Suppressing decoherence and improving entanglement by quantum-jump-based feedback control in two-level systems

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We study the quantum-jump-based feedback control on the entanglement shared between two qubits with one of them subject to decoherence, while the other qubit is under the control. This situation is very relevant to a quantum system consisting of nuclear and electron spins in solid states. The possibility to prolong the coherence time of the dissipative qubit is also explored. Numerical simulations show that the quantum-jump-based feedback control can improve the entanglement between the qubits and prolong the coherence time for the qubit subject directly to decoherence.

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I. INTRODUCTION

Superposition of states and entanglement make quantum information processing much different from its classical counterpart. But a quantum state would unavoidably interact with its environment, resulting in a degradation of coherence and entanglement. For example, spontaneous emission in atomic qubits [1] would spoil the coherence of quantum states and limit the entanglement time.

Recent experimental advances have enabled individual systems to be monitored and manipulated at quantum level [2]. This makes the quantum feedback control realizable. Among the feedback controls, The homodyne-mediated feedback [3, 4] and quantum-jump-based feedback controls have been proposed to generate steady state entanglement in a cavity [5, 6]. These two feedback schemes are Markovian, namely, a feedback information proportional to the quantum-jump detection is synchronously used. Besides, these control scheme can also be used to suppress decoherence [7–10].

Meanwhile, researchers are looking for proper systems for experimental implementation of quantum information processing. Among the various candidates, solid-states quantum devices based on superconductors [11] and lateral quantum dots [12] are promising ones, however, the decoherence from intrinsic noise originating from two-level fluctuatons is hard to engineered [13]. For this reason, the nuclear spins have attracted considerable attention [14] due to its long coherence times [15]. But their weak interactions to others make the preparation, control, and detection on them difficult. Thanks to its intrinsic interactions with electron spins, electron spin can be used as an ancilla to access single nuclear spin. This naturally leads us to rise the following question: can feedback strategy be used to suppress decoherence, prepare and protect entanglement between the nuclear and electron spins by controlling the electron spin? In this paper, we will study this problem by considering a nuclear spin (as a qubit) coupled to electron spin (as the other qubit) that is exposed to its environment. We show that a Markovian feedback based on quantum-jump can be used to suppress decoherence, produce entanglement and protect it.

The paper is organized as follows: In Sec.II, we describe our model and present the dynamics in absence of feedback. In Sec.III, we introduce the quantum-jump-based feedback and give the dynamical equation under the feedback control. The effect of feedback control on decoherence and entanglement is discussed in Sec.IV and Sec.V, respectively. Sec.VI concludes our results.

II. MODEL

Our system consists of a pair of two-level systems, called qubit 1 and qubit 2, where only the qubit 2 interacts with its environment. We present a scheme employing quantum-jump-based feedback control on the qubit 2 to affect the decoherence of the qubit 1 and increase entanglement between the two qubits. The Hamiltonian of the system reads

\[ H = \frac{1}{2} \hbar \omega_1 \sigma_1^z + \frac{1}{2} \hbar \omega_2 \sigma_2^z + \hbar g (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+). \] (1)

The first two terms represent the free Hamiltonian of the two qubits, the last term describes their interactions under the rotating-wave approximation. \( \omega_1 \) and \( \omega_2 \) are the transition frequency of the two qubits, respectively. \( g \) is the coupling strength of the two qubits. \( \sigma_z \) is the Pauli matrix, i.e., \( \sigma_z = |e\rangle \langle e| - |g\rangle \langle g| \), and \( \sigma^+ = |e\rangle \langle g| \), \( \sigma^- = |g\rangle \langle e| \).

The state of this quantum system can be described by the density operator \( \rho \) which is obtained by tracing out the environment. The dynamics of open quantum systems can be described by quantum master equations. The most general form of master equation for the density operator is [16, 17]

\[ \dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \mathcal{L}(\rho), \] (2)

where \( H \) is the system Hamiltonian and \( \mathcal{L} \) is a superoperator defined by \( \mathcal{L}(\rho) = \sum_k \gamma_k (L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k) \).
\[ \frac{1}{\hbar} \rho L_k^\dagger L_k \], in which different \( k \) characterizes different dissipative channels.

In our system, the first qubit is assumed to be isolated from environment. The decoherence comes from the spontaneous emission of the qubit 2 (the second qubit). This situation is of relevance to a system consisting of nuclear and electron spins in aforementioned solid state devices. The dynamics of such a system takes

\[ \dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \gamma (\sigma_2^+ \rho \sigma_2^- - 2 \sigma_2^+ \sigma_2^- \rho - \frac{1}{2} \rho \sigma_2^+ \sigma_2^-). \quad (3) \]

Here \( \sigma_2^\pm = I_1 \otimes \sigma_2^\pm \). The second part of Eq. (3) describes the dissipation of our system with \( \gamma \) the decay rate.

Though the first qubit is assumed to be isolated from environment, it still loss coherence due to the coupling to the second qubit. The decoherence process can be showed by the decay of off-diagonal elements of the reduced density matrix for the first qubit.

In order to investigate this decoherence, we calculate the evolution of system density operator \( \rho \) and then trace out the second qubit to get the reduced matrix

\[ \rho_1 = \text{Tr}_2(\rho) = \sum_{k=\text{e,g}} 2 \langle k | \rho | k \rangle_2 = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}. \quad (4) \]

The diagonal elements are the populations in the excited and ground states of the first qubit. And the off-diagonal elements represent the coherence of the qubit 1.

### III. QUANTUM-JUMP-BASED FEEDBACK CONTROL

Quantum feedback controls play an increasingly important role in quantum information processing. It is widely used to create and stabilize entanglement as well as combat with decoherence [6,8,11]. In our model, the second qubit is used as an ancilla through which the feedback can affect the dynamics of the first qubit, i.e., by employing a feedback control on the second qubit, we control the first qubit. The goal is to suppress the decoherence of the first qubit and enhance the entanglement between the two qubits by a feedback control on the second qubit [6].

Our feedback control strategy is based on quantum-jump detection. The master equation with feedback can be derived from the general measurement theory [4]. In our paper, Eq. (3) is equivalent to the general master equation

\[ \dot{\rho}(t + dt) = \sum_{\alpha=0,1} \Omega_\alpha(T) \rho(t) \Omega_\alpha^\dagger(T). \quad (5) \]

with

\[ \Omega_1(dt) = \sqrt{\gamma dt} \sigma_2^- \]

\[ \Omega_0 = 1 - \frac{i}{\hbar} (H + \frac{1}{2} \gamma \sigma_2^+ \sigma_2^-) dt. \quad (6) \]

When the measurement result is \( \alpha = 1 \), a detection occurs, which causes a finite evolution in the system via \( \Omega_1(dt) \). This is called a quantum jump. Then the un-normalized density matrix becomes \( \tilde{\rho}_{\alpha=1} = \sigma_2^+ \rho(t) \sigma_2^- dt \). The feedback control is added by giving \( \tilde{\rho}_{\alpha=1} \) a finite unitary evolution, then \( \tilde{\rho}_{\alpha=1} \) become \( \dot{\rho}_{\alpha=1} = F \sigma_2^+ \rho(t) \sigma_2^- F^\dagger dt \). In the limit that the feedback acts immediately after a detection and in a very short time (much smaller than the time scale of the system’s evolution), the master equation is Markovian,

\[ \dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \gamma (F \sigma_2^- \rho \sigma_2^+ F^\dagger - \frac{1}{2} \sigma_2^+ \sigma_2^- \rho - \frac{1}{2} \rho \sigma_2^+ \sigma_2^-). \quad (8) \]

Here \( F = e^{iH_f} \) and \( H_f = -\frac{1}{\hbar} H_f t_f \). We see that the operator \( H_f \) contains a relatively large operator \( H_f \) multiplied by a very short time \( t_f \) (Markovian assumption), but the product represents a certain amount of evolution, so it is convenient to discuss \( H_f \) instead of \( H_f^f \) and \( t_f \). Here \( H_f \) is a 2 × 2 hermitian operator which can be decomposed by Pauli matrices \( A_f = A_x \sigma_x + A_y \sigma_y + A_z \sigma_z \) (\( A_x, A_y, A_z \) are real numbers). So we have

\[ F = I_1 \otimes e^{i\vec{A} \cdot \vec{\sigma}} = I_1 \otimes (|\vec{A}| + i \sin |\vec{A}| \vec{A} \cdot \vec{\sigma}). \quad (9) \]

Here \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) and \( \vec{A} = (A_x, A_y, A_z) \) representing the amplitude of \( \sigma_x, \sigma_y, \sigma_z \) control.

In order to understand the physical meaning of feedback operator \( F \), we rewrite it as \( F = I_1 \otimes e^{-i\vec{n} \cdot \vec{\sigma}} \) where \( \vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) and \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \), this feedback operator is equivalent to a time-evolution with evolution operator \( F = I_1 \otimes e^{iH_f} \). And it is clear that the operator \( F \) rotate the Bloch vector of the second qubit with the angle \( \omega \) around the \( \vec{n} \) axis. The relationship between the two forms of \( F \) are \( A_x = -\frac{\omega}{\sin \theta} \sin \phi A_y = -\frac{\omega}{\sin \theta} \sin \phi A_z = -\frac{\omega}{\sin \theta} \cos \theta \). So a \( \sigma_z \) control (\( A_y = 0, A_z = 0 \)) means rotating the Bloch vector with a certain amount of angle around the \( x \) axis of Bloch sphere, so does the \( A_x \) and \( A_x \) control. Different \( \vec{A} \) represents different feedback evolution i.e., rotate the Bloch vector with a particular angle around a particular direction in the Bloch sphere. For simplicity, we discuss the \( \sigma_x, \sigma_y, \sigma_z \) control one by one in the following.

This control mechanism has the advantage of being simple to apply in practice, since it does not need real time state estimation as Bayesian feedback control does[13]. The emission of the second qubit is measured by a photo detector, whose signal provides the information to design the control \( F \). In this kind of monitoring, the absence of signal predominates the dynamics and the control is triggered only after a detection click, i.e. a quantum jump, occurs.

### IV. DECOHERENCE SUPPRESSION

Before investigating the influence of the feedback control, we first analyze the evolution of our system with-
out control. Assume that the two qubits are initially in the same pure superposition state, for example, $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle_1 + |g\rangle_1) \otimes \frac{1}{\sqrt{2}}(|e\rangle_2 + |g\rangle_2)$. The corresponding density matrix is,

$$\rho_0 = |\psi\rangle\langle\psi| = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{10}$$

We assign the Planck constant $\hbar$ to be 1, $\omega_1 = \omega_2 = \omega$ in Eq. (1), and $g/\omega = 1, \gamma/\omega = 0.5$. After numerical calculation, we get the evolution of the density matrix for the first qubit without control. Since $\rho_{eg} = \rho_{ge}^*$, $\rho_{ec} + \rho_{ge} = 1$, we only discuss coherence $|\rho_{eg}|$ and excited state population $\rho_{ee}$ for simplicity. The evolution of $|\rho_{eg}|$ and $\rho_{ee}$ without control is depicted in Fig. 1 (a) and (b) (dashed lines).

In Fig. 1 (a), a fast decay of $|\rho_{eg}|$ (dashed line) can be found. This demonstrates that the first qubit lost coherence due to the second qubit’s spontaneous emission and their interaction. Meanwhile, the first qubit lost energy due to couplings to the second qubit (Fig. 1 (b) (dashed line)). The results also show that the populations in excited state and ground state decay away. This is because the first qubit exchange energy with the second qubit, see Eq. (1).

Now we add feedback control $F$ to our system, the master equation then becomes Eq. (8). Our system is initially in the state $\rho_0$, other parameters remain unchanged.

We first analyze the $\sigma_x$ control by choosing feedback amplitude $A_x = 0 \sim \pi, A_y = A_z = 0$. Note that when $A_y = A_z = 0$, the feedback amplitude $A_x$ influence the system’s evolution with a period of $\pi$ which comes from the term $F\rho_0 F^\dagger$ in Eq. (8). It can be analytically proved that $e^{iA_x \sigma_x - \rho_{2g}^* e^{-iA_x \sigma_x}} = e^{i(A_x + \pi) \sigma_x} - e^{i(A_x + \pi) \sigma_x}$ and $e^{iA_y \sigma_y - \rho_{2g}^* e^{-iA_y \sigma_y}} = e^{i(A_y + \pi) \sigma_y} - e^{i(A_y + \pi) \sigma_y}$ where $A_y$ and $A_z$. Here $p_2$ is the reduced density matrix of the second qubit. The absolute value for the first qubit’s off-diagonal density matrix element evolves as showed in Fig. 2 (a). The figure indicates that, for an appropriate feedback amplitude, $A_x \approx 1.3$ and $A_y \approx 1.9$, the absolute value of off-diagonal element can be evidently enhanced compared with the uncontrolled case ($A_x = 0$). That means the decoherence is partially suppressed. The improvement of coherence caused by feedback is shown explicitly in Fig. 1 (a). We plot $|\rho_{eg}|$, representing the coherence of the first qubit, as a function of time with $A_x = 1.2, A_y = A_z = 0$ (a selected controlled case). In comparison with the uncontrolled case, a stronger oscillation amplitude and longer decoherence time appears. Meanwhile, the $\rho_{ee}$ decays slowly compared to the uncontrolled case as shown in Fig. 2 (b).

Similarly, the $\sigma_y$ control is also able to slow down the decay of $|\rho_{eg}|$. We make $A_y = 0 \sim \pi, A_x = A_z = 0$. The numerical results of $|\rho_{eg}|$ is shown in Fig. 2 (b). Unlike the $\sigma_x$ and $\sigma_z$ control, the $\sigma_y$ control ($A_z = 0 \sim \pi, A_x = A_y = 0$) has no effect on the evolution of the system as shown in Fig. 2 (c). This is because $e^{iA_y \sigma_y - \rho_{2g}^* e^{-iA_y \sigma_y}} = p_2$ for any $A_z$. The physics behind this result is that after emitting a photon, the controlled qubit must stay in the ground state with the Bloch vector pointing the bottom of the Bloch sphere, so the rotation around z axis does not change the Bloch vector, i.e., the state of the qubit remains unchanged.

The present results show that decoherence of the first qubit can be suppressed by controlling its partner. The decoherence source in our system is the spontaneous emission of the second qubit, once the detector detects a photon, i.e., a quantum jump of the second qubit happens, the feedback beam instantaneously act on the second qubit and then the first qubit is impacted through the coupling of the two qubit. The feedback control scheme can reduce the destructive effects of coherence and slow down the dissipation of energy.

The control effect is relevant to the coupling strength $g$. When $g$ is small, the first qubit is hard to be impacted by the second qubit, so it’s hard to prepare, measure and control the state of the first qubit. As the interaction goes stronger, the effect of feedback control becomes more evident.

For the case discussed in Fig. 1 the first qubit is dissipative. We found that when the control parameters is chosen as: $A_x = \frac{\pi}{2}, A_y = A_z = 0$, or $A_y = \frac{\pi}{2}, A_x = \frac{\pi}{2}$.
$P$ vector components

g/ω
diagonal element with different control parameters, for

FIG. 2: The evolution of absolute value of the first qubit’s off-

A

x

π,A

σ
diagonal element is remarkably enhanced. The

A

1.3 and 1.9 for both

σ
doesn’t work in our model.

FIG. 3: Polarization vector evolution in a bloch sphere

for feedback amplitude $A_x = \frac{\pi}{2}, A_y = A_z = 0$ (solid line) and

$A_y = A_x = A_y = 0$ (dashed line). The parameters

are $g/\omega = 1, \gamma/\omega = 0.5$, the initial state is $|\psi\rangle = \frac{1}{\sqrt{2}}(|e\rangle_1 + |g\rangle_1) \otimes \frac{1}{\sqrt{2}}(|e\rangle_2 + |g\rangle_2)$.

V. ENTANGLEMENT CONTROL

Quantum feedback control has been recently used to

improve the creation of steady state entanglement in

open quantum systems. A highly entangled states of two

qubits in a cavity can be produced with an appropriate

selection of the feedback Hamiltonian and detection

strategy [6, 20]. We will show that the quantum-jump-

based feedback scheme can produce and improve entan-

glement in our model. We choose the concurrence

$C$ as a measure of entanglement. For a mixed state repre-

sented by the density matrix $\rho$ the "spin-flipped" density

operator reads

$\rho = (\sigma_y \otimes \sigma_y) \rho (\sigma_y \otimes \sigma_y)^{*}$ (11)

where $*$ denotes complex conjugate of $\rho$ in the bases

of $\{|gg\rangle, |ge\rangle, |eg\rangle, |ee\rangle\}$, and $\sigma_y$ is the usual Pauli matrix. The

concurrence of the density matrix $\rho$ is defined as

$C(\rho) = \max \left(\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\right)$. (12)

where $\lambda_i$ are eigenvalues of matrix $\rho \rho^*$ and sorted in

decreasing order $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. The range of concurrence is from

0 to 1, and $C = 1$ represents the maximum

etanglement.

In absence of spontaneous emission, i.e. $\gamma = 0$, the

system evolves without dissipation. We find that for the

system initially in a separable state except $|\psi\rangle = |e\rangle_1 |e\rangle_2$ or

$|\psi\rangle = |g\rangle_1 |g\rangle_2$ (the eigenstates of system Hamiltonian $H$), an entangled state can be generated due to the interaction between the two qubits. The amount of entan-
glement depends on initial states of the system and the
coupling strength $g$. But when the spontaneous emission effect is taken into account, the performance of entanglement preparation get worse considerably.

Now we investigate if our feedback control strategy can improve the entanglement preparation with the effect of spontaneous emission of the second qubit. The master equation with control is Eq. (8). The effect of feedback control lies in different choices for the feedback parameters $A_x, A_y, A_z$, the coupling strength $g$ and different initial states. Here we present two typical results with two different states.

Our first choice is the initial state $|\psi\rangle = |g\rangle_1|e\rangle_2$ with $\sigma_y$ control for $A_y = 0 \sim \pi, A_x = 0, A_z = 0$. The concurrence evolution is plotted as a function of time and feedback amplitude $A_y$ in Fig.4 (a), and Fig.4 (b) denotes the concurrence evolution with a selected feedback amplitude compared with the uncontrolled case. We see that entangled states can be generated with any feedback parameters, but it decreases with time because of the dissipative effect. When an appropriate feedback amplitude $A_y \approx 0.9$ is chosen, the concurrence amplitude is remarkably enhanced, and the entanglement lasts for a long time. For the system initially in the state $|\psi\rangle = |e\rangle_1|e\rangle_2$ with $\sigma_y$ control, the dynamics of the concurrence is shown in Fig.4 (a). Note that in this case if there is no spontaneous effect, this is a steady state of the system, the density matrix elements does not change

![Figure 4](image1.png)

**FIG. 4:** (a) Concurrence as a function of time and $A_y$. The system is initially in the state $|\psi\rangle = |g\rangle_1|e\rangle_2$, for the parameters $g/\omega = 1, \gamma/\omega = 0.5$. (b) A controlled evolution for $A_y = 0.5\pi, A_x = A_z = 0$ vs. uncontrolled case. The entanglement is improved by choosing an appropriate feedback. $t$ is in the unit of $\frac{1}{\omega}$ for (a) and (b).

with time. Fig.5 (a) demonstrates that the dissipation and feedback can produce entanglement. We show this explicitly in Fig.5 (b) by choosing feedback amplitude $A_y = 1.2$. We can see that for a proper feedback amplitude, after an entanglement death, a larger amount entanglement is regenerated.

The above results shows the feedback control strategy can be used to prepare and protect entanglement in our model. The effect of entanglement control strongly depends on the initial state. For a certain initial state, we found that the $\sigma_z$ control and $\sigma_y$ control has the similar effect but the $\sigma_z$ control does not work.

**VI. CONCLUSION AND REMARKS**

In this paper, we studied the effect of quantum jump based feedback control on a system consisting of two qubits where only one of them subject to decoherence. By numerical simulation, we found that it is possible to suppress decoherence of the first qubits by a local control on the second qubits. We observed that the decoherence time of the first qubit is increased remarkably. The control scheme can also used to protect the entanglement between the two qubits. These features can be understood as that the feedback control changes the dissipative dynamics of the system through the quantum-jump operators. We would like to note that Hamiltonian Eq. (1) does not describe the hyperfine interaction. However, by the recent technology we can simulate Hamiltonian Eq. (1) in
nuclear-electron spin systems, in this sense, the scheme presented here is available for nuclear-electron spin systems. On the other hand, by using the hyperfine interaction Hamiltonian, our further simulations show that we can obtain results similar to that for Hamiltonian Eq.[1].

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