ON GENERALIZED DOLD MANIFOLDS

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Abstract. Let $X$ be a smooth manifold with a (smooth) involution $\sigma : X \to X$ such that $\text{Fix}(\sigma) \neq \emptyset$. We call the space $P(m,X) := S^m \times X/\sim$ where $(v,x) \sim (-v,\sigma(x))$ a generalized Dold manifold. When $X$ is an almost complex manifold and the differential $T\sigma : TX \to TX$ is conjugate complex linear on each fibre, we obtain a formula for the Stiefel-Whitney polynomial of $P(m,X)$ when $H^1(X;\mathbb{Z}_2) = 0$. We obtain results on stable parallelizability of $P(m,X)$ and a very general criterion for the (non) vanishing of the unoriented cobordism class $[P(m,X)]$ in terms of the corresponding properties for $X$. These results are applied to the case when $X$ is a complex flag manifold.

1. Introduction

Let $P(m,n)$ denote the space obtained as the quotient by the cyclic group $\mathbb{Z}_2$-action on the product $S^m \times \mathbb{C}P^n$ generated by the involution $(u,L) \mapsto (-u,\bar{L})$, $u \in S^m$, $L \in \mathbb{C}P^n$ where $\bar{L}$ denotes the complex conjugation. The spaces $P(m,n)$, which seem to have first appeared in the work of Wu, are called Dold manifolds, after it was shown by Dold [6] that, for suitable values of $m,n$, the cobordism classes of $P_{m,n}$ serve as generators in odd degrees for the unoriented cobordism algebra $\mathfrak{M}$. Dold manifolds have been extensively studied and have received renewed attention in recent years; see [9], [15] and also [14], [20], and [4].

The construction of Dold manifolds suggests, among others, the following generalization. Consider an involution on a Hausdorff topological space $\sigma : X \to X$ with non-empty fixed point set and consider the space $P(m,X,\sigma)$ obtained as the quotient of $S^m \times X$ by the action of $\mathbb{Z}_2$ defined by the fixed point free involution $(v,x) \mapsto (-v,\sigma(x))$. We obtain a locally trivial fibre bundle with projection $\pi : P(m,X,\sigma) \to \mathbb{R}P^m$ and fibre space $X$. If $x_0$ is a fixed point of $\sigma$, then the bundle admits a cross-section $s : \mathbb{R}P^m \to P(m,X,\sigma)$ defined as $s([v]) = [v,x_0]$. If $X$ is a smooth manifold and if $\sigma$ is smooth, then the above bundle and the cross-section are smooth.

In this paper we study certain manifold-properties of $P(m,X,\sigma)$ (or more briefly $P(m,X)$) where $X$ is a closed connected smooth manifold with an almost complex structure $J : TX \to TX$ and $\sigma$ is a conjugation, that is, the differential $T\sigma : TX \to TX$ and $J$ anti-commute: $T\sigma \circ J = -J \circ T\sigma$. We give a description of the tangent bundle.

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of $P(m, X)$. Assuming that $\text{Fix}(\sigma) \neq \emptyset$ and $H^1(X; \mathbb{Z}_2) = 0$, we obtain a formula for the Stiefel-Whitney classes of $P(m, X)$ (Theorem 3.1) and a necessary and sufficient condition for $P(m, X)$ to admit a spin structure (Theorem 3.2). We also obtain results on the stable parallelizability of the $P(m, X)$ (Theorem 3.3) and the vanishing of their (unoriented) cobordism class in the cobordism ring $\mathcal{R}$ (Theorem 3.7).

Recall that a smooth manifold $M$ is said to be parallelizable (resp. stably parallelizable) if its tangent bundle $\tau M$ (resp. $\epsilon \mathbb{R} \oplus \tau M$) is trivial.

By the celebrated work of Adams [1] on the vector field problem for spheres, one knows that the (additive) order of the element $([\zeta] - 1) \in KO(\mathbb{R}P^m)$ equals $2^{\varphi(m)}$ where $\zeta$ is the Hopf line bundle over $\mathbb{R}P^m$ and $\varphi(m)$ is the number of positive integers $j \leq m$ such that $j \equiv 0, 1, 2, \text{ or } 4 \text{ mod } 8$.

The complex flag manifold $\mathbb{C}G(n_1, \ldots, n_r)$ is the homogeneous space $U(n)/(U(n_1) \times \cdots \times U(n_r))$, where the $n_j \geq 1$ are positive integers and $n = \sum_{1 \leq j \leq r} n_j$. These manifolds are well-known to be complex projective varieties. We denote by $P(m; n_1, \ldots, n_r)$ the space $P(m, \mathbb{C}G(n_1, \ldots, n_r))$. The complete flag manifold $\mathbb{C}G(1, \ldots, 1)$ is denoted $\text{Flag}(\mathbb{C}^n)$. Note that $\mathbb{C}G(n_1, n_2)$ is the complex Grassmann manifold $\mathbb{C}G_{n_1, n_2}$ of $n_1$-dimensional vector subspaces of $\mathbb{C}^n$.

We highlight here the results on stable parallelizability and cobordism for a restricted classes of generalized Dold manifolds as in these cases the results are nearly complete.

**Theorem 1.1.** Let $m \geq 1$ and $r \geq 2$.

(i) The manifold $P(m; n_1, \ldots, n_r)$ is stably parallelizable if and only if $n_j = 1$ for all $j$ and $2^{\varphi(m)}$ divides $(m + 1 + \binom{n}{2})$.

(ii) Suppose that $P := P(m; 1, \ldots, 1)$ is stably parallelizable. Then it is parallelizable if $\rho(m + 1) > \rho(m + 1 + n(n - 1))$. If $m$ is even, then $P$ is not parallelizable.

The case when the flag manifold is a complex projective space corresponds to the classical Dold manifold $P(m, n - 1)$. In this special case the above result is due to J. Korbaš [9]. See also [21] in which J. Ucci characterized classical Dold manifolds which admit codimension one embedding into the Euclidean space.

**Theorem 1.2.** Let $1 \leq k \leq n/2$ and let $m \geq 1$.

(i) If $\nu_2(k) < \nu_2(n)$, then $[P(m, \mathbb{C}G_{n,k})] = 0$ in $\mathcal{R}$.

(ii) If $m \equiv 0 \mod 2$ and if $\nu_2(k) \geq \nu_2(n)$, then $[P(m, \mathbb{C}G_{n,k})] \neq 0$.

The above theorem leaves out the case when $m \geq 1$ is odd and $\nu_2(k) \geq \nu_2(n)$. See Remark 3.8 for results on vanishing of $[P(m; n_1, \ldots, n_r)]$.

Our proofs make use of basic concepts in the theory of vector bundles and characteristic classes. We first introduce, in §2, the notion of a $\sigma$-conjugate complex vector bundle over $X$ where $\sigma$ is an involution on $X$ and associate to each such complex vector bundle a real vector bundle over $\hat{\omega}$. We establish a splitting principle to obtain a formula for the Stiefel-Whitney classes of $\hat{\omega}$ in terms of certain ‘cohomology extensions’ of Stiefel-Whitney classes of $\omega$, assuming that $H^1(X; \mathbb{Z}_2) = 0$. This leads to a formula for the Stiefel-Whitney classes
of $P(m, X)$ when $X$ is a smooth almost complex manifold and $\sigma$ is a complex conjugation. Proof of Theorem 1.1 uses the main result of [18], the Bredon-Kosiński’s theorem [3], and a certain functor $\mu^2$ introduced by Lam [11] to study immersions of flag manifolds. Proof of Theorem 1.2 uses basic facts from the theory of Clifford algebras, a result of Conner and Floyd [5, Theorem 30.1] concerning cobordism of manifolds admitting stationary point free action of elementary abelian 2-group actions, and the main theorem of [17].

2. Vector bundles over $P(m, X, \sigma)$

Let $\sigma : X \to X$ be an involution of a path connected paracompact Hausdorff topological space and let $\omega$ be a complex vector bundle over $X$. Denote by $\omega^\vee$ the dual vector bundle $\text{Hom}_C(\omega, \epsilon_C)$. Here $\epsilon_F$ denotes the the trivial $F$-line bundle over $X$ where $F = \mathbb{R}, \mathbb{C}$. Note that, since $X$ is paracompact, $\omega$ admits a Hermitian metric and so $\omega^\vee$ is isomorphic to the conjugate bundle $\overline{\omega}$. The following definition generalises the notion of a conjugation of an almost complex manifold in the sense of Conner and Floyd [5, §24].

**Definition 2.1.** Let $\sigma : X \to X$ be an involution and let $\omega$ be a complex vector bundle over $X$. A $\sigma$-conjugation on $\omega$ is an involutive bundle map $\hat{\sigma} : E(\omega) \to E(\omega)$ that covers $\sigma$ which is conjugate complex linear on the fibres of $\omega$. If such a $\sigma$ exists, we say that $(\omega, \hat{\sigma})$ (or more briefly $\omega$) is a $\sigma$-conjugate bundle.

Note that if $\omega$ is a $\sigma$-conjugate bundle, then $\overline{\omega} \cong \sigma^*(\omega)$.

**Example 2.2.** (i) Let $\sigma$ be any involution on $X$. When $\omega = n\epsilon_C$, the trivial complex vector bundle of rank $n$, we have $E(\omega) = X \times \mathbb{C}^n$. The standard $\sigma$-conjugation on $\omega$ is defined as $\hat{\sigma}(x, \sum z_j e_j) = (\sigma(x), \sum z_j e_j)$. Here $\{e_j\}_{1 \leq j \leq n}$ is the standard basis of $\mathbb{C}^n$. Thus $(n\epsilon_C, \hat{\sigma})$ is a $\sigma$-conjugate bundle.

(ii) Let $X = \mathbb{C}G_{n,k}$ and let $\sigma : X \to X$ be the involution $L \mapsto \bar{L}$. Then the standard $\sigma$-conjugation on $n\epsilon_C$ defines, by restriction, a $\sigma$-conjugation of the canonical $k$-plane bundle $\gamma_{n,k}$. Explicitly, $v \mapsto \bar{v}$, $v \in L \in \mathbb{C}G_{n,k}$, is the required involutive bundle map $\hat{\sigma} : E(\gamma_{n,k}) \to E(\gamma_{n,k})$ that covers $\sigma$. Similarly the orthogonal complement $\beta_{n,k} := \gamma_{n,k}^\perp$ is also a $\sigma$-conjugate bundle.

(iii) If $X \subset \mathbb{C}P^N$ is a complex projective manifold defined over $\mathbb{R}$ and $\sigma : X \to X$ is the restriction of complex conjugation $[z] \mapsto [\bar{z}]$, then the tangent bundle $\tau X$ of $X$ is a $\sigma$-conjugate bundle. Indeed the differential of $\sigma$, namely $T\sigma : TX \to TX$ is the required bundle map $\hat{\sigma}$ of $\tau X$ that covers $\sigma$. As mentioned above, this classical case was generalized by Conner and Floyd [5, §24] to the case when $X$ is an almost complex manifold.

(iv) If $\omega, \eta$ are $\sigma$-conjugate vector bundles over $X$, then so are $\Lambda^r(\omega), \text{Hom}_C(\omega, \eta), \omega \otimes \eta$, and $\omega \oplus \eta$. For example, if $\hat{\sigma}$ and $\tilde{\sigma}$ are $\sigma$-conjugations on $\omega$ and $\eta$ respectively, both covering $\sigma$, then $\text{Hom}_C(\omega, \eta) \ni f \mapsto \hat{\sigma} \circ f \circ \tilde{\sigma} \in \text{Hom}_C(\omega, \eta)$ is verified to be a conjugate complex linear bundle involution of $\text{Hom}_C(\omega, \eta)$ that covers $\sigma$.

(v) Any subbundle $\eta$ of a $\sigma$-conjugate complex vector bundle $\omega$ over $X$ is also $\sigma$-conjugate provided $\hat{\sigma} : E(\omega) \to E(\omega)$ satisfies $\hat{\sigma}(E(\eta)) = E(\eta)$. 

2.1. Vector bundle associated to \((\eta, \hat{\sigma})\). Let \(\eta\) be a real vector bundle over \(X\) with projection \(p_\eta : E(\eta) \to X\) and let \(\hat{\sigma} : E(\eta) \to E(\eta)\) be an involutive bundle isomorphism that covers \(\sigma\). We obtain a real vector bundle, denoted \(\hat{\eta}\), over \(P(m, X, \sigma)\) as follows: \((v, e) \mapsto (-v, \hat{\sigma}(e))\) defines a fixed point free involution of \(S^m \times E(\eta)\) with orbit space \(P(m, E(\eta), \hat{\sigma})\). The map \(p_\hat{\eta} : P(m, E(\eta), \hat{\sigma}) \to P(m, X, \sigma)\) defined as \([v, e] \mapsto [v, p_\eta(e)]\) is the projection of the required bundle \(\hat{\eta}\).

This construction is applicable when \(\eta = \rho(\omega)\), the underlying real vector bundle of a \(\sigma\)-conjugate complex vector bundle \((\omega, \hat{\sigma})\). If \(\beta\) is a (real) subbundle of \(\eta\) such that \(\hat{\sigma}(E(\beta)) = E(\beta)\), then the restriction of \(\hat{\sigma}\) to \(E(\beta)\) defines a bundle \(\hat{\beta}\) which is evidently a subbundle of \(\hat{\eta}\).

We shall denote by \(\xi\) the real line bundle over \(P(m, X, \sigma)\), often referred to as the Hopf bundle, associated to the double cover \(S^m \times X \to P(m, X, \sigma)\). Its total space has the description \(S^m \times X \times \mathbb{Z}_2 \mathbb{R}\) consisting of elements \([v, x, t] = \{(v, x, t), (-v, \sigma(x), -t) \mid v \in S^m, x \in X, t \in \mathbb{R}\}\). Denote by \(\pi : P(m, X, \sigma) \to \mathbb{R}P^m\) the map \([v, x] \mapsto [v]\). Then \(\pi\) is the projection of a fibre bundle with fibre \(X\). The map \(E(\xi) \to E(\xi)\) defined as \([v, x, t] \mapsto [v, t]\) is a bundle map that covers the projection \(\pi : P(m, X, \sigma) \to \mathbb{R}P^m\) and so \(\xi \cong \pi^*(\xi)\).

If \(\sigma(x_0) = x_0 \in X\), then we have a cross-section \(s : \mathbb{R}P^m \to P(m, X)\) defined as \([v] \mapsto [v, x_0]\). Note that \(s^*(\xi) = \zeta\).

2.2. Dependence of \(\hat{\omega}\) on \(\hat{\sigma}\). It should be noted that the definition of \(\hat{\eta}\) depends not only on the real vector bundle \(\eta\) but also on the bundle map \(\hat{\sigma}\) that covers \(\sigma\). For example, on the trivial line bundle \(\epsilon_{\mathbb{R}}\), if \(\hat{\sigma}(x, t) = (\sigma(x), t)\), then \(\hat{\epsilon}_{\mathbb{R}} \cong \epsilon_{\mathbb{R}}\), whereas if \(\hat{\sigma}(x, t) = (\sigma(x), -t)\), then \(\hat{\epsilon}_{\mathbb{R}}\) is isomorphic to \(\xi\).

When \(\omega = \tau X\) is the tangent bundle over an almost complex manifold \((X, J)\) and \(\hat{\sigma} = T\sigma\), where \(\sigma\) is a conjugation on \(X\), \((i.e., \text{satisfies } J_{\sigma(x)} \circ T_x \sigma = -T_x \sigma \circ J_x \forall x \in X)\), the vector bundle \(\hat{\tau}X\) is understood to be defined with respect to the pair \((\tau X, T\sigma)\).

Let \(k, l \geq 0\) be integers and let \(n = k + l \geq 1\) and let \(s_1, \ldots, s_n\) be everywhere linearly independent sections of the trivial bundle \(n \epsilon_{\mathbb{R}}\). Denote by \(\epsilon_{k,l} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n\) the involutive bundle map \(n \epsilon_{\mathbb{R}}\) covering \(\sigma\) defined as \(\epsilon_{k,l}(x, \sum_j t_j s_j(x)) = (\sigma(x), -\sum_{1 \leq j \leq k} t_j s_j(x) + \sum_{k < j \leq n} t_j s_j(x))\). Then the bundle over \(P(m, X, \sigma)\) associated to \((n \epsilon_{\mathbb{R}}, \epsilon_{k,l})\) is isomorphic to \(k \xi \oplus l \epsilon_{\mathbb{R}}\). When \(n = 2d, k = l = d, n \epsilon_{\mathbb{R}} = \rho(\delta \epsilon_{\mathbb{C}})\) then the standard conjugation on \(\delta \epsilon_{\mathbb{C}}\) equals \(\epsilon_{d,d}\) (for an obvious choice choice of \(s_j, 1 \leq j \leq n\)).

Let \((\omega, \hat{\sigma})\) be a \(\sigma\)-conjugate complex vector bundle and let \(\eta\) be a real vector bundle which is isomorphic to the real vector bundle \(\rho(\omega)\) underlying \(\omega\). Suppose that \(f : \rho(\omega) \to \eta\) is a bundle isomorphism that covers the identity map of \(X\). Set \(\hat{\sigma} := f \circ \hat{\sigma} \circ f^{-1}\). Then \(\hat{\sigma}\) is an involution of \(\eta\) that covers \(\sigma\) and hence defines a vector bundle \(\hat{\eta}\) over \(P(m, X, \sigma)\).

**Lemma 2.3.** We keep the above notations. (i) The real vector bundles \(\hat{\omega}\) and \(\hat{\eta}\) over \(P(m, X, \sigma)\) associated to the pairs \((\omega, \hat{\sigma})\) and \((\eta, \hat{\sigma})\) are isomorphic. In particular \(\hat{\omega} \cong \hat{\omega}\). (ii) Suppose that \(\rho(\omega) = \eta_0 \oplus \eta_1\) where \(\eta_j, j = 0, 1\) are real vector bundles. Suppose that
\( \hat{\sigma}(E(\eta_j)) = E(\eta_j) \), then \( \hat{\omega} \) is isomorphic to \( \hat{\eta}_0 \oplus \hat{\eta}_1 \) where \( \hat{\eta}_j \) is defined with respect to the pair \( (\eta_j, \hat{\sigma}|_{E(\eta_j)}) \), \( j = 0, 1 \).

(iii) Let \( n = k + l \geq 1 \). Suppose that \( \rho(\omega) \oplus n\epsilon_R \cong N\epsilon_R \), where \( N := 2d + n \), and that \( \epsilon_{d+k,d+l} \) on \( N\epsilon_R \) restricts to \( \hat{\sigma} \) on \( \rho(\omega) \) and to \( \epsilon_{k,l} \) on \( n\epsilon_R \). Then \( \hat{\omega} \oplus k\xi \oplus l\epsilon_R \cong (d+k)\xi \oplus (d+l)\epsilon_i \).

Proof. We will only prove (i); the proofs of remaining parts are likewise straightforward. Consider the map \( \phi : S^m \times E(\omega) \to S^m \times E(\eta) \) defined as \( \phi(v, e) = (v, f(e)) \forall v \in S^m, e \in E(\omega) \). The \( \phi((-v, \sigma(e))) = (-v, f(\sigma(e))) = (-v, \tilde{\sigma}(f(e))) \). Thus \( \phi \) is \( \mathbb{Z}_2 \)-equivariant and so induces a vector bundle homomorphism \( \tilde{\phi} : P(m, E(\omega), \hat{\sigma}) \to P(m, E(\eta), \hat{\sigma}) \) that covers the identity map of \( P(m, X, \sigma) \). Restricted to each fibre, the map \( \tilde{\phi} \) is an \( \mathbb{R} \)-linear isomorphism since this is true of \( f \). Therefore \( \hat{\omega} \) and \( \hat{\eta} \) are isomorphic vector bundles. Finally, let \( \eta = \omega, \hat{\sigma} = \hat{\sigma} \) and \( f = id \). Then \( \hat{\omega} \cong \hat{\omega} \).

Example 2.4. (i) Consider the Riemann sphere \( S^2 = \mathbb{C}P^1 \). Let \( \gamma \subset 2\epsilon_C \) be the tautological (complex) line bundle over \( \mathbb{C}P^1 \) and let \( \beta \) be its orthogonal complement. As complex line bundles one has the isomorphism \( \beta \cong \bar{\gamma} \). It follows that from the above lemma that \( \gamma \cong \hat{\beta} \cong \hat{\xi} \oplus \epsilon_R \).

(ii) Suppose that \( X = \mathbb{C}G_{n,k} \) and let \( \sigma : X \to X \) be the conjugation \( L \to \bar{L} \). As seen in Example 2.2(ii), \( v \to \bar{v} \) define conjugations of \( \gamma_{n,k}, \beta_{n,k} \) that cover \( \sigma \). Note that \( \gamma_{n,k} \oplus \beta_{n,k} = n\epsilon_C \). By the above lemma we obtain that \( \gamma_{n,k} \oplus \beta_{n,k} \cong n\epsilon_C \). Also, the conjugations on \( \gamma_{n,k}, \beta_{n,k} \) induce an involution, denoted \( \sigma \), on \( \omega := \text{Hom}(\gamma_{n,k}, \beta_{n,k}) \); see Example 2.2(iv). One has the isomorphism \( \tau\mathbb{C}G_{n,k} \cong \omega \) of complex vector bundles (11). Under this isomorphism, the bundle involution \( \hat{\sigma} \) corresponds to \( T\sigma : T\mathbb{C}G_{n,k} \to T\mathbb{C}G_{n,k} \). Therefore \( \hat{\omega} \cong \hat{\tau}\mathbb{C}G_{n,k} \).

2.3. Splitting principle. Denote by \( \text{Flag}(\mathbb{C}^r) \) the complete flag manifold \( \mathbb{C}G(1, \ldots, 1) \). Let \( \omega \) be a complex vector bundle over \( X \) of rank \( r \geq 1 \) endowed with a Hermitian metric and let \( q : \text{Flag}(\omega) \to X \) be the \( \text{Flag}(\mathbb{C}^r) \)-bundle associated to \( \omega \). Thus the fibre over an \( x \in X \) is the space \( \{(L_1, \ldots, L_r) \mid L_1 + \cdots + L_r = p^{-1}_x(x), L_j \perp L_k, 1 \leq j < k \leq r, \dim \mathbb{C} L_j = 1\} \cong \text{Flag}(\mathbb{C}^r) \) of complete flags in \( p^{-1}_x(x) \subset E(\omega) \). The vector bundle \( q^*(\omega) \) splits as a Whitney sum \( q^*(\omega) = \bigoplus_{1 \leq j \leq r} \omega_j \) of complex line bundles \( \omega_j \) over \( \text{Flag}(\omega) \) with projection \( p_j : E(\omega_j) \to \text{Flag}(\omega) \). The fibre over a point \( L = (L_1, \ldots, L_r) \in \text{Flag}(\omega) \) of the bundle \( \omega_j \) is the vector space \( L_j \subset p^{-1}_x(q(L)) \).

Suppose that \( \sigma : X \to X \) is an involution and that \( \hat{\sigma} : E(\omega) \to E(\omega) \) is a \( \sigma \)-conjugation on \( \omega \). We shall write \( \bar{e} \) for \( \hat{\sigma}(e), e \in E(\omega) \). One has the involution \( \theta : \text{Flag}(\omega) \to \text{Flag}(\omega) \) defined as \( \text{L} = (L_1, \ldots, L_r) \mapsto (\bar{L}_1, \ldots, \bar{L}_r) =: \bar{L} \). Here \( \bar{V} \) denotes the subspace \( \sigma(V) \subset p^{-1}_x(\sigma(x)) \) when \( V \subset p^{-1}_x(x) \). Then \( \theta : E(q^*(\omega)) \to E(q^*(\omega)) \) defined as \( \hat{\theta}(L, e) = (\bar{L}, \bar{e}) \) is a \( \theta \)-conjugation on \( q^*(\omega) \). Moreover, it restricts to a \( \theta \)-conjugation \( \hat{\theta}_j \) on the subbundle \( \omega_j \) for each \( j \leq r \).

Recall from §2.1 that \( \hat{\omega} \) is the real vector bundle with projection \( p_\omega : P(m, E(\omega), \hat{\omega}) \to P(m, X, \sigma) \). Likewise, we have the real 2-plane bundle \( \hat{\omega}_j \) over \( P(m, \text{Flag}(\omega), \theta) \) with projection \( p_{\omega_j} : P(m, E(\omega_j), \hat{\omega}_j) \to P(m, \text{Flag}(\omega), \theta) \). Since \( q \circ \theta = \sigma \circ q \), we have the induced map \( \hat{q} : P(m, \text{Flag}(\omega), \theta) \to P(m, X, \sigma) \) defined as \( [v, \text{L}] \mapsto [v, q(\text{L})] \). The map
\( \hat{q} \) is in fact the projection of a fibre bundle with fibre the flag manifold \( \text{Flag}(\mathbb{C}^r) \). Since \( \hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_r) \), applying Lemma 2.3 (ii) we see that \( \hat{q}^*(\hat{\omega}) \cong \oplus_{1 \leq j \leq r} \hat{\omega}_j \).

Recall that the first Chern classes mod 2 of the canonical complex line bundles \( \xi_j \) over \( \text{Flag}(\mathbb{C}^r) \), \( 1 \leq j \leq r \), generate the \( \mathbb{Z}_2 \)-cohomology algebra \( H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2) \). In fact \( H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_r]/I \) where \( I \) is the ideal generated by the elementary symmetric polynomials in \( c_1, \ldots, c_r \). Here the generators \( c_j + I \) may be identified with the (integral) Chern class \( c_1(\xi_j) \). In particular \( H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}) \) is isomorphic, as an \( \mathbb{Z}_2 \)-module, to a free \( \mathbb{Z}_2 \)-module with basis \( a^{c_1(\xi_j)} \) and so a similar isomorphism holds for mod 2 cohomology.

Since \( \hat{\omega}_j \) restricts to the (real) 2-plane bundle \( \rho(\xi_j) \), we have \( c_1(\xi_j) = i^*(w_2(\omega_j)) \) where \( i : \text{Flag}(\mathbb{C}^r) \cong \hat{q}^{-1}([v, x]) \to P(m, \text{Flag}(\omega), \theta) \) is fibre inclusion, we see that the \( \text{Flag}(\mathbb{C}^r) \)-bundle \( (P(m, \text{Flag}(\omega), \theta), P(m, X, \sigma), \hat{q}) \) admits a \( \mathbb{Z}_2 \)-cohomology extension of the fibre. By Leray-Hirsch theorem [19, §7, Ch.V], we have \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \cong H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2) \). Thus \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \) is a free module over the algebra \( H^*(P(m, X, \sigma); \mathbb{Z}_2) \) of rank \( \dim_{\mathbb{Z}_2} H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2) = r! \). In particular, it follows that \( \hat{q} \) induces a monomorphism in mod 2 cohomology.

The symmetric group \( S_r \) operates on \( \text{Flag}(\omega) \) by permuting the components of each flag \( L = (L_1, \ldots, L_r) \) and the projection \( q : \text{Flag}(\omega) \to X \) is constant on the \( S_r \)-orbits. Moreover, \( \theta \circ \lambda = \lambda \circ \theta \) for each \( \lambda \in S_r \). This implies that the \( S_r \) action on \( \text{Flag}(\omega) \) extends to an action \( (P(m, \text{Flag}(\omega), \theta), P(m, X, \sigma), \hat{q}) \) where \( \lambda([v, L]) = [v, \lambda(L)] \). The projection \( \hat{q} : P(m, \text{Flag}(\omega), \theta) \to P(m, X, \sigma) \) is constant on \( S_r \)-orbits. It follows that the image of the ring homomorphism \( \hat{q}^* : H^*(P(m, X, \sigma); \mathbb{Z}_2) \to H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \) is contained in the subring \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2)^{S_r} \) of elements fixed by the induced action of \( S_r \) on \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \). As the \( S_r \)-action induces the identity map of \( P(m, X, \sigma) \) we see that it acts as \( H^*(P(m, X, \sigma); \mathbb{Z}_2) \)-module automorphisms on \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \).

Since \( H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2)^{S_r} = H^0(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2) \cong \mathbb{Z}_2 \), we have \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2)^{S_r} \cong H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2)^{S_r} = H^*(P(m, X, \sigma); \mathbb{Z}_2) \otimes H^0(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2) \cong H^*(P(m, X, \sigma); \mathbb{Z}_2) \).

We summarise the above discussion in the proposition below.

**Proposition 2.5.** (Splitting principle) Let \( \omega \) be a \( \sigma \)-conjugate complex vector bundle of rank \( r \) and let \( q : \text{Flag}(\omega) \to X \) be the associated \( \text{Flag}(\mathbb{C}^r) \)-bundle over \( X \). Then, with the above notations,

(i) the \( \omega_j \) are \( \theta \)-conjugate line bundles for \( 1 \leq j \leq r \), and, \( \hat{q}^*(\hat{\omega}) = \oplus_{1 \leq j \leq r} \hat{\omega}_j \).

(ii) \( \hat{q} : P(m, \text{Flag}(\omega), \theta) \to P(m, X, \sigma) \) induces monomorphism in cohomology, moreover, \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \) is isomorphic, as an \( H^*(P(m, X, \sigma); \mathbb{Z}_2) \)-module, to a free module with basis a \( \mathbb{Z}_2 \)-basis of \( H^*(\text{Flag}(\mathbb{C}^r); \mathbb{Z}_2) \).

(iii) The image of \( \hat{q}^* \) equals the subalgebra invariant under the action of the symmetric group \( S_r \) on \( H^*(P(m, \text{Flag}(\omega), \theta); \mathbb{Z}_2) \).

We end this section with the following lemma which will be used in the sequel.
Lemma 2.6. We keep the above notations. Let \( \omega \) be a \( \sigma \)-conjugate complex vector bundle over \( X \). Suppose that \( \text{Fix}(\sigma) \neq \emptyset \) and that \( H^1(X;\mathbb{Z}_2) = 0 \). Then \( \text{Fix}(\theta) \neq \emptyset \) and \( H^1(P(m, \text{Flag}(\omega), \theta);\mathbb{Z}_2) \cong H^1(P(m, X, \sigma);\mathbb{Z}_2) \cong H^1(\mathbb{R}P^m;\mathbb{Z}_2) \cong \mathbb{Z}_2 \).

Proof. Let \( \sigma(x) = x \in X \) and set \( V := p_\omega^{-1}(x) \). Then \( \hat{\sigma} \) restricts to a conjugate complex isomorphism \( \hat{\sigma}_x \) of \( V \) onto itself. Thus \( V \cong V \). Then, setting \( \text{Fix}(\hat{\sigma}_x) := U \subset V \), we see that \( V \) is the \( \mathbb{C} \)-linear extension of \( U \), that is, \( V = U \otimes \mathbb{C} \). The Hermitian product on \( V \) restricts to a (real) inner product on \( U \). Let \((K_1, \ldots, K_r)\) be a complete real flag in \( U \) and define \( L_j := K_j \otimes \mathbb{R} \subset V \). Then it is readily seen that \( L = (L_1, \ldots, L_r) \) belongs to \( \text{Flag}(\omega) \) and is fixed by \( \theta \).

Since \( H^1(X;\mathbb{Z}_2) = 0 \), we have \( H^1(P(m, X, \sigma);\mathbb{Z}_2) \cong H^1(\mathbb{R}P^m;\mathbb{Z}_2) \cong \mathbb{Z}_2 \), using the Serre spectral sequence of the \( X \)-bundle with projection \( \pi : P(m, X, \sigma) \to \mathbb{R}P^m \). The same argument applied to the \( \text{Flag}(\mathbb{C}^r) \)-bundle with projection \( q : \text{Flag}(\omega) \to X \) yields that \( H^1(\text{Flag}(\omega);\mathbb{Z}_2) \cong H^1(X;\mathbb{Z}_2) = 0 \). Now using the \( \text{Flag}(\omega) \)-bundle with projection \( \hat{q} : P(m, \text{Flag}(\omega), \theta) \to P(m, X, \sigma) \), we obtain that \( H^1(P(m, \text{Flag}(\omega), \theta);\mathbb{Z}_2) \cong H^1(P(m, X, \sigma);\mathbb{Z}_2) \cong \mathbb{Z}_2 \). \( \square \)

We shall identify \( H^1(P(m, \text{Flag}(\omega), \theta);\mathbb{Z}_2), H^1(P(m, X, \sigma);\mathbb{Z}_2), H^1(\mathbb{R}P^m;\mathbb{Z}_2) \) and denote the generator of any one of them by \( x \). \(^1\)

2.4. A formula for Stiefel-Whitney classes of \( \hat{\omega} \). Denote the Stiefel-Whitney polynomial \( \sum_{0 \leq i \leq q} w_i(\eta)t^i \) of a rank \( q \) real vector bundle \( \eta \) by \( w(\eta); t \) and similarly the Chern polynomial \( \sum_{0 \leq i \leq q} c_j(\alpha)t^i \) of a complex vector bundle \( \alpha \) of rank \( q \) by \( c(\alpha); t \). Recall that when \( \alpha \) is regarded as a real vector bundle, we have \( w(\alpha); t = c(\alpha); t^2 \) mod 2. (See [13].)

We shall make no notational distinction between \( c_j(\alpha) \in H^{2j}(X;\mathbb{Z}) \) and its reduction mod 2 in \( H^{2j}(X;\mathbb{Z}_2) \). In fact, we will mostly be working with \( \mathbb{Z}_2 \)-coefficients.

Since \( \hat{\omega} \) restricted to any fibre of \( \pi : P(m, X, \sigma) \to \mathbb{R}P^m \) is isomorphic to \( \omega \) (regarded as a real vector bundle), we obtain that, the total Stiefel-Whitney polynomial \( j^*(w(\hat{\omega}; t)) = w(\omega; t) = c(\omega; t^2) \) where \( j : X \to P(m, X, \sigma) \) is the fibre inclusion.

The following proposition yields the Stiefel-Whitney classes of \( \hat{\omega} \) when \( \omega \) is a complex line bundle. Using this and the splitting principle, we will obtain a formula for the Stiefel-Whitney classes when \( \omega \) is of arbitrary rank. The proposition was obtained in the special case of Dold manifolds in [21, Prop. 1.4]. Recall that \( \xi \) is the line bundle associated to the double cover \( S^m \times X \to P(m, X, \sigma) \) and is isomorphic to \( \pi^*(\xi) \).

Lemma 2.7. Let \( \sigma : X \to X \) be an involution with non-empty fixed point set and let \( \omega \) be a complex vector bundle of rank \( r \) over \( X \). With the above notations, we have \( \hat{\omega} \cong \xi \otimes \hat{\omega} \).

Proof. The total space of the bundle \( \xi \otimes \hat{\omega} \) has the description \( E(\xi \otimes \hat{\omega}) = \{(v, x; t \otimes e) | (v, x) \in P(m, X; \sigma), t \in \mathbb{R}, e \in p_\omega^{-1}(x)\} \) where \( (v, x; t \otimes e) = ((v, x; t \otimes e), (-v, \sigma; -t \otimes \hat{\sigma}(e)) \}; \) here \( \hat{\sigma} : E(\omega) \to E(\omega) \) is an involutive bundle map that covers \( \sigma \) and is conjugate linear isomorphism on each fibre. Thus we have the equality \( \hat{\sigma}(\sqrt{-1}te) = \sqrt{-1}t\hat{\sigma}(e) \).

\(^1\)This should however cause no confusion with the notation for a typical point of \( X \).
Observe that \([v, x; \sqrt{-1}e] = [-v, \sigma(x); \hat{\sigma}(\sqrt{-1}e)] = [-v, \sigma(x), -\sqrt{-1}\hat{\sigma}(e)]\) and so the map \(h : E(\xi \otimes \hat{\omega}) \to E(\hat{\omega}), [v, x; t \otimes e] \mapsto [v, x; \sqrt{-1}e] = [-v, \sigma(x); -\sqrt{-1}\hat{\sigma}(e)]\) is a well-defined isomorphism of real vector bundles. □

**Simplifying assumptions.** We shall make the following simplifying assumptions. 

(a) \(\sigma : X \to X\) has a fixed point. As observed already, the \(X\)-bundle \(\pi : P(m, X, \sigma) \to \mathbb{R}P^m\) admits a cross-section \(s : \mathbb{R}P^m \to P(m, X, \sigma)\). It follows that \(\pi^* : H^*(\mathbb{R}P^m; \mathbb{Z}_2) \to H^*(P(m, X, \sigma); \mathbb{Z}_2)\) is a monomorphism. We shall identify \(H^*(\mathbb{R}P^m; \mathbb{Z}_2)\) with its image under \(\pi^*\).

(b) \(H^1(X; \mathbb{Z}_2) = 0\). This implies that \(H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{Z}_2)\) induced by the homomorphism \(\mathbb{Z} \to \mathbb{Z}_2\) of the coefficient rings is surjective.

**Example 2.8.** (i) Let \(X\) be the complex flag manifold \(\mathbb{C}G(n_1, \ldots, n_r)\) and let \(\sigma : X \to X\) be defined by the complex conjugation on \(\mathbb{C}^n\), \(n = \sum n_j\). Then \(\text{Fix}(\sigma)\) is the real flag manifold \(\mathbb{R}G(n_1, \ldots, n_r) = O(n)/(O(n_1) \times \cdots \times O(n_r))\) so assumption (a) holds. Since \(X\) is simply connected, (b) also holds.

(ii) Let \(\omega\) be a \(\sigma\)-conjugate complex vector bundle of rank \(r\). Suppose that \(\text{Fix}(\sigma) \neq \emptyset\) and that \(H^1(X; \mathbb{Z}_2) = 0\). Let \(\theta : \text{Flag}(\omega) \to \text{Flag}(\omega)\) be the associated involution of the \(\text{Flag}(\mathbb{C}^r)\)-manifold bundle over \(X\). (See §2.3.) Then \(\text{Fix}(\theta) \neq \emptyset\) and \(H^1(\text{Flag}(\omega); \mathbb{Z}_2) = 0\).

In the Serre spectral sequence of the bundle \((P(m, X), \mathbb{R}P^m, X, \pi)\), we have \(E^{0,k}_2 = H^0(\mathbb{R}P^m; H^k(X; \mathbb{Z}_2))\) where \(H^k(X; \mathbb{Z}_2)\) denotes the local coefficient system on \(\mathbb{R}P^m\). The action of the fundamental group of \(\mathbb{R}P^m\) on \(H^*(X; \mathbb{Z}_2)\) is generated by the involution \(\sigma^* : H^*(X; \mathbb{Z}_2) \to H^*(X; \mathbb{Z}_2)\). Hence \(E^{0,2}_2 = H^2(X; \mathbb{Z}_2)\mathbb{Z}_2 = \text{Fix}(\sigma^*)\). In order to emphasise the dimension, we shall write \(H^2; \mathbb{Z}_2\) instead of \(\sigma^*\). Also (b) implies that \(E^{0,2}_3 = E^{0,2}_2\) and (a) implies that the transgression \(E^{0,2}_1 = \text{Fix}(H^2(\sigma; \mathbb{Z}_2))\) \(\to E^{3,0}_3 = H^3(\mathbb{R}P^3; \mathbb{Z}_2)\) is zero. It follows that \(E^{0,2}_3 = E^{0,2}_\infty\) and that the image \(j^* : H^2(P(m, X); \mathbb{Z}_2) \to H^2(X; \mathbb{Z}_2)\) equals \(\text{Fix}(H^2(\sigma; \mathbb{Z}_2))\), where \(j : X \hookrightarrow P(m, X)\) is the fibre inclusion. We have the exact sequence:

\[
0 \to H^2(\mathbb{R}P^m; \mathbb{Z}_2) \xrightarrow{\pi^*} H^2(P(m, X, \sigma); \mathbb{Z}_2) \xrightarrow{j^*} \text{Fix}(H^2(\sigma; \mathbb{Z}_2)) \to 0.
\] (1)

The homomorphism \(s^* : H^2(P(m, X, \sigma); \mathbb{Z}_2) \to H^2(\mathbb{R}P^m; \mathbb{Z}_2)\) yields a splitting and allows us to identify \(\text{Fix}(H^2(\sigma; \mathbb{Z}_2))\) as a subspace of \(H^2(P(m, X, \sigma); \mathbb{Z}_2)\), namely the kernel of \(s^*\). We shall denote the image of an element \(u \in \text{Fix}(H^2(\sigma; \mathbb{Z}_2))\) by \(\bar{u}\).

**Lemma 2.9.** Suppose that \(\sigma(x_0) = x_0\) and \(H^1(X; \mathbb{Z}_2) = 0\). Let \(s : X \to P(m, X, \sigma)\) be defined as \(v \mapsto [v, x_0]\) and let \(\omega\) be a \(\sigma\)-conjugate complex vector bundle over \(X\) of rank \(r\). Then (i) \(s^* (\hat{\omega}) \cong r e_\mathbb{R} \oplus r \zeta\), (ii) \(c_k(\omega) \in \text{Fix}(H^{2k}(\sigma; \mathbb{Z}_2))\), \(k \leq r\), and, (iii) if \(r = 1\), then \(w(\hat{\omega}) = 1 + x + c_1(\omega)\).

**Proof.** (i) Since \(\sigma(x_0) = x_0\), \(\hat{\sigma}\) restricts to a conjugate complex linear automorphism \(\hat{\sigma}_0\) of \(V := p_0^{-1}(x_0)\). Let \(U \subset V\) is the eigenspace of \(\hat{\sigma}_0\) corresponding to eigenvalue 1 of \(\hat{\sigma}_0\). Then \(\sqrt{-1}U\) is the \(-1\) eigenspace. The vector bundle \(s^*(\hat{\omega})\) is isomorphic to the
Whitney sum of the bundles $S^m \times_{Z_2} U \rightarrow \mathbb{R}P^m$ and $S^m \times_{Z_2} \sqrt{-1}U \rightarrow \mathbb{R}P^m$. Evidently these bundles are isomorphic to $r \epsilon$.

(ii) Since $\tilde{r} : E(\omega) \rightarrow E(\omega)$ is a conjugate complex linear bundle map covering $\sigma$, we have $\sigma^*(\omega) \cong \tilde{\omega}$. So $\sigma^*(c_k(\omega)) = c_k(\sigma^*(\omega)) = (c_k(\tilde{\omega})) = (-1)^k c_1(\omega) \in H^{2k}(X; \mathbb{Z})$. Therefore $c_k(\omega) \in \text{Fix}(H^{2k}(\sigma; \mathbb{Z}_2))$, $k \leq r$.

(iii) Using the isomorphism $s^* : H^1(P(m, X); \mathbb{Z}_2) \cong H^1(\mathbb{R}P^m; \mathbb{Z}_2)$, it follows from (i) that $w(1) = w(1) = x$. Since $c_1(\omega) \in \text{Fix}(H^2(\sigma; \mathbb{Z}_2))$, the element $\tilde{c}_1(\omega)$ is meaningful. It remains to show that $w(\tilde{\omega}) = \tilde{c}_1(\omega)$. Since $j^*(\tilde{\omega}) = \omega$, we see that $j^*(w(\tilde{\omega})) = w(\omega) = c_1(\omega) \in \text{Fix}(H^2(\sigma; \mathbb{Z}_2))$. On the other hand, $w(\tilde{s}^*(\tilde{\omega})) = 0$. So, under our identification of $\text{Fix}(H^2(\sigma; \mathbb{Z}_2))$ with the kernel of $s^*$, we have $w(\tilde{\omega}) = \tilde{c}_1(\omega)$. □

**Remark 2.10.** The above lemma shows that the element $\tilde{c}_1(\omega) \in H^2(P(m, X); \mathbb{Z}_2)$ is independent of the choice of the fixed point $x_0 \in X$ (used in the definition $s^*$) since it equals $w(\tilde{\omega})$.

Suppose that $\omega$ is a $\sigma$-conjugate complex vector bundle of rank $r$ over $X$. Since $q^*(\omega)$ splits as a Whitney sum $q^*(\omega) = \oplus_{1 \leq j \leq r} \omega_j$, where $q : \text{Flag}(\omega) \rightarrow X$ is the Flag($\mathbb{C}^r$)-bundle, in view of Example 2.8, we have $c_1(\omega_j) \in \text{Fix}(H^2(\theta; \mathbb{Z}_2))$. Therefore we obtain their ‘lifts’ $\tilde{c}_1(\omega_j) \in H^2(P(m, \text{Flag}(\omega); \theta; \mathbb{Z}_2))$. The bundle $\tilde{q}^*(\tilde{\omega})$ splits as $\tilde{q}^*(\tilde{\omega}) = \oplus_{1 \leq j \leq r} \tilde{\omega}_j$ (see Proposition 2.5(i)), where $\tilde{q} : P(m, \text{Flag}(\omega), \theta) \rightarrow P(m, X, \sigma)$ is the projection of the Flag($\mathbb{C}^r$)-bundle. Therefore $e_j(\tilde{c}_1(\omega_1), \ldots, \tilde{c}_1(\omega_r)) = e_j(w(\tilde{\omega}_1), \ldots, w(\tilde{\omega}_r))$ is in $H^2j(P(m, X, \sigma); \mathbb{Z}_2)$. Here $e_j$ stands for the $j$-th elementary symmetric polynomial.

**Notation:** Set $\tilde{c}_j(\omega) := e_j(w(\tilde{\omega}_1), \ldots, w(\tilde{\omega}_r)) \in H^2j(P(m, X, \sigma; \mathbb{Z}_2), 1 \leq j \leq r$.

When $j > r$, $\tilde{c}_j = 0$. Observe that $\tilde{c}_j(\omega)$ restricts to $c_j(\omega) \in H^{2j}(X; \mathbb{Z}_2)$ on any fibre of $\pi : P(m, X, \sigma; \mathbb{Z}_2) \rightarrow \mathbb{R}P^m$.

We have the following formula for the Stiefel-Whitney classes of $\tilde{\omega}$.

**Proposition 2.11.** We keep the above notations. Let $\omega$ be a $\sigma$-conjugate complex vector bundle over $X$. Suppose that $H^1(X; \mathbb{Z}_2) = 0$ and that $\text{Fix}(\sigma) \neq 0$. Then,

$$w(\tilde{\omega}; t) = \sum_{0 \leq j \leq r} (1 + xt)^{r-j} \tilde{c}_j(\omega) t^{2j}. \quad (2)$$

**Proof.** The case when $\omega$ is a line bundle was settled in Lemma 2.9. In the more general case, we apply the splitting principle, Proposition 2.5(i). The bundle isomorphism $q^*(\tilde{\omega}) = \tilde{\omega}_1 \oplus \cdots \tilde{\omega}_r$ given in Proposition 2.5(i) leads to the formula

$$w(\tilde{\omega}; t) = \prod_{1 \leq j \leq r} (1 + xt + \tilde{c}_1(\omega_j) t^2).$$

The proposition follows from Lemma 2.9 and the definition of $\tilde{c}_j(\omega)$ since $w_2(\tilde{\omega}_j) = \tilde{c}_1(\omega_j)$. □
3. The tangent bundle of $P(m, X)$

Let $X$ be a connected almost complex manifold and let $\sigma : X \to X$ be a complex conjugation. Thus $\sigma = T\sigma$ is a $\sigma$-conjugation. The manifold $P(m, X, \sigma)$ will be more briefly denoted $P(m, X)$. The bundle $\tilde{\tau}X$ restricts to the tangent bundle along any fibre of $\pi : P(m, X) \to \mathbb{R}P^m$ and so is a subbundle of $\tau P(m, X)$. Clearly $\tilde{\tau}X$ is contained in the kernel of $T\pi : TP(m, X) \to T\mathbb{R}P^m$. In fact $\tilde{\tau}X = \ker(T\pi)$ since their ranks are equal. Therefore we have a Whitney sum decomposition

$$\tau P(m, X) = \pi^*(\tau \mathbb{R}P^m) \oplus \tilde{\tau}X. \quad (3)$$

We assume that $\text{Fix}(\sigma)$ is non-empty and hence a smooth manifold of dimension $d = (1/2)\dim X$. Also we assume that $H^1(X; \mathbb{Z}_2) = 0$. Using the fact that $w(\mathbb{R}P^m) = (1 + x)^{m+1}$, and applying Proposition 2.11, we have

**Theorem 3.1.** Let $X$ be a connected compact almost complex manifold with complex conjugation $\sigma$. Suppose that $\text{Fix}(\sigma) \neq \emptyset$ and that $H^1(X; \mathbb{Z}_2) = 0$. Then:

$$w(P(m, X); t) = (1 + xt)^{m+1} \sum_{0 \leq j \leq d} (1 + xt)^{d-j}c_j(X)t^j. \quad (4) \Box$$

As an application of the above theorem we obtain

**Corollary 3.2.** (i) $P(m, X)$ is orientable if and only if if $m + d$ is odd.

(ii) $P(m, X)$ admits a spin structure if and only if $X$ admits a spin structure and $m+1 \equiv d \pmod{4}$.

**Proof.** Since $P(m, X) = (\mathbb{S}^m \times X)/\mathbb{Z}_2$, it is readily seen that $P(m, X)$ is orientable if and only if the antipodal map of $\mathbb{S}^m$ and the conjugation involution $\sigma$ on $X$ are simultaneously either orientation preserving or orientation reversing. The latter condition is equivalent to $m + 1 \equiv d \pmod{2}$. Alternatively, from Theorem 3.1, we obtain that $w_1(P(m, X)) = (m+1+d)x$, which is zero precisely if $m + d$ is odd.

Using the same formula, we have $w_2(P(m, X)) = \left(\binom{m+1}{2} + \binom{d}{2}\right)x^2 + c_1(X)$. The existence of a spin structure being equivalent to vanishing of the first and the second Stiefel-Whitney classes, we see that $P(m, X)$ admits a spin structure if and only if $X$ admits a spin structure and $\left(\binom{m+1}{2} \equiv \binom{d}{2} \pmod{2} \right.$ with $m + d$ odd. The latter condition is equivalent to $m + 1 \equiv d \pmod{4}$. \Box

The notions of of stable parallelizability and parallelizability were recalled in the Introduction. Recall from §2.2 the $\sigma$-conjugation $\varepsilon_{k,n-k} : X \times \mathbb{R}^n \to X \times \mathbb{R}^n$, defined with respect to a set of everywhere linearly independent sections $s_1, \ldots, s_n$.

**Theorem 3.3.** Let $\sigma$ be a conjugation on a connected almost complex manifold $X$ and let $\dim_{\mathbb{R}} X = 2d$. Suppose that $\text{Fix}(\sigma) \neq \emptyset$. Then:

(i) If $P(m, X)$ is stably parallelizable, then $X$ is stably parallelizable and $2^{\omega(m)}(m+1+d)$.

(ii) Suppose that $\rho(\tau X) \oplus n\varepsilon_{\mathbb{R}} \cong (2d+n)\varepsilon_{\mathbb{R}}$ as real vector bundle. Suppose that the bundle map $\varepsilon_{d+k,d+n-k}$ of $(2d+n)\varepsilon_{\mathbb{R}}$ covering $\sigma$ restricts to $\tilde{\sigma} = T\sigma$ on $TX$ and to $\varepsilon_{k,n-k}$ on $n\varepsilon_{\mathbb{R}}$. 
If $2^{c(m)}|(m+1+d)$, then $P(m, X)$ is stably parallelizable.

(iii) Suppose that $m$ is even and that $P(m, X)$ is stably parallelizable. Then $P(m, X)$ is parallelizable if and only if $\chi(X) = 0$.

**Proof.** (i) If $E \to B$ is any smooth fibre bundle with fibre $X$, the normal bundle to the fibre inclusion $X \hookrightarrow E$ is trivial. So if $E$ is stably parallelizable, then so is $X$. It follows that stable parallelizability of $P(m, X)$ implies that of $X$.

Let $x_0 \in \text{Fix} (\sigma)$ and let $s : \mathbb{R}^m \to P(m, X)$ be the corresponding cross-section defined as $[v] \mapsto [v, x_0]$. In view of Lemma 2.9 and the bundle isomorphism (3), we see that $s^*(\tau P(m, X)) = s^*(\pi^3 \mathbb{R}^m \oplus \hat{\tau} X) = \tau \mathbb{R}^m \oplus d \epsilon \oplus d \zeta \cong (m+1+d)\zeta \oplus (d-1)\epsilon$. Thus the stable parallelizability of $P(m, X)$ implies that $(m+1+d)(\lfloor \zeta \rfloor - 1) = 0$ in $KO(\mathbb{R}^m)$. By the result of Adams [1] (recalled in §1) it follows that $2^{c(m)}|(m+1+d)$.

(ii) Our hypothesis implies, using Lemma 2.3, that $\hat{\tau} X \oplus (k \xi \oplus (n-k)\epsilon) \cong (d+n-k)\epsilon \oplus (d+k)\zeta$. Therefore, using the isomorphism (3), $\tau P(m, X) \oplus k \xi \oplus (n-k+1)\epsilon \cong k \xi \oplus (n-k+1)\epsilon \oplus \pi^3 (\mathbb{R}^m) \oplus \hat{\tau} X \cong (m+1)\xi \oplus \hat{\tau} X \oplus (n-k)\epsilon \cong (m+1)\xi \oplus (d+k)\zeta \oplus (d+n-k)\epsilon$. Since dim $P(m, X) = 2d+m < 2d+n+1+m$, we may cancel the factor $k \xi \oplus (n-k)\epsilon$ on both sides [7, Theorem 1.1, Ch. 9], leading to an isomorphism $\tau P(m, X) \oplus \epsilon \cong (a+m+1)\xi \oplus d \epsilon$. Since $\xi = \pi^3(\zeta)$, again using Adams’ result it follows that $P(m, X)$ is stably parallelizable if $2^{c(m)}$ divides $(m+d+1)$.

(iii) Since $m$ is even, $P(m, X)$ is even dimensional. By Brez Kunski’s theorem [3], it follows that $P(m, X)$ is parallelizable if and only if its span is at least 1. By Hopf’s theorem, span $P(m, X) \geq 1$ if and only if $\chi(P(m, X))$ vanishes. Since $\chi(P(m, X)) = \chi(\mathbb{R}^m). \chi(X) = \chi(X)$ as $m$ is even, the assertion follows. $\square$

The stable span of a smooth manifold $M$ is the largest number $s \geq 0$ such that $\tau M \oplus \epsilon \cong (s+1)\epsilon \oplus \eta$ for some real vector bundle $\eta$. We extend the notion of span and stable span to a (real) vector bundle $\gamma$ over a base space $B$ in an obvious manner; thus span($\alpha$) is the largest number $r \geq 0$ so that $\gamma \cong \alpha \oplus r \epsilon$ for some vector bundle $\alpha$. If rank of $\gamma$ equals $n$ and if $B$ is a CW complex of dimension $d \leq n$, then span($\gamma$) $\geq n-d$. See [7, Theorem 1.1, Ch. 9]. It follows that if $n > d$, then span($\gamma$) = stable span($\gamma$).

**Remark 3.4.** (i) Suppose that $P(m, X)$ is stably parallelizable. If $m$ is odd, then $\chi(P(m, X)) = 0$ as $\chi(\mathbb{R}^m) = 0$. Consequently we obtain no information about $\chi(X)$ from the equality $\chi(P(m, X)) = \chi(\mathbb{R}^m). \chi(X)$. Let us suppose that $\chi(X) \neq 0$. Since span($\mathbb{R}^m$) = span($\mathbb{S}^m$), we obtain the lower bound span($P(m, X)$) $\geq$ span($\mathbb{S}^m$) = $\rho(m+1)-1$, where $\rho(m+1)$ is the Hurwitz-Radon function defined as $\rho(2^a+b)(2c+1) = 8a+2b$, $0 \leq b < 4$, $a, c \geq 0$. From Bredon-Kosiński’s theorem [3], we obtain that $P(m, X)$ is parallelizable if $\rho(m+1) > \rho(m+2d+1)$. For example if $m = (2c+1)2^r-1$ and $d = 2^r(2k+1)$ with $s < r - 1$ then $m+1 + 2d = ((2c+1)2^{r-1-s} + 2k + 1)2^{s+1}$ and so $\rho(m+1) = \rho(2^r) > \rho(2^{s+1}) = \rho(m+2d+1)$; consequently $P(m, X)$ is parallelizable.

(ii) The following bounds for the span and stable span of $P(m, X)$ are easily obtained.

- stable span($P(m, X)$) $\leq \min\{d + \text{span}(m+2d+1)\zeta, m + \text{stable span}(X)\}$,
\bullet \text{span}(P(m, X)) \geq \text{span}(\mathbb{R}^m).

If \( m \) is even and \( \chi(X) = 0 \), then \( \chi(P(m, X)) = 0 \). Since \( \dim P(m, X) \) is even, it follows by [10, Theorem 20.1], that \( \text{span}(P(m, X)) = \text{stable span}(P(m, X)) \).

We illustrate Theorem 3.3 in the case when \( X \) is the complex flag manifold \( \mathbb{C}G(n_1, \ldots, n_r) \), where the \( n_j \geq 1 \) are positive integers and \( n = \sum_{1 \leq j \leq r} n_j \), with its usual differentiable structure. It admits an \( U(n) \)-invariant complex structure and the smooth involution \( \sigma : X \to X \) defined by the complex conjugation on \( \mathbb{C}^n \) is a conjugation, as remarked in Example 2.8(i). We assume, without loss of generality, that \( n_1 \geq \cdots \geq n_r \). We denote by \( P(m; n_1, \ldots, n_r) \) the space \( P(m, \mathbb{C}G(n_1, \ldots, n_r)) \). Note that \( \mathbb{C}G(1, \ldots, 1) \) is the complete flag manifold \( \text{Flag}(\mathbb{C}^n) \).

The classical Dold manifold corresponds to \( r = 2 \) and \( n_1 \geq n_2 = 1 \). Theorem 1.1 in this special case is due to J. Korošn [9]. (Cf. [21], [12].)

**Proof of Theorem 1.1.** When \( n_j > 1 \) for some \( j \), the flag manifold \( X = \mathbb{C}G(n_1, \ldots, n_r) \) is well-known to be not stably parallelizable; see, for example, [18]. (Cf. [8].) So, by Theorem 3.3, the non-trivial part of theorem concerns the case when the flag manifold is stably parallelizable, namely, \( n_j = 1 \) for all \( j \). It remains to determine the values of \( m \) for which \( P = P(m; 1, \ldots, 1) \) is stably parallelizable. This is done in Proposition 3.5 below.

The manifold \( X = \mathbb{C}G(1, \ldots, 1) \) has non-vanishing Euler characteristic; in fact, \( \chi(X) = n! \), the order of the Weyl group of \( U(n) \). When \( m \) is even, it follows that \( \chi(P) = n! \) and so \( \text{span}(P) = 0 \).

Suppose that \( \rho(m + 1) > \rho(m + 1 + \binom{n}{2}) \). Then \( \text{span}(P) \geq \text{span}(\mathbb{R}^m) \geq \rho(m + 1) - 1 \) whereas the span of the sphere of dimension \( \dim P = m + 2d = m + n(n - 1) \) equals \( \rho(m + 1 + n(n - 1)) - 1 \). So, by Bredon-Kosiński theorem [3], \( P \) is parallelizable if it is stably parallelizable and \( \rho(m + 1) > \rho(m + 1 + n(n - 1)) \). \( \square \)

It is known that \( \text{Flag}(\mathbb{C}^n) \) is stably parallelizable, but not parallelizable, as a real manifold (Cf. [11, p.313].) (The non-parallelizability of \( \text{Flag}(\mathbb{C}^n) \) follows immediately from the fact that \( \chi(\text{Flag}(\mathbb{C}^n)) \neq 0 \).)

As a preparation for the proof of Proposition 3.5 we recall a certain functor \( \mu^2 \) introduced by Lam [11, §§4-5]. This allows us to apply Lemma 2.3(iii).

The functor \( \mu^2 = \mu^2_{\mathbb{C}} \) associates a real vector bundle to a complex vector.\(^2\) We assume the base space to be paracompact so that every complex vector bundle over it admits a Hermitian metric. If \( V \) is any complex vector space \( \mu^2(V) \) is defined as \( \mu^2(V) = \bar{V} \otimes_{\mathbb{C}} V / \text{Fix}(\theta) \) where \( \theta : \bar{V} \otimes V \to \bar{V} \otimes V \) is the conjugate complex linear automorphism defined as \( \theta(u \otimes v) = -v \otimes u \). As with any continuous functor ([13, §3(f)]), \( \mu^2 \) is determined by its restriction to the category of finite dimensional complex vector spaces and their isomorphisms. The functor \( \mu^2 \) has the following properties where \( \omega, \omega_1, \omega_2 \) are all complex vector bundles over a base space \( X \). The first three were established by Lam.

(i) \( \text{rank}(\mu^2(\omega)) = n^2 \) where \( n \) is the rank of \( \omega \) as a complex vector bundle.

\(^2\)Lam defined \( \mu^2 \) in a more general setting that includes (left) vector bundles over quaternions as well.
Remark 3.6.  

(ii) $\mu^2(\omega) \cong \epsilon_R$ if $\omega$ is a complex line bundle. Indeed, choosing a positive Hermitian metric on $\omega$, the map $E(\mu^2(\omega)) \ni [u \otimes zu] \mapsto (p_\omega(u), Re(z)||u||^2) \in X \times \mathbb{R}$, $z \in \mathbb{C}$ is a well-defined real vector bundle homomorphism. It is clearly non-zero and since the ranks agree, it is an isomorphism.

(iii) $\mu^2(\omega_1 \oplus \omega_2) = \mu^2(\omega_1) \oplus (\omega_1 \otimes \omega_2) \oplus \mu^2(\omega_2)$.

(iv) If $\hat{\sigma} : E(\omega) \to E(\omega)$ is a conjugation of $\omega$ covering an involution $\sigma : X \to X$, then $\mu^2(\hat{\sigma}) : E(\mu^2(\omega)) \to E(\mu^2(\omega))$ is a bundle map covering $\sigma$. In particular $\mu^2(\hat{\omega}) \cong \mu^2(\omega)$.

(v) If $\hat{\sigma}$ is a conjugation of a complex line bundle $\omega$ with a Hermitian metric $(\cdot, \cdot)$ covering an involution $\sigma$ such that $\langle u, v \rangle_x = \langle \hat{\sigma}(u), \hat{\sigma}(v) \rangle_{\sigma(x)}$, $u, v \in p_\omega^{-1}(x), x \in X$, then $\mu^2(\hat{\sigma}) : \mu^2(\omega) \to \mu^2(\omega)$ is the identity on each fibre under the isomorphism $\mu^2(\omega) \cong \epsilon_R$ of (ii) since $||\hat{\sigma}(u)|| = ||u||$.

Proposition 3.5.  The manifold $P(m; 1, \ldots, 1) = P(m, Flag(\mathbb{C}^n))$ is stably parallelizable if and only if $2^{\rho(m)}$ divides $(m + 1 + \binom{n}{2})$.

Proof. Recall ([11, Corollary 1.2]) that $\tau CG(n_1, \ldots, n_r) \cong \bigoplus_{1 \leq i < j \leq r} \gamma_i \otimes \gamma_j$ where $\gamma_j$ is the $j$-th canonical bundle of rank $n_j$ whose fibre over $(L_1, \ldots, L_r) \in CG(n_1, \ldots, n_r)$ is the complex vector space $L_j$. We have

$$\gamma_1 \oplus \ldots \oplus \gamma_r \cong n\epsilon_C.$$ 

Applying $\mu^2$ and using the above description of $\tau CG(n_1, \ldots, n_r)$ we obtain the following isomorphism of real vector bundles by repeated use of property (iii) of $\mu^2$ listed above:

$$\bigoplus \mu^2(\gamma_j) \oplus \tau(\mathbb{C}G(n_1, \ldots, n_r)) \cong n\epsilon_R \oplus \left( \bigoplus_{1 \leq i < j \leq n} \epsilon_C(e_i \otimes e_j) \right) \cong n^2\epsilon_R. \quad (5)$$

(Cf. [11, Theorem 5.1].) Specialising to the case of $X = Flag(\mathbb{C}^n)$ we have $\mu^2(\gamma_j) \cong \epsilon_R$. The involution $\sigma : X \to X$ defined as $L \mapsto \overline{L}$ induces a complex conjugation of $\hat{\sigma} = T\sigma$ on $\tau X$ which preserves the summands $\omega_{ij} := \gamma_i \otimes \gamma_j, i < j$, yielding a conjugation $\hat{\sigma}_{ij}$ on it. The bundle involution $\epsilon_{d,d}$ (covering $\sigma$) on the summand on the right $\bigoplus_{1 \leq i < j \leq n} \rho(\epsilon_C)$, defined with respect to the basis $\bar{e}_i \otimes e_j, \bar{e}_i \otimes \sqrt{-1}e_j, 1 \leq i < j \leq n$, and $\epsilon_{0,n}$ on the summand $\bigoplus_{1 \leq i < n} \epsilon_R(e_i \otimes e_i)$ defined with respect to $\bar{e}_i \otimes e_i, 1 \leq i \leq n$, together define an involution, denoted $\hat{\epsilon}$, that covers $\sigma$. Under the isomorphism, $\hat{\epsilon}$ restricts to $T\sigma$ on $\tau X$ and to $\epsilon_{0,n}$ on $\bigoplus_{1 \leq i \leq n} \mu^2(\gamma_i)$ defined with respect to a basis $\bar{u}_i \otimes u_i, 1 \leq i \leq n$, where $u_i \in L_i$ with $||u_i|| = 1$. It follows, by using (v) above and Lemma 2.3, that

$$n\epsilon_R \oplus \hat{\tau}Flag(\mathbb{C}^n) \cong n\epsilon_R \oplus \left( \binom{n}{2} \right) (\epsilon_R \oplus \hat{\xi}).$$

Therefore $(n + 1)\epsilon_R \oplus \tau P \cong (m + 1)\xi \oplus \hat{\tau}Flag(\mathbb{C}^n) \oplus n\epsilon_R \cong (m + 1 + \binom{n}{2})\xi \oplus \left( \binom{n+1}{2} \right) \epsilon_R$. Hence $\tau P$ is stably trivial if and only if $(m + 1 + \binom{n}{2})\xi$ is stably trivial if and only if $(m + 1 + \binom{n}{2})\xi$ on $\mathbb{R}P^m$ is stably trivial if and only if $2^{\rho(m)}$ divides $(m + 1 + \binom{n}{2})$. This completes the proof. \[\square\]

Remark 3.6. It is clear that for a given $n \geq 2$, there are only finitely many values $m \geq 1$ for which $P = P(m, Flag(\mathbb{C}^n))$ is parallelizable. In fact, since $2^{\rho(m)} \geq 2m$ for $m \geq 8$, we must have $m \leq \max\{8, \binom{n}{2}\}$. However the required values of $m$ are highly restricted.
For example when \( n = 2^s, s \geq 4 \), \( P \) is parallelizable only when \( m \in \{1, 3, 7\} \) and when \( n = 2^s - 2, s \geq 5, m \in \{2, 6\} \). When \( n = 6, P \) is not parallelizable for any \( m \).

### 3.1. More examples of parallelizable generalized Dold manifolds.

We give examples of parallelizable manifolds \( P(m, X) \) for some other classes of \( X \). Specifically, we take \( X \) to be certain (i) Hopf manifold, (ii) complex torus, and (iii) compact Clifford-Klein form of a (non-compact) complex Lie group. In all these case, it turns out that \( \text{Fix}(\sigma) \neq \emptyset \) and \( \hat{\tau}X \cong d\xi \oplus d\epsilon \). In particular \( \text{span}(P(m, X)) \geq d \). If \( 2^e(m) \) divides \((m + 1 + d)\), then \( P(m, X) \) is stably parallelizable. Furthermore, if \( d > \rho(m + 2d) \), then \( P(m, X) \) is parallelizable.

(i) Let \( \lambda > 1 \). The infinite cyclic subgroup \( \langle \lambda \rangle \) of the multiplicative group \( \mathbb{R}_{>0}^\times \) acts on \( \mathbb{C}^d_0 := \mathbb{C}^d \setminus \{0\} \) via scalar multiplication. Consider the Hopf manifold \( X = X_\lambda := \mathbb{C}^d_0/\langle \lambda \rangle \). Then \( X \cong S^1 \times S^{2d-1} \) is parallelizable. Although \( X_\lambda \) is defined for any complex number \( \lambda \) with \( |\lambda| \neq 1 \), our hypothesis that \( \lambda \) is real implies that complex conjugation on \( \mathbb{C}^d \) induces an involution \( \sigma \) on \( X \). Moreover \( \text{Fix}(\sigma) = (\mathbb{R}^d \setminus \{0\})/\langle \lambda \rangle \) is non-empty. Indeed \( \text{Fix}(\sigma) \cong S^1 \times S^{d-1} \). We claim that \( \tau X \) is isomorphic to \( d\xi \oplus d\epsilon \) as a complex vector bundle. Indeed, scalar multiplication \( \lambda : \mathbb{C}^d_0 \rightarrow \mathbb{C}^d_0 \) induces multiplication by \( \lambda \) on the tangent space \( T_x \mathbb{C}^d_0 \) for any \( x \in \mathbb{C}^d_0 \). Therefore \( TX = (\mathbb{C}^d_0 \times \mathbb{C}^d)/\langle \lambda \rangle \) where \( \langle \lambda \rangle \) acts diagonally. The required isomorphism \( \phi : TX \rightarrow X \times \mathbb{C}^n \) is then obtained as \( [z, v] \mapsto ([z], v/||z||) \). We observe that this is well-defined since \( \lambda \) is positive. Moreover, \( \phi(T_x(\lambda z)) = \phi(\lambda [z, v]) = ([z], v/||z||) \). Thus \( T_x \sigma \) corresponds to complex conjugation on \( d\xi \oplus d\epsilon \) and so \( \hat{\tau}X \cong d\xi \oplus d\epsilon \) by Theorem 3.3(ii).

(ii) Let \( X = X_\Lambda \cong (S^1)^{2d} \) be the complex torus \( \mathbb{C}^d/\Lambda \) where \( \Lambda \cong \mathbb{Z}^{2d} \) is stable under conjugation; equivalently \( \Lambda = \Lambda_0 + \sqrt{-1}\Lambda_0 \) where \( \Lambda_0 \) is a lattice in \( \mathbb{R}^d \). Then complex conjugation on \( \mathbb{C}^d \) induces a conjugation \( \sigma \) on \( X \). It is readily seen that \( \text{Fix}(\sigma) = (\mathbb{R}^d + \sqrt{-1}\Lambda_0)/\Lambda_0 \). Also \( \tau X \cong d\xi \oplus d\epsilon \) as a complex vector bundle. As in (i) above, \( \hat{\tau}X \cong d\xi \oplus d\epsilon \).

(iii) More generally, suppose that \( G \subset GL(N, \mathbb{C}) \) is a connected complex linear Lie group such that \( G \) is stable by conjugation \( A \mapsto \tilde{A} \in GL(n, \mathbb{C}) \). Suppose that \( \Lambda \) a discrete subgroup of \( G \) such that \( X = G/\Lambda \) is compact; that is, \( \Lambda \) is a uniform lattice in \( G \). Assume that \( \tilde{\Lambda} = \Lambda \). (For example, \( G \) is the group of unipotent upper triangular matrices in \( GL(N, \mathbb{C}) \) with \( \Gamma \) the subgroup of \( G \) consisting matrices with entries in \( \mathbb{Z}[\sqrt{-1}] \).) Then \( X = G/\Lambda \) is holomorphically parallelizable, i.e., \( \tau X \) is trivial as a complex analytic vector bundle. See [2]. In particular, \( \tau X \cong d\xi \). Let \( p : G \rightarrow X \) be the covering projection. Denoting by \( \mathfrak{g} \) the Lie algebra of \( G \), viewed as the space of vector fields on \( G \) invariant under right translation, we have a bundle isomorphism \( f : X \times \mathfrak{g} \rightarrow TX \) defined as \( (g\Gamma, V) \mapsto Tp_g(V_g) \forall V \in \mathfrak{g} \). This is well-defined since \( V \) is invariant under right-translation. Under this isomorphism, \( T\sigma \) is the standard \( \sigma \)-conjugation on \( d\xi \). So \( \hat{\tau} X \cong d\xi \oplus d\epsilon \). As the identity coset is fixed by \( \sigma \), \( \text{Fix}(\sigma) \neq \emptyset \).

### 3.2. Unoriented cobordism.

Recall from the work of Thom and Pontrjagin ([13, Ch. 4]) that the (unoriented) cobordism class of a smooth closed manifold is determined
by its Stiefel-Whitney numbers. Let $\sigma$ be a complex conjugation on a connected almost complex manifold $X$ and let $\dim_\mathbb{R} X = 2d$. Assume that $\text{Fix}(\sigma) \neq \emptyset$ and that $H^1(X; \mathbb{Z}_2) = 0$. Proposition 2.11 allows us to compute certain Stiefel-Whitney numbers of $P(m, X)$ in terms of those of $X$, even without the knowledge of the cohomology algebra $H^*(P(m, X); \mathbb{Z}_2)$. Let $s : \mathbb{R}^m \to P(m, X)$ be the cross-section corresponding to an $x_0 \in \text{Fix}(\sigma)$. We identify $\mathbb{R}^m$ with its image under $s$ and $X$ with the fibre over $[e_{m+1}] \in \mathbb{R}^m$. Then $X \cap \mathbb{R}^m = \{ [e_{m+1}, x_0] \}$ and the intersection is transverse. Denoting the mod 2 Poincaré dual of a submanifold $M \hookrightarrow P(m, X)$ by $[M]$, we have $[\mathbb{R}^m].[X] = [\mathbb{R}^m \cap X] = \{ [e_{m+1}, x_0] \}$, which is the generator of $H^{m+2d}(P(m, X); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

We claim that the class $[X] \in H^m(P(m, X); \mathbb{Z}_2)$ equals $x^m$. To see this, let $S_j$ be the sphere $S_j = \{ v \in S^m \mid v \perp e_j \}, 1 \leq j \leq m$. and let $X_j$ be the submanifold $\{ [v, x] \mid v \in S_j, x \in X \} \cong P(m-1, X)$. Let $u_0 = (e_1 + \ldots + e_m)/\sqrt{m}$. Then $C := \{ (\cos(t)u_0 + \sin(t)e_{m+1}, x_0) \in P(m, X) \mid 0 \leq t \leq \pi \} \cong \mathbb{R}P^1$ meets $X_j$ transversally at $[e_{m+1}, x_0]$. So $[C].[X]$ $\neq 0$. It follows that $[X] = x, 1 \leq j \leq m$, since $H^1(P(m, X); \mathbb{Z}_2) = \mathbb{Z}_2x$. Also (i) $\cap_{1 \leq j \leq m} X_j$ intersects $X_j$ transversely for any $j \leq m$, and (ii) $\cap_{1 \leq j \leq m} X_j = X$. It follows that $[X] = [X_1] \cdots [X_m] = x^m$ as claimed.

Denote by $\mu_X, \mu_{P(m, X)}$ the mod 2 fundamental classes of $X, P(m, X)$ respectively. Note that $w_{2j}(P(m, X))$ is of the form $w_{2j}(P(m, X)) = \tilde{c}_j(X) + a_1x^2\tilde{c}_{j-1}(X) + \ldots + a_kx^{2k}\tilde{c}_{j-k}(X)$ for suitable $a_i \in \{0, 1\}, 1 \leq i \leq k$, where $k = \min\{\lfloor m/2 \rfloor, j\}$. Similarly $w_{2j+1}(P(m, X)) = b_0x\tilde{c}_j(X) + b_1x^3\tilde{c}_{j-1}(X) + \ldots + b_kx^{2k+1}\tilde{c}_{j-k}, b_i \in \{0, 1\}, 0 \leq i \leq k$, with $k = \min\{\lfloor (m-1)/2 \rfloor, j\}$. A straightforward calculation using Theorem 3.1 reveals that $b_0 = m+1+d-j$. Let $J = j_1, \ldots, j_r$ be a sequence of positive integers among $j_k, 1 \leq k \leq r$, exceeds $m$.

Suppose that $I = i_1, \ldots, i_k; J = 1^m.2I = 1^m, 2i_1, \ldots, 2i_k$, (i.e., $j_k = 1, 1 \leq t \leq m$) and $P(m, X)$ is non-orientable, so that $w_I (P(m, X)) = x$, we have $w_I (P(m, X)) = x^m.\tilde{c}_I(X)$. Using $j^*(\tilde{c}_I(X)) = c_I(X) = w_{2I}(X)$, we obtain that $w_I [P(m, X)] := \langle w_I (P(m, X)), \mu_{P(m, X)} \rangle = \langle x^m. w_{2I}(P(m, X)), \mu_X \rangle = w_{2I}(X, \mu_X) = w_{2I}[X] \in \mathbb{Z}_2$.

**Theorem 3.7.** Suppose that $H^1(X; \mathbb{Z}_2) = 0$ and that $\text{Fix}(\sigma) \neq \emptyset$.

(i) Assume that $m \equiv d \mod 2$. If $[X] \neq 0$ in $\mathcal{M}$, then $[P(m, X)] \neq 0$.

(ii) If $[P(1, X)] \neq 0$, then $[X] \neq 0$.

**Proof.** (i) Since $m \equiv d \mod 2$, we have $w_I (P(m, X)) = x$. Since the odd Stiefel-Whitney classes $w_{2j}(X)$ vanish (as $X$ is an almost complex manifold), $[X] \neq 0$ implies that we must have that $w_{2I}[X] \neq 0$ for some $I$ with $|I| = d$. Then, by our above discussion $w_I [P(m, X)] \neq 0$ where $J = 1^m.2I$. This proves the first assertion.

(ii) Let $m = 1, \dim P(1, X) = 1 + 2d$ is odd. Using $x^2 = 0$, we have, from the above discussion, that $w_{2j}(P(1, X)) = \tilde{c}_j(X)$ and $w_{2j+1}(P(1, X)) = (d-j)x\tilde{c}_j(X)$. Suppose that $w_I [P(1, X)] \neq 0$. Then we see that exactly one term, say $j_k$, in $J$ must be odd. Write
$j_k = 2s + 1$ where $s \geq 0$. If $d - s$ is even, then $w_J[P(1, X)] = 0$. So $d - s$ is odd and we have $w_J(P(1, X)) = x\bar{c}_I(X)$ where $2I$ is obtained from $J$ by replacing $j_k$ by $j_k - 1$. Therefore $w_{2I}[X] = w_J[P(1, X)] \neq 0$. This completes the proof. \hfill $\square$

We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. We shall use the structure of complex Clifford algebras to obtain an action of $G := \mathbb{Z}_2^r$ on $P(m, X)$ with $X := \mathbb{C}G_{n,k}$ such that $P(m, X)$ has no $G$-fixed points. This implies, by [5, Theorem 30.1], that $[P(m, X)] = 0$. The required action of $G$ on $P(m, X)$ arises from such an action on $X$ via a linear representation of $G$ on $\mathbb{C}^n$. In order to ensure the $G$-action on $X$ leads to an action on $P(m, X)$, we need ensure that the representation of $G$ is real, that is, it arises by extension of scalars from an action on $G$ on $\mathbb{R}^n$.

Let $\nu_2(n) = r$. It is a basic fact that there exist orthogonal transformations $\phi_1, \ldots, \phi_r$ of $\mathbb{R}^{2r}$ such that $\phi_i^2 = -id$ and $\phi_i \circ \phi_j = -\phi_j \circ \phi_i$, $1 \leq i < j \leq r$. The $\mathbb{R}$-subalgebra of $M_{2r}(\mathbb{R})$ generated by these transformations is the Clifford algebra $C_r$ associated to the quadratic module $(\mathbb{R}^{2r}, -|| \cdot ||^2)$. See [7, Ch. 12]. We shall denote by $C_r^n$ the complex Clifford algebra $C_r \otimes_{\mathbb{R}} \mathbb{C}$. Evidently $\mathbb{R}^{2r}$ is a $C_r$-module and $\mathbb{C}^{2r}$ is a $C_r^n$-module. Then $\mathbb{C}^n = (\mathbb{C}^{2r})^s$ is a $C_r^n$-module where $s := n/2^r$.

We denote by the same symbol $\phi_j : \mathbb{C}^{2r} \to \mathbb{C}^{2r}$ the $\mathbb{C}$-linear extension of $\phi_j$. We further abuse notation by using the same symbol to denote the (diagonal) action of $\phi_j$ on $\mathbb{C}^n$. Since the $\phi_j$ are complexifications of real linear transformations, we have $\phi_j(\bar{z}) = \overline{\phi_j(z)}$, $\forall z \in \mathbb{C}^n$. Therefore $\phi_j(L) = \overline{\phi_j(L)}$ for all complex vector subspaces $L \subset \mathbb{C}^n$. It follows that $[v, L] \mapsto [v, \phi_j(L)]$ is a well-defined smooth self-map $f_j : (P(m, X) \to P(m, X)$, where $X := \mathbb{C}G_{n,k})$. We observe that the $f_j, 1 \leq j \leq r$, are pairwise commuting involutions. Therefore we obtain an action of $G = \mathbb{Z}_2^r$ on $P(m, X)$.

We claim that there are no $G$-fixed points for this action. Indeed $f_j([v, L]) = [v, \phi_j(L)] = [v, L]$ if and only if $L = \phi_j(L)$. So the $G$-fixed points $[v, L]$ are in bijective correspondence with $C_r^n$ submodules $L \subset \mathbb{C}^n$. But $C_r^n$ is isomorphic to $M_{2r}(\mathbb{C})$ or to $M_{2r}(\mathbb{C}) \oplus M_{2r}(\mathbb{C})$. See [7, §5, Ch. 12]. It follows that any non-zero module over $C_r^n$ has complex dimension divisible by $2^r$. Our assumption that $\nu_2(k) < \nu_2(n) = r$ implies that there is no $C_r^n$-submodule of $\mathbb{C}^n$ having dimension equal to $k$. This establishes the claim and the assertion of the lemma follows.

(ii) Suppose that $\nu_2(k) \geq \nu_2(n)$. Then $[\mathbb{C}G_{n,k}] \neq 0$ by the main theorem of [17]. (See also [16].) Note that dim$_{\mathbb{C}} \mathbb{C}G_{n,k}$ is even in this case. If $m$ is also even, then it follows that $[P(m, \mathbb{C}G_{n,k})] \neq 0$ by Theorem 3.7(i). \hfill $\square$

Remark 3.8. It appears to be unknown precisely which (real or complex) flag manifolds are unoriented boundaries. Let $n_1, \ldots, n_r \geq 1$ be integers and let $n = \sum_{1 \leq j \leq r} n_j$. Proceeding as in the case of the $P(m; n, k)$ it is readily seen that $[\mathbb{C}G(n_1, \ldots, n_r)]$ and $[P(m; n_1, \ldots, n_r)]$ in $\mathcal{M}$ are zero if $\nu_2(n) > \nu_2(n_j)$ for some $j$. Also, if $n_i = n_j$ for some $i \neq j$, then $X := \mathbb{C}G(n_1, \ldots, n_r)$ admits a fixed point free involution $t_{i,j}$ which swaps the $i$-th and the $j$-component of each flag $L$ in $X$. Clearly $t_{i,j}(L) = t_{i,j}(L), L \in X,$
and so we obtain an involution \([v, L] \mapsto [v, t_{i,j}(L)]\) on \(P(m; n_1, \ldots, n_r)\), which is again fixed point free. It follows that \([P(n_1, \ldots, n_r)] = 0\) in this case. If \(m \equiv d \mod 2\) where \(d = \dim \mathcal{C} X = \sum_{1 \leq i < j \leq r} n_i n_j\) and if \([X] \neq 0\), then \([P(m; n_1, \ldots, n_r)] \neq 0\) by Theorem 3.7.

For example, it is known that \(\chi(X) = n!/n_1! \ldots n_r!\). So if \(m\) and \(d\) are even and if \(n!/n_1! \ldots n_r!\) is odd, then \(\chi(P(m; n_1, \ldots, n_r))\) is also odd and so \([P(m; n_1, \ldots, n_r)] \neq 0\).

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