Topologies on the Space of Holomorphic Functions

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1 Introduction

One of the remarkable features of the space of holomorphic functions (in either one or several variables) is that the standard Frechet space topologies—say, for example, the $L^2$ or Bergman norm—control a stronger (and simpler) and more useful topology, namely uniform convergence on compact sets. This simple fact lies at the heart of many key results in basic complex function theory—for example the completeness of many important function spaces.

It is natural to wonder whether this property is universal. Is it the case that any Frechet space topology on the space of holomorphic functions implies uniform convergence on compact sets (equivalently, convergence in the compact-open topology)? The surprising answer to this question—suitably formulated—is “yes”, and that is the main result of the present paper.

It is a pleasure to thank Peter Pflug for posing the question that led to this paper, and for early discussions of the matter.

2 Basic Concepts

A domain is a connected open set. We typically denote a domain with the symbol $\Omega$. Let $O(\Omega)$ denote the complex linear space of holomorphic functions on $\Omega$.

A Frechet space topology $T$ on $O(\Omega)$ is a collection $\{F_a\}_{a \in A}$ of semi-norms on $O(\Omega)$ so that the resulting topology is complete. Many of the most useful topologies on $O(\Omega)$ are Frechet space topologies; some, however, are Banach space topologies.

Our basic result is this:

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1\ The author thanks the American Institute of Mathematics for its hospitality and support during a portion of this work.
Theorem 1 Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain. Let $T$ be a Frechet space topology on $\mathcal{O}(\Omega)$. Assume that each semi-norm $F_\alpha$ in $T$ is majorized by a semi-norm $G_\alpha$ on $C^\infty(\Omega)$. If a sequence of functions $\{f_j\} \subseteq \mathcal{O}(\Omega)$ converges in the topology $T$, then it converges uniformly on compact sets.

The main tool in the proof of the theorem is a fairly general version of the Hahn-Banach theorem, for which see [YOS, p. 105, ff.]:

Theorem 2 Let $V$ be a vector space over the field $\mathbb{C}$. A function $N : V \to \mathbb{R}$ is called sublinear if

$$N(ax + by) \leq |a|N(x) + |b|N(y)$$

for all $x, y \in V$ and all complex scalars $a, b$. Let $U$ be a complex linear subspace of $V$ (not necessarily closed) and let $\varphi : U \to \mathbb{C}$ be a linear functional (not necessarily bounded) on $U$. If $\varphi$ is dominated by $N$ in the sense that $|\varphi(x)| \leq N(x)$ for all $x \in U$, then there is a linear extension $\tilde{\varphi} : V \to \mathbb{C}$ of $\varphi$ to all of $V$ (meaning that $\tilde{\varphi}|_U = \varphi$) which is also dominated by $N$.

Since the next ideas do not seem to be well-documented in the literature, we close this section by reviewing a few elementary concepts from Frechet space theory.

If $F_\alpha$ is a semi-norm on our Frechet space $X$, and if $\lambda$ is a linear functional on $X$, then we may define the dual norm of $\lambda$ by

$$F_\alpha^*(\lambda) = \sup_{\|f\| \leq 1} |\langle f, \lambda \rangle|.$$ 

This definition is of course analogous to the norm of a functional on a Banach space.

With this language and notation, we have the following result:

Lemma 3 If $f \in X$ and $F_\alpha$ is a semi-norm on $X$, then

$$F_\alpha(f) = \sup_{\lambda \in X^*, F_\alpha^*(\lambda) \leq 1} |\langle f, \lambda \rangle|.$$ 

Proof: The proof follows classical lines. See [ZEI, p. 6].
3 Proof of the Main Result

Let $\lambda$ denote a typical element of the dual space of $\mathcal{O}(\Omega)$ equipped with the Frechet space topology $T$. According to Lemma 3, we must examine the expressions $\tau_{\lambda} : f \mapsto \langle f, \lambda \rangle$. Such a function is of course a linear functional on $\mathcal{O}(\Omega)$. And it is bounded with respect to the topology of $\mathcal{F}_\alpha$, any $\alpha$. Thus it is dominated (in the sense defined above) by some semi-norm $\mathcal{G}_\alpha$ on $C^\infty(\Omega)$. Thus the Hahn-Banach theorem applies and there is an extension $\hat{\lambda}$ of $\lambda$ to $C^\infty(\Omega)$ which is also dominated by $\mathcal{G}_\alpha$. It follows that the extension $\hat{\lambda}$ is a distribution.

We then know that $\hat{\lambda}$ is a finite sum of derivatives of measures. But the Cauchy estimates tell us that, when the derivative of a measure is acting on holomorphic functions, the action is dominated by the measure itself acting on holomorphic functions. So we may as well take it that $\hat{\lambda}$ is integration against a measure $\mu_\alpha$ on $\Omega$.

Thus for each $\alpha$ we have associated to the functional $\lambda$ a measure $\mu_\alpha$. Now we must think about the support $K_\alpha$ of each $\mu_\alpha$. Clearly the hull of holomorphy of $\bigcup_\alpha K_\alpha$ must be all of $\Omega$, otherwise the original Frechet space topology on $\mathcal{O}(\Omega)$ will not be complete. But that says immediately that the topology majorizes uniform convergence on compact sets. That is the desired conclusion.

4 Concluding Remarks

Not surprisingly, the key fact about holomorphic functions that is used in the proof of the main result here is the Cauchy estimates. The Cauchy estimates are logically equivalent to the analyticity of holomorphic functions, so they are at the heart of the subject. It would be interesting to identify other natural spaces of functions for which results like these hold.
REFERENCES

[YOS] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, Berlin, 1980.

[ZEI] E. Zeidler, *Applied Functional Analysis: Main Principles and their Applications*, Springer, New York, 1995.