Master integrals for the NNLO virtual corrections to $q\bar{q} \to t\bar{t}$ scattering in QCD: the non-planar graphs

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ABSTRACT: We complete the analytic evaluation of the master integrals for the two-loop non-planar box-diagrams contributing to the top-pair production in the light-quark initiated channel, at next-to-next-to-leading order in QCD. We adopt the method of differential equations and the Magnus exponential to determine a canonical set of integrals, finally expressed as a Taylor series around four space-time dimensions, with coefficients written as a combination of generalised polylogarithms. Initial-state quarks are treated as massless, while we retain full dependence on the mass of the top quark. We discuss the analytic continuation of the planar and the non-planar master integrals, which contribute to $q\bar{q} \to t\bar{t}$ in QCD, as well as to the companion QED scattering processes $ee \to \mu\mu$ and $e\mu \to e\mu$. 
1 Introduction

The top quark is the most heavy elementary matter particle. Its large production rate at the CERN LHC enables precision studies, thereby testing the Standard Model of particle physics at an unprecedented level, and potentially uncovering indirect evidence for new physics effects. Precision measurements of top quark observables [1–4] must be confronted with equally precise theory predictions, thereby requiring the perturbation theory descriptions of these observables to be extended to high orders.

The available second-order (next-to-next-to-leading order, NNLO) QCD corrections to the top quark pair production process (initially for the total cross section [5] and subsequently for differential distributions [6–9]) deliver a competitive level of theory predictions, and are enabling a multitude of precision studies with top quark pairs. A key ingredient to these calculations are the two-loop QCD corrections to the matrix elements for top pair production in quark-antiquark annihilation and gluon fusion. While numerical representations of these two-loop matrix elements were derived already some time ago [10, 11], only partial analytical results are available for them up to now [12–15] (and used in partial validations [16, 17] of the total NNLO cross section calculation). Such analytical results allow in-depth investigations into the structure of the matrix elements, enabling to investigate limiting behaviours and analyticity structures, as well as providing an independent approach to their numerical evaluation. Up to now, full results for the two-loop top quark
production matrix elements could not be derived due to incomplete knowledge on the relevant two-loop Feynman integrals.

In this work, we complete the task of determining all the non-planar two-loop functions that are needed for the evaluation of the scattering amplitudes of the process $q\bar{q} \to t\bar{t}$ at NNLO in QCD. Owing to the large value of the top mass compared to the mass of the two incoming quarks, which we suppose to have different flavour, we treat the latter as massless.

The results of this paper represent an additional milestone of the research programme dedicated to the two-loop QCD and QED corrections to the interaction of two fermionic currents, that was initiated with the study of the muon-electron scattering, in the context of the MUOne experiment \cite{18, 19}, which is currently under evaluation at CERN.

By following the same the path of the calculation of two-loop integrals for $\mu e \to \mu e$ and the crossing-related processes considered in \cite{20, 21}, we adopt a consolidated strategy \cite{22, 23}, which was proven to be particularly effective in the context of multi-loop integrals that involve multiple kinematic scales \cite{20, 21, 23–26}. By means of integration-by-parts identities (IBPs) \cite{27–29}, we identify a set of 52 master integrals (MIs) that we evaluate analytically, through the differential equation method \cite{30–33}. In particular, we consider an initial set of MIs that obey a system of first-order differential equations (DEQs) in two independent kinematic variables that is linear in the space-time dimensions $d$. The system is subsequently cast in canonical form \cite{22} by means of the Magnus exponential matrix \cite{23}. The matrix associated to the canonical system, where the dependence on $(d-4)$ is factorized from the kinematics, is a logarithmic differential form, which – once parametrized in terms of suitable variables – has a polynomial alphabet, constituted of twelve letters. Therefore, the canonical MIs are found to admit a Taylor series representation around $d = 4$, whose coefficients are combinations of generalised polylogarithms (GPLs) \cite{34–37}. The otherwise unknown integration constants are determined either from the knowledge of the analytic expression of the MIs in special kinematic configurations or by imposing their regularity at the pseudo-thresholds of the DEQs. Finally, we show how the MIs, that we initially compute in an unphysical region, can be analytically continued to the top-pair production region. Due to the non-trivial structure of their branch-cuts, the analytic continuation of the two-loop functions considered in this paper represents a paradigmatic case, that can be useful for the study of other planar and non-planar diagrams that involve massive particles. As a byproduct of the current analysis, we obtain the analytic continuation to the physical region of the functions required for the $\mu e \to \mu e$ and $ee \to \mu\mu$ scattering in QED \cite{20, 21}\footnote{The evaluation of the master integrals for the di-muon production in lepton-pair scattering, within the physical region, has been recently considered in \cite{38}.}.

In the completion of this calculation, we benefited from publicly available software dedicated to multi-loop calculus. The IBPs decomposition and the generation of the dimension-shifting identities and DEQs obeyed by the MIs have been performed with the packages Kira \cite{39}, LiteRed \cite{40, 41} and Reduze \cite{42, 43}. The analytic expressions of the MIs have been numerically evaluated with the help of GiNaC \cite{44}, and were successfully tested against the numerical values provided either by the computer code SecDec \cite{45} or, for the
most complicated two-loop non-planar topologies, with 7 denominators, by an in-house algorithm.

Beside these important validations, our results have been successfully compared against the computation of the master integrals relevant to the same integral topologies expressed in a different basis set, independently obtained by Becchetti, Bonciani, Casconi, Ferroglia, Lavacca and von Manteuffel [46], published in tandem to the current manuscript.

The remainder of the paper is organized as follows: in section 2 we set our notation and conventions for the non-planar two loop functions. In section 3, we present the system of DEQs obeyed by the MIs and their solutions in the unphysical region. In section 4, we study the analytic continuation of the MIs to the physical region. Finally, in section 5, we discuss the numerical evaluation of the 7-denominator integrals. Appendix A contains the coefficients of the linear combinations of MIs that satisfy a canonical system of DEQs.

The analytic expressions of the considered MIs are given in the ancillary files accompanying the arXiv version of this publication.

2 The non-planar four-point topology

In this paper, we consider the $q\bar{q} \to t\bar{t}$ scattering process

$$q(p_1) + \bar{q}(p_2) \to t(p_3) + \bar{t}(p_4),$$

(2.1)

i.e. with kinematics specified by

$$p_1^2 = p_2^2 = 0, \quad p_3^2 = p_4^2 = m^2,$$
\[ s = (p_1 + p_2)^2, \quad t = (p_2 - p_3)^2, \quad u = (p_1 - p_3)^2 = 2m^2 - t - s, \]

(2.2)

where \( m \) is the top quark mass. Representative two-loop non-planar diagrams contributing to this process are shown in the top row of figure 1. In the bottom row, we also show the integral families onto which we map those diagrams. Massive propagators and external legs are depicted using thick lines. The MIs for the QED-like family \( A_1 \) are already available in the literature, as they have been studied in the context of the NNLO QED corrections to \( ee\mu\mu \) processes (with suitable redefinitions of the momenta and of the Mandelstam invariants). In particular, they have been first computed in [21] (in an unphysical region, to be analytically continued), and later in [38] (directly in the heavy-fermion-production kinematic region). As for the genuine QCD contributions, the MIs for family \( N_1 \) have been computed in [48], while the MIs for family \( N_2 \) are the subject of the present publication.

The calculation involves the evaluation, in \( d \) dimensions, of Feynman integrals of the type

\[ I^{(d)}(n_1, \ldots, n_9) \equiv \int \frac{d^d k_1 d^d k_2}{D_1^{n_1} \cdots D_9^{n_9}}, \]

(2.3)

where \( D_i \) are inverse scalar propagators. The analytic calculation described in section 3 is performed expanding around \( d = 4 \), while the numerical evaluation presented in section 5 is carried over around \( d = 6 \). We set \( \epsilon \equiv (d_* - d)/2 \), where \( d_* = 4 \) and \( d_* = 6 \) according to the case considered, and define our integration measure as

\[ \tilde{d^d k} = \frac{d^d k}{i \pi^{d/2} \Gamma(1 + \epsilon)} \left( \frac{m^2}{\mu^2} \right)^\epsilon, \]

(2.4)

where \( \mu \) is the 't Hooft scale of dimensional regularisation.

For the non-planar four-point topology \( N_2 \), we choose the following set of propagators:

\[
\begin{align*}
D_1 &= k_1^2, \quad D_2 = k_2^2 - m^2, \quad D_3 = (k_2 - p_3)^2, \quad D_4 = (k_1 - p_2)^2, \\
D_5 &= (k_1 - p_3 - p_4)^2, \quad D_6 = (k_1 - k_2)^2 - m^2, \quad D_7 = (k_1 - k_2 - p_4)^2, \\
D_8 &= (k_2 - p_3)^2, \quad D_9 = (k_2 - p_2)^2,
\end{align*}
\]

(2.5)

where \( k_1 \) and \( k_2 \) denote the loop momenta. We note that, due to the definition of eq. (2.4), the tadpole integral \( \epsilon^2 I^{(4-2\epsilon)}(0, 2, 0, 0, 2, 0, 0, 0) \) is normalised to 1.

### 3 Solution of the system of differential equations

By means of IBPs, we reduce the two-loop integrals that belong to the integral family defined in eq. (2.5) to a basis of 52 distinct MIs. In order to determine the analytic expressions of the latter, we derive their DEQs in the kinematic variables \( s, t \) and \( m^2 \), by enforcing momentum conservation and requiring the external legs to be on-shell. The evaluation of the MIs can be further facilitated by parametrising the Mandelstam invariants in terms of two independent dimensionless variables, \( w \) and \( z \), which are defined by

\[
\begin{align*}
\frac{u - m^2}{t - m^2} &= -\frac{z^2}{w}, \\
\frac{s}{m^2} &= -\frac{(1 - w)^2}{w},
\end{align*}
\]

(3.1)

where the Mandelstam constraint \( s + t + u = 2m^2 \) is understood. This change of variables rationalises the canonical DEQs.
3.1 Differential equations for master integrals

We identify a canonical basis of MIs in $d = 4 - 2\epsilon$ by making use of the algorithm described in [23, 24]. Namely, we choose an initial set of MIs $F_i$ fulfilling DEQs that depend linearly on the dimensional regularisation parameter $\epsilon$,

$$F_1 = \epsilon^2 T_1, \quad F_2 = \epsilon^2 T_2, \quad F_3 = \epsilon^2 T_3.$$
Figure 3: The remaining 27 MIs $T_{26,..,32}$ for the two-loop non-planar topology $N_2$ of figure 1. The conventions are the same as in figure 2.

$$F_4 = \epsilon^2 T_4, \quad F_5 = \epsilon^2 T_5, \quad F_6 = \epsilon^2 T_6,$$
$$F_7 = \epsilon^2 T_7, \quad F_8 = \epsilon^2 T_8, \quad F_9 = \epsilon^2 T_9,$$
$$F_{10} = \epsilon^2 T_{10}, \quad F_{11} = \epsilon^3 T_{11}, \quad F_{12} = \epsilon^2 T_{12},$$
$$F_{13} = \epsilon^3 T_{13}, \quad F_{14} = \epsilon^2 T_{14}, \quad F_{15} = \epsilon^3 T_{15},$$
$$F_{16} = \epsilon^2 T_{16}, \quad F_{17} = \epsilon^2 T_{17}, \quad F_{18} = \epsilon^3 T_{18},$$
$$F_{19} = \epsilon^3 T_{19}, \quad F_{20} = \epsilon^4 T_{20}, \quad F_{21} = \epsilon^2 T_{21},$$
$$F_{22} = \epsilon^3 T_{22}, \quad F_{23} = \epsilon^2 T_{23}, \quad F_{24} = \epsilon^3 T_{24},$$

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\[
\begin{align*}
F_{25} &= e^2 T_{25}, & F_{26} &= e^2 T_{26}, & F_{27} &= e^3 T_{27}, \\
F_{28} &= e^3 T_{28}, & F_{29} &= e^4 T_{29}, & F_{30} &= e^3 T_{30}, \\
F_{31} &= e^3 T_{31}, & F_{32} &= e^4 T_{32}, & F_{33} &= e^3 T_{33}, \\
F_{34} &= e^4 T_{34}, & F_{35} &= e^3 T_{35}, & F_{36} &= e^4 T_{36}, \\
F_{37} &= (1 + 2\epsilon)e^2 T_{37}, & F_{38} &= e^4 T_{38}, & F_{39} &= e^3 T_{39}, \\
F_{40} &= e^3 T_{40}, & F_{41} &= e^3 T_{41}, & F_{42} &= e^4 T_{42}, \\
F_{43} &= e^4 T_{43}, & F_{44} &= e^4 T_{44}, & F_{45} &= e^4 T_{45}, \\
F_{46} &= e^4 T_{46}, & F_{47} &= e^4 T_{47}, & F_{48} &= e^4 T_{48}, \\
F_{49} &= e^4 T_{49}, & F_{50} &= e^4 T_{50}, & F_{51} &= e^4 T_{51}, \\
F_{52} &= e^4 T_{52},
\end{align*}
\]

where the \( T_i \) are the integrals depicted in figures 2 and 3.

The numerators of integrals \( F_{49,\ldots,52} \), are found by following the ideas in [49], i.e. by determining a set of canonical integrals through the inspection of their four-dimensional maximal-cuts. To this aim, we first localize the integral over \( k_2 \), which corresponds to the non-planar part of the diagram specified by \( D_{2,3,6,7} \). By enforcing the additional constraints \( k_1^2 = 0 \) and \( s = 2k_1 \cdot (p_3 + p_4) \) (which originate from the cut of the propagators depending on \( k_1 \)), we obtain

\[
\int d^4k_1 \frac{N(k_1)}{D_1D_4D_5} \int d^4k_2 \delta_2 \delta_3 \delta_6 \delta_7 = \int d^4k_1 \frac{N(k_1)}{D_1D_4D_5} \frac{1}{((k_1-p_3)^2-m^2)((k_1-p_4)^2-m^2)},
\]

(3.3)

where we have denoted \( \delta_i = \delta(D_i) \) and assumed the integral to contain some arbitrary numerator depending on \( k_1 \). From eq. (3.3), we observe that the maximal-cut of the one-loop subdiagram exposes two hidden propagators, \( D_{10} = (k_1-p_3)^2-m^2 \) and \( D_{11} = (k_1-p_4)^2-m^2 \). The latter, together with the residual uncut propagators \( D_{1,4,5} \), form an one-loop pentagon integral, known to obey a non-canonical DEQ. Therefore, we choose the numerator factor \( N(k_1) \) such that it cancels one or both of the hidden propagators, resulting in either a box or triangle integral, which both satisfy canonical DEQ. In this way, we determine three out of the four numerators corresponding to the integrals \( F_{49,\ldots,51} \), as they are displayed in figure 3. The last numerator \( F_{52} \) is defined to contain, besides the factor \( D_{10} \), also the auxiliary denominator \( D_9 \), which, depending on \( k_2 \), ensures the linear independence from the other three basis integral of the sector, without spoiling the properties of the DEQs.

Once a basis of MIs with \( \epsilon \)-linear DEQs has been determined, we apply the Magnus exponential in order rotate the integrals of eq. (3.2) into a new basis of MIs \( I_i \) that satisfy canonical DEQs in both variables \( w \) and \( z \),

\[
\begin{align*}
I_1 &= F_1, & I_2 &= -s F_2, \\
I_3 &= m^2 F_4, & I_4 &= t F_4, \\
I_5 &= -2m^2 F_4 - (m^2 - t) F_5, & I_6 &= -u F_6,
\end{align*}
\]
\[ I_7 = 2m^2 F_6 - (u - m^2) F_7, \]
\[ I_9 = -s F_8 + \lambda_s \left( \frac{1}{2} F_8 + F_9 \right), \]
\[ I_{11} = -(m^2 - t) F_{11}, \]
\[ I_{13} = -(u - m^2) F_{13}, \]
\[ I_{15} = \lambda_s F_{15}, \]
\[ I_{17} = (s - \lambda_s) \left( \frac{3}{2} F_{15} + m^2 F_{16} \right) - m^2 s F_{17}, \]
\[ I_{19} = \lambda_s F_{19}, \]
\[ I_{21} = (s - \lambda_s) \left( F_{15} + 2m^2 F_{16} + F_{18} - 2F_{20} \right) - m^2 s F_{21}, \]
\[ I_{22} = -s t F_{22}, \]
\[ I_{24} = u s F_{24}, \]
\[ I_{26} = -(u - m^2) F_{26}, \]
\[ I_{28} = -(m^2 - t) \lambda_s F_{28}, \]
\[ I_{30} = -(u - m^2) \left( m^2 F_{30} + \frac{2m^2 + \lambda_s - s}{2} F_{31} \right), \]
\[ I_{32} = -(m^2 - t) F_{32}, \]
\[ I_{34} = -(u - m^2) F_{34}, \]
\[ I_{36} = \lambda_s F_{36}, \]
\[ I_{38} = \lambda_s F_{38}, \]
\[ I_{40} = -m^2 (u - m^2) F_{40}, \]
\[ I_{42} = (m^2 - t) s F_{42}, \]
\[ I_{44} = -\sqrt{m^2(m^2 - t)(u - m^2)(-s)} F_{44}, \]
\[ I_{45} = -\lambda_s F_{38} - (m^2 - t) \left( (m^2 - s) F_{44} + F_{45} - F_{46} \right), \]
\[ I_{46} = (u - m^2) \left( F_{38} + m^2 F_{44} - F_{46} \right), \]
\[ I_{48} = -s \lambda_s F_{48}, \]
\[ I_{50} = (m^2 - t) s F_{50}, \]
\[ I_{52} = \sum_{i=2}^{8} c_{i,52} F_i + c_{10,52} F_{10} + c_{11,52} F_{11} + c_{13,52} F_{13} + c_{15,52} F_{15} + c_{17,52} F_{17}, \]
\[ + c_{18,52} F_{18} + \sum_{i=38}^{35} c_{i,52} F_i + \sum_{i=44}^{41} c_{i,52} F_i + \sum_{i=49}^{46} c_{i,52} F_i + \sum_{i=49}^{52} c_{i,52} F_i, \]

where we introduced the abbreviation \( \lambda_s = \sqrt{-s\sqrt{4m^2 - s}} \). The expressions given in eq. (3.2) and eq. (3.4) are provided in electronic format in the ancillary files of the arXiv version of the manuscript.
By combining the two DEQs in $w$ and $z$ into a single total differential, we write

$$dI = \epsilon dA I,$$

(3.5)

where $I$ is a vector that collects the 52 MIs and

$$dA = \sum_{i=1}^{12} M_i \, d\log(\eta_i),$$

(3.6)

with $M_i$ being the constant matrices with rational entries (provided as ancillary files in the arXiv submission of this paper). The arguments $\eta_i$ of this $d\log$-form, which define the so-called alphabet of the DEQs, can be taken to be the same 12 letters that appear in the calculation of the MIs for the QED-like topology $A_1$ presented in ref. [21]:

$$\eta_1 = w, \quad \eta_2 = 1 + w,$$
$$\eta_3 = 1 - w, \quad \eta_4 = z,$$
$$\eta_5 = 1 + z, \quad \eta_6 = 1 - z,$$
$$\eta_7 = w + z, \quad \eta_8 = z - w,$$
$$\eta_9 = z^2 - w, \quad \eta_{10} = 1 - w + w^2 - z^2,$$
$$\eta_{11} = 1 - 3w + w^2 + z^2, \quad \eta_{12} = z^2 - w^2 - w z^2 + w^2 z^2.$$

(3.7)

For the MIs of the $N_2$ topology computed in this paper, the matrix $M_{11}$ vanishes identically. Nevertheless, we adopted the above notation for consistency with ref. [21]. In the present work, we compute the MIs in the kinematic region where all the letters are real and positive,

$$0 < w < 1 \land \sqrt{w} < z < \sqrt{1 - w + w^2},$$

(3.8)

which corresponds to the unphysical region

$$t < 0 \land s < 0.$$

(3.9)

Since all the integrals defined in eq. (3.4) are finite in the $\epsilon \to 0$ limit, the vector $I(\epsilon, w, z)$ can be Taylor expanded in $\epsilon$ as

$$I(\epsilon, w, z) = I^{(0)}(w, z) + \epsilon I^{(1)}(w, z) + \epsilon^2 I^{(2)}(w, z) + \ldots,$$

(3.10)

where the $n$-th order coefficient is given by

$$I^{(n)}(w, z) = \sum_{i=0}^{n} \Delta^{(n-i)}(w, z; w_0, z_0) I^{(i)}(w_0, z_0),$$

(3.11)

with $I^{(i)}(w_0, z_0)$ being a constant vector and $\Delta^{(k)}$ the weight-$k$ operator

$$\Delta^{(k)}(w, z; w_0, z_0) = \int_{\gamma} dA \ldots dA, \quad \Delta^{(0)}(w, z; w_0, z_0) = 1,$$

(3.12)

that iterates $k$ ordered integrations of the 1-form $dA$ along a piecewise-smooth path $\gamma$ in the $wz$-plane. Since the rational alphabet given in eq. (3.7) has only algebraic roots, we...
can directly express the iterated integrals of eq. (3.12) in terms of GPLs, which are defined as
\[
G(\vec{a}_n; x) \equiv G(a_1, \vec{a}_{n-1}; x) \equiv \int_0^x dt \frac{1}{t - a_1} G(\vec{a}_{n-1}; t),
\]
(3.13)
\[
G(\vec{0}_n; x) \equiv \frac{1}{n!} \log^n(x).
\]
(3.14)

The length \(n\) of the vector \(\vec{a}_n\) corresponds the transcendental weight of \(G(\vec{w}_n; x)\) and it amounts to the number of iterated integrations that define the GPL. The GPLs in our solution were obtained by first integrating in \(w\) and then in \(z\) and therefore fall into two classes, namely GPLs in \(w\), with weights drawn from the set
\[
\left\{ 0, \pm 1, \pm z, z^2, \frac{z \left( z \pm \sqrt{4 - 3z^2} \right)}{2(z^2 - 1)}, \frac{1}{2} \left( 1 \pm \sqrt{4z^2 - 3} \right) \right\},
\]
(3.15)
and GPLs in \(z\), with weights drawn from
\[
\{ 0, \pm 1 \}.
\]
(3.16)

In the region defined by eq. (3.8), the explicit imaginary parts of our solution \(I(\epsilon, w, z)\) originate exclusively from the integration constants \(I^{(i)}(w_0, z_0)\).

### 3.2 Boundary conditions

Once a general solution of the system of DEQs in terms of GPLs has been obtained from the integration of eq. (3.5), it must be complemented by a suitable set of boundary conditions. The boundary conditions can be determined either from the knowledge of the analytic expression of the MIs in special kinematic configurations or by imposing their regularity at the pseudo-thresholds of the DEQs. In the case under study, the regularity conditions express the boundary constant as combinations of GPLs of argument 1, with complex weights, which arise from the kinematic limits imposed on the alphabet given in eq. (3.7). We made use of GiNaC to numerically verify that, for each MI, and at every order in \(\epsilon\), the corresponding combination of constant GPLs is proportional to a uniform combination of the transcendental constants \(\pi, \zeta_k\) and \(\log 2\).

In the following, we specify how the boundary constants of each integral have been derived:

- the integrals \(I_1, \ldots, 7, 10, \ldots, 14, 32, \ldots, 37\) are either common to the two-loop topologies discussed ref [20, 21], to which we refer the reader for the discussion of the boundary fixing, or related to them by simple kinematic crossing, i.e. by some interchange of the Mandelstam invariants;
- the boundary constants of \(I_8, 9, 18\) have been fixed by demanding finiteness of the MIs in the limit \(s \to 0\).

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• the boundary constants of $I_{15,16,17,20,21,38,47}$ have been obtained by demanding finiteness of the MIs in the limit $s \rightarrow 4m^2$. Additional constraints for the integrals $I_{15,16,17,20,21}$ have been obtained by requiring their corresponding boundary constants to be real-valued;

• the boundary constants of $I_{22,...,25,39,...,44,46}$ have been fixed by imposing the finiteness of the MIs in the limit $t \rightarrow m^2 \sqrt{4m^2 - s - \sqrt{s}}$;

• the boundary constants of $I_{26,...,31}$ have been determined by demanding the finiteness of the MIs the limits $s \rightarrow 0$ and $t \rightarrow m^2 \frac{\sqrt{4m^2 - s - \sqrt{s}}}{\sqrt{4m^2 - s + \sqrt{s}}}$;

• the boundary constants of $I_{49,...,52}$ have been obtained by demanding the finiteness of the MIs in the limits $s \rightarrow 4m^2$ and $t \rightarrow m^2 \frac{\sqrt{4m^2 - s - \sqrt{s}}}{\sqrt{4m^2 - s + \sqrt{s}}}$;

• the boundary constant of $I_{48}$ has been fixed by demand the massless limit, as described in [20].

The analytic expressions of the MIs are given in electronic form in the ancillary files attached to the arXiv version of the manuscript.

4 Analytic continuation

The results of section 3 have been obtained in the unphysical region $s,t < 0$. Therefore, the analytic continuation of such expressions to the $t\bar{t}$ production kinematics must be performed. In terms of the Mandelstam invariants, this region is defined by

$$s \geq 4m^2 \land m^2 - \frac{s}{2} - \frac{1}{2} \sqrt{s - 4m^2 \sqrt{s}} \leq t \leq m^2 - \frac{s}{2} + \frac{1}{2} \sqrt{s - 4m^2 \sqrt{s}},$$

where the boundaries of the allowed interval for $t$ are in one-to-one correspondence, in the center-of-mass frame, with the minimum and maximum scattering angles of the $t\bar{t}$ pair with respect to the beam line. For completeness, we also quote the physical region for elastic scattering, corresponding to the crossed $t$-channel process:

$$t \geq m^2 \land -t \left(1 - m^2 \frac{2}{t}\right)^2 \leq s \leq 0 \land 2m^2 - t \leq u \leq \frac{m^2}{t}.$$  

In the case of non-planar four-point integrals, the analytic continuation of the GPLs is quite non trivial. As originally noted in [50, 51], thresholds associated with all the Mandelstam invariants appear simultaneously, and $s, t, u$ should be treated as independent variables when discussing the analytic continuation of the expressions for the MIs. On the other hand, the approach we follow enforces the constraint $s + t + u = 2m^2$ from the outset, yielding a system of DEQs in two variables, e.g. $s$ and $t$. One way out could be to enforce the Mandelstam constraint only at a later stage (see e.g. [52]), at the price of considerably complicating the problem to be solved due to the presence of an extra scale.
In this paper we address the analytic continuation in a different way by exploiting the iterated-path-integral nature of our canonical MIs, together with the so-called first-entry condition [53, 54], in order to devise an effective prescription. Our approach allows to analytically continue the MIs everywhere in the kinematic plane, and in particular to evaluate our results in the $t\bar{t}$ production region. As a byproduct of our analysis, we also obtain the analytic continuation of the MIs for $\mu e$ scattering, presented in [21], both to the di-muon production region and to the elastic scattering region.

We already observed in section 3 that our canonical basis of MIs can be expressed, order by order in $\epsilon$, as a linear combination of GPLs and constants of uniform weight. From the first-entry condition it follows that only the innermost integration contributes to the discontinuities of the MIs. Strong restrictions on the analytic structure of the MIs are imposed already at the level of the canonical DEQs (by the coefficient matrices in the $d \log$ form), but knowledge of the boundary conditions is essential to fully pin it down. By inspection of our result (computed up to weight 4 in the region $s,t<0$), we find that the GPLs originating from the innermost integration are the following: $G_0(z), G_0(w), G_1(w), G_{z2}(w)$. This can be traced back to the fact that, of all the letters $\eta_i$ that appear in $dA$ (see eq.(3.7)), only four contribute to the first integration, namely $\eta_1, \eta_3, \eta_4, \eta_9$. Quite remarkably, one can build four combinations of the $\eta_1, \eta_3, \eta_4, \eta_9$ that correspond to simple functions of the Mandelstam invariants, whose logarithms exhibit the branch cuts expected from the normal thresholds of the four sunrise sub-topologies. If we define

$$\eta_1 = \theta_1 = \frac{\sqrt{4-s/m^2} - \sqrt{-s/m^2}}{\sqrt{4-s/m^2} + \sqrt{-s/m^2}},$$

(4.3)

$$\eta_2/\eta_9 = \theta_2 = 1 - t/m^2,$$

(4.4)

$$\eta_3/\eta_1 = \theta_3 = -s/m^2,$$

(4.5)

$$(\eta_3\eta_1)^2/(\eta_1\eta_9) = \theta_4 = u/m^2 - 1,$$

(4.6)

where use has been made of the relation $s + t + u = 2m^2$, then one can easily find the following GPL representations for the corresponding logarithmic differentials

$$d \log \theta_1 = dG_0(w),$$

(4.7)

$$d \log \theta_2 = 2dG_1(w) - 2dG_0(z) - dG_{z2}(w),$$

(4.8)

$$d \log \theta_3 = 2dG_1(w) - dG_0(w),$$

(4.9)

$$d \log \theta_4 = 2dG_1(w) - dG_0(w) - dG_{z2}(w).$$

(4.10)

We refer to log $\theta_{1,2,3,4}$ as to the physical logarithms. In figure 4 we show the physical regions for the $s$-channel production and the $t$-channel scattering processes, together with the region in which we solved the system of differential equations, and the thresholds of the physical logarithms. For completeness we also give a more transparent rearrangement of the other letters:

$$\eta_{12}/(\eta_1\eta_9) \equiv \theta_5 = u/m^2$$

(4.11)

$$\eta_{10}/\eta_9 \equiv \theta_6 = -t/m^2$$

(4.12)
Figure 4: In the plot we show a representative portion of the kinematic phase space, parametrized in terms of \((s,t)\). The region in which we solved the system of differential equations, \(s,t < 0\), is marked in green. The physical region for the \(s\)-channel production process, given in eq. (4.1), is highlighted in blue. In orange we also show the physical region for the \(t\)-channel process, given in eq. (4.2), which is relevant for \(\mu e\) scattering. The dashed lines indicate the thresholds of the physical logarithms in eq. (4.7)-(4.10).

\[
\eta_2^2/\eta_1 \equiv \theta_7 = 4 - s/m^2 \\
\eta_2^3/\eta_1^2 \eta_7^2 \eta_8/(\eta_1 \eta_6) \equiv \theta_8 = 1 - tu/m^4
\]

\[
\eta_6/\eta_5 \equiv \theta_{10} = \frac{1 - \sqrt{u-m^2}}{\sqrt{v-m^2}} \frac{\sqrt{4m^2-s+\sqrt{s}}}{\sqrt{4m^2-s-\sqrt{s}}} \\
1 + \frac{\sqrt{u-m^2}}{\sqrt{v-m^2}} \frac{\sqrt{4m^2-s-\sqrt{s}}}{\sqrt{4m^2-s+\sqrt{s}}}
\]

\[
\eta_8/\eta_7 \equiv \theta_{11} = \frac{1 - \sqrt{u-m^2}}{\sqrt{v-m^2}} \frac{\sqrt{4m^2-s-\sqrt{s}}}{\sqrt{4m^2-s+\sqrt{s}}} \\
1 + \frac{\sqrt{u-m^2}}{\sqrt{v-m^2}} \frac{\sqrt{4m^2-s+\sqrt{s}}}{\sqrt{4m^2-s-\sqrt{s}}}
\]

\[
\eta_{11}/\eta_9 \equiv \theta_{12} = 2 - t/m^2
\]

One can prove that, in the region \(s,t < 0\), eqs. (4.7)-(4.10) also hold if the total differential operator is dropped, without adding any integration constants. By choosing a suitable
analytic continuation prescription on the Mandelstam invariants one can then evaluate
the integrated expressions in the full kinematic plane, in an unambiguous way. One can
then check whether those expressions reproduce the imaginary parts of the corresponding
physical logarithms. The simple prescription we adopt is defined in two steps:

1. As for the Mandelstam invariants, we express the real part of \( u \) in terms of the real
   parts of \( s \) and \( t \), for which we use the standard Feynman prescription, but we give \( u \)
an independent prescription (i.e. before using \( u(s, t) = 2m^2 - s - t \) in the definition
of \( z \), eq. (3.1))

   \[
   \begin{align*}
   s &\rightarrow s + i\varepsilon \theta(s), \\
   t &\rightarrow t + i\varepsilon \theta(t - m^2), \\
   u(s, t) &\rightarrow 2m^2 - s - t - i\varepsilon,
   \end{align*}
   \]

   where \( i\varepsilon \) is an infinitesimal positive imaginary part, \( \theta(x) \) is the Heaviside step func-
tion, and the presence of the constant \( i\varepsilon \) term in the last equation guarantees that the
evaluation of the GPLs on spurious branch cuts (that are developed even for \( s, t < 0 \))
is always unambiguous. It can be shown, by repeated application of the identity

   \[
   \log ab = \log a + \log b + 2\pi i \left[ \theta(\text{Im } a)\theta(\text{Im } b)\theta(\text{Im } ab) \\
   - \theta(\text{Im } a)\theta(\text{Im } b)\theta(-\text{Im } ab) \right],
   \]

   that the above prescription is sufficient to reproduce, with the GPL representation
in the variables \((w, z)\) of eqs (4.7)-(4.10), the physical logarithms \( \log \theta_1,2,3 \) (that, for
instance, completely determine the analytic structure of the \( s \)-channel sunrise MIs
and the \( t \)-channel sunrise MIs).

2. It remains to be verified whether the above prescription allows to correctly repro-
duce also the imaginary part of \( \log \theta_4 \) (the one that, for instance, determines the
discontinuity of the \( u \)-channel sunrise MIs across the one-particle branch cut). This
is not guaranteed since, as stressed at the beginning of this section, we only have
two independent variables at our disposal, while having to deal simultaneously with
thresholds in all the three channels. The virtue of our prescription (4.18)-(4.20) is
that the representation of \( \log \theta_4 \) in terms of GPLs (corresponding to eq. (4.10))

   - for \( u > m^2 \) is always on the physical Riemann sheet;
   - for \( u < m^2 \) always ends up on the wrong side of the branch cut, i.e. the imaginary
     part is always \(-i\pi\) instead of \(+i\pi\).

We can therefore apply, as a second step, a simple correction to the combination of
GPLs corresponding to \( \log \theta_4 \) (and \textit{not} to the other three combinations) to bring our
GPL representation for the MIs to the correct Riemann sheet. At weight one one can
perform the replacement

   \[
   2G_1(w) - G_0(w) - G_{z2}(w) \rightarrow 2G_1(w) - G_0(w) - G_{z2}(w) + 2\pi i\theta(m^2 - u),
   \]

   which is then propagated iteratively to higher weights.
All the explicit imaginary constants in our solution, as stated in section 3, originate from the (iterated) integrations over $d\log \theta_4$. Indeed we integrate our DEQs in the region where $s, t < 0$, so that $u > 2m^2$. The combinations of GPLs corresponding to such integrations (at any weight) are then always accompanied by a constant term, namely an additional $-i\pi$. Therefore, the net effect of the correction (4.22) is to flip the sign of the imaginary constants in our solution. In summary, our effective way of implementing the analytic continuation of the result for the full set of MIs is

1. to use the prescription on the Mandelstam invariants of equations (4.18)-(4.20),

2’. to replace $i\pi \rightarrow -i\pi$ everywhere in the solution, whenever the latter is evaluated for $u(s, t) < m^2$.

Remarkably, once the analytic continuation of the four physical logarithms (i.e. of the weight-1 contribution to the canonical MIs for the four sunrise topologies) is taken care of explicitly [51], the first-entry condition guarantees that the analytic continuation of the full set of MIs at all weights is also correctly obtained. In particular, it is not necessary to make sure the GPL representation also reproduces the imaginary parts coming from the evaluation close to the branch cuts of the logarithms of the unphysical letters, eqs. (4.11)-(4.17). It is instead sufficient to always introduce an “auxiliary” $\epsilon\pi$ prescription in order to avoid the ambiguous evaluation on such branch cuts (in our case this is inherited from (4.18)-(4.20)). Our strategy for the analytic continuation has been validated by thorough numerical checks performed either with the help of SecDec or with the techniques outlined in section 5.

It is now clear that we can also obtain, by the same argument, the analytic continuation of the results for the MIs of the QED-like topology $A_1$ (see figure 1) presented in [21]. The only difference with respect to the present case (besides a trivial relabeling of the Mandelstam invariants $s \leftrightarrow t$ to match the notations), is that the letter $\eta_{11}$ contributes to the $d\log$ form with a non-zero coefficient matrix (but never appears in the first entry), and that $\eta_1 = \theta_1$ is not a physical logarithm anymore (as expected due to the absence of a two-massive-particles normal threshold). Since the analytic continuations of $\log \theta_1$ and $\log \theta_3$ are not independent, in practice this difference does not change the situation, and the same procedure described above can then be used for the analytic continuation of the MIs from $s, t < 0$ to the physical regions for the $\mu e \rightarrow \mu e$ and $e^+e^- \rightarrow \mu^+\mu^-$ processes, as confirmed also in this case by our numerical checks.

We stress that the method outlined above is fully general, as it only relies on the analyticity properties of the canonical MIs, and on their iterated-integral representation. It is in particular independent of the presence of massive propagators or massive external legs.

5 Numerical validation of the non-planar box integrals

Using the analytic continuation as described in the previous section the expressions for our MIs have been numerically evaluated in several points across the whole phase space,
including the euclidean region $s, t < 0$ and the physical region, eq. (4.1). In order to cross-check our analytic calculation, we numerically computed the MIs (or linear combinations of the latter) in some benchmark points with an alternative method, namely by integrating directly their Feynman-parameter representation. In particular, the integrals $I_i$ with $i = 1, \ldots, 48$ were computed with the package SecDec. For the most complex topologies, corresponding to the non-planar box integrals $I_i$ with $i = 49, \ldots, 52$, we used Reduze to identify an alternative set of independent MIs that are quasi finite in $d = 6$. On the one hand the latter have been computed semi-numerically by means of an in-house algorithm [21]. On the other hand these integrals are analytically related to our set of MIs by dimension-shifting identities [55, 56] and IBPs, implemented in LiteRed.

The definition of the 4 non-planar 7-denominator MIs that are finite in $d = 6$ dimensions, together with our results at the phase-space point $s = -1/7$, $t = -1/3$, $m^2 = 1$, are collected in table 1. In the next subsection, we use the first of those integrals as an example to describe our evaluation strategy. Throughout this section, we set $m^2 = 1$ and $u = 2 - s - t$.

5.1 The non-planar box in $d = 6$ dimensions

As an example, we describe the numerical evaluation of the non-planar scalar integral

$$I^{[d]}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0),$$

(5.1)

carried out in two steps.
5.1.1 Analytic integrations

By using Feynman parametrisation, the integral can be written as

\[ \Gamma(7) \int \frac{d^4k_1d^4k_2}{\prod x_i} \int_0^1 dx_1 \ldots \int_0^1 dx_7 \frac{\delta(1 - x_{1234567})}{D_{\text{tot}}}, \]  

(5.2)

where

\[ D_{\text{tot}} = x_1D_1 + x_2D_2 + x_3D_4 + x_4D_5 + x_5D_6 + x_6D_3 + x_7D_7 + i\varepsilon. \]  

(5.3)

After integrating over \( k_1 \) and \( k_2 \), one finds

\[ \Gamma^2(7 - d) \int_0^1 dx_1 \ldots \int_0^1 dx_7 \frac{\delta(1 - x_{1234567})}{A_{\alpha-\beta}^{\frac{d}{2}-d}} \]  

(5.4)

where

\[ A_0 = x_{34}x_{56} + x_5x_6 + x_{346}x_7 + x_2x_{3457} + x_1x_{2567}, \]

\[ \Delta = x_2^2x_{3457} + x_1x_{25} + x_2x_5(2x_{34} + x_5) + x_2^2x_{346} + (t - 1)(x_3(-x_5x_6 + x_2x_7) + s(x_2(-x_4x_5 + x_3x_7) - x_1(x_4x_{26} + x_4x_5)) - i\varepsilon A_0, \]  

(5.5)

where we used the notation \( x_{i_1i_2 \ldots i_n} = x_{i_1} + x_{i_2} + \ldots + x_{i_n} \) and the abbreviation \( \Gamma_\epsilon \equiv \Gamma(1+\epsilon) \). We perform as many analytic integrations as possible. In particular, we integrate over \( x_3 \) eliminating the \( \delta \)-function, and we make the changes of variables \( x_6 \rightarrow x_{26} - x_2, x_7 \rightarrow x_{57} - x_5 \). In this way, the polynomial \( \Delta \) becomes linear in \( x_4 \), so that eq. (5.4) becomes

\[ \Gamma^2(7 - d) \int_0^1 dx_{26} \int_0^{1-x_{26}} \frac{dx_{57}}{A_{\alpha-\beta}^{\frac{d}{2}-d}} \int_0^{1-x_{26}57} dx_1 \int_0^{x_{26}} dx_2 \int_0^{x_{57}} dx_5 \times \]

\[ \int_0^{1-x_{12657}} \frac{dx_4}{(C_{41}x_4 + C_{40})^{7-d}}, \]  

(5.6)

where

\[ A = x_{2657}(1 - x_{2657}) + x_{26x_{57}}, \]

\[ C_{41} = (t - 1)(x_{26}x_5 - x_2x_{57}) - s(x_2x_{57} + x_1x_{2657}), \]

\[ C_{40} = x_2^2(1 - x_{26}) + x_2^2(1 - x_{57}) + 2x_2x_5(1 - x_{2657}) + (t - 1)(1 - x_{12657})(x_{26}x_5 - x_2x_{57}) + s(x_{57} - x_5)(x_2(1 - x_{2657}) - x_1x_{26}) - i\varepsilon A. \]  

(5.7)

The integral over \( x_4 \) is eq. (5.6) is finite for \( d \rightarrow 6 \), and, in this limit, we get

\[ \Gamma^2(7 - d) = - \int_0^1 dx_{26} \int_0^{1-x_{26}} \frac{dx_{57}}{A^2} \int_0^{x_{57}} dx_5 \int_0^{x_{26}} dx_2 \int_0^{1-x_{26}57} dx_1 \ln \frac{g_2^2x_1^2 + g_1x_1 + g_0}{f_2 - g_2x_1} \left( \frac{g_2x_1^2 + g_1x_1 + g_0}{f_3x_1 + P_6} \right), \]  

(5.8)
where

\[ f_1 = (t - 1)x_{26}x_5 - (s + t - 1)x_{26}x_{57} - sx_{2657}(1 - x_{2657}), \]
\[ f_2 = (t - 1)x_{26}x_5 - (s + t - 1)x_{26}x_{57}, \]
\[ f_3 = (s + t - 1)x_{26}x_5 + x_2x_{57} - (tx_2 + s x_{26})x_{57}, \]
\[ P_6 = x_3^2(1 - x_{26}) + x_5^2(1 - x_{57}) + (2 - s)x_2x_3(1 - x_{2657}) - f_2(1 - x_{2657}) - i \varepsilon A, \]
\[ g_0 = P_6 + f_2(1 - x_{2657}), \]
\[ g_1 = s(x_{26}x_5 + x_2x_{57} - A), \]
\[ g_2 = sx_{2657}, \]
\[ P_2 = \frac{g_1 + \sqrt{g_1^2 - 4g_0g_2}}{2g_2}, \]
\[ P_4 = \frac{g_1 - \sqrt{g_1^2 - 4g_0g_2}}{2g_2}, \]
\[ P_1 = P_2 + 1 - x_{2657}, \]
\[ P_3 = P_4 + 1 - x_{2657}, \]
\[ P_5 = P_6 + f_3(1 - x_{2657}) = sx_{2657}P_1P_3. \]

(5.9)

Finally, we integrate over \(x_1\), and reduce eq. (5.8) to

\[
\Gamma_2 P_6 \Gamma_{x_{26}x_5}^{d=6} x_{26} x_{57} x_{2657} = \int_0^1 dx_{26} \int_0^{1-x_{26}} dx_{57} A^2 x_{2657} x_2 \int_0^{1-x_{26}} dx_{2} \int_0^{x_{57}} dx_{5} \times
\[
\left( \text{Li}_2 \left( \frac{Q_1}{R_1} \right) - \text{Li}_2 \left( \frac{Q_2}{R_1} \right) + \text{Li}_2 \left( \frac{Q_3}{R_2} \right) - \text{Li}_2 \left( \frac{Q_4}{R_2} \right) - \text{Li}_2 \left( \frac{Q_5}{R_3} \right) + \text{Li}_2 \left( \frac{Q_6}{R_3} \right) \right),
\]

(5.10)

where

\[ Q_1 = Q_3 = f_1, \quad Q_2 = Q_4 = f_2, \quad Q_5 = f_1f_3, \quad Q_6 = f_2f_3, \]
\[ R_1 = Q_1 + P_1sx_{2657} = Q_2 + P_2sx_{2657}, \]
\[ R_2 = Q_3 + P_3sx_{2657} = Q_4 + P_4sx_{2657}, \]
\[ R_3 = Q_5 + P_5sx_{2657} = Q_6 + P_6sx_{2657} = R_1R_2. \]

(5.11)

Differently from [21], we have 6 dilogarithms, and \(P_1, P_2, P_3, P_4, R_1\) and \(R_2\) are algebraic functions which contain the square root of the same polynomial.

5.1.2 Numerical integrations

The four remaining integration variables in eq. (5.10) are rescaled, and mapped onto a four-dimensional hypercube of unit side,

\[ x_{26} = t_1, \quad x_{57} = (1 - x_{26})t_2, \quad x_2 = x_{26}t_3, \quad x_5 = x_{57}t_4; \]

(5.12)
so that the new variables $t_i$ have to be integrated over $[0, 1]$. At this point, we have to consider the branch points of the dilogarithms that appear in eq. (5.10), which correspond the hypersurfaces defined by the equations

$$R_i(t_1, t_2, t_3, t_4) = 0, \quad P_j(t_1, t_2, t_3, t_4) = 0, \quad i = 1, \ldots 3, \quad j = 1, \ldots 6. \quad (5.13)$$

It is necessary to sample carefully the integrand near these branch points. Therefore, for the integration over $t_4$, we split the integration interval at the $N_4(t_1, t_2, t_3)$ real solutions $z_{4j}(t_1, t_2, t_3)$ of eq. (5.13) which are on the interval $[0, 1]$,

$$\int_0^1 dt_4 = \sum_{j=0}^{N_4-1} \int_{z_{4j}(t_1, t_2, t_3)}^{z_{4j+1}(t_1, t_2, t_3)} dt_4, \quad z_{40} = 0, \quad z_{4N_4} = 1. \quad (5.14)$$

Analogously, for the integration over $t_3$, we split the integration interval at the $N_3(t_1, t_2)$ real zeros $z_{3j}(t_1, t_2)$ of the discriminants (polynomials in $(t_1, t_2, t_3)$) that appear in the zeros $z_{4j}$. These are the points where the hypersurfaces of eq. (5.13) are tangent to the hyperplane $t_4 = \text{constant}$,

$$\int_0^1 dt_3 = \sum_{j=0}^{N_3-1} \int_{z_{3j}}^{z_{3j+1}} dt_3, \quad z_{30} = 0, \quad z_{3N_3} = 1. \quad (5.15)$$

Analogously, for the integration over $t_2$, we split the integration interval at the $N_2(t_1)$ zeros $z_{2j}(t_1)$ of the discriminants (polynomials in $(t_1, t_2)$) that appear in the zeros $z_{3j}$,

$$\int_0^1 dt_2 = \sum_{j=0}^{N_2-1} \int_{z_{2j}}^{z_{2j+1}} dt_2, \quad z_{20} = 0, \quad z_{2N_2} = 1. \quad (5.16)$$

We proceed in a similar way for the last integration,

$$\int_0^1 dt_1 = \sum_{j=0}^{N_1-1} \int_{z_{1j}}^{z_{1j+1}} dt_1, \quad z_{10} = 0, \quad z_{1N_1} = 1. \quad (5.17)$$

To carry out the integration over a generic interval $[t_a, t_b]$, we perform the change of variables $t_i \to u_i$, with

$$t_i = t_{ai} + \frac{e^{u_i}}{e^{u_i} + 1}(t_{bi} - t_{ai}), \quad i = 1, \ldots 4, \quad (5.18)$$

in order to deal with possible singularities at the endpoints. The variable $u_i$ should be integrated in $(-\infty, \infty)$ but we actually truncate the integration domain to $(-M, +M)$, with $M$ suitably large (typically $M \sim 4$), and we use Gauss-Legendre integration over 16 points. Note that all the singularities in the integrands are logarithmic, and therefore integrable, so we can safely set a very small value of $\varepsilon$, like $10^{-30}$.

By using 16 subdivisions in each interval and in every variable we find that our integral, in the phase space point $s = -1/7, t = -1/3, m^2 = 1$, amounts to
\[
\left. \begin{array}{c}
\text{(5.19)} \\
\end{array} \right. \\
\]

A similar procedure is adopted for the other integrals in Tab. 1. Case-by-case, after the analytic integrations, the corresponding integrands, in the \( d \to 6 \) limit, are found to be combinations of logarithms, so that the decomposition of the integration domain, and the numerical integration can be carried out along the same lines as for the non-planar scalar box integral.

6 Conclusions

In this paper, we presented the analytic expressions of the master integrals for a set of non-planar two-loop Feynman graphs, with two quark currents exchanging massless gauge bosons. Our results are the last missing ingredients required for the analytic evaluation of the double-virtual contribution to the scattering amplitude for the process \( q\bar{q} \to t\bar{t} \) at NNLO in QCD, which was so far known only numerically. The present computation completes the calculation of all the required master integrals and, hence, proves that the analytic evaluation of such amplitude is indeed feasible.

The two-loop integrals were evaluated by means of the differential equation method, which, combined with the ideas of the Magnus exponential matrix and of the canonical basis, yielded a representation of the master integrals in terms of generalised polylogarithms. The canonical systems of differential equations was conveniently solved in a non-physical region. Subsequently, we studied, in the presence of massive internal lines and of a non-trivial structure of branch cuts, the analytic continuation of the two-loop functions to the physical region relevant for the process under consideration.

The results of this paper represent an important milestone of the research programme dedicated to the evaluation of integrals originating from the planar and non-planar two-loop four-point diagrams that contribute to both QED and QCD processes, which include, beside the top-pair production at hadron colliders, also di-muon production at lepton colliders, as well as muon-electron elastic scattering, which is the investigation target of the novel, proposed CERN experiment MUone.

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A Canonical master integrals for $q\bar{q} \rightarrow t\bar{t}$

In this appendix we give the explicit expression of the kinematic coefficients appearing in the definition of canonical master integrals defined by eq. (3.4). The coefficients of the canonical MIs are given by

\[ c_{2,52} = \frac{\lambda_s - s}{2}, \]
\[ c_{3,52} = \frac{7m^2(m^2 - t)\lambda_s}{4(u - m^2)s}, \]
\[ c_{4,52} = -\frac{(3m^2 + 4t)(\lambda_s - s)}{8s}, \]
\[ c_{5,52} = -\frac{3(m^2 - t)(\lambda_s - s)}{16s}, \]
\[ c_{6,52} = -\frac{1}{24(u - m^2)s}[(\lambda_s - s)(33m^4 - 45m^2t + 12t^2 + 25m^2s - 4st - 16s^2) + 14(m^2 - t)s(m^2 + 2u)], \]
\[ c_{7,52} = \frac{(9m^2 - 9t + 5s)\lambda_s + 5s(u - m^2)}{48s}, \]
\[ c_{8,52} = -\frac{\lambda_s - s}{4}, \]
\[ c_{9,52} = -\frac{(m^2 - t)(\lambda_s - s)}{4s}, \]
\[ c_{10,52} = -\frac{2m^2 - 2t + s)\lambda_s + s(u - m^2)}{4(\lambda_s - s)} , \]
\[ c_{11,52} = \lambda_s - s, \]
\[ c_{12,52} = -\frac{\lambda_s - s}{2}, \]
\[ c_{13,52} = \frac{m^2\lambda_s - s}{2}, \]
\[ c_{14,52} = \frac{m^2(m^2 - t)\lambda_s}{6} , \]
\[ c_{15,52} = \frac{m^2}{6}(\lambda_s - s), \]
\[ c_{16,52} = -\frac{5m^2 - 5t + s)\lambda_s + s(u - m^2)}{6} , \]
\[ c_{17,52} = \frac{m^2}{2}(\lambda_s - s), \]
\[ c_{18,52} = \frac{m^2}{4s}(\lambda_s - s) , \]
\[ c_{19,52} = \frac{3(m^2 - t)(\lambda_s - s)}{4s}, \]
\[ c_{20,52} = \frac{(2m^2 - 2t + s)\lambda_s + s(u - m^2)}{4s} , \]
\[ c_{21,52} = \frac{(m^2 - t)(2m^2 - s)(\lambda_s - s)}{8s}, \]
\[ c_{22,52} = \frac{(m^2 - 2t + s)\lambda_s - s(u - m^2)}{6}, \]
\[ c_{23,52} = -\frac{(5m^2 - 5t + s)\lambda_s + s(u - m^2)}{6}, \]
\[ c_{24,52} = \frac{(m^2 - 2t + s)\lambda_s - s(u - m^2)}{6} , \]
\[ c_{25,52} = \frac{m^2}{2}(\lambda_s - s), \]
\[ c_{26,52} = \frac{(m^2 - t)(2m^2 - s)(\lambda_s - s)}{8s}, \]
\[ c_{27,52} = -\frac{3(m^2 - t)(\lambda_s - s)}{4s}, \]
\[ c_{28,52} = -\frac{(m^2 - t + s)\lambda_s - s(u - m^2)}{4s} , \]
\[ c_{29,52} = -\frac{(m^2 - t)(\lambda_s - s)}{4s}, \]
\[ c_{30,52} = \frac{(m^2 - 2t + s)\lambda_s + s(u - m^2)}{4s} , \]
\[ c_{31,52} = \frac{2m^2 - 2t + s)\lambda_s + s(u - m^2)}{8s}, \]
\[ c_{32,52} = \frac{(m^2 - t)(2m^2 - 2t + s)\lambda_s + s(u - m^2)}{(u - m^2)s} , \]
\[ c_{33,52} = \frac{(3m^2 - 3t + s)\lambda_s + s(u - m^2)}{2(u - m^2)} , \]
\[ c_{34,52} = \frac{(t - u)\lambda_s + s(u - m^2)}{s} , \]
\[ c_{35,52} = \frac{m^2\lambda_s - s}{2}, \]
\[ c_{36,52} = \frac{(m^2 - t)(\lambda_s - s)}{4s}, \]
\[ c_{37,52} = \frac{m^2}{2}(\lambda_s - s), \]
\[ c_{38,52} = \frac{(m^2 - t)(\lambda_s - s)}{4s}, \]
\[ c_{39,52} = \frac{m^2}{4s}(\lambda_s - s), \]
\[ c_{40,52} = m^2 (m^2 - u + 2s) \lambda_s + 3s(u - m^2) \frac{\lambda_s}{4s}, \]
\[ c_{44,52} = -\frac{(\lambda_s - s)(2m^2u + st)}{2s}, \]
\[ c_{46,52} = (u - t) \frac{\lambda_s - s}{2s}, \]
\[ c_{50,52} = (m^2 - t) \frac{\lambda_s - s}{2s}, \]
\[ c_{52,52} = \lambda_s, \]

\[ c_{41,52} = (m^2 - t) (m^2 - u + s) \lambda_s + 2s(u - m^2) \frac{\lambda_s}{4s}, \]
\[ c_{45,52} = -(m^2 - t) \frac{\lambda_s - s}{2s}, \]
\[ c_{49,52} = \frac{(3m^2 - u) \lambda_s + u - m^2}{2}, \]
\[ c_{51,52} = \frac{\lambda_s - s}{2}, \]

(A.1)

References

[1] ATLAS collaboration, Measurements of top-quark pair differential cross-sections in the lepton+jets channel in pp collisions at \( \sqrt{s} = 8 \) TeV using the ATLAS detector, Eur. Phys. J. C\textbf{76} (2016) 538 [1511.04716].

[2] ATLAS collaboration, Measurements of top-quark pair differential cross-sections in the lepton+jets channel in pp collisions at \( \sqrt{s} = 13 \) TeV using the ATLAS detector, JHEP \textbf{11} (2017) 191 [1708.00727].

[3] CMS collaboration, Measurement of the differential cross section for top quark pair production in pp collisions at \( \sqrt{s} = 8 \) TeV, Eur. Phys. J. C\textbf{75} (2015) 542 [1505.04480].

[4] CMS collaboration, Measurement of differential cross sections for the production of top quark pairs and of additional jets in lepton+jets events from pp collisions at \( \sqrt{s} = 13 \) TeV, Phys. Rev. D\textbf{97} (2018) 112003 [1803.08856].

[5] M. Czakon, P. Fiedler and A. Mitov, Total Top-Quark Pair-Production Cross Section at Hadron Colliders Through \( O(\alpha_4^3) \), Phys. Rev. Lett. \textbf{110} (2013) 252004 [1303.6254].

[6] M. Czakon, P. Fiedler and A. Mitov, Resolving the Tevatron Top Quark Forward-Backward Asymmetry Puzzle: Fully Differential Next-to-Next-to-Leading-Order Calculation, Phys. Rev. Lett. \textbf{115} (2015) 052001 [1411.3007].

[7] M. Czakon, D. Heymes and A. Mitov, High-precision differential predictions for top-quark pairs at the LHC, Phys. Rev. Lett. \textbf{116} (2016) 082003 [1511.00549].

[8] M. Czakon, D. Heymes and A. Mitov, Dynamical scales for multi-TeV top-pair production at the LHC, JHEP \textbf{04} (2017) 071 [1606.03350].

[9] S. Catani, S. Devoto, M. Grazzini, S. Kallweit, J. Mazzitelli and H. Sargsyan, Top-quark pair hadroproduction at next-to-next-to-leading order in QCD, Phys. Rev. D\textbf{99} (2019) 051501 [1901.04005].

[10] P. Barrreuther, M. Czakon and P. Fiedler, Virtual amplitudes and threshold behaviour of hadronic top-quark pair-production cross sections, JHEP \textbf{02} (2014) 078 [1312.6279].

[11] L. Chen, M. Czakon and R. Poncelet, Polarized double-virtual amplitudes for heavy-quark pair production, JHEP \textbf{03} (2018) 085 [1712.08075].

[12] R. Bonciani, A. Ferroglia, T. Gehrmann, D. Maitre and C. Studerus, Two-Loop Fermionic Corrections to Heavy-Quark Pair Production: The Quark-Antiquark Channel, JHEP \textbf{07} (2008) 129 [0806.2301].
[13] R. Bonciani, A. Ferroglia, T. Gehrmann and C. Studerus, Two-Loop Planar Corrections to Heavy-Quark Pair Production in the Quark-Antiquark Channel, *JHEP* **08** (2009) 067 [0906.3671].

[14] R. Bonciani, A. Ferroglia, T. Gehrmann, A. von Manteuffel and C. Studerus, Two-Loop Leading Color Corrections to Heavy-Quark Pair Production in the Gluon Fusion Channel, *JHEP* **01** (2011) 102 [1011.6661].

[15] R. Bonciani, A. Ferroglia, T. Gehrmann, A. von Manteuffel and C. Studerus, Light-quark two-loop corrections to heavy-quark pair production in the gluon fusion channel, *JHEP* **12** (2013) 038 [1309.4450].

[16] G. Abelof, A. Gehrmann-De Ridder, P. Maierhofer and S. Pozzorini, NNLO QCD subtraction for top-antitop production in the $q\bar{q}$ channel, *JHEP* **08** (2014) 035 [1404.6493].

[17] G. Abelof, A. Gehrmann-De Ridder and I. Majer, Top quark pair production at NNLO in the quark-antiquark channel, *JHEP* **12** (2015) 074 [1506.04037].

[18] C. M. Carloni Calame, M. Passera, L. Trentadue and G. Venanzoni, A new approach to evaluate the leading hadronic corrections to the muon $g-2$, *Phys. Lett.* **B746** (2015) 325 [1504.02228].

[19] G. Abbiendi et al., Measuring the leading hadronic contribution to the muon $g-2$ via $\mu e$ scattering, *Eur. Phys. J.* **C77** (2017) 139 [1609.08987].

[20] P. Mastrolia, M. Passera, A. Primo and U. Schubert, Master integrals for the NNLO virtual corrections to $\mu e$ scattering in QED: the planar graphs, *JHEP* **11** (2017) 198 [1709.07435].

[21] S. Di Vita, S. Laporta, P. Mastrolia, A. Primo and U. Schubert, Master integrals for the NNLO virtual corrections to $e$ scattering in QED: the non-planar graphs, *JHEP* **09** (2018) 016 [1806.08241].

[22] J. M. Henn, Multiloop integrals in dimensional regularization made simple, *Phys.Rev.Lett.* **110** (2013) 251601 [1304.1806].

[23] M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk et al., Magnus and Dyson Series for Master Integrals, *JHEP* **1403** (2014) 082 [1401.2979].

[24] S. Di Vita, P. Mastrolia, U. Schubert and V. Yundin, Three-loop master integrals for ladder-box diagrams with one massive leg, *JHEP* **09** (2014) 148 [1408.3107].

[25] R. Bonciani, S. Di Vita, P. Mastrolia and U. Schubert, Two-Loop Master Integrals for the mixed EW-QCD virtual corrections to Drell-Yan scattering, *JHEP* **09** (2016) 091 [1604.08581].

[26] S. Di Vita, P. Mastrolia, A. Primo and U. Schubert, Two-loop master integrals for the leading QCD corrections to the Higgs coupling to a W pair and to the triple gauge couplings ZWW and $\gamma^*WW$, *JHEP* **04** (2017) 008 [1702.07331].

[27] F. V. Tkachov, A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions, *Phys. Lett.* **100B** (1981) 65.

[28] K. Chetyrkin and F. Tkachov, Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops, *Nucl.Phys.* **B192** (1981) 159.

[29] S. Laporta, High precision calculation of multiloop Feynman integrals by difference equations, *Int.J.Mod.Phys.* **A15** (2000) 5087 [hep-ph/0102033].

– 23 –
[30] G. Barucchi and G. Ponzano, Differential equations for one-loop generalized Feynman integrals, *J. Math. Phys.* **14** (1973) 396.

[31] A. Kotikov, Differential equations method: New technique for massive Feynman diagrams calculation, *Phys.Lett. B254* (1991) 158.

[32] E. Remiddi, Differential equations for Feynman graph amplitudes, *Nuovo Cim. A110* (1997) 1435 [hep-th/9711188].

[33] T. Gehrmann and E. Remiddi, Differential equations for two loop four point functions, *Nucl. Phys. B580* (2000) 485 [hep-ph/9912329].

[34] A. Goncharov, Polylogarithms in arithmetic and geometry, *Proceedings of the International Congress of Mathematicians 1,2* (1995) 374.

[35] E. Remiddi and J. Vermaseren, Harmonic polylogarithms, *Int.J.Mod.Phys. A15* (2000) 725 [hep-ph/9905237].

[36] T. Gehrmann and E. Remiddi, Numerical evaluation of harmonic polylogarithms, *Comput.Phys.Commun. 141* (2001) 296 [hep-ph/0107173].

[37] J. Vollinga and S. Weinzierl, Numerical evaluation of multiple polylogarithms, *Comput.Phys.Commun. 167* (2005) 177 [hep-ph/0410259].

[38] R. N. Lee and K. T. Mingulov, Master integrals for two-loop C-odd contribution to $e^+e^- \rightarrow \ell^+\ell^-$ process, 1901.04441.

[39] P. Maierhofer, J. Usovitsch and P. Uwer, Kirra Feynman integral reduction program, *Comput. Phys. Commun. 230* (2018) 99 [1705.05610].

[40] R. N. Lee, Presenting LiteRed: a tool for the Loop InTEgrals REDuction, 1212.2685.

[41] R. N. Lee, LiteRed 1.4: a powerful tool for reduction of multiloop integrals, *J. Phys. Conf. Ser. 523* (2014) 012059 [1310.1145].

[42] A. von Manteuffel and C. Studerus, Reduce 2 - Distributed Feynman Integral Reduction, 1201.4330.

[43] A. von Manteuffel, E. Panzer and R. M. Schabinger, A quasi-finite basis for multi-loop Feynman integrals, *JHEP 02* (2015) 120 [1411.7392].

[44] C. W. Bauer, A. Frink and R. Kreckel, Introduction to the GiNaC framework for symbolic computation within the C++ programming language, cs/0004015.

[45] S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk and T. Zirke, SecDec-3.0: numerical evaluation of multi-scale integrals beyond one loop, *Comput. Phys. Commun. 196* (2015) 470 [1502.06595].

[46] M. Becchetti, R. Bonciani, V. Casconi, A. Ferroglia, S. Lavacca and A. von Manteuffel, to appear, 1904.xxxx.

[47] J. C. Collins and J. A. M. Vermaseren, Azodraw version 2, 1606.01177.

[48] A. von Manteuffel and C. Studerus, Massive planar and non-planar double box integrals for light-nf contributions to $gg \rightarrow j j$, *JHEP 10* (2013) 037 [1306.3504].

[49] J. M. Henn, Lectures on differential equations for Feynman integrals, *J. Phys. A48* (2015) 153001 [1412.2296].

[50] T. Gehrmann and E. Remiddi, Two loop master integrals for $\gamma^* \rightarrow 3$ jets: The Planar topologies, *Nucl.Phys. B601* (2001) 248 [hep-ph/0008287].
[51] T. Gehrmann and E. Remiddi, *Analytic continuation of massless two loop four point functions*, *Nucl.Phys.* **B640** (2002) 379 [hep-ph/0207020].

[52] J. Tausk, *Nonplanar massless two loop Feynman diagrams with four on-shell legs*, *Phys.Lett.* **B469** (1999) 225 [hep-ph/9909506].

[53] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Pulling the straps of polygons*, *JHEP* **12** (2011) 011 [1102.0062].

[54] S. Abreu, R. Britto and H. Gronqvist, *Cuts and coproducts of massive triangle diagrams*, *JHEP* **07** (2015) 111 [1504.00206].

[55] O. V. Tarasov, *Connection between Feynman integrals having different values of the space-time dimension*, *Phys. Rev.* **D54** (1996) 6479 [hep-th/9606018].

[56] R. N. Lee, *Space-time dimensionality D as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to D*, *Nucl. Phys.* **B830** (2010) 474 [0911.0252].