New integrable string-like fields in 1+1 dimensions

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Abstract

The symmetry classification method is applied to the string-like scalar fields in two-dimensional space-time. When the configurational space is three-dimensional and reducible we present the complete list of the systems admitting higher polynomial symmetries of the 3rd, 4th and 5th-order.

1 Introduction

It is known that many integrable by the inverse scattering method nonlinear systems possess infinitely many higher symmetries. And vice versa, if a nonlinear system admits higher symmetries then it is integrable as a rule. Therefore, search of the higher symmetries is one of the methods for the classification of nonlinear integrable systems. In this report we deal with the higher local symmetries for the string-like two-dimensional scalar fields:

\[ L = \frac{1}{2}g_{\alpha\beta}(u)u_\alpha^\beta u_\nu^\mu + f(u). \]  

(1)

Here \( f \) and \( g_{\alpha\beta} = g_{\beta\alpha} \) are some differentiable functions, \( \alpha = 1, \ldots, m \), and \( \det(g_{\alpha\beta}) \neq 0 \). Summation over the repeated indices is implied throughout the article. The field equations take the following form

\[ u_\alpha^\mu + \Gamma^\alpha_{\nu\mu}(u)u_\nu^\nu u_\mu^\mu = f^\alpha(u), \]  

(2)

where \( \Gamma^\alpha_{\nu\mu} \) are the Christoffel symbols for the metric \( g_{\alpha\beta} \), \( f^\alpha = g^\alpha_\beta f_\beta \), \( f_\alpha = \partial_\alpha f = \partial f/\partial u^\alpha \).

We assume that the Riemann tensor

\[ R^\alpha_{\nu\beta\mu} = \partial_\beta\Gamma^\alpha_{\nu\mu} - \partial_\mu\Gamma^\alpha_{\nu\beta} + \Gamma^\lambda_{\beta\mu}\Gamma^\alpha_{\nu\lambda} - \Gamma^\alpha_{\mu\lambda}\Gamma^\lambda_{\beta\nu} \]

of the configurational space \( \mathbb{V}_m \) does not vanish. The higher symmetries and higher conservation laws for the the systems (2) were studied in [1] and [2]. In the first article the polynomial conserved densities of the system (2) in \( \mathbb{V}_2 \) were studied in [1] and [2]. In the article [3] the structure of the polynomial higher symmetries of the system (2) in \( \mathbb{V}_m \) was investigated, and it was proved that the systems (2) in \( \mathbb{V}_2 \) admitting the polynomial higher symmetries are the same as in [1]. Moreover some preliminary results were obtained for the reducible space \( \mathbb{V}_3 \) (\( w^\alpha = \{u, v, w\} \)):

\[ ds^2 = du^2 + 2\psi(v, w)dvdw \]  

(3)

Here we present the systems (2) in the space (3) admitting Lie-Bäcklund symmetries of the 3rd, 4th and 5th-order.
2 General consideration

Writing out the defining equation for the Lie-Bäcklund symmetries (see [3], for example), one can rewrite it in the following matrix form:

\[(VW + \Phi)\sigma = 0.\]  \hfill (4)

Here \(\sigma\) is the Lie-Bäcklund vector field (the symmetry), \(\Phi_\beta = R^\alpha_{\nu\beta}\mu u^\mu_1 u^\nu_2 - f^\alpha_{\nu\beta},\) and

\[V^\alpha_\beta = \delta^\alpha_\beta D_x + \Gamma^\alpha_{\beta\nu} u^\nu_x, \quad W^\alpha_\beta = \delta^\alpha_\beta D_t + \Gamma^\alpha_{\beta\nu} u^\nu_t.\]  \hfill (5)

The semicolon denotes the covariant differentiation on \(\mathbb{V}_m\). Symbols \(D_x\) and \(D_t\) are the total differentiation operators, for example,

\[D_x F(x, u, u_x, u_t) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u^\alpha} u^\alpha_x + \frac{\partial F}{\partial u^\alpha} u^\alpha_{xx} + \frac{\partial F}{\partial u^\alpha} (f^{\alpha} - \Gamma^{\alpha}_{\mu\nu} u^\nu u^\mu).\]

Let us consider the manifold \(\mathbb{M}\) on the jet space corresponding to the system (2). It is obvious that quantities \(t, x, u^\alpha, \tilde{u}^\alpha\) are independent variables on \(\mathbb{M}\), and quantities \(u^\alpha_{ij} = \partial_x^i \partial_t^j u^\alpha\) are dependent.

One of the authors proved earlier [2] the following statements:

**Theorem 1** Any symmetry \(\sigma\) of the system (3) independent on \(x, t\) takes the following form:

\[\sigma(u, \tilde{u}) = \tau(u) + \omega(\tilde{u}),\]

where the vector fields \(\tau\) and \(\omega\) satisfy the following equations:

\[A_\nu(u)\tau^\alpha + R^\alpha_{\beta\gamma\nu} u^\beta_1 \tau^\gamma = p^\alpha_{\nu}(u_0) - h^\alpha_{\beta\nu}(u_0) u^\beta_1, \quad B(u)\tau^\nu - f^\alpha_{\nu\gamma} \tau^\gamma = q^\alpha_{\nu}(u_0) u^\nu - g^\alpha(u_0),\]

\[A_\nu(\tilde{u})\omega^\alpha + R^\alpha_{\beta\gamma\nu} \tilde{u}^\beta_1 \omega^\gamma = h^\alpha_{\nu\beta}(u_0) \tilde{u}^\beta_1 - q^\alpha_{\nu}(u_0), \quad B(\tilde{u})\omega^\nu - f^\alpha_{\nu\gamma} \omega^\gamma = g^\alpha(u_0) - p^\alpha_{\nu}(u_0) \tilde{u}^\nu,\]

where \(A_\nu\) and \(B\) are some linear differential operators.

**Theorem 2** If \(\sigma(u, \tilde{u}) = \tau(u) + \omega(\tilde{u})\) is a polynomial of \(u_i\) and \(\tilde{u}_i\), then \(\tau(u)\) and \(\omega(\tilde{u})\) are independent symmetries.

Therefore we may investigate \(\tau(u)\) only.

**Theorem 3** If the system (3) admits the polynomial symmetry \(\tau(u)\), then it admits the homogeneous polynomial symmetries:

\[\tau(u) = a^\alpha_{\beta;\gamma} u^\beta_n + A^\alpha_{\beta\gamma} u^\beta_1 u^\gamma_1 + (B^\alpha_{\beta\gamma} u^\beta_2 + C^\alpha_{\beta\gamma\nu} u^\nu_1 u^\gamma_1) u^\beta_3 + \ldots,\]  \hfill (6)

where the coefficients depend on the variables \(u^\nu\) only.

**Theorem 4** If the symmetry (2) is admitted, then the following system

\[a^\alpha_{\beta;\gamma} = 0, \quad X^\alpha_{\beta\gamma;\nu} = a^\alpha_{\sigma} R^\sigma_{\beta\gamma\nu}, \quad Y^\alpha_{\beta\gamma;\nu} = a^\sigma_{\beta} R^\alpha_{\gamma\nu\sigma},\]  \hfill (7)

must be solvable.

Here the tensors \(X\) and \(Y\) depend on the coefficients \(A\) and \(B\) from (3). We have not the general solution of the system (3) for any \(\mathbb{V}_m\). But the case of \(\mathbb{V}_2\) is investigated completely.

**Theorem 5** There are two and only two spaces \(\mathbb{V}_2\) where the system (3) has the nontrivial solution. They have the following metrics

\[(a) \quad ds^2 = 2\frac{dv dw}{vw + c}, \quad (b) \quad ds^2 = 2\frac{dv dw}{v + w},\]  \hfill (8)

where \(c \neq 0\) is a constant.

**Theorem 6** System (3) has the nontrivial solution in the reducible space \(\mathbb{V}_3\) if and only if the two-dimensional part of the metric (3) is one of the metric (8).
3 Systems admitting higher symmetries

In accordance with the results of the section 2 we are going to consider the field models with the following Lagrangian

$$L = \frac{1}{2} [u_t u_x + \psi(v,w)(v_t w_x + v_x w_t)] + f(u, v, w),$$

where $\psi = (vw + c)^{-1}$ or $\psi = (v + w)^{-1}$. The field equations take the following form:

$$u_{tx} = f_u, \quad v_{tx} = (f_w - \psi v_t v_x)/\psi, \quad w_{tx} = (f_v - \psi w_t w_x)/\psi,$$

where the subscripts of $f$ and $\psi$ denote the partial derivatives.

Let us mention from the first, that any system (10) where $\psi$ is arbitrary function, $f = g(v,w) \exp( ku )$, $k = \text{const}$, and $g$ is arbitrary function, admits the following higher symmetry:

$$\sigma^u = (D_x + ku_1)F, \quad \sigma^v = kv_1 F, \quad \sigma^w = kw_1 F, \quad D_t F = 0.$$

The equation $D_t F = 0$ is the unique constraint for $F$. It admits infinitely many solutions and the simplest integral takes the following form:

$$F = u_2 - k\psi v_1 w_1 - (k/2)u_1^2.$$

It is obvious that arbitrary function $\Phi(x,F,D_x F,D_x^2 F,\ldots)$ is the integral too. But there exist another integrals with higher order that are not expressed by the above formula. All these symmetries do not lead to the integrability and we shall not consider them below. Moreover we did not consider the case $f(u, v, w) = f_1(u) + f_2(v, w)$ as the independent equation appear in the system (10). Calculations of the higher symmetries are very cumbersome therefore we used the computer, and we did not calculate the symmetries with the order more than 5.

There are no the systems (10) admitting the 2nd-order symmetry. If $\psi = (v + w)^{-1}$ then the system (10) does not also admit nontrivial symmetries of the 3rd, 4th or 5th order.

If $\psi = (vw + c)^{-1}$, then the following general form of the 3rd-order symmetry follows from the equation (10):

$$\sigma^u = a_1 u_3 + (v_2 w_1 c_2 + c_3 w_2 v_1) \psi + c_6 u_1^3 + c_7 \psi u_1 v_1 w_1 - \psi^2 c_3 w_1^2 v_1 w_1 - \psi^2 c_2 v_1 w_1^2,$$

$$\sigma^v = a_2 v_3 + u_2 c_4 v_1 + v_2 (c_5 u_1 - 3a_2 v_1 w_1) + u_1^2 c_8 v_1$$
$$-2c_5 v_1 w_1 v_1 w_1 + c_9 v_1^2 w_1 + 3a_2 v_1^2 v_1 w_1^2,$$

$$\sigma^w = a_2 w_3 + u_2 (c_4 - c_3) v_1 - w_2 (c_5 u_1 + 3a_2 v_1^2 w_1) + u_1^2 c_8 w_1$$
$$+2w_5 v_1 w_1 w_1 + 3w_1^2 v_1^2 v_1 w_1 + c_9 v_1 w_1^2,$$

where $a_i$ and $c_i$ are constants. Substituting these functions into the equation (10) we obtained about 60 equations for the function $f$. Here is the solutions:

$$f = av \exp(\sqrt{2}u) + bw \exp(-\sqrt{2}u),$$

$$f = av^2 \exp(2u) + bw^2 \exp(-2u),$$

$$f = av^2 \exp(2u) + bw \exp(-u),$$

$$f = av \exp(u) + bw \exp(-u),$$

$$f = (vw + c/2)[a \exp(\sqrt{2}u) + b \exp(-\sqrt{2}u)],$$

$$f = a(vw + c/2) \exp(\sqrt{2}u) + b \exp(-\sqrt{2}u),$$

$$f = a(vw + c/2) \exp(\sqrt{2}u) + b \exp(-2\sqrt{2}u).$$
The system (10) corresponding to the function (13) admits 2-parametric 3rd-order symmetry with arbitrary \( a_1 \) and \( a_2 \). Other systems admit unique 3rd-order symmetry, with \( a_2 \neq 0 \). But when \( a = 0 \) or \( b = 0 \), then an additional 3rd-order symmetry appear for each system. Two systems corresponding to the functions (14) and (16) admit the 4th-order symmetry.

The system (10) corresponding to the function (11) with \( b = 0 \) admits the 7-parametric 5th-order symmetry and does not admit it when \( ab \neq 0 \). The systems (10) corresponding to the functions (12), (14) or (16) with \( b = 0 \) admit the 4-parametric 5th-order symmetries and do not admit them when \( ab \neq 0 \). The system (10) corresponding to the function (16) with \( ab \neq 0 \) admits the unique 5th-order symmetry. Besides there are only two new systems admitting the 5th-order symmetries. The systems (10) corresponding to the functions

\[
\begin{align*}
  f &= a(v^2w + 2vc/3)\exp(\sqrt{2}u), \\
  f &= av\exp(\sqrt{2}/3u)
\end{align*}
\]

admit the 2-parametric 5th-order symmetries.

We believe that the systems corresponding to the functions (11) – (19) are integrable by the inverse scattering method and we are going to find the zero curvature representations for these systems.

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References

[1] Getmanov B. S. Theor. Group Meth. Phys. Proc. Int. Sem. Zvenigorod, 22–26 November, 1982. Moskow, Nauka, 1983, v.2, 333.

[2] Meshkov A.G. On the symmetry of the two-dimensional scalar fields of chiral type. Preprint no. 28. Tomsk Scientific Center. Tomsk, 1991.

[3] Ibragimov N. H. Transformation Groups Applied to Mathematical Physics. Moscow, Nauka, 1983. (English translation published by Reidel, Dordrecht, 1985).