CYCLOTOMIC EXPANSION FOR THE COLORED HOMFLY-PT INVARIANTS OF DOUBLE TWIST KNOTS

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ABSTRACT. In this short note, we prove the cyclotomic expansion formula for the colored HOMFLY-PT invariant of double twist knots, it confirms the cyclotomic expansion conjecture for $SU(N)$-invariants proposed in [2].

1. Introduction

Colored HOMFLY-PT invariant of link is an important quantum invariant in mathematics and physics, we refer to [8] and the references therein for a recent review of the colored HOMFLY-PT invariant and related topics. During the past several years, we spent a lot of efforts to study the general structures for colored HOMFLY-PT invariants. Motivated by the congruence skein relations discovered in [1] and the celebrated cyclotomic expansion formula for colored Jones polynomial due to Habiro [3], we proposed the cyclotomic expansion conjecture in [2] (see also Conjecture 2.4 in version 2 of [7]) for the colored $SU(n)$ invariants $J_{SU(n)}^{N}(K; q)$ of a knot $K$ which is defined as the $a = q^n$ specialization of the normalized colored HOMFLY-PT invariant

$$H_N^{(n)}(K; q,a) := \frac{W_N^{(n)}(K; q,a)}{W_N^{(n)}(U; q,a)},$$

where $W_N^{(n)}(K; q,a)$ denotes the colored HOMFLY-PT invariant of $K$ with the symmetric representation labeled by partition $(N)$, and $U$ denotes the unknot. $\mathcal{H}_N(K; q,a)$ is normalized so that it is to be 1 for the unknot $U$.

Conjecture 1.1. For any knot $K$, there exist Laurent polynomials $H_k^{(n)}(K) \in \mathbb{Z}[q,q^{-1}]$, independent of $N$ ($N \geq 0$). Such that

$$J_{SU(n)}^{N}(K; q) = \sum_{k=0}^{N} C_{N+1,k}^{(n)} H_k^{(n)}(K),$$

where $C_{N+1,k}^{(n)} = \{N-(k-1)\} \{N-(k-2)\} \cdots \{N-1\} \{N\} \{N+n\} \{N+n+1\} \cdots \{N+n+(k-1)\}$, for $k = 1, ..., N$, and $C_{N+1,0}^{(n)} = 1$. In particular, $J_{SU(n)}^{N}(K; q) = H_0^{(n)}(K) = 1$.

For a given knot or link, it is difficult to calculate its colored HOMFLY-PT invariant in general. In [4], Kawagoe provided some formulae for the colored HOMFLY-PT invariants with symmetric representations based on the HOMFLY skein theory. Such formulae are useful to compute these invariants of the knots and links with twisted strands of opposite orientations. Furthermore, in the recent work [5], Kawagoe obtained a rigorous single sum formula for the colored HOMFLY-PT polynomial $\mathcal{H}_N(K_p; q,a)$ of the twist knot $K_p$. 
In this short note, based on Kawagoe’s recent results [5], we apply the techniques in [6] to derive the following cyclomotic expansion formula for $\mathcal{H}_N(K_{p,s}; q, a)$ of double twist knot $K_{p,s}$

\[ \mathcal{H}_N(K_{p,s}; q, a) = \sum_{k=0}^{N} (-1)^k (a^k q^{k(k-1)/2(p+s)}) \tilde{C}^{(p)}(k,k) \tilde{C}^{(s)}(k,k) \left[ \begin{array}{c} N \\ k \end{array} \right] \{N+k-1; a\} \{k-2; a\}. \]

where the coefficient $\tilde{C}^{(p)}(k,k) \in \mathbb{Z}[q, q^{-1}]$, whose explicit expression is given by formula (3.12).

In particular, let $a = q^n$ in the above formula (1.3), we obtain

\textbf{Corollary 1.3.} The Conjecture 1.1 holds for the SU($n$)-invariant $J_N^{SU(n)}(K_{p,s}; q)$ of the double twist knot $K_{p,s}$.

\section{Preliminaries}

In this section, we briefly review Kawagoe’s recent work [5] and fix the notations. Let $a$ and $q$ be two non-zero variables in $\mathbb{C}$. For an integer $n$, we define the symbols by

\[ [n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad \{n\} = q^n - q^{-n}, \quad \{n; a\} = aq^n - a^{-1}q^{-n}. \]

For integers $n > 0, i \geq 0$, we introduce the products of $i$ terms of these symbols by

\[ [n]_i = [n][n-1] \cdots [n-i+1], \quad \{n\}_i = \{n\}\{n-1\} \cdots \{n-i+1\}, \]
\[ \{n; a\}_i = \{n; a\}\{n-1; a\} \cdots \{n-i+1; a\}, \quad \{-n; a\}_i = \{-n; a\}\{-n+1; a\} \cdots \{-n+i-1; a\}, \]

which are defined to be 1 if $i = 0$. Furthermore, we let

\[ [n]! = [n]_n, \quad \{n\}! = \{n\}_n, \quad \left[ \begin{array}{c} n \\ i \end{array} \right] = \frac{[n]!}{[i]![n-i]!}. \]
The HOMFLY skein module $\mathcal{S}(M)$ of a oriented 3-manifold $M$ is the free $\mathbb{C}$-module modulo the submodule generated by the following HOMFLY-PT skein relations:

1. $L \cup U = \left\{0; a\right\} L$, and $\emptyset = 1$,

2. $\bigotimes - \bigotimes = (q - q^{-1}) \bigotimes$,

3. $\bigotimes = a \bigotimes$, $\bigotimes = a^{-1} \bigotimes$.

For an integer $n \geq 1$, we recursively define the $n$-th $q$-symmetrizer by Figure 2, where the $q$-symmetrizer is denoted by a white rectangle. The integer $n$ beside an arc means $n$ copies of the arc.

Using the $m$-th and $n$-th $q$-symmetrizers, Kawagoe introduced the $(m,n)$-th $q$-symmetrizer by Figure 3, where $x_{m,n}^i$ is given by

$$x_{m,n}^i = (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} \frac{\{i\}!}{\{m+n-2; a\}_i}.$$

According to their definitions, the $n$-th $q$-symmetrizer and $(m,n)$-th $q$-symmetrizer carry the following properties

(2.1) \[ \begin{array}{c} \end{array} = \begin{array}{c} \end{array} = q \begin{array}{c} \end{array} \]

(2.2) \[ \begin{array}{c} \end{array} = \begin{array}{c} \end{array} = \begin{array}{c} \end{array} \]

(2.3) \[ \begin{array}{c} \end{array} = \begin{array}{c} \end{array} \]
We denote the skein module of the solid torus $S^1 \times D^2$ by $S$. For a circle along $S^1$ of the solid torus, let $H_n \in S$ be the $n$ copies of the circles inserted by the $n$-th $q$-symmetrizer, and $H_{n,n}$ be two copies of $H_n$, one is anticlockwise and the other is clockwise. Let $D_{m,n}$ by $m$ copies of the anticlockwise circle and $n$ copies of the clockwise inserted by the $(m,n)$-th $q$-symmetrizer. Kawagoe introduced two submodule of $S$, $H_{n,n}$ and $D_{n,n}$, which are spanned by $H_{i,i}$ and $D_{i,i}$ for $i = 0,\ldots,n$, respectively. Then, he proved that $D_{n,n} = H_{n,n}$.

Let $\langle \rangle$ be the linear map on $S$ defined by evaluating it in $S^3$.

As in [6], Kawagoe introduced the twist map $t : S \rightarrow S$ induced by one right-handed twist on the solid torus as shown in the right-hand side of Figure 4, where the twist is acting on the bottom of the solid torus. Similarly, let $t^{-1}$ be the twist map induced by one left-hand twist. For $x \in S$, let $e_x : S \rightarrow S$ be the encircling map which encircles an element of $S$ by $x$ as shown in the left-hand side of Figure 4, where $x$ slides into the bottom of the solid torus and encircles the element.

\begin{equation}
\omega = \begin{array}{c}
\includegraphics[width=1in]{encircling_map}
\end{array}
\end{equation}

**Figure 4.** The encircling map $e_\omega$ and a positive full twist

We need the following lemmas which are borrowed from [5] directly.

**Lemma 2.1** (cf. Lemma 2.6 in [5]). For positive integers $m \geq n$, we have the following:

$$\langle D_{m,n} \rangle = \frac{\{m + n - 1; a\} \{m - 2; a\} \{n - 2; a\} \{n; a\}}{\{m\}! \{n\}!}.$$ 

For $i = 0,\ldots,n$, Kawagoe recursively defined elements $R_i \in S$ as follows:

$$R_0 = H_0 = 1,$$

$$R_n = H_n - \sum_{i=0}^{n-1} \frac{\{n - 1 + i; a\} \{n - i; a\}}{\{n - i\}!} R_i, \text{ for } n \geq 1.$$ 

Conversely, each $H_i$ can be expressed by

$$H_i = \sum_{j=0}^{i} \frac{\{i - 1 + j; a\} \{n - i; a\}}{\{i - j\}!} R_j.$$ 

These elements $R_i$ play important role in calculating the colored HOMFLY-PY polynomial of double twist knot. Actually, the behavior of $H_n$ concerning the encircling map is very complicated, so Kawagoe introduced $R_n$ as a linear combination of $H_i(i = 0,\ldots,n)$ which simplifies the computations.

**Lemma 2.2** (cf. Proposition 3.4 and Corollary 3.5 in [5]). For an integer $i \geq 0$, we have $e_{R_i}(D_{n,n}) = \theta_{n,i} D_{n,n}$, where $\theta_{n,i} = \{n\}_i = \{n\}_i \{n_i - 2; a\}_i$, and for $i > n$, $e_{R_i}(D_{n,n}) = 0.$
Lemma 2.3 (cf. Lemma 2.1 in [5]). We have the following formula

\[
\begin{align*}
\sum_{i=0}^{\min\{m,n\}} \alpha_{m,n}^i(q,a) & = \min\{m,n\} \sum_{i=0}^{\min\{m,n\}} \alpha_{m,n}^i(q,a),
\end{align*}
\]  

where

\[
\alpha_{m,n}^i(a,q) = (-a)^{-i} q^{-i(m+n)+\frac{(i+1)}{2}} \frac{\{m\}_i\{n\}_i}{\{i\}!}.
\]

Remark 2.4. In formula (2.6), when all the negative crossings change into positive crossings, the coefficients \(\alpha_{m,n}^i(q,a)\) will change into \(\alpha_{m,n}^i(q^{-1},a^{-1})\).

Lemma 2.5 (cf. Lemma 4.1 in [5]). The following holds:

\[
\sum_{i=0}^{\min\{m,n\}} y_{m,n}^i = \min\{m,n\} \sum_{i=0}^{\min\{m,n\}} y_{m,n}^i,
\]

where \(y_{m,n}^i\) is defined by

\[
y_{m,n}^i = \frac{\{m\}_i\{n\}_i}{\{i\}!\{m+n-i-1;a\}_i}.
\]

We set the element \(\omega_n^p \in S\) by

\[
\omega_n^p = \sum_{i=0}^{n} t_{i,p} R_i,
\]

for some \(t_{i,p} \in \mathbb{C}\), Kawagoe determined the coefficients \(t_{i,p}\) so that \(e_{\omega_n^p}(x) = t^p(x)\) for \(x \in D_{n,n}\) as follows.

By the idempotent and vanishing properties of \((n,n)\)-the \(q\)-symmetrizer, the left-hand side of Figure 5 is transformed into \(D_{n,n}\) with the multiplication of some \(T_{n,p} \in \mathbb{C}\).

To determine \(t_{n,p}\), Kawagoe calculated \(T_{n,p}\) in two ways as in [6]. One is to make use of the \((n-i,n-i)\)-th \(q\)-symmetrizer \((i = 0, \ldots, n)\), and the other is to encircle \(p\) full twists.
by \( \omega_p \). Finally, he obtained

\[
T_{n,p} = (a^{-n}q^{-n(n-1)})^{2p}\{n\}! \sum_{i=0}^{n} (-1)^i (a^i q^{i(i-1)})^{2p} \frac{\{n\}_{n-i}}{\{n-i\}! \{n+i-1; a\}_{n+1}},
\]

and

\[
T_{n,p} = (-1)^n (a^n q^{n(n-1)})^{-2p}(\{n\}!)^2 t_{n,p}.
\]

Hence,

\[
t_{n,p} = (-1)^n \sum_{i=0}^{n} (-1)^i (a^i q^{i(i-1)})^{2p} \frac{\{2i-1; a\}}{\{i\}! \{n-i\}! \{n+i-1; a\}_{n+1}}.
\]

### 3. Proof of the Theorem 1.2

The crucial step to prove the Theorem 1.2 is to give another explicit formula for \( T_{n,p} \) which shows that \( T_{n,p} \) is a Laurent polynomial divisible by \( \{n\}_k \).

First, we have the following lemma.

**Lemma 3.1.** We have the expansion formula

\[
\begin{aligned}
\sum_{k=0}^{n} C_{n,k}^{(p)} &= \sum_{k=0}^{n} \frac{\{2i-1; a\}_{n+1}}{\{n-i\}_{n+1}} \prod_{i=0}^{\lfloor k/2 \rfloor} (a^i q^{i(i-1)})^{2p} \frac{\{n\}_{n-k}}{\{n-k\}_k} \prod_{i=0}^{\lfloor k/2 \rfloor} (a^i q^{i(i-1)})^{2p} \frac{\{n\}_{2n-k}}{\{2n-k\}_{2n-k}}
\end{aligned}
\]

where the coefficient \( C_{n,k}^{(p)} \) is a Laurent polynomial divisible by \( \{n\}_k \).

**Proof.** After applying Lemma 2.3, one term index by \( k \) in sigma sum reads

\[
\begin{aligned}
&= (a^k q^{k(k-1)})^{-2} \prod_{i=0}^{\lfloor k/2 \rfloor} (a^i q^{i(i-1)})^{2p} \frac{\{n\}_{n-k}}{\{n-k\}_k} \prod_{i=0}^{\lfloor k/2 \rfloor} (a^i q^{i(i-1)})^{2p} \frac{\{n\}_{2n-k}}{\{2n-k\}_{2n-k}}
\end{aligned}
\]

where we have used the properties (2.1) and (2.2). Then, applying Lemma 2.3 to the \( n - k \) twisted strands in the last diagram of formula (3.2), we obtain the expansion (3.1) for \( p = 2 \), and

\[
C_{n,k}^{(2)} = \sum_{l_1+l_2=k} \alpha_{n,n}^{l_1} a^{-2l_1} q^{2l_1(l_1-2n+1)} \alpha_{n-l_1,n-l_1}^{l_2}.
\]

The formula (2.7) for \( \alpha_{n,n}^{l} \) implies that \( C_{n,k}^{(2)} \) is a Laurent polynomial divisible by \( \{n\}_k \). The proof for general \( p \) is similar.

\[\square\]
Remark 3.2. In formula (3.1), when all the negative crossings change into positive crossings, the coefficients also change so that $a, q$ are replaced by $a^{-1}, q^{-1}$, i.e.

\begin{equation}
C^{(p)}_{n,k}(q, a) = C^{(p-1)}_{n,k}(q^{-1}, a^{-1}), \text{ for } p < 0.
\end{equation}

Lemma 3.3. The following holds:

\begin{equation}
T_{n,p} = C^{(p)}_{n,n}.
\end{equation}

Proof. By the idempotent and vanishing properties (2.3) and (2.4) for the $(n,n)$-th $q$-symmetrizer, we have

\begin{equation}
\sum_{k=0}^{n} C^{(p)}_{n,k} = C^{(p)}_{n,n}.
\end{equation}

Comparing to the Figure 5, we obtain $T_{n,p} = C^{(p)}_{n,n}$. \qed

Actually, we can present the explicit formula for $C^{(p)}_{n,n}$ as follows. Note that the formula (2.7) gives

\begin{equation}
\alpha^i_{n,n} = (-a)^{-i}q^{-2ni + \frac{(i+3)q}{2}} \binom{n}{i} \{n\}_i.
\end{equation}

When $p \geq 1$, we consider the sum over all multi-indices denoted by $l = (l_1, \ldots, l_p)$ such that $l_i \geq 0$ for all $i$, and $\sum_{i=1}^{p} l_i = n$. We put $s_i = l_1 + \cdots + l_i$ for $i = 1, \ldots, p$ and define

\begin{equation}
\varphi(l) = \sum_{i=1}^{p-1} 2s_i(s_i - 2n + 1) + \sum_{i=1}^{p} l_i(l_i + 3) + 2 \sum_{i=1}^{p-1} s_il_{i+1},
\end{equation}

and

\begin{equation}
\binom{n}{l} = \frac{[n]!}{[l_1]! \cdots [l_p]!}.
\end{equation}

Then

\begin{equation}
C^{(p)}_{n,n} = \sum_{l} \alpha^{l_1}_{n,n} a^{-2l_1} q^{2l_1(l_1-2n+1)} \alpha^{l_2}_{n-l_1,n-l_1} a^{-2s_2} q^{2s_2(s_2-2n+1)} \cdots
\end{equation}

\begin{equation}
\times \alpha^{l_{p-1}}_{n-s_{p-2},n-s_{p-2}} a^{-2s_{p-1}} q^{2s_{p-1}(s_{p-1}-2n+1)} \alpha^{l_p}_{n-s_{p-1},n-s_{p-1}}
\end{equation}

\begin{equation}
= \sum_{l} a^{-2 \sum_{i=1}^{p-1} s_i} q^{2 \sum_{i=1}^{p-1} 2s_i(s_i-2n+1)} \alpha^{l_1}_{n,n} \alpha^{l_2}_{n-s_1,n-s_1} \cdots \alpha^{l_p}_{n-s_{p-1},n-s_{p-1}}
\end{equation}

\begin{equation}
= (-1)^n a^{-n} q^{-2n^2} \{n\}! \sum_{l} a^{-2 \sum_{i=1}^{p-1} (p-i)l_i} q^{\varphi(l)} \frac{n!}{[l]}.
\end{equation}
For $p \leq -1$, by Remark 3.2, we can define

\begin{equation}
C_{n,n}^{(p)}(q, a) = C_{n,n}^{(-p)}(q^{-1}, a^{-1}).
\end{equation}

Finally, we put

\begin{equation}
\tilde{C}_{k,k}^{(p)} := \frac{C_{k,k}^{(p)}}{\{k\}!} = (-1)^ka^{-k}q^{-2k^2} \sum_{l} a^{-2} \sum_{i=1}^{p-1} (p-i)_{i} q^\varphi(l) \left[ \begin{array}{c} k \\ l \end{array} \right],
\end{equation}

then $\tilde{C}_{k,k}^{(p)}$ is Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$.

For examples,

\begin{equation}
\tilde{C}_{1}^{(1)} = (-1)^k a^{-k} q^{-3k(k-1)/2}, \quad \tilde{C}_{1}^{(-1)} = a^k q^{3k(k-1)/2},
\end{equation}

\begin{equation}
\tilde{C}_{k,k}^{(2)} = (-1)^k a^{-k} q^{-3k(k-1)/2} \sum_{l=0}^{k} a^{-2l} q^{-3kl+l(l+2)} \left[ \begin{array}{c} k \\ l \end{array} \right],
\end{equation}

\begin{equation}
\tilde{C}_{k,k}^{(-2)} = a^k q^{3k(k-1)/2} \sum_{l=0}^{k} a^{2l} q^{3kl-l(l+2)} \left[ \begin{array}{c} k \\ l \end{array} \right].
\end{equation}

Now, we are ready to calculate the colored HOMFLY-PT polynomial of the double twist knots $K_{p,s}$. By its definition, the $N$-th colored HOMFLY-PT polynomial for a knot $K$ is given by

\begin{equation}
\mathcal{H}_N(K) = \frac{\langle K(H_N) \rangle}{\langle U(H_N) \rangle} = \frac{\{N\}!}{\{N-1; a\}^N} \langle K(H_N) \rangle,
\end{equation}

where $K(H_N)$ denotes the knot $K$ cabled by $H_N$ with compatible orientations. Note that if we assume that the framing of the knot $K$ is to be 0, then this definition of $\mathcal{H}_N(K)$ is equal to the formula (1.1).

For two integers $p, s$, the double twist knot $K_{p,s}$ is described in Figure 1. Note that $K_{-p,-s}$ is the mirror of $K_{p,s}$, we will assume $p \in \mathbb{Z}$ and $s \geq 1$. From this, in particular, we can see that $K_{1,1}$ is a left-handed trefoil, $K_{-1,1}$ is a figure-eight knot and $K_{2,1}$ is the $5_2$ knot.

We observe that the techniques used in [6, 5] are also workable to compute the double twist knot $K_{p,s}$. First, by using the definition of the encircling map and $\omega^p$, we have the following description for double twist knot with framing zero.
Let us calculate $\langle K_{p,s}(H_N) \rangle$ by using the above description for $K_{p,s}$. Since $H_N$ has the following presentation:

$$H_N = \sum_{k=0}^{N} \frac{\{N+k-1; a\}_{N-k}}{\{N-k\}!} R_k,$$

one can insert this presentation along the above description instead of $H_N$. Consider a pair $R_i \in K_{p,s}(H_N)$ and $R_j$ from the expression $\omega_{p}^{k}$ (cf. formula (2.8)), the key observation is that this pair is a cabling of the (twisted) Whitehead link, of which one component pierces the other twice in the opposite direction each other. Moreover, $R_i$ and $R_j$ are a linear combination of $D_{k,k}$ with $k \leq i$ and $k \leq j$, respectively. According to Lemma 2.2, this pair vanishes if $i \neq j$. Therefore, we can calculate $\langle K_{p,s}(H_N) \rangle$ as follows:

$$\langle K_{p,s}(H_N) \rangle = \sum_{k=0}^{N} \frac{\{N+k-1; a\}_N}{\{N-k\}!} t_{k,p}$$
and since $R_k - H_k$ is a linear combination of $R_j$ with $j < k$, we have

$$ (a^k q^{k(k-1)})^{2s} \left( \sum_{j=0}^{k-1} \frac{\theta_{j,k} \langle D_{j,k} \rangle}{k!} \right). $$

Substituting it back to formula (3.16), we obtain Theorem 1.2.

As an application of our main Theorem 1.2, we present several concrete calculations for the twist knots with small crossings.
Example 3.4. For the trefoil knot $3_1$ and figure-8 knot $4_1$ which are $K_{1,1}$ and $K_{-1,1}$ respectively,

$$H_N(3_1; q, a) = \sum_{k=0}^{N} (-1)^k a^{2k} q^{k(k-1)} \left[\frac{N}{k}\right] \{N + k - 1; a\}_k \{k - 2; a\}_k.$$ \hfill (3.18)

$$H_N(4_1; q, a) = \sum_{k=0}^{N} \left[\frac{N}{k}\right] \{N + k - 1; a\}_k \{k - 2; a\}_k.$$ \hfill (3.19)

which are the formulae given by Corollary 5.2 in [5].

Example 3.5. For the $5_2$ knot $K_{2,1}$, we have

$$H_N(5_2; q, a) = \sum_{k=0}^{N} \left((-1)^k a^{4k} q^{3k(k-1)} \sum_{l=0}^{k} a^{-2l} q^{-3kl+3l+2} \left[\frac{k}{l}\right]\right) \cdot \left[\frac{N}{k}\right] \{N + k - 1; a\}_k \{k - 2; a\}_k.$$ \hfill (3.20)

In particular, for $a = q^2$, we have

$$\left[\frac{N}{k}\right] \{N + k - 1; a\}_k \{k - 2; a\}_k = \frac{\{N + k + 1\} \{N + k\} \cdots \{N - k + 1\}}{\{N + 1\}}.$$  

Then

$$H_N(5_2; q, q^2) = \sum_{k=0}^{N} \left((-1)^k q^{3k^2+5k} \sum_{l=0}^{k} q^{-3kl+3l+2} \left[\frac{k}{l}\right]\right) \cdot \left[\frac{N}{k}\right] \{N + k + 1\} \{N + k\} \cdots \{N - k + 1\}.$$  

which is equal to $J'_{K_2}(N+1)$ in [6] (cf. Theorem 5.1 in [6] for $p = 2$, where the variable $a$ is just the variable $q$ in our notation).

Example 3.6. For the $6_1$ knot $K_{-2,1}$, we have

$$H_N(6_1; q, a) = \sum_{k=0}^{N} \left(a^{-2k} q^{-2k(k-1)} \sum_{l=0}^{k} a^{2l} q^{3kl-l(l+2)} \left[\frac{k}{l}\right]\right) \cdot \left[\frac{N}{k}\right] \{N + k - 1; a\}_k \{k - 2; a\}_k.$$ \hfill (3.22)

In particular, for $a = q^2$, we obtain

$$H_N(6_1; q, q^2) = \sum_{k=0}^{N} \left(q^{-2k(k+1)} \sum_{l=0}^{k} q^{3kl-l(l+2)} \left[\frac{k}{l}\right]\right) \cdot \left[\frac{N}{k}\right] \{N + k + 1\} \{N + k\} \cdots \{N - k + 1\}.$$  

which is equal to $J'_{K_{-2}}(N+1)$ in [6] (cf. Theorem 5.1 in [6] for $p = -2$, where the variable $a$ is just the variable $q$ in our notation).
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