Numerical solution of nonlinear mixed Volterra-Fredholm integro-differential equations by two-dimensional block-pulse functions

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Abstract: This paper proposed an effective numerical method to obtain the solution of nonlinear two-dimensional mixed Volterra-Fredholm integro-differential equations. For this purpose, the two-dimensional block-pulse functions (2D-BPFs) operational matrix of integration and differentiation has been presented. The 2D-BPFs method converts nonlinear two-dimensional mixed Volterra-Fredholm integro-differential equations to an algebraic system of equations which is computable as well. Error analysis and some numerical examples are presented to illustrate the effectiveness and accuracy of the method.

Subjects: Differential Equations; Integral Transforms & Equations; Mathematical Numerical Analysis; Applied Mathematics

Keywords: two-dimensional Volterra–Fredholm integral equations; operational matrix; nonlinear equations; two-dimensional block-pulse functions

MR Subject classifications: 65Gxx; 45G10; 65R20

1. Introduction
The modeling of the most phenomena in real life from engineering and physics to mechanics and etc leads to nonlinear equations. Meanwhile, the analytical solution of them in most of the time are not exist, so we need a powerful method to approximate the exact solutions.

The mixed Volterra-Fredholm integro-differential equation is one of the most important equation release in theory of parabolic boundary value problems, population dynamics, the mathematical

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PUBLIC INTEREST STATEMENT
The solutions of Partial Differential and Integral Equations (PDEs and IEs) with numerical methods play the most important role among researchers and scientific communities due to the fact that the analytical solution of them in most of the time does not exist. The concept of approximation almost along with error estimation, accuracy, efficiency, the reliability of the methods and etc. In this approach, we use the method which known as Black-Pulse Functions (BPFs) to find the solution of an equation which release in theory of parabolic boundary value problems, population dynamics, the mathematical modeling of the spatio-temporal development and in many physical and biological models.
modeling of the spatio-temporal development and in many physical and biological models (Diekmann, 1978; Khajehnasiri, 2016; Pachpatte, 1986; Thieme, 1977).

In 1977, Harmuth (Harmuth, 1969) presented the Block-Pulse Functions (BPFs) as a mathematical tool for approximate the problems. The BPFs which are defined in the time interval $[0, T_1]$ are a set of orthogonal functions with piecewise constant values such as:

$$
    \psi_i(t) = \begin{cases} 
    1, & (i - 1) \frac{T_1}{m} \leq t \leq i \frac{T_1}{m}, \\
    0, & \text{otherwise},
    \end{cases}
$$

(1.1)

where $i = 0, \ldots, m - 1$ with $m$ as a positive integer.

The solution of Fredholm and Volterra integral equations of the second kind have been approximated by using BPFs in (Kung & Chen, 1978). Maleknejad and Mahmoudi in (Maleknejad & Mahmoudi, 2004) have applied a combination of Taylor and Block-Pulse Functions to solve linear Fredholm integral equation.

The BPFs and Lagrange-interpolating polynomials have been used to approximate the solution of Volterra’s population model by Marzban et al. in (Marzban, Hoseini, & Razzaghi, 2009). Maleknejad and Mahdiani have applied two dimensional Block-Pulse functions 2D-BPFs for solving nonlinear mixed Volterra-Fredholm integral equations (Maleknejad & Mahdiani, 2011), Aghazadeh also developed the Block-pulse functions technique to solve the nonlinear two dimensional Volterra integro-differential equation (Aghazadeh & Khajehnasiri, 2013), Khajehnasiri et al. have extended the BPFs method for the solving systems of higher-order nonlinear Volterra integro-differential equations (Ebadian & Khajehnasiri, 2014), Nemati in (Nemati, Lima, & Ordokhani, 2013) solving a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials. Hesameddini et al. used hybrid Bernstein Block-Pulse functions for solving system of Volterra-Fredholm integral equations (Hesameddini & Shahbazi, 2017).

In the past decay, scientists have been used some methods to approximate the solution of the mixed Volterra-Fredholm integro-differential equation such as Legendre collocation method (Rohaninaso, Maleknejad, & Ezzati, 2018), Fixed point techniques (Berenguer et al., 2013), Bernstein polynomials (Yuzbas, 2016), A Bernstein operational matrix (Maleknejad, Basirat, & Hashemizadeh, 2012), Bessel collocation method (Yuzbas, 2015), Tau method (Shahmorad, 2005). But on the other hand, there are not suitable work on the partial mixed volterra-fredholm integral equations in this paper we study this kind of equations.

$$
a(t, x) \frac{\partial^p u(t, x)}{\partial x^p} + b(t, x) \frac{\partial^{n+m} u(t, x)}{\partial x^n \partial t^m} + c(t, x) \frac{\partial^m u(t, x)}{\partial t^m} + u(t, x) = f(t, x) + \int_{\Omega} k(t, s, x, y) G(u(s, y)) dy ds, \quad (t, x) \in [0, T] \times \Omega.
$$

(1.2)

with given supplementary conditions, $a(t, x), b(t, x), c(t, x)$ are given continuous functions, where $u(t, x)$ is an unknown function which should be determined, $f(t, x)$ and $k(t, s, x, y)$ are analytical functions on $D = [0, T] \times \Omega$ and $D \times \Omega^2$, respectively; $\Omega$ is a closed bounded region in $\mathbb{R}^n$ ($n = 1, 2, 3$) with piecewise smooth boundary $\partial \Omega$ (Brunner, 2004). In this paper, we consider the nonlinear function $G(u(s, y))$ in the following form

$$
G(u(s, y)) = u^p(s, y).
$$

where $p$ is a positive integer. With regard to the fact that every finite interval can be transformed to $[0, 1]$ by linear map, without loss of generality, we can consider $\Omega = [0, 1]$ and $[0, T] = [0, 1]$.

This paper is organized as follows. In Section 2, definition and some properties of the 2D-BPFs has presented. The 2D-BPFs are applied to solve Equation (1.2) in Section 3. The error analysis of
the proposed method has been investigated in Section 4. Some numerical results has been presented in Section 5 to show accuracy and efficiency of the proposed method. Finally, some concluding remarks are given in section 6.

2. Properties of the 2D-BPFs

We usually call the block-pulse functions containing two variables as two-dimensional block-pulse functions 2D-BPFs. An \((m_1, m_2)\)-set of 2D-BPFs are defined in region \(t \in [0, T_1)\) and \(x \in [0, T_2)\) as:

\[
\psi_{i_1,i_2}(t, x) = \begin{cases} 1, & (i_1 - 1)h_1 \leq t < i_1h_1 \text{ and } (i_2 - 1)h_2 \leq x < i_2h_2, \\ 0, & \text{otherwise.} \end{cases}
\]  

(2.1)

where \(i_1 = 1, 2, \ldots, m_1\) and \(i_2 = 1, 2, \ldots, m_2\) with positive integer values for \(m_1, m_2\), and \(h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}\). There are some properties for 2D-BPFs; e.g. disjointness, orthogonality, and completeness.

1. Disjointness

The two-dimensional block-pulse functions are disjoined with each other, i.e.

\[
\psi_{i_1,i_2}(t, x)\psi_{j_1,j_2}(t, x) = \begin{cases} \psi_{i_1,i_2}(t, x), & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise.} \end{cases}
\]  

(2.2)

2. Orthogonality

The two-dimensional block-pulse functions are orthogonal with each other, i.e.

\[
\int_0^{T_1} \int_0^{T_2} \psi_{i_1,i_2}(t, x)\psi_{j_1,j_2}(t, x)dxdt = \begin{cases} h_1h_2, & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise,} \end{cases}
\]  

(2.3)

in the region of \(t \in [0, T_1)\) and \(x \in [0, T_2)\) where \(i_1, j_1 = 1, 2, \ldots, m_1\) and \(i_2, j_2 = 1, 2, \ldots, m_2\).

3. Completeness

For every \(g \in L^2([0, T_1] \times [0, T_2])\) when \(m_1\) and \(m_2\) goes to infinity, Parseval identity holds:

\[
\int_0^{T_1} \int_0^{T_2} g(t, x)dxdt = \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \|g_{i_1,i_2}(t, x)\|^2 ,
\]  

(2.4)

where

\[
g_{i_1,i_2} = \frac{1}{h_1h_2} \int_0^{T_1} \int_0^{T_2} g(t, x)\psi_{i_1,i_2}(t, x)dxdt.
\]  

(2.5)

The set of 2D-BPFs may be written as a \((m_1, m_2)\)-vector \(\psi(t, x)\):

\[
\Psi(t, x) = [\psi_{1,1}(t, x), \psi_{1,2}(t, x), \ldots, \psi_{m_1,1}(t, x), \ldots, \psi_{m_1,m_2}(t, x)]^T ,
\]  

(2.6)

where \((t, x) \in [0, T_1) \times [0, T_2]\). From the above representation and disjointness property, it follows:

\[
\Psi(t, x)\Psi^T(t, x) = \begin{pmatrix}
\psi_{1,1}(t, x) & 0 & \cdots & 0 \\
0 & \psi_{2,2}(t, x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \psi_{m_1,m_2}(t, x)
\end{pmatrix} ,
\]  

(2.7)

\[
\Psi^T(t, x)\Psi(t, x) = 1,
\]  

(2.8)
\[ \Psi(t, x)\Psi^T(t, x)\Lambda = \tilde{\Lambda}\Psi(t, x), \]  

(2.9)

where \( \Lambda \) is an \( m_1m_2 \)-vector and \( \tilde{\Lambda} = \text{diag}(\Lambda) \). Moreover, it can be clearly concluded that for every \((m_1m_2) \times (m_1m_2) \) matrix \( B \):

\[ \Psi(t, x) B\Psi(t, x) = \tilde{B}\Psi(t, x) \]  

(2.10)

where \( \tilde{B} \) is an \( m_1m_2 \)-vector with elements equal to the diagonal entries of matrix \( B \).

### 3. 2D-BPFs expansion

A function \( g(t, x) \in L^2([0, T_1] \times [0, T_2]) \) may be expanded by the 2D-BPFs as:

\[ g(t, x) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g_{i,j} \Psi_{i,j}(t, x) = G^T \Psi(t, x) = \Psi^T(t, x)G, \]

(2.11)

where \( G \) is an \( (m_1m_2) \)-vector given by

\[ G = [g_{1,1}, \ldots, g_{1,m_2}, \ldots, g_{m_1,1}, \ldots, g_{m_1,m_2}]^T, \]

and \( \Psi(t, x) \) is defined in (2.6).

The block-pulse coefficients \( g_{i,j} \) are obtained as

\[ g_{i,j} = \frac{1}{h_1h_2} \int_{j-1/2}^{j+1/2} \int_{i-1/2}^{i+1/2} g(t, x) dx dt. \]

(2.13)

such that the error between \( g(t, x) \), and its block-pulse expansion (2.11) in the region of \( t \in [0, T_1) \), \( x \in [0, T_2) \); i.e.,

\[ \varepsilon = \frac{1}{T_1T_2} \int_0^{T_1} \int_0^{T_2} \left( g - \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} g_{i,j} \Psi_{i,j}(t, x) \right)^2 dx dt \]

(2.14)

is minimal. Since each two-dimensional block-pulse function takes only one value in its subregion, the 2D-BPFs can be expressed by the two 1D-BPFs:

\[ \Psi_{i,j}(t, s) = \Psi_{i}(t)\Psi_{j}(s), \]

(2.15)

where, \( \Psi_{i}(t) \) and \( \Psi_{j}(s) \) are 1D-BPFs related to the variables \( t \) and \( s \), respectively.

A function of four variables \( k(t, s, x, y) \), on \([0, T_1] \times [0, T_2] \times [0, T_3] \times [0, T_4] \) may be approximated with respect to BPFs such as:

\[ k(t, s, x, y) = \Psi^T(t, x)K\Psi(s, y), \]

(2.16)

where \( \Psi(t, x) \) and \( \Psi(s, y) \) are 2D-BPFs vectors of dimension \( m_1m_2 \) and \( m_3m_4 \), respectively, and \( K \) is a \((m_1m_2) \times (m_3m_4) \) two dimensional block-pulse coefficients matrix. Also, the positive integer powers of a function \( u(s, y) \) may be approximated by 2D-BPFs as

\[ [u(s, y)]^p = \Psi^T(s, y)U^p = \Psi^T(s, y)\Theta, \]

(2.17)

where \( \Theta \) is a column vector, whose elements are \( p \)th power of the elements of the vector \( U \).

### 3.1. Operational matrix of integration

The integration of the vector \( \Psi(t, x) \) defined in (2.1) may be obtained as
\[
\int_0^t \int_0^x \Psi(r_1, r_2) dr_1 dr_2 \simeq E \Psi(t, x) \tag{2.18}
\]
\[
= [Y_{(m_1 \times m_1)} \otimes Y_{(m_2 \times m_2)}] \Psi(t, x).
\tag{2.19}
\]
where \( E \) is a \((m_1 \times m_2) \times (m_1 \times m_2)\) operational matrix of integration for 2D-BPFs and \( Y \) is the operational matrix of 1D-BPFs defined over \([0, 1]\) with \( h = \frac{1}{m} \) as follows
\[
Y = \frac{h}{2} \begin{pmatrix}
1 & 2 & 2 & \ldots & 2 \\
0 & 1 & 2 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\tag{2.20}
\]
In (2.19), \( \otimes \) denotes the Kronecker product defined as:
\[
A \otimes B = (a_{ij}B).
\tag{2.21}
\]
So, the 2D integral of every function \( g(t, x) \) can be approximated as follows:
\[
\int_0^t \int_0^x g(r_1, r_2) dr_1 dr_2 \simeq \int_0^t \int_0^x G^T \Psi(r_1, r_2) dr_1 dr_2 \simeq G^T E \Psi(t, x) \tag{2.22}
\]

3.2. Operational matrix of differentiation

We now need to compute the operational matrix of differentiation. For this, let
\[
\begin{align*}
u(t, x) &= U^T \Psi(t, x), \\
u(t, 0) &= U^T_{10} \Psi(t, x), \\
u(0, x) &= U^T_{0} \Psi(t, x), \\
\upsilon_s(t, x) &= U^T_s \Psi(t, x), \\
\upsilon_t(t, x) &= U^T_t \Psi(t, x), \\
\upsilon_s(0, x) &= U^T_{0s} \Psi(t, x), \\
\upsilon_t(0, x) &= U^T_{10} \Psi(t, x), \\
\upsilon_{sx}(t, x) &= U^T_{sx} \Psi(t, x), \\
\upsilon_{tx}(t, x) &= U^T_{tx} \Psi(t, x).
\end{align*}
\tag{2.23}
\]
Now, we can write:
\[
u(t, x) - \nu(t, 0) = \int_0^x \upsilon_s(t, \tau) d\tau,
\tag{2.24}
\]
then from (2.23) and (2.24), we obtain
\[
U^T \Psi(t, x) - U^T_{10} \Psi(t, x) = \int_0^x U^T_s \Psi(t, \tau) d\tau
\]
\[
= U^T_s \int_0^x \Psi(t, \tau) d\tau
\]
\[
= U^T_{ts} \Psi(t, x).
\]
So we get
\[
U^T - U^T_{10} = U^T_{ts} Y.
\]
\[ U^T_t = (U^T_t - U^T_{0t}) Y^{-1}. \]  

(2.25)

Similarly, for the partial derivative of \( u(t, x) \) with respect to \( t \), it can be shown that
\[ U^T_t = (U^T_t - U^T_{0t}) Y^{-1}. \]  

(2.26)

Moreover, for the second-order partial derivatives of \( u(t, x) \), the following equations can be written:
\[ u_x(t, x) - u_x(t, 0) = \int_0^x u_{xx}(t, \tau) d\tau. \]  

(2.27)

by using (2.23) and (2.27), we have
\[ U^T_t \psi(t, x) - U^T_{0t0} \psi(t, x) = \int_0^t U^T_{tx} \psi(t, \tau) d\tau, \]
\[ = U^T_{xx} \int_0^t \psi(t, \tau) d\tau, \]
\[ = U^T_{xx} \psi(t, x), \]
so we get
\[ U^T_t - U^T_{0t0} = U^T_{xx} \psi(t, x), \]
then
\[ U^T_{xx} = (U^T_t - U^T_{0t0}) Y^{-1}. \]  

(2.28)

In the similar way, to approximate the second-order partial derivatives of \( u(t, x) \) with respect to \( t \), the following equation has been obtained:
\[ U^T_{tt} = (U^T_t - U^T_{0tx}) Y^{-1}. \]  

(2.29)

Finally, the following procedure can be applied to approximate \( u_{tx}(t, x) \),
\[ u_{tx}(t, x) - u_{tx}(t, 0) = \int_0^x u_{tx}(t, \tau) d\tau. \]  

(2.30)

hence
\[ U^T_t \psi(t, x) - U^T_{0tx} \psi(t, x) = \int_0^t U^T_{tx} \psi(t, \tau) d\tau, \]
\[ = U^T_{tx} \int_0^t \psi(t, \tau) d\tau, \]
\[ = U^T_{tx} \psi(t, x), \]
so we can obtain,\[ U^T_t - U^T_{0tx} = U^T_{tx} \psi(t, x), \]
then we have
\[ U^T_{tx} = (U^T_t - U^T_{0tx}) Y^{-1}. \]  

(2.31)
4. Applying the method

In this section, we use the 2DBPFs to solve the nonlinear two-dimensional mixed Volterra-Fredholm integro-differential equations. According to the previous section we have:

\[ u(t, x) = U^T \Psi(t, x), \]
\[ f(t, x) = F^T \Psi(t, x), \]
\[ |u(s, y)|^p = \Psi^T(s, y) \Lambda, \]
\[ u_k(t, x) = U^T_k \Psi(t, x), \]
\[ u_i(t, x) = U^T_i \Psi(t, x), \]
\[ u_{xx}(t, x) = U^T_{xx} \Psi(t, x), \]
\[ u_{lt}(t, x) = U^T_{lt} \Psi(t, x), \]
\[ u_{tx}(t, x) = U^T_{tx} \Psi(t, x), \]
\[ k(t, s, x, y) = \Psi^T(t, x) K \Psi(s, y), \]

where the \( m_1 m_2 \)-vectors \( U, F, \Lambda, U_x, U_l, U_{xx}, U_{lt}, U_{tx} \) and matrix \( K \) are the BPFs coefficients of \( u(t, x), f(t, x), |u(s, y)|^p, u_k(t, x), u_i(t, x), u_{xx}(t, x), u_{lt}(t, x), u_{tx}(t, x) \) and \( k(x, y, s, t) \) respectively, and \( \Theta \) is a column vector whose elements are \( p \)th power of the elements of the vector \( U \). Now, we consider the following equation,

\[ u_{xx} + u_{tx} + u_{lt} + u(t, x) = f(t, x) + \int_0^1 \int_0^1 k(t, s, x, y) u^p(s, y) dy ds, \quad (t, x) \in [0, T] \times \Omega. \]

Using the presented equations in the previous section to evaluated the partial derivatives, Initially, we considered the integral part in (1.2). From Equation (3.1) we have,

\[ \int_0^1 \int_0^1 k(t, s, x, y) u(s, y) dy ds = \int_0^1 \int_0^1 \Psi^T(t, x) K \Psi(s, y) \Psi^T(s, y) \Lambda dy ds \]
\[ = \Psi^T(t, x) K \left( \int_0^1 \int_0^1 \Psi(s, y) \Psi^T(s, y) dy ds \right) \Lambda. \]

Using Equations (2.15) and (2.7), denoting \( L_j \) for the \( j \)th row of the conventional integration operational matrix \( E \) and considering \( \int_0^1 \Psi(t) \, dt = P \) follow:

\[ \int_0^1 \Psi(s, y) \Psi^T(s, y) dy ds = \]

\[ = \begin{pmatrix}
\int_0^1 \int_0^1 \Psi_1(s) \Psi_1(y) dy ds & 0 & \cdots & 0 \\
0 & \int_0^1 \int_0^1 \Psi_2(s) \Psi_2(y) dy ds & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \int_0^1 \int_0^1 \Psi_{m_1}(s) \Psi_{m_1}(y) dy ds
\end{pmatrix}. \]

\[ = \begin{pmatrix}
PL_1 \Psi(t) & 0 & \cdots & 0 \\
0 & PL_2 \Psi(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & PL_n \Psi(t)
\end{pmatrix}. \]
Substituting (3.3) into (3.2) and setting it in Equation (1.2) gives:
\[
\Psi^T(t, x)(U_{xx} + U_{tx} + U_{tt} + U) = \Psi^T(t, x)F + \Psi^T(t, x)QG
\]
Finally, we can determine the block-pulse coefficients of \((u(t, x))^P\) by solving:
\[
U_{xx} + U_{tx} + U_{tt} + U = F + QG, \tag{3.5}
\]
where \(g_{ij} = (u_{ij})^P\) and \(Q = (Q_{iz})\) is a lower triangular block matrix with \(1 \leq l \leq m_2\) and \(1 \leq z \leq m_1\) and
\[
Q_{iz} = \begin{cases} 
0, & l < z, \\
\frac{P^3}{2}(k_j), & l = z, \\
\frac{P^2}{2}(k_j), & l > z,
\end{cases} \tag{3.6}
\]
for \(m_1(l - 1) + 1 \leq i \leq m_1l\) and \((z-1)m_2 + 1 \leq j \leq zm_2\). \(0\) is a zero matrix. If we have \(u(t, x) \geq 0\) for every \((t, x) \in D = [0, 1] \times [0, 1]\), then an approximate solution \(u(t, x) = \Psi^T(t, x)U\) can be computed for Equation (1.2) by setting \(u_{ij} = (g_{ij})^P\), where \(u_{ij}\) and \(g_{ij}\) are the elements of vectors \(U\) and \(G\) respectively.

Now, by using the Equations (2.25), (2.26), (2.28), (2.29), (2.31) and (3.7), we can obtain:
\[
AU = G, \tag{3.8}
\]
where \(A\) and \(G\) are the combination of block-pulse coefficient matrix, After which without using any projection method, we can evaluate the approximation of the solution of \(u = U^T\Psi(x, y)\) for Equation (1.2). The Equation (3.8) can be solved by using Newton iterative method.

5. The error analysis
In this section we evaluate the representation error of a differentiable function \(u(t, x)\) while it is in a series form of 2D-BPFs on \(D = [0, 1] \times [0, 1]\). For this purpose, we review and use some results from (Maleknejad & Mahdiani, 2011; Maleknejad, Sohrabi, & Baranji, 2010) briefly. We set \(m_1 = m_2 = m\), so \(h_1 = h_2 = \frac{1}{m}\).

We can define as well as evaluate the representation error between \(u(t, x)\) and its 2D-BPFs expansion, on any subregion \(D_{h_{i,j}}\) as follows.
\[
e_{h_{i,j}}(t, x) = g_{h_{i,j}}\psi_{h_{i,j}}(t, x) - g(t, x) = g_{h_{i,j}}(t, x) - g(t, x), \quad t, x \in D_{h_{i,j}}, \tag{4.9}
\]
where
\[
D_{h_{i,j}} = \left\{(t, x): \frac{i_1 - 1}{m} \leq t < \frac{i_1}{m}, \frac{i_2 - 1}{m} \leq t < \frac{i_2}{m}\right\} \tag{4.10}
\]
Using mean value theorem, it can be shown that
\[
\|e_{h_{i,j}}\|^2 \leq \frac{2}{m^2} M^2, \tag{4.11}
\]
where \(\|g(t, x)\| \leq M\) (Maleknejad & Mahdiani, 2011; Maleknejad et al., 2010). Error of \(g(t, x)\) and \(g_m(t, x)\), on \(D\), can be shows by:
\[
e(t, x) = g_m(t, x) - g(t, x). \tag{4.12}
\]
Using Equations (4.11) and (4.12), it can be shown that (see (Maleknejad & Mahdiani, 2011; Maleknejad et al., 2010)):
\[ \| e(t, x) \|^2 \leq \frac{2}{m^2} M^2. \]  

(4.13)

Hence, \( \| e(t, x) \| = O(\frac{1}{m}) \). According to (Maleknejad & Mahdiani, 2011; Maleknejad et al., 2010), assume that \( u(t, x) \) is evaluated by,

\[ g_m(t, x) = \sum_{i=1}^{m} \sum_{j=1}^{m} g_{i,j} \varphi_{i,j}(t, x). \]

We set \( \bar{g}_{i,j} \), the approximation of \( g_{i,j} \), and

\[ \bar{g}_m(t, x) = \sum_{i=1}^{m} \sum_{j=1}^{m} \bar{g}_{i,j} \varphi_{i,j}(t, x). \]

then from Equation (4.13) for \( (t, x) \in D_{\bar{g}_{i,j}} \) we have

\[ \| \bar{g}_{i,j} \varphi_{i,j} - g(t, x) \| \leq \frac{\sqrt{2} M}{m} + \frac{\| g_m - g \|_{\infty}}{m}. \]

(4.14)

Therefore, from Equation (4.14), it can be concluded that:

\[ \lim_{m \to \infty} g_m(t, x) = g(t, x). \]

For an error estimation, reconsider the following mixed Volterra-Fredholm integro-differential equations

\[ u_{xx}(t, x) + u_{tx}(t, x) + u_{tt}(t, x) + u(t, x) = f(t, x) + \int_0^t \int_0^1 k(t, s, x, y)u^p(s, y)dyds, \quad (t, x) \in [0.1] \times [0.1]. \]

(4.15)

Let \( e_m^p(t, x) = u^p(t, x) - \tilde{u}_m^p(t, x) \) be the error function of the approximate solution \( u_m(t, x) \) to \( u(t, x) \), where \( u(t, x) \) is the exact solution of Equation (4.15). Then, we consider

\[ R_m(t, x) = (u_{xx}(t, x) + u_{tx}(t, x) + u_{tt}(t, x) + u(t, x))_m = f(t, x) + \int_0^t \int_0^1 k(t, s, x, y)u^p_m(s, y)dyds, \]

(4.16)

where \( R_m(t, x) \) is the perturbation function that depends on \( u_m(t, x), (u_{xx}(t, x))_m, (u_{tx}(t, x))_m \) and \( (u_{tt}(t, x))_m \). It can be obtained by substituting \( u_m(t, x), (u_{xx}(t, x))_m, (u_{tx}(t, x))_m \) and \( (u_{tt}(t, x))_m \) into the Equation (4.15) as:

\[ R_m(t, x) = f(t, x) + \int_0^t \int_0^1 k(t, s, x, y)u^p_m(s, y)dyds - (u_{xx}(t, x) + u_{tx}(t, x) + u_{tt}(t, x) + u(t, x))_m. \]

by subtracting (4.16) from (4.15) we have:

\[ \int_0^t \int_0^1 k(t, s, x, y)e^p_m(s, y)dyds = -R_m(t, x) - (e_{xx}(t, x) + e_{tx}(t, x) + e_{tt}(t, x) + e(t, x))_m. \]

(4.17)

Finally, the proposed method in this paper, can be applied to approximate \( e_m(t, x) \) in the Equation (4.17).

6. Numerical results

In this section, we will use the 2D-BPFs to nonlinear mixed Volterra-Fredholm integro-differential equations with variable coefficients. To demonstrate the superiority and the practicability of this approach, two test examples are carried out in this section.
Example 1 In this example, we consider a two-dimensional mixed Volterra-Fredholm integro-differential equation:

$$\frac{\partial^2 u(t, x)}{\partial t^2} + u(t, x) + \int_0^1 \int_0^1 e^{y}u(s, y)dyds = g(t, x), \quad t, x \in [0, 1],$$

where

$$g(t, x) = -\frac{1}{12} + \frac{1}{3}e^t - \frac{1}{4}e^t\cos(t) + \frac{1}{2}e^t\sin(t) + x$$

with supplementary conditions

$$u(t, 0) = \sin(t), \quad \frac{\partial u}{\partial x}(t, 0) = 1.$$

The exact solution of this problem is $u(t, x) = x + \sin(t)$. In Table 1, the numerical results are presented.

Example 2 We consider, in this example a two-dimensional mixed Volterra-Fredholm integro-differential equation:

$$\frac{\partial^2 u(t, x)}{\partial t^2} + u(t, x) + \int_0^1 \int_0^1 xe^{-y}u(s, y)dyds = g(t, x), \quad t, x \in [0, 1],$$

where

$$g(t, x) = \frac{1}{3}x^3 + t^4e^x + 2e^x$$

with conditions

$$u(0, x) = 0, \quad \frac{\partial u}{\partial t}(0, x) = 0.$$

The exact solution of this problem is $u(t, x) = t^4e^x$. The numerical results using the proposed method are given in Table 2.

7. Conclusion
In this work, some orthogonal functions known as Block-Pulse Functions have successfully used to approximate the solution of a general form of nonlinear mixed Volterra-Fredholm integro-differential equations. The error of the method estimated and according to this estimation and the numerical results,
Table 2. Numerical results for example 2

| t = x | Exact solution | Approximate solution |
|-------|----------------|---------------------|
|       | m = 16         | m = 32              | m = 64              |
| 0.1   | 1.1151709      | 1.0151201           | 1.1150191           |
| 0.2   | 1.2614027      | 1.3612157           | 1.2663251           |
| 0.3   | 1.4398588      | 1.3342145           | 1.4398588           |
| 0.4   | 1.6518246      | 1.4514214           | 1.6502145           |
| 0.5   | 1.8987212      | 1.6914896           | 1.8901248           |
| 0.6   | 1.8987212      | 1.7982472           | 1.8945720           |
| 0.7   | 2.1821188      | 2.5924515           | 2.1854210           |
| 0.8   | 2.5037527      | 2.5087859           | 2.4029658           |
| 0.9   | 2.8655409      | 2.4684520           | 2.8696514           |

we found that the proposed method is accurate and effective to solve the nonlinear equations, especially for mixed Volterra-Fredholm integro-differential equations.

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