Seebeck and Nernst coefficients of a magnetized hot QCD medium with number conserving kernel

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Abstract

We study the thermoelectric response of a hot and magnetized QCD medium created in the noncentral events at heavy-ion collider experiments. The collisional aspects of the medium have been embedded in the relativistic Boltzmann transport equation (RBTE) using Bhatnagar-Gross-Krook (BGK) collision integral, which insures the particle number conservation, unlike the commonly used relaxation time approximation (RTA). We have incorporated the thermal medium effects in the guise of a quasiparticle model, where the interaction among the quarks and gluons is assimilated in the medium dependent masses of the quarks, which have been evaluated using imaginary-time formalism of thermal QCD with a background magnetic field. In the absence of $B$, the Seebeck coefficient for individual quark flavors gets slightly reduced in the BGK term in comparison to naive RTA, while it gets enhanced for the composite partonic medium. In the strong magnetic field ($B$), the BGK term enhances the Seebeck coefficient for the individual flavors as well as that for medium. The medium Seebeck coefficient rises with the strength of quark chemical potential ($\mu$) in the absence as well as that in the strong $B$. We observe chirality dependence in the transport coefficients in the weak $B$ as the masses of chiral modes become nondegenerate. In the case of the $L$ modes, the BGK collision term causes slight reduction in the Seebeck coefficient, while for $R$ modes both the collision integral produces same results. Nernst coefficient gets reduced (enhanced) for $L$ ($R$) chiral modes in the BGK term.

1 Introduction

A transition from the hadronic matter to a deconfined phase of quark gluon plasma (QGP) takes place in heavy-ion collision experiments at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC). In non-central collisions, a magnetic field (around $m_\pi^2$ at RHIC [1] and

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$15m^2$ at LHC [2]) is also produced, which persists in the medium for a considerable amount of time due to the finite electrical conductivity of the medium. This magnetic field leads to the modification in the thermodynamical [3, 4] and transport properties [5–11] of the hot and dense quark matter and also induces novel phenomena such as the chiral magnetic effect [1, 12], magnetic and inverse magnetic catalysis [13–16], axial magnetic effect [17, 18], chiral vortical effect in rotating QGP [19, 20], the conformal anomaly and production of soft photons [21, 22]. In addition to this, the dilepton production rate [23–25], dispersion relations [26], refractive indices and decay constants [21, 27] have been explored in the magnetic field background.

Transport coefficients are crucial input parameters needed in the dissipative hydrodynamics and transport simulation to describe the evolution of the partonic medium created post-collision at RHIC and LHC. Shear viscosity quantifies the response of the medium to the transverse momentum gradients while bulk viscosity to the pressure gradients. Both shear and bulk viscosities have been studied in the magnetic field extensively in different models [5, 7–10]. Electrical and thermal conductivities measure the response of the system to the electromagnetic fields and thermal gradients in the medium, respectively. Electrical conductivity plays an important role in the elongation of the lifetime of the magnetic field created in non-central collisions, while thermal conductivity controls the attenuation of sound through the Prandtl number. Both electrical and thermal conductivities have been extensively studied in phenomenological models as well as using perturbative methods [6, 28–33]. On the other hand, the transport coefficient corresponding to the thermoelectric response is known as Seebeck coefficient, which measures the ability of any material to convert the thermal gradient into the electric current. Thermoelectric properties of the materials have been mainly studied in the context of condensed matter physics over the years. There have been numerous studies regarding the thermoelectric properties of the various condensed matter systems such as superconductors [34–37], the graphene superconductor junction [38], correlated quantum dots coupled to superconducting electrode [39], high temperature cuprates [40], ferromagnet-superconductor hybrid junction [41] and low dimensional correlated organic metals [42].

We have recently explored the charge, heat, and momentum transport coefficients [43, 44]. Motivated by our earlier studies, we are now interested in thermoelectric effects in the strongly interacting matter produced in the heavy-ion collisions where a thermal gradient is present between the central and peripheral regions of the fireball. In addition to the temperature gradient, a finite baryon chemical potential is also needed to observe the thermoelectric effect in strongly interacting matter unlike the condensed matter systems, where only one type of the charge carriers participate in the transport process. Contrary to that, in strongly interacting medium, both positive and negative charge carriers take part in the transport phenomena. In the absence of the quark chemical potential, both particles and anti-particles have equal numbers, so no net thermoelectric effect is observed. The Seebeck effect in the absence of a magnetic field has been studied recently for a hot hadron gas in hadron resonance gas model [45] and for the QGP phase in the ambit of Nambu–Jona Lasinio (NJL) model [46]. In the magnetic field background, the thermoelectric response of the hot QCD medium has been explored earlier in [47–52]. In the earlier works, authors have used the relativistic kinetic theory approach, where the collisional effects of the medium have been incorporated with RTA. But, the widely used RTA collision integral has a drawback that it violates the conservation of the particle number and current. Taking this fact into consideration, we have used a more realistic BGK-type collision integral, which insures the particle number and
current conservation in the medium. The BGK collision integral has been used earlier to study dielectric functions, dispersion relations and damping rates of longitudinal and transverse modes of a photon in the electromagnetic plasma [53]. The authors noticed a small shift in the dispersion relations towards the lower energies for the collisional case in comparison to collisionless case. Schenke et al. [54] have studied the effects of the collisions using the BGK kernel on the collective modes of a gluon in the anisotropic QCD medium and have observed that incorporation of BGK collision integral slows down the growth of the unstable modes. The gluonic collective modes have been also studied in anisotropic medium within the effective fugacity model [55] and suppression of the instabilities were reported there also. The effects of the collision has been investigated using BGK term on the square of the refractive index ($n^2$) and Depine-Lakhtakia index ($n_{DL}$) for the QGP medium in ref. [56]. It was noticed that the real and imaginary parts of the $n^2$ gets changed dramatically compared to collisionless case. For a small collision rate, $n_{DL}$ becomes negative in certain frequency range and as the collision rate increases, the frequency range for $n_{DL} < 0$ becomes narrower. The wakes phenomenon has been explored for both isotropic [57] as well as anisotropic medium [58] and it is observed that the wake structure becomes less pronounced in both the cases in comparison to the collisionless plasma. The effect of collisions on the heavy quark energy loss has been investigated via BGK kernel and it is found that for the same momentum and collision frequency, energy loss gets increased in the BGK case in comparison to the collisionless case for both charm and bottom quarks and further increases as the collision rate is increased [59]. Authors in [60] perform similar study using the effective fugasity model and considering both RTA as well as BGK collision terms. They observed that the energy loss gets reduced in the BGK case as compared to RTA. In addition to these works, the response of stationary and homogenous quark gluon plasma to the background electromagnetic field has been studied in [61]. It was found that the latetime magnetic field is mainly determined by the static electrical conductivity of the medium. A similar kind of study was made for the electron positron plasma with time and space dependent magnetic fields [62]. The electric charge transport in a weakly magnetized hot QCD medium in the presence of time varying electric field has been investigated in [63]. Both Ohmic and Hall conductivities get enhanced in the BGK term as compared to RTA. Similar observations were noticed in the strong magnetic field, where longitudinal electrical conductivity becomes larger in the BGK term [43]. The momentum transport coefficients have been studied in strong magnetic field with BGK kernel by us in ref. [44] and we notice that the shear viscosity gets enhanced while bulk viscosity is reduced slightly in comparison to RTA.

Motivated by the earlier works, our main objective here is to investigate how current conserving BGK collision integral modifies the thermoelectric transport coefficients namely Seebeck and Nernst coefficients of the hot QCD medium. We include the medium effects in the framework of a quasi-particle model [64], where medium dependence enters through the dispersion relations of the quark and gluon quasi-particles. Quasi-particle models are widely used to study the thermodynamical and transport properties of the hot QCD medium. The masses of the quarks have been computed from the pole of the propagator resummed using the Dyson-Schwinger equation. We have employed the perturbative thermal QCD in magnetic field background to calculate the self-energy of the quark. We compare the BGK results with those obtained using RTA. We have explored two regimes of the magnetic field, the strong ($|q_iB| \gg T^2 \gg m_i^2$) and the weak magnetic field regime ($T^2 \gg |q_iB| \gg m_i^2$). In the magnetic field, the motion of the quarks is quantized in the transverse direction leading to the discrete energy spectrum in terms of the Landau levels. When the strength of the magnetic field is large, the energy separation between the consecutive
Landau levels become large (of the order $\sqrt{|qB|}$), consequently, the quarks get confined in the lowest Landau level (LLL) only. Moreover, in the weak magnetic field case, the magnetic field dependence enters through the cyclotron frequency. We found that in the absence of the magnetic field, the magnitude of the Seebeck coefficient for the individual $u$, $d$ and $s$ quarks gets reduced in the BGK collision integral while for the composite medium, it gets enhanced. In the strong $B$, it gets enhanced in BGK collision term for individual flavors as well as for medium. In case of the weak magnetic field, the Seebeck coefficient is not much sensitive to the collision integral and found to be almost similar in both collision terms. In addition, a hall type Nernst coefficient also appears, which quantifies the thermoelectric response in the transverse direction. Nernst coefficient gets reduced (enhanced) in the BGK term in the case of $L$ ($R$) modes for individual flavors as well as for the medium.

The present manuscript has been organized as follows: In section 2, we have discussed the quasi-particle model and thermal mass of the quarks in the thermal and magnetic field background obtained using the perturbative thermal QCD. In subsection 3.1 and 3.2, we have calculated the Seebeck coefficient without and with magnetic field background, respectively. We discussed the results in section 4 and finally we conclude in section 5.

## 2 Quasi-particle model

The central feature of quasiparticle models is that a strongly interacting system of massless quarks and gluons can be described in terms of the massive, weakly interacting quasiparticles originated due to the collective excitations in the medium. There are many quasi-particle models such as NJL and PNJL models [65–68], which are based on the respective effective QCD models, effective fugacity model [69] and model based on the Gribov–Zwanziger approach [70–72]. Such a kind of model was first proposed by Goloviznin and Satz [73] to study the gluonic plasma and then by Peshier et al. [74, 75] to study the equation of state of QGP obtained from lattice QCD at finite temperature. At the same time, authors in Refs. [76–79] used quasi-particle picture to explain the lattice data by using a suitable quasiparticle description for QGP with temperature and chemical potential dependent masses. These results suggest that the high-temperature QGP phase is suitably described by a thermodynamically consistent quasiparticle model. In the present study, we have used quasiparticle model by Bannur [64] where the total effective mass of the $i^{th}$ quark flavor with bare quark mass $m_{i,0}$ has been parametrized as [64,80,81]

$$m_i^2 = m_{i,0}^2 + \sqrt{2}m_{i,0}m_{i,T} + m_{i,T}^2,$$

(1)

to explain the lattice data with finite bare quark masses. The thermal mass ($m_{i,T}$) of the quark in Eq. (1) can be calculated using the HTL perturbation theory as [82]

$$m_{i,T}^2 = \frac{g'^2 T^2}{6} \left( 1 + \frac{\mu^2}{\pi^2 T^2} \right),$$

(2)

where $g' = \sqrt{4\pi \alpha_s}$ refers to the coupling constant which depends on the temperature as

$$\alpha_s(T) = \frac{g'^2}{4\pi} = \frac{6\pi}{(33 - 2N_f) \ln \left( \frac{Q}{\Lambda_{QCD}} \right)},$$

(3)
and $Q$ is set at $2\pi \sqrt{T^2 + \frac{E^2}{\pi^2}}$.

Now, we will include the strong magnetic field in the quasiparticle description. The quasiparticle mass in the presence of strong $B$ can be generalized as

$$m^2_{i,s} = m^2_{i,0} + \sqrt{2} m_0 m_{i,B,T} + m^2_{i,B,T},$$

where $m_{i,B,T}$ is the medium dependent quark mass, which is obtained from the pole the resummed propagator. We know from Dyson-Schwinger equation

$$S^{-1}(p_\parallel) = \gamma^\mu p_{\parallel\mu} - \Sigma(p_\parallel),$$

where $\Sigma(p_\parallel)$ refer to the self-energy of the quark at finite $T$ and strong $B$, which has been calculated as [83]

$$\Sigma(p_\parallel) = \frac{g^2 |q_i B|}{3\pi^2} \left[ \frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2 \zeta(3)}{8\pi^2 T^2} - \frac{31\mu^4 \zeta(5)}{32\pi^4 T^4} \right] \left[ \frac{\gamma^0 p_0}{p_\parallel^2} + \frac{\gamma^3 p_z}{p_\parallel^2} + \frac{\gamma^0 \gamma^3 p_z}{p_\parallel^2} + \frac{\gamma^3 \gamma^5 p_0}{p_\parallel^2} \right], (6)$$

where $g = \sqrt{3\pi \alpha_s}$ is the running coupling which depends on $T$, $B$ and $\mu$ as

$$\alpha_s(\Lambda^2, eB) = \frac{g^2}{4\pi} = \frac{\alpha_s(\Lambda^2)}{1 + b_1 \alpha_s(\Lambda^2) \ln(\frac{\Lambda^2}{\Lambda_{\overline{MS}}^2})},$$

with

$$\alpha_s(\Lambda^2) = \frac{1}{b_1 \ln(\frac{\Lambda^2}{\Lambda_{\overline{MS}}^2})}, \quad (8)$$

and $\Lambda$ is set at $2\pi \sqrt{T^2 + \frac{E^2}{\pi^2}}$ for quarks, $b_1 = \frac{11N_c - 2N_f}{12\pi}$ and $\Lambda_{\overline{MS}} = 0.176$ GeV.

Due to the heat bath and magnetic field, the Lorentz (boost) and rotational invariance of the system get broken. In such a non-trivial background, the covariant form of the quark-self energy $\Sigma(p_\parallel)$ can be written as [4, 84]

$$\Sigma(p_\parallel) = A_1 \gamma^\mu u_\mu + A_2 \gamma^\mu b_\mu + A_3 \gamma^5 \gamma^\mu u_\mu + A_4 \gamma^5 \gamma^\mu b_\mu. \quad (9)$$

Here $u^\mu(1, 0, 0, 0)$ and $b^\mu(0, 0, 0, -1)$ correspond to the heat bath and the magnetic field, respectively. $A_1$, $A_2$, $A_3$ and $A_4$ refer to the structure functions, which are given in the LLL approximation as [83]

$$A_1 = \frac{1}{4} \text{Tr}[\Sigma \gamma^\mu u_\mu] = \frac{g^2 |q_i B|}{3\pi^2} \left[ \frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2 \zeta(3)}{8\pi^2 T^2} - \frac{31\mu^4 \zeta(5)}{32\pi^4 T^4} \right] \frac{p_0}{p_\parallel^2}, \quad (10)$$

$$A_2 = -\frac{1}{4} \text{Tr}[\Sigma \gamma^\mu b_\mu] = \frac{g^2 |q_i B|}{3\pi^2} \left[ \frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2 \zeta(3)}{8\pi^2 T^2} - \frac{31\mu^4 \zeta(5)}{32\pi^4 T^4} \right] \frac{p_z}{p_\parallel^2}, \quad (11)$$

$$A_3 = \frac{1}{4} \text{Tr}[\gamma^5 \Sigma \gamma^\mu u_\mu] = -\frac{g^2 |q_i B|}{3\pi^2} \left[ \frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2 \zeta(3)}{8\pi^2 T^2} - \frac{31\mu^4 \zeta(5)}{32\pi^4 T^4} \right] \frac{p_0}{p_\parallel^2}, \quad (12)$$

$$A_4 = -\frac{1}{4} \text{Tr}[\gamma^5 \Sigma \gamma^\mu b_\mu] = -\frac{g^2 |q_i B|}{3\pi^2} \left[ \frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2 \zeta(3)}{8\pi^2 T^2} - \frac{31\mu^4 \zeta(5)}{32\pi^4 T^4} \right] \frac{p_z}{p_\parallel^2}, \quad (13)$$

5
where $\zeta(3)$ and $\zeta(5)$ correspond to the Riemann zeta functions. We can further cast the quark self-energy (9) using the chirality projection operators as

$$\Sigma(p_\parallel) = P_R[(A_1 - A_2)\gamma^\mu u_\mu + (A_2 - A_1)\gamma^\mu b_\mu]P_L + P_L[(A_1 + A_2)\gamma^\mu u_\mu + (A_2 + A_1)\gamma^\mu b_\mu]P_R,$$

where $P_R$ and $P_L$ are the right- and left-handed chiral projection operators, respectively,

$$P_R = \frac{(1 + \gamma^5)}{2} \quad (15)$$
$$P_L = \frac{(1 - \gamma^5)}{2} \quad (16)$$

We obtain the resummed quark propagator in terms of $P_R$ and $P_L$ from (5)

$$S(p_\parallel) = \frac{1}{2} \left[ P_L \gamma^\mu X_\mu P_R + P_R \gamma^\mu Y_\mu P_L \right], \quad (17)$$

where

$$\gamma^\mu X_\mu = \gamma^\mu p_\parallel - (A_2 - A_1)\gamma^\mu b_\mu - (A_1 - A_2)\gamma^\mu u_\mu, \quad (18)$$
$$\gamma^\mu Y_\mu = \gamma^\mu p_\parallel - (A_2 + A_1)\gamma^\mu b_\mu - (A_1 + A_2)\gamma^\mu u_\mu. \quad (19)$$

and

$$\frac{X^2}{2} = X_1^2 = \frac{1}{2} \left[ p_0 - (A_1 - A_2) \right]^2 - \frac{1}{2} \left[ p_z + (A_2 - A_1) \right]^2, \quad (20)$$
$$\frac{Y^2}{2} = Y_1^2 = \frac{1}{2} \left[ p_0 - (A_1 + A_2) \right]^2 - \frac{1}{2} \left[ p_z + (A_2 + A_1) \right]^2. \quad (21)$$

The static limit ($p_0 = 0, \ p_z \to 0$) of the poles of the propagator (17) (of either $X_1^2$ or $Y_1^2$ ) gives the mass of the quark as

$$m_{i,B}^2 = g^2|q_iB| \left[ \frac{\pi T}{2m_{i,0}} - \ln(2) + \frac{7\mu^2\zeta(3)}{8\pi^2T^2} - \frac{31\mu^4\zeta(5)}{32\pi^4T^4} \right], \quad (22)$$

which depends on the magnetic field, temperature and quark chemical potential.

The effective quark mass for $i^{th}$ flavor in the case of a weak magnetic field can be parameterized like the earlier cases as

$$m_{i,w}^2 = m_{i,0}^2 + \sqrt{2}m_{i,0}m_{i,L/R} + m_{i,L/R}^2, \quad (23)$$

where $m_{i,L/R}$ refers to the thermal mass for the left- or right-handed chiral mode of $i^{th}$ flavor which can be evaluated from the Dyson-Schwinger equation

$$S^{-1}(P) = \mathcal{P} - \Sigma(P). \quad (24)$$

Here $\Sigma(P)$ represents the self-energy of the quark in the weakly magnetized thermal medium which can be written in the covariant form at finite $T$ and $B$ as [85]

$$\Sigma(P) = -a_1\mathcal{P} - a_2\mathcal{P}' - a_3\gamma_5\mathcal{P}' - a_4\gamma_5\mathcal{P}', \quad (25)$$
where \( a_1, a_2, a_3, a_4 \) are the structure functions, which can be evaluated by taking the appropriate contractions of Eq. (25) as [85]

\[
a_1(p_0, |p|) = \frac{m^2_{th}}{|p|^2} Q_1 \left( \frac{p_0}{|p|} \right),
\]

\[
a_2(p_0, |p|) = -\frac{m^2_{th}}{|p|} \left[ \frac{p_0}{|p|} Q_1 \left( \frac{p_0}{|p|} \right) - Q_0 \left( \frac{p_0}{|p|} \right) \right],
\]

\[
a_3(p_0, |p|) = -4g^2 C_F M^2 |p| Q_1 \left( \frac{p_0}{|p|} \right),
\]

\[
a_4(p_0, |p|) = -4g^2 C_F M^2 |p| Q_0 \left( \frac{p_0}{|p|} \right),
\]

where

\[
M^2(T, \mu, B) = \frac{|q_i B|}{16\pi^2} \left( \frac{\pi T}{2m_{i0}} - \ln 2 + \frac{7\mu^2 \zeta(3)}{8\pi^2 T^2} \right),
\]

and \( Q_0 \) and \( Q_1 \) are given by

\[
Q_0(t) = \frac{1}{2} \ln \left( \frac{t+1}{t-1} \right),
\]

\[
Q_1(t) = \frac{t}{2} \ln \left( \frac{t+1}{t-1} \right) - 1 = tQ_0(t) - 1.
\]

Self-energy (25) can be written in the basis of right and left-hand chiral projection operators as

\[
\Sigma(P) = -P_R (a_1 \hat{P} + (a_2 + a_3) \hat{\bar{\psi}} + a_4 \hat{\bar{b}}) P_L - P_L (a_1 \hat{P} + (a_2 - a_3) \hat{\psi} - a_4 \hat{b}) P_R.
\]

We calculate the effective quark propagator from (24)

\[
S^*(P) = \frac{1}{2} \left[ P_L \frac{L}{L^2/2} P_R + P_R \frac{R}{R^2/2} P_L \right],
\]

where

\[
L^2 = (1 + a_1)^2 P^2 + 2(1 + a_1)(a_2 + a_3)p_0 - 2a_4(1 + a_1)p_2 + (a_2 + a_3)^2 - a_4^2,
\]

\[
R^2 = (1 + a_1)^2 P^2 + 2(1 + a_1)(a_2 - a_3)p_0 + 2a_4(1 + a_1)p_2 + (a_2 - a_3)^2 - a_4^2.
\]

Now in order to get the quark thermal mass in weakly magnetized thermal QCD medium, we take the static limit \((p_0 = 0, |p| \to 0)\) of \(L^2/2\) and \(R^2/2\) modes,\(^1\) we get

\[
\frac{L^2}{2} \big|_{p_0=0, |p| \to 0} = m^2_{th} + 4g^2 C_F M^2,
\]

\[
\frac{R^2}{2} \big|_{p_0=0, |p| \to 0} = m^2_{th} - 4g^2 C_F M^2.
\]

\(^1\)We have expanded the Legendre functions appearing in the structure functions in power series of \(\frac{p}{p_0}\) and have considered only upto \(\mathcal{O}(g^2)\).
The masses of the left- and right-handed modes are given by

\[ m_L^2 = m_{th}^2 + 4g^2 C_F M^2, \]
\[ m_R^2 = m_{th}^2 - 4g^2 C_F M^2, \]

respectively. We will use these medium generated masses in the dispersion relation of the quarks to calculate the Seebeck and Nernst coefficients in the forthcoming sections.

3 Thermoelectric response of a thermal QCD medium

In the kinetic theory approach, the evolution of the phase space distribution function is given by RBTE, which reads as

\[ p_\mu \frac{\partial f}{\partial x^\mu} + q F^{\rho\sigma} p_\sigma \frac{\partial f}{\partial p^\rho} = C[f], \]

where \( f = f_{eq} + \delta f \); \( \delta f \) is small deviation from the equilibrium and \( F^{\rho\sigma} \) corresponds to the electromagnetic field strength tensor. \( C[f] \) corresponds to the collision integral, which provides microscopic input to the RBTE. In general \( C[f] \) is non-linear in \( f \), but Anderson and Witten proposed a simple collision integral, which is known as RTA

\[ C[f] = -\frac{p_\mu u_\mu}{\tau} (f - f_{eq}), \]

where \( \tau \) is the relaxation time. The RTA collision term violates the particle number and current conservation. This shortcoming is the artifact of the linearization of the collision term otherwise in principle the full collision term respects all the conservation laws. Later this shortcoming was circumvented by Bhatnagar, Gross and Krook (BGK) by modifying the RTA as [54, 86]

\[ C[f] = -\frac{p_\mu u_\mu}{\tau} \left( f - \frac{n}{n_{eq}} f_{eq} \right), \]

where \( n \) and \( n_{eq} \) are the out of equilibrium and equilibrium number densities, respectively. The collision term (43) respects the conservation of the particle number i.e.

\[ \int \frac{d^3p}{(2\pi)^3} C[f] = 0. \]

In what follows, we will apply the framework discussed herewith to examine the thermoelectric response of the thermal medium of quarks and gluons with and without external magnetic field background.

3.1 Seebeck coefficient in the absence of the magnetic field

In this subsection, we will evaluate the Seebeck coefficient of the thermal QCD medium composed of \( u, d \) and \( s \) quarks (and their anti-particles). In the presence of the thermal gradient, the charge
carriers will move from the hotter regions to the colder ones. As a result, a current is induced in
the medium which can be written as
\[ J_\mu = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{p_\mu}{\omega_i} (q_i \delta f_i(x, p) + \bar{q}_i \delta \bar{f}_i(x, p)), \]
where \( \delta f_i(x, p) \) (\( \delta \bar{f}_i(x, p) \)) refers to the infinitesimal deviation in the phase space density of quarks
(anti-quarks) of \( i \)th flavor and \( g_i \) corresponds to the degeneracy factor.

The Boltzmann transport equation (41) in the presence of the temperature gradient with BGK
collision integral can be written as
\[ \vec{p} \cdot \frac{\partial f_i}{\partial \vec{r}} + q_i \vec{E} \cdot \frac{\partial f_i}{\partial p_0} + q_i p_0 \vec{E} \cdot \frac{\partial f_i}{\partial \vec{p}} = -p_\mu u_\mu (f_i - n_i n_{eq,i} f_{eq,i}), \]
where \( f_i = f_{eq,i} + \delta f_i \) and
\[ n_i = g_i \int \frac{d^3p}{(2\pi)^3} (f_{eq,i} + \delta f_i), \]
\[ n_{eq,i} = g_i \int \frac{d^3p}{(2\pi)^3} f_{eq,i}. \]
\( f_{eq,i} \) is the Fermi-Dirac distribution function and \( \nu_i \) is the collision frequency, which is estimated
by inverse of the relaxation time [87]
\[ \tau_i(T) = \frac{1}{5.1T \alpha_s^2 \log \left( \frac{1}{\alpha_s} \right) [1 + 0.12(2N_f + 1)]}, \]
where \( \alpha_s \) is the running coupling constant (3).

The RBTE (46) can be recast after some simplification as (see Appendix A)
\[ \delta f_i - g_i n_{eq,i} f_{eq,i} \int_p \delta f_i = \frac{\vec{p}}{\omega_i} (\omega_i - \mu) \tau_i \left( -\frac{1}{T^2} \right) f_{eq,i} (1 - f_{eq,i}) \nabla \vec{r} T(\vec{r}) \]
\[ + 2 q_i \beta \tau_i \frac{\vec{E} \cdot \vec{p}}{\omega_i} f_{eq,i} (1 - f_{eq,i}), \]
which can be further solved for \( \delta f_i \) as
\[ \delta f_i = \delta f_i^{(0)} + g_i n_{eq,i} f_{eq,i} \int_{p'} \delta f_i^{(0)}, \]
where
\[ \delta f_i^{(0)} = \frac{\vec{p}}{\omega_i} (\omega_i - \mu) \tau_i \left( -\frac{1}{T^2} \right) f_{eq,i} (1 - f_{eq,i}) \nabla \vec{r} T(\vec{r}) + \frac{2 q_i \beta \tau_i}{\omega_i} \vec{E} \cdot \vec{p} f_{eq,i} (1 - f_{eq,i}). \]
Following the similar steps, \( \delta \bar{f}_i \) can be calculated as
\[ \delta \bar{f}_i = \delta \bar{f}_i^{(0)} + g_i n_{eq,i} \bar{f}_{eq,i} \int_{p'} \delta \bar{f}_i^{(0)}, \]
where

$$\delta f_i^{(0)} = \frac{\vec{p}}{\omega_i} \left( \omega_i + \mu \right) \tau_i \left( -\frac{1}{T^2} \right) \bar{f}_{eq,i} (1 - f_{eq,i}) \nabla_T(r) + 2\beta \bar{q}_i \tau_i \frac{\vec{E} \cdot \vec{p}}{\omega_i} \bar{f}_{eq,i} (1 - \bar{f}_{eq,i}). \quad (54)$$

Now substituting $\delta f_i$ and $\delta \bar{f}_i$ in the Eq. (45) to obtain the space part of induced current due to a single quark flavor, which reads

$$J_{k,i} = q_i g_i \tau_i \int \frac{d^3p}{(2\pi)^3} \left\{ \left[ \frac{p_k^2}{\omega_i^2} (\omega_i - \mu) \left( -\frac{1}{T^2} \right) f_{eq,i} (1 - f_{eq,i}) \nabla_T(r) + 2\bar{p}_k \bar{q}_i \beta_i f_{eq,i} (1 - f_{eq,i}) \right] \right\}$$

$$+ \frac{g_i}{n_{eq,i}} \frac{p_k}{\omega_i} f_{eq,i} \int \frac{d^3p}{(2\pi)^3} \left\{ \left[ \frac{p_k^2}{\omega_i^2} (\omega_i + \mu) \left( -\frac{1}{T^2} \right) \bar{f}_{eq,i} (1 - \bar{f}_{eq,i}) \nabla_T(r) + 2\bar{p}_k \bar{q}_i \beta_i \bar{f}_{eq,i} (1 - \bar{f}_{eq,i}) \right] \right\}$$

$$+ q_i g_i \tau_i \int \frac{d^3p}{(2\pi)^3} \left\{ \left[ \frac{p_k^2}{\omega_i^2} (\omega_i + \mu) \left( -\frac{1}{T^2} \right) \bar{f}_{eq,i} (1 - \bar{f}_{eq,i}) \nabla_T(r) + 2\bar{p}_k \bar{q}_i \beta_i \bar{f}_{eq,i} (1 - \bar{f}_{eq,i}) \right] \right\}. \quad (55)$$

In the state of equilibrium, the resultant current due to $i^{th}$ quark flavor becomes zero i.e. $\vec{J}_i = 0$. Putting the induced current (55) to zero, we get a relation between the thermal gradient in the medium and electric field as

$$\vec{E} = \frac{1}{2Tq} \frac{\left( L_1 + L_2 \right)}{\left( L_3 + L_4 \right)} \nabla_T(r),$$

$$\equiv S \nabla_T(r). \quad (56)$$

Here $S$ is the Seebeck coefficient, which is given by

$$S = \frac{1}{2Tq} \frac{\left( L_1 + L_2 \right)}{\left( L_3 + L_4 \right)}, \quad (57)$$

where

$$L_1 = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p_k^2}{3\omega_i^2} (\omega_i - \mu) f_{eq} (1 - f_{eq}) + \frac{g_i}{n_{eq,i}} \frac{p_k}{\omega_i} f_{eq} \int \frac{d^3p'}{(2\pi)^3} \left( \omega' - \mu \right) f_{eq} (1 - f_{eq}) \right\}, \quad (58)$$

$$L_2 = -\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p_k^2}{3\omega_i^2} (\omega_i + \mu) \bar{f}_{eq} (1 - \bar{f}_{eq}) + \frac{g_i}{n_{eq,i}} \frac{p_k}{\omega_i} \bar{f}_{eq} \int \frac{d^3p'}{(2\pi)^3} \left( \omega' + \mu \right) \bar{f}_{eq} (1 - \bar{f}_{eq}) \right\}, \quad (59)$$

$$L_3 = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p_k^2}{3\omega_i^2} f_{eq} (1 - f_{eq}) + \frac{g_i}{n_{eq,i}} \frac{p_k}{\omega_i} f_{eq} \int \frac{d^3p'}{(2\pi)^3} \left( \omega' - \mu \right) f_{eq} (1 - f_{eq}) \right\}, \quad (60)$$

$$L_4 = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p_k^2}{3\omega_i^2} \bar{f}_{eq} (1 - \bar{f}_{eq}) + \frac{g_i}{n_{eq,i}} \frac{p_k}{\omega_i} \bar{f}_{eq} \int \frac{d^3p'}{(2\pi)^3} \left( \omega' + \mu \right) \bar{f}_{eq} (1 - \bar{f}_{eq}) \right\}. \quad (61)$$

Upto this point, we have only considered a single quark flavor, we will now focus on the hot QCD medium with multiple quark flavors. In our case, we have considered three flavour ($u,d$ and $s$)
quarks and their anti-particles) quark gluon plasma. The total induced current can be written as

\[ \vec{J} = \sum_i \vec{J}_i \]

\[ = \left( \frac{q_1 g_1 \tau_1}{T} (L_3 + L_4)_1 + \frac{q_2 g_2 \tau_2}{T} (L_3 + L_4)_2 + \ldots \right) \vec{E} \]

\[ - \left( \frac{q_1 g_1 \tau_1}{T^2} (L_1 + L_2)_1 + \frac{q_2 g_2 \tau_2}{T^2} (L_1 + L_2)_2 + \ldots \right) \nabla T(\vec{r}). \] (62)

In the steady state condition, the total induced current vanishes \( \vec{J} = 0 \). As a result, we get

\[ \vec{E} = \frac{1}{2T} \sum_i q_i g_i \tau_i (L_1 + L_2)_i \nabla T(\vec{r}), \] (63)

which gives the seebeck coefficient for the medium as

\[ S_{\text{tot}} = \frac{1}{2T} \sum_i q_i g_i \tau_i (L_1 + L_2)_i. \] (64)

Since all the flavors have same relaxation time and degeneracy factor, so the total Seebeck coefficient for the medium can be expressed in terms of the Seebeck coefficient of the individual flavor as

\[ S_{\text{tot}} = \frac{\sum_i S_i q_i^2 (L_3 + L_4)_i}{\sum_i q_i^2 (L_3 + L_4)_i}. \] (65)

In the next subsection, we will explore how the presence of the background magnetic field modulates the thermoelectric response of the hot QCD medium.

### 3.2 Seebeck and Nernst coefficient in the presence of the magnetic field

Now we will calculate the Seebeck coefficient in the magnetic field background. Firstly, we will consider the strong field regime, where the the energy of the quark is quantized via Landau quantization. Then, we will explore the weak field limit, where the magnetic field dependence in the transport coefficients enters through the cyclotron frequency, which manifests a classical description of the motion of charged particle in the magnetic field.

#### 3.2.1 The strong magnetic field case

In the presence of strong \( B \), the quark energy gets quantized as [88]

\[ \omega_i = \sqrt{p_i^2 + m_i^2} + 2n|q_i B|, \] (66)
where \( n = 0, 1, 2, \ldots \) correspond to the discrete Landau levels and the phase space factor takes the form \([88]\)

\[
\int \frac{d^3p}{(2\pi)^3} \to \sum_{n=0}^{\infty} \frac{|q_i B|}{2\pi} \int \frac{dp_3}{2\pi} (2 - \delta_{n0}).
\] (67)

Since we are interested in the strong magnetic field limit with scale hierarchy \((|qB| >> T^2 >> m_i^2)\), the quarks are confined to the lowest Landau levels \(i.e.\ n = 0\). A dimensional reduction in the quark dynamics takes place from \(3 + 1\) to \(1 + 1\) dimensions rendering the induced current along the \(z\) direction as

\[
J_3 = \sum_i g_i \frac{|q_i B|}{4\pi^2} \int dp_3 \frac{p_3}{\omega_i} (q_i \delta f_i^B + \bar{q}_i \delta \bar{f}_i^B),
\] (68)

where \(\delta f_i^B\) and \(\delta \bar{f}_i^B\) are the deviations in the quark and anti-quark distribution functions, respectively. The RBTE \((46)\) in the strong \(B\) becomes

\[
p^0 \frac{\partial f_i^B}{\partial x^0} + p^i \frac{\partial f_i^B}{\partial x^i} + q_i F^{\alpha i} \frac{\partial f_i^B}{\partial p^\alpha} + \bar{q}_i F^{\bar{i} \alpha} p_0 \frac{\partial \bar{f}_i^B}{\partial p^\alpha} = -p^\mu n_\mu \, \delta f_i^B + \bar{q}_i \delta \bar{f}_i^B \left( f_i^B - n^B_i \bar{n}^{B, -1} f_{eq,i} \right),
\] (69)

where \(f_i^B = f_{eq,i} + \delta f_i^B\); \(f_{eq,i}\) is given by

\[
f_{eq,i}^B = \frac{1}{e^{\beta \omega_i} + 1}.
\] (70)

Here \(\omega_i = \sqrt{p_3^2 + m_i^2}\); \(m_i\) is the quasiparticle mass \((4)\) and \(\nu_i^B\) is computed by the inverse of the relaxation time \([89]\)

\[
\tau_i^B (p_3; T, |q B|) = \frac{\omega_i (e^{\beta \omega_i} - 1)}{\alpha_s C_T m_i^2 (e^{\beta \omega_i} + 1)} \left( \int \frac{dp_3}{\omega_i (e^{\beta \omega_i} + 1)} \right)^{-1}.
\] (71)

In Eq. (69), \(n_{eq,i}^B\) and \(n_i^B\) are the equilibrium and non-equilibrium number densities of quarks in strong \(B\), which are given as

\[
n_{eq,i}^B = \frac{g_i |q_i B|}{4\pi^2} \int dp_3 f_{eq,i}^B,
\] (72)

\[
n_i^B = \frac{g_i |q_i B|}{4\pi^2} \int dp_3 (f_{eq,i}^B + \delta f_i^B).
\] (73)

In order to obtain \(\delta f_i^B\), we simplify Eq. (69) as

\[
\delta f_i^B - g_i n_{eq,i}^B \, \delta f_{eq,i}^B \int_{p_3} \delta f_i^B = \tau_i^B \frac{p_3}{\omega_i} (\omega_i - \mu) \left( \frac{-1}{T} \right) f_{eq,i}^B (1 - f_{eq,i}^B) (\nabla T)_3 + 2 q_i \beta \tau_i^B \frac{p_3}{\omega_i} E_3 f_{eq,i}^B (1 - f_{eq,i}^B),
\] (74)

which is further solved for \(\delta f_i^B\) upto first order in iteration as

\[
\delta f_i^B = \delta f_i^{B(0)} + g_i n_{eq,i}^B \, \delta f_{eq,i}^{B(0)} \int_{p_3} \delta f_i^{B(0)},
\] (75)
where
\[ \delta f_i^{B(0)} = \frac{p_3}{\omega_i} t^B(\omega_i - \mu) \left( -1 \frac{1}{T^2} \right) f_{eq,i}^B (1 - f_{eq,i}^B) (\nabla T)_3 + \frac{2q_i \beta \tau_i^B}{\omega_i} p_3 E_3 f_{eq,i}^B (1 - f_{eq,i}^B). \]  
(76)

Similarly, we can write for the anti-quarks as
\[ \delta \bar{f}_i^B = \delta \bar{f}_i^{B(0)} + g_i n_{eq,i}^{-1} \bar{f}_{eq,i} \int_{p'_3} \delta \bar{f}_i^{B(0)}, \]  
(77)

where
\[ \delta \bar{f}_i^{B(0)} = \frac{p_3}{\omega_i} \bar{t}^B(\omega_i + \mu) \left( -1 \frac{1}{T^2} \right) \bar{f}_{eq,i}^B (1 - \bar{f}_{eq,i}^B) (\nabla T)_3 + \frac{2q_i \beta \tau_i^B}{\omega_i} p_3 E_3 \bar{f}_{eq,i}^B (1 - \bar{f}_{eq,i}^B). \]  
(78)

Now substituting \( \delta f_i^B \) and \( \delta \bar{f}_i^B \) in Eq. (68) to obtain the induced current in the strong magnetic field due to a single \( i \)th quark flavor
\[ J_{3,i} = \frac{q_i g_i}{4 \pi^2} \int dp_3 \left[ \left\{ \frac{p_3^2}{\omega_i^2} t^B(\omega_i - \mu) \left( -1 \frac{1}{T^2} \right) f_{eq,i}^B (1 - f_{eq,i}^B) (\nabla T)_3 + \frac{2q_i \beta \tau_i^B}{\omega_i} p_3 E_3 f_{eq,i}^B (1 - f_{eq,i}^B) \right\} \times \bar{f}_{eq,i}^B (1 - \bar{f}_{eq,i}^B) (\nabla T)_3 + \frac{p_3^2}{\omega_i^2} \beta \bar{t}^B(\omega_i + \mu) \left( -1 \frac{1}{T^2} \right) \bar{f}_{eq,i}^B (1 - \bar{f}_{eq,i}^B) \right\} \right]. \]
(79)

In the state of equilibrium \( J_{3,i} \) becomes zero and we get the relation (omitting label \( i \) for simplicity)
\[ E_3 = \frac{1}{2qT} \left( \frac{H_1 + H_2}{H_3 + H_4} \right) (\nabla T)_3, \]
(80)
\[ \equiv S_B (\nabla T)_3. \]
(81)

\( S_B \) here corresponds to the Seebeck coefficient in the strong \( B \) background, which reads
\[ S_B = \frac{1}{2qT} \left( \frac{H_1 + H_2}{H_3 + H_4} \right), \]
(82)

where the integrals \( H_1, H_2, H_3 \) and \( H_4 \) are given by
\[ H_1 = \frac{|qB|}{4 \pi^2} \int dp_3 \left\{ \frac{p_3^2}{\omega^2} t^B(\omega - \mu) f_{eq}^B (1 - f_{eq}^B) + g p_3 f_{eq}^B \int_{p'_3} \frac{p_3^2}{\omega} \omega' t^B(\omega' - \mu) f_{eq}^B (1 - f_{eq}^B) \right\}, \]
(83)
\[ H_2 = -\frac{|qB|}{4 \pi^2} \int dp_3 \left\{ \frac{p_3^2}{\omega^2} t^B(\omega + \mu) \bar{f}_{eq}^B (1 - \bar{f}_{eq}^B) + g p_3 \bar{f}_{eq}^B \int_{p'_3} \frac{p_3^2}{\omega} \omega' \bar{t}^B(\omega' + \mu) \bar{f}_{eq}^B (1 - \bar{f}_{eq}^B) \right\}, \]
(84)
\[ H_3 = \frac{|qB|}{4 \pi^2} \int dp_3 \left\{ \frac{p_3^2}{\omega^2} \bar{t}^B f_{eq}^B (1 - f_{eq}^B) + g p_3 f_{eq}^B \int_{p'_3} \frac{p_3^2}{\omega} \omega' \bar{t}^B f_{eq}^B (1 - f_{eq}^B) \right\}, \]
(85)
\[ H_4 = \frac{|qB|}{4 \pi^2} \int dp_3 \left\{ \frac{p_3^2}{\omega^2} \bar{t}^B \bar{f}_{eq}^B (1 - \bar{f}_{eq}^B) + g p_3 \bar{f}_{eq}^B \int_{p'_3} \frac{p_3^2}{\omega} \omega' \bar{t}^B \bar{f}_{eq}^B (1 - \bar{f}_{eq}^B) \right\}. \]
(86)
For the hot QCD medium consisting of $u$, $d$ and $s$ quarks, the 3rd component of the induced current can be written as the vector sum of the individual currents as

$$J_3 = \sum_i J_{3,i}$$

(87)

$$= \left( \frac{q_1^2 g_1 |q_1 B|}{T} (H_3 + H_4)_1 + \frac{q_2^2 g_2 |q_2 B|}{T} (H_3 + H_4)_2 + .... \right) E_3$$

$$- \left( \frac{q_1 g_1 |q_1 B|}{T^2} (H_1 + H_2)_1 + \frac{q_2 g_2 |q_2 B|}{T^2} (H_1 + H_2)_2 + .... \right) (\nabla T)_3,$$

(88)

which vanishes in the steady state i.e. $J_3 = 0$ and gives

$$E_3 = \frac{1}{2T} \sum_i \frac{q_i g_i (H_1 + H_2)_i}{q_i^3 g_i (H_3 + H_4)_i} (\nabla T)_3.$$  

(89)

We extract the Seebeck coefficient for the composite medium as

$$S_B^{tot} = \frac{1}{2T} \sum_i \frac{q_i |q_i B|(H_1 + H_2)_i}{q_i^3 |q_i B|(H_3 + H_4)_i}$$

(90)

which can be expressed in terms of the Seebeck coefficient of the single quark flavor as

$$S_B^{tot} = \sum_i S_i \frac{|q_i|^3 (H_3 + H_4)_i}{\sum_i |q_i|^3 (H_3 + H_4)_i}.$$ 

(91)

In the strong magnetic field, there is no current in the transverse direction due to the one-dimensional (LLL) quark dynamics. Hence the Nernst coefficient which measures the thermoelectric response in the transverse direction, vanishes. In the next subsection, we will explore the weak magnetic field regime, where the quark dynamics is not affected by the Landau quantization, rather the magnetic field dependence enters via the cyclotron frequency, which manifests the semi-classical description. In this scenario, the Nernst coefficient would also appear.

### 3.2.2 The weak magnetic field case

In the weak magnetic field, the dispersion relation of the charged particle is not directly affected by the magnetic field rather $B$ acts as a perturbation. So the 1+1 dimensional Landau level kinematics is not applicable. The induced four current in the medium is given by

$$J_\mu = \sum_i g_i \int \frac{d^3 p}{(2\pi)^3 \epsilon_i} \frac{p_\mu}{\epsilon_i} [q_i \delta f_i + \bar{q}_i \delta \bar{f}_i],$$

(92)

where $\epsilon_i = \sqrt{p^2 + m_i^2}$. The RBTE (41) in the presence of the Lorenz force can be written as (see the Appendix)

$$\frac{\partial f_i}{\partial t} + \vec{v} \cdot \frac{\partial f_i}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f_i}{\partial \vec{p}} = -\frac{1}{\tau_i} \left( f_i - \frac{n_i}{n_{0,i}} f_{0,i} \right),$$

(93)
where \( \vec{F} = q_i \left( \vec{E} + \vec{v} \times \vec{B} \right) \). Since magnetic field is not a dominant energy scale here, we have used the same relaxation time calculated in the absence of \( B \). The magnetic field and chemical potential dependence in \( \tau_i \) will enter through the strong coupling constant \( \lambda \). \( n_i \) and \( n_{0,i} \) are given by Eqs. (47) and (48), respectively except the mass in the dispersion relation will be replaced by the thermo-magnetic mass calculated in weak magnetic field in section 3.

Without the loss of generality, let us consider the electric field in the \( xy \) plane i.e. \( \vec{E} = E_x \hat{x} + E_y \hat{y} \) and magnetic field in \( z \) direction i.e \( \vec{B} = B \hat{z} \). Then for QCD medium, which is homogeneous in time, Eq. (93) takes the form

\[
q_i B \tau_i \left( v_x \frac{\partial f_i}{\partial p_y} - v_y \frac{\partial f_i}{\partial p_x} \right) - \tau_i v_i \frac{\partial f_i}{\partial \vec{v}} - \tau_i \epsilon_0 q_i \vec{E} \cdot \frac{\partial f_i}{\partial \vec{p}} = \left( \delta f_i - g_i n_{0,i} f_{0,i} \int \delta f_i \right),
\]

which can be solved for \( \delta f_i \) up to first order as

\[
\delta f_i = \delta f_i^{(a)} + g_i n_{0,i} f_{0,i} \int \delta f_i^{(a)},
\]

where

\[
\delta f_i^{(a)} = q_i B \tau_i \left( v_x \frac{\partial f_i}{\partial p_y} - v_y \frac{\partial f_i}{\partial p_x} \right) - \tau_i v_i \frac{\partial f_i}{\partial \vec{v}} - \tau_i \epsilon_0 q_i \vec{E} \cdot \frac{\partial f_i}{\partial \vec{p}}.
\]

In order to solve Eq. (95), we take an ansatz [90]

\[
\delta f_i = \delta f_i^{(b)} + g_i n_{0,i} f_{0,i} \int_{p'} \delta f_i^{(b)},
\]

where

\[
\delta f_i^{(b)} = f_i - f_{0,i} = -\tau_i q_i \vec{E} \cdot \frac{\partial f_{0,i}}{\partial \vec{p}} - \vec{\lambda} \cdot \frac{\partial f_{0,i}}{\partial \vec{p}}.
\]

Equating Eqs. (95) and (97), we get

\[
\vec{\lambda} \cdot \frac{\partial f_{0,i}}{\partial \vec{p}} - q_i B \tau_i \left( v_y \frac{\partial f_i}{\partial p_x} - v_x \frac{\partial f_i}{\partial p_y} \right) - \tau_i v_i \frac{\partial f_i}{\partial \vec{v}} = g_i n_{0,i} f_{0,i} \left( \int_{p'} \delta f_i^{(a)} + \int_{p'} \delta f_i^{(b)} \right).
\]

We calculate \( \frac{\partial f_i}{\partial p_y} \) and \( \frac{\partial f_i}{\partial p_x} \) using the ansatz (97) as

\[
\begin{align*}
\left( v_x \frac{\partial f_i}{\partial p_y} - v_y \frac{\partial f_i}{\partial p_x} \right) &= \left( v_y \lambda_x + v_y \tau_i q_i E_x - v_x \lambda_y - v_x \tau_i q_i E_y \right) \frac{\partial f_{0,i}}{\partial \epsilon_i}.
\end{align*}
\]

where we have retained terms which are linear in the velocity only.

Now substituting Eq. (100) in (99) and after doing some re-arrangement, we get

\[
v_x \left[ \frac{\lambda_x}{\tau_i} - \omega^c \tau_i q_i E_y - \omega^c \lambda_y + \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial x} \right] + v_y \left[ \frac{\lambda_y}{\tau_i} + \omega^c \tau_i q_i E_x + \omega^c \lambda_x + \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial y} \right] + g_i T \frac{1}{n_{0,i} \tau_i} \left( \int_{p'} \delta f_i^{(a)} + \int_{p'} \delta f_i^{(b)} \right) = 0,
\]

15
\[ \lambda_x = \frac{\omega_c^2 \tau_i^3}{1 + \omega_c^2 \tau_i^2} q_i q_x - \omega_c \lambda_y + \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial x} = 0, \]  
\[ \lambda_y = \frac{\omega_c^2 \tau_i^3}{1 + \omega_c^2 \tau_i^2} q_i q_y + \omega_c \lambda_x + \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial y} = 0. \]

We solve above Eqs. (102) and (103) to get \( \lambda_x \) and \( \lambda_y \) as

\[ \lambda_x = -\frac{\omega_c^2 \tau_i^3}{1 + \omega_c^2 \tau_i^2} q_i q_x - \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial x} + \frac{\omega_c^2 \tau_i^3}{1 + \omega_c^2 \tau_i^2} q_i q_y - \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial y}, \]  
\[ \lambda_y = -\frac{\omega_c^2 \tau_i^3}{1 + \omega_c^2 \tau_i^2} q_i q_y + \frac{\omega_c^2 \tau_i^3}{1 + \omega_c^2 \tau_i^2} q_i q_x - \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) \frac{\partial T}{\partial y}. \]

Now we substitute \( \lambda_x \) and \( \lambda_y \) in (97) to obtain \( \delta f \) which reads

\[ \delta f = \delta f^{(b)} + g_i \alpha_{0,i}^{-1} f^0, i \int_{\nu'} \delta f^{(b)} \]  
where

\[ \delta f^{(b)} = \frac{\partial f_{0,i}}{\partial \epsilon} \left[ -\frac{\tau_i}{1 + \omega_c^2 \tau_i^2} q_i v_x + \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} q_i v_y \right] E_x \]

\[ + \frac{\partial f_{0,i}}{\partial \epsilon} \left[ -\frac{\tau_i}{1 + \omega_c^2 \tau_i^2} q_i v_y - \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} q_i v_x \right] E_y \]

\[ + \frac{\partial f_{0,i}}{\partial \epsilon} \left[ \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_x - \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_y \right] \frac{\partial T}{\partial x} \]

\[ + \frac{\partial f_{0,i}}{\partial \epsilon} \left[ \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_y + \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_x \right] \frac{\partial T}{\partial y}. \]

Similarly, deviation in the anti-quark distribution function can be evaluated as (replacing \( q_i \) with \(-q_i\) and \( \omega_c \) with \(-\omega_c\) in Eq. (107))

\[ \delta \bar{f} = \delta \bar{f}^{(b)} + g_i \tilde{\alpha}_{0,i}^{-1} \tilde{f}_{0,i} \int_{\nu'} \delta \bar{f}^{(b)} \]

where

\[ \delta \bar{f}^{(b)} = \frac{\partial \bar{f}_{0,i}}{\partial \epsilon} \left[ \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} q_i v_x + \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} q_i v_y \right] E_x \]

\[ + \frac{\partial \bar{f}_{0,i}}{\partial \epsilon} \left[ \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} q_i v_y - \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} q_i v_x \right] E_y \]

\[ + \frac{\partial \bar{f}_{0,i}}{\partial \epsilon} \left[ \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_x + \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_y \right] \frac{\partial T}{\partial x} \]

\[ + \frac{\partial \bar{f}_{0,i}}{\partial \epsilon} \left[ \frac{\tau_i}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_y - \frac{\omega_c \tau_i^2}{1 + \omega_c^2 \tau_i^2} \left( \frac{\epsilon_i - \mu}{T} \right) v_x \right] \frac{\partial T}{\partial y}. \]
We substitute $\delta f_i$ and $\delta \bar{f}_i$ in (92) to get the $x$ and $y$ components of the induced current density due to $i^{th}$ quark flavor

$$
J_{x,i} = q_i g_i \left[ (q_i \beta I_{1,i}) E_x + (q_i \beta I_{2,i}) E_y + (\beta^2 I_{3,i}) \frac{\partial T}{\partial x} + (\beta^2 I_{4,i}) \frac{\partial T}{\partial y} \right],
$$

$$
J_{y,i} = q_i g_i \left[ -(q_i \beta I_{2,i}) E_x + (q_i \beta I_{1,i}) E_y + (-\beta^2 I_{4,i}) \frac{\partial T}{\partial x} + (\beta^2 I_{3,i}) \frac{\partial T}{\partial y} \right].
$$

The integrals $I_1, I_2, I_3$ and $I_4$ in the above equations are given by (omitting label $i$ for simplicity)

$$
I_1 = I_{1q} + I_{1\bar{q}},
$$

$$
I_2 = I_{2q} + I_{2\bar{q}},
$$

$$
I_3 = I_{3q} + I_{3\bar{q}},
$$

$$
I_4 = I_{4q} + I_{4\bar{q}},
$$

where

$$
I_{1q} = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\tau}{(1 + \omega_{c}^2\tau^2)^2} f_0(1 - f_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\tau}{e'(1 + \omega_{c}^2\tau^2)^2} f_0(1 - f_0) \right\},
$$

$$
I_{1\bar{q}} = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\tau}{(1 + \omega_{c}^2\tau^2)^2} \bar{f}_0(1 - \bar{f}_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\tau}{e'(1 + \omega_{c}^2\tau^2)^2} \bar{f}_0(1 - \bar{f}_0) \right\},
$$

$$
I_{2q} = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\omega_{c}\tau^2}{(1 + \omega_{c}^2\tau^2)^2} f_0(1 - f_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\omega_{c}\tau^2}{e'(1 + \omega_{c}^2\tau^2)^2} f_0(1 - f_0) \right\},
$$

$$
I_{2\bar{q}} = -\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\omega_{c}\tau^2}{(1 + \omega_{c}^2\tau^2)^2} \bar{f}_0(1 - \bar{f}_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\omega_{c}\tau^2}{e'(1 + \omega_{c}^2\tau^2)^2} \bar{f}_0(1 - \bar{f}_0) \right\},
$$

$$
I_{3q} = -\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\tau}{(1 + \omega_{c}^2\tau^2)^2} (\epsilon - \mu) f_0(1 - f_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\tau}{e'(1 + \omega_{c}^2\tau^2)^2} (\epsilon' - \mu) f_0(1 - f_0) \right\},
$$

$$
I_{3\bar{q}} = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\tau}{(1 + \omega_{c}^2\tau^2)^2} (\epsilon - \mu) \bar{f}_0(1 - \bar{f}_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\tau}{e'(1 + \omega_{c}^2\tau^2)^2} (\epsilon' - \mu) \bar{f}_0(1 - \bar{f}_0) \right\},
$$

$$
I_{4q} = -\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\omega_{c}\tau^2}{(1 + \omega_{c}^2\tau^2)^2} (\epsilon - \mu) f_0(1 - f_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\omega_{c}\tau^2}{e'(1 + \omega_{c}^2\tau^2)^2} (\epsilon' - \mu) f_0(1 - f_0) \right\},
$$

$$
I_{4\bar{q}} = -\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{p^2}{3e^2} \frac{\omega_{c}\tau^2}{(1 + \omega_{c}^2\tau^2)^2} (\epsilon + \mu) \bar{f}_0(1 - \bar{f}_0) + \frac{g}{n_0} \frac{p}{\epsilon} \int_{p'} \frac{\omega_{c}\tau^2}{e'(1 + \omega_{c}^2\tau^2)^2} (\epsilon' + \mu) \bar{f}_0(1 - \bar{f}_0) \right\}.
$$

In the state of equilibrium, the components of the induced current density along $x$ and $y$ direction vanishes i.e. $J_{x,i} = J_{y,i} = 0$. We can write from Eqs. (110) and (111)

$$
C_1 E_x + C_2 E_y + C_3 \frac{\partial T}{\partial x} + C_4 \frac{\partial T}{\partial y} = 0,
$$

$$
-C_2 E_x + C_1 E_y - C_4 \frac{\partial T}{\partial x} + C_3 \frac{\partial T}{\partial y} = 0,
$$

provided $C_1 = q I_1$, $C_2 = q I_2$, $C_3 = \beta I_3$, $C_4 = \beta I_4$. Thermoelectric transport coefficients are related to the electric field components and temperature gradients via a matrix equation

$$
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix} = \begin{pmatrix}
S & N |B| \\
-N |B| & S
\end{pmatrix} \begin{pmatrix}
\frac{\partial T}{\partial x} \\
\frac{\partial T}{\partial y}
\end{pmatrix}.
$$
We solve Eqs. (116) and (117) for $E_x$ and $E_y$ as

$$E_x = \left( \frac{-C_1C_3 + C_2C_4}{C_1^2 + C_2^2} \right) \frac{\partial T}{\partial x} + \left( \frac{-C_2C_3 - C_1C_4}{C_1^2 + C_2^2} \right) \frac{\partial T}{\partial y},$$

(119)

$$E_y = \left( \frac{-C_1C_3 + C_2C_4}{C_1^2 + C_2^2} \right) \frac{\partial T}{\partial y} - \left( \frac{-C_2C_3 - C_1C_4}{C_1^2 + C_2^2} \right) \frac{\partial T}{\partial x},$$

(120)

which give the Seebeck and Nernst coefficients

$$S = -\frac{(C_1C_3 + C_2C_4)}{C_1^2 + C_2^2},$$

(121)

$$N|B| = \frac{(C_2C_4 - C_1C_4)}{C_1^2 + C_2^2},$$

(122)

respectively. The integrals $C_2$ and $C_4$ vanishes in the absence of the magnetic field, as a result Nernst coefficient would also vanish.

Now we will compute the Seebeck and Nernst coefficients for the medium. In the medium composed of $u$ and $d$ light quarks, the $x$ and $y$ components of the current can be written as the sum of the individual contributions as

$$J_x = \sum_{i=u,d} \left[ q_i(I_1)E_x + q_i(I_2)E_y + \beta(I_3) \frac{\partial T}{\partial x} + \beta(I_4) \frac{\partial T}{\partial y} \right],$$

(123)

$$J_y = \sum_{i=u,d} \left[ q_i(I_2)E_x + q_i(I_1)E_y - \beta(I_4) \frac{\partial T}{\partial x} + \beta(I_3) \frac{\partial T}{\partial y} \right].$$

(124)

The Seebeck and Nernst coefficients of the medium can be extracted by imposing the steady state condition (i.e. putting $J_x = J_y = 0$) as

$$S_{tot}^{B'} = -\frac{(K_1K_3 + K_2K_4)}{K_1^2 + K_2^2},$$

(125)

$$N|B| = \frac{(K_2K_3 - K_1K_4)}{K_1^2 + K_2^2}.$$ 

(126)

where

$$K_1 = \sum_{i=u,d} q_i(I_1)_i, \quad K_2 = \sum_{i=u,d} q_i(I_2)_i,$$

(127)

$$K_3 = \sum_{i=u,d} \beta(I_3)_i, \quad K_4 = \sum_{i=u,d} \beta(I_4)_i.$$ 

(128)

4 Results and discussion

In this section, we will discuss the results obtained in the previous sections numerically. In Fig. 1 (a), we display the variation of the Seebeck coefficient with $T$ in the BGK and RTA collision terms
for $u$ quarks at $\mu = 60$ MeV. It was found that the magnitude of the Seebeck coefficient decreases with $T$ in both the collision terms. We have computed the ratio of the Seebeck coefficients in BGK collision term to that calculated with the RTA (BGK/RTA) to get the numerical estimates of the relative competition between the two collision integrals. The ratio is found to be around $\sim 0.98$ for the individual flavors in the temperature domain $160 - 400$ MeV, which indicates that Seebeck coefficient is slightly reduced in the BGK term. The sign of the Seebeck coefficient for $d$ and $s$ quarks get reversed due to their negative charges [see Fig. 1 (b) and Fig. 2 (a)]. In the case of the composite medium, the Seebeck coefficient ($S_{tot}$) [see Fig. 2 (b)] has been found to be positive. We notice a considerable enhancement in the magnitude of $S_{tot}$ in BGK collision term, which is around $12\%$ at lower $T$ (160 MeV) and $26\%$ at high $T$ (400 MeV) [see Fig. 3 (b)]. We further study the effects of the quark chemical potential ($\mu$) on the medium Seebeck coefficient in the BGK collision term in Fig. 3 (a) taking the strength of $\mu = 40, 60$ and 80 MeV and have found that $S_{tot}$ increases as we raise $\mu$. Similar results of $\mu$ dependence in Seebeck coefficient have been found in Ref. [50] with RTA collision integral. We can conclude that both BGK collision term and baryon asymmetry in the medium enhance its ability to convert the temperature gradient into the current.

![Graphs showing temperature dependence of the Seebeck coefficient for $u$ quarks and $d$ quarks](image)

Figure 1: Temperature dependence of the Seebeck coefficient in $B = 0$ case: (a) for $u$ quarks and (b) for $d$ quarks
Figure 2: Temperature dependence of the Seebeck coefficient for $B = 0$ case: (a) for $s$ quarks and (b) for composite medium.

Figure 3: Left panel: Temperature dependence of Seebeck coefficient for the medium with BGK collision term in absence of magnetic field for different strengths of $\mu$. Right panel: Ratio of Seebeck coefficients in BGK to that in RTA collision integral with temperature.

In Figs. 4 and 5, we explore the effects of the BGK collision integral on the Seebeck coefficient of a strongly magnetized hot QCD medium. We have chosen the strength of the magnetic field as $eB = 15m_T^2$ and $10m_T^2$ with $\mu = 60$ MeV. The Seebeck coefficient for the individual quark flavors as well as for the combined medium gets enhanced in the BGK collision integral considerably. The enhancement is around $18\% - 25\%$ (for $u$ quarks) and $16\% - 27\%$ (for $d$ quarks) in the temperature
range $160 < T < 400$ MeV. For the $s$ quark, the enhancement is between 21% to 37% in the same domain of $T$. In the case of the medium, it is around 16% near $T_c$ but decreases as we go towards higher temperatures [see Fig. 6 (b)]. $S^{B}_{\text{tot}}$ increases with the strength of $\mu$ like the $B = 0$ case [see Fig. 6 (a)], which is in agreement with the study made in Ref. [51] in the RTA framework.

Figure 4: Temperature dependence of the Seebeck coefficient in the strong $B$: (a) for $u$ quarks and (b) for $d$ quarks

Figure 5: Temperature dependence of the Seebeck coefficient in strong $B$: (a) for $s$ quarks and (b) for composite medium
In Fig. 7, we investigate how the BGK collision term modifies the thermoelectric response in the presence of weak magnetic field ($eB = 0.3 \, m_e^2$) for $u$ (left panel), $d$ (right panel) quarks. We notice that the magnitude of the Seebeck coefficient depends on the chirality of the quark quasiparticles. For the left-handed chiral modes, the ratio BGK/RTA is less than one for individual quark flavours as well as for composite medium. The ratio is found in the range $0.97 - 0.98$ for the $u$ quarks in the temperature range $160 < T < 400$ MeV. In case of $d$ quarks and medium, this ratio is around $\sim 0.98$ [seen in Fig. 9 (a)]. This concludes that the BGK collision integral causes reduction in the Seebeck coefficients in comparison to the RTA [52]. On the other hand, BGK to RTA ratio is very close to unity in case of right-handed modes, which manifests that both the collision terms produce almost similar results [seen in Fig. 9 (b)]. We have also studied the effects of the baryon asymmetry on the thermoelectric phenomenon in Fig. 8 (b) for $\mu = 40, 60$ and 80 MeV and have noticed an increase in Seebeck coefficient with $\mu$ for both $L$ and $R$ modes.

Since BGK collision integral shows an improvement over RTA, it gives more realistic estimates of the transport coefficients like electrical conductivity ($\sigma_{el}$), thermal conductivity ($\kappa$), shear ($\eta$) and bulk ($\zeta$) viscosities as compared to RTA [43,44]. The nonzero value of the Seebeck coefficient modifies the electric current ($J = \sigma_{el}E - \sigma_{el}S\nabla T$) and thermal conductivity ($\kappa = \kappa_0 - T\sigma_{el}S^2$) of the medium. Therefore, the BGK collision term will indirectly influence the charge and heat transport in the medium. Electrical conductivity plays an important role in the time evolution of the electromagnetic fields produced in the noncentral collisions. Hence, the estimation of the electrical conductivity with realistic collision integrals is of paramount importance to understand the strength and life-span of the magnetic field during the various stages of its evolution in the medium. The magnetic field influences the particle production, dynamics of the heavy quarks and their bound states (quarkonium) and many aspects of the QCD phase diagram [91, 92]. Similarly, a more accurate understanding of the thermal conductivity is necessary to study the dynamics of
the first order phase transition \cite{93} and the chiral critical point in the heavy ion collisions \cite{94}. It also govern the attenuation of the sound in the medium via the Prandtl number. In addition to charge and heat transport, momentum transport also gets affected by the BGK collision integral so the hydrodynamic evolution of the medium may get influenced as shear and bulk viscosities act as input to the dissipative hydrodynamical equations. In principle BGK collision term can affect the phenomenology of the heavy ion collision in many ways.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure7a.png}
\includegraphics[width=0.4\textwidth]{figure7b.png}
\caption{Variation of the Seebeck coefficient with $T$ in the weak magnetic field (a) for $u$ quarks, (b) for $d$ quarks.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure8a.png}
\includegraphics[width=0.4\textwidth]{figure8b.png}
\caption{(a) Variation of the Seebeck coefficient with $T$ for the composite medium, (b) Seebeck coefficient with respect to $T$ in the BGK collision term for different strengths of $\mu$.}
\end{figure}
Figure 9: (a) Ratio of Seebeck coefficient in BGK to that in RTA with $T$ for $L$ modes,  (b) Ratio of Seebeck coefficient in BGK to that in RTA with $T$ for $R$ modes.

Figure 10: Variation of Nernst coefficient with respect to $T$ in the weak magnetic field (a) for $u$ quarks, (b) for $d$ quarks.
Figure 11: (a) Variation of Nernst coefficient with respect to $T$ for the composite medium, (b) Nernst coefficient with respect to $T$ in the BGK collision term for different strengths of $\mu$.

Figure 12: (a) Ratio of Nernst coefficient in BGK to that in RTA with temperature for $L$ modes, (b) Ratio of Nernst coefficient in BGK to that in RTA with temperature for $R$ modes.

In Fig. 10, we have examined the collision integral dependence of the Nernst coefficient for $u$ (left panel) and $d$ (right panel) quarks. We observe that magnitude of the Nernst coefficient gets changed drastically in the BGK collision term in comparison to RTA. The ratio BGK/RTA for the Nernst coefficient is less than one in the case of left-handed modes, which shows that BGK collision integral causes a reduction in the magnitude of the Nernst coefficient [see Fig. 12 (a)].
The ratio is in the range $0.67 - 0.65$ for the $u$ quarks, while in the range $0.67 - 0.64$ and $0.68 - 0.65$ for the $d$ quarks and for the medium, respectively. In the case of $R$ modes, ratio becomes greater than unity and its value is found to be in the range $1.89 - 1.23$ for $u$ quarks, $2.00 - 1.26$ for $d$ quarks and $1.83 - 1.23$ for the medium [see Fig. 12 (b)]. One important observation we notice that there is no difference in the Nernst coefficient corresponding to the $L$ and $R$ modes in RTA for $u$ and $d$ quarks but in the BGK the magnitude of the Nernst coefficient is greater for $R$ modes. In Fig 11 (b), we study the $\mu$ dependance of the medium Nernst coefficient in BGK collision integral. It is not much visible in the case of $L$ modes except near the transition temperature, where it gets slightly reduced as $\mu$ increases. In the case of $R$ modes, the Nernst coefficient increases with $\mu$ at a fixed value of temperature.

5 Conclusion

To conclude, we have investigated the thermoelectric response of a hot and magnetized QCD medium produced in the non-central collisions at RHIC and LHC. We have employed Boltzmann transport equation linearized by the BGK collision integral, which conserves the particle number and current instantaneously. We incorporate the medium effects via dispersion relation wherein $T$, $\mu$ and $B$ dependent masses have been calculated using the imaginary-time formalism of the finite temperature QCD. In the absence of $B$, the Seebeck coefficient gets reduced in the BGK collision term for $u$, $d$ and $s$ quarks whereas it gets enhanced for the composite medium. In the strong $B$ background, the magnitude of the individual as well as medium Seebeck coefficient get lifted in the BGK collision term. Seebeck coefficient of the medium gets enhanced with the quark chemical potential in both the cases. In addition to the Seebeck coefficient, Nernst coefficient also appears in the weak $B$. The Seebeck coefficient gets slightly reduced in the BGK collision integral for $L$ modes, while for $R$ modes, both the collision integral give same results. On the other hand, Nernst coefficient gets changed drastically in the BGK collision term and its magnitude gets reduced (enhanced) for $L$ ($R$) modes in comparison to RTA. Both Seebeck and Nernst coefficients increase with $\mu$ for both $L$ and $R$ modes.

A nonvanishing Seebeck coefficient will modify the electric as well as heat current in the medium. The electric current in the presence of Seebeck effect becomes $J = \sigma_{el}E - \sigma_{el}S\nabla T$, while thermal conductivity gets modified as $\kappa = \kappa_0 - T\sigma_{el}S^2$. Both electrical and thermal conductivities should take positive values in accordance with the second law of thermodynamics. i.e. $T\partial_\mu S^\mu > 0$. Hence a positive Seebeck coefficient will always reduce the electric current and the thermal conductivity. It will be also interesting to take the thermoelectric effects into the account in the calculation of the entropy production, which has been completely neglected in [95, 96]. Moreover, thermoelectric coefficients could also be relevant in the context of the spin Hall effect (SHE). In SHE, a transverse spin current is generated due to the external electric field but the life-time of such electric field produced in the heavy ion collisions could be too small to observe the SHE. The electric field produced due to the temperature gradients in the medium may induce spin Hall effect in a hot and dense strongly interacting matter produced in heavy-ion collisions [97]. So the study of the various implications of thermoelectric effects in the hot and dense medium needs further investigation.
Acknowledgements

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Appendices

A  Derivation of equation (50)

BGK collision term is given by (46) as

\[
C[f_i] = -p^\mu u_\mu f_i \left( f_i - \frac{n_i}{n_{eq,i}} f_{eq,i} \right)
\]

\[
= -p^\mu u_\mu f_i \left( f_i - \frac{g_i \int f_{eq,i} + \delta f_i}{n_{eq,i}} \right)
\]

\[
= -p^\mu u_\mu f_i \left( f_i - \frac{g_i \int f_{eq,i} + g_i \int \delta f_i}{n_{eq,i}} \right)
\]

\[
= -p^\mu u_\mu f_i \left( \delta f_i - g_i n_{eq,i}^{-1} f_{eq,i} \int \delta f_i \right).
\]

(A.129)

B  Boltzmann Equation in the weak magnetic field

The RBTE (41) can be written with the BGK collision integral as

\[
p^\mu \frac{\partial f_i}{\partial x^\mu} + q_i F^{\sigma\rho} \frac{\partial f_i}{\partial p_\sigma} = -p^\mu u_\mu f_i \left( f_i - \frac{n_i}{n_{eq,i}} f_{eq,i} \right)
\]

(B.130)

where \( F^{\sigma\rho} p_\rho = (q^0 \vec{v}, F^i \vec{p}, p^0 \vec{F}) \), is the covariant form of the Lorenz force \( \vec{F} = q (\vec{E} + \vec{v} \times \vec{B}) \).

We can write Eqn. (B.130) using \( F^{0i} = -E^i \) and \( 2F_{ij} = \epsilon_{ijk} B^k \) (\( \epsilon_{ijk} \) is anti-symmetric Levi-Civita tensor) as

\[
\frac{\partial f_i}{\partial t} + \vec{v} \cdot \frac{\partial f_i}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f_i}{\partial \vec{p}} = -\nu_i \left( f_i - \frac{n_i}{n_{eq,i}} f_{eq,i} \right)
\]

(B.131)

considering \( p^0 \) as an independent variable

\[
\frac{\partial}{\partial \vec{p}} \rightarrow \frac{\partial}{\partial \vec{p}} \frac{\partial}{\partial p^0} + \frac{\partial}{\partial \vec{p}} \frac{\partial}{\partial \vec{p}} + \frac{\partial}{\partial \vec{p}}
\]

(B.132)

Eqn. (B.131) takes the form

\[
\vec{v} \cdot \frac{\partial f_i}{\partial \vec{r}} + \vec{F} \cdot \frac{\partial f_i}{\partial \vec{p}} = -\nu_i \left( f_i - \frac{n_i}{n_{eq,i}} f_{eq,i} \right)
\]

(B.133)
C  Seebeck coefficient in relaxation time approximation

The Seebeck coefficient in the RTA collision term in $B = 0$ case has been calculated as

$$S = \frac{1}{2Tq} \left( \frac{L_1}{L_2} \right)$$  \hspace{1cm} (B.134)

$$L_1 = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\omega^2} \left\{ (\omega - \mu) f_{eq} (1 - f_{eq}) + (\omega + \mu) \bar{f}_{eq} (1 - \bar{f}_{eq}) \right\}$$  \hspace{1cm} (B.135)

$$L_2 = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\omega^2} \left\{ f_{eq} (1 - f_{eq}) + \bar{f}_{eq} (1 - \bar{f}_{eq}) \right\}$$  \hspace{1cm} (B.136)

and for the case of strong $B$ as

$$S_B = \frac{1}{2qT} \left( \frac{H_1}{H_2} \right)$$  \hspace{1cm} (B.137)

where

$$H_1 = \int \frac{dp_3}{(2\pi)^3} \frac{p_3^2}{\omega_3^2} r^B \left\{ (\omega - \mu) f_{eq}^B (1 - f_{eq}^B) + (\omega + \mu) \bar{f}_{eq}^B (1 - \bar{f}_{eq}^B) \right\}$$  \hspace{1cm} (B.138)

$$H_2 = \int \frac{dp_3}{(2\pi)^3} \frac{p_3^2}{\omega_3^2} r^B \left\{ f_{eq}^B (1 - f_{eq}^B) + \bar{f}_{eq}^B (1 - \bar{f}_{eq}^B) \right\}$$  \hspace{1cm} (B.139)

In other work, the thermoelectric response of the hot QCD medium has been studied in weak magnetic field, where Seebeck and Nernst coefficients are found to be

$$S = -\frac{(C_1 C_3 + C_2 C_4)}{C_1^2 + C_2^2}, \hspace{1cm} (B.140)$$

$$N|B| = \frac{(C_2 C_3 - C_1 C_4)}{C_1^2 + C_2^2}, \hspace{1cm} (B.141)$$

provided $C_1 = qI_1$, $C_2 = qI_2$, $C_3 = \beta I_3$ and $C_4 = \beta I_4$. The integrals $I_1$, $I_2$, $I_3$ and $I_4$ are given by

$$I_1 = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\epsilon^2} \frac{\tau}{1 + \omega_c^2 \tau^2} \left\{ f_0 (1 - f_0) + \bar{f}_0 (1 - \bar{f}_0) \right\}$$  \hspace{1cm} (B.142)

$$I_2 = \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\epsilon^2} \frac{\omega_c \tau^2}{1 + \omega_c^2 \tau^2} \left\{ f_0 (1 - f_0) - \bar{f}_0 (1 - \bar{f}_0) \right\}$$  \hspace{1cm} (B.143)

$$I_3 = -\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\epsilon^2} \frac{\tau}{1 + \omega_c^2 \tau^2} \left\{ (\epsilon - \mu) f_0 (1 - f_0) - (\epsilon + \mu) \bar{f}_0 (1 - \bar{f}_0) \right\}$$  \hspace{1cm} (B.144)

$$I_4 = -\int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\epsilon^2} \frac{\omega_c \tau^2}{1 + \omega_c^2 \tau^2} \left\{ (\epsilon - \mu) f_0 (1 - f_0) + (\epsilon + \mu) \bar{f}_0 (1 - \bar{f}_0) \right\}$$  \hspace{1cm} (B.145)
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