On Domination Number and Distance in Graphs

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Abstract

A vertex set $S$ of a graph $G$ is a dominating set if each vertex of $G$ either belongs to $S$ or is adjacent to a vertex in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of $S$ as $S$ varies over all dominating sets of $G$. It is known that $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1)$, where $\text{diam}(G)$ denotes the diameter of $G$. Define $C_r$ as the largest constant such that $\gamma(G) \geq C_r \sum_{1 \leq i < j \leq r} d(x_i, x_j)$ for any $r$ vertices of an arbitrary connected graph $G$; then $C_2 = \frac{1}{2}$ in this view. The main result of this paper is that $C_r = \frac{1}{r(r-1)}$ for $r \geq 3$. It immediately follows that $\gamma(G) \geq \mu(G) = \frac{1}{n(n-1)} W(G)$, where $\mu(G)$ and $W(G)$ are respectively the average distance and the Wiener index of $G$ of order $n$. As an application of our main result, we prove a conjecture of DeLaViña et al. that $\gamma(G) \geq \frac{1}{2}(\text{ecc}_G(B) + 1)$, where $\text{ecc}_G(B)$ denotes the eccentricity of the boundary of an arbitrary connected graph $G$.

Key Words: domination number, distance, diameter, spanning tree

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1 Introduction

We consider finite, simple, undirected, and connected graphs $G = (V(G), E(G))$ of order $|V(G)| \geq 2$ and size $|E(G)|$. For $W \subseteq V(G)$, we denote by $(W)_G$ the subgraph of $G$ induced by $W$. For $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the closed neighborhood of $v$ is $N[v] = N_G(v) \cup \{v\}$. Further, let $N(S) = \cup_{v \in S}N(v)$ and $N[S] = \cup_{v \in S}N[v]$ for $S \subseteq V(G)$. The degree of a vertex $v \in V(G)$ is $\deg_G(v) = |N_G(v)|$. The distance between two vertices $x, y \in V(G)$ in the subgraph $H$, denoted by $d_H(x, y)$, is the length of the shortest path between $x$ and $y$ in the subgraph $H$. The diameter $\text{diam}(H)$ of a graph $H$ is $\max\{d_H(x, y) \mid x, y \in V(H)\}$.

A set $S \subseteq V(G)$ is a dominating set (resp. total dominating set) of $G$ if $N[S] = V(G)$ (resp. $N(S) = V(G)$). The domination number (resp. total domination number) of $G$, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is the minimum cardinality of $S$ as $S$ varies over all dominating sets (resp. total dominating sets) in $G$; a dominating set (resp. total dominating set) of $G$ of minimum cardinality is called a $\gamma(G)$-set (resp. $\gamma_t(G)$-set).

Both distance and (total) domination are very well-studied concepts in graph theory. For a survey of the myriad variations on the notion of domination in graphs, see [4].

It is well-known that $\gamma(G) \geq \frac{1}{3}(\text{diam}(G) + 1)$ ($\ast$): a “proof” to ($\ast$) can be found on p.56 of the authoritative reference [3]. However, the “proof” contained therein is logically flawed. We provide a counterexample to a crucial assertion in the “proof” and then present a correct proof to ($\ast$). Upon some
reflection, we see that (*) is the two parameter case of a family of inequalities existing between \( \gamma(G) \) and the distances in \( G \), in the following way: \( \gamma(G) \geq \frac{1}{2}(diam(G)+1) = \frac{1}{3r}\left(\left(\begin{array}{c} r \\ 1 \end{array}\right)diam(G) + \left(\begin{array}{c} r \\ 2 \end{array}\right)\right) \). The inequality \( \gamma(G) \geq \frac{1}{3r}\left(\sum_{1 \leq i < j \leq r} d(x_i, x_j)\right) \) naturally brings up the question: What is the largest constant \( C_r \), such that \( \gamma(G) \geq C_r \left(\sum_{1 \leq i < j \leq r} d(x_i, x_j)\right) \), for all connected graphs \( G = (V, E) \) and arbitrary vertices \( x_1, \ldots, x_r \in V \), where \( r \geq 2 \)? Taking this viewpoint, we have \( C_2 = \frac{1}{3} \) by (*).

The main result of this paper is that \( C_r = \frac{1}{r(r-1)} \) for \( r \geq 3 \). Since, for a graph \( G \) of order \( n \), \( W(G) = \sum_{1 \leq i < j \leq n} d(x_i, x_j) \) is the Wiener index of \( G \) (see [1]) and \( \mu(G) = \frac{1}{n(n-1)} W(G) \) is the average distance (per definition found in [1]), it follows that \( \gamma(G) \geq \mu(G) = \frac{1}{r(r-1)} W(G) \). As an application of our main result, we prove a conjecture in [3] by DeLaViña et al. that \( \gamma(G) \geq \frac{1}{r}ecc_G(B) + 1 \), where \( ecc_G(B) \) denotes the eccentricity of the boundary of an arbitrary connected graph \( G \) (to be defined in Section 4).

This paper is motivated by the work of Henning and Yeo in [5], where they obtained similar inequalities for total domination number \( \gamma_t \) (rather than domination number \( \gamma \)). Given the close relation between the two graph parameters, we expect the techniques used in [5] to be readily adaptable towards the results of this paper. However, in striking contrast to [5], we avoid the painstaking case-by-case, structural analysis employed there by making use of the easy and well-known Lemma 5.1: this results in a much simpler and shorter paper. Further, we are able to obtain (in domination) the exact value of \( C_r \) for every \( r \), rather than only a bound (in total domination, c.f. [5]) for \( C_r \) for all but the first few values of \( r \).

2 An Error in the proof of \( \gamma(G) \geq \frac{1}{3}(diam(G) + 1) \) in FoDiG

For readers’ convenience, we first reproduce Theorem 2.24 and its incorrect proof as it appears on p.56 of [1], the authoritative reference in the field of domination titled Fundamentals of Domination in Graphs.

**Theorem 2.1.** For any connected graph \( G \), \[ \left\lfloor \frac{diam(G)+1}{3} \right\rfloor \leq \gamma(G). \]

“Proof” (as found on p.56 of [1]). Let \( S \) be a \( \gamma \)-set of a connected graph \( G \). Consider an arbitrary path of length \( diam(G) \). This diametral path includes at most two edges from the induced subgraph \( \langle N[v]\rangle \) for each \( v \in S \). Furthermore, since \( S \) is a \( \gamma \)-set, the diametral path includes at most \( \gamma(G) - 1 \) edges joining the neighborhoods of the vertices of \( S \). Hence, \( diam(G) \leq 2\gamma(G) + \gamma(G) - 1 = 3\gamma(G) - 1 \) and the desired result follows.” \( \square \)

![Figure 1: a counter-example](image-url)
in Figure 1 notice that $S = \{u, v\}$ is a $\gamma$-set and the vertices $1, 2, 3, 4$ form a diametral path containing 3 edges joining $\langle N[u] \rangle$ with $\langle N[v] \rangle$, whereas $\gamma(G) - 1 = 1$.

### 3 Domination number and distance in graphs

The following lemma can be proved by exactly the same argument given in the proof of Lemma 2 in [2]; it was also observed on p.23 of [1].

**Lemma 3.1.** [1, 2] Let $M$ be a $\gamma(G)$-set. Then there is a spanning tree $T$ of $G$ such that $M$ is a $\gamma(T)$-set.

Now, we apply Lemma 3.1 to give a correct proof of Theorem 2.1.

**Proof of Theorem 2.1.** Given $G$, take a spanning tree $T$ of $G$ such that $\gamma(G) = \gamma(T)$. Suppose, for the sake of contradiction, $\gamma(G) < \frac{4}{3}(\text{diam}(G) + 1)$. Since $\gamma(T) = \gamma(G)$ and $\text{diam}(T) \geq \text{diam}(G)$, we have

$$\gamma(T) < \frac{1}{3}(\text{diam}(T) + 1) \tag{1}$$

Take a path $P$ of $T$ with length equal to $\text{diam}(T)$. If (1) holds, there must exist a vertex $u$ of $T$ such that $|V(P) \cap N[u]| \geq 4$. Since $P$ is a path of $T$ (a tree), this is impossible. □

**Theorem 3.2.** Given any three vertices $x_1, x_2, x_3$ of a connected graph $G$, we have

$$\gamma(G) \geq \frac{1}{6}(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)). \tag{2}$$

Further, if equality is attained in (2), then $d_G(u, v) \equiv 2 \pmod{3}$ for any pair $u, v \in \{x_1, x_2, x_3\}$.

**Proof.** By Lemma 3.1 there exists a spanning tree $T$ of $G$ with $\gamma(T) = \gamma(G)$. Since $d_T(u, v) \geq d_G(u, v)$ for any two vertices $u, v \in V(T) = V(G)$, it suffices to prove (2) on $T$. If $x_1, x_2$, and $x_3$ all lie on one geodesic, then the inequality (2) obviously holds by Theorem 2.1. Thus, let $d_T(x_1, y) = a$, $d_T(x_2, y) = b$, and $d_T(x_3, y) = c$, with $0 \notin \{a, b, c\}$, as shown in Figure 2. Then, the inequality (2) on $T$ becomes

$$\gamma(T) \geq \frac{1}{6}((a + b) + (a + c) + (b + c)) = \frac{1}{3}(a + b + c). \tag{3}$$

Let $y'$ be the vertex lying on the $x_2$-$y$ path and adjacent to $y$. Let $P^1$ and $P^2$ denote the $x_1$-$x_3$ path and the $x_2$-$y'$ path, respectively. If there exists a $\gamma(T)$-set $M$ not containing $y$, then $M$ must contain a neighbor $z$ of $y$. Suppose, WLOG, $z \neq y'$. Then, inequality (3) follows immediately from applying Theorem 2.1 to $P^1$ and $P^2$. If $y$ belongs to every $\gamma(T)$-set $M$, then $\gamma(T) \geq 1 + \frac{4}{3}(a - 1) + \frac{1}{3}(b - 1) + \frac{1}{3}(c - 1) = \frac{1}{3}(a + b + c)$, and (3) again follows.

![Figure 2: $r = 3$ case](image-url)
Next, suppose equality is attained in \( \mathcal{E} \). Again, let \( T \) be a spanning tree with \( \gamma(T) = \gamma(G) \). Since \( d_G(x_i, x_j) \leq d_T(x_i, x_j) \) and \( \gamma(T) \geq \frac{1}{6}(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3)) \) holds, we have \( \gamma(G) = \frac{1}{6}(d_G(x_1, x_2) + d_G(x_1, x_3) + d_G(x_2, x_3)) \). Thus, we deduce that \( \gamma(T) = \frac{1}{6}(d_T(x_1, x_2) + d_T(x_1, x_3) + d_T(x_2, x_3)) \) and \( d_G(x_1, x_2) = d_T(x_1, x_2) \) for each pair \( (x_i, x_j) \). With \( a, b, c \) defined as above, the present assumption implies \( \gamma(T) = \frac{1}{3}(a + b + c) \). Observe, in light of Theorem 3.2, that the equality \( \gamma(T) = \frac{1}{3}(a + b + c) \) is only possible if the following “optimal domination” of \( T \) occurs: there is a \( \gamma(T) \)-set \( M \) which contains \( y \), a degree-three vertex in \( \langle V(P_1) \cup V(P_2) \rangle \); which dominates four or more vertices in \( T \); every other vertex of \( M \) dominates three or more vertices in \( T \); no vertex of \( T \) is dominated by more than one vertex of \( M \). (Note that Figure 2 only shows \( \langle V(P_1) \cup V(P_2) \rangle \), which may be a strict subgraph of \( T \).) This “optimal domination” condition clearly implies that each member of \( \{a, b, c\} \) must equal 1 (mod 3), which yields our second claim. \( \square \)

Next, we determine the largest \( C_r \) for \( r \geq 3 \) with the method deployed in \( [5] \). However, rather than just getting a bound on \( C_r \) in the case of total domination there, we obtain the exact value of \( C_r \) for every \( r \).

**Theorem 3.3.** For \( r \geq 3 \), \( C_r = \frac{1}{r^2 - r} \).

**Proof.** First, we prove \( C_r \leq \frac{1}{r^2 - r} \). Let \( G = K_1, r \) be a star with \( r \) leaves labeled \( x_1, \ldots, x_r \). Then \( \gamma(G) = 1 \) and
\[
\sum_{1 \leq i < j \leq r} d(x_i, x_j) = \binom{r}{2} \cdot 2 = r(r - 1).
\]
So, \( C_r \leq \frac{1}{r(r - 1)} \).

Next, we show that \( C_r \geq \frac{1}{r^2 - r} \). Notice that \( C_3 = \frac{1}{3(3-1)} = \frac{1}{6} \) is given by Theorem 3.2. Thus, let \( x_1, x_2, \ldots, x_r \) be any arbitrary \( r \geq 4 \) vertices of \( G \). Since \( \gamma(G) \geq \frac{1}{6}(d_G(x_i, x_j) + d_G(x_i, x_k) + d_G(x_j, x_k)) \) holds for any triplet \( \{x_i, x_j, x_k\} \subseteq \{x_1, x_2, \ldots, x_r\} \), we have
\[
\binom{r}{3} \gamma(G) \geq \sum_{1 \leq i < j < k \leq r} \frac{1}{6}(d_G(x_i, x_j) + d_G(x_i, x_k) + d_G(x_j, x_k)) = \frac{r - 2}{6} \sum_{1 \leq i < j \leq r} d(x_i, x_j);
\]
note that the last equality comes from the fact that there are \( r - 2 \) triplets containing any given pair of vertices. Thus, \( C_r \geq \frac{1}{r(r - 1)} \) as well. \( \square \)

### 4 Applying Theorem 3.2 to a Conjecture of DeLaViña et al.

We need a few more definitions. The **eccentricity** of a vertex \( v \) in \( G \), denoted by \( ecc_G(v) \), is \( \max \{d_G(v, x) \mid x \in V(G)\} \). The **boundary** of \( G \) is defined as the set \( B(G) = \{v \in V(G) \mid ecc_G(v) = diam(G)\} \); we denote it simply as \( B \) hereafter. The distance between a vertex \( v \in V(G) \) and a set \( S \subseteq V(G) \) is defined as \( d_G(v, S) = \min \{d_G(v, x) \mid x \in S\} \). Further, the eccentricity of \( S \subseteq V(G) \) is defined as \( ecc_G(S) = \max \{d_G(x, S) \mid x \in V(G)\} \).

In \([3]\), DeLaViña et al. proved, for a tree \( G \), that \( \gamma(G) \geq \frac{1}{2}(ecc_G(B) + 1) \). They further conjectured that the inequality holds for any connected graph \( G \). As an application of Theorem 3.2, we prove this conjecture. Our proof follows the arguments given by Henning and Yeo in \([5]\) proving the analogous Graffiti.pc conjecture \( \gamma_t(G) \geq \frac{1}{2}(ecc_G(B) + 1) \).

**Theorem 4.1.** Let \( G \) be a connected graph. Then \( \gamma(G) \geq \frac{1}{2}(ecc_G(B) + 1) \).
Proof. If $B = V(G)$, then $ecc_G(B) = 0$ and the desired inequality obviously holds. So, suppose $B \neq V(G)$; this implies that $|V(G)| \geq 3$ and $|B| \geq 2$. Pick vertices $x$ and $y$ with $d(x, y) = diam(G)$; then, $x, y \in B$. Let $ecc_G(B) = R$. Pick $z \in V(G) \setminus B$ such that $d(z, B) = R$. We have $d(x, z) \geq R$, $d(y, z) \geq R$ and $d(x, y) = diam(G) \geq R + 1$. Hence, we have $d(x, y) + d(x, z) + d(y, z) \geq 3R + 1$ (♠). If equality holds in (♠), then $R = d(x, z) = d(y, z) = d(x, y) - 1$, and we can Not have both $d(x, z)$ and $d(x, y)$ be congruent to 2 mod 3. In this case, by Theorem 3.2 we have that $\gamma(G) > \frac{1}{2}(d(x, y) + d(x, z) + d(y, z)) = \frac{1}{2}(3R + 1) = \frac{1}{2}R + \frac{1}{2}$, which implies $\gamma(G) \geq \frac{1}{2}R + \frac{1}{2}$. On the other hand, if the inequality (♠) is strict, again by Theorem 3.2 we have that $\gamma(G) \geq \frac{1}{2}(d(x, y) + d(x, z) + d(y, z)) > \frac{1}{2}(3R + 1) = \frac{1}{2}R + \frac{1}{2}$, which again implies $\gamma(G) \geq \frac{1}{2}R + \frac{1}{2}$. \qed

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