A nanoscale, cylindrical, semiconductor based superconductor-normal-superconductor (SNS) junction is considered theoretically by use of an idealized model. A magnetic field is applied along the axis of the cylinder. The Bogoliubov-de Gennes equations for Andreev bound states are solved while considering the transverse subbands in the $N$-section. It is seen that the energy-versus-phase curves of the Andreev bound states are shifted in phase, as the $N$-section Andreev quasiparticles occupying subbands of non-zero orbital angular momentum couple to the axial magnetic flux. A similar phase shift is observed in the continuum current of the junction. An intuitive, semi-classical picture of the phase shift is presented to complement the quantum mechanical calculation. The critical current of the junction is studied when multiple subbands carrying non-zero orbital angular momentum are occupied. Oscillations of the critical current versus the axial magnetic flux are identified as a novel form of Josephson interference. Regimes in which this effect is prominently visible are identified.

The Josephson effect is characterized by a current-phase relation (CPR) linking macroscopic current flow to the phase gradient of the superconducting order parameter $I_c \Phi$. The precise form of the CPR for a superconducting weak link depends on intrinsic factors such as junction geometry, material properties, coherence lengths, etc., in addition to extrinsic variables like temperature and magnetic field. In superconductor-normal-superconductor (SNS) junctions in which the $N$-section is long enough to suppress direct tunnelling of Cooper pairs, but shorter than the phase coherence length in the $S$-section, a supercurrent may be carried by quasiparticles undergoing Andreev reflection at the $S$-$N$ interfaces [2-5]. Planar SNS junctions of width large compared to the $S$-section superconducting coherence length have been studied in great detail [6] (width refers to the dimension perpendicular to the current). These have revealed, for example, Fraunhofer oscillations of the critical current $I_c$ with respect to an externally applied out-of-plane magnetic field [7,9]. For junction widths comparable to the $S$-section coherence length, i.e. the narrow junction limit, this becomes a quasi-Gaussian, monotonic decay of $I_c$ [7,10,11].

Recently, attention has been given to nanoscale, quasi one-dimensional (1D) SNS junctions, such as those readily engineered by contacting semiconductor nanowires with superconducting leads [12,15]. Gating the semiconducting $N$-section allows for modulating the supercurrent by controlling the chemical potential [12,16]. The oscillations of the magnetoresistance of a nanowire SNS junction in the voltage-carrying state (i.e. no supercurrent) versus an axial magnetic field have been studied [10]. Efforts to realize Majorana fermion quasiparticles in 1D semiconductors with strong spin-orbit interaction and proximity coupling to a superconductor [17,20] have further raised interest in this type of junction. Theoretical results have indicated that the behaviour of the critical current in such a junction versus magnetic field and chemical potential can be used to identify topological phases [21].

Previous theoretical descriptions of quasi-1D SNS junctions [10,11,22,23] have not fully considered the effects of nanoscale confinement on the CPR, in particular the implications of orbital angular momentum coupling to an external magnetic flux. Here we provide a quantum mechanical description of an idealized junction with a flux applied along the nanowire axis (parallel to the current). The results show significant modifications of the CPR and critical current versus magnetic field, due to both the trivial effect of nanowire subbands depopulating and also, surprisingly, to a non-trivial effect which we identify as a previously unknown form of Josephson interference. The latter effect is due to the coupling between Andreev quasiparticles (bound states and continuum states) with non-zero angular momentum and the applied flux, which results in phase shifts of the energy-versus-phase for these current carrying states. The total current summed over all occupied states can display interference. Qualitatively similar to aforementioned Fraunhofer interference, but in contrast to the wide planar junction, the flux is aligned with the current and the oscillations are not periodic in the flux quantum. We note the effect is only present in nanoscale junctions with lateral size (i.e. diameter) smaller than the London penetration depth. This is a regime in which the general theorem of Byers and Yang [24] does not apply. It is shown that the supercurrent from continuum states can also contribute to this interference. For certain junction parameters, the interference effect can dominate the $I_c$ vs $\Phi$ characteristics. Semiclassically, the effect can be thought of as due to a magnetic phase acquired by Andreev pairs with an
azimuthal velocity component. The aim of this paper is to lay a theoretical groundwork for describing this type of Josephson interference and to show, in an idealized model with simplifying assumptions, that the effect is present. In particular, we consider the case where the diameter is smaller than the superconducting coherence length in the S-section, so that the phase of the order parameter is uniform around the S-section circumference in any magnetic field. Furthermore, spin-orbit and Zeeman effects in the N-section are neglected, and no barriers are assumed at the S-N interfaces. The resulting model is therefore of limited applicability to real devices, and we discuss extensions to the model that would be needed to properly describe experimentally relevant junctions.

I. MODEL

Consider an SNS junction created by a semiconducting nanowire contacted by superconducting leads. We use parameters corresponding to Nb for the leads, and InAs for the semiconductor. Particularly, we use below a shell conduction model relevant to low band-gap semiconductor nanowires with downward surface band bending. However, the descriptive power of the model is not restricted to these materials. We use cylindrical coordinates \( r = (x, \rho, \theta) \), with the nanowire axis along \( \hat{x} \). The junction is modelled as a cylinder of radius \( R_0 \). The diameter \( d = 2R_0 \) of the cylinder is assumed smaller than the S-section London penetration depth \( \lambda_s \), and the S-section superconducting coherence length \( \xi_s \). We divide the cylinder into three regions (figure Ia), with region 1 the superconducting section corresponding to the left lead \((x < -L/2)\), region 2 the normal section corresponding to the nanowire \((|x| < L/2)\), and region 3 the S-section corresponding to the right lead \((x > L/2)\). The variations of the electrostatic potential within each section are neglected (i.e., no scattering potential is included, and the ballistic regime assumed), and no potential barrier assumed at the S-N interfaces (we consider the implications of finite barriers in section IV). A magnetic field \( B = B_\parallel \hat{z} \) penetrates the cylinder. Any screening of the magnetic field in the S sections is neglected, as we have \( d < \lambda_s \). In Coulomb gauge, the vector potential is \( A = A_\rho \hat{\theta} = (B_\parallel \rho / 2) \hat{\theta} \). Using the superscript \( \alpha = 1, 2, 3 \) to refer to the three regions (with the N-section corresponding to \( \alpha = 2 \)), the single-electron Hamiltonian (excluding the superconducting pairing potential) in the presence of the axial magnetic field can be written as:

\[
H_\theta = \sum_{\alpha=1,2,3} H^\alpha_x + H^\alpha_\theta + V^\alpha(\rho) - \mu; \tag{1a}
\]

\[
H^\alpha_x = -\frac{\hbar^2}{2m^\alpha} \frac{\partial^2}{\partial x^2}, \tag{1b}
\]

\[
H^\alpha_\theta = \frac{1}{2m^\alpha} \left(-i\hbar \frac{1}{\rho} \frac{\partial}{\partial \theta} - eA_\theta \right)^2. \tag{1c}
\]

FIG. 1. a) Schematic of the considered SNS junction. The junction is modelled as a cylinder with a nanoscale diameter \( d \) smaller than the London penetration length and the phase coherence length in the S-section. The junction length is \( L \). An axial magnetic field \( B = B_\parallel \hat{z} \) penetrates the cylinder. b) The superconducting order parameter has the magnitude \( \Delta_0 \) in the S-section, and is zero in the normal section, with a jump-like variation at the boundaries.

Here, \( H^\alpha_x \) describes the kinetic energy of motion along the axis of the cylinder, \( H^\alpha_\theta \) the kinetic and magnetic energies of the azimuthal motion around the cylinder, and \( V^\alpha(\rho) \) the radial confining potential of the cylinder, all corresponding to section \( \alpha \). Nanoscale radial confinement results in charge carriers occupying transverse subbands denoted by a pair of quantum numbers. We use the pair of numbers \((n, l)\) in the N-section, and \((p, l)\) in the S-section, where \( n \) and \( p \) are the radial quantum numbers, and \( l \) the orbital angular momentum quantum number. The chemical potential in the cylinder (i.e., the energy difference between the bottom of the lowest subband and the Fermi energy; see figure 2) is denoted by \( \mu \). The effective mass of electrons in section \( \alpha \) is denoted by \( m^\alpha \). In the calculations below, we use the free electron mass \( m_e \) in the leads (justified in metallic superconductors like Nb). In the N-section, we use the InAs value \( m^* = 0.023 m_e \). Spin-orbit and Zeeman effects in the InAs N-section of the junction (studied in Ref. [28]) are neglected, in order to focus on the effects of orbital angular momentum of interest here. We discuss this further in section IV.

In this paper we do not write out an explicit form for \( V^\alpha \), and do not solve for the radial wavefunctions corresponding to the subbands \((n, l)\) in any section of the cylinder. Instead, we use a simplified shell conduction model for the N-section. This is relevant to InAs or InN nanowires, wherein the charge carriers are confined near the surface of the nanowire due to a large positive surface potential, i.e. a surface accumulation layer [28, 29]. Assuming a strong downward surface band bending \((\sim 100 - 200 \text{ meV})[28]\), the radial position of the carriers in all subbands \((n, l)\) is taken to be \( R \lesssim R_0 \). This simplifies the calculation of the eigenvalues of \( H_\theta \).
is shown that energy eigenstates corresponding to these
in Ref. \[31\]. In this paper, we assume the critical current
flow in the
S
of the superconducting leads. Otherwise, the superfluid
SNS
junction is much smaller than the critical current
discussed in section IV. (ii) The injected current in the
d < ξ
order parameter \[30\], so ∆ must be uniform. The valid-
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ued (i.e., Fluxoid quantization in a non-zero magnetic
π
integer multiples of 2
 sets the length-scale for the spatial variations of the
order parameter \[30\], so ∆ must be uniform. The validity
of the assumption d < ξS in experimental devices
is discussed in section IV (ii) (ii) The injected current in the
SNS junction is much smaller than the critical current
of the superconducting leads. Otherwise, the superfluid
in the S-sections cannot be neglected, and a self-
consistent determination of ∆ is required, as performed
in Ref. \[31\]. In this paper, we assume the critical current
of the junction is bottle-necked in the N section.

\[\Delta(r) = \begin{cases} \Delta_0 e^{i\chi_L}, & x < -L/2 \\ 0, & |x| < L/2 \\ \Delta_0 e^{i\chi_R}, & x > L/2 \end{cases} \] (2)

Outside the radius of the cylinder, the order parameter is zero: ∆(r) = 0 when r > R0. Here, ∆0 is the super-
conducting energy gap value in the leads, and χL(R) is the
phase of the superconducting condensate in the
left (right) lead. The order parameter is zero in the N
section because of the lack of phonon mediated electron-
electron interactions (possible repulsive interactions are
neglected.)

In Eq. \[2\] a spatially uniform ∆ is assumed in the S
sections in all magnetic fields. This is justified as long as:
(i) the diameter of the cylinder is smaller than the su-
perconducting coherence length in the S section, d < ξS
The change in the phase of the order parameter around
the circumference of the cylinder, δχ, is constrained to
integer multiples of 2π, because ∆ has to be single val-
ued (i.e., Fluxoid quantization in a non-zero magnetic
field \[29\]). When d < ξS, one can assume δχ = 0, since
ξS sets the length-scale for the spatial variations of the
order parameter \[30\], so ∆ must be uniform. The validity
of the assumption d < ξS in experimental devices
is discussed in section IV (ii) (ii) The injected current in the
SNS junction is much smaller than the critical current
of the superconducting leads. Otherwise, the superfluid
flow in the S-sections cannot be neglected, and a self-
consistent determination of ∆ is required, as performed
in Ref. \[31\]. In this paper, we assume the critical current
of the junction is bottle-necked in the N section.

II. THEORY

We wish to calculate the current-phase relationship
(CPR) of the junction in the presence of an axial mag-
netic field. First, the spectrum of discrete levels (And-
dreev Bound states) in the junction is obtained. Next,
the current from the discrete levels, as well the “con-
tinuum” levels with energy |E| > ∆0 is calculated. It
is shown that energy eigenstates corresponding to these
energy levels (both the bound states and the continuum
states) follow the single-electron subband structure
imposed by the Hamiltonian \(H_0\) (Eq. \[4\]). In particular, we
show that the CPR is modified by the axial magnetic flux
in a way that depends on the orbital angular momentum of
the subbands. This leads to a form of Josephson inter-
ference when one or more subbands of non-zero orbital
angular momentum are occupied.

A. Bogoliubov-de Gennes Equations

The wavefunctions of the elementary excitations of the
SNS junction are identified as the solutions to the
Bogoliubov-de Gennes \[30\] (BdG) equations:

\[
\begin{pmatrix} H_0 & \Delta(r) \\ \Delta^*(r) & -H_0^* \end{pmatrix} \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = E \begin{pmatrix} u(r) \\ v(r) \end{pmatrix},
\] (3)

where \(H_0\) is given by Eq. \[1\] and u(r) and v(r) are particle-
and hole-like wavefunctions. The asterisk (*)
 denotes complex conjugation.

The solution strategy of Eq. \[3\] starts with finding the
solutions to \(H_0\). Given the simple forms of \(H^2_0\) and \(H_0^*\)
in Eq. 4 it is clear that the single-particle eigenfunctions are plane-waves in the \( x, \theta \) directions. The solutions \( u(r), v(r) \) to Eq. 3 can be expanded over these single-particle solutions. In the -section we have

\[
n(x, \rho, \theta) = \sum_{k,n,l} u_{kl} e^{i k x} e^{i \ell \rho} \phi_{n,l}(\rho). \tag{4}
\]

Here, \( u_{kl} \) are complex-valued expansion coefficients. There is an exactly analogous expression for \( v(x, \rho, \theta) \) with expansion coefficients \( v_{kl} \). The value \( h\kappa \) is the linear momentum along the axis of the cylinder, and \( \phi_{n,l}(\rho) \) is the radial eigensolution for the particle in the transverse subband \((n, l)\). As discussed above, we use a shell conduction model for the \( N \)-section, so \( \phi_{n,l} \) is not written out explicitly, but assumed to result in a radial position \( R \lesssim R_0 \) for the carriers. The single particle energies corresponding to Eq. 4 for a given set of quantum numbers \((k, n, l)\) are:

\[
H_0 u(r) = \left\{ \hbar^2 k^2 / 2m^* + (\hbar^2 / 2m^* R^2)(l^2 + \Phi^2) \right\} u(r), \quad \text{(5a)}
\]

\[
-H_0^2 v(r) = \left\{ -\hbar^2 h^2 / 2m^* - (\hbar^2 / 2m^* R^2)(l^2 + \Phi^2) \right\} v(r). \tag{5b}
\]

Here, \( \Phi = (\pi B / R^2) / (\hbar / e) \) is the reduced magnetic flux enclosed by the charge carriers, \( \epsilon_\ell = [\hbar^2 / (2m^* R^2)] (2\Phi) \), and \( \zeta_{n,l} \) is the radial confinement energy associated with \( \phi_{n,l}(\rho) \). The electron subband energies (i.e. the eigenvalues of \( u \) at \( k = 0 \)) are plotted in figure 2 and are parabolic in shape versus the magnetic flux. For the corresponding eigenvalues for \( v \), the parabolas are inverted. This gives \( v(r) \) its hole-like character: its group velocity \( v_g = -\hbar \nabla E \) is opposite to its wave vector \( k \) (see Ref. 31). This indicates that, for a given \( k \), charge is transported in opposite directions by the two wavefunctions. Notice the eigenvalues associated with \( u(r), v(r) \) are not exactly negative each other, as the term \( \epsilon_\ell \) has the same sign in both lines of Eq. 5. Mathematically, this follows because of the complex conjugation of the diagonal term on the second row of Eq. 3. It is the manifestation of the breaking of time-reversal symmetry in the problem in the presence of a magnetic field: since the electron and the hole have opposite charges, they couple to the magnetic field with opposite signs.

### B. Andreev bound states

Following the original work of Kulik 33 we calculate the spectrum of the bound states of the nanowire \( SNS \) junction, however, here we allow the solutions to carry finite orbital angular momentum.

Suppose there is a solution \( \Psi(r) = (u(r), v(r))^T \) to Eq. 3 with energy \( E \) within the gap, \( |E| < \Delta_0 \). Since there are no barriers at the \( S-N \) interface, the right- and left-moving solutions \( \Psi^\pm \) can be separated. We disallow superpositions of \((n, l)\) subbands in the \( N \)-section. This is justified because: (i) we have assumed the ballistic regime, so no scattering-induced subband-mixing occurs, and (ii) the pairing potential, Eq. 2 is zero in the \( N \)-section, and so it does not mix the \((n, l)\) states (see Ref. 33). The quantum numbers \( n, l \) are thus good quantum numbers for the \( N \)-section.

In the \( S \)-sections, Cooper pairing can, and generally does, mix subbands with different \( n \) numbers [32], but the orbital angular momentum number \( l \) is still a good quantum number. The latter follows from the cylindrical symmetry of Eq. 3 which in turn follows from a cylindrically symmetric \( H_0 \) and a spatially uniform \( \Delta \). So, the radial eigensolutions \( \phi \) can be identified by a pair of numbers \((p, l)\), and are mixed transverse subbands with the same \( l \). At a given quantum number \( l \) and energy \( E \), the most generic single-particle wavefunction in the left (right) lead is given by \( e^{i \theta} \sum_p \beta_p e^{i k_p x} \phi_{p,l} := e^{i \theta} \psi_{l}(x, \rho) \). In each term of the sum, \( k_p \) adjusts itself such that the energy of that term is \( E \). There is freedom in the choice of the expansion coefficients \( \beta_p \), modulo normalization.

The wavefunction \( \Psi^\pm \) can be written as:

\[
\Psi_{n,l,E}^\pm = \begin{cases} 
A^\pm e^{i \theta} x^{\pm i k_0 \rho} \phi_{n,l}(\rho) & 1 \\
B^\pm e^{i \theta} x^{\pm i k_1 \rho} \phi_{n,l}(\rho) & 0 \\
C^\pm e^{i \theta} \psi_{l}^R(x, \rho) & 1 \\
D^\pm e^{i \theta} \psi_{l}^L(x, \rho) & 0 
\end{cases}, \quad |x| < L/2
\]

\[
\gamma^\pm \begin{cases} 
\gamma^+ \quad x > L/2 \\
\gamma^- \quad x < -L/2
\end{cases}
\]

Here, \( \gamma^+ = \Delta_0 \left( E + i \sqrt{\Delta_0^2 - E^2} \right)^{-1} \) is the BCS coherence factor in the leads, and \( \gamma^- \) is its complex conjugate. Since no barrier is assumed at the \( S-N \) interfaces, \( \Psi^\pm \) has to be continuous at \(|x| = L/2\). In order for this to be possible, we must have \( \psi_{l}^R(L/2, \rho) = \psi_{l}^L(-L/2, \rho) = \phi_{n,l}(\rho) \). In words, the coefficients \( \beta_p \) must be chosen such that the radial wavefunction in the \( S \) section is an expansion of \( \phi_{n,l} \) in the basis of the \((p, l)\) states. There is a very large number of \((p, l)\) states available in a metallic superconductor (for example in Nb, the effective electron mass is a factor \( \sim 50 \) larger than in InAs, so the spacing between the transverse subbands is smaller by about the same factor). It is therefore assumed that the expansion of \( \phi_{n,l} \) in terms of the \((p, l)\) states is possible, and the form given in Eq. 6 for \( \Psi^\pm \) is valid.

We now concentrate on the \( N \)-section, and derive the quantization rules for the energies of the bound states. Asserting that each term of \( \Psi^\pm \) in the \( N \)-section has energy \( E \), the wavenumbers \( k_0, k_1 \) are obtained as a func-
tion of energy:

\[
k_0(E) = \frac{\sqrt{2m^*}}{\hbar} \sqrt{\mu_{n,l} + E}, \quad (7a)
\]

\[
k_1(E) = \frac{\sqrt{2m^*}}{\hbar} \sqrt{\mu_{n,l} - E}, \quad (7b)
\]

where we have defined an effective chemical potential for an electron-like (hole-like) particle in the subband \((n,l)\) in the \(N\)-section \(\mu_{n,l} = \mu - \frac{\hbar^2}{2m^* R^2} (l \mp \Phi)^2 - \xi_{n,l}.\) The minus (plus) sign in the brackets refers to the electron-like (hole-like) particle, and \(\xi_{n,l}\) is the radial confinement energy due to \(\phi_{n,l}.\) The effective chemical potential is the difference between the energy of the subband \((n,l)\) and the Fermi energy at a given magnetic field (see figure 2), and is a positive quantity for any subband that is occupied. The energy quantization rules can be obtained \[35\] by finding the set of coefficients \(\{A^\pm, \ldots, D^\pm\}\) that make Eq. (6) continuous at \(|x| = L/2.\) This is only possible if the following relation holds:

\[
\gamma^2 e^{i(k_0 - k_1) L} e^{\mp i\chi} = 1, \quad (8)
\]

where \(\chi = \chi_R - \chi_L\) enters with a minus sign when for \(\Psi^+\), and a plus sign for \(\Psi^-\). The complex phase of the right-hand side of Eq. (8) must equal \(2m\pi,\) with \(m = 0, 1, 2,\) etc. Since \(k_0, k_1\) depend explicitly on the bound-state energy \(E\) (Eq. 7), this results in a quantization rule for \(E,\) and yields the bound-state spectrum. This procedure is carried out in section \[11\] to numerically solve for \(E\) at a given \(\chi.\)

Quite generally, if \(\Psi^+ = (u, v)^T\) is an eigensolution of the BdG equation (Eq. \[3\]) with energy \(E,\) then \(\Psi^- = (-v^*, u^*)^T\) is an eigensolution with energy \(-E\) \[30\]. Here, the superscript \(s\) denotes the right- and left-moving solutions, \(s = +, -\) respectively. Let \(s\) denote the conjugate of \(s.\) The wavefunctions \(\Psi^+\) and \(\Psi^-\) are degenerate at zero field, for all \(\chi\) (figure 3a). This degeneracy is lifted in the presence of the magnetic field, which induces a finite phase shift, as will be discussed below. The pair of right-moving solutions \((\Psi^+ , \Psi^-)\) are phase shifted together in one direction, while the opposite moving pair \((\Psi^-, \Psi^+)\) are phase shifted in the opposite direction (figure 3b).

In a short junction, \(L \ll \xi_{n,l}\), there is only one \(m\) value allowed per solution, and there are two or four bound states per subband \((n,l).\) At zero field, there are two positive, and two negative solutions at any given \(\chi\) (figure 3b). For long junctions there are more than four bound-state energies per subband, with different \(m\) numbers \[36, 40\].

C. Andreev Approximation and Reduction to a Semiclassical Model

Deriving an analytical expression for the bound-state spectrum by inserting Eq. (7) in Eq. (8) becomes intractable, because of the complicated dependence of \(k_0 - k_1\) on \(E.\) So, in order to gain insight about the behaviour of the bound-states, we invoke below the well-known Andreev approximation \[2, 3, 41\], in which \(k_0 - k_1\) is considered a small quantity compared to \(k_0\) and \(k_1.\) This approximation is widely used in the literature for a variety of situations \[35, 36, 41, 42\], but can be violated in nanoscale, semiconductor based SNS junctions. In particular, when the subband energy is close to the Fermi energy (the parabola is about to cross the upper dashed line in figure 2), \(k_0\) and \(k_1\) become small and the assumption \(k_0 - k_1 \ll k_0, k_1\) is not justified. Keeping these restrictions in mind, we look at how the CPR of the junction is modified in the presence of the axial magnetic field.

The effective chemical potential for electron-like (hole-like) particles in the subband \((n,l)\) in the \(N\) section can be written \(\mu_{n,l} = \mu - \frac{\hbar^2}{2m^* R^2} (l^2 + \Phi^2) \pm \xi_{n,l},\) where \(\xi_{l} = [\hbar^2/(2m^* R^2)] (2\Phi)\) enters with a plus sign for electron-like particles. It reflects the coupling of the orbital motion and the axial field. The Andreev approximations translates to the following condition: \(E + \xi_{l} \pm \mu_e, \mu_h, i.e.\) the quasi-particle energy and the coupling to the field are small perturbations on the single-particle energies. Eq. (7a, 7b) can be expanded in a Taylor series of the powers of \((E + \xi_{l}).\) We calculate \(k_0 - k_1\) to the first order:

\[
(k_0 - k_1) \approx \frac{2}{\hbar} \frac{E + \xi_{l}}{v_{n,l}}\]

where \(v_{n,l} = (2 \mu - \hbar^2/(2m^* R^2)(l^2 + \Phi^2) - \xi_{n,l}) / m^*\) is the velocity of a particle in the subband \((n,l)\) travelling along the cylinder axis in the \(N\)-section. By inserting \((k_0 - k_1)\) above into Eq. (8) and equating the complex phase of the left hand side of Eq. (8) with \(2m\pi,\) where \(m = 0, 1, 2,\) etc., we obtain the following expression for the spectrum of bound states:

\[
\left( \frac{L}{\xi_{n,l}} \right) \frac{E_{n,l,m}^\pm}{\Delta_0} - 2 \arccos \left( \frac{E_{n,l,m}^\pm}{\Delta_0} \right) \mp \chi + \left( \frac{L}{\xi_{n,l}} \right) \left( \frac{\epsilon_l}{\Delta_0} \right) = 2\pi m, \quad (10)
\]

where \(\xi_{n,l} = \hbar v_{n,l}/(2\Delta_0)\) is the healing length \[36\] for the subband \((n,l),\) and the energy of the bound state depends on three quantum numbers \(n, l, m,\) and the junction phase difference \(\chi.\) \(E^+(E^-)\) refers to the eigenenergy of the right-moving (left-moving) solution.

For the case of no magnetic field \((\xi_{l} = 0),\) we recover a result similar to the well known Andreev levels of a clean SNS junction \[35, 36\], but with different values of the healing length for the different subbands. An example is shown in figure 3b. For finite \(\xi_{l},\) we see in figure 3b that, in the subband \((n,l),\) the energy-versus-phase curves of the bound levels are phase shifted by an amount \(\delta_{n,l} = \left( \frac{L}{\xi_{n,l}} \right) \left( \frac{\xi_{l}}{\Delta_0} \right), \) i.e. \(E_{n,l,m}^\pm(\chi + \delta_{n,l})\).
This phase shift can understood semiclassically as the phase picked up by azimuthal travel around the axis of the cylinder in the presence of the magnetic field. In this picture, for a subband \((n,l)\) with \(l \neq 0\), the particles (both electron- and hole-like) travel in a spiral path as they traverse the junction length \(L\). In the shell-conduction model the spiral has radius \(R\). The velocity along the axis is \(v_{n,l}\), while the azimuthal velocity is \(v_\theta(l) = \hbar l/(m^*R)\). The semiclassical phase \(\delta_{sc}\) is due to the coupling of \(v_\theta\) and the vector potential \(\mathbf{A} = (B_0\rho/2\theta)\), and is calculated from the Ginzburg-Landau formula for the phase:

\[
\delta_{sc} = (2e/\hbar) \int \mathbf{A} \cdot d\mathbf{l} = (2e/\hbar) \int \mathbf{A} \cdot \mathbf{v} dt, \tag{11}
\]

where the differential element \(d\mathbf{l}\) is along the spiral path, \(\mathbf{v} = v_{n,l} \hat{x} + v_\theta \hat{\theta}\) is the velocity, \(t\) is time, and the second integral is taken from \(t = 0\) corresponding to the particle leaving one \(S\)-section, to \(t = L/v_{n,l}\), when it arrives at the other \(S\)-section. The result is \(\delta_{sc} = eL B_0/(m^*v_{n,l})\), which equals \(\delta_{n,l}\). This confirms that the quantum-mechanical description of the phase shifts can be reduced to a simpler semiclassical picture, assuming shell-conduction and the Andreev approximation. Note in the expression for \(\delta_{sc}\) there is no explicit dependence on \(R\). The dependence of the phase shift on the radius comes through \(v_{n,l}\), and is a weak effect in the Andreev approximation.

Shifts in the energy-versus-phase curves are observed in the general quantum-mechanical picture, when the spectrum is calculated by considering Eq. 7 in its full form, without the Andreev approximation. The phase shifts are proportional to the length of the junction \(L\), and the angular momentum quantum number \(l\). The exact value needs to be calculated numerically, and reduces to the analytical value \(\delta_{n,l}\) when the Andreev approximation holds.

D. Bound-state and continuum currents

The current due to the Andreev bound states in the subband \((n,l)\) at temperature \(T\) is calculated \cite{30,39,43} from the formula

\[
I_{n,l}(\chi) = \frac{e}{h} \sum_{s,m} f(E^s_{n,l,m}) \frac{dE^s_{n,l,m}(\chi)}{d\chi}, \tag{12}
\]

where \(f(E^s_{n,l,m}) = 1/(\exp(E^s_{n,l,m}/(k_B T)) + 1)\) is the Fermi-Dirac occupation probability of a given energy level \((k_B\) is the Boltzmann constant). Energies corresponding to both types of wavefunctions \(\Psi_{n,l}^+\) and \(\Psi_{n,l}^-\) must be inserted into Eq. 12. The total bound-state current is

\[
I(\chi) = \sum_{n,l} I_{n,l}(\chi). \tag{13}
\]

Equation 13 is an incoherent sum of the contribution of each subband \((n,l)\) to the bound-state current. This is justified because we expect the mesoscopic charge transport across the SNS junction to be the result of a probabilistic mixture of subbands occupied according to the Fermi-Dirac distribution. Indeed, we have assumed here the ballistic regime (no scattering-induced subband-mixing), and zero a pairing potential in the \(N\) section (no

![FIG. 3. Eigenenergies of the Andreev bound states, and the supercurrents of a short (\(L = 25\) nm), cylindrical SNS junction with no barriers at the S-N interfaces, vs the superconducting phase difference \(\chi\) of the Nb leads. Two values of reduced magnetic flux, \(\Phi = \pi R^2 B_0/(h/e) = 0, 2.5\), are shown. We concentrate on one subband \((n,l)\) with \(n = 0\) and \(l = 1\). a) Zero magnetic field, \(\Phi = 0\). The energies correspond to the four allowed wavefunctions (defined in the main text), and the states are pairwise degenerate for all \(\chi\). b) \(\Phi = 2.5\), where the degeneracies are lifted, and there is a phase shift, with opposite directions for the left- and right-moving states. The phase shift is small because of the short length of the junction. c,d) Bound state current \(I_{n,l}\), continuum current \(J_{n,l}\), and the sum \(I_{n,l} + J_{n,l}\) for the subband. Since the junction is short, \(L = 25\) nm \(< \xi_0\), \(\sim 200\) nm, the bound state current dominates over the continuum current. c) Zero magnetic field, \(\Phi = 0\). The sum \(I_{n,l} + J_{n,l}\) is maximal at \(\chi = \pi\). d) \(\Phi = 2.5\). The bound bound state and continuum currents are phase shifted. The small jumps in the variances in \(I_{n,l}, J_{n,l}\) at \(\chi = 0.05, 0.95\) are due to bound Andreev levels crossing the gap edge into the continuum levels, i.e. the number of \(m\) values for which eigenenergies \(E_{n,l,m}^s\) exist changes by one. Two discontinuities develop in \(I_{n,l} + J_{n,l}\) because of the phase shifts, and the maximal value no longer occurs at \(\chi = \pi\). These parameters were used for all panels: \(\mu = 200\) meV, \(R = 30\) nm, \(T = 100\) mK.]
subband-mixing due to Cooper pairing).

The continuous spectrum of states with energies $|E| > \Delta_0$ also contributes to the junction current. A continuum level can be viewed as a “leaky” solution 46 47 to the Andreev bound state problem described above, with a complex-valued eigenenergy $E = E_R + iE_I$, with $|E_R| > \Delta_0$. The leaky level follows the same subband structure as the Andreev bound states. The imaginary component of energy results in a finite lifetime for the continuum level, reducing its contribution to the junction current, but for a long Junction $L > \xi_{n,l}$ this contribution is significant, and cannot be ignored 36.

The continuum current due to the subband $(n,l)$, $J_{n,l}(\chi)$, is calculated using the transmission formalism 38 40 44. We calculate the transmission coefficient for the electrical currents carried by electron-like and hole-like excitations incident on the S-N interfaces, resulting in four types of leaky solutions in the N-section, $\Psi^+, \Psi^-, \bar{\Psi}^+, \bar{\Psi}^-$. The calculation is based on that given in Ref. 36, but generalized so the transmission coefficient of each type of wavefunction is considered separately. These coefficients can then be used to calculate the continuum current. The result is the following formula for $J_{n,l}(\chi)$:

$$J_{n,l}(\chi) = \frac{e}{\hbar} \left( \int_{-\Delta_0}^{\Delta_0} + \int_{-\infty}^{\infty} \right) \left| u_0^2 - v_0^2 \right| \left( \frac{1}{D^+(E, -\chi)} - \frac{1}{D^-(E, -\chi)} - \frac{1}{D^+(E, +\chi)} + \frac{1}{D^-(E, +\chi)} \right) f(E) dE,$$

(14)

where $E$ is the energy of the continuum level and $f(E)$ is the Fermi-Dirac distribution at temperature $T$, and $u_0, v_0$ are real-valued BCS coherence factors:

$$u_0^2 = \frac{1}{2} \left( 1 + \sqrt{E^2 - \Delta_0^2/E} \right),$$

(15a)

$$v_0^2 = \frac{1}{2} \left( 1 - \sqrt{E^2 - \Delta_0^2/E} \right).$$

(15b)

The functions $D^\pm(E, \chi)$ depend on energy, through the wavenumbers $k_0$ and $k_1$ (Eq. 7) as well as the coherence factors $u_0, v_0$: $D^\pm(E, \chi) = u_0^2 + v_0^2 - 2u_0v_0\cos\left[(k_0(\pm E) - k_1(\pm E)L + \chi)\right]$. Eq. 14 can be intuitively understood in terms of the leaky solutions to the BdG equation: the terms containing $D^+$ pertain to the contribution of leaky states of type $\Psi^+, \bar{\Psi}^+$, while those containing $D^-$ pertain to $\bar{\Psi}^+, \Psi^-$. The phase difference $\chi$ enters with a plus (minus) sign for left (right) moving solutions. At zero magnetic field, $D^+(E, \chi) = D^-(E, -\chi)$. This is analogous to the degeneracy of $\Psi^+, \bar{\Psi}^+$ in figure 3a. Eq. 14 reduces to Eq. 17 in Ref. 36, up to an application of the Andreev approximation.

In the presence of the magnetic field, the terms $(k_0 - k_1)L$ shift the functions $D^\pm(E, \chi)$ in phase relative to the zero-field case, in the same manner as the bound-state energy-versus-phase curves. As an example, let us consider the subband $(n,l)$ and employ the Andreev approximation (Eq. 9). We obtain

$$D^\pm(E, \chi) = u_0^4 + v_0^4 - 2u_0^2v_0^2\cos\left[\left(\frac{E \pm \xi_l}{\Delta_0}\right) \left( \frac{L}{\xi_{n,l}} \right) + \chi \right],$$

(16)

which is shifted in phase with respect to the zero-field case: $D^\pm(E, \chi) \rightarrow D^\pm(E, \chi \pm \delta_{n,l})$, with $\delta_{n,l}$ defined above.

The total continuum current of the junction is

$$J(\chi) = \sum_{n,l} J_{n,l}(\chi).$$

(17)

The critical current $I_c$ of the junction is defined as the maximum of total bound state + continuum currents with respect to $\chi$:

$$I_c = \max_{\chi \in [0, 2\pi]} [I(\chi) + J(\chi)].$$

(18)

III. NUMERICAL RESULTS

We numerically solve the continuum and bound state currents of an SNS junction at finite magnetic fields, using the shell conduction approximation with the shell at a radius $R = 30$ nm, the temperature is $T = 100$ mK. In the N-section, only $(n,l)$ subbands with $n = 0$ are assumed to be occupied. The Andreev approximation is not used in calculating the CPR. The critical current of the junction is calculated from Eq. 18, and its behaviour versus axial magnetic flux $\Phi = \pi R^2 B_{||}/(\hbar/e)$ is studied.

Let us first discuss the behaviour of the obtained CPR for one subband. The bound state current $I_{n,l}$ and the continuum current $J_{n,l}$ of a subband $(n,l)$ with $l = 1$ are shown in figure 3b-d. The chemical potential is $\mu = 20$ meV and the junction length $L = 25$ nm, much shorter than the subband’s healing length $\xi_{0,n,l} \sim 200$ nm. In this short junction limit, the bound state current $I_{n,l}$ dominates over $J_{n,l}$. For a long junction ($L \gg \xi_{0,n,l}$), $I_{n,l}$ and $J_{n,l}$ would be the same order of magnitude, and the CPR takes on the well known 36 37 triangular shape. At zero magnetic field (figure 3a), the sum $I_{n,l} + J_{n,l}$ is always maximal at $\chi = \pi$, regardless of the junction length 36 (the continuum current is zero at $\chi = \pi$). When multiple subbands are occupied, the critical current of the junction at zero field occurs at $\chi = \pi$, due to the maximal contributions from all subbands.

At a finite magnetic flux $\Phi$, two discontinuities develop in $I_{n,l}$, as shown in figure 3. This is due to the eigenenergies corresponding to $(\Psi^+, \bar{\Psi}^+)$ being shifted in phase in the opposite direction to those of $(\Psi^-, \bar{\Psi}^-)$, see figure 3. A similar process happens for the continuum current: in Eq. 14 the terms containing $D^+(E, \chi), D^+(E, -\chi)$ are shifted in phase in one direction, while the other two terms are shifted in the opposite direction. The sum $I_{n,l} + J_{n,l}$ is no longer necessarily maximal at $\chi = \pi$. The amounts of the phase shifts in $I_{n,l}$ and $J_{n,l}$ depend
FIG. 4. Normalized critical current of the junction versus the reduced magnetic flux $\Phi = (\pi B_0 R^2)/(h/e)$, for $\mu = 20$ meV (panels a,b,c) and $\mu = 100$ meV (panels d,e,f), and several values of the junction length $L$. The following parameters were used: $R = 30$ nm, $T = 100$ mK. In panels a,b,c, the vertical lines at $\Phi = 0.2, 1.2, 2.2$ indicate the depopulation of the $|l| = 3, 2, 1$ subbands, respectively. Similarly in panels d,e,f, the vertical lines indicate the depopulation of the $|l| = 7, 6, 5$ subbands.

on $k_0, k_1$, which are subband parameters, and well as the junction length $L$. The total current of the junction is modulated as $\Phi$ is increased, since each subband contributes with a different amount of phase shift. This carries over to modulations of the critical current $I_c$ versus $\Phi$, which we identify as a new type of the Josephson interference effect: the CPR of a bound state or continuum Andreev quasiparticle in a given subband is modulated due to the coupling of its orbital angular momentum with the axial magnetic flux, resulting in oscillations in the critical current which need not be periodic in either $\Phi_0 = (h/e)$ or $\Phi_0/2$, since the underlying phase shifts depend on $(k_0 - k_1)L$ which can take on any value.

Figure 4 shows the numerically obtained $I_c$ versus $\Phi$ for several values of the chemical potential $\mu$ and junction length $L$. The critical current is normalized to its value at zero magnetic flux. Two processes affect the behaviour of $I_c$: (i) the depopulation of subbands as $\Phi$ is increased, and (ii) the Josephson interference effect described above. We identify regimes in which each effect dominates. Let us consider the case of the lower chemical potential first. In the left column, $\mu = 20$ meV. The subbands $(n,l)$ with $n = 0$ and $|l| \leq 3$ are occupied at zero field. Figure 4a corresponds to a short junction with $L = 25$ nm. The quasiparticles in subbands $|l| = 3, 2, 1$ depopulate at $\Phi = 0.2, 1.2, 2.2$ respectively (vertical dashed lines in figure 4 panels a,b,c). This can be seen as a step-like discontinuity in $I_c$ at those flux values. The $I_c$ vs. $\Phi$ curve is dominated by the effect of the depopulation of subbands. The phase shifts in the CPR of each subband are generally small (because of the short length of the junction), but become more significant close to the field at which the subband depopulates, where the axial velocities of the quasiparticles are smaller and their time of flight across the junction longer. This results in the observed decrease in $I_c$ before each step-like discontinuity. In figure 4b, the length is $L = 50$ nm, so the phase shift in the CPR of each subband is larger. This results in a modulation of $I_c$ through the interference mechanism described above, on top of the subband-depopulation effect. Most notably, the peak in $I_c$ at $\Phi = 1$ and the dip at $\Phi = 1.65$ are due to this interference effect. As the length of the junction is made even larger, $L = 200$ nm in panel e, the oscillations due to the interference effect dominate the $I_c$ versus $\Phi$ characteristics.

In the right column, $\mu = 100$ meV. Subbands with $|l|$ up to 7 are occupied at zero field. The depopulation of the $|l| = 7, 6, 5$ subbands occur at $\Phi = 0.34, 1.34, 2.34$, respectively (vertical dashed lines in figure 4 panels d,e,f). A common feature at this higher value for $\mu$ is the decrease of $I_c$ to about 20% of its initial value as the field is increased. The main cause of this is not the depopulation of subbands, but the interference effect. At zero field, all subbands contribute maximal currents at $\chi = \pi$. This condition for a “constructive interference” of subband currents is broken at non-zero field, where the phase at which a subband contributes maximal current depends on $|l|$. This lack of constructive interference reduces $I_c$. For a larger $\mu$, more subbands are constructively interfering at $\Phi = 0$, and the initial decrease of $I_c$ versus $\Phi$ becomes more visible. Since the phase shifts are proportional to the junction length $L$, the initial decrease is more rapid in $\Phi$ for $L = 50$ nm (figure 4c) than $L = 25$ nm (figure 4b); and even more rapid for $L = 200$ nm (figure 4f). Oscillations in $I_c$, smaller in amplitude than the low-$\mu$ case, can be observed at higher fields. These occur at $\Phi > 2$ for $L = 25$ nm. As the junction length is increased, the oscillations become progressively more prominent, occur at lower fields, and have shorter periods in $\Phi$. At $L = 200$ nm, rapid oscillations of $I_c$ are observed for $\Phi > 0.25$.

In summary, for a short junction with a low chemical potential (only a few transverse subbands occupied, e.g. an SNS point-contact [31, 45, 46]), the depopulation of subbands is more visible than the interference effect. However, as the junction length is increased, oscillation...
of \( I_c \) due to the interference effect become more visible. A long junction with a low chemical potential is therefore the optimal experimental device for observing the predicted oscillations of \( I_c \). At a high chemical potential (many transverse subbands occupied), interference causes a large decrease in \( I_c \) at low fields. Oscillations of \( I_c \) follow at higher fields, albeit with smaller amplitudes relative to the zero-field \( I_c \) than the low-\( \mu \) case.

IV. DISCUSSION

We have described the theory of a new form of the Josephson interference effect that can occur in nanoscale SNS junctions due to the coupling of the orbital angular momentum of the transverse subbands with an axial magnetic flux. We found in section III the regimes in which this interference effect dominates \( I_c \) vs \( \Phi \) characteristics of the junction. An idealized model of an SNS junction was used, with several simplifying assumptions. We discuss below these assumptions, and the applicability of this model to experimental situations.

First, a spatially uniform pairing potential \( \Delta \) was assumed in Eq. 2 at all magnetic fields. We justified this by assuming a cylindrical S-section, and restricting the cylinder diameter to be smaller than the superconducting coherence length in the S-section. However, experimental fabrication of nanoscale SNS junctions is usually done by evaporating or sputtering metallic (e.g. Al or Nb) thin film contacts onto a semiconductor nanowire. In this case, the geometry of the S-sections are not cylindrical but \( \Omega \)-shaped. The lack of cylindrical symmetry necessitates, in principle, a 3-dimensional numerical calculation of \( \Delta(r) \) using self-consistent methods. However, as long as the N-section can be assumed to be cylindrically symmetric, our model approximates the experimental situation. This is because the interference effect depends mainly on the eigensolutions in the N-section, particularly the orbital angular momentum states and their coupling to the flux. The details of the eigensolutions in the S-section do not play a direct role, other than asserting the form of the wavefunction ansatz (Eq. 6) is valid. On the other hand, the axial field can induce a non-uniformity in \( \Delta \). If the thickness of the metallic film is larger than its coherence length, fluxoid quantization can result in a \( \theta \)-dependent phase for \( \Delta \) in the presence of the field. Inserting a \( \theta \)-dependent \( \Delta \) in the BdG equations (Eq. 3) will affect the bound state solutions, requiring a 3-dimensional numerical solution. However, the qualitative behaviour of the junction is expected to be similar as long as the N-section is cylindrical.

The shell-conduction model used throughout our calculations was invoked in order to simplify computations, and to help gain intuitive insight into the problem; it is not strictly necessary for the main arguments of the paper. In future work, the radial wavefunctions in the N-section, \( \phi_{n,l} \), and the corresponding single particle energies in the presence of the field can be numerically calculated, yielding the appropriate wavenumbers \( k_0, k_1 \) for electron- and hole-like solutions. The term \((k_0-k_1)L\) appearing in Eqs. 8, 14 will continue to result in phase shifts. It is possible, however, that the radial eigenfunctions could be concentrated in the center of the cylinder (e.g. upward surface band bending). In that case, the phase shifts are expected to be small.

Spin-orbit and Zeeman effects in the InAs N-section were neglected in our model in order to unambiguously isolate the Josephson interference effect due to orbital angular momentum. In semiconductor based SNS junctions, both spin-orbit and Zeeman effects are also predicted to result in oscillations of \( I_c \) vs magnetic flux [25]. In a junction based on InAs or InSb for example, with a large Landé \( g \)-factor and strong spin-orbit coupling, these oscillations should be visible in conjunction with the interference effect studied here. Whereas the spin-orbit + Zeeman oscillations are predicted to be periodic in \( \Phi_0/2 = h/(2e) \), the Josephson oscillations are not, and can occur at much lower fields. Answering the question “which effect is more prominent?” depends on the detailed device parameters.

The inclusion of barriers at the S-N interfaces is the main necessity for a realistic description of semiconductor junctions, because of the large Fermi velocity mismatch between the semiconductor and the metallic superconductor. The CPR of a junction with barriers has been previously calculated at zero magnetic field [16, 47]. We expect the Josephson interference effect to be present when barriers are included. However, when barriers are present, the shape of the CPR resemble a sinusoidal curve [36, 47], and therefore the shape of the oscillations in \( I_c \) might appear more sinusoidal. Furthermore, normal reflection of the quasiparticle wavefunctions from the S-N barriers can result in the quasiparticle spending more time in the junction region [35, 14], accumulating greater phase shifts, leading to the interference effect at lower fields. These considerations are beyond the scope of this paper, and are left to future work.

The idealized model studied here serves to demonstrate the existence of a novel form of Josephson interference. Extensions to the model discussed above, most importantly barriers at the S-N interfaces, can pave the way toward a complete theoretical understanding of experimentally relevant junctions.

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