HEAT TRACE ASYMPTOTICS DEFINED BY TRANSFER
BOUNDARY CONDITIONS

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Abstract. We compute the first 5 terms in the short-time heat trace asymptotics expansion for an operator of Laplace type with transfer boundary conditions using the functorial properties of these invariants.

1. Introduction

Let \( M := (M^+, M^-) \) be a pair of compact smooth manifolds of dimension \( m \) which have a common smooth boundary \( \Sigma := \partial M^+ = \partial M^- \). A structure \( \Xi \) over \( M \) will be a pair of corresponding structures \( \Xi := (\Xi^+, \Xi^-) \) over the manifolds \( M^\pm \). Let \( g \) be a Riemannian metric on \( M \); we assume henceforth that \( g^+|\Sigma = g^-|\Sigma \), but do not assume any matching condition on the normal derivatives. Let \( V \) be a smooth vector bundle over \( M \); we do not assume any relationship between \( V^+|\Sigma \) and \( V^-|\Sigma \); in particular, we can consider the situation when we have \( \dim V^+ \neq \dim V^- \).

Let \( D \) be an operator of Laplace type on \( C^\infty(V) \). The operator \( D \) determines a natural connection \( \nabla \) and a natural 0th order operator \( E \) so that

\[
D = -(g^{ij} \nabla_i \nabla_j + E).
\]

Let the inward unit normals \( \nu^\pm \) of \( \Sigma \subset M^\pm \) determine \( \nu \); note that \( \nu^+ = -\nu^- \).

Assume given auxiliary impedance matching terms \( S = \{S^{++}, S^{+-}, S^{-+}, S^{--}\} \) where \( S^\nu : V^0|\Sigma \to V^\nu|\Sigma \). The transfer boundary operator \( B_T(S) \) is defined by:

\[
B_T(S)\phi := \left\{ \left( \begin{array}{cc}
\nabla^{++} + S^{++} & S^{-+} \\
S^{+-} & \nabla^{-+} + S^{--}
\end{array} \right) \left( \begin{array}{c}
\phi^+ \\
\phi^-
\end{array} \right) \right\} \bigg|_{\Sigma}.
\]

The terms \( S^{++} \) and \( S^{--} \) connect the structures on \( M^+ \) and \( M^- \) and are crucial to our investigation. These boundary conditions arise physically in heat transfer problems (see to Carslaw and Jaeger [6]), some problems of quantum mechanics [1], and in conformal field theory [2]. More on various spectral problems appearing in the string theory context can be found in [12].

Let \( D_{B_T(S)} \) be the associated realization of \( D \) with the boundary condition \( B_T(S)\phi = 0 \). Let \( Q \) be a smooth endomorphism of \( V \) which we use to localize the heat trace. As \( t \downarrow 0 \), there is a complete asymptotic expansion with locally computable coefficients:

\[
Tr_{L^2} \left( Q e^{-tD_{B_T(S)}} \right) \sim \sum_{n \geq 0} a_n(Q, D, B_T(S)) t^{(n-m)/2}.
\]

In a formal limiting case \( S^{++} - S^{--} = S^{--} - S^{++} \to \infty \) while \( v = 2(S^{++} + S^{--}) \) is kept finite one arrives at transmittal boundary conditions: \( \phi^+ = \phi^- \), \( \nabla^{++} + \nabla^{+-} = v \phi^+ \). The heat trace asymptotics for these boundary conditions have been studied in [8], [9], [11]. Some other particular cases of the boundary operator \( B_T(S) \) have been considered in [3], [4].

Let \( R_{ijkl} \) be the components of the Riemann curvature tensor, let \( \Omega \) be the curvature of \( \nabla \), and let the second fundamental forms \( L^\pm \) of \( \Sigma \subset M^\pm \) determine \( L \).

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We let Roman indices $i, j, k,$ and $l$ range from 1 to $m$ and index a local orthonormal frame for the tangent bundle of $M$ and let Roman indices $a, b, c$ range from 1 to $m - 1$ and index a local orthonormal frame for the tangent bundle of $\Sigma$. We adopt the Einstein convention and sum over repeated indices. Let $\text{Tr}^\pm$ be the fiber trace in $V^\pm$, let ‘;’ denote multiple covariant differentiation with respect to the Levi-Civita connection on $M$ and $\nabla$, and let ‘;’ denote multiple covariant differentiation with respect to the Levi-Civita connection of $\Sigma$ and $\nabla$. Let $S = (S^{++}, S^{--})$.

Local formulae which decouple can be written in the following format:

**Definition 1.1.** Let $\mathcal{E}(\nabla^* R, \nabla^* E, \nabla^* \Omega)$ and $\mathcal{F}(\nabla^* R, \nabla^* E, \nabla^* L, \nabla^* S)$ be local invariants on $M$ and $\partial M$, respectively. Set:

\[
\int_M \text{Tr}(\mathcal{E}) := \int_{M^+} \text{Tr}^+(\mathcal{E}^+) + \int_{M^-} \text{Tr}^-(\mathcal{E}^-), \\
\int_{\partial M} \text{Tr}(\mathcal{F}) := \int_{\partial M^+} \text{Tr}^+(\mathcal{F}^+) + \int_{\partial M^-} \text{Tr}^-(\mathcal{F}^-) = \int_S \{\text{Tr}^+(\mathcal{F}^+) + \text{Tr}^-(\mathcal{F}^-)\}.
\]

What is crucial is that the invariants $\mathcal{E}^\pm$ and $\mathcal{F}^\pm$ involve only structures on $M^\pm$.

We illustrate these two types in the following examples:

\[
\int_M \text{Tr}(Q R_{ijji} E) = \int_{M^+} \text{Tr}^+(Q^+ R_{ijji}^+ E^+) + \int_{M^-} \text{Tr}^-(Q^- R_{ijji}^- E^-), \\
\int_{\partial M} \text{Tr}(Q S L_{aa}) = \int_{\partial M^+} \text{Tr}^+(Q^+ S^{++} L_{aa}^+) + \int_{\partial M^-} \text{Tr}^-(Q^- S^{--} L_{aa}).
\]

There are, however, invariants which intertwine the two structures and which do not decouple; for example, the following invariant is a ‘mixed’ invariant which measures the interactions of these two structures:

\[
\int_S \{\text{Tr}^+(Q^+ S^{++} S^{--}) + \text{Tr}^-(Q^- S^{--} S^{++})\}.
\]

The main result of this letter is the following:

**Theorem 1.2.** With transfer boundary conditions, we have that:

1. $a_0(Q, D, \mathcal{B}_T(S)) = (4\pi)^{-m/2} \int_M \text{Tr}(Q)$.
2. $a_1(Q, D, \mathcal{B}_T(S)) = (4\pi)^{(1-m)/2} \frac{1}{6} \int_{\partial M} \text{Tr}(Q)$.
3. $a_2(Q, D, \mathcal{B}_T(S)) = (4\pi)^{-m/2} \frac{1}{6} \int_M \text{Tr}\{Q(R_{ijji} + 6E)\}$
   \[+ (4\pi)^{-m/2} \frac{1}{6} \int_{\partial M} \text{Tr}\{Q(2L_{aa} + 12S) + 3Q_{iv}\}.
\]
4. $a_3(Q, D, \mathcal{B}_T(S))$
   \[(4\pi)^{(1-m)/2} \frac{1}{384} \int_{\partial M} \text{Tr}\{Q(96E + 16R_{ijji} - 8R_{avva} + 13L_{aa} L_{bb} + 2L_{ab} L_{ab} + 96S L_{aa} + 192S^2 + Q_{iv}(6L_{aa} + 96S) + 24Q_{iv})\}$
   \[+ (4\pi)^{(1-m)/2} \frac{1}{384} \int_{\Sigma} \{\text{Tr}^+(192Q^+ S^{++} S^{--}) + \text{Tr}^-(192Q^- S^{--} S^{++})\}.
\]
5. $a_4(Q, D, \mathcal{B}_T(S))$
   \[(4\pi)^{-m/2} \frac{1}{384} \int_M \text{Tr}\{Q(60E_{kk} + 60R_{ijji} E + 180E^2 + 309^2 + 12R_{ijjj} kkk + 5R_{ijji} R_{kk} k - 2R_{ikjj} R_{ijji} + 2R_{ikjk} R_{ijji})\}$
   \[+ (4\pi)^{-m/2} \frac{1}{384} \int_{\partial M} \text{Tr}\{Q(240E_{iv} + 42R_{ijji,c} + 24L_{aa,bb} + 120E_{L_{aa}} + 20R_{ijji} L_{aa} + 4R_{avva} L_{bb} - 12R_{avva} L_{ab} + 4R_{abcb} L_{ac} + 4L_{aa} L_{bb} L_{cc} + 8L_{ab} L_{bc} L_{ac} + 360(S E + ES) + 120S R_{ijji} + 144S L_{aa} L_{bb} + 48S L_{ab} L_{ab} + 480S L_{aa} + 480S^3 + 120S L_{aa} + Q_{iv}(180E + 30R_{ijji} + 12L_{aa} L_{bb} + 12L_{ab} L_{ab} + 72S L_{aa} + 240S^3) + Q_{iv}(24L_{aa} + 120S) + 30Q_{ivw}\}.$
This completes the proof of Theorem 1.2 (1)-(3).

We may decompose the heat trace invariants in the form:

\[ a_n(Q, D, B_T(S)) = a_n^M(Q, D) + a_n^{\partial M}(Q, D, S) + a_n^\Sigma(Q, D, S). \]

The invariants \( a_n^M \) and \( a_n^{\partial M} \) decouple and can be expressed as local integrals of the form given in Definition 1.1; the invariant \( a_n^\Sigma \) involves integrals of mixed structures. Theorem 1.2 reflects this decomposition. We shall prove Theorem 1.2 by analyzing the 3 terms appearing in Equation (1.3) separately. Here is a brief guide to the remainder of this letter. In Section 2, we apply results of Branson and Gilkey concerning the heat trace asymptotics with Robin boundary conditions to determine \( a_n^M \) and \( a_n^{\partial M} \). In Section 3, we express \( a_n^\Sigma \) in terms of certain invariants with universal undetermined coefficients (see Lemma 3.1); these new terms which measure the interaction between the structures on \( M^\pm \) are the heart of the matter. The proof of Theorem 1.2 is then completed in Sections 4 and 5 by determining the universal coefficients of Lemma 3.1. In Section 4, we derive a new functorial property by doubling the manifold; in Section 5, we use conformal variations. We refer to [8] for an analogous computation of the heat content asymptotics with transfer boundary conditions.

2. Robin boundary conditions

Let \( D \) be an operator of Laplace type on a compact Riemannian manifold \( N \) with smooth boundary \( \partial N \) and let \( S \) be an auxiliary endomorphism defined on the boundary. Robin boundary conditions are defined by the operator:

\[ B_R(S)\phi := (\nabla \phi + S\phi)|_{\partial N}. \]

If we take \( S^{+-} = 0 \) and \( S^{-+} = 0 \), then the boundary conditions decouple so

\[
\begin{align*}
a_n(Q, D, B_T(S)) &= a_n(Q^+, D^+, B_R(S^{++})) + a_n(Q^-, D^-, B_R(S^{-})) \\
&= a_n(Q, D, B_R(S)).
\end{align*}
\]

Thus we may use Branson-Gilkey-Vassilevich [8] (Theorem 4.1) to determine the invariants \( a_n^M(Q, D) \) and \( a_n^{\partial M}(Q, D, S) \). Furthermore, we see that all the terms in the mixed integrals defining \( a_n^\Sigma(Q, D, B_T(S)) \) must contain either \( S^{+-} \) or \( S^{-+} \) and hence, since we are taking traces and have not identified \( V^+ \) with \( V^- \), both of these terms must appear in every mixed monomial as these are the only structures relating \( M^+ \) to \( M^- \).

As the boundary integrands describing \( a_n^\Sigma \) are homogeneous of weight \( n - 1 \) and as the variables \( S^{**} \) have weight 1, monomials which contain both \( S^{-+} \) and \( S^{+-} \) have weight at least 2 and thus do not appear in the expansion of \( a_n \) for \( n \leq 2 \). This completes the proof of Theorem 1.2 (1)-(3).

3. The mixed invariants

We can identify the general form of the invariants \( a_n^\Sigma \) for \( n \leq 4 \) as follows:

**Lemma 3.1.** There exist universal constants so that:

1. \( a_n^\Sigma(Q, D, B_T(S)) = (4\pi)^{-m/2} \frac{1}{384} \int_{\Sigma} \omega_0(\text{Tr}^+(Q^+ S^{++} S^{-}) + \text{Tr}^-(Q^- S^{-+} S^{+-})). \)
2. \( a_4^2(Q, D, \mathcal{B}_T(S)) = (4\pi)^{-m/2} \frac{1}{360} \int_{\Sigma} \{ \)
\[ \begin{align*}
\frac{1}{2} c_1 \text{Tr}^+(Q^+ S^+ S^+ S^-) &+ \frac{1}{2} c_1 \text{Tr}^-(Q^- S^- S^+ S^+) \\
+ \frac{1}{2} c_2 \text{Tr}^+(S^+ Q^+ S^- S^-) &+ \frac{1}{2} c_2 \text{Tr}^-(S^- Q^- S^+ S^+) \\
+ \alpha_2 \text{Tr}^+(S^+ S^- Q^+ S^-) &+ \alpha_2 \text{Tr}^-(S^- S^+ Q^+ S^+ ) \\
+ \alpha_3 L^{\pm}_{a a} \text{Tr}^+(Q^+ S^+ S^-) &+ \alpha_3 L^{\pm}_{a a} \text{Tr}^-(Q^- S^- S^+) \\
+ \alpha_4 L^{\pm}_{a a} \text{Tr}^+(Q^+ S^+ S^-) &+ \alpha_4 L^{\pm}_{a a} \text{Tr}^-(Q^- S^- S^+) \\
+ \alpha_5 \text{Tr}^+(Q^+ S^+ S^-) &+ \alpha_5 \text{Tr}^-(Q^- S^- S^+) \} .
\end{align*} \]

3. \( c_1 = c_2 \).

Proof. We observe first that the heat trace coefficient must be symmetric with respect to interchanging the labels “+” and “−”. Since we have written down a complete basis of invariants of weight 2 and 3 which contain both \( S^+ \) and \( S^- \), assertions (1) and (2) now follow.

We generalize an argument from [5] to prove assertion (3). If \( D, Q \) and \( S^{\pm} \) are real, then \( \text{Tr} (Q e^{-tD}) \) is real. This shows that all universal constants given above are real. Suppose now that the bundles \( V^\pm \) are equipped with Hermitian inner products and that the operators \( D^\pm \) are formally self-adjoint. This means that the associated connections \( \nabla^\pm \) are unitary and the endomorphisms \( E^\pm \) are symmetric. Suppose that \( S^{++} \) and \( S^{−−} \) are self-adjoint, and that \( S^{+-} \) is the adjoint of \( S^{-+} \). It then follows that \( D \) is self-adjoint. Therefore, \( \text{Tr} (Q e^{-tD}) \) is real; this implies necessarily that \( c_1 = c_2 \).

We remark in passing that it is exactly this argument which shows that the term \( \int_M \text{Tr}(720QSE) \) appearing in \( [3] \) for scalar \( Q \) must be replaced by the term \( \int_M \text{Tr}(360Q(SE + ES)) \) for endomorphism valued \( Q \) \([6]\).

Since \( c_1 = c_2 \), the lack of commutativity involved in dealing with endomorphisms plays no role; thus it suffices to consider the scalar case where everything is commutative. We assume therefore for the remainder of this letter that the bundles \( V^\pm = M^\pm \times \mathbb{C} \) are trivial line bundles and that the operators \( D^\pm \) are scalar. Thus we may drop ‘Tr’ from the notation. We set \( \alpha_1 := c_1 = c_2 \) - the symmetrization term then becomes

\[
(4\pi)^{-m/2} \frac{1}{360} \int_{\Sigma} \alpha_1 (Q^+ S^+ S^+ S^- + Q^- S^- S^+ S^+) .
\]

4. Doubling the Manifold

In Section 5, we related the heat trace asymptotics for transfer and Robin boundary conditions by taking \( S^{++} = S^{−−} = 0 \). We now give a different relationship between transfer and Robin boundary conditions related to doubling the manifold.

Lemma 4.1. Let \( M^\pm := M^0 \) be a m-dimensional Riemannian manifold with boundary \( \partial M^0 = \Sigma \) and let \( D^\pm = D^0 \) be a scalar operator of Laplace type. Fix an angle \( 0 < \theta < \frac{\pi}{2} \). Let \( S^{++} \) and \( S^{+-} \) be arbitrary. Set:

\[
\begin{align*}
S^{++} &:= S^+ , \\
S^{−−} &:= S^{++} + (\tan \theta - \cot \theta) S^{+-} , \\
S^{ϕ} &:= S^{++} + \tan \theta S^{+-} = S^{−−} + \cot \theta S^{−+} , \\
S^{ψ} &:= S^{++} - \cot \theta S^{+-} = S^{−−} - \tan \theta S^{−+} .
\end{align*}
\]

Then:

\[
a_n(Q, D, \mathcal{B}_T(S)) = a_n(\cos^2 \theta Q^+ + \sin^2 \theta Q^- , D^0, \mathcal{B}_{R(S_ϕ)}) \\
+ a_n(\sin^2 \theta Q^+ + \cos^2 \theta Q^- , D^0, \mathcal{B}_{R(S_ψ)}) .
\]
Proof. If \( u, v \in C^\infty (V^0) \), define \( u^\phi, v^\psi \in C^\infty (M) \) by setting
\[
\begin{align*}
u^\phi(x^+) &= \cos \theta u(x), \quad u^\phi(x^-) = \sin \theta u(x) \\
v^\psi(x^+) &= -\sin \theta v(x), \quad v^\psi(x^-) = \cos \theta v(x).
\end{align*}
\]
The conditions \( \mathcal{B}_T(S)u^\phi = 0 \) and \( \mathcal{B}_T(S)v^\psi = 0 \) are equivalent to the conditions:
\[
(\nabla_{\nu^0} + S^{++} + \tan \theta S^+)u|_{\partial M^0} = 0, \quad (\nabla_{\nu^0} + S^{--} + \cot \theta S^{-})u|_{\partial M^0} = 0,
\]
\[
(\nabla_{\nu^0} + S^{++} - \cot \theta S^{--})v|_{\partial M^0} = 0, \quad (\nabla_{\nu^0} + S^{--} - \tan \theta S^{-})v|_{\partial M^0} = 0,
\]
or equivalently to the conditions \( (\nabla_{\nu^0} + S_\phi)u|_{\partial M^0} = 0 \) and \( (\nabla_{\nu^0} + S_\phi)v|_{\partial M^0} = 0 \).

Let \( \{\lambda_i, u_i^\phi\} \) and \( \{\mu_j, v_j^\psi\} \) be discrete spectral resolutions for \( D^0 \) for Robin boundary conditions \( \mathcal{B}_R(S_\phi) \) and \( \mathcal{B}_R(S_\psi) \). Since
\[
Du_i^\phi = \lambda_i u_i^\phi, \quad Dv_j^\psi = \mu_j v_j^\psi, \quad \mathcal{B}_T(S)u_i^\phi = 0, \quad \text{and} \quad \mathcal{B}_T(S)v_j^\psi = 0,
\]
and since \( \{u_i^\phi, v_j^\psi\} \) is a complete orthonormal basis for \( L^2(M) \), \( \{\lambda_i, u_i^\phi\} \cup \{\mu_j, v_j^\psi\} \) is a discrete spectral resolution of \( D \) with transfer boundary conditions \( \mathcal{B}_T(S) \). Thus we may compute:
\[
\begin{align*}
\Tr_{L^2}(Qe^{-tD_{\mathcal{B}_T(S)}}) &= \int_M \sum_i Q e^{-t\lambda_i} |u_i^\phi|^2 + \int_M \sum_j Q e^{-t\mu_j} |v_j^\psi|^2 \\
&= \int_{M^0} \sum_i (\cos^2 \theta Q^+ + \sin^2 \theta Q^-)|u_i|^2 e^{-t\lambda_i} \\
&\quad + \int_{M^0} \sum_j (\sin^2 \theta Q^+ + \cos^2 \theta Q^-)|v_j|^2 e^{-t\mu_j} \\
&= \Tr_{L^2}(Q^+ + \cos^2 \theta Q^- e^{-tD^0_{\mathcal{B}_R(\phi)}} \\
&\quad + \Tr_{L^2}(\sin^2 \theta Q^+ + \cos^2 \theta Q^- e^{-tD^0_{\mathcal{B}_R(\psi)}}.
\end{align*}
\]

We use Lemma [4,1] as follows. We set \( Q^- = 0 \). (The case \( Q^- \neq 0 \) may be used as a check, but no additional information is obtained.) We use \[5\] (Theorem 1.2), Lemma [5,1] and Lemma [4,1] to derive the following relations,
\[
\begin{align*}192Q^+(\cos^2 \theta S^2_\phi + \sin^2 \theta S^2_\psi) &= 192Q^+(S^{++}S^{++} + S^{--}S^{--}) \\
&= 192Q^+S^{++}S^{++} + \alpha_0Q^+S^{+-}S^{--}, \\
480Q^+(\cos^2 \theta S^3_\phi + \sin^2 \theta S^3_\psi) &= 480Q^+(S^{++}S^{++}S^{++} + 3S^{++}S^{--}S^{--} + S^{++}S^{--}S^{++} - \tan \theta - \cot \theta) \\
&= 480Q^+S^{++}S^{++}S^{++} + \alpha_1Q^+S^{++}S^{--}S^{--} + \alpha_2Q^+[S^{++} + S^{--}(\tan \theta - \cot \theta)]S^{++}S^{--}, \\
480L_{aa}(\cos^2 \theta S^2_\phi + \sin^2 \theta S^2_\psi) &= 480L_{aa}(S^{++}S^{++} + S^{++}S^{--}) \\
&= 480L_{aa}S^{++}S^{++} + (\alpha_3 + \alpha_4)Q^+L_{aa}S^{--}S^{++}, \\
240L_{\nu_+}(\cos^2 \theta S^2_\phi + \sin^2 \theta S^2_\psi) &= 240Q^+(S^{++}S^{++} + S^{++}S^{--}) \\
&= 240Q^+S^{++}S^{++} + \alpha_5Q^+S^{++}S^{--}.
\end{align*}
\]
This implies that:
\[
(4.1) \quad 192 = \alpha_0, \quad 960 = \alpha_1, \quad 480 = \alpha_2, \quad 480 = \alpha_3 + \alpha_4, \quad \alpha_5 = 240.
\]
5. Conformal variations

The missing information about \(\{\alpha_3, \alpha_4\}\) is obtained via conformal transformations. As before, we deal only with the scalar situation. Given \((M,D)\) and \(\psi^+ \in C^\infty(M^+)\), we vary the structures on \(M^+\) to define the one-parameter family of operators

\[
D(\varepsilon) := (e^{2\varepsilon \psi^+} D^+, D^-)
\]

with associated structures \(g^+(\varepsilon) := e^{2\varepsilon \psi^+} g^+, \nabla^+(\varepsilon), \) and \(E^+(\varepsilon)\). To ensure that \(g^+(\varepsilon)|_\Sigma = g^-|_\Sigma\), we assume \(\psi^+\) vanishes on \(\Sigma\). Let \(Q = (Q^+, 0)\), \(\psi := (\psi^+, 0)\), and \(S(\varepsilon) := B_T(S(0)) - \nabla(\varepsilon)\Id\).

The following Lemma is a purely formal computation; see [3] for details.

**Lemma 5.1.** Adopt the notational conventions established above. Then

1. \(\partial_{|\varepsilon=0} a_n(1, D(\varepsilon), B_T(S(\varepsilon))) = (m-n) a_n(\psi, D, B_T(S))\).
2. \(\partial_{|\varepsilon=0} a_n(e^{-2\varepsilon \psi} Q, D(\varepsilon), B_T(S(\varepsilon))) = 0\) for \(m = n + 2\).

We use the following relations to apply Lemma 5.1:

\[
\begin{align*}
\partial_{|\varepsilon=0} S_{++}(\varepsilon) &= \frac{m-2}{2} \psi_{\alpha\beta}^+, & S_{++}^-(\varepsilon) &= S_{++}^-(0), \\
S_{--}^+(\varepsilon) &= S_{--}^+(0), & S_{--}^-(\varepsilon) &= S_{--}^-(0), \\
\partial_{|\varepsilon=0} L_{\alpha\alpha}^+(\varepsilon) &= -(m-1) \psi_{\alpha\beta}^+, & \partial_{|\varepsilon=0} \{\nabla_{\alpha}^+ (\varepsilon)(e^{-2\varepsilon \psi} Q)\} &= -2Q \psi_{\alpha\beta}^+.
\end{align*}
\]

Clearly Lemma 5.1 (1) yields no new information as the localizing function is continuous on \(\Sigma\) and thus cannot separate the contributions from \(\alpha_3\) and \(\alpha_4\). In fact, comparing the coefficient of the invariant \(\psi_{\alpha\beta}^+ S_{++}^- S_{--}^-\), one obtains

\[
\frac{m-2}{2}(\alpha_1 + \alpha_2) - (m-1)(\alpha_3 + \alpha_4) = (m-4)\alpha_5
\]

which is consistent with Equation (4.1). However, Lemma 5.1 (2) with \(m = 6\) yields the additional relation:

\[2\alpha_1 - 5\alpha_3 - 2\alpha_5 = 0.\]

We use Equation (4.1) to complete the proof of Theorem 1.2 by computing:

\[
\alpha_3 = 288, \quad \alpha_4 = 192.
\]

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