Nonlinear Quantum Transport and Current Noise

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We study nonequilibrium transport and noise in a generic dissipative tight-binding model. Within a real-time path integral approach, we derive formally exact series expressions in the number of tunneling events for the noise valid for arbitrary bias, frequency and temperature. At zero temperature, the low-frequency noise can be summed in analytic form. The resulting Shiba-like $|\omega|$-singularity is a consequence of the $1/t^2$ decay law for the current correlation function. At finite temperature, this singularity is smoothed out.

Low-energy excitations of 1D conduction electrons and of edge currents in fractional quantum Hall (FQH) systems \[1\] are bosonic and constitute prime realizations of a Luttinger liquid (LL) phase of interacting fermions \[2,3\]. A sensitive probe of this collective state is the tunneling conductance of electrons through an impurity or point contact in a 1D quantum wire \[4\], and of Laughlin quasiparticles through a narrow constriction for FQH edge currents \[5\]. In these systems, recent attention has been paid to current fluctuations with emphasis on dc noise. To second order in the tunneling amplitude, the dc noise $S(T=0)$ is classical shot noise, $S = 2e^*I$ \[6,7\]. Thus upon measuring dc current and dc noise in the weak-tunneling limit, one can determine the charge $e^*$ of the tunneling entity. In this way, the fractional charge of the Laughlin quasiparticles has been manifested \[8,9\]. In the strong tunneling regime, higher order tunneling processes cause a deviation of the dc noise from shot noise \[10,11\]. Furthermore, correlations between tunneling events lead to colored noise. Singular behavior $S(\omega) \propto |\omega|$ has been found within a nonperturbative analysis of the equivalent integrable Boundary Sine-Gordon (BSG) model \[12,13\]. In this treatise, the prefactor of the $|\omega|$-law depends on the particular treatment of the reflection matrix for quasiparticle scattering at the boundary coming from the Fermi sea (cf. the debate in Refs. \[12,13\]).

In this paper, we study the low-frequency noise upon employing real-time path integral techniques to a generic quantum transport model introduced by Schmid \[14\],

$$H = H_{TB} + \sum_{\alpha} \left[ \frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{1}{2} m_{\alpha} \omega_{\alpha}^2 \left( x_{\alpha} - \frac{c_{\alpha}}{m_{\alpha} \omega_{\alpha}^2} q \right)^2 \right].$$

(1)

The Hamiltonian (1) describes a particle which moves in the tilted tight-binding (TB) lattice $H_{TB} = -\frac{1}{2}\hbar\Delta \sum_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \hbar \epsilon q / q_0$ with lattice spacing $q_0$, tunneling matrix element $\Delta$, and biasing force $F = \hbar \epsilon / q_0$. The particle is bilinearly coupled via its position $q = q_0 \sum_n c_n^\dagger c_n$ to a bath of delocalized harmonic bosons. In the reduced density matrix description for the Brownian quantum particle, all effects of the environmental coupling are captured by the spectral density $J(\omega) = (q_0^2 / 2\hbar) \sum_{\alpha} (c_{\alpha}^2 / m_{\alpha} \omega_{\alpha}^2) \delta(\omega - \omega_{\alpha})$. The TB model (1) is dual to a corresponding weak-binding (WB) model, in which the particle moves in a tilted cosine potential. In the duality, the power $s$ in the generic form $J(\omega) \propto \omega^s$ is transformed into $2 - s$. Thus, super-Ohmic friction ($2 > s > 1$) is mapped on sub-Ohmic
friction \((0 < s < 1)\), and vice versa. In the Ohmic case, \(J(\omega) = 2K\omega\), the TB and WB models are self-dual to each other. The dimensionless Kondo parameter \(K\) is mapped onto the reciprocal value \(1/K\), and the bias \(\epsilon\) onto \(\epsilon/K\), as first observed by Schmid.

The TB and WB model describe many transport problems in solid state physics. Regarding applications of the model (1) to the beforementioned case of tunneling of a charge in a LL phase, the spectral density is strictly Ohmic for \(\omega \lesssim \omega_c\). In the scaling limit, \(\omega_c\) is by far the largest frequency of the problem. The interaction parameter \(g\) of the Luttinger model corresponds to \(1/K\), and the bias in the TB model (1) is related to the voltage drop \(V\) across the weak link or constriction by \(\hbar\epsilon = eV\), where \(e\) is the unit charge. Another important example where the model (1) applies is a voltage-biased Josephson junction [16, 17]. Here, the WB and TB limits correspond to the charge and phase representation, respectively.

Fluctuations are measured relative to a stationary state. Therefore, we proceed as follows. We first presuppose that the global system is in a factorized system-bath state in the infinite past. Then we switch on the bath coupling and the tilting force, and let the system evolve under reign of the full Hamiltonian (1). By the time \(t\), the system has reached a stationary nonequilibrium state, indicated by the index “ne”. The fluctuations of the current \(j(t) = e\hat{q}(t)/q_0\) are reflected in the noise spectrum, which is the Fourier transform of the symmetrized current correlation function,

\[
S(\omega) = \int_{-\infty}^{\infty} d\omega \, e^{i\omega t} \langle j(t)j(0) + j(0)j(t)\rangle_{ne}.
\] (2)

Observing that the current correlation function is related to the mean square displacement \(C_{ne}(t) = \langle [\hat{q}(t) - q(0)]^2\rangle_{ne}\) by

\[
\langle j(t)j(0) + j(0)j(t)\rangle_{ne} = \frac{e^2}{q_0^2} \frac{\partial^2}{\partial t^2} C_{ne}(t),
\] (3)

we can express the noise \(S(\omega)\) in terms of the spectral function of \(C_{ne}(t)\) as

\[
S(\omega) = -\frac{e^2}{q_0^2} \omega^2 \tilde{C}_{ne}(\omega), \quad \tilde{C}_{ne}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} C_{ne}(t).
\] (4)

In particular parameter regimes, the noise can directly be expressed in terms of transport quantities, e.g. the shot noise \(S = 2eVG(V)\) and Johnson-Nyquist noise \(S = 4k_B T G(V = 0)\), where \(G(V) = I(V)/V\) is the nonlinear conductance. For later convenience, we also introduce the differential conductance \(G_d(V) = \partial I(V)/\partial V\). The conductances are related to the mobilities of the model (1), \(\mu(F) = \langle \hat{q}(t \to \infty)\rangle_{tc}/F\) and \(\mu_d(F) = \partial\langle \hat{q}(t \to \infty)\rangle_{tc}/\partial F\), by \(G = e^2\mu/q_0^2\) and \(G_d = e^2\mu_d/q_0^2\). The index "tc" indicates that we may choose for the system-bath state at initial time zero a factorized form. This is possible since the leading contribution to mean values at asymptotic times \(t \to \infty\) does not depend on system-bath correlations in the initial state.

Within nonequilibrium linear response theory, the nonequilibrium susceptibility

\[
\chi_{ne}(t) = \frac{i}{\hbar} \Theta(t) \langle q(t)q(0) - q(0)q(t)\rangle_{ne}, \quad \tilde{\chi}_{ne}(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} \chi_{ne}(t)
\] (5)
comprises the differential mobility as
\[ \mu_a(F) = \lim_{\omega \to 0} \left[ -i\omega \chi_{ne}(\omega) \right] = \lim_{t \to \infty} \chi_{ne}(t) . \]  

Formally exact series expressions in \( \Delta \) for the above quantities can be derived upon employing a real-time path integral method for the reduced density matrix (RDM). Because of the discreteness of the TB model, a simple picture of a grandcanonical gas of charges can be developed. The charges are stringed at the flip times \( \{ t_j \} \), and the effects of the environment are in the charge interactions. For a path with \( n \) flips in the negative-time branch \( (t_1, t_2, \ldots, t_n < 0) \) and \( m \) flips in the positive-time branch \( (t_{n+1}, t_{n+2}, \ldots, t_{n+m} > 0) \) labelled by charges \( \{ \xi_j = \pm 1 \}, \{ \eta_j = \pm 1 \} \), the interaction factor or influence function is \( F_{n,m} = G_{n,m} H_{n,m}, \) where
\[
G_{n,m} = \exp \left\{ \sum_{k=2}^{n+m} \sum_{j=1}^{k-1} \xi_k \xi_j Q'(t_k - t_j) \right\}, \quad H_{n,m} = \exp \left\{ i \sum_{j=1}^{n+m-1} \eta_j \theta_{j,n+m} \right\}
\]
with \( \theta_{j,n+m} = \sum_{k=j+1}^{n+m} \xi_k Q''(t_k - t_j) \). The charges \( \{ \eta_j \} \) describe propagation on the infinite RDM in diagonal direction, whereas the charges \( \{ \xi_j \} \) indicate moves perpendicular to the diagonal, i.e. represent quantum fluctuations. The pair interaction \( Q(t) \) is the second integral of the autocorrelation function of the random force \( f(t) = \sum_{\alpha} c_{\alpha} x_{\alpha}(t) \) in thermal equilibrium, \( \dot{Q}(t) = (q_0/h)^2 \langle f(t)f(0) \rangle_{eq} \). In the Ohmic scaling limit, the pair interaction reads
\[ Q(t) = Q'(t) + iQ''(t) = 2K \ln \left[ \frac{(h/\beta)}{\pi} \sinh(\pi|t|/h\beta) \right] + ipK \sgn(t) . \]  

The influence of the bias \( \epsilon \) or voltage drop \( V \) is included in the bias phase factor
\[ b_{n,m} = \exp(-i\phi_{n,m}), \quad \phi_{n,m} = \epsilon \sum_{i=1}^{n+m} \xi_i t_i = \epsilon \sum_{i=1}^{n+m-1} g_{l,n+m} \rho_l . \]  

Here, we have introduced as an off-diagonal measure the cumulative charge after \( l \) jumps \( g_{l,n+m} = \sum_{k=l+1}^{n+m} \xi_k = -\sum_{k=1}^{l} \xi_k \), and the time intervals \( \rho_l = t_{l+1} - t_l \).

Following the lines of Ref. [15], it is straightforward to arrive at formally exact series expressions for the Laplace transforms of the above quantities in which the summation over all possible sequences \( \{ \eta_j \} \) has been performed already.

The first moment in the stationary nonequilibrium state with initial condition \( \langle q(t = 0) \rangle_{ne} = 0 \) is given by
\[ \langle \dot{q}(\lambda) \rangle_{ne} = \langle \dot{q}(\lambda) \rangle_{fc} + \dot{R}_{ne}(\lambda) . \]  

The term \( \langle \dot{q}(\lambda) \rangle_{fc} \) is the first moment with factorized system-bath initial state at time zero. This term has dynamics only in the positive-time branch. The residual term \( \dot{R}_{ne}(\lambda) \) describes the corrections due to the correlated initial state,
\[
\langle \dot{q}(\lambda) \rangle_{fc} = -\frac{i}{\lambda^2} \sum_{\{\xi\}} \sum_{m \text{ even}} \int_0^\infty \tilde{D}_{0,m}(\lambda) \sum_{\{\xi\}} q_{0,m}^{(1)} G_{0,m} \sin \phi_{0,m} ,
\]
\[
\dot{R}_{ne}(\lambda) = -\frac{i}{\lambda} \sum_{n,m=1}^\infty \int_0^\infty \tilde{D}_{n,m}(\lambda) \int_0^\infty d\tau_n \, ds_n \, e^{-\lambda s_n} \sum_{\{\xi\}} q_{n,m}^{(1)} G_{n,m} \sin \phi_{n,m} .
\]
The phase factor resulting from the interval lengths except for the interval \( \rho_n \),

\[
\int_0^\infty \tilde{D}_{n,m}(\lambda) \cdots = \Delta^{n+m} \left( \prod_{l=n+1}^{n+m-1} \int_0^\infty d\rho_l e^{-\lambda \rho_l} \right) \prod_{k=1}^{n-1} \int_0^\infty d\rho_k \cdots .
\]  

(13)

The interval \( \rho_n \) around \( t = 0 \) is divided into the negative-time part \( \tau_n = -t_n \) and the positive-time part \( s_n = t_{n+1} \) and is treated separately. Finally, the phase factor

\[
a_{n,m}^{(1)} = i \frac{q_0}{2} (-1)^{(n+m-2)/2} \prod_{j=1}^{n+m-1} \sin(\theta_{j,n+m})
\]

(14)

subsumes the interaction between the \( \{ \eta_j \} \) and \( \{ \xi_j \} \) charges. In the Ohmic scaling limit, the phase \( \theta_{j,n+m} \) is independent of the flip times and is \( \theta_{j,n+m} = \pi K g_{j,n+m} \).

Similarly, the series expression for the nonequilibrium susceptibility reads

\[
\hat{\chi}_{ne}(\lambda) = \frac{q_0}{i \hbar \lambda} \sum_{n,m=1}^\infty \int_0^\infty \tilde{D}_{n,m}(\lambda) \int_0^\infty dt_n ds_n e^{-\lambda s_n} \sum_{\{ \xi_l \}} g_{n,n+m} a_{n,m}^{(1)} G_{n,m} \cos \phi_{n,m} .
\]

The differential mobility can be obtained either from this form using the relation (6) or from its definition with eq. (11). In any event, we arrive at the series expression

\[
\mu_d(\epsilon) = \frac{q_0^2}{2 \hbar} \sum_{m \text{ even}} (-1)^{m/2-1} \int_0^\infty \tilde{D}_{0,m}(\lambda \to 0) \sum_{\{ \xi_l \}} \left( \prod_{l=1}^{m-1} \sin \theta_{l,m} \right) G_{0,m} B_{0,m} ,
\]

(15)

\[
B_{0,m} = \frac{\partial}{\partial \epsilon} \sin \phi_{0,m} = \left( \sum_{l=1}^{m-1} g_{l,m} \rho_l \right) \cos \phi_{0,m} .
\]

As well, the mean square displacement can be divided into the second moment with factorized initial state at time zero, \( \langle \hat{q}^2(\lambda) \rangle_{fc} \), and the correction term \( \hat{K}_{ne}(\lambda) \),

\[
\hat{C}_{ne}(\lambda) = \langle \hat{q}^2(\lambda) \rangle_{fc} + \hat{K}_{ne}(\lambda) ,
\]

(16)

\[
\langle \hat{q}^2(\lambda) \rangle_{fc} = \frac{1}{\lambda^2} \sum_{m \text{ even}} \int_0^\infty \tilde{D}_{0,m}(\lambda) \sum_{\{ \xi_l \}} a_{0,m}^{(2)} G_{0,m} \cos \phi_{0,m} ,
\]

(17)

\[
\hat{K}_{ne}(\lambda) = \frac{1}{\lambda} \sum_{n,m=1}^\infty \int_0^\infty \tilde{D}_{n,m}(\lambda) \int_0^\infty dt_n ds_n e^{-\lambda s_n} \sum_{\{ \xi_l \}} a_{n,m}^{(2)} G_{n,m} \cos \phi_{n,m} .
\]

(18)

The phase factor resulting from the \( \{ \eta_j \} \) summation differs from \( a_{n,m}^{(1)} \). We have

\[
a_{n,m}^{(2)} = -i q_0 a_{n,m}^{(1)} \sum_{k=n+1}^{n+m-1} \cot(\theta_{k,n+m}) .
\]

(19)
The formally exact series expression for the noise follows from (11) and (13) - (18) as

\[
S(\omega) = 2(e^2/q_0^2) \text{Re} \left[ \lambda^2 \hat{C}_{\text{ne}}(\lambda) \right]_{\lambda=-i\omega+0^+}.
\] (20)

Consider now the \( \lambda \)-expansions of eqs. (10) - (12) and eqs. (16) - (18) about \( \lambda = 0 \), which determine upon Laplace transformation the behavior of the respective quantities at long times. We first observe that the phase factor \( \alpha_{n,m}^{(1)} \) vanishes if the system reaches a diagonal state \( g_{l,n+m} = 0 \). Therefore, all paths that contribute to the first moment hit the diagonal of the RDM only in the initial and final state. Thus each interval \( \rho_l \) separates two charged clusters, and for \( \lambda = 0 \), the interaction factor \( G_{n,m} \) at \( T > 0 \) or the bias factor \( B_{n,m} \) at \( T = 0 \) ensures the convergence of the respective integration. For the first moment with factorized initial conditions, eq. (11), we find

\[
\lambda^2 \langle \hat{q}(\lambda) \rangle_{\text{fc}} = F\mu + \lambda q_\infty + \mathcal{O}(\lambda^2), \quad q_\infty = \lim_{\lambda \to 0} (\partial/\partial \lambda) \left[ \lambda^2 \langle \hat{q}(\lambda) \rangle_{\text{fc}} \right].
\] (21)

The series expression for \( \mu \) is as in (13), but with \( B_{0,m} = \sin(\phi_{0,m})/\epsilon \).

The series (12) can be rearranged upon using the relation (we put \( \rho_n = \tau_n + s_n \))

\[
\int_0^\infty d\tau_n \, ds_n \, e^{-\lambda s_n} \, A(\tau_n + s_n) = \frac{1}{\lambda} \int_0^\infty d\rho_n \left[ 1 - e^{-\lambda \rho_n} \right] A(\rho_n),
\] (22)

where \( A(\rho_n) \) subsumes all correlations across the intervall \( \rho_n \). It is then straightforward to see that the \( \lambda \)-dependence of the expression \( \lambda^2 \hat{R}_{\text{ne}}(\lambda) \) is fully cancelled by \( \lambda^2 \langle \hat{q}(\lambda) \rangle_{\text{fc}} \). In the end, we find using eq. (10) for all \( \lambda \)

\[
\lambda^2 \langle \hat{q}(\lambda) \rangle_{\text{ne}} = F\mu.
\] (23)

This verifies that the mean position with initial value \( \langle q(t = 0) \rangle_{\text{ne}} = 0 \) evolves in the stationary nonequilibrium state for all times as \( \langle q(t) \rangle_{\text{ne}} = F\mu t \).

After these preliminaries, consider now the mean square displacement, eq. (13). In the phase factor \( \alpha_{n,m}^{(2)} \), the factor \( \cos(\pi K g_{l,n+m}) \) can be allocated to any interval in the positive-time branch. However, upon employing the relation (22), the series for \( \hat{C}_{\text{ne}}(\lambda) = \langle \hat{q}^2(\lambda) \rangle_{\text{fc}} + \hat{K}_{\text{ne}}(\lambda) \) can be rearranged so that most contributions of the sum in eq. (19) cancel out. The only remaining contribution is the one in which the cosine factor covers the first interval in the positive-time branch, labelled by \( \rho_n \). Thus, this is the only interval which is allowed to be occupied by a diagonal state, \( g_{n,n+m} = 0 \). Since \( \rho_n \) separates two neutral \( \{\xi_j\} \)-clusters, the attractive bias and charge interaction factors are missing, and hence this interval gets very long as \( \lambda \to 0 \). It is convenient to factorize the interaction term \( G_{n,m} \) (\( n, m \) even) into the intra-cluster interaction factors \( G_{n,0} \) and \( G_{0,m} \) and the inter-cluster interaction factor \( G_{\text{int}} \), \( G_{n,m} = G_{n,0} G_{0,m} G_{\text{int}} \). Expanding the inter-cluster interaction stretching over the long interval \( \rho_n \), we get

\[
G_{\text{int}} = 1 - \hat{Q}'(\rho_n) \sum_{l=1}^{n-1} g_{l,n+m} \rho_l \sum_{k=n+1}^{n+m-1} g_{k,n+m} \rho_k.
\] (24)

The first term is the contribution in the absence of the cluster interaction, and the second term represents the dipole-dipole interaction. Accordingly, we get two contributions to
\( \dot{C}_{ne}(\lambda) \), called \( \dot{C}_{ne}^{(a)}(\lambda) \) and \( \dot{C}_{ne}^{(b)}(\lambda) \). In the first case, the cluster to the left of \( \rho_n \) is just \( F\mu = \lambda^2 \langle \dot{q}(\lambda) \rangle_{ne} \), whereas the cluster to the right of \( \rho_n \) represents \( \lambda^2 \langle \dot{q}(\lambda) \rangle_{fc} \) [Note that each cluster is actually an infinite series in \( \Delta^2 \)]. Since the two clusters are noninteracting, the integration over \( \rho_n \) gives a factor \( 1/\lambda \). Thus we find

\[
\lambda^2 \dot{C}_{ne}^{(a)}(\lambda) = 2 \left[ F\mu \right] \left[ \lambda^2 \langle \dot{q}(\lambda) \rangle_{fc} / \lambda \right] = 2F^2 \mu^2 / \lambda + 2F\mu q_{\infty} + \mathcal{O}(\lambda) .
\] (25)

In the second form, we have used the result (24).

Consider next the dipole contribution \( \dot{C}_{ne}^{(b)}(\lambda) \). Upon inserting the second term of eq. (24) into the series for \( \dot{C}_{ne}(\lambda) \), we observe that the two sums can be generated by suitable differentiations with respect to the bias. Thus we find

\[
\lambda^2 \dot{C}_{ne}^{(b)}(\lambda) = -\frac{q_0^2}{2\lambda^2} \sum_{n \text{ even}} \sum_{m \text{ even}} \int_0^\infty d\rho_n \exp(-\lambda\rho_n) \dot{Q}'(\rho_n) \int_0^\infty \hat{D}_{n,m}(\lambda)
\]

\[
\times \sum_{\langle \xi_l \rangle} \sum_{\langle \xi_k \rangle_m} G_{n,0} G_{0,m} \left( \frac{\partial}{\partial \epsilon} \sin\phi_{n,0} \right) \left( \frac{\partial}{\partial \epsilon} \sin\phi_{0,m} \right) \prod_{l=1, l\neq n}^{m+n-1} \sin \theta_{l,n+m} .
\] (26)

The sums \( \sum_{\langle \xi_l \rangle} \) and \( \sum_{\langle \xi_k \rangle_m} \) cover all neutral charge clusters, i.e., they are constrained by \( \sum_{l=1}^n \xi_l = 0 \) and \( \sum_{l=n+1}^{m+n} \xi_l = 0 \).

At \( T = 0 \), the dipole-dipole-interaction is found from eq. (8) as \( \dot{Q}'(t) = -2K/t^2 \). Substituting this form, the integration over \( \rho_n \) in the limit \( \lambda \to 0 \) yields

\[
\int_0^\infty d\rho_n \exp(-\lambda\rho_n) \dot{Q}'(\rho_n) \approx -2K \lambda \ln(\lambda/\lambda^*) .
\] (27)

Now, in leading order of \( \lambda \) both the cluster on the left and the cluster on the right of \( \rho_n \) can be identified with the differential mobility \( \mu_d \), eq. (13). Thus we find

\[
\lambda^2 \dot{C}_{ne}^{(b)}(\lambda) = -(4\hbar^2 K/q_0^2) \mu_d^2 \lambda \ln(\lambda/\lambda^*) \left[ 1 + \mathcal{O}(\lambda) \right] .
\] (28)

So far, we have considered the contributions to \( \dot{C}_{ne}(\lambda) \) in which the period \( \rho_n \) the system dwells in a diagonal state is very long. However, there are also contributions in which the diagonal state is either occupied for a short period only or not occupied at all, \( g_{n,n+m} \neq 0 \). In both cases, the charges form a single neutral cluster with regular behavior in the limit \( \lambda \to 0 \). These terms are combined in the expression

\[
\lambda^2 \dot{C}_{ne}^{(c)}(\lambda) = 2D' + \mathcal{O}(\lambda) .
\] (29)

Putting all contributions together, the mean square displacement takes the form

\[
\lambda^2 \dot{C}_{ne}(\lambda) = \frac{2}{\lambda} F^2 \mu^2 + (2F\mu q_{\infty} + 2D') - \frac{4\hbar^2 K}{q_0^2} \mu_d^2 \lambda \ln(\lambda/\lambda^*) + \mathcal{O}(\lambda) .
\] (30)

Turning to the time regime, the expression (30) gives the behavior at long times,

\[
C_{ne}(t) = F^2 \mu^2 t^2 + 2Dt + (4\hbar^2 K/q_0^2) \mu_d^2 \ln(t/t^*) .
\] (31)
Here, all terms linear in \( t \) are combined in the diffusion constant

\[
D = \frac{1}{2} \lim_{t \to \infty} \frac{\partial}{\partial t} [C_{\text{ne}}(t) - \langle q(t) \rangle_{\text{ne}}^2] = D' + F \mu q_\infty .
\]  

(32)

The current correlation function at long times is found from eqs. (3) and (31) as

\[
\langle j(t)j(0) + j(0)j(t) \rangle_{\text{ne}} = 2(eF \mu / q_0)^2 - \frac{2 \hbar e^2 \mu_0}{\pi q_0^2} \frac{[\tilde{\mu}_d]^2}{t^2} .
\]

(33)

Here, we have normalized the differential mobility on the mobility of a free Brownian particle,

\[
\tilde{\mu}_d = \mu_d / \mu_0 \quad \text{with} \quad \mu_0 = q_0^2 / 2 \pi \hbar K .
\]

Using (20), it is easy to commute the \( \lambda \)-expansion (30) into the low-frequency noise spectrum. The 1/\( \lambda \)-term in the expansion (30) does not contribute to the noise. The term of order \( \lambda^0 \) directly relates the dc noise \( S(\omega = 0) \) to the diffusion constant,

\[
S(\omega = 0) = 4e^2 D / q_0^2 .
\]

(34)

In the integrable BSG model, a basis of interacting quasiparticles which scatter off the impurity or point contact one by one can be introduced. Upon generalizing a Landauer-type approach familiar from transport of free electrons to this interacting case, it has been argued that the dc noise at \( T = 0 \) is connected with the conductance by differentiation with respect to the voltage \([10]\) or the tunneling amplitude \([11,20]\)

\[
S(\omega = 0) = eV \Delta \partial G / \partial \Delta = |1/(K - 1)| eV^2 \partial G / \partial V .
\]

(35)

In order \( \Delta^2 \), this formula reproduces shot noise, \( S(\omega = 0) = 2eVG = 2eI \). The higher order terms describe corrections due to quantum coherence for strong tunneling. The relation (35) can be proven on the basis of (13) - (18) and is a consequence of (i) a generally valid detailed balance relation (DBR) that follows from analytic properties of the pair interaction \( Q(t) \) and (ii) of a special DBR which is restricted to the scaling limit, i.e. relies on the particular form (8) for \( Q(t) \) \([21]\). The general DBR states that at \( T = 0 \) there is no real occupation of TB states higher than the initial state. The special DBR states that only those paths for which all \( \{ \eta_j \} \) charges have equal sign, i.e., which are free of backward hops parallel to the diagonal, contribute to the noise.

The low-frequency correction \( \Delta S(\omega) \) is captured by the term of order \( \lambda \ln(\lambda / \lambda^*) \) in (30). Using (20), we obtain for the noise the nonperturbative universal form

\[
\Delta S(\omega) = 2\hbar \mu_0 (e^2 / q_0^2) \tilde{\mu}_d^2 |\omega| = 2\hbar G_0(g) \tilde{G}_d^2 |\omega| .
\]

(36)

In the second form, we have used transport quantities of the related Luttinger model, i.e. the dimensionless differential conductance \( \tilde{G}_d = G_d / G_0(g) \), normalized with the static limit of the microwave conductance of a quantum wire or the Hall conductance, \( G_0(g) = ge^2 / h \). This result agrees with the findings of Ref. \([13]\).

The algebraic \( 1/t^2 \) decay law of the current correlation function (33) and the resulting \( |\omega| \)-singularity in the noise spectrum (36) are signatures of the unscreened dipole-dipole interaction. These forms are universal since they do not depend explicitly on the parameter \( K \) or \( g \) and the applied bias. The parameters are only in the mobilities or conductances which
determine the prefactor of the $1/t^2$- and $|\omega|$-law. The forms (33) and (36) are analogous to the $1/t^2$ decay law of the position correlation function \[18\] in the Ohmic two-state system and the equivalent Shiba relation in frequency space, respectively \[18\]. The difference, however, is in the prefactor. For the position autocorrelation function, each of the two sets of dipoles corresponds to the static susceptibility $\chi_0$, whereas in the present case each dipole set represents the differential mobility, which is related to the nonequilibrium susceptibility by eq. (6). In fact, in transport problems with a stationary nonequilibrium state, the static susceptibility diverges, and the differential mobility $\mu_d(F)$ is the proper quantity here.

At finite temperature, the dipole-dipole interaction is given by

$$\ddot{Q}(t) = -2K (\pi/\hbar\beta)^2 / \sinh^2(\pi t/\hbar\beta).$$

Performing the integration over the interval $\rho_n$ between the dipoles, eq. (27), and switching to the noise, the low-frequency contribution from $\lambda^2 \hat{C}^{(b)}_{ne}(\lambda)$ takes the form

$$\Delta S(\omega, T) = \frac{e^2}{\pi K} [\mu_d]^2 \omega \coth(\hbar\beta\omega/2) \approx \frac{2e^2}{\pi K \hbar \beta} [\bar{\mu}_d]^2 + \frac{e^2 \hbar \beta}{6\pi K} [\bar{\mu}_d]^2 \omega^2 + O(\omega^4).$$

Thus, the $|\omega|$-singularity is smoothed out at finite temperature. We get a contribution to the dc noise which is proportional to $T$ and a correction term of order $\omega^2$. However, there are additional contributions $\propto \omega^0$ and $\propto \omega^2$ from $\lambda^2 \hat{C}^{(a)}_{ne}(\lambda)$ and $\lambda^2 \hat{C}^{(c)}_{ne}(\lambda)$. However, the corresponding prefactors can not be given in analytic form for all $K$.

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[20] In the scaling limit, we have $G = G(\Delta^2 V^{2K-2})$ and thus $V \partial G/\partial V = [K - 1] \Delta \partial G/\partial \Delta$
[21] Details will be published elsewhere