Equivalence determination of unitary operations

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We study equivalence determination of unitary operations, a task analogous to quantum state discrimination. The candidate states are replaced by unitary operations given as a quantum sample, i.e., a black-box device implementing a candidate unitary operation, and the discrimination target becomes another black-box. The task is an instance of higher-order quantum computation with the black-boxes as input. The optimal error probability is calculated by semidefinite programs. Arbitrary quantum operations applied between the black-boxes in a general protocol provide advantages over protocols restricted to parallelized use of the black-boxes. We provide a numerical proof of such an advantage. In contrast, a parallelized scheme is analytically shown to exhibit the optimal performance of general schemes for a particular number of quantum samples of the candidates. We find examples of finite-sample equivalence determination that achieve the same performance as when a classical description of the candidates are provided, although an exact classical description cannot be obtained from finite quantum samples.

I. INTRODUCTION

A typical discrimination task constitutes a “candidate set” and “target object”. The target object is equivalent to an element in the candidate set. We are given the target object and informed of the candidate set. The goal then is to decide which of the candidates is actually given. Typically, the candidate set is enumerated and the aim becomes to guess the number assigned to the candidate corresponding to the given target object. Discrimination tasks are a simplified information processing task and, conversely, various information processing tasks can be rephrased in terms of discrimination.

In quantum state discrimination, the candidate set consists of quantum states. The target object is a quantum system prepared in a candidate state. Quantum state discrimination has been investigated for two candidate states [1], unambiguous discrimination [2], relations to the no-signaling principle and the no-cloning theorem [3–6], mixed-state candidates [7, 8], candidate states with geometric symmetries [9–11], bi- and multi-partite candidate states under local operations and classical communication [12–16], and the change point detection [17]. See Ref. [18] for a review.

The candidate set may consist of quantum operations. The target object is a quantum device, provided as a black-box that implements a candidate operation. The task is to determine which operation is performed by the target box. Although the state discrimination may be seen as a special case of an operation discrimination, these two types of discrimination tasks should be considered as separate problems. For instance, a perfect discrimination is not possible for any finite number of non-orthogonal quantum states with finite copies of the target state, but it is shown that a perfect discrimination is possible for a finite candidate set of unitary operations by using the target box for a finite number of times [19–21].

Discrimination of quantum operations can be considered as an information processing task taking quantum operations as an input. More generally, it is possible to imagine scenarios where a quantum operation is also the output of the task [22–26]. Conventionally, quantum operations are treated as a means to convert the states which represent the input quantum information. In contrast, the types of quantum information tasks that allow quantum operations to be the input and/or output belong to what may be called functional quantum computing [27] or higher-order quantum computation [28].

Quantum discrimination tasks assume that the candidate set is informed a priori. Most typically, a full classical description of the candidates is assumed to be given. On the other hand, it may be that the candidates are provided as a quantum object. These quantum objects are a quantum sample of the candidates. Quantum state discrimination with candidate states given as a quantum sample has been investigated under various settings [29–38]. A quantum sample in operation discrimination is another black-box implementing a candidate operation and labelled with the associated number. Generally, a full classical description may be obtained from quantum samples by quantum tomography, consuming an infinite number of copies of the samples for each candidate (see Ref. [39] and references therein for a review of quantum tomography).

A figure of merit for a discrimination task measures how well a discrimination protocol performs. Commonly, there is a probability distribution defined on the candidate set with which the target object is chosen. An optimal discrimination protocol minimizes the guessing error averaged over the candidate distribution. We may also impose “unambiguousness”, namely, that we allow no mistakes with our guesses. An unambiguous discrimination protocol is designed to declare “inconclusive”, whenever the employed discrimination strategy fails to
single out the correct candidate. Typically in the literature, “minimum-error” tasks in quantum discrimination accept incorrect guesses.

Discrimination of quantum operations with a full classical description of the candidates has been investigated when the candidates are unitary operations [19–21], non-unitary quantum channels [40–45], and quantum measurements [46, 47]. Both minimum-error [19, 40, 44] and unambiguous discrimination [41, 47] have been studied, in addition to error-free i.e., perfect discrimination [21, 42, 43, 45–47]. Especially for minimum-error discrimination of two unitary operations with full classical description, the optimal average success probability is derived as a closed formula for unitary operations in SU(2) [19] and SU(d) for an arbitrary dimension d [20].

In this paper, we analyze quantum operation discrimination with candidates presented as a quantum sample. More specifically, our goal is equivalence determination of quantum operations, i.e., to determine the quantum sample equivalent to the target box. For simplicity, the candidate set consists of two single-qubit unitary operations, $U_1$ and $U_2$ in SU(2), each distributed according to the Haar measure. The reference box $j$ implements $U_j$ for $j = 1, 2$, while the target box implements either $U_1$ or $U_2$ with probability 1/2. An $(N_1, N_2)$-equivalence determination task allows $N_j$ samples of $U_j$ and a single use of the target box. Otherwise, any quantum states and operations may be employed without any cost. The comparison of unitary operations [48, 49] and the pattern matching [49] are (1,0)- and (1,1)-equivalence determination with restriction, respectively. Reference [50] investigates the comparison of quantum measurements.

In general, an arbitrary quantum operation of our choice can be used in between each use of black-boxes. Some of the black-boxes may be used concurrently. It is known that general schemes outperform the parallelized schemes [41–43, 51], but parallelized schemes are more efficient in terms of circuit depth. In addition to concurrency, the quantum circuit used for equivalent determination introduces an ordering on the black-boxes, which is another degree of freedom to exploit.

The paper is organized as follows. In Sec. II A, we review quantum testers [52, 53] as generalized POVM measurements on quantum operations. The necessary properties of irreducible representations of SU(2) are given in Sec. II C. Section III discusses (1,1)-equivalence determination and analytically derives the optimal average success probability, both in parallelized and general schemes. Section III B investigates the effect of the entanglement in the initial state. In Sec. IV, we derive the optimal average success probability when a classical description is given for one of the candidates. Section V numerically analyzes the optimal average success probability for (2, 1)-equivalence determination under all possible orderings of the black-boxes. We conclude in Sec. VI.

## II. PRELIMINARY

### A. Quantum testers

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Let $\mathcal{M}$ be a completely positive and trace-preserving (CPTP) map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$. References [52, 53] introduce quantum testers, which may be interpreted as a quantum measurement on CPTP maps.

Denote a positive operator-valued measure (POVM) as $\{\Pi_i\}_{i=1}^L$ satisfying $\Pi_i \succeq 0$ for $i = 1, \ldots, L$ and $\sum_{i=1}^L \Pi_i = I_{KL}$, where $I_{KL}$ is the identity operator on $\mathcal{K} \otimes \mathcal{L}$. We define a quantum 2-tester $\{\Pi_i\}_{i=1}^2$ by

$$\Pi_i := (I_K \otimes \sqrt{X}) \Pi_i (I_K \otimes \sqrt{X}),$$

where $X$ is a positive semidefinite operator with unit trace.

Measuring $\{\Pi_i\}$ on $\mathcal{M}$ corresponds to applying $\mathcal{M} \otimes I$ on $|\psi\rangle = I \otimes \sqrt{X}|I\rangle$, where $|I\rangle := \sum_{d=1}^{\dim H} |i\rangle |i\rangle_{HA}$ with the computational basis $\{|i\rangle\}_{d=1}^{\dim H}$, and then measuring $\{\Pi_i\}$ on the resulting state. The probability $q_i$ of obtaining the outcome $i$ is

$$q_i = Tr[M\Pi_i],
$$

where $M$ is the Choi operator of $\mathcal{M}$ defined by

$$M := (\mathcal{M} \otimes I)(|I\rangle\langle I|).$$

We consider $N − 1$ CPTP maps $M_i$ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K}_i)$ for $i = 1, 2, \ldots, N − 1$. $M_i$ appears before $M_{i+1}$ in the quantum circuit. A generalized POVM measurement on the $N − 1$ CPTP maps can be described by quantum $N$-tester and its definition is given as follows.

**Definition 1.** Quantum $N$-tester is a set of operators $\{\Pi_i\}$ when $\Pi_i \in \mathcal{L}(\otimes_{j=1}^{N-1} \mathcal{K}_j \otimes \otimes_{j=1}^{N-1} \mathcal{H}_j)$ satisfy

$$\Pi_i \geq 0,$$

$$\sum_i \Pi_i = I_{KN-1} \otimes Y^{(N-1)},$$

$$Tr_{\mathcal{K}_i} Y^{(j)} = I_{K_{j-1}} \otimes Y^{(j-1)}, \text{ for } j = 2, \ldots, N − 1,$$

$$Tr Y^{(1)} = 1,$$

for some positive semidefinite operators $Y^{(j)} \in \mathcal{L}(\otimes_{i=1}^{j-1} \mathcal{K}_i \otimes \otimes_{i=1}^{j} \mathcal{H}_i)$ for $j = 2, \ldots, N − 1$ and $Y^{(1)} \in \mathcal{L}(H_1)$. When the quantum $N$-tester $\{\Pi_i\}$ is applied on $N − 1$ CPTP maps $\{M_j\}_{j=1}^{N-1}$, the probability of obtaining the outcome $i$ is given by

$$q_i = Tr\left[\Pi_i \bigotimes_{j=1}^{N-1} M_j\right].$$

A quantum tester is a special case of quantum comb [52, 53] or quantum strategy [54] and can be realized by a quantum circuit (Fig. 1). The details of quantum testers are given in Refs. [52, 53]. We often abbreviate $Y^{(j)}$ with the largest $j$ in the range as $Y$.
B. Relaxing ordering constraint

Equations (5) and (6) in general imply that the input CPTP maps are applied in a particular order, for example $M_1$ must be used before $M_2$. When a quantum N-tester defined in Def. 1 satisfies additional conditions, the ordering constraint is relaxed. Especially for the quantum testers in this paper, the first two uses of black-boxes can be parallelized. The necessary and sufficient condition for the parallelization is

$$Y^{(1)} = I_{K_1} \otimes Y'^{(1)}$$

for a positive semidefinite operator $Y'^{(1)}$. This condition implies that the quantum operation between $M_1$ and $M_2$ can be substituted by a swap operation as given in Fig. 2.

![Fig. 2: Relaxation of ordering constraint.](image)

C. Irreducible representation of SU(2)

Let $K_i$ ($i = 1, 2, 3$) be any two-dimensional Hilbert space whose computational basis is \{0\rangle, \{1\rangle\}. We define the following basis of the three-qubit system $K := K_1 \otimes K_2 \otimes K_3$,

$$|v_1\rangle = \left|\frac{1}{\sqrt{2}}\right|0\rangle_2; \frac{1}{\sqrt{2}}\left|\frac{3}{2}\right\rangle_3 = \sqrt{\frac{1}{3}}(\langle 001 | + | 010 \rangle),$$

$$|v_2\rangle = \left|\frac{1}{\sqrt{2}}\right|0\rangle_2; \frac{1}{\sqrt{2}}\left|\frac{1}{2}\right\rangle_3 = \sqrt{\frac{1}{3}}(\langle 011 | - | 100 \rangle),$$

$$|v_3\rangle = \left|\frac{1}{\sqrt{2}}\right|0\rangle_2; \frac{1}{\sqrt{2}}\left|\frac{3}{2}\right\rangle_3 = \sqrt{\frac{1}{3}}(\langle 001 | - | 010 \rangle) - \sqrt{\frac{1}{3}}(\langle 011 | + | 100 \rangle),$$

$$|v_4\rangle = \left|\frac{1}{\sqrt{2}}\right|0\rangle_2; \frac{1}{\sqrt{2}}\left|\frac{1}{2}\right\rangle_3 = \sqrt{\frac{1}{3}}(\langle 001 | - | 010 \rangle) + \sqrt{\frac{1}{3}}(\langle 011 | + | 100 \rangle),$$

and

$$|v_5\rangle = \left|\frac{1}{\sqrt{2}}\right|0\rangle_2; \frac{1}{\sqrt{2}}\left|\frac{3}{2}\right\rangle_3 = |000\rangle.$$
relations of these three bases

$$|0\rangle_\frac{1}{2} = \frac{1}{2} |0\rangle_{\frac{1}{2}} + \frac{\sqrt{3}}{2} |1\rangle_{\frac{1}{2}},$$ (27)

$$|1\rangle_\frac{1}{2} = \frac{x_3}{2} |0\rangle_{\frac{1}{2}} - \frac{1}{2} |1\rangle_{\frac{1}{2}},$$ (28)

$$|0\rangle_\frac{1}{2} = \frac{1}{2} |0\rangle_{\frac{1}{2}} + \frac{\sqrt{3}}{2} |1\rangle_{\frac{1}{2}},$$ (29)

$$|1\rangle_\frac{1}{2} = \frac{x_3}{2} |0\rangle_{\frac{1}{2}} + \frac{1}{2} |1\rangle_{\frac{1}{2}}.$$ (30)

A linear operator $$\rho \in \mathcal{L}(\mathcal{K})$$ satisfying

$$[\rho, A^{\otimes 2}] = 0$$ (31)

for all $$A \in SU(2)$$ has the form

$$\rho = \bigoplus_{J=\frac{1}{2}}^\frac{1}{2} \frac{I_J}{d_J} \otimes \rho_J,$$ (32)

from Schur’s lemma, where $$I_J$$ is the identity operator on $$\mathcal{U}_J$$ and $$\rho_J = Tr_{\mathcal{U}_J} \rho$$.

The reduced operator $$\sigma = Tr_{\mathcal{K}_3} \rho$$ satisfies

$$[\sigma, A^{\otimes 2}] = 0,$$ (33)

and therefore

$$\sigma = \bigoplus_{J=\frac{1}{2}}^\frac{1}{2} \frac{I_J}{d_J} \otimes \sigma_J.$$ (34)

Here we define the basis of the two-qubit system as

$$|w_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$ (35)

$$|w_2\rangle = |00\rangle,$$ (36)

$$|w_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$ (37)

$$|w_4\rangle = |11\rangle.$$ (38)

The basis of the irreducible subspace $$\mathcal{U}_1$$ and the multiplicity subspace $$\mathcal{V}_1^{[2]}$$ are denoted as $$\{|i\rangle_1^{[2]}\}_{i=0}^2$$ and $$\{|0\rangle_1^{[2]}\}$$. The bases are chosen so that

$$|0\rangle_1^{[2]} = |w_2\rangle,$$ (39)

$$|1\rangle_1^{[2]} = |w_3\rangle,$$ (40)

$$|2\rangle_1^{[2]} = |w_4\rangle.$$ (41)

Finally, $$\mathcal{U}_0 = span\{|0\rangle_1^{[2]}\}$$ and $$\mathcal{V}_0^{[2]} = span\{|0\rangle_1^{[2]}\}$$ and

$$|0\rangle_0^{[2]} = |w_1\rangle.$$ (42)

The elements of the multiplicity subspaces of $$\sigma$$ are given by

$$\sigma_0 = \langle \hat{0} | \rho_\frac{1}{2} | \hat{0} \rangle_{\frac{1}{2}}, \quad \sigma_1 = \langle \hat{1} | \rho_\frac{1}{2} | \hat{1} \rangle_{\frac{1}{2}} + \rho_\frac{1}{2}.$$ (43)

The operator $$\sigma \otimes I_{\mathcal{K}_3}$$ satisfies Eq. (31) and

$$\sigma \otimes I_{\mathcal{K}_3} = (I_0 \otimes I_{\mathcal{K}_3} \otimes \frac{\sigma_0}{d_0}) + (I_1 \otimes I_{\mathcal{K}_3} \otimes \frac{\sigma_1}{d_1})$$ (44)

$$= I_\frac{1}{2} \otimes \frac{\sigma_0}{d_0} |0\rangle_{\frac{1}{2}} \langle 0| + I_\frac{1}{2} \otimes \frac{\sigma_1}{d_1} |1\rangle_{\frac{1}{2}} \langle 1| + I_\frac{1}{2} \otimes \frac{\sigma_1}{d_1} |0\rangle_{\frac{1}{2}} \langle 0|$$ (45)

$$= I_\frac{1}{2} \otimes \left( \frac{\sigma_0}{d_0} |0\rangle_{\frac{1}{2}} \langle 0| + \frac{\sigma_1}{d_1} |1\rangle_{\frac{1}{2}} \langle 1| \right) + I_\frac{1}{2} \otimes \frac{\sigma_1}{d_1} |0\rangle_{\frac{1}{2}} \langle 0|.$$ (46)

III. (1,1)-EQUIVALENCE DETERMINATION

In this section, we consider the simplest case, (1,1)-equivalence determination of unitary operations. We denote input and output Hilbert spaces of the reference box $$i$$ by $$\mathcal{H}_i$$ and $$\mathcal{K}_i$$, respectively for $$i = 1, 2$$ and input and output spaces of the target box by $$\mathcal{H}_3$$ and $$\mathcal{K}_3$$. For simplicity, we define $$\mathcal{H} := \otimes_{j=1}^3 \mathcal{H}_j$$ and $$\mathcal{K} := \otimes_{j=1}^3 \mathcal{K}_j$$. We focus on qubit systems and therefore assume $$\mathcal{H}_i = \mathcal{K}_i \cong \mathbb{C}^2$$. For a given quantum tester $$\{\Pi_1, \Pi_2\}$$, the success probability of obtaining the correct answer $$p_{U_1, U_2}$$ is given by

$$p_{U_1, U_2} := \frac{1}{2} Tr[|W_1\rangle \langle W_1| \Pi_1 + |W_2\rangle \langle W_2| \Pi_2],$$ (48)

where $$W_i := U_1 \otimes U_2 \otimes I_i$$ for $$i = 1, 2$$ and $$|W_i\rangle = (W_i \otimes I_i)|I_i\rangle$$. In other words, (1,1)-equivalence determination is to determine which unitary operation, $$W_1$$ or $$W_2$$, is implemented.

The success probability above depends on the specific unitary operations $$U_1$$ and $$U_2$$. Therefore we adopt the average success probability (ASP) over the Haar measure as a figure of merit of equivalence determination of unitary operations. ASP $$p_{ave}$$ is given by

$$p_{ave} = \frac{1}{2} \text{Tr}[M_1 \Pi_1 + M_2 \Pi_2],$$ (49)

where $$M_i$$ are

$$M_i := \int d\mu(U_1) \int d\mu(U_2) |W_i\rangle \langle W_i|,$$ (50)

for the Haar measure $$d\mu(U)$$.

A. Parallelized schemes

First we consider the parallelized schemes, in which all of the black-boxes are applied simultaneously. A circuit representation of equivalence determination under parallelized schemes is given in Fig. 3. The equivalence determination is to determine which unitary operation, $$W_1$$ or $$W_2$$, is implemented.
FIG. 3: The quantum circuit for $(1, 1)$-equivalence determination of unitary operations under parallelized schemes.

Within parallelized schemes, a quantum tester $\{\Pi_1, \Pi_2\}$ with $\Pi_i \in \mathcal{L}(\mathcal{K} \otimes \mathcal{H})$ is a set of positive semidefinite operators satisfying

$$\begin{align*}
\Pi_i &\geq 0, \quad i = 1, 2 \\
\Pi_1 + \Pi_2 &= I_K \otimes X \\
\text{Tr} X &= 1,
\end{align*}$$

for some $X \in \mathcal{L}(\mathcal{H})$.

**Theorem 1.** The optimal average success probability of $(1, 1)$-equivalence determination under parallelized schemes is $7/8$ when unitary operations are chosen from the Haar measure.

The optimal ASP for $(1, 1)$-equivalence determination under parallelized schemes is given by a semidefinite program (SDP)

$$\begin{align*}
\text{maximize} \quad & p_{\text{ave}} = \frac{1}{2} \text{Tr} \left[ M_1 \Pi_1 + M_2 \Pi_2 \right], \\
\text{subject to} \quad & \Pi_i \geq 0, \quad i = 1, 2, \\
& \Pi_1 + \Pi_2 = I_K \otimes X, \\
& \text{Tr} X = 1,
\end{align*}$$

where $M_i$ are given by Eq. (50).

Due to the symmetry introduced by averaging over the Haar measure, the following lemma can be proven (Appx. B).

**Lemma 1.** The optimal average success probability of $(1, 1)$-equivalence determination can be achieved with $X$ satisfying

$$[A^{\otimes 3}, X] = 0,$$

for any unitary operator $A \in \text{SU}(2)$.

Since the target box is chosen among $U_1$ and $U_2$ with the same probability, we may assume an additional symmetry on $X$.

**Lemma 2.** Let $S_{H_{12}}$ be the swap operator between $H_1$ and $H_2$. The optimal average success probability of $(1, 1)$-equivalence determination is obtained by $X$ satisfying

$$[S_{H_{12}} \otimes I_{H_3}, X] = 0.$$

**Proof.** Suppose that a set of positive semidefinite operators $\{\Pi_1, \Pi_2\}$ gives the success probability $p$. By using a tensor product of the swap operators $S_{K_{12}} \otimes S_{H_{12}}$, where $S_{K_{12}}$ acts on $K_1 \otimes K_2$ as $S_{K_{12}}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle$ for any $|\psi\rangle \in K_1$ and $|\phi\rangle \in K_2$, and $S_{H_{12}}$ acts similarly on $H_1 \otimes H_2$, we define $\Pi'_i$ as

$$\Pi'_i := \frac{1}{2} (\Pi_i + (S_{K_{12}} \otimes S_{H_{12}} \otimes I) \Pi_i (S_{K_{12}} \otimes S_{H_{12}} \otimes I)),$$

where $\Pi_i = 1$ and $\Pi_2 = I$. Then we have

$$\Pi'_1 + \Pi'_2 = I_K \otimes X'_H,$$

where $X'_H = (X_H + (S_{H_{12}} \otimes I) X_H (S_{H_{12}} \otimes I))/2$ satisfying $\text{Tr} X'_H = 1$. The set $\{\Pi'_1, \Pi'_2\}$ is also a quantum 2-tester, which gives the same success probability $p$ since

$$\frac{1}{2} \text{Tr} \left[ M_1 \Pi'_1 + M_2 \Pi'_2 \right] = \frac{1}{2} \text{Tr} \left[ M_1 \Pi_1 + M_2 \Pi_2 \right].$$

The equality is derived by using

$$(S_{K_{12}} \otimes S_{H_{12}} \otimes I) M_i (S_{K_{12}} \otimes S_{H_{12}} \otimes I) = M_i,$$

for $i = 1, 2$. By definition of $X'_H$, if $[S_{H_{12}} \otimes I_{H_3}, X'_H] = 0$ holds. Therefore we can always choose $X$ satisfying Eq. (54). □

By the above argument, $(1, 1)$-equivalence determination under parallelized schemes reduces to a discrimination of two (known) random unitary channels $M_1$ and $M_2$,

$$M_i(\rho) := \int d\mu(U_1) \int d\mu(U_2) (U_1 \otimes U_2 \otimes U_i) \rho (U_1^\dagger \otimes U_2^\dagger \otimes U_i^\dagger),$$

for $i = 1, 2$. The optimal ASP $p_{\text{ave}}^{\text{opt}}$ of discriminating two channels is represented in terms of the diamond norm $\| \cdot \|_\diamond$ as

$$p_{\text{ave}}^{\text{opt}} = \frac{1}{2} + \frac{1}{4} \| M_1 - M_2 \|_\diamond$$

by optimization of $X$.

**Proof of Theorem 1.** From Lemma 1, $X$ can be chosen as

$$X = \frac{I_2}{2} \otimes p X_2 \otimes \frac{I_3}{4} \otimes (1 - p)|0\rangle\langle 0| \otimes |\bar{0}\rangle\langle \bar{0}|.$$
where \( X_\frac{1}{2} \) is a 2 \( \times \) 2 positive semidefinite operator on the multiplicity subspace \( \mathcal{V}_\frac{1}{2}^{[3]} \) with unit trace and \( 0 \leq p \leq 1 \).

In order to utilize Lemma 2, the basis \( \{ |0\rangle, |1\rangle \} \) of the multiplicity subspace \( \mathcal{V}_\frac{1}{2}^{[3]} \) satisfies

\[
|\hat{i}\rangle \rightarrow (-1)^{i+1}|\hat{i}\rangle
\]

for \( i = 0, 1 \) under application of \( S_{\mathcal{H}_2} \). Therefore the condition of Lemma 2, i.e., \( [S_{\mathcal{H}_2}, X] = 0 \), implies that \( X_\frac{1}{2} \) is diagonalized in the basis \( \{ |0\rangle, |1\rangle \} \), namely,

\[
X_\frac{1}{2} = q|0\rangle\langle 0| + (1 - q)|1\rangle\langle 1| =: X_q,
\]

where \( 0 \leq q \leq 1 \).

We have

\[
I_K \otimes X = \bigoplus_{j=\frac{1}{2}}^{\infty} \left\{ I_{\mathcal{I}_j} \otimes \frac{I_2^H}{d_2^2} \otimes I_{\mathcal{V}_{\mathcal{J}_j}} \otimes pX_\frac{1}{2} \right\} + \bigoplus_{j=\frac{1}{2}}^{\infty} \left\{ I_{\mathcal{I}_j} \otimes \frac{I_2^H}{d_2^2} \otimes I_{\mathcal{V}_{\mathcal{J}_j}} \otimes (1 - p)|0\rangle\langle 0| \right\},
\]

where \( I_{\mathcal{I}_j} \) is the identity operator on the irreducible subspace \( \mathcal{U}_{\mathcal{I}_j} \) of \( \mathcal{K}_j \) and \( I_2^H \) for \( \mathcal{U}_L \) in \( H = \bigotimes_{i=1}^3 \mathcal{H}_i \).

By substituting Eq. (61) and \( M_1^{[i]} \) given in Lemma 6 in Appx. C, the diamond norm \( \| M_1 - M_2 \|_\diamond \) in Eq. (59) is calculated as

\[
\| M_1 - M_2 \|_\diamond = \max_{0 \leq p, q \leq 1} \left\{ p (\Delta_q + \Delta_q') + (1 - p)\Delta'' \right\},
\]

where

\[
\Delta_q := \left\| \left( I_{\mathcal{V}_{\mathcal{J}_j}} \otimes \sqrt{X_q} \right) \left( M_{\mathcal{I}_j}^{(1)} - M_{\mathcal{I}_j}^{(2)} \right) \left( I_{\mathcal{V}_{\mathcal{J}_j}} \otimes \sqrt{X_q} \right) \right\|_1,
\]

\[
\Delta_q' := \frac{2}{3} \left\| \sqrt{X_q} \left( M_{\mathcal{I}_j}^{(1)} - M_{\mathcal{I}_j}^{(2)} \right) \right\|_1,
\]

\[
\Delta'' := \frac{1}{3} \left\| M_{\mathcal{I}_j}^{(1)} - M_{\mathcal{I}_j}^{(2)} \right\|_1.
\]

To maximize the diamond norm, we can assume \( p = 0 \) or \( p = 1 \). When \( p = 1 \), ASP is

\[
p_{ave} = \frac{1}{4} + \frac{1}{4} \max_{0 \leq q \leq \pi/2} \frac{2}{\sqrt{3}} (\sin t)(1 + \cos t) = 7/8,
\]

where \( t \) is defined as \( q := \sin^2 t \) and the maximization is achieved with \( t = \pi/3 \). When \( p = 0 \), ASP is

\[
p_{ave} = \frac{1}{2} + \frac{1}{12} \left\| |1\rangle\langle 1| - |\bar{1}\rangle\langle \bar{1}| \right\|_1 = \frac{1}{2} + \frac{\sqrt{3}}{12} < 7/8.
\]

Thus the optimal ASP \( p_{ave}^{opt} \) is given by 7/8.

\[\Box\]

**FIG. 4:** The quantum circuit for equivalence determination of unitary operations under parallelized schemes with restricted entanglement in the initial state.

### B. Parallelized schemes with restricted entanglement

The optimal ASP under parallelized schemes is obtained using an initial state entangled between the systems on which the reference boxes and target box act on. We prove that this entanglement is necessary. In particular we restrict the initial state to the form of

\[
|\psi\rangle \otimes |\phi\rangle = \sqrt{X_1} \otimes \sqrt{X_2} \otimes I_{\mathcal{H}} |I\rangle,
\]

where \( X_1 \) and \( X_2 \) are positive semidefinite operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) and \( \mathcal{H}_3 \), respectively, satisfying \( \text{Tr} X_1 = \text{Tr} X_2 = 1 \) and \( |I\rangle \) is an unnormalized maximally entangled vector in \( (\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3)^{\otimes 2} \) (see Fig. 4).

This imposes an extra restriction \( X = X_1 \otimes X_2 \) to the discussion in the previous subsection. From Lemma 1, \( [X, A^{\otimes 3} \otimes B^{\otimes 3}] = [X_1 \otimes X_2, A^{\otimes 3} \otimes B^{\otimes 3}] = 0 \) for arbitrary unitary operators \( A, B \in \text{SU}(2) \) and

\[
X_1 = q I_0 \otimes (1 - q) I_{\frac{1}{3}},
\]

\[
X_2 = I_{\mathcal{H}_3}^{\otimes 2}.
\]

Therefore we have

\[
X = I_{\frac{1}{2}}^{\otimes 2} \otimes \left( r|0\rangle\langle 0| + (1 - r) \frac{I_1}{3} \right)
\]

\[
\otimes \frac{I_\frac{1}{4}^{\otimes 2}}{2} (1 - r)|0\rangle\langle 0| \frac{1}{2}.
\]

Thus the optimal ASP is derived from maximizing

\[
p_{ave} = \frac{1}{2} + \frac{1}{4} \left( \frac{1}{3} \sin 2t + \frac{2 \cos^2 t}{3\sqrt{3}} + \frac{2 \cos t \sqrt{2 - \cos 2t}}{3\sqrt{3}} \right).
\]

The optimal ASP is numerically derived to be \( p_{ave}^{opt} \approx 0.746399 < 0.875 = 7/8 \). Hence, the entanglement in the initial state between the systems of the target and reference boxes is crucial for achieving the optimal ASP.
C. Optimality under general schemes

In general, arbitrary quantum operations can be applied between the black-boxes, which impose an ordering on the black-boxes in the quantum circuit. In this section, we show that the optimal ASP of (1,1)-equivalence determination under general schemes is 7/8.

Three different orderings can be considered. We assign the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{K}_1$ to first black-box used in the circuit. $\mathcal{H}_2$ and $\mathcal{K}_2$ are assigned to the second black-box, while $\mathcal{H}_3$ and $\mathcal{K}_3$ to the third. Each black-box is either a reference box or the target box (see Fig. 5). The number of independent orderings is three, because the probability of the target box being $U_1$ and $U_2$ are equal. The independent orderings are characterized by the location of the target box.

The success probability of obtaining the correct answer is given by

$$p_{U_1,U_2}^{(j)} = \frac{1}{2} \text{Tr} \left[ |W_1^{(j)}\rangle\langle W_1^{(j)}| \Pi_1 + |W_2^{(j)}\rangle\langle W_2^{(j)}| \Pi_2 \right],$$

(75)

where $|W_i^{(j)}\rangle$ defined by

$$|W_i^{(1)}\rangle_{KH} := |U_i\rangle \otimes |U_1\rangle \otimes |U_2\rangle,$$

(76)

$$|W_i^{(2)}\rangle_{KH} := |U_2\rangle \otimes |U_i\rangle \otimes |U_1\rangle,$$

(77)

$$|W_i^{(3)}\rangle_{KH} := |U_1\rangle \otimes |U_2\rangle \otimes |U_i\rangle,$$

(78)

correspond to the three orderings of the target box being used the first, second, and last, respectively. This success probability depends on the choice of $U_1$ and $U_2$. By taking the average over the Haar measure, we obtain the following proposition.

**Proposition 1.** The optimal average success probability for (1,1)-equivalence determination under general schemes is given as an SDP

$$\begin{align*}
\text{maximize} & \quad p_{U_1,U_2}^{(j)} = \frac{1}{2} \text{Tr} \left[ \Pi_1 M_1^{(j)} + \Pi_2 M_2^{(j)} \right], \\
\text{subject to} & \quad \Pi_1 \geq 0, \quad i = 1,2, \\
& \quad \Pi_1 + \Pi_2 = I_{K_3} \otimes Y, \\
& \quad \text{Tr}_{H_3}Y = I_{K_3} \otimes Y^{(1)}, \\
& \quad \text{Tr}_{H_2}Y^{(1)} = I_{K_2} \otimes Y^{(0)}, \\
& \quad \text{Tr}Y^{(0)} = 1,
\end{align*}$$

(79)

$$\begin{align*}
& \frac{1}{2} M_1^{(j)} - \Omega \leq 0, \\
& \frac{1}{2} M_2^{(j)} - \Omega \leq 0, \\
& \text{Tr}_{K_3} \Omega - I_{H_3} \otimes Y^{(1)} \leq 0, \\
& \text{Tr}_{K_2} \Omega^{(1)} - I_{H_2} \otimes Y^{(0)} \leq 0, \\
& \text{Tr}_{K_1} \Omega^{(0)} - \lambda I_{H_1} \leq 0.
\end{align*}$$

(80)

The Haar random sampling $U_1$ and $U_2$ demand the following constraints on the variables of the SDP in Proposition 1.

**Lemma 3.** The quantum 4-tester $\{\Pi_i\}$ and positive semidefinite operators $Y$, $Y^{(1)}$ and $Y^{(0)}$ can be chosen to satisfy

$$\begin{align*}
[\Pi_i, (A^{\otimes 3})_{K_3} \otimes (B^{\otimes 3})_{H_1}] & = 0, \\
[Y_i, (A^{\otimes 2})_{K_1} \otimes (B^{\otimes 2})_{H_1} \otimes (B^{\otimes 2})_{H_2}] & = 0, \\
[Y^{(1)}, A_{K_1} \otimes (B^{\otimes 2})_{H_1, H_2}] & = 0, \\
[Y^{(0)}, B_{H_1}] & = 0,
\end{align*}$$

(81)

for $i = 1, 2$ and arbitrary $A, B \in SU(2)$. The proof of the lemma is given in Appx. D.

**Proposition 1** and Schur’s lemma imply Eq. (9). Therefore quantum 3-testers described in Fig. 6 is sufficient. There are only two cases of non-trivial orderings, i.e., the target box being used the first or last.

Since we formulated the optimization problem as an SDP, there exists a dual SDP. A solution to the dual gives an upper bound of the primal [56]. A lower bound to the primal is 7/8 since the general schemes include the parallelized schemes. In the following, we give a feasible solution to the dual that achieves the value 7/8.

**Lemma 4.** A dual SDP of the primal SDP given in Eqs. (79) - (84) is expressed as

$$\begin{align*}
\text{minimize} & \quad \lambda, \\
\text{subject to} & \quad M_1^{(j)} / 2 - \Omega \leq 0, \\
& \quad M_2^{(j)} / 2 - \Omega \leq 0, \\
& \quad \text{Tr}_{K_3} \Omega - I_{H_3} \otimes Y^{(1)} \leq 0, \\
& \quad \text{Tr}_{K_2} \Omega^{(1)} - I_{H_2} \otimes Y^{(0)} \leq 0, \\
& \quad \text{Tr}_{K_1} \Omega^{(0)} - \lambda I_{H_1} \leq 0.
\end{align*}$$

(85)

(86)

(87)

(88)

(89)
This can be derived by introducing Lagrange multipliers [56] (Appx. E).

Lemma 5. The dual SDP given in Lemma 4 is equivalent to the following SDP on the multiplicity subspaces.

\[
\text{minimize } \lambda, \quad \text{subject to } \quad \Omega_{iL} - \frac{M[i]}{2} \geq 0, \quad \text{for } J, L = 1/2, 3/2 \text{ and } i = 1, 2, \tag{96}
\]

\[
\Omega^{(1)}_{00} - \Omega^{(1)}_{01} \geq 0, \quad \Omega^{(1)}_{01} - \Omega^{(1)}_{11} \geq 0, \quad \Omega^{(1)}_{10} - \Omega^{(1)}_{11} \geq 0, \quad \lambda - \Omega^{(1)}_{00} - \Omega^{(1)}_{11} \geq 0, \tag{97}
\]

and \( \Omega^{i-1/2} = \langle \hat{J} \rangle \frac{1}{2} \otimes |I\rangle |I\rangle \). The proof of this lemma is given in Appx. F.

Theorem 2. The optimal average success probability of (1, 1)-equivalence determination under general schemes is \( 7/8 \) when unitary operations are chosen from the Haar measure.

Proof of Theorem 2. The optimal ASP by the general schemes is at least \( 7/8 \) since the general schemes include the parallelized schemes. The dual SDP given in Lemmas 4 and 5 gives an upper bound of the primal SDP, whose answer gives the optimal ASP in the general schemes. The optimal ASPs coincide for \( M_i^{(1)} \) and \( M_i^{(2)} \). In Appx. G, we give a feasible set of parameters for \( \lambda = 7/8 \) for two nontrivial orderings of the black-boxes \( M_i^{(2)} \) and \( M_i^{(3)} \). Hence the optimal solution to the dual SDP is at most \( 7/8 \). This concludes the proof. \( \square \)

IV. WHEN \( U_1 \) IS KNOWN

In this section, we assume that a classical description of one of the reference boxes, \( U_1 \), is given hence we may optimize the choice of quantum operations based on the description. A classical description of \( U_1 \) is obtainable if there is an infinite number of quantum samples of \( U_1 \). Conversely, any number of quantum samples of \( U_1 \) can be generated whenever its classical description is available. Hence a classical description and infinite quantum samples are interchangeable resources.

A. No quantum sample for \( U_2 \)

First we consider the case in which only the target box is given without any quantum sample of \( U_2 \) or its classical description. Contrary to the difference in the resources, we show that the optimal ASP is still \( 7/8 \) if \( U_2 \) is distributed according to the Haar measure.

We denote the input and output space of the target box as \( \mathcal{H} \) and \( \mathcal{K} \), respectively. The ASP can always be attained with an initial state \( |\psi\rangle \in \mathcal{H} \otimes \mathcal{H} \) of the form \( |\psi\rangle = I \otimes \sqrt{X} |I\rangle \), with a positive semidefinite operator \( X \) with unit trace on \( \mathcal{H} \) and maximally entangled vector \( |I\rangle \) in \( \mathcal{H} \otimes \mathcal{H} \).

The equivalence determination in this case reduces to the state discrimination of \( U_1 \otimes |\psi\rangle \) and \( U_2 \otimes |\psi\rangle \). Without loss of generality, we may use the classical description of \( U_1 \) to apply \( U_1 \) before performing the measurement and retain the same success probability. For mathematical convenience, we assume that \( U_1 \otimes I \) maps \( \mathcal{K} \otimes \mathcal{H} \) to \( \mathcal{K} \otimes \mathcal{H} \). The POVM is denoted as \( \{ \Pi_{11}, \Pi_{12} \} \), which does not depend on \( U_2 \).

The ASP over \( U_2 \) is

\[
p_{U_2} = \frac{1}{2} \int d\mu(U_2) \text{Tr}[|\psi\rangle \langle \psi| \Pi_{11}^{U_1} + |U_2^{U_1} \otimes I| \Pi_{11}^{U_1}]
\]

\[
= \frac{1}{2} \text{Tr}[|\psi\rangle \langle \psi| \Pi_{11}^{U_1} + \bar{E} \Pi_{11}^{U_1}],
\]

where

\[
\bar{E} = \frac{I_K}{2} \otimes X_{\mathcal{H}}.
\]

Therefore, it suffices to find a POVM that optimally distinguishes \( |\psi\rangle \langle \psi| \) and \( \bar{E} \). To maximize ASP, we define a quantum 2-tester \( \bar{\Pi}_i = (I \otimes \sqrt{X}) \Pi_i (I \otimes \sqrt{X}) \) and obtain

\[
p_{\text{ave}} = \frac{1}{2} \text{Tr} \left[ |I\rangle \langle I| \bar{\Pi}_i + \left( \frac{I}{2} \otimes I \right) \bar{\Pi}_2 \right],
\]

where \( \bar{\Pi}_1 + \bar{\Pi}_2 \equiv I_K \otimes X_{\mathcal{H}} \).

For a given \( \{ \bar{\Pi}_1, \bar{\Pi}_2 \} \) realizing ASP of \( p_{\text{ave}} \), a quantum 2-tester

\[
\bar{\Pi}'_i = \int d\mu(A) (A \otimes A^*) \bar{\Pi}_i (A \otimes A^*)
\]

also achieve the same ASP \( p_{\text{ave}} \), since \( (A \otimes A^*) |I\rangle = |I\rangle \) for any \( A \in SU(2) \). By definition, \( \bar{\Pi}'_i \) satisfy \( \bar{\Pi}'_i, A \otimes \)
$A^* = 0$ for any $A \in \text{SU}(2)$. Thus the optimal APS can be obtained assuming this commutation relation.

The relation $\tilde{\Pi}_1' + \tilde{\Pi}_2' = I \otimes X$ and the commutation relation imply that
\[ [X, A] = 0, \quad (112) \]
for any $A \in \text{SU}(2)$. This implies that without loss of generality $X = I/2$. Moreover, we have
\[ \tilde{\Pi}_i' = \alpha_i \frac{|I_i\rangle\langle I_i|}{2} + \beta_i Q, \quad (113) \]
where $Q$ is the projector onto the subspace orthogonal to $|I_i\rangle\langle I_i|$ defined as $Q := I - |I\rangle\langle I|/2$ and $\alpha_i, \beta_i \geq 0$ for $i = 1, 2$. From the condition $\tilde{\Pi}_1' + \tilde{\Pi}_2' = I \otimes I/2$ we obtain
\[ \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \frac{1}{2}, \quad (114) \]
Hence, $p_{\text{ave}}$ satisfies
\[ p_{\text{ave}} = \frac{1}{2} \text{Tr} \left[ |I_i\rangle\langle I_i| \tilde{\Pi}_1 + \left( \frac{I}{2} \otimes I \right) \tilde{\Pi}_2 \right] \]
\[ = \frac{1}{4} (\alpha_2 + 3\beta_2 + 4\alpha_1), \quad (115) \]
\[ \leq \frac{7}{8}, \quad (116) \]
where the inequality saturates when $\alpha_1 = \beta_2 = 1/2$ and $\alpha_2 = \beta_1 = 0$.

### B. Single quantum sample for $U_2$

As discussed in the previous section, providing a complete classical description of $U_2$ implies an ability to prepare any number of its quantum samples. Nevertheless, the result shown in the previous subsection indicates that the classical description of reference box 1 alone without a quantum sample of the other candidate does not improve the optimal ASP. In this section we show that the classical description of $U_1$ increases the optimal ASP, compared to (1, 1)-equivalence determination, when a single quantum sample of the reference box 2 is provided.

Let $U_2$ be distributed according to the Haar measure. For simplicity, we employ a parallelized scheme. Repeating a similar argument made in the previous subsection, the equivalence determination under the said conditions reduces to a discrimination of unitary operations $U_2 \otimes U_2$ and $U_2 \otimes I$. ASP $p_{\text{ave}}$ is
\[ p_{\text{ave}} = \frac{1}{2} \text{Tr} |E_1\tilde{\Pi}_1 + E_2\tilde{\Pi}_2|, \quad (117) \]
where $E_1$ and $E_2$ are defined by
\[ E_1 = \frac{I_{K_1}}{2} \otimes I_{\mathcal{H}_1} \otimes I|I\rangle\langle I|_{K_2 \mathcal{H}_2}, \quad (118) \]
\[ E_2 = I_0^{K_1} \otimes I_0^{H_1} \otimes \frac{I_{K_1}^{K_2}}{3} \otimes I_1^{H_1} \otimes H_2, \quad (119) \]
while $\tilde{\Pi}_1, \tilde{\Pi}_2 \geq 0$ and $\tilde{\Pi}_1 + \tilde{\Pi}_2 = I_K \otimes X_{\mathcal{H}}$. Without loss of generality, we have
\[ X = (\sin^2 t) I_0 \oplus (\cos^2 t) I_1^t =: X_t, \quad (120) \]
with $0 \leq t \leq \pi$.

The optimal ASP is calculated as
\[ p_{\text{ave}}^{\text{opt}} = \frac{1}{2} \frac{1}{4} \left[ \max \{ \| (I_K \otimes \sqrt{X_t})(E_1 - E_2)(I_K \otimes \sqrt{X_t}) \|_1 \} \right. \]
\[ = \frac{5 \cos^2 t}{36} + \frac{3}{144} \sqrt{87 - 4 \cos 2t - 10 \cos 4t} \]
\[ + \frac{1}{36} \sqrt{357 - 352 \cos 2t + 20 \cos 4t}. \quad (121) \]
The maximization term can be calculated as
\[ \{ (I_K \otimes \sqrt{X_t})(E_1 - E_2)(I_K \otimes \sqrt{X_t}) \|_1 \]
\[ = \frac{5 \cos^2 t}{36} + \frac{3}{144} \sqrt{87 - 4 \cos 2t - 10 \cos 4t} \]
\[ + \frac{1}{36} \sqrt{357 - 352 \cos 2t + 20 \cos 4t}. \quad (122) \]
The above equation is derived by a symbolic calculation of Mathematica [57]. The eigenvalues consist of $\cos^2 t$ with 5-fold degeneracy,
\[ \frac{1}{72} (11 - 16 \cos 2t \pm \sqrt{357 - 352 \cos 2t + 20 \cos 4t}) \quad (124) \]
with 3-fold degeneracy, and non-degenerate
\[ \frac{1}{72} (-7 + 2 \cos 2t \pm \sqrt{87 - 4 \cos 2t - 10 \cos 4t}) \quad (125) \]
The rests are zero. The optimal ASP is numerically obtained as $p_{\text{ave}}^{\text{opt}} \approx 0.902127 > 0.875 = 7/8$.

### V. (2, 1)-EQUIVALENCE DETERMINATION

There are 12 distinct orderings of the reference and target boxes in the most general scheme for (2, 1)-equivalence determination, i.e., fully ordered case, given by
\[ \begin{align*}
&H_1 \quad U_a \quad K_1 \quad H_2 \quad U_b \quad K_2 \quad H_3 \quad U_c \quad K_3 \quad H_4 \quad U_d \quad K_4
\end{align*} \quad (126) \]
depending on how we assign the reference and target boxes to $U_a, U_b, U_c,$ and $U_d$.

It is expected that increasing the concurrency of black-boxes by using more of them simultaneously before applying the next quantum operation causes to lower the optimal ASP. We divide the orderings according to the concurrency pattern. The number of black-boxes in the first layer, i.e., after the initial state preparation and before the first quantum operation, is between one and four. The black-boxes in the first two layers can always be parallelized without sacrificing the optimal ASP if the first
layer contains only a single black-box, due to the symmetry of quantum testers induced by averaging over the Haar measure. Therefore, the most general scheme of Concurrency Pattern (126) is replaceable by

\[
\begin{align*}
\mathcal{H}_1 & \rightarrow U_a \\
\mathcal{H}_2 & \rightarrow U_b \\
\mathcal{H}_3 & \rightarrow U_c \\
\mathcal{H}_4 & \rightarrow U_d \\
\end{align*}
\]

which is abbreviated as

\[
U_a U_b U_c U_d.
\]

Other concurrency patterns are

\[
\begin{align*}
\mathcal{H}_1 & \rightarrow U_a U_b \\
\mathcal{H}_2 & \rightarrow U_c U_d \\
\end{align*}
\]

The optimal ASP is obtained for all concurrency patterns and assignments by numerically solving the relevant SDP. The results are summarized in Table I. The derivation of the Choi operators and SDPs corresponding to each ordering are given in Supplemental Material [58]. The SDPs are rewritten in terms of the multiplicity subspaces.

| Class 1: \( p_{\text{ave}}^{\text{opt}} \approx 0.910516 \) | Class 2: \( p_{\text{ave}}^{\text{opt}} \approx 0.902127 \) |
|------------------------------|------------------------------|
| \( U_1 \), \( U_5 \), \( U_4 \), \( U_3 \) | \( U_1 \), \( U_5 \), \( U_4 \), \( U_3 \) |
| \( U_2 \), \( U_3 \), \( U_4 \), \( U_5 \) | \( U_2 \), \( U_3 \), \( U_4 \), \( U_5 \) |
| \( U_1 \), \( U_5 \), \( U_4 \), \( U_3 \) | \( U_1 \), \( U_5 \), \( U_4 \), \( U_3 \) |
| \( U_2 \), \( U_3 \), \( U_4 \), \( U_5 \) | \( U_2 \), \( U_3 \), \( U_4 \), \( U_5 \) |

TABLE I: Numerical results of the optimal average success probability \( p_{\text{ave}}^{\text{opt}} \) for (2, 1)-equivalence determination. Each four-block group corresponds to a particular use of black-boxes indicated by the subscripts. The corresponding quantum circuit for each black-box ordering is given in Concurrency Patterns (127) - (131). The orderings are divided into two classes according to \( p_{\text{ave}}^{\text{opt}} \).
TABLE II: A comparison of the optimal average success probabilities of \((N_1, N_2)\)-equivalence determination. \(N_i\) are the number of quantum samples for \(U_i\). “known” indicates that a classical descriptions of \(U_i\) is given. “R” in the row “initial entanglement” implies that the initial entanglement is restricted and “G” otherwise. In the row “ordering”, “P” is for parallelized, “G” for general, and “C1” and “C2” for Class 1 and Class 2, respectively.

| Analyzed in: | Sec. III B | Sec. III A | Sec. III C | Sec. IV A | Refs. [48, 49] | Sec. IV B | Sec. V | Ref. [19], Appx. A |
|--------------|------------|------------|------------|------------|----------------|------------|--------|-------------------|
| \(N_1\)     | 1          | known      | 1          | known      | 2              | known      |        |                   |
| \(N_2\)     | 1          | 0          | 0          | 1          | known          |           |        |                   |
| Initial entanglement | G          | R          | G          | G = P      | P              | C2        | C1     | G = P             |
| Ordering     | P          | P          | G          | G = P      | P              | C2        | C1     | G = P             |
| \(p_{ave}^{opt}\) | \(\approx 0.746399\) | 7/8 = 0.875 | \(\approx 0.902127\) | \(\approx 0.910516\) | \(\frac{1}{2} + \frac{4}{\pi} \approx 0.924413\) |

VI. CONCLUSION

In this paper, we introduced \((N_1, N_2)\)-equivalence determination of unitary operations, which is a discrimination task with two candidate unitary operations, \(U_1\) and \(U_2\). Classical descriptions of \(U_i\) are not available, but \(N_i\) quantum samples are given. The optimal average success probability (ASP) obtained under each setting is summarized in Table II.

We derived the optimal ASP for \((1, 1)\)-equivalence determination in both parallelized and general schemes. The problem was formulated as a semidefinite program (SDP). The SDP was used for the parallelized schemes to reduce the number of degrees of freedom in the choice of the initial state. The optimal ASP under the parallelized schemes is 7/8. We also showed that 7/8 cannot be achieved when the entanglement of the initial state is restricted. For the general schemes, a dual SDP was derived, for which we found a feasible set of parameters establishing that the optimal ASP under general schemes is at most 7/8. Therefore, the parallelized schemes achieve the optimal ASP of the general schemes.

We investigated when a classical description of one of the candidates \(U_1\) is given. With no quantum sample of \(U_2\), the optimal ASP is analytically derived to be 7/8 in this case. The numerics shows that the probability increases to \(\approx 0.902127\) with a single quantum sample of \(U_2\).

In \((2, 1)\)-equivalence determination, the symmetry induced by averaging over the Haar measure reduces non-trivial orderings of the black-boxes to 15. From numerics, they divide into two classes according to the optimal ASP, i.e., Class 1 with \(p_{ave}^{opt} \approx 0.910516\) and Class 2 with \(p_{ave}^{opt} \approx 0.902127\).

The optimal ASP of 7/8 in \((1, 1)\)-equivalence determination has been obtained in the context of the comparison of unitary operations \([48, 49]\), which is a restricted \((1, 0)\)-equivalence determination. Therefore, one of the quantum samples does not contribute in \((1, 1)\)-equivalence determination. Contrasting the results obtained in Sec. IV A and Refs. \([48, 49]\), the optimal ASP for \((N_1, 0)\)-equivalence determination under the parallelized schemes can be achieved with \(N_1 = 1\). The optimal ASP does not increase with the additional \(N_1 - 1\) quantum samples. Similarly, the results obtained in Secs. IV B and V indicate that \((N_1, 1)\)-equivalence determination under the parallelized schemes can be achieved with \(N_1 = 2\).

The adaptive operations allowed in the general schemes provide advantages over the parallelized schemes in optimization \([41–43, 51]\). Indeed, the general schemes in \((2, 1)\)-equivalence determination outperform the parallelized. In contrast, the general schemes in \((1, 1)\)-equivalence determination do not give improvements over the parallelized. Moreover, an exact classical description of an unknown unitary operation implemented by a black-box cannot be determined by finite uses of the black-box. Nevertheless, finite quantum samples were sufficient to achieve the same performance as with a classical description given. Equivalence determination has revealed unexpected properties of resourcefulness of input quantum operations and their orderings in higher-order quantum computation.

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Appendix A: Discrimination of two unitary operations with full classical descriptions

We summarize the relevant results in Ref. [20] on minimum-error discrimination of two unitary operations with their full classical description given. Consider unitary operations \(U_1\) and \(U_2\) in SU\((d)\) acting on \(\mathcal{H}\) and a black-box implementing \(U_1\) and \(U_2\) with probability \(\eta_1\) and \(\eta_2\), respectively. We denote an initial state as \(|\psi\rangle_{\mathcal{H} \oplus \mathcal{A}}\) where \(\mathcal{H} \cong \mathcal{A}\). Then the two candidate states \(|\psi_1\rangle = U_1 \otimes I |\psi\rangle\) and \(|\psi_2\rangle = U_2 \otimes I |\psi\rangle\) are obtained after applying the unitary operation implemented by the black-box.

The optimal success probability for minimum-error dis-
crimination is derived as

\[ p_{\text{ave}}^{\text{opt}} = \begin{cases} 1 & \left( \theta_d - \theta_1 \geq \pi \right) \\ \frac{1}{2} + \frac{1}{\pi} \int_0^\pi dt \sin^2 t = \frac{1}{2} + \frac{4}{3\pi}, \end{cases} \]  

(\text{otherwise}),

(A1)

where \( \{\theta_i\}_{i=1}^d \) are defined by the spectral decomposition

\[ U_1^j U_2 = \sum_{j=1}^d e^{|i\theta_j|} |\zeta_j\rangle \langle \zeta_j| \] satisfying \( -\pi \leq \theta_i < \pi \) and \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_d. \)

For the case of SU(2), we can denote \( U = (\cos t)I + i(\sin t)(\sum_{j=1}^3 e^{ij}\sigma_j) \), where \( (v_1, v_2, v_3) \) is a normalized real vector and \( \{\sigma_j\}_{j=1}^3 \) are the Pauli operators. The optimal ASP \( p_{\text{ave}}^{\text{opt}} \) over the Haar measure is given by

\[ p_{\text{ave}}^{\text{opt}} = \frac{1}{2} + \frac{1}{\pi} \int_0^\pi dt \sin^2 t = \frac{1}{2} + \frac{4}{3\pi}, \]  

(A2)

for \( \eta_1 = \eta_2 = 1/2. \)

\section*{Appendix B: Proof of Lemma 1}

Suppose that a quantum 2-tester \( \{\Pi_1, \Pi_1\} \) gives ASP \( p_{\text{ave}} \), satisfying \( \Pi_i \geq 0 \) for \( i = 1, 2 \) and \( \Pi_1 + \Pi_2 = I_K \otimes X_H \) with \( \text{Tr} X = 1. \) Let us define an averaged operator of \( \Pi_i \) as

\[ \bar{\Pi}_i := \int d\mu(A) \int d\mu(B) \]  

\[ (A^{\otimes 3})_K \otimes (B^{\otimes 3})_H \Pi_i ((A^{\otimes 3})_K \otimes (B^{\otimes 3})_H). \]  

(B1)

We have

\[ \bar{\Pi}_1 + \bar{\Pi}_2 = I_K \otimes X_H', \]  

(B2)

where \( X_H' \) is defined as

\[ X_H' := \int d\mu(B) B^{\otimes 3} X_H B^{\otimes 3}. \]  

(B3)

For any unitary operator \( T, \) \( [X_H', T^{\otimes 3}] = 0, \) since

\[ T^{\otimes 3} X_H' T^{\otimes 3} = \int d\mu(B) (TB)^{\otimes 3} X_H (TB)^{\otimes 3} \]  

\[ = \int d\mu(B) (TB)^{\otimes 3} X_H (TB)^{\otimes 3} \]  

\[ = \int d\mu(B') (B')^{\otimes 3} X_H (B')^{\otimes 3} \]  

\[ = X', \]  

(B4)

(B5)

(B6)

(B7)

where we used the property of the Haar measure \( d\mu(AB) = d\mu(B) \) for arbitrary unitary operators \( A \) and \( B \) in SU(2).

Finally, \( \{\bar{\Pi}_i\} \) gives the same ASP as \( \{\Pi_i\}, \) because

\[ \text{Tr} \left[ M_1 \bar{\Pi}_1 + M_2 \bar{\Pi}_2 \right] = \text{Tr} \left[ M_1 \Pi_1 + M_2 \Pi_2 \right] \]  

(B8)

from

\[ (A^{\otimes 3})_K \otimes (B^{\otimes 3})_H M_i ((A^{\otimes 3})_K \otimes (B^{\otimes 3})_H) = M_i, \]  

(B9)

for \( i = 1, 2. \) Hence we may assume without loss of generality that \( [T^{\otimes 3}, X] = 0 \) for an arbitrary unitary operator \( T. \)

\section*{Appendix C: Lemma 6}

\textbf{Lemma 6.} \( M_i \) is represented as

\[ M_i = \frac{1}{2} \otimes I_2^H \otimes \left( |00\rangle \langle 00|_{\frac{1}{2} \frac{1}{2}} + \frac{1}{3} |11\rangle \langle 11|_{\frac{1}{2} \frac{1}{2}} \right) \]  

\[ \otimes \frac{I_2^K}{2} \otimes I_2^H \otimes \left( \frac{1}{3} |01\rangle \langle 01|_{\frac{1}{2} \frac{1}{2}} + \frac{2}{3} |10\rangle \langle 10|_{\frac{1}{2} \frac{1}{2}} \right) \]  

\[ \otimes \frac{I_2^K}{2} \otimes I_2^H \otimes \frac{2}{3} |00\rangle \langle 00|_{\frac{1}{2} \frac{1}{2}} \]  

(C1)

\( M_2 \) is obtained by transforming \( \{ |0\rangle, |1\rangle \} \rightarrow \{ |\tilde{0}\rangle, |\tilde{1}\rangle \} \) in \( M_1. \)

\textbf{Proof.} From Eq. (B9), we have

\[ [M_i, (A^{\otimes 3})_K \otimes (B^{\otimes 3})_H] = 0, \]  

(C2)

for any unitary operators \( A, B \) in SU(2) and \( i = 1, 2. \)

\( K \otimes H \) is decomposed as

\[ K \otimes H = \bigoplus_{J=\frac{1}{2}} \bigoplus_{L=\frac{1}{2}} U_J \otimes U_L \otimes V_J^{[3]} \otimes V_L^{[3]}. \]  

(C3)

Here we changed the order of the spaces for convenience. In terms of irreducible representation, the tensor products of unitary operators are given as

\[ (A^{\otimes 3})_K \otimes (B^{\otimes 3})_H = \bigoplus_{J=\frac{1}{2}} \bigoplus_{L=\frac{1}{2}} A_J \otimes B_L \otimes I_{V_J^{[3]} \otimes V_L^{[3]}}. \]  

(C4)

where \( A_J \) and \( B_L \) are the irreducible representations acting on \( U_J \) and \( U_L, \) respectively, and \( I_{V_J^{[3]} \otimes V_L^{[3]}} \) are the identity operator on \( V_J^{[3]} \otimes V_L^{[3]}. \)

From Schur’s lemma and Eq. (C2), \( M_i \) is represented as

\[ M_i = \bigoplus_{J=\frac{1}{2}} \bigoplus_{L=\frac{1}{2}} \frac{1}{2} I^J_{d_J} \otimes I^L_{d_J} \otimes M^{(i)}_{JL}, \]  

(C5)

where \( M^{(i)}_{JL} \) are linear operators on \( V_J^{[3]} \otimes V_L^{[3]} \) and \( d_J := 2J + 1. \)

The next step is to derive \( M^{(i)}_{JL} \) for \( i = 1, 2. \) Define \( \eta^{[N]} \)

\[ \eta^{[N]} = \int d\mu(U) |U^{\otimes N} \rangle \langle U^{\otimes N}|. \]  

(C6)
$M_1$ and $M_2$ are represented as
\[ M_1 = \eta^{[2]}_{K_1K_3H_1H_3} \otimes \eta^{[1]}_{K_2H_2}, \quad (C7) \]
\[ M_2 = \eta^{[1]}_{K_1H_1} \otimes \eta^{[2]}_{K_3K_3H_3H_3}, \quad (C8) \]

By inserting Eq. (C6), we obtain
\[ \eta^{[2]}_{K_1K_3H_1H_3} = I_0^{K_1K_3} \otimes I_0^{H_1H_3} + \frac{1}{3} I_1^{K_1K_3} \otimes I_1^{H_1H_3}, \quad (C9) \]
\[ \eta^{[1]}_{K_2H_2} = \frac{I_2}{2} \otimes \eta^{H_2}. \quad (C10) \]

Therefore, $M_1$ is decomposed as
\[ M_1 = \left( I_0^{K_1K_3} \otimes I_0^{H_1H_3} + \frac{1}{3} I_1^{K_1K_3} \otimes I_1^{H_1H_3} \right) \otimes \frac{1}{2} I_2^{K_2} \otimes I_2^{H_2} \]
\[ = \frac{I_2}{2} \otimes I_2^H \otimes \left( |00\rangle \langle 00| + \frac{1}{3} |11\rangle \langle 11| \right) \]
\[ + \frac{1}{3} |11\rangle \langle 11| \right). \]

\[ \Pi'_i := \int d\mu(U) \int d\mu(V) (U^\otimes 3)_K \otimes (V^\otimes 3)_h \Pi_i ((U^\dagger)^\otimes 3)_K \otimes (V^\dagger)^\otimes 3)_h, \quad (D1) \]
\[ Y' := \int d\mu(U) \int d\mu(V) (U^\otimes 2)_{K_1K_2} \otimes (V^\otimes 2)_h (Y(U^\dagger)^\otimes 3)_{K_1K_2} \otimes (V^\dagger)^\otimes 3)_h, \quad (D2) \]
\[ Y'^{(1)} := \int d\mu(U) \int d\mu(V) (U_{K_3} \otimes (V^\otimes 2)_h)_{K_3} \Pi_i (U^\dagger_{K_1} \otimes (V^\dagger)^\otimes 2)_h, \quad (D3) \]
\[ Y'^{(0)} := \int d\mu(V) YY'^{(0)}V^\dagger. \quad (D4) \]

The new operators $\{\Pi'_i\}$, $Y'$, $Y'^{(1)}$ and $Y'^{(0)}$ also satisfy Eqs. (80) - (84). Therefore $\{\Pi'_i\}$ is also a quantum tester. Similarly to the proof of Lemma 1, the new quantum tester $\{\Pi'_i\}$ can achieve the same ASP $p_{ave}$. From definition,
\[ ([\Pi'_i, (A^\otimes 3)_K \otimes (B^\otimes 3)_h] = 0, \quad (D5) \]
\[ [Y', (A^\otimes 2)_{K_1K_2} \otimes (B^\otimes 3)_h] = 0, \quad (D6) \]
\[ [Y'^{(1)}, A_{K_3} \otimes (B^\otimes 2)_h] = 0, \quad (D7) \]
\[ [Y'^{(0)}, B_{H_1}] = 0. \quad (D8) \]

\[ \square \]

Appendix E: Proof of Lemma 4

We derive the dual SDP using Lagrange multipliers. Lagrangian $L$ is defined as
\[ L := \frac{1}{2} \text{Tr} \left[ \Pi_1 M_1^{(j)} + \Pi_2 M_2^{(j)} \right] \]
\[ - \text{Tr} \left[ \Omega (\Pi_1 + \Pi_2 - I_K \otimes Y) \right] \]
\[ + \frac{1}{3} \text{Tr} \left[ Y \left( \text{Tr}_{K_3} \Omega - I_{H_3} \otimes \Omega^{(1)} \right) \right] \]
\[ + \frac{1}{3} \text{Tr} \left[ \Omega^{(0)} \left( \text{Tr}_{K_2} \Omega^{(1)} - I_{H_2} \otimes \Omega^{(0)} \right) \right] \]
\[ + \frac{1}{3} \text{Tr} \left[ Y^{(0)} \left( \text{Tr}_{K_1} \Omega^{(0)} - \lambda I_{H_1} \right) \right] \]
\[ + \lambda. \quad (E2) \]

Note that the trace of the product of two positive semidefinite operators is non-negative. Therefore, if the
following inequalities

\[
\begin{align*}
\frac{M^{(j)}_1}{2} - \Omega &\leq 0, & (E3) \\
\frac{M^{(j)}_2}{2} - \Omega &\leq 0, & (E4) \\
\text{Tr}_{K_3} \Omega - I_{H_3} \otimes \Omega^{(1)} &\leq 0, & (E5) \\
\text{Tr}_{K_2} \Omega^{(1)} - I_{H_2} \otimes \Omega^{(0)} &\leq 0, & (E6) \\
\text{Tr}_{K_1} \Omega^{(0)} - \lambda I_{H_1} &\leq 0, & (E7)
\end{align*}
\]

are satisfied, we obtain.

\[
L \leq \lambda. & \tag{E8}
\]

Therefore, minimizing \( \lambda \) under Conditions (E3) - (E7) is the desired dual SDP.

\[
\square
\]

### Appendix F: Proof of Lemma 5

First we assume that positive semidefinite operators \( \Omega, \Omega^{(1)}, \Omega^{(0)} \), and \( \lambda \) fulfill Eqs. (91) - (95). Then new positive semidefinite operators defined by

\[
\begin{align*}
\Omega' &:= \int \! d\mu(U) \int \! d\mu(V)((U^\otimes 3)_{K} \otimes (V^\otimes 3)_{H}) \Omega((U^\dag^\otimes 3)_{K} \otimes (V^\dag^\otimes 3)_{H}), & (F1) \\
\Omega'^{(1)} &:= \int \! d\mu(U) \int \! d\mu(V)((U^\otimes 2)_{K_1K_2} \otimes (V^\otimes 2)_{H_1H_2}) \Omega^{(1)}((U^\dag^\otimes 2)_{K_1K_2} \otimes (V^\dag^\otimes 2)_{H_1H_2}), & (F2) \\
\Omega'^{(0)} &:= \int \! d\mu(U) \int \! d\mu(V)(U_{K_1} \otimes V_{H_1}) \Omega^{(0)}(U_{K}^\dag \otimes V_{L}^\dag), & (F3)
\end{align*}
\]

also satisfy Eqs. (91) - (95). By definition, \( \Omega' \), \( \Omega'^{(1)} \), \( \Omega'^{(0)} \), and \( \lambda \) form a feasible set of parameters and satisfy

\[
\begin{align*}
[\Omega', A^\otimes 3 \otimes B^\otimes 3] & = 0, & (F4) \\
[\Omega'^{(1)}, A^\otimes 2 \otimes B^\otimes 2] & = 0, & (F5) \\
[\Omega'^{(0)}, A \otimes B] & = 0, & (F6)
\end{align*}
\]

for arbitrary unitary operators \( A \) and \( B \) in SU(2).

We can assume that \( \Omega' \), \( \Omega'^{(1)} \), and \( \Omega'^{(0)} \) are represented as

\[
\begin{align*}
\Omega &= \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} I_{J}^{K} \otimes I_{L}^{H} \otimes \Omega_{JL}, & (F7) \\
\Omega'^{(1)} &= \bigoplus_{J=0}^{1} \bigoplus_{L=0}^{1} I_{J}^{K_1K_2} \otimes I_{L}^{H_1H_2} \otimes \Omega'^{(1)}_{JL}, & (F8)
\end{align*}
\]

where \( I_{J} \) is the identity operator on the irreducible sub-

space \( \mathcal{U}_J \), \( \Omega_{JL} \) an operator on \( \mathcal{V}_J^{[3]} \otimes \mathcal{V}_L^{[3]} \) for \( J, L = 1/2, 3/2, \) and \( \Omega'^{(1)}_{JL} \) for \( J, L = 0, 1 \) and \( \Omega'^{(0)}_{JL} \) are some positive numbers. Note that \( \Omega_{JL}^{1/2} \) is a scalar.

We rewrite Eqs. (91) and (92) in terms of the operators on the multiplicity subspaces. The operators \( M^{(j)}_{i} \) are represented as

\[
M^{(j)}_{i} = \bigoplus_{J=\frac{1}{2}}^{\frac{3}{2}} \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \frac{I_{J}^{K} \otimes I_{L}^{H}_{2} \otimes M^{(j)(i)}_{JL}}{d_{J}}, & (F9)
\]

for \( i = 1, 2 \). Thus, Eqs. (91) and (92) are rewritten as

\[
\Omega_{JL} - \frac{M^{(j)(i)}_{JL}}{2} \geq 0, & (F10)
\]

for \( J, L = 1/2, 3/2 \) and \( i = 1, 2 \).

Next we rewrite Eq. (93). Using Eq. (43) for \( \Omega' \), we have

\[
\begin{align*}
\text{Tr}_{K_3} \Omega' &= \bigoplus_{L=\frac{1}{2}}^{\frac{3}{2}} \left[ \frac{K_{1}K_{2}}{d_{0}} \otimes I_{L}^{H} \otimes ((\vec{0}_{\frac{1}{2}} \otimes I_{V_{J}^{[3]}})(\vec{0}_{\frac{1}{2}} \otimes I_{V_{J}^{[3]}}) + \frac{K_{1}K_{2}}{d_{1}} \otimes I_{L}^{H} \otimes ((\vec{1}_{1/2} \otimes I_{V_{J}^{[3]}})(\vec{1}_{1/2} \otimes I_{V_{J}^{[3]}}) + \Omega_{JL}^{1/2}) \right]. \\
&= \bigoplus_{J=0}^{1} \bigoplus_{L=0}^{1} \frac{I_{J}^{K_1K_2} \otimes \Omega'^{(1)}_{JL}}{d_{J}} \otimes \bigoplus_{i=1}^{2} \left( I_{H_{3}} \otimes I_{H_{1}H_{2}}^{[3]} \otimes \Omega'^{(1)}_{JL} \right).
\end{align*}
\]

Using Eq. (47), we have

\[
I_{H_{3}} \otimes \Omega'^{(1)}_{JL} = \bigoplus_{J=0}^{1} \frac{I_{J}^{K_1K_2}}{d_{J}} \otimes \left[ (I_{H_{3}} \otimes I_{H_{1}H_{2}}^{[3]}) \otimes \Omega'^{(1)}_{J0} \right]
\]

Using Eq. (47), we have

\[
\begin{align*}
I_{H_{3}} \otimes \Omega'^{(1)}_{JL} &= \bigoplus_{J=0}^{1} \frac{I_{J}^{K_1K_2}}{d_{J}} \otimes \left[ (I_{H_{3}} \otimes I_{H_{1}H_{2}}^{[3]}) \otimes \Omega'^{(1)}_{J0} \right] \\
&= \bigoplus_{J=0}^{1} \bigoplus_{L=0}^{1} \frac{I_{J}^{K_1K_2}}{d_{J}} \otimes \left[ \left( I_{H_{3}} \otimes I_{H_{1}H_{2}}^{[3]} \otimes \Omega'^{(1)}_{J0} \right) \right].
\end{align*}
\]

Using Eq. (47), we have

\[
\begin{align*}
I_{H_{3}} \otimes \Omega'^{(1)}_{JL} &= \bigoplus_{J=0}^{1} \frac{I_{J}^{K_1K_2}}{d_{J}} \otimes \left[ (I_{H_{3}} \otimes I_{H_{1}H_{2}}^{[3]}) \otimes \Omega'^{(1)}_{J0} \right] \\
&= \bigoplus_{J=0}^{1} \bigoplus_{L=0}^{1} \frac{I_{J}^{K_1K_2}}{d_{J}} \otimes \left[ \left( I_{H_{3}} \otimes I_{H_{1}H_{2}}^{[3]} \otimes \Omega'^{(1)}_{J0} \right) \right].
\end{align*}
\]
\[
= \bigoplus_{j=0}^1 I_{\frac{K_1 K_2}{d_j}} \otimes \left[ I_{\frac{H_1}{2}} \otimes \left( \Omega_{00}^{(1)} \langle \hat{0} | \hat{0} \rangle_{\frac{1}{2}} + \Omega_{01}^{(1)} \langle \hat{1} | \hat{1} \rangle_{\frac{1}{2}} \right) \right.
\oplus I_{\frac{H_1}{2}} \otimes \Omega_{J_1}^{(1)} \langle 0 | 0 \rangle_{\frac{1}{2}} \right].
\]

Equation (93) is rewritten as
\[
\begin{align*}
\Omega_{00}^{(1)} \langle 0 | 0 \rangle_{\frac{1}{2}} + \Omega_{01}^{(1)} \langle 1 | 1 \rangle_{\frac{1}{2}} - \Omega_{00}^{0-1/2} & \geq 0, \\
\Omega_{01}^{(1)} - \Omega_{00}^{0-1/2} & \geq 0, \\
\Omega_{10}^{(1)} \langle 0 | 0 \rangle_{\frac{1}{2}} + \Omega_{11}^{(1)} \langle 1 | 1 \rangle_{\frac{1}{2}} - \Omega_{10}^{1-1/2} - \Omega_{2 \frac{1}{2}} & \geq 0, \\
\Omega_{11}^{(1)} - \Omega_{10}^{1-1/2} - \Omega_{2 \frac{1}{2}} & \geq 0,
\end{align*}
\]
where we define \( \Omega_{j \frac{1}{2}} := (\langle \hat{j} | \hat{j} \rangle \otimes I_{V_L}[3]) \Omega_{L,L} (\langle \hat{j} | \hat{j} \rangle \otimes I_{V_L}[3]). \)

In addition, we obtain
\[
\begin{align*}
\text{Tr}_{K_2} \Omega_{(1)}^{(1)} &= I_{\frac{K_1}{d_j}} \otimes \left[ (\Omega_{00} + \Omega_{10}) I_{H_1 L_2} + (\Omega_{01} + \Omega_{11}) I_{H_1 L_2} \right], \quad (F13)
\end{align*}
\]
and
\[
\begin{align*}
I_{H_2} \otimes \Omega_{(0)}^{(0)} &= \frac{I_{K_1}}{d_j} \otimes \left( \Omega_{00}^{(0)} I_{H_1 L_2} + \Omega_{11}^{(0)} I_{H_1 L_2} \right), \quad (F14)
\end{align*}
\]
Equations (94) and (95) become
\[
\begin{align*}
\Omega_{\frac{1}{2} \frac{1}{2}} - \Omega_{00}^{(0)} - \Omega_{01}^{(0)} & \geq 0, \quad (F21)
\Omega_{\frac{1}{2} \frac{1}{2}} - \Omega_{01}^{(0)} - \Omega_{11}^{(0)} & \geq 0, \quad (F22)
\lambda - \Omega_{\frac{0}{2} \frac{1}{2}} & \geq 0. \quad (F23)
\end{align*}
\]
We can assume \( \Omega_{\frac{1}{2} \frac{1}{2}} = \lambda \) without loss of generality. All in all, the dual SDP expressed in the multiplicity subspaces is
\[
\begin{align*}
\text{minimize} \quad & \lambda, \\
\text{subject to} \quad & \Omega_{J,L} - \frac{M_{J,L}^{(i)}}{2} \geq 0, \\
& \text{for } J, L = 1/2, 3/2 \text{ and } i = 1, 2, \\
& \Omega_{00}^{(1)} \langle 0 | 0 \rangle_{\frac{1}{2}} + \Omega_{01}^{(1)} \langle 1 | 1 \rangle_{\frac{1}{2}} - \Omega_{00}^{0-1/2} \geq 0, \\
& \Omega_{01}^{(1)} - \Omega_{00}^{0-1/2} \geq 0, \\
& \Omega_{10}^{(1)} \langle 0 | 0 \rangle_{\frac{1}{2}} + \Omega_{11}^{(1)} \langle 1 | 1 \rangle_{\frac{1}{2}} - \Omega_{10}^{1-1/2} - \Omega_{2 \frac{1}{2}} \geq 0, \\
& \Omega_{11}^{(1)} - \Omega_{10}^{1-1/2} - \Omega_{2 \frac{1}{2}} \geq 0, \\
& \lambda - \Omega_{00}^{(1)} - \Omega_{10}^{(1)} \geq 0, \\
& \lambda - \Omega_{01}^{(1)} - \Omega_{11}^{(1)} \geq 0.
\end{align*}
\]

### Appendix G: Feasible sets of the dual SDP in Theorem 2

The Choi operators \( M_{3}^{(3)} \) are the same as the Choi operators for the parallelized scheme in Eq. (C5). In the basis \( \{ \langle 0 | \frac{1}{2}, | 1 | \frac{1}{2} \rangle \} \),
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(1) = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & -\frac{\sqrt{3}}{3}
\end{pmatrix},
\]
(\text{G1)}
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(1) = \frac{1}{4} \begin{pmatrix}
1 & -\frac{1}{\sqrt{3}} \\
-\frac{1}{\sqrt{3}} & 1
\end{pmatrix},
\]
(\text{G2)}
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(2) = \frac{1}{2} \begin{pmatrix}
1 & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & 1
\end{pmatrix},
\]
(\text{G3)}
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(2) = \frac{2}{3},
\]
(\text{G4)}
from Eq. (C11). Note that \( \dim V_{L}[3] = 2 \) and \( \dim V_{L}[3] = 1 \). The swap operation on \( K_1 \otimes H_1 \) and \( K_2 \otimes H_2 \) transforms \( M_{1}^{(3)} \) to \( M_{3}^{(3)} \). The transformation corresponds to \( \{ 0 \langle \frac{1}{2}, | 1 | \frac{1}{2} \rangle \} \to \{ 0 \langle \frac{1}{2}, | 1 | \frac{1}{2} \rangle \} \) in the multiplicity subspaces in \( M_{1}^{(3)} \). In the basis \( \{ 0 \langle \frac{1}{2}, | 1 | \frac{1}{2} \rangle \} \),
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(2) = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & -\frac{\sqrt{3}}{3}
\end{pmatrix},
\]
(\text{G5})
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(2) = \frac{1}{4} \begin{pmatrix}
1 & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{3}} & 1
\end{pmatrix},
\]
(\text{G6})
\[
M_{\frac{3}{2} \frac{3}{2}}^{(3)}(2) = \frac{2}{3},
\]
(\text{G7})
\[
\lambda = \frac{7}{8},
\]
(\text{G8})
A feasible set of parameters of the dual SDP for \( M_{i}^{(3)} \) is
\[
\lambda = \frac{7}{8},
\]
(\text{G9})
\[
\Omega_{\frac{1}{2} \frac{1}{2}} = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix},
\]
(\text{G10})
\[
\Omega_{\frac{1}{2} \frac{1}{2}} = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix},
\]
(\text{G11})
\[
\Omega_{\frac{1}{2} \frac{1}{2}} = \frac{1}{4} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix},
\]
(\text{G12})
\[
\Omega_{00} = \frac{1}{8}, \quad \Omega_{01} = \frac{1}{4}, \quad \Omega_{10} = \frac{3}{4}, \quad \Omega_{11} = \frac{5}{8}.
\]
(\text{G13})
The Choi operator $M^{(2)}_1$ is

$$M^{(2)}_{ξ_3} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{array} \right), \quad \text{(G14)}$$

and $M^{(2)}_2 = M^{(3)}_2$. A feasible set of parameters of the dual SDP for $M^{(2)}_4$ is

$$\lambda = \frac{7}{8}, \quad \Omega^{i,j}_{12} = \frac{1}{4} \left( \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{2}{3} \end{array} \right), \quad \text{(G19)}$$

$$\Omega^{i,j}_{12} = \frac{1}{8} \left( \begin{array}{c} \frac{1}{\sqrt{3}} \\ 0 \end{array} \right)^T \left( \begin{array}{c} 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{2}{3} \end{array} \right), \quad \Omega^{i,j}_{12} = \frac{1}{2}, \quad \text{(G20)}$$

$$\Omega^{(i)}_{00} = \frac{1}{2}, \quad \Omega^{(i)}_{01} = \frac{1}{8}, \quad \Omega^{(i)}_{10} = \frac{3}{8}, \quad \Omega^{(i)}_{11} = \frac{3}{4}, \quad \text{(G22)}$$

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I. CHOI OPERATORS IN THE MULTIPLICITY SUBSPACES

First we consider the Choi operators $M_i$ corresponding to the ordering of the black-boxes where the first two are reference box 1, followed by the test box, and end with reference box 2. The input and output systems of the $i$-th black-box are denoted as $\mathcal{H}_i$ and $\mathcal{K}_i$, respectively. The Choi operators for the remaining 11 orderings of the black-boxes are calculated by introducing unitary operators on the multiplicity subspaces that represent the action of the swap operations on $\mathcal{H}_i \otimes \mathcal{H}_j$ and $\mathcal{K}_i \otimes \mathcal{K}_j$. We denote $\mathcal{K} := \bigotimes_{i=1}^3 \mathcal{K}_i$ and $\mathcal{H} := \bigotimes_{i=1}^3 \mathcal{H}_i$.

For $M_i$, we have

$$M_i := \int d\mu(U_1) \int d\mu(U_2) \langle U_1 \otimes U_1 \otimes U_2 \rangle |\langle U_1 \otimes U_1 \otimes U_2 \rangle|,$$

which have the form of

$$M_1 = \eta^{[3]}_{K_1 K_2 K_3} \otimes \eta^{[1]}_{K_4 H_4},$$

$$M_2 = \eta^{[2]}_{K_1 K_2 K_3} \otimes \eta^{[2]}_{K_4 H_4},$$

where

$$\eta^{[3]}_{K_1 K_2 K_3} = \frac{I^{K_1 K_2 K_3}}{2} \otimes I^{H_1 H_2 H_3} \otimes \langle \langle 00 \rangle \langle H_1 \rangle + \langle 11 \rangle \langle H_2 \rangle \rangle \otimes I^{K_1 K_2 K_3} \otimes I^{H_1 H_2 H_3} \otimes \langle \langle 00 \rangle \langle H_1 \rangle \rangle,$$

and $\langle \langle \psi \rangle \rangle := |\psi\rangle\langle\psi|$. We define bases of the multiplicity subspaces by

$$|0\rangle_{0}^{[6]} := |(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle - |v_2\rangle),$$

$$|0\rangle_{1}^{[6]} := |(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0)\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle + |v_2\rangle),$$

$$|0\rangle_{0}^{[1]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, -1, 0)\rangle = |v_1\rangle,$$

$$|1\rangle_{0}^{[1]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle + |v_2\rangle),$$

$$|2\rangle_{0}^{[1]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = |v_2\rangle,$$

$$|0\rangle_{1}^{[1]} := |(1, \frac{1}{2}, \frac{1}{2}, 1, -1, 0)\rangle = |v_3\rangle,$$

$$|1\rangle_{1}^{[1]} := |(1, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_1\rangle + |v_4\rangle),$$

$$|2\rangle_{1}^{[1]} := |(1, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = |v_4\rangle,$$

$$|0\rangle_{2}^{[1]} := |(\sqrt{2}, \frac{1}{2}, \frac{1}{2}, 1, -1, 0)\rangle = \frac{\sqrt{2}}{2}(|v_5\rangle |v_1\rangle - |v_2\rangle |v_4\rangle),$$

$$|1\rangle_{2}^{[1]} := |(\sqrt{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_5\rangle |v_1\rangle + |v_2\rangle |v_4\rangle),$$

$$|2\rangle_{2}^{[1]} := |(\sqrt{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = |v_5\rangle,$$

$$|0\rangle_{0}^{[2]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, -1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_5\rangle |v_1\rangle - |v_2\rangle |v_4\rangle),$$

$$|1\rangle_{0}^{[2]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_5\rangle |v_1\rangle + |v_2\rangle |v_4\rangle),$$

$$|2\rangle_{0}^{[2]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = |v_5\rangle,$$

$$|0\rangle_{1}^{[3]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, -1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_5\rangle |v_1\rangle - |v_2\rangle |v_4\rangle),$$

$$|1\rangle_{1}^{[3]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = \frac{1}{\sqrt{2}}(|v_5\rangle |v_1\rangle + |v_2\rangle |v_4\rangle),$$

$$|2\rangle_{1}^{[3]} := |(0, \frac{1}{2}, \frac{1}{2}, 1, 1, 0)\rangle = |v_5\rangle,$$
\begin{align*}
|1\rangle^{[2]}_{0}\rangle^{[4]}_{2} & := |(1\frac{1}{2})\frac{3}{2}; 2 - 1\rangle = \frac{1}{2}|v_6\rangle|1\rangle + \sqrt{\frac{3}{2}}|v_8\rangle|0\rangle, \\
|2\rangle^{[2]}_{0}\rangle^{[4]}_{2} & := |(1\frac{1}{2})\frac{3}{2}; 20\rangle = \frac{1}{\sqrt{3}}(|v_6\rangle|1\rangle + |v_7\rangle|0\rangle), \\
|3\rangle^{[2]}_{0}\rangle^{[4]}_{2} & := |(1\frac{1}{2})\frac{3}{2}; 21\rangle = \frac{\sqrt{3}}{2}|v_7\rangle|1\rangle + \frac{1}{2}|v_8\rangle|0\rangle, \\
|4\rangle^{[2]}_{0}\rangle^{[4]}_{2} & := |(1\frac{1}{2})\frac{3}{2}; 22\rangle = |v_8\rangle|1\rangle.
\end{align*}

where \(|(j_1j_2j_3j_4; jm)|\) represent a state with the spin angular momentum of \(m\) along the \(z\)-axis, obtained by first coupling the spin in \(K_1\) and \(K_2\) to form a spin-\(j_1j_2\), then coupled with the spin-\(j_3\) in \(K_3\) to form a spin-\(j_1j_2j_3\), and finally coupled with the spin-\(j_4\) in \(K_4\) to form a spin-\(j\). In these bases,

\[
M_1 = \frac{1}{4} I_0^K \otimes I_0^H \otimes |\langle 00\rangle_{00} + |11\rangle_{00}\rangle \otimes \frac{1}{2} I_0^K \otimes I_0^H \otimes |\langle 00\rangle_{01} + |11\rangle_{01}\rangle
\]
\[
\oplus \frac{1}{8} I_1^K \otimes I_1^H \otimes |\langle 00\rangle_{10} + |11\rangle_{10}\rangle \otimes I_1^K \otimes I_1^H \otimes |\langle 00\rangle_{11} + |11\rangle_{11}\rangle
\]
\[
\oplus \frac{1}{8} I_2^K \otimes I_2^H \otimes |\langle 00\rangle_{21}\rangle \oplus \frac{1}{8} I_1^K \otimes I_2^H \otimes |\langle 20\rangle_{12}\rangle
\]
\[
\oplus \frac{1}{8} I_2^K \otimes I_2^H \otimes |\langle 00\rangle_{22}\rangle \oplus \frac{1}{8} I_1^K \otimes I_2^H \otimes |\langle 00\rangle_{22}\rangle
\]
\[
= I_0^K \otimes I_0^H \otimes \frac{1}{4} |\langle 00\rangle_{00} + |11\rangle_{00}\rangle \oplus I_0^K \otimes I_1^H \otimes \frac{1}{4} |\langle 00\rangle_{01} + |11\rangle_{01}\rangle
\]
\[
\oplus \frac{1}{8} I_1^K \otimes I_1^H \otimes \frac{1}{2} |\langle 00\rangle_{10} + |11\rangle_{10}\rangle \oplus \frac{1}{8} I_2^K \otimes I_1^H \otimes \frac{1}{2} |\langle 00\rangle_{11} + |11\rangle_{11}\rangle
\]
\[
\oplus \frac{3}{8} I_1^K \otimes I_2^H \otimes \frac{1}{8} |\langle 00\rangle_{21}\rangle \oplus \frac{3}{8} I_1^K \otimes I_2^H \otimes \frac{1}{8} |\langle 20\rangle_{12}\rangle
\]
\[
= \frac{2}{\sqrt{12}} \sum_{j=0, L=0}^{2} \frac{I_1^K}{d_J} \otimes I_2^H \otimes M_{jL}^{(1)}.
\]

Therefore,

\[
M_{00}^{(1)} = \frac{1}{4} |\langle 00\rangle_{00} + |11\rangle_{00}\rangle,
\]
\[
M_{01}^{(1)} = \frac{1}{4} |\langle 00\rangle_{01} + |11\rangle_{01}\rangle,
\]
\[
M_{02}^{(1)} = 0,
\]
\[
M_{10}^{(1)} = \frac{3}{4} |\langle 00\rangle_{10} + |11\rangle_{10}\rangle,
\]
\[
M_{11}^{(1)} = \frac{3}{4} |\langle 00\rangle_{11} + |11\rangle_{11}\rangle + \frac{3}{8} |\langle 22\rangle_{11}\rangle,
\]
\[
M_{12}^{(1)} = \frac{3}{8} |\langle 20\rangle_{12}\rangle,
\]
\[
M_{20}^{(1)} = 0,
\]
\[
M_{21}^{(1)} = \frac{5}{8} |\langle 02\rangle_{21}\rangle,
\]
\[
M_{22}^{(1)} = \frac{5}{8} |\langle 00\rangle_{22}\rangle.
\]

For \(M_2\), we have

\[
\eta_{K_1K_2H_1H_2}^{[2]} = I_0^{K_1K_2} \otimes I_0^{H_1H_2} \otimes I_1^{K_1K_2} \otimes I_1^{H_1H_2},
\]
\[
\eta_{K_3K_4H_3H_4}^{[2]} = I_0^{K_3K_4} \otimes I_0^{H_3H_4} \otimes I_1^{K_3K_4} \otimes I_1^{H_3H_4}.
\]

We also define other bases of the multiplicity subspaces as

\[
|0\rangle^{[4]}_{0}\rangle_{0}^{[4]} := |0(\frac{1}{2}0; 00) = |w_1\rangle|w_1\rangle,
\]
\[
|0\rangle^{[4]}_{0}\rangle_{1}^{[4]} := |1(\frac{1}{2}0); 10\rangle \frac{1}{\sqrt{2}}(|v_2\rangle|v_4\rangle - |v_3\rangle|v_4\rangle + |v_4\rangle|w_2\rangle),
\]
\[
|0\rangle^{[4]}_{1}\rangle_{0}^{[4]} := |0(\frac{1}{2}1); 01\rangle = |w_1\rangle|w_2\rangle,
\]
\[
|1\rangle^{[4]}_{1}\rangle_{0}^{[4]} := |0(\frac{1}{2}1); 10\rangle = |w_1\rangle|w_3\rangle.
\]
The definition of the bases of the multiplicity subspaces corresponds to a composition of spin-1/2 particles starting from composing pair $\{K_1, K_2\}$ and $\{K_3, K_4\}$, individually, and followed by composition of the composed pairs. In these bases,

\[
M_2 = I_0^K \otimes I_0^H \otimes (\langle 00 \rangle_{00} + \frac{1}{9} I_0^K \otimes I_0^H \otimes (\langle 111 \rangle_{00})) \\
\quad \quad \oplus \frac{1}{3} I_0^K \otimes I_1^H \otimes (\langle 00 \rangle_{11} + \langle 111 \rangle_{11}) + \frac{1}{3} I_0^K \otimes I_1^H \otimes (\langle 00 \rangle_{12} + \langle 11 \rangle_{12}) + \frac{1}{3} I_0^K \otimes I_2^H \otimes (\langle 00 \rangle_{22} + \langle 11 \rangle_{22}) \\
\quad \quad \oplus \frac{1}{3} I_0^K \otimes I_1^H \otimes (\langle 11 \rangle_{01} + \langle 111 \rangle_{02}) + \frac{1}{3} I_0^K \otimes I_2^H \otimes (\langle 11 \rangle_{01} + \langle 111 \rangle_{02}) \\
\quad \quad \oplus \frac{1}{3} I_0^K \otimes I_2^H \otimes (\langle 11 \rangle_{12}) + \frac{1}{3} I_0^K \otimes I_3^H \otimes (\langle 00 \rangle_{21} + \langle 11 \rangle_{21}) (52)
\]

Therefore, we have

\[
M_{00}^{(2)} = \langle 00 \rangle_{00} + \frac{1}{9} \langle 111 \rangle_{00},
\]

\[
M_{01}^{(2)} = \frac{1}{9} \langle 112 \rangle_{01},
\]

\[
M_{02}^{(2)} = \frac{1}{9} \langle 110 \rangle_{02},
\]

\[
M_{03}^{(2)} = \frac{1}{3} \langle 121 \rangle_{03},
\]

\[
M_{11}^{(2)} = \langle 111 \rangle_{11} + \langle 00 \rangle_{11} + \frac{1}{3} \langle 12 \rangle_{11},
\]

\[
M_{12}^{(2)} = \frac{1}{3} \langle 120 \rangle_{12},
\]

\[
M_{13}^{(2)} = \frac{5}{9} \langle 120 \rangle_{13},
\]

\[
M_{21}^{(2)} = \frac{5}{9} \langle 02 \rangle_{21},
\]

\[
M_{22}^{(2)} = \frac{5}{9} \langle 00 \rangle_{22}.
\]
We define a $2 \times 2$ matrix $V(j_1, j_3, j_{13})$ by

\[
V(j_1, j_3, j_{13})_{11} := \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13} + 2)(2j_1 + 1)}} \sqrt{(j_3 + \frac{1}{2})^2 - (j_{13} - j_1 + \frac{1}{2})^2},
\]

(63)

\[
V(j_1, j_3, j_{13})_{12} := \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13} + 2)(2j_1 + 1)}} \sqrt{(j_{13} + j_1 + \frac{3}{2})^2 - (j_3 + \frac{1}{2})^2},
\]

(64)

\[
V(j_1, j_3, j_{13})_{21} := \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13} + 2)(2j_1 + 1)}} \sqrt{(j_{13} + j_1 + \frac{3}{2})^2 - (j_3 + \frac{1}{2})^2},
\]

(65)

\[
V(j_1, j_3, j_{13})_{22} := \frac{(-1)^{2(j_1+j_3+j_{13})}}{\sqrt{(2j_{13} + 2)(2j_1 + 1)}} \sqrt{(j_3 + \frac{1}{2})^2 - (j_{13} - j_1 + \frac{1}{2})^2}.
\]

(66)

Calculating Wigner’s $6j$ coefficients, the relation between the bases is

\[
|0\rangle_0 = V(0, 1/2, 0)_{12}|0\rangle_0,
\]

(67)

\[
|1\rangle_0 = V(0, 1/2, 0)_{21}|1\rangle_0,
\]

(68)

\[
|0\rangle_1 = V(0, 1/2, 1)_{22}|0\rangle_1,
\]

(69)

\[
|1\rangle_1 = V(1, 1/2, 1)_{11}|1\rangle_1 + V(1, 1/2, 1)_{12}|2\rangle_1,
\]

(70)

\[
|2\rangle_1 = V(1, 1/2, 1)_{21}|1\rangle_1 + V(1, 1/2, 1)_{22}|2\rangle_1,
\]

(71)

\[
|0\rangle_2 = V(1, 1/2, 2)_{22}|0\rangle_2.
\]

(72)

Swap operations on $\mathcal{H}_i \otimes \mathcal{H}_j$ and $\mathcal{K}_i \otimes \mathcal{K}_j$ are applied to change the ordering of the black-boxes. In the multiplicity subspaces, these swap operations correspond to unitary operations within each multiplicity subspace. Let $U^{i,j}_j$ be such a unitary operation on the multiplicity subspace $V^{[4]}_j$ when the $i$-th and $j$-th system are exchanged. By calculating Wigner’s $6j$ coefficients, we can derive

\[
U^{1:2}_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U^{1:2}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U^{1:2}_2 = 1,
\]

(73)

\[
U^{2:3}_0 = V(1/2, 1/2, 0)^T, \quad U^{2:3}_1 = \begin{pmatrix} V(1/2, 1/2, 0)^T & 0 \\ 0 & V(1, 1/2, 1/2)_{22} \end{pmatrix}, \quad U^{2:3}_2 = V(1/2, 1/2, 1)_{12},
\]

(74)

\[
U^{3:4}_0 = \begin{pmatrix} V(0, 1/2, -1/2)_{22} \\ 0 \\ V(1, 1/2, -1/2) \end{pmatrix}, \quad U^{3:4}_1 = \begin{pmatrix} V(0, 1/2, 1/2) \\ 0 \\ V(1, 1/2, 1/2)^T \end{pmatrix}, \quad U^{3:4}_2 = V(1, 1/2, 3/2)_{12}.
\]

(75)

We add a superscript $\langle j \rangle$ as $M^{(j)}_i$ to indicate the 12 orderings of the reference and target boxes. Different $M^{(j)}_i$ can be obtained by multiplying $U^{i:j'}_j \otimes U^{i:j'}_L$ and $U^{i:j+1} \otimes U^{i:j+1}_L$ on $M^{(j)}_L$.

II. SDP FOR (2, 1)-EQUIVALENCE DETERMINATION IN THE MULTIPlicity SUBSPACES

In this section, we provide the SDPs for Concurrency Patterns (127) to (131) .
A. Concurrency Pattern (127)

The SDP for Concurrency Pattern (127) is

\[
\text{maximize } \quad p_{\text{succ}} = \frac{1}{2} \text{Tr} \left[ M^{(j)}_1 \bar{\Pi}_1 + M^{(j)}_2 \bar{\Pi}_2 \right],
\]

subject to

\[
\bar{\Pi}_i \geq 0, \quad i = 1, 2, \tag{77}
\]
\[
\bar{\Pi}_1 + \bar{\Pi}_2 = I_{K_4} \otimes Y^{(3)}, \tag{78}
\]
\[
\text{Tr}_{H_4} Y^{(3)} = I_{K_3} \otimes Y^{(2)}, \tag{79}
\]
\[
\text{Tr}_{H_3} Y^{(2)} = I_{K_1 K_2} \otimes Y^{(1)}, \tag{80}
\]
\[
\text{Tr} Y^{(1)} = 1. \tag{81}
\]

We rewrite Eqs. (78) - (81) in terms of the multiplicity subspaces. To this end, we may assume without loss of generality that

\[
\bar{\Pi}_i = \bigoplus_{J,L=0} I^J_L \otimes \frac{I^H_L}{d_L} \otimes \bar{\Pi}^{(j)}_{J,L}, \tag{82}
\]
\[
Y^{(3)} = \bigoplus_{J=1, L=0} \bigoplus_{J=\frac{1}{2}, L=0} I^{J_1 K_2 K_3} \otimes \frac{I^H_L}{d_L} \otimes Y_{JL}^{(3)}, \tag{83}
\]
\[
Y^{(2)} = \bigoplus_{J=0, L=\frac{1}{2}} \bigoplus_{J=\frac{1}{2}, L=0} I^{J_2 K_3} \otimes \frac{I^H_{L H_3}}{d_L} \otimes Y_{JL}^{(2)}, \tag{84}
\]
\[
Y^{(1)} = \bigoplus_{L=0} I^{J_1 H_1 H_2} \otimes Y_{L}^{(1)}, \tag{85}
\]

for \( i = 1, 2 \). The positivity condition (77) is equivalent to

\[
\bar{\Pi}^{(i)}_{J,L} \geq 0, \tag{86}
\]

for \( i = 1, 2 \) and \( J, L = 0, 1, 2 \). We have

\[
\bar{\Pi}_1 + \bar{\Pi}_2 = \bigoplus_{J,L=0} I^J_L \otimes \frac{I^H_L}{d_L} \otimes (\bar{\Pi}^{(1)}_{J,L} + \bar{\Pi}^{(2)}_{J,L}), \tag{87}
\]
\[
I_{K_4} \otimes Y^{(3)} = \bigoplus_{L=0} I^K_L \otimes \frac{I^H_L}{d_L} \otimes Y^{(3)}_{L} \bigoplus \bigoplus_{L=0} I^K_L \otimes \frac{I^H_L}{d_L} \otimes (Y^{(3)}_{\frac{1}{2}L} \otimes Y^{(3)}_{\frac{1}{2}L}) \bigoplus \bigoplus_{L=0} I^K_L \otimes \frac{I^H_L}{d_L} \otimes Y^{(3)}_{\frac{1}{2}L}. \tag{88}
\]

Thus Eq. (78) is equivalent to

\[
\left( P_{0,1}^{[4]} \otimes I_{V_1^L} \right) \left( \bar{\Pi}^{(1)}_{0,1} + \bar{\Pi}^{(2)}_{0,1} \right) \left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) = Y^{(3)}_{\frac{1}{2}L}, \tag{89}
\]
\[
\left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) \left( \bar{\Pi}^{(1)}_{1,2} + \bar{\Pi}^{(2)}_{1,2} \right) \left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) = Y^{(3)}_{\frac{1}{2}L}, \tag{90}
\]
\[
\left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) \left( \bar{\Pi}^{(1)}_{1,2} + \bar{\Pi}^{(2)}_{1,2} \right) \left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) = Y^{(3)}_{\frac{1}{2}L}, \tag{91}
\]
\[
\left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) \left( \bar{\Pi}^{(1)}_{1,2} + \bar{\Pi}^{(2)}_{1,2} \right) \left( P_{1,2}^{[4]} \otimes I_{V_1^L} \right) = Y^{(3)}_{\frac{1}{2}L}. \tag{92}
\]
for $L = 0, 1, 2$, where

\begin{align}
P^{[4]}_{0,\frac{1}{2}} & := |0\rangle_{\frac{1}{2}} \langle 0|_{0}^{[4]} + |1\rangle_{\frac{1}{2}} \langle 1|_{0}^{[4]}, \\
P^{[4]}_{1,\frac{1}{2}} & := |0\rangle_{\frac{1}{2}} \langle 0|_{1}^{[4]} + |1\rangle_{\frac{1}{2}} \langle 1|_{1}^{[4]}, \\
P^{[4]}_{1,\frac{3}{2}} & := |0\rangle_{\frac{1}{2}} \langle 2|_{1}^{[4]}, \\
P^{[4]}_{2,\frac{3}{2}} & := |2\rangle_{\frac{1}{2}} \langle 0|_{1}^{[4]}. \\
\end{align}

We have

\begin{align}
\text{Tr}_{\mathcal{H}_3} Y^{[3]} = \frac{3}{2} \sum_{J=\frac{1}{2}} \sum_{L=\frac{1}{4}} \left[ \frac{I_{K_{1,2}K_3}^{L} I_{H_1 H_2 H_3}^{L}}{d_L} \otimes \left\{ \langle I_{V_{3}^{[1]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}} Y_{0}^{[3]} (I_{V_{3}^{[3]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}}) + (I_{V_{3}^{[3]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}}) Y_{0}^{[3]} (I_{V_{3}^{[3]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}}) \right\} \right] \\
+ \left[ \frac{I_{K_{1,2}K_3}^{L} I_{H_1 H_2 H_3}^{L}}{d_L} \otimes \left\{ \langle I_{V_{3}^{[2]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}} Y_{0}^{[3]} (I_{V_{3}^{[3]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}}) + (I_{V_{3}^{[3]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}}) Y_{0}^{[3]} (I_{V_{3}^{[3]} L_{0}}^{[4]} \otimes P^{[4]}_{0,\frac{1}{2}}) \right\} \right].
\end{align}

and

\begin{align}
I_{K_{3}} \otimes Y^{[2]} = \frac{3}{2} \sum_{L=\frac{1}{4}} \left[ \frac{I_{K_{1,2}K_3}^{L} I_{H_1 H_2 H_3}^{L}}{d_L} \otimes (Y_{0}^{[2]} \otimes Y_{1}^{[2]}) \right] + \frac{3}{2} \sum_{L=\frac{1}{4}} \left[ \frac{I_{K_{1,2}K_3}^{L} I_{H_1 H_2 H_3}^{L}}{d_L} \otimes (Y_{0}^{[2]} \otimes Y_{1}^{[2]}) \right].
\end{align}

Therefore Eq. (79) is equivalent to

\begin{align}
(P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} + (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} = Y^{[2]}_{0,\frac{3}{2}}, \\
(P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} + (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} = Y^{[2]}_{0,\frac{3}{2}}, \\
(P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} + (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} = Y^{[2]}_{0,\frac{3}{2}}, \\
(P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} + (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} = Y^{[2]}_{0,\frac{3}{2}}, \\
(P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} + (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} = Y^{[2]}_{0,\frac{3}{2}}, \\
(P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} + (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}}) Y^{[3]}_{0} (P^{[3]}_{\frac{3}{2},0} \otimes P^{[4]}_{0,\frac{1}{2}})^{\dagger} = Y^{[2]}_{0,\frac{3}{2}},
\end{align}

where

\begin{align}
P^{[3]}_{\frac{3}{2},0} & := |0\rangle_{\frac{3}{2}} \langle 0|_{\frac{3}{2}}, \\
P^{[3]}_{\frac{3}{2},0} & := |0\rangle_{\frac{3}{2}} \langle 1|_{\frac{3}{2}}, \\
P^{[3]}_{\frac{3}{2},0} & := |0\rangle_{\frac{3}{2}} \langle 1|_{\frac{3}{2}}.
\end{align}

We have

\begin{align}
\text{Tr}_{\mathcal{H}_2} Y^{[2]} = \frac{1}{2} \sum_{J=0} \sum_{L=\frac{1}{4}} I_{K_{1,2}K_3}^{J} \otimes \left[ \frac{I_{H_1 H_2}^{L} Y_{0}^{[3]} P^{[3]}_{0,\frac{3}{2}}}{d_{L}} + \frac{I_{H_1 H_2}^{L}}{d_{L}} \otimes \left\{ \langle P^{[3]}_{0,\frac{3}{2}} Y^{[2]} P^{[3]}_{0,\frac{3}{2}} \rangle + \langle P^{[3]}_{0,\frac{3}{2}} Y^{[2]} P^{[3]}_{0,\frac{3}{2}} \rangle \right\} \right],
\end{align}

\begin{align}
I_{K_{1,2}K_3} \otimes Y^{[1]} = \frac{1}{2} \sum_{J=0} \sum_{L=\frac{1}{4}} I_{K_{1,2}K_3}^{J} \otimes \left[ \frac{I_{H_1 H_2}^{L} Y_{0}^{[3]} P^{[3]}_{0,\frac{3}{2}}}{d_{L}} + \frac{I_{H_1 H_2}^{L}}{d_{L}} \otimes \left\{ \langle P^{[3]}_{0,\frac{3}{2}} Y^{[2]} P^{[3]}_{0,\frac{3}{2}} \rangle + \langle P^{[3]}_{0,\frac{3}{2}} Y^{[2]} P^{[3]}_{0,\frac{3}{2}} \rangle \right\} \right].
\end{align}

Therefore Eq. (80) is equivalent to

\begin{align}
P^{[3]}_{\frac{3}{2},0} Y^{[2]} P^{[3]}_{0,\frac{3}{2}} Y^{[1]} = 0, \\
P^{[3]}_{\frac{3}{2},0} Y^{[2]} P^{[3]}_{0,\frac{3}{2}} Y^{[1]} = Y^{[1]}.
\end{align}
for \( J = 0, 1 \). For Eq. (81), we have

\[
Y_0^{(1)} + Y_1^{(1)} = 1. \tag{113}
\]

B. Concurrency Pattern (129)

The SDP for Concurrency Pattern (129) is

\[
\begin{align*}
\text{maximize} & \quad p_{\text{succ}} = \frac{1}{2} \text{Tr} \left[ M_1^{(j)} \tilde{\Pi}_1 + M_2^{(j)} \tilde{\Pi}_2 \right] \\
\text{subject to} & \quad \tilde{\Pi}_i \geq 0, \ i = 1, 2 \\
& \quad \tilde{\Pi}_1 + \tilde{\Pi}_2 = I_{K_3K_4} \otimes Y^{(2)} \\
& \quad \text{Tr}_{H_3H_4} Y^{(2)} = I_{K_3K_4} \otimes Y^{(1)} \\
& \quad \text{Tr}Y^{(1)} = 1, 
\end{align*} \tag{115}
\]

We rewrite Eqs. (115) - (117) in terms of the multiplicity subspaces. We may assume without loss of generality that

\[
\tilde{\Pi}_i = \bigoplus_{J,L=0}^2 I_J^K \otimes \frac{I_L^H}{d_L} \otimes \tilde{\Pi}_i^{(i)} , \tag{118}
\]

\[
Y^{(2)} = \bigoplus_{j=0}^2 \bigoplus_{L=0}^2 I_J^{K_1K_2} \otimes \frac{I_L^H}{d_L} \otimes Y_{jL}^{(1)} , \tag{119}
\]

\[
Y^{(1)} = \bigoplus_{L=0}^2 \frac{I_L^{K_1H_2}}{d_L} \otimes Y^{(1)}_{jL} , \tag{120}
\]

for \( i = 1, 2 \). The positivity condition (114) is equivalent to

\[
\tilde{\Pi}_i^{(i)} \geq 0, \tag{121}
\]

for \( i = 1, 2 \) and \( J, L = 0, 1, 2 \). We have.

\[
\tilde{\Pi}_1 + \tilde{\Pi}_2 = \bigoplus_{J,L=0}^2 I_J^K \otimes \frac{I_L^H}{d_L} \otimes (\tilde{\Pi}_1^{(1)} + \tilde{\Pi}_2^{(2)}), \tag{122}
\]

\[
I_{K_3K_4} \otimes Y^{(2)} = \bigoplus_{j=0}^2 \bigoplus_{L=0}^2 I_J^0 \otimes \frac{I_L^H}{d_L} \otimes (Y_{0L}^{(2)} \oplus Y_{1L}^{(2)}) \tag{123}
\]

\[
\oplus \bigoplus_{j=0}^2 I_J^K \otimes \frac{I_L^H}{d_L} \otimes (Y_{0L}^{(2)} \oplus Y_{1L}^{(2)} \oplus Y_{2L}^{(2)}) \tag{124}
\]

\[
\oplus \bigoplus_{j=0}^2 I_J^2 \otimes \frac{I_L^H}{d_L} \otimes (Y_{1L}^{(2)}). \tag{125}
\]

Thus Eq. (115) is represented as is equivalent to

\[
(P_{3,1}^3 P_{0,1}^4) (\tilde{\Pi}_1^{(1)}) (\tilde{\Pi}_0^{(2)}) (P_{0,1}^4 P_{3,1}^3) \otimes I_{V_{L}^{(4)}} = Y_{0L}^{(2)} \tag{126}
\]

\[
(P_{3,0}^3 P_{0,2}^4) (\tilde{\Pi}_1^{(1)}) (\tilde{\Pi}_0^{(2)}) (P_{0,2}^4 P_{3,0}^3) \otimes I_{V_{L}^{(4)}} = Y_{1L}^{(2)} \tag{127}
\]

\[
(P_{3,0}^3 P_{1,2}^4) (\tilde{\Pi}_1^{(1)}) (\tilde{\Pi}_1^{(2)}) (P_{1,2}^4 P_{3,0}^3) \otimes I_{V_{L}^{(4)}} = Y_{0L}^{(2)} \tag{128}
\]

\[
(P_{3,1}^3 P_{1,2}^4) (\tilde{\Pi}_1^{(1)}) (\tilde{\Pi}_2^{(2)}) (P_{1,2}^4 P_{3,1}^3) \otimes I_{V_{L}^{(4)}} = Y_{1L}^{(2)} \tag{129}
\]

\[
(P_{3,0}^3 P_{2,2}^4) (\tilde{\Pi}_1^{(1)}) (\tilde{\Pi}_2^{(2)}) (P_{2,2}^4 P_{3,0}^3) \otimes I_{V_{L}^{(4)}} = Y_{1L}^{(2)} \tag{130}
\]

\[
(P_{3,1}^3 P_{2,2}^4) (\tilde{\Pi}_2^{(1)}) (\tilde{\Pi}_2^{(2)}) (P_{2,2}^4 P_{3,1}^3) \otimes I_{V_{L}^{(4)}} = Y_{1L}^{(2)} \tag{131}
\]
for $L = 0, 1, 2$. We have

$$
\text{Tr}_{H_3 H_4} Y^{(2)} = \bigoplus_{J=0}^1 I_J^{K_1 K_2} \otimes I_0^{H_1 H_2} \otimes \left[ P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{0,2}^{(2)} P_{0,2}^{[4]} + P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{0,2}^{(2)} P_{0,2}^{[4]} \right]
$$

(132)

$$
\otimes \bigoplus_{J=0}^1 I_J^{K_1 K_2} \otimes \frac{I_0^{H_1 H_2}}{d_1} \otimes \left[ P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{0,2}^{(2)} P_{0,2}^{[4]} + P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{0,2}^{(2)} P_{0,2}^{[4]} \right]
$$

$$
+ P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{1,2}^{(2)} P_{0,2}^{[4]} + P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{1,2}^{(2)} P_{0,2}^{[4]} \right],
$$

(133)

and

$$
I_K^{K_1 K_2} \otimes Y^{(1)} = \bigoplus_{J=0}^1 I_J^{K_1 K_2} \otimes \left[ I_0^{H_1 H_2} \otimes Y_0^{(1)} + \frac{I_0^{H_1 H_2}}{d_1} \otimes Y_1^{(1)} \right].
$$

(134)

Therefore Eq. (116) is equivalent to

$$
P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{0,2}^{(2)} P_{0,2}^{[4]} + P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{0,2}^{(2)} P_{0,2}^{[4]} = Y_0^{(1)}
$$

(135)

$$
P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{1,2}^{(2)} P_{0,2}^{[4]} + P_{0,2}^{[4]} P_{0,2}^{[4]} Y_{1,2}^{(2)} P_{0,2}^{[4]} = Y_1^{(1)}
$$

(136)

for $J = 0, 1$. Eq. (116) is equivalent to

$$
Y_0^{(1)} + Y_1^{(1)} = 1.
$$

(137)

C. Concurrency Pattern (130)

The SDP for Concurrency Pattern (130) is

$$\text{maximize } p_{\text{succ}} = \frac{1}{2} \text{Tr} \left[ M_1^{(j)} \Pi_1 + M_2^{(j)} \Pi_2 \right]
$$

subject to

$$\Pi_i \geq 0, \ i = 1, 2
$$

(138)

$$\Pi_1 + \Pi_2 = I_{K_1} \otimes Y^{(2)}
$$

(139)

$$\text{Tr}_{H_3} Y^{(2)} = I_{K_1 K_2 K_3} \otimes Y^{(1)}
$$

(140)

$$\text{Tr} Y^{(1)} = 1.
$$

(141)

We rewrite Eqs. (139) - (141) in terms of the multiplicity subspaces. We can assume without loss of generality that

$$
\Pi_i = \bigoplus_{J,L=0}^1 I_J^{K_1 K_2} \otimes \frac{I_L^{H_1 H_2}}{d_L} \otimes \Pi_{J,L}^{(i)}
$$

(142)

$$
Y^{(2)} = \bigoplus_{J,L=0}^1 I_J^{K_1 K_2 K_3} \otimes \frac{I_L^{H_1 H_2}}{d_L} \otimes Y_{J,L}^{(2)}
$$

(143)

$$
Y^{(1)} = \bigoplus_{L=0}^1 I_L^{H_1 H_2} \otimes Y_{L}^{(1)}.
$$

(144)

The positivity condition (138) is equivalent to

$$
\Pi_{J,L}^{(i)} \geq 0,
$$

(145)

for $i = 1, 2$ and $J, L = 0, 1, 2$. Equation (139) is same as Eq. (78) in Configuration 1. For Eq. (140), the LHS is the same as the on in Eq. (98) and we have

$$
I_{K_1 K_2 K_3} \otimes Y^{(1)} = \bigoplus_{J=rac{1}{2}}^1 \bigoplus_{L=rac{1}{2}}^1 I_J^{K_1 K_2} \otimes \frac{I_L^{H_1 H_2}}{d_L} \otimes (I_{Y_{J,L}}^{[3]} \otimes Y_{L}^{(1)}).
$$

(146)
Therefore Eq. (140) is equivalent to

\[
I_{V_j^3} \otimes Y^{(1)}_2 = (I_{V_j^3} \otimes P_{0, \frac{1}{2}}^4) Y^{(2)}_1 (I_{V_j^3} \otimes P_{0, \frac{1}{2}}^4) + (I_{V_j^3} \otimes P_{0, \frac{1}{2}}^4) Y^{(2)}_1 (I_{V_j^3} \otimes P_{0, \frac{1}{2}}^4)
\]

(147)

\[
I_{V_j^3} \otimes Y^{(1)}_2 = (I_{V_j^3} \otimes P_{1, \frac{2}{3}}^4) Y^{(2)}_1 (I_{V_j^3} \otimes P_{1, \frac{2}{3}}^4) + (I_{V_j^3} \otimes P_{1, \frac{2}{3}}^4) Y^{(2)}_1 (I_{V_j^3} \otimes P_{1, \frac{2}{3}}^4)
\]

(148)

for \( J = \frac{1}{2}, \frac{3}{2} \). Equation (141) is equivalent to

\[
\sum_{L=\frac{1}{2}}^3 \text{Tr} Y^{(1)}_L = 1.
\]

(149)

**D. Concurrency Pattern (131)**

The SDP for Concurrency Pattern (131) is

\[
\begin{align*}
\text{maximize} & \quad p_{\text{succ}} = \frac{1}{2} \text{Tr} \left[ M^{(j)} \tilde{\Pi}_1 + M^{(j)} \tilde{\Pi}_2 \right] \\
\text{subject to} & \quad \tilde{\Pi}_i \geq 0, \quad i = 1, 2 \\
& \quad \tilde{\Pi}_1 + \tilde{\Pi}_2 = I_{K_1K_2K_3K_4} \otimes X \\
& \quad \text{Tr} X = 1
\end{align*}
\]

(150)

(151)

(152)

We rewrite Eqs. (151) - (152) in the multiplicity subspaces. We may assume without loss of generality that

\[
\tilde{\Pi}_i = \bigoplus_{J,L=0}^2 I^r_J \otimes \frac{I^H_L}{d_L} \otimes \tilde{\Pi}^{(i)}_{JL}
\]

(153)

\[
X = \bigoplus_{L=0}^2 \frac{I^H_L}{d_L} \otimes X_L.
\]

(154)

The positivity condition (150) is equivalent to

\[
\tilde{\Pi}^{(i)}_{JL} \geq 0,
\]

(155)

for \( i = 1, 2 \) and \( J, L = 0, 1, 2 \). Equation (151) is equivalent to

\[
\tilde{\Pi}^{(1)}_{JL} + \tilde{\Pi}^{(2)}_{JL} = I_{V_j^4} \otimes X_L
\]

(156)

for \( J, L = 0, 1, 2 \) and Eq. (152) is to

\[
\sum_{L=0}^2 \text{Tr} X_L = 1.
\]

(157)