Dynamical system perspective of cosmological models minimally coupled with scalar field

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Abstract: The stability criteria for spatially flat homogeneous and isotropic cosmological dynamical system is investigated with the interaction of a scalar field endowed with a perfect fluid. In this paper, we depict the dynamical system perspective to study, qualitatively, the scalar field cosmology under two special cases, with and without potential. For analysis with potential we use simple exponential potential form, \( V_o e^{-\lambda \phi} \). We generate, by introducing new dimensionless variables, an autonomous system of ordinary differential equations (ASODE) for each case and obtain respective fixed points. We also analyse the type of fixed points, nature and stability of the fixed points and how their nature and behavior reflect towards the cosmic scenarios. Throughout the whole work, the investigation of this model has shown us the deep connection between these theories and cosmic acceleration phenomena. The phase plots of the system at different conditions and different values of \( \gamma \) have been analyzed in detail and their interpretations have been worked out. The perturbation plots of the dynamical system have also been studied and analyzed which emphasize our analytical findings.

Keywords: Dynamical system, ASODE, fixed points, phase space, late time attractor, past attractor, inflation etc.

1 Introduction

Dynamical systems approach, now-a-days, has become one of the most suitable and viable way for qualitative specification of various cosmic features, specially for studying the possible asymptotic states at early as well as late times of many a cosmological models. We do qualitative study of the system rather than finding the exact solutions. By qualitative study, we mean the study of the behavior obtaining information about the properties of the system. Considering the early Universe, inflation should be taken into account with which the expansion of the Universe with acceleration can be expounded through the dominance of potential energy density, \( V(\phi) \) of a scalar field (\( \phi \)) over its energy density. In fact, inflation has been regarded as a part of cosmological evolution. In terms of observational evidence, we can cite observations from supernova light curve data to Wilkinson Microwave Anisotropy Probe (WMAP) data[1] which strongly supports the accelerating Universe. We can also mention here f(R) theory of gravity for explaining the expanding Universe[2]. Several intricate observations have admitted that our Universe restarted expansion with acceleration around 5 Gyr ago, and that too at low energy scales of approximately \( 10^{-4} \) eV [3-5]. In recent years, it is well discovered that our Universe is in the phase of not just expansion but expansion with acceleration[6]. But, it is quite an accepted fact that inflation is one such theory of exponential expansion of the early Universe which corresponds to energy scales \( \sim 10^{16} \) GeV [7-14]. This large scale deviation in the energy scales have drawn many a researcher towards these two phenomena and how the difference has come up. This curiosity has led researchers to hunt new cosmic models that best suit the observations. So far, among all the recent works, the approaches that give appealing descriptions on the present scenario of the expanding Universe are firstly, "dark energy"
(DE) associated with a large negative pressure which is frequently described by a conventional vacuum
energy or scalar field, and secondly, we can mention the six-parameter base $\Lambda$CDM model which also
agrees with the current high-precision data\cite{6,14-15}, though there are some cosmological constant and
coincidence problems \cite{16,17,19-23}. Also, Dvali-Gabadadze-Porrati (DGP) braneworld model \cite{24-26}
has also put forward the cosmic accelerated expansion of the space of the present Universe. We can
also name the modified theories of gravity in this context\cite{27-29}. Despite roaming about all these
cosmological journeys, however, in this presentation we only stick to dynamical systems perspective to
cosmology. A cosmological model described by an autonomous system of ordinary differential equation
(ASODE) having a past time attractor, one or two saddle points and a late time attractor is regarded
as the most suitable and complete model. The past time attractor represents inflationary epoch, saddle
points correspond to the phase of the Universe where radiation and matter dominate and late time
attractor represents accelerated expansion phase of today’s Universe \cite{30,31}. Scalar field cosmology with
scalar potential forms such as flat, constant, simple exponential potential, inverse power law potential,
double-exponential potential, PNGB potential etc has been studied by many researchers also\cite{32-42].
Scalar field cosmology has also paved its grand way to study dark energy, which is taken as the main
entity for the cosmic expansion with acceleration at late time. Indeed, scalar field cosmological models
have owned their esteemed place in modern cosmology, specially in analyzing early inflation as well as
late time acceleration and dark energy scenarios for scalar field is such an exotic matter that provokes
the necessary negative pressure so as to cause acceleration \cite{43}. One can use relativistic equation of state
like that of a radiation ($\gamma = \frac{4}{3}$) or an ultra relativistic fluid ($\gamma = 2$). One can also choose $\gamma = 0$ to fetch
a vacuum energy density with no actual fluid where we obtain $\rho = -p = \text{constant}$. So, its worthy enough
to confer about the situation with a perfect fluid having the above equation of state for various values of $\gamma$.

This paper has been arranged as follows: In section 2, we depict how the system of gravitational
field and wave equations are developed. Then, with this field and wave equations, we introduce new
dimensionless variables to generate a dynamical system. We again categorize this section into two
sub-sections, one for analysis without potential and other for analysis with potential, the potential being
taken in simple exponential potential form shown before. From the perspective of dynamical system
which is an autonomous system of ordinary differential equation (ASODE), we are going to analyze
each sub-section deeply. The subsection containing analysis with potential will be observed thoroughly,
under two subcases depending on the value of $\gamma = 2$ or $\gamma \neq 2$, by sorting out the fixed points for each
case. We, then, study the nature of stability of the possible fixed points so obtained in respective cases
and discuss their stability along with the cosmic scenarios associated with them. We devote section 3 to
conclusion.
2 Dynamical system analysis

Dynamical system is a mathematical system that describes the time dependence of the position of a point in the space that surrounds it, termed as ambient space. There are different ways of approach to dynamical system, namely measure theory approach which is motivated by ergodic theory, real dynamical system, discrete dynamical system, etc. Here, we are approaching towards the system through an autonomous system of ordinary differential equations, (ASODE). ASODE is a system of ordinary differential equations which does not depend explicitly on time. As for our dynamical system, we will be using logarithmic time, \( N = \ln \left( \frac{a}{a_0} \right) \) as our independent variable, where \( a \) represents the scale factor and \( a_0 \) denotes the present scale factor value which, later on, will be taken to be unity for simplicity as doing so doesn’t affect the behavior of our system. Now, we consider a minimally coupled classical scalar field \( \phi \) represented by the classical equation of motion,

\[
\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0
\]

(1)

where the overhead dot represents the derivative with respect to cosmic time, \( t \). Here, in this paper, we consider the Universe is filled with a perfect fluid bearing an equation of state as follows:

\[
p = (\gamma - 1)\rho
\]

(2)

where \( \gamma \) is a constant, \( p \) is the pressure and \( \rho \) is the density of the fluid.

We also consider the Universe to be characterized by the following line element for an FRW space time as,

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dt^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2 \theta d\phi \right]
\]

(3)

we assume the Universe to be abounded with a perfect fluid where the energy momentum tensor of the perfect fluid is characterized by the following equation,

\[
T^{(F)}_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}
\]

(4)

where \( p \) and \( \rho \) are respectively thermodynamic pressure and density of the fluid. The unit time vector \( u_\mu \) for a co-moving system is specified by \( v^\mu v_\mu = -1 \) and \( v^\mu = \delta^\mu_0 \). We consider a minimally coupled classical scalar field \( \phi \) which contributes to \( T_{\mu\nu} \) as follows,

\[
T^{(\phi)}_{\mu\nu} = g_{\mu\nu} \left[ \frac{1}{2} \phi,\alpha \phi^\alpha \right]
\]

(5)

Equations 5 and 6 lead to the gravitational field equations shown below,

\[
3H^2 + \frac{k}{a^2} = \frac{1}{2} \dot{\phi}^2 + \rho
\]

(6)

\[
2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{1}{2} \dot{\phi}^2 - p
\]

(7)

where the overhead dot denotes differentiation w.r.t cosmic time \( t \), \( k \) is the curvature parameter which can take values -1, 0 or +1, according to open, flat or close Universe respectively and \( H = \frac{\dot{a}}{a} \) is the Hubble
parameter. Here, we choose the units in such a way that $8\pi G = 1$ and $c = 1$.[44] The conservation equation of the fluid is described by,

$$\dot{\rho} + 3H(\rho + p) = 0$$  (8)

2.1 Analysis without potential:

For studying without potential, we have

$$\ddot{\phi} + 3H\dot{\phi} = 0$$  (9)

We now present a new set of dimensionless variables, $x = \frac{\dot{\phi}^2}{\phi}$, $y = \frac{\rho}{\phi}$ and $N = \ln a$. Using these new variables, the above equations reduces to the following set of autonomous system of ordinary differential equations (ASODE)

$$x' = -\frac{4}{3}x^2 + 2(\gamma - \frac{2}{3})xy - \frac{14}{3}x$$  (10)

$$y' = 2(\gamma - \frac{2}{3})y^2 - \frac{4}{3}yx + (\frac{4}{3} - 3\gamma)y$$  (11)

where the overhead dash represents differentiation w.r.t logarithmic time, $N = \ln a$.

To find the fixed points, we equate $x' = 0$, $y' = 0$ and check the stability of the fixed points so obtained. For this, we determine the Jacobian matrix $J$ of the autonomous system at the respective fixed points. The fixed points along with the associated eigenvalues, according to which we have discussed stability analysis, have been shown in table 1. We have also analyze range value of $\gamma$ within which we have got the stable model, assuming $\gamma \neq \frac{4}{3}$ for fixed points in finite phase-plane. The jacobian matrix derived at the fixed point $a(0,0)$, $J_a$ is given by

$$J_a = \begin{bmatrix} -\frac{14}{3} & 0 \\ 0 & (\frac{4}{3} - 3\gamma) \end{bmatrix}$$

Since $J_a$ is a diagonal matrix, the eigenvalues are given by the diagonal entries which are $-\frac{14}{3}$ and $(\frac{4}{3} - 3\gamma)$. Both the eigenvalues are real and will have negative signs when $\gamma > \frac{4}{9}$. As long as $\gamma \neq \frac{4}{9}$, the fixed point $a$ is a hyperbolic fixed point as none of the eigenvalues vanishes and here, stability depends on the sign of the eigenvalues of the Jacobian matrix, $J_a$ at this fixed point $a$. Both the eigenvalues will be negative when $\gamma > \frac{4}{9}$, so the fixed point $a$ is stable for those values of $\gamma$ such that $\gamma > \frac{4}{9}$. So, when $\gamma > \frac{4}{9}$, the fixed point $a$ behaves as late time attractor representing the accelerated expansion phase of the observable Universe. For $\gamma < \frac{4}{9}$ the fixed point $a$ acts as a saddle point with one positive real eigenvalue and one negative real eigenvalue which is unstable. The existence of saddle point describes the radiation and matter dominated cosmic scenario of the Universe.[45] When $\gamma = \frac{4}{9}$, the fixed point $a$ becomes non-hyperbolic and stability depends on the sign of the remaining eigenvalue which is $-\frac{14}{3}$. Since, the remaining eigenvalue is negative, at $\gamma = \frac{4}{9}$, the fixed point $a$ is still stable.

We can also infer from Figures 1 and 2, showing the perturbation plots along x and y axes with respect to $N$ respectively, that projection of perturbation along y axis evolve to some fixed value. When we approach to the value of $\gamma$ from the right side of $\frac{4}{9}$, we find that the perturbation along y axis gradually
Fig. 1 shows variation of perturbation along x axis against N at $\gamma = \frac{4}{9}$ for fixed point $a$. Fig. 2 shows variation of perturbation along y-axis as $\gamma \to (\frac{4}{9})^+$ for fixed point $a$.

Fig. 3 shows phase plot of the system without potential for $\gamma = \frac{4}{9}$. Fig. 4 shows phase plot of the system without potential for $\gamma = 2$ showing $a (x=0, y \to 0)$ is a stable fixed point.

Fig. 5 shows phase plot of the system without potential for $\gamma = 0$ showing $c (x = 0, y \to 1)$ is a stable fixed point. Fig. 6 shows phase plot of the system without potential for $\gamma = \frac{4}{9} < \frac{2}{9}$ showing the stability nature of all the possible fixed points $a (x=0, y \to 0)$, $b (x=-3.5, y \to 0)$ and $c (x=0, y \to 0.5)$.
Fig. 7 shows phase plot of the system without potential for $\gamma = 0$ showing the stability nature of all the possible fixed points $a$ ($x=0, y\to0$), $b$ ($x=-3.5, y\to0$) and $c$ ($x=0, y\to1$).

evolves till it achieves a constant value and after achieving that fixed value even as $N$ tends to infinity the perturbation along $y$ axis fails to grow anymore even at $N$ tends to infinity. While the perturbation along $x$ axis decays to $x=0$ which is the fixed point value as $N \to \infty$. As a result, we conclude that the fixed point $a$ is still stable for $\gamma = \frac{4}{9}$ when it becomes non-hyperbolic. In addition to that, from another angle of analysis through phase plot shown in Fig. 3, it is found that all the trajectories closed enough to $a$ move towards it and decay to that point. Hence, the non-hyperbolic fixed point $a$ is stable for $\gamma = \frac{4}{9}$ also and thus our analytical findings have been supported by geometrical approach also. We refer to table I to study other fixed points. As the upper limit of $\gamma$ for any type of fluid is 2 [46], the fixed point is stable for all $\gamma \epsilon \left[\frac{4}{9}, 2\right]$. From Figure 4, we see that all the trajectories in the neighborhood of the fixed point $a$ are attracted towards $a$ and acts as attractor where we have drawn the phase plot for $\gamma = 2$. Figures 3, 5, 6 and 7 show the phase plot of the system at various conditions indicating the behavior of the other fixed points which strongly hold up our analytical findings from the geometrical point of view.

**TABLE I: Table for analysis without potential**

| Fixed points | $x$ | $y$ | Eigen Values | Stable range of $\gamma$ | Behavior |
|--------------|-----|-----|--------------|--------------------------|----------|
| $a$          | 0   | 0   | $\frac{1}{4}$, $(\frac{4}{9} - 3\gamma)$ | stable for $\gamma \epsilon \left[\frac{4}{9}, 2\right]$ | late time attractor for $\gamma > \frac{4}{9}$ |
| $b$          | $-\frac{7}{2}$ | 0   | $\frac{14}{9}$, $(6 - 3\gamma)$ | unstable for all $\gamma$ | unstable |
| $c$          | 0   | $(\frac{3\gamma - 6}{3\gamma - 4})$ | $(3\gamma - 6), (3\gamma - \frac{4}{3})$ | stable for $\gamma \in \left[0, \frac{4}{9}\right]$ | late time attractor for $\gamma < \frac{4}{9}$, saddle point for $\gamma > \frac{4}{9}$ |

Fig. 3 shows the qualitative nature of the system showing the stability behavior of all the fixed points for $\gamma = \frac{4}{9}$ showing all the trajectories converges towards $a$ and hence stable while $b(-3.5,0)$ acts as repeller as all the orbits moves away from it. Fig. 4 shows, geometrically, how the orbits in the neighborhood of the fixed point $a(x=0,y\to0)$ moves towards it as $N$ tends to infinity. Thus, Fig. 4 shows the phase plot of the system for $\gamma = 2$ showing $a$ as late time attractor which is stable. From the analytical finding, we see that the fixed point $c$ is stable for $\gamma \in \left[0, \frac{4}{9}\right]$. Now, Fig. 5 shows the phase plot of the system at $\gamma = 0$. At $x=0$, as $y$ approaches to 1, all the trajectories near the fixed point $c(x=0,y\to1)$ moves
towards $c$ showing that every solution that starts close enough to $c$ approaches to it in the limit $N \to \infty$ showing $c(x=0,y\to 1)$ is a stable fixed point which behaves as a late time attractor. Fig. 6 shows the stability nature of all the fixed points $a$, $b$ and $c$ of the system at $\gamma = \frac{1}{3} < \frac{4}{9}$. Here, all the orbits in the neighborhood of the fixed point $b(-3.5,y\to 0)$ move away from it as $N$ increases indicating that all the long term behavior moves away from the fixed point $c$ and hence unstable while near the fixed point $(x=0,y\to 0)$, some trajectories are attracted to it and some are repelled by it which indicates $a$ is a saddle fixed point whereas, all the other orbits near the fixed point $c(x=0,y\to 0)$ decays to it, ie, all the trajectories are attracted towards the fixed point $c$ for $\gamma = \frac{1}{3} < \frac{4}{9}$ which indicates that $c$ behaves as late time attractor. Thus, Fig. 6 supports the analytical findings shown in table I. When $\gamma = \frac{4}{9}$, the fixed points $a$ and $c$ become non-hyperbolic as one eigenvalue becomes zero. However, at $\gamma = \frac{4}{9}$, the fixed point $c$ $(\frac{3\gamma - 4}{(2\gamma - 3)})$ reduces to $a(x = 0, y = 0)$ which is stable for $\gamma \in [\frac{4}{9}, 2]$. The phase plot shown in Fig. 7 explains the geometrical study of the nature of the fixed points $a$, $b$ and $c$ at $\gamma = 0$. In this Fig. at $b$ $(x=-3.5,y\to 0)$, all the trajectories close enough to it are repelled away from it and hence unstable, but, near $a(x=0,y\to 0)$ some trajectories move towards it and some are repelled by it and hence becomes saddle fixed point. However, at $c(x=0,y\to 1)$, in the neighborhood of it, all the trajectories moves towards it and decays towards it as $N\to \infty$ and hence stable. We can see vividly, from the phase plot of the system shown in Figs. [3 - 7], the behavior of the fixed points $a$, $b$ and $c$ at different situations.

2.2 Analysis with potential:

The study of a scalar field coupled with a potential by dynamical system approach has many applications in GR specially to explain several cosmological features. Now, we introduce a scalar potential $V(\phi)$ which is taken in simple exponential potential form,

$$V(\phi) = V_0 e^{-\lambda \phi}$$

where $\lambda$ is a non-negative constant. We assume simple exponential potential of the form

$$V(\phi) = V_0 e^{-\lambda \phi}$$

where $V_0$ and $\lambda$ are non-negative constants. We refer to the work by CP Singh and Milan Srivastava for the choice of this potential form[47]. The impact of this potential on the energy momentum tensor is described as below:

$$T_{\mu\nu}^\phi = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu}\left[\frac{1}{2} \phi_{,\alpha} \phi^{,\alpha}\right] + V(\phi)$$

The field equations and wave equations now become,

$$3H^2 + \frac{k}{a^2} = \frac{1}{2} \dot{\phi^2} + V + \rho$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{1}{2} \dot{\phi^2} + V - p$$

$$\ddot{\phi} + 3H \dot{\phi} + \frac{\delta v}{\delta \phi} = 0$$

The conservation equation of fluid is given by:

$$\dot{\rho} + 3H(\rho + p) = 0$$
We consider the following dimensionless variables as: 

\[ x^2 = \frac{\dot{\phi}^2}{\rho H^2}, \quad y = \frac{\dot{\rho}}{\rho H^2}, \quad z = \frac{V}{\rho H^2} \]

With these new variables, the above system reduces to the following set of autonomous system of ordinary differential equation:

\[
\begin{align*}
\dot{x}' &= -\frac{2}{3}x^3 + (\gamma - \frac{2}{3})xy - \frac{2}{3}xz - \frac{7}{3}x + \frac{\sqrt{3}}{2}\lambda z \\
\dot{y}' &= 2(\gamma - \frac{2}{3})y^2 - \frac{4}{3}yx^2 - \frac{4}{3}yz + \left(\frac{4}{3} - 3\gamma\right)y \\
\dot{z}' &= -\frac{4}{3}z^2 - \lambda\sqrt{\frac{3}{6}}xz - \frac{4}{3}z^2 + 2(\gamma - \frac{2}{3})zy + \frac{4}{3}z
\end{align*}
\]  

(19) (20) (21)

To determine the fixed points, we equate \( \dot{x}' \), \( \dot{y}' \) and \( \dot{z}' \) to 0 and fixed points are studied. We tabulate, precisely, the fixed points and their stability analysis in table II. We consider two cases, one for \( \gamma = 2 \) and other for \( \gamma \neq 2 \).

**Case 1 : \( \gamma \neq 2 \)**

We obtained two fixed points on the finite phase-plane presuming \( \gamma \neq \frac{2}{3} \). The fixed point \( A(0, \frac{2\gamma - 4}{3\gamma - 2}) \) is non-hyperbolic for \( \gamma = 0 \) or \( \gamma = \frac{4}{9} \) or \( \gamma = 2 \), otherwise it is hyperbolic. For non-hyperbolic fixed points, we can’t use the usual stability analysis. There are some other famous methods like Center Manifold Theory [27,29], Lyapunov’s functions [48,49] and phase plot or numerical perturbation of solutions about the critical points [50]. However, we will perturb the system by a small amount and study the perturbation plots against \( N \). To draw the 3-D perturbation plot is difficult to study, so we will plot the variation of perturbation along each axis for each fixed point to determine the nature of stability and its contribution to the behavior of the system.

The Jacobian matrix for the system at this fixed point \( A \) is as follows:

| Fixed points | x   | y   | z     | stability and behaviour                          |
|--------------|-----|-----|-------|--------------------------------------------------|
| A            | 0   | 0   | 0     | saddle point for all \( \gamma > 0 \), unstable  |
| B            | 0   | 0   | 0     | saddle point for all \( \gamma > 0 \), unstable  |

\[
J_A = \begin{bmatrix}
\frac{3}{2}\gamma - 3 & 0 & \frac{\sqrt{3}}{2}\lambda \\
0 & (3\gamma - \frac{4}{3}) & -\frac{2}{3}(\frac{2\gamma - 4}{3\gamma - 2}) \\
0 & 0 & \frac{3\gamma}{3\gamma - 2}
\end{bmatrix}
\]

The above matrix \( J_A \) is an upper triangular matrix, so, diagonal entries represent the eigenvalues: \( n_1 = \frac{3\gamma}{2} - 3 \), \( n_2 = 3\gamma - \frac{4}{3} \), \( n_3 = 3\gamma \). Since, the upper limit of any fluid is 2, \( n_1 \) can have negative values till 0, ie, \( n_1 \) is negative for all \( \gamma \geq 0 \). For \( n_2 \), when \( \gamma = \frac{4}{9} \) it comes out to be 0, otherwise for \( \gamma < \frac{4}{9} \), \( n_2 \) is negative and for \( \gamma > \frac{4}{9} \), \( n_2 \) is positive. For \( n_3 \), for any value of \( \gamma > 0 \), \( n_3 \) is positive. As long as \( 0 \neq \gamma \neq \frac{4}{9} \), we get hyperbolic fixed points where all the eigenvalues are non-zero with some eigenvalues positive and some negative for corresponding value of \( \gamma \) particularly chosen. As such we found \( A \) to be a saddle point which is unstable. For the fixed point \( A \), when \( \gamma = 0 \) and \( \gamma = \frac{4}{9} \), it becomes non-hyperbolic as some eigenvalues become zeros. However, at \( \gamma = \frac{4}{9} \), the fixed point \( A \) reduces to \( B \) which is unstable.
Fig. 8 shows how perturbation along $x$ axis decays with the increase in $N$ for fixed point $B$ at $\gamma = \frac{4}{9}$.

Fig. 9 and Fig. 10 show the variation of perturbations along $y$ and $z$ axes at $\gamma = \frac{4}{9}$ for fixed point $B$.

at $\gamma = \frac{4}{9}$ as shown in Figures 8, 9 and 10 also. Figures 11, 12 and 13 shows how the perturbations along $x$, $y$, $z$ axes follow with the variation in $N$ at $\gamma = 0$. We see from Fig. 13 that the perturbation along $z$ axis grows so as to move the system away from the fixed point $A$ at $\gamma = 0$. So, from the perturbation plot we conclude that the fixed point $A$ is unstable at $\gamma = 0$.

The Jacobian matrix at the fixed point $B(0,0,0)$, $J_B$ is given by,

$$J_B = \begin{bmatrix} -\frac{7}{3} & 0 & \frac{\sqrt{3}}{3} \lambda \\ 0 & \left(\frac{4}{3} - 3\gamma\right) & 0 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

$J_B$ is an upper triangular matrix with the eigenvalues being $m_1 = -\frac{7}{3}$, $m_2 = \left(\frac{4}{3} - 3\gamma\right)$, $m_3 = \frac{4}{3}$. $B$ will be hyperbolic for all values of $\gamma$ except $\frac{4}{9}$. For non-hyperbolic case we consider phase plot for stability analysis. In case of hyperbolic type, since, $m_1$ is always negative and $m_3$ is always positive, the fixed point $B$ is always a saddle point no matter $m_2$ comes out to be positive or negative. The fixed point $B$ becomes non-hyperbolic when $\gamma$ takes the value $\frac{4}{9}$, as one of the eigenvalues vanishes thereat.

To study stability for this sort of fixed point, we refer to Figures 8, 9 and 10 which show the variation of perturbations along $x$, $y$ and $z$ axes against $N$. The fixed point $B$ becomes non-hyperbolic at $\gamma = \frac{4}{9}$. Figures 8, 9 and 10 show the projection of perturbations along each axis against $N$. We infer from the Figures that the perturbation along $x$ axis decays to zero as $N$ increases while the perturbations along $y$ and $z$ axes grow exponentially with the increase in $N$ and hence this fixed point $B(0,0,0)$ is unstable.
Fig. 11 and Fig. 12 show the variation of perturbations along x and y axes respectively against $N$ at $\gamma=0$ for fixed point A.

Fig. 13 shows the variation of perturbation along z axis against $N$ at $\gamma=0$ for fixed point A. Fig. 14 show variation of perturbations along x axis against $N$ at $\gamma = 2$ for fixed point C.

at $\gamma = \frac{4}{9}$ where it becomes non-hyperbolic. The fixed point $A$, at $\gamma = 0$, becomes non-hyperbolic and is unstable as seen from the perturbation plots shown in Figures 11, 12 and 13. Here also, the perturbations along z-axis grows with the increase in $N$ though the perturbations along $x$ and $y$-axes gradually decrease and decay to zero as $N \to \infty$. So from the perturbation plot, at $\gamma = 0$, the non-hyperbolic fixed point $A$ is unstable.

Case 2: When $\gamma = 2$:

Fig. 15 shows the variation of perturbation along y axis against $N$ at $\gamma=2$ for fixed point C. Fig.16 shows perturbation along z axis against $N$ at $\gamma=2$ for fixed point C.
We found the fixed point \( C \) as \( x = \text{undetermined}, \ y = \text{undetermined}, \ z = 0 \), i.e., \( C(x, y, 0) \). For further study we perturbed the system from the fixed point by a small amount and allow it to evolve. If the system returns back to the fixed point through the perturbations, the system is stable otherwise, if instead, the perturbation grows, the system is unstable. Figure 14 shows the projection of perturbation along \( x \) axis against \( N \). We find that the perturbation along \( x \) axis gradually decays and eventually evolves to a constant value near around \(-0.4\) while that of \( y \) axis evolves to a fixed value around \(-0.5\) as seen from Figure 15. But, the perturbation along \( z \) axis grows with the increase in the value of \( N \) as shown by Figure 16. Hence, this fixed point is unstable at \( \gamma = 2 \).

3 Conclusion:

We have presented, in this work, a dynamical system perspective of scalar field cosmology without potential as well as with potential, the potential being taken in simple exponential form. We have also found that scalar field cosmology, in the absence of potential, can be expressed as a two dimensional dynamical system and with potential we can extend the system to three dimensional one. For the analysis without potential, we have got three critical points out of which the critical point \( a \) is found to behave as late time attractor representing the late time behavior of the accelerated expansion Universe for radiation \( (\gamma = \frac{1}{3}) \) or dust \( (\gamma = 1) \) or stiff fluid \( (\gamma = 2) \) cosmological models. However, with false vacuum cosmological model with \( \gamma = 0 \), our developed model has become the most effective cosmic model that best describes the complete cosmic features[49]. The phase plots that we have enclosed have also supported our analytical findings. When \( \gamma = 0 \), one eigenvalue is positive and the other is negative at the critical point \( a \). So it contributes to our model with a saddle point while the fixed point \( b \) acts as past time attractor as both the eigenvalues thereat becomes positive representing the inflationary epoch and the fixed point \( c \) completes the model by adding late time attractor to represent the accelerated expansion phase with both the eigenvalues being negative. Fig. 7 also emphasizes our result at \( \gamma = 0 \) depicting all the stability nature of all the possible fixed points \( a, b, c \) so obtained thereby showing our cosmic model so developed completely describes all the three main cosmic features: inflation, saddle points, and late time attractor.

When we analyze with potential, we have got a 3 dimensional dynamical system and specially, we have considered two cases depending on the value of \( \gamma \), \( \gamma \neq 2 \) or \( \gamma = 2 \). For \( \gamma \neq 2 \), the fixed points \( A \) and \( B \) so obtained both represent saddle points for all \( \gamma > 0 \) which are unstable. So, here regardless of whether we consider stiff fluid or dust or radiation cosmological models, we get two saddle points. Besides, for false vacuum cosmology with \( \gamma = 0 \), the fixed point \( B \) is still saddle and unstable. As seen from the perturbation plots, it is found that the system deviates away from fixed point \( B \) as the perturbation grows along \( y \) and \( z \) axes with the increase in \( N \), \( B \) is unstable at \( \gamma = \frac{4}{9} \).

So, the present work reflects the fact that \( \text{GR} \) is a late time attractor[51-53] and our work as a whole reveals the present expanding scenario of the Universe which approaches to \( \text{GR} \) where we have analyzed for suitable values of \( \gamma \). We have also got all the three main features of cosmic evolution, i.e., inflation, saddle points and late time attractors. In fact, scalar field cosmology approximates to \( \text{GR} \) for suitable \( \gamma \) we have analyzed and shown in section 2 of our work. Our developed cosmic model shows a deep connection with the cosmic acceleration phenomena thereby supporting the fact that our
The Universe is in the phase of accelerated expansion. In terms of future perspective we are trying to extend our work, dynamical system approach, to higher derivative cosmological models. We are also currently working hard to head on to higher order derivative problems and related cosmic scenarios.

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