Accretion of the relativistic Vlasov gas onto a Bardeen regular black hole

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Abstract
We investigate the stationary spherically symmetric accretion of the collisionless Vlasov gas onto a Bardeen regular black hole. Compared with previous studies, we propose a model in which the total angular momentum of the gas particles is not uniformly distributed; instead, we assume a half-normal distribution. We find that the regularizing parameter in the Bardeen metric has no remarkable influence on the accretion of the gas, but the distribution of the total angular momentum can heavily suppress the mass accretion rate. This effect might be useful to understand the observed low luminosity of Sgr A* and other underluminous sources such as M87*.

Keywords Accretion · Vlasov gas · Regular black hole

1 Introduction

Black holes are the most fascinating objects in the universe. It is well known that black hole solutions in general relativity suffer from the so-called singularity problem, which tells us that the scalar curvature becomes divergent at the center of the black hole. Regular black holes are the solutions for which an event horizon is present, but the space-time is free from singularities. As a first model of a regular black hole, the Bardeen black hole (Bardeen 1968) is commonly considered as a solution of general relativity coupled to nonlinear electrodynamics (Ayon-Beato and Garcia 2000) or interpreted as a quantum-corrected Schwarzschild black hole (Maluf and Neves 2018). The properties of the Bardeen black hole have been studied extensively, such as gravitational lensing (Eiroa and Sendra 2011), quasi-normal modes (Fernando and Correa 2012), time-like and null geodesic structures (Zhou et al. 2012), the shadow they cast (Abdjabbarov et al. 2016), etc.

On the other hand, matter accretion is one of the most important processes in astrophysical systems, especially where black holes are involved, such as X-ray binaries, gamma-ray bursts, and active galactic nuclei. The theory of matter accretion onto compact objects can date back to pioneer works of Hoyle, Littleton, and Bondi (Hoyle and Lyttleton 1939; Bondi and Hoyle 1944). Bondi (Bondi 1952) studied the steady spherical accretion of a gas at rest at infinity onto a star in Newtonian dynamics. Later on, Michel generalized the work of Bondi and studied the perfect fluid accretion onto a Schwarzschild black hole in a relativistic setting (Michel 1972). Petrich et al. proposed an analytical solution for the steady state subsonic accretion of gas media onto Kerr black holes (Petrich et al. 1988). An extension to the accretion of the perfect ultrahard fluid on a moving Kerr–Newman black hole is given by Babichev et al. (2008). A fully general-relativistic modeling is necessary because the radiation due to accretion usually emanates from a region in the presence of strong gravitational field.

Numerous hydrodynamic accretion solutions, in which matter is described in the fluid approximation, have been obtained and accretion has been an extensively studied topic in the literature over the years (see, for example, (Petrich et al. 1989; Papadopoulos and Font 1998; Malec 1999; Mach and Malec 2013; Ganguly et al. 2014; Yang 2015; Cruz-Osorio et al. 2017; Rodrigues and Junior 2018; Abbas and Ditta 2018; Yang et al. 2019, 2021; Panotopoulos et al. 2021; Feng et al. 2022)). For a review with technical details, we refer to (Rezzolla and Zanotti 2013) and references therein.

However, the hydrodynamical approximation does not work for the accretion of low-collisional or collisionless matter, such as low-density and high-temperature plasma...
and dark matter near the supermassive black hole in the center of our galaxy, Sgr A* (Ghez et al. 2003; Gillessen et al. 2009; Falcke and Markoff 2013; Akiyama et al. 2022b,a) and M87* (Akiyama et al. 2019a,b). In these situations, to correctly describe the dynamics, one of the available methods is to consider the accreting flows in a kinetic way.

The kinetic theory approach to accretion problem is radically different from the technical point of view. For instance, the key point of analyzing the problem is not actual solving the Vlasov equation, but rather in analyzing the properties of accreting matter (gas) at infinity. Rioseco and Sarbach (2017a,b) systematically studied the accretion of a collisionless, relativistic kinetic gas (also known as Vlasov gas) into a Schwarzschild black hole. It is found that the tangential pressure at the event horizon is one order of magnitude larger than the radial pressure at low temperature and the mass accretion rate is lower than that of the Bondi–Michel model. Cieślik and Mach generalized the results of Rioseco and Sarbach to Reissner–Nordström black holes and found that the charge of the black hole would affect the mass accretion rate and particle current density (Cieślik and Mach 2020).

In the present work, we investigate the accretion of Vlasov gas onto a Bardeen regular black hole. The rest of the paper is organized as follows. We first give a brief review on the Hamiltonian description of the geodesic motion of a free particle coming from infinity on the Bardeen spacetime in Sect. 2. In Sect. 3, we introduce the distribution function of the Vlasov gas and the Vlasov function, derive expressions for the observables, and compute the particle current density and the mass accretion rate of the gas. Finally, conclusions and some remarks are given in Sect. 4.

Throughout the paper, we use geometric units with $c = G = 1$. The signature of the metric is assigned to be $(-, +, +, +)$.

## 2 The geodesic motion of a free particle from infinity on the Bardeen spacetime

The Bardeen spacetime can be described by the following line element:

$$ \text{ds}^2 = -(1 - \frac{2Mr^2}{(r^2 + r_0^2)^{3/2}}) \text{dr}^2 + r^2 \text{d} \Omega^2, $$

where $\text{d}\Omega^2 = \theta^2 + \sin^2 \theta \text{d} \phi^2$ denotes the line element of the two-dimensional unit sphere, $M$ may be interpreted as the ADM mass of the black hole, and $r_0$ is a parameter related to the magnetic charge of the nonlinear self-gravitating monopole (Ayon-Beato and Garcia 2000) or a length parameter proportional to the Planck length (Maluf and Neves 2018). For $r_0^2/2 \lesssim M^2$, the metric describes a black hole spacetime without curvature singularity.

To deal with the accretion problem, the horizon-penetrating Eddington–Finkelstein coordinates are usually introduced, in which the metric of the spacetime reads

$$ \text{ds}^2 = -(1 - \frac{2Mr^2}{(r_0^2 + r^2)^{3/2}}) \text{dr}^2 + \frac{4Mr^2}{(r_0^2 + r^2)^{3/2}} \text{dr} \text{d}r + \left(1 + \frac{2Mr^2}{(r_0^2 + r^2)^{3/2}}\right) \text{dr}^2 + r^2 \text{d} \Omega^2, $$

where the new coordinates $t$ and $r$ are defined by the transformations

$$ t = \tilde{t} + \int\tilde{r} \left[\frac{(r_0^2 + r^2)^{3/2}}{(r_0^2 + r^2)^{3/2} - 2Mr^2} - 1\right] \text{d}r, $$

$$ r = \tilde{r}. $$

It is convenient to investigate geodesic motion of particles in the Hamiltonian formulation. The Hamiltonian of a free particle with mass $m$ traveling along a timelike geodesics reads

$$ \mathcal{H}(x^\mu, p_\nu) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu = -\frac{1}{2} m^2 $$

with the components $g^{\mu\nu}(x)$ of the inverse metric at event $x$ and $p^\mu = dx^\mu/ds$. Because the Bardeen spacetime is static and spherically symmetrical, besides the Hamiltonian $\mathcal{H}$ itself (or, equivalently, $m = \sqrt{-2\mathcal{H}}$), there exist another three constants of motion for the particle: $E \equiv -p_t$, $L_\varphi \equiv p_\varphi$, and

$$ L \equiv \sqrt{p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta}}, $$

where $p_t$, $p_\theta$, and $p_\varphi$ are the corresponding components of canonical momenta of coordinates $t$, $\theta$, and $\varphi$, respectively. In terms of the above constants, we obtain from Eq. (5) that

$$ g^{\nu\mu} L^2 - 2g^{\nu\nu} E p_\nu + g^{\nu\mu} p_\mu + \frac{L^2}{r^2} + m^2 = 0. $$
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where

\[ g'' = -1 - \frac{2Mr^2}{(r^2 + r_0^2)^3/2}, \quad \text{(8)} \]

\[ g'^r = \frac{2Mr^2}{(r^2 + r_0^2)^3/2}, \quad \text{(9)} \]

\[ g''r = 1 - \frac{2Mr^2}{(r^2 + r_0^2)^3/2}. \quad \text{(10)} \]

By solving Eq. (7) with respect to the radial momentum \( p_r \), we can find that

\[ p_r^{(\pm)} = \frac{2Mr^2E \pm (r^2 + r_0^2)^{3/2}\sqrt{E^2 - V(r)}}{(r^2 + r_0^2)^3/2 - 2Mr^2}, \quad \text{(11)} \]

where the effective potential \( V(r) \) is defined as

\[ V(r) = \left(1 - \frac{2Mr^2}{(r^2 + r_0^2)^3/2}\right)\left(m^2 + \frac{L^2}{r^2}\right), \quad \text{(12)} \]

and the signs \( \pm \) correspond to the directions of motion along a geodesic. Clearly, \( V(r) \to m^2 \) as \( r \to \infty \). Consequently, only trajectories with \( E \geq m \) can reach infinity. We are interested in particle trajectories that originate from infinity and go toward the black hole. There are two possible endings for a free particle coming from infinity: one is that it is reflected back to infinity if it has high enough angular momentum; the other is that it is absorbed into the black hole.

The abbreviated action of the motion of a free particle along a geodesic \( \gamma \) parameterized by constants \( E, L_z, L, \) and \( m \) can be defined as

\[ S = \int_{\gamma} p_\mu dx^\mu = -Et + L_z \varphi + \int p_r dr + \int p_\theta d\theta, \quad \text{(13)} \]

which can be used as a generating function for the canonical transformation from \((x^\mu, p_\mu)\) to the so-called action-angle variables \((Q^\mu, P_\mu)\). The new momenta \( P_\mu \) are defined as \( P_0 = m, P_1 = E, P_2 = L_z, \) and \( P_3 = L, \) and the corresponding conjugate variables are \( Q^0 = \partial S/\partial m, Q^1 = \partial S/\partial E, Q^2 = \partial S/\partial L_z, \) and \( Q^3 = \partial S/\partial L. \) The main advantage of the new coordinates \((Q^\mu, P_\mu)\) is that they trivialize the Vlasov equation in the next sections.

Furthermore, it is also convenient to introduce dimensionless variables as follows (Rioseco and Sarbach 2017a):

\[ \xi = M^{-1}r, \quad \xi_0 = m^{-1}p_r, \quad \epsilon = m^{-1}E, \quad \lambda = M^{-1}m^{-1}L_z, \quad \xi \approx M^{-1}m^{-1}L_z, \quad a = M^{-1}r_0. \]

Then the dimensionless counterpart of the effective potential reads

\[ U(\lambda, a; \xi) = \left(1 - \frac{2\xi^2}{(\xi^2 + a^2)^{3/2}}\right)\left(1 + \frac{\lambda^2}{\xi^2}\right). \quad \text{(14)} \]

We can introduce a critical angular momentum \( \lambda_c(\epsilon) \) determined by the condition that the peak of the effective potential is equal to \( \epsilon^2 \), i.e.,

\[ \lambda_c(\epsilon) = \sqrt{\frac{\xi_m^6 - 2a^2\xi_m^4}{(a^2 + \xi_m^2)^{3/2} - 3\xi_m^4}}, \quad \text{(15)} \]

where \( \xi_m \) denotes the location of the peak of the effective potential and is the smaller positive real solution of the equation

\[ \xi_m^{12} + \left(6\epsilon^2 - 9\epsilon - 1\right)\xi_m^{10} + \left(15\epsilon^4 - 9\epsilon^2 - 6\epsilon + 16\right)\xi_m^8 + 24\epsilon^4 - 6(\epsilon^2 - 1)^2\xi_m^6 + 6\epsilon^2 a^{12} = 0. \quad \text{(16)} \]

The absorbed particles traveling from infinity are characterized by \( \lambda < \lambda_c(\epsilon) \) and \( \epsilon \geq 1 \), whereas the scattered particles occupy the region \( \lambda_c(\epsilon) < \lambda < \lambda_m(\epsilon, \xi) \) and \( \epsilon > \xi_m \) in the \((\epsilon, \lambda)\) phase space, where the maximal allowed angular momentum

\[ \lambda_m(\epsilon, \xi) = \xi \sqrt{\frac{(a^2 + \xi^2)^{3/2} - 2\xi^2}{(a^2 + \xi^2)^{3/2} - 2\xi^2} - 1}, \]

and the minimal allowed energy for scattered orbits

\[ \xi_{\text{min}} = \begin{cases} \sqrt{\frac{\xi_m^6 - 2a^2\xi_m^4}{(a^2 + \xi_m^2)^{3/2} - 3\xi_m^4}}, & \xi_{\text{ph}} < \xi < \xi_{\text{mb}}, \\ 1, & \xi \geq \xi_{\text{mb}}. \end{cases} \quad \text{(17)} \]

Here \( \xi_{\text{ph}} \) and \( \xi_{\text{mb}} \) are the dimensionless radii of the circular photon orbit (which can be considered as a limit of time-like orbits) and the marginally bound orbit \((U(\lambda, a; \xi_{\text{mb}}) = 1)\), respectively.

### 3 Vlasov gas in the Bardeen spacetime

A relativistic, collisionless kinetic gas in a curved spacetime can be locally described by the so-called one-particle distribution function \( f = f(x, p) \), where \( x \) is a spacetime event and \( p \) is the 4-momentum of the particle at \( x \). Using the distribution function, we can express many important
obtainable quantities as suitable integrals over momenta. For example, the particle current density can be expressed as

\[ J_\mu(x) = \int p_\mu f(x, p) d\text{vol}_x(p), \]  

(18)

where

\[ d\text{vol}_x(p) = \sqrt{-g} dp_0 dp_1 dp_2 dp_3 \]

(19)

with \( g \) the determinant of the contravariant metric of the spacetime. The corresponding invariant particle number density and mean four-velocity at \( x \) are then given by

\[ n(x) = \sqrt{-J_\mu(x) J^\mu(x)} \quad \text{and} \quad \bar{u}^\mu(x) = J^\mu(x)/n(x). \]

The distribution function \( f \) should be a solution to the Vlasov equation

\[ \frac{\partial H}{\partial p_\mu} \frac{\partial f}{\partial x^\mu} - \frac{\partial H}{\partial x^\mu} \frac{\partial f}{\partial p_\mu} = 0. \]

(20)

Note that the Vlasov equation (20) means that the distribution function \( f \) should remain unchanged along a geodesic. It is found that any distribution function, which is only a function of integrals of motion, satisfies Eq. (20) (Rioseco and Sarbach 2017a).

In view of the symmetry of the Bardeen spacetime, we restrict ourselves to the stationary and spherically symmetric solutions of the Vlasov equation (20), in which the distribution function of the gas takes the following form:

\[ f(x^\mu, p_\nu) = F(P_0, P_1, P_3) = F(m, \epsilon, M\lambda) = \mathcal{F}(m, \epsilon, \lambda). \]

(21)

For detailed discussions, we refer to (Rioseco and Sarbach 2017a; Cieslik and Mach 2020).

To obtain a solution describing the accretion onto a black hole by using the formalism presented above, we next need to choose an appropriate distribution function \( f \). In this work, without loss of generality, we assume that

\[ \mathcal{F}(m, \epsilon, \lambda) = \alpha\delta(m - m_\alpha)e^{-\beta^I\lambda}h(\lambda), \]

(22)

where the mass of the gas particles is specified to be a given value \( m_\alpha \), \( \alpha \) is the normalization constant related to the number density of particles at infinity, and the parameter \( \beta \) is determined by the thermodynamic properties of the gas. Here \( h(\lambda) \) is the distribution function on the total angular momentum of the gas particle \( \lambda \).

A model often considered in the literature is \( h(\lambda) = 1 \), which corresponds to the so-called Maxwell–Jüttner distribution (Israel 1963), if \( \beta \) is identified to be \( m_\alpha/(k_B T) \), where \( k_B \) is the Boltzmann constant. In the flat spacetime, this model describes a relativistic simple gas in thermal equilibrium with temperature \( T \).

Undeniably, it is too ideal to expect that the total angular momenta \( \lambda \) are uniformly distributed. To be a little more realistic, it is natural to assume that the total angular momenta of most gas particles are relatively small and only very few particles have a total angular momentum larger than a constant \( \lambda_*$ Then, more precisely, we can assume the half-normal probability distribution

\[ h(\lambda) = \exp\left(-\frac{\lambda^2}{\lambda_*^2}\right). \]

(23)

Clearly, this model goes back to the Maxwell–Jüttner distribution in the limit as \( \lambda_* \to \infty \).

### 3.1 Particle current density

The integrals over momentum in Eq. (19) can be computed conveniently with the help of new momentum coordinates \((\epsilon, m, \lambda, \chi)\), where \( \chi \) is defined by

\[ \pi_\theta = \lambda \cos \chi, \quad \zeta = \lambda \sin \theta \sin \chi. \]

(24)

Then the particle current density can be written as

\[ J_\mu(\xi) = \int p_\mu \mathcal{F}(m, \epsilon, \lambda)m^3 \lambda \xi^2 \sqrt{\epsilon^2 - U(\lambda, a; \xi)} d\epsilon dm d\lambda d\chi. \]

(25)

The components of \( J_\mu \) are computed by separating them into two parts: \( J_\mu = J_\mu^{\text{abs}} + J_\mu^{\text{scat}} \), where \( J_\mu^{\text{abs}} \) and \( J_\mu^{\text{scat}} \) denote the current densities of the absorbed and scattered particles and can be expressed as

\[ J_\mu^{\text{abs}} = \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon)} \int_0^\infty d\lambda \int_0^\infty d\epsilon \int_0^{\lambda_\text{max}(\epsilon)} m^3 \lambda d\lambda \int_0^{\lambda_\text{max}(\epsilon)} m^3 \lambda d\lambda \int_0^{\lambda_\text{max}(\epsilon)} m^3 \lambda d\lambda \int_0^{\lambda_\text{max}(\epsilon)} m^3 \lambda d\lambda \]

(26)

and

\[ J_\mu^{\text{scat}} = \sum_{\pm} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} m^3 \lambda d\lambda \int_0^\infty d\epsilon \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} m^3 \lambda d\lambda \int_0^\infty d\epsilon \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} \int_{\lambda_\text{min}}^{\lambda_\text{max}(\epsilon, \xi)} m^3 \lambda d\lambda \]

(27)

respectively. Note that for \( J_\mu^{\text{abs}} \) in (26), we only consider the incoming particles, since at infinity only incoming particles can be absorbed.

\[ \text{Note that the angular momentum is a vector and } \lambda \text{ here actually stands for the absolute value of the angular momentum. In the absence of additional information, it is reasonable to assume that the total angular momenta of the gas particles are normally distributed with mean zero in each direction; therefore, as the absolute value of the angular momentum, a half-normal distributed } \lambda \text{ is more physically natural than an exponentially decaying one, although the latter is simpler mathematically.} \]
Fig. 1 The nonvanishing components $J_t$ (left panels) and $J_r$ (right panels) of the current density $J_\mu$ of particles in the limit as $\lambda_\ast \to \infty$. From top to bottom, the dimensionless Bardeen parameter $a = 0, 0.2, 0.5,$ and the maximum possible value 0.76, respectively. In all panels, we set $\beta = 1$, and the gray lines mark the locations of the event horizon and the photon sphere of the black hole.

With the distribution function (22) together with Eq. (23) in hand, the current densities of the absorbed and scattered particles can be respectively rewritten as

$$J_{\mu}^{(\text{abs})} = \frac{2\pi \alpha m_\ast^4}{\xi^2} \int_1^\infty d\varepsilon e^{-\varepsilon \varepsilon_\ast} \times \int_0^{\lambda_\ast (\varepsilon)} d\lambda \frac{\lambda \exp(-\lambda^2/\lambda_\ast^2)}{\sqrt{\varepsilon^2 - U(\lambda, a; \xi)}}$$

and

$$J_{\mu}^{(\text{scat})} = \frac{2\pi \alpha m_\ast^4}{\xi^2} \int_{\varepsilon_{\text{min}}}^{\infty} d\varepsilon e^{-\varepsilon \varepsilon_\ast} \times \int_{\lambda_{\ast (\varepsilon)}}^{\lambda_{\ast (\varepsilon)}} d\lambda \frac{\lambda \exp(-\lambda^2/\lambda_\ast^2)}{\sqrt{\varepsilon^2 - U(\lambda, a; \xi)}} \times \left\{ -\varepsilon, \pi_{\xi}^\ast, 0, 0 \right\}$$

(28)
Fig. 2 The nonvanishing components \( J_t \) (left panels) and \( J_r \) (right panels) of the current density \( J_\mu \) of particles. From top to bottom, the absorbed, scattered, and total currents are plotted in turn. In each panel, the dimensionless Bardeen parameter \( a = 0.5 \), and the distribution parameters \( \beta = 1 \), \( \lambda_\ast = 30, 50, \) and \( \infty \), respectively. The gray lines mark the locations of the event horizon and the photon sphere of the black hole.

\[
\times \left\{ -2\epsilon, \sum_{\pm} \pi_\xi^{(\pm)}, 0, 0 \right\},
\]

where

\[
\pi_\xi^{(\pm)} = \frac{2\xi^2 \pm (\xi^2 + a^2)^{3/2} \sqrt{\epsilon^2 - U(\lambda, a; \xi)}}{(\xi^2 + a^2)^{3/2} - 2\xi^2}.
\]

In Fig. 1 the nonvanishing components \( J_t \) and \( J_r \) of the current density \( J_\mu \) of particles are illustrated for different values of dimensionless Bardeen parameter \( a \) in the limit as \( \lambda_\ast \rightarrow \infty \). From top panels to bottom, the dimensionless Bardeen parameter \( a \) are chosen to be 0, 0.2, 0.5, and the maximum possible value 0.76, respectively. The locations of the black-hole horizon and the photon sphere are marked with vertical lines. Note that the total currents are smooth, but both the contributions corresponding to absorbed and scattered particles are not smooth at the photon sphere. Below the photon sphere, the contributions corresponding to the scattered particles vanish. It is found that for both \( J_t \) and \( J_r \), with the increasing Bardeen parameter \( a \), the absorbed and total particle current densities increase significantly, especially in the region near the photon sphere of the black hole. However, the scattered currents are little changed, except that the maximum of \( J^{(\text{scat})} \) becomes getting larger. In Fig. 2, the components \( J_t \) and \( J_r \) (right panels) of the particles current density \( J_\mu \). From top to bottom, the absorbed, scattered, and total currents are plotted in turn. In each panel the dimensionless Bardeen parameter \( a = 0.5 \), and the distribution parameters \( \beta = 1 \), \( \lambda_\ast = 30, 50, \) and \( \infty \), respectively. Clearly, outside the photon sphere, the finite \( \lambda_\ast \) can dramatically suppress the amplitude of the scattered particle currents but has less effect on the absorbed ones. Consequently, both total particle currents \( J_t \) and \( J_r \) are decreased remark-
ably. However, inside the photon sphere, there is little effect of $\lambda_*$ on all the currents.

### 3.2 Mass accretion rate

Another key observable is the mass accretion rate through a sphere of a given radius $r = M\xi$. From the Vlasov equation Eq. (20) it is shown that

$$\nabla_\mu J^\mu = 0.$$  \hspace{1cm} (31)

In other words, for a distribution function $f$ satisfying the Vlasov equation, the conservation of the gas is automatically guaranteed (Cercignani and Kremer 2002). Therefore the conserved mass accretion rate of the spherical accretion can be defined by

$$\dot{M} = -m \int_0^{2\pi} d\phi \int_0^{\pi} d\theta r^2 \sin \theta J^\gamma,$$  \hspace{1cm} (32)

where $J^\gamma = g^{\gamma\rho} J^\rho + g^{\gamma\nu} J^\nu$. For model (22) with distribution (23), it is straightforward to obtain that

$$\dot{M} = 4\pi^2 m_\infty^2 M^2 \alpha \int_1^\infty e^{-\beta\varepsilon} \left(1 - \exp \left[-\frac{\lambda_\ast(e)^2}{\lambda_\ast^2}\right]\right) \lambda_\ast^2 d\varepsilon.$$  \hspace{1cm} (33)

We plot the mass accretion rate for different values of the Bardeen parameter $a$ and distribution parameter $\lambda_*$ in Fig. 3 for three different values of the asymptotic temperature corresponding to $\beta = 1, 5, \text{ and } 10$, respectively. It is easy to find that the finite value of $\lambda_*$ can remarkably reduce the mass accretion rate $\dot{M}$ of the Vlasov gas, and the lower the value of $\lambda_*$, the lower the mass accretion rate. Compared with that of $\lambda_*$, the influence of the Bardeen rate $a$ on the mass accretion rate is small. In the case that $\lambda_*$ is small, the effect of $a$ can be almost negligible. From panels in Fig. 3 we can see that this result does not depend on the variation of temperature.
4 Conclusions and discussions

In this work, we present a stationary, spherically symmetric accretion model of the relativistic Vlasov gas onto a Bardeen regular black hole. Unlike the models commonly used in the literature, the total angular momentum of the gas particles is assumed to take a half-normal distribution in this model. We find that, compared with the parameter in the distribution function of the total angular momentum $\lambda_s$, the regularizing parameter $a$ in the Bardeen metric has less influence on the accretion of the gas. Regularizing the singularity of the curvature of the black hole spacetime can only slightly change the mass accretion rate of the gas. In striking contrast, the nonuniform distribution of the total angular momentum can heavily suppress the mass accretion rate. This characteristic is little dependent on the asymptotic temperature.

A long-standing problem in the accretion physics is that the luminosity of Sgr A* and M87* measured via X-ray observations (Baganoff et al. 2003; Di Matteo et al. 2003) is far below that expected from the Eddington luminosity. Many solutions have been proposed in the past years (see Yuan and Narayan 2014) for a comprehensive review). One of the solutions is that the mass accretion rate is much lower than the Bondi rate, and this is supported by the measurement of the mean rotation measure (Bower et al. 2018) and observations by the Event Horizon Telescope (Akiyama et al. 2019a,b). Therefore the kinetic gas model proposed here may help us understand the above low-luminosity problem. Gamboa et al. (2021) argued that a kind of kinetic gas models that accretion from a finite region with a uniformly distributed angular momentum onto a black hole can also lead to a lower mass accretion rate. Hence, it is interesting to extend our model to the gas accreted from a finite region and other more realistic scenarios, such as those with collisionality, nonspherical symmetry, or black hole rotation.

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Author contributions All authors contributed to the study conception, design and calculation. Dao-Jun Liu wrote the main manuscript text. Jiawei Liao produced the figures. Both authors participated in the discussion and read the final manuscript.

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