ON THE KOLMOGOROV ENTROPY OF THE WEAK GLOBAL ATTRACTOR OF 3D NAVIER-STOKES EQUATIONS: I

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Abstract. One particular metric that generates the weak topology on the weak global attractor $A_w$ of three dimensional incompressible Navier-Stokes equations is introduced and used to obtain an upper bound for the Kolmogorov entropy of $A_w$. This bound is expressed explicitly in terms of the physical parameters of the fluid flow.

1. Introduction. In the study of the incompressible Navier-Stokes equations (NSEs), a frequently discussed problem is the understanding of the dynamics and the asymptotic behaviors of the solutions ([3, 5, 9, 10, 17, 18]). The asymptotic behaviors are often considered to be related with the so-called global attractor, which can usually be characterized as the compact set attracting all bounded sets with respect to the appropriate topology of some (phase) space $H$ (see [17, 18]).

In the two-dimensional case, the strong global attractor $A$ in the strong topology of the phase space can be easily defined due to the well-posedness of the incompressible NSEs. However, in three-dimensional case, one has to resort to the weak topology and define a weaker version of the global attractor, called weak global attractor, denoted by $A_w$ (see [11]). Further discussion of interesting topological properties of $A_w$ can be found in [8].

One significant difference between $A$ and $A_w$ is the complexities of these two geometric objects. It can be shown that $A$ has finite Hausdorff and fractal dimensions (see [2, 12]). However, to the best knowledge of the authors, no estimates are known regarding either the fractal dimension or the Hausdorff dimension of $A_w$. One can even show that the interior of $A_w$ is empty (see [1, 7]). It remains an open question to find a way to quantify the complexities of $A_w$. The concept of $\epsilon$-entropy was introduced by Kolmogorov [13] to measure the complexities for totally bounded sets in a metric space. Kolmogorov $\epsilon$-entropy can be viewed as a quantification of

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compactness property. It was defined to be the logarithm to the base 2 of the smallest number of elements in an \( \epsilon \)-covering for the set. Kolmogorov formulated such a definition in order to investigate the question of whether functions with \( n \geq 3 \) variables can be represented as compositions of functions with \( r < n \) variables. Kolmogorov successfully proved that each continuous function of several variables can be represented by sums and superpositions of continuous functions of one variable ([14]). The concept of Kolmogorov \( \epsilon \)-entropy is widely applied in branches of both pure and applied mathematics ([4, 14, 19, 20]).

The motivation for us to consider the Kolmogorov \( \epsilon \)-entropy for \( A_w \) comes from the fact proved in [8] that the weak topology on \( A_w \) is metrizable. Moreover, the weak topology on \( A_w \) can be generated by several different metric functions. The natural one will be the metric \( d_w \) (see (8) for its definition) induced by the norm of \( V' \), the dual of the space \( V \) defined in the next section. However, in this paper, we use another metric function, denoted by \( d_w \) (see (7) for its definition), which involves the exponentials of \( A_{1/2} \), where \( A \) denotes the Stokes operator. With the help of \( d_w \), we obtain an estimate on the upper bound of the Kolmogorov \( \epsilon \)-entropy of \( A_w \), denoted by \( H_\epsilon(A_w) \). This estimate is explicitly expressed in terms of the nondimensional number \( G_* \) (see Theorem 5.4). A consequence of this estimate which agrees with a general result given in [16] is that the functional dimension \( df(A_w) \) of \( A_w \) is finite and bounded above by 1. We remark that, in two-dimensional case, since \( A \) has finite fractal dimension, a quick application of Kolmogorov \( \epsilon \)-entropy and functional dimension on \( A_w \) is that \( H_\epsilon(A_w) \leq O(\ln \epsilon^{-1}) \) and \( df(A) \leq 1 \). So, using the metric function \( d_w \) gives us the same upper estimate on \( df(A_w) \) and \( df(A_w) \).

Following the same techniques, if \( d_n \) is used, we can only get \( df(A_w) \leq \infty \), which does not provide good enough information on the \( \epsilon \)-entropy of \( A_w \). So, one has to resort to different techniques. This will be done in ([6]).

2. The Navier-Stokes equations and Leray-Hopf weak solutions. The three-dimensional incompressible Navier-Stokes equations (NSEs) in Eulerian formulation are written as

\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0
\]

where, the variable \( u = (u_1, u_2, u_3) \) denotes the velocity vector field, \( f \) assumed to be time-independent represents the mass density of volume force applied to the fluid, the parameter \( \nu > 0 \) is the kinematic viscosity, and \( p \) is the kinematic pressure. The space variable is denoted by \( x = (x_1, x_2, x_3) \) and the time variable by \( t \).

We assume that the flow is periodic with period \( L \) in each spatial direction \( x_i, \ i = 1, 2, 3 \), and that the averages of the flow velocity and of the force \( f \) over \( \Omega = (0, L)^3 \) vanish,

\[
\int_\Omega u(x, t)dx = 0, \quad \int_\Omega f(x)dx = 0.
\]

Let us introduce the space \( H \) (respectively, \( V \)) as the subspace of \( L^2(\Omega)^3 \) (respectively, \( H^1(\Omega)^3 \)) which is the closure of the set of all \( \mathbb{R}^3 \)-valued trigonometric polynomials \( v \) such that

\[
\nabla \cdot v = 0 \quad \text{and} \quad \int_\Omega v(x)dx = 0.
\]
Denote the inner products in $H$ and $V$ respectively by
\[(u, v) = \int_{\Omega} u(x) \cdot v(x) dx, \quad ((u, v)) = \int_{\Omega} \sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i} dx,\]
and the associated norms by $|u| = (u, u)^{1/2}$, $||u|| = ((u, u))^{1/2}$. The phase space $H$ can be identified with its dual space $H'$, then it is easy to see that $V \subset H \subset V'$, with the injections being continuous and each space dense in the following one.

Let $\mathcal{P}: L^2(\Omega)^3 \to L^2(\Omega)^3$ be the orthogonal projection (called the Helmholtz-Leray projection) with range $H$, and define the Stokes operator as $A = -\mathcal{P} \Delta$ ($=-\Delta$, under periodic boundary conditions), which is positive, self-adjoint with a compact inverse. As a consequence, the space $H$ has an orthonormal basis \(\{w_j\}_{j=1}^\infty\) consisting of eigenfunctions of $A$; namely, $Aw_j = \lambda_j w_j$, with $0 < \lambda_1 = \kappa_0^2 := (2\pi/L)^2 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ (see [3]).

For any $\sigma \in \mathbb{R}$, the operator $A^\sigma$ is defined as
\[A^\sigma v = \sum_{j=1}^{\infty} \lambda_j^\sigma (v, w_j) w_j,\]
for $v$ in the domain of $A^\sigma$, denoted by
\[\mathcal{D}(A^\sigma) = \{v \in H : \sum_{j=1}^{\infty} \lambda_j^{2\sigma} (v, w_j)^2 < \infty\}.\]

We also recall the orthogonal projection $P_K : H \to \text{span} \{\omega_j : 1 \leq j \leq K\}$ which maps $v = \sum_{j=1}^{\infty} (v, w_j) w_j \in H$ to $P_K v = \sum_{j=1}^{K} (v, w_j) w_j$, where $K \geq 1$ is an integer.

The NSEs can be written as a differential equation (which will be referred to as the NSE) in the real Hilbert space $H$ in the following form
\[\frac{du}{dt} + \nu Au + B(u, u) = g, \quad u \in H, \tag{1}\]
where the bilinear operator $B$ and the driving force $g$ are defined as
\[B(u, v) = \mathcal{P}(u \cdot \nabla v) \text{ and } g = \mathcal{P} f.\]

The following inequalities will be needed in this paper
\[k_0 |w| \leq |A^{1/2} w|, \quad \text{for } w \in V, \tag{2}\]
\[||w||_{\infty} \leq c_A |A^{1/2} w|^{1/2} |Aw|^{1/2}, \quad \text{for } w \in \mathcal{D}(A), \tag{3}\]
known, respectively, as Poincaré and Agmon inequalities, with $c_A$ being a non-dimensional constant.

The definition of weak solutions used in this paper is the one given in [8].

**Definition 2.1.** A (Leray-Hopf) weak solution on a time interval $J \subset \mathbb{R}$ is defined as a function $u = u(t)$ on $J$ with values in $H$ satisfying the following properties:

i. $u \in L^\infty_{loc}(J; H) \cap L^2_{loc}(J; V)$;
ii. $\partial_t u \in L^{4/3}_{loc}(J; V')$;
iii. $u \in C(J; H_w)$, which means $u$ is weakly continuous in $H$ or equivalently for every $v \in H$, the function $t \mapsto (u(t), v)$ is continuous from $J$ into $\mathbb{R}$, where $H_w$ is the space $H$ endowed with its weak topology;
iv. $u$ satisfies (1) in the distribution sense on $J$ with values in $V'$. 
v. For almost all \( t' \) in \( J \), \( u \) satisfies the following energy inequality:

\[
\frac{1}{2} |u(t)|^2 + \nu \int_{t'}^t |A^{\frac{1}{2}} u(s)|^2 \, ds \leq \frac{1}{2} |u(t')|^2 + \nu \int_{t'}^t (g, u(s)) \, ds
\]

for all \( t \) in \( J \) with \( t > t' \). The allowed times \( t' \) are characterized as the points where \( u \) are continuous from the right in \( H \) and the set of all \( t' \) is of total Lebesgue measure and denoted by \( J'(u) \).

vi. If \( J \) is closed and bounded on the left, with its left end point denoted by \( t_0 \), then the solution is continuous in \( H \) at \( t_0 \) from the right, i.e., \( t_0 \in J'(u) \).

From now on, a weak solution will always mean a Leray-Hopf weak solution.

An important nondimensional parameter associated with the strength of the driving force \( g \) is the Grashof number \([8]\):

\[
G = \frac{|g|}{\nu^2 \kappa_0^{1/2}}.
\]

A related nondimensional parameter that will be used in our paper is

\[
G_* = \frac{|A^{-1/2}g|}{\nu^2 \kappa_0^{1/2}}.
\]

By Poincaré inequality (2), one has

\[
G_* = \frac{|A^{-1/2}g|}{\nu^2 \kappa_0^{1/2}} \leq \frac{\kappa_0^{-1} |g|}{\nu^2 \kappa_0^{1/2}} = G,
\]

where the equality occurs only if \( g \) is an eigenvector of \( A \).

3. Weak global attractor and Kolmogorov \( \epsilon \)-entropy. The weak global attractor \( \mathcal{A}_w \), introduced in [11], is defined as the set of all points in \( H \) each of which belongs to a global weak solution uniformly bounded in \( H \) on \( \mathbb{R} \), i.e.,

\[
\mathcal{A}_w := \{ u_0 \in H : \exists \text{ a global weak solution } u \text{ s.t., } \sup_{t \in \mathbb{R}} |u(t)| < \infty \text{ and } u(0) = u_0 \}.
\]

It is shown in [8] that

\[
\mathcal{A}_w \subset \{ u \in H : |u(t)| \leq G_* \nu \kappa_0^{-1/2}, \ \forall t \in \mathbb{R} \},
\]

and \( \mathcal{A}_w \) is a totally bounded set in \( H_w \). Therefore the weak topology of \( H \) is metrizable on \( \mathcal{A}_w \). Recall that a set \( F \) in the metric space \( (X, d) \) is totally bounded if, for any \( \eta > 0 \), there exists finitely many open balls of radius \( \eta \) whose union covers \( F \). It is well known that all compact sets are totally bounded and that a metric space is compact if and only if it is complete and totally bounded. We choose the metric function that generates the weak topology on \( \mathcal{A}_w \) as

\[
d_w(u, v) := \nu^{-1} \kappa_0^{1/2} |e^{-e^{-\kappa_0^{-1/2}} |A^{1/2}(u - v)|}|
\]

for \( u, v \in \mathcal{A}_w \).

Two other metric functions that will be used are

\[
d_n(u, v) = \nu^{-1} \kappa_0^{3/2} |A^{-1/2}(u - v)|,
\]

and

\[
d_s(u, v) = \nu^{-1} \kappa_0^{1/2} |u - v|.
\]

Note that the metric functions \( d_s, d_n, \) and \( d_w \) do not have physical dimensions.
For a metric space \((X,d)\), let \(B_d(x_0,\rho)\) denote the ball with radius \(\rho\) centered at \(x_0\),

\[
B_d(x_0,\rho) := \{ x \in X : d(x,x_0) < \rho \}.
\]

The following definitions were introduced by Kolmogorov in [15].

**Definition 3.1.** Suppose \(F\) is a totally bounded, non-empty set in a metric space \((X,d)\), and let \(\epsilon > 0\) be any real number.

(i). The system \(\gamma\) of sets \(U \subset X\) is said to be an \(\epsilon\)-covering of the set \(F\), if the diameter of any \(U \in \gamma\), \(\sup_{u_0,u_1 \in U} d(u_0,u_1)\), is no greater than \(2\epsilon\) and

\[
F \subset \bigcup_{U \in \gamma} U.
\]

(ii). The Kolmogorov \(\epsilon\)-entropy of the set \(F\), denoted by \(\mathcal{H}_\epsilon(F)\), is defined as

\[
\mathcal{H}_\epsilon(F) := \ln \mathcal{N}_\epsilon(F),
\]

and \(\mathcal{N}_\epsilon(F) = \min\{\text{card}(\gamma) : \gamma\text{ is an }\epsilon\text{-covering of }F\}\), where \(\text{card}(\gamma)\) is the cardinal of the system \(\gamma\).

(iii). The functional dimension of the set \(F\) is defined as

\[
df(F) := \lim_{\epsilon \to 0^+} \frac{\ln \mathcal{H}_\epsilon(A)}{\ln \ln 1/\epsilon}
\]

**Remark 1.** The above definition is not exactly the same as the one in [15], where Kolmogorov used the logarithm to the base 2 to define the \(\epsilon\)-entropy and functional dimension of a set \(F\). For the simplicity of the notations, we instead use natural logarithm.

4. **Two lemmas.** To estimate the Kolmogorov \(\epsilon\)-entropy \(\mathcal{H}_\epsilon(A_w)\), the next two lemmas will be useful.

**Lemma 4.1.** Let \(u(\cdot)\) be a solution of NSE in \(A_w\). Then for any \(t_0 \in \mathbb{R}\),

\[
\nu \int_{t_0}^t |A^{\frac{1}{2}}u(\tau)|^2 d\tau \leq G_s^2 \nu^2 \kappa_0^{-1} (1 + \nu \kappa_0^2 (t - t_0)), \quad t \geq t_0,
\]

where \(G_s\) is defined in (5).

In particular, for any \(m > 0\), there exists \(t_1 \in [t_0, t_0 + \frac{1}{m \nu \kappa_0}]\) such that \(u(t_1) \in \mathcal{D}(A^{\frac{1}{2}})\) and

\[
|A^{\frac{1}{2}}u(t_1)| \leq G_s \nu \kappa_0^{1/2} \sqrt{1 + m}.
\]

**Proof.** Applying the energy inequality (4) and invoking (6), we obtain that

\[
|u(t)|^2 + 2\nu \int_{t_0}^t |A^{\frac{1}{2}}u(\tau)|^2 d\tau \leq |u(t_0)|^2 + 2 \int_{t_0}^t (g, u(\tau)) d\tau
\]

\[
\leq G_s^2 \nu^2 \kappa_0^{-1} + 2 \frac{|A^{-\frac{1}{2}}g|}{\nu^{\frac{1}{2}}} \int_{t_0}^t \nu^{\frac{1}{2}} |A^{\frac{1}{2}}u(\tau)| d\tau
\]

\[
\leq G_s^2 \nu^2 \kappa_0^{-1} + 2 \frac{|A^{-\frac{1}{2}}g|}{\nu^{\frac{1}{2}}} (\int_{t_0}^t \nu^{\frac{1}{2}} |A^{\frac{1}{2}}u(\tau)|^2 d\tau)^{\frac{1}{2}}
\]

\[
\leq G_s^2 \nu^2 \kappa_0^{-1} + \frac{|A^{-\frac{1}{2}}g|^2}{\nu} (t - t_0) + \int_{t_0}^t \nu^{\frac{1}{2}} |A^{\frac{1}{2}}u(\tau)|^2 d\tau,
\]

for any \(t \geq t_0\).
It follows that
\[ \nu \int_{t_0}^{t} |A^\frac{1}{2} u(\tau)|^2 d\tau \leq G^*_\nu^2 \kappa_0^{-1}(1 + \nu \kappa_0^2 (t - t_0)), \quad \forall t \geq t_0. \] (11)

Choosing \( t = t_0 + \frac{1}{m \nu \kappa_0^2} \), we obtain from (11) that
\[ m \nu \kappa_0^2 \int_{t_0}^{t_0 + \frac{1}{m \nu \kappa_0^2}} |A^\frac{1}{2} u(\tau)|^2 d\tau \leq G^*_\nu^2 \kappa_0^- (1 + m). \]

Therefore, there exists at least one point \( t_1 \in [t_0, t_0 + 1/m \nu \kappa_0^2] \) where the inequality (10) is satisfied.

For any \( \delta > 0 \), let us introduce
\[ C_\delta = \{ u \in \mathcal{D}(A^{1/2}) : |A^{1/2} u| \leq G^*_\nu |1 + \frac{1}{\delta}\} , \] (12)
and
\[ F_\delta = C_\delta \cap \mathcal{A}_w. \]

Using Lemma 4.1, we can get a property about \( F_\delta \).

**Lemma 4.2.** For any \( u_0 \in \mathcal{A}_w \) and \( \delta > 0 \), there exists \( u_1 \in F_\delta \) satisfying
\[ d_w(u_0, u_1) \leq r_\delta, \]
where
\[ r_\delta := e^{-e(\epsilon A G^*_\nu^2 + 2G_\nu^*) \delta}. \] (13)

**Proof.** Let \( u(\cdot) \in \mathcal{A}_w \) be the weak solution of NSE with initial value \( u(0) = u_0 \), then (6) gives that \( |u(t)| \leq G^*_\nu \kappa_0^{-1/2} \), for all \( t \geq 0 \). From NSE, we see that
\[ \left| \frac{d}{dt} e^{-e^{-e_0^{-1} A^{1/2}}} g \right| = |e^{-e^{-e_0^{-1} A^{1/2}}} g - \nu A e^{-e^{-e_0^{-1} A^{1/2}}} u - e^{-e^{-e_0^{-1} A^{1/2}}} B(u, u)| \leq |e^{-e^{-e_0^{-1} A^{1/2}}} g| + \nu |A e^{-e^{-e_0^{-1} A^{1/2}}} u| + |e^{-e^{-e_0^{-1} A^{1/2}}} B(u, u)|. \]

For the first term of the right hand side, we have
\[ |e^{-e^{-e_0^{-1} A^{1/2}}} g| = |A^{1/2} e^{-e^{-e_0^{-1} A^{1/2}}} A^{-1/2} g| \leq \kappa_0 \sup_{K \geq 1} (|K e^{-e^K}|) |A^{-1/2} g| = \kappa_0 e^{-e} |A^{-1/2} g| = e^{-e} G_\nu^* \nu^2 \kappa_0^3/2, \]
the second term is estimated by
\[ \nu |A e^{-e^{-e_0^{-1} A^{1/2}}} u| \leq \nu \kappa_0^2 \sup_{K \geq 1} (|K^2 e^{-e^K}|) |u| = \nu \kappa_0^2 e^{-e} |u| \leq e^{-e} G_\nu^* \nu^2 \kappa_0^3/2. \]
and, the third term can be estimated by using (3); indeed, the inequality
\[ |(e^{-\epsilon_0^{-1}A_{1/2}^j}B(u, u), w)| = |(B(u, e^{-\epsilon_0^{-1}A_{1/2}^j}w), u)| \]
\[ \leq \left( c_A |Ae^{-\epsilon_0^{-1}A_{1/2}^j}w|^{1/2} |A^{3/2}e^{-\epsilon_0^{-1}A_{1/2}^j}w|^{1/2} \right) |u|^2 \]
\[ \leq c_A e^{-\epsilon \kappa_0^{5/2}} |u|^2 |w|, \]
implies that,
\[ |e^{-\epsilon_0^{-1}A_{1/2}^j}B(u, u)| \leq c_A e^{-\epsilon \kappa_0^{5/2}} |u|^2 \leq c_A e^{-\epsilon G^2 \nu^2 \kappa_0^{3/2}}. \]
For any \( t > t_0 := 0, \) we have
\[ \int_{t_0}^{t} |e^{-\epsilon_0^{-1}A_{1/2}^j} \frac{du}{dt}| d\tau \leq e^{-\epsilon (c_A G^2 + 2G \nu^2 \kappa_0^{3/2}) (t - t_0)}. \]  \hspace{2cm} (14)
Using Lemma 4.1, we could choose \( t_1 \in [0, \frac{\delta}{\nu \kappa_0^{3/2}}], \) such that \( u_1 := u(t_1) \in F_\delta. \) Then (14) implies that
\[ d_w(u_0, u_1) = \nu^{-1} \kappa_0^{1/2} \left| e^{-\epsilon_0^{-1}A_{1/2}^j} (u_1 - u_0) \right| \leq \nu^{-1} \kappa_0^{1/2} \int_{t_0}^{t_1} |e^{-\epsilon_0^{-1}A_{1/2}^j} \frac{du}{dt}| d\tau \leq \delta. \]

**Remark 2.** Lemma 4.2 can be also stated as follows, for any \( \delta > 0, \)
\[ A_w \subset \bigcup_{u \in F_\delta} B_{d_w}(u, r_\delta), \]
where \( r_\delta \) is given in (13).

This remark has the following consequence,

**Lemma 4.3.** For any \( \delta > 0, \) and \( r > 0. \) If \( C_\delta \) can be covered by \( b_r \) balls \( B_{d_w}(v_i, r), i = 1, \ldots, b_r, \) then \( A_w \) can be covered with \( m \) balls \( B_{d_w}(u_j, 3r + r_\delta), i = 1, \ldots, m, \) where \( m \leq b_r \) and \( u_j \in F_\delta, \) for \( i = 1, \ldots, m. \)

**Proof.** First consider the ball \( B_{d_w}(v_1, r), \) if \( F_\delta \cap B_{d_w}(v_1, r) \neq \emptyset, \) we choose \( u_j \in F_\delta \) such that \( u_j \in B_{d_w}(v_1, r). \) Otherwise, we begin with the next ball \( B_{d_w}(v_2, r). \)

Suppose we have dealt with \( B_{d_w}(v_i, r), i = 1, \ldots, j \) and obtained \( u_{j_n}, n = 1, \ldots, l. \) For \( B_{d_w}(v_{j+1}, r), \) if there is some point \( u_{j_n}, 1 \leq n \leq l \) such that \( u_{j_n} \in B_{d_w}(v_{j+1}, r) \) or there does not exist any other points of \( F_\delta \) in \( B_{d_w}(v_{j+1}, r), \) we will consider the next ball \( B_{d_w}(v_{j+2}, r). \) Otherwise we can choose one point in \( F_\delta, \) denoted as \( u_{j, n+1} \) which is contained in \( B_{d_w}(v_{j+1}, r) \) and is different from \( u_{j_n}, n = 1, \ldots, l. \) After having processed all \( b_r \) balls, we can get a set \( \{ u_{j_n}, n = 1, \ldots, m \} \subset F_\delta. \)

Clearly, the number \( m \) is finite and \( m \leq b_r. \)

We claim that \( A_w \subset \bigcup_{n=1}^{m} B_{d_w}(u_{j_n}, 3r + r_\delta). \) Indeed, for each \( u \in A_w, \) by Remark 2, there exist \( u_{j_n} \) such that \( u_{j_n} \in F_\delta \) and \( u \in B_{d_w}(u_{j_n}, r_\delta). \) Furthermore, there exist \( v_n \) and \( u_{j_n} \) such that \( u_j \in B_{d_w}(v_n, r) \) and \( u_{j_n} \in B_{d_w}(v_n, r). \) It follows that
\[ d_w(u, u_{j_n}) \leq d_w(u, u_j) + d_w(u_j, v_n) + d_w(v_n, u_{j_n}) \leq 2r + r_\delta. \]
That is \( u \in B_{d_w}(u_{j_n}, 2r + r_\delta) \subset B_{d_w}(u_{j_n}, 3r + r_\delta). \) This completes the proof. \( \square \)

In the next section, we will give an estimate on \( b_r, \) the number of balls with radius \( r > 0 \) covering \( C_\delta. \)
5. Kolmogorov $\varepsilon$-entropy of the weak global attractor.

5.1. Covering of $C_\delta$. According to Lemma 4.3, for any given $\delta > 0$, in order to get a covering of the weak global attractor $A_u$, it suffices to find a covering of $C_\delta = \{ u \in D(A^{1/2}) : |A^{1/2} u| \leq G_*K_0^{1/2}\sqrt{1 + 3/\delta} \}$ with balls of radius $r > 0$ in the metric $d_w$.

**Lemma 5.1.** Given $\delta > 0$ and $r > 0$. If $K$ is chosen to be the integer satisfying (18) and (19), then $N_{2r}(C_\delta) \leq N_r(C_\delta \cap P_K H)$.

**Proof.** For any $u_1, u_2 \in C_\delta$, denote $u = u_1 - u_2$. By the definition of $d_w$ in (7),

$$d_w(u_1, u_2)^2 = \nu^{-2}\kappa_0|e^{-e\phi_0^-A_{1/2}} u|^2$$

$$= \nu^{-2}\kappa_0|e^{-e\phi_0^-A_{1/2}} P_K u|^2 + \nu^{-2}\kappa_0|e^{-e\phi_0^-A_{1/2}} (I - P_K) u|^2$$

$$\leq \nu^{-2}\kappa_0|e^{-e\phi_0^-A_{1/2}} P_K u|^2 + \nu^{-2}\kappa_0(K_0 K)^{-2}e^{-2e\kappa} |(I - P_K) A^{1/2} u|^2$$

$$\leq \nu^{-2}\kappa_0|e^{-e\phi_0^-A_{1/2}} P_K u|^2 + 2K^{-2}e^{-2e\kappa} G_0^2(1 + 1/\delta),$$

where $K \geq 1$ is an integer, and $I$ denotes the identity operator.

Consequently, if $K$ is chosen to be large enough such that

$$2K^{-2}e^{-2e\kappa} G_0^2(1 + 1/\delta) \leq r^2$$

then

$$d_w(u_1, u_2)^2 \leq \nu^{-2}\kappa_0|e^{-e\phi_0^-A_{1/2}} P_K u|^2 + r^2. \tag{16}$$

The inequality (15) is equivalent to

$$K e^K \geq \sqrt{2G_*} \sqrt{1 + 1/\delta}. \tag{17}$$

A sufficient condition to guarantee that (17) holds is

$$e^K \geq \ln(\sqrt{2G_*} \sqrt{1 + 1/\delta}). \tag{18}$$

Taking the first integer $K \geq 1$ that satisfies (18), then

$$K \leq 1 + \ln(\ln(\sqrt{2G_*} \sqrt{1 + 1/\delta})). \tag{19}$$

By (16), it follows that for any $r$-covering of $C_\delta \cap P_K H$, we can find a $2r$-covering of $C_\delta$ having the same number of sets. This completes the proof.

A special finite covering of the set $C_\delta \cap P_K H$ with respect to the metric $d_s$, defined in (9), and an upper bound of the cardinal number of this covering are given in the following lemma.

**Lemma 5.2.** For any $\eta > 0$ and integer $K \geq 1$, we have,

$$C_\delta \cap P_K H \subset \bigcup_{u_0 \in S} B_{d_s}(u_0, \eta)$$

where $S \subset C_\delta$ and the cardinal of $S$ satisfies the estimate

$$\text{card} (S) \leq \left(\frac{2G_* \sqrt{1 + 1/\delta}}{\eta} + 1\right)^{\text{dim} P_K H}.$$
Proof. It follows from the definition of \( C_\delta \) in (12) and Poincaré inequality (2) that
\[
C_\delta \cap P_K H \subset \{ u \in P_K H : |u| \leq G_* \nu \kappa_0^{-1/2} \sqrt{1 + 1/\delta} \}.
\]

Notice that \( P_K H \) is a Banach space of finite dimension. For fixed \( R > 0 \), let \( u_1, \ldots, u_N \eta \) (\( N_\eta \) is called metric entropy, which is an upper bound for covering number) be a maximum set of points in \( B_{d_s}(0, R) \), the ball of radius \( R > 0 \) in \( P_K H \) with \( |u_i - u_j| > \eta \), for \( i \neq j \), then the closed balls of radius \( \eta/2 \) centered at the \( u_j \)'s are disjoint, and their union lies within the ball of radius \( R + \eta/2 \) centered at the origin. Consequently,
\[
N_\eta \cdot (\eta/2)^{\dim P_K H} \leq (R + \eta/2)^{\dim P_K H}.
\]
and thus,
\[
N_\eta(B_{d_s}(0, R)) \leq N_\eta \leq \left( \frac{R + \eta/2}{\eta/2} \right)^{\dim P_K H} = \left( 1 + \frac{2R}{\eta} \right)^{\dim P_K H}.
\]

The result follows by applying (20) with \( R = G_* \sqrt{1 + 1/\delta} \).

Remark 3. An estimate for \( \dim(P_K H) = \text{card} \{ k \in \mathbb{Z}^3 \setminus \{0\} : |k| \leq K \} \) is (see page 43-44 in [3]),
\[
2 \left( \frac{4\pi}{3} \left( K - \frac{\sqrt{3}}{2} \right)^3 - 1 \right) \leq \dim(P_K H) \leq 2 \left( \frac{4\pi}{3} \left( K + \frac{\sqrt{3}}{2} \right)^3 - 1 \right),
\]

Using (18), it follows that
\[
\dim(P_K H) \leq 2 \left( \frac{4\pi}{3} \left( \ln \ln \left( \frac{\sqrt{2}G_* \sqrt{1 + 1/\delta}}{r} \right) + \frac{\sqrt{3}}{2} + 1 \right)^3 - 1 \right).
\]

Lemma 5.3. For any \( v \in H \), and real number \( \rho > 0 \), the following holds,
\[
B_{d_s}(v, \rho) \subset B_{d_w}(v, \rho).
\]

Proof. The result follows from the following inequalities
\[
|e^{-\epsilon \alpha_0^{-1/2} A^{1/2}} v| \leq |v| \sup_{|k| \geq 1} e^{-\epsilon |k|} = e^{-\epsilon} |v| \leq |v|.
\]

Due to Lemma 5.2 and Lemma 5.3, the following covering of \( C_\delta \) using the metric \( d_w \) can be obtained,
\[
C_\delta \cap P_K H \subset \bigcup_{u_0 \in S} B_{d_w}(u_0, \eta).
\]

5.2. Kolmogorov \( \epsilon \)-entropy. Now, for any fixed \( \epsilon > 0 \), based on the above lemmas, we are ready to the get a estimate on the Kolmogorov \( \epsilon \)-entropy of the weak attractor \( A_w \) in the space \( H \) endowed with the weak topology generated by the metric \( d_w \).

Theorem 5.4. The Kolmogorov \( \epsilon \)-entropy for the weak global attractor \( A_w \), endowed with the weak topology generated by \( d_w \), of 3D Navier-Stokes equations is bounded above by the following explicit formula,
\[
\mathcal{H}_\epsilon(A_w) \leq 2 \left( \frac{4\pi}{3} \left( \frac{\sqrt{3}}{2} + 1 + \ln \ln \beta \right)^3 - 1 \right) \ln(\sqrt{2} \beta + 1),
\]
where
\[
\beta := \frac{\sqrt{2} G_* \sqrt{1 + 1/\delta_0}}{\eta_0} = \frac{12 \sqrt{2} G_*}{\epsilon} \sqrt{1 + \frac{2e^{-\epsilon}(c_A G_*^2 + 2G_*)}{\epsilon}}.
\]

Proof. Choose
\[
\delta = \delta_0 := \frac{\epsilon}{2e^{-\epsilon}(c_A G_*^2 + 2G_*)},
\]
and
\[
\eta = \eta_0 := \frac{\epsilon}{12}.
\]
Using (13), we have
\[
r_\delta \geq \frac{\epsilon}{2},
\]
which implies that
\[
r_\delta + 6\eta = \epsilon.
\]
By Lemma 4.3 and Lemma 5.1, for any \( \eta > 0 \), the following inequalities hold
\[
N_{6\eta + r_\epsilon}(A_w) \leq N_{2\eta}(C_\delta) \leq N_\eta(C_\delta \cap P_K H).
\]
That is
\[
A_w \subset \bigcup_{u_0 \in S} B_{d_w}(u_0, \epsilon),
\]
where the set \( S \) is as given in Lemma 5.2.

Taking into account Lemma 5.2 and Remark 3, we see that
\[
\mathcal{H}_\epsilon(A_w) = \ln(N_\epsilon(A_w)) \leq \ln(\text{card}(S))
\]
\[
\leq \dim P_K H \times \ln \left( \frac{2G_* \sqrt{1 + 1/\delta_0}}{\eta_0} + 1 \right)
\]
\[
\leq 2 \left( \frac{4\pi}{3} \left( \ln \left( \frac{\sqrt{2} G_* \sqrt{1 + 1/\delta_0}}{\eta_0} + \frac{\sqrt{3}}{2} + 1 \right)^3 - 1 \right) \right) \ln \left( \frac{2G_* \sqrt{1 + 1/\delta_0}}{\eta_0} + 1 \right).
\]
This completes the proof.

Remark 4. In Theorem 5.4, we have \( \delta_0 = O(\epsilon), \eta_0 = O(\epsilon) \) and \( \beta = O(\epsilon^{-3/2}) \), as \( \epsilon \to 0^+ \).

An immediate consequence of Theorem 5.4 is the following estimate regarding the functional dimension of \( A_w \).

Corollary 1. The functional dimension of \( A_w \), endowed with the weak topology generated by \( d_w \), is bounded above by 1, i.e., \( d_f(A_w) \leq 1 \).

Remark 5. The upper bound given in Corollary 1 is consistent with a general result obtained in [16].
Remark 6. If the natural metric $d_n$, defined in (8), is used for the weak topology on $A_w$, the above arguments will lead to an upper bound on the Kolmogorov-$\epsilon$ entropy that implies $d_f(A_w) \leq \infty$. Hence, if one consider $A_w$ endowed with the weak topology generated by the metric $d_n$, the techniques used in this paper will not provide good estimate on $H_\epsilon(A_w)$.

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