

**Abstract**

One of the most common and studied problems in machine learning is classification. While conventional algorithms for supervised classification rely on the determination of a function from features to labels, we propose a different approach based on the estimation of a probabilistic transformation from features to labels. Indeed, we determine a conditional probability distribution of the labels given the features and then features are classified as labels following such distribution. This randomized classification can result in improved performance as we show with simple intuitive examples.

In order to compute the conditional distribution, we follow a robust minimax approach, minimizing the worst-case expectation of the 0-1 loss. By doing so, we find the probabilistic transformation which achieves the minimum risk against an uncertainty set consistent with the training data. We show numerical results obtained by an implementation in python of this method and we compare its performance with state of the art techniques.

1 Introduction

One of the most common and studied problems in machine learning is classification. While conventional algorithms for supervised classification rely on the determination of a function from features to labels, we propose a different approach based on the estimation of a probabilistic transformation from features to labels. Indeed, we determine a conditional probability distribution of the labels given the features and then features are classified as labels following such distribution. This randomized classification can result in improved performance as we show with simple intuitive examples.

In order to compute the conditional distribution, we follow a robust minimax approach, minimizing the worst-case expectation of the 0-1 loss. By doing so, we find the probabilistic transformation which achieves the minimum risk against an uncertainty set. This uncertainty set contains all the distributions which are consistent with the training data, that are all the distributions whose expected statistics belong to a confidence region computed using the training data. Applying Lagrangian duality theory, this minimax problem can be decoupled, and the prediction phase can be done for each data point independently. In order to get this result, the constraints are moved in the objective function thanks to the Lagrange multipliers, which become the parameters of the classification method. Consequently, the Lagrange multipliers have to be computed in the learning phase where the convex minimax problem can be seen as the minimization of a mixed norm over a convex set with an infinite number of constraints. Finally, we apply discretization techniques either on the features or on the statistics in order to approximate this infinite number of constraints.

We show numerical results obtained by an implementation in python, using the package `cvxpy`, of this method, which can be applied to multiple classification problems. Even if the learning phase can be slow depending on how we deal with the constraints, the prediction phase is very fast, and the performances are comparable with some well-known classification methods like Random Forest.
2 Classification problem

Let \( \mathcal{D} \) be a dataset with data-points \((x, y) \in \mathcal{D}\), where \( x \) represents the features and \( y \) the corresponding label, we want to compute the conditional probability distributions of the labels given the features minimizing the worst-case expectation of the 0-1 loss. The 0-1 loss is defined for each feature \( x \) as

\[
\ell(y, \hat{y}(x)) = \begin{cases} 
0 & \text{if } y = \hat{y} \\
1 & \text{if } y \neq \hat{y}
\end{cases}
\]

where \( y \) is the actual label and \( \hat{y} \) the predicted one.

Without loss of generality, we can assume that the sets \( X \) of the features and \( Y \) of the labels are finite. Let \( q(y|x) \in \Delta(X, Y) \) be a conditional probability of the labels given the features and \( p(x, y) \in \Delta(X \times Y) \) be a joint distribution of features and labels. Moreover, let \( \phi: X \times Y \to \mathbb{R}^L \) be a vector of statistics, define an uncertainty set \( \mathcal{U} \), which is the set of all the distributions which are consistent with training data. We can distinguish two cases, depending on how we impose the constraints:

- **equality constraints**

  \[
  \mathcal{U} = \{ p \in \Delta(X \times Y) \mid \mathbb{E}_p[\phi(X, Y)] = \tau \} \tag{1}
  \]

- **inequality constraints**

  \[
  \mathcal{U} = \{ p \in \Delta(X \times Y) \mid \tau - \varepsilon \leq \mathbb{E}_p[\phi(X, Y)] \leq \tau + \varepsilon \} \tag{2}
  \]

where \( \tau \in \mathbb{R}^L \) is the empirical expectation of the statistics and \( \varepsilon \in \mathbb{R}^L \) represents the size of the intervals.

Then for each \( x \), if the true joint distribution is \( p \) and we choose as conditional distribution \( q \), the expected loss is

\[
\mathbb{E}[\ell] = \sum_y p(x, y) \mathbb{E}[\ell(y)] = \sum_y p(x, y) \mathbb{P}(\hat{y} \neq y) = \sum_y p(x, y)(1 - q(y|x)).
\]

Hence the convex optimization problem, which we want to solve, is

\[
\min_{q \in \Delta(X, Y)} \max_{p \in \mathcal{U}} \sum_x \sum_y p(x, y)(1 - q(y|x)). \tag{3}
\]

Using Lagrangian duality theory, the minimax problem can be decoupled and the prediction can be done for each feature independently [10][11]. The Lagrange multipliers became the parameters of our model and they have to be estimated in the learning phase. In the following sections we present these two phases for both the cases of equality and inequality constraints.

3 Equality constraints

3.1 Learning phase

Since \( p \) is a joint probability distribution, minimax problem [3] can be written in the following equivalent form

\[
1 - \max_{q \in \Delta(X, Y)} \min_{p \in \mathcal{U}} \sum_x \sum_y p(x, y)q(y|x)
\]

and, thanks to the fact that the sets, where \( p \) and \( q \) live, are closed and convex and the objective function is bounded, we can exchange the maximum and the minimum [9] to get

\[
1 - \min_{p \in \mathcal{U}} \max_{q \in \Delta(X, Y)} \sum_x \sum_y p(x, y)q(y|x).
\]

In the inner maximization problem

\[
\max_{q \in \Delta(X, Y)} \sum_x \sum_y p(x, y)q(y|x)
\]

for each fixed \( x \), we are computing a convex combination of \( p(x, \cdot) \), so the maximum value is \( \max_x p(x, y) \). Hence the optimization problem becomes the minimization of a mixed norm in a convex set

\[
\min_{p \in \mathcal{U}} \sum_x \max_y p(x, y) = \min_{p \in \mathcal{U}} \sum_x \max_y |p(x, y)| = \min_{p \in \mathcal{U}} ||p||_{\infty, 1}
\]
which can be written as

\[
\begin{align*}
\text{minimize} & \quad \|p\|_{\infty,1} + \ell^*(p) \\
\text{subject to} & \quad \sum_{x,y} p(x, y) = 1 \\
& \quad \sum_{x,y} \phi(x,y)p(x, y) = \tau.
\end{align*}
\]

Let \( \theta_0 \in \mathbb{R} \) and \( \theta \in \mathbb{R}^L \) be the Lagrange multipliers, then the dual problem is

\[
\begin{align*}
\text{maximize} & \quad -\theta_0 - \langle \theta, \tau \rangle \\
\text{subject to} & \quad \|(-\theta_0 - \langle \theta, \phi \rangle)^+\|_{1,\infty} \leq 1
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\text{maximize} & \quad -\theta_0 - \langle \theta, \tau \rangle \\
\text{subject to} & \quad \sum_y (-\theta_0 - \langle \theta, \phi(x,y) \rangle^+) \leq 1 \quad \forall x.
\end{align*}
\]  

Finally define \( \theta^*_0 \) and \( \theta^* \) the solution of problem (4).

A problem in (7) is that we have an infinite number of constraints given by all the possible values that the feature \( x \) can take. This can be solved using statistics that can take only values 0 and 1, indeed the feature \( x \) appears only inside the function \( \phi \), making the number of constraints finite. An example of statistics, which take only values 0 and 1, is the indicator function of some significant regions.

### 3.2 Prediction phase

Now, thanks to the Lagrange multipliers \( \theta_0^* \) and \( \theta^* \), problem (5) can be decoupled. Similarly to the learning phase, it can be written as

\[
1 - \max_{q \in \Delta(X,Y)} \min_{p(x,y) \geq 0} \sum_{x} \sum_{y} p(x,y)q(y|x)
\]

and, adding the redundant constraint \( \sum_y p(x,y) \leq 1 \) and the Lagrange multipliers, it is equivalent to solve

\[
\max_{q \in \Delta(X,Y)} \min_{p(x,y) \geq 0} \sum_{x} \sum_{y} p(x,y)q(y|x) + \theta_0^* \left( \sum_{x,y} p(x,y) - 1 \right) + \theta^* \left( \sum_{x,y} p(x,y)\phi(x,y) - \tau \right).
\]

We can neglect constant terms, so the solution \( q \) is the same as the following problem

\[
\max_{q \in \Delta(X,Y)} \min_{p(x,y) \geq 0} \sum_{x} \sum_{y} [q(y|x) + \theta_0^* + \theta^* \phi(x,y)] p(x,y).
\]

In the inner minimization problem

\[
\min_{p(x,y) \geq 0} \sum_{x} \sum_{y} [q(y|x) + \theta_0^* + \theta^* \phi(x,y)] p(x,y)
\]

for each fixed \( x \), we are computing a convex combination of \( q(\cdot|x) + \theta_0^* + \theta^* \phi(x, \cdot) \) and 0, so the minimum value is \( \min_0 \{ q(y|x) + \theta_0^* + \theta^* \phi(x,y) \} \). Hence the optimization problem becomes

\[
\max_{q \in \Delta(X,Y)} \sum_x \min_y \left\{ 0, \min_0 \{ q(y|x) + \theta_0^* + \theta^* \phi(x,y) \} \right\}
\]

and it can be solved for each \( x \) independently. Let \( \bar{x} \) be the features of a new data-point, we need to solve

\[
\max_{q \in \Delta(Y|\bar{x})} \min_y \left\{ 0, \min_0 \{ q(y|\bar{x}) + \theta_0^* + \theta^* \phi(\bar{x}, y) \} \right\}
\]
which is equivalent to the following linear problem

\[
\begin{align*}
\text{maximize} & \quad v \\
\text{subject to} & \quad q(y|x) \geq 0 \ \forall \ y \\
& \quad \sum_y q(y|x) = 1 \\
& \quad v \leq 0 \\
& \quad v \leq q(y|x) + \theta_0^* + \theta^T \phi(x, y) \ \forall \ y.
\end{align*}
\] (5)

This problem can be solved applying the so-called water-filling algorithm. The number of containers is equal to the number of classes and the initial quantity of water in each container is \( w(y) = \theta^*_0 + \theta^T \phi(x, y) \). Then we can add water in all containers in order to maximize the minimum quantity, and the total availability is 1. The algorithm can be implemented, in python using the package numpy, in the following way

\[
q = \text{numpy.zeros}(M) \\
\text{availability} = 1 \\
\text{while availability} > 0: \\
\quad w\_min = \text{numpy.min}(w) \\
\quad \text{indeces} = (w == w\_min) \\
\quad \text{if numpy.sum(indeces) == w.size:} \\
\quad \quad \text{quantity} = \text{availability/numpy.sum(indeces)} \\
\quad \text{else:} \\
\quad \quad w\_almost\_min = \text{numpy.min}(w[w > w\_min]) \\
\quad \quad \text{quantity} = \text{min}(w\_almost\_min - w\_min, \text{availability})/\text{numpy.sum(indeces)} \\
\quad q[\text{indeces}] += \text{quantity} \\
\quad w[\text{indeces}] += \text{quantity} \\
\quad \text{availability} -= \text{quantity*numpy.sum(indeces)}
\]

Finally the label \( \bar{y} \) corresponding to the features \( \bar{x} \) is sampled by the distribution \( \bar{q} \), which solves problem (5).

### 3.3 Equalizer

We say that the solution of the optimization problem \( q^* \) is an equalizer if the expectation of the loss is constant for all the distributions in the uncertainty set, that is when

\[
\mathbb{E}_p[\ell] = \sum_{x,y} p(x,y)(1 - q^*(y|x)) = 1 - \sum_{x,y} p(x,y)q^*(y|x) = \text{constant} \quad \text{for all } p \in \mathcal{U}.
\]

Now we want to write the probabilities using the vectorial formulation. Let \( p, q^* \in \mathbb{R}^{||X||Y} \) be the vectors, whose components represent respectively the joint and the conditional distributions, \( M \in \mathbb{R}^{(L+1)||X||Y} \) be a matrix defined as

\[
M_{ij} = \begin{cases} 
\phi((x,y)_j) & \text{if } i \neq L + 1 \\
1 & \text{if } i = L + 1.
\end{cases}
\]

where \((x,y)_j\) is the \( j \)-th couple feature/label after they have been reordered, and \( c \in \mathbb{R}^{L+1} \) be a vector defined as

\[
c_i = \begin{cases} 
\tau_i & \text{if } i \neq L + 1 \\
1 & \text{if } i = L + 1.
\end{cases}
\]

Then all the distributions \( p \) in the uncertainty set \( \mathcal{U} \) can be represented by the vectors \( p \) which satisfy the linear system \( Mp = c \) and the expected loss can be computed as

\[
\mathbb{E}[\ell] = 1 - \langle p, q^* \rangle.
\]

Therefore, \( q^* \) is an equalizer if and only if \( q^* \) can be written as a linear combination of the rows of \( M \), that is if there exist \( \theta_0 \in \mathbb{R} \) and \( \theta \in \mathbb{R}^{L} \) such that

\[
q^*_j = \theta_0 + \theta^T \phi((x,y)_j) \quad \text{for all } j = 1, \ldots, ||X||Y.
\]
which is equivalent to
\[ q^*(y|x) = \theta_0 + \theta^T \phi(x, y). \]
This can be obtained when \( \theta^*_0 + \theta^T \phi(x, y) \leq 0 \) for all \( y \in Y \) in the water-filling algorithm and we take \( \theta_0 = -\theta^*_0 \) and \( \theta = -\theta^* \).

On the other hand, if \( q^* \) is not an equalizer, we can show that the expected loss is a linear function of the parameters describing all the distributions inside the uncertainty set \( \mathcal{U} \). All the vectors \( p \) can be written as
\[ p = p^* + U \lambda \]
where
- \( p^* \in \mathbb{R}^{||X||Y} \) is the vector representing the maximum entropy distribution \( p^* \)
- \( U \in \mathbb{R}^{||X||Y \times (||X||Y - (L + 1))} \) is a matrix whose columns form a basis of the kernel of \( M \)
- \( \lambda \in \mathbb{R}^{||X||Y - (L + 1)} \) is a vector of parameters such that
\[ p^*_j + \sum_{i=1}^{||X||Y - (L + 1)} U_{ji} \lambda_i \geq 0 \quad \text{for all } j, \ldots, ||X||Y. \]

Then the expected loss for a distribution \( p \) is
\[ \mathbb{E}[\ell] = < p, 1 - q^* > = 1 - < p, q^* > = 1 - < p^*, q^* > = < U^T \lambda, q^* > = \ell^* - < \lambda, U^T q^* > \]
where \( \ell^* \) is the maximum loss, that is the worst case expectation computed by the algorithm. Therefore we must have
\[ < \lambda, U^T q^* > \geq 0 \]
for all the admissible \( \lambda \) and the expected loss is a linear function of \( \lambda \).

A simple example where \( q^* \) is not an equalizer is the following.

## 4 Inequality constraints

### 4.1 Estimation of \( \varepsilon \)

Let \( \delta \in [0, 1] \), we want to compute \( \varepsilon \), such that
\[ \mathbb{P}(\tau - \varepsilon \leq \phi(X, Y) \leq \tau + \varepsilon) \geq \delta \quad (6) \]
which is equivalent to
\[ \mathbb{P}(|\phi_l(X, Y) - \tau_l| \leq \varepsilon_l \; \forall \; l = 1, \ldots, L) \geq \delta. \]

In order to estimate the vector \( \varepsilon \) we can follow two approaches, which rely on:
- Central Limit Theorem
- Hoeffding’s inequality

and we want to apply both of them to each statistic separately, so, let \( \alpha \in [0, 1] \), we will have the following inequality for all \( l = 1, \ldots, L \)
\[ \mathbb{P}(|\phi_l(X, Y) - \tau_l| \leq \varepsilon_l) \geq 1 - \alpha. \]

Let \( A_l = \{|\phi_l(X, Y) - \tau_l| \leq \varepsilon_l\} \), then we have
\[ \mathbb{P}(|\phi_l(X, Y) - \tau_l| \leq \varepsilon_l \; \forall \; l = 1, \ldots, L) = \mathbb{P}\left( \bigcap_{l=1}^{L} A_l \right) \]
\[ = 1 - \mathbb{P}\left( \bigcup_{l=1}^{L} A_l^c \right) \]
\[ = 1 - \sum_{l=1}^{L} \mathbb{P}(A_l^c). \]

5
Since 
\[ P(A_l) \geq 1 - \alpha \]
then 
\[ -P(A_l^c) \geq -\alpha \]
which implies 
\[ P(\tau - \varepsilon \leq \phi(X, Y) \leq \tau + \varepsilon) \geq 1 - L\alpha. \]
Therefore taking \( \alpha = \frac{1 - \delta}{L} \) we get (6).

**Approach 1**: Central Limit Theorem.
We apply the Central Limit Theorem and compute an asymptotic confidence interval for the mean value of each statistic independently.
Let \( N \) be the dimension of the sample, for each \( l = 1, \ldots, L \), define
\[ \tau_l = \frac{1}{N} \sum_{n=1}^{N} \phi_l(x_n, y_n) \quad \text{and} \quad S_l^2 = \frac{1}{N-1} \sum_{n=1}^{N} [\phi_l(x_n, y_n) - \tau_l]^2 \]
then we have that
\[ \frac{\tau_l - \mathbb{E}[\phi_l(X, Y)]}{S_l/\sqrt{N}} \xrightarrow{d} N(0, 1). \]
Let \( \alpha \in [0, 1] \), we can construct an asymptotic confidence interval for \( \mathbb{E}[\phi_l(X, Y)] \) of level \( 1 - \alpha \) in the following way
\[
1 - \alpha = \mathbb{P}(-z_{\alpha/2} \leq N(0, 1) \leq z_{\alpha/2}) = \lim_{N \to \infty} \mathbb{P}\left(-z_{\alpha/2} \leq \frac{\tau_l - \mathbb{E}[\phi_l(X, Y)]}{S_l/\sqrt{N}} \leq z_{\alpha/2}\right) \\
\approx \mathbb{P}\left(-z_{\alpha/2} \leq \frac{\tau_l - \mathbb{E}[\phi_l(X, Y)]}{S_l/\sqrt{N}} \leq z_{\alpha/2}\right) = \mathbb{P}\left(\tau_l - \frac{S_l}{\sqrt{N}}z_{\alpha/2} \leq \mathbb{E}[\phi_l(X, Y)] \leq \tau_l + \frac{S_l}{\sqrt{N}}z_{\alpha/2}\right)
\]
where \( z_\beta \) is a quantile of the standard normal distribution defined as \( \beta = \mathbb{P}(N(0, 1) > z_\beta) \). Finally we define \( \varepsilon_l \in \mathbb{R}^L \) as
\[ \varepsilon_l = \frac{S_l}{\sqrt{N}}z_{\alpha/2} \]
which depends on the statistic \( \phi_l \), the dimension of the sample \( N \) and the confidence \( 1 - \alpha \), that we require.
Since we applied the Central Limit Theorem, this approximation is good when \( N \) is big.

**Approach 2**: Hoeffding’s inequality.
This idea can be employed when the statistics are bounded and in the following we assume that \( \phi_l \) takes values in \([0, 1]\) for all \( l \). Let \( N \) be the dimension of the sample, for all \( l \) we have
\[ \mathbb{P}(|\phi_l(X, Y) - \tau_l| \geq \varepsilon_l) \leq 2e^{-2\varepsilon_l^2 N}. \]
Therefore, imposing the right hand side equal to \( \alpha \), we get
\[ \varepsilon_l = \sqrt{\frac{1}{2N} \log \frac{2}{\alpha}}. \]
Differently from the first approach, now we do not obtain an asymptotic result, but this is true for any value of the dimension of the sample.

### 4.2 Learning phase
Since \( p \) is a joint probability distribution, minimax problem (3) can be written in the following equivalent form
\[
1 - \max_{q \in \Delta(X,Y)} \min_{p \in \mathcal{U}} \sum_{x} \sum_{y} p(x, y)q(y|x)
\]
and, thanks to the fact that the sets, where \( p \) and \( q \) live, are closed and convex and the objective function is bounded, we can exchange the maximum and the minimum \([9]\) to get

\[
1 - \min_{p \in \mathcal{U}} \max_{q \in \Delta(X,Y)} \sum_x \sum_y p(x,y)q(y|x).
\]

In the inner maximization problem

\[
\max_{q \in \Delta(X,Y)} \sum_x \sum_y p(x,y)q(y|x)
\]

for each fixed \( x \), we are computing a convex combination of \( p(x,\cdot) \), so the maximum value is \( \max_x p(x,y) \). Hence the optimization problem becomes the minimization of a mixed norm in a convex set \([11]\)

\[
\min_{p \in \mathcal{U}} \sum_x \max_y p(x,y) = \min_{p \in \mathcal{U}} \sum_x \max_y |p(x,y)| = \min_{p \in \mathcal{U}} \|p\|_{\infty,1}
\]

which can be written as

\[
\text{minimize} \quad \|p\|_{\infty,1} + I^+(p) \\
\text{subject to} \quad \sum_x p(x,y) = 1 \\
\sum_{x,y} \phi(x,y)p(x,y) \leq \tau + \varepsilon \\
\sum_{x,y} \phi(x,y)p(x,y) \geq \tau - \varepsilon.
\]

Let \( \theta_0 \in \mathbb{R} \) and \( \theta_L, \theta_R \in \mathbb{R}^k \) be the Lagrange multipliers, the dual problem is

\[
\text{maximize} \quad -\theta_0 - <\theta_R - \theta_L, \tau > - <\theta_R + \theta_L, \varepsilon > \\
\text{subject to} \quad \|(-\theta_0 - <\theta_R - \theta_L, \phi >)^+\|_{\infty,\infty} \leq 1 \\
\theta_L \geq 0 \\
\theta_R \geq 0
\]

which is equivalent to

\[
\text{maximize} \quad -\theta_0 - <\theta_R - \theta_L, \tau > - <\theta_R + \theta_L, \varepsilon > \\
\text{subject to} \quad \sum_y (-\theta_0 - <\theta_R - \theta_L, \phi(x,y)>)^+ \leq 1 \ \forall \ x \\
\theta_L \geq 0 \\
\theta_R \geq 0.
\]

Finally define \( \theta'_L, \theta'_r \) and \( \theta'_R \) the solution of problem \([7]\).

A problem in \([7]\) is that we have an infinite number of constraints given by all the possible values that the feature \( x \) can take. This can be solved using statistics that can take only values 0 and 1, indeed the feature \( x \) appears only inside the function \( \phi \), making the number of constraints finite. An example of statistics, which take only values 0 and 1, is the indicator function of some significant regions.

### 4.3 Prediction phase

Now, thanks to the Lagrange multipliers \( \theta'_L, \theta'_r \) and \( \theta'_R \), problem \([3]\) can be decoupled. Similarly to the learning phase, it can be written as

\[
1 - \max_{q \in \Delta(X,Y)} \min_{p \in \mathcal{U}} \sum_x \sum_y p(x,y)q(y|x)
\]

and, adding the redundant constraint \( \sum_y p(x,y) \leq 1 \) and the Lagrange multipliers, it is equivalent to solve

\[
\max_{q \in \Delta(X,Y)} \min_{p(X,Y), \sum_y p(x,y) \leq 1} \sum_{x,y} p(x,y)q(y|x) + \theta'_L \left( \sum_{x,y} p(x,y) - 1 \right) + \theta'_R \left( \sum_{x,y} p(x,y) - \tau - \varepsilon \right) + \theta'_R \left( \sum_{x,y} p(x,y) - \tau - \varepsilon \right).
\]
We can neglect constant terms, so the solution $q$ is the same as the following problem

$$\max_{q \in \Delta(X,Y)} \min_{p(x,y) \geq 0} \sum_x \sum_y [q(y|x) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(x,y)] p(x,y).$$

In the inner minimization problem

$$\min_{p(x,y) \geq 0} \sum_x \sum_y [q(y|x) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(x,y)] p(x,y)$$

for each fixed $x$, we are computing a convex combination of $q(y|x) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(x,\cdot)$ and $0$, so the minimum value is $\min \{0, \min_y [q(y|x) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(x,y)]\}$. Hence the optimization problem becomes

$$\max_{q \in \Delta(X,Y)} \sum_x \min_y \left\{0, \min_y [q(y|x) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(x,y)]\right\}$$

and it can be solved for each $x$ independently. Let $\bar{x}$ be the features of a new data-point, we need to solve

$$\max_{q \in \Delta(\bar{x})} \min_y \left\{0, \min_y [q(y|\bar{x}) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(\bar{x},y)]\right\}$$

which is equivalent to the following linear problem

$$\begin{align*}
\text{maximize} & \quad v \\
\text{subject to} & \quad q(y|\bar{x}) \geq 0 \quad \forall \ y \\
& \quad \sum_y q(y|\bar{x}) = 1 \\
& \quad v \leq 0 \\
& \quad v \leq q(y|\bar{x}) + \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(\bar{x},y) \quad \forall \ y.
\end{align*}$$

(8)

This problem can be solved applying the so-called water-filling algorithm. The number of containers is equal to the number of classes and the initial quantity of water in each container is $w(y) = \theta_0^y + (\theta_R^y - \theta_L^y)^T \phi(y)$. Then we can add water in all containers in order to maximize the minimum quantity, and the total availability is 1. The algorithm can be implemented, in python using the package numpy, in the following way

```python
q = numpy.zeros(M)
availability = 1

while availability > 0:
    w_min = numpy.min(w)
    indeces = (w == w_min)

    if numpy.sum(indeces) == w.size:
        quantity = availability/numpy.sum(indeces)
    else:
        w_almost_min = numpy.min(w[w > w_min])
        quantity = min(w_almost_min - w_min, availability/numpy.sum(indeces))

    q[indeces] += quantity
    w[indeces] += quantity
    availability -= quantity*numpy.sum(indeces)
```

Finally the label $\tilde{y}$ corresponding to the features $\bar{x}$ is sampled by the distribution $\tilde{q}$, which solves problem (8).

5 Numerical results

In this section we present some numerical results, obtained by an implementation in python, using the package cvxpy [5], of this method. It is tested on six different popular datasets of the UCI repository, and, for each of them, we
plot the performance as the percentage of correct predictions varying the maximum order of the statistics employed. We compare the cases with equality (in blue) and inequality (in red) constraints to a very well-known classification algorithm, Random Forest (in green). Depending on the dataset, our method can give better or worse results with respect to Random Forest, but, in general, we can say that the case with inequality constraints is more stable in relation to the increase of the statistics. Indeed, the case with equality constraints tends to overfit when the order of statistics becomes high.

The following plot gives an idea about how the algorithm works.
Finally we can see that the performance of our algorithm is comparable with state of the art techniques.

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