Quantum computation, quantum state engineering, and quantum phase transitions driven by dissipation

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We investigate the computational power of creating steady-states of quantum dissipative systems whose evolution is governed by time-independent and local couplings to a memoryless environment. We show that such a model allows for efficient universal quantum computation with the result of the computation encoded in the steady state. Due to the purely dissipative nature of the process, this way of doing quantum computation exhibits some inherent robustness and defies some of the DiVincenzo criteria for quantum computation. We show that there is a natural class of problems that can be solved with such a model—the preparation of ground states of frustration free quantum Hamiltonians. This allows for robust and efficient creation of exotic states that exhibit features like topological quantum order and the creation of PEPS and it proves the existence of novel dissipative phase transitions. In particular the latter can in principle be verified experimentally with present day technology such as with optical lattices.

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The strongest adversary in quantum information science is decoherence, which arises due to the coupling of a system with its environment. The induced dissipation tends to destroy and wash out the interesting quantum effects which give rise to the power of quantum computation, cryptography, and simulation. While this is certainly true for many forms of dissipation, we show here that dissipation can also have exactly the opposite effect: it can be a full-fledged resource for universal quantum computation and quantum state engineering without any coherent dynamics needed to complement it.

We consider a quantum system composed of $N$ particles (e.g. qubits) interacting with local environments giving rise to memoryless and time-independent dissipation processes. We will show first how to design the interactions with the environment to implement universal quantum computation. This new method, which we refer to as dissipative quantum computation (DQC), defies some of the standard criteria for quantum computation since it requires neither state preparation, nor unitary dynamics [1]. However, it is nevertheless as powerful as standard quantum computation.

Then we will show that dissipation can be engineered [2] to prepare all ground states of frustration free Hamiltonians. Those include matrix product states (MPS) [3] and projected entangled–pair states (PEPS) [4], like graph states [5] and Kitaev [6] and Levin–Wenn [7] topological codes. Both DQC and dissipative state engineering (DSE) are robust in the sense that, given the dissipative nature of the process, the system is driven towards its steady state independent of the initial state and hence of eventual perturbations along the way.

Apart from novel ways of performing quantum computation or state engineering, our results imply that quantum phase transitions [8] may be driven by dissipation alone. That is, the physical properties of the steady state of our system may change abruptly in the thermodynamical limit when we slightly modify the parameters characterizing the dissipative dynamics [3]. This immediately follows from the existence of frustration free quantum Hamiltonians that exhibit quantum phase transitions in one [11] and two dimensions [12].

In this work we will concentrate first on DQC, showing how given any quantum circuit one can construct a master equation whose steady state is unique, encodes the outcome of the circuit, and is reached in polynomial time (with respect to the one corresponding to the circuit). Then we will show how to construct dissipative processes which drive the system to the ground state of any frustration free Hamiltonian. We will prove that MPS and certain kinds of PEPS can be efficiently prepared using this method, with details given in the appendix. In this paper we will not consider specific physical setups where our ideas can be implemented. Nevertheless, the appendix will provide a universal way of engineering the master equations required for DQC and DSE, which can be easily adapted to current experiments based on, e.g. atoms in optical lattices [13] or trapped ions [14]. Thus, we expect that our predictions and in particular the existence of quantum phase transitions driven by dissipation may be experimentally tested in the near future.

Let us start with DQC by considering $N$ qubits in a line and a quantum circuit specified by a sequence of nearest-neighbor qubit operations $\{U_t\}_{t=1}^T$. We define $|\psi_T\rangle := U_T U_{T-1} ... U_1 |0\rangle_1 \otimes \ldots \otimes |0\rangle_N$, so that $|\psi_T\rangle$ is the final state after the computation. Our goal is to find a master equation $\dot{\rho} = \mathcal{L}(\rho)$ with Liouvillian in Lindblad form [15]

$$\mathcal{L}(\rho) = \sum_k L_k \rho L_k^\dagger - \frac{1}{2} \left\{ L_k^\dagger L_k, \rho \right\}$$

where the $L_k$ act locally and has a steady state, $\rho_0$; (i) which is unique; (ii) that can be reached in a time $\text{poly}(T)$; (iii) such that $|\psi_T\rangle$ can be extracted from it in a time $\text{poly}(T)$. As in Feynman’s construction of a quantum simulator [16], we consider another auxiliary register with states $\{|t\rangle\}_{t=0}^T$, which will rep-
resent the time. We choose the Lindblad operators
\[ L_i = |0\rangle_i \langle 1| \otimes |0\rangle_i \langle 0|, \quad (2) \]
\[ L_t = U_t \otimes |t+1\rangle \langle t| + U^\dagger_t \otimes |t\rangle \langle t+1|, \quad (3) \]
where \( i = 1, \ldots, N \) and \( t = 0, \ldots, T \). It is clear that the \( L \)'s act locally except for the interaction with the extra register, which can be made local as well. Furthermore,
\[ \rho_0 = \frac{1}{T+1} \sum_{t=0}^{T} |\psi_t\rangle \langle \psi_t| \otimes |t\rangle \langle t|. \quad (4) \]
is a steady state, i.e., \( \mathcal{L}(\rho_0) = 0 \). Given such a state, the result of the actual quantum computation can be read out with probability \( 1/T \) by measuring the time register. In the appendix we show that \( \rho_0 \) is the unique steady state and that the Liouvillian has a spectral gap \( \Delta = \pi^2/(2T+3)^2 \). This means indeed that the steady state will be reached in polynomial time in \( T \).

Note that this gap is independent of \( N \) as well as on the actual quantum computation which is performed (i.e., independent of the \( U_t \)). It is also shown that the same gap is retained if the clock register is encoded in the unary way proposed by Kitaev [20], making the Lindblad operators strictly local. A sketch of the proof is as follows: first, we do a similarity transformation on \( \mathcal{L} \) that replaces all gates \( U_t \) with the identity gates, showing that its spectrum is independent of the actual quantum computation. Second, another similarity transformation is done that makes \( \mathcal{L} \) hermitian and block-diagonal. Each block can then be diagonalized exactly leading to the claimed gap.

In some sense, the present formalism can be seen as a robust way of doing adiabatic quantum computation [18] (errors do not accumulate and the path does not have to be engineered carefully) and implementing quantum random walks [19], and it might therefore be easier to tackle interesting open questions, such as the quantum PCP theorem, in this setting [23]. Also, it seems that the dissipative way of preparing ground states is more natural than to use adiabatic time evolution, as nature itself prepares them by cooling.

Let us now turn to DSE and consider again a quantum system with \( N \) particles on a lattice in any dimension. We are interested in ground states \( \Psi \) of Hamiltonians
\[ H = \sum_{\lambda} H_{\lambda}, \quad (5) \]
which are frustration free, meaning that \( \Psi \) minimizes the energy of each \( H_{\lambda} \) individually, and local in the sense that \( H_{\lambda} \) acts non-trivially only on a small set \( \lambda \subset \{1, \ldots, N\} \) of sites (e.g., nearest neighbors) [21]. We can assume the \( H_{\lambda} \)'s to be projectors and we will denote the orthogonal projectors by \( P_{\lambda} = 1 - H_{\lambda} \). States \( \Psi \) of the considered form are, e.g., all PEPS (including MPS and stabilizer states [22]).

We will consider discrete time evolution generated by a trace preserving completely positive map (cp-map) instead of a master equation. These two approaches are basically equivalent [23] as every local cp-map \( T \) can be associated to a local Liouvillian via \( \mathcal{L}(\rho) = N[T(\rho) - \rho] \), which leads to the same fixed points and a gap that is \( N \) times larger [24]. We choose cp-maps of the form
\[ T(\rho) = \sum_{\lambda} p_{\lambda} \left( P_{\lambda} \rho P_{\lambda} + \frac{1}{m} \sum_{i=1}^{m} U_{\lambda,i} H_{\lambda} \rho H_{\lambda} U_{\lambda,i}^\dagger \right), \quad (6) \]
where the \( p_{\lambda} \)'s are probabilities and \( U_{\lambda,1}, \ldots, U_{\lambda,m} \) is a set of projectors acting non-trivially only within region \( \lambda \). They effectively rotate part of the high-energy space (with support of \( H_{\lambda} \)) to the zero-energy space, so that \( \text{tr}[T(\rho)\Psi] \geq \text{tr}[\rho \Psi] \) increases. As for Liouvillians [11] we could similarly take \( L_{\lambda,i} = U_{\lambda,i} H_{\lambda} \), or the ones associated to the cp-map.

We show now that for every frustration-free Hamiltonian the cp-map in Eq. (6) converges to the ground state space if we choose the unitaries \( U_{\lambda,i} \) to be completely depolarizing, i.e., \( T(\rho) \propto \sum_\lambda P_{\lambda} \rho P_{\lambda} + 1_\lambda \otimes \text{tr}[H_{\lambda} \rho]/\text{tr}[1_\lambda] \). For ease of notation we will explain the proof for the case of a one-dimensional ring with nearest-neighbor interactions labelled by the first site \( \lambda = 1, \ldots, N \). Assume \( \rho \) is such that its expectation value with respect to the projector \( \Psi \) is the ground state space of \( H \) is non-increasing under applications of \( T \). i.e., in particular \( \text{tr}[\rho \Psi] = \text{tr}[T^N(\rho)\Psi] \). Expressing this in the Heisenberg picture in which \( T^\ast(\Psi) = 1 \Psi + \sum_{\lambda} H_{\lambda} \text{tr}_{\lambda}(\Psi)/(d^2 N) \) we get
\[ \text{tr}[\rho \Psi] \geq \text{tr}[\rho \Psi] + \frac{1}{(d^2)^N} \text{tr} \left[ \rho \sum_{\mu=1}^{N} \prod_{\lambda=1}^{\mu} (H_{\lambda+\mu} \text{tr}_{\lambda+\mu}) (\Psi) \right] \]
\[ \geq \text{tr}[\rho \Psi] + \frac{\nu^N}{(d^2)^N} \text{tr}[\rho H], \quad (7) \]
where the first inequality comes from discarding (positive) terms in the sum and the second one is due to bounding all partial traces of \( H_{\lambda} \) from below by the respective smallest eigenvalue \( \nu \). Note that the latter is strictly positive unless \( H \) has a product state as ground state (in which case the statement becomes trivial). Hence, we must have \( \text{tr}[\rho H] = 0 \), i.e., \( \rho \) is a ground state of \( H \). It is easily seen that the same argumentation goes through for more general interactions on arbitrary lattices.

The above procedure implies the existence of quantum phase transitions driven by dissipation. By changing the parameters in the cp–map (or the master equation) one can obtain that some physical properties of the steady state abruptly change, in as much the same way as in Refs. [11, 12].

We have shown that it is possible to engineer dissipative processes which prepare ground states of frustration free Hamiltonians in steady state. However, in the above proof the time for this preparation scales as \( N^N \), which may be an issue for experiment with large number of particles. In the following we give much more efficient method for certain classes of frustration free Hamiltonians: commuting Hamiltonians and MPS.

We consider first frustration free Hamiltonians for which \( [H_{\lambda}, H_{\mu}] = 0 \) and show that the corresponding ground states can be prepared in a time that only scales polynomially with
the number of particles. The corresponding set of ground states contains important families, like stabilizer states (e.g. cluster states and topological codes), or certain kinds of PEPS. Note that there was no known way of efficient preparation for the latter.

Loosely speaking there are two classes of Hamiltonians of this type: (i) Hamiltonians for which all excitations can be locally annihilated. In this case the time of convergence scales as $\tau = \mathcal{O}(\log N)$, (ii) Interactions where excitations have to be moved along the lattice before they can annihilate and $\tau = \mathcal{O}(N \log N)$.

In order to see how the first case can occur consider Eq. (6) and note that it can be interpreted as randomly choosing a region $\lambda$ (according to $p_\lambda$ which we may set equal to $1/N$), then measuring $P_\lambda$ and applying a correction according the $U^{\lambda}$'s if the outcome was negative. Assume now that when iterating $T$ the correction on $\lambda$ does not change the outcome of previous measurements on neighboring regions since

$$\forall \lambda \neq \lambda' : \ [U^{\lambda}, H^{\lambda'}] = 0. \tag{8}$$

In fact, this can always be achieved by regrouping the regions into larger ones having an interior $I(\lambda) \subset \lambda$ on which only $H_\lambda$ acts non-trivially and letting the $U^{\lambda,i}$ solely act on $I(\lambda)$. Denote by $q$ the largest probability for obtaining twice a negative measurement outcome on the same region $\lambda$. The energy $\text{tr} [HT^M (\rho)]$ after $M$ applications of $T$ decreases then as $N(1 - (1 - q)/N)^M$ such that it takes $\mathcal{O}((N \log N)/(1 - q))$ steps to converge to a ground state. The relaxation time of the corresponding Liouvillian is thus $\tau = \mathcal{O}(\log N^{1/\gamma})$. Clearly, this is only a reasonable bound if $q < 1$, a condition possibly incompatible with Eq. (8).

Note that for all stabilizer states we can achieve $q = 0$, since there always exists a local unitary (acting on a single qubit) so that $H_\lambda U_\lambda H_\lambda = 0$. A class of stabilizer states where this is compatible with Eq. (8) are the so-called graph states [5]. In this case $\lambda$ labels (with some abuse of notation) a vertex of a graph and $H_\lambda = (1 - \sigma_z^{(\lambda)}) \prod_{(\lambda,\mu) \in E} \sigma_z^{(\mu)}/2$ where $\sigma_z^{(\lambda)}$ is a Pauli operator acting on site $\lambda$ and $E$ is the set of edges of the graph. Obviously, $U^{\lambda} = \sigma_z^{(\lambda)}$ does the job. In this special case we can get even faster convergence when using the Liouvillian

$$\mathcal{L}(\rho) = \left( \sum_\lambda U_\lambda H_\lambda \rho H_\lambda U_\lambda^\dagger \right) - \frac{1}{2} \{H, \rho\}_+. \tag{9}$$

The corresponding relaxation time can be determined exactly by realizing that the spectrum of $\mathcal{L}$ equals that of $-\{H \otimes 1 + 1 \otimes H\}/2$ so that $\tau = 1$ [4].

Let us now discuss the second type of commuting Hamiltonians—those for which Eq. (8) and $q < 1$ are incompatible. For this class we can still prove fast convergence by making explicit use of the fact that frustration-free ground states of commuting Hamiltonians have an efficient PEPS representation. That is, when expanded in computational product basis, the coefficients are given by a tensor-network whose geometry resembles the lattice structure of the interactions. A generic property of PEPS is injectivity [29] of local regions which is, in fact, a sufficient condition for the state to be the unique ground state of its parent Hamiltonian. Consider cases of commuting Hamiltonians for which the ground state has this property. To specify the cp-map in Eq. (7) we need to sort the regions of interactions $\lambda_1, \lambda_2, \ldots$ such that the union $\Lambda_k = \bigcup_{i=1}^k \lambda_i$ has an intersection with $\lambda_{k+1}$ but does not entirely cover it, i.e., $I_{k+1} := \lambda_{k+1} \setminus \Lambda_k \neq \emptyset$. Such an ordering is always achievable by possibly regrouping the interactions into slightly larger regions. The reason for this ordering is that any $U_{\lambda_k}$ which acts only on $I_{k}$, will not alter the energies in all regions $\lambda_i$ with $i < k$. That is, we have a weakened version of Eq. (8). Injectivity of the regions $I_k$ in the PEPS representation implies then that there is always a unitary $U_{\Lambda_k}$ (a depolarizing set would work as well) such that $q < 1$ [35] and we can thus follow the above lines of argumentation. The only difference is that due to the weakening of Eq. (8) the energy does not decrease homogeneously, but a low-energy region will grow stochastically according to the $\Lambda_k$'s which requires extra time proportional to the systems size, so that $\tau = \mathcal{O}(N \log N)$.

There are frustration-free ground states which belong to the second class of commuting Hamiltonians but for which injectivity does not hold (e.g. due to a degenerate ground state space). A paradigmatic example is Kitaev’s toric code state [6] where one has a four-fold degeneracy. Due to lack of injectivity, we have to prove $q < 1$ separately which is, however, trivial in this case since it is a stabilizer state so that $q = 0$. The action of the cp-map (or respective Liouvillian) can be understood as moving all the excitations towards a single point where they can mutually annihilate.

We turn now to another family of ground states of frustration free Hamiltonians, namely MPS [3]. Clearly, one possible efficient way of preparing them using dissipation is to exploit the fact that they can be obtained via a sequential application of quantum gates [25] together with the above DQC scheme. In the following we will, however, focus on a different way which does not require an additional time register.

For the sake of clearness, we will consider translationally invariant Hamiltonians, although the analysis can be straightforwardly extended to systems without that symmetry. We will specify a cp-map to prepare states of the form

$$|\Psi\rangle = \sum_{i=1}^d \text{tr} (A_{i_1} \cdots A_{i_d}) |i_1 \ldots i_N\rangle \tag{10}$$

where, the $A_i$'s are $D \times D$ matrices. As before, we assume the injectivity property which implies that $\Psi$ is the unique ground state of a nearest neighbor frustration free 'parent' Hamiltonian which has a gap. Denoting by $\rho$ the reduced density operator corresponding to particles $k$ and $k + 1$, $H_k$ and $P_k = 1 - H_k$ will denote the projectors onto its kernel and range, respectively. Note that $\text{tr}(P_k) = D^2$. We take $N = 2^n$ for simplicity, but this is clearly not necessary. We construct the channel $T$ in several steps. We first define a channel acting
on two neighboring particles \( k, k + 1 \), as follows

\[
\mathcal{R}_{r,c}(X) := P_k X P_k + \frac{P_k}{D^2} \text{tr}(H_k X).
\]

Here, \( k = 2^{r-1}(2c - 1) \) where \( r = 1, \ldots, n \) and \( c = 1, \ldots, 2^{n-r} \). The action of these maps has a tree structure, where the index \( r \) indicates the row in the tree, whereas \( c \) does it for the column. Now we define recursively,

\[
S_{r,c} := \frac{1 - \epsilon_r}{2}(S_{r-1,2c} + S_{r-1,2c+1}) + \epsilon_r \mathcal{R}_{r,c}. \tag{11}
\]

Here, \( r = 2, \ldots, n \), \( c = 1, \ldots, 2^{n-r} \), \( S_{1,c} := R_{1,c} \), and \( \epsilon_r = 1/M^r \) where \( M = CN^2 \) and \( C \gg 1 \) (see appendix). Note that \( S_{r,1} \) acts on the first \( 2^r \) particles, \( S_{r,2} \) on the next \( 2^r \), and so on. We finally define

\[
T := (1 - \epsilon_{n+1})S_{n,1} + \epsilon_{n+1} \mathcal{R}_{n,2}. \tag{12}
\]

In the appendix we show that this map achieves the fixed point (up to an exponentially small error in \( C \)) in a time \( O(N^{\log_2(N)}) \). The intuition behind the cp-map \( T \) is that the channels \( S_{1,c} \), which are the ones that most often applied, project the state of every second nearest neighbors onto the right subspace. Then \( S_{2,c} \) do the same with half of the pairs which have not been projected. Then \( S_{3,c} \) does the same on half of the rest, and so on.

In conclusion, we have investigated the computational power of purely dissipative processes, and proven that it is equivalent to that of the quantum circuit model of quantum computation. We have also shown that dissipative dynamics can be used to create ground states (like MPS or PEPS) of frustration free Hamiltonians of strongly correlated quantum spin systems. This implies the existence of dissipatively driven quantum phase transitions, something which could be experimentally tested using atoms or ions.

Let us stress that we have been concerned here with a proof-of-principle demonstration that dissipation provides us with an alternative way of carrying out quantum computations or state engineering. We believe, however, that much more efficient and practical schemes can be developed and adapted to specific implementations. We also think that these results open up some interesting questions which deserve further investigation. For example, how the use of fault tolerant computations can make our scheme more robust. Or how one can design translationally invariant cp-maps that prepare MPS more efficiently. Or the importance and generality of the set of commuting Hamiltonians, which we believe to be intimately connected to the fixed points of the renormalization group transformations on PEPS (as it happens with MPS \[26\]).

Here, \( \mathcal{L} \) is gapped. More specifically, it holds that for any initial condition \( \rho(0) \), we can show that \( \| \rho(t) - \rho(\infty) \| \leq \epsilon \) in a time \( t \) that scales logarithmically in \( 1/\epsilon \) and quadratically in \( 1/T \).

The Liouvillian is defined as the (non-symmetric) matrix

\[
\mathcal{L} = \sum_{\alpha} L_{\alpha} \otimes \tilde{L}_{\alpha} - \frac{1}{2} \left( \sum_{\alpha} L_{\alpha}^* L_{\alpha} \otimes I + I \otimes \sum_{\alpha} L_{\alpha}^* L_{\alpha} \right)
\]

where \( \alpha \) runs over the labels of all Lindblad operators. We will bring this Liouvillian into a simpler form by doing two eigenvalue-preserving similarity transformations. First, we apply the unitary

\[
W = \sum_t U_t U_{t-1} \cdots U_1 \otimes |t\rangle\langle t|
\]

and observe that the spectrum of the Liouvillian is the same as if the quantum circuit would only have consisted of identity gates. The spectrum of the Liouvillian is hence independent of the actual computation that we want to do (note that this was also the case in the context of adiabatic quantum computation). Without loss of generality, we therefore assume \( \forall t : U_t = I \). Second, we diagonalize the part acting on the logical qubits by the similarity transformation \( \mathcal{L}' = X \mathcal{L} X^{-1} \) where

\[
X = (X_1 \otimes X_2 \otimes \ldots X_N) \otimes |00\rangle_4 \langle 00| + I \otimes (I - |00\rangle_4 \langle 00|)
\]

\[
X_t = |00\rangle\langle 00| + |00\rangle\langle 11| - |11\rangle\langle 11| + |01\rangle\langle 01| + |10\rangle\langle 10|.
\]
Note that the double indices arising in those expressions reflect the fact that the Liouvillian acts on the tensor product of the physical space with itself. The part acting on the logical space is now completely diagonal, and $L'$ becomes

$$L' = \sum_{i_1, i_2, ... i_N}^{1} |i_1 ... i_N\rangle\langle i_1 ... i_N| \otimes \tilde{L}_{ij}$$

$$\tilde{L}_{ij} = \tilde{L} + \lambda_i |0\rangle\langle 0| \otimes I_t + \lambda_j I_t \otimes |0\rangle\langle 0|$$

$$\lambda_i = \sum_{k=1}^{N} i_k$$

$$\dot{\tilde{L}} = \sum_{t} \dot{L}_t \otimes \dot{\tilde{L}}_t - \frac{1}{2} \left( \sum_{t} \tilde{L}_t \dot{L}_t \otimes I_t + I_t \otimes \sum_{t} \tilde{L}_t \dot{L}_t \right)$$

$$\dot{\tilde{L}}_t = |t\rangle\langle t| + |t+1\rangle\langle t|$$

The problem of calculating the eigenvalues of $L$ is therefore reduced to the problem of calculating the eigenvalues of the $T^2 \times T^2$ matrices $\tilde{L}_{ij}$ for all possible positive integer numbers $\lambda_i, \lambda_j$. It happens that $\tilde{L}_{ij}$ is block-diagonal, and consists entirely of diagonal elements of the form $(-2), (-1 - \lambda_i), (-1 - \lambda_j)$, of $2 \times 2$ blocks of the form

$$\begin{pmatrix} -1 - \lambda_i & 1 \\ 1 & -1 - \lambda_j \end{pmatrix} \text{ and of tridiagonal matrices of the form}$$

$$- (1 + \lambda_i + \lambda_j) |0\rangle\langle 0| - 2 \sum_{t=2}^{T-1} |t\rangle\langle t| - |T\rangle\langle T|$$

$$+ \sum_{t=1}^{T-1} (|t\rangle\langle t| + |t+1\rangle\langle t|)$$

The first classes of diagonal blocks have all eigenvalues strictly smaller than $-1$, and therefore correspond to terms that converge extremely fast. The gap is determined by the last class of tridiagonal matrices, and that one corresponds exactly to the class of matrices that has been extensively studied in the context of random walks. The only matrix with eigenvalue $0$ is the one with $\lambda_i = \lambda_j = 0$, which proves that the fixed point is unique. The gap of $L_{00}$ can be calculated exactly, and is given by

$$2 \left( \cos \left( \frac{\pi}{T + 1} \right) - 1 \right) \approx -\frac{\pi^2}{(T + 1)^2}.$$  

The largest eigenvalue for the cases $(ij) \neq (00)$ is obtained for $\lambda_i + \lambda_j = 1$, and is given by

$$2 \left( \cos \left( \frac{\pi}{2T + 3} \right) - 1 \right) \approx -\frac{\pi^2}{(2T + 3)^2}.$$  

It follows that the gap of the Liouvillian is larger than $1/T^2$, which we set out to prove. In principle, this does not suffice to prove that convergence will be reached at a time in the order of $1/T^2$, as the Liouvillian might have exponentially large Jordan blocks; it can readily be checked that this is not the case here.

Let us next check what happens if we use Kitaev’s unary encoding [23] of the time register such as to make all Lindblad operators strictly local. We will have to replace $L_t$ by

$$L_t = I \otimes |1\rangle\langle 1| \otimes X \otimes |0\rangle\langle 0| \otimes I$$

where $X$ is acting on the $t$‘th qubit in the unary encoding, and add Lindblad terms that force the system to converge into the allowed subspace for the time Hamiltonians:

$$L_q = I \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 1| \otimes I.$$  

Furthermore, we have to replace

$$L_i = (I \otimes |0\rangle\langle 1| \otimes I) \otimes (|0\rangle\langle 0| \otimes I).$$  

It now happens that the relevant spectrum of the new corresponding Liouvillian is unchanged; all the terms $L_i, L_t, L_q$ are such that they map density operators that are in the right subspace of the unary encoding into the right subspace, and furthermore the only terms that connect the right with the wrong subspace are $L_q$ which can map wrong states right ones. The Liouvillian is therefore block-upper-diagonal, and the eigenvalues are completely determined by the diagonal blocks. All the eigenvalues in the wrong block turn out to be smaller than $-1$, and therefore the relevant eigenvalue lie in the right block. The gap is therefore left unchanged.

Making use of the recent results of Oliveira and Terhal [31], and of Aharonov, Gottesman and Kempe [32], it can also be shown that the same computational power is retained if the Lindblad operators are only acting on nearest neighbour qubits in a 2-D square lattice or on nearest neighbours in a 1-dimensional spin chain of 12-level systems.

Let us conclude this section by explaining why the gap will considerably change if we were to use the master equation approach to solve general problems in the class NP. The idea would be to put a penalty term on one of the output qubits corresponding to getting the right answer, and relaxing the constraints on some input bits. First note that the construction of the Feynman Hamiltonian [14] is only possible when all quantum gates are unitary; hence the whole circuit is reversible. This implies that the problem only makes sense if some input qubits to the quantum circuit are initialized, as otherwise any output can be obtained. Because of the fact that both input and output qubits must be initialized, we cannot replace the actual gates with the identity gate, and as a result the Liouvillian does not have a nice block-diagonal structure anymore with only blocks of polynomial size, but we get exponentially large blocks. Such blocks will typically lead to exponentially small gaps.
APPENDIX: ENGINEERING DISSIPATION

Here we show how to engineer the local dissipation which gives rise to the master equations (1) and cp–maps (3). They are composed of local terms, involving few particles (typically two), so that we just have to show how to implement those. In order to simplify the exposition, we will treat those particles as a single one and assume that one has full control over its dynamics (e.g. one can apply arbitrary gates).

Let us start with the cp-maps. It is clear that by applying a quantum gate to the particle and a ‘fresh’ ancilla and then tracing the ancilla one can generate any physical action (i.e. cp-map) on the system. Furthermore, by repeating the same process with short time intervals one can subject the system to an arbitrary time independent master equation. This last process may not be efficient. An alternative way works as follows. Let us assume that the ancilla is a qubit interacting with a reservoir such that it fulfills a master equation with Liouville operator $L_a = \sqrt{\Gamma} \sigma_\alpha$, where $\sigma_\alpha = |\alpha\rangle\langle \alpha |$ (1). Now, we couple the ancilla to the system with a Hamiltonian $H = \Omega (\sigma_y L^z + \sigma_x L^z)$. In the limit $\Gamma \gg \Omega$ one can adiabatically eliminate the level [1] of the ancilla [27] by applying second order perturbation theory to the Liouvillian. This is done as follows: the unperturbed Liouvillian can be written as $L_0 = QDQ^{-1}$ with $Q = Q^{-1}$ the eigenvectors of $L_0$; writing the perturbed eigenvectors as $Q \exp(\Omega/\sqrt{\Gamma}X)$, we solve the equation $-XD + DX + Q \mathcal{L}_H Q = 0$ with $\mathcal{L}_H$ the perturbation arising from the Hamiltonian part; the effective Liouvillian is then given by $-\Omega^2/\Gamma X DX$. In this way we obtain a master equation for $\rho$ describing the system alone, with Liouville operator $\Omega/\sqrt{\Gamma}X$. By using several ancillas with Hamiltonians $H = \Omega (\sigma_y L^z + \sigma_x L^z)$ and following the same procedure we obtain the desired master equation. Although we have not specified here a physical system, one could use atoms. In that case, the ancilla could be an atom itself with $|0\rangle$ and $|1\rangle$ an electronic ground and excited level, respectively, so that spontaneous emission gives rise to the dissipation. The coupling to the system (other atoms) could be achieved using standard techniques [28].

In order to simplify the discussion, we will take into account the following: $S^L_{r,s}\rho_s$ with $M := L_{r+1} \gg 1$ will be a sum of contributions which will typically have the form

$$S^L_{r,s} S^L_{r+1,s+1} \ldots$$

where $L_i \sim (1 - \epsilon_i^r)/2\epsilon_i^r$ and where the channel $R_{r+1}$ appears $\sim L_{r+1}$ times. We define: $L_r = 1/2\epsilon_r^r$; $f_r = S^L_{r+1,s+1} ; \Gamma_r = R_{r+1} R_{r+1} \ldots ; \rho_r = \rho_{r+1} \rho_{r+1} \ldots$; $\rho_l = \rho_{r+1} \rho_{r+1} \ldots$; $X_l = \text{tr}(\rho_{r+1} \rho_l)$; $Y_l = \text{tr}(\rho_{r+1} \rho_l)$. We will approximate $S^L_{r+1,s} = \Gamma R_{r+1}$. Using the fact that for $k < 2^r$, $P_k P_{k+1} = P_k P_{k+1}$ we have that $|\rho_{r+1} \rho_{r+1} \rho_{r+1} \rho_{r+1} \sigma | > |\rho_{r+1} \rho_{r+1} \rho_{r+1} \rho_{r+1} \sigma |$ for $i = 1, 2$, and thus $X_l > Y_l$. It is easy to show using the same thing that

$$Y_l = X_l + \frac{1}{D^2} \text{tr}[\rho_{r+1} \rho_0 (Q_0 Q)].$$

Here we have simplified the notation: $\tau_0$ denotes the trace with respect to the particles $k_0, k_0 + 1$, and $Q = K_{k_0, k_0+1}$.

Now, using that for any projectors $P$ and $Q = 1 - P$, $(\sqrt{\Gamma} P \pm Q/\sqrt{\Gamma}) \rho (\sqrt{\Gamma} P \pm Q/\sqrt{\Gamma}) \geq 0$ it follows that $\rho_l \geq P_l P_{l+1} - 3 \mu_l^r$. Finally, using the properties of MPS [3] it is easy to show that $|\text{tr}[\rho_{r+1} \rho_0 (Q_0 P_{r+1} P_{r+1} Q)]/D^2|$ is lower bounded by a $(1 - \mu_l^r)(1 - X_l)$, where $z$ is a constant. Thus we obtain

$$X_{l+1} \geq X_l - 3 \sqrt{\mu_l^r} - z(1 - \mu_l^r) (1 - X_l).$$

Iterating this expression we obtain

$$\mu_{l+1} \leq \frac{3}{z(1 - \mu_l^r)} \sqrt{\mu_l^r} + [1 - z(1 - \mu_l^r)] L_{l+1}. (18)$$

After some lengthy algebra, we obtain that if we choose $\epsilon_l/\epsilon_{r+1} = M = CN^2$ and $L_r = 1/\epsilon_r$ the final error will be $3z^2 (1 - z)^C$ whereas the number of applications of the map $N_{log_2 N} = N^2 log_2 N + \log_2 C$. Thus, by choosing $C$ sufficiently large we can always make the error arbitrarily small with a subexponential number of applications of the map.

APPENDIX: EFFICIENCY OF DSE FOR MATRIX PRODUCT STATES

We show that the cp-map defined in the text for the creation of MPS converges in sub–exponential time.

Let us denote by $\mathcal{H}_r$ the ground subspace of the Hamiltonian $\sum_{k=1}^r H_k$ corresponding to the zero eigenvalue, and by $q_n$ the projector onto the orthogonal subspace. The basic idea is that by applying the map $S_{r+1}$ a sufficiently large number of times, $L_r$, to any density operator, $\rho$, we obtain a state which is practically supported on $\mathcal{H}_r$. We will show that the error $\mu_r^L := \text{tr}(q_n S_{r+1}^L (\rho)$ can be made arbitrarily small. In particular, for $n = \log_2 N$ we just have to choose $L = O(N^2 \log_2(N))$.

[1] David P. DiVincenzo, The physical implementation of quantum computation, arXiv:quant-ph/0002077
[2] J. F. Poyatos, J. I. Cirac and P. Zoller, Quantum Reservoir Engineering with laser cooled trapped ions, Phys. Rev. Lett. 77, 4728 (1996).
[3] M. Fannes, B. Nachtergaele, and R. F. Werner, Finitely Correlated States on quantum spin chains, Comm. Math. Phys., 144:443, 1992; D. Perez-Garcia, F. Verstraete, M.M. Wolf, J.I. Cirac, Matrix Product State Representations, Quantum Inf. Comput. 7, 401 (2007).
[4] F. Verstraete and J. I. Cirac, Renormalization algorithms for Quantum–Many Body Systems in two and higher dimensions, arXiv:cond-mat/0407066v1, 2004.
[5] D.M. Schlingemann, Cluster states, algorithms and graphs, Quant. Inf. Comp. 4, 287 (2004); H.J. Briegel, R. Raussendorf, Persistent Entanglement in Arrays of Interacting Qubits, Phys. Rev. Lett. 86, 910 (2001).
[6] A. Yu. Kitaev, Fault-tolerant quantum computation by anyons, Annals Phys. 303, 2 (2003).
[7] Michael A. Levin, Xiao-Gang Wen, String-net condensation: A physical mechanism for topological phases, Phys.Rev. B 71, 045110 (2005).
[8] S. Sachdev, Quantum Phase Transitions, Cambridge University Press (1999).

9 This mechanism is different to the one usually considered in condensed matter physics, where instead of the system’s steady state, the ground state of the system–plus–environment experiences abrupt changes when the interaction is modified [10].

[10] P. Werner, K. Völlker, M. Troyer, S. Chakravarty, Phase Diagram and Critical Exponents of a Dissipative Ising Spin Chain in a Transverse Magnetic Field, Phys. Rev. Lett. 94, 047201 (2005).

[11] Michael M. Wolf, Gerardo Ortiz, Frank Verstraete, J. Ignacio Cirac, Quantum phase transitions in matrix product systems, Phys. Rev. Lett. 97, 110403 (2006).

[12] F. Verstraete, M. M. Wolf, D. Perez-Garcia, Criticality, the area law, and the computational power of PEPS, Phys. Rev. Lett. 96, 220601 (2006).

[13] I. Bloch, J. Dalibard, W. Zwerger, Many-Body Physics with Ultra-cold Gases, Rev. Mod. Phys. (to be published), e-print arXiv: 0704.3011.

[14] D. Leibfried, R. Blatt, C. Monroe, D. Wineland, Quantum dynamics of single trapped ions, Rev. Mod. Phys. 75, 281 (2003).

[15] G. Lindblad, On the generators of quantum dynamical semigroups, Comm. Math. Phys. 48, 119 (1976)

[16] R.P. Feynman, Simulating physics with computers, Int. J. Theor. Phys. 21 (1982)

[17] A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi, Classical and Quantum Computation, Amer. Math. Soc. (2002)

[18] Dorit Aharonov, Wim van Dam, Julia Kempe, Zeph Landau, Seth Lloyd, Oded Regev, Adiabatic Quantum Computation is Equivalent to Standard Quantum Computation, SIAM Journal of Computing 37, 166 (2007).

[19] Julia Kempe, Quantum random walks - an introductory overview, Contemporary Physics 44, 307 (2003).

[20] S. Arora and S. Safra, Probabilistic checking of proofs: A new characterization of NP, J. ACM, 45, 70 (1998).

[21] To simplify matters we assume that the number of interactions is proportional to the number of sites, and denote it, with some abuse of notation, by N as well.

[22] Daniel Gottesman, A Theory of Fault-Tolerant Quantum Computation, Phys.Rev. A 57, 127 (1998).

[23] M.M. Wolf, J.I. Cirac, Dividing Quantum Channels, Comm. Math. Phys. 279, 147 (2008).

[24] Note that the reduction in the time of convergence comes from the fact that the cp-map T applies one operation with probability p, at a time, whereas in L all sites couple to their respective baths simultaneously.

[25] C. Schoen, E. Solano, F. Verstraete, J. I. Cirac, M. M. Wolf, Sequential generation of entangled multi-qubit states, Phys. Rev. Lett. 95, 110503 (2005).

[26] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, Renormalization-Group Transformations on Quantum States, Phys. Rev. Lett. 94, 140601 (2005).

[27] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, Atoms-Photon interactions, John Wiley and Sons, New York, 1992.

[28] J. I. Cirac and P. Zoller, New Frontiers in Quantum Information With Atoms and Ions, Physics Today 57, Issue 3, 38 (2004).

[29] D. Perez-Garcia, F. Verstraete, J.I. Cirac, M.M. Wolf, PEPS as unique ground states of local Hamiltonians, Quant. Inf. Comp. 8, 0650 (2008).

[30] A. Yu. Kitaev, A. H. Shen, M. N. Vyalyi, Classical and Quantum Computation, Amer. Math. Soc. (2002).

[31] Roberto Oliveira, Barbara M. Terhal, The complexity of quantum spin systems on a two-dimensional square lattice, [arXiv:quant-ph/0504050]

[32] D. Aharonov, D. Gottesman, J. Kempe, The power of quantum systems on a line, [arXiv:0705.4077]

[33] For a 1D system with nearest-neighbor interaction we can for instance choose $H_{1,2,3} = H_{1,2} + H_{2,3}$ with site 2 as interior.

[34] That the first part (denote it by $\mathcal{F}$) of the Liouvillian in Eq.5 does not contribute to the spectrum can be seen by showing that $tr[(t\mathcal{F} + (L - \mathcal{F}))^n]$ with trace in Liouville space is independent of $t$ for all $n \in \mathbb{N}$. This can in turn be seen by observing that all the contributions which are not powers of $L - \mathcal{F}$ are nilpotent. As $tr[(t\mathcal{F} + (L - \mathcal{F}))^n]$ determines the spectrum of $L$ we can thus set $t = 0$ without changing the spectrum.

[35] More precisely there exists a unitary $U$ on $I_k$ such that if $|\Phi\rangle$ has zero energy on $\Lambda_{k-1}$, the rotated state $U|\Phi\rangle$ has non-zero overlap with the zero-energy subspace corresponding to $\Lambda_k$. We argue by contradiction. Assume that for all $U$ on $I_k$ this overlap and therefore the one of the depolarized state would be zero. Every normal vector in the support of the depolarized state has the form $|\phi\rangle = \sum_{\alpha,\beta} \text{tr}[A^{\alpha} X^{\beta}] |\alpha\rangle |\varphi\rangle |\beta\rangle$, where $\alpha, \beta$ are product bases for $\Lambda_{k-1}$ and the complement of $\Lambda_k$ respectively, $\varphi$ is a vector in the Hilbert space of $I_k$ and $tr$ denotes the contraction of the tensors $A$ and $X$. The tensors $A$ are already the one of the target state. Incorporating $U$ into $\varphi$, zero-overlap would mean that $\langle \phi | \phi \rangle = 0$ for all $|\phi\rangle = \sum_{\alpha,\beta} \text{tr}[A^{\alpha} B^{\beta}] |\alpha\rangle |j\rangle |\beta\rangle$, with $B^{\beta}$ being the tensor corresponding to region $I_k$ in the target state and $\bar{Y}$ arbitrary (in particular $\bar{Y} = X$). By the definition of injectivity there is, however, always a constant $c$ and a vector $\varphi$ such that $c^{-1} \sum_j \langle j | \varphi \rangle B^{\beta} = 1$ and thus $\langle \phi' | \phi \rangle = c$. 
