New Fantastic Curves Discovered from Rectangular Hyperbola

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

By means of constructing on the hyperbola \(xy = N\) a triangle whose shape is changed under the rule that one vertex is fixed at the vertex of the hyperbola, one vertex is moving on the hyperbola and the two laterals with respect to the fixed and the moving vertices respectively keep their directions unchanged, it is discovered that the loci of the triangle’s centroid and orthocenter are respectively a hyperbola and a line, the locus of the circumcenter is a new cubic algebraic curve, and those of the incenter and ex-centers are planar curves that have not been reported before. All the loci of the centers form a fantastic graph like a flying insect. Meanwhile, the discovered hyperbola and curves are merely \(N\)-dependent and can be used to estimate the distribution of \(y\) divided by \(x\) with respect to \(xy = N\).

Keywords: Rectangle hyperbola; locus; centroid; circumcenter; incenter; orthocenter.

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1 Introduction

The hyperbola was early studied by Menaechmus, Euclid and Aristaeus, as reviewed in [1]. It was formally and particularly studied systematically in Apollonius’s book of conics [2] and [3]. Since

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the analytic geometry came into being, it has been a primary knowledge for middle school students and even college students of engineering to learn. These can be seen in either "older" textbooks like [4][5] or "newer" textbooks like [6][7]. Everyone who learned the analytic geometry knows that all the textbooks focus a lot on the so-called standard form in the Cartesian coordinate system while another form $xy = N$ has been less exploited because the later is thought to be a transformed form from the standard form via coordinate transformations. Our recent study on integer factorization problem came across the literatures [8] and [9]. Accordingly we were aroused to have a study on the kind of the hyperbola. After the study, we gained windfall benefit of discovering several new fantastic curves. This paper introduces these curves as well as some of their applications. This paper is composed of 6 sections. Section 1 is this introductory part, the section 2 lists and proves some primary and fundamental properties that are cited in later sections, section 3 presents the newly found curves, section 4 shows the key points to draw multiple figures into one graph with Maple software, section 5 shows an application in analyzing divisor-ratio of a semiprime and the last section is the prospect for the future work.

2 Preliminaries

This section presents the necessary preliminaries for later descriptions, including symbols, notations and fundamental geometrical elements that support later researches in this paper.

2.1 Symbols and notations

In this whole paper, symbol $A \Rightarrow B$ means statement $A$ can derive out statement $B$, symbol $|AB|$ means the length of line segment $AB$, symbol $P : (x, y)$ means $x$ and $y$ are coordinates of point $P$ and symbol $\Gamma : f(x, y) = 0$ means $\Gamma$ is defined by the equation $f(x, y) = 0$. An odd interval $[a, b]$ means the interval contains odd integer from $a$ to $b$; for example, $[3, 10]$ is equivalent to the set $3, 5, 7, 9, 11$.

2.2 Necessary mathematical foundations

Given a real number $N > 0$, this paper mainly investigates the hyperbola $H: xy = N$ with $x > 0$ and $y > 0$ in the Cartesian coordinate system. For convenience, the capital letter $H$ is sometimes of the same meaning as $H : xy = N$. It is easy to establish the following basic properties, which are utilized in later sections.

**Property 1.** In the rectangular coordinate system $XOY$, as shown in Fig. 1, the vertex and focus of the hyperbola $H : xy = N$ are $P : (\sqrt{N}, \sqrt{N})$ and $P : (\sqrt{2N}, \sqrt{2N})$, respectively. The tangent line at $P$ to $H$ is $x + y = 2\sqrt{N}$.

**Property 2.** Let $Q$ be the intersection of $H$ with the line $y = 2x$, $U$ and $V$ be respectively the intersections of the horizontal line passing through $Q$ with $Y$-axis $x = 0$ and the line $y = x$, as shown in Fig. 2; then $U$ is coincided with the focus of $H$ and $|QU| = |QV|$.

**Proof.** The coordinates of $Q, U$ and $V$ are simply calculated to be

$$Q : (\sqrt{\frac{N}{2}}, \sqrt{2N}),$$

$$U : (\sqrt{2N}, \sqrt{2N})$$

and

$$V : (0, \sqrt{2N}).$$
By Property 1, $U$ is surely coincided with the focus of $H$ and it follows

$$|QU| = \sqrt{2N} - \sqrt{\frac{N}{2}} = \frac{\sqrt{2N}}{2} = |QV|$$

□

**Remark 1.** The divisor-ratio of a semiprime $N = pq$ with odd prime divisors $p$ and $q$ satisfying $2 < p < q$ is defined to be $\alpha = q/p$. $\alpha > 2$ or $\alpha < 2$ is critical for a very big semiprime like the RSA numbers, as studied in [11] and [10]. Therefore, $P$ and $Q$ are two critical points in analyzing the divisor-ratio of the big semiprimes. Actually, as stated in the introductory section, this whole paper is an extra gain that originated from a research to determine whether the divisor-ratio of a given RSA number is bigger or smaller than 2. For this reason and future citations, we list this property and the following Properties 3 and 4.

**Property 3.** Given real numbers $\alpha_1 = \left(\frac{71}{24} + \frac{17\sqrt{6}288}{24}\right)^2$, $\alpha_2 = \left(\frac{71}{24} - \frac{17\sqrt{6}288}{24}\right)^2$; assume $S_1$ and $S_2$ are respectively the intersections of $H$ with the lines $y = \alpha_1x$ and $y = \alpha_2x$. Construct two horizontal lines passing through $S_1$ and $S_2$, respectively; then the ellipse $\Gamma_x$ defined by

$$\Gamma_x : 150x^2 - 276xy + 150y^2 - 71\sqrt{N}x - 71\sqrt{N}y + 159N = 0$$

(2.1)
is tangent to the two horizontal lines, as shown in Fig. 3.

![Fig. 3. Ellipse tangent to two lines](image)

**Proof.** Direct calculations show the coordinates of $S_1$ and $S_2$ are respectively

$$S_1: (x_1, y_1) = \left( \sqrt{\frac{N}{\alpha_1}}, \sqrt{\frac{N}{\alpha_1}N} \right) = \left( \sqrt{\frac{N}{\frac{21}{24} + \frac{175\sqrt{6}}{288}}} \right), \left( \frac{71}{24} + 175\sqrt{6} \right) \sqrt{\frac{N}{\alpha_1}}$$

and

$$S_2: (x_2, y_2) = \left( \sqrt{\frac{N}{\alpha_2}}, \sqrt{\frac{N}{\alpha_2}N} \right) = \left( \sqrt{\frac{N}{\frac{21}{24} - \frac{175\sqrt{6}}{288}}} \right), \left( \frac{71}{24} - 175\sqrt{6} \right) \sqrt{\frac{N}{\alpha_2}}.$$ 

Substituting $y$ in (2.1) with $y_0$ yields an equation of $x$ by

$$150x^2 - 276x \sqrt{\alpha_1}N - 71x \sqrt{N} + 150\alpha_1N - 71N \sqrt{\alpha_1} + 159N = 0 \quad (2.2)$$

Since the equation 2.2 has a unique solution $x = \left( \frac{71}{24} + \frac{161\sqrt{6}}{288} \right) \sqrt{\frac{N}{\alpha_1}}$, it is known that the point $T_1: \left( \left( \frac{71}{24} + \frac{161\sqrt{6}}{288} \right) \sqrt{\frac{N}{\alpha_1}}, \left( \frac{71}{24} + \frac{175\sqrt{6}}{288} \right) \sqrt{\frac{N}{\alpha_1}} \right)$ is a tangent point of $\Gamma_e$. Similarly, it can be shown that $T_2: \left( \left( \frac{71}{24} - \frac{161\sqrt{6}}{288} \right) \sqrt{\frac{N}{\alpha_1}}, \left( \frac{71}{24} - \frac{175\sqrt{6}}{288} \right) \sqrt{\frac{N}{\alpha_1}} \right)$ is a tangent point of $\Gamma_e$. 

\[\square\]

**Remark 2.** It can verified that, the center of $\Gamma_e$ is $O_e: \left( \frac{71}{24} \sqrt{N}, \frac{71}{24} \sqrt{N} \right)$, the length of its major axis is $\frac{35\sqrt{N}}{24}$, length of its minor axis is $\frac{35\sqrt{N}}{24}$, and two foci are respectively $\left( \frac{71}{24} \sqrt{N}, \frac{71}{24} \sqrt{N} \right)$ and $\left( -\frac{71}{24} \sqrt{N}, -\frac{71}{24} \sqrt{N} \right)$. Since the major axis is coincided with the line $y = x$, its parametric equation is given by

$$\begin{align*}
x &= \frac{71}{24} \sqrt{N} + \frac{35\sqrt{N}}{24} \cos t + \frac{35\sqrt{6}N}{288} \sin t, \\
y &= \frac{71}{24} \sqrt{N} + \frac{35\sqrt{N}}{24} \cos t - \frac{35\sqrt{6}N}{288} \sin t,
\end{align*} \quad 0 \leq t \leq 2\pi \quad (2.3)
$$

For convenience to construct the ellipse $\Gamma_e$ and to calculate with its characteristic points, here choose several important points and mark them with Fig. 4. In the figure, $V_1$ and $V_2$ are two
vertices on the major axis, $U_1$ and $U_2$ are two vertices on the minor axis, $M_1$ and $M_2$ are middle points of the half major axis, $J_1J_2\perp V_1V_2$ and $K_1K_2\perp V_1V_2$, and $X_1$ and $X_2$ that are symmetric to $T_1$ and $T_2$ respectively by the major axis. The coordinates of these points are given as follows.

\[
V_1 : \left(\frac{53}{12}\sqrt{N}, \frac{53}{12}\sqrt{N}\right),
\]

\[
T_1 : \left(\frac{71}{24} + \frac{161\sqrt{6}}{288}\sqrt{N}, \frac{71}{24} + \frac{175\sqrt{6}}{288}\sqrt{N}\right),
\]

\[
J_1 : \left(\frac{59\sqrt{N}}{16} - \frac{35\sqrt{2}N}{192}, \frac{59\sqrt{N}}{16} + \frac{35\sqrt{2}N}{192}\right),
\]

\[
U_1 : \left(\frac{71}{24} - \frac{35\sqrt{6}}{288}\sqrt{N}, \frac{71}{24} + \frac{35\sqrt{6}}{288}\sqrt{N}\right),
\]

\[
K_1 : \left(\frac{107\sqrt{N}}{48} - \frac{35\sqrt{2}N}{192}, \frac{107\sqrt{N}}{48} + \frac{35\sqrt{2}N}{192}\right),
\]

\[
X_2 : \left(\frac{71}{24} - \frac{175\sqrt{6}}{288}\sqrt{N}, \frac{71}{24} - \frac{161\sqrt{6}}{288}\sqrt{N}\right),
\]

\[
V_2 : \left(\frac{3}{2}\sqrt{N}, \frac{3}{2}\sqrt{N}\right),
\]

\[
T_2 : \left(\frac{71}{24} - \frac{161\sqrt{6}}{288}\sqrt{N}, \frac{71}{24} - \frac{175\sqrt{6}}{288}\sqrt{N}\right),
\]

\[
K_2 : \left(\frac{107\sqrt{N}}{48} + \frac{35\sqrt{2}N}{192}, \frac{107\sqrt{N}}{48} - \frac{35\sqrt{2}N}{192}\right),
\]

\[
J_2 : \left(\frac{59\sqrt{N}}{16} + \frac{35\sqrt{2}N}{192}, \frac{59\sqrt{N}}{16} - \frac{35\sqrt{2}N}{192}\right),
\]

\[
U_2 : \left(\frac{71}{24} + \frac{35\sqrt{6}}{288}\sqrt{N}, \frac{71}{24} - \frac{35\sqrt{6}}{288}\sqrt{N}\right),
\]

\[
X_1 : \left(\frac{71}{24} + \frac{175\sqrt{6}}{288}\sqrt{N}, \frac{71}{24} + \frac{161\sqrt{6}}{288}\sqrt{N}\right).
\]

**Property 4.** Let $S$ be the intersection of $H$ with the line $l_2 : y = 2x$, $l_3$ be the line passing through $S$ and perpendicular to $l_2$ and $P_1$ be the ellipse defined by (1); transform $P_1$ to be $P_2$ such that $P_2$ passes through $S$ and its major axis is coincident with $l_3$, as shown in Fig. 5. Let $P_2^*$ be the reflection of $P_1$ with respect to $l_2$. Then equations of $P_2$ and $P_2^*$ are given by (2.4) and (2.5) respectively.

\[
28x^2 + 92xy + 97y^2 - \sqrt{2N}(120 + \frac{35\sqrt{5}}{6})x - \sqrt{2N}(240 - \frac{35\sqrt{5}}{12})y + 300N = 0 \tag{2.4}
\]

\[
28x^2 + 92xy + 97y^2 - \sqrt{2N}\left(120 - \frac{35\sqrt{5}}{6}\right)x - \sqrt{2N}\left(240 + \frac{35\sqrt{5}}{12}\right)y + 300N = 0 \tag{2.5}
\]

And $H$ has a part lying inside $P_2$ and a part lying inside $P_2^*$.

**Proof.** The equations (2.4) and (2.5) are easily obtained through coordinate transformations from $P_1$ to $P_2$ and from $P_2$ to $P_2^*$. Here simply show $H$ has a part lying inside the ellipse $P_2$. Let $ST$ be the tangent line at $S$ of $H$; then slope of $ST$ is $k_s = -\frac{N}{x_S} < 0$, where $x_S$ is the horizontal coordinate.
Fig. 4. Points on ellipse $\Gamma_e$

of $S$. Since $S$ is also on $l_2$ whose slope is 2, there is an angle between $ST$ and $l_2$. The condition that $l_2$ is the tangent line at $S$ of the ellipse $\Gamma_s$ yields that $H$ crosses $\Gamma_s$, which means $H$ has a part lying inside $\Gamma_s$. Similarly, $H$ has a part lying inside $\Gamma_s^*$. □

Remark 3. The ellipses $\Gamma_e$ to be $\Gamma_s$ are highly related with $H$. Referring to Remark 1, it is known they are useful in analyzing the divisor-ratio of a semiprime.

3 Research Method and Results

In the Cartesian coordinate system, consider the hyperbola $H: xy = N$ and two lines $l_1: y = x$ and $l_{\alpha}: y = \alpha x$ with $\alpha > 0$; let $P$ and $Q$ be respectively the intersections of $H$ with $l_1$ and $l_{\alpha}$, as shown in Fig. 6. Construct at $P$ a line parallel to $l_{\alpha}$, at $Q$ a line parallel to $l_1$; denote $R$ to be the intersection of the two constructed lines. The triangle $PQR$ is obviously $\alpha$ dependant because its shape changes with the change of $\alpha$. Since $P$, $Q$ and $R$ are collinear when $\alpha = 1$, $\alpha \neq 1$ is assumed by default in this section unless particularly mentioned.

3.1 New provable results induced from the triangle PQR

Property 5. The triangle $PQR$ is an obtuse one with $\angle Q$ being the obtuse angle.

Proof. Let the coordinates of $P$ and $Q$ be $P: (x_P, y_P)$ and $Q: (x_Q, y_Q)$; direct calculations show that $R$’s coordinate is $R: (x_R, y_R) = (x_P + x_Q, y_P + y_Q)$. That is

$$\begin{align*}
x_R &= x_P + x_Q = \sqrt{N}(1 + \frac{1}{\sqrt{\alpha}}) \\
y_R &= y_P + y_Q = \sqrt{N}(1 + \sqrt{\alpha}),
\end{align*}$$

(3.1)

Accordingly,
Fig. 5. Ellipses $\Gamma_e$, $\Gamma_s$ and $\Gamma_S^*$

\[ |QR| = \sqrt{x_P^2 + y_P^2} = \sqrt{2N} \]
\[ |PR| = \sqrt{x_Q^2 + y_Q^2} = \sqrt{\left(\frac{1}{\sqrt{\alpha}} + \alpha\right)N} \]
\[ |PQ| = \sqrt{\left(\frac{1}{\sqrt{\alpha}}\right)^2 + (\sqrt{\alpha} - 1)^2N} \]

Note that
\[ \sqrt{\left(\frac{1}{\sqrt{\alpha}} + \alpha\right)} - \sqrt{\left((1 - \frac{1}{\sqrt{\alpha}})^2 + (\sqrt{\alpha} - 1)^2\right)} = \frac{2\left(\frac{1}{\sqrt{\alpha}} + \sqrt{\alpha}\right) - 2}{\sqrt{\left(\frac{1}{\sqrt{\alpha}} + \alpha\right)^2 + (\sqrt{\alpha} - 1)^2}} < 0 \]

\[ \frac{1}{\sqrt{\alpha}} + \sqrt{\alpha} > 2 \]

Since
\[ \cos \angle Q = \frac{|QR|^2 + |PQ|^2 - |PR|^2}{2|PQ||QR|} = \frac{2 - 1\left(\frac{1}{\sqrt{\alpha}} + \sqrt{\alpha}\right)}{\sqrt{2\left(\left(1 - \frac{1}{\sqrt{\alpha}}\right)^2 + (\sqrt{\alpha} - 1)^2\right)}} < 0, \]

it yields $\angle Q > 90^\circ$

Property 6. The vertex $R$ is on hyperbola $(x - \sqrt{N})(y - \sqrt{N}) = N$, as illustrated in Fig. 7.

Proof. Direct calculations by (3.1) follows

\[ \begin{cases} x_R - \sqrt{N} = \frac{\sqrt{N}}{\sqrt{\alpha}} \\ y_R - \sqrt{N} = \frac{\sqrt{N}}{\sqrt{\alpha}} \end{cases} \Rightarrow (x_R - \sqrt{N})(y_R - \sqrt{N}) = N \]

which surely means $R: (x_R, y_R)$ is on the hyperbola $(x - \sqrt{N})(y - \sqrt{N}) = N$. 

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Remark 4. Seen from Property 6, the locus of the vertex R forms a hyperbola when \( \alpha \) changes, as described in Fig. 8. For convenience, this hyperbola is denoted by \( H^* \) and from now on called a companion hyperbola of \( H \) or simply companion and \( \Delta PQR \) is called a companion triangle.

Property 7. The centroid of \( \Delta PQR \) is on the hyperbola \((x - \frac{2}{3} \sqrt{N})(y - \frac{2}{3} \sqrt{N}) = \frac{4}{9} N\).

Proof. Let \( G \) be the centroid. First consider \( P, Q \) and \( R \) are not collinear. Then \( G \)'s coordinates are given by

\[
\begin{align*}
\begin{cases}
  x_G = (x_P + x_Q + x_R)/3 \\
  y_G = (y_P + y_Q + y_R)/3
\end{cases}
\end{align*}
\]

By (3.1) it follows

\[
\begin{align*}
\begin{cases}
  x_G = 2(x_P + x_Q)/3 = 2(\sqrt{N} + \sqrt{\frac{N}{\alpha}})/3 \\
  y_G = 2(y_P + y_Q)/3 = 2(\sqrt{N} + \sqrt{\alpha N})/3
\end{cases}
\end{align*}
\]

That is

\[
(x_G - \frac{2}{3} \sqrt{N})(y_G - \frac{2}{3} \sqrt{N}) = \frac{4}{9} N
\]

which is surely on the hyperbola \((x - \frac{2}{3} \sqrt{N})(y - \frac{2}{3} \sqrt{N}) = \frac{4}{9} N\)

When \( P, Q \) and \( R \) are collinear, it yields \( \alpha = 1 \). This time by (3.1) it holds

\[
\begin{align*}
\begin{cases}
  x_G = \frac{4}{3} \sqrt{N} \\
  y_G = \frac{4}{3} \sqrt{N}
\end{cases}
\Rightarrow
\begin{cases}
  x_G - \frac{2}{3} \sqrt{N} = \frac{2}{3} \sqrt{N} \\
  y_G - \frac{2}{3} \sqrt{N} = \frac{2}{3} \sqrt{N}
\end{cases}
\Rightarrow
\begin{cases}
  (x_G - \frac{2}{3} \sqrt{N})(y_G - \frac{2}{3} \sqrt{N}) = \frac{4}{9} N
\end{cases}
\end{align*}
\]
Fig. 7. Hyperbola H: \( xy = N \) and locus of \( R \)

which means \( G \) is still on the hyperbola \((x - \frac{\sqrt{2}}{3}\sqrt{N})(y - \frac{\sqrt{2}}{3}\sqrt{N}) = \frac{4}{9}N\).

\[\square\]

**Remark 5.** Seen from (3.2) and (3.3), the centroid \( G \) and the vertex \( R \) are on the line \( y = \sqrt{\alpha}x \) because \( \frac{y_R}{x_R} = \frac{y_G}{x_G} = \sqrt{\alpha} \).

**Property 8.** The orthocenter of \( \triangle PQR \) is on line \( l : x + y = 2\sqrt{N} \), which is tangent to H: \( xy = N \) at \( P : (\sqrt{N}, \sqrt{N}) \).

**Proof.** Let \( h \) be orthocenter. Direct calculations yields its coordinates by

\[
\begin{align*}
x_h &= -\frac{\alpha^2 - 2\alpha\sqrt{\alpha} + 1}{(\alpha - 1)\sqrt{\alpha}} \times \sqrt{N} \\
y_h &= \frac{\alpha^2 - 2\alpha\sqrt{\alpha} + 1}{(\alpha - 1)\sqrt{\alpha}} \times \sqrt{N} + 2\sqrt{N}
\end{align*}
\]

(3.4)

It immediately follows \( x_h + y_h = 2\sqrt{N} \), saying that \( H \) is on the line \( l \). By Property 1, \( l \) is tangent to \( H \) at \( P \).

\[\square\]

**Remark 6.** Take arbitrary two lines \( l_\alpha : y = \alpha x \) and \( l_\beta : y = \beta x \) with \( 1 < \beta \leq \alpha \), construct their companion triangles and draw the orthocenters \( I_\alpha \) and \( I_\beta \) respectively, as illustrated in Fig 9; it is seen that \( I_\alpha \) and \( I_\beta \) are surely on the tangent line at \( P \) of \( H \), provided that \( P, Q \) and \( R \) are not collinear.

### 3.2 New cubic algebraic curve derived from the triangle PQR

**Theorem 1.** The locus of the circumcenter of \( \triangle PQR \) is a new planar algebraic curve whose algebraic equation is given by

\[
x^3 + y^3 + (x^2 + y^2)\sqrt{N} - (x + 2\sqrt{N} + y)xy - (x + y)N + 3N\sqrt{N} = 0
\]

(3.5)
The curve is symmetric with respect to the line $y = x$ and passes through the point $V : (\frac{3}{2} \sqrt{N}, \frac{3}{2} \sqrt{N})$.

**Proof.** Let $C$ be circumcenter and $\Gamma_C$ be the locus formed by $C$’s changing with the change of $\Delta PQR$. By definition, $C$ is on the intersection of the three perpendicular bisectors of the laterals $PQ$, $QR$ and $RP$, as shown in Fig. 10. This means $\Gamma_C$ passes the point $V : (\frac{3}{2} \sqrt{N}, \frac{3}{2} \sqrt{N})$ because $C$ is on the line $y = x$ in the case that the line $OQ$ is limiting to and coincided with $OP$.

After mathematical reasoning, $C$’s coordinates are calculated by

$$
\begin{align*}
x_C &= \left( \frac{1}{1 + \sqrt{\alpha}} + \frac{\alpha + 1}{2\sqrt{\alpha}} \right) \times \sqrt{N} \\
y_C &= \left( \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} + \frac{\alpha + 1}{2\sqrt{\alpha}} \right) \times \sqrt{N}
\end{align*}
$$

(3.6)

With Maple command `implicitize` of the `algcurves` package, the implicit cubic algebraic equation (3.5) is derived out.

□

**Remark 7.** The parametric equation (3.6) is easily obtained by simple mathematical reasoning. However, its algebraic equation (3.5) is hard to obtain without the help of the Maple software. It took us several months to gain the equation (3.5). Drawing (3.6) with the Maple software yields an ellipse-arc like curve as seen in Fig.11.

Then we tried to find out an ellipse to match it. By taking two values $\alpha_1$ and $\alpha_2$ with $1 < \alpha_1 < \alpha_2$, we first calculated four points with $\frac{1}{\alpha_2}, \frac{1}{\alpha_1}, \alpha_1$ and $\alpha_2$. With the calculated four points and the vertex $V : (\frac{3}{2} \sqrt{N}, \frac{3}{2} \sqrt{N})$, we calculated an ellipse with Maple. After many times of experiments, we judged that the calculated ellipse was $\Gamma_e$ defined by (2.1). When we drew $\Gamma_C$ with $\frac{1}{\alpha_2} \leq \alpha \leq \alpha_2$ and half of $\Gamma_e$ with the Maple software we got Fig. 12. In the figure, the little red squares are points on $\Gamma_e$, the black-colored curve is from $\Gamma_C$, the pink curve is H and the green curve is H*. It is seen that

![Fig. 8. Hyperbola H and its companion H*](image-url)
the head of $\Gamma_C$ looks highly coincided with $\Gamma_\alpha$ in $\alpha$’s domain from $\frac{\alpha_1}{\alpha_2}$ to $\alpha_2$. Accordingly we took it for granted that $\Gamma_C$ was a part of the ellipse (1). That is why we spent a lot time on studying $\Gamma_\epsilon$.

However, drawn with another software Geogebra, the locus is not all coincided with $\Gamma_\epsilon$, as shown in Fig. 13. It has a head and a tail with the head looking coincided with a part of $\Gamma_\epsilon$ and the tail looking like two lines.

Actually, letting $t = \sqrt{\alpha}$ turns the formula (3.6) into the following (3.7)

$$
\begin{aligned}
\begin{cases}
  x_C = (\frac{1}{1+t} + \frac{t}{2t} + \frac{t}{2}) \times \sqrt{N}, \\
  y_C = (\frac{t}{1+t} + \frac{1}{2t} + \frac{t}{2}) \times \sqrt{N}
\end{cases},
\end{aligned}
$$

0 < t < \infty

(3.7)
Then it follows

\[
\begin{align*}
x_C' &= \left( 1 - \frac{1}{2t^2} - \frac{1}{(1+t)^2} \right) \times \sqrt{N} \\
y_C' &= \left( 1 + \frac{1}{2t^2} - \frac{1}{(1+t)^2} \right) \times \sqrt{N}
\end{align*}
\]

and

\[
\begin{align*}
x_C'' &= \left( \frac{1}{t^3} + \frac{2}{(1+t)^3} \right) \times \sqrt{N} \\
y_C'' &= \left( \frac{1}{t^3} - \frac{2}{(1+t)^3} \right) \times \sqrt{N}
\end{align*}
\]

The curvature of \( \Gamma_C \) is calculated with

\[
\kappa(t) = \frac{|x'y'' - x''y'|}{|x''^2 + y''^2|^{3/2}} = \frac{4|t^2 - t + 1| \cdot (t + 1)^5 t^4 \sqrt{2t}}{(t^8 + 4t^7 + 4t^6 - 4t^5 - 6t^4 - 4t^3 + 4t^2 + 4t + 1)^{3/2}}
\]

This yields \( \lim_{t \to -\infty} \kappa(t) = 0 \) and \( \lim_{t \to 0} \kappa(t) = 0 \), saying that \( \Gamma_C \) is tending to be like a line when \( \alpha \) is either bigger or smaller than a certain value. This is surely fit for what is seen in Fig. 13. Looking
into the professional books such as books [12][13], there is not a curve matching to the shape of $\Gamma_C$. It is surely a new curve.

### 3.3 New fantastic curves derived from the triangle $PQR$

**New Curve 1.** The locus of the incenter of $\Delta PQR$ is a new planar curve passing through the point $V: \left(\frac{3}{2}\sqrt{N}, \frac{3}{2}\sqrt{N}\right)$. It has two branches symmetric with respect to the line $y = x$ and has a fantastic shape that has never been recorded.

**Comment.** Let $I$ be the incenter and $\Gamma_I$ be the locus formed by $I$’s changing with the change of $\Delta PQR$. Consider $\Delta PQR$ is over the line $y = x$, as shown in Fig.14. Since by definition $\angle Q > \angle R$ and $\angle Q > \angle P$ in $\Delta PQR$, the distance from $I$ to $Q$ is smaller than that from $I$ to $P$ or that from $I$ to $R$. As a result, $I$ is infinitesimally close to $P$ when $Q$ is infinitesimally close to $P$, or $\Delta PQR$ is tending to be collinear with the line $y = x$. In another word, $P$ is coincided with $I$ when $Q$ is limiting to $P$.
Denote \(a = |PQ|\), \(b = |PR|\) and \(c = |QR|\); then it follows
\[
\begin{align*}
    c &= \sqrt{x_P^2 + y_P^2} = \sqrt{2N} \\
    b &= \sqrt{x_Q^2 + y_Q^2} = \sqrt{\left(\frac{1}{\alpha} + \alpha\right)N} \\
    a &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{(1 - \sqrt{\frac{1}{\alpha}})^2 + (\sqrt{\alpha} - 1)^2}N
\end{align*}
\]
(3.8)

Direct calculations knows that the coordinates of the incenter \(I\) are given by
\[
\begin{align*}
    x_I &= \frac{ax_P + bx_Q + cx_P}{a + b + c} \\
    y_I &= \frac{ay_P + by_Q + cy_P}{a + b + c}
\end{align*}
\]
(3.9)
which is simplified to be
\[
\begin{align*}
    x_I &= \sqrt{N}(1 + \frac{1}{\alpha} - \frac{\sqrt{1 + \alpha^2 + \sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{\alpha} + 1 + \sqrt{1 + \alpha^2 + \sqrt{2}}\alpha}), \quad 0 < \alpha \neq 1
\end{align*}
\]
(3.10)

Drawing this locus with Maple yields the black curve in Fig.15.

Fig. 15. Locus of the incenter \(I\)

Enlarged and marked with some of its points, it is an \(S\)-shaped curve with two branches coincided at the point \(V\), as Fig. 16 shows. Looking into the professional books such as books [12][13], there is not a curve or even a similar one matching to the shape of \(\Gamma_I\). So far, there has not been such a record.

**Remark 8.** We tried to find out the algebraic equation for (3.10) but failed even with Maple and Geogebra. So far we have not known the algebraic equations for (3.10), (3.12), (3.14) and (3.16).

**New Curve 2.** The locus of the ex-center of the excircle touching \(PQ\) of \(\Delta PQR\) is a new planar curve that has a fantastic shape. It is symmetric with respect to the line \(y = x\) and passes through the point \(V : \left(\frac{2\sqrt{N}}{3}, \frac{2\sqrt{N}}{3}\right)\) at which it touches \(\Gamma_I\) defined by (3.10).
Comment. Take the ex-center $W_{PQ}$ that touches $PQ$ as an example. Let $T_{W-PQ}$ be the locus formed by $W_{PQ}$ that changes with the change of $\Delta PQR$. Consider $\Delta PQR$ is over the line $y = x$, as shown in Fig.17.

Since by definition $\angle Q > \angle P$ in $\Delta PQR$, it knows $\angle WQP < \angle QPW$ in $\Delta QPW$ and the distance from $W$ to $Q$ is bigger than that from $W$ to $P$. As a result, $W$ is infinitesimally close to $P$ when $Q$ is infinitesimally close to $P$, or $\Delta QPW$ is tending to be collinear with the line $y = x$. In another word, $P$ is coincided with $W$ when $Q$ is limiting to $P$.

Let $a$, $b$ and $c$ be defined as those in (3.8); then the coordinates of $W$ are calculated by

\[
\begin{align*}
    x_W &= \frac{bx_Q + cx_P - ax_R}{b + c - a} \\
    y_W &= \frac{by_Q + cy_P - ay_R}{b + c - a}
\end{align*}
\]  

(3.11)
which is simplified to be

\[
\begin{aligned}
x_{W} &= \sqrt{N}(1 + \frac{1}{\sqrt{\alpha}} + \frac{\sqrt{1 + \alpha^2 + \sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{1 + \alpha - \sqrt{1 + \alpha^2 - \sqrt{2}\alpha}}}), \quad 0 < \alpha \neq 1 \\
y_{W} &= \sqrt{N}(1 + \sqrt{\alpha} + \frac{\sqrt{1 + \alpha^2 + \alpha\sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{1 + \alpha - \sqrt{1 + \alpha^2 - \sqrt{2}\alpha}}})
\end{aligned}
\]

(3.12)

Drawing this locus \( L_{W-PQ} \) together with \( L_{I} \), leads to an interesting figure as Fig. 18 shows. The two loci share the point \( V \) and look like an \( X \). By the way, \( L_{W-PQ} \) is also \( S \)-shaped with two branches coincided at \( V \).

![Fig. 18. Loci of incenter and ex-center like an X](image)

**New Curve 3.** The locus of the ex-center of the excircle touching \( PR \) of \( \Delta PQR \) is a new planar curve that has a fantastic shape. It is symmetric with respect to the line \( y = x \).

**Comment.** The coordinates of the ex-center are calculated by

\[
\begin{aligned}
x &= \frac{ax_R + cx_P - bx_Q}{a + c - b} \\
y &= \frac{ay_R + cy_P - by_Q}{a + c - b}
\end{aligned}
\]

(3.13)

where \( a, b \) and \( c \) are defined in (3.8). Simplifying (3.13) yields

\[
\begin{aligned}
x &= \sqrt{N}(1 + \frac{1}{\sqrt{\alpha}} + \frac{\sqrt{1 + \alpha^2 + \sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{1 + \alpha - \sqrt{1 + \alpha^2 + \sqrt{2}\alpha}}}), \quad 0 < \alpha \neq 1 \\
y &= \sqrt{N}(1 + \sqrt{\alpha} + \frac{\sqrt{1 + \alpha^2 - \alpha\sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{1 + \alpha - \sqrt{1 + \alpha^2 + \sqrt{2}\alpha}}})
\end{aligned}
\]

(3.14)

Drawing the locus yields its shape as the green curve in Fig. 19.

**New Curve 4.** The locus of the ex-center of the excircle touching \( QR \) of \( \Delta PQR \) is a new planar curve that has a fantastic shape. It is symmetric with respect to the line \( y = x \).
**Comment.** The coordinates of the ex-center are calculated by

\[
\begin{aligned}
 x &= \frac{ax_R + bx_Q - cx_P}{a + b - c} \\
 y &= \frac{ay_R + by_Q - cy_P}{a + b - c}
\end{aligned}
\]  

(3.15)

where \(a, b\) and \(c\) are defined in (3.8).

Simplifying (3.15) yields

\[
\begin{aligned}
 x &= \sqrt{N}(1 + \frac{1}{\sqrt{\alpha}} - \frac{\sqrt{1 + \alpha^2 - \sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{1 + \alpha + \sqrt{1 + \alpha^2 - \sqrt{2}}} \alpha}) \\
 y &= \sqrt{N}(1 + \sqrt{\alpha} - \frac{\sqrt{1 + \alpha^2 - \sqrt{2}}}{|\sqrt{\alpha} - 1|\sqrt{1 + \alpha + \sqrt{1 + \alpha^2 - \sqrt{2}}} \alpha}), \quad 0 < \alpha \neq 1
\end{aligned}
\]  

(3.16)

Drawing the locus yields its shape as the blue curve in Fig. 20.

### 3.4 Fantastic figure consisted of the new curves

Drawing together the hyperbola \(H: xy = N\), the companion hyperbola \(H^*: (x - \sqrt{N})(y - \sqrt{N}) = N\), all the loci of the centers as well as the ellipse \(E\), we have a very fantastic figure like something of a beetle, as seen in Figs. 21. It looks like a flying insect!

### 4 Keys to Drawing the Loci with Maple

The loci introduced in the last section can be tested with the software such as Matlab, Mathematica, Geogebra and Maple. It is known that any curve can be drawn in any of the software if the equation of the curve is known in either algebraic form or parametric form. Geogebra is easy to draw several curves in one work sheet no matter what form of the curves’ equations are. However, it is better to use the same form for Maple to draw different curves in one graph. Seen in Maple’s online help, it is known that Maple command \(\text{plot}\) can plot a list of curves all expressed by parametric equations and command \(\text{implicitplot}\) can plot a list of curves all expressed in algebraic equations.
Since merely a small part of the new curves introduced in the previous sections have obtained their algebraic equations, it is better to use parametric equations to draw their loci. Considering that the equations (2.3), (3.1), (3.2), (3.4), (3.6), (3.10), (3.12), (3.14) and (3.16) are all in parametric forms, parameterization of the hyperbola $H_{xy} = N$ is necessary. With Maple command `solve`, parameterized solutions for the hyperbola $H$, the companion hyperbola $H^*$, the line $x + y = \sqrt{N}$ and the centroid hyperbola $G : (x - \frac{2}{3}\sqrt{N})(y - \frac{2}{3}\sqrt{N}) = \frac{4}{9}N$ can all be parameterized into their parametric equations. For convenience, the following Table 1 lists all the parametric equations related with this paper. Drawing all these curves yields Fig. 21.

5 Application to Analyze Divisor-ratio of Semiprime

Consider $N$ is a composite integer with integers $p$ and $q$ being its divisors with $pq = N$ and $1 < p \leq q$. We are intending to know the range of $\alpha = \frac{q}{p}$ before $N$ is factorized. Since $1 < p \leq \sqrt{N}, \sqrt{N} \leq q$ and the point $(p, q)$ is on the hyperbola $xy = N$ and over the line $y = x$. Assume $\alpha$ is to be checked if it is smaller than 2, for example; then referring to Property 4 and the ellipse (2.4), which is for convenience called an $\alpha$-detecting ellipse, yields the following Theorem 2.

**Theorem 2.** Let $N = pq$ be a semiprime with $1 < p < q$ and $\alpha = \frac{q}{p}$; then $1 < \alpha < 2$ if and only if $p$ and $q$ are respectively in the integer solutions of $x$ and $y$ of the following inequalities (5.1)

\[
\begin{align*}
xy &= N \\
1 < x \leq \sqrt{N} &\leq y \\
28x^2 + 92xy + 97y^2 - (120\sqrt{2N} + \frac{35\sqrt{10N}}{6})x - (240\sqrt{2N} - \frac{35\sqrt{10N}}{12})y + 300N &< 0
\end{align*}
\]

while $2 < \alpha < \frac{N}{x^2}$ if and only if $p$ and $q$ are respectively in the integer solutions of $x$ and $y$ of the
Fig. 21. Fantastic pattern consisted of known hyperbolas and new curves following inequalities (5.2)

\[
\begin{align*}
\left\{ \begin{array}{l}
xy = N \\
1 < x \leq \sqrt{N} \leq y \\
28x^2 + 92xy + 97y^2 - \sqrt{2N} \left(120 - \frac{35\sqrt{5}}{6}\right)x - \sqrt{2N} \left(240 + \frac{35\sqrt{5}}{12}\right)y + 300N < 0
\end{array} \right. \\
(5.2)
\end{align*}
\]

where \( \chi \) is the \( x \)-coordinate of the intersection of the hyperbola \( H \) with the ellipse (2.5).

Remark 8.

(1) The inequalities (5.1) can be converted into two inequalities of \( x \) and \( y \) separately as follows

\[
\begin{align*}
\left\{ \begin{array}{l}
28x^4 - (120 + \frac{35\sqrt{5}}{6})\sqrt{2N}x^3 + 392Nx^2 - (240 - \frac{35\sqrt{5}}{12})N\sqrt{2N}x + 97N^2 < 0 \\
1 < x \leq \sqrt{N}
\end{array} \right. \\
(5.3)
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
97y^4 - (240 - \frac{35\sqrt{5}}{12})\sqrt{2N}y^3 + 392Ny^2 - (120 + \frac{35\sqrt{5}}{6})N\sqrt{2Ny} + 28N^2 < 0 \\
y \geq \sqrt{N}
\end{array} \right. \\
(5.4)
\end{align*}
\]

The inequalities (5.3) and (5.4) are called discriminant inequalities of \( 1 < \alpha < 2 \) for semiprime \( N = pq \), or simply discriminant. For the case \( 2 < \alpha < \frac{N}{\chi} \), the discriminant can also be defined.
Table 1. Loci and their parametric equations

| Locus       | Parametric Equation                     | Parameter Scope |
|-------------|-----------------------------------------|-----------------|
| H           | $x = \sqrt{\frac{2}{t}}$, $y = \sqrt{tN}$ | $0 < t < \infty$|
| $\Gamma_x$  | $x = \frac{N}{2\sqrt{N}} \cos t + \frac{\sqrt{N}}{2\sqrt{N}} \sin t$, $y = \frac{N}{2\sqrt{N}} \cos t - \frac{\sqrt{N}}{2\sqrt{N}} \sin t$ | $0 \leq t \leq 2\pi$|
| $H^*$       | $x = \frac{\sqrt{N(N+1)}}{2\sqrt{N}}$, $y = \sqrt{N(t+1)}$ | $0 < t < \infty$|
| Centroid    | $x = \frac{2\sqrt{N(N+1)}}{N}$, $y = \frac{2\sqrt{N(N+1)}}{2N}$ | $0 < t < \infty$|
| Orthocenter | $x = \frac{N}{\sqrt{\frac{1}{N}}} + \frac{5}{2} + \frac{3}{2}$, $y = \frac{N}{\sqrt{\frac{1}{N}}} + \frac{5}{2} + \frac{3}{2}$ | $0 < t < \infty$|
| Circumcenter| $x = \frac{\sqrt{N(N+1)}}{N}$, $y = \frac{\sqrt{N(N+1)}}{2N}$ | $0 < t < \infty$|
| Incenter    | $x_1 = \frac{N(N+1)}{2\sqrt{N}}$, $y_1 = \frac{N(N+1)}{2\sqrt{N}} - \frac{N^2}{N}$ | $0 < t \neq 1$|
| PQ side ex-center | $x_2 = \frac{N(N+1)}{2\sqrt{N}}$, $y_2 = \frac{N(N+1)}{2\sqrt{N}} - \frac{N^2}{N}$ | $0 < t \neq 1$|
| PR side ex-center | $x_3 = \frac{N(N+1)}{2\sqrt{N}}$, $y_3 = \frac{N(N+1)}{2\sqrt{N}} - \frac{N^2}{N}$ | $0 < t \neq 1$|
| QR side ex-center | $x_4 = \frac{N(N+1)}{2\sqrt{N}}$, $y_4 = \frac{N(N+1)}{2\sqrt{N}} - \frac{N^2}{N}$ | $0 < t \neq 1$|

(2) Theorem 2 and its derived process provide a way to construct a geometric object to detecting the range of the divisor-ratio $\alpha$. For example, we can construct an $\alpha$-detecting hyperbola by transforming the companion hyperbola $H^*$, or the locus of the centroid.

(3) By Maple, $\chi$ is calculated by

$$
\chi = -\frac{504}{\sqrt{60801876N^2 + 56796110N\sqrt{10N} + 1260N\sqrt{29718699546N + 12907950048N^5}}} + \frac{504}{\sqrt{60801876N^2 + 56796110N\sqrt{10N} + 1260N\sqrt{29718699546N + 12907950048N^5}}} \\
+ \frac{53\sqrt{2N}}{42} - \frac{5\sqrt{10N}}{72}
$$

Example 1. Let $N = 1333$; then odd integer solution of (5.3) is $x \in [27, 35]$ and that for (5.4) is $y \in [35, 51]$. It is seen $x = 31$ and $y = 43$ holds $xy = N$. Consequently, $N = 1333 = 31 \times 43$ and $1 < \alpha < 2$ holds for $N$.

Example 2. Let $N = 4171$; the odd integer solution of (5.3) is $x \in [47, 63]$ and the odd integer solution of (5.4) is $y \in [65, 91]$. Since there is not a pair $(x, y)$ such that $xy = N$, it is known that
there is no odd integer solution fit for $N$, which means $\alpha > 2$. Actually, $N = 4171 = 43 \times 97$ and $\alpha > 2$.

6 Conclusions and Future Work

By means of constructing the companion triangle on the hyperbola $xy = N$ and through study of the loci of the triangle’s centroid, incenter, circumcenter and ex-centers, we discover several new planar curves. Except for their fantastic shapes, the loci also can help us to solve certain problems. It is seen that this paper is just a very rough report of the study. There are something worthy of further investigating. For example, the intersections between two curves in Fig. 21 might be an amusing topic related with the divisor-ratio for which we had started the research work. In the future, we will work on finding their properties and solving more problem.

Competing Interests

Authors have declared that no competing interests exist.

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