Replica solution of the random energy model

V. Dotsenko

LPTMC, Université Paris VI - 75252 Paris, France, EU and
L.D. Landau Institute for Theoretical Physics - 119334 Moscow, Russia

received 19 April 2011; accepted in final form 19 July 2011
published online 22 August 2011

PACS 05.20.-y – Classical statistical mechanics
PACS 75.10.Nr – Spin-glass and other random models

Abstract – The alternative replica technique which involves summation over all integer momenta of the partition function and which does not require analytic continuation to non-integer values of the replica parameter \( n \) is discussed. In terms of this technique (which does not involve any replica symmetry breaking “magic operations”), a well-known solution for the average free energy of the random energy model is recovered in a very simple way.

Copyright © EPLA, 2011

Introduction. – In recent years there has been a renewed interest in the mathematical status of the replica method widely used in disordered systems during the last four decades (see, e.g, [1,2]). For the calculation of thermodynamic quantities averaged over disorder parameters (e.g., average free energy) the method assumes, first, the calculation of the averages of an integer \( n \)-th power of the partition function \( Z(n) \), and second, the analytic continuation of this function in the replica parameter \( n \) from integer to arbitrary non-integer values (and in particular, taking the limit \( n \rightarrow 0 \)). Usually one is facing difficulties at both stages of this program. First of all, in realistic disordered systems the calculations of the replica partition function \( Z(n) \) can be done only using some kind of approximations, and in this case the status of further analytic continuation in the replica parameter \( n \) becomes rather indefinite since the terms neglected at integer \( n \) could become essential at non-integer \( n \) (in particular the limit \( n \rightarrow 0 \))[3,4]. On the other hand, even in rare cases when the derivation of the replica partition function \( Z(n) \) can be done exactly, further analytic continuation to non-integer \( n \) appears to be ambiguous.

The classical example of this situation is provided by the Derrida’s random energy model (REM) [5]. At present this is one of the best studied models of spin-glasses (see, e.g., [6] and references therein) which exhibits non-trivial solution. It can be easily shown that in this system the partition function momenta \( Z(n) \) grow as \( \exp(n^2) \) at large \( n \), and in this case there exist many different distributions yielding the same values of \( Z(n) \), but providing different values for the average free energy of the system [5]. In this situation the replica solution which is generally believed to be correct is obtained via the “magic operations” of the Parisi replica symmetry breaking (RSB) scheme [1] (which in the case of REM reduces to the special case of the so-called one-step RSB). Unfortunately, this is not more than a heuristic procedure which at present has no rigorous mathematical grounding. On the other hand, it should be noted that during last decade remarkable progress have been achieved in mathematically rigorous derivations of various results previously obtained in terms of the replica method. A number of rigorous results have been obtained which prove the validity of the cavity method for the entire class of the random satisfiability problems revealing the physical phenomena similar to what happens in REM and which are described by the one-step RSB solution (see, e.g., [7] and references therein). The results obtained in terms of the continuous RSB scheme developed for mean-field spin glasses have been also confirmed by independent mathematically rigorous calculations [8].

Recently a notable progress has been achieved in the replica calculation technique itself [9,10]. This technique does not require performing analytic continuation to non-integer values of the replica parameter \( n \), and formally makes it possible to compute an entire free-energy distribution function summing over all integer momenta \( Z(n) \). In the recent paper [11] the replica calculations involving only integer momenta of the partition function have been consiered for systems exibiting one-step RSB. Unlike the present approach, to obtain RSB solutions these calculations are performed within a conventional saddle point...
approach taking into account an infinite number of saddle point solutions for the \((n \times n)\) replica order parameter matrix.

In this letter I would like to present very simple replica calculations which do not involve such tricks like RSB “magic operations”, and which, nevertheless, recover a well-known result for the average free energy of REM at all temperatures including the phase transition into the low-temperature phase (which is usually called the one-step RSB state).

**Replica technique.** – By definition the partition function \(Z\) of a given sample is related to its free energy \(F\) via

\[
Z = \exp(-\beta F).
\] (1)

The free energy \(F\) is defined for a specific realization of the disorder and thus represent a random variable. Taking the \(n\)-th power of both sides of this relation and performing the disorder averaging we obtain

\[
\bar{Z}^{n} \equiv Z(n) = \exp(-\beta n F),
\] (2)

where the quantity in the lhs of the above equation is called the replica partition function. The averaging in the rhs of the above equation can be represented in terms of the free-energy distribution function \(P(F)\). In this way we arrive to the following general relation between the replica partition function \(Z(n)\) and the distribution function of the free energy \(P(F)\):

\[
Z(n) = \int_{-\infty}^{+\infty} dF P(F) \exp(-\beta n F).
\] (3)

The above equation is the bilateral Laplace transform of the function \(P(F)\), and it looks as if, at least formally, it allows to restore this function via inverse Laplace transform of the replica partition function \(Z(n)\):

\[
P(F) = \int_{-\infty}^{+\infty} \frac{d(n\beta)}{2\pi i} Z(n) \exp(\beta n F).
\] (4)

In order to do so, first one has to compute \(Z(n)\) for an arbitrary integer \(n\) and then perform analytical continuation of this function from integer to arbitrary complex values of \(n\). This is the standard strategy of the replica method in disordered systems where it is well known that very often the procedure of such analytic continuation turns out to be a rather controversial point [3,5].

Usually the free energy of a given random system is expected to be an extensive quantity: \(F = V f\), where \(V\) is the volume of the system and \(f\) is (random) free-energy density described by some distribution function \(P_V(f)\) (which in general depends on the volume \(V\)). Substituting this into eq. (4) and introducing a new integration parameter \(s = \beta n V\) we get

\[
P_V(f) = \int_{-\infty}^{+\infty} \frac{ds}{2\pi i} Z_V(s) \exp(sf),
\] (5)

where \(P_V(f) = V P(V f)\) and \(Z_V(s) = Z(s/\beta V)\). If in the thermodynamic limit \(V \to \infty\) the phenomenon of self-averaging takes place, the limiting free-energy distribution function becomes \(\delta\)-like: \(\lim_{V \to \infty} P_V(f) = \delta(f - f(\beta))\), where \(f(\beta)\) is the mean free-energy density which is the quantity of the first interest in the disordered systems. According to the above equation, this means that the limiting replica partition function is expected to take the form \(\lim_{V \to \infty} Z_V(s) = \exp(-sf(\beta))\), where the parameter \(s\) remains finite. Since, by its definition, \(s = \beta n V\), this implies that in the limit \(V \to \infty\), the replica parameter must go to zero, \(n \sim 1/V \to 0\).

The problem is that before taking the thermodynamic limit, the replica partition function \(Z_V(s)\) has to be computed for finite volume \(V\). It is well known that in many cases the finite-size distribution functions of random quantities are extremely singular objects, and only in the thermodynamic limit they converge to a smooth and nice shape. The typical example is provided by the eigenvalues distribution functions in the random matrix theory (see, e.g., [12]). For that reason it could be easier, instead of the distribution function itself, to study its integral representation, namely,

\[
W(x) = \int_{-\infty}^{\infty} df P(f).
\] (6)

By definition, the function \(W(x)\) gives the probability that the random quantity \(f\) is bigger than a given value \(x\). It is clear that this function is much “smoother” object than the distribution function itself: even in the case that the finite system size function \(P_V(f)\) represents a set of delta functions, its integral representation would be only a kind of step-like continuous curve.

Formally the thermodynamic limit probability function \(W(x)\) can be defined as follows:

\[
W(x) = \lim_{V \to \infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp(\beta n V x) Z_V^{\bar{n}}
\]

\[
= \lim_{V \to \infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp(\beta n V x - \beta n V f)
\]

\[
= \lim_{V \to \infty} \exp[\beta n \left( x - f + \theta(f - x) \right)]
\]

\[
= \theta(f - x)
\] (7)

which coincides with the definition, eq. (6). Thus, according to eq. (7), the probability function \(W(x)\) can be computed in terms of the above replica partition function \(Z(n)\) by summing over all replica integers,

\[
W(x) = \lim_{V \to \infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \exp(\beta n V x) Z(n).
\] (8)

**Random energy model.** –

**Definition.** The random energy model is defined as a set of \(M = 2^N\) states, characterized by random energies...
\{E_i\}(i = 1, 2, \ldots, M) which are considered as independent quenched random parameters described by the Gaussian distribution

\[ P[E_1, E_2, \ldots, E_M] = \prod_{i=1}^{M} \left[ \frac{1}{\sqrt{2\pi N}} \exp\left(-\frac{E_i^2}{2N}\right) \right]. \tag{9} \]

Correspondingly, the partition function of REM is

\[ Z = \sum_{i=1}^{M} \exp(-\beta E_i) \tag{10} \]

which is a random quantity depending on \(M\) random parameters \(E_1, E_2, \ldots, E_M\). The choice for the value of \(M = 2^N\) is motivated by the fact that the free energy of this system (as will be shown below) is extensive in \(\ln M \propto N\), and thus it is the parameter \(N\) which plays the role of the effective volume of the system (which is taken to infinity in the thermodynamic limit). The particular form \(2^N\) (instead of say, \(\exp(N)\)) is motivated by the idea to imitate a random Ising system consisting of \(N\) spins (having \(2^N\) energy states, which, of course, are not independent).

**Naive** solution. Naively, one could propose a very simple derivation for the average value of the free energy of this system. Since the partition function, eq. (10), is given by the sum of a large number of independent parameters \(E_i\), it could be approximated as follows:

\[ Z \simeq M \times \exp(-\beta E). \tag{11} \]

Performing simple Gaussian averaging and substituting \(M = 2^N\) we get

\[ Z \simeq 2^N \exp\left(\frac{1}{2} N \beta^2\right) = \exp(-\beta N f(\beta)), \tag{12} \]

where

\[ f(\beta) = -\frac{1}{2} \beta - \frac{1}{\beta} \ln 2 \tag{13} \]

is the entropy density of the system. Correspondingly, for the entropy we get

\[ S(T) = \beta^2 \frac{d}{d\beta} f(\beta) = -\frac{1}{2} \beta^2 + \ln 2. \tag{14} \]

Since we are dealing with the discrete system, one can immediately note that something is very wrong, as the entropy becomes negative for \(\beta > \sqrt{2 \ln 2}\). In fact, it turns out that there is the phase transition in the considered system at \(\beta_c = \sqrt{2 \ln 2}\), such that at \(\beta > \beta_c\) (in the low-temperature phase) the system occupies only a finite number of the lowest energy states. For that reason the original hypothesis of the above derivation, that partition function, eq. (10), contains a macroscopic number of random terms, turns out to be wrong at low enough temperatures.

**Rigorous solution.** The result of the rigorous (non-replica) derivation of the average free-energy density of REM \([5]\) is given as follows:

\[ f(\beta) = \begin{cases} \frac{1}{2} \beta - \frac{1}{\beta} \ln 2, & \text{at } \beta \leq \beta_c = \sqrt{2 \ln 2}, \\ -\frac{1}{2} \beta + \ln 2, & \text{at } \beta \geq \beta_c. \end{cases} \tag{15} \]

Correspondingly, for the entropy density one gets

\[ S(\beta) = \begin{cases} -\frac{1}{2} \beta^2 + \ln 2, & \text{at } \beta \leq \beta_c, \\ 0, & \text{at } \beta \geq \beta_c. \end{cases} \tag{16} \]

The above result for the entropy demonstrates that indeed in the low-temperature phase the system effectively occupies only a finite number of the lowest energy states.

**Integer replicas solution.** In terms of the replica approach for the \(n\)-th momentum of the partition function, eq. (10), we find

\[ Z(n) = \left[ \sum_{i=1}^{M} \exp(-\beta E_i) \right]^n = \sum_{\{m_i\}=0}^{n} \frac{n!}{m_1! \cdots m_M!} \delta_{\Sigma, m_i, n} \prod_{i=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dE_i}{\sqrt{2\pi N}} \exp\left(-\frac{E_i^2}{2N}\right) \right], \tag{17} \]

where \(\delta_{p,q}\) is the Kronecker symbol and \(M = 2^N\). Redefining the integration variables, \(E_i = N \xi_i\) and substituting the above expression into eq. (8) (with \(V = N\)) for the probability distribution function we get

\[ W(x) = \lim_{N \to \infty} \sum_{n=0}^{\infty} \sum_{\{m_i\}=0}^{\delta_{\Sigma, m_i, n}} \frac{(-1)^n}{m_1! \cdots m_M!} e^{\beta N x n} \delta_{\Sigma, m_i, n} \prod_{i=1}^{M} \left[ \int_{-\infty}^{+\infty} d\xi_i e^{-\frac{1}{2} \xi_i^2 - \beta N \xi_i m_i} \right], \tag{18} \]

where, due to the presence of the Kronecker symbol, the summations over \(m_i\)'s are extended to infinity. Summing over \(n\) we can lift the constraint \(n = \sum_{i=1}^{M} m_i\) which provide independent summations over \(m_i\)'s:

\[ W(x) = \lim_{N \to \infty} \left[ G(N, x) \right]^{2^N}, \tag{19} \]

where

\[ G(N, x) = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{+\infty} d\xi \sum_{m=0}^{\delta_{\Sigma, \xi, m}} \frac{(-1)^m}{m!} e^{-\frac{1}{2} \xi^2 + \beta N (x - \xi) m} = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{+\infty} d\xi e^{-\frac{1}{2} N \xi^2 - \exp(\beta N (x - \xi))}. \tag{20} \]
Taking the limit $N \to \infty$ one can easily obtain the following result (see appendix):

1) In the region $\beta \leq \sqrt{2 \ln 2} \equiv \beta_c$:

$$W(x) = \begin{cases} 1, & \text{for } x < -\left(\frac{3}{2} \beta + \frac{1}{2} \ln 2\right), \\ 0, & \text{for } x > -\left(\frac{3}{2} \beta + \frac{1}{2} \ln 2\right). \end{cases}$$

(21)

2) In the region $\beta \geq \sqrt{2 \ln 2}$:

$$W(x) = \begin{cases} 1, & \text{for } x < -\sqrt{2 \ln 2}, \\ 0, & \text{for } x > -\sqrt{2 \ln 2}. \end{cases}$$

(22)

According to the definition in eq. (6), the above result means that in the thermodynamic limit the free-energy distribution function of the considered model is the delta function (which means that the system is self-averaging):

$$P(f) = \delta(f - f(\beta)),$$

(23)

where the free energy $f(\beta)$ coincides with the one obtained in the rigorous (non-replica) solution, eq. (15).

Conclusions. – In its traditional formulation the replica method procedure is as follows: first, one has to calculate the disorder average of the integer $n$-th power of the partition function, $Z^n = Z(n)$; second, one has to perform an analytic continuation of this function for arbitrary real or complex values of $n$; and third, one has to take the limit $n \to 0$ (to get the average free energy) or integrate over complex $n$ (to derive the free-energy distribution function). The third step is usually accompanied by taking the thermodynamic limit, which assumes that the system size $L$ is taken to infinity. The prescription of the replica method indicates that the two limits, $n \to 0$ and $L \to \infty$, have to be taken simultaneously such that the product $nL^\omega$ (where the exponent $\omega$ defines the scaling of the free energy with the system size) is kept finite.

In fact, the whole experience of the replica calculations in disordered systems shows that, except for trivial cases, this program, as it is formulated above, is never followed (for more detailed discussion of this issue see [13]). The typical illustration is provided by the studies of the Sherrington-Kirkpatrick model of spin glasses [14]. The replica solution of this model, which is generally believed to be correct, is derived in terms of the RSB technique [1] in which all the above three steps, (computing $Z(n)$, analytic continuation in $n$ and the limits $n \to 0$ and $L \to \infty$) are performed simultaneously.

In this paper an alternative replica technique has been discussed. In terms of this approach no analytic continuation for non-integer values of the replica parameter $n$ is required, and instead the summation over all positive integer momenta of the partition function has to be performed. Earlier this method has been successfully applied for solving the one-dimensional directed polymer problem [9, 10]. In this paper it has been demonstrated that in terms of this technique the derivation of the well-known solution of the random energy model takes just a few lines. Of course, a real challenge would be to find an alternative solution for the SK model. Unfortunately, here the situation is much more complicated, as already at the stage of calculation of the replica partition function $Z(n)$ (for integer $n$) the saddle point approximation is required, which does not seem to be legitimate in terms of the present technique. In any case, further systematic studies of the considered approach are required.

***

I am grateful to S. Nechaev, B. Derrida and H. Spohn for fruitful discussions of this work. This work was supported in part by the IRSES grant 269139.

Appendix

Let us study the properties of the function $G(N, x)$, eq. (20), in the limit of large $N$. First of all one can easily see that at $x > 0$ and $N \gg 1$

$$G(N, x) \sim \exp\left(-\frac{1}{2} N x^2\right) \to 0.$$  \hspace{1cm} (A.1)

Substituting this into eq. (19), we find that

$$W(x > 0) = 0.$$  \hspace{1cm} (A.2)

At $x < 0$ we can represent the function $G(N, x)$, eq. (20), as the sum of two contributions:

$$G(N, x) = G_1(N, x) + G_2(N, x),$$  \hspace{1cm} (A.3)

where

$$G_1(N, x) = \sqrt{\frac{N}{2\pi}} \int_{-\infty}^{-|x|} d\xi e^{-\frac{1}{2} N x^2 - \exp(-\beta N (|x| + \xi))}$$  \hspace{1cm} (A.4)

and

$$G_2(N, x) = \sqrt{\frac{N}{2\pi}} \int_{|x|}^{\infty} d\xi e^{-\frac{1}{2} N x^2 - \exp(-\beta N (|x| + \xi))}.$$  \hspace{1cm} (A.5)

Simple analysis shows that

$$G_1(N, x) \bigg|_{N \gg 1} \sim \exp\left(-\frac{1}{2} N x^2\right).$$  \hspace{1cm} (A.6)

The function $G_2(N, x)$ in the limit of large $N$ can be estimated as follows:

$$G_2(N, x) \sim \sqrt{\frac{N}{2\pi}} \int_{-|x|}^{\infty} d\xi e^{-\frac{1}{2} N x^2 \left[1 - e^{-\beta N (|x| + \xi)}\right]} \approx 1 - \sqrt{\frac{N}{2\pi}} \int_{-|x|}^{\infty} d\xi e^{-\frac{1}{2} N x^2 - \beta N |x|}.$$  \hspace{1cm} (A.7)

The function $\varphi(\xi) = -\frac{1}{2} N x^2 - \beta N |x|$ has the maximum at $\xi_\star = -\beta$. Thus

$$\sqrt{\frac{N}{2\pi}} \int_{-|x|}^{\infty} d\xi e^{-\frac{1}{2} N x^2 - \beta N |x|} \sim \begin{cases} e^{-\frac{1}{2} N x^2}, & \text{for } |x| < \beta, \\ e^{-\beta N (|x| - \frac{1}{2} \beta)}, & \text{for } |x| > \beta. \end{cases}$$  \hspace{1cm} (A.8)
The results, eqs. (A.6), (A.7) and (A.8), demonstrate that at $N \gg 1$ and negative $x$ the function $G(N, x)$, eq. (20), takes the form

$$G(N, x) \simeq 1 - g(N, x), \quad (A.9)$$

where

$$g(N, x) \sim \begin{cases} e^{-N x^2}, & \text{for } |x| < \beta, \\ e^{-\beta N (|x| - 1/2)}, & \text{for } |x| > \beta. \end{cases} \quad (A.10)$$

Substituting eqs. (A.9) and (A.10) into eq. (19) we get

$$W(x) = \lim_{N \to \infty} \exp[-\psi(N, x)], \quad (A.11)$$

where $\psi(N, x) = g(N, x)2^N$ so that

$$\psi(N, x) \sim \begin{cases} e^{-N x^2 + N \ln 2}, & \text{for } |x| < \beta, \\ e^{-\beta N(|x| - 1/2 - \ln 2)}, & \text{for } |x| > \beta. \end{cases} \quad (A.12)$$

Simple analysis of this expression yields

a) at $|x| < \beta$,

$$\lim_{N \to \infty} \psi(N, x) = \begin{cases} 0, & \text{for } |x| > \sqrt{2 \ln 2}, \\ +\infty, & \text{for } |x| < \sqrt{2 \ln 2}. \end{cases} \quad (A.13)$$

b) at $|x| > \beta$,

$$\lim_{N \to \infty} \psi(N, x) = \begin{cases} 0, & \text{for } |x| > \frac{1}{2} \beta + \frac{1}{2} \ln 2, \\ +\infty, & \text{for } |x| < \frac{1}{2} \beta + \frac{1}{2} \ln 2. \end{cases} \quad (A.14)$$

Substituting eqs. (A.13) and (A.14) into eq. (11) we find

a) at $|x| < \beta$,

$$W(x) = \begin{cases} 1, & \text{for } |x| > \sqrt{2 \ln 2}, \\ 0, & \text{for } |x| < \sqrt{2 \ln 2}. \end{cases} \quad (A.15)$$

b) at $|x| > \beta$,

$$W(x) = \begin{cases} 1, & \text{for } |x| > \frac{1}{2} \beta + \frac{1}{2} \ln 2, \\ 0, & \text{for } |x| < \frac{1}{2} \beta + \frac{1}{2} \ln 2. \end{cases} \quad (A.16)$$

One can easily see that the above results, eq. (A.15) (valid for $|x| < \beta$), and eq. (A.16) (valid for $|x| > \beta$), are equivalent to eqs. (21), (22).

REFERENCES

[1] MEZARD M., PARISI G., AND VIRASORO M. A., *Spin Glass Theory and Beyond* (World Scientific, Singapore) 1987.

[2] DOTSENKO V. S., *Introduction to the Replica Theory of Disordered Statistical Systems* (Cambridge University Press) 2001.

[3] VERBAARSCHOT J. J. M. AND ZIRNBAUER M. R., *J. Phys. A: Math. Gen.*, 17 (1985) 1093.

[4] ZIRNBauer M. R., *Another Critic of the Replica Trick*, arXiv:cond-mat/9903338 (1999).

[5] DERRI DA B., *Phys. Rev. B*, 24 (1981) 2613.

[6] FYODOROV YAN V. AND BOUCHAUD J.-P., *J. Phys. A: Math. Theor.*, 41 (2008) 372001.

[7] BAYATI M. AND NAIR C., *A rigorous proof of cavity method for counting matchings*, in *Proceedings of the 44th Annual Allerton Conference on Communication, Control and Computing*, Allerton, 2006.

[8] BARRA A., DI BIASIO A. AND GUERRA F., *J. Stat. Mech.*, (2010) P09006.

[9] CALABRESE P., LE DOUSSAL P., AND ROSSO A., *EPL*, 90 (2010) 20002.

[10] DOTSENKO V., *EPL*, 90 (2010) 20003; *J. Stat. Mech.*, (2010) P07010.

[11] CAMELONI M., PARISI G., AND VIRASORO M. A., *J. Stat. Phys.*, 138 (2010) 20.

[12] MEHTA M. L., *Random Matrices* (Elsevier, Amsterdam) 2004.

[13] DOTSENKO VICTOR, *One more discussion of the replica trick: the examples of exact solutions*, arXiv:1010.3913 (2010).

[14] SHERRINGTON D. AND KIRKPATRICK S., *Phys. Rev. Lett.*, 35 (1975) 1792.