EM algorithm converges to.

To state the theoretical results in Theorem 2, we first define some notations, while collect the proof of Theorem 2 in Section S.2 of the Supplementary document.

For any vector $a = (a_1, \ldots, a_p)$, let $g(a)$ be the element-wise evaluation of $g(\cdot)$ at $a$. Let $r_i = (r_{i1}, \ldots, r_{im})$, where $r_{ij} = (y_i - x_i^T\beta_j)\tau_j$. Define

$$
\Phi\{x_i, y_i, g(r_i), \theta\} = \begin{bmatrix} 
\Phi_1\{x_i, y_i, g(r_i), \pi, \tau, \beta_1\} \\
\vdots \\
\Phi_m\{x_i, y_i, g(r_i), \pi, \tau, \beta_m\}
\end{bmatrix},
$$

where $\pi = (\pi_1, \ldots, \pi_{m-1})^T$, $\tau = (\tau_1, \ldots, \tau_m)^T$, and

$$
\Phi_j\{x_i, y_i, g(r_i), \pi, \tau, \beta_j\} = \left[ \frac{g(r_{ij})\tau_j}{\sum_{j=1}^m \pi_j g(r_{ij})\tau_j} - 1, \frac{\pi_j g(r_{ij}) + g'(r_{ij})r_{ij}}{\sum_{j=1}^m \pi_j g(r_{ij})\tau_j}, \frac{\pi_j g'(r_{ij})\tau_j x_i^T}{\sum_{j=1}^m \pi_j g(r_{ij})\tau_j} \right]^T.
$$

Also, let

$$
\Psi\{t, g(t), g(r_i), \theta\} = \frac{\sum_{j=1}^m \pi_j g(r_{ij})\tau_j K_h(r_{ij} - t)}{\sum_{j=1}^m \pi_j g(r_{ij})\tau_j} - g(t).
$$

Let $g(\cdot, \theta)$ satisfy $E[\Psi\{t, g(t), g(r_i), \theta\}] = 0$ for all $\theta$ and $t$. Define

$$
r_{21}(t) = E \left[ \frac{\partial \Psi\{t, g(t), g(r_i), \theta\}}{\partial \theta} \bigg|_{g(\cdot) = g(\cdot, \theta)} \right],
$$

$$
r_{22}(t) = E \left[ \frac{\partial \Psi\{t, g(t, \theta), g(r_i, \theta), \theta\}}{\partial g(t, \theta)} \right].
$$
Further let \( r_{21}(x, y_i) = \{ r_{21}(r_{i1}), \ldots, r_{21}(r_{im}) \} \), \( r_{22}(x, y_i) = \text{diag}\{ r_{22}(r_{i1}), \ldots, r_{22}(r_{im}) \} \), \( r_{21} = \{ r_{21}(x_1, y_1), \ldots, r_{21}(x_n, y_n) \} \), \( r_{22} = \text{diag}\{ r_{22}(x_1, y_1), \ldots, r_{22}(x_n, y_n) \} \). Also let \( R_i(x, y_i) = \{ R_i(r_{i1}), \ldots, R_i(r_{im}) \} \), \( R_i = \{ R_i^T(x, y_i), \ldots, R_n^T(x, y_i) \}^T \), \( R = (R_1, \ldots, R_n) \). Let

\[
\mathbf{r}_3(x, y_i) = \begin{bmatrix}
\frac{\partial \Phi\{x, y_i, g(r_{i1}, \theta), \theta\}}{\partial g(r_{i1}, \theta)}, \\
\vdots \\
\frac{\partial \Phi\{x, y_i, g(r_{im}, \theta), \theta\}}{\partial g(r_{im}, \theta)}
\end{bmatrix},
\]

and write \( \mathbf{r}_3 = \{ \mathbf{r}_3(x_1, y_1), \ldots, \mathbf{r}_3(x_n, y_n) \} \). Now we define

\[
\mathbf{M} = E\{ n^{-1}\mathbf{r}_3(r_{22} + n^{-1}\mathbf{R})^{-1}\mathbf{r}_2^T \} - E\frac{\partial \Phi\{X_i, Y_i, g(Y_i - X_i^T\beta_1), \ldots, g(Y_i - X_i^T\beta_m), \theta\}}{\partial \mathbf{\theta}^T}.
\]

Define also

\[
\mathbf{u}(x, y_i) = \sum_{j=1}^{m} \sum_{k=1}^{m} E\left[ \frac{\sigma_j \pi_k \tau_k g(r_{ik}, \theta)}{\sum_{s=1}^{m} \pi_s \tau_s g(r_{is}, \theta)} \right] \frac{\partial \Phi\{X, X^T\beta_j, g(\gamma_{ijk}), \theta\}}{\partial g(r_{ik}, \theta)} \left[ \sum_{s=1}^{m} \pi_s \tau_s g(r_{is}, \theta) \right] - \sum_{k=1}^{m} E\left[ \frac{\partial \Phi\{X, Y_i, g(r_{ik}, \theta), \theta\}}{\partial g(r_{ik}, \theta)} g(r_{ik}, \theta) \right],
\]

and

\[
\mathbf{v}(x, y_i) = \sum_{j=1}^{m} \sum_{k=1}^{m} E\left[ \frac{\partial \Phi\{X, \sigma_j r_{ik} + X^T\beta_j, g(\gamma_{ijk}), \theta\}}{\partial g(r_{ik}, \theta)} \right] f_{Y_i} \left( \sigma_j r_{ik} + X^T\beta_j, X \right) g(r_{ik})
\]

\[
- \sum_{j=1}^{m} \sum_{k=1}^{m} \sum_{s=1}^{m} \sum_{t=1}^{m} E\left[ \frac{\partial \Phi\{X_1, \eta_{ijkst}, g(\tau_s (\eta_{ijkst} - X_1^T\beta_1)), \ldots, g(\tau_m (\eta_{ijkst} - X_1^T\beta_m), \theta)\}}{\partial g(\tau_s (\sigma_k r_{it} + X_1^T\beta_k - X_1^T\beta_s))} \right] \frac{\gamma_{ijkst} \tau_q g(\tau_q (\eta_{ijkst} - X_1^T\beta_q))}{\sum_{q=1}^{m} \pi_q \gamma_q g(r_{iq})}
\]

\[
\times \frac{1}{\sum_{q=1}^{m} \pi_q \gamma_q g(r_{iq})} \left\{ \sum_{q=1}^{m} \pi_q \gamma_q g(r_{iq}) \right\}.
\]
where \( \sigma_j = \tau_j^{-1}, \gamma_{ijkl} = \tau_i(X^T\beta_j + \sigma_j r_{ik} - X^T\beta_l), \gamma_{ijk} = (\gamma_{ijk1}, \ldots, \gamma_{ijkm}), \) and \( \eta_{jkst} = \sigma_j \tau_s (\sigma_k r_{st} + X_j^T \beta_k - X_j^T \beta_s) + X_j^T \beta_j. \)

**Theorem 2.** Under the Assumptions A1-A7, the regression parameter estimator \( \hat{\theta} \) obtained from Algorithm 1 is consistent and satisfies

\[
\sqrt{n}(\hat{\theta} - \theta_0) \rightarrow N(0, V)
\]

in distribution when \( n \rightarrow \infty \), where

\[
V = M^{-1} \text{var}\{\Phi(x_i, y_i, g(r_{i1}, \theta), \ldots, g(r_{im}, \theta), \theta) + u(x_i, y_i) + v(x_i, y_i)\} M^{-1T}.
\]

In addition, \( \hat{g}(t) - g_0(t) = O_p\{h^2 + (nh)^{-1/2}\}, \) for any \( t. \)

Theorem 2 establishes the theoretical properties of the estimator \( \hat{\theta} \) in Algorithm 1. It also shows that the nonparametric density estimator \( \hat{g}(t) \) has the same convergence properties as the classical nonparametric density estimator. The proof of Theorem 2 is lengthy and quite involved.

Let \( \tilde{\theta} \) and \( \tilde{g} \) be the resulting estimators of Algorithm 2 under the assumption of homogeneous component scales considered in Hunter and Young (2012). In Theorem 3 below, we show the consistency of \( \tilde{\theta} \) and \( \tilde{g} \) and provide their convergence rate properties, which have remained unsolved and are considered to be difficult to obtain in the literature. We benefit from the proof of Theorem 2, which sheds light on the problem and provides the basic approach to the proof. We first define some notations, while collect the detailed proof of Theorem 3 in the Supplement.
Define

\[ \tilde{\Phi}\{x_i, y_i, g(\epsilon_i), \theta\} = \begin{bmatrix} \tilde{\Phi}_1\{x_i, y_i, g(\epsilon_i), \pi, \beta_1\} \\ \vdots \\ \tilde{\Phi}_m\{x_i, y_i, g(\epsilon_i), \pi, \beta_m\} \end{bmatrix}, \]

where \( \epsilon_{ij} = y_i - x_i^T \beta_j \) and

\[ \tilde{\Phi}_j\{x_i, y_i, g(\epsilon_i), \pi, \beta_j\} = \begin{bmatrix} g(\epsilon_{ij}) \\ \sum_{j=1}^{m} \pi_j g(\epsilon_{ij}) - 1, \sum_{k=1}^{m} \pi_k g(\epsilon_{ik}) \end{bmatrix}^T. \]

Also let

\[ \tilde{\Psi}\{t, g(t), g(\epsilon_i), \pi\} = \frac{\sum_{j=1}^{m} \pi_j g(\epsilon_{ij}) K_h(\epsilon_{ij} - t)}{\sum_{j=1}^{m} \pi_j g(\epsilon_{ij})} - g(t). \quad (6) \]

Let \( \tilde{M}, \tilde{u}(x_i, y_i), \) and \( \tilde{v}(x_i, y_i) \) be defined similarly in Theorem 2 by replacing \( \{\Phi(\cdot), \Psi(\cdot)\} \) with \( \{\tilde{\Phi}(\cdot), \tilde{\Psi}(\cdot)\} \) and replacing \( r_{ij} \) with \( \epsilon_{ij}, i = 1, \ldots, n, j = 1, \ldots, m. \) (See Section S.2 of the Supplementary document for more detail.)

**Theorem 3.** Under the Assumptions A1-A7, \( \tilde{\theta} \) is consistent and satisfies

\[ \sqrt{n}(\tilde{\theta} - \theta_0) \to N(0, \tilde{V}) \]

in distribution when \( n \to \infty, \) where

\[ \tilde{V} = \tilde{M}^{-1} \text{var}[\tilde{\Phi}\{x_i, y_i, g(\epsilon_{i1}, \theta), \ldots, g(\epsilon_{im}, \theta), \theta\} + \tilde{u}(x_i, y_i) + \tilde{v}(x_i, y_i)](\tilde{M}^{-1})^T. \]

In addition, \( \tilde{g}(t) - g_0(t) = O_p\{h^2 + (nh)^{-1/2}\}, \) for any \( t. \)

Next, we establish the consistency and the asymptotic normality of the \( \tilde{\theta}, \) which is
the least squares version of Algorithm 2 (Hunter and Young, 2012). To formally state the theoretical results in Theorem 4, we need to define some notations, while give the proof of Theorem 4 in the Supplement.

Let

$$\Phi \{x_i, y_i, g(\epsilon_{i1}), \ldots, g(\epsilon_{im}), \theta \} = \begin{bmatrix} \Phi_1 \{x_i, y_i, g(\epsilon_{i1}), \ldots, g(\epsilon_{im}), \pi, \beta_1 \} \\ \vdots \\ \Phi_m \{x_i, y_i, g(\epsilon_{i1}), \ldots, g(\epsilon_{im}), \pi, \beta_m \} \end{bmatrix},$$

where

$$\Phi_j \{x_i, y_i, g(\epsilon_{i1}), \ldots, g(\epsilon_{im}), \pi, \beta_j \} = \begin{bmatrix} g(\epsilon_{ij}) \\ \sum_{j=1}^m \pi_j g(\epsilon_{ij}) \\ \sum_{j=1}^m \pi_j g(\epsilon_{ij}) \end{bmatrix}^T.$$

Let $\tilde{M}, \tilde{u}(x_i, y_i)$, and $\tilde{v}(x_i, y_i)$ be defined similarly in Theorem 2 by replacing $\{\Phi, \Psi\}$ with $\{\tilde{\Phi}, \tilde{\Psi}\}$ and replacing $r_{ij}$ with $\epsilon_{ij}, i = 1, \ldots, n, j = 1, \ldots, m$. (See the Supplement for more detail.)

**Theorem 4.** Under the Assumptions A1-A7, $\tilde{\theta}$ is consistent and satisfies

$$\sqrt{n}(\tilde{\theta} - \theta_0) \to N(0, \tilde{V})$$

in distribution, where

$$\tilde{V} = \tilde{M}^{-1} \text{var}\{\tilde{\Phi} \{x_i, y_i, g(\epsilon_{i1}, \theta), \ldots, g(\epsilon_{im}, \theta), \theta \} + \tilde{u}(x_i, y_i) + \tilde{v}(x_i, y_i)\}(\tilde{M}^{-1})^T.$$

In addition, $\tilde{g}(t) - g_0(t) = O_p \{h^2 + (nh)^{-1/2}\}$, for any $t$.

From the proof of Theorem 4 in Section S.4 of the Supplementary document, it is easy to see that we can also get consistent mixture regression parameter estimates if we
replace the least squares criterion in the M step by other robust criteria such as Huber’s \( \psi \) function (Huber, 1981) or Tukey’s bisquare function. The consistency of the estimators can be retained mainly because there is no modeling misspecification when estimating classification probabilities in the E step.

3. Simulation study

We conduct a series of simulation studies to demonstrate the effectiveness of the proposed estimator KDEEM under different scenarios of error distributions and compare them with the traditional normal assumption based MLE via the EM algorithm (MLEEM). For the proposed estimator, we use the traditional MLE as the initial values and select the bandwidth of the kernel density estimation of \( g(\cdot) \) based on the method proposed by Sheather and Jones (1991). Better estimation results might be obtained if more sophisticated methods were used to select the bandwidth. See, for example, Sheather and Jones (1991) and Raykar and Duraiswami (2006). For illustration purpose, we also include KDEEM.H presented in Algorithm 2 and the corresponding least squares version KDEEM.LSE proposed by Hunter and Young (2012) for comparison.

We generate the independent and identically distributed (i.i.d.) data \( \{(x_i, y_i), i = 1, \ldots, n\} \) from the model

\[
Y = \begin{cases} 
-3 + 3X + \epsilon_1, & \text{if } Z = 1; \\
3 - 3X + \epsilon_2, & \text{if } Z = 2,
\end{cases}
\]

where \( Z \) is the component indicator of \( Y \) with \( \text{pr}(Z = 1) = 0.5 \), and \( X \sim U(0, 1) \).

We consider the following cases for the error distribution \( \epsilon_1 \) and \( \epsilon_2 \):

Case I: \( \epsilon_1 \sim N(0, 1) \),

Case II: \( \epsilon_1 \sim U(-3, 3) \),
Case III: $\epsilon_1 \sim 0.5N(-1.5, 0.5^2) + 0.5N(1.5, 0.5^2)$,

Case IV: $\epsilon_1 \sim 0.5N(-1, 0.5^2) + 0.5N(1, 1.5^2)$,

Case V: $\epsilon_1 \sim \Lambda(0, 1^2)$,

Case VI: $\epsilon_1 \sim \text{Gamma}(2, 0.5)$,

Case VII: $\epsilon_1 \sim \text{Rayleigh}(3)$,

and $\epsilon_2$ has the same distribution as $0.5\epsilon_1$, i.e. $\epsilon_2 \sim 0.5\epsilon_1$. We use Case I to check the efficiency loss of the new semiparametric mixture regression estimators compared to MLE when the error distribution is indeed normal. The distribution in Case III is bimodal and the distribution in Case IV is right skewed. Cases II, V, VI, and VII are non-normal error densities and are used to check the adaptiveness of the new method to various densities.

In Tables 1 to 6, we report the mean absolute bias (MAB) and the root mean squared error (RMSE) of the regression parameter estimates based on 1,000 replicates for all seven cases and for $n = 250, 500, 1000$. For convenience of reading, all the values are multiplied by a factor of $10^2$. In addition, for better comparison, we also report the relative efficiency (RE) for each estimator when compared to the classical method MLEEM. For example, RE of KDEEM is calculated as

$$RE = \left( \frac{\text{RMSE}(\text{MLEEM})}{\text{RMSE}(\text{KDEEM})} \right)^2.$$ 

A larger value of $RE$ indicates better performance of the proposed method. Based on the simulation results, in Case I, when the error distribution is normal, MLE is the most efficient one as expected. However, for Cases II to VII, where the error pdf is not normal, KDEEM outperforms MLEEM and the improvement is very substantial, especially for the slope parameters. In addition, for all cases, KDEEM performs better than KDEEM.H and
KDEEM.LSE, which is expected since the data generation models have the heterogeneous component scales.

4. Real data analysis

As data collection techniques improve in molecular virology, an increasing number of data sets were collected and stored, whose prominent features are mixture and non-normality. These features, if not approached properly, might result in efficiency loss in statistical inference. In this section, we evaluate our proposed KDEEM approach by analyzing the EIAV data set collected from the experiments of Harbin Veterinary Research Institute (HVRI) conducted by Equine Infectious Disease Research Team in March 2017. EIAV is commonly used for Human immunodeficiency virus (HIV) research because both EIAV and HIV are lentivirus of the retrovirus family, with similar genomic structure, protein species, infection and replication style. In March 2017, some Chinese molecular virologists of HVRI developed a new attenuated vaccine successfully, which could induce excellent immune protection and control the spread of EIAV. Based on the experimental results of Equine Infectious Disease Research Team, 45 observations were obtained from 8 mixed-gender horses. All the horses were inoculated with EIAV infectious clone and 5 horses were inoculated with vaccine strain. The horses were monitored daily for clinical symptoms, and blood was drawn at regular intervals (weekly) for assays of platelets, viral replication, sequencing and virus-specific immune responses. After the 15 days immunization period, the five horses inoculated with vaccine (39 observations, ID 1 to 39) were normal and immune from the virulent strains. The three horses that were not vaccinated (6 observations, ID 40 to 45) had fever and two of them died at the end of the experiment. To test the immunization mechanism of the vaccine strains, the outcome variable of interest is the log value of viral loads, which measures the immune ability of the infected horses, and the explanatory variables include three antiviral agents (SLFN11,
Table 1: Case I-IV: Mean absolute bias (MAB) and root mean squared error (RMSE) of regression parameter estimates when \( n = 250 \)

| Error distributions | Sample Size \( n = 250 \) | MLEEM | KDEEM | KDEEM.H | KDEEM.LSE |
|---------------------|-----------------------------|-------|--------|---------|-----------|
|                     | \( n \) | MAB | RMSE | RE | MAB | RMSE | RE | MAB | RMSE | RE | MAB | RMSE | RE |
| Case I              | \( \beta_{1,0} \) | 15.01 | 18.65 |     | 16.91 | 21.34 | 0.76 | 19.03 | 23.70 | 0.62 | 17.50 | 21.68 | 0.74 |
| \( \epsilon_1 \sim N(0,1) \) | \( \beta_{1,1} \) | 27.79 | 34.90 |     | 30.05 | 37.98 | 0.84 | 36.81 | 46.14 | 0.57 | 35.03 | 43.34 | 0.65 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,0} \) | 7.45 | 9.28 |     | 8.39 | 10.61 | 0.77 | 11.10 | 13.90 | 0.45 | 12.20 | 15.55 | 0.36 |
|                     | \( \beta_{1,1} \) | 14.85 | 18.75 | | 16.45 | 20.87 | 0.81 | 17.11 | 21.80 | 0.74 | 17.69 | 22.45 | 0.70 |
| Case II             | \( \beta_{1,0} \) | 33.64 | 42.97 | | 25.88 | 35.56 | 1.46 | 42.05 | 54.74 | 0.62 | 80.13 | 89.61 | 0.23 |
| \( \epsilon_1 \sim U(-3,3) \) | \( \beta_{1,1} \) | 56.53 | 70.61 | | 30.29 | 40.70 | 3.01 | 65.74 | 84.36 | 0.70 | 78.68 | 98.68 | 0.51 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,0} \) | 22.70 | 29.52 | | 16.54 | 23.60 | 1.57 | 43.59 | 48.96 | 0.36 | 69.45 | 73.51 | 0.16 |
|                     | \( \beta_{1,1} \) | 42.06 | 51.97 | | 20.49 | 28.02 | 3.44 | 37.94 | 50.16 | 1.07 | 80.33 | 89.44 | 0.34 |
| Case III            | \( \beta_{1,0} \) | 27.02 | 34.24 | | 15.90 | 21.04 | 2.65 | 34.94 | 41.92 | 1.43 | 93.80 | 114.74 | 0.37 |
| \( \epsilon_1 \sim 0.5 N(-1.5,0.5^2) + 0.5 N(1.5,0.5^2) \) | \( \beta_{1,1} \) | 56.13 | 69.55 | | 15.24 | 19.35 | 12.93 | 42.35 | 50.08 | 1.43 | 93.80 | 114.74 | 0.37 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,0} \) | 17.39 | 21.88 | | 8.18 | 10.60 | 4.27 | 13.09 | 16.47 | 1.76 | 38.82 | 44.60 | 0.24 |
|                     | \( \beta_{1,1} \) | 36.91 | 44.91 | | 7.91 | 10.01 | 20.15 | 8.92 | 11.39 | 15.55 | 45.81 | 55.26 | 0.66 |
| Case IV             | \( \beta_{1,0} \) | 44.76 | 52.12 | | 39.57 | 49.34 | 1.12 | 47.99 | 54.13 | 0.93 | 39.77 | 45.90 | 1.29 |
| \( \epsilon_1 \sim 0.5 N(-1,0.5^2) + 0.5 N(1,1.5^2) \) | \( \beta_{1,1} \) | 52.02 | 66.36 | | 27.47 | 35.96 | 3.41 | 33.89 | 46.81 | 2.01 | 39.98 | 52.20 | 1.62 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,0} \) | 27.92 | 36.19 | | 12.72 | 17.59 | 4.23 | 16.03 | 20.40 | 3.15 | 29.36 | 34.25 | 1.12 |
|                     | \( \beta_{1,1} \) | 50.32 | 60.89 | | 16.82 | 22.19 | 7.53 | 17.01 | 21.84 | 7.78 | 46.60 | 55.40 | 1.21 |
Table 2: Case V-VII: Mean absolute bias (MAB) and root mean squared error (RMSE) of regression parameter estimates when \( n = 250 \)

| Sample Size \( n = 250 \) | MLEEM | KDEEM | KDEEM.H | KDEEM.LSE |
|--------------------------|-------|-------|---------|-----------|
| **Error distributions**  | **MAB** | **RMSE** | **RE** | **MAB** | **RMSE** | **RE** | **MAB** | **RMSE** | **RE** |
| Case V                   |       |       |         |           |           |       |       |           |           |       |       |       |       |
| \( \epsilon_1 \sim \Lambda(0, 1^2) \) | \( \beta_{1,0} \) | 96.01 | 108.35 | 92.62 | 100.58 | 1.16 | 77.56 | 86.56 | 1.57 | 43.75 | 53.13 | 4.15 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,1} \) | 67.90 | 95.97 | 22.36 | 32.31 | 8.83 | 24.20 | 43.24 | 4.93 | 57.95 | 104.06 | 0.85 |
| \( \beta_{1,1} \) | 52.97 | 71.19 | 10.65 | 15.75 | 20.34 | 10.20 | 14.01 | 25.81 | 42.02 | 59.38 | 1.44 |
| Case VI                  |       |       |         |           |           |       |       |           |           |       |       |       |       |
| \( \epsilon_1 \sim \text{Gamma}(2, 0.5) \) | \( \beta_{1,0} \) | 10.46 | 13.29 | 8.98 | 11.43 | 1.35 | 12.17 | 14.93 | 0.79 | 9.88 | 12.57 | 1.12 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,1} \) | 20.83 | 26.44 | 12.26 | 15.74 | 2.82 | 15.77 | 20.37 | 1.68 | 20.21 | 25.46 | 1.08 |
| \( \beta_{1,1} \) | 11.96 | 15.03 | 7.93 | 10.34 | 2.11 | 9.89 | 12.86 | 1.37 | 11.24 | 14.20 | 1.12 |
| Case VII                 |       |       |         |           |           |       |       |           |           |       |       |       |       |
| \( \epsilon_1 \sim \text{Rayleigh}(3) \) | \( \beta_{1,0} \) | 55.29 | 65.54 | 47.52 | 58.24 | 1.27 | 56.70 | 67.13 | 0.95 | 74.27 | 81.90 | 0.64 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) | \( \beta_{1,1} \) | 73.80 | 92.91 | 59.89 | 76.64 | 1.47 | 74.95 | 93.86 | 0.98 | 83.38 | 104.17 | 0.80 |
| \( \beta_{1,1} \) | 24.56 | 31.46 | 22.51 | 28.79 | 1.97 | 24.40 | 30.71 | 1.05 | 34.98 | 40.90 | 0.59 |
| \( \beta_{1,1} \) | 51.11 | 63.07 | 44.70 | 56.14 | 1.26 | 45.69 | 56.51 | 1.25 | 47.63 | 59.32 | 1.13 |
Table 3: Case I-IV: Mean absolute bias (MAB) and root mean squared error (RMSE) of regression parameter estimates when \( n = 500 \)

| Sample Size \( n = 500 \) | MLEEM | KDEEM | KDEEM.H | KDEEM.LSE |
|-----------------------------|-------|-------|---------|-----------|
| Error distributions        | MAB   | RMSE  | RE      | MAB       | RMSE  | RE   | MAB | RMSE | RE |
| Case I                     |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 10.22 | 12.89 | 11.25   | 12.74     | 16.08 | 0.64 | 13.17 | 16.38 | 0.62 |
| \( \beta_{1,1} \)          | 19.14 | 23.91 | 20.17   | 25.50     | 32.29 | 0.55 | 24.14 | 30.60 | 0.61 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 5.40  | 6.80  | 5.83    | 8.24      | 10.33 | 0.43 | 10.74 | 13.42 | 0.26 |
| \( \beta_{1,1} \)          | 11.09 | 13.75 | 11.79   | 13.17     | 16.38 | 0.62 | 13.09 | 16.43 | 0.70 |
| Case II                    |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 22.09 | 27.59 | 15.28   | 26.15     | 34.01 | 0.66 | 74.04 | 79.69 | 0.12 |
| \( \beta_{1,1} \)          | 44.78 | 55.58 | 17.75   | 49.75     | 63.75 | 0.76 | 58.39 | 72.61 | 0.59 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 18.84 | 22.80 | 10.27   | 37.80     | 40.27 | 0.32 | 69.76 | 71.84 | 0.10 |
| \( \beta_{1,1} \)          | 34.76 | 41.81 | 11.14   | 30.79     | 38.57 | 1.18 | 85.73 | 90.27 | 0.21 |
| Case III                   |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 21.21 | 26.38 | 10.87   | 29.19     | 34.62 | 0.58 | 25.15 | 31.88 | 0.68 |
| \( \beta_{1,1} \)          | 51.42 | 61.93 | 10.33   | 34.21     | 47.27 | 1.72 | 84.91 | 104.96 | 0.35 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 15.35 | 18.37 | 5.95    | 11.30     | 13.75 | 1.79 | 36.07 | 39.71 | 0.21 |
| \( \beta_{1,1} \)          | 31.42 | 37.30 | 5.36    | 6.31      | 8.01  | 21.67 | 41.85 | 48.12 | 0.60 |
| Case IV                    |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 41.69 | 46.67 | 35.19   | 47.46     | 51.19 | 0.83 | 35.50 | 39.58 | 1.39 |
| \( \beta_{1,1} \)          | 41.49 | 52.11 | 18.47   | 22.78     | 28.71 | 3.30 | 31.04 | 38.59 | 1.82 |
| \( \epsilon_2 \sim 0.5 \epsilon_1 \) |       |       |         |           |       |      |     |      |     |
| \( \beta_{1,0} \)          | 25.96 | 32.73 | 9.56    | 12.25     | 15.27 | 4.60 | 28.94 | 31.69 | 1.07 |
| \( \beta_{1,1} \)          | 45.13 | 53.33 | 11.31   | 12.10     | 15.30 | 12.14 | 46.60 | 51.88 | 1.06 |
Table 4: Case V-VII: Mean absolute bias (MAB) and root mean squared error (RMSE) of regression parameter estimates when $n = 500$

| Sample Size $n = 500$ | MLEEM | KDEEM | KDEEM.H | KDEEM.LSE |
|-----------------------|-------|-------|---------|-----------|
| **Error distributions** | **MAB** | **RMSE** | **RE** | **MAB** | **RMSE** | **RE** | **MAB** | **RMSE** | **RE** |
| **Case V** | | | | | | | | | |
| $\epsilon_1 \sim \Lambda(0, 1^2)$ | $\beta_{1,0}$ | 104.26 | 113.13 | 92.64 | 97.05 | 1.36 | 92.32 | 98.95 | 1.31 | 39.85 | 44.29 | 6.52 |
| $\epsilon_2 \sim 0.5 \epsilon_1$ | $\beta_{1,1}$ | 62.67 | 87.65 | 20.11 | 27.61 | 10.08 | 31.48 | 54.61 | 2.58 | 41.19 | 59.35 | 2.18 |
| **Case VI** | | | | | | | | | | | | |
| $\epsilon_1 \sim Gamma(5, 1)$ | $\beta_{1,0}$ | 15.90 | 21.67 | 7.59 | 9.53 | 3.93 | 10.27 | 13.18 | 2.05 | 16.98 | 20.74 | 0.83 |
| $\epsilon_2 \sim 0.5 \epsilon_1$ | $\beta_{1,1}$ | 4.11 | 5.18 | 4.65 | 5.67 | 0.83 | 12.11 | 13.33 | 0.15 | 3.98 | 4.94 | 1.10 |
| **Case VII** | | | | | | | | | | | | |
| $\epsilon_1 \sim Rayleigh(3)$ | $\beta_{1,0}$ | 48.95 | 56.32 | 38.61 | 47.02 | 1.44 | 51.31 | 58.90 | 0.91 | 73.40 | 77.61 | 0.53 |
| $\epsilon_2 \sim 0.5 \epsilon_1$ | $\beta_{1,1}$ | 63.12 | 77.70 | 43.56 | 55.11 | 1.99 | 64.28 | 79.20 | 0.96 | 73.98 | 90.71 | 0.73 |
| $\beta_{1,0}$ | 19.66 | 25.49 | 17.51 | 22.94 | 1.24 | 19.47 | 24.92 | 1.05 | 34.12 | 38.15 | 0.45 |
| $\beta_{1,1}$ | 40.52 | 50.07 | 34.32 | 43.35 | 1.33 | 36.00 | 45.62 | 1.21 | 41.65 | 51.05 | 0.96 |
Table 5: Case I-IV: Mean absolute bias (MAB) and root mean squared error (RMSE) of regression parameter estimates when \( n = 1000 \)

| Sample Size \( n = 1000 \) | MLEEM | KDEEM | KDEEM.H | KDEEM.LSE |
|-----------------------------|-------|-------|---------|-----------|
| Error distributions        | \( MAB \) | \( RMSE \) | \( RE \) | \( MAB \) | \( RMSE \) | \( RE \) | \( MAB \) | \( RMSE \) | \( RE \) | \( MAB \) | \( RMSE \) | \( RE \) |
| Case I \( \varepsilon_1 \sim N(0,1) \) | \( \beta_{1,0} \) | 7.80 | 13.74 | 8.23 | 16.14 | 0.72 | 9.94 | 17.51 | 0.62 | 10.51 | 16.18 | 0.72 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 13.62 | 19.98 | 14.45 | 25.58 | 0.61 | 18.45 | 29.40 | 0.46 | 18.15 | 25.07 | 0.64 |
| Case II \( \varepsilon_1 \sim U(-3,3) \) | \( \beta_{1,0} \) | 4.10 | 10.02 | 4.13 | 7.87 | 1.62 | 6.57 | 10.04 | 0.99 | 11.03 | 15.19 | 0.44 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 7.93 | 12.71 | 8.02 | 10.10 | 1.58 | 8.47 | 10.57 | 1.45 | 9.16 | 13.94 | 0.83 |
| Case III \( \varepsilon_1 \sim 0.5N(-1.5,0.5^2) + 0.5N(1.5,0.5^2) \) | \( \beta_{1,0} \) | 15.52 | 19.44 | 9.75 | 12.42 | 2.45 | 16.32 | 20.68 | 0.88 | 71.51 | 74.38 | 0.07 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 34.94 | 42.34 | 9.93 | 12.66 | 11.19 | 35.73 | 44.23 | 0.92 | 41.75 | 51.57 | 0.67 |
| Case IV \( \varepsilon_1 \sim 0.5N(-1,0.5^2) + 0.5N(1,1.5^2) \) | \( \beta_{1,0} \) | 15.39 | 19.21 | 8.41 | 10.46 | 3.37 | 24.57 | 28.82 | 0.44 | 20.34 | 25.37 | 0.57 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 45.20 | 51.43 | 7.42 | 9.22 | 31.11 | 24.56 | 33.70 | 2.33 | 74.26 | 90.50 | 0.32 |
| \( \varepsilon_1 \sim N(0.5,0.5^2) \) | \( \beta_{1,0} \) | 16.55 | 19.21 | 6.79 | 8.69 | 4.89 | 34.08 | 35.29 | 0.30 | 70.60 | 71.68 | 0.07 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 30.70 | 35.63 | 6.48 | 8.30 | 18.45 | 28.35 | 33.69 | 1.12 | 90.65 | 92.80 | 0.15 |
| \( \varepsilon_1 \sim U(-1,1) \) | \( \beta_{1,0} \) | 41.91 | 44.53 | 35.76 | 42.36 | 1.11 | 48.64 | 50.73 | 0.77 | 34.32 | 36.40 | 1.50 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 45.13 | 51.81 | 12.96 | 16.58 | 6.36 | 15.93 | 19.98 | 4.38 | 26.74 | 32.12 | 1.69 |
| \( \varepsilon_1 \sim N(0,1) \) | \( \beta_{1,0} \) | 27.99 | 33.30 | 8.73 | 10.90 | 9.34 | 9.67 | 11.88 | 7.85 | 29.63 | 31.16 | 1.14 |
| \( \varepsilon_2 \sim 0.5\varepsilon_1 \) | \( \beta_{1,1} \) | 45.49 | 52.40 | 8.25 | 10.46 | 25.08 | 9.46 | 13.73 | 19.94 | 48.44 | 51.44 | 1.04 |
Table 6: Case V-VII: Mean absolute bias (MAB) and root mean squared error (RMSE) of regression parameter estimates when $n = 1000$

| Error distributions | Sample Size $n = 1000$ | MLEEM | KDEEM | KDEEM.H | KDEEM.LSE |
|----------------------|------------------------|-------|-------|---------|-----------|
|                      |                        | MAB   | RMSE  | RE      | MAB       | RMSE  | RE   | MAB   | RMSE  | RE   |
| Case V               | $\beta_{1,0}$ | 109.23 | 115.24 | 95.77 | 97.98 | 1.38 | 94.41 | 97.92 | 1.39 | 34.80 | 38.02 | 9.19 |
| $\epsilon_1 \sim \Lambda(0, 1^2)$ | $\beta_{1,1}$ | 60.03  | 81.28  | 15.31 | 21.52 | 14.27 | 23.80 | 40.64 | 4.00 | 34.22 | 46.90 | 3.00 |
| $\epsilon_2 \sim 0.5\epsilon_1$ | $\beta_{1,0}$ | 48.93  | 53.06  | 29.25 | 31.07 | 2.98 | 24.68 | 26.76 | 4.01 | 13.93 | 17.97 | 8.90 |
|                      | $\beta_{1,1}$ | 49.10  | 62.31  | 11.13 | 13.71 | 20.64 | 7.70  | 10.27 | 36.78 | 42.53 | 48.00 | 1.69 |
| Case VI              | $\beta_{1,0}$ | 6.03   | 7.40   | 4.52  | 5.71  | 1.68 | 10.17 | 11.34 | 0.43 | 4.95  | 6.21  | 1.42 |
| $\epsilon_1 \sim \text{Gamma}(5, 1)$ | $\beta_{1,1}$ | 11.29  | 14.03  | 5.19  | 6.47  | 4.70 | 7.25  | 9.11  | 2.37 | 14.32 | 17.25 | 0.66 |
| $\epsilon_2 \sim 0.5\epsilon_1$ | $\beta_{1,0}$ | 2.82   | 3.59   | 3.88  | 4.59  | 0.61 | 11.81 | 12.44 | 0.08 | 2.75  | 3.45  | 1.08 |
|                      | $\beta_{1,1}$ | 7.34   | 8.59   | 4.13  | 5.11  | 3.07 | 6.08  | 7.41  | 1.46 | 5.42  | 6.85  | 1.71 |
| Case VII             | $\beta_{1,0}$ | 46.58  | 52.06  | 31.38 | 39.36 | 1.75 | 49.39 | 54.55 | 0.91 | 72.97 | 75.46 | 0.48 |
| $\epsilon_1 \sim \text{Rayleigh}(3)$ | $\beta_{1,1}$ | 58.42  | 69.16  | 30.49 | 40.63 | 2.90 | 56.44 | 67.83 | 1.04 | 64.53 | 76.89 | 0.81 |
| $\epsilon_2 \sim 0.5\epsilon_1$ | $\beta_{1,0}$ | 16.32  | 23.66  | 13.76 | 19.61 | 1.46 | 16.53 | 22.19 | 1.14 | 35.87 | 38.84 | 0.37 |
|                      | $\beta_{1,1}$ | 32.35  | 41.51  | 26.36 | 33.69 | 1.52 | 30.50 | 38.49 | 1.16 | 41.31 | 48.87 | 0.72 |
Viperin and Tetherin), which can be used in immunodiffusion assay to confirm whether an animal was protected. Antiviral agents belong to a type of cell-intrinsic protein which can potentially prevent the virus intrusion at every step of genes replication. In practice, most antiviral agents inside a protected animal’s body should have negative effects on the viral loads.

We apply the proposed semi-parametric mixture of linear regression models to help evaluate the lentivirus pathogenesis and immune protection mechanism. We obtain the MLEEM and KDEEM estimates without using the vaccine strain information of the horses (or the correlations among the observations). It is expected that the protected group and unprotected group might have different relationship between the response variable and explanatory variables. Table 7 displays mixture regression parameter estimates and the correct classification percentages (CCP) based on the leave-one-out cross validation for the two methods. The coefficient estimates indicate that the immunodiffusion mechanisms for two groups are significantly different. For the group of horses inoculated with vaccine, both methods demonstrate that all three antiviral agents have negative effects on the amount of viral loads inside the animals’ bodies, which verifies the effectiveness of the vaccine. In contrast, for the group of horses that were not vaccinated, two of the three antiviral agents have positive effects on the amount of viral loads, which is undesirable but sensible since these horses were not protected before the experiment. The results of CCP demonstrate that the new method KDEEM provides more accurate classification results than the classical MLE. In Figure 1, we also plot the classification probabilities that the observation is from the protected/vaccinated group versus the ID for different estimates. Based on the experiment setup, the observations with ID from 1 to 39 belong to the protected group and the ones with ID from 40 to 46 are from unprotected group. The red triangle points in Figure 1 are the observations that are wrongly classified. The correct classification percentage (CCP) of MLEEM is about 93.33%, and the
CCP of the proposed KDEEM is 100%. Therefore, the proposed semi-parametric mixture of linear regression models can reduce the modelling bias and have better classification performance for this dataset.

Table 7: The coefficient estimates and the correct classification percentages (CCP) based on MLEEM and KDEEM.

| Covariate | MLEEM   | KDEEM   |
|-----------|---------|---------|
|           | $\hat{\beta}_1$ | $\hat{\beta}_2$ | $\hat{\beta}_1$ | $\hat{\beta}_2$ |
| Intercept | 6.68    | 14.37   | 3.50    | 11.81   |
| SLFN11    | -0.07   | 3.69    | -0.06   | 3.75    |
| viperin   | -0.33   | -9.13   | -0.42   | -9.25   |
| Tetherin  | -0.59   | 2.24    | -0.32   | 2.26    |
| CCP       | 93.33%  | 100%    |

5. Discussion

Traditional mixture of regression models assume that the component error densities have normal distributions, and the subsequent analysis through MLE will be invalid if the normality assumption is violated. In this article, we propose a semiparametric mixture regression estimator with unspecified error densities. We establish the identifiability of the semi-parametric mixture of regression models and provide the asymptotic properties of the proposed estimators. Simulation studies and real data application demonstrate that the proposed estimators work well for different error densities and provide substantial improvement over the classic MLE when the component error densities are non-normal.

To stay focused, we only considered the mixture of linear regressions. It will be interesting to extend the results in this paper to some other mixture regression models such as semiparametric mixture regression models proposed by Huang and Yao (2012) and Xiang and Yao (2018) and nonparametric mixture regression models proposed by Huang
et al. (2013). In our semiparametric regression model (3), we assumed that the number of components is known. It will be also interesting to choose the number of components data adaptively for (3). For parametric finite mixture models, the information-based criteria methods, such as AIC and BIC (Fraley and Raftery, 1998; Keribin, 2000), and hypothesis testing methods (Li and Chen, 2010; Chen et al., 2012) are commonly used to choose the number of components. It will be useful to adapt the above procedures to the semiparametric mixture model framework.
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