Generic modules for the category of filtered by standard modules

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Abstract

Here we show that, given a finite homological system \((\mathcal{P}, \leq, \{\Delta_u\}_{u \in \mathcal{P}})\) for a finite-dimensional algebra \(\Lambda\) over an algebraically closed field, the category \(\mathcal{F}(\Delta)\) of \(\Delta\)-filtered modules is tame if and only if, for any \(d \in \mathbb{N}\), there are only finitely many isomorphism classes of generic \(\Lambda\)-modules adapted to \(\mathcal{F}(\Delta)\) with endolength \(d\). We study the relationship between these generic modules and one-parameter families of indecomposables in \(\mathcal{F}(\Delta)\). This study applies in particular to the category of filtered by standard modules for standardly stratified algebras.

1 Introduction

Denote by \(k\) a fixed field and let \(\Lambda\) be a finite-dimensional \(k\)-algebra. Given a \(\Lambda\)-module \(G\), recall that by definition the endolength of \(G\) is its length as a right \(\text{End}_\Lambda(M)^\text{op}\)-module. The \(\Lambda\)-module \(G\) is called generic if it is indecomposable, of infinite length as a \(\Lambda\)-module, but with finite endolength. The algebra \(\Lambda\) is called generically tame if, for each \(d \in \mathbb{N}\), there is only a finite number of isoclasses of generic \(\Lambda\)-modules with endolength \(d\). This notion was introduced by W. W. Crawley-Boevey in [15], where he proved that, in case \(k\) is algebraically closed, the class of generically tame algebras coincides with the class of tame algebras. Recall that an algebra \(\Lambda\), over an algebraically closed field \(k\), is tame if the pairwise non-isomorphic indecomposable modules in each dimension can be parametrized by a finite number of one-parameter families. Thus, the notion of generic tameness generalizes the notion of tameness for arbitrary fields.

In order to be more precise, we recall some definitions from [6].

**Definition 1.1.** For any \(k\)-algebra \(R\) and any \(R\)-module \(M \in R\text{-Mod}\) we write \(E_M = \text{End}_R(M)^\text{op}\). Any \(M \in R\text{-Mod}\) is naturally a right \(E_M\)-module. By

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definition, the endolength of $M$, denoted by $\text{endol}(M)$, is its length as an $E_M$-module. We say that $M$ is endofinite if it has finite endolength. The $R$-module $M$ is called pregeneric iff it is an endofinite indecomposable with infinite dimension over $k$.

If $R$ is a finite-dimensional $k$-algebra, the notion of pregeneric $R$-module coincides with the usual notion of generic $R$-module.

In this article we study generic $\Lambda$-modules adapted to the category of $\Delta$-filtered modules $\mathcal{F}(\Delta)$ in the case where $\Lambda$ is a finite-dimensional algebra, over an algebraically closed field, and $(\mathcal{P}, \leq, \{\Delta_u\}_{u \in \mathcal{P}})$ is a homological system. In the following, we recall the basic terminology from [22] and [9].

A preordered set $(\mathcal{P}, \leq)$ is a non-empty set $\mathcal{P}$ together with a reflexive transitive relation $\leq$. We have the equivalence relation in $\mathcal{P}$ defined by $u \sim v$ iff $u \leq v$ and $v \leq u$. The set $\mathcal{P} := \mathcal{P}/\sim$ of equivalence classes is partially ordered by the relation $\pi \leq \tau$ iff $\pi \leq v$, where $\pi$ and $\tau$ denote the equivalence classes of $u$ and $v$, respectively.

Definition 1.2. Given a finite-dimensional $k$-algebra $\Lambda$, a (finite) homological system $(\mathcal{P}, \leq, \{\Delta_u\}_{u \in \mathcal{P}})$ for $\Lambda$ consists of a finite preordered set $(\mathcal{P}, \leq)$ and a family of pairwise non-isomorphic indecomposable finite-dimensional $\Lambda$-modules $\{\Delta_u\}_{u \in \mathcal{P}}$ satisfying the following two conditions:

1. $\text{Hom}_\Lambda(\Delta_u, \Delta_v) \neq 0$ implies $u \leq v$;
2. $\text{Ext}_\Lambda^1(\Delta_u, \Delta_v) \neq 0$ implies $u \leq v$ and $u \not\sim v$.

Given such an homological system, we set $\Delta := \{\Delta_u \mid u \in \mathcal{P}\}$ and denote by $\mathcal{F}(\Delta)$ the full subcategory of $\Lambda$-mod consisting of the trivial module and all those $M \in \Lambda$-mod which admit a $\Delta$-filtration, that is a filtration of submodules

$$0 = M_\ell \subseteq M_{\ell-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that $M_j/M_{j+1}$ is isomorphic to some module in $\Delta$, for each $j \in [0, t - 1]$.

A finite homological system $\mathcal{H} = (\mathcal{P}, \leq, \{\Delta_u\}_{u \in \mathcal{P}})$ for a finite-dimensional $k$-algebra $\Lambda$ is called admissible if $\Lambda \in \mathcal{F}(\Delta)$ and the number of isoclasses of indecomposable projective $\Lambda$-modules coincides with the cardinality of $\mathcal{P}$. In this case, the modules in $\Delta$ are determined up to isomorphism and are known as the standard modules; $\Lambda$ is called a pre-standardly stratified algebra, see [9](2.11).

A pre-standardly stratified algebra $\Lambda$, with preordered index set $(\mathcal{P}, \leq)$, is called standardly stratified if $(\mathcal{P}, \leq)$ is a partial order. A standardly stratified algebra $\Lambda$, equipped with the poset $(\mathcal{P}, \leq)$, is quasi-hereditary iff $\text{End}_\Lambda(\Delta_u) \cong k$, for each $u \in \mathcal{P}$.

Definition 1.3. Given a finite-dimensional $k$-algebra $\Lambda$ and a finite homological system $(\mathcal{P}, \leq, \{\Delta_u\}_{u \in \mathcal{P}})$ we will also consider the full subcategory $\widetilde{\mathcal{F}}(\Delta)$ of $\Lambda$-Mod with objects the $\Lambda$-modules $M$ which admit a filtration

$$0 = M_\ell \subseteq M_{\ell-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that $M_j/M_{j+1}$ is isomorphic to a (possibly infinite) direct sum of modules in the family $\{\Delta_u\}_{u \in \mathcal{P}}$, for all $j \in [0, \ell - 1]$. 
Definition 1.4. Let \((P, \leq, \{\Delta_u\}_{u \in P})\) be a homological system for a finite-dimensional algebra \(\Lambda\). Then,

1. A \(\Lambda\)-module \(G\) is called a \textit{generic module for} \(\mathcal{F}(\Delta)\) iff \(G\) is an endofinite indecomposable module with infinite length as a \(\Lambda\)-module and \(G \in \tilde{\mathcal{F}}(\Delta)\).

2. The category \(\mathcal{F}(\Delta)\) is called \textit{generically tame} iff, for each \(d \in \mathbb{N}\), there is only a finite number of non-isomorphic generic modules for \(\mathcal{F}(\Delta)\) with endolength \(d\).

In 2021, we uploaded to arXiv a preprint with the results for \(\mathcal{F}(\Delta)\) of this article, in the particular case of the homological system of standard modules over a quasi-hereditary algebra. This article is a revised version of that one, where some of the arguments are simplified and the results are generalized for arbitrary homological systems over finite-dimensional algebras. The main results of this article are the following two statements.

Theorem 1.5. Assume that the ground field \(k\) is algebraically closed and that \(\Lambda\) is a finite-dimensional algebra. Let \((P, \leq, \{\Delta_u\}_{u \in P})\) be any homological system for \(\Lambda\). Then, if \(\mathcal{F}(\Delta)\) is not wild and \(G\) is a generic \(\Lambda\)-module for \(\mathcal{F}(\Delta)\), the following statements hold.

1. There is a rational algebra \(\Gamma_G\) and a \(\Lambda\)-\(\Gamma_G\)-bimodule \(Z_G\) such that as a right \(\Gamma_G\)-module \(Z_G\) is free of rank equal to the endolength of \(G\). If \(Q_G\) is the field of fractions of \(\Gamma_G\), then \(G \cong Z_G \otimes_{\Gamma_G} Q_G\).

2. The functor \(Z_G \otimes_{\Gamma_G} - : \Gamma_G\text{-Mod} \rightarrow \tilde{\mathcal{F}}(\Delta)\) preserves isoclasses and indecomposability.

3. The functor \(Z_G \otimes_{\Gamma_G} - : \Gamma_G\text{-mod} \rightarrow \mathcal{F}(\Delta)\) preserves Auslander-Reiten sequences.

In the following, when we say that almost all objects in a class \(C\) of modules satisfy some property, we mean that every object in this class has this property, with the exception of those lying in a finite union of isoclasses in \(C\).

Theorem 1.6. Assume that the ground field \(k\) is algebraically closed and that \(\Lambda\) is a finite-dimensional algebra. Let \((P, \leq, \{\Delta_u\}_{u \in P})\) be any homological system for \(\Lambda\). Then, \(\mathcal{F}(\Delta)\) is generically tame iff it is tame.

Moreover, if \(\mathcal{F}(\Delta)\) is generically tame, for any \(d \in \mathbb{N}\), almost every indecomposable \(\Lambda\)-module \(M \in \mathcal{F}(\Delta)\) with \(\dim_k M \leq d\) is of the form \(M \cong Z_G \otimes_{\Gamma_G} N\), for some generic \(\Lambda\)-module \(G\) for \(\mathcal{F}(\Delta)\) with endol \((G) \leq d\) and some indecomposable \(N \in \Gamma_G\text{-mod}\), with the notation of (1.5).

The proof of the preceding results rely, in case the given homological system is admissible, on the construction of a weak ditalgebra \(A(\Delta)\) associated to the category \(\mathcal{F}(\Delta)\), in such a way that \(\mathcal{F}(\Delta)\) is generically tame iff \(A(\Delta)\) is pregenerically tame in the sense of the following definition. The construction of \(A(\Delta)\) in the case of quasi-hereditary algebras is due to Koenig, Külschammer,
and Ovsienko, see [21]. The construction for a general pre-standardly stratified algebra can be found in [9]. The proofs of our main results in the case of a general homological system are reduced to the admissible case using a construction due to Mendoza, Sáenz, and Xi, which associates to a general homological system $\mathcal{P}, \leq, \{\Theta_v\}_{v \in \mathcal{P}}$ a pre-standardly stratified algebra with an admissible homological system $\mathcal{P}, \leq, \{\Delta_v\}_{v \in \mathcal{P}}$ such that $\mathcal{F}(\Theta)$ and $\mathcal{F}(\Delta)$ are equivalent as exact categories, see [22] and [8]§11.

We will assume some familiarity with the language introduced in the first sections of [8]. So $\mathbb{A}$-Mod (resp. $\mathbb{A}$-mod) denotes the category of modules (resp. finite-dimensional modules) over an interlaced weak ditalgebra $\mathbb{A}$ and, given $M \in \mathbb{A}$-Mod, we denote by $\text{End}_\mathbb{A}(M)$ its endomorphism algebra in $\mathbb{A}$-Mod.

**Definition 1.7.** Let $\mathbb{A} = (A,I)$ be an interlaced weak ditalgebra, with layer $(R,W)$, see [8](4.1). Given $M \in \mathbb{A}$-Mod, denote by $E_M := \text{End}_\mathbb{A}(M)^{\text{op}}$ the opposite of its endomorphism algebra. Then, $M$ admits a structure of $R$-$E_M$-bimodule, where $m \cdot (f^0,f^1) = f^0(m)$, for $m \in M$ and $(f^0,f^1) \in E_M$. By definition, the endolength of $M$, denoted by $\text{endol}(M)$, is the length of $M$ as a right $E_M$-module.

A module $M \in \mathbb{A}$-Mod is called **pregeneric** iff $M$ is indecomposable, with finite endolength but with infinite dimension over the ground field $k$. The interlaced weak ditalgebra $\mathbb{A}$ is called **pregenerically tame** iff, for each natural number $d$, there are only finitely many isoclasses of pregeneric $\mathbb{A}$-modules of endolength $d$.

The preceding notion extends the one introduced in [8]§2 for layered ditalgebras. As remarked there, if $B$ is any $k$-algebra, we have the corresponding regular ditalgebra $\mathbb{A}$ with layer $(B,0)$ and we can identify canonically the categories $\mathbb{A}$-Mod with $B$-Mod. Thus, if $B$ is a finite-dimensional algebra, the notions of **pregeneric module** and **pregeneric tameness** for the algebra $B$ coincide with the usual notions of **generic module** and **generic tameness**.

The interlaced weak ditalgebra $\mathbb{A}(\Delta)$ associated to the category $\mathcal{F}(\Delta)$, mentioned before, is a $\mathcal{P}$-oriented interlaced weak ditalgebra, which means that it has the special geometrical characteristics related to the preordered set $\mathcal{P}$ explained below.

**Definition 1.8.** A **biquiver** $\mathbb{B}$ is a triple $\mathbb{B} = (\mathcal{P}, \mathbb{B}_0, \mathbb{B}_1)$ formed by finite sets. The elements of $\mathcal{P}$ are called **points**, the elements of $\mathbb{B}_0$ are **full arrows**, and the elements of $\mathbb{B}_1$ are **dashed arrows**. Each arrow $\alpha \in \mathbb{B}_0 \cup \mathbb{B}_1$ has a **starting point** $s(\alpha) \in \mathcal{P}$ and a **terminal point** $t(\alpha) \in \mathcal{P}$.

Given a biquiver $\mathbb{B}$ with $n$ points, we consider the $k$-algebra $R$ defined as the product $R$ of $n$ copies of the ground field $k$. Thus the unit of $R$ is a sum of primitive orthogonal idempotents $1 = \sum_{u \in \mathcal{P}} e_u$, where $e_u$ corresponds to the unit of the $u$-copy of the ground field in the product $R$. Then, we consider vector spaces $W_0$ and $W_1$ with basis $\mathbb{B}_0$ and $\mathbb{B}_1$, respectively, and set $W := W_0 \oplus W_1$. If we define $e_u \alpha e_v = \delta_{u,s(\alpha)} \delta_{t(\alpha),v} \alpha$, we get a natural structure of $R$-$R$-bimodule on the space $W$, and $W = W_0 \oplus W_1$ is an $R$-$R$-bimodule decomposition. We have the graded tensor algebra $T = T_R(W)$, with
\[ [T]_0 = T_R(W_0) \text{ and } [T]_1 = [T]_0 W_1 [T]_0, \] which is called the tensor algebra of the biquiver \( \mathbb{B} \).

The algebra \( T \) can be identified with the path algebra \( k\mathbb{B} \) of the biquiver \( \mathbb{B} \), with underlying vector space with basis the set of paths (of any kind of arrows) of \( \mathbb{B} \) (including one trivial path for each point \( u \in P \)). Each primitive idempotent \( e_u \) of \( R \) is identified with the trivial path at the point \( u \). The homogeneous elements \( [k\mathbb{B}]_d \) of degree \( d \) are the linear combinations of paths containing exactly \( d \) dashed arrows.

**Definition 1.9.** Let \( P = (P, \leq) \) be a finite preordered set and \( \mathbb{B} \) a biquiver with set of points \( P \), then we say that \( \mathbb{B} \) is \( P \)-oriented iff

1. Whenever there is a dashed arrow from \( u \) to \( v \), we have \( u \leq v \);
2. Whenever there is a solid arrow from \( u \) to \( v \), we have \( \overline{u} < \overline{v} \).

A weak ditalgebra \( A = (T, \delta) \) will be called a \( P \)-oriented weak ditalgebra iff its underlying tensor algebra \( T \) is the tensor algebra of a \( P \)-oriented biquiver.

A biquiver \( \mathbb{B} \) is called directed iff it admits no oriented (non-trivial) cycle (composed by any kind of arrows). In a \( P \)-oriented biquiver \( \mathbb{B} \), the only (non-trivial) oriented cycles (composed of any kind of arrows) consist of dashed arrows between points in the same \( \sim \) class. Such a biquiver \( \mathbb{B} \) is not necessarily directed.

The proof of the main theorem (1.6) relies on the following result.

**Theorem 1.10.** Assume that the ground field \( k \) is algebraically closed and let \( P \) be a finite preordered set. Suppose that \( \mathcal{A} = (\mathcal{A}, I) \) is a \( P \)-oriented triangular interlaced weak ditalgebra, where \( I \) is an ideal of \( A \) contained in the radical of \( A \). Then, \( \mathcal{A} \) is pregenerically tame iff it is tame.

The proof of this last statement will follow the line of reasoning of [8], adapting the arguments to handlepregeneric modules. This strategy is similar to the one used by Crawley-Boevey to transit from [14] to [15]. In order to present a readable argument for the proof of our main results, we could not avoid to recall many definitions and constructions from [15] and [8].

2 Endolength and pregeneric tameness

In the following lemmas, we recall elementary reduction procedures on triangular interlaced weak ditalgebras studied in [8]§6, and we describe their effect on the endolength. The first four of them correspond to the lemmas (6.2), (6.3), (6.4), and (6.5) of [8]. For the statements on the endolength in all of these cases, see the proof of [6](2.2).

Here, as in [6], given a layered interlaced weak ditalgebra \( \mathcal{A} = (\mathcal{A}, I) \) with underlying weak ditalgebra \( A = (T, \delta) \) and layer \( (R, W) \), we denote by \( A = [T]_0 = T_R(W_0) \) the subalgebra of \( T = T_R(W) \) formed by the zero-degree elements and, by \( V = [T]_1 = AW_1 A \) the \( A \)-\( A \)-subbimodule of \( T \) formed by the elements with degree one.
Proposition 2.1 (deletion of idempotents). Suppose that $\mathcal{A} = (A, I)$ is a triangular interlaced weak ditalgebra with layer $(R, W)$. Assume that $e \in R$ is a central idempotent of $R$. Consider the canonical projections $\phi_e : R \rightarrow eRe$, and $\phi : W \rightarrow eWe$, for $r \in \{0, 1\}$. Set $T^d := T_{eRe}(eWe)$. Thus, we have a morphism of graded $k$-algebras $\phi : T \rightarrow T^d$ and we have the ideal $I^d = \phi(I)$ of $A^d$. Then, there is a triangular interlaced weak ditalgebra $A^d = (A^d, I^d)$ with layer $(R^d, W^d)$ where $R^d = eRe$, $W^d_0 = eW_0$, and $W^d_1 = eW_1$. The morphism $\phi : A \rightarrow A^d$ of interlaced weak ditalgebras induces a full and faithful functor $F^d := F_{\phi} : A^d\text{-Mod} \rightarrow A\text{-Mod}$ whose image consists of the $\mathcal{A}$-modules annihilated by $1 - e$. Moreover, the functor $F^d$ preserves endolength. Therefore, $A^d$ is pregenerically tame whenever $A$ is so.

We say that $A^d$ is obtained from $A$ by deletion of the idempotent $1 - e$.

Proposition 2.2 (regularization). Let $\mathcal{A} = (A, I)$ be a triangular interlaced weak ditalgebra with underlying weak ditalgebra $A = (T, \delta)$ and triangular layer $(R, W)$. Assume that we have $R$-$R$-bimodule decompositions $W_0 = W_0^r \oplus W_0^s$ and $W_1 = \delta(W_0^s) \oplus W_1^s$, set $W^r := W_0^r \oplus W_1^s$. Consider the identity map $\phi_1 : R \rightarrow R$, the canonical projections $\phi_j : W_j \rightarrow W^r_j$, for $j \in \{0, 1\}$, and the tensor algebra $T^r = T_R(W^r)$. Then, we have a morphism of graded algebras $\phi : T \rightarrow T^r$ and the ideal $I^r = \phi(I)$ of $A^r$. Then, there is a triangular interlaced weak ditalgebra $A^r = (A^r, I^r)$ with layer $(R^r, W^r)$, where $R^r = R$, $W^r_0 = W_0^r$, and $W^r_1 = W_1^s$. The morphism $\phi : A \rightarrow A^r$ of interlaced weak ditalgebras induces a full and faithful functor $F^r := F_{\phi} : A^r\text{-Mod} \rightarrow A\text{-Mod}$ which preserves endolength. So $A^r$ is pregenerically tame whenever $A$ is so. Moreover, if $A$ is a Roiter interlaced weak ditalgebra, as in (11.1), then $M \in A\text{-Mod}$ is isomorphic to an object in the image of $F^r$ iff $\ker \delta \cap W_0^0$ annihilates $M$. In particular, if this intersection is zero, $F^r$ is an equivalence of categories.

Lemma 2.3 (factoring out a direct summand of $W_0$). Let $\mathcal{A} = (A, I)$ be a triangular interlaced weak ditalgebra with underlying weak ditalgebra $A = (T, \delta)$ and triangular layer $(R, W)$. Assume that there is a decomposition of $R$-$R$-bimodules $W_0 = W_0^0 \oplus W_0^s$, such that $W_0^0 \subseteq I$ and $\delta(W_0^s) \subseteq AW_0^0 + VW_0^sA$. Set $T^q = T_R(W^q)$, where $W^q_0 = W_0^0$, $W^q_1 = W_1$, and $W^q = W^q_0 \oplus W^q_1$. Then, there is a derivation $\delta^q$ on $T^q$ such that $A^q := (T^q, \delta^q)$ is a weak ditalgebra with triangular layer $(R, W^q)$. The tensor algebra $A^q = T_R(W^q_0)$ can be identified with the quotient algebra $A/I'$, where $I' = AW_0^0A$ is the ideal of $A$ generated by $W_0^0$, so we can consider $I' := I/I'$ as an ideal of $A^q$. Then $A^{q} = (A^q, I^q)$ is a triangular interlaced weak ditalgebra, and there is a morphism of interlaced weak ditalgebras $\phi : A^q \rightarrow A^q$, with $\phi(I) = I^q$, which induces an endolength preserving equivalence of categories $F^q := F_{\phi} : A^q\text{-Mod} \rightarrow A\text{-Mod}$. So, $A^q$ is pregenerically tame iff $A$ is so.

Lemma 2.4 (absorption). Let $\mathcal{A} = (A, I)$ be a triangular interlaced weak ditalgebra with underlying weak ditalgebra $A = (T, \delta)$ and triangular layer $(R, W)$. Assume that there is a decomposition of $R$-$R$-bimodules $W_0 = W_0^0 \oplus W_0^s$ with $\delta(W_0^0) = 0$. Then, we can consider another layer $(R^0, W^r)$ for the same weak
ditalgebra $\mathcal{A}$, where $R^a$ is the subalgebra of $T$ generated by $R$ and $W_0'$, thus we can identify it with $T_R(W_0')$, $W_0'' := R^a W_0'' R^a$, and $W_1' := R^a W_1 R^a$. We denote by $\mathcal{A}^a$ the same weak ditalgebra $\mathcal{A}$ equipped with its new layer $(R^a, W^a)$; in particular, we have $\delta^a = \delta$. The layer $(R^a, W^a)$ is triangular and we obtain a triangular interlaced weak ditalgebra $\mathcal{A}^a = (\mathcal{A}^a, I^a)$, with $I^a = I$. The identity $a$ is pregenerically tame iff $\mathcal{A}$ is so.

We say that $\mathcal{A}^a$ is obtained from $\mathcal{A}$ by absorption of the bimodule $W_0'$.

In order to fix notation and terminology, we recall from [8] (6.6–6.7) the following two propositions.

**Proposition 2.5.** Assume that $\mathcal{A} = (T, \delta)$ is a weak ditalgebra with layer $(R, W)$ such that there is an $R-R$-bimodule decomposition $W_0 = W_0' \oplus W_0''$ with $\delta(W_0') = 0$. Suppose that $X$ is an admissible $B$-module, where $B = T_R(W_0')$. This means that the algebra $\Gamma = \text{End}_B(X)^{op}$ admits a splitting $\Gamma = S \oplus P$, where $S$ is a subalgebra of $\Gamma$, $P$ is an ideal of $\Gamma$, the direct sum is an $S-S$-bimodule decomposition of $\Gamma$, and the right $S$-modules $X$ and $P$ are finitely generated projective. By $(x_1, \nu_1) \in I$ and $(p_1, \gamma_1) \in J$ we denote finite duals of the right $S$-modules $X$ and $P$, respectively.

There is a comultiplication $\mu : P^* \rightarrow P^* \otimes_S P^*$ induced by the multiplication of $P$, which is coassociative (see [10] (11.7)). For $\nu \in P^*$, we have

$$\mu(\nu) = \sum_{i,j \in J} \gamma(p_ip_j) \gamma_j \otimes \gamma_i.$$

The action of $P$ on $X$ induces (see [10] (10–11)) a morphism of $S$-$R$-bimodules $\lambda : X^* \rightarrow P^* \otimes_S X^*$ and a morphism of $R$-$S$-bimodules $\rho : X \rightarrow X \otimes_S P^*$ with

$$\lambda(\nu) = \sum_{i,j \in J} \nu(x_ip_j) \gamma_j \otimes \nu_i \text{ and } \rho(x) = \sum_{j \in J} xp_j \otimes \gamma_j.$$

Set $W_0 = BW_0'' B$ and $W_1 = BW_1 B$. Recall that we can identify $A = T_R(W_0)$ with $T_B(W_0)$. We have $W = W_0 \oplus W_1$, see [10] (12.2).

We have the $S$-$S$-bimodules

$$W^X_0 = X^* \otimes_B W_0 \otimes_B X = X^* \otimes_R W_0'' \otimes_R X,$$

$$W^X_1 = (X^* \otimes_B W_1 \otimes_B X) \oplus P^* = (X^* \otimes_R W_1 \otimes_R X) \oplus P^*.$$

Consider the tensor algebra $T^X = T_S(W^X)$, where $W^X = W^X_0 \oplus W^X_1$. Following [10] (12.8), observe that for $\nu \in X^*$ and $x \in X$, there is a linear map

$$\sigma_{\nu, x} : T \rightarrow T^X$$

such that $\sigma_{\nu, x}(b) = \nu(bx)$, for $b \in B$, and given $w_1, w_2, \ldots, w_n \in W$, we have $\sigma_{\nu, x}(w_1 w_2 \cdots w_n)$ is given by

$$\sum_{i_1, \nu_1, x_1, \ldots, i_{n-1}} \nu \otimes w_1 \otimes x \otimes w_2 \otimes x_2 \otimes \nu_1 \otimes w_3 \otimes \cdots \otimes x_{i_{n-1}} \otimes \nu_{i_{n-1}} \otimes w_n \otimes x.$$
There is a derivation $\delta^X$ on $T^X$ determined by $\delta^X(\gamma) = \mu(\gamma)$, for $\gamma \in P^*$ and $\delta^X(\nu \otimes w \otimes x) = \lambda(\nu) \otimes w \otimes x + \sigma_{\nu,x}(\delta(w)) + (-1)^{\deg w + 1}\nu \otimes w \otimes \rho(x)$, for $w \in W_0 \cup W_1$, $\nu \in X^*$, and $x \in X$.

Then, $A^X = (T^X, \delta^X)$ is a weak ditalgebra. If the layer $(R, W)$ of $A$ is triangular and $X$ is a triangular admissible $B$-module, as in [17](14.6), then the layer $(S, W^X)$ of $A^X$ is also triangular.

**Proposition 2.6** (reduction with an admissible module $X$). Let $A = (A, I)$ be a triangular interlaced weak ditalgebra with underlying weak ditalgebra $A = (T, \delta)$ and triangular layer $(R, W)$. Suppose that there is an $R$-$R$-bimodule decomposition $W_0 = W'_0 \oplus W''_0$ with $\delta(W'_0) = 0$ and that $X$ is a triangular admissible $B$-module, where $B = T_R(W'_0)$. As usual, denote by $A^X = [T^X]_a$. Consider the ideal $I^X$ of $A^X$ generated by the elements $\sigma_{\nu,x}(h)$, for $\nu \in X^*$, $h \in I$, and $x \in X$. Then,

1. The pair $A^X := (A^X, I^X)$ is a triangular interlaced weak ditalgebra;
2. There is a functor $F^X : A^X$-$\text{Mod} \longrightarrow A$-$\text{Mod}$ such that for $M \in A^X$-$\text{Mod}$, the underlying $B$-module of $F^X(M)$ is $X \otimes_S M$ and $a \cdot (x \otimes m) = \sum x_i \otimes \sigma_{\nu,x,a}(m)$, for $a \in A$, $x \in X$, and $m \in M$. Moreover, given the morphism $f = (f^0, f^1) \in \text{Hom}_A(M, N)$, we have $F^X(f) = (F^X(f^0), F^X(f^1))$ given by

$$F^X(f^0)[x \otimes m] = x \otimes f^0(m) + \sum_j \sigma_{\nu,x}(\delta^X)^j(m)[m]$$
$$F^X(f^1)[x \otimes m] = \sum_i x_i \otimes f^1(\sigma_{\nu,x}(\delta^X))$$

for $v \in V$, $x \in X$, and $m \in M$.
3. For any $N \in A$-$\text{Mod}$ which is isomorphic as a $B$-module to some $B$-module of the form $X \otimes_S M$, for some $M \in S$-$\text{Mod}$, we have $N \cong F^X(M)$ in $A$-$\text{Mod}$, for some $M \in A^X$-$\text{Mod}$.

We also need to recall from [8] the following additional condition on the admissible $B$-module $X$ which guarantees that $F^X$ is full and faithful.

**Remark 2.7.** Assume that $B = T_R(W'_0)$ is a tensor algebra and let $X$ be an admissible $B$-module. Thus, we have a splitting $\Gamma = \text{End}_B(X)^{op} = S \oplus P$, as in (2.5). We have the ditalgebra $(B, 0)$, with trivial differential, and the ditalgebra $(B, 0)^X = (T_S(P^*), \delta)$, where $\delta$ is the differential determined by the counit $\mu : P^* \longrightarrow P^* \otimes_S P^*$. Recall from [10], that the admissible $B$-module $X$ is called complete if the functor $F^X : (B, 0)^X$-$\text{Mod} \longrightarrow B$-$\text{Mod}$ is full and faithful. From (17.4), (17.5), and (17.11) of [10], we know that we obtain complete admissible $B$-modules $X$ in the following cases:
1. $X$ is a finite direct sum of non-isomorphic finite-dimensional indecomposables in $B$-mod;
2. $X$ is the $B$-module obtained from the regular $S$-module $S$ by restriction through a given epimorphism of $k$-algebras $\phi : B \rightarrow S$;
3. $X = X_1 \oplus X_2$, where $X_1$ and $X_2$ are complete triangular admissible $B$-modules such that $\text{Hom}_B(I_{X_i}, I_{X_j}) = 0$, for $i \neq j$, and $I_{X_i}$ denotes the class of $B$-modules of the form $X_i \otimes_S N$, for some $N \in S$-Mod.

All the complete admissible $B$-modules we shall consider in this paper are constructed using 1, 2, and 3.

Again, from [8], we have the following.

**Proposition 2.8.** Under the assumptions of (2.6), consider the ideal $I_0 := B \cap I$ of $B = T_R(W_0')$. Then, we can consider the triangular interlaced weak ditalgebra $(B, I_0)$, where $B = (B, 0)$. Then, for any complete admissible $B$-module $X$, we have a full and faithful functor $F^{\times X} : (B^X, I_0^X)\text{-Mod} \rightarrow (B, I_0)\text{-Mod} = (B/I_0)\text{-Mod}$, and a full and faithful functor $F^X : A^X\text{-Mod} \rightarrow A\text{-Mod}$.

**Definition 2.9.** An interlaced weak ditalgebra $(A, I)$ over the field $k$ is called wild iff there is an $A/I\times k\langle x, y \rangle$-bimodule $Z$, which is free of finite rank as a right $k\langle x, y \rangle$-module, such that the composition functor $k\langle x, y \rangle\text{-Mod} \xrightarrow{Z \otimes k\langle x, y \rangle \cdot -} \xrightarrow{A/I\text{-Mod}} L_{(A, I)}(A, I)\text{-Mod}$ preserves isoclasses of indecomposables. Here $L_{(A, I)}$ denotes the canonical embedding functor mapping each morphism $f^0$ onto $(f^0, 0)$. In this case, we say that $Z$ produces the wildness of $(A, I)$.

We recall from [8](6.14) the following statement.

**Proposition 2.10.** Assume that a triangular interlaced weak ditalgebra $(A^z, I^z)$ is obtained from a Roiter interlaced weak ditalgebra $(A, I)$ by some of the procedures described in this section, that is $z \in \{d, r, q, a, X\}$. Then, $(A^z, I^z)$ is a Roiter interlaced weak ditalgebra. The associated full and faithful functor $F^z : (A^z, I^z)\text{-Mod} \rightarrow (A, I)\text{-Mod}$ preserves isoclasses and indecomposables. Moreover, if $(A^z, I^z)$ is wild (with wildness produced by an $(A^z/I^z)\times k\langle x, y \rangle$-bimodule $Z$) then so is $(A, I)$ (with wildness produced by the $(A/I)\times k\langle x, y \rangle$-bimodule $F^z(Z)$).

**Proposition 2.11.** Any pregenerically tame triangular interlaced weak ditalgebra $A$ is not wild.

**Proof.** The proof of [8](2.9) readily adapts to this situation. \qed
3 Bimodules and reduction functors

We recall some basic notions of [13], adapted to the language used here.

Definition 3.1. Given an interlaced weak ditalgebra $\mathcal{A} = (\mathcal{A}, I)$ and a $k$-algebra $E$, an $\mathcal{A}$-$E$-bimodule is an object $M \in \mathcal{A}$-$\mathrm{Mod}$, together with a $k$-algebra morphism $\alpha_M : E \to \mathrm{End}_\mathcal{A}(M)^{\mathcal{A}}$. If $N$ is another $\mathcal{A}$-$E$-bimodule, then

$$\hom_{\mathcal{A}}(M, N) = \{ f \in \hom_{\mathcal{A}}(M, N) \mid f\alpha_M(e) = \alpha_N(e)f, \text{ for all } e \in E\}.$$ 

The category of $\mathcal{A}$-$E$-bimodules, where the rule of composition is the same as for $\mathcal{A}$-$\mathrm{Mod}$, is denoted by $\mathcal{A}$-$E$-$\mathrm{Mod}$.

We shall denote $\mathcal{A} := \mathcal{A}/I$, where $\mathcal{A}$ is the tensor algebra generated by the degree zero elements in the underlying tensor algebra of $\mathcal{A}$. An $\mathcal{A}$-$E$-bimodule $M$ is proper iff $\alpha_M$ factors through the canonical embedding map

$$\mathrm{End}_{\mathcal{A}}(M)^{\mathcal{A}} \longrightarrow \mathrm{End}_{\mathcal{A}}(M)^{\mathcal{A}}$$

such that $f \mapsto (f, 0)$.

The full subcategory of $\mathcal{A}$-$E$-$\mathrm{Mod}$ formed by the proper $\mathcal{A}$-$E$-bimodules will be denoted by $\mathcal{A}$-$E$-$\mathrm{Mod}_p$. The canonical embedding functor $L_{\mathcal{A}} : \mathcal{A}$-$\mathrm{Mod} \to \mathcal{A}$-$\mathrm{Mod}$, which maps each $\mathcal{A}$-morphism $f$ to $(f, 0)$, induces a functor $L_{\mathcal{A}} : \mathcal{A}$-$E$-$\mathrm{Mod} \to \mathcal{A}$-$E$-$\mathrm{Mod}$. Clearly, the proper $\mathcal{A}$-$E$-bimodules coincide with the images of the $\mathcal{A}$-$E$-bimodules under the faithful functor $L_{\mathcal{A}}^E$, we will identify this class of objects with the $\mathcal{A}$-$E$-bimodules.

The following lemma is shown as in the classical case, see [10](21.3).

Lemma 3.2. Assume that $E$ is a $k$-algebra and $\mathcal{A}$, $\mathcal{A}'$ are interlaced weak ditalgebras. Then any functor $F : \mathcal{A}'$-$\mathrm{Mod} \longrightarrow \mathcal{A}$-$\mathrm{Mod}$ induces a functor

$$F_E : \mathcal{A}'$-E$-$\mathrm{Mod} \longrightarrow \mathcal{A}$-$E$-$\mathrm{Mod},$$

which maps any object $M \in \mathcal{A}'$-$E$-$\mathrm{Mod}$ onto $F(M) = F_E(M)$, where the structure of $\mathcal{A}$-$E$-bimodule on $F(M)$ is given by the composition morphism

$$\alpha_{F(M)} : E \cdot \mathcal{A}(M)^{\mathcal{A}} \longrightarrow \mathrm{End}_{\mathcal{A}}(M)^{\mathcal{A}} \longrightarrow \mathrm{End}_{\mathcal{A}}(F(M))^\mathcal{A}.$$ 

Moreover, if $F$ is full and faithful, then $F_E$ is so. In this case, if an $\mathcal{A}$-$E$-bimodule $M$ is isomorphic as an $\mathcal{A}$-module to $F(N)$, for some $\mathcal{A}'$-module $N$, then $N$ admits a natural structure of $\mathcal{A}'$-$E$-bimodule such that $M \cong F_E(N)$ in $\mathcal{A}$-$E$-$\mathrm{Mod}$.

Definition 3.3. Let $\mathcal{A}$ be a interlaced weak ditalgebra with layer $(R, W)$ and $E$ any $k$-algebra. Given any $\mathcal{A}$-$E$-bimodule $M$, the composition of $\alpha_M : E \longrightarrow \mathrm{End}_{\mathcal{A}}(M)^{\mathcal{A}}$ with the projection $\pi_M : \mathrm{End}_{\mathcal{A}}(M)^{\mathcal{A}} \longrightarrow \mathrm{End}_R(M)^{\mathcal{A}}$, which maps $(f^0, f^1)$ onto $f^0$, determines a structure of $R$-$E$-bimodule on $M$. We will denote by $\ell_E(M)$ the length of this right $E$-module.

The full subcategory of $\mathcal{A}$-$E$-$\mathrm{Mod}$ formed by the finite $E$-length bimodules is denoted by $\mathcal{A}$-$E$-$\mathrm{mod}$ and its intersection with $\mathcal{A}$-$E$-$\mathrm{Mod}_p$ by $\mathcal{A}$-$E$-$\mathrm{mod}_p$. 

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Suppose that $A$ and $A'$ are layered interlaced weak ditalgebras and $E$ is a $k$-algebra. Then we say that a functor $F_E : A-E-\text{Mod} \to A'-E-\text{Mod}$ is length controlling iff $\ell_E(M)$ finite implies $\ell_E(F(M))$ finite, and, furthermore, $\ell_E(M) \leq \ell_E(F(M))$, for all $M \in A-E-\text{Mod}$. If $\ell_E(M) = \ell_E(F(M))$, for all $M$, then $F$ is called length preserving.

**Lemma 3.4.** Let $\phi : A \to A'$ be a morphism of triangular interlaced weak ditalgebras and $F_\phi : A'-\text{Mod} \to A-\text{Mod}$ the associated restriction functor. Then, for any $k$-algebra $E$, the induced functor $F_E^\phi : A'-E-\text{Mod} \to A-E-\text{Mod}$ is length preserving and it sends proper bimodules to proper bimodules.

**Proof.** The proof of [10](21.6) immediately adapts to this situation. \qed

**Lemma 3.5.** Assume that $A$ is a triangular interlaced weak ditalgebra, that $A^d$ is obtained by deletion of an idempotent $e$, and let $F^d : A^d-\text{Mod} \to A-\text{Mod}$ be the associated functor, as in (2.6). Then, for any $k$-algebra $E$, we have:

1. $F^d_E : A^d-E-\text{Mod} \to A-E-\text{Mod}$ is a length preserving functor. It induces an equivalence from $A^d-E-\text{Mod}$ to the full subcategory of $A-E-\text{Mod}$ formed by the bimodules $M$ with $eM = 0$.

2. $F^d_E$ restricts to an equivalence from $A^d-E-\text{Mod}_p$ to the full subcategory of $A-E-\text{Mod}_p$ formed by the proper bimodules $M$ satisfying $eM = 0$.

**Proof.** It follows from (122) and (124), because $F^d = F_\phi$, where $\phi : A \to A^d$ is the morphism constructed in (2.1). \qed

**Proposition 3.6.** Let $A$ be a triangular interlaced weak ditalgebra with triangular layer $(R,W)$. Assume that $A^q$ is obtained from $A$ factoring out a direct summand of $W_0$, as in (2.3), and consider the associated functor $F^q : A^q-\text{Mod} \to A-\text{Mod}$. Then, for any $k$-algebra $E$, we have:

1. $F^q_E : A^q-E-\text{Mod} \to A-E-\text{Mod}$ is a length preserving equivalence.

2. The functor $F^q_E$ restricts to an equivalence $F^q_E : A^q-E-\text{Mod}_p \to A-E-\text{Mod}_p$.

**Proof.** It follows from (122) and (124), because $F^q = F_\phi$, where $\phi : A \to A^q$ is the morphism constructed in (2.3). \qed

**Lemma 3.7.** Let $A = (A,I)$ be a triangular interlaced weak ditalgebra and $X$ a complete triangular admissible $B$-module, where $B = T_R(W'_0)$ as in (2.0). Consider the triangular interlaced weak ditalgebra $A^N$ and the associated reduction functor $F^X : A^N-\text{Mod} \to A-\text{Mod}$.

Given any $k$-algebra $E$, assume that $M \in A-E-\text{Mod}_p$, $N \in S-E-\text{Mod}$, and $X \otimes_S N \cong M$ in $B-E-\text{Mod}$. Then, there is some $N' \in A^N-E-\text{Mod}_p$ with underlying $S-E$-bimodule $N$ and such that $F^X_E(N') \cong M$ in $A-E-\text{Mod}_p$.

**Proof.** Similar to the proof of [10](25.5), now using (2.6)(3). \qed
Definition 3.8. Recall that a minimal algebra $R$ is a finite product of algebras $R = R_1 \times \cdots \times R_n$, where each $R_i$ is either a rational $k$-algebra or is isomorphic the field $k$.

Proposition 3.9. Assume that $\mathcal{A} = (A, I)$ is a triangular interlaced weak ditalgebra with layer $(R, W)$. Under the assumptions of (2.7), consider the ideal $I_0 := B \cap I$ of $B = T_B(W_0)$. Assume that $\mathcal{A}^X$ is the triangular interlaced weak ditalgebra obtained from $\mathcal{A}$ by reduction, using a complete triangular admissible $B$-module $X$, thus $\mathcal{A}^X$ has layer $(S, W^X)$, where $\Gamma = \text{End}_B(X)^{op} = S \oplus P$. Assume that $S$ is a minimal algebra. Consider the associated full and faithful functor $F^X : \mathcal{A}^X - \text{Mod} \rightarrow \mathcal{A} - \text{Mod}$. Denote by $\mu(X)$ the number of generators in a set of generators of the right $S$-module $X$ with minimal cardinality. Then, the following holds.

1. For all $N \in \mathcal{A}^X - \text{Mod}$, we have that $\text{endol}(F^X(N)) \leq \mu(X) \times \text{endol}(N)$. Then $\mathcal{A}^X$ is pregenerically tame, whenever $\mathcal{A}$ is so.

2. For any $k$-algebra $E$, the associated functor $F^X_E : \mathcal{A}^X - E - \text{Mod} \rightarrow \mathcal{A} - E - \text{Mod}$ is length controlling, full and faithful. For any $N \in \mathcal{A}^X - E - \text{Mod}$, the inequality $\ell_E(F^X(E)(N)) \leq \mu(X) \times \ell_E(N)$ holds.

Proof. The argument is essentially the same as [10](4.7), an adaptation of the proof of (2.7), we sketch it here.

We first prove the second item. Take an $\mathcal{A}^X - E$-bimodule $N$, with bimodule structure $\alpha_N : E \rightarrow \text{End}_{\mathcal{A}^X}(N)^{op}$, so the $\mathcal{A}^X - E$-bimodule $F^X(N)$, has bimodule structure $\alpha_{F^X(N)} : E \rightarrow \text{End}_{\mathcal{A}^X}(F^X(N))^{op}$, as in (3.2).

As in the proof of [10](25.7), it can be shown that $\ell_E(X \otimes_S N)$ coincides with the length of the right $E$-module $X \otimes_S N$, where $E$ acts as usual on the right tensor factor $N$. Since $S$ is a minimal algebra, from this it follows that $\ell_E(N) \leq \ell_E(F^X(N))$, see the argument in the proof of [10](25.7).

Moreover, from an epimorphism $S^{\mu(X)} \rightarrow X$ of right $S$-modules, we obtain an epimorphism of right $E$-modules $S^{\mu(X)} \otimes_S N \rightarrow X \otimes_S N$. Therefore, we obtain $\ell_E(F^X(N)) = \ell_E(X \otimes_S N) \leq \ell_E(N^{\mu(X)}) = \mu(X) \times \ell_E(N)$, as wanted.

For the first item, given $N \in \mathcal{A}^X - \text{Mod}$, set $E := \text{End}_{\mathcal{A}^X}(N)^{op}$. Then, we have the $\mathcal{A}^X$-bimodule $N$, with bimodule structure $\alpha_N : E \rightarrow \text{End}_{\mathcal{A}^X}(N)^{op}$ given by the identity map. Since $F^X$ is full and faithful, we have the isomorphism of algebras $E \cong \text{End}_{\mathcal{A}^X}(F^X(N))^{op}$ induced by $F^X$. It provides, by restriction, a structure of right $E$-module on $F^X(N)$. Clearly, $\text{endol}(N) = \ell_E(N)$ and $\text{endol}(F^X(N)) = \ell_E(F^X(N))$. So, the formula in 1 follows from 2.

Finally, if we assume that $\mathcal{A}^X$ is not pregenerically tame, we have an infinite family of pairwise non-isomorphic pregeneric $\mathcal{A}^X$-modules with bounded endolength. Then, applying the full and faithful functor $F^X$ to them, we obtain an infinite family of pairwise non-isomorphic pregeneric $\mathcal{A}$-modules with bounded endolength. Hence, the interlaced weak ditalgebra $\mathcal{A}$ is not pregenerically tame. □
4 Reduction to minimal ditalgebras

In this section we show that the study of bimodules with bounded length over any non-wild ℵ-oriented interlaced weak ditalgebra $A$, as in (1.10), can be reduced to the study of the bimodules over a minimal ditalgebra obtained from $A$ by a finite number of reductions.

**Lemma 4.1** (multiple d-unravelling). Let $A = (A, I)$ be a triangular interlaced weak ditalgebra with triangular layer $(R, W)$, where $R$ is a minimal algebra. We have $R = \prod_{u \in J} R_u e_u$, where each $R_u$ is either isomorphic to $k$ or to some rational algebra. We can assume that $P = J \sqcup J'$ where $R_u = k e_u$, for $u \in J$, and $R_v = Re_v$, with $R_v = k[x]_{\delta_v}$, for $v \in J'$.

Consider $d \in \mathbb{N}$ and non-zero elements $h_v \in R_v$, for $v \in J'$. Then, there is complete triangular admissible $R$-module $X$ such that $A^X := (A^X, I^X)$ is a triangular interlaced weak ditalgebra with triangular layer $(S, W^X)$, where $S$ is a minimal algebra of the form

$$S = \left( \prod_{w \in J''} k f_w \right) \times \left( \prod_{v \in J'} e_v(R_v) h_v \right) \times \prod_{u \in J} k e_u,$$

where $\{f_w\}_{w \in J''}$ is a new finite family of primitive idempotents of $S$. The associated full and faithful functor $F^X : A^X \text{-Mod} \longrightarrow A \text{-Mod}$ is such that, for any $k$-algebra $E$, the following holds:

1. The functor $F^X : A^X \text{-Mod} \longrightarrow A \text{-Mod}$ induces a length controlling equivalence to a full subcategory of $A \text{-Mod}$ which contains all the $A \text{-E}$-bimodules with length $\leq d$.

2. If $E$ is a division $k$-algebra, $F^X$ induces an equivalence from $A^X \text{-Mod}_p$ to a full subcategory of $A \text{-E}$-Mod which contains all the proper $A \text{-E}$-bimodules with length $\leq d$.

**Proof.** Fix $d \in \mathbb{N}$, consider the functor $F^X : A^X \text{-Mod} \longrightarrow A \text{-Mod}$ constructed in $\mathfrak{S}(7.5)$. The additional hypothesis in $\mathfrak{S}(7.5)$ requiring a starlarity property for $A$ assumed there is irrelevant for the construction of $A^X$. Set $e = \sum_{u \in J} e_u$ and $e' = \sum_{v \in J'} e_v$. We have the algebra $C = \prod_{v \in J'} e_v R_v / (h_v^{E'})$, which admits only a finite number of isoclasses of indecomposable finite-dimensional $C$-modules represented by the $C$-modules $\{Z_w\}_{w \in J''}$. By definition, we have $Z := \bigoplus_{w \in J''} Z_w$ and the $R$-module

$$X = Z \oplus [\oplus_{v \in J'} e_v (R_v) h_v] \oplus Re.$$

Then, we have the splitting $\text{End}_R(X)^{op} = S \oplus P$, where

$$S = \left[ \prod_{w \in J''} k f_w \right] \times \left[ \prod_{v \in J'} e_v (R_v) h_v \right] \times Re,$$

$f_w \in \text{End}_R(Y)^{op}$ is the idempotent corresponding to the indecomposable direct summand $Z_w$ of $X$, and $P = \text{radEnd}_R(Z)^{op}$.
From [27] and [39], we already know that $F_E^X : \mathcal{A}^X - \text{Mod} \rightarrow \mathcal{A}^E - \text{Mod}$ is a length controlling, full and faithful functor. Then, its restriction to proper bimodules $F_E^X : \mathcal{A}^X - \text{Mod}_p \rightarrow \mathcal{A}^E - \text{Mod}_p$ is full and faithful too.

Let us show that the functor $F_E^X$ is dense on a full subcategory of $\mathcal{A}^E - \text{Mod}$ which contains all bimodules with length $\leq d$. The argument is essentially the same given in [14] (2.11) (see also [10] (25.9)). We only have to adapt it to the more general case of a simultaneous unravellings at all the points $e_v$ with $v \in J'$, with associated polynomials $h_v$, in the general case of triangular interlaced weak ditalgebras. We only give a sketch.

For $v \in J'$, set $H_v := e_v R_v = k[x]_{g_v}$ and consider the prime divisors $\pi_{v,1}, \ldots, \pi_{v,s_v}$ of $h_v$ in $H_v$. Then, write $Z_{v,s,a} := H_v/\langle \pi_{v,s,a} \rangle$, for $s \in [1,s_v]$ and $a \in [1,d]$. So $\{Z_{v,s,a}\}_{s,a}$ is a complete set of representatives of the isoclasses of the indecomposable $H_v/\langle h_v^a \rangle$-modules. When we consider the union of all these families of indecomposables, for $v \in J'$, we recover the family of indecomposable $C$-modules $\{Z_v\}_{v \in J'}$. Accordingly, we replace the preceding notation $\{f_v\}_{v \in J'}$, for the idempotents associated to $\{Z_v\}_{v \in J'}$, respectively, by the more descriptive one which uses $f_{v,s,a}$ to denote the idempotent in $S$ corresponding to the indecomposable $Z_{v,s,a}$.

Proof of (1): It is enough to see that any $M \in \mathcal{A}^E - \text{Mod}$ with length $\leq d$ is isomorphic in $\mathcal{A}^E - \text{Mod}$ to $F^X(N)$, for some $N \in \mathcal{A}^X - \text{Mod}$, and then we can apply [32]. Clearly, $\ell_E(e_v M) \leq d$, for all $v \in J'$. Since $x \in R e_v$, we have that multiplication by $h_v$ on $e_v M$ is an endomorphism of $E$-modules, so by Fitting’s Lemma, we have a decomposition of $E$-modules $e_v M = M_v^e \oplus M_v$, which is also a decomposition of $H_v$-modules, where $h_v$ acts invertibly on $M_v^e$ and $h_v^d$ annihilates $M_v^e$. (indeed, we have that $\ell_E(e_v M) \leq d$). So, $M_v$ is a $H_v/\langle h_v^d \rangle$-module and admits a decomposition of the form $M_v = \bigoplus_{s=1}^{s_v} \bigoplus_{a=1}^{d} M_v(s,a)$, where $M_v(s,a)$ is isomorphic to a possibly infinite direct sum of copies of $H_v/\langle \pi_{v,s,a} \rangle$. This follows, for instance, from [2], for $H_v/\langle h_v^d \rangle$ has finite representation type. We have that $M_v(s,a) \cong H_v/\langle \pi_{v,s,a} \rangle \oplus k V_v(s,a)$, for an appropriate $k$-vector space $V_v(s,a)$.

Then, given $M \in \mathcal{A}^E - \text{Mod}$, we can consider the $S$-module $N$ defined by the following: for $v \in J'$, set $e_v N := M_v$ and, for $s \in [1,s_v]$ and $a \in [1,d]$, define $f_{v,s,a} N := V_v(s,a)$; and, for $u \in J'$, we set $e_u N := e_u M$. Notice that we have isomorphisms of $R$-$E$-bimodules $X e_v \otimes_{S e_v} e_v N \cong (H_v)_{h_v} \otimes (e_u)_u M_u \cong M_u$, for $v \in J'$; $X e_u \otimes_{S e_u} e_u N \cong R e_u \otimes_{R e_u} e_u M \cong e_u M$, for $u \in J'$; and, finally, $X f_{v,s,a} \otimes_{S f_{v,s,a}} f_{v,s,a} N \cong H_v/\langle \pi_{v,s,a} \rangle \otimes_k V_v(s,a) \cong M_v(s,a)$.

Hence, we have an isomorphism of $R$-modules

\[
M = \bigoplus_{v \in J'} e_v M \oplus \bigoplus_{u \in J} e_u M \\
\cong \bigoplus_{v \in J'} \left( (X \otimes_S e_v N) \oplus \bigoplus_{s=1}^{s_v} \bigoplus_{a=1}^{d} X \otimes_S f_{v,s,a} N \right) \oplus \bigoplus_{u \in J} e_u N,
\]

that is we have an isomorphism of $R$-modules $X \otimes_S N \cong M$, and we can complete the proof of (1), using [6] and [32].

Proof of (2): Assume that $E$ is a division algebra and proceed as in the first part of the proof of [10] (25.9), using Crawley-Boevey’s argument of [15] (2.11),
adapted to the multiple unravelling situation in the context of triangular interlaced weak ditalgebras, using \ref{ide}. 

Lemma 4.2 (ideal-reduction). Let $\mathcal{A} = (\mathcal{A}, I)$ be a triangular interlaced weak ditalgebra with triangular layer $(R, W)$, where $R$ is a minimal algebra. We have $R = \prod_{u \in P} R_u e_u$, where each $R_u$ is either isomorphic to $k$ or to some rational algebra. Assume that $\mathcal{P} = J \cup J'$ where $R_u = k e_u$, for $u \in J$, and $R_v = R_v e_v$ with $R_v = k[x]_{g_v}$, for $v \in J'$. Assume furthermore that $I$ contains no primitive idempotent $e_u$ of $R$, but that $I_0 := I \cap R \neq 0$. Set $V = \{v \in \mathcal{P} \mid I_0 e_v \neq 0\}$.

Then, there is a complete triangular admissible $R$-module $X$ such that $\mathcal{A}_X := (\mathcal{A}_X, I^X)$ is a triangular interlaced weak ditalgebra with triangular layer $(S, W^X)$, where $S$ is a minimal algebra of the form

$$S = \left[ \prod_{w \in J''} k f_w \right] \times \prod_{u \in P \setminus V} k e_u,$$

where $\{f_w\}_{w \in J''}$ is a new finite family of primitive idempotents of $S$ and we have $I^X \cap S = 0$. The associated functor $F^X : \mathcal{A}_X \text{-Mod} \rightarrow \mathcal{A}_X \text{-Mod}$ is such that, for any $k$-algebra $E$, the following holds:

1. The functor $F^X_E : \mathcal{A}_X^E \text{-Mod} \rightarrow \mathcal{A}_E \text{-Mod}$ induces a length controlling equivalence.

2. If $E$ is a division $k$-algebra, $F^X_E$ induces an equivalence from $\mathcal{A}_X \text{-Mod}_p$ to $\mathcal{A}_E \text{-Mod}_p$.

Proof. Consider the functor $F^X : \mathcal{A}_X \text{-Mod} \rightarrow \mathcal{A}_X \text{-Mod}$ constructed in the proof of \ref{ide}(7.11)(case 2), so we have that $S \cap I^X = 0$. The additional hypothesis requiring a stellarity property for $\mathcal{A}_X$ assumed there is irrelevant for the construction of $\mathcal{A}_X^X$. Notice that $V \subseteq J'$. We need to recall some features of this construction. Set $e = \sum_{v \in V} e_v$, thus $I_0 \subseteq Re$ and the quotient algebra $Re/I_0 \cong \prod_{v \in V} Re_v/I_0 e_v$ has finite representation type. Consider a complete family of representatives $\{Z_w\}_{w \in J''}$ of the isoclasses of indecomposable $Re/I_0$-modules and set $Z := \bigoplus_{w \in J''} Z_w$. Then, we have the $R$-module $X := Z \oplus R(1-e)$ and the splitting $\Gamma = \text{End}_R(X)^{op} = S \oplus P$, where $P = \text{radEnd}_R(Z)^{op}$ and $S$ is the minimal algebra specified in the statement of this lemma. We know that $Re/I_0 = \prod_{v \in V} Re_v/(h_v)$, where $I_0 e_v = (h_v)$, for some $h_v \in Re_v = k[x]_{g_v}$.

For $v \in V$, we set $H_v := Re_v = k[x]_{g_v}$ and consider the prime factorization $h_v = \pi_{v,1} \pi_{v,2} \ldots \pi_{v,t_v}$ of $h_v$ in $H_v$. Then, write $Z_{v,t,a} := H_v/(\pi_{v,t}^a)$, for $t \in [1,t_v]$ and $a \in [1,a_{v,t}]$. So $\{Z_{v,t,a}\}_{t,a}$ is a complete set of representatives of the isoclasses of the indecomposable $H_v/(h_v)$-modules. When we consider the union of all these families of indecomposables, for $v \in V$, we recover the family of indecomposable $Re/I_0$-modules $\{Z_w\}_{w \in J''}$. Accordingly, we replace the preceding notation $\{f_w\}_{w \in J''}$, for the idempotents associated to $\{Z_w\}_{w \in J''}$, respectively, by the more descriptive one which uses $f_{v,t,a}$ to denote the idempotent in $S$ corresponding to the indecomposable $Z_{v,t,a}$.
From \[27\] and \[33\], we already know that \(F^X_E : \mathcal{A}^X \to \mathcal{A}^E\) is a length controlling, full and faithful functor. Then, its restriction to proper \(\mathcal{A}^E\) bimodules \(F^X_E : \mathcal{A}^X \to \mathcal{A}^E\) is full and faithful too.

Let us show that the functor \(F^X_E\) is dense. The argument is very similar to the proof of the preceding lemma. We only give a sketch.

**Proof of (1):** It is enough to see that any \(M \in \mathcal{A}^E\) isomorphic in \(\mathcal{A}^E\) to \(F^X(E)\), for some \(N \in \mathcal{A}^X\), and then we can apply \[32\].

Given \(v \in \mathcal{V}\), since multiplication by \(h_v\) annihilates the left \(H_v\)-module \(e_v M\), it is a \(H_v/\langle h_v \rangle\)-module and admits a decomposition of the form

\[
e_v M = \bigoplus_{t=1}^{a_v} \bigoplus_{a=1}^{t_v} M_v(t,a),
\]

where \(M_v(t,a)\) is isomorphic to a possibly infinite direct sum of copies of \(H_v/\langle \pi_v^{t,a} \rangle\). Again, this follows from \[2\]. We have \(M_v(t,a) \cong H_v/\langle \pi_v^{t,a} \rangle \otimes_k V_v(t,a)\), for an appropriate \(k\)-vector space \(V_v(t,a)\).

Then, given \(M \in \mathcal{A}^E\), we can consider the \(\mathcal{S}\)-module \(\mathcal{N}\) defined by the following: For \(u \in \mathcal{P} \setminus \mathcal{V}\), set \(e_u N := e_u M\) and, for \(v \in \mathcal{V}\), \(t \in [1,t_v]\), and \(a \in [1,a_v]\), set \(f_v,t,a N := V_v(t,a)\).

Notice that we have the following isomorphisms of \(R\)-modules. For \(v \in \mathcal{P} \setminus \mathcal{V}\), we have \(X e_u \otimes_{S e_u} e_u N \cong R e_u \otimes_{S e_u} e_u M \cong e_u M\), and, for a triple of indexes \((v,t,a)\), we have \(X f_v,t,a \otimes_{S f_v,t,a} f_v,t,a N \cong H_v/\langle \pi_v^{t,a} \rangle \otimes_k V_v(t,a) \cong M_v(t,a)\). Hence, we have an isomorphism of \(R\)-modules

\[
M = \bigoplus_{u \in \mathcal{P} \setminus \mathcal{V}} (X \otimes_e e_u N) \oplus (\bigoplus_{v \in \mathcal{V}} \bigoplus_{t=1}^{a_v} \bigoplus_{a=1}^{t_v} X \otimes_S f_v,t,a N),
\]

that is: we have an isomorphism of \(R\)-modules \(X \otimes S N \cong M\), and we can complete the proof of (1), using \[24\] and \[32\].

**Proof of (2):** Assume that \(E\) is a division algebra and proceed as in the last proof, using again \[37\].

**Lemma 4.3** (edge-reduction). Let \(\mathcal{A} = (\mathcal{A},I)\) be a triangular interlaced weak ditalgebra with triangular layer \((R,W)\), where \(R\) is a minimal algebra. We have \(\mathcal{R} = \prod_{u \in \mathcal{P}} R_u e_u\), where each \(R_u e_u\) is either isomorphic to \(k\) or to some rational algebra. We can assume that \(\mathcal{P} = \mathcal{J} \cup \mathcal{J}'\) where \(R_u e_u = k e_u\), for \(u \in \mathcal{J}\), and \(R_v e_v = R_v e_v\) with \(R_v = k[x]_{g_v}\), for \(v \in \mathcal{J}'\). Moreover, assume that \(W_0 = W_0' \oplus W_0''\), and that the \(R-R\)-bimodule \(W_0'\) is freely generated by some element \(e \in e_v W_0 e_u\), where \(u_0\) and \(v_0\) are two different elements in \(\mathcal{J}\). Set \(e = e_u + e_v\).

Then, there is a complete triangular admissible \(R\)-module \(X\) such that \(\mathcal{A}^X := (\mathcal{A}^X, I^X)\) is a triangular interlaced weak ditalgebra with triangular layer \((S,W^X)\), where \(S\) is a minimal algebra of the form

\[
S = k f_{v_1} \times k f_{\bar{z}} \times k f_{u_0} \times (1 - e) R,
\]

where \(f_{v_1}, f_{\bar{z}}, f_{u_0}\) are new primitive idempotents of \(S\). Moreover, there is a full and faithful functor \(F^X : \mathcal{A}^X \to \mathcal{A}^E\) such that, for any \(k\)-algebra \(E\), the following holds:
1. The functor $F^X_E : \Delta^X - E\text{-Mod} \to \Delta - E\text{-Mod}$ induces a length controlling equivalence.

2. If $E$ is a division $k$-algebra, the functor $F^X_E$ induces an equivalence of categories $\Delta^X - E\text{-Mod}_p \to \Delta - E\text{-Mod}_p$.

Proof. This is similar to [10](25.8), an adaptation of Crawley-Boevey’s argument of [15](2.10). We recall the argument. Here, we consider the subalgebra $B := eB \times (1-e)R$, where $eB$ is identified with the path algebra of the quiver $u_0 \to v_0$.

Thus, $eB$ admits only three classes of indecomposables represented by the simple injective $H_{u_0}$, the simple injective $H_{v_0}$, and the injective-projective $H_z$, which are naturally considered as $B$-modules. We consider the $B$-module $X = H_{u_0} \oplus H_z \oplus H_{v_0} \oplus (1-e)B$. Then, we have the splitting $\text{End}_B(X)^{\text{op}} = S \oplus P$, where $S$ has the form specified in the statement of this lemma, where $f_v, f_z, f_u \in \text{End}_B(X)^{\text{op}}$ are the idempotents corresponding to the indecomposable direct summands $H_{u_0}$, $H_z$, and $H_{v_0}$ of $X$, and $P = \text{rad}\text{End}_B(H_{u_0} \oplus H_z \oplus H_{v_0})^{\text{op}}$.

From (2.7) and (3.9), we know that the functor $F^X_E : \Delta^X - E\text{-Mod} \to \Delta - E\text{-Mod}$ is length controlling, full and faithful. Then, its restriction to proper bimodules $F^X_E : \Delta^X - E\text{-Mod}_p \to \Delta - E\text{-Mod}_p$ is full and faithful too. Assume that $E$ is a division algebra and let us show that the restricted one $F^X_E$ is dense.

Since the algebra $eB$ can be identified with the path algebra of the quiver $u_0 \to v_0$, given any $B$-$E$-$bimodule $M$, we can consider the multiplication by $\alpha$ map $M_\alpha : e_{u_0}M \to e_{v_0}M$. Since $E$ is semisimple, there is a right $E$-$module direct sum decomposition $e_{u_0}M = M'_{u_0} \oplus M'_1$, where $M'_{u_0} := \text{Ker}M_\alpha$. Again, semisimplicity of $E$ entails a right $E$-$module decomposition $e_{v_0}M = \alpha M'_1 \oplus M'_{u_0}$. Thus, $eM = e_{u_0}M \oplus e_{v_0}M = M'_{u_0} \oplus (M'_1 \oplus \alpha M'_1) \oplus M'_{u_0} \cong [H_{u_0} \otimes_k M'_{u_0}] \oplus [H_z \otimes_k M'_1] \oplus [H_{v_0} \otimes_k M'_{u_0}]$, where the last isomorphism is an $eB$-$E$-$module isomorphism.

Now, consider the $S$-$E$-$bimodule $N$ defined by $f_u, N := M'_{u_0}$, $f_z, N := M'_1$, $f_v, N := M'_{u_0}$, and, for $w \not\in \{u_0, v_0\}$, we take $f_w, N := M_w$ (recall that we are identifying $f_w, S$, where $f_w$ is the idempotent in $\Gamma$ associated to the direct summand $\text{Re}_w$, with $e_w R$ through the isomorphism of algebras $f_w S \cong \text{End}_R(\text{Re}_w) \cong \text{Re}_w$). Notice that we have the isomorphisms of $B$-$E$-$bimodules: $X f_u \otimes s f_u$, $N \cong H_{u_0} \otimes_k M'_{u_0}$, $X f_z \otimes s f_z$, $N \cong H_z \otimes_k M'_1$, $X f_v \otimes s f_v$, $N \cong H_{v_0} \otimes_k M'_{u_0}$, and $X f_w \otimes s f_w$, $N \cong \text{Re}_w \otimes_{\text{Re}_w} M_w \cong M_w$, for $w \not\in \{u_0, v_0\}$. Hence, we have an isomorphism $M \cong \otimes_{u \in \Gamma} X f_u \otimes s f_u$, $f_u, N$ of $B$-$E$-$bimodules, and we can apply (3.7) to obtain $F^X_E(\mathbf{N}) \cong M$ in $\Delta - E\text{-Mod}_p$ for some $\mathbf{N} \in \Delta^c - E\text{-Mod}_p$ with underlying $S$-$E$-$bimodule $N$. We have completed the proof of (2).

To order to show (1), it is enough to verify that $F^X : \Delta^X - \text{Mod} \to \Delta - \text{Mod}$ is dense, and then we can apply (3.2). The density of $F^X$ can be obtained with the same argument as before, taking in $E = k$. □

Remark 4.4. We have similar lemmas to the preceding ones corresponding to absorption and regularization, as in (2.2) and (2.4), we just need to recall that the associated reduction functors are equivalences and apply (3.2).

Remark 4.5. The minimal ditalgebras appearing in the next result are, by definition, ditalgebras $B$ with triangular layer $(R, W)$, where $R$ is a minimal algebra,
$W_0 = 0$, and $W_1$ is freely generated by a finite directed subset, see [10](23.2). Their module category is tame and well understood. A crucial fact in the proof of Drozd’s theorem on the tame-wild dichotomy for finite-dimensional algebras in [18] is that, for any seminested non-wild ditalgebra $A$ and any dimension $d$, there is a composition of reduction functors $F : B\text{-Mod} \rightarrow A\text{-Mod}$, where $B$ is a minimal ditalgebra such that any $M \in A\text{-Mod}$ with $\dim M \leq d$ is of the form $M \cong F(N)$, for some $N \in B\text{-Mod}$. This is proved in [4]§8 using bocses and in [10]§28, using the equivalent language of ditalgebras. Using bocses, almost the same statement was proved insightfully in [15], among other important results, but using a finite number of minimal ditalgebras. The mentioned fact plays also a crucial role in the construction of the family of functors in the next theorem.

**Theorem 4.6.** Assume that the ground field $k$ is algebraically closed and that $A = (A, I)$ is a $P$-oriented triangular interlaced weak ditalgebra, as in (1.9), where $I \subseteq \text{rad} A$. Suppose that $A$ is not wild and take $d \in \mathbb{N}$. Then, there is a finite sequence of reductions

$$A = (A, I) \mapsto (A^{z_1}, I^{z_1}) \mapsto \cdots \mapsto (A^{z_{1;2\cdots;2^t}}, I^{z_{1;2\cdots;2^t}})$$

of type $z_i \in \{a, d, r, q, X\}$ such that $A^{z_{1;2\cdots;2^t}}$ is a minimal ditalgebra, we have $I^{z_{1;2\cdots;2^t}} = 0$ and, for any $k$-algebra $E$ and every $(A, I)$-$E$-bimodule $M$ with $\ell_E(M) \leq d$ has the form $M \cong F_{E}^{z_1} \cdots F_{E}^{z_t}(N)$, for some $(A^{z_{1;2\cdots;2^t}}, I^{z_{1;2\cdots;2^t}})$-$E$-bimodule $N$. Moreover, if $E$ is a division algebra and $M$ is a proper bimodule, then $N$ can be chosen to be proper.

**Proof.** Consider the finite sequence of reductions

$$A = (A, I) \mapsto (A^{z_1}, I^{z_1}) \mapsto \cdots \mapsto (A^{z_{1;2\cdots;2^t}}, I^{z_{1;2\cdots;2^t}})$$

of type $z_i \in \{a, d, r, q, X\}$ such that $A^{z_{1;2\cdots;2^t}}$ is a minimal ditalgebra and $I^{z_{1;2\cdots;2^t}} = 0$, constructed inductively in the proof of [8](9.2), for a fixed $d \in \mathbb{N}$. The same sequence of functors $F^{z_1}, \ldots, F^{z_t}$ consists of functors $F^z$ type $z \in \{a, d, r, q, X\}$, as described in [22]. Moreover, those of type $X$ are either multiple $d$-unravellings, edge-reductions, or ideal-reductions, as described in the preceding lemmas. This is a consequence of the fact that these are the type of functors appearing in the reduction of non-wild seminested ditalgebras to minimal ditalgebras, see [10](28.22) and [8](8.9), which are used in the inductive argument of the proof.

Then, the theorem follows as a consequence of [3.3], [3.6], [4.2], [4.1], and [4.3].

**Definition 4.7.** If $B$ is a minimal ditalgebra with layer $(R, W)$, we have the decomposition of the unit element of $R$ as a sum $1 = \sum_{i=1}^{n} e_i$ of primitive orthogonal idempotents. If $Be_i \neq k$, we denote by $Q_i$ the field of fractions of the rational $k$-algebra $Be_i$, considered as a $B$-module.

**Theorem 4.8.** Assume that the field $k$ is algebraically closed and that $A$ is a $P$-oriented triangular interlaced weak ditalgebra, as in (1.17). Suppose that $A$
is not wild and take \( d \in \mathbb{N} \). Consider the functor \( F : \mathcal{B}\text{-Mod} \rightarrow \mathcal{A}\text{-Mod} \) associated to the composition of reductions of \( (4.3) \), where \( \mathcal{B} \) is a minimal ditalgebra. Denote by \( \mathcal{B} \) the underlying algebra of degree zero elements of \( \mathcal{B} \). The following holds.

1. The \( \mathcal{A}\text{-}\mathcal{B}\text{-bimodule} \ T := F(\mathcal{B}) \) is finitely generated by the right and the composition

   \[ \mathcal{B}\text{-Mod} \xrightarrow{L_{\mathcal{B}}} \mathcal{B}\text{-Mod} \xrightarrow{F} \mathcal{A}\text{-Mod} \]

   is naturally isomorphic to \( L_{\mathcal{A}}(T \otimes_B -) \). So, \( F(N) \cong T \otimes_B N \), for any \( N \in \mathcal{B}\text{-Mod} \). If \( N \) is a proper \( \mathcal{B}\text{-}\mathcal{E}\text{-bimodule} \), for some \( \mathcal{K}\text{-algebra} \mathcal{E} \), then we have an isomorphism of proper \( \mathcal{A}\text{-}\mathcal{E}\text{-bimodules} \ F_{\mathcal{E}}(N) \cong T \otimes_B (\mathcal{N}_{\mathcal{E}}) \).

2. The \( \mathcal{A}\text{-modules} \) of the form \( G \cong T \otimes_B Q_i \) are pregeneric.

3. If \( M \) is an indecomposable \( \mathcal{A}\text{-module} \) with \( \text{end}(M) \leq d \), there is an indecomposable \( \mathcal{B}\text{-module} \ N \) with finite endolength such that \( F(N) \cong M \). Moreover, if \( M \) is pregeneric, we have \( \text{End}_{\mathcal{A}}(M)^{\text{op}} = \mathcal{Q}_M \oplus \text{radEnd}_{\mathcal{A}}(M)^{\text{op}} \), for some subalgebra \( \mathcal{Q}_M \cong k(x) \) and the ideal \( \text{radEnd}_{\mathcal{A}}(M)^{\text{op}} \) is nilpotent.

4. If the given \( \mathcal{A}\text{-module} \ M \) is pregeneric, the \( \mathcal{B}\text{-module} \ N \) is generic and has a natural structure of \( \mathcal{B}\text{-}k(x)\text{-bimodule} \). The morphism of algebras \( \mu : k(x) \rightarrow \text{End}_{\mathcal{A}}(T \otimes_B N)^{\text{op}} \) given by \( \mu(q) = (\text{id}_T \otimes \mu_q, 0) \) gives \( M \) the structure of a proper \( \mathcal{A}\text{-}k(x)\text{-bimodule} \ T \otimes_B N \cong F(N) \cong M \), such that \( \text{end}(M) = \dim_{k(x)} M \).

Proof. (1) follows from \( [8] \) (6.13). The proof of (2) is similar to the proof of \( [15] \) (4.3), because \( F_{k(x)} \) is length controlling.

(3) and (4): Here, \( M \) is an \( \mathcal{A}\text{-}\mathcal{E}\text{-bimodule} \) with finite length \( \ell_E(M) \), where \( E = \text{End}_{\mathcal{A}}(M)^{\text{op}} \) and its bimodule structure map \( \alpha_M : E \rightarrow \text{End}_{\mathcal{A}}(M)^{\text{op}} \) is the identity. From \( (4.3) \), we know that there is a \( \mathcal{B}\text{-}\mathcal{E}\text{-bimodule} \ N \) with finite length \( \ell_E(N) \) such that \( F_{\mathcal{E}}(N) \cong M \). For the sake of simplicity, we can assume that \( F_{\mathcal{E}}(N) = M \). The bimodule structure \( \alpha_N : E \rightarrow \text{End}_{\mathcal{G}}(N)^{\text{op}} \) of \( N \) is given by the composition

\[ E \xrightarrow{\alpha_M} \text{End}_{\mathcal{A}}(M)^{\text{op}} \xrightarrow{F_{\mathcal{E}}^{-1}} \text{End}_{\mathcal{G}}(N)^{\text{op}}, \]

so \( \alpha_N \) is an isomorphism.

Since \( E \cong \text{End}_{\mathcal{G}}(N)^{\text{op}} \) and \( \ell_E(N) \) is finite, the module \( N \in \mathcal{B}\text{-}\text{Mod} \) has finite endolength. By \( [10] \) (31.6), the module \( N \) over the minimal algebra \( \mathcal{B} \) has finite endolength.

If \( M \) is finite-dimensional, so is \( N \). If \( M \) is infinite-dimensional, \( N \) has the same property, so, \( N \) is a pregeneric \( \mathcal{B}\text{-module} \).

For each point \( i \) of \( \mathcal{B} \), we have that \( B\epsilon_i = k\epsilon_i \) or \( \Gamma_i = B\epsilon_i \neq k\epsilon_i \). In the last case, we have \( Q_i \cong k(x) \) the field of fractions of \( \Gamma_i \), considered as a \( \Gamma_i\text{-module} \). It is known that the \( \mathcal{B}\text{-modules} \ Q_i \) provide a complete set of representatives of
the isomorphism classes of the pregeneric $B$-modules, see [10](31.3). So, we can assume that $N \cong Q_i$, for some $i$. Moreover, by [10](31.6), we have
\[
\text{End}_B(N)^{\text{op}} = Q_N \bigoplus \text{radEnd}_B(N)^{\text{op}},
\]
for some subalgebra $Q_N$ of $\text{End}_B(N)^{\text{op}}$ such that $Q_N \cong k(x)$ and the ideal $\text{radEnd}_B(N)^{\text{op}}$ is nilpotent. Moreover, $N$ is a proper $B$-$k(x)$-bimodule. Then $M = F(N)$ is a proper $\mathcal{A}$-$k(x)$-bimodule and, since $F : \text{End}_B(N)^{\text{op}} \rightarrow \text{End}_\mathcal{A}(M)^{\text{op}}$ is an isomorphism, we obtain that
\[
\text{End}_\mathcal{A}(M)^{\text{op}} = \overline{Q}_M \bigoplus \text{radEnd}_\mathcal{A}(M)^{\text{op}},
\]
where $\overline{Q}_M := F(Q_N)$ is a subalgebra of $\text{End}_\mathcal{A}(M)^{\text{op}}$ such that $\overline{Q}_M \cong k(x)$ and the ideal $\text{radEnd}_\mathcal{A}(M)^{\text{op}}$ is nilpotent. Statement (4) follows from this. □

**Proof of (1.10)** We know that $\mathcal{A}$ is tame if it is not wild, by §(10.4). From Lemma 5.1, we know that if $\mathcal{A}$ is pregenerically tame, then it is not wild. So, it remains to show that if $\mathcal{A}$ is not wild, then it is pregenerically tame. So, take $d \in \mathbb{N}$ and a pregeneric $\mathcal{A}$-module $G$ with $\text{end}(G) \leq d$. Apply (4.8) to $d$ to obtain a functor $F : B\text{-Mod} \rightarrow \mathcal{A}\text{-Mod}$, and a generic $B$-module $G'$ with $F(G') \cong G$. Since there are only finitely possible generic $B$-modules $G'$ up to isomorphism, it follows that $\mathcal{A}$ is pregenerically tame. □

5 On the category of modules for a special $\mathcal{A}$

In this section, we consider a special triangular interlaced weak ditalgebra $\mathcal{A} = (\mathcal{A}, I)$, as defined in §12. That is such that its layer $(S, W)$ satisfies: $S$ is a finite product of copies of the field $k$, $W$ is finite-dimensional, and so is $\mathcal{A} = A/I$, where $A = T_3(W_0)$. Thus, $\mathcal{A}$ is a Roiter interlaced weak ditalgebra.

It is known that, in this case, the category $\mathcal{A}\text{-Mod}$ admits an exact structure $\mathcal{E}$ where the conflations are the composable pairs of morphisms $M \longrightarrow E \longrightarrow N$ such that $gf = 0$ and the sequence $0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0$ is exact in $S\text{-Mod}$, see §11.

The **right algebra** of $\mathcal{A}$ is the finite-dimensional $k$-algebra $\Gamma := \text{End}_\mathcal{A}(\mathcal{A})^{\text{op}}$. The composition $\mathcal{A} \longrightarrow \text{End}_\mathcal{A}(\mathcal{A})^{\text{op}} \longrightarrow \text{End}_\mathcal{A}(\mathcal{A})^{\text{op}} = \Gamma$, is an embedding of $k$-algebras, so $\Gamma$ is naturally an $\mathcal{A}$-algebra. The functor $H : \mathcal{A}\text{-Mod} \longrightarrow \Gamma\text{-Mod}$, given by $H = \text{Hom}_{\mathcal{A}}(\cdot, -)$ is an exact functor, because $\mathcal{A}$ is $\mathcal{E}$-projective, see [12] and §12.

**Lemma 5.1.** Let $\mathcal{A}$ be a special triangular interlaced weak ditalgebra. Then the functor $H = \text{Hom}_{\mathcal{A}}(\cdot, -)$ preserves arbitrary direct sums. More precisely, for any family $\{M_i\}_{i \in I}$ of $\mathcal{A}$-modules, there is an isomorphism of $\Gamma$-modules
\[
\Theta : \text{Hom}_{\mathcal{A}}(\bigoplus_{i \in I} M_i) \longrightarrow \bigoplus_{i \in I} \text{Hom}_{\mathcal{A}}(A, M_i)
\]
such that $f \mapsto \sum_{i \in I} \pi_i f$, where $\pi_i : \bigoplus_{i \in I} M_i \longrightarrow M_i$ denotes the projection on the factor $M_i$ in $\mathcal{A}\text{-Mod}$. This is a natural isomorphism.
\textbf{Proof.} Among other properties, we will verify in a moment that the sum $\sum_{i \in I} \pi_i f$ is finite.

First recall that whenever $Q$ is a finitely generated $R$-module over some algebra $R$, we have the morphism $\theta : \text{Hom}_R(Q, \bigoplus_{i \in I} M_i) \longrightarrow \bigoplus_{i \in I} \text{Hom}_R(Q, M_i)$, such that $f \longmapsto \sum_{i \in I} \pi_i f$, where $\pi_i : \bigoplus_{i \in I} M_i \longrightarrow M_i$ denotes the projection on the factor $M_i$. Indeed, since $Q$ is finitely generated, the image of each morphism $f$ lies in a finite subsum $\bigoplus_{i \in I} M_i$ of $\bigoplus_{i \in I} M_i$, the sum $\sum_{i} \pi_i f$ is finite, and we have the commutative square

$$\begin{align*}
\text{Hom}_R(Q, \bigoplus_{i \in I} M_i) & \longrightarrow \bigoplus_{i \in I} \text{Hom}_R(Q, M_i) \\
\uparrow & \\
\text{Hom}_R(Q, \bigoplus_{i \in I} M_i) & \longrightarrow \bigoplus_{i \in I} \text{Hom}_R(Q, M_i),
\end{align*}$$

where the horizontal arrows have the same recipe, the right vertical arrow is an inclusion and the first vertical arrow is induced by an inclusion. This implies that the first row is an isomorphism too. We denote by $V$ the homogeneous component of degree 1 of the underlying tensor algebra of $A$. Then, we have isomorphisms

$$\begin{align*}
\text{Hom}_S(A, \bigoplus_{i \in I} M_i) \times \text{Hom}_A(V, \text{Hom}_k(A, \bigoplus_{i \in I} M_i)) & \longrightarrow \bigoplus_{i \in I} \text{Hom}_A(V, \bigoplus_{i \in I} \text{Hom}_k(A, M_i)) \\
\downarrow_{\theta_1 \times \theta_2} & \\
\bigoplus_{i \in I} \text{Hom}_S(A, M_i) \times \text{Hom}_A(V, \bigoplus_{i \in I} \text{Hom}_k(A, M_i)) & \longrightarrow \bigoplus_{i \in I} \text{Hom}_A(V, \bigoplus_{i \in I} \text{Hom}_k(A, M_i)).
\end{align*}$$

Let us show that the composition $\Theta$ of these isomorphisms restricts to an isomorphism as in the statement of this lemma. So we take $f \in \text{Hom}_A(A, \bigoplus_{i \in I} M_i)$ and let us show that $\Theta(f) \in \bigoplus_{i \in I} \text{Hom}_A(A, M_i)$. We have

$$\Theta(f) = \Theta(f^0, f^1) = (\theta_1(f^0), \theta_2(\theta_2^*(f^1))) = (\sum_{i} \pi_i f^0, \sum_{j} p_j \theta_2^*(f^1)) = (\sum_{i} \pi_i f^0, \sum_{j} p_j \theta_2 f^1),$$

where each $p_j : \bigoplus_{i \in I} \text{Hom}_k(A, M_j) \longrightarrow \text{Hom}_k(A, M_j)$ is the canonical projection. Thus, for $v \in \bigoplus_{i} V$, we have $\sum_{i} \sum_{j} p_j \pi_i f^1(v) = \sum_{i} \pi_i f^1(v)$. Since $f$ is a morphism in $A\text{-Mod}$, for $a, m \in A$, we have $a f^0(m) = f^0(\pi_i f^1(v))$. Then, we have $\sum_{i} \pi_i a f^0(m) = \sum_{i} \pi_i f^0(\sum_{j} p_j \theta_2 f^1)$, since each $\pi_i$ is a morphism of $A$-modules, we get $\sum_{i} a \pi_i f^0(m) = \sum_{i} \pi_i f^0(\sum_{j} p_j \theta_2 f^1)$ or, equivalently,

$$a \Theta(f)^0[m] = \Theta(f)^0(\sum_{i} \pi_i f^0(\sum_{j} p_j \theta_2 f^1))[m].$$

So, $\Theta : \text{Hom}_A(\bigoplus_{i \in I} M_i) \longrightarrow \bigoplus_{i \in I} \text{Hom}_A(A, M_i)$ is a natural isomorphism. Moreover, it is easy to see that indeed we have $\Theta(f) = \sum_{i \in I} \pi_i f$. \hfill \Box

\textbf{Lemma 5.2.} The functor $H : A\text{-Mod} \longrightarrow \Gamma\text{-Mod}$ is exact, full and faithful.
Proof. We can adapt the argument of the proof of [10](7.12), using that $H$ commutes with direct sums by (5.1). We use the commutative square of isomorphisms

$$
\begin{array}{ccc}
\text{Hom}_A(A^{(I)}, A^{(J)}) & \overset{H}{\rightarrow} & \text{Hom}_\Gamma(H[A^{(I)}], H[A^{(J)}]) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_A(A^{(I)}, A^{(J)}) & & \text{Hom}_\Gamma(H[A^{(I)}], H[A^{(J)}]) \\
\prod_I \text{Hom}_A(A, A) & \overset{\cong}{\rightarrow} & \prod_I \text{Hom}_\Gamma(H[A], H[A]) \\
\prod_I \prod_J \text{Hom}_A(A, A) & & \prod_I \prod_J \text{Hom}_\Gamma(H[A], H[A])
\end{array}
$$

\[ \square \]

Lemma 5.3. For each $M \in A$-Mod, there is a natural isomorphism

$$\sigma_M : \Gamma \otimes_A M \longrightarrow \text{Hom}_A(A, L_A(M))$$

of functors from $A$-Mod onto $\Gamma$-Mod. So the functor $H$ restricts to an equivalence of categories $H : A$-Mod $\longrightarrow \widetilde{I}$, where $\widetilde{I}$ denotes the full subcategory of $\Gamma$-Mod of modules induced from $A$-Mod, that is by the class of $\Gamma$-modules isomorphic to some $\Gamma \otimes_A N$, with $N \in A$-Mod.

Proof. The proof of [9](12.5) can be adapted to this situation, using again (5.1) and the fact that both $H$ and $\Gamma \otimes_A -$ are exact functors from $A$-Mod to $\Gamma$-Mod. See also [10](7.14). \[ \square \]

Lemma 5.4. The category $\widetilde{I}$ is closed under extensions and under direct summands.

Proof. Consider the equivalence $H : A$-Mod $\longrightarrow \widetilde{I}$. In order to show that $\widetilde{I}$ is closed under extensions, take an exact sequence

$$0 \longrightarrow M \overset{f}{\longrightarrow} E \overset{g}{\longrightarrow} N \longrightarrow 0$$

in $\Gamma$-Mod with $M, N \in \widetilde{I}$. Then $M \cong \Gamma \otimes_A M'$ and $N \cong \Gamma \otimes_A N'$, and we may assume that $M = H(M')$ and $N = H(N')$. Consider an exact sequence

$$0 \longrightarrow K' \overset{s}{\longrightarrow} P' \overset{t}{\longrightarrow} N' \longrightarrow 0$$

in $A$-Mod with $P'$ projective. It determines a conflaction $K' \overset{k}{\longrightarrow} P' \overset{t}{\longrightarrow} N'$ in $A$-Mod and, applying $H$, we get an exact sequence

$$0 \longrightarrow H(K') \overset{H(s)}{\longrightarrow} H(P') \overset{H(t)}{\longrightarrow} H(N') \longrightarrow 0$$

in $\Gamma$-Mod with $H(P')$ projective. Then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & H(K') & H(P') & H(N') & \longrightarrow & 0 \\
\downarrow h & & \downarrow H(s) & \downarrow H(t) & \| & \| & \| \\
0 & \longrightarrow & M & F & E & g & N & \longrightarrow & 0.
\end{array}
$$
Now, there is some morphism $h' : K' \to M'$ in $\mathcal{A}$-Mod with $H(h') = h$. Since $\mathcal{A}$-Mod is an exact category, we have a pushout diagram in $\mathcal{A}$-Mod of the form

$$
\begin{array}{ccc}
K' & \xrightarrow{s} & P' \\
\downarrow h' & & \downarrow t \\
M' & \xrightarrow{f'} & E' \\
\end{array}
\begin{array}{ccc}
\xrightarrow{\|} & & \xrightarrow{\|} \\
M' & \xrightarrow{f} & E' \\
\end{array}
\begin{array}{ccc}
\xrightarrow{\|} & & \xrightarrow{\|} \\
M' & \xrightarrow{\|} & N'.
\end{array}
\]

Applying the functor $H$ to this diagram, we obtain a pushout diagram in $\Gamma$-Mod

$$
\begin{array}{ccc}
0 & \to & H(K') & \xrightarrow{H(s)} & H(P') & \xrightarrow{H(t)} & H(N') & \to & 0 \\
0 & \to & M & \xrightarrow{H(f')} & H(E') & \xrightarrow{H(g')} & N & \to & 0.
\end{array}
$$

By the uniqueness of pushouts, we obtain $E \cong H(E')$ and $E \in \mathcal{Z}$.

Now, in order to show that $\mathcal{Z}$ is closed under direct summands, take $M \in \mathcal{Z}$. We may assume that $M = H(M')$. Take a direct summand $N$ of $M$, consider the inclusion $i : N \to M$, the projection $p : M \to N$, and the idempotent $e = ip : M \to M$, which satisfy $pi = id_N$. Then, there is an idempotent $e' : M' \to M'$ in $\mathcal{A}$-Mod with $H(e') = e$. Since $\mathcal{A}$ is a Roiter interlaced weak ditalgebra, idempotents split in $\mathcal{A}$-Mod, see [9](11.4), so there is an $\mathcal{A}$-module $N'$ and morphisms $q : M' \to N'$ and $j : N' \to M'$, such that $jq = e'$ and $qj = id_{N'}$ in $\mathcal{A}$-Mod. The images $H(q) : M \to H(N')$ and $H(j) : H(N') \to M$ satisfy that $H(j)H(q) = H(e') = e = ip$ and $H(q)H(j) = id_{H(N')}$. This implies that $H(N') \cong N$, indeed we have the isomorphism $pH(j) : H(N') \to N$ with inverse $H(q)i : N \to H(N')$.

For the proof of the following results, the use of the right algebra $\Gamma$ associated to the special triangular interlaced weak ditalgebra $\mathcal{A}$ and the exact equivalence $H : \mathcal{A}$-mod $\to \mathcal{Z}$ is crucial. The first ideas in this direction can be traced back to [5], [12], and [13]. These results were already used in [8], but we prefer to state them here explicitly.

**Theorem 5.5.** If $\mathcal{A}$ is a special triangular interlaced weak ditalgebra, then it is a Roiter interlaced weak ditalgebra and the category $\mathcal{A}$-mod has a natural exact structure $\mathcal{E}_\mathcal{A}$, see [9](11.11). The corresponding exact category $(\mathcal{A}$-mod, $\mathcal{E}_\mathcal{A})$ has almost split conflations.

**Proof.** The proof of [10](7.18) can be adapted to this situation, using the equivalence $H : \mathcal{A}$-mod $\to \mathcal{Z}$, where $\mathcal{Z}$ denotes the full subcategory of $\Gamma$-mod of modules induced from $\mathcal{A}$-mod.

**Lemma 5.6.** Let $\mathcal{A}$ be a special triangular interlaced weak ditalgebra. For any non $\mathcal{E}_\mathcal{A}$-projective indecomposable object $M \in \mathcal{A}$-mod, there is an almost split conflation $N \to E \to M$. Thus, the indecomposable object $N$ is uniquely determined up to isomorphism. We call $N$ the translate of $M$ and denote it by $\tau(M)$. There is a number $c_\mathcal{A}$ such that, for any non $\mathcal{E}_\mathcal{A}$-projective indecomposable $M \in \mathcal{A}$-mod, the following inequality holds:

$$
\dim_k \tau(M) \leq c_\mathcal{A} \dim_k M.
$$

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Proof. The proof of \[10\] (32.5) can be adapted to this situation. See \[8\] (Proof of (1.4), in the admissible homological case, Steps 1 and 2).

**Theorem 5.7.** Let $\mathcal{A}$ be a non-wild special triangular interlaced weak ditalgebra. Thus, $(\mathcal{A}\text{-mod}, E_\mathcal{A})$ has almost split conflations and a translation $\tau$. Then, for every $d \in \mathbb{N}$, almost all $d$-dimensional indecomposable modules $M \in \mathcal{A}\text{-mod}$ satisfy $\tau(M) \cong M$.

**Proof.** The proof of \[10\] (32.6) can be adapted to this situation, using (5.6). See \[8\] (Proof of (1.4), in the admissible homological case, Steps 2 and 3).

### 6 Covering $\Delta'$-filtered modules

In this section we consider the $\mathcal{P}$-oriented interlaced weak ditalgebra $\mathcal{A} = \mathcal{A}(\Delta)$ associated to an admissible homological system $(\mathcal{P}, \leq, \{\Delta_v\}_{v \in \mathcal{P}})$ for a finite-dimensional algebra $\Lambda$, see for instance \[9\] (5.22 and 13.2). Thus $\mathcal{A}$ is a special triangular interlaced weak ditalgebra and §5 applies to it. Moreover, $\mathcal{A}$ satisfies the hypothesis of (4.8).

**Definition 6.1.** Let $\Gamma$ be the right algebra of $\mathcal{A}$. Consider the family of $\Gamma$-modules $\{\Delta'_v = \Gamma \otimes_{\mathcal{A}} S_v\}_{v \in \mathcal{P}}$, where $\{S_v\}_{v \in \mathcal{P}}$ is a complete set of representatives of the isoclasses of the simple $\mathcal{A}$-modules.

From \[9\] (13.10), we know that $(\mathcal{P}, \leq, \{\Delta'_v\}_{v \in \mathcal{P}})$ is an admissible homological system for $\Gamma$. Thus, we have the full subcategory $\tilde{F}(\Delta')$ of $\text{Gamma-Mod}$, as in \[10\]. Moreover, there is a Morita equivalence $\Omega : \Gamma\text{-mod} \rightleftharpoons \Lambda\text{-mod}$ such that $\Omega(\Delta'_v) \cong \Delta_v$, for $v \in \mathcal{P}$. The existence of the interlaced weak ditalgebra $\mathcal{A}(\Delta)$ and the equivalence $\Omega$ will play a crucial role in our arguments, when we transfer the covering results of this section to the general case of arbitrary homological systems in §7.

We fix the preceding notation for all this section, but, in the next ones, we apply the results to a particular algebra $\Lambda$ with a specific homological system $(\mathcal{P}, \leq, \{\Delta_v\}_{v \in \mathcal{P}})$.

**Proposition 6.2.** We have $\mathcal{I} = \tilde{F}(\Delta')$. So, the functor $H$ restricts to an exact equivalence $H : \mathcal{A}\text{-mod} \rightleftharpoons \mathcal{I} = \tilde{F}(\Delta')$, see (5.3) and (5.7).

**Proof.** We show first that any $M \in \mathcal{I}$ admits a filtration

$$0 = M_\ell \subseteq M_{\ell-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$$

such that $M_\ell/M_{\ell+1}$ is isomorphic to a (possibly infinite) direct sum of modules in the family $\{\Delta'_v = \Gamma \otimes_{\mathcal{A}} S_v\}_{v \in \mathcal{P}}$.

For this, we may assume that $M = \Gamma \otimes_{\mathcal{A}} M'$, for some $\mathcal{A}$-module $M'$. Since $\mathcal{A}$ is finite-dimensional, we have a filtration of $M'$ of the form

$$0 = M'_\ell \subseteq M'_{\ell-1} \subseteq \cdots \subseteq M'_1 \subseteq M'_0 = M'.$$
with $M' = J'M'$, where $J$ denotes the Jacobson radical of the finite-dimensional algebra $A$. Since $A/J$ is semisimple, each $A/J$-module $L$ is semisimple. So, we have $L \cong \bigoplus_{v \in P} S_v \otimes_k V_v$, where each $V_v$ is a vector space over the field $k$ (not necessarily finite-dimensional). The preceding filtration gives rise to exact sequences

$$0 \to M'_{\ell} \to M'_{\ell-1} \to L'_{\ell-1} \to 0$$

$$0 \to M'_{\ell-1} \to M'_{\ell-2} \to L'_{\ell-2} \to 0$$

$$\cdots$$

$$0 \to M'_1 \to M'_0 \to L'_0 \to 0$$

with $L'_0, \ldots, L'_{\ell-1}$ semisimple. Applying the exact functor $\Gamma \otimes_A -$ we obtain exact sequences

$$0 \to \Gamma \otimes_A M'_\ell \to \Gamma \otimes_A M'_{\ell-1} \to \Gamma \otimes_A L'_{\ell-1} \to 0$$

$$0 \to \Gamma \otimes_A M'_{\ell-1} \to \Gamma \otimes_A M'_{\ell-2} \to \Gamma \otimes_A L'_{\ell-2} \to 0$$

$$\cdots$$

$$0 \to \Gamma \otimes_A M'_1 \to \Gamma \otimes_A M'_0 \to \Gamma \otimes_A L'_0 \to 0.$$

Then, the module $\Gamma \otimes_A M'_0 = M$ admits a filtration with factors of the form $\Gamma \otimes_A L$, where $L \cong \bigoplus_{v \in P} S_v \otimes_k V_v$ is a semisimple $A$-module. So, these factors have the form

$$\Gamma \otimes_A L \cong \bigoplus_{v \in P} S_v \otimes_k V_v \cong \bigoplus_{v \in P} \Gamma \otimes_A S_v \otimes_k V_v = \bigoplus_{v \in P} \Delta'_v \otimes_k V_v.$$

In order to prove the converse, assume that a $\Gamma$-module $M \in \mathcal{F}(\Delta')$ has a filtration as in the beginning of this proof. Since direct sums of $\Gamma$-modules of the family $\{\Gamma \otimes_A L'_\ell\}_{\ell \in P}$ have the form $\Gamma \otimes_A L'$, where $L'$ is a semisimple $A$-module, we have exact sequences

$$0 \to M_{\ell} \to M_{\ell-1} \to \Gamma \otimes_A L'_{\ell-1} \to 0$$

$$0 \to M_{\ell-1} \to M_{\ell-2} \to \Gamma \otimes_A L'_{\ell-2} \to 0$$

$$\cdots$$

$$0 \to M_1 \to M_0 \to \Gamma \otimes_A L'_0 \to 0.$$

From the first sequence, we get that $M_{\ell-1} \cong \Gamma \otimes_A L'_{\ell-1}$. From the second one, using that $\mathcal{I}$ is closed under extensions, by [5, 4], we get that $M_{\ell-2} \in \mathcal{I}$. Proceeding inductively, in a finite number of steps, we get that $M = M_0 \in \mathcal{I}$. $\Box$

In [15], the results on generic modules for a finite-dimensional algebra $A$ are obtained from the corresponding objects of $D$-Mod, where $D$ is the Drozd’s ditalgebra of $A$. Here, the results on generic modules for $\mathcal{F}(\Delta')$ are obtained from the properties of the corresponding objects of $\mathcal{A}(\Delta)$-Mod. The passage between both categories is done with the help of the equivalence $H$. In the following we study first their effect on endolength and then transfer the results of [15,8] onto $\mathcal{F}(\Delta').
Lemma 6.3. Let $\Gamma$ be the right algebra of $\mathcal{A}$ and consider the exact equivalence $H : \mathcal{A}-\text{Mod} \longrightarrow \mathcal{F}(\Delta')$. If $E$ is any $k$-algebra and $N$ is a proper $\mathcal{A}$-$E$-bimodule, then the map $\sigma_N : \Gamma \otimes_{\mathcal{A}} N \longrightarrow H(N)$ is an isomorphism of $\Gamma$-$E$-bimodules, where the compared $E$-module structures are given by $\Gamma \otimes_{\mathcal{A}} (N_E)$ and

$$E \xrightarrow{\alpha_N} \text{End}_{\mathcal{A}}(N)^{\text{op}} \xrightarrow{H} \text{End}_{\Gamma}(H(N))^{\text{op}}.$$ 

As a consequence, we get $\ell_E(N) \leq \ell_E(H(N))$.

Proof. Once we know that $H(N) \cong \Gamma \otimes_{\mathcal{A}} (N_E)$, we have $\ell_E(N) \leq \ell_E(H(N))$. Indeed, we know that $\mathcal{A}$ is a direct summand of the right $\mathcal{A}$-module $\Gamma$, see for instance the proof of [9](12.8). Hence, we get

$$\ell_E(N) = \ell_E(\mathcal{A} \otimes_{\mathcal{A}} N) \leq \ell_E(\Gamma \otimes_{\mathcal{A}} N) = \ell_E(H(N)).$$

In order to prove the first statement, consider the structure of proper $\mathcal{A}$-$E$-bimodule $\alpha_N : E \longrightarrow \text{End}_{\mathcal{A}}(N)^{\text{op}}$ of $N$, which is such that $\alpha_N(e) = (\alpha_N(e)^0, 0)$, for all $e \in E$. Recall that, for any $n \in N$ and $e \in E$, we have $ne = \alpha_N(e)n[\ell]$. The structure map $\sigma_{H(N)}$ for the $\Gamma$-$E$-bimodule $H(N)$ is given by the composition

$$E \xrightarrow{\alpha_N} \text{End}_{\mathcal{A}}(N)^{\text{op}} \xrightarrow{H} \text{End}_{\Gamma}(H(N))^{\text{op}}.$$ 

It will be enough to show that, for all $f \in \Gamma$, $n \in N$, and $e \in E$, we have

$$\sigma_N(f \otimes ne) = \sigma_N(f \otimes n)e.$$ 

With the notation of [9](12.5), we have $\sigma_N(f \otimes ne) = f_{ne}$ and

$$\sigma_N(f \otimes n)e = f_ne = \alpha_{H(N)}(e)(f_n) = (H \circ \alpha_N)(e)[f_n] = H(\alpha_N(e))[f_n] = \alpha_N(e) \circ f_n.$$ 

So, we need to show that $f_{ne} = \alpha_N(e) \circ f_n$. We start by looking at the first components. Since $N$ is an $\mathcal{A}$-$E$-bimodule, for $a \in \mathcal{A}$, we get

$$(\alpha_N(e) \circ f_n)^0(a) = \alpha_N(e)^0(f_n^0(a)) = (f^0_n(a))e = f^0_n(ne)(a) = f_{ne}^0(a).$$ 

Now, we look at the second components and recall that $\alpha_N(e)^1 = 0$, because $N$ is a proper $\mathcal{A}$-$E$-bimodule. Then, for $v \in \mathcal{V}$, we have

$$(\alpha_N(e) \circ f_n)^1(v)[a] = \alpha_N(e)^0 f_n^1(v)[a] = f_n^1(v)[a]e = (f^1_n(v)[a])e = (f^1(v)[a]ne) = f_{ne}^1(v)[a].$$ 

So, indeed $f_{ne} = \alpha_N(e) \circ f_n$, as we wanted to show. 

Lemma 6.4. Assume that $\mathcal{A}$ is not wild, let $\Gamma$ be its right algebra, and consider the exact equivalence $H : \mathcal{A}-\text{Mod} \longrightarrow \mathcal{F}(\Delta')$. If $M$ is an indecomposable in $\mathcal{A}$-Mod with finite endolength, then the $\Gamma$-module $H(M) \cong \Gamma \otimes_{\mathcal{A}} M$ is an indecomposable $\Gamma$-module with finite endolength. Moreover, we have

$$\text{endol}(M) \leq \text{endol}(H(M)).$$
Therefore, the multiplicity of the simple $A$-module $v$ is $\dim \Gamma \otimes A M = \dim A H(M) = \text{endol}(H(M))$.

If $M$ is infinite-dimensional, then $M$ is a pregeneric $A$-module. From (1.8), it is a proper $A/k(x)$-bimodule. Then, $M$ is an $A/k(x)$-bimodule, that is an $A \otimes_k k(x)$-module. Then, $\Gamma \otimes A M$ has a structure of a $\Gamma/k(x)$-bimodule, that is of a left $\Gamma \otimes_k k(x)$-module.

Set $A^{k(x)} := A \otimes_k k(x)$ and $\Gamma^{k(x)} := \Gamma \otimes_k k(x)$. Notice that there is an isomorphism of $\Gamma/k(x)$-bimodules: $\Gamma \otimes A M \cong \Gamma^{k(x)} \otimes A^{k(x)} M$, through this isomorphism we identify both bimodules.

From (1.8), we know that the $A^{k(x)}$-module $M$ is finite-dimensional over $k(x)$. Then, it has finite length and so, it admits a filtration

$$0 = M_l \subseteq M_{l-1} \subseteq \cdots \subseteq M_1 \subseteq M_0 = M,$$

where each factor $M_i/M_{i+1}$ is isomorphic to some simple $A^{k(x)}$-module. Since $k(x) : k$ is a MacLane separable field extension, $\text{rad}(A^{k(x)}) = \text{rad}(A) \otimes_k k(x)$, see (20)(3.3) and (11), so the simple $A^{k(x)}$-modules have the form $S_v \otimes_k k(x)$, for some $v \in P$. Then, we have a filtration of the $\Gamma^{k(x)}$-module $\Gamma^{k(x)} \otimes A^{k(x)} M$ of the form

$$0 = \Gamma^{k(x)} \otimes A^{k(x)} M_l \subseteq \cdots \subseteq \Gamma^{k(x)} \otimes A^{k(x)} M_1 \subseteq \Gamma^{k(x)} \otimes A^{k(x)} M,$$

where the factors have the form

$$\Delta'_v \otimes_k k(x) \cong \Gamma \otimes A S_v \otimes_k k(x) \cong \Gamma^{k(x)} \otimes A^{k(x)} (S_v \otimes_k k(x)).$$

The multiplicity of the simple $A^{k(x)}$-module $S_v \otimes_k k(x)$ in the given filtration for $M$ is $\dim_{k(x)} e_v M$, hence the multiplicity of the $\Gamma^{k(x)}$-module $\Delta'_v \otimes_k k(x)$ in the considered tensored filtration for $\Gamma^{k(x)} \otimes A^{k(x)} M$ is the same number. Hence,

$$\dim_{k(x)} \Gamma^{k(x)} \otimes A^{k(x)} M = \sum_v \dim_{k(x)} e_v M \times \dim_{k(x)} (\Delta'_v \otimes k(x)).$$

Therefore,

$$\text{endol}(\Gamma \otimes A M) = \dim_{k(x)} \Gamma^{k(x)} \otimes A^{k(x)} M = \sum_v \dim_{k(x)} e_v M \times \dim_{k(x)} (\Delta'_v).$$

Then, we get

$$\text{endol}(M) = \dim_{k(x)} M = \sum_v \dim_{k(x)} e_v M \leq \sum_v \dim_{k(x)} e_v M \times \dim_{k(x)} (\Delta'_v) = \text{endol}(\Gamma \otimes A M),$$

as claimed.

\begin{proposition}
Consider the exact equivalence $H : A \text{-Mod} \longrightarrow \tilde{F}(\Delta')$. Take $N \in A \text{-Mod}$ and consider $M := H(N) \in \tilde{F}(\Delta')$. Then, if $M$ is a generic $\Gamma$-module, we have that $N$ is a pregeneric $A$-module.
\end{proposition}
Proof. Set $E := \text{End}_A(N)^{op}$, then $N$ is an $A$-$E$-bimodule and $M = H(N)$ is a $\Gamma$-$E$-bimodule. The action of $E$ by the right on $N$ is given by $n\phi = \phi^h(n)$, for $n \in N$ and $\phi \in E$, and the action of $E$ by the right on $M$ is given by $h\phi = H(\phi)[h] = \phi^h$, the composition in $A$-$\text{Mod}$, for $h \in \text{Hom}_A(A, N) = H(N) = M$ and $\phi \in E$.

Notice that the right $\text{End}_R(M)^{op}$-module $M$ is a right $E$-module by restriction using the isomorphism of algebras $H : E \longrightarrow \text{End}_R(M)^{op}$. This action coincides with the preceding one. We have that $\ell_E(M) = \text{endol}(M)$, which is finite by assumption. We will show that $\ell_E(N)$ is finite too. Since $\text{endol}(N) = \ell_E(N)$, we will get from this that $N$ is a pregeneric $A$-module, as claimed.

Set $\tau := \text{rad}E$. From [15] (4.2; 4.4), we know that $E \cong \text{End}_R(M)^{op}$ is a local ring and that $\tau$ is nilpotent. Thus, for some natural number $l$, we have $\tau^l = 0$. So, we have a filtration of right $E$-modules

$$0 = M_l = M\tau^l \subseteq M_{l-1} = M\tau^{l-1} \subseteq \cdots \subseteq M_1 = M\tau^1 \subseteq M_0 = M,$$

and we can consider the division $k$-algebra $D := E/\tau$. Each $M_i/M_{i+1}$ is a $D$-vector space. Since $\ell_E(M)$ is finite, we know that each $M_i/M_{i+1}$ is a $D$-vector space with finite dimension, say $d_{i+1}$. Consider $d := \max\{d_1, \ldots, d_l\}$.

We also have a filtration of right $E$-modules

$$0 = N_l = N\tau^l \subseteq N_{l-1} = N\tau^{l-1} \subseteq \cdots \subseteq N_1 = N\tau^1 \subseteq N_0 = N,$$

where each quotient $N_i/N_{i+1}$ is a $D$-vector space. We will show that

$$\dim_D(N_i/N_{i+1}) \leq d,$$

for each $i$.

So, we fix $i \in [0, l - 1]$. Then, $N_i/N_{i+1}$ is generated by classes modulo $N_{i+1}$ of elements of the form $n\rho$, with $n \in N$ and $\rho \in \tau$. In order to prove our claim, it is enough to show that the classes of any set with $d + 1$ such elements $n_1\rho_1, \ldots, n_d+1\rho_{d+1}$ are $D$-linearly dependent in $N_i/N_{i+1}$.

For each $j \in [1, d + 1]$ and $n_j$ as before, we have the morphism of $A$-modules $h^0_j : A \longrightarrow N$ with $h^0_j(1) = n_j$, which determines the morphism $h_j = (h^0_j, 0) \in \text{Hom}_A(A, N) = H(N) = M$.

Since, $h_1\rho_1, \ldots, h_{d+1}\rho_{d+1} \in M\tau^i = M_i$, their classes modulo $M_{i+1}$ are $D$-linearly dependent. So, there are elements $\phi_1, \ldots, \phi_{d+1} \in E$ such that their classes modulo $\tau$ are not all zero, and $g := h_1\rho_1\phi_1 + \cdots + h_{d+1}\rho_{d+1}\phi_{d+1} \in M_{i+1} = M\tau^{i+1}$.

The element $g \in M\tau^{i+1}$ is a finite sum of elements of the form $h\zeta$, where $h \in M = H(N)$ and $\zeta = \zeta_1\zeta_2\cdots\zeta_{i+1}$, where $\zeta_1, \ldots, \zeta_{i+1} \in \tau$.

Then, $g^0(1)$ is a finite sum of elements of the form $\zeta_1^0\zeta_2^0h^0(1)$, which implies that $g^0(1) \in N\tau^{i+1} = N_{i+1}$.

But, we also have

$$(h_1\rho_1\phi_1 + \cdots + h_{d+1}\rho_{d+1}\phi_{d+1})^0(1) = (n_1\rho_1\phi_1 + \cdots + (n_{d+1}\rho_{d+1})\phi_{d+1}) = (n_1\rho_1)\phi_1 + \cdots + (n_{d+1}\rho_{d+1})\phi_{d+1}.$$ 

Therefore, we get $(n_1\rho_1)\phi_1 + \cdots + (n_{d+1}\rho_{d+1})\phi_{d+1} = g^0(1) \in N_{i+1}$. This implies that $n_1\rho_1, \ldots, n_{d+1}\rho_{d+1}$ are $D$-linearly dependent modulo $N_{i+1}$, as we wanted to show. 

\[\square\]
Remark 6.6. If $\mathcal{A}$ is wild, so is $\mathcal{F}(\Delta')$ (and so is $\mathcal{F}(\Delta)$). Indeed, a composition functor

$$k(x,y)-\text{Mod} \xrightarrow{Z\otimes k(x,y)-} \mathcal{A}-\text{Mod} \xrightarrow{L_{\mathcal{A}}} \mathcal{A}-\text{Mod}$$

which preserves isoclasses and indecomposables as in (2.9) composed with the equivalence $H : \mathcal{A}-\text{mod} \rightarrow \mathcal{F}(\Delta')$, determines a functor

$$HL_{\mathcal{A}}(Z \otimes k(x,y)-) : k(x,y)-\text{mod} \rightarrow \mathcal{F}(\Delta')$$

which preserves isoclasses and indecomposables, see [8](1.2). Moreover, by (5.3), it is such that $HL_{\mathcal{A}}(Z \otimes k(x,y)-) \cong \Gamma \otimes \mathcal{A} Z \otimes k(x,y)-$, where $\Gamma \otimes \mathcal{A} Z$ is a finitely generated projective $k(x,y)$-module. So $\mathcal{F}(\Delta')$ is wild. If we compose with the equivalence $\Omega : \mathcal{F}(\Delta') \rightarrow \mathcal{F}(\Delta)$, we obtain a functor $\Omega HL_{\mathcal{A}}(Z \otimes k(x,y)-)$ and a bimodule producing the wildness of $\mathcal{F}(\Delta)$.

Theorem 6.7. Assume that $\mathcal{F}(\Delta')$ is not wild and take $d \in \mathbb{N}$. Consider the functor $F : \mathcal{B}-\text{Mod} \rightarrow \mathcal{A}-\text{Mod}$ associated to the composition of reductions of (4.6), where $\mathcal{B}$ is a minimal ditalgebra, and the equivalence $H : \mathcal{A}-\text{Mod} \rightarrow \mathcal{F}(\Delta')$. Then, we have:

1. The $\Gamma \cdot \mathcal{B}$-bimodule $T' := \Gamma \otimes \mathcal{A} F(B)$ is finitely generated by the right and the composition

$$\mathcal{B}-\text{Mod} \xrightarrow{L_{\mathcal{B}}} \mathcal{B}-\text{Mod} \xrightarrow{F} \mathcal{A}-\text{Mod} \xrightarrow{H} \tilde{\mathcal{F}}(\Delta')$$

is naturally isomorphic to $T' \otimes \mathcal{B} -$.

In particular, $HF(N) \cong T' \otimes \mathcal{B} N$, for any $N \in \mathcal{B}-\text{Mod}$. If $N$ is a proper $\mathcal{B} \cdot \mathcal{E}$-bimodule, for some $k$-algebra $\mathcal{E}$, then we have an isomorphism of proper $\Gamma \cdot \mathcal{E}$-bimodules $HF_{\mathcal{E}}(N) \cong T' \otimes \mathcal{B} (N_{\mathcal{E}})$.

2. The $\Gamma$-modules of the form $G \cong T' \otimes \mathcal{B} Q_1$ are generic modules for $\mathcal{F}(\Delta')$.

3. Every indecomposable $\Gamma$-module $M \in \tilde{\mathcal{F}}(\Delta')$ with $\text{endol}(M) \leq d$ is isomorphic to $T' \otimes \mathcal{B} N$, for some indecomposable $\mathcal{B}$-module $N$ with finite endolength.

Moreover, if $M$ is generic, we have $\text{End}_\Gamma(M)^{op} = Q_M^l \oplus \text{radEnd}_\Gamma(M)^{op}$, for some subalgebra $Q_{\mathcal{M}} \cong k(x)$ of $\text{End}_\Gamma(M)^{op}$.

4. If the given $\Gamma$-module $M$ is generic, the $\mathcal{B}$-module $N$ is pregeneric and has a natural structure of $\mathcal{B} \cdot k(x)$-bimodule. The morphism of algebras $\mu : k(x) \rightarrow \text{End}_\Gamma(T' \otimes \mathcal{B} N)^{op}$ given by $\mu(q) = (id_{T'} \otimes q, 0)$ gives $M$ the structure of a proper $\Gamma \cdot \mathcal{B}(x)$-bimodule $T' \otimes \mathcal{B} N \cong HF(N) \cong M$ such that $\text{endol}(M) = \dim_{k(x)} M$.

Proof. (1): As remarked in (5.6), $\mathcal{A}$ is not wild. From (1.8), we already know that $\mathcal{B}-\text{Mod} \xrightarrow{L_{\mathcal{B}}} \mathcal{B}-\text{Mod} \xrightarrow{F} \mathcal{A}-\text{Mod}$ is naturally isomorphic to $L_{\mathcal{A}}(T \otimes \mathcal{B} -)$. From (5.3), we know that $HL_{\mathcal{A}} \cong \Gamma \otimes \mathcal{A} -$. So, applying $H$ to the first isomorphism, we get a natural isomorphism

$$HFL_{\mathcal{B}} \cong HL_{\mathcal{A}}(T \otimes \mathcal{B} -) \cong \Gamma \otimes \mathcal{A}(T \otimes \mathcal{B} -) = T' \otimes \mathcal{B} -.$$
If $N$ is a proper $B$-$E$-bimodule, from (4.8), so is $F_E(N) \cong T \otimes_B N$. Hence, from (6.3), so is $H_E F_E(N) \cong T' \otimes_B N$.

(2): It follows from (4.8) and the fact that $H$ maps pregeneric $A$-modules on generic $\Gamma$-modules in $\tilde{F}(\Delta')$, see (6.4).

(3)&(4): Suppose first that $M$ is a finite-dimensional indecomposable in $\tilde{F}(\Delta')$ with $\operatorname{dim}_k M = \operatorname{endol}(M) \leq d$. Since $H : A\text{-Mod} \rightarrow \tilde{F}(\Delta')$ is an equivalence, there is a finite-dimensional indecomposable $N' \in A\text{-Mod}$ with $H(N') \cong M$. From (6.3), we know that $\operatorname{endol}(N') = \dim_k N' \leq \dim_k H(N') = \dim_k M \leq d$. Thus, from (4.8), there is a finite-dimensional indecomposable $B$-module $N$ such that $F(B) \otimes_B N \cong N'$.

Now, assume that $M$ is a generic $\Gamma$-module in $\tilde{F}(\Delta')$ with $\operatorname{endol}(M) \leq d$. Since $H : A\text{-Mod} \rightarrow \tilde{F}(\Delta')$ is an equivalence, there is an indecomposable infinite-dimensional $N \in A\text{-Mod}$ with $H(N) \cong M$. By (4.8), $N$ is a pregeneric $A$-module. By (6.4), we have $\operatorname{endol}(N) \leq \operatorname{endol}(H(N)) = \operatorname{endol}(M) \leq d$. Applying (4.8)(4) to $N$, we get a pregeneric $B$-module $N$ such that $F(B) \otimes_B N \cong N'$, so $T' \otimes_B N \cong \Gamma \otimes_A N' \cong M$. Since $H : \operatorname{End}_A(N) \rightarrow \operatorname{End}_\Gamma(M)$ is an isomorphism of algebras, the splitting of the radical for the second algebra, follows immediately from the same property for the first one. The fact that $M$ es a proper $\Gamma$-$k(x)$-bimodule follows from the corresponding property for $N$, stated in (4.8)(4), and (6.3).

7 Covering $\Theta$-filtered modules

This section is devoted to show that the generic modules for the category $F(\Theta)$, in case it is not wild, where $(\mathcal{P}, \leq, \{\Theta_v\}_{v \in \mathcal{P}})$ is any homological system over a finite-dimensional algebra $\Upsilon$, can be covered with a similar procedure to the one described in (6.7) for the case of generic modules for $\tilde{F}(\Delta')$. This extension relies on the results of the preceding sections and the following adaptation (7.6) of a theorem, due to Mendoza, Sáenz, and Xi, see [22]§3 and [8]§11.

Remark 7.1. In this section, $(\mathcal{P}, \leq, \{\Theta_v\}_{v \in \mathcal{P}})$ denotes a fixed general homological system for a finite-dimensional algebra $\Upsilon$. It is well known that the subcategory $F(\Theta)$ of $\Upsilon\text{-mod}$ is closed under direct summands, see [22]. A short proof of this statement was given in [23]. The argument used in [23] can be adapted to show that the subcategory $\tilde{F}(\Theta)$ of $\Upsilon\text{-Mod}$ is closed under direct summands, see [24]. As a consequence of this, the indecomposable objects in $\tilde{F}(\Theta)$ are indecomposable $\Upsilon$-modules.

The following statement is crucial in the proof of (7.6), its dual is verified in the first part of the proof of [22](3.12).

Lemma 7.2. For each $v \in \mathcal{P}$, there is an exact sequence

$$0 \rightarrow V_v \rightarrow U_v \rightarrow \Theta_v \rightarrow 0$$
such that $U_v$ is an indecomposable $\mathcal{F}(\Theta)$-projective and $V_v \in \mathcal{F}(\Theta)$ has a $\Theta$-filtration with factors of the form $\Theta_u$ with $u > v$.

We fix a family of special exact sequences, as provided by the last lemma, for the rest of this section.

**Lemma 7.3.** For each $M \in \widetilde{\mathcal{F}}(\Theta)$, there is an exact sequence

$$0 \rightarrow K \rightarrow W \rightarrow M \rightarrow 0$$

in $\widetilde{\mathcal{F}}(\Theta)$, such that $W$ is a direct sum of copies of the modules in $\{U_v \mid v \in \mathcal{P}\}$.

The family $\{U_v\}_{v \in \mathcal{P}}$ is a complete set of representatives of the isomorphism classes of the indecomposable $\widetilde{\mathcal{F}}(\Theta)$-projective modules.

**Proof.** Since each $U_v$ is an $\mathcal{F}(\Theta)$-projective module, we know that $\text{Ext}^1_\mathcal{F}(U_v, \Theta_u) = 0$, for all $u \in \mathcal{P}$. Since $U_v$ is a finite-dimensional $\Lambda$-module, the functor $\text{Ext}^1_\mathcal{F}(U_v, -)$ commutes with direct limits, see for instance [19](3.1.6). But direct sums are direct limits of their finite subdirect sums, so $\text{Ext}^1_\mathcal{F}(U_v, V) = 0$ for any direct sum $V$ of copies of the modules in $\{\Theta_v \mid v \in \mathcal{P}\}$. A simple induction argument shows that $\text{Ext}^1_\mathcal{F}(U_v, M) = 0$, for any $M \in \widetilde{\mathcal{F}}(\Theta)$. So, any such $U_v$ is an $\widetilde{\mathcal{F}}(\Theta)$-projective.

If $M$ is any indecomposable $\widetilde{\mathcal{F}}(\Theta)$-projective, we have an exact sequence as we have just mentioned, so $M$ is an indecomposable direct summand of $W$. As a consequence of Crawley-Jønsson-Warfield and Azumaya Theorems, see [1](12.6; 26.5; and 26.6), we obtain that $M \cong U_v$, for some $v \in \mathcal{P}$.

Finally, the fact that the family $\{U_v\}_{v \in \mathcal{P}}$ consists of pairwise non-isomorphic $\Upsilon$-modules is proved in [22](3.7).

**Definition 7.4.** Define $U := \bigoplus_{v \in \mathcal{P}} U_v$ and $\Lambda := \text{End}_\Upsilon(U)^{op}$. Moreover, consider the family $\{\Delta_v\}_{v \in \mathcal{P}}$ of $\Lambda$-modules given by $\Delta_v := \text{Hom}_\Upsilon(U, \Theta_v)$, for $v \in \mathcal{P}$.

It was shown in [22] that $(\mathcal{P}, \leq, \{\Delta_u\}_{u \in \mathcal{P}})$ is an admissible homological system for $\Lambda$ such that $\mathcal{F}(\Theta)$ is equivalent to $\mathcal{F}(\Delta)$ as exact categories, see also [22]§11. In the following, we extend a little their results to show that $\mathcal{F}(\Theta)$ is equivalent to $\widetilde{\mathcal{F}}(\Delta)$ as exact categories. The quasi-inverse equivalences will be realized by the appropriate restrictions of the functors $\mathcal{H} := \text{Hom}_\Upsilon(U, -) : \Upsilon\text{-Mod} \rightarrow \Lambda\text{-Mod}$ and $\mathcal{F} := U \otimes_\Lambda - : \Lambda\text{-Mod} \rightarrow \Upsilon\text{-Mod}$. We start with the following.

**Lemma 7.5.** The functor $\mathcal{H}$ restricts to a full and faithful exact functor

$$\mathcal{H} : \widetilde{\mathcal{F}}(\Theta) \rightarrow \Lambda\text{-Mod}.$$
Proof. Assume that \( V = \bigoplus_{j \in J} V_j \) is a direct sum of modules \( V_j \) which are isomorphic to some modules in \( \{ U_v \mid v \in \mathcal{P} \} \). We have the direct sums

\[
\{ s_i : V_i \rightarrow \bigoplus_j V_j \}_i \text{ and } \{ \sigma_i : \mathcal{H}(V_i) \rightarrow \bigoplus_j \mathcal{H}(V_j) \}_i
\]

in \( \mathcal{Y}\text{-Mod} \) and \( \Lambda\text{-Mod} \), respectively. For \( N \in \mathcal{Y}\text{-Mod} \), consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{Y}(\bigoplus_j V_j, N) & \xrightarrow{\phi} & \text{Hom}_\mathcal{Y}(\mathcal{H}(V), \mathcal{H}(N)) \\
\Downarrow \Theta_s & & \Downarrow \Theta_s \\
\prod_j \text{Hom}_\mathcal{Y}(V_j, N) & \xrightarrow{\prod \mathcal{H}(V_j, N)} & \prod_j \text{Hom}_\mathcal{Y}(\mathcal{H}(V_j), \mathcal{H}(N))
\end{array}
\]

where: \( \phi : \bigoplus_j \mathcal{H}(V_j) = \bigoplus_j \text{Hom}_\mathcal{Y}(U, V_j) \rightarrow \text{Hom}_\mathcal{Y}(U, \bigoplus_j V_j) = \mathcal{H}(\bigoplus_j V_j) \) is the isomorphism given by \( \phi(\sum_j h_j) = \sum_j s_j h_j ; \Theta_s \) is the isomorphism given by \( \Theta_s(f) = \{ fs_j \}_j \) and, similarly, \( \Theta_s \) is the isomorphism given by \( \Theta_s(h) = \{ h s_j \}_j \).

It follows that the morphism \( \mathcal{H} \) in the diagram is an isomorphism.

Consider the full subcategory \( \text{add}(U) \) of \( \mathcal{Y}\text{-mod} \) (resp. the full subcategory \( \text{Add}(U) \) of \( \mathcal{Y}\text{-Mod} \)) formed by the direct summands of finite direct sums of \( U \) (resp. by the direct summands of direct sums of \( U \)). Here, we adapt the argument used in \( (\mathcal{B}) \) to show that \( \mathcal{H} \) restricts to an equivalence of categories \( \text{add}(U) \rightarrow \text{add}(\Lambda) \). Using the fact that \( \mathcal{H} \) commutes with direct sums, we have verified that, for each \( W \in \text{Add}(U) \) and \( N \in \mathcal{Y}\text{-Mod} \), the functor \( \mathcal{H} \) restricts to an isomorphism \( \text{Hom}_\mathcal{Y}(W, N) \rightarrow \text{Hom}_\mathcal{Y}(\mathcal{H}(W), \mathcal{H}(N)) \). Then, \( \mathcal{H} \) restricts to an equivalence of categories \( \text{Add}(U) \rightarrow \text{Add}(\Lambda) \), where \( \text{Add}(\Lambda) \) consists of the projective \( \Lambda\text{-modules} \).

Given \( M \in \mathcal{F}(\Theta) \), from \( (\mathcal{B}) \), there is an exact sequence of the form

\[
W' \rightarrow W \xrightarrow{\pi} M \rightarrow 0,
\]

where \( W, W' \in \text{Add}(U) \). Since \( U \) is \( \mathcal{F}(\Theta)\text{-projective} \), applying \( \mathcal{H} \), we obtain an exact sequence

\[
\mathcal{H}(W') \rightarrow \mathcal{H}(W) \xrightarrow{\pi} \mathcal{H}(M) \rightarrow 0.
\]

Then, for any \( N \in \mathcal{Y}\text{-Mod} \), we have the commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \text{Hom}_\mathcal{Y}(M, N) & \rightarrow & \text{Hom}_\mathcal{Y}(W, N) & \rightarrow & \text{Hom}_\mathcal{Y}(W', N) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Hom}_\mathcal{Y}(\mathcal{H}(M), \mathcal{H}(N)) & \rightarrow & \text{Hom}_\mathcal{Y}(\mathcal{H}(W), \mathcal{H}(N)) & \rightarrow & \text{Hom}_\mathcal{Y}(\mathcal{H}(W'), \mathcal{H}(N))
\end{array}
\]

where the second and third vertical arrows are isomorphisms, then so is the first one. This shows that the restriction of \( \mathcal{H} \) to the category \( \mathcal{F}(\Theta) \) is full and faithful. This restriction is exact because \( U \) is an \( \mathcal{F}(\Theta)\text{-projective} \) module.

**Theorem 7.6.** Given any homological system \( (\mathcal{P}, \leq, \{ \Theta_v \}_{v \in \mathcal{P}}) \) for \( \mathcal{G} \), consider the algebra \( \Lambda = \text{End}_U(U)^{op} \) as before and the triple \( (\mathcal{P}, \leq, \{ \Delta_v \}_{v \in \mathcal{P}}) \), where \( \Delta_v = \text{Hom}_{\mathcal{Y}}(U, \Theta_v) \), for all \( v \in \mathcal{P} \). Then, \( (\mathcal{P}, \leq, \{ \Delta_v \}_{v \in \mathcal{P}}) \) is an admissible homological system for \( \Lambda \) and the functors \( \mathcal{H} = \text{Hom}_{\mathcal{Y}}(U, -) \) and \( \Xi = U \otimes_{\Lambda} - \) induce quasi-inverse equivalences of exact categories between \( \mathcal{F}(\Theta) \) and \( \mathcal{F}(\Delta) \).
Proof. With the preceding notations, we consider the following natural transformations \( \alpha : U \otimes_{\Lambda} \mathcal{F} \rightarrow id_{\mathcal{F}(\Theta)} \) and \( \beta : id_{\mathcal{F}(\Theta)} \rightarrow \mathcal{F}(U \otimes \Lambda -) \) given by

1. For \( M \in \mathcal{Y}_{\text{Mod}} \), the morphism \( \alpha_M : U \otimes_{\Lambda} \mathcal{F}(M) \rightarrow M \) has the recipe \( \alpha_M(u \otimes g) = g(u) \).

2. For \( N \in \Lambda_{\text{Mod}} \), the morphism \( \beta_N : N \rightarrow \mathcal{F}(U \otimes \Lambda N) \) has the recipe \( \beta_N(n)[u] = u \otimes n \).

We already know that \((\mathcal{P}, \leq, \{\Delta_v\}_{v \in \mathcal{P}})\) is an admissible homological system, see \([9](13.7)\). We have to show that we have quasi-inverse exact equivalences \( \overline{\mathcal{F}(\Theta)} \xrightarrow{\overline{\beta}} \overline{\mathcal{F}(\Delta)} \) and \( \overline{\mathcal{F}(\Delta)} \xrightarrow{\overline{\alpha}} \overline{\mathcal{F}(\Theta)} \).

**Step 1.** \( \alpha_M \) is an isomorphism, for all \( M \in \overline{\mathcal{F}(\Theta)} \).

Since \( \alpha_U \) is the composition \( U \otimes_{\Lambda} \mathcal{F}(U) = U \otimes_{\Lambda} \Lambda \cong U \), we know that \( \alpha_U \) is an isomorphism. It follows that \( \alpha_W \) is an isomorphism, for each \( W \in \text{Add}(U) \). For a given \( M \in \overline{\mathcal{F}(\Theta)} \), consider an exact sequence \( W' \rightarrow W \rightarrow M \rightarrow 0 \), as before, with \( W, W' \in \text{Add}(U) \). Then, we have a commutative diagram

\[
\begin{array}{cccccc}
U \otimes_{\Lambda} \mathcal{F}(W') & \rightarrow & U \otimes_{\Lambda} \mathcal{F}(W) & \rightarrow & U \otimes_{\Lambda} \mathcal{F}(M) & \rightarrow & 0 \\
\downarrow \alpha_{W'} & & \downarrow \alpha_W & & \downarrow \alpha_M & & \\
W' & \rightarrow & W & \rightarrow & M & \rightarrow & 0
\end{array}
\]

with exact rows, and \( \alpha_M \) is an isomorphism.

**Step 2.** The functor \( \mathcal{T} = U \otimes_{\Lambda} - \) restricts to an exact functor \( \overline{\mathcal{F}(\Delta)} \rightarrow \overline{\mathcal{F}(\Theta)} \).

As in \([8](11.5)(\text{step 2})\), we have that \( \text{Tor}^1(U, \Delta_v) = 0 \), for all \( v \in \mathcal{P} \). Since \( \text{Tor}^1(U, -) \) commutes with direct sums, see \([22](5.2.50)\), we get \( \text{Tor}^1(U, V) = 0 \), for any \( V \in \text{Add}(\bigoplus \Delta_v) \). A simple induction argument shows that \( \text{Tor}^1(U, N) = 0 \), for any \( N \in \overline{\mathcal{F}(\Delta)} \). With almost the same argument used in \([8](11.5)(\text{step 2})\), we derive from this that \( U \otimes_{\Lambda} N \in \overline{\mathcal{F}(\Theta)} \), for any \( N \in \overline{\mathcal{F}(\Delta)} \).

Moreover, if \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is an exact sequence in \( \overline{\mathcal{F}(\Delta)} \), using that \( \text{Tor}^1(U, N) = 0 \), we obtain that \( 0 \rightarrow U \otimes_{\Lambda} L \rightarrow U \otimes_{\Lambda} M \rightarrow U \otimes_{\Lambda} N \rightarrow 0 \) is an exact sequence.

**Step 3.** \( \beta_N \) is an isomorphism, for all \( N \in \overline{\mathcal{F}(\Delta)} \).

As in \([8](11.5)(\text{step 3})\), we show that \( \beta_{\Delta_v} \) is an isomorphism for \( v \in \mathcal{P} \). Then, using again that \( \mathcal{F} \) and \( \mathcal{T} \) commute with direct sums, we derive that \( \beta_N \) is an isomorphism for all \( N \in \overline{\mathcal{F}(\Delta)} \).

We have shown that we can restrict the functors \( \mathcal{F} \) and \( \mathcal{T} \) to functors \( \mathcal{F} : \overline{\mathcal{F}(\Theta)} \rightarrow \overline{\mathcal{F}(\Delta)} \) and \( \mathcal{T} : \overline{\mathcal{F}(\Delta)} \rightarrow \overline{\mathcal{F}(\Theta)} \) and that these restrictions are exact. Moreover, the restrictions of the natural transformations \( \alpha \) and \( \beta \) determine isomorphisms of functors \( \alpha : \mathcal{T} \mathcal{F} \rightarrow id_{\overline{\mathcal{F}(\Theta)}} \) and \( \beta : id_{\overline{\mathcal{F}(\Delta)}} \rightarrow \mathcal{F} \mathcal{T} \). \( \square \)
Lemma 7.7. With the preceding notation, for any \( N \in \tilde{\mathcal{F}}(\Theta) \), we have

\[
\text{endol}(\delta(N)) \leq \dim_k U \times \text{endol}(N)
\]

and, for any \( M \in \tilde{\mathcal{F}}(\Delta) \), we have \( \text{endol}(U \otimes \Lambda M) \leq \dim_k U \times \text{endol}(M) \).

Therefore, the functors \( \delta : \tilde{\mathcal{F}}(\Theta) \longrightarrow \tilde{\mathcal{F}}(\Delta) \) and \( \varepsilon : \tilde{\mathcal{F}}(\Delta) \longrightarrow \tilde{\mathcal{F}}(\Theta) \) preserve generic modules.

Proof. Given \( N \in \tilde{\mathcal{F}}(\Theta) \), set \( E := \text{End}_\Lambda(\delta(N))^{\text{op}} \), and consider the \( \Upsilon \)-\( E \)-bimodule structure on \( N \), where \( E \) acts by the right through the algebra isomorphism \( \alpha_N := H^{-1} : E \longrightarrow \text{End}_\Upsilon(\Upsilon^{b})^{\text{op}} \).

If \( b = \dim_k U \), there is an epimorphism \( \Upsilon^{b} \longrightarrow U \) of left \( \Upsilon \)-modules. So, there is a monomorphism \( H(N) = \text{Hom}_\Upsilon(U, N) \longrightarrow \text{Hom}_\Upsilon(\Upsilon^{b}, N) \cong \text{Hom}_\Upsilon(\Upsilon, N)^{b} \cong N^{b} \) of right \( E \)-modules. Hence,

\[
\text{endol}(\delta(N)) = \ell_{E}(\delta(N)) \leq b \times \ell_{E}(N) = b \times \text{endol}(N).
\]

The other claim in the statement of this lemma is similar (and well known).

Corollary 7.8. \( \mathcal{F}(\Theta) \) is generically tame iff \( \mathcal{F}(\Delta) \) is so.

Proof. From (7.7), we know that, given \( d \in \mathbb{N} \), whenever \( \tilde{\mathcal{F}}(\Delta) \) has only finitely many generic \( \Lambda \)-modules with endolength bounded by \( \dim_k U \times d \), then \( \tilde{\mathcal{F}}(\Theta) \) has only finitely many generic \( \Upsilon \)-modules with endolength bounded by \( d \). So, \( \mathcal{F}(\Theta) \) is generically tame whenever \( \mathcal{F}(\Delta) \) is so. The proof of converse statement is similar.

Remark 7.9. \( \mathcal{F}(\Theta) \) is wild iff \( \mathcal{F}(\Delta) \) is so. Indeed, if \( \mathcal{F}(\Theta) \) is wild, we have the composition functor

\[
k(x, y)\text{-mod} \longrightarrow Z \otimes_{k(x, y)} \mathcal{F}(\Theta) \longrightarrow \tilde{\mathcal{F}}(\Delta),
\]

where \( Z \) is the \( \Upsilon \)-\( k(x, y) \)-bimodule which produces the wildness of \( \mathcal{F}(\Theta) \). This composition is exact and preserves arbitrary direct sums, so by Watts theorem, it is isomorphic to a tensor product \( Z' \otimes_{k(x, y)} \), where \( Z' \) is finitely generated by the right. This implies (after an adjustment of \( Z' \)) the wildness of \( \mathcal{F}(\Delta) \).

The verification of the converse is simpler.

In the following, we keep the preceding notation and consider the context studied in §6. Then, we have the following.

Theorem 7.10. Assume that \( (\mathcal{P}, \leq, \{\Theta_v\}_{v \in \mathcal{P}}) \) is any homological system for a finite-dimensional algebra \( \Upsilon \). Consider the algebra \( \Lambda = \text{End}_\Upsilon(U)^{\text{op}} \) and the admissible homological system \( (\mathcal{P}, \leq, \{\Delta_v\}_{v \in \mathcal{P}}) \). Assume that \( \mathcal{F}(\Theta) \) is not wild, so, neither is \( \mathcal{F}(\Delta) \), and take any \( d \in \mathbb{N} \). Consider the equivalence \( H : \Lambda\text{-Mod} \longrightarrow \tilde{\mathcal{F}}(\Delta') \) and the Morita equivalence \( \Omega : \Gamma\text{-Mod} \longrightarrow \Lambda\text{-Mod} \), as in \([9](13.10)\). Assume that \( \Omega \) is given as a tensor product functor \( P \otimes_{\Gamma} \), for an appropriate progenerator \( P \in \text{mod}\Gamma \), and that a quasi-inverse for \( \Omega \) is given as a tensor product functor \( P' \otimes_{\Lambda} \).
If first isomorphism, we get a natural isomorphism $F$ decomposable in $\Gamma$-finite endolength $\text{endol}(N)$ is naturally isomorphic to $M$ that $T$ endol $(\text{endol}(\Delta)) = \text{dim}_k \text{End}_{\text{mod}}(\Delta)$ with $\text{endol}(\Delta)$ is a proper $E$-bimodule. Hence, we get $\text{dim}_k U \times \text{dim}_k P' \times d$. Then, we have:

1. The $\Gamma$-$B$-bimodule $T := U \otimes_A P \otimes_{\Gamma} \Gamma \otimes_\Delta F(B)$ is finitely generated by the right and the composition

$$B \text{-Mod} \xrightarrow{L_B} B \text{-Mod} \xrightarrow{F} \Delta \text{-Mod} \xrightarrow{H} \vec{\mathcal{F}}(\Delta') \xrightarrow{\Omega} \vec{\mathcal{F}}(\Delta) \xrightarrow{\sigma} \vec{\mathcal{F}}(\Theta)$$

is naturally isomorphic to $T \otimes_B -$.

In particular, $\mathfrak{T} \Omega H F(N) \cong T \otimes_B N$, for any $N \in B \text{-Mod}$. If $N$ is a proper $B$-$E$-bimodule, for some $k$-algebra $E$, then we have an isomorphism of $\Gamma$-$E$-bimodules $\mathfrak{T} E \text{End}_{\text{mod}}(H E F_E(N)) \cong T \otimes_B (N_E)$.

2. The $\Gamma$-modules of the form $G \cong T \otimes_B Q_i$ are generic modules for $\mathcal{F}(\Theta)$.

3. Every indecomposable $\Gamma$-module $M \in \vec{\mathcal{F}}(\Theta)$ with $\text{endol}(M) \leq d$ is isomorphic to $T \otimes_B N$, for some indecomposable $B$-module $N$ with finite endolength.

Moreover, if $M$ is generic we have $\text{End}_{\Gamma}(M)^{op} = Q_M \bigoplus \text{rad}\text{End}_{\Gamma}(M)^{op}$, for some subalgebra $Q_M \cong k(x)$ of $\text{End}_{\Gamma}(M)^{op}$.

4. If the given module $M$ is generic, the $B$-module $N$ is pregeneric and has a natural structure of $B$-$k$-bimodule. The morphism of algebras $\mu : k(x) \xrightarrow{\text{id}_T \otimes \mu_q} \text{End}_{\Gamma}(T \otimes_B N)^{op}$ given by $\mu(q) = (\text{id}_T \otimes \mu_q, 0)$ gives $M$ the structure of a proper $\Gamma$-$k$-bimodule $T \otimes_B N \cong \mathfrak{T} \Omega H F(N) \cong M$ such that $\text{endol}(M) = \text{dim}_k(x) M$.

Proof. (1): From (6.7), we know that $B \text{-Mod} \xrightarrow{L_B} B \text{-Mod} \xrightarrow{F} \Delta \text{-Mod} \xrightarrow{H} \vec{\mathcal{F}}(\Delta')$ is naturally isomorphic to $T' \otimes_B -$. So, applying $\mathfrak{T} \Omega = U \otimes_A P \otimes_{\Gamma} -$ to the first isomorphism, we get a natural isomorphism

$$\mathfrak{T} \Omega H F L_B \cong \mathfrak{T} \Omega (T' \otimes_B -) \cong T \otimes_B -$$. If $N$ is a proper $B$-$E$-bimodule, from (6.7), we have that $H E F_E(N) \cong T' \otimes_B N$ as $\Gamma$-$E$-bimodules. Hence, we get $\mathfrak{T} \mathfrak{E} \text{End}_{\text{mod}}(H E F_E(N)) \cong T \otimes_B N$, as $\Gamma$-$E$-bimodules.

(2): It follows from (6.7) and the fact that $\mathfrak{T}$ and $\Omega$ preserve generic modules.

(3)&(4): Suppose first that $M$ is a finite-dimensional indecomposable in $\vec{\mathcal{F}}(\Theta)$ with $\text{dim}_k M = \text{endol}(M) \leq d$. Then, $\mathfrak{T} \mathfrak{S}(M)$ is a finite-dimensional indecomposable in $\vec{\mathcal{F}}(\Delta)$ with $\text{endol}(\mathfrak{T} \mathfrak{S}(M)) = \text{dim}_k \mathfrak{S}(M) \leq \text{dim}_k U \times d$. Then, $M' := P' \otimes_{\Delta} \mathfrak{S}(M)$ is a finite-dimensional indecomposable in $\vec{\mathcal{F}}(\Delta')$ such that $\text{endol}(M') = \text{dim}_k M' \leq d'$.

From (6.7), there is a finite-dimensional indecomposable $B$-module $N$ such that $T' \otimes_B N \cong M'$. So, we get $M \cong U \otimes_A P \otimes_{\Gamma} M' \cong T \otimes_B N$, where $N$ has finite endolength $\text{endol}(N) = \text{dim}_k N$.

Now, assume $M$ is a generic $\Gamma$-module in $\vec{\mathcal{F}}(\Theta)$ with $\text{endol}(M) \leq d$. Since $\mathfrak{T} \Omega = U \otimes_A P \otimes_{\Gamma} - : \vec{\mathcal{F}}(\Delta') \xrightarrow{\sigma} \vec{\mathcal{F}}(\Theta)$ is an equivalence, the infinite-dimensional
indecomposable \( M' := P' \otimes \Lambda \mathfrak{F}(M) \in \mathcal{F}(\Delta') \) satisfies \( \mathcal{F}(M') \cong \mathcal{F}(M) \). We know that \( \text{endol}(M') \leq \dim_k P' \times \text{endol}(\mathfrak{F}(M)) \leq \dim_k P' \times \dim_k U \times d = d' \), see for instance [7](11.1). So, \( M' \) is a generic \( \Gamma \)-module in \( \mathcal{F}(\Delta') \) with \( \text{endol}(M') \leq d' \). By [6.7], there is a pregeneric \( B \)-module \( N \) such that \( T' \otimes_B N \cong M' \). Therefore, we obtain \( T \otimes_B N \cong U \otimes \Lambda P \otimes \Gamma \mathfrak{F}(B) \otimes_B N \cong M \). Since \( \mathcal{F} : \text{End}_\Gamma(M')^{\text{op}} \longrightarrow \text{End}_\Gamma(M)^{\text{op}} \) is an isomorphism, the splitting of the radical for the second algebra follows from the same property for the first one. The fact that \( M \) is a proper \( \Upsilon \)-\( k(x) \)-bimodule follows from the corresponding property for \( M' \).

\[ \square \]

8 Generic modules for \( \mathcal{F}(\Theta) \)

This section is devoted to the proofs of the main results (1.5) and (1.6). We keep the notation of the preceding section. So, we assume that \( \Upsilon \) is a finite-dimensional algebra over an algebraically closed field and \( (\mathcal{P}, \leq, \{ \Theta_x \}_{x \in \mathcal{P}}) \) is any homological system for \( \Upsilon \), and we will develop the proofs with this notation.

**Proof of Theorem (1.5).** We assume that \( \mathcal{F}(\Theta) \) is not wild and that \( G \) is a generic \( \Upsilon \)-module for \( \mathcal{F}(\Theta) \). Let \( d := \text{endol}(G) \) and apply (7.10) to obtain the minimal ditalgebra \( B \) and the composition of functors

\[
B\text{-Mod} \xrightarrow{L_{\Upsilon}} B\text{-Mod} \xrightarrow{\mathcal{F}} \Lambda\text{-Mod} \xrightarrow{H} \mathcal{F}(\Delta') \xrightarrow{\Omega} \mathcal{F}(\Delta) \xrightarrow{T} \mathcal{F}(\Theta),
\]

isomorphic to a suitable tensor functor \( T \otimes_B = \).

By (7.10)(2), there is a generic \( B \)-module \( N \in B\text{-Mod} \) with \( \mathcal{F} \Omega HF(N) \cong T \otimes_B N \cong G \). In fact, \( N = B_{e_i} \otimes_{B_{e_i}} Q_i \), where \( Q_i \) is the field of fractions of \( \Gamma_i = B_{e_i} \) considered as a \( \Gamma_i \)-module. Here \( B_{e_i} \) is a rational \( k \)-algebra and \( Q_i = k(x) \). We have

\[
G \cong T \otimes_B N = T \otimes_B B_{e_i} \otimes_{B_{e_i}} Q_i \cong T e_i \otimes_{B_{e_i}} Q_i,
\]

where \( T e_i \) is a finitely generated right \( B_{e_i} \)-module, so there is some \( g \in B_{e_i} \) such that \( Z_G := T e_i \otimes_{B_{e_i}} (B_{e_i})_g \) is a free finitely generated right \( (B_{e_i})_g \)-module. Moreover, the field of fractions \( Q_G \) of the rational algebra \( \Gamma_G := (B_{e_i})_g \) coincides with \( k(x) = Q_i \), and we have

\[
Z_G \otimes_{\Gamma_G} Q_G = T e_i \otimes_{B_{e_i}} (B_{e_i})_g \otimes_{(B_{e_i})_g} Q_i \cong T e_i \otimes_{B_{e_i}} Q_i \cong G.
\]

From the last statement of (7.10), we know that \( \dim_{k(x)} G = \text{endol}(G) \) and from the last displayed formula, the rank of \( Z_G \) is \( \dim_{k(x)} G \), so (1) is proved.

(2): Since \( B \) is a minimal ditalgebra, we know that \( L_B : B\text{-Mod} \longrightarrow B\text{-Mod} \) preserves isoclasses, indecomposability and maps almost split sequences onto almost split conflations, see for instance [10](32.3). Since \( \mathcal{T}, \Omega, H, \) and \( F \) are full and faithful functors, they all preserve isoclasses and indecomposability. Then, the functor \( T \otimes_B \longrightarrow \mathcal{F} \Omega HF L_B \) preserves isoclasses and indecomposability. The localization morphism \( \phi : B_{e_i} \longrightarrow (B_{e_i})_g = \Gamma_G \) is an epimorphism which

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induces a full and faithful functor $F_{\phi} : \Gamma_G\text{-Mod} \rightarrow B_{\epsilon_i}\text{-Mod}$ which can be composed with the embedding functor $E : B_{\epsilon_i}\text{-Mod} \rightarrow B\text{-Mod}$. Then the functor $Z_G \otimes_{\Gamma_G} \simeq \cong \mathfrak{I} \Omega HFL_B E F_\phi$ preserves isoclasses and indecomposables.

(3): We know that, given any Roiter interlaced weak ditalgebra $\mathcal{A}$, the category $\mathcal{A}\text{-Mod}$ has a natural exact structure $\mathcal{E}_\mathcal{A}$, see [9](11.11). From [2](10), we know that $\mathcal{A}^x$ is a Roiter interlaced weak ditalgebra whenever $\mathcal{A}$ is so, for $z \in \{a, r, q, d, X\}$. Moreover, by [5](5) we know that $\mathcal{A}(\Delta)$-mod has almost split conflations.

By [9](11.5), the functor $F^X : \mathcal{A}^X\text{-Mod} \rightarrow \mathcal{A}\text{-Mod}$ is exact, when $\mathcal{A}$ is a Roiter interlaced weak ditalgebra, see [10](16.6). The same observations of [10][(9.8),(9.9),(9.10)] show that the functors $F^z : \mathcal{A}^z\text{-Mod} \rightarrow \mathcal{A}\text{-Mod}$, for $z \in \{a, r, q, d\}$, as in [2](1), [2](2), [2](3), [2](4), are exact, when $\mathcal{A}$ is a Roiter interlaced weak ditalgebra. These statements, and the corresponding ones for reduction functors $F^z$ of type $z \in \{a, r, d, e, u\}$ between module categories of Roiter ditalgebras, imply that the functor $F$ is exact (since it is a composition of such exact functors). So the composition $F : B\text{-mod} \rightarrow \mathcal{A}(\Delta)$-mod is a full and faithful exact functor between Krull-Schmidt categories with exact structures which admit almost split conflations.

From [7], we know that there are only finitely many indecomposable $\Gamma_G$-modules of the form $E_{\lambda} := \Gamma_G/(x-\lambda)$ such that $F_{\phi}E_{\phi}(E_{\lambda}) \not\simeq \tau F_{\phi}E_{\phi}(E_{\lambda})$. If this occurs for $\lambda_1, \ldots, \lambda_i \in k$, we can replace the polynomial $g$ used at the beginning of this proof by $g' := g(x-\lambda_1) \cdots (x-\lambda_i)$ to obtain a rational algebra $\Gamma_G' = (E_{\epsilon_i})_{g'}$ such that $F_{\phi}E_{\phi}(E_{\lambda}) \simeq \tau F_{\phi}E_{\phi}(E_{\lambda})$, for all such indecomposables $E_{\lambda}$ of $\Gamma_G'$, and we adjust $Z_G' := T_{\epsilon_i} \otimes_{B_{\epsilon_i}} (B_{\epsilon_i})_{g'}$ accordingly.

Moreover, (1) and (2), still hold for this $Z_G'$. So we can assume that $\Gamma_G$ and $Z_G$ already have this property and we can apply [10](32.7), to obtain that $F_{\phi}E_{\phi}$ preserves almost split conflations. Since $H, \Omega$, and $\mathfrak{T}$ are exact equivalences, so does the composition $Z_G \otimes_{\Gamma_G} \simeq \cong \mathfrak{I} \Omega HFL_B E F_\phi$. $\square$

**Proposition 8.1.** If $\mathcal{F}(\Theta)$ is not wild, then it is generically tame.

Moreover, in this case, for any $d \in \mathbb{N}$, almost all indecomposable $\mathcal{Y}$-modules $M \in \mathcal{F}(\Theta)$ with $\dim_k M \leq d$ are of the form $M \cong Z_G \otimes_{\Gamma_G} N$, for some generic module $G$ for $\mathcal{F}(\Theta)$ with $\text{endol}(G) \leq d$ and some indecomposable $N \in \Gamma_G\text{-mod}$, with the notation of [1](7). 

**Proof.** If $\mathcal{F}(\Theta)$ is not wild and $d \in \mathbb{N}$, we can apply [7](10) to obtain a minimal ditalgebra $B$ and a composition of functors

$$B\text{-Mod} \xrightarrow{L} B\text{-Mod} \xrightarrow{F} \mathcal{A}\text{-Mod} \xrightarrow{H} \mathcal{F}(\Delta') \xrightarrow{\Omega} \mathcal{F}(\Delta) \xrightarrow{\mathcal{T}} \mathcal{F}(\Theta),$$

isomorphic to a suitable tensor functor $T \otimes_B -$.

Then, from [7](11), we know that the generic $\mathcal{Y}$-modules in $\mathcal{F}(\Theta)$ with endolength $\leq d$, up to isomorphism, are $G_1, \ldots, G_n$, with $G_i = \mathfrak{I} \Omega HFL_B Q_i$, for $i \in [1, n]$, where $n$ is the number of rational factors of the minimal algebra of the layer of $B$. Here $Q_i$ is the $B$-module described in [1](7).

From [7](10)(3), we get that any indecomposable $\mathcal{Y}$-module $M \in \mathcal{F}(\Theta)$ with $\dim_k M \leq d$ is isomorphic to $T \otimes_B N$, for some indecomposable $B_{\epsilon_i}$-module $N$ with dimension bounded by some integer $d'$, which depends only on
Now, recall from the preceding proof that each bimodule $Z_i = T e_i \otimes_{B e_i} (B e_i)_g$, associated to a given generic module $G_i$ for $\mathcal{F}(\Theta)$, involves a localization of $k$-algebras $\phi : B e_i \rightarrow (B e_i)_g$ whose restriction functor $F_{\phi} : \Gamma G_i \text{-mod} = (B e_i)_g \text{-mod} \rightarrow B e_i \text{-mod}$ hits all isoclasses of indecomposable $B e_i$-modules with dimension bounded by $d'$, with only finitely many exceptions. Then, the second statement of this theorem holds.

**Proposition 8.2.** If $\mathcal{F}(\Theta)$ is wild, then it is not generically tame.

**Proof.** We adapt the argument of the proof of [15](4.4). From (7.9) and (7.8), we already know that $\mathcal{F}(\Theta)$ is wild (resp. generically tame) iff $\mathcal{F}(\Delta)$ is so. Moreover, $\mathcal{F}(\Delta)$ is wild (resp. generically tame) iff $\mathcal{F}(\Delta')$ is wild (resp. generically tame), see [8](1.8) and [10](22.6), it will be enough to prove the statement for $\mathcal{F}(\Delta')$.

Assume $\mathcal{F}(\Delta')$ is wild, then there is a $\Gamma k\langle x, y \rangle$-bimodule $Z$, which is free finitely generated as a right module, such that the functor

$$W = Z \otimes k\langle x, y \rangle : k\langle x, y \rangle \text{-mod} \rightarrow \mathcal{F}(\Delta')$$

preserves isoclasses and indecomposables. From [8](10.4) and the proof of [8](1.3), we know that $\mathcal{F}(\Delta')$ is wild iff $A$ is so. Then, we can assume that the bimodule $Z$ producing the wildness of $\mathcal{F}(\Delta')$ comes from a bimodule producing the wildness of $A$, see the proof of [8](1.3). So, by (6.2), we may assume that the bimodule $Z$ determines a functor

$$W = Z \otimes k\langle x, y \rangle : k\langle x, y \rangle \text{-Mod} \rightarrow \tilde{\mathcal{F}}(\Delta'),$$

which preserves isoclasses and idecomposables, see [8](6.12). As in [10](31.3) and (31.5)], we consider the $k\langle x, y \rangle$-modules $H_\lambda = k[x]/(y - \lambda)$, for $\lambda \in k$, where $k\langle x, y \rangle$ acts by restriction through the epimorphism of algebras

$$k\langle x, y \rangle \twoheadrightarrow k[x, y] \twoheadrightarrow k[x, y]_{k[x]} \cong k(x).$$

The family $\{H_\lambda\}_{\lambda \in k}$ is an infinite family of pairwise non-isomorphic generic $k\langle x, y \rangle$-modules with endolength 1. Set $G_\lambda := W(H_\lambda)$, for $\lambda \in k$. So $\{G_\lambda\}_{\lambda \in k}$ is an infinite family of pairwise non-isomorphic generic $\Gamma$-modules in $\tilde{\mathcal{F}}(\Delta')$ with bounded endolength. So $\mathcal{F}(\Delta')$ is not generically tame.

**Theorem 8.3.** $\mathcal{F}(\Theta)$ is generically tame iff $\mathcal{F}(\Theta)$ is tame.

**Proof.** From [8](1.3), (7.9), and (7.8), we already know that $\mathcal{F}(\Theta)$ is generically tame (resp. tame) iff $\mathcal{F}(\Delta)$ is tame. So it is enough to show the corresponding statement for $\mathcal{F}(\Delta)$. From (1.10), we already know that $A$ is pregenerically tame iff $\tilde{A}$ is not wild.

If $\mathcal{F}(\Delta)$ is tame then, by [8](1.3), $\mathcal{F}(\Delta)$ is not wild. So, by (6.6), we know that $A$ is not wild and, as mentioned before, this means that it is pregenerically tame. Thus given $d \in \mathbb{N}$, by (6.5) and (6.4), any generic module $G$ for $\mathcal{F}(\Delta)$ with $\text{endol}(G) \leq d$ is the image under $\Omega H$ of a pregeneric $A$-module $M$ with $d$. Theorem 8.3.
bounded endolength, so there are only a finite number of non-isomorphic generic modules for \( \mathcal{F}(\Delta) \) with endolength \( \leq d \). Hence, \( \mathcal{F}(\Delta) \) is generically tame.

Conversely, assume that \( \mathcal{F}(\Delta) \) is generically tame. By [8.2], it can not be wild. So, by [8.1], it is tame.

**Remark 8.4.** In [25], H. Treffinger constructs a family of examples of non-admissible homological systems for finite-dimensional algebras \( \mathcal{Y} \) with size which can be arbitrarily large in comparison to the number of isomorphism classes of simple \( \mathcal{Y} \)-modules. Our main results apply to \( \mathcal{F}(\Theta) \) for each one of these homological systems.

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