1. Introduction

Phase transitions are often described in terms of an order parameter, which in the electroweak theory is the background Higgs field. The states with lowest energy are conveniently analyzed with the help of the effective potential, which has lately been the subject of intensive studies in connection with baryogenesis in the electroweak theory [1]. It is however rather obvious that to investigate the dynamics of the order parameter in detail one should use the complete effective action, which includes also terms depending on the derivatives of the background field. Although the ground state has a constant value of the background field, locally there are fluctuations about this value, and they might turn out to be important for local phenomena such as bubble nucleation in first order phase transitions.

To study the evolution of background fields one should solve the equation of motion for the fluctuating background as derived from the effective action, including possible dissipation. As the full effective action is an object that is very difficult to estimate, one might resort to a derivative expansion of the effective action, cut it off after the first few terms and solve the resulting equation of motion. This approach does not always work at finite temperature because the derivative expansion does not necessarily exist [2] in the neighborhood of the origin of the \((p_0, p)\)-plane.

If the background field is assumed to be close to a constant value so that we may consider just small space–time dependent perturbations about that constant field, it is possible to expand the effective action in terms of the perturbation while retaining all the derivatives. In this approach one is able to see not only the dispersion of the fluctuation but also which fields, or modes of the field, are unstable and what are their actual decay rates. In what follows we shall adopt this approach for \(\lambda \phi^4\)–theories. We shall compute the effective action in this approximation up to two loops and study the evolution and stability of the fluctuations, comparing eventually the decay rates with the Hubble rate to demonstrate how a realistic system might actually behave.
The high temperature limit of the effective action of the \( \lambda \phi^4 \)–model has recently been considered at one–loop level by Moss et al. \cite{3} (see also \cite{4}). There are several reasons for reconsidering their work. First we want an independent check of their results. They used an approach based on the local-momentum space method for curved spacetime introduced by Bunch and Parker \cite{5}, while we calculate it in a much simpler way through Feynman diagrams. Secondly, in \cite{3} the effective action was expanded in powers of derivatives about a constant field. It is known \cite{2} that the finite temperature correction to the two-point function is non-analytic at \( p_\mu = 0 \), having different limits when, for instance, \( \{ p_0 = 0, \mathbf{p} \to 0 \} \) and \( \{ p_0 \to 0, \mathbf{p} = 0 \} \). It is, therefore, in general not meaningful to expand about the origin, but only in fixed directions in the \(( p_0, \mathbf{p})\)–plane. These expansions can be used to analyze e.g. space or time independent solutions. Recalling that the variation of the effective action gives an equation of motion for \( \langle \phi(x) \rangle \) we shall estimate the Fourier spectrum of a solution to the equation of motion. As an example we might consider fluctuations about the minimum of the effective potential and therefore shift the field by a constant to get to the minimum. Then the deviation from the constant field satisfies the on-shell condition \( p_0^2 = \mathbf{p}^2 + m^2 \), so for a spatially almost constant field we could expand around \( \{ p_0 = m, \mathbf{p} = 0 \} \). Here we have neglected corrections from interactions and quantum fluctuations but they have to be small in order for perturbation theory to work. If we were interested in other solutions than plane waves (such as solitons) we should study the size of the derivatives anew.

A third reason for studying the \( \lambda \phi^4 \)–theory anew is the existence of a two–loop diagram of order \( \lambda^2 \) which dominates over the one-loop diagram at high temperature and when the constant part of the background field, \( \sigma_c \), is close to zero. The one–loop tadpole in the \( \lambda \phi^4 \)–theory, depicted in Fig. 1, goes like \( \lambda T^2 \) for high \( T \) but it is momentum independent so that the only one–loop correction to the kinetic term comes from the diagram in Fig. 2. It has a high \( T \) behavior like \( \lambda^2 \sigma_c^2 T \). A naive estimate of the \( T \) dependence of the two–loop diagram in Fig. 3 is \( \lambda^2 T^2 \) (though we shall see in Sect. 3.2 that the mass correction goes like \( \lambda^2 T^2 \ln T \)) so it dominates for large enough \( T \). Also, it turns out that an
on-shell imaginary part of the effective action, which is important for studying the evolution of perturbations, first arises at two loops.

The paper is organized as follows. In Section 2 we present the derivative expansion and the generalized tadpole method for computing the effective action. In Section 3 we evaluate the effective action at one and two loop orders for small non–constant fluctuations about a constant background. We also comment on the existence of a derivative expansion of the effective action. In Section 4 we discuss the dispersion relations and the decay rates for the fluctuating modes. We show that although most fluctuations will decay during the course of the cosmic expansion, some will still be present at the phase transition. Section 5 contains our conclusions and a discussion of some open problems.

2. Methods for computing effective action

2.1 The derivative expansion

The effective action $\Gamma[\phi]$ is the Legendre transformation of the generating functional $W[J]$

$$e^{iW[J]} = \mathcal{N} \int D\Phi e^{iS[\Phi] + i \int d^4x J(x)\Phi(x)} ,$$  \hspace{1cm} (1)

where $S[\Phi]$ is the tree level action of the quantum field $\Phi$, and $\mathcal{N}$ is a normalization constant. Thus the effective action of the classical background field $\phi$ is defined by

$$\Gamma[\phi] = W[J] - \int d^4x \frac{\delta W[J]}{\delta J(x)} J(x) ,$$  \hspace{1cm} (2)

where $J$ has been eliminated using the definition of classical field $\phi(x) = \frac{\delta W[J]}{\delta J(x)}$, assumed to be invertible. From a calculational point of view it is often better to express the effective action by an equivalent functional integral form

$$e^{i\Gamma[\phi]} = \mathcal{N} \int D\Phi e^{iS[\Phi + \phi] + i \int d^4x J(\phi)\Phi} ,$$  \hspace{1cm} (3)
where now $J(\phi) = -\frac{\delta \Gamma}{\delta \phi}$, as can be directly verified from the definition of effective action. One can then expand the effective action $\Gamma[\phi]$ about an arbitrary field $\sigma(x)$:

$$\Gamma[\phi(x)] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4 x \Gamma^{(n)}[\sigma; x_i] \prod_i [\phi(x_i) - \sigma(x_i)] ,$$

where the Taylor coefficients of the expansion are given by

$$\Gamma^{(n)}[\sigma; x_i] = \frac{\delta^n \Gamma}{\delta \phi(x_1) \ldots \delta \phi(x_n)} \bigg|_{\phi=\sigma} ,$$

which are 1PI Green functions with a background field $\sigma$. In general $\Gamma^{(n)}[\sigma; x_i]$ is a functional of $\sigma(x)$ and a non–local function of $x_i$. If we want to write it as a local function we need to expand in the derivatives of the fields as well. We should however remember that the effective action is genuinely non-local both in time and space (e.g. through logarithmic terms from loop corrections).

Let us assume for the moment that the effective action can be written as a derivative expansion of the classical field $\phi$, which is equivalent to the statement that the Green functions are analytical at the origin, when expressed in momentum space. Then

$$\Gamma[\phi(x)] = \sum_{k=0}^{\infty} \int d^4 x \Gamma_{2k}(\phi, \partial \phi, \Box \phi, \ldots) ,$$

where each term $\Gamma_{2k}[\phi, \partial \phi, \Box \phi, \ldots]$ is of the order $2k$ with respect the partial derivative operators. $\Gamma_2$ can always be written in the unique form $\Gamma_2 = \frac{1}{2} Z[\phi] \partial_\mu \phi \partial^\mu \phi$; all functionals with two derivatives can always be cast into this form by partial integration [4] (note that $\Gamma_2$ cannot in general be written as $\frac{1}{2} Z(\phi) \phi \Box \phi$). We may thus write down the effective action to second order in derivatives by noting that by definition $Z$ and $V_{eff} \equiv -\Gamma^{(0)}[\phi]$ do not depend on derivatives, and thus their functional forms do not depend on whether $\phi$ is a constant or not. The easiest way to compute them is therefore to consider a constant background field $\sigma_c$ and afterwards replace $\sigma_c \to \phi$ in $V_{eff}[\sigma_c]$ and $Z[\sigma_c]$.

It is possible to follow a similar procedure for an arbitrary $k$ (and also for fields other than scalars). One must first find a basis, in which it is possible to express all the terms containing $k$ derivatives in a unique fashion.
For example, in the case $k = 4$, we find that the basis is formed by the set 
\{\partial_\mu \phi \partial^\mu \phi \partial_\nu \phi \partial^\nu \phi, \partial_\mu \phi \partial^\mu \phi \partial_\nu \phi \partial_\nu \phi, \partial_\mu \partial_\nu \phi \partial^\nu \phi \partial^\nu \phi\}. \text{ Thus } \Gamma_4 \text{ is uniquely represented by the linear combination}

$$
\Gamma_4 = \Gamma_{4,1}[\phi] \partial_\mu \phi \partial^\mu \phi \partial_\nu \phi \partial^\nu \phi + \Gamma_{4,2}[\phi] \partial_\mu \phi \partial^\mu \partial_\nu \phi \partial_\nu \phi + \Gamma_{4,3}[\phi] \partial_\mu \partial_\nu \phi \partial^\nu \phi \partial^\nu \phi . \quad (7)
$$

Clearly, different physical problems warrant different expansions. The derivative expansion is suitable for e.g. slowly varying background fields (but possibly with large amplitudes), whereas modes that approximately obey the on–shell condition are best described in terms of an expansion in small perturbations.

### 2.2 The tadpole method

A frequently used method for computing the effective potential is to first calculate the tadpole of the theory where the field $\phi$ has been shifted by a constant, $\sigma_c$, and then integrate it with respect to $\sigma_c$ [6]. In this way an infinite class of diagrams is resummed. The reason is that the mass gets a contribution from shifting the field and that contribution can be resummed to all orders using the effective $\sigma_c$–dependent mass. We shall here extend the tadpole method to the whole effective action.

Suppose the effective action is expanded about some field $\sigma(x)$ (we consider only a real scalar field for simplicity, but the method can easily be extended to other fields). Since $\Gamma[\phi(x)]$, given by Eq. (4), is independent of the expansion point $\sigma(x)$ we can take the functional derivative of Eq. (4) with respect to $\sigma(y)$ to obtain

$$
\sum_{n=0}^{\infty} \int d^4 x_i \left\{ \frac{\delta \Gamma^{(n)}[\sigma(x); x_i]}{\delta \sigma(y)} \prod_{i=1}^{n} (\phi(x_i) - \sigma(x_i)) - \Gamma^{(n)}[\sigma(x); x_i] \sum_{j=1}^{n} \delta(x_j - y) \prod_{i \neq j} (\phi(x_i) - \sigma(x_i)) \right\} = 0 . \quad (8)
$$

If we put $\phi(y) = \sigma(y)$ in Eq. (8) and use $\Gamma[\sigma(x)] = \Gamma^{(0)}[\sigma(x)]$ we get

$$
\frac{\delta \Gamma[\sigma(x)]}{\delta \sigma(y)} = \Gamma^{(1)}[\sigma(x); y] , \quad (9)
$$
which is the tadpole equation for the effective action. The right hand side of Eq. (9) is just the tadpole for a theory where the field has been shifted by a non-constant field $\sigma(x)$.

To see how the calculations work out in practise let us consider the Lagrangian

$$L = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right) - \frac{\lambda}{4!} \phi^4 .$$

(10)

After shifting the field to $\phi(x) + \sigma(x)$ we get effectively a space–time dependent mass term that cannot be dealt with exactly. This means also that we cannot resum the same infinite set of diagrams as for the effective potential. To proceed we have to consider a perturbation series in the non-constant part of $\sigma(x)$. Thus we write

$$\sigma(x) = \sigma_c + b(x) ,$$

(11)

where $\sigma_c$ is constant and $b(x)$ is small. Higher orders are not important when the fluctuation energy is dominated by the non–interacting part or when $b^2 \lesssim (p^2 + m^2)/\lambda$, where at finite temperature one should replace $m$ by the plasma mass $m(T)$. (When there is no risk of confusion, we shall denote the Fourier transform of $b(x)$ simply by $b(p)$).

The Lagrangian for the shifted field is

$$L(\phi(x) + \sigma(x)) = L(\sigma(x)) + \frac{\delta \Gamma_{cl}}{\delta \sigma(x)} \phi(x) + \frac{1}{2} \left( (\partial \phi(x))^2 - (m^2 + \frac{\lambda^2 \sigma_c}{2}) \phi^2(x) \right)$$

$$- \frac{\lambda}{2} (\sigma_c b(x) + \frac{b^2(x)}{2}) \phi^2(x) - \frac{\lambda (\sigma_c + b(x))}{3!} \phi^3(x) - \frac{\lambda}{4!} \phi^4(x) ,$$

(12)

where $\Gamma_{cl}[\sigma(x)]$ is the classical action for the Langrangian in Eq. (10). Because of the space dependence in $b(x)$ there are some new Feynman rules that do not conserve momentum. The rules are given in Fig. 4.

Note that the Fourier transform $(b^2)(q)$ is not equal to $(b(q))^2$ but is the Fourier transform of $b(x)^2$. If we can choose $\int d^4xb(x) = 0$, then $b(x)$ only includes symmetric fluctuations around a constant field. Then we still have $(b^2)(q = 0) = \int d^4xb^2(x) > 0$ if $b(x)$ is not identically zero but measures the average size of the fluctuation. In a diagram with all the external momenta equal to zero (i.e. for the effective potential) there still remains a contribution from $(b^2)(q = 0)$. 
Let us now take a look at the structure of the perturbative expansion of the effective action. We can consider an expansion in $\lambda$, $\beta$ and $\bar{h}$, but since we are interested in quantum effects we set the loop expansion parameter $\bar{h}$ equal to one. Higher loops are also suppressed by higher powers of $\lambda$. The tree-level tadpole just gives the classical action. At one-loop and to $\mathcal{O}(b)$ we have the diagrams in Fig. 5. The first diagram can be integrated with respect to $\sigma_c$ and we get the usual one-loop effective potential $V_{\text{eff}}(\sigma_c)$. If we expand the vertex function in $p$ in the second and third diagrams we have to zeroth order in $p$ and first order in $b$

\[ i\Gamma_{1\text{-loop}}^{(1)}[\sigma(x)] - i\Gamma_{1\text{-loop}}^{(1)}[\sigma_c] \simeq \int \frac{d^4k}{(2\pi)^4} \left( \frac{1}{2} \frac{\lambda}{k^2 - M^2} + \frac{1}{2} \frac{\lambda^2 \sigma_c^2}{(k^2 - M^2)^2} \right) b(-p) \]

\[ = \frac{1}{2} \frac{\partial}{\partial \sigma_c} \int \frac{d^4k}{(2\pi)^4} \frac{\lambda \sigma_c}{k^2 - M^2} b(-p) = -i \frac{\partial^2}{\partial \sigma_c^2} V_{\text{eff}}(\sigma_c) b(-p) , \]

where

\[ M^2 = m^2 + \lambda \sigma_c^2 / 2 . \]

This is what we get if we first replace $\sigma_c$ by $\sigma_c + b(x)$ in $\partial V_{\text{eff}}(\sigma_c)/\partial \sigma_c$ and then expand to first order in $b$. We can easily understand this from the derivative expansion of the effective action, as was discussed in Sect. 2.1.

The tadpole diagrams that are linear in $b(p)$ are, of course, essentially the two-point functions for the theory with shifted $\sigma_c$. In the following we calculate instead $\Gamma^{(2)}$ directly and one can construct the effective action from Eq. (4). The effective action in momentum space to $\mathcal{O}(b^2)$ is

\[ \Gamma[\sigma_c, b] = -V_{\text{eff}}(\sigma_c) + \Gamma^{(1)}(\sigma_c) b(k = 0) + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} b(-k) \Gamma^{(2)}(\sigma_c; k) b(k) , \]

and $\Gamma^{(1)}$ is zero at the minimum of $V_{\text{eff}}$.

### 3. Finite temperature effective action

There are at least two formalisms for calculating vertex functions at finite temperature, the imaginary and real time formalisms (ITF and RTF). They should
both give the same result when used correctly. In ITF there is a naive way of extracting the leading high $T$ behavior by only including the mode with zero Matsubara frequency ($n=0$) (for bosons). For the diagrams under consideration here this method is not reliable. The correct method should include a summation over all Matsubara frequencies and an analytic continuation at the end. RTF is easy to use for one-loop diagrams but for the two-loop diagrams we have employed instead ITF with the convenient method developed in [7–9].

3.1 One-loop diagrams

Let us start by giving the well known result for the zero temperature part. The effective potential can be found in [10] and we repeat it here for completeness:

$$V_{\text{eff}}(\sigma_c) = \frac{\sigma_c^2}{2} m^2 + \frac{\sigma_c^4}{4!} \lambda + \frac{M^4}{64\pi^2} \ln \left( \frac{M^2}{m^2} \right) + A + B\sigma_c^2 + C\sigma_c^4 .$$

(16)

The renormalized two-point function is (see Fig. 2)

$$\Gamma^{(2)}_{1\text{-loop}}[\sigma(x)] = \frac{\lambda^2\sigma_c^2}{32\pi^2} \left[ -2 \sqrt{\frac{4M^2 - p^2}{p^2}} \arctan \sqrt{\frac{p^2}{4M^2 - p^2}} - \ln \left( \frac{M^2}{4\pi\mu^2} \right) \right]$$

(17)

\begin{align*}
\simeq \left\{ \begin{array}{l}
\frac{\lambda^2\sigma_c^2}{32\pi^2} \left[ -2 + \frac{p^2}{6M^2} - \ln \left( \frac{M^2}{4\pi\mu^2} \right) \right] + \mathcal{O}(p^4) , \\
\frac{\lambda^2\sigma_c^2}{32\pi^2} \left[ (2 - \frac{\pi}{\sqrt{3}}) - \left( \frac{2\pi}{3\sqrt{3}} - 1 \right) \frac{p^2 - M^2}{M^2} - \ln \left( \frac{M^2}{4\pi\mu^2} \right) \right] + \mathcal{O}((p^2 - M^2)^2) .
\end{array} \right.
\end{align*}

While the expansion around $p^2 = 0$ agrees with [5,11], we should like to point out that the expansion around $p^2 - M^2$ is actually better suited for solutions close to the mass shell.

The finite temperature effective potential has been calculated in several papers (see e.g. [12]) and we just give the result (see Appendix A for the definition of $F_1^4$)

$$V_{\text{eff},\beta}(\sigma_c) = -\frac{1}{6\pi^2} F_1^4(T,M^2) \simeq -\frac{T^4\pi^2}{90} + \frac{T^2m^2}{24} + \frac{\sigma_c^2 \lambda T^2}{24} - \frac{TM^3(\sigma_c)}{12\pi} .$$

(18)
The dominant mass correction at high $T$ is $\lambda T^2/24$.

Next we look at the effective action at finite $T$ and here we notice some problems if we estimate the high $T$ limit using only the zero Matsubara frequency mode. In that approximation we would find

$$\Gamma^{(2)}_{\beta,n=0}(p_0, p) = \frac{\lambda^2 \sigma_c T}{16\pi} \frac{1}{p} \left[ \arcsin \left( \frac{p^2 + p_0^2}{p\sqrt{4M^2 + p^2 - 2p_0^2 + p_0^4/p^2}} \right) \right. 
\left. - \arcsin \left( \frac{p^2 - p_0^2}{p\sqrt{4M^2 + p^2 - 2p_0^2 + p_0^4/p^2}} \right) \right],$$

where $p = |p|$. This function has a well defined limit when $p_\mu \to 0$ and a threshold at $\{p_0 = M, p = 0\}$. These properties are however not shared by the correct two-point function. In particular the threshold is unphysical since there is no decay process allowed at that momentum [13]. To get the correct answer we can use either ITF or RTF to obtain

$$\Gamma^{(2)}_{\beta}(p_0, p) = -\frac{\lambda^2 \sigma_c^2}{16\pi^2} \int_0^\infty \frac{dk k f_B(\omega)}{\omega} \frac{1}{p} \ln \left( \frac{(p_0^2 - p^2 + 2pk)^2 - 4p_0^2\omega^2)}{(p_0^2 - p^2 - 2pk)^2 - 4p_0^2\omega^2)} \right),$$

where

$$\omega = \sqrt{k^2 + M^2}, \quad f_B(\omega) = \frac{1}{e^\beta\omega - 1}.$$

If we set $p_0 = 0$, expand in small $|p|$ and take the high $T$ limit we get

$$\Gamma^{(2)}_{\beta}(0, p) = -\frac{\lambda^2 \sigma_c^2}{16\pi^2} \int_0^\infty \frac{dk k f_B(\omega)}{\omega} \frac{2}{p} \ln \left( \frac{|p - 2k|}{|p + 2k|} \right) \approx \frac{\lambda^2 \sigma_c^2 T}{16\pi M} \left(1 - \frac{p^2}{12M^2} \right).$$

where the $p$–term coincides with the result of [3].* A possibility of avoiding the problem of non-analyticity at the origin is to expand around the mass shell. When we expand around $\{p_0^2 = M^2, p = 0\}$ we find

$$\Gamma^{(2)}_{\beta}(p_0, p) \approx \frac{\lambda^2 \sigma_c^2 T}{16\pi} M \left( 2(2 - \sqrt{3}) + \frac{7\sqrt{3} - 12}{3} \frac{p_0^2 - M^2}{M^2} - \frac{2(2 - \sqrt{3})}{3} \frac{p^2}{M^2} \right).$$

* Note that there is a misprint in Eq. (24) in [3].
One can, of course, obtain higher derivative terms by simply expanding to higher order in \((p_0^2 - M^2)\) and \(p^2\).

The expansion in Eq. (22) could be used to write down the dispersion relation near mass shell. We shall return to dispersion relations in Sect. 4. After performing the resummation of the tadpole to get the thermal mass of \(\lambda T^2/24\), no particularly interesting feature is found in the high \(T\) limit. The only appreciable effect is the thermal mass. For instance, on-shell there is no imaginary part at the one-loop level. Such an imaginary part, which is of essential importance when one wants to study the evolution of the perturbations, is first generated at the two-loop level.

### 3.2 Two-loop diagrams

For \(T/M\) large, higher loop diagrams go as higher powers of \(T\) than the one-loop diagram. On the other hand they are also suppressed by higher powers of \(\lambda\). At two-loop order the diagrams in Figs. 3 and 6 are of the order \(\lambda^2\), all other are of higher order. The leading diagram in \(T/M\) to each order in \(\lambda\) is the one with as many tadpole insertions as possible. As can be seen from Eq. (18) it gives a \(\lambda T^2/24\) contribution which can be resummed by just shifting to an effective temperature dependent mass \(M^2 + \lambda T^2/24\). This is well known at the one-loop level [12] but at the two-loop level one must be careful not to double count diagrams. Parts of the leading “double-bubble” (Fig. 6) are already included in the one-loop tadpole when the mass is shifted. An easy way of keeping track of the counting is to introduce a finite, \(T\) dependent counter term as in [14]. In the “setting sun” (Fig. 3), however, the leading corrections can be resummed using a \(T\)-dependent mass. In this Section we perform the calculation without resumming tadpoles, and defer the resummation to Sect. 4.

The diagram in Fig. 6 does not depend on the momentum so it can be absorbed in the effective potential. It has been calculated both at zero [11] and
finite [15] temperature. As for the one-loop case we state the result for the effective potential for completeness

$$V_{\text{eff}}^{(2)}(\sigma_c) = \frac{\lambda m^2}{8(4\pi)^4} \left\{ \frac{\lambda \sigma_c^2}{m^2} \left[ \ln^2 \left( \frac{M^2}{m^2} \right) - 5 \ln \left( \frac{M^2}{m^2} \right) \right] + \frac{M^4}{m^4} \ln^2 \left( \frac{M^2}{m^2} \right) \right\} + A + B\sigma_c^2 + C\sigma_c^4.$$  

(23)

At finite $T$ the result is

$$V_{\text{eff,}\beta}^{(2)}(\sigma_c) = \frac{\lambda}{32\pi^4} \left( F_1^2(T, M) \right)^2 + \frac{\lambda^2 \sigma_c^2 M^2}{128\pi^4} I(M/T)$$

$$+ \frac{\lambda M^2}{128\pi^4} \left( \frac{1}{2} + \ln \left( \frac{M^2}{m^2} \right) \right) F_1^2(T, M)$$

$$+ \frac{\lambda^2 \sigma_c^2}{128\pi^4} \left( \frac{\pi}{\sqrt{3}} - \frac{1}{2} + \ln \left( \frac{M^2}{m^2} \right) \right) F_1^2(T, M) - \frac{\nu_1''(\sigma_c)}{256\pi^4} F_1^2(T, M).$$

(24)

Here $\nu_1''(\sigma_c)$ is a second order polynomial of $\sigma_c$ that depends on the renormalization condition at one-loop, and $F_1^2(T, M)$ and $I(M/T)$ are defined in Appendix A. We have not verified the expressions in Eqs. (23) and (24), which can be found in [11,15], but instead concentrated on the high temperature expansion of Eq. (24). Most of it was derived in [15] but here we also show that (see Appendix A)

$$I(T, M) \simeq \frac{T^2}{M^2} \frac{5\pi^2}{24} \ln \left( \frac{M^2}{T^2} \right) + \mathcal{O}(T^2).$$

(25)

There is also a subleading term that goes like $T^2$ which need not be numerically small compared to the one given in Eq. (25).

When it comes to the effective action the finite temperature part of the “setting sun” is particularly interesting since it dominates over the one-loop “bubble” at high temperature $T \gg \sigma_c^2/M$. In the “setting sun” there is a $T$ dependent UV divergent constant term but that cancels against the “double-bubble” diagram so that there are no $T$ dependent infinities when all $\mathcal{O}(\lambda^2)$ diagrams are included.

The derivative term in the “setting sun” is finite (since there is no $p$-dependent term in the “double-bubble” that can cancel infinities) and one might be tempted
to try high $T$ expansion using the zero mode approximation. Expanding in $p_0$ and $p$ one finds, after some algebra, that

$$\Gamma^{(2)} = \frac{\lambda^2 T^2}{6} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 l}{(2\pi)^3} \frac{1}{k^2 + M^2 (k + 1)^2 + M^2 - p_0^2 + (p - l)^2 + M^2}$$

$$\simeq \frac{\lambda^2}{576\pi^2} \frac{T^2}{M^2} \left( A + p_0^2 - \frac{1}{9} p^2 \right), \quad (26)$$

where $A$ is an UV divergent momentum and temperature independent constant.

Our experience from the one–loop calculation in Sect 3.1 shows, however, that we cannot trust the zero mode approximation. In fact, although the imaginary part of $\Gamma^{(2)}$ at \{\(p_0 = M + i\epsilon, p = 0\}\} is UV finite, when it is calculated from Eq. (26) it is zero, whereas the correct value is \[14\]

$$\text{Im} \Gamma^{(2)}(p_0 = M, p = 0) = \frac{\lambda^2 T^2}{768\pi}. \quad (27)$$

Also, the leading $T$ dependence of the “setting sun” goes like $T^2 \ln T$ as we shall show below. We must therefore sum over all Matsubara frequencies and the easiest method to do that is the one described in [7–9] and [16].

The two-point function can then be written as

$$\Gamma^{(2)\text{-}\text{loop}}(p_0, p) = \frac{\lambda^2}{6} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 l}{(2\pi)^3} \int_0^\beta d\tau e^{i\nu\tau} \Delta(\tau, k) \Delta(\tau, l) \Delta(\tau, p + k + 1), \quad (28)$$

where

$$\Delta(\tau, k) = \frac{1}{2\omega_k} \left( e^{\beta \omega_k} - 1 \right) e^{-\omega_k |\tau|} + \frac{1}{e^{\beta \omega_k} - 1} e^{\omega_k |\tau|}. \quad (29)$$

After carrying out the $\tau$–integral we find

$$\Gamma^{(2)\text{-}\text{loop}}(p_0, p) = -\frac{\lambda^2}{8} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 l}{(2\pi)^3} \frac{1}{\omega_k \omega_l} p_0^2 - (\omega_k + \omega_l + \omega_{p + k + 1})^2$$

$$- \frac{\lambda^2}{4} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 l}{(2\pi)^3} \omega_k \omega_l f_B(\omega_k) \sum_{\pm} \frac{1}{(p_0 \pm \omega_k)^2 - (\omega_l + \omega_{p + k + 1})^2}$$

$$- \frac{\lambda^2}{8} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 l}{(2\pi)^3} \omega_k \omega_l f_B(\omega_k) f_B(\omega_l) \sum_{\pm, \pm} \frac{1}{(p_0 \pm \omega_k \pm \omega_l)^2 - \omega_{p + k + 1}^2}, \quad (30)$$
where the summations are over the two and four possible combinations of ±. We keep only the terms with one and two distribution functions \( f_B(\omega) \) since we are only interested in the finite temperature part. The term with one distribution function has a temperature dependent, but momentum independent, infinite part which cancels when all contributions are added. We also subtract this part to get a finite answer. All the angular integrations can then be performed explicitly but we leave one integral for notational simplicity. \( \Gamma_{2-loop}^{(2)}(p_0, p) \) has also an imaginary part which we determine through an analytical continuation corresponding to the retarded two-point vertex \((p_0 \to p_0 + i\epsilon)\). The final result is

\[
\Gamma_{2-loop}^{(2)}(p_0, p) = -\frac{\lambda^2}{8(2\pi)^4} \int_0^\infty \frac{dk dl kl}{\omega_k \omega_l} \left[ f_B(\omega_k) \left( G_1^{(2)}(p_0, p) + \delta G_1^{(2)}(p_0, p) \right) \right. \\
\left. + f_B(\omega_k) f_B(\omega_l) G_2^{(2)}(p_0, p) \right],
\]

where \( \delta G_1^{(2)}(p_0, p) \) regulates the infinity [14], and we have separated the terms with one and two distribution functions:

\[
G_1^{(2)}(p_0, p) = \frac{1}{2p} \int_{|k-p|}^{k+p} dz \sum_{\pm, \pm} \left\{ \ln \left| \frac{p_0^2 - (\pm\omega_k + \omega_1 + \omega_{l-z})^2}{p_0^2 - (\pm\omega_k + \omega_1 + \omega_{l+z})^2} \right| \\
+ i\pi \left[ \theta \left( p_0^2 - (\pm\omega_k + \omega_1 + \omega_{l-z})^2 \right) - \theta \left( p_0^2 - (\pm\omega_k + \omega_1 + \omega_{l+z})^2 \right) \right] \right\},
\]

\[
G_2^{(2)}(p_0, p) = \frac{1}{2p} \int_{|k-p|}^{k+p} dz \sum_{\pm, \pm} \left\{ \ln \left| \frac{(p_0 \pm \omega_k \pm \omega_1)^2 - m^2 - (z-l)^2}{(p_0 \pm \omega_k \pm \omega_1)^2 - m^2 - (z+l)^2} \right| \\
+ i\pi \text{sign}(p_0 \pm \omega_k \pm \omega_1) \left[ \theta \left( (z-l)^2 + m^2 - (p_0 \pm \omega_k \pm \omega_1)^2 \right) \\
- \theta \left( (z+l)^2 + m^2 - (p_0 \pm \omega_k \pm \omega_1)^2 \right) \right] \right\},
\]

\[
\delta G_1^{(2)}(p_0, p) = \frac{2l}{p} \int_{|k-p|}^{k+p} \frac{dz}{z}.
\]

It is interesting to note that the two–loop result Eq. (31) is analytic at the origin, unlike the one–loop action Eq. (20). The leading mass correction has been calculated explicitly by Parwani [14] at high \( T \) and at \( \{p_0 = M, p = 0\} \) (see
Appendix A for details):

\[ \Gamma_{2-loop}^{(2)}(p_0 = M, p = 0) \simeq -\frac{2}{3} \frac{\lambda^2}{128\pi^2} T^2 \ln \left( \frac{M}{T} \right). \]  

(33)

One can now construct the effective action using Eq. (15).

4. Evolution of perturbations

When studying the evolution of perturbations about a constant field, the knowledge about the two-point function is usually not enough. The action has, apart from some special cases, also a linear part, so that the equation of motion is of the form \( \Gamma^{(2)}(\sigma_c) b = -\Gamma^{(1)}(\sigma_c) \). The right hand side of this equation can be eliminated by writing the field in the form \( b(x) = b_h(t) + \delta b(x) \). The first part is only time dependent and satisfies the equation of motion with vanishing spatial derivatives. It describes the homogeneous rolling of the field on the effective potential surface as long as \( b_h(t) \) remains small. The second part, \( \delta b \), describes the fluctuations on the homogeneous field, and satisfies the dispersion relation.

As an application we shall now consider the evolution of local perturbations \( b(x) \) around the high \( T \) minimum \( \sigma_c = 0 \), which is the initial state for the phase transition, and for which \( \Gamma^{(1)} \equiv 0 \). The one–loop contribution to \( \Gamma^{(2)} \), Eq. (20), vanishes, and the lowest order correction is provided by the two–loop diagram of Fig. 3 as given by Eq. (31). For small \( b(x) \), when we may neglect terms higher than \( \mathcal{O}(b^2) \) in the expansion for the effective action, the evolution of different Fourier modes is independent and given by the dispersion relation

\[ p_0^2 = p^2 + m_R^2 + \Sigma_R(p_0, p; m) + i\Sigma_I(p_0, p; m), \]  

(34)

with

\[ \Sigma_R(p_0, p; m) = -\text{Re} \left( \Gamma^{(2)}(p_0, p; m) - \Gamma^{(2)}(m, 0; m) \right), \]

\[ \Sigma_I(p_0, p; m) = -\text{Im} \Gamma^{(2)}(p_0, p; m), \]  

(35)
where the renormalized mass $m_R$ has been introduced to absorb the order $\mathcal{O}(\lambda^2)$ mass correction, part of which comes from $\Gamma^{(2)}(m, 0; m)$. We have also indicated explicitly the mass-dependence of the two-point function. We may write

$$m_R^2 = m^2 + \frac{\lambda T^2}{24} + \mathcal{O}(\lambda^2) \equiv m(T)^2 + \mathcal{O}(\lambda^2), \quad \text{(36)}$$

where $m(T)$ is the familiar one-loop plasma mass. In fact, to order $\mathcal{O}(\lambda^2)$ we may simply replace the mass parameter $m$ by $m(T)$ in $\Gamma^{(2)}(p_0, p; m)$. This amounts to plasma resummation of the propagators in the loop diagrams. (As discussed before, there is no double counting in this case). Thus to lowest order we may write, denoting $\text{Im} p_0 = -\gamma/2$ and $\text{Re} p_0 = \omega$, the dispersion relation as

$$\omega^2 = p^2 + m(T)^2 - \frac{1}{4} \gamma^2 + \Sigma_R(\omega, p; m(T)), \quad \gamma = -\Sigma_I(\omega, p; m(T))/\omega. \quad \text{(37)}$$

We may solve Eq. (37) approximately by setting $\omega^2 \simeq \omega_p^2 \equiv p^2 + m(T)^2$. To this end we have studied the behavior of $\Sigma_R(p_0, p; m)$ and $\Sigma_I(p_0, p; m)$ numerically. It turns out that when $p \to \infty$, the contribution from the part with two distribution functions becomes unimportant. Roughly, numerically we find that $\gamma \sim T^3/p^2$ which is also an upper limit that can be derived analytically. We also find that for large $p$ the real part is independent of $p$ and goes roughly like $(2\pi)^{-4} \lambda^2 \ln T$, so that for $\lambda \ll 1$ this part is never important for the dispersion relations. The limit $p \to \infty$ is thus completely dominated by the part with one distribution function, $G^{(2)}_1 + \delta G^{(2)}_1$ in Eq. (31). Scaling the relevant part of Eq. (31) by $p$ and taking the limit, it is possible to show that (see Appendix B for details)

$$\Gamma^{(2)}(p_0, p) \simeq \frac{\lambda^2}{8(2\pi)^4} F^2_1(T, m(T)) \left[ \ln \frac{p^2}{m(T)^2} + i\pi \right], \quad \text{(38)}$$

where $F^2_1$ has been defined in Appendix A, and for large $T$ and $\lambda \ll 1$, we have $F^2_1 \sim \pi^2 T^2/6$. In the high temperature limit the dispersion relation Eq. (37) for the real part reads then $p_0^2 = p^2 + m(T)^2 + 3(2\pi)^{-4} \lambda m(T)^2 \ln(p^2/m(T)^2)$ so that $\Sigma_R(p_0, p; m)$ never contributes significantly. Thus $\omega = \omega_p$ is a self-consistent approximate solution.
As discussed in [13], $\gamma$ is the rate by which a given plane wave mode thermalizes. We have computed $\text{Im } \Gamma^{(2)}$ from Eq. (31) numerically, and the resulting thermalization rate is drawn in Fig. 7 for a few temperatures. The imaginary part exists for all four–momenta obeying the (approximate) dispersion relation, and this means that every small perturbation $b(x)$ about the high $T$ minimum $\sigma_c = 0$ will eventually smoothen out and thermalize.

The essential question then is, how rapid is the rate of thermalization as compared with the expansion rate of the Universe, given by the Hubble parameter $H$. Assuming the Standard Model degrees of freedom, at high temperature we may write

$$H = \left(\frac{8\pi^3 g_*}{90}\right)^{\frac{1}{4}} \frac{T^2}{M_P} \simeq 17 \frac{T^2}{M_P}, \quad (39)$$

where $M_P = 1.22 \times 10^{22}$GeV is the Planck mass. Comparing (38) and (39) we find that all momenta obeying

$$p \lesssim 2.8 \lambda^2 \times 10^{14} \text{GeV}, \quad (40)$$
decay in less than a Hubble time.

Recall that the critical temperature and the intrinsic mass scale (the inverse of the wall thickness of a bubble) are typically related by $\sqrt{\lambda T_c} \sim M$. This means that perturbation modes of the size of the critical bubble have not yet decayed by the onset of phase transition provided $\lambda \lesssim 10^{-5}(M/100 \text{ GeV})$. For such small $\lambda$ the inherent local perturbations would certainly be important for bubble nucleation.

5. Conclusions

We have found that the finite temperature effective action can be relatively easily computed using the tadpole method generalized for space–time dependent fields in Sect. 2.2. In that case resummation is however not possible except for constant background fields $\sigma_c$. Nevertheless, one can do perturbation theory in
small fluctuations $b(x)$ about a constant background, which in effect amounts to computing Green functions for the shifted theory. To lowest non–trivial order one then just evaluates the two–point function, which we did to two–loop order in Sect. 3. After that it is straightforward to write down dispersion relations and to study the evolution of the fluctuations. Since this only requires an expansion of the effective action about the mass shell we do not encounter the problem of non–analyticity at $p_\mu = 0$. We also observe that the zero Matsubara frequency approximation for the leading high temperature terms is not correct in general. In our case, even the UV–finite imaginary part needs a summation over all Matsubara frequencies.

Our main conclusion is that small space–time dependent perturbations about the high temperature minimum do not automatically thermalize before the phase transition starts. This would naturally affect bubble formation in a first order phase transition, although to what extent is not clear. In spite of the fact that this conclusion holds strictly speaking only for the $\lambda\phi^4$–model, we could view the $\lambda\phi^4$–model as an effective theory of the order parameter of a more realistic theory, such as the Standard Model. The gross features of the present study would then presumably show up also in the full Standard Model.

It is however clear that the estimate at the end of Sect. 4 is very rough and has to be improved for an application to a real physical system. The main issue is the strength of the effective coupling. Some recent estimates seem to indicate that the temperature dependence can be quite strong, and that the four–point interaction for zero momentum particles goes in fact to zero at the critical temperature of a second order phase transition [17,18]. This is caused by the vanishing mass at the critical temperature. The coupling constant decreases at the same time as the mass and thus the relevant $p$ also decreases. It is not clear what the net effect of the coupling constant renormalization is. Furthermore, the electroweak phase transition is likely to be first order, albeit possibly very weakly so. These questions can only be addressed after a more detailed study of the damping rates in the full electroweak theory [19].

We have only considered small perturbations. It is conceivable that large perturbations could, because of the non–linear coupling of the modes, contain
instabilities so that some perturbation modes would actually get amplified. Further studies in this direction are certainly needed.
Appendix A

In this Appendix we give some useful expansions employed in Sect. 3 and calculate the leading $T$ dependence of Eq. (31) as well as the function $I(T/M)$ in Eq. (24).

The function

$$F_n^m(T, M) = \int_0^\infty \frac{dk k^m}{\omega^n} f_B(\omega), \quad \omega = \sqrt{k^2 + M^2}, \quad (A1)$$

often occurs in loop calculation results at finite temperature (cf Eqs. (24), (38)). It satisfies the relation

$$\frac{\partial F_n^m}{\partial M^2} = -\frac{m-1}{2} F_n^{m-2}, \quad m > 1. \quad (A2)$$

In this paper we need $F_1^2$ and $F_1^4$ and they have the high $T$ expansions

$$F_1^2(T, M) = \frac{\pi^2 T^2}{6} - \frac{\pi T M}{2} - \frac{M^2}{4} \ln\left(\frac{M}{4\pi T}\right) + \ldots$$

$$F_1^4(T, M) = \frac{\pi^4 T^4}{15} - \frac{\pi^2 T^2 M^2}{4} + \frac{\pi T M^3}{2} + \frac{3M^4}{16} \ln\left(\frac{M}{4\pi T}\right) + \ldots \quad (A3)$$

Let us now turn to the leading $T$ dependence of Eq. (31) and the function $I(T/M)$ in Eq. (24). The leading high $T$ expansion of Eq. (31) was derived in [14]. We have checked the coefficients of the $T^2 \ln T$ terms in the part with two distribution functions, using a different method, and found them to be correct.

The essential integral in the part with two distribution functions of Eq. (31) is

$$T^2 I_1(T/M) = \int_0^\infty \frac{dk dl kl}{\omega_k \omega_l} f_B(\omega_k) f_B(\omega_l) \ln \left|\frac{k-l}{k+l}\right|, \quad (A4)$$

where $a = M/T$ and $\{p_0 = M, p = 0\}$. We expect $I_1$ to go like $\ln a$ for small $a$ so we compute the derivative with respect to $a^2$. Using

$$\frac{d}{da^2} \left( \frac{f(\omega_x) f(\omega_y)}{\omega_x \omega_y} \right) = \frac{f(\omega_y)}{\omega_y} \frac{d}{dx} \left( \frac{f(\omega_x)}{\omega_x} \right) + \frac{f(\omega_x)}{\omega_x} \frac{d}{dy} \left( \frac{f(\omega_y)}{\omega_y} \right), \quad (A5)$$
where \( \omega_x = \sqrt{x^2 + a^2} \) and \( f(\omega_x) = (e^{\omega_x} - 1)^{-1} \), we find after partial integration

\[
\frac{dI_1}{da^2} = \int_0^\infty dx \int_0^\infty dy \frac{f(\omega_x)f(\omega_y)}{\omega_x \omega_y} \approx \frac{\pi^2}{a^2} \frac{\pi^2}{4}
\]

for small \( a \). After integration with respect to \( a^2 \) we get

\[
T^2 I_1(T/M) = -\frac{\pi^2}{2} T^2 \ln \left( \frac{T}{M} \right) + O(T^2) . \tag{A6}
\]

The function \( I(T/M) \) in Eq. (24) can be expanded in a similar manner. It is defined as [15]

\[
I(T/M) = \frac{T^2}{M^2} \int_0^\infty dx \int_0^\infty dy \frac{x y}{\omega_x \omega_y} f(\omega_x)f(\omega_y) \ln \left| \frac{(a^2 + 2xy)^2 - 4\omega_x^2 \omega_y^2}{(a^2 - 2xy)^2 - 4\omega_x^2 \omega_y^2} \right| . \tag{A7}
\]

The same trick works and we find

\[
I(T/M) = -\frac{5}{6} \frac{\pi^2}{2} \frac{T^2}{M^2} \ln \left( \frac{T}{M} \right) + O(T^2) , \tag{A8}
\]

where the factor of 5/6 comes from

\[
\int_0^\infty \frac{dx \, dy}{(x^2 + 1)(y^2 + 1)} \frac{2x^4 - x^2y^2 + 2y^4}{(x^2 - xy + y^2)(x^2 + xy + y^2)} = \frac{5}{6} \frac{\pi^2}{2} . \tag{A9}
\]

It is the same diagram, the “setting sun”, that leads to \( I \) and \( I_1 \) but \( I \) is computed at \( p_0 = 0 \) since it occurs in the effective potential while \( I_1 \) is computed on-shell, \( p_0 = M \). We therefore do not expect them to be equal but only of the same order of magnitude and to have the same sign.

**Appendix B**

The asymptotic behavior of \( \Gamma^{(2)} \) can be calculated from Eq. (31) for all temperatures in terms of the functions \( F_n^m \). In the effective action the part with one distribution function, \( G_1^{(2)} + \delta G_1^{(2)} \) in Eq. (31), dominates over \( G_2^{(2)} \), and thus the behavior of the two–point function in the limit \( p \to \infty \) is solely determined by \( G_1^{(2)} + \delta G_1^{(2)} \). We define

\[
I_2 = \int_0^\infty \frac{dk \, dl \, k}{\omega_k} f_B(\omega_k) \int_{|k-p|}^{k+p} \frac{dz}{2p} \sum_\pm \left\{ \frac{1}{\omega_1} \ln \frac{p_0^2 + i \epsilon - (\pm \omega_k + \omega_1 + \omega_{l-z})^2}{p_0^2 + i \epsilon - (\pm \omega_k + \omega_1 + \omega_{l+z})^2} + \frac{4z}{\omega_1} \right\} . \tag{B1}
\]
$I_2$ can be evaluated in the limit $p \to \infty$. After scaling the integration variables $k$, $l$ and $z$ by $p$ we can extract the leading behavior of the integral $I_2$. The logarithmic piece has now a finite, non-zero limiting value, and to the leading order we can take this limit in the integrand. We obtain

$$I_2 \simeq \int_0^\infty \frac{dk \, dl \, k}{\omega_{pk}} p^3 f_B(\omega_{pk})$$

$$\times \int_{|1-l|}^{1+k} dz \frac{1}{2} \sum_{\pm} \left\{ \ln \frac{1+i\epsilon - (l + |l-z| \pm k)^2}{1+i\epsilon - (2l + z \pm k)^2} + \frac{4z}{\sqrt{l^2 + M^2/p^2}} \right\}.$$  \hspace{1cm} (B2)

Note that we cannot take the limit in the whole integral, because both the $k$ and $l$–integrations are singular at the origin when $p \to \infty$. The easiest way to proceed is to first perform the $l$–integration and after that the $z$–integration. This yields

$$I_2 \simeq -\int_0^\infty dk \frac{p^3 k}{\sqrt{p^2 k^2 + M^2}} f_B(\omega_{pk}) \left[ 2k \ln \frac{p^2}{M^2} + i2\pi k \right], \hspace{1cm} (B3)$$

which, apart from prefactors, is $\Gamma^{(2)}$ in Eq. (31). By rescaling $k \to k/p$ we obtain the result Eq. (38). The logarithmic behavior of the real part of the two point function is due to the IR–singularity in the UV–regulating term.
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Figure Captions

**Fig. 1** The one–loop tadpole.

**Fig. 2** One–loop diagram for $\Gamma^{(2)}$.

**Fig. 3** Two–loop diagram for $\Gamma^{(2)}$ (”setting sun”).

**Fig. 4** Feynman rules for momentum–dependent external legs.

**Fig. 5** One–loop diagrams for the effective action in the generalized tadpole method.

**Fig. 6** Two–loop tadpole (”double bubble”).

**Fig. 7** The thermalization rate $\gamma$ for on–shell configurations $\omega \simeq \omega_p$ for $T = 10 \ m(T)$ (dash-dotted curve), $100 \ m(T)$ (dotted curve) and $1000 \ m(T)$ (solid curve).
Finite temperature effective action
and thermalization of perturbations

Per Elmfors†, Kari Enqvist‡ and Iiro Vilja⋆

Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

The effective action is computed for the $\lambda \phi^4$–theory at finite temperature for small perturbations about a constant background field, using a generalized tadpole method. We find the complete effective action, including the real and imaginary parts, to all orders in derivatives and to order $O(\lambda^2)$. We demonstrate that the high $T$ approximation, where only the zero Matsubara frequency is included, is incorrect for the imaginary part even though it is UV-finite. The solutions of the dispersion relations show that initial perturbations do not necessarily thermalize fast enough to be absent at the onset of phase transition.

† internet: elmfors@nordita.dk
‡ internet: enqvist@nbivax.nbi.dk
⋆ internet: iiro@nordita.dk