Heisenberg Uncertainty Relations for Non-Hermitian Systems

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Robertson’s formalized version of the Heisenberg uncertainty relation contains a state of interest and two incompatible observables that are Hermitian operators. We construct uncertainty relation for PT invariant non-Hermitian quantum systems by introducing a more general condition of “good observable”. Our construction is not limited to the PT-symmetric phase but also valid in the PT-broken phase. In contrast to the usual quantum theory a good observable can also be a non-Hermitian operator for such systems. We show that the non-Hermitian Hamiltonian itself qualifies as a good observable in the PT-symmetric phase, but not in the broken phase. Consequently, this fact can be used as a diagnostic tool to detect the PT phase transition in any arbitrary finite-dimensional system.

Introduction: Heisenberg uncertainty principle [1–3] is one of the most important tenets of quantum mechanics. It encapsulates the impossibility of simultaneous measurement of two incompatible observables. In the standard quantum mechanics, the real spectrum, complete set of orthonormal eigenstates with positive definite norm and unitary evolution of the system are guaranteed by Hermitian Hamiltonian. Operators for the physical observables are Hermitian with the measurement result as one of its real eigenvalues. The Heisenberg uncertainty relation rigorously proved by Robertson [2] for the two general incompatible observables is

\[ \Delta A \Delta B \geq \frac{1}{4} |\langle \psi | [A, B] |\psi \rangle|^2. \] (1)

It has been actively debated in the last decade under the two different outlooks of ‘preparation’ and ‘measurement’ of a quantum state [4–6]. The Eq. (1) is valid for all the states unless the observables \(A\) and \(B\) are incompatible on the state of the system, i.e., the state of the system \(|\psi\rangle\) is an eigenstate of either \(A\) or \(B\) [7]. The remedy to this triviality issue gave rise to another class of uncertainty relation involving sum of variances for the incompatible observables known as ‘sum uncertainty relation’ [6–8]. The experimental realization of various forms of uncertainty relation has been registered from time to time [9–11].

Inequality (1) saturates for the minimum uncertainty states (MUS). MUS are vital because these states are expected to reproduce, as closely as possible, the classical motion, and those have been of utmost importance in the broad area of physics starting from the quantum optics [12] to the theoretical developments in quantum gravity [13]. If one defines the ratio of LHS and RHS of the inequality (1) as \(\eta\), then \(\eta = 1\) correspond to MUS. It has also been shown that these MUS are the eigenstates of the operator \(A + i\lambda B\) for \(\lambda = \Delta A/\Delta B\) [14].

Over the past two decades there has been great interest in the certain class of non-Hermitian quantum theories where Hermiticity condition on the Hamiltonian of the system is replaced with a physical and rather less constraining condition of PT-symmetry [15–22]. It has been well established that such a class of non-Hermitian theories can lead to consistent quantum theories with complete real spectrum, unitary time evolution and probabilistic interpretation in a modified Hilbert space equipped with a positive definite inner product, namely \(\mathbb{C}PT\) inner product [16]. Such PT-invariant non-Hermitian systems generally exhibit a phase transition, more specifically PT symmetry breaking transition that separates two regions, (i) PT-symmetric phase in which the entire spectrum is real and the eigenfunctions of the Hamiltonian respect PT symmetry and (ii) PT-broken phase in which the entire spectrum (or a part of it) is in complex conjugate pairs and the eigenstates of the Hamiltonian are not the eigenstates of PT operator. The phase transition happens at the exceptional point (EP) for the particular Hamiltonian. Even though we have fully consistent quantum theory in the unbroken phase with \(\mathbb{C}PT\) inner product very little has been explored in the broken phase which also plays significant role in developing non-Hermitian theories [17–21, 23]. While the \(\mathbb{C}PT\) inner product is restricted to the PT-symmetric phase only, recently proposed G-metric inner product based on the geometry of the quantum states by defining a connection-compatible positive definite metric operator \(G\) [24–27], can be extended even in the PT-broken phase.

The questions arises, what is the fate of the Heisenberg uncertainty relation for the system described by PT-invariant non-Hermitian Hamiltonian? This question is critical because the uncertainty relation is the most basic principle of quantum mechanics and its behaviour can be highly validating of the newly proposed theory of non-Hermitian quantum mechanics. In this letter, we show that the Hermiticity condition on the operator \(O\) can be replaced by a more general condition we call “good observable” meaning \(O^\dagger G = GO\) and use it to construct the modified uncertainty relation for two such good observables \(A\) and \(B\)

\[ \Delta A_G^2 \Delta B_G^2 \geq \frac{1}{4} |\langle \psi | G[A, B] |\psi \rangle|^2. \] (2)
Here, $\Delta A_G^2 = \langle \psi | G A^2 | \psi \rangle - (\langle \psi | G A | \psi \rangle)^2$ and $\Delta B_G^2 = \langle \psi | G B^2 | \psi \rangle - (\langle \psi | G B | \psi \rangle)^2$. A good observable can be either Hermitian or non-Hermitian, and $A$ and $B$ being good observables guarantees that the uncertainty relation holds in both the $\mathcal{PT}$-symmetric as well as $\mathcal{PT}$-broken phase. We also show that the non-Hermitian Hamiltonian itself is one of such good observables in the $\mathcal{PT}$-symmetric phase but not in the $\mathcal{PT}$-broken phase, thus indicating the EP for the system. The pictorial representation of the good observable condition on the Hamiltonian in Fig. 1 clearly demonstrates that the usual Hermitian quantum mechanics is a special case of non-Hermitian $\mathcal{PT}$-symmetric quantum mechanics under the $G$-metric inner product formalism in the context of the observables. Also, MUS in this context are represented by $\eta_G = 1$ for $\eta_G$ defines as the ratio of the LHS and RHS of the Eq. 2 and $\eta_G = \eta$ for Dirac inner product ($G = 1$).

$2 \times 2$ Non-Hermitian System: To realize the above modified uncertainty relation, we begin with a model of $2 \times 2$ one parameter $\mathcal{PT}$-invariant system described by the Hamiltonian

$$H(\gamma) = \begin{pmatrix} i\gamma & 1 \\ 1 & -i\gamma \end{pmatrix},$$

whose eigenvectors are denoted as $|E_1\rangle$ and $|E_2\rangle$. By tuning the parameter $\gamma$, one can go from $\mathcal{PT}$-symmetric phase to the $\mathcal{PT}$-broken phase, the EP is at $\gamma = \gamma_{EP} = 1$.

A state is called $\mathcal{PT}$ symmetric state if it respects the $\mathcal{PT}$ symmetry and $\mathcal{PT}$ broken state otherwise. Thus in the $\mathcal{PT}$-symmetric phase $|E_{1,2}\rangle$ are also eigenstates of $\mathcal{PT}$ operator. The state we choose to understand the behaviour of uncertainty relation is a general superposition of the eigenstates of the Hamiltonian $H$ in both the $\mathcal{PT}$-symmetric and $\mathcal{PT}$-broken phase

$$|\Psi\rangle = N(|E_1\rangle + pe^{i\theta} |E_2\rangle),$$

where $N$ is the normalization factor and $p, \theta$ are the state parameters. We firstly investigate the uncertainty relation within the Dirac product framework where the observables are Hermitian operators. Further we use the $G$-metric inner product (eigenstates remain orthonormal) to address the Hermiticity condition on the operators by the means of violation of the uncertainty relation.

**Dirac Inner Product:** Within the Dirac product, one can choose Hermitian pairs of non-commuting observables for the parametrized state in (4) to construct a particular uncertainty relation. For an example, we choose the non-commuting Pauli spin operators $\sigma_x$ and $\sigma_z$ over the state (4) and see no violation of the uncertainty relation (1) for any value of the state parameters. This is anticipated because the uncertainty relation must be valid under the Dirac inner product for Hermitian operators for any quantum state (see Fig. 2).

Additionally, we focus on MUS because they are distinguishably important and mark a boundary for the violation of the uncertainty relation. In the $p, \theta$ parameter space, we spot six different lines corresponding to the MUS lying on them in the $\mathcal{PT}$-symmetric phase and five different lines in the $\mathcal{PT}$-broken phase. The MUS are marked by solid (blue) and dashed (red) lines in the Fig 2(a) for the $\mathcal{PT}$-symmetric and in Fig 2(b) for the $\mathcal{PT}$-broken phase, respectively. It is important to note that a state on the MUS line in the $\mathcal{PT}$-symmetric or broken phase can be either $\mathcal{PT}$ symmetric state or $\mathcal{PT}$ broken state.

Given that MUS for the two incompatible observables $A$ and $B$ are the eigenstates of the operator $A + i\lambda B$ [14], it is straightforward to see that for the above pair of incompatible observables, the operator $\sigma_x + i\lambda \sigma_z$ is nothing but $H(\lambda)$ and one can use $\lambda$ to infer about the states lying on the MUS lines. However, at the EP ($\lambda = 1$), it is not trivial and requires one to separately check if the state is an eigenstate of $\mathcal{PT}$ operator or not in order to know if the state is $\mathcal{PT}$ symmetric state or $\mathcal{PT}$ broken state. In the $\mathcal{PT}$-symmetric phase, when the generic state $|\psi\rangle$ reduces to one of the eigenstates of the Hamiltonian (for $p = 0$, $p = \infty$), the MUS are $\mathcal{PT}$ symmetric states. The lines $\theta = 0, \pi, 2\pi$ lines also contain $\mathcal{PT}$ symmetric MUS because these are the eigenstates of $H(\lambda)$ with $\lambda < 1$. All the states on the MUS line $p = 1$ are $\mathcal{PT}$ broken states except for $\theta = 0, \pi, 2\pi$ which are at the EP ($\lambda = 1$). MUS located at the EP are the eigenstates of the $\mathcal{PT}$ operator and thus $\mathcal{PT}$ symmetric states. In the $\mathcal{PT}$-broken phase, the MUS lying on the lines $p = 0, p = \infty$ are the $\mathcal{PT}$ broken states. The MUS on $\theta = \pi/2, 3\pi/2$ lines are also $\mathcal{PT}$ broken states as $\lambda > 1$. The MUS spotted at $p = 1$ is $\mathcal{PT}$ symmetric states excluding the states located at the EP $\theta = \pi/2, 3\pi/2$. We now explicitly introduce the
matrix form of $G$ is defined as $\langle \text{eigenvector of } H \rangle$, the operators in the frame of uncertainty relation. $G$ can be chosen as a positive definite metric operator $G$ invariant orthonormal. The inner product defined with a positive definite inner product makes the eigenvectors of this Hamiltonian, it is necessary to define a new inner product for 2 systems in the $G$-metric inner product framework.

FIG. 2: Contour plots for $\eta$ in the uncertainty relation (1) for the observables $\sigma_x$ and $\sigma_z$ and generic state (4) in case of (a) $\mathcal{PT}$-symmetric phase, and (b) $\mathcal{PT}$-broken phase. Solid blue lines correspond to $\mathcal{PT}$ symmetric MUS and red dashed lines correspond to $\mathcal{PT}$ broken MUS.

FIG. 3: Contour plots for $\eta_G$ in the uncertainty relation (2) for observables $\sigma_x$ and $\sigma_z$ and generic state (4) in case of (a) $\mathcal{PT}$ symmetric phase, and (b) $\mathcal{PT}$ broken phase. Violation is seen because they are not good observables in the $G$-metric inner product framework.

$G$-metric inner product to learn about the behaviour of the operators in the frame of uncertainty relation.

$G$-Metric Inner Product: For the non-Hermitian $\mathcal{PT}$-invariant Hamiltonian, it is necessary to define a new inner product that makes the eigenvectors of this Hamiltonian orthonormal. The inner product defined with a positive definite metric operator $G$ satisfies this property and can be chosen as $G = \sum_i \langle L_i \rangle \langle L_i \rangle^* |L_i\rangle$ being the $i^{th}$ left-eigenvector of $H$. The dual vector of $|\psi\rangle_G$ in this formalism is defined as $\langle \psi | G = \langle \psi | G$ and for the corresponding $|\psi\rangle_G$, the inner product reads $\langle \psi | \psi\rangle_G = \langle \psi | G | \psi\rangle$. The matrix form of $G$ for the $2 \times 2$ systems in the $\mathcal{PT}$-symmetric phase is given by

$$G = \frac{1}{\sqrt{1 - \gamma^2}} \begin{bmatrix} 1 & -i\gamma \\ i\gamma & 1 \end{bmatrix},$$

(5)

In the $\mathcal{PT}$-broken phase $G$ reads

$$G = \frac{1}{\sqrt{\gamma^2 - 1}} \begin{bmatrix} \gamma & -i \\ i & \gamma \end{bmatrix},$$

(6)

On choosing the observables $\sigma_x$ and $\sigma_z$ for the generic state (4) with this new inner product, the uncertainty relation (2) shows violation for a rather big segment of states for both $\mathcal{PT}$-symmetric and $\mathcal{PT}$-broken phase as shown in Fig. 3. As it turns out, despite being Hermitian operators $\sigma_x$ and $\sigma_z$ are not good observables for the $G$-metric inner product.

In the usual quantum mechanics, the real expectation value of an operators imposes the ‘Hermiticity’ condition on the operators, i.e., $O = O^\dagger$. However, in the non-Hermitian quantum mechanics equipped with $G$-inner product, making the $G$-expectation value of an operator $O$ for any state $|\psi\rangle$ to be real leads to

$$O^\dagger G = GO.$$  

(7)

We coin the term “good observable” for the operators that satisfy this condition. Since $G$ is Hermitian, for Hermitian operators the condition (7) automatically means $[G, O] = 0$. For the Dirac inner product ($G = 1$), the condition (7) trivially reduces to the condition of Hermiticity of the operators.

A good observable that is also hermitian is $\sigma_y$ for the $G$ mentioned in (5) and (6). However, it is impossible to find any other Hermitian good observables incompatible with $\sigma_y$ for the purpose of constructing the uncertainty relation for $2 \times 2$ systems. Noticeably, if one allows a more general class of observables, i.e., non-Hermitian operators ($O \neq O^\dagger$), it is easy to construct them for both $\mathcal{PT}$-symmetric phase and $\mathcal{PT}$-broken phase. The structure of such operators in the $\mathcal{PT}$-symmetric phase ($\gamma < 1$) is

$$O^s = \begin{pmatrix} i\gamma x & x - iy \\ x + iy & -i\gamma x \end{pmatrix},$$

(8)

and in the $\mathcal{PT}$ broken phase ($\gamma > 1$) is

$$O^b = \frac{1}{\gamma} \begin{pmatrix} ix & \gamma(x - iy) \\ \gamma(x + iy) & -ix \end{pmatrix},$$

(9)

where $x, y$ are real numbers. As an example, we find the set of incompatible good observables $H(\gamma), \sigma_y$ in the $\mathcal{PT}$-symmetric and $H(1/\gamma), \sigma_y$ in the $\mathcal{PT}$-broken phase. Note that $H(\gamma)$ does not satisfy the good observables condition in Eq. (7) for the $\mathcal{PT}$-broken phase. As evident from Fig. 4, the uncertainty relation (2) holds good as long as one chooses the good observables.

Using the similar analysis as for the Dirac product, we obtain that in the $\mathcal{PT}$-symmetric phase, the MUS line...
In this section, we demonstrate that the above criterion can be used to detect \( PT \) phase transition. We use the two non-Hermitian Hamiltonians \( H^A \) with the non-zero matrix entries \( H_{j,j+1} = 1 \), \( H_{j,j+2} = -H_{j,j+1} \), and \( H^B \) with the non-zero matrix entries \( H_{j,j+1} = 1 \), \( H_{j,j+2} = -H_{j,j+3} \) for the Hamiltonian \( H^A \) and \( H^B \) respectively. We now define an operator, \( K = H^A G - G H \) followed by the identifier \( \kappa = N^2 \sum_{i,j} |K_{ij}|/\sum_{i,j} |\tilde{H}_{ij}| \), where \( \tilde{H} \) stands for either \( H^A \) or \( H^B \) depending on the choice of the Hamiltonian and \( G \) represents the corresponding \( G \)-metric. The identifier \( \kappa \) is defined such that \( \kappa = 0 \) for good observable and non-zero otherwise. We now plot \( \gamma \) vs. \( \kappa \) and is evident from the Fig. 5 that \( \kappa = 0 \) in the \( PT \)-symmetric phase characterized by \( \gamma < 1 \) for the Hamiltonian \( H^A \), and \( \gamma < 0.48 \) for the Hamiltonian \( H^B \). Hence, it is apparent that \( \kappa \) can be used as a parameter to detect \( PT \)-transition.

Although, for \( 2 \times 2 \) systems in the \( PT \)-broken phase \( H(1/\gamma) \) is a good observable but this is not extendable for the \( N \times N \) systems. It remains a challenge to construct a “good observable” for a \( N \times N \) Hamiltonian in the \( PT \)-broken phase.

Conclusion: The Heisenberg uncertainty relation (1) does not apply to the non-Hermitian systems. In this letter, we have shown that for \( PT \) non-Hermitian systems the observables need not necessarily be Hermitian operators rather these are good observables (7). We have constructed the uncertainty relation (2) for these good observables and identified MUS which mark a boundary for the violation of the uncertainty relation. Finding such good observables is in general a tedious tasks, however our work suggests that the non-Hermitian Hamiltonian itself qualifies as one such good observable as long as we remain in the \( PT \)-symmetric phase. This fact can be used as a diagnostic tool to detect the EP for a general \( N \times N \) Hamiltonian that can also be experimentally realized in...
an ultracold fermionic system as a Hamiltonian with gain and loss term. It is of great scope to look at the time evolution dynamics of non-Hermitian systems in the context of uncertainty relation and also the behaviour of tighter sun uncertainty relation for such systems.

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Is Hamiltonian a good observable?

In this section, we prove that for any arbitrary finite dimensional system, the non-Hermitian Hamiltonian itself is a good observable in the $\mathcal{PT}$-symmetric phase but not in the broken phase. In order to find whether the
non-Hermitian Hamiltonian $H$ is a good observable or not, one needs to check whether $H$ satisfies the condition of good observable, i.e., $H^\dagger G = GH$, where $G = \sum_i \langle L_i | L_i \rangle$. $| L_i \rangle$ and $| R_i \rangle$ are left eigenvector and right eigenvector of $H$, and they satisfy the eigenvalue equations $H^\dagger | L_i \rangle = h_i | L_i \rangle$ and $H | R_i \rangle = h_i^* | R_i \rangle$, respectively. The vectors $| L_i \rangle$ and $| R_i \rangle$ also satisfy biorthonormalization condition, i.e., $\langle R_i | L_j \rangle = \delta_{ij}$ and it automatically implies $\sum_i \langle R_i | L_i \rangle = 1$. All $h_i$s are completely real ($h_i = h_i^*$) when $H^\dagger$ (or $H$) belongs to the $PT$-symmetric phase and in the broken phase at least two of the eigenvalues must be complex. Now,

$$H^\dagger G = \sum_i H^\dagger | L_i \rangle \langle L_i | = \sum_i h_i | L_i \rangle \langle L_i |$$ (10)

On the other hand,

$$GH = \sum_i | L_i \rangle \langle L_i | H = \sum_i | L_i \rangle \langle L_i | H \sum_j | R_j \rangle \langle L_j |$$
$$= \sum_{ij} h_i^* | L_i \rangle \langle L_j | \delta_{ij} = \sum_i h_i^* | L_i \rangle \langle L_i |$$ (11)

It says that $H^\dagger G = GH$ will be satisfied only if $h_i = h_i^*$, i.e., $H$ belongs to the $PT$-symmetric phase. On the other hand in the broken phase, the following condition can not hold. Hence, $H$ will be always a good observable in the symmetric phase but not in the broken phase.

**Uncertainty relation for good observables**

We prove the uncertainty relation for the $G$-inner product and good observables in this section. In the usual quantum mechanics where the observables are Hermitian, the Heisenberg Uncertainty relation for two incompatible observables $A$ and $B$ is given by

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} \langle \psi | [A, B] | \psi \rangle^2,$$ (12)

where $| \psi \rangle$ is a state in the Hilbert space, and $\Delta A^2 = \langle \psi | A^2 | \psi \rangle - (\langle \psi | A | \psi \rangle)^2$, $\Delta B^2 = \langle \psi | B^2 | \psi \rangle - (\langle \psi | B | \psi \rangle)^2$. On relaxing the “Hermiticity” condition the variances are given by

$$\Delta A^2 = \langle \psi | A^\dagger A | \psi \rangle - \langle \psi | A^\dagger | \psi \rangle \langle \psi | A | \psi \rangle,$$
$$\Delta B^2 = \langle \psi | B^\dagger B | \psi \rangle - \langle \psi | B^\dagger | \psi \rangle \langle \psi | B | \psi \rangle,$$ (13)

and the commutator term on the RHS of Eq. (12) can be written as

$$\frac{1}{4} \langle \psi | [A, B] | \psi \rangle^2$$
$$= -\frac{1}{4} \langle \psi | (AB - BA) | \psi \rangle \langle \psi | (B^\dagger A^\dagger - A^\dagger B^\dagger) | \psi \rangle.$$ (14)

In the $G$-inner product formalism the adjoint the operators and the inner product need to be redefined [25] under the rule $A^\dagger \rightarrow G^{-1} A^\dagger G$, $B^\dagger \rightarrow G^{-1} B^\dagger G$ and the inner product under $\langle \psi | Q | \psi \rangle \rightarrow \langle \psi | Q G | \psi \rangle$. RHS and RHS of the Eq. 13 can now be written as

$$\Delta A^2_G = \langle \psi | A^\dagger G A | \psi \rangle - \langle \psi | A^\dagger | \psi \rangle \langle \psi | G A | \psi \rangle,$$
$$\Delta B^2_G = \langle \psi | B^\dagger G B | \psi \rangle - \langle \psi | B^\dagger | \psi \rangle \langle \psi | G B | \psi \rangle,$$ (15)

$$\frac{1}{4} \langle \psi | G [A, B] | \psi \rangle^2$$
$$= \frac{1}{4} \langle \psi | G (AB - BA) | \psi \rangle \langle \psi | (B^\dagger A^\dagger - A^\dagger B^\dagger) G | \psi \rangle.$$ (16)

Additional constraint of $A$ and $B$ being good observables, i.e., $A^\dagger G = GA$ and $B^\dagger G = GB$ leads to

$$\Delta A^2_G = \langle \psi | G A^2 | \psi \rangle - (\langle \psi | G A | \psi \rangle)^2,$$
$$\Delta B^2_G = \langle \psi | G B^2 | \psi \rangle - (\langle \psi | G B | \psi \rangle)^2,$$ (17)

$$\frac{1}{4} \langle \psi | G [A, B] | \psi \rangle^2$$
$$= -\frac{1}{4} \langle \psi | G (AB - BA) | \psi \rangle^2 = \left[ \frac{1}{2i} \langle \psi | G [A, B] | \psi \rangle \right]^2.$$ (18)

For the uncertainty relation to hold for the two good observables $A$ and $B$, the following inequality must satisfy

$$\Delta A^2_G \Delta B^2_G \geq \frac{1}{4} \langle \psi | G [A, B] | \psi \rangle^2,$$ (19)

where $\Delta A^2_G = \langle \psi | G A^2 | \psi \rangle - (\langle \psi | G A | \psi \rangle)^2$ and $\Delta B^2_G = \langle \psi | G B^2 | \psi \rangle - (\langle \psi | G B | \psi \rangle)^2$.

It is straightforward to show that the above inequality holds true for good observables in the following subsequent steps. Let us define two vectors in a vector space

$$| f \rangle = (A - \langle \psi | G A | \psi \rangle | \psi \rangle),$$
$$| g \rangle = (B - \langle \psi | G B | \psi \rangle | \psi \rangle).$$ (20)

The dual vectors are

$$\langle f | = \langle \psi | (A^\dagger - \langle \psi | G A^\dagger | \psi \rangle),$$
$$\langle g | = \langle \psi | (B^\dagger - \langle \psi | G B^\dagger | \psi \rangle).$$ (21)

Given that $A$ and $B$ are good observables, $\langle \psi | G A^\dagger | \psi \rangle$, $\langle \psi | G A | \psi \rangle$, $\langle \psi | G B^\dagger | \psi \rangle$, $\langle \psi | G B | \psi \rangle$ are all real quantities. On plugging in $| f \rangle$, and $| g \rangle$ in the Cauchy–Schwarz inequality,

$$\langle f | G | f \rangle \langle g | G | g \rangle \geq | \langle f | G | g \rangle |^2$$
$$\geq \frac{1}{2} | \langle f | G | g \rangle|^2.$$ (22)
Different terms in the above inequality can be simplified to
\[
\langle f|G|f \rangle = \langle \psi|(A^2 - r')G(A - r)|\psi \rangle = \langle \psi|GA^2|\psi \rangle - r^2,
\]
\[
(g|G|g) = \langle \psi|(B^2 - p')G(B - p)|\psi \rangle = \langle \psi|GB^2|\psi \rangle - p^2,
\]
\[
\langle f|G|g \rangle = \langle \psi|(A^2 - r')G(B - p)|\psi \rangle = \langle \psi|GAB|\psi \rangle - rp,
\]
\[
\langle g|G|f \rangle = \langle \psi|(B^2 - p')G(A - r)|\psi \rangle = \langle \psi|GBA|\psi \rangle - rp,
\]
for \( r = \langle \psi|GA|\psi \rangle, \) \( r' = \langle \psi|GA^\dagger|\psi \rangle, \) \( p = \langle \psi|GB|\psi \rangle, \)
\( p' = \langle \psi|GB^\dagger|\psi \rangle \) and \( |\psi|G|\psi \rangle = 1. \) For the real \( r, p \) and \( (GAB)\dagger = GBA \) we have \( (f|G|g)^* = (g|G|f). \) Collecting these relation in Eq. 22 leads to Eq. 19.

Therefore, the Heisenberg’s uncertainty relation holds true for \( G \)-metric inner product and good observables. However, the following derivation does not go through for any arbitrary operators and the violation of the inequality in Eq. 19 can be seen for not good observables as shown in the main text.

**Minimum Uncertainty states (MUS)**

On the lines of a result in the usual quantum mechanics[14], one would expect that for the \( G \)-inner product formalism and good observables \( A \) and \( B \), the MUS are the eigenstates of operator \( A + i\lambda_G B \). We prove it here in this section. The minimum uncertainty states in the \( G \)-metric formalism and good observables saturate the inequality in Eq. 19, i.e.,

\[
\Delta A^2_G \Delta B^2_G = \frac{1}{4} |\langle \psi|GA, B|\psi \rangle|^2
\]

In the context of Cauchy-Schwarz inequality for any two vectors the above equality corresponds to

\[
\langle f|G|f \rangle \langle g|G|g \rangle = |\langle f|G|g \rangle|^2 = \frac{1}{2} |\langle f|G|g \rangle - \langle g|G|f \rangle|^2.
\]

This implies

\[
\frac{1}{2} |\langle f|G|g \rangle + \langle g|G|f \rangle|^2 = 0,
\]

\[
\langle \psi|GAB|\psi \rangle + \langle \psi|GBA|\psi \rangle = 2rp.
\]

Separately, one can infer that \( |f \rangle = \mu|g \rangle \) from \( \langle f|G|f \rangle \langle g|G|g \rangle = |\langle f|G|g \rangle|^2 \) for a complex number \( \mu. \) Therefore,

\[
(A - r)|\psi \rangle = \mu(B - p)|\psi \rangle.
\]

Using the conditions in Eq. 25 and Eq. 26, one can show that MUS are the eigenstates of the operator \( A + i\lambda_G B \), i.e.,

\[
(A + i\lambda_G B)|\psi \rangle = (r + i\lambda_G p)|\psi \rangle,
\]

where \( \lambda_G \) is a real number given by

\[
\lambda^2_G = \frac{((\langle \psi|GA^2|\psi \rangle - r^2))/((\langle \psi|GB^2|\psi \rangle - p^2))}
\]

Note that \( \langle \psi|GB^2|\psi \rangle \neq p^2 \).

Hence, for the \( G \)-inner product formalism and good observables the MUS condition is perfectly aligned with the usual quantum mechanics for \( G = 1 \).

It is possible to infer some MUS discussed for specific cases in the main text, from the above result. We have shown in Fig. 4 that the general state \( |\psi \rangle = p e^{i\theta} |E_1 \rangle \) are MUS for \( p = 1 \) and \( \theta = \pi/2, 3\pi/2, \) for the operators \( A = H(\gamma), \) and \( B = \sigma_y \) in the \( PT \)-symmetric phase. Here \( N \) is the normalization constant such that \( \langle \psi|G|\psi \rangle = 1. \) Note that both \( A \) and \( B \) here are good observables as per our definition. At least for \( \theta = \pi/2, i.e., |\psi \rangle = N|E_1 \rangle + i|E_2 \rangle \), it is straightforward to check that indeed the state is one of the MUS in the following steps.

\[
[H(\gamma) + i\lambda_G \sigma_y]|E_1 \rangle + i|E_2 \rangle = N(\epsilon|E_1 \rangle - i\lambda_G \sigma_y|E_2 \rangle + i\lambda_G \sigma_y|E_1 \rangle + i|E_2 \rangle),
\]

where \( \pm \epsilon \) are the eigenvalues of \( H(\gamma). \) Since \( \sigma_y|E_1 \rangle = -|E_2 \rangle \) and \( \sigma_y|E_1 \rangle = |E_2 \rangle, \) \( N(|E_1 \rangle + i|E_2 \rangle) \) is an eigenstate of \( [H(\gamma) + i\lambda_G \sigma_y] \) if \( \epsilon = -\lambda_G \) and

\[
\lambda^2_G = \epsilon^2 = \frac{((\langle \psi|GA^2|\psi \rangle - r^2))/((\langle \psi|GB^2|\psi \rangle - p^2))}
\]

Therefore, the condition for the state \( |\psi \rangle = N(|E_1 \rangle + i|E_2 \rangle) \) to be one of the MUS is \( \epsilon = -\lambda_G. \) In fact, the condition for the state \( N(|E_1 \rangle + p e^{i\theta}|E_2 \rangle) \) to be MUS is \( \epsilon = -\lambda_G \sin \theta. \)

**MUS: symmetry of the state**

We have observed for the Dirac inner product formalism and the observables \( A = \sigma_x \) and \( B = \sigma_z \) that the generic state \( |E_1 \rangle + p e^{i\theta}|E_2 \rangle \) corresponds to the lines \( p = 1, 0, \infty \) (for all values of \( \theta \)) or \( \theta = 0, \pi, 2\pi \) (for all values of \( p \)) in the \( PT \)-symmetric phase.

It is established that these MUS must be eigenstates of the operator \( A + i\lambda B \), where \( \lambda = \Delta A/\Delta B. \) If one chooses \( A = \sigma_x \) (or \( \sigma_y \)), and \( B = \sigma_z, \) then \( A + i\lambda B \) will itself be non-Hermitian \( PT \)-invariant operator with all the real eigenvalues for \( \lambda < 1 \) and complex eigenvalues for \( \lambda > 1. \) Therefore, it is self explanatory that for given MUS if \( \lambda < 1(>1) \) the MUS are \( PT \) symmetric (broken) states. However, for \( \lambda = 1 \) nothing can be concluded from this analysis. One such example is \( N(|E_1 \rangle + |E_2 \rangle) \) in the \( PT \)-symmetric phase. One needs to check separately if such states are eigenstates of the \( PT \) operator.