INFINITE ENERGY SOLUTIONS FOR WEAKLY DAMPED QUINTIC WAVE EQUATIONS IN $\mathbb{R}^3$

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Abstract. The paper gives a comprehensive study of infinite-energy solutions and their long-time behavior for semi-linear weakly damped wave equations in $\mathbb{R}^3$ with quintic nonlinearities. This study includes global well-posedness of the so-called Shatah-Struwe solutions, their dissipativity, the existence of a locally compact global attractors (in the uniformly local phase spaces) and their extra regularity.

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1. Introduction

We study the following weakly damped wave equation:

$$\partial^2_t u + \gamma \partial_t u + (1 - \Delta_x) u + f(u) = g(t), \quad \{u, \partial_t u\}|_{t=0} = \{u_0, u'_0\}$$

in a whole space $\mathbb{R}^3$. Here $u(t,x)$ is the unknown function, $\Delta_x$ is the Laplacian with respect to variable $x$, $\gamma$ is a positive constant, $f: \mathbb{R} \to \mathbb{R}$ is a given non-linearity which is assumed to be of quintic growth ($f(u) \sim u^5$) and to satisfy some natural conditions (stated in (4.2)) and $g$ belonging to the space $L^1_{loc}(\mathbb{R}^+, L^2_{loc}(\mathbb{R}^3))$ or its closed subspace $L^1_b(\mathbb{R}^+, L^2_b(\mathbb{R}^3))$, see Section 2 for definitions of key functional spaces.

Dispersive or/dissipative semilinear wave equations of the form (1.1) model various oscillatory processes in many areas of modern mathematical physics including electrodynamics, quantum mechanics, nonlinear elasticity, 2000 Mathematics Subject Classification. 35B40, 35B45, 35L70.
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etc. and are of a big permanent interest, see [30, 4, 47, 13, 43, 45, 42] and references therein.

It is believed that the analytic properties and the dynamics as $t \to \infty$ of solutions for damped wave equations (1.1) strongly depend on the growth rate of the non-linearity $f(u)$ as $u \to \infty$. Indeed, in the most studied case of cubic and sub-cubic growth rate, the control of the energy norm which follows from the basic energy identity is sufficient to get the well-posedness of the problem in a natural energy space, dissipativity and further regularity of solutions as well as to develop the corresponding attractors theory in both autonomous and non-autonomous cases as well as in bounded and unbounded domains, see [2, 4, 13, 21, 28, 30, 34, 47, 50] and references therein.

We recall that the standard energy identity

$$E(\xi_u(t)) - E(\xi_u(\tau)) = -\gamma \int_{\tau}^{t} ||\partial_t u(s)||_{L^2}^2 \, ds + \int_{\tau}^{t} (\partial_t u(s), g(s)) \, ds, \quad \xi_u(t):=\{u(t),\partial_t u(t)\}$$

(1.2)

can be formally obtained by multiplying equation (1.1) by $\partial_t u$ and integrating over $t$ and $x$. Here

$$E(\xi_u) := \frac{1}{2} \left( ||\partial_t u||_{L^2}^2 + ||\nabla_x u||_{L^2}^2 + ||u||_{L^2}^2 + 2\langle F(u), 1 \rangle \right),$$

$$F(u) := \int_{0}^{u} f(z) \, dz \quad \text{and} \quad (u, v) := \int_{\mathbb{R}^3} u(x)v(x) \, dx.$$ This identity motivates the natural choice of the energy phase space and the class of energy solutions (as the solutions for which the energy functional is finite) and also gives the control of the energy norm of the solution. Namely, if the non-linearity has a sub-quintic or quintic growth rate, due to the Sobolev embedding theorem $H^1 \subset L^6$, the energy space is given by $\mathcal{E} := H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and in the supercritical case $f(u) \sim u|u|^q$ with $q > 4$, we need to take $\mathcal{E} := (H^1(\mathbb{R}^3) \cap L^{4+2/q}(\mathbb{R}^3)) \times L^2(\mathbb{R}^3)$ in order to guarantee the finiteness of the energy functional.

The case of super-cubic but sub-quintic growth rate ($2 < q < 4$) is a bit more complicated since the well-posedness of energy solutions is still an open problem here (at least in the case of bounded domains). However, this problem can be overcome using slightly more regular solutions than the energy ones for which, say, the mixed $L^4(0, T; L^{12}(\mathbb{R}^3))$ space-time norm is finite for every $T > 0$. These are the so-called Shatah-Struwe (or Strichartz) solutions for which the well-posedness is known. The existence of such solutions is strongly based on the Strichartz estimates for the linear wave equation which are now available not only for the whole space $\mathbb{R}^3$ or the torus $\mathbb{T}^3$, but also for bounded domains with Dirichlet or Neumann boundary conditions, see [6, 9, 10, 11, 43, 44, 45]. Moreover, crucial for the attractor theory is the following energy-to-Strichartz estimate for such solutions

$$\|u\|_{L^4(t,t+1;L^{12})} \leq Q(\|\xi_u(t)\|_{\mathcal{E}}) + Q(\|g\|_{L^1(t,t+1;L^2)}),$$

(1.3)
where $Q$ is monotone increasing function which is independent of $t$ and the solution $u$. In the sub-quintic case this estimate is a straightforward corollary of the linear Strichartz estimate and perturbation arguments. Energy-to-Strichartz estimate (1.3) allows us to deduce the control and establish the dissipativity of $u$ in the Strichartz norm based on the standard energy estimate. Since the control of this norm is enough for the uniqueness, the obtained control gives the well-posedness, dissipativity and the existence of global/uniform attractors in the way which is similar to the classical cubic case, see [19], [26] and [22] for the case of $\mathbb{R}^3$, $\mathbb{T}^3$ and a bounded domain endowed with the Dirichlet boundary conditions respectively (see also [37] for the case of damped wave equations with fractional damping).

In contrast to this, very few is known about the solutions of (1.1) in the supercritical (superquintic) growth rate of the non-linearity $f$. In this case the situation is somehow close to 3D Navier-Stokes problem, namely, we have the global existence of weak energy solutions for which the uniqueness is not known and the local existence of more regular solutions for which we do not know the global existence. It is expected that smooth solutions may blow up in finite time even in the defocusing case, but to the best of our knowledge there are no such examples. In this case the existing attractor theory is related to multivalued semigroups or/and the so-called trajectory dynamical systems and trajectory attractors, see [13, 12, 34, 51] (see also references therein).

We now turn to the most interesting borderline case of critical quintic non-linearity $f$ which is the main object of our study in this paper. In this case, the energy-to-Strichartz estimate (1.3) does not follow any more from the Strichartz estimate for the linear equation (at least in a straightforward way), so the proof of global existence for Shatah-Struwe solutions is usually based on the so-called non-concentration arguments and Pohozaev-Morawetz equality, see [6, 20, 23, 24, 25, 41, 40, 42, 45] (see also [10, 11] for the case of bounded domains with Dirichlet or Neumann boundary conditions). This approach allows us to construct a Shatah-Struwe solution $u$ such that the $L^4(0,T; L^{12})$-norm is finite for all $T$, but does not allow to get any control of this norm through the energy norm or to verify that the Strichartz norm does not grow as $T \to \infty$. This is clearly not sufficient for the attractors. Indeed, without the uniform control of the Strichartz norm as $T \to \infty$, this extra regularity may a priori be lost in the limit and the attractor may contain the solutions which are less regular than the Shatah-Struwe ones (for which we do not have the uniqueness theorem). Thus, the uniform control of the Strichartz norm is crucial for the attractor theory. This problem has been overcome in [22] where the asymptotic regularity and existence of global attractors for autonomous quintic wave equations in bounded domains of $\mathbb{R}^3$ has been established. The method suggested there is heavily based on the existence of global Lyapunov function and on the related convergence of the trajectories to the set of equilibria and, by this reason cannot be extended to the non-autonomous case or to the case of infinite-energy solutions.

An alternative method of verifying the asymptotic smoothing property for the quintic wave equation (1.1) has been recently suggested in [39]. This
method is based on a proper generalization of a direct energy-to-Strichartz estimate for the model quintic wave equation
\begin{equation}
\partial_t^2 u - \Delta_x u + u^5 = 0
\end{equation}
in \( \mathbb{R}^3 \) which in turn has been obtained earlier in [7] (see also [46]) via the profile decomposition technique. This method allowed us (in [39]) to build up more or less complete attractors theory for weakly damped quintic wave equation (1.1) with periodic boundary conditions in both autonomous and non-autonomous cases. Note that this result cannot be extended to the case of Dirichlet or Neumann boundary conditions since the analogue of energy-to-Strichartz estimates for equation (1.4) is still an open problem for this case.

In the present paper, which can be considered as a continuation of [39], we give a detailed study of the case where equation (1.1) is considered in the whole space \( x \in \mathbb{R}^3 \). Note first of all that the finite-energy case \( \xi u(t) \in \mathcal{E} \) can be treated exactly as in [39] and, by this reason, is not very interesting. The only difference is that, due to the non-compactness of Sobolev’s embedding \( H^1(\mathbb{R}^3) \subset L^2(\mathbb{R}^3) \), the sole asymptotic smoothing property will not give the asymptotic compactness (which is crucial for the existence of the attractor) and should be combined with the so-called tail estimates, see [15, 34] and references therein for more details.

However, the assumption that \( \xi u(t) \in \mathcal{E} \) is a big restriction since it assumes implicitly that the solution \( u(t,x) \) should decay sufficiently fast as \( |x| \to \infty \), so many physically relevant solutions (such as homogeneous equilibria, space or space-time periodic or/and quasi-periodic patterns as well as all solutions bifurcating from them) are automatically out of consideration. In addition, the extra conditions which we need to pose in order to get tail estimates are also restrictive and, in particular, for natural non-linearities like \( f(u) = u|u|^q - \kappa u \), a global attractor in \( \mathcal{E} \) does not exist if \( \kappa > 1 \).

By these reasons, it is natural, following [5, 18, 33, 34, 54] (see also references therein), to consider infinite energy solutions for which \( \xi u(t) \in \mathcal{E}_{\text{loc}} \) only, in other words, only the restrictions of \( \xi u(t) \) to bounded domains should have finite energy and the total energy may be infinite. In this case the key energy equality makes no sense any more (the energy is infinite) and a number of extra difficulties arises. We note from the very beginning that these difficulties are not only technical, for instance, in contrast to the case of finite energy, the corresponding attractors usually have infinite Hausdorff and fractal dimensions and infinite topological entropy, so principally new types of limit dynamics appear, see [55] for more details.

We will overcome the problem with infinite total energy by localizing the energy estimates using the machinery of weighted and uniformly local energy estimates, see [3, 15, 54, 34], as well as the finite speed of propagation property which is the fundamental property of wave equations, see e.g., [42], and which allows us to reduce the well-posedness result to the case of finite-energy solutions. This leads to our first main result.

**Theorem 1.1.** Let the non-linearity \( f \) satisfy some natural assumptions (see (4.2)), \( \xi u(0) \in \mathcal{E}_{\text{loc}} \) and \( g \in L^1_{\text{loc}}(\mathbb{R}^+, L^2_{\text{loc}}) \). Then, problem (1.1) possesses a unique global Shatah-Struwe solution \( u \) such that \( \xi u(t) \in \mathcal{E}_{\text{loc}} \) for all \( t \geq 0 \)
and, in addition,
\begin{equation}
(1.5) \quad u \in L^4_{\text{loc}}(\mathbb{R}^+, L^{12}_{\text{loc}}).
\end{equation}

We note that the local energy and Strichartz norms of $u$ can be estimated by the proper norms of the initial data and the external force $g$. However, these norms may grow in time if no extra assumptions on the growth of initial data and $g$ as $|x| \to \infty$ are posed, so we need to put extra restrictions if we want to speak about dissipativity and attractors. The natural choice of phase spaces for this is given by the so-called uniformly local phase spaces. The rigorous definitions of them will be given in Section 2 below and here we just mention that the uniformly local phase space $L^p_b(\mathbb{R}^3)$ consists of functions from $L^p_{\text{loc}}(\mathbb{R}^3)$ for which the following norm is finite:
\begin{equation}
\|u\|_{L^p_b} := \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^p(B^R_{x_0})},
\end{equation}
where $B^R_{x_0}$ stands for a ball of radius $R$ in $\mathbb{R}^3$ centered in $x_0$. The uniformly local version of Sobolev spaces and the energy space $E_b$ are defined analogously.

Our next result gives the dissipativity of the Shatah-Struwe solutions in uniformly local energy spaces.

**Theorem 1.2.** Let the assumptions of Theorem 1.1 hold and let, in addition, $\xi_u(0) \in \mathcal{E}_b$ and $g \in L^1_b(\mathbb{R}^+, L^2_{\text{loc}})$. Then the solution $u(t)$ constructed in Theorem 1.1 belongs to $E_b$ for all $t \geq 0$ and possesses the following dissipative estimate:
\begin{equation}
(1.6) \quad \|\xi_u(t)\|_{E_b} + \|u\|_{L^4(t,t+1; L^{12}_{\text{loc}})}^2 \leq Q(\|\xi_u(0)\|_{E_b}) e^{-\beta t} + Q(\|g\|_{L^1_b(\mathbb{R}^+, L^2_{\text{loc}})}),
\end{equation}
where the positive constant $\beta$ and monotone function $Q$ are independent of $t$, $u$ and $g$.

The analogue of this estimate for the energy norm $\|\xi_u(t)\|_{E_b}$ is well-known (see [31, 54]) and holds even in the case where $f$ has a super-critical growth rate, so the main novelty of (1.6) is exactly the dissipative control of the Strichartz norm which is crucial for the uniqueness and attractors.

We now turn to the attractors. For simplicity, we restrict ourselves to the autonomous case only
\begin{equation}
(1.7) \quad g(t) \equiv g \in L^2_b(\mathbb{R}^3).
\end{equation}
In this case, thanks to Theorem 1.2, the solution operators $S(t) : \mathcal{E}_b \to \mathcal{E}_b$ defined via
\begin{equation}
S(t)\xi_0 = \xi_u(t), \quad \xi_0 \in \mathcal{E}_b,
\end{equation}
where $\xi_u(t)$ is a Shatah-Struwe solution of (1.1) with the initial condition $\xi_u|_{t=0} = \xi_0$ generate, a dissipative semigroup in the phase space $\mathcal{E}_b$ and we may speak about its global attractor. We recall that, in contrast to the case of bounded domains, a compact global attractor usually does not exists even in the simplest cases if we work in uniformly local spaces, so the so-called locally compact global attractor is used instead, see [34] and also Section 6 below. By definition, a locally compact global attractor is a bounded closed set in $\mathcal{E}_b$ which is compact in $\mathcal{E}_{\text{loc}}$ only, strictly invariant and attracts the images of bounded sets in $\mathcal{E}_b$ also in the topology of $\mathcal{E}_{\text{loc}}$ only.

The next theorem can be considered as the third main result of the paper.
Theorem 1.3. Let the assumptions of Theorem 1.2 hold and let, in addition, \((1.7)\) is satisfied. Then, the solution semigroup \(S(t) : \mathcal{E}_b \to \mathcal{E}_b\) associated with equation \((1.1)\) possesses a locally compact global attractor \(A\) in \(\mathcal{E}_b\). This attractor is a bounded set of \(\mathcal{E}^1_b := H^2_b(\mathbb{R}^3) \times H^1_b(\mathbb{R}^3)\). Moreover, if the initial data \(\xi_u(0) \in \mathcal{E}^1_b\), then \(\xi_u(t) \in \mathcal{E}^1_b\) for all \(t \geq 0\) and the following estimate holds:

\[
\|\xi_u(t)\|_{\mathcal{E}^1_b} \leq Q(\|\xi_u(0)\|_{\mathcal{E}^1_b})e^{-\beta t} + Q(\|g\|_{L^2_b}),
\]

where the positive constant \(\beta\) and monotone function \(Q\) are independent of \(t\), \(u\) and \(g\). In other words, problem \((1.1)\) is globally well-posed and dissipative in \(\mathcal{E}^1_b\) as well.

As usual, the proof of this theorem is based on a decomposition of a solution \(u(t) = v(t) + w(t)\), where \(v(t)\) is exponentially decaying and \(w(t)\) is more regular and bootstrapping arguments. Similarly to [39], we establish the extra regularity \(w(t) \in \mathcal{E}^\alpha_b\) with \(\alpha \in (0, \frac{2}{5}]\) at the first step. And jump from \(\mathcal{E}^\alpha_b\) to \(\mathcal{E}^1_b\) at the second step. Although our proof follows in general the scheme suggested in [39], there are essential new difficulties here related with localization of Kato-Ponce type inequalities and the old scheme does not work directly. To overcome this difficulty, we introduce a new scheme of splitting \(u(t) = \tilde{v}(t) + \tilde{w}(t)\) of the solution \(u\) into a small and regular components which has an independent interest, see Remark 5.7 for the details.

The paper is organized as follows.

Section 2 gives an overview of weighted and uniformly local Sobolev spaces which are used in the paper. A special attention is paid to the localization of fractional Lebesgue-Sobolev spaces (=Bessel potential spaces) which are necessary for estimating the fractional norms of the differences \(f(u) - f(v)\) via the Kato-Ponce inequality. Some commutator estimates which are necessary to treat these spaces are proved in Appendix B.

The energy and Strichartz estimates for the linear equation \((1.1)\) (with \(f = 0\)) which are necessary for our study of the non-linear case are collected in Section 3.

Well-posedness and dissipativity of quintic wave equation \((1.1)\) is studied in Section 4. The proofs of Theorems 1.1 and 1.2 are also given there.

Decomposition of a solution \(\xi_u(t) \in \mathcal{E}_b\) into exponentially decaying and more regular (bounded in \(\mathcal{E}^\alpha_b, \alpha \leq \frac{2}{5}\)) parts is verified in Section 5. This is the most difficult part in the proof of Theorem 1.3. Some estimates for the fractional norms of the difference \(f(u) - f(v)\) are collected in Appendix A.

Finally, the existence and \(\mathcal{E}^1_b\) regularity of a locally compact global attractor for the considered equation \((1.1)\) is established in Section 6. At the end of this section we also discuss briefly some corollaries of the proved Theorem 1.3 as well as its possible generalizations including entropy estimates, exponential attractors and extensions to the non-autonomous case.

2. Weighted and uniformly local spaces

In this section we introduce a family of weighted and uniformly local Sobolev spaces which will be used throughout of the paper and briefly discuss useful relations between them, see e.g. [15, 34, 52] for more detailed
exposition. We start by introducing the class of admissible weight functions and the corresponding weighted Lebesgue spaces.

**Definition 2.1.** Let $\mu > 0$ be arbitrary. A function $\phi \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ to be called a weight function of exponential growth $\mu$ iff $\phi(x) > 0$ and there holds inequality

$$
\phi(x + y) \leq C_\phi e^{\mu |y|} \phi(x),
$$

for every $x, y \in \mathbb{R}^n$. Let $\phi$ be a weight of an exponential growth. Then the norm in the weighted Lebesgue space $L^p_{\phi}(\mathbb{R}^n)$, $1 \leq p \leq \infty$ is defined via

$$
\|u\|_{L^p_{\phi}} := \left(\int_{\mathbb{R}^n} \phi^p(x)|u(x)|^p \, dx\right)^{1/p}.
$$

The uniformly local analogue $L^p_{\phi, b}(\mathbb{R}^n)$ is defined by the following norm:

$$
\|u\|_{L^p_{\phi, b}} := \sup_{x_0 \in \mathbb{R}^n} \left\{ \phi(x_0)\|u\|_{L^p(B_{x_0}^R)} \right\},
$$

where $B_{x_0}^R$ stands for a ball of radius $R$ in $\mathbb{R}^n$ centered at $x_0$. We will write $L^p_b$ instead of $L^p_{b,1}$. The Sobolev spaces $W^{l,p}_{\phi}(\mathbb{R}^n)$ (resp. $W^{l,p}_{b,\phi}(\mathbb{R}^n)$) for $l \in \mathbb{N}$ are defined as spaces of distributions whose derivatives up to order $l$ belong to $L^p_{\phi}(\mathbb{R}^n)$ (resp. $L^p_{b,\phi}(\mathbb{R}^n)$).

**Remark 2.2.** One can easily check that if function $\phi$ is of exponential growth $\mu$ then so is the function $1/\phi$ with the same constant $C_\phi$. In other words (2.1) implies

$$
\phi(x + y) \geq C^{-1}_\phi e^{-\mu |x|} \phi(y),
$$

for every $x, y \in \mathbb{R}^n$. It is also not difficult to see that a sum and a product of two weights of exponential growth is also a weight of an exponential growth, see [15] for details.

The key examples of weight functions of exponential growth are $e^{-\varepsilon|x-x_0|}$, its smooth analogue $e^{-\varepsilon \sqrt{1+|x-x_0|^2}}$ and $(1+|x-x_0|^2)^\alpha$ where $\varepsilon$ and $\alpha$ belong to $\mathbb{R}$. It is easy to see that the first two examples are functions of exponential growth $|\varepsilon|$ and the last one is the weight function of exponential growth $\mu$ for arbitrary $\mu > 0$. In particular, the weights $\phi_{\varepsilon,x_0}(x) = e^{-\varepsilon \sqrt{1+|x-x_0|^2}}$ possess an extra important property

$$
|D^k_x \phi_{\varepsilon,x_0}(x)| \leq C_k \varepsilon^k \phi_{\varepsilon,x_0}(x), \quad x, x_0 \in \mathbb{R}^n,
$$

where $D^k_x$ stands for a collection of all partial derivatives of order $k$ and the constant $C_k$ depends only on $k$. This property allows us to reduce the study of weighted spaces to non-weighted ones. Indeed, let us define the multiplication operator:

$$
T_{\phi_{\varepsilon,x_0}} u := \phi_{\varepsilon,x_0} u.
$$

Then, as a corollary of (2.5), we get the following result, see [34].
Proposition 2.3. The operator $T_{\phi, x_0}$ realizes isomorphisms between the non-weighted space $W^{l,p}(\mathbb{R}^n)$ and its weighted analogue $W^{l,p}_{\phi, x_0} (\mathbb{R}^n)$ for any $l \in \mathbb{N}$. Moreover,

$$\|T_{\phi, x_0}\|_{\mathcal{L}(W^{l,p}_{\phi, x_0}, W^{l,p})} + \|T^{-1}_{\phi, x_0}\|_{\mathcal{L}(W^{l,p}, W^{l,p}_{\phi, x_0})} \leq C_{l,p},$$

where the constant $C_{l,p}$ is independent of $x_0$ and $\varepsilon$ such that $|\varepsilon| \leq 1$. The analogous result holds also for the spaces $W^{l,p}_{b,\phi, x_0}$ and $W^{l,p}_b$.

The next standard proposition gives more convenient equivalent norms in weighted and uniformly local spaces.

Proposition 2.4. Let $\phi(x)$ be a weight function of an exponential growth rate $\mu$ and let $\varepsilon > \mu$. Then, for every $u \in L^p_\phi(\mathbb{R}^n)$, $1 \leq p < \infty$, the following estimate holds:

$$\|u\|_{L^p_\phi}^p \leq \int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{L^p_{\phi, x_0}}^p \, dx_0 \leq C_2 \|u\|_{L^p_\phi}^p,$$

where the constants $C_1$ and $C_2$ depend only on $\mu$ and $\varepsilon$ and are independent of $u$ and $\phi$. Analogously, for every $u \in L^p_{b,\phi}(\mathbb{R}^n)$, we have

$$C_1 \|u\|_{L^p_{b,\phi}} \leq \sup_{x_0 \in \mathbb{R}^n} \left\{ \phi(x_0) \|u\|_{L^p_{b,\phi, x_0}} \right\} \leq C_2 \|u\|_{L^p_{b,\phi}}.$$

The proof of these estimates can be found in [15, 52].

Proposition 2.4 gives us a machinery for verifying various regularity estimates for linear PDEs by reducing them to the analogous non-weighted ones. We illustrate it on the following classical example:

$$\left(1 - \Delta\right)u(x) = g(x), \quad x \in \mathbb{R}^n. \tag{2.9}$$

Corollary 2.5. Let $\phi$ be a weight function of sufficiently small exponential growth $\mu$ ($\mu \leq \mu_0 \ll 1$) and let $g \in L^p_\phi(\mathbb{R}^n)$ for some $1 < p < \infty$. Then equation (2.9) possesses a unique solution $u \in W^{2,p}_\phi(\mathbb{R}^n)$ and the following estimate holds:

$$\|u\|_{W^{2,p}_\phi} \leq C_p \|g\|_{L^p_\phi}, \tag{2.10}$$

where the constant $C$ depends on $p$ and on the constant $C$ from inequality (2.1). Analogously, if $g \in L^p_{b,\phi}(\mathbb{R}^n)$ then the solution $u \in W^{2,p}_{b,\phi}(\mathbb{R}^n)$ and

$$\|u\|_{W^{2,p}_{b,\phi}} \leq C_p \|g\|_{L^p_{b,\phi}}. \tag{2.11}$$

Proof. We restrict ourselves to verifying the estimates only. The existence of a solution can be obtained using the standard approximation arguments.

Step 1. We start with the classical non-weighted maximal regularity estimate for the solutions of the elliptic equation (2.9), namely,

$$\|u\|_{W^{2,p}} \leq C_p \|g\|_{L^p}, \tag{2.12}$$

see e.g., [48].

Step 2. We get the analogue of (2.12) for the space $L^p_\phi$ with special weights $\phi = \phi_{\varepsilon, x_0}$ for small $\varepsilon > 0$ and arbitrary $x_0 \in \mathbb{R}^n$. To this end, we write $v = T_{\phi, x_0} u$ for the new variable $v$ which satisfies the equation
(2.13) \((1 - \Delta_x)v - B_{\varepsilon,x_0}v = T_{\phi_{\varepsilon,x_0}}g := \tilde{g},\)
\[B_{\varepsilon,x_0}v := 2\phi_{\varepsilon,x_0}\nabla_x \phi_{-\varepsilon,x_0} \nabla_x v + \phi_{\varepsilon,x_0}\Delta_x \phi_{-\varepsilon,x_0}v.\]

Then, according to Proposition 2.3, it is enough to verify the non-weighted \((L^p,W^{2,p})\)-estimate for equation (2.13). On the other hand, due to estimate (2.5), we have
\[(2.14) \|B_{\varepsilon,x_0}v\|_{L^p} \leq C\varepsilon \|v\|_{W^{1,p}},\]
so for sufficiently small \(\varepsilon > 0\), equation (2.13) is a small regular perturbation of equation (2.9), so the regularity estimate for this equation is an immediate corollary of (2.12), namely,
\[\|v\|_{W^{2,p}} \leq \|\tilde{g} + B_{\varepsilon,x_0}v\|_{L^p} \leq \|\tilde{g}\|_{L^p} + C\varepsilon \|v\|_{W^{2,p}},\]
and assuming that \(\varepsilon\) is small enough that \(C\varepsilon \leq 1/2\) we get the desired estimate for \(v\). Returning back to the variable \(u\) (and using Lemma 2.3 again), we arrive at
\[(2.15) \|u\|_{W^{2,p}_{\phi_{\varepsilon,x_0}}} \leq C_p \|\tilde{g}\|_{L^p_{\phi_{\varepsilon,x_0}}}.\]

**Step 3.** The case of arbitrary weight \(\phi\). We essentially use that the constant \(C_p\) in (2.15) is independent of \(x_0 \in \mathbb{R}\). Therefore, multiplying (2.15) by \(\phi(x_0)\) (where the exponential growth rate \(\mu\) of the weight \(\phi\) satisfies \((\mu < \varepsilon)\), taking \(p\)th power from both sides of the obtained inequality, integrating over \(x_0 \in \mathbb{R}^n\) and using (2.7) we get the desired estimate (2.11). Analogously, replacing integration by taking supremum over \(x_0 \in \mathbb{R}^n\), we get the desired estimate (2.11). This finishes the proof of the corollary. \(\square\)

**Remark 2.6.** The scheme described above works not only for the Laplace equation, but for many other types of equations (elliptic, parabolic, etc.), see [34] and references therein. It also works for obtaining higher regularity and regularity in fractional Sobolev spaces. We give above the detailed derivation of the simplest regularity estimate for the reader’s convenience only and will use the analogous results in what follows without further explanations.

The next useful estimate is actually a combination of (2.1) and Minkowski inequality.

**Corollary 2.7** (see, e.g., [15]). Let \(u \in L^p_{\phi}(\mathbb{R}^n)\), where \(\phi\) is a weight function of exponential growth \(\mu > 0\). Then for any \(1 \leq q \leq \infty\) and every \(\varepsilon > \mu\), the following estimate is valid:
\[(2.16) \left( \int_{\mathbb{R}^n} \phi(x_0)^{pq} \left( \int_{\mathbb{R}^n} \phi_{\varepsilon,x_0}^p(x)|u(x)|^p \, dx \right)^q \, dx_0 \right)^{1/q} \leq C \|u\|_{L^p_{\phi_{\varepsilon,x_0}}}},\]
where the constant \(C\) depends only on \(\varepsilon\), \(\mu\) and \(C_\phi\) from (2.1).

Indeed, thanks to Minkowski inequality and (2.1),
\[(2.17) \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \phi(x_0)\phi_{\varepsilon,x_0}(x)|u(x)|^p \, dx \right)^q \, dx_0 \right)^{1/q} \leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( (\phi(x_0)\phi_{\varepsilon,x_0}(x)|u(x)|)^{pq} \, dx \right)^{1/q} \, dx_0 \right)^{1/q} \leq \int \int \left( (\phi(x_0)\phi_{\varepsilon,x_0}(x)|u(x)|)^{pq} \, dx \right)^{1/q} \, dx_0 \right)^{1/q} \]
\[ \leq C_0^p \int_{\mathbb{R}^n} \phi(x)^p |u(x)|^p \left( \int_{\mathbb{R}^n} e^{pq(\mu-\varepsilon)|x-x_0|} \, dx_0 \right)^{1/q} \, dx = C \|u\|_{L_p}^p. \]

The next proposition gives another way to reduce the study of weighted spaces to the non-weighted case.

**Proposition 2.8.** Let \( \phi \) be a function of exponential growth \( \mu \) and let \( R > 0 \) be a fixed number. Then for any \( p \in [1; \infty) \) the following estimates are valid:

\[
(2.18) \quad C_1 \|u\|^p_{L_p^\phi} \leq \int_{\mathbb{R}^n} \phi^p(x_0) \|u\|^p_{L_p(B_{R_0}^0)} \, dx_0 \leq C_2 \|u\|^p_{L_p^\phi},
\]

where the constants \( C_1 \) and \( C_2 \) depend on \( R \), \( C_\phi \), \( p \), and \( \mu \) only.

For the proof of this estimate, see e.g., [15].

As an immediate corollary of this estimates, we get the equivalent norms in Sobolev spaces \( W_l^p(\mathbb{R}^n) \) for integer \( l > 0 \).

**Corollary 2.9.** Let \( l \in \mathbb{N}, 1 \leq p \leq \infty \). Then an equivalent norm in \( W_l^p(\mathbb{R}^n) \) is given by the following expression:

\[
(2.19) \quad \|u\|_{W_l^p} := \left( \int_{\mathbb{R}^n} \phi^p(x_0) \|u\|^p_{W_l^p(B_{R_0}^0)} \, dx_0 \right)^{1/p}.
\]

In particular we obtain that norms \((2.19)\) are equivalent for different \( R > 0 \).

We see that representation \((2.19)\) reduces weighted Sobolev norm to Sobolev norm on bounded domains. Particularly, this gives the benefit of using standard Sobolev embeddings theorems for bounded domains (see [34]). Moreover, in analogy to \((2.19)\) we are able to define fractional weighted Besov-Sobolev spaces. We recall that, for any domain \( V \) with smooth boundary and any \( s > 0 \), \( s \notin \mathbb{N} \), the space \( W^{s,p}(V) = B^s_{p,p}(V) \) is defined via the following norm:

\[
\|u\|^p_{W^{s,p}(V)} = \|u\|^p_{W_s^p(V)} + \sum_{|\alpha|=|s|} \int_{x \in V} \int_{y \in V} \left| \frac{\partial^\alpha u(x) - \partial^\alpha u(y)}{|x-y|^{n+\{s\}p}} \right|^p dx dy,
\]

where \([s]\) and \( \{s\}\) denote integer and fractional part of \( s \) respectively. As usual, for negative non-integer \( s \), the space \( W^{s,p}(V) \) is defined by duality, see [48] for the details.

**Definition 2.10.** Let \( s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \) and \( R > 0 \) be fixed numbers and let \( \phi \) be a weight function with an exponential growth \( \mu \). The equivalent norms in the space \( W^{s,p}_{\hat{\phi}}(\mathbb{R}^n) \) are defined by

\[
(2.20) \quad \|u\|^p_{W^{s,p}_{\hat{\phi}}} := \left( \int_{\mathbb{R}^n} \phi^p(x_0) \|u\|^p_{W^{s,p}(B_{R_0}^0)} \, dx_0 \right)^{1/p},
\]

where \( R > 0 \) is arbitrary. We will write \( \|u\|_{W^{s,p}_{\hat{\phi}}} \) instead of \( \|u\|_{W^{s,p}_{\hat{\phi},1}} \).

It is not difficult to check that norms defined by \((2.20)\) are indeed equivalent for different \( R > 0 \) as well as \((2.20)\) gives usual norm for \( W^{s,p}(\mathbb{R}^n) \) if we take \( \phi \equiv 1 \) (see [15]). Hence the above definition is natural. It is also straightforward to check that, analogously to \((2.7)\),

\[
(2.21) \quad \|u\|^p_{W^{s,p}_{\hat{\phi}}} \sim \int_{\mathbb{R}^n} \phi(x_0)^p \|\phi_{\varepsilon,x_0} u\|^p_{W^{s,p}(\mathbb{R}^n)} \, dx_0,
\]
if \( \mu < \varepsilon \), so the analogues of Proposition 2.3 and Corollary 2.5 hold for fractional weighted Besov-Sobolev spaces as well.

**Remark 2.11.** It is useful to introduce the following notation for the above mentioned equivalent norms in \( W_b^{s,p}(\mathbb{R}^n) \):

\[
\|u\|_{W_b^{s,p}} := \sup_{x_0 \in \mathbb{R}^n} \|u\|_{W^{s,p}(B_{R}^{x_0})}.
\]

Then, the equivalence means that, for any \( R_1, R_2 > 0 \),

\[
C_{R_1,R_2}^{-1} \|u\|_{W_b^{s,p},R_1} \leq \|u\|_{W_b^{s,p},R_2} \leq C_{R_1,R_2} \|u\|_{W_b^{s,p},R_1}
\]

for some positive constant \( C_{R_1,R_2} \). In particular, the case \( R_1 = 2R_2 = R \geq 1 \) is especially interesting for us. Note that in this case the constant \( C_{R_1,R_2} \) is actually independent of \( R_1 \) and \( R_2 \). The last fact can be easily verified using scaling arguments.

We also need the scale of weighted Lebesgue-Besov spaces \( H_\phi^{s,p}(\mathbb{R}^n) \) (or Bessel potential spaces). Recall that in the non-weighted case they are usually defined via the Fourier transform:

\[
H_\phi^{s,p}(\mathbb{R}^n) := \{ u \in S'(\mathbb{R}^n), \|u\|_{H_\phi^{s,p}} := \|F^{-1}((1 + |\xi|^2)^{s/2}F u)\|_{L^p} < \infty \},
\]

where \( F \) is a Fourier transform, \( s \in \mathbb{R} \) and \( 1 < p < \infty \). Alternatively, these spaces can be defined as domains of fractional powers of the operator \( 1 - \Delta_x \) in \( L^p \):

\[
H_\phi^{s,p}(\mathbb{R}^n) = D((1 - \Delta_x)^{s/2}).
\]

It is well-known that \( W_\phi^{s,p}(\mathbb{R}^n) = H_\phi^{s,p}(\mathbb{R}^n) \) if \( p = 2 \) or \( s \in \mathbb{Z} \). But for non-integer \( s \geq 0 \), we have the proper inclusion

\[
H_\phi^{s,p}(\mathbb{R}^n) \subset W_\phi^{s,p}(\mathbb{R}^n) \quad \text{if } p > 2
\]

and the opposite proper inclusion if \( p < 2 \), see \[48\] for details.

The space \( H_\phi^{s,p}(V) \), where \( V \) is a smooth bounded domain in \( \mathbb{R}^n \) (we will consider in this paper only the case \( V = B_{x_0}^R \)), is usually defined as a restriction of \( H_\phi^{s,p}(\mathbb{R}^n) \) to \( V \):

\[
H_\phi^{s,p}(V) = \{ v \in D'(V), \exists u \in H_\phi^{s,p}(\mathbb{R}^n), \ u|_\Omega = v \}
\]

endowed with the standard factor-norm. It is also known that the restriction operator \( u \to u|_V \) is a retraction and the corresponding co-retraction (extension operator) can be chosen independently of \( 1 < p < \infty \) and \( |s| \leq N \) for every fixed \( N \in \mathbb{N} \), see \[48\]. Mention also a useful relation

\[
(2.22) \quad H_\phi^{s,p}(V) = [L^p(V), W^{1,p}(V)], \ 0 < s < 1,
\]

where \([,\cdot,\cdot]\) means complex interpolation, see \[48\]. Throughout of the paper we will write below \( H^s \) instead of \( H^{s,2} \).

The main reason for us to use fractional Lebesgue-Sobolev spaces is the following Kato-Ponce estimate which is crucial for obtaining the further regularity of solutions for the considered damped wave equation and which is naturally formulated exactly in these spaces.
Proposition 2.12. Let $V$ be a bounded domain with smooth boundary and let $0 < \alpha < 1$ and $1 < r < \infty$. Then,
\[
\|uv\|_{H^{\alpha,r}(V)} \leq C\|u\|_{L^{p_1}(V)}\|v\|_{H^{\alpha,q_1}(V)} + C\|v\|_{L^{p_2}(V)}\|u\|_{H^{\alpha,q_2}(V)},
\]
where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1}$, $1 < p_i, q_i < \infty$.

The proof of this estimate can be found, e.g., in [6] for $V = \mathbb{R}^n$. The general case is reduced to the case $V = \mathbb{R}^n$ using the extension operator.

Remark 2.13. Mention also one more obvious, but useful property of the introduced norms. Namely, let $\psi_{x_0} \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\psi_{x_0}(x) \equiv 1$ for $|x - x_0| \leq 1$ and $\psi_{x_0}(x) \equiv 0$ if $|x - x_0| \geq 3/2$. Then
\[
\|
\phi_{\varepsilon, x_0}(1 - \Delta_x)^{s/2}u - (1 - \Delta_x)^{s/2}(\phi_{\varepsilon, x_0}u)\|_{L^p} \leq C_p\varepsilon\|u\|_{L^p_{\phi_{\varepsilon, x_0}}},
\]
where the constant $C_p$ depends on $p$ and $s$. Moreover, for any $\psi \in C_0^\infty(\mathbb{R}^n)$ and sufficiently small $\varepsilon > 0$,
\[
\|
\psi(1 - \Delta_x)^{s/2}u - (1 - \Delta_x)^{s/2}(\psi u)\|_{L^p} \leq C_p\varepsilon, \psi, x_0\|u\|_{L^p_{\phi_{\varepsilon, x_0}}}.
\]

Although these estimates are more or less standard, we sketch the proof in Appendix A below.

Corollary 2.15. Let $0 < s < 1$ and $|\varepsilon|$ be small enough. Then
\[
C_1\|
\phi_{\varepsilon, x_0}(1 - \Delta_x)^{s/2}u\|_{H^{s,p}} \leq \|
\phi_{\varepsilon, x_0}(1 - \Delta_x)^{s/2}u\|_{H^{s,p}} \leq C_2\|
\phi_{\varepsilon, x_0}u\|_{H^{s,p},}
\]
where the constants $C_1$ are independent of $x_0 \in \mathbb{R}^n$.

Indeed, according to (2.25),
\[
(1 - C_p|\varepsilon|)\|
\phi_{\varepsilon, x_0}u\|_{H^{s,p}} \leq \|
\phi_{\varepsilon, x_0}(1 - \Delta_x)^{s/2}u\|_{L^p} \leq (1 + C_p|\varepsilon|)\|
\phi_{\varepsilon, x_0}u\|_{H^{s,p}}.
\]

We are now ready to define the spaces $H^{s,p}_\phi(\mathbb{R}^n)$ and $H^{s,p}_{b,\phi}(\mathbb{R}^n)$.

Definition 2.16. Let $s \in \mathbb{R}$ and $1 < p < \infty$ and let $\phi$ be a weight of sufficiently small exponential growth rate $\mu$. Then the norms in the spaces $H^{s,p}_\phi(\mathbb{R}^n)$ and $H^{s,p}_{b,\phi}(\mathbb{R}^n)$ are defined by
\[
\|u\|_{H^{s,p}_\phi} := \int_{\mathbb{R}^n} \phi(x_0)^{p}\|u\|_{H^{s,p}(B_{x_0})}^p \, dx_0
\]
and
\[
\|u\|_{H^{s,p}_{b,\phi}} := \sup_{x_0 \in \mathbb{R}^n} \left\{ \phi(x_0)\|u\|_{H^{s,p}(B_{x_0})} \right\}
\]
respectively.
Corollary 2.17. Let $s \in \mathbb{R}$, $1 < p < \infty$ and let $\phi$ be a weight function of a sufficiently small exponential growth $\mu$. Then, for any $R > 0$ and sufficiently small $\varepsilon > 0$, we have
\[
\tag{2.30}
C_1 \int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{H^{s,p}(B_R^0)}^p \, dx_0 \leq 
\leq \int_{\mathbb{R}^n} \phi(x_0)^p \|\phi_{\varepsilon,x_0} u\|_{H^{s,p}(\mathbb{R}^n)}^p \, dx_0 \leq C_2 \int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{H^{s,p}(B_R^0)}^p \, dx_0,
\]
where the constants $C_1$ and $C_2$ may depend on $R$.

Proof. We give the proof for the case $0 < s < 1$ only (since we have $W^{s,p} = H^{s,p}$ for integer $s$, the general case can be reduced to this particular one). Moreover, analogously to (2.24), we have
\[
\tag{2.31}
\|u\|_{H^{s,p}(B_R^0)} \leq C \|\phi_{\varepsilon,x_0} u\|_{H^{s,p}(B_R^0)} \leq C \|\phi_{\varepsilon,x_0} u\|_{H^{s,p}(\mathbb{R}^n)},
\]
so the left inequality of (2.30) is obvious. To prove the right inequality, we assume for simplicity that $R = 2$ and use (2.27), (2.26) with $\psi = \psi_{x_0}$ and together with (2.18) to get
\[
\tag{2.32}
\int_{\mathbb{R}^n} \phi(x_0)^p \|\phi_{\varepsilon,x_0} u\|_{H^{s,p}(\mathbb{R}^n)}^p \, dx_0 \leq 
\leq C \int_{\mathbb{R}^n} \phi(x_0)^p \|\phi_{\varepsilon,x_0} (1 - \Delta_x)^{s/2} u\|_{L^p(\mathbb{R}^n)}^p \, dx_0 \leq 
\leq C \int_{\mathbb{R}^n} \phi(x_0)^p \|\phi_{\varepsilon,x_0} (1 - \Delta_x)^{s/2} u\|_{L^p(\mathbb{B}_{1,0}^1)}^p \, dx_0 
\leq C \int_{\mathbb{R}^n} \phi(x_0)^p \|\phi_{\varepsilon,x_0} u\|_{H^{s,p}(\mathbb{R}^n)}^p \, dx_0 + C \int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{L^p_{\phi,\varepsilon,x_0}} \, dx_0 
\leq 2C \int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{H^{s,p}(B_2^0)}^p \, dx_0.
\]
Here we have implicitly used that, due to (2.7) and (2.18),
\[
\int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{L^p_{\phi,\varepsilon,x_0}} \, dx_0 \leq C' \|u\|_{L^p_{\phi}} \leq C'' \int_{\mathbb{R}^n} \phi(x_0)^p \|u\|_{L^p_{\phi}}^p \, dx_0
\]
and the corollary is proved. \hfill \Box

Remark 2.18. Estimates obtained in Proposition 2.14 and Corollaries 2.15 and 2.17 show that the results concerning embeddings and regularity in the weighted spaces $H^{s,p}_\phi$ and $H^{s,p}_\theta$ can be obtained in the same way as for the spaces $W^{s,p}_\phi$ and $W^{s,p}_\theta$. We will use this fact in a sequel without further details.

Note also that the above mentioned scheme gives also the weighted Kato-Ponce estimate which has an independent interest:
\[
\tag{2.33}
\|uv\|_{H^{\alpha+r,p_1,q_1}} \leq C \|u\|_{L^{p_1}_{\phi_1}} \|v\|_{H^{\alpha+r,q_1}_\phi} + C \|v\|_{L^{p_2}_{\phi_2}} \|u\|_{H^{\alpha+r,q_2}_{\phi_2}},
\]
where the exponents $\alpha, r, p_i, q_i$ are the same as in Proposition 2.12 and $\phi_i$ are weights of sufficiently small exponential growth rate. We need not this result for what follows, so we leave its rigorous proof to the reader.
We will systematically use in what follows the spaces of functions \( u(t, x) \) which have different regularity with respect to time \( t \) and space \( x \) variables, for instance \( L^p(\mathbb{R}, L^q(\mathbb{R}^n)) \), \( L^p(\mathbb{R}, H^{s,q}(\mathbb{R}^n)) \) or/and their weighted and uniformly local analogue. In slight abuse of notations we denote by \( L^p(A, B; L_b^p(\mathbb{R}^n)) \) and \( L_b^p(A, B; L_b^p(\mathbb{R}^n)) \). where \(-\infty \leq A < B \leq \infty\), the spaces generated by the following norms:

\[
\|u\|_{L^p(A, B; L_b^p)} := \sup_{x_0 \in \mathbb{R}^n} \|u\|_{L^p(A, B; L_b^p(\mathbb{R}^n))}
\]

and

\[
\|u\|_{L_b^p(A, B; L_b^p)} := \sup_{x_0 \in \mathbb{R}^n} \sup_{T \in [A, B]} \|u\|_{L^p(T, \min\{B, T+1\}; L^q(\mathbb{R}^n))},
\]

respectively. The spaces \( L^q(0, T; H^{s,p}_b) \) and \( L_b^p(0, T; H^{s,p}_b) \) as well as spaces \( L_b^p(t, t+1; H^{s,p}_b) \) are defined analogously. Crucial is that the supremum with respect to \( x_0 \) or/and \( t \in \mathbb{R}_+ \) is always taken after the integration in time. We will not consider other type of spaces in our paper.

3. Linear wave equation: preliminaries and basic estimates

In this section we give the weighted analogues of the regularity result for the following damped wave equation:

\[
(3.1) \quad \partial_t^2 v + \gamma \partial_t v + (-\Delta_x + 1)v = g(t), \quad \xi_v|_{t=0} = \xi_0
\]

in the whole space \( x \in \mathbb{R}^3 \). Here and below \( \xi_v \) stands for the pair of functions \( v \) and \( \partial_t v \) \( (\xi_v := \{v, \partial_t v\}) \). The initial data \( \xi_0 \) will be taken from the energy spaces

\[
(3.2) \quad \mathcal{E}_\alpha^\alpha := H^{1+\alpha}_\text{loc}(\mathbb{R}^3) \times H^{\alpha}_\text{loc}(\mathbb{R}^3), \quad \alpha \in \mathbb{R},
\]

or from their weighted and uniformly local analogues \( (\mathcal{E}^\alpha_\theta^\theta \text{ and } \mathcal{E}_\theta^\theta) \) respectively. We will write \( \mathcal{E} \) instead of \( \mathcal{E}^\alpha_\theta \).

We will always assume here that \( \gamma > 0 \) is a fixed constant and the external force \( g \) satisfies

\[
(3.3) \quad g \in L^1(\mathbb{R}_+, H^\alpha_\text{loc}(\mathbb{R}^3)).
\]

Let us start by recalling the classical energy estimate for solutions of \( (3.1) \) in the non-weighted case and \( \alpha = 0 \).

**Proposition 3.1.** Let \( \xi_0 \in \mathcal{E} \) and \( g \in L^1(\mathbb{R}_+, L^2(\mathbb{R}^3)) \). Then problem \( (3.1) \) possesses a unique solution \( \xi_v \in C(\mathbb{R}_+, \mathcal{E}) \) and the following estimate holds:

\[
(3.4) \quad \|\xi_v(t)\|^2_{L^2} \leq C\|\xi_v(0)\|^2_{L^2} e^{-\beta t} + C \left( \int_{0}^{t} e^{-\beta(t-s)} \|g(s)\|_{L^2} ds \right)^2
\]

for some positive constants \( C \) and \( \beta \) depending only on \( \gamma \). Moreover, the function \( t \to \|\xi_v(t)\|^2_{L^2} \) is absolutely continuous and the following energy identity:

\[
(3.5) \quad \frac{1}{2} \frac{d}{dt}\|\xi_v(t)\|^2_{L^2} + \gamma \|\partial_t v(t)\|^2_{L^2} = (g(t), \partial_t v)
\]

holds for almost all \( t \in \mathbb{R}_+ \). Here and below \( (f, g) := \int_{\mathbb{R}^3} f(x)g(x) \, dx \) stands for the standard inner product in \( L^2(\mathbb{R}^3) \).
For the proof of this result, see e.g., [4, 47].

The next technical tool is the so-called Strichartz estimates which are crucial for the study of the non-linear case.

**Proposition 3.2.** Under the assumptions of Proposition 3.1 the solution \( v \) satisfies the following estimate:

\[
\|v\|_{L^p(0,1; L^{6p/5})} \leq C_p \left( \|\xi v(0)\|_E + \|g\|_{L^1(0,1; L^2)} \right)
\]

for all \( p \in (2, \infty) \).

For the proof of this estimate, see [42, 43, 45].

**Remark 3.3.** The most important case for us is \( p = 4 \) which gives \( L^4(L^{12}) \)-estimate for the solution \( v \). To control the nonlinearity we also need \( p = 5 \) which however can be derived from \( p = 4 \) and the energy estimate by using the following interpolation inequality:

\[
\|v\|_{L^5(L^{10})} \leq C \|v\|_{L^4(L^{12})} \|v\|_{L^\infty(L^6)},
\]

so we will state below the estimates for \( p = 4 \) only.

Combining Propositions 3.1 and 3.2, we get the following result.

**Corollary 3.4.** Let \( \xi v(0) \in E^\alpha \) and \( g \in L^1_{\text{loc}}(\mathbb{R}, H^{\alpha}(\mathbb{R}^3)) \) for some \( \alpha \in \mathbb{R} \). Then, for all \( \beta \in (0, \beta_0] \), the solution \( v(t) \) of problem (3.1) possesses the following estimate:

\[
\|\xi v(t)\|_{E^\alpha} + \left( \int_0^t e^{-\beta(t-s)} \|v(s)\|_{H^{\alpha,12}}^4 ds \right)^{1/4} \leq C \left( \|\xi v(0)\|_{E^\alpha} e^{-\beta t} + \int_0^t e^{-\beta(t-s)} \|g(s)\|_{H^{\alpha}} ds \right),
\]

where the positive constants \( C \) and \( \beta_0 \) are independent of \( t \geq 0 \), \( v \) and \( g \).

The proof of this estimate is straightforward and can be found in [39]. We mention here only that the general case \( \alpha \in \mathbb{R} \) is reduced to the case \( \alpha = 0 \) by applying the operator \((1 - \Delta_x)^{\alpha/2}\) to both sides of equation (3.1).

We will also need the finite speed propagation estimate for solutions of (3.1).

**Proposition 3.5.** Let the assumptions of Proposition 3.1 hold, \( x_0 \in \mathbb{R}^3 \) and \( R \in \mathbb{R}_+ \). Then the solution \( v \) of problem (3.1) satisfies the following estimate:

\[
\|\xi v(t)\|_{E(B^{R-t}_R)} \leq C \|\xi v(0)\|_{E(B^R_0)} + C \int_0^t \|g(s)\|_{L^2(B^{R-s}_{R_0})} ds,
\]

where \( 0 \leq t < R \) and the constant \( C \) is independent of \( R \), \( x_0 \) and \( t \).

See e.g., [42] for the proof of this estimate.

**Remark 3.6.** Let us define a cone

\[
C^R_{x_0} = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, |x - x_0| \leq R - t\}.
\]
Then, estimate (3.9) shows that the values of $v|_{C^R_{x_0}}$ depend only on the values of $\xi_\varepsilon(0)|_{B^R_{x_0}}$ and the values of $g|_{C^R_{x_0}}$. In particular, if for two solutions $v_1, v_2 \in C^1(\mathbb{R}^+, \mathcal{E}_{loc})$ of equation (3.1) and we know that
\[ \xi_{v_1}(0)|_{B^R_{x_0}} = \xi_{v_2}(0)|_{B^R_{x_0}} \quad \text{and} \quad g_1|_{C^R_{x_0}} = g_2|_{C^R_{x_0}}, \]
then $v_1|_{C^R_{x_0}} = v_2|_{C^R_{x_0}}$. In particular, this property allows us to verify the existence and uniqueness of solution $\xi_\varepsilon \in C^1(\mathbb{R}^+, \mathcal{E}_{loc})$ (with the initial data $\xi_0 \in \mathcal{E}_{loc}$ and $g \in L^1_{loc}(\mathbb{R}^+, L^1_{loc})$) using Proposition 3.1 for square integrable case. We will use this idea in the non-linear case as well.

We conclude the section by the weighted analogue of estimate (3.8).

**Corollary 3.7.** Let $\varepsilon \in \mathbb{R}$ be a sufficiently small positive number, $x_0 \in \mathbb{R}^3$ and $\alpha \in [0, 1]$. Let also $\xi_0 \in \mathcal{E}^\alpha_{\phi_{\varepsilon,x_0}}$ and $g \in L^1_{loc}(\mathbb{R}, H^\alpha_{\phi_{\varepsilon,x_0}}(\mathbb{R}^3))$. Then the solution $v(t)$ of problem (3.1) possesses the following estimate:

\[
(3.10) \quad \|\xi_\varepsilon(t)\|_{\mathcal{E}^\alpha_{\phi_{\varepsilon,x_0}}} + \left( \int_0^t e^{-4\beta(t-s)} \|v(s)\|^4_{H^\alpha_{\phi_{\varepsilon,x_0}}} \, ds \right)^{1/4} \leq C\|\xi_0\|_{\mathcal{E}^\alpha_{\phi_{\varepsilon,x_0}}} e^{-\beta t} + C\int_0^t e^{-\beta(t-s)} \|g(s)\|_{H^\alpha_{\phi_{\varepsilon,x_0}}} \, ds,
\]

where positive constants $C$ and $\beta$ are independent of $t \geq 0$ and $\xi, g$ and $\varepsilon$.

**Proof.** We give the proof for the fractional case $\alpha \in (0, 1)$ only. The case $\alpha = 0$ is much simpler and we leave the proof to the reader and the case $\alpha = 1$ can be reduced to $\alpha = 0$ by differentiation of the equation in $x$.

We use the trick with isomorphism $T_{\varepsilon,x_0} : H^\alpha_{\phi_{\varepsilon,x_0}}(\mathbb{R}^3) \to H^{\alpha,p}(\mathbb{R}^3)$, see estimate (2.27) and (2.7), described in the proof of Corollary 2.5. Namely, for proving the weighted estimate (3.10), it is sufficient to establish its non-weighted analogue (3.8) for the function $V = \phi_{\varepsilon,x_0} v$ which satisfies the perturbed analogue of (3.1):

\[
(3.11) \quad \partial_t^2 V + \gamma \partial_t V + (-\Delta_x + 1) V = \phi_{\varepsilon,x_0} g(t) + B_{\varepsilon,x_0} V, \quad \xi_v|_{t=0} = \phi_{\varepsilon,x_0} \xi_0,
\]

where the operator $B_{\varepsilon,x_0}$ is the same as in (2.13) and, therefore, satisfies the estimate
\[
\|B_{\varepsilon,x_0} V\|_{H^{\alpha}} \leq C \varepsilon \|V\|_{H^{1+\alpha}}.
\]

Thus, estimate (3.8) for $V$ follows from the analogous estimate for $v$ (treating the term $B_{\varepsilon,x_0} V$ as a perturbation) if $\varepsilon$ is small enough. This proves the corollary. \qed

4. **Quintic Wave Equation: Well-Posedness and Dissipativity**

The aim of this section is to study the infinite-energy solutions to the following semi-linear weakly damped wave equation:

\[
(4.1) \quad \partial_t^2 u + \gamma \partial_t u + (-\Delta_x + 1) u + f(u) = g(t), \quad \xi_u|_{t=0} = \xi_0
\]

in the whole space $x \in \mathbb{R}^3$. It is assumed that the nonlinearity $f \in C^2(\mathbb{R})$ has quintic growth rate:

\[
(4.2) \quad f(u) = u^5 + h(u), \quad |h''(u)| \leq C (1 + |u|^5), \quad h(0) = 0
\]
for some exponent $0 \leq q < 3$. We start with a general case where the initial data $\xi_0 \in \mathcal{E}_{\text{loc}}$ and $g \in L^1_{\text{loc}}((0,\infty);L^2_{\text{loc}}(\mathbb{R}^3))$, so we do not pose up to the moment any restrictions on the growth of the solution as $|x| \to \infty$.

**Definition 4.1.** A function $u(t)$ such that $\xi_u(t) \in C_{\text{loc}}(0,\infty)\cap \mathcal{E}_{\text{loc}}$ is a Shatah-Struwe (SS) solution of problem (4.1) if $\xi_u(t)|_{t=0} = \xi_0$,

$$
(4.3) \quad -\int_0^T (\partial_t u, \partial_t \phi)dt + \gamma \int_0^T (\partial_t u, \phi)dt + \int_0^T (\nabla u, \nabla \phi)dt + \int_0^T (u, \phi)dt + \int_0^T (f(u), \phi)dt = \int_0^T (g, \phi)dt,
$$

for all test functions $\varphi \in C_0^\infty((0,\infty) \times \mathbb{R}^3)$ and, in addition, the following extra space-time regularity holds:

$$
(4.4) \quad u \in L^4_{\text{loc}}([0,\infty), L^{12}_{\text{loc}}(\mathbb{R}^3)).
$$

**Remark 4.2.** As in the case of finite-energy solutions, extra regularity (4.4) is crucial for the uniqueness of the solution $u$. To the best of our knowledge the uniqueness of energy solutions is not known without this assumption even in the finite-energy case. Moreover, this assumption is also used in order to derive finite speed propagation inequalities which are crucial for the existence result as well.

We mention that, due to estimate (3.7) and growth restriction on $f$ this extra regularity gives us also that

$$
(4.5) \quad f(u) \in L^1_{\text{loc}}([0,\infty), L^2_{\text{loc}}(\mathbb{R}^3))
$$

and, therefore, we may treat the non-linearity $f(u)$ as an external force and use estimate (3.9) for the obtained linear equation.

The last remark allows us to verify the uniqueness of SS-solutions.

**Proposition 4.3.** Let the function $f$ satisfy (4.2). Then, for every two SS-solutions $u_1$ and $u_2$ of equation (4.1) (which correspond to different initial data and external forces) and every $R > 0$, $x_0 \in \mathbb{R}^3$, the following analogue of estimate (3.9) holds:

$$
(4.6) \quad \|\xi_{u_1}(t) - \xi_{u_2}(t)\|_{\mathcal{E}(B^R_{x_0})} \leq C\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{E}(B^R_{x_0})} + C \int_0^t \|g_1(s) - g_2(s)\|_{L^2(B^R_{x_0})} ds,
$$

where $0 < t < R$ and the constant $C$ depends on $R$, $x_0$ and the proper Strichartz norms of $u_1$ and $u_2$. In particular, SS-solution of (4.1) is unique.

**Proof.** Let $v(t) = u_1(t) - u_2(t)$. Then this function solves the equation

$$
(4.7) \quad \partial_t^2 v + \gamma \partial_t v + (1 - \Delta_x) v = g_1(t) - g_2(t) - [f(u_1(t)) - f(u_2(t))].
$$

Equation (4.7) has the form of (3.1) with the right-hand side belonging to $L^1_{\text{loc}}(\mathbb{R}^+, L^2_{\text{loc}}(\mathbb{R}^3))$, therefore, estimate (3.9) is applicable and gives

$$
(4.8) \quad \|\xi_v(t)\|_{\mathcal{E}(B^R_{x_0})} \leq C\|\xi_v(0)\|_{\mathcal{E}(B^R_{x_0})} + C \int_0^t \|g_1(s) - g_2(s)\|_{L^2(B^R_{x_0})} ds +
$$
Using assumptions (4.2) together with Hölder inequality and Sobolev embedding $H^1 \subset L^6$, we get

\begin{equation}
\|f(u_1(s)) - f(u_2(s))\|_{L^2(B_{R_0}^{R-a})} \leq C \int_0^s \|f(u_1(s)) - f(u_2(s))\|_{L^2(B_{R_0}^{R-a})} \, ds.
\end{equation}

Due to the extra regularity assumption (4.3), we have

\begin{equation}
\|f(u_1(s)) - f(u_2(s))\|_{L^2(B_{R_0}^{R-a})} \leq C \|v(s)\|_{L^2(B_{R_0}^{R-a})} \lesssim \|v\|_{H^1(B_{R_0}^{R-a})} \lesssim l(s)\|\xi_0\|_{E(B_{R_0}^{R-a})},
\end{equation}

where

\begin{equation}
l(s) := C(1 + \|u_1(s)\|_{L^2(B_{R_0}^{R-a})}^4 + \|u_2(s)\|_{L^2(B_{R_0}^{R-a})}^4) \in L^1(0, R),
\end{equation}

due to the extra regularity assumption (4.4). Inserting the obtained estimate into the right-hand side of (4.8) and applying the Gronwall inequality, we end up with the desired estimate (4.6) and finish the proof of the proposition. \hfill \Box

As in the linear case, estimate (4.6) allows to reduce the study of a general infinite energy case to the case of finite-energy solutions where the global well-posedness is known. Namely, we need the following result for the finite-energy case which is proved in [39].

**Proposition 4.4.** Let the function $f$ satisfy (4.2), $\xi_0 \in E$ and the external force $g \in L^1_{loc}(\mathbb{R}^+, L^2(\mathbb{R}^3))$. Then problem (4.1) possesses a unique global SS solution $u(t)$ and the following estimate holds:

\begin{equation}
\|\xi_u(t)\|_E + \|v\|_{L^4(0, t+1; L^2)} \leq Q(\|\xi_0\|_E) e^{-\alpha t} + Q(\|g\|_{L^1(0, t+1; L^2)}),
\end{equation}

where the positive constant $\alpha$ and monotone increasing function $Q$ are independent of $\xi_0$, $t \geq 0$ and $g$.

Combining Propositions 4.4 and 4.3, we get the following result.

**Theorem 4.5.** Let $\xi_0 \in E_{loc}$ and $g \in L^1_{loc}(\mathbb{R}^+, L^2_{loc}(\mathbb{R}^3))$ and let the nonlinearity $f$ satisfy (4.2). Then problem (4.1) possess a unique globally defined SS-solution $u(t)$. Moreover, this solution satisfies the following estimate:

\begin{equation}
\|\xi_u(t)\|_{E(B_{R_0}^{R-a})} + \|u(t)\|_{L^4(0, t+1; L^2(\mathbb{R}^3))} \leq Q(\|\xi_0\|_{E(B_{R_0}^{R-a})}) e^{-\alpha t} + Q(\|g\|_{L^1(0, t+1, L^2(\mathbb{R}^{R+a+1}))}),
\end{equation}

where the constant $\alpha > 0$ and monotone increasing function $Q$ are independent of $R > 0$, $x_0 \in \mathbb{R}^3$, $\xi_0$, $g$ and $t > 0$.

**Proof.** Indeed, the uniqueness of a solution is verified in Proposition 4.3. To construct the desired solution $u$ we utilize this proposition again. Namely, to get the value of $u$ in a cone $C_0^R$ for a given $R > 0$, we construct the initial data $\xi_0 \in E$ using the extension operator from $E(B_{R_0}^R)$ to $E$, take $\tilde{g}$ as zero extension of $g$ from the cone $C_0^R$ to the whole space and solve equation (4.1) with the data $\tilde{\xi}_0$ and $\tilde{g}$, where $\tilde{\xi}_0$ is the corresponding finite energy SS solution which exists due to Proposition 4.4.
Then, \( u = \tilde{u}|_{C_R^0} \) is a desired SS solution of the initial problem (4.1) in the cone \( C_R^0 \). Moreover, due to Proposition 4.3, this definition is independent of the choice of \( R \) (the solutions defined using different cones will coincide on a smaller cone). Therefore, increasing \( R \), we get the required global SS-solution \( u(t, x) \) of problem (4.1). Thus, the existence of a solution is also verified. Estimate (4.11) is also an immediate corollary of (4.10), (4.6) and the cut-off procedure described above and the theorem is proved. \( \square \)

We turn now to study the dissipativity of equation (4.1). We first note that even in the linear case this problem is not dissipative if we consider the initial data with sufficiently rapid growth rate as \(|x| \to \infty \) (this can be easily seen using the explicit formula for solutions in the linear case), so at least some restrictions on this growth rate should be posed in order to avoid growing in time solutions. Following the standard approach (see [34] and references therein for more details), we will consider problem (4.1) in the properly chosen uniformly local spaces. Namely, we assume from now on that

\[
\xi_0 \in \mathcal{E}_b := H^1_b(\mathbb{R}^3) \times L^2_b(\mathbb{R}^3), \quad g \in L^1_b(\mathbb{R}_+, L^2_b(\mathbb{R}^3))
\]

and study problem (4.1) in the uniformly local energy phase space \( \mathcal{E}_b \). The following theorem can be considered as the main result of this section.

**Theorem 4.6.** Let the assumptions of Theorem 4.5 hold and let, in addition, (4.12) be satisfied. Then the SS-solution \( u(t) \) of problem (4.1) belongs to \( \mathcal{E}_b \) for all \( t \geq 0 \) and satisfies the following estimate:

\[
(4.13) \quad \|\xi_u(t)\|_{\mathcal{E}_b}^2 + \|u\|_{L^4(t,t+1;L^{12}_b)}^4 \leq Q(\|\xi_0\|_{\mathcal{E}_b})e^{-\beta t} + Q(\|g\|_{L^1_b(\mathbb{R}_+;L^2_b)}),
\]

for some constant \( \beta > 0 \) and monotone nondecreasing function \( Q \) which are independent of \( u \) and \( t \geq 0 \).

**Proof.** Estimate (4.13) can be deduced also from the basic estimate (4.11), but then we need the explicit form of the function \( Q \) there, so we prefer to argue in a slightly different way. Namely, we first note that the regularity \( \xi_u(t) \in \mathcal{E}_b \) follows immediately from (4.11). Moreover, the following energy-to-Strichartz estimate is also guaranteed by this estimate:

\[
(4.14) \quad \|u\|_{L^4(t,t+1;L^{12}_b)} \leq Q(\|\xi_u(t)\|_{\mathcal{E}_b}) + Q(\|g\|_{L^1(t,t+1;L^2_b)}).
\]

Thus, it is enough to verify the dissipative estimate (4.13) for the energy norm \( \|\xi_u(t)\|_{\mathcal{E}_b} \) only. This can be done in a standard way using the weighted energy estimates. Namely, we need to multiply equation (4.1) by \( \phi^2_{\varepsilon,x_0}(\partial_t u + \delta u) \) for some small positive \( \delta \), get the weighted analogue of the standard energy estimate and finally take a supremum over \( x_0 \in \mathbb{R}^3 \) to derive the desired uniformly local energy estimate

\[
(4.15) \quad \|\xi_u(t)\|_{\mathcal{E}_b}^2 \leq Q(\|\xi_u(0)\|_{\mathcal{E}_b})e^{-\alpha t} + Q(\|g\|_{L^1_b(\mathbb{R}_+;L^2_b)}),
\]

see [31, 54] for the details. Since these arguments will be repeated in more details in the next section, we omit these details here. Combining (4.15) and (4.14) we get the desired dissipative estimate and finish the proof of the theorem. \( \square \)
5. Asymptotic smoothing property

In this section we verify that any SS-solution of our problem (4.1) can be split to exponentially decaying and more regular parts. For simplicity, we will consider only the case of autonomous equation, so we assume from now on that

\[ g \in L^2_1(\mathbb{R}^3). \]

The general case, say \( g \in L^1_1(\mathbb{R}^3, H^1_b(\mathbb{R}^3)) \) can be treated analogously, but we need not this since the study of non-autonomous attractors is out of scope of this paper.

Following [4, 39, 50], we split the solution \( u(t) \) of problem (4.1) as follows:

\[ u(t) = v(t) + w(t). \]

The decaying component \( v(t) \) is chosen to satisfy the following equation:

\[ \partial_t^2 v + \gamma \partial_t v + (1 - \Delta_x) v + L v + f(v) = 0, \quad \xi_v|_{t=0} = \xi_u(0), \]

where \( L \) is a sufficiently big positive number which will be fixed below. Finally, the smooth reminder \( w(t) \) solves the equation

\[ \partial_t^2 w + \gamma \partial_t w + (1 - \Delta_x) w + f(u) - f(v) = L v + g, \quad \xi_w|_{t=0} = 0. \]

We start with the \( v \)-component.

**Proposition 5.1.** Let the assumptions of Theorem 4.6 holds and let, in addition, \( \xi_u(0) \in E_0 \) and \( g \) enjoys (5.1). Then, for sufficiently large \( L = L(f) \), the SS-solution \( v(t) \) of problem (5.3) satisfies the following estimate:

\[ \|\xi_v(t)\|_{E_0} + \|v\|_{L^4(t; L^2; L^4)} \leq Q(\|\xi_u(0)\|_{E_0}) e^{-\beta t}, \]

where the constant \( \beta > 0 \) and non-decreasing function \( Q \) are independent of \( \xi_u(0) \in E_0, t \geq 0 \) and \( g \in L^2_1 \).

**Proof.** Let us fix sufficiently small \( \varepsilon > 0 \) and \( x_0 \in \mathbb{R}^3 \). Then, multiplying equation (5.3) by \( \phi_{\varepsilon, x_0}^2 \partial_t v + \kappa \phi_{\varepsilon, x_0}^2 v \) (with small \( \kappa > 0 \) which will be fixed later) and integrating over \( \mathbb{R}^3 \), we find

\[ \frac{d}{dt} E_v(t) + \kappa E_v(t) + P_v(t) = 0, \]

where

\[ E_v(t) := \frac{1}{2} \|\xi_v(t)\|^2_{E_{\varepsilon, x_0}} + \left( \phi_{\varepsilon, x_0}^2 F(v(t)) + \frac{L}{2} \|v\|^2 \right) + \kappa \left( \phi_{\varepsilon, x_0}^2 \partial_t v + \frac{\kappa^2}{2} \|v\|_{L^2_{\varepsilon, x_0}}^2 \right) \]

and

\[ P_v(t) := \frac{1}{2} \left( 2\gamma - 3\kappa \right) \|\partial_t v(t)\|^2_{L^2_{\varepsilon, x_0}} + \kappa \|\nabla v(t)\|^2_{L^2_{\varepsilon, x_0}} \]

\[ + \frac{1}{2} \kappa(1 - \gamma) \|v\|^2_{L^2_{\varepsilon, x_0}} - \kappa^2 \left( \phi_{\varepsilon, x_0} v, \phi_{\varepsilon, x_0} \partial_t v(t) \right)_{L^2} + \kappa \left( \phi_{\varepsilon, x_0}^2, f(v(t))v(t) + \frac{L}{2} \|v\|^2(t) - F(v(t)) \right) + 2(\phi_{\varepsilon, x_0} \nabla v(t), \nabla \phi_{\varepsilon, x_0} \partial_t v(t)) + 2\kappa(\phi_{\varepsilon, x_0} \nabla v(t), \nabla \phi_{\varepsilon, x_0} v(t)). \]
Note that our assumptions (4.2) on the nonlinearity $f$ give the following inequalities:

$$-Kv^2 \leq F(v) \leq f(v)v + Kv^2$$

for some positive $K$. By this reason, all terms in the definitions of $E_v(t)$ and $P_v(t)$ which contain the nonlinearity will be non-negative if we take $L \geq 2K$.

Fixing now $\kappa > 0$ and $\varepsilon > 0$ small enough and using inequality (2.5), we conclude that

$$P_v(t) \geq 0$$

and

$$\frac{1}{4}\|\xi_v(t)\|_{L^2_{\phi_\varepsilon-x_0}}^4 \leq E_v(t) \leq C(\|\xi_v(t)\|_{L^2_{\phi_\varepsilon-x_0}}^2 + (\phi_{\varepsilon,x_0}^2, F(v)),$$

we deduce from (5.6) that

$$\frac{d}{dt}E_v(t) + \kappa E_v(t) \leq 0.$$  

The Gronwall inequality together with the fact that $f(v)$ has a quintic growth rate and the embedding $H^1 \subset L^6$ now give

$$\|\xi_v(t)\|_{L^2_{\phi_\varepsilon-x_0}}^2 \leq C\left(\|\xi_v(0)\|_{L^2_{\phi_\varepsilon-x_0}}^2 + C\|\xi_v(t)\|_{L^6_{\phi_\varepsilon-x_0}}^6\right)e^{-\beta t},$$

for some positive constants $C$ and $\beta$ which are independent of $x_0$. Taking the supremum over $x_0 \in \mathbb{R}^3$, we arrive at the desired dissipative estimate (5.5) for the energy norm $\|\xi_v(t)\|_{E_v}^2$. So, it only remains to obtain its analogue for the Strichartz norm. We will do it in two steps.

**Step 1.** We apply energy-to-Strichartz estimate (4.14) (where $g = 0$ and $f$ is replaced by $f_L(v) = f(v) + Lv$) to get

$$\|v\|_{L^6_{(t,t+1;L^2_0)^2}} \leq Q(\|\xi_v(t)\|_{E_v}) \leq Q(\|\xi_v(0)\|_{E_v}).$$

Here we are unable to get the decaying estimate since we do not know that $Q(0) = 0$ (it is likely so, but to check this we need to revise the proof given in [39] as well as the proof of energy-to-Strichartz estimates given in [7] or [46] which we prefer not to do). So we need one more step.

**Step 2.** We apply *linear* Strichartz estimate to equation (5.3) treating the term $f_L(v)$ as a perturbation to get:

$$\|v\|_{L^6_{(t,t+1;L^2_0)^2}} \leq C(\|\xi_v(t)\|_{E_v} + C\|f_L(v)\|_{L^1_{(t,t+1;L^6_0)}})$$

Using estimate (3.7) and our assumptions (4.2) on the non-linearity, we deduce

$$\|f_L(v)\|_{L^1_{(t,t+1;L^6_0)}} \leq C(\|\xi_v(t)\|_{L^\infty_{(t,t+1;E_v)}} \left(1 + \|v\|_{L^4_{(t,t+1;L^1_0)}^4}^4\right)).$$

Combining these estimates with the already proved decaying estimate for the energy norm, we get the desired estimate (5.5) and finish the proof of the proposition. \(\square\)

We now turn to the most complicated $w$-component. Note first of all that estimates (4.13) and (5.5) give

$$\|\xi_w(t)\|_{E_v} + \|w\|_{L^4_{(t,t+1;L^1_0)}^4} \leq Q(\|\xi_v(0)\|_{E_v})e^{-\beta t} + Q(\|g\|_{L^6_0}),$$

but we need an analogue of this estimate in $E_v^\alpha$ for some $\alpha > 0$. We will derive it in two steps. At the first step we derive the exponentially divergent
analogue of this higher energy estimate which will be improved at the next step.

**Proposition 5.2.** Let the above assumptions hold and let \( \alpha \in (0, 2/5) \). Then the \( w \)-component of the SS-solution \( u \) of problem (4.1) satisfies the following estimate:

\[
\| \xi_w(t) \|_{L^3} + \| w \|_{L^4(\Omega \times [t, t+1]; H^{\alpha - \frac{1}{2}}_0)} \leq e^{Kt} \left( Q(\| \xi_u(0) \|_{L^3}) + Q(\| g \|_{L^\infty_1}) \right),
\]

for some monotone function \( Q \), which does not depend on \( \xi_u(0) \) and \( g \).

**Proof.** Let the cut-off function \( \psi(x) \in C_0^\infty(\mathbb{R}^3) \) be the same as in Remark 2.13 and let

\[
\psi_{R,x_0}(x) := \psi_0 \left( \frac{x - x_0}{R} \right),
\]

where the parameter \( R \geq 1 \) will be specified below (for the proof of this proposition we may fix \( R = 1 \), for what follows later we need \( R \gg 1 \)). Then, we have obvious estimates

\[
| \nabla_x \psi_{R,x_0}(x) | + | \Delta_x \psi_{R,x_0}(x) | \leq CR^{-1},
\]

where \( C \) is independent of \( R \) and \( x_0 \). Let us set

\[
v_{x_0} = \psi_{R,x_0}v, \quad w_{x_0} = \psi_{R,x_0}w \quad \text{and} \quad u_{x_0} := \psi_{R,x_0}u.
\]

Then \( w_{x_0} \) solves

\[
\partial_t^2 w_{x_0} + \gamma \partial_t w_{x_0} - \Delta_x w_{x_0} + w_{x_0} = -\psi_{R,x_0}(f(u) - f(v)) - \Delta_x \psi_{R,x_0}w - 2\nabla \psi_{R,x_0} \nabla w + Lv_{x_0} + \psi_{R,x_0}g, \quad \xi_{w_{x_0}}|_{t=0} = 0.
\]

Also, without loss of generality, we may assume that \( f'(0) = 0 \). Indeed, in general case we may just replace \( f(u) \) by \( \tilde{f}(u) := f(u) - f'(0)u \) and the extra term \( f'(0)(u - v) \) which will appear in the right-hand side of equation (5.4) is under the control due to estimates (4.13) and (5.12).

The right-hand side of this equation contains the term \( \psi_{R,x_0}g \) which is only \( L^2 \) and not \( H^\alpha \) and this prevents us to use the linear Strichartz estimate (3.8) directly. To overcome this difficulty, we introduce the functions

\[
\theta_{x_0} := (1 - \Delta_x)^{-1}(\psi_{R,x_0}g) \quad \text{and} \quad \tilde{w}_{x_0} := w_{x_0} - \theta_{x_0}.
\]

Then the last function solves

\[
\partial_t^2 \tilde{w}_{x_0} + \gamma \partial_t \tilde{w}_{x_0} + (1 - \Delta_x)\tilde{w}_{x_0} = -\psi_{R,x_0}(f(u) - f(v)) - \Delta_x \psi_{R,x_0}w - 2\nabla \psi_{R,x_0} \nabla w + Lv_{x_0}, \quad \xi_{w_{x_0}}|_{t=0} = \{-\theta_{x_0}, 0\}
\]

and we may apply (3.8) to this equation instead. Note also that due to the elliptic regularity and Sobolev embedding theorem,

\[
\| \theta_{x_0} \|_{H^2} + \| \theta_{x_0} \|_{H^{\alpha,12}} \leq C\| g \|_{L^\infty_1}
\]

(here we have used that \( \alpha \leq 2/5 \), there is no difference in estimation the corresponding energy and Strichartz norms of \( w_{x_0} \) and \( w_{x_0} \).

Applying linear Strichartz estimates (3.8) to equation (5.16) and using (4.13) and (5.5) together with (5.14), we find

\[
\| \xi_{w_{x_0}}(t) \|_{L^\infty_3} + \left( \int_0^t e^{-4\beta(t-s)} \| w_{x_0}(s) \|_{H^{\alpha,12}}^4 \, ds \right)^{\frac{1}{4}} \leq \]

\[
\frac{1}{2} \sum_{k=1}^K Q(\| \xi_u(0) \|_{L^3}) + Q(\| g \|_{L^\infty_1}) \cdot e^{Kt},
\]

where \( K = \max_{1 \leq k \leq K} \left( \frac{\sum_{k=1}^K Q(\| \xi_{w_{x_0}}(t) \|_{L^\infty_3})}{\sum_{k=1}^K Q(\| \xi_{w_{x_0}}(t) \|_{L^\infty_3})} \right) \).
The key problem is to estimate the integral in the right-hand side of (5.17) which contains non-linearity \( f \). To this end, we use estimate (A.5), see Appendix A, together with (4.13) and (5.5) to derive

\[
\|\psi_{x_0}(f(u) - f(v))\|_{H^\alpha(\mathbb{R}^3)} \leq C \left( 1 + \|u\|_{L^{12}(B_{t_0}^{R})} + \|v\|_{L^{12}(B_{t_0}^{R})} \right)^{4-\alpha} \left( 1 + \|u\|_{H^1(B_{t_0}^{R})} + \|v\|_{H^1(B_{t_0}^{R})} \right)^{\alpha} \\
\|w_{x_0}\|_{H^{1+\alpha}(\mathbb{R}^3)} \|w_{x_0}\|_{H^{\alpha,12}(\mathbb{R}^3)} \leq m_{R,x_0(t)} \left( 1 + \|u(t)\|_{L^{12}(B_{t_0}^{R})} + \|v(t)\|_{L^{12}(B_{t_0}^{R})} \right)
\]

and the constant \( K_R = K_R(\|\xi_u(0)\|_{E_{x}}, \|\eta\|_{L^2}) \).

Using now Holder’s inequality in time with exponents \( \frac{4}{4-\alpha} \) and \( \frac{1}{\alpha} \) we have the chain of inequalities as follows

\[
\left( \int_0^t e^{-\beta(t-s)} \|\psi_{x_0}(f(u) - f(v))\|_{H^\alpha} \, ds \right)^{1/4} \leq \left( \int_0^t e^{-k_0^\alpha \beta(t-s)} m_{R,x_0(s)} \|w_{x_0}\|_{H^{1+\alpha}} \, ds \right)^{1/4} \times
\]

\[
\left( \int_0^t e^{-4\beta(t-s)} \|w_{x_0}\|_{H^{\alpha,12}}^4 \, ds \right)^{1/4} \leq C \left( \int_0^t e^{k_0^\alpha \beta(t-s)} m_{R,x_0} \|w_{x_0}\|_{H^{1+\alpha}} \, ds \right)^{1/4} + \frac{1}{4} \left( \int_0^t e^{-4\beta(t-s)} \|w_{x_0}\|_{H^{\alpha,12}}^4 \, ds \right)^{1/4},
\]

where \( k_0 := \frac{4\alpha}{4-\alpha} \) and at the last step we used Young’s inequality with exponents \( \frac{1}{\alpha} \) and \( \frac{1}{1-k_0} \).

The second integral will be cancelled with the left-hand side of (5.17) and for the first one we continue the estimate using Young’s inequality in time with exponents \( \frac{1}{k_0} \) and \( \frac{1}{1-k_0} \):

\[
\left( \int_0^t e^{-k_0^\alpha \beta(t-s)} m_{R,x_0(s)} \|\xi_{w_{x_0}(s)}\|_{E_{x}} \, ds \right)^{1/k_0} \leq \frac{1}{k_0} \left( \int_0^t e^{-k_0^\alpha \beta(t-s)} m_{R,x_0(s)} \|\xi_{w_{x_0}(s)}\|_{E_{x}} \, ds \right)^{1/k_0} \leq \frac{1}{1-k_0} \left( \int_0^t e^{-k_0^\alpha \beta(t-s)} m_{R,x_0(s)} \, ds \right)^{1/1-k_0}.
\]
\[ \leq K_R \int_0^t e^{-k_\alpha \beta(t-s)} m_{R,x_0}(s)\|\xi_{w,x_0}(s)\|_{\mathcal{E}_\alpha} \, ds, \]

for some constant \( K_R \) depending on \( R \), \( \|\xi_u(0)\|_{\mathcal{E}_b} \) and \( \|g\|_{L^2_0} \) (here we have also implicitly used estimates \((5.5)\) and \((4.13)\)). Inserting these estimates into the right-hand side of \((5.17)\), we arrive at

\[
(5.22) \quad \|\xi_{w,x_0}(t)\|_{\mathcal{E}_\alpha} + \left( \int_0^t e^{-4\beta(t-s)} \|w_{x_0}(s)\|^4_{H^{1.12}} \, ds \right)^\frac{1}{2} \leq \\
\leq K_R \int_0^t e^{-k_\alpha \beta(t-s)} m_{R,x_0}(s)\|\xi_{w,x_0}(s)\|_{\mathcal{E}_\alpha} \, ds + \\
+ CR^{-1} \int_0^t e^{-\beta(t-s)}\|\xi_w(s)\|_{\mathcal{E}^b_{2.2R}} \, ds + Q_R(\|\xi_u(0)\|_{\mathcal{E}_b})e^{-\beta t} + Q_R(\|g\|_{L^2_0}).
\]

To complete the proof we need the following version of the Gronwall lemma.

**Lemma 5.3.** Let the function \( Y \in C_{loc}([\tau, \infty)) \) satisfies

\[ Y(t) \leq H(t) + \int_\tau^t e^{-\beta_0(t-s)}(l(s)Y(s) + G(s)) \, ds, \quad t \geq \tau \]

for some constant \( \beta_0 \), some functions \( H \in L^\infty_{loc}([\tau, \infty)), \ G \in L^1_{loc}([\tau, \infty)) \) and non-negative function \( l(t) \geq 0 \) such that \( l \in L^1_{loc}([\tau, \infty)) \). Then, the following estimate holds:

\[
(5.23) \quad Y(t) \leq H(t) + \int_\tau^t e^{-\beta_0(t-s)} + \int_\tau^t l(s)H(s) + G(s)) \, ds
\]

for all \( t \geq \tau \).

The proof of this lemma is standard and is left to the reader.

Using estimates \((4.13)\) and \((5.5)\), we see that

\[
(5.24) \quad K_R \|m_{R,x_0}\|_{L^1_b} \leq K,
\]

where the constant \( K \) depends on \( R \) and norms \( \|\xi_u(0)\|_{\mathcal{E}_b} \) and \( \|g\|_{L^2_0} \) (but is independent of \( x_0 \)). Applying the Gronwall inequality \((5.23)\) with

\[
Y(t) = \|\xi_{w,x_0}(t)\|_{\mathcal{E}_\alpha}, \ l(t) = K_R m_{R,x_0}(t), \ \beta_0 = k_\alpha \beta,
\]

\[
G(t) = \|\xi_w(t)\|_{\mathcal{E}^b_{2.2R}} \quad \text{and} \quad H(t) = Q_R(\|\xi_u(0)\|_{\mathcal{E}_b})e^{-\beta t} + Q_R(\|g\|_{L^2_0})
\]

to \((5.22)\) after the straightforward estimations, we arrive at

\[
(5.25) \quad \|\xi_{w,x_0}(t)\|_{\mathcal{E}_\alpha} \leq CR^{-1} \int_0^t e^{K(t-s)}\|\xi_w(s)\|_{\mathcal{E}^b_{2.2R}} \, ds + Qe^{Kt},
\]

where the constants \( Q \) and \( K \) depend on \( R \), \( \|\xi_u(0)\|_{\mathcal{E}_b} \) and \( \|g\|_{L^2_0} \). Note that in this estimate the constant \( C \) depends on \( K \) since we have used an obvious estimate

\[
\int_s^t l(\kappa) \, d\kappa \leq K + K(t - s),
\]

where we cannot avoid the first term \( K \) in the right-hand side. Thus, the constant \( C \) depends on \( R \) as well, but only through the constant \( K \). In the sequel we modify this estimate in such a way that \( K \) will be small and independent of \( R \), then the constant \( C \) will automatically be independent.
of $R$ as well (this observation is not important for the proof of the current proposition since we can just fix $R = 1$ here, but will be crucial for what follows later).

Taking the supremum with respect to $x_0 \in \mathbb{R}^3$ from both parts and using the standard inequalities

$$C_1 \|\xi_w\|_{\mathcal{E}_{b,R}} \leq \|\xi_w\|_{E_{b,2R}} \leq C_2 \|\xi_w\|_{\mathcal{E}_{b,R}},$$

where the constants $C_i$ are independent of $R$, we conclude that

$$\|\xi_w(t)\|_{\mathcal{E}_{b,R}} \leq C'R^{-1} \int_0^t e^{K(t-s)} \|\xi_w(s)\|_{\mathcal{E}_{b,R}} \, ds + Q_R(\|\xi_u(0)\|_{\mathcal{E}_b} + \|g\|_{L^2_b}) e^{Kt},$$

(5.26)

Applying the Gronwall inequality again, we end up with the desired inequality (5.13) for the $E_b$-energy. To get the estimate for the Strichartz part it is now enough to use (5.22). Thus, the proposition is proved.

We now want to improve estimate (5.13) and get its dissipative analogue. To this end, we need to obtain the analogue of estimate (5.22), where the $L^1_b$-norm of $m_{R,x_0}(t)$ will be small. Fixing also $R$ large enough, the Gronwall inequality would give us the desired dissipative estimate. The key idea is to split the solution $u$ in a sum

(5.27) $u(t) = \tilde{v}(t) + \tilde{w}(t)$

of more regular ($\tilde{w}$) and small ($\tilde{v}$) parts using already proved propositions 5.1 and 5.2 and then replace the function $u$ in (5.19) by its small part $\tilde{v}$ (estimating the term $f(u) - f(v)$ in a more accurate way). To this end, we need the following lemma.

**Lemma 5.4.** Let the assumptions of Proposition 5.1. Then, for every $\delta > 0$ and $\alpha \in (0, \frac{5}{4})$ there exists time $T_{\delta}$ depending on $\|\xi_u(0)\|_{\mathcal{E}_b}$ such that the SS-solution $u(t)$ of problem (4.1) possesses decomposition (5.27) such that, for all $t \geq T_{\delta}$,

(5.28) $\|\xi_{\tilde{v}}(t)\|_{\mathcal{E}_b} + \|\tilde{v}\|_{L^4(t,t+1;L^2_b)} \leq \delta$

and

(5.29) $\|\xi_{\tilde{w}}(t)\|_{\mathcal{E}_{b,12}} + \|\tilde{w}\|_{L^4(t,t+1;H^\alpha_{b,12})} \leq M_{\delta}$,

where the constant $M_{\delta}$ depends on $\|g\|_{L^2_b}$ and $\delta$, but is independent of $t$ and the norm $\|\xi_u(0)\|_{\mathcal{E}_b}$ of the initial data.

**Proof.** We first note that, due to the dissipative estimate (4.13), the solution $u(t)$ satisfies the estimate

(5.30) $\|\xi_u(t)\|_{\mathcal{E}_b} + \|u\|_{L^4(t,t+1;L^2_b)} \leq M_0 := 2Q(\|g\|_{L^2_b})$

for all $t \geq T'$, where $T'$ depends on the norm of the initial data only. By this reason, we may assume without loss of generality that $T' = 0$ and the solution $u$ satisfies (5.30) from the very beginning.
Let us now fix a big number $T = T(\delta)$ and consider decompositions $u(t) = v_n(t) + w_n(t)$, $t \geq nT$ which are defined by equations (5.3) and (5.4), but starting with the time moment $T_n = T(n - 1)$ with the initial data

$$\xi_{v_n}|_{t=T_n} = \xi_u|_{t=T_n}, \quad \xi_{w_n}|_{t=T_n} = 0.$$  

Then, due to estimates (5.5) and (5.13) we get

$$(5.31) \quad \|\xi_{v_n}(t)\|_{\mathcal{E}_b} + \|v_n\|_{L^4(t,t+1;L^6_h)} \leq Q(M_0)e^{-\beta t} \leq \delta$$

if $t \geq T_n + T = Tn$ and $T$ is chosen to satisfy

$$Q(M_0)e^{-\beta T} = \delta$$

and

$$(5.32) \quad \|\xi_{w_n}(t)\|_{\mathcal{E}_b} + \|w_n\|_{L^4(t,t+1;L^6_h)} \leq e^{Kt}(Q(M_0) + Q(\|g\|_{L^6_h})) \leq M_\delta$$

if $t \leq T_n + 2T = T(n + 1)$ and $M_\delta := e^{2Kt}(Q(M_0) + Q(\|g\|_{L^6_h}))$.

Finally, we define the desired functions $\tilde{v}(t)$ and $\tilde{w}(t)$ for $t \geq T_\delta := T$ as piece-wise continuous hybrid functions:

$$(5.33) \quad \tilde{v}(t) := v_n(t), \quad \tilde{w}(t) := w_n(t), \quad t \in [nT, (n + 1)T).$$

The desired properties of $\tilde{v}$ and $\tilde{w}$ are now guaranteed by estimates (5.31) and (5.32) and the lemma is proved. Crucial for the above construction is that we define the functions $v_n(t)$ and $w_n(t)$ starting from the initial time $T_n = T(n - 1)$, but using these functions in (5.33) on the time interval $t \in [Tn, T(n + 1)]$ only and this time shift of length $T$ guarantees that $v_n(t)$ is already small for $t \in [Tn, T(n + 1)]$ due to exponential decay of the $v_n$-component. Thus, the proposition is proved. \hfill \square

Since the function $v(t)$ defined by (5.3) is exponentially decaying, we may also assume that $T_\delta$ is large enough that

$$(5.34) \quad \|\xi_v(t)\|_{\mathcal{E}_b} + \|v\|_{L^4(t,t+1;L^6_h)} \leq \delta, \quad t \geq T_\delta.$$  

We are now ready to obtain a non-growing estimate for $w$.

**Proposition 5.5.** Let assumptions of Proposition 5.1 hold. Then the solution $w$ of problem (5.4) obeys the estimate

$$(5.35) \quad \|\xi_w(t)\|_{\mathcal{E}_b} + \|w\|_{L^4([t,t+1];H^2_b)} \leq Q(\|\xi_u(0)\|_{\mathcal{E}_b})e^{-\beta t} + Q(\|g\|_{L^2_b}),$$

for some positive constant $\beta$ and monotone function $Q$ which do not depend on $t$ and $\xi_u(0)$.

**Proof.** We will utilise again estimate (5.17), but will estimate the difference $f(u) - f(v)$ in a more accurate way. To this end, we first note that without loss of generality we may assume that $T_\delta = 0$ in Lemma 5.4. Indeed, in general case we just apply Strichartz estimate (3.8) to equation for $w$ starting not from $t = 0$, but from $t = T_\delta$ and estimate the initial data $\xi_w|_{t=T_\delta}$ using Proposition 5.2. This will give us the same type of estimate (5.17) up to maybe different function $Q_R$. As in the proof of Proposition 5.2, we also assume that $f'(0) = 0$.

We split the difference $f(u) - f(v)$ as follows

$$f(u) - f(v) = [f(\tilde{v} + \tilde{w}) - f(\tilde{v})] + [f(\tilde{v}) - f(v)].$$
The first term can be estimated using inequality (A.5) exactly as in the proof of Proposition (5.2). Thus, due to estimates (5.28) and (5.29), we have

\[ \int_0^t e^{-\beta(t-s)} \| \psi_{R,x_0} (f(\tilde{v} + \tilde{w}) - f(\tilde{v})) \|_{H^s} \, ds \leq Q_R = Q_R(M_0, \delta). \]

To estimate the second term we note that

\[ \tilde{v} - v = (u - \tilde{w}) - (u - w) = w - \tilde{w} \]

and, therefore, due to (A.8),

\[ \| \psi_{R,x_0} (f(\tilde{v}) - f(v)) \|_{H^s} \leq C(1 + \|\tilde{v}\|_{L^{12}(B_{2R}^0)} + \|v\|_{L^{12}(B_{2R}^0)})^{4-\alpha} \times \left\| \tilde{v} \right\|_{H^{1} (B_{2R}^0)} + \|v\|_{H^{1} (B_{2R}^0)})^{\alpha} \times \left( \|\psi_{R,x_0} w\|_{H^s} 1^{1-\alpha} \|\psi_{R,x_0} w\|_{H^{12}} + \|\psi_{R,x_0} \tilde{w}\|_{H^{12}} 1^{1-\alpha} \|\psi_{R,x_0} \tilde{w}\|_{H^{12}} \right). \]

The term containing \( \tilde{w} \) can be estimated exactly as in (5.36) and using (5.28) and (5.34) for estimating the first term, we arrive at

\[ \int_0^t e^{-\beta(t-s)} \| \psi_{R,x_0} (f(\tilde{v}) - f(v)) \|_{H^s} \, ds \leq \delta^\alpha \int_0^t e^{-\beta(t-s)} \tilde{m}_{R,x_0} (s)^{1-\alpha/4} \|\psi_{R,x_0} w\|_{L^{12}(B_{2R}^0)}^{1-\alpha} \|\psi_{R,x_0} w\|_{H^{12}}^{\alpha} \, ds + Q_R(M_0, \delta), \]

where

\[ \tilde{m}_{R,x_0} (t) := K_R (1 + \|v(t)\|_{L^{12}(B_{2R}^0)} + \|\tilde{v}(t)\|_{L^{12}(B_{2R}^0)})^4. \]

Inserting the obtained estimates into (5.17) and using the Hölder inequality (exactly as in the proof of Proposition 5.2), we end up with

\[ \int_0^t e^{-4\beta(t-s)} \| w_{x_0} (s) \|_{H^{12}}^4 \, ds \leq \delta^{\frac{7\alpha}{12}} \int_0^t e^{-k_a \beta(t-s)} \tilde{m}_{R,x_0} (s)^{1-\alpha/4} \|\psi_{R,x_0} w\|_{L^{12}(B_{2R}^0)}^{1-\alpha} \|\psi_{R,x_0} w\|_{H^{12}}^{\alpha} \, ds + \]

\[ + CR^{-1} \int_0^t e^{-\beta(t-s)} \|\xi_w (s)\|_{L^{12}}^4 \, ds + Q_R(\|\xi_u (0)\|_{L^4}^4) e^{-\beta t} + Q_R(\|g\|_{L^4}^4), \]

where in contrast to (5.22) we have an extra small parameter \( \delta \). Crucial for us that the \( L^1 \)-norm of \( \tilde{m}_{R,x_0} \) is independent of \( \delta \). Therefore, for every fixed \( R \) we may fix \( \delta = \delta(R) \) such that

\[ \delta^{\frac{7\alpha}{12}} \|\tilde{m}_{R,x_0}\|_{L^1} \leq \frac{1}{2} k_a \beta \]

and, therefore, the Gronwall inequality (5.23) applied to (5.39) gives the dissipative analogue of (5.25)

\[ \|\xi_{w_{x_0}} (t)\|_{L^\infty} \leq \frac{C'}{R} \int_0^t e^{-k_a \beta(t-s)/2} \|\xi_w (s)\|_{L^{12}}^2 \, ds + \]

\[ + Q_R(\|\xi_u (0)\|_{L^4}^4) e^{-k_a \beta/2} + Q_R(\|g\|_{L^4}^4). \]

It is important that the constant \( C' \) here is independent of \( R \). Taking the supremum over \( x_0 \in \mathbb{R}^3 \) from both sides of this inequality, analogously to (5.26) we arrive at
(5.41) \[ \| \xi_w(t) \|_{\mathcal{E}_b^\alpha} \leq \frac{C''}{R} \int_0^t e^{-k_{\alpha} \beta(t-s)/2} \| \xi_w(s) \|_{\mathcal{E}_b^\alpha} \, ds + \] \[ + Q_R(\| \xi_u(0) \|_{\mathcal{E}_b^\alpha}) e^{-k_{\alpha} \beta t/2} + Q_R(\| g \|_{L^2_b}). \]

Fixing now \( R > 0 \) large enough that \( C''/R \leq \frac{1}{4} k_{\alpha} \beta \) and applying the Gronwall inequality again, we get the desired dissipative estimate for \( \| \xi_w(t) \|_{\mathcal{E}_b^\alpha} \):
\[ \| \xi_w(t) \|_{\mathcal{E}_b^\alpha} \leq Q(\| \xi_u(0) \|_{\mathcal{E}_b^\alpha}) e^{-k_{\alpha} \beta t/4} + Q(\| g \|_{L^2_b}). \]

The dissipative estimate for the Strichartz norm follows now from (5.39) exactly as in Proposition 5.2. So, the proposition is proved. \( \square \)

The next corollary gives the well-posedness and dissipativity of solutions of equation (4.1) in higher energy spaces.

**Corollary 5.6.** Let the assumptions of Proposition 5.1 hold and let, in addition, \( \xi_u(0) \in \mathcal{E}_b^\alpha \) for some \( \alpha \in (0, \frac{2}{3}] \). Then the corresponding solution \( u(t) \in \mathcal{E}_b^\alpha \) for all \( t \geq 0 \) and the following estimate holds:
\[ (5.42) \quad \| \xi_u(t) \|_{\mathcal{E}_b^\alpha} + \| u \|_{L^4(t,t+1;H^{12}_b)} \leq Q(\| \xi_u(0) \|_{\mathcal{E}_b^\alpha}) e^{-\beta t} + Q(\| g \|_{L^2_b}), \]

where the positive constant \( \beta \) and monotone increasing function \( Q \) may depend on \( \alpha \), but is independent on \( g, u \) and \( t \).

Indeed, the proof of this estimate follows word by word to the proof of Proposition 5.5 and even slightly simpler since we may take \( v(t) \equiv 0 \), so we leave it to the reader.

**Remark 5.7.** To the best of our knowledge the idea to split the solution \( u \) into a sum (5.27) of regular and small components which are constructed using the previously obtained splitting into decaying and regular, but exponentially growing components has been suggested in [50] for the study of cubic non-autonomous damped wave equations. It has been widely used later in various modifications, see e.g., [14, 49] and, in particular, in [39] which is most close to our work and where the finite energy solutions of quintic wave equation have been studied. However, only the smallness in mean for the \( \tilde{v} \) component has been obtained there
\[ \int_{\tau}^t \| \tilde{v}(\tau) \|^4_{L^4} + \| \xi_u(\tau) \|_{\mathcal{E}} \, d\tau \leq C_\delta + \delta (t - \tau) \]
for all \( \delta > 0 \). The extra term \( C_\delta \) is not dangerous for finite energy solutions where the second integral in the right-hand side of (5.39) is absent, but is not acceptable in our case since it leads to the dependence of the constant \( C'' \) in (5.40) on \( \delta = \delta(R) \) and, as a result, we will be unable to make the constant \( C''R^{-1} \) small no matter how big \( R \) is.

Thus, the result of Lemma 5.4 is an essential and useful improvement of the scheme which has been somehow overseen in the previous papers and which has an independent interest.

6. **Attractors and concluding remarks**

The aim of this section is to build up the attractor theory for the SS-solutions of the damped quintic wave equation (4.1) and to discuss its natural generalizations. We restrict ourselves to consider the autonomous case only,
so assumption (5.1) is assumed to be satisfied (see Remark 6.7 for a brief discussion of the non-autonomous case). In this case, due to Theorem 4.6, equation (4.1) defines a dissipative semigroup $S(t)$, $t \geq 0$, in the uniformly local phase space $E_b$:

$$(6.1) \quad S(t)\xi_0 := \xi_u(t), \quad t \geq 0, \quad S(t) : E_b \to E_b,$$

where $u(t)$ is a uniquely defined SS-solution of problem (4.1) with the initial data $\xi_0 \in E_b$. Moreover, estimate (4.13) now reads

$$(6.2) \quad \|S(t)\xi_0\|_{E_b} \leq Q(\|\xi_0\|_{E_b})e^{-\beta t} + Q(\|g\|_{L^2_b})$$

and guarantees the existence of an absorbing ball for the solution semigroup in $E_b$. However, in contrast to the case of bounded domains or/and finite energy solutions, the solution semigroup $S(t)$ does not possess in general a compact global attractor in the space $E_b$, so the concept of the so-called \textit{locally compact} attractor is naturally used instead, see [34] and references therein for more details.

We recall that, by definition, a set $\mathcal{A}$ is a locally compact global attractor of a semigroup $S(t)$ acting in the uniformly local space $E_b$ if

1. $\mathcal{A}$ is bounded in $E_b$ and is compact in $E_{\text{loc}}$. The latter means that for any ball $B_{R_0}^E$, the restriction $\mathcal{A}|_{B_{R_0}^E}$ is compact in $E(B_{R_0}^E)$.
2. $\mathcal{A}$ is strictly invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.
3. $\mathcal{A}$ attracts the images of all bounded in $E_b$ sets in the topology of $E_{\text{loc}}$. This means that, for every bounded set $B \subset E_b$ and every neighbourhood $O(\mathcal{A})$ of the attractor $\mathcal{A}$ in the topology of $E_{\text{loc}}$, there exists $T = T(B, O)$ such that

$$S(t)B \subset O(\mathcal{A}), \quad \text{for } t \geq T.$$

The existence of such an attractor can be verified using the following standard attractor’s existence result.

**Proposition 6.1.** Let the semigroup $S(t) : E_b \to E_b$ be continuous for every fixed $t \geq 0$ in the topology of $E_{\text{loc}}$ and possesses a bounded in $E_b$ and compact in $E_{\text{loc}}$ attracting set $B$. Then there exists a (locally compact global) attractor $\mathcal{A} \subset B$ for this semigroup. Moreover, this attractor is generated by all bounded trajectories of the semigroup $S(t)$ defined for all $t \in \mathbb{R}$:

$$(6.3) \quad \mathcal{A} = \mathcal{K}|_{t=0},$$

where

$$(6.4) \quad \mathcal{K} := \{\xi_u \in L^\infty(\mathbb{R}, E_b), \quad \xi_u(t + \tau) = S(t)\xi_u(\tau), \quad \tau \in \mathbb{R}, \quad t \in \mathbb{R}_+\}$$

is the set of all bounded complete trajectories of the semigroup $S(t)$ (the kernel of $S(t)$ in the terminology of Chepyzhov and Vishik, see [13]).

For the proof of this criterion, see e.g., [4, 13].

Applying this criterion, to the solution semigroup $S(t)$ generated by equation (4.1), we get the following result.

**Theorem 6.2.** Let the nonlinearity $f$ satisfy (4.2) and the external force $g$ enjoy (5.1). Let also the semigroup $S(t)$ associated with equation (4.1) be defined by (6.1). Then this semigroup possesses a (locally compact) global attractor $\mathcal{A}$ which is a bounded set of $E_b^\alpha$ for $\alpha \in (0, 1)$]. Moreover, the
representation formula (6.3) holds and the set $\mathcal{K}$ of all bounded complete solutions of (4.1) possesses the following estimate:

$$\|u\|_{L^\infty(\mathbb{R}, \mathcal{E}_b^1)} + \|\xi\|_{L^4(\mathbb{R}, H^3_b; \mathcal{E}_b^1)} \leq Q(\|g\|_{L^2_b}), \quad \xi \in \mathcal{K},$$

where the function $Q$ is independent of $\xi \in \mathcal{K}$ and $\alpha \leq \frac{1}{2}$.

**Proof.** Indeed, the continuity of the operators $\mathcal{S}(t)$ in $\mathcal{E}_{\text{loc}}$ is an immediate corollary of estimate (4.6). In addition, estimates (5.5) and (5.35) guarantee that the set

$$\mathcal{B}_R := \{\xi \in \mathcal{E}_b^1, \|\xi\|_{\mathcal{E}_b^1} \leq R\}$$

is an attracting set for $\mathcal{S}(t)$ if $R = R(\|g\|_{L^2_b})$ is large enough and $\alpha \in (0, \frac{1}{2}]$. Obviously this set is bounded and closed in $\mathcal{E}_b$ and is compact in $\mathcal{E}_{\text{loc}}$. Thus, the existence of an attractor $\mathcal{A} \subset \mathcal{B}_R$ follows from Proposition 6.1. Finally, estimate (6.5) is also an immediate corollary of (5.35) and the theorem is proved.

As usual, the further regularity of the attractor $\mathcal{A}$ can be obtained by the standard bootstrapping arguments and is restricted by the regularity of $f$ and $g$ only. In particular, under our assumptions we may guarantee that the solutions are $\mathcal{E}_b^1$-regular.

**Theorem 6.3.** Let the assumptions of Theorem 6.2 hold. Then the attractor $\mathcal{A}$ of problem (4.1) constructed in the previous theorem is a bounded set of $\mathcal{E}_b^1$. Moreover, problem (4.1) is globally well-posed in the higher energy space $\mathcal{E}_b^1$ and the following dissipative estimate holds:

$$\|\xi_u(t)\|_{\mathcal{E}_b^1} \leq Q(\|\xi_u(0)\|_{\mathcal{E}_b^1})e^{-\beta t} + Q(\|g\|_{L^2_b}),$$

where the positive constant $\beta$ and monotone function $Q$ are independent of $u$, $g$ and $t$.

**Proof.** Actually, one extra step of bootstrapping is enough to improve the regularity of the attractor from $\mathcal{E}_b^\alpha$ ($\alpha > \frac{1}{2}$) to $\mathcal{E}_b^1$. Moreover, the non-linear decomposition (5.3) and (5.4) is no more necessary and much simpler linear splitting works. Namely, let now $u(t) = v(t) + w(t)$ where, in contrast to Section 5, the function $v$ solves the linear equation

$$\partial_t v + \gamma \partial_t v + (1 - \Delta_x)v = 0, \quad v|_{t=0} = \xi_u|_{t=0}$$

and the smooth component $w$ solves

$$\partial_t w + \gamma \partial_t w + (1 - \Delta_x)w = g - f(u), \quad w|_{t=0} = 0.$$

Indeed, applying estimate (3.10) to equation (6.7) and taking the supremum over $x_0 \in \mathbb{R}^3$, we arrive at the decaying estimate

$$\|\xi_v(t)\|_{\mathcal{E}_b} \leq C\|\xi_u(0)\|_{\mathcal{E}_b}e^{-\beta t}.$$

On the other hand, as not difficult to verify using the growth restriction of $f$ together with the Sobolev embedding theorem and proper interpolation inequalities,

$$\|f(u)\|_{L^4(\mathbb{R}, L^4_b)} \leq C \left(1 + \|u\|_{L^4_t \cap L^4_H^3}^{4} \right)\|\xi_u\|_{L^\infty(\mathbb{R}, \mathcal{E}_b^1)},$$

where $\alpha > \frac{1}{2}$. Therefore, we may apply estimate (3.10) with $\alpha = 1$ to equation (6.8) and obtain with the help of estimate (5.42) and the trick...
with function \( \theta \) described at the beginning of the proof of Proposition 5.2 that

\[
\| \xi_w(t) \|_{\mathcal{E}_b^1} \leq Q(\| \xi_u(0) \|_{\mathcal{E}_b^1}) e^{-\beta t} + Q(\| g \|_{L^2_b}).
\]

Estimates (6.9) and (6.11) guarantee that the attractor \( \mathcal{A} \) is a bounded set in \( \mathcal{E}_b^1 \). Finally, in order to get estimate (6.6), it is sufficient to take \( v \equiv 0 \) and repeat the derivation of (6.11). Thus, the theorem is proved. \( \square \)

**Remark 6.4.** Arguing in a standard way (e.g., using the energy method, see [8, 31]) one can easily show that the attractor \( \mathcal{A} \) is a compact set in \( \mathcal{E}_b^{1+\epsilon} \) for some positive \( \epsilon \). However, the inclusion \( \mathcal{A} \subset \mathcal{E}_b^{1+\epsilon} \) for some positive \( \epsilon \) is not true in general if \( g \in L^2_b \) only (we need more regularity of \( g \) to get this result).

**Remark 6.5.** Since \( H^2_b \subset C_b \), the growth rate of \( f \) is no more important if Theorem 6.3 is proved (we may just cut off the non-linearity \( f \) outside of the attractor), so all further results about the properties of the attractor obtained for energy subcritical (sub-cubic) growth rate of the non-linearity are automatically extended to the quintic case.

In particular, as known (see e.g., [34] and references therein), in contrast to the case of bounded domains, locally compact attractors in uniformly local spaces usually have infinite Hausdorff and fractal dimensions. By this reason, one usually replaces the dimension estimates by the proper estimates of Kolmogorov’s \( \varepsilon \)-entropy.

We recall that if \( K \) is a compact set in a metric space \( X \), then by Hausdorff criterion it can be covered by finitely many of \( \varepsilon \)-balls for any \( \varepsilon > 0 \). Let \( N_{\varepsilon}(K, X) \) be the minimal number of such balls. Then, by definition, the Kolmogorov’s entropy of \( K \) in \( X \) is the following number:

\[
H_{\varepsilon}(K, X) := \log_2 N_{\varepsilon}(K, X),
\]

see [27] for details. In particular, the case of finite fractal dimension corresponds to the estimate

\[
H_{\varepsilon}(K, X) \leq d_f(K) \log_2 \frac{1}{\varepsilon} + o(\log_2 \frac{1}{\varepsilon}).
\]

Since the attractor \( \mathcal{A} \) is not compact in \( \mathcal{E}_b \), but only in \( \mathcal{E}_{b\text{loc}} \), it is natural introduce the quantities \( H_{\varepsilon}(\mathcal{A}|_{B_{R_0}^x}, \mathcal{E}(B_{R_0}^x)) \) and study their dependence on two parameters \( R \) and \( \varepsilon \). It is known, see [34, 52] and references therein that, for many classes of dissipative PDEs in unbounded domains, these quantities possess the following universal estimates:

\[
H_{\varepsilon}(\mathcal{A}|_{B_{R_0}^x}, \mathcal{E}(B_{R_0}^x)) \leq C(R + \log_2 \frac{1}{\varepsilon})^3 \log_2 \frac{1}{\varepsilon},
\]

where \( C \) is independent of \( \varepsilon \) and \( R \) and \( \varepsilon > 0 \) (the exponent 3 here is the space dimension \( x \in \mathbb{R}^3 \)) and these estimates are sharp, see [17].

For the case of damped wave equation (4.1) with sub-cubic growth rate of the non-linearity \( f \) they are obtained in [54] (see also [32]). As explained above, the result of Theorem 6.3 allows us to extend this estimate to the case of quintic wave equations in \( \mathbb{R}^3 \).

**Remark 6.6.** Similarly to the case of bounded domains, we may introduce exponential attractors for the problem (4.1). Since the global attractor is already infinite-dimensional, the properly defined exponential attractor must
be also infinite dimensional, so in order to control its size it is natural (following [16]) to use universal entropy estimates (6.12). Namely, by definition, \( \mathcal{M} \) is an exponential attractor for the semigroup \( S(t) : \mathcal{E}_b \to \mathcal{E}_b \) if

1. The set \( \mathcal{M} \) is bounded in \( \mathcal{E}_b \) and compact in \( \mathcal{E}_{loc} \).
2. The set \( \mathcal{M} \) enjoys the universal entropy estimates (6.12).
3. The set \( \mathcal{M} \) is semi-invariant, i.e., \( S(t)\mathcal{M} \subset \mathcal{M} \) for \( t \geq 0 \).
4. The exponential attraction property

\[
dist_{\mathcal{E}_b}(S(t)B, \mathcal{M}) \leq Q(\|B\|_{\mathcal{E}_b})e^{-\beta t}
\]

holds for every bounded set \( B \) in \( \mathcal{E}_b \). Here \( \text{dist}_{\mathcal{E}_b}(X,Y) \) stands for the non-symmetric Hausdorff distance between sets \( X \) and \( Y \) in \( \mathcal{E}_b \) and the positive constant \( \beta > 0 \) and monotone function \( Q \) are independent of \( B \) and \( t \).

The existence of such an object in the case of reaction-diffusion equations in unbounded domains in uniformly local phase spaces is verified in [16]. The estimates for differences between solutions for damped wave equations allows to expect the same result to be true for equation (4.1) as well. We return to this question somewhere else.

It also worth to emphasize that the attraction to the exponential attractor holds in a uniform topology of the space \( \mathcal{E}_b \) and this is one of extra advantages of the exponential attractors approach. It is well-known, that for the global attractor \( \mathcal{A} \) we have the attraction property in a local topology of \( \mathcal{E}_{loc} \) only (there are natural examples where the attraction property in \( \mathcal{E}_b \) fail, see [34] for more details).

**Remark 6.7.** To conclude we note that the autonomous case of equation (4.1) has been chosen just for simplicity. All of the asymptotic smoothing results hold for general non-autonomous external forces \( g(t) \) as well if we pose some extra regularity assumptions on \( g \), for instance,

\[
g \in L^1_b(\mathbb{R}, H^1_b) \text{ or } g \in W^{1,1}_b(\mathbb{R}, L^2_b).
\]

The only difference is that we will need to consider instead of global attractors their proper generalizations to the non-autonomous case (e.g., uniform or pull-back attractors). We also expect that most part of the results obtained in [39] for the case of periodic boundary conditions can be naturally extended to the case of infinite-energy solutions in the whole space. We return to this problem somewhere else.

**Appendix A. Estimates in fractional Sobolev spaces**

In this Appendix we discuss the estimates in fractional Sobolev spaces which are necessary to treat the nonlinear term \( f(u) \) in equation (4.1). We start with the corollary of Kato-Ponce inequality which is proved in [39].

**Proposition A.1.** Let \( \alpha \in (0, 2/5] \) and let the functions \( v \) and \( w \) be such that

\[
(A.1) \quad v \in L^{12}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \quad w \in H^{\alpha,12}(\mathbb{R}^3) \cap H^{1+\alpha}(\mathbb{R}^3).
\]

Assume also that the function \( h \in C^1(\mathbb{R}) \), satisfies \( h(0) = 0 \) and

\[
(A.2) \quad |h'(v)| \leq C(1 + |v|^3)
\]
for some constant $C > 0$ and all $v \in \mathbb{R}$. Then $h(v)w \in H^\alpha(\mathbb{R}^3)$ and the following estimate holds:

\begin{equation}
\|h(v)w\|_{H^\alpha} \leq C_\alpha \left(1 + \|v\|_{L^{12}(\mathbb{R}^3)}^{4-\alpha}\right) \|v\|_{H^1} \|w\|_{H^{1+\alpha}} \|w\|_{H^{\alpha,12}},
\end{equation}

for some positive constant $C_\alpha$.

We need the analogue of this estimate for a bounded domain $V \subset \mathbb{R}^3$ (used in the paper for $V = B_{x_0}^R$ only).

**Corollary A.2.** Let $V$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary and let the assumptions of Proposition A.1 hold. Then the following estimate holds:

\begin{equation}
\|h(v)w\|_{H^\alpha(V)} \leq C_\alpha \left(1 + \|v\|_{L^{12}(\mathbb{R}^3)}^{4-\alpha}\right) \|v\|_{H^1(V)} \|w\|_{H^{1+\alpha}(V)} \|w\|_{H^{\alpha,12}(V)},
\end{equation}

Indeed, this is an immediate corollary of (A.3), the definition of the spaces $H^\alpha(V)$ and the existence of an extension operator from $V$ to $\mathbb{R}^3$.

We now turn to the estimates of $f(u) - f(v)$ which are crucial for our proof of asymptotic smoothing property.

**Corollary A.3.** Let $f \in C^2$ satisfy assumptions (4.2) and $f'(0) = 0$. In addition, let the functions $u$ and $v$ satisfy (A.1). Assume also that the cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$ be such that $\psi(x) \equiv 1$ for $x \in B^1_0$ and $\psi(x) \equiv 0$ for $x \not\in B^1_0$.

Let finally $\psi_{R,x_0}(x) = \psi(R^{-1}(x-x_0))$ for some $R > 1$ and $x_0 \in \mathbb{R}$. Then the following estimate holds:

\begin{equation}
\|\psi_{R,x_0}(f(u) - f(v))\|_{H^\alpha} \leq C \left(1 + \|u\|_{L^{12}(B_{x_0}^R)} + \|v\|_{L^{12}(B_{x_0}^R)}\right)^{4-\alpha} \times \left(\|u\|_{H^1(B_{x_0}^R)} + \|v\|_{H^1(B_{x_0}^R)}\right)^{4-\alpha} \|\psi_{R,x_0}(u-v)\|_{H^{1+\alpha}} \|\psi_{R,x_0}(u-v)\|_{H^{\alpha,12}},
\end{equation}

where the constant $C$ is independent of $R$ and $x_0$.

**Proof.** Indeed, using the analogue of estimates (2.24) for the scaled functions $\psi_{R,x_0}$, we get

\begin{equation}
\|\psi_{R,x_0}(f(u) - f(v))\|_{H^\alpha} \leq C\|\psi_{R,x_0}(f(u) - f(v))\|_{H^\alpha(B_{x_0}^R)} = \|\psi_{R,x_0}\int_0^1 f'(\lambda u + (1 - \lambda)v)(u-v)\ d\lambda\|_{H^\alpha(B_{x_0}^R)} \leq C \int_0^1 \|f'(\lambda u + (1 - \lambda)v)\|_{H^\alpha(B_{x_0}^R)} d\lambda.
\end{equation}

Note that the function $h(u) = f'(u)$ satisfies all assumptions of Proposition A.1, so we may use (A.4) to estimate the right-hand side of (A.6). Using also that, by the definition of the space $H^\alpha(B_{x_0}^R)$,

\begin{equation}
\|\psi_{R,x_0}u\|_{H^\alpha(B_{x_0}^R)} \leq \|\psi_{R,x_0}u\|_{H^\alpha(\mathbb{R}^3)},
\end{equation}

we get the desired estimate and finish the proof of the corollary. \qed

We conclude this section by stating one more useful corollary of the key estimate (A.4).
Corollary A.4. Let the assumptions of Corollary (A.3) hold and let, in addition,
\[(A.7)\]
\[u(x) - v(x) = w_1(x) + w_2(x)\]
for some functions \(w_1\) and \(w_2\) satisfying (A.1). Then the following estimate holds:
\[(A.8)\]
\[\|\psi_{R,x_0}(f(u) - f(v))\|_{H^s} \leq C \left(1 + \|u\|_{L^2(B_{2R}^0)} + \|v\|_{L^2(B_{2R}^0)}\right)^{4-\alpha} \times \times \left(\|u\|_{H^1(B_{2R}^0)} + \|v\|_{H^1(B_{2R}^0)}\right)\]
\[\times (\|\psi_{R,x_0}w_1\|_{H^{1+\alpha}} + \|\psi_{R,x_0}w_1\|_{L^{\infty}} + \|\psi_{R,x_0}w_2\|_{H^{1+\alpha}} + \|\psi_{R,x_0}w_2\|_{L^{\infty}}),\]
where the constant \(C\) is independent of \(R\) and \(x_0\).

Indeed, to verify (A.8), we just need to put (A.7) into the right-hand side of (A.6) and apply estimate (A.4) to every of two obtained terms separately.

**APPENDIX B. PROOF OF COMMUTATOR ESTIMATES**

In this Appendix we give the brief proof of estimates (2.25) and (2.26) stated in Proposition 2.14, see also [38] for the analogous proof in the particular case \(p = 2\). To this end, we will use the following formula for fractional powers:
\[(B.1)\]
\[(1 - \Delta_x)^{\alpha} u := \frac{1}{\Gamma(-\alpha)} \int_0^\infty (e^{-t(1-\Delta_x)}u - u) \frac{dt}{t^{1+\alpha}}\]
for \(\alpha \in (0,1)\), see e.g., [48]. Remind that in our case \(\alpha = s/2 \in (0,1/2)\). Let now \(u \in C_0^\infty(\mathbb{R}^n)\) and \(\psi\) be either also from \(C_0^\infty(\mathbb{R}^n)\) or \(\psi = \phi_{\varepsilon,x_0}\). Then
\[\psi(1 - \Delta_x)^{\alpha} u - (1 - \Delta_x)^{\alpha}(\psi u) = \]
\[
\frac{1}{\Gamma(-\alpha)} \int_0^\infty U(t) \frac{dt}{t^{1+\alpha}} = \frac{-1}{\Gamma(1-\alpha)} \int_0^\infty \frac{\partial_t U(t)}{t^\alpha} dt,
\]
where the function \(U(t) := \psi e^{-t(1-\Delta_x)}u - e^{-t(1-\Delta_x)}(\psi u)\) solves the following parabolic problem:
\[(B.2)\]
\[\begin{cases}
\partial_t U + (1 - \Delta_x) U = -2 \nabla_x \psi \nabla_x \bar{u} - \Delta_x \psi \bar{u} := h_\psi(t), & U|_{t=0} = 0, \\
\partial_t \bar{u} + (1 - \Delta_x) \bar{u} = 0, & \bar{u}|_{t=0} = u.
\end{cases}
\]
Note also that
\[\|h_\psi(t)\|_{L^p} \leq C_{\psi,x_0} \|\bar{u}(t)\|_{W^{1,p}_{\phi_{\varepsilon,x_0}}}
\]
and in the case \(\psi = \phi_{\varepsilon,x_0}\) we have \(C_{\psi,x_0} = C|\varepsilon|\). Moreover, applying the weighted parabolic smoothing property to the second equation of (B.2), we arrive at
\[t^{1/2}\|h_\psi(t)\|_{L^p} \leq C e^{-\kappa t}\|u\|_{L^p_{\phi_{\varepsilon,x_0}}}
\]
for some positive \(\kappa\) (the weighted smoothing property follows immediately from the classical non-weighted one and the trick with multiplication operator \(T_{\varepsilon,x_0}\)). Thus, for every \(\delta \in (0,1)\), we have
\[
\int_0^\infty e^{\kappa t}\|h_\psi(t)\|_{L^p_{\phi_{\varepsilon,x_0}}}^{2-\delta} dt \leq C_{\mu,\delta} C_{\psi,x_0}^2 \|u\|_{L^p_{\phi_{\varepsilon,x_0}}}^{2-\delta}.
\]
It is well-known that the heat equation
\[ \partial_t W + (1 - \Delta_x)W = h(t), \quad W|_{t=0} = 0 \]
possesses the following anisotropic \( L^q(L^p) \)-regularity estimate
\[ \|\partial_t W\|_{L^q(R_+; L^p(\mathbb{R}^3))} + \|W\|_{L^q(R_+; H^2,p(\mathbb{R}^3))} \leq C_{p,q}\|h\|_{L^q(R_+; L^p(\mathbb{R}^3))} \]
for all \( 1 < p, q < \infty \), see e.g., \[29\]. Applying this regularity result to the first equation of (B.2), we arrive at
\[ \int_0^\infty e^{\kappa t} \|\partial_t U(t)\|^2_{L^p} dt \leq C_{p,\delta} C^{2-\delta}_{\psi,x_0} \|u\|_{L^p_{\psi,x_0}}^{2-\delta} \]
and finally
\[ \|\psi(1 - \Delta_x)^\alpha u - (1 - \Delta_x)^\alpha (\psi u)\|_{L^p} \leq C \int_0^\infty \|\partial_t U(t)\|_{L^p} \frac{dt}{t^{\frac{\alpha}{2}}} \leq \]
\[ \leq \left( \int_0^\infty e^{\kappa t} \|\partial_t U(t)\|^2_{L^p} dt \right)^{\frac{1}{2}} \leq \left( \int_0^\infty \frac{e^{-\kappa t} dt}{t^\alpha} \right)^{\frac{1}{2}} \leq \]
\[ \leq C_{p,\delta} C^{2-\delta}_{\psi,x_0} \|u\|_{L^p_{\psi,x_0}}^{2-\delta} \]
if \( \delta > 0 \) is small enough that \( \alpha^{\frac{2-\delta}{\delta}} < 1 \) and the commutator estimates are proved.

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