Viscosity Solutions to First Order Path-Dependent Hamilton-Jacobi-Bellman Equations in Hilbert Space *

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Abstract

In this article, a notion of viscosity solutions is introduced for first order path-dependent Hamilton-Jacobi-Bellman (PHJB) equations associated with optimal control problems for path-dependent evolution equations in Hilbert space. We identify the value functional of optimal control problems as unique viscosity solution to the associated PHJB equations. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property.

Key Words: Path-dependent Hamilton-Jacobi-Bellman equations; Viscosity solutions; Optimal control; Path-dependent evolution equations

AMS Subject Classification: 49L20; 49L25; 93C23; 93C25; 93E20.

1 Introduction

Viscosity solutions for first order Hamilton-Jacobi-Bellman (HJB) equations in infinite dimensions have been investigated by Crandall and Lions in [4, 5, 6] for the case without unbounded term, in [7, 8, 10] for the case with unbounded linear term, and in [9] for the case with unbounded nonlinear term. Soon after, Gozzi, Srinathan, and Świȩch [16] studied Bellman equations associated to control problems for variational solutions of the Navier-Stokes equation. We also mention the work of Li and Yong [17], where the general unbounded first order HJB equations in infinite dimensional Hilbert spaces are studied.

Fully nonlinear path-dependent first order Hamilton-Jacobi equations have been studied by Lukoyanov [18]. The existence and uniqueness theorems are obtained when Hamilton function $H$ is $d_p$-locally Lipschitz continuous in the path function. In our paper [21], we extended the results in [18] to $d_\infty$-Lipschitz continuous case. Bayraktar and Keller [1] proposed notions of minimax and viscosity solutions for a class of fully nonlinear path-dependent HJB (PHJB) equations with nonlinear, monotone, and coercive operators on Hilbert space and proved the existence, uniqueness and stability for minimax solutions. For the second order path-dependent case, a viscosity solution approach has been successfully initiated by Ekren, Keller, Touzi and Zhang [13] in the semilinear context, and further extended to fully nonlinear equations by Ekren, Touzi, and Zhang [14, 15].

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elliptic equations by Ren [19], obstacle problems by Ekren [12], and degenerate second-order equations by Ren, Touzi, and Zhang [20]. Cosso, Federico, Gozzi, Rosestolato, and Touzi [3] studied a class of semilinear second order PHJB equations with a linear unbounded operator on Hilbert space.

In this paper, we consider the following controlled path-dependent evolution equation (PEE):

\[
\begin{align*}
\dot{X}^{\gamma_t,u}(s) &= AX^{\gamma_t,u}(s) + F(X^{\gamma_t,u}, u(s)), \quad s \in [t, T], \\
X^{\gamma_t,u}_t &= \gamma_t \in \Lambda_t.
\end{align*}
\]  

(1.1)

In the above equation, \( \Lambda_t \) denotes the set of all continuous \( H \)-valued functions defined over \([0, t] \) and \( \Lambda = \bigcup_{t \in [0, T]} \Lambda_t \); the unknown \( X^{\gamma_t,u}(s) \), representing the state of the system, is an \( H \)-valued process; the control process \( u \) takes values in some metric space \((U, d); A\) is the generator of a \( C_0 \) semigroup of bounded linear operator in Hilbert space \( H \) and the coefficient \( F \) is assumed to satisfy \( d_{\infty} \)-Lipschitz condition.

We try to minimize a cost functional of the form:

\[
J(\gamma_t, u) := \int_t^T q(X^{\gamma_t,u}_s, u(\sigma))d\sigma + \phi(X^{\gamma_t,u}_T), \quad (t, \gamma_t) \in [0, T] \times \Lambda,
\]  

(1.2)

over all admissible controls \( U[t, T] \). Here \( q \) and \( \phi \) are given real functionals on \( \Lambda \times U \) and \( \Lambda_T \), respectively. We define the value functional of the optimal control problem as follows:

\[
V(\gamma_t) := \inf_{u \in U[t, T]} J(\gamma_t, u), \quad (t, \gamma_t) \in [0, T] \times \Lambda.
\]  

(1.3)

The goal of this article is to characterize this value functional \( V \). We consider the following PHJB equation:

\[
\begin{align*}
\partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H + H(\gamma_t, \partial_x V(\gamma_t)) &= 0, \quad (t, \gamma_t) \in [0, T] \times \Lambda, \\
V(\gamma_T) &= \phi(\gamma_T), \quad \gamma_T \in \Lambda_T;
\end{align*}
\]  

(1.4)

where

\[
H(\gamma_t, p) = \sup_{u \in U} [(p, F(\gamma_t, u))_H + q(\gamma_t, u)], \quad (\gamma_t, p) \in \Lambda \times H.
\]

\( A^* \) denotes the adjoint operator of \( A \). The definitions of \( \partial_t \) and \( \partial_x \) will be introduced in Section 2.

In this paper we will develop a concept of viscosity solutions to PHJB equations on the space of \( H \)-valued continuous paths (see Definition 3.2 for details) and show that the value functional \( V \) defined in (1.3) is unique viscosity solution to the PHJB equation (1.4).

The main challenge comes from the both facts that the path space \( \Lambda_T \) is an infinite dimensional Banach space in which the maximal norm \( \|\cdot\|_0 \) is not smooth, and the operator \( A \) is unbounded. In [21] we gave a definition of viscosity solutions in a sequence of bounded and uniformly Lipschitz continuous functions spaces \( C^\mu_{M_0} \) which are compact subsets in \( R^d \)-valued path-dependent case, and proved that the value functional is unique viscosity solution to the associated PHJB equation. However, \( C^\mu_{M_0} \) are not compact in \( H \)-valued path-dependent case. By studying the \( B \)-continuity of the value functional, Li and Yong [17] proved the value functional is unique viscosity solution to the associated HJB equation in Hilbert space under the assumption (A.4) on page 231 of [17]. However, the value functional does not have the \( B \)-continuity under the assumption (A.4) in path-dependent case. Therefore, the techniques introduced in [21] and [17] are not applicable in our case.

The main contribution of this paper is the introduction of an appropriate functional \( \Upsilon^2(\cdot, \cdot) \) on \( \Lambda \times \Lambda \). The functional is the key to prove the stability and uniqueness of viscosity solutions.
Using Lemma 2.6, we can define an auxiliary function $\Psi$ which includes the functional $\overline{\gamma}^2(\cdot, \cdot)$ (see Step 1 in the proof of Theorem 4.1). More importantly, we can use $\overline{\gamma}^2(\cdot, \cdot)$ to define a smooth gauge-type function and apply a modification of Borwein-Preiss variational principle (see Lemma 2.3) to get a maximum of a perturbation of the auxiliary function $\Psi$. Then we prove the uniqueness of viscosity solutions without the assumption (A.4) of [17]. Regarding existence, we prove that the value functional $V$ defined in (1.3) is a viscosity solution to PHJB (1.4) under our definition by functional Itô formula and dynamic programming principle. We also emphasize that, with respect to the standard viscosity solution theory in infinite dimension, the assumption (A.4) of [17] is completely bypassed in our framework. Therefore, even in the case without path-dependent, our well-posedness result applies to equations which cannot be treated, up to now, with the known theory of viscosity solutions.

The paper is organized as follows. In the following section, we introduce preliminary results on path-dependent optimal control problems in Hilbert space, and prove Theorem 2.2 and Lemmas 2.4 and 2.6 which are the key of the existence and uniqueness results of viscosity solutions. In Section 3, we introduce our notion of viscosity solutions to equation (1.4) and prove that the value of classical solutions and the stability result. In Section 4, the uniqueness of viscosity solutions for equation (1.4) is proven and Section 5 is devoted to proving $(\hat{A}^t, d_{\infty})$ and $(\Lambda^t, d_{\infty})$ are two complete metric spaces.

## 2 Preliminary work

We list some notations that are used in this paper. We use the symbol $| \cdot |$ to denote the norm in a Banach space $\Xi$, with a subscript if necessary. Let $H$ denote a real separable Hilbert space, with scalar products $(\cdot, \cdot)_H$. The operator $A$ is the generator of a strongly continuous semigroup $\{e^{tA}, t \geq 0\}$ of bounded linear operators in the Hilbert space $H$. The domain of the operator $A$ is denoted by $D(A)$. $A^*$ denotes the adjoint operator of $A$. Let $T > 0$ be a fixed number. For each $t \in [0, T]$, define $\hat{\Lambda}_t := D([0, t]; H)$ as the set of càdlàg $H$-valued functions on $[0, t]$. We denote $\hat{\Lambda} := \bigcup_{s \in [t, T]} \hat{\Lambda}_s$ and let $\hat{\Lambda}$ denote $\hat{\Lambda}^0$.

As in [11], we will denote elements of $\hat{\Lambda}$ by lower case letters and often the final time of its domain will be subscripted, e.g., $\gamma \in \hat{\Lambda}$ will be denoted by $\gamma_t$. Note that, for any $\gamma \in \hat{\Lambda}$, there exists only one $t$ such that $\gamma \in \hat{\Lambda}_t$. For any $0 \leq s \leq t$, the value of $\gamma_t$ at time $s$ will be denoted by $\gamma_t(s)$. Moreover, if a path $\gamma_t$ is fixed, the path $\gamma_t|_{[0, s]}$, for $0 \leq s \leq t$, will denote the restriction of the path $\gamma_t$ to the interval $[0, s]$.

For convenience, define for $x \in H, \gamma_t, \tilde{\gamma}_t \in \hat{\Lambda}$, $0 \leq t \leq \tilde{t} \leq T$,

$$\begin{align*}
g_t^x(s) &:= \gamma_t(s)1_{[0,t]}(s) + (\gamma_t(t) + x)1_{[t,\tilde{t}]}(s), \quad s \in [0, \tilde{t}]; \\
g_{t,\tilde{t}}(s) &:= \gamma_t(s)1_{[0,\tilde{t}]}(s) + \gamma_{\tilde{t}}(t)1_{[t,\tilde{t}]}(s), \quad s \in [0, \tilde{t}]; \\
g_{t,\tilde{t},A}(s) &:= \gamma_t(1)1_{[0,\tilde{t}]}(s) + e^{A(s-t)}\gamma_{\tilde{t}}(t)1_{[t,\tilde{t}]}(s), \quad s \in [0, \tilde{t}].
\end{align*}$$

We define a norm and a metric on $\hat{\Lambda}$ as follows: for any $0 \leq t \leq \tilde{t} \leq T$ and $\gamma_t, \tilde{\gamma}_t \in \hat{\Lambda}$,

$$
||\gamma_t||_0 := \sup_{0 \leq s \leq t} |\gamma_t(s)|, \quad d_{\infty}(\gamma_t, \tilde{\gamma}_t) := |t - \tilde{t}| + ||\gamma_{t,T} - \tilde{\gamma}_{t,T}||_0.
$$

Then $(\hat{\Lambda}_t, ||\cdot||_0)$ is a Banach space, and $(\hat{\Lambda}^t, d_{\infty})$ is a complete metric space by Lemma 5.1. Following Dupire [11], we define spatial derivatives of $f : \hat{\Lambda} \to R$, if exist, in the standard sense: if there
exists a $B \in H$ such that

$$\lim_{|h| \to 0} \frac{|f(\gamma_t^h) - f(\gamma_t) - (B, h)_H|}{|h|} = 0,$$

we say $\partial_x f(\gamma_t) = B$, and the right time-derivative of $f$, if exists, as:

$$\partial_t f(\gamma_t) := \lim_{t \to 0, f > 0} \frac{1}{t} \left[ f(\gamma_{t,t+i}) - f(\gamma_t) \right], \quad t < T.$$  \hfill (2.2)

**Definition 2.1.** Let $t \in [0, T]$ and $f : \Lambda^t \to R$ be given.

(i) We say $f \in C^0(\Lambda^t)$ if $f$ is continuous in $\gamma_s$ on $\Lambda^t$ under $d_\infty$.

(ii) We say $f \in C^1(\Lambda^t) \subset C^0(\Lambda^t)$ if $\partial_t f$ and $\partial_x f$ exist and are continuous.

Let $\Lambda_t := C([0,t], H)$ be the set of all continuous $H$-valued functions defined over $[0,t]$. We denote $\Lambda^t := \bigcup_{s \in [t,T]} \Lambda_s$ and let $\Lambda$ denote $\Lambda^0$. Clearly, $\Lambda := \bigcup_{t \in [0,T]} \Lambda_t \subset \hat{\Lambda}$, and each $\gamma \in \Lambda$ can also be viewed as an element of $\hat{\Lambda}$. $(\Lambda_t, ||| \cdot |||_0)$ is a Banach space, and $(\Lambda^t, d_\infty)$ is a complete metric space by Lemma 5.1. Let $F : \Lambda \to R$ and $\hat{f} : \hat{\Lambda} \to R$ are called consistent on $\Lambda$ if $f$ is the restriction of $\hat{f}$ on $\Lambda$.

**Definition 2.2.** Let $t \in [0, T]$ and $f : \Lambda^t \to R$ be given.

(i) We say $f \in C^0(\Lambda^t)$ if $f$ is continuous in $\gamma_s$ on $\Lambda^t$ under $d_\infty$.

(ii) We say $f \in C^1(\Lambda^t)$ if there exists $\hat{f} \in C^1(\hat{\Lambda}^t)$ which is consistent with $f$ on $\Lambda^t$.

Let $(U, d)$ is a metric space. An admissible control $u = \{u(r), r \in [t,s]\}$ on $[t,s]$ (with $0 \leq t \leq s \leq T$) is a measurable function taking values in $U$. The set of all admissible controls on $[t,s]$ is denoted by $U[t,s]$, i.e.,

$$U[t,s] := \{u(\cdot) : [t,s] \to U | u(\cdot) \text{ is measurable}\}.$$  

Now, we describe some continuous properties of the solutions of state equation (1.1) and value functional (1.23). First let us assume that functionals $F : \Lambda \times U \to H$, $q : \Lambda \times U \to R$ and $\phi : \Lambda_T \to R$ satisfy the following assumption.

**Hypothesis 2.3.** (i) The operator $A$ is the generator of a $C_0$ contraction semigroup $\{e^{tA}, t \geq 0\}$ of bounded linear operators in the Hilbert space $H$.

(ii) For every fixed $\gamma_t \in \Lambda$, $F(\gamma_t, \cdot)$ and $q(\gamma_t, \cdot)$ are continuous in $u$.

(iii) There exists a constant $L > 0$ such that, for all $(t, \gamma_t, \zeta_T, u)$, $(s, \eta_s, \zeta_T' u) \in [0,T] \times \Lambda \times \Lambda_T \times U$,

$$|F(\gamma_t, u)|^2 \leq L^2(1 + ||\gamma_t||_0^2), \quad |F(\gamma_t, u) - F(\eta_s, u)| \leq Ld_\infty(\gamma_t, \eta_s); \quad (2.3)$$

$$|q(\gamma_t, u) - q(\eta_s, u)| \leq Ld_\infty(\gamma_t, \eta_s), \quad |q(\gamma_t, u)| \leq L(1 + ||\gamma_t||_0); \quad (2.4)$$

$$|\phi(\zeta_T) - \phi(\zeta_T')| \leq L||\zeta - \zeta_T'||_0, \quad |\phi(\zeta_T)| \leq L(1 + ||\zeta||_0). \quad (2.5)$$

We say that $X$ is a mild solution of equation (1.1) if $X \in C^0(\Lambda)$ and it satisfies:

$$X(s) = e^{(s-t)A} \gamma_t(t) + \int_t^s e^{(s-\sigma)A} F(X_\sigma, u(\sigma))d\sigma, \quad s \in [t,T]; \quad \text{and } X(s) = \gamma_t(s), \quad s \in [0,t).$$

The following lemma is standard.
Lemma 2.1. Assume that Hypothesis 2.3 (iii) holds. Then for every $u \in U[t,T]$, $\gamma_t \in \Lambda$, (2.1) admits a unique mild solution $X^\gamma_t,u$. Moreover, if we let $X^\eta_t,u$ be the solutions of (1.1), corresponding $\eta_t \in \Lambda$ and $u \in U[t,T]$. Then the following estimates hold:

$$
||X^\gamma_t,u - X^\eta_t,u||_0 \leq C_1||\gamma_t - \eta_t||_0, \quad ||X^\gamma_t,u||_0 \leq C_1(1 + ||\gamma_t||_0).
$$

The constant $C_1$ depending only on $T$, $L$ and $M_1 = \sup_{s \in [0,T]}|e^{sA}|$.

Proof. By Picard iteration, we can obtain the existence and the uniqueness of the mild solution. By Gronwall’s inequality, together with assumption 2.3, we can prove (2.6). □

The next result contains the local boundedness and the continuity of the trajectory $X^\gamma_t,u$ and value functional $V$. In what follows, $C$ is an absolute constant, that can be different in different places.

Lemma 2.2. Assume that Hypothesis 2.3 (iii) holds. Then, for any $0 \leq t \leq \tilde{t} \leq T$, $\gamma_t, \eta_t \in \Lambda$ and $u \in U[t,T]$,

$$
\sup_{s \in [t,T]} |X^\gamma_t,u(s) - e^{(s-t)A}\gamma_t(t)| \leq C(1 + ||\gamma_t||_0)|s - t|, \quad s \in [t,T];
$$

(2.7)

$$
||X^\gamma_t,u - X^\tilde{\gamma}_{t,A},u||_0 \leq C(1 + ||\eta_t||_0)(\tilde{t} - t) + C||\eta_t - \gamma_t||_0;
$$

(2.8)

$$
|V(\gamma_t)| \leq C(1 + ||\gamma_t||_0);
$$

(2.9)

$$
|V(\eta_{t,A}) - V(\eta_t)| \leq C(1 + ||\eta_t||_0)(\tilde{t} - t) + C||\eta_t - \gamma_t||_0.
$$

(2.10)

Proof. For any $\gamma_t \in \Lambda$, by (2.3) and (2.6), we obtain the following result:

$$
|X^\gamma_t,u(s) - e^{(s-t)A}\gamma_t(t)| \leq LM_1(1 + C_1(1 + ||\gamma_t||_0))|s - t|.
$$

Taking the supremum in $U[t,T]$, we obtain (2.7). For any $0 \leq t \leq \tilde{t} \leq T$, $\gamma_t, \eta_t \in \Lambda$ and $u \in U[t,T]$, by (2.3) and (2.6), we have

$$
\sup_{t \leq s \leq \sigma} |X^\eta_{t,A}(s) - X^\gamma_{t,A}(s)|
$$

\leq M_1|\eta_t(t) - \gamma_t(t)| + \int_t^\tilde{t} |e^{(s-r)A}F(X^\eta_r,u, u(r))|dr

+ \sup_{t \leq s \leq \sigma} \int_t^\tilde{t} |e^{(s-r)A}(F(X^\eta_r,u, u(r))) - F(X^\gamma_{t,A},u, u(r))|dr

\leq M_1|\eta_t(t) - \gamma_t(t)| + LM_1(1 + C_1(1 + ||\eta_t||_0))(\tilde{t} - t) + LM_1 \int_t^\sigma ||X^\eta_{\gamma_t,u} - X^\gamma_{t,A,u}||_0 dr.
$$

Thus,

$$
||X^\gamma_{\sigma,u} - X^\gamma_{t,A,u}||_0 \leq C||\eta_t - \gamma_t|| + C(1 + ||\eta_t||_0)(\tilde{t} - t) + C \int_t^\sigma ||X^\eta_{\gamma_t,u} - X^\gamma_{t,A,u}||_0 dr.
$$

Then, by Gronwall’s inequality, we obtain (2.8). Next, by (2.4), (2.5) and (2.8), we get

$$
|J(\gamma_{t,A}, u) - J(\eta_t, u)|
$$

(2.11)
Thus, taking the infimum in \( u(\cdot) \in \mathcal{U}[t, T] \), we obtain (2.10). By the similar procedure, we can show (2.9) holds true. The lemma is proved. \( \square \)

Next, we present the dynamic programming principle (DPP) for optimal control problems (1.1) and (1.3).

**Theorem 2.1.** Assume the Hypothesis 2.3 (ii) and (iii) hold true. Then, for every \((t, \gamma_t) \in [0, T) \times \Lambda\) and \( s \in [t, T] \), we have that

\[
V(\gamma_t) = \inf_{u \in \mathcal{U}[t, T]} \left[ \int_t^s q(X_{\sigma}^{\gamma_t,u}, u(\sigma))d\sigma + V(X_s^{\gamma_t,u}) \right].
\]

(2.11)

The proof is very similar to the case without path-dependent (see Theorem 1.1 of Chapter 6 in page 224 of [17]). For the convenience of readers, here we give its proof.

**Proof.** First of all, for any \( u \in \mathcal{U}[s, T], \ s \in [t, T] \) and any \( u \in \mathcal{U}[t, s] \), by putting them concatenatively, we get \( u \in \mathcal{U}[t, T] \). Let us denote the right-hand side of (2.11) by \( \overline{V}(\gamma_t) \). By (1.3),

\[
V(\gamma_t) \leq \int_t^s q(X_{\sigma}^{\gamma_t,u}, u(\sigma))d\sigma + J(X_s^{\gamma_t,u}, u(\cdot) \in \mathcal{U}[t, T]).
\]

Thus, taking the infimum over \( u(\cdot) \in \mathcal{U}[s, T] \), we obtain

\[
V(\gamma_t) \leq \int_t^s q(X_{\sigma}^{\gamma_t,u}, u(\sigma))d\sigma + V(X_s^{\gamma_t,u}).
\]

Consequently,

\[
V(\gamma_t) \leq \overline{V}(\gamma_t).
\]

On the other hand, for any \( \varepsilon > 0 \), there exists a \( u^\varepsilon \in \mathcal{U}[t, T] \) such that

\[
V(\gamma_t) + \varepsilon \geq \int_t^s q(X_{\sigma}^{\gamma_t,u^\varepsilon}, u^\varepsilon(\sigma))d\sigma + J(X_s^{\gamma_t,u^\varepsilon}, u^\varepsilon) \geq \int_t^s q(X_{\sigma}^{\gamma_t,u^\varepsilon}, u^\varepsilon(\sigma))d\sigma + V(X_s^{\gamma_t,u^\varepsilon}) \geq \overline{V}(\gamma_t).
\]

Hence, (2.11) follows. \( \square \)

The following lemma is needed to prove the existence of viscosity solutions.

**Theorem 2.2.** Suppose \( X \) is a solution of (1.7), \( \varphi \in C^1(\Lambda^t) \) and \( A^*\partial_x \varphi \in C^0(\Lambda^t) \) for some \( t \in [t, T) \). Then for any \( s \in [t, T] \):

\[
\varphi(X_s) = \varphi(X_t) + \int_t^s \partial_x \varphi(X_\sigma)d\sigma + \int_t^s (A^*\partial_x \varphi(X_\sigma), X(\sigma))_H + (\partial_x \varphi(X_\sigma), F(X_\sigma, u(\sigma)))_Hd\sigma. \tag{2.12}
\]

**Proof.** Denote \( X^n = X^1_{[0,t]} + \sum_{i=0}^{n-1} X(t_{i+1})1_{[t_i,t_{i+1}]} + X(s)1_{\{s\}} \) which is a càdlàg piecewise constant approximation of \( X \). Here \( t_i = t + \frac{i(s-t)}{2^n} \). For every \( \gamma \in \tilde{\Lambda} \), define \( \gamma_{\sigma^-} \in \tilde{\Lambda} \) by

\[
\gamma_{\sigma^-}(\theta) = \gamma_{\sigma}(\theta), \ \theta \in [0, \sigma), \ \text{and} \ \gamma_{\sigma^-}(\sigma) = \lim_{\theta \uparrow \sigma} \gamma_{\sigma}(\theta).
\]
We start with the decomposition
\[
\varphi(X^0_{i+1,-}) - \varphi(X^0_{i,-}) = \varphi(X^0_{i+1,-}) - \varphi(X^0_{i,-}) + \varphi(X^0_{i,-}) - \varphi(X^0_{i,-}).
\]  
(2.13)
Let \( \psi(l) = \varphi(X^0_{i,-}) \), we have \( \varphi(X^0_{i+1,-}) - \varphi(X^0_{i,-}) = \psi(h) - \psi(0) \), where \( h = \frac{\delta_i t}{\delta^+} \). Since \( \varphi \in C^1(\Lambda^t) \), the right derivative of \( \psi \) is continuous, therefore,
\[
\varphi(X^0_{i+1,-}) - \varphi(X^0_{i,-}) = \int_{t_i}^{t_{i+1}} \partial_t \varphi(X^0_{i,t}) dt.
\]
The term \( \varphi(X^0_{i,-}) - \varphi(X^0_{i,-}) \) in (2.13) can be written \( \pi(X(t_{i+1})) - \pi(X(t_i)) \), where \( \pi(l) = \varphi(X^0_{i,-} + (l - X(t_i))1_{(t_i)}) \). Since \( \varphi \in C^1(\Lambda^t) \), \( \pi \) is a \( C^1 \) function and \( \nabla_x \pi(l) = \partial_x \varphi(X^0_{i,-} + (l - X(t_i))1_{(t_i)}) \).
Thus, by Proposition 5.5 in Chapter 2 of [17], we have that:
\[
\pi(X(t_{i+1})) - \pi(X(t_i)) = \int_{t_i}^{t_{i+1}} (A^* \partial_x \varphi(X^0_{i,-} + (X(\sigma) - X(t_i))1_{(t_i)}), X(\sigma))_H
+ (\partial_x \varphi(X^0_{i,-} + (X(\sigma) - X(t_i))1_{(t_i)}), F(X_\sigma, u(\sigma)))_H d\sigma.
\]
Summing over \( i \geq 0 \) and denoting \( i(\sigma) \) the index such that \( \sigma \in [t_{i(\sigma)}, t_{i(\sigma)+1}) \), we obtain
\[
\varphi(X^n_k) - \varphi(X^n_k) = \sum_{i=0}^{n} \delta_i \varphi(X^n_{t_{i(\sigma)}}, \sigma) d\sigma + \sum_{i=0}^{n} (A^* \partial_x \varphi(X^n_{t_{i(\sigma)}}, \sigma) + (X(\sigma) - X(t_{i(\sigma)}))1_{(t_{i(\sigma)})}, X(\sigma))_H
+ (\partial_x \varphi(X^n_{t_{i(\sigma)}}, \sigma) + (X(\sigma) - X(t_{i(\sigma)}))1_{(t_{i(\sigma)})}), F(X_\sigma, u(\sigma)))_H d\sigma.
\]
\( \varphi(X^n_k) \) and \( \varphi(X^n_k) \) converge to \( \varphi(X_k) \) and \( \varphi(X_k) \), respectively. Since all approximations of \( X \) appearing in the integrals have a \( |||\cdot|||_0 \)-distance from \( X_k \) less than \( \|X^n_k - X_k\|_0 \to 0 \), \( \varphi \in C^1(\Lambda^t) \) and \( A^* \partial_x \varphi \in C^0(\Lambda^t) \) imply that the integrands appearing in the above integrals converge respectively to \( \delta_i \varphi(X_k), A^* \partial_x \varphi(X_k) \) and \( \partial_x \varphi(X_k) \) as \( n \to \infty \). By \( X \) is continuous, and \( \varphi \in C^1(\Lambda^t) \) and \( A^* \partial_x \varphi \in C^0(\Lambda^t) \), the integrands in the various above integrals are bounded. The dominated convergence then ensure that the Lebesgue integrals converge to the terms appearing in (2.12) as \( n \to \infty \). □

We conclude this section with the following four lemmas which will be used to prove the uniqueness and stability of viscosity solutions.

**Definition 2.4.** Let \( t \in [0, T] \) be fixed. We say that a continuous functional \( \rho : \Lambda^t \times \Lambda^t \to [0, +\infty) \) is a gauge-type function provided that:
(i) \( \rho(\gamma_s, \gamma_s) = 0 \) for all \( (s, \gamma_s) \in [t, T] \times \Lambda^t \),
(ii) for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( \gamma_s, \eta_l \in \Lambda^t \), we have \( \rho(\gamma_s, \eta_l) \leq \delta \) implies that \( d_\infty(\gamma_s, \eta_l) < \varepsilon \).

The following lemma is a modification of Borwein-Preiss variational principle (see Theorem 2.5.2 in Borwein & Zhu [2]). It will be used to get a maximum of a perturbation of the auxiliary function in the proof of uniqueness. The proof is completely similar to the finite dimensional case (see Lemma 2.12 in [22]). Here we omit it.

**Lemma 2.3.** Let \( t \in [0, T] \) be fixed and let \( f : \Lambda^t \to R \) be an upper semicontinuous functional bounded from above. Suppose that \( \rho \) is a gauge-type function and \( \{\delta_{li}\}_{i \geq 0} \) is a sequence of positive number, and suppose that \( \varepsilon > 0 \) and \( (t_0, \gamma_{t_0}) \in [t, T] \times \Lambda^t \) satisfy
\[
f(\gamma_{t_0}) \geq \sup_{(s, \gamma_s) \in [t, T] \times \Lambda^t} f(\gamma_s) - \varepsilon.
\]
Then there exist \( (i, \gamma_i) \in [t, T] \times \Lambda^t \) and a sequence \( \{(t_i, \gamma_{t_i})\}_{i \geq 1} \subset [t, T] \times \Lambda^t \) such that
(i) \(\rho(\gamma_i^0, \hat{\gamma}_i) \leq \frac{\varepsilon}{\delta_0}\), \(\rho(\gamma_i^1, \hat{\gamma}_i) \leq \frac{\varepsilon}{\delta_0}\) and \(t_i \uparrow \hat{t}\) as \(i \to \infty\),

(ii) \(f(\hat{\gamma}_i) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma_i^1, \hat{\gamma}_i) \geq f(\gamma_i^0)\), and

(iii) \(f(\gamma_s) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma_i, \gamma_s) < f(\hat{\gamma}_i) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma_i, \hat{\gamma}_i)\) for all \((s, \gamma_s) \in \hat{t}, T \times \Lambda \setminus \{(\hat{t}, \hat{\gamma}_i)\}\).

We define, for every \(\gamma_t \in \hat{A}\),

\[
S(\gamma_t) = \begin{cases} 
\frac{(||\gamma_t||^2 - |\gamma_t(t)|^2)^2}{||\gamma_t||^2}, & ||\gamma_t|| \neq 0; \\
0, & ||\gamma_t|| = 0.
\end{cases}
\]

Lemma 2.4. \(S(\cdot) \in C^1(\hat{A})\). Moreover,

\[
||\gamma_t||^2 \leq S(\gamma_t) + 2|\gamma_t(t)|^2 \leq 3||\gamma_t||^2. \quad (2.14)
\]

Proof. First, we prove \(S \in C^0(\hat{A})\). For any \(\gamma_t, \eta_s \in \hat{A}\), if \(s \geq t\),

\[
|\gamma_t(t) - \eta_s(s)| \leq |\gamma_t(t) - e^{(s-t)A}\gamma_t(t)| + |e^{(s-t)A}\gamma_t(t) - \eta_s(s)|,
\]

and

\[
|||\gamma_t|| - ||\eta_s||| \leq ||\gamma_t,A,s|| - ||\gamma_t|| + ||\gamma_t,A,s - \eta_s||
\]

if \(s < t\),

\[
|\gamma_t(t) - \eta_s(s)| \leq |\gamma_t(t) - e^{(s-t)A}\eta_s(s)| + |e^{(s-t)A}\eta_s(s) - \gamma_t(s)| + |e^{(s-t)A}(\gamma_t(s) - \gamma_t(t))|
\]

\[
+ |(e^{(s-t)A} - I)\gamma_t(t)| + |\gamma_t(t) - \gamma_t(s)| + |\gamma_t(s) - \eta_s(s)|,
\]

and

\[
|||\gamma_t|| - ||\eta_s||| \leq ||\gamma_t,A,s|| - ||\eta_s|| + ||\gamma_t,A,s - \gamma_t||
\]

\[
\leq \sup_{s \leq t \leq s} ||(e^{(s-t)A} - I)\gamma_t(t) + |(e^{(s-t)A} - I)||\gamma_t(t) - \eta_s(s) + ||\gamma_t,A,s - \gamma_t|||.
\]

Then we have \(S(\eta_s) \to S(\gamma_t)\) as \(\eta_s \to \gamma_t\) under \(d_\infty\). Thus \(S \in C^0(\hat{A})\). Second, by the definition of \(S(\cdot)\), it is clear that \(\partial_s S(\gamma_t) = 0\) for all \(\gamma_t \in \hat{A}\). Next, we consider \(\partial_x S\). For every \(\gamma_t \in \hat{A}\), let \(||\gamma_t||_0 = \sup_{0 \leq s < t} ||\gamma_t(s)||\). Then, if \(||\gamma_t(t)|| < ||\gamma_t||_0\),

\[
\lim_{|h| \to 0} \frac{|S(\gamma_t^h) - S(\gamma_t) + \frac{4(||\gamma_t||^2 - |\gamma_t(t)|^2)(\gamma_t(t), h)_H}{||\gamma_t||_0^2}|}{|h|}
\]

\[
= \lim_{|h| \to 0} \frac{|(|\gamma_t||_0^2 - |\gamma_t(t)|^2)^2 - (||\gamma_t||_0^2 - |\gamma_t(t)|^2)^2 + 4(||\gamma_t||^2 - |\gamma_t(t)|^2)(\gamma_t(t), h)_H|}{|h| \times ||\gamma_t||_0^2}
\]

\[
= \lim_{|h| \to 0} \frac{|(2|\gamma_t||_0^2 - |\gamma_t(t)|^2 - |\gamma_t(t)|^2(2(\gamma_t(t), h)_H + |h|^2) + 4(||\gamma_t||^2 - |\gamma_t(t)|^2)(\gamma_t(t), h)_H|}{|h| \times ||\gamma_t||_0^2}
\]

\[
= 0.
\]

Thus,

\[
\partial_x S(\gamma_t) = -4\frac{(||\gamma_t||^2 - |\gamma_t(t)|^2)\gamma_t(t)}{||\gamma_t||_0^2}. \quad (2.15)
\]
If $|\gamma_t| > ||\gamma_t||_0^-$,
\[ \partial_x S(\gamma_t) = 0; \] (2.16)
if $|\gamma_t| = ||\gamma_t||_0^- \neq 0$, since
\[
||\gamma_t^h||_0^2 - |\gamma_t(t) + h|^2 = \begin{cases} 0, & |\gamma_t(t) + h| \geq |\gamma_t(t)|, \\ |\gamma_t(t)|^2 - |\gamma_t(t) + h|^2, & |\gamma_t(t) + h| < |\gamma_t(t)|, \end{cases}
\]
we have
\[
0 \leq \lim_{|h| \to 0} \frac{|S(\gamma_t^h) - S(\gamma_t)|}{|h|} \leq \lim_{|h| \to 0} \frac{|h|^2(|h| + 2|\gamma_t(t)|)^2}{|h||\gamma_t^h||_0^2} = 0; \] (2.17)
if $|\gamma_t(t) - a_t(t)| = ||\gamma_t - a_{t,t}||_0^- = 0,$
\[ \partial_x S(\gamma_t) = 0. \] (2.18)
From (2.15), (2.16), (2.17) and (2.18) we obtain that
\[
\partial_x S(\gamma_t) = \begin{cases} 4(||\gamma_t||_0^2 - |\gamma_t(t)|^2)\gamma_t(t), & ||\gamma_t||_0^2 \neq 0, \\ 0, & ||\gamma_t||_0^2 = 0. \end{cases}
\]
It is clear that $\partial_x S \in C^0(\hat{\Lambda})$. Thus, we have show that $S(\cdot) \in C^1(\hat{\Lambda})$.

Now we prove (2.14). It is clear that
\[
S(\gamma_t) + 2|\gamma_t(t)|^2 \leq 3||\gamma_t||_0^2.
\]
On the other hand, if $||\gamma_t||_0 \neq 0$,
\[
S(\gamma_t) + 2|\gamma_t(t)|^2 = \frac{(||\gamma_t||_0^2 - |\gamma_t(t)|^2)^2 - 2||\gamma_t||_0^2|\gamma_t(t)|^2}{||\gamma_t||_0^2} = \frac{|\gamma_t(t)|^4 + ||\gamma_t||_0^4}{||\gamma_t||_0^2} \geq ||\gamma_t||_0^2.
\]
Thus, we have (2.14) holds true. The proof is now complete. \qed

Define, for every $M \in R$,
\[
\Upsilon^M(\gamma_t) := S(\gamma_t) + M|\gamma_t(t)|^2, \quad \gamma_t \in \Lambda;
\]
\[
\Upsilon^M(\gamma_t, \eta_s) := \Upsilon^M(\eta_s, \gamma_t) := \Upsilon^M(\eta_s - \gamma_t, \gamma_t), \quad 0 \leq t \leq s \leq T, \ \gamma_t, \eta_s \in \Lambda;
\]
and
\[
\Upsilon^M(\gamma_t, \eta_s) := \Upsilon^M(\eta_s, \gamma_t) := \Upsilon^M(\eta_s, \gamma_t) + |s - t|^2, \quad 0 \leq t \leq s \leq T, \ \gamma_t, \eta_s \in \Lambda.
\]
The proof of the following Lemma is completely similar to the finite dimensional case (see Lemma 2.13 in [22]). Here we omit it.

**Lemma 2.5.** For $M \geq 2$, we have that
\[ 2\Upsilon^M(\gamma_t) + 2\Upsilon^M(\eta_t) \geq \Upsilon^M(\gamma_t + \eta_t), \quad (t, \gamma_t, \eta_t) \in [0, T] \times \hat{\Lambda} \times \hat{\Lambda}. \] (2.19)

**Lemma 2.6.** Assume the Hypothesis 2.3 holds true. For every $t \in [0, T)$, $\eta_t \in \Lambda$ and $M \geq 2$, we have that
\[
\Upsilon^M(X^{\gamma_t, u} - \eta_t, s, A) \leq \Upsilon^M(X^{\gamma_t, u} - \eta_t) + \int_t^s \left( \partial_x \Upsilon^M(X^{\gamma_t, u} - \eta_t, s, A), F(X^{\gamma_t, u} - \eta_t, s, A) \right)_H ds. \] (2.20)
Remark 2.1. Let $A_\mu = \mu A(\mu I - A)^{-1}$ be the Yosida approximation of $A$ and let $X^\mu$ be the solution of the following:

$$X^\mu(s) = e^{(s-t)A_\mu} \gamma_\mu(t) + \int_t^s e^{(s-\sigma)A_\mu} F(X^\mu_\sigma, u(\sigma)) d\sigma, \ s \in [t, T]; \ X^\mu(s) = \gamma_\mu(s), \ s \in [0, t).$$

Define $y^\mu$ by $y^\mu(s) = X^\mu(s) - e^{(s-t)A_\mu} \eta_\mu(t), \ t \leq s \leq T$ and $y^\mu(s) = \gamma_\mu(s) - \eta_\mu(s), \ 0 \leq s < t$, then $y^\mu$ be the solution of the following:

$$y^\mu(s) = e^{(s-t)A_\mu} \eta^\mu_\mu(t) + \int_t^s e^{(s-\sigma)A_\mu} F(X^\mu_\sigma, u(\sigma)) d\sigma, \ s \in [t, T]; \ y^\mu(s) = \gamma_\mu(s) - \eta_\mu(s), \ s \in [0, t).$$

By Theorem 2.2 and Lemma 2.3, we have,

$$\Upsilon^M(y^\mu_\mu) = \Upsilon^M(y^\mu_t) + \int_t^s (\partial_x \Upsilon^M(y^\mu_\sigma), A_\mu y^\mu(\sigma) + F(X^\mu_\sigma, u(\sigma))_H d\sigma.$$

Noting that $A$ is the infinitesimal generator of a $C_0$ contraction semigroup, we have, if $||y^\mu||_0^2 \neq 0,$

$$\left(2M y^\mu(\sigma) - \frac{4(||y^\mu||_0^2 - ||y^\mu(\sigma)||_2^2)}{||y^\mu||_0^2}, A_\mu y^\mu(\sigma) \right) \leq 0, \text{ for } M \geq 2.$$

Thus,

$$\Upsilon^M(y^\mu_\mu) \leq \Upsilon^M(y^\mu_t) + \int_t^s (\partial_x \Upsilon^M(y^\mu_\sigma), F(X^\mu_\sigma, u(\sigma))_H d\sigma.$$

Letting $\mu \to \infty$, by Proposition 5.4 in Chapter 2 of [17], we obtain

$$\Upsilon^M(y_\mu) \leq \Upsilon^M(y_t) + \int_t^s (\partial_x \Upsilon^M(y_\sigma), F(X_\sigma, u(\sigma))_H d\sigma,$$

where $y(s) = X^{\gamma_{\mu, u}}(s) - e^{A(s-t)\eta_\mu}(t), \ t \leq s \leq T$ and $y(s) = \gamma_\mu(s) - \eta_\mu(s), \ 0 \leq s < t$. That is (2.21).

The proof is now complete. □

Remark 2.1. (i) Since $||\cdot||_0^2$ is not belongs to $C^1(\Lambda)$, then, for every $a_{\tilde{t}} \in \Lambda$, $||\gamma_{\tilde{t}} - a_{\tilde{t}, A}||_0^2$ cannot appear as an auxiliary functional in the proof of the uniqueness and stability of viscosity solutions. However, by the above lemma, we can replace $||\gamma_{\tilde{t}} - a_{\tilde{t}, A}||_0^2$ with its equivalent functional $\Upsilon^2(\gamma_{\tilde{t}} - a_{\tilde{t}, A})$. Therefore, we can get the uniqueness result of viscosity solutions without the assumption (A.4) on page 231 of [17].

(ii) It follows from (2.1) that $\Upsilon^2(\cdot, \cdot)$ is a gauge-type function. We can apply it to Lemma 2.3 to get a maximum of a perturbation of the auxiliary functional in the proof of uniqueness.

3 Viscosity solutions to PHJB equations: Existence theorem.

In this section, we consider the first order path-dependent Hamilton-Jacobi-Bellman (PHJB) equation (1.4). As usual, we start with classical solutions.

Definition 3.1. (Classical solution) A functional $v \in C^1(\Lambda)$ with $A^* \partial_x v \in C^0(\Lambda)$ is called a classical solution to the PHJB equation (1.4) if it satisfies the PHJB equation (1.4) point-wisely.
We will prove that the value functional $V$ defined by (1.3) is a viscosity solution of PHJB equation (1.4). Define

$$\Phi = \{ \varphi \in C^1(\Lambda) | A^* \partial_x \varphi \in C^0(\Lambda) \};$$

$$G_t = \left\{ g \in C^0(\Lambda^t) | \exists h \in C^1([0, T] \times R), \delta_i > 0, \gamma^i_{t_i} \in \Lambda, t_i \leq t, i = 0, 1, 2, \ldots, N > 0, \text{ with } \nabla_x h \geq 0, \right\}$$

For notational simplicity, if $g \in G_t$, we use $\partial_t g(\gamma_s)$ and $\partial_x g(\gamma_s)$ to denote $h_t(s, \Upsilon^2(\gamma_s)) + 2 \sum_{i=0}^{\infty} \delta_i (s-t_i)$ and $\nabla_x h(s, \Upsilon^2(\gamma_s))\partial_x \Upsilon^2(\gamma_s) + \sum_{i=0}^{\infty} \delta_i \partial_x \Upsilon^2(\gamma_s - \gamma^i_{t_i, s_\Lambda}),$ respectively.

Now we can give the following definition for viscosity solutions.

**Definition 3.2.** $w \in C^0(\Lambda)$ is called a viscosity subsolution (resp., supersolution) to (1.4) if the terminal condition, $w(\gamma_T) \leq \phi(\gamma_T)$ (resp., $w(\gamma_T) \geq \phi(\gamma_T)$), $\gamma_T \in \Lambda_T$ is satisfied, and for any $\varphi \in \Phi$ and $g \in G_t$ with $t \in [0, T)$, whenever the function $w - \varphi - g$ (resp., $w + \varphi + g$) satisfies

$$0 = (w - \varphi - g)(\gamma_t) = \sup_{\eta_s \in \Lambda^t} (w - \varphi - g)(\eta_s),$$

(resp., $0 = (w + \varphi + g)(\gamma_t) = \inf_{\eta_s \in \Lambda^t} (w + \varphi + g)(\eta_s),$)

where $\gamma_t \in \Lambda$, we have

$$\partial_t \varphi(\gamma_t) + \partial_t g(\gamma_t) + (A^* \partial_x \varphi(\gamma_t), \gamma_t(t))_H + H(\gamma_t, \partial_x \varphi(\gamma_t) + \partial_x g(\gamma_t)) \geq 0,$$

(resp., $-\partial_t \varphi(\gamma_t) - \partial_t g(\gamma_t) - (A^* \partial_x \varphi(\gamma_t), \gamma_t(t))_H + H(\gamma_t, -\partial_x \varphi(\gamma_t) - \partial_x g(\gamma_t)) \leq 0.$)

$w \in C^0(\Lambda)$ is said to be a viscosity solution to (1.4) if it is both a viscosity subsolution and a viscosity supersolution.

**Theorem 3.1.** Suppose that Hypothesis (2.3) holds. Then the value functional $V$ defined by (1.3) is a viscosity solution to (1.4).

**Proof.** First, Let $\varphi \in \Phi$ and $g \in G_t$ with $t \in [0, T)$ such that

$$0 = (V - \varphi - g)(\gamma_t) = \sup_{\eta_s \in \Lambda^t} (V - \varphi - g)(\eta_s),$$

where $\gamma_t \in \Lambda$. Thus, for fixed $u \in U$, by the DPP (Theorem 2.1), we obtain that, for all $t \leq t + \delta \leq T$,

$$\begin{align*}
(\varphi + g)(\gamma_t) &= V(\gamma_t) \leq \int_t^{t+\delta} q(X^\gamma_{t+\delta}, u) d\sigma + V(X^\gamma_{t+\delta}) \\
&\leq \int_t^{t+\delta} q(X^\gamma_{t+\delta}, u) d\sigma + (\varphi + g)(X^\gamma_{t+\delta}).
\end{align*}$$

Applying Theorem 2.2 and Lemma 2.6 we show that

$$0 \leq \lim_{\delta \to 0} \left[ \frac{1}{\delta} \int_t^{t+\delta} q(X^\gamma_{t+\delta}, u) d\sigma + \frac{1}{\delta} [\varphi(X^\gamma_{t+\delta}) - \varphi(\gamma_t)] + \frac{1}{\delta} [g(X^\gamma_{t+\delta}) - g(\gamma_t)] \right]$$

$$\leq q(\gamma_t, u) + \partial_t \varphi(\gamma_t) + \partial_t g(\gamma_t) + (A^* \partial_x \varphi(\gamma_t), \gamma_t(t))_H + (\partial_x \varphi(\gamma_t) + \partial_x g(\gamma_t), F(\gamma_t, u))_H.$$
By Theorem 2.2, the inequality above implies that

$$0 = (V + \varphi + g)(\gamma_t) = \inf_{\eta \in \Lambda} (V + \varphi + g)(\eta_t),$$

where $\gamma_t \in \Lambda$. Then, for any $\varepsilon > 0$, by the DPP (Theorem 2.1), one can find a control $u^\varepsilon(\cdot) \equiv u^\varepsilon,\delta(\cdot) \in U[t, T]$ such that, for any $t \leq t + \delta \leq T$,

$$\varepsilon \delta \geq \int_t^{t+\delta} q(X_{\sigma}^\gamma, u^\varepsilon) d\sigma + V(X_{t+\delta}^\gamma) - V(\gamma_t)$$

$$\geq \int_t^{t+\delta} q(X_{\sigma}^\gamma, u^\varepsilon) d\sigma - (\varphi + g)(X_{t+\delta}^\gamma) + (\varphi + g)(\gamma_t).$$

Then, by Theorem 2.2 and Lemma 2.6, we obtain that

$$\varepsilon \geq \frac{1}{\delta} \int_t^{t+\delta} q(X_{\sigma}^\gamma, u^\varepsilon) d\sigma - \frac{\varphi(X_{t+\delta}^\gamma) - \varphi(\gamma_t)}{\delta} + \frac{g(X_{t+\delta}^\gamma) - g(\gamma_t)}{\delta}$$

$$\geq -\partial_t \varphi(\gamma_t) - \partial_t g(\gamma_t) - (A^* \partial_x(\gamma_t), \gamma_t(t))_H + \frac{1}{\delta} \int_t^{t+\delta} q(\gamma_t, u^\varepsilon) d\sigma$$

$$- (\partial_x \varphi(\gamma_t) + \partial_x g(\gamma_t), F(\gamma_t, u^\varepsilon))_H d\sigma + o(1)$$

$$\geq -\partial_t \varphi(\gamma_t) - \partial_t g(\gamma_t) - (A^* \partial_x(\gamma_t), \gamma_t(t))_H + \inf_{u \in U} [q(\gamma_t, u) - (\partial_x \varphi(\gamma_t) + \partial_x g(\gamma_t), F(\gamma_t, u))_H] + o(1).$$

Letting $\delta \downarrow 0$ and $\varepsilon \to 0$, we show that

$$0 \geq -\partial_t \varphi(\gamma_t) - \partial_t g(\gamma_t) - (A^* \partial_x(\gamma_t), \gamma_t(t))_H + H(\gamma_t, -\varphi(\gamma_t) - g(\gamma_t)).$$

Therefore, $V$ is also a viscosity subsolution of equation (1.4). This completes the proof. □

Now, let us give the result of classical solutions, which show the consistency of viscosity solutions.

**Theorem 3.2.** Let $V$ denote the value functional defined by (1.3). If $V \in C^1(\Lambda)$ and $A^* \partial_x V \in C^0(\Lambda)$, then $V$ is a classical solution of equation (1.4).

**Proof.** First, using the definition of $V$ yields $V(\gamma_T) = \phi(\gamma_T)$ for all $\gamma_T \in \Lambda_T$. Next, for fixed $(t, \gamma_t, u) \in [0, T) \times \Lambda \times U$, from the DPP (Theorem 2.1), we obtain the following result:

$$0 \leq \int_t^{t+\delta} q(X_{t+\delta}^\gamma, u) d\sigma + V(X_{t+\delta}^\gamma) - V(\gamma_t), \quad 0 < \delta < T - t.$$  \hspace{1cm} (3.2)

By Theorem 2.2, the inequality above implies that

$$0 \leq \lim_{\delta \to 0^+} \frac{1}{\delta} \left[ \int_t^{t+\delta} q(X_{t+\delta}^\gamma, u) d\sigma + V(X_{t+\delta}^\gamma) - V(\gamma_t) \right]$$

$$= \partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H + (F(\gamma_t, u), \partial_x V(\gamma_t))_H + q(\gamma_t, u).$$

Taking the minimum in $u \in U$, we have that

$$0 \leq \partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H + H(\gamma_t, \partial_x V(\gamma_t)).$$  \hspace{1cm} (3.3)

On the other hand, let $(t, \gamma_t) \in [0, T) \times \Lambda$ be fixed. Then, by (2.11) and Theorem 2.2, there exists an $\bar{u} \equiv u^\varepsilon,\delta \in U[t, T]$ for any $\varepsilon > 0$ and $0 < \delta < T - t$ such that

$$\varepsilon \delta \geq \int_t^{t+\delta} q(X_{t+\delta}^\gamma, \bar{u}(s)) d\sigma + V(X_{t+\delta}^\gamma) - V(\gamma_t).$$
\[
\begin{align*}
&= \partial_t V(\gamma_t) \delta + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H \delta + \int_0^{t+\delta} q(\gamma_t, \tilde{u}(\sigma)) + (\partial_x V(\gamma_t), F(\gamma_t, \tilde{u}(\sigma))_H d\sigma + o(\delta) \\
&\geq \partial_t V(\gamma_t) \delta + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H \delta + H(\gamma_t, \partial_x V(\gamma_t)) \delta + o(\delta).
\end{align*}
\]

Then, dividing through by \(\delta\) and letting \(\delta \to 0^+\), we obtain that
\[
\varepsilon \geq \partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H + H(\gamma_t, \partial_x V(\gamma_t)).
\]

The desired result is obtained by combining the inequality given above with (3.3). \(\square\)

We conclude this section with the stability of viscosity solutions.

**Theorem 3.3.** Let \(F, q, \phi\) satisfy Hypothesis 2.3 and \(v \in C^0(\Lambda)\). Assume

(i) for any \(\varepsilon > 0\), there exist \(F^\varepsilon, q^\varepsilon, \phi^\varepsilon\) and \(v^\varepsilon \in C^0(\Lambda)\) such that \(F^\varepsilon, q^\varepsilon, \phi^\varepsilon\) satisfy Hypothesis 2.3 and \(v^\varepsilon\) is a viscosity subsolution (resp., supersolution) of equation (1.4) with generators \(F^\varepsilon, q^\varepsilon, \phi^\varepsilon\);

(ii) as \(\varepsilon \to 0\), \((F^\varepsilon, q^\varepsilon, \phi^\varepsilon, v^\varepsilon)\) converge to \((F, q, \phi, v)\) uniformly in the following sense:

\[
\lim_{\varepsilon \to 0} \sup_{(t, \gamma_t, u) \in [0, T] \times \Lambda} \sup_{v \in \mathcal{V}(\gamma_t)} \left[ |(F^\varepsilon - F) + |q^\varepsilon - q|)(\gamma_t, u) + |\phi^\varepsilon - \phi|)(\eta T) + |v^\varepsilon - v)(\gamma_t)| \right] = 0. \tag{3.4}
\]

Then \(v\) is a viscosity subsolution (resp., supersolution) of equation (1.4) with generators \(F, q, \phi\).

**Proof.** Without loss of generality, we shall only prove the viscosity subsolution property. First, from \(v^\varepsilon\) is a viscosity subsolution of equation (1.4) with generators \(F^\varepsilon, q^\varepsilon, \phi^\varepsilon\), it follows that

\[
v^\varepsilon(\gamma_T) \leq \phi^\varepsilon(\gamma_T), \quad \gamma_T \in \Lambda_T.
\]

Letting \(\varepsilon \to 0\), we have

\[
v(\gamma_T) \leq \phi(\gamma_T), \quad \gamma_T \in \Lambda_T.
\]

Next, Let \(\varphi \in \Phi\) and \(g \in G_t\) with \(t \in [0, T]\) such that

\[
0 = (V - \varphi - g)(\hat{\varphi}) = \sup_{\eta \in \Lambda^t} (V - \varphi - g)(\eta),
\]

where \(\hat{\varphi} \in \Lambda\). Denote \(g_1(\gamma_t) := g(\gamma_t) + \hat{\varphi}(\gamma_t, \hat{\varphi})\) for all \((t, \gamma_t) \in [\hat{t}, T] \times \Lambda\). Then we have \(g_1 \in G_{\hat{t}}\).

Define a sequence of positive numbers \(\{\delta_i\}_{i \geq 0}\) by \(\delta_i = \frac{1}{i}\) for all \(i \geq 0\). For every \(\varepsilon > 0\), since \(v^\varepsilon - \varphi - g_1\) is a upper semicontinuous functional and \(\hat{\varphi}((\cdot), \cdot)\) is a gauge-type function, from Lemma 2.3 it follows that, for every \((t_0, \gamma_{t_0}) \in [\hat{t}, T] \times \Lambda^t\) satisfy

\[
(v^\varepsilon - \varphi - g_1)(\gamma_{t_0}) \geq \sup_{(s, \gamma_s) \in [\hat{t}, T] \times \Lambda^t} (v^\varepsilon - \varphi - g_1)(\gamma_s) - \varepsilon, \quad \text{and} \quad (v^\varepsilon - \varphi - g_1)(\gamma_{t_0}) \geq (v^\varepsilon - \varphi - g_1)(\hat{\varphi})
\]

there exist \((t_i, \gamma_{t_i}) \in [\hat{t}, T] \times \Lambda^t\) and a sequence \(\{(t_i, \gamma_{t_i})\}_{i \geq 1} \subset [\hat{t}, T] \times \Lambda^t\) such that

(i) \(\hat{\varphi}^2(\gamma_{t_0}, \gamma_{t}) \leq \varepsilon\), \(\hat{\varphi}^2(\gamma_{t_i}, \gamma_{t_i}) \leq \frac{\varepsilon}{2^i}\) and \(t_i \uparrow \hat{t}\) as \(i \to \infty\),

(ii) \((v^\varepsilon - \varphi - g_1)(\gamma_{t_i}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \hat{\varphi}^2(\gamma_{t_i}, \gamma_{t_i}) \geq (v^\varepsilon - \varphi - g_1)(\gamma_{t_0})\), and

(iii) \((v^\varepsilon - \varphi - g_1)(\gamma_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} \hat{\varphi}^2(\gamma_{t_i}, \gamma_s) < (v^\varepsilon - \varphi - g_1)(\gamma_{t_0}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \hat{\varphi}^2(\gamma_{t_i}, \gamma_{t_i})\) for all \((s, \gamma_s) \in [\hat{t}, T] \times \Lambda^t\).
We claim that
\[ d_\infty(\gamma_\varepsilon^\varepsilon, \hat{\gamma}_\varepsilon^\varepsilon) \to 0 \text{ as } \varepsilon \to 0. \] (3.5)

Indeed, if not, by [2,14], we can assume that there exists an \( \nu_0 > 0 \) such that
\[ \Upsilon^2(\gamma_\varepsilon^\varepsilon, \hat{\gamma}_\varepsilon^\varepsilon) \geq \nu_0. \]

Thus, we obtain that
\[
0 = (v - \varphi - g)(\hat{\gamma}_\varepsilon^\varepsilon) = \lim_{\varepsilon \to 0}(v^\varepsilon - \varphi - g_1)(\hat{\gamma}_\varepsilon^\varepsilon) \leq \lim_{\varepsilon \to 0}\left[ (v^\varepsilon - \varphi - g_1)(\gamma_\varepsilon^\varepsilon) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^{2}(\gamma_i^\varepsilon, \hat{\gamma}_\varepsilon^\varepsilon) \right]
\]
\[
\leq \lim_{\varepsilon \to 0}[(v - \varphi - g)(\gamma_\varepsilon^\varepsilon) + (v^\varepsilon - v)(\gamma_\varepsilon^\varepsilon)] - \nu_0 \leq (v - \varphi - g)(\hat{\gamma}_\varepsilon^\varepsilon) - \nu_0 = -\nu_0,
\]
contradicting \( \nu_0 > 0 \). We notice that, by the property (i) of \((t_\varepsilon, \gamma_\varepsilon^\varepsilon)\),
\[
2 \sum_{i=0}^{\infty} \frac{1}{2^i}(t_\varepsilon - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\varepsilon}{2^i} \right) \leq 4\varepsilon \frac{1}{2}.
\]

and
\[
|\partial_x \Upsilon^2(\gamma_\varepsilon^\varepsilon - \hat{\gamma}_\varepsilon^\varepsilon, t_\varepsilon, A)| \leq 4|c(t_\varepsilon - t_0)A\hat{\gamma}_\varepsilon^\varepsilon(t_\varepsilon) - \gamma_\varepsilon^\varepsilon(t_\varepsilon)|.
\]

Then for any \( \varrho > 0 \), by (3.4) and (3.5), there exists \( \varepsilon > 0 \) small enough such that
\[
\hat{t} \leq t_\varepsilon < T, \quad 2|t_\varepsilon - \hat{t}| + 2 \sum_{i=0}^{\infty} \frac{1}{2^i}(t_\varepsilon - t_i) + |\partial_t \varphi(\gamma_\varepsilon^\varepsilon)| + |\partial_t g(\gamma_\varepsilon^\varepsilon)| \leq \frac{\varrho}{4},
\]
\[
|(A^* \partial_x \varphi(\gamma_\varepsilon^\varepsilon), \gamma_\varepsilon^\varepsilon(t_\varepsilon))_H - (A^* \partial_x \varphi(\hat{\gamma}_\varepsilon^\varepsilon), \hat{\gamma}_\varepsilon^\varepsilon(\hat{t}))_H| \leq \frac{\varrho}{4}, \text{ and } |I| + |II| \leq \frac{\varrho}{4},
\]
where
\[
I = H^\varepsilon(\gamma_\varepsilon^\varepsilon, \partial_x \varphi(\gamma_\varepsilon^\varepsilon) + \partial_x g_2(\gamma_\varepsilon^\varepsilon)) - H(\gamma_\varepsilon^\varepsilon, \partial_x \varphi(\gamma_\varepsilon^\varepsilon) + \partial_x g(\gamma_\varepsilon^\varepsilon)),
\]
\[
II = H(\gamma_\varepsilon^\varepsilon, \partial_x \varphi(\gamma_\varepsilon^\varepsilon) + \partial_x g_2(\gamma_\varepsilon^\varepsilon)) - H(\gamma_\varepsilon^\varepsilon, \partial_x \varphi(\hat{\gamma}_\varepsilon^\varepsilon) + \partial_x g(\hat{\gamma}_\varepsilon^\varepsilon)),
\]
\[
g_2(\gamma_\varepsilon^\varepsilon) = g(\gamma_\varepsilon^\varepsilon) + \Upsilon^2(\gamma_\varepsilon^\varepsilon - \hat{\gamma}_\varepsilon^\varepsilon, t_\varepsilon, A) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^2(\gamma_i^\varepsilon - \hat{\gamma}_\varepsilon^\varepsilon, t_i, A),
\]
\[
\partial_x g_2(\gamma_\varepsilon^\varepsilon) = \partial_x g(\gamma_\varepsilon^\varepsilon) + \partial_x \Upsilon^2(\gamma_\varepsilon^\varepsilon - \hat{\gamma}_\varepsilon^\varepsilon, t_\varepsilon, A) + \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \Upsilon^2(\gamma_i^\varepsilon - \hat{\gamma}_\varepsilon^\varepsilon, t_i, A),
\]
\[
\partial_t g_2(\gamma_\varepsilon^\varepsilon) = \partial_t g(\gamma_\varepsilon^\varepsilon) + 2(t_\varepsilon - \hat{t}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i}(t_\varepsilon - t_i),
\]
and
\[
H^\varepsilon(\gamma_\varepsilon^\varepsilon, p) = \sup_{u \in U}(p, F^\varepsilon(\gamma_\varepsilon^\varepsilon, u))_H + q^\varepsilon(\gamma_\varepsilon^\varepsilon, u), \quad (t, \gamma_\varepsilon^\varepsilon, p) \in [0, T] \times \Lambda \times H.
\]
Since \( v^\varepsilon \) is a viscosity subsolution of PHJB equation (1.4) with generators \( F^\varepsilon, q^\varepsilon, \phi^\varepsilon \), we have
\[
\partial_t \varphi(\gamma^\varepsilon_t) + \partial_x g(\gamma^\varepsilon_t) + (A^* \partial_x \varphi(\gamma^\varepsilon_t), \gamma^\varepsilon_t(t_x)) \mathbf{H} + H^\varepsilon(\gamma^\varepsilon_t, \partial_x \varphi(\gamma^\varepsilon_t) + \partial_x g(\gamma^\varepsilon_t)) \geq 0.
\]
Thus
\[
0 \leq \partial_t \varphi(\gamma^\varepsilon_t) + \partial_x g(\gamma^\varepsilon_t) + 2(t_x - \hat{t}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (t_x - t_i) + (A^* \partial_x \varphi(\gamma^\varepsilon_t), \gamma^\varepsilon_t(t_x)) \mathbf{H} + \mathbf{H}(\gamma^\varepsilon_t, \partial_x \varphi(\gamma^\varepsilon_t) + \partial_x g(\gamma^\varepsilon_t)) + I + II
\]
\[
\leq \partial_t \varphi(\hat{\gamma}_i) + \partial_x g(\hat{\gamma}_i) + (A^* \partial_x \varphi(\hat{\gamma}_i), \hat{\gamma}_i(\hat{t})) + \mathbf{H}(\hat{\gamma}_i, \partial_x \varphi(\hat{\gamma}_i) + \partial_x g(\hat{\gamma}_i)) + \varrho.
\]
Letting \( \varrho \downarrow 0 \), we show that
\[
\partial_t \varphi(\hat{\gamma}_i) + \partial_x g(\hat{\gamma}_i) + (A^* \partial_x \varphi(\hat{\gamma}_i), \hat{\gamma}_i(\hat{t})) + \mathbf{H}(\hat{\gamma}_i, \partial_x \varphi(\hat{\gamma}_i) + \partial_x g(\hat{\gamma}_i)) \geq 0.
\]
Since \( \varphi \in \Phi \) and \( g \in G_t \) with \( t \in [0, T] \) are arbitrary, we see that \( v \) is a viscosity subsolution of PHJB equation (1.4) with generators \( F, q, \phi \), and thus completes the proof. \( \square \)

4 Viscosity solutions to PHJB equations: Uniqueness theorem.

This section is devoted to a proof of uniqueness of viscosity solutions to (1.4). This result, together with the results from the previous section, will be used to characterize the value functional defined by (1.3).

We now state the main result of this section.

**Theorem 4.1.** Suppose Hypothesis 2.3 holds. Let \( W_1 \in C^0(\Lambda) \) (resp., \( W_2 \in C^0(\Lambda) \)) be a viscosity subsolution (resp., supsolution) to equation (1.4) and let there exist constant \( L > 0 \) such that, for any \( 0 \leq t \leq s \leq T \) and \( \gamma_t, \eta_t \in \Lambda \),
\[
|W_1(\gamma_t) \lor |W_2(\gamma_t)| \leq L(1 + ||\gamma_t||_0); \quad (4.1)
\]
\[
|W_1(\gamma_{t,s,A}) - W_1(\eta_t)| \lor |W_2(\gamma_{t,s,A}) - W_2(\eta_t)| \leq L(1 + ||\gamma_t||_0 + ||\eta_t||_0)|s - t| + L||\gamma_t - \eta_t||_0. \quad (4.2)
\]
Then \( W_1 \leq W_2 \).

Theorems 3.1 and 4.1 lead to the result (given below) that the viscosity solution to PHJB equation given in (1.4) corresponds to the value functional \( V \) of our optimal control problem given in (1.1) and (1.3). 

**Theorem 4.2.** Let Hypothesis 2.3 hold. Then the value functional \( V \) defined by (1.3) is the unique viscosity solution to (1.4) in the class of functionals satisfying (4.1) and (4.2).

**Proof.** Theorem 3.1 shows that \( V \) is a viscosity solution to equation (1.4). Thus, our conclusion follows from Lemma 2.2 and Theorem 4.1 \( \square \)

Next, we prove Theorem 4.1. Let \( W_1 \) be a viscosity solution of PHJB equation (1.4). We note that for \( \delta > 0 \), the functional defined by \( \tilde{W} := W_1 - \frac{\delta}{t} \) is a subsolution for
\[
\begin{cases}
\partial_t \tilde{W}(\gamma_t) + \mathbf{H}(\gamma_t, \partial_x \tilde{W}(\gamma_t)) = \frac{\delta}{t^2}, & \gamma_t \in \Lambda, \\
\tilde{W}(\gamma_T) = \phi(\gamma_T).
\end{cases}
\]
As $W_1 \leq W_2$ follows from $\tilde{W} \leq W_2$ in the limit $\delta \downarrow 0$, it suffices to prove $W_1 \leq W_2$ under the additional assumption given below:

$$\partial_t W_1(\gamma_t) + H(\gamma_t, \partial_x W_1(\gamma_t)) \geq c, \quad c := \frac{\delta}{T^2}, \quad \gamma_t \in \Lambda.$$

**Proof of Theorem 4.1** The proof of this theorem is rather long. Thus, we split it into several steps.

**Step 1.** Definitions of auxiliary functionals.

We only need to prove that $W_1(\gamma_t) \leq W_2(\gamma_t)$ for all $(t, \gamma_t) \in [T - \bar{a}, T) \times \Lambda$. Here,

$$\bar{a} = \frac{1}{16L} \wedge T.$$

Then, we can repeat the same procedure for the case $[T - i\bar{a}, T - (i - 1)\bar{a})$. Thus, we assume the converse result that $(\hat{t}, \hat{\gamma}) \in [T - a, T) \times \Lambda$ exists such that $\tilde{m} := W_1(\hat{\gamma}) - W_2(\hat{\gamma}) > 0$.

Consider that $\varepsilon > 0$ is a small number such that

$$W_1(\hat{\gamma}) - W_2(\hat{\gamma}) - 2\varepsilon \frac{\nu T - \hat{t}}{\nu T} Y^2(\hat{\gamma}) > \frac{\tilde{m}}{2},$$

and

$$\frac{\varepsilon}{\nu T} \leq \frac{c}{2}, \quad (4.3)$$

where

$$\nu = 1 + \frac{1}{16TL}.$$

Next, we define for any $(t, \gamma_t, \eta_t) \in (T - \tilde{a}, T] \times \Lambda \times \Lambda$,

$$\Psi(\gamma_t, \eta_t) = W_1(\gamma_t) - W_2(\eta_t) - \beta Y^2(\gamma_t, \eta_t) - \varepsilon \frac{\nu T - t}{\nu T} (Y^2(\gamma_t) + Y^2(\eta_t)).$$

Define a sequence of positive numbers $\{\delta_i\}_{i \geq 0}$ by $\delta_i = \frac{1}{2^i}$ for all $i \geq 0$. Since $\Psi$ is a upper semicontinuous function bounded from above and $Y^2(\cdot, \cdot)$ is a gauge-type function, from Lemma 2.3 it follows that, for every $(t_0, \gamma_{t_0}, \eta_{t_0}) \in [\tilde{t}, T] \times \Lambda^i \times \Lambda^i$ satisfy

$$\Psi(\gamma_{t_0}, \eta_{t_0}) \geq \sup_{(s, \gamma_s, \eta_s) \in [\tilde{t}, T]} \Psi(\gamma_s, \eta_s) - \frac{1}{\beta}, \quad \text{and} \quad \Psi(\gamma_{t_0}, \eta_{t_0}) \geq \Psi(\hat{\gamma}^i, \hat{\gamma}^i) > \frac{\tilde{m}}{2},$$

there exist $(\tilde{t}, \hat{\gamma}^i, \hat{\eta}^i) \in [\tilde{t}, T] \times \Lambda^i \times \Lambda^i$ and a sequence $\{(t_i, \gamma_{t_i}, \eta_{t_i})\}_{i \geq 1} \subset [\tilde{t}, T] \times \Lambda^i \times \Lambda^i$ such that

(i) $Y^2(\gamma_{t_0}, \hat{\gamma}_i^i) + Y^2(\eta_{t_0}, \hat{\eta}_i^i) + |\hat{t} - t_0|^2 \leq \frac{1}{\beta}$, $Y^2(\gamma_{t_i}, \hat{\gamma}_i^i) + Y^2(\eta_{t_i}, \hat{\eta}_i^i) + |\hat{t} - t_i|^2 \leq \frac{1}{\beta \nu^i}$ and $t_i \uparrow \hat{t}$ as $i \to \infty$,

(ii) $\Psi(\hat{\gamma}^i, \hat{\eta}^i) = \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^2(\hat{\gamma}_i^i, \hat{\gamma}^i) + Y^2(\hat{\eta}_i^i, \hat{\eta}^i) + |\hat{t} - t_i|^2] \geq \Psi(\gamma_{t_0}, \eta_{t_0})$, and

(iii) for all $(s, \gamma_s, \eta_s) \in [\tilde{t}, T] \times \Lambda^i \times \Lambda^i \setminus \{(\hat{t}, \hat{\gamma}^i, \hat{\eta}^i)\}$,

$$\Psi^1(\gamma_s, \eta_s) < \Psi^1(\hat{\gamma}^i, \hat{\eta}^i). \quad (4.4)$$

where we define

$$\Psi^1(\gamma_t, \eta_t) := \Psi(\gamma_t, \eta_t) - \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^2(\gamma_{t_i}, \gamma_t) + Y^2(\eta_{t_i}, \eta_t) + |t - t_i|^2], \quad (t, \gamma_t, \eta_t) \in [\tilde{t}, T] \times \Lambda^i \times \Lambda^i.$$
We should note that the point \((\hat{t}, \hat{\gamma}_i, \hat{\eta}_i)\) depends on \(\beta\) and \(\varepsilon\).

**Step 2.** There exists \(M_0 > 0\) such that

\[
||\hat{\gamma}_i||_0 \vee ||\hat{\eta}_i||_0 < M_0,
\]

\[
\beta ||\hat{\gamma}_i - \hat{\eta}_i||_0^2 \to 0 \text{ as } \beta \to \infty.
\]

Let us show the above. First, noting \(\nu\) is independent of \(\beta\), by the definition of \(\Psi\), there exists an \(M_0 > 0\) that is sufficiently large that \(\Psi(\gamma_t, \eta_t) < 0\) for all \(t \in [T - a, T]\) and \(||\gamma_t||_0 \vee ||\eta_t||_0 \geq M_0\). Thus, we have \(||\hat{\gamma}_i||_0 \vee ||\hat{\eta}_i||_0 \vee ||\gamma_t^0||_0 \vee ||\eta_t^0||_0 < M_0\).

Second, by (4.4), we have

\[
2\Psi(\hat{\gamma}_i, \hat{\eta}_i) - 2\sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon^2(\gamma_t^i, \hat{\gamma}_i) + \Upsilon^2(\eta_t^i, \hat{\eta}_i) + |\hat{t} - t_i|^2]
\]

\[
\geq \Psi(\hat{\gamma}_i, \hat{\gamma}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon^2(\gamma_t^i, \hat{\gamma}_i) + \Upsilon^2(\eta_t^i, \hat{\gamma}_i) + |\hat{t} - t_i|^2]
\]

\[
+ \Psi(\hat{\eta}_i, \hat{\eta}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon^2(\gamma_t^i, \hat{\eta}_i) + \Upsilon^2(\eta_t^i, \hat{\eta}_i) + |\hat{t} - t_i|^2].
\]

This implies that

\[
2\beta \Upsilon^2(\hat{\gamma}_i, \hat{\eta}_i) \leq |W_1(\hat{\gamma}_i) - W_1(\hat{\eta}_i)| + |W_2(\hat{\gamma}_i) - W_2(\hat{\eta}_i)| + \sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon^2(\gamma_t^i, \hat{\gamma}_i) + \Upsilon^2(\eta_t^i, \hat{\eta}_i)].
\]

On the other hand, by Lemma 2.5

\[
\sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon^2(\gamma_t^i, \hat{\eta}_i) + \Upsilon^2(\eta_t^i, \hat{\eta}_i)]
\]

\[
\leq 2\sum_{i=0}^{\infty} \frac{1}{2^i} [\Upsilon^2(\gamma_t^i, \hat{\eta}_i) + \Upsilon^2(\gamma_t^i, \hat{\eta}_i) + 2\Upsilon^2(\hat{\gamma}_i, \hat{\eta}_i)] \leq \frac{4}{\beta} + 8\Upsilon^2(\hat{\gamma}_i, \hat{\eta}_i).
\]

Thus we have

\[
(2\beta - 8)\Upsilon^2(\hat{\gamma}_i, \hat{\eta}_i) \leq |W_1(\hat{\gamma}_i) - W_1(\hat{\eta}_i)| + |W_2(\hat{\gamma}_i) - W_2(\hat{\eta}_i)| + \frac{4}{\beta}
\]

\[
\leq 2L(2 + ||\hat{\gamma}_i||_0 \vee ||\hat{\eta}_i||_0) + \frac{4}{\beta} \leq 4L(1 + M_0) + \frac{4}{\beta}.
\]

Letting \(\beta \to \infty\), we get

\[
\Upsilon^2(\hat{\gamma}_i, \hat{\eta}_i) \to 0 \text{ as } \beta \to +\infty.
\]

Then from (2.14) and (4.10) it follows that (4.6) holds.

**Step 3.** There exists \(N_0 > 0\) such that \(\hat{t} \in [T - a, T]\) for all \(\beta \geq N_0\).

By (4.6), we can let \(N_0 > 0\) be a large number such that

\[
L ||\hat{\gamma}_i - \hat{\eta}_i||_0 \leq \hat{m} \frac{m}{4},
\]
for all $\beta \geq N_0$. Then we have $\hat{t} \in [T - \bar{a}, T)$ for all $\beta \geq N_0$. Indeed, if $\hat{t} = T$, we will deduce the following contradiction:

$$\frac{\hat{m}}{2} \leq \Psi(\hat{\gamma}_t, \hat{\eta}_t) \leq \phi(\hat{\eta}_t) \leq L\|\hat{\gamma}_t - \hat{\eta}_t\|_0 \leq \frac{\hat{m}}{4}.$$

**Step 4. Completion of the proof.**

From above all, for the fixed $N_0 > 0$ in step 3, we find $(\hat{t}, \hat{\gamma}_t), (\hat{t}, \hat{\eta}_t) \in [\bar{t}, T] \times \Lambda$ satisfying $\hat{t} \in [T - \bar{a}, T)$ for all $\beta \geq N_0$ such that

$$\Psi^1(\hat{\gamma}_t, \hat{\eta}_t) \geq \Psi(\hat{\gamma}_t, \hat{\eta}_t) \quad \text{and} \quad \Psi^1(\hat{\gamma}_t, \hat{\eta}_t) \geq \Psi^1(\gamma_t, \eta_t), \quad (t, \gamma_t, \eta_t) \in [\hat{t}, T) \times \Lambda \times \Lambda.$$}

Now we consider the function, for $(t, \gamma_t), (s, \eta_s) \in [\hat{t}, T] \times \Lambda$,

$$\Psi_{\delta}(\gamma_t, \eta_s) = W_1^1(\gamma_t) - W_2^2(\eta_s) - 2\beta(\Psi^2(\gamma_t, \xi_t) + \Psi^2(\eta_s, \xi_s)) - \frac{1}{\delta}|s - t|^2,$$

where

$$W_1^1(\gamma_t) = W_1(\gamma_t) - \epsilon \frac{\nu T - t}{\nu T} \Psi^2(\gamma_t, \hat{\gamma}_t) = \epsilon \Psi^2(\gamma_t, \hat{\gamma}_t) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Psi^2(\gamma^i_{t_i}, \gamma_t),$$

and

$$W_2^2(\eta_s) = W_2(\eta_s) + \epsilon \frac{\nu T - s}{\nu T} \Psi^2(\eta_s, \hat{\eta}_t) = \epsilon \Psi^2(\eta_s, \hat{\eta}_t) + \sum_{i=0}^{\infty} \frac{1}{2^i} \Psi^2(\eta^i_{s_i}, \eta_s),$$

and

$$\xi_t = \frac{\hat{\gamma}_t + \hat{\eta}_t}{2}.$$

Define a sequence of positive numbers $\{\delta_i\}_{i \geq 0}$ by $\delta_i = \frac{1}{\nu T}$ for all $i \geq 0$. For every $\delta > 0$, from Lemma 2.3 it follows that, for every $(t_0, \gamma^0_{t_0}), (s_0, \eta^0_{s_0}) \in [\hat{t}, T] \times \Lambda$ satisfy

$$\Psi_{\delta}(\gamma^0_{t_0}, \eta^0_{s_0}) \geq \sup_{(t, \gamma_t), (s, \eta_s) \in [\hat{t}, T] \times \Lambda} \Psi_{\delta}(\gamma_t, \eta_s) - \delta,$$

there exist $(\hat{t}, \hat{\gamma}_t), (\hat{s}, \hat{\eta}_s) \in [\hat{t}, T] \times \Lambda$ and two sequences $\{(i, \tilde{t}, \tilde{\gamma}_i, \tilde{\eta}_i)\}_{i \geq 1}, \{(s_i, \tilde{\eta}_s)\}_{i \geq 1} \subset [\hat{t}, T] \times \Lambda$ such that

(i) $\Psi^2(\tilde{\gamma}_{s_i}, \tilde{\eta}_{s_i}) \leq \delta, \Psi^2(\tilde{\gamma}_{s_i}, \tilde{\eta}_{s_i}) \leq \frac{\delta}{2^i} \leq \frac{\delta}{2}$ and $\tilde{t} \uparrow \hat{t}, \tilde{t} \uparrow \hat{s}$ as $i \to \infty$,

(ii) $\Psi_{\delta}(\gamma_t, \eta_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} [\Psi^2(\gamma^i_{t_i}, \gamma_t) + \Psi^2(\eta^i_{s_i}, \eta_s)] \geq \Psi_{\delta}(\gamma^0_{t_0}, \eta^0_{s_0}),$ and

(iii) for all $(t, \gamma_t, \eta_s) \in [\hat{t}, T] \times \Lambda \times \Lambda$ such that

$$\Psi_{\delta}(\gamma_t, \eta_s) < \Psi_{\delta}(\gamma^0_{t_0}, \eta^0_{s_0}) - \sum_{i=0}^{\infty} \frac{1}{2^i} [\Psi^2(\gamma^i_{t_i}, \gamma_t) + \Psi^2(\eta^i_{s_i}, \eta_s)].$$

(4.13)
By the following Lemma 4.1, we have
\[
\lim_{\delta \to 0} [\mathcal{Y}^2(\tilde{\gamma}_t, \tilde{\gamma}_i) + \mathcal{Y}^2(\tilde{\eta}_s, \tilde{\eta}_i)] = 0. \tag{4.14}
\]

From (4.14) and \( \hat{t} < T \) for \( \beta > N_0 \), it follows that, for every fixed \( \beta > N_0 \), constant \( K_\beta > 0 \) exists such that
\[
|\hat{t}| \vee |s| < T, \quad \text{for all } 0 < \delta < K_\beta.
\]

Now, for every \( \beta > N_0 \) and \( 0 < \delta < K_\beta \), from the definition of viscosity solutions it follows that
\[
-c \frac{\nu T}{\nu T} \mathcal{Y}^2(\tilde{\eta}_s) - 2\varepsilon(\tilde{s} - \hat{t}) - 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\tilde{s} - \tilde{s}_i) + \frac{2}{\delta} (\hat{t} - \hat{s})
\]
\[
+ \mathcal{H}(\tilde{\gamma}_t, 2\beta \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\xi}_t,i,A) + \varepsilon \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\gamma}_i,i,A) + \varepsilon \frac{\nu T - \hat{t}}{\nu T} \partial_x \mathcal{Y}^2(\tilde{\gamma}_t)
\]
\[
+ \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\gamma}_i,i,A) + \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \gamma_t,i,i,A)) \geq c;
\tag{4.15}
\]
and
\[
-c \frac{\nu T}{\nu T} \mathcal{Y}^2(\tilde{\eta}_s) - 2\varepsilon(\tilde{s} - \hat{t}) - 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\tilde{s} - \tilde{s}_i) + \frac{2}{\delta} (\hat{t} - \hat{s})
\]
\[
+ \mathcal{H}(\tilde{\eta}_s, -2\beta \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\xi}_s,i,A) + \varepsilon \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\eta}_s,i,A) - \varepsilon \frac{\nu T - \tilde{s}}{\nu T} \partial_x \mathcal{Y}^2(\tilde{\eta}_s)
\]
\[
- \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\eta}_s,i,A) - \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\eta}_s,i,i,A)) \leq 0. \tag{4.16}
\]

We notice that, by the property (i) of \((\hat{t}, \tilde{\gamma}_t, \tilde{\gamma}_i, \tilde{\eta}_s)\),
\[
2 \sum_{i=0}^{\infty} \frac{1}{2^i} [(\tilde{s} - \tilde{s}_i) + (\hat{t} - \hat{t}_i)] \leq 4 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\delta}{2^i} \right)^{\frac{1}{2}} \leq 8 \delta^{\frac{1}{2}};
\]
\[
|\partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\gamma}_i,i,i,A)| + |\partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\eta}_s,i,A)| \leq 4|e^{(t - \hat{t})}A\tilde{\gamma}_t(\hat{t}) - \tilde{\gamma}_t(\hat{t})| + 4|e^{(s - \tilde{s})}A\tilde{\eta}_s(\tilde{s}) - \tilde{\eta}_s(\tilde{s})|;
\]
and
\[
\left| \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\gamma}_i,i,i,A) + \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\eta}_s,i,i,A) \right|
\]
\[
\leq 4 \sum_{i=0}^{\infty} \frac{1}{2^i} \left| \left| e^{(t - \hat{t})}A\tilde{\gamma}_t(\hat{t}) - \tilde{\gamma}_t(\hat{t}) \right| + \left| e^{(s - \tilde{s})}A\tilde{\eta}_s(\tilde{s}) - \tilde{\eta}_s(\tilde{s}) \right| \right| \leq 8 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\delta}{2^i} \right)^{\frac{1}{2}} \leq 16 \delta^{\frac{1}{2}}.
\]

Combining (4.15) and (4.16), and letting \( \delta \to 0 \), we obtain
\[
c + \frac{c}{\nu T} (Y^2(\tilde{\gamma}_t) + Y^2(\tilde{\eta}_i)) \leq H_1 - H_2 + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\hat{t} - \hat{t}_i), \tag{4.17}
\]
where
\[
H_1 = H(\tilde{\gamma}_t, 2\beta \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\xi}_t) + \varepsilon \frac{\nu T - \hat{t}}{\nu T} \partial_x \mathcal{Y}^2(\tilde{\gamma}_t) + \sum_{i=0}^{\infty} \frac{1}{2^i} \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \gamma_t,i,i,A));
\]
and
\[
H_2 = H(\tilde{\eta}_s, -2\beta \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\xi}_s) + \varepsilon \partial_x \mathcal{Y}^2(\tilde{\eta}_s - \tilde{\eta}_s) - \varepsilon \frac{\nu T - \tilde{s}}{\nu T} \partial_x \mathcal{Y}^2(\tilde{\eta}_s) + H(\tilde{\gamma}_t, 2\beta \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \tilde{\xi}_t) + \varepsilon \partial_x \mathcal{Y}^2(\tilde{\gamma}_t - \gamma_t,i,i,A));
\]
\[ H_2 = H(\tilde{\eta}_i, -2\beta \partial_x \gamma^2(\tilde{\eta}_i - \tilde{\xi}_i) - \varepsilon \frac{\nu T - \hat{i}}{\nu T} \partial_x \gamma^2(\tilde{\eta}_i) - \sum_{i=0}^{\infty} \frac{1}{2} \partial_x \gamma^2(\tilde{\eta}_i - \eta_i^i).) \]

On the other hand, by a simple calculation we obtain
\[ H_1 - H_2 \leq \sup_{u \in U'} (J_1 + J_2), \quad (4.18) \]

where
\[ J_1 = (F(\hat{\gamma}_i, u), 2\beta \partial_x \gamma^2(\hat{\gamma}_i - \hat{\xi}_i) + \varepsilon \frac{\nu T - \hat{i}}{\nu T} \partial_x \gamma^2(\hat{\gamma}_i) + \sum_{i=0}^{\infty} \frac{1}{2} \partial_x \gamma^2(\hat{\gamma}_i - \gamma_i^i)) \]
\[ -(F(\check{\eta}_i, u), -2\beta \partial_x \gamma^2(\check{\eta}_i - \check{\xi}_i) - \varepsilon \frac{\nu T - \hat{i}}{\nu T} \partial_x \gamma^2(\check{\eta}_i) - \sum_{i=0}^{\infty} \frac{1}{2} \partial_x \gamma^2(\check{\eta}_i - \eta_i^i)) \]
\[ \leq 4\beta L||\hat{\gamma}_i - \check{\eta}_i||_0^2 + 8\varepsilon \frac{\nu T - \hat{i}}{\nu T} L(1 + ||\gamma_i||_0^2 + ||\check{\eta}_i||_0^2) 
+ 4L \sum_{i=0}^{\infty} \frac{1}{2i} ||e^{(i-t_i)} A \gamma_i^i(t_i) - \hat{\gamma}_i(\hat{t})|| + ||e^{(i-t_i)} A \eta_i^i(t_i) - \check{\eta}_i(\hat{t})||(1 + ||\gamma_i||_0 + ||\check{\eta}_i||_0); \quad (4.19) \]
\[ J_2 = q(\hat{\gamma}_i, u) - q(\check{\eta}_i, u) \leq L||\hat{\gamma}_i - \check{\eta}_i||_0; \quad (4.20) \]

We notice that, by the property (i) of \((\hat{\gamma}_i, \check{\eta}_i, \hat{\xi}_i),\)
\[ 2 \sum_{i=0}^{\infty} \frac{1}{2i} (\hat{t} - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2i} \left( \frac{1}{2\beta} \right)^{\frac{i}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{1}{2}}, \]

and
\[ \sum_{i=0}^{\infty} \frac{1}{2i} ||e^{(i-t_i)} A \gamma_i^i(t_i) - \hat{\gamma}_i(\hat{t})|| + ||e^{(i-t_i)} A \eta_i^i(t_i) - \check{\eta}_i(\hat{t})|| \leq 2 \sum_{i=0}^{\infty} \frac{1}{2i} \left( \frac{1}{2\beta} \right)^{\frac{i}{2}} \leq 4 \left( \frac{1}{\beta} \right)^{\frac{1}{2}}. \]

Combining \((4.17), (4.20),\) and letting \(\beta \to \infty,\) by \((4.3)\) and \((4.6)\) we obtain
\[ c \leq \lim_{\beta \to \infty} \left[ - \frac{\varepsilon}{\nu T} (\gamma^2(\hat{\gamma}_i) + \gamma^2(\check{\eta}_i)) \right. \]
\[ \left. + 8\varepsilon \frac{\nu T - \hat{i}}{\nu T} L(1 + ||\gamma_i||_0^2 + ||\check{\eta}_i||_0^2) \right]. \quad (4.21) \]

Recalling \(\nu = 1 + \frac{1}{16L^2}\) and \(\tilde{a} = \frac{1}{16L} \wedge T,\) by \((4.3),\) the following contradiction is induced:
\[ c \leq \frac{\varepsilon}{\nu T} \leq \frac{c}{2}. \]

The proof is now complete. \(\square\)

To complete the previous proof, it remains to state and prove the following lemma.

**Lemma 4.1.** The maximum points \((\hat{\eta}_i, \check{\eta}_i, \hat{\gamma}_i, \check{\eta}_i)\) of \(\Psi_{\beta}(\gamma_t, \eta_t) = \sum_{i=0}^{\infty} \frac{1}{2i} (\gamma^2(\hat{\gamma}_i, \gamma_t) + \gamma^2(\check{\eta}_i, \eta_t))\]

defined by \((4.12)\) in \([\hat{\gamma}, T] \times \Lambda^4 \times [\hat{\eta}, T] \times \Lambda^4\) satisfy condition \((4.14).\)
Proof. Without loss of generality, we assume \( \bar{t} \leq s \). By (4.13), we have

\[
2\Psi_{\delta}(\tilde{\gamma}_t, \tilde{\eta}_s) - 2\sum_{i=0}^{\infty} \frac{1}{2^i} [T^2(\tilde{\gamma}_i, \tilde{\gamma}_t) + T^2(\tilde{\eta}_i, \tilde{\eta}_s)] \\
\geq \Psi_{\delta}(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\gamma}_{\bar{t},s,\bar{A}}) + \Psi_{\delta}(\tilde{\eta}_s, \tilde{\eta}_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} [T^2(\tilde{\gamma}_i, \tilde{\gamma}_{\bar{t},s,\bar{A}}) + T^2(\tilde{\eta}_i, \tilde{\gamma}_{\bar{t},s,\bar{A}}) + T^2(\tilde{\eta}_i, \tilde{\eta}_s)] \\
+ \sum_{i=0}^{\infty} \frac{1}{2^i} [T^2(\tilde{\eta}_i, \tilde{\eta}_s)].
\]

This implies that

\[
\frac{2}{\delta} |\bar{t} - \bar{s}|^2 \leq |W'_1(\tilde{\gamma}_{\bar{t},s,\bar{A}}) - W'_1(\tilde{\eta}_s)| + |W'_2(\tilde{\gamma}_{\bar{t},s,\bar{A}}) - W'_2(\tilde{\eta}_s)| + \sum_{i=0}^{\infty} \frac{1}{2^i} [T^2(\tilde{\gamma}_i, \tilde{\gamma}_{\bar{t},s,\bar{A}}) + T^2(\tilde{\eta}_i, \tilde{\gamma}_{\bar{t},s,\bar{A}}) + T^2(\tilde{\eta}_i, \tilde{\eta}_s)].
\]

(4.23)

Letting \( \delta \to 0 \), we have

\[
|\bar{t} - \bar{s}| \to 0 \quad \text{as} \quad \delta \to 0.
\]

By the properties of \( \Psi_{\delta} \), we get that

\[
\Psi_{\delta}(\tilde{\gamma}_t, \tilde{\eta}_s) \geq \Psi_{\delta}(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\eta}_{\bar{t},s,\bar{A}}) \geq \sup \{\Psi_{\delta}(\gamma_t, \eta_s) - \delta : (t, \gamma_t) \in [\bar{t}, \bar{s}] \times \Lambda \}
\]

\[
\geq \Psi_{\delta}(\tilde{\gamma}_t, \tilde{\eta}_s) - \delta = \Psi^1(\tilde{\gamma}_t, \tilde{\eta}_s) - \delta \geq \Psi^1(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\eta}_{\bar{t},s,\bar{A}}) - \delta
\]

\[
= W'_1(\tilde{\gamma}_{\bar{t},s,\bar{A}}) - W'_2(\tilde{\eta}_s) - \beta \Upsilon^2(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\xi}_s) + \varepsilon [T^2(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\xi}_s) + T^2(\tilde{\eta}_s, \tilde{\xi}_s)] - \delta
\]

\[
= \Psi_{\delta}(\tilde{\gamma}_t, \tilde{\eta}_s) + \frac{1}{\delta} |\bar{t} - \bar{s}|^2 + W'_1(\tilde{\gamma}_{\bar{t},s,\bar{A}}) - W'_1(\tilde{\gamma}_t) + 2\beta \Upsilon^2(\tilde{\gamma}_t, \tilde{\xi}_s) + \Upsilon^2(\tilde{\eta}_s, \tilde{\xi}_s) - \beta \Upsilon^2(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\eta}_s)
\]

\[
+ \varepsilon [T^2(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\xi}_s) + T^2(\tilde{\eta}_s, \tilde{\xi}_s)] - \delta.
\]

Noting that, since \( \{e^{tA}, t \geq 0\} \) is a \( C_0 \) contraction semigroup, for all \( \gamma_t \in \Lambda \) and \( s \in [t, T] \),

\[
\Upsilon^2(\gamma_t) = ||\gamma_t||^2_0 + \frac{||\gamma_t(t)||^4_0}{||\gamma_t||^2_0} \geq ||\gamma_{t,s,A}||_0 + \frac{||e^{(s-t)}A\gamma_t(t)||^4_0}{||\gamma_t||^2_0} = \Upsilon^2(\gamma_{t,s,A}) \geq ||\gamma_{t,s,A}||^2_0 = ||\gamma_t||^2_0 \geq \frac{1}{3} \Upsilon^2(\gamma_t).
\]

Letting \( \delta \to 0 \), we obtain that

\[
\frac{1}{\delta} |\bar{t} - \bar{s}|^2 + \varepsilon [T^2(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\gamma}_t) + T^2(\tilde{\eta}_s, \tilde{\eta}_s)] \to 0 \quad \text{as} \quad \delta \to 0,
\]

Notice that

\[
\Upsilon^2(\tilde{\gamma}_{\bar{t},s,\bar{A}}, \tilde{\gamma}_t) \geq ||\tilde{\gamma}_{\bar{t},s,\bar{A}} - \tilde{\gamma}_t||^2_0 = ||\tilde{\gamma}_t - \tilde{\gamma}_{\bar{t},s,\bar{A}}||^2_0 \geq \frac{1}{3} \Upsilon^2(\tilde{\gamma}_t, \tilde{\gamma}_t),
\]

we get that (4.14) holds true. The proof is now complete. \( \square \)

5 Appendix

In this Appendix, we prove \((\Lambda^f, d_\infty)\) and \((\Lambda^f, d_\infty)\) are two complete metric spaces.

Lemma 5.1. \((\Lambda^f, d_\infty)\) and \((\Lambda^f, d_\infty)\) are two complete metric spaces for every \( t \in [0, T] \).
Proof. Assume \( \{\gamma^n_{t_n}\}_{n \geq 0} \) is a cauchy sequence in \((\hat{A}, d)\), then for any \( \varepsilon > 0 \), there exists \( N(\varepsilon) > 0 \) such that, for all \( m, n \geq N(\varepsilon) \), we have
\[
d_{\infty}(\gamma^n_{t_n}, \gamma^m_{t_m}) = |t_n - t_m| + \sup_{0 \leq s \leq e} |e^{((s-t_n)\gamma^n_{t_n})s} - e^{((s-t_m)\gamma^m_{t_m})s}| < \varepsilon.
\]
Therefore, there exists \( \hat{t} \in [t, T] \) such that \( \lim_{n \to \infty} t_n = \hat{t} \). Moreover, for all \( s \in [0, T] \),
\[
|e^{((s-t_n)\gamma^n_{t_n})s} - e^{((s-t_m)\gamma^m_{t_m})s}| < \varepsilon, \quad (\forall m, n \geq N(\varepsilon)).
\]
For fixed \( s \in [0, T] \), we see that \( \{e^{((s-t_n)\gamma^n_{t_n})s}\} \) is a cauchy sequence, thereby the limit \( \lim_{n \to \infty} e^{((s-t_n)\gamma^n_{t_n})s} \) exists and denoted by \( \gamma_T(s) \). Letting \( m \to \infty \) in (5.1), we obtain that
\[
|\gamma_T(s) - e^{((s-t_n)\gamma^n_{t_n})s}| \leq \varepsilon, \quad (\forall n \geq N(\varepsilon)).
\]
Taking the supremum over \( s \in [0, T] \), we get
\[
\sup_{s \in [0, T]} |\gamma_T(s) - e^{((s-t_n)\gamma^n_{t_n})s}| \leq \varepsilon, \quad (\forall n \geq N(\varepsilon)).
\]
We claim that \( \gamma_T(s) = e^{(s-\hat{t})\gamma_{t_n}(s)} \) for all \( s \in (\hat{t}, T] \). In fact, if there exists a subsequence \( \{t_{ni}\}_{i \geq 0} \) of \( \{t_n\}_{n \geq 0} \) such that \( t_{ni} \geq \hat{t} \), then we have, for every \( s \in (\hat{t}, T] \),
\[
\gamma_T(s) = \lim_{n \to \infty} e^{((s-t_n)\gamma^n_{t_n})s} = \lim_{i \to \infty} e^{((s-t_{ni})\gamma^{ni}_{t_{ni}})(t_{ni} - \hat{t})} = e^{(s-\hat{t})\gamma_T(\hat{t})}.
\]
Otherwise, we may assume \( \{t_n\}_{n \geq 0} \geq \hat{t} \). For all \( s \in (\hat{t}, T] \), we can let \( m > n \) be large enough such that \( t_m \leq s < t_n \), and letting \( s = t_m \) in (5.1),
\[
|e^{(s-t)\gamma^n_{t_n}t_m} - e^{(s-t)\gamma^m_{t_m}t_m}| \leq M_1 |\gamma^n_{t_n}(t_m) - \gamma^m_{t_m}(t_m)| < M_1 \varepsilon,
\]
Letting \( m \to \infty \), we obtain
\[
|e^{(s-\hat{t})\gamma^n_{t_n}t_m} - \gamma_T(s)| \leq M_1 \varepsilon.
\]
Letting \( n \to \infty \), we have
\[
|e^{(s-\hat{t})\gamma^n_{t_n}t_m} - \gamma_T(s)| \leq M_1 \varepsilon, \quad \text{for all} \quad s \in (\hat{t}, T].
\]
Then, by (5.2) we obtain
\[
d_{\infty}(\eta_{\hat{t}}, \gamma^n_{t_n}) \to 0 \text{ as } n \to \infty. \quad (5.3)
\]
Here we let \( \eta_{t} \) denote \( \gamma_T|_{[0, t]} \). Now we prove \( \eta_{t} \in \hat{A}^t \). First, we prove \( \eta_{t} \) is right-continuous. For every \( 0 < s < \hat{t} \) and \( 0 < \delta < \hat{t} - s \), we have
\[
|\eta_{t}(s + \delta) - \eta_{\hat{t}}(s)| \leq |\gamma_T(s + \delta) - \gamma^n_{t_n}((s + \delta) \wedge t_n)| + |\gamma^n_{t_n}((s + \delta) \wedge t_n) - \gamma^m_{t_m}((s + \delta) \wedge t_n)| + |\gamma^n_{t_n}(s \wedge t_n) - \gamma_T(s)|.
\]
For every \( \varepsilon > 0 \), by (5.2), there exists \( n_0 > 0 \) be large enough such that
\[
|\gamma_T(s + \delta) - \gamma^n_{t_n}((s + \delta) \wedge t_n)| + |\gamma^n_{t_n}(s \wedge t_n) - \gamma_T(s)| < \frac{\varepsilon}{2}.
\]
For the fixed \( n \), since \( \gamma^n_{t_n} \in \hat{A}^t \), there exists a constant \( 0 < \Delta < \hat{t} - s \) such that, for all \( 0 \leq \delta < \Delta \),
\[
|\gamma^n_{t_n}((s + \delta) \wedge t_n) - \gamma^n_{t_n}(s \wedge t_n)| < \frac{\varepsilon}{2}.
\]

Then \( |\eta(t + \delta) - \eta(t)| < \varepsilon \) for all \( 0 \leq \delta < \Delta \). Next, let us prove \( \eta \) has left limit in \((0, \hat{t}]\). For every \( 0 < s \leq \hat{t} \) and \( 0 \leq s_1, s_2 < s \), we have
\[
|\eta(s_1) - \eta(s_2)| \leq |\gamma(t(s_1) - \gamma^n_{t_n}(s_1 \wedge t_n)| + |\gamma(t(s_2) - \gamma^n_{t_n}(s_2 \wedge t_n)| + |\gamma^n_{t_n}(s_1 \wedge t_n) - \gamma^n_{t_n}(s_2 \wedge t_n)|.
\]

For every \( \varepsilon > 0 \), by \([5,2]\), there exists \( n > 0 \), which is independent of \( s_1, s_2 \), be large enough such that
\[
|\gamma(t(s_1) - \gamma^n_{t_n}(s_1 \wedge t_n)| + |\gamma(t(s_2) - \gamma^n_{t_n}(s_2 \wedge t_n)| < \frac{\varepsilon}{2}.
\]

For the fixed \( n \), since \( \gamma^n_{t_n} \in \hat{A}^t \), there exists a constant \( \Delta > 0 \) such that, for all \( s_1, s_2 \in [s - \Delta, s) \),
\[
|\gamma^n_{t_n}(s_1 \wedge t_n) - \gamma^n_{t_n}(s_2 \wedge t_n)| < \frac{\varepsilon}{2}.
\]

Then there exists a constant \( \Delta > 0 \) such that \( |\eta(t) - \eta(s)| < \varepsilon \) for all \( s_1, s_2 \in [s - \Delta, s) \).

Finally, we prove \( \eta \in \hat{A} \) if \( \{\gamma^n_{t_n}\}_{n \geq 0} \) is a cauchy sequence in \((\hat{A}, d_\infty)\). By the similar proof process of right continuous above, we get that \( \eta \) is continuous in \([0, \hat{t}]\). Now we have to prove that \( \eta \) is left continuous at \( \hat{t} \). For every \( 0 \leq t < \hat{t} \), we have, if \( t_n \geq \hat{t} \),
\[
|\eta(t) - \eta(t_n)| \leq |\gamma(t) - \gamma^n_{t_n}(t \wedge t_n)| + |\gamma^n_{t_n}(t \wedge t_n) - \gamma^n_{t_n} (\hat{t} \wedge t_n)| + |\gamma^n_{t_n} (\hat{t} \wedge t_n) - \gamma(t)|.
\]

For every \( \varepsilon > 0 \), by the similar process above, there exists a constant \( \Delta > 0 \) such that, for all \( |t - \hat{t}| < \Delta \),
\[
|\eta(t) - \eta(t_n)| < \varepsilon.
\]

If \( t_n < \hat{t} \),
\[
|\eta(t) - \eta(t_n)| \leq |\gamma(t) - e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)}| + |e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)} - e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)}| + |e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)} - \gamma(t)|.
\]

For every \( \varepsilon > 0 \), by \([5,2]\), there exists \( n > 0 \) be large enough such that
\[
|\gamma(t) - e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)}| + |e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)} - \gamma(t)| < \frac{\varepsilon}{2}.
\]

For the fixed \( n \), since \( \{e^{tA}, t \geq 0 \} \) is a \( C_0 \) semigroup, there exists a constant \( 0 < \Delta \leq |\hat{t} - t_n| \) such that, for all \( |t - \hat{t}| < \Delta \leq |\hat{t} - t_n| \),
\[
|e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)} - e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)}| = |(e^{((t-t)^0)A} - I)e^{((t-t_n)^0)A_{t_n}(t \wedge t_n)}| < \frac{\varepsilon}{2}.
\]

Then \( |\eta(t) - \eta(t_n)| < \varepsilon \) for all \( |t - \hat{t}| < \Delta \). The proof is now complete. \( \Box \)

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