A SEMISIMPLE SERIES FOR $q$-WEYL AND $q$-SPECHT MODULES

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Abstract. In [41], the authors studied the radical filtration of a Weyl module $\Delta_\zeta(\lambda)$ for quantum enveloping algebras $U_\zeta(e\mathfrak{g})$ associated to a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. There $\zeta^2 = \sqrt{T}$ and $\lambda$ was, initially, required to be $e$-regular. Some additional restrictions on $e$ were required—e. g., $e > h$, the Coxeter number, and $e$ odd. Translation to a facet gave an explicit semisimple series for all quantum Weyl modules with singular, as well as regular, weights. That is, the sections of the filtration are explicit semisimple modules with computable multiplicities of irreducible constituents. However, in the singular case, the filtration conceivably might not be the radical filtration. This paper shows how a similar semisimple series result can be obtained for all positive integers $e$ in case $e\mathfrak{g}$ has type $A$, and for all positive integers $e \geq 3$ in type $D$. One application describes semisimple series (with computable multiplicities) on $q$-Specht modules. We also discuss an analogue for Weyl modules for classical Schur algebras and Specht modules for symmetric group algebras in positive characteristic $p$. Here we assume the James Conjecture and a version of the Bipartite Conjecture.

1. Introduction

In the modular representation theory of a reductive group $G$ (or a quantum enveloping algebra $U_\zeta(e\mathfrak{g})$, with $\zeta$ a primitive $e$th root of 1), the general failure of complete reducibility has given rise, in the past 40 years, to a rich cohomology theory for both $G$ and $U_\zeta(e\mathfrak{g})$. See [27] for a compilation of many results. The related question of better understanding important filtrations of certain modules, e. g., Weyl modules, also has attracted considerable attention. See, for example, [28, pp. 445, 455], [1, §8], [43], [41] on filtrations with semisimple sections as well as [29, §3], [17, 2], [15, §6], and [42] for the somewhat analogous $p$-filtrations.

Interesting filtrations can take many forms, but a basic filtration for any finite dimensional module $M$ is its radical filtration $M \supseteq \text{rad } M \supseteq \text{rad}^2 M \supseteq \cdots$. In this case, the sections $\text{rad}^i M / \text{rad}^{i+1} M$ are, of course, semisimple (i. e., completely reducible), so that $(\text{rad}^i M)$ is an example of a “semisimple series,” mentioned in the title of this paper. In recent work, the authors [41] succeeded in calculating the multiplicities of the irreducible constituents for the radical sections in the quantum Weyl modules associated to regular weights. It was required that $e > h$, the Coxeter number of $\mathfrak{g}$. In addition, $e$ was required to be odd (and there were some other mild conditions on $e$, depending on the root system). For such “large” $e$, we also could describe the sections in a semisimple series for quantum Weyl modules with singular highest weights (but we were unable to show the series was the radical series, though this seems likely to be the usual case). Our methods also were applicable for Weyl modules in sufficiently large positive characteristics having highest weights in the Janzten region.

This paper completes part of this project by giving, for types $A$ and $D$, an explicit semisimple series for quantum Weyl modules for all positive integers $e$, except that in type $D_{2m+1}$ it is required $e \geq 3$. Explicit formulas for the multiplicities of the irreducible modules for each semisimple section are also obtained. In particular, in type $A$, our previous results are extended to all small $e$ and all even $e$. Interestingly, these previous results, given in [41] for $e$ odd and $>h$, play a key role here in obtaining the results for $e$ even and/or small. Extensions of these results to other types would be possible provided there were improvements in the Kazhdan-Lusztig correspondence as quoted in [47, p. 273]. This paper is organized so as to make such extensions easy to obtain once such improvements are known.

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The method involves passing, for suitably large $e$, to an equivalent category of modules for the (untwisted) affine Lie algebra $\mathfrak{g}$ attached to $\hat{\mathfrak{g}}$. Category equivalences at the affine Lie algebra level provides the flexibility to treat small/even values of $e$ (and then pass back to the quantum case, using the work [34] of Kazhdan-Lusztig and our own results [41]). Our approach is non-trivial and takes up §§3–7. It requires the interaction of several highest weight categories of Lie algebra modules (some of them new) and exact functors between them. In particular, we treat (various versions of) categories of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$-modules which are integrable in the direction of $\hat{\mathfrak{g}}$, and we also study their associated standard and costandard modules. Section 7 contains several contributions to further understanding these categories; see, for instance, Theorem 7.3 which both mirrors and uses the filtration results of [41] Thms. 8.4, Cor. 8.5], and whose proof requires the combinatorial equivalences obtained by Feibig [23 Thm. 11]. All of this work is done when $\mathfrak{g}$ is an arbitrary complex semisimple Lie algebra. Much of what we need for the quantum case (in particular, the entire odd case) could be done by working with the translation functor theory we provide, which gives many categorial equivalences without the need to construct inverses at a Verma flag level, as in [22], or to construct explicit combinatorial deformations, as in [23]. However, the latter theory of Feibig is theoretically very satisfying and has many additional practical advantages. In particular, it allows us, in our quantum situation, to deal with the $e$ even case.

One huge advantage of our extension of the results of [41] to small $e$ is that the results can be used to obtain, in type $A$ and working with the $q$-Schur algebras $S_q(n, r)$ with $q = q^2$, semisimple series and multiplicity formulas for the Specht modules of the Hecke algebras $H_q(r)$. Small $e$ results are required because the contravariant Schur functor from $S_q(n, r)$-mod $\to$ mod-$H_q(r)$, taking Weyl modules to Specht modules, is only exact when $r \leq n$. On the other hand, the treatment of meaningful cases (i.e., $H_q(r)$ not semisimple) requires $e \leq r$, so that $e \leq n = h$. Thus, $e$ is “small” in the sense of this paper. (Also, except when $e = r = h$, we have $e < h$, and all weights are singular.)

Another application is to Weyl modules for classical Schur algebras $S(n, r)$ in characteristic $p > 0$. The weights $\lambda$ are required to be viewed as partitions of a positive integer $r$ satisfying $r < p^2$. Also, we assume (the defining characteristic version of) the James conjecture [20] and a Schur algebra version of the Bipartite Conjecture [41]; see §8.3. With these assumptions, we show that both Weyl modules and corresponding Specht modules have explicit semisimple series, with multiplicities of irreducible modules explicitly given in terms of inverse Kazhdan-Lusztig polynomials.

Returning to the quantum case, there is an interesting overlap, in type $A$ with $e \neq 3$, between our results and methods and those of Peng Shan [35]. Her focus is on the Jantzen filtration and ours is on a semisimple series. In the case of regular weights, the sections of the Jantzen filtration are semisimple; in fact, Theorem 7.3(a), together with the multiplicities given in [45], imply, in the regular weight case, that the Jantzen filtration is the radical filtration. Semisimplicity of the sections of the Jantzen filtration remains unknown for singular weights. However, semisimplicity is likely, since the section multiplicities in [45] agree with those for the semisimple series studied in this paper.

In §9, Appendix I, we provide (apparently new) equivalences in the affine case between $\uparrow$-style orders [27], [31], and the Bruhat-Chevalley order. The proofs in this section are all combinatorial. The results are used in our proofs here, and Theorem 9.6 has also been used in [23] to complete an argument in [4], relevant to the Koszulity of some of the algebras $A$ we consider in the regular case. See footnote 7. In Theorem 7.3 for example, we prove only that $\text{gr} A_\Gamma$ is Koszul, not the stronger property that $A_\Gamma$ is Koszul. Although the Koszulity of $\text{gr} A_\Gamma$ is all that is needed in the semisimple series results in this paper, it is still interesting to know about the Koszulity of $A_\Gamma$, as argued in footnote 7. For a (non-Lie theoretic) example when $\text{gr} A$ is Koszul, but $A$ is not Koszul, see [11].

2. Notation: Lie algebras

The following notation is standard, mostly following [30], [32], and [33] with cosmetic differences. (For example, our root system is denoted $\Phi$ rather than $\Delta$. The classical finite root system is denoted $\hat{\Phi}$, and the maximal long and short roots of $\Phi$ are denoted $\theta_l$ and $\theta_s$, respectively.) If $V$ is a complex vector space
and $V^*$ is its dual, the natural pairing $V^* \times V \to \mathbb{C}$ between $V^*$ and $V$ is usually denoted $\langle \phi, v \rangle = \phi(v)$. Unexplained notation is very standard.

Finite notation:

1. $\mathfrak{g}$ is finite dimensional, complex, simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$, Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$.
2. $\Phi, \Pi = \{\alpha_1, \cdots, \alpha_r\}, \Phi^+: roots of $\mathfrak{h}$ in $\mathfrak{g}$, simple roots determined by $\mathfrak{b}$, positive roots determined by $\Pi$. These are subsets of $\mathfrak{h}^*$, which is identified with $\mathfrak{h}$ using the restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{h}$ normalized so that the induced form on $\mathfrak{h}^*$ satisfies $\langle \theta_i, \theta_i \rangle = 2$ if $\theta_i$ is the maximal root in $\Phi$.
3. $\hat{\Phi} = \{\alpha^\vee | \alpha \in \Phi\}$: coroot system of $\mathfrak{h}$, identifies with $\{\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle | \alpha \in \Phi\}$.
4. $\hat{W} = \langle s_{\alpha_2}, \cdots, s_{\alpha_r} \rangle$: Weyl group of $\mathfrak{g}$, generated by fundamental reflections $s_{\alpha_i}: \mathbb{E} \to \mathbb{E}$, where $\mathbb{E} = \mathfrak{h}^*_\mathbb{R}$ is the Euclidean space associated to the Killing form on $\mathfrak{h}$.
5. $\varpi_1, \cdots, \varpi_r$: fundamental dominant weights; thus, $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}, 1 \leq i, j \leq r$.
6. $\hat{P}, \hat{P}^+$: the weight lattice $\bigoplus_{i=1}^{r} \mathbb{Z} \varpi_i$, set $\bigoplus_{i=1}^{r} \mathbb{N} \varpi_i$ of dominant weights.
7. $\hat{h}, \hat{g}$: Coxeter and dual Coxeter numbers; thus $\hat{h} - 1 = \langle \hat{\rho}, \vartheta_i^\vee \rangle$ and $\hat{g} - 1 = \langle \hat{\rho}, \theta_i^\vee \rangle, i = 1, \cdots, r$.

Affine notation:

1. $\mathfrak{g} := (\mathbb{C}t, t^{-1}) \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$: affine Lie algebra attached to $\mathfrak{g}$, with central element $c$.
2. $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the “Cartan subalgebra” of $\mathfrak{g}$. Following [17], p. 268, consider $\chi, \delta \in \mathfrak{h}^*$ defined by

$$\begin{align*}
\chi(h) &= \delta(h) = 0 \\
\chi(c) &= \delta(d) = 1 \\
\chi(d) &= \delta(c) = 0.
\end{align*}$$

Thus,

$$\mathfrak{h}^* := \mathfrak{h} \oplus \mathbb{C}\chi \oplus \mathbb{C}\delta.$$

Here $\mathfrak{h}^*$ identifies with a subspace of $\mathfrak{h}^*$ by making it vanish on $d$ and $c$.

3. $\Phi^{im} := \{n\delta | 0 \neq n \in \mathbb{Z}\}$, the imaginary roots.
4. $\Phi^{re} := \{j\delta + \alpha, j \in \mathbb{Z}, \alpha \in \hat{\Phi}\}$, the real roots.
5. $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi^{re}\}$, affine roots.
6. $\Phi = \Phi^+ \cup \Phi^{im}$, the root system of $\mathfrak{g}$.
7. $\Phi^+ = \{j\delta + \alpha | j \in \mathbb{Z}^+, \alpha \in \hat{\Phi} \} \cup \hat{\Phi}^+$: positive roots. If $\alpha \in \hat{\Phi}$, $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the $\alpha$-root space. The algebra $\mathfrak{g}$ has Borel subalgebra $\mathfrak{b} := \langle \mathfrak{h}, \mathfrak{g}_\alpha, \alpha \in \Phi^+ \rangle$. Let $\alpha_0 = \delta - \theta_i$, so that $\Pi := \{\alpha_0, \alpha_1, \cdots, \alpha_r\}$ is the set of simple roots for $\mathfrak{g}$. Then $\alpha_0^\vee = c - \theta_i^\vee$. Put $\rho = \hat{\rho} + g\chi$. $Q = \mathbb{Z}\alpha_0 + \cdots + \mathbb{Z}\alpha_r, Q^+ = \mathbb{N}\alpha_0 + \cdots + \mathbb{N}\alpha_r$: root lattice, positive root lattice.
8. $W := \langle s_\alpha | \alpha \in \Phi^{re} \rangle = \langle s_\alpha | \alpha \in \Pi \rangle$: Weyl group of $\mathfrak{g}$.
9. $\Phi(\lambda) := \{\alpha \in \Phi^{re} | \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$.
10. For $\lambda \in \mathfrak{h}^*$, $\Phi(\lambda) := \{\alpha \in \Phi^{re} | \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\}$.
11. $\mathcal{C} := \{\lambda \in \mathfrak{h}^* | \langle \lambda + \rho, (c) \rangle = \langle \lambda + \rho, c \rangle \neq 0\}$, the non-critical region.
12. $\mathcal{C}^- := \{\lambda \in \mathcal{C} | \langle \lambda + \rho, \alpha^\vee \rangle \leq 0, \text{ for all } \alpha \in \Phi^+ \}$, the “non-critical” anti-dominant chamber.
3. Module categories

Following [33], let $\mathcal{O} = \mathcal{O}(g)$ be the category of $g$-modules $M$ which are weight modules for $\mathfrak{h}$ having finite dimensional weight spaces $M_\lambda$, $\lambda \in \mathfrak{h}^*$, and which have the property that, given $\xi \in \mathfrak{h}^*$, the weight space $M_{\xi + \sigma} \neq 0$ for only finitely many $\sigma \in \mathbb{Q}^+$.

Any $\lambda \in \mathfrak{h}^*$ defines a one-dimensional module (still denoted $\lambda$) for the universal enveloping algebra $U(\mathfrak{b})$ of $\mathfrak{b}$. Let $M(\lambda) := U(\mathfrak{g}) \otimes U(\mathfrak{h})$. $\lambda$ be the Verma module for $\mathfrak{g}$ of highest weight $\lambda$. It has a unique irreducible quotient module $L(\lambda)$. Both $M(\lambda)$ and $L(\lambda)$ belong to $\mathcal{O}$.

The category $\mathcal{O}$ has a semisimple action (in addition to the local nilpotence of $\Omega$). Clearly, $\mathcal{O}$ has a contravariant, exact duality $\star$. 

Following [33], let $\mathcal{O}_k$ be the full subcategory of $\mathcal{O}$ consisting of all modules for which the central element $c$ acts by multiplication by the scalar $k$. For example, $M(\lambda), L(\lambda) \in \mathcal{O}_k$ for $k := \lambda(c) \in \mathbb{Q}$. In addition, the duality $M \rightarrow M^*$ restricts to a duality on $\mathcal{O}_k$.

Put
$$\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathbb{C}c,$$
the derived subalgebra $\mathfrak{g}$. Let $\tilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}c$, so that $\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}$. The Casimir operator $\Omega$, defined in [30, §2.5], belongs to a completion $U_c(\mathfrak{g})$ of $U(\mathfrak{g})$ [30, p. 229]. In particular, $\Omega$ defines a locally finite operator on each object in $\mathcal{O}(\mathfrak{g})$, commuting with the action of $\mathfrak{g}$. For $a \in \mathbb{C}$, the category $\mathcal{O}_{k,a}(\mathfrak{g})$ denotes the full subcategory of $\mathcal{O}(\mathfrak{g})$ consisting of objects $M$ in $\mathcal{O}(\mathfrak{g})$ upon which $\Omega - a$ acts locally nilpotently, and let $\mathcal{O}_{k,a}(\mathfrak{g})$ be the full subcategory of $\mathcal{O}_{k,a}(\mathfrak{g})$ having objects on which $d$ has a semisimple action (in addition to the local nilpotence of $\Omega - a$).

**Proposition 3.1.** (Kac-Polo) For $k \neq -g$ and $a \in \mathbb{C}$, there is a full embedding
$$F_{k,a} : M_{k}^{\text{res}}(\tilde{\mathfrak{g}}) \rightarrow M_{k}^{\text{res}}(\tilde{\mathfrak{g}})$$
of abelian categories, inducing an equivalence of $M_{k}^{\text{res}}(\tilde{\mathfrak{g}}) \rightarrow M_{k,a}^{\text{res}}(\mathfrak{g})$. Moreover, the inverse of the equivalence is given by restriction.

That is, for each $M \in \mathcal{O}_{k}^{\text{res}}(\tilde{\mathfrak{g}})$, $F_{k,a}(M) \in \mathcal{O}_{k,a}^{\text{res}}(\mathfrak{g})$ and $F_{k,a}(M)|_{\tilde{\mathfrak{g}}} \cong M$ (naturally). Also, any object in $\mathcal{O}_{k,a}^{\text{res}}(\mathfrak{g})$ is isomorphic to $F_{k,a}(M)$, for some $M \in \mathcal{O}_{k}^{\text{res}}(\tilde{\mathfrak{g}})$.

**Proof.** A brief outline of the proof may be found in Soergel [30, pp. 446-447]. We fill in some details. First, the algebra $U_c(\mathfrak{g})$ injects naturally into $U_c(\mathfrak{g})$, since tensor induction takes $M^{\text{res}}(\tilde{\mathfrak{g}})$ into $M^{\text{res}}(\mathfrak{g})$. If $T_0 \in U_c(\tilde{\mathfrak{g}})$ denotes the (0th) Sugawara operator, the discussion in [30, p. 228-229] shows that the equation
$$T_0 = -2(c + g)d + \Omega$$
holds in $U_c(\mathfrak{g})$. By [30, Lemma 12.8], the equation $[T_0, x] = -2(c + g)d, x$, for $x \in \tilde{\mathfrak{g}}$, holds in $U_c(\tilde{\mathfrak{g}})$. If we consider the corresponding equation of operators on an object $M \in \mathcal{O}_{k}^{\text{res}}(\tilde{\mathfrak{g}})$, we may replace $c + g$ by $k + g \neq 0$. Letting $d$ act as the operator $\tau := \frac{T_0 - \xi}{2(k + g)}$ gives an action of $\mathfrak{g}$ on $M$. Equivalently, $\tau x - x\tau$ acts...
as \([d, x]\) on \(M\) for each \(x \in \tilde{g} \subseteq U_{c}(\tilde{g})\). The operator \(T_0\) is locally finite on \(M\) \([30, \text{p.}229]\), as is \(\tau\). For any complex number \(\epsilon\), and any positive integer \(n\), let
\[
M_{n, \epsilon} := \{m \in M \mid (\tau - \epsilon)^{n} m = 0\}.
\]
If \(x \in g\) is a \(\gamma\)-eigenvector for \(ad\ d\), we easily find, by induction on \(n\), that \(xM_{\epsilon, n} \subseteq M_{\gamma + \epsilon, n}\). (Alternatively, see \([10, \text{Prop.} \ 2.7]\).) Let \(\tau_s\) be the semisimple part of the locally finite operator \(\tau\) on \(M\). The operator \(\tau_s\) acts as multiplication by \(\epsilon\) on \(M_{\epsilon, n}\), and by \(\gamma + \epsilon\) on \(M_{\gamma + \epsilon, n}\). For \(m \in M_{\epsilon, n}\) and \(x\) as above, we have
\[
\tau_s(xm) = (\gamma + \epsilon)xm = \gamma xm + c \epsilon x m = [d, x]m + x\tau_sm
\]
and so \(\tau_s x - x\tau_s\) acts as \([d, x]\) on \(M\). This, letting \(d\) act as \(\tau_s\), gives an action of \(\tilde{g}\) on \(M\), extending that of \(\tilde{g}\) (Note that \(\tilde{g}\) is spanned by the eigenvectors of \(ad\ d\)).

The constructed \(g\)-module belongs to \(M_{k, \epsilon}^{\text{res}}(g)\) and \(d\) acts semisimply (as \(\tau_s\)). The equation
\[
\tau = d + \frac{\Omega - a}{-2(k+g)}
\]
shows that \(\Omega - a\) acts as a nonzero scalar multiple of the locally nilpotent part of \(\tau\). So it is itself locally nilpotent. Finally, the assignment of \(M\) to the constructed \(g\)-module is clearly functorial providing a functor \(F_{k, a} : M_{k, \epsilon}^{\text{res}}(\tilde{g}) \rightarrow M_{k, \epsilon}^{\text{res}}(g)\) with \(F_{k, a}(M)_{|\tilde{g}} = M\). The construction shows that \(F_{k, a}(M) \in M_{k, \epsilon}^{\text{res}, d}\), and, clearly, any object \(N \in M_{k, \epsilon}^{\text{res}, d}\) satisfies \(N \cong F_{k, a}(N|\tilde{g})\). (Note that \(d\) must act as the semisimple part of \(\frac{\Omega - a}{-2(k+g)}\) and \(\Omega - a\) must act as the locally nilpotent part.) This completes the proof. \(\square\)

We will for the rest of this paper, unless otherwise explicitly stated to the contrary, that \(k\) is a rational number with \(k + g < 0\). We next define below, for such a \(k \in \mathbb{Q}\), a category \(O_{k}\) of \(\tilde{g}\)-modules. The definition is taken from \([47]\), adapted from \([34]\). In Corollary \([34, \text{Cor.} \ 3.2]\) \(O_{k}\) is shown to be equivalent to a category of \(g\)-modules, and is more fully integrated into the \(g\)-module theory in \(\S 5\). (See Remark \([5.6, \text{b)}\].)

Given any \(\tilde{g}\)-module \(M\) and positive integer \(n\), let \(M(n)\) be the subspace of all \(m \in M\) such that \(x_{1} \cdots x_{n} m = 0\) for any choice of \(x_{1}, \ldots, x_{n} \in t \mathbb{C}[t] \otimes \tilde{g}\). Now define \(O_{k}\) to be the full subcategory of \(\tilde{g}\)-modules \(M\) such that \((a)\ c\ acts\ as\ multiplication\ by\ \kappa; (b)\ each\ \(M(n)\)\ is\ finite\ dimensional;\ and\ (c)\ \(M = \bigcup_{n \geq 1} M(n)\). Since each \(M(n)\) is evidently a \(\tilde{g}\)-submodule of \(M|_{\tilde{g}}\), (c) implies that \(M\) is a locally finite, hence semisimple, \(\tilde{g}\)-module. Then, by \((a)\), \(M\) is a weight module for \(\tilde{g}\), in the sense that it decomposes into weight spaces for \(\tilde{g}\). In addition, \(O_{k}\) is a full subcategory of \(M_{k, \epsilon}^{\text{res}}(\tilde{g})\).

In \([34, \text{Defn.} \ 2.15, \text{Thm.} \ 3.2]\), it is shown that all objects in \(O_{k}\) have finite length. The irreducible modules involved are all generated by a highest weight vector having weight \(\lambda\) satisfying \(\langle \lambda, \alpha_{i}^{\vee} \rangle \geq 0\), for \(i = 1, \ldots, r\) and \(\lambda(c) = k\). These irreducible modules are non-isomorphic for distinct \(\lambda\) above. Conversely, any \(\tilde{g}\)-module with a finite composition series having irreducible quotients of this form belong to \(O_{k}\).

At the level of \(g\)-modules define \(O^{+} = O^{+}(\tilde{g})\) to be the full subcategory of \(\tilde{g}\) consisting of all objects \(M\) such that \(M : L(\mu) \neq 0\) implies \(\langle \mu, \alpha_{i}^{\vee} \rangle \in \mathbb{N}\) for \(i = 1, \ldots, r\). Let \(O^{+, \text{finite}}\) be the full category of \(O^{+}\) consisting of objects which have finite length.

Similarly, for any \(k \in \mathbb{Q}\) with \(k < -g\), let \(O_{k} = O_{k}(\tilde{g})\) be the full subcategory of \(O_{k}\) consisting of all objects on \(\tilde{g}\) by multiplication by \(k\), and let \(O_{k}^{+}\) be the full subcategory of \(O_{k}\) consisting of all objects in both \(O_{k}\) and \(O^{+}\). If \(a \in \mathbb{C}\), let \(O_{k,a}^{+}\) be the full subcategory of \(O_{k}^{+}\) consisting of all objects upon which \(\Omega\ acts\ with\ generalized\ eigenvalue\ \(a\). Finally, let \(O_{k, \epsilon}^{+, \text{finite}}\) and \(O_{k,a}^{+, \text{finite}}\) be the full subcategories of \(O_{k}^{+}\) and \(O_{k,a}^{+}\), respectively, consisting of objects of finite length.

**Corollary 3.2.** Suppose \(k \in \mathbb{Q}\), \(k < -g\). Then the restriction to \(\tilde{g}\) of any object in \(O_{k,a}^{+, \text{finite}}\) belongs to \(O_{k}\). Conversely, if \(M \in O_{k}\), then \(F_{k,a} M\) belongs to \(O_{k,a}^{+, \text{finite}}\). These two functors are mutually inverse, up to a

\[3\text{In this reference, the authors define a category } O_{k}\text{ which turns out to be } O_{\kappa}\text{ for } \kappa = k - g.\text{ The discussion is given only for the simply laced root system case, but this restriction is not necessary.}\]

\[4\text{Any indecomposable object of } O_{k}^{+, \text{finite}}\text{, or of } O_{k}\text{, or any object in a single block of } O_{k}\text{, already has finite length. The argument is given below.}\]
natural isomorphism, and provide an equivalence

\[ \mathcal{O}_{k,a}^{\text{finite}} \cong \mathcal{O}_k, \]

Given any weight \( \mu \in \mathfrak{h}^* \), if \( \mu \) is a Coxeter group with fundamental system consisting of the \( s_i \) reflections \( N \mathcal{L} \), define \( \tilde{\mu} \) to be the projection of \( \mu \) into \( \mathfrak{h}^* \), and, for \( k \in \mathbb{Q} \), put \( \mu^k = \tilde{\mu} + k\lambda \in (\mathfrak{h})^* \). Also, \( k \neq -g \) (the dual Coxeter number) put

\[ \mu^{k,a} := \mu^k + b \mu, \]

where \( b = \frac{a-2\mu_{\alpha}}{2(k+g)} \) depends on \( k \) and \( \mu \), as well as \( a \). Note that \( \mu|_{\mathfrak{h}^*} = \tilde{\mu} \) and \( \mu|_{\mathfrak{h}^*} = \mu^k \) if and only if \( \mu \) has level \( k \). Also, \( \mu = \mu^{k,a} \) if and only if \( \mu \) has level \( k \) and the Casimir operator \( \Omega \) acts with eigenvalue \( a \) on \( L(\mu) \), the irreducible \( \mathfrak{g} \)-module of high weight \( \mu \). (This is an easy calculation from [10, Prop. 11.36]. See also [20, p. 229].) Since we regard \( (\mathfrak{h})^* \) and \( \mathfrak{h}^* \) as contained in \( \mathfrak{h}^* \), \( \mu^k \) and \( \mu^{k,a} \) are defined for \( \mu \) in these spaces as well.

As a corollary of this discussion, we have the following.

**Proposition 3.3.** Suppose \( \mu, \mu' \in \mathfrak{c} \) and \( \mu' = w \cdot \mu \) for some element \( w \in W \). (Here \( w \cdot \mu := w(\mu + \rho) - \rho \) is the usual dot action of \( W \).) Then, if \( \mu \neq \mu' \), we have \( \mu|_{\mathfrak{h}^*} \neq \mu'|_{\mathfrak{h}^*} \).

**Proof.** The levels of \( \mu, \mu' \) have the value \( k = (\mu + \rho, \delta) = (w(\mu + \rho), \delta) = (\mu' + \rho, \delta) \), after noting that \( (\rho, \delta) = 0 \). By [10, Prop. 11.36], \( \Omega \) acts on \( L(\mu) \) and \( L(\mu') \) by multiplication by

\[
\begin{align*}
a &= (\mu + \rho, \mu + \rho) - (\rho, \rho) \\
&= (w(\mu + \rho), w(\mu + \rho)) - (\rho, \rho) \\
&= (\mu' + \rho, \mu' + \rho) - (\rho, \rho).
\end{align*}
\]

If \( \mu|_{\mathfrak{h}^*} = \mu'|_{\mathfrak{h}^*} \), then \( \tilde{\mu} = \tilde{\mu}' \) and (by above)

\[ \mu = \mu^{k,a} = \mu^{b_{\mu}} = \mu', \]

as required. \( \square \)

4. Weyl groups and linkage classes

Maintain the above notation. For \( \alpha \in \Phi^{re} \), form the reflection \( s_\alpha : \mathfrak{h}^* \rightarrow \mathfrak{h}^* \), \( x \mapsto x - (x, \alpha^\vee)\alpha \). The Weyl group

\[ W = \langle s_\alpha \mid \alpha \in \Phi^{re} \rangle \]

for \( \mathfrak{g} \) is a Coxeter group with fundamental reflections \( S = \{ s_\alpha \mid i = 0, \ldots, r \} \). For \( \lambda \in \mathfrak{h}^* \),

\[ \Phi(\lambda) := \{ \alpha \in \Phi^{re} \mid \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z} \} \]

is a subroot system of \( \Phi \), in the sense of [33]. The subgroup

\[ W(\lambda) = \langle s_\alpha \mid \alpha \in \Phi(\lambda) \rangle \]

is a Coxeter group with fundamental system consisting of the \( s_\alpha \), \( \alpha \in \Phi^+(\lambda) = \Phi^+ \cap \Phi(\lambda) \), such that

\[ s_\alpha(\Phi^+(\lambda) \setminus \{ \alpha \}) = \Phi^+(\lambda) \setminus \{ \alpha \} \].

Then \( \tilde{W}(\lambda) = \{ w \in W(\lambda) \mid w \cdot \lambda = \lambda \} \) is also a Coxeter system, generated by reflections \( s_\alpha \) with \( \Phi_0(\lambda) = \{ \alpha \in \Phi^{re} \mid \langle \lambda + \rho, \alpha^\vee \rangle = 0 \} \). Let \( [\lambda] := W(\lambda) \cdot \lambda \) and \( [\lambda]^+ = \{ \mu \in [\lambda] \mid \langle \mu, \alpha_i^\vee \rangle \in \mathbb{N}, 1 \leq i \leq r \} \). An element \( \lambda \in C^- \) is called regular if \( \tilde{W}(\lambda) = \{ 1 \} \).

Let \( \mathcal{O}[\lambda] \) be the full subcategory of \( \mathcal{O} \) consisting of modules \( M \) such that \( [M : L(\mu)] \neq 0 \) implies \( \mu \in [\lambda] \).

Let \( \lambda \in C^- \). Then \( \lambda \) is the smallest element in \( [\lambda] \), in the sense that all weights in \( [\mu] \) belong to \( W(\lambda) \cdot \lambda \subseteq \lambda + \mathbb{Q}^+ \). In particular, all objects of \( \mathcal{O}[\lambda] \) have finite length. As shown in [33, Prop. 3.1], using work of Kac-Kazhdan [31], if \( \mu \in [\lambda] \), the Verma module \( M(\mu) \) belongs to \( \mathcal{O}[\lambda] \). Also, \( M^*(\mu) \) also belongs to \( \mathcal{O}[\lambda] \). If \( M \in \mathcal{O} \) is indecomposable with \( [M : L(\mu)] \neq 0 \), then \( M \in \mathcal{O}[\lambda] \) [33, Prop. 3.2]. Or if \( M \) is an indecomposable module linked to \( L(\mu) \) via a chain of indecomposable modules \( M_i \) in \( \mathcal{O} \), \( i = 0, \ldots, n \) and irreducible modules \( L(\mu_i) \) satisfying \( M = M_0 \) and \( [M_i : L(\mu_i)] \neq 0 \neq [M_{i+1} : L(\mu_{i+1})] \), and \( \mu - \mu_a \), then \( M \in \mathcal{O}[\lambda] \). Moreover, if
$M = L(\nu)$ with $\nu \in [\lambda]$, there is such a chain, using [33 Prop. 31] (which quotes [31]). That is, $\mathcal{O}[\lambda]$ is the “block” of $\mathcal{O}$ associated to $L(\mu)$, and the irreducible modules $L(\nu)$, $\nu \in [\lambda]$, constitute a linkage class.

The inclusion $i_\nu : \mathcal{O}^+ \to \mathcal{O}$ admits a right adjoint $i^!$ and a left adjoint $i^*$. Explicitly, given $M \in \mathcal{O}$, $M_\nu = i^!M$ (resp., $M^\nu = i^*M$) is the largest submodule (resp., quotient module) lying in $\mathcal{O}^+$. For $\lambda \in \mathcal{C}^-$, $\mathcal{O}^+ \langle \lambda \rangle$ will denote the full subcategory of $\mathcal{O}[\lambda]$ consisting of all $M$ such that $[M : L(\mu)] \neq 0$ implies $\mu \in [\lambda]^+$. Consider the parabolic subalgebra

$$p := (C[t] \otimes \hat{g}) \oplus Cc \oplus Cd$$

of $\mathfrak{g}$. Its Levi factor is denoted

$$\mathcal{L} := \hat{g} \oplus Cc \oplus Cd = \hat{g} \oplus \mathfrak{h}.$$ 

Any $\lambda \in \mathfrak{h}^*$ determined an irreducible $\mathcal{L}$-module $s(\lambda)$ whose restriction to $\hat{g}$ is always irreducible. Writing $\lambda = \hat{\lambda} + k\chi + b\delta$, $k, b \in \mathbb{C}$, $s(\lambda)$ is finite dimensional if and only if $\hat{\lambda} \in X^+$ (and in this case its restriction to $\hat{g}$ is then the finite dimensional irreducible module $V(\hat{\lambda})$ of highest weight $\hat{\lambda}$).

Proposition 4.1. Let $\mu = \hat{\mu} + k\chi + b\delta$, with $k < -g$, and $\hat{\mu}$ dominant on $\hat{g}$. Then

$$M(\mu)^+ \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} s(\mu);$$

$$M(\mu)^- \cong \Hom_{U(\mathfrak{p})}(U(\mathfrak{g}), s(\mu)).$$

Also, $(M(\mu)^+)^* \cong M^*(\mu)_+.$

Proof. Each subspace $t^n \otimes \hat{g} \subset \mathfrak{g} \subset U(\mathfrak{g})$, $n \in \mathbb{Z}$, is a $\hat{g}$-submodule of $U(\mathfrak{g})$ under the adjoint action, isomorphic to the adjoint module $\hat{g}$. Obviously, $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} s(\mu)$, as a $\mathfrak{g}$-module under multiplication, is the homomorphic image of a direct sum of modules $(t^{-n} \otimes \hat{g}) \otimes (t^{-n} \otimes \hat{g}) \otimes \cdots (t^{-n} \otimes \hat{g}) \otimes s(\mu)$, $n_1, \ldots, n_m \in \mathbb{Z} \leq 0$, $m \in \mathbb{N}$. All these tensor products are finite dimensional $\hat{g}$-modules. Thus, if $L(\nu)$ is a $\mathfrak{g}$-composition factor of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} s(\mu)$, its highest weight space must generate a finite dimensional $\hat{g}$-module. In particular, $(\nu, \alpha_i^\vee) \in \mathbb{N}$, for $i = 1, \ldots, n$.

Let $U(\mathcal{L}) \otimes_{U(\mathcal{L} \oplus \mathfrak{b})} \mu$ be the Verma module for $\mathcal{L}$ with highest weight $\mu$. It inflates naturally to $p$ as $U(\mathfrak{p}) \otimes_{U(\mathfrak{b})} \mu$. There is an exact sequence

$$(4.1) \ 0 \to N \to U(\mathfrak{p}) \otimes_{U(\mathfrak{b})} \mu \to s(\mu) \to 0$$

of $p$-modules. From the classical theory of $\hat{g}$-Verma modules, the highest weight $\varpi$ of any composition factor $Y$ of $N$ has the property that $(\varpi, \alpha_i^\vee) < 0$ for some $i$. The section $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} Y$ of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$ has an irreducible head with the same highest weight $\varpi$.

It follows now, by tensor inducing the exact sequence $(4.1)$ of $p$-modules, that $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} s(\mu)$ is the largest quotient of $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mu$ with all composition factors in $\mathcal{O}^+$. That, $M(\mu)^+ \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} s(\mu)$.

A similar argument establishes the assertion for $M(\mu)^+$ and the final assertion is obvious.

5. Highest weight categories

Throughout this section, fix $\lambda \in \mathcal{C}^-$. The set $[\lambda] = W(\lambda) \cdot \lambda$ is a poset, putting $\mu \leq \nu$ if and only if $\nu - \mu \in Q^+ := \sum_{i=0}^n N\alpha_i$. The set $[\lambda]$ has a unique minimal element, namely, $\lambda$. If $\nu \in [\lambda]$, then $\{\mu \in [\lambda] \mid \mu \leq \nu\}$ is a finite poset ideal in $[\lambda]$. If $\Gamma \subseteq [\lambda]$, let $\mathcal{O}[\Gamma]$ be the full subcategory of $\mathcal{O}$ consisting of objects with have “composition factors” (in the sense of [30] p.151) $L(\gamma)$, $\gamma \in \Gamma$. It will often convenient (in §§6.7) to denote $\mathcal{O}[\Gamma]$ by $\mathcal{O}^\Gamma[\lambda]$ where $\lambda$ needs to be mentioned.

We will consider the following categories of $\mathfrak{g}$-modules. In each case, the irreducible modules are indexed, up to isomorphism, by a poset $\Gamma$ in $[\lambda]$ or $[\lambda]^+$. In case $\Gamma \subseteq [\lambda]^+$, we let $\mathcal{O}^+ \langle \Gamma \rangle = \mathcal{O}^\Gamma[\lambda]$ be the full subcategory of $\mathcal{O}^+$ consisting of modules composition factors $L(\nu)$, $\nu \in \Gamma$. Also, for $\nu \in \Gamma$, there are given two modules $\Delta(\nu)$ and $\nabla(\nu)$ in the category.

5If $\Gamma$ is a poset ideal in a poset $\Lambda$, i.e., if $\nu \leq \gamma \in \Gamma \Rightarrow \nu \in \Gamma$, we write $\Gamma \subseteq \Lambda$. We will also consider other partial orders on $[\lambda]$ in this section.
An object $X$ in $O^\infty[\lambda]$ (resp., $O^{+,\infty}[\lambda]$) is by definition a directed union $\{X^\Gamma\}_\Gamma$ of modules $X^\Gamma \in O[\lambda]$ ranging over finite $\Gamma \subseteq [\lambda]$ (resp., $\Gamma^+ \subseteq [\lambda]^+$).

**Theorem 5.1.** For $\lambda \in C^-$, each of the categories listed in Table 1 below is a highest weight category (in the sense of [9]) with standard (Weyl) modules $\Delta(\nu)$, costandard modules $\nabla(\nu)$ and indicated poset. In particular, each of these categories has enough injective objects.

**Proof.** The fact that $O[\Gamma]$ is a highest weight category follows from the dual of the definition [9] and the fact that projectives in suitable “truncated” categories have Verma module filtrations [44, Lemma 10]. The latter reference does not use an arbitrary poset ideal $\Gamma$, but every such $\Gamma$ is contained in one of theirs, which is sufficient, see [22, Lemma 2.3] for the case when $\Gamma$ is any ideal generated by a single element.

In [44], the authors also treat the parabolic case, and generalized Verma modules. Thus, it follows similarly that $O^{+,\infty}[\lambda]$ is a highest weight category.

If $\Gamma \leq \Gamma'$ are two poset ideals, then the injective hull $I_{\Gamma'}(\nu)$ of any given irreducible $L(\nu)$ in $O[\Gamma']$ embeds in the injective hull $I_{\Gamma}(\nu)$ in $O[\Gamma']$. Taking a directed union over the chain of poset ideals $\Gamma_n$, $n \in \mathbb{N}$, with $\bigcup \Gamma_n \supseteq [\lambda]$ gives an injective hull in $O^\infty[\lambda]$. It follows easily from [9] that $O^\infty[\lambda]$ is a highest weight category. Similarly, $O^{+,\infty}[\lambda]$ is also a highest weight category. $\square$

**Remark 5.2.** (a) The category $O[\lambda]$, $\lambda \in C^-$, is the union $\bigcup_{\Gamma} O[\Gamma]$ over all finite poset ideals $\Gamma \subseteq [\lambda]$ (with respect to $\leq$). The category $O[\lambda]$ satisfies most of the axioms in [9] for a highest weight category, though not all. (There are not enough injective objects.) However, each full subcategory $O[\Gamma]$ does have enough injective and projective objects, and is a highest weight category. As we have seen, the categories $O[\Gamma]$ can be used to formally complete $O[\lambda]$ to a highest weight category $O^\infty[\lambda]$. Some authors, speaking more informally, simply call such categories (like $O[\lambda]$) a highest weight category.

Similar remarks apply to $O^{+,\infty}[\lambda]$, defined to be $O^+ \cap O[\lambda]$, as well as many of the other categories we have introduced.

(b) For $\lambda \in C^-$, there is another poset structure $\leq W = \leq W(\lambda)$ on $[\lambda] = W(\lambda) \cdot \lambda$. Explicitly, given $\mu \in W(\lambda) \cdot \lambda$, write $\mu = w_\mu \cdot \lambda$ where $w_\mu \in W(\lambda)$ is the unique element $w \in W(\lambda)$ of minimal length such that $w \cdot \lambda = \mu$. Then, for $\mu, \nu \in [\lambda]$, put $\mu \leq W \nu$ if and only if $w_\mu \leq w_\nu$ in the Bruhat-Chevalley order on the Coxeter group $W(\lambda)$. We will show in Remark 5.6(a) that the above categories are highest weight categories using $\leq W$ (or its restriction to $[\lambda]^+$) with the same standard and costandard objects

**Definition 5.3.** For $\mu, \nu \in \tilde{H}^*$, put $\nu \leq^{\sim} \mu$ provided that the following two conditions hold:

1. $\mu, \nu$ have the same level $k \neq -g$, i.e., $\mu(c) = \nu(c) \neq -g$;
2. $\nu^{k,a} \leq^{\sim} \mu^{k,a}$ for some $a \in C$.

Condition (2) does not actually depend on $a$. Indeed, writing $\mu = \tilde{\mu} + k\chi, \nu = \tilde{\nu} + k\chi$, then $\mu^{k,a} = \tilde{\mu} + k\chi + b\delta$ and $\nu^{k,a} = \tilde{\nu} + k\chi + b'\delta$, then

$$
\begin{align*}
b &= \frac{a - (\tilde{\mu} + 2\tilde{\nu}, \tilde{\mu})}{k + g} \\
b' &= \frac{a - (\tilde{\nu} + 2\tilde{\mu}, \tilde{\nu})}{k + g}.
\end{align*}
$$

Thus, $\nu^{k,a} \leq^{\sim} \mu^{k,a}$ if and only if

1. $b - b' \in \mathbb{N}$.

| Category | poset | $\Delta(\nu)$ | $\nabla(\nu)$ |
|----------|-------|---------------|---------------|
| $O[\Gamma]$ | $\Gamma \leq [\lambda]$ | $M(\nu)$ | $M^*(\nu)$ |
| $O^+[\Gamma]$ | $\Gamma \leq [\lambda]^+$ | $M(\nu)^+$ | $M^*(\nu)^+$ |
| $O^\infty[\lambda]$ | $[\lambda]$ | $M(\nu)$ | $M^*(\nu)$ |
| $O^{+,\infty}[\lambda]$ | $[\lambda]^+$ | $M(\nu)^+$ | $M^*(\nu)^+$ |

Table 1. Various categories
The parameter \( a \) drops out of \( b - b' \) which depends only on \( \mu, \nu \) and \( k \). Thus, any \( a \) may be used in defining \( \bar{\nu} \).

For \( \lambda \in C^- \), we temporarily write \( [\lambda]^{-} \) for the collection of all \( \bar{\mu} := \mu|_{\bar{\mathfrak{g}}} \), for \( \mu \in [\lambda] \). As a consequence of the discussion, we have:

**Proposition 5.4.** The restriction map \( [\lambda] \to [\lambda]^{-} \) is a poset isomorphism, if \( [\lambda] \) is given its usual poset structure via \( \leq \) above, and if \( [\lambda]^{-} \) is given its poset structure via \( \leq^{-} \).

**Proof.** The bijectivity of restriction has already been established in Proposition 5.3. As noted in its proof, an inverse on \( [\lambda]^{-} \) of restriction is provided by \( \bar{\mu} \mapsto \bar{\mu}^{k,a} = \mu^{k,a} \), where \( \lambda = \lambda^{k,a} \). As the definition of \( \leq^{-} \) shows, this inverse is order preserving, as is the restriction map itself. \( \square \)

We introduce some further categories, obtained by restricting to \( \bar{\mathfrak{g}} \) all the categories listed in Table 1, as well as \( O[\lambda] \) and \( O^{+}[\lambda] \), decorating the resulting strict image category with a “tilde” (i.e., changing \( O \) to \( \bar{O} \)). Each of these \( \bar{\mathfrak{g}} \)-categories has an associated poset, given in Table 1 for the corresponding \( \mathfrak{g} \)-category. (We can view the posets as abstract sets, useful for labeling irreducible, standard, and costandard modules.)

As such, there is no need to pass to version using \( ([\lambda]^{-}, \leq^{-}) \) in view of the poset isomorphism in Proposition 5.4. Keeping the Table 1 version eases our notational burden.) Thus, in the proposition below, we use the \( \leq \) partial order on \( [\lambda] \) and \( [\lambda]^{-} \), whether or not we are dealing with categories of \( \mathfrak{g} \) or \( \bar{\mathfrak{g}} \)-modules. We will extend the proposition to the partial ordering \( \leq^{W} \) in Remark 5.6(a), as well as to additional partial orders \( \leq_{\text{nat}} \) discussed there.

For use below, define

\[
C_{\text{nat}}^{-} := \{ \lambda \in C^- \mid (\lambda, \alpha_i) \in \mathbb{Z}, \text{ for each } i = 1, \ldots, r \text{ and } (\lambda,c) \in \mathbb{Q} \}.
\]

In particular, each \( \lambda \in C_{\text{nat}}^{-} \) has level a rational number \( k \) and \( k < -g \) since \( \lambda \in C^- \). Note that, for \( \lambda \in C^- \), \( [\lambda]^{+} = \emptyset \) unless \( \lambda \in C_{\text{rat}}^{-} \).

**Proposition 5.5.** Let \( \lambda \in C_{\text{rat}}^{-} \) and let \( \Gamma \subseteq [\lambda] \) (resp., \( \Gamma^{+} \subseteq [\lambda]^{+} \)) be finite. The categories \( \bar{O}[\Gamma], \bar{O}^{+}[\Gamma^{+}], \bar{O}^{\infty}[\lambda], \) and \( \bar{O}^{+,\infty}[\lambda]^{+} \) are all highest weight categories with weight posets \( \Gamma, \Gamma^{+}, [\lambda] \) and \( [\lambda]^{+} \), respectively. Each of these categories is equivalent to its counterpart for \( \mathfrak{g} \)-modules, as is each of the categories \( \bar{O}[\lambda] \) and \( \bar{O}^{+}[\lambda] \). The functors providing these equivalences are, in each case, given by the restriction functor of \( \mathfrak{g} \)-modules to \( \bar{\mathfrak{g}} \)-modules, and by applying \( F_{k,a} \) to objects in, say, a category \( \bar{O}^{+}[\lambda] \), with a determined by \( \lambda \). There is a natural common extension of the functors \( F_{k,a} \) to the functors \( F_{k,a}^{\infty} \) on \( \bar{O}^{\infty}[\lambda] \), which is also inverse to the restriction functor.

**Proof.** Let \( M_{\mathfrak{g}}^{\text{res,\infty}}(\bar{\mathfrak{g}}), M_{\mathfrak{g}}^{\text{res,d,\infty}}(\bar{\mathfrak{g}}) \) denote the categories of \( \bar{\mathfrak{g}} \)-modules, respectively, which are directed unions of objects in \( M_{\mathfrak{g}}^{\text{res}}(\bar{\mathfrak{g}}), M_{\mathfrak{g}}^{\text{res,d}}(\bar{\mathfrak{g}}) \), respectively. Then the functors \( F_{k,a} \) extend in an obvious way to functors \( F_{k,a}^{\infty} \) on the direct union categories, giving equivalences inverse to restriction. In particular, \( F_{k,a}^{\infty} \) provides, \( \bar{O}^{\infty}[\lambda] \) and \( \bar{O}^{+,\infty}[\lambda]^{+} \) equivalences inverse to restriction to the versions without the “tilde.” The remaining equivalences are obvious. \( \square \)

Each of the \( \bar{\mathfrak{g}} \)-modules categories above has irreducible, standard and costandard modules. These modules will be denoted by placing a “tilde” over their \( \mathfrak{g} \)-counterparts. Thus, \( \bar{L}(\mu), \bar{M}(\mu) \) and \( \bar{M}_{\mathfrak{g}}(\mu) \) are the irreducible, standard and costandard modules off \( \bar{O}[\lambda] \), if \( \mu \in [\lambda] \). If \( \mu \in [\lambda]^{+} \), then \( \bar{M}^{+}(\mu), \bar{M}_{\mathfrak{g}}(\mu) \) are the standard and costandard modules for \( \bar{O}^{+}[\lambda]^{+} \). All of these modules are the restrictions to \( \bar{\mathfrak{g}} \) of their \( \mathfrak{g} \)-counterparts.

**Remark 5.6.** (a) Let \( \mathcal{E} \) be an abstract highest weight category \( \mathcal{E} \) with weight poset \( (\Lambda, \leq, \leq) \), and with costandard modules \( \nabla(\lambda), \lambda \in \Lambda \). Assume there also exist standard modules \( \Delta(\lambda), \lambda \in \Lambda, \) and assume that \( \Delta(\lambda) \) and \( \nabla(\lambda) \) have the same composition factors (with multiplicities). There is a natural partial order \( \leq_{\text{nat}} \), at least when \( \Delta(\lambda) \) and \( \nabla(\lambda) \) have the same composition factors (which holds for all the categories above).
More precisely, $\leq_{\text{nat}}$ is the partial order generated by the requirement that $\mu \leq_{\text{nat}} \nu$ when $[\Delta(\nu) : L(\mu)] \neq 0$. Let $\Lambda_{\text{nat}} = (\Lambda, \leq_{\text{nat}})$ denote this new poset. Then $\mathcal{E}$ is also a highest weight category with respect to $\Lambda_{\text{nat}}$ with the same standard and costandard modules. If $\nu \leq_{\text{nat}} \mu$, then clearly $\nu \leq \mu$. F Now suppose that $\nleq$ is a partial order on the set $\Lambda$ such that, for each $\mu, \nu \in \Lambda$, $\mu \leq_{\text{nat}} \nu \implies \mu \nleq \nu$. Then it is easily seen that $\mathcal{E}$ is a highest weight category with respect to $(\Lambda, \nleq)$, with the same standard and costandard modules. Moreover, if $\Gamma$ is a $\nleq$-ideal in $\Lambda$, it is also a $\leq_{\text{nat}}$-ideal in $\Lambda$.

Returning to the situation of Proposition 5.5, we have, in addition to the partial orders $\leq_{\text{nat}}$ and $\nleq$ on $[\lambda]$, there is also the partial order $\leq_W$ discussed in Remark 5.3(b) and a further partial ordering $\uparrow$ (discussed in §9, Appendix I). The orders $\leq_W$ and $\uparrow$ are shown to be the same on $[\lambda]$ in Proposition 9.1 below. It is also true that $\mu \leq_{\text{nat}} \nu$ implies in an obvious way, using the remarks above Proposition 9.1, that $\mu \uparrow \nu$, and, thus, now using Proposition 9.1, $\mu \leq_W \nu$. In turn, $\mu \leq_W \nu$ implies $\mu \leq \nu$, when $\lambda \in C^-$. We summarize this discussion as follows.

\begin{equation}
(5.1) \quad \mu \leq_{\text{nat}} \nu \implies \mu \leq_W \nu \implies \mu \leq \nu (\mu, \nu \in [\lambda]).
\end{equation}

There is a version of these implications for $[\lambda]^+$, when $\lambda \in C^-_{\text{rat}}$.

\begin{equation}
(5.2) \quad \mu \leq_{\text{nat}} \nu \implies \mu \leq_W \nu \implies \mu \leq \nu (\mu, \nu \in [\lambda]^+).
\end{equation}

The meaning of $\leq_{\text{nat}}$ in (5.2) is not quite the same as it is in (5.1) since the $\leq_{\text{nat}}$ for $[\lambda]^+$ is computed with respect to different standard modules. However, if $\mu \leq_{\text{nat}} \nu$ in (5.2), then $\mu \leq_{\text{nat}} \nu$ in (5.1). This implies the validity of (5.2). We will mostly be using (5.2), so we have preferred not to use separate notations for the two orders denoted $\leq_{\text{nat}}$. (Finally, it is interesting to observe that in (5.1) we have $\leq_{\text{nat}} = \leq_W$, though we will not need this fact.)

The main conclusion to be drawn is that Proposition 5.5 holds as written, if the order $\leq$ is replaced by $\leq_{\text{nat}}$ or by $\leq_W$, although it must be understood that the meaning of $\leq_{\text{nat}}$ varies between $\leq_W$ and $\leq_{\text{nat}}$. A similar observation holds regarding Theorem 5.1.

(b) We can also define categories $\tilde{O}_k$, $\tilde{O}_k^+$ and $\tilde{O}_k^{+,\text{finite}}$, just as the (strict) images under the restriction functor of the corresponding categories $O_k$, $O_k^+$ and $O_k^{+,\text{finite}}$ of $g$-modules. Each such strict image is a full subcategory of $g$-modules, by Proposition 5.1 (All of their objects belong to $M_{\text{res}}^+(\bar{\mathfrak{g}})$.) Also, $\tilde{O}_k^{+,\text{finite}} \cong \tilde{O}_k$. But it is not true that any of these new categories is equivalent to the category of $g$-modules from which its name is derived, since, in particular, no generalized eigenvector of $\Omega$ has been specified. Thus, while $O_k \cong \tilde{O}_k^{+,\text{finite}}$, the category $O_k$ is not equivalent to $\tilde{O}_k^{+,\text{finite}}$. Instead, $O_k$ is equivalent to $\tilde{O}_k^{+,\text{finite}}$ as Corollary 5.2 shows. (It is true that $\tilde{O}_k^{+,\text{finite}}$ is equivalent to a direct sum of copies of $O_k$.) This is not an issue, however, with the $\tilde{g}$-categories in Theorem 5.1 since for example the generalized eigenvalue of the Casimir operator $\Omega$ on objects of $O^\infty[\lambda]$ is determined by $\lambda$.

In [47], Tanisaki described the group $W(\lambda)$ explicitly for any $\lambda \in C^-_{\text{rat}}$. He also describes the dot action of $W(\lambda)$ on $[\lambda]$ in more explicit terms. Implicit in his discussion is a description of $\Phi(\lambda)$, together with a set of fundamental roots (which can in any event be easily calculated). We return to this in §7.1.

6. Translation functors

Let $P$ be a fixed $\mathbb{Z}$-lattice $\mathfrak{h}^*$ whose projection onto $\mathfrak{h}_+^*$ relative to the decomposition (2.1) is $\tilde{P} + \mathbb{Z} \chi$, where $\tilde{P}$ is the weight lattice of $\tilde{g}$ in $\mathfrak{h}_+^*$. Let $P^+$ be the set of all $\lambda \in P$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{N}$ for all $i = 0, \ldots, r$.

Given $\lambda, \mu \in C^-$ with $\mu - \lambda \in WP^+$, Kashiwara-Tanisaki [83] define an exact translation functor

$$T_\lambda^\mu : O[\lambda] \to O[\mu].$$

The definition is the familiar one, taking a “block” projection, after tensoring with an irreducible module $L(\gamma), \gamma \in P^+ \cap W(\mu - \lambda)$. We summarize the key properties they prove in the following proposition.

Proposition 6.1. ([83] Prop. 3.6, 3.8) Assume $\lambda, \mu \in C^-$ satisfy $\mu - \lambda \in WP^+$ and $\Phi_0(\lambda) \subseteq \Phi_0(\mu)$.

(a) For any $w \in W(\lambda)$,

$$T_\lambda^\mu M(w \cdot \lambda) \cong M(w \cdot \mu).$$
(b) Also,

\[ T_\mu^\lambda L(w \cdot \lambda) = \begin{cases} L(w \cdot \mu), & \text{if } w(\Phi_0^+(\mu) \setminus \Phi_0^+(\lambda)) \subseteq \Phi^+(\lambda); \\ 0, & \text{otherwise.} \end{cases} \]

**Remark 6.2.** Recall that if \( \lambda' \in [\lambda] \) we let \( w_{\lambda'} \in W(\lambda) \) be the shortest element \( x \in W(\lambda) \) such that \( w \cdot \lambda = \lambda' \). The condition \( w(\Phi_0^+(\mu) \setminus \Phi_0^+(\lambda)) \subseteq \Phi^+(\lambda) \) in Proposition 6.1(b) is equivalent to \( w_{\lambda'} = w_{\mu'} \) where \( \lambda' = w \cdot \lambda \) and \( \mu' = w \cdot \mu \). In case \( \lambda \) is regular, the condition is that \( w = w_{\mu'} \).

We add some additional useful properties of \( T_\mu^\lambda \) after extending these functors in two ways. First, the “block” projection definition of translation extends easily to \( \Omega^\infty \). We use the same notation \( T_\mu^\lambda \) for this extension, so that now we have the functor \( T_\mu^\lambda : \Omega^\infty \to \Omega^\infty \).

Second, \( T_\mu^\lambda \) and the natural equivalences \( \tilde{\Omega}[\lambda] \cong \Omega^\infty \) and \( \Omega^\infty \cong [\mu] \) define a composite functor \( \tilde{T}_\mu^\lambda : \Omega^\infty \to \Omega^\infty \).

Therefore, there is a commutative diagram

\[
\begin{array}{ccc}
\Omega^\infty[\lambda] & \xrightarrow{T_\mu^\lambda} & \Omega^\infty[\mu] \\
\downarrow \sim & & \downarrow \sim \\
\tilde{\Omega}[\lambda] & \xrightarrow{\tilde{T}_\mu^\lambda} & \tilde{\Omega}[\mu]
\end{array}
\]

where the vertical maps are just restriction of functors.

We now list these additional properties of \( T_\mu^\lambda \).

**Proposition 6.3.** Assume \( \lambda, \mu \in \mathcal{C}^- \) satisfy the properties \( \Phi_0(\lambda) \subseteq \Phi_0(\mu) \) and \( \mu - \lambda \in WP^+ \) of Proposition 6.1. Then the following statements hold.

(a) \( T_\mu^\lambda M^\ast(w \cdot \lambda) \cong M^\ast(w \cdot \mu) \) and \( \tilde{T}_\mu^\lambda \tilde{M}^\ast w \cdot \lambda \cong \tilde{M}^\ast w \cdot \mu \).

(b) If \( \Phi_0(\lambda) = \Phi_0(\mu) \), then \( T_\mu^\lambda \) and \( \tilde{T}_\mu^\lambda \) give equivalences of categories

\[
\begin{cases}
T_\mu^\lambda : \Omega^\infty[\lambda] \xrightarrow{\sim} \Omega^\infty[\mu] \\
\tilde{T}_\mu^\lambda : \tilde{\Omega}[\lambda] \xrightarrow{\sim} \tilde{\Omega}[\mu].
\end{cases}
\]

(c) Again assume that \( \Phi_0(\lambda) = \Phi_0(\mu) \). If \( \Gamma \subseteq [\lambda] \), let \( \Gamma' = \{ w : \mu \mid w \cdot \lambda \in \Gamma \} \). Then \( \Gamma \) is a poset ideal in \( ([\lambda], \leq W) \) if and only if \( \Gamma' \) is a poset ideal in \( ([\mu], \leq W) \). In this case, the posets \( \Gamma \) and \( \Gamma' \) are isomorphic by the evident map \( w : \lambda \mapsto w \cdot \mu \), and the functors \( T_\mu^\lambda \) and \( \tilde{T}_\mu^\lambda \) induce (by restriction) category equivalences

\[
\begin{cases}
T_\mu^\lambda : \Omega^\Gamma[\lambda] \xrightarrow{\sim} \Omega^\Gamma[\mu] \\
\tilde{T}_\mu^\lambda : \tilde{\Omega}^\Gamma[\lambda] \xrightarrow{\sim} \tilde{\Omega}^\Gamma[\mu].
\end{cases}
\]

**Proof.** We first prove (a). Because \( L(\mu)^\ast \cong L(\mu) \), the duality on \( \Omega[\mu] \) preserves blocks in the category \( \Omega[\lambda] \). It follows thus by construction that the translation functor \( T_\mu^\lambda \) commutes with duality. Now apply Theorem 6.1(a).

Next, we consider (c). First, the assertions concerning \( \Gamma \) and \( \Gamma' \) follow from Remark 6.2. Because the exact functor \( T_\mu^\lambda \) takes standard (resp., costandard) modules \( \Delta(w \cdot \lambda) \) (resp., \( \nabla(w \cdot \lambda) \)), \( w \cdot \lambda \in \Gamma \), in \( \Omega[\lambda] \) to standard (resp., costandard) modules \( \Delta(w \cdot \mu) \) (resp., \( \nabla(w \cdot \mu) \)) in \( \Omega[\mu] \), the comparison theorem [39, Thm. 5.8] implies it is an equivalence of categories. A similar argument applies to \( \tilde{T}_\mu^\lambda \). This proves (c).

Finally, (b) follows from (c), writing \( [\lambda] \) as a (directed) union of finite ideals \( \Gamma \). \( \Box \)

**Lemma 6.4.** Let \( \lambda, \mu \in \mathcal{C}^- \) satisfying the conditions \( \mu - \lambda \in WP^+ \) and \( \Phi_0(\lambda) \subseteq \Phi_0(\mu) \). Let \( w \in W(\lambda) = W(\mu) \).

(a) If \( w \cdot \mu \in [\mu]^+ \), then \( w \cdot \lambda \in [\lambda]^+ \). Also, when \( w = w_{\mu'} \), for \( \mu' \in [\mu]^+ \), then \( w = w_{\lambda'} \) for \( \lambda' = w \cdot \mu \).

(b) Assume \( \lambda' := w \cdot \lambda \in [\lambda]^+ \). Put \( \mu' = w \cdot \mu \). If \( w_{\lambda'} = w_{\mu'} \), then \( \mu' \in [\mu]^+ \).
Proof. To prove (a), it must be shown that if $\alpha \in \mathfrak{p}$, then $\langle w \cdot \lambda, \alpha \rangle > 0$. By hypothesis, $0 \leq \langle w \cdot \mu, \alpha \rangle = \langle w(\mu + \rho), \alpha \rangle - 1$ for any simple root $\alpha \in \mathfrak{p}$. Hence, $(\mu + \rho, w^{-1}(\alpha)) > 1$. Since $\mu \in C^-$, this forces $w^{-1}(\alpha) < 0$. Now suppose that $\langle w \cdot \lambda, \alpha \rangle < 0$, for some $\alpha \in \mathfrak{p}$. Then, arguing as before, $\langle \lambda + \rho, w^{-1}(\alpha) \rangle \leq 0$. If $\langle \lambda + \rho, w^{-1}(\alpha) \rangle < 0$, then, since $\lambda \in C^-$, $w^{-1}(\alpha) > 0$, which is impossible. Thus, $w^{-1}(\alpha) \in \Phi_0(\lambda) \subseteq \Phi_0(\mu)$, again impossible.

Next, for the last assertion of (a), notice that $w = w_{\mu'}$ means that $w_{\mu'}$ is of minimal length in its coset $wW_0(\mu)$, so it is certainly of minimal length in $wW_0(\lambda) \subseteq wW_0(\mu)$, giving $w = w_{\lambda'}$. Now (a) is completely established.

Now we prove (b). We can assume that $w = w_{\mu'} = w_{\lambda'}$. Assume, for some $\alpha \in \mathfrak{p}$, $\langle w \cdot \mu, \alpha \rangle = 0$. In other words, $(\mu + \rho, \alpha) < 1$, i.e., $(\mu + \rho, w^{-1}(\alpha)) \leq 0$.

But $\langle w \cdot \lambda, \alpha \rangle \geq 0$, so that $\langle \lambda + \rho, w^{-1}(\alpha) \rangle \geq 0$. Since $\lambda \in C^-$, this means that $w^{-1}(\alpha) > 0$, so that $l(s_{\alpha}w) > l(w)$. Thus, $s_{\alpha}w \cdot \mu = w \cdot \mu$. Since $\mu \in C^-$, we have $\langle \mu + \rho, w^{-1}(\alpha) \rangle = 0$.

Combining the two previous paragraphs, $\langle \mu + \rho, w^{-1}(\alpha) \rangle = 0$. However, this means that $s_{\alpha}w(\mu + \rho) = \mu + \rho$, or equivalently $s_{\alpha}w \cdot \mu = w \cdot \mu$, a contradiction.

Proposition 6.5. Let $\lambda, \mu \in C_{rat}^-$ satisfy $\mu - \lambda \in WP^+$. Then the functor $T_{\mu}^\lambda$ takes objects in $\mathcal{O}^+\tau \lambda$ to objects in $\mathcal{O}^+\tau \mu$. Also, $T_{\mu}^\lambda$ maps objects in $\mathcal{O}^+\tau \mu$ to objects in $\mathcal{O}^+\tau \mu$. If $\tau \in [\mu]^+$, then $L(w, \lambda)$ is the unique irreducible module $L(\nu)$ for which $T_{\mu}^\lambda L(\nu) = L(\tau)$. A similar statement holds for $\tilde{L}(\tau)$ and $\tilde{\mathcal{O}}^+\tau \lambda$.

Proof. We wish to show that $L' := T_{\mu}^\lambda L(\lambda) \in \mathcal{O}^+\tau \mu$. If $L' \neq 0$, then it is $L(\mu')$ where $\mu' = w \cdot \mu$ and $w_{\lambda'} = w_{\mu'}$ by Remark 6.2. Thus, $\mu' \in [\mu]^+$ by Lemma 6.4(b), as desired. This proves the first assertion of the proposition. The second assertion concerning $\tilde{T}_{\mu}^\lambda$ is an easy consequence.

The final assertion follows from Lemma 6.4(a) and Remark 6.2.

We also obtain

Lemma 6.6. Let $\lambda, \mu \in C_{rat}^-$ be satisfy $\mu - \lambda \in WP^+$. Then $T_{\mu}^\lambda$ is a quotient of $M(w \cdot \mu)$. We can compare their characters using Proposition 6.1 and Weyl’s character formula (applied for $\mathfrak{p}$ on the irreducible modules $V(w \circ \lambda)$, $V(w \circ \mu)$). This shows (after inducing $s(w \cdot \lambda)$, $s(w \cdot \mu)$ from $\mathfrak{p}$ to $\mathfrak{g}$) that $T_{\mu}^\lambda M(w \cdot \mu)$ and $M(w \cdot \mu)$ have the same character. Hence, they are isomorphic. A similar argument applies to show $\tilde{T}_{\mu}^\lambda M(w \cdot \mu)$ is isomorphic.

Now the following analogue of Proposition 6.5 follows as in the proof of the latter.

Proposition 6.7. Let $\lambda, \mu \in C_{rat}^-$ satisfy the conditions of Proposition 6.1, $\mu - \lambda \in WP^+$, and $\Phi_0(\lambda) \subseteq \Phi_0(\mu)$. Then the following statements hold.

(a) $T_{\mu}^\lambda M(w \cdot \lambda) + \cong M(w \cdot \mu) +$ and $\tilde{T}_{\mu}^\lambda M(w \cdot \lambda) + \cong \tilde{M}(w \cdot \mu) +$.

Also, when $w = w_{\lambda'}$, for $\mu' \in [\mu]^+$, then $w = w_{\lambda'}$, for $\lambda' = w_{\lambda'}$.

(b) If $\Phi_0(\lambda) = \Phi_0(\mu)$, then $T_{\mu}^\lambda$ and $\tilde{T}_{\mu}^\lambda$ give equivalences of categories

$$
\begin{align*}
T_{\mu}^\lambda : \mathcal{O}^+\tau \lambda & \cong \mathcal{O}^+\tau \mu \\
\tilde{T}_{\mu}^\lambda : \mathcal{O}^+\tau \lambda & \cong \mathcal{O}^+\tau \mu.
\end{align*}
$$

(c) Again assume that $\Phi_0(\lambda) = \Phi_0(\mu)$. If $\Gamma \subseteq [\mu]^+$, set $\Gamma' = \{w \cdot \mu | w \cdot \lambda \in \Gamma\}$. Then $\Gamma$ is a poset ideal in $([\mu]^+, \leq W)$ if and only if $\Gamma'$ is a poset ideal in $([\mu]^+, \leq W)$. In this case, the posets $\Gamma$ and $\Gamma'$ are isomorphic.
by the evident map \( w \cdot \lambda \mapsto w \cdot \mu \), and the functors \( T^\lambda_\mu \) and \( \tilde{T}^\lambda_\mu \) induce (by restriction) category equivalences

\[
\begin{align*}
T^\lambda_\mu & : \mathcal{O}^{G,+}[\lambda] \xrightarrow{\sim} \mathcal{O}^{G,+}[\mu] \\
\tilde{T}^\lambda_\mu & : \tilde{\mathcal{O}}^{G,+}[\lambda] \xrightarrow{\sim} \tilde{\mathcal{O}}^{G,+}[\mu].
\end{align*}
\]

7. Quantum enveloping algebras and category equivalences

We continue to work with the indecomposable root system \( \Phi \), and we let \( \ell \) be a positive integer. Set \( D = (\theta_1, \theta_2)/(\theta_s, \theta_a) \in \{1, 2, 3\} \). Let

\[
\ell := \begin{cases} 
\ell, & \text{if } \ell \text{ is odd;} \\
\ell/2, & \text{if } \ell \text{ is even.}
\end{cases}
\]

There is a natural dot action of the affine Weyl group \( \hat{W}_e = \hat{W} \times e\hat{Q} \) on the set of integer weights \( \hat{P} \subseteq \hat{h}^* \), given by \( w \cdot \hat{\mu} = w(\hat{\mu} + \hat{\rho}) - \hat{\rho} \) for \( w \in \hat{W}_e \). The action without the “dot” \( \cdot \) is the usual action of \( \hat{W} \), and is translation on \( e\hat{Q} \). The fundamental reflections \( s_0, s_1, \cdots, s_r \) for \( W \) consist of the usual reflections \( s_i \) associated to fundamental roots \( \alpha_i \in \Pi, i = 1, \cdots, r \), together with the reflections \( s_0 \) in the affine hyperplane \( \{ x \in \hat{h}^* | (x + \rho, \theta'_s) = -e \} \).

The following proposition is an easy calculation, similar to those given in [47] p. 269. The first observations of this kind are likely those of [35]. We state it only for \( (D, e) = 1 \). (In particular, this condition holds when \( \Phi \) is simply laced.) A somewhat similar result holds without the assumption \( (D, e) = 1 \), though the group \( \hat{W}_e \) must be modified; see [47] Lemma 6.3.

**Proposition 7.1.** Let \( \ell \) be a positive integer and let \( e \) be as in \( (7.1) \). Let \( \lambda \in C_{\text{rat}} \) with \( \lambda(c) = k \), and assume that \( -(k + g) = \ell/2D \) and that \( (e, D) = 1 \) with \( e \) as above. There is an isomorphism \( \phi_\ell : \hat{W}_e \xrightarrow{\sim} W(\lambda) \) sending \( s_0, \cdots, s_n \) to the fundamental reflections defined by \( 2\delta - \theta_s, \alpha_1, \cdots, \alpha_r \in \Phi(\lambda) \) if \( \ell \) is odd, and to the fundamental reflections defined by \( \delta - \theta_s, \alpha_1, \cdots, \alpha_r \) if \( \ell \) is even. In both cases,

\[
\begin{align*}
\phi_\ell(s_i) & : (\hat{\mu} + k\chi) = w \cdot \hat{\mu} + k\chi, \quad w \in \hat{W}_e; \\
\phi_\ell(\ell\gamma) & : (\hat{\mu} + \gamma) = \hat{\mu} + e\gamma + k\chi \mod \mathbb{Z}\delta, \gamma \in \hat{Q}.
\end{align*}
\]

In particular, if \( \mu \in \hat{h}^* \) with \( \mu(c) = k \), then

\[
\phi_\ell(w) \cdot \mu = w \cdot \hat{\mu} + k\chi \mod \mathbb{C}\delta.
\]

More generally, if \( -(k + g) = e/m \) for some positive integers \( e, m \) with \( (m, e) = 1 \) and \( D|m \), then there is an isomorphism \( \phi : \hat{W}_e \xrightarrow{\sim} W(\lambda) \), where \( \phi(s_0) = s_{m\delta - \theta_s} \), and, for \( i = 1, \cdots, r \), \( \phi(\alpha_i) \) is equal to the fundamental reflection defined by \( \alpha_1, \cdots, \alpha_r \). The roots \( m\delta - \theta_s, \alpha_1, \cdots, \alpha_r \) are the standard (positive) fundamental roots in \( \Phi(\lambda) \).

**Remark 7.2.** The maps \( \phi \) and \( \phi_\ell \) agree when \( \ell/2D = e/m \), which is our major interest in this paper (the “quantum case”, at least when \( (D, e) = 1 \)). The exact description of \( \phi_\ell(w) \cdot \mu \) above (or of \( \phi(w) \cdot \mu \)) is

\[
\phi_\ell(w) \cdot \mu = (w \cdot \hat{\mu})^{k,a} = (w \cdot \hat{\mu} + k\chi)^{k,a} = w \cdot \hat{\mu} + k\chi + b\delta,
\]

where \( b \) is chosen so that the Casimir operator \( \Omega \) acts on \( L(\phi_\ell(w) \cdot \mu) \) with the same action as on \( L(\mu) \). That is, \( a = (\mu + 2\rho, \mu) \) and \( b = \frac{a-w \cdot \hat{\mu} + 2\delta \cdot \hat{\mu}}{2(k+2)} \). This reader is cautioned that the projections onto \( \mathbb{C}\delta \) for \( \mu \) and for \( \phi_\ell(w) \cdot \mu \) will generally be different. In particular, if \( \gamma \in \hat{Q} \), then \( \phi_\ell(\ell\gamma) \) acts as a translation by \( \ell\gamma \mod \mathbb{Z}\delta \) on the elements \( \mu \) of level \( k \) in \( \hat{h}^* \). That is, \( \phi_\ell(\ell\gamma) \cdot \mu = \mu + e\gamma \mod \mathbb{C}\delta \). However, it is not true in general that \( \phi_\ell(\ell\gamma) \cdot \mu = \mu + e\gamma \) exactly, even if \( \gamma \) is replaced on the right by any fixed element of \( \gamma + \mathbb{Z}\delta \).

One consequence of having to work mod \( \mathbb{C}\delta \) with level \( k \) weights is that the meaning of dominance orders in the correspondence between \( \hat{P} \) and \( \hat{P} + k\chi \mod \mathbb{C}\delta \) is lost. However, the Bruhat-Chevalley order is
preserved. See the §9, Appendix I for a discussion of the Bruhat-Chevalley orders relative to the often used partial orders ↑ of strong linkage.

Let \( \zeta \in \mathbb{C} \) be a primitive \( \ell \)th root of unity and set \( q = \zeta^2 \). Let \( U_\zeta = U_\zeta(\tilde{\Phi}) \) be the (Lusztig) quantum enveloping algebra at \( \zeta \) for the root system \( \tilde{\Phi} \). Let \( Q_\ell \) be the category of type 1, integrable, finite dimensional \( U_\zeta \)-modules. According to Tanisaki [47, Thm. 7.1], summarizing work of Kazhdan-Lusztig [34] and Lusztig [37], there is a category equivalence

\[
F_{\ell} : O_{-\ell/2D - g} \rightarrow Q_\ell
\]

(7.2)

between the category \( O_{-\ell/2D - g} \) of \( \tilde{\Phi} \)-modules and the category \( Q_\ell \). This holds for all positive integers \( \ell \), when \( \tilde{\Phi} \) has type \( A \) or \( D \), but restrictions are required in the other cases; see [47, Thm. 7.1 and Rem. 7.2]—note that Rem. 7.2(a) should be replaced by \( r > h \), the Coxeter number.

In the notation of the previous section, letting \( k := -\ell/2D - g \), \( O_k \) identifies with the category

\[
\bigcup_{\lambda \in C_{\text{rat}}, \lambda(c) = k} O_{\text{finite}}[\lambda]^+.
\]

In this notation, \( \tilde{\Phi}[\lambda + b\Phi] \) identifies with \( O_{\Phi}[\lambda]^+ \); indeed, these are identical subcategories of the category of \( \tilde{\Phi} \)-modules. For \( \mu \in [\lambda]^+ \), \( F_{\ell}L(\mu) = L_\zeta(\tilde{\mu}) \). Also, as noted in [47, Thm. 7], Kazhdan-Lusztig prove that \( F_{\ell}\Phi(\mu) \cong \Delta_{\zeta}(\tilde{\mu}) \). (Here \( \mu \equiv \tilde{\mu} + k\chi \mod \mathbb{C}\delta \)).

Let \( \lambda \in C_{\text{rat}} \), and let \( \Gamma \) be a finite ideal in \( [\lambda]^+ \) with respect to \( \leq W(\lambda) \) (see Remark 5.2(b)). Recall that every object \( M \) in the highest weight category \( O_{\Gamma}^{\Phi}[\lambda] \) has a finite composition series. Let \( P \) be a projective generator for \( O_{\Gamma}^{\Phi}[\lambda] \). If \( A_{\Gamma} := \text{End}(P) \), then \( O_{\Gamma}^{\Phi}[\lambda] \cong A_{\Gamma} \)-mod. It is convenient to choose \( P \) so that the modules in \( O_{\Gamma}^{\Phi}[\lambda] \) are actually \( A_{\Gamma} \)-modules. Put \( gr A_{\Gamma} = \bigoplus \text{rad}^n A_{\Gamma} / \text{rad}^{n+1} A_{\Gamma} \) be the positively graded algebra obtained from \( A_{\Gamma} \) using its radical filtration. Let \( gr O_{\Phi}[\lambda] \) the category \( gr A_{\Gamma} \)-grmod of graded \( gr A_{\Gamma} \)-modules.

**Theorem 7.3.** Let \( \lambda \in C_{\text{rat}} \) be regular, and let \( \Gamma \) be a finite ideal in the poset \( ([\lambda]^+, \leq W(\lambda)) \).

(a) The category \( gr O_{\Phi}[\lambda] = gr A_{\Gamma} \)-grmod is a highest weight category with poset \( \Gamma \) (or \( \Gamma_{\text{rat}} \)) and standard objects \( gr M(\mu)^+, \mu \in \Gamma \).

(b) The algebra \( gr A_{\Gamma} \) is a Koszul algebra. Also, \( gr O_{\Phi}[\lambda] \) has a graded Kazhdan-Lusztig theory with respect to the length function on \( \Gamma \) defined by the Coxeter length.

(c) For \( \mu = x \cdot \lambda \in [\lambda]^+ \), form the radical filtration \( M(\mu)^+ = F^0(\mu) \supseteq F^1(\mu) \supseteq \cdots \supseteq F^m(\mu) = 0 \) of \( M(\mu)^+ \). For \( \nu = y \cdot \lambda \in [\lambda]^+ \), \( [F^i(\mu)/F^{i+1}(\mu) : L(\mu)] \) is the coefficient of \( t^{(x) - (y) - i} \) in the inverse Kazhdan-Lusztig polynomial \( Q_{x,y} \) for the Coxeter group \( W(\lambda) \).

**Proof.** Suppose that \( \lambda \in C_{\text{rat}} \) is regular and that \( k = \lambda(c) \). Write \( -(k + g) = e/m \), where \( (e, m) = 1 \).

**Case 1.** \( D \) divides \( m \). Proposition [21] gives an isomorphism \( \phi : W_e \cong W(\lambda) \) of Coxeter groups, matching up indicated sets of fundamental reflections. For an integer \( k' \), put \( \lambda' = \lambda + k'\chi \), so that \( k' = \lambda'(c) \). We can choose \( k' \) so that \( \lambda' \in C_{\text{rat}} \) is regular. We can also choose \( k' \) so that \( -(k' + g) = \ell'/2D \) for an integer \( \ell' \) not divisible by 2 or 3 (if \( \tilde{\Phi} \) has type \( G_2 \)). Defining \( e' \) as in (7.1) (using \( \ell' \) for \( \ell \) and \( e' \) for \( e \)), we have \((D, e') = 1 \). Proposition 7.1 then gives an isomorphism \( \phi : W_{e'} \cong W(\lambda') \), again matching up fundamental reflections. Thus, there is an isomorphism \( W(\lambda) \rightarrow W(\lambda') \) preserving fundamental reflections. Since \( \lambda \) and \( \lambda' \) are both regular, \( W_0(\lambda) = W_0(\lambda') \) is trivial. Therefore, by [23, Thm. 11], there is a category equivalence \( O[\lambda] \cong O[\lambda'] \). (See Proposition 5.5.) Since the orders \( \leq W(\lambda) \) and \( \leq W(\lambda') \) obviously correspond, standard modules correspond (implicit in [23]). Similar comments apply to costandard modules.

The sets \( [\lambda]^+ \) and \( [\lambda']^+ \) are easily characterized (when \( \lambda, \lambda' \) are regular) in terms of representing elements \( w \cdot \lambda, w' \cdot \lambda' \) by requiring \( w \) to be of maximal length in \( W_0w \subseteq W(\lambda) \). Thus, we get an equivalence

\[ \text{The algebra } A_{\Gamma} \text{ is only determined up to Morita equivalence. However, if } A \text{ and } B \text{ are two Morita equivalent finite dimensional algebras, then } gr A \text{ is Morita equivalent to } gr B, \text{ and, in addition, these two graded algebras have equivalent graded module categories. See Appendix II.} \]
\(\mathcal{O}^+[\lambda] \cong \mathcal{O}^+\o[\lambda]\), as well as an equivalence of \(\overline{\mathcal{O}}^+[\lambda] \cong \overline{\mathcal{O}}^+\o[\lambda]\). (See Proposition 5.5.) Since the partial orderings correspond, so do their ideals. Let \(\Gamma' \leq [\lambda]^t\) correspond to \(\Gamma \leq [\lambda]^t\). Clearly, \(\overline{\mathcal{O}}^{\Gamma',+}[\lambda] \cong \overline{\mathcal{O}}^{\Gamma,+}[\lambda]\).

Adjusting \(k'\) further we can assume that \(F_{\ell'}\) is a category equivalence. In particular, we can assume that \(\ell'\) is odd, not divisible by 3 in type \(G_2\), and \(> h\). If \(\Gamma'\) is an ideal in the \(\leq_{\mathcal{W}(\lambda')}\) partial order, then \(\Gamma\) is an ideal in the \(\uparrow_{\mathcal{W}(\lambda')}\)-order, by Appendix I, Theorem 9.7. Therefore, \(\Gamma\) is an ideal in the natural order \(\leq_{\mathcal{N}}\) on \(\mathcal{P}\). (As is well-known, when \(\tilde{\mu} \leq_{\mathcal{N}} \tilde{\nu}\), then \(\mu \uparrow_{\mathcal{W}(\lambda')} \nu\).) We have \(F_{\ell'}\overline{\mathcal{O}}^{\Gamma',+}[\lambda] = Q_{\ell}[\Gamma']\). Combining this with the previous paragraph gives an equivalence \(\overline{\mathcal{O}}^{\Gamma',+}[\lambda] \cong Q_{\ell}[\Gamma']\) of highest weight categories. Notice that by Proposition 6.7(c), the posets \(\Gamma'\) and \(\Gamma\) are isomorphic. Now apply [41] Cor. 8.5.

Case 2: \(D\) does not divide \(e\). In this case, \(W(\lambda) \cong W_{e}\langle e \rangle = \mathcal{W} \times Q_{e}\), which is the affine Weyl group for the dual root system \(\Phi_{\mathcal{W}}\). By [23] Thm. 11 again, \(\overline{\mathcal{O}}^{\Gamma,+}[\lambda]\) is equivalent to a similar category, but replacing \(g\) by the affine Lie algebra associated to the dual root system. Hence, Case 2 reduces to Case 1. (Notice that [23] Thm. 11 does not require the underlying Lie algebras to be the same!)

**Theorem 7.4.** Let \(\lambda = \tilde{\lambda} + k\chi + b\tilde{\delta} \in C_{\mathcal{N}}\). For \(\mu \in [\lambda]^t\), \(M(\mu)^+\) has a filtration \(M(\mu)^+ = F^{0}(\mu) \supseteq F^{1}(\mu) \supseteq \cdots \supseteq F^{m}(\mu) = 0\) in which each section \(F^{i}(\mu)/F^{i+1}(\mu)\) is a semisimple \(\mathcal{G}\)-module, and such that, given any \(\nu \in [\lambda]^t\), the multiplicity \(\langle F^{i}(\mu)/F^{i+1}(\mu) : L(\nu) \rangle\) is the coefficient of \(t^{2\ell(w_{\nu})-\ell(w_{\nu})-i}\) in the inverse Kazhdan-Lusztig polynomial \(Q_{w_{\nu}, w_{\nu}}\) associated to \(W(\lambda)\). If \(\lambda\) is regular, then the filtration \(F^{\bullet}(\mu)\) is the radical filtration of \(M(\mu)^+\).

**Proof.** We prove this result in the “quantum case” (discussed in the proof of Theorem 7.3) in which \(-(k+g) = \ell/2D\) with \((D, e) = 1\). We leave the “non-quantum case” to the reader. The case in which \(\lambda\) is regular is handled in the previous theorem, so assume that \(\lambda\) is not regular. The weight \(\lambda' := -\tilde{2}k\chi + b\tilde{\delta}\) is obviously regular (so \(\Phi_{0}(\lambda') = 0 \subseteq \Phi_{0}(\lambda)\)), and it lies in \(C_{\mathcal{N}}\). Also, \(\lambda' = \lambda + 2\tilde{\rho} \in \mathcal{W}P^+ \subseteq WP^+\). In addition, \(W(\lambda) = W(\lambda')\).

Let \(\mu \in [\lambda]^t\), so that \(\mu = w_{\mu} \cdot \lambda\), where \(w_{\mu} \in W(\lambda)\) has minimal length among all \(w \in W(\lambda)\) for which \(\mu = w \cdot \lambda\). Thus, by Lemma 6.3 \(\mu' := w_{\mu} \cdot \lambda' \in [\lambda]^t\), and so \(T_{\lambda}^{2\mu} M(\mu)^+ = M(\mu)^+.\) Put \(F^{i}(\mu) = T_{\lambda}^{2\mu} \text{ rad }^{i} M(\mu)^{+}\). Then \(F^{\bullet}(\mu)\) is a filtration of \(M(\mu)^+\). Since the functor \(T_{\lambda}^{2\mu}\) is exact we have

\[
F^{i}(\mu)/F^{i+1}(\mu) \cong T_{\lambda}^{2\mu}(\text{ rad }^{i} M(\mu)^{+}/\text{ rad }^{i+1} M(\mu)^{+})
\]

is semisimple. To determine the multiplicity \(\langle F^{i}(\mu)/F^{i+1}(\mu) : L(\nu) \rangle\), we can assume that \(\nu \in [\lambda]^t\). Write \(\nu = w_{\nu} \cdot \lambda'\). Since

\[
\text{ rad }^{i} M(\mu)^{+}/\text{ rad }^{i+1} M(\mu)^{+} : L(w_{\nu} \cdot \lambda') = [\text{ rad }^{i} M(\mu)^{+}/\text{ rad }^{i+1} M(\mu)^{+} : L(w_{\nu} \cdot \lambda')],
\]

which is the coefficient of \(t^{2\ell(w_{\nu})-\ell(w_{\nu})-i}\) in the inverse Kazhdan-Lusztig polynomial \(Q_{w_{\nu}, w_{\nu}}\) of \(W(\lambda)\), the result follows.

Now we consider an analogous result for the quantum enveloping algebras \(U_{\zeta}\) of type \(A_{n}\) or \(D_{n}\). Let \(C^-\) be the anti-dominant chamber. Given a dominant weight \(\nu\), let \(w_{\nu} \in W_{e}\) have minimal length so that \(w_{\nu}^{-1} \nu \in C^-\).

**Theorem 7.5.** Assume that \(\mathcal{W}\) has type \(A\) or \(D\). Also, for type \(D_{2n+1}\), it is required that \(e \geq 3\). (a) Assume that \(\mu, \nu \in P^+\) are dominant weights which \(W_{e}\)-conjugate. Then the standard module \(\Delta_{\zeta}(\mu)\) has filtration

\[
\mathcal{O}^{+}[\lambda] \cong \mathcal{O}^{+}[\lambda']
\]

Results claimed in [3] §9.5 (para. 1), Lem. 9.10.5, Thm. 9.10.2, together with the order compatibility result given in Theorem 7 in Appendix I, imply that \(A_{\mu'}\) itself is Koszul. (According to [24] Appendix, a result like our Theorem 9.6 is required in [41] Lem. 9.10.5) to make its proof work. In turn, this lemma is required for [3] Thm. 9.10.1; a main result. However, we only need that \(\text{ gr } A_{\mu'}\) is Koszul for the results below in the singular weight case, and no better result is obtained in these cases by knowing \(A_{\mu'}\) is Koszul here. (Koszulity is not generally preserved under exact functors.)

There is no other restriction on the positive integer \(e\). In case \(e\) is odd, the arguments in Theorems 7.3 and 7.4 can be rearranged to treat all quantum cases, using translation functors alone, without recourse to [23]. Of course, use of [23] not only handles the \(e\) even case, but also allows a treatment for affine Lie algebras at all weights in \(C_{\mathcal{N}}\) in Theorems 7.3 and 7.4.
\[ \Delta_{\xi}(\mu) = F^0(\mu) \supseteq \cdots \supseteq F^m(\mu) = 0 \] by \( U_{\xi} \)-submodules with each section \( F^i(\mu)/F^{i+1}(\mu) \) a semisimple \( U_{\xi} \)-module. Further, the multiplicity of \( [F^i(\mu)/F^{i+1}(\mu) : U_{\xi}(\nu)] \) can be taken to be the coefficient of \( q^{l(w)} - q^{l(w) - 1} \) in the inverse Kazhdan-Lusztig polynomial \( Q_{w_\tau, \nu} \) associated to \( W(\lambda) \). If \( \mu \) is regular, then the filtration \( F^\bullet(\mu) \) is the radical filtration of \( \Delta_{\xi}(\mu) \).

(b) Assume that \( e \geq h \). Let \( \Gamma \) be a finite ideal of \( e \)-regular weights. Let \( B_{\Gamma} \) be the finite dimensional algebra whose module category identifies with the category of \( U_{\xi} \)-modules having highest weights in \( \Gamma \). Then the algebra \( \text{gr} B_{\Gamma} \) is a Koszul algebra. Also, the category \( \text{gr} B_{\Gamma} - \text{grmod} \) has a graded Kazhdan-Lusztig theory with respect to the length function (defined on \( W_e \)-orbits in \( \Gamma \) by the Coxeter length).

Proof. This follows from Theorem 7.4 since for types \( A \) and \( D \) as indicated, the functor \( F_{\Gamma} \) is an equivalence of categories, preserving standard modules. \( \square \)

The argument above, traced through from the proof of Theorem 6.3 gives the additional result that, under the hypotheses of Theorem 7.3(b), there is an isomorphism \( \text{Ext}^n_{B_{\Gamma}}(L, L') \cong \text{Ext}^n_{\text{gr} B_{\Gamma}}(L, L') \), valid for all \( n \geq 0 \) and for all irreducible \( B_{\Gamma} \)-modules \( L, L' \) (which are naturally irreducible \( \text{gr} B_{\Gamma} \)-modules); see [11] Cor. 8.5(a). Consequently, the homological algebra of \( B_{\Gamma} \) in Theorem 7.3(b) is very close to that of \( \text{gr} B_{\Gamma} \). Of course, \( B_{\Gamma} \) is even isomorphic to \( \text{gr} B_{\Gamma} \), if we grant the Koszulity of \( A_{\Gamma} \) argued in footnote 7.

8. Applications

In this section, we reinterpret Theorem 7.5 for the \( q \)-Schur algebras and then pass to a similar result for Specht modules for Hecke algebras. Then we briefly raise some open questions. Finally, we obtain some similar results for classical Schur algebras in positive characteristic, involving the James conjecture and the bipartite conjecture.

8.1. \( q \)-Schur and Hecke algebras. Given a Coxeter system \((W, S)\), let \( \widetilde{H} = \widetilde{H}(W) \) be the Hecke algebra over \( \mathbb{Z} = \mathbb{Z}[q, q^{-1}] \) (Laurent polynomials in a variable \( q \)) with basis \( \{ \tau_w \mid w \in W \} \) and defining relations

\[ \tau_s \tau_w = \begin{cases} \tau_{sw}, & \text{if } l(sw) = l(w) + 1 \\ q \tau_w + (q - 1) \tau_{sw}, & \text{otherwise} \end{cases} \]

for \( s \in S, w \in W \).

Let \( \Psi : \widetilde{H} \rightarrow H \) be the \( \mathbb{Z} \)-algebra involution defined by \( \Psi(\tau_w) = (-q)^{l(w)} \tau_w^{-1} \). If \( \widetilde{M} \) is a \( \widetilde{H} \)-module, then \( \widetilde{M}^\Psi \) denotes the module obtained by making \( \widetilde{H} \) act through \( \Psi \).

For example, let \( \mathcal{S}_r \) be the symmetric group of degree \( r \), and let \( S = \{(1, 2, \ldots, (r - 1, r)\} \). Then \( (\mathcal{S}_r, S) \) is a Coxeter system. In this case, denote \( \widetilde{H}(W) \) simply by \( \widetilde{H} \), or \( H(r) \) if \( r \) needs to be mentioned. Let \( \Lambda(n, r) \) (resp., \( \Lambda^+(n, r) \)) be the set of compositions (resp., partitions) \( \lambda \) of \( r \) with at most \( n \) parts; let \( \Lambda(r) \) (resp., \( \Lambda^+(r) \)) be the set of all compositions (resp., partitions) of \( r \). For \( \lambda \in \Lambda(n, r) \), let \( \overline{T}_\lambda \) be the right “permutation” module for \( \widetilde{H} \) defined by \( \lambda \), and \( \overline{T}(n, r) := \bigoplus_{\lambda \in \Lambda(n, r)} \overline{T}_\lambda \). The (integral) \( q \)-Schur algebra (of bidegree \( (n, r) \)) is the endomorphism algebra

\[ S_q(n, r) := \text{End}_{\overline{T}}(\overline{T}(n, r)) \]

Given any commutative \( \mathbb{Z} \)-algebra \( K \) (e. g., a field), let \( S_q(n, r)_K \) (or just \( S_q(n, r) \) if \( K \) is clear) denote the \( K \)-algebra \( S_q(n, r) \otimes K \) —it has a description similar to (8.1.1), replacing \( \widetilde{H} \) and \( \overline{T}(n, r) \) by \( H = \widetilde{H}_K \) and \( T(n, r) = \overline{T}(n, r)_K \), respectively.

From now on assume that \( K \) contains \( \mathbb{Q}(\zeta) \), where \( \zeta \) is a primitive \( \ell \)th root of 1. Put \( q = \zeta^2 \), a primitive \( \ell \)th root of 1, in the notation of the previous section. (No restriction is placed on \( e \), except as otherwise noted.) The triple \( (S_q(n, r), T, H) \) satisfies the “ATR” set-up prosylhetized in [12]. In particular, given \( M \in \text{mod} - H \) (right modules), put \( M^\circ := \text{Hom}_{\overline{T}}(M, T) \in S_q(n, r) - \text{mod} \), and, given \( N \in S_q(n, r) - \text{mod} \), let \( N^\circ := \text{Hom}_{S_q(n, r)}(N, T) \). In this way, there is a contravariant functor \( M \mapsto M^\circ \) (resp., \( N \mapsto N^\circ \)) from \( \text{mod} - H \) to \( S_q(n, r) - \text{mod} \) (resp., \( S_q(n, r) - \text{mod} \) to \( \text{mod} - H \)). The convenience of denoting them by the same symbol overcomes the annoyance of denoting them by the same symbol!

If \( U_{\xi} \) is the quantum enveloping algebra of type \( A_{n-1} \) over \( K \), there is a surjective homomorphism \( U_{\xi} \rightarrow S_q(n, r) \). In this way, \( S_q(n, r) - \text{mod} \) is embedded in \( U_{\xi} - \text{mod} \). In addition, \( S_q(n, r) - \text{mod} \) is a highest weight
category with poset \((\Lambda^+(n, r), \preceq)\) defined by the dominance order on partitions. Irreducible modules \(L_q(\lambda)\), standard modules \(\Delta_q(\lambda)\), and costandard modules \(\nabla_q(\lambda)\) are all indexed by \(\Lambda^+(n, r)\). When regarded as \(U_\mathbb{C}\)-modules, \(L_q(\lambda)\) gets relabeled as \(L_\mathbb{C}(\lambda)\), where \(\bar{\lambda} \in \bar{P}^+\) is defined as follows: write \(\lambda = (\lambda_1, \cdots, \lambda_r)\), \(\lambda_1 \geq \cdots \geq \lambda_r\), and put \(\bar{\lambda} = a_1 \varpi_1 + \cdots + a_{r-1} \varpi_{r-1}\) with \(a_i := \lambda_i - \lambda_{i+1}\). (In this expression, we label the simple roots for \(A_{r-1}\) in the usual way, as in [5].) Each \(\lambda \in \Lambda^+(r)\), thus determines \(w_\lambda \in \mathfrak{S}_r\) which has minimal length among all \(w\) satisfying \(w^{-1} \cdot \lambda \in C^-\) (the anti-dominant chamber for \(U_\mathbb{C}\)).

In particular, for \(\lambda \in \Lambda^+(n, r)\), we have

\[
\begin{cases}
\Delta_q(\lambda)^\circ \cong S^\lambda, \\
\nabla_q(\bar{\lambda})^\circ \cong S^\lambda_{\bar{\lambda}}.
\end{cases}
\]

In this expression, \(\lambda'\) denotes the conjugate partition to \(\lambda \in \Lambda^+(r)\). In addition, the irreducible \(H\)-modules are indexed by the set \(\Lambda^+(r)_{\text{row-reg}}\) of (row) \(e\)-regular partitions (i.e., no row is repeated \(e\)-times). If \(\lambda \in \Lambda^+(r)\), then \(\lambda\) is \(e\)-restricted (i.e., it has all coefficients of fundamental dominant weights positive and \(< e\)) if and only if \(\lambda'\) is \(e\)-regular. Then for \(\lambda \in \Lambda^+_e(r)\) (the \(e\)-restricted partitions),

\[
L(\lambda)^\circ \cong \begin{cases}
D^\lambda_{\bar{\lambda}}, & \lambda \in \Lambda^+_e(r); \\
0, & \text{otherwise}.
\end{cases}
\]

**Theorem 8.1.1.** Assume that \(K\) is a field containing \(\mathbb{Q}(\zeta)\).

(a) For \(\lambda \in \Lambda^+(n, r)\), the \(q\)-Weyl module \(L_q(\lambda)\) for the \(q\)-Schur algebra \(S_q(n, r)\) has a filtration \(\Delta_q(\lambda) = F^0(\lambda) \supseteq F^1(\lambda) \supseteq \cdots \supseteq F^m(\lambda) = 0\) with semisimple sections \(F^i(\lambda)/F^{i+1}(\lambda)\) in which, given \(\nu \in \Lambda^+(r)\), the multiplicity of \(L_q(\nu)\) in \(F^i(\lambda)/F^{i+1}(\lambda)\) is the coefficient of \(t^{(\nu}(w_\lambda)^{-i}-(\nu_\lambda)^{-i}\) in the inverse Kazhdan-Lusztig polynomial \(Q_{\nu_\lambda, \nu_\lambda}\) associated to the affine Weyl group \(W_e\) of type \(A_{r-1}\).

(b) For \(\lambda \in \Lambda^+(r)\), the \(q\)-Specht module \(L^\lambda\) for the Hecke algebra \(H\) has a filtration \(0 = G^0(\lambda) \subseteq G^1(\lambda) \subseteq \cdots \subseteq G^m(\lambda) = S^\lambda\) with semisimple sections \(G^{i+1}(\lambda)/G^i(\lambda)\) in which, given \(\nu \in \Lambda^+_e(r)\), the multiplicity of the irreducible \(H\)-module \(D^\lambda_\nu\) in the section \(G^{i+1}(\lambda)/G^i(\lambda)\) is the coefficient of \(t^{(\nu}(w_\lambda)^{-i}-(\nu_\lambda)^{-i}\) in the inverse Kazhdan-Lusztig polynomial \(Q_{\nu_\lambda, \nu_\lambda}\) associated to the affine Weyl group \(W_e\) of type \(A_{r-1}\).

Proof. (a) is merely a translation into the language of \(q\)-Schur algebras of Theorem 7.3(a).

As for (b), we can take \(n = r\). We first observe \(T := T(r, r) \cong S_q(n, r) f\) for an idempotent \(f \in S_q(n, r)\) [40, p. 664], and so \(T\) is projective. In particular, \(T\) is a tilting module for \(S_q(n, r)\) and is therefore self-dual. See [18, Thm. 8.4]. Thus, \(T\) is also an injective \(S_q(r, r)\)-module and so the “diamond functor”

\[
(-)^\circ = \text{Hom}_{S_q(r, r)}(-, T) : S_q(r, r)\text{-mod} \to \text{mod-}\text{H}
\]

is exact. Hence, (a) implies (b), putting \(G^i(\lambda) = F^{m-i}(\lambda)^\circ\). \(\square\)

**8.2. Open questions.** We raise some open questions.

**Question 8.2.1.** Given \(\lambda \in \Lambda^+(r)\), when is it true that the filtration described in the proof of Theorem 8.1.1(b) is the socle filtration of \(S^\lambda\)? One should at least assume that \(\lambda\) is restricted, and the case where \(\bar{\lambda}\) is regular in the sense of alcove geometry is already interesting.

**Question 8.2.2.** When is there a positive grading on \(H\) (with grade 0 semisimple) such that for each \(\lambda \in \Lambda^+(r)\) there is a graded \(H\)-module structure on \(S^\lambda\) so that the multiplicities of irreducible \(H_0 \cong H/\text{rad } H\)-modules in each grade are as predicted by Theorem 8.1.1(b)? The same question may be asked for the quotient algebras \(H(n, r)\) defined in [18] and for the \(H(n, r)\)-modules \(S^\lambda, \lambda \in \Lambda^+(n, r)\).

**Question 8.2.3.** In [7], a \(Z\)-grading on Specht modules is given with respect to a \(Z\)-grading of the Hecke algebra. Since this grading is not, in general, a positive grading with the grade 0 term a semisimple algebra, individual grades of a given graded module are not necessarily semisimple modules. Nevertheless, it appears from the form of the graded multiplicities in [7], together with [18], that these multiplicities are the same coefficients which appear in our Theorem 8.1.1(b). The question, therefore, arises as to when it is possible to “regrade” the Hecke algebra \(H\) (shifting grades of projective indecomposable summands and passing to an endomorphism algebra) to achieve a positively graded algebra with grade 0 term semisimple in such a way
that the induced regradings of the Specht modules agree with our filtration sections as in Question 8.2.2. When this is possible, it answers Question 8.2.2 in a very specific way.

**Question 8.2.4.** For \( \lambda \in \Lambda^+(n,r) \), when is the filtration for \( \Delta_q(\lambda) \) described in Theorem 8.1.1(a) given by the radical series? The same question can be asked in all types; \([11]\) Thm. 8.4(c) gives a positive answer for regular highest weights. Lin \([36]\) Rem. 2.9(1)] suggests a positive answer in the singular case, at least for generic weights.

**Question 8.2.5.** When is there a positive grading on \( S_q(n,r) \) (with grade 0 semisimple) and on the standard modules \( \Delta_q(n,r) \) so that the grade \( i \) section multiplicities are predicted by those in Theorem 8.1.1(a) \( \text{for all } i \)? From the general theory of graded quasi-hereditary algebras \([11]\), if \( S_q(n,r) \) has a positive grading, its standard modules will automatically be graded.

**Question 8.2.6.** In \([3]\), Ariki gives a \( \mathbb{Z} \)-grading on \( S_q(n,r) \) and the standard modules under mild restrictions on \( e \). One can ask when some regrading process in this case serves to give a positive question in Question 8.2.5 above. When \( n \geq r \), \([3]\) computes the multiplicities of graded irreducible modules in his graded standard modules giving an answer involving (inverse) Kazhdan-Lusztig polynomials. Is there some positive regrading of the grading in \([3]\) possible so that the multiplicities in each grade agree with those in Theorem 8.1.1(a)?

**Question 8.2.7.** When is \( \text{gr} S_q(n,r) \) a quasi-hereditary algebra? When is it Koszul? One can also ask when \( \text{gr} S_q(n,r) \) has a Kazhdan-Lusztig theory in the sense of \([11]\), though it should be stated that the same question is open for singular blocks of \( S_q(n,r) \) itself.

Of course, all the above questions for \( q \)-Schur algebras can be asked in other types, i.e., for generalized \( q \)-Schur algebras.

### 8.3. Positive characteristic

Now assume that \( k \) is an algebraically closed field of positive characteristic \( p \). For positive integers \( n, r \), let \( S(n,r) = S_1(n,r) \) be the classical Schur algebra over \( k \) of bidegree \( (n,r) \); see \([20]\) for a detailed discussion in this special case. The irreducible \( S(n,r) \)-modules \( L(\lambda) \) are indexed by partitions \( \lambda \in \Lambda^+(n,r) \).

Form the PID \( \mathbb{Z}' = \mathbb{Z}[q]/[q^2 = \sqrt{1}] \). The \( q \)-Schur algebra \( S_q(n,r) \), taken over \( \mathbb{Q}(q) \), has a standard integral \( \mathbb{Z}' \)-form \( S_q(n,r)' \) such that \( S_q(n,r) = S_q(n,r)' \otimes_{\mathbb{Z}'} k \). For \( \lambda \in \Lambda^+(n,r) \), choose a \( S(n,r)' \)-stable \( \mathbb{Z}' \)-lattice \( L_q(\lambda)' \) in \( L_q(\lambda) \), and let \( \overline{L_q(\lambda)} = L_q(\lambda)' \otimes_{\mathbb{Z}'} k \) be the \( S(n,r) \)-module obtained by base change to \( k \). The following is a special case of a conjecture of James \([26]\).

**Conjecture 8.3.1.** (James Conjecture, defining characteristic case) If \( p^2 > r \), \( \overline{L_q(\lambda)} \cong L(\lambda) \) for all \( \lambda \in \Lambda^+(n,r) \).

The (full) James conjecture has been verified for all \( r \leq 10 \) \([26]\). It is also known that the James conjecture holds, for a fixed \( n \) and all \( r \), provided that \( p \) is sufficiently large. The conjecture is trivial unless \( p \leq r \), so that \( r \) must grow with \( p \) for the conclusion to be substantive.

For \( \lambda \in \Lambda^+(n,r) \), let \( \overline{\lambda} \) be the dominant weight for \( SL_n \) determined by \( \lambda \). Let \( w_\chi \) be the unique element \( x \) in the affine Weyl group \( W_p \) such that \( x \cdot \overline{\lambda} \in C^- \), the anti-dominant chamber. (See the discussion two paragraphs above Theorem 8.1.1.) In the same spirit, but motivated by \([21] \), \([8]\) Thm. 6.3], and, especially, the notion of an abstract Kazhdan-Lusztig theory given in \([11]\), we conjecture the following.

**Conjecture 8.3.2.** (Schur Algebra Bipartite Conjecture, explicit form) For \( p^2 > r \), the \( \text{Ext}^1 \)-quiver of \( S(n,r) \) is a bipartite graph. Explicitly, the decomposition

\[ E = \{ \lambda | l(w_\chi) \in 2\mathbb{Z} \}, \quad O = \{ \lambda | l(w_\chi) \in 2\mathbb{Z} + 1 \} \]

of \( \Lambda^+(n,r) \) is compatible with the bipartite \( \text{Ext}^1 \)-quiver.

The first sentence in Conjecture 8.3.2 means that \( \Lambda^+(n,r) \) decomposes into a disjoint union \( \Lambda^+(n,r) = E \cup O \) (the “even” and the “odd” partitions) such that \( \text{Ext}^1_{S_q(n,r)}(L(\lambda), L(\nu)) \neq 0 \) whenever \( \lambda, \nu \in \Lambda^+(n,r) \) are either both in \( E \) or both in \( O \). The second sentence provides explicit \( E \) and \( O \).

---

\(^9\)An advantage of using the James conjecture (over the Lusztig conjecture) is that it does not require that \( p \geq n \).
Theorem 8.3.3. Consider the Schur algebra $S(n, r)$ in positive characteristic $p$. Assume that $p^2 > r$ and that the James Conjecture 8.3.1 and the Bipartite Conjecture 8.3.2 are true.

(a) For $\lambda \in \Lambda^+(n, r)$, let $\{F^i(\lambda)\}$ be the semisimple series of $\Delta_\lambda(\lambda)$, $q = q_2 = \sqrt[q]{q}$, given in Theorem 8.3.1. Let $\mathcal{L}$ be an $S_q(n, r)^{\prime}$-stable $\mathbb{Z}^{\prime}$-lattice in $\Delta_\lambda(\lambda)$. Set $F^i(\lambda)^{\prime} = F^i(\lambda) \cap \mathcal{L}$ and $F^{i+1}(\lambda) = F^i(\lambda)^{\prime} \otimes_{\mathbb{Z}^{\prime}} k$. Then $\{F^i(\lambda)^{\prime}\}$ is a semisimple series in $\Delta(\lambda)$. Furthermore, for $\mu \in \Lambda^+(n, r)$,

$$[F^i(\lambda)/F^{i+1}(\lambda) : L(\mu)] = [F^i(\lambda)/F^{i+1}(\lambda) : L_q(\mu)].$$

In particular, this multiplicity is given by an inverse Kazhdan-Lusztig polynomial, as in Theorem 8.3.1.

(b) For $\lambda \in \Lambda^+(n, r)$, the Specht module $S^\lambda$ for $\mathfrak{S}_\lambda$ has a filtration $0 = G^0(\lambda) \subseteq G^1(\lambda) \subseteq \cdots \subseteq G^m(\lambda)$, in which, given $\mu \in \Lambda_{\text{type}}^+$, the multiplicity of the irreducible $\mathfrak{S}_\lambda$-module $D^\mu_{\lambda, \nu}$ in the section $G^{i+1}(\lambda)/G^i(\lambda)$ is the coefficient of $t^{q(\nu_\lambda) - q(\nu_\lambda)}$ in the inverse Kazhdan-Lusztig polynomial $Q_{\nu_\lambda, \nu_\lambda}$ associated to the affine Weyl group $W_e$ of type $A_{r-1}$. 

Proof. Since Conjecture 8.3.1 is assumed to hold, a section $F^i(\lambda)/F^{i+1}(\lambda)$ reduces mod $p$ to an $S(n, r)$-module whose composition factor multiplicities are given by 8.3.1. To check that $\{F^i(\lambda)\}$ is a semisimple series $\Delta(\lambda)$, it suffices to check that these sections are semisimple. But, if $L(\tau)$ and $L(\sigma)$ both appear in a composition series in $F^i(\lambda)/F^{i+1}(\lambda)$, then the coefficients of $t^{q(\nu_\lambda) - q(\nu_\lambda)}$ in $Q_{\nu_\lambda, \nu_\lambda}$ and of $t^{q(\nu_\lambda) - q(\nu_\lambda)}$ in $Q_{\nu_\lambda, \nu_\lambda}$ are nonzero. However, these are polynomials in $q = q_2 = \sqrt[q]{q}$, so that $l(\nu) - l(\nu') - i \equiv l(\nu) - l(\nu') - i \pmod{2}$.

It follows that $\nu$ and $\nu'$ have the same parity, so that $F^i(\lambda)/F^{i+1}(\lambda)$ is semisimple by Conjecture 8.3.2 as required for (a).

Part (b) is proved in the same way as Theorem 8.3.1(b). □

Remark 8.3.4. James also conjectured 28 a version of Conjecture 8.3.1 for $q$-Schur algebras, with $q$ an even root of 1 and $c p > r$. It is at least reasonable to ask, as a question, if the analog of Conjecture 8.3.2 holds under this assumption. Positive answers (to this question, and to this version of the James conjecture), would have consequences for the finite general linear groups $G$ in a non-defining characteristic $p$. See 14 §9, which shows the group algebra of $G$ has a large quotient which is a sum of tensor products of $q$-Schur at various roots of unity $q$. Another reference is 6.

For a different (though possibly related) use of Weyl module filtrations for (general) finite Chevalley groups, see 28 and 43.

9. Appendix I

If $\nu, \mu \in \mathcal{C}$, write $\nu \uparrow \mu$ if $\nu = \mu$, or, if there are positive real roots $\beta_1, \cdots, \beta_m$ (allowing repetition) such that the corresponding reflections $s_{\beta_i}$ satisfy

$$\nu = s_{\beta_m} s_{\beta_{m-1}} \cdots s_{\beta_1} \cdot \mu \leq \cdots \leq s_{\beta_1} \cdot \mu \leq \mu.$$  

(Observe that this forces each $\beta_i \in \Phi(\nu) = \Phi(\mu)$.) According to a result of Kac-Kazhdan (see 33 Thm. 3.1), $\nu \uparrow \mu$ if and only if $[M(\mu) : L(\nu)] \neq 0$. Here is another equivalence:

Proposition 9.1. Suppose that $\lambda \in \mathcal{C}^-$ and that $\mu, \nu \in [\lambda]$. Write $\nu = y \cdot \lambda$ and $\mu = w \cdot \lambda$ with $y, w \in W(\lambda)$ of minimal length. Then $\nu \uparrow \mu$ if and only if $y \leq w$ in the Bruhat-Chevalley order on $W(\lambda)$.

Proof. Suppose that $\nu \uparrow \mu$ and let $s_{\beta_1}, \cdots, s_{\beta_m}$ be as above. We can assume that all the inequalities in (9.1) are strict, i. e., putting $\mu_0 = \mu$ and, for $1 \leq i \leq m$, $\mu_i = s_{\beta_i} \cdot \mu_{i-1}$ then $\mu_i - \mu_{i-1} = n_i \beta_i$ for some $n_i \in \mathbb{Z}^+$. In particular, $\beta_i \in \Phi^+(\lambda)$, for each $i = 1, \cdots, m$.

Write a reduced expression $w = t_1 \cdots t_u$, where $t_j = s_{\alpha_j}$, for $\alpha_j$ a fundamental root of $\Phi(\lambda)$, $j = 1, \cdots, u$, $u \in \mathbb{N}$. (Actually, $u \neq 0$ since $y \cdot \lambda > \lambda$.) A standard argument shows that $s_{\beta} w \cdot \lambda < w \cdot \lambda$ implies that $\beta = t_1 \cdots t_{j-1}(\alpha_j)$, for some $j$, $1 \leq j \leq u$. Let $w_1$ be of minimal length with $w_1 \cdot \lambda = s_{\beta} w \cdot \lambda$. That is, $w_1$ is a distinguished member of the left-coset $s_{\beta} W(\lambda)$. Thus, $w_1 \leq s_{\beta} w$, while $s_{\beta} w = t_1 \cdots t_j \cdots t_m < w$. So $w_1 < w$. Continuing this way, eventually gives $y < w$, as desired.
Next, we start from the assumption $y < w$ and show that $\nu \uparrow \mu$ using induction on $l(w) - l(y)$. By Prop., p. 122, there exists $x \in W(\lambda)$ with $y < x < w$ and $l(w) - l(x) = 1$. Put $x = w'v$, where $v \in W_0(\lambda)$ and $w' \in xW_0(\lambda)$ is the element of shortest length. Then $y = x'v'$ where $x' < x$ and $v' \leq v$. The minimality of $y$ implies that $y = x' \leq w'$. Possibly, $y = w'$, but, in any case, induction implies that $\nu \uparrow w' \cdot \lambda = x \cdot \lambda$. However, $x$ may be obtained from a reduced expression $w = t_1 \cdots t_n$ by removing one of the fundamental reflections (with respect to $W(\lambda)$) reflections $t_j$. The discussion in the previous paragraph now shows that $x = s_\beta w$ for suitable $\beta \in \Phi^+(\lambda)$ with $s_\beta w \cdot \lambda < w \cdot \lambda$. Thus, $x \cdot \lambda \uparrow w \cdot \lambda \leq \mu$, so $\nu \uparrow x \cdot \lambda \uparrow \mu$, as desired. \hfill $\square$

**Lemma 9.2.** Suppose $\lambda \in C^-_{\text{rat}}, \mu \in [\lambda]^\ast$. Then there is a distinguished (i.e., minimal length) \((W, W_0(\lambda))\) double coset representative $d \in W(\lambda)$ with

$$
\mu = w_0d \cdot \lambda.
$$

The element $d$ is unique; write $d = d(\mu)$.

**Proof.** Choose an element $d \in W(\lambda)$ of minimal length with $d \cdot \lambda = w_0 \cdot \mu$. Since $\mu \in [\lambda]^\ast$, $s_\alpha \cdot \mu < \mu$ for all $\alpha \in \Phi^+$, whereas $s_\alpha \cdot (d \cdot \lambda) = s_\alpha(w_0 \cdot \mu) = s_\alpha(w_0(\mu + \zeta - \rho)) > w_0 \cdot \mu$. Thus $l(w_0d) \geq |\Phi^+| + l(d)$, by a count of separating hyperplanes. However, $l(w_0d) \leq l(w_0) + l(d) = |\Phi^+| + l(d)$, and so necessarily $l(w_0d) = l(w_0) + l(d)$.

It follows $d$ has the form $d'w'$ where $d'$ is a distinguished double coset representation of $W_0dW(\lambda)$ and $w' \in W(\lambda)$. Minimality of $l(d)$ gives $d = d'$, as desired. \hfill $\square$

In general, $w_0d$ will not be the element $\epsilon \in W(\lambda)$ of minimal length with $\epsilon \cdot \lambda = \mu$. We write $\epsilon = \epsilon(\mu)$ for such an element and note $d(\mu) \leq \epsilon(\mu) \leq w_0d(\mu)$.

**Proposition 9.3.** Suppose $\mu, \nu \in [\lambda]^\ast$ with $\lambda \in C^-_{\text{rat}}$. The the following are equivalent:

(a) $d(\mu) \leq d(\nu)$;
(b) $\epsilon(\mu) \leq \epsilon(\nu)$;
(c) $w_0d(\mu) \leq w_0d(\nu)$.

**Proof.** Note $w_0d(\mu) = \epsilon(\mu)w'(\mu)$ for some $w'(\mu) \in W(\lambda)$. By definition, $\epsilon(\mu)$ is the element of minimal length in $\epsilon(\mu)W_0(\lambda)$.

Thus, if (c) holds, then $\epsilon(\mu) \leq w_0d(\mu) \leq w_0d(\nu) = \epsilon(\mu)w'(\nu)$, and it follows that $\epsilon(\mu) \leq d(\nu)$, since $w_0d(\nu)w'(\nu)^{-1} = w_1d(\nu)w_2$ with $w_1 \in W_0$, $w_2 \in W_0(\nu)$ and $l(w_1d(\nu)w_2) = l(w_1)l(d(\nu)) + l(w_2)$. Then $d(\mu) = w'_1d(\nu')w'_2$ with $w'_1 \leq w_1$, $d(\nu') \leq d(\nu)$, $w'_2 \in w$, and $l(w'_1) + l(d(\nu')) + l(w'_2) = l(d(\mu))$. Minimality of $l(d(\mu))$ gives $d(\mu) = d(\nu') \leq d(\nu)$, which is (a).

Clearly, (a) implies (c), and the proposition is proved. \hfill $\square$

There is a further equivalence to add to the list. For $y, w \in W(\lambda)$, write $y \leq' w$, if $w_0yw_0 \leq w_0yw_0$, and put $l'(y) = l(w_0yw_0)$. In the above geometry, these operations amount to a change in generating fundamental reflections from the walls of the standard anti-dominant alcove to the walls of the standard dominant alcove.

Define $f(\mu)$, for $\mu \in [\lambda]^\ast$ and $\lambda \in C^-_{\text{rat}}$, to be the element $f \in W(\lambda)$ with minimal $l'$-length satisfying $f \cdot (w_0 \cdot \lambda) = w_0 \cdot \mu$.

**Proposition 9.4.** Suppose $\mu \in [\lambda]^\ast$, $\lambda \in C^-_{\text{rat}}$. Then $f(\mu) \cdot w_0d(\mu)w_0$. Moreover, if also $\nu \in [\lambda]^\ast$, then

$$
\text{if } f(\mu) \leq' f(\nu) \iff d(\mu) \leq d(\nu).
$$

**Proof.** By definition, $l(w_0d\nu) = l'(f)$. Thus, for $f = f(\mu)$, the element $w_0f\nu$ is the (unique) element $f'$ at minimal length $(f')$ satisfying $w_0f' \cdot \lambda = \mu$. However, $w_0d(\mu) \cdot \lambda = \mu, so w_0f'W_0(\lambda) = w_0d(\lambda)W_0(\lambda). Thus, l(d(\mu)) \leq l(f'), since d(\mu)W_0(\lambda). Therefore, d(\mu) = f' \cdot w_0f(\mu)w. So f(\mu) = w_0d(\mu)w_0$.

This last assertion is now obvious, and the proof is complete. \hfill $\square$

Finally, we need to compare the order $\leq'$ with the strong linkage order, in the sense of [27, II, §6.7]. Sections 6.1–6.11 of the latter reference apply for any positive integer $p$, which take to be $e$. For brevity, we refer the reader to those sections for details on the alcove notation used below. “Alcoves” are regarded as
certain open subsets of $\mathbb{R}P$, and the closure of an alcove is then a fundamental domain for the “dot” action of the affine Weyl group $W_e = \hat{W} \times e\hat{Q}$. The “standard alcove” $C$ satisfies
\[0 < \langle x + \rho, \alpha^\vee \rangle < 1,\]
for all $x \in C$, and this inequality defines $C$. For any alcove $C_1$, there are unique integers $n_\alpha = n_\alpha(C)$, for each $\alpha \in \Phi^+$, defined by
\[n_\alpha e < \langle x + \rho, \alpha^\vee \rangle < (n_\alpha + 1)e\]
and the function $d(C_1)$ is defined as $\sum_{\alpha \in \Phi^+} n_\alpha$. Allowing the right hand inequality “$\leq (n_\alpha + 1)e$” above to be the weaker “$\leq (n_\alpha + 1)e$”, defines the elements of the upper closure $\hat{C}_1$ of $C_1$. There is a “dot” action of $W_e$ on alcoves, agreeing with its “dot” action on $\hat{P}$, which is generated by reflections in the walls of $C$. For $y \in W_e$, write $l_e(y)$ for the length of $y$ with respect to this set of generating reflections, and $y \leq e w$ when $y, w \in W_e$ and $y \leq w$ in the Bruhat-Chevalley order with respect to these generating reflections. If $\mu \in P^+$, define $f(\mu) = f_e(\mu)$ to be the unique element $f \in W_e$ with $l_e(f)$ minimal satisfying $f \cdot x = \mu$ for some $x \in \overline{C}$, the closure of $C$. Equivalently, $\mu \in f \cdot \overline{C}$ [27 II, 6.11]. From separating hyperplane considerations,
\[(9.2)\]
\[l_e(f) = d(f \cdot C)\]
Using this identity, the following lemma is mostly an easy exercise.

**Lemma 9.5.** Let $\xi \in \overset{\circ}{P}^+ \cap \overset{\circ}{Q}$, i. e., a dominant weight of $\overset{\circ}{g}$ lying in the root lattice, and let $z \in W_e$ correspond to $e\xi \in e\hat{Q}$ (i. e., $z \cdot x = x + e\xi$, for all $x \in \mathbb{R}P$). Then, for any $\mu \in P^+$,
\[
\begin{cases}
  f(z \cdot \mu) = z f(\mu) \\
  l_e(z f(\mu)) = l_e(z) + l_e(f(\mu)).
\end{cases}
\]

**Proof.** Note that $\hat{C}_1 + e\xi = \overline{C}_1 + e\xi$, just by the definition of the upper closure. Thus, $f(z \cdot \mu) = z f(\mu)$. The length additivity is an easy calculations with the identity (9.2) and is left to the reader. \( \square \)

Now we can prove our main result on strong linkage in the sense of [27 II, 6.1–6.11]. To avoid conflict with our notation on $h^*$, we use $\uparrow_e$ to denote the $\uparrow$ ordering in [27 II, Ch. 6]. That is, if $\mu, \nu \in P$, write $\mu \uparrow_e \nu$ to mean that $\mu = \nu$ or there exists a chain $\mu = \mu_0 \leq \mu_1 \leq \cdots \leq \mu_m = \nu$ in $\overset{\circ}{P}$ and reflections $s_{i_1}, \cdots, s_{i_m}$ (not necessarily fundamental) in $W_e$ such that $s_i \cdot \mu_i = \mu_{i+1}$, for $i = 1, \cdots, m$. The order $\leq$ is used is the usual dominance order on $\overset{\circ}{P}$.

**Theorem 9.6.** Let $\mu, \nu \in P^+$ lie in the same orbit of $W_e$ under the dot action. Then
\[\mu \uparrow_e \nu \iff f(\mu) \leq_e f(\nu)\]
where $f(\mu)$ and $f(\nu)$ are the elements of $W_e$ described above.

**Proof.** First, note the general Coxeter group fact that if $u, w, z \in W_e$, and if the lengths of $zy$ and of $zw$ are obtained by adding the length of $z$ to that of $y$ and to that of $w$, respectively, then
\[y \leq_e w \iff zy \leq_e zw.\]
The implication that $y \leq_e w \implies zy \leq_e zw$ is obvious and the reverse implication reduces immediately to the case $e(z) = 1$, where it is obvious. (If $zy \leq_e zw$, then $y \leq_e zy \leq_e w$; otherwise, $zy = zw'$, where $w' \leq_e w$, whence $y = w' \leq_e w$.)

This fact, together with the preceding lemma, allows us to replace $y = f(\mu)$ and $w = f(\nu)$ by $zy, zw$ with $z \in W_e$ corresponding to an element $e\xi$ with $\xi \in \overset{\circ}{Q}$ and also dominant (i. e., in $P^+$). At the same time, we can replace $\mu, \nu$ by $\mu + e\xi, \nu + e\xi$, respectively. Obviously $\mu \uparrow_e \nu \iff \mu + e\xi \uparrow_e \nu + e\xi$. This equivalence holds for arbitrary weights $\mu, \nu \in P$ and thus may also be applied to intermediate instances of $\uparrow_e$. 
So starting from $\mu \uparrow_e \nu$ and making such an adjustment, we may assume all elements $\mu_0 \leq \mu_1 \leq \cdots \leq \mu_m$ in the defining chain are dominant, and also $l_\alpha(f(\mu_i)) = l_\alpha(f(\mu_{i-1})) + 1$, for $i = 1, \cdots, m$. (See the argument in [27, II, 6.10].) This implies that $\mu_{i-1}$ is obtained from $\mu_i$ by striking out a simple reflection, and so, in particular, $f(\mu_{i-1}) < f(\mu_i)$ for each $i = 1, \cdots, m$, and $f(\mu) \leq f(\nu)$.

Next, suppose that $f(\mu) \leq f(\nu)$ with $\mu, \nu \in P^+$ in the same $W_e$-orbit under the dot action. We want to show that $\mu \uparrow_e \nu$. By construction, $\mu, \nu$ belong to $f(\mu) \cdot C$ and $f(\nu) \cdot C$, respectively. This is equivalent to $f(\mu) \cdot C \uparrow_e f(\nu) \cdot C$, adapting the notation of [27, II, 6.11(3)]. As in [27], we say that an alcove $C_1$ is dominant if $n_\alpha(C_1) \geq 0$ for all $\alpha \in \Phi^+$. Both $f(\mu) \cdot C$ and $f(\nu) \cdot C$ are dominant. So, it suffices to prove that $y_1 \cdot C \uparrow_e y_2 \cdot C$, whenever $y_1 \leq y_2$ and $y_1 \cdot C, y_2 \cdot C$ are both dominant $(y_1, y_2 \in W_e)$.

We note that $y \cdot C$ is dominant if and only if $y$ is of minimal length $l_\alpha(y)$ in the coset $W \cdot y$. We now proceed by induction on the difference $m = l_\alpha(y_0) - l_\alpha(y_1)$. Without loss, $m \neq 0$ (where the desired result is trivially true). By [25, Prop. p. 122], there exists $y \in W_e$ with $y_1 \leq y <_e y_2$ and $l_\alpha(y) + 1 = l_\alpha(y_2)$. Thus, $y = sy_2$, where $s$ is an (affine) reflection in the hyperplane $H = H_{\alpha,n} = \{ x \in \mathbb{R}^P \; | \; \langle x + \rho, \alpha^\vee \rangle = ne \}$. The hyperplane $H$ must be one of those separating the dominant alcove $y \cdot C$ from $C$. See [25, Thm. p 58, Ex. 1, p. 58]. Since $y \cdot C$ is dominant, $n > 0$ for the parameter $n$ in $H = H_{\alpha,n}$.

Now [27, II, 6.8 Prop.] can be applied, to give a (unique) $w \in \hat{W}$ with $wy \cdot C$ dominant and $wy \cdot C \uparrow y_2 \cdot C$. (27 actually proves much more.) We have $y_1 \leq y$, and so $y_1 \leq wy$. Since $wy$ is of minimal length in the coset $W \cdot y$. We use that $y = w^{-1}(wy)$ with $l_\alpha(y) = l_\alpha(w^{-1}) + l_\alpha(wy)$. Thus a reduced expression of $y_1$ can be obtained from one for $y$ by omitting suitable reflections from $w^{-1}$, and from $wy$. However, $y_1$ is minimal in $W y_1$, so $y_1 \leq wy$. We have $l_\alpha(y_1) \leq l_\alpha(wy) \leq l_\alpha(y) < l_\alpha(y_2)$, so $y_1 \cdot C \uparrow wy \cdot C$ by induction. It follows that $y_1 \cdot C \uparrow y_2 \cdot C$, and the theorem is proved.

We can now give the promised relation of the order $\preceq'$ with the strong linkage order $\uparrow_e$. The order $\preceq'$ is on $W(\lambda), \lambda \in \mathcal{C}_{\text{rat}}$, whereas $\uparrow_e$ is on $\hat{P}$. We have already compared $\uparrow_e$ on $\hat{P}$ with $\preceq_e$ on $W_e$ in Theorem 9.6. It remains now only to compare $\preceq_e$ on $W_e$ with $\preceq'$ on $W(\lambda)$. For simplicity, we state the comparison only in the simply laced case, through a similar result holds in all types.

**Theorem 9.7.** Assume $\Phi$ is simply laced and let $\ell$ be a positive integer, $e = \ell$ if $\ell$ is odd and $e = \ell/2$ if $\ell$ is even. For $y, w \in W_e$, $y \preceq_e w$ if and only if $\phi_e(y) \preceq' \phi_e(w)$. Also,

(a) If $\lambda \in C$ is dominant, and $y, w \in W_e$ are minimal with $y \cdot \lambda, w \cdot \lambda$ dominant, then $y \preceq_e w$ if and only if $y \cdot \lambda \uparrow_e w \cdot \lambda$.

(b) In the affine case, let $\lambda \in \mathcal{C}_{\text{rat}}$ and let $\mu, \nu \in [\lambda]$. Write $\mu = y \cdot \lambda$ and $\nu = w \cdot \lambda$, with $y' := w_0 y w_0$ and $w' = w_0 y w_0$ both minimal with respect to $\preceq'$ (or, equivalently, $y, w$ are minimal with respect to $\preceq$). Then $y' \preceq w'$ if and only if $\mu \uparrow \nu$. (Also, equivalent is $y \preceq w$.)

10. APPENDIX: II: Graded algebras and Morita equivalence

We show that if $A$ is a finite dimensional algebra which is Morita equivalent to another finite dimensional algebra $B$, then the graded algebras $gr A$ and $gr B$ obtained from the radical filtrations of $A$ and $B$ are also Morita equivalent. Also, the corresponding categories of finite dimensional graded modules categories are equivalent.

Let $A$ be a finite dimensional algebra over a field. Let $P$ be any (finite dimensional) projective $A$-module. Any $A$-map $x : P \to P$ which doesn’t send $P$ to rad $P$ can be multiplied by another $A$-map $P \to P$ to get the identity on an irreducible summand of $P/\text{rad } P$. So the original map $x$ can’t be in the radical of $End P$.

On the other hand, if the original map $x$ does send $P$ into rad $P$, so does any element in the ideal $N$ that $x$ generates in $End P$. So a power of the ideal $N$ is zero. Consequently, $x$ must be in rad $End P$. We have now characterized the radical rad $End P$ of $End P$ as the space of all maps $x : P \to P$ with $xP$ contained in rad $P$.

Next, let’s add the condition that $P$ is a projective generator. Then rad $P$ is clearly the span of images $xP$ of elements $x$ in $End P$ such that $xP$ is contained in rad $P$. That is, rad $P = (\text{rad } End P)P$. It follows
inductively that \( \text{rad}^r P = (\text{rad} \text{End} P)^r P \). (Assume this isomorphism for \( r - 1 \) and multiply both sides by \( \text{rad} A \).) Thus, when \( P \) is viewed as a left \( A' \)-module, and we make it into a gr \( A'' \)-module—call it gr \( P' \)—by using the radical series of \( A'' \), we get a module natural identification of gr \( P' \) with gr \( P \) as a vector space. If we let \( A' \) be the opposite algebra of \( A'' \), then the vector space gr \( P \) provides a gr \( A, \text{gr} A' \)-bimodule through this identification.

However, Morita theory tells us that, if we similarly regard \( P \) as an \((A, A')\)-bimodule, it provides a Morita context. In particular, \( P \) is a projective generator as right \( A' \)-module or left \( A'' \)-module. Thus gr \( P \) is a projective generator as a left \( A'' \) or right gr \( A' \)-module. We have thus recaptured the Morita context provided by \( P \) for \( A \) and \( A' \) by one provided by gr \( P \) for gr \( A \) and gr \( A' \), and so the latter algebras are Morita equivalent. Moreover, since the bimodule providing this equivalence is graded, we obtain an equivalence between the categories of graded gr \( A \)-modules and graded gr \( A' \)-modules. In particular gr \( A \) is Koszul and only if gr \( A \) is Koszul.

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