Abstract. We propose a novel mathematical framework to address the problem of automatically solving large jigsaw puzzles. This problem assumes a large image which is cut into equal square pieces that are arbitrarily rotated and shifted and asks to recover the original image given the transformed pieces. The main contribution of this work is a theoretically-guaranteed method for recovering the unknown orientations of the puzzle pieces by using the graph connection Laplacian associated with the puzzle. Iterative application of this method and other methods for recovering the unknown shifts result in a solution for the large jigsaw puzzle problem. This solution is not greedy, unlike many other solutions. Numerical experiments demonstrate the competitive performance of the proposed method.

1. Introduction

Solving jigsaw puzzles is an entertaining task, which is commonly explored by children and adults. It is also a challenging mathematical and engineering problem that occupies researchers in computer science, mathematics and engineering. The solution of this problem is useful for several industrial applications. One example is reassembling archaeological artifacts [6, 21, 33, 41, 44], where one tries to recover the shape of an archaeological object from damaged pieces. Another example is recovering shredded documents or photographs [10, 20, 23, 26], where one tries to recover a document or a picture from small pieces of it. Additional applications appear in biology [25] and speech descrambling [49].

The automatic solution of puzzles, without having any information on the underlying image, is known to be NP hard [1, 11]. The first algorithm that attempted to automatically solve general puzzles was introduced by Freeman and Garder [13] in 1964. It was designed to solve puzzles with 9 pieces by only considering the geometric shapes of the pieces.

A recent setting of “jigsaw” puzzles assumes an image cut into equal square pieces. The mathematical problem is to recover the original image from the given pieces, which are possibly rotated and shifted along the puzzle grid. Gallagher [14] categorized these jigsaw puzzles into three types. In type 1 puzzles, the pieces are not rotated, but shifted. In type 3 puzzles, the pieces are not shifted, but rotated. In type 2 puzzles, the pieces are both shifted and rotated. We later formulate a more general mathematical setting; however, our current work addresses this special setting of jigsaw puzzles and focuses on type 2 and type 3 puzzles.

Many proposals for solving the latter jigsaw puzzles are based on greedy methods [2, 7, 13, 28, 32, 34, 42, 43]. However, greedy algorithms can easily get trapped in locally optimal solutions, which are not global. Some proposals also involve non-greedy constructive methods [8, 35, 37], which are often combined with greedy procedures.

SOLVING JIGSAW PUZZLES BY THE GRAPH CONNECTION LAPLACIAN

VAHAN HUROYAN, GILAD LERMAN, AND HAU-TIENG WU
This work proposes a constructive, non-greedy framework for recovering rotations between puzzle pieces. It also relies on a previous method for recovering locations, though in our implementation we try to avoid greedy procedures.

1.1. Previous Work. Several algorithms have been recently proposed for the automatic solution of the latter jigsaw puzzles \cite{2,7,14,28,32,34,35,36,37,38,42,43,45,47}. The problem becomes more challenging when the number of puzzle pieces increases and the sizes of puzzle pieces decrease. Some of these algorithms only consider type 1 puzzles (see e.g., \cite{2,8,34,35,49}), since recovering orientations increases the possible comparisons between two pieces by four and may also decrease the accuracy of solving the puzzle. The rest of these algorithms focus on type 2 puzzles, where \cite{14} also separately discusses type 3 puzzles. Other models of jigsaw puzzles and probabilistic results for their solutions are discussed in \cite{5,27,29}.

Cho et al. \cite{8} proposed a probabilistic, graphical model approach to the jigsaw puzzle problem and discussed different compatibility metrics between puzzle pieces. Yang et al. \cite{45} proposed another probabilistic solution by using a particle filter and a state permutations framework. Pomeranz et al. \cite{34} proposed a greedy method, discussed a few compatibility metrics and included some analysis on how to pick the correct compatibility metric for their method. Gallagher \cite{14} proposed a tree-based reassembly algorithm, which greedily merges components while respecting the geometric consistence constraints. It runs in three steps: building a constrained tree, trimming and filling. Mondal et al. \cite{28} used the algorithm of Gallagher \cite{14}, but they replaced its proposed metric with a combination of two existing metrics. They claimed to achieve a more robust metric using this technique. Andaló et al. \cite{2} proposed a quadratic assignment approach, which maximizes a constrained quadratic function via constrained gradient ascent. Jin et al. \cite{19} proposed a scoring approach that, in addition to considering edge similarity, also takes into account content similarity between puzzle pieces. Paikin and Tal \cite{32} proposed a greedy algorithm for handling puzzles of unknown size and with missing entries. Sholomon et al. \cite{35,36,37} proposed a genetic algorithm. Sholomon et al. \cite{38} proposed a new Deep Neural Network-Based approach for the prediction of the likelihood of two puzzle piece edges in the correct puzzle configuration.

Son et al. \cite{42} incorporated the “geometric structure” of the jigsaw puzzle by searching for small loops (4-cycles) of puzzle pieces, which form consistent cycles, and then hierarchically combining these small loops with higher order loops in a bottom-up fashion. They argued that loop constraints could effectively eliminate pairwise matching outliers. Son et al. \cite{43} proposed a growing consensus approach that assembles pieces by multiple modest bonds and uses a new objective function that maximizes consensus configurations. Yu et al. \cite{47} proposed a linear programming based formulation, which combines global and greedy approaches. Their proposed solver simultaneously exploits all the pairwise matches and globally computes the location of each piece/component at each step of the algorithm. Chen et al. \cite{7} proposed a greedy algorithm and combined several metrics to improve the performance of this algorithm.

A procedure for solving type 3 puzzles was only considered by Gallagher \cite{14} using a greedy method. We are not aware of any previous constructive and non-greedy procedure for solving type 3 puzzles. More importantly, we are not aware of a previous general method for finding the orientations of puzzle pieces with unknown locations. Such a procedure can enhance the solution of type 2 puzzles.
1.2. Our Contribution. In this paper we propose a novel approach to address type 2 and type 3 jigsaw puzzles. For type 3 puzzles, we suggest a fast, robust and constructive solution that uses the graph connection Laplacian (GCL) [40] (discussed in §3.1). Since the locations of puzzle pieces are given for type 3 puzzles, there is no need to find the metric between puzzle pieces, but only between neighboring pieces. Therefore the complexity of our proposed algorithm for type 3 puzzles is relatively low.

For type 2 puzzles we propose a novel iterative algorithm, which solves the following two subproblems (SPs) iteratively:

- **SP1** Finding the orientations of all puzzle pieces.
- **SP2** Finding the locations of all puzzle pieces.

These two steps are iteratively repeated until the desired result is achieved. We solve SP1 by using the GCL. We solve SP2 by incorporating an improved metric, obtained from the solution of SP1, within any state-of-the-art solution of type 1 puzzles. Some information inferred from the solution of SP2 is used to improve the solution of SP1.

All previous algorithms for solving type 2 puzzles simultaneously addressed both subproblems. On the other hand, this work separately solves the two subproblems and iteratively updates the solutions. The proposed procedure is also faster than the previous simultaneous procedures as long as the solver for SP1 is relatively accurate. Indeed, SP2 asks to solve type 1 puzzles, which are easier than type 2 puzzles. Moreover, most previous algorithms are greedy, whereas the one proposed here is not.

1.3. Structure of This Paper. This paper is organized as follows: §2 discusses a general mathematical setting for the jigsaw puzzle problem and a special case of it, which is studied later in the paper; §3 presents a solution for SP1, which assumes the existence of a “connection graph”; §4 shows how to construct the connection graph for type 2 and type 3 puzzles; §5 relies on existing solutions to SP2 and explains how to update the solution of SP1 based on the solution of SP2; §6 concludes with numerical experiments that test the proposed algorithm using digital images; finally, §7 concludes with a short discussion that includes possible extensions of this work. Modifications of our proposed algorithm that we find interesting but could not make competitive in practice are discussed in the appendix.

2. The Mathematical Setting for Jigsaw Puzzles

Here we mathematically formalize the jigsaw puzzle problem. We first formulate a general abstract setting of this problem in §2.1. We then describe a specific special case of interest in §2.2, where we also discuss a possible generalization and the direct application to the discrete setting of the application area of this paper. Lastly, §2.3 discusses the main challenge of addressing the specific setting.

2.1. A General Mathematical Formulation. Our general mathematical formulation assumes a $d$-dimensional compact manifold $M$ embedded in $\mathbb{R}^q$ via the inclusion map $\iota$. For simplicity, we refer to the embedded manifold by $M$ instead of $\iota(M)$. For this embedded $M$, we further consider a sufficiently smooth function $f : M \to \mathbb{R}^k$, where $k \geq 1$. The required smoothness of $f$ depends on the application domain. For the application we consider, which has a discrete setting with discontinuities of $f$, the assumption $f \in L^2(M, \mathbb{R}^k)$ seems natural. Note that $M$
serves as the “physical space” and \( f \) as an “image” defined on this space, where \( k > 1 \) can correspond to a multi-spectral image.

We will first discuss the notion of patches partitioning the embedded \( M \) as well as image patches. Generally, a patch is a subset of the embedded \( M \). Since our mathematical setting is continuous, we assume that patches are open sets. We later explain how this assumption does not matter to the discrete setting of this paper. An image patch on the embedded \( M \) is a pair of a patch and the restriction of \( f \) on it. For simplicity, we denote a patch by \((P, f|_P)\). Similarly, we denote an image patch by \((P, f|_P), \) even though \((i(P), f|_{i(P)})\) is more precise.

We partition \( M \) into open patches \( \{P_i\}_{i=1}^n \) so that \( M = \bigcup_{i=1}^n \bar{P}_i \), where for \( 1 \leq i \leq n \), \( P_i \subset M \) and \( \bar{P}_i \) is the closure of \( P_i \), and also for \( 1 \leq i \neq j \leq n \), \( P_i \cap P_j = \emptyset \). When defining the corresponding image patches we allow local rigid transformations, such as rotations and translations. We make the problem formulation even more general by considering local diffeomorphic transformations. For each \( 1 \leq i \leq n \), consider a transform \( D_i \) on \( \mathbb{R}^q \), so that \( D_i(P_i) \subset \mathbb{R}^q \) is diffeomorphic to \( P_i \). For \( x \in \mathbb{R}^q \), define \( (D_i \circ f|_{P_i})(x) := f(D_i^{-1}(x)) \) when \( D_i^{-1}(x) \in P_i \) and 0 otherwise.

There are three jigsaw puzzle problems we can formulate:

- **P0**: Given a set of image patches \( Q := \{ (D_i(P_i), D_i \circ f|_{P_i}) \}_{i=1}^n \) and \( M \), recover \( f \).
- **P1**: Given a set of image patches \( Q := \{ (D_i(P_i), D_i \circ f|_{P_i}) \}_{i=1}^n \) and \( M \), recover \( f \) and \( M \).
- **P2**: Given a set of patches \( P := \{ D_i(P_i) \}_{i=1}^n \), recover \( M \).

In general these are ill-defined and challenging problems, since more conditions may be needed. For example, if \( f \) is a constant function on a sufficiently large region of \( M \) and the shapes of the puzzle patches are not sufficient to uniquely determine neighboring patches, then there is no information available for reconstructing \( f \). Similarly, estimating the unknown local diffeomorphic functions is a challenging problem, and it makes sense to further restrict them. On the other hand, there are simplified, well-defined versions of these problems. In this paper we address P0 in the very specific setting of the two-dimensional square jigsaw puzzle problem, which we describe next. Examples of Problems P1 and P2 with different physical spaces of different dimensions appear in [15,16,17,24,30,31,46]. Note that in these papers, the patch boundary is not fixed and is used in solving P1 and P2.

### 2.2. A Special Setting and its Generalization

In the two-dimensional square jigsaw puzzle problem, \( M \) is a rectangle in \( \mathbb{R}^2 \), \( M = [a_1, b_1] \times [a_2, b_2] \) and \( \{P_i\}_{i=1}^n \) form a square tiling of \( M \). That is, the open patches partitioning \( M \) are shifted versions of the same square. In general, one may assume that \( f \in L^2(M, \mathbb{R}^k) \) and \( k \geq 1 \). We think of the graph \( \{(x, f(x)) : x \in M \subset \mathbb{R}^2\} \) as a continuous version of an image. One may use \( k = 1 \) for gray-scale images, \( k = 3 \) for color images and higher \( k \) for multispectral and hyperspectral images. Since a main challenge of the practical problem is dealing with discrete images, we further assume that \( f \) is piecewise constant with discrete values in the following way. Each patch is divided by a uniformly spaced grid to \( r \times r \) subsquares and the vector-valued \( f \) is constant on each subsquare, where each coordinate of the constant vector is discrete; for example, it lies in \( 0, \ldots, 255 \). We further assume that the diffeomorphic transformations \( D_i, 1 \leq i \leq n, \) are proper rigid transformations from one patch.
to another. That is, they are combinations of rotations and translations, where a rotation is by 0°, 90°, 180° or 270°, and translations of patches \( \{P_i\}_{i=1}^n \) can be described as \( \{P_{\sigma(i)}\}_{i=1}^n \), where \( \sigma \) is a permutation of degree \( n \). This assumes that the grid is labeled by numbers and the goal is to find the correct permutation for the indices of all patches that would place each square in the correct position of the tile. Therefore, we can write the set of image patches as \( Q = \{(R_{\sigma_i}(P_{\sigma_i}), R_{\sigma_i} \circ f|_{p_{\sigma_i}})\}_{i=1}^n \), where \( \sigma \) is a permutation of degree \( n \), \( R_{\sigma_i} \) is an element of the cyclic group \( \mathbb{Z}_4 \) and the action \( \circ \) was defined above. The problem of interest in this setting is P0. Note that its solution requires recovering \( \{R_{\sigma_i}\}_{i=1}^n \) and \( \sigma \). We also remark that in this case finding \( M \) in P1 is unique up to a proper rigid transformation; however, the extra component of P1 is artificial for this setting. Furthermore, in this setting, P2 is ill-defined as it has many possible solutions. In general, a solution of P2 requires stronger assumptions, for example, on the shape of puzzle patches or on the manifold that can be asked to be closed.

One can consider an equivalent formulation, where instead of having patches initially on the grid, the patches are arbitrarily shifted and rotated within \( \mathbb{R}^2 \). In this new formulation, \( D_i = T_{x_i} \circ R_{\theta_i} \), where \( x_i \) is an arbitrary vector in \( \mathbb{R}^2 \), \( T_{x_i}(x) = x_i + x \) for each \( x \in M \subset \mathbb{R}^2 \), and \( R_{\theta_i} \) is an arbitrary element of \( SO(2) \). The equivalence of the two formulations is evident. Indeed, given patches with any choice of centers and rotations, one can arbitrarily assign them to a grid and use the former formulation, and vice versa. Nevertheless, the latter formulation can apply to more general settings. Examples include settings with more complicated shapes of patches, such as polygonal shapes, which are common in tangram puzzles, or shapes with curvy edges, which are common in commercial jigsaw puzzles. Mathematical ideas for solving these two kinds of puzzles appear in [22] and [17], respectively. We remark that there are cases of more complicated shapes that are easier to solve. For example, if the shapes of the patches lead to unique determination of the neighboring patches, then exact reconstruction is easier. We will refer to the four nearest neighbors of the given patch (left, right, top or bottom) in the two-dimensional square jigsaw puzzle as neighboring patches. On the other hand, there are clearly very difficult cases of complicated shapes with many possibilities of aligning them together. In general, one may also consider various 3D puzzles or more complicated problems. Note that most of the ideas discussed in this paper can be well suited for puzzles with non-square patches and a higher-dimensional non-flat manifold.

Figure 1 demonstrates the particular instance of the two-dimensional square jigsaw puzzle problem we discuss in this paper with RGB images, where \( k = 3 \). We remark that the last column of this figure illustrates the image patches \( Q = \{(R_{\sigma_i}(P_{\sigma_i}), R_{\sigma_i} \circ f|_{p_{\sigma_i}})\}_{i=1}^n \) discussed above. We assumed above that \( f \) is a piecewise constant function. In this figure, \( f \) has constant values on squares corresponding to image pixels. Since the resolution is relatively high, one cannot notice that \( f \) is piecewise constant. However, this is noticeable in the low-resolution demonstration of patches of another puzzle at the top right image of Figure 2.

2.3. The Main Challenge of the Special Setting. We recall that the formulation of the two-dimensional square jigsaw puzzle problem requires finding a permutation \( \sigma \) and rotations \( \{R_{\theta_i}\}_{i=1}^n \subset SO(2) \). Equivalently, one may solve for locations \( \{x_i\}_{i=1}^n \) on a uniform grid, representing the centers of the patches, and rotations \( \{R_{\theta_i}\}_{i=1}^n \). In order to estimate these from \( Q \) for general functions \( f \), one needs to
rely on the similar function values on edges of neighboring patches. In our setting of digital images, we should often expect discontinuities in values of $f$ on neighboring edges. The top right image of Figure 2 demonstrates this phenomenon for two patches selected from the puzzles shown in top left image with lower resolution. Such discontinuity can result in loss of information for determining neighbors and may lead to ill-posed problems.

There are also special images for which the puzzle problem is ill-posed. For example, the bottom left image of Figure 2 demonstrates a case where several patches look very similar to each other and it is impossible to determine the right permutation. Nevertheless, the output of common algorithms given this particular puzzle is often visually acceptable. On the other hand, the bottom right image of Figure 2 demonstrates a case where the image consists of two parts that are disconnected by a uniform background. In this particular instance, the background is the white sky, one part is the main scene of the image and the other part includes two short branches of another tree at the top left corner of the image. In this case it would be impossible to figure out the exact position of the latter part of the image.

The following definition quantifies an ideal type of metric between edges of image patches that, if exists (i.e., if the problem is well-posed), can be used to solve the two-dimensional square jigsaw puzzle problem.

**Definition 2.1.** Fix an image $I$ and a set of image patches $Q := \{P_i, f|_{P_i}\}_{i=1}^n$. A metric defined on $Q$ is called perfect if there exists $c > 0$ so that two neighboring patches have a distance less than $c$ and two non-neighboring patches have a distance greater than $c$.

The main challenge of solving reasonable instances of the two-dimensional square jigsaw puzzle problem is to find a nearly perfect metric. Empirically, we have found that the Mahalanobis Gradient Compatibility (MGC) metric, described in §4.1, is often near perfect in well-posed cases.
Figure 2. Examples of two-dimensional square jigsaw puzzles, where the comparison of two neighboring patches is challenging or impossible. The top left image shows a puzzle with 432 pieces, each of size $28 \times 28$. The top right image demonstrates an example of 2 neighboring patches in the latter puzzle that have different pixel values around the boundaries due to the discrete nature of a digital image. These patches are circled with red in the original puzzle (top left image) and their nearby edges are circled with red in the top right image. The bottom two images demonstrate examples of puzzles that have patches with uniformly white edges (circled with red in the bottom left image) and also have some uniformly white patches. Natural solutions of the bottom left puzzle seem to yield visually correct images that may not coincide with the original assignment. However, there are natural solutions of the bottom right puzzle that result in different images than the original one. Indeed, the small component of the image circled with red can be placed in different area within the skies.

3. Frameworks for Recovering Rotations of Puzzle Pieces

This section applies the framework of \cite{12,40} for recovering the global orientations of puzzle patches. It also mentions another framework. These frameworks require the construction of a graph whose vertices correspond to the puzzle patches and whose edges connect neighboring patches. The rest of the section is organized as follows: §3.1 forms the graph connection Laplacian (GCL) and explains how to estimate the rotations of puzzle patches by this graph; §3.2 describes an equivalent
framework for solving this problem: \[3.3\] theoretically justifies the method described in §3.1.

3.1. Estimation of Orientations Using the Connection Graph. The general connection graph \([40]\) \(G = (V,E,W,R)\) consists of four components: vertices \(V\), edges \(E\), the affinity function (or weight function) \(W : E \to [0,1]\) and the connection function \(R : E \to \mathbb{G}\), where \(\mathbb{G}\) is a given group. The first three components are determined by the weighted graph and the fourth depends on the application in which the graph is used. In the case of a two-dimensional square jigsaw puzzle with a perfect metric (recall Definition \[2.1\]), the ideal connection graph is formed as follows. The vertices represent patches in \(Q\) with a perfect metric (recall Definition 2.1), the ideal connection graph is formed as which the graph is used. In the case of a two-dimensional square jigsaw puzzle determined by the weighted graph and the fourth depends on the application in

For possibly imperfect scenarios of the two-dimensional square jigsaw puzzles, the vertices are formed as above, but one needs to construct meaningful edges, affinity function and connection function (with \(\mathbb{G} = \mathbb{Z}_4\)). A heuristic construction of these is suggested for type 2 and type 3 puzzles in §4.3 and §4.2, respectively. Here we propose a general heuristic that uses a given connection graph of two-dimensional square jigsaw puzzles to estimate the unknown orientations of the patches. This heuristic is later justified in §4.3 under special assumptions. The main idea of this heuristic is to use the GCL for inferring global information (in the form of a certain eigendecomposition) from local information (needed to form the GCL).

Next, we review several matrices associated with a general connection graph. Recall that the functions \(W\) and \(R\) are defined on the set \(\{1,\ldots,n\} \times \{1,\ldots,n\}\), where \(n\) is the number of puzzle pieces. Thus, from now on, we denote these functions by their corresponding matrices \(W \in \mathbb{R}^{n \times n}\) and \(R \in \mathbb{R}^{2n \times 2n}\), respectively. Note that \(R\) is a block matrix whose \(2 \times 2\) blocks represent two-dimensional rotations. For \(1 \leq i,j \leq n\), we denote by \(R[i,j]\) the \((i,j)\)-th \(2 \times 2\) block of \(R\). The connection graph is thus \(G = (V,E,W,R)\). The connection adjacency matrix is an \(n \times n\) block matrix \(S\) with \(2 \times 2\) submatrices, where for \(1 \leq i,j \leq n\) the \((i,j)\)-th submatrix is

\[
S_{ij} = \begin{cases} \quad W(i,j)R[i,j], & \text{when } (i,j) \in E; \\
\quad 0, & \text{otherwise.}
\end{cases}
\]

The degree matrix is an \(n \times n\) block diagonal matrix \(D\), where for \(1 \leq i \leq n\), its \(i\)-th diagonal submatrix is

\[
D_{ii} = \sum_{j \neq i} W(i,j)I_2,
\]

where \(I_2\) is the \(2 \times 2\) identity matrix. We define \(C := D^{-1}S\) and \(\tilde{C} := D^{-1/2}SD^{-1/2}\) and refer to them as the GCL matrix and the normalized GCL matrix, respectively. We remark that in some other works, such as \[3\], the GCL matrix and normalized GCL matrix instead refer to \(I - C\) and \(I - \tilde{C}\), respectively.
The GCL matrix is associated with a random walk, whose transition probability matrices are \( W(i,j) \), \( 1 \leq i, j \leq n \). This can be seen by its action on a block vector \( v \in \mathbb{R}^{2n \times 2} \), whose \( n \)-th \( 2 \times 2 \) submatrices are

\[
v[j] = \begin{bmatrix} v_{2j-1,1} & v_{2j-1,2} \\ v_{2j,1} & v_{2j,2} \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad 1 \leq j \leq n,
\]

in the following way

\[
(Cv)[i] = \sum_{j: (i,j) \in E} \left[ \sum_{k: (i,k) \in E} W(i,k) \right] R[i,j]v[j].
\]

That is, a block vector \( v[j] \) is rotated by \( R[i,j] \) and assigned to the \( i \)-th patch with probability \( W(i,j) / \sum_{k: (i,k) \in E} W(i,k) \).

To recover the global orientations of the puzzle patches, we follow the procedures of [3, 40]. First, we form the block vector \( U \in \mathbb{R}^{2n \times 2} \) whose columns are the top 2 eigenvectors of \( C \). Then, we project each of the \( 2 \times 2 \) blocks of \( U \) onto \( \mathbb{Z}_4 \) and use the resulting blocks as the global orientations. Algorithm 1 summarizes the above straightforward procedure of recovering the unknown orientations of the image patches for a given two-dimensional square jigsaw puzzle.

**Algorithm 1** The GCL Algorithm

**Input:** Connection graph: \( G = (V, E, W, R) \)
- Construct the Connection Adjacency Matrix \( S \) by (1)
- Construct the degree matrix \( D \) by (2)
- Let \( C = D^{-1}S \)
- Form \( U \in \mathbb{R}^{2n \times 2} \) whose columns are the top 2 eigenvectors of \( C \)
- For \( 1 \leq i \leq n \), let \( R_i \in \mathbb{Z}_4 \) be the projection of the \( i \)-th block of \( U \) onto \( \mathbb{Z}_4 \)

**Return:** Global rotation matrices \( R_1, \ldots, R_n \)

We emphasize that the GCL algorithm for recovering the orientations of patches is non-greedy. Indeed, it directly constructs the orientation of patches using the information in the connection graph via diffusion. On the other hand, other methods, such as [14, 42, 43], try to greedily match pieces based on their relative orientations. We also mention that the GCL algorithm does not use any knowledge of the size of the puzzle image, or equivalently, of the number of puzzle pieces per length or width of the image.

### 3.2. Another Formulation

The general problem we have addressed in §3.1 is referred to as synchronization. That is, one assumes a connection graph \( G = (V, E, W, R) \) and needs to estimate for all vertices \( i \in V \) a group element \( g_i \in G \) from noisy or wrong measurements of \( g_i g_j^T \in G \). In the particular case of the two-dimensional square jigsaw puzzle, the graph is \( G = (V, E, W, R) \), \( G = \mathbb{Z}_4 \) and \( g_i g_j^T = R(i,j) \), which was defined in §3.1. In this case the synchronization problem is referred to as angular.

We describe here a least-squares formulation for this problem, review two common solutions for it and discuss the similarities and differences of one such solution with the method above. Using the matrix \( S \) defined in (1), the least-squares formulation for angular synchronization asks to solve the optimization problem

\[
\argmin_{u \in \mathbb{Z}_4^n} \|uu^T - S\|^2,
\]
or equivalently,

\[ \arg\max_{u \in \mathbb{Z}^4} \text{tr}(u^T S). \]  

We remark that sometimes the problem above is formulated with \( R \) instead of \( S \) when the affinities are ones for edges in \( E \) and zeros otherwise, so that \( R = S \).

This problem is NP-hard [50]. Nevertheless, approximate solutions were proposed, in particular, semidefinite programming and spectral relaxation [39]. The semidefinite programming method suggests to remove the rank 2 constraint on the PSD (positive semi-definite) matrix \( uu^T \) in (4) and consequently solve

\[ \arg\max_{H \succeq 0, H(i,i) = I_2} \text{tr}(HS). \]  

The solution \( v \in \mathbb{R}^{2n \times 2} \) is recovered by projecting the blocks of the top 2 eigenvectors of solution (4) into \( \mathbb{Z}_4 \).

The spectral relaxation method suggests to relax the set \( \mathbb{Z}_n^4 \) into \( \mathbb{R}^{2n \times 2} \) and solve the following eigenvalue/eigenvector problem

\[ \arg\max_{u \in \mathbb{R}^{2n \times 2}, \|u\| = 2n} \text{tr}(u^T Su). \]  

The block vector \( \tilde{v} \), which contains the top 2 eigenvectors of \( S \), solves (6). To recover the \( g_i \)'s one can project the \( 2 \times 2 \) blocks of \( \tilde{v} \) into \( \mathbb{Z}_4 \). The spectral relaxation is faster and better suited for higher-volume data. The SDP relaxation is often more accurate than the spectral relaxation for SO(2); however, for the special case of \( \mathbb{Z}_4 \) their accuracy should be comparable, since there are only four, well-separated elements of \( \mathbb{Z}_4 \). We note that the spectral relaxation method is very similar to the method described in §3.1 but directly uses the matrix \( S \) instead of \( C \). In fact, the method proposed in §3.1 is a spectral relaxation of (3) when \( S \) is replaced by \( C \).

One of the advantages of using \( C \) instead of \( S \) is that it gives rise to a natural diffusion distance, which is discussed later in §A.1.

### 3.3. Theoretical Justification of the GCL Algorithm.

In this section we show that the GCL algorithm for two-dimensional square jigsaw puzzles is robust to noise and incorrect measurements, where incorrect measurements are mistakes in estimating the connection graph. The three puzzles in Figure 2 exemplify cases where incorrect measurements are expected due to indistinguishability of some patches or low-resolution of patches.

For a given two-dimensional square jigsaw puzzle, let \( G_{\text{true}} = (V, E_{\text{true}}, W_{\text{true}}, R_{\text{true}}) \) denote the true connection graph. Note that the graph \( (V, E_{\text{true}}) \) is a grid, the true affinity function \( W_{\text{true}} \) is defined by

\[ W_{\text{true}}(i, j) = \begin{cases} 1, & \text{if } \{i, j\} \in E_{\text{true}}; \\ 0, & \text{otherwise}, \end{cases} \]  

and the true connection function \( R_{\text{true}} \) is defined by

\[ R_{\text{true}}(i, j) = \begin{cases} R_i R_j^T, & \text{if } \{i, j\} \in E_{\text{true}}; \\ 0, & \text{otherwise}, \end{cases} \]  

where \( R_1, \ldots, R_n \) are the rotation matrices of the rotations \( R_1, \ldots, R_n \) defined in §2.2. Let \( G_{\text{est}} = (V, E_{\text{est}}, W_{\text{est}}, R_{\text{est}}) \) denote the estimated connection graph.
Finally, denote by $C_{\text{est}}$ and $C_{\text{true}}$ the GCL matrices corresponding to $G_{\text{est}}$ and $G_{\text{true}}$, respectively.

The following lemma shows that if the estimated connection graph is a good approximation of the true connection graph, then the estimated GCL matrix is a good approximation of the true GCL matrix. It is analogous to Lemmas 2.1 and 2.2 of El Karoui and Wu [12] but assumes a different noise model. In fact, its proof is parallel to the proofs of the latter lemmas and is thus omitted here.

**Lemma 3.1.** Suppose that $G_{\text{true}} = (V, E_{\text{true}}, W_{\text{true}}, R_{\text{true}})$ and $G_{\text{est}} = (V, E_{\text{est}}, W_{\text{est}}, R_{\text{est}})$ are the true and estimated connection graphs, respectively, where $W_{\text{true}}$ and $R_{\text{true}}$ are defined in (7) and (8). Assume that there exists a set $E' \in E_{\text{true}} \cap E_{\text{est}}$ such that $(V, E')$ is a connected graph and

$$R_{\text{est}}(i, j) = \begin{cases} R_{\text{true}}(i, j), & \text{if } \{i, j\} \in E'; \\ \text{arbitrary element of } \mathbb{Z}_4, & \text{otherwise.} \end{cases}$$

Assume further that there exist $\epsilon > 0$ and $f_1, \ldots, f_n > 0$ such that

$$\sup_{\{i, j\} \in E'} \left| \frac{W_{\text{true}}(i, j) - W_{\text{est}}(i, j)}{f_i} \right| < \epsilon \quad \text{and} \quad \sup_{\{i, j\} \notin E'} \left| \frac{W_{\text{est}}(i, j)}{f_i} \right| < \epsilon$$

and there exists $\gamma > \epsilon$ such that $\inf_i \sum_{j \neq i} w_{ij} / n > \gamma$. Then

$$\|C'_{\text{true}} - C_{\text{est}}\|_2 \leq \frac{2\epsilon}{\gamma} + \frac{4\epsilon}{\gamma(\gamma - \epsilon)},$$

where $C'_{\text{true}}$ is the GCL matrix corresponding to $G'_{\text{true}} = (V, E', W_{\text{true}}, R_{\text{true}})$.

Note that if two patches are wrongly connected as neighbors in the estimated graph, then (9) enforces their affinity function to be small. Also note that each of $f_i$ cannot be too large, otherwise (9) cannot be satisfied when two patches are neighbors.

Recall that according to [3][40] the top 2 eigenvectors of the true GCL matrix $C'_{\text{true}}$ recover the global orientations of puzzle patches up to global rotation. Thus, if the top 2 eigenvectors of the estimated GCL matrix approximate well the top 2 eigenvectors of the true GCL matrix, they would recover the global orientations of puzzle patches. The Davis-Kahan sin $\Theta$ Theorem [9][18] guarantees such a good approximation when (10) holds for a small enough $\epsilon > 0$. Indeed, the matrix $V'_{\text{true}}$ of the top 2 column eigenvectors of $C'_{\text{true}}$ and the matrix $V_{\text{est}}$ of the top 2 column eigenvectors of $C_{\text{est}}$ satisfy

$$\|\sin \Theta(V'_{\text{true}}, V_{\text{est}})\|_2 \leq \frac{\|C'_{\text{true}} - C_{\text{est}}\|_2}{\lambda_2(C'_{\text{true}}) - \lambda_3(C_{\text{est}})},$$

where $\lambda_3(C_{\text{est}})$ is the third largest eigenvalue of $C_{\text{est}}$. To bound $\lambda_3(C_{\text{est}})$, we combine the triangle inequality with Weyl’s inequality [4] to achieve

$$|\lambda_3(C_{\text{est}})| \leq |\lambda_3(C'_{\text{true}}) - \lambda_3(C_{\text{est}})| + |\lambda_3(C'_{\text{true}})| \leq \|C_{\text{true}} - C_{\text{est}}\|_2 + |\lambda_3(C'_{\text{true}})|.$$

Combining (11) and (12) yields

$$\|\sin \Theta(V'_{\text{true}}, V_{\text{est}})\|_2 \leq \frac{\|C_{\text{true}} - C_{\text{est}}\|_2}{\lambda_2(C'_{\text{true}}) - |\lambda_3(C'_{\text{true}})| - \|C_{\text{true}} - C_{\text{est}}\|_2}.$$

We thus conclude by (10) and (13) that if $\epsilon$ is sufficiently small then $V_{\text{est}}$ closely approximates $V_{\text{true}}$. We remark that this analysis generalizes to other puzzles while using their corresponding GCL.
4. Connection Graph Construction For Type 2 and Type 3 Puzzles

As we have discussed in §3.1, if we are given a perfect metric, we can easily construct the connection graph. However, there is no perfect metric that would work for all images. For example, if part of the image contains a region with a uniform color, such as sky or ocean (see the images on the second row of Figure 2), the metric between the edges of the image patches from this region will be close to zero. Thus, all these patches should be wrongly identified by a perfect metric as neighbors. Furthermore, if all edges of patches are similar to each other, then the patches are indistinguishable and the problem is ill-posed. Therefore, the idea of finding a perfect metric and using a threshold to identify neighbors may not lead to a correct affinity graph. Instead, we suggest to iteratively update the graph construction, while identifying possibly incorrect edges and reassigning zero or small affinities to them. The rest of this section is organized as follows: §4.1 reviews the Mahalanobis Gradient Compatibility (MGC) metric that is used for the proposed graph construction; §4.3 describes a construction of the connection graph for type 2 puzzles; lastly, §4.2 proposes a construction of the connection graph for type 3 puzzles.

4.1. Approximate a Perfect Metric Between Image Patches. To automatically assemble a jigsaw puzzle, no matter what algorithm is used, one needs to have a measure that can indicate whether two patches are neighbors or not. As we can see in the first row of Figure 2, it can be challenging to compare patches. We recall that the discrete values of the digital image at two sides of an edge between two patches are not the same. The right image of the first row of Figure 2 shows two neighboring patches at high resolution, where the difference between the image values at the two sides of the edge (left and right) is noticeable. On the other hand, in the left image of the first row of Figure 2, this difference is hard to notice in the printed resolution. Nevertheless, we emphasize here the existing difference of numerical values at two sides of edges, which is challenging for any algorithm that needs to align patches.

In this work, we align patches by using the MGC metric, which was proposed in [14]. It is based on two main ideas. The first idea is that the derivatives of RGB values in the perpendicular direction to the edge are similar in both sides of that edge. The second idea is that these values can be compared by using the covariance between the color channels and the corresponding Mahalanobis distance.

To review Gallagher’s precise definition [14], we assume two neighboring image patches $P_i$ and $P_j$ of size $s \times s$. There are four different relative positions of $P_i$ and $P_j$, namely, left-right, right-left, top-bottom and bottom-top, and the computation needs to adapt to each case. We assume without loss of generality the left-right relative position, that is, $P_i$ on the left and $P_j$ on the right, and compute the corresponding MGC, which we denote by $\text{MGC}_{lr}(P_i, P_j)$, as follows. For each color channel $c$ (red, green and blue) and each row $r$, $1 \leq r \leq s$, of the $s \times s$ patch $P_i$, we find the derivatives near the right edge of the image patch $P_i$, that is, at the last column indexed by $s$, in the direction left-right as follows:

$$G_{iL}(r, c) = P_i(r, s, c) - P_i(r, s - 1, c).$$

The subscript $L$ in the above equation indicates that patch $P_i$ is on the left side of the patch $P_j$. Note that the matrix $G_{iL}$ is in $\mathbb{R}^{s \times 3}$ and can be singular. Gallagher [14] suggests regularizing it by adding the following 9 additional rows $(0,0,0)$,
The resulting regularized matrix in $\mathbb{R}^{(s+9)\times 3}$ is denoted by $\tilde{G}_{iL}$.

Next, for each color channel $c$ we define the mean distribution for those derivatives on the right side of the $s \times s$ patch $P_i$ as

$$\mu_{iL}(c) = \frac{1}{s} \sum_{r=1}^{s} G_{iL}(r,c).$$

The regularized covariance matrix $\Sigma_{iL} \in \mathbb{R}^{3\times 3}$ between color channels is

$$\Sigma_{iL} = \frac{1}{s+8} \left( \tilde{G}_{iL} - \text{mean}(\tilde{G}_{iL}) \right)^T \left( \tilde{G}_{iL} - \text{mean}(\tilde{G}_{iL}) \right),$$

where

$$\text{mean}(\tilde{G}_{iL}) = \frac{1}{s+9} \sum_{r=1}^{s+9} \tilde{G}_{iL}(r,c) = \frac{1}{s+9} \sum_{r=1}^{s} G_{iL}(r,c).$$

We also define $G_{ijLR}(p)$, the derivative from the left $s \times s$ image patch $P_i$ to the right $s \times s$ image patch $P_j$ at row $r$ and color $c$, by

$$G_{ijLR}(r,c) = P_j(r,1,c) - P_i(r,s,c).$$

The left-to-right compatibility measure from $P_i$ to $P_j$ is defined by

$$D_{LR}(P_i,P_j) = \sum_{r=1}^{s} (G_{ijLR}(r) - \mu_{iL}) \Sigma_{iL}^{-1} (G_{ijLR}(r) - \mu_{iL})^T.$$ 

Similarly, one can define the right-to-left compatibility measure from $P_j$ to $P_i$ in the same left-right setting, where $P_i$ is to the left of $P_j$. The left-right MGC metric then has the symmetrized form

$$MGC_{lr}(P_i,P_j) = D_{LR}(P_i,P_j) + D_{RL}(P_j,P_i).$$

The right-left, top-bottom and bottom-top MGC’s, denoted by $MGC_{rl}(P_i,P_j)$, $MGC_{tb}(P_i,P_j)$ and $MGC_{bt}(P_i,P_j)$, respectively, are similarly computed.

4.2. Connection Graph Construction for Type 3 Puzzles. For type 3 puzzles, the locations of patches are given. Furthermore, edges are drawn between neighboring patches. The affinity function is set by $W(i,j) = 1$ for all $(i,j) \in E$. One need only find the unknown orientations, that is, the unknown connection matrix $R$.

To construct the connection function we propose to use the MGC metric, described in 4.1. For all neighboring patches $P_i$ and $P_j$, we calculate the possible 16 values of the MGC metric (for all possible 16 relative positions) and select the smallest of these numbers and its corresponding rotation $R(i,j)$. If there is no unique minimum among these 16 values we suggest assigning $W(i,j) = 1/2$ (or another value smaller than 1) and letting $R(i,j)$ be the mean of the candidate rotations that obtain the minimal value.

4.3. Connection Graph Construction for Type 2 Puzzles. We propose the following step-by-step procedure for constructing the affinity graph, the affinity function and the connection function for type 2 puzzles and then summarize this procedure in Algorithm 2. The rest of this section is organized as follows: 4.3.1 discusses the initial step of constructing the connection graph; 4.3.2 discusses the Jaccard index and explains how to use it to update the affinity function; lastly,
4.3.3 describes how to deal with the cases when the connection graph is disconnected: 4.3.4 describes how to find and use diagonal neighbors in order to construct a more reliable connection graph.

4.3.1. Initial Step. We start with an initial construction of the directed graph $G = (V, E_{est})$. The vertex set $V$ contains the patches in $Q$. The edge set $E_{est}$ is updated by the following procedure. In order to describe it, we denote by $R \cdot P$ the action of the rotation $R \in Z_4$ on the patch $P$. For a patch $P_i$, we find the patches $P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}$ and the corresponding rotations $R(i, i_1), R(i, i_2), R(i, i_3), R(i, i_4) \in Z_4$ such that

$$\begin{align*}
\{P_{i_1}, R(i, i_1)\} &\in \arg\min_{P \in Q, R \in Z_4} \text{MGC}_{bt}(P, R \cdot P), \\
\{P_{i_2}, R(i, i_2)\} &\in \arg\min_{P \in Q, R \in Z_4} \text{MGC}_{dt}(P, R \cdot P), \\
\{P_{i_3}, R(i, i_3)\} &\in \arg\min_{P \in Q, R \in Z_4} \text{MGC}_{tb}(P, R \cdot P), \\
\{P_{i_4}, R(i, i_4)\} &\in \arg\min_{P \in Q, R \in Z_4} \text{MGC}_{b}(P, R \cdot P).
\end{align*} \tag{15}$$

The set $E_{est}$ of directed edges contains the edges that connect each vertex with index $i$, $1 \leq i \leq n$, to the vertices with indices $i_1, i_2, i_3$ and $i_4$. Note that these indices solving (15) may not be unique and we consider all solutions of (15) when forming $E_{est}$.

We next modify the directed graph $G = (V, E_{est})$ into an undirected graph. We fix two values of weights, $w_1 = 1$ and $w_2 = 0.01$, and then we define an initial affinity function $W_{init}$ by setting $W_{init}(i, j) = W_{init}(j, i) = w_1$ if both $(i, j)$ and $(j, i) \in E_{est}$ and $W_{init}(i, j) = W_{init}(j, i) = w_2$ if only one of $(i, j)$ or $(j, i)$ is in $E_{est}$. Figure 3 demonstrates this construction for a fixed patch.

Next, we enforce the constraint that, for two-dimensional square jigsaw puzzles, each patch can have at most one neighbor for each direction by trimming some edges that are likely not neighbors. This is done as follows. Assume without loss of generality that patch $P_i$ has more than one neighbor in the top direction and denote these neighbors by $P_{i_1}, \ldots, P_{i_k}$ where $k > 1$. Then we solve the minimization problem

$$j \in \arg\min_{1 \leq j \leq k} \text{MGC}_{bt}(P_i, R(i, i_j) \cdot P_{i_j}). \tag{16}$$

If (16) has a unique solution, we keep the edge $\{i, i_j\}$ and remove the rest of the edges. Otherwise, we remove all edges $\{i, i_j\}_{j=1}^k$ from $E$. The procedure is analogous if $P_i$ has more than one neighbor from left, bottom or right. If edges were eliminated from $E_{est}$, then the matrix $W_{init}$ is updated so it is zero on the corresponding indices. This process results in the following initial connection graph $G = (V, E_{est}, W_{init}, R)$.

The construction of this graph uses a nearest-neighbor construction. For a high-noise regime, El Karoui and Wu 12 recommend avoiding a nearest-neighbor construction. However, due to the special lattice structure of the true graph, the nearest-neighbor initial construction is natural for the two-dimensional square jigsaw puzzle problem.

4.3.2. Use of Jaccard Index to refine the graph. Next, we refine the connection graph by trying to assess the validity of the edges and decrease the weights of
Figure 3. Demonstration of the initial step for the construction of the connection graph. The left figure demonstrates the best matches for a given patch from the four directions: top, left, bottom and right. For each matching patch it records the rotation whose application to this patch results in correct matching with the central patch. The right figure shows the application of these rotations to the matching patches and demonstrates how to assign the weights to the undirected graph. In this example, the matching patches from top and left were originally connected by a single direction; their weights in the undirected graph are thus 0.01. On the other hand, the patches from right and bottom are connected in both directions and thus their weights in the undirected graph are 1.

edges that do not seem valid; that is, they may not appear in the true connection graph. The idea is to check after removing an edge whether its neighbors are still connected in some weak sense to each other. If so, then the edge seems to be valid, and otherwise, it may not be valid. For this purpose, we use the Jaccard index [18].

The description of this index uses the following notation in a graph $G = (V, E)$. Given a vertex $i$, $1 \leq i \leq |V|$, let $N_{G,i}^1$ denote the set of vertices in $V$ which are connected to vertex $i$, that is, $N_{G,i}^1 = \{ j \in V | \{i,j\} \in E \}$. Using our terminology, $N_{G,i}^1$ contains the neighbors of $i$. The set $N_{G,i}^2$ contains all vertices that are at most 2 steps away from vertex $i$, except vertex $i$. That is, $N_{G,i}^2 = \bigcup_j N_{G,j}^1 \setminus \{i\}$.

Finally, let $G^{\setminus (i,j)} = (V, E \setminus \{(i,j)\})$ denote the graph with the edge $(i,j)$ removed. The sets $N_{G,i}^1$ and $N_{G,i}^2$ are demonstrated in Figure 4.

By using this notation, we define the Jaccard index between vertices $i$ and $j$ as

$$
\mu_{\text{Jaccard}}(i, j) = \frac{|N_{G,(i,j)}^2 \cap N_{G,(i,j)}^2|}{|N_{G,(i,j)}^2|},
$$

where $|\cdot|$ denotes the cardinality of a set. This definition is similar to the one in [18], but there are two differences. The first one is that we consider the graph $G^{\setminus (i,j)}$ instead of $G$ to emphasize the common neighbors, while excluding the obvious pair $(i,j)$. The second one is that we do not divide by $|N_{G,(i,j)}^2 \cup N_{G,(i,j)}^2|$. The latter division does not matter to us as we only care about the positivity of this index. Figure 5 demonstrates calculation of the Jaccard index for a special example. Note that the chance of two vertices $i$ and $j$ to be neighbors in the graph $(V, E_{\text{est}})$ is
Figure 4. Demonstration of the sets $N_{G,i}^1$ and $N_{G,i}^2$. A given vertex $i$ is colored in red, the elements of the set $N_{G,i}^1$ are colored in blue and the elements of the set $N_{G,i}^2$ are colored in blue and orange.

Figure 5. Demonstration of Jaccard index. Vertex $i$ is denoted by a red circle and vertex $j$ is denoted by a red cross. The edge between these vertices was removed from the grid. The elements of $N_{G \setminus (i,j),i}^2$ are denoted by blue circles and the elements of $N_{G \setminus (i,j),j}^2$ by orange crosses. The Jaccard index is four since there are four elements in $N_{G \setminus (i,j),i}^2 \cap N_{G \setminus (i,j),j}^2$ (denoted by blue circles filled with orange crosses).

higher if $\mu_{\text{Jaccard}}(i,j) > 0$ than if $\mu_{\text{Jaccard}}(i,j) = 0$. Thus, we propose to use the Jaccard indices to refine the connection graph. We use another weight matrix $W_{\text{Jaccard}} \in \mathbb{R}^{n \times n}$, defined as

\begin{align}
W_{\text{Jaccard}}(i,j) = W_{\text{Jaccard}}(j,i) = \begin{cases} 
0, & \{i,j\} \in E_{\text{est}} \text{ and } \mu_{\text{Jaccard}}(i,j) = 0; \\
W_{\text{init}}(i,j), & \text{otherwise.}
\end{cases}
\end{align}

(18)

Since this procedure might also remove many correct edges by zeroing out the corresponding values of the affinity function, we propose a linear combination of $W_{\text{init}}$ and $W_{\text{Jaccard}}$ with larger coefficient given to $W_{\text{Jaccard}}$. In our experiments we set

\begin{align}
W_{\text{nb}} = 0.2 \times W_{\text{init}} + 0.8 \times W_{\text{Jaccard}}
\end{align}

(19)
Figure 6. Demonstration of a disconnected affinity graph and the way it got connected. The left figure shows an example where the resulting affinity graph using our method is disconnected. Indeed, the two top right patches are not connected to any of the other patches. The black edges connect between true neighbors and the only red edge is a wrongly determined edge. The right figure demonstrates the result of the simple procedure described in §4.3.3. The connected graph has two new blue edges. While these blue edges connect between non-neighboring patches, the originally disconnected patches are uniformly white and thus their rotations do not matter for the reconstruction of the image.

and use the following affinity graph $G = (V, E, W_{nh}, R)$. For simplicity, the weight in this linear combination, and some other parameters, are chosen in an ad hoc fashion. It is possible to carefully select these parameters, for example, by applying cross validation with test images.

4.3.3. Making the Affinity Graph Connected. The procedures described in §4.3.1 and §4.3.2 might result in a disconnected affinity graph $G$ as demonstrated in the left image of Figure 6. To complete $G$ so it is connected, we first find all connected components of $G$. Assume that they are $k$ connected components with corresponding vertices $V_1, \ldots, V_k$ that partition the set of vertices $V$. Assume further that they are labeled by descending size order, i.e., $|V_1| \geq |V_2| \cdots \geq |V_k|$. Next, we find vertices $i \in V_1$ and $j \in V \setminus V_1$ that minimize the MGC metric between the patches $P_i$ and $P_j$. Mathematically, we find

$$\{i, j, O\} \in \arg\min_{i \in V \setminus V_1, j \in V_1, O \in \mathbb{Z}_4} \min\{\text{MGC}_{lr}(P_i, O \cdot P_j), \text{MGC}_{tb}(P_i, O \cdot P_j), \text{MGC}_{rl}(P_i, O \cdot P_j), \text{MGC}_{bt}(P_i, O \cdot P_j)\}. \tag{20}$$

If the solution of (20) is not unique, we randomly choose one solution. We then add the edge $\{i, j\}$ of the chosen solution to $E_{est}$ and update the weight as follows: $W(i, j) = W(j, i) = w_3$, where $w_3 = w_2/2 = 0.005$, $R[i, j] = O$ and $R[j, i] = O^T$. We iterate the procedure described above for $V_2, \ldots, V_k$ until the graph becomes connected. The number of iterations needed is $k - 1$ since there are $k$ connected components and at each iteration we connect the largest component with a remaining component.
4.3.4. Taking Advantage of 4-Loops. We refine the constructed connection graph by using the following property of the two-dimensional square jigsaw puzzle: If two patches \( P_i \) and \( P_j \) are diagonal neighbors, then there exist exactly two other patches \( P_n \) and \( P_m \) and a cycle containing the vertices \( i, n, j \) and \( n_2 \). This idea is demonstrated in Figure 7. Such a cycle of 4 vertices is referred to as a 4-loop by [42]. In this latter work, 4-loops were used to solve the puzzle problem. We use them to define a better connection graph. As we have already discussed for the true grid, each patch can have at most 4 direct neighbors (right, top, left or bottom). Furthermore, each patch has at most 4 diagonal neighbors. Exactly four diagonal neighbors are obtained for a patch in the interior of the puzzle, a single diagonal neighbor occurs for a corner patch and there are 2 diagonal neighbors for a patch that lies on the boundary of the grid but not on a corner.

For patches \( P_i \) and \( P_j \) we define

\[
\delta_{\text{diag}}(i, j) = |N^1_{G,i} \cap N^1_{G,j}|.
\]

We observe that patches \( P_i \) and \( P_j \) are diagonal neighbors in the true grid if and only if \( \delta_{\text{diag}}(i, j) = 2 \). To find the diagonal neighbors for graph \( G = (V, E) \) we propose a two step procedure. First, we find the set of all pairs of vertices \( \{i, j\} \in V \times V \) for which \( \delta_{\text{diag}}(i, j) = 2 \). For each such pair \( \{i, j\} \) there exists another pair \( \{n_1, n_2\} \) such that

\[
N^1_{G,i} \cap N^1_{G,j} = \{n_1, n_2\},
\]

or equivalently, \( i, n_1, j \) and \( n_2 \) are contained in a 4-loop. We set

\[
W_{\text{diag}}(i, j) = \begin{cases} 
1, & \text{when } \delta_{\text{diag}}(i, j) = 2 \text{ and } R[i, n_1]R[n_1, j] = R[i, n_2]R[n_2, j]; \\
0, & \text{otherwise}. 
\end{cases}
\]

The condition \( R[i, n_1]R[n_1, j] = R[i, n_2]R[n_2, j] \) in (23) is explained below after the whole procedure is clarified. We further update blocks of the matrix \( W_{\text{nb}} \), as follows, where we denote by \( W_{\text{nb}}([i, j], [n_1, n_2]) \) the \( 2 \times 2 \) submatrix of \( W_{\text{nb}} \) indexed by \( (i, n_1), (i, n_2), (j, n_1) \) and \( (j, n_2) \):

\[
W_{\text{nb}}([i, j], [n_1, n_2]) = \begin{cases} 
\frac{W_{\text{nb}}([i, j], [n_1, n_2])}{3}, & \text{if } \delta_{\text{diag}}(i, j) = 2 \text{ and } R(i, n_1)R(n_1, j) \neq R(i, n_2)R(n_2, j); \\
1_{2 \times 2}, & \text{if } \delta_{\text{diag}}(i, j) = 2 \text{ and } R(i, n_1)R(n_1, j) = R(i, n_2)R(n_2, j); \\
\frac{2W_{\text{nb}}([i, j], [n_1, n_2])}{3}, & \text{otherwise}. 
\end{cases}
\]

Again, the weights \( 1/3, 2/3 \) and \( 1 \) here are chosen in an ad-hoc fashion and can be chosen, for example, via cross validation with representative test images.

We note that the support sets of \( W_{\text{nb}} \) and \( W_{\text{diag}} \) are disjoint. We set

\[
W = W_{\text{nb}} + W_{\text{diag}},
\]

and this is the final step of constructing the connection graph \( G = (V, E_{\text{est}}, W, R) \) for two-dimensional square jigsaw puzzles. The full algorithm of this construction is summarized in Algorithm 2.
Figure 7. Demonstration of finding diagonally neighboring vertices in the grid. Two vertices $i$ and $j$ are denoted by a red circle and a red cross, respectively. The elements of the sets $N_{G,i}^1$ and $N_{G,j}^1$ are colored by blue and orange, respectively. The intersection of these sets yields the two diagonally neighboring vertices to $i$ and $j$.

Figure 8. Intuition for the condition in (23) and (24). The two vertices $i$ and $j$ are diagonal neighbors and the vertices $n_1$ and $n_2$ satisfy (22). Thus, $i$, $j$, $n_1$ and $n_2$ form a cycle of size 4, that is, a 4-loop. The relative rotations between vertices are indicated on the corresponding edges. We note that both $R[i,n_1]R[n_1,j]$ and $R[i,n_2]R[n_2,j]$ are equal to the relative rotation $R[i,j]$ shown on edge $(i,j)$. In particular, $R[i,n_1]R[n_1,j] = R[i,n_2]R[n_2,j]$. The assigned weights thus try to encourage this constraint on rotations and penalize cases where it is not satisfied.

The condition for rotations in (23) and (24), that is, $R[i,n_1]R[n_1,j] = R[i,n_2]R[n_2,j]$, is naturally satisfied in a true graph as demonstrated in Figure 8. Therefore, when it is satisfied and also $\delta_{\text{diag}}(i,j) = 2$, the maximal weight of 1 is assigned to the corresponding diagonal edge. Equation (24), on the other hand, considers the case where this condition is violated, but $\delta_{\text{diag}}(i,j) = 2$. In this case, there is evidence for a mismatch between puzzle pieces, and therefore the weight is reduced by a factor of 3 so that the diagonal edge is less valid.
Algorithm 2 Connection Graph Construction for Type 2 puzzles

**Input:** Puzzle Patches: \( \{ P_i \}_{i=1}^n \subset \mathbb{R}^{p \times p \times 3} \)
- For all \( 1 \leq i < j \leq n \) calculate the 16 MGC metric values between patches \( P_i \) and \( P_j \) as explained in §4.1
- Construct \( G = (V, E_{\text{est}}, W_{\text{init}}, R) \) according to the procedure described in §4.3.1 with the following three stages: nearest-neighbors construction based on (15), symmetrization of \( W_{\text{init}} \) and pruning extra neighbors with the use of (16)
- For all \( \{i,j\} \in E_{\text{est}} \), calculate \( \mu_{\text{Jaccard}}(i,j) \) according to (17)
- For all \( \{i,j\} \in E_{\text{est}} \), if \( \mu_{\text{Jaccard}}(i,j) = 0 \), set \( W_{\text{Jaccard}}(i,j) = 0 \); otherwise, \( W_{\text{Jaccard}}(i,j) = 1 \)
- Set \( W_{\text{nb}} = 0.8 \times W_{\text{Jaccard}} + 0.2 \times W_{\text{init}} \)
- If the graph \( G \) is disconnected, iteratively connect the largest connected component to smaller connected components as explained in §4.3.3
- For all \( i, j \in V \), calculate \( \delta_{\text{diag}}(i,j) \) according to (21) and if \( \delta_{\text{diag}}(i,j) = 2 \), calculate \( n_1 \) and \( n_2 \) according to (22)
- Form \( W_{\text{diag}} \) according to (23) and update \( W_{\text{nb}} \) according to (24)
- Set \( W = W_{\text{nb}} + W_{\text{diag}} \)

**Return:** \( G = (V, E_{\text{est}}, W, R) \), MGC values for all pairs of patches

5. Solution for Type 2 Puzzles via GCL and Location Solver

This section completes the solution of type 2 jigsaw puzzles. It assumes any preferable solution to type 1 puzzles that is applied after forming the connection graph according to Algorithm 2 and after estimating the correct orientations by Algorithm 1. The new component is a procedure for updating the affinity function and the connection function based on the estimated rotations and locations. One can then estimate again the orientations and locations and repeat this procedure several times. This procedure and the straightforward solution of type 2 puzzles implied by it are summarized below in §5.1. Then §5.2 summarizes the time complexity of this solution.

5.1. Updating the Affinity and Connection Functions and the Resulting Solution. By now there are many successful solutions to type 1 puzzles. While we comment in the appendix on another possible solution, we are not sure how to practically implement it. According to our numerical tests, the algorithms of both Gallagher [14] and Yu et al. [47] for solving type 1 puzzles are highly competitive. We have often noticed a slight advantage of the latter algorithm, which applies a linear programming procedure. Therefore, we use the algorithm of Yu et al. [47] as a default solver for type 1 puzzles in our algorithm. One could use instead any algorithm that solves type 1 puzzles.

The basic idea for updating the values of the affinity and connection functions is that a given estimated solution for the orientations and locations can be used to infer possible mismatches. These identified mismatches could be used to reassign values for the affinity and connection functions that may lead to a more accurate solution.

First, we figure out which patches are wrongly placed in the assembled puzzle and remove them from the grid. For this purpose we use the following kinds of
metrics, which we refer to as NAM (Neighbor-Averaged Metric). For each patch $i$, $1 \leq i \leq n$, with neighbors $i_t$, $i_l$, $i_b$, and $i_r$, from top, left, bottom and right, respectively, we define

\begin{equation}
NAM_{all}(i) = \frac{MGC_{bt}(i,i_t) + MGC_{rl}(i,i_l) + MGC_{tb}(i,i_b) + MGC_{lr}(i,i_r)}{4}.
\end{equation}

If a patch $i$ is at the edge or corner of the puzzle grid, then it has 3 or 2 neighbors, respectively. In this case, we only sum up the respective MGC values and divide the sum by the number of neighbors. Similarly we define the following four metrics:

\begin{align*}
NAM_{ltr}(i) &= \frac{MGC_{rl}(i,i_l) + MGC_{bt}(i,i_t) + MGC_{lr}(i,i_r)}{3}, \\
NAM_{trb}(i) &= \frac{MGC_{bt}(i,i_t) + MGC_{lr}(i,i_r) + MGC_{tb}(i,i_b)}{3}, \\
NAM_{blt}(i) &= \frac{MGC_{tb}(i,i_b) + MGC_{rl}(i,i_l) + MGC_{bt}(i,i_t)}{3}, \\
NAM_{lbr}(i) &= \frac{MGC_{rl}(i,i_l) + MGC_{tb}(i,i_b) + MGC_{lr}(i,i_r)}{3}.
\end{align*}

Again, if the patch is at the edge or corner of the puzzle grid, then we sum the appropriate MGC values and divide by the corresponding number of neighbors.

If the puzzle is correctly assembled, the NAM values of all patches are relatively small as demonstrated for $NAM_{all}$ values in last figure of the first row of Figure 9. Otherwise, if there are some wrongly placed patches, their corresponding NAM values should be relatively higher. This is demonstrated in first and second figures of the first row of Figure 9 where the spikes of $NAM_{all}$ values correspond to wrongly orientated or placed patches. Based on this observation, we suggest to find all patches for which the corresponding $NAM_{all}$ value and at least one of the $NAM_{ltr}$, $NAM_{trb}$, $NAM_{blt}$ and $NAM_{lbr}$ values exceeds 1.5 times the median of all corresponding NAM values. We remove the corresponding edges from the grid. For example, for $NAM_{ltr}$, we remove the edges connecting vertex $i$ with its left, top and right neighbors. We refer to a location as empty if all edges connecting the patch in this location to its top, bottom, left and right neighbors were removed. Patches at empty locations at each iteration of this procedure are demonstrated in the second row of Figure 9. Their removal, which literally creates empty locations, is demonstrated in the last row of this figure.

Next, we identify all the empty locations for which at least 2 of 4 neighboring locations are not empty. We consider the neighboring locations in the puzzle grid, regardless of edges that were removed in the current process. For each fixed empty location, denote the set of neighboring patches (according to locations) by $S_{nb}$. Note that $S_{nb}$ contains either 2, 3 or 4 indices of patches. We identify the empty location with the set $S_{nb}$. For the same empty location, find an oriented patch that minimizes the averaged MGC metric with respect to the vertices in $S_{nb}$. The corresponding minimal value of the averaged MGC metric for a specified empty location with neighboring patches $S_{nb}$ is denoted by $NAM_{S_{nb}}$. Assuming the index of this latter patch is $i$, we denote its pairwise orientation with respect to patch $j \in S_{nb}$ by $R_{i,j}$. Let med($NAM_{all}$) denote the median value of all $NAM_{all}$ values. We update the affinity and connection functions for $i$ and $j \in S_{nb}$ as above by

\begin{equation}
W_{est}(i,j) = W_{est}(j,i) = \begin{cases} 
0.6, & \text{if } NAM_{S_{nb}} < \text{med}(NAM_{all}); \\
0.3, & \text{if } \text{med}(NAM_{all}) < NAM_{S_{nb}} < 2 \text{med}(NAM_{all})
\end{cases}
\end{equation}
Figure 9. Demonstration of the update step for 2 iterations, described in § 5.1, of a type 2 puzzle with 540 pieces each with sizes of 28 × 28. The first row shows the histograms of the NAM_{all} metric values for all patches, defined in (26). The second row shows the solution of the puzzle after each iteration of assembling the puzzle, and the third row shows the remaining patches of an assembled puzzle after removing the patches that are wrongly placed or oriented.

and

\begin{equation}
    R_{est}[i,j] = R_{i,j} \quad \text{and} \quad R_{est}[i,j] = R_{i,j}^T \quad \text{if med}(\text{NAM}_{all}) < 2\text{med}(\text{NAM}_{all}).
\end{equation}

Algorithm 3 summarizes this update procedure.

Finally, our proposed algorithm for reassembling two-dimensional square jigsaw puzzles is summarized in Algorithm 4. It iteratively solves the puzzle by repeating the following 3 steps: finding the orientations of all patches, finding the locations of all patches and updating the connection function and the affinity function. To measure how good the solution is at each iteration and to pick the better one among
Algorithm 3  Updating the affinity function and the connection function at a given iteration

**Input:** MGC metric values between all patches, current solution to the puzzle problem

- Calculate the NAM values for all patches according to (26) and (27)
- Remove all edges from the solution grid for which the corresponding NAM\text{all} value and at least one of the NAM\text{ltr}, NAM\text{trb}, NAM\text{blt} and NAM\text{lbr} values exceeds 1.5 times the median of all corresponding NAM values
- For any empty location (that is, for any location whose all edges were removed) which has at least two non-empty neighboring locations (according to the natural grid of locations), find the patch with the correct rotation which best fits in that position and update the affinity function and the connection function according to (28) and (29)

**Return:** $G = \{V, E, W, R\}$

Then, we recommend using the following metric

\[ \text{Err}(\{R_i\}_{i=1}^n, \sigma) = \sum_{i=1}^n (\text{MGC}_{lt}(R_i \cdot P_i, R_i{\sigma,r} \cdot P_i{\sigma,r}) + \text{MGC}_{tb}(R_i \cdot P_i, R_i{\sigma,b} \cdot P_i{\sigma,b})) \]

Algorithm 4  Solution of type 2 puzzles

**Input:** Puzzle Patches: $\{P_i\}_{i=1}^N \subset \mathbb{R}^{p \times p \times 3}$

- Apply Algorithm 2 with $\{P_i\}_{i=1}^n$ to construct the Affinity Graph $G = (V, E, W, R)$ and obtain MGC values between all patches
- Run Algorithm 1 with $G = (V, E, W, R)$ to find the orientations $\{R_i\}_{i=1}^N$
- Apply the type 1 jigsaw puzzle solver of [47] to solve the type 1 puzzle with patches $\{R_i \cdot P_i\}_{i=1}^n$ and obtain their estimated permutation vector $\sigma$
- Compute and record $\text{Err}(\{R_i\}_{i=1}^n, \sigma)$ by (30)

**for** iterations 1:5  **do**

- Apply Algorithm 3 with $\sigma$, $\{R_i\}_{i=1}^N$ and the MGC values to obtain the updated connection graph $G = (V, E, W, R)$
- Apply Algorithm 1 with $G = (V, E, W, R)$ to recover the orientations $\{R_i\}_{i=1}^N$
- Apply the type 1 jigsaw puzzle solver of [47] to solve the type 1 puzzle with patches $\{R_i \cdot P_i\}_{i=1}^n$ and obtain their estimated permutation vector $\sigma$
- Compute and record $\text{Err}(\{R_i\}_{i=1}^n, \sigma)$ by (30)

**end for**

**Return:** $\{R_i\}_{i=1}^n$ and $\sigma$, which minimize $\text{Err}(\{R_i\}_{i=1}^n, \sigma)$ among all the above choices

We remark that most state-of-the-art methods use a greedy step to make final corrections to the solved puzzle. On the other hand, the step discussed here only
updates the connection graph and is thus non-greedy. It is possible to incorporate greedy procedures that may improve the performance of our algorithm, however, we would like to show that a more principled method can be competitive.

5.2. **Time Complexity of Algorithm** 4. The most time consuming step is to find the MGC metric between all puzzle pieces. The order of operations for this step is $O(n^2d)$, where $n$ is the number of image patches and $d$ is the size of the square image patches. However, one can parallelize this procedure and achieve faster computation. We would like to mention that this step is vital for all jigsaw puzzle solvers.

After finding the MGC metric between all puzzle pieces, our proposed algorithm finds the orientations of all patches and converts a type 2 puzzle into a type 1 puzzle. To find the orientations of puzzle patches we need only construct the connection graph, which requires nearest neighbors computation for each patch. The worst case complexity for this is $O(n^2)$ and the average complexity is $O(n \log(n))$. Then, it finds the top eigenvector of a sparse symmetric matrix with 4 nonzero elements in each column and row, which would take $O(n)$ time. For the type 1 puzzle, the complexity depends on the state-of-the-algorithm being used. It is faster than using the latter algorithm for directly solving the type 2 puzzle.

One may suggest using subsampling to speed up the computation. That is, instead of calculating the MGC metric between a given patch and all other patches, one may only consider a fraction $p$ of the other patches. However, this procedure would only speed up the computation by a constant factor, so the order of time complexity will still remain the same. Also, one needs to be cautious when applying this idea because each patch of a two-dimensional square jigsaw puzzle has at most 4 neighbors, and subsampling, for instance, 50% of patches for each patch will produce on average only 2 neighbors for a central patch. This may yield a disconnected graph and may also result in sensitivity to individual mistakes.

6. **Numerical Experiments**

We apply our proposed algorithm to solve two-dimensional square jigsaw puzzles of the following standard image datasets: the MIT dataset from Cho et al. [8], which contains 20 images, each with 432 patches, and three datasets from Pomeranz et al. [34], where the first two, which are referred to as McGill and Pomeranz, include 20 images with 540 and 805 patches, respectively, and the third one has 3 images with 3300 patches, which is also referred to as Pomeranz or large Pomeranz. For all datasets, the patches are of size $28 \times 28$. Figure 10 demonstrates the application of our proposed algorithm to four images that represent the four datasets. To test the accuracy of our proposed algorithm we use the following four metrics, defined in Gallagher [14] and Cho et al. [8]: the direct comparison, the neighbors comparison, the largest component and the perfect reconstruction. The direct comparison measures the percentage of image patches whose location and orientation are correct. The neighbors comparison calculates the percentage of pairs of image patches that are matched correctly. The largest component calculates the percentage of patches in the largest correctly assembled component of the solved puzzle. Finally, the perfect reconstruction of a puzzle is 1 if it is solved correctly and 0 otherwise.

We compared our algorithm with those of Gallagher [14] and Yu et al. [47] for type 2 puzzles since they were the only algorithms with available codes (we have requested codes from all authors of published algorithms). One of the many
Figure 10. Reconstruction results of our algorithm for type 2 puzzles representing the four datasets. The images in the left column are the inputs for the algorithm and the ones in the right column are the outputs generated by our proposed algorithm. All the patches are of size 28 × 28. The puzzle in first row is from the MIT dataset with 432 patches, the puzzle in the second row is from the McGill dataset with 540 patches, the puzzle in the third row is from the Pomeranz dataset with 805 patches and the puzzle in the fourth row is from the Pomeranz dataset with 3300 patches.
Table 1. Comparison of results for type 2 puzzles for the four datasets. For the first three metrics, we report the mean values and standard deviations over all the images in a dataset. For the fourth metric, we report the sum over all images in a dataset. Due to randomness, the results of the algorithms of Gallagher [14] and Yu et al. [47] are averaged over 20 instances of solving a given puzzle.

| Dataset                  | Method              | Direct | Neighbor | Largest | Perfect |
|--------------------------|---------------------|--------|----------|---------|---------|
|                          | mean    | std    | mean     | std     | mean    | std    |
| MIT dataset, 20 images,  | Gallagher [14]     | 84.2   | 19.7     | 89.1    | 12.4    | 87.2   | 14.3   | 9       |
| 432 patches (28 × 28)    | Yu et al. [47]     | 95.5   | 13.0     | 96.4    | 8.7     | 95.4   | 13.2   | 13      |
|                          | Our method         | 94.8   | 11.3     | 95.2    | 9.2     | 95.4   | 9.1    | 13      |
| McGill dataset, 20 images, | Gallagher [14]   | 77.2   | 35.3     | 85.8    | 19.8    | 84.6   | 21.3   | 7       |
| 540 patches (28 × 28)    | Yu et al. [47]     | 92.9   | 24.6     | 93.5    | 14.8    | 93.1   | 15.4   | 13      |
|                          | Our method         | 88.3   | 25.6     | 92.2    | 15.2    | 91.4   | 17.2   | 13      |
| Pomeranz dataset, 20 images, | Gallagher [14] | 77.5   | 27.8     | 85.3    | 15.5    | 79.3   | 22.6   | 5       |
| 805 patches (28 × 28)    | Yu et al. [47]     | 91.8   | 14.2     | 92.7    | 13.0    | 91.7   | 14.2   | 9       |
|                          | Our method         | 86.8   | 21.4     | 90.0    | 14.2    | 89.3   | 15.4   | 9       |
| Pomeranz dataset, 3 images | Gallagher [14]  | 82.9   | 15.6     | 84.2    | 14.2    | 82.8   | 15.7   | 1       |
| 3300 patches (28 × 28)   | Yu et al. [47]     | 89.7   | 12.3     | 90.2    | 11.0    | 89.7   | 12.3   | 1       |
|                          | Our method         | 86.4   | 14.0     | 88.1    | 11.7    | 86.4   | 14.0   | 1       |

procedures in our algorithm is random and described in § 4.3.3 We have also noticed some randomness in the results of the other two algorithms with which we compare. Therefore, for each puzzle we run each algorithm 20 times and report the averaged result. To get an idea of the randomness of the three algorithms we report the averaged standard deviations when applying these algorithms 20 times to each of the 20 puzzles in the MIT dataset, where an average is taken over the 20 puzzles. These averaged standard deviations for Gallagher [14], Yu et al. [47] and our algorithm, are 6.5, 1.5 and 0.17, respectively. In this and other experiments, we notice that the randomness of our algorithm is not significant.

Table 1 compares the four metrics of our proposed algorithm with some state-of-the-art algorithms. For the first three metrics, which obtain percentages, we report the means and standard deviations among each of the four datasets. We clarify that here the means and standard deviations are with respect to the results of the various images in the datasets, whereas for Gallagher [14] and Yu et al. [47] they are averaged over the 20 instances mentioned above. On the other hand, the standard deviations mentioned above are with respect to these 20 instances, while we averaged them over the images in the MIT dataset. For the fourth metric of perfect reconstruction, we report its sum, that is, the numbers of perfectly solved images in each dataset. Figure 11 presents histograms of the metrics of accuracy of the algorithm for the first three datasets with 20 images. The fourth dataset is excluded from this figure since it only has three images.

As we can see, our results are comparable with those of state-of-the-art methods. The mean errors of the first three metrics are slightly better for [47] but with relatively large standard deviations. The histograms in Figure 11 indicate that our results are comparable to those of the state-of-the-art methods. In general,
we noted that when most of the patches have non-zero gradients around their boundaries, our algorithm obtained perfect recovery. On the other hand, we noted that images with low percentages of recovered puzzle pieces by any algorithm have large portions of patches with the same uniform color. In some cases, for example, in the MIT dataset, the puzzle that any of the three algorithms assembled with lowest percentage of 65% contains a lot of patches that are uniformly white and are identical. This puzzle, with the solution of our proposed algorithm, is presented in the first row of Figure 11. In this scenario, there is no way to find the exact original positions of all patches. However, the solution obtained by any of the three algorithms is visually identical to the original one. We remark that the methods of [14] and [47] also achieved 65% accuracy for this puzzle. Nevertheless, the two puzzles from the McGill dataset and the two from the Pomeranz dataset, with worst percentages of correctly assembled pieces by our proposed algorithm, do not look visually identical to the original ones. We remark that the pairs of two puzzles with worst percentages of correctly assembled pieces by [14] and [47] are the same ones, and the assembled puzzles by these algorithms are also not visually identical to the original ones.

Finally, we would like to mention that our proposed algorithm for recovering the unknown orientations of the puzzle patches has no assumption on the shape.

**Figure 11.** Histograms of the percentages of the recovered patches for type 2 jigsaw puzzle. The three rows correspond to results by our proposed algorithm, the algorithm of [47] and the algorithm of [14], respectively. The three columns correspond to the MIT, McGill and (small) Pomeranz datasets, respectively.
and size of the puzzle, unlike [14, 47]. Furthermore, it is non-greedy. On the other hand, most state-of-the-art algorithms, in particular [14, 42, 47], use a greedy step to make final corrections. We believe that by using that final step of corrections we could further improve our results; however, we would like to avoid any greedy or semi-greedy procedure.

7. DISCUSSION AND CONCLUSION

This paper introduces a novel, non-greedy mathematical approach for solving two-dimensional square jigsaw puzzles. More specifically, its main contribution is a theoretically-guaranteed strategy for recovering the unknown orientations of type 2 puzzle patches. Furthermore, it also suggests a non-greedy step for updating the full puzzle solution based on the latter strategy for solving orientations. Some components of the proposed algorithm, in particular, the strategy for recovering orientations, are relatively fast. Nevertheless, the main bottleneck in the computational complexity, that is, calculating a metric such as MGC between puzzle pieces, is shared by all existing algorithms. Numerical experiments on datasets of two-dimensional square jigsaw puzzles indicate that our results are at least comparable to state-of-the-art methods.

We expect some possible extensions of the proposed algorithm. First of all, we believe that the ideas pursued in this work could be extended to puzzles that come from more complicated manifolds, such as the two-dimensional sphere or a three-dimensional cube jigsaw puzzle, or puzzles with more complicated shapes of patches, such as tangrams. The GCL algorithm should be the same; however, instead of considering the group $\mathbb{Z}_4$, one needs to consider the corresponding rotation group. Two challenges though are defining a good metric between puzzle pieces and constructing the connection graph. By doing this, one will extend the applicability of this work to various real-world applications, such as three-dimensional image reconstruction from two-dimensional images.

In terms of theory, it is interesting to analyze our proposed GCL algorithm with more complicated noise models. Additional theoretical questions arise from different ideas discussed in the appendix that we cannot make practical. For example, we are interested to find out if one can effectively utilize the vector diffusion distances that are described in the appendix or a modification of them. In particular, we are interested to know whether one may modify the VDD distances and consequently resolve the problem with them described in the appendix. Moreover, we would like to know if one can better estimate the locations of the patches by using a quadratic assignment problem formulation, which is discussed in the appendix.

8. ACKNOWLEDGEMENT

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Appendix A. Optional Steps for The Proposed Solution of Type 2 Puzzles

We describe here optional steps to improve the algorithm. Implementation of these ideas did not obtain the desired improvement, but we believe that they might be interesting and useful for future explorations. In §A.1 we review the vector diffusion map and distance and explain how to use them for updating the MGC metric after applying Algorithm 2. We also note that these distances are only informative for sufficiently far away vertices and not for nearby vertices. In §B we discuss the problem of recovering the location of patches after updating the MGC metric in §A.1. We propose a mathematical idea for solving the problem by applying a quadratic assignment formulation with respect to the affinity function \( W \) defined in §4.3. However, the solver of the combinatorial optimization problem is not sufficiently fast and accurate. Lastly, §C suggests using other top eigenvectors of the GCL matrix.

A.1. Updating the Metric between Puzzle Pieces by Vector Diffusion Distances. The MGC metric defined in §4.1 is usually not a perfect metric, but it provides some information whether two patches are neighbors or not. However, if two patches are not neighbors, the MGC metric between them does not provide any information about their distance in the image. Such information can be helpful since the estimated information on neighboring patches can be wrong. For this purpose, we suggest updating the MGC metric by considering the diffusion process associated with the random walk determined by \( C \). The diffusion vector framework for doing this was suggested in [40]. This part is performed after the rotations of the patches were estimated according to Algorithm 2.

For \( t > 0 \), the vector diffusion map (VDM) [40] in our setting is a function \( V_{t,n} : \mathbb{Q} \to \mathbb{R}^{2n \times 2n} \) defined by

\[
V_{t,n} : P_i \mapsto ((\mu_{C,l}^C, l, v_{C,l}^r[i], v_{C,r}^l[i]))_{l,r=1}^{2n} \in \mathbb{R}^{(2n)^2},
\]

where \( \mu_{C,l}^C \) and \( v_{C,l}^r \) are the \( l \)-th eigenvalue and eigenvector, respectively, of \( C \), and \( v_{C,l}^r[i] \) is a two-dimensional vector containing the \( (2(i - 1) + 1) \)-th and \( (2i) \)-th entries of \( v_{C,l}^r \). The vector diffusion distance (VDD) at time \( t > 0 \) [40] between two patches indexed by \( i \) and \( j \) is

\[
d_{C,t,n}(i,j) := \|V_{t,n}(P_i) - V_{t,n}(P_j)\|_{\mathbb{R}^{(2n)^2}}.
\]

This distance converts the local information into global information and provides an estimate of the distance between patches in the original grid. Based on this distance one can infer whether two patches are close to each other in the original image or far away. As demonstrated in Figure 12, this distance is not sufficiently accurate to infer nearness when patches have comparable distances. Specifically, a problem arises when two neighboring patches, represented by \( i \) and \( j \) in Figure 12, were not estimated to be neighbors by any metric, and therefore the only paths connecting them are through their neighbors. In this case, the diffusion distance between them, in particular, between \( i \) and \( j \) in Figure 12 is larger than that between one of them and its diagonal neighbor, for example, between \( i \) and \( k \) in Figure 12. Therefore, this distance does not completely reflect the true underlying geometry. As a result, in general it cannot be used to infer whether two patches are neighbors or diagonal neighbors. We remark that this is due to the discrepancy between the metric we design and the true underlying metric.
Figure 12. An example where VDD fails to reflect the distance between nearby patches. The graph is a grid with one missing edge between vertices \( i \) and \( j \) (all other neighboring edges are connected by an edge). Due to the structure of the grid, the shortest path between vertices \( i \) and \( j \) is of length 3, whereas the shortest path between vertices \( i \) and \( k \) is 2. Thus the use of VDD leads to the wrong conclusion that vertex \( i \) is closer to vertex \( k \) than to vertex \( j \).

Nevertheless, it is still possible to use the VDD for improving the MGC metric in the following way. We first compute the VDD between all patches. For each image patch \( P_i \in Q \), we sort the VDD distances of all other patches to \( P_i \) and record the patches in the 0.1-quantile of largest distances. The MGC metric between these patches and \( P_i \) is then increased by the factor \( \alpha = 2 \). This ensures that patches that are not likely neighbors of \( P_i \) are penalized by a larger distance and thus have a smaller chance of becoming neighbors in the final solution.

Our numerical tests did not indicate any significant improvement when using this procedure. In order to reduce the computational time of the algorithm, we do not apply it in practice and mention it as an optional step for the whole algorithm.

Appendix B. Possible Estimation of Locations by Quadratic Assignment

As we have already mentioned, the main contribution of this work is to introduce a new approach for the recovery of the unknown orientations of patches in type 2 puzzles by using the GCL. Nevertheless, one can try to take advantage of the affinity function \( W_{\text{est}} \), whose construction is described in §4.3 and the fact that for two-dimensional square jigsaw puzzles the true affinity function \( W_{\text{true}} \) is known, whereas the shuffling of patches is unknown. Therefore, one may try to match \( W_{\text{est}} \) with \( W_{\text{true}} \). This gives rise to the problem of finding a permutation matrix \( P \), which corresponds to the permutation \( \sigma \) described in §4.3.3, such that \( W_{\text{est}} \) and \( P^T W_{\text{true}} P \) match. The desired permutation can be expressed as the solution of the following optimization problem:

\[
(32) \quad \underset{P \in \text{Perm}(n)}{\text{argmin}} \| W_{\text{est}} - P^T W_{\text{true}} P \|_2^2.
\]
Note that (32) is the Quadratic Assignment Problem (QAP) for the matrices $W_{true}$ and $W_{est}$. However, existing solvers are slow as the number of patches increases and thus we are not sure how to make this procedure practical for large puzzles. A similar idea has been proposed by Andalo et al. [2] for solving type 1 puzzles. They suggest solving a QAP with different weight matrices by using constrained gradient descent. It is unclear to us if their procedure is applicable to the QAP problem in (32). Formally, the modified algorithm for type 2 puzzles is Algorithm 4 where right after running Algorithm 1 which occurs twice, one needs to update the MGC metric according to the above procedure.

**Appendix C. Using Other Top Eigenvectors of the GCL Matrix**

As we have discussed in §3.3, if the constructed connection graph is good enough, the top 2 eigenvectors of the GCL matrix can recover the orientations of puzzle patches. However, when it is impossible to construct an accurate affinity graph (e.g., Figure 2), one might consider the top few eigenvectors, as they might also contain some useful information about the orientations of patches. For some puzzles and poorly-estimated connection graphs, the orientations recovered by the top 3-rd and 4-th eigenvectors are more accurate than the ones recovered by the top 2 eigenvectors. We thus suggest two candidate solutions, one where in the initial iteration (before applying the updates described in §5.1) we use the top 2 eigenvectors and another one where in the initial iteration we use the top 3-rd and 4-th eigenvectors.

We remark that this procedure of using the top 3-rd and 4-th eigenvectors is not needed at the later updates of §5.1 since the connection graphs are then nicely approximated. Our experiments indicate that even for the initial stage, the use of this procedure is beneficial only for few images. We thus leave this step as optional. This procedure is summarized in Algorithm 5.

**Algorithm 5** Variation on the solution of type 2 puzzles

**Input:** Puzzle Patches: $\{P_i\}_{i=1}^N \subset \mathbb{R}^{p \times p \times 3}$
- Apply Algorithm 1 with $\{P_i\}_{i=1}^N$ to obtain the solution $\{R_{i,1}\}_{i=1}^N$ and $\sigma_1$
- Apply Algorithm 4 with $\{P_i\}_{i=1}^N$, but for the initial iteration (which applies Algorithm 1 in step 2 of Algorithm 4) use the top 3-4 eigenvectors instead of the top 2 (step 4 of Algorithm 1) to obtain the orientations $\{R_{i,2}\}_{i=1}^N$ and $\sigma_2$
- Compute the Err values for $\{\{R_{i,1}\}_{i=1}^N, \sigma_1\}$ and $\{\{R_{i,2}\}_{i=1}^N, \sigma_2\}$ according to (30) and let $\{\{R_i\}_{i=1}^N, \sigma\}$ be the one that produces the smaller Err value

**Return:** $\{R_i\}_{i=1}^N$ and $\sigma$

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