ON CERTAIN LARGE ADDITIVE FUNCTIONS

ALEKSANDAR IVIĆ

Dedicated to the memory of Paul Erdős (1913-1996)

ABSTRACT. Let $P(n)$ denote the largest prime factor of an integer $n \geq 2$, $P(1) = 1$, and let

$$
\beta(n) = \sum_{p|n} p, \quad B(n) = \sum_{p^a||n} \alpha p, \quad B_1(n) = \sum_{p^a||n} p^a
$$

denote “large” additive functions. A survey of results on these functions is presented, as well as some new results and open problems.

1. Introduction

Let $P(n)$ denote the largest prime factor of an integer $n \geq 2$, $P(1) = 1$, and let

$$
(1.1) \quad \beta(n) = \sum_{p|n} p, \quad B(n) = \sum_{p^a||n} \alpha p, \quad B_1(n) = \sum_{p^a||n} p^a
$$

denote “large” additive functions, in contrast with the well-known “small” additive functions

$$
(1.2) \quad \omega(n) = \sum_{p|n} 1, \quad \Omega(n) = \sum_{p^a||n} \alpha.
$$

As usual $p$ will denote primes, $p^a||n$ means that $p^a$ divides $n$ but $p^{a+1}$ does not, and a function $f(n)$ is additive if $f(mn) = f(m) + f(n)$ whenever $(m, n) = 1$. From the pioneering works of Alladi and Erdős [1]-[2], P. Erdős’s perspicacity and insight have been one of the main driving forces in the research that brought on many results on summatory functions of large additive functions and $P(n)$. The functions $\omega(n)$ and $\Omega(n)$ may be successfully investigated by various analytical methods. In fact, it is Erdős who in two classical works with M. Kac [14], [15] established the Gaussian distribution law for these functions. A general principle is that $zf(n)$ is a multiplicative function whenever $f(n)$ is an additive function. Thus from the Euler product representation

$$
(1.3) \quad \sum_{n=1}^\infty z^{\omega(n)} n^{-s} = \prod_p \left(1 + \frac{z}{p^s - 1}\right) = \zeta^z(s)G(s, z) \quad (\text{Re } s > 1, \ z \in \mathbb{C}),
$$

where the Dirichlet series for $G(s, z)$ converges absolutely for $\text{Re } s > \frac{1}{2}$, one can obtain various results involving the distribution of values of $\omega(n)$ (and similarly of $\Omega(n)$ and $\Omega(n) - \omega(n)$; see e.g., [22, Chapter 13]). However, from the analogue of (1.3) for $\beta(n)$, namely

$$
(1.4) \quad \sum_{n=1}^\infty z^{\beta(n)} n^{-s} = \prod_p \left(1 + \frac{z^p}{p^s - 1}\right) \quad (\text{Re } s > 1, |z| \leq 1)
$$

one cannot factor out a power of $\zeta(s)$ that will dominate the Euler product (because of the difficulties inherent in handling the factor $z^p$), as was the case in (1.3). Therefore (1.4) does not appear to be very useful in dealing with problems involving $\beta(n)$.
For this reason other methods of approach seemed more appropriate to use. They involve a combination of various analytic and elementary methods. It transpired that in many problems a decisive rôle is played by the function

\[
\psi(x, y) = \sum_{n \leq x, P(n) \leq y} 1,
\]

the number of integers not exceeding \(x\) all of whose prime factors do not exceed \(y\). Results by Hildebrand, Tenenbaum (see [17]–[19] and [34]) and others brought on great progress. In more ways than one this progress on \(\psi(x, y)\) is reflected on the results on large additive functions and the largest prime factor of an integer.

The purpose of this paper is to present an overview of some of the results on large additive functions and the largest prime factor of an integer. This topic is motivated by the joint works of P. Erdős and the author [6]–[13], where the majority of published papers deals precisely with large additive functions and \(P(n)\). The span of the research covers a period of more than fifteen years, and besides P. Erdős and the author involves works of J.-M. De Koninck [3]–[5], C. Pomerance [15], [26], Smati and Wu [32], [33], Tizou Xuan [35], [36] and others. As already mentioned, it was P. Erdős who was the driving force behind this research, always ready to listen to ideas and problems, and always prepared to pour out new problems of his, new methods, and new ideas.

2. Some results on summatory functions

The first results on the summatory functions of \(\beta(n)\) and \(B(n)\) we obtained by Alladi–Erdős [1], [2]. Later research refined some of their results. Now we know that

\[
\sum_{n \leq x} \beta(n) = \sum_{j=1}^{M} A_j \frac{x^{2j-1}}{\log^j x} + O\left(\frac{x^{2j-1}}{\log^{j+1} x}\right),
\]

\[
A_j = (-1)^{j-1} \frac{\zeta(j+1)\zeta(s) s^{-1}}{s^{j-1}} \bigg|_{s=2}, \quad A_1 = \frac{\pi^2}{12},
\]

for any fixed integer \(M \geq 1\) (see [4], and [5] for the analogues for large additive functions over primes of positive density). The asymptotic formula (2.1) holds if \(\beta(n)\) is replaced by \(P(n)\) or \(B(n)\), and it also holds if \(\beta(n)\) is replaced by \(B_1(n)\), since one has

\[
\sum_{n \leq x} B_1(n) = \sum_{n \leq x} \beta(n) + O(x^{3/2}).
\]

The summatory functions of quotients of large additive functions were extensively investigated. The work of P. Erdős and the author [8] contains proofs of

\[
\sum_{2 \leq n \leq x} \frac{f(n)}{B_1(n)} = x + O\left(\frac{x \log \log x}{\log x}\right) \quad (f(n) \in \{P(n), \beta(n), B(n)\}),
\]

\[
\sum_{2 \leq n \leq x} \frac{B_1(n)}{B(n)} = Dx + O\left(\frac{x}{\log^{1/3} x}\right) \quad (D > 0),
\]

\[
\sum_{2 \leq n \leq x} \frac{B_1(n)}{g(n)} = e^{-1} x \log x + O(x),
\]

where \(\gamma\) is Euler’s constant, and \(g(n) \in \{P(n), \beta(n)\}\). The “closeness” of \(\beta(n)\) and \(B(n)\) is also evident in the asymptotic formula (see [25])

\[
\sum_{n \leq x} (B(n) - \beta(n)) = x \log x + x \sum_{j=0}^{M} \frac{C_j}{\log^j x} + O\left(\frac{x}{\log^{M+1} x}\right),
\]
which is valid for any fixed integer $M \geq 1$ and suitable constants $C_j$.

3. Local densities of $B(n) - \beta(n)$

The “local density” of a nonnegative, integer-valued arithmetic function $f(n)$ is the quantity

$$d_k := \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x, f(n) = k} 1 \quad (k \in \mathbb{N} \cup \{0\}),$$

provided that the limit exists. A classical problem of analytic number theory are the local densities of $\Omega(n) - \omega(n)$, which is known as “Rényi’s problem” (see [22, Chapter 13]). For a discussion on local densities of a fairly wide class of arithmetic functions the reader is referred to Ivić–Tenenbaum [27]. Here we shall complement the results on $B(n)$ and $\beta(n)$ by presenting a new result. This is the following

**THEOREM.** For suitable constants $d_k$ we have, uniformly in $k \geq 0$,

(3.1) $$\sum_{n \leq x, B(n) - \beta(n) = k} 1 = d_k x + O(x^{\frac{1}{2}} \log x),$$

(3.2) $$\sum_{n \leq x, B(n) - \beta(n) \geq k} 1 \ll \frac{x}{k} \quad (k \geq 1),$$

and also for any given $r > 0$

(3.3) $$\sum'_{n \leq x} \frac{1}{(B(n) - \beta(n))^r} = D_r x + O(x^{\frac{1}{2} - \frac{1}{2^n}} \log x) + O(x^{\frac{1}{2} \log x}),$$

where $\sum'$ denotes summation over numbers which are not squarefree and

(3.4) $$D_r = \sum_{k=1}^{\infty} \frac{d_k}{kr}.$$

**Proof.** The asymptotic formula in (3.1) follows similarly as the proof of the author’s result [20] (see also [22, Chapter 13]) for the function $a(n)$ (the number of non-isomorphic Abelian groups with $n$ elements). One has only to replace $a(n)$ by $B(n) - \beta(n)$, since

$$B(pm) - \beta(pm) = B(m) - \beta(m)$$

if $(p, m) = 1$, and the method of proof of [20] goes through. If $q$ denotes squarefree and $s$ denotes squarefull numbers, then since every $n$ can be written uniquely as $n = qs$, $(q, s) = 1$, the sum in (3.2) becomes

$$\sum_{qs \leq x, (q, s) = 1, B(s) - \beta(s) \geq k} 1 \leq x \sum_{s=1}^{\infty} \sum_{B(s) - \beta(s) \geq k} \frac{1}{s}.$$ 

But by induction on $\omega(s)$ one has

(3.5) $$k \leq B(s) - \beta(s) = \sum_{p^\alpha || s} (\alpha - 1)p \leq 2 \prod_{p^\alpha || s} p^{\frac{\alpha}{2}} = 2\sqrt{s},$$

hence $s \geq k^2/4$, which because of $\sum_{s \leq x} 1 \ll \sqrt{x}$ gives

$$\sum_{s=1}^{\infty} \frac{1}{s} \leq \sum_{s \geq k^2/4} \frac{1}{s} \ll \frac{1}{k}.$$
To prove (3.3) note that the sum on the left hand-side is, in view of (3.1) and (3.5),

\[
\sum_{1 \leq k \leq 2\sqrt{x}} \frac{1}{k^r} \left( \sum_{n \leq x, B(n) = \beta(n) = k} 1 \right) = \sum_{1 \leq k \leq 2\sqrt{x}} \frac{1}{k^r} \left( d_k x + O(x^{\frac{1}{k}} \log x) \right)
\]

\[
= \sum_{k=1}^{\infty} \frac{d_k}{k^r} \left( x \sum_{n \leq x, B(n) = \beta(n) = k} k^{-r} \right) + O \left( x^{\frac{1}{k}} \log x \sum_{k \leq 2\sqrt{x}} k^{-r} \right)
\]

\[
= D_r x + O \left( x^{1 - \frac{1}{k}} \log x \right) + O(x^{\frac{1}{k}} \log x),
\]

with \(D_r\) as in (3.4).

We shall conclude this section by stating some open problems.

**Problem 1.** Which density \(d_k\) is the largest one for \(k > 1\)? (We have \(d_0 = 6/\pi^2\) (= the density of squarefree numbers), \(d_1 = 0\) and \(d_k > 0\) for \(k > 1\)).

**Problem 2.** The proof of (3.2) shows that \(d_k \ll 1/k\). Is \(d_k\) decreasing for \(k \geq k_0\)? Can one find an asymptotic formula for \(d_k\)?

**Problem 3.** Is the density of \(n\) for which \(\beta(n) > \beta(n + 1)\) (or \(B(n) > B(n + 1), B_1(n) > B_1(n + 1)\)) equal to 1/2? What about the density of \(n\) for which, say, \(\beta(n) > \beta(n + 1) > \beta(n + 2)\)? These are the analogues of Erdős’s classical problem to prove that the density of \(n\) for which \(P(n) > P(n + 1)\) is 1/2.

**Problem 4.** For which \(n\) is it possible to have \(\beta(n) = \beta(n + 1)\) (like \(\beta(5) = \beta(6)\)), \(B(n) = B(n + 1)\) \((B(714) = B(715))\) and \(B_1(n) = B_1(n + 1)?\) It was proved by Erdős–Pomerance [16] that

\[
\sum_{n \leq x, B(n) = B(n+1)} 1 = O \left( \frac{x}{\log x} \right).
\]

One could look either for asymptotic estimates such as (3.6), or try to give an arithmetic characterization of the numbers in question.

**Problem 5.** Can one improve the \(O\)-term in (3.1) by taking into account the arithmetic structure of \(k\)?

It may be remarked that in the analogous problem for the local densities of \(a(n)\) (the number of non-isomorphic Abelian groups with \(n\) elements) this was done by Krätzel–Wolke [28].

4. Sums of reciprocals

It is a classical result of prime number theory that

\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + C + O \left( \frac{1}{\log x} \right).
\]

Sums of reciprocals of large additive functions and \(P(n)\) are much more difficult to handle. They were investigated by De Koninck, Erdős, Pomerance, Xuan and the author. It was proved by Erdős, Pomerance and the author [13] that

\[
\sum_{n \leq x} \frac{1}{P(n)} = \frac{x}{\log x} \left(1 + O \left( \frac{\log \log x}{\log x} \right) \right)
\]
with
\[ \delta(x) := \int_2^x \rho \left( \frac{\log x}{\log t} \right) \frac{dt}{t^2}, \]
where the Dickman–de Bruijn function \( \rho(u) \) is the continuous solution to the differential delay equation
\[ u \rho'(u) = -\rho(u - 1), \quad \rho(u) = 1 \text{ for } 0 \leq u \leq 1, \quad \rho(u) = 0 \text{ for } u < 0. \]

It is known (see [34]) that
\[ \rho(u) = \exp \left\{ -u \left( \log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O \left( \left( \frac{\log_2 u}{\log u} \right)^2 \right) \right) \right\}. \]

It was also proved in [13] that
\[ \sum_{n \leq x} P(n)^{-\omega(n)} = \exp \left\{ (4 + o(1)) \frac{\sqrt{\log x}}{\log \log x} \right\}, \]
\[ \sum_{n \leq x} P(n)^{-\Omega(n)} = \log \log x + D + O \left( \frac{1}{\log x} \right) \text{ (} D > 0) \text{,} \]
with effectively computable \( D \), showing the difference in behaviour between \( \omega(n) \) and \( \Omega(n) \). Of these two formulas it is (4.3) that is deeper than (4.4).

**Problem 6.** What is the shape of the above asymptotic formulas if we replace \( P(n) \) by \( \beta(n) \) and \( B(n) \)?

It was shown by Pomerance and the author [26] that one has asymptotically
\[ \delta(x) = \exp \left\{ -(2 \log x \log_2 x)^{1/2} \left( 1 + \log(x) + O \left( \frac{\log_3 x}{\log_2 x} \right) \right) \right\}, \]
where
\[ g_r(x) = \frac{\log_3 x + \log(1 + r) - 2 - \log 2}{2 \log_2 x} \left( 1 + \frac{2}{\log_2 x} \right) \]
\[ - \frac{(\log_3 x + \log(1 + r) - \log 2)^2}{8 \log_2 x}, \]
and the expression for \( \delta(x) \) was sharpened by the author [24]. Already in 1977 Erdős told the author that the function \( \delta(x) \) is slowly varying in the sense of J. Karamata (see [29], [31]), namely that for any \( C > 0 \) one has
\[ \lim_{x \to \infty} \frac{\delta(Cx)}{\delta(x)} = 1, \]
but it is only in 1986 in that (4.5) was established in [13], by the use of (4.2) and properties of the function \( \rho(u) \). The asymptotic formula (4.1) remains valid if \( P(n) \) is replaced by \( \beta(n) \) or \( B(n) \), and the asymptotic formula for the summatory function of \( \frac{B(n)}{\beta(n)} - 1 \) is of the same shape as the right-hand side of (4.1). Furthermore we have
\[ \sum_{2 \leq n \leq x} \left( \frac{1}{\beta(n)} - \frac{1}{B(n)} \right) \]
\[ = x \exp \left\{ -2(\log x \log_2 x)^{1/2} \left( 1 + g_1(x) + O \left( \frac{\log_3 x}{\log_2 x} \right) \right) \right\}. \]
Based on his joint work with Erdős and Pomerance [13], the author [23] sharpened some of the formulas and obtained e.g.

$$\sum_{n \leq x} \frac{\Omega(n) - \omega(n)}{P(n)} = \left\{ \sum_{p} \frac{1}{p^2 - p} + O\left( \left( \frac{\log_2 x}{\log x} \right)^{1/2} \right) \right\} \sum_{n \leq x} \frac{1}{P(n)},$$

$$\sum_{n \leq x} \frac{\omega(n)}{P(n)} = \left\{ \left( \frac{2 \log x}{\log_2 x} \right)^{1/2} \left( 1 + O\left( \frac{\log_3 x}{\log_2 x} \right) \right) \right\} \sum_{n \leq x} \frac{1}{P(n)},$$

and this remains valid if \( \omega(n) \) is replaced by \( \Omega(n) \),

$$\sum_{n \leq x} \frac{\mu^2(n)}{P(n)} = \left\{ \frac{6}{\pi^2} + O\left( \left( \frac{\log_2 x}{\log x} \right)^{1/2} \right) \right\} \sum_{n \leq x} \frac{1}{P(n)}.$$

The relevant contribution to the last three sums comes from \( n \) for which

$$L(-2, x) \leq P(n) \leq L(2, x),$$

where

$$L(c, x) := \exp\left\{ \left( \frac{1}{2} \log \log x \log 2 \right)^{1/2} \left( 1 + c \frac{\log_3 x}{\log_2 x} \right) \right\}.$$

During many years of collaboration on the problems discussed in this section Erdős was fond of saying “take \( \log P(n) \approx \sqrt{\log n} \) ”, but the above discussion shows that in this, as on countless other occasions, he was right.

5. Sums in residue classes

During the Conference on Analytic Number Theory in June 1993 in Lillafüred, P. Erdős asked the author to evaluate asymptotically \( S_0(x) \), where for fixed \( r \geq 0 \) and fixed integers \( 1 \leq \ell \leq k \), \((\ell, k) = 1\) one defines

$$S_r(x) = \sum_{n \leq x, P(n) \equiv \ell (\text{mod} k)} \frac{1}{P^r(n)}.$$

In the author’s work [24] it is shown that that

$$S_0(x) = \frac{x}{\varphi(k)} + O\left( x \exp\left( -\left( \log x \right)^{3/8-\varepsilon} \right) \right)$$

for any given \( \varepsilon > 0 \), and for \( r > 0 \) and any fixed integer \( J \geq 0 \)

$$(5.1) \quad S_r(x) = \frac{x}{\varphi(k)} \int_2^x \rho\left( \frac{\log t}{\log \ell} \right) \left\{ \sum_{j=0}^J Q_{j,r}(\log t) \left( \frac{\log^j \ell x}{\log \ell} \right) + O\left( \left( \frac{\log t}{\log x} \right)^{J+1} \right) \right\} \frac{dt}{t^{r+1}},$$

for suitable polynomials \( Q_{j,r}(x) \) of degree \( j \) in \( x \) whose coefficients depend on \( r \). In particular

$$Q_{0,r}(x) = r, \quad Q_{1,r}(x) = (r-r\gamma)(rx-1).$$

Let

$$T_r(x) := \sum_{n \leq x, P(n) \equiv \ell (\text{mod} k), P^2(n) \mid n} \frac{1}{P^r(n)} \quad (r \geq -1),$$

\( r \in \mathbb{R} \) fixed, \( 1 \leq \ell \leq k \), \((\ell, k) = 1\) fixed. For any given \( \varepsilon > 0 \)

$$T_{-1}(x) = \frac{C x}{\varphi(k)} + O\left\{ x \exp\left( -\left( \log x \right)^{3/8-\varepsilon} \right) \right\},$$

$$C = \int_0^\infty \frac{\rho(v)}{v+2} dv \quad (1).$$
For \( r > -1, r \in \mathbb{R} \) fixed and \( J \in \mathbb{N} \) fixed

\[
T_r(x) = \frac{x}{\varphi(k)} \int_2^x \rho\left(\frac{\log x}{\log t}\right) \left( \sum_{j=0}^J R_{j+1,r}(\log t) \frac{\log^{j+2} t}{\log^{j+1} x} + O\left(\frac{\log^{j+2} t}{\log^{j+1} x}\right) \right) \frac{dt}{t^{r+2}}
\]

for suitable polynomials \( R_{j,r}(x) \) \((j \in \mathbb{N})\) of degree \( j \) in \( x \) whose coefficients depend on \( r \). In particular, \( R_{1,r}(x) = (r + 1)^2 x - r - 1 \).

The formulas (5.1) and (5.2) sharpen the results of [26] (in the case when \( k = 1 \)), where one had

\[
S_r(x) = x \exp\left\{-\left(\frac{2r \log x \log 2}{\log x} x\right)^{1/2} \left(1 + g_{r-1}(x) + O\left(\frac{\log x}{\log 2 x}\right)^3\right)\right\},
\]

when \( r > 0 \), and

\[
T_r(x) = x \exp\left\{-\left(\frac{2r + 2}{\log 2 x}\right)^{1/2} \left(1 + g_r(x) + O\left(\frac{\log x}{\log 2 x}\right)^3\right)\right\},
\]

when \( r > -1 \). The proofs given in [24] use the sharp approximation of E. Saias [30], namely

\[
\psi(x,y) = \Lambda(x,y) \left\{1 + O\left(\exp\left(-\log^{1/3 + \varepsilon} y\right)\right)\right\},
\]

where

\[
e^{(\log \log x)^{1/3 + \varepsilon}} \leq y \leq x, \quad x \geq x_0(\varepsilon),
\]

for \( x, y \geq 1, x \not\in \mathbb{N} \)

\[
\Lambda(x,y) := x \int_{1-\varepsilon}^{\infty} \rho\left(\frac{\log x - \log t}{\log y}\right) d\left(\frac{[t]}{t}\right),
\]

and for \( x \in \mathbb{N} \) we define \( \Lambda(x,y) = \Lambda(x+0,y) \). Note that in the range (5.4) A. Hildebrand [17] had

\[
\psi(x,y) = x \rho(u) \left(1 + O\left(\frac{\log(u + 2)}{\log y}\right)\right), \quad u = \frac{\log x}{\log y}.
\]

Although (5.3) has the sharper error term than (5.6), the function on the right-hand side of (5.5) is discontinuous ([\( t \]), the greatest integer part of \( t \) has jumps when \( t \in \mathbb{N} \)), and there are technical difficulties in applying this formula.

As one of the corollaries of the above results we single out the following formula:

\[
\sum_{n \leq x, P^2(n) \mid n} 1 = \left\{\frac{\log x}{2} \left(\log_2 x + \log_3 x - \log 2 + \frac{\log_3 x - \log 2}{\log_2 x} + O\left(\frac{\log_2^2 x}{\log_2 x}\right)\right)\right\}^{1/2} \sum_{n \leq x} \frac{1}{P(n)}.
\]

The error term, like in most previous results, could be further sharpened at the cost of more technical elaboration.
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Aleksandar Ivić
Katedra Matematike RGF-a
Univeristeta u Beogradu
Djušina 7, 11000 Beograd
Serbia (Yugoslavia)
e-mail: aivic@matf.bg.ac.yu, aivic@rgf.bg.ac.yu