DIFFUSION APPROXIMATIONS FOR SELF-EXCITED SYSTEMS WITH APPLICATIONS TO GENERAL BRANCHING PROCESSES

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In this work, several convergence results are established for nearly critical self-excited systems in which event arrivals are described by multivariate marked Hawkes point processes. Under some mild high-frequency assumptions, the rescaled density process behaves asymptotically like a multi-type continuous-state branching process with immigration, which is the unique solution to a multi-dimensional stochastic differential equation with dynamical mechanism similar to that of multivariate Hawkes processes. To illustrate the strength of these limit results, we further establish diffusion approximations for multi-type Crump-Mode-Jagers branching processes counted with various characteristics by linking them to marked Hawkes shot noise processes. In particular, an interesting phenomenon in queueing theory, well-known as state space collapse, is observed in the behavior of the population structure at a large time scale. This phenomenon reveals that the rescaled complex biological system can be recovered from its population process by a lifting map.

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1. Introduction. This paper is concerned with developing a diffusion approximation for a stochastic dynamical system enjoying self-exciting property. In such a system, events are likely to not only occur in clusters but also mutually depend on each other. To capture both the self-exciting property and the clustering effect, we model the event arrivals with a multivariate marked Hawkes point measure with homogeneous immigration (multivariate MHPI-measure) on \((0, \infty) \times \mathbb{U}\), denoted by \(N_H(dt, du) := (N_i(dt, du))_{i \in \mathcal{H}}\), where \(\mathbb{U}\) is a measurable space and \(\mathcal{H} := \{1, 2, \ldots, d\}\) for some \(d \in \mathbb{Z}_+\). To be precise, the random point measure \(N_i(dt, du)\) has a predictable intensity \(\lambda_i(t- \cdot) \cdot dt \cdot \nu_i(du)\) in which \(\nu_i(du)\) is a probability law on \(\mathbb{U}\) and

\[
\Lambda_i(t) := \mu_i(t) + \int_0^t \int_{\mathbb{U}} \phi_i(t-s, u)N_I(ds, du) + \sum_{j \in \mathcal{H}} \int_0^t \int_{\mathbb{U}} \phi_i(t-s, u)N_j(ds, du), \quad t \geq 0
\]

for some non-negative functional-valued random variable \(\mu_H(t) := (\mu_i(t))_{i \in \mathcal{H}} : t \geq 0\), kernel\(^1\) \(\phi_H := (\phi_i)_{i \in \mathcal{H}} : \mathbb{R}_+ \times \mathbb{U} \rightarrow \mathbb{R}^d_+\), Poisson random measure \(N_I(dt, du)\) on \((0, \infty) \times \mathbb{U}\) with intensity \(\lambda_I \cdot dt \cdot \nu_I(du)\) for some constant \(\lambda_I \geq 0\) and probability law \(\nu_I(du)\) on \(\mathbb{U}\); more accurate definitions can be found in Section 2. Usually, \(\mu_H\) is interpreted as the impact of all events prior to time 0 on the arrivals of future events. This multivariate MHPI-measure includes both self/mutually-excited jumps \((N_H)\) and externally excited jumps \((N_I)\), which respectively model the impact of endogenous and exogenous factors of the underlying system. It can be considered as an extension of marked Hawkes processes (i.e., \(U = \mathbb{R}_+, \mu_H\) is a vector and \(\lambda_I = 0\)) introduced by Ogata [58] for the study of different effects of earthquakes of different magnitudes on the arrivals of the future earthquakes; see also [14, 15] for the case of abstract-valued marks. Specially, when \(U = \mathbb{R}_+\) and \(\phi_i(t, u) = ue^{-\beta_I t}\) for some \(\beta_I > 0\), the embedded point process \(\{N_H(t) := N_H([0, t], \mathbb{R}_+) : t \geq 0\}\) turns to be a multivariate version of dynamical contagion process given in [19]. Moreover, when \(\mu_H\) is a vector, \(\lambda_I = 0\) and the kernel is mark-independent, the point process \(N_H\) reduces to a classical multivariate Hawkes process, which was firstly introduced by Hawkes [30, 31].

\(^1\) For different kernels \((\phi_i)_{i, i \in \mathcal{H}}\) and \((\phi_i)_{i \in \mathcal{H}}\), we can extend the mark space to \(\tilde{U} := U \times \{1, \ldots, d, I\}\) and take \(\tilde{\nu}_j(du) := \nu_j(du_1)\delta_j(du_2)\) and \(\tilde{\phi}(t, \tilde{u}) := \sum_{i \in \mathcal{H}} \phi_i(t, \tilde{u}_1)1_{\{\tilde{u}_2=I\}} + \phi_i(t, \tilde{u}_1)1_{\{\tilde{u}_2=I\}}\) for \(i \in \mathcal{H}\) and \(j \in \mathcal{H} \cup \{I\}\).
As Hawkes processes are always able to provide convincing interpretations of the cascade phenomenon and clustering effect that have been widely observed in various fields (e.g., financial contagion (see [1]) and credit contagion (see [49])), their applications have nowadays gone far beyond the original purpose of modeling earthquakes and their aftershocks; readers may refer to [9] for reviews on Hawkes processes and their applications. In particular, since they were firstly used in the estimation of value-at-risk (see [16]) and modelling market events (see [11]), various financial models have been established in the Hawkes framework to investigate the foreign exchange rates (see [33]), mid-quote prices (see [6, 7]), limit order books (see [35, 56]), stochastic volatility (see [21, 46, 47]) and so on. Readers are suggested to refer to the seminal references of Bacry, Rosenbaum and their coauthors for various microstructure models and macroscopic models.

Different from Hawkes-based models, stochastic models driven by marked Hawkes processes/measures are individual-based models, also called agent-based models, in which a high degree of complexity and differences of events is allowed, and each event has a set of state variables or attributes and behaviors. An advantage of marked Hawkes-based models over Hawkes-based models is that they can incorporate any number of event-level mechanisms. Therefore, they are usually more effective in the modelling of complex dynamical systems. For instance, Horst and Xu [37] used a class of MHPI-measures with exponential kernel to study stochastic volatility models with self-exciting jump dynamics. In this case, each order is associated with a mark from the space $\mathbb{U} = \mathbb{Z} \times \mathbb{R}_+$ that describes the changes of the price in ticks caused by the order along with its impact on the arrival dynamics of future orders. Because of the significant impact of some orders on the arrivals of future orders, jumps occur in the high-frequency limit volatility models. This never happens in the high-frequency limits of Hawkes-based models; see [21, 46]. In another example that illustrates the advantage of marked Hawkes-based models, Xu [73] generalized a classical second Ray-Knight theorem to a spectrally positive stable process by linking the intrinsic branching structure of its local time to a MHPI-measure with kernel being a unit step function. More precisely, each individual in the population is endowed with a mark from the space $\mathbb{U} = \mathbb{R}_+$ to represent its life-length and its survival state is described by the kernel of the form $\mathbf{1}_{\{u > t\}}$ (i.e., it is alive when its life-length $u$ is larger than its age $t$). Furthermore, to emphasize the necessity of setting $\mathbb{U}$ to be an abstract space, in Section 4 we develop a new way to study the general branching particle systems by linking them to MHPI-measures, in which the abstract-valued marks represent individuals’ characteristics, e.g., life-length, reproduction process, impact on host and so on.

Similar to Hawkes processes, the MHPI-measure $N_\mathcal{H}(dt, du)$ can be constructed in collaboration with a labeled birth-immigration particle system, in which the embedding multi-type Galton-Watson process with immigration (GWI-process) has mean matrix $\|\phi_\mathcal{H}\|_{L^1}$ that is the $L^1$ norm of $\{\phi_{\mathcal{H}^2}(t) := (\phi_{ij}(t))_{i,j \in \mathcal{H}} : t \geq 0\}$ with $\phi_{ij}(t) := \int_{\mathbb{U}} \phi_i(t, u)\nu_j(du)$. By the elementary theory of branching processes; see Chapter V in [5], the GWI-process is subcritical, critical or supercritical if the mean matrix $\|\phi_\mathcal{H}\|_{L^1}$ has spectral radius $\varrho < 1$, $= 1$ or $> 1$ respectively. Furthermore, the stationary distribution exits if it is subcritical or critical with sparse immigrants (in this case the stationary distribution does not have finite mean). In the supercritical regime, it grows exponentially to infinity. Therefore, analogous to Hawkes processes; see [13], the condition $\varrho < 1$ is necessary for $N_\mathcal{H}(dt, du)$ to own an asymptotically stationary intensity process with finite first moment. Nowadays, because a significant part of financial transactions is carried out through electronic order books, high-frequency trading has enjoyed a growing popularity. This has made...
high-frequency financial models including Hawkes-based models receive considerable attention in the probability and financial mathematics literature in recent years. Two types of typical and important limit theorems have been widely established to study the behavior of Hawkes-based models at a large time scale. The first one mainly consists of functional law of large numbers (FLLN) and functional central limit theorem (FCLT), which were firstly established by Bacry et al. [7] for a multivariate Hawkes process whose kernel enjoys short-memory property and spectral radius $\rho$ is strictly smaller than one; also see [36] for the case of MHPI-measures. Recently, Horst and Xu [38, 39] established a full FLLN and FCLT for subcritical and critical uni-variate Hawkes processes. The second kind, usually known as scaling limit theorem, was firstly investigated by Jaisson and Rosenbaum [46] in the study of asymptotic behavior of Hawkes-based price models in the context of high-frequency trading. Their results state that under some short-memory condition, the rescaled intensity of nearly unstable Hawkes process converges weakly to a Feller diffusion (also known as CIR-model in finance). Different from the deterministic limit in FLLN and the Brownian motion in FCLT, CIR-model inherits not only the randomness but also the self-exciting property from Hawkes processes. Under a heavy-tailed condition, they also proved that the rescaled Hawkes point process converges weakly to the integral of a rough fractional diffusion; see [47]. A more refined convergence result has recently been established by Horst et al. [40], which proved the weak convergence of the rescaled intensities, instead of their integrals, to a rough fractional diffusion. An analogous scaling limit was later established by El Euch et al. [21] for multivariate Hawkes processes with positive and diagonalizable kernel, see [60] for the case of trigonalizable kernel. In addition, a jump-diffusion limit was provided in [37] for MHPI-measures with real-valued mark and exponential kernel.

In the first part of this work, we mainly investigate the behavior at a large time scale of stochastic dynamical systems driven by asymptotically critical multivariate MHPI-measures, which corresponds to the assumption that $\|\phi H\|_{L^1}$ is close to a limit matrix with unit spectral radius. In addition, compared to the uni-variate case the asymptotic criticality for multivariate Hawkes processes/measures is much more complicated, since there are infinite possibilities for the limit matrix and meanwhile the limit of rescaled Hawkes based models varies greatly for different limit matrix. For instance, in the case of limit matrix being positive and diagonalizable, El Euch et al. [21] proved that the rescaled intensity process of multivariate Hawkes process gradually concentrates in one direction of $\mathbb{R}^d$ and the limit process is the multiplication of one-dimensional CIR-model by a vector. By contrast, the mean matrix $\|\phi H\|_{L^1}$ in our setting is assumed to converge to an identity matrix. Under some short-memory conditions, we show that the rescaled intensity process asymptotically behaves like a multi-type continuous-state branching process with immigration (CBI-process) that is defined as the unique strong solution of a $d$-dimensional stochastic differential equation (SDE) with linear drift and $1/2$-Hölder continuous diffusion.

Our assumption of the convergence of $\|\phi H\|_{L^1}$ to an identity matrix stems from three main reasons. Firstly, under this assumption both proofs and statements can be dramatically simplified. Moreover, by using the rotation method developed in [21, 60], our proofs can be generalized to the case of trigonalizable limit matrices and the corresponding limit theorems can be established similarly. Secondly, in practice, the self-excitation is generally much stronger than the mutual-excitation, e.g. market, limit and cancel orders in financial market are likely to effect themselves (see [8]); the foregoing topic in social media communities is preferred to be discussed continuously (see [59]). These are consistent with our assumption, i.e., $\|\phi_{ij}\|_{L^1} \ll \|\phi_{ii}\|_{L^1}$ for $i \neq j$. It seems that the mutual-excitation in our setting can be asymptotically ignored, but its impact on the underlying system still can be observed in limit process. Finally, the diagonal entries of drift matrix in the limit $d$-dimensional SDE represents the net self-excitation. Moreover, the off-diagonal entries are non-negative and can be
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interpreted as the mutual-excitation. Hence, compared with the limit model established in [21] as the multiplication of one-dimensional CIR-model by a vector, our limit model is a more natural continuous version analogous in form to the stochastic dynamical system (1).

The main results in the first part of this paper are proved by extending the method developed in [46], in which the rescaled intensity of Hawkes process is rewritten in the form of an Itô's SDE and then the limit theorem is proved by using the convergence results established in [52] for finite-dimensional stochastic integrals. However, in addition to the technical difficulties encountered in [21], three main and new challenges are induced by our setting. Firstly, the high degree of complexity and differences of events gives rise to not only some additional perturbations in the dynamics of intensity processes but also severe fluctuation in error processes. These make it much more difficult to establish a rigorous connection between multivariate MHPI-measures and multi-type CBI-processes. Secondly, as two sufficient conditions for the convergence results in [52], the weak convergence and uniform tightness of driving noises were proved easily based on the fact that intensities of nearly unstable Hawkes processes in [46] are uniformly strictly larger than zero. By comparison, they are extremely difficult to be identified in our setting, since the exogenous intensity $\mu_H$ varies as time goes and the intensity process $\Lambda_H$ may hit zero in finite time. Thirdly, the stability condition, widely considered in the Hawkes literature (see [7, 21, 46, 47]), is not assumed in this work and MHPI-measures are allowed to be unstable ($\rho > 1$). In this case, both the resolvent and the intensity process may grow exponentially to infinity, which make the error estimates more difficult. To overcome the first two difficulties, we start by reconstructing MHPI-measures in collaboration with Poisson random measures. The mutual-excitation is interpreted as the non-local branching mechanism in the corresponding birth-immigration particle system. Under the assumption that $\|\phi_{H^2}\|_{L^1}$ converges to an identity matrix, we further consider it as a state-dependent immigration and translate the mutually-excited jumps into another kind of externally excited jumps. Inspired by the computations and techniques applied in [36, 37], associated to the resolvent we introduce a two-parameter function to describe the average impact of an event with some mark on the future intensity. It enables us to write the stochastic equation (1) approximately as an Itô's SDE driven by an infinite-dimensional semimartingale. In particular, this semimartingale mainly consists of several compensated Poisson random measures whose weak convergence and uniform tightness follow immediately from their orthogonal increments. Moreover, with the help of the foregoing two-parameter function, our error analysis is successfully carried out through investigating the exact perturbations of each event of various marks in the error processes. The desired limit theorem for intensity processes is finally obtained by using the convergence results of infinite-dimensional stochastic integrals established by Kurtz and Protter [53]. For the third difficulty, the exponential growth of the resolvent and intensity processes of supercritical MHPI-measures encourages us to modify the self-excited dynamical system by an exponential function. Due to the multiplicative property of exponential functions, the preceding representations and asymptotic analysis remain valid with some slight modifications.

As mentioned above, the main contribution in the second part of this work is to illustrate the strength of the foregoing limit results for MHPI-measures by applying them to study the behavior at a large time scale of multi-type Crump-Mode-Jagers branching processes with immigration (CMJI-processes). In the realistic pattern, as Peter Jagers [45] pointed out, population models “must be ultimately stochastic [...] individual based [...] life span can have an arbitrary distribution [...] reproduction should be modelling as it actually occurs”. As a result, CMJI-processes, as a class of continuous-time and discrete-state stochastic population models with age-dependent reproduction mechanism, have received considerable attention in the probability and mathematical biology literature since they were firstly introduced in [17, 18, 43]. However, because they are generally neither Markov nor semimartingales, the
instruments provided by modern probability theory are almost out of work and hence re-
searches concerning CMJI-processes are relatively less than those of Markovian population
models; see [5, 44]. To the best of our knowledge, only few asymptotic results have been
established for CMJI-processes up to now, e.g., a scaling limit was established by Lambert
et al. [55] for homogeneous, binary and single-type CMJI-processes without immigration via
connecting them to the local time of compound Poisson processes. However, this connection
can not be generalized to CMJI-processes with general branching mechanism and complex
individual characteristics.

Here we develop a new way to investigate multi-type CMJI-processes by linking them
to multivariate MHPI-measures. More precisely, we translate random point measures
\( N_I(dt, du) \) and \( N_R(dt, du) \) into the arrivals of immigrants and offspring respectively, and
marks from an abstract space into individuals’ information, e.g., size, type, life-length, re-
production process and characteristic. Different to the existing literature (e.g. [55, 62]) in
which the population size is usually studied in the first place, we start by considering the
asymptotic behavior of total reproduction rate of alive individuals that coincides with the
intensity process of \( N_R(dt, du) \). Using the preceding limit results for multivariate MHPI-
measures, we show that with a suitable scaling, the total reproduction rate process behaves
asymptotically as a multi-type CBI-process. Additionally, for a multi-type CMJI-process
counted with various characteristics (e.g., population size and total progeny), we link it to a
shot noise process driven by \( N_R(dt, du) \) and then show that with a suitable scaling, it can be
well approximated by a functional of the multi-type CBI-process. Furthermore, an interesting
phenomenon, known as state space collapse in queueing theory, is observed in the population
structure at a large time scale. In precise, the joint distribution of age and residual life of alive
individuals can be recovered directly from the population size by an appropriate lifting map.
This indicates that when the life-length and the reproduction process enjoy short-memory
property, more detailed information about the population, except the life-length distribution
and the mean/variance of offspring, is not necessary for the study of complex biological
systems.

Organization of this paper. In Section 2 we provide a branching representation as well
as a stochastic Volterra integral representation for multivariate MHPI-measures. A criticality
criterion for their stationarity is also given as a byproduct. In Section 3, we establish sev-
eral limit theorems for stochastic dynamical systems driven by multivariate MHPI-measure,
including the weak convergence of rescaled intensity processes to a multi-type CBI-process
and scaling limits for marked Hawkes shot noise processes. In Section 4 we apply these limit
results for self-excited systems to establish diffusion approximations for multi-type CMJI-
processes. Section 5 is devoted to the proofs for all limit theorems given in this work.

Notation. Denote by \([x]\) the integer part of the real number \( x \in \mathbb{R} \) and \( z_\mathcal{H} = (z_i)_{i \in \mathcal{H}} = (z_1, \ldots, z_d) \) with \( |z_\mathcal{H}| := |z_1| + \cdots + |z_d| \). Let \( f \ast g \) be the convolution of two functions \( f, g \)
on \( \mathbb{R}_+ \) and \( f^{*n} \) be the \( n \)-th convolution of \( f \). For \( h > 0 \), we write
\[
\Delta_-(f)(x) := f(x) - f(x-) \quad \text{and} \quad \Delta_h f(x) := f(x + h) - f(x).
\]
Let \( \|f\|_{TV} \) denote the total variation of \( f \). For any \( p, q \in (0, \infty] \), let \( L^{p,q}(\mathbb{R}_+) = L^p(\mathbb{R}_+) \cap L^q(\mathbb{R}_+) \) with norm \( \|f\|_{L^{p,q}} := \|f\|_{L^p} + \|f\|_{L^q} \). For \( T > 0 \), let
\[
\|f\|_{L^p_T} := \sup_{t \in [0,T]} |f(t)| \quad \text{and} \quad \|f\|_{L^q_T}^{\, q} := \int_0^T |f(t)|^q \, dt
\]
Denote by \( u.c. \), \( u.c.p. \), \( a.s. \), \( a.d. \), \( p \) and \( f.d.d. \) the compact convergence, uniform convergence
on compacts in probability, almost sure convergence, convergence in distribution, conver-
genesis in probability and convergence in the sense of finite-dimensional distributions. We
also use \( a.s. \), \( d \) and \( \mathbb{P} \) to denote almost sure equality, equality in distribution and equality in probability.

Given a measurable space \((\mathcal{V}, \mathcal{F})\), let \(B(\mathcal{V}), C(\mathcal{V})\) and \(C_0(\mathcal{V})\) be the spaces of measurable functions on \(\mathcal{V}\) that are bounded, continuous and continuous as well as vanishing at infinity respectively. For \(T \in [0, \infty)\), denote by \(D([0, T], \mathcal{V})\) the space of càdlàg functions from \([0, T]\) to \(\mathcal{V}\) furnished with the Skorokhod topology. Let \(\mathcal{M}(\mathcal{V})\) be the space of finite Borel measures on \(\mathcal{V}\) equipped with the weak convergence topology. Let \(\delta_a\) be the Dirac measure at point \(a \in \mathcal{V}\). For any \(v \in \mathcal{M}(\mathcal{V})\) and \(f \in B(\mathcal{V})\), we write

\[
v(f) := \int_{\mathcal{V}} f(x)v(dx) \quad \text{and} \quad f * v := \int_{\mathcal{V}} f(x-y)v(dy).
\]

Throughout this paper, we assume the generic constant \(C\) may vary from line to line.

2. Preliminaries. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space endowed with a filtration \(\{\mathcal{F}_t : t \geq 0\}\) satisfying the usual hypotheses and \((\mathbb{U}, \mathcal{W})\) be a measurable space. Let \(\mathcal{H} := \{1, 2, \cdots, d\}\) for some \(d \in \mathbb{Z}_+\). For \(i \in \mathcal{H}\), let \(\{\tau_{i,k}\}_{k \geq 1}\) be a sequence of increasing, \((\mathcal{F}_t)\)-stopping times and \(\{\xi_{i,k}\}_{k \geq 1}\) be a sequence of i.i.d. \(\mathbb{U}\)-valued random variables with distribution \(\nu_i(du)\) satisfying that \(\xi_{i,k}\) is independent of \(\mathcal{F}_{\tau_{i,k}}\) for any \(k \geq 1\). Associated to these two sequences we define an \((\mathcal{F}_t)\)-random point measure on \((0, \infty) \times \mathbb{U}\)

\[
N_i(dt, du) := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_{i,k} \in dt, \xi_{i,k} \in du\}}.
\]

We say the random point measure \(N_{\mathcal{H}}(dt, du) := (N_i(dt, du))_{i \in \mathcal{H}}\) is a multivariate marked Hawkes point measure (multivariate MHP-measure) on \((0, \infty) \times \mathbb{U}\) with embedded point process \(\{N_{\mathcal{H}}(t) := (N_i((0, t], \mathbb{U}))_{i \in \mathcal{H}} : t \geq 0\}\), if \(N_i(dt, du)\) has \((\mathcal{F}_t)\)-intensity \(\Lambda_i(t) \cdot dt \cdot \nu_i(du)\) and the \((\mathcal{F}_t)\)-intensity process \(\Lambda_i\) is of the form

\[
\Lambda_i(t) = \Lambda_{0,i}(t) + \sum_{j \in \mathcal{H}} \sum_{k=1}^{N_i(t)} \phi_i(t - \tau_{j,k}, \xi_{j,k}), \quad t \geq 0, \quad i \in \mathcal{H},
\]

for some non-negative, locally integrable, \((\mathcal{F}_t)\)-progressive exogenous intensity \(\Lambda_{0,\mathcal{H}} := (\Lambda_{0,i})_{i \in \mathcal{H}}\), and some kernel \(\phi_{\mathcal{H}} := (\phi_{ij})_{i \in \mathcal{H} : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{R}_+}\). Specially, we call \(N_{\mathcal{H}}(dt, du)\) a multivariate MHP-measure with homogeneous immigration (multivariate MHPI-measure) if \(\Lambda_{0,\mathcal{H}}\) admits the representation:

\[
\Lambda_{0,i}(t) = \mu_i(t) + \sum_{k=1}^{N_i(t)} \phi_i(t - \tau_{i,k}, \xi_{i,k}), \quad t \geq 0, \quad i \in \mathcal{H},
\]

where \(N_i\) is a Poisson process with rate \(\lambda_i\) and arrival times \(\{\tau_{i,k}\}_{k \geq 1}\), \(\{\xi_{i,k}\}_{k \geq 1}\) is a sequence of i.i.d. \(\mathbb{U}\)-valued random variables with distribution \(\nu_i(du)\) and independent of \(N_i\), and \(\mu_{\mathcal{H}} := (\mu_i)_{i \in \mathcal{H}}\) is an \(\mathcal{F}_0\)-measurable \(\mathcal{D}([0, \infty), \mathbb{R}_+^d)\)-valued random variable. For \(i \in \mathcal{H}\), let

\[
\mathcal{D} := \mathcal{H} \cup \{I\}, \quad \mathcal{H}_i := \mathcal{H} \setminus \{i\} \quad \text{and} \quad \mathcal{D}_i := \mathcal{D} \setminus \{i\}.
\]

For simplicity, we assume that \(\lambda_i = 1\) and \(\tau_{i,k} \neq \tau_{j,l}\) a.s. for \((i, k), (j, l) \in \mathcal{D} \times \mathbb{Z}_+\) with \((i, k) \neq (j, l)\). We also refer all externally excited jumps as type \(I\) events. Let \(\phi_{\mathcal{H}^2} := (\phi_{ij})_{i,j \in \mathcal{H}}\) and \(\phi_{\mathcal{H}^1} := (\phi_{ij})_{i \in \mathcal{H}}\) with

\[
\phi_{ij}(t) := \int_{\mathbb{U}} \phi_i(t, u)\nu_j(du), \quad t \geq 0, \quad i \in \mathcal{H}, \quad j \in \mathcal{D}
\]

be the mean impact functions of a type-\(j\) event on the future arrivals of type-\(i\) events. In the sequel, we always assume that

\[
\left\|\phi_{\mathcal{H}^2}\right\|_{L^1} := \left(\left\|\phi_{ij}\right\|_{L^1}\right)_{i,j \in \mathcal{H}} < \infty \quad \text{and} \quad \left\|\phi_{\mathcal{H}^1}\right\|_{L^1} := \left(\left\|\phi_{ij}\right\|_{L^1}\right)_{i \in \mathcal{H}} < \infty.
\]
2.1. Branching representation. In this section we show that the foregoing construction of multivariate MHPI-measure \( N_{\mathcal{H}}(dt, du) \) can be done in collaboration with a multi-type birth-immigration particle system defined on the probability basis \( (\Omega, \mathcal{F}, \mathcal{F}_1, P) \) by the following properties:

(A1) There is an ancestor at time 0, whose successive ages arrive according to a Cox point process with intensity process \( \{\mu_{\mathcal{H}}(t) : t \geq 0\} \). Only one child is born at each successive age. Conditioned on the birth time \( t \), the child is type-\( i \) and has mark \( u \in U \) with probability \( \mu_i(t) \cdot |\mu_{\mathcal{H}}(t)|^{-1} \cdot \nu_i(du) \);

(A2) Immigrants enter into the population according to a Poisson process with rate 1 and are endowed with a mark randomly and independently according to the probability law \( \nu_i(du) \);

(A3) For each individual (except the ancestor) with mark \( u \in U \), it gives birth to a child at the rate \( \phi_{\mathcal{H}}(t, u) \) at age \( t \). Moreover, the child has probability \( \phi_i(t, u) \cdot \phi_{\mathcal{H}}(t, u)^{-1} \) to be type-\( i \) and it picks up a mark according to the law \( \nu_i(du') \). Moreover, all individuals produce their offspring independently.

Denote by \( A \) the collection of all individuals except the ancestor. Associated with each individual \( x \in A \) is a random triple \( (t'_x, \tau'_x, u'_x) \) that represents its type, birth time and mark respectively. Define an \( (\mathcal{F}_1) \)-random point measure \( N'_i(dt, du) := (N'_i(dt, du))_{i \in \mathcal{H}} \) on \( (0, \infty) \times U \) with

\[
N'_i(dt, du) := \sum_{x \in A} 1_{\{t'_x = i, \tau'_x = dt, u'_x \in du\}}.
\]

The intensity of its embedded random point process \( \{N'_i(t) := (N'_i([0, t], U) : t \geq 0\} \), denoted by \( \{\Lambda'_i(t) := (\Lambda'_i(t))_{i \in \mathcal{H}} : t \geq 0\} \), equals to the total birth rate of children of various types. In addition, by the branching property, it admits the following representation

\[
\Lambda'_i(t) = \mu_i(t) + \sum_{x \in A_i} \phi_i(t - \tau'_x, u'_x) + \sum_{j \in \mathcal{H}} \sum_{x \in A_j} \phi_i(t - \tau'_x, u'_x), \quad t \geq 0, i \in \mathcal{H},
\]

where \( A_i \) and \( A_j \) are the collections of all immigrants and type-\( j \) offspring respectively. The following result can be obtained immediately by comparing (6) with (3)-(4).

**Proposition 2.1.** The random point measure \( N'_i(dt, du) \) is a realization of the multivariate MHPI-measure defined by (2)-(4).

The embedded point process \( N_{\mathcal{H}} \) (or \( N'_{\mathcal{H}} \)), also can be considered as a cluster process in which the process of cluster centres is the random point process formed by the arrivals of immigrants and the successive ages of the ancestor. The cluster at each centre is formed by all the descendants of an immigrant or a child of the ancestor. These clusters are mutually independent and identically distributed. Denote by \( \{X_{n, \mathcal{H}} := (X_{n, i})_{i \in \mathcal{H}} : n = 1, 2, \ldots\} \) the embedded multi-type Galton-Watson process (GW-process) of a cluster produced by an immigrant. It is easy to see that elements of \( X_{1, \mathcal{H}} \) are mutually independent and \( X_{1, i} \) is Poisson distributed with rate \( \|\phi_i\|_{L^1} \). For \( n \geq 2 \), \( X_{n, i} \) is the number of type-\( i \) individuals in the \( n \)-th generation, which can be written as

\[
X_{n, i} = \sum_{j \in \mathcal{H}} \sum_{k=1}^{X_{n-1, j}} \xi_{n, j, k, i}, \quad i \in \mathcal{H},
\]

with \( \xi_{n, j, k, i} \) being the number of type-\( i \) children born by the \( k \)-th type-\( j \) individual in the \((n - 1)\)-th generation, which is Poisson distributed with parameter \( \|\phi_{j}\|_{L^1} \). Let \( q \) be the
spectral radius of the matrix $\|\phi_{\mathcal{H}}\|_{L^1}$ and $I$ be an $d$-dimensional identity matrix. The mean cluster size equals to the mean of total progeny

$$
\sum_{n=1}^{\infty} \mathbb{E}[X_{n,H}] = \sum_{n=1}^{\infty} \|\phi_{\mathcal{H}^n}\|_{L^1} \cdot \|\phi_{\mathcal{H}}\|_{L^1} = \left( I - \|\phi_{\mathcal{H}}\|_{L^1} \right)^{-1} \cdot \|\phi_{\mathcal{H}}\|_{L^1},
$$

which is finite if and only if $\varrho < 1$. The next proposition follows immediately from Theorem 3 and Corollary 3.2 in [69].

**Proposition 2.2.** If $\mu_H$ is a non-negative constant vector and $\varrho < 1$, the embedded point process $N_H$ is asymptotically stationary.

Drawing from the criticality criterion for multi-type GW-processes, we say the multivariate MHPI-measure $N_H(dt, du)$ is subcritical, critical or supercritical if $\varrho < 1, = 1$ or $> 1$. These correspond to the three phases of a classical Hawkes process: stationary, quasi-stationary or non-stationary; see [9].

### 2.2. Stochastic Volterra representation.

We now provide a stochastic Volterra representation for the intensity process $\Lambda_H$, which will play a considerably important role in the following asymptotic analysis. Associated to the sequence $\{(\tau_{I,k}, \xi_{I,k})\}_{k \geq 1}$ we define an $(\mathcal{F}_t)$-Poisson random measure

$$
N_I(ds, du) := \sum_{k=1}^{\infty} 1_{\{\tau_{I,k} \in ds, \xi_{I,k} \in du\}}
$$
on $(0, \infty) \times \mathbb{U}$ with intensity $dt \cdot \nu_I(du)$ and then rewrite the intensity process $\Lambda_H$ under the form (1). Moreover, following the argument in [41, p.93], on an extension of the original probability space we can define $d$ mutually orthogonal Poisson point measures $N_{0,i}(dt, du, dz)$, $i \in \mathcal{H}$ on $(0, \infty) \times \mathbb{U} \times \mathbb{R}_+$ independent of $N_I(ds, du)$ such that $N_{0,i}(dt, du, dz)$ has intensity $dt \cdot \nu_i(du) \cdot dz$ and

$$
\int_0^t \int_{\mathbb{U}} f(u) N_I(ds, du) = \int_0^t \int_{\mathbb{U}} \int_0^{\Lambda_i(s-)} f(u) N_{0,i}(ds, du, dz), \quad t \geq 0, i \in \mathcal{H},
$$

for any $f \in B(\mathbb{U})$. We can thus rewrite the last stochastic integral in (1) as

$$
\int_0^t \int_{\mathbb{U}} \int_0^{\Lambda_i(s-)} \phi_i(t-s, u) N_{0,i}(ds, du, dz), \quad j \in \mathcal{H}.
$$

Actually, we can always construct multivariate MHPI-measures in collaboration with some Poisson random measures on the probability basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$; see Section 2 in [36].

**Remark 2.3.** If $\mu_H$ is a positive constant vector and $\nu_I(\mathbb{U}) = 0$, then $N_H(dt, du, H)$ reduces to a classical multivariate MHP-measure (without immigration) on $(0, \infty) \times \mathbb{U}$. We now link it to a special multivariate MHPI-measure. For each $i \in \mathcal{H}$, let $\Lambda_i^0 := \Lambda_i - \mu_i$ and

$$
N_i^0(dt, du) := N_{0,i}(dt, du, [0, \Lambda_i^0(t-)]),
$$

$$
N_I^0(dt, du) := N_{0,i}(dt, du, [\Lambda_i^0(t-), \Lambda_i(t-)]).
$$

Let $N_i^0(dt, du) := \sum_{j=1}^{d} N_{I,j}^0(dt, du)$, which is a Poisson random measure on $(0, \infty) \times \mathbb{U}$ with intensity $ds \cdot \sum_{j \in \mathcal{H}} \mu_j \cdot \nu_j(du)$. It is obvious that

$$
N_H(dt, du) = N_i^0(dt, du) + N_{I,H}^0(dt, du)$$
and $N^\omega_H(dt, du)$ is a multivariate MHPI-measure on $(0, \infty) \times U$ with intensity process $\Lambda^\omega_H$ being of the form

$$\Lambda^\omega_i(t) = \int_0^t \int_U \phi_i(t - s, u) N^\omega_j(ds, du) \, \mu_i(t) + \sum_{j \in H} \int_0^t \int_U \phi_i(t - s, u) N^\omega_j(ds, du), \quad t \geq 0, i \in H.$$ 

To get the desired stochastic Volterra representation, we need several important quantities associated to $\phi_i$ and $\phi_{ij}$ for $i \in H$ and $j \in D$. Let $R_{ii}$ be the resolvent of $\phi_{ii}$ defined by

$$R_{ii}(t) = \phi_{ii}(t) + R_{ii} \ast \phi_{ii}(t), \quad t \geq 0.$$

(8)

The existence and uniqueness of the solution $R_{ii}$ follow directly from the assumption $\|\phi_{ii}\|_{L^1} < \infty$ and Theorem 3.1 in [27, p.32]. It is easy to identify that $R_{ii}$ admits the representation

$$R_{ii}(t) = \sum_{k=1}^{\infty} \phi^{*k}_{ii}(t), \quad t \geq 0.$$

(9)

It is usual to interpret $R_{ii}$ as the mean impact of a type-$i$ event and its triggered events on the future arrivals of type-$i$ events. In addition, the mean impact of a type-$j \in D$ event and its triggered events on the future arrivals of type-$i$ events also can be described as

$$R_{ij}(t) := \phi_{ij}(t) + R_{ii} \ast \phi_{ij}(t), \quad t \geq 0.$$

Similarly, associated to the kernel $\phi_i$ we define a two-parameter function

$$R_i(t, u) := \phi_i(t, u) + R_{ii} \ast \phi_i(t, u), \quad (t, u) \in \mathbb{R}_+ \times U,$$

(10)

to recount the mean impact of an event with mark $u$ and its triggered events on the future arrivals of type-$i$ events. An argument similar to the one in [36, Section 2] deduces the next proposition immediately.

**Proposition 2.4 (Martingale representation).** The intensity process $\Lambda_H$ is the unique solution to the following stochastic Volterra integral equation

$$\Lambda_i(t) = \mu_i(t) + R_{ii} \ast \mu_i(t) + \sum_{j \in H} R_{ij} \ast \Lambda_j(t)$$

$$+ \int_0^t R_{ii}(s) ds + \sum_{j \in D} \int_0^t \int_U R_i(t - s, u) \tilde{N}_j(ds, du), \quad i \in H,$$

(11)

where $\tilde{N}_I(ds, du) := N_I(ds, du) - ds \cdot \nu_I(du)$ and $\tilde{N}_j(ds, du) := N_j(ds, du) - \Lambda_j(s-) \cdot ds \cdot \nu_j(du)$ for $j \in H$. Moreover, the last stochastic integral with $j \in H$ can be replaced by

$$\int_0^t \int_U \int_0^{\Lambda_j(s-)} R_i(t - s, u) \tilde{N}_{0,j}(ds, du, dz)$$

with $\tilde{N}_{0,j}(ds, du, dz) := N_{0,j}(ds, du, dz) - ds \cdot \nu_j(du) \cdot dz$. 

2.3. Examples. In this section, we consider three specific examples, which will be revisited when analyzing scaling limits.

Example (Exponential type). For $i \in \mathcal{H}$ and $j \in \mathcal{D}$, let $\beta_i > 0$, $\hat{u}_i \geq 0$ and $\nu_j(du_\mathcal{H})$ be a probability law on $\mathbb{R}_+^d$. The multivariate MHPI-measure $N_\mathcal{H}(dt, du_\mathcal{H})$ on $(0, \infty) \times \mathbb{R}_+^d$ is said to be of exponential type with parameter $(\hat{u}_\mathcal{H}, \beta_\mathcal{H}, \nu_\mathcal{H}, \nu_\mathcal{I})$ if
\[
\mu_i(t) = \hat{u}_i e^{-\beta_i t} \quad \text{and} \quad \phi_i(t, u_\mathcal{H}) = u_i \beta_i e^{-\beta_i t}, \quad i \in \mathcal{H}, \ t \geq 0, \ u_\mathcal{H} \in \mathbb{R}_+^d.
\]

Let $\mathcal{M}(\mathbb{R}_+)$ be the space of finite measures on $\mathbb{R}_+$ equipped with the weak convergence topology and a $\sigma$-algebra $\mathcal{M}(\mathbb{R}_+)$. Let $\mathcal{M}_0(\mathbb{R}_+)$ be the subspace of $v(dx) \in \mathcal{M}(\mathbb{R}_+)$ with $xv(dx) \in \mathcal{M}(\mathbb{R}_+)$ and $\mathcal{M}_0(\mathbb{R}_+)$ be the corresponding $\sigma$-algebra. For any $v \in \mathcal{M}_0(\mathbb{R}_+)$, denote by $L_v$ the Laplace transform of $xv(dx)$
\[
L_v(t) := \int_0^\infty e^{-tx}xv(dx), \quad t \geq 0.
\]
By Bernstein’s theorem; see Theorem 1.4 in [63, p.3], the function $L_v$ is completely monotone on $\mathbb{R}_+$.

Example (Completely monotone type). For $i \in \mathcal{H}$ and $j \in \mathcal{D}$, let $\hat{u}_i \in \mathcal{M}_0(\mathbb{R}_+)$ and $\nu_j(du_\mathcal{H})$ be a probability measure on $\mathcal{M}_0(\mathbb{R}_+)^d$. The multivariate MHPI-measure $N_\mathcal{H}(dt, du_\mathcal{H})$ on $(0, \infty) \times \mathcal{M}_0(\mathbb{R}_+)^d$ is said to be of completely monotone type with parameter $(\hat{u}_\mathcal{H}, \nu_\mathcal{H}, \nu_\mathcal{I})$ if
\[
\mu_i(t) = L_{\hat{u}_i}(t) \quad \text{and} \quad \phi_i(t, u_\mathcal{H}) = L_{\hat{u}_i}(t), \quad i \in \mathcal{H}, \ t \geq 0, \ u_\mathcal{H} \in \mathcal{M}_0(\mathbb{R}_+)^d.
\]

Example (Convolution type). For $i \in \mathcal{H}$ and $j \in \mathcal{D}$, let $\hat{u}_i \in \mathcal{M}(\mathbb{R}_+)$, $\rho_\mathcal{I}$ be a non-negative, bounded, integrable function on $\mathbb{R}_+$ and $\nu_j(du_\mathcal{H})$ be a probability measure on $\mathcal{M}(\mathbb{R}_+)^d$. The multivariate MHPI-measure $N_\mathcal{H}(dt, du_\mathcal{H})$ on $(0, \infty) \times \mathcal{M}(\mathbb{R}_+)^d$ is said to be of convolution type with parameter $(\hat{u}_\mathcal{H}, \rho_\mathcal{H}, \nu_\mathcal{H}, \nu_\mathcal{I})$ if
\[
\mu_i(t) = \rho_\mathcal{I} * \hat{u}_i(t) \quad \text{and} \quad \phi_i(t, u_\mathcal{H}) = \rho_\mathcal{I} * u_i(t), \quad i \in \mathcal{H}, \ t \geq 0, \ u_\mathcal{H} \in \mathcal{M}(\mathbb{R}_+)^d.
\]

3. Limit theorems for self-excited dynamical systems. We consider in this section the weak convergence of stochastic dynamical systems driven by nearly critical multivariate MHPI-measures, which are defined on the common filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. We start by presenting some basic setting on the self/mutual-excitation. In the $n$-th model, we assume that the MHPI-measure $N_\mathcal{H}^{(n)}(dt, du)$ has intensity process $\Lambda_\mathcal{H}^{(n)}$ and parameter$^2$ $(\mu_\mathcal{H}^{(n)}, \phi_\mathcal{I}^{(n)}, \nu_\mathcal{D}^{(n)})$. For $i \in \mathcal{H}$ and $j \in \mathcal{D}$, the mean impact function $\phi_{ij}^{(n)}$ is defined as (5). Here we are interested in the case in which the impact of each event on the future intensity enjoys short-memory property and does not fluctuate drastically. In precise,

(H1) There exist a constant $\alpha \in (1, 2)$ and a function $\Phi$ on $\mathbb{U}$ such that for any $i \in \mathcal{H}, j \in \mathcal{D}$ and $u \in \mathbb{U}$,
\[
\int_0^\infty t \cdot \phi_i(t, u) dt + \|\phi_i(u)\|_{TV} \leq \Phi(u) \quad \text{and} \quad \sup_{n \geq 1} \int_\mathbb{U} |\Phi(u)|^{2\alpha} \nu_{ij}^{(n)}(du) < \infty.
\]

$^2$Actually, the kernel $\phi_\mathcal{H}$ is allowed to be different in various models, similarly as in footnote 1 we also can unify them by extending the mark space.
We now give the detailed asymptotic assumptions on the matrix

\[ \lambda_0 e^{-\lambda_1 t} \] with \( 0 < \lambda_0 \leq (\lambda_1 \wedge K) \).

It is obvious that \( \mathcal{R}_K \) comprises exponential functions in the form of \( e^{-\lambda_1 t} \) with \( 0 < \lambda_0 \leq (\lambda_1 \wedge K) \). It also contains the following two kinds of non-negative functions in the form of

- \( f_\lambda \ast v \) in which \( v \in \mathcal{M}(\mathbb{R}_+) \) with \( v(\mathbb{R}_+) \leq 1 \) and \( f_\lambda \) is the probability density function of exponential distribution with rate \( \lambda \leq K \); see Lemma 4.1 in [50];
- \( \sum_{k=1}^{\infty} (-1)^{k+1} h_*^k \) in which \( h \) is a positive, continuous, non-increasing and log-convex function with \( \|h\|_{L^1} < \infty \), e.g., \( h \) is completely monotone; see Theorem 1 in [26].

In order to simplify the following asymptotic analysis and error estimates, we also assume an additional technical hypothesis on the mean self-excitation.

(H2) There exist two constants \( \beta \geq 0 \), \( K > 0 \) and a non-negative function \( \bar{\phi} \) with

\[ \int_0^\infty t \cdot \bar{\phi}(t) dt < \infty \]

such that for any \( n \geq 1 \) and \( t \geq 0 \),

\[ \phi_{\beta,ii}^{(n)}(t) := e^{-\beta t/n} \phi_{ii}^{(n)}(t) \in \mathcal{R}_K \quad \text{and} \quad \phi_{\beta,ii}^{(n)}(t) \leq \bar{\phi}(t). \]

3.1. Scaling limit for intensity processes. We provide in this section a limit theorem for the rescaled intensity processes, which plays a key role in studying the asymptotics of self-excited dynamical systems. Before giving the theorem, we offer an intuitive description on how to derive it under the following asymptotic assumptions. The detailed and accurate proof can be found in Section 5.1.

3.1.1. Asymptotic assumptions. By the criticality for multivariate MHPI-measures; see Proposition 2.2, the sequence \( \{N^{(n)}_H(dt, du)\}_{n \geq 1} \) is asymptotically critical when the matrix \( \|\phi_{H,n}^{(n)}\|_{L^1} \) converges to a limit matrix with spectral radius equals to one. However, compared to the uni-variate case, the asymptotic criticality for multivariate Hawkes processes/measure is much more complicated because of the infinite possibilities for the limit matrix. In this work we mainly consider a special case in which the limit matrix is an identity matrix \( I \).

Compared to classical Hawkes processes, the random marks make additional perturbations in the convergence of rescaled intensity process via the variances of total self-excitation

\[ \epsilon_{i}^{(n)} := \int_{\mathcal{H}} \|\phi_i(u)\|_{L^2}^2 \nu_i^{(n)}(du), \quad i \in \mathcal{H}, \quad n \geq 1. \]

We now give the detailed asymptotic assumptions on the matrix \( \phi_{H,n}^{(n)} \) and vector \( \phi_{H1}^{(n)} \).

**Condition 3.1.** There exist a matrix \( b_{H,2} := (b_{ij})_{i,j \in \mathcal{H}} \) and three vectors \( a_{H} := (a_{i})_{i \in \mathcal{H}} \in [0, \infty)^{d}, \sigma_{H} := (\sigma_i)_{i \in \mathcal{H}} \in (0, \infty)^{d}, \gamma_{H} := (\gamma_i)_{i \in \mathcal{H}} \in (0, \infty)^{d} \) such that as \( n \to \infty \),

\[ n(\|\phi_{H,n}^{(n)}\|_{L^1} - 1) \to b_{H,2}, \quad \|\phi_{i}^{(n)}\|_{L^1} \to a_i \]
and
\[ \sigma_i^{(n)} := \int_0^\infty t \phi_{ii}^{(n)}(t) dt \to \sigma_i, \quad c_i^{(n)} \to c_i^2. \]

**Remark 3.2.** Under this condition, the self-excitation in the underlying system is much stronger than the mutual-excitation, which is consistent to many practical applications. For instance, the main influence of market, limit and cancel orders in financial market is on themselves, which is linked to the well-known persistence of order flows and to the splitting of meta-orders into sequences of orders; see [8]. In social media communities, it is usual that the discussion of a topic is likely to prompt further discussion as people reply to each other; see Figure 2(a,b) in [59]. Moreover, the mutual-excitation seems to be asymptotically ignorable, but it would dominate the limit process by \( b_{ij} \neq 0 \) with \( i \neq j \).

**Remark 3.3.** It is obvious that \( b_{ii} \in \mathbb{R} \) and \( b_{ij} \geq 0 \) for \( i, j \in \mathcal{H} \) with \( i \neq j \). Moreover, the pre-limit models are allowed to be supercritical or unstable. Indeed, if \( b_{11} > 0 \) we have \( \| \phi_{11}^{(n)} \|_{L^1} > 1 \) for large \( n \) and
\[ \Lambda_1^{(n)}(t) \geq \mu_1^{(n)}(t) + \int_0^t \int_\mathcal{U} \phi_1(t - s, u) N_1(ds, du) + \int_0^t \int_\mathcal{U} \phi_1(t - s, u) N_1(ds, du). \]

Similarly as in the proof of [32, Theorem 1], we have \( \mathbb{P}(\Lambda_1^{(n)}(t) \to \infty) > 0 \) and \( N_1(dt, du) \) is unstable.

**Remark 3.4.** Our asymptotic setting is different to that in Rosenbaum et al.’s works [21, 46]. In their setting, the kernel matrix in the \( n \)-th multivariate Hawkes process has the form of \( \{a_n \cdot \Phi(t) : t \geq 0\} \), where \( \{a_n\}_{n \geq 1} \) is a positive sequence increasing to one, \( \Phi(t) \) is a diagonalizable, positive matrix for each \( t \geq 0 \) and \( \| \Phi \|_{L^1} \) has spectral radius equal to one\(^3\). Because the eigenvalue of largest absolute value of \( \| \Phi \|_{L^1} \) is simple and equals to one, the pre-limit model can be understood as a multivariate Hawkes process with a common intensity. This gives rise to the weak convergence of the rescaled intensity to the multiplication of one-dimensional CIR-model by a vector.

For \( i \in \mathcal{H} \) and \( j \in \mathcal{D} \), the resolvent \( R_{ij}^{(n)} \) and \( R_i^{(n)} \) associated to the mean impact function \( \phi_{ij} \) are defined as (8)-(10), i.e., for any \( (t, u) \in \mathbb{R}_+ \times \mathcal{U} \),
\[
R_{ij}^{(n)}(t) = \phi_{ij}^{(n)}(t) + R_{ii}^{(n)} \ast \phi_{ij}^{(n)}(t),
\]
\[
R_i^{(n)}(t, u) = \phi_i(t, u) + R_{ii}^{(n)} \ast \phi_i(t, u).
\]
An argument similar to that in [21, 46] induces that the expectation \( \mathbb{E}[\Lambda_1^{(n)}(nt)] \) is of the order of \( n \) and hence it is natural to consider the weak convergence of rescaled intensity process \( \{Z_H^{(n)}(t) := (Z_i^{(n)}(t))_{i \in \mathcal{H}} : t \geq 0\} \) with \( Z_i^{(n)}(t) := \Lambda_1^{(n)}(nt)/n \). From Proposition 2.4, we see that \( Z_i^{(n)} \) satisfies the following \( d \)-dimensional stochastic Volterra system
\[
Z_i^{(n)}(t) = \frac{\mu_i^{(n)}(nt)}{n} + R_{ii}^{(n)} + \sum_{j \in \mathcal{H}} \int_0^t nR_{ij}^{(n)}(nt - s)Z_j^{(n)}(s)ds
\]
\[^3\text{The matrix } \| \Phi \|_{L^1} \text{ is also assumed to be asymmetric in [46].} \]
\begin{equation}
(15) \quad + \int_0^t R_{ii}^{(n)}(ns)ds + \sum_{j \in \mathcal{D}} \int_0^t \int_{\mathcal{U}} \tilde{N}_i^{(n)}(n(ds,du), i \in \mathcal{H},
\end{equation}

where $\tilde{N}_i^{(n)}(n(ds,du)) := N_i^{(n)}(n(ds,du)) - n(ds,du)$ and $\tilde{N}_j^{(n)}(n(ds,du)) := N_j^{(n)}(n(ds,du)) - n^2 \cdot Z_j^{(n)}(s-) \cdot ds, du$ for $j \in \mathcal{H}$.

We now give some asymptotic assumptions on the impact of events prior to time 0 on the arrivals of future events. Based on our previous argument that the mutual-excitation usually can be asymptotically ignored, it is understandable to assume that type-$i$ events prior to time 0 make the main contribution to $\mu_i^{(n)}$. Denote by $\tau_x \leq 0$ and $u_x$ the arrival time and the mark of a typical type-$i$ event $x$ prior to time 0. Because of the lack of information, we may assume it arrives uniformly before time 0. Then its mean impact function would have the form of

$$I_{\phi,i}(t) := \mathbb{E}[\phi(t + \tau_x, u_x)] = \int_0^t ds \int_{\mathcal{U}} \phi(t + s, u) \nu_i^{(n)}(du) = \int_t^\infty \phi_i^{(n)}(s) ds, \ t \geq 0.$$ 

As the number of events prior to time 0 goes to infinity, by the law of large numbers it is natural to assume the following condition holds. Recall the constant $\alpha \in (1, 2)$ in the hypothesis (H1).

**Condition 3.5.** Assume that

$$\sup_{n \geq 1} \mathbb{E}\left[\frac{\|\mu_i^{(n)} / n\|_{L^{1.\infty}}^{2\alpha}}{L^{1.\infty}}\right] < \infty \ and \ \frac{\mu_i^{(n)} / n - \mu_i^{(n)}}{L^{1.\infty}} \to 0,$$

as $n \to \infty$ with $\mu_i^{(n)} := (Z_i^{(n)}(0), I_{\phi,i}^{(n)})_{i \in \mathcal{H}}$ for some random variable $Z_i^{(n)}(0) \in \mathbb{R}_+^d$.

### 3.1.2. Asymptotic analysis in intuition.

We begin this section with some asymptotic analysis for the time-scaled resolvents

$$\{R_{ij}^{(n)}(nt): t \geq 0\}_{i \in \mathcal{H}, j \in \mathcal{D}} \quad and \quad \{R_{ii}^{(n)}(nt, u): t \geq 0, u \in \mathcal{U}\}_{i \in \mathcal{H}}.$$ 

From (13)-(14), it is necessary to study $\{R_{ij}^{(n)}(n\cdot)\}_{i \in \mathcal{H}}$ first. Integrating both sides of (13) over $\mathbb{R}_+$ with $j = i$, we have

$$\|R_{ii}^{(n)}\|_{L^1} = \|\phi_i^{(n)}\|_{L^1} + \|R_{ii}^{(n)}\|_{L^1},$$

and hence

$$\int_0^\infty R_{ii}^{(n)}(nt)dt = \frac{\|\phi_i^{(n)}\|_{L^1}}{n(1 - \|\phi_i^{(n)}\|_{L^1})},$$

which is finite for large $n$ if and only if

$$n\left(1 - \|\phi_i^{(n)}\|_{L^1}\right) \to -b_{ii} > 0.$$ 

Otherwise, $R_{ii}^{(n)}(nt)$ may increase to infinity. To overcome this difficulty, we first adjust the kernel as follows. Choosing the constant $\beta$ in the hypothesis (H2) larger than $\lambda_0 := \max_{j \in \mathcal{H}} b_{jj} / \sigma_j$, we define

$$\phi_{\beta,i}^{(n)}(t, u) := e^{-\beta t / n} \phi_i(t, u) \quad and \quad \phi_{\beta,j}^{(n)}(t) := e^{-\beta t / n} \phi_{ij}^{(n)}(t),$$
for \((t, u) \in \mathbb{R}_+ \times U, i \in \mathcal{H}\) and \(j \in \mathcal{D}\). Their resolvents \(R_{\beta,ij}^{(n)}\) and \(R_{\beta,i}^{(n)}\) are defined as in (13) and (14) respectively. The following relationships are obvious

\[
R_{\beta,ij}^{(n)}(t) = e^{-\beta t/n}R_{ij}^{(n)}(t) \quad \text{and} \quad R_{\beta,i}^{(n)}(t,u) = e^{-\beta t/n}R_{i}^{(n)}(t,u), \quad (t, u) \in \mathbb{R}_+ \times U.
\]

We first consider the modified process \(\{Z_{\beta,\mathcal{H}}^{(n)}(t) : t \geq 0\}\) with

\[
Z_{\beta,\mathcal{H}}^{(n)}(t) := e^{-\beta t}Z_{\mathcal{H}}^{(n)}(t).
\]

Let \(\mu_{\beta,\mathcal{H}}^{(n)}(t) := e^{-\beta t/n}\mu_{\mathcal{H}}^{(n)}(t)\) for \(t \geq 0\). From (15) and the foregoing notation, it is easy to identify that \(Z_{\beta,\mathcal{H}}^{(n)}\) satisfies the following stochastic system

\[
Z_{\beta,\mathcal{H}}^{(n)}(t) = \frac{\mu_{\beta,i}^{(n)}(nt)}{n} + \int_0^t R_{\beta,ii}^{(n)}(nt)\mu_{\beta,i}^{(n)}(nt) + \int_0^t R_{\beta,ij}^{(n)}(nt)\mu_{\beta,j}^{(n)}(nt)\mu_{\beta,j}^{(n)}(nt)ds
\]

\[
+ \sum_{j \in \mathcal{H}} \int_0^t \int_0^u R_{\beta,ij}^{(n)}(nt)Z_{\beta,j}^{(n)}(s)ds
\]

\[
+ \sum_{j \in \mathcal{D}} \int_0^t \int_0^u R_{\beta,ij}^{(n)}(nt)Z_{\beta,j}^{(n)}(s)ds,
\]

for \(i \in \mathcal{H}\).

We now start to consider the convergence of the sequence \(\{R_{\beta,ii}^{(n)}(\cdot)\}_{n \geq 1}\) for each \(i \in \mathcal{H}\). Notice that

\[
n(1 - \|\phi_{\beta,ii}^{(n)}\|_{L^1}) = n(1 - \|\phi_{ii}^{(n)}\|_{L^1}) + n(\|\phi_{ii}^{(n)}\|_{L^1} - \|\phi_{\beta,ii}^{(n)}\|_{L^1}).
\]

Applying the dominated convergence theorem together with the hypothesis (H2) and Condition 3.1, we have \(n(\|\phi_{ii}^{(n)}\|_{L^1} - \|\phi_{\beta,ii}^{(n)}\|_{L^1}) \to \sigma_i\beta\) and hence

\[
n(1 - \|\phi_{\beta,ii}^{(n)}\|_{L^1}) \to \sigma_i\beta - b_{ii} > 0
\]

as \(n \to \infty\), which immediately induces that for large \(n,\)

\[
\int_0^\infty R_{\beta,ii}^{(n)}(nt)dt = \frac{\|\phi_{\beta,ii}^{(n)}\|_{L^1}}{n(1 - \|\phi_{\beta,ii}^{(n)}\|_{L^1})} < \infty.
\]

Without loss of generality, in the sequel we will always assume that

\[
\|\phi_{\beta,ii}^{(n)}\|_{L^1} < 1, \quad n \geq 1, \ i \in \mathcal{H}.
\]

Denote by \(\tilde{\phi}_{\beta,ii}^{(n)}\) and \(\tilde{R}_{\beta,ii}^{(n)}\) the Fourier transforms of \(\phi_{\beta,ii}^{(n)}\) and \(R_{\beta,ii}^{(n)}\) respectively. Taking the Fourier transform of both sides of (13) and then using the convolution theorem, we have

\[
\tilde{R}_{\beta,ii}^{(n)}(\lambda) = \tilde{\phi}_{\beta,ii}^{(n)}(\lambda)(1 + \tilde{R}_{\beta,ii}^{(n)}(\lambda))\]

for any \(\lambda \in \mathbb{R}\) and hence

\[
\int_0^\infty e^{i\lambda t}R_{\beta,ii}^{(n)}(nt)dt = \frac{1}{n} \tilde{R}_{\beta,ii}^{(n)}(\lambda/n) = \frac{\tilde{\phi}_{\beta,ii}^{(n)}(\lambda/n)}{n(1 - \tilde{\phi}_{\beta,ii}^{(n)}(\lambda/n))}.
\]

By the hypothesis (H2) and the dominated convergence theorem, the numerator goes to 1 as \(n \to \infty\). Moreover, the dominator can be written as

\[
n(1 - \|\phi_{\beta,ii}^{(n)}\|_{L^1}) - n(1 - e^{i\lambda t/n} - i\lambda t/n)\phi_{\beta,ii}^{(n)}(t)dt.
\]
By the inequality $|e^{i \Delta t} - 1 - i \frac{\Delta t}{n}| \leq \frac{|\Delta t|}{n} \wedge \frac{|\Delta t|^2}{n^2}$ and the dominated convergence theorem, the last integral vanishes as $n \to \infty$. By Condition 3.1, we have $n(1 - \hat{\phi}_{\beta,ii}(\lambda/n)) \to \sigma_i \beta - b_{ii} - i\sigma_i \lambda$, and hence

$$\int_0^\infty e^{i \lambda t} R_{\beta,ii}^{(n)}(nt) dt \to \int_0^\infty e^{i \lambda t} \frac{1}{\sigma_i \beta - b_{ii} - i\sigma_i \lambda} e^{-(\beta-b_{ii}/\sigma_i)t} dt,$$

which shows that $R_{\beta,ii}^{(n)}(nt)$ can be approximated by $\sigma_i^{-1} e^{-(\beta-b_{ii}/\sigma_i)t}$.

To analyze the asymptotics of the sequence $\{ R_{\beta,ii}^{(n)}(n \cdot, u) \}_{n \geq 1}$ for any $u \in \mathbb{U}$, we take the Fourier transform of both sides of (14) and obtain with some simple calculations that

$$\int_0^\infty e^{i \lambda t} R_{\beta,ii}^{(n)}(nt, u) dt = \frac{\hat{\phi}_{\beta,ii}(\lambda/n, u)}{n(1 - \hat{\phi}_{\beta,ii}(\lambda/n))}, \quad \lambda \in \mathbb{R},$$

where $\hat{\phi}_{\beta,ii}(\lambda, u)$ is the Fourier transform of $\phi_{\beta,ii}(\cdot, u)$. Like the previous argument, we have as $n \to \infty$,

$$\int_0^\infty e^{i \lambda t} n R_{\beta,ii}^{(n)}(nt, u) dt \to \int_0^\infty e^{i \lambda t} \frac{\| \phi_{ii}(u) \|_{L^1}}{\sigma_i} e^{-(\beta-b_{ii}/\sigma_i)t} dt,$$

which induces that $R_{\beta,ii}^{(n)}(nt, u)$ can be well approximated by $\frac{b_{ii}}{\sigma_i} e^{-(\beta-b_{ii}/\sigma_i)t}$. For the rescaled resolvents $n R_{\beta,ii}^{(n)}(n \cdot)$ with $i, j \in \mathcal{H}$ and $i \neq j$, by the dominated convergence theorem we have

$$\int_0^\infty e^{i \lambda t} n R_{\beta,ii}^{(n)}(nt) dt = \int_\mathcal{U} n \hat{\phi}_{\beta,ii}^{(n)}(\lambda/n) \int_0^\infty e^{i \lambda t} n R_{\beta,ii}^{(n)}(nt, u) dt
$$

$$= \frac{n \hat{\phi}_{\beta,ii}^{(n)}(\lambda/n)}{n(1 - \hat{\phi}_{\beta,ii}^{(n)}(\lambda/n))} \to \frac{b_{ij}}{\sigma_i \beta - b_{ii} - i\sigma_i \lambda} = \int_0^\infty e^{i \lambda t} \frac{b_{ij}}{\sigma_i} e^{-(\beta-b_{ii}/\sigma_i)t} dt,$$

as $n \to \infty$ and hence it can be approximated by $\frac{b_{ij}}{\sigma_i} e^{-(\beta-b_{ii}/\sigma_i)t}$. The same argument also induces that $R_{\beta,ij}(nt)$ can be approximated by $\frac{b_{ij}}{\sigma_i} e^{-(\beta-b_{ii}/\sigma_i)t}$.

We now turn to analyze the asymptotics of the impact of events prior to time 0 on the future intensity. A simple calculation together with the hypothesis (H2) shows that as $n \to \infty$,

$$\sup_{t \geq 0} \int_t^\infty |1 - e^{-\frac{\lambda}{n}(s-t)}| \phi_{ii}^{(n)}(s) ds \to 0, \quad i \in \mathcal{H}$$

and the first two terms on the right side of (16) can be approximated by $Z_i^{(n)}(0) \bar{\mu}_{\beta,i}^{(n)}(nt)$ with

$$\bar{\mu}_{\beta,i}^{(n)}(t) := I_{\phi,\beta,ii}^{(n)}(t) + R_{\beta,ii}^{(n)} * I_{\phi,\beta,ii}^{(n)}(t) \quad \text{and} \quad I_{\phi,\beta,ii}^{(n)}(t) := \int_t^\infty \phi_{\beta,ii}^{(n)}(s) ds, \quad t \geq 0.$$

Integrating both sides of (13) on $[t, \infty)$ and then using Fubini’s lemma, we have

$$\int_t^\infty R_{\beta,ii}^{(n)}(s) ds = I_{\phi,\beta,ii}^{(n)}(t) + \int_t^\infty R_{\beta,ii}^{(n)}(s) ds \cdot \| \phi_{\beta,ii}^{(n)} \|_{L^1} + R_{\beta,ii}^{(n)} * I_{\phi,\beta,ii}^{(n)}(t),$$

which induces that

$$\bar{\mu}_{\beta,i}^{(n)}(nt) = n(1 - \| \phi_{\beta,ii}^{(n)} \|_{L^1}) \int_t^\infty R_{\beta,ii}^{(n)}(ns) ds.$$
From Condition 3.1 and (18) we can approximate \( \hat{\mu}^{(n)}(nt) \) with \( e^{-(\beta-b_i)/\sigma_i)t} \) and hence the sum of first two terms on the right side of (16) is asymptotically equivalent to \( Z^{(n)}(0)e^{-(\beta-b_i)/\sigma_i)t}. \)

Plugging all approximations above back into (16), we may have the following asymptotic equivalence for the process \( Z^{(n)}_{\beta,i} \) for \( i \in \mathcal{H} \).

\[
Z^{(n)}_{\beta,i}(t) \sim Z^{(n)}_i(0)e^{-(\beta-b_i)/\sigma_i)t} + \int_0^t e^{-(\beta-b_i)/\sigma_i)(t-s)} \cdot \frac{a_i}{\sigma_i} e^{-\beta s} ds + \sum_{j \in \mathcal{H}} \int_0^t \frac{b_{ij}}{\sigma_i} e^{-(\beta-b_i)/\sigma_i)(t-s)} Z^{(n)}_{\beta,j}(s) ds + \sum_{j \in \mathcal{D}} \int_0^t \int_\Omega e^{-(\beta-b_i)/\sigma_i)(t-s)} \cdot \frac{\|\phi(u)\|_{L^1}}{n} e^{-\beta s} \tilde{N}^{(n)}_j(n \cdot ds, du).
\]

Using the fact that \( e^{-(\beta-b_i)/\sigma_i)(t-s)} = 1 - (\beta - b_i)/\sigma_i \int_s^t e^{-(\beta-b_i)/\sigma_i)(r-s)} dr \) and Fubini’s theorem, we can rewrite it into the following convenient form:

\[
Z^{(n)}_{\beta,i}(t) \sim Z^{(n)}_i(0) + \int_0^t \left( \frac{a_i}{\sigma_i} e^{-\beta s} - \beta Z^{(n)}(s) + \sum_{j \in \mathcal{H}} \frac{b_{ij}}{\sigma_i} Z^{(n)}_{\beta,j}(s) \right) ds + \sum_{j \in \mathcal{D}} M^{(n)}_{\beta,i,j}(t),
\]

where \( M^{(n)}_{\beta,i} \) is an \((\mathcal{F}_n)_t\)-local martingale with representation

\[
M^{(n)}_{\beta,i,j}(t) := \int_0^t \int_\Omega \frac{\|\phi(u)\|_{L^1}}{n} e^{-\beta s} \tilde{N}^{(n)}_j(n \cdot ds, du), \quad t \geq 0.
\]

By Condition 3.1, for any \( i \in \mathcal{H} \) and \( j \in \mathcal{D}_i \), we will show the quadratic variation of \( M^{(n)}_{\beta,i,j} \) goes to 0 as \( n \to \infty \) and hence the sequence \( \{M^{(n)}_{\beta,i,j}\}_{n \geq 1} \) converges to 0. On the other hand, the quadratic variation of \( M^{(n)}_{\beta,i,i} \) admits the form of

\[
[M^{(n)}_{\beta,i,i}]_t = \int_0^t \frac{c^{(n)}_i}{\sigma_i^2} e^{-\beta s} Z^{(n)}_{\beta,i}(s) ds + \int_0^t \int_\Omega \frac{\|\phi(u)\|_{L^1}}{n} e^{-2\beta s} \tilde{N}^{(n)}_i(n \cdot ds, du), \quad t \geq 0.
\]

Applying Doob’s martingale inequality to the last stochastic integral, we see that it converges to 0 uniformly on compacts in probability as \( n \to \infty \). If \( Z_{\beta,H} \) is a possible cluster point of the sequence \( \{Z^{(n)}_{\beta,H}\}_{n \geq 1} \), by Condition 3.1 we will show that

\[
[M^{(n)}_{\beta,i,i}]_t \to \int_0^t \frac{c^2_i}{\sigma_i^2} e^{-\beta s} Z_{\beta,i}(s) ds, \quad t \geq 0, i \in \mathcal{H}.
\]

By Theorem III-7 in [22], we can find an \((\mathcal{F}_1)\)-Gaussian white noise \( W_i(ds,dz) \) on \((0,\infty)^2\) with intensity \( dsdz \) such that

\[
M^{(n)}_{\beta,i,i}(t) \overset{d}{\to} \int_0^t \int_0^t e^{\beta s} Z_{\beta,i}(s) \frac{c_i}{\sigma_i} e^{-\beta s} W_i(ds,dz),
\]

in \( D([0,\infty), \mathbb{R}) \). Additionally, the conditional orthogonality of \( \{\tilde{N}^{(n)}_i(ds,dz) : i \in \mathcal{H}\} \) induces the mutual independence among the Gaussian white noises \( \{W_i(ds,dz) : i \in \mathcal{H}\} \).
3.1.3. Weak convergence of rescaled intensity processes. With all preparations above, we are ready to consider the weak convergence of the sequence \( \{Z_{\beta,\mathcal{H}}^{(n)}\}_{n \geq 1} \). Letting \( n \to \infty \), we may expect the limit process \( Z_{\beta,\mathcal{H}} \) to be the unique solution to the following stochastic system: for \( i \in \mathcal{H} \),

\[
Z_{\beta,i}(t) = Z_i(0) + \int_0^t \left( \frac{a_i}{\sigma_i} e^{-\beta s} - \beta Z_{\beta,i}(s) + \sum_{j \in \mathcal{H}} \frac{b_{ij}}{\sigma_i} Z_{\beta,j}(s) \right) ds \\
+ \int_0^t \int_0^s c_i e^{-\beta s} W_i(ds,dz).
\]

By the fact that \( Z_{\mathcal{H}}^{(n)}(t) = e^{\beta t} Z_{\beta,\mathcal{H}}^{(n)}(t) \) for any \( t \geq 0 \) and using Itô’s formula to \( e^{\beta t} Z_{\beta,\mathcal{H}}(t) \), we can get the following main theorem immediately.

**Theorem 3.6.** Under Condition 3.1 and 3.5, if \( Z_{\mathcal{H}}^{(n)}(0) \xrightarrow{d} Z_{\mathcal{H}}(0) \in \mathbb{R}^d_+ \), we have

\( Z_{\mathcal{H}}^{(n)} \xrightarrow{d} Z_{\mathcal{H}} \),

in \( \mathcal{D}([0, \infty), \mathbb{R}^d_+) \) as \( n \to \infty \) with the limit process \( Z_{\mathcal{H}} \) being the unique strong solution to

\[
(21) \quad Z_i(t) = Z_i(0) + \int_0^t \left( \frac{a_i}{\sigma_i} + \sum_{j \in \mathcal{H}} \frac{b_{ij}}{\sigma_i} Z_j(s) \right) ds + \int_0^t \int_0^s \frac{c_i}{\sigma_i} W_i(ds,dz), \quad i \in \mathcal{H}.
\]

**Remark 3.7.** By the martingale representation theorem in [41, p.84], Theorem 3.6 remains valid with the stochastic integral in the limit model (21) replaced by

\[
\int_0^t \frac{c_i}{\sigma_i} \overline{Z_i(s)} dB_i(s), \quad i \in \mathcal{H},
\]

where \( B_\mathcal{H} \) is a standard \( d \)-dimensional Brownian motion.

**Remark 3.8.** By comparing (21) with (1) or (11), we see that the limit model \( Z_{\mathcal{H}} \) is a natural high-frequency version analogous to the intensity process of multivariate MHPI-measure. More precisely, we can translate the term \( \int_0^t \frac{b_{ij}}{\sigma_i} Z_i(s) ds \) into the net impact of type-\( i \) events on themselves, i.e., as time goes, the impact of past events decreases while new impact is added. On the other hand, notice that \( b_{ij} \geq 0 \) for \( j \neq i \), the term \( \int_0^t \frac{b_{ij}}{\sigma_i} Z_i(s) ds \) can be interpreted as the mutual-excitation of type-\( j \) events on the future arrivals of type-\( i \) events. Clearly, in the short-memory setting, the intensity process \( \Lambda_\mathcal{H} \) can be successfully recovered from the limit process \( Z_{\mathcal{H}} \). The evolution dynamic of \( Z_{\mathcal{H}} \) is much simpler than that of \( \Lambda_\mathcal{H} \). Moreover, compared with the non-parametric estimation for the exogenous density and kernel in (1), parameters in (21) are much easier to be estimated from the data; see [72].

**Remark 3.9.** From Condition 3.5, the direct impact of events prior to time 0 in the scaling limit \( Z_{\mathcal{H}} \) vanishes immediately after time 0, i.e., \( \mu_i^{(n)}(nt)/n \sim Z_i^{(n)}(0) f_{\phi,ii}^{(n)}(nt) \sim Z_i(0) \cdot 1_{\{t=0\}} \). If the exogenous density decays slowly in the pre-limit model, then events prior to time 0 may continue dominating the scaling limit \( Z_{\mathcal{H}} \) after time 0. For instance, for \( i \in \mathcal{H} \) and a non-negative, integrable function \( g_i \) on \( \mathbb{R}_+ \), let \( g_i^{(n)}(t) := g_i(t/n) \) for \( t \geq 0 \) and \( n \geq 1 \). If Condition 3.5 holds with \( \mu_i^{(n)} = Z_i^{(n)}(0) f_{\phi,ii}^{(n)} + g_i^{(n)} \), then Theorem 3.6 holds with the first term on the right side of (21) replaced by \( Z_i(0) \int_0^t g_i(s) ds \).
### Remark 3.10
When the identity matrix $I$ in Condition 3.1 is replaced by a diagonal matrix $\text{diag}(\lambda_i)$ with $\lambda_i \in [0, 1]$, our previous asymptotic analysis remains valid with $R_{ij}^{(n)} \to 0$ for $i \in \mathcal{H}_{<1} := \{ l \in \mathcal{H} : \lambda_l < 1 \}$ and $j \in \mathcal{D}$. Moreover, if $Z_i(0) = 0$ for $i \in \mathcal{H}_{<1}$ then the weak convergence in Theorem 3.6 still holds with $Z_i \equiv 0$ for $i \in \mathcal{H}_{<1}$.

### Remark 3.11
Jaisson and Rosenbaum [46] established a scaling limit for nearly unstable uni-variate Hawkes processes with a common and constant exogenous density $\mu_0$. They approximated the rescaled intensity $Z^{(n)}$ with the solution of an Itô’s SDE with driving noise of the form $\int_0^1 \beta \cdot Z^{(n)}(s-)^{-1/2} {\tilde N}^{(n)}(ds)$, and then obtained the scaling limit by using Theorem 5.4 in [52]. As the key condition in [52, Theorem 5.4], the weak convergence and uniform tightness of driving noises were identified easily by the fact that their jumps are uniformly bounded; see the second paragraph in [46, p.623]. By contrast, the exogenous intensity in our setting may vanish as time goes and the intensity process may hit 0 in finite time. In this case, driving noises may have unbounded jumps and infinite moments, which makes the proof of their weak convergence and uniform tightness difficult. To get around these problems, we first approximate the rescaled intensity processes with a sequence of Itô’s SDEs driven by Poisson random measures, which are then rewritten under the form of Itô’s SDEs driven by infinite-dimensional semimartingales that have been studied in Kurtz and Protter [53]. Notably, the infinite-dimensional semimartingales are mainly determined by a sequence of compensated Poisson random measures whose weak convergence and uniform tightness can be identified by their orthogonal increments; see Section 5.1.4. Finally, the weak convergence of rescaled intensity processes follows directly from [53, Theorem 7.5]; see Section 5.1.

From the argument in [41, p.163-166], the unique strong solution $\{ Z_{\mathcal{H}}(t) : t \geq 0 \}$ is a $d$-dimensional non-negative strong Markov process with infinitesimal generator $\mathcal{L}$ given by

$$
\mathcal{L} f(x) := \sum_{i \in \mathcal{H}} \left( \frac{a_i}{\sigma_i} f(x_i) + \sum_{j \in \mathcal{H}} \frac{b_{ij}}{\sigma_i} x_j \frac{\partial f(x)}{\partial x_i} \right) + \sum_{i \in \mathcal{H}} \frac{c_i^2}{2\sigma_i^2} x_i \frac{\partial^2 f(x)}{\partial x_i^2},
$$

for any $f \in C^2(\mathbb{R}^d_+)$. Here $C^2(\mathbb{R}^d_+)$ is the space of all twice differentiable functions on $\mathbb{R}^d_+$ with the first two derivatives being continuous. Define a mapping $\varphi_{\mathcal{H}} := (\varphi_i)_{i \in \mathcal{H}}$ from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ by

$$
\varphi_i(z_{\mathcal{H}}) := - \sum_{j \in \mathcal{H}} \frac{b_{ij}}{\sigma_i} z_j + \frac{c_i^2}{2\sigma_i^2} z_i^2, \quad z_{\mathcal{H}} \in \mathbb{R}^d_+.
$$

From Theorem 2.7 in [20], $Z_{\mathcal{H}}$ is a regular affine process with Feller transition semigroup $(Q_t)_{t \geq 0}$ on $\mathbb{R}^d_+$ defined by

$$
\begin{align*}
\int_{\mathbb{R}^d_+} e^{-\langle z_{\mathcal{H}}, y_{\mathcal{H}} \rangle} Q_t(x_{\mathcal{H}}, dy_{\mathcal{H}}) &= \exp \left\{-\langle x_{\mathcal{H}}, v_{\mathcal{H}}(t, z_{\mathcal{H}}) \rangle - \int_0^t \langle (a_i/\sigma_i)_{i \in \mathcal{H}}, v_{\mathcal{H}}(s, z_{\mathcal{H}}) \rangle ds \right\},
\end{align*}
$$

where $x_{\mathcal{H}}, z_{\mathcal{H}} \in \mathbb{R}^d_+$ and $v_{\mathcal{H}} := (v_i)_{i \in \mathcal{H}}$ is the unique solution to the Riccati equation

$$
\frac{\partial}{\partial t} v_{\mathcal{H}}(t, z_{\mathcal{H}}) = -\varphi_{\mathcal{H}}(v_{\mathcal{H}}(t, z_{\mathcal{H}})) \quad \text{with} \quad v_{\mathcal{H}}(0, z_{\mathcal{H}}) = z_{\mathcal{H}}.
$$

---

4 Jumps of driving noise are proportional to $1/\sqrt{n \cdot Z^{(n)}}$ and uniformly bounded by $1/\sqrt{\mu_0}$, since $Z^{(n)} \geq \mu_0/n$ uniformly.

5 If $\mu_{\mathcal{H}} = 0$, we have $P(N_{\mathcal{H}}([0, 1]) + N_I([0, 1]) = 0) > 0$ and hence $P(A_i(t) = 0, t \in [0, 1]) > 0$. 

---
Moreover, the conservative Markov process $Z_\mathcal{H}$ is also known as a multi-type continuous-state branching process with immigration, which has branching mechanism $\phi_\mathcal{H}$ and immigration rate $a_\mathcal{H}$; see [68, 72].

### 3.1.4. Examples

In this section, we provide scaling limits for the self-excited dynamical systems driven by multivariate MHPI-measures considered in Section 2.3. For $n \geq 1$, define

$$c^{(n)}_\mathcal{H} := c^2_\mathcal{H} + \frac{1}{n} \quad \text{and} \quad b^{(n)}_{\mathcal{H}^2} := \frac{b_{\mathcal{H}^2}}{n} + I,$$

which is a positive matrix for large $n$. For simplicity, we assume $b^{(n)}_{\mathcal{H}^2}$ is positive for any $n \geq 1$.

**Example (Exponential type).** For each $n \geq 1$, let $N^{(n)}_\mathcal{H}(dt, du\mathcal{H})$ be a multivariate MHPI-measure of exponential type on $(0, \infty) \times \mathbb{R}_+^d$ with parameter $(\hat{u}^{(n)}_\mathcal{H}, \beta_\mathcal{H}, \nu^{(n)}_\mathcal{H}, \nu_I)$ defined by: for $i \in \mathcal{H},$

$$\hat{u}^{(n)}_i = Z_i(0) \cdot n, \quad \sup_{n \geq 1} \int_{\mathbb{R}_+^d} |u_\mathcal{H}|^{2\alpha} \nu^{(n)}_i (du\mathcal{H}) + \int_{\mathbb{R}_+^d} \hat{u}_\mathcal{H}^{(n)} \nu_I (du\mathcal{H}) < \infty,$$

$$\int_{\mathbb{R}_+^d} u_\mathcal{H} \nu_I (du\mathcal{H}) = a_\mathcal{H}, \quad \int_{\mathbb{R}_+^d} u_\mathcal{H} \nu^{(n)}_i (du\mathcal{H}) = b^{(n)}_i, \quad \int_{\mathbb{R}_+^d} u^2 \nu^{(n)}_i (du\mathcal{H}) = c^{(n)}_i.$$

In this case, we have $\|\phi_i(u_\mathcal{H})\|_{L^1} = u_i, \|\phi_i(u_\mathcal{H})\|_{TV} = u_i \beta_i, \int_0^\infty t \cdot \phi_i(t, u_\mathcal{H}) dt = u_i / \beta_i$ and $\phi^{(n)}_{ij}(t) = b^{(n)}_{ij} \beta_i e^{-\beta_i t}$ for $i, j \in \mathcal{H}$. It is easy to identify that the two hypotheses (H1)-(H2) and Condition 3.1-3.5 hold. Hence Theorem 3.6 holds with $\sigma_\mathcal{H} = (b_{ij} / \beta_i)_{i \in \mathcal{H}}$.

**Remark 3.12.** The state 0 is a polar for the intensity processes of multivariate MHPI-measures of exponential type, i.e., $P(L^{(n)}_i(t) > 0,\forall t \geq 0) = 1$ for each $n \geq 1$ and $i \in \mathcal{H}$, but may be not for the limit process $Z_\mathcal{H}$, i.e. $P(Z_i(t) = 0, \exists t \geq 0) > 0$ for some $i \in \mathcal{H}$. For instance, when $b_{ij} = 0$ for $j \neq i$, then $Z_i$ is a classic CIR-model. In this case, we have $P(Z_i(t) = 0, \exists t \geq 0) > 0$ if and only if the Feller Condition holds $(2a_i \sigma_i < c_i^2)$; see [24]. For the general multi-type CBI-processes, several sufficient conditions are given in [25] for them to not hit zero in finite time, but their polarity still remains unclear up to now.

Let $\chi(x) := x + 1/x$ for $x > 0$ and $\hat{\nu}(du)$ be a probability measure on $\mathcal{M}_0(\mathbb{R}_+)$ satisfying

$$\hat{u}(dx) := \int_{\mathcal{M}_0(\mathbb{R}_+)} \frac{u(dx)}{x} \hat{\nu}(du) \in \mathcal{M}(\mathbb{R}_+),$$

$$\int_{\mathcal{M}_0(\mathbb{R}_+)} |u_\mathcal{H}|^{2\alpha} \hat{\nu}(du) < \infty \quad \text{and} \quad \int_{\mathcal{M}_0(\mathbb{R}_+)} u_\mathcal{H}(\mathcal{H}) \hat{\nu}(du) = 1.$$

Let $\hat{\nu}_L$ be a positive function on $\mathbb{R}_+$ defined by

$$\hat{\nu}_L(t) := \int_{\mathcal{M}_0(\mathbb{R}_+)} L_u(t) \hat{\nu}(du) = \int_0^\infty e^{-tx} \int_{\mathcal{M}_0(\mathbb{R}_+)} xu(dx) \hat{\nu}(du), \quad t \geq 0,$$

which is completely monotone with $\|\hat{\nu}_L\|_{L^1} = 1$ and $\int_0^\infty t \hat{\nu}_L(t) dt = \hat{u}(\mathbb{R}_+) < \infty$.

**Example (Completely monotone type).** For each $n \geq 1$, let $N^{(n)}_\mathcal{H}(dt, du\mathcal{H})$ be a multivariate MHPI-measure of completely monotone type on $(0, \infty) \times \mathcal{M}_0(\mathbb{R}_+)^d$ with parameter $(\hat{u}^{(n)}_\mathcal{H}, \nu^{(n)}_\mathcal{H}, \nu_I)$, where $\hat{u}^{(n)}_\mathcal{H}(dx) = Z_\mathcal{H}(0) \cdot n \cdot \hat{u}(dx)$ and

$$\nu_I (du\mathcal{H}) = \int_{\mathcal{M}_0(\mathbb{R}_+)} \delta_{\alpha_\mathcal{H} \cdot u}(du\mathcal{H}) \hat{\nu}(du),$$

for them
\[ \nu_j^{(n)}(du_H) = \int_{\mathcal{M}_0(\mathbb{R}^n_+)} \delta_{\nu_j^{(n)}}(du_H) \hat{\nu}(du), \quad j \in \mathcal{H}. \]

For each \( i, j \in \mathcal{H} \), we have \( \| \phi_i(u_H) \|_{TV} + \int_0^\infty t \phi_i(t, u_H) dt \leq Cu_i(\chi), \| \phi_i(u_H) \|_{L^1} = u_i(\mathbb{R}^n_+), \phi_{ij}^{(n)}(t) = b_{ij}^{(n)} \hat{\nu}_L(t), \| \phi_{ij}^{(n)} \|_{L^1} = b_{ij}^{(n)}, \| \phi_{ij}^{(n)} \|_{L^1} = a_i \) and
\[
\mu_{ij}^{(n)}(t) = Z_H(0) \cdot n \cdot \int_{\mathcal{M}_0(\mathbb{R}^n_+)} \int_0^{\infty} e^{-tx} u(dx) \hat{\nu}(du) = Z_H(0) \cdot n \cdot \int_t^\infty \hat{\nu}_L(s) ds. \]

These imply that the two hypotheses (H1)-(H2) and Condition 3.1-3.5 hold. Hence Theorem 3.6 holds with
\[
c_i^2 = \int_{\mathcal{M}_0(\mathbb{R}^n_+)} |u(\mathbb{R}^n_+)|^2 \hat{\nu}(du), \quad i \in \mathcal{H}. \]

**Example (Convolution type).** For each \( i \in \mathcal{H} \), let \( \rho_i \) be the probability density function of exponential distribution with rate \( \beta_i > 0 \). Let
\[
\bar{\phi}_i(t) = \int_{\mathcal{M}_0(\mathbb{R}^n_+)} \rho_i * u(t) \hat{\nu}(du) \quad \text{and} \quad \bar{\phi}(t) = \sum_{i \in \mathcal{H}} \bar{\phi}_i(t), \quad t \geq 0.
\]

For each \( n \geq 1 \), let \( N_{ij}^{(n)}(dt, du_H) \) be a multivariate MHPI-measure of convolution type on \((0, \infty) \times \mathcal{M}_0(\mathbb{R}^n_+)^d\) with parameter \((\bar{u}_{ij}^{(n)}, \rho_H, \nu_H^{(n)}, \nu_I)\), where \( \nu_I \) and \( \nu_H^{(n)} \) are defined as in Example 3.1.4,
\[
\bar{u}_i^{(n)}(ds) = Z_i(0) \cdot n \cdot \int_{\mathcal{M}_0(\mathbb{R}^n_+)} (\beta_i^{-1}u(\mathbb{R}^n_+))\delta_0(ds) + u(s, \infty)ds \hat{\nu}(du), \quad i \in \mathcal{H}.
\]

For \( i, j \in \mathcal{H} \), we have \( \| \phi_i(u_H) \|_{TV} + \int_0^\infty t \phi_i(t, u_H) dt \leq Cu_i(\chi), \| \phi_i(u_H) \|_{L^1} = u_i(\mathbb{R}^n_+), \phi_{ij}^{(n)}(t) = b_{ij}^{(n)} \bar{\phi}_i(t), \| \phi_{ij}^{(n)} \|_{L^1} = b_{ij}^{(n)}, \int_0^{\infty} t \cdot \bar{\phi}_i(t) dt = \hat{\nu}_L(0) + \beta_i^{-1} \) and \( \mu_i(t) = Z_i(0) \cdot n \cdot \int_t^\infty \bar{\phi}_i(s) ds \). These imply that the two hypotheses (H1)-(H2) and Condition 3.1-3.5 hold. Thus Theorem 3.6 holds with
\[
\sigma_i = \hat{\nu}_L(0) + \beta_i^{-1} \quad \text{and} \quad c_i^2 = \int_{\mathcal{M}_0(\mathbb{R}^n_+)} |u(\mathbb{R}^n_+)|^2 \hat{\nu}(du), \quad i \in \mathcal{H}.
\]

### 3.2. Scaling limits for marked Hawkes shot noise processes

In this section we provide several limit theorems for shot noise processes driven by multivariate MHPI-measures, which are widely used to model the impact of events of various types on the underlying dynamical system, e.g., price models [37], risk reserve models [51], workload input models [57] and so on. They also play an important role in establishing diffusion approximations for the general branching particle systems in the next section. In the \( n \)-th model, we denote by \( S_D^{(n)}(t) := (S_i^{(n)}(t))_{i \in \mathcal{D}} \) the total impact of all events of various types at time \( t \) with
\[
S_i^{(n)}(t) := \int_0^t \int_U \zeta_i(t - s, u) X_s^{(n)}(ds, du),
\]
where \( \zeta_i : \mathbb{R}_+ \times \mathbb{U} \to \mathbb{R} \), usually known as shape function or response function, is cádlág in time and can be interpreted as the impact of each type-\( i \) event on the dynamical system. Specially, if \( \zeta_i(t, u) := 1_{\{t \geq 0\}} \) for \( u \in \mathbb{U} \), the shot noise process \( S_D^{(n)} \) reduces to the embedded point process \( N_D^{(n)} \).

As a typical application, the shot noise process (22) is usually considered as a natural model for the delay in claim settlement. Indeed, the process \( S_i^{(n)} \) can be interpreted as the
amount process of type-$i$ claims, in which the response function, being the form of $\{u_{i,k}(t - \tau_{i,k}) : t \geq 0\}$, represents the pay-off process of the $k$-th type-$i$ claim with arrival time $\tau_{i,k}$. In particular, if $u_{i,k}$ is a random non-null, finite measure on $\mathbb{R}_+$ with $u_{i,k}(t) = u_{i,k}([0,t])$ for $t \geq 0$, then $S_i^{(n)}$ turns to be the total amount process of type-$i$ claims. Moreover, when $u_{i,k}$ is differentiable, it is usual to translate the derivative process of $S_i^{(n)}$ into the total rate at which the insurance company pays to the type-$i$ claims. In conclusion, here we are mainly interested in the following two kinds of response functions:

- **Cumulative response function**: $\zeta_i$ is non-negative and non-decreasing in time $t$;
- **Instantaneous response function**: $\zeta_i$ is non-negative and integrable in time $t$.

By Condition 3.1 and Theorem 3.6, the arrival rates of external, mutually-triggered and self-triggered events in the $n$-th model are of the order of $1, 1$ and $n$ respectively. Thus compared to that of self-triggered events, the impact of external events and mutually triggered events on the dynamical system can be asymptotically ignored. Hence we mainly consider the marked Hawkes shot noise process $S_i^{(n)}$. Denote by $\{\zeta^{(n)}_{ii}(t) : t \geq 0\}$ the mean response function of a type-$i$ event in the $n$-th system with

$$\zeta^{(n)}_{ii}(t) := \int_{U} \zeta_i(t, u) \nu^{(n)}_i(du), \quad i \in \mathcal{H}.$$  

In this section we always assume that the two hypotheses (H1)-(H2) and Condition 3.1-3.5 hold.

3.2.1. **Cumulative response function.** In this section, we establish a limit theorem for the cumulative impact of events of various types on the dynamical system. Recall the constant $\alpha \in (1, 2)$ in the hypothesis (H1). For each $i \in \mathcal{H}$, we assume that the total impact of a type-$i$ event with mark $u \in U$ is finite, i.e., $\zeta_i(\infty, u) := \lim_{t \to \infty} \zeta_i(t, u) < \infty$, and satisfies the following condition.

**CONDITION 3.13.** For each $i \in \mathcal{H}$, assume that

$$\sup_{n \geq 1} \int_{U} |\zeta_i(\infty, u)|^{\alpha} \nu^{(n)}_i(du) < \infty \quad \text{and} \quad \lim_{n \to \infty} \zeta^{(n)}_{ii}(\infty) = b_{c,i} \geq 0.$$

Taking expectations on both sides of (22) and then letting $n \to \infty$, we may have

$$\mathbb{E}[S_i^{(n)}(nt)] = n^2 \int_{0}^{t} \zeta^{(n)}_{ii}(n(t - s)) \mathbb{E}[Z_i^{(n)}(s)] ds \sim n^2 \cdot \zeta^{(n)}_{ii}(\infty) \cdot \int_{0}^{t} \mathbb{E}[Z_i^{(n)}(s)] ds, \quad i \in \mathcal{H},$$

which is of the order of $n^2$. Thus a natural scaling in time and space leads us to consider the rescaled process $\{S_{c,H}^{(n)}(t) : t \geq 0\}$ with

$$S_{c,H}^{(n)}(t) := \frac{1}{n^2} \cdot S_i^{(n)}(nt).$$

By the locally stochastic boundedness of the sequence $\{Z^{(n)}\}_{n \geq 1}$, we see that the foregoing asymptotic equivalence holds if the mean residual impact $\{\zeta^{(n)c}_{ii}(t) := \zeta^{(n)}_{ii}(\infty) - \zeta^{(n)}_{ii}(t) : t \geq 0, i \in \mathcal{H}\}$ satisfies the next condition.

**CONDITION 3.14.** For each $i \in \mathcal{H}$, assume that $\sup_{n \geq 1} \zeta^{(n)c}_{ii}(t) \to 0$ as $t \to \infty$. 

THEOREM 3.15. Under Condition 3.13 and 3.14, we have

\[ S_{C,H}^{(n)} \overset{d}{\to} S_{C,H} \]

in \( D([0,\infty),\mathbb{R}^d_+) \) as \( n \to \infty \) with the limit process \( S_{C,H} \) given by

\[ S_{C,i}(t) = b_{C,i} \int_0^t Z_i(s) \, ds, \quad t \geq 0, i \in \mathcal{H}. \]

**Remark 3.16.** When \( \zeta_i(t,u) = 1_{\{t \geq 0\}} \), the shot noise process \( S_{C,H}^{(n)} \) reduces to the rescaled embedded point process \( S_{C,H}^{(n)} \), and also the mean response functions satisfying the next condition.

\[ \{ N_{C,H}^{(n)}(nt) / n^2 : t \geq 0 \} \]

with \( \zeta_i \equiv 1 \) for \( i \in \mathcal{H} \) and \( N_{C,H}^{(n)}(nt) / n^2 \overset{d}{\to} \int_0^t Z_H(s) \, ds \) in \( D([0,\infty),\mathbb{R}^d_+) \) as \( n \to \infty \).

**Example.** For each \( i \in \mathcal{H} \), let \( \mathcal{P}_{C,i} \) be a probability law on \( \mathcal{M}(\mathbb{R}_+) \) satisfying that

\[ \int_{\mathcal{M}(\mathbb{R}_+)} |u(\mathbb{R}_+)|^n \mathcal{P}_{C,i}(du) < \infty. \]

Suppose that the pay-off processes of claims of various types in the \( n \)-th insurance model are mutually independent and distributed as \( \mathcal{P}_{C,H} \). It is easy to identify that Condition 3.13 and 3.14 are satisfied. Then the rescaled total claim amount process converges weakly to \( S_{C,H} \) with

\[ b_{C,i} = \int_{\mathcal{M}(\mathbb{R}_+)} u(\mathbb{R}_+) \mathcal{P}_{C,i}(du), \quad i \in \mathcal{H}. \]

3.2.2. **Instantaneous response function.** In this section, we consider the convergence of \( \{ S_{H}^{(n)} \}_{n \geq 1} \) with instantaneous response function that has low volatility and enjoys short-memory property, i.e.,

**Condition 3.17.** For each \( i \in \mathcal{H} \), assume that

\[ \sup_{n \geq 1} \int_{\mathbb{R}_+} \left( \| \zeta_i(u) \|^2_{TV} + \| \zeta_i(u) \|^2_{L^1} \right) \nu_i^{(n)}(du) < \infty. \]

Taking expectations on both sides of (22) and then letting \( n \to \infty \), we may have

\[ \mathbb{E}[S_i^{(n)}(nt)] = \mathbb{E}[Z_i^{(n)}(t - s/n)] ds \approx n \cdot \mathbb{E}[Z_i^{(n)}(t)] \cdot \| \zeta_i^{(n)} \|_{L^1}, \quad t > 0, i \in \mathcal{H}. \]

It is reasonable to consider the rescaled shot noise process \( \{ S_{i,\mathcal{H}}^{(n)}(t) : t \geq 0 \} \)

\[ S_{i,\mathcal{H}}^{(n)}(t) = \frac{1}{n} \cdot S_{i,\mathcal{H}}^{(n)}(nt) \]

and also the mean response functions satisfying the next condition.

**Condition 3.18.** For each \( i \in \mathcal{H} \), there exist a non-negative, integrable function \( \tilde{\zeta} \) on \( \mathbb{R}_+ \) and a constant \( b_{1,i} \geq 0 \) such that for any \( t \geq 0 \),

\[ \sup_{n \geq 1} \| \zeta_i^{(n)}(t) \| \leq \tilde{\zeta}(t) \quad \text{and} \quad \lim_{n \to \infty} \| \zeta_i^{(n)} \|_{L^1} \to b_{1,i}. \]
From the intuitive analysis above, we may conjecture that $S_{1,H}^{(n)}$ can be approximated by $\hat{S}_{1,H}^{(n)}$ in which

$$\hat{S}_{1,i}^{(n)}(t) := \int_0^{nt} \zeta_{ii}^{(n)}(s) Z_i^{(n)}(t-s/n) ds, \quad t \geq 0, i \in H.$$ 

Since $Z_{H}^{(n)} \xrightarrow{d} Z_H$ in $D([0, \infty), \mathbb{R}_+^d)$; see Theorem 3.6, it is natural to expect that

$$S_{1,i}^{(n)}(t) \xrightarrow{d} S_{1,i}(t) := b_{1,i} \cdot Z_i(t), \quad t \geq 0, i \in H.$$ 

Unfortunately, this convergence may fail around time 0, because $S_{1,H}^{(n)}(0) \xrightarrow{a.s.} 0$ but $Z_H(0)$ may be positive.

**Theorem 3.19.** Under Condition 3.17 and 3.18, we have for any $\delta > 0$,

$$S_{1,H}^{(n)} \xrightarrow{d} S_{1,H}$$

in $D([\delta, \infty), \mathbb{R}_+^d)$ as $n \to \infty$. Moreover, if $Z_H(0) \xrightarrow{a.s.} 0$, this convergence also holds for $\delta = 0$.

This convergence result fails around time 0 mainly because the shot noise process (22) excludes the impact of events prior to time 0 on the dynamical system. In the $n$-th model, denote by $\psi_{1,H}^{(n)}(t)$ the total instantaneous impact of events of various types prior to time 0 at time $t \geq 0$. An argument similar to that before Condition 3.5 deduces that in the $n$-th model, the mean instantaneous response function of each event prior to time 0 admits the form of

$$I_{\zeta,ii}^{(n)}(t) := \int_t^\infty \zeta_{ii}^{(n)}(s) ds, \quad t \geq 0, i \in H.$$ 

Applying the law of large numbers again, it is natural to assume the next condition holds for $\{\psi_{1,H}^{(n)}\}_{n \geq 0}$.

**Condition 3.20.** Assume that $|\psi_{1,H}^{(n)}(n) - \hat{\psi}_{1,H}^{(n)}| \xrightarrow{d} 0$ in $D([0, \infty), \mathbb{R}_+^d)$ as $n \to \infty$ with

$$\hat{\psi}_{1,H}^{(n)} := (Z_{1}^{(n)}(0) \cdot I_{\zeta,ii}^{(n)})_{i \in H}.$$ 

**Theorem 3.21.** Under Condition 3.17, 3.18 and 3.20, we have $\psi_{1,H}^{(n)}(n) / n + S_{1,H}^{(n)} \xrightarrow{d} S_{1,H}^{(n)}$ in $D([0, \infty), \mathbb{R}_+^d)$ as $n \to \infty$.

Let $L_{TV}^1(\mathbb{R}_+)$ be the space of non-negative, integrable and càdlàg functions on $\mathbb{R}_+$ with bounded variation. It is endowed with the norm $\| \cdot \|_{TV} + \| \cdot \|_{L^1}$. For each $i \in H$, let $P_{1,i}$ be a probability measure on $L_{TV}^1(\mathbb{R}_+)$ satisfying that

$$\int_{L_{TV}^1(\mathbb{R}_+)} (\| u \|_{TV}^{2\alpha} + \| u \|_{L^1}^{\alpha}) P_{1,i}(du) < \infty.$$ 

**Example.** In the $n$-th insurance model, suppose that there are $[Z_H(0) \cdot n]$ claims at time 0 and the pay-off rate of claims of various types is distributed as $P_{1,H}$. Here we are interested in the total rate at which the insurance company pays to claims of various types. It is described as $S_{H}^{(n)}$ with $\zeta_{i}(t, u) = u(t)$ for $u \in L_{TV}^1(\mathbb{R}_+)$ and $t \geq 0$. In this case, we see that Condition 3.17 and 3.18 hold with $\hat{\zeta}_{ii}^{(n)}(t) = \int_{L_{TV}^1(\mathbb{R}_+)} u(t) P_{1,i}(du)$. Additionally, the pay-off rate of a typical claim $x$ prior to time 0 with arrival time $\tau_x < 0$ is $u_x(t - \tau_x)$ at time $t \geq 0$. 
By the law of large numbers and an argument similar to that before Condition 3.5, the total
pay-off rate process of type-\(i\) claims prior to time 0 can be approximated by
\[
Z_i(0) \int_0^\infty \int_{L^1_v(\mathbb{R}^+)} u(s) \mathcal{P}_{L,i}(du) = Z_i(0)I^{(n)}_{\xi,i}(t), \quad t \geq 0.
\]
Hence Condition 3.20 holds and the rescaled total pay-off rate process converges weakly to
\(S_{1,H}\) with
\[
b_{L,i} = \int_{L^1_v(\mathbb{R}^+)} \|u\|_{L^1} \mathcal{P}_{L,i}(du), \quad i \in H.
\]

4. Limits theorems for multi-type CMJI-processes. In this section, we apply our limit
theorems for self-excited dynamical systems to establish diffusion approximations for multi-
type Crump-Mode-Jagers branching processes (CMJI-processes). In order to clarify the con-
nection between multi-type CMJI-processes and multivariate MHPI-measures, we try to use
the same notation to represent quantities of population that play the similar roles in MHPI-
measures.

4.1. Multi-type CMJI-processes. A \(d\)-type CMJI-process is a general branching process
with \(d\) kinds of distinguishable individuals. These are usually to be called type-1, 2, \(\cdots\), \(d\).
In order to illustrate its considerable importance in biology, we give its definition with budding
microbes as a typical example. It is usual to assume that the observation on the population
starts from the appearance of symptoms on the host. The microbes alive at time 0 are consid-
ered as ancestors.

(P1) (Ancestors) The population starts with \(\Xi_H(0) := (\Xi_i(0))_{i \in H} \in \mathbb{N}^d\) ancestors at time 0.

Compared to binary fission microbes, budding microbes usually live much longer before
dying, being killed or spreading out of the host. Moreover, their life-lengths are rarely expo-
nentially distributed; see [34, Table 4] and [71, Figure 2-4].

(P2) (Life-length) Individuals of type-\(i\) have a common life-length distribution
\(\mathcal{P}_{L,i}(dy)\) on \(\mathbb{R}^+_+\) with finite first and second moments
\[
m_{L,i} := \int_0^\infty y \mathcal{P}_{L,i}(dy). \quad \text{and} \quad v_{L,i} := \int_0^\infty y^2 \mathcal{P}_{L,i}(dy).
\]

Moreover, different to binary fission in which the fully grown parent cell either dies or
splits into equally sized daughter cells, the mother budding microbes usually produce buds
several times during their lifetime. As so often, the budding rate is low during the growth
stage and then increases to the highest level after separating from the mother cell. As the bud
scars accumulate on the surface, the microbe enters into the senescence state and the budding
rate starts to decrease; see [48, Figure 2]. We collect the possible budding rate functions in
\[
B := \{B : \mathbb{R}^+_+ \mapsto \mathbb{R}^+: B(t,y) = 0 \text{ if } t \geq y \text{ and } \|B(y)\|_{TV} + \int_0^\infty t \cdot B(t,y)dt < \infty \}
\]
and describe the reproduction process of each budding microbe by a Cox process with inten-
sity process selected randomly in \(B\); see the following properties.

(P3) (Budding rate) Each type-\(i\) individual is endowed with a budding rate function ran-
domly according to the probability law \(\mathcal{P}_{B,i}(dB)\) on \(B\). The mean budding rate is finite and
light-tailed, i.e.,
\[
B_i(t) := \int_0^\infty \mathcal{P}_{L,i}(dy) \int_B B(t,y) \mathcal{P}_{B,i}(dB) < \infty \quad \text{and} \quad d_{B,i} := \int_0^\infty t \cdot B_i(t)dt < \infty.
\]
(P4) (Successive ages) Conditioned on the life-length $y$ and budding rate function $B$, the successive ages $0 < t_1 < t_2 < \cdots < y$ at which the individual gives birth to offspring are described by an inhomogeneous Poisson process on $(0, y)$ with intensity $B(t, y)$ and the mean number of successive ages is $\|B(y)\|_L := \int_0^y B(t, y) dt$. The first and second moments of successive ages are finite, i.e., for $i \in \mathcal{H}$,

$$m_{B,i} := \int_0^\infty \mathcal{P}_{L,i}(dy) \|B(y)\|_L \mathcal{P}_{B,i}(dB),$$

$$v_{B,i} := \int_0^\infty \mathcal{P}_{L,i}(dy) \|B(y)\|_L^2 \mathcal{P}_{B,i}(dB).$$

Usually, only one bud forms on the mother cell at each successive age. But multiple-budding is also widely observed in enveloped virus such as HIV and COVID-19; see [65, p.384].

(P5) (Branching mechanism) At each successive age, a type-$i$ individual gives birth to a random number of offspring of various types according to a probability law $p_i := \{p_i(k_H) : k_H \in \mathbb{N}^d\}$, where $p_i(k_H)$ is the probability to produce $k_1$ children of type-1, $k_2$ of type-2, ..., $k_d$ of type-$d$. The first and second moments of offspring of various types are finite

$$m_{ij} := \sum_{k_H \in \mathbb{N}^d} k_i \cdot p_j(k_H) \quad \text{and} \quad v_{ij} := \sum_{k_H \in \mathbb{N}^d} k_i^2 \cdot p_j(k_H), \quad j \in \mathcal{H}.$$

In addition to budding, microbes may enter into the host from the external environment or the neighboring hosts. For simplicity, we assume that

(P6) (Immigration rate) The arrivals of immigrants follow a Poisson point process with a unit rate;

(P7) (Immigration mechanism) The number of invading microbes of various types in each immigration is distributed as a probability law $p_I := \{p_I(k_H) : k_H \in \mathbb{N}^d\}$, where $p_I(k_H)$ is the probability that $k_1$ immigrants of type-1, $k_2$ of type-2, ..., $k_d$ of type-$d$ enter into the population. The mean number of immigrants is finite

$$m_I := \sum_{k_H \in \mathbb{N}^d} k_i \cdot p_I(k_H), \quad i \in \mathcal{H}.$$

It is the impact of microbes on the host that has been widely considered in mathematical biology literature, e.g., releasing toxins and attacking the host cell. For instance, Candida albicans in the gastrointestinal and genitourinary tract do not only release a kind of toxins called Candidiasis but also alkalinate phagosome by physical rupture. We refer the impact of each microbe on the host as its characteristic, which usually is described as a non-negative function of its age and life-length. Denote by $\Upsilon$ the measurable space of all possible characteristic functions on $\mathbb{R}_+^d$.

(P8) (Characteristic) Each type-$i$ individual is endowed with a characteristic function randomly according to a probability law $\mathcal{P}_{T,i}(d\Upsilon)$ on $\Upsilon$ with

$$T_i(t) := \int_0^\infty \mathcal{P}_{L,i}(dy) \int_\Upsilon T(t, y) \mathcal{P}_{T,i}(d\Upsilon) < \infty, \quad t \geq 0.$$

The branching particle system defined by these properties is a multi-type CMJ-process with initial state $\Xi_H(0)$ and parameter $(p_D, \mathcal{P}_{L,H}, \mathcal{P}_{B,H}, \mathcal{P}_{T,H})$. In particular, when $p_D(1) = 1$ and $\mathcal{P}_{B,H}(B(t, y) = 1_{(y \geq t)}) = 1$, it is often known as a homogeneous, binary CMJ-process. Denote by $I_i$ the collection of all type-$i$ individuals in the population. Associated with each individual $x \in I_i$ is a quadruple $(\tau_x, \ell_x, B_x, T_x)$ that represents its birth/immigrating time,
life-length, budding rate function and characteristic function. We are usually interested in the multi-type CMJI-process counted with random characteristic $T$ (T-CMJI-process), denoted by $\{T_{\mathcal{H}}(t) := (T_{i}(t))_{i \in \mathcal{H}} : t \geq 0\}$ with

$$T_{i}(t) := \sum_{x \in I_{i}} T_{x}(t - \tau_{x}, \ell_{x}).$$

Specially, if $\mathcal{P}_{T,i}(T(t, y) = 1_{\{y > t\}}) = 1$, then the T-CMJI-process reduces to the process of population size, denoted as $\{\Xi_{\mathcal{H}}(t) : t \geq 0\}$ with

$$\Xi_{i}(t) := \sum_{x \in I_{i}} 1_{\{\ell_{x,t-y} \land \tau_{x} \geq 0\}}, \quad t \geq 0, i \in \mathcal{H}.$$

We end this section with several typical characteristic functions that are widely considered in biology and mathematics.

**Example.** For each $i \in \mathcal{H}$, if $\mathcal{P}_{T,i}$ is a probability measure on $\mathcal{M}(\mathbb{R}_{+})$ and $T(t, y) := T(t \land y)$ is the mass of $T$ on $[0, t \land y]$, then $T_{i}(t) = \sum_{x \in I_{i}} T_{x}((t - \tau_{x}) \land \ell_{x})$ is a multi-type CMJI-process counted with random measure. In particular,

1. If $\mathcal{P}_{T,i}(T(t, y) = 1_{\{t \geq y\}}) = 1$, then $T_{i}(t) = \sum_{x \in I_{i}} 1_{\{t \geq \tau_{x}\}}$ is known as the total progeny of type-$i$ up to time $t$;
2. If $\mathcal{P}_{T,i}(T(t, y) = t^{+} \land y) = 1$, then $T_{i}$ is the integral of type-$i$ population, i.e.,

$$T_{i}(t) = \sum_{x \in I_{i}} (t - \tau_{x})^{+} \land \ell_{x} = \sum_{x \in I_{i}} \int_{0}^{t} 1_{\{\ell_{x,s} - \tau_{x} \geq 0\}} ds = \int_{0}^{t} \Xi_{i}(s) ds, \quad t \geq 0.$$

**Example.** For each $i \in \mathcal{H}$, let $\mathcal{P}_{T,i}$ be a probability measure on $L^{1}_{T_{\mathcal{V}}}(\mathbb{R}_{+})$. Then $T_{i}(t) = \sum_{x \in I_{i}} T_{x}(t - \tau_{x})1_{\{t - \tau_{x} \leq \ell_{x}\}}$ is a multi-type CMJI-process counted with random integrable function. In particular, for some constant $\eta > 0$,

1. If $\mathcal{P}_{T,i}(T(t, y) = 1_{\{0 \leq t < \eta \land y\}}) = 1$, then $T_{i}(t) = \sum_{x \in I_{i}} 1_{\{t - \tau_{x} \in [0, \eta \land \ell_{x}\]}\}$ is the total type-$i$ population alive at time $t$ which is younger than $\eta$;
2. If $\mathcal{P}_{T,i}(T(t, y) = 1_{\{\eta \leq t < y\}}) = 1$, then $T_{i}(t) = \sum_{x \in I_{i}} 1_{\{\eta \leq t - \tau_{x} < \ell_{x}\}}$ is the total type-$i$ population alive at time $t$ which is older than $\eta$;
3. If $\mathcal{P}_{T,i}(T(t, y) = 1_{\{0 < y - t \leq \eta\}}) = 1$, then $T_{i}(t) = \sum_{x \in I_{i}} 1_{\{0 < \ell_{x} - (t - \tau_{x}) \leq \eta\}}$ is the total type-$i$ population alive at time $t$ with residual life less than $\eta$.

**4.2. Hawkes representation.** In this section, we link the foregoing multi-type CMJI-process to a self-excited dynamical system driven by multivariate MHPI-measures. Different to the early literature in which the population size is often studied first, we start by considering the total budding rate process $\{B_{\mathcal{H}}(t) := (B_{i}(t))_{i \in \mathcal{H}} : t \geq 0\}$, where $B_{i}(t)$ is the total budding rate of all type-$i$ individuals alive at time $t$, i.e.,

$$B_{i}(t) = \sum_{x \in I_{i}} B_{x}(t - \tau_{x}, \ell_{x}).$$

Obviously, the process $B_{\mathcal{H}}$ is a B-CMJI-process. However, this representation fails to clarify the population evolution dynamics and is not helpful to explore the long-term behavior of the population. We now establish a new representation based on a finer classification for individuals of various types. Denote by $\{T_{i,k}\}_{k \geq 1}$ the immigrating times. For $i \in \mathcal{H}$, let $\{\tau_{i,k}\}_{k \geq 1}$ be the successive ages of all type-$i$ individuals. From property (P4) and the mutual independence among individuals, we have $\tau_{i,k} < \tau_{i,k+1}$ and $\tau_{i,k} \neq \tau_{j,l}$ a.s. for any $(i, k), (j, l) \in \mathcal{D} \times \mathbb{Z}_{+}$ with $(i, k) \neq (j, l)$. According to the origin of each type-$i$ individual, we can split $I_{i}$ into three kinds of disjoint sets: for each $j \in \mathcal{H}$ and $k \geq 1$,
\[ \mathcal{I}_{i,0}: \text{Ancestors of type-}i \text{ at time 0;} \]
\[ \mathcal{I}_{i,1,k}: \text{Immigrants of type-}i \text{ entering into the population at the immigrating time } \tau_{i,1,k}; \]
\[ \mathcal{I}_{i,j,k}: \text{Offspring of type-}i \text{ produced by a type-}j \text{ mother individual at the successive age } \tau_{j,k}. \]

Notice that for each individual \( x \) in \( \mathcal{I}_{i,0} \) or \( \mathcal{I}_{i,j,k} \) with \( i \in \mathcal{H}, j \in \mathcal{D} \) and \( k \geq 1 \), we have \( \tau_x = 0 \) or \( \tau_{j,k} \) respectively. Thus we can write the total budding rate of all type-\( i \) individuals alive at time \( t \) as
\[
B_t(t) = \sum_{x \in \mathcal{I}_{i,0}} B_x(t - \tau_x, \ell_x) + \sum_{\tau_{i,j,k} \leq t} \sum_{x \in \mathcal{I}_{i,j,k}} B_x(t - \tau_{i,j,k}, \ell_x)
\]
\[
+ \sum_{j \in \mathcal{H}} \sum_{\tau_{j,k} \leq t} \sum_{x \in \mathcal{I}_{i,j,k}} B_x(t - \tau_{j,k}, \ell_x).
\]
(23)

Here the first sum on the right side of this equation is the total budding rate of all type-\( i \) ancestors. The inner-sum in the second term is the total budding rate of all type-\( i \) immigrants entering into the population in the \( k \)-th immigration. Similarly, the second inner-sum in the last term is the total budding rate of all type-\( i \) offspring born at time \( \tau_{i,k} \). Repeating the previous progress, we also can give representation analogous to (23) for \( T_t(t) \) by replacing the budding rate function \( B_x \) with the characteristic function \( T_x \).

To get a Hawkes representation for the CMJ-process, it remains to construct two random point measures to describe the arrivals and characteristics of immigration and reproduction respectively. Let \( \mathbb{U} := (\mathbb{N} \times \mathbb{R}^+_0 \times \mathbb{N} \times \mathbb{T}^N)^d \). For each \( j \in \mathcal{D} \) and \( k \geq 1 \), we introduce a notation
\[
\mathbf{u}_{j,k} := (k_i, y_i, B_i, T_i, i \in \mathcal{H}) := ((k_i)_i, (y_i)_i, (B_i)_i, (T_i)_i) \in \mathbb{U},
\]
to describe the new individuals getting into the population at time \( \tau_{j,k} \), where
- \( k_i \in \mathbb{N} \): Number of type-\( i \) offspring/immigrants;
- \( y_i := (y_{i,l})_{l=1}^{k_i} \in \mathbb{R}^k_i \): Life-lengths of type-\( i \) offspring/immigrants;
- \( B_i := (B_{i,l})_{l=1}^{k_i} \in \mathbb{B}^k_i \): Budding rate functions of type-\( i \) offspring/immigrants;
- \( T_i := (T_{i,l})_{l=1}^{k_i} \in \mathbb{T}^k_i \): Characteristic functions of type-\( i \) offspring/immigrants.

At time \( t \), the total budding rate and total characteristic of these new born/immigrating individuals of type-\( i \) can be written as
\[
\sum_{x \in \mathcal{I}_{i,j,k}} B_x(t - \tau_{j,k}, \ell_x) = \sum_{l=1}^{k_i} B_{i,l}(t - \tau_{j,k}, y_{i,l}) =: \phi_i(t - \tau_{j,k}, \mathbf{u}_{j,k}),
\]
(24)
\[
\sum_{x \in \mathcal{I}_{i,j,k}} T_x(t - \tau_{j,k}, \ell_x) = \sum_{l=1}^{k_i} T_{i,l}(t - \tau_{j,k}, y_{i,l}) =: \zeta_i(t - \tau_{j,k}, \mathbf{u}_{j,k}).
\]
(25)

For each \( j \in \mathcal{D} \), associated to the sequence \( \{(\tau_{j,k}, \mathbf{u}_{j,k})\}_{k \geq 1} \) we define an \((\mathcal{F}_t)\)-random point measure on \((0, \infty) \times \mathbb{U}\)
\[
N_j(ds, d\mathbf{u}) = \sum_{k=1}^{\infty} 1_{\{\tau_{j,k} \in ds, \mathbf{u}_{j,k} \in d\mathbf{u}\}}.
\]

From properties (P3)-(P4) and (P6)-(P7), we see that \( N_j(ds, d\mathbf{u}) \) has intensity \( ds \cdot \nu_j(d\mathbf{u}) \) when \( j = I \) or \( B_j(s-) \cdot ds \cdot \nu_j(d\mathbf{u}) \) when \( j \in \mathcal{H} \), where \( \nu_j(d\mathbf{u}) \) is a probability law on \( \mathbb{U} \).
defined by
\[ \nu_j(du) := \sum_{n_H \in N^d} p_j(n_H) \cdot \delta_{n_H}(dk_H) \prod_{i \in H} \prod_{l=1}^{n_i} \mathcal{P}_{L,i}(d\gamma_{i,l}) \mathcal{P}_{B,i}(dB_{i,l}) \mathcal{P}_{T,i}(dT_{i,l}), \quad j \in D. \]

We now give a more detailed description for ancestors. For each \( i \in H \) and ancestor \( x \in \mathcal{I}_{i,0} \), its life-length equals to the sum of its age \( a_{i,x} \) and residual life \( r_{i,x} \) at time 0. Thus we can write the total budding rate and total characteristic of all type-\( i \) ancestors at time \( t \geq 0 \) as
\[
\mu_i(t) := \sum_{x \in \mathcal{I}_{i,0}} B_x(t + a_{i,x}, r_{i,x} + a_{i,x}),
\]
\[
\psi_i(t) := \sum_{x \in \mathcal{I}_{i,0}} T_x(t + a_{i,x}, r_{i,x} + a_{i,x}).
\]

Based on all preparations above, an argument similar to that in Section 2.1 gives the following Hawkes representations for the total budding rate process \( B_H \) and the T-CMJI-process \( T_H \).

**Proposition 4.1.** \( N_H(ds, du) \) is a multivariate MHPI-random measure on \( \mathbb{R}_+ \times \mathbb{U} \) with mark distribution \( \nu_H(du) \) and intensity process \( B_H \) admitting the form of
\[
B_i(t) = \mu_i(t) + \sum_{j \in D} \int_0^t \int_\mathbb{U} \phi_i(t - s, u) N_j(ds, du), \quad t \geq 0, i \in H.
\]
Moreover, an analogous representation for \( T_H \) can be obtained by replacing \( (\mu_H, \phi_H) \) with \( (\psi_H, \zeta_H) \).

4.3. **Scaling limit theorems.** In practice, the microbial population usually is very large and birth/death events occur at a high-frequency. These make the low-frequency biological models (e.g. CMJ-processes and GW-processes) inefficient and the high-frequency models popular in the study of microbial population. We introduce a parameter \( n \in \mathbb{Z}_+ \) to scale the population size and assume that individuals are weighted by \( 1/n \). Under some mild scaling assumptions, we now establish several limit theorems for multi-type CMJI-processes by using the convergence results in Section 3. In the \( n \)-th model, the multi-type CMJI-process starts from \( \mathcal{I}_{i,0}^{(n)} \) ancestors and has parameter \( (p_D^{(n)}, p_{L,H}^{(n)}, p_{B,H}^{(n)}, p_{T,H}^{(n)}) \). Quantities in the last two sections are defined similarly with superscript \( (n) \). Recall the constant \( \alpha \in (1, 2) \) in the hypothesis (H1).

4.3.1. **Scaling limit for total budding rate processes.** We first give some sufficient conditions on the initial state and parameters such that the rescaled CMJI-process converges to a non-degenerate limit. For any \( i \in H \) and \( u \in \mathbb{U} \), by (24) we have
\[
\| \phi_i(u) \|_{L^1} = \sum_{l=1}^{k_i} \| B_{i,l}(y_{i,l}) \|_{L^1}, \quad \| \phi_i(u) \|_{TV} \leq \sum_{l=1}^{k_i} \| B_{i,l}(y_{i,l}) \|_{TV}
\]
and
\[
\int_0^\infty t \cdot \phi_i(t, u) dt = \sum_{l=1}^{k_i} \int_0^\infty t \cdot B_{i,l}(t, y_{i,l}) dt.
\]
By Hölder’s inequality, the hypothesis (H1) is satisfied under the following condition.
CONDITION 4.2. For $i \in \mathcal{H}$ and $j \in \mathcal{D}$, there exists a constant $C > 0$ such that for any $n \geq 1$,
\[
\nu_{L,i}^{(n)} + \sum_{k \in \mathbb{N}} |k|^{2\alpha} p_j^{(n)}(kH) \leq C
\]
and
\[
\int_0^\infty \mathcal{P}_{L,i}^{(n)}(dy) \int_{\mathbb{R}} \left( \int_0^\infty t \cdot B(t,y)dt + \|B(y)\|_{TV} \right)^{2\alpha} \mathcal{P}_{B,i}^{(n)}(dB) \leq C.
\]
This condition means that both the branching and immigration mechanisms satisfy the light-tailed condition. More precisely, the number of new individuals in each immigration or born at each successive age is light-tailed distributed. Meanwhile, each individual is likely to give birth to its offspring in youth. Since individuals give birth to their offspring independently, we have for each $i \in \mathcal{H}$ and $j \in \mathcal{D}$,
\[
\phi_{ij}^{(n)}(t) = b_i^{(n)}(t) \cdot m_{ij}^{(n)}, \quad \|\phi_{ij}^{(n)}\|_{L^1} = m_{B,i}^{(n)} \cdot m_{ij}^{(n)}, \quad \int_0^\infty t \cdot \phi_{ij}^{(n)}(t)dt = d_{B,i}^{(n)} \cdot m_{ij}^{(n)}
\]
and
\[
\int_{\mathbb{R}} \|\phi_{i}(u)\|_{L^2}^2 \nu_{i}^{(n)}(du) = \nu_{B,i}^{(n)} \cdot m_{i}^{(n)} + (v_{ii}^{(n)} - \bar{v}_{ii}^{(n)}) \cdot \|m_{B,i}^{(n)}\|^2.
\]
We now provide some asymptotic assumptions on the branching mechanism and immigration mechanism.

CONDITION 4.3. Assume that hypothesis (H2) holds and as $n \to \infty$,

1. for each $i \in \mathcal{H}$, there exist constants $m_{L,i}^*, v_{ii}^*, d_{B,i}^*, v_{E,i}^*$, $m_{ii}^*, m_{ii}^*, m_{iI}^*$ such that
\[
\nu_{L,i}^{(n)} \to m_{L,i}^*, \quad v_{ii}^{(n)} \to v_{ii}^*, \quad d_{B,i}^{(n)} \to d_{B,i}^*,
\]
and
\[
v_{B,i}^{(n)} \to v_{E,i}^*, \quad m_{ii}^{(n)} \to m_{ii}^*, \quad m_{iI}^{(n)} \to m_{iI}^*;
\]

2. there exists a matrix $b_{H^2}^* := \left(b_{ij}^*\right)_{i,j \in \mathcal{H}}$ such that
\[
n \left( m_{H^2}^{(n)} - \text{diag}(1/m_{B,H}^{(n)}) \right) \to b_{H^2}^*.
\]

The essence of this condition is that the rescaled branching and immigration mechanisms converge to a non-degenerate limit, i.e., immigrants enter into the population at the rate $m_{H^2}^*$ and the net growth rate of the population is $b_{H^2}^*$. We now provide some sufficient conditions on ancestors. For each $i \in \mathcal{H}$ and ancestor $x \in T_{i,0}^{(n)}$, denote by $A_{i,x}^{(n)}$ and $R_{i,x}^{(n)}$ its age and residual life at time 0 respectively. Enlightened by the inspection paradox relating to the fact that observing a renewal interval at time $t$ gives an interval with average value larger than that of an average renewal interval; see Chapter 7.7 in [61, p.460], we may assume that the residual life $R_{i,x}^{(n)}$ is distributed as the excess life-length distribution of $\mathcal{P}_{L,i}^{(n)}$, also called forward recurrence time, which is defined by
\[
\mathcal{P}_{L,i}^{(n)}(dy) := \mathcal{P}_{L,i}^{(n)}(y, \infty) \cdot \frac{dy}{m_{L,i}^{(n)}},
\]
For an individual getting into the population at time \(-t < 0\), it stays alive at time 0 with probability \(\mathcal{P}_{i,j}^{(n)}[t, \infty]\). Since the ancestor \(x\) may get into the population at any time prior to time 0, we may assume that its age \(\tau^{(n)}_{i,x}\) is distributed as

\[
P(\tau^{(n)}_{i,x} \in dt) = \mathcal{P}_{L,i}^{(n)}[t, \infty] \cdot \frac{dt}{m^{(n)}_{L,i}} = \hat{\mathcal{P}}_{L,i}^{(n)}(dt).
\]

Taking these together, we assume that the next condition holds for ancestors.

**CONDITION 4.4.** For each \(n \geq 1\) and \(i \in \mathcal{H}\), assume that \(\tau^{(n)}_{i,x}\) and \(\tau^{(n)}_{i,x}\) have joint distribution

\[
\hat{\mathcal{P}}_{AR,i}^{(n)}(dt, dy) := P(\tau^{(n)}_{i,x} \in dt, \tau^{(n)}_{i,x} \in dy) = \frac{dt}{m^{(n)}_{L,i}} \cdot \mathcal{P}_{L,i}^{(n)}(t + dy).
\]

Actually, Condition 4.4 is consistent with our previous assumptions on the age and residual-life distributions of ancestors. Indeed, it is easy to identify that the marginal distribution is

\[
P(\tau^{(n)}_{i,x} \in dt) = \hat{\mathcal{P}}_{AR,i}^{(n)}(dt, \mathbb{R}^+),
\]

Moreover, for any \(y \geq 0\) we also have

\[
P(\tau^{(n)}_{i,x} \geq y) = \hat{\mathcal{P}}_{AR,i}^{(n)}([y, \infty)) = \int_{0}^{\infty} \mathcal{P}_{L,i}^{(n)}[t + y, \infty] \frac{dt}{m^{(n)}_{L,i}}
\]

\[
= \int_{y}^{\infty} \mathcal{P}_{L,i}^{(n)}[t, \infty] \frac{dt}{m^{(n)}_{L,i}}
\]

and hence \(P(\tau^{(n)}_{i,x} \leq dy) = \hat{\mathcal{P}}_{L,i}^{(n)}(dy)\). We now give a scaling limit theorem for the total budding rate process.

**THEOREM 4.5.** Under Condition 4.2, 4.3 and 4.4, if \(\sup_{n \geq 1} \mathbb{E}[\Xi_{\mathcal{H}}^{(n)}(0)/n^{2\alpha}] < \infty\) and \(\Xi_{\mathcal{H}}^{(n)}(0)/n \xrightarrow{d} \Xi_{\mathcal{H}}^{(0)} \in \mathbb{R}^d\), then the rescaled process \(\{B_{\mathcal{H}}^{(n)}(nt)/n : t \geq 0\}\) converges weakly to \(\{B_{\mathcal{H}}^{(n)}(t) : t \geq 0\}\) in \(D([0, \infty), \mathbb{R}^d)\) as \(n \to \infty\), where \(B_{\mathcal{H}}^{(n)}\) is the unique strong solution to (21) with

\[
Z_i(0) = \Xi_i^{(0)}, \quad a_i = \frac{m_{iL}^{(*)}}{m_{ii}^{(*)}}, \quad b_{ij} = \frac{b_{ij}^{(*)}}{m_{ii}^{(*)}}
\]

and

\[
c_i^2 = \nu_{\mathcal{H},i} \cdot m_{ii}^{(*)} + \frac{\nu_{i}^{(*)} - m_{ii}^{(*)}}{|m_{ii}^{(*)}|^2}, \quad \sigma_i = \frac{d_{\mathcal{H},i}^{(*)} \cdot m_{ii}^{(*)}}{m_{ii}^{(*)}}, \quad i, j \in \mathcal{H}.
\]

4.3.2. Scaling limits for T-CMJI-processes. In this section we study the convergence of rescaled multi-type CMJI-processes counted with random characteristic by using the limit results in Section 3.2. For each \(i \in \mathcal{H}\) and \(j \in \mathcal{D}\), the mutual independence among individuals induces that in the \(n\)-th model, the mean total impact of type-\(i\) offspring produced by a type-\(j\) individual at each successive age is

\[
\zeta_{ij}^{(n)}(t) = \int_{U} \zeta_i(t, u) \nu_{i}^{(n)}(du) = T_{ij}^{(n)}(t) \cdot m_{ij}^{(n)}, \quad t \geq 0.
\]
Condition 4.3 tells that the mean arrival rate of type-\(i\) immigrants is of the order of 1. From Theorem 4.5, The mean rate of type-\(j\) individuals giving birth to type-\(i\) offspring is of the order of \(n\) if \(j = i\) and 1 otherwise. Thus the main contribution to the T-CMJI-process is made by individuals whose types are same to those of their parents.

We first establish a convergence result for the rescaled process \(\{T_{1,\mathcal{H}}^{(n)}(t) : t \geq 0\}\) with
\[
T_{1,\mathcal{H}}^{(n)}(t) := \frac{1}{n} \cdot T_{\mathcal{H}}^{(n)}(nt),
\]
in which the characteristic function of each individual represents the instantaneous rate at which it affects the host, e.g., toxin release rate and population size. From (25), we have
\[
\|\zeta_i^{(n)}(u)\|_{L^1} = \sum_{l=1}^{k_i} \|T_{i,l}(y,l)\|_{L^1},
\]
for any \(u \in \mathbb{U}\) and \(i \in \mathcal{H}\). It is easy to identify Condition 3.17 and 3.18 by the next condition.

**CONDITION 4.6.** Assume that
\[
\sup_{n \geq 1} \int_0^\infty P_{L,i}^{(n)}(dy) \int_T \left(\|T(y)\|_{L^1}^2 + \|T(y)\|_{TV}^2\right) P_{T,i}^{(n)}(dT) < \infty, \quad i \in \mathcal{H}.
\]
Moreover, there exist a constant \(a_{i,\mathcal{H}}^* \in \mathbb{R}_+^d\) and a non-negative function \(\bar{T} \in L^1(\mathbb{R}_+)\) such that
\[
\lim_{n \to \infty} \|T_{\mathcal{H}}^{(n)}\|_{L^1} = a_{i,\mathcal{H}}^* \quad \text{and} \quad \sup_{n \geq 1} \|T_{\mathcal{H}}^{(n)}(t)\|_{TV} \leq \bar{T}(t), \quad t \geq 0.
\]

**THEOREM 4.7.** Under Condition 4.2, 4.3, 4.4 and 4.6, we have \(T_{1,\mathcal{H}}^{(n)} \xrightarrow{d} T_{1,\mathcal{H}}^*\) in \(\mathcal{D}([0, \infty), \mathbb{R}_+^d)\) as \(n \to \infty\) with \(T_{i,\mathcal{H}}^* := a_{i,\mathcal{H}}^* \cdot m_i^* \cdot B_i^*\) for each \(i \in \mathcal{H}\).

As a corollary, we next give a scaling limit theorem for the population size process \(\Xi_{\mathcal{H}}^{(n)}\), which is a T-CMJI-process with \(P_{L,i}^{(n)}(T(t,y) = 1_{(y>t)}) = 1\). In this case, we have for any \(t \geq 0\) and \(y > 0\),
\[
\|T(y)\|_{TV} = 1, \quad \|T(y)\|_{L^1} = y, \quad T_{\mathcal{H}}(t) = P_{L,\mathcal{H}}^{(n)}(t, \infty), \quad \|T_{\mathcal{H}}^{(n)}\|_{L^1} = m_{\mathcal{H}}^{(n)}.
\]

**COROLLARY 4.8.** Under Condition 4.2, 4.3 and 4.4, we have

1. The rescaled process \(\{\Xi_{\mathcal{H}}^{(n)}(nt)/n : t \geq 0\}\) converges weakly to \(\{\Xi_{\mathcal{H}}^*(t) := (m_{\mathcal{H}}^{(n)} m_{\mathcal{H}}^{(n)} \cdot B_i^*(t))_{i \in \mathcal{H} : t \geq 0}\}\) in \(\mathcal{D}([0, \infty), \mathbb{R}_+^d)\) as \(n \to \infty\);
2. If \(P_{L,\mathcal{H}}^{(n)} \xrightarrow{d} P_{L,\mathcal{H}}^*\) as \(n \to \infty\), then for any constant \(\eta > 0\), the rescaled processes of total population which is younger than \(\eta\), is older than \(\eta\) or has residual life less than \(\eta\) converge weakly to
\[
\left(\int_0^{\infty} (y \wedge \eta) P_{L,i}^{(n)}(dy) \cdot \Xi_i^{(n)} \right)_{i \in \mathcal{H}} \quad \text{or} \quad \left(\int_0^{\infty} (y \wedge \eta) P_{L,i}^{(n)}(dy) \cdot \Xi_i^{(n)} \right)_{i \in \mathcal{H}}
\]
respectively in \(\mathcal{D}([0, \infty), \mathbb{R}_+^d)\) as \(n \to \infty\).
We now consider the behavior at a large time scale of the cumulative impact of microbes on the host, e.g., cumulative toxin release and total progeny. By Corollary 4.8(1), the cumulative impact of individuals with same type as their parents on the host is of the order of $n^2$. However, the assumption $\Xi_{\mathcal{H}}^{(n)}(0) \sim \Xi_{\mathcal{H}}^{(n)}(0) \cdot n$ induces that the cumulative impact of ancestors is of the order of $n$ and can be asymptotically ignored. Consequently, we have

$$T_i^{(n)}(nt) \sim \int_0^{nt} \int_\mathbb{U} \xi_i(nt-s,u)N_i^{(n)}(ds,du), \quad t \geq 0, i \in \mathcal{H}$$

as $n \to \infty$ and hence $E[T_i^{(n)}(nt)]$ is of the order of $n^2$. Thus it is natural to consider the weak convergence of the rescaled process $\{T_i^{(n)}(t) : t \geq 0\}$ by using Theorem 3.15, where

$$T_{c,\mathcal{H}}(t) := \frac{1}{n^2} \cdot T_i^{(n)}(nt).$$

From (25), we have

$$\xi_i(\infty, u) = \sum_{l=1}^{k_i} T_{i,l}(y_{i,l}, y_{i,l}), \quad u \in \mathbb{U}.$$  

It is easy to see that Condition 3.13 and 3.14 are satisfied under the following condition.

**CONDITION 4.9.** Assume that $T_{\mathcal{H}}^{(n)}(\infty) \to a^{*}_{c,\mathcal{H}} \in \mathbb{R}^d_+$ as $n \to \infty$ and $\sup_{n \geq 1} |T_{\mathcal{H}}^{(n)}(\infty) - T_{\mathcal{H}}(t)| \to 0$ as $t \to \infty$. Moreover, assume that

$$\sup_{n \geq 1} \int_T P_{T,i}^{(n)}(d\tau) \int_0^\infty |T(y, y)|^\alpha P_{L,i}^{(n)}(dy) < \infty, \quad i \in \mathcal{H}.$$  

**THEOREM 4.10.** Under Condition 4.2, 4.3, 4.4 and 4.9, we have $T_{c,\mathcal{H}}^{(n)} \overset{d}{\to} T_{c,\mathcal{H}}^*$ in $\mathbb{D}([0, \infty), \mathbb{R}^d_+)$ as $n \to \infty$ with

$$T_{c,i}^*(t) := a^{*}_{c,i} \cdot m^{*}_{i} \cdot \int_0^t B^*_i(s)ds, \quad t \geq 0, i \in \mathcal{H}.$$  

**COROLLARY 4.11.** Under Condition 4.2, 4.3 and 4.4, the two rescaled processes of total progeny and integral of population converge weakly in $\mathbb{D}([0, \infty), \mathbb{R}^d_+)$ to

$$\left(\frac{1}{m^*_{L,i}} \cdot \int_0^t \Xi^*_i(s)ds\right)_{i \in \mathcal{H}} \quad \text{and} \quad \left(\int_0^t \Xi^*_i(s)ds\right)_{i \in \mathcal{H}}.$$

**4.3.3. Scaling limits for population structures.** In this section, we give some asymptotic results for the population structure of nearly critical multi-type CMJI-processes under the following condition.

**CONDITION 4.12.** Assume that $P_{k,i}^{(n)} \overset{d}{\to} P_{L,i}^*$ as $n \to \infty$ and

$$\sup_{n \geq 1} \int_0^\infty y^{2\alpha} P_{L,\mathcal{H}}^{(n)}(dy) < \infty.$$  

In mathematical biology, the population structure is usually described by the age-distribution and residual-life distribution of all alive individuals in the population. In precise,
for $i \in \mathcal{H}$, denote by $\mathcal{A}R_{i,t}^{(n)}(ds,dz)$ the joint distribution of age and residual life of all type-$i$ individuals alive at time $t$ in the $n$-th model, i.e.

$$\mathcal{A}R_{i,t}^{(n)}(ds,dz) := \sum_{x \in \mathcal{X}^{(n)}} 1_{\{0 \leq t - \tau_x < \ell_x\}} \cdot \delta(t - \tau_x, \ell_x - (t - \tau_x))(ds,dz)$$

is a measure on $\mathbb{R}^2_+$ with unit mass at the age and residual-life of each type-$i$ individual alive at time $t$. The two marginal measures

$$\mathcal{A}_{i,t}^{(n)}(ds) := \mathcal{A}R_{i,t}^{(n)}(ds, \mathbb{R}_+) \quad \text{and} \quad \mathcal{R}_{i,t}^{(n)}(dz) := \mathcal{A}R_{i,t}^{(n)}(\mathbb{R}_+, dz)$$

are the corresponding age distribution and residual-life distribution at time $t$. Similarly, the life-length distribution $\mathcal{L}_{i,t}^{(n)}$ of all type-$i$ individuals alive at time $t$ is given by

$$\mathcal{L}_{i,t}^{(n)}(dy) := \sum_{x \in \mathcal{X}^{(n)}} 1_{\{0 \leq t - \tau_x < \ell_x\}} \delta_{\ell_x}(dy).$$

We establish a scaling limit for the population structure in the next theorem in collaboration with the following three probability laws

$$\hat{\mathcal{P}}_{\mathcal{A}R,i}^{*}(ds,dz) := \frac{ds}{|m_{\mathcal{L},i}^*|} \cdot \mathcal{P}_{\mathcal{L},i}^*(s + dz),$$
$$\hat{\mathcal{P}}_{\mathcal{A}R,i}^{*}(dy) := \mathcal{P}_{\mathcal{L},i}^*[y, \infty) \cdot \frac{dy}{|m_{\mathcal{L},i}^*|},$$
$$\hat{\mathcal{P}}_{\mathcal{L},i}^{*}(dy) := \frac{y}{|m_{\mathcal{L},i}^*|} \cdot \mathcal{P}_{\mathcal{L},i}^*(dy).$$

Wherein, $\hat{\mathcal{P}}_{\mathcal{A}R,i}^*$ is usually known as the size-biased distribution of $\mathcal{P}_{\mathcal{L},i}^*$.  

**Theorem 4.13.** Under Condition 4.2, 4.3, 4.4 and 4.12, we have as $n \to \infty$,

1. $\{\mathcal{A}R_{H,n,t}^{(n)} / n : t \geq 0\} \overset{d}{\longrightarrow} \{(\Xi^*_i(t) \cdot \hat{\mathcal{P}}_{\mathcal{A}R,i}^*)_{i \in \mathcal{H}} : t \geq 0\}$ in $\mathcal{D}([0, \infty), \mathcal{M}(\mathbb{R}^2_+)^d)$;
2. both the two rescaled processes $\{\mathcal{A}_{H,n,t}^{(n)} / n : t \geq 0\}$ and $\{\mathcal{R}_{H,n,t}^{(n)} / n : t \geq 0\}$ converge weakly to $\{(\Xi^*_i(t) \cdot \hat{\mathcal{P}}_{\mathcal{L},i}^*)_{i \in \mathcal{H}} : t \geq 0\}$ in $\mathcal{D}([0, \infty), \mathcal{M}(\mathbb{R}_+)^d)$;
3. $\{\mathcal{L}_{H,n,t}^{(n)} / n : t \geq 0\} \overset{d}{\longrightarrow} \{(\Xi^*_i(t) \cdot \hat{\mathcal{P}}_{\mathcal{L},i}^*)_{i \in \mathcal{H}} : t \geq 0\}$ in $\mathcal{D}([0, \infty), \mathcal{M}(\mathbb{R}_+)^d)$.

**Remark 4.14.** The essence of Theorem 4.13 is that as the rescaled measure-valued process $\mathcal{A}R_{H,n,t}^{(n)} / n$ approaches to the limit, it can be recovered from the diffusion scaled population process $\Xi_{H,n}^{(n)} / n$ by the lifting map $\hat{\mathcal{P}}_{\mathcal{A}R,H}^*$ from $\mathbb{R}^d_+$ to $\mathcal{M}(\mathbb{R}^2_+)^d$. In other words, in a complex biological system enjoying short-memory property, the evolution of population can be fully described by the process of population size together with the life-length distribution, with more detailed information about the population not being necessary. This type of asymptotic behavior is well known as state space collapse. It was first systematically investigated in [12, 70] in the study of multi-class queueing systems and since then has been widely observed in heavy traffic limits of various queueing systems; see [28, 64, 67].

**Remark 4.15.** Compared to the complex structure of CMJI-processes, the approximating models in the preceding theorems can be useful for several reasons. Firstly, they have simpler structures and are easier to be understood than CMJI-models counted with random characteristic. Each coefficient in the limit models has an intuitive and understandable interpretation. Secondly, their properties are usually consistent with those of CMJI-processes, e.g. criticality, extinction and stationarity; see [44, 54, 72]. Thirdly, compared to the non-parametric estimation of CMJI-models, the approximating models are computationally more tractable and only few parameters are needed to be estimated.
5. Proofs. In this section, we give the detailed proofs for the main results in the previous sections including Theorem 3.6, 3.15, 3.19, 3.21, 4.5, 4.7, 4.10 and 4.13.

5.1. Proof for Theorem 3.6. By the argument at the beginning of Section 3.1.3, it suffices to prove the weak convergence of the sequence \{Z_{\beta,\mathcal{H}}^{(n)}\}_{n \geq 1} to \(Z_{\beta,\mathcal{H}}\). In order to simplify the following statements and notation, we prove this result with \(\lambda_0 < 0\) (equivalently, \(bii < 0\) for all \(i \in \mathcal{H}\)) and \(\beta = 0\). The general case can be proved similarly. The asymptotic analysis in Section 3.1.2 has shown that the time-scaled functions \(R_{ii}(n \cdot, n \cdot, u)\), \(R_{ij}(n \cdot, n \cdot, u)\), \(R_{ij}(n \cdot, n \cdot, u)\) and \(R_{ij}(n \cdot, n \cdot, u)\) can be approximated respectively by the corresponding exponential functions. The errors are denoted as: for \(i, j \in \mathcal{H}\) with \(i \neq j\) and \((t, u) \in \mathbb{R}_+ \times \mathbb{U},
\[
\begin{align*}
\varepsilon_{R_{ii}}^{(n)}(t) &:= R_{ii}(nt) - \frac{1}{\sigma_i} e^{bii t}, \\
\varepsilon_{R_{ij}}^{(n)}(t) &:= R_{ij}(nt) - \frac{b_{ij}}{\sigma_i} e^{bii t}, \\
\varepsilon_{R_{ij}}^{(n)}(t) &:= R_{ij}(nt) - \frac{3}{\sigma_i} e^{bii t}, \\
\varepsilon_{R_{ij}}^{(n)}(t) &:= R_{ij}(nt) - \frac{\|\phi_i(u)\|_1}{\sigma_i} e^{bii t}.
\end{align*}
\]
The sum of the first two terms on the right side of (15) can be approximated by \(Z_i^{(n)}(0)e^{bii /\sigma_i \cdot t}\) and other terms can be approximated respectively by the corresponding (stochastic) integrals with the integrand replaced by the limit exponential function. Meanwhile, the error processes have the following representations respectively: for \(i, j \in \mathcal{H}\) and \(t \geq 0\),
\[
\begin{align*}
\varepsilon_{\mu_i}^{(n)}(t) &:= \mu_{i}^{(n)}(nt) + R_{ii}(nt) - \frac{\mu_{i}^{(n)}(nt)}{n} - Z_i^{(n)}(0)e^{bii t}, \\
\varepsilon_{\mu_j}^{(n)}(t) &:= \int_0^t \varepsilon_{R_{ij}}^{(n)}(s) ds, \\
\varepsilon_{\mu_j}^{(n)}(t) &:= \int_0^t \int_{\mathbb{U}} \varepsilon_{R_{ij}}^{(n)}(t-s,u) \tilde{N}_j^{(n)}(n \cdot ds, du), \\
\varepsilon_{\mu_j}^{(n)}(t) &:= \int_0^t \int_{\mathbb{U}} \varepsilon_{R_{ij}}^{(n)}(t-s,u) \tilde{N}_j^{(n)}(n \cdot ds, du).
\end{align*}
\]
Let \(E_i^{(n)} := \varepsilon_{\mu_i}^{(n)} + \sum_{j \in D} \varepsilon_{\mu_j}^{(n)} + \sum_{j \in D} \varepsilon_{\mu_j}^{(n)}\). Based on these notation, we can write (15) under the form
\[
Z_i^{(n)}(t) = Z_i^{(n)}(0)e^{bii t} + E_i^{(n)}(t) + \int_0^t \int_{\mathbb{U}} \varepsilon_{R_{ij}}^{(n)}(t-s,u) \tilde{N}_j^{(n)}(n \cdot ds, du) + \sum_{j \in \mathcal{H}} \int_0^t \int_{\mathbb{U}} \varepsilon_{R_{ij}}^{(n)}(t-s,u) \tilde{N}_j^{(n)}(n \cdot ds, du),
\]
Using the fact that \(e^{bii (t-s)} = 1 + bii \int_s^t e^{bii (r-s)} dr\) and Fubini’s theorem, we also can write the foregoing equation into the following convenient form:
\[
Z_i^{(n)}(t) = Z_i^{(n)}(0) + E_i^{(n)}(t) + \sum_{j \in D} M_{ij}^{(n)}(t)
\]
\[
+ \int_0^t \left( \frac{b_{ii}}{\sigma_i} E_i^{(n)}(s) + \frac{b_{ij}}{\sigma_i} \frac{\|\phi_i(u)\|_1}{\sigma_i} \tilde{N}_{i,j}^{(n)}(n \cdot ds, du, n \cdot dz) \right) ds, \quad i \in \mathcal{H},
\]
where \(M_{ij}^{(n)}\) is an \((\mathcal{F}_t)\)-local martingale defined as (20) with \(\beta = 0\). By (7) and Proposition 2.4, for each \(j \in \mathcal{H}\) we also can write \(M_{ij}^{(n)}\) under the form
\[
M_{ij}^{(n)}(t) = \int_0^t \int_{\mathbb{U}} \tilde{Z}_{ij}^{(n)}(s-u) \tilde{N}_{i,j}^{(n)}(n \cdot ds, du, n \cdot dz), \quad t \geq 0,
\]
(31)
where \( \tilde{N}_{0,j}^{(n)}(n \cdot ds, du, n \cdot dz) \), \( j \in \mathcal{H} \) are d mutually orthogonal compensated Poisson random measures on \( (0, \infty) \times U \times \mathbb{R}_+ \) with intensity \( n^2 \cdot ds \nu_j^{(n)}(du)dz \) respectively and also independent of \( N_{i}^{(n)}(ds, du) \).

We now start to prove Theorem 3.6 by using the convergence results for infinite-dimensional stochastic differential equations established by Kurtz and Protter [53, Theorem 7.5]. The existence and uniqueness of solutions to (21) follow from Theorem 1 in [74]. We now write (31) into the form of a stochastic integral and differential equation driven by an infinite-dimensional semimartingale; see Appendix 5.7. Let \( \mathbb{H} := \mathbb{R} \times (L^2(\mathbb{R}_+))^d \) be a separable Banach space endowed with a norm \( \| \cdot \|_H \) defined by \( \|x\|_H = |x_0| + \sum_{i=1}^d \|x_i\|_{L^2} \) for \( x := (x_0, x_1, \ldots, x_d) \in \mathbb{H} \). For each \( n \geq 1 \), we define a process \( U_i^{(n)} \) by

\[
U_i^{(n)}(t) = Z_i^{(n)}(0) + E_i^{(n)}(t) + \int_0^t \frac{b_{ij}}{\sigma_i} E_i^{(n)}(s)ds + \sum_{j \in D_i} \tilde{M}_{ij}^{(n)}(t) + \frac{a_i}{\sigma_i} \cdot t, \quad t \geq 0, \ i \in \mathcal{H}
\]

and a standard \( \mathbb{H}^\# \)-semimartingale \( Y_i^{(n)} := (Y_0^{(n)}, Y_1^{(n)}, \ldots, Y_d^{(n)}) \) by \( Y_0^{(n)}(t) := t \) and

\[
W_i^{(n)}(t) := \int_0^t \int_U \frac{\phi_{i,j}(u)}{n} \frac{1}{c_i} \tilde{N}_{0,j}^{(n)}(n \cdot ds, du, n \cdot dz), \quad t \geq 0, \ i \in \mathcal{H}.
\]

We can rewrite the rescaled intensity process (31) as

\[
Z_i^{(n)}(t) = U_i^{(n)}(t) + F_H(Z_i^{(n)}(-)) \cdot Y_i^{(n)}(t),
\]

where \( F_H := (F_i)_{i \in \mathcal{H}} : \mathbb{R}_+^d \rightarrow \mathbb{H}^d \) with the function \( F_i \) defined by

\[
F_i(x_H) := \left( \sum_{j \in \mathcal{H}} \frac{b_{ij}}{\sigma_i} x_j, 0, \ldots, 0, \frac{c_i}{\sigma_i} \cdot \mathbf{1}(z < x_i), 0, \ldots, 0 \right) \in \mathbb{H}, \quad x_H \in \mathbb{R}_+^d.
\]

By [52, Example 5.3], the function \( F_H \) satisfies conditions in [53, Theorem 7.5]. The desired weak convergence in Theorem 3.6 follows immediately if the sequence of \( \mathbb{H}^\# \)-semimartingales \( \{Y_i^{(n)}\}_{n \geq 1} \) is uniformly tight; see Definition A.25, and \( (U_i^{(n)}, Y_i^{(n)}) \Rightarrow (U_i, Y) \) as \( n \rightarrow \infty \), where

\[
U_i(t) := \left( Z_i(0) + \frac{a_i}{\sigma_i} \cdot t \right)_{i \in \mathcal{H}} \quad \text{and} \quad Y_i(t) := (t, W_1(t, \cdot), \ldots, W_d(t, \cdot)).
\]

Actually, they follow directly from the next three claims:

- The two processes \( E_i^{(n)} \) and \( \int_0^t E_i^{(n)}(s)ds \) converge weakly to 0 in \( \mathcal{D}([0, \infty), \mathbb{R}^d) \); see Section 5.1.2.
- For each \( i \in \mathcal{H} \) and \( j \in D_i \), the local martingale \( \tilde{M}_{ij}^{(n)} \) converges weakly to 0 in \( \mathcal{D}([0, \infty), \mathbb{R}) \); see Section 5.1.3.
- The sequence of \( ((L^2(\mathbb{R}_+))^d)^\# \)-local martingales \( \{W_i^{(n)}\}_{n \geq 1} \) is uniformly tight and \( W_i^{(n)} \Rightarrow W_i \); see Section 5.1.4.

5.1.1. Negligible error functions \( \{\varepsilon_{R_i}^{(n)} \}_{i \in \mathcal{H}, j \in D} \). In this section, we prove the convergence of error functions \( \{\varepsilon_{R_i}^{(n)} \}_{i \in \mathcal{H}, j \in D} \) and \( \{\varepsilon_{R_i}^{(n)} \}_{i \in \mathcal{H}} \) to 0 in \( L^1 \) or \( L^2 \). From the hypothesis (H2), we have

\[
\sup_{n \geq 1} \|R_{ii}^{(n)}\|_{L^\infty} < \infty.
\]
Moreover, by extending the proofs of Lemma 4.2 and 4.4 in [46], we can get the following helpful estimates for the Fourier transform $\hat{\phi}^{(n)}_{ii}$ with $i \in \mathcal{H}$; readers may refer to Appendix 5.7 for the detailed proof.

**Proposition 5.1.** There exist constants $C_1, C_2, n_0 > 0$ such that for any $i \in \mathcal{H}$, $n \geq n_0$ and $\lambda \in \mathbb{R}$,

\[ |\hat{\phi}^{(n)}_{ii}(\lambda)| \leq C_1 (|\lambda|^{-1} \wedge 1) \quad \text{and} \quad |1 - \hat{\phi}^{(n)}_{ii}(\lambda)| \geq C_2 (|\lambda| \wedge 1). \tag{33} \]

**Lemma 5.2.** For each $i \in \mathcal{H}$, we have $\|\hat{\varepsilon}^{(n)}_{R_{ii}}\|_{L^2} \to 0$ as $n \to \infty$.

**Proof.** We first provide an upper bound for the Fourier transform of $R^{(n)}_{ii}(\cdot)$. By Condition 3.1,

\[ \left| \int_0^\infty e^{i\lambda t} R^{(n)}_{ii}(nt) dt \right| \leq \int_0^\infty |R^{(n)}_{ii}(nt)| dt = \frac{\|\hat{\phi}^{(n)}_{ii}\|_{L^1}}{n(1 - \|\hat{\phi}^{(n)}_{ii}\|_{L^1})}, \]

which converges to $-1/b_{ii} > 0$ as $n \to \infty$ and hence there exist constants $C, n_0 > 0$ such that

\[ \sup_{n \geq n_0} \left| \int_0^\infty e^{i\lambda t} R^{(n)}_{ii}(nt) dt \right| \leq C. \]

On the other hand, by Proposition 5.1 we also have for large $n$,

\[ \left| \int_0^\infty e^{i\lambda t} R^{(n)}_{ii}(nt) dt \right| \leq \frac{|\hat{\phi}^{(n)}_{ii}(\lambda/n)|}{n|1 - \hat{\phi}^{(n)}_{ii}(\lambda/n)|} \leq C \frac{n^{\frac{1}{2}}}{|\lambda| \wedge n} = \frac{C}{|\lambda|}. \]

Putting these two estimates together, there exist two constants $C, n_0 > 0$ such that for any $\lambda \in \mathbb{R}$,

\[ \sup_{n \geq n_0} \left| \int_0^\infty e^{i\lambda t} R^{(n)}_{ii}(nt) dt \right| \leq C (|\lambda|^{-1} \wedge 1). \tag{34} \]

Thus the Fourier transforms of $R^{(n)}_{ii}(\cdot)$ and $\varepsilon^{(n)}_{R_{ii}}$ are square integrable. By the Fourier isometry,

\[ \|\varepsilon^{(n)}_{R_{ii}}\|^2_{L^2} = \int_0^\infty |\varepsilon^{(n)}_{R_{ii}}(t)|^2 dt = \int_{\mathbb{R}} \left| \int_0^\infty e^{i\lambda t} R^{(n)}_{ii}(nt) dt \right|^2 d\lambda. \]

By the dominated convergence theorem, (34) and (18), we can get the desired result immediately. \qed

**Lemma 5.3.** For any $T > 0$, $i \in \mathcal{H}$ and $j \in \mathcal{D}$, we have $\|\varepsilon^{(n)}_{R_{ij}}\|_{L^1_T} \to 0$ as $n \to \infty$.

**Proof.** Here we just prove this lemma with $i, j \in \mathcal{H}$ and $i \neq j$. For the case of $j = I$, it can be proved in the same way. By Hölder’s inequality,

\[ \|\varepsilon^{(n)}_{R_{ij}}\|_{L^1_T} \leq \int_0^T n\phi^{(n)}_{ij}(nt) dt + \int_0^T n \int_0^{nt} |R^{(n)}_{ii}(nt - s)\phi^{(n)}_{ij}(s)| ds \frac{b_{ij}}{\sigma_i} e^{\frac{bs}{\sigma_i}} dt \leq \|\phi^{(n)}_{ij}\|_{L^1} + \sqrt{T} \left( \int_0^\infty n \int_0^{nt} |R^{(n)}_{ii}(nt - s)\phi^{(n)}_{ij}(s)| ds \frac{b_{ij}}{\sigma_i} e^{\frac{bs}{\sigma_i}} dt \right)^{1/2}. \]
Here the first term on the right side of the last inequality vanishes as \( n \to \infty \); see Condition 3.1. By the convolution theorem and Condition 3.1, we have as \( n \to \infty \),

\[
\int_0^\infty e^{i\lambda t} n \int_0^{nt} \hat{R}_{ii}^{(n)}(nt - s) \phi_{ij}^{(n)}(s) ds = \frac{\hat{\phi}_{ii}^{(n)}(\frac{\lambda}{n}) \cdot n \hat{\phi}_{ij}^{(n)}(\frac{\lambda}{n})}{n(1 - \hat{\phi}_{ii}^{(n)}(\frac{\lambda}{n}))} \to -\frac{b_{ij}}{b_{ii} + i\sigma_i \lambda} = \int_0^\infty e^{i\lambda t} \frac{b_{ij}}{b_{ii} + i\sigma_i \lambda} dt.
\]

Moreover, notice that \( \sup_{n \geq 1} |n \hat{\phi}_{ij}^{(n)}(\lambda/n)| \leq \sup_{n \geq 1} n \|\phi_{ij}^{(n)}\|_{L^1} < \infty \), by Proposition 5.1 we have

\[
\sup_{n \geq 1} \left| \int_0^\infty e^{i\lambda t} n \int_0^{nt} \hat{R}_{ii}^{(n)}(nt - s) \phi_{ij}^{(n)}(s) ds \right| \leq C(\lambda)^{-1} \wedge 1.
\]

By the Fourier isometry,

\[
\int_0^\infty |n \int_0^{nt} \hat{R}_{ii}^{(n)}(nt - s) \phi_{ij}^{(n)}(s) ds - \frac{b_{ij}}{\sigma_i} e^{-\frac{\lambda u}{\sigma_i}}|^2 dt = \int_0^\infty \left| \frac{\hat{\phi}_{ii}^{(n)}(\lambda/n) \cdot n \hat{\phi}_{ij}^{(n)}(\lambda/n)}{n(1 - \hat{\phi}_{ii}^{(n)}(\lambda/n))} + \frac{b_{ij}}{b_{ii} + i\sigma_i \lambda} \right|^2 d\lambda.
\]

By the dominated convergence theorem, it vanishes as \( n \to \infty \) and the proof is completed. \( \square \)

**Lemma 5.4.** There exists a sequence \( \{\varepsilon_n\}_{n \geq 1} \) vanishing as \( n \to \infty \) such that for any \( n \geq 1 \), \( u \in \mathcal{U} \) and \( i \in \mathcal{H} \)

\[
\|\varepsilon_{R_i}^{(n)}(u)\|_{L^2} \leq \varepsilon_n \cdot \Phi(u).
\]

**Proof.** Notice that \( |\varepsilon_{R_i}^{(n)}(t, u)| \leq \phi_i(t, u) + |A_1^{(n)}(t, u)| + |A_2^{(n)}(t, u)| \), where

\[
A_1^{(n)}(t, u) := \int_0^t n \phi_i(t - s, u) \varepsilon_{R_i}^{(n)}(s) ds,
\]

\[
A_2^{(n)}(t, u) := \int_0^t n \phi_i(t - s, u) \frac{1}{\sigma_i} e^{-\frac{\lambda u}{\sigma_i}} ds - \frac{\|\phi_i(u)\|_{L^1}}{\sigma_i} e^{-\frac{\lambda u}{\sigma_i}} t.
\]

By the hypothesis (H1) we first have

\[
\int_0^\infty |\phi_i(nt, u)|^2 dt \leq \frac{\|\phi_i(u)\|_{TV} \cdot \|\phi_i(u)\|_{L^1}}{n} \leq \frac{\|\Phi(u)\|^2}{n}.
\]

Moreover, by Young’s convolution inequality and the hypothesis (H1),

\[
\|A_1^{(n)}(u)\|_{L^2} \leq \|\phi_i(u)\|_{L^1} \cdot \|\varepsilon_{R_i}^{(n)}\|_{L^2} \leq \|\varepsilon_{R_i}^{(n)}\|_{L^2} \cdot \Phi(u).
\]

Taking Fourier transform of \( A_2^{(n)}(t, u) \), we have

\[
\int_0^\infty e^{i\lambda t} A_2^{(n)}(t, u) dt = \frac{\|\phi_i(u)\|_{L^1} - \hat{\phi}_i(\lambda/n, u)}{b_{ii} + i\sigma_i \lambda}
\]

and by the Fourier isometry,

\[
\|A_2^{(n)}(u)\|_{L^2}^2 = \int_\mathbb{R} \left| \frac{\|\phi_i(u)\|_{L^1} - \hat{\phi}_i(\lambda/n, u)}{b_{ii} + i\sigma_i \lambda} \right|^2 d\lambda.
\]
From the facts that $|e^{\frac{\lambda t}{n}} - 1| \leq (|\lambda|t/n) \wedge 2$ and $|b_{ii} + i\sigma_l \lambda|^{-1} \leq C/(1 + |\lambda|)$, we also have

$$\|A_2^{(n)}(u)\|_{L^2} \leq C \int_\mathbb{R} \left| \int_0^\infty \left( \frac{|\lambda|t}{n} \wedge 1 \right) \phi_i(t, u) dt \right|^2 \frac{d\lambda}{(1 + |\lambda|)^2} \leq C \int_\mathbb{R} \left( \frac{|\lambda|}{n} \wedge 1 \right)^2 \frac{d\lambda}{(1 + |\lambda|)^2} \cdot |\Phi(u)|^2 \leq \frac{C}{n} \cdot |\Phi(u)|^2.$$ 

Putting these estimates together, we can immediately get the desired result with $\epsilon_n := C \cdot (\|\varepsilon_{R_1}^{(n)}\|_{L^2} \lor n^{-1/2})$ for some large constant $C > 0$. \hfill $\square$

5.1.2. Weak convergence of error processes. If $E_{H}^{(n)} \xrightarrow{d} 0$ in $D([0, \infty), \mathbb{R}^d)$, by Proposition 1.17(b) in [42, p.328] we have

$$E_{H}^{(n)} \xrightarrow{u.c.p.} 0 \quad \text{and hence} \int_0^t E_{H}^{(n)}(s) d\sigma \xrightarrow{u.c.p.} 0.$$ 

For the weak convergence of $E_{H}^{(n)}$ to 0, by Corollary 3.33 in [42, p.353] it suffices to prove separately that $\varepsilon_{R_i}^{(n)}$, $\varepsilon_{ij}^{(n)}$ and $\varepsilon_{ij}^{(n)}$ converge weakly to 0 in $D([0, \infty), \mathbb{R})$ as $n \to \infty$ for each $i \in H$ and $j \in \mathcal{D}$.

**Lemma 5.5.** For each $i \in H$, we have $\varepsilon_{R_i}^{(n)} \xrightarrow{u.c.p.} 0$ as $n \to \infty$.

**Proof.** Let $\varepsilon_{R_i}^{(n)}(t) := \mu_{R_i}^{(n)}(t)/n - \hat{\mu}_{R_i}^{(n)}(t)$ for $t \geq 0$. We can write $\varepsilon_{R_i}^{(n)}$ under the form

$$\varepsilon_{R_i}^{(n)}(t) = Z^{(n)}(0) \cdot n(1 - \|\phi_{R_i}^{(n)}\|_{L^1}) \int_t^\infty R_{R_i}^{(n)}(ns) ds.$$ 

By (32) and Condition 3.5, we have $\|\varepsilon_{R_i}^{(n)}\|_{L^\infty} \xrightarrow{d} 0$ and $\|R_{R_i}^{(n)} \ast \varepsilon_{R_i}^{(n)}\|_{L^\infty} \leq C\|\varepsilon_{R_i}^{(n)}\|_{L^1} \xrightarrow{d} 0$ as $n \to \infty$. By (19) with $\beta = 0$,

$$\hat{\mu}_{R_i}^{(n)}(nt) + R_{R_i}^{(n)} \ast \hat{\mu}_{R_i}^{(n)}(nt) = Z^{(n)}(0) \cdot n(1 - \|\phi_{R_i}^{(n)}\|_{L^1}) \int_t^\infty R_{R_i}^{(n)}(ns) ds.$$ 

By (18) with $\beta = 0$ and Condition 3.1, we have

$$\int_t^\infty R_{R_i}^{(n)}(ns) ds \xrightarrow{u.c.p.} \int_t^\infty \sigma^{-1} e^{b_{ii}/\sigma \cdot t} ds = \frac{1}{b_{ii}} \cdot e^{b_{ii}/\sigma \cdot t}$$

and hence $\hat{\mu}_{R_i}^{(n)}(nt) + R_{R_i}^{(n)} \ast \hat{\mu}_{R_i}^{(n)}(nt) - Z^{(n)}(0) e^{b_{ii}/\sigma \cdot t} \xrightarrow{u.c.p.} 0$. The desired result follows by putting these estimates together. \hfill $\square$

**Lemma 5.6.** There exist constants $C, \vartheta > 0$ such that for any $t \geq 0$ and $n \geq 1$,

$$E[|Z_{H}^{(n)}(t)|^{2\alpha}] \leq C e^{\vartheta t}.$$ 

**Proof.** Obviously, it suffices to prove $E[|Z_{\vartheta, H}^{(n)}(t)|^{2\alpha}] \leq C$ for any $t \geq 0$ and $n \geq 1$. Taking expectations on the both sides of (16) with $\beta = \vartheta$, we have

$$E[Z_{\vartheta, H}^{(n)}(t)] = E\left[\frac{\mu_{\vartheta, H}^{(n)}(nt)}{n}\right] + \int_0^t R_{\vartheta, H}^{(n)}(nt - s) \cdot E\left[\frac{\mu_{\vartheta, H}^{(n)}(s)}{n}\right] ds$$

where $Z_{\vartheta, H}^{(n)}(t) := \mu_{\vartheta, H}^{(n)}(nt)/n - \hat{\mu}_{\vartheta, H}^{(n)}(nt)$ for $t \geq 0$. We have

$$\|\varepsilon_{\vartheta, H}^{(n)}\|_{L^\infty} \xrightarrow{d} 0$$

and $\|R_{\vartheta, H}^{(n)} \ast \varepsilon_{\vartheta, H}^{(n)}\|_{L^\infty} \leq C\|\varepsilon_{\vartheta, H}^{(n)}\|_{L^1} \xrightarrow{d} 0$ as $n \to \infty$. By (19) with $\beta = \vartheta$,

$$\hat{\mu}_{\vartheta, H}^{(n)}(nt) + R_{\vartheta, H}^{(n)} \ast \hat{\mu}_{\vartheta, H}^{(n)}(nt) = Z_{\vartheta, H}^{(n)}(0) \cdot n(1 - \|\phi_{\vartheta, H}^{(n)}\|_{L^1}) \int_t^\infty R_{\vartheta, H}^{(n)}(ns) ds.$$ 

By (18) with $\beta = \vartheta$ and Condition 3.1, we have

$$\int_t^\infty R_{\vartheta, H}^{(n)}(ns) ds \xrightarrow{u.c.p.} \int_t^\infty \sigma^{-1} e^{b_{ii}/\sigma \cdot t} ds = \frac{1}{b_{ii}} \cdot e^{b_{ii}/\sigma \cdot t}$$

and hence $\hat{\mu}_{\vartheta, H}^{(n)}(nt) + R_{\vartheta, H}^{(n)} \ast \hat{\mu}_{\vartheta, H}^{(n)}(nt) - Z_{\vartheta, H}^{(n)}(0) e^{b_{ii}/\sigma \cdot t} \xrightarrow{u.c.p.} 0$. The desired result follows by putting these estimates together. \hfill $\square$
\[
+ \int_0^t R_{\vartheta,i}^{(n)}(ns)ds + \sum_{j \in \mathcal{H}} \int_0^t nR_{\vartheta,j}^{(n)}(n(t-s))E[Z_{\vartheta,j}^{(n)}(s)]ds.
\]

From Condition 3.5 and (32), the first two terms on the right side of this equality are uniformly bounded. Moreover, by Young’s convolution inequality,

\[
\sup_{t \geq 0} \int_0^t nR_{\vartheta,j}^{(n)}(n(t-s)) \cdot E[Z_{\vartheta,j}^{(n)}(s)]ds \leq \|R_{\vartheta,j}^{(n)}\|_{L^1} \cdot \sup_{s \geq 0} E[Z_{\vartheta,j}^{(n)}(s)].
\]

From Condition 3.1 and (17), we have as \(n \to \infty\),

\[
\|R_{\vartheta,j}^{(n)}\|_{L^1} = \frac{n\|\phi_{\vartheta,j}^{(n)}\|_{L^1}}{n(1 - \|\phi_{\vartheta,j}^{(n)}\|_{L^1})} \to \frac{\|\vartheta\|}{\sigma_i \vartheta - b_{ii}} < \infty.
\]

Choosing \(\vartheta\) large enough such that \(\|R_{\vartheta,j}^{(n)}\|_{L^1} \leq \frac{1}{2d}\) for any \(n \geq 1\), we have

\[
\sup_{t \geq 0} E[Z_{\vartheta,j}^{(n)}(t)] \leq C + \frac{1}{2d} \sum_{j \in \mathcal{H}} \sup_{s \geq 0} E[Z_{\vartheta,j}^{(n)}(s)]
\]

and hence

\[
\sum_{i \in \mathcal{H}} \sup_{t \geq 0} E[Z_{\vartheta,i}^{(n)}(t)] \leq C + \frac{1}{2} \sum_{j \in \mathcal{H}} \sup_{s \geq 0} E[Z_{\vartheta,j}^{(n)}(s)],
\]

which immediately induces that \(\sup_{t \geq 0} E[|Z_{\vartheta,H}^{(n)}(t)|] \leq C\). Here the constant \(C\) is independent of \(n\) and \(t\). We now start to give an upper bound for the second moment. Squaring both sides of (16), using the Cauchy-Schwarz inequality and then taking expectations, we have

\[
E[|Z_{\vartheta,i}^{(n)}(t)|^2] \leq 2^{d+2} E\left[\left(\frac{\mu_{\vartheta,i}^{(n)}(nt)}{n}\right)^2\right] + 2^{d+2} \left|\int_0^t R_{\vartheta,i}^{(n)}(ns)ds\right|^2
\]

\[
+ 2^{d+2} E\left[\int_0^{nt} R_{\vartheta,i}^{(n)}(nt - s) \cdot \frac{\mu_{\vartheta,i}^{(n)}(s)}{n} ds\right]^2
\]

\[
+ \sum_{j \in \mathcal{H}} 2^{d+2} E\left[\left|\int_0^t nR_{\vartheta,j}^{(n)}(n(t-s))Z_{\vartheta,j}^{(n)}(s)ds\right|^2\right]
\]

\[
+ \sum_{j = 1}^d 2^{d+2} E\left[\left|\int_0^t \int_\mathbb{U} R_{\vartheta,i}^{(n)}(n(t-s), u) \frac{e^{-\vartheta s}}{n} \tilde{N}_j^{(n)}(n \cdot ds, du)\right|^2\right].
\]

Like the previous argument, we can prove that the first and third terms on the right side of this inequality are uniformly bounded. Notice that the integrand in the stochastic integral satisfies that for any \(u \in \mathbb{U}\),

\[
\sup_{n \geq 1} \|R_{\vartheta,i}^{(n)}(u)\|_{L^\infty} \leq \|\phi_i(u)\|_{TV} + C \cdot \|\phi(u)\|_{L^1} \leq C \cdot \Phi(u).
\]

By the Burkholder-Davis-Gundy inequality and the uniform bound for the first moment of \(\{Z_{\vartheta,H}^{(n)}\}_{n \geq 1}\),

\[
E\left[\left|\int_0^t \int_\mathbb{U} R_{\vartheta,i}^{(n)}(n(t-s), u) \frac{e^{-\vartheta s}}{n} \tilde{N}_j^{(n)}(n \cdot ds, du)\right|^2\right] \leq C \int_0^t e^{-\vartheta s} E[Z_{\vartheta,j}^{(n)}(s)] ds \int \Phi(u)^2 \mu_j^{(n)}(du) \leq C.
\]
Here the constant $C > 0$ is independent of $n$ and $t$. Moreover, by Hölder’s inequality and then Young’s convolution inequality,
\[
\sup_{t \geq 0} E \left[ \left| \int_0^t nR_{\vartheta,ij}^{(n)} (n(t-s))Z_{\vartheta,j}^{(n)}(s)ds \right|^2 \right] \\
\leq \left\| R_{\vartheta,ij}^{(n)} \right\|_{L^1} \cdot \sup_{t \geq 0} \int_0^t nR_{\vartheta,ij}^{(n)} (n(t-s))E \left[ \left| Z_{\vartheta,j}^{(n)}(s) \right|^2 \right] ds \\
\leq \left\| R_{\vartheta,ij}^{(n)} \right\|_{L^1}^2 \cdot \sup_{t \geq 0} E \left[ \left| Z_{\vartheta,j}^{(n)}(t) \right|^2 \right].
\]

Putting all estimates above together, we also have
\[
\sup_{t \geq 0} E \left[ \left| Z_{\vartheta,i}^{(n)}(t) \right|^2 \right] \leq C + \sum_{j \in \mathcal{H}} 2^{2d+2} \left\| R_{\vartheta,ij}^{(n)} \right\|_{L^1}^2 \cdot \sup_{t \geq 0} E \left[ \left| Z_{\vartheta,j}^{(n)}(t) \right|^2 \right].
\]

From (35), we choose $\vartheta > 0$ large enough such that $2^{2d+2} \left\| R_{\vartheta,ij}^{(n)} \right\|_{L^1}^2 \leq \frac{1}{4d}$ and then
\[
\sum_{i=1}^d \sup_{t \geq 0} E \left[ \left| Z_{\vartheta,i}^{(n)}(t) \right|^2 \right] \leq C + \frac{1}{2} \sum_{j=1}^d \sup_{t \geq 0} E \left[ \left| Z_{\vartheta,j}^{(n)}(t) \right|^2 \right].
\]

These induce that $\sup_{t \geq 0} E \left[ \left| Z_{\vartheta,i}^{(n)}(t) \right|^2 \right] \leq C$ with the constant $C$ independent of $n$ and $t$. Similarly, we also can prove that for some $\vartheta > 0$,
\[
\sup_{n \geq 1} \sup_{t \geq 0} E \left[ \left| Z_{\vartheta,i}^{(n)}(t) \right|^{2\alpha} \right] \leq C.
\]

\begin{lemma}
For $i, j \in \mathcal{H}$ with $i \neq j$, we have $\tilde{\varepsilon}_{ij}^{(n)} \uhr 0$ and $\tilde{\varepsilon}_{ij}^{(n)} \uhr 0$ as $n \to \infty$.
\end{lemma}

\begin{proof}
The first convergence follows directly from Lemma 5.3. For the second one, by Hölder’s inequality we have for any $T > 0$,
\begin{equation}
\sup_{t \in [0,T]} \left| \frac{\varepsilon_{ij}^{(n)}}{\varepsilon_{R_{ij}}^{(n)}} \right|^{\alpha} \leq \left\| \varepsilon_{R_{ij}}^{(n)} \right\|_{L^1}^{\alpha-1} \cdot \sup_{t \in [0,T]} \varepsilon_{ij}^{(n)} \ast Z_j^{(n)}(t).
\end{equation}

By Young’s inequality and Lemma 5.6,
\[
E \left[ \sup_{t \in [0,T]} \left| \frac{\varepsilon_{ij}^{(n)}}{\varepsilon_{R_{ij}}^{(n)}} \right| \ast \left| Z_j^{(n)}(t) \right|^\alpha \right]^2 \leq \left\| \varepsilon_{R_{ij}}^{(n)} \right\|_{L^2}^2 \cdot \int_0^{2T} E \left[ \left| Z_j^{(n)}(s) \right|^{2\alpha} \right] ds \leq C \left\| \varepsilon_{R_{ij}}^{(n)} \right\|_{L^2}^2.
\]

From (13), (32) and Condition 3.1, there exists a constant $C > 0$ independent of $n$ such that
\[
\int_0^{2T} \left| nR_{ij}^{(n)} (ns) \right|^2 ds \leq Cn \int_0^\infty \left| \phi_{ij}^{(n)}(s) \right|^2 ds + C \int_0^{2T} \left| nR_{ij}^{(n)} * \phi_{ij}^{(n)}(nt) \right|^2 dt \leq Cn \left\| \phi_{ij}^{(n)} \right\|_{L^1} \cdot \left\| \phi_{ij}^{(n)} \right\|_{TV} + Cn^2 \left\| \phi_{ij}^{(n)} \right\|_{L^1}^2 < C
\]

and hence $\sup_{n \geq 1} \left\| \varepsilon_{R_{ij}}^{(n)} \right\|_{L^2} < \infty$. Taking this back into (37), from Lemma 5.3 we have as $n \to \infty$,
\[
E \left[ \sup_{t \in [0,T]} \left| \frac{\varepsilon_{ij}^{(n)}}{\varepsilon_{R_{ij}}^{(n)}} \right|^{2\alpha} \right] \leq C \left\| \varepsilon_{R_{ij}}^{(n)} \right\|_{L^2}^{2\alpha-2} \to 0
\]

and this proof is completed.
\end{proof}
For each $i \in \mathcal{H}$ and $j \in \mathcal{D}$, we now start to prove the error $\varepsilon_{ij}^{(n)}$ vanishes as $n \to \infty$. Like the proof of Lemma 5.7 in [36], we can prove the following moment estimate for the stochastic integral driven by MHPI-measures by using the Burkholder-Davis-Gundy inequality.

**Proposition 5.8.** For any $T > 0$, there exists a constant $C > 0$ such that for any $\kappa \in [1, \alpha]$, $i \in \mathcal{D}$, $r, h \in [0, T]$ and measurable function $f(t, s, u)$ defined on $\mathbb{R}^2_+ \times U$,

$$
\mathbb{E} \left[ \left| \int_r^{r+h} \int_U f(t, s, u) \tilde{N}_i^{(n)}(n \cdot ds, du) \right|^{2\kappa} \right] \\
\leq C n^2 \int_U \nu_i^{(n)}(du) \int_r^{r+h} |f(t, s, u)|^{2\kappa} ds \\
+ C n^2 \int_U \nu_i^{(n)}(du) \int_r^{r+h} |f(t, s, u)|^2 ds,
$$

(38)

**Corollary 5.9.** For each $i \in \mathcal{H}$ and $j \in \mathcal{D}$, we have $\varepsilon_{ij}^{(n)} \xrightarrow{L_d} 0$ as $n \to \infty$.

**Proof.** Applying (38) together with Lemma 5.4 and (H1) to $\mathbb{E} \left[ |\varepsilon_{ij}^{(n)}(t)|^2 \right]$, we have for any $T > 0$,

$$
\sup_{t \in [0, T]} \mathbb{E} \left[ |\varepsilon_{ij}^{(n)}(t)|^2 \right] \leq C \int_U \|\varphi_i^{(n)}(u)\|_{L^1} \frac{b_{ij}(t-s)}{\sigma_i} \frac{1}{n} \tilde{N}_i^{(n)}(n \cdot ds, du),
$$

which goes to 0 as $n \to \infty$ and the desired result follows. \hfill \Box

We now start to prove the tightness of the sequence $\{\varepsilon_{ii}^{(n)}\}_{n \geq 1}$. The tightness of other sequences can be proved similarly. By Corollary 3.33 in [42, p. 353] and the definition of $\varepsilon_{ii}^{(n)}$, it suffices to prove that $\{I_{ii}^{(n)}\}_{n \geq 1}$ is tight and $\{J_{ii}^{(n)}\}_{n \geq 1}$ is $C$-tight, where

$$
I_{ii}^{(n)}(t) := \int_0^t \int_U R_i^{(n)}(n(t-s), u) \frac{1}{n} \tilde{N}_i^{(n)}(n \cdot ds, du),
$$

(39)

$$
J_{ii}^{(n)}(t) := \int_0^t \int_U \frac{\|\varphi_i(u)\|_{L^1}}{\sigma_i} \frac{b_{ii}(t-s)}{n} \frac{1}{n} \tilde{N}_i^{(n)}(n \cdot ds, du),
$$

(40)

From the fact that $\exp \left\{ \frac{b_{ii}(t-s)}{\sigma_i} \right\} = 1 + \frac{b_{ii}}{\sigma_i} \int_s^t \exp \left\{ \frac{b_{ii}}{\sigma_i} (r-s) \right\} dr$, we can write $J_{ii}^{(n)}$ as

$$
J_{ii}^{(n)}(t) = \frac{b_{ii}}{\sigma_i} \int_0^t J_{ii}^{(n)}(s) ds - \int_0^t \int_U \frac{\|\varphi_i(u)\|_{L^1}}{\sigma_i^n} \frac{1}{n} \tilde{N}_i^{(n)}(n \cdot ds, du), \quad t \geq 0.
$$

Obviously, $J_{ii}^{(n)}$ is an $(\mathcal{F}_n)$-semimartingale.

**Proposition 5.10.** The sequence $\{J_{ii}^{(n)}\}_{n \geq 1}$ is $C$-tight.

**Proof.** As a preparation, we firstly give some moment estimates for $J_{ii}^{(n)}$. The exists a constant $C > 0$ such that for any $n \geq 1$ and $T > 0$,

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |J_{ii}^{(n)}(t)|^2 \right] \leq CT \int_0^T \mathbb{E} \left[ \sup_{s \in [0, t]} |J_{ii}^{(n)}(s)|^2 \right] dt \\
+ CE \left[ \int_0^T \frac{\|\varphi_i(u)\|_{L^1}^2}{\sigma_i^n} \frac{1}{n^2} \tilde{N}_i^{(n)}(n \cdot ds, du) \right]
$$
Here the first inequality follows from Hölder’s inequality and (38). The second one follows from Lemma 5.6 and (H1). By Gronwall’s inequality,

\[
\sup_{n \geq 1} E \left[ \sup_{t \in [0,T]} |J_{ii}^{(n)}(t)|^2 \right] < \infty.
\]

We now prove the tightness of \( \{ J_{ii}^{(n)} \} \). For any bounded stopping time \( \tau \leq T \) and \( h \in (0,1) \), we have

\[
E \left[ | \Delta_h J_{ii}^{(n)}(\tau) |^2 \right] \leq C E \left[ \int_{\tau}^{\tau+h} |J_{ii}^{(n)}(s)|^2 \, ds \right]
\]

\[
+ C E \left[ \int_{\tau}^{\tau+h} \left( \int_{[n]} \frac{\| \phi_i(u) \|_{L^1}}{\sigma_i n} \hat{N}^{(n)}_i(n \cdot ds, du) \right)^2 \, ds \right].
\]

From Hölder’s inequality and (41), we have

\[
E \left[ \int_{\tau}^{\tau+h} |J_{ii}^{(n)}(s)|^2 \, ds \right] \leq \frac{h}{1} E \left[ \int_{\tau}^{\tau+h} |J_{ii}^{(n)}(s)|^2 \, ds \right]
\]

\[
\leq h \int_{\tau}^{\tau+1} E \left[ |J_{ii}^{(n)}(s)|^2 \right] \, ds \leq C h.
\]

Moreover, applying (38) again to the last expectation in (42), it can be bounded by

\[
C E \left[ \int_{\tau}^{\tau+h} \left( \int_{[n]} \frac{\| \phi_i(u) \|_{L^1}^2}{n^2} \hat{N}^{(n)}_i(n \cdot ds, du) \right) \right] \leq C E \left[ \int_{\tau}^{\tau+h} \left( \int_{[n]} Z_i^{(n)}(s) \, ds \right) \cdot \left( \int_{[n]} \| \phi_i(u) \|_{L^1}^2 \nu_i^{(n)}(du) \right) \right].
\]

Applying Hölder’s inequality, Jensen’s inequality and then using Lemma 5.6, we also have

\[
E \left[ \int_{\tau}^{\tau+h} Z_i^{(n)}(s) \, ds \right] \leq \sqrt{h} E \left[ \left( \int_{\tau}^{\tau+h} |Z_i^{(n)}(s)|^2 \, ds \right)^{1/2} \right]
\]

\[
\leq \sqrt{h} \left( \int_{\tau}^{\tau+1} E \left[ |Z_i^{(n)}(s)|^2 \right] \, ds \right)^{1/2} \leq C \sqrt{h}.
\]

Putting all estimates above together, we have \( E[| \Delta_h J_{ii}^{(n)}(\tau) |^2] \leq C \sqrt{h} \) with the constant \( C \) independent of \( n \) and \( \tau \). The criterion of Aldous; see [2], yields the tightness of \( \{ J_{ii}^{(n)} \} \) directly.

It remains to prove the continuity of cluster points. For any cluster point \( J_{ii}^* \), it suffices to prove that \( E[\sum_{t \in [0,T]} |\Delta J_{ii}^*(t)|^{2\alpha}] = 0 \) for any \( T > 0 \). There exists a subsequence of \( \{ J_{ii}^{(n)} \} \), still denoted by itself, such that \( J_{ii}^{(n)} \overset{d}{\to} J_{ii}^* \) in \( D([0, \infty), \mathbb{R}) \). For \( \epsilon > 0 \), let \( g_\epsilon \) be a continuous function on \( \mathbb{R} \) vanishing in a neighborhood of 0 and satisfying that \( g_\epsilon(x) \) increases to \( |x|^{2\alpha} \) as \( \epsilon \to 0 \) for any \( x \in \mathbb{R} \). By the monotone convergence theorem and then Proposition 3.16 in [42, p.349],

\[
E \left[ \sum_{t \in [0,T]} |\Delta J_{ii}^*(t)|^{2\alpha} \right] = \lim_{\epsilon \to 0^+} E \left[ \sum_{t \in [0,T]} g_\epsilon(\Delta J_{ii}^*(t)) \right]
\]

\[
= \lim_{\epsilon \to 0^+} \lim_{n \to \infty} E \left[ \sum_{t \in [0,T]} g_\epsilon(\Delta J_{ii}^{(n)}(t)) \right]
\]

\[
\leq \lim_{\epsilon \to 0^+} \lim_{n \to \infty} E \left[ \sum_{t \in [0,T]} |\Delta J_{ii}^{(n)}(t)|^{2\alpha} \right] \leq C \sqrt{h}.
\]
\[
\leq \lim_{n \to \infty} \mathbb{E} \left[ \sum_{t \in [0,T]} \left| \Delta_n f_{ii}^{(n)}(t) \right|^{2\alpha} \right].
\]

By the hypothesis (H1) and the properties of stochastic integrals with respect to a random point measure,
\[
\mathbb{E} \left[ \sum_{t \in [0,T]} \left| \Delta_n f_{ii}^{(n)}(t) \right|^{2\alpha} \right] = \mathbb{E} \left[ \int_0^T \int_\mathcal{U} \frac{\phi_i(u)}{|\sigma_i u|^{2\alpha}} N_i^{(n)}(n \cdot ds, du) \right] \leq \frac{C}{n^{2\alpha - 2}},
\]

which goes to 0 as \( n \to \infty \) and hence the sequence \( \{ J_{ii}^{(n)} \}_{n \geq 1} \) is \( C \)-tight.

We now start to prove the tightness of \( \{ J_{ii}^{(n)} \}_{n \geq 1} \). For some \( \theta > 2 \), let \( I_{ii,\theta}^{(n)} \) be a linear interpolation of \( I_{ii}^{(n)} \) defined as follows

\[
I_{ii,\theta}^{(n)}(t) := I_{ii}^{(n)} \left( \frac{[n \theta t]}{n^\theta} \right) + \left( n^\theta t - [n \theta t] \right) \left[ I_{ii}^{(n)} \left( \frac{[n \theta t] + 1}{n^{\theta}} \right) - I_{ii}^{(n)} \left( \frac{[n \theta t]}{n^{\theta}} \right) \right],
\]

for \( t \geq 0 \). We now start to prove that the sequence \( \{ I_{ii,\theta}^{(n)} \}_{n \geq 1} \) is tight and a good approximation for \( \{ I_{ii}^{(n)} \}_{n \geq 1} \). This will induce the tightness of \( \{ I_{ii}^{(n)} \}_{n \geq 1} \) immediately. As a preparation, the next proposition gives some upper bound estimates for the shifted resolvent.

**Proposition 5.11.** For any \( \kappa \geq 1 \), there exists a constant \( C > 0 \) such that for any \( u \in \mathcal{U} \) and \( h \in [0,1] \),

\[
\sup_{n \geq 1} \int_0^\infty \left| \Delta_nh R_i^{(n)}(nt, u) \right|^{2\kappa} dt \leq C |\Phi(u)|^{2\kappa} \cdot h.
\]

**Proof.** We first prove this result with \( \kappa = 1 \). By the Fourier isometry,

\[
\int_0^\infty \left| \Delta_h R_i^{(n)}(nt, u) \right|^2 dt = \int_\mathbb{R} \left| (e^{i \lambda h} - 1) \int_\mathbb{R} e^{i \lambda t} R_i^{(n)}(nt, u) dt \right|^2 d\lambda.
\]

Similarly as in the proof of Lemma 5.2, we can prove that for any \( u \in \mathcal{U} \) and \( \lambda \in \mathbb{R} \),

\[
\left| \int_0^\infty e^{i \lambda t} \phi_i^{(n)}(t, u) dt \right| + \left| \int_0^\infty e^{i \lambda t} R_i^{(n)}(nt, u) dt \right| \leq C \cdot |\Phi(u)| \left( \frac{1}{|\lambda|} \wedge 1 \right).
\]

From this and the fact that \( |e^{i \lambda h} - 1| \leq |\lambda h| \wedge 2 \),

\[
\int_0^\infty \left| \Delta_h R_i^{(n)}(nt, u) \right|^2 dt \leq C |\Phi(u)|^2 \cdot \int_\mathbb{R} \left( |\lambda h|^2 \wedge 1 \right) \left( \frac{1}{|\lambda|^2} \wedge 1 \right) d\lambda.
\]

A simple calculation deduces that the last integral can be bounded by \( 6h \) and hence the inequality (44) holds for \( \kappa = 1 \). When \( \kappa > 1 \), we have

\[
\int_0^\infty \left| \Delta_h R_i^{(n)}(nt, u) \right|^{2\kappa} dt \leq 2^{2\kappa - 2} \left\| R_i^{(n)}(u) \right\|_{L^\infty}^{2\kappa - 2} \int_0^\infty \left| \Delta_h R_i^{(n)}(nt, u) \right|^2 dt.
\]

Then (44) with \( \kappa > 1 \) follows directly from (36) and the previous result.

**Proposition 5.12.** We have \( |I_{ii}^{(n)} - I_{ii,\theta}^{(n)}| \overset{u.c.-P.}{\to} 0 \) as \( n \to \infty \).
Proof. Here we just prove \( |I_{i,i}^{(n)}(t) - I_{i,i,\theta}^{(n)}(t)| \rightarrow 0 \) uniformly on \([0, 1]\). From the definition of \( I_{i,i,\theta} \) and the triangle inequality, we have for any \( k \geq 0 \) and \( t \in [k/n, (k + 1)/n] \),

\[
|I_{i,i}^{(n)}(t) - I_{i,i,\theta}^{(n)}(t)| \leq |I_{i,i}^{(n)}(t) - I_{i,i}^{(n)}(kn^{-\theta})| + |I_{i,i}^{(n)}(t) - I_{i,i}^{(n)}((k + 1)n^{-\theta})|,
\]

which can be bounded by \( 3 \sup_{h \leq n^{-\theta}} |\Delta_h I_{i,i}^{(n)}(kn^{-\theta})| \) and hence

\[
\sup_{t \in [0, 1]} |I_{i,i}^{(n)}(t) - I_{i,i,\theta}^{(n)}(t)| \leq 3 \sup_{k=0,\ldots,[n^\theta];h \leq n^{-\theta}} |\Delta_h I_{i,i}^{(n)}(kn^{-\theta})|.
\]

For any \( u \in \mathbb{U} \), since \( \|\phi_i(u)\|_{TV} < \infty \), we have \( \phi_i(t, u) = \phi_i^+(t, u) - \phi_i^-(t, u) \), where \( \phi_i^+(t, u) \) and \( \phi_i^-(t, u) \) are two non-negative, non-decreasing functions\(^6\) on \( \mathbb{R} \) with \( \phi_i^+(t, u) = \phi_i^+(0, u) \) for \( t < 0 \). From (39), we have \( |\Delta_h I_{i,i}^{(n)}(t)| \leq A_1(n, t, h) + A_2(n, t, h) + A_3(n, t, h) + A_4(n, t, h) \) for any \( t, h \in [0, 1] \) with

\[
A_1(n, t, h) := n \int_{\mathbb{U}} \nu_i^{(n)}(du) \int_0^{t+h} Z_i^{(n)}(s) |\Delta_{nh} R_i^{(n)}(n(t-s), u)| ds,
\]

\[
A_2(n, t, h) := \int_0^t \int_{\mathbb{U}} \left| \frac{\Delta_{nh} \left( R_i^{(n)} \ast \nu_i(n(t-s), u) \right)}{n} \right| K_i^{(n)}(n, ds, du),
\]

\[
A_3(n, t, h) := \int_0^{t+h} \int_{\mathbb{U}} \left| \frac{\Delta_{nh} \phi_i^+(n(t-s), u)}{n} \right| K_i^{(n)}(n, ds, du),
\]

\[
A_4(n, t, h) := \int_0^{t+h} \int_{\mathbb{U}} \left| \frac{\Delta_{nh} \phi_i^-(n(t-s), u)}{n} \right| K_i^{(n)}(n, ds, du).
\]

Thus it suffices to prove that for any \( \eta > 0 \) and \( j \in \{1, 2, 3, 4\} \),

\[
\lim_{n \to \infty} P \left( \sup_{k=0,\ldots,[n^\theta];h \leq n^{-\theta}} A_j^{(n)}(kn^{-\theta}, h) \geq \eta \right) = 0.
\]

In the sequel of this proof, the constant \( C > 0 \) is independent of \((n, t, u, h)\) and may vary from line to line.

Step 1. We first prove (47) with \( j = 1 \). By Young’s convolution inequality and Proposition 5.11,

\[
\sup_{t \in [0, 1]} \left| \int_0^{t+h} Z_i^{(n)}(s) \cdot |\Delta_{nh} R_i^{(n)}(n(t-s), u)| ds \right|^2 \leq \int_0^2 |Z_i^{(n)}(r)|^2 dr \int_0^\infty \left| \Delta_{nh} R_i^{(n)}(ns, u) \right|^2 ds \leq C \int_0^2 \left| Z_i^{(n)}(s) \right|^2 ds \cdot \Phi(u)^2 \cdot h
\]

and hence

\[
\sup_{h \leq n^{-\theta}, t \in [0, 1]} |A_1^{(n)}(t, h)|^2 \leq C \int_0^2 \left| Z_i^{(n)}(s) \right|^2 ds \cdot n^{2-\theta}.
\]

\(^6\) For any non-negative function \( f \) on \( \mathbb{R}_+ \) with \( \|f\|_{TV} < \infty \), by the Jordan decomposition there exist two non-negative, non-decreasing functions \( f_0^+ \) and \( f_0^- \) on \( \mathbb{R}_+ \) such that \( f = f_0^+ - f_0^- \), \( f^+_0 \geq 1 \) and \( \|f\|_{TV} = f_0^+(\infty) - f_0^-(0) + f_0^-(\infty) - f_0^-(0) < \infty \). Thus \( f = f^+ + f^- \), where \( f^+ := \|f\|_{TV} - f_0^- \) and \( f^- := \|f\|_{TV} - f_0^- \) are non-negative, non-increasing on \( \mathbb{R}_+ \).
From Chebyshev’s inequality and Lemma 5.6,
\[ P\left( \sup_{k=0,\ldots,[n^θ]:h<n^{-θ}} A_1^{(n)}(kn^{-θ}, h) \geq η \right) \leq \frac{1}{η^2} E\left[ \sup_{h \leq n^{-θ}, t \in [0,1]} A_1^{(n)}(t, h) \right]^2 \leq \frac{C}{η^2} n^{2-θ}, \]
which vanishes as \( n \to ∞ \) since \( θ > 2 \).

**Step 2.** We now prove (47) with \( j = 2 \). By (32), we have for any \( t, h \in [0,1] \),
\[ |Δ_{nh}(R_{ii}^{(n)} * φ_i(nt, u))| \leq \int_0^{nt} R_{ii}^{(n)}(s)|Δ_{nh}φ_i(nt-s, u)|ds \]
\[ + \int_{nt}^{(n+t)h} R_{ii}^{(n)}(s)φ_i(n(t+h)-s, u)ds \]
\[ \leq C \int_0^{nh} φ_i(s, u)ds + C \int_0^{nt} |φ_i(nh+s, u)-φ_i(s, u)|ds, \]
The first term on the right side of the last inequality can be bounded by \( C\|φ_i(u)\|_{TV} \cdot nh \). By the preceding decomposition of \( φ_i \), the second term can be bounded by
\[ \int_0^{nt} [φ_i^+(s, u) - φ_i^-(nh+s, u)]ds + \int_0^{nt} [φ_i^-(s, u) - φ_i^-(nh+s, u)]ds, \]
which can be bounded by \( 4\|φ_i(u)\|_{TV} \cdot nh \). Putting these estimates together, we have
\[ |Δ_{nh}(R_{ii}^{(n)} * φ_i)(nt, u)| \leq C\|φ_i(u)\|_{TV} \cdot nh \]
and hence
\[ \sup_{t \in [0,1]:h \leq n^{-θ}} A_2^{(n)}(t, h) \leq \frac{C}{n^θ} \int_0^2 \|φ_i(u)\|_{TV} N_i^{(n)}(n \cdot ds, du). \]

By Chebyshev’s inequality and hypothesis (H1),
\[ P\left( \sup_{k=0,\ldots,[n^θ]:h \leq n^{-θ}} A_2^{(n)}(kn^{-θ}, h) \geq η \right) \leq \frac{1}{η^2} E\left[ \sup_{t \in [0,1]:h \leq n^{-θ}} A_2^{(n)}(t, h) \right] \leq \frac{C}{η^2} \cdot n^{2-θ}, \]
which goes to 0 as \( n \to ∞ \) since \( θ > 2 \).

**Step 3.** We now prove (47) with \( j = 3 \). For the case of \( j = 4 \), it can be proved in the same way. Notice that \( sup_{h \leq n^{-θ}} A_3^{(n)}(t, h) = A_{3,1}^{(n)}(t) + A_{3,2}^{(n)}(t) \) with
\[ A_{3,1}^{(n)}(t) := n \int_U ν_i^{(n)}(du) \int_0^{t+n^{-θ}} Z_i^{(n)}(s) \cdot |Δ_{n^{-θ}φ_i^+(n(t-s), u)|ds, \]
\[ A_{3,2}^{(n)}(t) := \int_0^{t+n^{-θ}} \int_U \frac{|Δ_{n^{-θ}φ_i^+(n(t-s), u)|}{n} N_i^{(n)}(n \cdot ds, du). \]

By Young’s convolution inequality, we can bound \( sup_{t \in [0,1]} A_{3,1}^{(n)}(t) \) by
\[ n \int_U ν_i^{(n)}(du) \cdot \left( \int_0^2 |Δ_{n^{-θ}φ_i^+(ns, u)|^2 ds \right)^{1/2} \cdot \left( \int_0^2 Z_i^{(n)}(s)^2 ds \right)^{1/2}. \]

Since \( φ_i^+ \) is non-increasing, we have
\[ \int_0^2 |Δ_{n^{-θ}φ_i^+(ns, u)|^2 ds \leq 2\|φ_i(u)\|_{TV} \int_0^2 [φ_i^+(ns, u) - φ_i^+(n(s+n^{-θ}), u)]ds \]
\[ = 2\|φ_i(u)\|_{TV} \left( \int_0^2 φ_i^+(ns, u)ds - \int_{n^{-θ}}^{2+n^{-θ}} φ_i^+(ns, u)ds \right) \]
\[ \leq \frac{4}{n^θ} \cdot \|φ_i(u)\|_{TV}^2. \]
and hence by hypothesis (H1),

\[ \sup_{t \in [0, 1]} |A_{3,1}^{(n)}(t)|^2 \leq \frac{C}{n^{\theta/2}} \cdot \int_0^2 |Z_i^{(n)}(s)|^2 ds. \]

Applying Chebyshev’s inequality again, we have as \( n \to \infty \),

\[ P \left( \sup_{t \in [0, 1]} A_{3,1}^{(n)}(t) \geq \eta \right) \leq \frac{C}{\eta^2} \mathbb{E} \left[ \sup_{t \in [0, 1]} |A_{3,1}^{(n)}(t)|^2 \right] \leq \frac{C}{\eta^2} \frac{1}{n^{\theta/2}} \to 0. \]

Applying (38) to \( A_{3,2}^{(n)}(t) \), similarly as in (49) we also have

\[ \sup_{t \in [0, 1]} \mathbb{E} \left[ |A_{3,2}^{(n)}(t)|^{2\alpha} \right] \leq C \left( \int_U \nu_i^{(n)}(du) \int_0^{2\alpha} |\Delta_{n^{1-\theta}} \phi_i^+(ns, u)|^2 ds \right)^\alpha \]

\[ + C n^{2-2\alpha} \int_U \nu_i^{(n)}(du) \int_0^{2\alpha} |\Delta_{n^{1-\theta}} \phi_i^+(ns, u)|^{2\alpha} ds \]

\[ \leq C \cdot (n^{-\alpha \theta} + n^{2-2\alpha}). \]

From this and Chebyshev’s inequality,

\[ P \left( \sup_{k=0, \ldots, [n^\theta]} |A_{3,2}^{(n)}(kn^{-\theta})| \geq \eta \right) \leq \frac{1}{\eta^{2\alpha}} \sum_{k=0}^{[n^\theta]} \mathbb{E} \left[ |A_{3,2}^{(n)}(kn^{-\theta})|^{2\alpha} \right] \]

\[ \leq \frac{C}{\eta^{2\alpha}} (n^{\theta(1-\alpha)} + n^{2-2\alpha}), \]

which vanishes as \( n \to \infty \) since \( \alpha \in (1, 2) \).

\[ \square \]

**Proposition 5.13.** For any \( T > 0 \), there exists a constant \( C > 0 \) such that for any \( h \in [0, 1] \) and \( n \geq 1 \),

\[ \sup_{t \in [0, T]} \mathbb{E} \left[ |\Delta_h I_{ii}^{(n)}(t)|^{2\alpha} \right] \leq C \cdot (n^{2-2\alpha} \cdot h + h^\alpha). \]

**Proof.** Applying the inequality (38) to \( \Delta_h I_{ii}^{(n)}(t) \) and then using Proposition 5.11,

\[ \mathbb{E} \left[ |\Delta_h I_{ii}^{(n)}(t)|^{2\alpha} \right] \leq C \left( \int_U \nu_i^{(n)}(du) \int_0^\infty |\Delta_n h R_i^{(n)}(ns, u)|^2 ds \right)^\alpha \]

\[ + C n^{2-2\alpha} \int_U \nu_i^{(n)}(du) \int_0^\infty |\Delta_n h R_i^{(n)}(ns, u)|^{2\alpha} ds \]

\[ \leq C h^\alpha \left( \int_U |\Phi(u)|^2 \nu_i^{(n)}(du) \right)^\alpha + \frac{C h}{n^{2\alpha-2}} \int_U |\Phi(u)|^{2\alpha} \nu_i^{(n)}(du) \]

and the desired result follows directly from hypothesis (H1).

\[ \square \]

**Proposition 5.14.** The sequence \( \{ I_{ii,\theta}^{(n)} \}_{n \geq 1} \) is tight.

**Proof.** From Proposition 10.3 in [23, p.149], it suffices to prove that there exist constants \( C > 0 \) and \( \epsilon \in (0, (2\alpha - 2)/\theta) \) such that for any \( t, h \in [0, 1] \),

\[ \sup_{n \geq 1} \mathbb{E} \left[ |\Delta_h I_{ii,\theta}^{(n)}(t)|^{2\alpha} \right] \leq C \cdot h^{1+\epsilon}. \]
If \(jn^{-\theta} \leq t < t + h \leq (j + 1)n^{-\theta}\) for some \(j \geq 0\), from (43) and Proposition 5.13 we have 
\[
\Delta_h \tilde{I}^{(n)}_{ii,\theta}(t) = n^{\theta}h \cdot \Delta_{n^{-\theta}} \tilde{I}^{(n)}_{ii}(jn^{-\theta})
\]
and 
\[
E \left[ \left| \Delta_h I^{(n)}_{ii,\theta}(t) \right|^{2\alpha} \right] = n^{2\alpha\theta}h^{2\alpha}E \left[ \left| \Delta_{n^{-\theta}} \tilde{I}^{(n)}_{ii}(jn^{-\theta}) \right|^{2\alpha} \right] 
\leq Cn^{2\alpha\theta - 2\alpha - \theta + 2\alpha}h^{2\alpha} \leq Ch^{1+\epsilon}.
\]
Here the constant \(C > 0\) is independent of \(n\) and \(h\). Similarly, if \(jn^{-\theta} \leq t \leq (j + 1)n^{-\theta} \leq t + h \leq (j + 2)n^{-\theta}\) for some \(j \geq 0\), we also have

\[
E \left[ \left| \Delta_h I^{(n)}_{ii,\theta}(t) \right|^{2\alpha} \right] \leq C \epsilon E \left[ \left| I^{(n)}_{ii,\theta}(t + h) - I^{(n)}_{ii,\theta}((i + 1)n^{-\theta}) \right|^{2\alpha} \right] 
+ C \epsilon E \left[ \left| I^{(n)}_{ii,\theta}((i + 1)n^{-\theta}) - I^{(n)}_{ii,\theta}(t) \right|^{2\alpha} \right] \leq C \cdot h^{1+\epsilon}.
\]
Finally, if \(jn^{-\theta} \leq t \leq (j + 1)n^{-\theta} \) and \(ln^{-\theta} \leq t + h \leq (l + 1)n^{-\theta}\) for some \(j < l\), we have

\[
E \left[ \left| \Delta_h \tilde{I}^{(n)}_{ii,\theta}(t) \right|^{2\alpha} \right] \leq C \epsilon E \left[ \left| I^{(n)}_{ii,\theta}(t + h) - I^{(n)}_{ii,\theta}(l \cdot n^{-\theta}) \right|^{2\alpha} \right] 
+ C \epsilon E \left[ \left| I^{(n)}_{ii,\theta}(l \cdot n^{-\theta}) - I^{(n)}_{ii,\theta}((j + 1)n^{-\theta}) \right|^{2\alpha} \right].
\]
From the foregoing two results, the first two terms on the right side of this inequality can be bounded by \(C \cdot h^{1+\epsilon}\). For the third term, notice that

\[
I^{(n)}_{ii,\theta}(ln^{-\theta}) - I^{(n)}_{ii,\theta}((j + 1)n^{-\theta}) = I^{(n)}_{ii,\theta}(ln^{-\theta}) - I^{(n)}_{ii,\theta}((j + 1)n^{-\theta}),
\]
by Proposition 5.13 it can be bounded by \(C(h^{\alpha} + n^{2-2\alpha}h) \leq Ch^{1+\epsilon}\). Here we have finished the proof. \(\square\)

We now summarize the results in Corollary 5.9, Proposition 5.10, 5.12 and 5.14 to get the weak convergence of \(\{\varepsilon^{(n)}_{ij}\}_{n \geq 1}\) to 0.

**Lemma 5.15.** For each \(i \in \mathcal{H}\) and \(j \in \mathcal{D}\), we have \(\varepsilon^{(n)}_{ij} \xrightarrow{d} 0\) in \(D([0, \infty), \mathbb{R})\) as \(n \to \infty\).

5.1.3. Weak convergence of \((M^{(n)}_{ij})_{i \in \mathcal{H}, j \in \mathcal{D}}\) to 0. For each \(i \in \mathcal{H}\) and \(j \in \mathcal{D}\), we now prove the weak convergence of \(\{M^{(n)}_{ij}\}_{n \geq 1}\) to 0 by using the Burkholder-Davis-Gundy inequality.

**Lemma 5.16.** For any \(i \in \mathcal{H}\) and \(j \in \mathcal{D}\), we have \(M^{(n)}_{ij} \xrightarrow{u.c.,p} 0\) as \(n \to \infty\).

**Proof.** For any \(T \geq 0\), by the Burkholder-Davis-Gundy inequality and the hypothesis (H1) we have

\[
E \left[ \sup_{t \in [0,T]} \left| M^{(n)}_{ij}(t) \right|^2 \right] \leq C \frac{1}{n} \int_{U} \left\| \phi_i(u) \right\|_{L^2}^2 \nu^{(n)}_i(du),
\]
which goes to 0 as \(n \to \infty\). Similarly, for \(j \in \mathcal{H}\), by Lemma 5.6 we also have

\[
E \left[ \sup_{t \in [0,T]} \left| M^{(n)}_{ij}(t) \right|^2 \right] \leq C \epsilon E \left[ \int_{0}^{T} \int_{U} \left\| \phi_i(u) \right\|_{L^2}^2 N^{(n)}_j(n \cdot ds, du) \right] 
\leq C \epsilon \int_{U} \left\| \phi_i(u) \right\|_{L^2}^2 \nu^{(n)}_j(du).
\]
For any $K > 0$, let $U_K := \{ u \in U : \| \phi_i(u) \|_{L^1} \leq K \}$ and $U_K^c$ be its complement. We have
\[
\int_U \| \phi_i(u) \|_{L^1}^2 (du) \leq K \int_{U_K^c} \| \phi_i(u) \|_{L^1}^2 (du) + \frac{1}{K^{2\alpha - 2}} \int_{U_K} \| \phi_i(u) \|_{L^1}^2 (du).
\]
The first term on the right side of this inequality can be bounded by $K \| \phi_i \|_{L^1}$, which goes to 0 as $n \to \infty$; see Condition 3.1. By the hypothesis (H1), the second term can be uniformly bounded by $C/K^{2\alpha - 2}$, which can be ignored for large $K$ and hence $E[\sup_{t \in [0,T]} |M_{ij}^{(n)}(t)|^2] \to 0$ as $n \to \infty$. \hfill \Box

5.1.4. Uniform tightness and weak convergence of $\{W_{i}^{(n)}\}_{n \geq 1}$. By the mutual independence among $W_{i}^{(n)}$, $i \in \mathcal{H}$, it suffices to prove the uniform tightness and weak convergence of $L^2(\mathbb{R}_+)\#$-martingales $\{W_{i}^{(n)}\}_{n \geq 1}$ separately for each $i \in \mathcal{H}$.

**Lemma 5.17.** For each $i \in \mathcal{H}$, we have $W_{i}^{(n)} \Rightarrow W_{i}$ as $n \to \infty$.

**Proof.** By the continuity of $W_{i}$ and Corollary 3.33 in [42, p.353], it suffices to prove $W_{i}^{(n)}(f) \overset{d}{\to} W_{i}(f)$ in $\mathcal{D}([0,\infty), \mathbb{R})$ for any $f \in L^2(\mathbb{R}_+)$. Similarly as in the proof of Proposition 5.10, we can prove that $\{W_{i}^{(n)}(f)\}_{n \geq 1}$ is $C$-tight and
\[
\sup_{n \geq 1} E \left[ \sup_{t \in [0,T]} |W_{i}^{(n)}(f,t)|^2 \right] < \infty, \quad T \geq 0.
\]

We now start to characterize the cluster points. Without loss of generality, we may assume $W_{i}^{(n)}(f)$ converges to a limit process $X_{f}$ weakly and hence uniformly on compacts in probability. By the Skorokhod representation theorem, we may assume $W_{i}^{(n)}(f) \overset{\text{u.c.}}{\to} X_{f}$ a.s. and hence in $L^2([0,T])$, which induces that
\[
|W_{i}^{(n)}(f,t)|^2 - \frac{c_i^2}{K^2} \cdot \| f \|_{L^2}^2 \cdot t \overset{\text{u.c.}}{\to} \| f \|_{L^2}^2 \cdot t,
\]
a.s. and hence in $L^1([0,T])$. Thus both $X_{f}$ and $\{|X_{f}(t)|^2 - \| f \|_{L^2}^2 \cdot t : t \geq 0\}$ are martingales. In conclusion, we have $X_{f}$ is a continuous martingale with quadratic variation $\langle X_{f} \rangle_t = \| f \|_{L^2}^2 \cdot t$ for $t \geq 0$. By Theorem III-7 in [22], there exists a Gaussian white noise $W_{i}(ds,dz)$ on $(0,\infty)^2$ with intensity $dsdz$ such that
\[
X_{f}(t) = \int_{0}^{t} \int_{0}^{\infty} f(z)W_{i}(ds,dz) = W_{i}(f,t), \quad t \geq 0.
\]
\hfill \Box

**Lemma 5.18.** For each $i \in \mathcal{H}$, the sequence of $L^2(\mathbb{R}_+)\#$-martingales $\{W_{i}^{(n)}\}_{n \geq 1}$ is uniformly tight.

**Proof.** Let $\mathcal{S}$ be the collection of all $(\mathcal{F}_t)$-predictable $L^2(\mathbb{R}_+)$-valued processes. By the definition of uniform tightness, it suffices to prove that for any $T > 0$,
\[
\bigcup_{n=1}^{\infty} \left\{ \sup_{t \in [0,T]} |W_{i}^{(n)}(X,t)| : X \in \mathcal{S} \text{ with } \sup_{t \in [0,T]} \| X(t) \|_{L^2} \leq 1 \right\}
\]
is stochastically bounded. Actually, using Chebyshev’s inequality and then the Burkholder-Davis-Gundy inequality together with the hypothesis (H1), we have for any $\eta > 0$, 

$$
\begin{align*}
\mathbb{P}\left( \sup_{t \in [0,T]} |W_i^{(n)}(X,t)| \geq \eta \right) & \leq \eta^{-2} \mathbb{E}\left[ \sup_{t \in [0,T]} \left| \int_0^t \int_0^\infty X(s,z)W_i^{(n)}(ds,dz) \right|^2 \right] \\
& \leq \frac{C}{\eta^2} \int_0^T \mathbb{E}[\|X(s)\|^2_{L^2}]ds \leq \frac{C}{\eta^2} \cdot T.
\end{align*}
$$

This upper bound holds uniformly in $n \geq 1$ and $X \in \mathcal{H}$ with $\sup_{t \in [0,T]} \|X(t)\|_{L^2} \leq 1$. Thus the sequence $\{W_i^{(n)}\}_{n \geq 1}$ is uniformly tight.

5.2. Proof for Theorem 3.15. By Condition 3.14, the rescaled process $S_{c,H}^{(n)}$ can be well approximated by $\hat{S}_{c,H}^{(n)}$ with

$$
\hat{S}_{c,H}^{(n)}(t) := \int_0^t \int_U \hat{\zeta}(t,s,n^{-1/2}N_i^{(n)}(n \cdot ds,du), \quad t \geq 0, i \in \mathcal{H}.
$$

The error process is denoted as $\varepsilon_{c,H}^{(n)} := \hat{S}_{c,H}^{(n)} - S_{c,H}^{(n)}$. By Corollary 3.33 in [42, p.353], Theorem 3.15 follows directly from the following weak convergence results for the two sequences $\{\hat{S}_{c,H}^{(n)}\}_{n \geq 1}$ and $\{\varepsilon_{c,H}^{(n)}\}_{n \geq 1}$. The first one has been widely studied in [42, Chapter IX] under Condition 3.13.

**Lemma 5.19.** Theorem 3.15 holds with $S_{c,H}^{(n)}$ replaced by $\hat{S}_{c,H}^{(n)}$.

**Lemma 5.20.** We have $\varepsilon_{c,H}^{(n)} \overset{d}{\to} 0$ in $D([0,\infty), \mathbb{R}^d)$ as $n \to \infty$.

**Proof.** For $t \geq 0$ and $u \in U$, let $\zeta_i(t,u) := \zeta_i(\infty,u) - \zeta_i(t,u)$. For any $\epsilon \in (0,1)$, we split $\varepsilon_{c,i}^{(n)}(t)$ into the following two parts:

$$
\varepsilon_{c,i}^{(n)}(t,\epsilon) := \int_0^{(t-\epsilon)^+} \int_U \zeta_i(n(t-s),u,n^{-1/2}N_i^{(n)}(n \cdot ds,du),
$$

$$
\varepsilon_{c,i}^{(n)}(t,\epsilon) := \int_{(t-\epsilon)^+}^t \int_U \zeta_i(n(t-s),u,n^{-1/2}N_i^{(n)}(n \cdot ds,du).
$$

The monotonicity of $\zeta_i(\cdot,u)$ induces that

$$
|\varepsilon_{c,i}^{(n)}(t,\epsilon)| \leq \int_0^t \int_U |\zeta_i((n \epsilon,u)|n^{-1/2}N_i^{(n)}(n \cdot ds,du).
$$

From Lemma 5.6 and Condition 3.14, we have for any $T > 0$,

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} |\varepsilon_{c,i}^{(n)}(t,\epsilon)| \right] \leq \int_0^T \mathbb{E}[Z_i^{(n)}(s)]ds \cdot \int_U |\zeta_i(n \epsilon,u)|\nu_i^{(n)}(du) \leq C \cdot \zeta_i^{(c)}(n \epsilon),
$$

which vanishes as $n \to \infty$. Moreover, by the monotonicity of $\zeta_i(\cdot,u)$ again,

$$
\sup_{t \in [0,T]} |\varepsilon_{c,i}^{(n)}(t,\epsilon)| \leq \sup_{t \in [0,T]} \int_{(t-\epsilon)^+}^t \int_U \zeta_i((n \epsilon,u)|n^{-1/2}N_i^{(n)}(n \cdot ds,du)
$$

$$
\leq \sup_{0 \leq j \leq [T/\epsilon]} \int_i^{(j+2)\epsilon} \int_U \zeta_i((n \epsilon,u)|n^{-1/2}N_i^{(n)}(n \cdot ds,du).$$
By Chebyshev’s inequality, we have for any \( \eta > 0 \),
\[
P \left( \sup_{t \in [0, T]} |\xi_{i}^{(n)}(t, \epsilon)| \geq \eta \right) \leq \sum_{j=0}^{[T/\epsilon]} P \left( \int_{j\epsilon}^{(j+2)\epsilon} \int_{U} \frac{\zeta_{i}(\infty, u) N_{i}^{(n)}(n \cdot ds, du) \geq \eta}{n^{2}} \right)
\]
(50)
\[
\leq \frac{1}{\eta^{n}} \sum_{j=0}^{[T/\epsilon]} E \left[ \left( \int_{j\epsilon}^{(j+2)\epsilon} \int_{U} \frac{\zeta_{i}(\infty, u) N_{i}^{(n)}(n \cdot ds, du) }{n^{2}} \right)^{\alpha} \right].
\]
Using the Cauchy-Schwarz inequality inequality to the last expectation, it can be bounded by
\[
C \cdot E \left[ \left( \int_{j\epsilon}^{(j+2)\epsilon} \int_{U} \frac{\zeta_{i}(\infty, u) N_{i}^{(n)}(n \cdot ds, du) }{n^{2}} \right)^{\alpha} \right] + C \cdot E \left[ \left( \int_{j\epsilon}^{(j+2)\epsilon} \int_{U} \frac{\zeta_{i}(\infty, u) N_{i}^{(n)}(n \cdot ds, du) }{n^{2}} \right)^{\alpha} \right],
\]
which also, by (38) and then Hőlder’s inequality and Lemma 5.6, can be bounded by
\[
C \epsilon^{\alpha-1} \int_{j\epsilon}^{(j+2)\epsilon} E \left[ |Z_{i}^{(n)}(s)|^{\alpha} \right] ds + \frac{C}{n^{2\alpha-2}} \int_{j\epsilon}^{(j+2)\epsilon} E \left[ Z_{i}^{(n)}(s) \right] ds \leq C \left( \epsilon^{\alpha} + \frac{\epsilon}{n^{2\alpha-2}} \right).
\]
Taking this back into (50),
\[
P \left( \sup_{t \in [0, T]} |\xi_{i}^{(n)}(t, \epsilon)| \geq \eta \right) \leq C \epsilon^{\alpha-1} + C n^{2-2\alpha},
\]
which vanishes as \( n \rightarrow \infty \) and then \( \epsilon \rightarrow 0^{+} \). We have finished the proof.

5.3. Proofs for Theorem 3.19 and 3.21. Based on our asymptotic analysis before Theorem 3.19, the rescaled process \( S_{1,\mathcal{H}}^{(n)} \) can be well approximated by \( \hat{S}_{1,\mathcal{H}}^{(n)} \) and the error process is denoted as \( \epsilon_{1,\mathcal{H}}^{(n)} \). Firstly, we prove the weak convergence of \( \{\hat{S}_{1,\mathcal{H}}^{(n)}\}_{n \geq 1} \) and \( \{\epsilon_{1,\mathcal{H}}^{(n)}\}_{n \geq 1} \) in the next two lemmas.

**Lemma 5.21.** Theorem 3.19 holds with \( S_{1,\mathcal{H}}^{(n)} \) replaced by \( \hat{S}_{1,\mathcal{H}}^{(n)} \).

**Proof.** By the Skorokhod representation theorem, we may assume that \( Z_{\mathcal{H}}^{(n)} \) converges to \( Z_{\mathcal{H}} \) a.s. in \( \mathbb{D}([0, \infty), \mathbb{R}_{+}^{d}) \) and hence uniformly on compacts. Thus it suffices to prove that for each \( i \in \mathcal{H} \),
\[
\int_{0}^{nt} \zeta_{ii}^{(n)}(s)Z_{i}^{(n)}(t-s/n)ds - b_{1,i} \cdot Z_{i}(t)
\]
goesto 0 a.s. in \( \mathbb{D}([\delta, 1], \mathbb{R}_{+}) \). Subtracting the integral \( \int_{0}^{nt} \zeta_{ii}^{(n)}(s)ds \cdot Z_{i}(t) \) and then adding it back, we can write the preceding quantity into
\[
(51) \int_{0}^{nt} \zeta_{ii}^{(n)}(s)|Z_{i}^{(n)}(t-s/n) - Z_{i}(t)|ds - I_{i}^{(n)}_{ii}(nt) \cdot Z_{i}(t) + \left( \|\zeta_{ii}^{(n)}\|_{L^{1}} - b_{1,i} \right)Z_{i}(t).
\]
Condition 3.18 implies that the last two terms go to 0 as \( n \rightarrow \infty \) uniformly on \([\delta, 1]\) and \([0, 1]\) respectively. For any \( \epsilon \in (0, 1) \), the first term can be bounded by
\[
\|\zeta_{ii}^{(n)}\|_{L^{1}} \cdot \left( \sup_{t \in [0,1]} |Z_{i}^{(n)}(t) - Z_{i}(t)| + \sup_{t \in [0,1]} \sup_{s \in [0,\epsilon]} |\Delta_{s}Z_{i}(t)| \right)
\]
\[ + I_{\varsigma,ii}^{(n)}(n\epsilon) \cdot \sup_{t \in [0,1]} \left( Z_i^{(n)}(t) + Z_i(t) \right), \]

which goes to 0 a.s. as \( n \to \infty \), since \( Z_i^{(n)} \) \( \overset{a.s.}{\to} \) \( Z_i \) uniformly on \([0,1]\) and \( Z_i \) is uniformly continuous on \([0,2]\). Putting these estimates together, we can get the first desired result. For the second one, like the preceding argument it suffices to prove that the second term in (51) converges to 0 uniformly on \([0,1]\) as \( n \to \infty \). Indeed, by the fact that \( \sup_{n \geq 1} \| \zeta_{ii}^{(n)} \|_{L^1} + \sup_{t \in [0,1]} Z(t) < \infty \) a.s., we have for any \( \epsilon \in (0,1) \),

\[
\sup_{t \in [0,1]} I_{\varsigma,ii}^{(n)}(nt) \cdot Z_i(t) \leq \| \zeta_{ii}^{(n)} \|_{L^1} \cdot \sup_{t \in [0,\epsilon]} Z_i(t) + I_{\varsigma,ii}^{(n)}(n\epsilon) \cdot \sup_{t \in [0,1]} Z_i(t)
\]

\[
\leq C \sup_{t \in [0,\epsilon]} Z_i(t) + C \int_0^\infty \zeta(s)ds,
\]

which goes to 0 a.s. as \( n \to \infty \) and then \( \epsilon \to 0+ \) because of the continuity of \( Z_i \) and the integrability of \( \overline{\zeta} \). Here we have finished the proof. \( \square \)

**Lemma 5.22.** We have \( \varepsilon_{1,i}^{(n)} \overset{d}{\to} 0 \) in \( D([0,\infty), \mathbb{R}^d) \) as \( n \to \infty \).

**Proof.** By the Burkholder-Davis-Gundy inequality, the inequality \( (x+y)^{\alpha/2} \leq |x|^{\alpha/2} + |y|^{\alpha/2} \) and Lemma 5.6, we have for any \( t \geq 0 \),

\[
\mathbb{E} \left[ |\varepsilon_{1,i}^{(n)}(t)|^\alpha \right] \leq C \cdot \mathbb{E} \left[ \left( \int_0^{nt} \int_U \frac{|\zeta_i(nt-s,u)|^2}{n^2} N_i^{(n)}(ds,du) \right)^{\alpha/2} \right]
\]

\[
\leq C \cdot \mathbb{E} \left[ \int_0^{nt} \int_U \frac{|\zeta_i(nt-s,u)|^\alpha}{n^\alpha} N_i^{(n)}(ds,du) \right]
\]

\[
\leq C \int_U \left\| \zeta_i(u) \right\|_{L^\alpha} \nu_i^{(n)}(du).
\]

Notice that

\[
\left\| \zeta_i(u) \right\|_{L^\alpha} \leq \left\| \zeta_i(u) \right\|_{L^\alpha} \leq \left\| \zeta_i(u) \right\|_{L^\alpha} \leq C \left( \left\| \zeta_i(u) \right\|_{TV} + \left\| \zeta_i(u) \right\|_{\alpha} \right)
\]

By Condition 3.17, we have \( \mathbb{E} \left[ |\varepsilon_{1,i}^{(n)}(t)|^\alpha \right] \to 0 \) as \( n \to \infty \) and hence \( \varepsilon_{1,i}^{(n)} \overset{d.f.d.}{\to} 0 \). We now start to prove the tightness of \{\( \varepsilon_{1,i}^{(n)} \)\} \( \overset{n \geq 1}{\to} \) on \([0,1]\) and the general case can be proved similarly. Since \( \left\| \zeta_i(u) \right\|_{TV} \leq C \) for any \( u \in U \), similarly as in the proof of Proposition 5.12, it suffices to prove the case in which \( \zeta_i(t, u) \) decreases in \( t \).

**Step 1.** We first show that \( \varepsilon_{1,i}^{(n)} \) can be well approximated by its linear interpolation \( \varepsilon_{1,i,\theta}^{(n)} \) defined as (43), i.e. \( \varepsilon_{1,i}^{(n)} - \varepsilon_{1,i,\theta}^{(n)} \to 0 \). Like (45)-(46), we also have

\[
\sup_{t \in [0,1]} \left| \varepsilon_{1,i}^{(n)}(t) - \varepsilon_{1,i,\theta}^{(n)}(t) \right| \leq 3 \sup_{k=0,\ldots,[n^\theta]; h \leq n^{-\theta}} \left[ A_{1,1}^{(n)}(kn^{-\theta}, h) + A_{1,2}^{(n)}(kn^{-\theta}, h) \right],
\]

where

\[
A_{1,1}^{(n)}(t, h) := n \int_0^{t+h} Z_i^{(n)}(s)ds \int_U |\Delta_{nh} \zeta_i(n(t-s), u)| \nu_i^{(n)}(du),
\]

\[
A_{1,2}^{(n)}(t, h) := \int_0^{t} \int_U |\Delta_{nh} \zeta_i(n(t-s), u)| N_i^{(n)}(n \cdot ds, du).
\]
For any $t, h \in [0,1]$, we first have

$$A_{1,1}^{(n)}(t, h) \leq \sup_{r \in [0, 2]} Z_i^{(n)}(r) \cdot \int \nu_i^{(n)}(du) \cdot \left( \int_0^\infty |\Delta_{nh} \zeta_i(s, u)| ds. \right)$$

Similarly as in (48), we have

$$\int_0^\infty |\Delta_{nh} \zeta_i(s, u)| ds \leq 2\|\zeta_i(u)\|_{TV} \cdot nh.$$ 

By Condition 3.17, we have uniformly in $h \in [0,1]$,

$$\sup_{t \in [0,1]} A_{1,1}^{(n)}(t, h) \leq C \sup_{r \in [0, 2]} Z_i^{(n)}(r) \cdot nh.$$ 

Since $Z_i$ is continuous, we have $\sup_{n \geq 1} \sup_{r \in [0, 2]} Z_i^{(n)}(r) < \infty$ a.s. and hence as $n \to \infty$,

$$\sup_{k=0,\ldots,\lceil n^{\theta} \rceil; h \leq n^{-\theta}} A_{1,1}^{(n)}(kn^{-\theta}, h) \leq \sup_{r \in [0, 2]} Z_i^{(n)}(r) \cdot n^{1-\theta} \to 0.$$ 

Moreover, like Step 3 in the proof of Proposition 5.12, we can prove that for any $\eta > 0$,

$$\lim_{n \to \infty} P \left( \sup_{k=0,\ldots,\lceil n^{\theta} \rceil; h \leq n^{-\theta}} A_{1,2}^{(n)}(kn^{-\theta}, h) \geq \eta \right) = 0.$$ 

Taking these two estimates back into (52), we have $\sup_{t \in [0, 1]} |\epsilon_{1, i}^{(n)}(t) - \epsilon_{1, i, \theta}^{(n)}(t)| \leq \mathbb{P} 0$ as $n \to \infty$.

**Step 2.** We now prove the tightness of the sequence $\{\epsilon_{1, i, \theta}^{(n)}\}_{n \geq 1}$. By (38), there exists a constant $C > 0$ such that for any $n \geq 1$ and $t, h \in [0, 1]$,

$$\mathbb{E} \left[ |\Delta_{h} \epsilon_{1, i}^{(n)}(t)|^{2\alpha} \right] \leq C \left( \int \nu_i^{(n)}(du) \int_0^{t+h} |\Delta_{nh} \zeta_i(n(t-s), u)|^2 ds \right)^\alpha$$

$$+ \frac{C}{n^{2\alpha-2}} \int \nu_i^{(n)}(du) \int_0^{t+h} |\Delta_{nh} \zeta_i(n(t-s), u)|^{2\alpha} ds.$$ 

Notice that $\int_0^{t+h} |\Delta_{nh} \zeta_i(n(t-s), u)|^{2\alpha} ds \leq 2\|\zeta_i(u)\|_{TV}^{2\alpha} \cdot h$ uniformly in $t, h \in [0, 1]$. By Condition 3.17, there exists a constant $C > 0$ such that

$$\mathbb{E} \left[ |\Delta_{h} \epsilon_{1, i}^{(n)}(t)|^{2\alpha} \right] \leq C \cdot (n^{2-2\alpha} h + h^\alpha),$$

for any $t, h \in [0, 1]$ and $n \geq 1$. Like the proof of Proposition 5.14, we can prove the tightness of the sequence $\{\epsilon_{1, i, \theta}^{(n)}\}_{n \geq 1}$ in the same way. Consequently, the sequence $\{\epsilon_{1, i}^{(n)}\}_{n \geq 1}$ is tight in $\mathcal{D}([0, \infty), \mathbb{R})$ and the whole proof is end. 

**Proofs for Theorem 3.19 and 3.21.** By Corollary 3.33 in [42, p.353], we can get Theorem 3.19 directly from Lemma 5.21 and 5.22. For Theorem 3.21, by Condition 3.20 and Theorem 3.19 it suffices to prove that

$$\psi_{1, i}^{(n)}(nt) + S_{1, i}^{(n)}(t) \xrightarrow{d} b_{1, i} \cdot Z_i(t),$$

in $\mathcal{D}([0, 1], \mathbb{R}_+)$ as $n \to \infty$. Like the proof of Lemma 5.21, it suffices to prove that

$$Z_{i}^{(n)}(0)I_{\zeta_{i, i}}^{(n)}(nt) + \int_0^{nt} \zeta_{i, i}^{(n)}(s)Z_{i}^{(n)}(t-s/n)ds - b_{1, i} \cdot Z_i(t)$$

$$\begin{align*}
= I_{\zeta_{i, i}}^{(n)}(nt) \cdot (Z_{i}^{(n)}(0) - Z_i(t)) + \left( \|\zeta_{i, i}^{(n)}\|_{L_1} - b_{1, i} \right)Z_i(t) \\
+ \int_0^{nt} \zeta_{i, i}^{(n)}(s) \cdot \left[ Z_{i}^{(n)}(t-s/n) - Z_i(t) \right] ds
\end{align*}$$

\[\square\]
goes to 0 a.s. in $D([0,1],\mathbb{R})$. From the proof of Lemma 5.21, the last two terms on the right side of this equality go to 0 uniformly on compacts. For any $\epsilon \in (0,1)$, the first term can be bounded by

$$ I_{z,ii}^{(n)}(ne) \cdot \sup_{t \in [e,1]} |Z_i^{(n)}(0) - Z_i(t)| + \sup_{t \in [0,e]} I_{z,ii}^{(n)}(nt) \cdot |Z_i^{(n)}(0) - Z_i(t)|. $$

Since $\sup_{t \in [e,1]} |Z_i^{(n)}(0) - Z_i(t)| < \infty$ a.s. and $I_{z,ii}^{(n)}(ne) \to 0$, the first term in this sum goes to 0 as $n \to \infty$. Moreover, the second term can be bounded by

$$ \sup_{t \in [0,e]} I_{z,ii}^{(n)}(nt) \cdot (|Z_i^{(n)}(0) - Z_i(0)| + |Z_i(0) - Z_i(t)|) \leq C \cdot |Z_i^{(n)}(0) - Z_i(0)| + C \sup_{t \in [0,e]} |Z_i(0) - Z_i(t)|, $$

which goes to 0 a.s. as $n \to \infty$ and then $\epsilon \to 0+$. The proof is end. \hfill \square

5.4. Proof for Theorem 4.5. Before proving Theorem 4.5 by using Theorem 3.6, it remains to identify the total budding rate function of ancestors satisfies Condition 3.5. Indeed, by Condition 4.4, the mean budding rate function of each type-$i$ ancestor, denoted as $B_i^{(n)}$, is

$$ B_i^{(n)}(t) = \int_{\mathbb{B}} P_{B,i}^{(n)}(db) \int_{\mathbb{R}_+^2} B(t + s, y + s) P_{AR,i}^{(n)}(ds, dy), \quad t \geq 0. $$

By Fubini’s theorem and the fact that $B(t, y) = 0$ for $t \geq y,$

$$ \int_{\mathbb{R}_+^2} B(t + s, y + s) P_{Ar,i}^{(n)}(ds, dy) = \frac{1}{m_{L,i}} \int_0^\infty ds \int_s^\infty B(t + s, y) P_{L,i}^{(n)}(dy) $$

(54)

and hence

$$ B_i^{(n)}(t) = \frac{1}{m_{L,i}^{(n)}} \int_t^\infty ds \int_{\mathbb{B}} P_{B,i}^{(n)}(db) \int_0^\infty B(s, y) P_{L,i}^{(n)}(dy) = \frac{1}{m_{L,i}^{(n)}} \int_t^\infty B_i^{(n)}(s) ds $$

(55)

By the law of large numbers, it is natural to believe that $\mu_i^{(n)}/n$ can be well approximated by

$$ \bar{\mu}_i^{(n)} := \left( Z_i^{(n)}(0) \cdot I_{\phi,ii}^{(n)} \right)_{i \in \mathcal{H}} \quad \text{with} \quad Z_i^{(n)}(0) := \frac{\Xi_i^{(n)}(0)/n}{m_{L,i}^{(n)}}. \quad i \in \mathcal{H}. $$

**Lemma 5.23.** We have $\|\mu_i^{(n)}/n - \bar{\mu}_i^{(n)}\|_{L_{1,\infty}} \overset{d}{\to} 0$ as $n \to \infty$.

**Proof.** For each $i \in \mathcal{H}$, let $\varepsilon_i^{(n)} := \mu_i^{(n)}/n - \bar{\mu}_i^{(n)}$. For any $\eta > 0$ and $K > 0$, we have

$$ \mathbf{P}(\|\varepsilon_i^{(n)}\|_{L_{1,\infty}} > \eta) \leq \mathbf{P}(Z_i^{(n)}(0) > K) + \mathbf{P}(\|\varepsilon_i^{(n)}\|_{L_{1,\infty}} > \eta, Z_i^{(n)}(0) \leq K). $$

By Condition 4.3 and the assumption that $\Xi_i^{(n)}(0)/n \overset{d}{\to} \Xi_i^*(0)$ as $n \to \infty$,

$$ \lim_{K \to \infty} \sup_{n \geq 1} \mathbf{P}(Z_i^{(n)}(0) > K) = 0. $$

5.5. Proof for Theorem 4.6. Before proving Theorem 4.6 by using Theorem 3.6, it remains to identify the total budding rate function of ancestors satisfies Condition 3.5. Indeed, by Condition 4.4, the mean budding rate function of each type-$i$ ancestor, denoted as $B_i^{(n)}$, is
Thus it suffices to prove this lemma with \( \{ \Xi_i^{(n)}(0)/n \}_{n \geq 1} \) being deterministic and uniformly bounded. The following proof follows closely that of Theorem 4.1 in [4]. In detail, let \( \{ \tilde{Y}_{i,k}^{(n)} : k = 1, 2, \ldots, \Xi_i^{(n)}(0) \} \) be a sequence of i.i.d. function-valued random variables with

\[
\tilde{Y}_{i,k}^{(n)}(t) := \tilde{F}_{i,k}^{(n)}(t + A_{i,k}^{(n)} + A_{i,k}^{(n)} + \tilde{B}_{i,k}^{(n)}(t)), \quad t \geq 0.
\]

From (27), we have

\[
\tilde{\mu}_{i, \xi}^{(n)}(t) = \frac{\mu_i^{(n)}(t)}{n} - \frac{\Xi_i^{(n)}(0)}{n} \cdot \tilde{B}_{i}^{(n)}(t) = \frac{1}{n} \sum_{k=1}^{\Xi_i^{(n)}(0)} Y_{i,k}^{(n)}(t), \quad t \geq 0, i \in \mathcal{H}
\]

From (53)-(55), we have \( E[Y_{i,k}^{(n)}(t)] = 0 \) for any \( t \geq 0 \). By (55), Fubini’s theorem and Condition 4.2, there exists a constant \( C > 0 \) such that for any \( n \geq 1 \) and \( i \in \mathcal{H} \),

\[
\| \tilde{B}_{i}^{(n)} \|_{L^1} = \int_{0}^{\infty} \frac{g \cdot B_{i}^{(n)}(s)}{m_{L, i}^{(n)}} ds \leq C \quad \text{and} \quad \| \tilde{B}_{i}^{(n)} \|_{L^\infty} = \tilde{B}_{i}^{(n)}(0) \leq C.
\]

Notice that

\[
\int_{0}^{\infty} B_{i,k}^{(n)}(t + A_{i,k}^{(n)} + A_{i,k}^{(n)} + \tilde{A}_{i,k}^{(n)}) dt \leq \| B_{i,k}^{(n)}(R_{i,k}^{(n)} + A_{i,k}^{(n)}) \|_{L^1}
\]

and

\[
\sup_{t \geq 0} \left| B_{i,k}^{(n)}(t + A_{i,k}^{(n)} + A_{i,k}^{(n)} + \tilde{A}_{i,k}^{(n)}) \right| \leq \| B_{i,k}^{(n)}(R_{i,k}^{(n)} + A_{i,k}^{(n)}) \|_{TV}.
\]

By (54), Fubini’s theorem, the inequality \( 2 |xy| \leq x^2 + y^2 \) and Condition 4.2,

\[
E[\| B_{i,k}^{(n)}(R_{i,k}^{(n)} + A_{i,k}^{(n)}) \|_{L^1}] = \int_{B} \mathcal{P}_{B}^{(n)}(dB) \int_{\mathbb{R}^2} \| B(s + y) \|_{L^1} \mathcal{P}_{B}^{(n)}(ds, dy)
\]

\[
= \int_{B} \mathcal{P}_{B}^{(n)}(dB) \int_{0}^{\infty} \frac{ds}{m_{L, i}^{(n)}} \int_{s}^{\infty} \| B(y) \|_{L^1} \mathcal{P}_{B}^{(n)}(dy)
\]

\[
= \int_{B} \mathcal{P}_{B}^{(n)}(dB) \int_{0}^{\infty} \frac{y}{m_{L, i}^{(n)}} \mathcal{P}_{B}^{(n)}(dy)
\]

\[
\leq \int_{0}^{\infty} \frac{y^2}{m_{L, i}^{(n)}} \mathcal{P}_{B}^{(n)}(dy) + \int_{B} \mathcal{P}_{B}^{(n)}(dB) \int_{0}^{\infty} \frac{\| B(y) \|_{L^1}^2}{m_{L, i}^{(n)}} \mathcal{P}_{B}^{(n)}(dy)
\]

which is bounded uniformly in \( n \geq 1 \) and \( i \in \mathcal{H} \). Similarly, we also have

\[
\sup_{n \geq 1} E[\| B_{i,k}^{(n)}(R_{i,k}^{(n)} + A_{i,k}^{(n)}) \|_{TV}] < \infty, \quad i \in \mathcal{H}.
\]

Putting all these estimates together and then using Minkowski’s inequality, we have

\[
\sup_{n \geq 1} E[\| \tilde{Y}_{i,k}^{(n)} \|_{L^1, \infty}] < \infty, \quad i \in \mathcal{H}.
\]

For \( K > 0 \), by Chebyshev’s inequality and Minkowski’s inequality,

\[
\sup_{n \geq 1} \mathbb{P}(\| \tilde{\mu}_{i, \xi}^{(n)} \|_{L^1, \infty} \geq K) \leq \frac{1}{K} \sup_{n \geq 1} E[\| \tilde{\mu}_{i, \xi}^{(n)} \|_{L^1, \infty}]
\]

\[
\leq \frac{1}{K} \sup_{n \geq 1} \frac{\Xi_i^{(n)}(0)}{n} \sup_{n \geq 1} E[\| \tilde{Y}_{i,k}^{(n)} \|_{L^1, \infty}] \leq \frac{C}{K},
\]
which vanishes as \( K \to \infty \). Thus the sequence \( \{\hat{\varepsilon}^{(n)}_{\mu}\}_{n \geq 1} \) is tight and hence flatly concentrated; see Definition 2.1 in [3]. Notice that \( L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+) \) is the dual space of \( L^{1,\infty}(\mathbb{R}_+) \). For any \( f \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+) \), we see that \( \{f(Y_{i,k}^{(n)}) : k = 1, \cdots, \Xi^{(n)}_i(0)\}_{n \geq 1} \) is an array of row-wise independent random variables with

\[
E[|f(Y_{i,k}^{(n)})|^\alpha] \leq CE[\|Y_{i,k}^{(n)}\|_{L^{1,\infty}}^\alpha] \leq C,
\]

uniformly in \( n \geq 1 \) and hence this array is uniformly integrable. By using the main theorem in [29], we have as \( n \to \infty \),

\[
f(\varepsilon^{(n)}_{\mu}) = \frac{1}{n} \sum_{k=1}^{\Xi^{(n)}_i(0)} f(Y_{i,k}^{(n)}) \xrightarrow{P} 0.
\]

By Theorem 2.4 in [3], it follows that \( \|\varepsilon^{(n)}_{\mu}\|_{L^{1,\infty}} \xrightarrow{d} 0 \) as \( n \to \infty \).

\[\square\]

5.5. Proof for Theorem 4.7. It is obvious that this theorem follows directly from Theorem 3.21 and the following two auxiliary results.

(1) The impact of immigrants and offspring born by parents of different type can be ignored, i.e., for each \( i \in \mathcal{H} \) and \( j \in \mathcal{D}_i \), the rescaled process \( \varepsilon^{(n)}_{N,ij} \) defined as below converges weakly to 0 in \( D([0, \infty), \mathbb{R}) \):

\[
\varepsilon^{(n)}_{N,ij}(t) := \frac{1}{n} \int_0^{nt} \int_{\mathbb{U}} \zeta_i(nt-s, u)N^{(n)}_j(ds, du), \quad t \geq 0.
\]

Here we prove it with \( j = I \), it can be proved similarly. By Lemma 5.6 and Condition 4.6, there exist two constants \( C, \vartheta > 0 \) such that \( \sup_{n \geq 1} E[|B^{(n)}_{ij}(t)|^{2\vartheta}] \leq Ce^{\vartheta t} \) for any \( t \geq 0 \) and

\[
E[|\varepsilon^{(n)}_{N,ij}(t)|] \leq C \int_{\mathbb{U}} |\zeta_i(u)|_{L^1} N^{(n)}_j(d\nu_j^{(n)}(du))
\]

\[
= C \cdot m_{ij}^{(n)} \int_0^{\infty} P_{L,i}^{(n)}(dy) \int_T |T(y)|_{L^1} P_{T,i}^{(n)}(dT) \leq C \cdot m_{ij}^{(n)},
\]

which goes to 0; see Condition 4.3. Hence \( \varepsilon^{(n)}_{N,ij} \xrightarrow{f.d.d.} 0 \). We now prove the tightness of \( \{\varepsilon^{(n)}_{N,ij}\}_{n \geq 1} \). Because of \( |T(y)|_{TV} < \infty \) for any \( \xi \in \mathbb{T} \) and \( y \geq 0 \), we have \( |\zeta_i(u)|_{TV} < \infty \) for any \( u \in \mathbb{U} \). Similarly as in the proof of Lemma 5.22, it suffices to consider the case with \( \zeta_i(t, u) \) being non-increasing in \( t \). Let \( \varepsilon^{(n)}_{N,ij,\theta} \) be the linear interpolation of \( \varepsilon^{(n)}_{N,ij} \) defined as (43). Like the argument in (45)-(46), we have

\[
\sup_{t \in [0,1]} \left| \varepsilon^{(n)}_{N,ij}(t) - \varepsilon^{(n)}_{N,ij,\theta}(t) \right| \leq 3 \sup_{k=0,\cdots,\lfloor n^\theta \rfloor, h \leq n^{-\theta}} \int_0^{k_n-h} \int_{\mathbb{U}} |\Delta_{nh}\zeta_i(n(k_n-h-s), u)|_{n} N^{(n)}_j(n \cdot ds, du).
\]

Proceeding as in Step 3 of the proof for Lemma 5.12, we have the foregoing supremum goes to 0 in probability as \( n \to \infty \). Like the proof of Proposition 5.14, we now turn to prove the \( C \)-tightness of the sequence \( \{\varepsilon^{(n)}_{N,ij,\theta}\}_{n \geq 1} \). For any \( t, h \in [0,1] \), using the Cauchy-Schwarz inequality and then (38),

\[
E\left[ |\Delta_{h}\varepsilon^{(n)}_{N,ij}(t)|^{2\alpha} \right] \leq C(n \int_0^{t+h} ds \int_{\mathbb{U}} |\Delta_{nh}\zeta_i(n(t-s), u)|_{\nu_j^{(n)}(du)}^{2\alpha}
\]
The monotonicity of $\zeta_i(\cdot, u)$ induces that
\[
\int_0^{t+h} |\Delta_nh\zeta_i(n(t-s), u)| ds = \int_t^{t+h} \zeta_i(ns, u) ds \leq \|\zeta_i(u)\|_{TV} \cdot h.
\]
By (30), the first term on the right side of (56) can be bounded by
\[
C\left(n \cdot h \int \|\zeta_i(u)\|_{TV} \nu_j^{(n)}(du)\right)^{2\alpha}
\leq Ch^{2\alpha} \left(n \cdot m_{ij}^{(n)} \int_0^\infty \mathcal{P}_{L,i}^{(n)}(dy) \int_T \|T(y)\|_{TV} \mathcal{P}_{T,i}^{(n)}(dT)\right)^{2\alpha},
\]
which can be uniformly bounded by $Ch^{2\alpha}$; see Condition 4.3 and 4.6. Similarly, the second term on the right side of (56) also can be bounded by
\[
C \left(\int_0^\infty \|\zeta_i(u)\|_{TV} \nu_j^{(n)}(du)\right)^\alpha \cdot h^\alpha \leq Ch^\alpha
\]
and the third term can be bounded by $Ch/n^{2\alpha-2}$. Putting all estimates above together, we have $E[|\Delta_h \varepsilon_{N,ij}(t)|^{2\alpha}] \leq C(h^\alpha + h/n^{2\alpha-2})$. Like the proof of Proposition 5.14, we have $\{\varepsilon_{n,ij}\}_{n \geq 1}$ is $C$-tightly bounded and so is the sequence $\{\varepsilon_{n,ij}\}_{n \geq 1}$.

2. The impact of ancestors, $\psi_H^{(n)}$, satisfies Condition 3.20. Similarly as in (53)-(55), the mean instantaneous characteristic of a type-$i$ ancestor at time $t$, denoted as $\check{T}_i^{(n)}(t)$, equals to
\[
\check{T}_i^{(n)}(t) := \int_T \mathcal{P}_{T,i}^{(n)}(dT) \int_T T(t+s,y+s)\check{P}_{H,i}^{(n)}(ds,dy) = \frac{1}{m_{i}^{(n)}} \int_0^\infty \check{T}_i^{(n)}(s) ds
\]
and hence $\psi_i^{(n)}/n$ can be well approximated by $\check{\psi}_i^{(n)}$ with
\[
\check{\psi}_i^{(n)}(t) := \frac{\mathbb{E}_i^{(n)}(0)/n}{m_{i}^{(n)}} \int_0^\infty \check{T}_i^{(n)}(s) ds = \frac{\varepsilon_{i}^{(n)}(0)}{n} \int_0^\infty \check{\psi}_i^{(n)}(s) ds = \frac{\varepsilon_{i}^{(n)}(0)}{n} \int_0^\infty \check{\psi}_i^{(n)}(s) ds
\]
Similarly as in proof of Lemma 5.23, we can prove $\|\psi_{H}^{(n)}/n - \check{\psi}_i^{(n)}\|_{L^{1,\infty}} \to 0$ as $n \to \infty$.

5.6. Proof of Theorem 4.10. We first consider the cumulative characteristic of all ancestors. From Condition 4.4, for any $i \in H$ we have $\Xi_i^{(n)}(0) = O(n)$ as $n \to \infty$ and hence
\[
\mathbb{E}_{i} \left[\sup_{t \geq 0} \psi_i^{(n)}(nt)/n^2\right] \leq \mathbb{E}_{i} [n^{-2}\Xi_i^{(n)}(0)] \cdot T_i^{(n)}(\infty) \to 0.
\]
For $i \in H$ and $j \in H$, we now prove that the cumulative impact of all type-$i$ offspring born by type-$j$ parents can be ignored. Indeed, since $\zeta_i(t, u)$ is non-decreasing in $t$, by Lemma 5.6 we have for $j \in H$.
\[
\mathbb{E}_{i} \left[\sup_{t \in [0,1]} \frac{1}{n^2} \int_0^{nt} \zeta_i(n t - s, u) N_j^{(n)}(ds, du)\right] \leq C \int_0^\infty \psi_i^{(\infty)}(s, u)d\nu_j^{(n)}(du)
\]
which goes to 0 as $n \to \infty$ since $m_{ij}^{(n)} \to 0$; see Condition 4.3. Similarly, the cumulative characteristic of immigrants also can be ignored, i.e.,

$$
\mathbb{E}\left[\sup_{t \in [0,1]} \frac{1}{n^2} \int_0^t \int_U \zeta_i(nt - s, u) N_i^{(n)}(ds, du)\right] \leq \frac{C}{n} \int_U \zeta_i(\infty, u) \nu_i^{(n)}(du) = \frac{C}{n} \cdot T_i^{(n)}(\infty) \cdot m_i^{(n)},
$$

which goes to 0 as $n \to \infty$. In conclusion, the asymptotic behavior of $T_i^{(n)}(nt)/n^2$ is fully determined by

$$
\frac{1}{n^2} \int_0^t \int_U \zeta_i(nt - s, u) N_i^{(n)}(ds, du),
$$

whose weak convergence can be obtained by using Theorem 3.15.

5.7. Proof for Theorem 4.13. Let $C_{\text{Lip}}(\mathbb{R}^2_+)$ be the space of Lipschitz continuous functions on $\mathbb{R}^2_+$ with compact support, which is dense in $C_0(\mathbb{R}^2_+)$. From Theorem 9.1 in [23, p.142], it suffices to prove that the following two claim hold.

(a) The sequence $\{AR_{i,nt}^{(n)} : t \geq 0\}_{n \geq 1}$ satisfies the compact containment condition, i.e., for any $\eta, T > 0$ there exists a compact set $\Gamma_{\eta,T} \subset \mathcal{M}(\mathbb{R}^2_+)$ such that

$$
\inf_{n \geq 1} \mathbf{P}\left(\frac{1}{n} \cdot AR_{i,nt}^{(n)} \in \Gamma_{\eta,T} \text{ for } t \in [0, T]\right) \geq 1 - \eta;
$$

(b) For any $f \in C_{\text{Lip}}(\mathbb{R}^2_+)$, the sequence $\{AR_{i,nt}^{(n)}(f) : t \geq 0\}_{n \geq 1}$ converges weakly to $\{\Xi_i^t(t) \cdot \tilde{P}_{AR,i}^*(f) : t \geq 0\}$ in $\mathcal{D}([0, \infty), \mathbb{R})$ as $n \to \infty$.

Indeed, for any $f \in C_{\text{Lip}}(\mathbb{R}^2_+)$ or $f \equiv 1$, we have $AR_{i,nt}^{(n)}(f) = T_i^{(n)}(t)$ with $P_{i,t}^{(n)}(T(t, y) = f(t, y - t) \cdot 1_{\{y - t > 0\}}) = 1$. In this case, we have $\|T(y)\|_{L^1} + \|T(y)\|_{TV} \leq C(1 + y)$ for any $y \geq 0$. By Condition 4.12, we see that Condition 4.6 is satisfied with

$$
\begin{align*}
\|T_i^{(n)}\|_{L^1} &= \int_0^\infty dt \int_0^\infty f(t, y - t) \cdot 1_{\{y - t > 0\}} \mathcal{P}_{L,i}^{(n)}(dy) \\
&= \int_0^\infty dt \int_0^\infty f(t, z) \mathcal{P}_{L,i}^{(n)}(t + dz) \\
&\to \int_0^\infty dt \int_0^\infty f(t, z) \mathcal{P}_{L,i}^*(t + dz) = m_{L,i}^* \cdot \tilde{P}_{AR,i}^*(f),
\end{align*}
$$

as $n \to \infty$. By Theorem 4.7 and Corollary 4.8, we have

$$
\frac{1}{n} \cdot AR_{i,nt}^{(n)}(f) \Rightarrow m_{L,i}^* \cdot \tilde{P}_{AR,i}^*(f),
$$

in $\mathcal{D}([0, \infty), \mathbb{R})$ as $n \to \infty$. Here we have got claim (b). By the Skorokhod representation theorem, we may assume

$$
\{AR_{i,nt}^{(n)}(f) : t \geq 0\}_{n \geq 1} \overset{\text{a.s.}}{\Rightarrow} \Xi_i^t \cdot \tilde{P}_{AR,i}^*(f),
$$

in $\mathcal{D}([0, \infty), \mathbb{R})$. In particular, by Proposition 1.17 in [42, p.328] and the continuity of $\Xi_i^t$,

$$
\{AR_{i,nt}^{(n)}(1) : t \geq 0\}_{n \geq 1} \overset{\text{a.s.}}{\Rightarrow} \Xi_i^t \cdot \tilde{P}_{AR,i}^*(1),
$$
uniformly on compacts. For any \( T > 0 \) and \( \eta \in (0, 1) \), there exists a constant \( K > 0 \) such that
\[
P \left( \sup_{t \in [0, T]} \frac{1}{n} \cdot \mathcal{A}R_{i,t}^{(n)}(1) \leq K \right) \geq 1 - \eta
\]
and hence claim (a) holds. Here we have got the first convergence result in Theorem 4.13. Similarly as in (28), we have \( \mathcal{P}^{*}_{AR,i}(dy, \mathbb{R}_{+}) = \mathcal{P}^{*}_{AR,i}(\mathbb{R}_{+}, dy) = \mathcal{P}^{*}_{L,i}(dy) \). Hence the second desired convergence result follows from the fact that \( \mathcal{A}^{(n)}_{i,t} \) and \( \mathcal{R}^{(n)}_{i,t} \) are marginal measures of \( \mathcal{A}R_{i,t}^{(n)} \). Finally, the third desired convergence result follows directly from the first one together with the facts that
\[
\mathcal{L}^{(n)}_{i,t}(dy) = \int_{\mathbb{R}_{+}^{1}} \delta_{s+z}(dy) \mathcal{A}R_{i,t}^{(n)}(ds, dz) \quad \text{and} \quad \int_{\mathbb{R}_{+}^{1}} \delta_{s+z}(dy) \mathcal{P}^{*}_{AR,i}(ds, dz) = \mathcal{P}^{*}_{L,i}(dy).
\]

**APPENDIX: STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY \( \mathbb{H} \text{-SEMIMARTINGALES} \)**

In this section we give a brief introduction to stochastic differential equations driven by infinite-dimensional semimartingales; readers may refer to [53] for more details. Let \( \mathbb{H} \) be an arbitrary, separable Banach space endowed with norm \( \| \cdot \|_{\mathbb{H}} \). We now give the definition of \( \mathbb{H} \text{-semimartingales} \).

**Definition A.24.** \( Y \) is an \((\mathcal{F}_{t})\)-adapted \( \mathbb{H} \text{-semimartingale} \), if it is a stochastic process indexed by \( \mathbb{H} \times \mathbb{R}_{+} \) such that

- for each \( f \in \mathbb{H} \), \( \{Y(f, t) : t \geq 0\} \) is a cádlág \((\mathcal{F}_{t})\)-semimartingale with \( Y(f, 0) \overset{a.s.}{=} 0 \);
- for each \( t \geq 0 \), \( \alpha_{1}, \cdots, \alpha_{m} \in \mathbb{R} \) and \( f_{1}, \cdots, f_{m} \in \mathbb{H} \),
\[
Y \left( \sum_{k=1}^{m} \alpha_{k} f_{k}, t \right) \overset{a.s.}{=} \sum_{k=1}^{m} \alpha_{k} Y(f_{k}, t).
\]

Let \( \mathbb{H}_{0} \) be a dense subset of \( \mathbb{H} \) and \( S_{0} \) the collection of \( \mathbb{H} \)-valued stochastic processes of the form
\[
X(t) := \sum_{k=1}^{m} \xi_{k}(t) \varphi_{k} \quad \text{with} \quad \xi_{k}(t) := \sum_{i=0}^{\infty} \eta_{i}^{k} \cdot 1_{(t_{i}, t_{i+1})}(t), \quad t \geq 0,
\]
where \( m \geq 1 \), \( \varphi_{1}, \cdots, \varphi_{m} \in \mathbb{H}_{0} \), \( \{t_{i}\}_{i \geq 0}^{\infty} \) is a sequence of non-decreasing \((\mathcal{F}_{t})\)-stopping times and \( \eta_{i}^{k} \in \mathbb{R}^{d} \) is \( \mathcal{F}_{t_{i}} \)-measurable. For any \( X \in S_{0} \), we define
\[
X_{-} \cdot Y(t) := \sum_{k=1}^{m} \int_{0}^{t} \xi_{k}(s-) dY(t, \varphi_{k}), \quad t \geq 0.
\]
The \( \mathbb{H} \text{-semimartingale} \( Y \) is standard if
\[
\mathcal{H}_{t} := \left\{ \sup_{s \leq t} |X_{-} \cdot Y(s)| : X \in S_{0}, \sup_{s \leq t} \|X(s)\|_{\mathbb{H}} \leq 1 \right\}
\]
is stochastically bounded for each \( t \geq 0 \). In this case, for any \( \mathbb{H} \)-valued cádlág process \( X \), we can find a sequence \( \{X^{\epsilon}\}_{\epsilon \geq 0} \subset S_{0} \) such that as \( \epsilon \to 0 \),
\[
\sup_{t \in [0, T]} \|X^{\epsilon}(t) - X(t)\|_{\mathbb{H}} \overset{a.s.}{\to} 0 \quad \text{and} \quad X_{-} \cdot Y \equiv \lim_{\epsilon \to 0+} X^{\epsilon}_{-} \cdot Y
\]
exists a.s. in the sense that \( \sup_{t \in [0,T]} |X_- \cdot Y(t) - X_- \cdot Y(t)| \xrightarrow{p} 0 \). Moreover, the limit process \( X_- \cdot Y \) is càdlàg, independent of \( \{X^\epsilon\}_{\epsilon > 0} \) and called the stochastic integral of \( X \) with respect to \( Y \). For any \( (\mathcal{F}_t) \)-stopping time \( \sigma \), we have
\[
X_- \cdot Y(t \land \sigma) = X_-'^\sigma \cdot Y(t) \quad \text{and} \quad X_-'^{\sigma}(t) := X_- \cdot 1_{[0,\sigma]}(t), \quad t \geq 0.
\]

**Definition A.25.** We say a sequence of \( \mathbb{H}^\# - \) semimartingales \( \{Y_n\}_{n \geq 1} \) is uniformly tight if \( \{H_{n,t}\}_{n \geq 1} \) is uniformly stochastically bounded for any \( t \geq 0 \) with \( H_{n,t} \) is defined as \( H_t \) with \( Y \) replaced by \( Y_n \).

Moreover, we say it converges weakly to \( Y \) and write \( Y_n \Rightarrow Y \) if
\[
(Y_n(f_1), \ldots, Y_n(f_m)) \xrightarrow{d} (Y(f_1), \ldots, Y(f_m)),
\]
in \( D([0, \infty), \mathbb{R}^m) \) as \( n \to \infty \), for any \( m \geq 1 \) and \( f_1, \ldots, f_m \in \mathbb{H} \).

**APPENDIX: PROOF FOR PROPOSITION 5.1**

We first have \( |\hat{\phi}_{ii}^{(n)}(\lambda)| \leq \|\phi_{ii}^{(n)}\|_{L^1} \) for any \( n \geq 1 \) and \( \lambda \in \mathbb{R} \). Moreover, for any \( \epsilon > 0 \) we can find a nonnegative smooth function \( g_\epsilon(t) \) on \( \mathbb{R}_+ \) satisfying that \( \|\phi_{ii}^{(n)} - g_\epsilon\|_{L^1} \leq \epsilon \) and \( \|g_\epsilon\|_{TV} \leq \|\phi_{ii}^{(n)}\|_{TV} \). Denote by \( \hat{g}_\epsilon \) the Fourier transform of \( g_\epsilon \). Then we have
\[
|\hat{\phi}_{ii}^{(n)}(\lambda)| \leq |\hat{\phi}_{ii}^{(n)}(\lambda) - \hat{g}_\epsilon(\lambda)| + |\hat{g}_\epsilon(\lambda)| \leq \|\phi_{ii}^{(n)} - g_\epsilon\|_{L^1} + |\hat{g}_\epsilon(\lambda)|.
\]

By the differentiation property of Fourier transform, we have
\[
\hat{g}_\epsilon(\lambda) = \frac{1}{\lambda} \int_0^\infty e^{\lambda t} \frac{\partial}{\partial t} g_\epsilon(t) dt \quad \text{and hence} \quad |\hat{g}_\epsilon(\lambda)| \leq \frac{\|g_\epsilon\|_{TV}}{|\lambda|} \leq \frac{\|\phi_{ii}^{(n)}\|_{TV}}{|\lambda|}.
\]

From these estimates and the arbitrariness of \( \epsilon \), we have \( \hat{\phi}_{ii}^{(n)}(\lambda) \leq \|\phi_{ii}^{(n)}\|_{TV} / |\lambda| \) and hence
\[
|\hat{\phi}_{ii}^{(n)}(\lambda)| \leq \|\phi_{ii}^{(n)}\|_{L^1} \land \frac{\|\phi_{ii}^{(n)}\|_{TV}}{|\lambda|}.
\]

The first inequality in (33) follows directly from (12).

We now start to prove the second inequality in (33). From the hypothesis (H2) and Condition 3.1, there exist constants \( n_0 \geq 1 \) and \( T_0 > 0 \) such that for any \( n \geq n_0 \),
\[
\int_{T_0}^\infty t \phi_{ii}^{(n)}(t) dt \leq \int_{T_0}^\infty \frac{1}{t} \phi_{ii}^{(n)}(t) dt \leq \frac{1}{8} \cdot \sigma_i \quad \text{and} \quad \int_0^\infty t \phi_{ii}^{(n)}(t) dt \geq \frac{3}{8} \cdot \sigma_i.
\]

Since \( \cos(x) \geq 1/2 \) for any \( |x| \leq 1 \), we have for any \( |\lambda| \leq 1/T_0 \),
\[
\frac{\partial}{\partial \lambda} \int_0^\infty \sin(\lambda t) \phi_{ii}^{(n)}(t) dt = \int_0^\infty \cos(\lambda t) \cdot t \cdot \phi_{ii}^{(n)}(t) dt \geq \int_0^{T_0} \frac{1}{2} \cdot t \phi_{ii}^{(n)}(t) dt - \int_{T_0}^\infty t \phi_{ii}^{(n)}(t) dt \geq \frac{3}{16} \cdot \sigma_i.
\]

By the mean value theorem, we have for any \( |\lambda| \leq 1/T_0 \),
\[
|1 - \hat{\phi}_{ii}^{(n)}(\lambda)| \geq \left| \int_0^\infty \sin(\lambda t) \phi_{ii}^{(n)}(t) dt \right| \geq \frac{3}{16} \cdot \sigma_i \cdot |\lambda|.
\]

Here we have proved the desired result for \( |\lambda| \leq 1/T_0 \). For \( |\lambda| > 1/T_0 \), from Proposition 5.1, there exists a constant \( \lambda_0 > 0 \) such that
\[
|\hat{\phi}_{ii}^{(n)}(\lambda)| \leq \frac{1}{2} \quad \text{and hence} \quad |1 - \hat{\phi}_{ii}^{(n)}(\lambda)| \geq \frac{1}{2},
\]
for any \( n \geq 1 \) and \( |\lambda| \geq \lambda_0 \). It is obvious that the desired result follows if \( 1/T_0 \geq \lambda_0 \) and then the proof ends. If \( 1/T_0 < \lambda_0 \), it suffices to prove (33) holds for \( \lambda \in [1/T_0, \lambda_0] \). Notice that

\[
|1 - \phi_{ii}^{(n)}(\lambda)| \geq 1 - \int_0^\infty \cos(\lambda t)\phi_{ii}^{(n)}(t)dt =: 1 - F^{(n)}(\lambda).
\]

The continuity of \( F^{(n)} \) induces that \( \lambda_n := \arg \max_{|\lambda| \in [1/T_0, \lambda_0]} F^{(n)}(\lambda) \) is well defined. For any \( T > 0 \), since \( \cos(t) \leq 1 \) we have

\[
F^{(n)}(\lambda_n) \leq \int_0^T \cos(\lambda_n t)\phi_{ii}^{(n)}(t)dt + \int_T^\infty \phi_{ii}^{(n)}(t)dt.
\]

Using the hypothesis (H2) again, we can choose \( T > 0 \) large enough such that

\[
\sup_{n \geq 1} \int_T^\infty \phi_{ii}^{(n)}(t)dt \leq \frac{1}{T} \int_T^\infty t \cdot \tilde{g}(t)dt \leq \frac{1}{2}
\]

and hence

\[
\inf_{n \geq 1} \int_0^T \phi_{ii}^{(n)}(t)dt \geq \frac{1}{2}.
\]

By the periodicity of \( \cos(\lambda_n t) \), we have

\[
\int_0^T \cos(\lambda_n t)\phi_{ii}^{(n)}(t)dt \leq \sum_{k=0}^{[T\lambda_n/(2\pi)]} \int_{(2k\pi+\pi/2)/\lambda_n}^{(2k\pi+\pi/2)/\lambda_n} \cos(\lambda_n t)\phi_{ii}^{(n)}(t)dt.
\]

We now start to analyze the maximum of the sum above. Notice that \( \cos(\lambda_n t) \) is unimodal on each interval \([2k\pi - \pi/2)/(\lambda_n, (2k\pi + \pi/2)/(\lambda_n)\) for any \( k \geq 0 \) with the maximum arrived at the point \( 2k\pi/\lambda_n \). Thus the more weight of \( \phi_{ii}^{(n)} \) is distributed around the local maximum points, the larger the sum above will be. To obtain the maximum of the summation in (58) we should split the weight of \( \int_0^T \phi_{ii}^{(n)}(t)dt \) uniformly around these maximum points. In precise, we choose \( T > 0 \) large enough such that

\[
R_{\lambda_n} := \frac{\lambda_n \int_0^T \phi_{ii}^{(n)}(t)dt}{2\|\phi_{ii}^{(n)}\|_{TV} \cdot ([T\lambda_n/(2\pi)] + 1)} < 1.
\]

From the previous observation and the fact that \( \cos(\lambda_n t) \leq 1 \), we have for any \( k \geq 0 \),

\[
\int_{(2k\pi+\pi/2)/\lambda_n}^{(2k\pi+\pi/2)/\lambda_n} \cos(\lambda_n t)\phi_{ii}^{(n)}(t)dt \leq \|\phi_{ii}^{(n)}\|_{TV} \int_{(2k\pi+\pi/2)/\lambda_n}^{(2k\pi+\pi/2)/\lambda_n} \cos(\lambda_n t)dt
\]

\[
= \|\phi_{ii}^{(n)}\|_{TV} \int_{-R_{\lambda_n}}^{R_{\lambda_n}} \cos(t)dt
\]

\[
= \frac{\int_0^T \phi_{ii}^{(n)}(t)dt}{[T\lambda_n/(2\pi)] + 1} \cdot \frac{\sin(R_{\lambda_n})}{R_{\lambda_n}}.
\]

Taking this back into (58) and then (57), we have

\[
F^{(n)}(\lambda_n) \leq \frac{\sin(R_{\lambda_n})}{R_{\lambda_n}} \int_0^T \phi_{ii}^{(n)}(t)dt + \int_T^\infty \phi_{ii}^{(n)}(t)dt
\]

and hence

\[
\inf_{|\lambda| > 1/T_0} |1 - \phi_{ii}^{(n)}(\lambda)| \geq 1 - \|\phi_{ii}^{(n)}\|_{L^1} + \left(1 - \frac{\sin(R_{\lambda_n})}{R_{\lambda_n}}\right) \cdot \int_0^T \phi_{ii}^{(n)}(t)dt
\]

\[
\inf_{|\lambda| > 1/T_0} |1 - \phi_{ii}^{(n)}(\lambda)| \geq 1 - \|\phi_{ii}^{(n)}\|_{L^1} + \left(1 - \frac{\sin(R_{\lambda_n})}{R_{\lambda_n}}\right) \cdot \int_0^T \phi_{ii}^{(n)}(t)dt
\]
\[ \geq \frac{1}{2} \left( 1 - \frac{\sin(R_{\lambda_n})}{R_{\lambda_n}} \right). \]

From the fact that \( \lambda_n \in (1/T_0, \lambda_0) \) for any \( n \geq 1 \) and sup\( n \geq 1 \| \phi_{ii}^{(n)} \|_{TV} < \infty \), we have

\[ \inf_{n \geq 1} R_{\lambda_n} > 0 \quad \text{and hence} \quad \sup_{n \geq 1} \frac{\sin(R_{\lambda_n})}{R_{\lambda_n}} < 1. \]

Consequently, there exists a constant \( C_0 > 0 \) such that for any

\[ \inf_{n \geq n_0} \inf_{|\lambda| > 1/T_0} \left| 1 - \phi_{ii}^{(n)}(\lambda) \right| \geq C_0 (|\lambda| \wedge 1). \]

Putting all estimates above together, we can immediately get the desired result with \( C_2 := (3\sigma_i/16) \wedge (1/2) \wedge C_0. \)

\[ \square \]

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