MINIMAL SECTIONS OF CONIC BUNDLES

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Abstract

Let \( X \) be a smooth conic bundle over the projective plane, and let \((J(X), \Theta)\) be the principally polarized intermediate jacobian of \( X \). In this paper we relate the Wirtinger description of \((J(X), \Theta)\) as a Prym variety, and the description of maximal subbundles of rank 2 vector bundles over curves, to prove the existence of two canonical families \( C^+ \) and \( C^- \) of curves on \( X \) such that the Abel-Jacobi map sends one of these families onto a copy of the theta divisor \( \Theta \subset J(X) \), and the other – onto \( J(X) \). The elements of \( C^+ \) and \( C^- \) correspond to the non-isolated and to the isolated minimal sections of a naturally defined system of conic bundle surfaces on \( X \). In either of the two possible cases we describe the general fiber of the Abel-Jacobi map for \( C^+ \) and for \( C^- \).

Preliminaries

In the classification of algebraic threefolds, conic bundles take a special place. One of the important applications of the Mori theory of minimal models, due especially to investigations of Miyaoka (see e.g. \([Miy]\)), is the obtained classification of regular threefolds of negative Kodaira dimension. In general, any such threefold \( V \) is birational to a threefold \( X \) with at most terminal singularities and of (at least) one of the following types: 1. Q-Fano threefolds; 2. Conic bundles \( X \to S \) over normal surfaces \( S \); 3. Del-Pezzo bundles \( X \to C \) over smooth curves \( C \) (see \([Isk]\) for more details).

Especially, if \( X \to S \) is a conic bundle, and if the base surface \( S \) is not rational, then \( X \) is not rational. Therefore, from the point of view of the problem of rationality, it makes sense to study conic bundles with a rational base \( S \). Since any rational surface \( S \) is birational to the projective plane \( \mathbb{P}^2 \), letting \( S = \mathbb{P}^2 \) is not a substantial restriction.

From some other point of view, any smooth algebraic threefold \( Y \) admitting a rational map with connected and rational general fibers \( f : Y \to T \) to a rational surface \( T \), is birational to a standard conic bundle \( p : X \to S \) over a rational surface \( S \) (see \([Z]\), \([S]\)).

Let \( X \) be a smooth projective threefold, and let \( S \) be a smooth surface. By definition, \( X \) is a standard conic bundle over \( S \), if there exists a surjective morphism \( p : X \to S \) such that: (i). The general fiber \( p^{-1}(s) \) is a smooth connected rational curve (a conic); (ii). For any irreducible curve \( C \subset S \), the preimage \( p^{-1}(C) \subset X \) is irreducible.

Let \( X \) be standard, let \( \Delta = \{ s \in S : p^{-1}(s) \text{ is singular} \} \) be the discriminant set of \( p \), and let \( \Delta \neq \emptyset \). Then (see e.g. \([B1]\)) \( \Delta \) is a curve with at most double points, and if \( s \in \Delta \) then \( p^{-1}(s) \) is either: (i). a union
of two smooth rational curves (lines) intersecting each other simply in a unique point, and $s$ is a smooth point of $\Delta$; or (ii) $p^{-1}(s)$ is a rational curve counted twice (a double line), and $s$ is a double point of $\Delta$.

As follows from the preceding, it makes sense to study the standard conic bundles over rational surfaces $S$ (esp. $S = \mathbb{P}^2$). Moreover, standard argument regarding general positions gives that the case $\text{Sing}(\Delta) \neq \emptyset$ can be regarded as a “degeneration of the general case” $\Delta$ — non-singular.

Taking in mind all this, we shall study in detail the standard conic bundles $p : X \to \mathbb{P}^2$ with smooth discriminant curves $\Delta$.

Let $p : X \to \mathbb{P}^2$ be such a conic bundle. Being a rational fibration over a rational surface, $X$ is a threefold with a non-effective canonical class, i.e. $h^{3,0}(X) = h^0(X, \Omega^1_X) = 0$, i.e. the complex torus (the Griffiths intermediate jacobian) $J(X)$ of $X$ does not contain a $(3, 0)$-part. In particular $J(X) = H^{2,1}(X)^* / (H_2(X, \mathbb{Z}) : \text{mod. torsion})$ is a principally polarized abelian variety (p.p.a.v.) with a principal polarization (p.p.) defined by the intersection of real 3-chains on $X$ (see [CG]). The divisor $\Theta$ of this polarization is called the theta divisor of $J(X)$.

Since $p : X \to \mathbb{P}^2$ is standard and $\Delta$ is smooth, the splitting $p^{-1}(s) = \mathbb{P}^1 \cup \mathbb{P}^1, s \in \Delta$ defines an unbranched double covering $\pi : \tilde{\Delta} \to \Delta$ of the smooth discriminant curve $\Delta$. Therefore the pair $(\tilde{\Delta}, \Delta)$ defines in a natural way the p.p.a.v. $P(\tilde{\Delta}, \Delta)$ — the Prym variety of $\pi : \tilde{\Delta} \to \Delta$ (see section 4); and the well-known result of Beauville [B1] tells that $(J(X), \Theta)$ and $P(\tilde{\Delta}, \Delta)$ are isomorphic as p.p.a.v.

The Beauville’s approach to conic bundles is based on the Wirtinger description of the p.p. Prym variety $P(\tilde{\Delta}, \Delta)$ by sheaves on the double discriminant curve $\tilde{\Delta}$ — this way, this approach is “nearer” to the associated curve data, and “far” from the geometry of the threefold $X$. In this paper we study conic bundles from an alternative point of view, based on the properties of some special families of curves on the threefold $X$ — the sections of the conic bundle map $p$. The advantages of this second approach are that it makes possible to describe the pair $(J(X), \Theta)$ in terms of families of curves on $X$ and their Abel-Jacobi images.

More concretely, let $X$ be a smooth threefold with $h^{3,0} = 0$, let $(J(X), \Theta)$ be the p.p. intermediate jacobian of $X$, and let $A_1(X)$ be the group of rational equivalence classes of algebraic 1-cycles $C$ on $X$ which are homologous to 0. Then the integrating over the real 3-chains $\gamma$ s.t. $\delta(\gamma) = (\text{the boundary of } \gamma) = C$, $C \in A_1(X)$ defines the natural map: $\Phi : A_1(X) \to J(X)$ — the Abel-Jacobi map for $X$ (see e.g. [CG]). In addition, if $C$ is a smooth family of homologous cycles $C$ on $X$, and $C_0$ is a fixed element of $C$, then the composition of $\Phi$ and the cycle-class map $F : A_1(X), C \to [C - C_0]$, defines a map $\Phi_C : C \to J(X)$.

Let $\text{Alb}(\mathcal{C})$ be the Albanese variety of $F$. By the universal property of the Albanese map $a : \mathcal{C} \to \text{Alb}(\mathcal{C})$, $\Phi_C$ can be factorized through $a$, and defines the map $\Phi_C : \text{Alb}(\mathcal{C}) \to J(X)$.

Both $\Phi_C$ and $\Phi_C'$ are called the Abel-Jacobi maps for the family of 1-cycles $\mathcal{C}$.

For a large class of such threefolds $X$ (esp. — for conic bundles), the transpose $^t\Phi$ of the Abel-Jacobi map for $X$ defines an isomorphism between the Chow group $A_1(X)$ and $J(X)$ (see [BM]), and one may expect that for some “rich” families of curves $\mathcal{C}$ on $X$ the Abel-Jacobi map $\Phi_C$ will be surjective. Moreover, one can set the following problem:

(*) The problem of parameterization of $\Theta$.

Find an appropriate family $\mathcal{C}_\theta$ of algebraic 1-cycles on $X$ such that the Abel-Jacobi map $\Phi_{\mathcal{C}_\theta}$ sends $\mathcal{C}_\theta$ surjectively onto a copy of the theta divisor $\Theta$.

Assume the existence of such a family $\mathcal{C}_\theta$. One can formulate the following additional questions:

(**). Describe, in terms of $\mathcal{C}_\theta$ and $X$, the structure of the general fiber of $\Phi_{\mathcal{C}_\theta}$;

(***). Describe, in terms of $\mathcal{C}_\theta$ and $X$, (the) components of the set $\text{Sing}(\Theta)$ of singular points of the theta divisor $\Theta$, and the tangent cones to $\Theta$ in these singularities.
Especially, the answer of (****) is closely related to the Torelli problem for $X$ (see [CG], [B1], [B2], [Vo], [De], [I1] etc.).

In this paper we give a positive answer of the problems (*) and (**) for the general conic bundle $p : X \to \mathbf{P}^2$. More concretely, we prove the existence of two naturally defined families of connected surfaces $C_+$ and $C_-$ on the conic bundle $X$, such that the Abel-Jacobi map sends one of these two families onto a copy of the theta divisor $\Theta$, and the second – onto the intermediate jacobian $J(X)$. Let $\deg(\Delta) = d > 3$. It turns out that these two families of curves are the two components of the canonical family $C_{\text{min}}$ of minimal sections of the conic bundle map $p : X \to \mathbf{P}^2$ – the elements of $C_{\text{min}}$ being the the minimal sections of the induced system of ruled surfaces \{${p^{-1}(C_0)}$: $C_0$ – a plane curve of degree $d - 3$\}.

The answer of the question: (Which one of $C_+$ and $C_-$ parameterizes $\Theta$) closely depends on the geometry of the given conic bundle. In particular: the “theta”-family in Example (6.5) (the bidegree (2,2) threefold) is $C_+$, while the “theta”-family in Example (6.6) (the nodal quartic double solid) is $C_-$.

Our particular approach to the minimal sections of the conic bundle $p : X \to \mathbf{P}^2$ uses the relation between: 1. the families of sections on ruled surfaces, and 2. the families of subbundles of rank 2 vector bundles describing these surfaces; and also – their deformation theory (see [LN], [Se]). The general observation is that it is possible to relate the effective divisors from the linear systems of the sheaves on $\tilde{\Delta}$ representing ($\Theta$) these surfaces; and also – their deformation theory (see [LN], [Se]). The general observation is that it is possible to relate the effective divisors from the linear systems of the sheaves on $\tilde{\Delta}$ representing ($J(X), \Theta$) as a Prym variety, and the families of minimal sections on some special non-minimal ruled surfaces $S \subset X$.

As regards the question (***) we give its answer in various examples: the cubic threefold, the Fano threefold $X_{16}$, the bidegree (2,2) divisor, the nodal quartic double solid (see (6.3),(6.4),(6.5),(6.6)). Although the structure of $Sing(\Theta)$ for these threefolds is well-known (see e.g. [GH], [Tju], [B2], [Ve], [I1], [Vo], [De]), the approach presented here suggests to study these special subvarieties of $\Theta$ by studying special subfamilies of degenerate 1-cycles, in the families of minimal sections $C_+$ and $C_-$. In addition, the presented approach can be easily generalized to arbitrary standard conic bundles $X \to S$ over rational surfaces $S$. Moreover, the general Prym variety can be represented (non-uniquely) by the intermediate jacobian of such a conic bundle (see [Isk, Lemma 1(iv)]). Therefore the minimal sections can be used for future studying of associated Prym varieties (see e.g. Example (6.5) – originated from the Verra’s counterexample to the Torelli theorem for Prym varieties [Ve] – where the Prym variety, of a double covering of a general plane sextic $\Delta$, is represented by the jacobian of a bidegree (2,2) threefold $X_{2,2}$; this way – involving additional information about $Sing(\Theta)$; see also [I1]).

0. Structure of the paper.

Let $S$ be a smooth rational surface. It is well-known (see e.g. [B1]) that the principally polarized intermediate jacobian $(J(X),\Theta)$ of a smooth conic bundle $p : X \to S$ is isomorphic to the Prym variety $P$ of the induced discriminant pair $(\Delta, \Delta)$.

We shall study in detail the conic bundles $p : X \to \mathbf{P}^2$.

We let $S = \mathbf{P}^2$, and $p : X \to \mathbf{P}^2$; $d = \deg(\Delta)$, $\Delta \subset \mathbf{P}^2$.

In this paper we prove the existence of two naturally defined families $C_-$ and $C_+$ of homologically equivalent algebraic 1-cycles (curves) on $X$ (see e.g. (4.1.2)) with the following properties ($d = \deg(\Delta) \geq 4$):

Let $\Phi_- : C_- \to J(X)$ be, as above, the Abel-Jacobi map for $C_\epsilon \epsilon \in \{-, +\}$. Then one of the following two alternatives is true (see (4.3), (4.4)):

$(\mathbb{A},+)\Phi_-$ is surjective, and $\Phi_+$ maps $C_+$ onto a copy of the theta divisor $\Theta \subset J(X)$, or

$(\mathbb{A},-)\Phi_+$ is surjective, and $\Phi_-$ maps $C_+$ onto a copy of the theta divisor $\Theta \subset J(X)$.
(B) The general element \( C \in C_- \) (resp. – of \( C_+ \)) is mapped isomorphically onto a smooth curve \( p(C) \) such that \( p(C) \cap \Delta \) is an effective divisor of the canonical system \( |\omega_\Delta| \) of the curve \( \Delta \). Moreover, if \( C \in C_+ \) (resp. – if \( C \in C_- \)) then \( C \) is an isolated (resp. – a non-isolated) minimal section of the non-minimal ruled surface \( p : p^{-1}(p(C)) \to p(C) \) (see the definitions in (2.2)).

In sections 1 – 5 we prove the existence, and describe the general properties, of the two canonical families of minimal sections \( C_- \) and \( C_+ \).

More concretely, let \( p : X \to \mathbb{P}^2 \) be a general conic bundle, let \( \Delta \) be the discriminant curve of \( p \), and let \( \text{deg}(\Delta) = d \). Let \( C \) be e.g. a reduced plane curve of degree \( k < d \), and let \( S_C = p^{-1}(C) \). Then \( p \) induces a conic bundle structure \( p : S_C \to C \). This way \( p \) defines, for any \( k < d \), the family \( S[k] \) of conic bundle surfaces over the base space \( |O(k)| \) of plane curves of degree \( k \).

Theorem (2.4) tells that if \( X \) is sufficiently general then the general element \( S_C \) of any of the families \( S[k], k < d \), can be regarded as a general element of a versal deformation of a conic bundle surface over a plane curve of degree \( k \). The minimal sections of such a surface \( S_C \) are of two types – isolated and non-isolated – and correspond to the minimal sections of the general element \( S_\sigma \) in any of the two types of a versal deformation of ruled surfaces over a plane curve of degree \( k \) (resp. – of genus \( g = (k - 1)(k - 2)/2 \)). These two types are separated by the parity of the invariant \( e(S_\sigma) \). The general element \( S_\sigma \) of the even type: \( e \equiv 0 \pmod{g} \) has a 1-dimensional family of minimal sections, while the general element of the odd type has a finite number of minimal sections. By collecting all these sections we find, for any \( k < d \), two naturally defined families – \( C_-(k) \) and \( C_+(k) \) – of isolated and non-isolated minimal sections of the elements of the family of surfaces \( S[k] \).

In section 3 we prove Theorem (2.4). The proof is based on: (a). the versality of the family \( S[1] \to \{ \text{the lines in } \mathbb{P}^2 \} \) (see section 1); (b). the versality of the families \( S_k \subset S[k] \) of conic bundle surfaces over degenerate curves \( C_0 = \text{a union of } k \text{ lines in } \mathbb{P}^2 \) (see (3.3)). This involves the versality of \( S[k] \).

Let \( C_- \) and \( C_+ \) be the two families of minimal sections for \( k = d - 3 \) – the canonical families of isolated and non-isolated minimal sections of \( p : X \to \mathbb{P}^2 \) (see (4.1.2), the definitions in (2.2), (5.2) – (5.7)). Theorem (4.4) tells that the Abel-Jacobi map sends one of these two families subjectively onto the intermediate jacobian \( J(X) \), while the second family is mapped onto a translate of the theta divisor \( \Theta \).

Theorem (5.8) describes the general fibers of the Abel-Jacobi maps \( \Phi_+ \) and \( \Phi_- \) for \( C_+ \) and \( C_- \), depending on which one of these two families parameterizes \( \Theta \).

In section 6 we study some of the most tipical examples of conic bundles with \( \text{deg}(\Delta) = 4,5 \) and 6.

In particular, in example (6.4) we find a new parameterization of the theta divisor \( \Theta \) for the cubic threefold \( X_3 \) – via the Abel-Jacobi image of the 6-dimensional family \( C_3^0 \) of rational cubic curves on \( X_3 \). Comparing with the known parameterization of \( \Theta \) by the Abel-Jacobi image of the 4-dimensional family \{differences of two lines on \( X_3 \)} \), the new parameterization has the advantage that the fiber of the Abel-Jacobi map for \( C_3^0 \) is connected, and the degree of the Gauss map for \( \Theta \) can be easily checked to be \( 72 = \) the number of \( \mathbb{P}^2 \)-systems of rational cubics on a general cubic surface (see (6.3.4)).

In examples (6.4), (6.5) and (6.6), we study the conic bundle structures on the Fano 3-fold \( X_{16} \subset \mathbb{P}^{10} \), on the bidegree \( (2,2) \) divisor \( X_{2,2} \) in \( \mathbb{P}^2 \times \mathbb{P}^2 \), and on the nodal quartic double solid. As an application we give another proof of the result of Tikhomirov: The family \( \mathcal{R} \) of Reye sextics (sextics of genus 3) on \( X_2^* \) parameterizes the theta divisor of the general quartic double solid \( X_2^* \). Moreover, we find a natural family of curves which parameterizes (via the Abel-Jacobi map) the intermediate jacobian of \( X_2^* \) (see (6.6.6) and (6.6.7)(ii)).

It follows from [T] that any connected component of the general fiber of the Abel-Jacobi map \( \Phi_\mathcal{R} : \mathcal{R} \to \Theta \) is isomorphic to \( \mathbb{P}^3 \). Now, the “minimal section” approach tells that the number of these fibers is two, i.e.
the Stein quotient of $\Phi_R$ is a finite morphism of degree 2 (see (6.6.7)(i)).

1. The two Fano families $F_+$ and $F_-$ on $X \to \mathbb{P}^2$.

(1.1) **The conic bundle surfaces $S_l$ and their relatively minimal models $S(L)$.

Let $p : X \to \mathbb{P}^2$ be a general smooth conic bundle, let $\Delta = \{ x \in \mathbb{P}^2 : \text{the fiber } p^{-1}(x) \text{ is singular} \} \subset \mathbb{P}^2$ be the discriminant curve of $p$, and let $\pi : \hat{\Delta} \to \Delta$ be the double covering induced by the splitting $p^{-1}(x) = \mathbb{P}^1 \cup \mathbb{P}^1$ of the fiber $p^{-1}(x)$, $x \in \Delta$. Let $d = \text{deg}(\Delta)$. Since $X$ is assumed to be general, we can suppose that $\Delta$ is a general plane curve of degree $d$ (see [Isk], Lemma 1 (iv)). In addition, we shall always assume that $d = \text{deg}(\Delta) \geq 3$, disregarding the trivial cases $d = 1, 2$.

The general line $l \subset \mathbb{P}^2$ intersects $\Delta$ in $d$ points $(x_1, ..., x_d)$. Let $S_l = p^{-1}(l)$. Then $S$ is a non-minimal ruled surface over $l \cong \mathbb{P}^1$. Indeed, $S$ inherits the conic bundle structure $p : S \to l$ from $p : X \to \mathbb{P}^2$. Let $p^{-1}(x_i) = \tilde{l}_i + \tilde{7}_i$, $i = 1, ..., d$ be the singular fibers of $S$. Since $l$ is general, $S_l$ is smooth, and $\tilde{l}_i$ and $\tilde{7}_i$ are (-1)-curves on $S_l$. The effective divisor $L = \tilde{l}_1 + ... + \tilde{l}_d \in S^d(\Delta)$ defines a morphism $\sigma(L) : S_l \to S(L)$, $\sigma(L) = \Pi_{i=1, ..., d} \sigma(\tilde{l}_i)$, where $\sigma(\tilde{l}_i)$ is the blow-down of $\tilde{l}_i$. The map $p$ defines on $S(L)$ a structure of a ruled surface $p(L) : S(L) \to l$.

**Definition.** Let $p : S \to C_0$ be a ruled surface over the curve $C$, and let $e = e(S)$ be the invariant of the ruled surface $S$, i.e. $e = \min \{ C^2 : C \text{ – a section of } p \}$.

Call the section $C$ of $S$ a minimal section of $S$ (resp. of $p$) if $C^2 = e(L)$.

By definition, the set of minimal sections of $S$ is non-empty – finite or not.

**Definition.** Call the minimal section $C$ an isolated minimal section if the set of minimal sections of $S$ is finite. Call the minimal section $C$ a isolated minimal section if the set of minimal sections of $S$ is infinite.

Let $\pi : \hat{\Delta} \to \Delta$ be double covering induced by $p$, and let $\pi_d = S^d(\pi) : S^d(\hat{\Delta}) \to S^d(\Delta)$ be the $d$th symmetric power of $\pi$. We identify $| \mathcal{O}_{\mathbb{P}^2}(1) |$ and $| \mathcal{O}_\Delta(1) |$ by the natural isomorphism $\cap : \mathbb{P}^2^* \to S^d(\Delta)$, $\cap(l) = l_\Delta$.

(1.2) **The components $F_+$ and $F_-$ in $S^d(\hat{\Delta})$, and their subsets $F^0_\epsilon$.**

Let $F = \pi_d^{-1}(\cap \mathbb{P}^2^*) \subset S^d(\hat{\Delta})$. It is well-known (see e.g. [B3]) that the 2-dimensional family $F$ splits into two connected components: $F = F_+ \cup F_-$. We shall find two natural families $F_+$ and $F_-$ of curves on $X$ which correspond to the components $F_+$ and $F_-$. Let $L \in F$, and let $l \in \mathbb{P}^2^*$ be the unique line such that $\cap(l) = \pi_d(L)$. Let $U \subset \mathbb{P}^2^*$ be the open subset: $U = \{ l : l \text{ intersects } \Delta \text{ in a set of } d \text{ disjoint points, and } S_l = p^{-1}(l) \text{ is smooth} \}$. Let $F^0 \subset F$ be the open subset $F^0 = \pi_d^{-1}(U)$. The map $e : L \to e(L) = e(S(L))$, (see (1.1)), is well defined on $F^0$. We introduce the subsets $F^\epsilon_0 = \{ L \in F^0 : e(L) = \epsilon \} \subset F^0$.

We shall prove the following

(1.3) **Lemma.** There exists an open subset $V \subset U$ such that $\pi_d^{-1}(V) \subset F^\epsilon_0 \cup F^0$.

**Proof.** Let $e^+ = \max \{ e(L) : L \in F^0 \}$. Clearly, $e^+ \leq 0$, and we have to see that $e^+ = 0$.

Assume that $e^+ \leq -1$, and let $L \in F^0$. Then the rational ruled surface $S(L) \cong \mathbb{P}^{-\epsilon}$, and $S(L)$ has only one minimal section – a section $C$ such that $(C^2)_{S(L)} = e$. Let $L = l_1 + ... + l_d$, and let $L_i = L + \tilde{l}_i - \tilde{l}_i$, $i \in \{1, ..., d\}$. Since $e(L) = e^+$ is maximal, $e_\epsilon = e(L_\epsilon) = e^+$. The ruled surface $S(L_\epsilon)$ is obtained from $S(L)$ by an elementary transformation $elm_i$ – centered in the point $\sigma(\tilde{l}_i) \in S(L)$. In particular, $| e_\epsilon - e_\epsilon | = 1$. 

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Since \( e_i \leq e_+ \), \( e_i = e^+-1 \). Therefore the unique minimal section \( C = C(L) \subset S(L) \) passes through the point \( \sigma(\tilde{t}_i) \) (see e.g. [LN, Lemma (4.3)]). Denote by \( C, C \subset X \) also the proper preimage of \( C \subset S(L) \) in \( S_l \subset X \). It follows that \( C \subset X \) intersects the component \( \tilde{t}_i \), and we have to see that \( C \) does not intersect the component \( \tilde{t}_i \). In fact, if \( C \) intersects also the component \( \tilde{t}_i \), then the line \( l \) must be tangent to \( \Delta \) in the point \( x_i = p(l_i) = p(\tilde{t}_i) \), which is impossible since \( l \in U \). If we repeat the last argument for any \( i \in \{1,...,d\} \), we obtain that (the proper preimage \( C \subset X \) of) the unique minimal section \( C \subset S(L) \) intersects any of the components \( \tilde{t}_i, i = 1,...,d \), and \( C \) does not intersect \( l_i \) for \( i = 1,...,d \). Clearly, \( C \subset X \) is also the proper preimage of the unique minimal section \( C_i \) of \( S(L_i) \), \( i = 1,...,d \). Let \( \tilde{T} = \tilde{t}_1 + ... + \tilde{t}_d \). It follows from the preceding that the map \( \sigma : S_l \to S(\tilde{T}) \) sends \( C \) isomorphically onto the unique minimal section of \( S(\tilde{T}) \), and \( e(\tilde{T}) = e^+ - d \).

Note that the same arguments as above imply that the subset \( F_{e^+}^0 \) must contain an open subset of \( F^0 \). In fact, the image of the map \( e : F^0 \to \mathbb{Z}_{\leq 0} \) is discrete, and \( e(L) \) must be constant on an open subset of any of the components of \( F^0 \). Let \( e_0 \) be the minimal among these values of \( e \) for which \( e(L) = e_0 \) on an open set. Just as above, if \( L = l_{01} + ... + l_{0d} \in F_{e_0}^0 \) is general, then the minimal section \( C_0 \) of \( S(L_0) \) does not intersect any of the components \( \tilde{t}_{0i}, i = 1,...,d \). In particular, \( e(L_0 + l_{0i} - l_{0i}) = e_0 + 1 \), for any \( i = 1,...,d \). Therefore \( e = e_0 + 1 \) over an open subset of \( F^0 \). By repeating the same argument we obtain by induction that \( e = e^+ \) over an open subset \( V \) of \( F^0 \).

Now, the assumption \( e_0 < 0 \) implies that \( e(L) = e^+ \) for exactly one \( L \in \pi_d^{-1}(l), \ L \in V \). Indeed, if \( L = l_1 + ... + l_d \in V \) and \( L \in F_{e^+}^0 \), then the combinatorial arguments from above imply: \( e(L') = e^+ - d + \#\{L \cap L'\} < e^+ \), for any \( L' \in \pi_d^{-1}(\pi_d(L)) - \{L\} \). Therefore the map \( \pi_d : F \to \mathbb{P}^{2*} \) admits an inverse, over the open subset \( V \subset F^0 \) (by construction, \( V \) is a subset of \( F_{e^+}^0 \)).

It follows that the closure of \( V \), in \( F_{e^+}^0 \), contains a 2-dimensional rational family. Therefore the map \( L \to l \), where \( l \) is the unique line such that \( \pi_d(L) = \Delta \cdot l \), defines a linear system of degree \( d \) and of dimension 2 on \( \Delta \). The uniqueness of the inverse \( L \in V \) of \( l \) implies that \( \Delta \) must be projected onto a plane curve of degree \( d \), and the degree of the projection is 1. In particular, \( g(\Delta) \leq g(\Delta) \) – contradiction. Therefore, \( e^+ \) cannot be negative, i.e. \( e^+ = 0 \).

Let \( F^0_+ = \{ L \in F : e(L) = 0 \} \). Clearly, \( F^0 \) contains an open subset of \( F \); and we can assume, without any confusion, that \( F^0 \) is open. Since \( S(L) \cong F_0 \) for \( L \in F^0_+ \), the ruled surface \( S(L) \to l \) has an infinite set of minimal sections. In fact, \( S(L) \) is a quadric. Let \( L_i \) be as above. Then \( e(L_i) = -1 \). Therefore, there exists an open subset \( F^0_+ \subset F \) such that \( e(L) = -1 \) for any \( L \in F^0_+ \). q.e.d.

**Corollary.** Among the sets \( F^0_+, F^0_0 \) and \( F^0_{-1} \) are the only sets in \( F^0 \) which contain open subsets of \( F = \pi_d^{-1}(\cap \mathbb{P}^{2*}) \).

In fact, \( F^0_+ \cap F^0_0 \) dominate over an open subset of \( \mathbb{P}^{2*} \). Therefore, the set \( U = \{ l \in \mathbb{P}^{2*} : e(L) \in \{0,1\} \text{ for any } L \in \pi^{-1}(l(\Delta)) \} \) contains an open subset of the plane. q.e.d.

**The Fano families of minimal sections \( \mathcal{F}_+ \) and \( \mathcal{F}_- \).**

Let \( F_+ \) and \( F_- \) be the closures of \( F^0_+ \) and \( F^0_- \) in \( F \). It follows from (1.2) and (1.3) that \( F = F_+ \cup F_- \), i.e., outside a closed subset of codimension \( \geq 1 \), the invariant \( e(L) \) is either 0 or \( -1 \). If \( L \) is general and \( e(L) = -1 \) then \( S(L) \cong F_1 \), and \( S(L) \) has exactly one minimal section. Moreover (see the proof of (1.2)), the proper preimage \( C \subset X \) of this minimal section does not intersect any of the components \( \tilde{t}_i \) of the elements \( l_i \) of \( L \), \( i = 1,...,d \). Therefore, the proper preimage \( C \) coincides with the preimage of the minimal section (the preimage does not contain components of degenerate fibers of \( p : X \to \mathbb{P}^2 \)). Therefore we can define...
\( \mathcal{F}_+ = (\text{the closure of}) \{ C \subset X : C = C(L) \text{ is (the isomorphic preimage of) the unique minimal section of } S(L) \text{ for some } L \in F^0_+ \}. \)

Similarly, we define \( \mathcal{F}_+ = (\text{the closure of}) \{ C \subset X : C \text{ is a minimal section of } S(L) \text{ for some } L \in F^0_+ \}. \)

Clearly, the family \( \mathcal{F}_- \) is 2-dimensional, and \( p \) defines the the finite map \( p_- : \mathcal{F}_- \to \mathbb{P}^2 \) of degree \( \deg(p_-) = 2^{d-1} \). The family \( \mathcal{F}_+ \) is 3-dimensional, and \( p \) defines a map \( p_+ : \mathcal{F}_+ \to \mathbb{P}^2 \). The general fiber \( p_+^{-1}(l) \) consists of a finite number \( (=2^{d-1}) \) of smooth rational 1-dimensional families of curves on \( X \).

Keeping in mind all this we postulate:

**Definition.** Call: 1. the 3-dimensional family \( \mathcal{F}_+ - \text{the Fano family of non-isolated minimal sections of } p \); 2. the 2-dimensional family \( \mathcal{F}_- - \text{the Fano family of isolated minimal sections of } p \).

**2. The versality of the family of conic bundle surfaces \( S[k] \).**

**2.1 Remarks.**

Let \( F_+ \) and \( F_- \) be the components of \( F \), and let \( F^\text{reg}_+ = F_+ - \text{Sing } F_+ \) and \( F^\text{reg}_- = F_- - \text{Sing } F_- \). Consider e.g. the component \( F_- \). Let \( C \subset F_- \) be a smooth algebraic curve, and assume that \( C \) is otherwise general; in particular, we can assume that the general point \( z \in C \) belongs to \( F^\text{reg}_- \). The general point \( z \in C \) represents an effective divisor \( L = L(z) \in S_0(\Delta) \) such that \( \pi_d(L) = l \) is a line in \( \mathbb{P}^2 \). Let \( p_L : S(L) \to l \cong \mathbb{P}^1 \) be the ruled surface defined by \( L \). This way, the curve \( Z \) defines the algebraic family of ruled surfaces \( S \to Z \), such that the general fiber \( S(z) = S(L(z)) \) is isomorphic to \( F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \).

The same arguments can be repeated for \( C \subset F_+ \). It follows from section 1 that the general fiber \( S(z) \) of the corresponding families \( S \to Z \) is a ruled surface of type \( F_0 \) – a smooth quadric with a fixed ruling. Remember that the general member of a versal deformation of a rational ruled surface must be either of type \( F_1 \) or of type \( F_0 \). Moreover, two rational ruled surfaces \( S_1 \) and \( S_2 \) can be deformed into each other iff their invariants \( e(S_1) \) and \( e(S_2) \) have the same parity (see [Se, Th.5, Th.13]). Now, the results from section 1 imply that the general members of the families \( S \to Z \) are members of a versal deformation of a rational ruled surface.

**2.2 Versal deformations of 2-dimensional conic bundles.**

Let \( \rho : S \to Y \) be an algebraic family of surfaces such that the general element \( S(y) = \rho^{-1}(y) \) has a structure of a smooth conic bundle \( p(y) : S(y) \to C(y) \) over a (smooth) curve \( C(y) \) of genus \( g \), with \( d \) degenerated fibers.

Clearly, the general surface \( S(y) \) admits exactly \( 2^d \) morphisms \( \sigma : S(y) \to S_\sigma \), where \( S \) is a ruled surface \( p_\sigma : S_\sigma \to C(y) \), such that \( p(y) = \sigma \circ p_\sigma \). The map \( \sigma \to e(S_\sigma) \) (= the invariant of the ruled surface \( S_\sigma \)) defines a map

\[ e : Y \to \{ \text{the subsets of } \mathbb{Z} \}. \]

**Definition.** Isolated and non-isolated minimal sections of a conic bundle surface.

Let \( p : S \to C_0 \) be a smooth conic bundle surface over the curve \( C_0 \). Let \( C \subset S \) be a non-singular section of \( p \) (i.e. \( p \) maps \( C \) isomorphically onto the base \( C_0 \), and \( C \) does not intersect the finite set of singular points of the degenerate fibers of \( p \)), and let \( p(C) : S_C \to C_0 \) be the ruled surface defined by \( C \subset S \) (i.e. \( S \to S(C) \) is the contraction of the components of the singular fibers of \( p \) non-intersecting \( C \), and \( p(C) \) is the induced \( \mathbb{P}^1 \)-bundle map).

Call \( C - \text{a minimal section of the conic bundle surface } S \) if \( C \) is a minimal section of the ruled surface \( S(C) \). Call the minimal section \( C \) of \( S \) isolated (resp. – non-isolated) if \( C \) is an isolated (resp. – non-isolated) minimal section of \( S(C) \).
As it follows from [Se, Th. 13] (see also (5.2) – (5.7)) the general member of a versal deformation of a ruled surface, over a curve of genus \(g\), is a ruled surface \(S\) with invariant \(e \in \{g - 1, g\}\), and

(i) if \(g = 1\) and \(e = 0\) then \(S\) is decomposable;

(ii) if \(g = 1\) and \(e = 1\), or if \(g \geq 2\), then \(S\) is indecomposable.

In particular (see (5.2) – (5.7)): If \(C\) is a minimal section of the general versal element \(S\), then \(C\) is isolated or not – depending on \(e = e(S) = g - 1\) or \(e = g\). This suggests the following

**Definition.** Call the algebraic family of conic bundles \(\rho : S \to Y\) versally embedded, if the invariant \(e\) of the general \(S_\sigma\) belongs to \(\{g - 1, g\}\) and \(S_\sigma\) fulfills (i), (ii).

Equivalently, the family \(S\) is versally embedded if there exists an open subset \(U \subset Y\) such that \(e(U) \subset \{g - 1, g\}\), and the ruled surfaces \(S_\sigma = \sigma(S(u)), u \in U\), fulfill (i),(ii).

In particular (see (5.2) – (5.7)): If \(C\) is a (non-singular) minimal section of the general versal element \(S\), then \(C\) is isolated or not – depending on \(e = e(S(C)) = g - 1\) or \(e = g\).

**2.3 Corollary.** Let \(X\) be a smooth threefold, which admits a regular map \(p : X \to P^2\) which is a conic bundle structure on \(X\). Then the family of conic bundle surfaces \(S \to P^2\), with fibers \(S(l) = p^{-1}(l)\), is versally embedded.

Let \(|O_{P^2}(k)|\) be the family of plane curves of degree \(k\), \(k < d = deg(\Delta)\). The intersection defines an isomorphism \(\cap : |O_{P^2}(k)| \to |O_{\Delta}(k)|\).

Let \(C[k] = \pi_{kd}^{-1}(|O_{P^2}(k)|) = \pi_{kd}^{-1} \cap^{-1}(|O_{P^2}(k)|).\) It follows from the definition that \(C[k]\) is a \((k + 1)(k + 2)/2 - 1\)-dimensional subset of \(S^k(\Delta)\), since the natural map \(\pi_d : C[k] \to |O_{P^2}(k)| \cong P^{(k+1)(k+2)/2 - 1}\) is finite of degree \(2^kd\).

Let \(C \subset P^2\) be a general plane curve of degree \(k\), and let \(S_C = p^{-1}(C)\). The conic bundle structure \(p : X \to P^2\) defines a conic bundle structure \(p(C) : S(C) \to C\), on the surface \(S(C)\).

This way we obtain the family of conic bundle surfaces \(S[k] \to |O_{P^2}(k)|\), parameterized by the space of plane curves of degree \(k\).

**2.4 Theorem.** Let \(p : X \to P^2\) be a smooth conic bundle. Let \(\pi : \Delta \to \Delta\) be the corresponding double covering of the discriminant curve (\(\Delta\) is assumed to be smooth, and \(\pi\) – unbranched and non-trivial). Let \(\mathcal{F}_-\) be the 2-dimensional family (the Fano surface of \(X\)) defined in section 1, and assume that the following is true:

(*) The curves of the Fano surface \(\mathcal{F}_-\) sweep \(X\) out.

Then the family \(S[k] \to |O_{P^2}(k)|\) is versally embedded, for any positive integer \(k < d = deg(\Delta)\).

3. Proof of Theorem (2.4).

**3.1 The level families \(\mathcal{F}_-(m)\) and \(\mathcal{F}_+(m)\).**

Let \(S_\Delta = p^{-1}(\Delta)\). Obviously, the map \(p : X \to P^2\) defines an isomorphism \(p_\Delta : Sing(S_\Delta) \to \Delta \subset P^2\).

The inverse \(s\) of \(p_\Delta\) is the Steiner map: \(s : \Delta \to Sing(S_\Delta), s : x \mapsto sing p^{-1}(x)\).

Let \(\mathcal{F}_+\) and \(\mathcal{F}_-\) be the Fano families defined in section 1. The general element \(C \in \mathcal{F}_-\) does not intersect \(s(\Delta)\) and defines by intersection with \(S_\Delta = p^{-1}(\Delta)\) the effective divisor without multiple points \(L = L(C) \subset \mathcal{F}_-\). Let \(l \subset P^2\) be the line defined by \(L\). The morphism \(\sigma(L) : S_l = p^{-1}(L) \to S(L)\) defined by \(L\) (see section 1) maps \(C\) isomorphically onto the (-1) section \(C(L)\) of \(S(L) \cong \mathbb{F}_1\). Since \(C\) does not intersect \(s(\Delta), (C^2)_{S_l} = (C(L)^2)_{S(L)} = -1\), i.e. \(C\) is a (-1)-section also on the conic bundle surface \(S_l\).
Similar arguments imply that the general \( C \in \mathcal{F}_+ \) is a 0-section of the conic bundle surface \( S_l \) defined by \( L \).

Let \( S_l = p^{-1}(l) \) be as above, and let \( C \subset S_l \) be a curve such that \( p : C \to l \) is an isomorphism. We call such a curve a \textit{section of the conic bundle surface} \( S_l \). Call \( C \) nonsingular if \( C \cap s(\Delta) = \emptyset \). Any nonsingular section \( C \) of \( S_l \) defines – via an intersection with \( S_\Delta \) – a nonsingular element \( L \) of \( \mathcal{F} \), i.e. \( L \) has no multiple points. The curve \( C \subset X \) can be regarded also as a section of the ruled surface \( S(L) \). Let \( \epsilon(L) = \epsilon(S(L)) \) be the invariant of \( S(L) \). Since \( C \) is a nonsingular section of \( S_l \), the intersection number \( (C^2)_{S_l} \) is well defined, and \( (C^2)_{S_l} \equiv 0 \pmod{\epsilon(L)} \). Since \( \epsilon(L) \in \{0, -1\} \), the parity of \( C^2 \) defines uniquely \( \epsilon(L) \). Let \( C_0 \) be a minimal section of \( S(L) \). The general choice of \( l \) implies that \( C_0 \) can be considered as a nonsingular section of \( S_l \).

Denote by \( f \) the general fiber of \( S(L) \). The fiber \( f \) can be considered as a fiber (a conic) \( q = p^{-1}(x) \) in \( l \). In particular, \( C \) can be considered both: as an element of the linear system \( | C(L) + m.f | \) on \( S(L) \), and an element of the linear system \( | C + m.q | \) on \( S_l \); here \( m = m(L) \) is defined by \( 2m = (C^2)_{S_l} - \epsilon(L) \). This suggests to define the families

\[ \mathcal{F}_-(m) = \text{the closure of } \{ C - \text{a curve on } X : C \text{ is a nonsingular section of some (nonsingular) } S_l, \text{ and } (C^2)_{S_l} = 2m - 1 \}. \]

We call the elements \( C \) of \( \mathcal{F}_-(m) \) \textit{odd sections} (of level \( m \)).

The same definition, up to parity, is used to define the families \( \mathcal{F}_+(m) \) of \textit{even} Fano sections of level \( m \) (the number \( C^2 \), for a general even section \( C \) of level \( m \), is \( 2m \)).

Let \( l \) be a line in \( P^2 \). It follows from the preceding that:

\textit{Any section} \( C \) of \( S_l \) (i.e., a curve on \( S_l \) which is mapped isomorphically to \( l \) by \( p \)) \textit{is an element of some of the level families} \( \mathcal{F}_-(m) \) or \( \mathcal{F}_+(m) \).

\textbf{(3.2) Corollary.} \textit{The conic bundle projection} \( p : X \to P^2 \) \textit{defines, for any nonnegative integer} \( m \), \textit{the surjective morphisms} \( p_-(m) : \mathcal{F}_-(m) \to P^{2^*} \) and \( p_+(m) : \mathcal{F}_+(m) \to P^{2^*} \).

Moreover, if \( d = \text{deg}(\Delta) \) and \( l \) is a general line in \( P^2 \) then the fiber \( (p_-(m))^{-1}(l) \) has \( 2d - 1 \) irreducible components, and any such a component is isomorphic to the projective space \( P^{2m} \). Similarly, \( (p_+(m))^{-1}(l) \) has \( 2d - 1 \) irreducible components isomorphic to \( P^{2m+1} \).

\textbf{Proof.} Consider the fiber of \( p_-(m) \). Let \( C \in (p_-(m))^{-1}(l) \) be general (esp. nonsingular), and let \( L = L(C) \in F \) be the divisor defined by \( C \). The minimal section \( C(L) \) is nonsingular. Therefore \( C(L) \subset S(L) \) can be regarded also as a curve on \( X \); in particular

\( C \in | C(L) + m.q | \equiv | C(L) + m.f \mid \equiv P^{2m} \).

Clearly, the nonsingularity of \( C \) implies the uniqueness of the minimal section \( C(L) \) in the representation \( C \in | C(L) + m.q | \). Therefore, the components of \( (p_-(m))^{-1}(l) \) are in (1:1)-correspondence with the \( 2d - 1 \) sections of \( \mathcal{F}_- \) “above” \( l \), i.e. – with the \( 2d - 1 \) elements of \( (p_-(0))^{-1}(l) \).

Similarly – for \( (p_+(m))^{-1}(l) \); note that \( (p_+(0))^{-1}(L) \) consists of \( 2d - 1 \) components, and any of these components is isomorphic to a ruling of a quadric defined by a nonsingular minimal section \( C(L)(t) \in \mathcal{F}_+ \) over \( l \).

\textbf{(3.3) Minimal sections of degenerate conic bundle surfaces in} \( X \).

We shall use the level families in the study of the minimal sections of the singular conic bundle surfaces \( S_C = p^{-1}(C) \), where \( C \) is a reduced plane curve of degree \( k \) all the component of which are lines.

Let \( S_k \subset | \mathcal{O}_{P^2}(m) | \) be the \( 2k \)-dimensional locus of plane curves of degree \( k \), all of the components of which are (possibly multiple) lines. We shall see, after an appropriate completion of the definitions, that the
induced family of degenerate conic bundle surfaces \( S \to S_k \) is versally embedded, provided the condition (*) of (2.4) takes place.

Let \( x \) be a point of \( \mathbb{P}^2 \), and let \( q(x) \subset X \) be the conic \( q(x) = p^{-1}(x) \). Let \( \sigma(x) = \sigma_{10}(x) = \{ l \in \mathbb{P}^{d*} : x \in l \} \) be the "Schubert line" of lines through \( x \). Let \( F_{\sigma}(x) = F_{\sigma}(0)(x) \) be the set of all the odd Fano curves \( C \) of level 0 such that the line \( l = p(C) \) passes through \( x \). Clearly, \( F_{\sigma}(x) = \bigcup \{ F_{\sigma}(l) : l \in \sigma(x) \} \), where \( F_{\sigma}(l) = \{ C \in F_{\sigma} : p(C) = l \} \).

If \( l \) is general, then the set \( F_{\sigma}(l) \) consists of \( 2^{d-1} \) elements: \( C_1(l), ..., C_{2^{d-1}}(l) \). Any of the curves \( C_i(l) \) intersects the conic \( q(x) \) in a point \( \xi_i(l), i = 1, ..., 2^{d-1} \).

It follows from (2.4)(*) that any point \( \xi = q(x) \) can be represented in this way, i.e., there exists \( l \in \sigma(p(\xi)) \) such that \( \xi = C \cap q(p(\xi)) \) for some \( C \in F_{\sigma}(p(\xi)) \). It is easy to see that (2.4)(*) also implies that the set \( \{ \xi \in F_{\sigma} : \xi \in C \} \) is finite for the general \( \xi \in X \).

Let \( C_0 = n_1 + ... + n_k \) be a general element of \( S_k \), and let \( S_{C_0} \subset X \) be the singular conic bundle surface \( S_{C_0} = p^{-1}(C_0) \). The surface \( S_{C_0} \) has \( k \) irreducible components – the rational conic bundle surfaces \( S_i = p^{-1}(n_i), i = 1, ..., k \). Since \( C_0 \) is general, all the \( S_i \) can be assumed to be smooth.

**Definition.** Call a section of \( S_{C_0} \) any connected curve \( C = C_1 + ... + C_k \subset X \) such that \( C_i \) is a section of \( S_i \). Respectively, we call the section \( C \) nonsingular if \( C_i \) is nonsingular, \( i = 1, ..., k \). Clearly, the nonsingular section \( C \) can be singular as a curve.

Since \( C_0 \) is general, we can assume that any of the ruled surfaces \( S_i(\sigma) \) is either \( F_0 \) or \( F_1 \). We shall find the minimal sections of \( S_{C_0} \).

Fix an order, say \([1, 2, ..., k]\). Let \( C_1 \in F_{\sigma} \) be any of the \( 2^{d-1} \) odd Fano sections of level 0 such that \( p(C) = n_1 \). Let \( x_{12} = n_1 \cap n_2 \), and let \( q(x_{12}) = p^{-1}(x_{12}) \). The general choice of \( C_0 \) implies that \( q(x_{12}) \) is smooth, and that (because of (2.4)(*)) \( \xi_{12} = C_1 \cap q(x_{12}) \) is a general point of \( q(x_{12}) \). It follows from (3.2) that the set \( \{ C \in F_{\sigma} : p(C) = n_2, \xi_{12} \in C \} \) is finite (it has \( 2^{d-1} = \# \{ F^o : F^o \text{ - a component of } F_{\sigma}(n_2) \} \) elements, any of which can be assumed to be a nonsingular section of \( S_2 \)). Let \( C_2 \) be any of these curves. Obviously, \( C_1 + C_2 \) is a minimal section of \( S_1 + S_2 \), and the invariant of the corresponding reducible ruled surface \( S(C_1 + C_2) = S(C_1) + S(C_2) \) is \( (C_1)^2 + (C_2)^2 = -1 = p_{\sigma}(n_1 + n_2) - 1 \); i.e. the singular (reducible) ruled surface \( S(C_1 + C_2) \) is of type \( F_1 \). We shall prove by induction the following:

**Lemma.** If \( C_0 = n_1 + ... + n_k \) is general, then the surface \( S_{C_0} \) has only a finite number of sections \( C_1 + ... + C_k \) such that

(i). if \( 1 \leq i \leq k \) and \( i \in 2\mathbb{Z} \) then \( C_i \in F_{\sigma}(i/2) \);

(ii). if \( 1 \leq i \leq k \) and \( i \in 2\mathbb{Z} + 1 \) then \( C_i \in F_{\sigma}((i-1)/2) \).

**Proof.** Assume that (3.3.1) is true for \( i = 1, 2, ..., k - 1 \). Let \( x_{ik} = n_k \cap n_i \), let \( q(x_{ik}) = p^{-1}(x_{ik}) \), and let \( \xi_{ik} = C_i \cap q(x_{ik}), i = 1, 2, ..., k - 1 \). Since \( C_0 \) is general, we can assume that \( n_k \) is a general line in \( \mathbb{P}^2 \), and \( (x_{1k}, ..., x_{k-1,k}) \) is a general \((k-1)\)-tuple of points on \( n_k \). Moreover, since \( n_i \in \sigma(x_{ik}) \) are general, the points \( \xi_{ik} \in q(x_{ik}) \) can be assumed to be general (see above).

Let, for definiteness, \( k \) be even. It follows from (3.2) that there is only a finite number of sections \( C \in F_{\sigma}(k/2) \) such that \( p(C) = n_k \), which pass through the points \( \xi_{1k}, ..., \xi_{k-1,k} \). Indeed, the general \((k-1)\)-tuple \( (\xi_{1k}, ..., \xi_{k-1,k}) \) imposes \( k - 1 \) independent conditions on the elements of any of the \( \mathbb{Z} \) finite number of linear systems \( | C_0 + (k/2).q | \) on \( \mathbb{S}_k = p^{-1}(n_k) \), where \( C_0 \) is a \((-1)\)-section of \( S_k \).

By induction, \( e(C_1 + ... + C_k) = e(C_1 + ... + C_{k-1}) + (C_k)^2 = (p_{\sigma}(n_1 + ... + n_{k-1}) - 1) + (-1 + 2k/2) = p_{\sigma}(n_1 + ... + n_k) - 1 \).

If \( k \) is odd, the proof is similar. Lemma (3.3.1) is proved.

(*) **Definition of level and weight.**
Let \( C = C_1 + \ldots + C_k \) be a section of \( S_1 + \ldots + S_k \), and let \( m_i \) be the level of \( C_i \), \( i = 1, \ldots, k \). Call the vector \( \text{level}(C) = (m_1, \ldots, m_k) \) the \textit{level} of \( C \). Call the \textit{weight} of \( C \) the number \( \text{weight}(l) = m_1 + \ldots + m_k \).

In particular, if \( C \) is a section of type described in (3.3.1) and \( k = 2k_0 \) is even, then \( \text{weight}(C) = (k_0 - 1).k_0 \); if \( k = 2k_0 + 1 \) is odd then \( \text{weight}(C) = k_0^2 \).

Call a section \( C \) of \( S_{C_0} \) \textit{isolated} if the linear system \( |C| \) on \( S_{C_0} \) is trivial.

\textbf{(3.3.2) Corollary.} If \( C_0 = n_1 + \ldots + n_k \in S_k \) is general then the surface \( S_{C_0} \) has only a finite number of isolated sections. These are the sections \( C \) of \( S_{C_0} \) for which

\[ \text{weight}(C) = (k_0 - 1).k_0 \text{ if } k = 2k_0 \text{ is even}; \]

\[ \text{weight}(C) = k_0^2 \text{ if } k = 2k_0 + 1 \text{ is odd}. \]

The proof of (3.3.2) is purely combinatorial, and similar to the proof of (3.3.1), where the isolated sections of \( S_{C_0} \) for which \( m_{2i} = m_{2i+1} = i, 1 \leq i \leq k \), are described.

Let \( C = C_1 + \ldots + C_k \) be a nonsingular section of \( S_{C_0} \). Call \( C \) \textit{minimal} if \( C \) is a section of a minimal self-intersection \( C^2 = C_1^2 + \ldots + C_k^2 \) on \( S_{C_0} \).

It is not hard to see also if \( C \) is an isolated section of such a \( S_{C_0} \) then \( C \) is minimal and \( e(C) = \mu(n_1 + \ldots + n_k) - 1 = (k - 1).4/2 - 1 = e(S(C_1 + \ldots + C_k)) \).

By using just the same arguments as in the proof of (3.3.1) (based on the assumption (2.4)(*)), we obtain the following

\textbf{(3.3.3) Corollary.} Let \( C_0 \in S_k \) be general, and let \( C \) be a non-isolated minimal section of \( S_{C_0} \). Then

\[ \text{weight}(C) = (k_0 - 1).k_0 \text{ if } k = 2k_0 \text{ is even}; \]

\[ \text{weight}(C) = k_0^2 + 1 \text{ if } k = 2k_0 + 1 \text{ is odd}. \]

Moreover, the set of non-isolated (continual) minimal sections of (the general) \( S_{C_0} \) form a 1-dimensional algebraic family, the components of which are finite number of rational curves.

\textbf{(3.4). End of the proof of Theorem(2.4).}

Assume that the family \( S[k] \to |\mathcal{O}_{P^2}(k)| \) is not versally embedded. By construction, \( S_k \subset |\mathcal{O}_{P^2}(k)| \).

Let \( S_k \to S_k \) be the restriction of \( S_k \) on the preimage of \( S_k \).

Let \( g = g(k) = (k - 1).4/2 \) be the genus of the general plane curve of degree \( k \). Then (3.3.1) - (3.3.3) imply: \( e(S) = \{ g - 1, g \} \), for the general \( S \in \mathcal{S}_k \). Fix the general k-tuple of lines \( C_0 = n_1 + \ldots + n_k \) such that \( e(S(C_0)) = \{ g(k) - 1, g(k) \} \). Let \( U \subset |\mathcal{O}_{P^2}(k)| \) be a sufficiently small analytic neighborhood of \( C_0 \). The general curve \( C \in U \) is a smooth plane curve of degree \( k \), and the invariant function \( e : U \to \{ \text{the subsets of } \mathbb{Z} \} \) takes discrete values, on the elements \( C \) of the continuous set \( U \). Now, standard arguments including the general position of \( C \in U \) imply that \( e(C) = e(S(C)) = e(C_0) = \{ g - 1, g \} \) for the general element \( C \in U \). Theorem (2.4) is proved.

\textbf{4. The intermediate jacobian \( (J(X), \Theta) \) as a Prym variety \( P(\Delta, \Delta) \), and the family \( S[d - 3] \).}

\textbf{(4.1). The Wirtinger description of \( (J(X), \Theta) \) as a Prym variety, and the canonical families \( C_+ \) and \( C_- \) of minimal sections.}

\textbf{(4.1.1) Let \( (\Delta, \Delta) \) be the discriminant pair of \( p : X \to P^2 \), and let \( \pi : \tilde{\Delta} \to \Delta \) be the induced double covering. Without any essential restriction we assume that the discriminant curves are smooth, and \( \pi \) is unbranched. It is well-known that the principally polarized intermediate jacobian \( (J(X), \Theta) \) can be identified with the Prym variety \( P(\Delta, \Delta) \) defined by the double covering \( \pi : \tilde{\Delta} \to \Delta \) (see e.g.,[B1]). Here we remember the Wirtinger description of \( P(\Delta, \Delta) \) by sheaves on \( \tilde{\Delta} \) (see e.g., [W2]).}
Let \( d = deg(\Delta) \), and let \( g = (d-1)(d-2)/2 = g(\Delta) \) be the genus of \( \Delta \). The map \( \pi \) induces the natural map \( Nm : \text{Pic}(\Delta) \to \text{Pic}(\Delta) \).

Let \( \omega_\Delta \) be the canonical sheaf of \( \Delta \). Then the fiber \( Nm^{-1}(\omega_\Delta) \) splits into two components:

\[
P^+ = \{ \mathcal{L} \in \text{Pic}^{2g-2}(\Delta) : Nm(\mathcal{L}) = \omega_\Delta, h^0(\mathcal{L}) \text{ even} \},
\]
\[
P^- = \{ \mathcal{L} \in \text{Pic}^{2g-2}(\Delta) : Nm(\mathcal{L}) = \omega_\Delta, h^0(\mathcal{L}) \text{ odd} \}.
\]

Both \( P^+ \) and \( P^- \) are translates of the Prym variety \( P = P(\Delta, \Delta) \subset J(\Delta) = \text{Pic}^0(\Delta) ; P \) is the connected component of \( \mathcal{O} \) in the kernel of \( Nm^0 : \text{Pic}^0(\Delta) \to \text{Pic}^0(\Delta) \).

The general sheaf \( \mathcal{L} \in P^+ \) is non effective, i.e. the linear system \( | \mathcal{L} | \) is empty. The set \( \Theta = \{ \mathcal{L} \in P^+ : | \mathcal{L} | \neq \emptyset \} = \{ \mathcal{L} \in P^+ : h^0(\mathcal{L}) \geq 2 \} \) is a copy of the theta divisor of the p.p.a.v. \( P_+ \cong P \). Since the general sheaf \( \mathcal{L} \in P^- \) is effective, this suggests to introduce the following two subsets of \( \text{Pic}^{2g-2}(\Delta) \):

\[
\text{Supp}(\Theta) = \{ l \in | \mathcal{L} | : \mathcal{L} \in \Theta \}, \text{Supp}(\Theta^c) = \{ l \in | \mathcal{L} | : \mathcal{L} \in \Theta^c \}.
\]

Clearly, \( \dim \text{Supp}(\Theta) = \dim \text{Supp}(\Theta^c) = \dim (P) = g - 1 \). Indeed, the general fiber \( \phi_{\mathcal{L}} : \text{Supp}(\Theta) \to \Theta \) coincides with the linear system \( | \mathcal{L} | \cong \mathbb{P}^1 \), and the general fiber of \( \phi_{\mathcal{L}} : \text{Supp}(\Theta^c) \to \Theta^c \) is \( | \mathcal{L} | \cong \mathbb{P}^0 \).

The Gieseker-Petri inequalities for the codimension of the special subsets of \( P^+ \) and \( P^- \) (see [W2]) imply that the general point \( L \) of \( \text{Supp}(\Theta) \) lies on the 1-dimensional fiber over the general point \( \mathcal{L} \) of \( \Theta \) (similarly for \( \text{Supp}(\Theta^c) \)).

We identify the effective sheaf \( \mathcal{L} \) and the set of effective divisors \( L : L \in | \mathcal{L} | \).

Let \( \text{Pic}^{2g-2}(\pi) : \text{Pic}^{2g-2}(\Delta) \to \text{Pic}^{2g-2}(\Delta) \) be the \((2g-2)\)-th symmetric power of \( \pi \), and let \( | \omega_\Delta | \cong | \mathcal{O}_{\Delta}(d-3) | \cong | \mathcal{O}_{\Delta}(d-3) | \cong | \mathcal{O}_{\Delta}(d-3) | \cong | \mathbb{P}^{d-1} \) be the canonical system of \( \Delta \). We shall use equivalently any of the different interpretations of the elements of this system, as it is written just above.

**The canonical families \( C_+ \) and \( C_- \) of non-isolated and isolated minimal sections of \( p : X \to \mathbb{P}^2 \).**

Let \( C_+ \) be the closure of the set

\[
\mathcal{C}_+ = \{ C - a nonsingular section of \ p : X \to \mathbb{P}^2 : C_0 = p(C) \text{ is a curve from the canonical system for } \Delta, \ C_0 \text{ intersects transversely } \Delta \}
\]

Similarly, let \( C_- \) be the closure of \( \mathcal{C}_- = \{ \ldots \} \), and \( C \) is an isolated (minimal) section of \( S_{C_0} \).

In the definitions of \( C_+ \) and \( C_- \), we use implicitly Theorem 2.4. In fact, if the family \( S(d-3) \) is not versally embedded, then any minimal section of the general conic bundle surface \( S_{C_0} \), \( C_0 \) - a plane curve of degree \( d-3 \), will be isolated (see e.g. [Se] or [LN]).

Let \( S_\Delta = p^{-1}(\Delta) \), let \( \psi_0 : \mathcal{C}_0 \cup \mathcal{C}_0 \to \text{Pic}^{2g-2}(\Delta), \psi(C) \mapsto L(C) = C \cap S_\Delta, \) and let \( \psi : \mathcal{C}_+ \cup \mathcal{C}_- \to \text{Pic}^{2g-2}(\Delta) \) be the completion of \( \psi_0 \).

Denote by \( C_+ = \psi(C_+) \) and \( C_- = \psi(C_-) \) the \( \psi \)-images of \( C_+ \) and \( C_- \).

**Lemma.** The non-ordered pairs \( \{ C_+, C_- \} \) and \( \{ \text{Supp}(\Theta), \text{Supp}(\Theta^c) \} \) of subsets of \( \text{Pic}^{2g-2}(\Delta) \) coincide.

**Proof.** It rests only to be seen that the sets \( C_+ \cup C_- \subset \text{Pic}^{2g-2}(\Delta) \) and \( \text{Pic}^{2g-2}(\Delta) \) coincide. q.e.d.

**The Abel-Jacobi images of the families \( C_+ \) and \( C_- \).**

Let \( J(X) = (H^{2,1}(X))^*/H_3(X, \mathbb{Z}) \text{ mod torsion} \), be the intermediate jacobian of \( X \), provided with the principal polarization \( \Theta_X \) defined by the intersection of 3-chains on \( X \). It is well known (see [B1]) that \( J(X, \Theta_X) \) is isomorphic, as a p.p.a.v., to the Prym variety \( (P, \Theta) \) of the discriminant pair \( (\Delta, \Delta) \). Let

\[
\Phi_+ : C_+ \to J(X) \cong P \text{ and } \Phi_- : C_- \to J(X) \cong P
\]
be the Abel-Jacobi maps for the families \( C_+ \) and \( C_- \) of algebraically equivalent 1-cycles on \( X \). Let \( Z_+ = \Phi_+(C_+) \) and \( Z_- = \Phi_-(C_-) \) be the images of \( \Phi_+ \) and \( \Phi_- \). We shall prove the following

**Theorem**. One of the following two alternatives always takes place:

1. \( h^0(\psi(C)) = 2 \) for the general \( C \in C_+ \) \( \Leftrightarrow \) \( h^0(\psi(C)) = 1 \) for the general \( C \in C_- \), and then
   - (i). \( Z_+ \) is a copy of the theta divisor \( \Theta_X \),
   - (ii). \( Z_- \) coincides with \( J(X) \);

2. \( h^0(\psi(C)) = 1 \) for the general \( C \in C_+ \) \( \Leftrightarrow \) \( h^0(\psi(C)) = 2 \) for the general \( C \in C_- \), and then
   - (i). \( Z_+ \) coincides with \( J(X) \),
   - (ii). \( Z_- \) is a copy of the theta divisor \( \Theta_X \).

**Proof.** The map \( \phi = \phi_C : Supp(\Theta) \cup Supp(P^-) \to \Theta \cup P^- \) introduced above, can be regarded as the (Prym)-Abel-Jacobi map for the sets of algebraically equivalent \((2g-2)\)-tuples of points \( Supp(\Theta) \subset S^{2g-2}(\Delta) \) and \( Supp(P^-) \subset S^{2g-2}(\Delta) \), to the Prym variety \( P \cong J(X) \).

According to Lemma (4.2), \( C_+ = \psi(C_+) \) coincides either with \( Supp(\Theta) \), or with \( Supp(P^-) \). Alternatively, \( C_- = \psi(C_-) \) coincides either with \( Supp(P^-) \), or with \( Supp(\Theta) \).

Let e.g. \( C_+ = Supp(\Theta) \). Then \( h^0(\psi(C)) = 2 \) for the general \( C \in C_+ \), \( h^0(\psi(C)) = 1 \) for the general \( C \in C_- \); and we have to see that \( Z_+ \cong \Theta_X \), and \( Z_- = J(X) \cong P \).

Let \( C \in C_+ \) be general, and let \( z = \Phi_+(C) \in J(X) \) be the Abel-Jacobi image of \( C \). Since \( C \) is general, \( C \) is a nonsingular section of the conic bundle surface \( S_{p(C)} \subset X \), and the effective divisor \( L = L(C) = \psi(C) \in Supp(\Theta) \) is well defined.

We can also assume that \( p(C) \) is nonsingular, and \( p(C) \) intersects \( \Delta \) transversely. In particular, the effective divisor \( L = L(C) \) does not contain multiple points. We shall prove the following

**Lemma.** Let \( C' \) and \( C'' \in C_+ \) be such that \( \psi(C') = \psi(C'') = L \), and let \( z' = \Phi_+(C') \), \( z'' = \Phi_+(C'') \).

Then \( z' = z'' \).

**Proof of (\ast).** Since \( \psi(C') = \psi(C'') \), the curves \( C' \) and \( C'' \) have the same \( \phi \)-image \( C_0 = p(C') = p(C'') \), and \( C' \) and \( C'' \) are non-isolated sections of the conic bundle surface \( S_{C_0} = p^{-1}(C_0) \). Let \( L = l_1 + \ldots + l_{2g-2} \), and \( x_i = p(l_i) \), \( i = 1, \ldots, 2g-2 \). The degenerate fibers of \( p : S_{C_0} \to C_0 \) are the singular conics \( q(x_i) = p^{-1}(x_i) = l_i + \frac{t_i}{\theta} \). By assumption \( C' \) and \( C'' \) intersect simply any of the components \( l_i \), and does not intersect any of \( t_i \).

Let \( C \) be any nonsingular section of \( S_{C_0} \) such that \( \psi(C) = C \cap S_\Delta = L \), e.g. \( C = C' \). Then \( Div(S_{C_0}) = p^*(Div(C_0)) + Z.l_1 + \ldots + Z.l_{2g-2} + Z.C \).

Since \( (C' - C'').q = 1 - 1 = 0 \), \( (C' - C'').l_i = 0, (i = 1, \ldots, 2g-2) \), the divisor \( C' - C'' \) belongs to \( p^*(Div(C_0)) \); i.e. \( C' - C'' = p^\ast \delta \) for some \( \delta \in Div(C_0) \).

Obviously, \( deg(\delta) = 0 \). Represent \( \delta \) as a difference of two effective divisors (of the same degree): \( \delta = \delta_1 - \delta_2 \). Without loss of the generality we can assume that the sets \( Supp(\delta_1) \) and \( Supp(\delta_2) \) are disjoint. Therefore, \( p^*(C' - C'') = p^{-1}(\delta_1) - p^{-1}(\delta_2) \) is a sum of fibers of \( p \), with positive and negative coefficients, and of total degree 0.

Since all the fibers of \( p : X \to P^2 \) are rationally equivalent, the rational cycle class \( [p^{-1}(\delta)] \), of \( p^{-1}(\delta) \), is 0, in the Chow ring \( A_*(X) \). Since the Abel-Jacobi map for any family of algebraically equivalent 1-cycles on \( X \) factors through the cycle class map, the curves \( C' \) and \( C'' \) have the same Abel-Jacobi image, i.e. \( z' = z'' \). This proves (\ast).

It follows from (\ast) that the Abel-Jacobi map \( \Phi_+ \) factors through \( \psi \), i.e., there exists a well-defined map \( \overline{\Phi}_+ : Supp(\Theta) \to Z_+ \), such that \( \Phi = \overline{\Phi}_+ \circ \psi \).
Let $C \subset C_+$ be general, and let $L = L(C) = \psi(C)$. Let $\mathcal{L} = \phi(L)$ be the sheaf defined by the 1-dimensional linear system of effective divisors linearly equivalent to $L$. Let $C_+(\mathcal{L}) = \psi^{-1}(\{\mathcal{L}\})$ be the preimage of $\{\mathcal{L}\}$ in $C_+$. Since $\Phi_+$ factors through $\psi$, and $\Phi_+$ is a map to an abelian variety (the intermediate jacobian $J(X)$ of $X$), the map $\overline{\Phi}_+$ contracts rational subsets of $Supp(\Theta)$ to points. However, $\psi(C_+(\mathcal{L})) \cong |\mathcal{L}| \cong \mathbb{P}^1$. Therefore, there exists a point $z = z(\mathcal{L}) \in Z_+$ such that $\Phi_+(\phi^{-1}(\mathcal{L})) = \Phi_+(C_+(\mathcal{L})) = \overline{\Phi}_+(|\mathcal{L}|) = \{z\} \subset Z_+$.

Clearly $z = \Phi_+(C)$, and the uniqueness of the sheaf $\mathcal{L}$ defined by $C$, implies that the correspondence $\Sigma = \{(z, \mathcal{L}): z = \Phi_+(C), \mathcal{L} = \phi \circ \psi(C), C \in C_+\}$ is generically $(1:1)$.

Let $i: \Sigma \rightarrow Z_+$ and $j: \Sigma \rightarrow \Theta$ be the natural projections. The general choice of $C \in C_+$, and the identity $\psi(C_+) = Supp(\Theta)$, imply that $j$ is surjective. Therefore $Z_+$ and $\Theta$ are birational. In particular, $Z_+$ is a divisor in $J(X) \cong P$. It is not hard to see that the map $i \circ j^{-1}: \Theta \rightarrow Z_+$ is regular. In fact, let $\mathcal{L}$ be any sheaf which belongs to $\Theta$. The definition of $\phi$ implies that $\phi^{-1}(\mathcal{L})$ coincides with the linear system $|\mathcal{L}|$, which is an (odd dimensional) projective space. Therefore, $\overline{\Phi}_+$ contracts the connected rational set $\psi^{-1}(\mathcal{L})$ to a unique point $z = z(\mathcal{L})$, i.e. $i \circ j^{-1}$ is regular in $\mathcal{L}$. It follows that $Z_+$ is birational to the divisor of principal polarization $\Theta$, i.e. $Z_+$ is a translate of $\Theta$.

The coincidence $Z_- = J(X)$ follows in a similar way.

In case (2), the only difference is that the general fiber of $\psi$ is discrete, since the minimal sections $C \in C_-$ which majorate the general $L \in Supp(\Theta)$, are isolated. Theorem 4.4 is proved.

5. The fibers of the Abel-Jacobi maps $\Phi_+$ and $\Phi_-$. We shall describe explicitly the general fibers of $\Phi_+$ and $\Phi_-$ in either of the cases (4.4)(1) and (4.4)(2).

5.1 It follows from Theorem (4.4) that the fibers of $\Phi_+$ and $\Phi_-$ depend closely on the alternative conclusions: $Z_+ = \Theta$, or $Z_- = \Theta$. The examples show that any of the two alternatives (4.4)(1) – (4.4)(2) can be true, depending on the choice of the conic bundle $p: X \rightarrow \mathbb{P}^2$ (see section 6). Notwithstanding, theorems (2.4) and (4.4), and some results regarding versal deformations of ruled surfaces (see [Se]) and subbundles of rank 2 vector bundles over curves (see e.g. [LN]), make it possible to describe these fibers in each of the cases (1) and (2).

5.2 Minimal sections of ruled surfaces and maximal subbundles of rank 2 vector bundles on curves.

Here we collect some known facts about ruled surfaces and rank 2 vector bundles over curves.

Any ruled surface $S$ over a smooth curve $C$ can be represented as a projectivization $\mathbb{P}_C(E)$ of a rank 2 vector bundle $E$ over $C$. Clearly, $\mathbb{P}_C(E)$ is a ruled surface for any such $E$, and $\mathbb{P}_C(E) \cong \mathbb{P}_C(E')$ iff $E = E' \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$; here we identify vector bundles and the associated free sheaves.

Call the bundle $E$ normalized if $h^0(E) \geq 1$, but $h^0(E \otimes \mathcal{L}) = 0$ for any invertible $\mathcal{L}$ such that $deg(\mathcal{L}) < 0$ (see [H, ch.5,#2]).

The question is:

(*). How many normalized rank 2 bundles represent the same ruled surface?

The answer depends on the choice of the curve $C$ (esp. on the genus $g = g(C)$ of $C$), and on the choice of the ruled surface $S$ over $C$. Let $p: S \rightarrow C$ be the natural fiber structure on $S$. We shall reformulate the question (*) in the terms of sections of $p$.

DEFINITION. Call the section $C \subset S$ minimal if $C$ is a section on $S$ for which the number $(C.C)_S$ is minimal. Let $C$ be a minimal section of $S$. The number $e = e(S) = (C.C)_S$ is an integer invariant of the ruled surface $S$. The number $e(S)$ coincides with $deg(E) := deg(det(E))$, where $E$ is any normalized rank 2 bundle which represents $S$ (i.e. such that $S \cong \mathbb{P}_C(E)$) (see e.g. [H, ch.5,#2]). We call the number $e = e(S)$ the invariant of $S$. 14
**Remark.** Here, in contrast with the definition in use, we let $e(S) := - (\text{the invariant of } S)$.

The new question is:

(***). How many minimal sections lie on the same ruled surface?

The two questions are equivalent in the following sense: Let $E$ be normalized and such that $\mathbf{P}(E) = S$. By assumption $h^0(E) \geq 1$. Therefore $E$ has at least one section $s \in H^0(E)$. The bundle section defines (and is defined by) an embedding $0 \to \mathcal{O}_C \to E$. The sheaf $L$, defined by the cokernel of this injection, is invertible, and $L$ defines in a unique way a minimal section $C = C(s)$ of the ruled surface $S = \mathbf{P}_C(E)$ (see e.g. [H, ch.5, 2.6, 2.8]). If $h^0(E) = 1$, the bundle section $s \in H^0(E)$ is unique, and the corresponding minimal section $C(s)$ is unique. In contrary, if $h^0(E) \geq 2$, the map $\mathbf{P}(H^0(E)) \to \{\text{the minimal sections of } S, s \mapsto C(s)\}$ defines a nontrivial linear system of minimal sections of $S$ (e.g., if $S$ is a quadric). Therefore, the set of minimal sections of $S$ is the same as the set of the bundle sections of normalized bundles which represent $S$.

In fact, if $g(C) \geq 1$ and $S$ is general, then $h^0(E) = 1$ for any normalized $E$ which represents $S$. In this case the questions (*) and (***) are equivalent. This is exactly the setting of Theorem (2.4) for $k \geq 3$.

**Definition.** Call the line subbundle ($= \text{the invertible subsheaf}$) $M \subset E$ a maximal subbundle of $E$, if $M$ is a line subbundle of $E$ of a maximal degree.

Let $E$ be a fixed normalized bundle which represents $S$, and let $M \subset E$ be a maximal subbundle of $E$. Clearly $deg(M) \geq 0$, since $\mathcal{O}_C \subset E$. Assume that $deg(M) > 0$. Then, after tensoring by $M^{-1}$, we obtain the embedding $\mathcal{O}_C \subset E \otimes M^{-1}$.

In particular, $h^0(E \otimes M^{-1}) \geq 0$, $E \otimes M^{-1}$ represents $S$, and $deg(E \otimes M^{-1}) < deg(E)$. However $E$ is normalized, hence $deg(E \otimes M^{-1})$ cannot be less than $deg(E) - \text{contradiction}$. Therefore $deg(M) = 0$, and the maximal subbundle $M$ of $E$ defines the normalized bundle $E \otimes M^{-1}$ which also represents $S$.

Therefore, we can reduce the question (*) to the following question:

(****). How many maximal subbundles has a fixed normalized rank 2 bundle $E$ which represents a given ruled surface $S$?

The answer of (*) - (***) for $S$ decomposable, is given in [H, ch.5, Examples 2.11.1, 2.11.2, 2.11.3]. In particular, this implies the well known description of the set of minimal sections of a rational ruled surface $p: S \to \mathbf{P}^1$. In particular, if $d = deg(\Delta) = 4$ or 5, Theorem (2.4) implies that the ruled surface $S(L)$ is either a quadric (if $L \in \mathcal{C}_+$), or the surface $\mathbf{F}_1$ (if $L \in \mathcal{C}_-$). If $d = 6$, the minimal sections $C \in \mathcal{C}_+ \cup \mathcal{C}_-$ are elliptic curves.

If $d = deg(\Delta) = 6$, the general $C \in \mathcal{C}_+ \cup \mathcal{C}_-$ is a smooth elliptic curve. It follows from the proof of Theorem (2.4) that $S(L)$ is a general element of a versal deformation of ruled surface over an elliptic base. Similar conclusion takes place if $deg(\Delta) \geq 7$. Therefore, the description of the general fibers of $\Phi_+$ and $\Phi_-$ will follow from the description of the set of minimal sections of the general member in a versal deformation of a ruled surface.

**5.3 Lemma** (see [Se, Theorem 5]). Let $S \to C$ and $S' \to C'$ be two ruled surfaces. Then $S$ and $S'$ can be deformed into each other iff $C$ and $C'$ have the same genus, and the invariants $e(S)$ and $e(S')$ have the same parity.

In particular, the parity of $e(S)$ is an invariant of the deformations of the ruled surface $S$.

**5.4 Lemma** (see [Se, Theorem 13]). The general surface in the versal deformation of a rational ruled surface is a quadric if $e$ is even, and the surface $\mathbf{F}_1$ if $e$ is odd.
The general surface of a versal deformation of a ruled surface over elliptic base is a surface represented by the unique indecomposable rank 2 vector bundle of degree 1 if \( e \) is odd, and a decomposable ruled surface represented by a sum of two (non-incident) line bundles of degree 0 if \( e \) is even.

The general surface of a versal deformation of a ruled surface over a curve of genus \( g \geq 2 \) is indecomposable. The invariant of such \( S \) is \( g - 1 \) if \( e \equiv g \mod 2 \), or \( g \) if \( e \equiv g - 1 \mod 2 \).

(5.5) Lemma (see [H, ch.5, Example 2.11.2 and Exer.2.7]). Let \( C \) be an elliptic curve, and let \( S \) be the unique indecomposable ruled surface over \( C \) with invariant \( e(S) = 0 \). Then the set \( C + (S) \) of minimal sections of \( S \) form a 1-dimensional family parameterized by the points of the base \( C \). In particular, all the minimal sections of \( S \) are linearly non equivalent.

Let \( C \) be an elliptic curve, and let the ruled surface \( S \) be represented by the normalized bundle \( E = O_C \oplus L \), where \( \text{deg}(L) = 0 \) and \( L \neq O_C \). Then \( S \) has exactly two minimal sections: the section \( C = C(s_E) \) defined by the unique bundle section \( s_E \) of \( E \), and the section \( C \) defined by the unique section \( s_E \) of the second normalized bundle \( E = O_C \oplus L^{-1} \) which represents \( S \).

(5.6) Lemma (see [LN, Proposition 2.4]). Let \( S \) be an indecomposable ruled surface over a curve \( C \) of genus \( g \geq 2 \). Let \( E \) be a fixed normalized rank 2 bundle over \( C \) which represents \( S \), and let \( (E) \in \mathbb{P}(H^0(K_C \oplus L)) \) be the point which corresponds to the extension \( 0 \to O_C \to E \to L \to 0 \) defined by \( E \). Let \( \alpha : C \to \mathbb{P}(H^0(K_C \oplus L)) \) be the map defined by the linear system \( | K_C \oplus L | \), and let \( \alpha(C) \) be the image of \( C \). Then the set of maximal line subbundles \( M \) of \( E \), which are different from \( O_C \), is naturally isomorphic to the set of \( e \)-secant line bundles of \( \alpha(C) \) which pass through the point \( [E] \).

Remarks.
(1) Here \( e = e(S) = \text{deg}(L) \) is the invariant of \( S \).
(2) The line bundle \( D \) on \( C \) of degree \( e \) is an \( e \)-secant line bundle of \( \alpha(C) \) which passes through \( [E] \), if the linear system \( | D | \) contains an effective divisor \( D \) such that the space \( \text{Span}(\alpha(D)) \) passes through the point \( [E] \).

(5.7) The general position of the ruled surfaces \( S_L \).

Let \( p : X \to \mathbb{P}^2 \) be a smooth conic bundle which induces a nontrivial unbranched double covering \( \pi : \tilde{\Delta} \to \Delta \) of the smooth discriminant curve \( \Delta \) of degree \( \text{deg}(\Delta) = d \). Without any restriction, we can assume that \( d \geq 4 \) (otherwise the jacobian \( J(X) \) will be trivial). Let \( C_0 \subset \mathbb{P}^2 \) be a general smooth plane curve of degree \( d - 3 \), and let \( S_{C_0} = p^{-1}(C_0) \subset X \) be, as usual, the induced conic bundle surface over \( C_0 \). In particular, we can assume that \( C_0 \) is smooth, and \( C_0 \) intersects \( \Delta \) in \( 2g - 2 = d(d - 3) \) disjoint points \( (x_1, \ldots, x_{2g-2}) \). Let \( L_0 = x_1 + \ldots + x_{2g-2} \), and let \( F(L_0) = (S_{2g-2}^-)^{-1}(L_0) \subset S_{2g-2}^- \).

As it follows from Theorem (2.4), the assumption \((2.4)^*\) and the general choice of \( C_0 \) imply that the set \( F(L_0) \) splits into two disjoint sets \( F(L_0)_+ \) and \( F(L_0)_- \), of equal cardinality \( = 2^{g-3} \), with the following properties:
1. If \( L = l_1 + \ldots + l_{2g-2} \in F(L_0)_- \) then the ruled surface \( S_L \) (obtained from \( S_{C_0} \) by blowing down the \( 2g - 2 \) complimentary fibers \( \overline{l}_1, \ldots, \overline{l}_{2g-2} \)) has invariant \( e = e(L) := e(S_L) = g - 1 \).
2. If \( L = l_1 + \ldots + l_{2g-2} \in F(L_0)_+ \) then the ruled surface \( S_L \) (obtained from \( S_{C_0} \) by blowing down the \( 2g - 2 \) complimentary fibers \( \overline{l}_1, \ldots, \overline{l}_{2g-2} \)) has invariant \( e = e(L) := e(S_L) = g \).

Moreover, as it follows from the proof of (2.4), any such \( S_L \) has to be a general element of a versal deformation of a ruled surface over plane curve of degree \( d - 3 \), provided \( C_0 \) is chosen to be sufficiently general. In particular, we can assume:
(5.7.1). (If \( d \geq 7 \), then)

The general \( S(L) \) is indecomposable; and if \( L \in F(C_0)_- \) (see above), then:

(1.i). \( e(S(L)) = g - 1 \);

(1.ii). Let \( E \) be a general normalized rank 2 bundle which represents \( S_L \). Let \( s : 0 \to \mathcal{O}_{C_0} \to E \) be the embedding defined by the bundle section \( s \) of \( E \), and let \( \mathcal{L}(s) \) be the cokernel of \( s \). Then the point \([E] \in \mathbb{P}(H^0(K_{C_0} \otimes \mathcal{L}(s)))\) (which represents the extension class of \( s \)) is in general position with respect to the set of \((g - 1)\)-secant line bundles of \( \alpha(C_0) \) (see (5.6)).

Remark.

The condition (ii) is open, on the set of extension classes \([E]\). Therefore, it has to be fulfilled for at least one \( C_0 \in |\mathcal{O}_{\mathbb{P}^2}(d - 3)| \).

Let e.g. \( C_0 \in S_{d-3} \) (see (3.3)). The proof of Lemma (3.3.1) implies that if such an \( C_0 \) is general, then the centers \( \xi_{ik} \) of the elementary transformations \( elm(\xi_{ik}) \) which interchange the ruled surfaces \( S_L : L \in F(C_0) \) can be chosen to be general (provided \((2,4)(^*)\) takes place). We can assume that \( C_0 \in S_{d-3} \) has reduced components, which intersect \( \Delta \) transversely. Now, it is not hard to complete the definition of (normalized) bundles \( E \), bundle sections \( s \), extension classes \([E]\), etc., for such a curve \( C_0 \). In particular, the possibility of the general choice of the centers \( \xi_{ik} \) implies that (ii) takes place for such a curve \( C_0 \). Therefore, the same is true also for the general \( C_0 \in |\mathcal{O}_{\mathbb{P}^2}(d - 3)| \).

In a more exact setting, if \((d - 3) \leq 3 \), the genus of \( C_0 \) is 0 or 1; and (i) and (ii) have to be reformulated (in the obvious way) in the context rational ruled surfaces and ruled surfaces over elliptic base (see e.g. (5.5)).

Similarly,

(5.7.2) Let \( L \in F(C_0)_+ \), and let \( C_0 \) be general. Then the following takes place:

(If \( d \geq 7 \), then)

(2.i). \( S_L \) is indecomposable, and \( e(S(L)) = g \);

(2.ii). The point which is defined by the general extension class \([E]\) is in general position with respect to the set of \( g \)-secant line bundles of \( \alpha(C_0) \) (see (5.7.1)(1.ii)).

(5.7.3). For \( d \leq 6 \) – see the last remark.

Now, (5.1) - (5.7) and Theorems (2.4) and (4.4) imply the following

(5.8). Description of the general fibers of the Abel-Jacobi maps \( \Phi_+ \) and \( \Phi_- \).

Theorem. Let \( p : X \to \mathbb{P}^2 \) be a conic bundle, as in (5.1). Let \( C_+ \) and \( C_- \) be the families of non-isolated and isolated minimal sections, and let \( \phi : C_+ \to C_+ \), \( \phi : C_- \to C_- \), \( \psi : \text{Supp}(\Theta) \to \Theta \), and \( \psi : \text{Supp}(P^-) \to P^- \) be the families and the natural maps defined in (4.1). Let \( \Phi_+ : C_+ \to J(X) \) and \( \Phi_- : C_- \to J(X) \) be the Abel-Jacobi maps for \( C_+ \) and \( C_- \), and let \( Z_+ \) and \( Z_- \) be the images of \( \Phi_+ \) and \( \Phi_- \).

Then one of the following two alternatives is true:

(A). \( C_+ = \text{Supp}(\Theta) \), \( Z_+ \) is a translate of \( \Theta \) (\( \Leftrightarrow C_- = \text{Supp}(P^-) \), \( Z_- = J(X) \equiv P \)).

Let \( z \in Z_+ \) is general, and let \( \mathcal{L} = \psi \circ i^{-1}(z) \in \Theta \) be the sheaf which corresponds to \( z \) (see the proof of (4.4)). Then:

1. The fiber \( C_+(z) := \Phi_+^{-1}(z) \) is 2-dimensional.
2. The map \( \psi \) defines on \( C_+(z) \) the natural fibration \( \psi(z) : C_+(z) \to |\mathcal{L}| \equiv \mathbb{P}^1 \).
3. The general fiber \( C_+(L) := \psi(z)^{-1}(L) \) of \( \psi(z) \) can be described as follows \((d \geq 4)\):

Let \( C_0(L) \subset \mathbb{P}^2 \) be the plane curve of degree \( d - 3 \) defined by \( L \). Then
(i). If \( d = \deg(\Delta) = 4 \) or 5, then \( S(L) \cong \mathbf{P}^1 \times \mathbf{P}^1 \), and \( C_+(L) \cong \mathbf{P}^1 \) of the projection \( p(L) : S(L) \to C_0(L) \cong \mathbf{P}^1 \) induced by \( p \);

(ii). If \( d = \deg(\Delta) = 6 \), then \( p(L) : S(L) \to C_0(L) \) is the only indecomposable ruled surface over the elliptic case \( C_0(L) \), and the fiber \( C_+(L) \) of the \( \psi(z) : C_+(z) \to |L| \cong \mathbf{P}^1 \) is isomorphic to \( C_0(L) \). In particular, \( C_+(z) \) is an elliptic fibration over the rational base curve \( |L| \);

(iii). Let \( d = \deg(\Delta) \geq 7 \), let \( g = \frac{d(d-3)}{2} + 1 \) be the genus of \( C_0(L) \), let \( C \in C_+(L) \) be general, and let \( 0 \to \mathcal{O}_{C_0} \to E \to 
abla \to 0 \) be the extension defined by the section \( C \). Let \( \alpha(C_0) \subset \mathbf{P}(\mathcal{H}^0(K_{C_0} \otimes \mathcal{N})) \) be the image of \( C_0 \) defined by the sheaf \( K_{C_0} \otimes \mathcal{N} \).

Then \( \mathbf{P}(\mathcal{H}^0(K_{C_0} \otimes \mathcal{N})) \cong \mathbf{P}^{2g-2} \), \( \alpha \) is a regular morphism of degree 1, and the point \([E]\) defined by this extension is in general position with respect to the set of \( g \)-secant line bundles of \( \alpha(C_0) \). Moreover, \( C_+(L) \) is birational to the 1-dimensional set \( \{ \text{particular, if } C \} \) of the linear system \( \mathbf{P} \).

Then \( |L| = \mathbf{P}^1 \) is surjective, and the fiber of this fiber, let \( \Phi(z) \) be general, and \( \mathbf{P}(\mathcal{H}^0(K_{C_0} \otimes \mathcal{N})) \cong \mathbf{P}^{2g-3} \), \( \alpha \) is generically of degree 1, and \([E]\) does not lie on an infinite set of \((g-1)\)-secant planes of \( \alpha(C_0) \). Moreover, the cardinality of \( C_-(z) \) is equal to \#\{\((g-1)\)-secant planes of \( \alpha(C_0)\)\} + 1 (see (5.6)).

(A′−). \( C_- = \mathrm{Supp}(\Theta) \), \( Z_- \cong \Theta \) (\( \Leftrightarrow C_+ = \mathrm{Supp}(P^-) \), \( Z_- = J(X) \cong P \)).

Then the description of the general fibers of \( \Phi_- \) and \( \Phi_+ \) is similar to this from (A.+)-(1)-(4). We shall mark only the differences:

(1)-(2)-(3). The fiber \( C_-(z) \) is 1-dimensional. The map \( \psi(z) : C_-(z) \to |L(z)| \cong \mathbf{P}^1 \) is finite and surjective, and the fiber of \( \psi(z) \) has the same description as the fiber \( C_-(L) = \psi^{-1}(L) \) described in (A.+)-(4).

(4). The fiber \( C_+(z) \) is 1-dimensional. Let \( \mathcal{L} = \mathcal{L}(z) = j \circ i^{-1}(z) \), and let \( L = L(z) \) be the unique element of the linear system \( |L| \). Then the sets \( C_+(z) \) and \( C_+(L) \) coincide. In particular, the fiber \( C_+(z) \) has the same description as the fiber \( C_+(L) \) described in (A.+)-(1)-(4). Note that here \( L = L(z) \) is unique.

(5.9) Corollary. \( \dim(C_+) = \dim J(X) + 1 = \dim (|\mathcal{O}_{\mathbf{P}^2}(d-3)|) + 1 = (d-2)(d-1)/2 \).

\( \dim(C_-) = \dim J(X) = \dim (|\mathcal{O}_{\mathbf{P}^2}(d-3)|) = (d-2)(d-1)/2 - 1 \).
6. Examples.

(6.1) The double covering of $\mathbf{P}^2 \times \mathbf{P}^1$.

(6.1.1) Definition. Let $Y = \mathbf{P}^2 \times \mathbf{P}^1 \subset \mathbf{P}^5$, let $Q$ be a general smooth quadric in $\mathbf{P}^5$, and let $\xi : X \to Y$ be the double covering branched along the smooth surface $S = Y \cap Q$.

Clearly, $X$ is a Fano 3-fold with rank $\text{Pic}(X) = 2$. We call the threefold $X$ the double covering of $\mathbf{P}^2 \times \mathbf{P}^1$.

(6.1.2) The conic bundle structure on $X$, and the families $\mathcal{C}_{1,0}$ and $\mathcal{C}_{1,1}$.

Let $p_o : Y \to \mathbf{P}^2$ and $q_o : Y \to \mathbf{P}^1$ be the projections, and let $p = p_o \circ \xi$, $q = q_o \circ \xi$.

Then $p : X \to \mathbf{P}^2$ is a conic bundle, and the discriminant $\Delta$ is a smooth plane curve of degree 4. Let $\eta \in \text{Pic}^2(\Delta)$ be the torsion sheaf which defines the double covering $\pi : \widetilde{\Delta} \to \Delta$. Then the Prym-canonical system $|K_{\Delta} + \eta|$ is naturally isomorphic to $\mathbf{P}^1$. Indeed, the elements of the system $|K_{\Delta} + \eta| = |O_{\Delta}(1+\eta)|$ are in $(1:1)$-correspondence with the $\mathbf{P}^1$-family of conics $q$ which are totally tangent to $\Delta$ along effective divisors $\delta$ of the linear system $|O_{\Delta}(1+\eta)|$. The family of all these conics is parameterized naturally by the points $x \in \mathbf{P}^1$. Indeed, let $x \in \mathbf{P}^1$ be general, and let $Q_x = q^{-1}(x)$. Then $Q_x$ is a quadric, and the projection $p$ defines a double covering $q : Q_x \to \mathbf{P}^2$ branched along a conic $q(x)$ which is totally tangent to $\Delta$ along a degree 4 divisor $\delta(x)$. It is not hard to see that $\delta(x) \in |O_{\Delta}(1+\eta)|$. In particular, the pencil of quadrics $q: X \to \mathbf{P}^1$ has 6 degenerations - the cones $Q_{x_i}, i = 1, \ldots, 6$ which correspond to the 6 degenerated conics $q(x_i)$ in the quadratic pencil $\{q(x)\}$ of conics totally tangent to the plane quartic $\Delta$. Let $x \in \mathbf{P}^1$ be general, and let $p : Q_x \to \mathbf{P}^2$ be the double covering branched along the smooth conic $q(x)$. The double covering $p$ maps the elements of the two generators $\Lambda(x)$ and $\overline{\Lambda}(x)$ of $Q_x$ to lines $l = p(C)$ tangent to $q(x)$. Clearly, these elements $C$ are (1,0)-lines on $X$, i.e. curves of bidegree (1,0) on $X$ (with respect to the bidegree map induced by the projections $p$ and $q$). Moreover, any such $C$ is a section of the conic bundle surface $S_l = p^{-1}(l)$, where $l = l(C)$. It follows that the 2-dimensional family $\mathcal{C}_{1,0}(X)$ of (1,0)-lines on $X$ coincides with the family $\mathcal{F}_{-} = \mathcal{C}_{-}$ of isolated sections over the canonical system of $\Delta$. The including of any (1,0)-line $C$ in a generator of a quadric $Q_x$ implies that the effective divisor $L(C)$ of any (1,0)-line $C$ in a generator of a quadric $Q_x$ moves in a non-trivial linear system of dimension one. Therefore $\psi(C) = C \cap p^{-1}(\Delta)$ moves in a non-trivial linear system of dimension one. Therefore $\psi(C_0(1,1)(X)) = \text{Supp}(\Theta)$, and the set of fibers of $\psi$ is in (1,1)-correspondence with the set of generators $\text{Gen}(X) = \text{(the closure of $\{(\Lambda(x),\overline{\Lambda}(x)) : x \in \mathbf{P}^1 - \{x_1,\ldots,x_6\}\}$}}$. Clearly, these generators can be treated also as sheaves on $\widetilde{\Delta}$ which belong to $\Theta$. Moreover, since $\text{dim}(\text{J}(X)) = 2$, $\text{J}(X)$ is an jacobian of a curve $C_X$ of genus 2. In particular $C_X \cong \Theta \cong \text{Gen}(X)$, and the natural double covering $q : \text{Gen}(X) \to \mathbf{P}^1$ defines a double covering $q' : C_X \to \mathbf{P}^1$, branched in the 6 points $x_1,\ldots,x_6$ over which the $(\mathbf{P}^1 \times \mathbf{P}^1)$-bundle $q : X \to \mathbf{P}^1$ degenerate.

Now, it is not hard to see that:

(6.1.3) The 3-dimensional family $\mathcal{C}_{+}$ of non-isolated sections (over the canonical system of $\Delta$) coincides with the family $\mathcal{C}_{1,1}(X)$ of bidegree $(1,1)$-conics on $X$. In particular, the Abel-Jacobi map $\Phi_{+} : \mathcal{C}_{1,1}(X) \to J(X)$ is surjective, and the general fiber of $\Phi_{+}$ is isomorphic to $\mathbf{P}^1$.

(6.2) The intersection of two quadrics.

(6.2.1) Let $X = X_{2,2} \subset \mathbf{P}^5$ be a smooth intersection of two quadrics. The threefold $X$ is a Fano 3-fold with rank $\text{Pic}(X) = 1$. In particular, there are no birational conic bundle structures on $X$ (otherwise, rank $\text{Pic}(X) \geq 2$). However, $X$ admits various natural birational conic bundle structures indexed by the general elements of the 4-dimensional family of conics on $X$. 19
The conic bundle structure defined by a conic \( q_o \) on \( X \), and the families \( C_1^0(X) \) and \( C_2^0(X)_{q_o} \).

Let \( q_o \) be a sufficiently general fixed conic on \( X = X_{2,2} \subset \mathbb{P}^5 \), and let \( p_o : X \to \mathbb{P}^2 \) be the rational projection centered in the plane of the conic \( q_o \). Then \( p_o \) defines a birational structure of a conic bundle \( p : X \to \mathbb{P}^2 \), and the non-trivial component \( \Delta \) of the determinant locus of \( p \) is a smooth plane quartic.

The canonical system of \( \Delta \) is defined by the system of lines \( l \) in \( \mathbb{P}^2 \); and it is not hard to see that the “canonical” family \( \mathcal{S}[1] \) of conic bundle surfaces over the lines in \( \mathbb{P}^2 \) is versally embedded. Let \( \mathcal{C}_- = \mathcal{F}_- \) and \( \mathcal{C}_+ = \mathcal{F}_+ \) be the families of isolated and non-isolated minimal sections of the elements of the system \( \mathcal{S}[1] \). By construction, the general curve \( C = \mathcal{C}_- \cup \mathcal{C}_+ \) is projected isomorphically (by \( p_o \)) onto a line \( l \subset \mathbb{P}^2 \). The same property have the general elements \( C \) of the families

\[
C_1^0(X) = \{ \text{the lines in } X \} \quad \text{and} \quad C_2^0(X)_{q_o} = \{ \text{the conics in } X \text{ which intersect } q_o \}.
\]

Moreover, \( \dim(C_+) = 3 = \dim(C_2^0(X)_{q_o}) \), and \( \dim(C_-) = 2 = \dim(C_1^0(X)) \).

Therefore, by disregarding the necessary corrections caused by the non-standard form of the conic bundle projection \( p \), we can assume that \( C_+ \cong C_2^0(X)_{q_o} \), and \( C_- \cong C_1^0(X) \).

The parameterization of \( (J(X), \Theta) \) via minimal sections.

It is well known (see e.g. [Do]) that the family \( C_1^0(X) \) is isomorphic to the intermediate jacobian \( J(X) \). Therefore, \( \Phi_- \) sends the 3-dimensional family \( C_+ \) onto a copy of the theta divisor \( \Theta(X) \), i.e. (in this case) the theta divisor is described as an Abel-Jacobi image of the family of non-isolated minimal sections of the canonical system of conic bundle surfaces \( \mathcal{S}[1] \).

In fact, the conics on \( X = X_{2,2} \) are described by the elements of the generators of the \( \mathbb{P}^1 \)-system of quadrics in \( \mathbb{P}^5 \) which contain \( X \). The general of these quadrics is a quadric \( Q(x) \) of rank 6, and \( Q(x) \) has two generators \( \Lambda(x) \cong \overline{\Lambda}(x) \cong \mathbb{P}^2 \). Any of these two generators define a \( \mathbb{P}^2 \)-system of conics in \( X \). In particular, any conic on \( X \) is rationally equivalent to a conic on \( X \) which intersects the fixed conic \( q_o \).

Therefore the Abel-Jacobi image of the 4-dimensional family \( C_2^0(X) \) of conics on \( X \) isomorphic to the Abel-Jacobi image of \( C_2^0(X)_{q_o} \). Moreover, the induced (by the generators) double covering of \( \mathbb{P}^1 \) is branched over the set of 6 points which represent the 6 singular quadrics through \( X \). Just as in (6.1), the so defined curve \( Gen \) is isomorphic to \( \Theta \), and (since \( \dim(J(X)) = 2 \) ) \( J(X) \cong J(Gen) \) as p.p.a.v.

The cubic threefold.

Let \( X = X_3 \subset \mathbb{P}^4 \) be a general smooth cubic threefold, and let \( \mathcal{F} \) be the 2-dimensional family of lines on \( X \). It is well known (see e.g. [CG] or [Be2]) that the Abel-Jacobi map

\[
\Phi^{+,-} : \mathcal{F} \times \mathcal{F} \to J(X), \quad \Phi^{\pm,-} : [l,m] \mapsto \Phi([l-m])
\]

is a 6-sheeted covering of a copy of the theta divisor \( \Theta \) of \( J(X) \), and the general fiber of \( \Phi^{+,,-} \) is a Schl"afli's double-six of lines on a hyperplane section of \( X \).

We shall see (by using appropriate conic bundle structures on \( X \)) that the Abel-Jacobi image of the 6-dimensional family \( C_2^0(X) \) of twisted cubics on \( X \) is also a copy of the theta divisor \( \Theta \) of the intermediate jacobian \( J(X) \). The advantage of the second parameterization of \( \Theta \) is that the fibers of the Abel-Jacobi map \( \Phi : C_2^0 \to \Theta \) are connected (the general one is, in fact, a projective plane – see below).
(6.3.2) The families $C_+$ and $C_-$ for the birational conic bundle structure defined by a line $l_0 \subset X$.

Let $l_0$ be a sufficiently general fixed line on $X$, and let $p_o : X \to \mathbb{P}^2$ be the rational projection from the line $l_0$. Let $\tilde{X}$ be the blow-up of $X$ along $l_0$. The map $p_o$ defines a conic bundle structure $p : \tilde{X} \to \mathbb{P}^2$ on $\tilde{X}$, and the nontrivial component $\Delta$ of the discriminant curve of $p$ is a smooth plane quintic.

In this case, the families $C_-$ and $C_+$ are the families of isolated and non-isolated sections of the 5-dimensional family $\mathcal{S}[2]$ of conic bundle surfaces over the $\mathbb{P}^5$-space of plane conics. Just as in (6.2), we see that (outside the components and subsets caused by the blow up and some coincidences with $l_0$):

1. The family $C_+$ can be identified with the 6-dimensional family $C_4^0(X)_{l_0,l_0}$ of rational quartics on $X$ which intersect twice the line $l_0$.

2. The family $C_-$ can be identified with the 5-dimensional family $C_5^0(X)_{l_0}$ of twisted cubics on $X$ which intersect the line $l_0$.

Let $C \subset X$ be a general twisted cubic on $X$ which intersects the line $l_0$, and let $x_0$ be the common point of $C$ and $l_0$. Let $H = \text{Span}(C)$ be the hyperplane spanned on $C$, and let $S_H = X \cap H$ be the hyperplane section defined by $H$. The general choice of $C$ implies that $S_H$ is a smooth cubic hypersurface. Since $C$ is a twisted cubic on $S_H$, the curve $C$ moves in a 2-dimensional linear system on $S_H$. Again by the general choice of $C$, the non-complete linear system $|\mathcal{O}_{S_H}(C) - x|$ is isomorphic to $\mathbb{P}^1$.

Therefore the natural “intersection” map:

$$\psi : C_- \cup C_+ \to \text{Supp}(\Theta) \cup \text{Supp}(P^-)$$

sends the general element $C \in C_-$ onto an element $L(C) = \psi(C)$ of a non-trivial linear system.

Therefore $\Phi_+(C_-) = \Theta$, $\Phi_+(C_+) = J(X)$.

The same argument as in (6.2) (based on the existence of a rational deformation to an element of a subfamily) implies

(6.3.3). Proposition. Let $X$ be a general smooth cubic hypersurface in $\mathbb{P}^4$. Let $C_5^0(X)$ be the 6-dimensional family of twisted cubics on $X$, and let $C_4^0(X)$ be the family of rational quartics on $X$. Then

1. The Abel-Jacobi map $\Phi_3$ for $C_5^0(X)$ sends $C_5^0(X)$ onto a copy of the theta divisor $\Theta$ of $J(X)$.

Moreover, if $C \in C_5^0$ is general, and $z = \Phi_3^o(C)$, then the fiber $(\Phi_3^0)^{-1}(z)$ coincides with the 2-dimensional complete linear system $|\mathcal{O}_S(C)|$ on the smooth cubic surface $S = X \cap \text{Span}(C)$.

2. The Abel-Jacobi map $\Phi_4$ for $C_4^0(X)$ sends the family $C_4^0(X)$ onto the intermediate jacobian $J(X)$.

(6.3.4) Remarks.

1. The degree of the Gauss map for $\Theta$ via the family of twisted cubics.

Let $\gamma$ be rational Gauss map

$$\gamma : \Theta - \text{Sing}(\Theta) \to \mathbb{P}(T_{J(X)})_0^* \cong \mathbb{P}^4^*, \gamma : z \mapsto \mathbb{P}(\text{the tangent space of } \Theta \text{ in } z),$$

and let $C_4^0 \to \mathbb{P}^4^*$ be the (rational) span-map defined, on the general element $C$ of $C_4^0$, by: $C \mapsto \text{Span}(C)$.

It can be seen (see e.g. [Vo], [T], [CG]) that $\text{Span} = \gamma \circ \Phi_4^o$. In particular, we obtain:

The degree of the Gauss map $\gamma_{|_\Theta}$, for the theta divisor of a cubic threefold, is $72 = \text{the number of linear systems of twisted cubics on a smooth cubic surface}$.

In connection with the parameterization of $\Theta$ by $F \times F$ (see (6.3.1)), it is not hard to find a $(1:1)$-correspondence between the set of Schl"afli’s double-six’s on a fixed cubic surface $S$, and the set of linear systems of twisted cubics on $S$. In fact, the double-six’s on $S$ are in $(1:1)$-correspondence with the set of morphisms $\sigma : S \to \mathbb{P}^2$. The linear system $\{C\}(\sigma)$ of twisted cubics which corresponds to such $\sigma$ is the $\sigma$-preimage of the system $|\mathcal{O}_{p^2}(1)|$. 

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(2). The point $\text{Sing}(\Theta)$.

It is well-known (see e.g. [CG], [Tju]) that $\Theta$ has unique singular point $o$, and the base of the tangent cone to $\Theta$ in $o$ is isomorphic to the cubic $X$. The point $o$ coincides with the image of the 5-dimensional subfamily $D \subset C_3^0$ of nodal plane cubics on $X$. Indeed, $D$ coincides with the set of these $C \in C_3^0$ where the rational Gauss map $\gamma_C$ is not regular on $\Phi(C)$ (see (1)), etc.

(6.4) The Fano threefold $X_{16} \subset P^{10}$.

(6.4.1) The conic bundle structure on $X_{16}$.

Let $W \subset P^{10}$ be a general hyperplane section of the Segre 5-fold $P^2 \times P^3 \subset P^{11}$, and let $X \subset W$ be a general smooth divisor of bidegree $(1, 2)$. Let $p : X \to P^2$ and $q : X \to P^3$ be the standard projections. It can be easily verified that the map $p$ defines a standard conic bundle structure on $X$, and the map $q$ is a blow up of a smooth curve $B \subset P^3$ of genus 5, and of degree 7. Denote by $l$ and $h$ be the generators of $\text{Pic}(P^2)$ and $\text{Pic}(P^3)$ (and also their preimages on the various subsets of $P^2 \times P^3$).

The invariants of $X$ can be computed in many ways, e.g.:

- By adjunction $-K_X = (l+h)$ is a hyperplane section of $X$, i.e. $X$ is a Fano threefold with $\text{rank}(\text{Pic}(X)) = 2$.
- The identities $l^3 = 0$, $h^3 = 0$, $l^2 h^3 = 1$, and $[X] = (\text{the class of } X) = (l+h)(l+2h)$, imply that $(-K_X)^3 = 16$. The invariant $h^{2,1}(X) = 5$ can be computed from the tangent bundle sequence for the embedding $X \subset P^2 \times P^3$. In the list of Mori and Mukai (see [MM]), the only Fano 3-fold with these invariants is the blow up of $P^3$ along a smooth curve $B$ of degree 7 and of genus 6, s.t. $B \subset P^3$ is a (non-complete) intersection of cubics.

In fact, the blow up coincides with $q$, and the map $q \circ p^{-1}$, $l \mapsto q(S_l) = q(p^{-1}(l)) \subset P^3$ defines an isomorphism between $P^2^*$ and the non-complete linear system $|O_{P^3}(3-B)|$ of cubics through $B$. The degree $\text{deg}(\Delta)$ of the discriminant curve $\Delta$ of the standard conic bundle $p : X \to P^2$ can be computed by the formula: $-4.K_{P^2} \equiv p_*([-K_X]^2) + \Delta$,

where $\equiv$ is a num.equivalence = a lin.equivalence, on $P^2$ (see [MM]). Therefore $\text{deg}(\Delta) = 12 - l(-K_X)^2[X] = 12 - 2l(l+h)(2l+h) = 5$. The discriminant curve $\Delta$ is smooth, since $5 = h^{2,1}(X) = \text{dim}(J(X)) = \text{dim}(P(\Delta, \Delta)) = g(\Delta) - 1$.

(6.4.2) The families $C_+$ and $C_-$.

Let $x \in P^2$ be general, and let $f_x$ be the conic $f_x = p^{-1}(x)$. Let $l$ and $m$ be two lines such that $l \cap m = x$. Then the curve $q(S_l).q(S_m) = B + q(f_x)$ is a complete intersection of two cubics. The adjunction formulae for the arithmetical genus of a complete intersection imply that the conic $q(f_x)$ intersects $B$ in 6 points. Now, it is not hard to see that the $P^2$-system of $(0, 2)$-conics on $X$ is mapped isomorphically onto a system of 6-secant conics of $B$. Similar computations imply that:

(1). $\tilde{\Delta} = C_{0,1}^0 = (\text{the set of } (0, 1)\text{-lines on } X) \cong (\text{the set of } 3\text{-secant lines of } B)$.

The 5-dimensional family $C_-$ of isolated minimal sections of the system $S[2]$ coincides with the family $C_{0,3}^0(X)$ of rational curves of bidegree $(2, 3)$ on $X$, and $q$ maps the last family on a component $C_{10}^0[7](B)$ of twisted cubics which are 7-secant of $B$;

Similarly, $C_+ = C_{2,3}^0$, and $q$ defines an isomorphism between this family and a component $C_{10}^0[10](B)$ of the family of rational quartic curves which are 10-secant of $B$.

In contrast with the example (6.3), the set $\text{Supp}(\Theta)$ coincides with $C_+ = \psi(C_+)$, i.e.:
(2). The theta divisor $\Theta$ of $J(X)$ is parameterized by the Abel-Jacobi image of the 6-dimensional family of non-isolated minimal sections of the system $S[2]$.

Proof of (2). The verification of the coincidence (2) is reduced to the verification of the fact that the general twisted cubic $C_3$, which is 7-secant of $B$, cannot move in a rational system in $C_0[7](B)$.

Let, in contrary, $\text{Supp}(\Theta) = C_-$, and let $\{C(t) : t \in \mathbb{P}^1\}$ be a general (possibly non-linear) pencil in $C_{0,3}$, which represents a point of $\Theta$. The 1-dimensional family $\{C(t)\}$ has a finite number of degenerations of type (section + section). It is easy to see that all these degenerations have to be of the form $C_{0,1,1} + C_{0,1,2}$, where $C_{0,1,1}$ is a conic on $X$ of bidegree $(1,1)$ (which is mapped onto a bisecant line of $B$), and $C_{0,1,2}$ is a twisted cubic on $X$ of bidegree $(1,2)$ (which is mapped onto a 5-secant conic of $B$). Any of these twisted cubics $C = C_{1,2}$ is a component of the 1-cycle $w(C) = p^{-1}(p(C)) \cap q^{-1}(q(C))$ on $X$.

On the one hand, the computation in the Chow ring $A.(\mathbb{P}^2 \times \mathbb{P}^3)$ implies that the class $[w(C)] \in A.(\mathbb{P}^2 \times \mathbb{P}^3)$, of the cycle $w(C)$, is $4lh^3 + 6l^2h^2$ (in $A.(X)$ this class is $2lh$).

On the other hand, $w(C) = C + C$, where $C$ is a $(1,1)$-conic on $X$; in particular, $C$ is an isolated section of the system $S[1]$, i.e. $C \in F_-$ is an isolated Fano section on $X$. Therefore, $|C| = [C] - 2lh$. By applying this to the general pencil in $C_-$, and by exploring the fact that the Abel-Jacobi maps are factorized through the cycle-class map, we obtain that $\Theta = \Phi_+(C_{0,3})$ is a translate of $\Phi_+((F_- \times F_-))$.

where $\Phi_+ : (C_1, C_2) \mapsto \Phi([C_1 - C_2])$, and $\Phi$ is the quotient Abel-Jacobi map for the corresponding cycle-classes. By using the fact that the Fano family $F_-$ is mapped onto the family of bisecants of $B$, and by the obvious isomorphism $(J(X), \Theta) \cong (J(B), \Theta(B))$, we obtain that $\Theta(B)$ is parameterized by (the linear equivalence classes of the divisors of) the 4-dimensional family $\{x_1 + x_2 - x_3 - x_4 : x_i \in B\}$.

However, this is impossible. Indeed, the initial general nontrivial linear system $\{C(t)\} \subset C_-$ is projected via $p$ to a non trivial system of plane conics $\{p(C(t))\}$. This system forms a rational curve $A$ of degree $d \geq 1$, in the $\mathbb{P}^5$-space of plane conics. Therefore, the curve $A$ intersects the determinantal cubic $\text{Det}$ of all degenerate plane conics in $3d$ points. Therefore, if $x_1, i = 1, \ldots, 4$ is a general 4-tuple of points of $B$, then there exist at least two fourtuples $y_i, (i = 1, \ldots, 4)$ different from $x_1, (i = 1, \ldots, 4)$, and such that $x_1 + x_2 - x_3 - x_4$ and $y_1 + y_2 - y_3 - y_4$ are linearly equivalent on $B$. The last is true (for general $x_i$’s) iff $\{x_1, x_2\} = \{y_1, y_2\}$ and $\{x_3, x_4\} = \{y_3, y_4\}$. This coincidence of non-ordered 2-point sets implies the coincidence of the corresponding degenerate conics – contradiction. Therefore, $\text{Supp}(\Theta) = \psi(C_+)$, q.e.d. Therefore.

(3). The theta divisor $\Theta$ of $J(X)$ is a translate of the Abel-Jacobi image of the 6-dimensional family $C_{0,4}(X)$ of non-isolated minimal sections of the system $S[2]$.

Just as in the proof of the contradiction from above, it follows:

(4). $\Theta$ is a translate of $\Phi_+((F_- \times F_-))$, where $\Phi_+ : (C_1, C_2) \mapsto \Phi([C_1 + C_2])$.

6.4.3 Parameterization of $\text{Sing}(\Theta)$

The well-known from the theory of jacobians of curves set $\text{Sing}(\Theta) \cong \text{Sing}(\Theta(B))$ can be expressed also in terms of the Prym variety $P(\Delta, \Delta) \cong J(B)$.

The first which has to be noted is that the planes of the conics $q(f_x), x \in \mathbb{P}^2$ have a common point $\xi_0 \in B$. Indeed, any such $q(f_x)$ is a 6-secant of the curve $B$, and $\text{deg}(B) = 7$. Therefore the plane $< q(f_x) > \subset \mathbb{P}^3$ intersects $B$ in an additional point, different from the 6 common points of $B$ and $q(f_x)$, and we have to see that this point does not depend on $x$.

Let $l$ be a general line in $\mathbb{P}^2$. Then the map $\phi_l : l \to B, x \mapsto B \cap \text{Span}(q(f_x)) - B \cap q(f_x)$ is regular, and the curve $B$ is non-rational. Therefore the image $\phi_l(l) \subset B$, of the rational curve $l$, is a point. Clearly, this point $\xi_0$ does not depend on the choice of $l$ (since any two lines in $\mathbb{P}^2$ intersect each other).
Now, it is easy to see that $K_B = (\text{the hyperplane section of } B) + \xi_0$.

In particular $B$ is a projection of the canonical curve $B_{can}$ from $\xi_0$; and the the Riemann-Roch theorem implies that the singularities of $\Theta(B)$ correspond to the non-trivial linear systems defined by some family of fourtuples of coplanar points on $B$. It is not hard to see, that in the terms of the attached Prym variety, these singularities of $\Theta$ can be described by the family of degenerations of $C \in C_{2,4}^0$ to quasi-sections of type $C = C_{2,3}^0 + f_x$, where $C_{2,3}^0$ is a rational curve on $X$ of bidegree $(2,2)$. Since all the $(0,2)$ conics on $X$ are rationally equivalent, we obtain:

**The Abel-Jacobi image $\Phi(C_{2,2}^0)(X)$ of the family of rational curves on $X$ of bidegree $(2,2)$ coincides with the non-trivial 1-dimensional component of $\text{Sing}(\Theta)$**.

By the theory of jacobians of curves, this component is naturally isomorphic to the curve $\tilde{\Delta}$ (see e.g. [ACGH, Ch.VI, Ex.F]).

We shall see that similar degenerations describe components of $\text{Sing}(\Theta)$ for the bidegree $(2,2)$ threefold and for the quartic double solid (see (6.5) and (6.6)).

**The bidegree $(2,2)$ threefold $X$** (see [Ve], [I]).

**The two conic bundle structures on $X$.**

Let $W \subset P^8$ be the Segre fourfold $P^2 \times P^2$, and let $X$ be an intersection of $W$ with a general quadric, i.e. $X$ is a bidegree $(2,2)$ threefold.

Let $p$ and $q$ be the two standard projections from $W$ to $P^2$, (resp. from $W$ to $P^2$). Clearly, $p$ and $q$ define conic bundle structures on $X$.

Let $l = [p^*(O(1))]$ and $h = [q^*(O(1))]$ be the generators of $\text{Pic}(W)$ (resp. of $\text{Pic}(X)$).

Call the 1-cycle $C$ on $X$ a bidegree $(m,n)$-cycle, if $C$ has degree $m$ with respect to $l$, and degree $n$ - w.r. to $h$.

**The families $C_+$ and $C_-$ for $p$.**

Fix the projection, say $p$. Then $p : X \to P^2$ is a standard conic bundle, and the discriminant $\Delta$ is a smooth general plane sextic.

Therefore, the jacobian $J(X)$ is a 9-dimensional Prym variety. The two marked families $C_-$ and $C_+$ are the families of isolated and non-isolated minimal sections of the system $S[3]$ of conic bundle surfaces, over the $P^9$-space of plane cubics. Theorem(4.4) tells that the Abel-Jacobi image of one of these two families is a copy of $\Theta$. It is proven in [I] that the family which parameterizes the theta divisor is $C_+$. Since the proof of this coincidence and the description of the families $C_-$ and $C_+$ are purely technical, and do not differ substantially from the study in (6.1) - (6.4), we shall only sketch in brief the results (see [I]):

Let $C_{m,n}^g$ be the family, the general element of which is a smooth connected curve $C \subset X$ of genus $g$ and of bidegree $(m,n)$; e.g., $C_{0,1}^0 \cong \Delta; C_{0,3}^0 = \emptyset$, etc. Then the following takes place:

1. Let $C_+$ be the 10-dimensional family of non-isolated minimal sections, and let $C_-$ be the 9-dimensional family of isolated minimal sections of the system of conic bundle surfaces $S[3]$. Then: $C_+ = C_{3,7}^1$, $C_- = C_{3,5}^1$.

2. Parameterization of $\Theta$:
   (i) $\text{Supp}(\Theta) = C_+$. Therefore $\Phi_+(C_{3,7}^1)$ is a copy of $\Theta$;
   (ii) $\Phi_-(C_{3,6}^1)$ coincides with $J(X)$.

In particular (1), (2), and Theorem (5.8) imply:
passes through the 18 unique effective divisor in $S$ be general. Then the fiber $\psi^{-1}(L)$ is isomorphic to the plane cubic $C_0(L) =$ (the only plane cubic which passes through the 18 points $p(L) \cap \Delta$).

The surjective map $\Phi_+ : C^1_{3,6} \rightarrow J(X)$ is generically finite of degree 2. If $L \in P^-$ is general, and $L$ is the unique effective divisor in $|L |$, then $(\Phi_+)^{-1}(L) \cong \psi^{-1}(L) =$ the two minimal sections of the ruled surface $S(L)$.

(4). Parameterization of $Sing^* \Theta$ via degenerate sections:

Let $C^1_{3,3}$ be the (6-dimensional) family of elliptic curves of bidegree $(3, 3)$ on $X$. Then the Abel-Jacobi image $Z = \Phi(C^1_{3,3})$ is a 3-dimensional component of stable singularities of $\Theta$.

An elliptic sextic $C_{3,3}$ of bidegree $(3, 3)$ can be treated also as a component of a (connected) quasi-section $C_{3,7} = C_{3,3} + \text{two fibers of } p$.

Let $C_{\text{sing}} \subset C^1_{3,7}$ be the set of all such quasi-sections. Since all the fibers of $p$ are rationally equivalent, the subset $Z \subset J(X)$ can be also treated as a translate of the set $\Phi_+(C_{\text{sing}})$.

We shall see in (6.6.5) that similar quasi-sections describe a singular component of $\Theta$ for the nodal quartic double solid. The difference is that the theta divisor of the nodal q.d.s. is described by isolated sections. This last result is mentioned implicitly in [C2], in a different context.

(6.6) The nodal quartic double solid.

(6.6.1) By definition, a quartic double solid is a double covering $\rho : X \rightarrow P^3$ branched along a quartic surface $B \subset P^3$.

The quartic double solid (see [C1], [C2], [W1], [Vo], [T], [De]) is the most popular example of a Fano threefold, together with the cubic and the intersection of three quadrics. In particular, the nodal q.d.s. has a birational structure of a conic bundle (see e.g. (6.6.3)).

Here we shall use the theory from sections 1 – 5, in order to find additional information about the known parameterization (see (A)) of $\Theta$ for the general q.d.s. (see Corollary (6.6.6)(*)).

(6.6.2). Summary of some known results (see [T], [C2], [De], [Vo]).

Let the branch locus $B$ has $0 \leq \delta \leq 7$ simple nodes, which impose independent conditions on the $P^9$-space of quadrics in $P^3$. Let $R \cong B$ be the ramification divisor, and let $\Sigma = $ Sing$(R) = $ Sing$(X) \cong $ Sing$(B)$ be the set of nodes of $X$. Let $\mathcal{R}_\Sigma$ be the $(12 - \delta)$-dimensional family of Reye sextics (sextics of genus 3) on $X$, which pass through all the points of $\Sigma$ (and also the proper preimage of $\mathcal{R}_\Sigma$ on the desingularization $\tilde{X}$ of $X$). Let $(J, \Theta)$ be the $(10 - \delta)$-dimensional p.p. intermediate jacobian of $\tilde{X}$. Then:

(A). The Abel-Jacobi map $\Phi_{\mathcal{R}} : \mathcal{R}_\Sigma \rightarrow J$ sends $\mathcal{R}_\Sigma$ onto a copy of $\Theta$, and the connected components of the general fiber of $\Phi_{\mathcal{R}}$ are isomorphic to the projective space $P^3$.

(B). Let $\delta \leq 5$, and let $\mathcal{E}_\Sigma$ be the $(8 - \delta)$-dimensional family of elliptic quartics on $X$ which pass through all the points of $\Sigma$ (and also the proper preimage of $\mathcal{E}_\Sigma$ on $\tilde{X}$). Then:

The Abel-Jacobi map $\Phi_{\mathcal{E}} : \mathcal{E}_\Sigma \rightarrow J$ sends $\mathcal{E}_\Sigma$ onto a $(5 - \delta)$-dimensional component $Z$ of Sing $\Theta$. Let $\tilde{B} \subset P^{9-\delta} \cong P(\text{Tang.space of } J \text{ in } o)$ be the image of $S$ by the non-complete system $|O(2 - \text{Sing}(B))|$ of quadrics through Sing$(B)$. Then:
The general \( z \in \mathbb{Z} \) is a quadratic singularity of \( \Theta \); the tangent cone \( \text{Cone}_z(\Theta) \) has rank 5, and \( \text{Cone}_z \) passes through \( \hat{B} \). Moreover, any quadric of rank 5 through \( \hat{B} \) arises from some \( z \in \mathbb{Z} \), and the intersection of all such cones \( \text{Cone}_z \) coincides with \( \hat{B} \) (= the Torelli theorem for \( \tilde{X} \)).

Comments.

Let \( X \) be smooth, i.e. \( \delta = 0 \). It follows from \([T]\) and \([C2]\) that the connected components of the general fiber of \( \Phi_\Sigma \) are \( \mathbb{P}^3 \)-spaces, and any such a component coincides with a complete linear system of Reye sextics on \( X \) which lie on a fixed \( K3 \)-surface \( S \in |\mathcal{O}_X(2)| \cong \mathbb{P}(\rho^*H^0(\mathcal{O}_{\mathbb{P}_3}(2)) + R) \) (i.e. \( S \) is a quadratic section of \( X \)).

It is also not hard to see that the family of quadratic sections of \( X \) containing Reye sextics, form a codimension 1 subvariety of the complete linear system \( \mathbb{P}^9 \) of all quadratic sections of \( X \).

(6.6.3). The conic bundle structure on the nodal q.d.s.

Let \( S \) has a simple node \( o \). Denote by \( o \) also the node of \( X \) — “above” \( o \). Let \( \tilde{B} \subset \tilde{\mathbb{P}} \subset \mathbb{P}^8 \) be the image of \( B \subset \mathbb{P}^3 \) by the system of quadrics through \( o \), and let \( \hat{\rho} : \tilde{X} \to \tilde{\mathbb{P}} \) be the induced double covering branched along \( \tilde{B} \). (The threefold \( \tilde{\mathbb{P}} \) is a projection of the Veronese image \( \mathbb{P}^3_3 \subset \mathbb{P}^9 \) of \( \mathbb{P}^3 \), through the image of \( o \). In particular, \( \tilde{\mathbb{P}} \) contains a plane \( \mathbb{P}^2_0 \), and the inverse map \( \sigma : \tilde{\mathbb{P}} \to \mathbb{P}^3 \) is a blow-down of \( \mathbb{P}^2_0 \) to \( o \). The restriction \( \sigma : \tilde{B} \to B \) is a blow-down of a smooth conic \( q_o \subset \tilde{B} \) to the node \( o \).

The threefold \( \tilde{\mathbb{P}} \cong \mathbb{P}(\mathcal{O}(0) + (1)) \) has a natural projection \( p_o \) to \( \mathbb{P}^2 = \{ \text{the lines} \ l \in \mathbb{P}^3 \text{ through } o \} \), and \( \mathbb{P}^2_0 \) is the exceptional section of the projectivized bundle \( \tilde{\mathbb{P}} \). The general fiber \( p^{-1}(l) \) of the composition \( p = p_o \circ \hat{\rho} : \tilde{X} \to \mathbb{P}^2 \) is a smooth conic \( q(l) = p^{-1}(l) \cong \text{(the desingularization of } p^{-1}(l) \text{ in } o) \).

The restriction \( p_o |_{\tilde{B}} : \tilde{B} \to \mathbb{P}^2 \) desingularizes the projection from the quartic \( B \) through the node \( o = \text{Sing}(B) \). Therefore, \( p_o |_{\tilde{B}} \) is a double covering branched along a smooth plane sextic \( \Delta \), and the conic \( q_o \) is totally tangent to \( \Delta \). Clearly, the fiber \( p^{-1}(x) \) is singular for any \( x \in \Delta \), and the natural Abel-Jacobi map \( \hat{\Delta} \to J = J(\tilde{X}) \) induces an isomorphism of p.p.a.v. \( P(\hat{\Delta}, \Delta) \cong J \) (see \([B1]\)).

(6.6.4) The families \( C_+ \) and \( C_- \).

It is not hard to find the families \( C_+ \) and \( C_- \) for \( p \). Since this description does not differ substantially from the general one, we shall state it in a brief:

Since \( \tilde{\mathbb{P}} \subset \mathbb{P}^8 \), the degree map \( \text{deg} : \{ \text{subschemes of } \tilde{\mathbb{P}} \} \to \mathbb{Z} \) is well defined. In particular, \( \text{deg}(\tilde{\mathbb{P}}) = \text{deg}(\mathbb{P}^3_3) - 1 = 7 \).

Let \( Z \subset \tilde{X} \) be a subscheme of \( \tilde{X} \). Define \( \text{deg}(Z) := \text{deg}(\hat{\rho}_*(Z)) \).

Example. Let \( l \subset \mathbb{P}^3 \) be a line through \( o \), let \( x = [l] \in \mathbb{P}^2 \) be the point representing \( l \), and let \( q(x) \) be the “conic” \( q(x) = p^{-1}(x) \). Then \( \text{deg}(p^{-1}(x)) = 2 \). Indeed, \( \hat{\rho}_*(q(x)) = 2l' \) where \( l' = l^{-1}(x) \subset \tilde{\mathbb{P}} \) is the line in \( \mathbb{P}^8 \) which represents the “bundle-fiber” in \( \tilde{\mathbb{P}} \) over \([l]\). Note also that \( l' \) is the proper preimage of the line \( l \subset \mathbb{P}^3 \) under the blow-down \( \sigma_o : \tilde{\mathbb{P}} \to \mathbb{P}^3 \).

Proposition. Let \( S[3] \) be the canonical family of conic bundle surfaces over plane cubics, induced by \( p \), and let \( C_+ \) and \( C_- \) be the 10-dimensional family of non-isolated minimal sections and the 9-dimensional family of isolated minimal sections of the system \( S[3] \). Then:

(1). \( C_+ \) is a component \( C_{10} \) of the family of elliptic curves of degree 10 on \( \tilde{X} \).

(2). \( C_- \) is a component \( C_9 \) of the family of elliptic curves of degree 9 on \( \tilde{X} \).

(3). \( \psi(C_-) = \text{Supp}(\Theta) \). Therefore \( \Phi_-(C_9) \) is a copy of \( \Theta \); \( \Phi_+(C_{10}) = J \).
(4). The general fiber $\Phi^{-1}_- (L)$ is isomorphic to a disjoint union of two lines $P'$ and $P''$; and the elements of $P'$ and $P''$ are fibers of elliptic fibrations on two fixed $K3$-surfaces $S'$ and $S''$ on $\tilde{X}$.

**Proof.** The proof of (1) and (2) is standard.

Proof of (3). The general element $C \in C_-$ lies in a unique $S$ of the 9-dimensional system $| O_\tilde{X}(1) | = | \tilde{\rho}^* O_p(1) |$. Since $S = S(C)$ is a $K3$-surface, and $C$ is an elliptic curve on $S$, the base of the linear system $| C |$ on $S$ is isomorphic to $P^1$.

The map $\psi : | C | \to Supp(\Theta) \cup Supp(P^-)$ sends the elliptic fibration $| C |$ onto a non-trivial linear system $L$. Therefore, $\psi(C) \in Supp(\Theta)$ q.e.d.

Proof of (4). It follows from (3), and from Theorem (5.8), that $\psi$ is generically of degree 2. Moreover, if $L \in \Theta$ and $L \in | L |$ are general, then $\psi^{-1}(L) = \{ C', C'' \}$, where $C'$ and $C''$ are the two minimal sections (zero-sections) of the defined by $L$ ruled surface $S(L)$. It is not hard to see that the general $C'$ (respectively $C''$) lies on a unique $K3$-surface $S'$ (resp. $S''$), and defines there an elliptic fibration $\{ C'(t) \}$ (resp. $\{ C''(t) \}$). Now, (4) follows from the preceding and from Theorem (5.8). q.e.d.

(6.6.5). The component $Z \subset Sing(\Theta)$

**Proposition.** Let $X$ has a simple node $o$. Let $E$ be the 8-dimensional family of elliptic quartics on $X$, and let $E_0$ be the 7-dimensional subfamily of elliptic quartics on $X$ through $o$. Then the proper preimage of $E_0$ on $X$ is a component $C_1^+ \subset C_1^+$ of elliptic curves of degree 7 on $X$. The conic bundle projection $p : X \to P^2$ maps the general $C \in C_1^+$ isomorphically onto a plane cubic $p(C)$. Therefore the Abel-Jacobi map $\Phi : C_1^+ \to J = J(X)$ sends $C_1^+$ onto a 4-dimensional component $Z$ of $Sing(\Theta)$.

**Proof.** Since the proof of (8) can be obtained by a standard degeneration argument from (B), we shall only sketch it in brief.

The elements of $C_1^+$ form a family of isolated minimal sections on surfaces of the system $S[3]$. However, this family is not globally defined on the system $S[3]$ (i.e. the general surface of this system does not admit such a section). The general element $C \in C_1^+$ can be completed (by $#(p(C) \cap \Delta)$ ways) to a connected quasi-section $(C + \text{ fiber}) \in C_1^+$. Since all the fibers of $p$ are rationally equivalent, it follows that $Z$ can be treated as a translate of the set $\Phi_-(C_{\text{sing}})$ where $C_{\text{sing}} = \{ C + \text{ fiber} \} \cap C_1^+$.

It is not hard to see that the points $z$ of the 4-dimensional set $Z := \Phi(C_1^+)$ describe stable singularities $L = L(z)$ of $\Theta$ regarded as a theta divisor on a Prym variety.

Indeed, the general $C \in C_1^+$ lies on a $P^2$-system of $K3$-surfaces $S(u, v)$ of the system $| O_\tilde{X}(1) |$, and any embedding $C \subset S(u, v)$ includes the curve $C$ in an elliptic pencil $C(u, v; t)$ on $S(u, v)$. Now, the map $\psi : \{ C(u, v; t) \} \to Supp(\Theta) \cup Supp(P^-)$ defines a 3-dimensional linear system $\{ L(u, v; t) \}$ on $\Delta$. q.e.d.

(6.6.6). (c). From nodal q.d.s to general q.d.s.

The parameterization of $\Theta$ for the general q.d.s. can be obtained from the parameterization of $\Theta$ for the nodal q.d.s. The consideration offered here is based on the Clemens’s suggestion to prove the parameterization of $\Theta$ of the general q.d.s by using degeneration to a nodal q.d.s. ([see [C2, p.98]]).

Let $X_t$ be a general Lefschetz pencil of quartic double solids, st. $X_0$ is a nodal q.d.s. One can consider $\{ X_t \}$ as a pencil of hyperplane sections in a fourfold $\pi : W \to P^4$, which is a double covering of $P^4$ branched along a smooth quartic 3-fold.

Let $X_0 \to P^2$ be the induced conic bundle, and let $C_-$ be the family of isolated minimal sections on $\tilde{X}_0$.

Define the degree of the effective 1-cycle $C \subset W$ as the degree of the effective 1-cycle $\pi_*(C)$.

It is not hard to see that image on $X_0$ of the family $C_-$ coincides with the 9-dimensional family $C_1^+ [3]$ of curves $C \subset X_0$ s.t. $deg(C) = 6, p_a(C) = 3$, and $C$ has a triple point in the node $o$ of $X_0$. The Abel-Jacobi
image of this family coincides with the Abel-Jacobi image of the 12-dimensional family $R(X_0)$ of Reye sextics on $X_0$, since any Reye sextic on $X_0$ can be included in a rational family (≈ $\mathbb{P}^3$) of Reye sextics containing an element of $C_+^3$ [3] (see e.g. [C2]). Indeed, rationally equivalent 1-cycles have the same Abel-Jacobi image. The rest repeats the arguments from [C2].

However, the obtained description (4) of the fiber of $\Phi$ adds a new information to the known results (A) and (B). More concretely, let $X = X_t$ be a general q.d.s., and let $R$ be the family of Reye sextics on $X$. According to (A), any connected component of the general fiber of $\Phi: R \to \Theta$ is isomorphic to $\mathbb{P}^3$. Now, (4) implies that these components are exactly two.

In just the same way one can see that the image on $X_0$ of family $C_+$ is a subset of the 14-dimensional family $D(X_0)$ of septics of genus 4 on $X_0$. This, in addition, proves the existence of such curves on the general $X_t$. Now, the same argument as above imply that the Abel-Jacobi map sends the corresponding family $D(X_t)$ surjectively onto the intermediate jacobian $J(X_t)$.

(6.6.7). Corollary. Let $X$ be a general smooth quartic double solid, and let $(J(X), \Theta)$ be the principally polarized intermediate jacobian of $X$. Then

(i). The Abel-Jacobi map $\Phi_R$ sends the 12-dimensional family $R$ of Reye sextics (sextics of genus 3) on $X$ onto a copy of $\Theta$. Moreover, the general fiber of $\Phi_R$ has two connected components – each isomorphic to the projective space $\mathbb{P}^3$.

(ii). The Abel-Jacobi map $\Phi_D$ sends the 14-dimensional family $D$ of septics of genus 4 on $X$ surjectively onto the intermediate jacobian $J(X)$.

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