Heat Kernel and Loop Currents by the Generating-Function Method

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Abstract

The generating function method is applied to the trace of the heat kernel and the one-loop effective action derived from the covariant perturbation theory. The basis of curvature invariants of second order for the heat kernel (Green function) is built and simple rules for form factor manipulations are proposed. The results are checked by deriving the Schwinger-DeWitt series of the heat kernel and divergences of one-loop currents.

1 The generating function method

The present talk reviews a progress on calculations of heat kernel and relevant quantum objects in the framework of the covariant perturbation theory [1, 2, 3]. As generating functions we take the one-loop effective action

\[ W = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \text{Tr}K(s) \tag{1} \]

and the functional trace of the heat kernel

\[ \text{Tr}K(s) = \int dx \ g^{1/2}(x) \text{tr}K(s|x, x) \tag{2} \]

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while the heat kernel itself
\[ \hat{K}(s|x, y) = e^{s F(\nabla)} \delta(x, y) \]
is defined by the second order operator
\[ F = \Box \hat{1} + \hat{P} - \frac{1}{6} R \hat{1}, \quad \Box = g^{\mu\nu} \nabla_\mu \nabla_\nu \]
acting on any column of fields \( \varphi^a(x) \), an overhat indicating the matrix-valued nature of the corresponding field objects: \( \hat{1} = \delta^a_b, \hat{P} = P^a_b \) etc. This operator contains three independent fields: the spacetime metric \( g^{\mu\nu} \), the arbitrary connection entering the covariant derivative \( \nabla_\mu \) and the potential term \( \hat{P} \) which correspond to the three curvatures: Riemann tensor \( R^{\mu\nu\alpha\beta} \), the potential term \( \hat{P} \), and the commutator curvature
\[ (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) \varphi^A = R^{A}_{B\mu\nu} \varphi^B. \]

Our purpose is to derive coincidence limits of the heat kernel of the most general form
\[ \nabla^x \ldots \nabla^x \nabla^y \ldots \nabla^y \hat{K}(s|x, y) \big|_{y=x} \]
that bear all essential information of quantum field theory [4, 5, 6]. From (6) Green’s functions are defined via Schwinger’s proper-time representation [5, 6]
\[ G(x, y) \equiv \nabla^x \ldots \nabla^x \nabla^y \ldots \nabla^y \hat{1} \frac{1}{F(\nabla)} \delta(x, y) = -\int_0^\infty ds \nabla^x \ldots \nabla^x \nabla^y \ldots \nabla^y \hat{K}(s|x, y). \]

Actually our final goal is a computation of two-loop effective action [7, 8]
\[ W_{\text{two-loop}} = -\frac{1}{2} - \frac{1}{8} \]
for Yang-Mills and gravity theories. Building blocks for “fish” and “eight” diagrams [8] are “tadpoles” (universal functional traces [6, 7])
\[ G(x, x) = \]
and polarization operators
\[ G_1(x, y)G_2(x, y) = \]
Having these two sets of basic elements the two-loop graphs can be easily composed, e.g.

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\(^4\) Our conventions are: \( R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^\lambda_{\mu\lambda\nu} = g^{\mu\nu}(\partial_\lambda \Gamma^\lambda_{\mu\nu} - ...) \).
\[
\int dx G_1(x, x) G_2(x, x) = \bigcirc \cdot \bigcirc
\]

Briefly the generating expressions approach originates from a simple principle [9]
\[
\frac{1}{s} \delta \left( \text{Tr} e^{s \hat{F}} \right) = \text{Tr} \left( \delta \hat{F} e^{s \hat{F}} \right)
\]
which implies
\[
\frac{1}{s} \delta \left( \text{Tr} K(s) \right) = \int dx \text{tr} \left[ \delta \hat{F} (\nabla^x) \hat{K}(s|x, y) \right]_{y=x}.
\]

Because there are three independent curvatures in the differential operator \( \mathcal{F} \) two variational equations can be written down immediately:
\[
\hat{K}(s|x, x) = \frac{1}{s} \frac{\delta}{\delta P(x)} \text{Tr} K(s),
\]
\[
\nabla_\alpha \hat{K}(s|x, y) |_{y=x} = \frac{1}{s} \frac{\delta}{\delta \Gamma^\alpha(x)} \text{Tr} e^{s (\mathcal{F} + \Gamma^\mu \nabla_\mu)} \big|_{\Gamma^\nu=0},
\]

where a new operator \( \hat{F} \to \hat{F} + \hat{\Gamma}^\mu \nabla_\mu \) in (13) is generated by redefinition of the covariant derivative connection \( \nabla_\mu \to \nabla_\mu + \hat{\Gamma}_\mu \), consequently \( \hat{P} \) and \( \hat{\mathcal{R}}_{\mu\nu} \) are redefined. Third variational equation over the metric \( g^{\mu\nu} \) give us a combination of \( \nabla_{(\alpha} \nabla_{\beta)} \text{tr} \hat{K}(s) \) and \( g^{\alpha\beta} \bullet \text{tr} \hat{K}(s) \), so the desired object
\[
\nabla_\alpha \nabla_\beta \hat{K}(s|x, y) |_{y=x}
\]
can be expressed using the outcome of (12) and the heat equation itself
\[
\frac{\partial}{\partial s} \hat{K}(s) = \hat{F} (\nabla) \hat{K}(s).
\]

An antisymmetric part \( \nabla_{[\alpha} \nabla_{\beta]} \hat{K}(s) \) is recovered from (3) and (12). Fortunately, the terms in (14) removed by the matrix trace operation are absent in first order as proved by the covariant perturbation theory [9].

The method of generating expressions first has been proposed [7] for analysis of divergences of \( G(x, x) \) and \( G_1(x, y) G_2(x, y) \). However, even local expansions of two-loop diagrams require knowledge of finite nonlocal structures of one-loop generating functions [7]; thus we need to resort to the covariant perturbation theory.

The covariant perturbation theory allows one to obtain an expansion of (the trace of) the heat kernel in powers of the curvatures to a given accuracy \( O[\mathcal{R}^n] \) with a collective notation \( \mathcal{R} = (R^{\beta\mu\nu}, \hat{\mathcal{R}}_{\mu\nu}, \hat{P}) \). In Refs. [1, 2, 3] \( \text{Tr} K(s) \) and \( W \) were calculated in a \( 2\omega \)-dimensional asymptotically flat Euclidean spacetime to third order in curvature
\[
\text{Tr} K(s) = \frac{1}{(4\pi s)^{\omega}} \int dx g^{1/2} \text{tr} \left\{ \hat{1} + s \hat{P} + s^2 \sum_{i=1}^5 f_i(-s \square_2) \mathcal{R}_1 \mathcal{R}_2(i) \right\}
\]
\[ + s^3 \sum_{i=1}^{11} F_i(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) \]
\[ + s^4 \sum_{i=12}^{25} F_i(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) \]
\[ + s^5 \sum_{i=26}^{28} F_i(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(i) \]
\[ + s^6 F_{29}(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_3(29) + O[\mathcal{R}^4] \],
\( (15) \)

where quadratic tensor structures and terms of third order in curvature make up a basis of nonlocal curvature invariants \([2, 3, 4]\), and the notation on the left-hand side of \((15)\) assumes that \(\Box_i\) acts on \(\mathcal{R}_i\).

The only important feature of the form factors \(f_i\) and \(F_i\) required here is that they are functions of the operator
\[ \xi_i = -s \Box_i \]
and do not depend explicitly on the curvatures \(\mathcal{R}\). This is an artifact of the method which always destroys one curvature reducing an accuracy \(O[\mathcal{R}^n]\) by one order. In this circumstance the variation of form factors is not required.

The covariant perturbation theory represents the high-energy approximation so its validity is restricted by the condition
\[ \nabla \nabla \mathcal{R} \gg \mathcal{R}^2. \]

All results of next two sections obtained with the generating function methods have been checked by the covariant perturbation theory formulae \([5]\).

## 2 The heat kernel to second order in curvatures

We shall not elaborate on results for \(\nabla_\alpha \hat{K}^\alpha(s)\) and \(\nabla_\alpha \nabla_\beta \hat{K}^\alpha(s)\) \([5]\), but will consider the heat kernel \(\hat{K}(s)\) in more detail. One can easily observe that only twelve of 29 cubic tensor structures \([15]\) contain the matrix \(\hat{P}\) and therefore contribute to the heat kernel. The variational derivative of \(\text{Tr}K(s)\) over \(\hat{P}\) results in the following expression
\[ \hat{K}(s) = \frac{1}{(4\pi s)^{\omega}} g^{1/2} \{ \hat{1} + s \left( g_1(-s \Box) \hat{P} + g_2(-s \Box) \hat{R} \hat{1} \right) \]
\[ + s^2 \sum_{i=1}^{5} G_i(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2[i] \]
\[ + s^3 \sum_{i=6}^{10} G_i(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2[i] \]
\[ + s^4 G_{11}(-s \Box_1, -s \Box_2, -s \Box_3) \mathcal{R}_1 \mathcal{R}_2[11] + O[\mathcal{R}^3] \}, \]
\( (16) \)

where first order form factors are formed by the second order ones of \((17)\) as
\[ g_1(\xi) = 2f_4(\xi), \]
\[ g_2(\xi) = f_3(\xi), \]
\( (17) \)
and quadratic tensor structures $\\mathcal{R}_1 \mathcal{R}_2(i)$ are

\[
\begin{align*}
\mathcal{R}_1 \mathcal{R}_2[1] &= \hat{\mathcal{R}}_1 \hat{\mathcal{R}}_2, \\
\mathcal{R}_1 \mathcal{R}_2[2] &= \mathcal{R}_1^{\mu \nu} \mathcal{R}_2_{\mu \nu}, \\
\mathcal{R}_1 \mathcal{R}_2[3] &= \hat{\mathcal{R}}_1 \mathcal{R}_2, \\
\mathcal{R}_1 \mathcal{R}_2[4] &= R_1 R_2, \\
\mathcal{R}_1 \mathcal{R}_2[5] &= R_1^{\mu \nu} R_{2 \mu \nu}, \\
\mathcal{R}_1 \mathcal{R}_2[6] &= \nabla_\mu \hat{\mathcal{R}}_1^{\mu \nu} \nabla^\alpha \hat{\mathcal{R}}_2_{\alpha \nu}, \\
\mathcal{R}_1 \mathcal{R}_2[7] &= [\nabla_\alpha \hat{\mathcal{R}}_1, \nabla_\beta \hat{\mathcal{R}}_2^{\beta \alpha}], \\
\mathcal{R}_1 \mathcal{R}_2[8] &= \nabla_\mu \nabla_\nu \hat{\mathcal{R}}_1 R_2^{\mu \nu}, \\
\mathcal{R}_1 \mathcal{R}_2[9] &= \nabla_\alpha \mathcal{R}_1_{\mu \nu} \nabla^\mu \mathcal{R}_2^{\alpha \mu \nu}, \\
\mathcal{R}_1 \mathcal{R}_2[10] &= \nabla_\mu \nabla_\nu \mathcal{R}_1 R_2^{\mu \nu}, \\
\mathcal{R}_1 \mathcal{R}_2[11] &= \nabla_\alpha \nabla_\beta \mathcal{R}_1_{\mu \nu} \nabla^\mu \nabla^\nu \mathcal{R}_2^{\alpha \beta \mu \nu}. \tag{18}
\end{align*}
\]

Without gravity, the basis (18) reduces to only four curvature structures. Taking into account the symmetries of form factors (15) the second order form factors $G_i, i = 1$ to 11 are expressed via $F_i$ in the following way,

\[
\begin{align*}
G_1(\xi_1, \xi_2, \xi_3) &= 3 F_1(\xi_1, \xi_2, \xi_3), \\
G_2(\xi_1, \xi_2, \xi_3) &= F_3(\xi_1, \xi_2, \xi_3), \\
G_3(\xi_1, \xi_2, \xi_3) &= F_6(\xi_1, \xi_2, \xi_3)|_{\xi_2 \leftrightarrow \xi_3} + F_6(\xi_1, \xi_2, \xi_3)|_{\xi_1 \leftrightarrow \xi_3, \xi_2 \leftrightarrow \xi_1} \\
&\quad - \left(\frac{1}{2} \xi_2 + \frac{1}{2} (\xi_3 - \xi_2 - \xi_1)\right) F_{17}(\xi_1, \xi_2, \xi_3)|_{\xi_1 \leftrightarrow \xi_2, \xi_2 \leftrightarrow \xi_3, \xi_3 \leftrightarrow \xi_1}, \\
G_4(\xi_1, \xi_2, \xi_3) &= F_4(\xi_1, \xi_2, \xi_3) + \frac{1}{4} (\xi_3 - \xi_2 - \xi_1) F_{15}(\xi_1, \xi_2, \xi_3), \\
G_5(\xi_1, \xi_2, \xi_3) &= F_5(\xi_1, \xi_2, \xi_3), \\
G_6(\xi_1, \xi_2, \xi_3) &= F_{14}(\xi_1, \xi_2, \xi_3), \\
G_7(\xi_1, \xi_2, \xi_3) &= -F_{13}(\xi_1, \xi_2, \xi_3)|_{\xi_1 \leftrightarrow \xi_2}, \\
G_8(\xi_1, \xi_2, \xi_3) &= F_{17}(\xi_1, \xi_2, \xi_3)|_{\xi_1 \leftrightarrow \xi_2} + F_{17}(\xi_1, \xi_2, \xi_3)|_{\xi_1 \leftrightarrow \xi_2, \xi_2 \leftrightarrow \xi_3, \xi_3 \leftrightarrow \xi_1}, \\
G_9(\xi_1, \xi_2, \xi_3) &= F_{16}(\xi_1, \xi_2, \xi_3), \\
G_{10}(\xi_1, \xi_2, \xi_3) &= -F_{15}(\xi_1, \xi_2, \xi_3)|_{\xi_1 \leftrightarrow \xi_2}, \\
G_{11}(\xi_1, \xi_2, \xi_3) &= F_{20}(\xi_1, \xi_2, \xi_3). \tag{19}
\end{align*}
\]

It should be emphasized again that these rules are applicable to any of the tabulated form factors of Ref. [2] related to the heat kernel trace; i.e., they can be in the explicit or $\alpha$-polynomial representations, or even large and short time expansions. Due to the special status of the matrix $\hat{P}$, which is a curvature and perturbation simultaneously, the variation (12) was performed at the level of perturbations of the metric and covariant derivative as well.

The explicit form of a few first form factors is as follows

\[
g_1 = f(\xi), \tag{20}
\]
with the polynomial of the presence of $\Delta$ in denominators. Only representation with help of fortunately we can not manage this in the given form factor representation (22)-(23) because form factors admit significantly more compact form point for the explicit representation [2] can be used for this purpose. In this representation naturally arises from the covariant perturbation theory and, in fact, serves as a starting

$$g_2 = \frac{1}{12} f(\xi) + \frac{1}{2} \frac{f(\xi) - 1}{\xi}, \quad (21)$$
$$G_1 = F(\xi_1, \xi_2, \xi_3), \quad (22)$$
$$G_2 = F(\xi_1, \xi_2, \xi_3) \left[ \frac{2\xi_1\xi_2}{\Delta^2} (\xi_3 + \xi_2 - \xi_1)(\xi_3 + \xi_1 - \xi_2) + \frac{2}{\Delta} (\xi_3 - \xi_2 - \xi_1) \right]$$

$$\quad - f(\xi_1) \frac{4\xi_1\xi_2}{\Delta^2} (\xi_3 + \xi_1 - \xi_2)$$

$$\quad - f(\xi_2) \frac{4\xi_1\xi_2}{\Delta^2} (\xi_3 + \xi_2 - \xi_1)$$

$$\quad - f(\xi_3) \frac{1}{\Delta^2} (-6\xi_1\xi_2\xi_3 - 3\xi_1\xi_3^2 - 3\xi_2\xi_3^2 + 3\xi_3\xi_1^2 + 3\xi_3\xi_2^2$$

$$\quad + \xi_3^3 + \xi_1\xi_2^2 + \xi_2\xi_1^2 - \xi_2^3 - \xi_1^3), \quad (23)$$

where $f(\xi)$ is the basic second-order form factor [1, 2].

$$f(\xi) \equiv \left\langle \exp(-\alpha_1\alpha_2\xi) \right\rangle_2 \equiv$$

$$\int_{\alpha \geq 0} d^2\alpha \delta(1 - \alpha_1 - \alpha_2) \exp(-\alpha_1\alpha_2\xi) = \int_0^1 d\alpha e^{-\alpha(1-\alpha)} \xi \quad (24)$$

and $F_1(\xi_1, \xi_2, \xi_3)$ - the basic third-order form factor [2, 3]

$$F(\xi_1, \xi_2, \xi_3) \equiv \left\langle e^{\Omega} \right\rangle_3 \equiv \int_{\alpha \geq 0} d^3\alpha \delta(1 - \alpha_1 - \alpha_2 - \alpha_3) \exp(\Omega), \quad (25)$$

$$\Omega = -\alpha_1\alpha_2\xi_3 - \alpha_2\alpha_3\xi_1 - \alpha_1\alpha_3\xi_2 \quad (26)$$

with the polynomial

$$\Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3. \quad (27)$$

The statement of the full set of form factors takes several pages and will be published elsewhere. A noncovariant perturbation theory first was formulated for a scalar field in Ref. [11] where $g_1$ and $G_1$ can be found.

The apparent way to verify (16) is to implement the functional trace operation (2). Unfortunately we cannot manage this in the given form factor representation (22)-(23) because of the presence of $\Delta$ in denominators. Only representation with help of $\alpha$-polynomials which naturally arises from the covariant perturbation theory and, in fact, serves as a starting point for the explicit representation [2] can be used for this purpose. In this representation form factors admit significantly more compact form

$$g_1 = \left\langle e^{-\alpha_1\alpha_2\xi} \right\rangle_2, \quad (28)$$
$$g_2 = \left\langle \left( \frac{\alpha_1^2}{12} \right) e^{-\alpha_1\alpha_2\xi} + \frac{e^{-\alpha_1\alpha_2\xi} - 1}{\xi} \right\rangle_2, \quad (29)$$
$$G_1 = \left\langle e^{\Omega} \right\rangle_3, \quad (30)$$
$$G_2 = \left\langle 2\alpha_1\alpha_2 e^{\Omega} \right\rangle_3. \quad (31)$$

The $\alpha$-representation (28-31) is not unique, i.e. some form factors $G_i$, that in fact vanish, can present in the heat kernel [4]. Due to this fact there is an additional quadratic structure linear in $R_{\mu\nu}$

$$R_1 R_2[12] = \nabla_{\nu} R_{1\mu}^{\nu} \nabla_{\mu} R_2$$

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absent in the final expression (16). All integrals over the whole spacetime should be discarded and all second order form factors reduced to first order ones by identities like

\[
\operatorname{tr} \int dx \frac{1}{2} \left( \frac{1}{2} \left( x \right) \mathcal{F} \left( \xi_1, \xi_2, \xi_3 \right) \right) = \frac{1}{2} \operatorname{tr} \int dx \frac{1}{2} \left( x \right) \mathcal{F} \left( \xi \right) \mathcal{R} + O[\mathcal{R}^3].
\]

We have verified that the form factor tables of (16) are in full agreement with ones of \( \operatorname{Tr} K(s) \) (17).

From the variational principle and (2) we can conclude that there exists a remarkable link between two neighboring orders in the curvature of the heat kernel trace, namely, each lower order is completely defined by the higher order

\[
\mathcal{K}_{n-1} = \operatorname{tr} \int dx \frac{\delta}{\delta P} \mathcal{K}_n,
\]

where

\[
\mathcal{K}_n = \operatorname{tr} \int dx \frac{1}{2} \sum_i F_i(\nabla_1, \ldots, \nabla_n) \mathcal{R}_1 \ldots \mathcal{R}_n(i).
\]

Another good check is short time expansions for the form factors of the heat kernel. They can be treated in either explicit form or the form of \( \alpha \)-representation. Of course, the expanded form factors are nonlocal but the local Schwinger-DeWitt coefficients can be restored using the identity [2, 3]

\[
\Box R^{\alpha\beta\mu\nu} = \frac{1}{2} \left( \nabla^\alpha \nabla^\beta R^{\mu\nu} + \nabla^\alpha \nabla^\mu R^{\beta\nu} - \nabla^\nu \nabla^\alpha R^{\mu\beta} - \nabla^\alpha \nabla^\nu R^{\mu\beta}
\right.
\]

\[
- \nabla^\mu \nabla^\beta R^{\alpha\nu} - \nabla^\beta \nabla^\mu R^{\alpha\nu} + \nabla^\nu \nabla^\beta R^{\mu\alpha} + \nabla^\beta \nabla^\nu R^{\mu\alpha}
\]

\[
+ R^{[\mu}_{\lambda} R^{\lambda\beta\nu]} + R^{[\alpha}_{\lambda} R^{\lambda\beta\nu]} - 4 R^{[\alpha}_{\lambda} R^{\mu\nu]} |_{\lambda} - R^{\alpha\beta}_{\sigma\lambda} R^{\mu\nu\sigma\lambda}. \tag{34}
\]

What we get finally takes the shape of the local Schwinger-DeWitt series [3, 4]

\[
\hat{K}(s) = \frac{g^{1/2}}{(4\pi s)^{\omega}} \sum_{n=0}^{\infty} s^n \hat{a}_n(x, x).
\]

Terms \( \hat{a}_n \) up to \( n = 3 \) are well-established [3, 4, 10, 12] but the fourth coefficient is of special interest. Now available results for \( \hat{a}_4 \) are to be further reduced [12] or restricted to scalar field case [11, 13] lacking gauge fields and matrix structure itself. Therefore even limited form (\( O[\mathcal{R}^3] \)) of \( \hat{a}_4 \) deserves to be reproduced completely here

\[
\hat{a}_4(x, x) = \frac{1}{840} \Box^3 \hat{P} + \frac{1}{15120} \Box^3 \hat{R}
\]

\[
+ \frac{1}{360} \Box (\hat{P} \Box \hat{P}) + \frac{1}{360} \Box \hat{P} \Box \hat{P} + \frac{1}{360} (\Box \hat{P} \hat{P})
\]

\[
+ \frac{1}{360} \Box^2 (\hat{P} \hat{P}) + \frac{1}{360} \Box^2 \hat{P} \hat{P} + \frac{1}{360} \hat{P} \Box^2 \hat{P}
\]

\[
+ \frac{1}{3360} \Box^2 R^{\mu\nu} \hat{R}_{\mu\nu} + \frac{1}{3360} R^{\mu\nu} \Box^2 \hat{R}_{\mu\nu} + \frac{1}{2520} \Box^2 \hat{R}^{\mu\nu} \hat{R}_{\mu\nu}
\]

\[
+ \frac{1}{1680} \Box (\Box \hat{R}^{\mu\nu} \hat{R}_{\mu\nu}) + \frac{1}{1120} \Box^2 (\hat{R}^{\mu\nu} \hat{R}_{\mu\nu}) + \frac{1}{1680} \Box (\hat{R}^{\mu\nu} \Box \hat{R}_{\mu\nu})
\]
+ \frac{1}{15120} \Box (\Box \hat{P} R) - \frac{1}{3024} \Box^2 \hat{P} R + \frac{1}{3780} \hat{P} \Box^2 R \\
+ \frac{1}{3780} \Box^2 (\hat{P} R) + \frac{1}{15120} \Box \hat{P} \Box R + \frac{1}{3780} \Box (\hat{P} \Box R) \\
+ \frac{1}{2520} \Box (\nabla_\mu \hat{R}^{\mu\nu} \nabla^\alpha \hat{R}_{\alpha\nu}) + \frac{1}{2520} \Box \nabla_\mu \hat{R}^{\mu\nu} \Box \nabla^\alpha \hat{R}_{\alpha\nu} + \frac{1}{2520} \Box \nabla_\mu \hat{R}^{\mu\nu} \nabla^\alpha \hat{R}_{\alpha\nu} \\
- \frac{1}{630} \Box [\nabla_\alpha \hat{R}, \nabla_\beta \hat{R}^{\alpha\beta}] - \frac{1}{1260} [\nabla_\alpha \hat{R}, \nabla_\beta \hat{R}^{\alpha\beta}] \\
+ \frac{1}{840} \Box \nabla_\mu \nabla_\nu \hat{P} R^{\mu\nu} + \frac{1}{2520} \Box \nabla_\mu \nabla_\nu \hat{P} \Box R^{\mu\nu} + \frac{1}{840} \Box (\nabla_\mu \nabla_\nu \hat{P} R^{\mu\nu}) \\
+ \left[ \frac{1}{12600} \Box^2 (R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta}) + \frac{1}{3150} \Box (R^{\mu\nu\alpha\beta} \nabla_\mu \nabla_\alpha R_{\nu\beta}) + \frac{1}{9450} R^{\mu\nu\alpha\beta} \Box \nabla_\mu \nabla_\alpha R_{\nu\beta} \\
+ \frac{1}{12600} \Box \nabla_\beta R_{\mu\nu} \nabla^\mu \nabla^\nu \nabla^\alpha \nabla^\beta - \frac{1}{6300} \Box (\nabla_\alpha R_{\mu\nu} \nabla^\mu R^{\alpha\nu}) + \frac{1}{18900} \nabla_\alpha R_{\mu\nu} \Box \nabla^\mu R^{\alpha\nu} \\
+ \frac{1}{15120} \Box (\nabla_\mu \nabla_\nu R R^{\mu\nu}) + \frac{1}{15120} \Box \nabla_\mu \nabla_\nu R R^{\mu\nu} + \frac{1}{75600} \Box \nabla_\nu R \Box R^{\mu\nu} \\
+ \frac{1}{50400} \Box^2 (R^{\mu\nu} R_{\mu\nu}) - \frac{1}{75600} \Box (R^{\mu\nu} \Box R_{\mu\nu}) + \frac{1}{37800} R^{\mu\nu} \Box^2 R_{\mu\nu} \\
- \frac{1}{15120} \Box R^{\mu\nu} \Box R_{\mu\nu} - \frac{1}{56700} \Box^2 R R + \frac{1}{453600} \Box (\Box R R) \\
- \frac{1}{453600} \Box \Box R R + \frac{1}{129600} \Box^2 (R R) \right] \hat{1} + O[\Re^3]. \tag{36} \]

We have checked that terms of (36) containing matter fields are equivalent to a reduced form of this coefficient found in \[12\]. The expression in the square brackets disagrees with \(a_4\) of Ref. \[13\] in several of its coefficients. It is straightforward to get from (36) the functional trace \(\int dx \hat{a}_4\) known from \[2, 3, 14\]. Even after implementation of the trace there remain differences with Ref. \[13\] in the quadratic terms.

Needless to say that from (33) a similar relationship for the Schwinger-DeWitt coefficients \(a_{4k}\) follows \[13\]

\[\hat{a}_{n-1}(x, x) = g^{-1/2} \frac{\delta}{\delta \hat{P}} \int dx g^{1/2} \text{tr} \hat{a}_n(x, x). \tag{37} \]

### 3 One-loop currents in lowest order in the curvature

Now it would be a simple task to proceed from heat kernels \(\Box\) to universal functional traces \(\Box\) with aid of the proper time equation \(\Box\). Of course, according to the generating function principle we could take as a generating expression the four dimensional one-loop effective action \(\Box\) which looks like \[2, 3\]

\[-W = \frac{1}{2(4\pi)^2} \int dx g^{1/2} \text{tr} \left\{ \sum_{i=1}^{\mu} \gamma_i (\Box_1 \Re_1 \Re_2(i) \\
+ \sum_{i=1}^{\nu} \Gamma_i (\Box_1, \Box_2, \Box_3) \Re_1 \Re_2 \Re_3(i) + O[\Re^4] \right\}. \tag{38}\]
where the second order and third order form factors generated from ones of (15) by the proper
time integration, and ultraviolet divergences within \( \gamma_i(-\Box) \) are extracted by the dimensional
regularization at \( \omega \to 2 \), while \( \Gamma_i(-\Box_1, -\Box_2, -\Box_3) \) are proved to be finite \([2, 3]\).

Either of these two ways gives the same result \([9]\)

\[
\hat{F}(\nabla) \frac{\delta(x, y)}{|y - x|} = \frac{1}{16\pi^2} g^{1/2} \left[ -\gamma(-\Box) \hat{P} + \frac{1}{18} R \hat{P} \right] + O[\Re^2],
\]

(39)

\[
\nabla_\alpha \frac{\hat{F}(\nabla)}{\nabla_\alpha \hat{F}(\nabla)} \frac{\delta(x, y)}{|y - x|} = \frac{1}{16\pi^2} g^{1/2} \left[ \frac{1}{6} \left( \gamma(-\Box) + \frac{2}{3} \nabla^\beta \hat{R}_{\beta\alpha} \right) - \frac{1}{2} \gamma(-\Box) \nabla_\alpha \hat{P} \\
+ \frac{1}{36} \nabla_\alpha R \hat{P} \right] + O[\Re^2],
\]

(40)

\[
\nabla_\alpha \nabla_\beta \frac{\hat{F}(\nabla)}{\nabla_\alpha \hat{F}(\nabla)} \frac{\delta(x, y)}{|y - x|} = \frac{1}{16\pi^2} g^{1/2} \left[ \frac{1}{12} \left( \gamma(-\Box) + \frac{2}{3} \right) g_{\alpha\beta} \Box \hat{P} \\
- \frac{1}{60} \left( \gamma(-\Box) + \frac{16}{15} \right) \Box R_{\alpha\beta} \hat{P} + \frac{1}{360} \left( \gamma(-\Box) + \frac{1}{15} \right) g_{\alpha\beta} \Box R \hat{P} \\
+ \frac{1}{6} \left( \gamma(-\Box) + \frac{2}{3} \right) \nabla_\alpha \nabla^\mu \hat{R}_{\mu\beta} - \frac{1}{3} \left( \gamma(-\Box) + \frac{1}{6} \right) \nabla_\alpha \nabla_\beta \hat{P} \\
+ \frac{1}{180} \left( \gamma(-\Box) + \frac{61}{15} \right) \nabla_\alpha \nabla_\beta R \hat{P} \right] + O[\Re^2].
\]

(41)

The basic form factor \( \gamma(-\Box) \) is derived from the basic form factor \( f(-\Box) \) (24) and possesses
divergent and finite nonlocal pieces

\[
\gamma(-\Box) = \frac{1}{2 - \omega} + \ln 4\pi + 2 + C - \ln(-\Box) + O[2 - \omega], \quad \omega \to 2,
\]

C is the Euler constant. The divergences of (39)–(41) coincide with ones obtained earlier \([7]\).

Let us note that the metric variation of the effective action itself is of great importance
since it is nothing but the expectation value of the energy-momentum tensor \([1, 4]\)

\[
\langle T_{\alpha\beta} \rangle \equiv 2g^{-1/2} \frac{\delta W}{\delta g_{\alpha\beta}}.
\]

Having (41) in explicit form gives us \( \langle T_{\alpha\beta} \rangle \) as well.

The spectral representation \( \gamma(-\Box) \) is of the form

\[
\gamma(-\Box) = \int_0^\infty \left( \frac{1}{m^2 - \Box} - \frac{1}{m^2 - \mu^2} \right),
\]

(42)

where the parameter \( \mu^2 > 0 \) describes the ultraviolet renormalization arbitrariness. These
one-loop Euclidean radiation currents should be analytically continued to the physical Lorentzian
spacetime \([1]\). In the Lorentzian spacetime the unique Euclidean Green function with zero
boundary conditions at infinity corresponds to two different setups having the in–out and
in–in (out–out) vacuum boundary conditions. The analytic continuation consists in substituting
\( 1/(m^2 - \Box) \) by the Feynman or retarded (advanced) Green functions \([1, 4]\). The
spectral representations for form factors of second order Green functions (read – third order
effective action) even in Euclidean spacetime is rather involved and given in full account in Ref. [2].

The universal functional traces (39)–(41) themselves are generating expressions for polarization operators (10) as seen from

\[ \delta \left( \frac{1}{F} \right) = -\frac{1}{F} \delta F \frac{1}{F}. \]

The curvature power is again reduced by one and resulting expressions are

\[ \frac{\delta_B^A}{F(\nabla)} \delta(x, y) \frac{\delta_D^C}{F(\nabla)} \delta(x, y) = \frac{g^{1/2}}{16\pi^2} \gamma(-\Box) \delta^A_B \delta^C_D \delta(x, y) + O[\Re], \]

\[ \nabla_{\alpha} \frac{\delta_B^A}{F(\nabla)} \delta(x, y) \frac{\delta_D^C}{F(\nabla)} \delta(x, y) = \frac{g^{1/2}}{2 \cdot 16\pi^2} \gamma(-\Box) \delta^A_B \frac{\delta_D^C}{F(\nabla)} \nabla_{\alpha} \delta(x, y) + O[\Re], \]

and three more are readily computed [9]

\[ \nabla_{\alpha} \nabla_{\beta} \frac{\delta_B^A}{F(\nabla)} \delta(x, y) \frac{\delta_D^C}{F(\nabla)} \delta(x, y), \quad \nabla_{\alpha} \frac{\delta_B^A}{F(\nabla)} \delta(x, y) \nabla_{\beta} \frac{\delta_D^C}{F(\nabla)} \delta(x, y), \]

\[ \nabla_{\alpha} \nabla_{\beta} \frac{\delta_B^A}{F(\nabla)} \delta(x, y) \nabla_{\mu} \frac{\delta_D^C}{F(\nabla)} \delta(x, y). \]

Again divergences of the expressions above were established in [7]. These and other polarization operators are made up of derivatives and basic form factors \( \gamma(-\Box) \) acting on delta functions.

At this point we can observe an elegant hierarchy

| trace of heat kernel | (integration over proper time) | one-loop effective action |
|----------------------|-------------------------------|---------------------------|
| ↓ (variation)        |                               | ↓ (variation)             |
| heat kernels         | →                             | universal functional traces |
| ↓                    |                               | ↓                         |
| two-point heat kernel loops | →          | polarization operators.   |

4 Conclusions

In this report we have aimed to outline the interplay among the whole class of quantum objects including one-loop effective action, universal functional traces, and polarization operators whose properties are entirely defined by the trace of the heat kernel derived from the covariant perturbation theory [2].

We have shown that the generating function method is simple and efficient while dealing with elements of two-loop graphs in quantum gravity and gauge theories. In this way some “tadpoles” and polarization operators have been obtained as well as heat kernels themselves. However to start we should be provided by the generating functions \( \text{Tr}K(s) \) and \( W \) with qualitatively higher accuracy of perturbation theory. Its present state is limited to third order in curvature and any further progress seems formidable and unnecessary in quantum gravity at least, though in pure gauge field theory next orders might be manageable and needed for measurements at hadron colliders such as one-loop two-quark \( n \)-gluon QCD amplitudes [13].
Another disadvantage of the method is its inability to produce Green function with more than two derivatives since the operator $F(\nabla)$ contains no more than two of them acting on the metric (1). But the ordinary covariant perturbation theory encompasses all possible cases of (1).

We have presented second order manipulations only for the heat kernel (undifferentiated Green function) but to reproduce just two-loop local divergences requires such knowledge for all the discussed radiation currents as well as for Green functions with mixed derivatives [4, 8]. Despite the fact that our program is far from being complete, we have indicated here its basic features.

Our concluding remark is that now the covariant perturbation theory approach and correspondingly all derivations above look even more promising in view of the fact that the same conventional perturbational rules were obtained for low-energy limit of string theory [10].

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