On the Properties of the
Power Systems Nodal Admittance Matrix

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Abstract—This letter provides conditions determining the rank of the nodal admittance matrix, and arbitrary block partitions of it, for connected AC power networks with complex admittances. Furthermore, some implications of these properties concerning Kron Reduction and Hybrid Network Parameters are outlined.

Index Terms—Nodal Admittance Matrix, Rank, Block Form, Network Partition, Kron Reduction, Hybrid Network Parameters

I. INTRODUCTION

The contributions of this letter are threefold. First, the rank of the nodal admittance matrix of a generic AC power system with complex admittances is determined in the absence and presence of shunt elements (Section III). Second, it is shown that the diagonal blocks of the nodal admittance matrix given by an arbitrary partition of the network’s nodes always have full rank (Section IV). Third, some implications concerning the existence of the Kron Reduction and the Hybrid Network Parameters are outlined (Section V).

II. FOUNDATIONS

A. Graph Theory

Let \( |S| \) denote the cardinality of a set \( S \). A directed graph with vertices \( V = \{v_1, \ldots, v_{|V|}\} \) and edges \( E = \{e_1, \ldots, e_{|E|}\} \) is denoted by \( (V, E) \). The connectivity of \( (V, E) \) is defined by the incidence matrix \( A_{(V, E)} \). As known from graph theory \([1]\),

\[
\text{Lemma 1. If } (V, E) \text{ is connected, } \text{rank}(A_{(V, E)}) = |V| - 1.
\]

B. Circuit Theory

Let \( (V, E) \) define the topology of a power network. Note that \( V = N \cup G \), where \( G = \{v_1\} \) is the reference (ground) node, and \( N = \{v_1, \ldots, v_{|V|-1}\} \) are generic nodes. All sources and nodal voltage phasors \( V_n (n \in N) \) are referenced to \( G \). Say \( V_N / I_N \) the vectors of nodal voltage / current phasors. They are linked by the nodal admittance matrix, i.e. \( I_N = Y_N V_N \). Connections between any pair of nodes in \( N \) are represented by passive and reciprocal two-port equivalents. So \( E = L \cup T \), where \( L \) are the branches and \( T \) are the shunts. The former correspond to the longitudinal, and the latter to the transversal electrical parameters of the two-port equivalents. Say \( Y_L \) and \( Y_T \) the associated admittance matrices, then \([1]\)

\[
Y_N = A_{(N, L)}^T Y_L A_{(N, L)} + Y_T
\]

where \( Y_T \) is diagonal with \( \text{rank}(Y_T) \leq |N| \). For \( Y_L \), assume

**Hypothesis 1.** The branches are not coupled and have nonzero admittances. Therefore, \( Y_L \) is diagonal with \( \text{rank}(Y_L) = |L| \).

Define \( y_T = \text{diag}(Y_T) \). As known from circuit theory \( [2] \)

**Lemma 2.** \( \sum_k \text{row}_k(Y_N) = \sum_k \text{col}_k(Y_N) = y_T \).

C. Linear Algebra

Known equalities from linear algebra.

**Lemma 3.** For any matrix \( M \), \( \text{rank}(M^T M) = \text{rank}(M) \).

**Lemma 4.** For square matrices \( N_L, N_R \) with full rank and matching size, \( \text{rank}(N_L M) = \text{rank}(M) = \text{rank}(M N_R) \).

III. RANK

**Theorem 1.** If \((N', L')\) is connected and Hypothesis \([7]\) holds

\[
\text{rank}(Y_{N'}) = \begin{cases} |N'| - 1 & \text{if } Y_T = 0 \\ |N'| & \text{otherwise} \end{cases}
\]

In other words, \( Y_{N'} \) has full rank if there is at least one shunt.

Proof (Case \( Y_T = 0 \), see Fig. 1a) Observe that (1) reduces to \( Y_{N} = A_{(N', L')}^T Y_L A_{(N', L')} + Y_T \). Under Hypothesis \([1] \) \( N' \) is diagonal and with full rank, such that \( Y_L = B^T B \). Lemma 3 & 4 give \( \text{rank}(Y_{N'}) = \text{rank}(B A_{(N', L')}) = \text{rank}(A_{(N', L')}) \). Finally, Lemma 1 implies that the claim is correct.

Proof (Case \( Y_T \neq 0 \), see Fig. 1b) Let \( G' \) be a virtual ground. Define \( N' = N' \cup G', L' = L \cup T, T' = \emptyset \). Let \( Y'' = N'' \cup G' \) and \( E'' = L' \cup T' \) form the graph \((V', E')\). Redefine the voltages

\[
\frac{V'_{N'}}{V'_{|N'|+1}} = \begin{bmatrix} I & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V''_N \\ V'_{|N'|+1} \end{bmatrix}
\]

Accordingly, the circuit equations may be written as follows

\[
\frac{I_N}{I_{|N'|+1}} = \begin{bmatrix} Y_N & -y_T \\ -y_T^T & Y_T \end{bmatrix} \begin{bmatrix} V'_N \\ V'_{|N'|+1} \end{bmatrix}
\]
where $Y_T = \sum_k (y_T)_k$. Eliminate $I_{N^t+1} = -\sum_k (I_N)_k$ (Kirchhoff’s Law), and apply Lemma 2 on the right-hand side

$$\begin{bmatrix} I_N \\ 0 \end{bmatrix} = \begin{bmatrix} Y_N & -y_T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_N' \\ V_{|N|+1} \end{bmatrix}$$

(5)

Substitute $I_N$, and use Lemma 2 again on the right-hand side

$$\begin{bmatrix} I_N \\ 0 \end{bmatrix} = \begin{bmatrix} Y_N & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_N' \\ V_{|N|+1} \end{bmatrix}$$

(6)

Note that $(N', \mathcal{L}')$ is connected, and $Y_{\mathcal{L}'} = 0$ by construction. Hence, the matrix in (6) has rank $|N|$. Since the step (4)–(5) only involves elementary row operations, it preserves the rank. Lemma 3 implies that the matrix in (5) has full rank. Due to the zero rows and columns in (6), it follows that rank($Y_N$) = $|N|$. □

IV. BLOCK RANK

Consider a partition $\{N_p\} (p \in \mathcal{P})$ of $N$ into $|\mathcal{P}|$ subsets. Arrange $Y_N$ in block form by reordering rows and columns, where $Y_{N,i,j}$ is the block of $Y_N$ which relates $I_{N,i}$ and $V_{N,j}$.

Theorem 2. If $(\mathcal{V}, \mathcal{E})$ is connected and Hypothesis 2 holds, all the diagonal blocks $Y_{N,pp}$ ($p \in \mathcal{P}$) of $Y_N$ have full rank.

Proof. By definition, $Y_{N,pp}$ relates $I_{N,p}$ and $V_{N,p}$ when all $N_q$ ($q \in \mathcal{Q}$, $q \neq p$) are grounded. This defines a modified network, where branches between $N_p$ and $N_q$ become shunts. Say $(N_p, L_p)$, where $L_p = \{(v_i,v_j) \in \mathcal{L} \mid v_i,v_j \in N_p\}$, the part of $(N, \mathcal{L})$ associated with $N_p$. $(N_p, L_p)$ may not be connected, but it consists of components $(N_p,c, \mathcal{L}_c)$ ($c \in \mathcal{L}$) that are connected (see Fig. 2). Thus, $Y_{N,pp}$ is block diagonal, i.e. $Y_{N,pp} = \text{diag}(\{Y_{N,pp,cc}\})$. Since $(N, \mathcal{L})$ is connected, $\exists e \in \mathcal{L}$ acting as shunts $\forall N_{p,c}$, so any $N_{p,c}$ is connected to $G$ (see Fig. 2). Thus, by Theorem 1 any $Y_{N,pp,cc}$ has full rank. Since $Y_{N,pp}$ is block diagonal, it thus has full rank, too. □

V. IMPLICATIONS

A. Kron Reduction

Corollary 1. Suppose that the assumptions of Theorem 2 hold. Let $V_t$ ($t \in \mathcal{P}$) be a set of zero injection nodes, i.e. $I_{N,t} = 0$. Then, the $V_{N,s}$ ($s \in \mathcal{P}$, $s \neq t$) uniquely define $V_{N,t}$, so that $I_N = Y_N V_N$ may be reduced without loss of information.

This reduction technique is known as Kron Reduction [5], and is widely used in the field, e.g. for Power Flow studies or State Estimation. However, it is hardly ever verified whether the reduction is indeed feasible. For instance, the inventor [5] does not consider this issue at all, and [4] only studies simple cases (purely resistive / inductive connections). In this regard, Corollary 1 ensures that Kron Reduction can be performed.

Proof. Expand block row $t$ of $I_N = Y_N V_N$

$$I_N^t = Y_{N,tt} V_{N,t} + \sum_{k \neq t} Y_{N,tk} V_{N,k} = 0$$

(7)

According to Theorem 2, $Y_{N,tt}$ has full rank, so it possesses a unique inverse. Solve the above equation for $V_{N,t}$, and use the result to express the $I_{N,s}$ ($s \in \mathcal{P}$, $s \neq t$).

$$I_{N,s} = \sum_{k \neq t} (Y_{N,sk} - Y_{N,st} Y_{N,tt}^{-1} Y_{N,tk}) Y_{N,k}$$

(8)

which defines the reduced nodal admittance matrix $\hat{Y}$. □

B. Hybrid Network Parameters

Corollary 2. Suppose that the assumptions of Theorem 2 hold. Then one may swap $I_{N,p}$ and $V_{N,p}$ in $I_N = Y_N V_N \forall p \in \mathcal{P}$. Therefore, it exists a hybrid network parameter matrix $H$.

The existence of hybrid network parameters lies in Voltage Stability Assessment, namely some Voltage Stability Indices [3, 9]. In this regard, Corollary 2 ensures the existence of the required $H$ matrix.

Proof. Since Theorem 2 guarantees that $Y_{N,p}$ has full rank, one can solve block row $p$ of $I_N = Y_N V_N$ for $V_{N,p}$

$$V_{N,p} = Y_{N,pp}^{-1} I_{N,p} - \sum_{k \neq p} Y_{N,pp}^{-1} Y_{N,pk} V_{N,k}$$

(9)

which defines $H_{pp}$ and $H_{pk}$ ($k \neq p$). Substitute $V_{N,p}$ into the expressions for $I_{N,q}$ ($q \in \mathcal{Q}$, $q \neq p$)

$$I_{N,q} = \left\{ \begin{array}{l}
+ \sum_{k \neq p} (Y_{N,qk} - Y_{N,qp} Y_{N,pp}^{-1} Y_{N,pk}) V_{N,k} \\
\end{array} \right.$$

(10)

which defines $H_{qp}$ and $H_{qk}$ ($q \neq p, k \neq p$). So $H$ exists. □

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