Star-based a Posteriori Error Estimator for Convection Diffusion Problems

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Abstract In this paper, we derive an a posteriori error estimator, for nonconforming finite element approximation of convection-diffusion equation. The a posteriori error estimator is based on the local problems on stars. Finally, we prove the reliability and the efficiency of the estimator without saturation assumption nor comparison with residual estimator.

Keywords A posteriori error estimator, nonconforming finite elements method, convection diffusion equations

AMS subject classification: 65D05, 65D15, 65N50

1 Introduction

A posteriori error estimators provide the basis for adaptive mesh refinement and quantitative error control [1, 8, 4, 5, 12, 19, 14, 10]. One of the most successful estimators was proposed by Bank and Weiser and extended by many authors [2, 3, 7, 13, 16, 20, 21], it is based on the solution of local Neumann problems on elements, which seem to allow for cancelation and thus lead to better results than the residual estimators. The classical proof of equivalence with the energy error require the saturation assumption: this says that the solution can be approximated asymptotically better with quadratic than with linear finite elements. The saturation assumption is shown to be superfluous by Nochetto in [17]. However, removing this assumption requires comparison with residual estimators. More recently, a new a posteriori error estimators on stars was proposed in [15], and the proof of the equivalence with energy error it applies directly without reference to residual estimators.

In this paper, we extended the results of [15] to the case of nonconforming finite elements and the convection diffusion case. A new a posteriori error estimator is introduced based on the solution of a small discrete problem in stars. We prove the reliability and the efficiency of the estimator without saturation assumption nor comparison with residual estimator. We consider the simpler case of nonconforming approximations for convection diffusion problem, and we introduced a technique which allowed us to define a new a posteriori error estimator which are equivalent to the energy error.

2 Setting the problem

We consider here the convection-diffusion problem:

\[ \begin{align*} 
-\varepsilon \Delta u + \beta \cdot \nabla u &= f \quad \text{in } \Omega, \\
\quad u &= 0 \quad \text{on } \Gamma = \partial \Omega 
\end{align*} \]

(P)

In the following we assume that \( \Omega \in \mathbb{R}^2 \) a simply connected polygon domain, \( 0 < \varepsilon \ll 1 \), \( \beta \in (W^{1,\infty}(\Omega))^2 \), such that \( -\frac{1}{2} \text{div } \beta \geq a > 0 \) and \( f \in L^2(\Omega) \). Let \( \mathcal{T}_h \) be a family of conforming shape-regular triangulation of \( \Omega \) by triangular, we denoted by \( E_I \) the set of interior edges and by \( E_f \) the set of all edges included in \( \Gamma \). Let \( V_h \) be the
lowest order non-conforming Crouzeix-Raviart finite element space defined by:

\[
V_h = \{ v_h \in L^2(\Omega) : \forall T \in \mathcal{T}_h, \ v_h|_T \in P_1(T), \ \\
\forall E \in E_I, \ \int_E [v_h]_E d\sigma = 0 \text{ and } \forall E \in E_f, \ \int_E v_h d\sigma = 0 \}.
\]

where \([.]_E\) denoted the jump of the function across \(E\). For each \(T \in \mathcal{T}_h\), we denote by \(P_k(T)\) the polynomial space of degree less than or equal to \(k\).

For all \(T \in \mathcal{T}_h\), we define \(\partial T^{-}\) such the part of the frontier of \(T\) such that \(\beta n_T < 0\) where \(n_T\) stands for the unit outward normal vector to \(T\) on \(\partial T\).

In the sequel, we consider \(u_h^{NC} \in V_h\) be a solution of the stabilized nonconforming approximation problem:

\[
(P_h)^{NC} \begin{cases} \\
\forall v_h \in V_h, \\
\sum_{T \in \mathcal{T}_h} \int_T [\epsilon \nabla u_h^{NC} \cdot \nabla v_h + \beta \nabla u_h^{NC} v_h] dx \\
+ \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T^{-}} \beta n[u_h^{NC}] v_h d\sigma = \int_{\Omega} f v_h dx.
\end{cases}
\]

2.0.1 The a posteriori error estimator

For the a posteriori error analysis of the considered approximation, we need to define some local spaces and problems. We denoted by \(\{x_i\}_{i \in \mathcal{N}}\) the set of all nodes of the triangulation \(\mathcal{T}_h\). For each \(i \in \mathcal{N}\), \(\phi_i\) denoted the canonical continuous piecewise linear basis function corresponding to \(x_i\). The star \(\omega_i\) is the interior relative to \(\Omega\) of the support of \(\phi_i\), and \(h_i\) is the maximal size of the elements forming \(\omega_i\). Finally, \(\Gamma_i\) will denote the union of the sides touching \(x_i\) that are contained in \(\Omega\), and \(\overline{\Gamma}_i\) will denote the union of the sides touching \(x_i\) that are contained in \(\overline{\Omega}\).

For each star \(\omega_i\), \(i \in \mathcal{N}\), if \(x_i\) we introduce the space \(V(\omega_i)\) defined by

\[
V(\omega_i) = \{ v \in H^{1}_{loc}(\omega_i) : \int_{\omega_i} v \phi_i dx = 0 \},
\]

if \(x_i\) is an interior node, and

\[
V(\omega_i) = \{ v \in H^{1}_{loc}(\omega_i) : v = 0 \text{ on } \partial \omega_i \cap \Gamma \},
\]

if \(x_i\) is a boundary node.

We have the following result of ([15])

**Proposition 1** There exists a constant \(C\), only depending on the minimum angle of the triangulation but independent of the star being considered, such that:

\[
\forall v \in V(\omega_i) \quad \|v\|_{0, \omega_i} \leq Ch_i (\int_{\omega_i} |\nabla v|^2 \phi_i dx)^{1/2}. \tag{1}
\]

We define the finite dimensional local spaces \(P^2(\omega_i)\) as follows,

**Definition 2** For \(i \in \mathcal{N}\), let \(P^2(\omega_i)\) denote the space of continuous piecewise quadratic functions on star \(\omega_i\) that vanish on \(\partial \omega_i\). The spaces \(P^2(\omega_i)\) is defined by \(P^2(\omega_i) = P^2(\omega_i) \cap V(\omega_i)\).

In the following we consider the energy norm:

\[
\|u\|^2_{\varepsilon, \omega_i} = \varepsilon \|\nabla u\|^2_{0, \omega_i} + \|u\|^2_{0, \omega_i}.
\]

Let \(u_h^{NC} \in V_h\) be fixed and we denoted by \(\nabla_h u_h\) the vector belonging to \((L^2(\Omega))^2\) defined by

\[
\forall T \in \mathcal{T}_h; \quad \nabla_h v_h = \nabla v_h \text{ on } T.
\]
For each \( i \in \mathcal{N} \), we consider the local problems:

\[(P_1)_i \left\{ \begin{array}{l}
\text{Find } \eta_i \in \mathcal{P}_0^2(\omega_i) \text{ such that } \forall u_i \in \mathcal{P}_0^2(\omega_i) \\
\int_{\omega_i} (\varepsilon \nabla \eta_i, \nabla u_i) \phi_i dx = \int_{\omega_i} (\varepsilon \nabla h_{\nu} u^N_{\nu} \cdot \nabla \mu_i) \phi_i dx \\
+ \frac{1}{2} \sum_{T \in T_h} \int_{\partial T} \beta \cdot [u^N_{\nu}]_{\mu_i} \phi_i ds - \int_{\omega_i} f \phi_i dx
\end{array} \right.\]

and

\[(P_2)_i \left\{ \begin{array}{l}
\text{Find } \alpha_i \in \mathcal{P}_0^2(\omega_i) \text{ such that } \forall u_i \in \mathcal{P}_0^2(\omega_i) \\
\int_{\omega_i} (\varepsilon \nabla \alpha_i, \nabla u_i) \phi_i dx = \int_{\omega_i} \varepsilon \nabla h_{\nu} u^N_{\nu} \cdot \text{Curl}(\mu_i \phi_i) dx.
\end{array} \right.\]

Using Lax-Milgram theorem, we prove that the discrete problems have unique solution. The problems \((P_1)_i\), estimate the approximation error, but the problems \((P_2)_i\), estimate the consistency error of the used method.

Finally we set:

\[\forall i \in \mathcal{N}, \quad E^1_{i,i}(u^N_{\nu}) = \int_{\omega_i} \varepsilon |\nabla \eta_i|^2 \phi_i dx,\]

and

\[\forall i \in \mathcal{N}, \quad E^2_{i,i}(u^N_{\nu}) = \int_{\omega_i} \varepsilon |\nabla \alpha_i|^2 \phi_i dx.\]

### 2.0.2 Upper bound of the error

In this section we prove the one of main results of this paper. First, we prove the upper bound of the error without oscillation. As in ([6], [9], [11]). Recall that [18]:

**Lemma 3** *(Discrete Poincar and Friedrichs inequalities).*

There is a positives constants \( C \) depending only on the minimum angle of \( T_h \) the \( \Omega \) such that:

\[\forall v \in H^1_0(\Omega) + V_h, \quad \|v\|^2_{0,\Omega} \leq C(\sum_{T \in T_h} \|\nabla v\|^2_{0,T}).\]  

(2)

We have the following global upper bound of the error:

**Theorem 4** Let \( u^N_{\nu} \in V_h \) such that \((P_h)^N_{\nu}\) holds. There is a positive constant \( C_1 \) depending only on the minimum angle of \( T_h \) such that

\[\|u - u^N_{\nu}\|_{\epsilon, \Omega} \leq C_1 \left( \sum_{i \in \mathcal{N}} (E^1_{i,i}(u^N_{\nu}) + E^2_{i,i}(u^N_{\nu}))^{\frac{1}{2}} \right)
\]

\[+ \|\beta\|_{1,\infty} h_i \left( \sum_{i \in \mathcal{N}} E^2_{i,i}(u^N_{\nu}) \right)^{\frac{1}{2}},\]

\[+ \left( \|\beta\|_{0,\infty} + \|\beta\|_{H^0(\partial \omega_i)} \right) h_i \|u^N_{\nu}\|_{0,\omega_i} \]

\[+ \text{osc}(f).\]

(3)

where \( \text{osc}(f) \) is the data oscillations defined by: \( \text{osc}(f) = \left( \sum_{i \in \mathcal{N}} \alpha_i^2 \|f - f_i\|^{\frac{1}{2}}_{\infty,\omega_i} \right)^{\frac{1}{2}}.\)

where \( f_i = \frac{\int_{\omega_i} f \phi_i dx}{\int_{\omega_i} \phi_i dx} \) and \( \alpha_i = \min(1, \frac{h_i}{\sqrt{\omega_i}}).\)

**Proof.** Remark that using Helmholtz-decomposition, we have

\[\nabla h^N_{\nu} - \nabla u = \nabla w + \text{Curl} \zeta,\]

(4)

with \( w \in H^1_0(\Omega), \zeta \in H^1(\Omega) \) and \( \int_{\Omega} \nabla w \cdot \text{Curl} \zeta dx = 0.\)

Let us remark also that the orthogonality implies the following error decomposition:

\[\varepsilon \|\nabla h^N_{\nu} - \nabla u\|^2_{0,\Omega} = \varepsilon \|\nabla w\|^2_{0,\Omega} + \varepsilon \|\text{Curl} \zeta\|^2_{0,\Omega}.\]

and the following equalities:

\[\varepsilon \|\nabla w\|^2_{0,\Omega} = \int_{\Omega} \varepsilon (\nabla h^N_{\nu} - \nabla u) \cdot \nabla w dx.\]  

(5)
Lemma 5 For each node $i \in N$ there exists an unique operator $\Pi_i : V(\omega_i) \rightarrow P_0^d(\omega_i)$, such that for any $v \in V(\omega_i)$ the following conditions hold:

1. For all edge $E \subset \Gamma_i$, $\int_E (v - \Pi_i v) \phi_i d\sigma = 0$.
2. $\int_{\omega_i} (v - \Pi_i v) \phi_i dx = 0$
3. $\epsilon \int_{\omega_i} |\nabla v|^2 \phi_i \leq C \int_{\omega_i} |\nabla \phi_i|^2$
4. $(\sum_{i \in N} \alpha_i^{-2} d E_{\omega_i}) \leq C \|v\|_{\epsilon, \Omega}$

where the constant $C$ depends only on the minimum angles of $T_h$.

The Lemma 5, is an adaptation of arguments given in [15], and so the proof will be skipped.

Lemma 6 For each node $i \in N$, $\Pi_i$ defined in the 5, functions $v \in V(\omega_i)$, $\zeta \in V(\omega_i)$ and $u_h^{NC} \in V_h$. We have

$$\int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla((\Pi_i v) \phi_i) dx = \int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla(v \phi_i) dx,$$

and

$$\int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla((\Pi_i \zeta) \phi_i) dx = \int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla(\zeta \phi_i) dx.$$

Proof. If we denoted by $\partial u_h^{NC}/\partial n_E \in P_0(E)$ the jump of the normal derivative across the edge $E$. By Green formula and by lemma 5, we have

$$\int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla((\Pi_i v) \phi_i) dx$$

$$= \sum_{E \subset \omega_i} \int_E \epsilon \nabla u_h^{NC} \cdot \nabla((\Pi_i v) \phi_i) dx$$

$$= \sum_{E \subset \omega_i} \int_E \epsilon \nabla u_h^{NC} \cdot \nabla(v \phi_i) dx = \int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla(v \phi_i) dx.$$

To prove the second equality, if we denoted by $\partial u_h^{NC}/\partial T_E \in P_0(E)$ the jump of the tangential derivative across the edge $E$. We have by Green formula and lemma 5

$$\int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla((\Pi_i \zeta) \phi_i) dx = \sum_{E \subset \omega_i} \int_E \epsilon \nabla u_h^{NC} \cdot \nabla((\Pi_i \zeta) \phi_i) dx$$

$$= \sum_{E \subset \omega_i} \int_E \epsilon \nabla u_h^{NC} \cdot \nabla(\zeta \phi_i) dx = \int_{\omega_i} \epsilon \nabla u_h^{NC} \cdot \nabla(\zeta \phi_i) dx.$$

The following Lemma give an estimate of expression (5):

Lemma 7 For $w \in H^1_0(\Omega)$ defined in (4), there exists positives constants $C^*$, depending only on the minimum angle of $T_h$ such that

$$\|w\|_{\epsilon, \Omega} \leq C^*[\left(\sum_{i \in N} (E_{1,i}(u_h^{NC}))\right)^{\frac{1}{2}} + \|\beta\|_{1, \infty} h_i \left(\sum_{i \in N} E_{2,i}(u_h^{NC})\right)^{\frac{1}{2}},$$

$$+ \sum_{i \in N} (h_i \sum_{T \subset \omega_i} \|u_h^{NC}\|_{L^2(T, \zeta)}) + \|\beta\|_{0, \infty}$$

$$+ \|\beta\|_{H(\text{div}, \omega_i)} h_i \|u_h^{NC}\|_{0, \omega_i},$$

$$+ \text{osc}(f).$$
Proof. For $w \in H^1_0(\Omega)$ defined in (4), we set

$$w_h = \sum_{i \in N} w_i \phi_i,$$

where $w_i = \frac{\int_{\Omega_i} w \phi_i \, dx}{\int_{\Omega_i} \phi_i \, dx}$ for interior nodes, and $w_i = 0$ otherwise.

By adapting standard arguments used in the analysis of finite element approximation, and by using the propriety $-\frac{1}{2} \text{div} \beta \geq a > 0$, we have:

$$a \left\| w \right\|^2_{\varepsilon, \Omega} \leq \int_{\Omega} \varepsilon \nabla w . \nabla w + \int_{\Omega} \beta \nabla w \varepsilon \varepsilon dx,$$

where: $\varepsilon \left\| \nabla w \right\|^2_{\Omega} = \int_{\Omega} \varepsilon (\nabla \phi_h^{NC} - \nabla u) \nabla w dx$ and $\int_{\Omega} \nabla w . \text{Curl} \, \zeta dx = 0$.

This gives,

$$\int_{\Omega} \varepsilon \nabla w . \nabla w + \int_{\Omega} \beta \nabla w w = \int_{\Omega} \varepsilon \nabla w . (\nabla \phi_h^{NC} - \nabla u) dx + \int_{\Omega} \beta \nabla w \phi_h^{NC} dx$$

$$- \int_{\Omega} \beta \varepsilon \nabla w . \nabla w dx - \int_{\Omega} \varepsilon \varepsilon \nabla w . \text{Curl} \, \zeta dx$$

$$= \int_{\Omega} \varepsilon \nabla h \phi_h^{NC} . \nabla w dx + \int_{\Omega} \beta \nabla h u_h^{NC} \varepsilon \varepsilon dx$$

$$- \int_{\Omega} \varepsilon \nabla u . \nabla w dx - \int_{\Omega} \beta \nabla u \varepsilon \varepsilon \varepsilon w dx$$

$$- \int_{\Omega} \beta \varepsilon \nabla w . \text{Curl} \, \zeta dx.$$

Since $w_h \in V_h \cap H^1_0(\Omega)$, $a(w, w_h) = 0$, we have: $a(w, w) = a(w, w - w_h)$. Then

$$a \left\| w \right\|^2_{\varepsilon, \Omega} \leq \int_{\Omega} \varepsilon \nabla h \phi_h^{NC} . \nabla (w - w_h) dx + \int_{\Omega} \beta \nabla h u_h^{NC} (w - w_h) dx$$

$$- \int_{\Omega} \varepsilon \nabla u . \nabla (w - w_h) dx - \int_{\Omega} \beta \nabla u (w - w_h) dx - \int_{\Omega} \beta (w - w_h) \text{Curl} \, \zeta dx.$$
On the other hand, since both of \((w - w_i)\) \(\in V(\omega_i)\), adding and removing same quantities in the two last terms give:

\[
    a||w||^2_{\Omega, \epsilon} \\
    \leq \sum_{i \in N} \int_{\omega_i} \varepsilon \nabla \eta_i \cdot \nabla \Pi_i (w - w_i) \phi_i dx \\
    + \int \beta \nabla u_h^{NC} \Pi_i (w - w_i) \phi_i dx \\
    + \sum_{i \in N} \left( \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} [\beta u_i^{NC}] (\Pi_i (w - w_i)) \phi_i d\sigma \right) \\
    - \sum_{i \in N} \left( \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} [\beta u_i^{NC}] (\Pi_i (w - w_i)) \phi_i d\sigma \right) \\
    + \sum_{i \in N} \int_{\omega_i} \beta \nabla u_h^{NC} (w - w_i - \Pi_i (w - w_i)) \phi_i dx \\
    - \sum_{i \in N} \int_{\omega_i} (w - w_i - \Pi_i (w - w_i)) \phi_i dx \\
    + \left[ \int_{\omega_i} \beta (w - w_i) \phi_i \text{Curl} \zeta dx \right].
\]

Using the definition of local problems \((P_1)\),

\[
    a||w||^2_{\Omega, \epsilon} \\
    \leq \sum_{i \in N} \int_{\omega_i} \varepsilon \nabla \eta_i \cdot \nabla \Pi_i (w - w_i) \phi_i dx \\
    - \sum_{i \in N} \int_{\omega_i} f (w - w_i - \Pi_i (w - w_i)) \phi_i dx \\
    - \sum_{i \in N} \left( \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} [\beta u_i^{NC}] (\Pi_i (w - w_i)) \phi_i d\sigma \right) \\
    + \sum_{i \in N} \int_{\omega_i} \beta \nabla u_h^{NC} (w - w_i - \Pi_i (w - w_i)) \phi_i dx \\
    - \sum_{i \in N} \int_{\omega_i} \beta (w - w_i) \phi_i \text{Curl} \zeta dx.
\]

We now process successively with each term of the right-hand side. On one hand, using Cauchy-Schwarz and item 3 of lemma 5 we have:

\[
    \int_{\omega_i} \varepsilon \nabla \eta_i \cdot \nabla \Pi_i (w - w_i) \phi_i dx \\
    \leq \left( \sum_{i \in N} \int_{\omega_i} \varepsilon |\nabla \eta_i|^2 \phi_i dx \right)^{\frac{1}{2}} \left( \sum_{i \in N} \int_{\omega_i} \varepsilon |\nabla \Pi_i (w - w_i)|^2 \phi_i dx \right)^{\frac{1}{2}} \\
    \leq C \left( \sum_{i \in N} E_{1,i}^2 (u_h^{NC}) \right)^{\frac{1}{2}} \left( \sum_{i \in N} \int_{\omega_i} \varepsilon |\nabla (w - w_i)|^2 \phi_i dx \right)^{\frac{1}{2}} \\
    \leq C \left( \sum_{i \in N} E_{1,i}^2 (u_h^{NC}) \right)^{\frac{1}{2}} ||w||_{\epsilon, \Omega, \Omega}.
\]

On the other hand, since both of \((w - w_i)\) and \(\Pi_i (w - w_i)\) belong to \(V(\omega_i)\), using definition of \(V(\omega_i)\) and coefficients \(f_i\) give:

\[
    \sum_{i \in N} \left[ \int_{\omega_i} f (w - w_i - \Pi_i (w - w_i)) \phi_i dx \right] \\
    = \sum_{i \in N} \left[ \int_{\omega_i} (f - f_i)(w - w_i - \Pi_i (w - w_i)) \phi_i dx \right]
\]
Using Cauchy-Schwarz, the proposition 1, the item 4 of lemma 5 and once more $\sum_{i \in N} \phi_i = 1$ we get
\[
\sum_{i \in N} \left[ \int_{\omega_i} f(w - w_i - \Pi_i(w - w_i))\phi_i dx \right]
\leq osc(f) \sum_{i \in N} \alpha_i^{-1} \|(w - w_i - \Pi_i(w - w_i))(\phi_i)^\frac{1}{2}\|_{0, \omega_i}^2 \frac{\|\phi_i\|_{0, \omega_i}^2}{\alpha_i},
\]
\[
\leq C \ osc(f) \|w\|_{r, \Omega}.
\]
C is a generic constant only depending on the minimum angle of triangulation.

We note:
\[
A = \sum_{i \in N} \left[ \int_{\omega_i} \beta_\nu h u_h^{NC}(w - w_i - \Pi_i(w - w_i))\phi_i dx \right]
\]
\[
- \left[ \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} |\beta_n||u_h^{NC}|(\Pi_i(w - w_i))\phi_i d\sigma \right]
\]

Using the Green formula we have:
\[
A = \sum_{i \in N} \left[ \int_{\omega_i} \beta_\nu h u_h^{NC}(w - w_i - \Pi_i(w - w_i))\phi_i dx \right]
\]
\[
- \left[ \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} |\beta_n||u_h^{NC}|(\Pi_i(w - w_i))\phi_i d\sigma \right]
\]
\[
\leq \sum_{i \in N} \left[ \int_{\omega_i} \beta_\nu h u_h^{NC}(w - w_i - \Pi_i(w - w_i))\phi_i dx \right]
\]
\[
+ \sum_{i \in N} \left[ \int_{\omega_i} \text{div} \beta_\nu h u_h^{NC}(w - w_i - \Pi_i(w - w_i))\phi_i dx \right]
\]
\[
- \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} |\beta_n||u_h^{NC}|(\Pi_i(w - w_i))\phi_i d\sigma \right].
\]

Using Cauchy-Schwarz and the proposition 1 we have:
\[
A = \sum_{i \in N} \left[ \int_{\omega_i} \beta_\nu h u_h^{NC}(w - w_i - \Pi_i(w - w_i))\phi_i dx \right]
\]
\[
- \left[ \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T} |\beta_n||u_h^{NC}|(\Pi_i(w - w_i))\phi_i \right]
\]
\[
\leq C \sum_{i \in N} \left( h_i \sum_{T \subset \omega_i} \|[u_h^{NC}]\|_{L^2(\partial T)} + (\|\beta\|_{0, \infty}
\]
\[
+ \|\beta\|_{H(\text{div}, \omega_i)} h_i \|u_h^{NC}\|_{0, \omega_i} + osc(f) \right).
\]

Finally:
\[
\sum_{i \in N} \left[ \int_{\omega_i} \beta(w - w_i)\text{Curl} \zeta dx \right]
\]
\[
\leq \|\beta\|_{1, \infty} \|[\text{Curl} \zeta]_{0, \omega_i}\|(w - w_i)\|_{0, \omega_i}
\]
\[
\leq C \|\beta\|_{1, \infty} \left( \sum_{i \in N} E^2_{(2,s)}(u_h^{NC}) \right)^\frac{1}{2} \|w\|_{r, \omega_i}.
\]

Summing up the different contributions in the estimate of $\|w\|_{r, \omega_i}$ and using the continuity of $a(\ldots)$ yield the result.

We have also the following result giving an estimation of expression (6):

**Lemma 8** For $\zeta \in H^1(\Omega)$ defined in (4), there exists a positive constant $C$, depending only on the minimum angle of $\mathcal{T}_h$ such that
\[
\varepsilon^\frac{1}{2} \|[\text{Curl} \zeta]\|_{0, \Omega} \leq C \left( \sum_{i \in N} E^2_{(2,s)}(u_h^{NC}) \right)^\frac{1}{4},
\]
Proof. Let \( \zeta \in H^1(\Omega) \) defined in (4), we note
\[
\zeta_h = \sum_{i \in N} \zeta_i \phi_i, \text{ where } \zeta_i = \frac{\int_{\omega_i} \zeta \phi_i \, dx}{\int_{\omega_i} \phi_i \, dx} \text{ for all nodes.}
\]
First, using (6) and the fact that
\[
\int_{\Omega} \varepsilon \nabla \omega \cdot \text{Curl} \zeta \, dx = 0.
\]
Which easy to verify that:
\[
\|\text{Curl} \zeta\|_{0, \Omega}^2 = \int_{\Omega} \varepsilon (\nabla_h u_h^{NC} - \nabla u) \cdot \text{Curl} \zeta \, dx
\]
and hence:
\[
\|\text{Curl} \zeta\|_{0, \Omega}^2 = \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} (\zeta - \zeta_h) \, dx + \int_{\Omega} \varepsilon \nabla_h u_h^{NC} \cdot \text{Curl} \zeta_h \, dx.
\]
On one hand, since: \( \text{Curl} \zeta_h \in H(\text{div}; \Omega), \) \( \text{div} \left( \text{Curl} \zeta_h \right) = 0, \) and \( \forall T \in T_h, \) \( \left( \text{Curl} \zeta_h \right)_T \) belonging the lowest Raviart-Thomas space \( RT_0(T) = (P_0(T))^2 + \mathbb{P}_0(T) \) and \( u_h^{NC} \in V_h, \) by Green formula we have
\[
\int_{\Omega} \nabla_h u_h^{NC} \cdot \text{Curl} \zeta_h \, dx = 0.
\]
On the other hand, since \( (\zeta - \zeta_i) \in V(\omega_i), \) using Lemma 5, and the definition of local problem \( (P_2)_i \) we have, for all \( i \in N : \)
\[
\sum_{i \in N} \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \text{Curl} ((\zeta - \zeta_i) \phi_i) \, dx
\]
\[
= \sum_{i \in N} \int_{\omega_i} \varepsilon \nabla_h u_h^{NC} \text{Curl} (\Pi_i(\zeta - \zeta_i) \phi_i) \, dx
\]
\[
= \sum_{i \in N} \int_{\omega_i} \varepsilon (\nabla \alpha_i \cdot \nabla \Pi_i(\zeta - \zeta_i)) \phi_i \, dx.
\]
Using the last equalities, \( \sum_{i \in N} \phi_i = 1 \) and the equality \( \|\text{Curl} \zeta\|_{0, \Omega} = \|\nabla \zeta\|_{0, \Omega} \), we obtain
\[
\begin{align*}
\varepsilon \|\text{Curl} \zeta\|_{0, \Omega}^2 &= \sum_{i \in N} \int_{\omega_i} \varepsilon (\nabla_h u_h^{NC}) \cdot \text{Curl} (\zeta - \zeta_h) \phi_i \, dx \\
&= \sum_{i \in N} \int_{\omega_i} \varepsilon (\nabla_h u_h^{NC}) \cdot \text{Curl} (\Pi_i(\zeta - \zeta_h)) \phi_i \, dx \\
&= \sum_{i \in N} \int_{\omega_i} \varepsilon (\nabla \alpha_i \cdot \nabla (\Pi_i(\zeta - \zeta_h))) \phi_i \, dx \\
&\leq C \left( \sum_{i \in N} E_{(2,1)}^2(u_h^{NC}) \right)^{\frac{1}{2}} \left( \sum_{i \in N} \int_{\omega_i} (\nabla \zeta_i^2 \phi_i) \, dx \right)^{\frac{1}{2}} \\
&\leq C \left( \sum_{i \in N} E_{(2,1)}^2(u_h^{NC}) \right)^{\frac{1}{2}} \varepsilon \|\text{Curl} \zeta\|_{0, \Omega}.
\end{align*}
\]
Which completes proof of the lemma.

The combination of the two lemmas 7 and 8, gives the result of the upper bound of the error of the theorem 4.

2.0.3 Lower bound of the error

In this section we prove a lower bound of the error without oscillation.

**Theorem 9** Let \( u_h^{NC} \in V_h, \) there exists a positives constants \( C_1 \) and \( C_2, \) depending on the minimum angle of the triangulation such that, for any \( i \in N, \)
\[
E_{1,i}(u_h^{NC}) \leq C_1 (1 + \|\beta\|_{1,\infty, h_i}) \|u - u_h^{NC}\|_{\varepsilon, \omega_i},
\]

where \( \beta \) depends on the lowest order Raviart-Thomas space.
and

\[ E_{2,i}(u_h^{NC}) \leq C_2 \| u - u_h^{NC} \|_{\varepsilon,w_i}. \]

**Proof.** Let \( i \in N \). We have

\[
E_{1,i}(u_h^{NC}) = \int_{\omega_i} (\varepsilon \nabla_h u_h^{NC} \cdot \nabla \eta_i) \phi_i dx + \int_{\omega_i} (\beta \nabla_h u_h^{NC} \cdot \eta_i) \phi_i dx - \int_{\omega_i} f \eta_i \phi_i dx + \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T_-} |\beta n| [u_h^{NC}]_{\eta_i} \phi_i d\sigma.
\]

Then

\[
E_{1,i}^2(u_h^{NC}) = \int_{\omega_i} (\varepsilon \nabla_h u_h^{NC} \cdot \nabla \eta_i) \phi_i dx + \int_{\omega_i} (\beta \nabla_h u_h^{NC} \cdot \eta_i) \phi_i dx - \int_{\omega_i} f \eta_i \phi_i dx + \frac{1}{2} \sum_{T \subset \omega_i} \int_{\partial T_-} |\beta n| [u_h^{NC}]_{\eta_i} \phi_i d\sigma.
\]

Using Cauchy-Schwarz theorem imply that:

\[
E_{2,i}^2(u_h^{NC}) \leq \varepsilon^{\frac{1}{2}} \| \nabla u - \nabla_h u_h^{NC} \|_{0,\omega_i} (\varepsilon \int_{\omega_i} |\nabla \eta_i|^2 \phi_i dx)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \| \nabla u - \nabla_h u_h^{NC} \|_{0,\omega_i} \varepsilon^{\frac{1}{2}} \| \eta_i \|_{0,\omega_i} \| \phi_i \|_{1,\omega_i} + \| \beta \|_{0,\omega_i} \| \nabla u - \nabla_h u_h^{NC} \|_{0,\omega_i} (\int_{\omega_i} |\eta_i|^2 \phi_i dx)^{\frac{1}{2}} + C \| [u_h^{NC}] \|_{L^2(\partial T)} \| \eta_i \|_{0,\omega_i}.
\]

Or:

\[
\varepsilon^{\frac{1}{2}} \| \eta_i \|_{0,\omega_i} \leq C \varepsilon \int_{\omega_i} (|\nabla \eta_i|^2 \phi_i dx)^{\frac{1}{2}}.
\]

Then

\[
\varepsilon^{\frac{1}{2}} \| \eta_i \|_{0,\omega_i} \leq C \varepsilon \frac{1}{h_i} E_{1,i}(u_h^{NC}).
\]

and since \( \eta_i \in V(\omega_i) \), we have \( |\phi_i|_{1,\omega_i} \leq \frac{C}{h_i} \).

And:

\[
\sum_{T \subset T_h} \int_{\partial T_-} |\beta n|[u_h^{NC}]_{\eta_i} \phi_i d\sigma \\
\leq C \| [u_h^{NC}] \|_{L^2(\partial T)} \| \eta_i \|_{0,\omega_i} \\
\leq C \varepsilon^{\frac{1}{2}} \alpha^{\frac{1}{2}} \| [u_h^{NC}] \|_{L^2(\partial T)} E_{1,i}(u_h^{NC}) \\
\leq C \| u - u_h^{NC} \|_{\varepsilon,w_i} E_{1,i}(u_h^{NC}).
\]
Summing up the different contributions in the estimate of $E_{1,i}(u_{h}^{NC})$ we get:

$$E_{1,i}(u_{h}^{NC}) \leq C(1 + \|\beta\|_{1,\infty}h_i)\|u - u_{h}^{NC}\|_{\varepsilon,\omega_i}.$$ 

To prove the second inequality, let us remark that, using Green formula, we have:

$$\forall \mu_i \in P_2^2(\omega_i): \int_{\omega_i} \varepsilon \nabla u.Curl(\mu_i \phi_i)dx = 0.$$ 

Then using Cauchy-Schwartz inequality we deduce

$$E_{2,i}(u_{h}^{NC}) = \int_{\omega_i} \varepsilon \nabla_h u_{h}^{NC}.Curl(\alpha_i \phi_i)dx$$

$$= \int_{\omega_i} \varepsilon(\nabla_h u_{h}^{NC} - \nabla u).Curl(\alpha_i \phi_i)dx$$

$$\leq \varepsilon^{\frac{1}{2}}\|\nabla u - \nabla u_{h}^{NC}\|_{0,\omega_i} \varepsilon^{\frac{1}{2}}\|Curl(\alpha_i \phi_i)\|_{0,\omega_i}.$$ 

Where

$$\varepsilon^{\frac{1}{2}}\|Curl(\alpha_i \phi_i)\|_{0,\omega_i} \leq CE_{(2,i)}(u_{h}^{NC}).$$

Then

$$E_{2,i}(u_{h}^{NC}) \leq C\varepsilon^{\frac{1}{2}}\|\nabla u - \nabla u_{h}^{NC}\|_{0,\omega_i} \leq C\|u - u_{h}^{NC}\|_{\varepsilon,\omega_i}.$$ 

3 \hspace{1em} Mesh adaptation procedure and Numerical results

Example 1. We now consider, on the same domain $\Omega = ]0,1[ \times ]0,1[$, the convection diffusion reaction problem:

$$-\varepsilon \Delta u + \beta \nabla u = f.$$ 

with $u = 0$ on $\partial \Omega$ such that $\beta = [3, 2]^t$, $\varepsilon = 0.01$, and $f = 1$.

This solution, for small $\varepsilon > 0$, exhibits boundary layers near the top ($y = 1$) and right ($x = 1$) boundaries. The adaptive finite element method correctly refines in these layers, yielding accurate solutions with a small number of unknowns (relative to uniform refinement). The meshes and the contour maps are omitted for brevity reasons (see Fig 1 and Fig 2).

The interesting point of this problem concerns the exactness of estimators as $\varepsilon$ goes to zero.

Figure 1. Adaptive mesh refinement using the error indicator.
**Example 2.** We now consider, on the same domain $\Omega = [0, 1] \times [0, 1]$, the convection diffusion reaction problem:

$$-\epsilon \Delta u + \beta \nabla u = f.$$  

With the source term $f$ given by the exact solution,

$$u = xy(x - 1)(y - 1)e^{-100(x-0.5)^2-100(y-0.5)^2}$$

which presents sharp curvature in the vicinity of point $(0.5, 0.5)$, and we perform a nonconforming finite element discretization on it. Successive iterations of adaptive mesh are represented in Figure 4. Computed and Exact solution are given in Figure 5, where the scaling of the height is the same for both pictures.

**Remarks.**

1. We can write a general framework regroups the conforming and nonconforming approximations of the convection diffusion problem, just write the approximate problem as:

$$\begin{cases} 
\text{Find } u_h \in W_h, \text{ such that: } & \forall v_h \in W_h, \\
\sum_{T \in T_h} \int_T \left[ \epsilon \nabla u_h \cdot \nabla v_h + \beta \nabla u_h \cdot \nabla v_h \right] dx + \frac{1}{2} \int_{\partial T} \beta n [u_h] v_h d\sigma = \int_\Omega f v_h dx. 
\end{cases}$$

with $W_h = V_h \cap H^1_0(\Omega)$ in the conforming case and $W_h = V_h$ in the non conforming case. The same arguments can be used to derive an a posteriori error estimate on stars with similar properties.

2. The three dimensional case can be made with some regularity assumptions of the domain $\Omega$, and by adapting the used technics in the two dimensional case.
3.1 Conclusion

In this work we analyzed an a posteriori error estimator for nonconforming convection diffusion problem, with the Helmholtz-Decomposition technics. These estimators are efficient and robust to respect to physical parameters of the problem.

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