MODULARITY OF CERTAIN MOD $p^n$ GALOIS REPRESENTATIONS

RAJENDER ADIBHATLA

Abstract. For a rational prime $p \geq 3$ and an integer $n \geq 2$, we study the modularity of continuous 2-dimensional mod $p^n$ Galois representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ whose residual representations are odd and absolutely irreducible. Under suitable hypotheses on the local structure of these representations and the size of their images we use deformation theory to construct characteristic 0 lifts. We then invoke modularity lifting results to prove that these lifts are modular. As an application, we show that certain unramified mod $p^n$ Galois representations arise from modular forms of weight $p^n-1(p-1)+1$.

1. Introduction

Let $p$ be an odd rational prime and $k$ be a finite field of characteristic $p$. Let $W$ denote the ring of Witt vectors of $k$. For an integer $n \geq 2$ suppose we are given a continuous Galois representation $\rho_n : G_{\mathbb{Q}} \rightarrow \text{GL}_2(W/p^n)$. The goal of this paper is to investigate local and global conditions under which $\rho_n$ lifts to a representation $\rho$ which is modular. The question of modularity of $\rho_n$ naturally leads one to wonder about statements analogous to Serre’s Modularity conjecture in the mod $p^n$ situation as well. However, naïve attempts at such generalizations are bound to fail because, a fortiori, a given mod $p^n$ representation might not even lift to characteristic 0. Nevertheless, we show that when the local structure of $\rho_n$ “mimics” the (mod $p^n$ reduction of) the local structure of the $p$-adic Galois representation attached to a $p$-ordinary modular eigenform then we can not only construct $p$-adic lifts but, indeed, we can also prove their modularity. Our main result is the following.

Theorem A. Let $\rho_n : G_{\mathbb{Q}} \rightarrow \text{GL}_2(W/p^n)$ be a continuous Galois representation whose residual representation $\overline{\rho} := \rho_n \mod p$ is odd and has squarefree, prime-to-$p$ Artin conductor $N := N(\overline{\rho})$. Assume the following hypotheses:

(C1) The representation $\rho_n$ has fixed determinant $\epsilon_n$ which lifts to $\epsilon = \psi \chi^{k-1}$ where $\psi$ is a finite order character unramified at $p$, $\chi$ is the $p$-adic cyclotomic character, and $k \geq 2$ is an integer.

(C2) The image of $\overline{\rho}$ contains $\text{SL}_2(k)$. In addition, if $p = 3$ then the image $\rho_n$ contains a transvection $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(C3) At $p$ the restriction of $\rho_n$ to the decomposition group $G_p$ is

$$\rho_n|_{G_p} \simeq \begin{pmatrix} \psi_{1p} \chi^{k-1} & * \\ 0 & \psi_{2p} \end{pmatrix}$$
where $\psi_1, \psi_2: G_F \rightarrow W^\times$ are some unramified characters lifting $\psi_{1p}, \psi_{2p}$, and $\psi_1\psi_2 = \psi$. Furthermore $\psi_{1p}x^{k-1} \neq \psi_{2p} \mod p$ and $\psi_{1p}x^{k-2} \neq \psi_{2p} \mod p$.

(C4) Suppose that $\rho_n$ is ramified at a place $q \nmid p$. If $p|\#(I_q)$ then

$$\rho_n|_{G_q} \simeq \begin{pmatrix} \chi^{k-1} & \ast \\ 0 & 1 \end{pmatrix} \delta$$

and if $p \nmid \#(I_q)$ then $\rho_n|_{G_q} \simeq \delta^{-1}\chi^{k-1}\psi|_{G_q} \oplus \delta$ for some unramified $\delta$. Additionally, if $q \equiv 1 \mod p$ and $\overline{p}$ is unramified at $q$ then $p|\#(\text{Frob}_q)$.

There is then a $p$-ordinary modular form $f$ of weight $k$ and level prime to $p$ such that its associated $p$-adic representation $\rho_f$ lifts $\rho_n$, has determinant $\psi\chi^{k-1}$, and

$$\rho_f|_{G_p} \simeq \begin{pmatrix} \psi_1\chi^{k-1} & \ast \\ 0 & \psi_2 \end{pmatrix}$$

with $\psi_2$ being an unramified lift of $\psi_{2p}$.

A priori, the requirement that $\rho_n$ have fixed determinant is forced upon us since we wish to reconcile $\rho_n$ with a Galois representation attached to a modular form. Likewise hypothesis C4 is also a necessary imposition because of the local structure, at ramified primes, of Galois representations attached to modular forms. We emphasize that some of the work that goes into proving Theorem A was done in a previous paper [1] of the author. The focus of that work however was to prove higher companion form theorems. This paper shifts emphasis to a more general setting of mod $p^n$ Galois representations and takes a closer look at the local deformation theory that is used.

The proof relies on being able to prescribe local deformation conditions for $\rho_n$ and using the methods of Ramakrishna [8], generalized by Taylor in [10], to piece this local information together to produce a lift in characteristic 0. Hypotheses C2 and the last part of C4 are technical assumptions that make the deformation theory work. This is done in Section 2. In Section 3, the lift that we construct is shown to be modular as a consequence of a modularity lifting theorem due to Skinner and Wiles [9]. We note that hypothesis C2 also ensures that $\overline{p}$ is absolutely irreducible and, since it is assumed to be continuous and odd, Serre’s Modularity Conjecture guarantees the modularity of $\overline{p}$. This, along with hypothesis C3, is crucial for the successful application of Skinner-Wiles. We conclude Section 3 by applying the methods used to prove Theorem A to also prove what can be considered a mod $p^n$ analog of [4, Theorem 2.1.1].

**Theorem B.** Set $k = p^n(p - 1) + 1$ Let $\rho_p : G_Q \rightarrow GL_2(W/p^n)$ be a continuous, odd Galois representation with fixed determinant as in Theorem A. Suppose that $G_p$ is unramified and in fact suppose that

$$\rho_n|_{G_p} \simeq \begin{pmatrix} \psi_{1p,x} & 0 \\ 0 & \psi_{2p,x} \end{pmatrix}$$

where $\psi_x$ is the unramified character sending an arithmetic Frobenius element to $x$. Additionally, suppose that $\rho_n$ satisfies conditions (C2)-(C4) of Theorem A. Then there is a modular eigenform $f$
of weight $p^n(p-1)+1$ and level prime to $p$ such that $\rho_{f,n} \simeq \rho_n$, where $\rho_{f,n}$ is the mod $p^n$ reduction of the $p$-adic Galois representation $\rho_f$ attached to $f$. Moreover,

$$\rho_f|_{G_p} \sim \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$$

with $T_pf = \psi_2(Frob_p)f$ for some unramified lift $\psi_2$ of $\psi_{2p}$.

2. Deformation Theory

In this section, we study the deformation theory that is needed for realizing locally any global obstructions to lifting the given mod $p^n$ representation. The method of Ramakrishna [8] (generalized by Taylor [10]) that we wish to adapt, relies on producing sufficiently well-behaved local deformations of a given residual representation and, at the cost of additional ramification, trivializing the dual Selmer group in order to remove the local obstructions and construct the desired lift in characteristic 0.

2.1. Substantial deformation conditions. Let $k$ be a finite field of characteristic $p \geq 3$ and $W$ be the ring of Witt vectors of $k$. We follow Mazur’s seminal work in [6] to briefly explain deformation conditions. Let $\mathcal{C}_k$ be the category of complete, local, Noetherian rings over $W$ with fixed residue field $k$. Given a residual representation $\overline{\rho} : G_{\mathbb{Q}} \rightarrow GL_2(k)$, a deformation condition $D$ on a deformation of $\overline{\rho}$ is a collection of lifts of $G_{\mathbb{Q}}$ defined over $W$-algebras in $\mathcal{C}_k$ which is closed under projections (morphisms of objects in $\mathcal{C}_k$) and satisfies a certain Mayer-Vietoris property. The former property ensures that $D$ defines a subfunctor while the latter implies that this subfunctor is relatively representable.

If $D$ is a deformation condition for $\overline{\rho}$ then there is a complete local Noetherian $W$-algebra $R$ with residue field $k$ and a lift $\rho : \Gamma \rightarrow GL_2(R)$ in $D$ with the property that if $\rho' : \Gamma \rightarrow GL_2(A)$ is a lift of $\overline{\rho}$ in $D$ then there is a morphism $R \rightarrow A$ which, when composed with $\rho$, gives a representation strictly equivalent to $\rho'$. We also insist that the morphism $R \rightarrow A$ is unique when $A$ is the ring of dual numbers $k[e]/(e^2)$. Letting $ad\overline{\rho}$ (resp. $ad^0\overline{\rho}$) be the vector space of $2 \times 2$-matrices (resp. traceless $2 \times 2$-matrices) over $k$ with $GL_2(k)$ acting by conjugation, we identify the tangent space $t_D$ with a subspace of $H^1(\Gamma, ad\overline{\rho})$ (and with a subspace of $H^1(\Gamma, ad^0\overline{\rho})$ when considering deformations with a fixed determinant.) The fundamental consequence then is the existence of a (uni)versal deformation and the (uni)versal deformation ring $R$ has a presentation $W[[T_1, \ldots, T_n]]/J$ where $n = \dim_k t_D$. A deformation ring is smooth when the ideal of relations $J$ is (0). We construct the smooth global deformation conditions that we are interested in by finding local (uni)versal deformation rings smooth in a number of variables. To this end, we say that a smooth local deformation condition for $\overline{\rho}$ is substantial if its tangent space $t$ satisfies the inequality $\dim_k t \geq \dim_k H^1(G_F, ad^0\overline{\rho}) + 1$. We refer the reader to [1] Definition 2.1 for the precise definition. We now describe substantial local deformation conditions which, as we shall see later, are relevant to the proof of Theorem A.
2.2. **Substantial deformation conditions at** $p$. Let $G_p$ denote the decomposition group at $p$. Suppose we are given an integer $k \geq 2$ and a representation $\overline{\rho} : G_p \rightarrow GL_2(\mathbb{k})$ such that

$$
\overline{\rho} = \begin{pmatrix}
\chi^{k-1} \psi_1 & \ast \\
0 & \psi_2
\end{pmatrix}
$$

where $\psi_1, \psi_2$ are unramified characters. Let $\psi$ be the Teichmüller lift of $\psi_1 \psi_2$. If $A$ is a coefficient ring, let $\rho_A : G_p \rightarrow GL_2(A)$ of $\overline{\rho}$ be strictly equivalent to a representation of the form

$$
\begin{pmatrix}
\psi_1 \chi^{k-1} & \ast \\
0 & \psi_2
\end{pmatrix}
$$

for some unramified characters $\psi_1, \psi_2 : G_p \rightarrow A^\times$ lifting $\psi_1, \psi_2$ and $\psi_1 \psi_2 = \psi$. We then have the following proposition

**Proposition 2.1.** Let $\overline{\rho} : G_p \rightarrow GL_2(\mathbb{k})$ be as above and further assume that $\chi^{k-1} \psi_1 \neq \chi \psi_2$. Then the deformation condition consisting of lifts $\rho_A$ of $\overline{\rho}$ is a smooth deformation condition. The dimension of its tangent space is equal to $1 + \dim_k H^0(G_p, ad^0 \overline{\rho})$.

**Proof.** See Example 3.4 in [5].

2.3. **Substantial deformation conditions at** $q \nmid p$. For the rest of this subsection let $G_q$ denote the decomposition group at $q$ for some place $q \nmid p$. Let $\overline{\rho} : G_q \rightarrow GL_2(\mathbb{k})$ be a residual representation with fixed determinant $d$. We describe substantial deformation conditions for $\overline{\rho}|_{G_q}$ in two cases according as $\overline{\rho}$ is ramified at $q$ or not. In the former case we have the following proposition.

**Proposition 2.2.** Suppose that $q | N(\overline{\rho})$ where $N(\overline{\rho})$ is the Artin conductor of $\overline{\rho}$. If $p \nmid \#(I_q)$ then the collection of lifts of $\overline{\rho}$ which factor through $G_q/(I_q \cap \ker \overline{\rho})$ and have fixed determinant lifting $d$ is a substantial deformation condition. The tangent space has dimension $\dim_k H^0(G_q, ad^0 \overline{\rho})$. If $p | \#(I_q)$ then suppose that

$$
\overline{\rho} \sim \begin{pmatrix}
\chi & * \\
0 & 1
\end{pmatrix} \overline{\varepsilon}
$$

for some character $\overline{\varepsilon} : G_q \rightarrow k^\times$. Moreover, assume that if $\overline{\rho}$ is semi-simple then $\chi$ is non-trivial. Fix a character $\varepsilon : G_q \rightarrow W^\times$ lifting $\overline{\varepsilon}$. Then the collection of lifts strictly equivalent to

$$
\begin{pmatrix}
\chi & * \\
0 & 1
\end{pmatrix} \varepsilon
$$

is a substantial deformation condition. The tangent space has dimension $\dim_k H^0(G_q, ad^0 \overline{\rho})$.

**Proof.** For the case $p \nmid \#(I_q)$ see Example E1 in [3] and for case $p | \#(I_q)$ see Example 3.3 in [5].
Now, for the remaining case, set $F = \mathbb{Q}_q$ and let $\overline{\rho} : G_F \to GL_2(k)$ be unramified at $q$ with $\overline{\rho}(\text{Frob}) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$. Of course, any lift of $\overline{\rho}$ necessarily factors through the maximal tamely ramified extension $F^{\text{tr}}$ of $F$. The group $\text{Gal}(F^{\text{tr}}/F)$ is topologically generated by two elements $\tau$ and $\sigma$ which satisfy the relation $\sigma \tau \sigma^{-1} = \tau^q$ and such that $\tau$ is a (topological) generator of the tame inertia subgroup and $\sigma$ is a lift of Frobenius to $\text{Gal}(F^{\text{tr}}/F)$.

First define polynomials $h_n(T) \in \mathbb{Z}[T], n \geq 1$, by the recursion $h_{n+2} = Th_{n+1} - h_n$ and initial values $h_1 := 1, h_2 := T$. The following properties of $h_n$ are easily verified by induction:

- $h_n(2) = n$
- If $M$ is a $2 \times 2$ matrix over any commutative ring with trace $t$ and determinant $1$ then, $M^n = h_n(t)M - h_{n-1}(t)I$
- $h_n^2 - Th_n h_{n-1} + h_{n-1}^2 = 1$

We then have the following proposition.

**Proposition 2.3.** Let $\overline{\rho}$ be as above with $q \alpha \neq \alpha^{-1}$. Denote by $\hat{\alpha}$ the Teichmuller lift of $\alpha$. Let $R$, resp. $\rho : \text{Gal}(F^{\text{tr}}/F) \to GL_2(R)$, be the versal deformation ring, resp. the versal representation, for lifts of $\overline{\rho}$ with determinant $\chi$.

(i) Suppose $\alpha^2 \neq 1$ and $q^2 \alpha^2 \neq 1$. If $q \equiv 1 \mod p$ then $R \cong W(k)[[S, T]]/((1 + T)^q - (1 + T))$ and

$$
\rho(\sigma) = \begin{pmatrix} q\hat{\alpha}(1 + S) & 0 \\ 0 & (\hat{\alpha}(1 + S))^{-1} \end{pmatrix},
\rho(\tau) = \begin{pmatrix} 1 + T & 0 \\ 0 & (1 + T)^{-1} \end{pmatrix}.
$$

If $q \equiv 1 \mod p$ then $R \cong W[[S]]$ and $\rho(\sigma) = \begin{pmatrix} q\hat{\alpha}(1 + S) & 0 \\ 0 & (\hat{\alpha}(1 + S))^{-1} \end{pmatrix}$. In any case, a deformation condition is substantial if and only if it is unramified.

(ii) If $\alpha^2 = 1$ and $q^2 \neq 1 \mod p$ then $R \cong W(k)[[S, T]]/(ST)$ and

$$
\rho(\sigma) = \hat{\alpha} \begin{pmatrix} 1 + S & 0 \\ 0 & (1 + S)^{-1} \end{pmatrix},
\rho(\tau) = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}.
$$

A deformation condition for $\overline{\rho}$ with determinant $\chi$ is substantial if and only if it is either unramified or of the type considered in Proposition 2.2.

(iii) If $\alpha^2 = 1$ and $q \equiv -1 \mod p$, then $R := W(k)[[S, T_1, T_2]]/J$ where

$$
J := \langle T_1(q(1 + S)^2 - h_q(2\sqrt{1 + T_1 T_2})), T_2(1 - q(1 + S)^2 h_q(2\sqrt{1 + T_1 T_2})) \rangle,
$$
and

$$
\rho(\sigma) = \hat{\alpha} \begin{pmatrix} q(1 + S) & 0 \\ 0 & (1 + S)^{-1} \end{pmatrix},
\rho(\tau) = \begin{pmatrix} \sqrt{1 + T_1 T_2} & T_1 \\ T_2 & \sqrt{1 + T_1 T_2} \end{pmatrix}.
$$

The only ramified substantial deformation condition for $\overline{\rho}$ is the of the type considered in Proposition 2.2 and it corresponds to the quotient $W(k)[[S, T_1, T_2]]/(S, T_2)$. 
Proof. Let $A$ be a coefficient ring with maximal ideal $\mathfrak{m}_A$, and let be $\rho_A$ a lifting of $\overline{\rho}$ with determinant $\chi$. By Hensel’s Lemma, we can assume that $\rho_A(\sigma)$ is diagonal. Let

$$
\rho_A(\sigma) = \begin{pmatrix} q\hat{\sigma}(1 + s) & 0 \\ 0 & (\hat{\sigma}(1 + s))^{-1} \end{pmatrix}, \quad \rho_A(\tau) = \begin{pmatrix} a & t_1 \\ t_2 & d \end{pmatrix}
$$

with $s, t_1, t_2, a - 1, d - 1 \in \mathfrak{m}_A$ and $ad - t_1t_2 = 1$. Since $\sigma\tau\sigma^{-1} = \tau^a$, we have

$$
\begin{pmatrix} a & t_1q(\hat{\sigma}(1 + s))^2 \\ t_2q^{-1}(\hat{\sigma}(1 + s))^{-2} & d \end{pmatrix} = \begin{pmatrix} ah_q(t) - h_{q-1}(t) & t_1h_q(t) \\ t_2h_q(t) & dh_q(t) - h_{q-1}(t) \end{pmatrix}
$$

where $t = a + d$ is the trace. Note that $t \equiv 2 \mod \mathfrak{m}_A$ and so $h_q(t) \equiv q \mod \mathfrak{m}_A$.

If $\alpha^2 \neq 1$ then $q(\hat{\sigma}(1 + s))^2 - h_q(t)$ is a unit and we get $t_1 = 0$. Similarly, if $q^2\alpha^2 \neq 1$ then $t_2 = 0$. The claims made in part (i) of the proposition are now immediate.

We now continue our analysis of $\rho_A$ under the assumption that $\alpha^2 = 1$ and $q \not\equiv \pm 1 \mod p$. For ease of notation, we shall in fact assume that $\hat{\sigma} = 1$. Since $1 - h_q(t)$ is a unit, taking the difference of the diagonal entries on both sides of (2.2) gives $a - d = 0$ and $a = d = \sqrt{1 + t_1t_2}$. Comparison of the off-diagonal entries of (2.2) (followed by multiplication) produces $t_1t_2(1 - h_q(t)^2) = 0$.

Suppose now that $q \not\equiv -1 \mod p$. Then $t_1t_2 = 0$ and so $t = 2, h_3(t) = t$. We can now simplify the two relations from the off-diagonal entries to get $t_1 = t_1(1 + s)^2$ and $t_2 = t_2q^2(1 + s)^2$, and finally deduce that $st_1 = 0, t_2 = 0$. Part (ii) of the proposition now follows easily.

Finally, we consider the case $q \equiv -1 \mod p$. The presentation for $R$ and $\rho$ follows from the presentation of an arbitrary lift along with the fact that $\dim_k H^1(G_\mathcal{Q}, \text{ad}^*\overline{\rho}) = 3$. We now indicate how to determine the substantial deformation conditions. Take $A$ to be characteristic $0$ (and $\hat{\sigma} = 1$). In the presentation (2.3) the trace of $\rho_A(\tau)$ is $2\sqrt{1 + t_1t_2}$. If $t_1t_2 \neq 0$ then $\rho_A(\tau)$ has distinct eigenvalues - contradicting the fact that $\rho_A$ is twist equivalent to $(\chi^*_1)$ over its field of fractions. Hence $t_1 = t_2 = 0$ and $s = 0$.

Let $\rho_n$ be as in Theorem A. We summarize the various substantial local deformation conditions described in this section to show that there exists a global deformation condition for $\rho_n$, i.e. a global deformation condition $\mathcal{D}$ for $\overline{\rho} := \rho_n$ with determinant $\epsilon$ such that $\rho_n$ is a deformation of type $\mathcal{D}$.

**Proposition 2.4.** There exists a global deformation condition $\mathcal{D}$ for $\rho_n$ such that each local component $\mathcal{D}_q$ is a substantial deformation condition.

**Proof.** Let $S$ denote the finite set of places of $\mathbb{Q}$ at which $\rho_n$ is ramified along with places dividing $pQ$. We define a global deformation condition $\mathcal{D}$ for $\overline{\rho}$ by the following requirements:

(a) Deformations are unramified outside $S$ and have determinant $\psi\chi^{k-1}$. By Proposition 2.3, $\rho_n$ is a substantial deformation condition at these primes.

(b) At $p$, the local condition $\mathcal{D}_p$ consists of deformations as in Proposition 2.1. Then this proposition, along with hypothesis C3 in Theorem A ensures that $\rho_n$ is a substantial deformation condition at these places.

(c) Let $q$ be a place in $S$. We need to distinguish two cases:

(i) If $|p| = \#(I_q)$ then $\overline{\rho}_{I_q} \sim \left( \begin{smallmatrix} \chi^*_1 \\ 0 \end{smallmatrix} \right)$ for some character $\tilde{\epsilon}$. We then take $\mathcal{D}_q$ to be local lifts with determinant $\psi\chi^{k-1}$ of the type considered in the first part of Proposition 2.2.
(ii) If \( p \nmid \# \mathcal{P}(I_4) \) we take \( D_q \) as in the second part of Proposition 2.2 i.e. lifts with determinant \( \psi \chi^{k-1} \) which factor through \( G_0/(I_4 \cap \ker \mathcal{P}) \).

In either case, hypothesis C4 of Theorem A and Proposition 2.2 imply that \( \rho_n \) is a substantial deformation condition at these places.

It then follows that \( \rho_n \) is a deformation of type \( D \). \( \square \)

2.4. Deformations of mod \( p^n \) representations to \( W \). We now show that any obstructions to lifting \( \rho_n \) to characteristic 0 can be realized locally and these, in turn, can be overcome by trivializing certain dual Selmer groups. We denote by \( \text{ad}^0 \) the the vector space of \( 2 \times 2 \)-matrices over \( k \) with \( GL_2(W/p^n) \) acting by conjugation, and by \( \text{ad}^0(i) \) its twist by the \( i \)-th power of the determinant. Given an global deformation condition \( D \) we denote, by \( D_q \), the local component at a prime \( q \), its tangent space by \( t_{D_q} \), and by \( t_{D_q}^+ \subseteq H^1(G_q, \text{ad}^0 \mathcal{P}(1)) \) the orthogonal complement of \( t_{D_q} \) under the pairing induced by \( \text{ad}^0 \mathcal{P} \times \text{ad}^0 \mathcal{P}(1) \) \( \text{trace} \ k(1) \). Let \( H^1(G_q, \text{ad}^0 \mathcal{P}) \) denote the preimage of \( \bigoplus_{q \in S} t_{D_q}^+ \) under the restriction map

\[
H^1(G_q, \text{ad}^0 \mathcal{P}) \to \bigoplus_{q \in S} H^1(G_q, \text{ad}^0 \mathcal{P}).
\]

Similarly we will let the dual Selmer group \( H^1(G_q, \text{ad}^0 \mathcal{P}(1)) \) denote the preimage of \( \bigoplus_{q \in S} t_{D_q}^+ \) under the restriction map

\[
H^1(G_q, \text{ad}^0 \mathcal{P}(1)) \to \bigoplus_{q \in S} H^1(G_q, \text{ad}^0 \mathcal{P}(1)).
\]

The tangent space for \( D \) is the Selmer group \( H^1_{(t_{D_q})}(G_q, \text{ad}^0 \mathcal{P}) \) (cf. [7] Definition 8.6.19). We also set

\[
\delta(D) := \dim_k H^1_{(t_{D_q})}(G_q, \text{ad}^0 \mathcal{P}) - \dim_k H^1_{(t_{D_q})}(G_q, \text{ad}^0 \mathcal{P}(1)).
\]

Before we state and prove the lifting result for \( \rho_n \) we record the following lemma about some properties of certain subgroups of \( GL_2(W/p^n) \).

**Lemma 2.5.** Let \( G \) be a subgroup of \( GL_2(W/p^n) \). Suppose the mod \( p \) reduction of \( G \) contains \( SL_2(k) \). Furthermore, assume that if \( p = 3 \) then \( G \) contains a transvection \((1 \ 1 \ 0 \ 1)\). Then the following statements hold.

(a) \( G \) contains \( SL_2(W/p^n) \).

(b) Suppose that \( p \geq 5 \). If \( k = \mathbb{F}_5 \) assume that \( G \mod 5 = GL_2(\mathbb{F}_5) \). Then \( H^1(G, \text{ad}^0(i)) = 0 \) for \( i = 0, 1 \).

(c) The restriction map \( H^1(G, \text{ad}^0(i)) \to H^1(\langle 1 \ 1 \rangle, \text{ad}^0(i)) \) is an injection (for all \( p \geq 3 \)).

**Proof.** This is Proposition 2.7 in [1]. We sketch a slightly simpler argument for parts (b) and (c). We first make the following observation:

For \( i = 0, 1 \) an \( p \geq 3 \), \( H^1(GL_2(k), \text{ad}^0(i)) = 0 \) if \( k = \mathbb{F}_5 \) and \( H^1(SL_2(k), \text{ad}^0(i)) = 0 \) if \( k \neq \mathbb{F}_5 \).

It is well known—see Lemma 2.48 of [2], for instance—that \( H^1(SL_2(k), \text{ad}^0) = 0 \) except when \( k = \mathbb{F}_5 \). We prove the exceptional case. Let \( B \supset U \) be the subgroups of \( GL_2(\mathbb{F}_5) \) consisting of matrices of the form \((* \ 1), (1 \ 1)\) respectively. We then need to verify that \( H^1(B, \text{ad}^0(i)) \cong H^1(U, \text{ad}^0(i))_{B/U} = (0) \). Let \( \sigma := (1 \ 1), \tau := (0 \ 1) \). It follows that \( (\sigma - 1)\text{ad}^0 \) is the subspace of upper triangular matrices in \( \text{ad}^0 \). Thus if \( 0 \neq \xi \in H^1(U, \text{ad}^0(i)) \) then we can assume that
\[ \xi(\sigma) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \] and \( \xi \) is fixed by \( B/U \) if and only if \( (\tau \ast \xi)(\sigma) - \xi(\sigma) \) is upper triangular. Now \( (\tau \ast \xi)(\sigma) = \tau \xi(\sigma^2) \tau^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) and so \( \xi \in H^1(U, \text{ad}^0(i))^{B/U} \) if and only if \( 1 + 3t = 0 \).

Now, to prove part (b), we use the Inflation-Restriction exact sequence

\[ 0 \to H^1(G/H, \text{ad}^0(i)^H) \to H^1(G, \text{ad}^0(i)) \to H^1(H, \text{ad}^0(i))^{G/H} \]

where \( H := \ker(G \to G \mod p^n) \). Then, the first term vanishes by the hypothesis and the observation made above while the last term vanishes because \( \text{ad}^0(i) \) has trivial fixed points under the action of \( SL_2(k) \) which \( G/H \) contains.

The sequence above also shows that all non-zero classes in \( H^1(G, \text{ad}^0(i)) \) must come from \( H^1(G \mod p, \text{ad}^0(i)) \) via inflation. Since \( H^1(SL_2(k), \text{ad}^0(i)) \) is trivial, (c) follows unless \( k = \mathbb{F}_5 \) in which case the assertion is the general fact from group cohomology that restriction to a \( p \)-Sylow subgroup is injective if the coefficients are \( p \)-primary.

Another crucial input that we will use is a well known formula due to Wiles which we state for the sake of convenience.

**Wiles’ formula.** The following formula is a specialization of [7, Theorem 8.7.9].

\[ \delta(D) = \dim_k H^0(G, \text{ad}^0(\overline{\rho})) - \dim_k H^0(G, \text{ad}^0(\overline{\tau})) + \sum_{q \leq \infty} (\dim_k \mathfrak{t}_{D_q} - \dim_k H^0(G_q, \text{ad}^0(\overline{\sigma}))). \]

In the context of our work \( H^0(G, \text{ad}^0(\overline{\rho})) = H^0(G, \text{ad}^0(\overline{\tau})) = 0 \) since \( \overline{\rho} \) is absolutely irreducible. Moreover it follows from easy group cohomological considerations that \( \dim_k \mathfrak{t}_D = 0 \) and \( \dim_k H^0(G_{\infty}, \text{ad}^0(\overline{\sigma})) = 1 \). Therefore we will only need the simpler formula:

\[ \delta(D) = \sum_{q \leq \infty} (\dim_k \mathfrak{t}_{D_q} - \dim_k H^0(G_q, \text{ad}^0(\overline{\sigma}))) - 1. \]

We are now ready to prove the following lifting result. (cf. [1, Proposition 3.1 and Theorem 3.2].)

**Theorem 2.6.** Suppose we are given a deformation condition \( D \) for \( \rho_n \) with determinant \( \epsilon \). Let \( S \) be a fixed finite set of primes of \( \mathbb{Q} \) including primes where \( D \) is ramified and all primes dividing \( \infty \).

If \( \delta(D) \geq 0 \) then we can find a smooth deformation condition \( E \) for \( \rho_n \) with determinant \( \epsilon \) such that the local conditions \( E_q \) and \( D_q \) differ only at primes \( q \notin S \) and, at these primes, \( E_q \) is a substantial deformation condition. Moreover, \( H^1_{\text{tr}}(\mathfrak{t}_E)(\mathbb{Q}, \text{ad}^0(\overline{\sigma})) = 0 \) and the universal deformation ring is a power series ring over \( W \) in \( \delta(D) \) variables. In particular, there is a representation \( \rho : G_{\mathbb{Q}} \to GL_2(W) \) of type \( E \) lifting \( \rho_n \).

**Proof.** Let \( K \) be the splitting field of \( \rho_n \) adjoined \( p^n \)-th roots of unity. We claim that we can find elements \( g, h \in \text{Gal}(K/\mathbb{Q}) \) such that

- \((R1) \ \rho_n(h) \sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \chi(h) = 1 \mod p^n \).
- \((R2) \ \rho_n(g) \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( \chi(g) = -1 \mod p^n \).

For \( R1 \), we can write \( \epsilon = \chi_0 \epsilon_1^2 \) where \( \epsilon_0 \) is a finite order character of order co-prime to \( p \). Lemma 2.5(a) then implies that the image of the twist of \( \rho_n \otimes \epsilon_1^{-1} \) contains \( SL_2(W/p^n) \). Thus we can find \( h_1 \in \text{Gal}(K/\mathbb{Q}) \) such that \( \rho_n(h_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \epsilon_1(h_1) \) and we get \( \epsilon_0(h_1) \chi(h_1) = 1 \). We can then take \( h \) to be \( h_1 p^{k-1} \) where \( p^{k} \) is the cardinality of \( k \). For \( R2 \), we can take \( g \) to be complex conjugation.

We define a deformation condition \( E_0 \) for \( \rho_n \) with determinant \( \epsilon \) as follows. If \( p \geq 5 \) and the projective image of \( \overline{\rho} \) strictly contains \( PSL_2(\mathbb{F}_5) \) then \( E_0 \) is \( D \). Now suppose that either \( p = 3 \) or...
the projective image of $\overline{\rho}$ is $A_5$ (so $k$ is necessarily $\mathbb{F}_5$). Using the Chebotarev Density Theorem and R1 above, we can find a prime $q_0 \notin S$ with $q_0 \equiv 1 \mod p^n$ and $\rho_n(\text{Frob}_{q_0}) = a \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$. Let $\mathcal{E}_0$ be the deformation condition of $\overline{\rho}$ with determinant $\epsilon$ such that $\mathcal{E}_{0q} = \mathcal{D}_q$ at primes $q \neq q_0$, and at $q_0$, $\mathcal{E}_{0q_0}$ consists of deformations of the form $\left( \begin{array}{cc} \chi & * \\ 0 & 1 \end{array} \right)$ where $\epsilon' : G_{q_0} \rightarrow W^\times$ is unramified and $\epsilon|_{G_{q_0}} = \chi \epsilon^2$. Proposition 2.2 then ensures that $\mathcal{E}_{0q_0}$ is a substantial local deformation condition and hence that $\mathcal{E}_0$ is a global deformation condition for $\rho_n$. Further, Wiles’ formula shows that $\delta(\mathcal{E}_0)$ remains invariant after this adjustment so that $\delta(\mathcal{E}_0) = \delta(D)$.

Next, we claim that the restriction maps

$$H^1(\{t_{\xi,q}\}, G, \text{ad}^0\overline{\rho}) \rightarrow H^1(G_K, \text{ad}^0\overline{\rho})$$

and

$$H^1(\{t_{\xi,q}^+\}, G, \text{ad}^0\overline{\rho}(1)) \rightarrow H^1(G_K, \text{ad}^0\overline{\rho}(1))$$

are injective. When $p \geq 5$ and the projective image of $\overline{\rho}$ strictly contains $A_5$ an easy calculation using Lemma [2,5](b) shows that the kernels $H^1(\text{Gal}(K/Q), \text{ad}^0\overline{\rho})$ and $H^1(\text{Gal}(K/Q), \text{ad}^0\overline{\rho}(1))$ are trivial, and so the injectivity follows. In the exceptional case $i.e.$ when $p = 3$ or the projective image of $\overline{\rho}$ is $A_5$, we observe that $\xi \in \ker \left( H^1(\{t_{\xi,q}\}, G, \text{ad}^0\overline{\rho}) \rightarrow H^1(G_K, \text{ad}^0\overline{\rho}) \right)$ is naturally an element of $H^1(\text{Gal}(K/Q), \text{ad}^0\overline{\rho})$. Thus $\xi$ is unramified at $q_0$ and so the restriction of $\xi$ to the decomposition group at $q_0$ must be trivial. Using Lemma [2,5](c) it follows that $\xi \in H^1(\text{Gal}(K/Q), \text{ad}^0\overline{\rho})$ is trivial. A similar argument works for $\text{ad}^0\overline{\rho}(1)$.

As previously noted $\delta(\mathcal{E}_0) = \delta(D)$ and since $\delta(D) \geq 0$ (by hypothesis) we conclude that if the dual Selmer group for $\mathcal{E}_0$ is non-trivial then we can find

$$0 \neq \xi \in H^1(\{t_{\xi,q}\}, G, \text{ad}^0\overline{\rho}), \quad 0 \neq \psi \in H^1(\{t_{\xi,q}^+\}, G, \text{ad}^0\overline{\rho}(1)).$$

Taking $g \in \text{Gal}(K/Q)$ as in R2 we apply Proposition 2.2 of [5] to find a prime $\tau \notin S \cup \{q_0\}$ lifting $g$ such that the restrictions of $\xi, \psi$ to $G_\tau$ are not in $H^1(G_\tau, N_1), H^1(G_\tau, N_2)$ for $\{(0, 0)\} = N_1 \subset M_1 = \text{ad}^0\overline{\rho}, \{(0, 0)\} = N_2 \subset M_2 = \text{ad}^0\overline{\rho}(1)$. We now adjust $\mathcal{E}_0$ to produce a new deformation condition $\mathcal{E}_1$ determinant $\epsilon$ such that $\mathcal{E}_1$ and $\mathcal{E}_0$ differ only at $\tau$ where the local component consists of deformations of the form $\left( \begin{array}{cc} \chi & * \\ 0 & 1 \end{array} \right)$ considered in Proposition 2.3. Here, $(\epsilon/\chi)^{1/2}$ is the unramified character determined by taking the square-root of $\epsilon(\text{Frob}_\tau) \chi^{-1}(\text{Frob}_\tau)$. Since $\text{Frob}_\tau$ lifts $g$ we have $\chi(\text{Frob}_\tau) \equiv -1 \mod p^n$, and consequently $\mathcal{E}_1$ is a substantial deformation condition for $\rho_n$. A dimension calculation identical to [5 Proposition 4.2] shows that $\dim_k H^1(\{t_{\xi,q}^+\}, G, \text{ad}^0\overline{\rho}(1)) = \dim_k H^1(\{t_{\xi,q}\}, G, \text{ad}^0\overline{\rho}(1)) - 1$. By arguing recursively, we find the desired global deformation condition $\mathcal{E}$ with trivial dual Selmer group. The existence of the lift $\rho$ follows from [10] Lemma 1.1]. Finally, the claim that the universal deformation ring is a power series ring over $W$ in $\delta(D)$ variables follows upon verifying that $\dim_k H^1(\{t_{\xi,q}\}, G, \text{ad}^0\overline{\rho}) = \delta(\mathcal{E}) = \delta(D)$. This is again done using Wiles’ formula.

3. Modularity of $\rho$: Proof of Theorems A and B

We now have all the tools required to prove Theorems A and B. On the one hand we have lifted $\rho_n$ to a representation in characteristic 0 and on the other we already know that its mod $p$ reduction is modular. This presents an ideal scenario for the application of a suitable modularity lifting theorem and, indeed, we invoke the theorem of Skinner and Wiles.
Proof of Theorems A and B: Let \( \rho_n \) be as in Theorem A or Theorem B. Proposition 2.4 implies that there exists a global deformation condition \( D \) for \( \rho_n \) such that each local component \( D_q \) is a substantial deformation condition. Therefore, if we knew that \( \delta(D) \geq 0 \), we may apply Theorem 2.6 to get a \( \rho : G \mathbb{Q} \to GL_2(W) \) lifting \( \rho_n \). We verify this inequality as follows. By Wiles’ formula
\[
\delta(D) = \sum_{q < \infty} (t_{D_q} - \text{dim}_kkkH^0(G_q, \text{ad}^1\mathcal{P})) - 1.
\]
However, \( t_{D_q} = \text{dim}_kkkH^0(G_q, \text{ad}^1\mathcal{P}) \) for \( q \neq p \) (Propositions 2.2 and 2.3) and \( t_{D_p} = 1 + \text{dim}_kkkH^0(G_q, \text{ad}^1\mathcal{P}) \) for \( q = p \) (Proposition 2.1). Consequently \( \delta(D) = 0 \). As noted in the Introduction, the irreducibility of \( \mathcal{P} \) is ensured by hypothesis C2 and so it is modular by Serre’s Modularity Conjecture. The \( p \)-distinguishedness property is part of hypothesis C3 and our choice of local deformations at \( p \) (Proposition 2.1 and the preceding paragraph) ensures that the image of \( \rho|_{I_p} \) has 1 as its lower left matrix entry. A direct application of the main theorem of [7] then shows that \( \rho \) arises from a \( p \)-ordinary modular form of weight \( k \) and level prime to \( N \).

We conclude with the following remarks.

Remark 3.1. All the work in Section 2, and consequently Theorems A and B, can be generalized to the setting of 2-dimensional Galois representations of \( G_F \) – where \( F \) is a totally real field in which \( p \) ramifies – and Hilbert modular forms of parallel weight. The caveat for Theorem A is that Serre’s Modularity Conjecture is known only for \( F = \mathbb{Q} \). Theorem B, however would become readily available because in this case the modularity of \( \mathcal{P} \) follows from [4, Theorem 2.1.1].

Remark 3.2. As we mentioned in the Section 1, Theorem B can be thought of as an analog of [4, Theorem 2.1.1] where it is used as a “first step” in proving a much deeper result about the existence of weight 1 companion forms. A mod \( p^n \) version of that result however would involve geometric arguments which are beyond the scope and intent of this work.

Acknowledgements

The author wishes to thank Jayanta Manoharmayum for his valuable inputs. The author is also grateful to Uwe Jannsen and DFG-GRK 1692, Universität Regensburg for postdoctoral support.

References

[1] R. Adibhatla and J. Manoharmayum. Higher congruence companion forms. \textit{Acta Arith.}, 156(2):159–175, 2012.
[2] H. Darmon, F. Diamond, and R. Taylor. Fermat’s last theorem. In \textit{Elliptic curves, modular forms & Fermat’s last theorem (Hong Kong, 1993)}, pages 2–140. Int. Press, Cambridge, MA, 1997.
[3] T. Gee. Companion forms over totally real fields. II. \textit{Duke Math. J.}, 136(2):275–284, 2007.
[4] T. Gee and P. Kassaei. Companion forms in parallel weight one. \textit{Compos. Math.}, To appear.
[5] J. Manoharmayum. Lifting Galois representations of number fields. \textit{J. Number Theory}, 129(5):1178–1190, 2009.
[6] B. Mazur. Deforming Galois representations. In \textit{Galois groups over \( \mathbb{Q} \) (Berkeley, CA, 1987)}, volume 16 of \textit{Math. Sci. Res. Inst. Publ.}, pages 385–437. Springer, New York, 1989.
[7] J. Neukirch, A. Schmidt, and K. Wingberg. \textit{Cohomology of number fields}, volume 323 of \textit{Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]}. Springer-Verlag, Berlin, second edition, 2008.
[8] R. Ramakrishna. Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur. \textit{Ann. of Math. (2)}, 156(1):115–154, 2002.
[9] C. M. Skinner and A. J. Wiles. Nearly ordinary deformations of irreducible residual representations. \textit{Ann. Fac. Sci. Toulouse Math. (6)}, 10(1):185–215, 2001.
[10] R. Taylor. On icosahedral Artin representations. II. \textit{Amer. J. Math.}, 125(3):549–566, 2003.