A note on the null condition for quadratic nonlinear Klein-Gordon systems in two space dimensions

Soichiro Katayama∗  Tohru Ozawa†  Hideaki Sunagawa‡

January 20, 2013

Dedicated to the memory of Professor Yujiro Ohya

Abstract

We consider the Cauchy problem for quadratic nonlinear Klein-Gordon systems in two space dimensions with masses satisfying the resonance relation. Under the null condition in the sense of J.-M. Delort, D. Fang, R. Xue (J. Funct. Anal. 211 (2004), 288–323), we show the global existence of asymptotically free solutions if the initial data are sufficiently small in some weighted Sobolev space. Our proof is based on an algebraic characterization of nonlinearities satisfying the null condition.

Key Words: Nonlinear Klein-Gordon systems; Null condition; Mass resonance.
Mathematics Subject Classification: 35L70, 35B40, 35L15
Running Title: Null condition for NLKG systems

1 Introduction

In the present paper we consider large time behavior of solutions to the Cauchy problem for nonlinear systems of Klein-Gordon equations in two space dimensions:

\[(\Box + m_j^2)u_j = F_j(u, \partial u, \partial^2 u), \quad (t, x) \in [0, \infty) \times \mathbb{R}^2, \quad j = 1, 2, \quad (1.1)\]
\[ u_j(0, x) = f_j(x), \quad \partial_t u_j(0, x) = g_j(x), \quad x \in \mathbb{R}^2, \ j = 1, 2, \quad (1.2) \]

where \( \Box = \partial_t^2 - \partial_x^2 - \partial_y^2 \), \( \partial = (\partial_x, \partial_y, \partial_t) \) with \( \partial_0 = \partial_t = \partial/\partial t, \ \partial_j = \partial/\partial x_j \) for \( j = 1, 2 \), and \( u = (u_k)_{k=1,2} \) is an \( \mathbb{R}^2 \)-valued unknown function, while \( \partial u = (\partial_a u_k)_{a=0,1,2} \) and \( \partial^2 u = (\partial_a \partial_b u_k)_{a,b=0,1,2} \) are its first and second order derivatives, respectively. The masses \( m_1, m_2 \) are positive constants. Without loss of generality we may always assume \( m_1 \leq m_2 \) throughout this paper. The nonlinear term \( F_j = F_j(\xi, \eta, \zeta) \) is a \( C^\infty \) function of \( (\xi, \eta, \zeta) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 3} \times \mathbb{R}^{2 \times 9} \) which vanishes of quadratic order at the origin, that is,

\[ F_j(\xi, \eta, \zeta) = O((|\xi| + |\eta| + |\zeta|)^2) \quad \text{as} \quad (\xi, \eta, \zeta) \to (0, 0, 0). \]

We always suppose that the system is quasi-linear. In other words we assume that

\[ F_j(u, \partial u, \partial^2 u) = \sum_{k=1}^{2} \sum_{a,b=0}^{3} \gamma_{ab}^{jk}(u, \partial u) \partial_a \partial_b u_k + \tilde{F}_j(u, \partial u) \quad (1.3) \]

with some functions \( \gamma_{ab}^{jk}(\xi, \eta) \) vanishing of first order at the origin, and \( \tilde{F}_j(\xi, \eta) \) vanishing of quadratic order. To ensure the hyperbolicity, we assume that

\[ \gamma_{ab}^{jk}(\xi, \eta) = \gamma_{ab}^{kj}(\xi, \eta), \quad j, k = 1, 2, \ a, b = 0, 1, 2, \ (\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 3}. \quad (1.4) \]

Without loss of generality, we may also assume that \( \gamma_{ab}^{jk}(\xi, \eta) = \gamma_{ba}^{kj}(\xi, \eta) \) and \( \gamma_{00}^{jk}(\xi, \eta) \equiv 0 \).

From the perturbative viewpoint, quadratic nonlinear interaction is of special interest for the Klein-Gordon equations in two space dimensions because large time behavior of the solution is actually affected by the structure of the nonlinearities and by the ratio of the masses even if the initial data are sufficiently small, smooth and localized. In the case of \( m_2 \neq 2m_1 \) (which we call the non-resonant case), it is shown by Sunagawa [17] and Tsutsumi [19] that the solution exists globally without any restrictions on \( F_1, \ F_2 \) (other than the hyperbolicity assumption) if \( f_j, g_j \) are sufficiently small in a suitable weighted Sobolev space. Moreover, the solution is asymptotically free in the sense that we can find a solution \( u^+(t) \) of the homogeneous linear Klein-Gordon equations satisfying

\[ \lim_{t \to +\infty} \| u(t) - u^+(t) \|_E = 0, \]

where the energy norm \( \| \cdot \|_E \) is defined by

\[ \| \phi(t) \|_E = \left( \sum_{j=1}^{2} \frac{1}{2} \int_{\mathbb{R}^2} (\partial_t \phi_j(t, x))^2 + |\nabla \phi_j(t, x)|^2 + m_j^2 \phi_j(t, x)^2 \, dx \right)^{1/2} \]

for \( \phi = (\phi_j)_{j=1,2} \). On the other hand, in the resonant case (i.e., the case where \( m_2 = 2m_1 \)) we must put some structural condition on the nonlinearities in order to obtain asymptotically
free solution because there are examples of \((F_1, F_2)\) such that the energy of the corresponding solution grows up as \(t \to \infty\) (see \[18\]). A sufficient condition on the nonlinearities is introduced by Delort-Fang-Xue \[3\], called the null condition (see Definition 2.1 below), which allows us to show the global existence of small amplitude solutions for (1.1)–(1.2) in the resonant case if the data are sufficiently small, smooth and compactly supported (see also Kawahara-Sunagawa \[13\]). A pointwise asymptotic profile of the global solution is also given in \[3\]. However, their result does not imply the existence of a free profile in the sense of the energy norm.

The aim of this paper is to show the existence of a free profile in the sense of the energy norm under the null condition. Our approach is based on an algebraic characterization of the nonlinearities satisfying the null condition (see Proposition 5.1 below), which will allow us to reduce the problem essentially to the case of cubic nonlinearity through a kind of normal form argument. Note also that, differently from \[3\] and \[13\], our proof does not require compactness of the support of the initial data because we do not use the hyperbolic coordinates at all. We have only to assume that the initial data belong to some weighted Sobolev space and are sufficiently small in its norm, as in the non-resonant case \[17\]. This is another advantage of our approach.

There is a large literature on the nonlinear Klein-Gordon equation. We refer the readers to \[1\]–\[5\], \[9\]–\[19\] and references therein.

## 2 Main result

Let us first recall the definition of the null condition for the resonant quadratic nonlinear Klein-Gordon systems. We will follow the reformulation by Kawahara-Sunagawa (see the condition (a) in \[13\]) instead of the original definition given by \[3\]. For \(j = 1, 2\), we denote by \(F_{quad}^j\) the quadratic homogeneous part of \(F_j\), that is,

\[
F_{quad}^j(\xi, \eta, \zeta) = \lim_{\lambda \to 0} \lambda^{-2} F_j(\lambda \xi, \lambda \eta, \lambda \zeta)
\]

for \((\xi, \eta, \zeta) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 3} \times \mathbb{R}^{2 \times 9}\). Also we set the unit hyperboloid

\[
\mathbb{H} = \{ \omega = (\omega_0, \omega_1, \omega_2) \in \mathbb{R}^3 : \omega_0^2 - \omega_1^2 - \omega_2^2 = 1 \}
\]

and

\[
\Phi_j(\omega) = \int_0^1 F_{quad}^j(U(\theta), V(\omega, \theta), W(\omega, \theta)) e^{-2\pi ij\theta} d\theta \quad \text{(2.1)}
\]
for \( \omega \in \mathbb{H} \), where

\[
U(\theta) = (\cos 2\pi k\theta)_{k=1,2},
\]
\[
V(\omega, \theta) = (-\omega_{a} m_{k} \sin 2\pi k\theta)_{k=1,2, a=0,1,2},
\]
\[
W(\omega, \theta) = (-\omega_{a} \omega_{b} m_{k} \cos 2\pi k\theta)_{k=1,2, a,b=0,1,2},
\]

and \( i = \sqrt{-1} \).

**Definition 2.1.** We say that the nonlinear term \((F_1, F_2)\) satisfies the null condition if \( \Phi_1(\omega) = \Phi_2(\omega) = 0 \) for all \( \omega \in \mathbb{H} \).

Examples of quasi-linear term \((F_1, F_2)\) which satisfies the null condition as well as the hyperbolicity assumption \((1.4)\) will be given in Remark 5.1 below.

In order to state the main result precisely, let us also introduce the weighted Sobolev space as follows:

\[
H^{s,k}(\mathbb{R}^2) = \{ \phi \in L^2(\mathbb{R}^2) : (1 + |\cdot|^2)^{k/2}(1 - \Delta)^{s/2} \phi \in L^2(\mathbb{R}^2) \}
\]
equipped with the norm

\[
\| \phi \|_{H^{s,k}(\mathbb{R}^2)} = \|(1 + |\cdot|^2)^{k/2}(1 - \Delta)^{s/2} \phi \|_{L^2(\mathbb{R}^2)}.
\]

For simplicity, we write \( H^s = H^{s,0} \) and \( \| \phi \|_{H^s} = \| \phi \|_{H^{s,0}} \).

Now we are in a position to state the main result.

**Theorem 2.1.** Let \( m_2 = 2m_1 \). Assume that \((F_1, F_2)\) satisfies the null condition in the sense of Definition 2.1. Assume also that \((1.3) - (1.4)\) is satisfied. Let \( f_j \in H^{s+1,s}(\mathbb{R}^2) \), \( g_j \in H^{s,s}(\mathbb{R}^2) \) for \( j = 1,2 \) with \( s \geq 29 \). There exists a positive constant \( \varepsilon \) such that if

\[
\sum_{j=1}^{2} \left( \|f_j\|_{H^{s+1,s}(\mathbb{R}^2)} + \|g_j\|_{H^{s,s}(\mathbb{R}^2)} \right) \leq \varepsilon,
\]

the problem \((1.1) - (1.2)\) admits a unique global solution \( u = (u_1, u_2) \) satisfying

\[
u \in \bigcap_{k=0}^{s} C^k([0, \infty); H^{s+1-k}(\mathbb{R}^2)).
\]

Furthermore, \( u \) has a free profile, i.e., there exists \((f_j^+, g_j^+) \in H^{s-3}(\mathbb{R}^2) \times H^{s-4}(\mathbb{R}^2)\) such that

\[
\lim_{t \to +\infty} \left( \|(u_j - u_j^+)(t, \cdot)\|_{H^{s-3}(\mathbb{R}^2)} + \|\partial_t(u_j - u_j^+)(t, \cdot)\|_{H^{s-4}(\mathbb{R}^2)} \right) = 0,
\]

where \( u_j^+ \) solves \((\Box + m_j^2)u_j^+ = 0\) with \((u^+_j, \partial_t u^+_j)|_{t=0} = (f^+_j, g^+_j)\) for \( j = 1,2 \).
Remark 2.1. In the paper by Kawahara-Sunagawa [13], another sufficient condition for global existence with $C_0^\infty$ small data is also introduced (see the condition (b) in [13]). Our proof in the present paper does not work for that case, and as pointed out in [13], some long-range effect should be taken into account. It is still an open problem to find out precise asymptotic profile of the global solution under the condition (b) even in the simplest case

$$\begin{align*}
F_1 &= u_1 u_2, \\
F_2 &= u_1^2,
\end{align*}$$

that is a typical example which satisfies the condition (b) but violates the null condition. For closely related works on nonlinear Schrödinger systems, see the recent papers by Hayashi-Li-Naumkin [6], [7] and by Hayashi-Li-Ozawa [8].

The rest of this paper is organized as follows: In the next section, we give some preliminaries mainly on the commuting vector fields and the null forms. In Section 4, we recall an algebraic normal form transformation developed in the previous papers. A characterization of the nonlinearities satisfying the null condition will be given in Section 5. After that, we will prove the main theorem in Section 6. Throughout this paper, we will frequently use the following conventions on implicit constants:

- $A \lesssim B$ (resp. $A \gtrsim B$) stands for $A \leq CB$ (resp. $A \geq CB$) with a positive constant $C$.

- The expression $f = \sum'_{a \in A} g_a$ means that there exists a family $\{C_a\}_{a \in A}$ of real constants such that $f = \sum_{a \in A} C_a g_a$.

Also, the notation $(y) = (1 + |y|^2)^{1/2}$ will be used for $y \in \mathbb{R}^N$ with a positive integer $N$.

3 Notations and preliminaries

We put $x_0 = -t$, $x = (x_1, x_2)$, $\Omega_{ab} = x_a \partial_b - x_b \partial_a$, $0 \leq a, b \leq 2$, and

$$Z = (Z_1, \ldots, Z_6) = (\partial_0, \partial_1, \partial_2, \Omega_{01}, \Omega_{02}, \Omega_{12}).$$

Note that the following commutation relations hold:

$$[\Box + m^2, Z_j] = 0,$$

$$[\Omega_{ab}, \partial_c] = \eta_{bc} \partial_a - \eta_{ca} \partial_b,$$

$$[\Omega_{ab}, \Omega_{cd}] = \eta_{ad} \Omega_{bc} + \eta_{bc} \Omega_{ad} - \eta_{ac} \Omega_{bd} - \eta_{bd} \Omega_{ac}$$

for $m \in \mathbb{R}$, $1 \leq j \leq 6$, $0 \leq a, b \leq 2$. Here $[\cdot, \cdot]$ denotes the commutator of linear operators, and $(\eta_{ab})_{0 \leq a, b \leq 2} = \text{diag}(-1, 1, 1)$. For a smooth function $\phi$ of $(t, x) \in \mathbb{R}^{1+2}$ and for a non-negative integer $s$, we define

$$|\phi(t, x)|_s := \sum_{|\alpha| \leq s} |Z^\alpha \phi(t, x)|$$
and

\[ \| \phi(t) \|_s := \sum_{|\alpha| \leq s} \| Z^\alpha \phi(t, \cdot) \|_{L^2(\mathbb{R}^2)}, \]

where \( \alpha = (\alpha_1, \ldots, \alpha_6) \) is a multi-index, \( Z^\alpha = Z_1^{\alpha_1} \cdots Z_6^{\alpha_6} \) and \( |\alpha| = \alpha_1 + \cdots + \alpha_6 \). Next we introduce the null forms

\[ Q_{ab}(\phi, \psi) = (\partial_a \phi)(\partial_b \psi) - (\partial_b \phi)(\partial_a \psi), \quad 0 \leq a, b \leq 2, \quad (3.2) \]

\[ Q_0(\phi, \psi) = (\partial_t \phi)(\partial_t \psi) - \nabla_x \phi \cdot \nabla_x \psi = -\sum_{a,b=0}^{2} \eta_{ab}(\partial_a \phi)(\partial_b \psi). \quad (3.3) \]

As pointed out in [4] (see also [12]), \( Q_{ab} \) has a certain compatibility with the Klein-Gordon operator \( \Box + m^2 \), while \( Q_0 \) is sometimes not so if \( m \neq 0 \). In what follows, we call \( Q_{ab} \) the strong null forms. From the identities

\[ x_c Q_{ab}(\phi, \psi) = (\Omega_{ca} \phi)(\partial_b \psi) + (\Omega_{bc} \phi)(\partial_a \psi) + (\partial_c \phi)(\Omega_{ab} \psi), \]

\[ \partial_c Q_{ab}(\phi, \psi) = Q_{ab}(\partial_c \phi, \psi) + Q_{ab}(\phi, \partial_c \psi), \]

\[ \Omega_{cd} Q_{ab}(\phi, \psi) = Q_{ab}(\Omega_{cd} \phi, \psi) + Q_{ab}(\phi, \Omega_{cd} \psi) + \eta_{ad} Q_{bc}(\phi, \psi) + \eta_{bd} Q_{ac}(\phi, \psi) - \eta_{bc} Q_{ad}(\phi, \psi), \]

we deduce the following properties on the strong null forms.

**Lemma 3.1.** Let \( \phi, \psi \) be smooth functions on \( (t, x) \in \mathbb{R}^{1+2} \). We have

\[ |Q_{ab}(\phi, \psi)| \lesssim \frac{1}{\langle t + |x| \rangle} \left( |\phi|_1 |\partial \psi| + |\partial \phi| \| \psi \|_1 \right) \]

for \( 0 \leq a, b \leq 2 \), and

\[ Z^\alpha Q_{ab}(\phi, \psi) = \sum_{c,d=0}^{2} \sum'_{|\beta| + |\gamma| \leq |\alpha|} Q_{cd}(Z^\beta \phi, Z^\gamma \psi) \]

for any multi-index \( \alpha \).

We close this section with the following decay estimate due to Georgiev [5].

**Lemma 3.2.** Let \( m \) be a positive constant and \( w \) be a solution of the inhomogeneous linear Klein-Gordon equation \( \Box + m^2 \) \( w = h \) for \( t \geq 0, \ x \in \mathbb{R}^2 \). Then we have

\[ \langle t + |x| \rangle \| w(t, x) \| \lesssim \sum_{j=0}^{\infty} \sum_{|\beta| \leq 4} \sup_{\tau \in [0, t]} \| \varphi_j(\tau) \| \| \tau + |y| \| \| Z^\beta h(\tau, y) \|_{L^2(\mathbb{R}^2)} \]

\[ + \sum_{j=0}^{\infty} \sum_{|\beta| \leq 5} \| \langle y \rangle \varphi_j(\| y \|) \| Z^\beta w \|_{L^2(\mathbb{R}^2)}, \]

6
provided that the right-hand side is finite. Here \( \{ \varphi_j \}_{j=0}^{\infty} \) is the Littlewood-Paley partition of unity, i.e.,

\[
\sum_{j=0}^{\infty} \varphi_j(\tau) = 1 \quad (\tau \geq 0); \quad \varphi_j \in C_{0}^{\infty}(\mathbb{R}), \varphi_j \geq 0 \text{ for } j \geq 0;
\]

\[
\text{supp } \varphi_j \subset [2^{j-1}, 2^{j+1}] \text{ for } j \geq 1, \quad \text{supp } \varphi_0 \cap \mathbb{R}_{\geq 0} \subset [0, 2].
\]

4 Algebraic normal form transformation

In this section, we recall an algebraic normal form transformation developed by [14], [11], [19], [17], etc. Let \( v_j \) and \( \tilde{v}_j \) be smooth functions on \((t, x) \in \mathbb{R}^{1+2}\). We set \( h_j = (2^{j} + m_j^{2})v_j \) and \( \tilde{h}_j = (\Box + m_j^{2})\tilde{v}_j \) for \( j = 1, 2 \). Throughout this section, we use the the following convention: We write

\[
\Phi \sim \Psi
\]

if \( \Phi - \Psi \) can be written as a linear combination of \( Q_{ab}(\partial^\alpha v_k, \partial^\beta \tilde{v}_l), (\partial^\alpha v_k)(\partial^\beta \tilde{h}_l), (\partial^\alpha h_k)(\partial^\beta \tilde{v}_l) \) or \( h_k \tilde{h}_l \) with \( |\alpha|, |\beta| \leq 1, 0 \leq a, b \leq 2 \) and \( 1 \leq k, l \leq 2 \), where \( Q_{ab} \) is given by \( (3.2) \).

**Proposition 4.1.** Put \( e_{kl} = v_k \tilde{v}_l, \tilde{e}_{kl} = Q_0(v_k, \tilde{v}_l) \) and \( L_j = \Box + m_j^{2} \), where \( Q_0 \) is given by \( (5.3) \). We have

\[
(L_j e_{kl}, L_j \tilde{e}_{kl}) \sim (e_{kl}, \tilde{e}_{kl}) A_{jkl},
\]

where

\[
A_{jkl} = \begin{pmatrix}
m_j^2 - m_k^2 - m_l^2 & 2m_j^2 m_k^2 \\
2 & m_j^2 - m_k^2 - m_l^2
\end{pmatrix}.
\]

**Proof.** This proposition is nothing but a paraphrase of Lemma 6.1 of [17]. However, for the convenience of the readers, we give a proof here. It is sufficient to show that

\[
\Box e_{kl} \sim -(m_k^2 + m_l^2) e_{kl} + 2\tilde{e}_{kl}
\]

(4.1)

and

\[
\Box \tilde{e}_{kl} \sim -(m_k^2 + m_l^2) \tilde{e}_{kl} + 2m_k^2 m_l^2 e_{kl}.
\]

(4.2)

Since \( \Box v_k = -m_k^2 v_k + h_k \) and \( \Box \tilde{v}_l = -m_l^2 \tilde{v}_l + \tilde{h}_l \), we have

\[
\Box e_{kl} = (\Box v_k) \tilde{v}_l + v_k \Box \tilde{v}_l + 2Q_0(v_k, \tilde{v}_l)
\]

\[
\sim (m_k^2 v_k) \tilde{v}_l + v_k (-m_l^2 \tilde{v}_l) + 2\tilde{e}_{kl}
\]

\[
= -(m_k^2 + m_l^2) e_{kl} + 2\tilde{e}_{kl},
\]

7
which yields the first relation (4.1). As for the second relation (4.2), we observe the relation
\[(\partial_a \partial_c v_k)(\partial_b \partial_d \tilde{v}_l) = (\partial_a \partial_b v_k)(\partial_c \partial_d \tilde{v}_l) + Q_{ab}(\partial_a v_k, \partial_d \tilde{v}_l) \sim (\partial_a \partial_b v_k)(\partial_c \partial_d \tilde{v}_l)\]
to obtain
\[
\Box \tilde{e}_{kl} = Q_0(\Box v_k, \tilde{v}_l) + Q_0(v_k, \Box \tilde{v}_l) + 2 \sum_{a,b} \sum_{c,d} \eta_{ab} \eta_{cd} (\partial_a \partial_c v_k)(\partial_b \partial_d \tilde{v}_l)
\approx Q_0(-m_k^2 v_k, \tilde{v}_l) + Q_0(v_k, -m_l^2 \tilde{v}_l) + 2 \sum_{a,b} \sum_{c,d} \eta_{ab} \eta_{cd} (\partial_a \partial_b v_k)(\partial_c \partial_d \tilde{v}_l)
= -(m_k^2 + m_l^2)Q_0(v_k, \tilde{v}_l) + 2(\Box v_k)(\Box \tilde{v}_l)
\approx -(m_k^2 + m_l^2)\tilde{e}_{kl} + 2(-m_k^2 v_k)(-m_l^2 \tilde{v}_l)
= -(m_k^2 + m_l^2)\tilde{e}_{kl} + 2m_k^2 m_l^2 \tilde{e}_{kl}.
\]
This completes the proof. 

Now we focus our attention on the structure of the matrix \(A_{jkl}\) for \(1 \leq j \leq 2\) and \(1 \leq k \leq l \leq 2\) under the resonance relation \(m_2 = 2m_1 > 0\). Since
\[
\det A_{jkl} = \prod_{\sigma_1, \sigma_2 \in \{\pm 1\}} (m_j + \sigma_1 m_k + \sigma_2 m_l),
\]
we can see that \(A_{jkl}\) is invertible if and only if \((j, k, l) = (1, 1, 1), (1, 2, 2), (2, 1, 2)\) or \((2, 2, 2)\). In this case, we have
\[
v_k \tilde{v}_l = (e_{kl} \tilde{e}_{kl}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}
= (e_{kl} \tilde{e}_{kl}) A_{jkl} \begin{pmatrix} p_{jkl} \\ \tilde{p}_{jkl} \end{pmatrix}
\sim (L_j e_{kl} \tilde{L}_j \tilde{e}_{kl}) \begin{pmatrix} p_{jkl} \\ \tilde{p}_{jkl} \end{pmatrix}
= (\Box + m_j^2) \begin{pmatrix} p_{jkl} v_k \tilde{v}_l + \tilde{p}_{jkl} Q_0(v_k, \tilde{v}_l) \end{pmatrix}
\]
with
\[
\begin{pmatrix} p_{jkl} \\ \tilde{p}_{jkl} \end{pmatrix} = A_{jkl}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
On the other hand, \(A_{jkl}\) is degenerate when \((j, k, l) = (1, 1, 2)\) or \((2, 1, 1)\). Indeed,
\[
\text{Image}(A_{112}) = \left\{ \kappa \begin{pmatrix} -2m_1^2 \\ 1 \end{pmatrix} : \kappa \in \mathbb{R} \right\}, \quad \text{Image}(A_{211}) = \left\{ \kappa \begin{pmatrix} m_1^2 \\ 1 \end{pmatrix} : \kappa \in \mathbb{R} \right\}.
\]
Remark that Image $(A_{112})$ and Image $(A_{211})$ correspond directly to the quadratic terms

$$G_1(v_1, \tilde{v}_2) := Q_0(v_1, \tilde{v}_2) - 2m_1^2 v_1 \tilde{v}_2 \sim (\Box + m_1^2) \left( \frac{v_1 \tilde{v}_2}{2} \right)$$  \hspace{1cm} (4.4)$$

and

$$G_2(v_1, \tilde{v}_1) := Q_0(v_1, \tilde{v}_1) + m_1^2 v_1 \tilde{v}_1 \sim (\Box + m_2^2) \left( \frac{v_1 \tilde{v}_1}{2} \right),$$  \hspace{1cm} (4.5)$$

respectively. The following quadratic terms should be also associated with Image $(A_{112})$:

$$H_{1,a}(v_1, \tilde{v}_2) := v_1 \partial_a \tilde{v}_2 + 2\tilde{v}_2 \partial_a v_1, \hspace{0.5cm} a = 0, 1, 2.$$  \hspace{1cm} (4.6)$$

In fact, it follows that

$$2m_1^2 H_{1,a}(v_1, \tilde{v}_2) + G_1(v_1, \partial_a \tilde{v}_2) = Q_0(v_1, \partial_a \tilde{v}_2) + 4m_1^2 (\partial_a v_1) \tilde{v}_2$$

$$= - \sum_{b,c} \eta_{bc} (\partial_b v_1) (\partial_c \partial_a \tilde{v}_2) + m_2^2 (\partial_a v_1) \tilde{v}_2$$

$$\sim - \sum_{b,c} \eta_{bc} (\partial_a v_1) (\partial_b \partial_c \tilde{v}_2) + m_2^2 (\partial_a v_1) \tilde{v}_2$$

$$= (\partial_a v_1) \tilde{h}_2 \sim 0,$$

whence

$$H_{1,a}(v_1, \tilde{v}_2) \sim - \frac{1}{2m_1^2} G_1(v_1, \partial_a \tilde{v}_2) \sim (\Box + m_1^2) \left( \frac{v_1 \partial_a \tilde{v}_2}{-4m_1^2} \right).$$  \hspace{1cm} (4.7)$$

Similarly

$$H_{2,a}(v_1, \tilde{v}_1) := v_1 \partial_a \tilde{v}_1 - \tilde{v}_1 \partial_a v_1, \hspace{0.5cm} a = 0, 1, 2,$$  \hspace{1cm} (4.8)$$

are associated with Image $(A_{211})$ since we can see that

$$H_{2,a}(v_1, \tilde{v}_1) \sim \frac{1}{m_1^2} G_2(v_1, \partial_a v_1) \sim (\Box + m_2^2) \left( \frac{v_1 \partial_a \tilde{v}_1}{2m_1^2} \right).$$  \hspace{1cm} (4.9)$$

The above observation will play the key role in the proof of Theorem 2.1.

5 Characterization of the null condition

The aim of this section is to give a characterization of the null condition in terms of $G_1$, $G_2$, $H_{1,a}$, $H_{2,a}$, and $Q_{ab}$ defined in the previous sections. What we are going to prove is the following.
Proposition 5.1. \((F_1, F_2)\) satisfies the null condition if and only if its quadratic homogeneous part can be written in the following form:

\[
F_1^{\text{quad}}(u, \partial u, \partial^2 u) = \sum_{|\alpha|+|\beta| \leq 1} G_1(\partial^\alpha u_1, \partial^\beta u_2) + \sum_{\alpha=0}^{2} \sum_{|\alpha|,|\beta| \leq 1} H_{1,\alpha}(\partial^\alpha u_1, \partial^\beta u_2) \\
+ \sum_{a,b=0}^{2} \sum_{|\alpha|,|\beta| \leq 1} Q_{ab}(\partial^\alpha u_1, \partial^\beta u_2) \\
+ \sum_{|\alpha|,|\beta| \leq 2; |\alpha|+|\beta| \leq 3} (\partial^\alpha u_1)(\partial^\beta u_2) + \sum_{|\alpha|,|\beta| \leq 2; |\alpha|+|\beta| \leq 3} (\partial^\alpha u_2)(\partial^\beta u_2),
\]

where \(G_1, G_2, H_{1,\alpha}, H_{2,\alpha},\) and \(Q_{ab}\) are given by (4.4), (4.5), (4.6), (4.8), and (3.2), respectively.

Remark 5.1. If we further assume (1.3)–(1.4) in addition to the null condition, some restriction on the coefficients for the terms including \(\partial^3 u\) is needed. For example, the following \((F_1, F_2)\) satisfies the null condition as well as (1.3)–(1.4):

\[
F_1 = \sum_{a=1}^{2} p_a G_1(u_1, \partial_a u_2) + \sum_{0 \leq a,b \leq 2; a+b \neq 0} q_{ab} H_{1,\alpha}(u_1, \partial_b u_2) \\
+ \sum_{a=1}^{2} G_1(\partial_a u_1, u_2) + \sum_{0 \leq a,b \leq 2; a+b \neq 0} H_{1,\alpha}(\partial_b u_1, u_2),
\]

\[
F_2 = \sum_{a=1}^{2} p_a G_2(u_1, \partial_a u_1) + \sum_{0 \leq a,b \leq 2; a+b \neq 0} q_{ab} H_{2,\alpha}(u_1, \partial_b u_1)
\]

with real constants \(p_a\) and \(q_{ab}\).

Proof of Proposition 5.1. For \(j = 1, 2\), we will write

\[
\Psi(u, \partial u, \partial^2 u) \sim \tilde{\Psi}(u, \partial u, \partial^2 u)
\]

if we have

\[
\int_0^1 (\Psi - \tilde{\Psi})(U(\theta), V(\omega, \theta), W(\omega, \theta)) e^{-2\pi ij\theta} d\theta = 0
\]
for $\omega \in \mathbb{H}$, where $U$, $V$ and $W$ are given by \((2.2)\), \((2.3)\) and \((2.4)\), respectively. We split $F^{\text{quad}}_j(u, \partial u, \partial^2 u)$ into the three parts:

$$F^{\text{quad}}_j(u, \partial u, \partial^2 u) = F^{11}_j(u, \partial u, \partial^2 u) + F^{12}_j(u, \partial u, \partial^2 u) + F^{22}_j(u, \partial u, \partial^2 u)$$

with

$$F^{kl}_j(u, \partial u, \partial^2 u) = \sum_{|\alpha|,|\beta| \leq 2; |\alpha|+|\beta| \leq 3} (\partial^\alpha u_k)(\partial^\beta u_l).$$

Then we can check that

$$F^{\text{quad}}_1(u, \partial u, \partial^2 u) \cong F^{12}_1(u, \partial u, \partial^2 u), \quad F^{\text{quad}}_2(u, \partial u, \partial^2 u) \cong F^{21}_2(u, \partial u, \partial^2 u)$$

by using the relation

$$\int_0^1 e^{2\pi i(k_1+k_2-j)}d\theta = \begin{cases} 1 & \text{if } (j, k_1, k_2) = (1, -1, 2) \text{ or } (2, 1, 1), \\ 0 & \text{otherwise}, \end{cases}$$

for $j = 1, 2$ and $k_1, k_2 \in \mathbb{Z}$ with $1 \leq |k_1| \leq |k_2| \leq 2$. Hence we only have to investigate $F^{12}_1$ and $F^{21}_2$ in order to check the null condition.

First we consider $F^{12}_1$. We rewrite $(\partial^\alpha u_1)(\partial^\beta u_2)$ with $|\alpha| = 0$ as follows:

$$u_1(\partial^\beta u_2) = \frac{1}{2m_1^2} (Q_0(u_1, \partial^\beta u_2) - G_1(u_1, \partial^\beta u_2)), \quad |\beta| \leq 1,$$

$$u_1(\partial_a \partial_b u_2) = -2(\partial_a u_1)(\partial_b u_2) + H_{1,a}(u_1, \partial_b u_2).$$

We can rewrite $(\partial^\alpha u_1)(\partial^\beta u_2)$ with $|\beta| = 0$ in a similar fashion. Moreover we have

$$(\partial_c u_1)(\partial_a \partial_b u_2) = -2(\partial_b u_2)(\partial_a \partial_c u_1) + H_{1,a}(\partial_c u_1, \partial_b u_2).$$

Hence we find

$$F^{12}_1(u, \partial u, \partial^2 u) = \sum_{a,b=0}^{2} \lambda_{ab}(\partial_a u_1)(\partial_b u_2) + \sum_{a,b,c=0}^{2} \mu_{abc}(\partial_c u_1)(\partial_a \partial_b u_1)$$

$$+ \sum_{|\alpha|+|\beta| \leq 1} G_1(\partial^\alpha u_1, \partial^\beta u_2) + \sum_{a=0}^{2} \sum_{|\alpha|,|\beta| \leq 1} H_{1,a}(\partial^\alpha u_1, \partial^\beta u_2)$$

with some real constants $\lambda_{ab}$ and $\mu_{abc}$. Since $G_1(\partial^\alpha u_1, \partial^\beta u_2) \cong 0$ and $H_{1,a}(\partial^\alpha u_1, \partial^\beta u_2) \cong 0$, it follows from the definition of $\Phi_1$ that

$$\Phi_1(\omega) = \frac{m_1 m_2}{4} \sum_{a,b=0}^{2} \lambda_{ab} \omega_a \omega_b - \frac{m_1^2 m_2}{4} \sum_{a,b,c=0}^{3} \mu_{abc} \omega_a \omega_b \omega c.$$
In order that this quantity vanishes identically on \( \mathbb{H} \), we must have \( \lambda_{aa} = 0 \) for \( 0 \leq a \leq 2 \), \( \lambda_{ab} = -\lambda_{ba} \) for \( 0 \leq a < b \leq 2 \), and so on. Now it is not difficult to see that \( \sum_{a,b} \lambda_{ab}(\partial_a u_1)(\partial_b u_2) \) and \( \sum_{a,b,c} \mu_{abc}(\partial_c u_2)(\partial_a \partial_b u_1) \) can be written in terms of the strong null forms. Hence we have (5.1). The converse is also true. Similarly, by writing

\[
    u_1(\partial^\alpha u_1) = -\frac{1}{m^2_1} (Q_0(u_1, \partial^\alpha u_1) - G_2(u_1, \partial^\alpha u_1)), \quad |\alpha| \leq 1,
\]

we have

\[
    F_2^{11}(u, \partial u, \partial^2 u) = \sum_{a,b=0}^2 \tilde{\lambda}_{ab}(\partial_a u_1)(\partial_b u_1) + \sum_{a,b=0}^2 \tilde{\mu}_{abc}(\partial_c u_1)(\partial_a \partial_b u_1)
\]

with appropriate real constants \( \tilde{\lambda}_{ab} \) and \( \tilde{\mu}_{abc} \), which leads to

\[
    \Phi_2(\omega) = -\frac{m^2_1}{4} \sum_{a,b=0}^2 \tilde{\lambda}_{ab}\omega_a\omega_b - i\frac{m^3_1}{4} \sum_{a,b,c=0}^2 \tilde{\mu}_{abc}\omega_a\omega_b\omega_c.
\]

As before, \( \Phi_2(\omega) \equiv 0 \) on \( \mathbb{H} \) implies that \( \sum_{a,b} \tilde{\lambda}_{ab}(\partial_a u_1)(\partial_b u_1) \) and \( \sum_{a,b,c} \tilde{\mu}_{abc}(\partial_c u_1)(\partial_a \partial_b u_1) \) can be written in terms of the strong null forms. This leads to (5.2). The converse is also true.

By Proposition 5.1, (4.3), (4.4), (4.5), (4.7), and (4.9) we obtain the following.

**Corollary 5.1.** Let \((u_1, u_2)\) be a smooth solution for (1.1). If \((F_1, F_2)\) satisfies the null condition, we have

\[
    F_j(u, \partial u, \partial^2 u) = (\Box + m^2_j)\Lambda_j + N_j + R_j
\]

for \( j = 1, 2 \), where

\[
    \Lambda_j = \sum_{k,l=1}^2 \sum_{|\alpha|,|\beta| \leq 3} (\partial^\alpha u_k)(\partial^\beta u_l),
\]

\[
    N_j = \sum_{k,l=1}^2 \sum_{a,b=0}^2 \sum_{|\alpha|,|\beta| \leq 3} Q_{ab}(\partial^\alpha u_k, \partial^\beta u_l),
\]

and \( R_j \) is a smooth function of \((\partial^\alpha u)_{|\alpha| \leq 5}\) with

\[
    R_j = O\left(|(\partial^\alpha u)_{|\alpha| \leq 5}|^3\right) \quad \text{near } (\partial^\alpha u)_{|\alpha| \leq 5} = 0.
\]
6 Proof of the main theorem

Now we are ready to prove Theorem 2.1. The main step of the proof is to get some \textit{a priori} estimate. From now on, we suppose that the null condition, as well as (1.3)–(1.4), is satisfied and let \( u = (u_j)_{j=1,2} \) be a solution to (1.1)–(1.2) for \( t \in [0,T) \). We define

\[
E(T) = \sup_{0 \leq t < T} \left[ \langle t \rangle^{-\delta} (\| u(t) \|_s + \| \partial u(t) \|_s) \\
+ \| u(t) \|_{s-4} + \| \partial u(t) \|_{s-4} + \sup_{x \in \mathbb{R}^2} \{ (t + |x|) |u(t,x)|_{s-12} \} \right]
\]

where \( s \geq 29 \) and \( 0 < \delta < 1 \). Then we have the following.

**Proposition 6.1.** Assume that \( f_j, g_j \) satisfy (2.5). Suppose that \( E(T) \leq 1 \). There exists a positive constant \( C_0 \), which is independent of \( \varepsilon \) and \( T \), such that

\[
E(T) \leq C_0 (\varepsilon + E(T)^2). \tag{6.1}
\]

**Proof.** The following argument is almost the same as that of the previous works ([11], [15], [17], etc.). First we note that Corollary 5.1 and the commutation relation (3.1) imply

\[
(\Box + m_j^2) Z^\alpha (u_j - \Lambda_j) = Z^\alpha (N_j + R_j) \tag{6.2}
\]

with

\[
|Z^\alpha \Lambda_j(t,x)| \lesssim |u|_{|\alpha|/2+3} (|u|_{|\alpha|+2} + |\partial u|_{|\alpha|+2}), \tag{6.3}
\]

\[
|Z^\alpha R_j(t,x)| \lesssim |u|^2_{|\alpha|/2+5} (|u|_{|\alpha|+4} + |\partial u|_{|\alpha|+4}), \tag{6.4}
\]

and

\[
|Z^\alpha N_j(t,x)| \lesssim \frac{1}{\langle t + |x| \rangle} |u|_{|\alpha|/2+4} (|u|_{|\alpha|+3} + |\partial u|_{|\alpha|+3}) \tag{6.5}
\]

by Lemma 3.1. We use Lemma 3.2 for (6.2) with \( |\alpha| \leq s-12 \) to obtain

\[
\sum_{|\alpha| \leq s-12} \langle t + |x| \rangle |Z^\alpha (u_j - \Lambda_j)(t,x)|
\]

\[
\lesssim \varepsilon + \sum_{j=0}^\infty \sum_{|\beta| \leq s-8} \sup_{\tau \in [0,t]} \varphi_j(\tau) \| \langle \tau + | \cdot | \rangle Z^\beta (N_j + R_j)(\tau, \cdot) \|_{L^2},
\]

where we have used

\[
\sum_{j=0}^\infty \sum_{|\beta| \leq s-7} \| \langle y \rangle \varphi_j(y) \left( Z^\beta (u_j - \Lambda_j) \right)(0,y) \|_{L^2(\mathbb{R}^2)} \lesssim \sum_{j=0}^\infty 2^{-j} (\| f \|_{H^{s-4,s-2}} + \| g \|_{H^{s-5,s-2}}) \lesssim \varepsilon
\]

13
by (2.5). From (6.4) and (6.5) it follows that
\[
\sum_{|\beta| \leq s-8} \| (t + |z|) Z^3 (N_j + R_j) (t, \cdot) \|_{L^2} \\
\lesssim \frac{1}{\langle t \rangle} \left( \| u(t) \|_{s-4} + \| \partial u(t) \|_{s-4} \right) \sup_{y \in \mathbb{R}^2} \left( \langle t + |y| \rangle |u(t, y)| \|_{[(s-8)/2]+5} \right) \\
+ \frac{1}{\langle t \rangle} \left( \| u(t) \|_{s-4} + \| \partial u(t) \|_{s-4} \right) \sup_{y \in \mathbb{R}^2} \left( \langle t + |y| \rangle |u(t, y)| \|_{[(s-8)/2]+5} \right)^2 \\
\lesssim \frac{E(T)^2}{\langle t \rangle}.
\]
Here we have used the relation \([ (s-8)/2 ] + 5 \leq s - 12 \) for \( s \geq 25 \). So we have
\[
\langle t + |x| \rangle |u_j(t, x) - \Lambda_j(t, x)|_{s-12} \lesssim \varepsilon + \sum_{j=0}^{\infty} \sup_{\tau \in [0, \varepsilon]} \varphi_j(\tau) \frac{E(T)^2}{\langle \tau \rangle} \\
\lesssim \varepsilon + E(T)^2 \sum_{j=0}^{\infty} 2^{-j} \\
\lesssim \varepsilon + E(T)^2.
\]
Also, (6.3) and the Sobolev embedding theorem yield
\[
\langle t + |x| \rangle |\Lambda_j(t, x)|_{s-12} \lesssim \langle t + |x| \rangle |u(t, x)|_{[(s-12)/2]+3} |u(t, x)|_{s-9} \\
\lesssim \langle t + |x| \rangle |u(t, x)|_{s-12} \| u(t) \|_{s-7} \\
\lesssim E(T)^2.
\]
Summing up, we obtain
\[
\langle t + |x| \rangle |u(t, x)|_{s-12} \lesssim \varepsilon + E(T)^2 \tag{6.6}
\]
for \((t, x) \in [0, T] \times \mathbb{R}^2\). Next we apply the standard energy inequality to (6.2) with \(|\alpha| \leq s-4\). Then we obtain
\[
\| (u_j - \Lambda_j)(t) \|_{s-4} + \| \partial (u_j - \Lambda_j)(t) \|_{s-4} \lesssim \varepsilon + \int_0^t \| (N_j + R_j)(\tau) \|_{s-4} \, d\tau.
\]
By using (6.4) and (6.5) again, we see that
\[
\| (N_j + R_j)(t) \|_{s-4} \lesssim \frac{1}{\langle t \rangle^2} \left( \| u(t) \|_{s} + \| \partial u(t) \|_{s} \right) \sup_{y \in \mathbb{R}^2} \left( \langle t + |y| \rangle |u(t, y)| \|_{[(s-4)/2]+5} \right) \\
+ \frac{1}{\langle t \rangle^2} \left( \| u(t) \|_{s} + \| \partial u(t) \|_{s} \right) \sup_{y \in \mathbb{R}^2} \left( \langle t + |y| \rangle |u(t, y)| \|_{[(s-4)/2]+5} \right)^2 \\
\lesssim \frac{E(T)^2}{\langle t \rangle^{2-s}}.
\]
Here we have used the relation \([ (s - 4)/2 ] + 5 \leq s - 12 \) for \( s \geq 29 \). So we have
\[
\|(u_j - \Lambda_j)(t)\|_{s-4} + \|\partial(u_j - \Lambda_j)(t)\|_{s-4} \lesssim \varepsilon + \int_0^t \frac{E(T)^2}{\langle \tau \rangle^{2-\delta}} d\tau \lesssim \varepsilon + E(T)^2.
\]
Also, (6.3) leads to
\[
\|\Lambda_j(t)\|_{s-3} + \|\partial\Lambda_j(t)\|_{s-4} \lesssim \frac{1}{(t)} \sup_{y \in \mathbb{R}^2} \left( |t + |y| |u(t, y)| \right) \left( \|u(t)\|_{s-1} + \|\partial u(t)\|_{s-1} \right) \lesssim E(T)^2(t)^{\delta-1}.
\]
To sum up, we have
\[
\|u(t)\|_{s-4} + \|\partial u(t)\|_{s-4} \lesssim \varepsilon + E(T)^2
\]
for \( t \in [0, T) \). Finally we apply \( Z^a \) to (1.1) with \( |\alpha| \leq s \) to obtain
\[
(\Box + m^2_j)(Z^a u_j) - \sum_{k=1}^2 \sum_{a,b=0}^3 \gamma_{ab}^k \partial_a \partial_b (Z^a u_k) = F_j^{(a)}, \quad j = 1, 2
\]
with \( F_j^{(a)} = Z^a \left( F_j(u, \partial u, \partial^2 u) \right) - \sum_{k=1}^2 \sum_{a,b=0}^3 \gamma_{ab}^k \partial_a \partial_b (Z^a u_k) \), where \( \gamma = (\gamma_{ab}^k) \) is from (1.3). Because of (1.4), we can use the energy inequality for hyperbolic systems with symmetric variable coefficients to estimate \( \|Z^a u(t)\|_{L^2} + \|\partial Z^a u(t)\|_{L^2} \), and we see that
\[
\|u(t)\|_{s} + \|\partial u(t)\|_{s} \lesssim \varepsilon + \int_0^t \|\partial (\gamma(u, \partial u))(\tau)\|_{L^\infty} \|\partial u(\tau)\|_{s} + \|F^{(a)}(\tau)\|_{s} d\tau.
\]
Since
\[
|F^{(a)}| \lesssim |u|_{s/2+2}(|u|_{s} + |\partial u|_{s}) \lesssim |u|_{s-12}(|u|_{s} + |\partial u|_{s})
\]
and
\[
|\partial (\gamma_{ab}^k(u, \partial u))| \lesssim |u|_{1} + |\partial u|_{1},
\]
we have
\[
\|u(t)\|_{s} + \|\partial u(t)\|_{s} \lesssim \varepsilon + \int_0^t E(T)^2(t)^{\delta-1} d\tau \lesssim \varepsilon + E(T)^2(t)^{\delta}
\]
for \( t \in [0, T) \). By (6.6), (6.7) and (6.9), we arrive at the desired estimate (6.1).}

Now we finish the proof of Theorem 2.1. The inequality (6.1) implies that there exists a constant \( M > 0 \), which does not depend on \( T \), such that
\[
E(T) \leq M
\]
15
if we choose \( \varepsilon \) sufficiently small. The unique global existence is an immediate consequence of this \textit{a priori} bound and the classical local existence theorem (see [10] etc.). To prove the existence of a free profile, we remember that

\[
(\Box + m_j^2)(u_j - \Lambda_j) = N_j + R_j
\]

with

\[
\| (N_j + R_j)(t, \cdot) \|_{H^{s-4}(\mathbb{R}^2)} \lesssim \langle t \rangle^{-2+\delta},
\]

\[
\| \Lambda_j(t, \cdot) \|_{H^{s-3}(\mathbb{R}^2)} + \| \partial_t \Lambda_j(t, \cdot) \|_{H^{s-4}(\mathbb{R}^2)} \lesssim \langle t \rangle^{-1+\delta}.
\]

Now we set

\[
f_j^+ = f_j - \Lambda_j|_{t=0} + \int_0^\infty \frac{\sin(-\tau \Omega_j)}{\Omega_j} (N_j + R_j)(\tau, \cdot) d\tau,
\]

\[
g_j^+ = g_j - \partial_t \Lambda_j|_{t=0} + \int_0^\infty (\cos(-\tau \Omega_j))(N_j + R_j)(\tau, \cdot) d\tau
\]

and

\[
u_j^+(t, \cdot) = (\cos(t \Omega_j)) f_j^+ + \frac{\sin(t \Omega_j)}{\Omega_j} g_j^+
\]

with \( \Omega_j = (m_j^2 - \Delta)^{1/2} \). Since the Duhamel formula yields

\[
u_j(t, \cdot) - \Lambda_j(t, \cdot) = (\cos(t \Omega_j))(f_j - \Lambda_j|_{t=0}) + \frac{\sin(t \Omega_j)}{\Omega_j}(g_j - \partial_t \Lambda_j|_{t=0})
\]

\[
+ \int_0^t \frac{\sin((t - \tau) \Omega_j)}{\Omega_j} (N_j + R_j)(\tau, \cdot) d\tau
\]

\[
= u_j^+(t, \cdot) - \int_0^\infty \frac{\sin((t - \tau) \Omega_j)}{\Omega_j} (N_j + R_j)(\tau, \cdot) d\tau,
\]

we have

\[
\| (u_j - u_j^+)(t, \cdot) \|_{H^{s-3}(\mathbb{R}^2)} + \| \partial_t (u_j - u_j^+)(t, \cdot) \|_{H^{s-4}(\mathbb{R}^2)}
\]

\[
\lesssim \| \Lambda_j(t, \cdot) \|_{H^{s-3}(\mathbb{R}^2)} + \| \partial_t \Lambda_j(t, \cdot) \|_{H^{s-4}(\mathbb{R}^2)} + \int_0^\infty \| (N_j + R_j)(\tau, \cdot) \|_{H^{s-4}(\mathbb{R}^2)} d\tau
\]

\[
\lesssim \langle t \rangle^{-1+\delta} + \int_0^\langle \tau \rangle^{-2+\delta} d\tau
\]

\[
\lesssim \langle t \rangle^{-1+\delta}.
\]

This completes the proof of Theorem 2.1. \qed
Acknowledgments

The authors would like to express their sincere gratitude to Professor Jalal Shatah for the fruitful discussion that motivates the present work, and also for his warm hospitality during their visit to the Courant Institute of Mathematical Sciences, New York University, where a part of this work was done.

The first author (S.K.) is partially supported by Grant-in-Aid for Scientific Research (C) (No.20540211), JSPS. The second author (T.O.) is partially supported by Grant-in-Aid for Scientific Research (A) (No.21244010), JSPS. The third author (H.S.) is partially supported by Grant-in-Aid for Young Scientists (B) (No.22740089), MEXT.

References

[1] J.-M. Delort, Normal forms and long time existence for semi-linear Klein-Gordon equations, Boll. Unione Mat. Ital. Sez. B 10 (2007), 1–23.

[2] J.-M. Delort and D. Fang, Almost global existence for solutions of semilinear Klein-Gordon equations with small weakly decaying Cauchy data, Comm. Partial Differential Equations 25 (2000), 2119–2169.

[3] J.-M. Delort, D. Fang and R. Xue, Global existence of small solutions for quadratic quasilinear Klein-Gordon systems in two space dimensions, J. Funct. Anal. 211 (2004), 288–323.

[4] V. Georgiev, Global solution of the system of wave and Klein-Gordon equations, Math. Z. 203 (1990), 683–698.

[5] V. Georgiev, Decay estimates for the Klein-Gordon equation, Comm. Partial Differential Equations 17 (1992), 1111–1139.

[6] N. Hayashi, C. Li and P.I. Naumkin, On a system of nonlinear Schrödinger equations in 2D, Differential Integral Equations, 24 (2011), 417–434.

[7] N. Hayashi, C. Li and P.I. Naumkin, Modified wave operator for a system of nonlinear Schrödinger equations in 2D, preprint.

[8] N. Hayashi, C. Li and T. Ozawa, Small data scattering for a system of nonlinear Schrödinger equations, preprint.

[9] N. Hayashi, and P.I. Naumkin, Wave operators to a quadratic nonlinear Klein-Gordon equation in two space dimensions, Nonlinear Anal. 71 (2009), 3826–3833.
[10] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*, Springer Verlag, Berlin, 1997.

[11] S. Katayama, *A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension*, J. Math. Kyoto Univ. 39 (1999), 203–213.

[12] S. Katayama, *Global existence for coupled systems of nonlinear wave and Klein-Gordon equations in three space dimensions*, to appear in Math. Z.

[13] Y. Kawahara and H. Sunagawa, *Global small amplitude solutions for two-dimensional nonlinear Klein-Gordon systems in the presence of mass resonance*, to appear in J. Differential Equations.

[14] R. Kosecki, *The unit condition and global existence for a class of nonlinear Klein-Gordon equations*, J. Differential Equations 100 (1992), 257–268.

[15] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions*, Math. Z. 222 (1996), 341–362.

[16] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. 38 (1985), 685–696.

[17] H. Sunagawa, *On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension*, J. Differential Equations 192 (2003), 308–325.

[18] H. Sunagawa, *A note on the large time asymptotics for a system of Klein-Gordon equations*, Hokkaido Math. J. 33 (2004), 457–472.

[19] Y. Tsutsumi, *Stability of constant equilibrium for the Maxwell-Higgs equations*, Funkcial. Ekvac. 46 (2003), 41–62.