**Controlled viscosity in dense granular materials**

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We experimentally investigate the fluidization of a granular material subject to mechanical vibrations by monitoring the angular velocity of a vane suspended in the medium and driven by an external motor. On increasing the frequency we observe a re-entrant transition, as a jammed system first enters a fluidized state, where the vane rotates with high constant velocity, and then returns to a frictional state, where the vane velocity is much lower. While the fluidization frequency is material independent, the viscosity recovery frequency shows a clear dependence on the material, that we rationalize by relating this frequency to the balance between dissipative and inertial forces in the system. Molecular dynamics simulations well reproduce the experimental data, confirming the suggested theoretical picture.

**Introduction.**– Granular systems can be found in solid-like states able to resist applied stresses [1], and in flowing fluid-like states [2,3]. The transition between these regimes is driven by changes in density and in applied stresses, as well as by changes in applied forcing. In this respect, the role of mechanical vibrations [4–32] in driving the unjamming transition is particularly relevant as related to many phenomena, from avalanche dynamics [33] and earthquake triggering [34] in geophysics, to the manufacturing process in material, food and pharmaceutical industries [35]. The influence of applied vibrations on the transition from a solid to a fluid-like state, and possibly from the fluid to the solid state, investigated in some numerical simulations [36,39], is an issue of great practical relevance. Indeed, its understanding might open the possibility of controlling the frictional resistance of granular media [19,37,38]. This problem has been addressed by Capozza et al. [30,39] on a prototypical model of particles confined between two rigid substrates in relative motion [37,40], the bottom substrates vertically vibrating. Their numerical simulations suggest that viscosity is reduced when the bottom plate vibrates in a range of frequencies, as rationalized through a general argument based on the reduction of effective interface contacts in the system. However, the validity of this argument lacks experimental verification in real granular media.

In this Letter we experimentally investigate the fluidization properties of different granular materials subject to periodic vertical vibrations, in a wide range of frequencies and amplitudes. We probe the viscosity features of the granular system by a vane suspended in it and driven by a motor, see Fig. 1. Measuring the average angular velocity of the vane as a function of the vibration frequency, we are able to explore a broad range of behaviors of the granular system, from fully jammed to unjammed-fluidized states. Our results confirm the existence of a frequency range in which the system is fluidized, the vane rotating with a finite speed. The transition from the solid to the fluid state occurs at a frequency which is very well estimated by the theory of Ref. [36], that we confirm also investigating particles of different materials. Conversely, we show that the viscosity recovery frequency, where the system transitions from the fluid to the highly viscous state, depends on the material properties. We argue that the material dependence of the viscosity recovery transition originates from a balance condition between dissipative and inertial forces acting in the system, and we support this claim through numerical simulations that allow us for a precise control of the dissipative forces.

**Experimental setup.**– We study the behavior of a granular system made of \( N = 2600 \) spheres, with diameter \( d = 4 \) mm, contained in a cylinder with a conical-shaped

![FIG. 1: Experimental setup. A vane (red rectangle) is coupled to a dc motor and is suspended in a dense granular system of spherical particles. The container is vertically vibrated with sinusoidal oscillations of frequency \( f \) and amplitude \( A \). On the top of the granular medium there is a plate, whose vertical displacement \( \Delta Z \) is measured with a laser sensor.](image-url)
floor (diameter 90 mm, minimum height 28.5 mm, maximum height 47.5 mm), see Fig. 1 with packing fraction \( \sim 49 \div 52\% \). The mass of each particle is \( m = 0.267 \) g for steel, \( m = 0.0854 \) g for glass, and \( m = 0.0462 \) g for delrin. The container is vertically vibrated by an electrodynamic shaker (LDS V450) following the protocol:

\[
z(t) = A \sin(2\pi ft)
\]  

(1)

where \( z \) is the vertical coordinate of the shaker plate. The maximal acceleration is \( z_{\text{max}} = A (2\pi f)^2 \). The explored frequency and amplitude ranges are \( 30 \div 700 \) Hz and \( 0.014 \div 0.053 \) mm, respectively. Higher values of \( f \) cannot be reached in our setup. Errors on the fixed vibration amplitudes are about 10\%. A Plexiglas vane (height 15 mm, width 6 mm, length 35 mm) is suspended in the medium, and is subject to an external torque. A dc motor coupled with the rotator is operated at 3 V, producing a torque \( \tau \sim 6 \times 10^{-5} \) Nm, see Supplemental Material (SM) [41]. Further details on the experimental setup are given in [39] [42] [43]. On the top of the granular medium we place a thick aluminum plate (mass \( M_{\text{top}} = 218 \) g). The vertical displacement of the plate \( \Delta Z \) can be measured by a laser device optoNCDT 1400, while the angular position of the vane \( \theta(t) \) is recorded by an encoder.

**Activated fluidization.**– In the absence of vibrations, the applied torque is not able to fluidize the system, which is in a static jammed configuration. We have considered how fluidization occurs when we drive the system, investigating the role of \( f \), for some fixed values of \( A \). In Fig. 2 (top panel), we show a typical time dependence of the rotator angular position for different experiments with steel spheres, shaken at \( A = 0.026 \) mm. From the signal \( \theta(t) \) we obtain the average angular velocity \( \omega = \langle d\theta(t)/dt \rangle \), where the average is taken over trajectories of 30 seconds. Considering the applied torque constant, the inverse of \( \omega \) is proportional to the macroscopic viscosity of the system. The values of \( \omega \) as a function of \( f \), for different amplitudes \( A = 0.014, 0.026, 0.053 \) mm are reported in the bottom panel of Fig. 2. The behavior of the system, as probed by the rotating vane, is characterized by three regimes. First, at low vibration frequencies, the vane velocity is zero, corresponding to infinite viscosity. This is due to the low energy fed into the system: the granular medium remains at rest in its jammed state, frictionally interacting with the vane. In the second regime, the vane angular velocity rapidly increases and reaches a maximum value \( \omega_{\text{max}} \). This behavior reflects the jammed-unjammed transition, induced by the mechanical vibrations, and corresponds to a viscosity reduction in the system. As detailed below, such a fluidized regime corresponds to the detachment condition from the vibrating substrate, where the granular medium expands and the top plate reaches its maximum height. In the third regime, for higher values of \( f \), \( \omega \) decreases, signaling an increasing viscosity. Our experimental setup is similar to the one used in [44], where the granular fluid was described as a thermalized system, displaying Brownian motion. In our case, the high density system leads to a more complex phenomenology [42]. Similar studies for granular suspensions are presented in [45] [46], where a model predicting their rheology is proposed. However, the dependence on the vibration frequency is not investigated.

The raise of \( \omega \) going from the first to the second regime, occurs at a well defined value of \( f \), that we denote by \( f_1 \). A quantitative estimation of this value can be obtained from the theoretical argument discussed in Refs. [39]. Indeed, the fluidization condition is realized when the largest force provided by the shaker, \( F = M \ddot{z}_{\text{max}} \), where \( M \) is the total mass of the system (granular particles and top plate), equals the weight \( Mg \), with \( g \) the gravity acceleration. According to this argument, from Eq. 1, the fluidization frequency \( f_1 \) is given by the following:  

\[
f_1 = \sqrt{\frac{Mz_{\text{max}}}{Mg}}
\]
FIG. 4: Power spectrum $S(F)$ of the signal $\Delta Z(t)$, measured in experiments with steel spheres at $A = 0.026$ mm, in the frictional regime (top panel) and in the fluidized regime (bottom panel).

relation:

$$2\pi f_1 = \sqrt{g/A}. \quad (2)$$

This expression predicts an explicit dependence of the fluidization frequency on the vibration amplitude, very well confirmed by our experimental data, as reported in the bottom panel of Fig. 2 for steel spheres (see vertical dashed lines). In order to demonstrate the generality of the fluidization mechanism, in Fig. 3 we report data obtained in experiments with different materials (steel, glass and delrin). We rescale the frequencies by $f > f_1$. Further insights are provided by the top plate vertical displacement $\Delta Z(t)$. In good agreement with what observed in the numerical simulations reported in Ref. [30], the power spectrum $S(F)$ of the signal $\Delta Z(t)$ shows pronounced peaks at integer multiples of $f$ for $f < f_1$, while in the fluidization region additional peaks at multiple values of $f/2$ do appear, see Fig. 1. As already observed in [39], this phenomenon is similar to the problem of period-doubling as a route to chaos in the bouncing ball and related models [49, 50].

Viscosity recovery. – Remarkably, the system exits the state of minimum viscosity at vibration frequencies $f \gtrsim f_2$, as shown in Fig. 3. Viscosity recovery at high frequencies has been previously observed in the model system of Ref. [30] and in numerical simulations of a driven spring-block model [16]. According to the argument of Ref. [30], the viscosity recovery is expected to occur when the detachment time from the bottom plate equals the period of the external oscillation. Since in our experiments there is no confining pressure on the top plate and the total normal force is simply $F_N = Mg$, the value of $f_2$ would be proportional to the fluidization frequency $f_2 = \sqrt{2\pi f_1}$, without any dependence on the materials. On the contrary, the recovery frequency $f_2$ observed in our experiments shows a marked dependence on the material: $\omega$ drops to about $0.5\omega_{\text{max}}$ at frequencies $f_2 \sim 3.5f_1$ and $5f_1$, for steel and glass, respectively (Fig. 5).

A similar phenomenology is observed in a simple spring-block model under vertical vibration [16]. There, the second transition to a state of larger viscosity originates from a balance between dissipative and inertial forces. More precisely, according to Ref. [16], $f_2$ depends on the dissipation rate. In our system, this quantity is affected by the elastic and dissipative forces characterizing the grain-grain and grain-interface interactions. To clarify the role of the dissipation at the medium-bottom interface in our system, we performed experiments where the bottom plate is covered with a thick layer of rubber tape, reducing the restitution coefficient in the collisions with the grains. As shown in Fig. 5, the recovery frequency is significantly reduced in this case (compare blue squares to green squares), namely $f_2$ decreases upon increasing the dissipation in the system.

Numerical simulations. – To confirm the above argument and obtain a quantitative explanation, we have performed molecular dynamics simulations of a granular medium of $N = 1000$ grains of unitary mass $m$ and diameter $d$, enclosed between two plates. The plates are made of closed packed grains whose relative positions are kept fixed during the dynamics. The system is confined by the gravitational force and has dimensions $L_x \times L_y = 20d \times 5d$, with periodic boundary conditions along the $x$ and $y$ directions. We use a standard model for the interparticle interaction, see SM for details [41]. The vertical distance $L_z$ between the plates is not fixed, as the system is allowed to expand under vibration, but typically $L_z \approx 10d$. Data for larger system size are reported in the SM [41] and show similar behaviors, suggesting that our results are robust and not a finite size effect.

The bottom plate moves according to the same protocol used in the experiments (i.e. $z(t) = A\sin(2\pi ft)$). To study the viscous properties of the system, we monitor the motion of a rigid cross-shaped subset of 5 grains, touching and glued to each other, and lying in the plane $z$-$y$. This probe, playing the role of the vane in the experiment, is subject to a constant force $F$ along the $x$ direction, while the positions of the 5 grains are kept fixed along $z$ and $y$. Time is measured in units of $t_0$ and the integration step is $5 \cdot 10^{-4}t_0$. Other parameters are $F = 500\,md/t_0^2$ and $g = 10\,d/t_0^2$. We employ a contact force model that captures the major features of granular interactions, known as linear spring-dashpot model, taking into account also the presence of static friction, as fully described in [16] [51, 52]. Our numerical simulations take into account both normal and tangential frictional forces among grains. We measure the velocity $v$ of the
probe along the x direction (averaged over trajectories of $3 \times 10^4 t_0$) for different values of $f$ and $A$, chosen in the range $f \in [10^{-3} t_0^{-1}, 10^{-1} t_0^{-1}]$ and $A \in [0.05 d, 0.2 d]$. Numerical results show a low frequency fluidization transition at $f_1$ in good agreement with Eq. (2), followed by a viscosity recovery at higher frequencies (Fig. 5). This complex behavior of the velocity is supported by the nonmonotonic behavior of the average particle coordination number, shown in SM [41]. A similar behavior of this quantity is also observed in the range of frequencies of acoustic fluidization [53].

These results, combined with the analysis of translational and rotational kinetic energies (see SM [41]), indicate that in the high frequency regime, the system attains high density values, and a relevant fraction of kinetic energy is rotational, in agreement with Refs. [54] [55], for systems under shear. In the following, we focus on the viscosity recovery transition observed at higher frequencies whose behavior is expected to depend on the dissipation mechanisms.

In numerical simulations we can change the viscoelastic properties of the system by tuning the rigidity of each grain, corresponding to a change in the restitution coefficient of each grain $e$ [61]. Frictional dissipation can be neglected, see SM [41]. More precisely, to reproduce the experimental setup, we consider two different restitution coefficients: $e_g$ for grain-grain collisions, and $e_b$ for collisions between grain and bottom plate. In Fig. 5 we show the value of $v/v_{\text{max}}$ as a function of $f/f_1$ for different values of $e_g$ and $e_b$. Results clearly indicate that $f_1$ is not affected by $e_b$ and $e_g$, whereas the recovery frequency $f_2$ depends on the dissipation. In particular, we expect that the smaller is the fraction of energy lost in a collision, the higher is the value of the recovery frequency $f_2$, since the system needs a larger number of collisions to dissipate the amount of energy necessary for the viscosity recovery. More specifically, the rate of energy dissipation can be estimated as $(1 - e^2) f$ [54] (the main assumption here being that the collision frequency is $\propto f$), giving a condition for the viscosity recovery of the kind $(1 - e^2) f_2 > \text{const}$. To verify this dependence of $f_2$ on $e$, we consider the simplest case $e = e_g = e_b$, and define $f_2$ as the value of the shaking frequency for which the velocity of the probe is $v = 0.25 v_{\text{max}}$. From the results reported in the inset of Fig. 5 we obtain a behavior $f_2/f_1 \sim (1 - e^2)^{-1}$, in agreement with the above argument. As in the experiments, these findings are not consistent with the scenario of [60], where the frequency $f_2$ is related to the rise time associated with the internal vibrations of the grains. Our numerical results also indicate that the main dissipation in the system occurs at the medium-bottom interface, as illustrated in Fig 5, where black and green symbols correspond to sets with $e_g$ and $e_b$ values interchanged. This scenario is confirmed by the direct measurement of the energy dissipated in the bulk and at the bottom interface, see SM [41]. Note that in the simulations we did not consider a dependence of the restitution coefficient on the impact velocity. Since the collision velocity changes with the vibration frequency, the restitution coefficient could increase up to 30%, depending on the material parameters [57] [60]. This could explain some differences between the experimental and numerical curves.

**Conclusion.** We have studied experimentally the viscous properties of dense granular materials under vertical vibration. Using a vane subject to an external torque as probe, we have observed different regimes in the system, from very large viscosity at low vibration frequencies, to fluidized states (corresponding to viscosity reduction) at intermediate $f$, with a viscosity recovery at higher values of $f$. The first transition to the fluidized state is well characterized by the detachment condition, and is independent of the material properties. The second transition, leading to viscosity recovery, turns out to be related to dissipation mechanisms in the medium and between the medium and the bottom plate, and therefore shows a strong dependence on materials. Our study suggests the possibility to control the viscous properties of confined granular media by tuning the shaking frequency in the system, with important practical application in several fields, from tribology to geophysics and material industry.

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**FIG. 5:** Numerical simulations. Rescaled velocity of the probe as a function of $f/f_1$. Data are obtained for systems with different values of $e_g$ and $e_b$, chosen in analogy with the experiments (see Fig. 3). Inset: The rescaled values of the recovery frequency $f_2$ as a function of the inverse of the dissipation factor $1 - e^2$ for systems with $e_g = e_b$. 
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Supplemental Material

I. RHEOLOGY EXPERIMENT

In our experimental setup we probe the viscosity features of the granular system measuring the average angular velocity of a vane suspended in it and driven by a dc motor. The motor is operated at 3 V, producing a torque $\tau \sim 6 \times 10^{-3}$ Nm. In Fig. 6 we show the dimensionless ratio $\tau/mgd$, where $m$ is the mass of one particle, $g$ the gravity acceleration and $d$ the particle diameter, for different materials. To give a physical meaning to the values obtained for the ratio $\tau/mgd$, we estimate a lower bound for the torque required to move the vane through the granular medium (in the absence of external vibration). For simplicity, we consider a disc with radius $R$ and height $l$ equal to those of the vane, and estimate the tangential frictional forces acting on the disc surface due to the medium. Then, the minimum torque necessary to rotate the disc is given by $\tau_0 \sim R S \mu P$, where $R = 17.5$ mm, $S$ is the surface of the disc ($2\pi R l = 1650$ mm$^2$, where $l = 15$ mm, we neglect the upper and lower faces of the disc), $\mu$ a friction coefficient and $P$ the pressure acting on the disc. The pressure $P$ can be estimated as $P \sim \phi \rho gh + Mg/\pi a^2$, where $\phi$ is the packing fraction ($\sim 50\%$) of the granular medium, $\rho$ the material density ($\rho = 0.008$ g/mm$^3$ for steel, $\rho = 0.0025$ g/mm$^3$ for glass and $\rho = 0.0014$ g/mm$^3$ for delrin), $g$ the gravity acceleration, $h$ the depth of the disc in the medium ($\sim 20$ mm), $M$ the mass of the top plate (218 g), and $a$ the radius of the container (45 mm). Then, we obtain that $\tau/mgd$ is smaller than the lower bound $\tau_0/mgd$, for realistic values of $\mu \geq 0.3$. This is consistent with our observation that in the absence of vibration the system remains in a jammed state, because the vane alone cannot fluidize the medium. In the figure we compare the ratio $\tau/mgd$ to $\tau_0/mgd$ for $\mu = 0.5$ for all materials.

![FIG. 6: Dimensionless ratio $\tau/mgd$ (black circles) and lower bound $\tau_0/mgd$ (red squares) for $\mu = 0.5$ for different materials.](image)

II. DETAILS OF THE NUMERICAL MODEL

We have performed molecular dynamics simulations of spherical frictional grains. We use a standard model for the interparticle interaction, detailed as model L2 in L. E. Silbert, D. Ertas, G. S. Grest, T. C. Halsey, D. Levine, and S. J. Plimpton Phys. Rev. E, 64, 051302, (2001). Two particles of diameter $d$ only interact when in contact, i.e. when the relative distance between their centers is $r < d$ and the overlap $\delta = d - r > 0$. In this case, the force has a normal and a tangential component. The normal component is

$$F_n = (k_n \delta - \gamma_n m \dot{\delta}) \vec{n}$$

where $k_n$ is the stiffness of the particle, $m$ the effective mass, $\gamma_n$ a normal damping coefficient. The restitution coefficient $e$ of this model does not depend on the impact velocity, and is a function of $k_n$ and $\gamma_n$:

$$e = \exp[-\pi/(2\sqrt{2k_n/(m\gamma_n^2)} - 1/4)].$$

In our simulations, we have fixed $\gamma_n = 50/t_0$ ($t_0 = 1$) and tuned the restitution coefficient acting on $k_n$, as experimentally we have considered particles with different stiffness. Specifically, we have introduced a restitution coefficient $e_g$ for particles in the bulk, and a restitution coefficient $e_b$ for the interaction between a particle in the bulk and one forming the rough bottom of the container.
The tangential interaction force is given by
\[ \vec{F}_t = k_t \vec{\delta}_t, \]
where \( k_t \) is a tangential stiffness and \( \vec{\delta}_t \) is the tangential shear displacement, which is defined as the integral of the relative velocity at the point of contact. However, to implement Coulomb’s condition, at every integration timestep the magnitude of \( \vec{\delta}_t \) is fixed to \( \mu|F_n|/k_t \) (\( \mu = 0.5 \)) if \( |F_t| > \mu|F_n| \). In our simulations we use \( k_t = (2/7)k_n \) as usual assumed in the literature. In the fluidized regime, contacts continuously form and break and therefore the average value of \( \vec{\delta}_t \) is much smaller than the average overlap \( \delta \). For this reason dissipation due to tangential friction is negligible.

III. DISSIPATED ENERGY

In Fig. 7 we show the energy dissipated per unit time, measured in numerical simulations, at the interface between the bottom plate and the granular material, \( \Delta E_{\text{bottom}} \), and in the bulk of the medium, \( \Delta E_{\text{bulk}} \). Results show that for \( f \gtrsim f_2 \), interaction with the boundary becomes the dominant dissipation mechanism. This effect becomes more relevant the softer are the bottom-plate grains.

![Energy dissipation graph](image)

**Fig. 7:** In numerical simulations we report the energy dissipated per unit time in the interaction between the bottom plate with the granular material, \( \Delta E_{\text{bottom}} \) (open squares), and in the bulk of the granular material, \( \Delta E_{\text{bulk}} \) (filled circles). Different colors correspond to different elastic properties of the grains.

IV. COORDINATION NUMBER

In Fig. 8 we report the coordination number defined as the average number of particles at distance smaller than one diameter from the center of a given particle. At low frequencies, softer grains present a higher coordination number. In all cases, for increasing \( f \), the coordination number presents a non-monotonic behavior with a minimum at the frequency \( f_{\text{max}} \) which corresponds to the frequency with maximum velocity in Fig. 5 of the manuscript. It is interesting to observe that the two curves (green diamonds and blue triangles) having the same value of \( e_g = 0.75 \), but different \( e_b \), are very similar up to \( f \approx 4f_1 \). This indicates that at small frequencies the dissipation with the bottom plate does not play a crucial role. At larger frequencies, conversely, curves separate drastically with grains in the system with the softer bottom plate presenting a larger coordination number. Results suggest that the system approaches a dense state at high frequencies where no translation is possible.

![Coordination number graph](image)

**Fig. 8:** Coordination number measured in numerical simulations. Different colors and symbols are used for the different elastic properties of grains in the bulk as well as in the bottom plate, with the same color and symbol code of Fig. 5 of the manuscript.
**V. TRANSLATIONAL AND ROTATIONAL KINETIC ENERGY**

In Fig. 9 we report the rotational kinetic energy per particle, \(E_{\text{rot}}\), and the translational kinetic energy per particle, \(E_{\text{tra}}\). The rotational energy presents a non-trivial behavior, as function of \(f\): An initial increase for \(f \lesssim f_{\text{max}}\) followed by a decay for \(f_{\text{max}} \lesssim f \lesssim f_2\) and a final increase for \(f \gtrsim f_2\). Also the translational energy presents a non-monotonic behavior, with a maximum value of the ratio \(E_{\text{rot}}/E_{\text{tra}}\) at \(f = f_{\text{max}}\). At larger frequencies \(f > f_{\text{max}}\), \(E_{\text{tra}}\) decreases quickly to zero indicating that for \(f > f_2\) translational motion is suppressed. The increase of \(E_{\text{rot}}\) for \(f > f_2\), conversely, indicates that, even if the system is approaching a denser state, the energy injected by the vibrating plate amplifies the grain rotation.

**VI. FINITE SIZE EFFECTS**

We study how the system size affects the dependence of viscous friction on the frequency of the vibrating plate. We simulate a system larger (see legend of Fig. 10) than the one presented in the manuscript, which better corresponds to the experimental set up. Results indicate only a weak dependence on the system size.

**FIG. 9**: In numerical simulations, we measure the rotational energy per particle \(E_{\text{rot}}\) (upper panel) and the translational kinetic energy per particle \(E_{\text{tra}}\) (lower panel), as a function of \(f/f_1\). The same color and symbol code of Fig.5 of the manuscript is used.

**FIG. 10**: Numerical simulations for systems with different sizes.