Sources of intrinsic rotation in the low-flow ordering

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Abstract
A low flow, $\delta f$ gyrokinetic formulation to obtain the intrinsic rotation profiles is presented. The momentum conservation equation in the low-flow ordering contains new terms, neglected in previous first-principles formulations, that may explain the intrinsic rotation observed in tokamaks in the absence of external sources of momentum. The intrinsic rotation profile depends on the density and temperature profiles and on the up–down asymmetry.

1. Introduction
Experimental observations have shown that tokamak plasmas rotate spontaneously without momentum input \cite{1}. This intrinsic rotation has been the object of recent work \cite{1,2} because of its relevance for ITER \cite{3}, where the projected momentum input from neutral beams is small, and the rotation is expected to be mostly intrinsic.

The origin of the intrinsic rotation is still unclear. There has been some theoretical work in turbulent transport of momentum using gyrokinetic simulations \cite{4–12}, and two main mechanisms have been proposed as candidates to explain intrinsic rotation. On the one hand, the momentum pinch due to the Coriolis drift \cite{4} has been argued to transport into the core momentum generated in the edge. On the other hand, up–down asymmetry generates intrinsic rotation \cite{7,8}. However, neither of these explanations are able to account for all experimental observations. The up–down asymmetry is only large in the edge, generating rotation in that region that then needs to be transported inwards by the Coriolis pinch. Thus, intrinsic rotation in the core could only be explained by the pinch. The pinch of momentum is not sufficient because it does not allow the toroidal rotation to change sign in the core as is observed experimentally \cite{13}.

In this paper we present a new model implementable in $\delta f$ flux tube simulations \cite{14–17}. This model is based on the low-flow ordering of \cite{18}, and self-consistently includes higher order contributions. As a result, new drive terms for the intrinsic rotation appear that depend on the gradients of the background profiles of density and temperature.

We recast the results from \cite{18} in a form similar to the equations in the high-flow ordering \cite{19,20}. These are the equations that have been implemented in most gyrokinetic codes that are employed to study momentum transport. For this reason, the new form of the equations is useful to identify the differences with previous models. In addition, we discuss how the new contributions drive intrinsic rotation and we show that the intrinsic rotation resulting from these new processes depends on density and temperature gradients.

In the rest of this paper we present the model, developed originally in \cite{18}, in a form more suitable for $\delta f$ flux tube simulation. In section 2, we give the complete model, and in section 3, we discuss its implications for intrinsic rotation. Appendix A contains details of the transformation from the equations in \cite{18} to the formulation in this paper. In appendix B, we discuss different forms of deriving the radial flux of toroidal angular momentum, showing that the final form presented here is convenient and has many advantages.

2. Transport of toroidal angular momentum
The derivation of the transport of toroidal angular momentum in the low-flow regime, including both turbulence and neoclassical effects, is described in detail in \cite{18}. To simplify the derivation, the extra expansion parameter $B_\psi/B = \varepsilon/q \ll 1$ was employed, assuming that the turbulence length scales and amplitudes do not depend strongly on $B_\psi/B$. Here $B$ is the total magnetic field and $B_\psi$ its poloidal component, $\varepsilon = r/R$ is the inverse aspect ratio of the flux surface, $q$ is the safety factor, $r$ is the minor radius of the flux surface and $R$ is the major radius. The ratio $B_\psi/B$ is below or around 0.1 across the core in most tokamaks ($\varepsilon$ is small near the magnetic axis and $q$ is large near the edge). In this section, we review the results of \cite{18} and we recast them in a more convenient form.

We assume that the turbulence is electrostatic and that the magnetic field is axisymmetric, i.e. $B = I\nabla\psi + \nabla\zeta \times \nabla\psi$, where $\psi$ is the poloidal magnetic flux, $\zeta$ is the toroidal angle, and we use a poloidal angle $\theta$ as our third spatial coordinate. With an axisymmetric magnetic field, in steady state and in the absence of momentum input, the equation that
determines the rotation profile is \( \langle R\ddot{\zeta}, \dot{P}_i, -\nabla\psi \rangle_T = 0 \), where \( \dot{P}_i = \int d^3v f_i M_v v \) is the ion stress tensor, \( M \) is the ion mass, \( \ddot{\zeta} \) is the unit vector in the toroidal direction, \( \langle \cdot \rangle_T = (v')^{-1} \int dv' d\zeta' (\cdot)/(B \cdot \nabla\psi) \) is the flux surface average, \( v' = dV/d\psi = \int dv' d\zeta' (B \cdot \nabla\psi)^{-1} \) is the derivative of the volume with respect to \( \psi \), and \( \langle \cdot \rangle_T \) is the coarse grain or ‘transport’ average over the time and length scales of the turbulence, assumed much shorter than the transport time scale \( \delta t a / v_a \) and the minor radius \( a \). Here \( \delta t = \rho_i / a \ll 1 \) is the ion gyroradius \( \rho_i \) over the minor radius of the tokamak \( a \) and \( v_a \) is the ion thermal speed. Note that we use the prime in \( v' \) to indicate that the velocity is measured in the laboratory frame. Later we will find the equations in a convenient rotating frame where the velocity is \( v = v' - R\Omega_1\ddot{\zeta} \).

In [18] we derived a method to calculate \( \langle R\ddot{\zeta}, P_i, -\nabla\psi \rangle_T \) to order \( (B/B_i)\delta^i_2 \rho_i R \nabla|\psi| \), with \( \rho_i \) the ion pressure. We present the method again in a different form to make it easier to compare with previous works in the high-flow regime [19, 20]. In section 2.1, we explain how we split the distribution function and the electrostatic potential into different pieces, and we present the equations to self-consistently obtain them. In section 2.2, we evaluate \( \langle R\ddot{\zeta}, P_i, -\nabla\psi \rangle_T \) employing the pieces of the distribution function and the potential obtained in section 2.1. Before presenting all of the results, we emphasize that our results and order of magnitude estimates are valid for \( \delta t \ll B_i / B \ll 1 \), assuming that the turbulence does not scale strongly with \( B_i / B \), and for collisionality in the range \( \delta t^2 \ll q R v_a / v_{\perp} \lesssim 1 \) [18], where \( v_{\perp} \) is the ion–ion collision frequency.

2.1. Distribution function and electrostatic potential

The electrostatic potential is composed to the order of interest by the pieces in table 1 [18]. The axisymmetric long wavelength pieces \( \phi_0(\psi, t), \phi^e_0(\psi, \theta, t) \) and \( \phi^e_2(\psi, \theta, t) \) are the zeroth, first and second order equilibrium pieces of the potential. The lowest order component \( \phi_0 \) is a flux surface function. The corrections \( \phi^{nc}_e \) and \( \phi^{nc}_2 \) give the electric field parallel to the flux surface, established to force quasi-neutrality at long wavelengths (the superscript \( \text{nc} \) refers to neoclassical because these are long wavelength contributions; however, turbulence can affect the final value of \( \phi^{nc} \)). We do not calculate \( \phi^{nc}_2 \) because it will not appear in the final expression for \( \langle R\ddot{\zeta}, \dot{P}_i, -\nabla\psi \rangle_T \). The piece \( \phi^{nc}(r, t) \) is turbulent and includes both axisymmetric components (zonal flow) and non-axisymmetric fluctuations.

| Potential | Size | Length scales | Time scales |
|-----------|------|---------------|-------------|
| \( \phi_0(\psi, t) \) | \( T_e / e \) | \( k_a \approx 1 \) | \( \omega / \delta t - \omega / v_a / a \) |
| \( \phi^e_0(\psi, \theta, t) \) | \( (B/B_i) T_e / e \) | \( k_a \approx 1 \) | \( \omega / \delta t - \omega / v_a / a \) |
| \( \phi^e_2(\psi, \theta, t) \) | \( (B/B_i) T_e^2 / e \) | \( k_a \approx 1 \) | \( \omega / \delta t - \omega / v_a / a \) |
| \( \phi^{\text{nc}}(r, t) \) | \( k_i / R \approx 1 \) | | | |

We use a new kinetic energy \( E \) obtained to order \( (B/B_i) \delta^2 \), and in the gyroaverage holding \( R, v, \mu \) and \( v_\perp \) fixed. When the ion distribution function is written as a function of these gyrokinetic variables, it does not depend on the gyrophase \( \psi \) up to order \( (B/B_i) \delta^2 (v R v_a / v_{\perp}) f_{\text{MAD}} \). In [18, 21], where \( f_{\text{MAD}} \) is the lowest order distribution function that is a Maxwellian. For the magnetic field, the gyrophase, only the first order corrections \( \mu_1 \) and \( \varphi_1 \) are needed because the lowest order distribution function \( f_{\text{MAD}} \) does not depend on \( \mu \) or \( \varphi \). Moreover, in [18] we expand for \( 1 \gg B_i / B \gg \delta t \), and the distribution function need only be known to order \( (B/B_i) \delta^2 f_{\text{MAD}} \). Consequently, the piece of the distribution function that depends on the gyrophase, of order \( (B/B_i) \delta^2 (q R v_a / v_{\perp}) f_{\text{MAD}} \), is negligible, and the gyrokinetic variables \( R \) and \( E \) only need to be obtained to order \( (B/B_i) \delta^2 \) and \( (B/B_i) \delta^2 v^2 / e \), respectively, implying that the corrections \( R_2 \) and \( E_2 \) are not needed for the final result. To change to the new reference frame, where the velocity is \( v = v' - R\Omega_1\ddot{\zeta} \), the distribution function that is independent of \( \psi \) has to be written as a function of the new gyrokinetic variables \( R, \varepsilon \) and \( \mu, v_\perp \) where \( \varepsilon \) is a new variable that will be defined shortly. Note that the gyrocentre position and the magnetic moment are the same in both reference frames to the order of interest. In the case of \( \mu \), the reason is that \( \mu \) is obtained such that its time derivative vanishes, \( d\mu / dt = 0 \), making its definition unique. For \( R, \varepsilon \), the reason is that the toroidal rotation has two components, one parallel to the magnetic field, \( \Omega_0 \ddot{\zeta} = \Omega_0 \ddot{\zeta} / (B/B_i) \delta^t \), and the other perpendicular, \( \Omega_0 \ddot{\zeta} / (B/B_i) \delta^t \). \( \delta^t \) enters to order \( \delta t a \), we can safely neglect the corrections due to the change of reference frame because they are of order \( \delta t a \).

In contrast, the kinetic energy \( E \), as defined in [18], cannot be used in the rotating frame because it includes the parallel velocity \( v^p_i = v_i + I\Omega_1 / \Omega_2 \). We use a new kinetic energy
variable $\varepsilon$ that is related to the old kinetic energy variable by $\varepsilon = E - f_{\Omega}^2 u/B$, where $u' = \pm \sqrt{2(E - \mu B)}$ is the gyrokinetic parallel velocity in the laboratory frame. It is easy to check that $u = \pm \sqrt{2\varepsilon - \mu B + (1/B^2)\Omega_2^2/2}$ is equal to $u = u' - 1/\Omega_2/B$ and it is the gyrokinetic parallel velocity in the rotating frame. With this relation, we find that another way to interpret the new energy variable

$$\varepsilon = \frac{u^2}{2} + \mu B - \frac{R^2\Omega_2^2}{2}$$

is realizing that it is the kinetic energy in the rotating frame plus the potential due to the centrifugal force. To write expression (1) we have used that $I/B = R + O(\rho_i/B^2 R)$ for $\rho_i/B < 1$. In appendix A, we rewrite the results in [18] using the new gyrokinetic kinetic energy $\varepsilon$.

The different pieces of the ion distribution function are given in table 2 [18]. The functions $f_{\Omega}^\text{Me}$, $H_{\Omega}^\text{nc}$, $H_{\Omega}^\text{ne}$ and $f_i^\text{b}$ are axisymmetric long wavelength contributions. The Maxwellian $f_{\Omega}^\text{Me}(\psi(R), \varepsilon) = n_i(\psi(R))\left[\frac{M}{2\pi T_i(\psi(R))}\right]^{3/2} \times \exp\left(\frac{-M\varepsilon}{T_i(\psi(R))}\right)$ (2) is uniform on a flux surface. The first and second order corrections $H_{\Omega}^\text{nc}$ and $H_{\Omega}^\text{ne}$ are neoclassical corrections, and they are not the functions $f_i^\text{b}$ and $f_i^\text{nc}$ in [18] because we are now working in the rotating frame. The function $H_{\Omega}^\text{ne}$ is an axisymmetric piece of the distribution function that originates from collisions acting on the ions transported by turbulent fluctuations into the volume between two adjacent flux surfaces [18]. The function $f_i^\text{b}$ is the turbulent contribution. It will be determined self-consistently up to order $(B/B_p)^2f_{\Omega}^\text{Me}$, i.e. $f_i^\text{b} = f_i^\text{b,nc} + f_i^\text{b,ne}$ with $f_i^\text{b,nc} \sim \delta f_{\Omega}^\text{Me}$ and $f_i^\text{b,ne} \sim (B/B_p)^2f_{\Omega}^\text{Me}$. It is convenient to combine both pieces of the turbulent distribution function into one function $f_i^\text{b}$.

The electron distribution function is very similar to the ion distribution function. It will have its own gyrokinetic variables that can be easily deduced from the ion counterparts. To the order of interest in this calculation, the electron distribution function is determined by the pieces in table 3. The long wavelength, axisymmetric pieces $f_{\Omega}^\text{Me}$ and $H_{\Omega}^\text{ne}$, are the lowest order Maxwellian and the first order neoclassical correction. The second order long wavelength neoclassical correction is not needed for transport of momentum because of the small electron mass. The piece $f_i^\text{b}$ is the short wavelength, turbulent component that will be self-consistently calculated to order $(B/B_p)^2f_{\Omega}^\text{Me}$.

We now proceed to describe how to find the different pieces of the distribution function and the potential. We use the equations in [18] but we change to the new gyrokinetic kinetic energy $\varepsilon$. The details of this transformation are contained in appendix A.

2.1.1. First order neoclassical distribution function and potential. The equation for $H_{\Omega}^\text{nc}$ is

$$u\hat{b} \cdot \nabla \hat{R}\left\{ H_{\Omega}^\text{nc} = \frac{Ze\varepsilon}{T_e}f_{\Omega}^\text{Me} + \frac{1}{p_i} \frac{\partial p_i}{\partial \psi} + \frac{M_e}{T_e} \frac{5}{2} \frac{1}{\Omega_1} \frac{\partial T_i}{\partial \psi} + \frac{1}{\Omega_1} \frac{\partial \varepsilon}{\partial \psi} \right\}$$

where $\varepsilon = \pm \sqrt{2(\varepsilon - \mu B)}$ is the gyrokinetic parallel velocity and $C^{(i)}$ is the linearized ion–ion collision operator. The correction $H_{\Omega}^\text{nc}$ gives the parallel component of the velocity [22, 23] $n_iW_{\Omega}^\text{nc} = b \cdot d^3v H_{\Omega}^\text{nc} = -\langle c\hat{b} \hat{e} n e \rangle \hat{b} \cdot \hat{e} \hat{e} d^3v$, where $k$ is a flux function that depends on the collisionality and the magnetic geometry.

The equation for $H_{\Omega}^\text{ne}$ is similar to (3) and is given by

$$u\hat{b} \cdot \nabla \hat{R}\left\{ H_{\Omega}^\text{nc} = \frac{e\varepsilon}{T_e}f_{\Omega}^\text{Me} - \frac{1}{p_i} \frac{\partial p_i}{\partial \psi} + \frac{M_e}{T_e} \frac{5}{2} \frac{1}{\Omega_1} \frac{\partial T_i}{\partial \psi} \right\}$$

where $m$ and $\Omega_e = eB/mc$ are the electron mass and gyrofrequency, $E^A$ is the electric field driven by the transformer, $C^{(i)}_{\Omega e}$ is the linearized electron–electron collision operator and $C^{(i)}_{\Omega i}$ is the linearized ion–ion collision operator. The lowest order solution for $H_{\Omega}^\text{nc}$ is the Maxwell–Boltzmann response $(e\varepsilon/T_e) f_{\Omega}^\text{Me} \sim (B/B_p)\delta f_{\Omega}^\text{Me}$. The rest of the terms are small because they are of order $(B/B_p)^2\delta f_{\Omega}^\text{Me} \sim (B/B_p)\sqrt{m/\delta e} f_{\Omega}^\text{Me} \ll (B/B_p)\delta f_{\Omega}^\text{Me}$, where $\delta e = \mu_e/\alpha$ is the ratio between the electron gyroradius $\rho_e$ and the minor radius $a$.

Finally the poloidal variation of the potential is determined by quasi-neutrality.

$$Z \int d^3v H_{\Omega}^\text{ne} = \frac{e\varepsilon}{T_e}$$

giving $e\varepsilon/T_e \sim (B/B_p)\delta_i$.  

2.1.2. Turbulent distribution function and potential. Using the turbulent piece of the ion distribution function is determined by the relation

$$\delta f_{\Omega}^\text{Me} + \delta f_{\Omega}^\text{ne}$$

Table 2. Pieces of the ion distribution function: $f_i = f_{\Omega}^\text{Me} + H_{\Omega}^\text{nc} + H_{\Omega}^\text{ne} + f_i^\text{b}$.  

| Distribution function | Size | Length scales | Time scales |
|-----------------------|------|---------------|-------------|
| $f_{\Omega}^\text{Me}(\psi(R), \varepsilon)$ | $f_{\Omega}^\text{Me}$ | $ka \sim 1$ | $\partial/\partial t \sim \delta^2 v_i/a$ |
| $H_{\Omega}^\text{nc}(\psi(R), \theta(R), \varepsilon)$ | $(B/B_p)\delta f_{\Omega}^\text{Me}$ | $ka \sim 1$ | $\partial/\partial t \sim \delta^2 v_i/a$ |
| $H_{\Omega}^\text{ne}(\psi(R), \theta(R), \varepsilon)$ | $(B/B_p)^2\delta f_{\Omega}^\text{Me}$ | $ka \sim 1$ | $\partial/\partial t \sim \delta^2 v_i/a$ |
| $f_i^\text{b}(R, \varepsilon)$ | $f_i^\text{b,nc}$ | $k_e \rho_i \sim 1$ | $\partial/\partial t \sim \delta v_i/a$ |

The function $f_i^\text{b}$ is the turbulent contribution. It will be determined self-consistently up to order $(B/B_p)^2f_{\Omega}^\text{Me}$, i.e. $f_i^\text{b} = f_i^\text{b,nc} + f_i^\text{b,ne}$ with $f_i^\text{b,nc} \sim \delta f_{\Omega}^\text{Me}$ and $f_i^\text{b,ne} \sim (B/B_p)^2f_{\Omega}^\text{Me}$. It is convenient to combine both pieces of the turbulent distribution function into one function $f_i^\text{b}$. The electron distribution function is very similar to the ion distribution function. It will have its own gyrokinetic variables that can be easily deduced from the ion counterparts. To the order of interest in this calculation, the electron distribution function is determined by the pieces in table 3. The long wavelength, axisymmetric pieces $f_{\Omega}^\text{Me}$ and $H_{\Omega}^\text{ne}$, are the lowest order Maxwellian and the first order neoclassical correction. The second order long wavelength neoclassical correction is not needed for transport of momentum because of the small electron mass. The piece $f_i^\text{b}$ is the short wavelength, turbulent component that will be self-consistently calculated to order $(B/B_p)^2f_{\Omega}^\text{Me}$.
the gyrokinetic equation (see appendix A)

\[
\frac{D f^{ib}}{D t} + (\hat{u} + v_M + v_C + v_{B1} + \hat{v}_b) \cdot \nabla f^{ib} - (C^{ib\mu}(h^{ib}))_\mu = \frac{1}{\eta_c} \left[ \frac{M e}{T_i} - \frac{3}{2} \right] \frac{1}{T_i} \frac{\partial T_i}{\partial \psi} \\
+ \frac{M I_T \delta \Omega_b}{B T_i} f_M - v^{ib} \cdot \nabla H^{nc}_{\mu i} \\
- \frac{Z e f_M}{T_i} (\hat{u} + v_M + v_C) \cdot \nabla R (\phi^{ib})_i \\
+ \frac{Z e \partial H^{nc}_R}{T_i} (\hat{u} + v_M + v_C) \cdot \nabla R (\phi^{ib})_i \\
+ \frac{Z e \partial H^{nc}_R}{T_i} \hat{u} \cdot \nabla R (\phi^{ib})_i,
\]

where \( \frac{D}{D t} \) is the time derivative in the rotating frame, \( u = \pm \sqrt{2\epsilon - \mu B + R^2 \Omega_b^2/2} \) is the parallel velocity in the rotating frame, \( v_M = \mu \Lambda_i \hat{b} \times \nabla R B + (u' \Omega_b) \hat{b} \times (\hat{b} \cdot \nabla R \hat{b}) \) are the \( \nabla B \) and curvature drifts, \( v_C = (2u \Omega_b / \Omega) \hat{b} \times (\nabla R \times \hat{b}) \times \hat{b} \) is the Coriolis drift, \( v_{B1} = -(c/B) \nabla R \phi^{nc}_R \times \hat{b} \) and \( v^{ib} = -(c/B) \nabla R (\phi^{ib})_i \times \hat{b} \) are the neoclassical and turbulent \( E \times B \) drifts, \( C^{ib\mu}(h^{ib})_\mu \) is the linearized ion–ion collision operator, and \( \{\cdot\}_i = (2\pi)^{-1} \int d\psi \langle \cdots \rangle_{R,\epsilon,\mu,t} \) is the gyroaveraging operation on the ion gyrokinetic variables \( R = r + \Omega^{-1}_b v \times \hat{b} + \cdots, \epsilon, \mu \) and \( t \) fixed. The function that enters in the collision operator is

\[
h^{ib}_\epsilon = f^{eb}_\epsilon + \frac{Z e (\phi^{ib} - (\phi^{ib})_\epsilon)}{m} \times \left( -\frac{M f_{M0}}{T_e} \frac{\partial H^{nc}_{0i}}{\partial \psi} + \frac{\partial H^{nc}_{1i}}{B} \frac{\partial \mu_0}{\partial \psi} \right).
\]

Here the subscript \( g \) in \( f^{ib}_g = f^{ib}(R_g, v^2/2, v^2/2B, t) \) indicates that we have replaced the variables \( R, \epsilon, \mu \), and \( v \) by \( R_g = r + \Omega^{-1}_b v \times \hat{b}, e_0 = v^2/2 \) and \( \mu_0 = v^2/2B \); similarly, the subscript \( 0 \) in \( f_{M0} = f_{M0}(\psi(r), v^2/2, t) \) and \( H^{nc}_{0i} = H^{nc}_i(\psi(r), \theta(r), v^2/2, v^2/2B, t) \) indicates that we have replaced the variables \( R, \epsilon, \mu \), and \( v \) by \( R_0, \epsilon_0, \mu_0, v^2/2 \) and \( v^2/2B \).

The equation for electrons is of the same form as the one for the ions, giving

\[
\frac{D f^{eb}}{D t} + (\hat{u} + v_M + v_C + v_{B1} + \hat{v}_b) \cdot \nabla f^{eb} - (C^{eb\mu}(h^{eb}))_\mu = \frac{1}{\eta_c} \left[ \frac{M e}{T_i} - \frac{3}{2} \right] \frac{1}{T_i} \frac{\partial T_i}{\partial \psi} \\
+ \frac{M I_T \delta \Omega_b}{B T_i} f_M - v^{eb} \cdot \nabla H^{nc}_{\mu i} \\
- \frac{Z e f_M}{T_i} (\hat{u} + v_M + v_C) \cdot \nabla R (\phi^{eb})_i \\
+ \frac{Z e \partial H^{nc}_R}{T_i} (\hat{u} + v_M + v_C) \cdot \nabla R (\phi^{eb})_i \\
+ \frac{Z e \partial H^{nc}_R}{T_i} \hat{u} \cdot \nabla R (\phi^{eb})_i,
\]

where \( v_{M} = -\mu \Omega / \Omega_b \hat{b} \times \nabla R B - u' / \Omega_b \hat{b} \times (\hat{b} \cdot \nabla R \hat{b}) \) are the \( \nabla B \) and curvature drifts for electrons, \( v^{eb}_i = -(c/B) \nabla R (\phi^{eb})_i \times \hat{b} \) is the turbulent \( E \times B \) drift, \( C^{eb\mu}(h^{eb})_\mu \) is the linearized electron–electron collision operator, \( C^{eb\mu}(h^{eb})_\mu \) is the linearized electron–ion collision operator, and \( \{\cdot\}_i = (2\pi)^{-1} \int d\psi \langle \cdots \rangle_{R,\epsilon,\mu,t} \) is the gyroaveraging operation holding the electron gyrokinetic variables \( R = r - \Omega^{-1}_b v \times \hat{b} + \cdots, \epsilon, \mu \) and \( t \) fixed. The electron distribution function that enters in the collision operator is

\[
h^{eb}_\epsilon = f^{eb}_\epsilon + \frac{e (\phi^{eb} - (\phi^{eb})_\epsilon)}{m} \times \left( -\frac{m f_{M0}}{T_e} \frac{\partial H^{nc}_{0i}}{\partial \psi} + \frac{\partial H^{nc}_{1i}}{B} \frac{\partial \mu_0}{\partial \psi} \right)
\]

where \( \phi^{eb} \) is the electron distribution function.
2.2. Calculation of the momentum transport

The radial transport of toroidal angular momentum \( \langle \hat{R} \hat{\zeta} \cdot \vec{P}_i \cdot \nabla \hat{\psi} \rangle_T \) is given in equation (39) of [18] that we reproduce here for convenience

\[
\langle \hat{R} \hat{\zeta} \cdot \vec{P}_i \cdot \nabla \hat{\psi} \rangle_T = M_c \left\langle \frac{\partial \phi}{\partial \zeta} \int \, d^3v \, \hat{f}_i \left( \nu \cdot \hat{\zeta} \right) \right\rangle_T,
\]

\[
+ \frac{M_c \langle R^2 \rangle}{2 \pi e} \frac{\partial p_i}{\partial \zeta} \int \, d^3v \, f_i \left( \nu \cdot \hat{\zeta} \right) \right\rangle_T,
\]

\[
+ \frac{M^2 c^2}{2 \pi e} \int \, d^3v \, \left\langle \hat{C}_i \left( \nu_1, \nu \right) \right\rangle_T \, R \left( \nu_1 \cdot \nu \right) \left( \nu_2 \cdot \nu \right) \right\rangle_T,
\]

\[
+ \frac{M^2 c^2}{6 \pi e^2} \int \, d^3v \, \left\langle \hat{C}_i \left( \nu_1, \nu \right) \right\rangle_T \, R \left( \nu_1 \cdot \nu \right) \left( \nu_2 \cdot \nu \right) \right\rangle_T.
\]

This expression is derived in [18]. In appendix B, we present an alternative proof that makes clear the convenience of using form (14).

Using that for \( B/B_0 \gg 1 \), \( R \nu \cdot \hat{\zeta} \simeq 1 v_{||}/B \), and employing the decomposition of the ion distribution function in section 2.1, we find that we can write (14) as

\[
\langle \hat{R} \hat{\zeta} \cdot \vec{P}_i \cdot \nabla \hat{\psi} \rangle_T = \Pi^{b_1} + \Pi^{b_0} + \Pi^{n_{c1}} + \Pi^{0},
\]

with

\[
\Pi^{b_1} = M_c \left\langle \frac{\partial \phi}{\partial \zeta} \int \, d^3v \, f_i \left( \nu_1 \cdot R^2 \Omega_1 \right) \right\rangle_T,
\]

\[
\Pi^{b_0} = \frac{M^2 c^2}{2 \pi e} \int \, d^3v \, V \left\langle \frac{\partial \phi}{\partial \zeta} \int \, d^3v \, f_i \left( \nu_1 \cdot R^2 \left| \hat{C} \right| \right) \right\rangle_T,
\]

\[
- \frac{M^2 c^2}{6 \pi e^2} \int \, d^3v \, \left\langle \hat{C}_i \left( \nu_1, \nu \right) \right\rangle_T \, R \left( \nu_1 \cdot \nu \right) \left( \nu_2 \cdot \nu \right) \right\rangle_T.
\]

3. Discussion

We finish by showing how this new formalism gives a plausible model for intrinsic rotation. Until now, models have only considered the contribution \( \Pi^{b_1} \), with \( f^b_0 \) and \( \phi^b \) obtained by employing equations (6) and (10) without the terms that contain \( H^a_\parallel \) and \( \phi^a \). This is acceptable for \( R^2 \Omega \gg \left( B/B_0 \right)^2 \nu_0 \), where \( v_{||}/v_0 \) is the turbulent pinch of momentum and \( \phi^b \) is the turbulent pinch of potential. To obtain this last expression we have linearized around \( a \Omega = \Omega = 0 \) and \( v_{||}/v_0 \ll 1 \). Here \( \nu_0 \) is the turbulent diffusivity, \( \Gamma \) is the turbulent pinch of momentum and \( \phi^b \) is the turbulent pinch of potential. This is acceptable for \( R^2 \Omega \gg \left( B/B_0 \right)^2 \nu_0 \), where \( v_{||}/v_0 \ll 1 \). Here \( \phi^b \) is the turbulent pinch of potential and \( \phi^b \) is the turbulent pinch of potential. This is acceptable for \( R^2 \Omega \gg \left( B/B_0 \right)^2 \nu_0 \), where \( v_{||}/v_0 \ll 1 \). Here \( \phi^b \) is the turbulent pinch of potential and \( \phi^b \) is the turbulent pinch of potential. To obtain this expression we have linearized around \( a \Omega = \Omega = 0 \) and \( v_{||}/v_0 \ll 1 \). Here \( \phi^b \) is the turbulent pinch of potential and \( \phi^b \) is the turbulent pinch of potential.
The equations. In addition, there are the new terms $\Pi^{ib}_1$, $\Pi^{ic}_1$ and $\Pi^{ib}_0$. As we did for $\Pi^{ib}_1$, we can linearize $\Pi^{ic}_1(\delta \Omega_i)$ around $\delta \Omega_i = 0$ to find $\Pi^{ic}_1 \simeq -\nabla \cdot \delta \Omega_i + \Pi^{ic}_1$, where $\Pi^{ic}_1 \sim \Delta_{ud}(B/B_i)(q R v_i/v_o) \delta^2 p_i R |\nabla \psi|$ and $\Pi^{ic}_1 \sim (B/B_i)^2(q R v_i/v_o) \delta^2 p_i R |\nabla \psi|$. Combining all of these results and imposing that $(R \tilde{C}_1 \nabla \psi)_{TT} = 0$, we obtain

$$
\Omega_i = -\int_{\psi}^{\psi_0} d\psi' \Pi^{iib}_1 |_{i=\phi, \psi} \nabla \cdot \frac{v}{v_o + v_{nc}} |_{\psi = \psi'} \times \exp \left( \int_{\psi}^{\psi_0} d\psi'' \frac{\Gamma^{ib}}{v_o + v_{nc}} |_{\psi = \psi''} \right) + \Omega_i |_{\psi = \psi_0} \exp \left( \int_{\psi}^{\psi_0} d\psi'' \frac{\Gamma^{ib}}{v_o + v_{nc}} |_{\psi = \psi''} \right),
$$

where $\psi_0$ is the poloidal flux at the edge. $\Omega_i |_{\psi = \psi_0}$ is the rotation velocity in the edge and $\Pi^{iib}_1 = \Pi^{ib}_1 + \Pi^{iic}_1 = \Pi^{ib}_0 + \Pi^{ic}_1 \sim \Pi^{ic}_1$. Note that this equation gives a rotation profile that depends on $\Pi^{ic}_1$ in that turn depends on the gradients of temperature and density, and the magnetic field profile. The typical size of the rotation is $\Omega_i \sim (B/B_i) \delta v_i / R$ for $\Delta_{ud} \lesssim (B/B_i) \delta$ and $\Omega_i \sim \Delta_{ud} v_i / R$ for $\Delta_{ud} \gtrsim (B/B_i) \delta$.

This new model for intrinsic rotation has been constructed such that the pinch and the up-down asymmetry drive, discovered in the high-flow ordering, are naturally included. By transforming to the frame rotating with $\Omega_i$, we have made this property explicit.

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**Appendix A. Equations in the rotating frame**

In this appendix, we derive equations (3), (6) and (11) for the different pieces of the electron distribution function, equations (4) and (8) for the different pieces of the electron distribution function, and equations (5) and (10) for the different pieces of the potential. These equations are valid in the frame rotating with angular velocity $\Omega_i$, and we deduce them from the results in [18], obtained in the laboratory frame.

In [18], we showed that in the time $B_i/B \ll 1$, assuming that the turbulence does not scale strongly with $B_i/B$, the ion distribution function is given by $f_i(R, \theta, \mu, r) = f_{mi}(\psi(R), \theta, \mu, r) + f^{inc}_i(\psi(R), \theta, \mu, r) + f^{reb}_i(\psi(R), \theta, \mu, r) + f^{reb}_i(\psi(R), \theta, \mu, r)$, where the size of these different pieces is $F^{iib}_{1i} \sim (B/B_i)^2 m_i$, $F^{iib}_{2i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{2i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$, $F^{iib}_{3i} \sim (B/B_i)^2 f_{mi}$. The equations for the different pieces were obtained from the gyrokinetic equation

$$
\frac{\partial f_i}{\partial t} + \mathbf{R} \cdot \nabla f_i + \frac{E}{\Omega_i} \frac{\partial f_i}{\partial E} = (C_i[f_i]),
$$

where the time derivative $\mathbf{R}$ is

$$
\dot{\mathbf{R}} = u \mathbf{b}(R) + v_M \times \frac{\nabla R \langle \phi \rangle_i \times \mathbf{b}}{B},
$$

and the time derivative $\dot{E}$ is

$$
\dot{E} = -\frac{Ze}{M} [u \mathbf{b}(R) + v_M \times \nabla R \langle \phi \rangle_i],
$$

Here, $u' = \pm \sqrt{2(E - \mu B)}$ is the gyrokinetic parallel velocity in the laboratory frame, and

$$
v_M = \frac{\mu}{\Omega_i} \mathbf{b}(R) + \frac{u'^2}{\Omega_i} \mathbf{b}(R) \times \mathbf{b} \times \mathbf{b}(R)
$$

are the $\nabla B$ and curvature drifts in the laboratory frame. Equations (19) and (20) of [18] for $\Pi^{ic}_0$ and equation (24) of [18] for $\Pi^{ic}_1$ are obtained from the long wavelength axisymmetric contributions to (A.1) of order $\delta_1 f_{mib}/a$ and $\delta_1 f_{mib}/a$, respectively. Equation (25) of [18] for $\Pi^{ic}_1$ is also a long wavelength axisymmetric component of (A.1). In particular, it is the contribution of order $\delta_1 f_{mib}/a$ that when the equation is orbit averaged does not vanish as $v_i \to 0$. Equation (55) of [18] for $f^{ic}_1$ is the sum of the short wavelength components of (A.1) of order $\delta_1 f_{mib}/a$ and $\delta_1 f_{mib}/a$.

In this paper, we write the formulation in [18] in the frame rotating with velocity $\Omega_i$, that is, we need to use the new gyrokinetic variable $\varepsilon = E - I \Omega_i u' / B$. Thus, the new gyrokinetic equation is

$$
\frac{\partial f_i}{\partial t} + \mathbf{R} \cdot \nabla f_i + i \frac{\partial f_i}{\partial \varepsilon} = (C_i[f_i]),
$$

The time derivative of the new gyrokinetic variable $\varepsilon$ is

$$
\dot{\varepsilon} = \dot{\mathbf{R}} \cdot \nabla R \mathbf{R} + \frac{\partial E}{\partial E}.
$$

In $\mathbf{R}$, using $u' = u + I \Omega_i / B$, with $u = \pm \sqrt{2(E - \mu B + R^2 \Omega_i^2 / 2)}$, leads to

$$
\dot{\mathbf{R}} = u \mathbf{b} + \frac{I \Omega_i}{B} \mathbf{b} + v_M + v_C - \frac{C}{B} \nabla \langle R \phi_i \rangle \times \mathbf{b} + O(B^2 / B_i)^2 \delta_{mib}/a.
$$

with

$$
v_M = \frac{\mu}{\Omega_i} \mathbf{b} \times \nabla R \mathbf{b} + \frac{u'^2}{\Omega_i} \mathbf{b} \times \mathbf{b} \times \mathbf{b}
$$

the $\nabla R$ and curvature drifts in the rotating frame, and $v_C = (2u \Omega_i / B \Omega_i) \mathbf{b} \times (\mathbf{b} \times \mathbf{b})$. The Coriolis drift. To obtain this expression for $\mathbf{R}$ we have used $(u'^2 = u'^2 + 2I \Omega_i u / B + O((B/B_i)^2 \delta_{mib}/a))$ to write $v_M = v_M + v_C + O((B^2 / B_i)^2 \delta_{mib}/a)$.

The usual result for the Coriolis drift $v_C = (2u \Omega_i / \Omega_i) \mathbf{b} \times \nabla \mathbf{R} \times \mathbf{b}$ can be recovered by realizing that for
where we have used $I/B = R + O([B^2_{0}/B^2] R)$ and $\hat{b} \cdot \nabla_R \hat{b} = -\nabla_R \ln R + O((B/B_R)^{-1})$; and

$$v'_M \cdot \nabla_R \left( \frac{1u'}{B} \right) = \frac{u'}{\Omega_e} \left[ \nabla_R \times (\nabla_R \times \hat{b}) \right] \sim \frac{B^2_{0}}{B^2} \delta \nu_i^2,$$

(A.15)

where we have used $\hat{b} \cdot \nabla_R \hat{b} \sim (B/B_R)^{-1} \hat{b} \cdot \nabla_R (Iu'/B) \sim (a/R)(r_{nu}/q_R) \sim (B/B_R)r_{nu}$ and $\nabla_R \times (\nabla_R \times \hat{b})$. To obtain (3), we take the long wavelength axisymmetric contribution to (A.5) to order $\delta f_{delta}/\nu_i$, giving

$$u_{\nu} \hat{b} \cdot \nabla_R H_{nu}^{d} + \nu_{M} \cdot \nabla_{R} f_{M} = \frac{Ze_{\delta} f_{M} \hat{b}}{\Omega_{M}} u_{\nu} \hat{b} \cdot \nabla_R H_{nu}^{d}$$

(A.16)

This equation differs from equations (19) and (20) of [18], and gives a function $H_{nu}^{d}$ different from the function $F_{nu}^{d}$ defined in [18]. The reason is that $F_{nu}^{d}(\psi(R), t, \zeta_B)$ and $H_{nu}^{d}$ must be equal to the function $f_{delta}(\psi(R), E) + F_{nu}^{d} + F_{nu}^{d}$ defined in [18] to the order of interest, but how the terms of first and second order in $\delta_i$ are assigned to one or the other piece differs depending on the frame. For this reason, we have changed the name of the functions. The final result in (3) is obtained from (A.16) by using $\nu_{M} \cdot \nabla_{R} \psi = u_{\nu} \hat{b} \cdot \nabla_{R} \psi (Iu'/\Omega_e) \sim \pm \sqrt{2}(\mu - \mu B)$.

Equation (6) is the sum of the short wavelength contributions to (A.5) of order $\delta f_{delta}/\nu_i$, and $(B/B_R)\delta_i^{2} f_{delta}/\nu_i$. The equation is straightforward if we apply the same methodology as in [18].

Equation (11) is found from the long wavelength axisymmetric components of (A.5) to order $\delta_i^{2} f_{delta}/\nu_i$. Note that to this order we have the time derivative $\delta_i f_{delta}$ [18]. Using $\delta_i f_{delta} = [n_i^{-1} \delta n_i + (M/\Omega_T - 3/2)T_i^{-1} \delta n_i]$ and realizing that $\delta n_i = (f d^3 v S^b_{M} \psi) + (f d^3 v S^d_{M} \psi)$, we find the final form in (11), (12), and (13). Here the integral $(f d^3 v S^d_{M} \psi)$ gives the divergence of the turbulent radial energy transport and the turbulent heating. Similarly, $(f d^3 v S^b_{M} \psi)$ gives the divergence of the neoclassical radial flux of energy. The equations for $H_{nu}^{b}$ and $H_{nu}^{b}$ are obtained in the same way as equations (24) and (25) in [18], i.e. the equation for $H_{nu}^{b}$ is the axisymmetric long wavelength component of order $(B/B_R)\delta_i^{2} f_{delta}/\nu_i$, and the equation for $H_{nu}^{b}$ is the axisymmetric long wavelength component of order $\delta_i^{2} f_{delta}/\nu_i$, which when it is orbit averaged does not vanish as $\nu_i \rightarrow 0$.

Equations (4) and (8) for the electron distribution function in the rotating frame are derived in the same way as the equations for the ion distribution function. The only differences are a subsidiary expansion in $\sqrt{m/M}$ (the Coriolis drift $v_c$ and the term in (A.13) that is proportional to $\nu_i \Omega_e$ are small by $\sqrt{m/M}$ and hence negligible), and that we include the electric field $E_d$ driven by the transformer, leading to a
modified time derivative for the energy
\[\dot{e} = -\frac{eF}{m} \cdot \mathbf{E} + \frac{e}{m} [\mathbf{u} \cdot \mathbf{b}(\mathbf{R}) + \mathbf{v}_M] \cdot (\nabla_R \phi^{\text{eq}} + \nabla_R (\phi^{\text{inh}})).\]
(A.17)

Finally, the equations for the different pieces of the electrostatic potential (5) and (10) are easily deduced from the results in [18] by realizing that moving to a rotating reference frame does not modify the quasineutrality equation.

**Appendix B. Derivation of equation (14)**

In this appendix, we derive equation (14) with a procedure different from the one employed in [18]. This new derivation shows the connection with calculations that split the off-diagonal components of the viscosity into gyroviscosity and perpendicular viscosity [25]. The derivation presented here and the one in [18] lead to identical results (as they should), but we believe that this new approach emphasizes the advantages of the formula in (14).

We begin by using \( \mathbf{R} \dot{\mathbf{C}} = \mathbf{b} \mathbf{I}/B - B^{-1} \mathbf{b} \cdot \nabla \psi \) to write

\[
\langle \langle \mathbf{R} \dot{\mathbf{C}} \cdot \hat{\mathbf{P}}_0 \cdot \nabla \psi \rangle \rangle_T = \left\langle \left\langle \frac{M}{B} \int d^3v_f v_{i'v_{i'}} \cdot \nabla \psi \right\rangle \right\rangle_T
\]

\[
\times \left\langle \langle [v_{i'} - (v' \times \mathbf{b}) \cdot \nabla \psi] \rangle \right\rangle_T
\]

\[
= \left\langle \left\langle \frac{M}{B} \int d^3v_f v_{i'v_{i'}} \cdot \nabla \psi \right\rangle \right\rangle_T
\]

\[
- \left\langle \left\langle \frac{M}{2B} \int d^3v_f \mathbf{v}_{i'} \cdot [v_{i'} v_{i'}]ight\rangle \right\rangle_T
\]

\[+(v' \times \mathbf{b})(v' \times \mathbf{b}) \cdot [(v' \times \nabla \psi) \langle v_{i'} - (v' \times \mathbf{b}) \cdot \nabla \psi] \rangle_T \right\rangle_T
\]

(B.1)

Since the transport of toroidal angular momentum needs to be known to order \( B/B_0 \delta v_i/\nabla \psi \), evaluating it directly from this equation would require knowing \( f_1 \) to order \( B/B_0 \delta v_i/\nabla \psi \), and it is easy to see from our decomposition of the ion distribution function given in section 2.1 that we cannot calculate the ion distribution function to that order. To circumvent this problem, we use exact moments of the Fokker–Planck equation to write the two integrals that appear in the transport of momentum as

\[M \int d^3v_f f_{i'v_{i'}}(v' \times \mathbf{b}) \]

\[+ \frac{M}{\Omega_i} \mathbf{b} \times \left\langle \left\langle \left( \int d^3v_f v'v'v' \right) \cdot \mathbf{b} \right\rangle \right\rangle_T
\]

\[+ \frac{M}{\Omega_i} \mathbf{b} \cdot \nabla \psi \int d^3v_f v'v'v' + (\nabla \phi \mathbf{b}) \int d^3v_f v'v_{i'}
\]

\[+ \frac{M}{\Omega_i} \int d^3v_C v_{i'}(v' \times \mathbf{b}),\]

(B.2)

where we have used the \( v_{i'}(v' \times \mathbf{b}) \) moment of the Fokker–Planck equation, and

\[
\frac{M}{2} \int d^3v_f \left\langle [v_{i'v_{i'}} - (v' \times \mathbf{b})v_{i'}] \right\rangle_T
\]

\[+ \frac{M}{4\Omega_i} \mathbf{b} \times \left\langle \left\langle \left( \int d^3v_f v'v'v' \right) \cdot (\mathbf{I} - b\mathbf{b}) \right\rangle \right\rangle_T
\]

\[= \frac{M}{4\Omega_i} \mathbf{b} \times \left\langle \left\langle \left( \int d^3v_f [v_{i'}(v' \times \mathbf{b}) + (v' \times \mathbf{b})v_{i'}] \right\rangle \right\rangle_T
\]

\[+ \frac{M}{4\Omega_i} \mathbf{b} \times \nabla \cdot \left\langle \left\langle \left( \int d^3v_f v'v'v' \right) \cdot (\mathbf{I} - b\mathbf{b}) \right\rangle \right\rangle_T
\]

We could use these two equations to evaluate \( \langle \langle \mathbf{R} \dot{\mathbf{C}} \cdot \hat{\mathbf{P}}_0 \cdot \nabla \psi \rangle \rangle_T \) instead of the form in (14). If we did so, we would still need to evaluate the first term on the right side of (B.4) and the first and second terms on the right side of (B.5) to order \( \delta v_i/\Delta \mathbf{c}_0 \mathbf{P}_0 R/\nabla \psi \).
for an up–down asymmetric tokamak with $\Delta_{ad} \gg (B/B_p)\delta_i$, and to order $(B/B_p)\delta_i \nu_i R |\nabla \psi|$, for an up–down symmetric tokamak with $\Delta_{ad} \leq (B/B_p)\delta_i$. In the up–down asymmetric case, the dominant contribution to the first term on the right side of (B.4) and (B.5) is due to $H_{nc}^{\psi}$:

$$-\left(\frac{M_1}{B_{\Omega_1}}(\hat{b} \times \nabla \psi) \cdot \left[ \nabla \cdot \left( \int d^3v \\frac{H_{nc}^{\psi}}{H_{nc}^{\psi,v}(v_2 - v_1)} \right) \right] \hat{b} \right)_\psi$$

$$\approx \left(\frac{M_1}{2B_{\Omega_1}} \hat{b} \cdot \nabla \hat{b} \cdot (\hat{b} \times \nabla \psi) \int d^3v \frac{H_{nc}^{\psi}}{H_{nc}^{\psi,v}(v_2 - v_1)} \right)$$

$$= \left(\frac{M_1^2}{2Ze} \int d^3v \frac{H_{nc}^{\psi}}{H_{nc}^{\psi,v}(v_2 - v_1)} \right)$$

$$= \frac{M_1^2}{2Ze} \int d^3v \frac{H_{nc}^{\psi}}{H_{nc}^{\psi,v}(v_2 - v_1)}$$

(B.5)

This term is of order $(B/B_p)\Delta_{ad} \delta_i^2 \nu_i R |\nabla \psi|$, suggesting that the transport due to the neoclassical piece $H_{nc}^{\psi}$ does not scale with collisionality and that it is larger than the turbulent piece $\Pi_{11}^{\psi}$ by a factor of $(B/B_p) \gg 1$ (see table 4). In fact, this contribution cancels to this order with other terms in (B.4) and (B.5), as we will show in equations (B.7), (B.8), (B.9), (B.10) and (B.11). The final result of the cancellation is that the neoclassical pieces of the distribution function give a contribution of order $(B/B_p)\Delta_{ad} \delta_i^2 \nu_i R |\nabla \psi|$ for $\Delta_{ad} \gg (B/B_p)\delta_i$.

In the up–down symmetric case, similar problems appear. Obtaining the transport of toroidal angular momentum to order $(B/B_p)\delta_i^2 \nu_i R |\nabla \psi|$ requires calculating the long wavelength piece of the ion distribution function $(f_i)_T$ to order $(B/B_p)\delta_i^2 f_{Mi}$ for the first term on the right side of (B.4) and to order $(B/B_p)^2 \delta_i^2 f_{Mi}$ for the first and second terms on the right side of (B.4). In section 2.1, we show how to calculate $H_{nc}^{\psi} \sim (B/B_p)\delta_i^2 f_{Mi}$ and $H_{nc}^{\psi} \sim (B/B_p)f_{Mi}(v_1/q R_v) \delta_i^2 f_{Mi}$, so in principle, it is possible to evaluate these terms to the appropriate order. Note, however, that equations (B.4) and (B.5) suggest misleading scalings for the transport of momentum. For example, using $H_{nc}^{\psi} \sim (B/B_p)^3 \delta_i^2 f_{Mi}$ in the first and second terms on the right side of (B.5), we obtain that a purely neoclassical piece can give momentum transport of order $(B/B_p)^3 \nu_i R |\nabla \psi|$, that is, transport that does not scale with collisionality. Similarly, integrating over $H_{nc}^{\psi} \sim (B/B_p)(v_1/q R_v) \delta_i^2 f_{Mi}$ in the first term on the right side of (B.4) gives momentum transport of order $(B/B_p)(v_1/q R_v) \delta_i^2 \nu_i R |\nabla \psi|$, i.e. it scales inversely with collisionality. In fact, these contributions to the transport of momentum vanish by themselves or when combined with other terms in (B.4) and (B.5), as we will show in (B.7), (B.8), (B.9), (B.10) and (B.11). In equation (14), these cancellations have already been taken into account. There are advantages to this. For example, if we decide to use (B.4) and (B.5) instead of (14), $H_{nc}^{\psi} \sim (B/B_p)((v_1/q R_v) \delta_i^2 f_{Mi}$ must be calculated to order $(B/B_p)\delta_i^2 f_{Mi} < (B/B_p)(v_1/q R_v) \delta_i^2 f_{Mi}$, that is, to more precision than is necessary. Equation (14), on the other hand, makes explicit that the first term in the right side of (B.4) vanishes when the lowest order piece of $H_{nc}^{\psi} \sim (B/B_p)(v_1/q R_v) \delta_i^2 f_{Mi}$ is integrated over and combined with other terms in (B.4) and (B.5).
Furthermore, we will use the \( (M_1^2/2\Omega_1)(v_i')^2 \) and \( (M/\nabla \psi^2/4B\Omega_1)(v_i')^2 \) moments of the Fokker–Planck equation to remove most of the \( \int d^4v' \langle f_i \rangle_T v^\perp v' v' \) moments,

\[
- \left\{ \frac{M_1^2}{2\Omega_1} \hat{b} \cdot \nabla \cdot \left( \int d^4v' \langle f_i \rangle_T v^\perp v' v' \right) \right\}_\psi \\
- \left\{ \frac{M c^2}{2B} \hat{b} \cdot \nabla \phi \int d^4v' f_i v_i' \right\}_T \\
+ \frac{M_1^2}{2\Omega_1} \int d^4v' \langle C_{i}(f_i) \rangle_T (v_i')^2 \\
= \frac{\partial}{\partial t} \left\{ \frac{M_1^2}{2\Omega_1} \int d^4v' \langle f_i \rangle_T (v_i')^2 \right\}_\psi = \left\{ \frac{I^2}{2B\Omega_1} \right\}_\psi \frac{\partial p_i}{\partial t}.
\]

and

\[
- \left\{ \frac{M}{4\Omega_1} \nabla \psi^2 \left( I - \hat{b} \hat{b} \right) \right\}_\psi \\
- \left\{ \frac{M c}{2B} \nabla \phi \int d^4v' f_i v_i' \nabla \phi \right\}_T \\
+ \frac{M}{4\Omega_1} \int d^4v' \langle C_{i}(f_i) \rangle_T (v_i')^2 \\
= \frac{\partial}{\partial t} \left\{ \frac{M}{4\Omega_1} \int d^4v' \langle f_i \rangle_T (v_i')^2 \right\}_\psi = \left\{ \nabla \psi^2 \right\}_\psi \frac{\partial p_i}{\partial t}.
\]

Combining (B.7), (B.8) and (B.9), and then using (B.10) and (B.11), we finally obtain

\[
\langle R \hat{c} \cdot \hat{P}_i \cdot \nabla \psi \rangle_T = \frac{M c}{2Ze} \left\{ \frac{R^2}{2} \frac{\partial p_i}{\partial t} \\
+ \frac{M^2 c^2}{2Ze} V \left\{ \frac{R^2}{2} \int d^3v' \langle f_i \rangle_T (v' \cdot \hat{c})^2 v' \cdot \nabla \phi \right\}_\psi \\
+ \frac{M c}{2Ze} \left\{ \frac{R^2}{2} \hat{c} \cdot \nabla \phi \int d^3v' f_i (v' \cdot \hat{c}) \right\}_T \\
- \frac{M^2 c^2}{2Ze} \left\{ \frac{R^2}{2} \int d^3v' \langle C_{i}(f_i) \rangle_T (v' \cdot \hat{c})^2 \right\}_\psi \right\}
\]

To obtain the final result in (14), we just need to rewrite the integral \( \int d^4v' \langle f_i \rangle_T (v' \cdot \hat{c})^2 v' \cdot \nabla \psi \) using the Fokker–Planck equation as is done in [18].

Finally, note that in writing (14) we have used the expressions (B.2) and (B.3) that give the gyroviscosity and perpendicular viscosity, and (B.10) and (B.11) that are equations for the parallel and perpendicular pressure. It is necessary to combine all of these equations to explicitly show the cancellations mentioned below equation (B.5).

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