Short-range oscillators in power-series picture

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Abstract

The class of short-range potentials $V^{[M]}(x) = \sum_{m=2}^{M} (f_m + g_m \sinh x) / \cosh^m x$ is considered as an asymptotically vanishing phenomenological alternative to the popular anharmonic long-range $V(x) = \sum_{n=2}^{N} h_n x^n$. We propose a method which parallels the analytic Hill-Taylor description of anharmonic oscillators and represents all the wave functions $\psi^{[M]}(x)$ non-numerically, in terms of certain infinite hypergeometric-like series. In this way the well known exact $M = 2$ solution is generalized to any $M > 2$.

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1 Introduction

A routine numerical solution of an asymmetric Schrödinger bound-state problem on the line \( x \in (-\infty, \infty) \) requires a careful verification [1]. One needs non-numerical asymmetric models. For this purpose we may use the shifted harmonic oscillator, Morse’s well and the two scarf-shaped hyperbolic forces. All of these models (cf. Table 1) are listed in review [2] as possessing the complete solution in closed form.

There exist incompletely solvable polynomials \( V(x) = a x + b x^2 + \ldots + z x^N \) and multi-exponentials \( V(x) = a e^{-x} + b e^{-2x} + \ldots + z e^{-Nx} \). They extend the possible tests and further non-numerical applications beyond \( N = 2 \). In a puzzling contrast, a natural generalization

\[
V^{[M]}(x) = \sum_{m=2}^{M} f_m \cosh^m x + \sinh x \sum_{n=1}^{M} g_n \cosh^n x
\]

of the remaining two items in Table 1 is not amenable to the similar elementary treatment [3]. This distracts attention from the hyperbolic oscillators [3] in spite of their obvious phenomenological as well as purely mathematical appeal.

In the present paper we shall return to several formal as well as descriptive parallels between the separate items in Table 1. On their basis we shall propose and describe a new semi-analytic approach to the “neglected” family (1).

In Section 2 we recall the harmonic and Morse oscillators and their \( N > 2 \) generalizations as our overall methodical guide. In the language of the well known Lanczos method [1] we underline the key role of simplicity of the repeated action of the Hamiltonian upon a suitable trial state \( |0\rangle \). An appropriate choice of this initial ket vector is able to inspire some of the existing non-numerical power series solutions. In this setting the Lanczos approach is shown to find its natural re-incarnations in the well known method of Hill determinants [7] as well as in the symmetric Jost-solution method of ref. [8].

In Section 3 we show that and how the latter two examples pave the way towards eq. (1) with any \( M \geq 2 \). In a full parallel to the polynomial case we construct the asymptotically correct bound state solutions which all retain a recurrently defined power-series structure. Via an appropriate \( D \)-dimensional partitioning of the basis...
we preserve their connection to the two remaining exactly solvable hyperbolic $M = 2$ examples of Table [1].

Section 4 illustrates the technical details at the first nontrivial $D = 2$. We contemplate there a spatially anti-symmetric $M = 2$ exercise (1) using $f_2 = g_1 = 0$. We detail the proof of the point-wise convergence of our “partitioned hypergeometric” wave functions. We show how the symmetry considerations significantly simplify the construction and matching of our wave functions near the origin.

Section 5 adds a short summary.

2 The method

2.1 Wave functions in the Lanczos basis

The Lanczos numerical eigenvalue method [3] works with a set $\{ |n\rangle \}$ of the basis ket vectors which are generated via a repeated action of the Hamiltonian $H$ upon an initial vector $|0\rangle$. In a slight generalization of this procedure one has to assume that the action of the full Schrödinger operator $H - z$ upon each ket $|n\rangle$ may be represented as a linear superposition over the same set of the kets [9],

$$
(H - z)|n\rangle = |0\rangle \cdot Q_{0,n}(z) + |1\rangle \cdot Q_{1,n}(z) + \ldots .
$$

(2)

With a matrix of functions $Q_{m,n}(z)$ (cf. [9], p. 257) we may abbreviate

$$
\left[ (H - z)|0\rangle, (H - z)|1\rangle, \ldots \right] \equiv (H - z) \left[ |0\rangle, |1\rangle, |2\rangle, \ldots \right] \equiv (H - z) \left| X \right\rangle,
$$

$$
(H - z) \left| X \right\rangle = \left| X \right\rangle \cdot Q(z)
$$

and solve any linear homogeneous equation $(H - E)|y\rangle = 0$ by the ansatz

$$
|y\rangle = \sum_{n=0}^{\infty} |n\rangle \cdot h_n \equiv |X\rangle \cdot \vec{h}.
$$

(3)

Provided that the separate lanczosean kets are linearly independent the resulting identity $|X\rangle Q(z)\vec{h} = 0$ may be interpreted as a system of conditions

$$
Q(z)\vec{h} = 0
$$

(4)
The practical applicability of this recipe relies upon several tacit assumptions. Most often one chooses the set \( \{ |n\rangle \} \) as a common harmonic oscillator basis \([11]\). It is orthonormal \( (\{X|X\} = I) \) and complete \( (|X\rangle \{X| = Id) \) and we may truncate the linear set \([4]\) to the mere routine matrix diagonalization

\[
\sum_{n=0}^{\mathcal{M}} [Q(0) - E I]_{m,n} h_n = 0, \quad m = 0, 1, \ldots, \mathcal{M}, \quad \mathcal{M} \gg 1.
\] (5)

This is a textbook variational recipe and its secular equation

\[
\det Q(E) = 0
\] (6)

determines the spectrum numerically \([12]\).

A non-variational and less numerical modification of the construction may be based on a more sophisticated choice of the Lanczos basis. Various linear algebraic algorithms of such a type are used to solve various Schrödinger equations in applications \([13]\). Let us recall two examples as our methodical guide.

### 2.2 Anharmonic example

Both the above-mentioned multi-exponential and polynomial oscillators prove mutually equivalent after a change of variables \([14]\). Their “canonical” \([15]\) representation

\[
V(x) = \frac{g_{-1}}{r^2} + g_1 r^2 + g_2 r^4 + \ldots + g_{2N-1} r^{4N-2}, \quad r \in (0, \infty)
\] (7)

is easily tractable by the variational algorithms. In the less numerical power-series approaches \([16]\) the harmonic kets are being replaced by their mere power-law components \( \langle r|n \rangle = \langle r|0^{\text{HO}} \rangle \cdot r^n \). This leads to an asymmetric matrix \( Q(z) \). Its linear algebraic eq. (3) proves often solvable as a very simple recurrent specification of the coefficients \( h_n \) in eq. (3) (cf. ref. \([17]\) for more details).

An even more ambitious reduction of \( Q \) may be achieved after an anharmonic choice of the initial \( |0\rangle \). According to Magyari \([3]\) this assigns a few elementary bound-state solutions to many multi-exponential and polynomial potentials at certain exceptional couplings. At arbitrary couplings and energies the same option \( |0\rangle \) may provide an extremely compact infinite-dimensional algebraic secular equation
For illustration let us consider the famous sextic oscillator example of ref. [18]. With $N = 2$ in eq. (7), denoting $g_3 = 16 \alpha^2$ and $g_2 = 16 \alpha \beta$ and using the WKB-inspired postulate

$$\langle r|n \rangle = r^{n+\ell+1} e^{-\alpha r^4 - \beta r^2}, \quad \alpha > 0$$

we get the tridiagonal quasi-Hamiltonian

$$Q(E) = \begin{pmatrix}
\alpha_0 & \gamma_1 & 0 & 0 & \cdots \\
\beta_0 & \alpha_1 & \gamma_2 & 0 & \cdots \\
0 & \beta_1 & \alpha_2 & \gamma_3 & \cdots \\
& & & & \ddots \\
& & & & & \ddots
\end{pmatrix}.$$  

(9)

Its equation (4) may safely be interpreted as an infinite-dimensional limit of the truncated diagonalization (5) provided only that $g_2 > 0$ [19]. The three nonzero diagonals in eq. (4) have to be compared with the seven-diagonal structure of the Hamiltonian in the usual orthogonalized harmonic oscillator basis.

For $g_2 \leq 0$ and at a special discrete set of the couplings $g_1$ the infinite-dimensional tridiagonal secular Hill determinant factorizes and the recipe reproduces a part of the spectrum correctly [18]. In all the other cases the WKB-compatible Lanczos basis ceases to be adequate. The Hill-determinant recipe (4) loses its relation to the correct asymptotic boundary conditions and the basis (8) must be regularized for certain hidden-symmetry reasons [20]. More diagonals necessarily appear in eq. (4). Otherwise, one gets wrong results from the truncated eq. (4) even in its infinite-dimensional limit [21].

Virtually no similar constructions of our short-range hyperbolic oscillators seem to appear in the current literature. In the present paper we intend to explain the difference and develop a new semi-analytic approach to eq. (4). Our construction will fairly closely parallel the formalism of the Hill-determinant method. In our second preparatory step the appropriately modified choice of the Lanczos basis will be illustrated via the symmetrized Rosen-Morse or scarf model of Table I.
2.3 Pöschl-Teller example

Formula (1) with \( M = 2 \), attraction \( f = -\lambda(\lambda - 1) \) and vanishing \( g = 0 \) defines the bell-shaped and spatially symmetric Pöschl-Teller well \( V^{(PT)}(x) = f/\cosh^2 x \) [22].

The functional form of the optimal lanczosean kets is more or less uniquely deduced, very much in the spirit of the “most ambitious” WKB-like choice in eq. (8) above, from the available exact solutions,

\[
\langle x|n \rangle = \xi_{n,p,q,\kappa}(x) = \frac{\sinh^{1-q} x}{\cosh^{\kappa+2n+p} x} \in L_2(-\infty, \infty). \tag{10}
\]

All these basis states possess the even or odd parity at \( q = 1 \) or \( q = 0 \), respectively.

Within this subsection let us fix \( p \equiv 1 - q \). Then, the action of the full Hamiltonian \( H^{(PT)} = -\partial_x^2 + V^{(PT)}(x) \) on our symmetrized/anti-symmetrized states (10) becomes particularly transparent. For energies \( E = -\kappa^2 \), it is characterized by the mere two-diagonal matrix

\[
Q(E) = \begin{pmatrix}
\alpha_0 & 0 & 0 & 0 & \cdots \\
\beta_0 & \alpha_1 & 0 & 0 & \cdots \\
0 & \beta_1 & \alpha_2 & 0 & \cdots \\
& & & & \ddots \\
& & & & \ddots
\end{pmatrix} \tag{11}
\]

with the vanishing uppermost element \( \alpha_0 = 0 \). The bound-state solutions (3) of our Schrödinger differential equation read

\[
\frac{1}{h_0} \langle x|y \rangle = |0\rangle - |1\rangle \cdot \frac{\beta_0}{\alpha_1} + |2\rangle \cdot \frac{\beta_0 \beta_1}{\alpha_1 \alpha_2} + \ldots \tag{12}
\]

As long as they are defined by the really elementary two-term recurrences (3),

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots \\
& & & \ddots \\
& & & \ddots \\
& & & \ddots \\
& & & \ddots
\end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ \vdots \end{pmatrix} = 0 \tag{13}
\]

our solution \( |y\rangle \) coincides with the Gauss hypergeometric series,

\[
\langle x|y \rangle = h_0 \tanh^p x \frac{1}{\cosh^\kappa x} \binom{\kappa + p + \lambda}{2} \binom{\kappa + p + 1 - \lambda}{2} ; 1 + \kappa; \frac{1}{\cosh^2 x}. \tag{14}
\]
It is defined on a half-axis, say, $x \geq 0$. Fortunately, due to the manifest symmetry or anti-symmetry of the physical solutions the necessary analytic continuation across the origin proves equivalent to the termination of this infinite series. The well known Jacobi polynomial solutions are obtained at each physical energy $\mathbb{E}$.

3 Partitioned expansions

We may conclude that the description of bound states by the infinite series (3) proves easy and efficient not only in the Hill-determinant setting of section 2.2 but also in an alternative Jost-solution spirit of section 2.3. We intend to extend the parallelism far beyond the trivial example of section 2.3.

The action of the kinetic energy $T = -\partial_x^2$ on the basis $|0\rangle$ conserves both the independent parity-like parameters $p$ and $q$. The same conservation law is obeyed by the single-term symmetric potentials $V_s^{(M)}(x) = f / \cosh^M x$ with the even exponents $M = 2K$. The rule is broken by the general Hamiltonians containing superpositions (4) of the symmetric and anti-symmetric components $V_s^{(M)}(x)$ and $V_a^{(N)}(x) = g \sinh x / \cosh^N x$, respectively. Nevertheless, the full basis $|\Xi\rangle$ numbered by a composite index $\mu = \mu(n, p, q) = 4n + 2p + q \geq 1$ (as $\Xi_{n,p,q,\kappa}(x) \equiv \langle x|\Xi_{\mu}\rangle$, $\mu = 1, 2, \ldots$) proves reducible for all the single-term potentials $V_{s,a}^{(N)}(x) = \pm V_{s,a}^{(N)}(-x)$ of a definite parity.

3.1 Symmetric potentials $V(x) = V(-x)$

We may choose the initial Lanczos ket $|0\rangle$ either as the spatially symmetric (and asymptotically correct) element $\langle x|\Xi_{\mu(0,0,1)}\rangle \equiv \cosh^{-\kappa} x$ with $p = 0$ and $q = 1$ or as its anti-symmetric analogue $\langle x|\Xi_{\mu(0,1,0)}\rangle \equiv \sinh x \cdot \cosh^{-\kappa-1} x$ with $p = 1$ and $q = 0$. In both these cases, all the Hamiltonian operators $T + V_s^{(2K)}(x)$ become compatible with recurrences (2) in the two alternative bases

$$|0\rangle, |1\rangle, |2\rangle, \ldots = |\Xi_{\mu(0,0,1)}\rangle, |\Xi_{\mu(1,0,1)}\rangle, |\Xi_{\mu(2,0,1)}\rangle \ldots \equiv |\Xi_1\rangle, |\Xi_5\rangle, |\Xi_9\rangle, \ldots,$$

$$|0\rangle, |1\rangle, |2\rangle, \ldots = |\Xi_{\mu(0,1,0)}\rangle, |\Xi_{\mu(1,1,0)}\rangle, |\Xi_{\mu(2,1,0)}\rangle \ldots \equiv |\Xi_2\rangle, |\Xi_6\rangle, |\Xi_{10}\rangle, \ldots.$$
with \( p = 1 - q = 0 \) or 1, respectively. After we abbreviate \( a_j = -j(2\kappa + j) \) and \( b_j = (\kappa + j)(\kappa + j + 1) \), this enables us to reproduce the two-diagonal Pöschl-Teller realization of \( Q = Q^{(p)} \) at \( K = 1 \),

\[
Q^{(0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & f + b_0, & a_2 & 0 & \ldots \\
0 & f + b_2, & a_4 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad Q^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & f + b_1, & a_2 & 0 & \ldots \\
0 & f + b_3, & a_4 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]

In the “first unsolvable” case with \( K = 2 \) the coupling \( f \) moves one step down,

\[
Q^{(0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & b_0, & a_2 & 0 & \ldots \\
0 & f + b_2, & a_4 & 0 & \ldots \\
0 & f + b_4, & a_6 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad Q^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & b_1, & a_2 & 0 & \ldots \\
0 & f + b_3, & a_4 & 0 & \ldots \\
0 & f + b_5, & a_6 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]

Partitioning indicated by the auxiliary lines tries to preserve the same two-diagonal pattern as above. At \( K = 3 \) we have, similarly,

\[
Q^{(0)} = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & b_0, & a_2 & 0 & \ldots \\
0 & b_2, & a_4 & 0 & \ldots \\
0 & f + b_4, & a_6 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad Q^{(1)} = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots \\
0 & b_1, & a_2 & 0 & \ldots \\
0 & b_3, & a_4 & 0 & \ldots \\
0 & f + b_5, & a_6 & 0 & \ldots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

and so on. The dimension of partitions grows linearly with \( M = 2K \) as \( D = K \).

The second series of the symmetric potentials \( V^{(2K+1)}(x) = f/\cosh^{2K+1}x \) with the odd powers \( M = 2K + 1 \) must be investigated separately. It acts on our parity-preserving basis in such a way that the conservation of the quantum number \( p \) is broken. The following \( q \)-preserving bases must be used,

\[
|0\rangle, |1\rangle, |2\rangle, \ldots \equiv |\Xi_2\rangle, |\Xi_4\rangle, |\Xi_6\rangle, |\Xi_8\rangle, \ldots, \quad q = 0,
\]

\[
|0\rangle, |1\rangle, |2\rangle, \ldots \equiv |\Xi_1\rangle, |\Xi_3\rangle, |\Xi_5\rangle, |\Xi_7\rangle, \ldots, \quad q = 1.
\]
At $K = 0$ the new lower triangular matrices $Q = Q^{[gl]}$ contain just the three nonzero neighboring diagonals. For a preservation of the two-diagonal notation it is sufficient to switch to the $D = 2$ partitioning. Similarly, a three-dimensional partitioning is needed at $K = 1$. With the further increase of $K$ the dimension $D = 2K + 1$ grows more quickly.

3.2 Anti-symmetric potentials $V(x) = -V(-x)$

The class of the anti-symmetric forces $V^{(2L)}_a(x)$ with $L \geq 1$ inter-relates the basis states with different parities $q$. The value of the index $p$ is conserved,

$$V^{(2L)}_a(x)\langle \Xi_{\mu(n,p,q)} | = (1 - q) g \cdot |\Xi_{\mu(n+L-1,p,1-q)} \rangle + (-1)^{1-q} g \cdot |\Xi_{\mu(n,L,p,1-q)} \rangle.$$  

The Hamiltonian $T + V^{(2L)}_a$ acts transitively on the following two reduced Lanczos bases,

$$|0\rangle, |1\rangle, |2\rangle, \ldots \equiv |\Xi_1\rangle, |\Xi_4\rangle, |\Xi_5\rangle; |\Xi_8\rangle, |\Xi_9\rangle; \ldots, \quad p = 0 \quad (15)$$

$$|0\rangle, |1\rangle, \ldots \equiv |\Xi_2\rangle, |\Xi_3\rangle; |\Xi_6\rangle, |\Xi_7\rangle; |\Xi_{10}\rangle, |\Xi_{11}\rangle; \ldots, \quad p = 1. \quad (16)$$

Marginally, we may note that at $L = 0$ the structure of the matrix $Q$ ceases to be triangular. This seems closely related to the asymptotic asymmetry of the $g_1 \neq 0$ potentials $V^{[M]}(-\infty) = -g_1 \neq V^{[M]}(\infty) = +g_1$ and to their anomalous non-Jost solvability via a change of variables at $M = 2$ (cf., e.g., [23]). In this subsection we shall assume that $g_1 \equiv 0$, therefore. This constraint is further supported by the observation that at $M = 1$ the monotonic $V^{(1)}_a(x) = g_1 \cdot \tanh x$ itself cannot generate any bound states at all. Thus, our study of the anti-symmetric models has to start at the exactly solvable $V^{(2)}_a(x) = g \sinh x / \cosh^2 x$ (cf. Table 1).

This anti-symmetric scarf (AS) potential $V^{(2)}_a(x) \equiv V^{[AS]}_a(x)$ is extremely suitable for methodical purposes. Its significance is connected to the fact that our basis (10) is not tailored precisely to its exact solvability. A $D = 2$ partitioning is needed. In the reduced bases (15) and (16) its recommended boundaries are marked by the semi-colons. For all the $L = 2, 3, \ldots$ descendants $V^{(2L)}_a(x)$ of the AS example the size $D$ of partitions will grow due to the downward shift of the constant $g$ again.
The action of the last class \( V_{a}^{(2L+1)}(x) = g \cdot \sinh x \cdot \cosh^{-2L-1} x \) of the simplified single-term potentials on the kets \([\Xi]\) looks irreducible. The impression is wrong. After we introduce a new quantum number \( I \equiv 2p+q \) (modulo 4), the basis elements with \( I = 0 \) and \( I = 3 \) never mix with their \( I = 1 \) and \( I = 2 \) counterparts. For both the initial choices of \( |0\rangle = |\Xi_1\rangle \) and \( |0\rangle = |\Xi_2\rangle \) we arrive at the same output,

\[
|0\rangle, |1\rangle, |2\rangle, \ldots \equiv |\Xi_{1 \ or \ 2}\rangle, |\Xi_5\rangle, |\Xi_6\rangle, |\Xi_9\rangle, |\Xi_{10}\rangle, |\Xi_{13}\rangle, \ldots
\]

The difference between the two matrices \( Q \) will only lie in their elements.

3.3 Asymmetric Lanczos kets

Asymmetric oscillators \([\Xi]\) admit a non-conservation of parity by each Lanczos element \( |n\rangle \) separately. The functions

\[
\langle x|n \rangle = \xi_{n,p,q,a,k}(x) = \frac{\sinh^{1-q} x}{\cosh^{\sigma+2n+p} x} e^{a \arctan(\sinh x)} \in L_2(-\infty, \infty)
\]

generalize their \( a = 0 \) predecessors \([\Xi]\) and represent a very good new candidate since, due to the presence of a new parameter \( a \), the number of the new terms in eq. (2) may be lowered, for any potential \([\Xi]\), more efficiently. First of all, this implies that we may admit the nonzero \( g_1 \) again. Via a suitable choice of the value of \( a \) we shall be able to reproduce all the “missing” (viz., Rosen Morse and scarf) terminating solutions of ref. [2] or Table [4].

At \( a \neq 0 \) also the action of an arbitrary hyperbolic Hamiltonian remains transparent and elementary in the purely kinetic limit,

\[
\frac{\xi_{n,p,q,a,k}''(x)}{\xi_{n,p,q,a,k}(x)} = (\sigma + q - 1)^2 + \frac{a^2 - \sigma(\sigma + 1) - (2\sigma + 1)a \sinh x}{\cosh^2 x} + (q - 1) \frac{q - 2a \sinh x}{\sinh^2 x}.
\]

Here, \( \sigma = \sigma(n, p) = \kappa + 2n + p \) and the prime denotes the differentiation with respect to \( x \). The action of the purely kinetic Hamiltonian \( T = -\partial_x^2 \) on our innovated kets \( \langle x|\Xi_\mu \rangle \equiv \xi_{n,p,q,a,k}(x) \) may employ the multi-indices \( \mu(n, p, q) = 4n + 2p + q \) again,

\[
T|\Xi_{\mu(n,p,0)}\rangle = -(\sigma - 1)^2 |\Xi_{\mu(n,p,0)}\rangle + (2\sigma - 1) a |\Xi_{\mu(n,p,1)}\rangle +
\]
\[
+ (\sigma^2 + \sigma - a^2) |\Xi_{\mu(n+1,p,0)}\rangle - (2\sigma + 1) a |\Xi_{\mu(n+1,p,1)}\rangle,
\]
$$T|\Xi_{\mu(n,p,1)}\rangle = -\sigma^2|\Xi_{\mu(n,p,1)}\rangle + (2\sigma + 1)\ a|\Xi_{\mu(n+1,p,0)}\rangle + (\sigma^2 + \sigma - a^2)|\Xi_{\mu(n+1,p,1)}\rangle.$$  

The kinetic matrix elements of $Q$ depend on $\sigma$ and $a$ and all of them increase with $n$. Due to the presence of the new parameter $a$ the kinetic operator $T$ inter-twins the states (17) with different parities $q = 0, 1$. The states with different $p = 0, 1$ stay decoupled.

### 3.4 Partitioned hypergeometric-like series

Our present proposal may be summarized as an application of expansions (3) to potentials (1) inspired by the analogies between the Pöschl-Teller and harmonic oscillators. The feasibility of our construction stems from the fact that the action of the present class of Hamiltonians on the suitable Lanczos kets may be characterized by the lower triangular matrices $Q(z)$. Their partitioning brings us back to the two-diagonal pattern of eq. (11) and replaces its scalars $\alpha_j$ and $\beta_j$ by the respective two-dimensional submatrices $A_j$ and $B_j$, 

$$Q = \begin{pmatrix} A_0 & 0 & 0 & 0 & \ldots \\ B_0 & A_1 & 0 & 0 & \ldots \\ 0 & B_1 & A_2 & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$  

(18)

In both the respective $a = 0$ and $a \neq 0$ bases (10) and (17) the $D$–plets of kets $(|m + 1\rangle, |m + 2\rangle, \ldots, |m + D\rangle)$ with $m = m(n) = nD - d_0$ and with any $d_0$ may be denoted as $|n\rangle$. In such an abbreviated notation our linear system (4) implies the recurrence relations 

$$F_n \equiv \begin{pmatrix} h_{m(n)+1} \\ \ldots \\ h_{m(n)+D} \end{pmatrix} = -(A_n)^{-1}B_{n-1}F_{n-1}, \quad n = 1, 2, \ldots$$  

(19)

which define the $D$–dimensional vectors of coefficients in terms of finite products of the certain $D \times D$–dimensional matrices. In place of $d_0 = 1$ in a consequently $D$–dimensional “democratic” partitioning we may use the shift $d_0 = D$. Both these
options appear in our AS example where we recommended \( d_0 = D - p \). The latter
one is globally preferable as it leaves the uppermost element of \( Q \) vanishing, \( A_0 = 0 \).
The initial array \( F_0 \) degenerates to the mere scalar norm then.

At any \( d_0 \) the formal solution \( (3) \) of the Schrödinger equation \((H - E)|y⟩ = 0\) may be re-written in the form of the double or partitioned sum,

\[
|y⟩ = \sum_{n=0}^{∞} D \sum_{j=1}^{D} |n⟩⟩_j [F_n]_j = \sum_{n=0}^{∞} |n⟩⟩ \cdot F_n .
\]

(20)

In a little bit vague sense it looks like an immediate hypergeometric-like general-
ization of eq. \((14)\). Equation \((13)\) defines all its coefficients in closed form. They
depend on the “measure of asymmetry” \( a \) and on the unknown energy \( E = -κ^2 \).

4 Example

Our recipe strongly resembles the Hill-determinant method which proves useful in
many (e.g., perturbative \([24]\)) applications. In the majority of similar applications
one must analyze, first of all, the convergence of infinite series \((3)\) or \((20)\). In
\( x \)-representation their point-wise convergence is basically controlled by the asympt-
totics of the coefficients. They are dominated by the purely kinetic terms which are
asymptotically increasing. All the characteristics of the potential itself (e.g., parity
mixing) will play, necessarily, a secondary role.

The first non-trivial asymmetric potential \( V^{[AS]}(x) \) seems best suited for a more
explicit illustration of this role. Its coefficients \( h_j = h_j^{(q)}(p) \) in both the \( p = 0 \) and
\( p = 1 \) solutions \((3)\) are easily derived from the respective recurrences. Choosing the
simplest $a = 0$ and using the same abbreviations $a_j$ and $b_j$ as above we have

$$Q^{(0)} = \begin{pmatrix} 0 & a_1 & & & & \\ g & a_2 & & & & \\ b_0 & g & a_3 & & & \\ 0 & b_2 & g & a_4 & & \\ -g & b_2 & g & a_5 & & \\ 0 & b_4 & g & a_6 & & \\ -g & b_4 & g & a_7 & & \\ & & & & & \vdots \\ & & & & & \vdots \\ & & & & & \vdots \\ & & & & & \vdots \end{pmatrix}, \quad (21)$$

$$Q^{(1)} = \begin{pmatrix} 0 & a_1 & & & & \\ g & a_2 & & & & \\ b_1 & g & a_3 & & & \\ -g & b_1 & g & a_4 & & \\ 0 & b_3 & g & a_5 & & \\ -g & b_3 & g & a_6 & & \\ & & & & & \vdots \\ & & & & & \vdots \\ & & & & & \vdots \end{pmatrix}. \quad (22)$$

The contribution of the coupling $g$ is clearly separated from the growing and energy-dependent kinetic terms.

After a return to the general $a \neq 0$ we just have to modify the values of the matrix elements accordingly. We may preserve the reduction of bases (15) and (16) as well as their $D = 2$ partitioning. It is obvious that the exact Jacobi polynomial solutions may be reproduced in our $D = 2$ language. It is an instructive exercise to show how this reproduction proceeds. Firstly, the variability of the parameter $a$ and of the energy or momentum $\kappa$ enables us to achieve a complete disappearance of the two-by-two submatrix $B_K = 0$ at an arbitrary optional $K$. The resulting series (20) then strictly terminates and reproduces the known Gauss hypergeometric solution. Its termination just reflects the factorization of the secular determinant.

Let us underline that the simpler, “termination-incompatible” basis (10) with $a = 0$ is an analogue of the non-WKB bases in Section 2.2. Hence, we may fix $a = 0$
and recall the same AS model also as one of the simplest illustrative examples of a general non-terminating solution.

4.1 AS oscillator in the $a = 0$ representation

The AS solutions (3) may be split in the two separate sums with the well defined parity,

$$|Y^{[AS]}⟩ = |Y^{[AS]}(p)⟩ = |Y^{(even)}(p)⟩ + |Y^{(odd)}(p)⟩.$$  \hfill (23)

The first few terms in the even partial sums with $q = 1$,

$$|Y^{(even)}(0)⟩ = |Ξ_1⟩ \cdot h_0^{(1)}(0) + |Ξ_5⟩ \cdot h_2^{(1)}(0) + |Ξ_9⟩ \cdot h_4^{(1)}(0) + \ldots,$$

$$|Y^{(even)}(1)⟩ = |Ξ_3⟩ \cdot h_1^{(1)}(1) + |Ξ_7⟩ \cdot h_3^{(1)}(1) + |Ξ_{11}⟩ \cdot h_5^{(1)}(1) + \ldots,$$  \hfill (24)

as well as their odd, $q = 0$ counterparts

$$|Y^{(odd)}(0)⟩ = |Ξ_4⟩ \cdot h_1^{(0)}(0) + |Ξ_8⟩ \cdot h_3^{(0)}(0) + |Ξ_{12}⟩ \cdot h_5^{(0)}(0) + \ldots,$$

$$|Y^{(odd)}(1)⟩ = |Ξ_2⟩ \cdot h_0^{(0)}(1) + |Ξ_6⟩ \cdot h_2^{(0)}(1) + |Ξ_{10}⟩ \cdot h_4^{(0)}(1) + \ldots,$$  \hfill (25)

are easily computed in the recurrent manner,

$$h_0^{(1)}(0) = 1, \quad h_1^{(0)}(0) = -g/a_1, \quad h_2^{(1)}(0) = -b_0/a_2 + g^2/(a_1a_2), \quad \ldots,$$

$$h_0^{(0)}(1) = 1, \quad h_1^{(1)}(1) = -g/a_1, \quad h_2^{(0)}(1) = -b_1/a_2 + g^2/(a_1a_2), \quad \ldots.$$  \hfill (26)

A compact general determinantal formula for these coefficients also exists \[15\]. It would enable us to re-write eq. (23), i.e.,

$$|Y^{[AS]}(p)⟩ = \sum_{j=0}^{∞} |Ξ_{µ(j,p,1)}⟩ \cdot h_{2j+p}^{(1)}(p) + \sum_{j=0}^{∞} |Ξ_{µ(j+1-p,p,0)}⟩ \cdot h_{2j+1-p}^{(0)}(p)$$  \hfill (27)

in the explicit form if needed. Here, we prefer the recurrent generation of the doublets of coefficients

$$F_{n+1-p} = F_{n+1-p}(p) = \begin{pmatrix} h_{2n+1-p}^{(0)}(p) \\ h_{2n+2-p}^{(1)}(p) \end{pmatrix}, \quad p = 0 \text{ or } 1, \quad n = 0, 1, \ldots$$

13
as a matrix product,

\[ F_j(p) = [-A_j(p)]^{-1} B_j(p) F_{j-1}(p), \quad j = 1, 2, \ldots . \quad (28) \]

In our partitioned notation with \( D = 2 \) the solution \(|y\rangle\) may be presented as a two-dimensional hypergeometric series since its matrix coefficients remain surprisingly elementary,

\[ [-A_j(p)]^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/a_{2j+p} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} 1/a_{2j+p-1} & 0 \\ 0 & 1 \end{pmatrix}. \]

As long as \( 0 > a_1 > a_2 > \ldots \) at any \( \kappa > 0 \), all our vectors of coefficients are well defined and unique. Their initialization is provided by the “model space” equation \( A_0(p) F_0(p) = 0 \) which depends on \( p \). At \( p = 0 \), we have the vanishing scalar \( A_0(0) \equiv 0 \) while the exceptional singlet \( F_0(0) = h_0^{(1)}(0) \) (conveniently put equal to one) is the norm. In the parallel two-dimensional initialization at \( p = 1 \), the first component \( h_0^{(0)}(1) = 1 \) of \( F_0(1) \) is the norm. The second component must be re-calculated, \( h_0^{(1)}(1) = g h_0^{(0)}(1)/(2\kappa + 1) \).

We are ready to prove the convergence. Its decisive simplification occurs in the \( j \gg 1 \) asymptotic domain. The upper and lower components of eq. (28) decouple there in a \( p \)-independent manner,

\[ [F_j(p)]_q \equiv h_{2j+p+q-1}(p) = \left[ 1 + \frac{1 - 4q}{2j} + \mathcal{O}\left(\frac{1}{j^2}\right) \right] [F_{j-1}(p)]_q, \quad q = 0 \text{ or } 1. \quad (29) \]

For both our infinite series (27) the proof is easy at \( x \neq 0 \). As long as \( \cosh x > 1 \) for all the nonzero and real coordinates \( x \), the ordinary geometric criterion together with the estimate (29) implies that our series (27) are convergent absolutely, i.e., for all the (complex) couplings \( g \) and energies \( -\kappa^2 \). The same geometric argument extends the validity of our conclusion to all the complex coordinates \( x + iy \) which lie out of a wiggly bounded domain such that \(|\cosh(x + iy)| = \sqrt{\sinh^2 x + \cos^2 y} \leq 1 \) or, in a cruder approximation, out of the fairly narrow strip with \(|x| \leq \ln(1 + \sqrt{2}) \) at least.

On the real axis, an indeterminate behaviour of the type \( 0 \times \infty \) emerges at the point \( x = 0 \). This follows from eq. (29) and from the slightly more sophisticated
Raabe criterion. Strictly speaking, this forces us to work on a punctured domain of \( x \in (-\infty, 0) \cup (0, \infty) \) in principle. As a consequence, logarithmic derivatives of our left and right Jost solutions have to be matched in the origin. This task is to be fulfilled numerically. Let us outline its two steps.

### 4.2 Generalized parity

Since our \( D = 2 \) hypergeometric AS series \( \langle x|Y^{[AS]}(p) \rangle \equiv \varphi^{(p)}(g, x, \kappa) \) (27) satisfy the differential Schrödinger equation on a punctured domain \((-\infty, 0) \cup (0, \infty)\) only, we necessarily have to match them in the origin. In the Pöschl-Teller example of section 2.3 where the non-matrix Gauss solutions also developed a certain discontinuity in the origin at a general unphysical energy \( E \), the point has easily been settled after an account of parity. As long as our potentials lose their spatial symmetry in general, the parity is broken and a matching of the two sub-intervals \((-\infty, 0) \cup (0, \infty)\) becomes nontrivial.

We have to employ a broader invariance of our model(s) with respect to the product \( \hat{P} \) of parity \( P \) with the reflections of couplings \( g_j \to -g_j \). The operator (such that \( \hat{P}^2 = 1 \)) commutes with our Hamiltonian(s), \( H = \hat{P} H \hat{P} \). Each physical bound state \( \psi(x) \) may be assigned an even or odd \( \hat{P} \)–parity, \( \hat{P} \psi(x) = \pm \psi(x) \).

In a way resembling the parity-breaking systems with \( \mathcal{PT} \) invariance [25], the assignment of the \( \hat{P} \)–parity to our AS states \( \chi(g, x) \) depends on their normalization,

\[
\{ \hat{P} \chi(g, x) = \pm \chi(g, x) \} \implies \{ \hat{P} [g \cdot \chi(g, x)] = \mp [g \cdot \chi(g, x)] \}.
\]

Fortunately, our AS coefficients \( h_n^{(q)}(p) = h_n^{(g)}(p, g) \) are explicitly defined by the triangularized Hamiltonians (21) and (22) and we immediately notice that

\[
h_j^{(q)}(p, -g) = (-1)^{p+q+1} h_j^{(g)}(p, g).
\]

Both our AS hypergeometric-like series \( \varphi^{(p)}(g, x, \kappa) = \langle x|Y^{[AS]}(p) \rangle \) (27) behave as eigenstates of our double-parity operator \( \hat{P} \),

\[
\hat{P} \varphi^{(p)}(g, x, \kappa) = \varphi^{(p)}(-g, -x, \kappa) = (-1)^p \varphi^{(p)}(g, x, \kappa).
\]
With a pair of some constants $\mathcal{M} \neq \mathcal{M}(g)$ and $\mathcal{N} \neq \mathcal{N}(g)$ we may postulate that the bound states read

$$\psi^{[\text{AS}]}(x) = \mathcal{M} \varphi^{(0)}(g, x, \kappa) + g \cdot \mathcal{N} \varphi^{(1)}(g, x, \kappa), \quad x \neq 0. \quad (30)$$

The same (conventionally, even) $\hat{P}$-parity may be assigned to all our physical solutions since their energy spectrum is non-degenerate.

### 4.3 Match in the origin

A return to the ordinary spatial parity $\mathcal{P}$ enables us to distinguish between the cosine-like (i.e., spatially even) and sine-like (i.e., spatially odd) components of our generalized hypergeometric functions (27),

$$
\begin{align*}
    c(x, \kappa) &= \frac{1}{2}[\varphi^{(0)}(g, x, \kappa) + \varphi^{(0)}(g, -x, \kappa)], \\
    \tilde{s}(x, \kappa) &= \frac{1}{2}[\varphi^{(0)}(g, x, \kappa) - \varphi^{(0)}(g, -x, \kappa)], \\
    \tilde{c}(x, \kappa) &= \frac{1}{2}[\varphi^{(1)}(g, x, \kappa) + \varphi^{(1)}(g, -x, \kappa)], \\
    s(x, \kappa) &= \frac{1}{2}[\varphi^{(1)}(g, x, \kappa) - \varphi^{(1)}(g, -x, \kappa)].
\end{align*}
$$

The tildas $\tilde{\cdot}$ marking the asymptotical subdominance are not too relevant since we dwell in a vicinity of the origin where $x = \pm \varepsilon \approx 0$. Wavefunctions must be continuous there,

$$
\lim_{\varepsilon \to 0^+} \psi^{[\text{AS}]}_{\text{(physical)}}(\varepsilon) = \lim_{\varepsilon \to 0^+} \psi^{[\text{AS}]}_{\text{(physical)}}(-\varepsilon).
$$

The even, cosine-like components of our solutions satisfy such a requirement identically. In the light of eq. (30) we are left with a reduced continuity condition

$$\mathcal{M} \tilde{s}(\varepsilon, \kappa) + g \cdot \mathcal{N} s(\varepsilon, \kappa) = 0, \quad \varepsilon \to 0. \quad (31)$$

In the same manner, the continuity of derivatives is required. In the upper-case notation with abbreviations

$$S(x, \kappa) = \frac{1}{2}[\partial_x \varphi^{(0)}(g, x, \kappa) + \partial_x \varphi^{(0)}(g, -x, \kappa)],$$
\( \tilde{C}(x, \kappa) = \frac{1}{2}[\partial_x \varphi^{(0)}(g, x, \kappa) - \partial_x \varphi^{(0)}(g, -x, \kappa)], \)

\( \tilde{S}(x, \kappa) = \frac{1}{2}[\partial_x \varphi^{(1)}(g, x, \kappa) + \partial_x \varphi^{(1)}(g, -x, \kappa)], \)

\( C(x, \kappa) = \frac{1}{2}[\partial_x \varphi^{(1)}(g, x, \kappa) - \partial_x \varphi^{(1)}(g, -x, \kappa)] \)

this leads to the second reduced matching condition

\[ \mathcal{M} S(\varepsilon, \kappa) + g \cdot \mathcal{N} \tilde{S}(\varepsilon, \kappa) = 0, \quad \varepsilon \to 0. \] (32)

In the limit \( \varepsilon \to 0 \) a root \( \kappa(\varepsilon) \) of the two-dimensional secular equation

\[
\det \begin{pmatrix}
\tilde{s}(\varepsilon, \kappa) & s(\varepsilon, \kappa) \\
S(\varepsilon, \kappa) & \tilde{S}(\varepsilon, \kappa)
\end{pmatrix} = 0
\]

will determine the physical energy. Matrix elements of this secular equation are convergent series in \( t = \cosh^{-2} \varepsilon < 1, \)

\[ \tilde{s}(\varepsilon, \kappa) = \sum_{n=1}^{\infty} h_n^{(0)}(0, g) t^n, \quad s(\varepsilon, \kappa) = \sum_{n=0}^{\infty} h_n^{(0)}(1, g) t^n, \]

\[ S(\varepsilon, \kappa) = \sum_{n=0}^{\infty} (\kappa + 2n) h_n^{(1)}(0, g) t^n, \quad \tilde{S}(\varepsilon, \kappa) = \sum_{n=0}^{\infty} (\kappa + 2n + 1) h_n^{(1)}(1, g) t^n. \]

Norms \( h_0^{(0)}(0, g) = h_0^{(0)}(1, g) = 1 \) are fixed and the higher coefficients carry the \( \kappa \)–dependence. An analogy with the spatially symmetric Pöschl-Teller example of section 2.3 is fully restored.
5 Summary

We described a new approach to the Schrödinger bound-state problem with any Rosen-Morse-like multi-term potential (1). For all these forces we have shown how

• the ordinary differential Schrödinger equation for the wave functions $\psi(x)$ may be reduced to a linear homogeneous algebraic problem $Q(E)\vec{\hbar} = 0$;

• an “inspired” choice of the Lanczos-like (i.e., Hamiltonian-dependent) basis makes the related infinite-dimensional secular determinant vanish identically, $\det Q(E) = 0$;

• the very special (viz., lower-triangular) structure of our quasi-hamiltonian matrices $Q(E)$ reduces the construction of the separate Taylor-like coefficients $h_n$ in our wave functions $\psi(x)$ to the mere (partitioned) two-term recurrences.

On a characteristic AS example we have illustrated that

• all our solutions $\psi(x)$ are convergent and may be understood as a certain generalization of the Gauss hypergeometric series (which further degenerates to the Jacobi polynomials at the physical energies in the solvable cases);

• a certain generalized parity symmetry of our forces enables us to determine Jost solutions which are compatible with both our asymptotic boundary conditions;

• via our final two-by-two condition (31) + (32), the values of the remaining two free parameters (viz., energy and $p$–mixing) in our Jost solutions may (and have to) be tuned to their necessary continuity and smoothness in the origin.
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Table 1: Shape-invariant potentials on the line \[2\]

| model       | \(V(x)\)                  | \(V(-\infty)\) | \(V(\infty)\) | polynomial \(\psi(x)\) |
|-------------|-----------------------------|-----------------|----------------|------------------------|
| harmonic    | \(\omega^2(x+b)^2\)        | \(\infty\)     | \(\infty\)    | Laguerre               |
| Morse       | \(a e^{-x} + be^{-2x}\)    | \(\infty\)     | 0              | Laguerre               |
| Rosen-Morse II | \(f/cosh^2 x + g \tanh x\) | -g              | g              | Jacobi                 |
| scarf II    | \((f + g \sinh x)/cosh^2 x\) | 0               | 0              | Jacobi                 |