\textbf{L}^p-ERROR BOUNDS OF TWO AND THREE-POINT QUADRATURE RULES FOR RIEMANN–STIELTJES INTEGRALS

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ABSTRACT. In this work, \(L^p\)-error estimates of general two and three point quadrature rules for Riemann-Stieltjes integrals are given. The presented proofs depend on new triangle type inequalities of Riemann-Stieltjes integrals.

1. Introduction

The Newton–Cotes formulas use values of function at equally spaced points. The same practice when the formulas are combined to form the composite rules, but this restriction can significantly decrease the accuracy of the approximation. In fact, these methods are inappropriate when integrating a function on an interval that contains both regions with large functional variation and regions with small functional variation. If the approximation error is to be evenly distributed, a smaller step size is needed for the large-variation regions than for those with less variation.

In numerical analysis, inequalities play a main role in error estimations. A few years ago, by using modern theory of inequalities and Peano kernel approach a number of authors have considered an error analysis of some quadrature rules of Newton-Cotes type. In particular, the Mid-point, Trapezoid, Simpson’s and other rules have been investigated recently with the view of obtaining bounds for the quadrature rules in terms of at most first derivative.

The number of proposed quadrature rules that provides approximation of Stieltjes integral \(\int_a^b f(t) \, du(t)\) using derivatives or without using derivatives are very rare in comparison with the large number of methods available to approximate the classical Riemann integral \(\int_a^b f(t) \, dt\).

The problem of introducing quadrature rules for RS-integral \(\int_a^b f(t) \, dg\) was studied via theory of inequalities by many authors. Two famous real inequalities used in this approach, which are the well known Ostrowski and Hermite-Hadamard inequalities and their modifications. For this purpose and in order to approximate the RS-integral \(\int_a^b f(t) \, du(t)\), a generalization of closed Newton-Cotes quadrature rules of RS-integrals without using derivatives provides a simple and robust solution to a significant problem in the evaluation of certain applied probability models was presented by Tortorella in [24].
In 2000, Dragomir [15] introduced the Ostrowski’s approximation formula (which is of One-point type formula) as follows:

\[ \int_a^b f(t) \, du(t) \approx f(x) \left( u(b) - u(a) \right) \quad \forall x \in [a, b]. \]

Several error estimations for this approximation had been done in the works [14] and [15].

From different point of view, the authors of [16] (see also [10, 11]) considered the problem of approximating the Stieltjes integral \( \int_a^b f(t) \, du(t) \) via the generalized trapezoid formula:

\[ \int_a^b f(t) \, du(t) \approx [u(x) - u(a)] f(a) + [u(b) - u(x)] f(b). \]

Many authors have studied this quadrature rule under various assumptions of integrands and integrators. For full history of these two quadratures see [5] and the references therein.

Another trapezoid type formula was considered in [19], which reads:

\[ \int_a^b f(t) \, du(t) \approx \frac{f(a) + f(b)}{2} [u(b) - u(a)] \quad \forall x \in [a, b]. \]

Some related results had been presented by the same author in [17] and [18]. For other connected results see [12] and [13].

In 2008, Mercer [22] introduced the following trapezoid type formula for the RS-integral

\begin{equation}
\int_a^b f(t) \, dg(t) \approx \left[ G - g(a) \right] f(a) + [g(b) - G] f(b),
\end{equation}

(1.1)

where \( G = \frac{1}{b-a} \int_a^b g(t) \, dt \).

Recently, Alomari and Dragomir [3], proved several new error bounds for the Mercer–Trapezoid quadrature rule (1.1) for the RS-integral under various assumptions involved the integrand \( f \) and the integrator \( g \).

Follows Mercer approach in [22], Alomari and Dragomir [9] introduced the following three-point quadrature formula:

\begin{align}
\int_a^b f(t) \, dg(t) \approx & \left[ G(a, x) - g(a) \right] f(a) + [G(x, b) - G(a, x)] f(x) \\
+ & [g(b) - G(x, b)] f(b)
\end{align}

(1.2)

for all \( a < x < b \), where \( G(\alpha, \beta) := \frac{1}{\beta - \alpha} \int_\alpha^\beta g(t) \, dt \).

Several error estimations of Mercer’s type quadrature rules for RS-integral under various assumptions about the function involved have been considered in [3] and [6].
Motivated by Guessab-Schmeisser inequality (see [21]) which is of Ostrowski's type, Alomari in [4] and [8] presented the following approximation formula for RS-integrals:

\[
\int_a^b f(t) \, du(t) \\
\approx \left[ u\left(\frac{a+b}{2}\right) - u(a) \right] f(x) + \left[ u(b) - u\left(\frac{a+b}{2}\right) \right] f(a+b-x),
\]

for all \( x \in [a, a+b/2] \). For other related results see [5]. For different approaches variant quadrature formulae the reader may refer to [7], [20] and [23].

Among others the \( L_\infty \)-norm gives the highest possible degree of precision; so that it is recommended to be ‘almost’ the norm of choice. However, in some cases we cannot access the \( L_\infty \)-norm, so that \( L_p \)-norm (\( 1 \leq p < \infty \)) is considered to be a variant norm in error estimations.

In this work, several \( L_p \)-error estimates (\( 1 \leq p < \infty \)) of general Two and Three points quadrature rules for Riemann-Stieltjes integrals are presented. The presented proofs depend on new triangle type inequalities for RS-integrals.

2. Two Lemmas

It is well known that the class of functions satisfying Lipschitz condition is a subset of the class of functions of bounded variation. More precisely, if \( f \) has the Lipschitz property, then \( f \) is of bounded variation. However, a continuous function of bounded variation need not have a Lipschitz property. For example, the series \( \sum_{k=1}^{\infty} \frac{\sin kt}{k \log k} \) (\( 0 \leq t \leq 1 \)) converges uniformly to the sum \( g \), which is absolutely continuous and hence is of bounded variation, however \( g \) does not satisfies Lipschitz property.

Not far away from this, very useful inequality regarding Lipschitz functions is the following: for a Riemann integrable function \( w : [a, b] \to \mathbb{R} \) and \( L \)-Lipschitzian function \( \nu : [a, b] \to \mathbb{R} \), one has the inequality

\[
\left| \int_a^b w(t) \, d\nu(t) \right| \leq L \|w\|_1.
\]

A generalization of this inequality to \( L_p \)-spaces is incorporated in the following lemma [1]:

**Lemma 1.** Let \( 1 \leq p < \infty \). Let \( w, \nu : [a, b] \to \mathbb{R} \) be such that is \( w \in L^p[a, b] \) and \( \nu \) has a Lipschitz property on \([a, b] \). Then the inequality

\[
\left| \int_a^b w(t) \, d\nu(t) \right| \leq L \|w\|_p,
\]

holds and the constant ‘1’ in the right hand side is the best possible. Provided that the RS-integral \( \int_a^b w(t) \, d\nu(t) \) exists, where

\[
\|w\|_p = \left( \int_a^b |w(t)|^p \, dt \right)^{1/p}, \quad (1 \leq p \leq \infty).
\]

**Remark 1.** Clearly, when \( p = 1 \) in (2.2) then we refer to (2.1).

Under weaker conditions we may state the following result [1]:
Lemma 2. Let $1 \leq p < \infty$. Let $w, \nu: [a, b] \to \mathbb{R}$ be such that $w \in L^p[a, b]$ and $
u$ is of bounded variation on $[a, b]$. Then the inequality

$$(2.3) \quad \left| \int_a^b w(t) \, d\nu(t) \right| \leq \left( \int_a^b \frac{1}{\nu'(t)} \cdot \|w\|_p \right)^{1/p}, \quad a.e.$$ holds. The constant ‘1’ in the right hand side is the best possible. Provided that the RS-integral $\int_a^b w(t) \, d\nu(t)$ exists.

Remark 2. If $\nu$ is $M$-Lipschitz then

$$\text{Lip}_M(\nu) = \sup_{x, y \in [a, b]} \left| \frac{\nu(y) - \nu(x)}{y - x} \right| < \infty.$$ Therefore, we rewrite the inequality (2.3) such as:

$$(2.4) \quad \left| \int_a^b w(t) \, d\nu(t) \right| \leq \text{Lip}_M(\nu) \left( \int_a^b \frac{1}{\nu'(t)} \cdot \|w\|_p \right)^{1/p},$$

which is valid everywhere and sharp.

3. A General Quadrature rule For RS-integrals

Let $\Phi_\alpha(f, u; x)$ is the general quadrature formula

$$(3.1) \quad \Phi_\alpha(f, u; x) := (1 - \alpha) \left\{ \left[ u \left( \frac{a + b}{2} \right) - u(a) \right] f(a + b - x) + \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] f(x) \right\}$$

Define the mapping

$$S_a(t; x) := \begin{cases} (1 - \alpha) [u(t) - u(a)] + \alpha [u(t) - u(x)], & t \in [a, x] \\ (1 - \alpha) [u(t) - u \left( \frac{a + b}{2} \right)] + \alpha [u(t) - u(x)], & t \in (x, a + b - x) \\ (1 - \alpha) [u(t) - u(b)] + \alpha [u(t) - u(x)], & t \in (a + b - x, b) \end{cases}$$

Using integration by parts formula, its not difficult to obtain that

$$(3.2) \quad \int_a^b S_a(t; x) \, df(t) = \Phi_\alpha(f, u; x) - \int_a^b f(t) \, du(t) = \mathcal{E}_\alpha(f, u; x).$$

where $\mathcal{E}_\alpha(f, u; x)$ is the error term.

Thus, the RS-integral $\int_a^b f(t) \, du(t)$ can be approximated by the quadrature rule

$$(3.3) \quad \int_a^b f(t) \, du(t) = \Phi_\alpha(f, u; x) - \mathcal{E}_\alpha(f, u; x).$$

In particular cases, we consider:

- If $\alpha = 0$, then the following general Two-point formula holds

$$(3.4) \quad \Phi_0(f, u; x) = \left[ u \left( \frac{a + b}{2} \right) - u(a) \right] f(x) + \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] f(a + b - x).$$
Theorem 1. Let $RS[\alpha]$ for all $\alpha$ in the relation:

\[ |E| a, b \leq H a, b \]

3.8

3.7

3.6

3.5

\[ \Phi_{1/2} (f, w; x) = \frac{1}{2} \left\{ (u(x) - u(a)) f (a) + (u(b) - u(x)) f (b) \right\} \]

\[ + \left( u \left( \frac{a + b}{2} \right) - u(a) \right) f (a) + \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] f (a + b - x) \right\} . \]

\[ \Phi_{1/2} (f, w; x) = \frac{1}{2} \left\{ (u(x) - u(a)) f (a) + (u(b) - u(x)) f (b) \right\} \]

\[ + \left( u \left( \frac{a + b}{2} \right) - u(a) \right) f (a) + \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] f (a + b - x) \right\} . \]

\[ \Phi_{1} (f, w; x) := |u(x) - u(a)) f (a) + |u(b) - u(x)) f (b) . \]

A convex combination between Trapezoid and Midpoint formulas is incorporated in the relation:

\[
\Phi_{\alpha} \left( f, w; \frac{a + b}{2} \right) = \alpha \left\{ u \left( \frac{a + b}{2} \right) - u(a) \right) f (a) + \left[ u(b) - u \left( \frac{a + b}{2} \right) \right] f (b) \right\} + (1 - \alpha) [u(b) - u(a)] f \left( \frac{a + b}{2} \right) ,
\]

for all $\alpha \in [0, 1]$. Furthermore, if $\alpha = \frac{1}{3}$, we get the Simpson’s formula for $RS$-integrals.

**Theorem 1.** Let $u : [a, b] \to [0, \infty)$ be a Hölder continuous of order $r \in (0, 1]$ on $[a, b]$ and belongs to $L_{p}[a, b] (p \geq 1)$. If $f : [a, b] \to \mathbb{R}$ $M$-Lipschitzian mapping on $[a, b]$, then for any $x \in [a, \frac{a+b}{2}]$ and $\alpha \in [0, 1]$, we have

\[
|E_{\alpha} (f, w; x)| \leq H \text{Lip}_{M} (f) \cdot \left( \sqrt[1/p]{\int a b} \right) \frac{1}{p} \cdot \left( 1 - \alpha \right) \left\{ \frac{2 (x - a)^{\frac{rp+1}{p}}}{(rp + 1)^{1/p}} + 2^{1/p} \frac{\left( \frac{a+b}{2} - x \right)^{\frac{rp+1}{p}}}{(rp + 1)^{1/p}} \right\} + \alpha \left\{ \frac{(x - a)^{\frac{rp+1}{p}}}{(rp + 1)^{1/p}} + \frac{(a + b - 2x)^{\frac{rp+1}{p}}}{(rp + 1)^{1/p}} + \left( \frac{(b - x)^{\frac{rp+1}{p}} - (a + b - 2x)^{\frac{rp+1}{p}}}{rp + 1} \right)^{1/p} \right\}.
\]
Proof. As \( f \) is of bounded variation on \([a, b]\), and \( u \) is Hölder continuous of order \( r \in (0, 1] \) which belongs to \( L^p[a, b] \), then by (2.4) we have

\[
\left| \int_a^b S_u(t; x) \, df(t) \right| \\
= \left| \int_a^x \{(1 - \alpha) \, [u(t) - u(a)] + \alpha \, [u(t) - u(x)]\} \, df(t) \right| \\
+ \left| \int_x^{a+b-x} \{(1 - \alpha) \, \left| \frac{a + b}{2} \right| + \alpha \, [u(t) - u(x)]\} \, df(t) \right| \\
+ \left| \int_{a+b-x}^b \{(1 - \alpha) \, [u(t) - u(b)] + \alpha \, [u(t) - u(x)]\} \, df(t) \right| \\
\leq \text{Lip}_M(f) \left\{ \left( \int_a^x (f) \right)^{1 - \frac{1}{p}} \times \left[ (1 - \alpha) \| u - u(a) \|_{p, [a, x]} + \alpha \| u - u(x) \|_{p, [a, x]} \right] \\
+ \left( \int_{a+b-x}^b (f) \right)^{1 - \frac{1}{p}} \times \left[ (1 - \alpha) \| u - u(b) \|_{p, [a+b-x, b]} + \alpha \| u - u(x) \|_{p, [a+b-x, b]} \right] \right\} \\
\leq \text{Lip}_M(f) \left( \int_a^b (f) \right)^{1 - \frac{1}{p}} \\
\times (1 - \alpha) \left[ \| u - u(a) \|_{p, [a, x]} + \| u - u(b) \|_{p, [a+b-x, b]} + \| u - u(x) \|_{p, [a+b-x, b]} \right] \\
+ \alpha \left\{ \| u - u(x) \|_{p, [a, x]} + \| u - u(x) \|_{p, [a+b-x, x]} + \| u - u(x) \|_{p, [a+b-x, b]} \right\}.
\]

Now, since \( u \) is Hölder continuous of order \( r \in (0, 1] \), then there exits a positive constant \( H > 0 \) such that

\[
|u(y) - u(z)| \leq H |y - z|^r
\]

for all \( y, z \in [a, b] \). Accordingly, since \( u \in L^p[a, b] \) then for all fixed \( z \in [c, d] \) we have

\[
\| u - u(z) \|^p_{p, [c, d]} = \int_c^d |u(y) - u(z)|^p \, dy \\
\leq H^p \int_c^d |y - z|^\frac{rp}{p+1} \, dy \\
= H^p \frac{(z-c)^{r+1} + (d-z)^{r+1}}{r+1}
\]
for every subinterval \([c, d] \subseteq [a, b]\) and \(y, z \in [c, d]\). Applying this step for each norm in the last inequality above, we get

\[
\left| \int_a^b S_u(t; x) \, df(t) \right| 
\leq H \text{Lip}_M(f) \cdot \left( \frac{b}{a} \right)^{1 - \frac{1}{p}} \cdot \left( 1 - \alpha \right) \left\{ 2 \frac{(x - a)^{\frac{r + 1}{p}}}{(r + 1)^{1/p}} + 2^{1/p} \frac{(a + b - x)^{\frac{r + 1}{p}}}{(r + 1)^{1/p}} \right\}
+ \alpha \left\{ \frac{(x - a)^{\frac{r + 1}{p}}}{(r + 1)^{1/p}} + \frac{(a + b - 2x)^{\frac{r + 1}{p}}}{(r + 1)^{1/p}} + \frac{(b - x)^{\frac{r + 1}{p}} - (a + b - 2x)^{\frac{r + 1}{p}}}{(r + 1)} \right\}^{1/p},
\]

and hence the proof is established. \(\square\)

Remark 3. In Theorem 1, if \(u \in L^2[a, b]\) then for all \(x \in \left[a, \frac{a + b}{2}\right]\) and \(\alpha \in [0, 1]\), we have

\[
|E_\alpha(f, u; x)| 
\leq H \text{Lip}_M(f) \cdot \left( \frac{b}{a} \right)^{\frac{1}{2}} \cdot \left( 1 - \alpha \right) \left\{ 2 \frac{(x - a)^{\frac{2r + 1}{2}}}{(2r + 1)^{1/2}} + 2^{1/2} \frac{(a + b - x)^{\frac{2r + 1}{2}}}{(2r + 1)^{1/2}} \right\}
+ \alpha \left\{ \frac{(x - a)^{\frac{2r + 1}{2}}}{(2r + 1)^{1/2}} + \frac{(a + b - 2x)^{\frac{2r + 1}{2}}}{(2r + 1)^{1/2}} + \frac{(b - x)^{2r + 1} - (a + b - 2x)^{2r + 1}}{2r + 1} \right\}^{1/2}.
\]

Moreover, if \(u\) is Lipschitzian mapping (i.e., \(r = 1\)), we get

\[
|E_\alpha(f, u; x)| 
\leq \frac{1}{\sqrt{3}} H \text{Lip}_M(f) \cdot \left( \frac{b}{a} \right)^{\frac{1}{2}} \cdot \left( 1 - \alpha \right) \left\{ 2 \frac{(x - a)^{\frac{3}{2}}}{2} + 2^{1/2} \frac{(a + b - x)^{\frac{3}{2}}}{2} \right\}
+ \alpha \left\{ (x - a)^{\frac{3}{2}} + (a + b - 2x)^{\frac{3}{2}} + \frac{(b - x)^3 - (a + b - 2x)^3}{2} \right\}^{1/2}.
\]

Remark 4. In very special interesting case if \(u\) is Hölder continuous of order \(r = \frac{1}{p}\) \((p \geq 1)\) and belongs to \(L^p[a, b]\), then (3.8) becomes

\[
|E_\alpha(f, u; x)| 
\leq \frac{H}{2^{1/p}} \text{Lip}_M(f) \cdot \left( \frac{b}{a} \right)^{1 - \frac{1}{p}} \cdot \left( 1 - \alpha \right) \left\{ 2 \frac{(x - a)^{\frac{2r + 1}{p}}}{(r + 1)^{1/p}} + 2^{1/p} \frac{(a + b - x)^{\frac{2r + 1}{p}}}{(r + 1)^{1/p}} \right\}
+ \alpha \left\{ (x - a)^{\frac{2r + 1}{p}} + (a + b - 2x)^{\frac{2r + 1}{p}} + \frac{(b - x)^{2r + 1} - (a + b - 2x)^{2r + 1}}{2r + 1} \right\}^{1/p}.
\]
Remark 5. In (3.8)–(3.10), choosing an appropriate \( x \in \left[ a, \frac{a+b}{2} \right] \) we get error estimations of several quadrature formulae for RS-integrals, such as: Trapezoid, several Two-points, Midpoint, Simpson’s, Three-point, Average Trapezoid-Midpoint quadrature formulae and others. In parallel, these inequalities may be considered as generalizations of Ostrowski’s type inequalities for RS-integrals for arbitrary \( x \in \left[ a, \frac{a+b}{2} \right] \).

4. More Error bounds in \( L^p \)-space

Let \( I \) be a real interval such that \( [a, b] \subseteq I^o \) the interior of \( I, a, b \in \mathbb{R} \) \( a < b \). Consider \( \mathcal{U}^p(I) \) \((p > 1)\) be the space of all positive \( n \)-th differentiable functions \( f \) whose \( n \)-th derivatives \( f^{(n)} \) is positive locally absolutely continuous on \( I^o \) with \( \int_a^b (f^{(n)}(t))^p \, dt < \infty \).

**Theorem 2.** Let \( u \in \mathcal{U}^p(I) \). Assume that \( f : [a, b] \to \mathbb{R} \) is \( M \)-Lipschitz on \([a, b]\), then for any \( x \in \left[ a, \frac{a+b}{2} \right] \) and \( \alpha \in [0, 1] \), we have

\[
|\mathcal{E}_\alpha (f, u; x)| \leq \text{Lip}_M(f) \cdot \left( \frac{b}{a} \right)^{1 - \frac{1}{p}} \cdot \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt[p]{p - 1}}^{n} \cdot \left( \frac{x - \frac{a+b}{4}}{\frac{3a+b}{4}} \right)^n \cdot \left( \frac{b-a}{2} \right)^{n} \cdot \left( \int_a^b (f(t))^p \, dt \right)^{1 - \frac{1}{p}} \cdot \|u^{(n)}\|_{p, [a,b]},
\]

for all \( x \in \left[ a, \frac{a+b}{2} \right] \). In particular, we have

\[
|\mathcal{E}_\alpha (f, u; \frac{a+b}{2})| \leq \text{Lip}_M(f) \cdot \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt[p]{p - 1}} \right)^n \cdot \left( \frac{b-a}{2} \right)^n \cdot \left( \int_a^b (f(t))^p \, dt \right)^{1 - \frac{1}{p}} \cdot \|u^{(n)}\|_{p, [a,b]}
\]

**Proof.** We repeat the proof of Theorem 1. Now, using the recent result proved by the first author of this paper; on generalization of Beesack-Wintinger inequality \([2]\) which reads: If \( h \in \mathcal{U}^p(I) \) then for all \( \xi \in [a, b] \) we have

\[
\int_a^b |h(t) - h(\xi)|^p \, dt \leq \left( \frac{p^p \sin \left( \frac{\pi}{p} \right)}{\pi^p (p - 1)} \right)^n \left[ \frac{b-a}{2} + \left| \xi - \frac{a+b}{2} \right| \right]^{np} \cdot \int_a^b (h^{(n)}(x))^p \, dx.
\]

In case \( n = 1 \), the inequality \((4.3)\) is sharp see \([2]\).

Therefore, since \( u^{(n)} \in L^p[a, b] \) then replacing \( h \) by \( u \) and the interval \([a, b]\) by the corresponding intervals defines \( u \) in the proof of Theorem 1 we get the required result we shall omit the details.

The dual assumptions on \( f \) and \( u \) are considered in the following two results.
Theorem 3. Let \( f \in L^p(I) \). Assume that \( u : [a, b] \to \mathbb{R} \) has \( M \)-Lipschitz property on \([a, b] \), then we have the inequality

\[
(4.4) \quad |E_0 (f, u; x)| \leq 2 \text{Lip}_M (u) \cdot \left( \frac{b-a}{2} \right)^{1/q} \cdot \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt[4]{p - 1}} \right)^n \left[ \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right]^n \cdot \|f^{(n)}\|_{p, [a, b]}
\]

for all \( x \in \left[ a, \frac{a+b}{2} \right] \). In particular, we have

\[
(4.5) \quad |E_0 \left( f, u; \frac{3a+b}{4} \right)| \leq \text{Lip}_M (u) \cdot \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt[4]{p - 1}} \right)^n \frac{(b-a)^{n+\frac{1}{q}}}{2^n + \frac{1}{q}} \cdot \|f^{(n)}\|_{p, [a, b]}.
\]

Proof. Using the integration by parts formula for \( RS \)-integral, we have

\[
\int_a^{\frac{a+b}{2}} [f(x) - f(t)] \, du(t) = f(x) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] - \int_a^{\frac{a+b}{2}} f(t) \, du(t),
\]

and

\[
\int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)] \, du(t) = f(a+b-x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_{\frac{a+b}{2}}^b f(t) \, du(t).
\]

Adding the above equalities, we have

\[
\int_a^{\frac{a+b}{2}} [f(x) - f(t)] \, du(t) + \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)] \, du(t) = f(x) \left[ u \left( \frac{a+b}{2} \right) - u(a) \right] + f(a+b-x) \left[ u(b) - u \left( \frac{a+b}{2} \right) \right] - \int_a^b f(t) \, du(t).
\]

Applying the inequality (2.1), and then applying the Hölder inequality we get

\[
|E_0 (f, u; x)| = \left| \int_a^{\frac{a+b}{2}} [f(x) - f(t)] \, du(t) + \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)] \, du(t) \right|
\]

\[
\leq \left| \int_a^{\frac{a+b}{2}} [f(x) - f(t)] \, du(t) \right| + \left| \int_{\frac{a+b}{2}}^b [f(a+b-x) - f(t)] \, du(t) \right|
\]

\[
\leq \text{Lip}_M (u) \cdot \int_a^{\frac{a+b}{2}} |f(x) - f(t)| \, dt + \text{Lip}_M (u) \cdot \int_{\frac{a+b}{2}}^b |f(a+b-x) - f(t)| \, dt
\]

\[
\leq \text{Lip}_M (u) \cdot \left( \frac{b-a}{2} \right)^{1/q} \left[ \left( \int_a^{\frac{a+b}{2}} |f(x) - f(t)|^p \, dt \right)^{1/p} + \left( \int_{\frac{a+b}{2}}^b |f(a+b-x) - f(t)|^p \, dt \right)^{1/p} \right].
\]
Utilizing (4.3) we can write
\[
\int_{a}^{b} |f(x) - f(t)|^p \, dt \\
\leq \left( \frac{p^p \sin^p \left( \frac{\pi}{p} \right)}{\pi^p (p-1)} \right)^n \cdot \left( \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right)^{np} \int_{a}^{\frac{a+b}{2}} |f^{(n)}(t)|^p \, dt,
\]
and
\[
\int_{\frac{a+b}{2}}^{b} |f(a + x - b) - f(t)|^p \, dt \\
\leq \left( \frac{p^p \sin^p \left( \frac{\pi}{p} \right)}{\pi^p (p-1)} \right)^n \cdot \left( \frac{b-a}{4} + \left| x - \frac{3a+b}{4} \right| \right)^{np} \int_{\frac{a+b}{2}}^{b} |f^{(n)}(t)|^p \, dt.
\]
Substituting in these two inequalities in the previous one and simplify we get the required result and thus the theorem is proved. \(\square\)

**Theorem 4.** Let \(f \in \mathcal{L}^p(I)\). Assume that \(u : [a, b] \to \mathbb{R}\) has \(M\)-Lipschitz property on \([a, b]\), then we have the inequality
\[
|\mathcal{E}_1(f, u; x)| \\
\leq 2 \text{Lip}_M(u) \cdot \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi^\frac{p}{p-1}} \right)^n \left[ \frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^{n+\frac{p}{p-1}} \cdot \left\| f^{(n)} \right\|_{p, [a, b]},
\]
for all \(x \in [a, b]\). In particular, we have
\[
|\mathcal{E}_1(f, u; \frac{a+b}{2})| \leq \text{Lip}_M(u) \cdot \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi^\frac{p}{p-1}} \right)^n \left( \frac{b-a}{2} \right)^{n+\frac{p}{p-1}} \cdot \left\| f^{(n)} \right\|_{p, [a, b]}.
\]

**Proof.** Using the integration by parts formula for \(\mathcal{R}S\)-integral, we have
\[
\int_{a}^{x} [f(a) - f(t)] \, du(t) = f(a) [u(x) - u(a)] - \int_{a}^{x} f(t) \, du(t),
\]
and
\[
\int_{x}^{b} [f(b) - f(t)] \, du(t) = f(b) [u(b) - u(x)] - \int_{x}^{b} f(t) \, du(t).
\]
Adding the above equalities, we have
\[
\int_{a}^{x} [f(a) - f(t)] \, du(t) + \int_{x}^{b} [f(b) - f(t)] \, du(t) \\
= f(a) [u(x) - u(a)] + f(b) [u(b) - u(x)] - \int_{a}^{b} f(t) \, du(t).
\]
Following the same steps in the proof of Theorem 3 we get the required result. \(\square\)

**Remark 6.** The general error term \(\mathcal{E}_\alpha(f, u; x)\) has the form
\[
\mathcal{E}_\alpha(f, u; x) = (1 - \alpha) \mathcal{E}_0(f, u; x) + \alpha \mathcal{E}_1(f, u; x).
\]
for all \(\alpha \in [0, 1]\) and \(x \in [a, \frac{a+b}{2}]\).
In particular, the error of Simpson-like quadrature formula is obtained from the identity

\[
E_4 \left( f, u; \frac{a + b}{2} \right) = \frac{2}{3} E_0 \left( f, u; \frac{a + b}{2} \right) + \frac{1}{3} E_1 \left( f, u; \frac{a + b}{2} \right).
\]

Thus, by (4.5) and (4.7) we get

\[
\left| E_4 \left( f, u; \frac{a + b}{2} \right) \right| \leq \text{Lip}_M (u) \frac{(b - a)^{n + \frac{1}{2}}}{2^{n + \frac{1}{2} - 1}} \left( \frac{p \sin \left( \frac{\pi}{p} \right)}{\pi \sqrt{p - 1}} \right)^n \| f^{(n)} \|_{p, [a, b]}.
\]

**Remark 7.** In all above results the best error estimates hold with \( L^2 \)-norm i.e., \( p = q = 2 \).

**Remark 8.** To get \( L^p \)-bounds with bounded variation integrators one may apply Lemma 2 instead of (2.1) in the proofs of Theorems 3 and 4. Also, we may apply Lemma 1 instead of Hölder inequality in the proofs of Theorems 3 and 4.

**Remark 9.** One may apply the unused results in Section 2 to obtain more error bounds.

**Remark 10.** In the presented quadrature, high degree of accuracy occurred significantly with less error estimations when higher derivatives are assumed on very small scale of intervals. Particularly, if one assumes that \( f^{(m)} \in L_2[a, b] \) and \( b - a \leq \frac{1}{2^m} \) \((m \in \mathbb{N})\) then as \( m \) increases all obtained error estimations become very small. Hence, the presented results are recommended to be applied for small scale of intervals or to be applied as composite rules.

Let \( f : [a, b] \to \mathbb{R} \), be a twice differentiable mapping such that \( f''(x) \) exists and bounded on \((a, b)\). Then the trapezoidal rule reads

\[
(4.8) \quad \int_a^b f(x) \, dx = (b - a) \frac{f(a) + f(b)}{2} - \frac{(b - a)^3}{12} f''(\xi), \quad \text{for some } a < \xi < b.
\]

To improve our Remark 10, we give a numerical example by comparing our formula (4.5) with Trapezoidal rule (4.8). It is unusual to compare two approximations evaluated by two different norms unless we get a very close estimations or we don’t have a well-know rule to compare with.

Let \( [a, b] = [0, \frac{1}{2^{16}}] \) with \( u(t) = t \). In viewing (4.5) we get

\[
(4.9) \quad \left| E_0 \left( f, t; \frac{1}{2^{n+2n!}} \right) \right| \leq \frac{21/2}{(2\pi)^n} \frac{1}{2^n n!} \| f^{(n)} \|_2.
\]

Employing (3.3) for the particular choice \( n = 2 \), we get

\[
(4.10) \quad \int_0^\frac{1}{8} f(t) \, dt = \Phi_0 \left( f, t; \frac{1}{32} \right) - E_0 \left( f, t; \frac{1}{32} \right).
\]

Consider \( f(t) = \exp(-t^2), t \in \left[ 0, \frac{1}{8} \right] \). Then, we have the exact value

\[
(4.11) \quad \int_0^\frac{1}{8} f(t) \, dt = \frac{\sqrt{\pi}}{2} \text{erf} \left( \frac{1}{8} \right) = 0.1243519988.
\]

Employing (4.10) we get \( \Phi_0 \left( f, t; \frac{1}{17} \right) = 0.1243920852 \) and \( E_0 \left( f, t; \frac{1}{32} \right) = 1.482678376 \times 10^{-15} \) so that \( \int_0^\frac{1}{8} f(t) \, dt = 0.1243920852 \). However, applying the Trapezoidal rule
we get \( \int_0^4 f(t) \, dt = 0.1243487939 \). By comparing the two evaluations, the absolute error in (4.10) is \( 4.00864 \times 10^{-5} \) and in (4.8) is \( 3.2049 \times 10^{-6} \). Taking into account that we compare two approximations via two different norms.

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