Staggered Polarization of Vertex Models
with $U_q(\widehat{sl}(n))$-Symmetry

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Abstract

In this paper we give an explicit formula for level 1 vertex operators related to $U_q(\widehat{sl}(n))$ as operators on the Fock spaces. We derive also their commutation relations. As an applications we calculate the one point functions of the one-dimensional spin chain associated with the vector representation of $U_q(\widehat{sl}(n))$, thereby extending the recent work on the staggered polarization of the $XXZ$-model.

1 Introduction

The Hamiltonian of the $XXZ$-model has $U_q(\widehat{sl}(2))$-symmetry in the thermodynamic limit. Recently, on the basis of this fact, the $XXZ$-model was formulated in the framework of representation theory of $U_q(\widehat{sl}(2))$. Let us explain the scheme described in \\cite{[1]} briefly.

First we recall the $XXZ$-model as it appears in physics. The space of states of the $XXZ$-model is the infinite tensor product $\cdots \otimes V \otimes V \otimes \cdots$, where $V = Cv_+ \oplus Cv_-$ is the two-dimensional vector space. The $XXZ$-Hamiltonian is the following operator formally acting the above space:

$$H_{XXZ} = -\frac{1}{2} \sum_{k \in \mathbb{Z}} \left( \sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \frac{q + q^{-1}}{2} \sigma_k^z \sigma_{k+1}^z \right),$$

where $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices on $V$, $\sigma_k^\alpha$ acting on the $k$-th component of $\cdots \otimes V \otimes V \otimes \cdots$. Let $U_q'(\widehat{sl}(2))$ denote the subalgebra of $U_q(\widehat{sl}(2))$ with the grading operator $d$ being dropped. It acts on $V$ as follows:

$$e_1.v_- = v_+, \quad f_1.v_+ = v_-, \quad t_1.v_\pm = q^{\pm 1}v_\pm,$$

$$e_0.v_+ = v_-, \quad f_0.v_- = v_+, \quad t_0.v_\pm = q^{\mp 1}v_\pm.$$
Furthermore, $U'_q(\hat{sl}(2))$ acts on $\cdots \otimes V \otimes V \otimes \cdots$ via the iterated coproduct $\Delta^{(\infty)}$.

$$\Delta^{(\infty)}(t_i) = \cdots \otimes t_i \otimes t_i \otimes \cdots,$$

$$\Delta^{(\infty)}(e_i) = \sum \cdots \otimes t_i \otimes e_i \otimes 1 \otimes 1 \otimes \cdots,$$

$$\Delta^{(\infty)}(f_i) = \sum \cdots \otimes 1 \otimes 1 \otimes f_i \otimes t_i^{-1} \otimes t_i^{-1} \otimes \cdots,$$

Formal manipulation shows that

$$[H_{XXZ}, U'_q(\hat{sl}(2))] = 0.$$

Letting $T$ denote the shift operator on $\cdots \otimes V \otimes V \otimes \cdots$, we can also check

$$\frac{2q}{1 - q^2} H_{XXZ} = T^2 d T - d.$$

The above observation holds only in the infinite lattice case. Of course, $H_{XXZ}$ and the action of $U_q(\hat{sl}(2))$ are not literally well-defined. Nevertheless, when we consider the model in the anti-ferroelectric regime $-1 < q < 0$, we can construct a well-defined theory on “the space of physical states”, which is the subspace consisting of finite excitations over the ground states in $\cdots \otimes V \otimes V \otimes V \otimes \cdots$. The formulation of [1] is based on the (hypothetical) identification

$$\text{“the space of physical states”} = \bigoplus_{0 \leq i, j \leq 1} V(\Lambda_i) \otimes V(\Lambda_j),$$

where $V(\Lambda_i)$ is the level 1 highest weight irreducible $U_q(\hat{sl}(2))$-module and $V(\Lambda_j)^*$ is the dual module of $V(\Lambda_j)$. The symbol $\otimes$ is to be understood with an appropriate completion, but we will not go into such details in the sequel. To motivate this hypothesis, consider the intertwiner of $U_q(\hat{sl}(n))$-modules

$$\tilde{\Phi}_{\Lambda_i}^{\Lambda_{i-1} V} : V(\Lambda_i) \longrightarrow V(\Lambda_{i-1}) \otimes V,$$

called vertex operators([2]). In fact, such an operator exists, is unique up to a scalar, gives an isomorphism. Iterating the vertex operators, we get the following isomorphism.

$$V(\Lambda_i) \otimes V(\Lambda_j)^* \cong V(\Lambda_{i-1}) \otimes V \otimes V(\Lambda_j)^* \cong V(\Lambda_{0 \text{ orig}}) \otimes V \otimes \cdots \otimes V \otimes V(\Lambda_j)^*.$$

It tells us that the local structure $\cdots \otimes V \otimes V \otimes \cdots$ in the naive picture is realized in the space $\bigoplus V(\Lambda_i) \otimes V(\Lambda_j)^*$. By composing (*) with a similar vertex operator

$$V \otimes V(\Lambda_i)^* \longrightarrow V(\Lambda_{1-i})^*,$$

we get

$$V(\Lambda_i) \otimes V(\Lambda_j)^* \cong V(\Lambda_{1-i}) \otimes V \otimes V(\Lambda_j)^* \cong V(\Lambda_{1-i}) \otimes V(\Lambda_{1-j})^*. $$
The resulting isomorphism can be identified with the shift operator $T$. In this manner we can build a well-defined theory on $\bigoplus V(\Lambda_i) \otimes V(\Lambda_j)^*$ that captures all the essential features expected from the physical definition.

It is straightforward to generalize the above formulation to the models related to any quantum affine algebra. In this paper, we consider a multi spin-analogue of the XXZ-model related to the vector representation of $U_q(\hat{sl}(n))$. Our main results are twofold. One is the bosonization of the level 1 vertex operators (Theorem 3.3, 3.4). The other is the exact calculation of the one-point functions (Theorem 5.2). The structure of this paper is the following. In §2, we review the construction of the level 1 irreducible highest weight $U_q(\hat{sl}(n))$-modules. In §3, we construct the vertex operators on the bosonic Fock space explicitly. In §4, we explain the mathematical formulation of models. In §5, first, we derive an integral representation for the one-point function by using the bosonization of the vertex operators. Next, by using the commutation relations of the vertex operators, we derive difference equations for the one-point functions. This equation can be solved easily. As a result, we obtain an explicit formula of the one-point functions extending the previous work on the spontaneous staggered polarization for the XXZ-model [13].

2 Vertex operator representations of $U_q(\hat{sl}(n))$

In this section, we review the construction of the level 1 irreducible highest weight modules following [13].

2.1 Notations

Throughout this paper, we fix a real number $q$ ($-1 < q < 0$) and a positive integer $n$. The $q$-integer and $q$-factorial are denoted by $[k] = (q^k - q^{-k})/(q - q^{-1})$ and $(a; q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$ respectively. Most notations concerning Lie algebras follow [14]. Let $P$ be a free $\mathbb{Z}$-module

$$P := \bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta.$$ 

We call it the weight lattice. We define $P^*$ as follows.

$$P^* := \text{Hom}(P, \mathbb{Z}) = \bigoplus_{i=0}^{n-1} \mathbb{Z}h_i \oplus \mathbb{Z}d.$$ 

The pairing is given by $\langle \Lambda_i, h_j \rangle = \delta_{ij}$, $\langle \Lambda_i, d \rangle = 0$, $\langle \delta, h_j \rangle = 0$, $\langle \delta, d \rangle = 1$. The indices are extended cyclically such as $\Lambda_i = \Lambda_{i+n}$, etc. Let $\alpha_0 = -\Lambda_{n-1} + 2\Lambda_0 - \Lambda_1 + \delta$, $\alpha_j = -\Lambda_{j-1} + 2\Lambda_j - \Lambda_{j+1}$ ($1 \leq j \leq n-1$) be the simple roots. The invariant bilinear form on $P$ is given by $\langle \alpha_i | \alpha_j \rangle = -\delta_{ij} + 2\delta_{ij} - \ldots$
\( \delta_{ij+1} \) and \( (\delta\delta) = 0 \). The projection to the classical weight lattice is given by \( \Lambda_i = \Lambda_i - \Lambda_0, \delta = 0 \). \( U_q(\mathfrak{sl}(n)) \) is the \( \mathbb{C} \)-algebra generated by the symbols \( \{ t^\pm_i (= q^{\pm h_i}), q^d, e_i, f_i, (i = 0, \cdots, n - 1) \} \) which satisfy the following defining relations.

\[
ti tj = tjt^\pm_i, \ ti e j t^\pm_i = q^{(\alpha_j, h_i)} e_j, ti fjt^\pm_i = q^{-(\alpha_j, h_i)} f_j
\]

\[
[e_i, f_j] = \delta_{ij} \frac{t_i - t^{-1}}{q - q^{-1}}
\]

\[
\sum_{k=0}^{b} (-1)^k \left[ \begin{array}{c} b \\ k \end{array} \right] e_i^k e_j e_i^{b-k} = 0, \sum_{k=0}^{b} (-1)^k \left[ \begin{array}{c} b \\ k \end{array} \right] f_i^k f_j f_i^{b-k} = 0
\]

where \( b = 1 - (\alpha_i, h_j) \), \( \left[ \begin{array}{c} b \\ k \end{array} \right] = \frac{[b]!}{[k]![b-k]!} \). Throughout this paper, we denote \( U_q(\mathfrak{sl}(n)) \) by \( U_q \). \( U_q' \) is the subalgebra of \( U_q \) generated by \( \{ t_i, e_i, f_i \} \). We denote the irreducible highest weight \( U_q'(or \ U_q')-module \) with highest weight \( \lambda \) by \( V(\lambda) \). We fix a highest weight vector of \( V(\lambda) \) and denote it by \( |\lambda\rangle \). The coproduct \( \Delta \) and antipode \( S \) are given as follows.

\[
\Delta(q^h) = q^h \otimes q^h, \ \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \ \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i,
\]

\[
S(q^h) = q^{-h}, \ S(e_i) = -t_i^{-1}e_i, \ S(f_i) = -f_it_i.
\]

When \( W \) is a \( U_q'(or \ U_q')-module \), we introduce the left module structure on the dual space \( W^* \) by \( x.u^*(v) = u^*(S(x).v) \) for \( x \in U_q'(or \ U_q') \), \( u^* \in W^* \) and \( v \in W \). If \( W \) has a weight decomposition \( \bigoplus_\lambda W_\lambda \), we define the completion \( \hat{W} = \prod_\lambda W_\lambda \). Normally we omit \( \hat{\cdot} \).

### 2.2 Drinfeld generators of \( U_q \)

We introduce another set of generators of \( U_q \) (\( \hat{\mathfrak{h}} \)).

**Definition.** \( \mathfrak{A} \) is the \( \mathbb{C} \)-algebra generated by the symbols \( \{ \gamma^\pm_i, K_i, a_i(k), x^\pm_i(k) (1 \leq i \leq n - 1, k \in \mathbb{Z}\setminus\{0\}, l \in \mathbb{Z}) \} \) which satisfy the following defining relations.

1. \( \gamma^\pm_i \in \text{Center of } \mathfrak{A}, \ \gamma^\pm_i \gamma^{-\pm}_j = 1 \),
2. \( [a_i(k), a_j(l)] = \delta_{k+l,0} \left[ (\alpha_i | a_j(k)] \frac{\gamma^k - \gamma^{-k}}{q - q^{-1}} \right] \),
3. \( [a_i(k), K_j] = 0 \),
4. \( K_ix^\pm_j(k)K_j^{-1} = q^{\pm(\alpha_i, h_j)}x^\pm_j(k) \),
5. \( [a_i(k), x^\pm_j(l)] = \pm \left[ (\alpha_j, h_i)k] \frac{\gamma^\pm_i \gamma^\mp_j}{k} \right] x^\pm_j(k + l) \),
the vertex operators it is convinient to define

$\Delta_{ij} = \frac{\delta_{ij}}{q - q^{-1}}(\gamma^{(k-l)/2}\psi_i(k+l) - \gamma^{(l-k)/2}\varphi_i(k+l))$

where

\[
\sum_{k=0}^{\infty} \psi_i(k)z^{-k} = K_i \exp \left( (q - q^{-1}) \sum_{k=1}^{\infty} a_i(k)z^{-k} \right),
\]

\[
\sum_{k=0}^{\infty} \varphi_i(-k)z^{-k} = K_i^{-1} \exp \left( -(q - q^{-1}) \sum_{k=1}^{\infty} a_i(-k)z^{-k} \right).
\]

8) $[x_i^+(k), x_j^-(l)] = 0$ for $\langle \alpha_i, h_j \rangle = 0$.

9) 

$\{x_i^+(k)x_j^+(l)x_j^-(m) - (q + q^{-1})x_i^+(k)x_j^+(m)x_i^-(l) + x_j^+(m)x_i^+(k)x_i^-(l)\}
\{x_i^+(l)x_j^+(k)x_j^+(m) - (q + q^{-1})x_i^+(l)x_j^+(m)x_i^-(k) + x_j^+(m)x_i^+(l)x_i^-(k)\}
= 0$ for $\langle \alpha_j, h_i \rangle = -1$.

\[\square\]

We know the following theorem.

**Theorem 2.2** The following correspondance gives an isomorphism $U_q^e \cong A$.

$t_j \mapsto K_j$ , $e_j \mapsto x_j^+(0)$ , $f_j \mapsto x_j^-(0)$ (1 \leq j \leq n) ,

$t_0 \mapsto \gamma K_1^{-1} \ldots K_{n-1}^{-1}$ ,

$e_0 \mapsto [x_{-1}^-(0), [x_{-2}^-(0), \ldots [x_{-1}^-(0), x_1^+(1)]_{q^{-1}}, \ldots ]_{q^{-1}}]_{q^{-1}}K_1^{-1} \ldots K_{n-1}^{-1}$,

$f_0 \mapsto K_1 \ldots K_{n-1}[[[x_{-1}^+(1), x_{-2}^+(0)]_{q^{-1}}, \ldots, x_{-1}^+(0)]_{q^{-1}}]_{q^{-1}}$.

Here we have set $[A, B]_q = AB - qBA$.

\[\square\]

### 2.3 Group algebra $C[\mathcal{P}]$

For the construction of representations, it is enough to consider only $C[\mathcal{Q}]$, where $\mathcal{Q} = \oplus_{j=1}^{\ell=n-1} \alpha_j$ is the classical root lattice. But, for the construction of the vertex operators it is convinient to define $C[\mathcal{P}]$ ($\mathcal{P} = \oplus_{j=0}^{\ell=n-1} \alpha_j \oplus \Lambda_{n-1}$: the classical weight lattice). In fact, we use a central-extension of the group algebra of $\mathcal{P}$.
\textbf{Definition.} \(C[\bar{P}]\) is the \(\mathcal{C}\)-algebra generated by the symbols \(\{e^{\alpha_2}, \ldots, e^{\alpha_n}, e^{\tilde{\alpha}_{n-1}}\}\) which satisfy the following defining relations.

\[e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle \alpha_i | \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i} \quad (2 \leq i \leq n - 1)\]

\[e^{\alpha_i} e^{\tilde{\alpha}_{n-1}} = (-1)^{\delta_{i,n-1}} e^{\tilde{\alpha}_{n-1}} e^{\alpha_i} \quad (2 \leq i \leq n - 1)\]

\[\square\]

For \(\alpha = m_2 \alpha_2 + \cdots + m_n \alpha_n - 1 + m_n \tilde{\alpha}_{n-1}(\in \bar{P})\), we denote \(e^{m_2 \alpha_2} \cdots e^{m_n \alpha_n - 1} e^{m_n \tilde{\alpha}_{n-1}}\) by \(e^\alpha\). For example, \(e^{\alpha_1} = e^{-2 \alpha_2} e^{-3 \alpha_3} \cdots e^{- (n-1) \alpha_n - 1} e^{n \tilde{\alpha}_{n-1}}\), \(e^{\tilde{\alpha}_i} = e^{- \alpha_i + 1} e^{-2 \alpha_{i+2}} \cdots e^{-(n-i-1) \alpha_n - 1} e^{n \tilde{\alpha}_{n-1}}\). A simple calculation shows the following.

\textbf{Proposition 2.3}

\begin{enumerate}
  \item \(e^{\alpha_i} e^{\alpha_j} = (-1)^{\langle \alpha_i | \alpha_j \rangle} e^{\alpha_j} e^{\alpha_i} \quad (1 \leq i, j \leq n - 1)\)
  \item \(e^{\alpha_1} e^{\tilde{\alpha}_{n-1}} = (-1)^{n-1} e^{\tilde{\alpha}_{n-1}} e^{\alpha_1}\)
  \item \(e^{\alpha_i} e^{\tilde{\alpha}_i} = (-1)^{\delta_{i,n}} e^{\tilde{\alpha}_i} e^{\alpha_i} \quad (1 \leq i \leq n - 1)\)
  \item \(e^{\tilde{\alpha}_i} e^{\tilde{\alpha}_{n-1}} = (-1)^{n} e^{\tilde{\alpha}_{n-1}} e^{\tilde{\alpha}_i}\)
\end{enumerate}

\[\square\]

\subsection{Construction of representations}

Let

\[W_i := \mathcal{C} [a_j (-k) (1 \leq j \leq n - 1, k \in \mathbb{Z}_{=0})] \otimes \mathcal{C} [\bar{Q}] e^{\tilde{\alpha}_i} \quad (0 \leq i \leq n - 1).\]

We define the operators \(a_j(k) (1 \leq j \leq n - 1), \partial_\alpha, e^\alpha (\alpha \in \bar{Q})\), \(d\) on \(W_i\) as follows:

for \(f \otimes e^\beta = a_i (-n_1) \cdots a_k (-n_k) \otimes e^\beta \in W_i\),

\[a_j(k). f \otimes e^\beta = \begin{cases} 
  a_j(k) f \otimes e^\beta \quad (k < 0) \\
  [a_j(k), f] \otimes e^\beta \quad (k > 0)
\end{cases}\]

\[\partial_\alpha . f \otimes e^\beta = (\alpha | \beta) f \otimes e^\beta\]

\[e^\alpha . f \otimes e^\beta = f \otimes e^\alpha e^\beta\]

\[d . f \otimes e^\beta = (- \sum_{l=1}^{k} n_l - \frac{(\beta | \beta)}{2} + \frac{(X_l | X_l)}{2}) f \otimes e^\beta.\]

Let

\[X_j^\pm (z) := \sum_{k \in \mathbb{Z}} x_j^\pm (k) z^{-k-1} \quad (1 \leq j \leq n - 1).\]
We define the action of $U_q$.

\[
\gamma \mapsto q \quad K_j \mapsto q^{\delta_{\alpha_j}} \quad (1 \leq j \leq n-1),
\]

\[
X_j^\pm(z) \mapsto \exp(\pm \sum_{k=1}^{\infty} \frac{a_j(-k)}{[k]} q^{\mp k} z^k) \exp(\mp \sum_{K=1}^{\infty} \frac{a_j(k)}{[k]} q^{\mp k} z^{-k}) e^{\pm \alpha_j z^{\pm}}.
\]

We know the following theorem.

**Theorem 2.4** ([5]) By the above action, $W_i$ becomes the irreducible highest weight module with highest weight $\Lambda_i$, and $1 \otimes e^{\Lambda_i}$ is a highest weight vector of $W_i$.

From now on, we identify $W_i$ and $1 \otimes e^{\Lambda_i}$ with $V(\Lambda_i)$ and $|\Lambda_i\rangle$ respectively.

### 3 Construction of Vertex operators

In this section, we construct the vertex operators on $W_i$ explicitly.

#### 3.1 Vertex operators

We review the definition and some properties of the vertex operators. ([3],[3]) Let $V$ be a finite dimensional representation of $U_q'$. The affinization of $V$ is the following $U_q$-module $V_z$.

\[
V_z = V \otimes \mathbb{C}[z, z^{-1}].
\]

We define the $U_q$-module structure on $V_z$ as follows.

\[
e_i.(v \otimes z^m) = e_i.v \otimes z^{m+\delta_{\alpha_0}} , \quad f_i.(v \otimes z^m) = f_i.v \otimes z^{m-\delta_{\alpha_0}}
\]

\[
t_i.(v \otimes z^m) = t_i.v \otimes z^m , \quad q^d.(v \otimes z^m) = mv \otimes z^m.
\]

**Definition.** The vertex operator is a $U_q$-homomorphism of the following form.

**Type I:**

\[
\tilde{\Phi}_\lambda^V(z) : V(\lambda) \longrightarrow V(\mu) \otimes V_z
\]

**Type II:**

\[
\tilde{\Phi}_\lambda^V(z) : V(\lambda) \longrightarrow V_z \otimes V(\mu)
\]

The symbol $\otimes$ means $W_1 \otimes W_2$. From now on, we omit it. We know the following theorem about the existence of the vertex operators.

**Theorem 3.1** ([3])

\[
\text{Hom}_{U_q}(V(\lambda), V(\mu) \otimes V_z)
\]

\[\cong \{v \in V \mid \text{the weight of } v = \lambda - \mu \mod \delta\]
and $e_i^{(\mu, h_i)} v = 0$ for $i = 0, \ldots, n - 1$,

where $\Phi \in \text{Hom}_{U_q}(V(\lambda), V(\mu) \otimes V_z)$ corresponds to $v$ via the relation $\Phi|\lambda\rangle = |\mu\rangle \otimes v^+$ (terms of positive powers in $z$).

We define the components of the vertex operators as follows.

$$\tilde{\Phi}^\mu_{\lambda V}(z)|u\rangle = \sum_{j=0}^{n-1} \tilde{\Phi}^\mu_{\lambda j}(z)|u\rangle \otimes v_j$$

for $|u\rangle \in V(\lambda)$, where $\{v_j\}$ is a set of basis of $V$. For the type II, the components are also defined similarly. Using the components, we define similar vertex operators

$$\tilde{\Phi}^\mu_{\lambda V}(z) : V(\lambda) \otimes V^*_z \rightarrow V(\mu) \otimes C[z, z^{-1}]$$

by

$$\tilde{\Phi}^\mu_{\lambda V}(z)(|v\rangle \otimes v_i) = \tilde{\Phi}^\mu_{\lambda j}(z)|v\rangle$$

for $|v\rangle \in V(\lambda)$.

Here $x \in U_q$ acts on $V(\mu) \otimes C[z, z^{-1}]$ as $x \otimes 1$.

Now, we specialize $V$ to the vector representation.

$$V = Cv_0 \oplus \cdots \oplus Cv_{n-1}$$

The $U'_q$-module structure on $V$ is the following.

$$e_i v_j = \delta_{ij} v_{i-1}, \quad f_i v_j = \delta_{i-1, j} v_i, \quad t_i v_j = q^{\delta_{ij+1}} v_{j+1}$$

$(V^*)_z$ is denoted by $V^*_z$. The action of $U_q$ on $V^*_z$ is the following.

$$e_i (v_j^* \otimes z^m) = -q^{-1} \delta_{i-1, j} v_i^* \otimes z^{m+\delta_{i0}}, \quad f_i (v_j^* \otimes z^m) = -q \delta_{i0} v_{i-1}^* \otimes z^{m-\delta_{i0}}$$

$$t_i v_j \otimes z^m = q^{-\delta_{i, j+1}} v_{j+1} \otimes z^m, \quad q^d v_j \otimes z^m = m v_j \otimes z^m.$$

In our case, by the above theorem, only

$$\tilde{\Phi}^\lambda_{\Lambda_i+1}(z), \tilde{\Phi}^{\Lambda_i+1\Lambda_i}(z), \tilde{\Phi}^{V^*\Lambda_i+1}(z)$$

are non-trivial. Furthermore, each of them is unique up to a scalar. Here, we take the following normalization.

$$\tilde{\Phi}^\lambda_{\Lambda_i+1}(z)|\Lambda_i+1\rangle = |\Lambda_i\rangle \otimes v_i + \text{ (terms of positive powers in } z)$$

$$\tilde{\Phi}^{\Lambda_i+1\Lambda_i}(z)|\Lambda_i\rangle = |\Lambda_i+1\rangle \otimes v_i^* + \text{ (terms of positive powers in } z)$$

For the type II, we take a similar normalization.
3.2 Coproduct of $a_i(k), x_i^\pm(l)$ and action of $a_i(k), x_i^\pm(l)$ on $V_z$

The coproduct of Drinfeld generators is not known in full. But the “main terms” are calculated in \[7\] for $U_q(\mathfrak{sl}(2))$. The case of $U_q(\mathfrak{sl}(n))$ is quite similar.

**Proposition 3.2.A** For $k \geq 0$, $l > 0$,

\[
\Delta(x_i^+(k)) = x_i^+(k) \otimes \gamma^k + \gamma^{2k} K_i \otimes x_i^+(k)
\]

\[
+ \sum_{j=0}^{k-1} \gamma^{(k-j)/2} \psi_i(-k+j) \otimes \gamma^{k-j} x_i^+(j) \mod UN_- \otimes UN_+^2
\]

\[
\Delta(x_i^+(l)) = x_i^+(l) \otimes \gamma^{-l} + K_i^{-1} \otimes x_i^+(l)
\]

\[
+ \sum_{j=1}^{l-1} \gamma^{(l-j)/2} \phi_i(-l+j) \otimes \gamma^{-l+j} x_i^+(j) \mod UN_- \otimes UN_+^2
\]

\[
\Delta(x_i^-(l)) = x_i^-(l) \otimes K_i + \gamma^k \otimes x_i^-(l)
\]

\[
+ \sum_{j=1}^{k-1} \gamma^{k-j} x_i^-(j) \otimes \gamma^{(j-k)/2} \psi_i(k-j) \mod UN_2 \otimes UN_+
\]

\[
\Delta(x_i^-(k)) = x_i^-(k) \otimes \gamma^{-2k} K_i^{-1} + \gamma^{-k} \otimes x_i^-(k)
\]

\[
+ \sum_{j=0}^{k-1} \gamma^{j-k} x_i^-(j) \otimes \gamma^{-(k+3j)/2} \phi_i(j-k) \mod UN_2 \otimes UN_+
\]

\[
\Delta(a_i(l)) = a_i(l) \otimes \gamma^k + \gamma^{-k} \otimes a_i(l) \mod UN_- \otimes UN_+
\]

\[
\Delta(a_i(-l)) = a_i(-l) \otimes \gamma^{-2k} + \gamma^{-k} \otimes a_i(-l) \mod UN_- \otimes UN_+
\]

where, $UN_-, UN_+$ are the left ideals generated by $\{x_i^+(k), x_i^+(l)\}$.

**Proposition 3.2.B** The action of $a_i(k), x_i^\pm(l)$ on $V_z$ is the following.

\[
x_i^+(k) \mapsto (q^l z)^k E_{ii-1}
\]

\[
x_i^-(k) \mapsto (q^l z)^k E_{i-1i}
\]

\[
a_i(l) \mapsto \frac{|k|}{k} (q^l z)^k (q^{-k} E_{i-1i} - q^k E_{ii})
\]

where $E_{ij}$ is the matrix unit of $\text{End}V$ such that $E_{ij}v_l = \delta_{ij} v_l$.
3.3 Vertex operators of type I

First, we consider the vertex operator \( \tilde{\Phi}^{A_iV}_{\Lambda_i+1}(z) : V(\Lambda_i+1) \rightarrow V(\Lambda_i) \otimes V_z \).

We can determine the \((n-1)\)-th component as follows. By prop.3.2, we get the following commutation relations.

\[
[a_j(k), \Phi_{n-1}^V(z)] = \delta_{j,n-1} q^{k} \cdot \Phi_{n-1}^V(z)
\]

\[
[a_j(-k), \Phi_{n-1}^V(z)] = \delta_{j,n-1} q^{-k} \cdot \Phi_{n-1}^V(z)
\]

for \( 1 \leq j \leq n-1 \).

The above conditions determine the form of \( \tilde{\Phi}^{A_iV}_{\Lambda_i+1n-1}(z) \) completely under the normalization conditions in §3.1. The other components are determined by one of the intertwining conditions.

\[
\tilde{\Phi}^{A_iV}_{\Lambda_i+1j-1}(z) = [ \Phi^{A_iV}_{\Lambda_i+1j}(z) , f_j ]_q.
\]

Hence, the other components are represented by the integral of the currents.

For the vertex operators \( \tilde{\Phi}^{A_iV^*}_{\Lambda_i+1V^*}(z) : V(\Lambda_i) \rightarrow V(\Lambda_{i+1}) \otimes V_z^* \), we have the similar commutation relations this time for the 0-th component. We summarize the results.

**Theorem 3.3**

1) \( \tilde{\Phi}^{A_iV}_{\Lambda_i+1n-1}(z) = \exp \left( \sum_{k=1}^{\infty} a_{n-1}^*(-k) q^{-\frac{2n+1}{2} k} z^k \right) \cdot \exp \left( \sum_{k=1}^{\infty} a_{n-1}^*(k) q^{-\frac{2n+1}{2} k} z^{-k} \right) \cdot \exp \left( \sum_{k=1}^{\infty} a_{n-1}^*(k) q^{-\frac{2n+1}{2} k} z^{-k} \right) \]

\[
\cdot e^{(q^{n+1})^{-1} \cdot \Phi_{n-1}^V q^{\frac{n-1}{2}} \cdot (-1)^{(n-1)(n-1)} \cdot \Phi_{n-1}^{V^*} q^{\frac{1}{2}(n-1)(n-1)}}
\]

\[
(i = 0, \ldots, n-1)
\]

\[
\tilde{\Phi}^{A_iV}_{\Lambda_i+1j-1}(z) = [ \tilde{\Phi}^{A_iV}_{\Lambda_i+1j}(z) , f_j ]_q \quad (j = 1, \ldots, n-1).
\]

2) \( \tilde{\Phi}^{A_iV^*}_{\Lambda_iV_0}(z) = \exp \left( \sum_{k=1}^{\infty} a_{1}^*(-k) q^{\frac{1}{2} k} z^k \right) \cdot \exp \left( \sum_{k=1}^{\infty} a_{1}^*(k) q^{-\frac{1}{2} k} z^{-k} \right) \cdot \exp \left( \sum_{k=1}^{\infty} a_{1}^*(k) q^{-\frac{1}{2} k} z^{-k} \right) \]

\[
\cdot e^{A_1((-1)^{n-1} q z)^{\frac{1}{2}} \cdot \Phi_{1}^{V^*} q^{i(n+1)^{i+1}} \cdot (-1)^{i(n-1)(n-1)}}
\]

\[
(i = 0, \ldots, n-1)
\]

\[
\tilde{\Phi}^{A_iV^*}_{\Lambda_ij-1}(z) = [ f_j , \tilde{\Phi}^{A_iV^*}_{\Lambda_i+1j-1}(z) ]_{q^{-1}} \quad (j = 1, \ldots, n-1).
\]
where $a_{n-1}^*(k) = \sum_{l=1}^{n-1} \frac{\text{[n][n/k]} a_l(k)}{\text{[n][n/k]}}$, $a_1^*(k) = \sum_{l=1}^{n-1} \frac{\text{[n-l][n]} a_l(k)}{\text{[n][n/k]}}$.

The coefficients of $a_{n-1}^*(k)$ and $a_1^*(k)$ are determined by the conditions

$$[a_i(k), a_{n-1}^*(-k)] = \delta_{i,n-1} \frac{[k]}{k}, \quad [a_i(k), a_1^*(-k)] = \delta_{i,1} \frac{[k]}{k}$$

\[ \square \]

### 3.4 Vertex operators of type II

We can also apply the same method for the vertex operators of type II.

**Theorem 3.4**

1) $\tilde{\Phi}^{V_{\Lambda_i}}_{\Lambda_i+1,j}(z) = \exp(-\sum_{k=1}^{\infty} a_1^*(-k)q^{\frac{2n+1}{2} z - k}) \exp(-\sum_{k=1}^{\infty} a_1^*(k)q^{-\frac{2n+1}{2} z - k})$

\[ \times e^{-\mathcal{R}_1((-1)^{n-1} q z)^{-\frac{n(n-1)}{2}}} q^{i - \frac{1}{2}n} q^{-\frac{n(n-1)}{2}} = \left( -1 \right)^{i n + \frac{1}{2} i (i+1)} \]

\[ (i = 0, \ldots, n - 1) \]

$\tilde{\Phi}^{V_{\Lambda_i}}_{\Lambda_i+1,j}(z) = [ \tilde{\Phi}^{V_{\Lambda_i}}_{\Lambda_i+1,j-1}(z), e_j ]_q \quad (j = 1, \ldots, n - 1).$

2) $\tilde{\Phi}^{V^*_{\Lambda_i+1}}_{\Lambda_i + n - 1}(z) = \exp(-\sum_{k=1}^{\infty} a_{n-1}^*(-k)q^{2n+1 z - k}) \exp(-\sum_{k=1}^{\infty} a_{n-1}^*(k)q^{-2n+1 z - k})$

\[ \times e^{-\mathcal{R}_1(q^{n+1} z)^{-\frac{n(n+1)}{2}}} \exp\left( -\sum_{k=1}^{\infty} a_{n-1}^*(k) q^{-\frac{n(n+1)}{2} z - k} \right) \]

\[ \times (-1)^{\frac{1}{2} n} (-1)^{i (n-i)} (-1)^{\frac{1}{2} (n-i)(n-i-1)} \]

\[ (i = 0, \ldots, n - 1) \]

$\tilde{\Phi}^{V^*_{\Lambda_i+1}}_{\Lambda_i + j-1}(z) = [ e_j, \tilde{\Phi}^{V^*_{\Lambda_i}}_{\Lambda_i+1,j}(z) ]_{q^{-1}} \quad (j = 1, \ldots, n - 1).$

\[ \square \]

### 3.5 Commutation relations of the vertex operators

In [12], by solving the $q$-KZ equations the authors get the commutation relations of the vertex operators of type I related to $U_q(\mathfrak{sl}(n))$. These formulas can be derived directly by using our explicit formulas for vertex operators. First, we write down the matrix coefficient of $\mathcal{R}_{V \cdot V}(z_1/z_2) \in \text{End}_C V^*_2 \otimes V^*_2$.

$$\mathcal{R}_{V \cdot V}(z)(v_i^* \otimes v_j) = v_i^* \otimes v_j \quad (i \neq j), \quad \mathcal{R}_{V \cdot V}(z)(v_i^* \otimes v_i) = \sum_{j=0}^{n-1} a_{ij} v_j^* \otimes v_j,$$
where \( a_{ij} = \begin{cases} \frac{(q - q^{-1})z}{1 - z} & (i > j) \\ \frac{q - q^{-1}z}{1 - z} & (i = j) \\ \frac{q - q^{-1}}{1 - z} & (i < j) \end{cases} \).

**Proposition 3.5**

1) \( \hat{\Phi}^{\Lambda_i}_{i+1} (z) \hat{\Phi}^{\Lambda_i}_{V} (z) = (q^{2n}; q^{2n})_{\infty} \text{id}_{V(\Lambda_i)} \)

2) \( \hat{\Phi}^{\Lambda_i}_{i} (z) \hat{\Phi}^{\Lambda_{i+1}}_{V} (z) = (q^{2n}; q^{2n})_{\infty} \text{id}_{V(\Lambda_i) \otimes V} \)

3) \( \hat{\Phi}^{\Lambda_i}_{i+1} (z_2) \hat{\Phi}^{\Lambda_{i+1}}_{V^*} (z_1) = -q^{(z_1 - z_2)} \delta_{u, 0} (z_1) F R V^* V (z_1) \hat{\Phi}^{\Lambda_{i-1}}_{V^*} (z_1) \hat{\Phi}^{\Lambda_i}_{V^*} (z_2) \)

where \( r(z) = \frac{(z; q^{2n})_{\infty} (q^{2n+2} z^{-1}; q^{2n})_{\infty}}{(q^2 z; q^{2n})_{\infty} (q^{2n} z^{-1}; q^{2n})_{\infty}} \), \( P v_i^* \otimes v_j = v_j \otimes v_i^* \).

**proof**) Formulas 1), 2) follow from simple calculations. We know the uniqueness of the vertex operator \( V(\Lambda_i) \rightarrow V(\Lambda_i) \otimes V_{z_1} \otimes V_{z_2} \). (For the details see [2],[3].)

So, the left and the right hand sides of 3) coincide up to a scalar factor. By comparing the \( v_i \otimes v_i^* \) component of both sides, we get the above equation. \( \square \)

## 4 Vertex model

In this section, we give a mathematical definition of the model treated in this paper. ([1],[10],[12])

### 4.1 Space of states

We know the integrable generalization of the \( XXZ \)-model related to any quantum affine algebra \( U_q(\hat{g}) \). Let \( V \) be a finite dimensional representation of \( U_q(\hat{g}) \) with a spectral parameter \( z \) and \( R(z_1/z_2) \in \text{End}(V_{z_1} \otimes V_{z_2}) \) be the \( R \)-matrix for \( U_q'(\hat{g}) \). We define the model on the infinite lattice \( \cdots \otimes V \otimes V \otimes V \otimes \cdots \) Let \( h \) be the operator on \( V \otimes V \) such that

\[ PR(z_1/z_2) = (1 + uh + \cdots) \times \text{const.} \quad (u \to 0), \]

\[ P : \text{ the transposition,} \quad e^u = z_1/z_2 \]
We define the Hamiltonian $H$ as follows.

$$H = \sum_{k \in \mathbb{Z}} h_{l+1l},$$

where $h_{l+1l}$ is $\cdots \otimes 1 \otimes h \otimes 1 \otimes \cdots$ acting the $l$-th component and $l+1$-th component. We can check immediately

$$[U'_q(\hat{g}), H] = 0.$$

When $g = sl(2)$ and $V_z$ is two-dimensional $U'_q(\hat{sl}(2))$-module, $H$ becomes $H_{XXZ}$.

From now on, we specialize $g$ to $sl(n)$ and $V_z$ to the vector representation of $U'_q$ in §3.1. Later, when we solve the difference equations for the one-point functions, we find it convenient to pass to an equivalent representation $V^pr_\zeta$ defined by

$$V = C u_1 \oplus \cdots \oplus C u_{n-1},$$

$$e_i u_j = \delta_{ij} u_{i-1} \zeta, \quad f_i u_j = \delta_{i-1j} u_i \zeta^{-1}, \quad t_i u_j = q^{\delta_{i-1j}-\delta_{ij}} u_j.$$

The equivalence is given by

$$V_z \rightarrow V^pr_\zeta, \quad v_i \mapsto u_i \zeta^{-i}, \quad z = \zeta^n.$$ We will refer to $V_z$ and $V^pr_\zeta$ as the homogeneous picture and the principal picture, respectively. As explained in the introduction, we take

$$\text{End}_C(\bigoplus_{i=0}^{n-1} V(\Lambda_i)) \cong \bigoplus_{i,j} V(\lambda_i) \otimes V(\lambda_j)^*$$
as the space of states $F$. $F$ is understood naively as the subspace of the infinite tensor product $\cdots \otimes V \otimes V \otimes \cdots$. We give the left and right action of $U$ on $F$ as follows.

$$x.f = \sum x(1) \circ f \circ S(x(2))$$

$$f.x = \sum S^{-1}(x(2)) \circ f \circ x(1)$$

where $f \in F$, $x \in U$, $\Delta(x) = \sum x(1) \otimes x(2)$.

The space $F$ regarded as the right module is denoted by $F^r$. Let

$$F_{ij} = \text{Hom}(V(\lambda_i), V(\lambda_j)) \cong V(\lambda_i) \otimes V(\lambda_j)^*.$$ $F_{ii}$ has the unique canonical element $id_{V(\Lambda_i)}$. We call it the vacuum and denote it by $|\text{vac}\rangle_i \in F_{ii}$, $i\langle \text{vac} | \in F^r_{ii}$. There is a natural inner product between $F^r_{ij}$ and $F_{ji}$ as follows.

$$\langle f | g \rangle = \frac{\text{tr}_{V(\Lambda_i)}(q^{-2\rho}fg)}{\text{tr}_{V(\Lambda_i)}(q^{-2\rho})}$$

for $f \in F^r_{ij}$, $g \in F_{ji}$, $\rho = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_{n-1}$.

It is invariant under the action of $U_q$:

$$\langle fx | g \rangle = \langle f | xg \rangle$$

for $\forall x \in U$. 

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4.2 Local structure and local operators

We use the vertex operator
\[ \tilde{\Phi}^{\Lambda_i - 1}_V(z) : V(\Lambda_i) \rightarrow V(\Lambda_i - 1) \otimes V_z \]
to incorporate the local structure into \( \mathcal{F} \).

Setting \( z = 1 \), we obtain the \( U'_q \)-homomorphism
\[ \tilde{\Phi}^{\Lambda_i - 1}_V : V(\Lambda_i) \rightarrow V(\Lambda_i - 1) \otimes V. \]

Let
\[ \tilde{\Phi}^{(m)}_{\Lambda_i} := \tilde{\Phi}^{\Lambda_i - m}_V \cdots \tilde{\Phi}^{\Lambda_i - 1}_V \tilde{\Phi}^{\Lambda_i}_V. \]

\( \tilde{\Phi}^{(m)}_{\Lambda_i} \) converges and gives the following isomorphism.
\[ \mathcal{F}_{ij} = V(\Lambda_i) \otimes V(\Lambda_j)^* \cong V(\Lambda_{i - m}) \otimes V \cdots \otimes V \otimes V(\Lambda_j)^* \]

By this isomorphism, the local structure is inserted into \( \mathcal{F} \). Next, we define the local operators. For \( L \in \text{End} V^\otimes m \), let
\[ L(i) := (\tilde{\Phi}^{(m)}_{\Lambda_i})^{-1} (id V(\Lambda_{i - m}) \otimes L)(\tilde{\Phi}^{(m)}_{\Lambda_i}). \]

By prop.3.6, we know
\[ (\tilde{\Phi}^{(m)}_{\Lambda_i})^{-1} = \left( \frac{q^2; q^{2n}}{q^{2n}; q^{2n}} \right)_{\infty}^{m} \tilde{\Phi}^{\Lambda_i}_{\Lambda_{i - m + 1}} \cdots \tilde{\Phi}^{\Lambda_i - 1}_V \tilde{\Phi}^{\Lambda_i}_V, \]

where \( \tilde{\Phi}^{\Lambda_i}_{\Lambda_{i - 1}} = \tilde{\Phi}^{\Lambda_i}_{\Lambda_{i - 1}}(1) \). The action of \( L \) on \( \mathcal{F}_{ij} \) is defined as follows.
\[ L.f := \mathcal{L}(i) \circ f. \]

We denote the correlator \( \langle \text{vac}|L|\text{vac}\rangle_i \) by \( \langle L \rangle^{(i)} \).

5 Staggered polarization

The aim of this section is to give \( \langle E_{m'm} \rangle^{(i)} \) explicitly.

5.1 Integral representations

In [9], the authors construct an integral representation of correlators of the XXZ-model by using the trace formula explained in [8] Appendix C. We can apply the same method to \( \langle E_{m'm} \rangle^{(i)} \).

Put
\[ P^m_{m'}(z_1, z_2 | x, y) := \frac{\left( q^2; q^{2n} \right)_{\infty} \text{tr}_{V(\Lambda_i)}(x^{-d} y^{\mathfrak{J}} \tilde{\Phi}^{\Lambda_i}_{\Lambda_{i - 1}} V_{m'}(z_1) \tilde{\Phi}^{\Lambda_i - 1}_V(z_2))}{\left( q^{2n}; q^{2n} \right)_{\infty} \text{tr}_{V(\Lambda_i)}(x^{-d} y^{\mathfrak{J}})}. \]
then \( \langle E_{m'm'} \rangle^{(i)} = P_{m'n}^{m'}(z, z|q^{2n}, q^{-1}|i) \).
Let

\[
\varphi(z; x) = \prod_{k=1}^{\infty} \frac{(zx^k; q^{2n})_{\infty}(q^{2n}x^{-1}z^{-1}; q^{2n})_{\infty}}{(q^2zx^k; q^{2n})_{\infty}(q^{2n+2}z^{-1}x^{-1}; q^{2n})_{\infty}}.
\]

\[
\theta_i(z_1, \ldots, z_{n-1}) = y^{(2p, 1)} \sum_{\alpha \in \mathbb{Q}} \frac{\varphi^{(\alpha)}}{(\alpha|\alpha)\int \varphi^{(\alpha)}z_1^{\langle \alpha \rangle} \cdots z_{n-1}^{\langle \alpha \rangle}.}
\]

We get the following.

\[
P_m^n(z_1, z_2|x, y|i) = c \times \delta_{m'\epsilon} \frac{\varphi(z; x)^{n-1}}{2\pi \sqrt{-1}} \frac{1}{2}^{n} \sum_{\xi_1, \ldots, \xi_{n-1}} \prod_{i=1}^{n} \left( \frac{q}{\xi_i} \right)^{1-\delta_{i0}}
\]

\[
\times \int \frac{d^2z}{q^{2} < z | < \scriptstyle t \neq m} \frac{d\xi_1 \cdots d\xi_{n-1}}{(2\pi)^{n}} \frac{1}{2} \left( \frac{q}{\xi_i} \right)^{1-\delta_{i0}}
\]

\[
\times h(w_0) \cdots h(w_{m-1}) h(zw_m) h(w_{m+1}) \cdots h(w_{n-1})
\]

\[
\times \theta_i(\eta_1, \ldots, \eta_{m-1}, \eta_{m}z^{-1}, \eta_{m+1}z, \ldots, \eta_{n-1})
\]

where \( w_0 = \frac{q^2}{\xi_1}, w_1 = \frac{q^2}{\xi_2}, \ldots, w_{n-2} = \frac{q^2}{\xi_{n-1}}, w_{n-1} = \frac{\xi_{n-1}}{q^n}, z = \frac{z_1}{z_2}, \eta_j = \frac{w_{j-1}}{w_j} \cdot y^2, c = \begin{cases} q^i & (m < i) \\ q^i & (m \geq i) \end{cases}.
\]

By this expression, we can verify that

\[
\varphi(z; x)^{-1} \frac{\partial}{\partial x} \varphi^{(z_1)} = q^{-2n} \varphi^{(z_2)}
\]

is a function of \( z(= z_1/z_2) \) and regular in \( q^{-2n} < |z| < q^{2n} \).

### 5.2 Staggered polarization

In this subsection we derive the difference equations for one point functions and solve them. These equations can be solved easily up to a pseudo-constant factor and we can determine the factor by the analyticity gained from the integral representations. Following [10], we explain how to derive the difference equations in our context.

Let

\[
\tilde{F}^{(i)}(\frac{z_1}{z_2} := \text{tr}_{(\Lambda_i)}(q^{-2p} \tilde{A}_{\Lambda_i}^{1V} (z_1) \tilde{A}_{\Lambda_i}^{1V} (z_2))
\]
then, we get the equation
\[
\hat{F}^{(i)}(z_1^2 q^{2n}) = \text{tr}_{V(\Lambda_i)}(q^{-2\rho} \tilde{\Phi}^{A_{i-1}V^*}(z_1^2 q^{2n}))
\]
\[
= P\text{tr}_{V(\Lambda_{i-1})}(q^{-2\rho} \tilde{\Phi}^{A_{i-1}V^*}(z_1^2 q^{2n})q^{-2\rho} \tilde{\Phi}^{A_{i-1}V^*}(z_1))
\]
\[
= (q^{-2\rho} \otimes 1) P\text{tr}_{\Lambda_{i-1}}(q^{-2\rho} \tilde{\Phi}^{A_{i-1}V}(z_2) \tilde{\Phi}^{A_{i-1}V}(z_1))
\]
\[
= -q(z_1 \otimes z_2)(1 \otimes q^{-2\rho}) R_{V^*V} \left( \frac{z_1}{z_2} \right)
\]
\[
\times \text{tr}_{V(\Lambda_{i-1})}(q^{-2\rho} \tilde{\Phi}^{A_{i-1}V^*}(z_1) \tilde{\Phi}^{A_{i-1}V^*}(z_2))
\]
or,
\[
\hat{F}^{(i)}(zq^{-2n}) = -qz^{-\delta_{i1}}(1 \otimes q^{-2\rho}) R_{V^*V}(z) \hat{F}^{(i-1)}(z).
\]

We show this equation reduces to scaler equations. Let
\[
\hat{F}^{(i)}(z) := \varphi(z; q^{2n})^{-1} \hat{F}^{(i)}(z),
\]
then
\[
\hat{F}^{(i)}(zq^{-2n}) = -qz^{-\delta_{i1}}(1 \otimes q^{-2\rho}) R_{V^*V}(z) \hat{F}^{(i-1)}(z).
\]

Let \( z = \zeta^n \) and \( \omega \) be an \( n \)-th primitive root of 1. We put
\[
\sum_{m=0}^{n-1} G^{(j)}(\zeta)v^*_m \otimes v_m := \sum_{i=1}^{n} q^{(n-i)i} \omega^{ij} \zeta^{n-1} \hat{F}^{(i)}(\zeta).
\]

Let further
\[
G^{(j,k)}(\zeta) := \zeta \sum_{m=0}^{n-1} \omega^{km} \zeta^{m-n} G_m^{(j)}(\zeta).
\]
Then, we find
\[
G^{(j,k)}(\zeta q^{-2}) = -q^2 \omega^j \zeta^{-1} G^{(j,k)}(\zeta).
\]
Let
\[
\Theta_p(z) = (p; p)_\infty(z; p)_\infty(z^{-1}p; p)_\infty.
\]
The reduced equation determines \( G^{(j,k)}(\zeta) \) as
\[
G^{(j,k)}(\zeta) = c_{jk}(\zeta) \frac{1 - \omega^k \zeta}{\Theta q^2(\omega^{-j} \zeta)} \quad (*).
\]
where $c_{jk}(\zeta)$ is a pseudo-constant (i.e. $c_{jk}(q^{-2}) = c_{jk}(\zeta)$). Let us show that $c_{jk}(\zeta)$ is independent of zeta. As $P(\zeta^n)$ is regular in $q^{-2} < |\zeta| < q^2$, so is $G^{(j,k)}(\zeta)$. So, when we set $\zeta = e^{2\pi \sqrt{-1}u}$ and $q = e^{\pi \sqrt{-1}r}$,

$$c_{jk}(\zeta) = G^{(j,k)}(\zeta) \frac{\Theta_{q^2}(\omega^{-j}\zeta)}{1 - \omega^k \zeta}$$

has at most a simple pole in the fundamental region $[0, 1] \times [0, \tau]$ in the $u$-plane. Hence, $c_{jk}(\zeta)$ is an absolute-constant $c_{jk}$. Moreover, it can be determined by calculating the residue at $\zeta = q^{-2}\omega^j$ of the both sides of the above equation (*). The result is the following.

$$G^{(j,k)}(\zeta) = \begin{cases} n\omega^j C \frac{1 - \omega^{-j}\zeta}{\Theta_{q^2}(\omega^{-j}\zeta)} & (j + k \equiv 0 \mod n) \\ 0 & \text{(otherwise)} \end{cases}$$

where $C = \frac{(q^2; q^2)_\infty^3 (q^{2n}; q^{2n})_\infty}{\varphi(1; q^{2n}) (q^2; q^{2n})_\infty} \text{tr}_{V(\Lambda_0)}(q^{-2\rho})$.

We get the following theorem.

**Theorem 5.2** Let $\omega$ be an n-th primitive root of 1 and $E_{ij}$ be the matrix unit, then

$$\sum_{m=0}^{n-1} \omega^{km} \langle E_{mm} \rangle^{(i)} = \frac{\omega^{(i-1)k} (q^2; q^2)_\infty^{i}}{(q^2\omega^k; q^2)_\infty (q^2\omega^{-k}; q^2)_\infty} .$$

\[ \square \]

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