On the divisibility of class numbers of cubic number fields with discriminants in a prescribed rational quadratic class

Ivan Chipchakov and Kalin Kostadinov

Abstract

Let $n$ be an odd number and $F$ an imaginary quadratic field with odd discriminant. We show that there exists infinitely many cubic fields $K$ such that the class number of $K$ is divisible by $n$ and the Galois closure of $K$ contains $F$.

§1. Introduction

This paper is devoted to the study of the problem of whether there exists a cubic extension $K_{n,d}$ of the field $\mathbb{Q}$ of rational numbers such that the class number $h(K_{n,d})$ is divisible by an arbitrary fixed natural number $n$ and the discriminant $\text{disc}(K_{n,d})$ is lying in the set $\{ds^2 : s \in \mathbb{N}\}$, for a given square-free integer $d$. It is known that this problem has a positive solution for any $n$ in case $d = 1$ [6] and in case $d = -3$ [4]. Our main result shows that the answer to the considered question is affirmative whenever $n$ is odd and $d$ is negative and congruent to 1 modulo 4. The field $K_{n,d}$ is associated with Uchida’s solution $K_{n,1}$ by a congruence type relation modulo a suitably chosen set of prime numbers splitting in $K_{n,1}$ (see sections 4 and 5).

§2. Preliminaries on units and ramification in a cubic field generated by a root of Uchida’s polynomial

Let $n, d$ and $a$ be integers such that $n$ is odd and positive, $d$ is negative, square-free and $d \equiv 1 \pmod{4}$, and $a$ is odd. With these notations, the

*The research of I. Chipchakov was supported in part by Grant MM1106/2001 from the Bulgarian Foundation for Scientific Research.
polynomial

\[ u(x) = x^3 + m(x + 1)^2, \quad \text{where } m = \frac{(3^6d^n a^{2n} + 2\mathcal{T})}{4}, \]

is irreducible over the field of rational numbers (cf. [6, Lemma 1]). Throughout this paper we denote by \( K \) an extension of \( \mathbb{Q} \), obtained by adjoining a root \( \pi \) of the polynomial \( u(x) \) and by \( F \) the quadratic field \( \mathbb{Q}(\sqrt{d}) \) \((\pi, \sqrt{d} \in \mathbb{Q})\). It is easily verified that the discriminant \( \text{disc}(u(x)) \) of \( u(x) \) is equal to \( m^2(4m - 27) \), i.e. the compositum \( KF = K(\sqrt{d}) \) is a root field of \( u(x) \). As \( \text{disc}(u(x)) < 0 \), \( K \) has only one embedding in the field of real numbers. Hence \( KF/\mathbb{Q} \) is an imaginary Galois extension of degree 6, and by Dirichlet’s unit theorem, the multiplicative group \( \mathfrak{O}_{KF}^* \) is of rank 2, \( \mathfrak{O}_{KF} \) being as usual the ring of algebraic integers in \( KF \).

The following proposition gives information about the discriminant \( \text{disc}(K) \) and the ramification of primes not equal to 3 in the ring \( \mathfrak{O}_{KF} \); the excluded case of \( p = 3 \) is considered in the next section.

**Proposition 2.2.** For a prime number \( p \) not equal to 3, the following is true:

(i) \( p \) is totally ramified in \( K \) if and only if it divides \( m \) and \( \text{disc}(K) \); this occurs if and only if the power of \( p \) in the canonical primary decomposition of \( m \) is not divisible by 3;

(ii) \( p.\mathfrak{O}_K = p_1^2 p_2 \), where \( p_1 \) and \( p_2 \) are different prime ideals of \( \mathfrak{O}_K \), if and only if the power of \( p \) in the canonical decomposition of the number \( 4m - 27 \) is odd.

**Proof.** (i) Consider the presentation \( m = b c^3 \) with \( b, c \) integers and \( b \) cube-free. The element \( \rho = \pi/c \) of \( K \) has a minimal polynomial

\[ g(x) = c^{-3}.u(cx) = x^3 + b(cx + 1)^2, \]
with discriminant $b^2(4m - 27)$. Note that it is enough to prove the following three implications:
1. If $p$ divides both $m$ and $\text{disc}(K)$, then $p$ divides $b$;
2. If $p$ divides $b$, then $p$ is totally ramified in $K$;
3. If $p$ is totally ramified in $K$, then $p$ divides both $m$ and $\text{disc}(K)$.

For the first, note that $\gcd(\text{disc}(K), m)$ is a divisor of $\gcd(b^2(4m - 27), m)$, and the last is a divisor of $27b^2$. From $p \neq 3$ it follows, that $p$ divides $b$.

For the second, let $p$ be a prime divisor of $b$ not equal to 3. If $p^2 \nmid b$, then $\rho$ is a root of an Eisensteinian polynomial relative to $p$ and in case $p^2 \mid b$ this is true for the minimal polynomial of $\rho^2/p$. Hence in both cases $p$ is totally ramified in $K$ (cf. [1, Ch. I, Theorem 6.1]).

The proof of implication 3 relies on the fact that the assumption on $p$ ensures that it divides $\text{disc}(K)$, which in turn divides the product $m^2(4m - 27)$. Note also that the norms $N^K_\mathbb{Q}(\pi + 3)$ and $N^K_\mathbb{Q}(\pi + m - 6)$ are equal to $27 - 4m$ and $(m - 8)(27 - 4m)$, respectively. It is therefore clear that if $p$ divides $4m - 27$, then the prime ideal $p$ of $\mathcal{O}_{KF}$ lying above $p$ must contain the elements $\pi + 3$, $\pi + m - 6$ and $4m - 27$. Since $4(\pi + 3 - (\pi + m - 6)) + (4m - 27) = 9$, this leads to the conclusion that $9 \in p$ in contradiction with the inequality $p \neq 3$. These observations prove that $p$ is a divisor of $m$.

(ii) It is clear from Proposition 2.2(i) that $p\mathcal{O}_K = p_1^2p_2$ ($p_1 \neq p_2$) if and only if $p$ divides $\text{disc}(K)$ and $4m - 27$. In view of Dedekind’s discriminant theorem [3, Ch. III] and the inequality $p > 3$ this occurs if and only if $p\text{disc}(K)$ and $p^2 \nmid \text{disc}(K)$. This combined with the fact that $\text{disc}(u(x))/\text{disc}(K)$ equals the square of the index $[\mathcal{O}_K : \mathbb{Z}[\pi]]$ (as additive groups), proves Proposition 2.2(ii).

§3. The canonical decomposition of the principal ideal $\frac{\pi - \pi^\sigma}{3\pi} \mathcal{O}_{KF}$

The purpose of this section is to show that the element $\alpha = \frac{\pi - \pi^\sigma}{3\pi}$, where $\sigma$ is an $F$-automorphism of the field $KF$ of order 3, is an algebraic integer and the ideal $\alpha\mathcal{O}_{KF}$ is an $n$-th power of an ideal in $\mathcal{O}_{KF}$.

Lemma 3.1. The minimal polynomial of the element $\alpha = \frac{\pi - \pi^\sigma}{3\pi}$ over the field $F$ is

$$h(x) = x^3 + \frac{2m - 9 + \sqrt{4m - 27}}{6} x^2 - \frac{2m - 9 + 3\sqrt{4m - 27}}{18} x + \frac{\sqrt{4m - 27}}{27}.$$  

The coefficients of $h(x)$ are integers in $F$ and therefore $\alpha$ is in $\mathcal{O}_{KF}$.
Proof. It is convenient first to compute the minimal polynomial $h_1(x)$ over $F$ of the element $\pi^\sigma/\pi$. The following equalities are true

\[
\begin{align*}
Tr_{F}^{KF}(\pi^\sigma/\pi) + Tr_{F}^{KF}(\pi/\pi^\sigma) &= \frac{\pi^\sigma}{\pi} + \frac{\pi^\sigma^2}{\pi^\sigma} + \frac{\pi}{\pi^\sigma} + \frac{\pi^\sigma}{\pi^\sigma^2} + \frac{\pi}{\pi} = \\
&= (\pi + \pi^\sigma + \pi^\sigma^2)(\frac{1}{\pi} + \frac{1}{\pi^\sigma} + \frac{1}{\pi^\sigma^2}) - 3 = -m.(-\frac{u'(0)}{u(0)}) - 3 = 2m - 3,
\end{align*}
\]

\[
Tr_{F}^{KF}(\pi^\sigma/\pi) - Tr_{F}^{KF}(\pi/\pi^\sigma) = \frac{\pi^\sigma - \pi^\sigma^2}{\pi} + \frac{\pi^\sigma^2 - \pi}{\pi^\sigma} + \frac{\pi - \pi^\sigma}{\pi^\sigma^2} = \\
= \frac{(\pi - \pi^\sigma)(\pi - \pi^\sigma^2)(\pi^\sigma - \pi^\sigma^2)}{\pi^\sigma^2} = \sqrt{m^2(4m - 27)}/(-m) = \sqrt{4m - 27}.
\]

From the above it follows that

\[
\begin{align*}
Tr_{F}^{KF}(\pi^\sigma/\pi) &= (2m - 3 + \sqrt{4m - 27})/2, \\
Tr_{F}^{KF}(\pi/\pi^\sigma) &= (2m - 3 - \sqrt{4m - 27})/2.
\end{align*}
\]

The second elementary symmetric polynomial in the three variables $\pi^\sigma, \pi^\sigma^2, \pi^\sigma^3$ is equal to $Tr_{F}^{KF}(\pi/\pi^\sigma)$, and the norm of $\pi^\sigma/\pi$ is equal to 1. Therefore

\[
h_1(x) = x^3 - \frac{2m - 3 + \sqrt{4m - 27}}{2}x^2 + \frac{2m - 3 - \sqrt{4m - 27}}{2}x - 1.
\]

The minimal polynomial of $\alpha$ is computed from $h(x) = \frac{1}{27}h_1(1 - 3x)$. So at the end one gets

\[
h(x) = x^3 + \frac{(2m - 9 + \sqrt{4m - 27})}{6}x^2 - \frac{2m - 9 + 3\sqrt{4m - 27}}{18}x + \frac{\sqrt{4m - 27}}{27}.
\]

The free term of $h(x)$ is in $\mathcal{O}_F$, since $3^6$ divides $4m - 27$. Since $27$ exactly divides $m$, the numerators of the coefficients of the first and second degree terms of $h(x)$ are exactly divisible by $9$ and simple parity checks proves that these coefficients are also in $\mathcal{O}_F$. So $h(x)$ is in $\mathcal{O}_F[x]$ and besides, the trace of $\alpha$ is a multiple of three, but the trace of $\alpha^\sigma$ is not.

The following two lemmas determine the decomposition in $\mathcal{O}_{KF}$ of the prime ideals of $\mathcal{O}_F$ containing the norm of $\alpha$ over $F$.

**Lemma 3.2.** (i) If $3 \nmid da$, then $\alpha$ is not contained in any ideal of $\mathcal{O}_{KF}$ with norm equal to a power of $3$.

(ii) If $3 | d$, then the following decompositions are true:

\[
3.\mathcal{O}_F = P_3^2,
\]

\[
P_3.\mathcal{O}_{KF} = \mathcal{P}_3'\mathcal{P}_3''\mathcal{P}_3''',
\]

The following two lemmas determine the decomposition in $\mathcal{O}_{KF}$ of the prime ideals of $\mathcal{O}_F$ containing the norm of $\alpha$ over $F$.

**Lemma 3.2.** (i) If $3 \nmid da$, then $\alpha$ is not contained in any ideal of $\mathcal{O}_{KF}$ with norm equal to a power of $3$.

(ii) If $3 | d$, then the following decompositions are true:

\[
3.\mathcal{O}_F = P_3^2,
\]

\[
P_3.\mathcal{O}_{KF} = \mathcal{P}_3'\mathcal{P}_3''\mathcal{P}_3''',
\]

The following two lemmas determine the decomposition in $\mathcal{O}_{KF}$ of the prime ideals of $\mathcal{O}_F$ containing the norm of $\alpha$ over $F$.
where \( P_3 \) and \( \mathfrak{P}_3, \mathfrak{P}_3', \mathfrak{P}_3'' \) are prime ideals respectively in \( \mathcal{O}_F \) and \( \mathcal{O}_{KF} \). The element \( \alpha \) belongs exactly to one of the ideals \( \mathfrak{P}_3, \mathfrak{P}_3', \mathfrak{P}_3'' \).

**Proof.** (i) This follows immediately from the fact that in that case the norm of \( \alpha, N_{KF}^Q(\alpha) = d^n a^{2n} \) is relatively prime to 3;
(ii) In this case the ideal 3\( \mathbb{Z} \) ramifies in \( \mathcal{O}_F \), since 3 divides \( \text{disc}(F) \). Since 3 does not divide the trace of \( \alpha \cdot \alpha^\sigma \), there are at least two different ideals of \( \mathcal{O}_{KF} \) with norm a power of 3 and at least one of them does not contain \( \alpha \). This, combined with the fact that \( KF/F \) is a cyclic cubic extension, implies that \( P_3 \) splits completely in \( \mathcal{O}_{KF} \). Note finally that if \( \alpha \) belongs to two of the ideals of \( \mathcal{O}_{KF} \) lying over \( P_3 \), then \( \alpha \cdot \alpha^\sigma \in P_3 \mathcal{O}_{KF} \), which implies that the trace of \( \alpha \cdot \alpha^\sigma \) is a multiple of 3. The obtained contradiction proves the concluding assertion of Lemma 3.2.

**Lemma 3.3.** Let \( P \) be a prime ideal of \( \mathcal{O}_F \), which contains \( 4m - 27 \) and does not contain 3. Then:
(i) The ideal \( P \) splits completely in \( \mathcal{O}_{KF} \);
(ii) The element \( \pi - \pi^\sigma \) belongs exactly to one of the ideals of \( \mathcal{O}_{KF} \) which lie above \( P \);

**Proof.** (i) Denote by \( p \) the prime number satisfying the equality \( p\mathbb{Z} := P \cap \mathbb{Q} \).
It is clear from the choice of \( P \) that \( p \mid 4m - 27 \) and \( p \neq 3 \). Therefore, \( p \) does not divide \( m \), and by Proposition 2.2, \( p \) is not totally ramified in \( K \). As \([KF : K] = 2\), this means that 3 does not divide the ramification index of \( p \) in \( \mathcal{O}_{KF} \), which amounts to saying that \( P \) does not ramify in \( \mathcal{O}_{KF} \). Note also that \( P \) is not inert in \( \mathcal{O}_{KF} \). Assuming the opposite, one obtains from the multiplicative property of inertia degrees in towers of finite extensions that then \( p \) must be inertial in \( \mathcal{O}_K \). This, however, contradicts the existence of at least two different prime ideals of \( \mathcal{O}_K \) containing \( p \) (established in the proof of Proposition 2.2), and so proves our assertion. As \( KF/F \) is a cubic cyclic extension, these observations indicate that \( P \) splits in \( \mathcal{O}_{KF} \), as claimed.
(ii) It is easily verified that \( u(x) \) has the following reduction modulo \( 4m - 27 \):

\[
u(x) \equiv (x + 3)^2(x + m - 6) \pmod{4m - 27} \tag{3.1}
\]
Let \( P \mathcal{O}_{KF} = \mathfrak{P} \mathfrak{P}^\sigma \mathfrak{P}^\sigma^2 \). It is clear from 3.1 that \( (\pi + 3)^2(\pi + m - 6) \) belongs to \( \mathfrak{P} \mathfrak{P}^\sigma \) and \( \mathfrak{P}^\sigma^2 \). Since these ideals are prime, each contains at least one multiple of the above product. Suppose for a moment that \( (\pi + 3) \) is contained in \( \mathfrak{P}, \mathfrak{P}^\sigma \) and \( \mathfrak{P}^\sigma^2 \), i.e. in their intersection \( P \mathcal{O}_{KF} \). Then the trace of \( \pi + 3 \) over \( \mathcal{O}_F \) must lie in \( P \mathcal{O}_F \). Since \( \text{Tr}_{KF}^F(\pi + 3) = -m + 9 \), this means that \( P \mathcal{O}_F \) contains the element \( 4m - 27 + 4(9 - m) = 9 \). This
contradicts the fact that the norm of \( P \mathcal{O}_F \) is relatively prime to 3 and so proves that \((\pi + 3)\) is contained in at most two of \( \mathfrak{P}, \mathfrak{P}^\sigma \) and \( \mathfrak{P}^\sigma^2 \). Suppose that \((\pi + m - 6)\) is contained in two of the ideals, for example, \((\pi + m - 6) \in \mathfrak{P} \cap \mathfrak{P}^\sigma\). Then \((\pi + m - 6)(\pi^\sigma + m - 6) \in \mathfrak{P} \mathfrak{P}^\sigma \mathfrak{P}^\sigma^2\), and therefore \( \text{Tr}_{F}^{\mathcal{O}_F}((\pi + m - 6)(\pi^\sigma + m - 6)) \in P \). On the other hand, direct calculations show that \( \text{Tr}_{F}^{\mathcal{O}_F}((\pi + m - 6)(\pi^\sigma + m - 6)) = 4^{-2}\{(4m - 61).\{(4m - 27) + 81\} \}. \) The obtained result implies that \( 81 \in P \), which again contradicts the fact that the norm of \( P \) is relatively prime to 3. Thus one concludes that \( \pi + m - 6 \) lies in exactly one of the ideals \( \mathfrak{P}, \mathfrak{P}^\sigma, \mathfrak{P}^\sigma^2 \). Assuming as we can that \( \pi + 3 \) is contained in \( \mathfrak{P} \) and \( \mathfrak{P}^\sigma \), we obtain consecutively that \( \pi + m - 6 \in \mathfrak{P}^\sigma^2, \pi - \pi^\sigma \in \mathfrak{P}^\sigma \) and \( \pi - \pi^\sigma \notin \mathfrak{P} \cup \mathfrak{P}^\sigma^2 \), which completes our proof.

The main result of this section can be stated as follows:

**Proposition 3.4.** The principal ideal \((\alpha) = \frac{\pi - \pi^\sigma}{3\pi} \mathcal{O}_{KF}\) is an \(n\)-th power in the semigroup of ideals of \( \mathcal{O}_{KF} \), i.e. there exist an ideal \( \mathfrak{A} \) such that \((\alpha) = \mathfrak{A}^n \).

**Proof.** Since the norm \( N_{KF}(\alpha) = \sqrt{4m - 27}/27 \) is an \(n\)-th power in \( F^* \), this can be deduced from Lemmas 3.2 and 3.3.

§4. Sets of prime numbers associated with Uchida’s cyclic cubic fields

In this section we present the main features of Uchida’s solution of the class number divisibility problem for cyclic cubic fields and prepare the basis for constructing similar solutions for the case pointed out in the introduction. Our main result in this direction is the following theorem:

**Theorem 4.1.** Let \( \bar{K} \) be an extension of \( \mathbb{Q} \) in \( \overline{\mathcal{O}} \) obtained by adjoining a root \( \bar{\pi} \) of the polynomial \( \bar{a}(X) = X^3 + \bar{m}X^2 + 2\bar{m}X + \bar{m} \), where \( \bar{m} = (\bar{a}^{3n} + 27)/4 \), for some positive integers \( \bar{a}, s \) and \( n \). Assume also that \( d \) is a square-free integer not equal to 1, \( \gcd(n, 6) = \gcd(\bar{a}, 2) = 1 \), \( \bar{\sigma} \) is an automorphism of \( \bar{K} \) of order 3, the ideal \( \bar{\alpha} \mathcal{O}_{\bar{K}} \), where \( \bar{\alpha} = (\bar{\pi} - \bar{\pi}^\bar{\sigma})/\bar{\pi} \), is not an \( l \)-th power of a principal ideal of \( \mathcal{O}_{\bar{K}} \), for any \( l \) in the set \( \mathcal{P}_{2n} \) of prime divisors of \( 2n \), and the subgroup of \( \mathcal{O}_{\bar{K}}^* \) generated by the elements \( \bar{e} = \bar{\pi} + 1 \) and \( \bar{e}^{\bar{\sigma}} = \bar{\pi}^\bar{\sigma} + 1 \) is of index relatively prime to \( 2n \).

Then there exist infinitely many pairs \((q_1(l; i, j), q_2(l; i, j))\) of prime numbers satisfying the following conditions, for each \( l \in \mathcal{P}_{2n} \) and each pair \((i, j)\) of nonnegative integers less than \( l \):

(i) \( q_1 \) and \( q_2 \) split in \( \bar{K} \);
(ii) \( q_1 \neq q_2 \) and \( d \) is a \( 2^s \)-th power residue modulo \( q_1 q_2 \);
(iii) 3 is a $2^n n$-th power residue modulo $q_1 q_2$;
(iv) $\tilde{\alpha}_{ij} = \tilde{\alpha}_i \tilde{\alpha}_j^{\tilde{\epsilon}^j}$ is not an $l$-th power residue modulo $q_1$ (i.e. modulo some prime ideal of $\mathcal{O}_K$ containing $q_1$);
(v) $\tilde{\epsilon}^i \tilde{\epsilon}^j$ is not an $l$-th power residue modulo $q_2$;
(vi) $q_1$ and $q_2$ do not divide $6\text{disc} (\tilde{u}(X))$.

Let us note that the existence of a cyclic cubic field $\tilde{K}$ satisfying the conditions of Theorem 4.1 for any pair of integers $(n, s)$, has been established by Uchida [6]. As a matter of fact, he has shown there that the fulfillment of these conditions guarantees that the class number of $\tilde{K}$ is divisible by $2^{s-1} n$.

To prove the theorem, we need the following lemma:

**Lemma 4.2.** With assumptions and notations being as in Theorem 4.1 fix a triple $(l; i, j)$ of admissible integers, and put

$$K_1 = \tilde{K}(\delta, \lambda, \zeta), \quad K_2 = \tilde{K}(\mu, \mu_\sigma, \zeta_0), \quad K_3 = \tilde{K}(\xi, \xi_\sigma, \zeta_0)$$

where $\delta, \lambda, \zeta, \mu, \mu_\sigma, \zeta_0, \xi, \xi_\sigma$ lie in $\bar{Q}$, $\delta^{2^n} = d, \lambda^{2^n} = 3, \mu^l = \tilde{\alpha}_{ij}, \mu_\sigma^l = \tilde{\alpha}_i \tilde{\alpha}_j^{\tilde{\epsilon}}, \xi^l = \tilde{\epsilon}, \xi_\sigma^l = \tilde{\epsilon}^j, \zeta$ and $\zeta_0$ are primitive roots of unity of degrees $2^n l$ and $l$ respectively. Then $K_2$ and $K_3$ are not subfields of $K_1$.

**Proof.** It is clearly sufficient to show that $K_1$ does not include the fields $L_2 = K_2(\zeta)$ and $L_3 = K_3(\zeta)$. Note first that $K_2$ and $K_3$ are normal extension of $Q$ with non-abelian Galois groups. Indeed, the fulfillment of the conditions of Theorem 4.1 ensures the equality $[\tilde{K}(\mu) : \tilde{K}] = [\tilde{K}(\xi) : \tilde{K}] = l$, whereas the degree $[\tilde{K}(\zeta_0) : \tilde{K}]$ is even and divides $l - 1$ by the elementary properties of cyclotomic extensions. This implies that $\zeta_0 \not\in \tilde{K}(\mu)$ and $\zeta_0 \not\in \tilde{K}(\xi)$, which leads to the conclusion that $\tilde{K}(\mu)/\tilde{K}$ and $\tilde{K}(\xi)/\tilde{K}$ are not normal extensions, and thereby reduces our assertion to an immediate consequence of Galois theory. Since the field $L_1 = \tilde{K}(\zeta)$ is abelian over $Q$, the obtained result shows that $K_2$ and $K_3$ are not included in $L_1$. Applying now Kummer theory to the extensions $L_2/L_1$ and $K_1/L_1$, one concludes that if $L_2 \subseteq K_1$, then there exist integers $n_1$ and $n_2$ such that $gcd(n_1, n_2, l) = 1$, $\tilde{\alpha}_{ij}^{n_1} \tilde{\alpha}_i^{n_2} \not\in L_1^l$ and $(\tilde{\alpha}_{ij}^{n_1} \tilde{\alpha}_i^{n_2})^{n_1} (\tilde{\alpha}_i^{n_1} \tilde{\alpha}_j^{n_2})^{n_2} \not\in L_1^l$. Observing, however, that $N_{Q}^{\tilde{K}}(\tilde{\alpha}_{ij}^{n_1} \tilde{\alpha}_i^{n_2})$ lies in $Q^l$, one verifies directly that $(\tilde{\alpha}_{ij}^{n_1} \tilde{\alpha}_i^{n_2})^{3} \in L_1^l$. Since $gcd(l, 3) = 1$, this yields $\tilde{\alpha}_{ij}^{n_1} \tilde{\alpha}_i^{n_2} \in L_1^l$, a contradiction, proving that $L_2$ is not a subfield of $K_1$. Replacing $\tilde{\alpha}_{ij}$ by $\tilde{\epsilon}_{ij}$ and arguing in the same way, one concludes that $L_3$ is not a subfield of $K_1$ either. \qed

**Proof of Theorem 4.1** Suppose that the fields $K_1, K_2, K_3$ are as in Lemma 4.2 and for each $i$ denote by $\text{Spl}(K_i)$ the set of rational prime numbers splitting
completely in \( K \). It is clear from the definition of \( K \) and the inclusion \( \bar{K} \subset K \) that the elements of \( \text{Spl}(K_1) \) satisfy conditions (i),(ii) and (iii). It follows from Kummer’s theorem that the elements of \( \text{Spl}(K_1) \setminus \text{Spl}(K_2) \) satisfy condition (iv) and the elements of \( \text{Spl}(K_1) \setminus \text{Spl}(K_3) \) satisfy condition (v) of Theorem 4.1 (in either case with at most finitely many exceptions). Note finally that \( K_1, K_2, K_3 \) are normal extensions of \( \mathbb{Q} \), so Bauer’s theorem (cf. [2]) and Lemma 4.2 imply that \( \text{Spl}(K_1) \setminus \text{Spl}(K_2) \) and \( \text{Spl}(K_1) \setminus \text{Spl}(K_3) \) are infinite sets. These observations prove Theorem 4.1.

**Corollary 4.3.** With assumptions and notations being as in Theorem 4.1, let \( \Omega_\nu = \{q(l; i, j) : l \in \mathcal{P}_{2n}, i, j \in \{0, 1, \ldots, l - 1\}\}, (\nu = 1, 2) \) be sets of prime numbers with the properties required by Theorem 4.1. Then there exists an integer number \( q \) satisfying the system of congruences

\[
3^6d^n a^{2^zn} \equiv \tilde{a}^{2^zn} \pmod{q_1(l; i, j)q_2(l; i, j)} : q_\nu(l; i, j) \in \Omega_\nu.
\]

**Proof.** In view of the Chinese remainder theorem, it is sufficient to prove the solvability of the congruence \( 3^6d^n a^{2^zn} \equiv \tilde{a}^{2^zn} \pmod{q} \), for an arbitrary element \( q \in \Omega_1 \cup \Omega_2 \). The choice of \( \Omega_1 \cup \Omega_2 \) ensures the existence of integers \( z_1 \) and \( z_2 \) satisfying the congruences \( z_1^{2^zn} \equiv 3 \pmod{q} \) and \( z_2^{2^zn} \equiv d \pmod{q} \). It is therefore clear that every solution of the linear congruence \( z_1^{6}z_2x \equiv \tilde{a} \pmod{q} \) is a solution of the required type.

\[\square\]

**§5. Proof of the main result**

Let \( n \) be a positive integer and \( d < 0 \) and \( d \equiv 1 \pmod{4} \). In this section we find an infinite sequence of cubic fields with discriminants lying in \( d\mathbb{Q}^{\ast} \) and with class numbers divisible by \( n \). This is realized by a construction preserving the main features of Uchida’s cyclic cubic fields.

**Theorem 5.1.** Assume that \( \bar{K}, \pi, m, \tilde{a}, n, s \) and \( \tilde{\sigma} \) satisfy the conditions of Theorem 4.1. \( \mathcal{P}_{2n} \) is the set of prime divisors of \( 2n \), \( \Omega_1 \) and \( \Omega_2 \) are sets of prime numbers determined in accordance with Corollary 4.3. \( d \) is a square-free negative integer congruent to 1 modulo 4, and \( K \) is an extension of \( \mathbb{Q} \) in \( \bar{Q} \) obtained by adjoining a root \( \pi \) of the polynomial \( f(X) = X^3 + mX^2 + 2mX + m \), where \( m = (3^6d^n a^{2^zn} + 27)/4 \), for some integer number \( a \) satisfying the system of congruences described in Corollary 4.3. Let \( \varepsilon = \pi + 1, F = \mathbb{Q}(\sqrt{d}) \) (\( \sqrt{d} \in \bar{Q} \)), and \( \alpha = (\pi - \pi^s)/(3\pi^s) \), where \( \sigma \) is a fixed automorphism of the compositum \( KF \) of order 3. Then the following assertions are true:

(i) The subgroup of \( \mathcal{O}_KF^\ast \) generated by \( \varepsilon \) and \( \varepsilon^s \) is of (finite) index relatively prime to \( n \) and not divisible by 4.

(ii) The ideals \( \alpha \mathcal{O}_KF \) and \( \alpha^s\alpha^s \mathcal{O}_K \) are perfect \( n \)-th powers of ideals of \( \mathcal{O}_KF \).
and $\mathcal{O}_K$, respectively;

(iii) The class groups of $KF$ and of $K$ contain elements of order $n$;

(iv) The class numbers $\text{Cl}(KF)$ and $\text{Cl}(K)$ are divisible by $3^n$, in case at least $6 + t$ prime numbers are totally ramified in $K$.

Remark. It should be noted that the system of congruences described in corollary 4.3 has a set $\{a_k : k \in \mathbb{N}\}$ of integer solutions, such that each $a_k$ gives rise to a cubic field satisfying the conditions of Theorem 5.1 for which one can find an integer $\Omega_2$ from Kummer’s theorem and Theorem 4.1(i):

The class numbers $\text{Cl}(KF)$ and $\text{Cl}(K)$ contain elements of order $n$; respectively;

(iii) and of $\text{Cl}(KF)$.

Conversely, every prime ideal of $\mathcal{O}_K$ contains the elements $\tilde{\pi}$, $\tilde{\pi}^\sigma$, $\tilde{\pi}^{\sigma^2}$, for any $\pi \in \mathcal{O}_K$. It is clear from (5.2)(ii) that if $q$ is an arbitrary element of $\Omega_1 \cup \Omega_2$, then $q$ contains the elements $\pi - \sigma_1, \pi^\sigma - \sigma_2, \pi^{\sigma^2} - \sigma_3$, for some bijection $\nu$ of the set $\{1, 2, 3\}$. As $q$ is an arbitrary element of $\Omega_1 \cup \Omega_2$, this observation, combined with (5.2)(i), leads to the following conclusions, for each admissible triple $(l, i, j)$:

(5.3)(i) $\varepsilon_{ij} = \varepsilon^l \varepsilon^{\sigma^j}$ is not an $l$-th power residue modulo any $q_k(l, i, j) \in \Omega_2$;

(ii) $\alpha_{ij} = \alpha \varepsilon_{ij}$ is not an $l$-th power modulo any $q_k(l, i, j) \in \Omega_2$.

Note that Theorem 5.1(i) is equivalent to the statement that every unit $\eta \in \mathcal{O}^*_K$ of norm $1$ over $F$ can be presented in the form $\eta = \varepsilon^l \varepsilon^{\sigma^j} \eta_1^{2n}$, for some $\eta_1 \in \mathcal{O}^*_K$ and suitably chosen integers $i, j$. Therefore, its validity can be deduced from (5.3)(ii) and Dirichlet’s unit theorem. Similarly, it follows from (5.3)(ii) and Theorem 5.1(i) that $\alpha \mathcal{O}_K$ is not an $l$-th power of a principal ideal, for any $l \in \mathcal{P}_{2n}$, $l \neq 2$.

Observing that $\pi^\sigma / \pi^{\sigma^2} \in \mathcal{O}^*_{KF}$, one sees that $N_K^{KF}(\alpha^\sigma).\mathcal{O}_K = (\alpha^\sigma)^2 \mathcal{O}_K$, which reduces Theorem (ii) to a consequence of Proposition 3.4. Note also
that the obtained results prove Theorem 5.1(iii). Applying finally the main result of [5], one completes the proof of Theorem 5.1(iv).

It is very likely that the field defined in Theorem 5.1 can be chosen so that \( h(K) \) is divisible by \( 2^{s-1}n \). A sufficient condition for this is that the ideal \( \sqrt{d} \mathcal{O}_{KF} + \alpha \mathcal{O}_{KF} \) is of even order in the class group of \( KF \).

References

[1] A. Fröhlich *Local Fields*, Algebraic Number Theory (J. W. Cassels and A. Fröhlich, eds.), Proc. of Instructional. Conf. held in Brighton 1965, Academic Press, London (1968) pp. 1-41.

[2] H. Hasse, Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper, Teil I: Klassenkörpertheorie, Teil Ia: Beweise zu Teil I, Teil II: Reziprozitätsgesetz. 3.ed. - Würzburg e.a.: Physika-Verlag, (1970).

[3] S. Lang, Algebraic Numbers, Addison-Wesley Series in Mathematics. Reading, Mass., (1964).

[4] S. Nakano, *Class numbers of pure cubic fields*, Proc. Japan Acad., Ser. A 59 (1983), pp. 263-265.

[5] P. Roquette and H. Zassenhaus, *A class of rank estimate for algebraic number fields*. J. London Math. Soc. 44 (1969), No. 1, pp.31-38.

[6] K. Uchida, *Class numbers of cubic cyclic fields*, J. Math. Soc. Japan 26 (1974), No. 3, 447-453.

Ivan Chipchakov
Institute of Mathematics
Bulgarian Academy of Sciences
Acad G. Bonchev, bl 8
1113 Sofia, Bulgaria
chipchak@math.bas.bg

Kalin Kostadinov
Department of Mathematics
Boston University
111 Cummington Str.
Boston, MA02215 USA
kost@math.bu.edu