A planar heat conduction element improving temperature-related simulation accuracy

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Abstract. In the numerical study of temperature field in the fields of freezing and thawing cycles in cold regions, geothermal energy and functionally graded materials, the selection of heat conduction elements of the finite element method has an important influence on the numerical calculation results. Based on the fundamental solution and heat conduction theory, the fundamental solutions of the two-dimensional heat conduction control equation were added in the temperature interpolation function of the conventional planar 4-node quadrilateral heat conduction element. A new planar heat conduction non-coordination element with the fundamental solutions was established. The element interpolation function and the stiffness matrix were derived, and the element convergence was examined. The calculation accuracy was compared among the plane heat conduction non-coordination element with the fundamental solutions, the conventional finite element method and the analytical solution. The results showed that the planar heat conduction non-coordination element with the fundamental solutions proposed in this paper is high in calculation method and precision.

1. Introduction

The temperature field related problems widely exist in the field of geotechnical engineering, such as developing underground space, freezing and thawing cycles in cold regions, exploiting deep resources[1]. The effective numerical simulation method to study the heat substitution problem of the temperature field is significant for temperature-related practical engineering problems. A large number of researchers devoted themselves to obtaining high-performance heat conduction elements with higher numerical simulation accuracy, shorter computer running time, and better simulation results. Li et al.[2] and Chen[3] established the singular-boundary method and solved the transient thermal substitution problem. Zhang et al.[4] introduced the cubic B-spline interpolation basis functions in the element mapping to derive a planar heat conduction element. Wang et al.[5-7], Liu et al.[8], Li[9], and Cao et al.[10] combined the fundamental solutions with the boundary element to establish the thermodynamic problem of the mesh-free method to generate functionally graded materials. However, studies that date the fundamental solutions as an additional term into the temperature interpolation function of the thermal cross-linking element are scarce.

By adding the fundamental solutions in the logarithmic form of the two-dimensional heat exchange control equation to the temperature interpolation function with conventional planar four nodes, a new planar heat conduction non-coordinated element containing fundamental solutions is established. The interpolation function, element stiffness matrix and convergence are studied. Finally, an example is used to verify the performance of the method.
2. Heat conduction equation and finite element equation

The planar heat conduction equation without an internal heat source is\[^{[11]}\]:

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad (1)
\]

\[
T_n = T_w, \quad (2)
\]

\[
-\lambda \frac{\partial T}{\partial n} = \bar{q}, \quad (3)
\]

\[
\lambda \frac{\partial T}{\partial n} = h(T_e - T_f), \quad (4)
\]

where, \( T_w \) is the known temperature on the temperature boundary, \( \bar{q} \) is the heat flux intensity on the heat flow boundary, \( \lambda \) is the thermal conductivity, \( h \) is the heat transfer coefficient, \( T_e \) is the temperature of the solid, and \( T_f \) is the temperature of the fluid medium.

The heat conduction equation of the planar steady-state heat conduction problem can be derived by the variational method:

\[
[\lambda ]^e \{ T \}^e = \{ R \}^e, \quad (5)
\]

where, \([\lambda ]^e\) is the element conduction matrix, \(\{ T \}^e\) is the element node temperature array, and \(\{ R \}^e\) is the heat flow array of the element node.

3. Planar heat conduction element with fundamental solutions

3.1. Temperature interpolation function

Fig. 1 shows a random four-node quadrilateral heat conduction element 1234. The sources A, B, C and D of the fundamental solutions are established on the perpendicular lines of lines 12, 23, 34 and 41. The sources located outside the quadrilateral element. The distances from the sources A, B, C and D to the corresponding cell edge \( R_1, R_2, R_3 \) and \( R_4 \) are variable.

Figure 1 A planar four-node quadrilateral element with fundamental solutions.

In the planar cartesian coordinate system, the coordinates of source A, B, C and D are \((x_{n1}, y_{n1})\), \((x_{n2}, y_{n2})\), \((x_{n3}, y_{n3})\) and \((x_{n4}, y_{n4})\). \( r_A \) is the position vector from the coordinate origin O to source A. \( r_e \) is the position vector from point O to any point E in the cell. \( r_i \) is the position vector from the source A to point E. Then,

\[
r_i = r_A - r_e \Rightarrow (x - x_{n1}, y - y_{n1}). \quad (6)
\]

The temperature interpolation function can be expressed as the function of undetermined coefficients \( \beta_1, \beta_2, \beta_3 \) and \( \beta_4 \).

\[
T = \beta_1 x + \beta_2 y + \beta_3 (xy + f(x, y)), \quad (7)
\]

where, \( f(x, y) = \sum_{i=1}^{4} f_i(x, y) \), and the source functions of the four sources are \( f_i(x, y) = \frac{1}{2\pi} \ln \frac{1}{r_{ei}} \).
\[
 substituted the source functions into equation (7), then,

\[
\begin{align*}
\beta_1 + \beta_2 x_i + \beta_3 y_i + \beta_4 (x_i y_i + f_i(x_i y_i)) &= T_i \\
\beta_1 + \beta_2 x_i + \beta_3 y_i + \beta_4 (x_i y_i + f_i(x_i y_i)) &= T_j \\
\beta_1 + \beta_2 x_i + \beta_3 y_i + \beta_4 (x_i y_i + f_i(x_i y_i)) &= T_k \\
\beta_1 + \beta_2 x_i + \beta_3 y_i + \beta_4 (x_i y_i + f_i(x_i y_i)) &= T_l
\end{align*}
\]

(8)

The parameters \(\beta_1, \beta_2, \beta_3\) and \(\beta_4\) are got:

\[
\begin{align*}
\beta_1 &= s_{1i} T_1 + s_{12} T_2 + s_{13} T_3 + s_{14} T_4 \\
\beta_2 &= s_{3i} T_1 + s_{32} T_2 + s_{33} T_3 + s_{34} T_4 \\
\beta_3 &= s_{5i} T_1 + s_{52} T_2 + s_{53} T_3 + s_{54} T_4 \\
\beta_4 &= s_{7i} T_1 + s_{72} T_2 + s_{73} T_3 + s_{74} T_4
\end{align*}
\]

(9)

In equation (9), \(s_i (i=1,2,3,4, j=1,2,3,4)\) are parameters which can be determined when the element temperatures equal node temperatures.

Substitute equation (9) in equation (1), the temperatures \(T\) is got:

\[
T = N_1 T_1 + N_2 T_2 + N_3 T_3 + N_4 T_4,
\]

(10)

where,

\[
\begin{align*}
N_1 &= s_{11} + s_{12} x + s_{13} y + s_{14} (xy + f(x,y)) \\
N_2 &= s_{21} + s_{22} x + s_{23} y + s_{24} (xy + f(x,y)) \\
N_3 &= s_{31} + s_{32} x + s_{33} y + s_{34} (xy + f(x,y)) \\
N_4 &= s_{41} + s_{42} x + s_{43} y + s_{44} (xy + f(x,y))
\end{align*}
\]

3.2. Element stiffness matrix

The gradient of the temperature field of the element can be got with the derivative of equation (10):

\[
\frac{\partial \{T\}^r}{\partial n} = \left\{ \frac{\partial T}{\partial x} \right\}^r = \left[ B \right]\{T\}^r,
\]

(11)

where, \([B]\) is the element temperature gradient matrix:

\[
[B] = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y}
\end{bmatrix}
\]

When the material is isotropic, and the thermal conductivity is constant, the element conduction matrix is:

\[
[\lambda] = \int_B [B]^T \lambda [B] \text{d}x \text{d}y,
\]

(12)

where, \(b\) is the cell thickness, and \([\lambda]\) is a \(4 \times 4\) element conduction matrix. The conduction matrix obtained by taking the fundamental solutions as the temperature interpolation function has an effect on the accuracy of the solution.

3.3. The convergence

The convergence of the proposed planar heat conduction non-coordinating element with the fundamental solutions should be verified with the patch test. The slice test is generally used for the finite element method. For each node in the element, when the temperature value corresponding to the constant temperature gradient state is given, we have:
Obviously, the planar heat conduction element with the fundamental solutions satisfies equation
(13). When the size of the heat conduction element continues to shrink, the solution of this element
can converge quickly to the theoretical solution. Therefore, the plane heat conduction inconsistent
element with the fundamental solutions is convergent.

4. Example
In this section, an example is given to compare the calculation results of the heat conduction non-
coordinated element with the fundamental solutions, the conventional finite element method, and the
analytical solution. The conventional finite element method is conducted with ANINAT.

In Fig. 2, for a square area with a side length of \( L = 1.0 \), the boundary temperature is shown in Fig.
2 (a), and the temperature field in this area is calculated. The temperature field is symmetrical about
AB, so the right half ABCD area can be taken to solve the problem. As shown in Figs. (b)-(d), the
ABCD area is divided into two, four and eight elements, respectively. The temperature of the nodes \( i, j, k \) and \( m \) affects with the newly proposed method in this paper and traditional finite element
method.

![Figure 2](image)

What’s more, the analytical solution of the temperature field in this area is:

\[
T(x, y) = T_i + (T_2 - T_1) \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi}{L} x - \frac{sh\frac{n\pi}{W}}{sh\frac{n\pi}{L}} y.
\]  

(14)

The analytical solution of the temperature of each node is also calculated with MATLAB and listed
in Table 1.

| Elements | Method     | Temperature \( ^\circ \) |
|----------|------------|--------------------------|
|          |            | \( i \)  | \( j \)  | \( k \)  | \( m \)  |
| Two      | New        | 0.2505 | /      | /      | /      |
|          | Traditional| 0.1250 | /      | /      | /      |
| Four     | New        | 0.2476 | 0.1855 | /      | /      |
|          | Traditional| 0.3377 | 0.2682 | /      | /      |
| Eight    | New        | 0.2501 | 0.1875 | 0.0982 | 0.5268 |
|          | Traditional| 0.2786 | 0.1929 | 0.1009 | 0.5848 |
|          | Analytical | 0.2500 | 0.1870 | 0.0980 | 0.5270 |

The data are plotted in Fig. 3 for better comparison. The errors of the calculation results and
analytical solutions of the two finite element methods decrease with the increase of the number of
elements. However, the node temperature calculated with the newly proposed method is closer to the
analytical solution, indicating that the calculation accuracy of this method is higher than the
conventional method. Therefore, the calculation accuracy of the planar heat conduction non-
coordinated element with the fundamental solutions is better than that of the conventional planar four-
node quadrilateral element.
5. Conclusions
In the four-node quadrilateral heat conduction element, by adding the fundamental solutions of the two-dimensional heat conduction control equation to the temperature interpolation function, the new planar heat conduction non-coordinated element containing fundamental solutions was established. The element interpolation function and stiffness matrix were given, and the problem of element convergence was studied. The calculation results were compared among the heat conduction non-coordinated element with the fundamental solutions, the conventional finite element method and the analytical solutions. The results showed that the performance of the proposed planar heat conduction non-coordinated element was significantly better than the traditional 4-node quadrilateral heat conduction element.

Acknowledgment
The authors acknowledge the financial support of the project KJQN201902504 supported by the Scientific and Technological Research Program of Chongqing Municipal Education Commission.

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