orvara: An Efficient Code to Fit Orbits Using Radial Velocity, Absolute, and/or Relative Astrometry

Timothy D. Brandt1, Trent J. Dupuy2, Yiting Li2, G. Mirek Brandt1,7, Yunlin Zeng3, Daniel Michalik4, Daniella C. Bardalez Gagliuffi5, and Virginia Raposo-Pulido6

1 Department of Physics, University of California, Santa Barbara, Santa Barbara, CA 93106, USA; tbrandt@ucsb.edu
2 Institute for Astronomy, University of Edinburgh, Royal Observatory, Blackford Hill, Edinburgh, EH9 3HJ, UK
3 School of Physics, Georgia Institute of Technology, 837 State Street, Atlanta, GA, USA
4 European Space Agency (ESA), European Space Research and Technology Centre (ESTEC), Keplerlaan 1, 2201 AZ Noordwijk, The Netherlands
5 American Museum of Natural History, 200 Central Park West, New York, NY 10024, USA
6 Space Dynamics Group, Technical University of Madrid, ETSIAE, Pza. Cardenal Cisneros 3, E-28040 Madrid, Spain

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Abstract

We present an open-source Python package, Orbits from Radial Velocity, Absolute, and/or Relative Astrometry (orvara), to fit Keplerian orbits to any combination of radial velocity, relative astrometry, and absolute astrometry data from the Hipparcos-Gaia Catalog of Accelerations. By combining these three data types, one can measure precise masses and sometimes orbital parameters even when the observations cover a small fraction of an orbit. The computational performance of orvara is achieved with an eccentric anomaly solver 5–10 times faster than commonly used approaches and low-level memory management to avoid Python overheads and by analytically marginalizing out parallax, barycenter proper motion, and instrument-specific radial velocity zero-points. Through its integration with the Hipparcos and Gaia intermediate astrometry package htof, orvara can properly account for the epoch astrometry measurements of Hipparcos and the measurement times and scan angles of individual Gaia epochs. We configure orvara with modifiable .ini configuration files tailored to any specific stellar or planetary system. We demonstrate orvara with a case study application to a recently discovered white dwarf/main-sequence system, HD 159062. By adding absolute astrometry to literature radial velocity and relative astrometry data, our comprehensive Markov Chain Monte Carlo analysis improves the precision of HD 159062B’s mass by more than an order of magnitude to $0.6083^{+0.0083}_{-0.0073}$ $M_\odot$. We also derive a low eccentricity and large semimajor axis, establishing HD 159062AB as a system that did not experience Roche lobe overflow.

Unified Astronomy Thesaurus concepts: Orbit determination (1175); Keplerian orbit (884); Astronomy software (1855); Exoplanet astronomy (486); Astrometric exoplanet detection (2130); Radial velocity (1332)

1. Introduction

The history of orbit fitting extends back to ancient times, when it culminated in the Ptolemaic model of the solar system. During the Age of Enlightenment, astronomers fit the first Keplerian orbits to visual binary stars. More recently, masses derived from orbital fits, known as dynamical masses, anchor models of stellar evolution (Torres et al. 2010). Precise dynamical masses are used to infer the dynamical evolution of objects that reside off of the main sequence and rapidly evolve in the Hertzsprung–Russell diagram, such as young stars, brown dwarfs, giant planets, and white dwarfs (WDs; Chabrier et al. 2000; Hillenbrand & White 2004; Pala & Gänsicke 2017).

Orbits have assumed a central role in exoplanet research, since exoplanets are often detected only through their effects on their host stars. For a planet detected by radial velocity (RV) monitoring, a Keplerian orbital fit may be the only observational result. Exoplanet demographics show that the mass distribution of companions has a gap between 10 and 100 $M_{\text{Jup}}$ (Zucker & Mazeh 2001; Schlaufman 2018), which suggests that exoplanets form very differently from low-mass stellar companions. Precise masses, along with luminosities and ages, provide a powerful way to test different planet formation mechanisms, substellar evolutionary models, and WD cooling and atmospheric models (e.g., Benvenuto & Althaus 1999; Martins & Palacios 2017). Brown dwarfs and planets cool and fade with time; evolutionary models predict their luminosities as a function of age and mass (e.g., Burrows et al. 1997; Baraffe et al. 2003; Saumon & Marley 2008). Systems with independent, dynamical masses provide the strongest tests and calibrations of these models (Crepp et al. 2016; Dupuy & Liu 2017; Brandt et al. 2019; Maire et al. 2020).

For many years, orbital analyses focused on stars and used techniques appropriate for the computing power available. Some approaches used a grid search over a small number of parameters combined with linear least-squares fitting of the remaining parameters to map out $\chi^2$ surfaces (e.g., Hartkopf et al. 1989; Schaefer et al. 2006). Other common approaches used nonlinear least-squares fitting to all parameters at once (e.g., Forveille et al. 1999; Gudehus 2001) until Markov Chain Monte Carlo (MCMC) methods became more widespread (e.g., North et al. 2007; Ireland et al. 2008; Liu et al. 2008). In the meantime, orbital analysis has become central to the study of exoplanets, inspiring the development of many orbit-fitting tools over the last several years, including ExoFast (Eastman et al. 2013), PyAstrOfit (Wertz et al. 2016), BATMAN (Kreidberg 2015), ExoSOFT (Mede & Brandt 2017), RadVel (Fulton et al. 2018), and orbitize! (Blunt et al. 2020). In many cases, only one type of data (astrometry or RV) is available to fit an orbit. Some of these tools, like BATMAN and RadVel, are designed with a particular data type in mind, while others, including ExoSOFT and orbitize!, are designed to deal with multiple data types. Multiple types of data can combine to offer much stronger
constraints than any one type alone. Absolute astrometry from Hipparcos and Gaia is now capable of offering meaningful constraints (e.g., Pearce et al. 2020), especially when combined with relative astrometry and RV measurements.

The Gaia spacecraft (Gaia Collaboration et al. 2016, 2018, 2021) has been surveying the nearby stellar or exoplanetary systems by taking photometric, spectroscopic, and astrometric measurements across the sky with onboard instruments since 2014. The conceptually similar Hipparcos satellite (ESA 1997; van Leeuwen 2007) obtained astrometric measurements from 1989 to 1993. Hipparcos and Gaia measured the positions and motions of stars in an inertial reference frame called the International Celestial Reference System (ICRS) defined by distant quasars (Ma et al. 1998; Fey et al. 2015). The difference in their separate measurements of proper motions indicates acceleration in an inertial frame, which can be used to refine the orbital parameters of the accelerators. However, neither Hipparcos nor Gaia achieved a perfect realization of the ICRS, and both have low-level systematics and uncertainties that are sometimes underestimated. Brandt (2018) cross-calibrated Hipparcos and Gaia DR2 to account for systematics as a function of position on the sky, putting them in the same reference frame. Brandt (2021) performed a similar cross-calibration for Gaia ED3.

Absolute astrometry can enable precise constraints on systems (Benedict et al. 2002, 2006) even when only a small fraction of the orbit is observed (Brandt et al. 2019). The combination of Gaia and Hipparcos proper motions provides a measurement of acceleration in the plane of the sky; an RV trend adds a third dimension to the accelerations. Finally, direct imaging provides projected separations and position angles of the companions, allowing orbital constraints even without observing a substantial fraction of an orbit. Long-baseline precision RV surveys can identify stellar, substellar, or planetary companions inducing accelerations on their host stars, while a growing number of exoplanets are being discovered and characterized via high-contrast imaging (Marois et al. 2008; Lagrange et al. 2010; Kuzuhara et al. 2013; Rameau et al. 2013; Macintosh et al. 2015; Bowler 2016).

Many authors have recently combined absolute astrometry with other data types to measure masses and orbits. These applications include RV-detected exoplanets (Feng et al. 2019; Damasso et al. 2020; De Rosa et al. 2020; Kervella et al. 2020; Xuan & Wyatt 2020), directly imaged planets and brown dwarfs (Calissendorff & Janson 2018; Snellen & Brown 2018; Brandt et al. 2019; Dupuy et al. 2019; Maire et al. 2020; Nielsen et al. 2020), and stars (Torres 2019; Torres et al. 2019a; Pearce et al. 2020). Our goal is to introduce a generalized, optimized, open-source, and flexible piece of software that can easily incorporate astrometric acceleration measurements.

In this paper, we present Orbits from Radial Velocity, Absolute, and/or Relative Astrometry (orvara), a Python package designed for fast orbit fitting and plotting of companions. We adopt a similar approach to Brandt et al. (2019) to jointly fit the absolute astrometry, relative astrometry, and/or RV of a given star and companion. Furthermore, we reduce the computational cost of our approach by introducing a more efficient eccentric anomaly solver, marginalizing out four or more nuisance parameters in the likelihood function, and using low-level memory management to avoid Python overheads. The orvara package Requires (and is distributed with) the Hipparcos-Gaia Catalog of Accelerations (HGCA).

The paper is structured as follows. Section 2 reviews the basics of Keplerian orbits, including equations that govern the measured positions and velocities as a function of time and corresponding orbital parameters. This is followed by a discussion of the implementation of these equations in orvara in Section 3. Section 4 describes the marginalized likelihood function over four or more parameters used in MCMC fitting. The computational performance of our implementation is examined in Section 5. The configuration and use of orvara, postprocessing of the output, and plotting of a suite of eight plots relevant to astrometry and RV are discussed in Section 6. We present a case study application of orvara to a recently discovered white dwarf/main-sequence (WD/MS) system, HD 159062, in Section 7, where new constraints on the companion mass and the eccentricity are presented. We conclude with our results and findings in Section 8.

2. Keplerian Orbits

A Keplerian orbit is a solution of the two-body problem for Newtonian gravity. It is fully described by six orbital elements plus the masses of the two components. We wish to use measured positions and velocities to derive posterior probability distributions on these eight parameters, marginalizing over several nuisance parameters (e.g., the position, parallax, and velocity of the system’s barycenter and astrophysical jitter in the measured RV). In this section, we briefly review the equations that give the measured positions and velocities as a function of time and the orbital parameters. We will describe our implementation of these equations and the likelihood in subsequent sections.

In a Keplerian orbit, the mean anomaly $M$ varies linearly with time as

$$M = \frac{2\pi}{P}(t - t_p),$$

where $P$ is the system period and $t_p$ is the epoch of periastron passage. The position and velocity may be computed using the eccentric anomaly, which is given implicitly by

$$M = E - \varepsilon \sin E,$$

where $\varepsilon$ is the eccentricity. The RV is given through the true anomaly $\nu$ by

$$\nu = 2 \arctan \left[ \sqrt{1 + \varepsilon} \sin \frac{E}{2}, \sqrt{1 - \varepsilon} \cos \frac{E}{2} \right],$$

$$\text{RV} = k (\cos(\nu + \omega) + \varepsilon \cos \omega),$$

where $\omega$ is the argument of periastron, $k$ is the RV amplitude, and $\arctan 2$ is the two-argument arctangent. We adopt a convention that the orbital parameters all refer to the companion(s). The orbital parameters for the primary are the same, except that $\omega_{\text{pri}} = \omega + \pi$.

The projected offset between the two bodies may be computed through the elliptical rectangular coordinates,

$$X = \cos E - \varepsilon,$$

$$Y = \sin E \sqrt{1 - \varepsilon^2},$$

and the Thiele–Innes constants,

$$A = \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i,$$

$$B = \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i,$$
\[ F = -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i, \]
\[ G = -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i. \]

In Equations (7)-(10), \( i \) is the inclination, and \( \Omega \) is the longitude of the ascending node. The projected offsets of the secondary with respect to the primary star in decl. \( \Delta \delta \) and R.A. \( \Delta \alpha^* = \Delta (\alpha \cos \delta) \) are then given by
\[
\Delta \delta = a(AX + FY),
\]
\[
\Delta \alpha^* = a(BX + GY),
\]
where \( a \) is the semimajor axis in angular units.

Constraints on a system’s orbit may come from measurements of the primary star’s RV over time, the projected angular offset of the two bodies, and/or the projected motion of either component relative to the system’s barycenter in an inertial reference frame. The orvara package is designed to account for all of these within the context of Gaussian uncertainties. While it is straightforward to apply Equations (4), (11), and (12) to compute predicted positions and velocities, implementing them as shown is typically inefficient and may require many trigonometric function calls per observational epoch. In the following sections, we describe our computational implementation of the equations given in this section and our calculation of the likelihood.

3. Computational Implementation

3.1. The Eccentric Anomaly Solver

Equation (2) for the eccentric anomaly is known as Kepler’s equation. Its importance, combined with its lack of an analytic solution, has inspired centuries worth of work on eccentric anomaly solutions. Its importance, combined with its lack of an analytic solution, has inspired centuries worth of work on eccentric anomaly solutions. It is the first-order Newton–Raphson approach requires an evaluation of sine and cosine at every iteration. Depending on the quality of the initial guess, this could result in anywhere from a few to \( \approx 10 \) trigonometric evaluations per epoch. We have developed a more efficient approach for orvara: an eccentric anomaly solver based closely on that of Raposo-Pulido & Peláez (2017, hereafter RPP), with several modifications.

The basic approach of RPP is to obtain a very good initial guess for the eccentric anomaly through fitting formulae, followed by a single step of a modified Newton–Raphson method. We begin by reducing the range of the mean anomaly \( M \) to \((-\pi, \pi]\); we further reduce the range to \([0, \pi]\) and save the sign of \( M \) for later use. We then adopt the piecewise quintic fitting function for the eccentric anomaly \( E \) as a function of the mean anomaly given in RPP, with the function values and the first two derivatives fixed where the eccentric anomaly is an integer multiple of \( \pi/12 \) (for 12 polynomials). Each polynomial \( i \) is then defined over a range of mean anomalies \( M_i, M_{i+1} \) given in Table 3 of RPP. To facilitate the calculation of the coefficients, we define the \( i \)th quintic polynomial as
\[
E \approx \sum_{k=0}^{5} a_{i,k}(M - M_i)^k,
\]
where \( M_i \) is the minimum mean anomaly in the domain of the \( i \)th quintic polynomial. With this definition,
\[
a_{i,0} = E(M_i),
\]
\[
a_{i,1} = E'(M_i),
\]
and
\[
a_{i,2} = \frac{1}{2} E''(M_i),
\]
where the values and derivatives are tabulated in Table 3 of RPP. The other three coefficients may be derived by matching the values at \( M_i, M_{i+1} \),
\[
E(M_{i+1}) = \sum_{k=0}^{5} a_{i,k}(M_{i+1} - M_i)^k,
\]
\[
E'(M_{i+1}) = \sum_{k=1}^{5} a_{i,k}k(M_{i+1} - M_i)^{k-1},
\]
and
\[
E''(M_{i+1}) = \sum_{k=2}^{5} a_{i,k}k(k-1)(M_{i+1} - M_i)^{k-2}.
\]

Since \( a_{i,0}, a_{i,1}, \) and \( a_{i,2} \) are known from Equations (14)-(16), Equations (17)-(19) form a linear system of three equations for the three remaining coefficients. We solve this linear system by hand.

With a good initial guess for the eccentric anomaly \( E \), we deviate very slightly from RPP and use Halley’s method to refine the solution for eccentricities, \( \varepsilon < 0.78 \). This requires the sine and cosine of the eccentric anomaly. We use a series expansion for one and a square root call for the other; these are significantly faster than trigonometric calls and remain within a factor of a few of double-precision floating point accuracy. If \( E < \pi/4 \), we use the Taylor series for sine up to 15th order for a cost of seven additions and nine multiplications. We then compute \( \cos E = \sqrt{1 - \sin^2 E} \), with no ambiguity in the sign given the range reduction. If \( E > 3\pi/4 \), we use \( \pi - E \) in the same Taylor expansion for sine. If \( \pi/4 < E < 3\pi/4 \), we use the identity \( \cos E = \sin(\pi/2 - E) \) and the Taylor expansion of sine to compute \( \cos E \); we then use a square root call to evaluate \( \sin E \).

For \( \varepsilon < 0.78 \), a single iteration of Halley’s method brings the accuracy of our computed eccentric anomaly to nearly double precision. We then update \( \sin E \) and \( \cos E \), which we initially computed using series and square roots, with the summation trigonometric identities for \( \sin(E + \delta E) \) and \( \cos(E + \delta E) \). We evaluate \( \sin E \) and \( \cos E \) to second order, in keeping with the second-order accuracy of Halley’s method. Finally, we multiply \( E \) and \( \sin E \) by the sign of the mean anomaly. This brings \( E, \sin E, \) and \( \cos E \) all close to double precision, with the worst-case errors below \( 10^{-15} \) for \( \varepsilon < 0.78 \).

At higher eccentricities, the initial polynomial guess behaves poorly when \( M \ll 1 \). If \( 2M + (1 - \varepsilon) < 0.2 \), we adopt the second-order series expansion derived in RPP (their Equation (19)) for our starting point and revert to the quintic polynomial fits otherwise. We find that at higher eccentricities and low mean anomalies, we sometimes require a third-order step to reach double-precision accuracy in a single iteration. We therefore use Householder’s third-order method if both \( \varepsilon > 0.78 \) and \( M < 0.4 \) and Halley’s second-order method otherwise. As before, we refine \( \sin E \) and \( \cos E \) using the summation identities. Where \( \varepsilon > 0.78 \) and \( M < 0.4 \), we use series expansions up to third order to match the overall accuracy of Householder’s (third-order) method. This approach is slightly slower than at lower eccentricities due to the higher-order variant of Newton’s method and the need to occasionally use a series expansion for the initial guess. Up to eccentricities of 0.9999, however, the absolute errors in \( E \) and \( \sin E \) remain...
below $2 \times 10^{-14}$ at all mean anomalies, and up to eccentricities of 0.99, they remain below $3 \times 10^{-15}$ for all mean anomalies.

Figure 1 summarizes the performance of the eccentric anomaly solver. The left panel shows the overall accuracy, both the average and maximum errors as a function of eccentricity for a uniform distribution of mean anomalies. Even at an eccentricity of 0.9999, the algorithm is accurate to $2 \times 10^{-14}$ in $E$ and $\sin E$ and $10^{-15}$ in $\cos E$. The increasing error at high eccentricity is a result of a cancellation between $E$ and $\sin E$ in Kepler’s equation; it could be overcome by rewriting the equation in terms of $1 - \varepsilon$ and expanding $\sin E$ as a series for small values of $E$. However, this would not solve the more fundamental problem that we can only compute the mean anomaly from the observation epoch with limited precision, and that $dE/dM$ can be large for $\varepsilon \approx 1$.

The right panel of Figure 1 shows the computational cost of our implementation of RPP compared to other widely used eccentric anomaly solvers, all evaluated on a single core of an Intel Xeon E5-2630 2.2 GHz processor. Our implementation gives $\sin E$ and $\cos E$ as by-products of the calculation. We add a call to sine and cosine when assessing other methods, as $\sin E$ and $\cos E$ are required to calculate RV and/or positional offsets. There is an overhead of $\approx 170$ ns associated with calculating the 72 polynomial coefficients at a given eccentricity, making the performance of our RPP implementation only slightly faster than other solvers for a single epoch. We have also implemented a version of this solver tailored to single-epoch calculations. The single-epoch version eliminates most of the overhead by constructing only one of the quintic polynomials (six coefficients). All methods shown in the right panel of Figure 1 also incur an overhead of a few hundred nanoseconds for the Cython function call. For 25 epochs at the same eccentricity, our implementation of RPP is around five times faster than these other approaches. With $\geq 100$ epochs at a given eccentricity, it can be faster than one call per epoch to both sine and cosine. Our implementation is $\sim 5$–8 times faster than the SDG code (RPP) because of its avoidance of explicit trigonometric calls and its approach to computing the polynomial coefficients.

3.2. Radial Velocities

Given the eccentric anomaly $E$, eccentricity $\varepsilon$, and RV amplitude $k$, the RV can be calculated by Equations (3) and (4). Written this way, the RV evaluation requires five trigonometric calls (four if range reduction is used to enable the use of the single-argument arctangent). Only one of these, $\cos \omega$, can be computed once for all epochs in an orbit. However, Equations (3) and (4) are equivalent by trigonometric identities to

$$g = \left(1 + \varepsilon \right) \left(1 - \cos E \over \sin E\right),$$

(20)

and

$$RV = k \left(1 - g^2 \over 1 + g^2 \right) \cos \omega \left(-2g \over 1 + g^2 \right) \sin \omega.$$

(21)

The quantities $\sin \omega$ and $\cos \omega$ appearing in Equation (21) only need to be computed once per orbit, not once per epoch. The quantities $\sin E$ and $\cos E$, needed for Equation (20), were already evaluated in the course of solving for the eccentric anomaly. Once $\sin \omega$ and $\cos \omega$ are computed for an orbit, computing the RV using Equations (20) and (21) does not require a single trigonometric evaluation. In comparison tests, we found the use of Equations (20) and (21) to be $\sim 10$–20 times faster than Equations (3) and (4).

Equation (20) can be problematic when $E \approx 0$. If $|E| < 0.015$, we use the series expansion up to fifth order (there is no sixth-order term); this gives a value of $g$ accurate to better than $10^{-15}$. If $E \approx \pm \pi$, Equation (21) presents no numerical problems unless $\sin E$ is precisely zero (in which case, we simply set $RV = k(\varepsilon - 1)\cos \omega$).
3.3 Absolute and Relative Projected Positions

The relative offsets of the secondary from the primary star in R.A. \( \alpha^* \) and decl. \( \delta^* \) (where \( \alpha^* = \alpha \cos \delta \)) are given by Equations (11) and (12). These equations present no computational difficulties. With \( \sin E \) and \( \cos E \) already computed from the eccentric anomaly solver, Equations (5)-(12) require five trigonometric evaluations plus one square root per orbit, regardless of the number of epochs.

We typically fit orbits using both absolute and relative astrometry. The displacement of the primary star from the system’s barycenter is related to the relative separations of Equations (11) and (12) by

\[
\Delta \delta^* = \left( \frac{-M_b}{M_A + M_b} \right) \Delta \delta
\]

and

\[
\Delta \alpha^* = \left( \frac{-M_b}{M_A + M_b} \right) \Delta \alpha^* ,
\]

where \( M_A \) is the mass of the primary star and \( M_b \) is the mass of its companion.

Astrometric missions like Hipparcos and Gaia measure the position of a star many times and fit an astrometric sky path. We use the Hundred Thousand Orbit Fitter (htof; Brandt et al. 2021) to compute synthetic Hipparcos and Gaia catalog positions and proper motions from the offsets given in Equations (22) and (23). We then compare the htof synthetic catalog values to the cross-calibrated absolute astrometry of the HGCA (Brandt 2018).8

3.4. Multiple Companions

Keplerian orbits only describe two-body systems. To fit more than one companion in orvara, we use a set of simplifying approximations that we describe here.

We approximate the star’s motion by a superposition of Keplerian orbits, one due to each companion. We use the orbital elements of each companion (shifting the argument of periastron by \( \pi \) to give the stellar orbit), but we modify the total mass. For the mutual orbit between the star and a given companion indexed by \( i \), we add the mass of all other companions that orbit closer to the star than the companion \( i \) itself. Effectively, we approximate the star and inner companions as a single body and solve for its motion about its barycenter due to the \( i \)th companion. We then solve for the orbital motion within the inner system when we account for those companions. We add the contributions to the host star’s motion from all companions for both absolute astrometry and RV.

We treat relative astrometry slightly differently. In this case, we ignore the influence of all companions orbiting beyond the companion of interest. We first compute the relative astrometry between this companion and the barycenter of the star and all inner companions. We then add the offset of the star relative to this barycenter due to the inner companions. This calculation is the same as the shift in absolute astrometry due to these inner companions.

These approximations do not fully capture interactions between companions. While incorrect in detail, our approach does capture some of the effects of a multibody system. It can be used to fit extremely accurate relative astrometry (like that from GRAVITY; Gravity Collaboration et al. 2020), though not with the fidelity of a full (and expensive) suite of \( N \)-body simulations.

4. The Likelihood

We compute the likelihood \( \mathcal{L} \) of an orbit as

\[
-2 \ln \mathcal{L} = \chi^2_{\mathrm{RV}} + \chi^2_{\mathrm{rel \ ast}} + \chi^2_{\mathrm{abs \ ast}} . \tag{24}
\]

We treat \( \chi^2_{\mathrm{RV}} \), the RV component, first. We then take the latter two terms together as we marginalize out the barycenter’s proper motion and the system’s parallax.

4.1. Radial Velocity

For the RV, we take

\[
\chi^2_{\mathrm{RV}} = \sum_{j=1}^{N_{\mathrm{obs}}} \sum_{k=1}^{N_{\mathrm{fit}}} \frac{\left( \text{RV}_k + \text{ZP}_j - \text{RV}_k \right)^2}{\sigma^2_{\text{RV}_k} + \sigma^2_{\text{jitter}}} + \ln \left( \sigma^2_{\text{RV}_k} + \sigma^2_{\text{jitter}} \right) , \tag{25}
\]

where \( \text{ZP}_j \) is the instrument-specific RV zero-point and \( \sigma^2_{\text{jitter}} \) is a jitter term. We use \( \text{RV}_k \) to denote the measured RV at epoch \( t_k \), \( \text{RV}_k \) for the model-predicted RV, and \( \sigma^2_{\text{RV}_k} \) for its variance.

Our approach differs from Brandt et al. (2019) in that we adopt a single jitter for all instruments, attributing it to stellar activity. This has the disadvantage of assuming that instrumental uncertainties are properly estimated and that instruments sample similar regimes of stellar activity (the latter assumption could be problematic if combining visible and near-infrared data). However, it makes the fit robust to the inclusion of instruments with just a few data points, and it reduces the number of parameters to fit. The orvara package does include the option of fitting a different jitter to every RV instrument.

Equation (25) is the only place in the likelihood where the RV zero-point appears. We therefore marginalize it out, assuming a flat prior, by integrating

\[
\mathcal{L} \propto \int_\infty^\infty d\text{ZP} \exp \left[ -\chi^2_{\text{RV}} \right] , \tag{26}
\]

We perform the integral separately for each instrument. This requires only one pass through the RV data set and adds a negligible amount of computation. Performing the integral results in replacing \( \chi^2_{\text{RV}} \) with

\[
\chi^2_{\text{RV}} = \sum_{j=1}^{N_{\text{fit}}} \left( -B_j^2 + C_j + \ln A_j \right) \]

\[
+ \sum_{j=1}^{N_{\text{fit}}} \sum_{k=1}^{N_{\text{obs}}} \ln \left[ \sigma^2_{\text{RV}_k} + \sigma^2_{\text{jitter}} \right] , \tag{27}
\]

where

\[
A_j = \sum_{k=1}^{N_{\text{obs}}} \frac{1}{\sigma^2_{\text{RV}_k} + \sigma^2_{\text{jitter}}} . \tag{28}
\]

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8 We refer the reader to the source code at https://github.com/gmbrandt/htof and Brandt et al. (2021) for further details.
\begin{align}
B_j &= \sum_{k=1}^{N_k} \frac{2(RV_k - RV[t_k])}{\sigma^2[RV_k] + \sigma^2_{\mu_k}}, \\
C_j &= \sum_{k=1}^{N_k} \frac{(RV_k - RV[t_k])^2}{\sigma^2[RV_k] + \sigma^2_{\mu_k}},
\end{align}
and the sums over \(k\) are restricted to RV data points from instrument \(j\). As shown, \(\sigma_{\mu_k}\) uses Equations (27)–(30).

### 4.2. Astrometry

The \(\chi^2\) for relative astrometry consists of two components: the relative separation \(\rho\) and the position angle \(\theta\) measured east of north. The model orbit’s relative separation is the product of the projected relative separation \(\rho\) in astronomical units and the system’s parallax \(\varpi\). In general, the measurements of separation and position angle may be covariant; we take \(c_{\varpi,\theta} \in (-1, 1)\) to be the correlation coefficient between the two measurements at epoch \(k\). The contribution to \(\chi^2\) is then

\[
\chi^2_{\text{rel ast}} = \sum_{k=1}^{N_k} \frac{[\theta_k - \theta[t_k] - (1 - c_{\varpi,\theta})\sigma[\theta_k]]^2}{\sigma^2[\theta_k]} + \sum_{k=1}^{N_k} \frac{(\rho_k - \varpi\rho[t_k])^2}{\sigma^2[\rho_k]}
- \frac{2}{\sigma^2}[c_{\rho,\theta}\{\theta_k - \theta[t_k]\}(\rho_k - \varpi\rho[t_k])}{(1 - c_{\varpi,\theta})\sigma[\theta_k]\sigma[\rho_k]},
\]

where \(\rho_{k}\) and \(\theta_{k}\) are the observed separation and position angle at time \(t_{k}\). \([\theta[t_k]\) and \([\theta[t_k]\] are the model-predicted values, and \([\theta_k - \theta[t_k]]\) is the difference between the measured and predicted position angles, reduced to the range \((-\pi, \pi]\).

The absolute astrometry in angular units is similarly proportional to parallax and also has a velocity zero-point. This is the proper motion of the system barycenter in the plane of the sky \(\varpi\), an almost perfect analog of the RV zero-point. This component of the likelihood then reads

\[
\chi^2_{HG} = (\mu_{H,0} - \varpi\mu_H)^T C_{H}^{-1}(\mu_{H,0} - \varpi\mu_H)
+ (\mu_{HG,0} - \varpi\mu_{HG})^T C_{HG}^{-1}(\mu_{HG,0} - \varpi\mu_{HG})
+ (\mu_{G,0} - \varpi\mu_G)^T C_{G}^{-1}(\mu_{G,0} - \varpi\mu_G),
\]

where, e.g., \(\mu_{H,0}\) is the observed Hipparcos proper motion and \(\mu_{H}\) is the model orbit’s predicted Hipparcos proper motion in \(\text{au yr}^{-1}\) (which must then be multiplied by the parallax to obtain a proper motion in angular units).

In some cases, Gaia obtains an astrometric solution for both stars in a binary. In this case, the observed epochs and scan angles will generally be the same for both stars, and the position and proper motion of the secondary may be easily computed from those of the primary. In the frame of the system barycenter (in which the orbit code operates), we have

\[
\mu_B = -\mu \left(\frac{M_s}{M_B}\right),
\]

where \(\mu\) is the proper motion of the primary star and \(\mu_B\) is the proper motion of its companion. Denoting the Gaia proper motion of the secondary by \(\mu_{G,0,B}\), this measurement contributes an extra term to the astrometric \(\chi^2\),

\[
\Delta\chi^2_{HG} = (\mu_{G,0,B} - \varpi\mu_{G,B})^T C_{G,B}^{-1}(\mu_{G,0,B} - \varpi\mu_{G,B}),
\]

where \(C_{G,B}^{-1}\) is the inverse of the Gaia covariance matrix for the secondary and \(\varpi\) remains the center-of-mass motion of the system’s barycenter. Including this additional term assumes that both stars have their proper motion in the same reference frame and have well-calibrated errors. Either assumption may break down in practice; Gaia DR2 has magnitude-dependent systematics in the reference frame of \(\sim 0.2\) mas \(\text{yr}^{-1}\) (Figure 4 of Lindegren et al. 2018), while the HGCA required spatially variable error inflation by a typical factor of \(\sim 1.7\) (Brandt 2018). Gaia EDR3 retains magnitude-dependent systematics of up to \(\approx 100\) \(\mu\text{as yr}^{-1}\) (Cantat-Gaudin & Brandt 2021). We also neglect the covariance between the secondary’s measured proper motion and the projected separation of the two bodies.

Equations (31) and (32) (with or without the additional term from Equation (34)) are quadratic equations in parallax and the two components of the proper motion of the system’s barycenter. We adopt a uniform prior on \(\varpi\) but use a Gaussian prior in parallax to incorporate the measured Gaia value. This is equivalent to adding one additional component to the log likelihood,

\[
\chi^2_{\varpi} = \frac{(\varpi - \varpi_{\text{Gaia}})^2}{\sigma^2_{\varpi,\text{Gaia}}},
\]

With the addition of Equation (35), we can write the design matrix of the system and solve for the maximum-likelihood values of the parameters and their covariance matrix. Integrating the likelihood over parallax and barycentric proper motion is equivalent to substituting these maximum-likelihood values into the expressions for \(\chi^2\) and multiplying the likelihood by the square root of the determinant of the covariance matrix. This requires solving a \(3 \times 3\) linear system and computing the determinant of a \(3 \times 3\) matrix, neither of which incurs a significant computational cost. We write out this linear system in the Appendix. We solve the \(3 \times 3\) linear system using the singular value decomposition but with a C routine to avoid the overheads associated with Python function calls and memory management.

### 4.3. The Marginalized Likelihood

The integrations discussed in this section remove four parameters from the fit for a system when all RVs are from a single instrument, and more parameters otherwise. The computational cost of doing so is small relative to the cost of computing the orbit. The maximum posterior probability values of the parallax and barycenter proper motion may be of interest and are difficult to reconstruct from an output chain. We therefore save these quantities at each step for later use, along with the maximum-likelihood RV zero points of all instruments. We also save the contributions to \(\chi^2\) from relative separation, position angle, and from each component of the absolute astrometry.

It is much more difficult to analytically integrate out other parameters. Wright & Howard (2009) showed that the log
likelihood can be linear in additional parameters if the parallax is accurately known. However, the priors on the combinations of parameters that enter the problem linearly can be complex, even for simple choices of the underlying Keplerian parameters; the product of the likelihood and prior is difficult to integrate. Introducing relative astrometry (as we do for many of our targets) breaks the linearity entirely. If we consider the parallax to be known, we may still integrate out the secondary mass if we adopt the total system mass as the other mass parameter in the problem, as long as we use a flat prior. Given these restrictions, we retain the parallax as a linear parameter and do not analytically integrate out any others.

Our likelihood, after marginalizing four parameters, is a function of the six Keplerian orbital elements, the masses of the two components, and an RV jitter, for a total of nine parameters. If there is more than one companion in the system, the likelihood becomes a function of two parameters (mass of the primary and RV jitter) plus seven per companion (the six Keplerian orbital elements and the companion’s mass). We sample from this marginalized likelihood using the parallel-temperring MCMC sampler ptemcee (Vousden et al. 2016), a fork of emcee (Foreman-Mackey et al. 2013).

5. Performance

The computational cost of a single step includes contributions from the computation of the eccentric anomaly, the RV and positional offsets, and the likelihood calculation. A typical data set consists of a few hundred measurements. Of these, ∼150–200 might be individual epochs from Hipparcos and Gaia. For a sample data set of GI 758 (Brandt et al. 2019), there are 652 RV measurements, 181 epochs for the absolute astrometry, and four epochs for relative astrometry. This data set requires 85 μs per iteration on a single core of a 2.2 GHz Intel Xeon E5-2630. Of this time, 44 μs is spent in the calculation of the eccentric anomalies, 5 μs on RV, 2 μs on absolute astrometry for individual epochs, 9 μs on fitting the Hipparcos and Gaia models to the epoch astrometry, and 10 μs on the likelihood calculation. The remaining 15 μs are in overheads associated with emcee and the main Python functions. For a sample data set of HD 4747, with 49 RV measurements, 159 epochs for the absolute astrometry, and eight epochs for relative astrometry, the total cost is ∼40 μs per iteration. Due to the smaller number of data points, the times spent calculating the eccentric anomaly, RV, and likelihood fall to 11, 1, and 5 μs, respectively. The cost of fitting the Hipparcos and Gaia models to the epoch astrometry remains ∼9 μs. By comparison, a single-planet fit to 216 epochs of RV data using RadVel takes 1 ms per iteration with one core of the same CPU, or about 25 times slower than a fit to the same number of epochs (but with a diversity of data types) using orvara.

Overheads associated with allocating numpy arrays could take the majority of the computational time if we operated exclusively on arrays. We therefore explicitly handle memory allocation in Cython and use C structures with pointers rather than numpy arrays. Replacing each pointer allocation with an ndarray and creating a memory view would more than double the run time for HD 4747. For similar reasons, we use a C implementation of the singular value decomposition rather than a Python call to the linear algebra routine in numpy. For a typical system with a few hundred observational epochs, the total computational cost to compute an orbit and evaluate its likelihood (marginalizing out parallax, RV zero-point, and barycenter proper motion) is equivalent to about three to seven trigonometric evaluations per epoch.

The orvara package uses the parallelization built into the packages emcee and pttemcee. The scaling under typical use is poor when using more than a handful of processors. Using five processors results in a speed-up of just a factor of 2 on our machines, while adding further processors gives almost no additional speed-up at all. Parallelization also works differently on different operating systems and may not be supported on all machines. Regardless, multiple instances of the program may be run simultaneously for significantly better performance.

6. Configuration, Use, and Plotting

The open-source Python package orvara performs parameter fits for multiplanetary or binary star systems using a combination of the Hipparcos-Gaia Catalog of Accelerations, literature RV, and/or relative astrometry. It includes Python plotting routines to produce a comprehensive orbit-fitting and plotting package. It is available on GitHub.9 The detailed installation procedure for orvara and its updated documentation can be found in the README.rst file.

Most applications of orvara use RV and/or relative astrometry data. Our example in Section 7 uses both. Each type of data is given in an input ASCII text file with fields separated by spaces or tabs; lines beginning with # are ignored. We provide examples in the repository and summarize their required structure in Table 1. The RV file must include BJD, RV, and RV error (both in units of meters per second) as its first three columns. An optional fourth column may include an integer >0 to distinguish instruments from one another; each instrument will have its own unique RV zero-point. If the fourth column is not supplied, orvara will assume that all RVs share the same zero-point. The relative astrometry file must include the date (either BJD or decimal year, with values <3000 interpreted as years), the separation and its error (both in arcseconds), and the position angle and its error (east of north, both in degrees) as its first five columns. An optional sixth column gives the correlation coefficient (∈(−1, 1)) between separation and position angle. An optional seventh column gives the ID of the measured companion. The first companion is indexed as zero, the second as 1, etc. If the sixth and seventh columns are not supplied, orvara will use default values of zero for each.

Orbit fitting and plotting can be accessed with the fit_orbit and plot_orbit commands from the command line, respectively. The fit_orbit output is a single Flexible Image Transport System (FITS) file (Wells et al. 1981) containing the MCMC chains and described in more detail below. The header of this FITS file includes the configuration parameters used to produce the chain. The plot_orbit output is a suite of up to eight plots relevant to RV and relative astrometry and also discussed individually below. A starspecific .ini configuration file containing the appropriate file directories and settings serves as the input for both commands. This configuration file is comprised of four main sections denoted in square brackets: the data paths to all related file directories ([data_paths]), settings for MCMC fitting ([mcmc_settings]), settings for the primary mass prior and uncertainty ([priors_settings]), and settings for

9 https://github.com/t-brandt/orvara
plotting ([plotting]) and saving a table of results (save_results). A complete list of file directories and customizable input parameters, including their descriptions and functionalities, is provided in Table 2. Sample configuration files for GI 758 and HD 4747, along with their sample data sets of RV and relative astrometry, are included in the package. A user can omit data (for example, if there is no relative astrometry) by providing a blank or invalid file path. A user can also omit absolute astrometry by providing an invalid Hipparcos ID and supplying an explicit file path. A user can choose to incorporate other knowledge of the star.

The first few lines in the .ini file specify the locations of data files and the Hipparcos ID of the primary star. The initial guesses for the nine MCMC parameters (plus seven for each companion beyond the first) may also be set in a file referenced in the .ini configuration file. Here orvara draws starting values for each walker from a normal or lognormal distribution with the means and variances specified. We assume log-flat priors for semimajor axis a, primary mass Mpri, companion mass Msec, and RV jitter σjitter; a prior of sin i for inclination; and uniform priors for all other fitted parameters. Alternatively, the prior on the primary mass and its uncertainty can be set in [priors_settings] to incorporate other knowledge of the star.

In addition to RV fitting, we typically fit orbits using both absolute and relative astrometry (though only absolute astrometry is strictly required). Relative astrometry can be utilized by providing the data path to the relative astrometry file containing the astrometric epochs from different high-resolution imaging instruments. For epoch astrometry, the htof (Brandt et al. 2021) package mentioned in Section 3.3 is employed in orvara to provide parameter fits to the astrometric intermediate data from either Hipparcos data reduction or Gaia. In order to use epoch astrometry, the file path to observational epochs and scan angles for Gaia or the intermediate data of Hipparcos (original or rereduction) must be provided. The Gaia-predicted scan epochs and angle can be queried using the Gaia Observation Forecast Tool, publicly available at https://gaia.esac.esa.int/gost/index.jsp. The htof parses the given intermediate data and extracts (inverse-)covariance matrices and epochs of observations to compute the synthetic Hipparcos and Gaia catalog positions and proper motions described in Section 4.2.

The output of orvara is a single FITS file with the chain and other calculated parameters in a FITS table in header-data unit (HDU) 1. The header of the first extension (HDU 0) contains the configuration parameters read from the .ini file and used to construct the chain. The FITS table in HDU 1 contains columns with names, arrays, and units (where appropriate) for each fitted parameter, the natural logarithm of the likelihood, and other quantities computed at each step of the chain. Each array has dimensions nwalkers × nstep//thin.

We have implemented a plotting routine in Python to visualize the results obtained from the FITS file containing the MCMC chains produced by a joint orbit fit. In the [plotting] section of the .ini configuration file, the file path to the FITS file and a burn-in length for the (thinned) MCMC chains must be specified. Also, users have the option to select which plots to generate using True and False values for the plot keywords. Here we briefly describe the plots that orvara is currently configured to produce. Section 7 will provide an example fit and sample plots.

### 6.1. Astrometric Orbits

The relative astrometric orbits for companions of stellar or planetary systems, shown in the left two panels of Figure 2, can be generated by setting Astrometry_orbits_plot to True. Several features are worth mentioning in this plot. The thick black line indicates the highest-likelihood orbits; thin lines are orbits randomly drawn from the posterior distributions and colored according to either the companion mass or eccentricity. This is the case for all of the plots described below, except for the astrometric prediction plot, which is a 2D contour. The black dashed line inside the most likely orbit is the line of nodes joining the ascending and descending nodes of the most likely orbit. This line of nodes indicates the position of the orbital plane of the system with respect to the sky plane. The unfilled circles plotted along the most likely orbit are the predicted positions of the companion.
### Table 2
Description of the Configuration File Contents

| Parameter Name                              | Data Type | Default | Description                                                                 |
|---------------------------------------------|-----------|---------|-----------------------------------------------------------------------------|
| [data_paths]                                |           |         |                                                                             |
| HipID                                       | Integer   | 0       | Hipparos number; if valid, load HGCA absolute astrometry                    |
| HGCAFile                                    | Data file | [required]| Hipparos-Gaia Catalog (either the DR2 or EDR3 edition)                      |
| RVFile                                      | Data file |         | File containing the RV time series for the star                            |
| AstrometryFile                              | Data file |         | File containing the relative astrometry for the companion(s)                |
| GaiaDataDir                                 | Directory |         | Path to the Gaia scans as output by GOST in .csv format                     |
| Hip1DataDir                                 | Directory |         | Path to the Hipparcos (original reduction) intermediate data                |
| Hip2DataDir                                 | Directory |         | Path to the Hipparcos (rereduction) intermediate data                       |
| start_file                                  | Data file | "none"  | File with the initial parameter guesses. If "none", use default guesses     |
| [mcmc_settings]                             |           |         |                                                                             |
| ntemps                                      | Integer   | 10      | Number of temperatures to use in the parallel-tempering chain               |
| nwalkers                                    | Integer   | 100     | Number of walkers, each with ntemps number of chains                        |
| nplanets                                    | Integer   | [required]| Number of companions to fit                                                |
| nstep                                       | Integer   | [required]| Number of steps contained in each chain                                   |
| thin                                        | Integer   | 50      | Thinning of the chain (keep every thin step)                               |
| nthreads                                    | Integer   | 1       | Number of threads to use (the built-in parallelization is poor)             |
| use_epoch_astrometry                        | Boolean   | False   | Use the epoch astrometry in GaiaDataDir, Hip1DataDir, etc.?                 |
| jit_per_inst                                | Boolean   | False   | Fit a separate RV jitter for each instrument?                               |
| [priors_settings]                           |           |         |                                                                             |
| mpri                                        | Float     | 1       | Mean of a Gaussian prior on primary mass (in $M_\odot$)                     |
| mpri_síg                                     | Float     | inf     | Uncertainty in the stellar prior/s; if inf, use the default 1/$M_\odot$ prior|
| minjitter                                   | Float     | 1e-5    | Minimum allowable RV jitter (in km s$^{-1}$), should be >0                  |
| maxjitter                                   | Float     | 1e3     | Maximum allowable RV jitter (in km s$^{-1}$); must be >0jitter               |
| m_secondary0                                | Float     | 1       | Mean (in m$\mu$) of the (Gaia) mass prior for companion zero                |
| m_secondary0_síg                            | Float     | 1       | Standard deviation of the (Gaia) mass prior for companion zero               |
| parallax                                    | Float     | None    | Parallax prior; required if star is not in the HGCA, ignored otherwise      |
| parallax_error                              | Float     | None    | Parallax prior if star is not in the HGCA                                   |
| [secondary_gaia]                            |           |         | Gaia data of the secondary (set companion_ID = -1 if undetected)           |
| companion_ID                                | Integer   | -1      | ID of the Gaia companion, should match entry in AstrometryFile             |
| pmra                                        | Float     | 0       | Gaia proper motion (R.A.) of companion                                      |
| pmdec                                       | Float     | 0       | Gaia proper motion (decl.) of companion                                     |
| epmra                                       | Float     | 1       | Gaia proper motion uncertainty (R.A.) of companion                          |
| epmdec                                      | Float     | 1       | Gaia proper motion uncertainty (decl.) of companion                         |
| corr_pmra_pmdec                             | Float     | 0       | Correlation ($\epsilon(-1,1)$) between pmra and pmdec                      |
| [plotting]                                  |           |         |                                                                             |
| McmcDataFile                                | Data file | [required]| Path to MCMC chain produced from the fit_orbit command                      |
| burn in                                     | Integer   | 0       | Burn-in length for thinned chains                                          |
| check_convergence                           | Boolean   | False   | Make diagnostic plots to help check for convergence?                       |
| iplanet                                     | Integer   | 0       | ID of the companion to plot (0 <= iplanet < nplanets)                      |
| target                                      | String    |         | Name of the target, used for file naming                                   |
| start_epoch                                 | Integer   | 1950    | Customized range of dates (fractional years)                               |
| end_epoch                                   | Integer   | 2030    | Customized range of dates (fractional years)                               |
| num_orbits                                  | Integer   | 50      | Number of random orbits drawn from the posterior distribution              |
| num_steps                                   | Integer   | 1000    | Points per plotted orbit (aliasing can occur if num_steps is too small)    |
| predicted_years                             | Float(s)  | 2010, 2020| Labeled year(s) on the best-fit orbit in the Astrometry plot              |
| position_predict                            | Float     | 2020    | Epoch (fractional year) for the Astrometric_prediction_plot               |
| Astrometry_orbits_plot                      | Boolean   | True    | Plot the astrometric orbits?                                               |
| Astrometric_prediction_plot                 | Boolean   | True    | Plot the density plot for the predicted epoch?                             |
| RV_orbits_plot                              | Boolean   | True    | Plot the full RV orbits?                                                   |
| RV_plot                                     | Boolean   | True    | Plot RV orbits versus epoch and $O - C$ over the baseline of RV data?      |
| RV_Instrument                               | Integer/string| All | All or instrument number                                                   |
| Relative_separation_plot                    | Boolean   | True    | Plot relative separation versus epoch and $O - C$?                         |
| Position_angle_plot                         | Boolean   | True    | Plot position angle versus epoch and $O - C$?                              |
| Proper_motion_plot                          | Boolean   | True    | Plot the proper motions in R.A. and decl. and $O - C$?                     |
| Proper_motion_separate_plots                | Boolean   | False   | True if two separate plots for the proper motions are desired              |
| Corner_plot                                 | Boolean   | True    | Plot a 2D corner plot from MCMC chain?                                     |
| set_limit                                   | Boolean   | False   | Use user-specified axis limits?                                            |
| xlim                                        | Floats    | None    | If set_limit is True, set x limits with two comma-separated values         |
Figure 2. Relative astrometric orbits and predicted future position for HD 159062B. Left: relative astrometric orbits colored according to the secondary mass. Middle: relative astrometric orbits colored by eccentricity. Right: astrometric prediction of the location of HD 159062B in 2040. In the left and middle panels, the thick black lines indicate the highest-likelihood orbit, and the colored thin lines from purple to yellow are 50 orbits drawn randomly from the posterior distributions colored according to the companion mass (left panel) or eccentricity (middle panel). The dotted lines connect the host star to the periastron passages. The open black circles along the maximum-likelihood deprojected orbit indicate epochs spaced by 10 yr from 1990 to 2030. The dashed lines are the line of nodes. The filled orange circles are plotted along the maximum likelihood at the epochs corresponding to the relative astrometry used in our analysis. The arrows mark the direction of motion of the secondary companion. In the right panel, the contour lines demonstrate the likelihood of the location of HD 159062B in 2040, ranging from light yellow (least likely) to red to black (most likely).

at specific user-defined epochs from predicted_years. If an astrometric file is provided from AstrometryFile, the observations will be plotted as filled circles. If there is more than one companion included in the fit, the user may specify which one is plotted using the keyword iplanet (with 0 ≤ iplanet < nplanets).

6.2. Astrometric Prediction

The orvara package can predict the location of a companion relative to its host star at a specified epoch. The resulting density plot, shown in the right panel of Figure 2, can be obtained by setting position_predict to the desired epoch and Astrometric_prediction_plot to True. This contour plot shows the posterior probability density of the predicted positions of the companion in terms of relative offsets from the primary star in R.A. and decl., with the inner contour being the most likely position of the companion at that future epoch. The 1σ, 2σ, and 3σ contours enclose 68.3%, 95.4%, and 99.7% of the posterior probabilities for the future location.

6.3. RV Orbits

The orvara package can make two plots of the stellar RV: one restricted to the observational time frame and one spanning a longer, user-specified range of dates to show longer-term behavior. To make the latter plot, shown in the top panels of Figure 3, the user may set RV_orbits_plot to True. The former plot, restricted to the observational baseline, requires the user to give the path for RVFile; it is shown in the bottom panels of Figure 3. The RVFile containing the observed RV time series must have the format specified by Table 1. The filled circles on both plots represent RV data from all of the RV instruments. The observed RV data are shifted by an offset according to the maximum-likelihood zero-point for each RV instrument; the maximum-likelihood offsets for each instrument are given by fields named, e.g., RV_ZP_0_ML for instrument zerp.

Also, orvara can make a plot of the RVs restricted to the time range sampled by the RV instruments; RV_plot = True enables this plot. This is essentially a zoomed-in view of the part of the RV orbit plot described above. It contains RV data from single- or multi-instrument RV observations, plus a corresponding observed
— calculated (hereafter $O - C$) residual shown underneath. To choose to plot the RVs from either a specific or all of the RV instruments, users may set $RV_{\text{Instrument}}$ to the instrument number or 'All.' The observed data are represented by the filled circles with error bars. The $O - C$ residual indicates both the deviation of the observed value from the most likely orbit and the variation in RV across the orbits randomly drawn from the posterior. The error bars include the RV jitter of the best-fit orbit.

### 6.4. Relative Separation and Position Angle

For relative astrometry, orvara offers two more plots in addition to the astrometric orbit plot: the relative separation (in arcseconds) and the position angle (in degrees) of the imaged companion relative to the primary star in a time range sampled by the relative astrometric instruments and their corresponding $O - C$ residuals. These two plots, shown in the upper four panels of Figure 4, can be generated if the user sets $\text{Relative\_separation\_plot}$ or $\text{Position\_angle\_plot}$ to True. In each case, the path to $\text{AstrometryFile}$ should be specified. The orbit’s relative separation in arcseconds is the product of the projected relative separation $\rho$ in astronomical units and the parallax $\varpi$; we use the best-fit parallax for each set of orbital parameters. We refer the reader to Section 3.3 for a detailed description of the equations and the Appendix for the calculation of the best-fit parallax. The filled circles with error bars indicate the relative astrometric data from direct imaging instruments. Due to each instrument’s having different uncertainties, data reduction methods, and variations in the field rotation and plate scale, users may choose to adjust the imaging data in $\text{AstrometryFile}$ to keep or discard a set of imaging data. Only one companion will be plotted. If there is more than one companion fit, the user may specify which one to plot using $\text{iplanet}$.

### 6.5. Proper Motion

Variations in proper motion induced by the companion(s) on the primary star as measured from absolute astrometry from Hipparcos and Gaia can be plotted with $\text{Proper\_motion\_plot}=\text{True}$. The orvara package computes these proper motions as instantaneous values using the time derivative of the
Figure 4. Observed and fitted relative and absolute astrometry for the HD 159062AB system. From top to bottom, the panels show the relative separation and position angle of HD 159062B and the proper motion of HD 159062A in R.A. and decl. The thick black lines represent the best-fit orbit in the MCMC chain, while the other 50 lines represent random draws from the chain, color-coded by either companion mass (left panels) or eccentricity (right panels). The small bottom panel below each plot shows the difference between observed and calculated astrometry. The mean proper motion is used to compute the MCMC chain but is not shown in the proper-motion plots. This constraint is an integral over the proper motion between the Hipparcos and Gaia epochs.
eccentric anomaly. In contrast, both Hipparcos and Gaia fit sky paths to the star’s position as measured over several years. As a result, the observed data points and error bars do not correspond to the plotted lines in the same way that they do for, e.g., RV. The orvara package does not plot the Hipparcos-Gaia mean proper motion, as this measurement does not represent an instantaneous proper motion at the mean epoch. It is, rather, a measurement of the integral of the proper motion. The \( O - C \) curves should be taken with caution. The actual \( \chi^2 \) values for each set of measurements (which are meaningful and also include measurement covariance) are available in the FITS file written by orvara.

To plot the two components of the proper motion as two separate, individual plots (one for R.A. and one for decl.), users may set \texttt{Proper motion separate plots} to True. The data with error bars are from the cross-calibrated absolute astrometry of the HGCA. The HGCA provides three proper motions: one near 1991.25, another near 2015.5, and the positional difference between the Hipparcos rereduction and the Gaia catalog (either the DR2 or EDR3 edition) scaled by the time between them. Here orvara plots data points for only the Hipparcos and Gaia proper motions. It adds the best-fit barycenter proper motion, calculated as described in the Appendix, to each orbit. The lower four panels of Figure 4 show examples of these plots.

6.6. Corner Plot

A 2D corner plot of the MCMC chain can be obtained by setting \texttt{Corner plot} to True. A burn-in phase for the thinned MCMC chain can be chosen for plotting. We have modified \texttt{corner.py} from Foreman-Mackey (2016) to format the titles displayed on top of the histograms in this plot. We chose to keep two significant figures in the uncertainties with a matching number of decimal places in the values.

The corner plot shows astrophysically meaningful Keplerian orbital elements, including semimajor axis \( a \), eccentricity \( \varepsilon \), inclination \( i \), and the masses of the primary star in solar mass and its companion in Jupiter mass from the joint RV and astrometric MCMC analysis. If more than one companion is fit, the corner plot will use the parameters from the companion designated by \texttt{iplanet}.

7. Case Study: Application to HD 159062B

In this section, we provide a case study application of orvara to a nearby WD/MS binary system, HD 159062. The WD was discovered by Hirsch et al. (2019, hereafter H19), who fit for its orbit and mass. We first review the system, a widely separated binary with a degenerate companion to a main-sequence star, before summarizing the results of H19. We then discuss our own orbital fit and its implications for the binary system’s past evolution.

7.1. Background on HD 159062

Main-sequence stars with masses \( \lesssim 8 \, M_\odot \) will evolve off the main sequence through the asymptotic giant branch (AGB) phase at the stellar evolutionary end point and cool over billions of years until they eventually become dense WDs (Dobbie et al. 2006). More than 97% of stars are expected to evolve into WDs (Renedo et al. 2010). The WDs are used as powerful tools to trace the evolution of the galaxy and provide constraints on global stellar populations. In binaries, they can also be potential progenitors of Type Ia supernovae (Hillebrandt & Niemeyer 2000).

Spectroscopic and photometric surveys such as the Sloan Digital Sky Survey, Pan-STARRS, and Gaia can reveal fundamental properties of WDs, including mass, cooling rate and age, and atmospheric and internal composition. The recent Gaia DR2 catalog has provided updated precise astrometric and photometric data that can be used to infer the local WD population (Gentile Fusillo et al. 2019). Since most stars end their lives as WDs, and most stars reside in binaries (Moe & Di Stefano 2017), it is estimated that around one-fourth of the more than 200 known WDs within 25 pc of the Sun reside in WD/MS or WD/WD binary systems (Holberg et al. 2016). Population synthesis modeling shows that this number is reasonable, but the observed rate of WD/MS binaries likely suffers from selection effects due to low detection sensitivity to faint WDs near bright main-sequence stars (Toonen et al. 2017).

The host star of the system we study here, HD 159062A, is revealed spectroscopically as an old main-sequence G-K dwarf star with low metallicity based on its log \( g \) and \( T_{\text{eff}} \) (H19). The observed Ca II HK emission \( R_{\text{HK}} \) and rotation period led to an age diagnostic of \( \sim 7 \) Gyr. Literature reports on HD 159062A’s age as derived from high-resolution spectroscopy range from approximately 5 Gyr (e.g., Raghavan et al. 2012; Torres et al. 2019b) to 8 Gyr (e.g., Pace 2013) to 14 Gyr (e.g., Luck 2017; Bai et al. 2018).

Before the discovery of the WD HD 159062B, Fuhrmann et al. (2017) predicted the existence of a degenerate companion around HD 159062A due to a barium overabundance of [Ba/Fe] = +0.4 dex in HD 159062A. This is based on an empirical relation of the barium-to-iron enrichment as a function of age derived from an all-sky local population study of ancient Population II and intermediate-disk stars. Deviants from this Ba/Fe ratio versus age relation included five known stars with degenerate companions; HD 159062A was the most extreme outlier with an overabundance of barium. These five known systems almost certainly underwent mass transfer. Fuhrmann et al. (2017) characterized them as field blue stragglers due to high rotation, excessive chromospheric activity, lithium depletion, low orbital eccentricity, and noticeable age discrepancies with activity-based age indicators (Fuhrmann & Bernkopf 1999).

Solar-type Population II stars are old; they rotate at low rotational velocities and show low levels of chromospheric and coronal activity (Barnes 2003; Mamajek & Hillenbrand 2008). Additional observed chromospheric activity could be explained by mass and angular momentum transfer from binary companions. Many of the stars with degenerate companions studied by Fuhrmann et al. (2017) had slight barium depletions and \( \lesssim 1000 \) day orbital periods, suggestive of mass transfer from first-ascent red giant progenitors. However, HD 159062B had an anomalously high Ba/Fe ratio unlikely to have arisen from a field anomaly at its formation epoch; it must have been instead due to the \( s \)-process nucleosynthesis product of a binary companion.

The overabundance of barium in HD 159062 suggests wind accretion of \( s \)-process elements from a more distant AGB progenitor, rather than Roche lobe overflow from a first-ascent red giant progenitor (Boffin & Jorissen 1988; Han et al. 1995; Jeffries & Stevens 1996), so HD 159062B must have a longer
period than \(\sim 1000\) days. Indeed, Fuhrmann et al. (2017) argued that barium enrichment via wind accretion only works for orbital periods from about 10 to 1000 yr. This claim is backed up by an example of HD 114174, a cool WD companion separated by 59.8 \(\pm\) 0.4 au from the G-type host star, whose orbital period is estimated to be between 154 and 881 yr for eccentricities in the range \(0 \leq e \leq 0.5\) (Matthews et al. 2014). A barium overabundance of \([\text{Ba}/\text{Fe}] = +0.24\) dex is observed for the G star. The projected separation excludes the possibility of any substantial past mass exchange via Roche lobe overflow; however, Matthews et al. (2014) speculated that its orbit may have been pushed significantly outward during dynamical events, such as the ejection of a third body.

The case of HD 114174 suggests a similar scenario for HD 159062 and the presence of a wide WD companion. Object HD 159062B could be located at a wider separation of tens of astronomical units away from HD 159062A, whose observed barium overabundance can be explained by wind accretion of \(\alpha\)-process materials from a former AGB primary that survived as a WD companion, as well as any other dynamical events that may have widened its orbit. Apart from HD 114174 and HD 159062, Fuhrmann et al. (2017) also identified two systems, HR 3578 and 104 Tau, from their Ba/Fe ratio studies, both of which are without known companions but with slight barium anomalies. However, both stars’ lack of strong astrometric accelerations is not suggestive of any orbiting WD companions at orbital periods of 10–1000 yr. System HR 3578 does show an \(\sim 3\sigma\) acceleration in decl. in the HGCA. System 104 Tau has been reported to be a visual binary (Eggen 1956), but more recent interferometric nondetections (Raghavan et al. 2010) and a stable RV with a shallow trend (Butler et al. 2017) strongly suggest otherwise. Both HR 3578 and 104 Tau represent good targets for follow-up imaging.

### 7.2. Data and Previous Fit

Object HD 159062B was discovered by H19 in 14 yr of precise RV data from the HIRES spectrograph on Keck (Vogt et al. 1994; Howard et al. 2010) and imaged using multi-epoch multiband imaging observations from the Shane AO system on the Lick Observatory 3 m Shane telescope (Gavel et al. 2014; Srinath et al. 2014), the PHARO AO system on the Palomar Observatory 5 m telescope (Hayward et al. 2001), and the NIRC2 AO system at the Keck II 10 m telescope (Wizinowich et al. 2000). A joint MCMC analysis was performed by H19 using 45 RV and three NIRC2 astrometric measurements to derive the best orbital parameters. They assumed Gaussian priors on the mass of the primary star, the difference between the pre-upgrade velocity zero-point and the post-upgrade velocity zero-point and parallax, and uniform priors on all other fitted parameters. They derived a new spectroscopic mass of 0.76 \(\pm\) 0.03 \(M_{\odot}\) but adopted a prior of 0.80 \(\pm\) 0.05 \(M_{\odot}\) based on literature values. They used 10 temperatures and 300 walkers in their MCMC in a total of \(10^5\) steps. After the initial MCMC analysis, an additional prior on the WD cooling age was added based on WD cooling models and photometric constraints from Shane AO, PHARO, and NIRC2 photometry measurements in \(J, K_s\), and \(L'\). They concluded that the companion is an old \(M_B = 0.65_{-0.02}^{+0.03} M_{\odot}\) with an orbital period of \(P = 250_{-130}^{+30}\) yr and a cooling age of \(\tau = 8.2_{-0.3}^{+0.7}\) Gyr.

More recently, Bowler et al. (2021) performed an orbital fit using similar data and orvara with the DR2 version of the HGCA. Our results are similar to theirs but slightly more precise with the updated absolute astrometry from Gaia EDR3. We omit the additional RVs from Bowler et al. (2021); which do not extend the observational baseline and the additional relative astrometry. This enables a direct comparison of our results with those of H19 and shows the power of absolute astrometry to constrain the orbit of HD 159062AB.

### 7.3. A New Fit

We use orvara to perform a comprehensive joint MCMC analysis of HD 159062B using the same RV and imaging data as H19 but adding the cross-calibrated absolute astrometry of the HGCA. Table 3 summarizes the HGCA proper motions for HD 159062AB.

| Source   | \(\mu_{\alpha*}\) (mas yr\(^{-1}\)) | \(\mu_{\delta}\) (mas yr\(^{-1}\)) | Corr. | \(t_{\alpha*}\) | \(t_{\delta}\) | \(\varpi\) (mas) | RV (km s\(^{-1}\)) |
|----------|---------------------------------|---------------------------------|-------|----------------|----------------|----------------|----------------|
| Hipparcos | 174.316 \(\pm\) 0.666 | 75.598 \(\pm\) 0.612 | 0.27 | 1991.20 | 1991.12 | | |
| HG       | 172.499 \(\pm\) 0.019 | 75.776 \(\pm\) 0.020 | 0.11 | | | | |
| Gaia EDR3 | 169.814 \(\pm\) 0.026 | 77.133 \(\pm\) 0.029 | 0.22 | 2016.07 | 2016.27 | 46.118 \(\pm\) 0.024 | \(-84.11 \pm 0.18\) |

Table 4 provides a complete list of the

| Parameter | Prior |
|-----------|-------|
| RV jitter \(\sigma_{\text{jit}}\) | \(1/\sigma\) (log-flat) |
| Primary mass \(M_{\text{pri}}\) | \(1/M\) (log-flat) |
| Secondary mass \(M_{\text{sec}}\) | \(1/M\) (log-flat) |
| Semimajor axis \(a\) | \(1/a\) (log-flat) |
| \(\sqrt{e} \sin \omega\) | Uniform |
| \(\sqrt{e} \cos \omega\) | Uniform |
| Inclination \(i\) | \(\sin(i), 0^\circ < i < 180^\circ\) |
| Ascending node \(\Omega\) | Uniform |
| Parallax \(\varpi\) | \(\exp\left(-\frac{(\varpi - \varpi_{\text{Gaia}})^2}{\sigma_{\varpi}^2}\right)\) |

#### Table 3
HGCA Proper Motions

#### Table 4
Basic Parameters and Default Priors
the observations, are displayed in Figure 3. The relative
separation and position angles of the companion HD 159062B with
respect to the host star HD 159062A and the absolute
astrometry of the host star from HGC A are illustrated in
Figure 4. Finally, to evaluate the full behavior of the joint
posterior distributions, we show a corner plot of the derived
parameters we fit and the priors we used. We have margin-
alized out four parameters: the parallax $\varpi$, proper motion
of the system barycenter in R.A. and Decl., and RV offset as
described in Section 4 and the Appendix.

We examine the chains of each parameter to make sure that the
burn-in phase is complete and all walkers have stabilized in the
mean and standard deviation of the posterior. We discard the
first 500 recorded steps (the first 25,000 overall, as we save
every 50th) as the burn-in phase so that the chains are ready to
be used for inference. The derived posterior probabilities from
our joint orbit fit are given in Table 5. With this case study, we
provide an example of the eight plots produced by orvara,
described in Section 6. Two types of reference schemes are
available for coloring the curves randomly drawn from the
posterior distributions, based on either the mass of the
secondary companion $M_{\text{comp}}$ or eccentricity $e$. We demonstrate
both reference schemes for HD 159062B.

Figure 2 shows the astrometric orbit of HD 159062B. The
RVs, both over a full orbit and restricted to the time frame of
the observations, are displayed in Figure 3. The relative
separation and position angles of the companion HD 159062B with
respect to the host star HD 159062A and the absolute
astrometry of the host star from HGC A are illustrated in
Figure 4. Finally, to evaluate the full behavior of the joint
posterior distributions, we show a corner plot of the derived
parameters of HD 159062B in Figure 5.

Our joint orbit fit yields a companion mass of $0.608^{+0.0083}_{-0.0073}$
$M_{\odot}$, an orbiting period of $P = 411^{+70}_{-70}$ yr, and an eccentricity of
$0.102^{+0.11}_{-0.066}$ for the WD HD 159062B. With our use of HGC A
astrometry, we improve H19’s constraint on HD 159062B’s mass by an order of magnitude. Our tight constraints on the
parameter mass and eccentricity help to refine our picture of
what the stellar system looks like. This can be used to infer the
mass of the progenitor via the initial–final mass relation
(Kalirai et al. 2008), which could serve as an input on the
barium enrichment of the primary star. We firmly place the
system on the long-period and low-eccentricity tail of the
distributions shown in Figure 6 of H19. This favors barium
enrichment via wind accretion and rules out mass transfer
by Roche lobe overflow. If the abundance measurement of
$[\text{Ba/Fe}] = 0.4$ dex is accurate, HD 159062 would add to the
few known cases of barium enrichment from wind accretion.
This new information can be used to guide studies on the local
stellar populations and Ba stars.

8. Conclusions

In this paper, we have presented the orbit-fitting package
orvara, designed to fit one or more Keplerian orbits to any
combination of RV, relative astrometry, and absolute
astrometry. The orvara package achieves high performance by
using a combination of a fast eccentric anomaly solver (faster
than a stand-alone call to sine and cosine for a large number of
data points), analytic marginalization of parallax and barycen-
ter proper motion, and low-level memory management. It is
free and open-source. It depends on other free packages,
including emcee (Foreman-Mackey et al. 2013), ptemcee
(Vousden et al. 2016), htof (Brandt et al. 2021), and
astropy (Astropy Collaboration et al. 2013; Price-Whelan
et al. 2018), along with numpy (Oliphant 2006; van der Walt
et al. 2011) and scipy (Virtanen et al. 2020). The orvara
package may be downloaded from github and installed by
pip. We have tested its installation and operation on
Windows, Mac, and Linux machines.

We have demonstrated orvara on the system HD 159062,
a WD/MS binary discovered by Hirsch et al. (2019). They
found a WD mass of $0.65^{+0.12}_{-0.04} M_{\odot}$ and a semimajor axis and
eccentricity that were compatible with a close pericenter
passage. By combining the absolute astrometry with the RV
and relative astrometry used by Hirsch et al. (2019), we
improve the precision of the mass by an order of magnitude, to
$0.617^{+0.013}_{-0.012} M_{\odot}$. We exclude orbits with a close pericenter
passage, establishing that the binary was a case of a wind
accretion system in its past.

The orvara package supports a variety of applications
beyond what we have demonstrated here with HD 159062. It
can account for multiple companions, it appropriately treats
epoch astrometry from Hipparcos and/or Gaia, and it can
account for a companion proper motion measured by Gaia. It is
also designed to produce publication-ready plots and may be
configured with a single .ini file. We plan to keep orvara
current, particularly when new Gaia data releases enable
substantial improvements in astrometric precision.10

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10 The version of orvara used in this publication is available at 10.5281/zenodo.5178227.
Appendix

Here we derive the multivariate Gaussian that describes the parallax and barycenter proper motion at fixed values of the other orbital parameters. Marginalizing over this multivariate Gaussian is equivalent to replacing the parallax and proper motion with their mean (i.e., best-fit) values and multiplying by the square root of the determinant of the covariance matrix.

We first write out the components of \( \chi^2 \) that involve the parallax \( \varpi \) and the proper motion of the system barycenter \( \mu \). We assume here that we have a measurement of the secondary star’s proper motion from Gaia; this measurement has a proper motion \( \mu_{G,B} \) and an inverse covariance matrix \( C_{G,B}^{-1} \). Measurements of the primary star use a suffix A, and the proper-motion model values refer to the proper motion of the primary star unless they carry the subscript B. If the secondary lacks a proper-motion measurement in Gaia (as is the case for the companions in Brandt et al. 2019 and Dupuy et al. 2019), then we set \( C_{G,B}^{-1} = 0 \). We also take model proper motions to be in physical units (e.g., au yr\(^{-1} \)), so that multiplying by a

\[
\begin{align*}
\chi^2 & = (\varpi - \bar{\varpi})^T C_{\varpi}^{-1} (\varpi - \bar{\varpi}) + (\mu - \bar{\mu})^T C_{\mu}^{-1} (\mu - \bar{\mu})
\end{align*}
\]

Figure 5. Corner plot of the derived parameters, including the mass of the primary star, mass of the secondary companion, semimajor axis, eccentricity, and inclination. The semimajor axis and eccentricity are highly covariant with one another and, to a lesser degree, with inclination.
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The covariance matrix of $\varpi$ and the two components of $\varpi$ are the inverse of $\mathbf{M}$. Integrating, or marginalizing, over $\varpi$ and $\varpi$ is equivalent to replacing $\varpi$ and $\varpi$ in Equation (36) with the solutions of Equation (37) and dividing by the square root of the determinant of $\mathbf{M}$ (multiplying by the square root of the determinant of the covariance matrix $\mathbf{M}^{-1}$).

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