Emergent Mott insulators and non-Hermitian conservation laws in an interacting bosonic chain with noninteger filling and nonreciprocal hopping

Zuo Wang, Li-Jun Lang, and Liang He

Guangdong Provincial Key Laboratory of Quantum Engineering and Quantum Materials, School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou 510006, China

We investigate the ground state and quantum dynamics of an interacting bosonic chain with the nonreciprocal hopping. In sharp contrast to its Hermitian counterpart, the ground state can support Mott insulators in systems with noninteger filling due to the competition between nonreciprocal hopping and the on-site interaction. For the quantum dynamics, conservation laws for non-Hermitian systems manifest a stark difference from their Hermitian counterpart. In particular, for any Hermitian operator that commutes with the Hamiltonian operator, its expectation value is guaranteed to be nonconserved in the non-Hermitian quantum dynamics. To systematically identify the non-Hermitian conservation law, we establish a generic approach for constructing the conserved quantities in non-Hermitian many-body quantum systems with completely real spectra, and illustrate it concretely by the system under study. The direct experimental observation of Mott insulators in systems with noninteger filling and non-Hermitian conservation laws can be performed by ultracold atoms in optical lattices with the engineered nonreciprocal hopping.

I. INTRODUCTION

In a very short time, the study of non-Hermitian systems has spread over almost all branches of physics in recent years, including both quantum and classical ones [1]. Besides the reason that the gain/loss is naturally inherited in open systems [2, 3], the introduction of non-Hermiticity can also give birth to exotic phenomena that cannot exist in Hermitian counterparts such as the spontaneous breaking of parity-time symmetry [4] and exceptional topology [5]. Among them, one of the most intriguing discussions is about the breakdown of the bulk-boundary correspondence (BBC) in non-Hermitian topological systems, which has stimulated many efforts in the construction of topological theories to rescue it [6–10]. Around 2018, many research groups paid special attention to a Su-Schrieffer-Heeger-like model [11–17], where a so-called non-Hermitian skin effect (NHSE) [11] that assembles bulk states via nonreciprocal hopping to localize at one end, was found to be responsible for the breakdown of BBC. This effect is unique to non-Hermitian systems and has been observed in diverse experimental platforms including photons [18–20], electrical circuits [21, 22], mechanical systems [23, 24], nitrogen-vacancy centers [25], and cold atoms [26].

At single-particle levels, while NHSEs have been widely studied to generalize basic notions of topology in non-Hermitian systems [1], they also affect prototypical quantum phenomena such as Anderson localization in non-Hermitian quasicrystals [27–29]. In comparison, at many-body levels, the discussion of NHSEs on many-body phenomena has just begun to draw attention [30–34]. First attempts have been made at the celebrated one-dimensional Hubbard-like models to understand how the on-site interaction interplays with the NHSE [35–39]. For fermionic models, the Fermi-Dirac statistics intrinsically prohibits the nonreciprocal-hopping-induced aggregation of fermions at one end, resulting in a Fermi-surface-like plateau in real space [32, 38]; for bosonic models, the on-site repulsive interaction can still drive a superfluid–Mott-insulator transition, but the superfluid is skinned to one end [30]. In spite of this progress, open questions are still left, for instance, it is well known that Mott insulators can only exist at integer fillings for the Hermitian Bose-Hubbard model [40, 41], therefore a natural question that arises in this context is whether the interplay between non-Hermitian nonreciprocal hopping and the interaction could give rise to a Mott insulator in systems with noninteger filling. As a matter of fact, even more questions are left open concerning the quantum many-body dynamics of these interacting non-Hermitian systems. A particularly fundamental one is the existence and identification of the conservation laws in non-Hermitian interacting quantum many-body systems, which not only plays a crucial role in simplifying and solving physical problems, but also influences strongly the macroscopic universal dynamical properties of the systems in the vicinities of possible continuous phase transitions via “slow-modes” [42].

In this paper, we address these questions for an interacting bosonic chain with nonreciprocal hopping and on-site repulsive interactions. To this end, we investigate both its ground state and quantum dynamical properties via exact diagonalization combined with analytic analyses, and find that the system manifests both distinct static (cf. Fig. 1) and dynamical (cf. Figs. 2 and 3) properties that are in sharp contrast to those of its Hermitian counterpart. More specifically, we find the following.

(i) Emergent Mott insulators in systems with noninteger filling due to the competition between nonreciprocal hopping and the on-site interaction (cf. Fig. 1), in sharp...
contrast to the system’s Hermitian counterpart where Mott insulator can only exist at integer filling: Due to nonreciprocal hopping, bosons aggregated near one end of the chain [cf. the red curve of Fig. 1(a)] could strongly repel each other in the presence of strong enough on-site interactions, giving rise to a Mott insulator region with strongly depressed particle number fluctuations [cf. the black and green curves of Figs. 1(a) and 1(b) and also Figs. 1(c) and 1(d)]. Moreover, this result is consistent with that of the noninteracting Fermi-Hubbard model with nonreciprocal hopping [35, 38], which corresponds to the hard-core limit of the system investigated here.

(ii) Non-Hermitian conservation laws of the system [cf. Eqs. (5)–(7) and Figs. 2 and 3] and a systematic approach to identify the non-Hermitian conservation laws in non-Hermitian many-body quantum systems with completely real spectra [cf. Eq. (4)]: We find that the conservation laws for the non-Hermitian system generically manifest a stark difference from their Hermitian counterpart, in particular, for any Hermitian operator that commutes with the Hamiltonian operator, its expectation value is guaranteed to be nonconserved in the non-Hermitian quantum dynamics. This clarifies a fundamental concept that is widely misused in the literature that the total particle number of this system is conserved because the corresponding operator $\hat{N}$ commutes with the system’s Hamiltonian, i.e., $[\hat{N}, \hat{H}] = 0$ [cf. Figs. 2(a), 2(b) and 3 for the time evolution of the expectation value of the $\hat{N}$, clearly manifesting time dependence]. By employing similarity transformations between the non-Hermitian Hamiltonians and the auxiliary Hermitian Hamiltonians, we establish a generic approach for identifying the conservation laws in non-Hermitian many-body quantum systems with completely real spectra [cf. Eq. (4)], and find a set of conservation laws of the system [cf. Eqs. (5)–(7) and Figs. 2 and 3].

II. SYSTEM AND MODEL

The system under consideration is an interacting bosonic chain with nonreciprocal hopping, which is described by the Hamiltonian

$$\hat{H} = -J \sum_{j=1}^{L-1} (e^{\alpha} \hat{b}_j^\dagger \hat{b}_{j+1} + e^{-\alpha} \hat{b}_{j+1}^\dagger \hat{b}_j) + \frac{U}{2} \sum_j \hat{n}_j (\hat{n}_j - 1),$$

where $\hat{b}_j^\dagger$ ($\hat{b}_j$) is the creation (annihilation) operator at site $j$ in the Wannier representation, $\hat{n}_j \equiv \hat{b}_j^\dagger \hat{b}_j$ is the particle number operator, and $L$ is the total number of the lattice sites. The first term in Eq. (1) describes the nonreciprocal hopping with $J > 0$ being the geometric mean and $\alpha \in \mathbb{R}$ characterizing the asymmetry of the hopping amplitudes in opposite directions. The second term describes the on-site interaction with strength $U$. In general, a non-Hermitian system has a complex spectrum and thus cannot reach a dynamical equilibrium; here, we focus on the system with the open boundary condition whose spectrum is completely real [36]. Most numerical results to be presented in the following are obtained by directly diagonalizing the system’s Hamiltonian at a finite size $L$ ($L = 12$), and without loss of generality we set $\alpha \geq 0$, that is, the left-biased hopping is considered.

When the hopping is reciprocal ($\alpha = 0$), the model Hamiltonian reduces to the conventional Bose-Hubbard model, where it is well known that at large enough on-site interaction $U$ the system supports a Mott insulator only at an integer filling, while a superfluid at a noninteger filling. Once $\alpha \neq 0$, the nonreciprocity of the hopping is introduced and makes the Hamiltonian non-Hermitian. Previous investigations in related noninteracting systems have revealed that nonreciprocal hopping generally can result in NHSEs [11], where particles tend to accumulate on one end of the chain due to the unbalanced hopping in opposite directions. In this regard, one naturally expects the competition between nonreciprocal hopping and the on-site interaction could give rise to physics that is absent in conventional Hermitian Bose-Hubbard-type systems and the noninteracting non-Hermitian bosonic chain. Indeed, as we shall see in the following, Mott insulators can emerge even in systems with noninteger filling due to the collaboration between nonreciprocal hopping and the on-site interaction.

III. EMERGENT MOTT INSULATORS IN SYSTEMS WITH NONINTEGER FILLING

To investigate the ground state properties of the system we mainly focus on the local density distributions $\langle \hat{n}_j \rangle = \langle \psi_\alpha^R | \hat{n}_j | \psi_\alpha^R \rangle$ and the local relative density fluctuations $\Delta_j \equiv \sqrt{\langle \hat{n}_j^2 \rangle - \langle \hat{n}_j \rangle^2} / \langle \hat{n}_j \rangle$, with $| \psi_\alpha^R \rangle$ being the right ground state of $\hat{H}$. The density distributions of a half-filled system, i.e., the filling factor $\rho \equiv N/L = 1/2$ with $N$ being the total number of particles, at different interaction strengths are shown in Fig. 1(a) (other system parameters are kept fixed with $|J| = 1$, $\alpha = 2$). We see that at weak on-site interaction $U/|J| \ll 1$, the nonreciprocal hopping term dominates, and the system shows a strong NHSE where particles accumulate at the left end of the bosonic chain [cf. the red curve in Fig. 1(a)].

As the on-site repulsive interaction increases, particles on the same site strongly repel each other hence decreasing the accumulation of particles at the end of the chain [cf. the blue curve in Fig. 1(a)]. At large enough interaction, one expects that the most favorable density distribution of the system is the one with a region where on average one particle per site (reducing the on-site energy cost) is located at one side of the chain (favored by the NHSE). Indeed, as we can see from the green, black and brown curves in Fig. 1(a), the system forms a pronounced density plateau once $U > 14.2$ (estimated by the density fluctuation on the first site $\sqrt{\langle \hat{n}_1^2 \rangle - \langle \hat{n}_1 \rangle^2} < 0.1$), where the local relative density fluctuations $\Delta_j$ are strongly
From the above discussion, we see that the emergence of a Mott insulator at $\rho = 1/2$ shown in Fig.(a) is generally induced by the competition between NHSE and the on-site repulsive interaction, therefore we expect Mott insulators could also emerge at other noninteger fillings. Indeed, as we can see from Figs.(c) and (d), systems can generally form a Mott insulator at noninteger fillings ($\rho = 1/3, 5/12, 1/2, 7/12, 2/3$) in the presence of the strong competition between NHSE and the on-site repulsive interaction. Moreover, it is worth noting that the density fluctuations are no longer strongly suppressed in the region away from the plateau indicating the whole system is not insulating.

### IV. NON-HERMITIAN CONSERVATION LAWS

Thus far we have investigated the static ground state properties of the system. Now let us turn to investigate a fundamental aspect of the dynamical property of the system, namely its conservation laws, which not only play a crucial role in simplifying and solving physical problems, but also influence strongly the macroscopic universal dynamical properties of the system in the vicinities of possible continuous phase transitions via “slow modes” \[42\]. For closed quantum systems, this can be usually done by checking the commutator between the possible candidate and the Hamiltonian operator of the system under consideration. However, this wisdom no longer applies to the open quantum systems described by non-Hermitian Hamiltonians. For instance, the total energy of an open quantum system is generically not conserved despite the apparent fact that the Hamiltonian operator of the system commutes with itself. This thus leads us first to investigate the general requirement for any observable being a conserved quantity in generic non-Hermitian quantum systems.

#### A. General requirement of non-Hermitian conservation laws

We first denote the Hermitian operator associated with a generic physical observable by $\hat{O}$, whose expectation value at time $t$ is denoted as $\hat{O}(t) \equiv \langle \hat{\psi}(t)|\hat{O}|\hat{\psi}(t)\rangle$, with $|\hat{\psi}(t)\rangle$ being the state of a generic time-independent non-Hermitian system described by the Hamiltonian $\hat{H}$. The formal solutions of $|\hat{\psi}(t)\rangle$ and $\langle \hat{\psi}(t)\rangle$ are $|\hat{\psi}(t)\rangle = e^{-i\hat{H}t}|\hat{\psi}(0)\rangle$ (units are chosen such that $\hbar = 1$) and $\langle \hat{\psi}(t)\rangle = \langle \hat{\psi}(t = 0)|e^{i\hat{H}t}\rangle$, respectively. If $\hat{O}$ is a conserved quantity, then $\partial_t \langle \hat{O}(t) \rangle = i\langle \hat{\psi}(0)|e^{i\hat{H}t}([\hat{H}, \hat{O}])|\hat{\psi}(0)\rangle = 0$ have to be satisfied, indicating $\hat{O}$ have to satisfy the requirement

$$\hat{H}^\dagger \hat{O} - \hat{O} \hat{H} = 0,$$

where we immediately notice that any Hermitian operator $\hat{O}$ that commutes with the Hamiltonian operator, i.e., $[\hat{O}, \hat{H}] = 0$, is guaranteed to be not conserved due to the non-Hermiticity, i.e., $\hat{H}^\dagger \hat{O} - \hat{O} \hat{H} = (\hat{H}^\dagger - \hat{H}) \hat{O} \neq 0$. This is in sharp contrast to the Hermitian quantum systems, where any Hermitian operator that commutes with the Hamiltonian operator is guaranteed to be a conserved quantity, and raises the fundamental question of the existence of conserved laws in non-Hermitian quantum systems and their concrete forms. Indeed, as we shall present in the following, such conserved quantities indeed exist and can be systematically constructed for non-Hermitian systems with completely real energy spectra.

We first notice that for any non-Hermitian diagonalizable Hamiltonian $\hat{H}$ with a completely real spec-
trum, one can always construct a similarity transformation $\hat{S}$ which transforms $\hat{H}$ into an auxiliary Hermitian Hamiltonian $\hat{H}_{\text{aux}}$ with the same spectrum (cf. Ref. 43 or a short derivation presented in the Appendix), i.e., $\hat{S}^{-1} \hat{H} \hat{S} = \hat{H}_{\text{aux}}$. Therefore, one could reformulate Eq. (2) in terms of the similarity transformation $\hat{S}$ and the corresponding auxiliary Hermitian Hamiltonian, i.e., $(\hat{S}^\dagger)^{-1} \hat{H}_{\text{aux}} \hat{S}^\dagger \hat{O} - \hat{O} \hat{S} \hat{H}_{\text{aux}} \hat{S}^{-1} = 0$, which is equivalent to

$$[\hat{S}^\dagger \hat{O} \hat{S}, \hat{H}_{\text{aux}}] = 0,$$

indicating if $\hat{S}^\dagger \hat{O} \hat{S}$ is a conserved quantity, denoted as $C_{\text{aux}}$, of the auxiliary Hermitian Hamiltonian $\hat{H}_{\text{aux}}$, then $\hat{O} = (\hat{S}^\dagger)^{-1} C_{\text{aux}} \hat{S}^{-1}$ is a conserved quantity of the non-Hermitian system, i.e.,

$$\partial_t \hat{O}(t) = 0 \text{ for } \hat{O} = (\hat{S}^\dagger)^{-1} C_{\text{aux}} \hat{S}^{-1}, \text{ if } [C_{\text{aux}}, \hat{H}_{\text{aux}}] = 0. \quad (4)$$

This thus provides a systematic way to identify the conservation laws in non-Hermitian quantum systems. In the following, we shall use the interacting bosonic chain with nonreciprocal hopping described by the Hamiltonian (1) as a concrete example to illustrate identifying the conservation laws in non-Hermitian quantum systems.

**B. Non-Hermitian conservation laws in an interacting bosonic chain with nonreciprocal hopping**

As one can notice from Eq. (1), as long as one finishes the construction of the similarity transformation $\hat{S}$, identification of the non-Hermitian conservation laws directly follows. For the bosonic non-Hermitian chain with nonreciprocal hopping under consideration, one can take the similarity transformation employed in the investigation of the “non-Hermitian skin effect” 11 as a convenient choice for $\hat{S}$, whose explicit form in the second quantization form reads $\hat{S} = A \exp(-\alpha \sum_{j=1}^{L} j \hat{n}_j)$, with $A$ being a nonzero “gauge” constant. One can straightforwardly show that the similarity transformation $\hat{S}$ can transform the non-Hermitian Hamiltonian (1) into the Hermitian Hamiltonian of the conventional Bose-Hubbard chain, i.e., $\hat{S}^{-1} \hat{H} \hat{S} = \hat{H}_{\text{BH}}$ with $\hat{H}_{\text{BH}} = -J \sum_{j=1}^{L-1} (\hat{b}_j^\dagger \hat{b}_{j+1} + \text{H.c.}) + \frac{U}{2} \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1)$. From the form of $\hat{H}_{\text{BH}}$, one can directly notice that the total particle number operator $\hat{N} \equiv \sum_{j=1}^{L} \hat{n}_j$, $\hat{H}_{\text{BH}}$ itself, and the identity operator $\hat{I}$ commute with $\hat{H}_{\text{BH}}$, corresponding to the total particle number, energy, probability conservation in the conventional Bose-Hubbard chain, respectively. Via Eq. (4), we thus can directly identify three conservation laws for the non-Hermitian system, namely, the observables that correspond to the Hermitian opera-

**FIG. 2. Quantum dynamics of the expectation values for the conserved observables $\hat{I}$, $\hat{N}$, $\hat{H}$ and the related operators $\hat{I}$, $\hat{N}$, $\hat{H}$ that commute with the system’s Hamiltonian. The parameters of the system are $\alpha = 2$, $Je^\alpha = 1$, $U = 1$, $L = 12$. The “gauge” constant $A$ is chosen in such a way that makes $\bar{I}(t = 0) = 1$. (a), (c), (e) Quantum dynamics of the system with the initial state $|\psi(t = 0)\rangle = \sum_{m,n=1}^{M} \langle \psi_n^R | \psi_m^L \rangle | \psi_m^L \rangle$, with $M = 3$ and the total particle number being 6. (b), (d), (f) Quantum dynamics of the system with the initial state being the same as the one for (a), (c), (e) except $M = 10$. See text for more details.**

Conservation laws ered.

To further corroborate the above analytic results, we numerically simulate the quantum dynamics of the system and calculate the time evolution of the expectation value of the conserved quantities, i.e., $\bar{N}(t)$, $\bar{H}(t)$, $\bar{I}(t)$, and also those of the related operators that commute with the Hamiltonian $\hat{H}$ of the system, i.e., $\bar{N}(t)$, $\bar{H}(t)$, $\bar{I}(t)$. More specifically, as shown in Fig. 2 we choose two initial states of the form $|\psi(t = 0)\rangle = \sum_{m,n=1}^{M} \langle \psi_n^R | \psi_m^L \rangle | \psi_m^L \rangle$ with $M = 3, 10$ and the total particle number being 6 ($|\psi_m^L \rangle$ is the nth lowest right eigenstate of $\hat{H}$) and simulate the corresponding quantum dynamics of the system with $L = 12$, $U = 1$, $\alpha = 2$, $Je^\alpha = 1$ kept fixed. As one can see from Fig. 2, ...
while $\tilde{\mathcal{N}}(t)$, $\tilde{\mathcal{H}}(t)$, $\tilde{\mathcal{I}}(t)$ manifest a strong time dependence despite the fact that their corresponding operators commute with the Hamiltonian. It is interesting to notice that $\tilde{\mathcal{N}}(t)/\tilde{\mathcal{I}}(t)$ manifests a time-independent behavior, however this mainly originates from the fact that the total particle number operator $\tilde{\mathcal{N}}$ commutes with the similarity transformation operator $\tilde{\mathcal{N}}$, $\tilde{\mathcal{H}}(t)$, $\tilde{\mathcal{I}}(t)$ indeed remain exactly at their constant values during the whole quantum dynamics.

Finally, we would like to remark on a crucial difference between the conservation laws in non-Hermitian and Hermitian systems. In non-Hermitian systems, the existence of conservation laws manifests a strong sensitivity on the system’s boundary condition, since its change can turn the completely real spectrum into a complex one in certain cases, which consequentially excludes the existence of the conserved quantity. For instance, systems described by noninteracting or interacting Hamiltonians similar to Hamiltonian \[1\] with periodic boundary conditions imposed assume complex energy spectra \[36\] responding Hermitian systems, as interacting bosonic chains with nonreciprocal hopping have shown: The systems with noninteger filling can support Mott insulators in particular their quantum dynamical properties.

**V. EXPERIMENTAL OBSERVABILITY**

We expect that the Mott insulators in systems with noninteger filling and the non-Hermitian conservation laws identified in this work can be observed experimentally. For instance, by employing the well-established experimental platform offered by ultracold atoms in optical lattices \[44\] \[45\], the nonreciprocal hopping can be effectively engineered by introducing atom loss \[26\] \[46\] \[48\]. The direct observation of the Mott insulator in systems with noninteger filling can be readily performed via measuring the local density distribution with a quantum gas microscope \[49\]. For the non-Hermitian conservation laws, we expect the one associated with $\mathcal{A}$ is particularly feasible to be observed in current experimental setups via measuring the local density $\hat{n}_j$ with a quantum gas microscope at different time points during the quantum dynamics, and reconstructing the expectation value of $\mathcal{A}$ according to Eq. \[5\]. For instance, one could choose the Fock state with the central part of the system uniformly filled as the initial state which is most accessible in experiments, and monitor the time evolution of $\tilde{\mathcal{N}}(t)$. In Fig. 3 the time evolutions of $\tilde{\mathcal{N}}(t)$ [also $\tilde{\mathcal{I}}(t)$] with two centrally filled Fock states $\langle 000011110000 \rangle$, $\langle 000111110000 \rangle$ of the system with $L = 12$ as the initial states are shown. As we can see, while the direct measurement of the total particle number $\tilde{\mathcal{N}}(t)$ is expected to manifest a strong time dependence, the experimental measurements of $\tilde{\mathcal{N}}(t)$ are expected to remain unchanged at different time points.

**VI. CONCLUSIONS**

Non-Hermiticity together with interactions can give rise to rich many-body physics beyond those of corresponding Hermitian systems, as interacting bosonic chains with nonreciprocal hopping have shown: The systems with noninteger filling can support Mott insulators in particular their quantum dynamical properties.

**ACKNOWLEDGMENTS**

We thank Xiancong Lu, Guoqing Zhang, and Danwei Zhang for helpful discussions. This work was supported by NSFC (Grants No. 11874017 and No. 11904109), GDSTC under Grant No. 2018A030313853, Guangdong Basic and Applied Basic Research Foundation (Grant No. 2019A1515111101), Science and Technology Program of Guangzhou (Grant No. 2019050001), and START grant of South China Normal University.

**Appendix A: Similarity transformations between non-Hermitian Hamiltonians with completely real spectra and Hermitian Hamiltonians**

To be self-contained, we review here a few mathematical properties of non-Hermitian diagonalizable Hamiltonians with completely real spectra presented in Ref. \[43\].

For a non-Hermitian Hamiltonian $\hat{H}$ with a completely

![FIG. 3. Quantum dynamics of the expectation value of the conserved quantity $\mathcal{N}$ and also the total particle number $\tilde{\mathcal{N}}$ with centrally filled Fock states as the initial states. System parameters are $\alpha = 0.1$, $J e^{\alpha} = 1$, $U = 1$, $L = 12$. The “gauge” constant $A$ is chosen in such a way that makes $\tilde{\mathcal{I}}(t = 0) = 1$. (a) Time evolutions of $\tilde{\mathcal{N}}(t)$ and $\tilde{\mathcal{N}}(t)$ with $\langle \psi(t = 0) \rangle = |000111110000 \rangle$ being the initial state. (b) Time evolutions of $\tilde{\mathcal{N}}(t)$ and $\tilde{\mathcal{N}}(t)$ with $\langle \psi(t = 0) \rangle = |0001111110000 \rangle$ being the initial state. See text for more details.](image)
real spectrum $E_n$, it assumes the formal form
\[ \hat{H} = \sum_{n} E_n |\psi_n^R \rangle \langle \psi_n^L|, \tag{A1} \]
where $|\psi_n^R \rangle$ and $|\psi_n^L \rangle$ are the right and left eigenstates of $\hat{H}$, respectively. These two sets of eigenstates satisfy the following relations,
\[ \hat{H} |\psi_n^R \rangle = E_n |\psi_n^R \rangle, \quad \hat{H} |\psi_n^L \rangle = E_n |\psi_n^L \rangle, \tag{A2} \]
\[ \langle \psi_m^L | \psi_n^R \rangle = \delta_{mn}, \quad \sum_n \langle \psi_m^L | \psi_n^R \rangle = \hat{I}, \tag{A3} \]
with $\delta_{mn}$ being the Kronecker delta function and $\hat{I}$ being the identity operator of the Hilbert space $\mathbb{H}$ associated with $\hat{H}$. Since the spectrum of the non-Hermitian Hamiltonian $\hat{H}$ is real, one can always construct a auxiliary Hermitian Hamiltonian $\hat{H}_{aux}$ that shares the same spectrum with $\hat{H}$, i.e.,
\[ \hat{H}_{aux} \equiv \sum_{n} E_n |n \rangle \langle n|, \tag{A4} \]
where $\{|n\rangle\}$ is a complete set of the orthonormal basis of the Hilbert space $\mathbb{H}$, i.e., $\sum_n |n \rangle \langle n| = \hat{I}$, $\langle m | n \rangle = \delta_{mn}$. By comparing Eqs. (A1) and (A4), one can notice that $\hat{H}$ can be transformed to $\hat{H}_{aux}$ via a similarity transformation, i.e.,
\[ \hat{S}^{-1} \hat{H} \hat{S} = \hat{H}_{aux}, \tag{A5} \]
with
\[ \hat{S} \equiv \sum_n |\psi_n^R \rangle \langle n|, \quad \hat{S}^{-1} = \sum_n |\psi_n^L \rangle \langle n|. \tag{A6} \]
[41] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, 2011).

[42] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[43] A. Mostafazadeh, J. Math. Phys. **43**, 2814 (2002).

[44] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature **415**, 39 (2002).

[45] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).

[46] L. Li, C. H. Lee, and J. Gong, Phys. Rev. Lett. **124**, 250402 (2020).

[47] Z. Ren, D. Liu, E. Zhao, C. He, K. K. Pak, J. Li, and G.-B. Jo, Nature Physics (2022).

[48] J. Li, A. K. Harter, J. Liu, L. de Melo, Y. N. Joglekar, and L. Luo, Nature Communications **10**, 855 (2019).

[49] W. S. Bakr, J. I. Gillen, A. Peng, S. Fölling, and M. Greiner, Nature **462**, 74 (2009).