Binomial formulas via divisors of numbers

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Abstract: The purpose of this note is to prove several binomial-like formulas whose exponents are values of the function \( \omega(n) \) counting distinct prime factors of \( n \).

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1 Introduction

Throughout the article, let \( n \geq 2 \) be an integer with canonical factorization

\[
n = \prod_{i=1}^{k} p_i^{a_i},
\]

where \( p_i \)'s are prime numbers and \( a_i \)'s are positive integers. We define the function \( \omega(n) \) (including \( n = 1 \) as an argument) counting the number of distinct prime factors \([1]\), that is,

\[
\omega(n) := \begin{cases} 
k, & n = \prod_{i=1}^{k} p_i^{a_i}, \\
0, & n = 1. \end{cases}
\] (1)

In the recent paper of Vassilev-Missana \([3]\) the following fact is provided.

Theorem 1.1. If \( n \) is a square-free number, then

\[
(1 + x)^{\omega(n)} = \sum_{d|n} x^{\omega(d)}. \] (2)
In particular, after substitution $x \to \frac{b}{a}$ the equation (2) leads to the binomial-like expansion

$$(a + b)^{\omega(n)} = \sum_{d|n} a^{\omega(n) - \omega(d)} b^{\omega(d)}. \quad (3)$$

In the paper we provide several generalizations of formulas (2) and (3). We prove some results for the sum of more than two terms case and also some results for non-square-free numbers.

2 Multinomial theorem for square-free number

In this section, we generalize formula (3) to the power of more than two terms. First, for a given integer $n \geq 1$ and any integer $m \geq 1$ we define the set

$$\text{Div}(n, m) = \{(d_0, d_1, \ldots, d_{m-1}, d_m) \in \mathbb{N}^{m+1} : d_0 = n, d_1|d_0, \ldots, d_{m-1}|d_{m-2}, d_m = 1\}.$$  

**Theorem 2.1.** Suppose $n$ is a square-free number. Then

$$(x_1 + \cdots + x_m)^{\omega(n)} = \sum_{\text{Div}(n, m)} \prod_{i=1}^{m} x_i^{\omega(d_{i-1}) - \omega(d_i)}. \quad (4)$$

Note that (3) is a special case of (4) for $m = 2$.

**Proof.** The proof goes by induction on $m$. First, we recall the proof for the case $m = 2$ adapted to our notation.

For arbitrary integer $n \geq 1$ and real $x$, set $f(n) = x^{\omega(n)}$. Then $f$ is multiplicative and so

$$F(n) = \sum_{d|n} f(d)$$

is multiplicative. Now suppose $n$ is a square-free number, that is $n = \prod_{i=1}^{\omega(n)} p_i$. Then

$$F(n) = \prod_{i=1}^{\omega(n)} F(p_i) = \prod_{i=1}^{\omega(n)} (f(1) + f(p_i)) = \prod_{i=1}^{\omega(n)} (1 + x) = (1 + x)^{\omega(n)}. \quad (5)$$

On the other hand,

$$F(n) = \sum_{d|n} f(d) = \sum_{d|n} x^{\omega(d)}. \quad (6)$$

Setting $x = \frac{x_2}{x_1}$ yields

$$\left(1 + \frac{x_2}{x_1}\right)^{\omega(n)} = \sum_{d|n} x_2^{\omega(d)} x_1^{-\omega(d)}.$$  

Multiplying by $x_1^{\omega(n)}$ we get the formula (4) with $m = 2$. Note that in this case

$$\text{Div}(n, 2) = \{(n, d, 1) : d|n\}$$

and the exponents of $x_1$ and $x_2$ are $\omega(n) - \omega(d)$ and $\omega(d)$, accordingly.

We now move to the induction step. Suppose (4) holds for $m > 1$. Then
where in (7) we apply (3) for $a = x_{m+1}$ and $b = x_1 + \cdots + x_m$, and in (8) we apply induction hypothesis. Notice that the set of indices of the double sum in (9) and the set $\text{Div}(n, m+1)$ are in one-to-one correspondence, that is

$$\{ (n, (d_0, d_1, \ldots, d_m)) : (d_0, \ldots, d_m) \in \text{Div}(d, m), \ d|n \}$$

and

$$\text{Div}(n, m+1) = \{ (d'_0, d'_1, \ldots, d'_m, d'_{m+1}) \in \mathbb{N}^{m+2} : d'_0 = n, \ d'_1|d'_0, \ldots, d'_m|d'_{m-1}, \ d'_{m+1} = 1 \}$$

are bijective and the bijection is set by

$$(n, (d_0, d_1, \ldots, d_m)) \mapsto (d'_0, d'_1, \ldots, d'_m, d'_{m+1}) = (n, d_0, d_1, \ldots, d_m).$$

We use the above reasoning to (9), which leads to the following formula

$$\sum_{d|n} \sum_{\text{Div}(d, m)} x^{\omega(n) - \omega(d)}_1 \prod_{i=1}^{m} x^{\omega(d_{i-1}) - \omega(d_i)}_i = \sum_{\text{Div}(n, m+1)} \prod_{i=1}^{m+1} x^{\omega(d_{i-1}) - \omega(d_i)}_i$$

and completes the induction. \hfill \Box

**Example 2.2.** Consider $m = 4$ and $n = 2 \cdot 3$ (here $\omega(n) = 2$). Then

$$\text{Div}(6, 4) = \{(6, 6, 6, 6, 1), (6, 6, 6, 3, 1), (6, 6, 6, 2, 1), (6, 6, 6, 1, 1), (6, 6, 3, 3, 1), (6, 6, 3, 1, 1), (6, 6, 2, 2, 1), (6, 6, 2, 1, 1), (6, 6, 1, 1, 1), (6, 3, 3, 3, 1), (6, 3, 3, 1, 1), (6, 3, 1, 1, 1), (6, 2, 2, 2, 1), (6, 2, 2, 1, 1), (6, 2, 1, 1, 1), (6, 1, 1, 1, 1) \}$$

and the corresponding terms of (4) with (for clarity) $a, b, c$ and $d$ instead of $x_1, x_2, x_3$ and $x_4$ are:

$$d^2, \quad cd, \quad cd, \quad e^2, \quad bd, \quad bc, \quad bd, \quad bc, \quad b^2, \quad ad, \quad ac, \quad ab, \quad ad, \quad ac, \quad ab, \quad a^2$$

It is clear that this corresponds to the multinomial expansion of $(a + b + c + d)^2$.

We note two immediate consequences of Theorem 2.1.

**Corollary 2.3.** If $n$ is a square-free number, then

$$\text{card Div}(n, m) = m^{\omega(n)}.$$

*Proof.* Apply Theorem 2.1 with $x_1 = \cdots = x_m = 1$. \hfill \Box

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Corollary 2.4. The number of non-increasing sequences $d_1, \ldots, d_m$ of length $m$, provided the numbers are from the set of factors of some square-free number $n$ and $d_{i+1}$ is a factor of $d_i$ for $i = 1, \ldots, m-1$, is equal to $(m + 1)^{\omega(n)}$.

3 Results for numbers that are not square-free

In Theorem 1.1, we assume that $n$ is a square-free number. It turns out that (2) and (3) are special cases of the following formula (see also [3]).

Theorem 3.1. For arbitrary integer $n > 0$ and any $x, y \in \mathbb{R}$ we have

$$\prod_{i=1}^{\omega(n)} (x + a_i y) = \sum_{d|n} x^{\omega(n) - \omega(d)} y^{\omega(d)}.$$

(10)

Proof. Notice that for prime $p$ and $a \geq 0$ we have

$$F(p^a) = f(1) + f(p) + \cdots + f(p^a) = 1 + ax,$$

where $F$ and $f$ are defined as in the proof of Theorem 2.1. Hence

$$F(n) = \prod_{i=1}^{\omega(n)} F(p_i^{a_i}) = \prod_{i=1}^{\omega(n)} (1 + a_i x)$$

and equation (6) is valid for arbitrary $n$. Therefore,

$$\prod_{i=1}^{\omega(n)} (1 + a_i x) = \sum_{d|n} x^{\omega(d)}$$

and substitution $x \rightarrow \frac{y}{x}$ leads to (10). \qed

Example 3.2. Let $n = 360 = 2^3 \cdot 3^2 \cdot 5$. Then the left-hand-side becomes a formula for

$$(x + 3y)(x + 2y)(x + y) = x^3 + 6x^2 y + 11xy^2 + 6y^3$$

in terms of the numbers related to divisors of 360. The terms corresponding to given divisor $d$ are gathered in Table 1.

| $d$ | 360 | 72 | 120 | 24 | 40 | 8 | 180 | 36 | 60 | 12 | 20 | 4 |
|-----|-----|----|-----|----|----|---|-----|----|----|----|----|---|
| $x^{\omega(n) - \omega(d)} y^{\omega(d)}$ | $y^3$ | $xy^2$ | $y^3$ | $xy^2$ | $x^2 y$ | $y^3$ | $xy^2$ | $y^3$ | $xy^2$ | $x^2 y$ | $x^2 y$ | $x^3$ |

| $d$ | 90 | 18 | 30 | 6 | 10 | 2 | 45 | 9 | 15 | 3 | 5 | 1 |
|-----|----|----|-----|----|----|---|-----|----|----|----|----|---|
| $x^{\omega(n) - \omega(d)} y^{\omega(d)}$ | $y^3$ | $xy^2$ | $y^3$ | $xy^2$ | $x^2 y$ | $y^3$ | $xy^2$ | $y^2$ | $x^2 y$ | $x^2 y$ | $x^3$ |

Table 1. Terms corresponding to all divisors $d$ or 360, ordered in decreasing order of the vector of powers of consecutive primes.

Note a trivial observation based on Theorem 3.1. If we substitute $x = y = 1$, then the right-hand side of (10) counts divisors of $n$, while the left-hand side of that formula is the usual formula for the number of divisors:
\[
\prod_{i=1}^{k} (1 + a_i).
\]

The following results search for the expansion of \((x + y)^{\omega(n)}\) for \(n\)'s that are not square-free numbers.

The next theorem is a binomial-like expansion for powers of square-free numbers. Here, to compensate changes in the formula, we have to include additional factor to the right-hand side.

**Theorem 3.3.** Suppose \(m\) is a square-free number and \(n = m^\ell\) for some integer \(\ell > 1\). Then

\[
(x + y)^{\omega(n)} = \sum_{d|n} \frac{x^{\omega(n) - \omega(d)} y^{\omega(d)}}{\ell^{\omega(d)}}.
\]

**Proof.** We apply previous results to obtain the following equations:

\[
(x + y)^{\omega(n)} = \left( x + \ell \cdot \frac{y}{\ell} \right)^{\omega(n)} = \prod_{i=1}^{\omega(n)} \left( x + \ell \cdot \frac{y}{\ell} \right) = \sum_{d|n} \frac{x^{\omega(n) - \omega(d)} y^{\omega(d)}}{\ell^{\omega(d)}},
\]

where (12) follows from (10).

Notice that equation (11) can also be written in one of the following fashion, resembling a binomial-like expansion:

\[
(x + y)^{\omega(n)} = \ell^{-\omega(n)} \sum_{d|n} (\ell \cdot x)^{\omega(n) - \omega(d)} y^{\omega(d)},
\]

\[
(\ell x + \ell y)^{\omega(n)} = \sum_{d|n} (\ell \cdot x)^{\omega(n) - \omega(d)} y^{\omega(d)}.
\]

**Example 3.4.** Consider \(n = 36 = (2 \cdot 3)^2\) (here \(\ell = 2\)). The terms corresponding to all divisors of \(n\) are in Table 2.

| \(d\) | \(36\) | \(18\) | \(12\) | \(9\) | \(6\) | \(4\) | \(3\) | \(2\) | \(1\) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \(x^{\omega(n) - \omega(d)} y^{\omega(d)}\) | \(y^2\) | \(y^2\) | \(y^2\) | \(xy\) | \(y^2\) | \(xy\) | \(xy\) | \(xy\) | \(x^2\) |
| \(\ell^{\omega(d)}\) | 4 | 4 | 4 | 2 | 4 | 2 | 2 | 2 | 1 |

Table 2. Analysis of \(n = 36\)

Interpreting second and third row of Table 2 as fractions we see that they add up to \(x^2 + 2xy + y^2\), as expected.

We now present a result for arbitrary number \(n\). Recall that if \(\mathbb{R}[X_1, \ldots, X_k]\) is a ring of polynomials in \(k\) variables over the field of real numbers, then elementary symmetric polynomials \(S_m(X_1, \ldots, X_k)\) are defined as the sums of all distinct products of \(m\) variables, that is:
\[ S_0(X_1, \ldots, X_k) = 1, \]
\[ S_1(X_1, \ldots, X_k) = X_1 + \cdots + X_k, \]
\[ S_2(X_1, \ldots, X_k) = \sum_{1 \leq i < j \leq k} X_i X_j, \]
\[ \vdots \]
\[ S_{k-1}(X_1, \ldots, X_k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq k} \prod_{j=1}^{k-1} X_{i_j}, \]
\[ S_k(X_1, \ldots, X_k) = X_1 \cdots X_k. \]

See [2] for further details concerning symmetric polynomials.

We now present the binomial-like expansion formula involving the function \( \omega(n) \) and symmetric polynomials.

**Lemma 3.5.** Suppose \( n = \prod_{i=1}^{\omega(n)} p_i^{a_i} \) is a canonical factorization of \( n \) and fix \( m \geq 0 \). Then
\[
\text{card}\{d \in \mathbb{N} : \omega(d) = m \text{ and } d | n\} = S_m(a_1, \ldots, a_{\omega(n)}).
\]

**Proof.** Suppose \( \omega(d) = m \). Then the number of divisors of \( n \) with that many distinct prime factors is, using combinatorial argument, equal to
\[
\sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq \omega(n)} \prod_{j=1}^{m} a_{i_j} = S_m(a_1, \ldots, a_{\omega(n)}).
\]

For example, if \( m = 2 \), then we choose two prime factors \( p_i \) and \( p_j \) with \( i \neq j \) and consider numbers of the form \( p_i^{b_i} p_j^{b_j} \), where \( b_i \in \{1, \ldots, a_i\} \) and \( b_j \in \{1, \ldots, a_j\} \). There are exactly
\[
\sum_{1 \leq i < j \leq \omega(n)} a_i a_j = S_2(a_1, \ldots, a_{\omega(n)})
\]
many divisors with two distinct prime factors. This generalizes to any number \( m \). \( \square \)

**Theorem 3.6.** Suppose \( n = \prod_{i=1}^{\omega(n)} p_i^{a_i} \) is a canonical factorization of \( n \). Then
\[
(x + y)^{\omega(n)} = \sum_{d | n} \frac{S_{\omega(d)}(a_1, \ldots, a_{\omega(n)})}{S_{\omega(d)}(a_1, \ldots, a_{\omega(n)})} x^{\omega(n)-\omega(d)} y^{\omega(d)}.
\]

**Proof.** Let \( k = \omega(n) \). Using classic binomial expansion we have
\[
(x + y)^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i = \sum_{i=0}^{k} \sum_{d | n \atop \omega(d) = i} C_i \binom{k}{i} x^{k-i} y^i.
\]
(13)

Equation (13) includes an additional factor that is a sum over divisors multiplied by a constant \( C_i \), fixed for given \( i \). In particular,
\[
\binom{k}{i} = \sum_{d | n \atop \omega(d) = i} C_i \binom{k}{i}.
\]
To find the constant, notice that for fixed \( i \) and by Lemma 3.5 we have

\[
C_i = \frac{\binom{k}{i}}{\sum_{d|n, \omega(d) = i} \frac{1}{\text{card}\{d \in \mathbb{N} : \omega(d) = i \text{ and } d|n\}}} = \frac{1}{S_i(a_1, \ldots, a_k)}.
\] (14)

Since \( k = \omega(n) \) and \( i = \omega(d) \), combining (14) with (13) we obtain

\[
(x + y)^{\omega(n)} = \sum_{d|n} \frac{\binom{k}{\omega(d)}}{S_{\omega(d)}(a_1, \ldots, a_{\omega(n)})} x^{\omega(n) - \omega(d)} y^{\omega(d)}. \tag{15}
\]

\[\blacksquare\]

Example 3.7. To illustrate Theorem 3.6, let \( n = 360 = 2^3 \cdot 3^2 \cdot 5^1 \). Then

\[
S_0(3, 2, 1) = 1,
S_1(3, 2, 1) = 6,
S_2(3, 2, 1) = 11,
S_3(3, 2, 1) = 6,
\]

and using the values in Table 1 in Example 3.2 we see that respective values coincide with the coefficients of the expansion of the polynomial. For example, there are 11 different divisors of \( n \) with \( \omega(d) = 2 \), each of them providing the same term \( \frac{\binom{3}{2}}{S_2(3, 2, 1)^2} x^2 y^2 = \frac{3}{11} x^2 y^2 \).

The above example inspires us to provide one more result. Using Theorem 3.1 and Lemma 3.5, we can easily deduce the following formula.

Corollary 3.8. Suppose \( n = \prod_{i=1}^{\omega(n)} p_i^{a_i} \) is a canonical factorization of \( n \). Then

\[
\prod_{i=1}^{\omega(n)} (x + a_i y) = \sum_{i=0}^{\omega(n)} S_i(a_1, \ldots, a_{\omega(n)}) x^{\omega(n) - i} y^i.
\]

4 Conclusion

We have derived several binomial-like expansions related to the function \( \omega(n) \). Our results also cover the cases where \( n \) need not be a square-free number. On the other hand, the formula provided in Theorem 3.6 is far from a very elegant formula (11). It would be interesting to find a simplified version of the former, perhaps without using binomial coefficients or symmetric polynomials.

References

[1] Jakimczuk, R. (2018). On the function \( \omega(n) \). International Mathematical Forum, 13(3), 107–116.

[2] Lang, S. (2002). Algebra, Graduate Texts in Mathematics, 211 (Revised third ed.), New York: Springer-Verlag.

[3] Vassilev-Missana, M. V. (2019). New form of the Newton’s binomial theorem. Notes on Number Theory and Discrete Mathematics, 25(1), 48–49.