On the Asymptotic Dynamics of 2-D Magnetic Quantum Systems

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Abstract. In this work, we provide results on the long-time localization in space (dynamical localization) of certain two-dimensional magnetic quantum systems. The underlying Hamiltonian may have the form $H = H_0 + W$, where $H_0$ is rotationally symmetric and has dense point spectrum and $W$ is a perturbation that breaks the rotational symmetry. In the latter case, we also give estimates for the growth of the angular momentum operator in time.

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1. Introduction

Consider a charged quantum particle subject to a time-independent electromagnetic field profile. The system may be described through a self-adjoint operator $H$ with domain $\mathcal{D}(H)$ in a Hilbert space $\mathcal{H}$. If we assume that the system is initially in a state $\varphi \equiv \varphi(0) \in \mathcal{D}(H)$, then, according to the Schrödinger equation, the state of the system at time $t$, $\varphi(t)$, is given by $e^{-iHt/\hbar}\varphi$. (Here, $\hbar > 0$ is Planck’s constant divided by $2\pi$.) A fundamental question is whether the system remains localized for long times and, if not, what is the speed of the wavepackage spreading in terms of the electromagnetic field profile.

These phenomena can be investigated, for instance, by looking at the long-time behaviour of the expected radius of the state

$$\langle \varphi(t), |x| \varphi(t) \rangle, \quad \text{for} \quad t \gg 1. \quad (1)$$

This, in turn, can sometimes be estimated if one has information on the spectral quality of the underlying Hamiltonian (see, for example, [8, 14, 20]). Let us assume that the initial state belongs to a finite energy region $I \subset \mathbb{R}$ (with $|I| < \infty$) of $\mathcal{H}$, i.e. $\varphi = E_I(H)\varphi$, with $E_I(H)$ being the spectral projection of $H$ on $I$. Consider now the case when $H$ is a Schrödinger-type operator. Then, one can easily check that if the spectrum of the Hamiltonian in $I$ is a discrete set, then the system remains localized in the sense that

$$\sup_{t \in \mathbb{R}} \langle \varphi(t), |x| \varphi(t) \rangle \leq C \|\varphi\|^2,$$

for some constant $C > 0$. Moreover, for one-dimensional systems, it is known that if the spectrum is absolutely continuous the wavefunction spreading is ballistic in time average. More precisely, there is a constant $c > 0$ such that

$$\frac{1}{T} \int_0^T \langle \varphi(t), |x| \varphi(t) \rangle \geq cT, \quad T > 1.$$

(See [14] for this and more general results of this type.) However, if the spectrum is dense point or singular continuous there is very little one can say a priori (see [10]). Indeed, for $H$ having point spectrum, it is only known in general that the system exhibits a sub-ballistic dynamical behaviour [19], i.e.

$$\lim_{t \to \infty} \langle \varphi(t), |x| \varphi(t) \rangle/t = 0.$$

Moreover, there are examples of Hamiltonians with pure point spectrum where the spreading rate is arbitrarily close to ballistic [10], i.e. for any $\varepsilon > 0$

$$\limsup_{t \to \infty} \langle \varphi(t), |x| \varphi(t) \rangle/t^{1-\varepsilon} = \infty,$$

for a large class of initial data $\varphi$.

In the work at hand, we shed some more light on this problem for the cases, (a) when $H = H_0$ is the two-dimensional magnetic Schrödinger operator with a radially symmetric magnetic field $B$ and has dense point spectrum and (b) when $H = H_0 + W$, with $H_0$ as before and $W$ being an electric perturbation,
smooth in the angular variable, that decays at infinity. Our conditions include the cases when

\[ \int_0^r B(s) s^\sigma \, ds = \lambda r^{\sigma}, \quad \lambda > 0, \sigma \geq 1. \]

Using the same arguments as in [16], one can easily show that for \( \sigma \in (1, 2) \) the spectrum of \( H_0 \) is dense pure point. Moreover, if \( \sigma = 1 \) there is a mobility edge at energy \( \lambda^2 \), i.e. the spectrum is dense pure point on \( [0, \lambda^2) \) and purely absolutely continuous on \( (\lambda^2, \infty) \). It follows directly from Theorem 1.5 below that when \( \sigma \in (1, 2) \) the dynamics generated by \( H_0 \) is localized in time, provided the initial data are sufficiently smooth. Moreover, we show an analogous result for the case \( \sigma = 1 \), whenever \( \varphi = E_{[0, \lambda^2)}(H) \varphi \) (see Sect. 5). Similar results have been obtained for Dirac operators in [5].

The problem for \( \sigma \in (1, 2) \) becomes much more delicate if we turn on the electric perturbation \( W \). In this case, we do not even know the quality of the spectrum. Indeed, through the perturbation, certain spectral subspaces may cease to be pure point and continuous spectrum (presumably singular) may appear (see, for example, [8,9]). In this case, we provide estimates on the wave package spreading in terms of the decay rate of \( W \). In particular, we show that if \( W \) decays exponentially fast, then the expected radius of the system grows at most logarithmically fast in time. Moreover, if \( W = O(1/|x|^p) \) for some \( p > 2\sigma \), then \( \langle \varphi(t), |x| \varphi(t) \rangle \) grows at most as \( t^\theta \) with \( \theta = (p - \sigma)^{-1} < 1 \) (see Theorem 1.8, below).

In order to prove that, we show, on the one hand, that one can control the growth of the radius (1) in terms of the expected, time-dependent, angular momentum. (This is the actual content of Theorem 1.5.) On the other hand, in Theorem 1.7, we provide estimates on the growth in time of the angular momentum operator in terms of the decay rate of \( W \). As an essential tool, we use certain novel tunnelling estimates (see Theorem 1.3) which are in turn derived from fairly general exponential decay estimates for the spectral projections \( E_I(H) \) given in Theorem 3.1.

The above discussion roughly summarizes our main results. We emphasize that we are not aware of other localization bounds of this type in such situations (perturbed dense point spectrum) for deterministic systems. Notice, however, that the subject is frequently addressed in the realm of random Schrödinger operators. In these cases, the randomness of the potential plays a fundamental role in the proofs of localization (see, for example, [2,13]).

This paper is organized as follows: in the rest of this section, we describe precisely the model and state most of our main results. We show Theorems 1.5 and 1.7 in Sect. 2. In Sect. 3 we state and prove the exponential decay estimates for the spectral projections Theorem 3.1. Finally, in Sect. 4, we apply the latter theorem to the model at hand and show the tunnelling estimates stated in Theorem 1.3.
1.1. The Model and Main Results

Let us introduce the Hamiltonian $H_0$ of a quantum particle moving in $\mathbb{R}^2$ that is interacting with a magnetic field $\mathbf{B}$ pointing perpendicularly to the plane. We denote by $\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a magnetic vector potential associated to the magnetic field through $\mathbf{B} = (\partial_1 A_2 - \partial_2 A_1) \mathbf{\hat{x}}_3$. Throughout this work, we use units such that $\hbar = 2m = 1$, where $m$ is the mass of the particle. For $\mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$, we define the sesquilinear form

$$q_0(\varphi, \psi) = \int_{\mathbb{R}^2} (-i \nabla - \mathbf{A}(\mathbf{x})) \varphi(\mathbf{x})(-i \nabla - \mathbf{A}(\mathbf{x})) \psi(\mathbf{x}) d\mathbf{x}, \quad \varphi, \psi \in D(q_0),$$

with domain

$$D(q_0) = \{ \varphi \in L^2(\mathbb{R}^2) \mid (-i \partial_j - A_j) \varphi \in L^2(\mathbb{R}^2), j \in \{1, 2\} \}.$$

It is well known [18] that $C_0^\infty(\mathbb{R}^2)$ is a form core for $q_0$. We denote by $H_0$ the self-adjoint operator corresponding to $q_0$ and by $D(H_0) \subset D(q_0)$ its domain.

We are interested in the particular case in which $H_0$ describes the dynamics of a particle in a rotationally symmetric magnetic field $\mathbf{B}(\mathbf{x}) = B(|\mathbf{x}|) \mathbf{\hat{x}}_3$. We choose the Poincaré gauge where (here $r := |\mathbf{x}|$, as usual)

$$\mathbf{A}(\mathbf{x}) = A(r) \hat{\theta} = \frac{\Phi(r)}{r^2} \left( -\frac{x_2}{x_1} \right) \quad \text{with} \quad \Phi(r) = A(r)r = \int_0^r B(s) s \, ds. \quad (4)$$

We will show that this choice of vector potential is locally square integrable, whenever the magnetic field is, see Lemma A.1 in the appendix. Notice that $\Phi(r)$ is, up to factor of $2\pi$, the magnetic flux through a disc of radius $r > 0$ centred at the origin.

One can show, see the discussion in “Appendix A”, that the quadratic form $q_0$ corresponding to $H_0$ is given by

$$q_0(\varphi, \psi) = \langle \partial_r \varphi, \partial_r \psi \rangle + \left\langle \frac{1}{r}(\Phi - L) \varphi, \frac{1}{r}(\Phi - L) \psi \right\rangle, \quad (5)$$

for all $\varphi, \psi \in D(q_0)$ since the magnetic flux $\Phi$ is radial. Here, $\partial_r = \frac{x}{|x|} \cdot \nabla$ is the radial derivative and $L = -i(x_1 \partial_2 - x_2 \partial_1)$ is the generator of rotations in $\mathbb{R}^2$.

We change variables by identifying the underlying Hilbert space $L^2(\mathbb{R}^2)$ with $\mathcal{H} := L^2(\mathbb{R}^+ \times S^1, r dr d\theta)$ through the unitary operator

$$U : L^2(\mathbb{R}^2) \rightarrow \mathcal{H}, \quad (6)$$

with $\varphi \mapsto U \varphi = \tilde{\varphi}$, where $\tilde{\varphi}(r, \theta) = \varphi(r \cos \theta, r \sin \theta)$, we define the self-adjoint angular momentum operator $J = ULU^{-1}$. It is easy to see that

$$J \tilde{\varphi} := -i \frac{\partial}{\partial \theta} \tilde{\varphi} \quad (7)$$

when $\tilde{\varphi} \in D(J) \cong L^2(\mathbb{R}^+, r dr) \otimes H^1(S^1)$. In this coordinates we have, for any $\varphi, \psi \in D(q_0) = \mathcal{Q}(H_0),

$$q_0(\varphi, \psi) = \left\langle \partial_r \tilde{\varphi}, \partial_r \tilde{\psi} \right\rangle_{\mathcal{H}} + \left\langle \frac{1}{r}(\Phi - J) \tilde{\varphi}, \frac{1}{r}(\Phi - J) \tilde{\psi} \right\rangle_{\mathcal{H}}. \quad (8)$$
where \( \tilde{\varphi} = U\varphi \) and \( \tilde{\psi} = U\psi \).

Notice that the spectrum of \( J \) coincides with the set of integers \( \mathbb{Z} \). In addition, the orthogonal projection onto the subspace of \( \mathcal{H} \) with constant angular momentum \( j \in \mathbb{Z}, P_j \) admits the representation

\[
(P_j \psi)(r, \theta) = (2\pi)^{-1/2} \psi_j(r)e^{ij\theta}, \quad r > 0, \ \theta \in [0, 2\pi),
\]

where \( (\psi_j)_{j \in \mathbb{Z}} \in \bigoplus_{j \in \mathbb{Z}} L^2(\mathbb{R}^+, rdr) \) is the Fourier series of \( \psi \in \mathcal{H} \), as defined in (99). If \( \varphi \in L^2(\mathbb{R}^2) \), we will simply write \( \varphi_j \) for \( (U\varphi)_j \). In this setting, the family \( (P_j)_{j \in \mathbb{Z}} \) decomposes \( L^2(\mathbb{R}^2) \) into blocks of constant angular momentum that diagonalize \( q_0 \). More precisely, for any \( \varphi, \psi \in D(q_0) \) we have

\[
q_0(\varphi, \psi) = \sum_{j \in \mathbb{Z}} \left( \left\langle \partial_r \varphi, \partial_r \psi \right\rangle_{L^2(\mathbb{R}^+, rdr)} + \left\langle \frac{1}{r} (\Phi - j) \varphi, \frac{1}{r} (\Phi - j) \psi \right\rangle_{L^2(\mathbb{R}^+, rdr)} \right) \\
= \sum_{j \in \mathbb{Z}} \left( \left\langle \partial_r \varphi_j, \partial_r \psi_j \right\rangle_{L^2(\mathbb{R}^+, rdr)} + \left\langle \varphi_j, V_j \psi_j \right\rangle_{L^2(\mathbb{R}^+, rdr)} \right) \\
= : \langle \partial_r \varphi, \partial_r \psi \rangle + \langle \varphi, V \psi \rangle,
\]

where \( V_j(r) := \frac{1}{r^2} (\Phi(r) - j)^2 \) is the effective potential. In what follows, we will identify \( J \) and \( L \) since they only differ by a change of variables.

We consider electric perturbations of \( H_0 \) through a potential \( W \) which is not necessarily rotationally symmetric but satisfies the following smoothness condition in the angular variable.

**Condition 1.** There are constants \( a > 0, 0 < \zeta \leq 1 \), and a function \( v \in L^2(\mathbb{R}^2) + L^{\infty}(\mathbb{R}^2) \) such that for all \( j \in \mathbb{Z} \) and almost every \( r > 0 \)

\[
\left| \hat{W}(r, j) \right| \leq b(r)e^{-a|j|^\zeta},
\]

where \( b(|x|) = v(x) \) for \( x \in \mathbb{R}^2 \) and, for \( j \in \mathbb{Z}, \)

\[
\hat{W}(r, j) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} W(r, \theta)e^{-ij\theta} d\theta, \quad \text{for a.e. } r > 0.
\]

is the Fourier transform of the potential \( W \) in the angular variable.

**Remark 1.1.** (i) The above condition is an analyticity condition in terms of the Gevrey scale in the angular variable \( \theta \). In particular, if \( \zeta = 1 \), the above condition is precisely the analyticity of \( W \) in \( \theta \), for almost every \( r > 0 \), which is also clear from the familiar Paley–Wiener theorem. Such an analyticity condition is not unusual, see [11,17].

(ii) In view of the diamagnetic inequality, see Theorems 2.4 and 2.5 from [3], we see that \( W \) is infinitesimally \( H_0 \)-bounded in the operator and form sense. In particular, we have that for any \( \varepsilon > 0 \) there exist constants \( C(\varepsilon) > 0 \) such that

\[
\|W \varphi\|^2 \leq \varepsilon \|H_0 \varphi\|^2 + C(\varepsilon) \|\varphi\|^2, \quad \varphi \in D(H_0)
\]

and

\[
\left| \int_{\mathbb{R}^2} W(x) |\varphi(x)|^2 dx \right| \leq \varepsilon q_0[\varphi] + C(\varepsilon) \|\varphi\|^2, \quad \varphi \in D(q_0).
\]
(iii) Notice that the diamagnetic inequality (form bounded with respect to the nonmagnetic kinetic energy implies the same for the magnetic kinetic energy) in this gauge, is an easy consequence of our analysis of the magnetic quadratic form, see “Appendix A”: Lemma A.5 and Remark A.6.

In the work at hand, we study the dynamics of a quantum particle governed by the Hamiltonian

$$H \phi := H_0 \phi + W \phi, \quad \phi \in \mathcal{D}(H) = \mathcal{D}(H_0). \quad (14)$$

In view of Remark 1.1.ii and the Kato-Rellich theorem, the operator $H$ is bounded from below and self-adjoint.

For $\phi, \psi$ in the form domain of $W$, $\mathcal{Q}(W)$, one checks that as quadratic forms

$$\langle \phi, W \psi \rangle = \sum_{j, k \in \mathbb{Z}} \left\langle \phi_j, \hat{W}(\cdot, j - k) \psi_k \right\rangle_{L^2(\mathbb{R}_+, r dr)} \quad (15)$$

and thus the quadratic form of the magnetic Schrödinger operator $H = H_0 + W$ is given by

$$q(\phi, \psi) = \langle \phi, H \psi \rangle = \sum_{j \in \mathbb{Z}} \left( \langle \partial_r \phi_j, \partial_r \psi_j \rangle_{L^2(\mathbb{R}_+, r dr)} + \langle \phi_j, V_j \psi_j \rangle_{L^2(\mathbb{R}_+, r dr)} \right) + \sum_{j, k \in \mathbb{Z}} \left\langle \phi_j, \hat{W}(\cdot, j - k) \psi_k \right\rangle_{L^2(\mathbb{R}_+, r dr)} = \langle \partial_r \phi, \partial_r \psi \rangle + \langle \phi, V \psi \rangle + \langle \phi, W \psi \rangle. \quad (16)$$

see also the discussion in “Appendix A” for more details.

In order to state our results in a concise way, we will work separately with the following two conditions on the magnetic flux $\Phi$ given in (4).

**Condition 2.** Let $\Phi \in L^2_{\text{loc}}(\mathbb{R}^+, dr/r)$ such that there are constants $\lambda_+ < \infty$ and $\sigma_+ > 1$ such that

$$|\Phi(r)| \leq \lambda_+ (1 + r^{\sigma_+}), \quad (17)$$

for all $r > 0$.

**Condition 3.** Let $\Phi \in L^2_{\text{loc}}(\mathbb{R}^+, dr/r)$ be such that there are constants $r_0 > 1$, $\lambda_0 > 0$ and $\sigma_- > 1$ such that

$$|\Phi(r)| \geq \lambda_0 r^{-\sigma_-} \quad r \geq r_0. \quad (18)$$

**Remark 1.2.** It is instructive to consider the model case where the magnetic field is bounded and asymptotically decays to zero as

$$B(r) \sim r^{-\alpha}, \quad r \to \infty, \quad (19)$$

for some $\alpha < 1$. Here, Conditions 2 and 3 are fulfilled with $\sigma_+ = \sigma_- = 2 - \alpha$. 
Our results concern the time evolution of a state $\varphi$ with energy on a bounded interval $I \subset \mathbb{R}$. In order to state them, we introduce some notation that will be used throughout this work. Let $e_0 := \inf \text{spec}(H)$, be the infimum of the spectrum of $H$ and let $E_0 \in (e_0, \infty)$ be a fixed constant. We set

$$I := [e_0, E_0]$$

and denote by $E_I(H)$ the spectral projection of $H$ onto the interval $I$. Let $U(t) := e^{-itH}$ be the time evolution operator associated to $H$. For any initial state $\varphi \in L^2(\mathbb{R}^2)$ we denote by $\varphi(t) := U(t)\varphi$ the state of the system at time $t$.

We are now ready to state our main results.

Theorem 1.3. Assume that Condition 1 is satisfied.

(i) (Interior tunnelling estimates). Under Condition 2 there exist constants $c_+ \in (0, 1]$ and $\delta_+ > 0$ such that

$$\sum_{j \in \mathbb{Z}} e^{\delta_+ |j|^\zeta} \| \mathbb{1}_{[0, c_+ |j|^\zeta/(\sigma_+ + 1)}(|x|) P_j E_I(H) \|^2 < \infty. \quad (20)$$

(ii) (Exterior tunnelling estimates). Under Condition 3 there exist constants $c_- \geq r_0$ and $\delta_- > 0$ such that

$$\sum_{j \in \mathbb{Z}} \| \mathbb{1}_{[c_- |j|^\zeta/(\sigma_-), \infty)}(|x|) e^{\delta_- |x|^\zeta_\sigma -} P_j E_I(H) \|^2 < \infty. \quad (21)$$

Here, $\mathbb{1}_A$ stands for the indicator function of the set $A$ and $\sigma_-$ and $\sigma_+$ are the parameters given in Condition 2 and 3.

Remark 1.4. (i) The interior and exterior tunnelling bounds above show strong decay of the spectral projection $E_I(H)$ for finite energy intervals $I$ into the classically forbidden region. They are derived from the exponential decay of the energy projections described in Sect. 3. Remarkably, these bounds are valid in a regime, where the unperturbed operator $H_0$ has dense point spectrum.

(ii) The bound (20) also immediately implies that there exists a positive constant $C$ such that

$$\| \mathbb{1}_{[0, c_+ |j|^\zeta/(\sigma_+ + 1)}(|x|) P_j E_I(H) \| \leq C e^{-\delta_+ |j|^\zeta}, \quad j \in \mathbb{Z}. \quad (22)$$

(iii) Consider the example of Remark 1.2. Theorem 1.3 indicates that a wavefunction with energies in the interval $I$ and angular momentum $j \in \mathbb{Z}$, $P_j E_I(H)\varphi$, is essentially localized in the annulus between $c_+ |j|^{1/(2-\alpha)}$ and $c_- |j|^{1/(2-\alpha)}$. This is in fact the scale where the classically allowed region (see (36) below) for $P_j E_I(H)\varphi$ is located. For further details, see Sect. 4, where the proof of Theorem 1.3 is given.

Our next theorem states that the expectation of $|x|$ in time is dominated by the expectation of the angular momentum operator in time to certain power. This power depends on the behaviour of the magnetic flux far from the origin, see Condition 3.
**Theorem 1.5.** \((J(t) \text{ controls } x(t))\) Let \(H\) be the Hamiltonian defined in (14) and assume that Conditions 1 and 3 are satisfied. Then, for any \(\nu > 0\), there exists a constant \(C > 0\) such that for all initial states \(\varphi \in E_I(H)L^2(\mathbb{R}^2)\)

\[
\langle \varphi(t), |x|^{\nu} \varphi(t) \rangle \leq C \left( \| \varphi \|^2 + \left\langle \varphi(t), |J|^{\nu/\sigma} \varphi(t) \right\rangle \right), \quad \text{for } t \in \mathbb{R}. \tag{23}
\]

**Remark 1.6.** (i) Assume further that \(W\) is rotationally symmetric. Then, the time evolution \(U(t)\) commutes with \(|J|\) (i.e. angular momentum is conserved); hence, Theorem 1.5 implies dynamical localization for any \(\varphi \in D(|J|^{\nu/(2\sigma)})\), i.e.

\[
\sup_{t \geq 0} \langle \varphi(t), |x|^{\nu} \varphi(t) \rangle < \infty.
\]

In order to formulate the next theorem, we define the symmetric and non-symmetric parts of the potential \(W\) by writing

\[
W = W_s + W_{ns},
\]

where \(W_s\) is the radially symmetric part of \(W\) given, for almost all \(r > 0\), by

\[
W_s(r) := \frac{1}{2\pi} \int_0^{2\pi} W(r, \theta) d\theta.
\]

If we assume some further decay in space for \(W_{ns}\), we obtain bounds for the expectation of the angular momentum in time. We state our results for two different classes of decay of \(W_{ns}\).

**Theorem 1.7 (Bounds on \(J(t)\)).** Assume that Conditions 1 and 2 are satisfied.

(i) Suppose that for \(p > \sigma_+ / \zeta\)

\[
W_{ns}(x) = O \left( \frac{1}{|x|^p} \right), \quad |x| \to \infty. \tag{24}
\]

Then, for any \(0 < \beta < (\zeta p - \sigma_+)/\sigma_+\) there exists \(C > 0\) such that for all initial states \(\varphi \in E_I(H)L^2(\mathbb{R}^2)\) we have

\[
\| |J|^{\beta/2} \varphi(t) \|^2 \leq \| |J|^{\beta/2} \varphi \|^2 + C t^{\gamma \beta} \| \varphi \|^2, \quad t > 1, \tag{25}
\]

where \(\gamma = \frac{\sigma_+}{\zeta p - \sigma_+}\).

(ii) Suppose that there exists \(\mu > 0\) and \(s > 0\) such that

\[
W_{ns}(x) = O \left( \exp(-\mu|x|^s) \right), \quad |x| \to \infty. \tag{26}
\]

Then, for any \(\beta > 0\), there exists \(C > 0\) such that for all initial states \(\varphi \in E_I(H)L^2(\mathbb{R}^2)\) we have

\[
\| |J|^{\beta/2} \varphi(t) \|^2 \leq \| |J|^{\beta/2} \varphi \|^2 + C (\ln(t)^{\theta \beta} + 1) \| \varphi \|^2, \quad t > 1, \tag{27}
\]

where \(\theta = 1/ \min\{\zeta, \zeta s / \sigma_+\}\).

The proofs of Theorems 1.5 and 1.7 are given in Sect. 2. Let us emphasize that while Theorem 1.5 uses Condition 3 through the exterior tunnelling estimate (21), Theorem 1.7 requires Condition 2 in order to apply (20). The next result is a direct combination of Theorems 1.5 and 1.7.
Theorem 1.8 (Bounds on $x(t)$). Assume that Conditions 1 and 2, and 3 are satisfied with $0 < \zeta \leq 1$, $1 < \sigma_- \leq \sigma_+$. Then,

(i) Assume that $W_{ns}(\mathbf{x}) = \mathcal{O}(\frac{1}{|\mathbf{x}|^p})$, as $|\mathbf{x}| \to \infty$, for some $p > \sigma_+ / \zeta$. Then, for any $0 < \nu < \frac{\sigma_-}{\zeta}(\frac{p - \sigma_+}{\sigma_+})$, there exists $C > 0$ such that, for all $\varphi \in E_1(H)L^2(\mathbb{R}^2)$,

$$\langle \varphi(t), |x|^{\nu} \varphi(t) \rangle \leq C \left( \langle \varphi, |J|^{\zeta^s / \sigma_-} \varphi \rangle + t^{\varepsilon_p} \| \varphi \|^2 \right), \quad t > 1. \quad (28)$$

where $\varepsilon_p = \frac{\zeta}{\sigma_-} \left( \frac{\sigma_+}{\zeta} \right) \nu < 1$.

(ii) Suppose that there exists $\mu > 0$ and $s > 0$ such that $W_{ns}(\mathbf{x}) = \mathcal{O}(\exp(-\mu |\mathbf{x}|^s))$, as $|\mathbf{x}| \to \infty$. Then, for any $\nu > 0$, there exists $C > 0$ such that, for all $\varphi \in E_1(H)L^2(\mathbb{R}^2)$,

$$\langle \varphi(t), |x|^{\nu} \varphi(t) \rangle \leq C \left( \langle \varphi, |J|^{\zeta^s / \sigma_-} \varphi \rangle + \ln(t)^{\theta_s} \| \varphi \|^2 \right), \quad t > 1, \quad (29)$$

where $\theta_s = \frac{1}{\sigma_- \min\{1,s/\sigma_+\}} \nu$.

2. From Tunnelling Estimates to Dynamical Bounds

We start with the proof of Theorem 1.7 and consider Theorem 1.5 at the end of this section. The proof of Theorem 1.7 is based upon certain dynamical bounds, that use Heisenberg’s equation for $J(t)$, combined with the tunnelling estimates given in Remark 1.4.ii. Before proceeding with the proof of Theorem 1.7, we establish the following.

Lemma 2.1. Assume that Conditions 1 and 2 are satisfied. Then, there is a constant $C_{ns} \in (0, \infty)$ such that

$$\|E_1(H)W_{ns}\| \leq C_{ns}.$$

Proof. In view of Eq. (11), for $j = 0$, and Remark 1.1.ii we see that the inequality (12) holds for $W_{ns}$ as well as for $W_s$. This implies that $\mathcal{D}(W_{ns}) \supseteq \mathcal{D}(H_0) = \mathcal{D}(H)$. By the Closed Graph Theorem, we have that $W_{ns}(H + \lambda)^{-1}$ is a bounded operator, for some $\lambda > -\epsilon_0$. We conclude by observing that

$$\|W_{ns}E_1(H)\| \leq \|W_{ns}(H + \lambda)^{-1}\| \|(H + \lambda)E_1(H)\| < \infty.$$

Proof of Theorem 1.7. For any $\varphi \in E_1(H)L^2(\mathbb{R}^2)$ and $M > 0$ we have

$$\| |J|^{\beta / 2} \varphi(t) \|^2 \leq M^\beta \| \varphi \|^2 + \sum_{|j| > M} |j|^\beta \| P_j \varphi(t) \|^2. \quad (30)$$

Using Heisenberg’s evolution equation, we get

$$\| P_j \varphi(t) \|^2 = \|P_j \varphi\|^2 + \int_0^t \langle \varphi(s), [W, P_j] \varphi(s) \rangle ds.$$

Notice that $[W, P_j] = [W_{ns}, P_j]$. The above equation combined with (30) yields

$$\| |J|^{\beta / 2} \varphi(t) \|^2 \leq M^\beta \| \varphi \|^2 + \| |J|^{\beta / 2} \varphi \|^2 + \sum_{|j| > M} |j|^\beta \int_0^t | \langle \varphi(s), [W_{ns}, P_j] \varphi(s) \rangle | ds$$
\[ \leq M^\beta \|\varphi\|^2 + \|J|^{\beta/2}\varphi\|^2 + 2 \sum_{|j| > M} |j|^\beta \int_0^t |\langle \varphi(s), W_{ns}P_j \varphi(s) \rangle| \, ds \]
\[ \leq M^\beta \|\varphi\|^2 + \|J|^{\beta/2}\varphi\|^2 + 2t \sum_{|j| > M} |j|^\beta \|E_I(H)W_{ns}P_j E_I(H)\| \|\varphi\|^2. \]

(31)

Using Lemma 2.1, we can estimate the norm appearing in the last sum as
\[ \|E_I(H)W_{ns}P_j E_I(H)\| \]
\[ \leq \left\| E_I(H)W_{ns} \mathbb{1}_{(0,c_+ |j| \zeta/\sigma_+)}(|x|) P_j E_I(H) \right\| + \left\| E_I(H)W_{ns} \mathbb{1}_{(c_+ |j| \zeta/\sigma_+, \infty)}(|x|) P_j E_I(H) \right\| \]
\[ \leq \|E_I(H)W_{ns}\| \left\| \mathbb{1}_{(0,c_+ |j| \zeta/\sigma_+)}(|x|) P_j E_I(H) \right\| + \left\| W_{ns} \mathbb{1}_{(c_+ |j| \zeta/\sigma_+, \infty)}(|x|) \right\| \]
\[ \leq C \|E_I(H)W_{ns}\| e^{-\delta |j| \zeta} + \left\| W_{ns} \mathbb{1}_{(c_+ |j| \zeta/\sigma_+, \infty)}(|x|) \right\|. \]

(32)

where in the last step we used Remark 1.4.ii.

We now study separately the two cases depending on the decay rate of the potential.

Case (i) Assume that \( W_{ns} \) decays as in (24). Then, for all sufficiently large \( M > 0 \)
\[ \left\| W_{ns} \mathbb{1}_{(c_+ |j| \zeta/\sigma_+, \infty)}(|x|) \right\| \lesssim |j|^{-s_\zeta/\sigma_+}, \quad |j| > M. \]

(33)

Combining this bound with (32), using that \( e^{-\delta |j| \zeta} \leq |j|^{-s_\zeta/\sigma_+} \) for large \( |j| \) this implies
\[ \sum_{|j| > M} |j|^\beta \|E_I(H)W_{ns}P_j E_I(H)\| \lesssim \sum_{|j| > M} |j|^\beta e^{-s_\zeta/\sigma_+} \lesssim M^{\beta+1-s_\zeta/\sigma_+}. \]

(34)

Since \( \beta - s_\zeta/\sigma_+ < -1 \). This together with (31) implies that there exists a constant \( C > 0 \) such that
\[ \left\| |J|^{\beta/2}\varphi(t)\right\|^2 \leq \left\| |J|^{\beta/2}\varphi\right\|^2 + M^\beta \|\varphi\|^2 + CtM^{\beta+1-s_\zeta/\sigma_+} \|\varphi\|^2. \]

The theorem holds provided we pick \( M(t) = t^{\frac{s_\zeta/\sigma_+}{s_\zeta/\sigma_+}} \).

Case (ii) We now turn to the case when \( W_{ns} \) decays as in (26). Then, analogously as above we conclude for all \( |j| \) large enough
\[ \|E_I(H)W_{ns}P_j E_I(H)\| \lesssim \exp(-\delta |j| \zeta) + \exp(-\mu(c_+)^s |j|^{s_\zeta/\sigma_+}) \lesssim \exp(-\eta |j| \zeta), \]
for \( \kappa := \min\{\zeta, s_\zeta/\sigma_+\} \) and \( \eta = \min\{\delta, \mu(c_+)^s\} > 0 \). Hence,
\[ \sum_{|j| > M} |j|^\beta \|E_I(H)W_{ns}P_j E_I(H)\| \lesssim \exp\left(-\frac{\eta}{2}M^\kappa\right). \]

(35)

for all large enough \( M > 0 \). This together with (31) yields
\[ \left\| |J|^{\beta/2}\varphi(t)\right\|^2 \leq \left\| |J|^{\beta/2}\varphi\right\|^2 + M^\beta \|\varphi\|^2 + 2Ct \exp\left(-\frac{\eta}{2}M^\kappa\right) \|\varphi\|^2. \]
for some constant $C > 0$. Finally, the claim follows by picking $M(t) = (2\ln(t)/\eta)^{1/\kappa}$.

**Remark 2.2.** The proof of Theorem 1.5 is a rigorous implementation of the intuition that the component of the wavefunction with angular momentum $j$, $P_j \varphi$, moves under the influence of an effective potential $V_j$ whose classical region $\{ x \in \mathbb{R}^2 : V_j(x) \leq E \}$ is concentrated inside an annular region of inner radius $\sim |j|^{1/\sigma_+}$ and outer radius $\sim |j|^{1/\sigma_-}$ (see Theorem 1.3 and Remark 1.4.iii).

**Proof of Theorem 1.5.** Let $t \geq 0$ and consider the following splitting

$$
\| |x|^{\nu/2} \varphi(t) \|^2 = \sum_{j \in \mathbb{Z}} \left\| \mathbb{1}_{[0, c_- |j|^{\zeta/\sigma_-}]} |x|^{\nu/2} P_j E_I(H) \varphi(t) \right\|^2 + \sum_{j \in \mathbb{Z}} \left\| \mathbb{1}_{[c_- |j|^{\zeta/\sigma_-}, \infty)} |x|^{\nu/2} P_j E_I(H) \varphi(t) \right\|^2
$$

where $c_-$ is the constant given by Theorem 1.3. For notational simplicity, we will drop the argument in the indicator function and simply write $\mathbb{1}_A \equiv \mathbb{1}_A(|x|)$ in the following. The first sum may be estimated in terms of the expectation values of the angular momentum as

$$
\sum_{j \in \mathbb{Z}} \left\| \mathbb{1}_{[0, c_- |j|^{\zeta/\sigma_-}]} |x|^{\nu/2} P_j E_I(H) \varphi(t) \right\|^2 \leq c_\nu \| |J|^{\nu \zeta/(2 \sigma_-)} \varphi(t) \|^2.
$$

As for the second sum, we may use Eq. (21) from Theorem 1.3 to obtain

$$
\sum_{j \in \mathbb{Z}} \left\| \mathbb{1}_{[c_- |j|^{\zeta/\sigma_-}, \infty)} |x|^{\nu/2} P_j E_I(H) \varphi(t) \right\|^2 \leq \| |x|^{\nu/2} e^{-\delta |x|^{\zeta/\sigma_-}} \| \sum_{j \in \mathbb{Z}} \| \mathbb{1}_{[c_- |j|^{\zeta/\sigma_-}, \infty)} e^{\delta |x|^{\zeta/\sigma_-}} P_j E_I(H) \|^2 \| \varphi \|^2.
$$

This completes the proof of the theorem. □

### 3. Exponential Decay of the Energy Projections

An essential tool in our approach is certain exponential decay estimates for the spectral projections $E_I(H)$ in the variables $j$ and $r$. They enable us to control the tunneling effect away from the classically allowed region. Note that since the perturbation $W$ is not rotationally symmetric, $H$ and $J$ cannot be simultaneously diagonalized. This combined with the fact that the unperturbed operator has dense point spectrum makes such decay bounds trickier to deal with.

It turns out that it suffices to work with the classical region associated to the unperturbed magnetic operator $H_0$, which is well defined for fixed angular momentum $j$. The analysis in this section is given for general magnetic fields which are rotationally symmetric such that the magnetic vector potential potential in the Poincaré gauge is locally square integrable.
For a given energy $E > 0$ and fixed $j \in \mathbb{Z}$, we define the \textit{classically allowed region} for angular momentum $j \in \mathbb{Z}$ as the set
\[ C_j(E) := \{ r \in \mathbb{R}^+ : V_j(r) \leq E \} \tag{36} \]
where $V = (V_j)_{j \in \mathbb{Z}}$ is the effective potential. Because of the fundamental role of the above sets in our analysis, we will use a distinctive notation for their characteristic functions. Namely, we let
\[ \chi_j(E) := 1_{C_j(E)} : \mathbb{R}^+ \to [0, 1] \]
together with its complement defined as $\chi_j^+(E) := 1 - \chi_j(E)$. Upcoming conditions will be stated in terms of the orthogonal projection $\chi(E) := \sum_{j \in \mathbb{Z}} \chi_j(E) P_j$.

In order to quantify the exponential decay of wave functions away from the classical regions $C_j(E)$, let us introduce appropriate spaces for the exponential weights. Let $F = (F_j)_{j \in \mathbb{Z}}$ be a sequence of measurable functions $F_j : \mathbb{R}^+ \to \mathbb{R}$ and consider the linear operator $e^F = \sum_{j \in \mathbb{Z}} e^{F_j} P_j$, defined on its maximal domain $\mathcal{D}(e^F)$. We denote by $PC^1(\mathbb{R}^+, \mathbb{R})$ the set of non-negative continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ that are piecewise continuously differentiable. We say that the sequence $F = (F_j)_{j \in \mathbb{Z}}$ is in $PC^1(\mathbb{R}^+, \mathbb{R})$ whenever $F_j \in PC^1(\mathbb{R}^+, \mathbb{R})$ for all $j \in \mathbb{Z}$. If, additionally, the norm
\[ \|F\|_\infty := \sup_{j \in \mathbb{Z}} \sup_{r > 0} |F_j(r)| \tag{37} \]
is finite, we say that $F = (F_j)_{j \in \mathbb{Z}}$ is in $PC^1_{bd}(\mathbb{R}^+, \mathbb{R})$. In this case, $e^{\pm F}$ are bounded and invertible operators satisfying $\|e^{\pm F}\| \leq e^{\|F\|_\infty}$. We prove in Lemma B.1 that $e^{\pm F}$ leave the quadratic form domain $\mathcal{D}(g_0)$ invariant, provided $|F_j|^2$ is $g_0$-bounded.

Finally, recall the constants $a > 0$ and $\zeta \in (0, 1]$ defined through Condition 1 and write $\xi(a, \zeta) = \sum_{m \in \mathbb{Z}} e^{-\frac{a}{2}|m|^\zeta}$. Since $v$ is a radial potential, Lemma A.5 in “Appendix A” shows that for any $a > 0$ and $0 < \zeta \leq 1$ there exists a constant $c_0 > 0$, used in the formulation of the upcoming theorem, such that
\[ \langle \partial_r \varphi, \partial_r \varphi \rangle - \xi(a, \zeta) \langle \varphi, v \varphi \rangle \geq -c_0 \|\varphi\|^2, \tag{38} \]
for all $\varphi$ in the quadratic form domain of the magnetic Schrödinger operator.

**Theorem 3.1** (Exponential decay of energy projections). \textit{Let $H$ be the perturbed magnetic Schrödinger operator defined by the quadratic form (16). Given $\delta_0 > 0$ and $E_0 \geq 0$ put}
\[ \tilde{E} := E_0 + c_0 + \delta_0, \tag{39} \]
\textit{with $c_0$ from (38) above. Then, for any sequence of weight functions $F = (F_j)_{j \in \mathbb{Z}}$ in $PC^1(\mathbb{R}^+, \mathbb{R})$ that satisfy}
\[ (F')^2 \leq V - \tilde{E} \chi^+(\tilde{E}), \tag{40} \]
\[ \|e^{F} \chi(\tilde{E})\|_\infty < \infty, \tag{41} \]
\[ \sup_{r > 0} |F_j(r) - F_k(r)| \leq \frac{a}{2} |j - k|^{\zeta}, \quad \text{for all } j, k \in \mathbb{Z}, \tag{42} \]
\textit{there exists a constant $C > 0$ such that}
\[ \|e^{F} \chi(\tilde{E})\|_\infty < C. \]
there exists $C = C(\delta_0) > 0$ such that
\[
\|e^F E_I(H)\| \leq C \|e^F \chi(\tilde{E})\|_\infty. \tag{43}
\]
If $W = 0$ the bound (43) holds without the requirement of (42).

Remark 3.2. (i) Note that the right-hand side of (43) stays finite as long as $e^F$ is bounded on $\text{supp}(\chi(\tilde{E})) = \bigcup_{j \in \mathbb{Z}} \text{supp}(\chi_j(\tilde{E}))$. Thus, we may approximate unbounded weight functions $F$, which may grow in both variables $r > 0$ and $j \in \mathbb{Z}$, by bounded ones and deduce from (43) that $\|e^F E_I(H)\|$ is finite as long as $F$ is bounded on the support of $\chi(\tilde{E})$. In particular, we will choose a family $(F_j)$ such that it provides exponential decay estimates of $E_I(H)$ away from the classical regions $C_j(\tilde{E})$. The difference $\tilde{E} - E_0 > 0$ is a price we pay for defining the classical region with respect to the operator $H_0$ instead of $H$.

(ii) In the case when $W = 0$ we may choose $c_0 = 0$ and we have almost optimality in the energy $\tilde{E} = E_0 + \delta_0$.

(iii) Our proof of Theorem 3.1 borrows ideas from [12] which are, in turn, inspired by the proof of exponential decay in QED systems given in [4] and the beautiful approach to exponential bounds for eigenfunctions in [1].

As a first step, we define a more convenient (smooth) version of the spectral projection $E_I$, denoted by $g_\Delta(H)$, which is given in formula (47) below. The constants $\tilde{E}, E_0, \delta_0$ and $c_0$ are as in Theorem 3.1.

Set $\Delta := [e_0 - \delta_0/2, E_0 + \delta_0/2]$, where $e_0 = \inf \sigma(H)$. Consider $g_\Delta \in C^{\infty}_0([0, 1])$ such that $\text{supp} g_\Delta \subseteq \Delta$ and $g_\Delta|_{[0, 1]} = 1$. Since $E_I(H) = g_\Delta(H) E_I(H)$ it is enough to prove the bound (43) with $E_I(H)$ replaced by $g_\Delta(H)$.

Next, we use the almost analytic functional calculus (see [7]) to write $g_\Delta(H)$ in terms of an integral over the resolvent of $H$. We denote by $\tilde{g}_\Delta$ an almost analytic extension of $g_\Delta$ with the property that $\text{supp}(\tilde{g}_\Delta)$ is a compact subset of $\Delta + i\mathbb{R}$ and
\[
|\partial z \tilde{g}_\Delta(z)| = O(|\text{Im}(z)|), \quad \text{Im}(z) \to 0. \tag{44}
\]
One can give a straightforward explicit construction of $\tilde{g}_\Delta$ with the above properties, however, see [7,12] for details. The following formula, known in the literature as the Helffer–Sjöstrand formula, then holds for any self-adjoint operator $H$
\[
g_\Delta(H) = -\frac{1}{\pi} \int_{\mathbb{C}} (z - H)^{-1} \partial z \tilde{g}_\Delta(z) \, dx \, dy \tag{45}
\]
where we have denoted $z = (x, y) \in \mathbb{C}$. We work with the comparison operator, more precisely, with the quadratic form corresponding to
\[
\tilde{H} := H + \tilde{E} \chi(\tilde{E}),
\]
Notice that $\tilde{H}$ is just $H$ except that it is boosted by $\tilde{E}$ in (a neighbourhood of) the classical region, i.e. as quadratic forms

$$\langle \varphi, \tilde{H} \varphi \rangle = \langle \varphi, \tilde{V} \varphi \rangle + \langle \varphi, (V + \tilde{E} \chi(\tilde{E})) \varphi \rangle + \langle \varphi, W \varphi \rangle$$

(46)

for all $\varphi \in \mathcal{D}(q_0)$

It is easy to check, see Lemma 3.3 below, that $g_\Delta(\tilde{H}) = 0$ and, therefore, by the resolvent identity

$$g_\Delta(H) = g_\Delta(H) - g_\Delta(\tilde{H}) = -\frac{1}{\pi} \int_{C}(z - \tilde{H})^{-1}(\tilde{H} - H)(z - H)^{-1} \frac{\partial g_\Delta}{\partial z} \, dx \, dy.$$  

(47)

Lemma 3.3. For the operator $\tilde{H}$ defined above, one has a quadratic form inequality

$$\tilde{H} \succeq E_0 + \delta_0.$$  

(48)

Furthermore, if the sequence of non-negative functions $F = (F_j)_{j \in \mathbb{Z}}$ in $PC_{bd}(\mathbb{R}^+, \mathbb{R})$ satisfies (40) and (42), then

$$\sup_{z \in \text{supp}(\tilde{g}_\Delta)} \|e^F(z - \tilde{H})^{-1}e^{-F}\| \leq 2/\delta_0.$$  

(49)

where $\delta_0 > 0$ is the fixed parameter from Theorem 3.1.

Proof. In order to show (48), notice that from (46) we get

$$\langle \varphi, \tilde{H} \varphi \rangle = \langle \varphi, V \varphi \rangle + \langle \varphi, (V + \tilde{E} \chi(\tilde{E})) \varphi \rangle + \langle \varphi, W \varphi \rangle$$

$$\geq \langle \varphi, V \varphi \rangle + \langle \varphi, (V + \tilde{E} \chi(\tilde{E})) \varphi \rangle - \xi(2a, \zeta) \langle \varphi, v \varphi \rangle$$

$$\geq \langle \varphi, (V + \tilde{E} \chi(\tilde{E}) - c_0) \varphi \rangle = \langle \varphi, (V + \tilde{E} \chi(\tilde{E}) - c_0) \varphi \rangle$$

(50)

where we also used the bound (102) from Lemma B.3 and then (38). Since by assumption $V \geq \tilde{E} \chi^\perp_j$ and $\tilde{E} - c_0 = \delta_0$, the bound (48) follows.

Next we turn to the proof of (49), which borrows ideas from [12]. Note first that because $\|F\|_\infty < \infty$, the operators $e^{\pm F}$ are invertible and bounded in $L^2(\mathbb{R}^2)$. Moreover, in view of Lemma B.1 and

$$\sum_{j \in \mathbb{Z}} \langle \varphi_j, |F_j|^2 \varphi_j \rangle \leq \sum_{j \in \mathbb{Z}} \langle \varphi_j, V_j - \tilde{E} \chi^\perp_j(\tilde{E}) \varphi_j \rangle \leq q_0[\varphi], \quad \forall \varphi \in \mathcal{D}(q_0)$$  

(51)

then $e^{\pm F}$ also leave the form domain of $H$ and $H_0$ invariant.

Consider now the linear operator $\tilde{H}_F := e^F \tilde{H} e^{-F}$ defined on $\mathcal{D}(\tilde{H}_F) := e^F \mathcal{D}(H_0) \subset \mathcal{D}(q_0)$. Thanks to the bijectivity of $e^{\pm F}$, we verify that $\rho(\tilde{H}_F) = \rho(\tilde{H})$ and that the identity

$$(\tilde{H}_F - z)^{-1} = e^F (\tilde{H} - z)^{-1} e^{-F}, \quad z \in \rho(\tilde{H}_F),$$
holds. Note that thanks to (48) we have that \( \text{supp} \tilde{g}_\Delta \subset \rho(\tilde{H}_F) \), since \( \text{Re} z \leq E_0 + \delta_0/2 \) for any \( z \in \text{supp} \tilde{g}_\Delta \). Moreover, if \( z \in \text{supp} \tilde{g}_\Delta \) is such that

\[
\text{Re} \langle \varphi, (\tilde{H}_F - z) \varphi \rangle \geq \frac{\delta_0}{2} \| \varphi \|^2, \quad \varphi \in \mathcal{D}(\tilde{H}_F),
\]

then we obtain the estimate

\[
\| (\tilde{H}_F - z)^{-1} \| \leq \frac{2}{\delta_0}
\]
since clearly \( \text{Re} \langle \varphi, (\tilde{H}_F - z) \varphi \rangle \leq \| (\tilde{H}_F - z) \varphi \| \varphi \). Thus, to show (49) it suffices to prove that, for any \( z \in \text{supp} \tilde{g}_\Delta \), (52) holds. For this we use the exponentially twisted version of (46) provided by Proposition B.4 in “Appendix B” which shows that as quadratic forms, for every \( \varphi \in \mathcal{D}(q_0) \supset \mathcal{D}(\tilde{H}_F) \),

\[
\text{Re} \langle \varphi, e^F \tilde{h} e^{-F} \varphi \rangle = \langle \partial_r \varphi, \partial_r \varphi \rangle + \langle \varphi, (V + \tilde{E} \chi(\tilde{E}) - (F')^2) \varphi \rangle
\]

\[
+ \text{Re} \langle e^F \varphi, W e^{-F} \varphi \rangle
\]

\[
\geq \langle \partial_r \varphi, \partial_r \varphi \rangle + \langle \varphi, (V + \tilde{E} \chi(\tilde{E}) - (F')^2) \varphi \rangle - \xi(a, \zeta) \langle \varphi, \psi \varphi \rangle
\]

\[
\geq \langle \varphi, (V - \tilde{E} \chi(\tilde{E}) - (F')^2 - c_0) \varphi \rangle
\]

\[
= \langle \varphi, (V - \tilde{E} \chi(\tilde{E}) - \tilde{E} - (F')^2 - c_0) \varphi \rangle,
\]

where we also used Lemma B.3 and (38). Thus, using also (40) we get

\[
\text{Re} \langle \varphi, e^F \tilde{h} e^{-F} \varphi \rangle \geq \langle \varphi, (V - \tilde{E} \chi(\tilde{E}) - \tilde{E} - (F')^2 - c_0) \varphi \rangle
\]

\[
= \langle \varphi, (V - \tilde{E} \chi(\tilde{E}) - (F')^2 + E_0 + \delta_0) \varphi \rangle \geq (E_0 + \delta_0) \| \varphi \|^2,
\]

which implies (52) since for all \( z \in \text{supp} (\tilde{g}_\Delta) \) we have \( \text{Re} z \leq E_0 + \delta_0/2 \). This finishes the proof of the lemma.

\[ \square \]

**Proof of Theorem 3.1.** We first prove the statement of the theorem for **bounded** exponential weights. Indeed, let \( F = (F_j)_{j \in \mathbb{Z}} \in PC_1^{bd}(\mathbb{R}^+, \mathbb{R}) \) be non-negative. Using Eq. (47), we may write

\[
e^F g_\Delta(H) = -\frac{1}{\pi} \int_{\mathbb{C}} e^F (z - \tilde{H})^{-1} e^{-F} e^F \tilde{E} \chi(\tilde{E}) (z - H)^{-1} \frac{\partial \tilde{g}_\Delta}{\partial z} \ dx \ dy.
\]

(53)

In view of (49) we have

\[
\| e^F g_\Delta(H) \| \leq \frac{\tilde{E}}{\pi} \| e^F \chi(\tilde{E}) \| \int_{\mathbb{C}} \| e^F (z - \tilde{H})^{-1} e^{-F} \| \| (z - H)^{-1} \| |\partial_z \tilde{g}_\Delta(z)| \ dx \ dy,
\]

(54)

\[
\leq \| e^F \chi(\tilde{E}) \| \frac{2\tilde{E}}{\pi \delta_0} \int_{\mathbb{C}} |\partial_z \tilde{g}_\Delta(z)| |y| \ dx \ dy
\]

(55)

where we used the standard bound \( \| (z - H)^{-1} \| \leq 1/|\text{Im}(z)| \). Thanks to (44), the above integral is finite. Thus, (43) is proven for the case in which \( \| F \|_\infty < \infty \).

Now we turn to the general case. Let \( F = (F_j)_{j \in \mathbb{Z}} \in PC_1(\mathbb{R}^+, \mathbb{R}) \) be non-negative and assume that it satisfies (40), (41) and (42). Define the family
of bounded functions $F_n = (F_{j,n})_{j \in \mathbb{Z}}$ by
\[ F_{j,n} := \frac{F_j}{1 + \frac{1}{n} F_j}, \quad n \in \mathbb{N}, j \in \mathbb{Z}. \] (56)

Indeed, the bound $F_{j,n} \leq n$ shows that $\|F_n\|_\infty \leq n$, that is, $F_n \in PC_{bd}^1(\mathbb{R}^+, \mathbb{R})$ for every $n \in \mathbb{N}$. Moreover, for each $j \in \mathbb{Z}$, there is pointwise, monotone convergence $F_{j,n} \nrightarrow F_j$ as $n \rightarrow \infty$. Notice now that $|F_{j,n}'| = (1 + \frac{1}{n} F_j)^{-2} |F_j'| \leq |F_j'|$ for $j \in \mathbb{Z}$. In addition, one checks that for each $n \in \mathbb{N}$ we also have $|F_{j,n} - F_{k,n}| \leq |F_j - F_k|$. Thus, for each $n \in \mathbb{N}$, $(F_{j,n})_{j \in \mathbb{Z}}$ is a sequence of bounded functions that satisfies conditions (40), (41) and (42).

To conclude the proof, we note that the bound (55) holds for each sequence $F_n$. Additionally, we note that $\|e^{F_n} \chi(E)\| \leq \|e^{F} \chi(E)\|$ implies that the right-hand side of (55) is uniformly bounded in $n \in \mathbb{N}$. Therefore, the Monotone Convergence Theorem shows that for any $\varphi \in L^2(\mathbb{R}^2)$,
\[
\|e^F g_\triangle(H)\varphi\|^2 = \lim_{n \to \infty} \|e^{F_n} g_\triangle(H)\varphi\|^2 \leq C_{\delta_0} \|\varphi\|^2.
\]
This finishes the proof. \qed

4. The Tunnelling Bounds

In this section, we apply Theorem 3.1 to derive the interior and exterior tunnelling bounds from Theorem 1.3. To do so, we need to construct suitable sequences of weights $(F_j)_{j \in \mathbb{Z}}$ that satisfy the requirements of Theorem 3.1.

In order to verify (40), it is important to estimate the value of the effective potential $V_j(r) = (\tilde{j} - \Phi(r))^2/r^2$ in the classically forbidden regions. One can get an intuition by considering the case $\tilde{j}(r)$ as either empty, or it is contained in an interval $[-|j|^{1/\sigma_-}, r_+ |j|^{1/\sigma_+}]$, for some $r_+ > r_- > 0$. Conditions 2 and 3 allow us to obtain estimates for $V_j - E$, to the left and to the right of the classically allowed region of the corresponding effective potential, respectively. This is shown in the next lemma, for any energy $E > 0$.

**Lemma 4.1.**

(i) Under Condition 2, there exists a constant $j_0 > 0$ and, given $E > 0$, a constant $\varepsilon_E > 0$ such that, for any $j \in \mathbb{Z}$ with $|j| \geq j_0$ and $r \leq \varepsilon_E |j|^{1/\sigma_+}$
\[
V_j(r) - E \geq |j|^{2 \sigma_+^{-1}}.
\] (57)

(ii) Under Condition 3, there exist for any $E > 0$, a constant $\eta_E > 1$ such that, for any $j \in \mathbb{Z}$ and $r \geq \eta_E (1 + |j|)^{1/\sigma_-}$,
\[
V_j(r) - E \geq \lambda_+^2 r^{2(\sigma_- - 1)},
\] (58)

where $\sigma_\mp, \lambda_\mp$ are parameters defined in Condition 2 and 3.

**Proof.** (i) From Condition 2, one sees that for any $\varepsilon > 0$ we have for all $r \leq \varepsilon |j|^{1/\sigma_+}$
\[
|\Phi(r)| \leq \lambda_+ (1 + r^{\sigma_+}) \leq \lambda_+ (1 + \varepsilon^{\sigma_+} |j|).
\] (59)
Thus
\[
\sqrt{V_j(r)} = \frac{1}{r} |j - \Phi(r)| \geq \frac{1}{r} (|j| - |\Phi(r)|) \geq \frac{1}{r} (|j| - \lambda_+ (1 + \varepsilon^{\sigma_+} |j|))
\]
\[
= \frac{1}{r} (|j| (1 - \lambda_+ \varepsilon^{\sigma_+}) - \lambda_+)
\]
for all \(0 < r \leq \varepsilon |j|^{1/\sigma_+}\). Thus, if we choose \(\varepsilon\) so small that \(1 - \lambda_+ \varepsilon^{\sigma_+} \geq 1/2\) we get
\[
\sqrt{V_j(r)} \geq \frac{1}{r} \left( \frac{1}{2} |j| - \lambda_+ \right) \geq \frac{|j|}{4r} \geq |j|^{1-\frac{1}{\sigma_+}}
\]
whenever \(0 \leq r \leq \varepsilon |j|^{1/\sigma_+}\) and \(|j| \geq 4\lambda_+\). So, by making \(\varepsilon\) so small that also \(16\varepsilon^2 \leq 1/(E + 1)\), we get, for any \(|j| \geq 4\lambda_+ > 0\) and \(r \leq \varepsilon |j|^{1/\sigma_+}\),
\[
V_j(r) - E \geq |j|^{2\sigma_+ - 1} \left( \frac{1}{16\varepsilon^2} - E \right) \geq |j|^{2\sigma_+_+ - 1}.
\]
This shows the claim.

(ii) Let \(\eta \geq \max\{r_0, (2/\lambda_-)^{1/\sigma_-}\}\) and \(j \in \mathbb{Z}\). Using Condition 3, we see that for any \(r \geq \eta(1 + |j|)^{1/\sigma_-}\),
\[
|\Phi(r)| \geq \lambda_- \eta^{\sigma_-} (1 + |j|) \geq 2|j|.
\]

Therefore,
\[
\sqrt{V_j(r)} = \frac{1}{r} |\Phi(r) - j| \geq \frac{1}{r} (|\Phi(r)| - |j|) \geq \frac{|\Phi(r)|}{2r} \geq \frac{\lambda_-}{2} r^{\sigma_- - 1}.
\]
Thus, for any \(r \geq \eta(1 + |j|)^{1/\sigma_-}\),
\[
V_j(r) - E \geq r^{2(\sigma_- - 1)} (\lambda_0^2 / 4 - E/\eta),
\]
so claim follows with the choice \(\lambda_0^2 = \lambda^2 / 4 - E/\eta\) by choosing \(\eta > 1\) sufficiently large. \(\square\)

**Remark 4.2.** The following simple observation is useful: If for any \(j \in \mathbb{Z}\), the sets \(M_j\) are subsets of \(\mathbb{R}^+\), possibly empty, and \(V_j - E > 0\) on \(M_j\), then
\[
1_{M_j}(r) \chi_j(E) = 0, \quad \text{and} \quad (V_j(r) - E) \chi_j^+(E) \geq (V_j(r) - E) 1_{M_j}(r).
\]

Now we come to the

**Proof of Theorem 1.3.** In order to show (20) and (21), we construct two different sequences and verify that they satisfy the requirements of Theorem 3.1, Eqs. (40), (41) and (42). Throughout this proof, we abbreviate \(E \equiv E\) and \(\chi_j(\vec{E}) \equiv \chi_j\).

Proof of (20). Let \(\varepsilon \in (0, \varepsilon_E]\) be a constant to be fixed below. In the following \(\varepsilon_E\) and \(j_0 > 0\) are the parameters from the first part of Lemma 4.1. We define, for any \(j \in \mathbb{Z}\) with \(|j| \geq j_0 + 1\),
\[
F_j(r) = |j|^{(1-\sigma_+^{-1})}(\varepsilon |j|^\zeta/\sigma_+ - r)_+, \quad r > 0,
\]
where $x_+ = \max\{x, 0\}$, and $F_j(r) = 0$ for all $r > 0$ when $|j| \leq j_0$. Note that $F_j$ is piecewise continuously differentiable and its derivative is given by

$$|F_j'(r)| = |j|^\zeta(1-\sigma_+^{-1})\mathbf{1}_{(0,\varepsilon]|j|^{\zeta/\sigma_+}}(r),$$

when $|j| \geq j_0 + 1$ and its derivative vanishes when $|j| \leq j_0$.

In view of (57) and (66), this choice of $F_j$ clearly satisfies (40), since on $\mathbb{R}^+$ we have

$$(V_j - E)\chi_j^+ \geq (V_j - E)\mathbf{1}_{(0,\varepsilon]|j|^{\zeta/\sigma_+}} \geq |j|^{2\zeta(1-\sigma_+^{-1})}\mathbf{1}_{(0,\varepsilon]|j|^{\zeta/\sigma_+}} \geq |F_j'|^2.$$ 

Moreover, using Remark 4.2, one sees $F_j \chi_j = 0$, for all $j \in \mathbb{Z}$, and hence, we also have (41).

To show that $F$ satisfies (42), it is enough to assume $|j| > |k|$, by symmetry. Also, if $|j|, |k| \leq j_0$, then $F_j = F_k = 0$, so (42) trivially holds in this case.

In the case $|j| \geq |k| \geq j_0 + 1$, we argue as follows: If $r \leq \varepsilon |k|^{\zeta/\sigma_+}$, then we have

$$\left(|j|^{\zeta(1-\sigma_+^{-1})} - |k|^{\zeta(1-\sigma_+^{-1})}\right) r \leq \varepsilon \left(|j|^{\zeta} - |k|^{\zeta}\right) \leq \varepsilon |j - k|^{\zeta},$$

since $0 < \zeta \leq 1$ and the map $\mathbb{Z} \ni j \to |j|^{\zeta}$ obeys the triangle inequality—this is recalled in Lemma 4.3 at the end of this section. Thus

$$|F_j(r) - F_k(r)| = \varepsilon \left(|j|^{\zeta} - |k|^{\zeta}\right) - \left(|j|^{\zeta(1-\sigma_+^{-1})} - |k|^{\zeta(1-\sigma_+^{-1})}\right) r \leq \varepsilon |j - k|^{\zeta}.$$ 

When $r \in [\varepsilon |k|^{\zeta/\sigma_+}, \varepsilon |j|^{\zeta/\sigma_+}]$ then

$$|F_j(r) - F_k(r)| = \varepsilon |j|^{\zeta} - |j|^{\zeta(1-\sigma_+^{-1})} r \leq \varepsilon |j|^{\zeta} - \varepsilon |j|^{\zeta(1-\sigma_+^{-1})} |k|^{\zeta/\sigma_+}$$

$$\leq \varepsilon (|j|^{\zeta} - |k|^{\zeta}) \leq \varepsilon |j - k|^{\zeta}.$$ 

Moreover, for $r > \varepsilon |j|^{\zeta/\sigma_+}$ both $F_j$ and $F_k$ vanish; thus, (42) will hold for all $|j|, |k| \geq j_0 + 1$, provided we pick $\varepsilon < a/2$.

If $|j| \geq j_0 + 1$ and $|k| \leq j_0$, then $|j - k| \geq 1$, so

$$|j| \leq |j - k| + |k| \leq |j - k| + j_0 \leq |j - k| + j_0 |j - k| = (j_0 + 1) |j - k|$$

Thus, in this case

$$|F_j(r) - F_k(r)| = |F_j(r)| \leq \varepsilon |j|^{\zeta} \leq \varepsilon (j_0 + 1)^{\zeta/2} |j| - |k|^{\zeta}.$$ 

Choosing $\varepsilon = a/(2(j_0 + 1)^{\zeta}) \leq a/2$ one sees that (40), (41) and (42) are satisfied by $F$.

Therefore, we may apply Theorem 3.1 to conclude

$$\sum_{j \in \mathbb{Z}} \|e^{F_j} P_j E_I(H)\|^2 < \infty.$$ 

Finally, notice that $F_j(r) \geq \frac{\varepsilon}{2} |j|^{\zeta}$ whenever $r \in (0, \frac{\varepsilon}{2} |j|^{\zeta/\sigma_+})$. Hence,

$$e^{F_j} \geq \mathbf{1}_{(0, \frac{\varepsilon}{2} |j|^{\zeta/\sigma_+})} e^{F_j} \geq \mathbf{1}_{(0, \frac{\varepsilon}{2} |j|^{\zeta/\sigma_+})} e^{\frac{\varepsilon}{2} |j|^{\zeta}},$$ 

for all $j \in \mathbb{Z}$ which yields (20).
Proof of (21): We consider another sequence of functions defined, for any \( j \in \mathbb{Z} \), by
\[
G_j(r) = c[r^{\zeta \sigma_0} - \eta^{\zeta \sigma_0} (1 + |j|)^{\zeta}]_+, \quad r > 0,
\]
for constants \( c > 0 \) and \( \eta \geq \eta_E \) to be fixed below, and \( 0 < \zeta \leq 1 \). Using Lemma 4.1.(ii), we have
\[
|G_j'|^2 = (c\zeta \sigma_0)^2 r^{2(\zeta \sigma_0 - 1)}[1+|j|]^{\zeta} \leq (V_j - E) \chi_{(\eta(1+|j|)^{\zeta}, \infty)} \leq (V_j - E) \chi_{(\eta(1+|j|)^{\zeta}, \infty)},
\]
provided \( \sigma_0 \leq \lambda_\zeta \). Thus, in view of Remark 4.2 this choice of \( G_j \) satisfies the requirements (40) and (41).

In a similar but easier fashion as above for \( F \), one can check
\[
|G_j - G_k| \leq c \eta^{\zeta \sigma_0} |j - k|^{\zeta},
\]
for any \( j,k \in \mathbb{Z} \). Hence, for the choice \( \eta = \eta_E \) and \( c = \min \{ \lambda_\zeta / \sigma_0, (a/(2\eta_E))^{1/(\zeta \sigma_0)} \} \), one sees that (40)–(42) are satisfied by \( G_j \). This implies \( \sum_{j \in \mathbb{Z}} \|e^{G_j} P_j E_t(H)\|^2 < \infty \).

Finally, we note that for \( r \geq \eta \{2(1 + |j|)^{1/\sigma_0} \} \) we have \( G_j(r) \geq \frac{c_0}{2} r^{\zeta \sigma_0} \).

Therefore,
\[
e^{G_j(r)} \geq \chi_{(\eta \{2(1+|j|)^{\zeta}/\sigma_0, \infty\})} e^{G_j(r)} \geq \chi_{(\eta \{4|j|^{\zeta}/\sigma_0, \infty\})} e^{\frac{c_0}{2} r^{\zeta \sigma_0}}.
\]

This concludes the proof of (21).

We recall here, the (reverse) triangle inequality for \( j \mapsto |j|^{\zeta} \), when \( 0 < \zeta \leq 1 \).

**Lemma 4.3.** Let \( \zeta \in (0, 1] \). Then, for all \( j,k \in \mathbb{Z} \) we have \( |j + k|^{\zeta} \leq |j|^{\zeta} + |k|^{\zeta} \) and, in particular, also \( |j|^{\zeta} - |k|^{\zeta} \leq |j + k|^{\zeta} \).

**Proof.** This is well known, we give the easy argument for the convenience of the reader(s). If \( \zeta = 1 \), this is the usual triangle inequality. So let \( 0 < \zeta < 1 \) and also \( j,k \neq 0 \). Then,
\[
|j + k|^{\zeta} \leq (|j| + |k|)^{\zeta} = \frac{|j| + |k|}{(|j| + |k|)^{1/\zeta}} \leq \frac{|j|}{|j|^{1/\zeta}} + \frac{|k|}{|k|^{1/\zeta}} = |j|^{\zeta} + |k|^{\zeta},
\]
and with the usual trick, the reverse triangle inequality \( ||j|^{\zeta} - |k|^{\zeta}| \leq |j + k|^{\zeta} \) follows.

\[
\Phi(r) = \lambda r, \quad r > 0
\]

5. **A Remark on a Model with Mobility Edge**

So far in the article, nothing has been said about the limiting case \( \sigma_- = \sigma_+ = 1 \). On this subsection, we give results on the localization of particles moving under such magnetic fields when no electric field is present. More precisely, we consider the situation in which the magnetic flux is given by
\[
\Phi(r) = \lambda r, \quad r > 0
\]
for some $\lambda > 0$, with the Hamiltonian $H_0$ being defined through (2). The spectral quality of this operator has already been determined, see [16] or [6, Theorem 6.2]. In particular, it is proven that $\sigma(H_0) = [0, \infty)$ and the spectrum is dense pure point in $[0, \lambda^2)$ and absolutely continuous in $(\lambda^2, \infty)$. For energies above $\lambda^2$ one may use the absolute continuity of the spectrum, similarly as it was first done in [14] for Schrödinger operators and then adapted in [15] to Dirac particles, with rotational symmetry to show ballistic dynamics. The long-time dynamics for high energies is therefore understood; we now settle the question about dynamics for low, positive energies.

First, note that the rotational symmetry of $H_0$ makes the dynamics of $J$ trivial. Therefore, to obtain an estimate on $|x(t)|$ it suffices to adapt Theorem 1.5 to the present case. One may go through its proof and realize that the exterior tunnelling estimate (21) is all that is needed. We state both of these adapted results in the following.

**Theorem 5.1.** Let $H_0$ be the Hamiltonian associated to the quadratic form (2) with $A \equiv \lambda > 0$, as given in (4). Let $E \in (0, \lambda^2)$ and $I = [0, E]$. Then, there exist constants $c_- \in (1, \infty)$ and $\delta > 0$ such that

$$
\sum_{j \in \mathbb{Z}} \| \prod_{|j|, \infty} e^{\delta|\cdot|} P_j E I (H_0) \| < \infty. 
$$

Consequently, for every $\nu > 0$ there exists a constant $C > 0$ such that for all $\varphi \in E I (H_0) L^2(\mathbb{R}^2)$

$$
\langle \varphi(t), |x|^\nu \varphi(t) \rangle \leq C \left( \| \varphi \|^2 + \langle \varphi, |J|^\nu \varphi \rangle \right), \text{ for } t \geq 0,
$$

holds.

We conclude that dynamical localization holds for energies $E \in (0, \lambda^2)$, provided the initial data are sufficiently regular in the angular variable, that is,

$$
\sup_{t \geq 0} \langle \varphi(t), |x|^\nu \varphi(t) \rangle < \infty.
$$

for all $\varphi \in E I (H_0) L^2(\mathbb{R}^2) \cap D(|J|^\nu/2)$.

**Proof of Theorem 6.** We adapt the argument used to prove Theorem 1.3, i.e. we construct an explicit sequence of functions satisfying (40) and (41) and apply Theorem 3.1. Since we assume $W = 0$, we can omit (42) and take $\epsilon_0 = c_0 = 0$ during the proof. Let $\delta_0 > 0$ and $E \equiv \tilde{E}$. First, notice that the classically allowed regions are simplified to

$$
C_j(E) = \begin{cases} 
\left[ \frac{j}{\lambda + E^{1/2}}, \frac{j}{\lambda - E^{1/2}} \right] & j > 0, \\
\emptyset & j \leq 0.
\end{cases}
$$

Note that the proof breaks down for $E > \lambda^2$ since then this structure is lost; one has in turn $C_j(E) = \left[ \frac{j}{\lambda + E^{1/2}}, \infty \right)$ for $j > 0$. Now, pick $\eta_1 = \eta_1(\lambda, E) > \frac{1}{\lambda^2 - E}$ big enough such that

$$
\frac{2\lambda}{\eta_1} \leq \frac{1}{2} (\lambda^2 - E)
$$

(75)
holds. Then, let \( \delta_1 = \delta_1(\lambda, E, \eta_1) < \min\{\sqrt{\frac{\lambda^2 - E}{2}}, \frac{\eta_1}{2\eta_1}\} \) and define \( H_j(r) = \delta_1(r - \eta_1|j|)_+ \). We estimate for \( r \in (\eta_1|j|, \infty) \)

\[
V_j(r) - E = \lambda^2 \left(1 - \frac{j}{\lambda r}\right)^2 - E \geq \lambda^2 \left(1 - \frac{2}{\lambda \eta_1}\right) - E = (\lambda^2 - E) - \frac{2\lambda}{\eta_1} \geq \frac{\lambda^2 - E}{2}
\]

where the last inequality follows from (75). Since \( |H_j'|^2 = \delta_1^2 1_{(\eta_1|j|, \infty)} \leq \frac{\lambda^2 - E}{2} 1_{(\eta_1|j|, \infty)} \) we have that (40) is satisfied. Equation (41) is fulfilled in view of \( \eta_1 > \frac{1}{\lambda^2 - E} \) and (74). We finish the proof using Theorem 3.1, putting \( c_- = 2\eta_1 \) together with \( \delta = \min\{\delta_1/2, \delta_0\} \) and arguing as in the end of the proof of Theorem 1.3. \( \square \)

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**Appendix A: The Magnetic Schrödinger Operator**

Here, we want to review the form definition of the magnetic Schrödinger operator \( H_0 \). We have to be a little bit careful, since we want to be able to handle rotationally symmetric, but possibly singular, magnetic fields \( B \). Recall that we choose the vector potential \( A \) in the Poincaré gauge given by (4).

**Lemma A.1.** If the magnetic field \( B : \mathbb{R}^2 \to \mathbb{R} \) is rotationally symmetric and locally square integrable, then the function

\[
\mathbb{R}^2 \ni x \mapsto \Phi(|x|)/|x| = \frac{1}{|x|} \int_0^{|x|} B(s) ds
\]

is locally square integrable. In particular, the magnetic vector potential \( A \) given by (4) is in \( L^2_{\text{loc}}(\mathbb{R}^2) \).

**Proof.** This is a simple consequence of Jensen’s inequality. Since \( \frac{2}{r^2} \int_0^r s ds = 1 \), Jensen’s inequality shows

\[
\left(\frac{2\Phi(r)}{r^2}\right)^2 \leq \left(\frac{2}{r^2} \int_0^r B(s) ds\right)^2 \leq \frac{2}{r^2} \int_0^r B(s)^2 ds
\]

for all \( r > 0 \). Thus

\[
\int_{|x| \leq R} \left(\frac{\Phi(|x|)}{|x|}\right)^2 dx = 2\pi \int_0^R \frac{r^2}{4} \left(\frac{2\Phi(r)}{r^2}\right)^2 r dr \leq \pi \int_0^R \int_0^r B(s)^2 ds dr dr,
\]

\[
= \frac{\pi}{2} \int_0^R B(s)^2(R^2 - s^2) ds < \infty,
\]

\( \Box \)
for all $R > 0$. By the definition (4) we have $|A(x)| = \frac{\Phi(|x|)}{|x|}$, so also the magnetic vector potential in the Poincaré gauge $A$ is locally square integrable. \hfill \Box

Denote by $p = -i\nabla$ the usual momentum operator. We will need a representation of the magnetic Schrödinger operator $(p - A)^2$, when $A$ is in the Poincaré gauge and the magnetic field is rotationally symmetric. This is well known, but we want to include singular magnetic fields, so we have to be a bit careful.

**Lemma A.2.** The quadratic form $q_0$ of the free magnetic Schrödinger operator $(p - A)^2$ is given by

$$q_0(\varphi, \varphi) = \langle (p - A)\varphi, (p - A)\varphi \rangle = \langle \partial_r \varphi, \partial_r \varphi \rangle + \left\langle \frac{1}{r}(\Phi - L)\varphi, \frac{1}{r}(\Phi - L)\varphi \right\rangle$$

(78)

for all $\varphi \in D(q_0)$. Here, $L = x_1 p_2 - x_2 p_1$ is the generator of rotations, i.e. the angular momentum operator, in $L^2(\mathbb{R}^2)$, $r = |x|$, and the radial derivative is given by $\partial_r = \frac{\mathbf{x}}{|x|} \cdot \nabla$. In particular, $D(q_0) = \mathcal{D}(\partial_r) \cap \mathcal{D}(\frac{1}{r}(\Phi - L))$.

Before we show this, we collect one more result, which is needed

**Lemma A.3.** The quadratic form corresponding to the kinetic energy $p^2$ in dimension $d \geq 2$ is given by

$$\langle p \varphi, p \varphi \rangle = \langle \partial_r \varphi, \partial_r \varphi \rangle + \sum_{1 \leq j < k \leq d} \left\langle \frac{1}{r}L_{j,k} \varphi, \frac{1}{r}L_{j,k} \varphi \right\rangle,$$

(79)

for all $\varphi \in H^1(\mathbb{R}^d)$, the form domain of $p^2$, where $r = |x|$, $\partial_r = \frac{\mathbf{x}}{|x|} \cdot \nabla$ is the radial derivative on $\mathbb{R}^d$ and $L_{j,k} = x_j p_k - x_k p_j$, $1 \leq j < k \leq d$ are the angular momentum generators.

In particular, $H^1(\mathbb{R}^d) = \mathcal{D}(\partial_r) \cap \mathcal{D}(\frac{1}{r}L_{j,k})$.

**Proof of Lemma A.2.** We assume that $\varphi \in C_0^\infty(\mathbb{R}^2)$, by density. Then, since $A$ is locally square integrable,

$$\langle \varphi, H_0 \varphi \rangle = \langle (p - A)\varphi, (p - A)\varphi \rangle,$$

$$= \langle p \varphi, p \varphi \rangle - 2\text{Re} \left\langle A_1 \varphi, p_1 \varphi \right\rangle - 2\text{Re} \left\langle A_2 \varphi, p_2 \varphi \right\rangle + \langle A \varphi, A \varphi \rangle.$$

Using the explicit form of vector potential in the gauge (4), one also sees that

$$\left\langle A_1 \varphi, p_1 \varphi \right\rangle = - \left\langle \frac{\Phi}{r^2} \varphi p_1 \varphi \right\rangle = - \left\langle \frac{\Phi}{r} \varphi \frac{1}{r} x_2 p_1 \varphi \right\rangle$$

and similarly for

$$\left\langle A_2 \varphi, p_2 \varphi \right\rangle = \left\langle \Phi \frac{1}{r} \varphi \frac{1}{r} x_1 p_2 \varphi \right\rangle,$$

where all terms are well defined, since Lemma A.1 shows that $\frac{\Phi(|x|)}{|x|}$ is locally square integrable over $\mathbb{R}^2$. Thus,
\[
\langle A_1 \varphi, p_1 \varphi \rangle + \langle A_2 \varphi, p_2 \varphi \rangle = \left\langle \frac{\Phi}{r} \varphi \frac{1}{r} (x_1 p_2 - x_2 p_1) \varphi \right\rangle = \left\langle \frac{\Phi}{r} \varphi \frac{1}{r} L \varphi \right\rangle 
\]
with \( L = x_1 p_2 - x_2 p_1 \). Since also
\[
\langle A \varphi, A \varphi \rangle = \left\langle \frac{\Phi}{r} \varphi \frac{1}{r} L \varphi \right\rangle,
\]
this yields
\[
\langle (p - A) \varphi, (p - A) \varphi \rangle = \langle p \varphi, p \varphi \rangle + 2 \text{Re} \left\langle \frac{\Phi}{r} \varphi \frac{1}{r} L \varphi \right\rangle + \langle A \varphi, A \varphi \rangle,
\]
which proves \((78)\) when \( \varphi \in C_0^\infty(\mathbb{R}^2) \). Since \( C_0^\infty(\mathbb{R}^2) \) is dense in the domain of \( q_0 \) [18] and all terms on the right-hand side of \((78)\) are non-negative, a standard density argument shows that the domain of \( D(q_0) \) is equal to the intersection of \( D(\partial_r) \) and \( D(\frac{1}{r} (\Phi - L)) \). This proves Lemma A.2. \( \square \)

**Proof of Lemma A.3.** We can use the same density argument as above to see that it is enough to assume that \( \varphi \in C_0^\infty(\mathbb{R}^d) \). Then,
\[
\sum_{1 \leq j < k \leq d} \langle L_{j,k} \varphi, L_{j,k} \varphi \rangle = - \sum_{1 \leq j < k \leq d} \langle \varphi, (x_j \partial_k - x_k \partial_j)^2 \varphi \rangle.
\]
Moreover,
\[
\sum_{1 \leq j < k \leq d} (x_j \partial_k - x_k \partial_j)^2 = \frac{1}{2} \sum_{j \neq k} (x_j \partial_k - x_k \partial_j)^2,
\]
\[
= \frac{1}{2} \sum_{j \neq k} (x_j \partial_k x_j \partial_k - x_j \partial_k x_k \partial_j - x_k \partial_j x_j \partial_k + x_k \partial_j x_k \partial_j),
\]
\[
= \frac{1}{2} \sum_{1 \leq j, k \leq d} (x_j^2 \partial_k^2 + x_k^2 \partial_j^2 - \partial_k x_j x_j \partial_j - x_j \partial_j x_k x_k \partial_k - x_j \partial_j - x_j x_k \partial_k - x_k \partial_k)
\]
\[
- \sum_j (x_j^2 \partial_j^2 - \partial_j x_j^2 \partial_j - x_j \partial_j),
\]
\[
= |x|^2 \Delta - (\nabla \cdot x)(x \cdot \nabla) + 2x \cdot \nabla = |x|^2 \Delta - (x \cdot \nabla)^2 - (d - 2)x \cdot \nabla.
\]
Thus, one obtains that
\[ p^2 = -\Delta = -\frac{1}{|x|^2} \left( x \cdot \nabla \right)^2 - \frac{(d-2)}{|x|^2} x \cdot \nabla + \frac{1}{|x|^2} \sum_{1 \leq j < k \leq d} L_{j,k}, \]
that is, as quadratic forms
\[ \langle p \varphi, p \varphi \rangle = -\left\langle \varphi \frac{1}{|x|^2} \left( x \cdot \nabla \right)^2 \varphi \right\rangle - \left\langle \varphi \frac{d-2}{|x|^2} (x \cdot \nabla) \varphi \right\rangle + \sum_{1 \leq j < k \leq d} \left\langle \varphi \frac{1}{|x|^2} L_{j,k} \varphi \right\rangle \]
(80)
at least when \( \varphi \in C_0^\infty(\mathbb{R}^d) \). For such \( \varphi \) we set
\[ \psi(r, \omega) := \varphi(r\omega), \]
when \( r \geq 0 \) and \( |\omega| = 1 \). That is, \( \varphi(x) = \psi(|x|, x/|x|) \). Then, clearly,
\[ x \cdot \varphi(x) = r \partial_r \psi(r, \omega) \]
(81)
with \( r = |x| \) and \( \omega = x/|x| \in S^{d-1} \). Then,
\[ -\left\langle \varphi \frac{1}{|x|^2} (x \cdot \nabla)^2 \varphi \right\rangle = -\int_{S^{d-1}} d\omega \int_0^\infty dr r^{d-1} \psi(r, \omega) r^{-2} (r \partial_r)^2 \psi(r, \omega). \] (82)
Now for fixed \( \omega \), we have
\[ -\int_0^\infty dr r^{d-1} \psi(r, \omega) r^{-2} (r \partial_r)^2 \psi(r, \omega) \]
\[ = -\left[ r^{d-1} \psi(r, \omega) \partial_r \psi(r, \omega) \right]_{r=0}^{r=\infty} + \int_0^\infty dr (\partial_r r^{d-2} \psi(r, \omega))(r \partial_r) \psi(r, \omega), \]
\[ = (d-2) \int_0^\infty dr r^{d-3} \psi(r, \omega) r \partial_r \psi(r, \omega) + \int_0^\infty dr r^{d-1} |\partial_r \psi(r, \omega)|^2. \]
The first term in the second line above vanishes, since \( \varphi \) has compact support and \( d \geq 2 \). Thus,
\[ -\left\langle \varphi \frac{1}{|x|^2} (x \cdot \nabla)^2 \varphi \right\rangle = (d-2) \left\langle \varphi \frac{1}{|x|^2} (x \cdot \nabla) \varphi \right\rangle + \langle \partial_r \varphi, \partial_r \varphi \rangle \]
with \( \partial_r = \frac{x}{|x|^2} \cdot \nabla \). Using this in (80) shows
\[ \langle p \varphi, p \varphi \rangle = \langle \partial_r \varphi, \partial_r \varphi \rangle + \sum_{1 \leq j < k \leq d} \left\langle \varphi \frac{1}{|x|^2} L_{j,k} \varphi \right\rangle, \]
\[ = \langle \partial_r \varphi, \partial_r \varphi \rangle + \sum_{1 \leq j < k \leq d} \left\langle \frac{1}{|x|^2} L_{j,k} \varphi \frac{1}{|x|} L_{j,k} \varphi \right\rangle, \]
since \( L_{j,k} \) commutes with multiplication with radial functions. This proves (79), at least when \( \varphi \in C_0^\infty(\mathbb{R}^d) \). On the other hand, since all the terms on the right-hand side of (79) are positive and \( C_0^\infty(\mathbb{R}^d) \) is dense in the Sobolev space \( H^1(\mathbb{R}^d) \), the form domain of \( p^2 \), it is an easy exercise to show that then \( \partial_r \varphi, \frac{1}{|x|^2} L_{j,k} \varphi \in L^2(\mathbb{R}^d) \) for all \( \varphi \in H^1(\mathbb{R}^d) \) and (79) holds for all \( \varphi \in H^1(\mathbb{R}^d) \). \( \square \)
To work in polar coordinates, we identify $L^2(\mathbb{R}^2)$ with the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^+ \times S^1, rdr\,d\theta)$ with the scalar product

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^+ \times [0, 2\pi]} f(r, \theta)g(r, \theta)\,rdr\,d\theta.$$ 

By means of the unitary map $U : L^2(\mathbb{R}^2) \to \mathcal{H}$ introduced in (6), the angular momentum component $L_{1,2}$ takes the form $J := -i \partial_{\theta} = UL_{1,2}U^*$. In these coordinates we have

**Proposition A.4.** Let $\varphi$ be in the domain of the quadratic form $q_0$ corresponding to $(P - A)^2$, and expand $\tilde{\varphi} = U\varphi$ as in (99). Then,

$$q_0(\varphi, \varphi) = \sum_{j \in \mathbb{Z}} \left( \langle \partial_r \varphi_j, \partial_r \varphi_j \rangle_{L^2(\mathbb{R}^+, rdr)} + \left\langle \varphi_j \frac{1}{r^2}(\Phi(r) - j)^2 \varphi_j \right\rangle_{L^2(\mathbb{R}^+, rdr)} \right).$$

(83)

So, the eigenspaces corresponding to $P_j$, introduced in (9), are invariant subspaces for the unperturbed magnetic Schrödinger operator with a rotationally symmetric magnetic field, when the magnetic vector potential is in the Poincaré gauge (4).

Because of the above identity, it is convenient to recall the definition of the effective potential, namely,

$$V_j(r) := \frac{1}{r^2}(\Phi(r) - j)^2.$$ 

(84)

By polarization, Proposition A.4 shows that when $\varphi, \psi$ are in the domain of the form $q_0$ corresponding to $(p - A)^2$ and $\tilde{\varphi} = U\varphi, \tilde{\psi} = U\psi$ are expanded as in (99) then

$$q_0(\varphi, \psi) = \sum_{j \in \mathbb{Z}} \left( \langle \partial_r \varphi_j, \partial_r \psi_j \rangle_{L^2(\mathbb{R}^+, rdr)} + \langle \varphi_j, V_j \psi_j \rangle_{L^2(\mathbb{R}^+, rdr)} \right).$$

(85)

We need one more result, concerning the form boundedness of potentials $W$ satisfying Condition 1 with respect to the radial kinetic energy.

**Lemma A.5.** Assume that $v$ is a rotationally symmetric potential which is form bounded with respect to $p^2$, that is, for any $0 < \varepsilon$ there exists $C(\varepsilon) < \infty$ with

$$| \langle \varphi, v\varphi \rangle | \leq \varepsilon \| \nabla \varphi \|^2 + C(\varepsilon) \| \varphi \|^2$$

(86)

for all $\varphi \in \mathcal{D}(p)$. Then, also

$$| \langle \varphi, v\varphi \rangle | \leq \varepsilon \| \partial_r \varphi \|^2 + C(\varepsilon) \| \varphi \|^2$$

(87)

for all $\varphi \in \mathcal{D}(\partial_r)$, where $\partial_r = \frac{r}{|x|} \cdot \nabla$ is the radial derivative.
Proof. We expand $\bar{\varphi} = U \varphi = \sum_{j \in \mathbb{Z}} \varphi_j e_j$, where $\varphi_j$ are purely radial functions and $e_j$ are the basis of complex exponentials. Then, for a radial potential $v$ we have

$$\langle \varphi, v \varphi \rangle = \sum_{j \in \mathbb{Z}} \langle P_j \varphi, v \varphi \rangle = \sum_{j \in \mathbb{Z}} \langle P_j^2 \varphi, v \varphi \rangle = \sum_{j \in \mathbb{Z}} \langle P_j \varphi, vP_j \varphi \rangle$$

$$= \sum_{j \in \mathbb{Z}} \langle \varphi_j, v \varphi_j \rangle_{L^2(\mathbb{R}_+, r dr)},$$

with the angular momentum projections $P_j$. Lifting each $\varphi_j$ back to $L^2(\mathbb{R}^2)$, by considering it to be constant in the angular coordinate, i.e. identifying it as the function $\mathbb{R}^2 \ni x \mapsto \varphi_j(|x|)$, we see have by assumption (86)

$$|\langle \varphi, v \varphi \rangle| \leq \varepsilon \sum_{j \in \mathbb{Z}} |\nabla \varphi_j, \nabla \varphi_j| + C(\varepsilon) \sum_{j \in \mathbb{Z}} |\varphi_j, \varphi_j|.$$

Now, since $\varphi_j$ lifted back to $\mathbb{R}^2$ is radial, we have

$$\langle \nabla \varphi_j, \nabla \varphi_j \rangle = \langle \partial_r \varphi_j, \partial_r \varphi_j \rangle = \langle \partial_r P_j \varphi, \partial_r P_j \varphi \rangle = \langle P_j \partial_r \varphi, \partial_r \varphi \rangle$$

since each angular momentum projection $P_j$ commutes with the radial part of the kinetic energy. We also have

$$\langle \varphi_j, \varphi_j \rangle = \langle P_j \varphi, P_j \varphi \rangle = \langle P_j \varphi, \varphi \rangle$$

so combining the above yields

$$|\langle \varphi, v \varphi \rangle| \leq \varepsilon \sum_{j \in \mathbb{Z}} \langle P_j \partial_r \varphi, \partial_r \varphi \rangle + C(\varepsilon) \sum_{j \in \mathbb{Z}} \langle P_j \varphi, \varphi \rangle = \varepsilon \langle \partial_r \varphi, \partial_r \varphi \rangle + C(\varepsilon) \langle \varphi, \varphi \rangle$$

which proves the claim. \qed

Remark A.6. The above result also shows that any radial potential $v$ which is form bounded with respect to the nonmagnetic kinetic energy is also form bounded with respect to the magnetic kinetic energy with a rotationally symmetric magnetic field, with the same constants, since by Lemma A.5 we have

$$|\langle \varphi, v \varphi \rangle| \leq \varepsilon \langle \partial_r \varphi, \partial_r \varphi \rangle + C(\varepsilon) \langle \varphi, \varphi \rangle \leq \varepsilon (\langle \partial_r \varphi, \partial_r \varphi \rangle + \langle \varphi, V \varphi \rangle) + C(\varepsilon) \langle \varphi, \varphi \rangle,$$

$$= \varepsilon q_0(\varphi, \varphi) + C(\varepsilon) \langle \varphi, \varphi \rangle,$$

since the effective potential $V = (V_j)_{j \in \mathbb{Z}} \geq 0$.

Appendix B: The Exponentially Twisted Magnetic Quadratic Forms

The sequences of weights $F = (F_j)_{j \in \mathbb{Z}}$ we have interest in, are bounded and have derivatives that can be dominated by the magnetic quadratic form $q_0$. This turns to be sufficient to prove invariance of the form domain.

In what follows, we denote by $\| \cdot \|_{q_0}^2 = q_0[\cdot] + \| \cdot \|_2^2$ the norm induced by $q_0$ on $D(q_0)$. 

Lemma B.1. Let $G = (G_j)_{j \in \mathbb{Z}} \subset PC_{bd}^1(\mathbb{R}^+, \mathbb{R})$ be a sequence of functions for which there exists $c > 0$ such that

$$\sum_{j \in \mathbb{Z}} \langle \varphi_j, |G'_j| \varphi_j \rangle \leq c \|\varphi\|_{q_0}^2 \quad \varphi \in \mathcal{D}(q_0). \quad (90)$$

Then, $e^G : \mathcal{D}(q_0) \to \mathcal{D}(q_0)$ is a continuous bijection with respect to $\| \cdot \|_{q_0}$.

Proof. Recall that $e^G$ is bounded in $L^2(\mathbb{R}^2)$ and $\mathcal{D}(q_0) = \mathcal{D}(\partial_r) \cap \mathcal{D}(r^{-1}(\Phi - L))$ because of Lemma A.2. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ and compute that

$$\|\partial_r(e^G \varphi)\|^2 \leq 2\|e^G\|^2(\|G'\varphi\|^2 + \|\partial_r \varphi\|^2) \leq 2\|e^G\|^2(cq_0[\varphi] + c\|\varphi\|^2 + \|\partial_r \varphi\|^2). \quad (91)$$

Since $C_0^\infty(\mathbb{R}^2) \subset \mathcal{D}(q_0) \subset \mathcal{D}(\partial_r)$, the right-hand side above is finite and we get $\partial_r(e^G \varphi) \in L^2(\mathbb{R}^2)$. By a similar argument, $r^{-1}(\Phi - L)e^G \varphi \in L^2(\mathbb{R}^2)$ as well. Therefore, we conclude that $e^G \varphi \in \mathcal{D}(q_0)$. Moreover, we compute for $\varphi \in C_0^\infty(\mathbb{R}^2)$

$$q_0[e^G \varphi] = \sum_{j \in \mathbb{Z}} \left\langle e^{G_j}(\partial_r + G'_j)\varphi_j, e^{G_j}(\partial_r + G'_j)\varphi_j + \|V_{j/2}e^{G_j}\varphi_j\|^2 \right\rangle,$$

$$\leq \sum_{j \in \mathbb{Z}} 2\left\langle e^{G_j}\partial_r \varphi_j, e^{G_j}\partial_r \varphi_j \right\rangle + 2\left\langle e^{G_j}G'_j\varphi_j, e^{G_j}G'_j\partial_r \varphi_j \right\rangle + \|V_{j/2}\varphi_j\|^2,$$

$$\leq 2e^{2\|G\|(1 + c)}(q_0[\varphi] + \|\varphi\|^2). \quad (94)$$

Now, recalling that $C_0^\infty(\mathbb{R}^2)$ is a form core for $H_0$, for a given $\varphi \in \mathcal{D}(q_0)$ we find a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^2)$ converging to $\varphi$ with respect to the norm $\| \cdot \|_{q_0}$. In view of the above estimate, we have that $(e^G \varphi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\| \cdot \|_{q_0}$, and so a limit $\psi \in \mathcal{D}(q_0)$ exists. Since $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^2)$, continuity of $e^G$ then ensures that $e^G \varphi_n \to e^G \varphi$ in $L^2(\mathbb{R}^2)$. Uniqueness of the limits then gives us that $e^G \varphi = \psi$, and we are able to conclude that $e^G \mathcal{D}(q_0) \subset \mathcal{D}(q_0)$. The same analysis holds for $e^{-G}$ as well; we conclude that $e^G \mathcal{D}(q_0) = \mathcal{D}(q_0)$. Continuity now follows from (94).

For sequences $F = (F_j)_{j \in \mathbb{Z}} \in PC_{bd}^1(\mathbb{R}^+, \mathbb{R})$ satisfying the bound (90), we define the exponentially twisted sesquilinear form

$$q_F(\psi, \varphi) := q_0(e^F \psi, e^{-F} \varphi), \quad \psi, \varphi \in \mathcal{D}(q_0). \quad (95)$$

In view of Lemma B.1, $q_F$ is well defined. Moreover, it is easily verified that

$$q_F(\psi, \varphi) = \langle \psi, e^F H_0 e^{-F} \varphi \rangle, \quad \psi \in \mathcal{D}(q_0), \varphi \in e^F \mathcal{D}(H_0). \quad (96)$$

Its explicit representation will be recorded here.
**Proposition B.2.** For the sesquilinear form $q_F$ defined above, we have for any $\varphi, \psi \in D(q_0)$
\[
q_F(\psi, \varphi) = q_0(\psi, \varphi) + \sum_{j \in \mathbb{Z}} \left( \langle F_j^* \psi_j, \partial_r \varphi_j \rangle - \langle \partial_r \psi_j, F_j^* \varphi_j \rangle - \langle F_j^* \psi_j, F_j^* \varphi_j \rangle \right).
\]

(97)

In particular, for any $\varphi \in D(q_0)$
\[
\text{Re } q_F[\varphi] = \langle \partial_r \varphi, \partial_r \varphi \rangle + \langle \varphi, (V - (F')^2) \varphi \rangle.
\]

(98)

**Proof.** A straightforward calculation starting from (85). \qed

To control a perturbation $W$ which is not rotationally symmetric, we recall that the Fourier transformation of the angular variable is given through the unitary operator
\[
\mathcal{F} : \mathcal{H} \longrightarrow \bigoplus_{j \in \mathbb{Z}} L^2(\mathbb{R}^+, dr)
\]
acting as the closure of the map
\[
\psi \mapsto (\mathcal{F} \psi)_j \equiv \psi_j := \left( \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \psi(\cdot, \theta) e^{-ij\theta} d\theta \right)_{j \in \mathbb{Z}},
\]
initially defined on $\mathcal{U}C_0^\infty(\mathbb{R}^2)$.

It is easy to check that, for any $j \in \mathbb{Z}$ and $\psi \in \mathcal{U}C_0^\infty(\mathbb{R}^2)$,
\[
[P_j \psi](r, \theta) = \psi_j e_j(\theta), \quad r > 0, \theta \in [0, 2\pi),
\]
with $e_j(\theta) = e^{ij\theta}/\sqrt{2\pi}$, since $P_j = 1 \otimes |e_j\rangle \langle e_j|$ on $L^2(\mathbb{R}^+) \otimes L^2(S^1) \approx \mathcal{H}$. Moreover, we have
\[
\langle P_j \varphi, WP_k \psi \rangle_{\mathcal{H}} = \langle \varphi_j, \hat{W}(\cdot, j - k) \psi_k \rangle_{L^2(\mathbb{R}^+)}. \tag{100}
\]

**Lemma B.3.** Assume that $W$ satisfies Condition 1 for some $a > 0$, $0 < \zeta \leq 1$. Let $F = (F_j)_{j \in \mathbb{Z}} \in PC_{bd}^1(\mathbb{R}^+, \mathbb{R})$ satisfy (42). Then, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$,
\[
\left| \langle \varphi, e^F W e^{-F} \varphi \rangle \right| \leq \xi(a, \zeta) \langle \varphi, v \varphi \rangle. \tag{101}
\]

Moreover, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$,
\[
\left| \langle \varphi, W \varphi \rangle \right| \leq \xi(2a, \zeta) \langle \varphi, v \varphi \rangle. \tag{102}
\]

Here, $v$ is defined through Condition 1 and $\xi(a, \zeta) := \sum_{k \in \mathbb{Z}} e^{-\frac{a}{2}|k|^\zeta}$.

**Proof.** We estimate using (11) for any $\varphi \in C_0^\infty(\mathbb{R}^2)$
\[
\left| \langle \varphi, e^F W e^{-F} \varphi \rangle \right| \leq \sum_{j, k \in \mathbb{Z}} \left| \langle e^F_j P_j \varphi, We^{-F_k} P_k \varphi \rangle \right|
\]
\[
= \sum_{j, k \in \mathbb{Z}} \left| \langle e^F_j \varphi_j, \hat{W}(\cdot, j - k) e^{-F_k} \varphi_k \rangle_{L^2(\mathbb{R}^+)} \right|
\]
\[
\leq \sum_{j, k \in \mathbb{Z}} e^{-a|j - k|^\zeta} \left| \langle e^F_j \varphi_j, b e^{-F_k} \varphi_k \rangle_{L^2(\mathbb{R}^+)} \right|
\]
\[
\leq \sum_{j,k \in \mathbb{Z}} e^{-a|j-k|^{\zeta}/2} \langle |\phi_j|, b |\phi_k| \rangle_{L^2(\mathbb{R}^+)} \\
\leq \sum_{j,k \in \mathbb{Z}} e^{-a|j-k|^{\zeta}/2} \left\| b^{1/2} \phi_j \right\|_{L^2(\mathbb{R}^+)} \left\| b^{1/2} \phi_k \right\|_{L^2(\mathbb{R}^+)}
\]
where in the last two inequalities we use (42) and Cauchy–Schwarz inequality for the scalar product, respectively. We can estimate the last sums applying Young’s inequality for convolutions to get

\[
\left| \langle \phi, e^F V e^{-F} \phi \rangle \right| \leq \left( \sum_{k \in \mathbb{Z}} e^{-a|k|^{\zeta}/2} \right) \left( \sum_{j \in \mathbb{Z}} \left\| b^{1/2} \phi_j \right\|_{L^2(\mathbb{R}^+)}^2 \right) = \xi(a, \zeta) \langle \phi, v \phi \rangle.
\]
This proves (101). In the case, \( F = 0 \), we clearly obtain the same estimate as above with \( a/2 \) replaced by \( a \). This concludes the proof of the lemma.

Since \( v \) is infinitesimally \( q_0 \)-bounded, as shown in “Appendix A”, we conclude that so is \( \langle \psi, e^F V e^{-F} \phi \rangle \). In particular, its form domain contains \( D(q_0) \) and the perturbed sesquilinear form

\[
q_{F,W}(\psi, \varphi) = q_F(\psi, \varphi) + \langle \psi, e^F V e^{-F} \phi \rangle, \quad \psi, \varphi \in D(q_0),
\]
is well defined and satisfies

\[
q_{F,W}(\psi, \varphi) = \langle \psi, e^F H e^{-F} \varphi \rangle, \quad \psi \in D(q_0), \varphi \in e^F D(H_0).
\]
We write here the explicit form of its real part, which will be used in the proof of Lemma 3.3.

**Proposition B.4.** Assume that \( W \) satisfies Condition 1 for some \( a > 0 \), \( 0 < \zeta \leq 1 \). Let \( F = (F_j)_{j \in \mathbb{Z}} \in PC_{bd}^1(\mathbb{R}^+, \mathbb{R}) \) satisfy (42) and (90). Then, the quadratic form \( q_{F,W} \) satisfies

\[
\text{Re} q_{F,W}[\varphi] = \langle \partial_r \varphi, \partial_r \varphi \rangle + \langle \varphi, (V - (F')^2) \varphi \rangle + \text{Re} \langle \varphi, e^F V e^{-F} \varphi \rangle
\]
as quadratic forms on \( D(q_0) \).

**Proof.** Follows directly from (97).

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