MASS CONCENTRATION FOR THE $L^2$-CRITICAL NONLINEAR
SCHRÖDINGER EQUATIONS OF HIGHER ORDERS

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Abstract. We consider the mass concentration phenomenon for the $L^2$-critical nonlinear Schrödinger equations of higher orders. We show that any solution $u$ to

$$
iu_t + (-\Delta)^{\frac{\alpha}{2}} u = \pm|u|^{2^{*}_\alpha}u, \quad (t,x) \in \mathbb{R}^d, \quad u(0,x) \in L^2$$

for $\alpha > 2$, which blows up in a finite time, satisfies a mass concentration phenomenon near the blow-up time. We verify that as $\alpha$ increases, the size of region capturing a mass concentration gets wider due to the stronger dispersive effect.

1. Introduction. We consider the $L^2$-critical Cauchy problem in $\mathbb{R}^d$, $d \geq 2$,

$$
\begin{cases}
iu_t + (-\Delta)^{\frac{\alpha}{2}} u = \pm|u|^{2^{*}_\alpha}u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
u(0,x) = u_0(x).
\end{cases}
$$

1. (1)

Here $(-\Delta)^{\frac{\alpha}{2}}$ is the pseudo-differential operator defined by

$$(-\Delta)^{\frac{\alpha}{2}} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} |\xi|^{\alpha} \hat{f}(\xi) d\xi$$

and $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x) d\xi$. The equation (1) is $L^2$-critical in the sense that the equation is invariant under the rescaling transformation $u \to u_\lambda$, $u_\lambda(t,x) = u(t/x^\lambda, x)$. The equation (1) is invariant under the rescaling transformation $u \to u_\lambda$, $u_\lambda(t,x) = u(t/x^\lambda, x)$. The equation (1) is invariant under the rescaling transformation $u \to u_\lambda$, $u_\lambda(t,x) = u(t/x^\lambda, x)$.
\[ \lambda^2 u(\lambda \alpha t, \lambda x), \] which preserves \( L^2 \) norm. The system conserves the mass \( M(t) \) and the energy \( E(t) \) a priori:

\[
\begin{align*}
M(t) &= \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx, \\
E(t) &= \int_{\mathbb{R}^d} |(-\Delta)^{\frac{d}{2}} u(t, x)|^2 \pm \left( \frac{\alpha}{d} + 1 \right)^{-1} |u(t, x)|^\frac{2d+2}{d} \, dx.
\end{align*}
\]

The Fourth order Schrödinger equations were initially studied by Karpman [7] and Karpman and Shagalov [8]. They considered the fourth order Schrödinger equation to take into account the role of small fourth order dispersion terms in the propagation of intense laser beams in a bulk medium with a cubic nonlinearity (Kerr nonlinearity). The fourth order \( L^2 \)-critical case with nonlinearity \( |u|^\frac{4d}{d-2} u \) in (1) was studied in [6]. The several aspects on the fourth order Schrödinger equations were studied in [14]. When \( \alpha \neq 2 \), it is unknown whether there exists a blow up solution of (1) except the numerical evidence of [6]; unlike the \( \alpha = 2 \) case a virial type inequality or a pseudoconformal type symmetry are not yet known to hold.

In this paper we are concerned with the mass concentration phenomena of blowup solutions to (1), especially when the initial datum \( u_0 \in L^2 \) and its mixed \( L^1 \times L^\infty \)-norm blows up in a finite time. When \( \alpha = 2 \) and \( d = 2 \), Bourgain in his seminal paper [3] showed that if the \( L^2 \)-wellposed solution in \( \mathbb{R}^2 \) breaks down at a maximal time \( 0 < T^* < \infty \) with \( \|u\|_{L^\infty_t L^2_x((0, T^*) \times \mathbb{R}^d)} = \infty \), then the blow-up solution has a mass concentration phenomenon:

\[
\limsup_{t \uparrow T^*} \sup_{x \in \mathbb{R}^d} \int_{B(x,(T^*-t)^{\frac{1}{2}})} |u(t, x)|^2 \, dx \geq \epsilon
\]

where \( \epsilon = C \|u_0\|^{-M}_2 \) for some \( M > 0 \). Later, this was extended to higher dimensions by Béguin and Vargas [1]. A generalization in mixed norm spaces \( L^q_t L^r_x \) was obtained in [4]. Similar phenomena for nonelliptic Schrödinger equation for \( d = 2 \) were studied by Rogers and Vargas in [15]. For the case \( u_0 \in H^2 \) a general \( L^2 \) concentration result of blow up solutions was obtained earlier by Merle and Tustumi [11] in the radial case, and further generalized by Nawa [13]. In [11] and [13] it was shown that the blow up solution accumulates at least a \( L^2 \) mass of the unique ground state solution at blow up time.

We consider the case \( \alpha > 2 \). The linear part of the equation (1) has stronger dispersion, compared to the case \( \alpha = 2 \), which may be explained using a following heuristics (see p. 59 in [18]). The plane wave \( u(t, x) = e^{ix\xi_0 + t|\xi_0|^\alpha} \) solves

\[
\begin{align*}
\begin{cases}
i\partial_t u + (-\Delta)^{\frac{d}{2}} u = 0, \\
\bar{u}_0(\xi) = \delta_{\xi_0}.
\end{cases}
\end{align*}
\]

In order to get a sufficiently broad band solution, still around \( \xi_0 \), we define

\[
u(t, x) = e^{ix\xi_0 + t|\xi_0|^\alpha} \phi(e(x + \alpha|\xi_0|^{\alpha-2}\xi_0))
\]

for a smooth bounded function \( \phi \), then \( u \) can be shown to satisfy \( i\partial_t u + (-\Delta)^{\frac{d}{2}} u = O_\phi(e^2) \). This means that the profile of \( |u| \) moves roughly at velocity \( -\alpha|\xi_0|^{\alpha-2}\xi_0 \). Assuming high frequency initial data \( (|\xi_0| \gg 1) \), the propagation speed increases as \( \alpha \) increases. In other words, the wave tends to spend shorter time in a fixed region. So, when \( \alpha > 2 \) it is reasonable to expect that we need a larger set for concentration...
region than \( B(x, (T^*-t)^{\frac{2}{d}}) \) to capture nonzero mass in the set. Now, considering the scaling invariance \( u \rightarrow u_\lambda \) of the equation (1), it is natural to expect that

\[
\limsup_{t \uparrow T^*} \sup_{x \in \mathbb{R}^d} \int_{B(x, (T^*-t)^{\frac{2}{d}})} |u(t,y)|^2 dy \geq \epsilon > 0
\]

(2)

for a solution \( u \) which blows up at \( T^* \). If we impose further condition that \( \epsilon \) depends only on \( \|u_0\|_{L^2} \), then among the power type sizes \( (T^*-t)^\beta \), we see that \( (T^*-t)^{\frac{2}{d}} \) is the only possible one. It can be shown by a simple scaling argument. Indeed, suppose that (2) holds with \( (T^*-t)^\beta \) for some \( \beta > 0 \) and \( \epsilon = \epsilon(\|u_0\|_{L^2}) \). Then for \( \lambda > 0 \), \( u_\lambda \) satisfies

\[
\limsup_{t \uparrow \lambda^{-\alpha}T^*} \sup_{x \in \mathbb{R}^d} \int_{B(x, (\lambda^{-\alpha}T^*-t)^{\beta})} |u_\lambda(t,y)|^2 dy \geq \epsilon > 0
\]

with \( \epsilon \) independent of \( \lambda \) because \( u_\lambda \) blows up at \( \lambda^{-\alpha}T^* \) and \( \|u_\lambda(0)\|_2 = \|u(0)\|_2 \).

From the change of variables \((t, y) \rightarrow (\lambda^{-\alpha}t, \lambda^{-1}y)\) it follows that

\[
\limsup_{t \uparrow T^*} \sup_{x \in \mathbb{R}^d} \int_{B(x, \lambda^{-\alpha\beta}(T^*-t)^{\beta})} |u(t,y)|^2 dy \geq \epsilon.
\]

Hence there is a \( t \in (0, T^*) \) such that

\[
\sup_{x \in \mathbb{R}^d} \int_{B(x, \lambda^{-\alpha\beta}(T^*-t)^{\beta})} |u(t,y)|^2 dy \geq \epsilon.
\]

Since \( |u(t, \cdot)|^2 \) is integrable and \( \lambda \) is arbitrary, by uniform absolute continuity of integrable function we see that \( \alpha \beta = 1 \).

Before giving precise statement of our results, we briefly clarify the issue of wellposedness of (1). The local well-posedness in \( H^s(\mathbb{R}^d), s \geq 0 \) relies on the space time estimate for the free propagator

\[
e^{it(-\Delta)^{\frac{d}{2}}} f(x) = \int_{\mathbb{R}^d} e^{ix\xi + it|\xi|^\alpha} \hat{f}(\xi)d\xi,
\]

which is called Strichartz’s estimate (see (12) in Lemma 2.1). We call that a pair \((q, r)\) is \( \alpha \)-admissible if

\[
\frac{\alpha}{q} + \frac{d}{r} = \frac{d}{2}, \quad q, r \geq 2 \quad \text{and} \quad r \neq \infty.
\]

Then by the usual argument it is possible to show the inhomogeneous Strichartz’s estimates (13) (Lemma 2.1) for \( \alpha \)-admissible \((q, r)\) and \((\tilde{q}, \tilde{r})\). By Duhamel principle the solution can be written as

\[
u(t, x) = e^{it(-\Delta)^{\frac{d}{2}}} u_0 + i \int_0^t e^{i(t-s)(-\Delta)^{\frac{d}{2}}} (|u|^\frac{2\alpha}{d}\varphi(s)u(s))ds.
\]

(3)

When the initial datum \( u_0 \in L^2(\mathbb{R}^d) \), following the standard argument for local wellposedness, we see that there exists the unique solution \( u(t, x) \) on a small time interval \([0, T]\) such that

\[
u \in C([0, T]; L^2(\mathbb{R}^d)) \cap L^q([0, T]; L^r(\mathbb{R}^d))
\]

whenever \((q, r)\) is \( \alpha \)-admissible and

\[
\max \left( \frac{d}{2(d+2\alpha)}, \frac{d-\alpha}{2d} \right) \leq \frac{1}{r} \leq \frac{d+\alpha}{2(d+2\alpha)}.
\]

(4)
Figure 1. The line segment \([a, b] = \left[\left(\frac{d-a}{2d}, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right)\right]\) stands for \((\frac{1}{r}, \frac{1}{q})\) of admissible pair \((q, r)\), and \([c, d] = \left[\left(\frac{1}{2}, 1\right), \left(\frac{d+a}{2d}, \frac{1}{2}\right)\right]\) stands for \((\frac{1}{\tilde{r}}, \frac{1}{\tilde{q}})\) of the dual exponents of admissible pairs \((\tilde{q}, \tilde{r})\). For \((\frac{1}{r}, \frac{1}{q})\) in the segment \([A, B]\) we can find admissible pairs \((\tilde{q}, \tilde{r})\) satisfying the relation (6).

The existence time interval \([0, T]\) extends as long as \(\|u\|_{L_q^t L_r^x([0,T] \times \mathbb{R}^d)} < \infty\). If the solution blows up at \(T^*\), then

\[
\|u\|_{L_q^t L_r^x([0, T^*] \times \mathbb{R}^d)} = \infty. \tag{5}
\]

Indeed, using (3), the inhomogeneous Strichartz’s estimate (13) and Hölder’s inequality, one get for any \(\alpha\)-admissible \((q, r)\) and \((\tilde{q}, \tilde{r})\)

\[
\left\| \int_0^T e^{it(-\Delta)^{\frac{\alpha}{2}}} (t-s) \left[|u(s)|^{\frac{2\alpha}{d}} u(s) - |v(s)|^{\frac{2\alpha}{d}} v(s)\right] ds \right\|_{L_q^t L_r^x} \\
\leq \||u(s)|^{\frac{2\alpha}{d}} u(s) - |v(s)|^{\frac{2\alpha}{d}} v(s)\|_{L_q^t L_r^x} \\
\leq C \left(\|u\|_{L_{q_0}^t L_{r_0}^x}^{\frac{2\alpha}{d}} + \|v\|_{L_{q_0}^t L_{r_0}^x}^{\frac{2\alpha}{d}}\right) \|u - v\|_{L_{q_0}^t L_{r_0}^x}
\]

where \(\frac{2\alpha + d}{d}(\frac{1}{q_0}, \frac{1}{r_0}) = \left(\frac{1}{q}, \frac{1}{r}\right)\). Hence the nonlinear map becomes a contraction map if there are \(\alpha\)-admissible pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\) satisfying

\[
\left(\frac{2\alpha}{d} + 1\right)\frac{1}{q} = \frac{1}{\tilde{q}}, \quad \left(\frac{2\alpha}{d} + 1\right)\frac{1}{r} = \frac{1}{\tilde{r}}. \tag{6}
\]

It is possible as long as the condition (4) is satisfied (see Figure 1).

The following is our first result.
Theorem 1.1. Let \((q, r)\) be an \(\alpha\)-admissible pair satisfying \(q > 2\) and (4). Suppose that the solution of (1) satisfies \(\|u\|_{L^q_t L^r_x([0,T] \times \mathbb{R}^d)} < \infty\) for \(0 < t < T^* < \infty\) and (5). Then (2) holds.

The results in [1, 3, 4] were obtained by making use of refinement of Strichartz’s estimates for \(e^{it\Delta} f\) which come from bilinear restriction estimate for the paraboloid [12, 10, 17, 19]. To deal with the case \(\alpha > 2\) we need similar estimates for \(e^{it(-\Delta)^{\alpha/2}}\). However, it turns out that the related analysis is simpler than those of [1, 3, 4] due to a stronger dispersion effect. So, we can obtain a refinement of Strichartz’s estimate for \(e^{it(-\Delta)^{\alpha/2}}\) by exploiting bilinear interaction of Schrödinger waves. In particular we have the refinement (Proposition 2.3) in terms of dyadic shells, instead of cubes as in the previous work [1, 3, 4] for which the Galilean invariance of the operator \(e^{it\Delta} f\) played an important role, which is no longer available when \(\alpha \neq 2\). It does not seem possible to extend the method of this paper to the case \(\alpha < 2\). In fact, for this one needs mixed norm estimates for \(e^{it(-\Delta)^{\alpha/2}} f\) on the region \(2/q + d/r > d/2\) because of the scaling structure. However, such estimates are known to fail by Knapp’s example.

Secondly, we consider the \(L^2\)-critical Hartree equation, which is given by for \(2 < \alpha < d\)

\[
\begin{cases}
    iu_t + (-\Delta)^{\alpha/2} u = \pm(|x|^{-\alpha} * |u|^2) u \\
    u(0, x) = u_0(x) \in L^2(\mathbb{R}^d), \quad d \geq 3.
\end{cases}
\]  

(7)

One can easily check that the equation (7) is also \(L^2\)-critical, that is, invariant under \(u \to u_\lambda\). One may be interested in a mass concentration for the finite time
blow-up solutions for (7). The local wellposedness can be established by following the standard argument. In fact, using the Strichartz estimates (13) and triangle inequality
\[
\left\| \int_0^T e^{i\alpha(t-s)} \left( |x|^{-\alpha} * |u|^2 \right) u - \left( |x|^{-\alpha} * |u|^2 \right) v \right\|_{L_t^q L_x^r} \\
\leq \left\| (|x|^{-\alpha} * |u|^2 - |v|^2) u \right\|_{L_t^q L_x^r} + \left\|||x|^{-\alpha} |v|^2 (v - u) \right\|_{L_t^q L_x^r}.
\]
By Hölder’s and Hardy-Littlewood-Sobolev inequality the last of the above is bounded by
\[
C \|u\|^2 - \|v\|^2 \|L_t^q L_x^r\| \|u\|_{L_t^q L_x^r} + C \|v\|^2 \|L_t^q L_x^r\| \|u - v\|_{L_t^q L_x^r} 
\]
for \(q_1, r_1\) satisfying \((\frac{1}{q_1}, \frac{1}{r_1}) + (\frac{1}{q}, \frac{1}{r}) = (\frac{1}{q'}, \frac{1}{r'}) + (0, \frac{d - \alpha}{d})\). Let us take \(q_1 = \frac{d}{2}\) and \(r_1 = \frac{d}{2}\). Then we find that the nonlinear map
\[
u \to e^{i(\alpha(t-s)(-\Delta)^{\frac{d}{q}})} (|x|^{-\alpha} * |u|^2) u ds
\]
is a contraction if there is an \(\alpha\)-admissible pair \((\tilde{q}, \tilde{r})\) such that
\[
\left(\frac{3}{q}, \frac{3}{r} \right) = \left(\frac{1}{q'}, \frac{1}{r'}\right) + \left(0, \frac{d - \alpha}{d}\right).
\]
An easy calculation shows that the line segment \([A, B]\) in Figure 2 is parallel to the segment \([e, f]\) corresponding to the set \(\{(1/q', 1/r') + (0, (d - \alpha)/d) : (\tilde{q}, \tilde{r}) \text{ is } \alpha \text{-admissible}\}\), and moreover \(|e - f| = 3|A - B|\). So it is possible to find \((\tilde{q}, \tilde{r})\) satisfying (9) as long as \((q, r)\) is contained in \([A, B]\), that is,
\[
\frac{6d}{3d - \alpha} \leq r \leq \frac{6d}{3d - 2\alpha}.
\]
For these \((q, r)\) we also get a blowup alternative; If \(T^* < \infty\), then (5) should be satisfied. As it was shown in [4], the mass concentration phenomenon is mostly involved with the homogeneous part of the solution. The argument used in [1, 3] works for (7) without much modifications if the nonlinear term can be controlled properly. This is actually equivalent to showing the local wellposedness of (7) under the condition (10).

**Theorem 1.2.** Let \(d \geq 3\). Let \((q, r)\) be an \(\alpha\)-admissible satisfying (10). Suppose that the solution \(u\) of (7) satisfies \(\|u\|_{L_t^q L_x^r([0,T) \times \mathbb{R}^d)} < \infty\) for \(0 < t < T^* < \infty\) and (5). Then (2) holds.

Throughout the paper, we assume \(\alpha > 2\) if it is not stated otherwise and \(C\) denotes positive constants of which values vary from line to line.

Finally, the paper is organized as follows. In Section 2 we obtain some preliminary estimates which are to be used for the proofs of Theorems. In Section 3 we give the proofs of Theorems 1.1 and 1.2.

2. **preliminary.** In this section we show several lemmas which will be used later for the proofs of the theorems. For \(q, r \geq 2\), \(r \neq \infty\) and \(\frac{2}{q} + \frac{2}{r} \leq \frac{d}{2}\), set
\[
\beta = \beta(\alpha, q, r) = \frac{d}{2} - \frac{d - \alpha}{r}.
\]
Let \( \rho \) be a smooth function supported in \([1/2, 4]\) and satisfying \( \sum_{-\infty}^{\infty} \rho(x/2^k) = 1 \) for all \( x > 0 \). Then we define a projection operator by

\[
\hat{P}_k f(\xi) = \rho(\|\xi\|^2/2^k)\hat{f}(\xi).
\]

The following lemma is a version of Strichartz estimates for Schrödinger equations of higher orders \( \alpha > 2 \). It seems well known but for the convenience of the readers we include a proof. The arguments are based on rescaling and Littlewood-Paley theorem.

Lemma 2.1. For \( q, r \geq 2 \), \( r \neq \infty \) and \( \frac{2}{q} + \frac{d}{r} \leq \frac{d}{2} \),

\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L_t^q L_x^r} \leq C \left( \sum_k 2^{k\left(\frac{d}{2} - \frac{d}{r} - \frac{2}{r} \right)} \| P_k f \|_{L_t^q L_x^r}^{\frac{2}{r}} \right)^{\frac{1}{2}}.
\]

(11)

In particular, if \((q, r)\) is \( \alpha \)-admissible, then

\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L_t^q L_x^r} \leq C \| f \|_2.
\]

(12)

Also if \((q, r)\) and \((\tilde{q}, \tilde{r})\) are \( \alpha \)-admissible, then we have

\[
\| \int_0^t e^{i(t-s)(-\Delta)^{\frac{\alpha}{2}}} F(s)ds \|_{L_t^q L_x^r} \leq C \| F \|_{L_{t}^{\tilde{q}'} L_{x}^{\tilde{r}'}}.
\]

(13)

Proof. Once we get (11), then (12) follows from Plancherel’s theorem. Also (13) can be shown by duality and the argument due to Christ and Kiselev ([5]).

We now show (11). Since \( \alpha > 2 \), by the stationary phase method (see p.344 in [16]), we see \( \int_{\mathbb{R}^d} e^{ix^2 + it\xi} \psi(\xi)d\xi \leq C t^{-\frac{d}{2}} \) for any \( \psi \) with compact support contained in \( \mathbb{R}^d \setminus \{0\} \). Hence, from the argument of Keel-Tao in [9], we have

\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_0 f \|_{L_t^q L_x^r} \leq C \| f \|_2
\]

(14)

whenever \( \frac{d}{2} + \frac{\alpha}{2} \leq \frac{d}{2} \), and \( r, q \geq 2 \) (with exception \( r \neq \infty \) when \( d = 2 \)). Then by rescaling we observe that

\[
e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f(x) = e^{it\alpha k(-\Delta)^{\frac{\alpha}{2}}} P_0 \left( f\left( \frac{x}{2^k} \right) \right) (2^k x).
\]

Therefore it follows that

\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f \|_{L_t^q L_x^r} \leq C 2^{j(\frac{d}{2} - \frac{d}{r} + \frac{2}{r})} \| f \|_2.
\]

(15)

Since \( f = \sum_k P_k f \) and \( q, r \geq 2 \), from Littlewood-Paley theorem followed by Minkowski’s inequality we have

\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L_t^q L_x^r} \leq C \left( \sum_k \| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f \|_{L_t^q L_x^r}^2 \right)^{\frac{1}{2}}.
\]

Putting (15) in the right hand side of the above, we get the desired. \( \square \)

2.1. Refinement of Strichartz’s estimates.

Lemma 2.2. Let \((q, r)\) satisfy \( 2 < q, r < \infty \) and \( \frac{2}{q} + \frac{d}{r} < \frac{d}{2} \). If \( M \leq N \) then there is \( \epsilon = \epsilon(q, r) > 0 \) such that

\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_N f e^{i\epsilon t(\Delta)^{\frac{\alpha}{2}}} P_M f \|_{L_t^{q/2} L_x^{r/2}} \leq C 2^{(M+N)\beta(\alpha, q, r)} (2^M - N) \| f \|_2 \| g \|_2
\]

This means that it is possible to get better bounds than the one trivially obtained by rescaling (Lemma 2.1) when the waves interact at different frequency levels. Such observation was first made by Bourgain [3].
Proof. By rescaling it is enough to show that
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} \hat{P}_0 e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{M-N} f\|_{L_t^q L_x^r} \leq C (2^{M-N})^\beta (\alpha,q,r) + \epsilon \|f\|_2 \|g\|_2. \] (16)

Let us set \( L = M - N \leq 0 \). Hence Fourier supports of \( P_0 f, P_r f \) are contained in the sets \( \{ |\xi| \sim 1 \} \), \( \{ |\xi| \sim 2^L \} \), respectively. For \( \frac{d}{r} + \frac{2}{q} \leq \frac{2}{d} \), and \( r,q \geq 2 \), by Hölder’s inequality and (15) one can see
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} \hat{P}_0 f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_L g\|_{L_t^q L_x^r} \leq C 2^L \beta (\alpha,q,r) \|f\|_2 \|g\|_2. \]

If one interpolates this with
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} P_0 f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_L g\|_{L_t^q L_x^r} \leq C 2^L (d-1)/2 \|f\|_2 \|g\|_2 \] (17)
which will be proven later, one get the desired estimate (16). Indeed, note that the bound in the above is better than the trivial bounds which follow from rescaling. That is,
\[ 2^L \beta (\alpha,4,4) = 2^L (d/4 - \alpha/4) > 2^L (d-1)/2 = 2^L (\beta (\alpha,4,4) + \epsilon) \]
for some \( \epsilon > 0 \). Hence via interpolation we get the estimate
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} P_0 f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_L g\|_{L_t^q L_x^r} \leq C 2^L (\beta (\alpha,q,r) + \epsilon) \|f\|_2 \|g\|_2 \]
with some \( \epsilon > 0 \) as long as \( d/r + 2/q < d/2 \) and \( 2 < q,r < \infty \). \( \square \)

Proof of (17). We may assume that \( \hat{f} \) is supported in the set \( \{ \xi : |\xi| \sim 1 \} \). When \( 2^L \sim 1 \), the estimate (17) is trivial from (15). So we also may assume \( 2^L \ll 1 \).

By decomposing the Fourier support of \( f \) into finite number of sets, rotation and mild dilution, it is enough to show that
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f e^{it(-\Delta)^{\frac{\alpha}{2}}} g\|_{L_t^q L_x^r} \leq C \lambda (d-1)/2 \|f\|_2 \|g\|_2 \]
whenever \( \hat{f} \) is supported in \( B(c_1,\epsilon) \) and \( \hat{g} \) is supported in \( \{ |\xi| \sim 2^L \} \). Here \( B(x,r) \) is the open ball centered at \( x \) with radius \( r \). We write
\[ e^{it(-\Delta)^{\frac{\alpha}{2}}} f(x) e^{it(-\Delta)^{\frac{\alpha}{2}}} g(x) = \int \int e^{i(x(\xi + \eta) + \xi |\xi| + |\eta|)} \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta. \]

Freezing \( \eta = (\eta_2, \ldots, \eta_d) \), we consider an operator
\[ B_\eta(f,g) = \int \int e^{i(x(\xi + \eta) + t(|\xi| + |\eta|))} \hat{f}(\xi) \hat{g}(\eta_1, \eta) d\xi d\eta. \]

We now make the change of variables
\[ \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{d+1}) = (\xi + \eta, |\xi|, |\eta|). \]

Then by a direct computation one can see that
\[ \left. \frac{\partial \zeta}{\partial (\xi, \eta)} \right| = \alpha |\eta_1| |\eta|_{\alpha-2} - \xi_1 |\xi|_{\alpha-2} \sim 1 \]
on the supports of \( \hat{f} \) and \( \hat{g} \). Hence making change of variables \( (\xi, \eta_1) \to \zeta \), applying Plancherel’s theorem and reversing the change variables \( (\zeta \to (\xi, \eta_1)) \), we get
\[ \|B_\eta(f,g)\|_{L_t^q L_x^r} \leq C \|\hat{f}(\xi) \hat{g}(\eta_1, \eta)\|_{L_t^q L_x^r}. \]

Since
\[ e^{it(-\Delta)^{\frac{\alpha}{2}}} f(x) e^{it(-\Delta)^{\frac{\alpha}{2}}} g(x) = \int B_\eta(f,g(\cdot, \eta)) d\eta, \]
by Minkowski’s inequality we see
\( \| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^q_{t} L^r_x} \leq C \sum_k \| \hat{P}_k f \|_{\dot{B}^{\alpha}_{q,r}} \).  
This gives the desired bound by Schwartz’s inequality, because \( |\eta| \leq 2^k \).

Now we prove the following refinement of the Stichartz’s estimate. This type of inequality for \( \alpha = 2 \) was first obtained by Moyua-Vargas-Vega in [12], and used by Bourgain in [3].

**Proposition 2.3.** If \((q, r)\) is an \( \alpha \)-admissible with \( q > 2 \), there are \( \theta \in (0, 1) \) and \( 1 < p < 2 \) such that
\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^q_{t} L^r_x} \leq C \left( \sup_k 2^{kd(\frac{1}{q} - \frac{1}{r})} \| \hat{P}_k f \|_{L^q} \right)^\theta \| f \|_{L^p}^{1-\theta}.
\]
Here \( B_k = \{ \xi : 2^{k-1} < |\xi| \leq 2^k \} \).

**Proof of Proposition 2.3.** In fact, for the proof it is sufficient to show that
\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^q_{t} L^r_x} \leq C \left( \sup_k 2^{kd(\frac{1}{q} - \frac{1}{r})} \| \hat{P}_k f \|_{L^q} \right)^\theta \| f \|_{L^p}^{1-\theta}.
\]
By dividing the support of \( \hat{P}_k f \) into three dyadic shells \( B_{k-1}, B_k, \) and \( B_{k+1}, \) we get the desired. This actually can be shown by using (11) and the following two estimates:

If \((q, r)\) is an \( \alpha \)-admissible with \( q > 2 \), then
\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^q_{t} L^r_x} \leq C \sum_k \| \hat{P}_k f \|_{\dot{B}^{\alpha}_{q,r}}^{\frac{1}{q}},
\]
and
\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^q_{t} L^r_x} \leq C \left( \sum_k \left( 2^{kd(\frac{1}{q} - \frac{1}{r})} \| \hat{P}_k f \|_{L^p} \right)^2 \right)^\frac{1}{2}
\]
with some \( \bar{p} < 2 \). Interpolation among (11) and these two estimates gives
\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} f \|_{L^q_{t} L^r_x} \leq C \left( \sum_k \left( 2^{kd(\frac{1}{q} - \frac{1}{r})} \| \hat{P}_k f \|_{L^p} \right)^2 \right)^\frac{1}{2}
\]
as long as \((1/p_0, 1/q_0)\) is contained in the triangle \( \Gamma \) with vertices \((1/2, 1/2), (1/2, 1/\bar{q})\) and \((1/\bar{p}, 1/2)\). Obviously one can find a point \((1/p_0, 1/q_0)\) contained in the interior of \( \Gamma \) so that it lies on the line segment joining \((1/2, 1/2)\) and \((1/p, 0)\) for some \( p < 2 \). Then by interpolation among the mixed norm spaces* \([2]) we see
\[
\left( \sum_k \left( 2^{kd(\frac{1}{q} - \frac{1}{r})} \| \hat{P}_k f \|_{L^p} \right)^2 \right)^\frac{1}{2} \leq C \left( \sup_k 2^{kd(\frac{1}{q} - \frac{1}{r})} \| \hat{P}_k f \|_{L^q} \right)^\theta \left( \sum_k \| \hat{P}_k f \|_{L^p}^2 \right)^{1-\theta}.
\]
Therefore, using (20) with \((q_0, p_0) = (q_0, p_0),\) the above inequality and Plancherel’s theorem we get the desired inequality. Now it remains to show (18) and (19).

We first show (19) which is easier. Note that \( \alpha > 2 \). By interpolation between (14) and the trivial \( L_1^1 \to L_\infty^\infty \) bound, one can see that for each \( \alpha \)-admissible \((q, r), q > 2,\) there is a \( p < 2 \) such that
\[
\| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_0 f \|_{L^q_{t} L^r_x} \leq C \| \hat{P}_0 f \|_{L^p}.
\]

*Here the mixed norm spaces are given with the norm \( \left( \sum_k (2^{kd(\frac{1}{q} - \frac{1}{r})} \| f_k \|_s)^2 \right)^{\frac{1}{2}} \).
Here we used the fact that $\alpha > 2$. Then by rescaling we see that
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f\|_{L_t^q L_x^r} \leq C 2^{k(d - \frac{\alpha}{2} - \frac{\alpha}{q})} \|\widehat{P_k f}\|_p. \]
By using Littlewood-Paley theorem, Minkowski’s inequality and the above we get
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L_t^q L_x^r} \leq C(\sum_k \|e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f\|_{L_t^q L_x^r}^2)^{\frac{1}{2}} \]
\[ \leq C(\sum_k 2^{2k(d - \frac{\alpha}{q} - \frac{\alpha}{d})} \|\widehat{P_k f}\|_p^2)^{\frac{1}{2}}. \]
In particular, when $(q, r)$ is $\alpha$-admissible we get (19).

Now we turn to (18). We start with the inequality (11) which reads as
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L_t^q L_x^r} \leq C(\sum_k 2^{2k\beta(q, r)} \|\widehat{P_k f}\|_2^2)^{\frac{1}{2}} \]
for $q, r \geq 2$, $r \neq \infty$ and $\frac{2}{q} + \frac{d}{r} \leq \frac{d}{2}$. However in the right hand side the norm in $k$ is $\ell^2$. We need to upgrade this slightly so that the norm in $k$ is replaced by $\ell^{\hat{q}}$ for some $\hat{q} > 2$. To do this it is enough to show that there is a pair $(q, r)$ satisfying $q, r \geq 2$, $r \neq \infty$ and $\frac{2}{q} + \frac{d}{r} \leq \frac{d}{2}$, such that
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L_t^{\hat{q}} L_x^2} \leq C(\sum_k 2^{2k\beta(q, r)} \|\widehat{P_k f}\|_2^2)^{\frac{1}{2}} \]
for some $\hat{q} > 2$. The interpolation between this and (11) gives the desired. In particular when $(q, r)$ is $\alpha$-admissible we get (18).

We show (21) with $q = r = 4$ and $\hat{q} = 4$. We write
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L_t^q L_x^r}^2 = \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L_t^{\hat{q}} L_x^2}^2 \]
and
\[ e^{it(-\Delta)^{\frac{\alpha}{2}}} f e^{it(-\Delta)^{\frac{\alpha}{2}}} \]
Then by triangle inequality
\[ \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f e^{it(-\Delta)^{\frac{\alpha}{2}}} \|_{L_t^{\hat{q}} L_x^2} \leq \sum_{j \geq 0} \|\sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f\|_{L_t^{\hat{q}} L_x^2} \]
\[ + \sum_{j > 0} \|\sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f\|_{L_t^{\hat{q}} L_x^2}. \]
By symmetry it is enough to deal with the first one because the second can be handled similarly. Hence it is enough to show that
\[ \|\sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f\|_{L_t^{\hat{q}} L_x^2} \leq C 2^{-\epsilon j}(\sum_k 2^{4k\beta(q, r)} \|\widehat{P_k f}\|_2^4)^{\frac{1}{2}} \]
for some $\epsilon > 0$. We consider separately the cases $j = 0, 1, 2$ and $j \geq 3$.

First we handle the case $j = 0, 1, 2$. By Cauchy-Schwarz’s inequality we have
\[ \|\sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f\| \leq \sum_k |e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f|^2. \]
So, squaring both sides we get
\[ \left| \sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f \right|^2 \leq C \sum_{l \geq 0} \sum_k \left| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+l} f \right|^2. \]

Hence it follows that
\[ \left\| \sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f \right\|_{L_t^2 L_x^2}^2 \leq C \left( \sum_{l \geq 0} \sum_k \left| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+l} f \right|^2 \right). \]

Then by Lemma 2.2 we see
\[ \sum_{l \geq 0} \sum_k \left| e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+l} f \right|_{L_t^2 L_x^2} \leq C \sum_{l \geq 0} \left( 2^{2k\beta(\alpha,4,4)} \|P_k f\|_2^2 \right)^{2(1/4)\beta(\alpha,4,4)} \|P_{k+l} f\|_2^2. \]

Therefore by Schwarz’s inequality and summation in \( l \) we get
\[ \left\| \sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f \right\|_{L_t^2 L_x^2}^2 \leq C \sum_k \left( 2^{2k\beta(\alpha,4,4)} \|P_k f\|_2^2 \right)^{2(1/4)\beta(\alpha,4,4)} \|P_{k+j} f\|_2^2. \]

Now we turn to case \( j \geq 3 \). Observe the Fourier supports of
\[ e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f, \quad -\infty < k < \infty \]
are boundedly overlapping. Hence by Plancherel’s theorem, we see that
\[ \left\| \sum_k e^{it(-\Delta)^{\frac{\alpha}{2}}} P_k f e^{it(-\Delta)^{\frac{\alpha}{2}}} P_{k+j} f \right\|_{L_t^2 L_x^2} \leq C \sum_k \left( 2^{2k\beta(\alpha,4,4)} \|P_k f\|_2^2 \right)^{2(1/4)\beta(\alpha,4,4)} \|P_{k+j} f\|_2^2. \]

Using Lemma 2.2, the right hand side is bounded by
\[ C 2^{-2\epsilon j} \sum_k \left( 2^{2k\beta(\alpha,4,4)} \|P_k f\|_2^2 \right)^{2(1/4)\beta(\alpha,4,4)} \|P_{k+j} f\|_2^2. \]

Therefore, Schwarz’s inequality gives us the desired bound (22).

Proposition 2.3 can be combined with the following elementary lemma to find out the region where the given \( L^2 \) function is not severely concentrating but still containing a moderate amount of mass.

**Lemma 2.4.** Let \( \epsilon > 0, f \in L^2(\mathbb{R}^2) \) and suppose that there is a measurable subset \( Q \) such that
\[ \epsilon \leq (|Q|^\frac{1}{2} - \frac{1}{2p} \|f\|_p^p)^\theta \|f\|_2^{1-\theta} \]
for some \( \theta \in (0,1) \) and \( p \in [1,2) \). Then if \( \lambda \sim |Q|^\frac{1}{2} \epsilon^{-\frac{1}{2p}} \|f\|_2^{\frac{1}{2p}+1} \), then \( f_\lambda = f \chi_{\{x \in Q : |f| \leq \lambda\}} \) satisfies
\[ \epsilon^{\frac{1}{2}} \|f_\lambda\|_2^{1-\frac{1}{2}} \lesssim |Q|^{\frac{1}{2} - \frac{1}{2p}} \|f_\lambda\|_p \leq \|f_\lambda\|_2. \]

Here all the implicit constants are independent of \( f, Q, \epsilon \) and \( \lambda \).

**Proof.** Changing \( |Q|^\frac{1}{2} f |Q|^{\frac{1}{2}} / \|f\|_2 \to f \), \( |Q|^\frac{1}{2} \lambda / \|f\|_2 \to \lambda \), and \( \epsilon / \|f\|_2 \to \epsilon \), we may assume \( |Q| = 1 \) and \( \|f\|_2 = 1 \). Since
\[ \epsilon^{\frac{1}{2}} \leq \int f^p dx \leq \int_{|f| \leq \lambda} f^p dx + \int_{|f| > \lambda} f^p dx, \]

it trivially follows that
\[ \epsilon^2 \leq \int_{|x| \leq \lambda} |f|^p + \lambda^{p-2} \]
because \( \|f\|_2 = 1 \). Now we only need to choose \( \lambda \) such that \( \lambda^{p-2} = \frac{1}{2} \epsilon^2 \). The remaining is easy to see by making the changes of \( f \rightarrow |Q|^{\frac{2}{d}} f (|Q|^{\frac{1}{d}} \cdot)/\|f\|_2 \), \( \lambda \rightarrow |Q|^{\frac{2}{d}} \lambda/\|f\|_2 \) and \( \epsilon/\|f\|_2 \rightarrow \epsilon \).

**Proof of Theorems.** As in the case \( \alpha = 2 \) ([1, 3, 4]) the following two lemmas play crucial roles in showing mass concentration. The first one is concerned with decomposition of the initial datum into functions of which Fourier transforms are spreading rather than concentrating. In view of uncertainty principle the spreading part of the initial datum may concentrate on some spatial region. The second one enables us to find regions where the linear Schrödinger wave concentrates in the mixed norm space \( L^q_x L^r_t \) (here \( q, r \) is \( \alpha \) admissible) when the Fourier transform of the initial data does not severely concentrate.

**Lemma 3.1.** Let \( (q, r) \) be an \( \alpha \)-admissible pair satisfying \( q > 2 \) and \( \alpha/q + d/r = d/2 \). Suppose \( f \in L^2(\mathbb{R}^d) \) and
\[
\|e^{it(-\Delta)^{\frac{\alpha}{2}}} f\|_{L^q_x L^r_t} \geq \epsilon \quad (23)
\]
for some \( \epsilon > 0 \). Then there exist a \( f_k \in L^2(\mathbb{R}^d) \) and a dyadic shell \( B_{nk} \) for \( k = 1, 2, \ldots, N \) with \( N = N(\|f\|_{L^2}, d, \epsilon) \) such that
(1) \( \sup \hat{f}_k \subset B_{nk} \), \( |\hat{f}_k| < C2^{-\frac{d}{2} + \frac{d}{\alpha}} \epsilon^{-\nu} \|f\|^q_{L^q_x} \) for all \( k = 1, \ldots, N \),
(2) \( \|e^{it(-\Delta)^{\frac{\alpha}{2}}} f - \sum_{k=1}^N e^{it(-\Delta)^{\frac{\alpha}{2}}} f_k\|_{L^q_x L^r_t} < \epsilon \),
(3) \( \|f\|^2_{L^q_x} = \sum_{k=1}^N \|f_k\|^2_{L^q_x} + \|f - \sum_{k=1}^N f_k\|^2_{L^q_x} \).

Here the constants \( C, \mu, \) and \( \nu \) depend only on \( d \).

**Lemma 3.2.** Let \( (q, r) \) be an \( \alpha \)-admissible pair satisfying \( 2 < q \leq r \). Suppose \( g \in L^2(\mathbb{R}^d) \) and
\[
\sup \hat{g} \subset B_k \quad \text{and} \quad |\hat{g}| < C_0 2^{-\frac{d}{2} + \frac{d}{\alpha}}
\]
for \( C_0 > 0 \). Then for any \( \epsilon > 0 \), there exist \( N_1 \in \mathbb{N} \), \( N_1 \leq C(d, C_0, \epsilon) \), and sets \( (Q_n)_{1 \leq n \leq N_1} \subset \mathbb{R} \times \mathbb{R}^d \) which is given by
\[
Q_n = \{(t, x) \in \mathbb{R} \times \mathbb{R}^d; t \in I_n \text{ and } x \in C_n\},
\]
where \( I_n \subset \mathbb{R} \) is an interval with \( |I_n| = 2^{-k} \alpha \) and \( C_n \) is a cube with the side length \( l(C_n) = 2^{-k} \) such that
\[
\|e^{it(-\Delta)^{\frac{\alpha}{2}}} g\|_{L^q_x L^r_t(\mathbb{R}^{d+1} \setminus \bigcup_{n=1}^{N_1} Q_n)} < \epsilon.
\]

**Notation.** Let \( E \) be a measurable set in \( \mathbb{R}^{d+1} \) and \( f : \mathbb{R}^{d+1} \rightarrow \mathbb{R} \) is a measurable function. If \( E_t = \{x : (t, x) \in E\} \) is measurable in \( \mathbb{R}^d \) for all \( t \in \mathbb{R} \), we define the mixed integral \( \|f\|_{L^q_x L^r_t(E)}^q \) by
\[
\|f\|_{L^q_x L^r_t(E)}^q = \int_{\mathbb{R}} (\int_{E_t} |f(t, x)|^r \, dx)^{\frac{q}{r}} \, dt.
\]

Once we have the refinement of Strichartz estimates (Proposition 2.3) the proofs of Lemma 3.1 and 3.2 can be given by a modification of the argument in [1, 3]. The proofs of lemmas are given in Appendix.
3.1. **Proof of Theorem 1.1.** The proof consists of following steps:

- Controlling the inhomogeneous part,
- Decomposition to the initial datum with non-concentration Fourier transforms,
- Figuring out the concentrating region,
- Determining the size of mass concentration region.

The two lemmas (Lemmas 3.1 and 3.2) will be incorporated into the second and the third steps, respectively.

To prove Theorem 1.1 it is enough to consider the case $q \leq r$ in which $q \leq 2(d + \alpha)/d$. From interpolation with the conserved mass it is clear that if $\|u\|_{L^q_0 L^r_x([0,T^*))} = \infty$ for some admissible $(q_0, r_0)$ then $\|u\|_{L^q_0 L^r_x([0,T^*))} = \infty$ for all admissible $(q, r)$ satisfying $r_0 \leq r$. Hence if one can show (2) with $\|u\|_{L^q_0 L^r_x([0,T^*))} = \infty$, the result for $\|u\|_{L^q_0 L^r_x([0,T^*))} = \infty$ automatically follows.

Let $u$ be the maximal solution to (1) over the maximal forward existence time interval $[0, T^*)$ so that (5) is satisfied for an $\alpha$-admissible pair $(q, r)$, $2 < q \leq r$ and $\|u\|_{L^q_0 L^r_x([0, t_1] \times \mathbb{R}^d)} < \infty$ for $0 < t < T^* < \infty$.

Then for a fixed small $\eta > 0$ there is a strictly increasing sequence $\{t_n\}_{n=1}^{\infty}$ in $[0, T^*)$ such that $\lim_{n \to \infty} t_n = T^*$ and for every $n \in \mathbb{N}$

$$\|u\|_{L^q_t L^r_x([t_n, t_{n+1}])} = \eta.$$  

By Duhamel’s formula, we have for $t \in (0, T^*)$

$$u(t, x) = e^{i(-\Delta)^{\frac{\sigma}{2}} \cdot (t-t_n)} u(t_n) + i \int_{t_n}^{t} e^{i(-\Delta)^{\frac{\sigma}{2}} \cdot (t-s)} |u(s)|^2 u(s) ds.$$ 

Applying Strichartz’ estimate with (25), we have

$$\|u - e^{i(-\Delta)^{\frac{\sigma}{2}} \cdot (t-t_n)} u(t_n)\|_{L^q_t L^r_x([t_n, t_{n+1}])} \leq C \|u\|_{L^{2\alpha + d}_t L^{\alpha + 1}_x([t_n, t_{n+1}])} = C \eta^{\frac{2\alpha + d}{\alpha}}$$

where (4) holds. Hence from (25), (26) and time translation invariance property we obtain

$$\|e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} u(t_n)\|_{L^q_t L^r_x([t_n, t_{n+1}])} \geq \eta - C \eta^{\frac{2\alpha + d}{\alpha}} > \eta^{\frac{2\alpha + d}{\alpha}}$$

for sufficiently small $\eta$.

Fix $n \in \mathbb{N}$ and the time interval $(t_n, t_{n+1})$. We denote $f = u(t_n)$ and then by the mass conservation we have

$$\|f\|_{L^2_x} = \|u_0\|_{L^2_x}.$$ 

Applying Lemma 3.1 to $f$ with $\epsilon = \eta^{\frac{2\alpha + d}{\alpha}}$, there exists $\{f_\sigma\}_{1 \leq \sigma \leq L}$ such that $\hat{f}_\sigma$ is supported in a dyadic shell $B_{n,\sigma}$,

$$|\hat{f}_\sigma| \leq C \epsilon^{-\frac{\alpha}{2}} 2^{-\frac{d}{2}}$$

and

$$\|e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f - \sum_{\sigma=1}^{L} e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f_\sigma\|_{L^q_t L^r_x([t_n, t_{n+1}])} < \eta^{\frac{2\alpha + d}{\alpha}},$$

where $L = L(\|f\|_{L^2}, d, \eta)$. 

By Hölder’s inequality with $\frac{2}{r} + \frac{r-2}{r} = 1$, we have
\begin{align*}
\int_{t_n}^{t_{n+1}} \left( \int_{\mathbb{R}^d} |u|^2 |u(t, x)| - \sum_{s=1}^{L} e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} dx \right)^{\frac{2}{r}} dt \\
\leq \left\| u \right\|_{L_t^q L_x^r((t_n,t_{n+1}) \times \mathbb{R}^d)}^{2q} \left\| e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} \right\|_{L_t^q L_x^r}^{2r}
\end{align*}
(29)

By using Hölder’s inequality with $\frac{2}{r} + \frac{r-2}{r} = 1$ again, the right hand side of (29) is bounded by
\begin{align*}
\left\| u \right\|_{L_t^q L_x^r((t_n,t_{n+1}) \times \mathbb{R}^d)}^{2q} \left\| e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} \right\|_{L_t^q L_x^r}^{2r}
\end{align*}
\begin{align*}
+ \left\| e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} \right\|_{L_t^q L_x^r}^{2r} = E + F.
\end{align*}

In order to estimate $E$ and $F$, we apply (25), (26) and (28). Since $\frac{(2\alpha+d)r-4\alpha}{d} > r$ for $r \geq q > 2$, we see that
\begin{align*}
E + F &\leq C \eta^{\frac{2q}{r}} \eta^{\frac{(2\alpha+d)(r-2)q}{dr}} < \frac{\eta^q}{2}.
\end{align*}
(30)

We may split (25) into two integrals such as
\begin{align*}
\eta^q &= \int_{t_n}^{t_{n+1}} \left( \int_{\mathbb{R}^d} |u|^2 dx \right)^{\frac{q}{r}} dt \\
&\leq \int_{t_n}^{t_{n+1}} \left( \int_{\mathbb{R}^d} |u|^2 \sum_{\sigma=1}^{L} e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} dx \right)^{\frac{q}{r}} dt \\
&+ \int_{t_n}^{t_{n+1}} \left( \int_{\mathbb{R}^d} |u|^2 |u - \sum_{\sigma=1}^{L} e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} dx \right)^{\frac{q}{r}} dt.
\end{align*}
(31)

From (30) and (31) we obtain that
\begin{align*}
\int_{t_n}^{t_{n+1}} \left( \int_{\mathbb{R}^d} |u|^2 \sum_{\sigma=1}^{L} e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} dx \right)^{\frac{q}{r}} dt \geq \frac{\eta^q}{2}.
\end{align*}

Since $L = L(\|u_0\|_{L^2(\mathbb{R}^d)}, \eta)$, there exist an $n_0$ and an $f_0 = f_{n_0}$ supported on a dyadic shell $B_k$ for some $k$ such that
\begin{align*}
\int_{t_n}^{t_{n+1}} \left( \int_{\mathbb{R}^d} |u(t, x)|^2 |e^{i(t-t_n)\alpha} f_{\sigma} |r^{-2} dx \right)^{\frac{q}{r}} dt \geq \epsilon_0,
\end{align*}
(32)

where $\epsilon_0 = \frac{1}{2} \frac{\eta^q}{L_{n_0}^{-2n/r}}$. Then from (27) we have $|\hat{f}_0| \leq C \epsilon^{-\nu/2} \frac{\eta^q}{r}$.

By Lemma 3.2, there are an $L_1 = L_1(\|f_0\|_{L^2}, \eta)$ and a set of regions $\{Q_n\}_{1 \leq n \leq L_1}$ defined by
\begin{align*}
Q_n = \{ (t, x) \in \mathbb{R} \times \mathbb{R}^d ; t \in I_n \text{ and } x \in C_n \},
\end{align*}

where $C_n$ is a cube of side length $l(C_n) = 2^{-k}$ and $I_n$ is an interval of length $|I_n| = 2^{-\kappa \alpha}$ such that
\begin{align*}
\| e^{i(t-t_n)\alpha} f_0 \|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d \setminus \bigcup_{n=1}^{L_1} Q_n)} < \frac{\epsilon_0}{2 \eta^{2q/r}}.
\end{align*}
Then by using Hölder’s inequality with $\frac{2}{r} + \frac{r-2}{\alpha} = 1$ repeatedly, we have

\[
\left\| |u|^2 |e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f_0|^{r-2} \right\|_q^{\frac{q}{r}} \leq \|u\|_{L_t^q L_x^r((t_n,t_{n+1}) \times \mathbb{R}^d \setminus \bigcup_{n=1}^m \mathbb{Q}_n)}^{q-2} \|e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f_0\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d \setminus \bigcup_{n=1}^m \mathbb{Q}_n)}^{q(r-2)} \leq \eta^{q/r} \frac{\epsilon_0}{2}\eta^{q/r} = \frac{\epsilon_0}{2}.
\]

Thus from (32) it follows that

\[
\left\| |u|^2 |e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f_0|^{r-2} \right\|_q^{\frac{q}{r}} \geq \frac{\epsilon_0}{2}.
\]

This implies that there is a region $\mathbb{Q}_0 \subset \{ \mathbb{Q}_n \}_{n=1}^m$ such that

\[
\int_{(t_n,t_{n+1}) \cap I_0} \mathcal{H}(t) \, dt \geq \frac{1}{2L_1} \epsilon_0 := \epsilon_1
\]  

(33)

where

\[
\mathcal{H}(t) = \left( \int_{\mathbb{Q}_0^t} |u(t,x)|^2 |e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f_0(x)|^{r-2} \, dx \right)^{\frac{r}{2}}.
\]

Since $|f_0| \leq C \epsilon^{-\alpha} 2^{-\frac{kd}{2}}$ and $\hat{f}_0$ is supported in a dyadic shell of measure $2^{kd}$, we have

\[
|e^{i(t-t_n)(-\Delta)^{\frac{\alpha}{2}}} f_0(x)|^{\frac{q(r-2)}{r}} \leq \left( \int_{B_0} |\hat{f}_0(\xi)|^2 \, d\xi \right)^{\frac{q(r-2)}{r}} \leq C 2^{k_0 \alpha} \epsilon^{-\frac{2\alpha}{r}} = C |I_0|^{-1} \epsilon^{-\frac{2\alpha}{r}},
\]

where we use $dq(r-2)/r = 2\alpha$ and $|I_0| = 2^{-k_0 \alpha}$. Thus we have

\[
\mathcal{H}(t) \leq C |I_0|^{-1} \epsilon^{-\frac{2\alpha}{r}} \|u_0\|_{L_t^2}^{2\alpha},
\]

(34)

and in view of (33)

\[
\epsilon_1 \leq |I_0|^{-1} \epsilon^{-\frac{2\alpha}{r}} \|u_0\|_{L_t^2}^{2\alpha} (t_{n+1} - t_n).
\]

Thus we find the lower bound

\[
t_{n+1} - t_n \geq C |I_0| \epsilon^{2\alpha/r} \epsilon_1 := C |I_0| \epsilon_2.
\]

We divide the integral in the left hand side of (33) into two integrals such that

\[
\left( \int_{t_n}^{t_{n+1} - A |I_0| \epsilon_2} + \int_{t_{n+1} - A |I_0| \epsilon_2}^{t_{n+1}} \right) \mathcal{H}(t) \, dt.
\]

By (34), we can similarly choose $A$ small enough so that

\[
\int_{t_{n+1} - A |I_0| \epsilon_2}^{t_{n+1}} \mathcal{H}(t) \, dt \leq A \epsilon_1 \|u_0\|_{L_t^2}^{2\alpha} \leq \frac{\epsilon_1}{2}.
\]

In view of this and (33), we obtain that

\[
\int_{(t_n,t_{n+1} - A |I_0| \epsilon_2) \cap I_0} \mathcal{H}(t) \, dt \geq \frac{\epsilon_1}{2}.
\]
The inequality (34) leads to us that
\[
\frac{\epsilon_1}{2} \leq C |I_0| \sup_{t \in (t_n, t_{n+1}-A|I_0|/2)} \mathcal{H}(t)
\]
\[
\leq C \epsilon_1 \epsilon_2^{-1} \left( \sup_{t \in (t_n, t_{n+1}-A|I_0|/2)} \int_{Q_0^*} |u|^2 \, dx \right)^{\frac{\alpha}{2}}.
\]
Hence we obtain that \[
\sup_{t \in (t_n, t_{n+1}-A|I_0|/2)} \int_{Q_0^*} |u|^2 \, dx \geq C \left( \frac{\epsilon_2}{2} \right)^{\frac{\alpha}{2}}.
\]
Thus, for each \(t_n\) there are \(t_0 \in (t_n, t_{n+1}-A|I_0|/2)\) and a cube \(Q_0^*\) such that \[
\int_{Q_0^*} |u(t_0, x)|^2 \, dx \geq C \left( \frac{\epsilon_2}{2} \right)^{\frac{\alpha}{2}}.
\]
Since \(l(Q_0^*) = |I_0|^\frac{1}{2}\), then \(Q_0^*\) is contained in a ball of radius \(C_{d/2} |I_0|^\frac{1}{2}\). Since \(t_{n+1} - t_0 \geq C \epsilon_2 |I_0|\), \[
\epsilon_2^{\frac{1}{2}} |I_0|^\frac{1}{2} \leq C (t_{n+1} - t_0)^\frac{1}{2} \leq C (T* - t_0)^\frac{1}{2}.
\]
Hence \(Q_0^*\) can be covered by a finite number (depending on \(\eta, d, \|u_0\|_2\)) of balls of radius \(r = (T^* - t_0)^\frac{1}{2}\).

Therefore, there exists \(x_0 \in \mathbb{R}^d\) such that \[
\int_{B(x_0, (T^* - t_0)^\frac{1}{2})} |u(t_0, x)|^2 \, dx \geq \varepsilon,
\]
where \(\varepsilon \in \varepsilon (\|u_0\|_{L^2(\mathbb{R}^d)}, d, \eta)\) and independent of \(t_n\). This completes the proof. \(\square\)

3.2. Proof of Theorem 1.2. We proceed as in proof of Theorem 1.1. Let \(u\) be the maximal solution to (7) over the maximal forward existence time interval \([0, T^*)\) so that (5) holds for some Strichartz admissible pairs \((q, r)\) satisfying (9), and \(\|u\|_{L^q_t L^r_x(0, T^*)} < \infty\) for \(0 < t < T^* < \infty\). Let \(\eta\) and sequence \(t_1, \ldots, t_n, \ldots\) be given as before such that \(t_n \nearrow T^*\) and (25) is satisfied for every \(n \in \mathbb{N}\). By the Duhamel's formula we may write for \(t \in (0, T^*)\)
\[
u(t) = e^{i(-\Delta)^\frac{\alpha}{2}}(t-t_n) \, u(t_n) \pm i \int_{t_n}^t e^{i(t-s)(-\Delta)^\frac{\alpha}{2}} [(|x|^{-\alpha} \ast |u(s)|^2) u(s)] \, ds.
\]
We need to show the similar estimate as (26) for the solution of Hartree equation. That is to say, for the solution \(u\) of (7) there is a constant \(C > 0\) such that
\[
\left\| \int_{t_n}^t e^{i(t-s)(-\Delta)^\frac{\alpha}{2}} [(|x|^{-\alpha} \ast |u(s)|^2) u(s)] \, ds \right\|_{L^q_t L^r_x([t_n, t_{n+1}] \times \mathbb{R}^d)} \leq C \eta^{1+\theta}
\]
for \((q, r)\) satisfying (9) and for some \(0 < \theta < 1\). We note that the inequality above is obtained by repeating the local well-posedness argument. See the argument around (8)\(^1\) in Section 1. After achieving this we only need to deal with the homogeneous part of the solution to show the mass concentration. Hence, the remaining parts are the same as those for Theorem 1.1. We omit the details.

\(^1\)In fact, with \(v = 0\) one can easily show (35) with \(\theta = 0\).
4. **Appendix.** To prove Lemma 3.1 and Lemma 3.2 we modify Bourgain’s arguments in [3] (also see [1]) for the Schrödinger operator of higher orders \( \alpha \) with \( \alpha > 2 \). The proof of Lemma 3.1 relies on Proposition 2.3 which is obtained in Section 2. For the Proof of Lemma 3.2 the required strengthened estimate is given by (14) because the \( \alpha \) admissible pairs are contained in the range \( 2/q + d/r < d/2 \).

**Proof of Lemma 3.1.** From (23) and Proposition 2.3, we see that there are \( 0 < \theta < 1 \) and \( p < 2 \) such that

\[
\epsilon \leq \left\| e^{it(-\Delta)^{\frac{a}{2}}} \| \right\|_{L^q_t L^r_x} \lesssim \left\| f \right\|_{L^q_t L^r_x}^{1-\theta} \left( \sup_k 2^{kd(\frac{1}{p} - \frac{1}{2})} \left\| \hat{f} \chi_{B_k} \right\|_{L^p} \right)^\theta.
\]

So there exists a dyadic shell \( B_{n_1} \) for some \( n_1 \) such that

\[
\left\| \hat{f} \right\|_{L^p(B_{n_1})} \geq (\epsilon^{\frac{1}{p}} 2^{n_1 d(\frac{1}{p} - \frac{1}{2})} \left\| f \right\|_{L^2}^{1-\frac{1}{p}})^p.
\] (36)

Applying Lemma 2.4 to \( \hat{f} \) and \( B_{n_1} \), we have

\[
\epsilon^{\frac{1}{p}} \left\| f \right\|_{L^2}^{1-\frac{1}{p}} \lesssim \left\| \hat{f} \chi_{B_{n_1}} \right\|_2
\]

when \( \lambda \sim |B_{n_1}|^{-\frac{1}{2}} \epsilon^{-\frac{a}{2}-\frac{a}{2p} \frac{\mu}{2p}} \left\| f \right\|_{L^2}^{\frac{a}{2p}} + 1 \). We now define \( f_1 \) by \( \hat{f}_1 = \hat{f} \chi_{B_{n_1}} \) and insert \( |B_{n_1}|^{-\frac{1}{2}} \epsilon^{-\frac{a}{2}-\frac{a}{2p} \frac{\mu}{2p}} \left\| f \right\|_{L^2}^{\frac{a}{2p}} + 1 \) into (36). If \( \left\| e^{it(-\Delta)^{\frac{a}{2}}} \right\| \left\| f_1 \right\|_{L^q_t L^r_x} \leq \epsilon \), we are done by setting \( \nu = \frac{\mu}{2(p-2)} \), \( \mu = \frac{\nu}{2(p-2)} \) + 1. The property (3) follows from disjoint supports of \( \hat{f}_1 \) and \( \hat{f} \). We now define

\[
\left\| f \right\|_{L^2}^{1-\frac{1}{p}} \lesssim \left\| f \right\|_2
\]

where the first inequality follows from (3) and \( \hat{f} = \hat{f}_1 \). On the other hand, if \( \left\| e^{it(-\Delta)^{\frac{a}{2}}} \right\| \left\| f - f_1 \right\|_{L^q_t L^r_x} \geq \epsilon \), we repeat the above argument for \( f - f_1 \) to find \( f_2, B_{n_2} \), \( \lambda \) such that \( \left\| \hat{f}_2 \right\| \leq \lambda \), \( \lambda \sim |B_{n_2}|^{-\frac{1}{2}} \epsilon^{-\nu} \left\| f \right\|_{L^2}^{\nu} \), and \( \epsilon \left\| f \right\|_{L^2}^{1-\frac{1}{p}} \leq \epsilon \left\| f - f_1 \right\|_{L^2}^{1-\frac{1}{p}} \lesssim \left\| f_2 \right\|_2 \). The \( L^2 \) orthogonality holds as well, \( \left\| f - f_1 \right\|_2 = \left\| f_1 \right\|_2 + \left\| f - f_1 \right\|_2 \).

Recursively we can find \( f_k \) supported on \( B_{n_k} \) in the frequency space for \( k = 1, 2, \ldots, N \) such that

\[
\left\| f_k \right\|_2 = \sum_{k=1}^{N} \left\| f_k \right\|_2^2 + \left\| f - \sum_{k=1}^{N} f_k \right\|_2^2.
\]

This process will stop when \( \epsilon \) and \( \left\| f \right\|_{L^2} \) because

\[
\left\| e^{it(-\Delta)^{\frac{a}{2}}} f - \sum_{j=1}^{n} e^{it(-\Delta)^{\frac{a}{2}}} f_j \right\|_{L^q_t L^r_x} \leq C \left\| f - \sum_{j=1}^{n} f_j \right\|_{L^2}^2
\]

\[
= C(\left\| f \right\|_{L^2}^2 - \sum_{j=1}^{n} \left\| f_j \right\|_{L^2}^2)
\]

\[
\leq C(\left\| f \right\|_{L^2}^2 - n \epsilon)\left\| f \right\|_{L^2}^2 \epsilon^b)\).
\]

This completes the proof.

**Proof of Lemma 3.2.** We follow closely the argument for the proof Lemma 3.3 in [1]. Let \( g' \in L^2(\mathbb{R}^d) \) be the normalized function of \( g \) defined by \( g'(\xi') = 2^{\frac{d-1}{2}} g(2^d \xi') \).
Then \( \text{supp} \hat{g'} \subset B_1 \), \( \| \hat{g'} \|_{L^2} = \| g \|_{L^2} \) and \( |\hat{g'}| < 1 \). We see that
\[
e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(2^k x) = 2^{\frac{kd}{2}} \int e^{i|x|+it2^k\alpha} \hat{g}(2^k \xi) d\xi = 2^{-\frac{kd}{2}} \int e^{i|x|+it\xi} \hat{g}(\xi) d\xi = 2^{-\frac{kd}{2}} e^{it(-\Delta)^{\frac{\alpha}{2}}} g(x).
\]
That is to say,
\[
e^{it(-\Delta)^{\frac{\alpha}{2}}} g(x) = 2^{\frac{kd}{2}} e^{it(-\Delta)^{\frac{\alpha}{2}}} g(x') \tag{37}
\]
by the change of variable \((t, x) \rightarrow (t', x') = (2^{k+n}t, 2^k x)\).

We will keep track of the free evolution of \( g' \). Let \( E \subset \mathbb{R} \times \mathbb{R}^d \) be the set \( \{(t', x') : |e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x')| < \lambda\} \) for a given \( \lambda \). We have
\[
\|e^{it(-\Delta)^{\frac{\alpha}{2}}} g'\|_{L^q_t L^r_x(E)}^q = \int \int_{\{x' : |e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x')| < \lambda\}} \|e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x')\|^{r} r dx' \ dt' \leq \lambda^{(r-r^*)\frac{q}{2}} \int \left(\|e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x')\|^{r} dx'\right)^{\frac{q}{r}} dt'.
\]
Since \( \alpha > 2 \), we now note that the \( \alpha \)-admissible line is contained in the region of \( 2/q + d/r < d/2 \). Hence, we can pick up a pair \((q', r')\) in the region \( \frac{d}{r'} + \frac{2}{q'} \leq \frac{d}{2} \) such that \( q' < q \), \( r' < r \) and \( r'/q' = r/q \). Such choice may not be possible for the end point \( \frac{1}{2} = \frac{d}{2r_{2a}} \), which case was excluded because we are assuming \( q > 2 \) and \( r \neq \infty \) (see (4)). Then for \( \alpha \)-admissible \((q, r)\), (14) yields
\[
\|e^{it(-\Delta)^{\frac{\alpha}{2}}} g'\|_{L^q_t L^r_x(E)}^q \leq C \lambda^{(r-r^*)\frac{q}{2}} \| \hat{g'} \|_{L^2} \leq C \lambda^{(r-r^*)\frac{q}{2}} \| \hat{g} \|_{L^\infty},
\]
where the second inequality follows from the fact that \( \text{supp} \hat{g'} \subset B_1 \). Since \( r^* < r \), by choosing \( \lambda = \lambda(C_0, \epsilon) \) small enough, we have
\[
\|e^{it(-\Delta)^{\frac{\alpha}{2}}} g'\|_{L^q_t L^r_x(E)}^q \leq \epsilon^q
\]
where \( \tilde{E} = \{(t', x') : |e^{it\Delta} g'(x')| < 2\lambda\} \).

Due to the normalization, \( \text{supp} \hat{g'} \subset B_1 \) and \( \| \hat{g'} \|_{L^\infty} \leq C_0 \). Hence the function \((x,t) \rightarrow e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x)\) is smooth with bounded derivatives. In particular, the map
\[
e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x) = \int_{\mathbb{R}^d} e^{2\pi i (x - 2\pi t |\xi|^\alpha)} \hat{g'}(\xi) d\xi
\]
is Lipschitz. That is,
\[
|e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x') - e^{it''(-\Delta)^{\frac{\alpha}{2}}} g'(x'')| \leq C (|t' - t''| + |x' - x''|),
\]
where \( C = C(C_0, d) \geq 1 \). Hence, if \((t', x') \in E\) and \( |x' - x''|, |t' - t''| \leq \frac{1}{2^{2-s}} < \frac{1}{2} \), then \((t'', x'')\) is in \( \tilde{E} \). In other words, for \((t', x') \in (\mathbb{R} \times \mathbb{R}^d) \setminus \tilde{E} \), there is a space-time cube \( P = J \times K \) centered at \((t', x')\) with \( |J| = \frac{1}{2^s} \) and \( l(K) = \frac{1}{2^s} \) such that \( P \in (\mathbb{R} \times \mathbb{R}^d) \setminus E \). Let us cover \((\mathbb{R} \times \mathbb{R}^d) \setminus \tilde{E} \) with the family of \((P_r)_{r \in I}\) such that \( \text{Int}(P_r) \cap \text{Int}(P_s) = \emptyset \) for \( r \neq s \), and
\[
\{ |e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x')| \geq 2\lambda \} \subset \bigcup_{r \in I} P_r \subset \{ |e^{it(-\Delta)^{\frac{\alpha}{2}}} g'(x')| \geq \lambda \}. \tag{38}
\]
Here $\text{Int}(P_r)$ denotes the interior of the set $P_r$. Note that the index set $I$ is finite. We set $N_1 = \mathfrak{g} I$. It follows from (38) and the Strichartz’s estimate that

$$\begin{align*}
N_1 \left( \frac{\lambda}{C} \right)^{d+1} &= \left| \bigcup_{r \in I} P_r \right| \leq \left| \left\{ \left| e^{it\left(-\Delta\right)^{\frac{q}{2}}} g'(x') \right| \geq \lambda \right\} \right| \\
&\leq \lambda^{-\frac{2(d+n)}{d}} \left\| e^{it\left(-\Delta\right)^{\frac{q}{2}}} g'(x') \right\|_{L^{2(d+n)}(\mathbb{R} \times \mathbb{R}^d)\frac{2(d+n)}{d}} \\
&\leq C \lambda^{-\frac{2(d+n)}{d}} \|g\|_{L^2_{\mathfrak{g}}}^{\frac{2(d+n)}{d}}.
\end{align*}$$

From this we deduce that $N_1 \leq C(\|g\|_{L^2}, d, C_0, \epsilon)$. Actually, since our hypothesis implies that $\|g\|_{L^2} \leq C_0$, we can also write $N_1 \leq C(d, C_0, \epsilon)$. For simplicity let $\{1, \ldots, N_1\}$ denote the index set $I$. For any integer $1 \leq n \leq N_1$, let $(t_n, x_n)$ be the center of $P_n$ and let $I_n \subset \mathbb{R}$ be the interval of center $\frac{t_n}{2^{n-1}}$ with $|I_n| = \frac{1}{2^{n-1}}$. Also let $C_n$ be the cube of center $2^{-k} x_n$ with $\ell(C_n) = 2^{-k} C_n^3 = 2^k C_n$.

Finally let $\mathcal{Q}_n$ be defined by (24). Then from the choice of $\lambda$ it follows that

$$\left\| e^{it'(\Delta)^{\frac{q}{2}}} g' \right\|_{L^q_{t'}L^r_{x'}(\mathbb{R}+1 \cup \bigcup_{n=1}^{N_1} I_n \times C_n')} < \epsilon^q.$$ 

By (37) and reversing the change of variables $(t', x') \to (t, x)$, we have

$$\begin{align*}
\left\| e^{it(\Delta)^{\frac{q}{2}}} g' \right\|_{L^q_{t}L^r_{x}(\mathbb{R}+1 \cup \bigcup_{n=1}^{N_1} \mathcal{Q}_n)} &= 2^{k(d/2-d/r-\alpha/q)} \left\| e^{it(\Delta)^{\frac{q}{2}}} g' \right\|_{L^q_{t'}L^r_{x'}(\mathbb{R}+1 \cup \bigcup_{n=1}^{N_1} I_n \times C_n')} < \epsilon^q.
\end{align*}$$

since $(q, r)$ is admissible. This concludes the proof of the lemma. 

\[ \square \]

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