Closed timelike curves and the second law of thermodynamics

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One out of many emerging implications from solutions of Einstein's general relativity equations are closed timelike curves (CTCs), which are trajectories through spacetime that loop back on themselves in the form of wormholes. Two main quantum models of computation with the use of CTCs were introduced by Deutsch (D-CTC) and by Bennett and Schumacher (P-CTC). Unlike the classical theory in which CTCs lead to logical paradoxes, the quantum D-CTC model provides a solution that is logically consistent due to the self-consistency condition imposed on the evolving system, whereas the quantum P-CTC model chooses such solution through post-selection. Both models are non-equivalent and imply nonstandard phenomena in the field of quantum computation and quantum mechanics. In this work we study the implications of these two models on the second law of thermodynamics – the fundamental principle which states that in the isolated system the entropy never decreases. In particular, we construct CTC-based quantum circuits which lead to decrease of entropy.

A concept of voyage through time has been puzzling modern physicists for a long time. Einstein’s general theory of relativity envisages the subsistence of closed timelike curves (CTCs) [1, 2], where an object could travel through a wormhole, come out back in time and interact with a former version of itself.

The possible existence of CTCs points to a variety of logical paradoxes, such as famous grandfather paradox [3]. However, such paradoxes can be efficiently eliminated in quantum theory. One of the models that does so was proposed by Deutsch [4]. His model of CTC (D-CTC) operates within the density matrix formalism to describe the states of the two registers involved: a chronology-respecting (CR) system and a CTC chronology-violating (CV) quantum system, which interact with each other via some unitary operation. Under a self-consistency condition [5], the state of a latter system prior and after the interaction is set to remain the same. Such condition ensures the exclusion of the arising grandfather-like paradoxes, but also introduces a nonlinear evolution. This, in turn, gives rise to peculiar phenomena, e.g., violation of no-cloning theorem [6, 7], increasing entanglement with LOCC [8], and distinguishing non-orthogonal quantum states [9] which has been experimentally simulated in [10]. In addition, there exist another unusual implications of this model concerning the possible enhancement of power of D-CTC-assisted computation, such as the equivalence of classical and quantum computing or the ability to efficiently solve NP-complete and PSPACE-complete problems [11–13].

Bennett et al. [14] questioned all of above implications and have shown that they stem from falling into a “linearity trap”, i.e., from the hasty generalization of the analysis made for pure input states to their mixture. Indeed, in computation, it is profitable to consider a classically-labeled unknown mixture of states as an input due to possible various state preparation procedures. When analyzing the evolution of such mixture in the Deutsch’s model, one gets that the output of the D-CTC-assisted circuit may become independent of the input data, in particular, all the correlations between CR and CV systems can be lost. This comes from the nonlinear evolution, where the map acting on the mixture of states is not equivalent to a mixture of states individually acted upon this map. In light of the fact that analysis should hold for an arbitrarily prepared input state, one necessarily needs to increase his awareness to avoid falling into such linearity trap. Whence, using this argument, Bennett et al. concluded that the use of D-CTC does not entail any computational benefits and that observation of the phenomena against quantum postulates is blurred.

Another idea for a computational model of time travel came forth initially from Bennett and Schumacher [15], and was subsequently developed in [16–18]. This model of P-CTC is based on post-selected teleportation. In the P-CTC-assisted circuit there are 3 inputs: qudit 1, and a pair of qubits (2 and 3) in a maximally entangled state $\Phi^{+}_{23}$. Firstly, qudit 1 interacts unitarily with qudit 2 from the maximally entangled pair. After interaction, a state of qudit 1 together with a state of a qudit 3 from the maximally entangled pair is projected onto a maximally entangled state $\Phi^{+}_{13}$, and the state of qudit 2 is post-selected. The post-selection is unsuccessful on set of states of measure 0. When it is successful, i.e., the projection does not yield identically zero, the final output of the circuit is a time-traveled version of qudit 1.

Some of the key implications of P-CTC model have been investigated by Brun and Wilde [19], namely,
the ability to distinguish linearly independent quantum states, as well as the potential use of a single P-CTC qubit to complete different computational tasks such as effective integers factorization or solving NP-problems. It has also been shown that the experimental realization of this model is possible [16], which is achievable due to a post-selection procedure involved.

The debate regarding which model, either D-CTC or P-CTC, if any, is correct is still ongoing [20–26]. Apart from the argument about linearity trap posed by Bennett et al., other criticisms concerning Deutsch’s approach appeared. Among them is that the self-consistency condition only seemingly avoids time travel paradoxes from an observer’s perspective [22]. There also appeared a critique about the very use of density matrix formalism in the CTC-assisted models [23]. Notwithstanding, the P-CTC does also have a weakness: it loses certain (unwanted) evolutions.

As pointed out, the intriguing idea of the “quantization” of the CTCs initiated by Deutsch opened a Pandora’s box of the questions forbidden by the standard quantum formalism. Therefore a physical criterion for the considered models seems to be desirable. In this work we consider aforementioned models of CTCs: D-CTC and P-CTC and explore their behavior in regards to one of the most elementary laws of physics: the second law of thermodynamics. We start by discussing the D-CTC model in detail and show that it allows the violation of the second law of thermodynamics, when one falls into the linearity trap. Then, we move on to the Bennett and Schumacher model of post-selected CTCs and show that for any input states it implies the violation of the second law of thermodynamics. We summarize by interpreting our results in the context of a long lasting debate between the enthusiasts of D-CTC and P-CTC model.

Deutsch’s model.— The well-known Deutsch’s model of CTCs operates within two spacetime regions: the one that preserves the time chronology, and the other where the chronology is violated. The input (the density matrix of a quantum system S) that enters the former region from the unambiguous past, interacts with the input (density matrix of CTC system) existing in the latter region, and afterwards, it continues its travel into an unambiguous future.

Before the interaction, the systems are treated as separated and therefore the initial state is described by a product of density matrices $\rho_S \otimes \rho_{CTC}$, where $\rho_S$ is an initial density matrix of a CR qudit S, whereas $\rho_{CTC}$ denotes the initial density matrix of a qudit traveling along the CTC. The interaction set by a unitary operation $U_{SCTC}$ causes the entering system to evolve into a state:

$$\rho'_{SCTC} = U_{SCTC}(\rho_S \otimes \rho_{CTC})U_{SCTC}^\dagger.$$  \hspace{1cm} (1)

Now, one imposes the self-consistency condition [4]

$$\rho_{CTC} = \text{Tr}_S(\rho'_{SCTC}).$$  \hspace{1cm} (2)

on the final state $\rho_{CTC}$ of a CTC qudit, that results from tracing out the system $\rho'_{SCTC}$ defined in Eq. (1) over subsystem S. Note that the self-consistency condition (2) enforces the initial and final state of the CTC qudit, that appear respectively on the RHS of Eq. (1) and LHS of Eq. (2), to be the same. Deutsch showed that the fixed point of transformation (2) always exists.

After interaction, the state of the CR qudit S is given by

$$\rho_S = \text{Tr}_{CTC}(U_{SCTC}(\rho_S \otimes \rho_{CTC})U_{SCTC}^\dagger),$$  \hspace{1cm} (3)

and is a nonlinear function of the initial state of system $\rho_S$, since it depends on both: $\rho_S$ itself and $\rho_{CTC}$ (2) that also depends on $\rho_S$.

Transformation of a mixture of states into a pure state.— Let us now assume that one prepares a statistical ensemble $\{p_k, |\psi_k\rangle\}$ of states $|\psi_k\rangle$ with respective probabilities $p_k$, which gives the initial state of the system $S$ in the form:

$$\rho_S = \sum_k p_k |\psi_k\rangle\langle \psi_k|.$$  \hspace{1cm} (4)

Now, following the Deutsch’s model, we show how to construct a quantum circuit that transforms the above state into a pure state $|\varphi\rangle$, i.e.,

$$\{p_k, |\psi_k\rangle\} \xrightarrow{U_{SCTC}} |\varphi\rangle.$$  \hspace{1cm} (5)

In our approach, we will evolve each component $|\psi_k\rangle$ of a mixture (4) separately and then average over obtained evolutions.

To this end, let us first choose the unitary operation in the form:

$$U_{SCTC}= \sum_i V_i \otimes |i\rangle\langle i| \left(\sum_{i'} |i'\rangle\langle i'| \otimes U_{i'}\right) \text{SWAP},$$  \hspace{1cm} (6)

FIG. 1: CTC quantum circuit based on Deutsch’s model. Two input states: $\rho_S$ and $\rho_{CTC}$ interact with each other via a unitary operation $U_{SCTC}$, which is a product of SWAP, controlled-$U_i$ and controlled-$V_i$ operations. The self-consistency condition enforces the output state $\rho'_{CTC}$ of a qudit traveling along CTC to match the corresponding input state $\rho_{CTC}$. The output state $\rho_S'$ of a CR qudit continues its travel into an unambiguous future.
for which an associated CTC quantum circuit is shown in Fig. 1. The desired transformation (5) is achieved, if we choose a set of \( \{ U_k \} \), \( \{ V_k \} \) from Eq. (6) such that each \( U_k \) and \( V_k \) satisfies

\[
U_k |\psi_k \rangle = |k\rangle, \quad (7) \\
V_k |k\rangle = |\varphi\rangle. \quad (8)
\]

This can be shown by following the component-wise evolution of the initial state of the whole system throughout the circuit:

\[
U_{SCTC}(|\psi_k \rangle \otimes |k\rangle) \\
= \left( \sum_i V_i \otimes |i\rangle \langle i| \right) \left( \sum_{i'} |i'\rangle \langle i'| \otimes U_{i'}\right) (\text{SWAP})(|\psi_k \rangle \otimes |k\rangle) \\
= \left( \sum_i V_i \otimes |i\rangle \langle i| \right) \left( \sum_{i'} |i'\rangle \langle i'| \otimes V_{i'}\right) (|k\rangle \otimes |\psi_k\rangle) \\
= \left( \sum_i V_i \otimes |i\rangle \langle i| \right) (|k\rangle \otimes |\psi_k\rangle) \\
= |\varphi\rangle \otimes |k\rangle, \quad (9)
\]

where in the third equality we used Eq. (7), and in the last equality Eq. (8). In Sec. I of the Supplemental Material we show that the transformation (5) is achieved while obeying the self-consistency condition (2).

Next step is to average over obtained evolutions, i.e., \( \sum_k p_k(U_{SCTC}(|\psi_k \rangle \otimes |k\rangle)) = \sum_k p_k(|\varphi\rangle \otimes |k\rangle \langle k| \otimes |k\rangle \langle k|) \), where we used Eq. (9). One can note that by tracing out the obtained state over subsystem CTC, one obtains the pure state \( |\varphi\rangle \) as an output state of the system \( S \), as desired.

Now, we will examine the change of entropy of the state of CR system due to the above transformation (5). In our calculations, the starting and end points are taken when the only object in the system is qudit \( S \). This is possible since CTC exists only for a finite period of time, and therefore it does not contribute to the entropy of a system in the appropriately chosen moments. Before interaction, the von Neumann entropy of the initial mixed state of the qudit \( S \) is bounded as \( S(\sum_k p_k |\psi_k \rangle \langle \psi_k|) \geq c \), where a constant \( c \neq 0 \) is a finite number. Since the entropy of a final pure state is zero, \( S(|\varphi\rangle \langle \varphi|) = 0 \), we observe the decrease of entropy which indicates the violation of the second law of thermodynamics.

To sum up, in our approach we have evolved each component \( \{ |\psi_k\rangle \} \) of a mixture separately and then averaged over these evolutions. By speaking the language of Bennett et al., we fell into a linearity trap, which led to the peculiar implication: the decrease in entropy. However, if we avoid falling into such a linearity trap, in the spirit of [14], i.e., by evolving the whole state of the system (Eq. (4)) in the first place rather than its components, then the second law of thermodynamics would still hold. This can be seen by analyzing the entropy of the whole SCTC system before, \( S(\rho_S \otimes \rho_{CTC}) \), and after, \( S(\rho'_{SCTC}) \), time travel. With the only assumption of the self-consistency condition being fulfilled, Deutsch showed [4] that the entropy of the system does not decrease i.e., that \( S(\rho'_S) \geq S(\rho_S) \) holds.

**CTC model with post-selection.**— The post-selected closed timelike curve (P-CTC) [15] is presented in Fig. 2. Qudit 1 interacts with a qudit 2 from the entangled pair through the unitary operation \( U_{12} \). In the end, qudits 1 and 3 are projected onto the state \( |\Phi^+\rangle_{13} \) and the state of qudit 2, denoted as \( \rho'_2 \), is post-selected.

Let us now show that by using a P-CTC as in Fig. 2, any set of quantum states can be transformed into any pure state of choice, e.g., \( |0\rangle \). This also applies for a mixture of states. To this end, we use two different approaches. Firstly, we let ourselves fall into a linearity trap, namely, we calculate how each component of the initial mixture of states evolves separately and then average over evolutions. Later, we evolve the whole mixture of states, and by doing so, avoid falling into linearity trap.

**Falling into a linearity trap.**— Consider state of the first qudit \( \rho_1 = \sum_i p_i |\psi_i\rangle \langle \psi_i| \) which follows from an statistically prepared ensemble of initial states \( |\psi_i\rangle = \sum_{j=0}^{d-1} a_{i,j} |j\rangle \) that satisfy the normalization condition \( \sum_{j=0}^{d-1} |a_{i,j}|^2 = 1 \); and the maximally entangled state of the second and the third qudits \( |\Phi^+\rangle_{23} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_2 \otimes |j\rangle_3 \). In this approach, we perform evolution on each component of a mixture separately, and then calculate the average of them. Hence, we begin with a pure state:

\[
|\psi\rangle_{123} = |\psi_1\rangle_1 \otimes |\Phi^+\rangle_{23} = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} a_{i,j} |i\rangle_1 \otimes |j\rangle_2 \otimes |j\rangle_3. \quad (10)
\]

The first and the second qudits evolve due to the controlled-unitary operation \( U_{12} \), whereas the third qu-
dit remains untouched. The overall evolution $U$ is given by:

$$U = U_{12} \otimes I_3 = \sum_{i=0}^{d-1} |i\rangle \langle i| \otimes U_i \otimes I_3,$$

(11)

where $U_i = \sum_{j=0}^{d-1} |f_i(j)\rangle \langle j|$. The initial state of the system (10) evolves then as:

$$U|\psi\rangle_{123} = \frac{1}{\sqrt{d}} \sum_{i,j=0}^{d-1} a_{li}|i\rangle_1 \otimes |f_i(j)\rangle_2 \otimes |j\rangle_3,$$

(12)

and the normalized state of the second qudit after post-selection is given by:

$$|\psi\rangle_2 = \frac{\langle \Phi^+ |_{13} U|\psi\rangle_{123}}{\langle \Phi^+ |_{13} U|\psi\rangle_{123}} = \frac{\sum_{i=0}^{d-1} a_{li}|f_i(i)\rangle_2}{\sum_{i,k=0}^{d-1} a_{li}a_{lk}^* \langle f_k(k)|f_i(i)\rangle_2}.$$

(13)

Let us now impose the following two conditions. First, as established earlier, we choose the post-measurement state to be $|\psi\rangle_2 = |0\rangle$, which requires that

$$f_i(i) = 0, \quad \forall_i. \quad (14)$$

Also, the operation $U$ must be unitary, i.e., $UU^\dagger = U^\dagger U = 1$, which simplifies to

$$U_iU_i^\dagger = U_i^\dagger U_i = 1, \quad \forall_i. \quad (15)$$

To fulfill the first condition (14), we choose functions $f_i(j)$ as permutations shown graphically in Fig. 3 and defined explicitly in Sec. II of the Supplemental Material. Indeed $f_i(i) = 0, \forall_i$, and so the condition (14) is obeyed. Moreover, since each $f_i(j)$ represents permutation, the second condition (15) is immediately satisfied.

Eventually, using a condition (14) in Eq. (13), one sees that the P-CTC assisted circuit can transform each component of a mixture into the same state (here: $|0\rangle$). After averaging over evolutions, we obtain that the initially mixed state becomes a pure state with entropy zero. Consequently, we can observe the decrease in entropy and that the second law of thermodynamics is violated.

Without falling into a linearity trap.— In the second approach, we evolve the whole state of the system, i.e.,

$$\rho = \frac{1}{d} \sum_{l=0}^{N} p_l \sum_{m,n} a_{lk}a_{lk}^* |k\rangle_1 \langle j|_2 \otimes |m\rangle_3 \langle n|_3, \quad (16)$$

where the coefficients satisfy the normalization condition $\sum_{k,j=0}^{d-1} |a_{lk}|^2 = 1$.

Following the evolution described by the unitary operator (11), the initial state (16) gets transformed into the final post-measurement state:

$$\rho' = \langle \Phi^+ |_{13} U\rho U^\dagger |\Phi^+\rangle_{13}$$

$$= \frac{1}{d} \sum_{l=0}^{N} p_l \sum_{k,j=0}^{d-1} a_{lk}a_{lk}^* |f_i(i)\rangle_2 \langle f_i(i)\rangle_2. \quad (17)$$

Using the condition (14), and after normalization, we eventually obtain a pure state: $\rho' = |0\rangle_2\langle 0|_2$.

Again, in this approach, we observe that the P-CTC assisted circuit with appropriately satisfied conditions (14) and (15), allows to transform a mixture of states into a state with zero entropy. If the entropy of initial state is nonzero, we observe the decrease of entropy throughout the procedure and therefore violation of the second law of thermodynamics.

**Conclusions.**— In our work, we were interested in examining whether the second law of thermodynamics holds under the two basic models of CTC-assisted quantum circuits: Deutsch’s (D-CTC) and post-selected (P-CTC).

One can analyze the implications of these models using two different approaches. In the first one, evolutions are performed on the components of mixed state and then the average of them is calculated. The second one assumes the evolution of the whole input state. We found that by using different approaches, different implications of these models can be found, and in particular how they affect the changes of entropy. For a D-CTC model we established that if we let ourselves fall into a linearity trap by following the first approach, then the total entropy of the system decreases and therefore the second law of thermodynamics is violated. On the other hand, when we avoid falling into such a trap, i.e., we use second approach, the total entropy does not decrease. In the P-CTC model, however, regardless of approach, the second law of thermodynamics is violated.

We have shown that careful consideration of the ap-
approach used in CTC-assisted quantum circuits may lead to different implications on other related theories, such as thermodynamics. Remarkably the later has been recently considered [27] as a consequence of information conservation [28, 29]. Then it cannot be excluded that the information conservation principle is a necessary condition for physical (nonlinear) extensions of the quantum theory. Furthermore, by evaluating the implications on the second law of thermodynamics, we could understand features of these models. Since in the Deutsch’s model, the assumption of avoiding the non-linearity trap guarantees that the second law is obeyed, D-CTC seems to be more reasonable than the post-selection model, where the violation seems inevitable. Of course, other questions remain as to what are the physical meanings of the Deutsch’s type models and whether their physical implications can be truly observed using experimental methodologies. Finally, the investigation of such models may also throw some light on the epistemic and formal gap between general relativity and quantum mechanics [30, 31].

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[1] K. Gödel, Rev. Mod. Phys. 21, 447 (1949).
[2] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. 61, 1446 (1988).
[3] P. J. Nahin, Time Machines: Time Travel in Physics, Metaphysics and Science Fiction (Springer-Verlag and AIP Press, New York, 1999).
[4] D. Deutsch, Phys. Rev. D 44, 3197 (1991).
[5] J. Friedman, M. S. Morris, I. D. Novikov, F. Echeverria, G. Klinkhammer, K. S. Thorne, and U. Yurtsever, Phys. Rev. D 42, 1915 (1990).
[6] D. Ahn, T. C. Ralph, and R. B. Mann, (2010), arXiv:1008.0221 [quant-ph].
[7] T. A. Brun, M. M. Wilde, and A. Winter, Phys. Rev. Lett. 111, 190401 (2013).
[8] S. R. Moullick and P. K. Panigrahi, Sci. Rep. 6, 37958 (2016).
[9] T. A. Brun, J. Harrington, and M. M. Wilde, Phys. Rev. Lett. 102, 210402 (2009).
[10] M. Ringbauer, M. A. Broome, C. R. Myers, A. G. White, and T. C. Ralph, Nat. Comm. 4145 (2014).
[11] T. A. Brun, Found. Phys. 16, 245 (2003).
[12] D. Bacon, Phys. Rev. A 70, 032309 (2004).
[13] S. Aranson and J. Watrous, Proc. R. Soc. A 465, 631 (2009).
[14] C. H. Bennett, D. Leung, G. Smith, and J. A. Smolin, Phys. Rev. Lett. 103, 170502 (2009).
[15] C. H. Bennett, QUPON talk (2005), http://web.archive.org/web/20070206131550/http://www.research.ibm.com/people/b/bennetc/QUPONBshort.pdf.
[16] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L. A. Rozema, A. Darabi, Y. Soudagar, L. K. Shalm, and A. M. Steinberg, Phys. Rev. Lett. 106, 040403 (2011).
[17] S. Lloyd, L. Maccone, R. Garcia-Patron, V. Giovannetti, and Y. Shikano, Phys. Rev. D 84, 025007 (2011).
[18] G. Svetlichny, Int. J. Theor. Phys. 50, 3903 (2011).
[19] T. A. Brun and M. M. Wilde, Found. Phys. 42, 341 (2012).
[20] T. C. Ralph, Phys. Rev. A 76, 012336 (2007).
[21] T. C. Ralph, G. J. Milburn, and T. Downes, Phys. Rev. A 79, 022121 (2009).
[22] J. J. Wallman and S. D. Bartlett, Found. Phys. 42, 656 (2012).
[23] E. G. Cavalcanti and N. C. Menicucci, (2010), arXiv:1004.1219 [quant-ph].
[24] T. C. Ralph and C. R. Myers, Phys. Rev. A 82, 062330 (2010).
[25] J.-M. A. Allen, Phys. Rev. A 90, 042107 (2014).
[26] X. Dong, H. Chen, and L. Zhou, (2017), arXiv:1711.06814 [quant-ph].
[27] M. Nath Bera, A. Riera, M. Lewenstein, and A. Winter, (2017), arXiv:1707.01750 [quant-ph].
[28] M. Horodecki, R. Horodecki, A. Sen(De), and U. Sen, Found. Phys. 35, 2041 (2005).
[29] F. Hulpke, U. V. Poulsen, A. Sanpera, A. Sen(De), U. Sen, and M. Lewenstein, Found. Phys. 36, 477 (2006).
[30] G. Brian, Freeman Dyson in : The Fabric of the Cosmos: Space, Time, and the Texture of Reality (Alfred Knopf, 2004).
[31] L. M. Krauss and F. Wilczek, Phys. Rev. D 89, 047501 (2014).

SUPPLEMENTAL MATERIAL

I. Proof of the self-consistency condition

Here, we show that the self-consistency condition introduced in the main text in Eq. (2) is indeed satisfied.
\[ \rho_{CTC} = \text{Tr}_S \left( \left( \sum_i V_i \otimes |i\rangle\langle i| \right) \left( \sum_{i'} |i'\rangle \langle i'| \otimes U_{i'} \right) \left( \sum_{m,n} \rho_{mn} |m\rangle \otimes |\psi_k\rangle \langle \psi_k| \right) \left( \sum_{j'} |j'\rangle \langle j'| \otimes U_{j'}^\dagger \right) \left( \sum_j V_j^\dagger \otimes |j\rangle \langle j| \right) \right) \]

\[ = \text{Tr}_S \left( \left( \sum_i V_i \otimes |i\rangle\langle i| \right) \left( \sum_{m,n} \rho_{mn} |m\rangle \otimes U_m |\psi_k\rangle \langle \psi_k| U_n^\dagger \right) \left( \sum_j V_j^\dagger \otimes |j\rangle \langle j| \right) \right) \]

\[ = \text{Tr}_S \left( \sum_{i,j,m,n} \rho_{mn} \langle n| V_j^\dagger m \rangle \langle i| U_m |\psi_k\rangle \langle \psi_k| U_n^\dagger |j\rangle \right) \]

\[ = \sum_{i,j,m,n} \rho_{mn} \langle n| V_j^\dagger V_i |m\rangle \langle i| U_m |\psi_k\rangle \langle \psi_k| U_n^\dagger |j\rangle. \]  
(18)

Now, we abbreviate \( \rho_{CTCij} \) as \( \rho_{ij} \), and the self-consistency condition reads as:

\[ \rho_{ij} = \sum_{i,j,m,n} \rho_{mn} \langle n| V_j^\dagger V_i |m\rangle \langle i| U_m |\psi_k\rangle \langle \psi_k| U_n^\dagger |j\rangle. \]  
(19)

If we are interested only in diagonal elements we can simplify this condition further:

\[ \rho_{ii} = \sum_{i,m,n} \rho_{mn} \langle n| V_i^\dagger V_i |m\rangle \langle i| U_m |\psi_k\rangle \langle \psi_k| U_i^\dagger |i\rangle \]

\[ = \sum_{m,n} \rho_{mn} \langle n| U_m |\psi_k\rangle \langle \psi_k| U_n^\dagger |i\rangle \]

\[ = \sum_{m,n} \rho_{mn} \langle n| U_m |\psi_k\rangle \langle \psi_k| U_n^\dagger |i\rangle. \]  
(20)

At this point we got to the Eq. (6) from [9] and now the rest analysis of Brun et al. follows.

II. Explicit form of the permutations \( f_i(j) \)

Below we provide an explicit form of the permutation functions \( f_i(j) \) represented graphically in the main text in Fig. 3. The functions \( f_i(j) \) are given by:

\[ f_{i \neq (1,d-1)}(j) = \begin{cases} 
  j \to j + 1, & \forall j \neq 0,i-1,d-1 \\
  i \leftrightarrow 0, & (i-1) \mod d \to i + 1, \\
  d - 1 \to 1, & 
\end{cases} \]  
(21)

and for special cases of \( i = 1 \) and \( i = d - 1 \), respectively by:

\[ f_1(j) = \begin{cases} 
  j \to j + 1, & \forall j \neq 0,1,d-1 \\
  1 \leftrightarrow 0, & \\
  d - 1 \to 2, & 
\end{cases} \]  
(22)

and

\[ f_{d-1}(j) = \begin{cases} 
  j \to j + 1, & \forall j \neq 0,d-2,d-1 \\
  d-1 \leftrightarrow 0, & \\
  d - 2 \to 1. & 
\end{cases} \]  
(23)