Conflict-free connection number and independence number of a graph

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Abstract An edge-colored graph $G$ is conflict-free connected if any two of its vertices are connected by a path, which contains a color used on exactly one of its edges. The conflict-free connection number of a connected graph $G$, denoted by $cfc(G)$, is defined as the minimum number of colors that are required in order to make $G$ conflict-free connected. In this paper, we investigate the relation between the conflict-free connection number and the independence number of a graph. We firstly show that $cfc(G) \leq \alpha(G)$ for any connected graph $G$, and an example is given showing that the bound is sharp. With this result, we prove that if $T$ is a tree with $\Delta(T) \geq \alpha(T)+2$, then $cfc(T) = \Delta(T)$.

Keywords edge-coloring, conflict-free connection number, independence number, tree

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1 Introduction

All graphs considered here are simple, finite and undirected. An edge-coloring of a graph $G$ is proper if any two adjacent edges in this coloring receive different colors. If $G$ is colored with a proper coloring, then we say that $G$ is properly colored.

The rainbow connection number was introduced by Chartrand et al. \cite{9}. An edge-colored graph $G$ is called rainbow connected if any two vertices are connected by a path whose edges have pairwise distinct colors. The rainbow connection number of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected. Chakraborty et al. \cite{5} showed that given a graph $G$, deciding if $rc(G) = 2$ is NP-complete. Bounds for the rainbow connection number of a graph have also been studied in terms of other graph parameters, see \cite{14, 17, 19, 18, 22} and the references therein.

As an extension of proper colorings and motivated by rainbow connections of graphs, Andrews et al. \cite{1} and independently Borozan et al. \cite{3} introduced the concept of proper connection of graphs. An edge-colored graph $G$ is called properly connected if any two vertices are connected by a path which is properly colored. The proper connection number of a connected graph $G$, denoted by $pc(G)$, is the smallest number of colors that are needed in order to make $G$ properly connected. One can find many results on proper connection, see \cite{4, 16, 20, 21} et al.

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Very recently, inspired by rainbow connection colorings and proper connection colorings of graphs and by conflict-free colorings of graphs and hypergraphs [10, 11, 15, 23], Czap et al. [12] introduced the concept of conflict-free connection of graphs. An edge-colored graph $G$ is conflict-free connected if any two vertices are connected by a path, which contains at least one color used on exactly one of its edges. This path is called a conflict-free path, and this coloring is called a conflict-free connection coloring of $G$. The conflict-free connection number of a connected graph $G$, denoted by $cfc(G)$, is defined as the minimum number of colors that are required in order to make $G$ conflict-free connected.

An easy observation is that a rainbow edge-coloring of a connected graph $G$ is a trivial conflict-free connection coloring, while the other way around is not true in general. Moreover, all above mentioned three parameters of a graph $G$ with order $n$ are bounded by $n - 1$, since one may color the edges of a given spanning tree of $G$ with distinct colors and color the remaining edges with already used colors. There is an extensive research concerning on this topic, see [6, 7, 8, 13, 23, 24].

Recall that an independent set in a graph $G$ is a set of vertices no two of which are adjacent. The cardinality of a maximum independence set in $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$. The observation follows immediately from the concept.

**Observation 1** Let $G$ be a connected graph of order $n$. Then $1 \leq \alpha(G) \leq n - 1$. Moreover, $\alpha(G) = 1$ if and only if $G = K_n$, $\alpha(G) = n - 1$ if and only if $G = K_{1,n-1}$.

Dong and Li [14] gave a relation between the rainbow connection number and the independence number of a graph, they showed that if $G$ is a connected graph without pendant vertices, then $rc(G) \leq 2\alpha(G) - 1$. Inspired by these results, we try to investigate the relation between the conflict-free connection number and the independence number of a graph and obtain our first main result.

**Theorem 1** Let $G$ be a connected graph of order $n$. Then

$$1 \leq cfc(G) \leq \alpha(G) \leq n - 1.$$ 

Moreover, $cfc(G) = 1$ if and only if $\alpha(G) = 1$, $cfc(G) = n - 1$ if and only if $\alpha(G) = n - 1$.

Czap et al. [12] proved that 2-connected graphs have conflict-free connection number 2, while deciding the conflict-free connection number of graphs with cut-edges is very difficult, including trees. Chang et al. [7] came up with a rapid approach to obtain the conflict-free connection number of a tree when its maximum degree is large. Motivated by these results, we find a method to determine the conflict-free connection number of a tree in terms of independence number and obtain our second main result.

**Theorem 2** Let $T$ be a tree with $\Delta(T) \geq \frac{\alpha(T)+2}{2}$. Then $cfc(T) = \Delta(T)$. 
We organize this paper as follows. Some useful preliminaries are presented in Section 2. Then, the proofs of Theorem 1 and Theorem 2 can be given in Section 3 and Section 4, respectively.

We end this section with some terminology. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $d_G(v)$, $N_G(v)$ and $\Delta(G)$ to denote the degree of $v$ in $G$, the set of neighbours of $v$ in $G$ and the maximum degree of $G$, respectively. For $e \in E(G)$, we denote by $G \setminus e$ the graph obtained from $G$ by deleting $e$. An edge $e$ is said to be a cut-edge of $G$ if $c(G \setminus e) = c(G) + 1$, where $c(G)$ is the number of components of $G$. Let $G$ and $F$ be two graphs, we use $F \subseteq G$ to denote that $F$ is a subgraph of $G$. For notation not explained here, readers are referred to [2].

2 Preliminaries

This section is devoted to state several results which concerning on the conflict-free connection number of graphs. Czap et al. [12] showed that it is easy to obtain the conflict-free connection number for 2-connected graphs.

Lemma 1 ([12]) If $G$ is a 2-connected and non-complete graph, then $cfc(G) = 2$.

Chang et al. [7] and independently Deng et al. [13] extended the result of Lemma 1 to 2-edge-connected graphs in the following.

Lemma 2 ([7], [13]) Let $G$ be a non-complete 2-edge-connected graph, then $cfc(G) = 2$.

Compared with 2-edge-connected graphs, the problem of determining the conflict-free connection number of graphs with cut-edges is very difficult. This fact arises many authors’ attention to obtain lower or upper bounds of $cfc(G)$ for a connected graph. Chang et al. [7] gave sharp lower and upper bound of $cfc(G)$ and characterized graphs $G$ for which $cfc(G) = 1$ or $cfc(G) = n - 1$.

Lemma 3 ([7]) Let $G$ be a connected graph of order $n$ ($n \geq 2$). Then $1 \leq cfc(G) \leq n - 1$. Moreover, $cfc(G) = 1$ if and only if $G = K_n$, $cfc(G) = n - 1$ if and only if $G = K_{1,n-1}$.

A block of a graph $G$ is a maximal connected subgraph of $G$ that has no cut-vertex. If $G$ itself is connected and has no cut-vertex, then $G$ is a block. An edge is a block if and only if it is a cut-edge. A block consisting of a cut-edge is called trivial. Note that any nontrivial block is 2-connected.

Let $C(G)$ be the subgraph of $G$ induced on the set of cut-edges of $G$, and let $h(G) = \max\{cfc(T) : T$ is a component of $C(G)\}$.
Lemma 4 ([12]) If \( G \) is a connected graph with cut-edges, then \( h(G) \leq cfc(G) \leq h(G) + 1 \). Moreover, these bounds are tight.

Chang et al. [7] gave a sufficient condition such that the lower bound in Lemma 4 is sharp for \( h(G) \geq 2 \).

Lemma 5 ([7]) Let \( G \) be a connected graph with \( h(G) \geq 2 \). If there exists a unique component \( T \) of \( C(G) \) satisfying (i) \( cfc(T) = h(G) \), (ii) \( T \) has an optimal conflict-free connection coloring which contains a color used on exactly one edge of \( T \), then \( cfc(G) = h(G) \).

It is seen from Lemma 4 that, to determine the conflict-free connection number of graphs relies on the conflict-free connection number of trees, with an error of only one. Thus, determining the conflict-free connection number of trees is of great importance. Here we list some known results concentrating on the conflict-free connection number of trees.

Lemma 6 ([12]) If \( P_n \) is a path on \( n \) edges, then \( cfc(P_n) = \lceil \log_2(n + 1) \rceil \).

Lemma 7 ([12]) If \( T \) is an \( n \)-vertex tree of maximum degree \( \Delta(T) \geq 3 \) and diameter \( d(T) \), then
\[
\max \{\Delta(T), \log_2 d(T)\} \leq cfc(T) \leq \frac{(\Delta(T) - 2) \cdot \log_2 n}{\log_2 \Delta(T) - 1}.
\]

The following result in [7] indicates that when the maximum degree of a tree is large, the conflict-free connection number is immediately determined by its maximum degree.

Lemma 8 ([7]) Let \( T \) be a tree of order \( n \), and \( t \) be a positive integer such that \( n \geq 2t + 2 \). Then \( cfc(T) = n - t \) if and only if \( \Delta(T) = n - t \).

We end this section with the following lemma, which is no more than an observation.

Lemma 9 Let \( T_1 \) and \( T_2 \) be two trees such that \( T_1 \subseteq T_2 \). Then \( cfc(T_1) \leq cfc(T_2) \).

3 The proof of Theorem 1

Proof of Theorem 1 By Observation 1 and Lemma 3, it suffices to prove that \( cfc(G) \leq \alpha(G) \) for a non-complete graph \( G \). Our main strategy is by induction on the number of cut-edges in \( G \). For simplicity, set \( k := |E(C(G))| \).

Since \( cfc(G) = 2 \) and \( \alpha(G) \geq 2 \) for a non-complete 2-edge-connected graph \( G \), we get that \( cfc(G) \leq \alpha(G) \) when \( k = 0 \) by Lemma 2 and Observation 1. Assume
that the statement holds for any graph with \( \leq k - 1 \) cut-edges, and let \( G \) be a graph with \( k \) cut-edges. We distinguish two cases.

**Case 1.** There exists a cut-edge, say \( e \), such that each component of \( G \setminus e \) is a subgraph of order greater than 1. 

W.l.o.g., let \( e = u_1u_2 \) and let \( G_1 \) and \( G_2 \) be two components of \( G \setminus e \) with \( u_i \in V(G_i), i \in \{1, 2\} \).

For \( i \in \{1, 2\} \), it is seen that \( |V(G_i)| \geq 2 \); and that the number of cut-edges in \( G_i \) must be no more than \( k - 1 \). By induction hypothesis, we have 

\[
cfc(G_i) \leq \alpha(G_i) \quad \text{for } i \in \{1, 2\}.
\]

W.l.o.g., assume that \( cfc(G_2) \leq cfc(G_1) \). Let \( S_1 \) be a maximum independent set in \( G_1 \). Moreover, since \( |V(G_2)| \geq 2 \), there must exist a vertex, say \( z \), such that \( z \in V(G_2) \setminus \{u_2\} \). Note that \( z \) is not adjacent to vertices in \( G_1 \), then \( S_1 \cup \{z\} \) is an independent set in \( G \) whose cardinality is 

\[
\alpha(G_1) + 1 \leq \alpha(G).
\]

Now, we are able to assign \( cfc(G_1) + 1 \) colors to all the edges of \( G \) in order to make \( G \) conflict-free connected: first we color each component of \( G \setminus e \) with at most \( cfc(G_1) \) colors, next we color the edge \( e \) with a fresh color. We only need to prove that any pair of distinct vertices \( x \) and \( y \) of \( G \) are connected by a conflict-free path. If the vertices \( x \) and \( y \) are from the same component of \( G \setminus e \), then such a path exists. If they are in different components of \( G \setminus e \), then there is a \( x-y \) path through the edge \( e \) with a unique color.

The analyses above imply that 

\[
cfc(G) \leq cfc(G_1) + 1 \leq \alpha(G_1) + 1 \leq \alpha(G).
\]

**Case 2.** Each cut-edge is a pendant edge.

Thus, each component of \( C(G) \) is a complete bipartite graph \( K_{1,r} \) where \( 1 \leq r \leq n - 1 \). Let \( \tilde{G} \) be the graph obtained from \( G \) by deleting all the pendant vertices. Note that \( |V(\tilde{G})| \neq 2 \), otherwise \( \tilde{G} \) is a non-pendant cut-edge in \( G \), a contradiction.

**Subcase 2.1.** \( |V(\tilde{G})| = 1 \).

That means \( G = K_{1,n-1} \). By Observation \( \Box \) and Lemma \( \Box \), \( cfc(G) = n - 1 = \alpha(G) \).

**Subcase 2.2.** \( |V(\tilde{G})| \geq 3 \).

W.l.o.g., let \( v \) be a vertex of \( C(G) \) such that 

\[
d_{C(G)}(v) = \max \{ d_{C(G)}(x) : x \in V(C(G)) \}.
\]

For simplicity, setting \( t := d_{C(G)}(v) \) and let \( y_1, \ldots, y_t \) be pendant vertices adjacent to \( v \) in \( G \). Thus,

\[
h(G) = cfc(K_{1,t}) = t.
\]
Since $|V(\tilde{G})| \geq 3$, we can choose a vertex, say $z$, such that $z \in V(\tilde{G}) \setminus \{v\}$. Note that $\{z, y_1, \cdots, y_t\}$ is an independent set in $G$ with cardinality $t+1$, obviously

$$t + 1 \leq \alpha(G). \quad (2)$$

Therefore, Lemma 4 together with Eq.(1) and Eq.(2) yield

$$cfc(G) \leq h(G) + 1 = t + 1 \leq \alpha(G).$$

Thus, $1 \leq cfc(G) \leq \alpha(G) \leq n - 1$ for a connected graph $G$.

Moreover, Observation 1 together with Lemma 3 imply that $cfc(G) = 1$ if and only if $\alpha(G) = 1$, and that $cfc(G) = n - 1$ if and only if $\alpha(G) = n - 1$. \hfill $\Box$

By Theorem 1, it is easy to obtain the conflict-free connection number of a graph whose independence number is 2.

**Corollary 1** Let $G$ be a connected graph with $\alpha(G) = 2$. Then $cfc(G) = 2$.

By Theorem 1 and Lemma 7 we can give an upper bound on the conflict-free connection number of trees. Moreover, a sufficient condition for which the conflict-free connection number of a tree equals to its maximum degree is obtained.

**Corollary 2** Let $T$ be a tree. Then $\Delta(T) \leq cfc(T) \leq \alpha(T)$. Moreover, if $\Delta(T) = \alpha(T)$, then $cfc(T) = \Delta(T)$.

At the end of this section, an example is given showing that there exists non-complete graph whose conflict-free connection number can be any integer no more than its independence number. Thus the bound $cfc(G) \leq \alpha(G)$ in Theorem 1 is tight.

**Example 1** Let $l, k$ be integers such that $3 \leq l \leq n - 2$ and that $2 \leq k \leq l$. There exists a graph $G_{l,k}$ of order $n$ for which $\alpha(G_{l,k}) = l$ and $cfc(G_{l,k}) = k$.

**Proof.** We will construct the desired graph $G_{l,k}$ by considering two cases: $k = l$ or $k < l$.

When $k = l$, let $G_{l,l}$ be a graph obtained by identifying a leaf vertex of $K_{1,l}$ with a vertex of the complete graph $K_{n-l}$. It is seen that $\alpha(G_{l,l}) = l = cfc(G_{l,l})$.

When $k < l$, we construct $G_{l,k}$ with vertex set

$$V(G_{l,k}) = \{w, v, u_1, \cdots, u_{n-2}\}$$

and edge set

$$E(G_{l,k}) = \{wu_i : 1 \leq i \leq n - 2\} \cup \{vu_i : k + 1 \leq i \leq n - 2\} \cup \{wv\} \cup \{u_iu_j : l \leq i \neq j \leq n - 2\}.$$ 

Note that the subgraph induced on vertices $\{v, w, u_l, \cdots, u_{n-2}\}$ is a clique on $n - l + 1$ vertices. We can get that $\alpha(G_{l,k}) = l$ since $\{u_1, \cdots, u_k, \cdots, u_l\}$ is a maximum independent set in $G_{l,k}$. Moreover, the subgraph induced on $\{w, u_1, \cdots, u_k\}$ is the unique component of $C(G_{l,k})$, thus $cfc(G_{l,k}) = k$ by Lemma 3 and Lemma $\Box$
4 The proof of Theorem 2

We firstly give some results on the conflict-free connection number of certain trees, which will be useful in the later discussions.

Lemma 10 Define $H_k$ ($k \geq 3$) be a tree obtained by subdividing each edge of the complete bipartite graph $K_{1,k}$ to a path of length two, see Figure 1. Then $cfc(H_k) = k$.

Proof. By the definition of $H_k$ and by Lemma 7, we have $cfc(H_k) \geq \Delta(H_k) = k$. To complete the proof, we only need to assign a conflict-free connected coloring $c : E(H_k) \to [k]$ as follows

$$c(e) = \begin{cases} 
i, & \text{if } e = uu_i, \ 1 \leq i \leq k; \\
k, & \text{if } e = u_1v_1; \\
i - 1, & \text{if } e = u_iv_i, \ 2 \leq i \leq k. \end{cases}$$

It is not difficult to check that $c$ is a conflict-free connected coloring of $H_k$, thus $cfc(H_k) \leq k$. The proof is done.

Now we are in a position to prove Theorem 2.
Proof of Theorem 2. We will prove the theorem by induction on \( k := \Delta(T) \).

Since a tree \( T \) satisfying \( \Delta(T) = 2 \) and \( \Delta(T) \geq (\alpha(T) + 2)/2 \) is the path \( P_2 \) or \( P_3 \), by Lemma 6, the theorem holds when \( k = 2 \). Assume that the result is true for any tree \( T' \) with \( \Delta(T') \leq k - 1 \) and \( \Delta(T') \geq (\alpha(T') + 2)/2 \). Now consider a tree \( T \) with \( \Delta(T) = k \) \( (k \geq 3) \) and \( \Delta(T) \geq (\alpha(T) + 2)/2 \).

Let \( u \) be a vertex of \( T \) such that \( d_T(u) = \Delta(T) \) and let \( N_T(u) = \{u_1, \cdots, u_k\} \). Firstly, we claim that

**Claim 1.** If there exists a vertex \( w \neq u \) such that \( d_T(w) = \Delta(T) \), then \( w \in N_T(u) \).

Proof of Claim 1. Suppose to contrary that there is a vertex \( w \notin N_T(u) \) such that \( d_T(w) = \Delta(T) \). Then \( N_T(u) \cup N_T(w) \) is an independent set in \( T \), whose cardinality is at least

\[
2k - 1 = 2\Delta(T) - 1 \geq \alpha(T) + 1,
\]

the last inequality holds since \( \Delta(T) \geq (\alpha(T) + 2)/2 \). A contradiction. \( \square \)

It is inferred from the proof of Claim 1 that, in \( T \), the vertices of maximum degree must be adjacent to each other. Since \( T \) is a tree, there is at most one vertex of \( \{u_1, \cdots, u_k\} \) can be of maximum degree. W.l.o.g., let

\[
d_T(u_1) = \max\{d_T(u_i) : 1 \leq i \leq k\}.
\]

Therefore, we claim that

**Claim 2.** For any vertex \( x \in V(T) \setminus \{u, u_1\} \), it has \( d_T(x) \leq \Delta(T) - 1 \).

For \( 1 \leq i \leq k \), let \( T_{i1} \) and \( T_{i2} \) be two components of \( T \setminus uu_i \) where \( u_i \in V(T_{i2}) \).

We discuss three cases.

**Case 1.** \( d_T(u_1) = 1 \).

Then \( T \) is the graph \( K_{1,k} \). By Lemma 3, \( cfc(T) = k = \Delta(T) \).

**Case 2.** \( d_T(u_1) = 2 \).

**Subcase 2.1.** \(|E(T_{i2})| \leq 1 \) for each \( 1 \leq i \leq k \).

Then \( T \) is a subgraph of \( H_k \) which is defined in Lemma 10. By Lemma 9 and Lemma 10, we have \( cfc(T) \leq cfc(H_k) = k \). On the other hand, by Lemma 7, \( cfc(T) \geq \Delta(T) = k \). Thus, \( cfc(T) = k = \Delta(T) \).

**Subcase 2.2.** \(|E(T_{i2})| \leq 2 \) for each \( 1 \leq i \leq k \), and there exists an integer \( i \) such that \( |E(T_{i2})| = 2 \).

Note that \( T_{i2} \) is a path \( P_2 \) when \( |E(T_{i2})| = 2 \). W.l.o.g., let \( w_i \) be an end vertex other than \( u_i \) in \( T_{i2} \). Since \( d_T(u_1) = 2 \), there are at most \( k - 2 \) integers \( i \) such that \( T_{i2} \) is a path \( P_2 \), otherwise \( S := \{u_1, \cdots, u_k\} \cup (I_i \{w_i\}) \) is an independent set in \( T \), moreover,

\[
|S| \geq k + k - 1 = 2\Delta(T) - 1 \geq \alpha(T) + 1,
\]
a contradiction.
Therefore, \( T \) is a subgraph of \( Q_k \) which is defined in Lemma 11. By Lemmas 7, 9 and 11, we have \( cf(T) = k = \Delta(T) \).

Subcase 2.3. There exists an integer \( i \) (1 \leq i \leq k) such that \( |E(T_{i2})| \geq 3 \).

Let \( e = uu_i \). Recall that \( T_{i1} \) and \( T_{i2} \) are two components of \( T \setminus e \) with \( u_i \in V(T_{i2}) \). Since \( d_T(u_i) = 2 < \Delta(T) \), thus \( \Delta(T_{i1}) = \Delta(T) - 1 \) and \( \Delta(T_{i2}) \leq \Delta(T) - 1 \) by Claim 2.

Firstly, we try to obtain the conflict-free connection number of \( T_{i1} \). Let \( S_1 \) be a maximum independent set in \( T_{i1} \). Since \( |E(T_{i2})| \geq 3 \), we always can choose at least two non-adjacent vertices, say \( x \) and \( y \), from \( V(T_{i2}) \setminus \{u_i\} \), such that \( S := S_1 \cup \{x, y\} \) is an independent set in \( T \). That means \( |S| = \alpha(T_{i1}) + 2 \leq \alpha(T) \)


\[
\Delta(T_{i1}) = \Delta(T) - 1 \geq \frac{\alpha(T)}{2} \geq \frac{\alpha(T_{i1}) + 2}{2},
\]

the above first inequality holds since \( \Delta(T) \geq \frac{\alpha(T) + 2}{2} \). By induction hypothesis, we have

\[
 cf(T_{i1}) = \Delta(T_{i1}) = \Delta(T) - 1.
\] (3)

Next, we consider the conflict-free connection number of \( T_{i2} \). Firstly, we claim that

Claim 3. \( \alpha(T_{i2}) \leq \alpha(T) - \Delta(T) + 1 \).

Proof of Claim 3. Suppose to contrary that \( \alpha(T_{i2}) > \alpha(T) - \Delta(T) + 1 \). Let \( S_2 \) be a maximum independent set in \( T_{i2} \), obviously, \( S' := S_2 \cup \bigcup \{u_j\} \) is an independent set in \( T \) with cardinality

\[
|S'| = |S_2| + k - 1 > \alpha(T) - \Delta(T) + 1 + k - 1 = \alpha(T),
\]
a contradiction. □

By Theorem 11 and Claim 3, we have

\[
 cf(T_{i2}) \leq \alpha(T_{i2}) \leq \alpha(T) - \Delta(T) + 1 \leq \Delta(T) - 1,
\] (4)

the last inequality holds since \( \Delta(T) \geq \frac{\alpha(T) + 2}{2} \).

By Eq. (3) and Eq. (4), we are now able to assign \( \Delta(T) \) colors to all the edges of \( T \) in order to make \( T \) is conflict-free connected: firstly we color \( T_{i1} \) and \( T_{i2} \) with at most \( \Delta(T) - 1 \) colors, next we color the edge \( e = uu_i \) with a fresh color. Therefore, \( cf(T) \leq \Delta(T) \). Combined this conclusion with Lemma 11, we have \( cf(T) = \Delta(T) \).

Case 3. \( d_T(u_1) \geq 3 \).

Let \( e = uu_1 \). Recall that \( T_{11} \) and \( T_{12} \) are two components of \( T \setminus e \) with \( u_1 \in V(T_{12}) \). By Claim 2, \( \Delta(T_{11}) = \Delta(T) - 1 \) and \( \Delta(T_{12}) \leq \Delta(T) - 1 \).

Using similar discussions in Subcase 2.3, we can get that \( cf(T_{11}) = \Delta(T) - 1 \) and that \( cf(T_{12}) \leq \Delta(T) - 1 \), moreover, \( cf(T) = \Delta(T) \).

The proof is completed. □
Remark 1 The sharpness example for Theorem 2 is given as follows. Let $T$ be a tree obtained from two copies of $K_{1,k-1}$ with $k \geq 3$ by identifying a leaf vertex in one copy with a leaf vertex in the other copy. It is seen that $\alpha(T) = 2k - 3$ and that $\Delta(T) = k - 1 = \frac{\alpha(T)+1}{2}$. Furthermore, Theorem 5.5 in [8] showed that $\text{cfc}(T) = k$. Thus $\text{cfc}(T) > \Delta(T)$.

Remark 2 Theorem 2 gives a sufficient condition for the conflict-free connection number of a tree equals to its maximum degree. However, this condition is not necessary. Define $G$ to be a tree obtained from two copies of $K_{1,k}$ ($k \geq 3$) by adding an edge joining a leaf vertex in one copy to a leaf vertex in the other copy. Figure 3 illustrates that we can assign $k = \Delta(G)$ colors to all the edges of $G$ in order to make it conflict-free connected, thus $\text{cfc}(G) = \Delta(G)$. On the other hand, we can testify that $\alpha(G) = 2k - 1$ and thus $\Delta(G) < \frac{\alpha(G)+2}{2}$.

![Figure 3: The graph $G$.](image)

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References

[1] E. Andrews, C. Lumduanhom, E. Laforge, P. Zhang, On proper-path colorings in graphs, J. Combin. Math. Combin. Comput., 97, (2016) 189-207.

[2] J.A. Bondy, U.S.R. Murty, Graph Theory, New York, Springer, (2008).

[3] V. Borozan, S. Fujita, A. Gerek, C. Magnant, Y. Manoussakis, L. Monter, Z. Tuza, Proper connection of graphs, Discrete Math., 312, (2012) 2550-2560.

[4] C. Brause, T. Doan, I. Schiermeyer, Minimum degree conditions for the proper connection number of graphs, Graphs and Combin., 33, (2017) 833-843.

[5] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, Hardness and algorithms for rainbow connectivity, J. Comb. Optim., 21, (2011) 330-347.

[6] H. Chang, T. Doan, Z. Huang, S. Jendrol’, X. Li, I. Schiermeyer, Graphs with conflict-free connection number two, Graphs and Combin., 34, (2018) 1553-1563.
[7] H. Chang, Z. Huang, X. Li, Y. Mao, H. Zhao, On conflict-free connection of graphs, Discrete Appl. Math., 255, (2019) 167-182.

[8] H. Chang, M. Ji, X. Li, J. Zhang, Conflict-free connection of trees, J. Comb. Optim., (in press), https://link.springer.com/content/pdf/10.1007%2Fs10878-018-0363-x.pdf.

[9] G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, Rainbow connection in graphs, Math. Bohem., 133, (2008) 85-98.

[10] P. Cheilaris, B. Keszegh, D. Pálvölgyi, Unique-maximum and conflict-free coloring for hypergraphs and tree graphs, SIAM J. Discrete Math., 27, (2013) 1775-1787.

[11] P. Cheilaris, G. Tóth, Graph unique-maximum and conflict-free colorings, J. Discrete Algorithms, 9, (2011) 241-251.

[12] J. Czap, S. Jendrol’, J. Valiska, Conflict-free connection of graphs, Discuss. Math. Graph Theory, 38, (2018) 911-920.

[13] B. Deng, W. Li, X. Li, Y. Mao, H. Zhao, Conflict-free connection numbers of line graphs, in: Proc. COCOA 2017, Shanghai, China, in: Lecture Notes in Computer Science, vol. 10627, 141-151.

[14] J. Dong, X. Li, Rainbow connection number and independence number of a graph, Graphs and Combin., 32, (2016) 1829-1841.

[15] I. Fabrici, F. Göring, Unique-maximum coloring of plane graphs, Discuss. Math. Graph Theory, 36, (2016) 95-102.

[16] F. Huang, X. Li, Z. Qin, C. Magnant, K. Ozeki, On two conjectures about the proper connection number of graphs, Discrete Math., 340, (2017) 2217-2222.

[17] N. Kamčev, M. Krivelevich, B. Sudakov, Some remarks on rainbow connectivity, J. Graph Theory, 83, (2016) 372-383.

[18] A. Kemnitz, J. Przybyło, I. Schiermeyer, M. Woźniak, Rainbow connection in sparse graphs, Discuss. Math. Graph Theory, 33, (2013) 181-192.

[19] H. Li, X. Li, S. Liu, Rainbow connection of graphs with diameter 2, Discrete Math., 312, (2012) 1453-1457.

[20] X. Li, C. Magnant, Properly colored notions of connectivity—a dynamic survey, Theory Appl. Graphs, 0(1), (2015) Article 2, 1-30.

[21] X. Li, C. Magnant, Z. Qin, Properly colored connectivity of graphs, Springer Briefs in Math., Springer, Switzerland, (2018).
[22] X. Li, Y. Sun, An updated survey on rainbow connections of graphs-a dynamic survey, Theory Appl. Graphs, 0(1), (2017), Article 3, 1-67.

[23] X. Li, Y. Zhang, X. Zhu, Y. Mao, H. Zhao, S. Jendrol’, Conflict-free vertex-connections of graphs, Discuss. Math. Graph Theory (in press), http://dx.doi.org/10.7151/dmgt.2116.

[24] Z. Li, B. Wu, On the maximum value of conflict-free vertex-connection number of graphs, arXiv:1709.01225V1 [math.CO].

[25] J. Pach, G. Tardos, Conflict-free colorings of graphs and hypergraphs, Comb. Probab. Comput., 18, (2009) 819-834.