Abstract

We provide an investigation of the $P$-wave hyperons employing the field correlator method in QCD. This method allows to derive the Effective Hamiltonian successfully applied to the meson and ground state baryon spectra. The hyperon spectrum appears to be expressed through two parameters relevant to QCD, the string tension $\sigma$, the strong coupling constant $\alpha_s$, and the bare strange quark mass $m_s$. Using these parameters a unified description of the ground and excited hyperon states is achieved. We also briefly consider the nucleon $P$-wave excitations. In particular, we predict that both the nucleon and hyperon states have the similar cost (in $\Delta L$) $\sim 460$ MeV.

1. Introduction

The advent of new ideas concerning quark-quark forces in QCD have led to revival of interest in baryon spectroscopy. The spectroscopy of heavy baryons has undergone a great renaissance in recent years, providing an exceptional window into tests of QCD, see e.g. [1]. As to the “old” $\Xi$ resonances, which we consider in this paper, nothing has changed since 1996. In the last Particle Data Group review [2], five resonances remain stuck with the same one star * or two stars **, meaning doubtful [1]. Most of the spin-parity values of $\Xi$ have not been measured but have been assigned in accord with expectation

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1The ambitious program of $\Xi$ spectroscopy has been proposed at JLab [3], but we do not have any finished results so far.
of the theory. Besides, knowledge of excited states is very much limited. Therefore, a powerful guideline for assigning quantum numbers to new states is required both by theory and experiment.

The reproduction of the baryon mass spectrum from first principles is an important challenge for QCD. Ground state spectroscopy on the lattice is by now a well understood problem and impressive agreement with experiments has been achieved. However, the lattice study of excited states is not so advanced, see [1] and references therein. The purpose of this paper is to present a consistent treatment of the $P$-wave hyperons within the alternative method in QCD, the field correlator method (FCM) [5]. The similar analysis of heavy $c$- and $b$- baryons will be given elsewhere [6].

In the FCM one derives the Effective Hamiltonian (EH), which comprises both confinement and relativistic effects, and contains only universal parameters: the string tension $\sigma$, the strong coupling constant $\alpha_s$, and the bare (current) quark masses. The simple local form of this Hamiltonian occurs for the objects with temporal scale larger than the vacuum gluon correlation scale $T_g \sim 0.2 \text{ fm}$, i.e. it is applicable to all states, perhaps with an exception of bottomonium. There is a lot of calculations of masses and wave functions of light mesons heavy quarkonia and heavy-light mesons [7], but only a few for S-wave baryons [8]. The present investigation was initially motivated as an attempt to extend an approach of Refs. [8] for the $P$-wave low-lying orbitally excited baryons. As in Ref. [8], we compute only the confinement energies (corrected by the perturbative one gluon exchange potential) and disregard the spin-spin and spin-orbit interactions.

The paper is organized as follows. In Section 2 we briefly review the EH method. In Section 3 we consider the hyperspherical approach which is a very effective numerical tool to solve the EH. In Section 4 our predictions for the P-wave nucleons and strange baryons are reported and compared with the results of other approaches. Section 5 contains our conclusions.

2. The Effective Hamiltonian in FCM

The application of the method for the baryons was described in detail elsewhere [8]. Here we give only a brief summary important for this particular calculation.

The EH has the following form:

$$H = \sum_{i=1}^{3} \left( \frac{m_{i}^{2}}{2\mu_{i}} + \frac{\mu_{i}}{2} \right) + H_0 + V.$$  \hfill (1)

Here $H_0$ is the nonrelativistic kinetic energy operator and $V$ is the sum of the string potential $V_Y(r_1, r_2, r_3)$ and a Coulomb interaction term arising from one gluon exchange.
The string potential considered in this work is
\[ V_Y(r_1, r_2, r_3) = \sigma r_{\text{min}}, \] (2)
where \( \sigma \) is the string tension and \( r_{\text{min}} \) is the minimal length string corresponding to the \( Y \)-shaped configuration. In this picture, strings start from each quark and meet at the Toricelli point of the triangle formed by the three quarks. This point is such that it minimizes the sum of the string lengths, and its position is a complicated function of the quark coordinates \( r_i \). The Coulomb interaction is
\[ V_{\text{Coulomb}} = -\frac{2}{3} \alpha_s \sum_{i<j} \frac{1}{r_{ij}}, \] (3)
where \( \alpha_s \) is the strong coupling constant and \( r_{ij} \) are the distances between quarks.

In Eq. (1) \( m_i \) are the bare quark masses, while \( \mu_i \) are the constant auxiliary einbein fields, initially introduced in order to get rid of the square roots appearing in the relativistic Hamiltonian [9]. The dynamics remains essentially relativistic, though being non-relativistic in form. The einbein fields are eventually treated as variational parameters. The eigenvalue problem is solved for each set of \( \mu_i \), then one has to minimize \( \langle H \rangle \) with respect to \( \mu_i \). Such an approach allows one a very transparent interpretation of einbeins: \( \mu_i \) can be treated as constituent masses of quarks of current mass \( m_i \). In this way the notion of constituent masses arises.

The baryon mass is then given by formula
\[ M_B = \sum_{i=1}^{3} \left( \frac{m_i^2}{2\mu_i} + \frac{\mu_i}{2} \right) + E_0(m_i, \mu_i) + C, \] (4)
where \( E_0(m_i, \mu_i) \) is an eigenvalue of the operator \( H_0 + V \), \( \mu_i \) are defined from the condition
\[ \frac{\partial}{\partial \mu_i} \left( \sum_{i=1}^{3} \left( \frac{m_i^2}{2\mu_i} + \frac{\mu_i}{2} \right) + E_0(m_i, \mu_i) \right) = 0, \] (5)
and \( C \) is the quark self-energy correction which is created by the color magnetic moment of a quark propagating through the vacuum background field \( T_g \). The effect of the quark self-energy is to shift the mass spectrum by a global negative amount \( C \):
\[ C = -\frac{2\sigma}{\pi} \sum_i \eta(t_i) \int_0^\infty z^2 K_1(tz) e^{-z} \, dz, \] (6)
where \( 1/T_g \) is the gluon correlation length. In this paper we use \( T_g = 1 \) GeV. The function \( \eta(t) \) is defined as
\[ \eta(t) = t \int_0^\infty z^2 K_1(tz) e^{-z} \, dz, \] (7)
\(^2\)Its negative sign is due to the paramagnetic nature of the particular mechanism at work in this case.
where $K_1$ is the McDonald function. The straightforward calculation yields

$$
\eta(t) = \frac{1 + 2t^2}{(1 - t^2)^2} - \frac{3t^2}{(1 - t^2)^{5/2}} \ln \frac{1 + \sqrt{1 - t^2}}{t}, \quad t < 1,
$$
$$
= \frac{1 + 2t^2}{(1 - t^2)^2} - \frac{3t^2}{(t^2 - 1)^{5/2}} \arctan(\sqrt{t^2 - 1}) \quad t > 1.
$$

(8)

Note that $\eta(0) = 1$ and $\eta(t) \sim 2/t^2$ as $t \to \infty$. For the values of the bare strange quark mass $m_s = 100$ MeV and 175 MeV used in this paper $\eta_s = 0.9486$ and 0.8882, respectively.

3. Hyperspherical formalism. Outline of the calculation.

In this section, we briefly review the hyperspherical method, which we use to calculate the masses of the ground and excited hyperon states.

The baryon wave function depends on the three-body Jacobi coordinates

$$
\rho_{ij} = \sqrt{\frac{\mu_{ij}}{\mu_0}} (\mathbf{r}_i - \mathbf{r}_j),
$$

(9)

$$
\lambda_{ij} = \sqrt{\frac{\mu_{ij,k}}{\mu_0}} \left( \frac{m_i \mathbf{r}_i + m_j \mathbf{r}_j - \mathbf{r}_k}{m_i + m_j} \right),
$$

(10)

(i, j, k cyclic), where $\mu_{ij}$ and $\mu_{ij,k}$ are the appropriate reduced masses

$$
\mu_{ij} = \frac{m_i m_j}{m_i + m_j}, \quad \mu_{ij,k} = \frac{(m_i + m_j)m_k}{m_i + m_j + m_k},
$$

(11)

$\mu_0$ is an arbitrary parameter with the dimension of mass which drops off in the final expressions. The coordinate $\rho_{ij}$ is proportional to the separation of quarks $i$ and $j$ and coordinate $\lambda_{ij}$ is proportional to the separation of quarks $i$ and $j$, and quark $k$. There are three equivalent ways of introducing the Jacobi coordinates, which are related to each other by linear transformations with the coefficients depending on quark masses and Jacobian equal to unity. In what follows we omit indices $i, j$.

In terms of the Jacobi coordinates the kinetic energy operator $H_0$ is written as

$$
H_0 = -\frac{1}{2\mu_0} \left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \lambda^2} \right) = -\frac{1}{2\mu} \left( \frac{\partial^2}{\partial R^2} + \frac{5}{R} \frac{\partial}{\partial R} + \frac{L^2(\Omega)}{R^2} \right),
$$

(12)

where $R$ is the six-dimensional hyperradius,

$$
R^2 = \rho^2 + \lambda^2, \quad \rho = R \sin \theta, \quad \lambda = R \cos \theta,
$$

(13)
Ω denotes five residuary angular coordinates, and $L^2(\Omega)$ is an angular operator

$$L^2 = \frac{\partial^2}{\partial \theta^2} + 4 \cot \theta \frac{\partial}{\partial \theta} - \frac{l^2_\rho}{\sin^2 \theta} - \frac{l^2_\lambda}{\cos^2 \theta},$$

(14)

whose eigenfunctions (the hyperspherical harmonics) satisfy

$$L^2(\Omega) Y_{[K]}(\Omega) = -K(K + 4) Y_{[K]}(\Omega),$$

(15)

with $K$ being the grand orbital momentum.

In terms of $Y_{[K]}$, the wave function $\psi(\rho, \lambda)$ can be written in a symbolical shorthand as

$$\psi(\rho, \lambda) = \sum_K \psi_K(R) Y_{[K]}(\Omega),$$

(16)

where the set $[K]$ is defined by the orbital momentum of the state and the symmetry properties. We truncate this set using the approximation $K = K_{\min}$ ($K_{\min} = 0$ for $L = 0$ and $K_{\min} = 1$ for $L = 1$). The corresponding hyperspherical harmonics are

$$Y_0(\Omega) = \sqrt{\frac{1}{\pi^3}}, \quad K = 0,$$

(17)

and

$$Y_\rho(\Omega) = \sqrt{\frac{6}{\pi^3}} \frac{\rho}{R}, \quad Y_\lambda(\Omega) = \sqrt{\frac{6}{\pi^3}} \frac{\lambda}{R}, \quad K = 1.$$  

(18)

The normalization coefficients in (18) are easily calculated using the relations

$$\int \rho_i \rho_j f(\theta, \cos \chi) d\Omega = \frac{1}{3} \delta_{ij} \int \rho^2 f(\theta, \cos \chi) d\Omega,$$

$$\int \lambda_i \lambda_j f(\theta, \cos \chi) d\Omega = \frac{1}{3} \delta_{ij} \int \lambda^2 f(\theta, \cos \chi) d\Omega,$$

(19)

where $d\Omega = d\mathbf{n}_\rho d\mathbf{n}_\lambda \sin^2 \theta \cos^2 \theta d\theta$.

For $\Lambda$ and $\Sigma$ we use the $uds$ basis in which the strange quark is singled out as quark 3 but in which the non strange quarks are still antisymmetrized. In the same way, for the $\Xi$ we use the $ssq$ basis with $q$ standing for $u$ or $d$ quarks, in which the non strange quark is singled out as quark 3. The $uds$ basis states diagonalize the confinement problem with eigenfunctions that correspond to separate excitations of the non strange and strange quarks ($\rho$- and $\lambda$- excitations, respectively). The nonsymmetrized $uds$ or $ssq$ basis usually provides a much simplified picture of the states. In particular, excitation of the $\lambda$ variable unlike excitation in $\rho$ involves the excitation of the “odd” quark ($s$ for $\Lambda$, $\Sigma$ or $q$ for $\Xi$).

We introduce the reduced function $u_\gamma(R)$ ($\gamma = 0$ for $L = 0$ and $\gamma = \rho$ or $\lambda$ for $L = 1)$

$$\Psi_\gamma(R, \Omega) = \frac{u_\gamma(R)}{R^{5/2}} \cdot Y_\gamma(\Omega),$$

(20)

\[^3\text{In what follows we do not write explicitly the magnetic quantum numbers of the vector hyperspherical harmonics.}\]
and average the interaction $U = V_C + V_{\text{string}}$ over the six-dimensional sphere $\Omega$ with the weight $|Y_\gamma(\Omega)|^2$. Then one obtains the Schrödinger equation for $u_\gamma(x)$

$$\frac{d^2u_\gamma}{dx^2} + 2 \left( E_0 - \frac{(K + 3/2)(K + 5/2)}{2x^2} - V_\gamma(x) \right) u_\gamma(x) = 0,$$

where

$$x = \sqrt{\mu_0} R.$$

In Eq. (21) $V_\gamma(x) = V^\gamma_Y(x) + V^\gamma_{\text{Coulomb}}(x)$,

$$V^\gamma_Y(x) = \int |Y_\gamma(\Omega)|^2 V_\gamma(r_1, r_2, r_3) d\Omega = \sigma b_\gamma R = \sigma \hat{b}_\gamma x,$$

and

$$V^\gamma_{\text{Coulomb}}(x) = -\frac{2}{3} \alpha_s \int |Y_\gamma(\Omega)|^2 \sum_{i<j} \frac{1}{r_{ij}} d\Omega = -\frac{2}{3} \alpha_s \frac{a_\rho}{R} = -\frac{2}{3} \alpha_s \frac{\hat{a}_\gamma}{x},$$

with

$$\hat{a}_\gamma = a \sqrt{\mu_0}, \quad \hat{b}_\gamma = \frac{b_\gamma}{\sqrt{\mu_0}}.$$

In what follows we denote

$$\mu_1 = \mu_2 = \mu, \quad \text{and} \quad \mu_3 = \kappa \mu.$$

The straightforward analytical calculation of the integrals in Eq. (25) yields

$$\hat{a}_0 = \frac{32 \sqrt{\mu}}{9\pi} \left( \frac{1}{\sqrt{2}} + 2 \sqrt{\frac{\kappa}{1 + \kappa}} \right),$$

$$\hat{a}_\rho = \frac{32 \sqrt{\mu}}{15\pi} \left( \sqrt{2} + \sqrt{\frac{\kappa}{\kappa + 1}} \frac{5\kappa + 6}{\kappa + 1} \right),$$

$$\hat{a}_\lambda = \frac{32 \sqrt{\mu}}{5\pi} \left( \frac{1}{\sqrt{2}} + \frac{1}{3} \sqrt{\frac{\kappa - 4 + 5\kappa}{1 + \kappa}} \right),$$

while the corresponding expressions for the string potential are more complicated. We relegate the details of the numerical procedure used to calculate the $\hat{b}_\gamma$ in Eq. (21) to Appendix.

For the doublet spin states one can introduce another possible basis corresponding to the fact that the $P$-wave hyperons contain both positive and negative parity two-quark subsystems. New basis states are given by the linear combinations

$$\xi_\sigma = \frac{\chi_\sigma(12,3) Y_\rho + \chi_s(12,3) Y_\lambda}{\sqrt{2}}, \quad \xi_\alpha = \frac{\chi_\sigma(12,3) Y_\lambda - \chi_s(12,3) Y_\rho}{\sqrt{2}},$$
where $\chi_a$ and $\chi_s$ are the doublet spin functions which are even and odd under the permutation of quarks 1 and 2, respectively. In the $SU(3)$ limit $\mu_1 = \mu_2 = \mu_3$ the spin-angular functions $\xi_s (\xi_a)$ are totally symmetric (antisymmetric). Using basis states (31) one has for the P-wave matrix elements

$$V_{\text{Coulomb}}(x) = -\frac{2}{3} \alpha_s \frac{\hat{a}}{x} , \quad V_{\text{Y}}(x) = \sigma \frac{\hat{b}}{x} , \quad (32)$$

where

$$\hat{a} = \frac{\tilde{a}_\rho + \tilde{a}_\lambda}{2} , \quad \hat{b} = \frac{\tilde{b}_\rho + \tilde{b}_\lambda}{2} . \quad (33)$$

4. RESULTS

In this Section we present the results obtained for $qqq$, $qqs$ and $ssq$ baryons. As already stated, we disregard the hyperfine interactions which give spin-doublet - spin-quartet splittings and the spin-orbit interactions, which describes the fine structure of states. Note that with its attractive $\delta$-function, the hyperfine interaction produces effects which are must stronger than those one would obtain from the lowest order perturbation theory, in which the $\Lambda$ and $\Sigma$ hyperons are almost degenerate. The large hyperfine effects in $\Delta - N$ or $\Sigma - \Lambda$ splittings are usually described by the smeared $\delta$-function [11] and/or meson exchanges between quarks (see e.g. [12]), and generally require additional model-dependent assumptions about the structure of interquark forces.

We do not perform a systematic study in order to determine the best set of parameters $\sigma$, $\alpha_s$ and $m_s$ to fit the hyperon spectra. Instead, we employ some typical values of $\sigma$ and $\alpha_s$ that have been used for the description of the ground state baryons: the string tension $\sigma$ is taken to be commonly used value of 0.15 GeV$^2$ and the strong coupling constant $\alpha_s = 0.39$. For the nucleon we slightly varied $\alpha_s$ to illustrate the sensitivity of the results to the chosen input. In our calculations we use the values of the current light quark masses $m_u = m_d = 7$ MeV, $m_s = 100$ and 175 MeV.

We begin with the discussion of the $qqq$ states with $L^P = 0^+$ and $1^-$. For $L = 1$ we use the spin-angular functions (31). In Table 1 we display the nucleon masses for the three choices of $\alpha_s$: 0.39, 0.5 and 0.6 $^4$. The last value have been used in the Capstick-Isgur model [11]. Increasing $\alpha_s$ by $\sim 0.1$ decreases the nucleon mass by $\sim 50$ MeV. We get $\frac{1}{2} (N + \Delta)_{\text{theory}} = 1228, 1181$ and 1131 MeV for $L = 0$ and $\alpha_s = 0.39, 0.5$ and 0.6, respectively, vs $\frac{1}{2} (N + \Delta)_{\text{exp}} = 1085$ MeV. For $L = 1$ we obtain 1770 MeV, 1653 MeV $^4$

$^4$The results for $L^P = 0^+$ state and $\alpha_s = 0.39$, as those for the $qqs$ and $ssq$ states with $m_s = 175$ MeV (shown in Table 2) have been previously obtained in [5], and we quote the (slightly updated) results here for comparison.
and 1613 MeV, respectively. The excitation energies only weakly depend on \( \alpha_s \); cost (in \( \Delta L \)) is 469 MeV for \( \alpha_s = 0.39 \), 472 MeV for \( \alpha_s = 0.5 \), and 482 MeV for \( \alpha_s = 0.6 \).

The \( qqs \) and \( ssq \) states which belong to the octet are structurally identical to the nucleon: only one and two, respectively, of the light quarks are replaced by a strange quark. Consequently, the analysis of these states and the results are only a variation of what has been found for the nucleon system. Table 2 displays the sensitivity of hyperon masses to the chosen value of \( m_s \). Increasing \( m_s \) by 75 MeV increases \( \mu_s \) by 30 MeV both for \( qqs \) and \( ssq \), but practically does not affect \( \mu_q \). As the result the masses of the \( qqs \) and \( ssq \) states are increased by 40 MeV and 75 MeV, respectively. We get \( \frac{1}{4}(\Lambda + \Sigma + 2\Sigma^*)_{theory} = 1278 \) and 1317 MeV for \( m_s = 100 \) and 175 MeV, respectively, vs \( \frac{1}{4}(\Lambda + \Sigma + 2\Sigma^*)_{exp} = 1267 \) MeV.

Table 3 displays our main results for the \( \rho \) and \( \lambda \) excitations of the \( P \) wave \( qqs \) and \( ssq \) hyperons. As in Table 2 we present here the dynamical quark masses, zero-order eigenenergies and the hyperon masses for \( m_s = 100 \) and 175 MeV. The dynamical quark masses \( \mu_i \) corresponding to the excitations \( \rho \) and \( \lambda \) are somewhat different. This is not surprising because these quantities can be considered as the average kinetic energies of the current quarks, which are larger for the quarks in the \( P \)-wave and smaller for the quarks in the \( S \)-wave. The eigenenergies \( E_0 \) of the \( \rho \) and \( \lambda \) excitations are degenerate, the masses of the corresponding states are nearly degenerate: they differ no more than 10 MeV\(^5\). This difference is due to the difference of the dynamical masses \( \mu_i \) which enter the mass formula (4) and the self energy contribution (6).

In Table 4 we present the zero-order \( qqs \) and \( ssq \) masses calculated for the mixture of the \( \rho \) and \( \lambda \) excitations, Eq. (31), for \( m_s = 100 \) MeV. Comparing the results with those of Table 3 we observe again the same eigenenergies \( E_0 \) and slightly different hyperon masses \( M \). The excitation energies \( \Delta = M(L = 1) - M(L = 0) \) are of the order of 460 MeV both for \( \Sigma \) and \( \Xi \) and also coincide with the excitation energies for the nucleon.

The discrepancy of the results shown in Tables 1-4 with the experimental ones hints that it is the chiral physics, missing our approach, that could shift nucleon and \( \Lambda \) states down [14]. However, the chiral effects are less important for the \( \Sigma \) states. Therefore we identify two approximately degenerate \( qqs \) excited states of negative parity with the \( P \)-wave \( \Sigma \) resonances. The PDG [2] lists \( D_{13} \), \( S_{11} \) and \( D_{15} \) resonances with \( I = 1, J^P = (3/2)^-, (1/2)^-, (5/2)^- \) and masses 1670, 1750 and 1775 MeV, respectively. The latter state corresponds to the \( \lambda \) - excitation with \( S = 3/2 \). The results are compatible with the known states, showing discrepancies with the experimental data of order 5% or

\(^5\)This degeneracy disappear for the heavy baryons, in which case the masses of the \( \rho \) and \( \lambda \) excitations differ by \( \sim 100 \) MeV [9].
We finally remark that our result for the negative parity ground state in the \( \Xi \) channel \( M(L = 1) = 1781 \) MeV exactly agrees with the recent finding from the lattice quenched calculation \[4\]. The other theoretical predictions for this state are listed in Table 5.

5. Conclusions

In this paper we have extended our previous study of the ground states of the \( qqq, qqs \) and \( ssq \) baryons to the description of their first angular excitations. We use the EH method. The three-quark problem has been solved using the hyperspherical approach. For each baryon we have calculated the dynamical quark masses \( \mu_i \) from Eq. (5), energy eigenvalues \( E_0 \) from Eq. (21), and the baryon masses \[4\] with the self-energy corrections (6). The main results are given in Tables 3 and 4. Our study suggests that a good description of the P-wave baryons can be obtained with a spin independent energy eigenvalues corresponding to the confinement plus Coulomb potentials. Moreover this comparative study gives a better insight into the quark model results where the constituent masses encode the QCD dynamics.

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Appendix. The string junction potential in the hyperspherical formalism

Recall the definition of the minimal length string Y–shaped configuration \[11\]. Let \( \varphi_{ijk} \) be the inner angle between the line from quark \( i \) to quark \( j \) and that from quark \( j \) to quark \( k \). One should distinguish two cases. If all the inner angles of the triangle formed by three constituent quarks sitting at the apexes are smaller than \( 120^\circ \), the junction point coincides with the so-called Torrichelli point of the triangle and

\[
    r_{\text{min}} = \left( \frac{1}{2} \sum_{i<j} r_{ij}^2 + \frac{\sqrt{3}}{2} \sqrt{(r_{12} + r_{31} + r_{23})(r_{12} - r_{31} + r_{23})(r_{12} - r_{31} + r_{23})} \right)^{1/2}
\]

The relative distances \( r_{ij} \) in (A.1) are expressed in terms of the Jacoby coordinates:
\[ r_{12} = \sqrt{\frac{2\mu_0}{\mu}} \rho, \]
\[ r_{31} = \sqrt{\frac{\mu_0}{2\mu}} \left( \rho^2 + \frac{\kappa + 2}{\kappa} \lambda^2 + 2 \sqrt{\frac{\kappa + 2}{\kappa}} \rho \lambda \cos \chi \right)^{1/2}, \quad (A.2) \]
\[ r_{23} = \sqrt{\frac{\mu_0}{2\mu}} \left( \rho^2 + \frac{\kappa + 2}{\kappa} \lambda^2 - 2 \sqrt{\frac{\kappa + 2}{\kappa}} \rho \lambda \cos \chi \right)^{1/2}, \]

where \( \cos \chi = n_\rho n_\lambda \). Substituting these expressions into (A.1), one obtains

\[ r_{\text{min}} = x l_0(\theta, \chi) = \frac{x}{\sqrt{\mu}} \left( \frac{3}{2} \sin^2 \theta + \sqrt{3} \sqrt{\frac{\kappa + 2}{\kappa}} \sin \theta \cos \sin \chi + \frac{\kappa + 2}{2\kappa} \cos^2 \theta \right)^{1/2}. \quad (A.3) \]

If \( \phi_{ijk} \) is equal to or greater than \( 2\pi/3 \), the lowest energy configuration has the junction at the apex connected with quark \( j \):

\[ r_{\text{min}} = r_{ij} + r_{jk}, \quad (A.4) \]

where

\[ r_{ij} = x l_{ij}(\theta, \chi). \quad (A.5) \]

Accordingly, in the \( \theta - \chi \) plane one should distinguish the four regions:

(i) region I: \( \cos \phi_{ijk} \geq -1/2 \), \( r_{\text{min}} = x l_0(\theta, \chi) \), where \( l_0(\theta, \chi) \) is defined in (A.3),

(ii) region II: \( \cos \phi_{312} \leq -1/2 \), \( \chi \geq \frac{2\pi}{3} \), \( r_{\text{min}} = r_{12} + r_{31} = x(l_{12}(\theta) + l_{31}(\theta, \chi)) \),

(iii) region III: \( \cos \phi_{123} \leq -1/2 \), \( \chi \leq \frac{\pi}{3} \), \( r_{\text{min}} = r_{12} + r_{23} = x(l_{12}(\theta) + l_{23}(\theta, \chi)) \), and

(iv) region IV: \( \cos \phi_{231} \leq -1/2 \), \( r_{\text{min}} = r_{31} + r_{23} = x(l_{31}(\theta, \chi) + l_{23}(\theta, \chi)) \).

The expressions for \( l_{12}(\theta) \), \( l_{31}(\theta, \chi) \) and \( l_{23}(\theta, \chi) \) follow from Eqs. (A.2). The boundaries between region I and regions II, III are given by

\[ \theta_{1,2}(\chi) = \arctan \left( \sqrt{\frac{\kappa + 2}{\kappa}} (-\cos \chi - \frac{1}{\sqrt{3}} \sin \chi) \right), \quad (A.6) \]

while the boundary between regions I and IV is given by

\[ \theta_3(\chi) = \arctan \left( \sqrt{\frac{1}{3}} \sqrt{\frac{\kappa + 2}{\kappa}} \left( \sin \chi + \sqrt{\sin^2(\chi) + 3} \right) \right). \quad (A.7) \]

The constants \( \hat{b}_\gamma \) in Eq. (24) are expressed in terms of the four integrals:
\[ \hat{b}_\gamma = \frac{1}{x} \left( \int_I + \int_{II} + \int_{III} + \int_{IV} \right) r_{\min} \, d\Omega, \] (A.8)

or, in the explicit form,

\[ \hat{b}_\gamma = \int_0^{\frac{\pi}{4}} \sin \chi \, d\chi \times \left( \int_0^{\theta_3(\chi)} \left( l_{12}(\theta, \chi) + l_{23}(\theta, \chi) \right) \, d\Omega \right) + \left( \int_0^{\theta_3(\chi)} l_0(\theta, \chi) \, d\Omega \right) + \left( \int_0^{\theta_3(\chi)} (l_{23}(\theta, \chi) + l_{31}(\theta, \chi)) \, d\Omega \right) \]

\[ + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin \chi \, d\chi \times \left( \int_0^{\theta_3(\chi)} \left( l_{12}(\theta, \chi) + l_{23}(\theta, \chi) \right) \, d\Omega \right) + \left( \int_0^{\theta_3(\chi)} l_0(\theta, \chi) \, d\Omega \right) + \left( \int_0^{\theta_3(\chi)} (l_{23}(\theta, \chi) + l_{31}(\theta, \chi)) \, d\Omega \right), \]

where

\[ d\Omega_0 = \frac{8}{\pi} \sin^2 \theta \cos^2 \theta \sin \chi \, d\chi \, d\theta \] (A.10)

for \( K = 0 \),

\[ d\Omega_\rho = \frac{1}{3} \cdot \frac{48}{\pi} \sin^4 \theta \cos^2 \theta \sin \chi \, d\chi \, d\theta, \quad d\Omega_\lambda = \frac{1}{3} \cdot \frac{48}{\pi} \sin^2 \theta \cos^4 \theta \sin \chi \, d\chi \, d\theta \] (A.11)

for \( K = 1 \).

**References**

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Table 1: Ground and excited state nucleon masses for $\alpha_s = 0.39, 0.5$ and $0.6$. For each case shown are the dynamical quark masses $\mu$, defined by Eq. (5), eigenenergies $E_0$ of Eq. (21), the nucleon masses given by Eq. (4), and $\Delta = M(L = 1) - M(L = 0)$ (all in units of MeV). P-wave eigenenergies and nucleon masses correspond to the spin angular functions (31).

| $\alpha_s$ | $L$ | $\mu$ | $E_0$ | $M_N$ | $\Delta$ |
|------------|-----|-------|-------|-------|---------|
| 0.39       | 0   | 408   | 1318  | 1228  |         |
| 0.5        | 0   | 425   | 1217  | 1181  |         |
| 0.6        | 0   | 442   | 1121  | 1131  |         |
| 0.39       | 1   | 457   | 1638  | 1697  | 469     |
| 0.5        | 1   | 469   | 1560  | 1653  | 472     |
| 0.6        | 1   | 481   | 1487  | 1613  | 482     |

Table 2: Ground state hyperons for $m_s = 175$ and 100 MeV. The notations are the same as in Table 1.

| $m_s$ | $\mu_1 = \mu_2 = \mu_3$ | $E_0$ | $M$ |
|-------|--------------------------|-------|-----|
| qqs   | 100                      | 410   | 424 | 1308 | 1278 |
|       | 175                      | 414   | 453 | 1291 | 1317 |
| ssq   | 100                      | 426   | 412 | 1298 | 1327 |
|       | 175                      | 458   | 419 | 1266 | 1402 |
| Hyperon | $m_s$ | Excitation | $\mu_1 = \mu_2$ | $\mu_3$ | $E_0$ | $M$ |
|---------|-------|------------|-----------------|--------|-------|------|
| qqs     | 100   | $\rho$    | 479             | 431    | 1627  | 1724 |
|         | 100   | $\lambda$ | 438             | 509    | 1629  | 1717 |
|         | 175   | $\rho$    | 482             | 457    | 1612  | 1774 |
|         | 175   | $\lambda$ | 440             | 532    | 1616  | 1758 |
| ssq     | 100   | $\rho$    | 491             | 419    | 1621  | 1745 |
|         | 100   | $\lambda$ | 452             | 500    | 1620  | 1752 |
|         | 175   | $\rho$    | 518             | 423    | 1594  | 1829 |
|         | 175   | $\lambda$ | 480             | 506    | 1591  | 1845 |

Table 3: Masses of the $\rho$ and $\lambda$ hyperon excitations. Shown are the dynamical quark masses $\mu_i$, the confinement energies $E_0$ and the hyperon masses $M$ (all in units of MeV).

| Hyperon | $L^P$ | $\mu_1 = \mu_2$ | $\mu_3$ | $E_0$ | $M$ | $\Delta$ |
|---------|-------|-----------------|--------|-------|-----|--------|
| qqs     | 0$^+$ | 410             | 424    | 1308  | 1278|      |
|         | 1$^-$ | 458             | 471    | 1630  | 1739| 461   |
| ssq     | 0$^+$ | 426             | 412    | 1298  | 1327|      |
|         | 1$^-$ | 472             | 460    | 1622  | 1781| 454   |

Table 4: Solutions of Eqs. (4), (21) for the hyperon states with $L = 0, 1$. $\alpha_s = 0.39$, $m_s = 100$ MeV. P-wave eigenenergies and the hyperon masses correspond to the spin-angular functions [31].
Table 5: Low-lying $\Xi$ spectrum of spin $L = 1$ predicted by the non-relativistic quark model of Chao, Isgur and Karl [13], relativized quark model of Capstick and Isgur [11], Glozman-Riska model [14], large $N_c$ analysis [16], algebraic model [17], QCD sum rules [18], and the Skirm model [20]. The question mark in the last column means that the $J^P$ quantum numbers are not identified by PDG. The mass is given in the unit of MeV.

| State | [13] | [11] | [14] | [16] | [17] | [18] | [19] | [20] | This work | PDG [2] |
|-------|------|------|------|------|------|------|------|------|-----------|---------|
| $\Xi(\frac{1}{2}^\pm)$ | 1785 | 1755 | 1758 | 1780 | 1869 | 1550 | (1630) | 1660 | 1781 | $\Xi(1690)$? |
| $\Xi(\frac{3}{2}^-)$ | 1800 | 1785 | 1758 | 1815 | 1828 | 1840 | 1820 | 1781 | $\Xi(1820)$ |

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