Proposal of texture shape optimization algorithm under constant load condition and considerations on new shape update equation
(Texture shape optimization for minimization of friction coefficient)

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Abstract
An algorithm to optimize texture shape under constant load conditions and a shape update equation of the design variables are proposed. The tribological properties are improved by machining grooves and holes, termed “texture”, on frictional surfaces that are lubricated by fluid. Improvement of tribological properties, such as the friction coefficient, is likely to lead to a reduction in energy loss and extension of machine life, resulting in major economic benefits. Because tribological properties depend on the shape of the texture, the focus of this study was on the dimensional shape of the texture. Most countermeasures to this problem involve size optimization rather than shape optimization. Conventionally, when the effect of the texture shape on the friction coefficient is evaluated experimentally, the load is kept constant. However, when the texture shape is changed as part of the analysis, the pressure field changes. The load, which is the integrated value of the pressure, also changes. Therefore, it is difficult to evaluate the friction coefficient accurately. At this time, the load can be kept constant by adjusting the basic oil film thickness, which is the distance between the frictional surfaces. This occurs naturally in real-world situations. In general, when the adjoint variable method is applied to determine the texture shape, constraint conditions are included in the Lagrange function. But, in this study, the constant load condition, i.e., the constraint condition, was simply added to keep the initial load, because it is difficult to calculate the gradient of the constraint condition with respect to the design variable. Considering the above, the purpose of this study was to find an appropriate oil film thickness for a texture by shape optimization and to reduce the friction coefficient by adding an algorithm that keeps the load constant by varying the basic oil film thickness. In addition, the shape update equation for the design variable was improved, and results based on the present method were compared with those based on the steepest descent and the conjugate gradient methods. This was achieved by replacing the interpretation of the update equation using the steepest descent method with a differential equation and by applying the differential to the step length of the design variable in the Taylor expansion equation of the design variable. By improving the shape update equation, a lower performance function was obtained. Texture shape optimization was performed by the adjoint variable method using the Reynolds equation as the governing equation. The performance function is defined by the frictional force, and the friction coefficient is optimized at the same time by keeping the load constant. FreeFEM++ was used to calculate the optimal shape.

Keywords : Shape optimization, Adjoint variable method, Finite element method, Texture, Friction coefficient, Shape update equation

1. Introduction
In this study, to minimize the friction coefficient, the shape optimization of the texture added to two surfaces under
fluid lubrication condition was carried out under constant load conditions. Frictional force is generated in most moving machine parts, so it is expected to bring about significant economic benefits because it leads to less energy loss, longer life of machine parts, reduction of lubricant use, etc. There are two main methods for reducing friction. One is to improve the material of the friction surface, and the other is to improve the shape of texture or the roughness on the frictional surface (El-Mahallawy, 2019; Dejun, 2020; Rosenkranz, 2019). Adding texture appears to provide such advantages as improved lubricant retention and increased dynamic pressure. However, in practice, the mechanism by which texture reduces friction has not been clarified. The effect of texture on tribological properties also depends on the shape of the texture. This has prompted some research into texture shapes (Codrignani, 2018; Sharma, 2019; Manser, 2020). However, these studies mostly involve parameter studies and size optimization. Few focused on shape optimization. Using the adjoint variable method is one method of bringing a shape closer to the intended target (Azegami, 1994, 2014; Katamine, 1994, 1995, 2005). Thus, the texture shape is optimized here based on the adjoint variable method. Because the friction coefficient is the ratio of the load to the frictional force, the load changes when the performance function is defined by the friction coefficient. However, the load is kept constant when the friction coefficients are compared experimentally. Therefore, the performance function is defined by the frictional force, and a calculation is applied to keep the load constant by adjusting the basic oil film thickness. In general, constraint conditions are included in the Lagrange function. But, it is difficult to include the constraint condition because the design variable is not explicitly included in the constraint condition, and the differentiation of the constant condition with respect to the design variable is not directly calculated. Based on the above, in this study, the question of whether the friction coefficient can be lowered by adding the above algorithm was investigated. In addition, investigations were performed by comparing results using shape update equations based on the steepest descent and the conjugate gradient methods. Here, by replacing the interpretation of the update equation with a differential equation and applying the differential to the step length of the design variable to the Taylor expansion equation of the design variable, the possibility of obtaining an optimal shape was investigated. Shape optimization using FreeFEM++ was also performed (Hecht, 2012; Ootsuka, 2014). Most textures were below millimeter size. Cutting processing using multiedge tools, ultrashort pulse laser processing, shot peening, etc. have been mentioned as methods for adding texture (Mao, 2020). It is difficult to add complex texture shapes using the above processing methods. Shapes obtained by shape optimization can be more complicated than those obtained by parameter study and size optimization. It is currently difficult, therefore, to realize texture shapes obtained by shape optimization. However, shapes obtained by topology optimization, which have previously been regarded as difficult, have been realized by the improvement of the processing technology used in 3D printers (DebRoy, 2018). Further improvement of processing technology is likely to lead to the creation of more complicated texture shapes.

2. Formulation
2.1. Nomenclature and calculation model

Table 1 shows the meanings of the symbols, and the calculation model is shown in Figure 1. In this paper, a model with nine circular planar textures is introduced. Here, $R$, $h_{dep}$ and $h_0$ indicate the radius of the texture, the depth of the texture and the basic oil film thickness. The distance between two surfaces is called the oil film thickness $h$, and the oil film thickness in the domain where no texture exists is called the basic oil film thickness $h_0$. The design variable is the oil film thickness $h$, and the shape optimization of the oil film thickness is performed. As shown in Figure 2, the whole domain is defined by $\Omega$, and its boundaries are defined by $\Gamma_1$. As shown in Figure 3, the domain which textures exist is defined by $\omega$, and its boundaries are defined by $\Gamma_2$. Here, the whole domain $\Omega$ contains the texture domain $\omega$. The definition of the domain and the boundary shown in Figure 2 is employed in the calculation of the governing and the adjoint equations. In addition, the definition of the domain and the boundary shown in Figure 3 is employed in the calculation of the smoothing of the gradient shown in Eqs. (36) and (37).
Table 1 Nomenclature.

| symbol | meaning                                      | symbol | meaning                                      |
|--------|----------------------------------------------|--------|----------------------------------------------|
| \( u \) | Velocity of the fluid in the \( x \) direction | \( v \) | Velocity of the fluid in the \( y \) direction |
| \( w \) | Velocity of the fluid in the \( z \) direction | \( U_1 \) | Velocity on the lower surface in the \( x \) direction |
| \( V_1 \) | Velocity on the lower surface in the \( y \) direction | \( W_1 \) | Velocity on the lower surface in the \( z \) direction |
| \( U_2 \) | Velocity on the upper surface in the \( x \) direction | \( V_2 \) | Velocity on the upper surface in the \( y \) direction |
| \( W_2 \) | Velocity on the upper surface in the \( z \) direction | \( n \) | Number of iterations |
| \( h \) | Oil film thickness                           | \( h_{dep} \) | Initial depth of a texture                   |
| \( R \) | Radius of a texture                          | \( p \) | Pressure                                      |
| \( \eta \) | Viscosity of fluid                           | \( \sigma \) | Normal stress                                 |
| \( \tau \) | Shear stress                                  | \( F_f \) | Frictional force                              |
| \( F_w \) | Load                                         | \( F_{tot} \) | Target load                                   |
| \( \epsilon \) | convergence criterion                        | \( \mu \) | Friction coefficient                          |
| \( \Omega \) | Whole domain including the domain \( \omega \) | \( \omega \) | The domain in which the texture exists        |
| \( \Gamma_1 \) | Boundary of the \( \Omega \)                | \( \Gamma_2 \) | Boundary of the \( \omega \)                 |
| \( J \) | Performance function                         | \( J^* \) | Lagrange function                            |
| \( K \) | Modified Lagrange function                   | \( g \) | Constraint function for the constant load condition |
| \( L \) | Integrand                                    | \( \lambda \) | Adjoint variable for the governing equation   |
| \( \Lambda \) | Adjoint variable for constraint condition    | \( \partial \) | Gradient with respect to oil film thickness   |
| \( \Delta \) | Modified \( \partial \)                      | \( \Delta \theta \) | Step length                                   |

Fig. 1 Calculation model and size. \( h_0 \), \( h_{dep} \) and \( R \) indicate the basic oil film thickness, depth of the texture and radius of the texture. This is a model with nine cylindrical textures added in a square domain.

2.2. Derivation of the Reynolds equation and frictional force

The frictional force and the Reynolds equation are derived based on the Reynolds assumption for considering the 3-dimensional model of fluid between two surfaces as 2-dimensinal model, as shown in Figure 4 (Hori, 2002). The derivation of the Reynolds equation and the frictional force is started from 3-dimensional model. The velocity of fluid is defined by \( u \), \( v \) and \( w \) for each direction, i.e., \( x \), \( y \), and \( z \), respectively. And velocities on lower and upper surfaces are defined by \( U_1 \), \( V_1 \), \( W_1 \), \( U_2 \), \( V_2 \) and \( W_2 \). The distance between lower and upper surfaces is the oil film thickness \( h \). In the 2-dimensional model, the oil film thickness \( h \) is the variable for each coordinate, and velocities on lower and upper surfaces are defined by Eqs. (1) and (2).
Domain $\Omega$ and boundary $\Gamma_1$. The whole domain is defined by $\Omega$. The boundary of the domain $\Omega$ is defined by $\Gamma_1$. The domain and the boundary shown in Figure 2 are employed in the calculation of the governing and the adjoint equations.

Domain $\omega$ and boundary $\Gamma_2$. The domain in which the texture exists is defined by $\omega$, and the whole domain $\Omega$ contains the texture domain $\omega$. The boundary of the domain $\omega$ is defined by $\Gamma_2$. In addition, the domain and the boundary shown in Figure 3 are employed in the calculation of the smoothing of the gradient shown in Eqs. (36) and (37).

Fig. 4 The 3-dimensional model and the 2-dimensional model of fluid between two surfaces. In the Reynolds equation, the 3-dimensional model can be considered as 2-dimensional model based on the Reynolds assumption. The velocities of fluid in the $x$-, $y$-, and $z$-directions are defined by $u$, $v$, and $w$. The velocity on the lower surface is defined by $U_1$, $V_1$ and $W_1$ for each direction, i.e., $x$-, $y$-, and $z$-directions, respectively. In addition, the velocity on the upper surface is defined by $U_2$, $V_2$, and $W_2$ for each direction, i.e., $x$-, $y$-, and $z$-directions, respectively. The distance between two surfaces is denoted by the oil film thickness $h$.

In the 2-dimensional model, the oil film thickness $h$ is the variable for each coordinate, and velocities of lower and upper surfaces are defined by $(U_1, V_1, W_1) = (U, 0, 0)$ and $(U_2, V_2, W_2) = (0, 0, 0)$.

$$U_1 = U, \quad V_1 = 0, \quad W_1 = 0$$

$$U_2 = 0, \quad V_2 = 0, \quad W_2 = 0$$

Here, we introduce the Reynolds assumptions shown below.

- The fluid flow is laminar.
- Gravity and inertial forces acting on the fluid are negligible compared with viscous forces.
- The compressibility of the fluid is negligible.
- The fluid is a Newtonian fluid with a constant viscosity.
- The fluid pressure does not change in the $z$-direction.
vi The rates of change in the \( x \)- and \( y \)-directions of the velocities \( u \) and \( v \) are negligible compared with those in the \( z \)-direction.

vii There is no slip between the fluid and the surfaces.

The balance of forces acting on the control volume in the fluid is represented by Eq. (3).

\[
\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0
\]  
(3)

where \( p = -\sigma_r \) is substituted.

\[
\frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right)
\]  
(4)

The shear force is given by Eqs. (5) and (6) based on the Reynolds assumptions i and iv.

\[
\tau_{yx} = \eta \frac{\partial u}{\partial y}
\]  
(5)

\[
\tau_{zx} = \eta \frac{\partial u}{\partial z}
\]  
(6)

Substituting Eqs. (5) and (6) for Eq. (3), Eq. (7) is obtained.

\[
\frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right)
\]  
(7)

The first term on the right side of Eq. (7) can be ignored based on the Reynolds assumption vi.

\[
\frac{\partial p}{\partial x} = \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right)
\]  
(8)

Based on the Reynolds assumption iv, the viscosity of fluid \( \eta \) is constant and Eq. (9) can be obtained.

\[
\frac{\partial^2 u}{\partial z^2} = \frac{1}{\eta} \frac{\partial p}{\partial x} 
\]  
(9)

Similarly, Eq. (10) is obtained from the balance of forces in \( z \)-direction.

\[
\frac{\partial^2 v}{\partial z^2} = \frac{1}{\eta} \frac{\partial p}{\partial y} 
\]  
(10)

By calculating double integral of Eqs. (9) and (10) with respect to \( z \)-direction, velocities \( u \) and \( v \) are obtained as shown in Eqs. (11) and (12). Here, \( C_1, C_2, C_3 \) and \( C_4 \) are integration constants, and boundary conditions are given as Eqs. (13) and (14) based on conditions with respect to velocities \( U_1, V_1, W_1, U_2, V_2 \) and \( W_2 \) shown in Figure 4 and the Reynolds assumption vii.

\[
u = \frac{1}{\eta} \frac{\partial p}{\partial x} dz\,dz = \frac{1}{\eta} \frac{\partial p}{\partial x} z^2 + C_1z + C_2
\]  
(11)

\[
u = \frac{1}{\eta} \frac{\partial p}{\partial y} = \frac{1}{\eta} \frac{\partial p}{\partial y} z^2 + C_3z + C_4
\]  
(12)

\[
u = U_1 \quad , v = V_1 \quad , w = W_1 \quad \text{on} \quad z = 0
\]  
(13)

\[
u = U_2 \quad , v = V_2 \quad , w = W_2 \quad \text{on} \quad z = h
\]  
(14)

On the other hand, Eq. (15) is obtained by substituting Eq. (11) for Eq. (5).

\[
\tau_{zx} = -\frac{1}{2} \frac{\partial p}{\partial x} (h - 2z) - \eta \frac{U_1 - U_2}{h}
\]  
(15)
Substituting Eqs. (1) and (2) for Eq. (15), the shear stress acting on the lower surface \( \tau_{z=0} \) is obtained as Eq. (16).

\[
\tau_{z=0} = -\frac{h}{2} \frac{\partial p}{\partial x} - \frac{\eta U}{h}
\]  

(16)

The frictional force acting on the lower surface \( F_f \) is obtained by the area integral of Eq. (16).

\[
F_f = \int_{\Omega} \left( -\frac{h}{2} \frac{\partial p}{\partial x} - \frac{\eta U}{h} \right) d\Omega
\]

(17)

The first term on the right side of Eq. (17) is the frictional force resulting from Poiseuille flow, and the second term is the frictional force caused by Couette flow. Poiseuille flow is based on the pressure gradient, and Couette flow is based on the velocity of the surface. The load is obtained by the area integral of the pressure, as shown in Eq. (18).

\[
F_w = \int_{\Omega} pd\Omega
\]

(18)

The equation of continuity for incompressible fluid is shown in Eq. (19).

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(19)

Integrating Eq. (19) from \( z=0 \) to \( z=h \) with respect to \( z \) direction, Eq. (20) is obtained.

\[
\int_{0}^{h} \frac{\partial u}{\partial x} dz + \int_{0}^{h} \frac{\partial v}{\partial y} dz + [w]_0^h = 0
\]

(20)

Consequently, Eq. (21) is obtained by arranging Eq. (20).

\[
\frac{\partial}{\partial x} \left( h \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{h^3}{12\mu} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{h^3}{12\mu} \frac{\partial p}{\partial z} \right) - \frac{h}{2} \left( U_1 + U_2 \right) - W_2 \frac{\partial h}{\partial x} + W_2 \frac{\partial h}{\partial y} + W_2 \frac{\partial h}{\partial z} = 0
\]

(21)

Considering boundary conditions shown in Eqs. (13) and (14), and substituting Eqs. (11) and (12) for Eq. (21), Eq. (22) is consequently obtained.

\[
\frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^3 \frac{\partial p}{\partial y} \right) - 6\eta U \frac{\partial h}{\partial x} = 0
\]

(22)

From boundary conditions shown in Eqs.(1) and (2), Eq. (23) can be obtained.

\[
\frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^3 \frac{\partial p}{\partial y} \right) - 6\eta U \frac{\partial h}{\partial x} = 0
\]

(23)

Eq. (23) is called the Reynolds equation.

### 2.3. Formulations for the shape optimization problem

Using the Reynolds equation shown in Eq. (23), a model of the sliding surface under fluid lubrication can be treated as the 2-dimensional model in \( x \)- and \( y \)- directions. The friction coefficient is defined by Eq. (24) and includes the load in the denominator.

\[
J = \mu = \frac{F_f}{F_w} = \frac{\int_{\Omega} \left( \frac{\partial U}{\partial y} + \frac{h}{2} \frac{\partial p}{\partial x} \right) d\Omega}{\int_{\Omega} pd\Omega}
\]

(24)

Therefore, when the performance function is defined by the friction coefficient, the performance function diverges owing to the effects of boundary conditions and negative pressure elimination when it is discretized. To prevent divergence, the performance function is defined by the frictional force as shown in Eq. (25) and a new calculation to make the load constant is added.

\[
J = F_f = \int_{\Omega} \left( \frac{\eta U}{h} + \frac{h}{2} \frac{\partial p}{\partial x} \right) d\Omega
\]

(25)

The Reynolds equation shown in Eq. (23) is employed as the governing equation in the domain \( \Omega \) (see Eq. (26)).

\[
\frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^3 \frac{\partial p}{\partial y} \right) - 6\eta U \frac{\partial h}{\partial x} = 0 \quad \text{in} \quad \Omega
\]

(26)
The fluid is sandwiched between free surfaces, and pressure is generated. The pressure becomes zero on \( \Gamma_1 \), because the boundary \( \Gamma_1 \) is assumed as free surface. Therefore, the boundary conditions of the governing equation are shown in Eq. (27).

\[
p = 0 \quad \text{on} \quad \Gamma_1
\]

(27)

The pressure is actually zero because the fluid flows into the domain where negative pressure is generated, and negative pressure is frequently eliminated in analysis using the Reynolds equation (Oda, 2018).

\[
p = 0 \quad \text{in} \quad p < 0
\]

(28)

Using the adjoint variables, the Lagrange function is defined by in Eq. (29).

\[
J^* = J + \int_{\Omega} \Lambda \left[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{3}{h} \left( \frac{\partial p}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial p}{\partial y} \frac{\partial h}{\partial y} \right) - \frac{6 \eta U}{h^3} \frac{\partial h}{\partial x} \right] d\Omega = \int_{\Omega} L_d \Omega
\]

(29)

where \( \lambda \) indicates the adjoint variable for the pressure. The Reynolds equation shown in Eq. (26) is arranged to simplify the calculation. In general, constraint conditions are included in the Lagrange function (Uchiyama, 2018). If the constant load is included as the constraint condition, the modified Lagrange function can be expressed by Eq. (30). Here, \( \Lambda \) indicates the adjoint variable for the constraint condition.

\[
K = J^* + \Lambda \left( \int_{\Omega} p d\Omega - F_{w1} \right) = J^* + \Lambda \cdot g
\]

(30)

Therefore, the update equation for the design variable, i.e., oil film thickness \( h \), is indicated as Eq. (31).

\[
h^{n+1} = h^n - \Delta \left[ \frac{\partial L}{\partial h} \right] + \Lambda \left( \frac{\partial J}{\partial \lambda} \right) \]

(31)

If the constant load condition is employed as the constraint condition for the Lagrange function, it is necessary to calculate \( \Lambda \left( \frac{\partial J}{\partial \lambda} \right) \). However, it is difficult to calculate \( \Lambda \left( \frac{\partial \phi}{\partial \gamma} \right) \), because the oil film thickness \( h \) is not explicitly included in the constraint condition. For example, considering the volume constraint condition in shape optimization and the topology optimization problems, the gradient of the constraint condition with respect to the design variable can be obtained due to include the design variable in the constraint condition explicitly. However, in the constant load condition, the design variable \( h \) is not explicitly included. Therefore, in this paper, the constant load condition is not added as the constraint condition in the Lagrange function, and the calculation to keep the load constant is added by adjusting the basic oil film thickness. The first variation of the Lagrange function must be zero to satisfy the stationary condition, and is represented by Eq. (32).

\[
\delta J^* = \int_{\Omega} \delta \lambda \left[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{3}{h} \left( \frac{\partial p}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial p}{\partial y} \frac{\partial h}{\partial y} \right) - \frac{6 \eta U}{h^3} \frac{\partial h}{\partial x} \right] d\Omega
\]

\[
+ \int_{\Omega} \delta p \left[ - \frac{1}{2h} \left( \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} - 3 \frac{\partial \lambda}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \lambda}{\partial y} \frac{\partial h}{\partial y} \right) + \frac{3 \lambda}{h^2} \left( \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial h}{\partial y} \right) - \frac{3 \lambda}{h} \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) \right] d\Omega
\]

\[
+ \int_{\Gamma_1} \delta h \left[ \frac{\partial \lambda}{\partial y} \frac{\partial p}{\partial h} - \frac{\partial \lambda}{\partial x} \right] + \delta \frac{\partial p}{\partial h} (\lambda) + \delta \frac{\partial p}{\partial h} (\lambda) \left[ \frac{3 \lambda}{h^2} \frac{\partial h}{\partial y} \right] n_x d\Omega
\]

\[
+ \int_{\Gamma_1} \delta h \left[ \frac{\partial \lambda}{\partial y} \frac{\partial p}{\partial h} - \frac{\partial \lambda}{\partial x} \right] + \delta \frac{\partial p}{\partial h} (\lambda) + \delta \frac{\partial p}{\partial h} (\lambda) \left[ \frac{3 \lambda}{h^2} \frac{\partial h}{\partial y} \right] n_x d\Omega
\]

\[
= \int_{\Omega} \delta L d\Omega
\]

(32)

Consequently, the adjoint equation for the adjoint variable is obtained as shown in Eq. (33).

\[
\frac{\partial L}{\partial p} = - \frac{1}{2h} \left( \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} - 3 \frac{\partial \lambda}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \lambda}{\partial y} \frac{\partial h}{\partial y} \right) + \frac{3 \lambda}{h^2} \left( \frac{\partial h}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} \frac{\partial h}{\partial y} \right) - \frac{3 \lambda}{h} \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = 0
\]

in \( \Omega \)

(33)

The boundary conditions for the adjoint analysis are shown in Eq. (34).

\[
\lambda = 0 \quad \text{on} \quad \Gamma_1
\]

(34)
From Eq. (32), the gradient of the Lagrange function with respect to the oil film thickness is obtained as Eq. (35).

$$\frac{\partial L}{\partial h} = -\eta U \frac{h}{\kappa^2} + \frac{1}{h^2} \frac{\partial p}{\partial x} - \frac{3}{h} \frac{\partial \lambda}{\partial x} \frac{\partial p}{\partial x} + \frac{3h}{h^2} \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) + \frac{6\eta U \lambda}{h^3} \frac{\partial \lambda}{\partial x}$$  (35)

The gradient obtained by Eq. (35) shows numerical vibration. To eliminate numerical vibration, the gradient is smoothed using the Poisson equation shown in Eq. (36). The Poisson equation was used for smoothing of distribution of the gradient with respect to the design variable. In the shape optimization analysis, the second-order differential equation is sometimes applied to get smoothing effect for the gradient oscillation (Jameson, 2003). The boundary condition is given in Eq. (37).

$$\left( \frac{\partial L}{\partial h} \right) = 0 \quad \text{on} \quad \Gamma_2$$  (37)

Normally the performance function is lowered by updating the oil film thickness at each node using the steepest descent method of Eq. (38).

$$h^{(n+1)} = h^n - \Delta \alpha \left( \frac{\partial L}{\partial h} \right)^n$$  (38)

where $\Delta \alpha$ indicates the step length of the steepest descent method. The step length is adjusted such that the performance function of the $(n+1)$ step is smaller than that of step $(n)$. In this study, a new design variable update equation is proposed by replacing the interpretation of Eq. (38) with a differential to the step length of the oil film thickness to the Taylor expansion equation for the oil film thickness. Eq. (38) is arranged as shown in Eq. (39).

$$\frac{h^{(n+1)} - h^n}{\Delta \alpha} = -\left( \frac{\partial L}{\partial h} \right)^n$$  (39)

When the left side of Eq. (39) is interpreted as the differential to the step length of the oil film thickness, Eq. (40) can be obtained as

$$\frac{\partial h}{\partial \alpha} = -\left( \frac{\partial L}{\partial h} \right)^n$$  (40)

Differentiating Eq. (40) for the step length $\alpha$ gives Eq. (41).

$$\frac{\partial^2 h}{\partial \alpha^2} = -\frac{\partial}{\partial \alpha} \left( \frac{\partial L}{\partial h} \right)^n$$  (41)

Taylor expansion is applied to the second order on $h^{(n+1)}$, and Eqs. (40) and (41) are substituted, as shown in Eq. (42).

$$h^{(n+1)} = h^n + \Delta \alpha \frac{\partial h^n}{\partial \alpha} + \frac{\Delta \alpha^2}{2} \frac{\partial^2 h^n}{\partial \alpha^2}$$

$$= h^n - \Delta \alpha \left( \frac{\partial L}{\partial h} \right)^n - \frac{\Delta \alpha^2}{2} \frac{\partial \left( \frac{\partial L}{\partial h} \right)^n}{\partial \alpha}$$

$$= h^n - \Delta \alpha \left( \frac{\partial L}{\partial h} \right)^n - \frac{\Delta \alpha^2}{2} \left[ \left( \frac{\partial L}{\partial h} \right)^n - \frac{\partial \left( \frac{\partial L}{\partial h} \right)^n}{\partial \alpha} \right]$$

$$= h^n - \Delta \alpha \left( \frac{\partial L}{\partial h} \right)^n - \frac{\Delta \alpha}{2} \left( \frac{\partial L}{\partial h} \right)^n - \frac{\Delta \alpha}{2} \left( \frac{\partial L}{\partial h} \right)^n$$

$$= h^n - \Delta \alpha \left( \frac{\partial L}{\partial h} \right)^n + \frac{\Delta \alpha}{2} \left( \frac{\partial L}{\partial h} \right)^n - \frac{\Delta \alpha}{2} \left( \frac{\partial L}{\partial h} \right)^n$$

$$= h^n - \Delta \alpha \left( \frac{\partial L}{\partial h} \right)^n$$

$$= h^n - \Delta \alpha \left\{ \frac{\partial L}{\partial h} \right\}^n - \frac{\Delta \alpha}{2} \left( \frac{\partial L}{\partial h} \right)^n - \frac{\Delta \alpha}{2} \left( \frac{\partial L}{\partial h} \right)^n$$  (42)

Eq. (42) is employed as the shape update equation.
2.4. Algorithm of the conjugate gradient method

In the algorithm of the conjugate gradient method, the update equation for the design variable is written as follows (Togawa, 1977).

\[ h^{n+1} = h^n + \Delta \alpha d^n \]  
(43)

\[ d^n = -\left(\frac{\partial L}{\partial h}\right)^n + \Delta \beta d^{n-1} \]  
(44)

In the numerical experiments, parameters \( \Delta \alpha \) and \( \Delta \beta \) are given as follows.

\[ \Delta \alpha = 10^{-3} \]  
(45)

\[ \Delta \beta = \frac{1}{2} \]  
(46)

From Eq. (44), the following equation is obtained.

\[ d^{n-1} = -\left(\frac{\partial L}{\partial h}\right)^{n-1} + \Delta \beta d^{n-2} \]  
(47)

In addition, substituting Eqs. (44) and (47) for Eq. (43), and using the coefficient \( \Delta \beta \) shown in Eq. (46), the following equation is obtained.

\[ h^{n+1} = h^n - \Delta \alpha \left(\frac{\partial L}{\partial h}\right)^n - \Delta \alpha \Delta \beta \left(\frac{\partial L}{\partial h}\right)^{n-1} + \Delta \alpha \Delta \beta^2 d^{n-2} \]

\[ = h^n - \Delta \alpha \left(\frac{\partial L}{\partial h}\right)^n + \Delta \alpha \left(\frac{\partial L}{\partial h}\right)^{n-1} + \frac{\Delta \alpha}{4} d^{n-2} \]  
(48)

The optimization analysis is carried out by using shape update equations shown in Eqs. (38), (42) and (48), and numerical results are compared.

2.5. Proposed computational algorithm

The computational flow of shape optimization in this paper is shown as follows. By increasing or decreasing this basic oil film thickness, the load is adjusted to a constant value in Step (3).

(1) Select the initial oil film thickness \( h \) in the whole domain \( \Omega \) and the “convergence criterion \( \epsilon \)”, and number of iterations \( n = 0 \).

(2) Solve the Reynolds equation, Eq. (26), using the finite element method under the boundary condition in Eq. (27) to obtain the pressure field.

(3) Compute the load \( F_w \) by integrating the pressure field. If the value of the load \( F_w \) is close to the target load, proceed to the next step. Otherwise, increase or decrease the value of the basic oil film thickness and go to Step (2).

(4) Compute the performance function, i.e., Eq. (25). If the value of the judgment equation \( \frac{F_{\text{target}}}{F_{\text{total}}} \) is lower than the convergence criterion \( \epsilon \), this computation finalizes. Otherwise, go to Step (5).

(5) Solve the adjoint equation in Eq. (33) under the boundary condition in Eq. (34).

(6) Compute the gradient of the Lagrange function with respect to the oil film thickness in Eq. (35).

(7) Compute the modified gradient based on the Poisson equation, Eqs. (36) and (37).

(8) Update the oil film thickness using the shape update equation, i.e., Eq. (38), (42) or (48). And return to Step (2).

3. Numerical experiments on shape optimization

3.1. Computational conditions

The computational conditions are listed in Table 2. The finite element mesh is shown in Figure 5. In the numerical test, the target load \( F_{\text{target}} \) is given as 10 N, the lower surface velocity in the \( x \)-direction \( U \) is given as 1000.0 mm/s, the viscosity of fluid \( \eta \) is given as 0.08 Pa·s, the radius of the texture \( R \) is given as 6.0 mm, and the initial depth of the texture \( h_{\text{dep}} \) is given as 0.01 mm.
Table 2  Computational conditions.

| Parameter                        | Value     |
|----------------------------------|-----------|
| Number of nodes                  | 13905     |
| Number of elements               | 27488     |
| Target Load $F_{w1}$             | 100[N]    |
| Lower surface velocity in $x$ direction $U$ | 1000.0[mm/s] |
| Viscosity of fluid $\eta$        | 0.08[Pa·s]|
| Radius of a texture $R$          | 6.0[mm]   |
| Initial depth of a texture $h_{dep}$ | 0.01[mm] |
| $\Delta r$                      | $10^{-1}$ |
| $\Delta \beta$                  | $\frac{1}{4}$ |

Fig. 5  Finite element mesh. The whole domain is divided by a triangular mesh with giving 13,905 nodes and 27,488 elements.

3.2. Comparison of update equations

Numerical results of the shape optimization using Eqs. (38), (42) and (48) for the calculation model shown in Figure 6 were compared. Figure 6 shows the changes in the friction coefficient calculated under the condition of load $F_{w1} = 10$ N using Eqs. (38), (42) and (48). The vertical axis shows the friction coefficient and the horizontal axis represents number of iterations. Blue and green lines show numerical results for shape update equations based on the steepest descent and the conjugate gradient methods. Orange line indicates the result by the proposed shape update equation. Gray line is the difference in the friction coefficient between the shape update equation based on the steepest descent and the proposed methods. When the proposed shape update equation was used, smaller friction coefficient was consequently obtained, and the number of iterations was lower than with other shape update equations. And, in Figure 6, it is found that the value of gray line increases in earlier iteration. Because the amount of shape update is large in the earlier iterations, the proposed shape update equation quickly approaches the optimal shape. In later iterations, the amount of shape update becomes small and converges easily. Figures 7(a), 7(b) and 7(c) show the final shape using Eqs. (38), (48) and (42), respectively. The contour shows the oil film thickness based on the basic oil film thickness. The initial shape of the texture was cylindrical with a flat bottom. Shape optimization caused the shape of the texture to resemble a paraboloid. The final shape of the deeper texture was obtained using the proposed shape update equation. The cross-sectional shape at $y = 0$ of textures I, II, and III, shown in Figure 7(c), are compared, and are shown in Figure 8. Figure 8 shows a very minor difference in the final shape of the texture between the upstream and downstream of the flow. These results mean that the value of the friction coefficient decreases with increasing the depth of texture.
Fig. 6 History of friction coefficient when $F_w=10$ N. Blue and green lines show the history of the friction coefficient for shape update equations based on the steepest descent and the conjugate gradient methods. Orange line indicates the history of the friction coefficient for the proposed shape update equation. Gray line is the difference in the friction coefficient between the shape update equation based on the steepest descent and the proposed methods.

Fig. 7 Final shapes at $F_w=10$ N with 3 types of shape update equation. The contour shows the oil film thickness. The flat part with no texture is taken as the minimum value, and the maximum value is obtained by adding 0.055 mm to the minimum value.
3.3. Influence of target load on texture shape

The gradient with respect to design variable shown in Eq. (35) has several terms, including the pressure gradient. Changing the target load also changes the pressure distribution and affects the pressure gradient. It appears that the final shape differs when changing the value of the pressure gradient. Optimization is performed when the target load is set to 30, 70, and 100 N. Figures 9-11 show the changes in friction coefficient calculated under the condition of load $F_w = 30$, 70, and 100 N. Each horizontal axis represents the number of iterations. The friction coefficient can be reduced. Figure 12 shows the final shape of the textures when $F_w = 30$, 70, and 100 N. The contour shows the oil film thickness based on the basic oil film thickness. The final shapes of the centered texture under each load were compared. Figure 13 shows the cross-sectional shapes of these textures at $y = 0$. The solid line shows the texture depth of the final shape for each load. The dotted line indicates the texture depth of the initial shape. As the target load increases, the texture depth decreases. As the target load increases, the texture depth on the downstream side becomes deeper than that on the upstream side. Table 3 shows the values of frictional force, friction coefficient and basic oil film thickness in the final shape under each load. The parenthesis indicates the value in the initial shape. The frictional force decreased under all load conditions, and the friction coefficient also decreased. In addition, it is found that when the load is large, the rate of the friction coefficient is small.

Table 3  Each parameter when the load is set at 10, 30, 70, or 100 N. Frictional force, friction coefficient, value of basic oil film thickness, and reduction rate of friction coefficient are shown. Initial values are shown in parentheses.

| Load $F_w$[N] | 10  | 30  | 70  | 100 |
|---------------|-----|-----|-----|-----|
| Frictional force $F_f$[N] | 1.4(2.2) | 2.1(3.1) | 3.2(4.2) | 3.7(4.7) |
| Friction coefficient $\mu$[-] | 0.14(0.22) | 0.07(0.10) | 0.045(0.06) | 0.037(0.047) |
| Reduction rate of friction coefficient [%] | 36 | 30 | 25 | 21 |
| Basic oil film thickness $h_0$[mm] | 0.079(0.054) | 0.048(0.036) | 0.032(0.026) | 0.026(0.023) |
Fig. 9 History of friction coefficient when $F_w = 30$ N. Blue and green lines show the history of the friction coefficient for shape update equations based on the steepest descent and the conjugate gradient methods. Orange line indicates the history of the friction coefficient for the proposed shape update equation. Gray line is the difference in the friction coefficient between the shape update equation based on the steepest descent and the proposed methods.

Fig. 10 History of friction coefficient when $F_w = 70$ N. Blue and green lines show the history of the friction coefficient for shape update equations based on the steepest descent and the conjugate gradient methods. Orange line indicates the history of the friction coefficient for the proposed shape update equation. Gray line is the difference in the friction coefficient between the shape update equation based on the steepest descent and the proposed methods.
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**Fig. 11** History of friction coefficient when $F_w = 100$ N. Blue and green lines show the history of the friction coefficient for shape update equations based on the steepest descent and the conjugate gradient methods. Orange line indicates the history of the friction coefficient for the proposed shape update equation. Gray line is the difference in the friction coefficient between the shape update equation based on the steepest descent and the proposed methods.

**Fig. 12** Final shape. The contour shows the oil film thickness. The flat part with no texture is taken as the minimum value, and the maximum value is obtained by adding 0.055 mm to the minimum value.
3.4. Consideration of target load

The influence of the variation of the load on the final shape of the texture, as described in Subsection 3.3, was examined.

3.4.1. Effect on texture depth

Figure 13 shows that the texture depth of the final shape decreases with an increase in the target load. Figure 14 shows the relationship between texture depth and friction coefficient at loads $F_w = 10$, 30, 70, and 100 N. Based on the figure 14, the texture depth at which the friction coefficient is minimum for each load is shown in Table 4. The calculated results agree with the trend of the shape optimization results. Eq. (24) shows the friction coefficient, and the denominator of Eq. (24), i.e., the load, is constant. Here, we focus on the fraction of Eq. (24). The first term of the fraction is the product of the oil film thickness $h$ and pressure gradient. The pressure gradient increases when the load $F_w$ is large. To reduce the value of the second term of the fraction, it is necessary to reduce the oil film thickness $h$. Therefore, it is considered that the optimal shape of the texture decreases as the load increases.

![Diagram of texture depth and friction coefficient](image)

Fig. 13  Cross-sectional shape of the centrally located texture of the final shape for each load. Purple line indicates a load of 10 N, Blue line indicates a load of 30 N, Green line indicates a load of 70 N, Red line indicates a load of 100 N, and Dotted line indicates the initial shape.

![Diagram of friction coefficient vs. texture depth](image)

Fig. 14  Relationship between the texture depth and the friction coefficient for each load. The loads are set at 10, 30, 70, and 100 N. It is found that when the load is large, the depth of the texture at which the friction coefficient is minimum is smaller.
Table 4  Texture depth at which the friction coefficient is minimum for each load in Figure 14.

| Load $F_w$ [N] | 10    | 30    | 70    | 100   |
|----------------|-------|-------|-------|-------|
| Texture depth at which the friction coefficient is minimum [mm] | 0.036 | 0.042 | 0.056 | 0.100 |

3.4.2. Reason why the oil film thickness on the right side of the texture increased

When the target load was increased, the depth on the right side of the texture became deeper than that on the left side. The gradient with respect to the oil film thickness shown in Eq. (35) affects the updated texture shape. Eq. (35) has five terms. The distribution of each term in the first iteration was investigated under loads of 10 and 100 N. Their distributions are shown in Figs. 15-19. The following Items 1 and 2 were noticed.

Item 1. The distribution of the gradient in the texture is not symmetrical.
Item 2. The proportion of the distribution when the load is changed.

It appears that the term satisfying Items 1 and 2 causes the depth variation of the texture. The results are shown in Figs. 15-19 and Table 5. The first and fifth terms do not satisfy Items 1 and 2. The second, third and fourth terms satisfy Items 1 and 2. The second, third and fourth terms have pressure $p$ in common. It appears that the distribution of the second, third and fourth terms changes on changing the target load, which is the integrated value of the pressure. It also appears that changes in these terms affect the final shape of the texture.

Table 5  Whether the distribution of the five terms in the gradient equation with respect to oil film thickness Eq. (35) satisfies Items 1 and 2. Item 1 is that the distribution in the texture is not symmetrical, and Item 2 is that the proportion of the distribution changes when the load is changed.

| Item | First term | Second term | Third term | Fourth term | Fifth term |
|------|------------|-------------|------------|-------------|------------|
| Item1 | -          | satisfy     | satisfy    | satisfy     | -          |
| Item2 | -          | satisfy     | satisfy    | satisfy     | -          |

Fig. 15  The distribution of the first-term in Eq. (35) at first iteration. The distribution in the texture is symmetrical, and the proportion of the distribution does not change when the load is changed.
Fig. 16 The distribution of the second-term in Eq. (35) at first iteration. The distribution in the texture is not symmetrical, and the proportion of the distribution changes when the load is changed.

Fig. 17 The distribution of the third-term in Eq. (35) at first iteration. The distribution in the texture is not symmetrical, and the proportion of the distribution changes when the load is changed.
Fig. 18  The distribution of the fourth-term in Eq. (35) at first iteration. The distribution in the texture is not symmetrical, and the proportion of the distribution changes when the load is changed.

Fig. 19  The distribution of the fifth-term in Eq. (35) at first iteration. The distribution in the texture is symmetrical, and the proportion of the distribution does not change when the load is changed.

4. Conclusions

In this study, an algorithm to keep the load constant by adjusting the basic oil film thickness was proposed. It is used to optimize the texture shape to minimize the friction coefficient. The findings are summarized below.

- The friction coefficient was reduced while maintaining a constant load.
- A parabolic texture was obtained as the final shape.
- There was little difference in the final shape of the texture located upstream and downstream of the lubricant flow. Moreover, a new shape update equation was proposed by introducing Taylor expansion into the shape update equation using the steepest descent method. The findings are summarized below.
  - When the proposed shape update equation is used, the convergence value of the performance function is lower than shape update equations based on the steepest descent and the conjugate gradient methods.
  - When the proposed shape update equation is used, the number of iterations is fewer than the shape update equations based on the steepest descent and the conjugate gradient methods.
To investigate the effect of a constant load, some calculations were made assuming a change in the load. The findings are summarized below.

- As the target load increases, the texture depth shrinks.
- As the target load increases, the texture depth on the downstream side becomes deeper than that on the upstream side.

Because it is difficult to give an initial shape other than a flat bottom surface using FreeFEM++, the analysis to change the initial shape was not performed. In addition, cavitation caused by a high negative pressure was not fully considered. Therefore, it is necessary to investigate the dependences of the initial shape and cavitation. These remain as future studies.

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