Orders of Oscillation Motivated by Sarnak’s Conjecture, Part II *†

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Abstract

I have investigated orders of oscillating sequences motivated by Sarnak’s conjecture in [13] and proved that an oscillating sequence of order $d$ is linearly disjoint from affine distal flows on the $d$-torus. One of the consequences is that an oscillating sequence of order $d$ in the arithmetic sense is linearly disjoint from affine flows with zero topological entropy on the $d$-torus. In this paper, I will extend these results to polynomial skew products on the $d$-torus, that is, given a polynomial skew product on the $d$-torus, there is a positive integer $m$ such that any oscillating sequence of order $m$ is linearly disjoint from this polynomial skew product. In particular, when all polynomials depend only on the first variable, I have that an oscillating sequence of order $m = d + k - 1$ is linearly disjoint from all polynomial skew products on the $d$-torus with polynomials of degree less than or equal to $k$. One of the consequences is the linear disjointness for flows which are automorphisms of the $d$-torus with absolute values of eigenvalues 1 plus a polynomial vector and oscillating sequences of order $m$ in the arithmetic sense. Furthermore, I will prove that an oscillating sequence of order $d$ is linearly disjoint from minimal mean attractable and minimal quasi-discrete spectrum of order $d$ flows. Finally, I define and give some examples of Chowla sequences from our paper [4].

1 Introduction

This paper uses $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ to mean natural numbers, integers, real numbers, and complex numbers. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. Suppose $d \geq 2$ is an integer. Let $\mathbb{R}^d$ be the $d$-Euclidean real vector space and let $\mathbb{Z}^d$ be the integer lattice in $\mathbb{R}^d$. Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the $d$-torus.

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We use $c = (c_n)_{n \in \mathbb{N}}$ to denote a sequence of complex numbers. An example is the Möbius sequence $u = (\mu(n))_{n \in \mathbb{N}}$, the one generated by the Möbius function defined as $\mu(1) = 1$, $\mu(n) = (-1)^r$ if $n$ is a product of $r$ distinct prime numbers, and $\mu(n) = 0$ if $p^2 \mid n$ for a prime number $p$.

Suppose $X$ is a compact metric space with a metric $d(\cdot, \cdot)$ and $f : X \to X$ is a (piece-wise) continuous map. We use $f^n$ to denote $n$ compositions of $f$ for $n \in \mathbb{N}$ and $f^0$ to denote the identity. Then $\{f^n\}_{n=0}^\infty$ is a (discrete) dynamical system. Following [14, 15], we call it or just $f$ a (discrete) flow. Let $C(X)$ be the space of all continuous complex-valued functions on $X$ with the maximum norm $\|\phi\| = \max_{x \in X} |\phi(x)|$.

**Definition 1.** We say $c$ is linearly disjoint from (or orthogonal to) $f$ if for any $\varphi \in C(X)$ and any $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n \varphi(f^n x) = 0. \tag{1}$$

Sarnak [14, 15] made a conjecture connecting number theory and ergodic theory as follows.

**Conjecture 1.** The Möbius sequence $u$ is linearly disjoint from all flows $f$ with zero topological entropy.

In the paper [13], I defined the order of an oscillating sequence $c$ as follows and showed why orders of oscillation on $c$ is important for understanding Conjecture 1.

**Definition 2.** We call $c = (c_n)_{n \in \mathbb{N}}$ an oscillating sequence of order $d \geq 2$ if for every real coefficient polynomial $P$ of degree $\leq d$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i P(n)} = 0 \tag{2}$$

with the technical condition (3),

$$\frac{1}{N} \sum_{n=1}^{N} |c_n|^\lambda \leq C, \quad \forall \ N \geq 1. \tag{3}$$

I also defined a stronger version in the arithmetic sense.

**Definition 3.** We call $c = (c_n)_{n \in \mathbb{N}}$ an oscillating sequence of order $d \geq 2$ in the arithmetic sense if for every real coefficient polynomial $P$ of degree $\leq d$ and every pair of integers $0 \leq l < k$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq n = mk + l \leq N} c_n e^{2\pi i P(n)} = 0 \tag{4}$$

with the technical condition (3).
Remark 1. An oscillating sequence of order 1 is just called an oscillating sequence which is defined in [8].

Due to Davenport [6] and Hua [11], we know that the Möbius sequence $u = (\mu(n))_{n \in \mathbb{N}}$ is an example of an oscillating sequence of order $d$ as well as in the arithmetic sense for all $d \geq 1$. In the view of expanding dynamical systems, we found in [4] that for almost all $\beta > 1$, sequences $(e^{2\pi i \beta n})_{n \in \mathbb{N}}$ are oscillating sequences of order $d$ as well as in the arithmetic sense for all $d \geq 1$. We actually proved a more general result (refer to Theorem 4 in section 4). It is still an interesting problem to find a concrete number $\beta > 1$ such that $(e^{2\pi i \beta n})_{n \in \mathbb{N}}$ is an example of an oscillating sequence of order $d$ as well as in the arithmetic sense for all $d \geq 1$.

We are interested in the following conjecture.

Conjecture 2. An oscillating sequence $c$ of order $d$ for all $d \geq 2$ in the arithmetic sense is linearly disjoint from flows $f$ on a torus with zero topological entropy.

I have studied Conjecture 2 for all affine flows with zero topological entropy on a torus in [13] as follows.

For $x \in \mathbb{T}^d$, let $x^t = (x_1, \ldots, x_d)$ mean the transpose of $x$. Let $GL(d, \mathbb{Z})$ be the space of all $d \times d$–matrices $A$ of integer entries with determinants $\det(A) = \pm 1$. For any $A \in GL(d, \mathbb{Z})$, the map $Ax$ is an automorphism of $\mathbb{T}^d$. For $a \in \mathbb{T}^d$ with $a^t = (a_1, \ldots, a_d)$, we have an affine flow $T_{A,a}(x) = Ax + a : \mathbb{T}^d \to \mathbb{T}^d$. (5)

A result of Sinai says that the flow $T_{A,a}$ has zero topological entropy if and only if the absolute values of all eigenvalues of $A$ are 1. Furthermore, a result of Kronecker implies that all eigenvalues of $A$ are roots of unity. Therefore, for some positive integer $m$, all eigenvalues of $A^m$ are 1. Without loss of generality, we only need to consider $A \in GL(d, \mathbb{Z})$ such that its all eigenvalues are 1. We have that

Lemma 1. Suppose $A \in GL(d, \mathbb{Z})$ such that its all eigenvalues are 1. Then we have a matrix $P \in GL(d, \mathbb{Z})$ whose determinant is 1 such that $L = P^{-1}AP \in GL(d, \mathbb{Z})$ is a lower triangular matrix.

Thus without loss of generality, we only need to consider a lower triangular matrix $A \in GL(d, \mathbb{Z})$ whose entries on the main diagonal are 1. Under this assumption, the map in (5) is an affine skew product,

$$T_{A,a}(x) = \begin{pmatrix} x_1 + a_1 \\ x_2 + b_{21}x_1 + a_2 \\ \vdots \\ x_d + b_{d(d-1)}x_{d-1} \cdots + b_{d1}x_1 + a_d \end{pmatrix}$$ (6)
where $b_{ij}$, $2 \leq i \leq d$, $1 \leq j \leq i-1$, are constants. In [13], I proved the following theorem.

**Theorem A.** An oscillating sequence $c$ of order $d \geq 2$ is linearly disjoint from affine skew products $T_{A,a}(x)$ in (6) on the $d$-torus.

One of the consequences of Theorem A is that

**Corollary A.** An oscillating sequence $c$ of order $d \geq 2$ in the arithmetic sense is linearly disjoint from affine flows $T_{A,a}$ on the $d$-torus with zero topological entropy.

One purpose of this paper is to extend these results to some nonlinear flows on a torus, which we will present in section 2. Another goal of this paper is to generalize a result of Abdalaoui in [1] about quasi-discrete spectrum flows, which we will present in section 3. Finally, I will define and construct Chowla sequences, which will be presented in section 5.

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### 2 Polynomial skew products.

We call a map $f$ on the $d$-torus a polynomial skew products of degree $k$ if

$$f(x) = \begin{pmatrix}
    x_1 + a \\
    x_2 + h_2(x_1) \\
    x_3 + h_3(x_1, x_2) \\
    \vdots \\
    x_d + h_d(x_1, x_2, \ldots, x_{d-1})
\end{pmatrix}$$

(7)

where $a$ is a constant and $h_2(x_1)$ is a polynomial of $x_1$ of degree $\leq k$, $h_3(x_1, x_2)$ is a polynomial of $x_1$ and $x_2$ of degree $\leq k$, $\ldots$, $h_d(x_1, \ldots, x_{d-1})$ is a polynomial of $x_1, \ldots, x_{d-1}$ of degree $\leq k$. We call $f$ a simple polynomial skew product if it is in the form

$$f(x) = \begin{pmatrix}
    x_1 + a \\
    x_2 + h_2(x_1) \\
    x_3 + h_3(x_1) \\
    \vdots \\
    x_d + h_d(x_1)
\end{pmatrix}$$

(8)
where $a$ and $b_{ij}$, $3 \leq i \leq d$, $2 \leq j \leq i - 1$, are constants and $h_i(x_1)$, $2 \leq i \leq d$, are polynomials of $x_1$ of degree $\leq k$. One of the results in this paper is the following theorem.

**Theorem 1.** An oscillating sequence $c$ of order $m = d + k - 1$ is linearly disjoint from all simple polynomial skew products $f(x)$ of degree $k$ in (8) on the $d$-torus for any $d \geq 2$ and $k \geq 2$.

**Corollary 1.** Suppose $A \in GL(d, \mathbb{Z})$ and all eigenvalues of $A$ have the absolute values 1. Let $q \geq 1$ be the smallest integer such that all eigenvalues of $A^q$ are 1. Suppose

$$f^q(x) = A^q x + h(x)$$

where $h^j(x_1) = (a, h_2(x_1), \ldots, h_d(x_1)) \in \mathbb{T}^d$ and $a$ is a constant and $h_i(x_1)$, $2 \leq i \leq d$, are polynomials of $x_1$ of degree $\leq k$ for $k \geq 2$. Then any oscillating sequence $c$ of order $m = d + k - 1$ in the arithmetic sense is linearly disjoint from the flow $f$.

**Remark 2.** For $k = 1$, they are Theorem A and Corollary A.

**Proof of Theorem 1.** Let $k \in \mathbb{Z}^d$ with $k^t = (k_1, \ldots, k_d)$, define

$$e(k \cdot x) = e^{2\pi i (k_1 x_1 + \cdots + k_d x_d)}, \quad x \in \mathbb{T}^d.$$  

From the Stone-Weierstrass theorem, the set $S = \{e(k \cdot x)\}_{k \in \mathbb{Z}^d}$ forms a dense subset in $C(\mathbb{T}^d)$. A linear combination $p$ of elements in $S$ is called a trigonometric polynomial. We can write $p$ as

$$p(x) = \sum_{m_1 \leq k_1 \leq s_1} \cdots \sum_{m_d \leq k_d \leq s_d} a_{k} e^{2\pi i (k_1 x_1 + \cdots + k_d x_d)}.$$  

For any $\phi \in C(\mathbb{T}^d)$, we have a sequence of trigonometric polynomials

$$p_q(x) = \sum_{m_1 q \leq k_1 \leq s_{1q}} \cdots \sum_{m_d q \leq k_d \leq s_{dq}} a_{k/q} e^{2\pi i (k_1 x_1 + \cdots + k_d x_d)}.$$  

such that $\|\phi - p_q\| \to 0$ as $q \to \infty$. The sequence $\{p_q\}_{q \in \mathbb{N}}$ is called the trigonometric approximation of $\phi$. Consider

$$S_N \phi(x) = \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^n x).$$  

For any $\epsilon > 0$, we have an integer $r > 0$ such that

$$\|\phi - p_r\| < \frac{\epsilon}{2C^r}.$$  

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Then

\[ S_N \phi(x) = \left( \frac{1}{N} \sum_{n=1}^{N} c_n (\phi(f^n x) - p_r(f^n x)) \right) + \left( \frac{1}{N} \sum_{n=1}^{N} c_n p_r(f^n x) \right) = I + II. \]

For the estimation of \( I \), we apply the Hölder inequality. Let \( \lambda \) and \( C \) be the numbers in (4) and \( \lambda' > 1 \) be the dual number of \( \lambda \), that is,

\[ \frac{1}{\lambda} + \frac{1}{\lambda'} = 1. \]

Then we have that

\[ |I| \leq \left( \frac{1}{N} \sum_{n=1}^{N} |c_n|^\lambda \right)^{\frac{1}{\lambda}} \left( \frac{1}{N} \sum_{n=1}^{N} |\phi(f^n x) - q_r(f^n x)|^{\lambda'} \right)^{\frac{1}{\lambda'}} \leq C^{\frac{1}{\lambda}} \frac{\epsilon}{2C^{\frac{1}{\lambda}}} = \frac{\epsilon}{2}. \]

For the estimation of \( II \), let \( x_n = f^n x \) for \( n = 0, 1, \ldots \). Denote

\[ x_n = (x_1^n, \ldots, x_d^n). \]

We have that

\[ x_1^n = x_1 + na, \]

\[ x_2^n = x_2 + \sum_{l=0}^{n-1} h_2(x_1 + la) = x_2 + \sum_{l=0}^{n-1} \sum_{i=0}^{k} r_i (x + la)^i \]

\[ = x_2 + \sum_{i=0}^{k} r_i \sum_{l=0}^{n-1} (x + la)^i = x_2 + \sum_{i=0}^{k} r_i \sum_{l=0}^{n-1} \binom{i}{j} x^{i-j}(la)^j \]

\[ = x_2 + \sum_{i=0}^{k} r_i \sum_{j=0}^{n-1} \binom{i}{j} x^{i-j}(la)^j = x_2 + \sum_{i=0}^{k} r_i \sum_{j=0}^{n-1} \binom{i}{j} x^{i-j} \sum_{l=0}^{n-1} (\sum_{l=0}^{n-1}l^j). \]

Since \( \sum_{l=0}^{n-1}l^j \) is a polynomial of \( n \) of degree \( j + 1 \) for \( 1 \leq j \leq k \), \( x_2^n \) is a polynomial of \( n \) of degree \( \leq k + 1 = 2 + k - 1 \). Next, we calculate

\[ x_3^n = x_3 + \sum_{j=1}^{n-1} b_{3j} x_2^n + b_{31} x_1^n + b_{33} \sum_{j=1}^{n-1} \sum_{l=0}^{n-2} h_2(x_1 + la) + \sum_{l=0}^{n-1} h_3(x_1 + la). \]

So we see that \( x_3^n \) is a polynomial of \( n \) of degree at most \( k + 2 = 3 + k - 1 \). In general,

\[ x_i^n = x_i^{n-1} + b_{i(i-1)} x_{i-1}^{n-1} + \cdots + b_{i2} x_2^{n-1} + \sum_{l=0}^{n-1} h_i(x_1 + la), \quad 4 \leq i \leq d. \]
Inductively, one can see that the term \( b_{2^i x_i^{n-1}} \) in \( x_i^n \) contains
\[
\sum_{j_1=0}^{n-2} \sum_{j_2=0}^{j_1-3} \cdots \sum_{j_i=0}^{j_{i-2}} \sum_{l=0}^{j_{i-2}} h_2(x_1 + l a).
\]
So \( x_i^n \) is a polynomial of degree \( i + k - 1 \). Thus we have that
\[
x_i^n = (P_1(n), \cdots, P_d(n)),
\]
where \( P_1(n) \) is polynomial of degree 1 and \( P_i(n) \) is a polynomial of degree at most \( i + k - 1 \) for every \( 2 \leq i \leq d \). Now using the formula (9), we have that
\[
p_r(f^n x) = \sum_{m_{1r} \leq k_1 \leq s_{1r}} \cdots \sum_{m_{dr} \leq k_d \leq s_{dr}} a_{k,r} e^{2\pi i P_{k,r}(n)},
\]
where
\[
P_{k,r}(n) = c_{k,r,d} n^{d+k-1} + \cdots + c_{k,r,1} n + c_{k,r,0}
\]
is a real coefficient polynomial of degree at most \( d + k - 1 \).

After the above calculations, we get
\[
|II| = \left| \frac{1}{N} \sum_{n=1}^{N} c_n \sum_{m_{1r} \leq k_1 \leq s_{1r}} \cdots \sum_{m_{dr} \leq k_d \leq s_{dr}} a_{k,r} e^{2\pi i P_{k,r}(n)} \right|
\]
\[
= \left| \sum_{m_{1r} \leq k_1 \leq s_{1r}} \cdots \sum_{m_{dr} \leq k_d \leq s_{dr}} a_{k,r} \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i P_{k,r}(n)} \right|.\]
Let
\[
L = \max\{|m_{1r}|, \cdots, |m_{dr}|, |s_{1r}|, \cdots, |s_{dr}|, |a_{k,r}| \mid m_{1r} \leq k_j \leq s_{jr}, 1 \leq j \leq d\}.
\]
Since \( c \) is an oscillating sequence of order \( d + k - 1 \), we can find an integer \( M > r \) such that for \( N > M \),
\[
\left| \frac{1}{N} \sum_{n=1}^{N} c_n e^{2\pi i P_{k,r}(n)} \right| < \frac{\epsilon}{2L d}, \quad \forall m_{1r} \leq k_1 \leq s_{1r}, \cdots, m_{dr} \leq k_n \leq s_{dr}.
\]
This implies that \( |II| < \epsilon/2 \). Therefore, we get that for all \( N > M \),
\[
|S_N \phi(x)| < \epsilon.
\]
This says that \( \lim_{N \to \infty} S_N \phi(x) = 0 \). We completed the proof.

Theorem [1] can be generalized to the following theorem.
Theorem 2. Suppose \( f \) is a polynomial skew products of degree \( k \) on the \( d \)-torus in the form \([7]\). Then there is a positive integer \( m = m(d, k, h_1, \cdots, h_d) \) such that any oscillating sequence \( c \) of order \( m \) is linearly disjoint from \( f \).

Proof. The most of the proof is similar to the proof of Theorem 1. Here we just outline some different steps. Let \( x_n = f^n x \) for \( n = 0, 1, \cdots \). Denote \( x^i_n = (x^n_1, \cdots, x^n_d) \). Then

\[
x^n_1 = x_1 + na
\]
is a linear map of \( n \),

\[
x^n_2 = x_2 + \sum_{l=0}^{n-1} h_2(x_1 + la),
\]
is a polynomial of \( n \) of degree \( k_2 \leq k + 1 \),

\[
x^n_3 = x_3 + \sum_{j=1}^{n-2} \sum_{l=0}^{j} h_2(x_1 + la) + \sum_{l=0}^{n-1} h_3(x_1 + la, x_2 + \sum_{s=0}^{r-1} h_2(x_1 + sa)).
\]
is a polynomial of \( n \) of degree \( k_3 \leq k^2 + 1 \), which depends on the terms of \( x_1 \) and \( x_2 \) appeared in \( h_3 \). Inductively, we have that

\[
x^n_i = x^{n-1}_i + \sum_{l=0}^{n-1} h_i(x^{n-1}_1, x^{n-1}_2, \cdots, x^{n-1}_{i-1})
\]
is a polynomial of \( n \) of degree \( k_i \leq k^{i-1} + 1 \) for \( 4 \leq i \leq d \). Thus we have that

\[
x^n_i = (P_1(n), \cdots, P_d(n)),
\]
where \( P_i(n) \) is polynomial of degree 1 and \( P_i(n) \) is a polynomial of degree at most \( k_i \) for every \( 2 \leq i \leq d \) and \( m = \max\{1, k_2, \cdots, k_d\} \).

Corollary 2. Suppose \( A \in GL(d, \mathbb{Z}) \) and all eigenvalues of \( A \) have the absolute values 1. Let \( q \geq 1 \) be the smallest integer such that all eigenvalues of \( A^q \) are 1. Suppose

\[
f^q(x) = A^q x + h
\]
where \( h^i = (a, h_2(x_1), h_3(x_1, x_2), \cdots, h_d(x_1, \cdots, x_{d-1})) \in \mathbb{Z}^d \) and \( h_i(x_1, \cdots, x_{i-1}) \), \( 2 \leq i \leq d \), are polynomials of \( x_1, \cdots, x_{i-1} \) of degree \( \leq k \). Then there is a positive integer \( m = m(d, k, h_1, \cdots, h_d) \) such that any oscillating sequence \( c \) of order \( m \) in the arithmetic sense is linearly disjoint from the flow \( f \).
3 Minimal Mean Attractable and Minimal Quasi-Discrete Spectrum Flows.

In [8], we extended the concept of the mean-Lyapunov-stability (abbreviated MLS) of Fomin [7] to the concept of minimal MLS (abbreviated MMLS). We also define a new concept called minimal attractability (abbreviated MMA) in [8]. We studied the linear disjointness of an oscillating sequence (of order 1) from MMA and MMLS flows using these two concepts. In this paper, we will extend the concept of the quasi-discrete spectrum (abbreviated QDS) of Abramov, Hahn, and Parry [3, 10] to the concept of the minimal QDS (abbreviated MQDS) and study the linear disjointness of an oscillating sequence of order $d$ from MMA and MQDS($d$) flows.

Suppose $K$ is a subset of $X$. We say $K$ is minimal (respective to $f$) if $f : K \to K$ and for any $x \in K$, the closure of the forward orbit of $x$ is $K$, that is,

$$\overline{\{f^n x\}_{n=0}^\infty} = K.$$ 

Definition 4 (MMA). Suppose $K \subseteq X$ is minimal. We say $x \in X$ is mean attracted to $K$ if for any $\epsilon > 0$, there is a point $z = z_{\epsilon, x} \in K$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} d(f^n x, f^n z) < \epsilon.$$  \hspace{5mm} (10)

The basin of attraction of $K$, denoted as $\text{Basin}(K)$, is defined to be the set of all points $x$ which are mean attracted to $K$. We call $f$ minimally mean attractable (abbreviated MMA) if

$$X = \bigcup_{K} \text{Basin}(K)$$ \hspace{5mm} (11)

where $K$ varies among all minimal subsets of $X$.

Remark 3. It is clear that $K \subseteq \text{Basin}(K)$. The union in the above definition could be uncountable.

We use [9] as a general reference in the reviewing of a QDS flow. Suppose $G$ is an Abelian group in multiplication and $\Lambda : G \to G$ is homomorphism. Consider another homomorphism $\Phi : G \to G$ defined as

$$\Phi g := g \cdot \Lambda g, \quad g \in G.$$
Then we have that
\[ \Phi^n g = \prod_{j=0}^{n} (\Lambda_j g)^{\binom{n}{j}}, \quad g \in G. \]

Let \( G_n = \ker(\Lambda^n) \) be a subgroup of \( G \). Then we have a filter
\[ \{1\} = G_0 \subseteq G_1 \subseteq G_n \subseteq G_{n+1} \subseteq \cdots \]
and \( \Lambda : G_n \to G_{n-1} \) for all \( n \geq 1 \). Note that \( \Phi : G_n \to G_n \) is an automorphism for every \( n \geq 1 \). The homomorphism \( \Lambda : G \to G \) is called nilpotent if \( G = G_n \) for some \( n \geq 0 \). It is called quasi-nilpotent if \( G = \bigcup_{n=0}^{\infty} G_n \).

A triple \( (G, \Lambda, \iota) \) is called a signature if \( G \) is an Abelian group, \( \Lambda : G \to G \) is a quasi-nilpotent homomorphism and \( \iota : G_1 \to \mathbb{T}^2 \) is an injective homomorphism (i.e., monomorphism), where \( G_1 = \ker(\Lambda) \). The order of the signature \( (G, \Lambda, \iota) \) is
\[ \text{ord}(G, \Lambda, \iota) := \inf\{n \in \mathbb{N} \mid G = G_n\} \in \mathbb{N} \cup \{\infty\}. \]

Note that the order is infinite if \( G \neq G_n \) for all \( n \in \mathbb{N} \), that is, \( G \) is not nilpotent.

Now let us return to \( f : X \to X \). Suppose \( K \subseteq X \) is minimal. Consider the Koopman operator \( \Phi : C(K) \to C(K) \) defined as
\[ \Phi \phi = \phi \circ f, \quad \phi \in C(K). \]

Let \( G = \{g \in C(K) \mid \|g\| = 1\} \).

Then it is an Abelian group in multiplication. Consider the homomorphism \( \Lambda : G \to G \) defined as
\[ \Lambda g = \Phi g \cdot \overline{f}, \quad g \in G. \]

Then
\[ \Phi g = g \cdot \Lambda g, \quad g \in G. \]

One can see that \( G_1 = \text{Fix}(\Phi) \cap G \) where \( \text{Fix}(\Phi) \) is the set of all fixed points of \( \Phi \). Since \( K \) is minimal, we have that \( G_1 = \mathbb{T} \) and
\[ \text{Fix}(\Phi) = \text{span}_{\mathbb{C}}(G_1) = \mathbb{C}. \]

Let us consider \( G_2 = \ker(\Lambda^2) \). For any \( g \in G_2 \), we have \( \Lambda^2 g = 1 \). This implies that
\[ \Phi^2 g = g \cdot (\Lambda g)^2 \cdot \Lambda^2 g = \Lambda g \cdot \Phi g \]

Since \( \Lambda g \in G_1 \), \( \lambda = \Lambda g \) with \( |\lambda| = 1 \) is a unimodular eigenvalue of \( \Phi \) with an unimodular eigenvector \( g' = \Phi g \in G_2 \). Conversely, suppose \( \lambda \) is a unimodular
eigenvalue of $\Phi$ with a unimodular eigenvector $g$, that is, $\Phi g = \lambda g$ and $|\lambda| = 1$. Then $|g| = 1 \in G_1$ and $\Lambda g = \Phi g \cdot \bar{g} = \lambda g \cdot \bar{g} = \lambda \in G_1$. This says that $g \in G_2$. Thus $G_2$ is the set of all unimodular eigenvectors of $\Phi$. In general, elements in $G_n$ are called unimodular quasi-eigenvectors of order $n - 1$. We say that $f : K \to K$ is a quasi-discrete spectrum (abbreviated QDS) flow if the linear hull of all unimodular quasi-eigenvectors is dense in $C(K)$, that is, $G = \cup_{n=0}^{\infty} G_n$ and

$$\text{span}_{C(G)}(G) = C(K).$$

We say that $f : K \to K$ is a quasi-discrete spectrum of order $d$ (abbreviated QDS(d)) if it is a QDS flow and $\text{ord}(G, \Lambda, \iota) = d + 1$.

**Definition 5** (MMA and MQDS). We call $f : X \to X$ a MMA and MQDS $(d)$ flow if $f$ is MMA and for every minimal set $K \subseteq X$, $f : K \to K$ is QDS $(k)$ for some $k \leq d$.

Now we can state another result in this paper.

**Theorem 3.** An oscillating sequence $c$ of order $d \geq 1$ is linearly disjoint from MMA and MQDS $(d)$ flows.

**Proof.** Let $\lambda > 1$ and $C$ be the numbers in (3). Let $\lambda' > 1$ be the dual number of $\lambda$, that is,

$$\frac{1}{\lambda} + \frac{1}{\lambda'} = 1.$$

For any $x \in X$ and any $\phi \in C(X)$, we need to prove that (1) holds. Since $f$ is MMA, $x \in \text{Basin}(K)$ for some minimal subset $K \subseteq X$. Assume that we already know that (1) holds for $z \in K$. For an arbitrarily small number $\epsilon > 0$, the uniform continuity of $\phi$ implies that there is a $\delta > 0$ such that

$$|\phi(u) - \phi(v)| < \frac{\epsilon}{2^{1+\lambda'}/C^\lambda}$$

whenever $d(u, v) < \delta$. Since $f$ is MMA, there exists $z = z_{\delta, x} \in K$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} d(f^n x, f^n z) < \delta^2.$$

This implies that

$$\mathcal{D}(E) = \limsup_{n \to \infty} \frac{\#(E \cap [1, N])}{N} \leq \delta$$

where $E = \{n \geq 1 \mid d(f^n x, f^n z) \geq \delta\}$, because

$$\delta \sharp(E \cap [1, N]) \leq \sum_{n=1}^{N} d(f^n x, f^n z).$$

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Consider
\[ S_N\phi(x) = \frac{1}{N} \sum_{n=1}^{N} c_n \phi(f^n x). \]

Write
\[ S_N\phi(x) = \left( S_N\phi(x) - S_N\phi(z) \right) + S_N\phi(z) = I + II \]

Since we assume that (1) holds for \( z \in K \), we have a \( N_0 > 0 \) such that \( |II| = |S_N\phi(z)| < \epsilon/2 \) for all \( N > N_0 \). From the Hölder inequality,
\[ |I| = |S_N\phi(x) - S_N\phi(z)| \leq \left( \frac{1}{N} \sum_{n=1}^{N} |c_n|^\lambda \right)^{\frac{1}{\lambda}} \left( \frac{1}{N} \sum_{n=1}^{N-1} |\phi(f^n x) - \phi(f^n y)|^{\lambda'} \right)^{1/\lambda'} \]
\[ \leq C^{\frac{1}{\lambda}} \left( \frac{1}{N} \sum_{n=1}^{N-1} |\phi(f^n x) - \phi(f^n y)|^{\lambda'} \right)^{1/\lambda'}. \]

This implies that
\[ |I| \leq C^{\frac{1}{\lambda}} \left( \frac{1}{N} \sum_{n \in [1,N] \setminus E} |\phi(f^n x) - \phi(f^n y)|^{\lambda'} + \frac{1}{N} \sum_{n \in E} |\phi(f^n x) - \phi(f^n y)|^{\lambda'} \right)^{1/\lambda'} \]
\[ \leq C^{\frac{1}{\lambda}} \left( \frac{\epsilon^{\lambda'}}{2^{1+\lambda'} C^\frac{\lambda}{\lambda'}} + (2\|\phi\|_\infty^{\lambda'})^{\lambda'} \right)^{\frac{1}{\lambda'}}. \]

Here we can take \( \delta \) small enough such that
\[ (2\|\phi\|_\infty^{\lambda'})^{\lambda'} \delta < \frac{\epsilon^{\lambda'}}{2^{1+\lambda'} C^\frac{\lambda}{\lambda'}}. \]

So we have that
\[ |S_N\phi(x)| \leq |I| + |II| \leq \epsilon \]
for all \( N > N_0 \). We proved (1) under the assumption.

Now we prove the assumption that for any \( z \in K \), (1) holds. Since \( f : K \to K \) is a QDS (d) flow, we have that
\[ C(K) = \text{span}_C(G_{d+1}). \]

Similar to the proof of Theorem 1 we need only to prove (1) for any \( g \in G_{d+1} \) and any \( z \in K \). Since \( \Lambda^{d+1} g = 1 \), For any \( n \geq d + 1 \), we have
\[ \Phi^n g(z) = g \circ f^n(z) = \prod_{j=0}^{n} (\Lambda^j g(z))^{\binom{n}{j}} = \prod_{j=0}^{d} (\Lambda^j g(z))^{\binom{n}{j}}. \]
Since \( \Lambda^j g \in G_{d+1-j} \) for \( 0 \leq j \leq d \), we have \( \Lambda^j g(z) = e^{2\pi i \theta_j} \) for some real number \( \theta_j \in [0, 2\pi] \). This implies that
\[
\Phi^n g(z) = e^{2\pi i \sum_{j=0}^d \theta_j \binom{n}{j}} = e^{2\pi i P(n)}
\]
where
\[
P(n) = \sum_{j=0}^d a_j n^j
\]
is a real-coefficient polynomial of \( n \) of degree \( \leq d \). Now
\[
S_N g(z) = \frac{1}{N} \sum_{n=1}^N c_n g \circ f^n = \frac{1}{N} \sum_{n=1}^d c_n \Phi^n g(z) + \frac{1}{N} \sum_{n=d+1}^N c_n \Phi^n g(z)
\]
\[
= \frac{1}{N} \sum_{n=1}^d c_n \Phi^n g(z) + \frac{1}{N} \sum_{n=d+1}^N c_n e^{2\pi i P(n)}.
\]
Since \( c \) is an oscillating sequence of order \( d \),
\[
\lim_{n \to \infty} \frac{1}{N} \sum_{n=d+1}^N c_n e^{2\pi i P(n)} = 0.
\]
Therefore, we have that
\[
\lim_{n \to \infty} S_N g(z) = 0.
\]
We completed the proof of the assumption and thus the proof of the theorem.

\[\square\]

**Remark 4.** From Theorem 1 and Theorem 2 and Theorem 3, we can see that in general the flow in the form (8) or (7) will not be a MQDS(d) flow if \( k \geq 2 \).

## 4 Sarnak’s conjecture

One of the consequences of this paper is the confirmation of Conjecture 1 for all polynomial skew products (for \( d = 2 \), it is a consequence of the result in [19]. Still, we have a completely different approach, and hopefully, this proof will lead to a new proof of the result of Wang in [19] and Conjecture 2 for \( d = 2 \). For \( d > 2 \), it is a new result) and for all MMA and MQDS (d) flows in this paper.
5 Chowla Sequences.

It has been a question that if it is enough to study Sarnak’s conjecture just through the study of orders of oscillation sequences? It turns out it is not the case. The paper [1] contains an example of an oscillating sequence of order $d$ for all $d \geq 2$, which is not linear disjoint from a flow with zero topological entropy. So we would also like to study Chowla sequences defined as follows.

**Definition 6 (Chowla Sequence).** We call a sequence $c$ of complex numbers a Chowla sequence if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} c_{n+l_1}^{k_1} c_{n+l_2}^{k_2} \cdots c_{n+l_r}^{k_r} = 0$$

(12)

for all $r \geq 1$, all choices of integers $0 \leq l_1 < l_2 < \cdots < l_r$ and $k_i \in \mathbb{N}$, $i = 1, 2, \cdots, r$, such that not all

$$c_{n+l_i}^{k_i} = |c_{n+l_i}|, \quad 1 \leq i \leq r$$

with the control condition (3).

Chowla’s conjecture [5] states that the M"obius sequence $u$ is a Chowla sequence. Due to the recent work of Sanark [14, 15], Veech [18] and Tao [16, 17] (see also [2]), Chowla’s conjecture is equivalent to Sarnak’s conjecture. In this section, we give some examples of Chowla sequences using our work in [4].

Take a non empty interval $I$ in the real line $\mathbb{R}$, which can be closed, open or semi-open. Let $C^k_+(I)$ be the space of all positive real valued $k$-times continuously differentiable functions on an interval $I$, whose $i$-th derivative is non-negative for $i \leq k$. Then it is closed under addition and multiplication, that is, if $f, g \in C^k_+(I)$, then $f + g, fg \in C^k_+(I)$. In what follows, we often use this closure property of $C^k_+(I)$.

Let $g$ be a function in $C^2_+((0, \infty))$ and $\alpha \neq 0$ and $\beta > 1$ be real numbers. Consider the sequence of complex number $c = (c_n)_{n \in \mathbb{N}}$ where

$$c_n = \exp\left(2\pi i (\alpha \beta^n g(\beta))\right) = e^{2\pi i (\alpha \beta^n g(\beta))}.$$ 

Then we have that

$$c_{n+l_1}^{k_1} c_{n+l_2}^{k_2} \cdots c_{n+l_r}^{k_r} = \exp\left(2\pi i \left(\sum_{i=1}^{r} k_i (\alpha \beta^{n+l_i} g(\beta))\right)\right)$$
This says that in this case, (12) relates with the uniformly distributed modulo 1 (abbreviated u. d. mod 1) of the sequence

$$\left( \sum_{i=1}^{r} k_i(\alpha \beta^n + l_i g(\beta)) \right)_{n \in \mathbb{N}}$$
on the unit interval $[0, 1]$ as follows.

For a real number $x$, let $\{x\} = x \mod 1$ be the fractional part of $x$.

**Definition 7 (Uniform Distribution).** We say a sequence $x = (x_n)_{n \in \mathbb{N}}$ of real numbers is u. d. mod 1 if for any $0 \leq a < b \leq 1$, we have

$$\lim_{N \to \infty} \frac{\#(\{n \in [1, N] \mid \{x_n\} \in [a, b]\})}{N} = b - a.$$  

**Theorem B (The Weyl Criterion).** The sequence $x = (x_n)_{n \in \mathbb{N}}$ is u. d. mod 1 if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \text{for all integers } h \neq 0.$$ 

The following theorem is essentially [4, Theorem 1] (that is, take $l = 0$ in the proof in [4]). For the reader’s convenience, I will also provide detailed proof in this paper.

**Theorem 4.** Let us take $g \in C^2_+((1, \infty))$. Then, for a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers $\alpha$) and for all $r \geq 1$, all choices of integers $0 \leq l_1 < l_2 < \cdots < l_r$ and $k_i \in \mathbb{N}$, $i = 1, 2, \cdots, r$, sequences

$$\left( \sum_{i=1}^{r} k_i(\alpha \beta^n + l_i g(\beta)) \right)_{n \in \mathbb{N}}$$

are u. d. mod 1.

The following corollary is a consequence of this theorem because of the Weyl criterion.

**Corollary 3.** Let us take $g \in C^2_+((1, \infty))$. Then, for a fixed real number $\alpha \neq 0$ and almost all real numbers $\beta > 1$ (alternatively, for a fixed real number $\beta > 1$ and almost all real numbers $\alpha$), sequences

$$c = \left( \exp(2\pi i (\alpha \beta^n g(\beta))) \right)_{n \in \mathbb{N}}$$

are Chowla sequences.
Remark 5. Again it is an interesting problem to find a concrete number of 
\( \beta > 1 \) such that 
\[
c = (e^{2\pi i \beta^n})_{n \in \mathbb{N}}
\]
is a Chowla sequence.

**Theorem C** (Koksma’s Theorem). Let \((y_n(x))_{n \in \mathbb{N}}\) be a sequence of real valued \(C^1\) functions defined on an interval \([a, b]\). Suppose \(y'_n(x) - y'_m(x)\) is monotone on \([a, b]\) for any two integers \(m \neq n\) and suppose 
\[
\inf_{m \neq n, x \in [a, b]} |y'_m(x) - y'_n(x)| > 0.
\]
Then for almost all \(x \in [a, b]\), the sequence \(y = (y_n(x))_{n \in \mathbb{N}}\) is u. d. mod 1.

The reader who is interested in Theorem B and Theorem C can find proofs in [12].

**Proof of Theorem 4.** Given a choice of \(0 \leq l_1 < l_2 < \cdots < l_r\) and \(k_i \in \mathbb{N}, i = 1, 2, \cdots, r\), consider the sequence
\[
\sum_{i=1}^{r} k_i \beta^{l_i} g(\beta) = \beta^n g(\beta) \sum_{i=1}^{r} k_i \beta^{l_i} = G(\beta) \beta^n
\]
where \(G(\beta) = g(\beta) \sum_{i=1}^{r} k_i \beta^{l_i} \in C^2_+((0, \infty))\) since both \(g(\beta)\) and \(\sum_{i=1}^{r} k_i \beta^{l_i}\) are in \(C^2_+((0, \infty))\). Define a function
\[
y_n(x) = G(x)x^n, \quad x \in (1, \infty).
\]

For \(n > m\), we have
\[
y'_n(x) - y'_m(x) = G(x)(nx^{n-1} - mx^{m-1}) + G'(x)(x^n - x^m)
\]
Since \(G', G'' \geq 0\) and since \(nx^{n-m} - m \geq n-m \geq 1\) for \(x > 1\), we see that every term in the expression of \(y'_n(x) - y'_m(x)\) are in \(C^1_+([a, \eta])\) for any \(1 < a < \eta < \infty\). By the closure property of \(C^1_+([a, \eta])\), we have that
\[
y'_n(x) - y'_m(x) \in C^1_+([a, \eta]) \quad \forall n > m. \quad (14)
\]
In particular, this imply that \(y'_n - y'_m\) is increasing for \(n > m\). Furthermore, we see that there is a constant \(L > 0\) such that
\[
|y'_n(x) - y'_m(x)| \geq L \quad \forall n > m \in \mathbb{N}, \forall a \leq x \leq \eta. \quad (15)
\]
(14) and (15) say that the sequence of real valued \(C^1\) functions \((y_n(x))_{n \in \mathbb{N}}\) satisfies all hypothesizes of Theorem C.
Theorem C implies that for almost all \( x \) in \( [(2^k + 1)/2^k, (2^{k-1} + 1)/2^{k-1}] \) or \([k, k+1]\) for \( k \geq 2 \), the sequence \( (\alpha y_n(x))_{n \in \mathbb{N}} \) is u. d. mod 1. Further, this implies that for almost all \( x \in (1, \infty) = \bigcup_{k=2}^{\infty} \left[ \frac{2^k + 1}{2^k}, \frac{2^{k-1} + 1}{2^{k-1}} \right] \cup \bigcup_{k=2}^{\infty}[k, k+1] \) the sequence \( (\alpha y_n(x))_{n \in \mathbb{N}} \) is u. d. mod 1.

Let

\[
A_{r,(l_1,\ldots,l_r),(k_1,\ldots,k_r)} = \{ \beta > 1 \mid \alpha \beta^n g(\beta) \sum_{i=1}^{r} k_i \beta^{l_i} \text{ is not u. d. mod 1 } \}.
\]

Then the one dimensional Lebesgue measure of \( A_{r,(l_1,\ldots,l_r),(k_1,\ldots,k_r)} \) is zero. Since the set

\[
U = \{(r, (l_1,\ldots,l_r), (k_1,\ldots,k_r)) \mid r, k_1, \ldots, k_r, l_2 < \cdots < l_r \in \mathbb{N}, l_1 \in \mathbb{N}\cup\{0\}, l_1 < l_2 \}
\]

is countable, the one dimensional Lebesgue measure of

\[
\bigcup_{(r, (l_1,\ldots,l_r), (k_1,\ldots,k_r)) \in U} A_{r,(l_1,\ldots,l_r),(k_1,\ldots,k_r)}
\]

is zero too.

For a fixed real number \( \alpha \neq 0 \) in the theorem, take a real number

\[
\beta \in (1, \infty) \setminus \bigcup_{(r, (l_1,\ldots,l_r), (k_1,\ldots,k_r)) \in U} A_{r,(l_1,\ldots,l_r),(k_1,\ldots,k_r)}.
\]

This says that the sequence

\[
\left( \sum_{i=1}^{r} \alpha \beta^{n+l_i} g(\beta) \right)_{n \in \mathbb{N}}
\]

is u. d. mod 1 for all \( (r, (l_1,\ldots,l_r), (k_1,\ldots,k_r)) \in U. \)

We proved the theorem. \( \square \)
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