THE MINIMALLY ANISOTROPIC METRIC OPERATOR IN QUASI-HERMITIAN QUANTUM MECHANICS

DAVID KREJČIŘÍK, VLADIMIR LOTOREICHIK, AND MILOS ZNOJIL

ABSTRACT. We propose a unique way how to choose a new inner product in a Hilbert space with respect to which an originally non-self-adjoint operator similar to a self-adjoint operator becomes self-adjoint. Our construction is based on minimizing a ‘Hilbert-Schmidt distance’ to the original inner product among the entire class of admissible inner products. We prove that either the minimizer exists and is unique, or it does not exist at all. In the former case we derive a system of Euler-Lagrange equations by which the optimal inner product is determined. A sufficient condition for the existence of the unique minimally anisotropic metric is obtained. The abstract results are supplied by examples in which the optimal inner product does not coincide with the most popular choice fixed through a charge-like symmetry.

1. Introduction

Quantum mechanics is a fundamental theory of modern science. In addition to its enormous success in describing physical phenomena and important technological impact, it is also mathematically exquisite through the coherent implementation of functional analysis of operators in Hilbert spaces. It is intrinsically linear and self-adjoint: physical observables are represented by linear self-adjoint operators and the time evolution is governed by unitary groups.

The self-adjointness of quantum theory does not mean that one need not deal with an analysis of non-self-adjoint operators when it comes to applications. The description of quantum scattering by means of a complex effective potential due to H. Feshbach in 1958 [7] is just an early example. However, the non-self-adjointness arises in such approaches as a result of a technical method or a useful approximation to attack a concrete physical problem involving observables still represented by self-adjoint operators.

A conceptually new point of view in this respect was suggested by nuclear physicists F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne in 1992 [15]: physical observables in quantum mechanics can be represented by non-self-adjoint operators provided that they are quasi-self-adjoint. They actually use the term quasi-Hermitian, which was also previously used by the mathematician J. Dieudonné in 1961 [6], but we have decided to use a more modern mathematically terminology in this paper. An operator $\mathcal{H}$ in the Hilbert space $\mathcal{H}$ equipped with the inner product $\langle \cdot, \cdot \rangle$ is called quasi-self-adjoint if it is densely defined and there exists a bounded, non-negative self-adjoint operator
\( \Theta: \mathcal{H} \to \mathcal{H} \) having a bounded inverse such that

\begin{equation}
H^* = \Theta H \Theta^{-1}.
\end{equation}

Here \( H^* \) denotes the adjoint of \( H \) in \( \mathcal{H} \) with respect to the inner product \( \langle \cdot, \cdot \rangle \).

A quasi-self-adjoint \( H \) is self-adjoint with respect to a modified (but topologically equivalent) inner product \( \langle \cdot, \Theta \cdot \rangle \) in \( \mathcal{H} \). For this reason, the operator \( \Theta \) is often called *metric*. Let us also remark that the quasi-self-adjointness of \( H \) is equivalent to the fact that \( H \) is similar to a self-adjoint operator in \( \mathcal{H} \) with respect to the original inner product \( \langle \cdot, \cdot \rangle \). Consequently, the spectrum of a quasi-self-adjoint operator \( H \) is purely real and \( H \) generates a unitary time evolution when considered in the Hilbert space with the modified inner product. In summary, by considering the larger class of quasi-self-adjoint operators, one gets a useful flexibility (overlooked for almost one century) in the mathematical description of physical observables in quantum mechanics. We refer to the recent book [2] for more information on this new concept in quantum mechanics and many references.

It is well known and easy to check that the choice of the metric operator \( \Theta \) in (1.1) is inevitably non-unique. In fact, this diversity occurs already in the self-adjoint situation \( H^* = H \) when one has a one-parametric sub-family of metrics \( \Theta_\alpha := \alpha I \) with \( \alpha > 0 \). A meaningful way of picking up a ‘good’ metric operator for a given quasi-self-adjoint operator is the subject of active ongoing research. This issue is addressed already in the pioneering work [15], where the uniqueness is partially settled by considering an irreducible set of quasi-self-adjoint operators. For just one operator, however, there is no canonical way how to choose the metric operator and the existing procedures are typically motivated by simplicity of calculation (see, e.g., [13, 12, 10]) or extra physical-like symmetries (see, e.g., [3, 1]).

The goal of this paper is to present a new promising way of selecting the metric, which relies on a certain minimality condition. We focus on quasi-self-adjoint operators having purely real simple discrete spectra. Moreover, we assume that the eigenfunctions constitute a basis quadratically close to an orthonormal one [8, §VI.2]. For a quasi-self-adjoint operator \( H \) in the Hilbert space \( \mathcal{H} \) satisfying these hypotheses, it follows from [14, §2.3] and Lemma 1 below that \( H \) possesses a sub-family of metrics of the form \( \Theta = I + K \), where \( K \) is a Hilbert-Schmidt operator in \( \mathcal{H} \) of a specific structure. The metric in this sub-family with the smallest Hilbert-Schmidt norm of \( K \) will be called *minimally anisotropic*.

Our main result formulated in Theorem 3 shows that either there exists a unique minimally anisotropic metric, or there is no such metric at all. In the former case we derive a system of Euler-Lagrange equations, by which the minimally anisotropic metric is uniquely determined. Furthermore, we provide in Proposition 4 a condition on the eigenfuctions of \( H^* \) which is sufficient for the existence of the unique minimally anisotropic metric for \( H \).
The abstract results are supported by finite- and infinite-dimensional examples. These examples illuminate the mechanism behind existence/non-existence of the minimally anisotropic metric. They show that the minimally anisotropic metric need not coincide with the metric constructed by means of the charge-symmetry operator.

Instead of minimising the Hilbert-Schmidt norm of $K$, it is also possible to consider the analogous minimisation problems in other Schatten classes (including the operator norm of the Hilbert space $\mathcal{H}$). The distinction of our choice is that the class of Hilbert-Schmidt operators forms a Hilbert-space structure.

2. Admissible class of quasi-self-adjoint operators

In what follows $(\mathcal{H}, \langle \cdot , \cdot \rangle)$ is a separable Hilbert space of dimension $N \in \mathbb{N} \cup \{\infty\}$. We adopt the physical convention that the inner product is linear in the second entry. The norm in $\mathcal{H}$, induced by the inner product $\langle \cdot , \cdot \rangle$, will be denoted by $\| \cdot \|$. We assume that the reader is familiar with the basics of the Hilbert space theory. Nevertheless, we provide definitions for some selected important concepts and recall their key properties.

For the sake of convenience, we define the set

$$\mathcal{K} := \begin{cases} 
\{1, 2, \ldots, N\}, & N < \infty, \\
\mathbb{N}, & N = \infty.
\end{cases}$$

A sequence of vectors $\{\phi_n\}_{n=1}^N$ in $\mathcal{H}$ is called a basis if any $\phi \in \mathcal{H}$ admits a unique expansion into the series $\phi = \sum_{n=1}^N c_n \phi_n$ with the complex coefficients $\{c_n\}_{n=1}^N$. This series is assumed to be norm-convergent if $N = \infty$.

Let $\{\chi_n\}_{n=1}^N$ be an orthonormal basis in $\mathcal{H}$ and let $A$ be a bounded and boundedly invertible operator in $\mathcal{H}$. By [8, §VI.2], the set of vectors $\phi_n := A\chi_n$, $n \in \mathcal{K}$, is also a basis in $\mathcal{H}$, called a Riesz basis. Let $\mathcal{X} = \{\chi_n\}_{n=1}^N$ be an orthonormal basis in $\mathcal{H}$ and the family $\Psi = \{\psi_n\}_{n=1}^N$ be a basis such that $\sum_{n=1}^N \|\psi_n - \chi_n\|^2 < \infty$. Such a basis is called quadratically close to an orthonormal alias Bari basis. By [8, §VI.2, Thm. 2.3], the Bari basis $\Psi$ is also a Riesz basis with respect to $\mathcal{X}$. For $N < \infty$, any basis is a Riesz as well as a Bari basis.

Now we introduce a class of quasi-self-adjoint operators.

**Hypothesis 1.** Let $\mathcal{H}$ be a quasi-self-adjoint operator in $\mathcal{H}$ with purely real simple discrete spectrum and assume that the set of its eigenfunctions $\Psi = \{\psi_n\}_{n=1}^N$ is a basis in $\mathcal{H}$. Additionally, assume that $\Psi$ is quadratically close to an orthonormal basis $\mathcal{X} = \{\chi_n\}_{n=1}^N$.

Recall that an operator is said to have a purely discrete spectrum if its resolvent is compact. The spectrum is said to be simple if the geometric and the algebraic multiplicities of all the eigenvalues are equal to one. While the condition on basis properties is automatically satisfied only in finite-dimensional Hilbert spaces, it also holds for a class of Schrödinger operators on a bounded interval with complex Robin boundary conditions [14]. On the other hand, there exist quasi-self-adjoint operators without the Bari property [11].
Clearly, the operator $H^*$ has the same simple discrete spectrum as $H$. Let $\Phi = \{ \phi_n \}_{n=1}^N$ be the set of the eigenfunctions of $H^*$. We adopt the normalisation for the families $\Psi$ and $\Phi$ such that:

(i) $\langle \psi_m, \phi_n \rangle = \delta_{nm}$ for $n, m \in K$;

(ii) $\| \phi_n \| = 1$ for $n \in K$.

The normalisation condition (i) implies convenient resolution-of-identity decompositions

\[ I = \sum_{n=1}^N \psi_n \langle \phi_n, \cdot \rangle = \sum_{n=1}^N \phi_n \langle \psi_n, \cdot \rangle. \]

According to [8, §VI.3], the family $\Phi$ is also a Bari basis in $K$, being quadratically close to $\mathcal{X}$. Moreover, there exists a bounded and boundedly invertible operator $A$ in $H$ such that $\chi_n = A^* \phi_n = A^{-1} \psi_n$ for all $n \in K$.

In view of the construction in [14, Sec. 2.3], any metric operator $\Theta$ for the quasi-self-adjoint operator $H$ as in Hypothesis 1 admits the following representation:

\[ \Theta = \sum_{n=1}^N C_n \langle \phi_n, \cdot \rangle \phi_n, \]

where $C_- \leq C_n \leq C_+$ for all $n \in K$ with some $C_-, C_+ \in (0, \infty)$, $C_- \leq C_+$. Note that the sum in (2.1) should be understood as the strong limit in the case that $N = \infty$. Evident ambiguity in the choice of the constants $C_n$ in the representation (2.1) reflects the non-uniqueness of the metric.

3. The Hilbert-Schmidt and the trace classes

We assume that the reader is familiar with the concept of the compact linear operator in a Hilbert space [4, §2.6]. For a compact linear operator $T: \mathcal{H} \to \mathcal{H}$ we define its module by $|T| := (T^*T)^{1/2}$. The eigenvalues $s_k(T)$, $k \in K$, of $|T|$ ordered non-increasingly and with multiplicities taken into account are called the singular values of $T$.

A compact operator $T: \mathcal{H} \to \mathcal{H}$ is Hilbert-Schmidt (respectively, of trace class) if, and only if, $\sum_{k=1}^N (s_k(T))^2 < \infty$ (respectively, $\sum_{k=1}^N s_k(T) < \infty$). We denote by $\mathcal{S}_2(\mathcal{H})$ and by $\mathcal{S}_1(\mathcal{H})$ the families of all Hilbert-Schmidt and of all trace class operators over $\mathcal{H}$, respectively. In particular, the inclusion $\mathcal{S}_1(\mathcal{H}) \subseteq \mathcal{S}_2(\mathcal{H})$ holds. Note that in a finite-dimensional Hilbert space ($N < \infty$) any linear operator is Hilbert-Schmidt as well as of trace class.

For any operator $T \in \mathcal{S}_1(\mathcal{H})$ we denote by $\lambda_k(T)$, $k \in K$, its eigenvalues repeated with the algebraic multiplicities taken into account. The trace mapping

\[ \mathcal{S}_1(\mathcal{H}) \ni T \mapsto \text{Tr} T := \sum_{k=1}^N \lambda_k(T) \]

is well defined and the sum on the right-hand side converges absolutely. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be an open set. The trace of an integral operator $T \in \mathcal{S}_1(L^2(\Omega))$ with the
integral kernel \( t: \Omega \times \Omega \rightarrow \mathbb{R} \) satisfying \( t \in C(\overline{\Omega} \times \Omega) \) can be computed as follows (cf. [9, Ex. X.1.18] and [5, Cor. 3.2])

\[
\text{Tr } T = \int_{\Omega} t(x, x) \, dx.
\]

The class \( \mathcal{S}_2(\mathcal{H}) \) of Hilbert-Schmidt operators over \( \mathcal{H} \) viewed as a linear space and endowed with the conventional inner product \( \langle S, K \rangle_2 := \text{Tr}(S^*K) \) turns out to be a Hilbert space; cf. [8, §III.9]. The norm on \( \mathcal{S}_2(\mathcal{H}) \) induced by the inner product \( \langle \cdot, \cdot \rangle_2 \) will be denoted by \( \| \cdot \|_2 \).

### 4. The main result

In this section we prove the main result of this paper. To this aim we need an auxiliary lemma on metric operators that can be represented as the sum of the identity and a Hilbert-Schmidt operator.

**Lemma 1.** Let \( N = \infty \) and let \( \mathcal{H} \) be a quasi-self-adjoint operator as in Hypothesis 1. Let \( \Theta \) be a metric operator for \( \mathcal{H} \) represented as in (2.1). Then \( 1 - \Theta \) is Hilbert-Schmidt if, and only if, \( C_n = 1 + \alpha_n \) with \( \alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N}) \).

**Proof.** Define the following auxiliary operators

\[
U := A^*(1 - \Theta)A, \quad V := \sum_{n=1}^{\infty} \alpha_n \chi_n \langle \chi_n, \cdot \rangle, \quad \text{and} \quad W := \sum_{n=1}^{\infty} \chi_n \langle A^*(\psi_n - \phi_n), \cdot \rangle.
\]

The operator \( U \) can be represented as

\[
U = \sum_{n=1}^{\infty} \left[ A^* \phi_n \langle A^* \psi_n, \cdot \rangle - [1 + \alpha_n] A^* \phi_n \langle A^* \phi_n, \cdot \rangle \right] = \sum_{n=1}^{\infty} \left[ \chi_n \langle \chi_n, \cdot \rangle + \chi_n \langle A^*(\psi_n - \phi_n), \cdot \rangle - [1 + \alpha_n] \chi_n \langle \chi_n, \cdot \rangle \right] = W - V.
\]

The Hilbert-Schmidt norm of \( V \) is given by \( \| V \|_2 = \| \alpha \|_{\ell^2(\mathbb{N})} \). For the operator \( W \) we can estimate the square of the Hilbert-Schmidt norm as follows

\[
\| W \|_2^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| A^*(\psi_n - \phi_n), \chi_m \|^2 = \sum_{n=1}^{\infty} \| A^*(\psi_n - \phi_n) \|^2 \leq \| A \|^2 \sum_{n=1}^{\infty} \| \psi_n - \phi_n \|^2 < \infty.
\]

Suppose that \( 1 - \Theta \) is a Hilbert-Schmidt operator. Then the operator \( U \) is Hilbert-Schmidt as well and we have

\[
\| \alpha \|_{\ell^2(\mathbb{N})} = \| V \|_2^2 \leq 2 \| U \|_2^2 + 2 \| W \|_2^2 \leq 2 \| U \|_2^2 + 2 \| A \|^2 \sum_{n=1}^{\infty} \| \psi_n - \phi_n \|^2 < \infty.
\]

Second, suppose that \( \Theta \) is as in (2.1) with \( C_n = 1 + \alpha_n \), where \( \alpha \in \ell^2(\mathbb{N}) \). Then

\[
\| 1 - \Theta \|_2^2 \leq \| A^{-1} \|^4 \| U \|_2^2 \leq 2 \| A^{-1} \|^4 \left( \| W \|_2^2 + \| V \|_2^2 \right)
\]

\[
\leq 2 \| A^{-1} \|^4 \| A \|^2 \sum_{n=1}^{\infty} \| \psi_n - \phi_n \|^2 + 2 \| A^{-1} \|^4 \| \alpha \|_{\ell^2(\mathbb{N})}^2 < \infty. \quad \square
\]
Next, we introduce the cone of operators

\[ \mathcal{C} := \left\{ \sum_{n=1}^{N} [1 + \alpha_n] \langle \phi_n, \cdot \rangle \phi_n : \alpha \in \ell^2(\mathbb{K}; J) \right\}, \quad \text{where} \quad J = [-1, \infty). \]

For \( \Theta \in \mathcal{C} \), we call \( \alpha = \alpha(\Theta) \in \ell^2(\mathbb{K}; J) \) its characteristic vector. It is straightforward to see that for any \( \omega \in [0, 1] \) and \( \Theta_1, \Theta_2 \in \mathcal{C} \) we have \( \omega \Theta_1 + (1 - \omega) \Theta_2 \in \mathcal{C} \). The boundary \( \partial \mathcal{C} \) and the interior \( \mathcal{C}^o \) of \( \mathcal{C} \) are defined by

\[ \partial \mathcal{C} := \{ \Theta \in \mathcal{C} : \min \alpha(\Theta) = -1 \} \quad \text{and} \quad \mathcal{C}^o := \{ \Theta \in \mathcal{C} : \min \alpha(\Theta) > -1 \}. \]

In view of Lemma 1 any metric for \( H \), which can be represented as \( \Theta = I + K \) with a Hilbert-Schmidt \( K \), belongs to \( \mathcal{C}^o \). Now, we consider the minimisation problem:

\[
\inf_{\Theta \in \mathcal{C}} \| \Theta - I \|_2.
\]

**Proposition 2.** Under Hypothesis 1, the minimisation problem (4.1) has a unique minimiser \( \Theta_* \in \mathcal{C} \).

**Proof.** We split the proof into two steps. First, we show existence of a minimiser, then we prove its uniqueness.

**Step 1 (existence).** Suppose that the infimum in (4.1) equals \( M \geq 0 \). Let \( (\Theta_m)_m \subset \mathcal{C} \) be a minimising sequence that is \( \delta_n := \| \Theta_m - I \|_2^2 - M^2 \to 0^+ \) as \( m \to \infty \). Employing the parallelogram identity in the Hilbert space \( \mathcal{S}_2(\mathcal{H}) \), we find

\[
\| \Theta_k - \Theta_l \|_2^2 = 2\| \Theta_k - I \|_2^2 + 2\| \Theta_l - I \|_2^2 - 4 \| \frac{\Theta_k + \Theta_l}{2} - I \|_2^2 \leq 2\delta_k + 2\delta_l \to 0^+, \quad k, l \to \infty.
\]

Hence, \( 1 - \Theta_m \) is a Cauchy sequence in \( \mathcal{S}_2(\mathcal{H}) \). Thus, \( 1 - \Theta_m \) converges in the norm \( \| \cdot \|_2 \) to some Hilbert-Schmidt operator \( K_* \). Moreover, by continuity we infer that \( \| K_* \|_2 = M \).

Next, we show that \( \Theta_* = I + K_* \in \mathcal{C} \). Indeed, for every \( n \in \mathbb{K} \) we have \( \Theta_* \psi_n = \lim_{m \to \infty} \Theta_m \psi_n = \text{span} \{ \phi_n \} \). Thus, we conclude that

\[
\Theta_* = \sum_{n=1}^{N} [1 + \alpha_n(\Theta_*)] \langle \phi_n, \cdot \rangle \phi_n,
\]

and it only remains to show that \( \alpha(\Theta_*) \in \ell^2(\mathbb{K}; J) \). Since \( \Theta_* - I \) is Hilbert-Schmidt,

\[
\| \alpha(\Theta_*) \|_{\ell^2(\mathbb{K})}^2 = \sum_{n=1}^{N} | \langle \psi_n, (\Theta_* - I) \psi_n \rangle |^2 = \sum_{n=1}^{N} | \langle \chi_n, \mathcal{A}^*(\Theta_* - I) \mathcal{A} \chi_n \rangle |^2 \leq \| \mathcal{A}^*(\Theta_* - I) \mathcal{A} \|_2^2 \leq \| \mathcal{A} \|_2^4 \| \Theta_* - I \|_2^2 < \infty.
\]

Finally, we check that \( \alpha_n(\Theta_*) \in J \). Indeed, we have

\[
1 + \alpha_n(\Theta_*) = \langle \phi_n, \Theta_* \psi_n \rangle = \lim_{m \to \infty} \langle \phi_n, \Theta_m \psi_n \rangle = \lim_{m \to \infty} (1 + \alpha_n(\Theta_m)) \geq 0.
\]
Step 2 (uniqueness). Suppose that for some \( \Theta_1, \Theta_2 \in \mathfrak{C} \) we have \( \| \Theta_1 - I \|_2 = \| \Theta_2 - I \|_2 = M \). For the arithmetic mean \( \Theta_3 := \frac{1}{2}(\Theta_1 + \Theta_2) \in \mathfrak{C} \), the parallelogram identity yields
\[
\| \Theta_3 - I \|_2^2 + \frac{1}{4} \| \Theta_1 - \Theta_2 \|_2^2 = \frac{1}{2} \left( \| \Theta_1 - I \|_2^2 + \| \Theta_2 - I \|_2^2 \right) = M^2.
\]
Hence, using that \( M = \inf_{\Theta \in \mathfrak{C}} \| \Theta - I \|_2 \) we get
\[
\| \Theta_1 - \Theta_2 \|_2^2 = 4(M^2 - \| \Theta_3 - I \|_2^2) \leq 0.
\]
Therefore, we conclude that \( \Theta_1 = \Theta_2 \). \( \square \)

Now we have all the tools to formulate and prove the main result of this paper.

Theorem 3. Let \( H \) be an operator as in Hypothesis 1 and let \( \Theta_\ast \in \mathfrak{C} \) be the unique minimiser for the problem (4.1). Then the following hold.

(i) If \( \Theta_\ast \in \mathfrak{C}^\circ \), then \( \Theta_\ast \) is the minimally anisotropic metric for \( H \) and its characteristic vector \( \alpha = \alpha(\Theta_\ast) \) satisfies the Euler-Lagrange equations
\[
\sum_{m=1}^{N} (\langle \phi_m, \phi_n \rangle)^2 \alpha_m + \sum_{m \neq n} (\langle \phi_m, \phi_n \rangle)^2 = 0, \quad \forall \ n \in \mathbb{K}.
\]

(ii) If the system of equations (4.2) has a solution \( \alpha \in \ell^2(\mathbb{K}; (-\infty, \infty)) \), then such solution is unique in \( \ell^2(\mathbb{K}; (-\infty, \infty)) \) and the operator \( \Theta \in \mathfrak{C}^\circ \) with the characteristic vector \( \alpha(\Theta) = \alpha \) coincides with \( \Theta_\ast \).

(iii) If \( \Theta_\ast \in \partial \mathfrak{C} \), then the minimally anisotropic metric operator for \( H \) does not exist.

Proof. (i) Clearly, \( \Theta_\ast \) is a metric operator for \( H \) by (2.1). By Lemma 1 it can be represented as \( \Theta_\ast = I + K \) with a Hilbert-Schmidt \( K \). Moreover, Proposition 2 implies \( \| \Theta_\ast - I \|_2 < \| \Theta - I \|_2 \) for any \( \Theta \in \mathfrak{C}^\circ \). Thus, \( \Theta_\ast \) is minimally anisotropic.

Now we derive the Euler-Lagrange equations in (4.2). The condition of minimality \( \| I - \Theta_\ast \|_2 \) implies that the functions
\[
f_n(\varepsilon) := \| I - \Theta_\ast + \varepsilon \langle \phi_n, \cdot \rangle \phi_n \|_2^2, \quad \forall \ n \in \mathbb{K},
\]
satisfy \( f_n(0) = 0 \). That is we have
\[
\frac{d}{d\varepsilon} \left( \| I - \Theta_\ast \|_2^2 + 2\varepsilon \text{Tr} \left[ (I - \Theta_\ast) \phi_n \langle \phi_n, \cdot \rangle \right] + \varepsilon^2 \| \phi_n \langle \phi_n, \cdot \rangle \|_2^2 \right) |_{\varepsilon = 0} = 2\text{Tr} \left[ (I - \Theta_\ast) \phi_n \langle \phi_n, \cdot \rangle \right] = 0, \quad \forall \ n \in \mathbb{K}.
\]
Hence, using the shorthand notation \( \alpha_k = \alpha_k(\Theta_\ast) \) we get
\[
0 = -\frac{\dot{f}_n(0)}{2} = \text{Tr} \left( \sum_{k=1}^{N} (\phi_k + \alpha_k \phi_k - \psi_k) \langle \phi_k, \phi_n \rangle \langle \phi_n, \cdot \rangle \right)
\]
\[
= \sum_{k=1}^{N} \text{Tr} \left( (\alpha_k \phi_k + \phi_k - \psi_k) \langle \phi_k, \phi_n \rangle \langle \phi_n, \cdot \rangle \right) = \sum_{k=1}^{N} \langle \phi_n, \alpha_k \phi_k + \phi_k - \psi_k \rangle \langle \phi_k, \phi_n \rangle
\]
\[
= \sum_{k=1}^{N} |\langle \phi_k, \phi_n \rangle|^2 + \sum_{k=1}^{N} \alpha_k |\langle \phi_k, \phi_n \rangle|^2 - ||\phi_n||^2 = \sum_{k \neq n} |\langle \phi_k, \phi_n \rangle|^2 + \sum_{k=1}^{N} \alpha_k |\langle \phi_k, \phi_n \rangle|^2.
\]
(ii) Let \( \alpha' \in \ell^2(\mathbb{K}; (-1, \infty)) \) be a solution of (4.2). Consider the operator \( \Theta_1 \in \mathcal{C}^0 \) with the characteristic vector \( \alpha(\Theta_1) = \alpha' \). As in the proof of (i), one has Tr \((\mathbb{I} - \Theta_1)\phi_n(\phi_n, \cdot)\) = 0 for all \( n \in \mathbb{K} \). Let \( \Theta_2 \in \mathcal{C}^0 \) be such that \( \Theta_2 \neq \Theta_1 \). Hence, we conclude that

\[
||l - \Theta_2||^2 = ||l - \Theta_1 + \Theta_1 - \Theta_2||^2 = ||l - \Theta_1||^2 + ||\Theta_1 - \Theta_2||^2 > ||l - \Theta_1||^2.
\]

Thus, \( \Theta_1 \) is indeed the minimally anisotropic metric. Existence of another solution \( \alpha'' \in \ell^2(\mathbb{K}; (-1, \infty)) \) for (4.2) contradicts uniqueness of the minimally anisotropic metric.

(iii) By (2.1) we infer that \( \Theta_* \) is not a metric for \( \mathbb{H} \). Proposition 2 yields that any metric \( \Theta \in \mathcal{C}^0 \) for \( \mathbb{H} \) satisfies the inequality \( M := ||\Theta_* - I|| < ||\Theta - I|| \). If \( N < \infty \), then it is easy to construct a sequence of metrics \( \Theta_m \in \mathcal{C}^0, m \in \mathbb{N} \), which converges in the Hilbert-Schmidt norm to \( \Theta_* \), and we omit this construction. If \( N = \infty \), then we consider the following sequence of operators

\[
\Theta_m := \sum_{n=1}^{m} [1 + \alpha_n(\Theta_*) + e^{-m}] \phi_n(\phi_n, \cdot) + \sum_{n=m+1}^{\infty} \phi_n(\phi_n, \cdot).
\]

It is easy to check that \( \Theta_m \in \mathcal{C}^0 \). Moreover, we get

\[
M = \lim_{m \to \infty} ||\Theta_m - I|| \leq ||\Theta_* - I|| + \lim_{m \to \infty} ||\Theta_m - \Theta_*|| = M + ||A^{-1}||^2 \lim_{m \to \infty} ||A^*(\Theta_m - \Theta_*)A||_2
\]

\[
= M + ||A^{-1}||^2 \lim_{m \to \infty} \left( me^{-2m} + \sum_{n=m+1}^{\infty} |\alpha_n(\Theta_*)|^2 \right)^\frac{1}{2} = M.
\]

Hence, we conclude that \( \lim_{m \to \infty} ||\Theta_m - I|| < ||\Theta_* - I|| \), and thus the minimally anisotropic metric does not exist.

Finally, we provide a sufficient condition for the unique minimiser \( \Theta_* \in \mathcal{C} \) of (4.1) to be indeed a metric for \( \mathbb{H} \), that is for \( \Theta_* \in \mathcal{C}^0 \) to hold.

**Proposition 4.** Let \( \mathbb{H} \) be an operator as in Hypothesis 1. In addition, assume that

\[
(4.5) \quad \sum_{n=1}^{N} \sum_{m \neq n} |\langle \phi_n, \phi_m \rangle|^2 < 1.
\]

Then the unique minimiser \( \Theta_* \) of (4.1) satisfies \( \Theta_* \in \mathcal{C}^0 \), thus, being a metric for \( \mathbb{H} \).

**Proof.** Clearly, the decomposition \( \mathcal{C} = \partial \mathcal{C} \cup \mathcal{C}^0 \) with \( \partial \mathcal{C} \cap \mathcal{C}^0 = \emptyset \) holds. For any \( \Theta \in \partial \mathcal{C} \) there exists \( n_0 \in \mathbb{K} \) such that \( \alpha_{n_0}(\Theta) = -1 \) and we get

\[
||\Theta - I|| \geq ||\Theta - I|| \geq \frac{\langle \psi_{n_0}, (\Theta - I)\psi_{n_0} \rangle}{||\psi_{n_0}||^2} = \frac{||\psi_{n_0}||^2}{||\psi_{n_0}||^2} = 1.
\]

For the constant-coefficient metric operator \( \Theta_0 = \sum_{n=1}^{N} \phi_n(\phi_n, \cdot) \in \mathcal{C}^0 \) we compute

\[
(l - \Theta_0)^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} (\psi_n - \phi_n)(\phi_n, \psi_m - \phi_m)(\phi_m, \cdot).
\]
Hence, we obtain
\[
\|I - \Theta_0\|_2^2 = \text{Tr} \left( I - \Theta_0 \right)^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} \langle \phi_n, \psi_m - \phi_m \rangle \langle \phi_m, \psi_n - \phi_n \rangle
\]
\[
\leq \sum_{n=1}^{N} \sum_{m=1}^{N} |\delta_{nm} - \langle \phi_n, \phi_m \rangle| |\delta_{nm} - \langle \phi_m, \phi_n \rangle| = \sum_{n=1}^{N} \sum_{m=1}^{N} |\delta_{nm} - \langle \phi_n, \phi_m \rangle|^2 = \sum_{n=1}^{N} \sum_{m \neq n} |\langle \phi_n, \phi_m \rangle|^2 < 1.
\]

Therefore, we have \(\|I - \Theta \|_2 \leq \|I - \Theta_0\|_2 < 1\) and infer that \(\Theta \notin \partial \mathcal{C}\). Thus, we conclude \(\Theta \in \mathcal{C}^\circ\). □

Remark 5. The sum in (4.5) can be interpreted as squared Hilbert-Schmidt norm of the linear operator in the Hilbert space \(l^2(\mathbb{K})\) induced by the matrix \(\{\delta_{nm} - \langle \phi_n, \phi_m \rangle\}_{n,m \in \mathbb{K}}\). We point out that the same matrix appears in [8, §VI.3]. In particular, for \(\mathbb{K} = \mathbb{N}\), finiteness of its Hilbert-Schmidt is necessary and sufficient for \(\Phi = \{\phi_n\}_{n=1}^{\infty}\) to be a Bari basis, provided that \(\Phi\) is \(\omega\)-linearly independent.

5. Finite-dimensional examples

In this section we provide a couple of toy examples in two- and four-dimensional Hilbert spaces. The example in \(\mathbb{C}^4\) is very special and its aim is to show that the minimally anisotropic metric indeed does not always exist.

5.1. 2 x 2 example

Let the vectors \(\phi_1, \phi_2 \in \mathbb{C}^2\) be linearly independent and normalised as \(\|\phi_1\|_{\mathbb{C}^2} = \|\phi_2\|_{\mathbb{C}^2} = 1\). We select the vectors \(\psi_1, \psi_2 \in \mathbb{C}^2\) so that \(\langle \psi_n, \phi_m \rangle_{\mathbb{C}^2} = \delta_{nm}, n, m \in \{1, 2\}\). Any matrix
\[
H = \lambda_1 \psi_1 \langle \phi_1, \cdot \rangle + \lambda_2 \psi_2 \langle \phi_2, \cdot \rangle, \quad -\infty < \lambda_1 < \lambda_2 + \infty,
\]
can be viewed as a quasi-self-adjoint operator in the Hilbert space \(\mathbb{C}^2\) satisfying Hypothesis 1. According to (2.1), any metric for \(H\) can be decomposed as
\[
\Theta = [1 + \alpha_1] \phi_1 \langle \phi_1, \cdot \rangle + [1 + \alpha_2] \phi_2 \langle \phi_2, \cdot \rangle,
\]
with \(\alpha_1, \alpha_2 \in (-1, \infty)\). Using the shorthand notation \(\gamma := |\langle \phi_1, \phi_2 \rangle_{\mathbb{C}^2}|^2\) we can write the system of Euler-Lagrange equations (4.2) as follows
\[
\begin{cases}
\gamma \alpha_1 + \alpha_2 = -\gamma, \\
\alpha_1 + \gamma \alpha_2 = -\gamma.
\end{cases}
\]
Solving the above linear system, we find
\[
\alpha_1 = \alpha_2 = -\frac{\gamma}{1 + \gamma} > -1.
\]
Hence, by Theorem 3 (ii) the minimally anisotropic metric always exists and is given by
\[ \Theta^\star = \frac{\phi_1 \langle \phi_1, \cdot \rangle}{1 + \gamma} + \frac{\phi_2 \langle \phi_2, \cdot \rangle}{1 + \gamma}. \]

In the special case
\[ \phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
and \( \lambda_1 = 1, \lambda_2 = 2, \)
we have
\[ \psi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}. \]

Thus, the quasi-self-adjoint Hamiltonian \( H \) is given by
\[ H = \psi_1 \langle \phi_1, \cdot \rangle + 2\psi_2 \langle \phi_2, \cdot \rangle = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}. \]

The definition of \( \gamma \) and the formula (5.1) yield \( \gamma = \frac{1}{2}, \alpha_1 = \alpha_2 = -\frac{1}{2}. \) Thus, the minimally anisotropic metric is explicitly given by
\[ \Theta^\star = \frac{2}{3} \phi_1 \langle \phi_1, \cdot \rangle + \frac{2}{3} \phi_2 \langle \phi_2, \cdot \rangle = \frac{2}{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]

5.2. 4 \times 4 example

Let us fix \( x \in (0, 1) \), set \( y := \sqrt{1 - x^2} \), and consider the following normalised vectors in \( \mathbb{C}^4 \)
\[ \phi_1 = (1, 0, 0, 0)^\top, \quad \phi_2 = (y, x, 0, 0)^\top, \quad \phi_3 = (y, 0, x, 0)^\top, \quad \phi_4 = (y, 0, 0, x)^\top. \]

We will show that for all sufficiently small \( x \in (0, 1) \) the minimally anisotropic metric for any quasi-self-adjoint Hamiltonian \( H = \sum_{j=1}^4 \lambda_j \langle \phi_j, \cdot \rangle \) with \( \lambda_i \in \mathbb{R} \) (\( \lambda_i \neq \lambda_j \) for \( i \neq j \)) does not exist. Suppose that the minimally anisotropic metric \( \Theta^\star \in \mathbb{C}^4 \) with the characteristic vector \( \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top \) exists. Taking the symmetries into account, we conclude that \( \alpha_2 = \alpha_3 = \alpha_4. \) By Theorem 3 we can write the system of Euler-Lagrange equations (4.2) with the notation \( a := \alpha_1 \) and \( b := \alpha_2 = \alpha_3 = \alpha_4 \) as follows
\[ \begin{cases} a + 3y^2 b = -3y^2, \\ y^2 a + (1 + 2y^4) b = -y^2 - 2y^4. \end{cases} \]

The system can be simplified as
\[ \begin{cases} a + 3y^2 b = -3y^2, \\ a + \frac{1 + 2y^4}{y^2} b = -1 - 2y^2. \end{cases} \]

Hence, we derive an equation on \( b \)
\[ \frac{y^4 - 1}{y^2} b = 1 - y^2. \]
Finally, we get
\[ a = -\frac{3y^2}{y^2 + 1} \quad \text{and} \quad b = -\frac{y^2}{y^2 + 1} \]
For a sufficiently small \( x \in (0, 1) \), the difference \( 1 - y \) is arbitrarily small, we have \( a < -1 \) and therefore the minimally anisotropic metric does not exist.

6. An \( \infty \)-dimensional example: \( \mathcal{PT} \)-symmetric Robin Laplacian

The feasibility of the construction of the minimally anisotropic metric for quantum systems with \( N = \infty \) may be illustrated via the \( \mathcal{PT} \)-symmetric Robin Laplacian on an interval. For this purpose, let us set \( \beta := (-\frac{\pi}{2}, \frac{\pi}{2}) \) and define the conjugation and parity operators on the Hilbert space \( (L^2(\beta), \langle \cdot, \cdot \rangle_\beta) \) as
\[
(\mathcal{T}\psi)(x) := \overline{\psi(x)} \quad \text{and} \quad (\mathcal{P}\psi)(x) := \psi(-x).
\]
Introduce the m-sectorial operator \( H_\beta, \beta \in \mathbb{R}, \) on \( L^2(\beta) \) as
\[
H_\beta \psi := -\psi'' \quad \text{dom} \, H_\beta := \{ \psi \in H^2(\beta) : \psi' (\pm \frac{\pi}{2}) + i\beta \psi (\pm \frac{\pi}{2}) = 0 \}.
\]
The spectral theory of \( H_\beta \) is developed in [13, 12, 14]. Below we recall basic spectral properties of this operator.

Proposition 6 ([13],[14, Prop. 2.4]). Let the operator \( H_\beta, \beta \in \mathbb{R}, \) be as in (6). Let \( \mathcal{P} \) and \( \mathcal{T} \) be as in (6.1). Then the following hold.

(i) \( H_\beta^* = H_{-\beta}, \quad H_\beta = \mathcal{PT} H_\beta^* \mathcal{P}, \quad \mathcal{PT} H_\beta \subseteq H_\beta \mathcal{PT}. \) In particular, \( H_\beta \) is \( \mathcal{PT} \)-self-adjoint.

(ii) \( \sigma(H_\beta) = \bigcup_{n \in \mathbb{N}_0} \{ \lambda_n \} \subset \mathbb{R}, \) where \( \lambda_0 = \beta^2 \) and \( \lambda_n = n^2, n \in \mathbb{N}. \)

(iii) If \( \beta \notin \mathbb{Z} \setminus \{ 0 \}, \) then \( H_\beta \) satisfies Hypothesis 1. The eigenfunctions \( \{ \psi_n^\beta \}_{n=0}^{\infty} \) and \( \{ \phi_n^\beta \}_{n=0}^{\infty} \) of \( H_\beta \) and \( H_\beta^* \), respectively, corresponding to \( \{ \lambda_n \}_{n \in \mathbb{N}_0} \) read as
\[
\psi_0^\beta(x) = A_0 e^{-i\beta(x+a)}, \quad \psi_n^\beta(x) = A_n \left[ \cos(n(x+a)) - \frac{i\beta}{n} \sin(n(x+a)) \right], \quad n \in \mathbb{N},
\]
\[
\phi_0^\beta(x) = B_0 e^{i\beta(x+a)}, \quad \phi_n^\beta(x) = B_n \left[ \cos(n(x+a)) + \frac{i\beta}{n} \sin(n(x+a)) \right], \quad n \in \mathbb{N},
\]
where \( a = \frac{\pi}{2} \) and the real positive constants \( \{ A_n \}_{n=0}^{\infty} \) and \( \{ B_n \}_{n=0}^{\infty} \) are chosen so that \( (\psi_n^\beta, \phi_n^\beta) = \delta_{nm} \) and \( \| \phi_n^\beta \| = 1. \) In particular,
\[
B_0 = \frac{1}{\sqrt{\pi}} \quad \text{and} \quad B_n = \sqrt{\frac{2}{\pi}} \frac{n}{\sqrt{n^2 + \beta^2}}, \quad n \in \mathbb{N}.
\]

(iv) If \( \beta \in \mathbb{Z} \setminus \{ 0 \}, \) then the eigenvalue \( \lambda_0 \) of \( H_\beta \) has the geometric multiplicity one and the algebraic multiplicity two, and all the other eigenvalues are simple.
Proposition 6 (iii), representation (2.1), and Lemma 1 imply that for $\beta \notin \mathbb{Z} \setminus \{0\}$ all the metrics for $H_\beta$, that admit the representation $I + K$ with a Hilbert-Schmidt $K$, can be written as

$$\Theta = \sum_{n=0}^{\infty} [1 + \alpha_n] \phi_n^\beta \langle \phi_n^\beta \rangle_\beta, \quad \alpha \in \ell^2(\mathbb{N}_0; \mathbb{J}), \ \min \alpha > -1.$$  

The coefficients $a_{nm} := \langle \phi_n^\beta, \phi_m^\beta \rangle$ can be explicitly computed. Clearly, $a_{nn} = 1$ holds for all $n \in \mathbb{N}_0$ and for $n, m \geq 1, n \neq m$ we have

$$a_{nm} = \frac{B_n B_m}{nm} \int_0^{\pi} \left[ n \cos(nx) - i \beta \sin(nx) \right] \left[ m \cos(mx) + i \beta \sin(mx) \right] dx$$

$$= \frac{B_n B_m (1 - e^{i\pi(n+m)})i\beta}{2nm} \left( \frac{n - m}{n + m} - \frac{n + m}{n - m} \right)$$

$$= \frac{B_n B_m (1 - e^{i\pi(n+m)})i\beta}{2nm} \frac{4mn}{n^2 - m^2} = \frac{2B_n B_m (1 - e^{i\pi(n+m)})}{m^2 - n^2}.$$  

For $n \geq 1$ we get

$$a_{n0} = \frac{B_n B_0}{n} \int_0^{\pi} \left[ n \cos(nx) - i \beta \sin(nx) \right] e^{i\beta x} dx$$

$$= \frac{B_n B_0}{n} \left( \frac{2(-1)^n n \beta \sin(\pi \beta)}{\beta^2 - n^2} + \frac{2(-1)^n n \beta ((-1)^n - \cos(\pi \beta))}{\beta^2 - n^2} \right)$$

$$= \frac{2B_n B_0 i\beta (1 - e^{i\pi(n+b)})}{\beta^2 - n^2}.$$  

The remaining coefficients can be recovered via the relation $a_{nm} = \pi_{mn}$. The Euler-Lagrange system in (4.2) reduces to

$$\sum_{m=0}^{\infty} |a_{nm}|^2 \alpha_m(\Theta_\star) + \sum_{m \neq n} |a_{nm}|^2 = 0, \quad n \in \mathbb{N}_0.$$  

It seems however that this system can not be explicitly solved.

On the other hand, using the above expressions for $a_{nm}$ and the formulae for $B_n$ in Proposition 6 (iii) we obtain the following bounds on the coefficients in (6.2):

$$|a_{nm}|^2 \leq \frac{64\beta^2}{\pi^2 (m^2 - n^2)^2}, \quad |a_{n0}|^2 \leq \frac{64\beta^2}{\pi^2 (\beta^2 - n^2)^2}, \quad n, m \in \mathbb{N}, n \neq m.$$  

Using these bounds we get for any $\beta \in (0, \frac{1}{2})$

$$\sum_{n=0}^{\infty} \sum_{m \neq n} |a_{nm}|^2 \leq \frac{64\beta^2}{\pi^2} \left( \sum_{n=1}^{\infty} \frac{2}{(\beta^2 - n^2)^2} + \sum_{n=1}^{\infty} \sum_{m > n} \frac{2}{(m^2 - n^2)^2} \right)$$

$$\leq \frac{64\beta^2}{\pi^2} \left( \sum_{n=1}^{\infty} \frac{2}{(n^2 - 1/4)^2} + \sum_{d=1}^{\infty} \sum_{n=1}^{\infty} \frac{2}{((n+d)^2 - n^2)^2} \right)$$

$$\leq \frac{64\beta^2}{\pi^2} \left( 2\pi^2 - 16 + \frac{1}{2} \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \right) = \frac{64\beta^2}{\pi^2} \left( 2\pi^2 - 16 + \frac{\pi^4}{72} \right).$$
Hence, for all $\beta \in (0, \frac{1}{2})$ small enough we have $\sum_{n=0}^{\infty} \sum_{m \neq n} |a_{nm}|^2 < 1$ and Proposition 4 yields that the unique minimizer of (4.1) in this particular setting is indeed a metric operator for $H_\beta$.

Another possibility to select a metric operator relies on the concept of the $C$-symmetry.

**Definition 7** (Charge operator $\mathcal{C}$). Assume that $H$ in $L^2(\mathbb{J})$ is $\mathcal{P}$-self-adjoint. We say that $H$ possesses the property of $C$-symmetry, if there exists a bounded linear operator $\mathcal{C}$ in $L^2(\mathbb{J})$ such that $[H, \mathcal{C}] = 0$, $\mathcal{C}^2 = I$, and $\mathcal{PC}$ is a metric for $H$.

By [14, Sec. 4.3] the operator $H_\beta$ with $\beta \notin \mathbb{Z} \backslash \{0\}$ possesses a $C$-symmetry with the uniquely determined charge operator $\mathcal{C}$: $L^2(\mathbb{J}) \rightarrow L^2(\mathbb{J})$ through the identities $\mathcal{C}^2 = I$ and $\mathcal{C} = \mathcal{P}\Theta$ with a metric operator $\Theta$ as in (2.1). The metric for $H_\beta$ induced by the charge operator $\mathcal{C}$ is given by $\Theta = \mathcal{P}C = I + K_\mathcal{C}$, where $K_\mathcal{C}$ is an integral operator in $L^2(\mathbb{J})$ with the kernel $k_\mathcal{C}(x, y) = \beta e^{-i\beta(y-x)} \left[ \tan \left( \frac{\pi \beta}{2} \right) - i \text{sign} (y-x) \right]$.

It is easy to see that $\|K_\mathcal{C}\|_2 < \infty$. Consider the function

$$\mathbb{R} \ni \varepsilon \rightarrow f_0(\varepsilon) = \|K_\mathcal{C} + \varepsilon \phi_0^\beta(\phi_0^\beta, \cdot)\|^2_2.$$  

To show that $\Theta_\mathcal{C}$ is not the minimiser for (4.1) it suffices to check that $f_0(0) \neq 0$. Differentiating $f_0(\cdot)$ at the point $\varepsilon = 0$ we find

$$f_0(0) = 2 \text{Tr} \left( K_\mathcal{C}\phi_0^\beta(\phi_0^\beta, \cdot) \right) = 2 \langle \phi_0^\beta, K_\mathcal{C}\phi_0^\beta \rangle_\mathbb{J} = 2 \int_\mathbb{J} \int_\mathbb{J} k_\mathcal{C}(x, y) \phi_0^\beta(x) \phi_0^\beta(y) dx dy$$

$$= \frac{2\beta}{\pi} \int_\mathbb{J} \int_\mathbb{J} \left[ \tan \left( \frac{\pi \beta}{2} \right) - i \text{sign} (y-x) \right] dx dy = 2\beta \pi \tan \left( \frac{\pi \beta}{2} \right) \neq 0.$$  

**Acknowledgements**

D. K. was partially supported by the GAČR grant No. 18-08835S and by FCT (Portugal) through project PTDC/MAT-CAL/4334/2014. V. L. acknowledges the support by the GAČR grant No. 17-01706S. M. Z. acknowledges the support by the GAČR grant No. 16-22945S.

**References**

1. S. Albeverio and S. Kuzhel, *One-dimensional Schrödinger operators with $\mathcal{P}$-symmetric zero-range potentials*, J. Phys. A: Math. Theor. 38 (2005), 4975–4988.
2. F. Bagarello, J.-P. Gazeau, F. Szafraniec, and M. Znojil, *Non-selfadjoint operators in quantum physics. Mathematical aspects*, John Wiley & Sons, 2015.
3. C. M. Bender, D. C. Brody, and H. F. Jones, *Complex extension of quantum mechanics*, Phys. Rev. Lett. 89 (2002), 270401.
4. M. Z. Birman, M. Sh. and Solomyak, *Spectral theory of self-adjoint operators in Hilbert spaces*, Dodrecht, Holland, 1987.
5. Ch. Brislawn, *Traceable integral kernels on countably generated measure spaces.*, Pac. J. Math. 150 (1991), 229–240.
6. J. Dieudonné, *Quasi-Hermitian operators*, Proceedings of the International Symposium on Linear Spaces (Jerusalem 1960), Jerusalem Academic Press, Pergamon, Oxford, 1961, pp. 115–123.
7. H. Feshbach, *Unified theory of nuclear reactions*, Ann. Phys. 5 (1958), 357–390.
8. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space*, Amer. Math. Soc., 1969.
9. T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1966.
10. M. Kolb and D. Krejčířík, *Spectral analysis of the diffusion operator with random jumps from the boundary*, Math. Z. 284 (2016), 877–900.
11. D. Krejčířík, *Complex magnetic fields: An improved Hardy-Laptev-Weidl inequality and quasi-self-adjointness*, arXiv:1802.05586 [math-ph] (2018).
12. D. Krejčířík, *Calculation of the metric in the Hilbert space of a $\mathcal{P}\mathcal{T}$-symmetric model via the spectral theorem*, J. Phys. A: Math. Theor. 41 (2008), 244012.
13. D. Krejčířík, H. Bíla, and M. Znojil, *Closed formula for the metric in the Hilbert space of a $\mathcal{P}\mathcal{T}$-symmetric model*, J. Phys. A 39 (2006), 10143–10153.
14. D. Krejčířík, P. Siegl, and Železný, *On the similarity of Sturm-Liouville operators with non-Hermitian boundary conditions to self-adjoint and normal operators*, Complex Anal. Oper. Theory 8 (2014), 255–281.
15. F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne, *Quasi-Hermitian operators in quantum mechanics and the variational principle*, Ann. Phys 213 (1992), 74–101.