Subgroupoids and Quotient Theories

Two applications of Moerdijk’s site description for equivariant sheaf toposes

Henrik Forssell *

November 15, 2011

Abstract

Moerdijk’s site description for equivariant sheaf toposes on open topological groupoids is used to give a proof for the (known, but apparently unpublished) proposition that if $\mathcal{H}$ is a strictly full subgroupoid of an open topological groupoid $\mathcal{G}$, then the topos of equivariant sheaves on $\mathcal{H}$ is a subtopos of the topos of equivariant sheaves on $\mathcal{G}$. This proposition is then applied to the study of quotient geometric theories and subtoposes. In particular, an intrinsic characterization is given of those subgroupoids that are definable by quotient theories. A self-contained presentation of Moerdijk’s site description is included for the case of open topological, rather than localic, groupoids. In the final section, the site description is used to give an intrinsic characterization of those open topological groupoids that induce toposes which have a generating set of compact objects and the property that a finite product of compact objects is compact. This generalizes a known characterization of coherent topological groups.

Contents

1 Introduction 2

2 Moerdijk’s site description for equivariant sheaf toposes 4

2.1 Equivariant sheaf toposes 6

2.2 Moerdijk-sites 8

*Thanks to Jiří Rosický and Steve Awodey for interesting and helpful conversations. This research was supported by the Eduard Čech Center for Algebra and Geometry, grant no. LC505.
If \( G \) is a topological groupoid, that is, a groupoid object in the category, \( \mathbf{Sp} \), of topological spaces and continuous maps,

\[
\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \xrightarrow{m} & G_1 \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{d} & G_0
\end{array}
\]

one can consider equivariant sheaves over \( G \), where an equivariant sheaf is a local homeomorphism \( r : R \to G_0 \) together with a continuous action, \( \rho : G_1 \times_{G_0} R \to R \). A morphism of equivariant sheaves is a morphism of sheaves that commutes with the action. The resulting category of equivariant sheaves is a (Grothendieck) topos with enough points, written \( \text{Sh}_{G_1}(G_0) \). Conversely, for any topos with enough points \( \mathcal{E} \), Butz and Moerdijk show in [1] that there exists a topological groupoid \( G \) such that

\[
\mathcal{E} \simeq \text{Sh}_{G_1}(G_0)
\]

We can refer to the groupoid constructed in [1] to obtain the representation as the \textit{Butz-Moerdijk groupoid} for \( \mathcal{E} \) (the construction depends on a choice of a set of points, but this is of no concern for the present purposes). It can be assumed to be open, in the sense that the domain (d) and codomain (c)
maps are open. We restrict attention to open groupoids and (Grothendieck) toposes with enough points.

A morphism, \( f : \mathcal{H} \to \mathcal{G} \), of topological groupoids induces a geometric morphism \( f : \text{Sh}_{\mathcal{H}_1}(\mathcal{H}_0) \to \text{Sh}_{\mathcal{G}_1}(\mathcal{G}_0) \). In \cite{2}, Moerdijk gives sufficient properties, in the more general localic case, for morphisms of groupoids to induce e.g. surjective or open geometric morphisms. Analogously, it is shown here, for the spatial case, that a strictly full subgroupoid inclusion induces a subtopos embedding. Although known by specialists (Moerdijk in particular) this result seems not to have been published. For certain groupoids \( \mathcal{G} \)—Butz-Moerdijk groupoids in particular—the subtoposes with enough points of \( \text{Sh}_{\mathcal{G}_1}(\mathcal{G}_0) \) are therefore precisely the subtoposes on the form \( \text{Sh}_{\mathcal{H}_1}(\mathcal{H}_0) \) for \( \mathcal{H} \) a strictly full topological subgroupoid of \( \mathcal{G} \). Call such groupoids ‘saturated’ (for lack of a better term).

If \( T \) is a geometric theory, i.e. a deductively closed set of sequents consisting of formulas constructed with the connectives \( \top, \bot, \land, \exists, \) and \( \lor \) (but with no more than finitely many free variables), then there exists a classifying topos, \( \text{Set}[T] \), with the property that the category of \( T \)-models in a topos, \( \mathcal{E} \), is equivalent to the category of geometric morphisms from \( \mathcal{E} \) to \( \text{Set}[T] \) (see \cite{3}). Any topos is the classifying topos of some geometric theory. Restricting attention to toposes with enough points means restricting attention to geometric theories with enough models, in the sense that a sequent is provable in the theory if it is true in all models in the category of sets and functions, \( \text{Sets} \). Finally, if a topos given as \( \text{Sh}_{\mathcal{G}_1}(\mathcal{G}_0) \) classifies a theory \( T \), then the topological groupoid \( \mathcal{G} \) can be regarded as consisting of \( T \)-models and isomorphisms.

A quotient theory of \( T \) is, briefly, a theory extension of \( T \) in the same language. Quotient theories of \( T \) are known to correspond to subtoposes of \( \text{Set}[T] \) (see \cite{3}). The preprint \cite{4} contains a detailed spelling out and analysis of this correspondence. Here, the correspondence is extended to subgroupoids. Suppose \( \text{Set}[T] \simeq \text{Sh}_{\mathcal{G}_1}(\mathcal{G}_0) \) so that we can regard \( \mathcal{G} \) as a groupoid of \( T \)-models and isomorphisms. Then for a strictly full—i.e. full and replete—subgroupoid, \( \mathcal{H} \hookrightarrow \mathcal{G} \), the induced subtopos classifies the theory obtained by adding to \( T \) all sequents true in the subset of models \( H_0 \subseteq G_0 \). Conversely, given a subtopos, \( \text{Set}[T'] \hookrightarrow \text{Set}[T] \simeq \text{Sh}_{\mathcal{G}_1}(\mathcal{G}_0) \), with \( \mathcal{G} \) an open “saturated” groupoid and with \( T' \) a quotient of \( T \) with enough models, we have that \( \text{Set}[T'] \simeq \text{Sh}_{\mathcal{H}_1}(\mathcal{H}_0) \) where \( \mathcal{H} \) is the strictly full subgroupoid of \( \mathcal{G} \) consisting of those \( T \)-models which also model \( T' \). An intrinsic characterization is given of the subgroupoids \( \mathcal{H} \hookrightarrow \mathcal{G} \) which are definable by a quotient theory in this way. Armed with the correspondence between quotient theories, subtoposes, and subgroupoids, a few examples
are given of how the analysis of [4] concerning the ‘logical’ meaning of the subtopos lattice operations and of notions such as open and closed subtoposes can be extended to include definable subgroupoids.

Section 2 recalls (briefly) the topos of equivariant sheaves on a topological groupoid. Since the site construction given in [2] is extensively used in the following sections a self-contained presentation of this construction in the topological (as opposed to localic) groupoid case is given in this section. In Section 3 a proof is supplied for the fact that subgroupoid inclusions induce subtopos inclusions. Section 4 applies this result to quotient theories. Section 5 uses Moerdijk’s site construction to give an intrinsic characterization—that is, one only mentioning properties of the groupoid—of those open topological groupoids that induce sheaf toposes which have a generating set of compact objects and the property that a finite product of compact objects is compact. This generalizes the known characterization of coherent topological groups. A rather more involved characterization of coherent topological groupoids—those that induce coherent sheaf toposes—is also included.

2 Moerdijk’s site description for equivariant sheaf toposes

2.1 Equivariant sheaf toposes

Let \( \mathcal{G} \) be a topological groupoid. Recall from e.g. [3] that the objects of the category of equivariant sheaves, \( \text{Sh}_{G_1}(G_0) \), on \( \mathcal{G} \) are pairs \( (r: R \to G_0, \rho) \) where \( r \) is a local homeomorphism—i.e. an object of \( \text{Sh}(G_0) \)—and \( \rho \) is a continuous action, i.e. a continuous map

\[
\rho : G_1 \times G_0 R \longrightarrow R
\]

with the pullback being along the domain map and such that \( r(\rho(f, x)) = c(f) \), satisfying a unit and an composition axiom:

1. \( \rho(e(r(x)), x) = x \)
2. \( \rho(m(g, f), x) = \rho(g, \rho(f, x)) \)

If \( \mathcal{G} \) is an open topological groupoid then it follows that the action \( \rho \) is an open map. A morphism of equivariant sheaves is a morphism of sheaves commuting with the actions. The category, \( \text{Sh}_{G_1}(G_0) \), of equivariant sheaves
on \( \mathcal{G} \) is a (Grothendieck) topos. The forgetful functors of forgetting the action,

\[
u : \text{Sh}_{G_1}(G_0) \longrightarrow \text{Sh}(G_0)
\]

and of forgetting the topology,

\[
v : \text{Sh}_{G_1}(G_0) \longrightarrow \text{Sets}^\mathcal{G}
\]

are both conservative inverse image functors. A \textit{continuous functor}, or morphism of topological groupoids, \( f : \mathcal{H} \longrightarrow \mathcal{G} \), i.e. a morphism of groupoid objects in \( \text{Sp} \)

\[
\begin{array}{ccc}
H_1 \times_{H_0} H_1 & \xrightarrow{f_1 \times f_1} & G_1 \times_{G_0} G_1 \\
\downarrow m & & \downarrow m \\
H_1 & \xrightarrow{f_1} & G_1 \\
\downarrow c & & \downarrow c \\
H_0 & \xrightarrow{f_0} & G_0
\end{array}
\]

induces a geometric morphism

\[
\begin{array}{ccc}
\text{Sh}_{H_1}(H_0) & \xrightarrow{\perp} & \text{Sh}_{G_1}(G_0) \\
\downarrow f^* & & \downarrow f_*
\end{array}
\]

where \( f^* \) can be quickly described by pointing to the commutativity of the following diagram.

\[
\begin{array}{ccc}
\text{Sets} & \xrightarrow{(-)\circ f} & \text{Sets}^\mathcal{G} \\
\downarrow v^* & & \downarrow v^* \\
\text{Sh}_{H_1}(H_0) & \xrightarrow{f^*} & \text{Sh}_{G_1}(G_0) \\
\downarrow u^* & & \downarrow u^* \\
\text{Sh}(H_0) & \xrightarrow{f_0^*} & \text{Sh}(G_0)
\end{array}
\]

A description of a canonical site for the topos of equivariant sheaves on an open localic groupoid was given by Moerdijk in [2]. We recall the construction here for, open \textit{topological} groupoids, with enough detail for it to be self-contained, and with a few added consequences that will be useful further on.
2.2 Moerdijk-sites

This section contains a presentation of Moerdijk’s site description from [2] for equivariant sheaves on an open topological groupoid, written out for reference and with enough details to be self-contained.

2.2.1 Objects in the Moerdijk-site

We restrict attention to topological groupoids that are open, in the sense that the domain and codomain maps are open maps (it follows that composition of arrows must also then be open, see [2]). Let $G$ be an open topological groupoid, fully written out as a groupoid object in $Sp$ as

\[
\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \xrightarrow{m} & G_1 \\
\downarrow & & \downarrow \\
G_1 & & G_0 \\
\end{array}
\]

with $m$ the composition, $e$ the insertion of identities, and $i$ the inverse. We also write $g \circ f$ for $m(g, f)$; $1_x$ for $e(x)$; and $f^{-1}$ for $i(f)$ as usual. As described in [2], the open subgroupoids of $G$ induce objects of $Sh_{G_1}(G_0)$, the set of which is a generating set, in the sense that any object $A \in Sh_{G_1}(G_0)$ is covered by morphisms the domains of which belong to the set.

**Definition 2.2.2** An open subgroupoid of $G$ is an open set $N \subseteq G_1$ which is closed under inverses and compositions.

Clearly, such an open set of arrows determines an open set $U := d(N) = c(N)$ of objects, since $G$ is open. Thus we can also define an open subgroupoid as an open set, $U \subseteq G_0$, of objects together with an open set, $N \subseteq G_1$ of arrows satisfying conditions:

a) $d(N), c(N) \subseteq U$;
b) $e(U) \subseteq N$;
c) $m(N \times_{G_0} N) \subseteq N$;
d) $i(N) \subseteq N$.

Such an open subgroupoid determines an object, written $\langle G, U, N \rangle$, of $Sh_{G_1}(G_0)$ as follows. First, define an equivalence relation of arrows with domain in $U$,

\[
\sim_N := \{ (g, f) \mid c(f) = c(g) \wedge g^{-1} \circ f \in N \} \subseteq d^{-1}(U) \times_{G_0} d^{-1}(U)
\]
readily seen to be an open subset of \( d^{-1}(U) \times_{G_0} d^{-1}(U) \), i.e. an open equivalence relation, since it is a pullback of the open set \( N \subseteq G_1 \)

\[
\begin{array}{ccc}
\sim_N & \subseteq & N \\
\downarrow & & \downarrow \\
\cdots & \subseteq & \cdots \\

d^{-1}(U) \times_{G_0} d^{-1}(U) & \xrightarrow{i \times 1} & c^{-1}(U) \times_{G_0} d^{-1}(U) \\
& \xrightarrow{m} & G_1 \\
\end{array}
\]

Therefore, if we consider the quotient, with the quotient topology,

\[
\begin{array}{ccc}
d^{-1}(U) & \xrightarrow{q} & d^{-1}(U)/\sim_N \\
\downarrow & \searrow & \downarrow \\
G_0 & \xrightarrow{c} & G_0 \\
\end{array}
\]

we have that \( q \) is an open surjection and that the codomain map

\[
c : d^{-1}(U)/\sim_N \longrightarrow G_0
\]

is a local homeomorphism. Composition can then be seen to be a continuous action,

\[
\begin{array}{ccc}
G_1 \times_{G_0} d^{-1}(U) & \xrightarrow{\quad m \quad} & d^{-1}(U) \\
\downarrow & \searrow & \downarrow \\
1_{G_1} \times q & \xrightarrow{\quad = \quad} & q \\
\downarrow & \searrow & \downarrow \\
G_1 \times_{G_0} d^{-1}(U)/\sim_N & \xrightarrow{\quad m \quad} & d^{-1}(U)/\sim_N \\
\end{array}
\]

Denote the resulting object of \( \text{Sh}_{G_1}(G_0) \) by \( \langle G, U, N \rangle \). Objects of the form \( \langle G, U, N \rangle \) can be seen to form a generating set for \( \text{Sh}_{G_1}(G_0) \). For given an equivariant sheaf \( \langle r : R \to G_0, \rho \rangle \) and a continuous section \( t : U \longrightarrow R \), we get a set of arrows

\[
N_t = \{ f \in d^{-1}(U) \cap c^{-1}(U) \mid \rho(f, t(d(f))) = t(c(f)) \}
\]

which is open, as it is the pullback of an open set

\[
\begin{array}{ccc}
N_t & \subseteq & t(U) \\
\downarrow & \searrow & \downarrow \\
\cdots & \subseteq & \cdots \\
d^{-1}(U) \cap c^{-1}(U) & \xrightarrow{1_{G_1} \times t \circ \rho} & G_1 \times_{G_0} R \\
& \xrightarrow{\rho} & R \\
\end{array}
\]
and which satisfies conditions (a)–(d). The section $t$ lifts to a morphism, $\hat{t} : \langle \mathcal{G}, U, N_t \rangle \longrightarrow R$, of $\text{Sh}_{G_1}(G_0)$,

$$d^{-1}(U)/\sim_{N_t} \xrightarrow{i} R \xleftarrow{n} U$$

such that $\hat{t}([f]) = \rho(f, t(d(f)))$. Here $n : U \to d^{-1}(U)/\sim_{N_t}$ is the continuous section that assigns an object, $x \in U$ to the equivalence class consisting of arrows in $N_t$ with codomain $x$, i.e. $x \mapsto [1_x]_{\sim_{N_t}}$, so that $n(U) = q(N_t) \subseteq d^{-1}(U)/\sim_{N_t}$. Moreover, $\hat{t}$ is 1–1, for if $\hat{t}([f]) = \hat{t}([g])$ then $\rho(g^{-1} \circ f, t(d(f))) = t(d(g))$ whence $f \sim_{N_t} g$. We summarize, for reference:

**Proposition 2.2.3** Any object $A \in \text{Sh}_{G_1}(G_0)$ is the join of its subobjects of the form $\langle \mathcal{G}, U, N \rangle \mapsto A$ for an open subgroupoid $N$ with $U = d(N)$.

**Proof** We show that $\hat{t}$ is continuous, the rest is clear from construction. Let $V \subseteq R$ be open. Pullback to obtain the open set $W''$ as follows:

$$\begin{array}{ccc}
W'' & \xrightarrow{\subseteq} & W' \\
\downarrow & & \downarrow \\
G_1 \times_{G_0} n(U) & \xrightarrow{1 \times e} & G_1 \times_{G_0} U \\
\subseteq & & \subseteq \\
\downarrow & & \downarrow \\
G_1 \times_{G_0} R & \xrightarrow{\rho} & R \\
\subseteq & & \subseteq \\
\end{array}$$

Since the action $m : G_1 \times_{G_0} d^{-1}(U)/\sim_{N_t} \longrightarrow d^{-1}(U)/\sim_{N_t}$ is open, the set $m(W'') \subseteq d^{-1}(U)/\sim_{N_t}$ is open, and we see that

$$m(W'') = \{ f \circ [1_{d(f)}] \mid \rho(f, t(d(f))) \in V \}$$

$$= \{ [f] \mid \rho(f, t(d(f))) \in V \}$$

$$= \hat{t}^{-1}(V)$$

The full subcategory of $\text{Sh}_{G_1}(G_0)$ consisting of objects of the form $\langle \mathcal{G}, U, N \rangle$ is, accordingly, a site for $\text{Sh}_{G_1}(G_0)$ when equipped with the canonical coverage. We refer to this as the **Moerdijk-site** for $\text{Sh}_{G_1}(G_0)$, and denote it $\text{MS}(\mathcal{G}) \longrightarrow \text{Sh}_{G_1}(G_0)$. Moerdijk sites are closed under subobjects. For consider an object $\langle \mathcal{G}, U, N \rangle$ and let $V \subseteq U$ be an open subset closed under $N$, that is, such that $x \in V$ and $f : x \to y$ in $N$ implies $y \in V$. Then

$$d^{-1}(V)/\sim_{N|V} = m(G_1 \times_{G_0} n(V)) \subseteq d^{-1}(U)/\sim_N$$
is an open subset closed under the action, and so a subobject. All subobjects are of this form:

**Lemma 2.2.4** Let \( \langle \mathcal{G}, U, N \rangle \) be an object of \( \text{Sh}_{G_1}(G_0) \) Then

\[
V \mapsto d^{-1}(V)/\sim_N|_V
\]

defines an isomorphism between the frame of open subsets of \( U \) that are closed under \( N \) and the frame of subobjects of \( \langle \mathcal{G}, U, N \rangle \).

**Proof** The inverse is given by

\[
V \mapsto d^{-1}(V)/\sim_N
\]

\[
U \mapsto d^{-1}(U)/\sim_N
\]

\[
\downarrow \quad \downarrow
\]

\[
\subseteq \quad \subseteq
\]

\[
\text{Lemma 2.2.5} \quad \text{Morphisms in the Moerdijk-site}
\]

The morphisms in the Moerdijk-site can be described in a manner similar to the objects in it; for consider a morphism

\[
\langle \mathcal{G}, U, N \rangle \xrightarrow{i} \langle \mathcal{G}, V, M \rangle
\]

Such a morphism determines and is determined by a section \( t : U \xrightarrow{d^{-1}(V)/\sim_M} \) with the property that for any \( f : x \rightarrow y \) in \( N \), we have that \( f \circ t(x) = t(y) \). We note this for reference.

**Lemma 2.2.6** Given two objects \( \langle \mathcal{G}, U, N \rangle \) and \( \langle \mathcal{G}, V, M \rangle \) in \( \text{Sh}_{G_1}(G_0) \), a morphism \( i : d^{-1}(U)/\sim_N \xrightarrow{} d^{-1}(V)/\sim_M \) determines and is determined by a section \( t : U \xrightarrow{d^{-1}(V)/\sim_M} \) with the property that for any \( f : x \rightarrow y \) in \( N \), we have that \( f \circ t(x) = t(y) \). We note this for reference.

**Proof** We obtain the section \( t \) from the morphism \( i \) by composing with the ‘canonical’ section, \( n : U \xrightarrow{d^{-1}(U)/\sim_N} \), that sends an object to the equivalence class of consisting of those arrows in \( N \) of which it is the codomain, \( n(x) = [1_x]_{\sim_N} \).
Conversely, a section $t: U \to d^{-1}(V)/\sim_M$ induces a $\hat{t}: d^{-1}(U)/\sim_{N_t} \to d^{-1}(V)/\sim_M$ as in Proposition 2.2.3 with $N_t = \{f \in d^{-1}(U) \cap e^{-1}(U) \mid f \circ t(x) = t(y)\}$. Thus $N \subseteq N_t$, whence we obtain a canonical surjection $d^{-1}(U)/\sim_N \to d^{-1}(U)/\sim_{N_t}$. Composing yields the required morphism. It is clear that these constructions are inverse to each other.

We spell this out also in terms of open sets of arrows. Given a morphism $\hat{t}: d^{-1}(U)/\sim_N \to d^{-1}(V)/\sim_M$ we can pull the corresponding section along the quotient map to obtain an open set, $T \subseteq d^{-1}(V)$,

$$
\begin{array}{ccc}
T & \subseteq & U \\
\downarrow & & \downarrow t \\
q & & \to \\
d^{-1}(V) & \to & d^{-1}(V)/\sim_M
\end{array}
$$

with the following properties: a) $m(T \times_{G_0} M) \subseteq T$, that is to say, $T$ is closed under $\sim_M$; b) $c(T) = U$, by construction; c) $m(T^{-1} \times_{G_0} T) \subseteq M$, since $t$ is a section, so any two arrows in $T$ sharing a codomain must be $\sim_M$-equivalent; and d) $m(N \times_{G_0} T) \subseteq T$, since for any $f : x \to y$ in $N$, we have that $f \circ t(x) = t(y)$. Verifying that these conditions are also sufficient gives us the following.

**Lemma 2.2.7** Given two objects, $(\mathcal{G}, U, N)$ and $(\mathcal{G}, V, M)$, in $\text{Sh}_{G_1}(G_0)$, morphisms between them,

$$
\begin{array}{ccc}
d^{-1}(U)/\sim_N & \to & d^{-1}(V)/\sim_M \\
\downarrow c & & \downarrow c \\
G_0 & \to & G_0
\end{array}
$$

are in one-to-one correspondence with open subsets $T \subseteq d^{-1}(V)$ satisfying the following properties:

i) $m(T \times_{G_0} M) \subseteq T$, i.e., $T$ is closed under $\sim_M$. 

10
ii) \( c(T) = U \);

iii) \( m(T^{-1} \times_{G_0} T) \subseteq M \), i.e., if two arrows in \( T \) share a codomain then they are \( \sim_M \)-equivalent;

iv) \( m(N \times_{G_0} T) \subseteq T \), i.e., if \( f : x \to y \) is in \( T \) and \( g : y \to z \) is in \( N \) then \( g \circ f \in T \).

Moreover, \( \hat{t} \) can be thought of as ‘precomposing with \( T \)’, in the sense that \( \hat{t}([f]_{\sim_N}) = [f \circ g]_{\sim_M} \) for some \( (any) \) \( g \in T \) such that \( c(g) = d(f) \).

**Proof** We have already indicated how a morphism from \( \langle \mathcal{G}, U, N \rangle \) to \( \langle \mathcal{G}, V, M \rangle \) determines an open subset of \( d^{-1}(V) \). For the converse construction, let \( T \subseteq d^{-1}(V) \) be given and assume \( T \) satisfies properties (i)–(iv). Map an object \( x \in U \) to the set \( t(x) := \{ f \in T \mid c(f) = x \} \). This yields a well-defined function \( t : U \to d^{-1}(V)/\sim_M \) by properties (ii) and (iii). And since \( t(U) = q(T) \) and \( q : d^{-1}(V) \to d^{-1}(V)/\sim_M \) is open, \( t(U) \) is open, so \( t \) is a continuous section. By Lemma 2.2.6 and property (iv), \( t : U \to d^{-1}(V)/\sim_M \) determines a morphism \( \hat{t} : d^{-1}(U)/\sim_N \to d^{-1}(V)/\sim_M \). It is easy to see that these constructions are inverse to each other. The final statement of the lemma is then clear from the fact that \( \hat{t} \) commutes with the actions. \( \dashv \)

We make a few further observations. Given a morphism \( \langle \mathcal{G}, U, N \rangle \to \langle \mathcal{G}, V, M \rangle \), with \( t : U \to d^{-1}(V)/\sim_M \) the corresponding section, we can write the regular epi-mono factorization of \( \hat{t} \) as

\[
\begin{array}{ccc}
\text{d}^{-1}(U)/\sim_N & \xrightarrow{i} & \text{d}^{-1}(V)/\sim_M \\
\downarrow{e} & \text{d}^{-1}(U)/\sim_{N_t} & \downarrow{\sim_M} \\
\end{array}
\]

(3)

where \( N_t = \{ f : x \to y \mid x, y \in U \land f \circ t(x) = t(y) \} \)—so that \( N \subseteq N_t \)—and \( e \) is the obvious quotient.

**Lemma 2.2.8** Given a morphism \( \hat{t} : \langle \mathcal{G}, U, N \rangle \to \langle \mathcal{G}, V, M \rangle \), corresponding to a section \( t : U \to d^{-1}(V)/\sim_M \) and an open subset \( T \subseteq d^{-1}(V) \).

a) \( \hat{t} \) is injective (and so monic) if and only if \( m(T \times_{G_0} T^{-1}) = N \). This is again equivalent to the property that for any \( f : x \to y \) such that \( x, y \in U \) we have that \( f \circ t(x) = t(y) \) implies \( f \in N \).

b) \( \hat{t} \) is surjective (and so epic) if and only if \( d(T) = V \) if and only if \( m(T^{-1} \times_{G_0} T) = M \).
c) \( \hat{t} \) is an isomorphism if and only if \( m(T \times_{G_0} T^{-1}) = N \) and \( m(T^{-1} \times_{G_0} T) = M \).

**Proof** a) We have from (3) that \( \hat{t} \) is 1–1 iff \( N = N_t = \{ f : u \to u' \mid u, u' \in U \land f \circ t(u) = t(u') \} \). This is again equivalent to the condition that \( m(T \times_{G_0} T^{-1}) \subseteq N \). That \( N \subseteq m(T \times_{G_0} T^{-1}) \) holds whether \( \hat{t} \) is 1–1 or not, since for any \( f : u \to u' \in N \) we can choose \( g : v \to u \in T \) and then \( f \circ g \in T \).

b) Let \( v \in V \) be given and suppose \( \hat{t} \) is surjective. Then there exists some \( f : u \to v \in d^{-1}(U) \) such that \( 1_v \sim_M = \hat{t}([f]_{\sim_N}) = f \circ t(u) \). Then for \( g \in t(u) \subseteq T \) we have \( 1_v \sim_M f \circ g \) whence \( f^{-1} \sim_M g \) whence \( f^{-1} \in T \), and \( d(f^{-1}) = v \). Conversely, given \( [f : v \to y]_{\sim_M} \in d^{-1}(V)/\sim_M \), choose \( g : v \to u \in T \). Then \( [f \circ g^{-1}]_{\sim_N} = d^{-1}(U)/\sim_N \) and

\[
\hat{t}([f \circ g^{-1}]_{\sim_N}) = (f \circ g^{-1}) \circ t(u) = (f \circ g^{-1}) \circ [g]_{\sim_M} = [f]_{\sim_M}
\]

The equivalence between \( d(T) = V \) and \( m(T^{-1} \times_{G_0} T) = M \) is straightforward.

**Lemma 2.2.9** Given two objects of \( \mathcal{G} \), \( \langle \mathcal{G}, U, N \rangle \) and \( \langle \mathcal{G}, V, M \rangle \), and suppose \( T \subseteq d^{-1}(V) \) is an open subset satisfying conditions (i), (iii), and (iv) of Lemma 2.2.7 and such that

\[
ii') \ c(T) \subseteq U
\]

Then \( T \) determines a morphism from the subobject \( \langle \mathcal{G}, c(T), N \mid_{c(T)} \rangle \) of \( \langle \mathcal{G}, U, N \rangle \) to \( \langle \mathcal{G}, V, M \rangle \),

\[
\langle \mathcal{G}, c(T), N \mid_{c(T)} \rangle \xrightarrow{i} \langle \mathcal{G}, V, M \rangle
\]

where \( N \mid_{c(T)} = N \cap d^{-1}(c(T)) \cap c^{-1}(c(T)) \).

**Proof** \( c(T) \) is closed under \( N \) by condition (iv), and the rest is straightforward.
2.3 Morphisms of Moerdijk-sites

A morphism \( f : \mathcal{H} \rightarrow \mathcal{G} \) of open topological groupoids induces a geometric morphism

\[
\text{Sh}_{H_1}(H_0) \xrightarrow{f^*} \text{Sh}_{G_1}(G_0)
\]

but the inverse image \( f^* \) does not necessarily restrict to a functor between the respective Moerdijk-sites. The following condition (somewhat simplified from [2]) ensures that it does.

**Definition 2.3.1** A morphism \( f : \mathcal{H} \rightarrow \mathcal{G} \) of open topological groupoids is *strictly full* if for all \((h : x \rightarrow f_0(y)) \in G_1\) there exists \( g \in H_1 \) such that \( c(g) = y \) and \( f_1(g) = h \).

If \( f : \mathcal{H} \rightarrow \mathcal{G} \) is a morphism of open topological groupoids and \( N \subseteq G_1 \) is open and closed under compositions and inverses, then so is \( f_1^{-1}(N) \subseteq H_1 \), and \( d(f_1^{-1}(N)) = f_0^{-1}(U) \). Therefore, \( \langle \mathcal{H}, f_0^{-1}(U), f_1^{-1}(N) \rangle \) is an object in the Moerdijk site of \( \text{Sh}_{H_1}(H_0) \). Moreover:

**Lemma 2.3.2** Let \( f : \mathcal{H} \rightarrow \mathcal{G} \) be a strictly full morphism of open topological groupoids, and let \( \langle \mathcal{G}, U, N \rangle \) be an object of the Moerdijk-site of \( \text{Sh}_{G_1}(G_0) \). Then

\[
\langle \mathcal{H}, f_0^{-1}(U), f_1^{-1}(N) \rangle \cong f^*(\langle \mathcal{G}, U, N \rangle)
\]

**Proof** Consider the diagram

\[
\begin{array}{ccc}
V = f_0^{-1}(U) & \overset{t}{\rightarrow} & H_0 \\
\downarrow & & \downarrow f_0 \\
\text{res} & \subseteq & G_0
\end{array}
\]

where \( t \) is the section obtained by pulling back the canonical section \( n : U \rightarrow d^{-1}(U)/N \) so that \( t(v) = \langle v, [1, [f_0(v)]_N] \rangle \) —and \( N_t \subseteq H_1 \) and \( t \) are the induced open subgroupoid and morphism as in [2] and Proposition [2.2.3]. Now, we have

\[
N_t = \{ g \in d^{-1}(V) \cap e^{-1}(V) \mid f_1(g) \circ [1, [f_0(d(g))]_N = [1, [f_0(c(g))]_N \}
\]

\[
= \{ g \in d^{-1}(V) \cap e^{-1}(V) \mid f_1(g) \in N \}
\]

\[
= f_1^{-1}(N)
\]
Lemma 2.3.3 Let $f : \mathcal{H} \to \mathcal{G}$ be a strictly full morphism of open topological groupoids, and let

\[ \hat{i} : \langle \mathcal{G}, U_1, N_1 \rangle \to \langle \mathcal{G}, U_2, N_2 \rangle \]

be a morphism in the Moerdijk-site of $\text{Sh}_G(G_0)$ corresponding to an open set $T \subseteq G_1$. Then

\[ f^*(\hat{i}) : \langle \mathcal{H}, f^{-1}_0(U_1), f^{-1}_1(N_1) \rangle \to \langle \mathcal{H}, f^{-1}_0(U_2), f^{-1}_1(N_2) \rangle \]

corresponds to the open set $f^{-1}_1(T) \subseteq H_1$.

**Proof** We verify, using Lemma 2.2.7, that $f^{-1}_1(T)$ defines a morphism from $\langle \mathcal{H}, f^{-1}_0(U_1), f^{-1}_1(N_1) \rangle$ to $\langle \mathcal{H}, f^{-1}_0(U_2), f^{-1}_1(N_2) \rangle$: First, we have

\[ f^{-1}_1(T) \subseteq f^{-1}_1(d^{-1}(U_2) \cap c^{-1}(U_1)) = d^{-1}(f^{-1}_0(U_2)) \cap c^{-1}(f^{-1}_0(U_1)) \]

For $c(f^{-1}_1(T)) = f^{-1}_0(U_1)$, we have $c(f^{-1}_1(T)) \subseteq f^{-1}_0(U_1)$ since given $h$ such that $f_1(h) \in T$ then $f_0(c(h)) \in U_1$. Conversely, given $y \in H_0$ such that $f_0(y) \in U_1$, then there exist $g \in T$ such that $y = c(g)$. But then, since $f$ is strictly full, there exist $h \in H_1$ such that $c(h) = y$ and $f_1(h) = g$. The remaining conditions (i), (ii), and (iv) are straightforward, and so $f^{-1}_1(T)$ does define a morphism $\hat{i}$. Remains to show that it is $f^*(\hat{i})$, that is, that the diagram

\[
\begin{array}{ccc}
\langle \mathcal{H}, f^{-1}_0(U_1), f^{-1}_1(N_1) \rangle & \xrightarrow{\sim} & H_0 \times_{G_0} d^{-1}(U_1)/\sim_{N_1} \\
\downarrow \hat{i} & & \downarrow f^*(\hat{i}) \\
\langle \mathcal{H}, f^{-1}_0(U_2), f^{-1}_1(N_2) \rangle & \cong & H_0 \times_{G_0} d^{-1}(U_2)/\sim_{N_2}
\end{array}
\]

commutes. Let $\langle g : u \to x \mapsto [f_1(g)]_{N_1} \rangle$ be given, with $u \in f^{-1}_0(U_1)$. Translating along the top isomorphism yields the element $\langle x, [f_1(g)]_{N_1} \rangle$. There exists a $h \in T$ with $c(h) = f_0(u)$, and

\[ \langle x, [f_1(g) \circ h]_{N_2} \rangle = f^*(\langle x, [f_1(g)]_{N_1} \rangle). \]
Similarly, we compute $\hat{t}([g])$ by precomposing with an arrow from $f_1^{-1}(T)$. But $f$ is strictly full, so there exists an $h' \in H_1$ with $c(h) = u$ such that $f_1(h') = h$. And translating the element

$$\hat{t}([g]_{f_1^{-1}(N_1)}) = [g \circ h']_{f_1^{-1}(N_2)}$$

along the bottom isomorphism yields

$$\langle x, [f_1(g \circ h')]_{N_2} \rangle = \langle x, [f_1(g) \circ h]_{N_2} \rangle.$$

\[ \square \]

3 Subgroupoids and Subtoposes

3.1 Subtoposes

In this section Moerdijk-sites are used to show that strictly full topological subgroupoids induce subtoposes. Again, all topological groupoids under consideration are considered to be open, in the sense that the domain and codomain maps, $d, c : G_1 \rightrightarrows G_0$, are open. Recall that a geometric morphism $f : \mathcal{E} \rightarrow \mathcal{F}$ is said to be an inclusion if the direct image functor is full and faithful. Among the number of equivalent conditions for a geometric morphism to be an inclusion, it is particularly relevant, for the purposes of this section, to highlight the following one. Say that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ which preserves monomorphisms is full on subobjects if the induced map $F : \text{Sub}_\mathcal{C}(A) \rightarrow \text{Sub}_\mathcal{D}(F(A))$ is surjective for any $A$ in $\mathcal{C}$. Say that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is essentially full if for any $B, C$ in $\mathcal{C}$ and morphism $f : F(B) \rightarrow F(C)$ in $\mathcal{D}$, there exists in $\mathcal{C}$ an object $B'$ with a zig-zag between $B$ and $B'$, and object $C'$ with a zig-zag between $C$ and $C'$, and a morphism $f' : B' \rightarrow C'$ such that: i) $F$ sends the morphisms in both zig-zags to isomorphisms; and ii) the resulting isomorphisms $F(B) \cong F(B')$ and $F(C) \cong F(C')$ form a commuting square with $f$ and $F(f)$:

$$\begin{array}{ccc}
F(B') & \xrightarrow{F(f')} & F(C') \\
\cong & \downarrow & \cong \\
F(B) & \xrightarrow{f} & F(C')
\end{array}$$

Note that if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a regular functor between regular categories then the condition for essential fullness can be simplified to say that given $f : F(A) \rightarrow F(B)$ in $\mathcal{D}$ there exists a span

$$\begin{array}{ccc}
B & \xrightarrow{\pi} & R \\
& \searrow & \downarrow f' \\
& & C
\end{array}$$
in $C$ such that $F(\pi)$ is an isomorphism and

\[
\begin{array}{ccc}
F(\pi) & \xrightarrow{F(R)} & F(f') \\
\downarrow & & \downarrow \\
F(B) & \xrightarrow{f} & F(C)
\end{array}
\]

commutes. Moreover, it is straightforward to verify the following.

**Lemma 3.1.1** If $F : C \rightarrow D$ is a regular functor between regular categories then $F$ is essentially surjective and full on subobjects if and only if $F$ is essentially surjective and essentially full.

For geometric morphisms we have, then:

**Lemma 3.1.2** With $f : \mathcal{E} \rightarrow \mathcal{F}$ a geometric morphism, the following are equivalent:

1. $f$ is an inclusion.

2. The inverse image functor $f^* : \mathcal{F} \rightarrow \mathcal{E}$ is essentially surjective and full on subobjects.

3. The inverse image functor $f^* : \mathcal{F} \rightarrow \mathcal{E}$ is essentially surjective and essentially full.

**Proof** (2) and (3) are equivalent as noted in Lemma 3.1.1. If $f$ is an inclusion then every counit component of the adjunction is an isomorphism, and it follows that $f^*$ is both essentially surjective and essentially full. Given (3) we can consider the surjection-inclusion factorization $f = m \circ e$. The inverse image functor of the surjection $e^*$ is faithful and reflects isomorphisms. Since (the counit of $m$ is an isomorphism and) $f^*$ is essentially full and essentially surjective, so is $e^*$. But since $e^*$ reflects isomorphisms, this means that $e^*$ must be full, and therefore an equivalence.

It is sufficient to check the third condition of Lemma 3.1.2 on sites, in the following sense:

**Lemma 3.1.3** Let $f : \mathcal{E} \rightarrow \mathcal{F}$ be a geometric morphism and $S_\mathcal{E}$, $S_\mathcal{F}$ small subcategories the objects of which form separating sets for $\mathcal{E}$ and $\mathcal{F}$ respectively, and suppose the inverse image functor $f^*$ restricts to a functor $F : S_\mathcal{F} \rightarrow S_\mathcal{E}$. If $F$ is essentially surjective and essentially full, then $f$ is an inclusion.
Proof Again, consider the surjective inclusion factorization

\[
\begin{array}{c}
\mathcal{E} \\
\downarrow^f \\
\mathcal{F}
\end{array}
\]

of \( f \). The full subcategory \( S_I \hookrightarrow \mathcal{I} \) consisting of the objects that are in \( m^*(S_F) \) is a site for \( \mathcal{I} \) when equipped with the canonical coverage inherited from \( \mathcal{I} \). The inverse image \( e^* \) restricts to a conservative functor of sites \( E : S_I \rightarrow S_E \) such that a sieve in \( S_I \) is covering if (and only if) the image of it under \( E \) generates a covering sieve in \( S_E \). But now \( E \) is also essentially surjective, because \( F \) is, and full, because it reflects isomorphisms and \( F \) is essentially full. So \( e \) is an equivalence.

\[ \qed \]

3.2 Subgroupoids

Using Moerdijk-sites, we apply Lemma 3.1.3 to the geometric morphism induced by a strictly full topological subgroupoid inclusion.

Definition 3.2.1 For a topological groupoid \( \mathcal{G} \), a topological subgroupoid, \( \mathcal{H} \), of \( \mathcal{G} \) is a subgroupoid equipped with the subspace topologies, i.e. such that the inclusion functor components

\[
\begin{array}{c}
H_1 \\
\downarrow^d \\
H_0
\end{array}
\quad \subseteq
\quad \begin{array}{c}
G_1 \\
\downarrow^c \\
G_0
\end{array}
\]

are subspace inclusions. We say that a topological subgroupoid is strictly full if it is strictly full as a subcategory, where ‘strictly full’ means full and replete, i.e. such that for any \( x \in H_0 \) if we have an arrow \( f : x \rightarrow y \) in \( G_1 \) then \( f \in H_1 \).

It is clear that a strictly full (topological) subgroupoid of an open groupoid is itself open. Thus a strictly full subgroupoid inclusion \( \iota : \mathcal{H} \rightarrow \mathcal{G} \) is a strictly full morphism between open groupoids, so that the induced inverse
image functor restricts to a functor $I$ between Moerdijk-sites by 2.3.2:

\[
\begin{array}{c}
\text{MS}(\mathcal{G}) \xrightarrow{I} \text{MS}(\mathcal{H}) \\
\text{Sh}_{G_1}(G_0) \xrightarrow{\iota^*} \text{Sh}_{H_1}(H_0)
\end{array}
\]

with

\[
\iota^*(\langle \mathcal{G}, U, N \rangle) = \langle \mathcal{H}, \iota_0^{-1}(U), \iota_1^{-1}(N) \rangle = \langle \mathcal{H}, U \cap H_0, N \cap H_1 \rangle
\]

The remainder of this section shows that $I$ is essentially surjective and essentially full, so that the geometric morphism $\iota$ is an inclusion. It is useful to note the following:

**Lemma 3.2.2** Let $\mathcal{H} \rightarrow \mathcal{G}$ be a strictly full subgroupoid of an open groupoid, and let $V, W \subseteq G_1$ be open sets. Then $m(V \times_{G_0} W)$ is open and

\[
m(V \times_{G_0} W) \cap H_1 = m(V \cap H_1 \times_{H_0} W \cap H_1)
\]

**Proof** Composition of arrows is an open map for all open groupoids (see [2]). The rest is a straightforward consequence of the inclusion being strictly full.

**Lemma 3.2.3** $I : \mathcal{G} \rightarrow \mathcal{H}$ essentially surjective

**Proof** Consider an object $\langle \mathcal{H}, V, M \rangle$. With $M$ an open set in the subspace $H_1 \subseteq G_1$, we have the open set

\[
N := \bigcup \{ K \in \mathcal{O}(G_1) \mid K \cap H_1 \subseteq M \} \subseteq G_1
\]

and since $\mathcal{G}$ is an open groupoid, the open set

\[
U := d(N) \cup c(N)
\]

We verify that these two sets give us an object in $\mathcal{G}$.

a) We have $d(N), c(N) \subseteq U$ by construction of $U$.

b) Next, $e(U) \subseteq N$ follows from (b) and (c) below together with the definition of $U$. 18
c) For $m(N \times_{G_0} N) \subseteq N$, let $\langle g, f \rangle$ be a composable pair of arrows in $N$, then we can choose open sets, $g \in L$ and $f \in K$ such that $L \cap H_1, K \cap H_1 \subseteq M$. By Lemma 3.2.2, we have $m(L \times_{G_0} K) \cap H_1 = m(L \cap H_1 \times_{H_0} K \cap H_1) \subseteq M$, so $g \circ f \in N$.

d) $i(N) \subseteq N$ follows from the fact that $i : G_1 \longrightarrow G_1$ is a homeomorphism and $M$ is closed under inverses. (Note that this means that $U = d(N) = c(N)$.)

Finally, $I(\langle \mathcal{G}, U, N \rangle) = \langle \mathcal{H}, U \cap H_0, N \cap H_1 \rangle = \langle \mathcal{H}, V, M \rangle$.

Denote the object constructed in the proof of Lemma 3.2.3 by $J(\langle \mathcal{H}, V, M \rangle)$.

**Lemma 3.2.4** For any object $G \in \text{MS}(\mathcal{G})$ there is a canonical morphism $G \longrightarrow J(I(G))$ such that $I$ sends this morphism to the identity

$$1_{I(G)} : I(G) \longrightarrow I(G) = I(J(I(G)))$$

**Proof** Let $\langle \mathcal{G}, U, N \rangle$ be given, and write

$$\langle \mathcal{H}, U, N \rangle := \langle \mathcal{H}, U \cap H_0, N \cap H_1 \rangle = I(\langle \mathcal{G}, U, N \rangle)$$

$$\langle \mathcal{G}, U, N \rangle := J(\langle \mathcal{G}, U, N \rangle)$$

Then $N \subseteq \overline{N}$ and, consequently, $U \subseteq \overline{U}$. We can therefore compose the canonical section $\bar{n} : \overline{U} \longrightarrow d^{-1}(\overline{U})$ with the inclusion $U \subseteq \overline{U}$,

$$d^{-1}(U)/_{\sim_N} \xrightarrow{i} d^{-1}(\overline{U})/_{\sim_N}$$

(4)

For any $f : x \to y$ in $N$, we have that

$$f \circ t(x) = f \circ \bar{n}(x) = f \circ [1_x]_{\sim_N} = [f]_{\sim_N} = [1_y]_{\sim_N} = t(y)$$

since $N \subseteq \overline{N}$. So $t$ induces the morphism $\hat{t}([f]_{\sim_N}) = [f]_{\sim_N}$ in (4). By Lemma 2.3.3 $\hat{t}$ is sent to the identity by $I$.

**Lemma 3.2.5** The functor $I : \mathcal{G} \to \mathcal{H}$ is essentially full.
Proof: Let two objects $\langle \mathcal{G}, U, N \rangle$ and $\langle \mathcal{G}, V, M \rangle$ be given, write

$$\langle \mathcal{H}, U, N \rangle = \langle \mathcal{H}, U \cap H_0, N \cap H_1 \rangle = I(\langle \mathcal{G}, U, N \rangle)$$
$$\langle \mathcal{H}, V, M \rangle = \langle \mathcal{H}, V \cap H_0, M \cap H_1 \rangle = I(\langle \mathcal{G}, V, M \rangle)$$

and suppose we have a morphism $\hat{t}: \langle \mathcal{H}, U, N \rangle \longrightarrow \langle \mathcal{H}, V, M \rangle$. Write $T \subseteq d^{-1}(V)$ for the corresponding open subset of arrows. Write $\langle \mathcal{G}, V, M \rangle = J(\langle \mathcal{H}, V, M \rangle)$, and recall the morphism $v: \langle \mathcal{G}, V, M \rangle \rightarrow \langle \mathcal{G}, V, M \rangle$ of Lemma 3.2.4. Now, consider the open set

$$S := c^{-1}(U) \cap \bigcup \{P \in O(G_1) \mid P \cap H_1 \subseteq T\}$$

That is, an arrow $f: x \rightarrow y$ is in $S$ if and only if $y \in U$ and there exists an open neighborhood $W_f$ of $f$ so that $W_f \cap H_1 \subseteq T$. Note that $S \cap H_1 = T$. First, we show that $m(S^{-1} \times_{G_0} S) \subseteq \overline{M}$. Given $\langle g^{-1}, f \rangle \in m(S^{-1} \times_{G_0} S)$, we can choose open sets, $W_g$ and $W_f$, around $g$ and $f$ respectively such that $W_g \cap H_1, W_f \cap H_1 \subseteq T$. We then have the open set $W_{g^{-1} \circ f} := m(i(W_g) \times_{G_0} W_f)$ around $g^{-1} \circ f$, with $i(W_g) \cap H_1 \subseteq T$, since $i(T) = T$. Now, since $m(T^{-1} \times_{H_0} T) \subseteq \overline{M}$, we have $m(i(W_g) \times_{G_0} W_f) \cap H_1 = m(i(W_g) \cap H_1 \times_{H_0} W_f \cap H_1) \subseteq \overline{M}$, so $g^{-1} \circ f \in \overline{M}$. Next, similar arguments establish that $m(S \times_{G_0} \overline{M}) \subseteq S$ and $m(N \times_{G_0} S) \subseteq S$. Having thus verified that:

i) $m(S \times_{G_0} \overline{M}) \subseteq S$;

ii) $c(S) \subseteq U$;

iii) $m(S^{-1} \times_{G_0} S) \subseteq \overline{M}$; and

iv) $m(N \times_{G_0} S) \subseteq S$.

Lemma 2.2.3 tells us that $S$ corresponds to a morphism $\hat{s}: \langle \mathcal{G}, c(S), N \mid c(S) \rangle \rightarrow \langle \mathcal{G}, V, M \rangle$,

$$\langle \mathcal{G}, c(S), N \mid c(S) \rangle \xrightarrow{\hat{s}} \langle \mathcal{G}, V, M \rangle$$

where (by inspection and Lemma 3.2.4 respectively) $I$ sends both vertical arrows to identities. Moreover, $S \cap H_1 = T$ and so by Lemma 2.3.3, $I(\hat{s}) = \hat{t}$.

We conclude:
**Theorem 3.2.6** Let $\mathcal{G}$ be an open groupoid and $\iota : \mathcal{H} \longrightarrow \mathcal{G}$ a strictly full subgroupoid inclusion. Then the induced geometric morphism

$$\iota : \text{Sh}(\mathcal{H}) \longrightarrow \text{Sh}(\mathcal{G})$$

is an inclusion.

**Proof** By Lemmas and 3.1.3, 2.3.3, 3.2.3, and 3.2.5.

We end this section by noting the consequence of Theorem 3.2.6 concerning the surjective-embedding factorization of geometric morphisms induced by morphisms of topological groupoids.

**Lemma 3.2.7** Let $f : \mathcal{H} \rightarrow \mathcal{G}$ be a topological functor between (not necessarily open) groupoids. If $f$ is essentially surjective (as a functor) then the geometric morphism induced,

$$f : \text{Sh}_{H_1}(H_0) \longrightarrow \text{Sh}_{G_1}(G_0)$$

is a surjection.

**Proof** The induced inverse image functor must clearly be faithful if $f : \mathcal{H} \rightarrow \mathcal{G}$ is (essentially) surjective.

For a topological groupoid, $\mathcal{G}$, and a subset $X \subseteq G_0$, we call the strictly full subgroupoid of $\mathcal{G}$ the objects of which are the elements of $G_0$ that are isomorphic to an element in $X$ for the subgroupoid generated by $X$. (If $X$ is already closed under $G_1$ in this sense and we want to draw attention to this fact, we say the full subgroupoid induced by $X$ instead.)

**Proposition 3.2.8** Let $f : \mathcal{F} \rightarrow \mathcal{G}$ be a topological functor between groupoids, where at least $\mathcal{G}$ is open. Set $\mathcal{H}$ to be the strictly full subgroupoid of $\mathcal{G}$ generated by $f_0(F_0)$, and set $\iota : \mathcal{H} \hookrightarrow \mathcal{G}$ to be the embedding. Then $f$ factors through $\iota$,

and the geometric morphisms induced by $e$ and $\iota$ form a surjective-embedding factorization of the morphism induced by $f$,

$$\text{Sh}_{F_1}(F_0) \longrightarrow \text{Sh}_{G_1}(G_0) \longrightarrow \iota_*$$

$$\text{Sh}_{H_1}(H_0)$$
Proof

By Theorem 3.2.6 and Lemma 3.2.7.

Consider a topological groupoid, \( \mathcal{G} \), and an element \( x \in G_0 \). The lone element \( x \) together with its identity arrow constitute a (topological) groupoid, the topos of equivariant sheaves over which is simply the topos \( \text{Sets} \), and the inclusion into \( \mathcal{G} \) therefore induces a point of \( \text{Sh}_{G_1}(G_0) \), that is a geometric morphism

\[
x : \text{Sets} \longrightarrow \text{Sh}_{G_1}(G_0)
\]

By Theorem 3.2.6 we have, then, the following.

**Corollary 3.2.9** For a topological groupoid, \( \mathcal{G} \), and an element \( x \in G_0 \), the image of the point of \( \text{Sh}_{G_1}(G_0) \) induced by \( x \) is the topos of equivariant sheaves on the subgroupoid, \( \mathcal{G}^x \), induced by \( x \),

\[
\begin{array}{ccc}
\text{Sets} & \xrightarrow{x} & \text{Sh}_{G_1}(G_0) \\
\downarrow & & \downarrow \iota \\
\text{Sh}_{G_1^x}(G_0) & & \\
\end{array}
\]

4 Quotient Theories and Subgroupoids

4.1 Quotient Theories and Subtoposes

It is convenient in what follows to stipulate that theories are always closed under consequence, so by a (geometric) theory we mean a deductively closed set of (geometric) sequents.

**Definition 4.1.1** For theories, \( T, T' \), over the same signature, \( \Sigma \), we say that \( T' \) is a quotient of \( T \) and write

\[
T' \subseteq T
\]

if \( T \) is contained in \( T' \) as a set of sequents.

Quotient theories of a theory \( T \) correspond to subtoposes of \( \text{Set}[T] \). Specifically, the subtoposes of a topos form a complete lattice, and if \( T \) is a geometric theory over a signature \( \Sigma \), then the assignment sending a quotient of \( T \) to its classifying topos defines a bijection between the quotients of \( T \) and the subtoposes of the classifying topos \( \text{Set}[T] \) of \( T \). Moreover, if \( i : \mathcal{E} \longrightarrow \text{Set}[T] \) is a subtopos embedding into the classifying topos of a geometric theory, \( T \)—so that \( \mathcal{E} \) is the classifying topos of a quotient, \( T' \), of \( T \)—and \( \mathcal{U} \) is the universal model of \( T \) in \( \text{Set}[T] \) then \( i^*(\mathcal{U}) \) is the universal model of \( T' \) in \( \mathcal{E} \). For more on subtoposes, classifying toposes, and geometric theories, see [3]. A detailed presentation and analysis of the correspondence between subtoposes and quotient theories can be found in the preprint [4].
4.2 Groupoids of Models

Let \( \mathcal{G} \) be a topological groupoid, \( \text{Sh}_{G_1}(G_0) \) the topos of equivariant sheaves on it. By [3, D3.1.13] there exists a geometric theory, \( T \), such that
\[
\text{Sh}_{G_1}(G_0) \cong \text{Set}[T]
\]

The forgetful functor \( u^* : \text{Sh}_{G_1}(G_0) \to \text{Sh}(G_0) \) is a faithful inverse image functor, whence \( \text{Sh}_{G_1}(G_0) \) (and so \( T \)) has enough points (models). Specifically, any \( x \in G_0 \) induces a point of \( \text{Sh}(G_0) \) and hence of \( \text{Sh}_{G_1}(G_0) \),
\[
\begin{array}{ccc}
\text{Sets} & \xrightarrow{x} & \text{Sh}_{G_1}(G_0) \\
\downarrow & & \downarrow u \\
x & \xrightarrow{u} & \text{Sh}(G_0)
\end{array}
\]

which determines a \( T \)-model, \( M_x \) in \( \text{Sets} \). If \( f : x \to y \) is an arrow in \( G_1 \), then it induces an invertible geometric transformation of points,
\[
\begin{array}{ccc}
\text{Sets} & \xrightarrow{y} & \text{Sh}_{G_1}(G_0) \\
\downarrow f & & \downarrow \\
x & \xrightarrow{f} & \text{Sh}(G_0)
\end{array}
\]

which in turn determines a homomorphism of \( T \)-models, \( f : M_x \to M_y \).

Fixing a classified theory \( T \) and a generic \( T \)-model \( U \in \text{Sh}_{G_1}(G_0) \) allows us, therefore, to regard \( \mathcal{G} \) as a topological groupoid consisting of \( T \)-models and isomorphisms. Since the points of \( \text{Sh}_{G_1}(G_0) \) induced by elements in \( G_0 \) are jointly surjective, in the sense that the inverse image functors are jointly faithful and thus jointly conservative, we can conclude that \( T \) has enough models in \( G_0 \), in the sense, then, that if \( \sigma \) is a sequent true in all models in \( G_0 \) then \( T \vdash \sigma \). Regarding \( \mathcal{G} \) thus as a groupoid of models and isomorphisms for some theory \( T \) we often denote elements of \( G_0 \) by boldface capital letters, \( M, N \), and elements of \( G_1 \) with boldface lower case letters, \( f, g \).

Now, suppose that \( \mathcal{G} \) is an open topological groupoid, that \( t : \mathcal{K} \to \mathcal{G} \) is a strictly full subgroupoid embedding, and that \( T \) is a theory classified by \( \text{Sh}_{G_1}(G_0) \). Then \( \text{Sh}_{H_1}(H_0) \) will classify a quotient of \( T \), and we verify that it is the expected one:

**Proposition 4.2.1** Let \( \mathcal{G} \) be an open topological groupoid and
\[
t : \mathcal{K} \to \mathcal{G}
\]
a strictly full topological subgroupoid. Suppose \( \text{Sh}_{G_1}(G_0) \) classifies a geometric theory, \( \mathcal{T} \), and write \( \mathcal{U}_T \) for the universal model of \( \mathcal{T} \) in \( \text{Sh}_{G_1}(G_0) \). Then \( \text{Sh}_{H_1}(H_0) \) classifies a quotient theory, \( \mathcal{T}' \) of \( \mathcal{T} \), and

\[
\iota^* : \text{Sh}_{G_1}(G_0) \longrightarrow \text{Sh}_{H_1}(H_0)
\]

takes \( \mathcal{U}_T \) to the universal model, \( \mathcal{U}_{T'} \), of \( \mathcal{T}' \) in \( \text{Sh}_{H_1}(H_0) \). Moreover, if we consider the elements of \( G_0 \) to be \( \mathcal{T} \)-models, then \( \mathcal{T}' \) is the theory given by

\[
\mathcal{T}' = \{ \sigma \in \mathcal{L} \mid M \models \sigma, \text{ for all } M \in H_0 \}
\]

where \( \mathcal{L} \) is the set of all geometric sequents in the language of \( \mathcal{T} \).

**Proof** An element \( M \in H_0 \subseteq G_0 \) corresponds to a point,

\[
m : \text{Sets} \longrightarrow \text{Sh}_{H_1}(H_0) \longrightarrow \text{Sh}_{G_1}(G_0)
\]

so \( M \models T' \), whence

\[
\mathcal{T}' \subseteq \{ \sigma \in \mathcal{L} \mid M \models \sigma, \text{ for all } M \in H_0 \}
\]

But the points of \( \text{Sh}_{H_1}(H_0) \) induced by elements of \( H_0 \) are jointly surjective, so this inclusion is an equality. \( \dashv \)

It is known that a join of subtoposes classifies the intersection of the corresponding theories, and so we note:

**Corollary 4.2.2** If \( \mathcal{G}_i, i \in I \) is a family of strictly full subgroupoids of \( \mathcal{G} \) and \( \mathcal{H} \) is the subgroupoid induced by \( \bigcup_{i \in I} \{ H_0^i \} \), then \( \text{Sh}_{H_1}(H_0) \hookrightarrow \text{Sh}_{G_1}(G_0) \) is the join of the subtoposes \( \text{Sh}_{H_1}(H_0^i) \).

Now, suppose that \( \mathcal{G} \) is an open topological groupoid, that \( \mathcal{T} \) is a theory classified by \( \text{Sh}_{G_1}(G_0) \), and that \( \mathcal{T}' \) is a quotient of \( \mathcal{T} \). Then \( \mathcal{T}' \) defines a subset,

\[
G_0^{\mathcal{T}'} = \{ M \in G_0 \mid M \models \mathcal{T}' \} \subseteq G_0
\]

This set defines a further quotient

\[
\mathcal{T}(G_0^{\mathcal{T}'}) = \{ \sigma \in \mathcal{L} \mid M \models \mathcal{T}', \text{ for all } M \in G_0^{\mathcal{T}'} \}
\]

which, in light of Proposition 4.2.1, is the theory classified by the topos of equivariant sheaves on the strictly full subgroupoid of \( \mathcal{G} \) induced by \( G_0^{\mathcal{T}'} \). We can say that an open topological groupoid has enough elements with respect to quotients of \( \mathcal{T} \) if \( \text{Sh}_{G_1}(G_0) \) classifies \( \mathcal{T} \) and if for any quotient, \( \mathcal{T}' \supseteq \mathcal{T} \) that has enough models we have that \( \mathcal{T}(G_0^{\mathcal{T}'}) = \mathcal{T}' \). This comes to saying that the subgroupoids with enough points are exactly the ones arising from subgroupoids, and so it does not depend on the fixed theory. We note this for reference.
**Lemma 4.2.3** Let $\mathcal{G}$ be an open topological groupoid. The following are equivalent:

1. There exists a geometric theory, $\mathcal{T}$, such that $\mathcal{G}$ has enough elements with respect to quotients of $\mathcal{T}$.

2. Any subtopos of $\text{Sh}_{\mathcal{G}_1}(G_0)$ which has enough points is of the form $\text{Sh}_{\mathcal{H}_1}(H_0)$ for a strictly full subgroupoid $\mathcal{H} \subseteq \mathcal{G}$.

3. Any subtopos, $\mathcal{E} \hookrightarrow \text{Sh}_{\mathcal{G}_1}(G_0)$ which has enough points is the join, in the lattice of subtoposes of $\text{Sh}_{\mathcal{G}_1}(G_0)$, of the subtoposes arising as the images of points of $\text{Sh}_{\mathcal{G}_1}(G_0)$ induced by elements of $G_0$ which factor through $\mathcal{E}$.

**Proof** Straightforward using \[3.2.9, 4.2.1, 4.2.2\] ⊣

Any topos with enough points is equivalent to an equivariant sheaf topos on an open topological groupoid by the construction of Butz and Moerdijk in \[1\]. (A somewhat simplified description of this groupoid directly in terms of models and isomorphisms of a fixed geometric theory was presented in \[3\], and a variant with a slightly different topology suitable for first-order classical theories was given in \[6\]). Such groupoids are (in all three cases) open and readily seen to satisfy the conditions of Lemma 4.2.3. Thus for all geometric theories $\mathcal{T}$ with enough models there exists an open topological groupoid $\mathcal{G}$ satisfying the conditions of 4.2.3 such that

$$\text{Sh}_{\mathcal{G}_1}(G_0) \simeq \text{Set}[\mathcal{T}]$$

Call, for current purposes, an open topological groupoid satisfying those conditions a *saturated groupoid* (with apologies for overloading terms). Call a strictly full subgroupoid of a saturated groupoid $\mathcal{G}$ *definable* if it is the strictly full subgroupoid induced by $G^T_1$ for a quotient theory $\mathcal{T}'$ of a theory $\mathcal{T}$ classified by $\text{Sh}_{\mathcal{G}_1}(G_0)$. Clearly, a definable subgroupoid (of a saturated groupoid) is saturated, but a strictly full subgroupoid of a saturated groupoid need not be saturated, and a saturated subgroupoid need not be definable. We proceed to characterize the definable subsets of a saturated groupoid in terms of the groupoid and without reference to theories.

Say that two $\mathcal{T}$-models are geometrically equivalent if they satisfy the same geometric sequents. Removing the reference to a theory $\mathcal{T}$, we formulate a corresponding notion for points of a groupoid and extend to sets of points in order to characterize definable sets.
**Definition 4.2.4** Say that two geometric morphisms

\[ \mathcal{F} \xrightarrow{f} \mathcal{E} \xleftarrow{g} \mathcal{E} \]

are *geometrically equivalent* (GE), written \( f \sim_{GE} g \), if for all objects \( A \in \mathcal{E} \) and subobjects \( P, R \in \text{Sub}_\mathcal{E}(A) \) we have

\[ f^*(P) \leq f^*(R) \iff g^*(P) \leq g^*(R) \]

**Remark 4.2.5** Note that if we have a generating set of objects \( S_\mathcal{E} \) of \( \mathcal{E} \) which is closed under subobjects, it is sufficient to check the condition for GE on the full subcategory of \( S_\mathcal{E} \). In particular, two geometric morphisms into a classifying topos

\[ \mathcal{F} \xrightarrow{f} \text{Set}[T] \]

are GE iff the \( T \)-models in \( \mathcal{F} \) corresponding to \( f \) and \( g \) satisfy the same geometric sequents in the language of \( T \).

We now apply Definition 4.2.4 to the case of two points

\[ \text{Sets} \xrightarrow{f_x} \text{Sh}_{G_1}(G_0) \xleftarrow{f_y} \text{Sh}_{G_1}(G_0) \]

induced by elements \( x, y \in G_0 \) of an open topological groupoid.

**Definition 4.2.6** Two elements \( x, y \in G_0 \) of an open topological groupoid \( G \) are said to be *geometrically equivalent* (GE), written \( x \sim_{GE} y \), if for all objects \( \langle G, U, N \rangle \in \text{Sh}_{G_1}(G_0) \)—i.e. all open subsets \( U \subseteq G_0 \) and \( N \subseteq G_1 \) satisfying conditions (a)–(d) of 2.2.1—and all open subsets \( V, W \subseteq U \) that are closed under \( N \) we have

\[ c^{-1}(x) \cap d^{-1}(V) \subseteq c^{-1}(x) \cap d^{-1}(W) \]

\[ \iff c^{-1}(y) \cap d^{-1}(V) \subseteq c^{-1}(y) \cap d^{-1}(W) \]

**Lemma 4.2.7** Two elements \( x, y \in G_0 \) of an open topological groupoid \( G \) are geometrically equivalent iff the induced points

\[ \text{Sets} \xrightarrow{f_x} \text{Sh}_{G_1}(G_0) \xleftarrow{f_y} \text{Sh}_{G_1}(G_0) \]

are geometrically equivalent.
It is sufficient to check the condition of Definition 4.2.4 on the
Moerdijk-site of Sh_{G_1}(G_0), and subobjects in the Moerdijk-site were charac-
terized in Lemma 2.2.4. Put together, the result is the condition of Definition 4.2.6.

We now extend this to sets of points.

**Definition 4.2.8**

1. For a geometric morphism \( g : \mathcal{F} \rightarrow \mathcal{E} \) and a set of geometric morphisms \( X = \{ f_j : \mathcal{F} \rightarrow \mathcal{E} \mid j \in J \} \), say that \( g \) (geometrically) dominates \( X \), written \( g \gg_{GD} X \), if for all objects \( A \in \mathcal{E} \) and subobjects \( P, R \in \text{Sub}_E(A) \) we have
   \[
   f_j^*(P) \leq f_j^*(R) \quad \text{for all } j \in J \Rightarrow g^*(P) \leq g^*(R)
   \]

2. For an open topological groupoid \( \mathcal{G} \), an element \( x \in G_0 \) and a subset \( H_0 \subseteq G_0 \), say that \( x \) (geometrically) dominates \( H_0 \), written \( x \gg_{GE} H_0 \), if for all objects \( (\mathcal{G}, U, N) \in \text{Sh}_{G_1}(G_0) \)—i.e. all open subsets \( U \subseteq G_0 \) and \( N \subseteq G_1 \) satisfying conditions (a)–(d) of 2.2.1—and all open subsets \( V, W \subseteq U \) that are closed under \( N \) we have
   \[
   c^{-1}(H_0) \cap d^{-1}(V) \subseteq c^{-1}(H_0) \cap d^{-1}(W) \Rightarrow c^{-1}(x) \cap d^{-1}(V) \subseteq c^{-1}(x) \cap d^{-1}(W)
   \]

**Proposition 4.2.9** Let \( \mathcal{G} \) be an open, saturated topological groupoid, and \( \mathcal{H} \) a strictly full subgroupoid. Then \( \mathcal{H} \) is definable iff \( H_0 \) is closed under domination, in the sense that for any \( x \in G_0 \) if \( x \gg_{GD} H_0 \) then \( x \in H_0 \).

**Proof** As with geometric equivalence, it is clear that geometric domination of geometric morphisms can be checked on generating full subcategories closed under subobjects. As in Lemma 4.2.7, given \( x \in G_0 \) and \( H_0 \subseteq G_0 \), spelling out the condition 4.2.8.1 of the induced points \( g_x : \text{Sets} \rightarrow \text{Sh}_{G_1}(G_0) \) and \( X = \{ f_y : \text{Sets} \rightarrow \text{Sh}_{G_1}(G_0) \mid y \in H_0 \} \) in terms of subobjects of the Moerdijk-site yields 4.2.8.2. Thus, thinking of elements of \( G_0 \) as \( T \)-models for a \( T \) such that \( \text{Set}[T] \simeq \text{Sh}_{G_1}(G_0) \), saying that \( H_0 \) is closed under domination is saying that if a model \( M \) (in \( G_0 \)) satisfies all sequents true in all models of \( H_0 \) then \( M \in H_0 \). And \( H_0 \) is definable if and only if

\[
H_0 = \left\{ N \in G_0 \mid N \models \bigcap_{M \in H_0} \text{Th}(M) \right\}
\]
since \( \mathcal{G} \) is saturated.

-
Given an open, saturated topological groupoid and a geometric theory $T$ classified by $\text{Sh}_{G_1}(G_0)$, the familiar Galois connection between subsets of $G_0$ (considered as the class of $T$-models) and quotients of $T$ therefore restricts to a contravariant equivalence between quotients of $T$ with enough models and definable subsets of $G_0$, where the latter can be characterized topologically in terms of $G$ as those subsets which are closed under domination. Moreover, such subsets are in 1-1 correspondence with subtoposes of $\text{Sh}_{G_1}(G_0)$ that have enough points. This opens up the possibility of extending the analysis of [4], concerning the correspondence between quotient theories and subtoposes, to include (definable) subgroupoids. We give a couple of (low-hanging) examples.

**Proposition 4.2.10** Let $\mathcal{H}$ be a definable subgroupoid of an open, saturated topological groupoid $\mathcal{G}$. Then the following are equivalent:

1. The induced geometric inclusion $\text{Sh}_{H_1}(H_0) \hookrightarrow \text{Sh}_{G_1}(G_0)$ is open.

2. $\text{Sh}_{G_1}(G_0)$ classifies a theory $T$ and $\text{Sh}_{H_1}(H_0)$ a quotient $T'$ such that $T'$ can be obtained from $T$ by adding a single geometric sentence as an axiom.

3. $H_0 \subseteq G_0$ is an open subset.

**Proof** That the quotient theories of $T$ that induce open inclusions into $\text{Set}[T]$ are exactly those that can be axiomatized by adding a single geometric sentence to $T$ follows from the fact that the inclusions are the geometric morphisms induced by slicing over a subterminal object (a full proof can be found in is shown in Section 7.1 of [4]). (3)$\Rightarrow$(1): Since $H_0 \subseteq G_0$ is open and closed under $G_1$, we can consider $H_0$ as a subterminal object, slicing over which produces the (inverse image part of) the induced geometric inclusion, which is thereby open. (1&2)$\Rightarrow$(3): The (inverse image part of) the induced geometric inclusion is up to equivalence, obtained by slicing over a subterminal object, and a subterminal object can be considered as an open subset $U \subseteq G_0$ closed under $G_1$. Now, $U$ must be definable—i.e. closed under domination—for if $x \gg_{\text{GE}} U$ then

$$c^{-1}(U) \cap d^{-1}(G_0) \subseteq c^{-1}(U) \cap d^{-1}(U)$$

implies that $x \in U$. But then $U = H_0$ since both are definable and they classify the same theory. \[\]
Proposition 4.2.11 Let \( \mathcal{H} \) be a definable subgroupoid of an open, saturated topological groupoid \( \mathcal{G} \). Then the following are equivalent:

1. \( \text{Sh}_{\mathcal{H}}(H_0) \hookrightarrow \text{Sh}_{\mathcal{G}}(G_0) \) is a closed subtopos.

2. \( \text{Sh}_{\mathcal{G}}(G_0) \) classifies a theory \( \mathcal{T} \) and \( \text{Sh}_{\mathcal{H}}(H_0) \) a quotient \( \mathcal{T}' \) such that \( \mathcal{T}' \) can be obtained from \( \mathcal{T} \) by adding a single geometric sequent \( \phi \vdash \bot \) where \( \phi \) is a geometric sentence.

3. \( H_0 \subseteq G_0 \) is a closed subset.

Proof A detailed proof that the quotient theories of \( \mathcal{T} \) that induce closed subtoposes of \( \text{Set}[\mathcal{T}] \) are exactly those that can be axiomatized by adding a single geometric sequent of the form \( \phi \vdash \bot \) for a geometric sentence \( \phi \) can be found in Section 7.2 of [4]. Now, by Proposition 4.2.10, \( H_0 \subseteq G_0 \) is closed if and only if there exists a single geometric sentence \( \phi \) such that \( H_0 \) is the set of \( \mathcal{T} \)-models (in \( G_0 \)) where \( \phi \) is false if and only if \( H_0 \) is defined by the theory (generated by) \( \mathcal{T} \cup \{ \phi \vdash \bot \} \) for a geometric sentence \( \phi \) (note for the last “if and only if” that if a theory has enough models, then so does any quotient obtained by adding a single sequence of the form \( \phi \vdash \bot \) for a sentence \( \phi \)).

5 Compactness, stability, and coherence

A theory \( \mathcal{T} \) is coherent if it can be axiomatized using only (sequents involving) finitary formulas, that is, formulas that do not contain infinite disjunctions. A topos is coherent if it classifies a coherent theory. We recall from [3, D3.3] that this is equivalent to the following condition, which we state as the definition.

Definition 5.0.12 A topos \( \mathcal{E} \) is coherent if there exists a site of definition \((\mathcal{C}, T)\) such that \( \mathcal{C} \) is small and Cartesian and \( T \) is generated by finite covering families.

Alternatively, \( \mathcal{E} \) is coherent if it has a generating set of compact objects which is closed under finite limits. Recall from loc. cit. that an object \( A \) in a topos \( \mathcal{E} \) is compact if the top element in the lattice \( \text{Sub}_\mathcal{E}(A) \) is compact; that an image of a compact object is compact; and that a coproduct \( A + B \) is compact if and only if \( A \) and \( B \) are. Products of compact objects are not necessarily compact. Recall that an object \( A \) is stable if all pullbacks of
compact objects \( B, C \)

\[
\begin{array}{ccc}
C \times_A B & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & A
\end{array}
\]

are compact, and that stable objects are closed under finite coproducts and subobjects. An object is \textit{coherent} if it is compact and stable.

It is clearly of interest to have an intrinsic characterization of those open topological groupoids \( \mathcal{G} \) such that \( \text{Sh}_{G_1}(G_0) \) is coherent. Although rather involved, it is possible to spell such a characterization out using Moerdijk-sites. An intrinsic characterization of those topological \textit{groups} that induce coherent toposes can be found in [3, D3.4]. Generalizing the relevant property of having finite bi-index to open topological groupoids, we arrive in Section [5.1] at a (not so involved) characterization of those open topological groupoids that induce toposes in which compact objects are closed under finite products and there is a generating set of compact objects. This is sufficient to have the characterization of coherent groups as a special case. The following sections [5.2] and [5.3] develop this characterization further to those groupoids that induce coherent decidable toposes and coherent toposes, respectively.

### 5.1 Compact groupoids

In addition to the notion of a compact object in a topos and the usual notion of a compact space, we introduce the notion of a compact open subgroupoid. Say that an open subgroupoid \( N \subseteq G_1 \) of an open topological groupoid is \textit{compact} if \( U = d(N) \) is compact in the lattice of open subsets of \( U \) that are closed under \( N \). Thus, by Lemma 2.2.4, an open subgroupoid \( N \) of \( \mathcal{G} \) is compact precisely when the induced object \( \langle \mathcal{G}, U, N \rangle \) in \( \text{MS}(G) \) is compact.

Accordingly, we make the following definition:

**Definition 5.1.1** Say that an open topological groupoid \( \mathcal{G} \) is \textit{compact} if \( G_0 \) is compact with respect to open subsets that are closed under \( G_1 \). Say that \( \mathcal{G} \) is \textit{locally compact} if for every open subgroupoid \( N \) and every \( x \in U = d(N) \) there is an open neighborhood \( x \in V \subseteq U \) such that \( V \) is closed under \( N \) and \( V \) is compact with respect to open subsets closed under \( N \).

Note that \( \langle \mathcal{G}, G_0, G_1 \rangle \) is the terminal object in \( \text{Sh}_{G_1}(G_0) \). Therefore, Lemma 2.2.4 immediately gives us the following.

**Lemma 5.1.2** \( \mathcal{G} \) is compact if and only if the terminal object in \( \text{Sh}_{G_1}(G_0) \) is compact. Furthermore, the following are equivalent:
- $\mathcal{G}$ is locally compact;
- every object $\langle \mathcal{G}, U, N \rangle$ in $\text{MS}(\mathcal{G}) \to \text{Sh}_{G_1}(G_0)$ is a join of compact subobjects;
- the compact objects in $\text{MS}(\mathcal{G})$ form a generating set for $\text{Sh}_{G_1}(G_0)$;
- $\text{Sh}_{G_1}(G_0)$ has a generating set of compact objects.

Say that a topos $\mathcal{E}$ is PCC if finite products of compact objects are compact and there exists a generating set of compact objects. For $\mathcal{E} \simeq \text{Sh}_{G_1}(G_0)$ this translates into the following property of of open subgroupoids of $G$.

Consider a pair of objects $\langle \mathcal{G}, U, N \rangle$ and $\langle \mathcal{G}, V, M \rangle$ in $\text{Sh}_{G_1}(G_0)$. Starting out with the (sub)space $c^{-1}(U) \cap d^{-1}(V)$, form the quotient space

$$\text{DC}(M, N) = c^{-1}(U) \cap d^{-1}(V)/_{N \sim M} \quad (5)$$

by the equivalence relation $N \sim_M$ defined by $(f : v_1 \to u_1) N \sim_M (g : v_2 \to u_2)$ if there exists arrows $n \in N$, $m \in M$ forming a commutative square:

$$
\begin{array}{ccc}
  u_1 & \xrightarrow{f} & v_1 \\
  \downarrow{n} & & \downarrow{m} \\
  u_2 & \xrightarrow{g} & v_2
\end{array}
$$

Call $\text{DC}(M, N)$ the double-coset space of the open subgroupoids $M$ and $N$.

**Lemma 5.1.3** Let $\langle \mathcal{G}, U, N \rangle$ and $\langle \mathcal{G}, V, M \rangle$ be two objects in $\text{Sh}_{G_1}(G_0)$. Then the product $\langle \mathcal{G}, U, N \rangle \times \langle \mathcal{G}, V, M \rangle$ is compact (in $\text{Sh}_{G_1}(G_0)$) if and only if the double-coset space $\text{DC}(M, N)$ is compact (as a topological space).

**Proof** Consider the square

$$
\begin{array}{ccc}
  d^{-1}(U) \times_{G_0} d^{-1}(V) & \xrightarrow{m \circ (i, 1)} & c^{-1}(U) \cap d^{-1}(V) \\
  \downarrow{q \times q} & & \downarrow{k} \\
  d^{-1}(U)/_{N \sim M} \times_{G_0} d^{-1}(V)/_{M \sim N} & \xrightarrow{p} & c^{-1}(U) \cap d^{-1}(V)/_{N \sim M}
\end{array}
$$

(6)

where $k$ is the quotient map, and $p$, as the top horizontal map, inverts the left arrow and composes:

$$p([f]_{N \sim M}, [g]_{M \sim N}) = [f^{-1} \circ g]_{N \sim M}$$

31
Then one easily sees that: i) $p$ is well-defined; ii) the square commutes; iii) all maps of the diagram (6) are open maps (so, in particular, $p$ is continuous); moreover, v) for all (open) sets $W \subseteq d^{-1}(U)/\sim_N \times_{G_0} d^{-1}(V)/\sim_M$ we have $(m \circ \langle i, 1 \rangle)((q \times q)^{-1}(W)) = k^{-1}(p(W))$; therefore, vi) the bottom horizontal map $p$ is also an open surjection; and, finally, vii) for a pair of arrows $f : u \to x \leftarrow v : g$ with $u \in U$, $v \in V$ and an arrow $h : x \to y$, we have $p([h \circ f]_{\sim_N}, [h \circ g]_{\sim_M}) = p([f]_{\sim_N}, [g]_{\sim_M})$.

From this, it is readily verified that $p^{-1}$ is a frame isomorphism between open subsets of $\text{DC}(M, N)$ and open sets of $d^{-1}(U)/\sim_N \times_{G_0} d^{-1}(V)/\sim_M$ which are closed under composing with arrows from $G_1$, with image along $p$ being the inverse. As such, it yields an isomorphism between open subsets of $\text{DC}(M, N)$ and subobjects of $\langle \mathcal{G}, U, N \rangle \times \langle \mathcal{G}, V, M \rangle$, and so the latter is compact if and only if the space $\text{DC}(M, N)$ is.

Putting this together with Lemma 5.1.2, we have:

**Proposition 5.1.4** $\text{Sh}_{G_1}(G_0)$ is PCC if and only if $\mathcal{G}$ is compact and locally compact and for any compact open subgroupoids $M, N \subseteq G_1$ the double-coset space $\text{DC}(M, N)$ is a compact space.

**Proof** By Lemma 5.1.2, Lemma 5.1.3 and the fact that the existence in a topos $\mathcal{E}$ of a generating set $S$ of compact objects such that $A \times B$ is compact for all $A, B \in S$ implies that a finite product of compact objects in $\mathcal{E}$ is compact.

**Remark 5.1.5** As a special case, we obtain the characterization of coherent groupoids from [3, D3.4]. For a topological group $G$ and open subgroups $M, N \subseteq G$, $\text{DC}(M, N)$ is the discrete space of double cosets, $\{NgM \mid g \in G\}$. Since $G$ is automatically compact and locally compact in the sense of Definition 5.1.1, the topos of continuous $G$-sets $\text{Cont}(G) \simeq \text{Sh}_{G}(\{\star\})$ is PCC if and only if these sets are finite for all open subgroups, i.e. if $G$ has finite bi-index, in the sense of loc.cit. And since $\text{Cont}(G)$ is Boolean, it is coherent if and only if it is PCC.

### 5.2 Coherent decidable groupoids

Say that an object $A$ in a topos $\mathcal{E}$ is *decidable* if the diagonal $\Delta : A \to A \times A$ is complemented in $\text{Sub}_\mathcal{E}(A \times A)$. Since the issue arises in the preprint [3], we briefly consider which open groupoids induce toposes of the following kind:
**Definition 5.2.1** A topos $\mathcal{E}$ is **coherent decidable** if there exists a generating set of compact and decidable objects which is closed under finite limits.

Equivalently, $\mathcal{E}$ is coherent decidable if there exists a generating set of compact and decidable objects and compact objects are closed under finite products in $\mathcal{E}$. We note the following:

**Lemma 5.2.2** An object of the form $\langle \mathcal{G}, U, N \rangle$ is decidable if and only if $N \subseteq d^{-1}(U) \cap c^{-1}(U)$ is clopen (that is, if $N$ is a closed subset of $d^{-1}(U) \cap c^{-1}(U)$).

**Proof** The bottom horizontal maps in the following diagram

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\subseteq} & \sim N \\
\downarrow & & \downarrow \\
q \times q & \xrightarrow{\subseteq} & N \\
\end{array}
\]

are both open surjections. ⊣

Say, accordingly, that an open subgroupoid $N$ is **decidable** if $N$ is a closed subset of $d^{-1}(U) \cap c^{-1}(U)$, where $U = d(N) = c(N)$ as usual. The notion of a generating set of objects of the form $\langle \mathcal{G}, U, N \rangle$ translates into the following:

**Lemma 5.2.3** Let $S = \{ N_i \subseteq G_1 \mid i \in I \}$ be a set of open subgroupoids such that for all open subgroupoids $M \subseteq G_1$ and all $v \in V = d(M)$ there exists $N_i \in S$ and open subset $T \subseteq d^{-1}(V) \cap c^{-1}(U_i)$ (where $U_i = d(N_i)$) satisfying the conditions of Lemma 2.2.7 such that $v \in d(T)$. Then the induced objects $\langle \mathcal{G}, U_i, N_i \rangle$ form a generating set for $\text{Sh}_{G_1}(G_0)$.

**Proof** Straightforward by Lemma 2.2.7 ⊣

Say, accordingly, that a set of open subgroupoids satisfying the conditions of Lemma 5.2.3 is **generating**.

**Proposition 5.2.4** $\text{Sh}_{G_1}(G_0)$ is coherent decidable if and only if $\mathcal{G}$ is compact and there exist a generating set $\{ N_i \subseteq G_1 \mid i \in I \}$ of compact decidable subgroupoids such that $\text{DC}(N_i, N_j)$ is a compact space for all $i, j \in I$.

**Proof** The only if direction follows by Proposition 2.2.3 the fact that a subobject of a decidable object is decidable, and that compact objects are closed under finite products in a coherent topos. The if direction follows
since the generating set of compact open subgroupoids induces a generating set $S$ of compact decidable objects in $\text{Sh}_{G_1}(G_0)$ such that $A \times B$ is compact for $A, B \in S$. In particular, therefore, $\text{Sh}_{G_1}(G_0)$ is PCC. Since a finite product of decidable objects is decidable and a complemented subobject of a compact object is compact, any finite limit of objects from $S$ is again compact and decidable.

5.3 Coherent groupoids

Consider a pullback

$$
\begin{array}{ccc}
P & \xrightarrow{\JOIN} & \langle \mathcal{G}, V, M \rangle \\
\downarrow s & & \downarrow s \\
\langle \mathcal{G}, U, N \rangle & \xrightarrow{t} & \langle \mathcal{G}, W, L \rangle
\end{array}
$$

in $\text{Sh}_{G_1}(G_0)$, with $S \subseteq d^{-1}(W) \cap c^{-1}(V)$ and $T \subseteq d^{-1}(W) \cap c^{-1}(U)$ open sets of arrows corresponding to the maps $s$ and $t$, as per Lemma 2.2.7. Consider the subspace

$$P = \{[f : v \to u]_{N \sim M} \mid \exists (h_t : w \to u) \in T, (h_s : w \to v) \in S. f \circ h_s = h_t\} \subseteq \text{DC}(M, N)$$

which is well-defined by Lemma 2.2.7.

**Lemma 5.3.1** The pullback $P$ is compact (as an object in $\text{Sh}_{G_1}(G_0)$) if and only if the space $P$ is compact (as a topological space).

**Proof** It is straightforward to see that the image of the underlying space $|P|$ of the pullback $P$ under the map $p$ from the proof of Lemma 5.1.3

$$|P| = \{(f)_{N \sim M}, (g)_{N \sim M} \mid t([f]) = s([g])\} \xrightarrow{\subseteq} P \xrightarrow{\subseteq} \text{DC}(M, N)$$

in the space $P$. It follows by the proof of Lemma 5.1.3 that $P$ is an open subset and that $p^{-1}$ induces an isomorphism between open subsets of $P$ and subobjects of $P$. 

34
Corollary 5.3.2 Let $\mathcal{G}$ be a locally compact open groupoid. An open subgroupoid $N \subseteq G_1$ induces a stable object $⟨\mathcal{G}, U, N⟩$ if (and only if) for all compact open subgroupoids $L, M \subseteq G_1$ and open sets $S \subseteq d^{-1}(U) \cap c^{-1}(d(M))$ and $T \subseteq d^{-1}(U) \cap c^{-1}(d(L))$ satisfying the conditions of Lemma 2.2.7, the quotient space
$$\{[f : v \to w]_{L \sim M} \mid \exists(h_t : u \to w) \in T, ((h_s : u \to v) \in S, f \circ h_s = h_t \}$$
$$\subseteq DC(M, L)$$
is compact.

Say, accordingly, that an open subgroupoid $N \subseteq G_1$ of a locally compact groupoid $\mathcal{G}$ is stable if the condition of Corollary 5.3.2 is satisfied.

Proposition 5.3.3 An open topological groupoid $\mathcal{G}$ induces a coherent topos $Sh_{G_1}(G_0)$ if and only if $\mathcal{G}$ is compact and there exists a generating set $S = \{N_i \subseteq G_1 \mid i \in I\}$ of open subgroupoids each of which is compact and stable and such that each double-coset space $DC(N_i, N_j)$ is compact and has the property that any finite intersection of compact open subsets is a compact subset.

Proof If $Sh_{G_1}(G_0)$ is coherent then, since a subobject of a stable object is stable, Proposition 2.2.3 guarantees a generating set of coherent objects of the form $⟨\mathcal{G}, U, N⟩$. And coherent objects are closed under finite limits in a coherent topos. For the only if direction: By Proposition 5.1.4 $Sh_{G_1}(G_0)$ is PCC. Moreover, the existence of a generating set $S$ of compact objects such that for all $A, B \in S$ the product $A \times B$ has the property that compact subobjects are closed under binary meets entails that a finite product of stable objects is again stable. A generating set of coherent objects can therefore be closed under finite limits to obtain a generating set of compact objects (which is closed under finite limits).

References

[1] C. Butz and I. Moerdijk, “Representing topoi by topological groupoids,” *Journal of Pure and Applied Algebra*, vol. 130, pp. 223–235, 1998.

[2] I. Moerdijk, “The classifying topos of a continuous groupoid. I,” *Transactions of the American Mathematical Society*, vol. 310, no. 2, pp. 629–668, 1988.
[3] P. T. Johnstone, *Sketches of an Elephant*, vol. 43 and 44 of *Oxford Logic Guides*. Oxford: Clarendon Press, 2002.

[4] O. Caramello, “Lattices of theories,” 2009. http://arxiv.org/PS_cache/arxiv/pdf/0905/0905.0299v1.pdf.

[5] H. Forssell, “Topological representation of geometric theories.” http://arxiv.org/abs/1109.0699.

[6] S. Awodey and H. Forssell, “First-order logical duality.” http://arxiv.org/PS_cache/arxiv/pdf/0906/0906.3061v1.pdf (submitted).