DARBOUX COORDINATES ON K-ORBITS AND THE SPECTRA OF CASIMIR OPERATORS ON LIE GROUPS

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Abstract

We propose an algorithm for obtaining the spectra of Casimir (Laplace) operators on Lie groups. We prove that the existence of the normal polarization associated with a linear functional on the Lie algebra is necessary and sufficient for the transition to local canonical Darboux coordinates \((p,q)\) on the coadjoint representation orbit that is linear in the "momenta." We show that the \(\lambda\)-representations of Lie algebras (which are used, in particular, in integrating differential equations) result from the quantization of the Poisson bracket on the coalgebra in canonical coordinates.

Introduction

The method of orbits discovered in the pioneering works of Kirillov [1, 2] (see also [3, 4]) is a universal base for performing harmonic analysis on homogeneous spaces and for constructing new methods of integrating linear differential equations. Steps towards implementing this program were described in [5]. The main result of this work is an algorithm for obtaining the spectra of Casimir (Laplace) operators on a Lie group via linear algebraic methods starting with the known structure constants and the information regarding the compactness of certain subgroups. We show that the existence of a normal polarization associated with a linear functional \(\lambda\) is necessary and sufficient for the existence of local canonical Darboux coordinates \((p,q)\) on the K-orbit \(O_\lambda\) such that the transition to these coordinates is linear in the "momenta." (We developed a computer program based on the computer algebra system Maple V to evaluate the canonical Darboux coordinates for a given functional and the corresponding normal polarization.) The subsequent "quantization" leads to the notion of \(\lambda\)-representations of Lie algebras consisting in assigning each K-orbit a special representation of the Lie algebra via differential operators. The \(\lambda\)-representations first appeared in the noncommutative integration method of linear differential equations [6] as a "quantum" analogue of the noncommutative Mishchenko-Fomenko integration method for finite-dimensional Hamiltonian systems [7]. The \(\lambda\)-representation operators are also implicitly involved as the generators of irreducible representations of Lie groups.
1 The description of K-orbits

Let $G$ be a real connected $n$-dimensional Lie group and $\mathcal{G}$ be its Lie algebra. The action of the adjoint representation $Ad^*$ of the Lie group defines a fibration of the dual space $\mathcal{G}^*$ into even-dimensional orbits (the K-orbits). The maximum dimension of a K-orbit is $n-r$, where $r$ is the index (ind $\mathcal{G}$) of the Lie algebra defined as the dimension of the annihilator of a general covector. We say that a linear functional (a covector) $\lambda$ has the degeneration degree $s$ if it belongs to a K-orbit $\mathcal{O}_\lambda$ of the dimension $\dim \mathcal{O}_\lambda = n-r - 2s$, $s = 0, \ldots, (n-r)/2$.

We decompose the space $\mathcal{G}^*$ into a sum of nonintersecting invariant algebraic surfaces $M_s$ consisting of K-orbits with the same dimension. This can be done as follows. We let $f_i$ denote the coordinates of the covector $f$ in the dual basis, $f = f_ie^i$ with $\langle e^i, e_j \rangle = \delta_j^i$, where $\{e_j\}$ is the basis of $\mathcal{G}$. The vector fields on $\mathcal{G}^*$

$$Y_i(f) \equiv C_{ij}(f) \frac{\partial}{\partial f_j}, \quad C_{ij}(f) \equiv C_{ij}^k f_k$$

are generators of the transformation group $G$ acting on the space $\mathcal{G}^*$, and their linear span therefore constitutes the space $T_f \mathcal{O}_\lambda$ tangent to the orbit $\mathcal{O}_\lambda$ running through the point $f$. Thus, the dimension of the orbit $\mathcal{O}_\lambda$ is determined by the rank of the matrix $C_{ij}$,

$$\dim \mathcal{O}_\lambda = \text{rank} C_{ij}(\lambda).$$

It can be easily verified that the rank of $C_{ij}$ is constant over the orbit. Therefore, equating the corresponding minors of $C_{ij}(f)$ to zero and ”forbidding” the vanishing of lower-order minors, we obtain polynomial equations that define a surface $M_s$,

$$M_0 = \{ f \in \mathcal{G}^* \mid \neg(F^1(f) = 0) \};$$

$$M_s = \{ f \in \mathcal{G}^* \mid F^s(f) = 0, \neg(F^{s+1}(f) = 0) \}, \quad s = 1, \ldots, \frac{n-r}{2} - 1;$$

$$M_{\frac{n-r}{2}} = \{ f \in \mathcal{G}^* \mid F^{\frac{n-r}{2}}(f) = 0 \}.$$ 

Here, we let $F^s(f)$ denote the collection of all minors of $C_{ij}(f)$ of the size $n-r-2s+2$, the condition $F^s(f) = 0$ indicates that all the minors of $C_{ij}(f)$ of the size $n-r-2s+2$ vanish at the point $f$, and $\neg(F^s(f) = 0)$ means that the corresponding minors do not vanish simultaneously at $f$.

The space $M_s$ can also be defined as the set of points $f$ where all the polyvectors of degree $n-r-2s+1$ of the form $Y_i(f) \wedge \ldots \wedge Y_{n-r-2s+1}(f)$ vanish, but not all the polyvectors of degree $n-r-2s-1$ vanish.

We note that in the general case, the surface $M_s$ consists of several nonintersecting invariant components, which we distinguish with subscripts as $M_s = M_{sa} \cup M_{sb} \ldots$. (To avoid stipulating each time that the space $M_s$ is not connected, we assume the convention that $s$ in parentheses, $(s)$, denotes a specific type of the orbit with the degeneration degree $s$.) Each component $M_{(s)}$ is defined by the corresponding set of homogeneous polynomials $F_{(s)}^s(\alpha)$ satisfying the conditions

$$Y_iF_{(s)}^s(\alpha)|_{F_{(s)}^s(\alpha) = 0} = 0. \quad (1.1)$$

Although the invariant algebraic surfaces $M_{(s)}$ are not linear spaces, they are star sets, i.e., $f \in M_{(s)}$, implies $tf \in M_{(s)}$ for $t \in R^1$.
Example 1 (The Poincare group $P^{1,3}$). The Poincare group $P^{1,3} = T^4 \triangleright SO(1, 3)$ is the group of motions in Minkowski space and is a semidirect product of the semisimple Lorentz rotation group $SO(1, 3)$ with the four-dimensional commutative group of translations $T^4$. The commutation relations of the Poincare algebra $\mathcal{P}^{1,3} = \{e_a, e_b\}$ can be written as

$$[e_{ab}, e_{cd}] = g_{ad}e_{bc} - g_{ac}e_{bd} + g_{bc}e_{ad} - g_{bd}e_{ac}; \quad [e_a, e_b] = g_{ab}e_c - g_{ac}e_b;$$

$$[e_a, e_b] = 0; \quad a, b, c, d = 0, 1, 2, 3; \quad g_{ab} = \text{diag}(1, -1, -1, -1).$$

We have $n = 10$ and $r = 2$ in this example. With $\{l_{ab}, p_a\}$ denoting the coordinates of a covector $f$ in the dual basis, we give the explicit form of the surfaces $M_{(a)}$:

$$(\mathcal{P}^{1,3})^* = M_0 \cup M_{1a} \cup M_{1b} \cup M_2 \cup M_4,$$

$$M_0 = \{f \in R^{10} \mid - (W_a p_b - W_b p_a = 0)\}; \quad \text{dim} M_0 = 10;$$

$$M_{1a} = \{f \in R^{10} \mid W_a = 0, - (p_a = 0)\}; \quad \text{dim} M_{1a} = 7;$$

$$M_{1b} = \{f \in R^{10} \mid W_a p_0 = W_0 p_a, - (W_a = 0)\}; \quad \text{dim} M_{1b} = 7;$$

$$M_2 = \{f \in R^{10} \mid p_a = 0, - (f = 0)\}; \quad \text{dim} M_2 = 6;$$

$$M_3 = \emptyset; \quad M_4 = \{f = 0\}; \quad \text{dim} M_4 = 0.$$

(We introduce the notation $W^a \equiv \frac{1}{2} e^{abcd} p_c p_d$, and the indices are raised and lowered by the diagonal matrix $g_{ab}$.) We note that in view of the identity $W^a p_a = 0$, there are only three independent functions among the four functions $W^a$.

The dual space $G^*$ has a degenerate linear Poisson bracket

$$\{\varphi, \psi\}(f) \equiv \langle f, [\nabla \varphi(f), \nabla \psi(f)] \rangle; \quad \varphi, \psi \in C^\infty(G^*). \quad (1.2)$$

The functions $K_{(s)}^\mu(f)$ that are nonconstant on $M_{(s)}$ are called the (s)-type Casimir functions if they commute with any function on $M_{(s)}$.

The (s)-type Casimir functions can be found from the equations

$$C_{ij}(f) \frac{\partial K_{(s)}^\mu(f)}{\partial f_j} \bigg|_{f \in M_{(s)}} = 0, \quad i = 1, \ldots, n. \quad (1.3)$$

It is obvious that the number $r_{(s)}$ of independent (s)-type Casimir functions is related to the dimension of the space $M_{(s)}$ as $r_{(s)} = \text{dim} M_{(s)} + 2s + r - n$. Because $M_{(s)}$ is star spaces, we can assume without loss of generality that the Casimir functions $K_{(s)}^\mu(f)$ are homogeneous,

$$\frac{\partial K_{(s)}^\mu(f)}{\partial f_i} f_i = m_{(s)}^\mu K_{(s)}^\mu(f) \iff K_{(s)}^\mu(\gamma f) = t^{m_{(s)}^\mu} K_{(s)}^\mu(f); \quad \mu = 1, \ldots, r_{(s)}. \quad \text{(2.3)}$$

In the general case, the Casimir functions are multivalued (for example, if the orbit space $G^*/G$ is not semiseparable, the Casimir functions are infinitely valued), hence, what follows, we use the term "Casimir function" to mean a certain fixed branch of the multivalued function $K_{(s)}^\mu$. In the general case, the Casimir functions $K_{(s)}^\mu$ are only locally invariant under the coadjoint representation, i.e., the equality $K_{(s)}^\mu(Ad^*_g f) = K_{(s)}^\mu(f)$ holds for the elements $g$ belonging to some neighborhood of unity in the group $G$. 

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Remark 1. Without going into detail, we note that the spaces \( M_s \) are critical surfaces for some polynomial \((s - 1)\)-type Casimir functions, which gives a simple and efficient way to construct the functions \( F^{(s)} \).

We now let \( \Omega^{(s)} \subset R^{r(s)} \) denote the set of values of the mapping \( K^{(s)} : M_s \rightarrow R^{r(s)} \) and introduce a locally invariant subset \( O^{(s)}_\omega \) as the level surface,

\[
O^{(s)}_\omega = \{ f \in M_s \mid K^{(s)}_\mu (f) = \omega^{(s)}_\mu, \; \mu = 1, \ldots, r(s); \; \omega^{(s)} \in \Omega^{(s)} \}.
\]

The dimension of \( O^{(s)}_\omega \) is the same as the dimension of the orbit \( O_\lambda \in M_s \), where \( \omega^{(s)} = K^{(s)}(\lambda) \).

If the Casimir functions are single valued, the orbit space in separable, and the set \( O^{(s)}_\omega \) then consists of a denumerable (typically, finite) number of orbits; accordingly, we call this level surface the class of orbits.

The space \( G^* \) thus consists of the union of connected invariant nonintersecting algebraic surfaces \( M_s \); these, in turn, are union of the classes of orbits \( O^{(s)}_\omega \):

\[
G^* = \bigcup M_s = \bigcup_{\omega^{(s)} \in \Omega^{(s)}} O^{(s)}_\omega.
\]  \hspace{1cm} (1.4)

**Example 2** (The Poincare group \( P^{1,3} \), continuation of Example 1). In this case, decomposition (1.4) (with \( \Delta_{1,3} \equiv p_a p^a \)) becomes

\[
(P^{1,3})^* = \bigcup_{\omega \in R^2} O^{(s)}_\omega \bigcup_{\omega \in R^1} O^{(s)}_\omega \bigcup_{\omega \in \{0\} \setminus R^1} O^{(s)}_\omega \bigcup_{\omega \in R^2} O^{(s)}_\omega \bigcup \{0\};
\]

where

\[
O^{0}_\omega = \{ f \in M_0 \mid W^a W_a = \omega_1^0, \Delta_{1,3} = \omega_2^0 \}; \; \Omega^0 = R^2;
\]

\[
O^{1a}_\omega = \{ f \in M_{1a} \mid \Delta_{1,3} = \omega_1^{1a} \}; \; \Omega^{1a} = R^1;
\]

\[
O^{1b}_\omega = \{ f \in M_{1b} \mid W_0 = \omega_1^{1b} \}; \; \Omega^{1b} = \{0\} \setminus R^1;
\]

\[
O^{2}_\omega = \{ f \in M_2 \mid I_{abc} l^{ab} = \omega_1^2, \varepsilon^{abed} l_{ab} l_{cd} = \omega_2^2 \}; \; \Omega^2 = R^2.
\]

In this example, each class consists of several K-orbits (with the number ranging from one to four depending on the signs of the \( \omega^{(s)} \) parameters).

We now consider the quotient space \( B_s = M_s / G \), whose points are the orbits \( O_\lambda \in M_s \). It is obvious that \( \dim B_s = r(s) \). We introduce local coordinates \( j \) on \( B_s \). For this, we parameterize an \( (s) \)-covector \( \lambda \in M_s \) by real-valued parameters \( j = (j_1, \ldots, j_{r(s)}) \), assuming that \( \lambda \) depends linearly on \( j \) (this can be done because \( M_s \) is a star surface):

\[
\lambda = \lambda(j)
\]

with

\[
F^{(s)}(\lambda(j)) \equiv 0, \quad K^{(s)}_\mu (\lambda(j)) = \omega^{(s)}_\mu (j), \quad \det \frac{\partial \omega^{(s)}_\mu (j)}{\partial j_\nu} \neq 0.
\]

In other words, \( \lambda(j) \) is a local section of the bundle \( M_s \rightarrow B_s \). We let \( \Theta^{(s)} \equiv (\omega^{(s)})^{-1}(\Omega^{(s)}) \subset R^{r(s)} \) denote the inverse image and \( \Gamma^{(s)} \) denote the discrete group of transformations of the set.
\( \Theta(s) : j \to j^\ast \) such that \( \omega^{(s)}_\mu(j^\ast) = \omega^{(s)}_\mu(j) \). Each point \( j \) from the domain \( J^{(s)} = \Theta(s) / \Gamma^{(s)} \subset R^{(s)} \) then corresponds to a single class \( O^{(s)}_\lambda \).

Elementary examples show that global parameterization does not exist on the whole of \( M^{(s)} \) in general, i.e., the manifold \( B^{(s)} \) is not covered by one chart. In this case, we define an atlas of charts on \( B^{(s)} \) and parameterize the corresponding connected invariant nonintersecting subsets \( M^{A(s)}_1, M^{B(s)}_1, \ldots \) with a nonvanishing measure in \( M^{(s)} \) as follows:

\[
M^{(s)} = M^{A(s)}_1 \cup M^{B(s)}_1 \cup \ldots
\]

The corresponding domains of values \( J^{A}, J^{B}, \ldots \) of the \( j \) parameters then satisfy the relation

\[
\Omega^{(s)} = \omega^{(s)}(J^{A}) \cup \omega^{(s)}(J^{B}) \cup \ldots
\]

In what follows, we show that each space \( M^{A(s)}_1 \) corresponds to its own type of the spectrum of Casimir operators on the Lie group. Therefore, we can say that decomposition (1.5) is the decomposition with respect to the \textit{spectral type} and an \( (s) \)-orbit \( O_\lambda \) belongs to spectral type \( A \) if \( O_\lambda \in M^{A(s)}_1 \). We illustrate decomposition (1.5) with a simple example.

Example 3 (The group \( SO(2, 1) \)). In the case of the group \( SO(2, 1) \), \( [e_1, e_2] = e_2, [e_2, e_3] = 2e_1, \) and \( [e_3, e_1] = e_3 \). Decomposition (1.4) becomes

\[
O^{(s)}_\omega = \{ f^2 + f_2 f_3 = \omega, - (f = 0) \}, \quad O^1 = \{ f = 0 \}.
\]

For \( \omega > 0 \), the class \( O^{(s)}_\omega \) consists of two orbits. For nondegenerate orbits, \( \Omega = R^1 \). There is no single parameterization in this case. Indeed, the most general form of the parameterization \( \lambda(j) = (a_1 j, a_2 j, a_3 j) \) (where \( a_i \) are some numbers) leads to \( \omega(j) = a j^2 \), where \( a = a_1^2 + a_2 a_3 \), and therefore (depending on the sign of \( a \)) \( \omega(j) \) is always greater than zero, less than zero, or equal to zero, i.e., \( \omega(R^1) \neq O \). We introduce two spectral types,

- type \( A \) : \( \lambda(j) = (0, j, j); \quad J^A = [0, \infty); \quad O^{(s)}_{\omega(j)} = \{ f^2 + f_2 f_3 = j^2, f \neq 0 \} \);
- type \( B \) : \( \lambda(j) = (0, j, -j); \quad J^B = (0, \infty); \quad O^{(s)}_{\omega(j)} = \{ f^2 + f_2 f_3 = -j^2, f \neq 0 \} \).

In what follows, we return to this example and show that types \( A \) and \( B \) respectively correspond to the continuous and the discrete spectra of the Laplace (Casimir) operator on the group \( SO(2, 1) \).

The classification of K-orbits obtained above allows describing the structure of the annihilator \( G^\lambda \) for an arbitrary (in general, degenerate) covector \( \lambda \) in more detail. Let \( \lambda \) be a covector of the \( (s) \)-type. It follows from the definition of the annihilator that \( \dim G^\lambda = \text{corank} C_{ij}(\lambda) = \text{codim} O_\lambda = 2s + r \).

It can be verified that the functions \( \Phi^{(s)}_\alpha(f) = (F^{(s)}_\alpha(f), K^{(s)}_\mu(f)), (a = 1, \ldots, 2s + r) \), have the Poisson brackets

\[
\{ F^{(s)}_\alpha(f), F^{(s)}_\beta(f) \} = C^{(s)}_{\alpha\gamma}(f) F^{(s)}_\gamma(f);
\]

\[
\{ F^{(s)}_\alpha(f), K^{(s)}_\mu(f) \} = C^{(s)}_{\alpha\beta}(f) F^{(s)}_\beta(f);
\]

\[
\{ K^{(s)}_\mu(f), K^{(s)}_\nu(f) \} = C^{(s)}_{\mu\nu}(f) F^{(s)}_\alpha(f).
\]

Because the functions \( \Phi^{(s)}_\alpha(f) \) are independent, recalling the definition of the space \( M^{(s)} \), we conclude that the gradients \( \nabla \Phi^{(s)}(\lambda) \) are linearly independent and constitute a basis of a \((2s + r)\)-dimensional Lie algebra \( G^\lambda \) with the commutation relations

\[
[\nabla F_\alpha(\lambda), \nabla F_\beta(\lambda)] = C^{(s)}_{\alpha\beta}(\lambda) \nabla F_\gamma(\lambda);
\]
\[ \nabla F_\alpha(\lambda), \nabla K_\mu(\lambda) = C^\beta_{\alpha\mu}(\lambda) \nabla F_\beta(\lambda); \]
\[ \nabla K_\mu(\lambda), \nabla K_\nu(\lambda) = C^\alpha_{\mu\nu}(\lambda) \nabla F_\alpha(\lambda); \]

(where we omit the superscript \((s)\) for brevity). From these commutation relations, we now have the following proposition.

**Proposition 1.** The annihilator \(\mathcal{G}^\lambda\) of an arbitrary \((s)\)-covector \(\lambda\) contains the ideal \(N_\lambda = \{\nabla F_\alpha(s)(\lambda)\}\). The quotient algebra \(\mathcal{K}_\lambda = \mathcal{G}^\lambda/N_\lambda = \{\nabla K_\mu(s)(\lambda) + N_\lambda\}\) is commutative and \(r_{(s)}\)-dimensional.

Closed subgroups of \(G\) corresponding to the Lie algebras \(\mathcal{G}^\lambda, N_\lambda\) and \(\mathcal{K}_\lambda\) are respectively denoted by \(G^\lambda, N_\lambda\) and \(K_\lambda = G^\lambda/N_\lambda\).

## 2 Darboux coordinates and \(\lambda\)-representations of Lie algebras

We let \(\omega_\lambda\) denote the Kirillov form on the orbit \(O_\lambda\). It defines a symplectic structure and acts on the vectors \(a\) and \(b\) tangent to the orbit as

\[ \omega_\lambda(a, b) = \langle \lambda, [\alpha, \beta]\rangle, \]

where \(a = ad^*_\alpha \lambda\) and \(b = ad^*_\beta \lambda\). The restriction of Poisson brackets (1.2) to the orbit coincides with the Poisson bracket generated by the symplectic form \(\omega_\lambda\). According to the well-known Darboux theorem, there exist local canonical coordinates (Darboux coordinates) on the orbit \(O_\lambda\) such that the form \(\omega_\lambda\) becomes

\[ \omega_\lambda = dp_a \wedge dq_a; \quad a = 1, \ldots, \frac{1}{2}\dim O_\lambda = \frac{n - r}{2} - s, \]

where \(s\) is the degeneration degree of the orbit.

Let \(\lambda\) be an \((s)\)-type covector and \(f \in O_\lambda\). It can be easily seen that the transition to canonical Darboux coordinates \((f_i) \to (p_a, q^a)\) amounts to constructing analytic functions \(f_i = f_i(q, p, \lambda)\) of the variables \((p, q)\) satisfying the conditions

\[ f_i(0, 0, \lambda) = \lambda_i; \]
\[ \frac{\partial f_i(q, p, \lambda)}{\partial p_a} \frac{\partial f_j(q, p, \lambda)}{\partial q^a} - \frac{\partial f_j(q, p, \lambda)}{\partial p_a} \frac{\partial f_i(q, p, \lambda)}{\partial q^a} = C^k_{ij} f_k(q, p, \lambda); \]
\[ F_\alpha^{(s)}(f(q, p, \lambda)) = 0, \quad K_\mu^{(s)}(f(q, p, \lambda)) = K_\mu^{(s)}(\lambda). \]

We require that the transition to the canonical coordinates (in other words, the \(gp\)-transition) be linear in \(p_a\);

\[ f_i(q, p, \lambda) = \alpha^a_i(q)p_a + \chi_i(q, \lambda); \quad \text{rank } \alpha^a_i(q) = \frac{1}{2}\dim O_\lambda. \]

Obviously, a transition of form (2.4) does not exist in the general case; however, assuming that \(\alpha^a_i(q)\) and \(\chi_i(q, \lambda)\) are holomorphic functions of the complex variables \(q\), we considerably broaden the class of Lie algebras and \(K\)-orbits for which this transition does exist. (We assume...
that functionals from $G^*$ can be continued to $G^c$ by linearity.) It seems that transition (2.4) exists for an arbitrary Lie algebra and any of its nondegenerate orbits.

**Theorem 1.** The linear transition to canonica coordinates on the orbit $O_\lambda$ exists if and only if there exists a normal polarization (in general, complex) associated with the linear functional $\lambda$, i.e., a subalgebra $H \subset G^c$ such that

$$\dim H = n - \frac{1}{2} \dim O_\lambda, \quad \langle \lambda, [H, H] \rangle = 0, \quad \lambda + H^\perp \subset O_\lambda.$$ 

Before discussing and proving this theorem, we digress into a subject that appears to be of independent interest. Let $X_i(x) = X^a_i(x)\partial_{x^a}$ be transformation group generators that generate an $n$-dimensional Lie algebra $G$ of vector fields on a homogeneous space $M = G/H : [X_i, X_j] = C^k_{ij}X_k$ (here and in what follows, $a^a$ $(a = 1, \ldots, m = \dim M)$, are local coordinates of a point $x \in M$); $H$ is the isotropy group of a base point $x_0$ and $H$ is its Lie algebra. Inhomogeneous first-order operators $\tilde{X}_i = X_i + \chi_i(x)$ are called the *continuations* of the generators $X_i$ if they still satisfy the commutation relations of the algebra $G$ ($\chi_i(x)$ and are smooth functions on $M$).

By definition, the $n$-component function $\chi(x)$ is to be found from the system of equations

$$X^a_i(x)\frac{\partial \chi^i(x)}{\partial x^a} - X^a_j(x)\frac{\partial \chi^j(x)}{\partial x^a} = C^k_{ij}\chi^k(x). \quad (2.5)$$

Solutions of this system span a linear space, in which we can single out the module of trivial solutions of the form

$$\chi^0 = \left\{ \chi^0_i(x) = X^a_i(x)\frac{\partial S(x)}{\partial x^a} \right\},$$

where $S(x)$ is an arbitrary smooth function. Constructing trivial continuations $\tilde{X}_i = e^{-S}X_i e^S$ is equivalent to performing the "gauge" transformation $\partial_{x^a} \rightarrow \partial_{x^a} + \partial_{x^a} S(x)$. In what follows, we are interested in only nontrivial continuations that generate the quotient space of all solutions of system (2.5) modulo trivial solutions.

**Proposition 2.** The space of nontrivial continuations is finite dimensional and is isomorphic to the quotient space $H^*/[H, H]^*$. 

We do not give the complete proof here; however, we give the explicit form of all nontrivial solutions of system (2.5), which implies the validity of Proposition 2. We relabel and change the basis in $G$,

$$X_a(x_0) = \frac{\partial}{\partial x^a}|_{x_0}, \quad a = 1, \ldots, m; \quad X_\alpha(x_0) = 0, \quad \alpha = m + 1, \ldots, n.$$ 

We consider the right action of $G$ on $M$. An arbitrary element $g \in G$ is represented as $g = hs(x)$, where $h \in H$ and $s(x)$ is a smooth Borel mapping $M \rightarrow G$ assigning the right coset class $Hs(x) \subset G$ to each point $x \in M$. In the coordinates $g = (h^a, x^\alpha)$ (assuming that $e = (0, x_0)$), the left-invariant vector fields $\xi_i(g)$ have the form $\xi_i(g) = X^a_i(x)\partial_{x^a} + \xi^\alpha_i(h, x)\partial_{h^\alpha}$. Direct calculation verifies that the continuations given by the operators

$$\tilde{X}_i = X_i + \xi^a_i(0, x)\lambda_a; \quad \lambda \in H^*, \quad \langle \lambda, [H, H] \rangle = 0,$$

are nontrivial. It can also be shown that there are no other nontrivial continuations.

We now outline the main points of the proof of Theorem 1.
Proof of Theorem 1. Using (2.4), we write Eq. (2.2) in more detail as

\[
\begin{align*}
\alpha_i^a(q)\partial^a_{q^a} \alpha_j^b(q) - \alpha_j^b(q)\partial^a_{q^a} \alpha_i^a(q) &= C_{ij}^k \alpha_k^b(q); \\
\alpha_i^a(q)\partial^a_{q^a} \chi_j(q; \lambda) - \alpha_j^a(q)\partial^a_{q^a} \chi_i(q; \lambda) &= C_{ij}^k \chi_k(q; \lambda).
\end{align*}
\] (6)

Equation (6) is equivalent to \([a_i, a_j] = C_{ij}^k a_k\), where \(a_i = \alpha_i^b(q)\partial^a_{q^a}\), and the operators \(a_i\) are therefore generators of the transformation group acting in the domain \(Q = G/H\) for a real polarization and \(Q = G^c/H\) for a complex polarization, where \(H\) is the Lie group corresponding to \(\mathcal{H}\). Because \(\text{rank } \alpha_i^a(q) = \dim \mathcal{O}_\lambda/2 = \dim \mathcal{Q}\), we conclude that whenever solutions of system (2.6) exist, there exists an isotropy algebra \(H\) of the point \(q = 0\) of the dimension \(n - \dim \mathcal{O}_\lambda/2\). It is obvious that the converse is also true: the existence of an \((n - \dim \mathcal{Q})\)-dimensional subalgebra \(H\) is sufficient for the existence of solutions to system (2.6).

As follows from Proposition 2, the existence of solutions to system (2.7) with initial conditions (2.1) (or the existence of nontrivial continuations of the operators \(a_i\)) is equivalent to the condition that the algebra \(H\) be adapted to the linear functional \(\lambda\). Therefore, the existence of the polarization \(H\) for a covector \(\lambda\) is necessary and sufficient for relations (2.1) and (2.2) to be satisfied.

We now show that the normality of the polarization, i.e., that the polarization satisfies the Pukanski conditions \(\lambda + H^\perp \subset \mathcal{O}_\lambda\), is necessary and sufficient for relations (2.3) to be satisfied. We let \(\{e_A\}\) denote a basis of the isotropy algebra \(H\) and \(\{e_a\}\) denote the complementary basis vectors in \(G^c\): \(e_i = \{e_A, e_a\}\). By definition of the isotropy algebra, \(\alpha_i^a(0) = 0\), whence \(\det \alpha_i^a(0) \neq 0\). Making a linear change of coordinates \(q\) and of the basis in \(G^c\), we can assume without loss of generality that \(\alpha_i^a(0) = \delta_i^a\). In our notation, \(H^\perp = \{(0, p_a)\}\), where \(p_a\) are arbitrary numbers, and the Pukanski condition becomes \((\lambda_A, p_a + \lambda_a) \in \mathcal{O}_\lambda\).

Let relations (2.3) be satisfied. Setting \(q = 0\) in (2.3), we then have

\[\Phi(\lambda_A, \lambda_a) = \Phi(f_A(q, p; \lambda), f_a(q, p; \lambda))|_{q=0} = \Phi(\lambda_A, p_a + \lambda_a).\]

(We recall the notation \(\Phi = (F(s), K(s))\)). This implies that for any value of \(p_a\), the point \((\lambda_A, p_a + \lambda_a)\) belongs to the same class of orbits as the point \((\lambda_A, \lambda_a)\); because the class of orbits consists of adenumerable number of K-orbits, we then conclude that these two points belong to the same orbit.

Conversely, let the Pukanski condition be satisfied, which means that \(f(0, p; \lambda) = (\lambda_A, p_a + \lambda_a) \in \mathcal{O}_\lambda\) and \(\Phi(\lambda_A, p_a + \lambda_a) = \Phi(\lambda_A, \lambda_a)\). We then show that Eq. (2.3) is satisfied. Because the functions \(\Phi(f)\) satisfy Eqs. (1.1) and (1.3), we have

\[\left.\left(\alpha_a^b(q) \frac{\partial \Phi(f(q, p; \lambda))}{\partial q^a} - \frac{\partial f_a(q, p; \lambda)}{\partial q^a} \frac{\partial \Phi(f(q, p; \lambda))}{\partial p_a}\right)\right|_{q=0} = \left.\frac{\partial \Phi(f(q, p; \lambda))}{\partial q^b}\right|_{q=0} = 0.\]

Therefore, \(\Phi(f(q, p; \lambda)) = \Phi(f(0, p; \lambda)) = \Phi(\lambda)\), and Theorem 1 is proved.

It is known that a solvable polarization exists for an arbitrary Lie algebra and any non-degenerate covector. On the other hand, if the algebra \(G\) is solvable, every functional has a polarization \(H \subset G^c\). In the classical method of orbits, the polarization appears as an \(n - \dim \mathcal{O}_\lambda/2\)-dimensional subalgebra \(H \subset G^c\), with its one-dimensional representation determined by the functional \(\lambda\). In our case, a normal polarization determines linear transition (2.4) to the canonical coordinates.
It can be easily seen that replacing the functional $\lambda$ with another covector belonging to the same orbit leads to replacing the polarization $H$ with the conjugate one $\tilde{H}$, with the Darboux coordinates corresponding to these two polarizations related by a point transformation, $\tilde{q}^a = \tilde{q}^a(q)$; $\tilde{p}_a = \frac{\partial \tilde{h}}{\partial \tilde{q}^a} p_b$. Therefore, the choice of a specific representative of the orbit is not essential. On the other hand, if the polarizations are not conjugate, the corresponding Darboux coordinates are related by a more general canonical transformation. With the “quantum” canonical transformation determined (with $q$ and $p$ being operators, see below), we can thus construct the intertwining operator between the two representations obtained via the method of orbits involving two polarizations.

In the case where no polarization exists for a given functional, the transition to Darboux coordinates (which is nonlinear in the $p$ variables) can still be constructed, and the $\lambda$-representation of $G$ can still be defined (see below); this representation is the basis for the harmonic analysis on Lie groups and homogeneous spaces (applications of the method of orbits to harmonic analysis go beyond the scope of this paper and are not considered here). In other words, the existence of a polarization is a useful property but is not necessary for the applicability of the method of orbits.

As already mentioned, the functional $\lambda$ can have several different polarizations; however, it is easy to verify the following proposition.

**Proposition 3.** If the normal polarization $H$ exists for a given $\lambda \in G^*$, then $G^\lambda \subset H$.

**Example 4** (The group $SO(2,1)$, continuation of Example 3). The Kirillov form on non-degenerate orbits is given by $\omega_\lambda = df_2 \wedge df_3/2f_1$. For different spectral types, we obtain

- **type A**: $f_1 = p$, $f_2 = e^q(-p + j)$, $f_3 = e^{-q}(p + j)$; $\lambda = (0, j, j)$; $H = \{e_2 + e_3, e_1 + e_2\}$; $(p, q) \in \mathbb{R}^2$;

- **type B**: $f_1 = p$, $f_2 = e^q(-ip + j)$, $f_3 = e^{-q}(ip + j)$; $\lambda = (0, j, -j)$; $H = \{e_2 - e_3, e_1 - ie_2\}$.

To find the domain of definition of the variables $(p, q)$ for type-B orbits, we decompose the complex variable $q$ into its real and imaginary components, $q = \alpha + i\beta$. Because the variables $f_i$ are real, we obtain

$$p = j \tan \beta; \quad f_1 = j \tan \beta, \quad f_2 = \frac{je^\alpha}{\cos \beta}; \quad f_3 = -\frac{e^{-\alpha}}{\cos \beta},$$

which implies that $p$ takes any real value and $q$ is a complex variable with its real part in $\mathbb{R}^1$ and the imaginary part in $S^1$, i.e., $q \approx q + 2\pi i$. Therefore, functions on a type-B K-orbit are analytic functions of a real variable $p$ and a complex variable $q$ that are $2\pi i$-periodic in $q$.

We define the notion of the quantization of K-orbits. This quantization is to be done separately for each spectral type of orbits; it consists in assigning the spectral type of the orbit a special representation of the Lie algebra (the $\lambda$-representation), with the orbits subject to the integrability condition considered in the next section.

We now view the transition functions $f_i(q, p; \lambda(j))$ to local canonical coordinates as symbols of operators that are defined as follows: the variables $p_a$ are replaced with derivatives, $p_a \rightarrow \hat{p}_a \equiv -i\hbar \frac{\partial}{\partial q^a}$, and the coordinates of a covector $f_i$ become the linear operators

$$f_i(q, p; \lambda(j)) \rightarrow \hat{f}_i \left(q, -i\hbar \frac{\partial}{\partial q}; \lambda(j) \right)$$
(with $\hbar$ being a positive real parameter). This quantization procedure is obviously ambiguous. This ambiguity is eliminated if we require that the operators $\hat{f}_i$ satisfy the commutation relations

$$\frac{i}{\hbar} [\hat{f}_i, \hat{f}_j] = C_{ij}^k \hat{f}_k. \quad (2.8)$$

If the transition to the canonical coordinates is linear, i.e., a normal polarization exists for orbits of a given type, it is obvious that

$$\hat{f}_i = -i\hbar \alpha^a_i(q) \frac{\partial}{\partial q^a} + \chi_i(q, \lambda(\tilde{j})),$$

and Eq. (2.8) is equivalent to conditions (2.6) and (2.7).

Under quantization, an arbitrary analytic function $\varphi(f)$ on the coalgebra is mapped into a symmetrized operator function $\varphi(\hat{f})$ of the operators $\hat{f}_i$. The parameters $j$ are related to the parameters $\tilde{j}$ of the orbit as $\tilde{j} = j + i\hbar\beta$, where $\beta$ is an $r^{(s)}$-dimensional real vector that is to be found from the condition that the functions

$$\kappa^{(s)}_{\mu}(j) = K^{(s)}_{\mu}(\hat{f}). \quad (2.9)$$

are real. We note that in the "classical" limit as $\hbar \to 0$, we have $\kappa^{(s)}_{\mu}(j) \to \omega^{(s)}_{\mu}(j)$, and the commutator of linear operators goes into the Poisson bracket on the coalgebra,

$$\frac{i}{\hbar} [\cdot, \cdot] \to \{\cdot, \cdot\}.$$

Because the functions $\kappa^{(s)}_{\mu}(j)$ are generally different from the functions $\omega^{(s)}_{\mu}(j)$ for $\hbar \neq 0$, we must redefine the domain of definition $J^{(s)}$ of the $j$ parameters such that each point $j \in J^{(s)}$ is in a one-to-one correspondence with the values of $\kappa^{(s)}_{\mu}(j)$, i.e., the functions $(\kappa^{(s)})^{-1}$ are unambiguous upon restrictions to $J^{(s)}$. The condition

$$\kappa^{(s)}(J^{(s)}) = \Omega^{(s)}. \quad (2.10)$$

must also be satisfied.

We introduce the operators

$$l_k(q, \partial_q, j) = \frac{i}{\hbar} \hat{f}_k(q, \hat{p}; \lambda(\tilde{j})).$$

It is obvious that

$$[l_i, l_j] = C_{ij}^k l_k; \quad F^{(s)}_\alpha(-i\hbar l(q, \partial_q, j)) \equiv 0, \quad K^{(s)}_{\mu}(-i\hbar l(q, \partial_q, j)) \equiv \kappa^{(s)}_{\mu}(j);$$

$$\kappa^{(s)}_{\mu}(j) = \kappa^{(s)}_{\mu}(\tilde{j}); \quad \det \frac{\partial \kappa^{(s)}_{\mu}(j)}{\partial J^\nu} \neq 0; \quad j \in J, \ q \in Q. \quad (2.11)$$

**Definition 1.** Let $f_i = f_i(q, p; \lambda(j))$ be a transition to canonical coordinates on the orbit $O_{\lambda(j)}$ and $\lambda(j)$ be a parameterized covector. The realization of the Lie algebra $\mathcal{G}$ by the respective operators $l_i(q, \partial_q, j)$ is called the $\lambda$-representation.
In what follows, we show that the quantities $\kappa_\mu^0(j)$ constitute the spectrum of Casimir operators on the Lie group $\hat{K}_\mu \equiv K_\mu(ih\xi(g))$, where $\xi_i(g)$ are left-invariant vector fields on $G$. Accordingly, $\kappa_\mu^{(s)}(j)$ are the eigenvalues of the Casimir operators $\hat{K}_\mu^{(s)} \equiv K_\mu^{(s)}(ihX)$, where $X_i$ are the transformation group generators, on the homogeneous $(s)$-type space [5].

**Example 5** (The group $SO(2,1)$, continuation of Example 3). \[\]

Type $A$: $\hat{f}_1 = -ih\frac{\partial}{\partial q}$, $\hat{f}_2 = e^q(ih\frac{\partial}{\partial q} + j + ih\beta)$, $\hat{f}_3 = e^{-q}(-ih\frac{\partial}{\partial q} + j + ih\beta)$;

$\kappa(j) = K(\hat{f}) = \hat{f}_1^2 + \hat{f}_2 \circ \hat{f}_3 = j^2 + h^2(1 - \beta) + ihj(2\beta - 1)$.

Because $\kappa(j)$ is real, we obtain $\beta = \frac{1}{2}$. Then the $\lambda$-representation for spectral type $A$ is given by

$l_1 = \frac{\partial}{\partial q}$, $l_2 = e^q(-\frac{\partial}{\partial q} + i\frac{h}{j} - 2)$, $l_3 = e^{-q}(\frac{\partial}{\partial q} + i\frac{h}{j} - 2)$; $\kappa(j) = j^2 + \frac{h^2}{4}$,

where $j \in J_A = [0, \infty)$ and $\Omega^A \equiv \kappa(J_A) = \left[\frac{h^2}{4}, \infty\right)$.

Type $B$: $\hat{f}_1 = -ih\frac{\partial}{\partial q}$, $\hat{f}_2 = e^q(-h\frac{\partial}{\partial q} + j + ih\beta)$, $\hat{f}_3 = e^{-q}(-h\frac{\partial}{\partial q} - j - ih\beta)$;

$\kappa(j) = K(\hat{f}) = \hat{f}_1^2 + \hat{f}_2 \circ \hat{f}_3 = -j^2 + h^2\beta^2 - hj - ih\beta(2j + h)$.

Because $\kappa(j)$ is real, we obtain $\beta = 0$; the $\lambda$-representation of spectral type $B$ is given by

$l_1 = \frac{\partial}{\partial q}$, $l_2 = e^q(-i\frac{\partial}{\partial q} + i\frac{h}{j})$, $l_3 = e^{-q}(-i\frac{\partial}{\partial q} - i\frac{h}{j})$; $\kappa(j) = -j(j + h)$,

where $j \in J_B = [-\frac{h}{2}, \infty)$ and $\Omega^B \equiv \kappa(J_B) = (-\infty, \frac{h^2}{4})$. Condition (2.10) is satisfied, $\Omega^A \cup \Omega^B = \Omega = R^1$.

**Example 6** (The group $St(1,R)$). Kirillov [3] gives the group $St(1,R)$ as an example illustrating the absence of the polarization for a general covector. We describe the structure of K-orbits of this group. The corresponding Lie algebra can be realized by the matrices

$$X(a, \xi, c) = \begin{pmatrix} 0 \xi_1 & \xi_2 & c \\ 0 \quad a_1 & a_2 & \xi_2 \\ 0 \quad a_3 & -a_1 & -\xi_1 \\ 0 & 0 & 0 \end{pmatrix} \equiv a_1e_1 + a_2e_2 + a_3e_3 + \xi_1e_4 + \xi_2e_5 + ce_6.$$

The basis elements $e_i$, have the nonvanishing commutation relations

\[
\begin{align*}
[e_1, e_2] &= 2e_2, & [e_1, e_3] &= -2e_3, & [e_1, e_4] &= -e_4, & [e_1, e_5] &= e_5, \\
[e_2, e_3] &= e_1, & [e_2, e_4] &= -e_5, & [e_3, e_5] &= -e_4, & [e_4, e_5] &= 2e_6.
\end{align*}
\]

\[\]

\[1\]

Here and in what follows, the symbol " $\circ$ " denotes the symmetrized product of operators, $A \circ B \equiv \frac{1}{2}(AB + BA)$.
We now describe orbit (1.4) for this group as

\[ O^0_\omega = \{ K_1(f) = \omega^0_1, K_2(f) \equiv f_6 = \omega^0_2; \ - (F^1(f) = 0) \}; \]

\[ O^{1a}_\omega = \{ F_{1a}^1(f) = F_{2a}^1(f) = F_{3a}^1(f) = 0, \ f_6 = \omega^{1a} \neq 0 \}; \]

\[ O^{1b}_\omega = \{ f \neq 0; \ f_4 = f_5 = f_6 = 0, \ f_2 f_3 + f^2_1/4 = \omega^{1b} \}; \]

\[ O^2 = \{ f = 0 \}; \ K_1(f) \equiv f_2 f_6 - f_1 f_4 f_5 + f^2_4 f_2 - f^2_5 f_3 + 4 f_2 f_3 f_6; \]

\[ F_{1a}^1(f) \equiv 2 f_1 f_6 - f_4 f_5; \ F_{2a}^1(f) \equiv 4 f_2 f_6 - f^2_4; \ F_{3a}^1(f) \equiv 4 f_3 f_6 + f^2_4. \]

It can be easily seen that the polarization does not exist for the degenerate orbits \( O^1_{\omega} \). The Kirillov form on orbits of this type is given by \( \omega_\lambda = df_4 \wedge df_5/2 f_6 \). The space \( M_{1a} \) is of the first spectral type, and we can introduce global coordinates on \( B_{1a} \), i.e., a parameterization \( \lambda(j) = (0, 0, 0, 0, 0, j) \). For this functional, we have \( G^\lambda = \{ e_1, e_2, e_3, e_6 \} \), \( N_\lambda = \{ e_1, e_2, e_3 \} \), and \( K_\lambda = \{ e_6 \} \). For the orbits under consideration, linear transition (2.4) to canonical coordinates is replaced by

\[ f_1 = qp, \ f_2 = q^2 j, \ f_3 = -p^2/4 j, \ f_4 = p, \ f_5 = 2 q j, \ f_6 = j; \quad (p, q) \in \mathbb{R}^2, \]

which is quadratic in the \( p \) variables. We now construct the corresponding \( \lambda \)-representation. We do not encounter problems with ordering the operators \( \hat{p} = -i \hbar \partial_q \) and \( \hat{q} = q \) involved in the operators \( \hat{f}_2 = q^2 j, \hat{f}_3 = -\hat{p}^2/4 j, \hat{f}_4 = \hat{p}, \hat{f}_5 = 2 q j, \hat{f}_6 = j \). The operator \( \hat{f}_1 \) is unambiguously found from the commutation relations as \( \hat{f}_1 = q \hat{p} - i \hbar/2 \). For orbits of this type, the \( \lambda \)-representation is therefore given by

\[ l_1 = q \partial_q + \frac{1}{2}, \ l_2 = \frac{i}{\hbar} q^2 j, \ l_3 = \frac{i \hbar}{4 j} \partial_q^2, \ l_4 = \partial_q, \ l_5 = \frac{i}{\hbar} 2 q j, \ l_6 = \frac{i}{\hbar} j; \ \kappa^{1a}(j) = j \in \mathbb{R}^1. \]

3 The integral-valuedness condition for the orbits and the spectra of Casimir operators

In this section, we describe the first stage in the explicit construction of harmonic analysis on homogeneous spaces. This is an involved subject, however, worthy of a separate investigation, and we practically do not consider it here. In this section, we show that the functions \( \kappa(j) \) introduced above are the eigenvalues of the Casimir operators, with the parameters \( j \) satisfying the integral-valuedness condition.

On a connected and simply-connected real Lie group \( G \), we introduce the quasi-invariant measure \( dg = \sqrt{d\sigma d\tau} g \), where \( d\sigma \) and \( d\tau \) are the left and right Haar measures. In the space \( L_2(G, dg) \), we define a unitary representation of \( G \times G \) via

\[ T_{(g_1, g_2)} u(g) = \sqrt{\frac{d(g_1^{-1} g g_2)}{dg}} u(g_1^{-1} g g_2), \quad u(g) \in L_2(G, dg). \quad (3.1) \]

The infinitesimal generators of representation (3.1) are the left-and right-invariant operators \( \xi_i \) and \( \eta_i \) given by

\[ \xi_i(g) = \xi_i(g) \partial_{g^i} + C_i, \quad \eta_i(g) = \eta_i(g) \partial_{g^i} + C_i; \quad C_i \equiv -\frac{1}{4} C_{ij}^j. \]
The generators $\xi_i$ and $\eta_i$, differ by additive constants from (for a unimodular group, coincide with) the corresponding left- and right-invariant vector fields on the group $G$, and we also refer to them as vector fields.

Because the eigenfunctions of unbounded operators with continuous spectra do not belong to $L_2(G, dg)$ and are instead linear functionals on a dense set (a nuclear space) $\Phi \subset L_2(G, dg)$, we must consider the Gelfand triplet $\Phi \subset L_2(G, dg) \subset \Phi'$, where $\Phi'$ is the space dual to $\Phi$.

Similarly to (1.4), we decompose the space $L_2(G, dg)$ into subspaces that are invariant with respect to representation (3.1) as $L_2(G, dg) = \cup_s L_s$, where

$$L_s = \{ \varphi(g) \in L_2(G, dg) \mid F^{(s)}_\alpha(\xi)\varphi(g) = 0, \ -\langle F^{s+1}(\xi)\varphi(g) = 0 \rangle \}. \quad (3.2)$$

We note that replacing the left-invariant fields $\xi_i$ in (3.2) with the right-invariant $\eta_i$ does not change the spaces $L_s$. (This can be seen, for example, using the mapping $g \to g^{-1}$, under which the left-invariant fields pass into right-invariant ones and vice versa, and the measure $dg$ remains invariant.) For each $L_s$, we also introduce the Gelfand triplet $\Phi_s \subset L_s \subset \Phi'_s$.

On each space $L_s$, there exist bi-invariant Casimir operators $K^{(s)}(\mu)(i\hbar\xi)(= K^{(s)}(\mu)(-i\eta))$, and the space $L_s$ can be decomposed into a direct sum (a direct integral) of eigensubspaces of the Casimir operators, i.e., we observe complete similarity with decomposition (1.4).

**Theorem 2.** Let $\lambda(j)$ be a parameterized $(s)$-covector. The quantities $\kappa^{(s)}(j)$, Eq. (2.9), are eigenvalues of the Casimir operators $K^{(s)}(\mu)(i\hbar\xi)$ on $L_s$, where the parameters $j$ satisfy the condition

$$\langle \lambda(j), e_{\mu} \rangle = 2\pi\hbar n_{\mu}/T_{\mu}; \quad n_{\mu} \in \mathbb{Z}, \quad (3.3)$$

where $e_{\mu}$ is the basis vector of the one-dimensional Lie algebra of the one-parameter compact subgroup of the commutative quotient group $K_{\lambda}$ and $T_{\mu}$ the period of the one-dimensional compact subgroup $(\exp(T_{\mu}e_{\mu}) = 1)$.

To find the spectra of the Casimir operators, it is therefore sufficient to find the functions $\kappa^{(s)}(j)$ (these functions are actually given by the structure constants and determine the spectra of the Casimir operators on the universal covering group $\hat{G}$) and impose quantization condition (3.3) on the parameters $j$. We shortly demonstrate that condition (3.3) is equivalent to the Kirillov *integral-valuedness* condition for K-orbits.

**Proof of Theorem 2.** We discuss the main points of the proof. Let $\lambda(j)$ be a parameterized $(s)$-covector and $l_i(q, \partial_q, j)$ be the corresponding $\lambda$-representation. We define the distributions $D^j_q(g)$ on $G$ by the equations

$$[\xi_i(g) + l_i(q, \partial_q, j)]D^j_q(g) = 0, \quad i = 1, \ldots, n. \quad (3.4)$$

From the definition of $\lambda$-representation (2.11), it is easy to obtain that

$$F^{(s)}_\alpha(\xi)D^j_q(g) = 0, \quad i.e., \quad D^j_q(g) \in \Phi'_s; \quad K^{(s)}(\mu)(i\hbar\xi)D^j_q(g) = \kappa^{(s)}(j)D^j_q(g).$$

Although system (3.4) is compatible, the functions $D^j_q(g)$ do not exist globally on the entire group for all values of the $j$ parameters. We restrict system (3.4) to the subgroup $G^\lambda$ and set $q = 0$. Recalling Proposition 1, we then obtain

$$[\xi_A(g_\lambda) + i/\hbar \lambda_A(j)]D^j_0(g_\lambda) = 0, \quad g_\lambda \in G^\lambda, \quad (3.5)$$
where the subscript $A$ labels the basis vectors of $G^\lambda$. It follows from (3.5) that the functions $D^i_\lambda(g)$ are globally defined only if the condition

$$\frac{1}{2\pi \hbar} \oint_{\gamma \in H_1(G^\lambda)} \omega^j_{\gamma} = n_\gamma \in \mathbb{Z}. \quad (3.6)$$

is satisfied, where $\omega^j_{G^\lambda} = \omega^A \lambda_A(j)$ is a closed left-invariant 1-form on the group $G^\lambda$. (We have thus rediscovered the Kostant criterion [8].) Condition (3.6) is equivalent to the Kirillov integral-valuedness condition for orbits [3]. Using the results in the previous sections, we can represent integral-valuedness condition (3.6) in a more detailed form. For a chosen parameterized covector $\lambda(j)$, the basis of left-invariant vector fields of $G^\lambda$ is spanned by the vectors ($\tilde{\lambda} \equiv \lambda(j)$)

$$\nabla F^{(s)}(g) \equiv (L_g)_* \nabla F^{(s)}(\tilde{\lambda}) = F^{(s)i}(\tilde{\lambda}) \xi_i(g),$$

$$\nabla K^{(s)}(g) \equiv (L_g)_* \nabla K^{(s)}(\tilde{\lambda}) = K^{(s)i}(\tilde{\lambda}) \xi_i(g),$$

where

$$F^{(s)i}(\tilde{\lambda}) = \frac{\partial F^{(s)}(f)}{\partial f_i} \bigg|_{f=\lambda}, \quad K^{(s)i}(\tilde{\lambda}) = \frac{\partial K^{(s)}(f)}{\partial f_i} \bigg|_{f=\lambda},$$

and $L_g$ is the left-invariant representation of $G \times G$ on $L_2(G, dg)$. We rewrite system (3.5) as

$$\nabla F^{(s)}(g) D^i_\lambda(g) = 0; \quad g_\lambda \in G^\lambda; \quad (3.7)$$

$$[\nabla K^{(s)}(g_\lambda) + \frac{i}{\hbar} m^{(s)}_\mu \omega^{(s)}(\tilde{j})] D^i_\lambda(g_\lambda) = 0; \quad g_\lambda \in G^\lambda. \quad (3.8)$$

The operators $\nabla F^{(s)}$ entering Eq. (3.7) are left-invariant vector fields on the normal subgroup $N_\lambda$, and the function $D^i_\lambda(g_\lambda)$ is therefore independent of the coordinates of the element $n$ entering the decomposition $g_n = nk; \ n \in N_\lambda, \ k \in K_\lambda$. Similarly, the operators $\nabla K^{(s)}$ are left-invariant vector fields on the commutative group $K_\lambda$, and system (3.8) is therefore an eigenvalue problem for $r^{(s)}$ commuting first-order operators. This problem is easily solved. It is obvious that if the one-parameter subgroup with the generator $\nabla K^{(s)}$ is noncompact, the corresponding eigenvalue $m^{(s)}_\mu \omega^{(s)}(\tilde{j})$ can take any value, i.e., the parameters $j$ are not quantized in that case. With a basis chosen to be independent of the parameters $j$ (such a basis always exists because the Casimir functions are homogeneous), system (3.8) for compact subgroups of $K_\lambda$ becomes

$$\left[ \frac{\partial}{\partial k^\mu} + \frac{i}{\hbar} \lambda_\mu(j) \right] D^i_\lambda(k) = 0.$$

Therefore, the integral-valuedness condition is given by Eq. (3.3).

We have thus shown that the functions $s^{(s)}_\mu(j)$ of the integer parameters $j$ belong to the spectrum of Casimir operators on $\mathcal{L}_s$. To complete the proof, it is necessary to show that the family of functions $D^i_\lambda(g)$ is dense in $\Phi^{(s)}$, which in turn implies the absence of other elements of the spectrum. There is convincing evidence that this is the case [5], but the discussion of this point takes us far beyond the subject of the present paper.

**Example 7** (Group $\text{SO}(2, 1)$, continuation of Example 3). For spectral type $A$, we have $\lambda(j) = (0, j, j)$ and $G^\lambda = \{e_2 + e_3\}$; for spectral type $B$, $\lambda(j) = (0, j, -j)$ and $G^\lambda = \{e_2 - e_3\}$. Therefore, to single out integral orbits, we must know whether the corresponding one-parameter
groups are compact. It is obvious that for spectral type A, the group $K_\lambda = \exp[(1/2)t(e_2 + e_3)]$ is noncompact, and all orbits of this type are therefore integral, i.e., the parameter $j$ is not quantized; $\kappa(j) = j^2 + \hbar^2/4$, $j \geq 0$. For spectral type B, the group $K_\lambda = \exp[(1/2)t(e_2 - e_3)]$ is compact and has the period $T = 2\pi$. From Eq. (3.3), we see that $\langle \lambda(j), (1/2)(e_2 - e_3) \rangle = j = \hbar n$, whence $j = n\hbar$, $n \in \mathbb{Z}$ end $\kappa(j) = -\hbar^2 n(n + 1)$ for $n = 0, 1, \ldots$.

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