Turán number for odd-ballooning of trees

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Abstract
The Turán number $\text{ex}(n, H)$ is the maximum number of edges in an $H$-free graph on $n$ vertices. Let $T$ be any tree. The odd-ballooning of $T$, denoted by $T_o$, is a graph obtained by replacing each edge of $T$ with an odd cycle containing the edge, and all new vertices of the odd cycles are distinct. In this paper, we determine the exact value of $\text{ex}(n, T_o)$ for sufficiently large $n$ and $T_o$ being good, which generalizes all the known results on $\text{ex}(n, T_o)$ for $T$ being a star, due to Erdős, Füredi, Gould, and Gunderson (1995), Hou, Qiu, and Liu (2018), and Yuan (2018), and provides some counter-examples with chromatic number three to a conjecture of Keevash and Sudakov (2004), on the maximum number of edges not in any monochromatic copy of $H$ in a 2-edge-coloring of a complete graph of order $n$.

KEYWORDS
decomposition family, odd-ballooning, tree, Turán number

1 | INTRODUCTION

Let $G = (V(G), E(G))$ be a graph and $e(G) = |E(G)|$. For $v \in V(G)$ and $S \subseteq V(G)$, define $N_S(v) = \{u : uv \in E(G) \text{ and } u \in S\}$ and $|N_S(v)| = d_S(v)$, and set $N_S(v) = N(v)$ and $|N(v)| = d(v)$ if $S = V(G)$. The maximum degree of $G$ is $\Delta(G) = \max\{d(v) : v \in V(G)\}$. For $X \subseteq V(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$. If $X, Y \subseteq V(G)$ and $X \cap Y = \emptyset$, then $G[X, Y]$ denotes the induced bipartite subgraph of $G$ with bipartition $(X, Y)$. Let $G, H$ be two graphs. We use $G \cup H$ to denote the disjoint union of $G$ and $H$. Let $G + H$ denote the join of $G$ and $H$, which is obtained from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$. A path, cycle, complete graph, empty graph and star on $n$ vertices are denoted by $P_n, C_n, K_n, E_n$ and $S_{n-1}$, respectively. For a family of graphs $\mathcal{H}$, $G$ is called $\mathcal{H}$-free if $G$ contains no member of $\mathcal{H}$ as a subgraph. Write $H$-free for $\mathcal{H}$-free if $\mathcal{H} = \{H\}$. A covering of $G$ is a set of vertices which
meet all edges. Let $\beta(G)$ denote the minimum number of vertices in a covering of $G$. Define $\nu(G)$ as the size of a maximum matching of $G$. Set

$$f(\nu, \Delta) = \max\{e(G) : \nu(G) \leq \nu \text{ and } \Delta(G) \leq \Delta\}.$$ 

The following is a celebrated theorem due to Chvátal and Hanson.

**Theorem 1** (Chvátal and Hanson [3]).

$$f(\nu, \Delta) = \nu \Delta + \left\lfloor \frac{\nu}{\Delta} \right\rfloor \leq \nu(\Delta + 1).$$

The special case when $\nu = \Delta = k - 1$ was first determined by Abbott, Hanson, and Sauer [1]:

$$f(k - 1, k - 1) = \begin{cases} k^2 - k & \text{if } k \text{ is odd}, \\ k^2 - \frac{3k}{2} & \text{if } k \text{ is even}. \end{cases}$$

The Turán number of $H$, denoted by $\text{ex}(n, H)$, is the maximum size of an $H$-free graph on $n$ vertices. Extremal graph theory dates back to the early 1940s when Turán [17] proved that $T_p(n)$ is the unique extremal graph of $\text{ex}(n, K_{p+1})$, where $T_p(n)$ is a complete $p$-partite graph on $n$ vertices in which each partite set has $\lfloor n/p \rfloor$ or $\lceil n/p \rceil$ vertices, called Turán graph. The famous Erdős–Stone–Simonovits Theorem [5, 6] states if $H$ is a graph with chromatic number $\chi(H) = p + 1 \geq 3$, then

$$\text{ex}(n, H) = e(T_p(n)) + o(n^2).$$

This means the Turán number $\text{ex}(n, H)$ is determined asymptotically for $H$ being a nonbipartite graph. Moreover, Erdős and Simonovits [5] showed that the “asymptotic structure” of the extremal graphs for the Turán number $\text{ex}(n, H)$ is also determined by the chromatic number $\chi(H)$ of $H$. This is to say, the extremal graphs for a nonbipartite $H$ are very similar to $T_p(n)$ when $n$ is sufficiently large. For convenience to describe the “asymptotic structure” of the extremal graphs for $\text{ex}(n, H)$, Simonovits [16] defined the following decomposition family.

**Definition 1.1.** The decomposition family $\mathcal{M}(H)$ is the set of minimal graphs $M$ such that if an $M$ is embedded into one partite set of $T_p(n)$, then the resulting graph contains $H$ as a subgraph, where the minimal means $M$ missing any edge has no this property, $\chi(H) = p + 1$ and $n$ is large enough.

This concept provides us an idea on how to consider the extremal graphs for $\text{ex}(n, H)$ via Turán graph $T_p(n)$, that is, one can embed a maximal $\mathcal{M}(H)$-free graph into one partite set of $T_p(n)$ to obtain the extremal graphs for $\text{ex}(n, H)$. Although it is still a challenging problem to determine the Turán number and all extremal graphs for many nonbipartite graphs, Simonovits' method is still a very useful tool for dealing with Turán problems. The following result, on the Turán number of friendship graph, denoted by $F_k$, consisting of $k$ triangles intersecting in one common vertex, can be viewed as a classical example using this idea.
Theorem 2 (Erdős, Füredi, Gould, and Gunderson [4]). For any $k \geq 1$ and $n \geq 50k^2$,
\[
ex(n, F_k) = ex(n, K_3) + f(k - 1, k - 1).
\]

Note that the decomposition family of the friendship graph is $\{kK_2, S_k\}$ and the maximum number of edges in a $\{kK_2, S_k\}$-free graph is exactly $f(k - 1, k - 1)$. It was also shown in [4] that the only extremal graph in Theorem 2 is the one obtained from $T_2(n)$ by embedding a maximum $\{kK_2, S_k\}$-free graph into one partite set.

Theorem 2 was generalized in different ways. Since a triangle is a $K_3$, Chen et al. [2] considered the same problem by replacing each triangle with a $K_p$, $p \geq 2$, in Theorem 2, and obtained a generalization as follows.

Theorem 3 (Chen, Gould, Pfender, and Wei [2]). For any $p \geq 2$ and $k \geq 1$, when
\[
nK_p k K n K f k k ex( , ) = ex( , ) + (−1, −1),
\]

Inspired by these results, Glebov [7] considered the same extremal problem for many $K_{p+1}$ intersecting in a different way: replace each edge of $P_k$ with a $K_{p+1}$. Shortly after that, Liu [13] introduced the concept of edge blow-up graph. Let $H$ be a graph and $p$ be an integer, the edge blow-up of $H$, denoted by $HP^{p+1}$, is one obtained by replacing each edge in $H$ with a clique $K_{p+1}$, where the new vertices are all different. Clearly, the graphs mentioned above are all special edge blow-up graphs: $S_k^3, S_k^{p+1}$ and $P_k^{p+1}$. Liu [13] also determined the Turán number for the edge blow-up of some trees and all cycles. Recently, Wang, Hou, Liu, and Ma [18] determined the extremal graphs for the edge blow-up of a large family of trees and Yuan [20] got a tight bound for any $G^{p+1}$ when $p \geq \chi(G)$.

Observe that a triangle is also a $C_3$, so Theorem 2 can be generalized in another direction: replace each $C_3$ with an odd cycle. Let $H$ be any graph. The odd-ballooning of $H$ is a graph obtained by replacing each edge of $H$ with an odd cycle containing the edge, and all new vertices of the odd cycles are distinct. Obviously, the friendship graph is a special odd-ballooning of a star. Hou, Qiu, and Liu [10] and Yuan [19] studied the Turán number of any odd-ballooning of a star and got the following.

Theorem 4 (Hou, Qiu, and Liu [10] and Yuan [19]). Let $T = S_k$ be a star and $T_o$ be an odd-ballooning of $T$. When $n$ is large enough,
\[
ex(n, T_o) = \begin{cases} n^2/4 & \text{if } T_o \neq F_k, \\ (k - 1)^2/2 & \text{if } T_o = F_k. \end{cases}
\]

Later, Zhu, Kang, and Shan [21] determined the Turán number for the odd-ballooning of a path.

No matter a star or a path, it is a special tree. This motivates us to consider the Turán number for the odd-ballooning of a tree $T$ in a more general situation. Before stating our result, we first introduce some additional notations. Throughout this paper, we use $T = T[A, B]$ to denote a tree $T$ with unique bipartition $A, B$, and assume $a = |A| \leq |B|$ and $\delta(A) = \min\{d(v) : v \in A\}$. An edge $uv$ is called a leaf-edge if at least one end of $uv$ is of degree 1. The odd-ballooning of $T$ is denoted by $T_o$. For any $uv \in E(T)$, we use $C(u, v)$ to denote the odd cycle containing $uv$ in $T_o$ and
P(u, v) = C(u, v) - uv. If C(u, v) is a triangle in To, then we say the edge uv is of Type I and \(uv\) is of Type II otherwise. An odd-ballooning \(T_0\) of \(T\) is good if all edges of Type I are leaf-edges and the leaf vertices of these leaf-edges are in \(B\).

We still use Simonovits’ idea to investigate the Turán number \(ex(n, T_0)\) and the extremal graphs. Let \(G(n, p, a) = E_{a-1} + T_p(n - a + 1)\), \(X = V(E_{a-1})\) and \(X_1, ..., X_p\) be the \(p\) partite sets of \(T_p(n - a + 1)\). We use \(G(n, p, a, F)\) and \(G(n, p, a, \mathcal{H}, F)\) to denote a graph obtained from \(G(n, p, a)\) by embedding a graph \(F\) into \(X_1\), and embedding a maximum \(\mathcal{H}\)-free graph into \(X\) and a graph \(F\) into \(X_i\), respectively. Moreover, we define a family of subgraphs \(B(T_0)\) based on the decomposition family \(\mathcal{M}(T_0)\) as follows.

**Definition 1.2.** If each \(M \in \mathcal{M}(T_0)\) has no covering with less than \(a\) vertices, then \(B(T_0) = \{K_a\}\) and otherwise,

\[B(T_0) = \{M[S] : M \in \mathcal{M}(T_0) \text{ and } S \text{ is a covering of } M \text{ with } |S| < a\}.\]

The main result of this paper is the following.

**Theorem 5.** Let \(T = T[A, B]\) be a tree with \(a = |A| \leq |B|\) and \(\delta(A) = k\), and \(T_0\) be a good odd-ballooning of \(T\). If \(n\) is sufficiently large, then

\[ex(n, T_0) = e(G(n, 2, a)) + ex(a - 1, B(T_0)) + \begin{cases} (k - 1)^2 & \text{if } T_0 \neq F_k, \\ f(k - 1, k - 1) & \text{if } T_0 = F_k. \end{cases}\]

Furthermore, if \(T_0 \neq F_k\), then \(G(n, 2, a, B(T_0), K_{k-1, k-1})\) is an extremal graph, and if \(T_0 = F_k\), then \(G(n, 2, 1, F)\) is an extremal graph, where \(F\) is the extremal graph of the function \(f(k - 1, k - 1)\).

Since a star \(S_k\) is a tree \(T[A, B]\) with \(|A| = a = 1\) and any odd-ballooning of \(S_k\) is good, one can see Theorem 5 generalizes the results obtained in [4, 9, 10, 19]. Moreover, let \(f(n, H)\) denote the maximum number of edges not in any monochromatic copy of \(H\) in a 2-edge-coloring of \(K_n\). As a by-product, Theorem 5 also provides some counterexamples with chromatic number 3 to the following conjecture:

**Conjecture 1** (Keevash and Sudakov [12]). For any \(H\) and sufficiently large \(n\),

\[f(n, H) = ex(n, H).\]

The details will be presented in Section 5. The other parts of this paper are organized as follows. Section 2 is devoted to characterizing the decomposition family of \(T_0\); Section 3 contains some preliminaries; and the proof of Theorem 5 is given in Section 4.

## 2 DECOMPOSITION FAMILY OF \(T_0\)

In this section, our task is to characterize the decomposition family of \(T_0\) through two operations. Let \(G\) be a graph, \(u \in V(G)\) and \(N(u) = \{v_1, ..., v_d\}\). A splitting on \(u\) is to replace \(u\) with an independent set \(\{u_1, ..., u_d\}\) and \(uv_i\) with a new edge \(u_iv_i, 1 \leq i \leq d\), and write
Let \( u_v \) be a leaf-edge. If \( d(u) = 1 \) and \( d(v) \geq 2 \), then peeling off \( u_v \) is to delete \( u_v \) and add a new vertex \( v' \) and a new edge \( u_v' \), and write \( p(u_v) = u_v' \). If \( d(u) = d(v) = 1 \), then peeling off \( u_v \) is to do nothing and \( p(u_v) = u_v \).

For a given odd-ballooning \( T_0 \) of a tree \( T \), we use \( \mathcal{SP}(T) \) to denote the family of graphs, each of which can be obtained from \( T \) through splitting the vertices in some independent set first, say the resulting graph \( T' \). Then peeling off some leaf-edges of \( T' \) which satisfies \( u_v \) or \( s^{-1}(u_v) \) is of Type II.

We have the following lemma.

**Lemma 1.** For any tree \( T \) and any odd-ballooning \( T_0, \mathcal{M}(T_0) = \mathcal{SP}(T) \). In particular, if \( T_0 \) is good, then a matching of size \( e(T) \) is in \( \mathcal{M}(T_0) \).

**Proof.** Let \( X, Y \) be two partite sets of \( T_0(n) \), where \( n \) is sufficiently large.

Let \( M \) be any graph in \( \mathcal{M}(T_0) \). We first show \( M \in \mathcal{SP}(T) \). Embed \( M \) into \( X \), then by the definition of \( \mathcal{M}(T_0) \), the resulting graph contains a copy of \( T_0 \). Color the vertices in \( X \) red and the vertices in \( Y \) blue, and call an edge red if its two ends are colored red. Clearly, \( E(M) \) are all red edges.

Because \( \chi(T_0) = 3 \) and \( \chi(T_0 - E(M)) = 2 \), hence \( |E(M) \cap E(C(u, v))| \geq 1 \). Suppose \( C(u, v) \) is an odd cycle in \( T_0 \) containing at least two red edges. Since \( X, Y \) are large enough, if \( u_v \) is a red edge, then we can replace \( P(u, v) = ua_1 \cdots a_v v \) by a new proper colored path \( ub_1 \cdots b_v v \) using vertices distinct with the original \( T_0 \), and if \( u \) is red and \( v \) is blue, then we can replace \( P(u, v) \) by a path which contains exactly one red edge of \( P(u, v) \). Obviously, the red edges in the new \( T_0 \) form a subgraph of \( M \), contradicting the minimality of \( M \). Therefore, \( C(u, v) \) contains exactly one red edge.

Now, consider the skeleton \( T \) of \( T_0 \). The blue vertices in \( T \) (if any) is an independent set. Split all blue vertices of \( T \). Suppose \( u \) is any blue vertex, \( N_T(u) = \{v_1, ..., v_d\} \) and \( u \) is split into \( \{u_1, ..., u_d\} \). Clearly, \( u_i v_i \) is a leaf-edge after splitting. Let \( w_1 w' \) be the only red edge in \( C(u, v_i) \). If \( w_1 w' \) is not incident \( v_i \), then \( C(u, v_i) \) is of order at least 5 and \( u_v \) is of Type II. In this case, we peel off \( u_i v_i \) and let \( p(u_i v_i) \) be the new edge. If \( w_1 w' \) is incident with \( v_i \), then we do not peel off \( u_i v_i \). Let \( T' \) be the resulting graph by peeling off all such edges. Then \( T' \in \mathcal{SP}(T) \). Now, let \( e \leftrightarrow e' \) if \( e \in E(M) \cap E(T'), w_1 w' \leftrightarrow p(u_i v_i) \) if \( w_1 w' \) is not incident with \( v_i \) and \( w_1 w' \leftrightarrow u_i v_i \) if \( w_1 w' \) is incident with \( v_i \). We can see that \( M \cong T' \), and so \( M \in \mathcal{SP}(T) \).

On the other hand, let \( T' \) be any graph in \( \mathcal{SP}(T) \), which is obtained from \( T \) by splitting some independent set \( U \) first, and then peeling off some leaf-edges satisfying \( u_v \) or \( s^{-1}(u_v) \) is of Type II, from the resulting graph. We will show \( T' \in \mathcal{M}(T_0) \).

For any \( u \in V(T) \), color \( u \) blue if \( u \in U \) or \( u_v \in E(T) \) is a leaf-edge with \( d(u) = 1 \) and \( v \notin U \), otherwise color \( u \) red. And then, color the vertices in \( V(T_0) - V(T) \) as follows. Let \( u_v \) be any edge of \( T \). If \( u_v \) is red, then give a proper red–blue coloring to \( P(u, v) \); If \( u \) is blue and \( v \) is red, then \( d_T(u) = 1 \) and \( v \notin U \), or \( u \in U \) is split and say \( s(uv) = u'v \). In this case, give \( P(u, v) \) a red–blue coloring such that it contains exactly one red edge \( w_1 w' \), and \( w_1 w' \) is incident with \( v \) if \( u_v \) or \( u'v \) is not peeled off and not incident \( v \) if \( u_v \) or \( u'v \) is peeled off. Because if \( u_v \) or \( u'v \) is peeled off, then \( u_v \) is of Type II and \( P(u, v) \) is an odd path of order at least 5, and so such a coloring exists and all blue vertices form an independent set. Assume that \( M' \) is the subgraph in \( T_0 \) induced by all red edges. Let \( e \leftrightarrow e' \) if \( e \in E(T) \cap E(M'), \ p(uv) \leftrightarrow w_1 w' \) if \( d_T(u) = 1 \) and \( u_v \) is peeled off, \( u'v \leftrightarrow w_1 w' \) if \( w_1 w' \) is incident with \( v \) and \( p(u'v) \leftrightarrow w_1 w' \) if \( w_1 w' \) is not incident with \( v \). We can see that \( T' \cong M' \).
Observe that each odd cycle in $T_o$ contains exactly one red edge, we can see that $T_o - E(M')$ is proper red–blue colored. Thus, we can embed $T_o - E(M')$ into $T_2(n)$ such that all red vertices of $T_o$ are in $X$ and all blue vertices of $T_o$ are in $Y$. Note that $T' \cong M'$ and $e(T') = e(T)$, we have $T' \in \mathcal{M}(T_o)$.

In particular, if $T_o$ is a good odd-ballooning of $T = T[A, B]$, then apply vertex splitting on $A$ first, the resulting graph is the disjoint union of stars and all edges of Type I become isolated edges. For any star other than a $K_4$ in the resulting graph, each edge $uv$ of it satisfying $uv$ or $s^{-1}(uv)$ is of Type II. Thus, peel off some edges from each such star, we can obtain a matching of size $e(T)$. Since $\mathcal{M}(T_o) = SP(T)$, this matching is in $\mathcal{M}(T_o)$.

### 3 | PRELIMINARIES

**Lemma 2** (König [11]). Let $G$ be a bipartite graph. Then $\beta(G) = \nu(G)$.

**Lemma 3** (Hall [8]). Let $G = G[X, Y]$ be a bipartite graph. Then $\nu(G) \geq |X|$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

**Lemma 4** (Wang, Hou, Liu, and Ma [18]). Let $T[A, B]$ be a tree with $\delta(A) = k \geq 2$. If a vertex in $A$ is split, then the resulting graph $T'$ satisfies $\nu(T') \geq a - 1 + k$.

**Lemma 5.** Let $T = T[A, B]$ be a tree and $T_o$ be any odd-ballooning of $T$. Then $B(T_o) = \{K_o\}$ if and only if $\beta(T) = a$. Furthermore, if $\delta(A) \geq 2$, then $\beta(T) = a$.

**Proof.** By Lemma 1, $\mathcal{M}(T_o) = SP(T)$. Let $T'$ be any graph in $SP(T)$. Since splitting vertices and peeling off leaf-edges do not decrease the size of maximum matching, we have $\nu(T') \geq \nu(T)$. If $\beta(T) = a$, then by Lemma 2, $\beta(T') = \nu(T') \geq \nu(T) = \beta(T) = a$. That is, $T'$ has no covering $S$ with $|S| < a$. By Definition 1.2, we have $B(T_o) = \{K_o\}$. On the other hand, if $B(T_o) = \{K_o\}$, then since $T \in \mathcal{M}(T_o)$, we get $\beta(T) = a$.

Furthermore, if $\delta(A) \geq 2$, then since $T[S, N(S)]$ is a forest for any $S \subseteq A$, we have

$$2|S| \leq e(T[S, N(S)]) \leq |S| + |N(S)| - 1,$$

which implies $|N(S)| \geq |S| + 1$. By Lemma 3, we have $\beta(T) = \nu(T) = a$.

**Lemma 6.** Let $G$ be an $\{S_k, kK_2, S_{k-1} \cup K_3\}$-free graph without isolated vertices. Then $e(G) \leq (k - 1)^2$ with equality if and only if $G = K_{k-1,k-1}$, or $G = 3K_3$ and $k = 4$.

**Proof.** Let $G$ be an $\{S_k, kK_2, S_{k-1} \cup K_3\}$-free graph with $e(G) \geq (k - 1)^2$. Obviously, $\Delta(G) \leq k - 1$. If $\Delta(G) \leq k - 2$, then $G$ is $\{kK_2, S_{k-1}\}$-free and hence we have

$$(k - 1)^2 \leq e(G) \leq f(k - 1, k - 2) = (k - 1)(k - 2) + \left|\frac{k - 2}{2}\right|\left|\frac{k - 1}{2(k - 2)/2}\right|,$$

which implies $k = 4$ and $G = 3K_3$. If $\Delta(G) = k - 1$, let $v$ be a vertex with $d(v) = k - 1$. Since $G$ is $S_{k-1} \cup K_2$-free, $G - N(v)$ has no edges, and so $e(G) \leq (k - 1)^2$, with equality if
and only if all vertices in \( N(v) \) have degree \( k - 1 \) and \( N(v) \) is an independent set. Moreover, all vertices in \( N(v) \) have the same neighborhoods for otherwise we can find an \( S_{k-1} \cup K_2 \) in \( G \). Therefore, \( G = K_{k-1,k-1} \).

**Lemma 7** (Erdős, Füredi, Gould, and Gunderson [4]). Let \( \Delta \) and \( b \) be two nonnegative integers such that \( b \leq \Delta - 2 \). If \( \Delta(G) \leq \Delta \), then

\[
\sum_{v \in V(G)} \min\{d(v), b\} \leq \nu(G)(b + \Delta).
\]

Suppose \( G \) is a graph with partition \( V(G) = V_0 \cup V_1 \), let \( G_0 = G[V_0], G_1 = G[V_1], G_{cr} = G[V_0, V_1], \) and \( d_{cr}(v) \) be the degree of the vertex \( v \) in \( G_{cr} \). We have the following.

**Lemma 8.** Let \( k \geq k_i \) be two nonnegative integers. If the following hold,

1. \( G_i \) is \( \{S_{k-\ell} \cup \ell K_2 : 0 \leq \ell \leq \min\{k - k_i, k - 2\}\} \)-free for \( i = 0, 1 \),
2. \( d_{cr}(v) + \nu(G_{1-i}[N_{V_i}(v)]) \leq k - 1 \) for any \( v \in V_i \),
3. If \( k > k_i \), then \( N_{V_{1-i}}(v) \) are isolated in \( G_{1-i} \) for any \( v \in V_i \) with \( d_{V_i}(v) = k - 1 \), then

\[
e(G_0) + e(G_1) - (|V_0||V_1| - e(G_{cr})) \leq \begin{cases} (k - 1)^2 & \text{for } k > k_i, \\ f(k - 1, k - 1) & \text{for } k = k_i. \end{cases}
\]

**Proof:** As it is shown in [4] that (1) holds for \( k = k_i \), we may assume that \( k > k_i \).

Choose a graph \( G \) with partition \( V(G) = V_0 \cup V_1 \) satisfying (1)–(3), such that \( e(G_0) + e(G_1) - (|V_0||V_1| - e(G_{cr})) \) is as large as possible, and subject to this, \( |G| \) is as small as possible.

If there is some \( v \in V_0 \) such that \( d_{V_0}(v) - (|V_1| - d_{cr}(v)) \leq 0 \), then since \( G - v \) with partition \( V(G - v) = (V_0 - \{v\}) \cup V_1 \) still satisfies (1)–(3), we have \( |G - v| < |G| \) and

\[
e(G_0) + e(G_1) - (|V_0 - \{v\}||V_1| - e(G_{cr} - v)) \\
\geq e(G_0) + e(G_1) - (|V_0||V_1| - e(G_{cr})),
\]

this contradicts the choice of \( G \). Thus, we have \( d_{V_0}(v) - (|V_1| - d_{cr}(v)) > 0 \) for any \( v \in V_1 \) by the symmetry of \( V_0 \) and \( V_1 \). Moreover, because \( d_{V_0}(v) + \nu(G_1[N_{V_1}(v)]) \leq k - 1 \), we have \( d_{V_0}(v) - (|V_1| - d_{cr}(v)) \leq k - 1 - \nu(G_1[N_{V_1}(v)]) \leq k - 1 - \nu(G_1) \). The second inequality holds since any matching in \( G_1 \) has at most \( \nu(G_1[N_{V_1}(v)]) \) edges in \( N_{V_1}(v) \) and at most \( |V_1| - d_{cr}(v) \) additional edges. Therefore, for any \( v \in V_i, i = 0, 1, \) we have

\[
0 < d_{V_i}(v) - (|V_{1-i}| - d_{cr}(v)) \leq k - 1 - \nu(G_{1-i}).
\]

By (2), we can deduce that

\[
\nu(G_i) \leq k - 2 \quad \text{for } i = 0, 1.
\]
If \( \nu(G_i) = 0 \) for some \( i \in \{0, 1\} \), then \( e(G_i) = 0 \). Observe that \( |V_0||V_i| - e(G_{cr}) = e(G[V_0, V_i]) \geq 0 \), we get \( e(G_0) + e(G_1) - (|V_0||V_i| - e(G_{cr})) \leq e(G_{1-i}) \leq (k - 1)^2 \) by Lemma 6, and hence (1) holds.

If \( \nu(G_i) = 1 \) for some \( i \in \{0, 1\} \), we assume that \( \nu(G_i) = 1 \) by symmetry. Clearly, \( G_1 \) is a star or a triangle with some isolated vertices and \( e(G_k) \leq \max \{ -1, 3 \} \leq 0 \). Since \( G_0 \) is \( S_k \)-free, \( G_k \Delta 0 \leq 0 \). If \( G_k \Delta 0 \leq -1 \), then \( e(G_k - 2) \leq 0 \) by (3) and Lemma 1. Let \( v \in V_0 \) with \( d_{V_i}(v) = k - 1 \). By the assumption (3), \( N_{V_i}(v) \) are isolated vertices of \( G_1 \), which implies the vertices of the star or the triangle in \( G_1 \) are nonadjacent to \( v \), and hence \( |V_0||V_i| - e(G_{cr}) \geq e(G_1) \). Thus, we have

\[
e(G_0) + e(G_1) - (|V_0||V_i| - e(G_{cr})) \leq (k - 2)k + e(G_1) - e(G_1) < (k - 1)^2.
\]

If \( \Delta(G_0) \leq k - 2 \), then \( e(G_0) \leq f(k - 2, k - 2) \) by Lemma 1 and

\[
e(G_0) + e(G_1) - (|V_0||V_i| - e(G_{cr})) \leq f(k - 2, k - 2) + k - 1 \leq (k - 1)^2.
\]

Now, assume that \( 2 \leq \nu(G_i) \leq k - 2 \) for each \( i \). By (2), we have

\[
2e(G_i) - (|V_0||V_i| - e(G_{cr})) = \sum_{v \in V_i} [d_{V_i}(v) - (|V_i - 1| - d_{cr}(v))]
\leq \sum_{v \in V_i} \min\{d_{V_i}(v), k - 1 - \nu(G_{1-i})\}.
\]

Applying Lemma 7 on \( G_i \) with \( \Delta = k - 1 \) and \( b = k - 1 - \nu(G_{1-i}) \leq \Delta - 2 \), we have

\[
2e(G_i) - (|V_0||V_i| - e(G_{cr})) \leq \nu(G_i) (2(k - 1) - \nu(G_{1-i})),
\]

and then

\[
2e(G_0) + 2e(G_1) - 2(|V_0||V_i| - e(G_{cr}))
\leq \nu(G_0)(2(k - 1) - \nu(G_1)) + \nu(G_1)(2(k - 1) - \nu(G_0))
= 2(k^2 - 2k + 1) - (k - 1 - \nu(G_0))(k - 1 - \nu(G_1)) < 2(k - 1)^2,
\]

the last inequality follows from (3). Thus, we finish the proof of Lemma 8.

\[\square\]

4 | Proof of Theorem 5

Let \( u \in A \) be any vertex with \( d(u) = \delta(A) = k \), and \( d_i(u) = k_i \) denote the number of edges of Type I incident with \( u \) in \( T \). It should be noted that \( d(u) = k = k_i = d_i(u) \) if and only if \( T_o = F_k \) by the definition of good odd-ballooning.

4.1 | Lower bound of \( \text{ex}(n, T_o) \)

It is not difficult to check the sizes of the two graphs described in Theorem 5 is the expected value of \( \text{ex}(n, T_o) \). So it suffices to show each of the two graphs is \( T_o \)-free to get the lower bound for \( \text{ex}(n, T_o) \).
As we have mentioned, if \( d_t(u) = k_t = k \), then \( T = S_k \) and \( T_0 = F_k \). By Lemma 1, \( \mathcal{M}(T_0) = \{ S_k, kK_2 \} \). Let \( F \) be an extremal graph of the function \( f(k - 1, k - 1) \). So \( F \) is \( \{ S_k, kK_2 \} \)-free. Note that embedding an \( F \) into one partite set of \( T_2(n) \) is \( G(n, 2, 1, F) \), by the definition of decomposition family, it is \( T_0 \)-free. Hence we may assume \( d_t(u) < k \).

It remains to show \( G = G(n, 2, a, B(T_0), K_{k-1,k-1}) \) is \( T_0 \)-free. Since \( G \) is obtained by embedding a \( K_{k-1,k-1} \) into the partite set \( X_i \) and a maximum \( B(T_0) \)-free graph into the partite set \( X \) of \( G(n, 2, a) \), by the definition of decomposition family, we need only to prove \( G[X \cup X_i] = G[X] + G[X_i] \) is \( \mathcal{M}(T_0) \)-free.

If \( k = 1 \), then \( e(G[X_i]) = 0 \). Since \( G[X] \) is \( B(T_0) \)-free, by the definition of \( B(T_0) \), we know \( G[X \cup X_i] \) is \( \mathcal{M}(T_0) \)-free, and so we may assume \( k \geq 2 \). In this case, we have \( G[X] = K_{a-1} \) by Lemma 5.

Suppose to the contrary that \( G[X \cup X_i] \) contains a \( T' \in \mathcal{M}(T_0) \). Choose \( T' \) such that \( T' \) is minimum. By Lemma 1, \( T' \in SP(T) \). Assume that \( T' \) is obtained from \( T \) by splitting an independent set \( U \) and then peeling off some leaf-edges satisfying \( uv \) or \( s^{-1}(uv) \) is of Type II. Clearly, \( T' \) is bipartite. Let \( A', B' \) be two partite sets of \( T' \) such that if \( u \in A \) or \( u \) is split from a vertex in \( A \), then \( u \in A' \), and the same is for \( B \) and \( B' \). Such a bipartition \( A', B' \) is unique. Let \( A'_1 = X \cap A', A'_2 = X \cap A', B'_1 = X \cap B', B'_2 = X \cap B' \) and \( T'[A'_2, B'_2] \) be the subgraph of \( T' \) induced by \( A'_2 \cup B'_2 \).

If some vertex in \( A \) is split, then \( \nu(T') \geq a - 1 + k \) by Lemma 4. Since \( A'_1 \cup B'_1 \) is a vertex covering of \( T' - E(T'[A'_2, B'_2]) \), \( \nu(T' - E(T'[A'_2, B'_2])) \) \( |A'_1 \cup B'_1| \leq a - 1 \) by Lemma 2. Note that \( \nu(T'[A'_2, B'_2]) \leq k - 1 \) since only \( K_{k-1,k-1} \) is in \( X_i \), we have

\[
\nu(T') \leq \nu(T' - T'[A'_2, B'_2]) + \nu(T'[A'_2, B'_2]) \leq a + k - 2,
\]
a contradiction. Thus, no vertex in \( A \) is split.

Assume that \( uv \) is a leaf-edge with \( d(v) = 1 \), which is peeled off after splitting \( U \), and \( p(uv) = u'v \). Because \( \delta(A) = k \geq 2 \) and no vertex in \( A \) is split, we have \( u \in A \) and \( u' \in A' \). Suppose that we get \( T' \) by peeling off \( uv \) from \( T'_* \). Then \( T'_* \in SP(T) \). We now show \( T'_* \subseteq G[X \cup X_i] \). If \( u \in A'_1 \) or \( u \in A'_2 \) and one of \( u' \), \( v \) is in \( X \), then \( uv \) or \( uuv \) is an edge of \( G[X \cup X_i] \). Add \( uv \) or \( uuv \), and delete \( u'v \), we can find a \( T'_* \) in \( G[X \cup X_i] \). If \( u, u' \in A'_2 \) and \( v \in B'_2 \), then since \( G[X_i] \) is a \( K_{k-1,k-1} \) with some isolated vertices, one of \( u' \), \( v \) is adjacent to all \( N_T(u) \). Replace \( u \) with \( u' \) or \( v \), we can find a \( T'_* \) in \( G[X \cup X_i] \). By Lemma 1, \( T'_* \in \mathcal{M}(T_0) \), which contradicts the choice of \( T' \) since \( |T'_*| < |T'| \). Thus, no edge is peeling off after splitting \( U \).

By the argument above, \( A = A' = A'_1 \cup A'_2 \). Since \( a = |A'_1| + |A'_2| \geq |A'_1| + |B'_1| + 1 \) and \( T'[A'_2, B'_2] \) is a forest, we have

\[
e(T'[A'_2, B'_2]) \geq k |A'_2| - (|A'_2| + |B'_1| - 1) \geq (k - 2)|A'_2| + 2.
\]

If \( k = 2 \), then \( e(T'[A'_2, B'_2]) \geq 2 \), which contradicts that \( e(G[X_i]) = 1 \), and hence \( k \geq 3 \). Because \( \Delta(G[X_i]) = k - 1 \), by the inequality above, \( A'_2 \) has two vertices of degree \( k - 1 \) in \( T'[A'_2, B'_2] \), that lie in the same partite set of \( K_{k-1,k-1} \). Thus, the two vertices have the same neighbors in \( B'_2 \), and so \( T'[A'_2, B'_2] \) has cycles, a contradiction.

Therefore, \( G[X] + G[X_i] \) is \( \mathcal{M}(T_0) \)-free, and so \( G \) is \( T_0 \)-free.
4.2 Upper bound of \( \text{ex}(n, T_0) \)

To establish the upper bound for \( \text{ex}(n, T_0) \), we need a result of Simonovits.

**Definition 4.1.** Denote by \( \mathcal{D}(n, p, r) \) the family of \( n \)-vertex graphs \( G \) satisfying the following symmetry condition:

- It is possible to omit at most \( r \) vertices of \( G \) so that the remaining graph \( G' \) is a join of graphs of almost equal order: \( G' = G_1 + \cdots + G_p \), where \( \left| V(G_i) \right| - \frac{n}{p} \leq r \) for all \( i \leq p \).
- For each \( i \leq p \), there exist connected graphs \( H_i \) such that \( G_i = k_i H_i \), where \( k_i = \frac{|V(G_i)|}{|V(H_i)|} \), and any two copies \( H'_i, H''_i \) of \( H_i \) in \( G_i \), are symmetric subgraphs of \( G \): there exists an isomorphism \( \phi : V(H'_i) \to V(H''_i) \) such that for any \( u \in V(H'_i) \) and \( v \in V(G) - V(G') \), \( uv \in E(G) \) if and only if \( \phi(u)v \in E(G) \).

The graphs \( H_i \) (\( 1 \leq i \leq p \)) will be called the *blocks* and the vertices in \( V(G) - V(G') \) will be called *exceptional vertices*.

**Theorem 6** (Simonovits [16]). For a given graph \( H \) with \( \chi(H) = p + 1 \), if \( \mathcal{M}(H) \) contains a linear forest, then there exist \( r = r(H) \) and \( n_0 = n_0(r) \) such that \( \mathcal{D}(n, p, r) \) contains an extremal graph of \( H \) for all \( n \geq n_0 \). Furthermore, if this is the only extremal graph in \( \mathcal{D}(n, p, r) \), then it is the unique extremal graph for every sufficiently large \( n \).

We now begin to show the upper bound of \( \text{ex}(n, T_0) \).

Let \( T_0 \) be any good odd-ballooning of a tree \( T = [A, B] \) with \( a = |A| \leq |B| \). By Lemma 1, \( \mathcal{M}(T_0) \) contains a linear forest \( e(T) \cdot K_2 \). By Theorem 6, \( \mathcal{D}(n, 2, r) \) contains an extremal graph \( G \) for sufficiently large \( n \). Omit at most \( r = r(T_0) \) exceptional vertices from \( G \), then it is the unique extremal graph for every sufficiently large \( n \).

Let \( A_i \) be the sets of vertices in \( A_1 \cup A_2 \), or to all vertices in \( A_3 \), but no vertices in \( A_3 \), or to no vertices in \( A_3 \). Let \( W \) and \( W' \) be the sets of vertices in \( V(G) - V(G') \) that are adjacent to all vertices and to no vertex in \( G' \), respectively, and \( B_i \) be the set of vertices in \( V(G) - V(G') \) that are adjacent to all vertices in \( A_3 \) but not any vertex in \( A_i \), \( i = 1, 2 \). If \( W' = \emptyset \), say \( v \in W' \), then we can delete the edges between \( v \) and all other exceptional vertices, and then add all edges between \( v \) and \( A_2 \). This does not decrease the number of edges because \( r \) is a constant and \( A_2 \sim \frac{n}{2} \). Also \( G \) still contains no \( T_0 \), for otherwise replace \( v \) with some vertex in \( A_1 \), we can find a \( T_0 \) in the original graph \( G \). Hence, \( W' = \emptyset \).

**Claim 1.** \( |W| = a - 1 \) and \( G[W] \) is \( B(T_0) \)-free.

**Proof.** If \( |W| \geq a \), then we can find a copy of \( T \) in \( G[W \cup A_1] \). By Lemma 1, \( T \in \mathcal{M}(T_0) \) and hence \( T_0 \subseteq G[W \cup A_1 \cup A_2] \), a contradiction. If \( |W| \leq a - 2 \), then
\[ e(G) \leq e(G(n, 2, a - 1)) + \left( \frac{|W|}{2} \right) + \left( \frac{|B_1|}{2} \right) + \left( \frac{|B_2|}{2} \right) \]
\[ = e(T_2(n - a + 2)) + (a - 2)(n - a + 2) + o(1) \]
\[ < e(T_2(n - a + 1)) + (a - 1)(n - a + 1) = e(G(n, 2, a)), \]
a contradiction. Therefore, \( |W| = a - 1 \).

If \( G[W] \) is not \( B(T_o) \)-free, then there is some \( T' \in \mathcal{M}(T_o) \) such that \( T' \subseteq G[W \cup A_1] \) by the definition of \( B(T_o) \), and hence \( T_o \subseteq G[W \cup A_1 \cup A_2] \), a contradiction. □

Let \( G^* = G - W, X_i = A_i \cup B_i \) and \( G^*_i = G[X_i] \) for \( i = 1, 2 \), and \( G^*_{cr} = G[X_1, X_2] \). Moreover, let \( T = T[A, B] \) be a tree with \( \delta(A) = k, u \in A \) with \( d(u) = k \) such that \( d(u) = k_1 \) is maximum. Let \( N(u) = \{u_1, u_2, ..., u_k\} \) and \( uu_i \) is of Type I for \( i \leq k_1 \). Obviously, \( k \geq k_1 \). We now show the graph \( G^* \) with partition \( V(G^*) = X_i \cup X_2 \) satisfying the conditions of Lemma 8 by the following three claims.

Claim 2. \( G^*_i \) is \( \{S_{k-\ell} \cup \ell K_2 : 0 \leq \ell \leq \min\{k - k_1, k - 2\}\} \)-free, \( i = 1, 2 \).

Proof: Suppose to the contrary that \( G^*_i \) contains an \( S_{k-\ell} \cup \ell K_2 \) for some \( \ell \).

If \( \ell = k - 1 \), then \( S_{k-\ell} \cup \ell K_2 = kK_2 \). Let \( T' \) be the forest obtained from \( T \) by splitting \( u \) first and then peeling off the leaf-edges of Type II which are incident with \( u \) in \( T \). Then \( T' \) is the union of \( kK_2 \) and a forest \( T'[A - \{u\}, B] \). Embed \( A - \{u\} \) into \( W \) and \( B \) into the vertices other than that of the \( kK_2 \) in \( A_1 \subseteq X_i \), we can find a \( T' \) in \( G^*[W \cup X_i] \). By Lemma 1, \( T' \in \mathcal{M}(T_o) \), a contradiction. Hence, \( G^*_i \) is \( kK_2 \)-free.

Assume that \( 0 \leq \ell \leq \min\{k - k_1, k - 2\} \). Let \( T' \) be the forest obtained from \( T \) by splitting the set \( \{u_1, u_2, ..., u_k\} \), and then peeling off \( \ell \) edges of Type II from the star \( S_{k-\ell} \) with center \( u \). Then \( T' \) is the union of \( S_{k-\ell} \cup \ell K_2 \) and a forest \( T'[A - \{u\}, B'] \). Embed \( A - \{u\} \) into \( W \) and \( B' \) into the vertices other than that of the \( S_{k-\ell} \cup \ell K_2 \) in \( A_1 \subseteq X_i \), we can get a \( T' \) in \( G^*[W \cup X_i] \). By Lemma 1, \( T' \in \mathcal{M}(T_o) \), a contradiction. So, \( G^*_i \) is \( S_{k-\ell} \cup \ell K_2 \)-free. □

Claim 3. For each \( i = 1, 2 \) and any vertex \( v \in X_i \),
\[ d_{X_i}(v) + \nu(G^*[N_{X_{-i}}(v)]) \leq k - 1. \]

Proof: By symmetry, we assume \( i = 1 \). Let \( N_{X_1}(v) = \{v_1, ..., v_t\} \) and \( \{x_1 y_1, ..., x_\ell y_\ell\} \) be a maximum matching in \( G[N_{X_1}(v)] \). If \( t + \ell \geq k \), then note that \( G^*[W, A_2] \) is a complete bipartite graph and \( |A_2| \) is sufficiently large, we can find a copy of \( T \) in \( G^* \) by embedding \( A - u \) into \( W \), \( u \) into \( v_1, ..., u_k \) into \( \{v_1, ..., v_t\} \cup \{x_1, ..., x_\ell\} \) and all other vertices of \( B \) into \( A_2 \).

For each \( xy \in E(T) \) with \( x \in W \) and \( y \in A_2 \), since \( G^*[A_1, A_2] \) is a complete bipartite graph and \( |A_2| \) is sufficiently large, we can use some vertices in \( \mathcal{A}_1 \cup \mathcal{A}_2 \) to form an odd cycle \( C(x, y) \). For each \( vv' \in E(T) \), if \( v \in \{v_1, ..., v_t\} \), then choose a vertex \( y' \in A_2 \), and if \( y = x_1 \) for some \( j \in [1, \ell] \), then let \( y' = y_j \), we can get a triangle \( C(v, y, y') \); If \( y \in \{u_k, ..., u_k\} \), then since both \( G^*[A_1, X_2] \) and \( G^*[A_2, X_1] \) are complete bipartite graphs and \( |A_2| \) is sufficiently large, we can get an odd cycle \( C(v, y) \) by using one of the \( t + \ell - k_1 \geq k - k_1 \) edges in \( \{vv_1, ..., vv_t, x_1 y_1, ..., x_\ell y_\ell\} \), which are not used to form a triangle \( vyy' \), together with some vertices in \( A_1 \cup A_2 \). Thus, \( G^* \) contains a \( T_o \), a contradiction. □
Claim 4. If \( k > k_1 \), then \( N_{X_3}(v) \) are isolated in \( G^*_3 \) for any \( v \in X_3 \) with \( d_{X_3}(v) = k - 1 \).

Proof. By symmetry, we assume \( i = 1 \). Let \( N_{X_3}(v) = \{v_1, \ldots, v_{k-1}\} \) and \( z \in N(v) \). Since \( k > k_1 \) implies \( k_1 \leq k - 1 \), \( G^*[W, A_2] \) is a complete bipartite graph and \( |A_2| \) is sufficiently large, we can find a copy of \( T \) in \( G^* \) by embedding \( A - \{u\} \) into \( W, u \) into \( v, \{u_1, \ldots, u_{k_1}\} \) into \( \{v_1, \ldots, v_{k-1}\} \) and all other vertices of \( B \) into \( A_2 \).

For each \( xy \in E(T) \) with \( x \in W \) and \( y \in A_2 \), since \( G^*[A_1, X_3] \) and \( G^*[A_2, X_3] \) are complete bipartite graphs and \( |A_i| \) is sufficiently large, we can get an odd cycle \( C_{xy}(, ) = (, ) \) by using one of the \( k - k_1 \) edges in \( \{vv_{k+1}, \ldots, vv_{k-1}, zz\} \) and some other vertices in \( A_1 \cup A_2 \). Hence, \( G^* \) contains a \( T_o \), a contradiction. \( \square \)

By the structure of \( G \), we have

\[
e(G) \leq e(G[W]) + |W||G^*| + e(G^*) = e(G[W]) + |W||G^*| + e(G^1) + e(G^2) + e(G^cr).
\]

By Claims 2–4, and applying Lemma 8 on \( G^* \), we have

\[
e(G^1) + e(G^2) + e(G^cr) \leq |X_1||X_2| + \begin{cases} (k - 1)^2, & \text{if } k > k_1, \\ f(k - 1, k - 1), & \text{if } k = k_1. \end{cases}
\]

By Claim 1, \( e(G[W]) \leq \text{ex}(a - 1, B(T_o)) \). Moreover, \( |W||G^*| + |X_1||X_2| \leq e(G(n, 2, a)) \) by the definition of the graph \( G(n, 2, a) \). Therefore, we have

\[
e(G) \leq e(G(n, 2, a)) + \text{ex}(a - 1, B(T_o)) + \begin{cases} (k - 1)^2, & \text{if } k > k_1, \\ f(k - 1, k - 1), & \text{if } k = k_1. \end{cases}
\]

The proof of Theorem 5 is complete.

5 Remark on Conjecture 1

It is clear that in a complete graph \( K_n \), if we color the edges of an extremal graph for \( \text{ex}(n, H) \) red and color the other edges blue, then any red edge is not in monochromatic \( H \), which implies \( f(n, H) \geq \text{ex}(n, H) \) for any \( H \). Keevash and Sudakov [12] proved that if \( H \) has an edge \( e \) such that \( \chi(H - e) = \chi(H) - 1 \) or \( H = C_4 \), then \( f(n, H) = \text{ex}(n, H) \). Later, Ma [15] and Liu, Pikhurko, and Sharifzadeh [14] confirmed Conjecture 1 for a large family of bipartite graphs, including cycles and some complete bipartite graphs. Recently, Yuan [20] found some counterexamples to Conjecture 1 with large chromatic number. However, it remains unknown if Conjecture 1 holds for all bipartite graphs or other graphs with a small chromatic number. In particular, is it true \( f(n, H) = \text{ex}(n, H) \) for \( \chi(H) = 3 \)?

Let \( T_o \) be a good odd-ballooning of \( T \). By Theorem 5, \( \text{ex}(n, T_o) \) is the sum of three terms, and if \( T_o \neq F_k \), then \( G(n, 2, a, B(T_o), K_{k-1,k-1}) \) is an extremal graph. Note that the size of the
maximum $B(T_o)$-free graph embedded into the set $X$ is the middle term $\text{ex}(a - 1, B(T_o))$. For a complete graph $K_n$, we divide the edges into two sets $R$ and $B$. The edges in $R$ induce the extremal graph for $\text{ex}(n, T_o)$ and the remaining edges are in the set $B$. Then we color the edges in $R$ red and the edges in $B$ blue. One can see the blue edges in the set $X$ are not covered by any monochromatic $T_o$, either. Hence, we have

$$f(n, T_o) \geq e(G(n, 2, a)) + \begin{cases} a - 1 \choose 2 & \text{if } T_o \neq F_k, \\ (k - 1)^2 & \text{if } T_o = F_k. \end{cases}$$

Compare the right-hand side of this inequality with $\text{ex}(n, T_o)$ in Theorem 5, we can see if $\text{ex}(a - 1, B(T_o)) \neq \begin{pmatrix} a - 1 \\ 2 \end{pmatrix}$, then Conjecture 1 is not true for $T_o$. By Lemma 5, we know if $\beta(T) < a$, then $B(T_o) \neq [K_a]$ and hence $\text{ex}(a - 1, B(T_o)) \neq \begin{pmatrix} a - 1 \\ 2 \end{pmatrix}$. It is easy to see there are many trees $T = T[A, B]$ with $\beta(T) < a$, for example, a double star. Let $T = S_{a,b}$ be a double star and $u, v$ two centers of $T$ with $d(u) = a, d(v) = b$ and $a \leq b$. For any good odd-balooning $T_o$, because $S_{a,b} \in \mathcal{M}(T_o)$ by Lemma 1 and $\{u, v\}$ is a covering of $S_{a,b}$, we have $uv \in B(T_o)$ by the definition of $B(T_o)$. Therefore, $\text{ex}(a - 1, B(T_o)) = 0$ and

$$\text{ex}(n, T_o) = e(G(n, 2, a)).$$

That is to say, any good odd-balloonning of a double star $S_{a,b}$ is a counterexample to Conjecture 1, and so $f(n, H) > \text{ex}(n, H)$ for many $H$ with $\chi(H) = 3$.

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