Research Article

2-Quasitotal Fuzzy Graphs and Their Total Coloring

V. N. Srinivasa Rao Repalle and Fekadu Tesgera Agama

Department of Mathematics, College of Natural and Computational Science, Wollega University, Nekemte, Ethiopia

Correspondence should be addressed to V. N. Srinivasa Rao Repalle; rvnrepalle@gmail.com

Received 20 July 2020; Revised 15 October 2020; Accepted 21 October 2020; Published 21 November 2020

Academic Editor: Katsuhiro Honda

Copyright © 2020 V. N. Srinivasa Rao Repalle and Fekadu Tesgera Agama. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Coloring of fuzzy graphs has many real-life applications in combinatorial optimization problems like traffic light system, exam scheduling, and register allocation. The coloring of total fuzzy graphs and its applications are well studied. This manuscript discusses the description of 2-quasitotal graph for fuzzy graphs. The proposed concept of 2-quasitotal fuzzy graph is explicated by several numerical examples. Moreover, some theorems related to the properties of 2-quasitotal fuzzy graphs are stated and proved. The results of these theorems are compared with the results obtained from total fuzzy graphs and 1-quasitotal fuzzy graphs. Furthermore, it defines 2-quasitotal coloring of fuzzy total graphs and which is justified.

1. Introduction

As of its emerging, the graph theory rapidly moved into the mainstream of mathematics. It has diverse applications in the fields of science and technology [1, 2]. In 1965, the total coloring of the graph was introduced by Behzad [3], which is followed by Harary, who contributed the concept of total graphs [4]. Jayaram studied the total chromatic number of total graphs [5]. Besides, Sastry and Raju defined quasitotal graphs [6], and Srinivasa Rao and Rao introduced 1-quasitotal graphs and bounds for its total chromatic number [7]. Nowadays, many real-world problems cannot be properly modeled by a crisp graph theory as the problems contain uncertain information. The fuzzy set theory, anticipated by Zadeh [8], is used to handle the phenomena of uncertainty and real-life situation. Coloring of fuzzy graphs plays a vital role in both theory and practical applications. It is mainly studied in combinatorial optimization problems such as traffic light control, exam scheduling, and register allocation.

After Zadeh’s paper on fuzzy sets [9], Rosenfeld introduced fuzzy graphs [10]. Later, Bhattacharya [11] gave some remarks on fuzzy graphs. Some operations on fuzzy graphs were introduced by Mordeson and Peng [12]. As an advancement, the fuzzy coloring of the fuzzy graph was defined by Eslahchi and Onagh in 2004 and later developed by themselves as fuzzy vertex coloring in 2006 [13]. Lavanay and Sattanathan extended the concept of fuzzy vertex coloring into a family of fuzzy sets [14]. Kavitha [15] defined the total fuzzy graph and studied the total chromatic number of total graphs of fuzzy graphs [1]. Kavitha derived fuzzy chromatic numbers for various graphs of complete fuzzy graphs [15]. Nevethana studied about fuzzy total coloring and its chromatic number of complete fuzzy graphs [16]. Sitara and Akram studied fuzzy graph structures and their applications [17]. The total coloring of 1-quasitotal graph for crisp graph was studied. Recently Fekadu and Srinivasa Rao Repalle have established the definition of 1-quasitotal fuzzy graph and its total coloring [18]. Koam and Akram described decision making analysis in the real-life applications like marine crimes and road crimes by using graph structures [19]. Akram and Sitara introduced the concept of Residue Product of Fuzzy Graph Structures and studied their properties [20]. Akram covers both theories and applications of an introduction to m-polar fuzzy graphs and m-polar fuzzy hypergraphs [21].

This paper is being organized as follows: In Section 2, some basic definitions and elementary concepts of the fuzzy set, fuzzy graph, and coloring of fuzzy graphs have been reviewed. In Section 3, 2-quasitotal fuzzy graph is defined and the concept is justified with numerous examples. Section
4 describes and proves some properties of 2-quasitotal fuzzy graphs and compares the result with the properties of total fuzzy graphs and 1-quasitotal fuzzy graphs. Furthermore, Section 5 introduces the concept of 2-quasitotal fuzzy coloring and elaborates some of its properties. Finally, the paper is concluded in Section 6.

2. Preliminaries

In this section, some basic definitions that are necessary for this paper have been included. Unless otherwise mentioned, the concepts are from Mordeson and Nair (see [22]).

Definition 1. Fuzzy Graph

A fuzzy graph is defined as an ordered triple $G = (V, \sigma, \mu)$, where $V$ is the set of vertices $\{v_1, v_2, \ldots, v_n\}$, $\sigma$ is a fuzzy subset of $V$, such that $\sigma : V \rightarrow [0, 1]$ and $\mu$ is a fuzzy relation on $\sigma$ with $\mu : V \times V \rightarrow [0, 1]$ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v) \forall u, v \in V$.

Definition 2. Crisp Graph

The underlying crisp graph of the fuzzy graph $G = (V, \sigma, \mu)$ is denoted by $G^* = (V, E)$, where $E \subseteq V \times V$. The crisp graph $(V, E)$ is a special fuzzy graph $G$ with each vertex, and each edge of $G$ has the same degree of membership equal to 1.

Definition 3. Order and Size of Fuzzy Graph

Let $G = (V, \sigma, \mu)$ be a fuzzy graph with the underlying set $V$. Then, the order of $G$ denoted by Order $(G)$ is defined as follows:

$$\text{Order}(G) = \sum_{u \in V} \sigma(u),$$

and the size of $G$ denoted by Size $(G)$ and defined as follows:

$$\text{Size}(u) = \sum_{u, v \in V} \mu(u, v).$$

Definition 4. Degree of a Vertex

Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The degree of a vertex $u \in V$ is defined as follows:

$$d_G(u) = \sum_{v \neq u, v \in V} \mu(u, v).$$

Definition 5. Busy Value of a Vertex

Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The busy value of the vertex $v$ in $G$ is $D(v) = \sum \sigma(v) \wedge \sigma(v_i)$ where $v_i$ are neighbors of $v$ and the busy value of $G$ is $D(G) = \sum D(v_i)$ where $v_i$ are the vertexes of $G$.

Definition 6. Adjacent Vertices

If $\mu(u, v) > 0$, then $u$ and $v$ are said to be adjacent to each other and lie on the edge, $e = (u, v)$. A path $p$ in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct nodes $v_0, v_1, v_2, \ldots, v_n$ such that $\mu(v_{i-1}, v_i) > 0$, $1 \leq i \leq n$. Here $n$ is called the length of the path.

Definition 7 (see [23]). Path in Fuzzy Graph

A path $P$ in a fuzzy graph $G = (\sigma, \mu)$ is a sequence of distinct vertices $u_0, u_1, \ldots, u_n$ (except possibly $u_0$ and $u_n$) such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, \ldots, n$. Here, $n$ is called the length of the path.

Definition 8. Connected Vertices

If $u, v$ are vertices in $G$ and if they are connected by means of a path, then the strength of that path is defined as $\land_{i=1}^{n} \mu(v_{i-1}, v_i)$. If $u, v$ are connected by means of paths of length $k$, then

$$\mu^k(u, v) = \sup\{\mu(u, v_1) \wedge \mu(v_1, v_2) \wedge \mu(v_2, v_3) \wedge \cdots \wedge \mu(v_{k-1}, v) : u, v_1, v_2, \ldots, v_{k-1}, v \in V\},$$

(4)

If $u, v \in V$, then, the strength of connectedness between $u$ and $v$, $\mu^{\infty}(u, v) = \sup\{\mu^k(u, v) : k = 1, 2, \ldots\}$

Definition 9. Connected Fuzzy Graph

Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then, $G$ is said to be connected if $\mu^{\infty}(u, v) > 0$ for all $u, v \in \sigma$. An arc $(u, v)$ is said to be a strong arc if $\mu(u, v) \geq \mu^{\infty}(u, v)$ and a node $u$ is said to be an isolated node, if $\mu(u, v) = 0$, for all $u \neq v$.

Definition 10 (see [24]). Cyclic Fuzzy Graph

$G = (V, \sigma, \mu)$ is a cyclic fuzzy if and only if $(\sigma^*, \mu^*)$ is a cycle and there does not exist a unique $(x, y) \in \mu^*$ such that $\mu(x, y) = \Lambda \{\mu(u, v) : (u, v) \in \mu^*\}$.

Definition 11 (see [25]). Total Coloring

A family $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k\}$ of fuzzy sets on $V \cup E$ is called a $\mu$-fuzzy total coloring of $G = (V, \sigma, \mu)$, if

(a) $\text{Max}\{\gamma_i(v) : \forall v \in V\} = \sigma(v)$ for all $v \in V$ and $\text{Max}\{\gamma_i(u, v) : \forall u, v \in E\} = \mu(u, v)$ for all edges $(u, v) \in E$

(b) $\gamma_i \Lambda \gamma_j = 0$

(c) For every adjacent vertex $u, v$ of $G$, $\text{Min}\{\gamma_i(u), \gamma_i(v)\} = 0$

The least value of $k$ for which there exists a $\mu$-fuzzy coloring is called the fuzzy total chromatic number of $G$ and is denoted by $\chi_T(G)$.

Definition 12 (see [18]). 1-Quasitotal Fuzzy Graph

Let $G = (V, \sigma, \mu)$ be a fuzzy graph with an underlying set $V$ and crisp graph $G^* = (\sigma^*, \mu^*)$. The pair $Q_T^*(G) = (\sigma_{Q_T}, \mu_{Q_T})$ of the fuzzy graph $G$ is defined as follows:

Let the node set of $Q_T(G)$ be $V \cup E$, where $V$ is the vertex set and $E$ is the edge set of the underlying crisp graph. The fuzzy subset $\sigma_{Q_T}$ is defined on $V \cup E$ as follows:

$$\sigma_{Q_T}(u) = \sigma(u), \quad \text{if } u \in V,$$

$$\sigma_{Q_T}(e) = \mu(e), \quad \text{if } e \in E.$$

(5)

The fuzzy relation $\rho_{Q_T}$ is defined on $(V \cup E) \times (V \cup E)$, called edges of $Q_T^*(G)$ as follows:
\[ \mu_{Q,T}(u,v) = \mu(u,v), \quad \text{if } u,v \in V \]
\[ \mu_{Q,T}(e_i,e_j) = \mu(e_i) \wedge \mu(e_j), \quad \text{if } e_i \text{ and } e_j \text{ have a node in common between them} \]
\[ = 0, \quad \text{Otherwise.} \]

By definition, \( \mu_{Q,T}(u,v) \leq \sigma_{Q,T}(u) \wedge \sigma_{Q,T}(v) \) for all \( u,v \in V \cup E \). Hence, \( \mu_{Q,T} \) is a fuzzy relation on the fuzzy subset \( \sigma_{Q,T} \). Thus, the pair \( Q,T \) is a fuzzy graph, and it is termed as 1-Quasitotal fuzzy graph of fuzzy graph. By definition of the fuzzy graph, \( \mu_{Q,T} \) is a fuzzy relation on the fuzzy edge set \( \sigma_{Q,T} \). Therefore, the pair \( Q,T \) is a fuzzy graph, and it is termed as 2-Quasitotal fuzzy graph of fuzzy graph.

### 3.2-Quasitotal Fuzzy Graph

This section introduces the definition of 2-Quasitotal fuzzy graph and sketches the 2-Quasitotal fuzzy graph of a given fuzzy graph.

**Definition 13.** Let \( G = (V, \sigma, \mu) \) be a fuzzy graph with its underlying set \( V \) and crisp graph \( G^* = (\sigma^*, \mu^*) \). The pair \( \mu_{Q,T}(u,v) = \mu(u,v), \quad \text{if } u,v \in V \]
\[ \mu_{Q,T}(u,v) = \sigma(u) \wedge \mu(e), \quad \text{if } u \in V, e \in E \text{ and the node } u \text{ lies on the edge } e \]
\[ = 0, \quad \text{otherwise.} \]

By the definition of the fuzzy graph, \( \mu_{Q,T}(u,v) \leq \sigma_{Q,T}(u) \wedge \sigma_{Q,T}(v) \) for all \( u,v \in V \cup E \). Hence, \( \mu_{Q,T} \) is a fuzzy relation on the fuzzy subset \( \sigma_{Q,T} \). Therefore, the pair \( Q,T \) is a fuzzy graph, and it is termed as 2-Quasitotal Fuzzy Graph of fuzzy graph.

**Example 1.** Let \( G = (V, \sigma, \mu) \) be a fuzzy graph with its underlying crisp graph \( G^* = (V, E) \), where \( V = \{v_1, v_2, v_3\} \) and edge set \( E = \{v_1v_2, v_2v_3, v_3v_1\} \). Let the fuzzy vertex set defined on \( V \) be as \( \sigma : S \rightarrow [0,1] \) such that
\[ \sigma(v_1) = \frac{1}{3}, \]
\[ \sigma(v_2) = \frac{1}{2}, \quad (9) \]
\[ \sigma(v_3) = \frac{1}{4} \]

Let the fuzzy relation defined on the fuzzy edge set be as \( \mu : SXS \rightarrow [0,1] \) such that
\[ \mu(v_1, v_2) = \frac{1}{3}, \]
\[ \mu(v_2, v_3) = \frac{1}{5}, \quad (10) \]
\[ \mu(v_3, v_1) = \frac{1}{4} \]

Then, we have \( \mu(v_i, v_j) \leq \sigma(v_i) \wedge \sigma(v_j) \) for all \( v_i, v_j \in V \), and hence the graph \( G = (V, \sigma, \mu) \) is a fuzzy graph and its graph is as shown in Figure 1.

Now, let us construct the 2-Quasitotal fuzzy graph of the fuzzy graph in Example 1 as follows.

**Example 1.** Let \( G = (V, \sigma, \mu) \) be a fuzzy graph with its underlying crisp graph \( G^* = (V, E) \), where \( V = \{v_1, v_2, v_3\} \) and edge set \( E = \{v_1v_2, v_2v_3, v_3v_1\} \). Hence, we define the fuzzy subset \( \delta_{Q,T} \) as follows:
\[ \sigma_{Q,T}(u) = \sigma(u), \quad \text{if } u \in V, \]
\[ \sigma_{Q,T}(e) = \mu(e), \quad \text{if } e \in E. \quad (12) \]

Thus, we have the following fuzzy subsets \( \sigma_{Q,T} : \)

\[ Q,T_{f}(G) = (\sigma_{Q,T}, \mu_{Q,T}) \] of the fuzzy graph \( G \) is defined as follows:

Let the node set of \( Q,T_{f}(G) \) be the union of the vertex set and the edge set of the underlying crisp graph. That is \( V \cup E \).

Let the fuzzy subset \( \sigma_{Q,T} \) be defined on \( V \cup E \) as follows:
\[ \sigma_{Q,T}(u) = \sigma(u), \quad \text{if } u \in V, \]
\[ \sigma_{Q,T}(e) = \mu(e), \quad \text{if } e \in E. \quad (7) \]

Let the fuzzy relation \( \mu_{Q,T} \) be defined on \( (V \cup E) \times (V \cup E) \), called edges of \( Q,T_{f}(G) \) as follows:

\[ Q_{T_{f}}(G) \] of the fuzzy graph \( G \) is defined as follows:

\[ \frac{1}{3} = \mu(v_1, v_2) \leq \sigma(v_1) \wedge \sigma(v_2) = \frac{1}{3} \wedge \frac{1}{2} = \frac{1}{3}, \]
\[ \frac{1}{5} = \mu(v_2, v_3) \leq \sigma(v_2) \wedge \sigma(v_3) = \frac{1}{2} \wedge \frac{1}{4} = \frac{1}{4}, \quad (11) \]
\[ \frac{1}{4} = \mu(v_3, v_1) \leq \sigma(v_3) \wedge \sigma(v_1) = \frac{1}{4} \wedge \frac{1}{3} = \frac{1}{4}. \]
\[
\begin{align*}
\sigma_{Q_1 T_f}(v_1) &= \sigma(v_1) = \frac{1}{3}, \\
\sigma_{Q_1 T_f}(v_2) &= \sigma(v_2) = \frac{1}{2}, \\
\sigma_{Q_1 T_f}(v_3) &= \sigma(v_3) = \frac{1}{4}, \\
\sigma_{Q_1 T_f}(v_1, v_2) &= \mu(v_1, v_2) = \frac{1}{3}, \\
\sigma_{Q_1 T_f}(v_2, v_3) &= \mu(v_2, v_3) = \frac{1}{5}, \\
\sigma_{Q_1 T_f}(v_3, v_1) &= \mu(v_3, v_1) = \frac{1}{4}.
\end{align*}
\]

(13)

The fuzzy relations \(\mu_{Q_1 T_f}\) will be as follows:

\[
\begin{align*}
\mu_{Q_1 T_f}(u, v) &= \mu(u, v), \quad \text{if } u, v \in V, \\
\mu_{Q_1 T_f}(u, v) &= \sigma(u) \land \mu(e), \quad \text{if } u \in V, \ v \in E \text{ and } e \text{ lies on the edge } e = 0, \quad \text{otherwise.}
\end{align*}
\]

(14)

Hence,

\[
\begin{align*}
\mu_{Q_1 T_f}(v_1, v_2) &= \mu(v_1, v_2) = \frac{1}{3}, \\
\mu_{Q_1 T_f}(v_2, v_3) &= \mu(v_2, v_3) = \frac{1}{5}, \\
\mu_{Q_1 T_f}(v_3, v_1) &= \mu(v_3, v_1) = \frac{1}{4}, \\
\mu_{Q_1 T_f}(v_1, v_1, v_2) &= \sigma(v_1) \land \mu(v_1, v_2) = \frac{1}{3}, \\
\mu_{Q_1 T_f}(v_1, v_1, v_3) &= \sigma(v_1) \land \mu(v_2, v_3) = \frac{1}{4}, \\
\mu_{Q_1 T_f}(v_1, v_2, v_3) &= 0, \\
\mu_{Q_1 T_f}(v_2, v_2, v_3) &= \sigma(v_2) \land \mu(v_2, v_3) = \frac{1}{5}, \\
\mu_{Q_1 T_f}(v_2, v_2, v_3) &= \sigma(v_2) \land \mu(v_3, v_3) = \frac{1}{3}, \\
\mu_{Q_1 T_f}(v_3, v_3, v_1) &= \sigma(v_3) \land \mu(v_3, v_1) = \frac{1}{4}, \\
\mu_{Q_1 T_f}(v_1, v_2) &= \sigma(v_3) \land \mu(v_3, v_2) = \frac{1}{5}, \\
\mu_{Q_1 T_f}(v_3, v_1) &= 0.
\end{align*}
\]

(15)

Thus, we conclude that \(\mu_{Q_1 T_f}(v_i, v_j) \leq \sigma_{Q_1 T_f}(v_i) \land \sigma_{Q_1 T_f}(v_j)\) for all \(v_i, v_j \in V \cup E\); thus the graph \(Q_1 T_f(\sigma_{Q_1 T_f}, \mu_{Q_1 T_f})\) is a fuzzy graph and is called 2-quasitotal fuzzy graph of the fuzzy graph \(G\) in Example 1.

Now, based on the node sets \(V \cup E\), fuzzy subsets \(\sigma_{Q_1 T_f}\), and fuzzy relations \(\mu_{Q_1 T_f}\), the 2-quasitotal fuzzy graph of \(G\) is as shown in Figure 2.

**Example 2.** Consider the following graph \(G = (V, \sigma, \mu)\) with the fuzzy vertex set:

\[
\begin{align*}
\sigma(v_1) &= 1, \quad \sigma(v_2) = 0.75, \quad \sigma(v_3) = 1, \quad \sigma(v_4) = 0.25 \quad \text{and} \\
\text{fuzzy edge set:} \quad &
\end{align*}
\]

\[
\begin{align*}
\mu(v_1, v_2) &= 0.5, \\
\mu(v_2, v_3) &= 0.5, \\
\mu(v_3, v_4) &= 0.25, \\
\mu(v_4, v_1) &= 0.5.
\end{align*}
\]

(17)

Clearly, \(\mu(v_i, v_j) \leq \sigma(v_i) \land \sigma(v_j)\) for all \(v_i, v_j \in V\), the graph \(G = (V, \sigma, \mu)\) is a fuzzy graph and its graph is as shown in Figure 3.

Now, the construction of 2-quasitotal fuzzy graph \(Q_1 T_f(\sigma_{Q_1 T_f}, \mu_{Q_1 T_f})\) of the graph \(G\) in Example 2 will be as follows.

(i) The node set of \(\sigma_{Q_1 T_f}\) will be as follows:

\[
V \cup E = \{v_1, v_2, v_3, v_4, v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_1\}.
\]

(18)

(ii) The fuzzy subset \(\sigma_{Q_1 T_f}(G)\) will be as follows:
Hence,

\[
\sigma_{Q,T_f} (u) = \sigma(u), \quad \text{if } u \in V, \\
\sigma_{Q,T_f} (e) = \mu(e), \quad \text{if } e \in E. 
\]

(iii) The fuzzy relation \(\mu_{Q,T_f}\) will be as follows:

\[
\begin{align*}
\mu_{Q,T_f} (u, v) &= \mu(u, v), \quad \text{if } u, v \in V, \\
\mu_{Q,T_f} (u, e) &= \sigma(u) \wedge \mu(e), \quad \text{if } u \text{ lies on the edge } e, \\
&= 0, \quad \text{otherwise.} 
\end{align*}
\]

Hence,

\[
\begin{align*}
\mu_{Q,T_f} (v_1, v_2) &= \mu(v_1, v_2) = 0.5, \\
\mu_{Q,T_f} (v_2, v_3) &= \mu(v_2, v_3) = 0.5, \\
\mu_{Q,T_f} (v_3, v_4) &= \mu(v_3, v_4) = 0.25, \\
\mu_{Q,T_f} (v_4, v_1) &= \mu(v_4, v_1) = 0.25, \\
\mu_{Q,T_f} (v_1, v_1 v_2) &= \sigma(v_1) \wedge \mu(v_1, v_2) = 1 \wedge 0.5 = 0.5, \\
\mu_{Q,T_f} (v_1, v_2 v_1) &= 0, \\
\mu_{Q,T_f} (v_1, v_3 v_4) &= 0, \\
\mu_{Q,T_f} (v_2, v_2 v_3 v_4) &= \sigma(v_2) \wedge \mu(v_2, v_3) = 0.75 \wedge 0.5 = 0.5, \\
\mu_{Q,T_f} (v_2, v_3 v_4) &= 0, \\
\mu_{Q,T_f} (v_3, v_4 v_1) &= \sigma(v_3) \wedge \mu(v_3, v_4) = 1 \wedge 0.25 = 0.25, \\
\mu_{Q,T_f} (v_3, v_4 v_2) &= 0, \\
\mu_{Q,T_f} (v_3, v_3 v_4) &= \sigma(v_3) \wedge \mu(v_3, v_4) = 0.75 \wedge 0.75 = 0.75, \\
\mu_{Q,T_f} (v_4, v_4 v_1) &= \sigma(v_4) \wedge \mu(v_4, v_1) = 0.25 \wedge 0.25 = 0.25, \\
\mu_{Q,T_f} (v_4, v_2) &= 0, \\
\mu_{Q,T_f} (v_4, v_3) &= 0, \\
\mu_{Q,T_f} (v_4, v_3 v_4) &= \sigma(v_4) \wedge \mu(v_3, v_4) = 0.25 \wedge 0.25 = 0.25.
\end{align*}
\]

Clearly, \(\mu_{Q,T_f} (v_i, v_j) \leq \sigma_{Q,T_f} (v_i) \wedge \sigma_{Q,T_f} (v_j)\) for all \(v_i, v_j \in V \cup E\) and hence the graph \(Q_2 T_f (\delta_{Q,T_f}, \mu_{Q,T_f})\) is a fuzzy graph and it is a 2-quasitotal fuzzy graph of a graph in the above Example 2, and its graph is as shown in Figure 4.

4. Properties of 2-Quasitotal Fuzzy Graph

Theorem 1. Let \(G = (V, \sigma, \mu)\) be a fuzzy graph. Then,

\[
\text{Order}(Q_2 T_f (G)) = \text{Order}(G) + \text{Size}(G).
\]
Theorem 2. Let $G = (V, \sigma, \mu)$ be a fuzzy graph, then
\[
\text{Size}(Q_2T_f(G)) = \text{Size}(G) + \sum_{u \in V, e \in E} (\sigma(u) \Lambda \mu(e)).
\] (27)

Proof. By the definition of the size of a fuzzy graph, we have the following:

\[
\text{Size}(Q_2T_f(G)) = \sum_{u \in V} \mu(Q_2T_f)(u, v) = \sum_{u \in V} \mu(Q_2T_f)(u, v) + \sum_{u \in V, e \in E} \mu(Q_2T_f)(u, e) + \sum_{e, f \in E} \mu(Q_2T_f)(f, e) = \sum_{u \in V} \mu(Q_2T_f)(u, v) + \sum_{u \in V, e \in E} \mu(Q_2T_f)(u, e) + 0.
\] (28)

(The third summation is zero since there is no fuzzy relation between $e, f \in E$ in 2-quasitotal fuzzy graph)

\[
\text{Size}(Q_2T_f(G)) = \text{Size}(G) + \sum_{u \in V} (\sigma(u) \Lambda \mu(e)).
\] (29)

Note 2. For any fuzzy graph $G = (V, \sigma, \mu)$,

1. \[\text{Size}(T(G)) = 3 \text{Size}(G) + \sum_{u \in V, e \in E} (\sigma(e) \Lambda \mu(e)), \text{ where } T(G) \text{ is the total fuzzy graph of } G\]
2. \[\text{Size}(Q_2T_f(G)) = \text{Size}(G) + \sum_{e, f \in E} (\sigma(e) \Lambda \mu(f)), \text{ where } Q_2T_f(G) \text{ is 1-quasitotal fuzzy graph of } G\]
3. \[\text{Size}(Q_2T_f(G)) = \text{Size}(G) + \sum_{u \in V} (\sigma(u) \Lambda \mu(e)), \text{ where } Q_2T_f(G) \text{ is 2-quasitotal fuzzy graph of } G\]

Theorem 3. Let $G = (V, \sigma, \mu)$ be a fuzzy graph; then,
\[
d(Q_2T_f(G)) = d_G(u) + \sum_{u \in V, e \in E} (\sigma(u) \Lambda \mu(e)), \text{ if } u \in V,
\] (30)

\[
d(Q_2T_f(G)) = d_G(u) + \sum_{u \in V, e \in E} (\sigma(u) \Lambda \mu(e)), \text{ if } e \in E.
\] (31)

Proof. By the definition of the degree of a vertex of a fuzzy graph, we have the following:

\[
d(Q_2T_f(G)) = d_G(u) + \sum_{u \in V, e \in E} (\sigma(u) \Lambda \mu(e)),
\] (32)

Case 1. Let $u \in V$. Then,

\[
d(Q_2T_f(G)) = d_G(u) + \sum_{u \in V, e \in E} (\sigma(u) \Lambda \mu(e)),
\] (33)

\[
d(Q_2T_f(G)) = d_G(u) + \sum_{u \in V, e \in E} (\sigma(u) \Lambda \mu(e)).
\] (34)
Case 2. Let $e_i \in E$; then,
\[
d( Q_2T_f(G)(e_i) ) = \sum_{u \in V} \mu_{Q_2T_f(G)}(e_i, u) + \sum_{e_j \in E} \mu_{Q_2T_f(G)}(e_i, e_j)
= \sum_{u \in V} \mu(e_i, u) + 0.
\]
(33)

(The second summation is zero since there is no fuzzy relation between $e_i, e_j \in E$ in 2-quasitotal fuzzy graph)
\[
d( Q_2T_f(G)(e_i) ) = \sum_{u \in V} (\mu(e_i) \land \sigma(u)),
\]
(34)

Note 3. For any fuzzy graph $G = (V, \sigma, \mu)$,
(1) $d( T_f(G)(u) ) = 2d_f(u)$, if $u \in V$, where $T_f(G)$ is the total fuzzy graph of $G = \text{busy value of } e_i \in T(G)$, if $u \in E$
(2) $d( Q_2T_f(G)(u) ) = d(G)(u)$, if $u \in V = \text{bussiy value of } e_i$ in $Q_2T_f(G)$, if $u \in E$, and where $Q_2T_f(G)$ is 1-quasitotal fuzzy graph of $G$
(3) $d( Q_2T_f(G)(u) ) = d(G)(u) + \sum_{u \in V \in E} (\sigma(u) \land \mu(e))$, if $u \in V = \sum_{u \in V \in E} (\mu(e) \land \sigma(u))$ and where $Q_2T_f(G)$ is 2-quasitotal fuzzy graph of $G$

The least number of colors possible is called 2-quasitotal fuzzy chromatic number of $Q_2T_f(G)$ and it is denoted by $\chi_{Q_2}(G)$.

Example 3. Consider a fuzzy graph $G = (V, \sigma, \mu)$ as shown in Figure 5.
From the graph, we have the vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$, whose membership functions can be expressed as follows from the graph:
\[
\sigma(v_i) = \begin{cases} 
0.2, & \text{for } i = 1, \\
0.7, & \text{for } i = 2, \\
0.5, & \text{for } i = 3, \\
0.4, & \text{for } i = 4, \\
0.6, & \text{for } i = 5, \\
0.3, & \text{for } i = 6, \\
0.2, & \text{for } (i, j) = (1, 2), \\
0.5, & \text{for } (i, j) = (2, 3), \\
0.1, & \text{for } (i, j) = (3, 4), \\
0.4, & \text{for } (i, j) = (4, 5), \\
0.3, & \text{for } (i, j) = (5, 6), \\
0.1, & \text{for } (i, j) = (6, 1).
\end{cases}
\]

The family of fuzzy sets $\Gamma = \{\gamma_1, \gamma_2\}$ on $V \cup E$ will be as follows:
\[
\gamma_1(v_i) = \begin{cases} 
0.2, & \text{for } i = 1, \\
0.5, & \text{for } i = 3, \\
0.6, & \text{for } i = 5, \\
0, & \text{otherwise},
\end{cases}
\gamma_2(v_i) = \begin{cases} 
0.7, & \text{for } i = 2, \\
0.4, & \text{for } i = 4, \\
0.3, & \text{for } i = 6, \\
0, & \text{otherwise},
\end{cases}
\]

5. 2-Quasitotal Fuzzy Coloring

In this section, we introduce the concept of 2-quasitotal fuzzy total coloring and discuss some of its properties.

Definition 14. A family $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of a fuzzy set on $V \cup E$ is called a 2-quasi-k-fuzzy total coloring of fuzzy graph $G = (V, \sigma, \mu)$, if the following three conditions are met.
(i) $\text{Max} \{\gamma_i(v)\} = \sigma(v)$ for all $v \in V$ and $\text{Max}\{\gamma_i(u, v)\} = \mu(u, v)$, for all edges $(u, v) \in E$. $\gamma_i \land \gamma_j = 0$
(ii) For every adjacent vertex $u, v$ of $Q_2T_f(G)$, $\text{Min}\{\gamma_i(u), \gamma_i(v)\} = 0$. 

Theorems and properties of 2-quasitotal fuzzy coloring are discussed in this section.
To justify that the family of fuzzy sets $\Gamma = \{y_1, y_2\}$ defined as above satisfies the definition of the total coloring of the fuzzy graph and determines its total chromatic number, $\chi^t_t(G)$, we use Table 1 to check for the three conditions of the total coloring of a fuzzy graph.

From Table 1, we observe that the family of the fuzzy set $\Gamma = \{y_1, y_2\}$ satisfies the definition of the total coloring of a fuzzy graph $G$. Hence, $\chi^t_t(G) = 2$.

When we come to our point of concern, we need to determine the chromatic number of 2-quasitotal fuzzy graph of the fuzzy graph in Example 3.

Now, to construct a 2-quasitotal fuzzy graph $Q_{\varphi, \sigma_{2Q,T}}(G) = (V \cup E, \sigma_{Q_{2Q,T}}, \mu_{Q_{2Q,T}})$, where $V \cup E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$. The fuzzy subset of $Q_{2Q,T}(G)$ will be as follows:

$$\sigma_{Q_{2Q,T}}(v_i) = \begin{cases} 0.2, & \text{for } i = 1, \\ 0.7, & \text{for } i = 2, \\ 0.5, & \text{for } i = 3, \\ 0.4, & \text{for } i = 4, \\ 0.6, & \text{for } i = 5, \\ 0.3, & \text{for } i = 6, \\ 0.2, & \text{for } (i, j) = (1, 2), \\ 0.5, & \text{for } (i, j) = (2, 3), \\ 0.1, & \text{for } (i, j) = (3, 4), \\ 0.4, & \text{for } (i, j) = (4, 5), \\ 0.3, & \text{for } (i, j) = (5, 6), \\ 0.1, & \text{for } (i, j) = (6, 1). \end{cases}$$

The fuzzy relation will be as follows:

$$\mu_{Q_{2Q,T}}(v_i, v_j) = \begin{cases} 0.2, & \text{for } (i, j) = (1, 2), \\ 0.5, & \text{for } (i, j) = (2, 3), \\ 0.1, & \text{for } (i, j) = (3, 4), \\ 0.4, & \text{for } (i, j) = (4, 5), \\ 0.3, & \text{for } (i, j) = (5, 6), \\ 0.1, & \text{for } (i, j) = (6, 1). \end{cases}$$

Hence, the 2-quasitotal fuzzy graph $Q_{\varphi, \sigma_{2Q,T}}(G) = (V \cup E, \sigma_{Q_{2Q,T}}, \mu_{Q_{2Q,T}})$ of the fuzzy graph $G$ in Example 3 is as shown in Figure 6.

Let $\Gamma = \{y_1, y_2\}$ be a family of fuzzy subset defined on $V \cup E$ as follows:

(i) For the vertex set:
For the edge set:

\[
\begin{align*}
\gamma_1(v_i, v_j) = & \begin{cases} 
0.2, & \text{for } i = 1, \\
0.5, & \text{for } i = 3, \\
0.6, & \text{for } i = 5, \\
0, & \text{Otherwise,}
\end{cases} \\
\gamma_1(v_i, v_j) = & \begin{cases} 
0.2, & \text{for } ij = 12, \\
0.1, & \text{for } ij = 34, \\
0.3, & \text{for } ij = 56, \\
0, & \text{Otherwise,}
\end{cases} \\
\gamma_1(v_i, v_j) = & \begin{cases} 
0.7, & \text{for } i = 2, \\
0.4, & \text{for } i = 4, \\
0.3, & \text{for } i = 6, \\
0, & \text{Otherwise,}
\end{cases} \\
\gamma_1(v_i, v_j) = & \begin{cases} 
0.5, & \text{for } (i, j) = (2, 3), \\
0.4, & \text{for } (i, j) = (4, 5), \\
0.1, & \text{for } (i, j) = (6, 1), \\
0, & \text{Otherwise.}
\end{cases}
\end{align*}
\]

Using Table 2 below, we can check whether \( \Gamma \) satisfies the definition of 2-quasitotal fuzzy coloring of \( G \).

As shown in Table 1, \( \Gamma = \{ \gamma_1, \gamma_2 \} \) satisfies the definition of 2-quasitotal fuzzy coloring of a fuzzy graph \( G \).

Therefore, \( x_{Q2T}^\Gamma(G) = 2 \).
Note 4. Unfortunately, \( \chi_{Q,T}^f(G) = \chi_{T}^f(G) = 2 \) for this example and no evidence that it is always true in this manuscript.

6. Conclusion

This article has introduced the new concept of 2-quasitotal fuzzy graph for a given fuzzy graph. The concept is clearly explained with the particle examples by giving a fuzzy graph and its 2-quasitotal fuzzy graph. Some properties of the 2-quasitotal fuzzy graphs have been proposed and proved. Further, the theorems and results obtained for 2-quasitotal fuzzy graphs are compared with the existing properties of total fuzzy graphs and 1-quasitotal fuzzy graphs. Lastly, it has been defined 2-quasitotal coloring for a fuzzy graph and its total coloring is exemplified.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors’ Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

References

[1] J. A. Bondy and U. S. R. Murthy, *Graph Theory with Applications*, The Macmillan Press Ltd, New York, NY, USA, 1976.
[2] R. Balakrishna, *A Text Book of Graph Theory*, Springer-Verlag, Berlin, Germany, 2000.
[3] M. Behzad, “Graphs and their chromatic numbers,” Doctoral Thesis, Michigan State University, East Lansing, MI, USA, 1965.
[4] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Boston, MA, USA, 1972.
[5] D. M. Jayaraman, “Total chromatic number of middle and total graph of path and sunlet graph,” *International Journal of Scientific and Innovative Mathematical Research*, vol. 6, no. 4, pp. 1–9, 2018.
[6] D. V. S. Sastry and B. S. P. Raju, “Graph equations for line graphs, total graphs, middle graphs and quasi-total graphs,” *Discrete Mathematics*, vol. 48, no. 1, pp. 113–119, 1984.
[7] R. V. N. Simhasarao and J. V. Rao, “A discussion for bounds for 1-quasi total colourings,” *International Journal of Mathematical Archive*, vol. 3, no. 6, pp. 2314–2320, 2012.
[8] L. A. Zadeh, “Fuzzy sets,” *Information and Computation*, vol. 8, pp. 338–353, 1965.
[9] R. Zurmuhl, “Matrizen und Ihre Technischen Anwendung,” *Mathematics of Computation*, vol. 19, no. 90, pp. 338–353, 1965.
[10] A. Rosenfield, “Fuzzy graphs, in fuzzy sets and their application to cognitive and decision process,” in *Proceedings of the U.S.–Japan Seminar on Fuzzy Sets and Their Applications*, Held at the University of California, Berkeley, CA, USA, July 1975.
[11] P. Bhattacharya, “Some remarks on fuzzy graphs,” *Pattern Recognition Letters*, vol. 6, no. 5, pp. 297–302, 1987.
[12] J. N. Mordeson and J. A. Peng, “Operations on fuzzy graphs,” *Information Sciences*, vol. 79, pp. 159–170, 1994.
[13] B. Eslahchi, “Vertex strength of fuzzy graphs,” *International Journal of Mathematics and Mathematical Science*, vol. 2006, Article ID 43614, 9 pages, 2006.
[14] S. Lavanya, “Fuzzy total coloring of fuzzy graphs,” *International Journal of Information Technology and Knowledge Management*, vol. 2, no. 3, pp. 37–39, 2009.
[15] S. Kavitha, “Fuzzy chromatic number of line, total and middle graphs of fuzzy complete graphs,” *Annals of Pure and Applied Mathematics*, vol. 8, no. 2, pp. 251–260, 2014.
[16] V. Neveithana, “Fuzzy total coloring and chromatic number of complete fuzzy graph,” *International Journal of Engineering and Development*, vol. 6, no. 3, pp. 377–384, 2013.
[17] M. Sitara, M. Akram, and B. Muhammad, “Fuzzy graph structures with application,” *Mathematics*, vol. 7, no. 1, p. 63, 2019.
[18] T. A. Fekadu and V. N. Srinivasarao Reaplle, “1-Quasi total fuzzy graph and its total coloring,” *Pure and Applied Mathematics Journal*, vol. 9, no. 1, pp. 9–15, 2020.
[19] A. N. A. Koam, M. Akram, and L. Peide, “Decision-making analysis based on fuzzy graph structures,” *Mathematical Problems in Engineering*, vol. 2020, Article ID 6846257, 2020.
[20] M. Yousaf BhattiSitaraM et al., “Residue Product of fuzzy graph structures,” *Journal of Multiple-Valued Logic and Soft Computing*, vol. 34, no. 3/4, pp. 365–399, 2020.
[21] M. Akram, “Polar fuzzy graphs theory, methods & applications,” in *Fuzziness and Soft Computing*, Springer, Berlin, Germany, 2019.
[22] N. M. John and P. Morderson, *Fuzzy Graphs and Fuzzy Hypergraphs*, Physica-Verlag, Berlin, Germany, 2000.
[23] M. S. M. Akram, “Certain fuzzy graph structures,” *Journal of Applied Mathematical Computation*, pp. 1–32, 2019.
[24] M. J. Mathew S, *Fuzzy Graph Theory*, Springer International Publishing, Berlin, Germany, 2018.
[25] M. Nithyakalyani, “Total coloring regular domination,” *International Journal of Engineering, Science and Mathematics*, vol. 7, no. 3, pp. 10–16, 2018.