THE MINKOWSKI \( ?(x) \) FUNCTION, A CLASS OF SINGULAR MEASURES, THETA-CONSTANTS, AND MEAN-MODULAR FORMS

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ABSTRACT. The Minkowski question mark function is a rich object which can be explored from the perspective of dynamical systems, complex dynamics, metric number theory, multifractal analysis, transfer operators, integral transforms, and as a function itself via analysis of continued fractions and convergents. Our permanent target, however, was to get arithmetic interpretation of the moments of \( ?(x) \) (which are relatives of periods of Maass wave forms) and to relate the function \( ?(x) \) to certain modular objects. In this paper we establish this link, embedding \( ?(x) \) not into the modular-world itself, but into a space of functions which are generalizations and which we call mean-modular forms. For this purpose we construct a wide class of measures, and also investigate modular forms for congruence subgroups which additionally satisfy the three term functional equation. From this perspective, the modular forms for the whole modular group as well as the Stieltjes transform of \( ?(x) \) (the dyadic period function) minus the Eisenstein series of weight 2 fall under the same uniform definition.

1. INTRODUCTION

The relation between continued fractions and modular functions is and old and deep subject; see, for example, [7, 10, 12]. In this paper we provide yet another example of this relation of a very different sort\(^1\).

The Minkowski question mark function \( ?(x) : [0, 1] \mapsto [0, 1] \) is defined by

\[
?(\{0, a_1, a_2, a_3, \ldots\}) = \sum_{\ell=1}^{\infty} (-1)^{\ell+1} 2^{-\sum_{j=1}^{\ell} a_j}, \quad a_j \in \mathbb{N};
\]

\( x = [0, a_1, a_2, a_3, \ldots] \) stands for a representation of \( x \) by a regular continued fraction. In view of the current paper, note that the Minkowski question mark function can be defined also in terms of semi-regular continued fractions. These are given by

\[
\{b_1, b_2, b_3, \ldots\} = \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \ddots}}};
\]

\(^1\) This is a short version of the forthcoming manuscript.
The Minkowski $\phi(x)$ function

where integers $b_i \geq 2$. Each real irrational number $x \in (0, 1)$ has a unique representation in this form, and rationals $x \in (0, 1)$ have two representations: one finite and one infinite which ends in $[[2, 2, 2, \ldots]] = 1$. It was proved in [4] that

$$\phi([b_1, b_2, b_3, \ldots]) = \sum_{\ell=1}^{\infty} 2^{\ell-\sum_{j=1}^{\ell} b_j}.$$  

The function $\phi(x)$ is continuous, strictly increasing, and singular. For $x \in [0, 1]$, it satisfies functional equations

$$\phi(x) = \begin{cases} 
1 - \phi(1 - x), \\
2\phi\left(\frac{x}{x+1}\right).
\end{cases}$$

These equations are responsible for the rich arithmetic nature of $\phi(x)$ and its relations (at least analogies) to the objects in the modular-world [1, 2]: for example, if we define

$$G(z) = \int_0^1 \frac{x}{1 - xz} \, d\phi(x),$$

then $G(z) = o(1)$ if $z \to \infty$ and the distance to $\mathbb{R}_+$ remains bounded away from 0, and

$$\frac{1}{z} + \frac{1}{z^2} G\left(\frac{1}{z}\right) + 2G(z + 1) = G(z), \quad z \in \mathbb{C} \setminus [1, \infty).$$

In this paper we exhibit explicitly the connection of $\phi(x)$ to the modular world. The factor “2” in the above formula - an intrinsic constant which comes from the dyadic nature of $\phi(x)$ - was always an obstacle which prevented an application of many techniques (Hecke operators, modularity, Fourier series) to the theory of $\phi(x)$. Now it appears that there exists a natural way to integrate $\phi(x)$ into the modular world, and this factor “2” is no longer an obstacle but rather the reason why this integration is possible. For this purpose, first, we construct a wide generalization of $\phi(x)$.

2. A CLASS OF FUNCTIONS

Here we present a new way to construct a wide class of continuous fractal functions which encode the self-similarity via semi-regular continued fractions.

Proposition 1. Let $q = \{q_\ell : 2 \leq \ell < \infty\}$ be the sequence of complex numbers such that

$$\sum_{\ell=2}^{\infty} q_\ell = 1, \quad \sum_{\ell=2}^{\infty} |q_\ell| < +\infty, \quad \sup_{\ell} |q_\ell| < 1.$$  

Then there exists the function $\mu = \mu_q : [0, 1] \to \mathbb{C}$ with the following properties.

1) It is continuous, $\mu(0) = 0$, $\mu(1) = 1$.

2) The function $\mu$ is of bounded variation. If all $q_\ell$ are real and non-negative, then $\mu$ is non-decreasing; if all $q_\ell$ are strictly positive, then $\mu$ is strictly increasing.

3) The function $\mu$ has the following self-similarity property:

$$\mu\left(\frac{1}{\ell - x}\right) = q_\ell \cdot \mu(x) + \sum_{j=\ell+1}^{\infty} q_j, \quad 2 \leq \ell < \infty, \quad x \in [0, 1].$$

4) if $q_\ell = 2^{1-\ell}$, $\ell \geq 2$, then $\mu(x) = \phi(x)$. 

Proof. To construct such a function, we use iterations. As an initial state, set
\( \mu_0(x) = x, \quad x \in [0, 1] \). Then define \( \mu_{w+1} \) piecewise recurrently by
\[
\mu_{w+1}(x) = q_\ell \cdot \mu_w \left( \ell - \frac{1}{x} \right) + \sum_{j=\ell+1}^{\infty} q_j, \quad x \in \left[ \frac{1}{\ell}, \frac{1}{\ell-1} \right], \quad w \geq 0.
\]
By induction we see that \( \mu_{w+1}(0) = 0, \mu_{w+1}(1) = 1 \), and that \( \mu_w \) is continuous. Now, consider the following series
\[
\mu_0(x) + \sum_{w=0}^{\infty} \left( \mu_{w+1}(x) - \mu_w(x) \right).
\] (1)
Let \( \sup_\ell |q_\ell| = \delta < 1 \), and \( \sup_{[0,1]} |\mu_1(x) - \mu_0(x)| = M \). By the very construction,
\[
\mu_{w+1}(x) - \mu_w(x) = q_\ell \cdot \left( \mu_w \left( \ell - \frac{1}{x} \right) - \mu_{w-1} \left( \ell - \frac{1}{x} \right) \right), \quad x \in \left[ \frac{1}{\ell}, \frac{1}{\ell-1} \right], \quad w \geq 1.
\]
So, for \( w \geq 1 \),
\[
\sup_{x \in [0,1]} |\mu_{w+1}(x) - \mu_w(x)| \leq \delta \cdot \sup_{x \in [0,1]} |\mu_w(x) - \mu_{w-1}(x)|.
\]
Thus, the series (1) is majorized by the series \( \sum_w M \delta^w \), and so the function
\[
\mu(x) = \lim_{w \to \infty} \mu_w(x)
\]
is continuous and satisfies all of the needed properties, as can be checked. \( \square \)

We call this function \( \mu_\mathbf{q} \) the \( \mathbf{q} \)-question mark function. For example, the Figures 1,2,3 shows the graph of these in cases \( \mathbf{q} = \left( \frac{2}{3}, \frac{1}{3}, 0, 0, \ldots \right), \mathbf{q} = \left( \frac{4}{7}, \frac{2}{7}, \frac{1}{7}, 0, 0, \ldots \right), \mathbf{q} = \left( -\frac{1}{7}, \frac{4}{7}, \frac{4}{7}, 0, 0, \ldots \right) \).

![Figure 1. \((\frac{2}{3}, \frac{1}{3})\)-question mark function, \(x \in [0,1]\)](image-url)
As an aside, let us define

\[ m_{q}(s) = \int_{0}^{1} e^{xs} d\mu_{q}(x), \quad p_{q}(s) = \sum_{\ell=2}^{\infty} q_{\ell} e^{-i\ell x}. \]
It is unknown whether $m_q(is)$ vanishes at infinity for $s \in \mathbb{R}$ in case of the Minkowski question mark function - this is the Salem’s problem [5, 6]. Most likely, all $m_q(is)$ vanish at infinity. It is out of the scope of the current paper, but we mention that the integral equation for the Laplace-Stieltjes transform of $?_q(x)$, defined by [1]

$$m(s) = \int_0^1 e^{xs} \, d?_q(x), \quad s \in \mathbb{C},$$

is compatible with this much more general construction. So, the function $m_q(s)$ is entire, and it satisfies the following integral equation

$$i \, m_q(is) \, p_q(s) = \int_0^\infty m_q'(it) \, J_0(2\sqrt{st}) \, dt, \quad s > 0;$$

the integral converges conditionally. On the other hand, the three term functional equation for the Stieltjes transform of $?_q(x)$ is compatible only with a narrow one parameter subclass of such $q$’s which we introduce now, since this is our main object.

3. A SPECIAL SUBCLASS

We will now focus on the important sequence $q_\ell$ given by $q_\ell = (1 - \kappa) \lambda^{\ell-2}, \ell \geq 2, \lambda \in \mathbb{C}, |\kappa| < 1, |1 - \kappa| < 1$. Let therefore $\mu_q = \mu_\lambda$ in this case. Let us define

$$G(\lambda, z) = \int_0^1 \frac{1}{x - z} \, d\mu_\lambda(x), \quad z \in \mathbb{C} \setminus [1, \infty).$$
Proposition 2. The function \( G(\kappa, z) \) satisfies the three term functional equation

\[
G(\kappa, z + 1) - \kappa G(\kappa, z) = \frac{(1 - \kappa)}{(1 - z)^2} G\left(\kappa, \frac{1}{1 - z}\right) + \frac{1 - \kappa}{1 - z}, \quad z \in \mathbb{C} \setminus [1, \infty).
\]

Moreover, \( G(\kappa, z) = o(1) \) if \( z \to \infty \) remaining bounded away from \([1, \infty)\).

Proof. First, we note the identity

\[
\int_0^1 f(x) \, d\mu(\kappa)(x) = \sum_{\ell=2}^{\infty} (1 - \kappa) \frac{1}{\ell-2} \int_0^1 f\left(\frac{1}{\ell - x}\right) \, d\mu(\kappa)(x),
\]

provided that all integrals are absolutely convergent. This follows from Proposition 1, the Property 3. In the special case, for \( f(x) = (\frac{1}{x} - z)^{-1} \), this reduces to

\[
G(\kappa, z) = \int_0^1 \frac{1}{x - z} \, d\mu(\kappa)(x) = \sum_{\ell=2}^{\infty} (1 - \kappa) \frac{1}{\ell-2} \int_0^1 \frac{1}{\ell - x - z} \, d\mu(\kappa)(x).
\]

Thus,

\[
G(\kappa, z + 1) - \kappa G(\kappa, z) = (1 - \kappa) \int_0^1 \frac{1}{1 - x - z} \, d\mu(\kappa)(x).
\]

Now, let us use the identity

\[
\frac{1}{1 - x - z} = \frac{1}{(1 - z)^2} \cdot \frac{1}{x - \frac{1}{1 - z}} + \frac{1}{1 - z}.
\]

This gives the functional equation (2). The regularity property is immediate. \( \square \)

Using the same method as in [3] we see that

\[
G(\kappa, z + 1) = (1 - \kappa) \sum_{a, b, c, d \geq 0, \frac{ad - bc}{b+d} = 1} \frac{\kappa^j (\frac{a+c}{b+d}) (1 - \kappa)^j (\frac{a+c}{b+d})}{[(a+c)z - (b+d)(cz - d)]};
\]

here \( i \) and \( j \) stand for the number of maps \( T \) and \( R \) (see the next section), respectively, needed to obtain the rational number \( \frac{a+c}{b+d} \) from the root \( \frac{1}{t} \) in the Calkin-Wilf tree [8]. So, \( G(\kappa, z) \) is holomorphic in both variables.

Let \( D = \{ \kappa \in \mathbb{C} : |\kappa| < 1, |1 - \kappa| \leq 1 \} \). This is the definition domain of the function \( G(\kappa, z) \) in variable \( \kappa \). If \( \kappa \in D \) and \( \kappa \to 0_+ \), then the function \( \mu(\kappa) \) tends pointwise to the function which is 0 in \([0, 1)\) and 1 at \( x = 1 \). Thus,

\[
\lim_{\kappa \to 0_+} G(\kappa, z) = G(0, z) = \frac{1}{1 - z}.
\]

This satisfies the functional equation (2) in case \( \kappa = 0 \). On the other hand, if \( \kappa \to 1_- \), then the function \( \mu(\kappa) \) tends pointwise to the function which is 0 at \( x = 0 \) and 1 in the interval \((0, 1]\). Thus, we also get

\[
\lim_{\kappa \to 1_-} G(\kappa, z) = G(1, z) \equiv 0.
\]
4. Mean-modular forms

Let \( \mathfrak{h} \) be the upper half plane, and let \( G_2(z) \) stands for the holomorphic quasi-modular Eisenstein series of weight 2 [14]:

\[
G_2(z) = \frac{\pi^2}{3} - \frac{8\pi^2}{3} \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz}.
\]

We know that for \( z \in \mathfrak{h} \),

\[
G_2(z + 1) = G_2(z), \quad G_2(-1/z) = z^2 G_2(z) - 2\pi iz.
\]

There exist several extensions of the space \( M_k \) of modular forms of weight \( k \). One of the extensions is the space of the so called quasi-modular forms, which are weight \( k \) elements of the ring \( \mathbb{C}[G_2, G_4, G_6] \), \( G_4 \) and \( G_6 \) being Eisenstein series of weights 4 and 6, respectively. Now we describe another \( ?(x) \)-related class of extensions of \( M_k \).

A direct calculation shows that \( \frac{i}{2\pi} G_2(z) \) satisfies the functional equation (2) for \( z \in \mathfrak{h} \). Let, as before, the number \( \varkappa \) belong to \( \mathcal{D} \). If \( G(\varkappa, z) \) is the function from the previous subsection, then, if we set

\[
g(\varkappa, z) = G(\varkappa, z + 1) - \frac{i}{2\pi} G_2(z),
\]

we see that this function is uniformly bounded for \( \Im(z) > \epsilon > 0, |\varkappa| < 1 - \epsilon, |1 - \varkappa| \leq 1 \), and for \( z \in \mathfrak{h}, \varkappa \in \mathcal{D} \), it satisfies the functional equation

\[
f(\varkappa, z) = \varkappa f(\varkappa, z - 1) + \frac{1 - \varkappa}{(1-z)^k} f\left(\varkappa, \frac{z}{1-z}\right)
\]

for \( k = 2 \). Let \( U, S, I, T, R \) be the standard \( 2 \times 2 \) matrixes:

\[
U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

The matrices \( U, S \) satisfy \( U^3 = S^2 = I \), and freely generate the modular group, while \( T = U^2 S \), \( R = US \) (all relations are considered modulo \( \pm I \)). Our main interest is the equation (4) in case \( \varkappa = \frac{1}{2}, k = 2 \), since this, as we have seen, is directly related to the Minkowski question mark function. Nevertheless, suppose \( f(\varkappa, z) \) satisfies (4), and let us consider this identity as the one for the function of two complex variables \( \varkappa \) and \( z \). At the one end of the real interval \( \varkappa \in [0, 1] \) (of course, \( 1 \notin \mathcal{D} \), but suppose we are allowed to plug it), the function \( f(\varkappa, z) \) is \( T \)-periodic:

\[
f(1, z) = f(1, z)|T^n, \quad n \in \mathbb{Z}.
\]

The \( R \)-periodicity holds at the other end:

\[
f(0, z) = f(0, z)|R^n, \quad n \in \mathbb{Z}.
\]

The two matrices \( T \) and \( R \) generate the whole modular group, and \( R \) and \( T \) are primitive elements there (i.e. not powers of other matrices) of infinite order. For example,

\[
g(1, z) = -\frac{i}{2\pi} G_2(z), \quad g(0, z) = -\frac{1}{z} - \frac{i}{2\pi} G_2(z),
\]
which are, respectively, $T$- and $R$-periodic. So, generally, the function $f(\kappa, z)$ cannot be called a modular form, but it rather as if interpolates a modular form, and we think that the name mean-modular form (that is, a modular form on average), is apt.

**Definition 1.** Let $k \in 2\mathbb{N}$. The function $f(\kappa, z)$ is called a weight $k$ mean-modular form, or MMF, if

1) it is bivariate holomorphic function and satisfies the functional equation (4) for $z \in \mathfrak{h}$, $\kappa \in D$;
2) for every $\epsilon > 0$ there exist a constant $C(\epsilon)$ such that $|f(\kappa, z)| < C(\epsilon)$ for $\Im(z) > \epsilon$, $|\kappa| < 1 - \epsilon$, $|1 - \kappa| \leq 1$.

We denote this $\mathbb{C}$-linear space by $\text{Mmf}_k$. Moreover, if $f(\kappa, z)$ is of the form

$$f(\kappa, z) = \sum_{j=0}^{L} \kappa^j A_j(z),$$

the functional equation is then satisfied for all $\kappa \in \mathbb{C}$) the function $f(\kappa, z)$ is called a weight $k$, height $L$ mean-modular form. We denote the $\mathbb{C}$-linear space of such forms by $\text{Mmf}_k^L$.

In fact, there are many functions, constant in variable $\kappa$, which satisfy the functional equation but fail the regularity condition. For example, when $k = 2$ such functions are $j'(z)P(j(z))$, where $j(z)$ is the $j$-invariant, and $P$ is any polynomial.

Let us denote the arc of the circle $|\kappa| = \epsilon > 0$, which is inside the disc $|1 - \kappa| \leq 1$, by $\Omega(\epsilon)$. When $\epsilon \to 0_+$, the length of the arc $\Omega(\epsilon)$ is $\pi \epsilon + O(\epsilon^2)$.

**Definition 2.** Let $Q \subset \mathfrak{h}$ be a compact set. The $j$-th coordinate of the mean-modular form $f(\kappa, z)$ is defined recurrently by

$$A_j(z) = \lim_{\epsilon \to 0_+} \frac{1}{\pi i} \int_{\Omega(\epsilon)} \frac{1}{\kappa^{j+1}} \left( f(\kappa, z) - \sum_{s=0}^{j-1} \kappa^s A_s(z) \right) d\kappa, \quad j \in \mathbb{N}_0, \quad z \in Q,$$

where the integral is taken via the arc $\Omega(\epsilon)$ from the bottom upwards, and empty sum is 0 by convention. The definition of $A_j(z)$ is extended to $\mathfrak{h}$ by expanding $Q$. Symbolically, we write

$$f(\kappa, z) \sim \sum_{j=0}^{\infty} \kappa^j A_j(z).$$

Comparing the corresponding coefficients at powers of $\kappa$ we obtain the basic relations among these functions:

$$A_{j+1}|(I - US) = A_j|(I - SU^2S);$$

(5)

this holds for $j \geq -1$, assuming $A_{-1}(z) \equiv 0$. This also holds assuming $A_{L+1}(z) \equiv 0$ if $f(\kappa, z) \in \text{Mmf}_k^L$. It is clear that

$$\text{Mmf}_k^0 = \text{Mm}_k,$$

where the latter stands for the space of modular forms of weight $k$ for the full modular group. In fact, we have a filtration

$$\text{Mmf}_k^0 \subseteq \text{Mmf}_k^1 \subseteq \text{Mmf}_k^2 \subseteq \cdots, \quad \bigcup_{L=0}^{\infty} \text{Mmf}_k^L = \text{Mmf}_k^\infty.$$
Moreover, also we have $x \cdot \text{Mmf}_k^L \subset \text{Mmf}_k^{L+1}$. We do not have any evidence yet, but hopefully each of spaces $\text{Mmf}_k^L$ is finite dimensional. Let $d_L = \dim \mathbb{C}(\text{Mmf}_k^L)$; we omit the second subscript $k$.

**Proposition 3.** The sequence $d_L$ satisfies the properties

$$
\begin{align*}
d_0 &= \dim \mathbb{C}(M_k), \\
d_{L+i} + d_{L-i} &\geq 2d_L, \quad 0 \leq i \leq L, \\
d_{i+j+1} &\geq d_i + d_j, \quad i, j \geq 0.
\end{align*}
$$

**Proof.** Let $p(x, z), q(x, z) \in \text{Mmf}_k^L$ be non-zero functions. Then $p(x, z), x^i q(x, z) \in \text{Mmf}_k^{L+i}$, $i \geq 0$. If these are linearly dependent, then

$$
c x^{-i} p(x, z) = q(x, z) \in \text{Mmf}_k^{L-i}, \quad c \neq 0.
$$

This gives

$$
d_{L+i} \geq 2d_L - d_{L-i}, \quad 0 \leq i \leq L.
$$

Also, if $p(x, z) \in \text{Mmf}_k^i$, $q(x, z) \in \text{Mmf}_k^j$, then $p(x, z), x^{i+1} q(x, z) \in \text{Mmf}_k^{i+j+1}$ and they are linearly independent. \hfill \Box

In this setting, however, most likely that the Minkowski question mark function itself cannot be described in finite terms. That is,

$$
g(x, z) \in \text{Mmf}_k \setminus \text{Mmf}_k^\infty.
$$

In other words, $g(x, z)$ seemingly contains all powers of $x$; see (3).

5. **Height 1**

Now we will briefly investigate the space $\text{Mmf}_k^1$. Let therefore $f(x, z) = A(z) + xB(z)$ be a mean-modular form of weight $k$. Plugging $f(x, z)$ into (4) and comparing the coefficients at $x^0, x^1, x^2$, we obtain:

$$
\begin{align*}
A &= A|R^{-1}, \\
B &= A|T^{-1} + B|R^{-1} - A|R^{-1}, \\
B|T^{-1} &= B|R^{-1}.
\end{align*}
$$

So,

$$
\begin{align*}
A &= A|R, \\
A + B &= (A + B)|T, \\
B &= B|\text{SUS}.
\end{align*}
$$

The first identity means that $A$ is $R$-periodic, and that $B$ has a three-fold symmetry, since $(\text{SUS})^3 = I$. The second identity implies that

$$
B(z) = -A(z) + p(z)
$$
for a certain $T$-periodic function $p$. The third identity then gives $A(I - SUS) = p(I - SUS)$. So, since $p$ is 1-periodic, we have

$$A(z) - \frac{1}{z^k}A(\frac{1}{z}) = p(z) - \frac{1}{z^k}p\left(\frac{1}{z}\right) = p(I - S),$$

$$A(z) - \frac{1}{z^k}A(\frac{1}{z}) = p(z) - \frac{1}{z^k}p\left(1 - \frac{1}{z}\right) = p(I - U^2).$$

So, this gives three identities for a function $A$:

$$\begin{cases}
A(I - US) = 0, \\
A(I - SUS)(I + S) = 0, \\
A(I - SUS)(I + U + U^2) = 0.
\end{cases} \tag{6}$$

The third one, however, is a consequence of the second - this is a pleasant fact which facilitates our computations. Indeed, multiply the second identity from the left by, respectively, $I, SU$, and $SU^2$:

$$A(I + S - SUS - SU) = 0,$$

$$A(SU + U - SU^2 - SUSU) = 0,$$

$$A(SU^2 + U^2 - S - SUSU^2) = 0.$$

If we add all three, we obtain the last identity in (6). Let us introduce $C = A|S$. Then the first two identities of (6) can be rewritten in terms of $C$:

$$C(z + 1) = C(z),$$

$$C(z + \frac{1}{z}) = \frac{1}{(z + 1)^k}C\left(-\frac{1}{z + 1}\right) + \frac{1}{(z - 1)^k}C\left(-\frac{1}{z - 1}\right), \tag{7}$$

We can write these in a form $C|(I - T) = 0$, $C|(I - U)(I + S) = 0$, or even in a more symmetric form as follows.

**Proposition 4.** The function $C(z)$ satisfies

$$C(z + 1) = C(z),$$

$$2C(z) = \frac{1}{z^k}C\left(-\frac{1}{z}\right) + \frac{1}{(z + 3)^k}C\left(-\frac{1}{z + 3}\right),$$

$$3C(z) = \frac{1}{z^k}C\left(-\frac{1}{z}\right) + \frac{1}{(z + 2)^k}C\left(-\frac{1}{z + 2}\right) + \frac{1}{(z + 4)^k}C\left(-\frac{1}{z + 4}\right).$$

This implies

$$C(z) = \frac{1}{(1 - 6z)^k}C\left(\frac{z}{1 - 6z}\right).$$

Conversely - for every $C$ satisfying these properties and which is bounded at infinity there exists a mean modular form of the form $C(z)|S + zB(z)$. If we denote the $\mathbb{C}$-linear space of such functions $C$ as $\tilde{M}_k$, then $M_k \subset \tilde{M}_k$.

**Proof.** Having in mind the periodicity $C(z) = C(z + 1)$, it is directly checked that (7) implies the last two identities of Proposition 4, and these two also imply (7) - this is just a combination of these identities for $C(z), C(z + 1), C(z + 2)$. \qed
Note that the group $\Gamma_1(6)$ is generated by three matrices \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -6 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix} \), and the first two generate a non-co-compact subgroup. As noted, in any case, $M_k \subset \text{MMF}_k$. It would be nice that in some cases we have $d_1 > 2d_0$. In particular, in connection with the Minkowski question mark function, I hope that $1 \leq \dim_{\mathbb{C}}(\text{MMF}_2) < \infty$. Thus, Proposition 4 provides a first step in modular approach towards the nature of $?\left(\frac{a}{b}\right)$.

6. Modular solutions

We will now show that the requirement that a mean-modular form is holomorphic in variable $\tau$ is essential and strong, since there exists too many functions which satisfy (4) for certain particular fixed $\tau$. Moreover, such functions can even be modular forms for congruence subgroups. Consequently, such “sporadic” solutions do not qualify as MMF.

Let $N \in \mathbb{N}, k \in 2\mathbb{N}$. Consider the space of modular forms $M_k(\Gamma(N))$. Let $u(z) = (u_1(z), u_2(z), \ldots, u_\ell(z))$ be the basis of this space. We know that for any $u(z) \in M_k(\Gamma(N))$, both $u(z-1)$ and $(1-z)^{-k}u(z/(1-z))$ belong to $M_k(\Gamma(N))$. This simply follows from the fact that $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$. So there exists two matrices $A$ and $B$ such that

$$u(z-1)^T = Au(z)^T, \quad (1-z)^{-k}u(z/(1-z))^T = Bu(z)^T.$$  

We want a function

$$\sum_{i=1}^\ell a_iu_i(z), \quad a_i \in \mathbb{C},$$

to satisfy (4). There exists a non-zero vector $(a_1, \ldots, a_\ell)$ if and only if the determinant of the matrix $I - \tau A - (1 - \tau)B$ vanishes:

$$P_{N,k}(\tau) := \det(I - B + \tau(B - A)) = 0.$$  

So, each pair $N \geq 2, k \in 2\mathbb{N}$, generates the polynomial $P_{N,k}(\tau)$, and each root of this polynomial produces the element of $M_k(\Gamma(N))$ that also satisfies (4). For example, let $N = 2, k = 2$. The space $M_2(\Gamma(2))$ is 2-dimensional and is spanned by $\vartheta^4(0,1/2; z)$ and $\vartheta^4(1/2,0; z)$, the Jacobi’s theta functions (see further). The polynomial $P_{2,2}(\tau) = 3\tau(1 - \tau)$. So, in this case only $\tau = 0$ belongs to $\mathcal{D}$. Anyway, using approach via theta constants, we have calculated many possible $\tau$, and there are plenty of whose which belong to $\mathcal{D}$; for example, $\tau = \frac{1}{2} + \frac{1}{2}i$ is one of them. The approach via theta consists consists of the following.

Let us define, for $a, b \in \mathbb{R}, k \in \mathbb{N}$ (no relation to the weight!), $z \in \mathfrak{h}$, the theta-constants \[9, 13]\n
$$\vartheta(a, b; z)_k = \sum_{n \in \mathbb{Z}} e^{k\pi i(a+n)^2z+2b(a+n)} = \vartheta(a, kb; kz)_1,$$

$$\vartheta(a, b; z)'_k = 2k\pi i \sum_{n \in \mathbb{Z}} (a+n) e^{k\pi i(a+n)^2z+2b(a+n)} = k\vartheta(a, kb; kz)'_1.$$  

The next identities are checked directly; they are either immediate, or follow from the Poisson summation formula.

**Proposition 5.** The functions $\vartheta(a, b; z)_k$ and $\vartheta(a, b; z)'_k$ for rational $a, b$ are modular forms of weights 1/2 and 3/2, respectively. Further, we have
1-1') \vartheta(a + 1, b; z)_k = \vartheta(a, b; z)_k;
2-2') \vartheta(a, b + \frac{1}{3}; z)_k = e^{2\pi i a} \vartheta(a, b; z)_k;
3-3') \vartheta(a, b; z + 1)_k = e^{-k\pi i (a^2 + b)} \vartheta(a, b + a + \frac{1}{2}; z)_k;
4') \vartheta(-a, -b; z')_k = \vartheta(a, b; z)_k;
5') \vartheta(a, b; -\frac{1}{2})_k = k^{-1/2}(-iz)^{1/2} e^{2k\pi i ab} \sum_{s=0}^{k-1} \vartheta(b + \frac{s}{k}, -a; z)_k.

1-1', 2-2' and 3-3' mean that the same transformation rules hold for \vartheta(a, b; z)_k and \vartheta(a, b; z)'_k.

So, we start from any product of these theta constants, that include only rational parameters \(a, b\), and which amount to the total weight of, say, 2. This function satisfies transformation properties under \(z \mapsto z + 1, z \mapsto -z^{-1}\). It belongs to the finite orbit, and thus this also reduces to the condition for the determinant. For example, let us consider the simplest case of weight 2 and when these products are in fact 4th powers of theta constants.

6.1. Theta functions \(\vartheta^4(a, b; z)_1\) for \(4a, 4b \in \mathbb{Z}\). There are three orbits in this case. First, the orbit-singleton \((\frac{1}{4}, \frac{1}{2})\), which produce a zero theta constant. Further, the 3-element orbit \((0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2})\), which was already investigated; these three functions are related via the Jacobi identity:

\[\vartheta^4(0, 0; z) = \vartheta^4(1/2, 0; z) + \vartheta^4(0, 1/2; z).\]

The third orbit consists of 6 elements \((0, \frac{1}{4}), (\frac{1}{4}, 0), (\frac{1}{4}, \frac{1}{4}), (\frac{3}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), \) and \((\frac{1}{2}, \frac{1}{4})\). Therefore,

\[\mathbf{u}(z)^T = \begin{pmatrix} \vartheta^4(0, 1/4; z) \\ \vartheta^4(1/4, 0; z) \\ \vartheta^4(1/4, 1/4; z) \\ \vartheta^4(3/4, 1/4; z) \\ \vartheta^4(1/4, 1/2; z) \\ \vartheta^4(1/2, 1/4; z) \end{pmatrix},\]

and the space generated by all six components is invariant under the action of \(T\) and \(S\).

6.2. Theta functions \(\vartheta^4(a, b; z)_1\) for \((6a, 6b) \in \mathbb{Z}^2, (2a, 2b) \notin \mathbb{Z}^2\). In this case the theta functions split into three orbits: \(Q_1\), consisting of 4 functions with rational pairs \((a, b) = (\frac{1}{6}, \frac{1}{6}), (\frac{5}{6}, \frac{1}{6}), (\frac{1}{6}, \frac{1}{3}), (\frac{1}{3}, 0)\); \(Q_2\), consisting of 4 rational pairs \((\frac{1}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, 0)\), \((\frac{1}{6}, \frac{1}{3}), (\frac{1}{3}, 0)\), \((\frac{1}{3}, \frac{1}{6})\), \((\frac{2}{3}, \frac{1}{6})\), \((\frac{3}{6}, \frac{1}{3})\), and \(Q_3\), consisting of 8 pairs \((0, \frac{1}{6}), (0, \frac{1}{3}), (\frac{1}{6}, 0), (\frac{1}{3}, 0)\), \((\frac{1}{3}, \frac{1}{6})\), \((\frac{2}{3}, \frac{1}{6})\), \((\frac{3}{6}, \frac{1}{3})\). For example,

\[\mathbf{u}(z)^T = \begin{pmatrix} \vartheta^4(1/6, 1/6; z) \\ \vartheta^4(5/6, 1/6; z) \\ \vartheta^4(1/6, 1/2; z) \\ \vartheta^4(1/2, 1/6; z) \end{pmatrix},\]

and the space generated by all four components is invariant under the action of \(T\) and \(S\); this is the subspace of \(M_2(\Gamma_2(18))\). In fact, we can use not only the fourth powers but products of different theta constants, this produces the plethora of solutions to (4) with many different algebraic \(x\).
7. Mean-modular sections

Definition 3. We call a function \( T(z) \) a mean-modular section, or MMS, of weight \( k \) and parameter \( \zeta_0 \), if \( \zeta_0 \in D \), and there exists a mean-modular form \( f(\zeta, z) \) of weight \( k \) such that

\[
T(z) = f(\zeta_0, z).
\]

So,

- if \( T(z) \) is a modular form for \( \text{PSL}_2(\mathbb{Z}) \), then \( T(z) \) is a MMS of the same weight.
- “Sporadic” solutions of the three term functional equation, which are also in \( M_k(\Gamma(N)) \) and which were described in the previous subsection, do not qualify MMS.
- Most importantly,

\[
\int_0^1 \frac{x}{1-x(z+1)} \, d\zeta(x) - \frac{i}{2\pi} G_2(z).
\]

is a central, corresponding to \( \zeta_0 = \frac{1}{2} \), MMS of weight 2.

Problem 1. Develop a theory of mean-modular forms and mean-modular sections. (Are there analogues of Hecke operators? Scalar product?)

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THE MINKOWSKI \( ?(x) \) FUNCTION

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