SYMPLECTIC CURVATURE FLOW

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ABSTRACT. We introduce a parabolic flow of almost Kähler structures, providing an approach to constructing canonical geometric structures on symplectic manifolds. We exhibit this flow as one of a family of parabolic flows of almost Hermitian structures, generalizing our previous work on parabolic flows of Hermitian metrics. We exhibit a long time existence obstruction for solutions to this flow by showing certain smoothing estimates for the curvature and torsion. We end with a discussion of the limiting objects as well as some open problems related to the symplectic curvature flow.

1. INTRODUCTION

In the past two decades the study of symplectic manifolds has been very active. New tools have been introduced and new insight has been provided, for instance, by Gromov’s work on pseudoholomorphic curves and symplectic topology \cite{Gromov99}, the Gromov-Witten invariants and their applications to mirror symmetry (see for instance \cite{Givental96} etc.), Taubes’ works on the Seiberg-Witten equations on symplectic manifolds \cite{Taubes01}, \cite{Taubes02}, invariants coming from studying Hamiltonian dynamics and Lagrangian intersections (see for instance \cite{McDuff98} and \cite{McDuff99}). These approaches have all had a profound impact on our understanding of symplectic manifolds, and are linked in the sense that they are all “topological” in nature. The purpose of this paper is to introduce a geometric approach to studying symplectic manifolds. Specifically we introduce a new curvature flow which preserves symplectic structures and evolves almost Kähler structures, which always exist on symplectic manifolds, towards certain canonical geometric structures on symplectic manifolds. Hopefully, this curvature flow provides us a very different approach to and enables us to apply the methods of geometric analysis to understanding the topology and geometry of symplectic manifolds from a different point of view.

To begin, let $(M^{2n}, \omega)$ denote a compact smooth manifold with closed, nondegenerate 2-form $\omega$. Any such $\omega$ admits compatible almost complex structures. Below we will define a coupled degenerate parabolic system of equations for a compatible pair $(\omega, J)$ preserving the symplectic condition for $\omega$. If the initial almost complex structure is in fact integrable, then the resulting one-parameter family of complex structures is fixed, i.e. $J(t) = J(0)$, and the family of Kähler forms $\omega(t)$ is a solution to Kähler Ricci flow. This parabolic system is furthermore a special instance of a general family of parabolic flows of almost Hermitian structures. We begin by describing this more general setup, then proceed to define the flow of almost Kähler structures.

Let $(M^{2n}, \omega, J)$ be an almost Hermitian manifold. Let $\nabla$ denote the Chern connection associated to $(\omega, J)$, which is the unique connection satisfying

\[ \nabla \omega \equiv 0, \quad \nabla J \equiv 0, \quad T^{1,1} \equiv 0 \]
where \( T^{1,1} \) refers to the \((1,1)\) component of the torsion of \( \nabla \) thought of as a section of \( \Lambda^2 \otimes TM \). Let \( \Omega \) denote the \((4,0)\)-curvature tensor associated to this connection, and let

\[
S_{ij} = \omega^{kl} \Omega_{klij}.
\]

Furthermore, let \( Q \) denote a \((1,1)\) form which is a quadratic expression in the torsion \( T \) of \( \nabla \). Let

\[
K_{ij}^k = \omega^{kl} \nabla^k N_{ij}^l.
\]

where \( N \) denotes the Nijenhuis tensor associated to \( J \). Also, let \( \mathcal{H} \) denote a generic quadratic expression in the Nijenhuis tensor which is an endomorphism of the tangent bundle which skew-commutes with \( J \). Finally, let

\[
H = \frac{1}{2} [\omega(K - \mathcal{H}, J) + \omega(J, K - \mathcal{H})].
\]

These definitions are spelled out in greater detail in the rest of the paper. Consider the initial value problem

\[
\begin{align*}
\frac{\partial}{\partial t} \omega &= -S + Q + H \\
\frac{\partial}{\partial t} J &= -K + \mathcal{H} \\
\omega(0) &= \omega_0 \\
J(0) &= J_0.
\end{align*}
\]

This is a degenerate parabolic system of equations for \((\omega, J)\), with degeneracy arising from the action of the diffeomorphism group. In section 3 we prove the general short-time existence of solutions of (1.2), a generalization of Theorem 1.1 of [18].

**Theorem 1.1.** Let \((M^{2n}, \omega_0, J_0)\) be a compact almost Hermitian manifold. There exists \( \epsilon > 0 \) and a unique one parameter family of almost Hermitian structures \((\omega(t), J(t))\) solving (1.2) with initial condition \((\omega_0, J_0)\). If \( J_0 \) is integrable, then \( J(t) = J_0 \) for all \( t \in [0, \epsilon) \). Furthermore, if \( J_0 \) is integrable and \( g_0 \) is Kähler, then \( g(t) \) is Kähler for all \( t \in [0, \epsilon) \) and \( g(t) \) solves the Kähler-Ricci flow with initial condition \( g_0 \).

**Remark 1.2.** It is important to note that equation (1.2) is defining a family of equations. Indeed, the choice of \( Q \) and \( \mathcal{H} \) are arbitrary in the definition of (1.2) and the proof of Theorem 1.1.

**Remark 1.3.** When \( J_0 \) is integrable, the one-parameter family of metrics \( \omega(t) \) is a solution to *Hermitian curvature flow*, as defined in [18]. Again, the torsion term \( Q \) can be arbitrary for the result of Theorem 1.1.

**Remark 1.4.** As will be clear from Proposition 5.5, it is possible to define a parabolic flow of metrics compatible with any given almost complex structure. Specifically, given \((M^{2n}, J)\) an almost complex manifold, one can set

\[
\frac{\partial}{\partial t} \omega = -S + Q
\]

where again \( Q \) is a \((1,1)\) form which is a quadratic expression in the torsion. This viewpoint was considered recently by Vezzoni [27]. When \( J \) is integrable, this is precisely the family of equations introduced in [18]. If one is interested in understanding metrics compatible with a given almost complex structure, (1.3) could be a useful tool.
We now proceed to define the flow of almost Kähler structures.

**Definition 1.5.** An almost Hermitian manifold \((M^{2n}, \omega, J)\) is **almost Kähler** if

\[ d\omega = 0. \]

This condition is a very natural extension of Kähler geometry, and one may consult [1] for a nice fairly recent survey of results on these structures. Due to their connection with symplectic geometry, almost Kähler structures have become a central area of mathematics (see for instance [6], [14]).

An almost Kähler structure has an associated Levi Civita connection \(D\), as well as a canonical Hermitian connection \(\nabla\) (which coincides with the Chern connection) with curvature \(\Omega\). Furthermore, one can define

\[ P_{ij} = \omega^{kl} \Omega_{ijkl}. \]

By Chern-Weil theory we know that \(P \in \pi c_1(M, J)\), and moreover \(dP = 0\). In analogy with Kähler Ricci flow, it is natural to expect that \(P\) is the right operator by which to flow a symplectic structure. However, \(P \notin \Lambda^{1,1}\), therefore one is forced to attach a flow of \(J\) as well to preserve compatibility of the pair. Set

\[ N^j_i = g^{jk} g_{mn} g^{pq} D_p J^m_r J^r_i D_q J^q_k, \]

\[ R^j_i = J^k_i \text{Rc}^j_k - \text{Rc}^k_i J^j_k, \]

and consider the initial value problem

\[
\begin{align*}
\frac{\partial}{\partial t} \omega &= -P \\
\frac{\partial}{\partial t} J &= -D^* DJ + N + R \\
\omega(0) &= \omega_0 \\
J(0) &= J_0.
\end{align*}
\]  

(1.4)

**Theorem 1.6.** Let \((M^{2n}, \omega_0, J_0)\) be a compact almost Kähler manifold. There exists \(\epsilon > 0\) and a unique one-parameter family of almost Kähler structures \((\omega(t), J(t))\) solving (1.4) for \(t \in [0, \epsilon)\). Moreover, the pair \((\omega(t), J(t))\) is a solution to an equation of the type (1.2), for appropriate choices of \(Q\) and \(H\), with \(H\) defined by (1.1). In particular, this instance of equation (1.2) preserves the almost Kähler condition. Finally, if \(J_0\) is integrable, \(J(t) = J(0)\) for all \(t\) and \(\omega(t)\) is a solution to Kähler Ricci flow.

**Remark 1.7.** In [13] a certain geometric evolution equation was studied on symplectic manifolds. There the perspective taken is that the symplectic structure \(\omega\) is fixed, and then one studies the gradient flow of the functional of compatible almost complex structures

\[ \mathcal{F}(J) := \int_M |DJ|^2 \, dV \]

where the metric defining the quantities above is that associated to \(J\) via \(\omega\). The proof of short time existence of this flow is already technical, due to certain local obstructions in prescribing the skew-symmetric part of the Ricci tensor. Our approach here is different, as we allow both \(\omega\) and \(J\) to change. This seems to have certain advantages, since for instance the diffeomorphism action is the only obstruction to parabolicity. Furthermore, our flow is a natural generalization of Kähler Ricci flow, whereas any Kähler metric is already a fixed point for this flow.
We are able to derive equations for the evolution of curvature and torsion under solutions of (1.2) and (1.4). The general theory is similar to the case of Hermitian curvature flow, where one requires bounds on the curvature, torsion, and first derivative of torsion to conclude long time existence of the flow. This result is obtained by proving smoothing estimates for higher derivatives which hold in the presence of these bounds. For a technical reason explained in section 8, one is forced to get $L^2$ smoothing estimates. Incidentally, this technical problem does not occur for (1.4), and one obtains the usual pointwise smoothing estimates (see Theorem 8.1).

**Theorem 1.8.** Given $m > 0$, there exists $C = C(m,n)$ such that if $(M^{2n}, \omega(t), J(t))$ is a solution to (1.2) on $[0, \frac{T}{m}]$ satisfying

$$\sup_{M \times [0, \frac{T}{m}]} \{ |Rm|, |T|^2, |DT| \} \leq K,$$

then

$$\sup_{M \times [0, \frac{T}{m}]} \{ ||D^m Rm||_{L^2}^2, ||D^{m+1} T||_{L^2}^2 \} \leq \frac{CK}{t^{\frac{m}{2}}}.$$

Using these we obtain the long time existence obstruction.

**Theorem 1.9.** Let $(M^{2n}, \omega_0, J_0)$ be an almost Hermitian manifold. There is a unique solution to (1.2) on a maximal time interval $[0, \tau)$. Furthermore, if $\tau < \infty$ then

$$\limsup_{t \to \tau} \{ |Rm|_{C^0}, |DT|_{C^0}, |T|_{C^0}^2 \} = \infty.$$

Furthermore, one can improve this regularity requirement in the case of symplectic curvature flow. This is because of an a priori estimate for $|DJ|^2$ which holds when the curvature is bounded.

**Theorem 1.10.** Let $(M^{2n}, \omega_0, J_0)$ be an almost Kähler manifold. There is a unique solution to (1.4) on a maximal time interval $[0, \tau)$. Furthermore, if $\tau < \infty$ then

$$\limsup_{t \to \tau} |Rm|_{C^0} = \infty.$$

Here is an outline of the rest of the paper. In § 2 we review some basic aspects of almost Hermitian geometry, and recall the Chern connection. We recall and generalize some known curvature identities in section § 3. We give basic calculations on variations of almost Hermitian structures in § 4. In § 5 and § 6 we give the proofs of Theorems 1.1 and 1.6. Evolution equations for the curvature tensor, torsion, and their derivatives are shown in § 7 and we use these to prove smoothing estimates which are used to prove Theorems 1.8, 1.9 and 1.10 in § 8. In § 9, we give a discussion of some special properties of the limiting metrics of (1.4). We end in § 10 by posing a number of problems related to symplectic curvature flow.

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2. Background on Almost Hermitian Geometry

In this section we review some basic material about almost Hermitian geometry and various associated connections. Let \((M^{2n}, J)\) be an almost complex manifold. This means that \(J\) is an endomorphism of \(TM\) satisfying

\[ J^2 = -\text{Id}. \]

By the theorem of Newlander-Nirenberg [15], the almost complex structure \(J\) is integrable, i.e. one can find local complex coordinates at each point, if and only if the Nijenhuis tensor vanishes. The Nijenhuis tensor is

\[ N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]. \] (2.1)

As in the case of complex manifold, the almost-complex structure \(J\) induces a decomposition of the space of differential forms on \(M\) via the eigenspace decomposition on \(TM\). In particular we will write

\[ \Lambda^r(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}. \]

Also, for a general two-tensor \(W \in T^*M \otimes T^*M\), let

\[ W^J(X, Y) = \frac{1}{2} (W(X, Y) + W(JX, JY)), \]
\[ W^{-J}(X, Y) = \frac{1}{2} (W(X, Y) - W(JX, JY)) \]

denote the projections of \(W\) onto \(J\)-symmetric and \(J\)-antisymmetric tensors.

The operator \(d\) acts on \(\Lambda^r\), but in general one does not have \(d\Lambda^{p,q} \subset \Lambda^{p+1,q} \oplus \Lambda^{p,q+1}\), due to the potential lack of integrability of \(J\). Finally, we will use the operator

\[ d^c : \Lambda^r \to \Lambda^{r+1}, \quad \psi \to -Jd\psi \]

where for a differential \(r\)-form \(\phi\) one has

\[ (J\phi)(X_1, \ldots, X_r) = \phi(JX_1, \ldots, JX_r). \]

Moving to the metric geometry, let \(g\) be an almost Hermitian metric on \(M\), i.e. \(g\) satisfies

\[ g(\cdot, \cdot) = g(J\cdot, J\cdot). \]

Associated to this pair is the Kähler form

\[ \omega(\cdot, \cdot) = g(J\cdot, \cdot). \]

Next we consider connections associated to almost Hermitian manifolds. A very thorough discussion of these connections can be found in [9]. A linear connection \(\nabla\) on \(TM\) is called Hermitian if

\[ \nabla \omega \equiv 0, \quad \nabla J \equiv 0. \]

These two conditions alone do not suffice to determine a unique connection in general. Indeed, there is freedom yet of \(\psi \in \Lambda^3(\mathbb{R}) \cap \Lambda^{2,1} \oplus \Lambda^{1,2}\) and \(B \in \Lambda^{1,1} \otimes TM\) satisfying a certain Bianchi identity (see [9] Proposition 2). Certain members of this family are chosen according to certain desirable properties of the torsion. Frequently, one chooses the Chern connection.
**Definition 2.1.** Given \((M^{2n}, \omega, J)\) an almost-Hermitian manifold, the *Chern connection* associated to \((\omega, J)\) is the unique connection \(\nabla\) satisfying

\[
\nabla \omega \equiv 0 \\
\nabla J \equiv 0 \\
T^{1,1} \equiv 0
\]

where \(T\) denotes the torsion tensor of \(\nabla\) and \(T^{1,1}\) is the projection of the vector-valued torsion two-form onto the space of \((1,1)\)-forms.

Gauduchon [9] has identified a canonical family of Hermitian connections associated to an almost Hermitian pair. Before describing it though let us introduce a further piece of notation. For \(\phi \in \Lambda^3\), let

\[
\phi^+ := \phi^{(2,1)+(1,2)} \\
\phi^- := \phi^{(3,0)+(0,3)}.
\]

By making certain natural assumptions about the torsion of a Hermitian connection (see [9] Definition 2), one identifies a one-parameter family of such connections ([9] 2.5.4)

\[
\langle \nabla_X Y, Z \rangle = \langle D_X Y, Z \rangle + \frac{1}{2} \langle (D_X J)Y, Z \rangle + \frac{t}{4} \left( (d\omega)^+_X Y, Z + (d\omega)^+_{X, JY, JZ} \right).
\] (2.2)

In the formula above \(D\) denotes the Levi-Civita connection. The choice \(t = 1\) corresponds to the *Chern connection*. As a final important remark we observe that in the case of almost Kähler manifolds, this family reduces to a single point, i.e. there is a *canonical* Hermitian connection on almost Kähler manifolds, taking the simple form

\[
\langle \nabla_X Y, Z \rangle = \langle D_X Y, Z \rangle + \frac{1}{2} \langle (D_X J)Y, Z \rangle.
\] (2.3)

### 3. Curvature Identities for Almost Hermitian Structures

In this section we collect some important identities for the curvature and torsion of almost Hermitian pairs. Fix \((\omega, J)\) an almost Hermitian pair, and let \(g\) denote the associated Riemannian metric. As usual, let \(\text{Ric}\) denote the usual Ricci curvature of the Levi-Civita connection, and let \(\text{Ric}^J\) denote the \(J\)-invariant part of the Ricci tensor of \(g\), i.e.

\[
\text{Ric}^J = \frac{1}{2} \left[ \text{Ric}(\cdot, \cdot) + \text{Ric}(J\cdot, J\cdot) \right].
\] (3.1)

Furthermore set

\[
\rho(\cdot, \cdot) = \text{Ric}^J(J\cdot, \cdot).
\] (3.2)

Note \(\rho \in \Lambda^{1,1}\). Next set

\[
\rho^* = R(\omega)
\] (3.3)

i.e., the Levi-Civita curvature operator acting on the Kähler form \(\omega\). One can see [11] for more information on these quantities.

Now let \(\nabla\) denote a Hermitian connection associated to an almost Hermitian pair. The connection \(\nabla\) induces a Hermitian connection on the anticanonical bundle, and we denote the curvature form of this connection by \(P\). Alternatively, if \(\Omega\) denotes the curvature of \(\nabla\), one has

\[
P_{ij} = \omega^{kj} \Omega_{ijkl}.
\] (3.4)
By the general Chern-Weil theory, \( P \) is a closed form and \( P \in \pi c_1(M, J) \). We record some lemmas relating these different curvature tensors. A key role is played in our analysis by the Weitzenböck formula for two-forms.

**Lemma 3.1.** Let \((M^{2n}, \omega, J)\) be an almost Hermitian manifold. Then
\[
\rho^* - 2\rho = (D^*D\omega - \Delta d\omega). \tag{3.5}
\]

**Proof.** By the Weitzenböck formula for 2-forms ([4] pg. 53) applied to \( \omega \) we conclude
\[
\Delta d\omega - D^*D\omega = \text{Ric}(\omega, \cdot) - \text{Ric}(\cdot, \omega) - R(\omega),
\]
where here the action of the Ricci tensor on the two form \( \omega \) is by raising the index on the Ricci tensor using the metric and letting the endomorphism act naturally. Phrasing this in terms of \( J \) one sees
\[
R(\omega) + [\text{Ric}(\cdot, J\cdot) - \text{Ric}(J\cdot, \cdot)] = D^*D\omega - \Delta d\omega.
\]
The Ricci curvature terms simplify to \(-2\rho\), and the result follows. \( \square \)

Furthermore (see [1]), for an almost Kähler structure one has the relation
\[
P = \rho^* - \frac{1}{2} N^1, \tag{3.6}
\]
where
\[
N^1(X,Y) = \langle D_JX\omega, D_Y\omega \rangle, \tag{3.7}
\]
or, in coordinates,
\[
N^1_{ab} = g^{kl} J_k^p J_l^m D_p J^a D_b J^m.
\]
We next derive a more general version of this formula.

**Lemma 3.2.** Let \((M^{2n}, \omega, J)\) be an almost Hermitian manifold. Let \( \nabla \) denote the canonical connection corresponding to \( t = 0 \) in the sense of ([2,2]). Then
\[
P = \rho^* - \frac{1}{2} N^1 + \frac{1}{2} W
\]
where
\[
W(X,Y) = \langle [JD_JJ, D_JX], D_YJ \rangle.
\]

**Proof.** Fix commuting vector fields \( X, Y \), and let \( e_i \) be local normal coordinates for \( g \) at some point. Then
\[
P(X,Y) = \Omega(X,Y, e_i, Je_i)
\]
\[
= \langle \nabla_Y \nabla_X e_i - \nabla_X \nabla_Y e_i, Je_i \rangle
\]
\[
= \langle D_Y \left( D_X e_i + \frac{1}{2} (D_X J) J e_i \right), D_Y J (D_X J) (e_i, Je_i) \rangle
\]
\[
- \text{symmetric term in } X \text{ and } Y
\]
\[
= R(\omega) + \frac{1}{2} \langle [(D_Y D_X - D_X D_Y) J] J e_i, Je_i \rangle
\]
\[
+ \frac{1}{2} \langle (D_X J)(D_Y J) e_i - (D_Y J)(D_X J) e_i, Je_i \rangle
\]
\[
+ \frac{1}{4} \langle (D_Y J)(D_X J J) Je_i - (D_X J)(D_Y J J) Je_i, Je_i \rangle. \tag{3.8}
\]
First we observe that
\[
g^{kl} \langle (D_iD_j - D_jD_i)Je_k, Je_l \rangle = g^{kl} g_{st} \left( R^p_{ijr} J^s_p - R^s_{ijp} J^p_r \right) J^r_k J^t_l = R^k_{ijr} J^r_k + g^{kl} g_{st} R^s_{ijk} J^t_l = R^k_{ijr} J^r_k - R^s_{ijk} J^s_k = 0.
\]
Therefore the second term in the last equality of (3.8) vanishes. Next we note that
\[
-\frac{1}{2} \langle (D_YJ)(D_XJ)e_i, Je_i \rangle = \frac{1}{2} \langle J(D_YJ)(D_XJ)e_i, e_i \rangle = -\frac{1}{2} \langle (D_YJ)J(D_XJ)e_i, e_i \rangle = +\frac{1}{2} \langle (D_YJ)(D_XJ + [JD_XJ,D_JX]) e_i, e_i \rangle = -\frac{1}{2} \langle D_JX\omega, D_Y\omega \rangle + \frac{1}{2} \langle [JD_XJ,D_JX], D_YJ \rangle.
\]
A similar calculation yields the same result for the skew symmetric term. For the remaining term we compute
\[
-\frac{1}{4} \langle (D_XJ)J(D_YJ)e_i, Je_i \rangle = \frac{1}{4} \langle J(D_XJ)(D_YJ)e_i, Je_i \rangle = -\frac{1}{4} \langle (D_XJ + [JD_XJ,D_JX]) (D_YJ)e_i, Je_i \rangle = \frac{1}{4} \langle D_JX\omega, D_Y\omega \rangle - \frac{1}{4} \langle [JD_XJ,D_JX], D_YJ \rangle.
\]
A similar calculation yields the same result for the skew symmetric piece, and the result follows.

It is relevant to us to know that the commutator term in the definition of \( W \) above is determined by \( d\omega \). We record the formula here.

\[\textbf{Lemma 3.3.} \text{ Let } (M^{2n},\omega,J) \text{ be an almost Hermitian manifold. Then}
\]
\[D_JX\omega(Y,Z) - D_X\omega(JY,Z) = (d\omega)^+ (JX,Y,Z) - (d\omega)^+ (JX,JY,JZ). \quad (3.9)\]
\[\text{In particular, if } d\omega = 0 \text{ then one has}
\]
\[D_JXJ = -J(D_XJ). \quad (3.10)\]

\[\textbf{Proof.} \text{ This is a restatement of [9] Proposition 1.iv.} \quad \Box\]

\[\textbf{Lemma 3.4.} \text{ Let } (M^{2n},\omega,J) \text{ be an almost Hermitian manifold. Let } \nabla \text{ denote the canonical connection corresponding to } t = 0 \text{ in the sense of [2,2]. Then}
\]
\[P^{(2,0)+(0,2)} = D^*D\omega - N^2 + \left( -\Delta_d\omega - \frac{1}{2} N^1 + \frac{1}{2} W \right)^{(2,0)+(0,2)} \quad (3.11)\]
\[\text{where}
\]
\[N^2(X,Y) = \langle (DJ)JX, (DJ)Y \rangle, \quad (3.12)\]
\[\text{or, in coordinates,}
\]
\[N^2_{ab} = g^{ij} g_{mn} D_i J^m_p J^p_a D_j J^b_n.\]
Lemma 4.1. Let \( (M^{2n}, J) \) be a complex manifold and suppose \( J(t) \) is a one-parameter family of endomorphisms of \( TM \) such that \( J(0) = J \). Then \( J(t) \) is a one-parameter family of almost-complex structures if and only if for all \( t \),

\[
J \left( \frac{\partial}{\partial t} J \right) + \left( \frac{\partial}{\partial t} J \right) J = 0.
\]

Proof. Combining Lemmas 3.1 and 3.2 yields

\[
P = 2\rho + D^*D\omega - \Delta_d\omega - \frac{1}{2} N + \frac{1}{2} W
\]

(3.13)

for an almost Kähler structure. Since \( \rho \in \Lambda^{1,1} \), it remains to compute the \((2,0) + (0,2)\) component of \( D^*D\omega \). We do this by computing the \((1,1)\) component, which we will compute in local coordinates.

\[
-(D^*D\omega)^{1,1}_{ab} = -\frac{1}{2} \left( (D^*D\omega)(J, J) + D^*D\omega \right)_{ab}
\]

\[
= \frac{1}{2} g^{ij} \left[ (D_iD_j\omega)_{pq}J^p_aJ^q_b + D_iD_j\omega_{ab} \right]
\]

\[
= \frac{1}{2} g^{ij} \left[ D_iD_j (\omega_{pq}J^p_aJ^q_b) + D_iD_j\omega_{ab} - 2D_i\omega_{pq}D_jJ^p_aJ^q_b - 2D_i\omega_{pq}J^p_aD_jJ^q_b 
\]

\[
- \omega_{pq} \left( (D_iD_jJ^p_a)J^q_b + D_iJ^p_aD_jJ^q_b \right) + D_iJ^p_aD_jJ^q_b + J^p_aD_iD_jJ^q_b \right) \right].
\]

Using compatibility of \( \omega \) with \( J \),

\[
D_iD_j (\omega_{pq}J^p_aJ^q_b) = D_iD_j\omega_{ab}.
\]

Also, we have that

\[
-\omega_{pq}(D_iD_jJ^p_a)J^q_b = -g_{pb}D_iD_jJ^p_a
\]

\[
= -D_iD_j (g_{pb}J^p_a)
\]

\[
= -D_iD_j (\omega_{ab}).
\]

Next we compute

\[
-\omega_{pq}J^p_aD_iD_jJ^q_b = g_{aq}D_iD_jJ^q_b
\]

\[
= -D_iD_j\omega_{ab}.
\]

Next note that

\[
D_i\omega_{pq}D_jJ^p_aJ^q_b = D_i \left[ -J^r_qg_{pr} \right] D_jJ^p_aJ^q_b
\]

\[
= -g_{pr} \left[ D_iJ^r_qJ^p_a \right] D_jJ^q_b
\]

\[
= g_{pr}J^r_qD_iJ^p_aD_jJ^q_b
\]

\[
= -\omega_{pq}D_jJ^p_aD_iJ^q_b.
\]

Likewise \( D_i\omega_{pq}J^p_aD_iJ^q_b = -\omega_{pq}D_iJ^p_aD_iJ^q_b \). It follows that

\[
(D^*D\omega)^{1,1}_{ab} = -g^{ij} \omega_{pq}D_iJ^p_aD_jJ^q_b = g^{ij} g_{mn}D_iJ^p_mD_jJ^q_nD_iJ^q_n.
\]

The lemma follows. \( \square \)

4. Variations of Almost Hermitian Structures

Lemma 4.1. Let \( (M^{2n}, J) \) be a complex manifold and suppose \( J(t) \) is a one-parameter family of endomorphisms of \( TM \) such that \( J(0) = J \). Then \( J(t) \) is a one-parameter family of almost-complex structures if and only if for all \( t \),

\[
J \left( \frac{\partial}{\partial t} J \right) + \left( \frac{\partial}{\partial t} J \right) J = 0.
\]
Proof. Assume $J(t)$ is an almost complex structure. Then $J(t)^2 = -\text{Id}$, thus
\[ 0 = \frac{\partial}{\partial t} J^2 = JK + KJ. \]
Conversely, if this equation holds for all time, we may integrate it to obtain that $J^2(t) = J^2(0) = -\text{Id}$, and so $J(t)$ is a one parameter family of almost complex structures. \qed

Lemma 4.2. Let $(M^{2n}, g(t), J(t))$ be a one-parameter family of metrics and almost complex structures, with $g(0)$ compatible with $J(0)$. Further suppose
\[ \frac{\partial}{\partial t} g = H + F \]
\[ \frac{\partial}{\partial t} J = K \]
where $F \in \text{Sym}^{(2,0)+(0,2)} T^* M$ and $H \in \text{Sym}^{1,1} T^* M$. Then $g(t)$ is compatible with $J(t)$ if and only if
\[ F = \frac{1}{2} (g(K,J) + g(J,K)). \]
Furthermore, assuming this holds one has
\[ \frac{\partial}{\partial t} \omega = H(J,\cdot) + \frac{1}{2} [g(K,\cdot) - g(\cdot,K)] \]

Proof. It suffices to show that the time derivative of the compatibility condition vanishes at time $t = 0$. We directly compute
\[ \frac{\partial}{\partial t} (g(J,\cdot) - g(\cdot,\cdot)) = (H + F) (J,\cdot) + g(K,\cdot) + g(J,\cdot) \]
\[ - (H + F) (\cdot,\cdot). \]
Note that $H(J,\cdot) - H(\cdot,\cdot) = 0$. Now let $F = \frac{1}{2} (g(K,J) + g(J,K))$. Observe that
\[ F(J \cdot J) - F(\cdot,\cdot) = \frac{1}{2} [g(KJ,JJ) + g(JJ,KJ) - g(K,J) - g(J,K)] \]
\[ = \frac{1}{2} [-g(KJ,\cdot) - g(\cdot,KJ) - g(K,J) - g(J,K)] \]
Using Lemma 4.1 and compatibility of $J$ and $g$ at time zero we note that
\[ -g(KJ,\cdot) = g(JK,\cdot) = g(JJK,J) = -g(K,J) \]
and likewise $-g(\cdot,KJ) = -g(J,K)$. Thus combining these calculations we conclude that the time derivative vanishes, and the lemma follows. \qed

Lemma 4.3. Let $(M^{2n}, \omega(t), J(t))$ be a one-parameter family of Kähler forms and almost-complex structures with $\omega(0)$ compatible with $J(0)$. Further suppose
\[ \frac{\partial}{\partial t} \omega = \phi + \psi \]
\[ \frac{\partial}{\partial t} J = K \]
where $\psi \in \Lambda^{(2,0)+(0,2)}$ and $\phi \in \Lambda^{1,1}$. Then $\omega(t)$ is compatible with $J(t)$ if and only if
\[ \psi = \frac{1}{2} [\omega(K,J) + \omega(J,K)]. \] (4.1)
Proof. We directly compute
\[
0 = \frac{\partial}{\partial t} (\omega(J\cdot, J\cdot) - \omega(\cdot, \cdot))
\]
\[
= (\phi + \psi)(J\cdot, J\cdot) + \omega(K\cdot, J\cdot) + \omega(J\cdot, K\cdot) - (\phi + \psi)(\cdot, \cdot)
\]
\[
= \psi(J\cdot, J\cdot) - \psi(\cdot, \cdot) + \omega(K\cdot, J\cdot) + \omega(J\cdot, K\cdot).
\]
Since \(\psi \in \Lambda^{(2,0)+(0,2)}\) the above equation is equivalent to
\[
2\psi = \omega(K, J) + \omega(J, K)
\]
as required.

Remark 4.4. Fix a point \(p \in M\) and choose some local coordinates. Certainly (4.1) holds if
\[
K^b_a = g^{bc} \psi_{ac}.
\]
Observe however that \(K\) is not determined by \(\psi\) alone. Indeed one may add to \(K\) an endomorphism of the form \(g^{-1}W - J\), where \(W\) is a symmetric two tensor, and (4.1) will still hold.

Lemma 4.5. Let \((M^{2n}, J)\) be an almost-complex manifold and let \(X\) be a vector field on \(M\). Then
\[
JL_X J + L_X J J = 0.
\]
Furthermore, if \(\omega\) is compatible with \(J\), we have
\[
(L_X \omega)^{(2,0)+(0,2)} = \frac{1}{2} (\omega(L_X J\cdot, J\cdot) + \omega(J\cdot, L_X J\cdot)).
\]
Proof. By [4] pg. 86, we have the formula for \(L_X J\):
\[
(L_X J)(Y) = [X, JY] - J[X, Y].
\]
Given this, the first equation follows by direct calculation. The second equation obviously must hold since it is just the linearized compatibility condition (4.1) and the action of a diffeomorphism preserves compatibility, but we just as well compute
\[
0 = L_X (\omega(\cdot, \cdot) - \omega(J\cdot, J\cdot))
\]
\[
= (L_X \omega)(\cdot, \cdot) - (L_X \omega)(J\cdot, J\cdot) - \omega(L_X J\cdot, J\cdot) - \omega(J\cdot, L_X J\cdot).
\]
Rearranging the above formula gives the result.

Lemma 4.6. Let \((M^{2n}, J)\) be an almost complex manifold and let \(X\) be a vector field on \(M\). Then
\[
(L_X J)^i_k = J^i_p \partial_k X^p - J_p^k \partial_p X^l + X^p \partial_p J^i_k.
\]
Proof. Choose local coordinate vector fields \(e^k\). Using (4.3) we see
\[
(L_X J)^i_k e^k = - \left( J e^k \right)^p \partial_p X^l + J^i_p \left[ e^k \partial_k X^p \right] + X^p \partial_p J^i_k
\]
\[
= - J^i_p \partial_p X^l + J^i_p \partial_k X^p + X^p \partial_p J^i_k,
\]
as required.
5. Parabolic Flows of Almost Hermitian structures

In this section we prove Theorem 1.1. Let us recall some definitions from the introduction used in (1.2). In particular, let \((M^{2n}, \omega, J)\) be an almost Hermitian manifold and let \(\nabla\) denote the associated Chern connection (see Definition 2.1). Let \(\Omega\) denote the \((4,0)\) curvature tensor associated to \(\nabla\), and consider

\[ S_{ij} = \omega_{kl}^{\beta} \Omega_{\beta klij}. \]

Furthermore, let \(N\) denote the Nijenhuis tensor associated to \(J\), and let

\[ K^i_j = \omega_{kl}^{\beta} \nabla^k N_{lj}^i. \]

Let \(Q\) denote a \((1,1)\) tensor which is quadratic expression in the torsion of \(\nabla\), and let \(\mathcal{H}\) denote a \(J\)-skew endomorphism of the tangent bundle which again is quadratic in the Nijenhuis tensor. Let

\[ H = \frac{1}{2} \left[ \omega(\mathcal{K} - \mathcal{H}, J) + \omega(J, \mathcal{K} - \mathcal{H}) \right]. \tag{5.1} \]

Consider the initial value problem

\[
\begin{align*}
\frac{\partial}{\partial t} \omega &= -S + Q + H \\
\frac{\partial}{\partial t} J &= -K + \mathcal{H} \tag{5.2} \\
\omega(0) &= \omega_0 \\
J(0) &= J_0.
\end{align*}
\]

We first show some preliminary lemmas which show that the right hand sides of (5.2) satisfy the linearized conditions for one parameter families of almost Hermitian pairs from Lemmas 4.1 and 4.3

**Lemma 5.1.** Let \((M^{2n}, J)\) be an almost-Hermitian manifold. Then, viewing the Nijenhuis tensor \(N\) as a section of \(\Lambda^1 \otimes \text{End}(TM)\),

\[ JN + NJ = 0. \]

**Proof.** We can derive this by direct calculation using the definition of the Nijenhuis tensor (2.1). First

\[
-JN = J ([X, Y] + J ([JX, Y] + [X, JY]) - [JX, JY]) \\
= J[X, Y] - [JX, Y] - [X, JY] - J[JX, JY].
\]

Next

\[
-NJ = [X, JY] + J ([JX, JY] + [X, JJJY]) - [JX, JYY] \\
= [X, JY] + J[JX, JY] - J[X, Y] + [JX, Y].
\]

The result follows adding these two calculations together. \(\square\)

**Lemma 5.2.** Let \((M^{2n}, \omega, J)\) be an almost Hermitian manifold. Then

\[ JK + KJ = 0. \]
Proof. We may write the result of Lemma 5.1 in coordinates as
\[ J^m_{k} N^l_{jm} + N^m_{jk} J^l_j = 0 \]
We differentiate this using the Chern connection. Since \( J \) is parallel we see
\[ 0 = J^m_{k} \nabla_i N^l_{jm} + \nabla_i N^m_{jk} J^l_j \]
we can now take the required contraction of indices using \( \omega \) to yield the statement of the lemma. \( \square \)

Definition 5.3. Let \((M^{2n}, \omega, J)\) be an almost Hermitian manifold. Let \(\nabla\) denote some fixed connection on \(TM\). Define a vector field
\[ X^p = (\omega, J, \nabla)^p = \omega k l \nabla_k N^i_{lj} \]
(5.3)

Proposition 5.4. Let \((M^{2n}, \omega, J)\) be an almost-Hermitian manifold and let \(\nabla\) denote some fixed connection on \(TM\). The map
\[ J \rightarrow K - L_{X(\omega, J, \nabla)} J \]
is a second order elliptic operator.

Proof. We recall a coordinate formula for the Nijenhuis tensor.
\[ N^i_{jk} = J^p_{k} \partial_p J^i_{j} - J^p_{j} \partial_p J^i_{k} - J^p_{j} \partial_k J^i_{p} + J^p_{k} \partial_j J^i_{p} \]
(5.4)
It follows that
\[ K^i_j = \omega^{kl} \nabla_k N^i_{lj} \]
\[ = \omega^{kl} \left( J^q_{k} \partial_q J^i_{j} - J^q_{j} \partial_q J^i_{k} - J^p_{j} \partial_k J^i_{p} + J^p_{k} \partial_j J^i_{p} \right) + O(\partial J, \partial \omega) \]
(5.5)
where the notation \( O(\partial J, \partial \omega) \) means an expression which only depends on at most first derivatives of \( J \) and \( \omega \) (possibly in a nonlinear fashion). In particular, Chern connection terms are of this form. Note that the matrix \( \omega \) is skew-symmetric, but coordinate derivatives are symmetric, therefore the middle term in the parentheses in the last line vanishes. Also, using (5.3) and Lemma 4.6 we express
\[ \left[ L_{X(\omega, J, \nabla)} J \right]^i_j = \omega^{kl} \left( J^q_{k} \partial_q J^i_{j} - J^q_{j} \partial_q J^i_{k} \right) + O(\partial J, \partial \omega) \]
Combining these two calculations yields
\[ \left[ K - L_{X(\omega, J, \nabla)} J \right]^i_j = -g^{kl} \partial_k J^i_j + O(\partial J, \partial \omega). \]
The claim follows immediately. \( \square \)

Proposition 5.5. Let \((M^{2n}, \omega, J)\) be an almost-Hermitian manifold. The map
\[ \omega \rightarrow S(\omega) \]
is a second order elliptic operator.
Proof. Fix a point \( p \in M \), and choose a local frame of \((1,0)\) vector fields \( \{e_i\} \) such that \( g_{\sigma}(p) = \delta_{ij} \). In this frame we compute using metric compatibility of \( \nabla \),

\[
S_{k\ell} = \omega \partial_j \Omega_{j\ell k} \\
= \omega \partial_j \left( \nabla_i \nabla_j e_k - \nabla_j \nabla_i e_k - \nabla_{[e_i,e_j]} e_k, e_\ell \right) \\
= \omega \partial_j \left( e_i \left( \nabla_j e_k, e_\ell \right) - \nabla_j \nabla_i e_k + e_j \left( \nabla_i e_k, e_\ell \right) \right) + O(\partial \omega, \partial J) \\
= \omega \partial_j \left( e_i \left( \nabla_j e_k, e_\ell \right) - e_j e_i \left( e_k, e_\ell \right) + e_j \left( e_k, \nabla_i e_\ell \right) \right) + O(\partial \omega, \partial J).
\]

Now using \( J \) compatibility of the connection and the fact that the torsion \( T \) has no \((1,1)\)-component we see that

\[
\left( \nabla_j e_k, e_\ell \right) = \left( \nabla_j e_k - \nabla_k e_j, e_\ell \right) = \left( T_{jk} + [e_j, e_k], e_\ell \right) = \left( [e_j, e_k], e_\ell \right) = O(\omega, \partial J).
\]

The last line follows since the basis \( e_i \) is constructed by projecting local coordinates onto \( T^{1,0} \) using \( J \), and therefore their Lie brackets will only contain derivatives of \( J \). Likewise one concludes that \( \langle e_k, \nabla e_i e_\ell \rangle = O(\omega, \partial J) \). Therefore

\[
S_{k\ell} = -g^b e_j e_i \omega_{k\ell} + O(\partial \omega, \partial^2 J) \]

\[
= -\frac{1}{2} g^{ab} \partial_a \partial_b \omega_{k\ell} + O(\partial \omega, \partial^2 J).
\]

The result follows.

We can now give the proof of Theorem 1.1.

Proof. First we show existence. This will be a two step process. First we will define a gauge-fixed flow which will define a strictly parabolic system. This flow equation will only be defined however for compatible pairs. Therefore to apply short time existence results from the theory of parabolic differential equations one needs to define a more general evolution equation which makes sense for arbitrary pairs \((\omega, J)\). Moreover, the desired flow on \( J \) takes place in a nonlinear manifold, therefore one must “pull back” the flow on \( J \) to a linear space, namely the tangent space to the space of almost complex structures at \( J_0 \). We define such a generalized version of our gauge-fixed flow which has short time existence. We can show that this flow preserves compatibility of the initial condition and produces a solution of the original gauge-fixed flow. We remove the gauge parameter to finally produce the required solution of the original flow.

Fix \( \nabla \) any connection on \( TM \) and let \( X \) be defined as in Definition 5.3. First consider the following gauge-fixed version of equation (5.2)

\[
\frac{\partial}{\partial t} \omega = -S + Q + H + L_{X_{(g,J)}} \omega = \mathcal{D}_1(\omega, J) \\
\frac{\partial}{\partial t} J = -K + \mathcal{H} + L_{X_{(g,J)}} J = \mathcal{D}_2(\omega, J).
\]
We observe that by definition the vector field $X(g, J)$ can be expressed completely in terms of first derivatives of $J$ and therefore $(L_{X(g, J)} \omega)_{ij}$ is a first order operator in $\omega$. Use $L_{\omega}, L_{J}$ to denote linearization in the $\omega$ and $J$ variables respectively. It follows from Proposition 5.5 that

$$\sigma \left[ L_{\omega} D_1 \right] (h)_{ij} = \sigma \left[ L(-2S) \right] (h)_{ij} = |\xi|^2 h_{ij}.$$  

Furthermore, from Proposition 5.3 we conclude that

$$\sigma \left[ L_{J} D_2 \right] (K)_{i}^{j} = |\xi|^2 K_{i}^{j}.$$  

We also need to check the linearization of $D_2$ in the variable $\omega$. Since by construction we have that $D_2$ only depends on first derivatives of $\omega$, we conclude

$$\sigma \left[ L_{\omega} D_2 \right] (h)_{ij} = 0.$$  

We note that second derivative terms of $J$ appear in the evolution of $\omega$, therefore these terms appear in the full linearized operator. Collecting these observations we conclude that the overall symbol is upper-triangular. In particular it takes the form

$$\sigma \left[ \hat{L} D \right] (h, K) = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix} \begin{pmatrix} h \\ K \end{pmatrix}.$$  

It follows that (5.6) is a strictly parabolic system of equations. However, as mentioned above we must define a more general flow defined for arbitrary pairs $(\omega, J)$ to apply short time existence theory. First note that the space of almost complex structures $\mathcal{J}$ near a fixed $J_0$ is a smooth Banach manifold with tangent space modeled on the space of $J_0$-skew endomorphisms (see Lemma 4.1). On obtains a diffeomorphism

$$\pi : U_0 \subset T_{\mathcal{J}} J_0 \to U_{J_0} \subset \mathcal{J}$$  

from a neighborhood of 0 in $T_{\mathcal{J}} J_0$ to a neighborhood of $J_0$ in $\mathcal{J}$. Moreover this diffeomorphism satisfies

$$D\pi_{0} = \text{Id}_{T_{\mathcal{J}} J_0}. \quad (5.7)$$  

Note that our desired flow for $J$ is moving through a certain (nonlinear) manifold in the space of endomorphisms of the tangent bundle. We will use the map $\pi$ to pull back the flow onto the linear space $T_{\mathcal{J}} J_0$.

Suppose now $(\omega_0, J_0)$ is a compatible pair and let $g_0$ denote the associated metric. Recall that if $J$ is an almost complex structure and $\omega \in \Lambda^2$, $\omega_J$ denotes the $J$-symmetric piece of $\omega$, while $\omega^{-J}$ denotes the $J$-anti-invariant piece. Consider now the initial value problem

$$\frac{\partial}{\partial t} \omega = D_1 (\omega^{\pi E}, \pi E) - D_2^* g_0 D_{g_0} (\omega^{-\pi E}) =: \tilde{D}_1 (\omega, E)$$  

$$\frac{\partial}{\partial t} E = (D \pi_{-E}^{-1}) (D_2 (\omega^{\pi E}, \pi E)) =: \tilde{D}_2 (\omega, E).$$  

$$\omega(0) = \omega_0$$  

$$E(0) = 0.$$  

We observed above that $D_2 (\omega^{\pi E}, \pi E)$ lies in $T_{\pi E} \mathcal{J}$, therefore the operator $\tilde{D}_2$ is well defined, and has image in $T(T_{\mathcal{J}} E) \cong T_{\mathcal{J}} J_0$ for arbitrary pairs $(\omega, J)$, therefore this equation defines a flow in the linear space $B := T_{\mathcal{J}} J_0 \oplus \Lambda^2 \mathbb{R}$. 


We want to compute the linearization of this system at \( t = 0 \). First we compute the linearization of \( \tilde{D}_1 \) in the \( \omega \) variable. Combining Proposition 5.5 with an obvious calculation of the symbol for \( D_{g_0}^* D_{g_0} (\omega^{-\pi E}) \) yields
\[
\sigma \left[ \mathcal{L}_\omega \left( D_1 (\omega^{\pi E}, \pi E) - D_{g_0}^* D_{g_0} (\omega^{-\pi E}) \right) \right] \wedge (h)_{ij} = |\xi|^2 h_{ij}^{\pi E} + |\xi|^2 h_{ij}^{-\pi E} = |\xi|^2 h_{ij}.
\]

Furthermore, the calculation of the linearization of \( \tilde{D}_2 (\omega, E) \) at \( t = 0 \) is identical to that for \( D_2 \) above using (5.7). It follows that
\[
\sigma \left[ \hat{\mathcal{L}} \tilde{D}_2 \right] (h, K) = \left( I^* 0 I \right) \left( h \right).
\]

Therefore the initial value problem (5.8) is a nonlinear strictly parabolic equation in the Banach space \( B \), and standard results imply the existence of a short time solution to (5.8).

Now let \( J = \pi E \). We claim that \((\omega, J)\) is in fact a solution to (5.6). First we compute
\[
\frac{\partial}{\partial t} J = (D\pi_E) \left( \frac{\partial}{\partial t} J \right) = (D\pi_E) \left( D\pi^{-1}_E (D_2 (\omega^J, J)) \right) = D_2 (\omega^J, J).
\]

Next we want to show compatibility of the pair \((\omega, J)\) is preserved. Note that since we have already computed that solutions to (5.6) satisfy the conditions of Lemma 4.3 one has
\[
\frac{\partial}{\partial t} \omega - J = \left[ -D_{g_0}^* D_{g_0} (\omega^{-J}) \right]^{-J}.
\]

Now let \( \tilde{g}(t) \) be a one-parameter family of metrics which is compatible with \( J(t) \). It follows that
\[
\frac{\partial}{\partial t} |\omega^{-J}|_{\tilde{g}} = 2 \left\langle D_{g_0}^* D_{g_0} \omega^{-J}, \omega^{-J} \right\rangle_{g_0} + \left( \frac{\partial}{\partial t} \tilde{g} \right)^* (\omega^{-J})^2
\]
\[
= 2 \left( \text{tr}_{g_0} \left( D_{g}^2 \tilde{g} + \tilde{R} + R_{0} \right) \omega^{-J}, \omega^{-J} \right) + \left( \frac{\partial}{\partial t} \tilde{g} \right)^* (\omega^{-J})^2\] (5.9)
\[
\leq \text{tr}_{g_0} D_{g}^2 |\omega^{-J}|_{g_0} - 2 D_{\tilde{g}} |\omega^{-J}|_{g_0}^2 + C |\omega^{-J}|_{\tilde{g}}^2.
\]

Applying the maximum principle to \( e^{-Ct} |\omega^{-J}|_{\tilde{g}}^2 \) we conclude that if \( \omega^{-J}(0) \equiv 0 \), then \( \omega^{-J}(t) \equiv 0 \) for all \( t \). Hence the pair \((\omega(t), J(t))\) is compatible for all \( t \), and we conclude that the one parameter family \((\omega(t), J(t))\) is a solution to (5.6).

Now we want to pull back our solution to (5.6) by the family of diffeomorphisms generated by \( X \). Specifically let \( \phi_t \) be a one-parameter family of diffeomorphisms of \( M \) defined by the ODE
\[
\frac{\partial}{\partial t} \phi_t = -X(\omega(t), J(t), \nabla) \quad \quad \phi_0 = \text{id}_M.
\] (5.10)
It follows that
\[
\frac{\partial}{\partial t} (\phi_t^* \omega(t)) = \frac{\partial}{\partial s} \bigg|_{s=0} \left( \phi_{t+s}^* \omega(t+s) \right) = \phi_t^* \left( \frac{\partial}{\partial t} \omega(t) \right) + \frac{\partial}{\partial s} \bigg|_{s=0} \left( \phi_{t+s}^* \omega(t) \right) = \phi_t^* (-S + Q + H + L_{X(\omega(t), J(t))} \omega) + \frac{\partial}{\partial s} \bigg|_{s=0} \left[ (\phi_t^{-1} \circ \phi_{t+s})^* \phi_t^* \omega(t) \right] = (-S + Q + H) (\phi_t^*(\omega), \phi_t^*(J)) + \phi_t^*(L_{X(\omega(t), J(t))} - L_{(\phi_t^{-1})_{X(\omega(t), J(t))}}) (\phi_t^* \omega(t)) = (-S + Q + H) (\phi_t^*(\omega), \phi_t^*(J)). \]

Likewise we may compute
\[
\frac{\partial}{\partial t} (\phi_t^* J(t)) = \frac{\partial}{\partial s} \bigg|_{s=0} \left( \phi_{t+s}^* J(t+s) \right) = \phi_t^* \left( \frac{\partial}{\partial t} J(t) \right) + \frac{\partial}{\partial s} \bigg|_{s=0} \left( \phi_{t+s}^* J(t) \right) = \phi_t^* (-\mathcal{K}(\omega, J) + \mathcal{H}(\omega, J) + L_{X(\omega(t), J(t))}) + \frac{\partial}{\partial s} \bigg|_{s=0} \left[ (\phi_t^{-1} \circ \phi_{t+s})^* \phi_t^* J(t) \right] = -\mathcal{K}(\phi_t^*(\omega), \phi_t^*(J)) + \mathcal{H}(\phi_t^*(\omega), \phi_t^*(J)) + \phi_t^*(L_{X(\omega(t), J(t))} - L_{(\phi_t^{-1})_{X(\omega(t), J(t))}}) (\phi_t^* J(t)) = -\mathcal{K}(\phi_t^*(\omega), \phi_t^*(J)) + \mathcal{H}(\phi_t^*(\omega), \phi_t^*(J)). \]

Therefore \((\phi_t^*(\omega(t), \phi_t^*(J(t)))\) is a solution to \(5.2\).

Next we show uniqueness. As in the proof of uniqueness for Ricci flow, we will show that the diffeomorphism ODE \(5.10\), when written with respect to the changing metric, is in fact a parabolic equation for \(\phi\). What is more, as we now show, our choice of \(X\) is essentially equivalent to that used for Ricci flow short-time existence. Let \(\Gamma_C, \Gamma, \Upsilon\) denote the connection coefficients of the Chern, Levi-Civita, and background connections respectively. Consider the following calculation:

\[
X^p = \omega^{kl} \nabla_k J_l^p = \omega^{kl} \partial_k J_l^p + \mathcal{O}(\omega, J) = \omega^{kl} \left( \nabla_k J_l^p + (\Gamma_C)^q_{kl} J_q^p - (\Gamma_C)_j^p J_q^i \right) + \mathcal{O}(\omega, J) = \omega^{kl} (\Gamma_C)^q_{kl} J_q^p + g^{kj} (\Gamma_C)_k^p J_q^j + \mathcal{O}(\omega, J). \tag{5.11}
\]

The first term is the contraction of the Chern connection coefficient with a skew-symmetric one-form, and hence vanishes. Specifically we compute

\[
\omega^{kl} (\Gamma_C)^q_{kl} J_q^p = \frac{1}{2} \omega^{kl} ((\Gamma_C)^q_{kl} - (\Gamma_C)^q_{lk}) J_q^p = \frac{1}{2} \omega^{kl} T_{kl}^q J_q^p \tag{5.12}
\]

since the torsion of \(\nabla\) has no \((1,1)\) component. Next observe that since \(g\) is symmetric, the contraction \(g^{kj} \Gamma_{kq}^p\) does not involve the torsion of the connection. In particular we
conclude

\[ g^{kq} (\Gamma^p_C)_{kq} = g^{kq} \Gamma^p_{kq} \]

In particular, combining these calculations we may conclude that

\[ X^p = g^{kl} [\Gamma^p_{kl} - \Gamma^p_{kl}] + O(\omega, J). \tag{5.13} \]

In particular we have shown that, up to lower order terms, the vector field we used in our short-time existence proof is the same as that used for Ricci flow. So, set \( \tilde{g} = \phi_t^* g(t) \), \( \tilde{J} = \phi_t^* J(t) \). It follows (see [5] pg. 89) that one may rewrite the solution to (5.10) as

\[ \frac{\partial}{\partial t} \phi_t = \Delta_{\tilde{g}(t), \phi_0} \phi_t + O(\partial \phi) \tag{5.14} \]

where \( \Delta_{\tilde{g}(t), \phi_0} \) is the harmonic map Laplacian taken with respect to the metrics \( \tilde{g}(t) \) and \( g_0 \).

We now proceed with the proof of uniqueness. Let \( \tilde{g}_1(t), \tilde{g}_2(t) \) be two solutions to (5.2) with \( \tilde{g}_1(0) = \tilde{g}_2(0) = g_0 \). Let \( \phi_i(t) \) be solutions to (5.14) with respect to \( \tilde{g}_i(t) \), which exists in general because (5.14) is strictly parabolic and \( M \) is compact. Now, pushing forward by these diffeomorphisms we observe that \( g_i(t) := (\phi_i(t))^* \tilde{g}_i(t) \) are both solutions of (5.6).

Since \( g_1(0) = g_2(0) \) and solutions to (5.6) are unique, it follows that \( g_1(t) = g_2(t) \) as long as these metrics are defined. But now one observes that \( \phi_1(t) \) and \( \phi_2(t) \) are both solutions to the same ODE (5.10) with the same initial condition, and are therefore equal. It follows that \( \tilde{g}_1(t) = \tilde{g}_2(t) \) as long as they are both defined and the result follows.

Now consider the case where \( J_0 \) is integrable. In this case one can consider the flow of Kähler forms

\[ \frac{\partial}{\partial t} \omega = -S + Q \]

\[ \omega(0) = \omega_0. \]

Here \( \omega_0 \) is compatible with an integrable complex structure, and so this equation is an example of Hermitian curvature flow, studied in [18]. In particular, there is a short time solution to this equation which remains compatible with \( J_0 \). If we let \( J(t) = J_0 \), it is easy to verify that \( (\omega(t), J(t)) \) is a solution to (5.2), and more to the point, the unique solution. If the initial structure \( (\omega, J) \) is Kähler one can verify that the solution to (5.2) is the Kähler Ricci flow by a similar argument.

6. Symplectic Curvature Flow

In this section we will motivate and investigate the equation (1.4). The general philosophical starting point is clear: one would like to define a flow of symplectic structures \( \frac{\partial}{\partial t} \omega = -P \), purely in analogy with Kähler Ricci flow. However, \( P \) is not a \((1, 1)\)-form, so \( \omega \) would not stay compatible with \( J \), and then the definitions fall apart. Thus one is naturally led to allowing \( J \) to flow as well. Lemma 4.3 suggests part, but not all of what should appear in the evolution equation for \( J \) (see Remark 4.4). It is a small miracle that in the almost Kähler setting there is a very natural choice for this component, i.e. the endomorphism \( R \), which ends up yielding a parabolic equation. These considerations lead to the definition of
which we repeat here:

\[
\frac{\partial}{\partial t}\omega = -P \\
\frac{\partial}{\partial t}J = -D^*DJ - N + R \\
\omega(0) = \omega_0 \\
J(0) = J_0.
\]

Before proving Theorem 1.6 we give two equivalent formulations of this flow, one putting it in the framework of equation (1.2), the other realizing this system as a flow coupling a parabolic for \(J\) with Ricci flow. The latter formulation is the appropriate viewpoint to use to show the short time existence of solutions to (1.4).

**Proposition 6.1.** Let \((M^{2n}, \omega(t), J(t))\) be a one-parameter family of almost Kähler structures solving (1.4). Then the family \((\omega(t), J(t))\) is a solution to

\[
\frac{\partial}{\partial t}\omega = -S + Q^1 + H \\
\frac{\partial}{\partial t}J = -K + H^1
\]

where \(Q^1\) and \(H^1\) are defined in (6.3) and (6.4) respectively, and \(H\) is defined according to (5.1). In particular, (6.1) is a degenerate parabolic equation for almost Hermitian pairs \((\omega, J)\) which preserves the almost Kähler condition.

**Proof.** Let \((\omega, J)\) be an almost Kähler structure, and let \(\Omega\) denote the curvature of the Chern connection. Let \(\{e_i\}\) denote a local orthonormal frame for \(T^{1,0}(M)\). First recall the Bianchi identity for a connection \(\nabla\):

\[
\Sigma_{X,Y,Z} [\Omega(X,Y)Z - T(T(X,Y),Z) - \nabla_X T(Y,Z)] = 0
\]

For our almost Kähler structure the torsion \(T\) is completely determined by the Nijenhuis tensor, which is a \((0,2)\) form with values in \((1,0)\) vectors. Using this we compute an expression for the \((1,1)\) part of \(P\).

\[
P(e_j, \bar{e}_k) = \Omega(e_j, \bar{e}_k, e_i, \bar{e}_i) \\
= \Omega(e_i, \bar{e}_k, e_j, \bar{e}_i) + \langle N(N(e_i, e_j), \bar{e}_k), \bar{e}_i\rangle + \langle \nabla_{\bar{e}_k}N(e_i, e_j), \bar{e}_i\rangle \\
= -\Omega(e_i, \bar{e}_k, e_j, \bar{e}_i) + \langle N(N(e_i, e_j), \bar{e}_k), \bar{e}_i\rangle + \langle \nabla_{\bar{e}_k}N(e_i, e_j), \bar{e}_i\rangle \\
= S(e_j, \bar{e}_k) - \langle N(N(\bar{e}_k, \bar{e}_i), e_i), e_j\rangle - \langle \nabla_{e_i}N(\bar{e}_k, \bar{e}_i), e_j\rangle \\
+ \langle N(N(e_i, e_j), \bar{e}_k), \bar{e}_i\rangle + \langle \nabla_{\bar{e}_k}N(e_i, e_j), \bar{e}_i\rangle.
\]

But since \(N\) takes values in \((1,0)\) vectors and \(\nabla\) is a Hermitian connection, it follows that

\[
\langle \nabla_{e_i}N(\bar{e}_k, \bar{e}_i), e_j\rangle = \langle \nabla_{\bar{e}_k}N(e_i, e_j), e_i\rangle = 0.
\]

It follows that

\[
P^{1,1} = S + Q^1
\]

where

\[
Q^1_{ij} = \omega^{k\ell} \left( g_{mj} N^m_{ki} N^p_{jl} - g_{mj} N^m_{ij} N^p_{jl} \right).
\]
Next we examine the evolution equation for $J$. Choose normal coordinates for the associated metric at a point $p$. Then, including the precise lower order terms in (5.5), we see that
\[
\omega^{kl} \nabla_k N^i_{lj} = - g^{kl} \partial_k \partial_l J^i_j + \omega^{kl} \left( J^q_j \partial_k \partial_j J^p_l - J^q_j \partial_k \partial_q J^i_l \right) \\
+ \omega^{kl} \left( D_k J^p_l D_p J^i_j - D_k J^q_j D_p J^p_l - D_k J^q_j D_l J^p_p + D_k J^p_p D_l J^q_j \right) \\
+ \frac{1}{2} \omega^{kl} \left( N^i_{kp} N^p_{lj} - N^p_{kl} N^i_{pj} - N^p_{kj} N^i_{lp} \right).
\]
Furthermore, by a direct calculation in normal coordinates at $p$ one has
\[
(-D^* DJ + N + R)^i_j = g^{kl} \partial_k \partial_l J^i_j + g^{kl} J^p_l \partial_p \Gamma^i_{kl} - g^{kl} J^l_j \partial_j \Gamma^p_{kl} + N^i_j.
\]
Furthermore, calculating as in (5.11), (5.12), again using the normal coordinates,
\[
g^{kl} J^p_l \partial_p \Gamma^i_{kl} = J^p_l \partial_p \left( g^{kl} \Gamma^i_{kl} \right) = J^p_l \partial_p \left( g^{kl} (\Gamma C)_{kl} + \omega^{kl} (\Gamma C)^q_{kl} \right) \\
= J^p_l \partial_p \left( \omega^{kl} \partial_k J^i_l - \omega^{kl} \nabla_k J^i_l \right) \\
= J^p_l \omega^{kl} \partial_p \partial_k J^i_l - J^p_l \omega^{kr} J^l_q D_p J^p_l D_k J^q_i.
\]
Likewise
\[
g^{kl} J^l_j \partial_j \Gamma^p_{kl} = \omega^{kl} J^l_j \partial_l \partial_j J^p_p - J^l_j \omega^{kr} J^q_i D_l J^p_q D_k J^q_i.
\]
Combining these calculations yields
\[
(-D^* DJ + N + R) = - \mathcal{K} + \mathcal{H}^l
\]
where
\[
(\mathcal{H}^l)^i_j = \omega^{kl} \left( D_k J^p_l D_p J^i_j - D_k J^q_j D_p J^p_l - D_k J^q_j D_l J^p_p + D_k J^p_p D_l J^q_j \right) \\
+ \frac{1}{2} \omega^{kl} \left( N^i_{kp} N^p_{lj} - N^p_{kl} N^i_{pj} - N^p_{kj} N^i_{lp} \right) \\
- J^p_l \omega^{kr} J^l_q D_p J^p_l D_k J^q_i + J^l_j \omega^{kr} J^q_i D_l J^p_q D_k J^q_i + g^{ik} \omega^{pq} \omega_{pq} D_p J^p_l D_l J^q_i.
\]
Finally, it is clear by construction that if we define $H$ so that (5.1) holds, it must equal $-p^{2.0+0.2}$. It follows that a solution to (1.4) is a solution to (6.1), and the proposition follows. The final statement of the proposition will follow once existence and uniqueness of solutions to (1.4) is established. \hfill \Box

**Proposition 6.2.** Let $(M^{2n}, \omega(t), J(t))$ be a one-parameter family of almost Kähler structures solving (1.4). Then the associated Riemannian metric $g(t)$ satisfies
\[
\frac{\partial}{\partial t} g = -2 \text{Ric} + \frac{1}{2} B^1 - B^2
\]
where
\[
B^1(\cdot, \cdot) = N^i(\cdot, J^i).\]

In coordinates one has
\[
B^1_{ij} = g^{kl} g_{mn} D_i J^m_k D_j J^n_l, \\
B^2_{ij} = g^{kl} g_{mn} D_k J^m_i D_l J^n_j.
\]
Proof. We begin with a general calculation using the notation of Lemma 4.3. Specifically, we have
\[
\frac{\partial}{\partial t} g(\cdot, \cdot) = \frac{\partial}{\partial t} [\omega(\cdot, J \cdot)]
= [\phi(\cdot, J \cdot) + \psi(\cdot, J \cdot) + \omega(\cdot, K \cdot)].
\]

Let us compute these three terms separately. First of all, since \(d\omega = 0\) it follows from (3.11) and (3.13) that
\[
\phi(\cdot, J \cdot) = -P^{1,1}(\cdot, J \cdot) = P^{2,0+0,2}(\cdot, J \cdot) - P(\cdot, J \cdot)
= \left( -2\rho + \frac{1}{2} N^1 - N^2 \right)(\cdot, J \cdot)
= -2 \text{Ric}^J(\cdot, \cdot) + \frac{1}{2} B^1(\cdot, \cdot) - B^2(\cdot, \cdot).
\]

Now observe that
\[
\psi(\cdot, J \cdot) = -P^{2,0+0,2}(\cdot, J \cdot).
\]

Next consider
\[
\omega(\cdot, K \cdot)_{ij} = \omega_{ik} K^k_j = \omega_{ik} \left( g^{kl} \left( -P^{2,0+0,2}_{ji} + J^l_j \text{Rc}^k_l - \text{Rc}_j^l J^k_l \right) \right).
\]
The first term simplifies to
\[
-\omega_{ik} g^{kl} P^{2,0+0,2}_{jl} = -J^p_l g_{pk} g^{kl} P^{2,0+0,2}_{jl}
= -J^p_l P^{2,0+0,2}_{jl}
= J^p_l \left( J^m_j P^{2,0+0,2}_{mp} \right)
= -J^p_l P^{2,0+0,2}_{mi}
= P^{2,0+0,2}(\cdot, J \cdot)_{ij}.
\]

Next we calculate
\[
\omega_{ik} \left( \text{Rc}^l_j J^k_l - J^l_j \text{Rc}^k_l \right) = J^p_l g_{pk} \left( J^l_j \text{Rc}^k_l - \text{Rc}_j^l J^k_l \right)
= \left( J^* \text{Ric} - \text{Rc} \right)_{ij}
= -2 \left( \text{Rc} - \text{Ric}^J \right)_{ij}.
\]

Combining the above calculations, the result follows. \(\square\)

Now we give the proof of Theorem 1.6.

Proof. As in the proof of Theorem 1.1, we must work in a more general setting to construct solutions to (1.4). It will be most convenient in this case to work with a coupled system of Riemannian metric and almost complex structure. The main goal will be to construct a
solution to
\[
\frac{\partial}{\partial t} g = -2 \text{Ric} + \left(\frac{1}{2} N^1(\cdot, J\cdot) - N^2(\cdot, J\cdot)\right)^J
\]
\[
\frac{\partial}{\partial t} J = -D^* DJ - N + \mathcal{R}
\]
\[
g(0) = g_0
\]
\[
J(0) = J_0
\]
for a compatible initial condition \((g_0, J_0)\). We will accomplish this in the same two step process employed in the proof of Theorem 1.1, first defining a gauge-fixed flow, then defining a flow for arbitrary pairs \((g, J)\) which preserves compatibility of the initial condition. In particular, let \(X\) be defined as in (5.3), and consider the gauge fixed flow
\[
\frac{\partial}{\partial t} g = -2 \text{Ric} + \left(\frac{1}{2} N^1(\cdot, J\cdot) - N^2(\cdot, J\cdot)\right)^J + L_X g =: \mathcal{D}_1(g, J)
\]
\[
\frac{\partial}{\partial t} J = -D^* DJ - N + \mathcal{R} + L_X J =: \mathcal{D}_2(g, J).
\]
(6.7)

Also as in Theorem 1.1 we must modify the flow and pull back to a linear space to construct the solution to (6.7). Using the notation of the proof of Theorem 1.1 let
\[
\frac{\partial}{\partial t} g = \mathcal{D}_1(g^\pi E, \pi E) - D^* g_0 D g^{-\pi E} =: \mathcal{\tilde{D}}_1(g, E)
\]
\[
\frac{\partial}{\partial t} E = (D \pi^{-1} \pi E) (\mathcal{D}_2(g^\pi E, \pi E)) =: \mathcal{\tilde{D}}_2(g, E)
\]
\[
g(0) = g_0
\]
\[
E(0) = 0.
\]
(6.8)

First note that in fact the evolution for \(E\) is well defined. Indeed, an endomorphism of the form \(g^{-1} \psi\) where \(\psi \in \Lambda^{(2,0) + (0,2)}\) automatically satisfies \(KJ + JK = 0\). Therefore by Lemma 3.4 the tensor \(-D^* DJ + \mathcal{N}\) is already \(J\)-skew, as of course is \(\mathcal{R}\). Therefore \(\mathcal{D}_2(g^E, \pi E) \in T_{\pi E}, \text{ and so } \mathcal{\tilde{D}}_2 \text{ is well defined.}

We compute the linearization of (6.8) at \(t = 0\). Here the small miracle of equation (5.13) is highly relevant, i.e. that the vector field \(X\) generates the same one parameter family of diffeomorphisms which appears in proving short time existence of Ricci flow. It follows from a well-known calculation (see [5] pg. 114) that
\[
\mathcal{L}_g ((-2 \text{Rc} + L_X g) (h)) = \Delta_L h + \text{lower order terms.}
\]
Since the terms \(N^1\) and \(N^2\) are both first order in both \(g\) and \(J\), we conclude that
\[
\sigma \left[ \mathcal{L}_g \mathcal{D}_1 \right] (h)_{ij} = |\xi|^2 h^\pi E_{ij} + |\xi|^2 h^{-\pi E}_{ij} = |\xi|^2 h_{ij}
\]
\[
\sigma \left[ \mathcal{L}_J \mathcal{\tilde{D}}_1 \right] (K)_{ij} = 0.
\]
Next we compute the linearization of $\tilde{D}_2$ at $t = 0$. We do this by first computing a coordinate formula for $D_2$ applied to a compatible pair $(g, J)$ in stages. First of all we have

$$-[D^*DJ]_k^l = g^{ij}\partial_k [DJ]_{jk}^l + O(\partial J, \partial g)$$

$$= g^{ij}\partial_k \left[ \partial_j J_k^l - \Gamma^l_{jk} J_p^i + \Gamma^l_{jp} J_k^i \right] + O(\partial J, \partial g) \quad (6.9)$$

$$= g^{ij} \left[ \partial_t \partial_i J_k^l - \partial_i \Gamma^l_{jk} J_p^i + \partial_i \Gamma^l_{jp} J_k^i \right] + O(\partial J, \partial g)$$

where here $\Gamma$ denotes the Levi Civita connection. Next, using Lemma 4.6 and (5.13) we compute

$$[\mathcal{L}_X J]_k^l = J_p^l \partial_k \left( g^{ij} \Gamma_{ij}^p \right) - J_p^l \partial_p \left( g^{ij} \Gamma_{ij}^t \right) + O(\partial J, \partial g)$$

$$= J_p^l g^{ij} \partial_k \Gamma_{ij}^p - J_p^l g^{ij} \partial_p \Gamma_{ij}^t + O(\partial J, \partial g).$$

Combining the above calculations we observe

$$D_2(g, J)_k^l = g^{ij} \left[ \partial_t \partial_j J_k^l - \partial_1 \Gamma^l_{jk} J_p^i + \partial_1 \Gamma^l_{jp} J_k^i \right] + J_p^l g^{ij} \partial_k \Gamma_{ij}^p - J_p^l g^{ij} \partial_p \Gamma_{ij}^t$$

$$+ J_p^l R_{ij}^l - R_{ij}^l J_p^i + O(\partial J, \partial g)$$

$$= g^{ij} \partial_t \partial_j J_k^l + g^{ij} J_p^l \left[ \partial_1 \Gamma_{jk}^l + \partial_1 \Gamma_{jp}^l + R_{kij} \right]$$

$$+ g^{ij} J_p^l \left[ \partial_k \Gamma_{ij}^p - \partial_p \Gamma_{ij}^k \right] + O(\partial J, \partial g) \quad (6.10)$$

It follows that

$$\sigma \left[ \mathcal{L}_g D_2 \right] (h)_i^j = 0$$

$$\sigma \left[ \mathcal{L}_J D_2 \right] (K)_i^j = |\xi|^2 K_i^j.$$  

Using that $D\pi_0 = \text{Id}_{\mathcal{T}J_0}$ it follows that the same formulas hold for the linearization of $\tilde{D}_2$. We conclude

$$\sigma \left[ \mathcal{L}\tilde{D} \right] (h, K) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} h \\ K \end{pmatrix}.$$  

It follows from standard parabolic theory that, starting from a compatible pair $(g_0, J_0)$, there is a unique short time solution to (6.8).

Now let $J = \pi E$. We want to show that the pair $(g, J)$ is a solution to (6.7). By a straightforward computation using that $g(D^*DJ - \mathcal{N}, \cdot) \in \Lambda^{(2,0)+(0,2)}$, one can show that the differential operators $D_1$ and $D_2$ satisfy, for a compatible pair $(g, J)$,

$$D_1(g^J, J)^{-1} + g(J, D_2(g^J, J) \cdot) + g(D_2(g^J, J) \cdot, J) = 0.$$  

It follows that a solution to (6.8) satisfies

$$\frac{\partial}{\partial t} g^{-J} = D_1 g^{-1} D_2 g^{-J},$$

and at this point one can follow the estimate of (5.9) and apply the maximum principle to obtain that compatibility of the pair $(g_0, J_0)$ is preserved along a solution to (6.8). It follows that $(g(t), J(t))$ is a solution of (6.7). If $\phi_t$ denotes the one parameter family of diffeomorphisms generated by $-X$ as in (5.10), then $(\phi_t^* g(t), \phi_t^* J(t))$ is a solution of (6.6).
Furthermore, the proof of uniqueness of solutions to (6.6) follows exactly the same lines as the uniqueness part of Theorem 1.1 since it is the same vector field $X$ we are using in the gauge-fixing technique.

Next we must show that the symplectic condition $d\omega = 0$ is preserved along our solution to (6.6). We want to reverse the steps of Proposition 6.2 except this time we need to use more general formula since our Kähler form $\omega$ is not a priori symplectic. We start with

$$\frac{\partial}{\partial t} \omega = \frac{\partial}{\partial t} g(J\cdot, \cdot)$$

$$= -2 \text{Re}(J\cdot, \cdot) + \frac{1}{2} N^1(\cdot, \cdot)' - N^2(\cdot, \cdot)' + g(-D^*DJ + N + \mathcal{R}, \cdot)$$

As $\mathcal{R}$ is the $J$-skew part of the Ricci tensor, one has

$$-2 \text{Re}(J\cdot, \cdot) + g(N, \cdot) = -2 \rho(\cdot, \cdot).$$

Furthermore, as the calculation of Lemma 3.4 shows, the tensor $N^2$ is in general already $J$-symmetric, and $N$ is just $g^{-1}N^2$, so it follows that

$$-N^2(\cdot, \cdot)' + g(N', \cdot) = 0.$$

Combining these calculations yields

$$\frac{\partial}{\partial t} \omega = -2 \rho + \left(\frac{1}{2} N^1\right)' - D^*D\omega$$

$$= -\rho' + \left(\frac{1}{2} N^1\right)' - \Delta_d\omega$$

$$= -P + \left(\frac{1}{2} N^1\right)' - \frac{1}{2} W - \Delta_d\omega.$$

Now note that it follows from Lemma 3.3 that $W = d\omega * DJ$, hence

$$\frac{\partial}{\partial t} d\omega = -d\Delta_d\omega + A * d\omega + B * \nabla d\omega$$

$$= -\Delta_d d\omega + A * d\omega + B * Dd\omega.$$

for some tensor quantities $A$ and $B$. It follows using the Böchner formula that

$$\frac{\partial}{\partial t} |d\omega|^2_g = -2 \langle \Delta_d d\omega, d\omega \rangle + 2 \langle A * d\omega + B * Dd\omega, d\omega \rangle$$

$$= -2 \langle D^* D\omega + \text{Rm} * d\omega, d\omega \rangle + 2 \langle A * d\omega + B * Dd\omega, d\omega \rangle$$

$$\leq \Delta |d\omega|^2 - 2 |Dd\omega|^2 + C |d\omega|^2 + 2B |Dd\omega| |d\omega|$$

$$\leq \Delta |d\omega|^2 - 2 |Dd\omega|^2 + C |d\omega|^2 + \left(\frac{C}{\epsilon} |Dd\omega|^2 + \frac{C}{\epsilon} |d\omega|^2 \right)$$

$$\leq \Delta |d\omega|^2 - |Dd\omega|^2 + C |d\omega|^2$$

where the last line follows by choosing $\epsilon$ small with respect to bounds on $A$ and $B$. Applying the maximum principle to $e^{-Ct} |d\omega|^2_g$, it follows that $d\omega \equiv 0$ is preserved along the solution to (6.6).

The final step is to show that if the initial structure is Kähler then the solution reduces to Kähler Ricci flow. Given $(\omega_0, J_0)$ Kähler, we can construct the solution to (1.4) by the above proof. However we can also construct the solution to Kähler Ricci flow with the initial condition $\omega_0$. Since the Kähler condition is preserved, one observes that in fact $(\omega(t), J_0)$
is a solution to (1.4). Since solutions to (1.4) are unique, it follows that our given solution to (1.4) must be the same as the solution to Kähler Ricci flow.

**Remark 6.3.** We close this section with an important remark regarding the uniqueness of equations satisfying the results of Theorem 1.6. In principle, there is a natural family of parabolic flows which preserve the almost Kähler condition for the pair \((\omega, J)\). To see this observe that if \(\mathcal{H}\) denotes any endomorphism of the tangent bundle consisting only of first order terms in \(\omega\) and \(J\), which further satisfies

\[
\omega(\mathcal{H}, J) + \omega(J, \mathcal{H}) = 0, \quad J\mathcal{H} + \mathcal{H}J = 0 \tag{6.11}
\]

then the proof of Theorem 1.6 carries through for the evolution equation

\[
\begin{align*}
\frac{\partial}{\partial t}\omega &= -P \\
\frac{\partial}{\partial t}J &= -D^*DJ + \mathcal{N} + \mathcal{R} + \mathcal{H} \\
\omega(0) &= \omega_0 \\
J(0) &= J_0 
\end{align*} \tag{6.12}
\]

Indeed, it follows from (6.11), the prior calculations and Lemma 4.3 that compatibility of the pair will still be preserved under this flow. One can think of this also as adding a certain \((2, 0) + (0, 2)\) tensor to the evolution of both \(g\) and \(J\), which cancels out and thus does not appear in the evolution of \(\omega\).

Furthermore, since the term \(\mathcal{H}\) is first order in \(\omega\) and \(J\) the discussion of parabolicity and short time existence is not affected. Observe that since we are staying within the class of almost Kähler structures, the only first order invariant of the pair \((\omega, J)\) is the Nijenhuis tensor. Therefore in principle we could let \(\mathcal{H}\) denote any expression in the Nijenhuis tensor which satisfies (6.11). To produce an evolution equation with a natural scaling property, it is most relevant to consider endomorphisms \(\mathcal{H}\) which are quadratic. We codify this discussion with the following proposition.

**Proposition 6.4.** Let \((M^{2n}, \omega_0, J_0)\) denote an almost Kähler structure. Suppose \(\mathcal{H} \in \text{End}(TM)\) is a quadratic expression in the Nijenhuis tensor satisfying (6.11). Then there exists a unique short time solution to (6.12) with initial condition \((\omega_0, J_0)\).

7. Curvature Evolution Equations

In this section we derive evolution equations for the curvature of the Chern connection, the Nijenhuis tensor, and their derivatives for solutions to (1.2). By Proposition 6.1, these general equations hold for solutions to the symplectic curvature flow as well. As one would expect from the calculation of the symbol of (1.2) in Theorem 1.1, the system of equations is upper triangular in the appropriate sense.

7.1. Evolution equations for almost Hermitian curvature flow. First we derive the evolution of the Nijenhuis tensor.

**Proposition 7.1.** Let \((M^{2n}, \omega(t), J(t))\) be a solution to (1.2). Then

\[
\frac{\partial}{\partial t}N = \Delta N + \Omega \ast T + \nabla T \ast T + T^* T. \tag{7.1}
\]
Proof. Choose normal coordinates for the induced metric \( g = \omega(J, \cdot, \cdot) \) at a fixed point in space and time. Note that this has the effect that any first coordinate derivative of \( \omega \) or \( J \) can be expressed in terms of the torsion of the Chern connection \( T \). Furthermore, any second coordinates derivative of \( J \) can be expressed using the curvature, torsion, and first derivative of torsion. Starting from (5.4) we compute
\[
\frac{\partial}{\partial t} N_{jk}^i = (\dot{J} \ast \partial J)_{jk}^i + J_p^p \partial_p \dot{J}_k^i - J_k^p \partial_p \dot{J}_j^i - J_j^p \partial_p \dot{J}_k^i + J_p^p \partial_k \dot{J}_j^i.
\]
Since \( \dot{J} = -K + H \) where \( H \) is quadratic in the torsion of \( \nabla \), it immediately follows that
\[
\dot{J} \ast \partial J = T^*^3
\]
where \( T \) denotes the full torsion tensor of the Chern connection. Of course in the symplectic setting this only depends on \( N \). Furthermore, it is clear that
\[
\partial H = \nabla T \ast T + T^*^3.
\]
Therefore it remains to calculate the highest order term
\[
\begin{align*}
\mathcal{W}_{jk}^i := & - J_p^p \partial_p K_k^i + J_k^p \partial_p K_j^i + J_p^p \partial_j J_k^i - J_p^p \partial_k J_j^i \\
= & - J_p^p \partial_p (\omega^s \nabla_r N_{sk}^i) + J_k^p \partial_p (\omega^s \nabla_r N_{sj}^i) + J_p^i \partial_j (\omega^s \nabla_r N_{sk}^p) - J_p^i \partial_k (\omega^s \nabla_r N_{sj}^p) \\
= & - J_p^p \partial_p (\omega^s \partial_r N_{sk}^i) + J_r^i \partial_j (\omega^s \partial_r N_{sk}^p) + J_p^i \partial_j (\omega^s \partial_r N_{sk}^p) + T^*^2 \\
+ & J_p^i \partial_j (\omega^s \partial_r N_{sk}^p) + T^*^2 - J_p^i \partial_k (\omega^s \partial_r N_{sj}^p + T^*^2) \\
= & - J_p^i \omega^s \partial_p \partial_r N_{sk}^i + J_k^p \omega^s \partial_p \partial_r N_{sj}^i + J_r^i \omega^s \partial_p \partial_r N_{sk}^p - J_p^i \omega^s \partial_p \partial_r N_{sj}^p + \nabla T \ast T.
\end{align*}
\]
Now plugging in (5.4) again we conclude
\[
\begin{align*}
\mathcal{W}_{jk}^i = & - J_p^i \omega^s \partial_p \partial_r (J_s^i \partial_r J_k^j - J_k^i \partial_r J_s^j - J_j^i \partial_r J_k^s + J_k^i \partial_k J_s^j) \\
+ & J_k^p \omega^s \partial_p \partial_r (J_s^i \partial_r J_k^j - J_k^i \partial_r J_s^j - J_j^i \partial_r J_k^s + J_k^i \partial_k J_s^j) \\
+ & J_p^i \omega^s \partial_j \partial_r (J_s^i \partial_r J_s^j - J_j^i \partial_r J_s^j + J_k^i \partial_k J_s^j) \\
+ & J_p^i \omega^s \partial_j \partial_r (J_s^i \partial_r J_s^j - J_j^i \partial_r J_s^j + J_k^i \partial_k J_s^j) \\
- & J_p^i \omega^s \partial_k \partial_r (J_s^i \partial_r J_s^j - J_j^i \partial_r J_s^j + J_k^i \partial_k J_s^j) + \nabla T \ast T \\
= & - J_p^i \omega^s (J_s^i \partial_p \partial_r J_k^j - J_k^i \partial_p \partial_r J_s^j - J_j^i \partial_p \partial_r J_k^s + J_k^i \partial_p \partial_r J_s^j) \\
+ & J_k^p \omega^s (J_s^i \partial_p \partial_r J_k^j - J_j^i \partial_p \partial_r J_s^j - J_k^i \partial_p \partial_r J_s^j + J_k^i \partial_p \partial_r J_j^j) \\
+ & J_p^i \omega^s (J_j^i \partial_p \partial_r J_s^j - J_j^i \partial_p \partial_r J_s^j + J_k^i \partial_p \partial_r J_s^j) \\
+ & J_p^i \omega^s (J_j^i \partial_p \partial_r J_s^j - J_j^i \partial_p \partial_r J_s^j + J_k^i \partial_p \partial_r J_s^j) \\
- & J_p^i \omega^s (J_k^i \partial_p \partial_r J_s^j - J_k^i \partial_p \partial_r J_s^j + J_k^i \partial_k \partial_r J_s^j) + \Omega \ast T + \nabla T \ast T.
\end{align*}
\]
Let us label the sixteen third derivative terms above as \( I - XVI \) in Roman numerals in the order in which they appear. Some cancellations are apparent, namely, using that \( \omega \) is skew symmetric and coordinate derivatives are symmetric, it follows that \( III = 0, VII = 0, XI = 0, XV = 0 \). Also, one observes that \( XII + XVI = 0, II + VI = 0, IV + XIV = 0 \).
Proposition 7.2. Let

\[ \mathcal{W}_{jk}^i = g^r \partial_r \partial_i \left( J^r_j \partial_k J_r^i - J^r_k \partial_r J^i_j - J^r_j \partial_j J^r_k + J^r_k \partial_k J^r_j \right) + \Omega \ast T + \nabla T \ast T \]

\[ = g^r \partial_r \partial_i N_{jk}^r + \Omega \ast T + \nabla T \ast T \]

\[ = \Delta N_{jk}^i + \Omega \ast T + \nabla T \ast T. \]

The result follows. \qed

Proposition 7.2. Let \((M^{2n}, \omega(t), J(t))\) be a solution to \((1.2)\). Then

\[ \frac{\partial}{\partial t} (d\omega)^+ = \Delta (d\omega)^+ + \nabla^2 N + \Omega \ast T + \nabla T \ast T + T \ast 3. \]  

(7.2)

Proof. We will use a specialized frame to make this calculation. We choose normal coordinates for the induced metric at a given point as in Proposition 7.1. Let \(\{\partial\}_{i=1,...,2n}\) be the associated coordinate vector fields. Now let

\[ e_j := \frac{\partial}{\partial x_j} - iJ(0) \left( \frac{\partial}{\partial x_j} \right). \]

Clearly \(\{e_j\}_{j=1}^{2n}\) contains a spanning set for \(T^{1,0}(M)\), and by relabeling we may assume that \(\{e_i\}_{i=1}^{n}\) is a basis for \(T^{1,0}(M)\). One has the corresponding basis for \(T^{0,1}(M)\) given by

\[ \tilde{e}_j = \frac{\partial}{\partial x_j} + iJ(0) \left( \frac{\partial}{\partial x_j} \right). \]

Now note that

\[ (d\omega)^+ = (d\omega + d\omega(\cdot, J\cdot, J\cdot) + d\omega(J\cdot, \cdot, J\cdot) + d\omega(J\cdot, J\cdot, Z)). \]

It follows that

\[ \frac{\partial}{\partial t} (d\omega)^+ = J \ast d\omega + \left( \frac{\partial}{\partial t} d\omega \right)^{(2,1)+(1,2)} = \left( \frac{\partial}{\partial t} d\omega \right)^{(2,1)+(1,2)} + \nabla T \ast T. \]

We now compute

\[ \left( \frac{\partial}{\partial t} d\omega \right)_{ij\tilde{k}} = \partial_i \tilde{\omega}_{jk} - \partial_j \tilde{\omega}_{ik} - \partial_{\tilde{k}} \omega_{ij} \]

\[ = \nabla_j (-S + Q + H)_{jk} - \nabla_j (-S + Q + H)_{\tilde{k}} \]

\[ - \Delta k (-S + Q + H)_{ij \tilde{k}} + T \ast \Omega + T \ast 3 + T \ast \nabla T \]

\[ = - \nabla_i S_{\tilde{k}j} + 2 \nabla_j S_{\tilde{k}i} + \nabla^2 N + T \ast \Omega + T \ast 3 + T \ast \nabla T \]

where the last line follows using that \(S \in \Lambda^{1,1}, Q\) is quadratic in the torsion, and \(H\) depends on a quadratic term in torsion and the derivative of the Nijenhuis tensor. Now we apply the Bianchi identity to conclude

\[ \nabla_j S_{\tilde{k}i} - \nabla_i S_{\tilde{k}j} = \omega^{\mu \tilde{\nu}} \left[ \nabla_j \Omega_{\mu \tilde{\nu} \tilde{k}} - \nabla_i \Omega_{\mu \tilde{\nu} \tilde{\mu}} \right] \]

\[ = \omega^{\mu \tilde{\nu}} \left[ \nabla_j \left( \Omega_{\mu \tilde{\nu} \tilde{k}} + \nabla_{\tilde{\nu}} T_{\mu \tilde{k}} \right) \right] - \nabla_i \left( \Omega_{\mu \tilde{\nu} \tilde{k}} + \nabla_{\tilde{\nu}} T_{\mu \tilde{k}} \right). \]

Next we apply the differential Bianchi identity and simplify

\[ \omega^{\mu \tilde{\nu}} \left( \nabla_j \Omega_{\mu \tilde{\nu} \tilde{k}} - \nabla_i \Omega_{\mu \tilde{\nu} \tilde{k}} \right) = \nabla_{\tilde{\nu}} \Omega_{\mu \tilde{k}} + T \ast \Omega \]

\[ = \nabla^2 N + \nabla T \ast T + T \ast \Omega \]

\[ = \Delta N_{\mu \tilde{k}} + \Omega \ast T + \nabla T \ast T. \]
where the last line follows because the \( (2, 0) + (0, 2) \) component of the Chern curvature only depends on a quadratic expression in the torsion and one derivative of the Nijenhuis tensor. We commute derivatives and apply the Bianchi identity a final time to conclude
\[
\omega^{mn} \left( \nabla_j \nabla^m T_{jnk} - \nabla_i \nabla^m T_{inm} \right) = \omega^{mn} \left( \nabla_j \left( \nabla_i T_{mjk} - \nabla_i T_{mj} \right) \right) + \Omega * T
\]
\[
= \omega^{mn} \left( \nabla_j \left( T_{mjk} + T^{*2} \right) \right) + \Omega * T
\]
\[
= \omega^{mn} \nabla_j \nabla_m T_{ijk} + \nabla T * T + \Omega * T
\]
\[
= \Delta T_{ijk} + \nabla T * T + \Omega * T.
\]
But, since \((d\omega)^-\) can be expressed using the Nijenhuis tensor, we conclude from (2.2) that
\[
\Delta T = \Delta (d\omega)^+ + \nabla^2 N.
\]
The result follows.

□

**Proposition 7.3.** Let \((M^{2n}, \omega(t), J(t))\) be a solution to \((1.2)\). Then
\[
\frac{\partial}{\partial t} T = \Delta T + \nabla^2 N + \Omega * T + \nabla T * T + T^{*3}.
\]
(7.3)

**Proof.** Since by (2.2) the torsion is determined algebraically by the Nijenhuis tensor and \((d\omega)^+\), the result follows immediately from Propositions 7.1 and 7.2. □

Next we will compute the evolution of the Riemannian curvature tensor. To do this we will first observe a general formula for \(S\) in terms of the Ricci tensor.

**Lemma 7.4.** Let \((M^{2n}, \omega, J)\) be an almost Hermitian manifold. Then
\[
S(J \cdot, \cdot) = 2 \text{Rc} + DT + T^{*2}.
\]

**Proof.** This is an immediate consequence of various lemmas above. □

**Proposition 7.5.** Let \((M^{2n}, \omega(t), J(t))\) be a solution to \((1.2)\). Then
\[
\frac{\partial}{\partial t} \text{Rm} = \Delta \text{Rm} + \text{Rm}^{*2} + \text{Rm} * T^{*2} + \text{Rm} * DT + T * D^2 T + DT * DT + D^3 T.
\]
(7.4)

**Proof.** Recall the general variational formula for the Riemannian curvature tensor. If \(\frac{\partial}{\partial t} g = h\) then
\[
\frac{\partial}{\partial t} \text{Rm}_{ijkl} = \frac{1}{2} (D_i D_k h_{jl} - D_i D_l h_{jk} - D_j D_k h_{il} + D_j D_l h_{ik})
\]
\[
+ \frac{1}{2} \left( R_{ijkl}^{p} h_{pl} - R_{ijkl}^{p} h_{pk} \right).
\]
(7.5)

First recall that for variation by \(-2 \text{Rc}\), one has
\[
\frac{\partial}{\partial t} \text{Rm} = \Delta \text{Rm} + \text{Rm}^{*2}.
\]
The precise form of the equation appears in the next subsection on the symplectic flow. Using Lemma 7.4, the result follows. □
Theorem 7.6. Let \((M^{2n}, \omega(t), J(t))\) be a solution to (1.2). Then

\[
\frac{\partial}{\partial t} D_k N = \Delta D_k N + \sum_{i=0}^{k-1} D^i \text{Rm} \ast D^{k-i} T + \sum_{i=1}^{k+1} D^i T \ast D^{k+1-i} T \\
+ \sum_{i=0}^k \sum_{j=0}^i D^i T \ast D^{i-j} T \ast D^{k-i} T,
\]

\[
\frac{\partial}{\partial t} D_k T = \Delta D_k T + \sum_{i=0}^{k-1} D^i \text{Rm} \ast D^{k-i} T + \sum_{i=1}^{k+1} D^i T \ast D^{k+1-i} T \\
+ \sum_{i=0}^k \sum_{j=0}^i D^i T \ast D^{i-j} T \ast D^{k-i} T + D^{k+2} N,
\]

\[
\frac{\partial}{\partial t} D_k \text{Rm} = \Delta D_k \text{Rm} + \sum_{i=0}^k D^i \text{Rm} \ast D^{k-i} \text{Rm} + D^{k+3} T + \sum_{i=0}^k D^{i+1} T \ast D^{k-i} \text{Rm} \\
+ \sum_{i=0}^k \sum_{j=0}^i D^i T \ast D^{i-j} T \ast D^{k-i} \text{Rm} + \sum_{i=0}^{k+2} D^i T \ast D^{k+2-i} T.
\]

Proof. We compute the first evolution equation, the case of \(D_k T\) being formally similar. Using Proposition 7.1 and Lemma 7.4 we compute

\[
\frac{\partial}{\partial t} D_k N = \frac{\partial}{\partial t} (\partial + \Gamma) \ldots (\partial + \Gamma) N \\
= \sum_{i=0}^k D^i \left( \frac{\partial}{\partial t} \Gamma \right) \ast D^{k-i} N + D^k \left( \frac{\partial}{\partial t} N \right) \\
= \sum_{i=0}^k D^i \left( \text{Rm} + DT + T^2 \ast D^{k-i} N + D^k \left( \Delta N + \text{Rm} \ast T + DT \ast T + T^3 \right) \right) \\
= \Delta D^k N + \sum_{i=0}^{k-1} D^i \text{Rm} \ast D^{k-i} T + \sum_{i=1}^{k+1} D^i T \ast D^{k+1-i} T + \sum_{i=0}^k \sum_{j=0}^i D^i T \ast D^{i-j} T \ast D^{k-i} T,
\]
as required. Next we compute
\[
\frac{\partial}{\partial t} D^k R_m = \frac{\partial}{\partial t} (\partial + \Gamma) \ldots (\partial + \Gamma) R_m \\
= \sum_{i=0}^{k-1} D^i \left( \frac{\partial}{\partial t} \Gamma \right) \ast D^{k-i-1} R_m + D^k \left( \frac{\partial}{\partial t} R_m \right) \\
= \sum_{i=0}^{k-1} D^{i+1} (R_m + DT + T^2) \ast D^{k-i-1} R_m \\
+ D^k (\Delta R_m + R_m T^2 + R_m DT + T D^2 T + DT D T + D^3 T) \\
= \Delta D^k R_m + \sum_{i=0}^{k} D^i R_m \ast D^{k-i} R_m + D^{k+3} T + \sum_{i=0}^{k} D^{i+1} T \ast D^{k-i} R_m \\
+ \sum_{i=0}^{k} \sum_{j=0}^{i} D^i T \ast D^{i-j} T \ast D^{k-i} R_m + \sum_{i=0}^{k+2} D^i T \ast D^{k+2-i} T.
\]

7.2. Evolution equations for symplectic curvature flow. We begin by deriving an evolution equation for the Levi Civita derivative of $J$.

**Proposition 7.7.** Let $(M^{2n}, \omega(t), J(t))$ be a solution to (1.4). Then
\[
\frac{\partial}{\partial t} (DJ)_{ij}^k = \Delta D_i J_j^k + D_i N_j^k - g^{kl} (D_t B_{jl} + D_j B_{il} - D_l B_{ij}) J_p^k \\
+ g^{kl} (D_i B_{pl} + D_p B_{il} - D_l B_{ip}) J_j^p \\
+ 2g^{rs} \left( R_{sij}^l D_r J_j^k - R_{sij}^k D_r J_j^l \right) - R_i D_t J_j^k + R_p D_r J_j^p - R_j^p D_r J_i^k \\
\tag{7.6}
\]

where
\[
B = \frac{1}{4} B^1 - \frac{1}{2} B^2.
\]

**Proof.** We recall the coordinate expression
\[
D J_{ij}^k = \partial_i J_j^k - \Gamma_{ij}^p J_p^k + \Gamma_{ip}^k J_j^p
\]
If we choose normal coordinates for $g(0)$ at some point and differentiate this expression with respect to $t$ this yields
\[
\frac{\partial}{\partial t} (DJ)_{ij}^k = D_i J_j^k - \dot{\Gamma}_{ij}^p J_p^k + \dot{\Gamma}_{ip}^k J_j^p.
\]

Next we recall that if $\frac{\partial}{\partial t} g = h$, one has
\[
\frac{\partial}{\partial t} \Gamma_{ij}^k = \frac{1}{2} g^{kl} (D_t h_{jl} + D_j h_{il} - D_l h_{ij}).
\]
To simplify the calculation, we set $B = \frac{1}{4}B^1 - \frac{1}{2}B^2$, then using Proposition 6.2 we derive

$$\frac{\partial}{\partial t} (D J)_{ij}^k = D_i \left( \mathcal{R}_{rs}^k \mathcal{D}_r D_s J_{ij}^k + \mathcal{N}_j^k + R_j^k \right)$$

$$+ g^{pl} \left( D_i \left( R_{ij} - B_{ij} \right) + D_j \left( R_{il} - B_{il} \right) - D_l \left( R_{ij} - B_{ij} \right) \right) J_p^k$$

$$- g^{kl} \left( D_i \left( R_{pl} - B_{pl} \right) + D_p \left( R_{il} - B_{il} \right) - D_l \left( R_{ip} - B_{ip} \right) \right) J_p^l.$$

Now we commute derivatives and apply the differential Bianchi identity to conclude

$$D_i (g^{rs} D_r D_s J_j^k) = g^{rs} \left( D_r D_i D_s J_j^k + R_{rjs}^i D_i J_j^k + R_{rij}^k D_s J_j^k - R_{rli}^k D_s J_j^k \right)$$

$$= g^{rs} \left( D_r \left( D_s D_i J_j^k + R_{rs}^i J_j^k - R_{sir}^i J_j^k \right) \right) + g^{rs} \left( R_{rjs}^i D_t J_j^k + R_{rs}^i J_j^k - R_{sir}^k J_j^k \right)$$

$$= \Delta D_i J_j^k + g^{rs} \left( D_r R_{rs}^i J_j^k - D_r R_{sir}^k J_j^k \right) + g^{rs} \left( R_{rs}^i D_t J_j^k - R_{sir}^k D_s J_j^k + R_{rs}^i D_t J_j^k - R_{sir}^k D_s J_j^k \right)$$

$$= \Delta D_i J_j^k + \left( D^t R_{rjs} - D_j R_{rjs}^i \right) J_j^k - \left( D^k R_{rli} - D_l R_{rli}^k \right) J_j^k$$

$$+ 2g^{rs} \left( R_{rjs}^i D_r J_j^k - R_{sir}^k D_s J_j^k \right) - R_{rli}^k D_t J_j^k. \quad (7.7)$$

Plugging this into the above line yields

$$\frac{\partial}{\partial t} (D J)_{ij}^k = \Delta D_i J_j^k + D_i N_j^k - g^{pl} \left( D_t B_{jl} + D_j B_{il} - D_l B_{ij} \right) J_p^k$$

$$+ g^{kl} \left( D_i B_{pl} + D_p B_{il} - D_l B_{ij} \right) J_p^l$$

$$+ 2g^{rs} \left( R_{rjs}^i D_r J_j^k - R_{sir}^k D_s J_j^k \right) - R_{rli}^k D_t J_j^k + D_i \left( J_p^k R_p^k - R_p^k J_p^k \right)$$

$$+ \left( D^t R_{rjs} - D_j R_{rjs}^i \right) J_j^k - \left( D^k R_{rli} - D_l R_{rli}^k \right) J_j^k$$

$$+ g^{pl} \left( D_i R_{jl} + D_j R_{il} - D_l R_{ij} \right) J_p^k$$

$$- g^{kl} \left( D_i R_{pl} + D_p R_{il} - D_l R_{ip} \right) J_p^l.$$

Indeed, slightly miraculously, all of the $DR$ terms drop out of the equation. This is the required result. \qed

Next we use this proposition to compute the evolution of $|D J|^2$. 
Proposition 7.8. Let \((M^{2n}, \omega(t), J(t))\) be a solution to \((1.4)\). Then
\[
\frac{\partial}{\partial t} |DJ|^2 = \Delta |DJ|^2 - 2 |D^2 J|^2 - \frac{1}{2} |B^1|^2 + \frac{3}{2} \langle B^1, B^2 \rangle
\]
\[
+ 8 R_{si,j} D_s J^p_i D_i J^k_j + 2 D_p D_i J^m_i D_k J^p_i D_i J^k_j + 2 D_p D_k D_m J^m_i D_i J^p_i D_i J^k_j \tag{7.8}
\]
\[
- 4 D_l D_m J^m_i D_m J^k_l D_i J^k_j - 4 D_l D_m J^k_l D_m J^m_i D_i J^k_j,
\]
where the result is interpreted in an orthonormal frame.

Proof. First we note for \(\frac{\partial}{\partial t} g = h\) one has
\[
\frac{\partial}{\partial t} |DJ|^2 = \frac{\partial}{\partial t} \left( g^{pq} g^{sqr} D_p J^r_j D_q J^k_j \right)
\]
\[
= 2 \left\langle \frac{\partial}{\partial t} DJ, DJ \right\rangle - \langle h, B^1 \rangle + \langle g^{ip} g^{jq} h_{kr} - g^{ip} h_{jq} g_{kr} \rangle D_i J^k_i D_p J^q_p
\]
\[
= 2 \left\langle \frac{\partial}{\partial t} DJ, DJ \right\rangle - \langle h, B^1 \rangle.
\]
First of all, from \((6.5)\), we have \(h = -2 \text{Rc} + \frac{1}{2} B^1 - B^2\) and so
\[
- \langle h, B^1 \rangle = 2 \left\langle \text{Rc}, B^1 \right\rangle - \frac{1}{2} |B^1|^2 + \langle B^1, B^2 \rangle. \tag{7.9}
\]
We simplify the contributions of to \(\frac{\partial}{\partial t} DJ\) from \((7.6)\) separately. First note that
\[
2 \left\langle \Delta DJ, DJ \right\rangle = \Delta |DJ|^2 - 2 |D^2 J|^2. \tag{7.10}
\]
Next we simplify
\[
2 \left( -R^k_i D_i J^k_j + R^k_i D_i J^p_j - R^p_i D_i J^k_j \right) \left( D_i J^k_j \right) = -2 \left\langle \text{Rc}, B^1 \right\rangle. \tag{7.11}
\]
Next we simplify, working in an orthonormal frame,
\[
4 g^{rs} \left( R^k_i D_r J^k_j - R^k_i D_r J^p_j \right) D_i J^k_j = 8 R_{si,j} D_s J^k_i D_i J^k_j. \tag{7.12}
\]
Next consider
\[
2 D_p N^m_j D_i J^k_j = 2 D_i \left( g^{pl} g_{mn} g^{pq} D_p J^m_i J^p_j D_q J^k_j \right) D_i J^k_j
\]
\[
= 2 \left[ D_i D_p J^m_i J^p_j D_p J^m_k + D_p J^m_i D_i J^p_j J^m_k + D_p J^m_i D_i J^p_j J^m_k \right] D_i J^k_j.
\]
Combining the first and last terms and applying the identity \(D(J^2) = 0\) yields
\[
2 \left[ -J^k_i D_i D_p J^m_i J^p_j J^m_k + D_p J^m_i D_i J^p_j J^m_k D_i J^k_j \right] = 0. \tag{7.13}
\]
Thus
\[
2 D_i N^m_j D_i J^k_j = 2 \left\langle B^1, B^2 \right\rangle. \tag{7.14}
\]
We simplify the remaining \(DB\) terms of \((7.6)\). We begin by considering the contribution of \(B^1\). First by a simplification like in line \((7.13)\) we have
\[
-\frac{1}{2} D_i B^1_{jp} J^k_i D_i J^k_j = -\frac{1}{2} D_i \left( D_j J^m_l D_p J^m_k \right) J^k_p D_i J^k_j
\]
\[
= -\frac{1}{2} \left[ D_i D_j J^m_l D_p J^m_k J^k_p D_i J^k_j + D_i D_p J^m_k J^k_p D_i J^k_j \right]. \tag{7.15}
\]
Combining the first and last terms and applying the identity \(D(J^2) = 0\) yields
\[
-\frac{1}{2} D_i B^1_{jp} J^k_i D_i J^k_j = 0. \tag{7.16}
\]
Next we note

\[
\frac{1}{2} D_{i} B_{i p}^1 J^p_{m p} D_i J_j^k = - \frac{1}{2} D_j (D_i J^m_{m p} D_p J^n_m) J^k_p D_i J_j^k \\
= - \frac{1}{2} D_j D_i J^m_{m p} D_p J^n_m J^k_p D_i J_j^k - \frac{1}{2} D_j D_p J^m_{m p} D_i J^n_m J^k_p D_i J_j^k. 
\] (7.16)

The next term is

\[
\frac{1}{2} D_i B_{i j}^1 J^k_j D_i J_j^k = \frac{1}{2} D_i (D_i J^m_{m j} D_j J^n_m) J^k_i D_i J_j^k \\
= \frac{1}{2} D_i D_i J^m_{m j} D_j J^n_m J^k_i D_i J_j^k + \frac{1}{2} D_i D_p J^m_{m j} D_i J^n_m J^k_p D_i J_j^k. 
\] (7.17)

Next

\[
\frac{1}{2} D_i B_{i k}^1 J^p_j D_i J_j^k = \frac{1}{2} D_i (D_p J^n_{m p} D_k J^m_m) J^p_j D_i J_j^k \\
= \frac{1}{2} D_i D_p J^n_{m p} D_k J^m_m J^p_j D_i J_j^k + \frac{1}{2} D_i J^m_{m p} D_i J^n_m J^p_j D_i J_j^k. 
\] (7.18)

Next we compute

\[
\frac{1}{2} D_j B_{i p}^1 J^p_j D_i J_j^k = \frac{1}{2} D_p (D_i J^m_{m p} D_k J^m_m) J^p_j D_i J_j^k \\
= \frac{1}{2} D_p D_i J^m_{m p} D_k J^m_m J^p_j D_i J_j^k + \frac{1}{2} D_p J^m_{m p} D_i J^n_m J^p_j D_i J_j^k. 
\] (7.19)

The final $B^1$ term is

\[
- \frac{1}{2} D_k B_{i p}^1 J^p_j D_i J_j^k = - \frac{1}{2} D_k (D_i J^m_{m p} D_p J^n_m) J^p_j D_i J_j^k \\
= - \frac{1}{2} D_k D_i J^m_{m p} D_p J^n_m J^p_j D_i J_j^k - \frac{1}{2} D_k J^m_{m p} D_i J^n_m J^p_j D_i J_j^k. 
\] (7.20)

Now we compute the $B^2$ terms. The first such is

\[
D_i B_{i p}^2 J^p_j D_i J_j^k = D_i (D_m J^n_{m p} D_p J^m_m) J^p_j D_i J_j^k \\
= D_i D_m J^n_{m p} D_p J^m_m J^p_j D_i J_j^k + D_i D_m J^n_{m p} D_p J^p_j D_i J_j^k \\
= D_i D_m J^n_{m p} D_p J^m_m J^p_j D_i J_j^k - D_i D_m J^n_{m p} D_p J^p_j D_i J_j^k. 
\] (7.21)

Next we note

\[
D_j B_{i p}^2 J^p_j D_i J_j^k = D_j (D_m J^n_{m p} D_p J^m_m) J^p_j D_i J_j^k \\
= D_j D_m J^n_{m p} D_p J^m_m J^p_j D_i J_j^k + D_j D_m J^n_{m p} D_p J^p_j D_i J_j^k. 
\] (7.22)

The next term is

\[
- D_i B_{i j}^2 J^k_j D_i J_j^k = - D_i (D_m J^n_{m j} D_m J^m_j) J^k_j D_i J_j^k \\
= - D_i D_m J^n_{m j} D_m J^m_j J^k_j D_i J_j^k - D_i D_m J^n_{m j} J^k_j D_i J_j^k. 
\] (7.23)
Next
\[-D_i B^2_{pk} J_p J^k = - D_i \left( D_m J^n_p D_m J^n_k \right) J^p J^k \]
\[= - D_i D_m J^n_p D_m J^n_k J^p J^k + D_i D_m J^n_k D_m J^n_p J^k J^p \]
\[= 0 \quad (7.24) \]

Next we compute
\[-D_p B^2_{ik} J^k = - D_p \left( D_m J^n_i D_m J^n_k \right) J^k \]
\[= - D_p D_m J^n_i D_m J^n_k J^k D^i D^k + D_p D_m J^n_k D_m J^n_i J^k D^i J^k. \quad (7.25) \]

The final $B^2$ term is
\[D_k B^2_{ip} J^p J_i J^k = D_k \left( D_m J^n_i D_m J^n_p \right) J^p J_i J^k \]
\[= D_k D_m J^n_i D_m J^n_p J^p J_i J^k + D_k D_m J^n_p D_m J^n_i J^p J^k. \quad (7.26) \]

By further applying the identity $DJ^2 = 0$ one may observe that lines (7.16), (7.17), (7.19), (7.20) are all equal. Likewise the lines (7.22), (7.23), (7.25), (7.26) are equal. Combining the calculations yields the result. □

Next we derive an evolution equation for the Riemannian curvature tensor.

**Proposition 7.9.** Let $(M^{2n}, \omega(t), J(t))$ be a solution to (1.4). Then
\[\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + R_{m}^{ijkl} - (R_{ip} R_{pjkl} + R_{jp} R_{ipkl} + R_{kp} R_{ijlp} + R_{lp} R_{ijkp}) \]
\[+ D_i D_k B_{jl} - D_i D_l B_{jk} + D_j D_k B_{il} + D_j D_l B_{ik} + B^p_i R_{ijkp} - B^p_k R_{ijlp}, \quad (7.27) \]

where
\[B = \frac{1}{4} B^1 - \frac{1}{2} B^2. \]

**Proof.** The starting point is the formula (7.25). A well-known calculation ([5] Lemma 2.51) shows that for $h = -2 R_c$,
\[\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + R_{m}^{ijkl} - (R_{ip} R_{pjkl} + R_{jp} R_{ipkl} + R_{kp} R_{ijlp} + R_{lp} R_{ijkp}) \]
\[+ D_i D_k B_{jl} - D_i D_l B_{jk} + D_j D_k B_{il} + D_j D_l B_{ik} + B^p_i R_{ijkp} - B^p_k R_{ijlp}, \]

where $R_{m}^2$ is the square of the curvature operator and $R_{m}^2$ is the Lie algebra square. Including the extra term of $h = 2B$, the result follows. □

In the next theorem we derive evolution equations for the covariant derivatives of the Nijenhuis tensor and curvature for solutions to (1.4).
Theorem 7.10. Let \((M^{2n}, \omega(t), J(t))\) be a solution to (1.4). Then
\[
\frac{\partial}{\partial t} D^k J = \Delta D^k J + \sum_{i=1}^k D^i J \ast D^{k-i} \text{Rm} + \sum_{i=1}^{k+1} D^i J \ast D^{k+2-i} J
\]
\[
+ \sum_{i=0}^{k-1} \sum_{j=0}^i D^{j+1} J \ast D^{i+1-j} J \ast D^{k-i} J,
\]
\[
\frac{\partial}{\partial t} D^k \text{Rm} = \Delta D^k \text{Rm} + \sum_{i=0}^k D^i \text{Rm} \ast D^{k-i} \text{Rm}
\]
\[
+ \sum_{i=0}^k \sum_{j=0}^i D^{k-i} \text{Rm} \ast D^{j+1} J \ast D^{i-j+1} J + \sum_{i=0}^k D^{i+1} J \ast D^{k-i+3} J.
\]

Proof. We express \(D^k J = D^{k-1} DJ\) and compute using the result of Proposition 7.7
\[
\frac{\partial}{\partial t} D^k J = \frac{\partial}{\partial t} (\partial + \Gamma) \ldots (\partial + \Gamma) DJ
\]
\[
= \sum_{i=0}^{k-2} D^i \left( \frac{\partial}{\partial t} \Gamma \right) \ast D^{k-i-1} DJ + D^{k-1} \left( \frac{\partial}{\partial t} DJ \right)
\]
\[
= \sum_{i=0}^{k-2} D^{i+1} (\text{Rm} + DJ^2) \ast D^{k-i-1} J + D^{k-1} (\Delta DJ + DJ^2 \ast DJ + \text{Rm} \ast DJ)
\]
\[
= \Delta D^k J + \sum_{i=1}^k D^i J \ast D^{k-i} \text{Rm} + \sum_{i=0}^{k-1} \sum_{j=0}^i D^{j+1} J D^{i+1-j} J D^{k-i} J
\]
\[
+ \sum_{i=1}^{k+1} D^i J \ast D^{k+2-i} J.
\]
Likewise we compute using the result of Proposition 7.9,
\[
\frac{\partial}{\partial t} D^k \text{Rm} = \frac{\partial}{\partial t} (\partial + \Gamma) \ldots (\partial + \Gamma) \text{Rm}
\]
\[
= \sum_{i=0}^{k-1} D^i \left( \frac{\partial}{\partial t} \Gamma \right) \ast D^{k-i-1} \text{Rm} + D^k \left( \frac{\partial}{\partial t} \text{Rm} \right)
\]
\[
= \sum_{i=0}^{k-1} D^{i+1} (\text{Rm} + DJ^2) \ast D^{k-i-1} \text{Rm}
\]
\[
+ D^k (\Delta \text{Rm} + \text{Rm}^2 \ast DJ + DJ^2 + \text{Rm} \ast DJ)
\]
\[
= \Delta D^k \text{Rm} + \sum_{i=0}^k D^i \text{Rm} \ast D^{k-i} \text{Rm}
\]
\[
+ \sum_{i=0}^k \sum_{j=0}^i D^{k-i} \text{Rm} \ast D^{j+1} J \ast D^{i-j+1} J + \sum_{i=0}^k D^{i+1} J \ast D^{k-i+3} J,
\]
as required. \(\square\)
8. Smoothing Estimates and a Long time existence obstruction

We begin with the proof of Theorem 1.8.

Proof. Arguments of this kind are standard. We give the proof for the case $m = 1$ to indicate why we are forced to deal with $L^2$ norms for this flow. Let $A$ denote a constant to be determined later, and consider

$$F(t) := t \left( ||D Rm||_{L^2}^2 + ||D^2 T||_{L^2}^2 + \alpha ||Rm||_{L^2}^2 + \beta ||DT||_{L^2}^2 + \gamma ||DN||_{L^2}^2 \right).$$

Using the result of Theorem 7.6, integrating by parts and repeatedly applying the Cauchy-Schwarz inequality yields

$$\frac{d}{dt} F \leq \left( ||D Rm||_{L^2}^2 + ||D^2 T||_{L^2}^2 \right) (1 + C t K) - 2 \alpha ||D Rm||_{L^2}^2 - 2 \beta ||D^2 T||_{L^2}^2 - 2 \gamma ||D^2 N||_{L^2}^2 + \alpha \int_M D^3 T * Rm + \beta \int_M D^3 N * DT + C (1 + t K) F.$$

The first two terms in the last line are reflecting the upper triangularity of the symbol of our system of equations, and are the reason for resorting to $L^2$ norms. Integrating by parts and applying the Cauchy Schwarz inequality, we observe that with $\alpha$ chosen large with respect to a uniform constant, $\beta$ chosen large with respect to $\alpha$, and $\gamma$ chosen large with respect to $\beta$ we yield the estimate

$$\frac{d}{dt} F \leq C((1 + t K)) F$$

which one may integrate over the required time interval to yield the required result. □

Next we claim pointwise smoothing estimates for solutions to (1.4).

**Theorem 8.1.** Given $m > 0$, there exists $C = C(m, n)$ such that if $(M^{2n}, \omega(t), J(t))$ is a solution to (1.4) on $[0, \frac{t}{K}]$ satisfying

$$\sup_{M \times [0, \frac{t}{K}]} \{|Rm|, |DJ|^2, |D^2 T| \} \leq K,$$

then

$$\sup_{M \times [0, \frac{t}{K}]} \{|D^m Rm|_{C^0}, |D^{m+2} J|_{C^0} \} \leq \frac{CK}{t^{\frac{m}{2}}}. $$

**Proof.** Though the quantities involved are different, the proof is formally identical to [18] Theorem 7.3, with $DJ$ playing the role of $T$. □

We can now give the proof of Theorem 1.9

**Proof.** This argument is also standard. First one notes that if the curvature, torsion, and first derivative of torsion are bounded on a finite time interval, then the metrics are uniformly equivalent along the flow, and hence the Sobolev constant of the manifold is also bounded. The $L^2$ derivative bounds of Theorem 7.6 then yield pointwise bounds on the derivatives of curvature and torsion. Then an application of [11] Theorem 2.3 yields smooth convergence of the almost Hermitian structures as $t \to \tau$, contradicting maximality of $\tau$. □
Next we want to prove Theorem 1.10 which will follow directly from Theorem 1.9 and the following proposition.

**Proposition 8.2.** Let \((M^{2n}, \omega(t), J(t))\) be a solution to (1.4) on \([0, T]\) satisfying

\[
\sup_{M \times [0, T]} |Rm| = K.
\]

There exists a constant \(C(K, n, \omega_0, J_0, T)\) such that

\[
\sup_{M \times [0, T]} |D^2 \! J| + |D^2 J| \leq C.
\]

**Proof.** From the Weitzenböck formula (Lemma 3.1) we note that for an almost Kähler manifold with \(|Rm| \leq K\), one has \(|D^* D \omega| \leq CK\) for a constant \(C\) depending only on \(n\). In the proof of Lemma 8.4, we noted that \((D^* D \omega)^{1,1} = N^2\). By a direct calculation one has that

\[
|D J|^2 = |\langle \omega, N^2 \rangle| = |\langle \omega, D^* D \omega \rangle| \leq C|D^* D \omega| \leq CK.
\]

Now fix some constants \(\alpha, \beta > 0\) and let

\[
F(x, t) = |D^2 J|^2 + \alpha |Rm|^2 + \beta |D J|^2.
\]

Using Proposition 7.9 and Theorem 7.10 we conclude that there is a constant \(C\) depending only on \(n\) such that

\[
\frac{\partial}{\partial t} F \leq \Delta F - 2 \left( |D^3 J|^2 + \alpha |D Rm|^2 + \beta |D^2 J|^2 \right)
+ C \left( |Rm| |D^2 J|^2 + |D J| |D Rm| |D^2 J| + |D J| |D^2 J| |D^3 J| + |D^2 J|^2 \right)
+ C \alpha \left( |Rm|^3 + |D J| |D^3 J| |Rm| + |D^2 J|^2 |Rm| + |D J|^2 |Rm|^2 \right)
+ C \beta \left( |D J|^4 + |Rm| |D J|^2 + |D^2 J| |D J|^2 \right).
\]

Using the estimates on \(|Rm|\) and \(|D J|^2\), and replacing \(K\) by \(\max\{K, 1\}\) if necessary we conclude by the Cauchy-Schwarz inequality

\[
\frac{\partial}{\partial t} F \leq \Delta F - 2 \left( |D^3 J|^2 + \alpha |D Rm|^2 + \beta |D^2 J|^2 \right)
+ \frac{1}{2} |D^3 J|^2 + C K |D^2 J|^2 + C K^2 \left( |D Rm|^2 + |D^2 J|^2 \right)
+ \frac{1}{2} |D^3 J|^2 + C \alpha K |D^2 J|^2 + C \alpha^2 K^3
+ \frac{1}{2} |D^2 J|^2 + C (\beta + \beta^2) K^2.
\]

It is clear then that if we choose \(\alpha\) large with respect to dimension dependent constants, and then choose \(\beta\) large with respect to \(\alpha\), we may conclude

\[
\frac{\partial}{\partial t} F \leq \Delta F + C K^3.
\]

The proposition follows from the maximum principle. \(\Box\)
9. The structure of critical metrics

In this section we record some results on the structure of the limiting objects of equations (1.4).

Definition 9.1. Let \((M^{2n}, \omega, J)\) be an almost Kähler manifold. We say that this manifold is \emph{static} if there exists \(\lambda \in \mathbb{R}\) such that

\[
P = \lambda \omega \tag{9.1}
\]

\[
D^* DJ - \mathcal{N} - \mathcal{R} = 0. \tag{9.2}
\]

Let us say a word on the definition of this condition. We want to understand the limiting behavior of equation (1.4), hence the first condition arises for solutions which simply rescale the metric. Observe though that even for solutions which are scaling the metric, one expects \(J\) to remain fixed as one cannot scale almost complex structures. Thus the static condition defined above is a natural expression of the expected smooth limit points of (1.4).

Lemma 9.2. Let \((M^{2n}, \omega, J)\) be a static structure. Then

\[
\text{Ric}^{-J} = 0, \tag{9.3}
\]

i.e. the Ricci tensor is \(J\)-invariant.

Proof. Equation (9.1) implies that \(P^{2,0+0,2} = 0\). Equation (9.2) may be expressed as

\[
g^{-1} [P^{2,0+0,2} + \text{Ric}^{-J}] = 0,
\]

and so the lemma follows.

Let us show some further structure in dimension 4. Let \((M^4, g)\) be an oriented Riemannian manifold. Since one may decompose \(\Lambda^2 = \Lambda^+ \oplus \Lambda^-\), the action of the curvature tensor on \(\Lambda^2\) decomposes accordingly, and is typically written

\[
R = \left( \begin{array}{c|c}
W^+ + \frac{a}{12} I & \text{Rc} \\
\hline
\text{Rc} & W^- + \frac{b}{12} I
\end{array} \right) \tag{9.4}
\]

where \(\text{Rc}\) is a certain action of the traceless Ricci tensor and \(W^+\) and \(W^-\) are the self-dual and anti-self-dual Weyl curvatures. If one further has \((M^4, \omega, J)\) an almost Hermitian manifold, then one can refine the decomposition of \(\Lambda^2\) as

\[
\Lambda^2 = ((\omega) \oplus \Lambda^{2,0}) \oplus \Lambda_0^{1,1} \tag{9.5}
\]

where \(\Lambda_0^{1,1}\) are real \((1,1)\) forms orthogonal to \(\omega\). Using this further decomposition one yields, adopting notation of [3],

\[
R = \left( \begin{array}{ccc}
a & W_F^+ & R_F \\
W_F^{++} & W_0^0 + \frac{1}{3} b I & R_0^0 \\
R_F^* & R_0^* & W_0^0 + \frac{3}{3} c I
\end{array} \right) \tag{9.6}
\]

where the tensors in this equation are defined by comparing with (9.4) and using the refined decomposition of forms of (9.5). The double bars indicate the original decomposition into self-dual and anti-self-dual forms. Now we recall a curvature calculation in [3] which decomposes the curvature tensor of the \emph{canonical} connection of an almost Kähler manifold according to (9.5).
Proposition 9.3. ([3] Proposition 2)

\[
\begin{pmatrix}
\frac{s^\nu}{P} & W_F^+ & R_F - 2C \\
0 & 0 & 0 \\
R_F^+ & R_{00} & W^- + \frac{3}{2}cI
\end{pmatrix}
\]
(9.7)

One may consult [3] for the precise definition of \( C \), which is not relevant to us here. All the other tensors are the same as what appears in (9.6). It is important to observe that this matrix acts from the right on two-forms. For instance, the image acting from the right lies entirely in \((1,1)\) forms, as required.

Proposition 9.4. Let \((M^4, \omega, J)\) be a static structure. Then

\[ W_F \equiv 0. \]
(9.8)

Proof. This immediate from (9.7) and the fact that \( P = \lambda \omega \).

Returning to (9.6) it follows that the \( \omega \) is an eigenvector for the action of \( W_4 \). This condition is related to delicate topological estimates of LeBrun [13] related to the Seiberg-Witten equations. Furthermore, a Theorem of Apostolov, Armstrong, and Drăghici states that compact almost Kähler four-manifolds satisfying (9.3) and (9.8) are automatically Kähler. Thus using Lemma 9.2 and Proposition 9.4 and [2] Theorem 2 it follows that compact static four-manifolds are Kähler Einstein.

Corollary 9.5. Let \((M^4, \omega, J)\) be a compact static structure. Then \((\omega, J)\) is Kähler-Einstein.

10. Remarks and open problems

Recall from [21], [25] we know that the solution to Kähler Ricci flow exists smoothly as long as the associated cohomology class is in the Kähler cone. Therefore it is natural, for purposes of understanding the long time existence and singularity formation of solutions to (10.1), to understand the corresponding cone \( C \) of symplectic forms in \( H^2(M, \mathbb{R}) \). Note that \( C \) consists of all cohomology classes in \( H^2(M, \mathbb{R}) \) which can be represented by a symplectic form. Any symplectic form \( \omega \) admits compatible almost complex structures, and moreover the space of these almost complex structures is contractible. Thus one may define the canonical class

\[ K = c_1(M, \omega) := c_1(M, J) \]

where \( J \) is any almost complex structure compatible with \( \omega \) and the orientation. It is clear that the homotopy classes of symplectic structures define the same canonical class. Therefore, associated to a solution of (1.4), one has the well-defined associated ODE in cohomology

\[ \frac{d}{dt}[\omega] = -K. \]
(10.1)

It is clear by the definition that given a solution to (1.4), the associated one parameter family of cohomology classes satisfies (10.1). Thus, we have

Lemma 10.1. Given \((M^{2n}, \omega(t), J(t))\) a solution to (1.4), let

\[ T^* := \sup\{t > 0| [\omega(t)] = [\omega(0)] - tK \in C\}. \]
Furthermore, let $T$ denote the maximal existence time of $(\omega(t), J(t))$. Then

$$T \leq T^*$$

It is natural to conjecture: The maximal existence time for $(1.4)$ with initial condition $\omega(0)$ is given by $T^*$. This is the analogue of the theorem of Tian-Zhang [21, 25] mentioned above for Kähler Ricci flow.

If the above $T^* < \infty$, then $(1.4)$ develops finite-time singularity. The second basic problem is to study the nature of such a singularity. Is it possible that such a singularity is caused by $J$-holomorphic spheres as we see in the case of Kähler manifolds? The case of 4-dimensional symplectic manifolds is of particular interest and may be easier to study. We expect that either $(\omega(t), J(t))$ collapses to a lower dimensional space or converges to a smooth pair $(\omega_f, J_T)$ outside a subvariety as $t$ tends to $T \leq T^*$. If so, we may do surgery and extend $(1.4)$ across $T$. By scaling, one may get ancient solutions for $(1.4)$. A basic problem is to classify all the ancient solutions. In dimension 4, it may be possible to classify.

Another natural problem is to find functionals which are monotonic along $(1.4)$. In particular one can ask, is $(1.4)$ a gradient flow like the Ricci flow and the pluriclosed flow of [19]? We showed in [20] that the parabolic flow of pluriclosed metrics of [19] is in fact a gradient flow. This was done by exhibiting that after change by a certain diffeomorphism solutions to this flow are equivalent to solutions to the $B$-field renormalization group flow of string theory. In light of Proposition 6.2 solutions to $(1.4)$ have the metric evolving by the Ricci flow plus certain lower order terms, therefore one expects to be able to add a certain Lagrangian to the Perelman functionals to obtain a gradient flow property for $(1.4)$, as in the $B$-field renormalization group flow. This will be the subject of future work.

Furthermore, we believe that this new symplectic curvature flow will be useful in studying the topology of symplectic manifolds, particularly in dimension 4. It follows from the results in section 6 that static solutions in dimension 4 are of anti-self-dual type, more precisely, the self-dual part of curvature for the canonical connection is determined by its scalar curvature. This gives a hope to use $(1.4)$ to prove a symplectic version of the Miyaoka-Yau inequality for complex surfaces. Such an inequality for symplectic 4-manifolds has been long speculated. For still further applications, we are led to studying limits of $(1.4)$ as time $t$ tends to $\infty$ and after appropriate scalings. The limits should include the above static metrics, soliton solutions as well as collapsed metrics which generalize the metrics studied by Song-Tian [17] for elliptic surfaces.

Finally, one could go still further and attempt to use symplectic curvature flow to understand general four-manifolds with $b_2^+ \geq 1$. A smooth 4-manifold $M$ with $b_2^+ \geq 1$, admits a “near-symplectic form,” roughly speaking a symplectic form which degenerates along certain disjoint circles in the manifold. Forms of this type have been studied for instance in [23]. By imposing the appropriate boundary condition, one may be able to construct solutions of $(1.4)$ for such near-symplectic structures. In this manner one envisions using symplectic curvature flow as a tool for understanding more general 4-manifolds.

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SYMPLECTIC CURVATURE FLOW

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