Third Generation Effects on Fermion Mass Predictions in Supersymmetric Grand Unified Theories†

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ABSTRACT

Relations among fermion masses and mixing angles at the scale of grand unification are modified at lower energies by renormalization group running induced by gauge and Yukawa couplings. In supersymmetric theories, the $b$ quark and $\tau$ lepton Yukawa couplings, as well as the $t$ quark coupling, may cause significant running if $\tan \beta$, the ratio of Higgs field expectation values, is large. We present approximate analytic expressions for the scaling factors for fermion masses and CKM matrix elements induced by all three third generation Yukawa couplings. We then determine how running caused by the third generation of fermions affects the predictions arising from three possible forms for the Yukawa coupling matrices at the GUT scale: the Georgi-Jarlskog, Giudice, and Fritzsch textures.

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1. Introduction

Thirteen is the number of independent parameters required to describe the fermion sector of the standard model: nine masses, three mixing angles, and a CP-violating phase. Although such a plethora of arbitrary parameters is usually regarded as a weakness, we could instead view the situation as an opportunity to reach beyond the standard model. Because the fermion parameters can take on arbitrary values in the standard model, any prediction of these parameters can only come from beyond the standard model. Conversely, their observed experimental values could provide a clue to new physics.

A step in this direction has been taken with the discovery of various phenomenological relations among fermion masses and mixing angles, which could be viewed as modern-day Balmer formulae. Two types of relations among fermion parameters have been explored. The first type links the masses of fermions within the same generation to one another. Such relations result naturally from grand unified theories (GUTs) when the fermions belong to a common grand unified representation and couple to a single Higgs field; group theory then dictates a relation between their masses [1–3]. The second type of relation connects Cabibbo-Kobayashi-Maskawa (CKM) matrix elements with ratios of masses of fermions in different generations. These relations arise naturally when certain entries of the Yukawa matrices vanish, perhaps as a result of discrete symmetries [4, 5].

These various relations among Yukawa couplings are presumably a consequence of new physics, and therefore hold at the energy scale of the new physics, e.g., the grand unification scale. But Yukawa couplings evolve in accord with the renormalization group (RG) equations, so relations among them that apply at one scale will not necessarily hold at another scale. Therefore, before they can be compared with low-energy data, GUT-scale relations must be corrected to account for RG running.

Most of the running of the Yukawa couplings is induced by the gauge couplings. For example, group-theoretic relations between quark and lepton masses at the GUT scale are greatly modified at low-energy scales because quarks and leptons
have different gauge couplings and their masses therefore run differently [1–3]. If they are sufficiently large, however, Yukawa couplings themselves induce further running [6], and therefore further modifications of relations between masses [7, 8]. Yukawa couplings also induce running of the CKM matrix elements [9], which are invariant under gauge coupling-induced running. The effect of the top quark Yukawa coupling on fermion mass and mixing angle relations in supersymmetric theories was recently investigated in refs. [10–14].

In the standard model, the Yukawa couplings of fermions other than the top quark are too small to cause significant running. In supersymmetric theories, however, the Yukawa couplings of the other third generation fermions, the $b$ quark and $\tau$ lepton, may be comparable to the $t$ quark Yukawa coupling, even though their masses are much smaller. This occurs when the expectation value of the Higgs field to which $b$ and $\tau$ are coupled is much less than that of the Higgs field to which $t$ is coupled; that is, when $\tan \beta$, the ratio of expectation values of the two Higgs fields, is large. In this regime, all three third generation fermions may cause significant running. This case has also been investigated recently by several authors [10, 11, 15–17], who solved the RG equations numerically.

In this paper, we would like to calculate the effect of RG running induced by the entire third generation of fermions analytically. An analytic result would have several advantages over a numerical solution. In addition to enhancing intuition about the effects of Yukawa coupling-induced running, it would allow one to see transparently how changes in the input parameters affect the predictions, without having to re-run the numerical routines for each new set of data. It would also simplify the error analysis. Although it is not possible to solve the RG equations exactly, we introduce an approximation (good to within a few percent) that includes the running induced by all three third generation fermions and that allows an analytic solution. We then use this approximate solution to determine the effects of RG running on several different sets of mass and mixing angle relations, and on the predictions that follow from those relations.
The RG running of the Yukawa couplings is logically independent of the fermion mass and mixing angle relations because the latter arise from new physics at the GUT scale whereas the running from the GUT scale to the low-energy scale only depends on the particle spectrum below the GUT scale. From the point of view of the RG equations, the only role of the fermion relations is to provide boundary conditions at the GUT scale. Because of this, we will be able to analyze the RG running of the Yukawa couplings independently of any particular set of fermion relations.

One approach is to evolve the matrices of Yukawa couplings down to the low-energy scale, and then diagonalize them to find the fermion masses and mixing angles. Many degrees of freedom of the Yukawa matrices are not physical, however, because of the freedom to perform unitary redefinitions of the fermion bases. Therefore, we instead begin with the RG equations for the smaller set of physical parameters, viz., the fermion masses, mixing angles, and CP-violating phase. This simplifies the computational task by reducing the number of equations, and allows us to deal only with physical quantities throughout.

In sect. 2, we review the one-loop RG equations for fermion masses and CKM matrix elements in the minimal supersymmetric standard model. Adopting a non-standard parametrization of the CKM matrix, we obtain explicit RG equations for the mixing angles and CP-violating phase. In sect. 3, we introduce an approximation that allows us to solve the RG equations analytically, including the effects of the entire third generation of fermions. We then analyze in sect. 4 the RG effects on three different sets of fermion mass and mixing angle relations that might result from new physics at the GUT scale. Sect. 5 contains our conclusions.
2. Running Masses and Mixing Angles

In this section, we review the renormalization group running of fermion masses and CKM matrix elements. We also include a discussion of phase choices for the CKM matrix so that we can obtain explicit RG equations for the mixing angles which parametrize it.

As noted in the introduction, the renormalization group analysis is independent of the new physics responsible for relations between fermion masses and mixing angles, and can therefore be applied to different sets of relations. We assume, however, that the relations result from some grand unified theory, and therefore that the three gauge couplings meet at a single scale $\mu$. This can be achieved in the context of the minimal supersymmetric standard model, with the supersymmetry breaking scale $\mu_{\text{susy}}$ between 100 GeV and 10 TeV [18]. We will assume that this framework describes physics up to the GUT scale $\mu$.

The one-loop RG equations for the gauge couplings are

$$16\pi^2 \frac{d}{dt} \ln g_i = b_i g_i^2, \quad t = \ln \mu, \quad (2.1)$$

and have solutions

$$\frac{1}{g_i^2(\mu)} = \frac{1}{g_i^2} - \frac{b_i}{8\pi^2} \ln \left( \frac{\mu}{\mu} \right). \quad (2.2)$$

Between $\mu_{\text{susy}}$ and the grand unified scale $\mu$, the coefficients are given by

$$(b_1, b_2, b_3) = \left( \frac{33}{5}, 1, -3 \right). \quad (2.3)$$

We choose $\mu_{\text{susy}} = 170$ GeV for convenience (close to the top quark mass); our results will be rather insensitive to the exact value of $\mu_{\text{susy}}$. Using [15]

$$\frac{g_1^2(\mu_{\text{susy}})}{4\pi} = \frac{1}{58.5}, \quad \frac{g_2^2(\mu_{\text{susy}})}{4\pi} = \frac{1}{30.1}, \quad \text{for } \mu_{\text{susy}} = 170 \text{ GeV}, \quad (2.4)$$
we obtain

\[ \frac{\overline{g}^2}{4\pi} = \frac{1}{25.0}, \quad \overline{\mu} = 1.2 \times 10^{16} \text{ GeV}. \] (2.5)

Throughout this paper, an overline denotes quantities evaluated at the GUT scale.

In the minimal supersymmetric standard model, the charge \( \frac{2}{3} \) quarks couple to a Higgs field with expectation value \( (v/\sqrt{2}) \sin \beta \), where \( v = 246 \text{ GeV} \) and \( \beta \) is arbitrary; the charged leptons and charge \( -\frac{1}{3} \) quarks couple to a Higgs field with expectation value \( (v/\sqrt{2}) \cos \beta \). The fermion masses come from the Yukawa couplings

\[
L_{\text{Yuk}} = \left( \frac{v}{\sqrt{2}} \sin \beta \right) U_L Y_U U_R + \left( \frac{v}{\sqrt{2}} \cos \beta \right) D_L Y_D D_R + \left( \frac{v}{\sqrt{2}} \cos \beta \right) E_L Y_E E_R + \text{h.c.,}
\]

where

\[
U = \begin{pmatrix} u \\ c \\ t \end{pmatrix}, \quad D = \begin{pmatrix} d \\ s \\ b \end{pmatrix}, \quad E = \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix},
\]

(2.6)

are the fermion fields, and \( Y_U, Y_D, \) and \( Y_E \) are arbitrary complex \( 3 \times 3 \) matrices. (In this paper, we take the neutrinos to be massless.) These Yukawa matrices obey the one-loop supersymmetric RG equations [7]

\[
16\pi^2 \frac{d}{dt} Y_U = \left[ -c_{\text{u}}^u g_t^2 + \text{Tr}(3Y_U Y_U^\dagger) + 3Y_U Y_U^\dagger + Y_D Y_D^\dagger \right] Y_U,
\]

\[
16\pi^2 \frac{d}{dt} Y_D = \left[ -c_{\text{d}}^d g_t^2 + \text{Tr}(3Y_D Y_D^\dagger + Y_E Y_E^\dagger) + 3Y_D Y_D^\dagger + Y_U Y_U^\dagger \right] Y_D,
\]

\[
16\pi^2 \frac{d}{dt} Y_E = \left[ -c_{\text{e}}^e g_t^2 + \text{Tr}(3Y_D Y_D^\dagger + Y_E Y_E^\dagger) + 3Y_E Y_E^\dagger \right] Y_E,
\]

(2.8)

where

\[
(c_{\text{u}}^u, c_{\text{d}}^d, c_{\text{e}}^e) = (\frac{13}{15}, 3, \frac{16}{3}), \quad (c_{\text{d}}^d, c_{\text{d}}^d, c_{\text{d}}^d) = (7, 3, \frac{16}{3}), \quad (c_{\text{e}}^e, c_{\text{e}}^e, c_{\text{e}}^e) = (\frac{27}{19}, 3, 0),
\]

(2.9)

between \( \mu_{\text{susy}} \) and \( \overline{\mu} \).
Not all the parameters of the Yukawa matrices are physical. Under an arbitrary unitary transformation on the fermion bases, \( F_L \rightarrow U F_L, \) \( F_R \rightarrow R F_R \) (where \( F = U, D, E \)), the Yukawa matrix undergoes a bi-unitary transformation, \( Y_F \rightarrow L_F Y_F R_F \), and the charged current becomes off-diagonal, with mixing matrix \( L_U \dagger L_D \). We may perform scale-dependent unitary transformations \( L_F(\mu) \) and \( R_F(\mu) \) on the fermion bases so as to diagonalize the Yukawa matrices at each scale. Thus

\[
\hat{Y}_F(\mu) = L_F(\mu)Y_F(\mu)R_F(\mu), \quad F = U, D, E, \tag{2.10}
\]

where \( \hat{Y}_F \) denotes the diagonalized Yukawa matrix, and

\[
V(\mu) = L_U(\mu) L_D(\mu) \tag{2.11}
\]

is the corresponding scale-dependent CKM matrix.

We now derive RG equations for the physically relevant quantities: the Yukawa eigenvalues \( \hat{Y}_F(\mu) \) and the CKM matrix \( V(\mu) \) [9]. The transformations on the right-handed fields are irrelevant to the CKM matrix, so we begin by writing \( \hat{Y}_F \) in terms of \( L_F \) only

\[
\hat{Y}_F(\mu) = L_F(\mu)Y_F(\mu)L_F(\mu), \quad F = U, D, E. \tag{2.12}
\]

Differentiating eq. (2.12), and using eqs. (2.8), we obtain

\[
\frac{d}{dt} \left( \hat{Y}_U^2 \right) = \left[ \hat{Y}_U^2, L_U \dagger \hat{L}_U \right] + \frac{1}{16\pi^2} \left[ 6 \text{ Tr}(\hat{Y}_U^2) \hat{Y}_U^2 + 6\hat{Y}_U^4 + V \hat{Y}_D^2 V^\dagger \hat{Y}_U^2 + \hat{Y}_D^2 V^\dagger \hat{Y}_D V \right],
\]

\[
\frac{d}{dt} \left( \hat{Y}_D^2 \right) = \left[ \hat{Y}_D^2, L_D \dagger \hat{L}_D \right] + \frac{1}{16\pi^2} \left[ 2 \text{ Tr}(3\hat{Y}_D^2 + \hat{Y}_E^2) \hat{Y}_D^2 + 6\hat{Y}_D^4 + V^\dagger \hat{Y}_D^2 V \hat{Y}_D^2 + \hat{Y}_D^2 V^\dagger \hat{Y}_D V \right],
\]

\[
\frac{d}{dt} \left( \hat{Y}_E^2 \right) = \left[ \hat{Y}_E^2, L_E \dagger \hat{L}_E \right] + \frac{1}{16\pi^2} \left[ 2 \text{ Tr}(3\hat{Y}_E^2 + \hat{Y}_E^2) \hat{Y}_E^2 + 6\hat{Y}_E^4 \right], \tag{2.13}
\]

where \( \hat{L}_F = (dL_F/dt) \). The commutator \( \left[ \hat{Y}_F^2, L_F \dagger \hat{L}_F \right] \) has vanishing diagonal elements because \( \hat{Y}_F^2 \) is diagonal. Thus the RG equations for the Yukawa eigenvalues...
\( y_\alpha \) follow immediately from the diagonal entries of eqs. (2.13). The remaining entries of eqs. (2.13) yield equations for the off-diagonal elements of \( L_U^\dagger \dot{L}_U \) and \( L_D^\dagger \dot{L}_D \), as long as there are no degeneracies among the quark masses:

\[
(L_U^\dagger \dot{L}_U)_{\alpha\beta} = \frac{1}{16\pi^2} \sum_{\gamma=d,s,b} \frac{y_\beta^2 + y_\alpha^2}{y_\beta^2 - y_\alpha^2} V_{\alpha\gamma} y_\gamma^2 V_{\gamma\beta}^\dagger, \quad \alpha \neq \beta, \quad \alpha, \beta = u, c, t,
\]

\[
(L_D^\dagger \dot{L}_D)_{\alpha\beta} = \frac{1}{16\pi^2} \sum_{\gamma=u,c,t} \frac{y_\beta^2 + y_\alpha^2}{y_\beta^2 - y_\alpha^2} V_{\alpha\gamma} y_\gamma^2 V_{\gamma\beta}^\dagger, \quad \alpha \neq \beta, \quad \alpha, \beta = d, s, b.
\]

(2.14)

The diagonal elements of \( L_U^\dagger \dot{L}_U \) and \( L_D^\dagger \dot{L}_D \) are not determined by eqs. (2.13). It is easy to see why. Equation (2.12) determines \( L_U \) and \( L_D \) only up to right multiplication by a diagonal matrix of (scale-dependent) phases. These undetermined phases contribute arbitrary imaginary functions to the diagonal elements of \( L_U^\dagger \dot{L}_U \) and \( L_D^\dagger \dot{L}_D \). (The off-diagonal elements are unambiguously determined because they receive no contribution from the phases.) We can, however, choose the phases to make the diagonal elements of \( L_U^\dagger \dot{L}_U \) and \( L_D^\dagger \dot{L}_D \), which are manifestly imaginary, vanish:

\[
(L_U^\dagger \dot{L}_U)_{\alpha\alpha} = (L_D^\dagger \dot{L}_D)_{\alpha\alpha} = 0 \quad \text{by an appropriate choice of phases.} \quad (2.15)
\]

The RG equations for the CKM matrix elements (2.11) are then [9]

\[
16\pi^2 \frac{d}{dt} V_{\alpha\beta} = 16\pi^2 \left( VL_D^\dagger \dot{L}_D - L_U^\dagger \dot{L}_U V \right)_{\alpha\beta}
= \sum_{\gamma=u,c,t} \sum_{\delta=d,s,b} \frac{y_\beta^2 + y_\alpha^2}{y_\beta^2 - y_\alpha^2} V_{\alpha\delta} V_{\delta\beta}^\dagger y_\gamma^2 V_{\gamma\beta} + \sum_{\gamma=u,c,t} \sum_{\delta=d,s,b} \frac{y_\alpha^2 + y_\beta^2}{y_\alpha^2 - y_\beta^2} V_{\alpha\delta} y_\delta^2 V_{\delta\gamma}^\dagger V_{\gamma\beta},
\]

(2.16)

as long as we choose the phases to guarantee eqs. (2.15).

We now neglect the contributions to the running caused by the first and second generation Yukawa couplings. If we further assume \( y_u^2 \ll y_c^2 \ll y_t^2 \) and \( y_d^2 \ll y_s^2 \ll y_b^2 \)
$y_b^2$, then the CKM matrix RG equations (2.16) reduce to

$$16\pi^2 \frac{d}{dt} V_{\alpha\beta} = y_t^2 \sum_{\delta=d,s,b} \varepsilon_{\delta\beta} V_{\alpha\delta} V_{\delta t}^\dagger V_{t\beta} + y_b^2 \sum_{\gamma=u,c,t} \varepsilon_{\gamma\alpha} V_{\alpha\gamma} V_{\gamma t}^\dagger V_{t\beta},$$

$$\varepsilon_{\alpha\beta} = \begin{cases} 1 & \text{if } y_\alpha < y_\beta, \\ 0 & \text{if } y_\alpha = y_\beta, \\ -1 & \text{if } y_\alpha > y_\beta. \end{cases}$$

The CKM matrix elements $V_{\alpha\beta}$ are not all independent because of the constraint of unitarity.

We prefer to go a step further than previous treatments by deriving RG equations for a set of independent quantities parametrizing the unitary CKM matrix: the mixing angles and CP-violating phase (but see ref. [19]). To do so, however, we must squarely face the issue of phases multiplying the matrix. We adopt the following (nonstandard) parametrization of the CKM matrix

$$V(\mu) = \begin{pmatrix} e^{i\phi_u} & 0 & 0 \\ 0 & e^{i\phi_c} & 0 \\ 0 & 0 & e^{i\phi_t} \end{pmatrix} \begin{pmatrix} s_{1s2}c_3 + c_{1c2}e^{i\phi} & c_{1s2}c_3 - s_{1c2}e^{i\phi} & s_{2s3} \\ c_{1c2}c_3 - s_{1s2}e^{i\phi} & c_{1c2}c_3 + s_{1s2}e^{i\phi} & c_{2s3} \\ s_{1s3} & -c_{1s3} & c_3 \end{pmatrix} \begin{pmatrix} e^{i\phi_d} & 0 & 0 \\ 0 & e^{i\phi_s} & 0 \\ 0 & 0 & e^{i\phi_b} \end{pmatrix},$$

(2.18)

where $c_i = \cos \theta_i$ and $s_i = \sin \theta_i$, and all the parameters are functions of the scale $\mu$. The middle matrix is chosen to have real elements in the third row and column. We cannot automatically eliminate the left and right phase matrices by rephasing the quark fields; we have already used that freedom to ensure eqs. (2.15), and those equations implicitly determine the functions $\dot{\phi}_\alpha(\mu)$. We can, however, impose the boundary condition $\phi_\alpha(\mu) = 0$; i.e., the initial values for the five matrix elements $V_{\alpha\beta}$ in the third row or column may be chosen to be real. One can show, using $V^\dagger V = 1$, that the RG equations (2.17) for this set of five matrix elements close on themselves. Since their initial values are real and the coefficients in the equations are real, these five matrix elements remain real (i.e., $\phi_\alpha(\mu) = 0$) for all $\mu$. In other words, the choice of quark phases that guarantees eqs. (2.15) also implies that the phases $\phi_\alpha(\mu)$ in the CKM parametrization (2.18) vanish (in the approximation that eq. (2.17) is valid). We then obtain the RG equations for the angles $\theta_i$ and
the CP-violating phase \( \phi \) by substituting eq. (2.18) into eq. (2.17)

\[
\begin{align*}
16\pi^2 \frac{d}{dt} \ln \tan \theta_1 &= -y_t^2 \sin^2 \theta_3, \\
16\pi^2 \frac{d}{dt} \ln \tan \theta_3 &= -y_t^2 - y_b^2, \\
16\pi^2 \frac{d}{dt} \ln \tan \theta_2 &= -y_b^2 \sin^2 \theta_3, \\
16\pi^2 \frac{d}{dt} \phi &= 0.
\end{align*}
\] (2.19)

The RG equations for the Yukawa eigenvalues [7, 9],

\[
\begin{align*}
16\pi^2 \frac{d}{dt} \ln y_u &= -c_i^u g_i^2 + 3y_t^2 + y_b^2 \cos^2 \theta_2 \sin^2 \theta_3, \\
16\pi^2 \frac{d}{dt} \ln y_c &= -c_i^u g_i^2 + 3y_t^2 + y_b^2 \sin^2 \theta_2 \sin^2 \theta_3, \\
16\pi^2 \frac{d}{dt} \ln y_t &= -c_i^u g_i^2 + 6y_t^2 + y_b^2 \cos^2 \theta_3, \\
16\pi^2 \frac{d}{dt} \ln y_d &= -c_i^d g_i^2 + y_t^2 \sin^2 \theta_1 \sin^2 \theta_3 + 3y_b^2 + y_\tau^2, \\
16\pi^2 \frac{d}{dt} \ln y_s &= -c_i^d g_i^2 + y_t^2 \cos^2 \theta_1 \sin^2 \theta_3 + 3y_b^2 + y_\tau^2, \\
16\pi^2 \frac{d}{dt} \ln y_b &= -c_i^d g_i^2 + y_t^2 \cos^2 \theta_3 + 6y_b^2 + y_\tau^2, \\
16\pi^2 \frac{d}{dt} \ln y_c &= -c_i^e g_i^2 + 3y_b^2 + y_\tau^2, \\
16\pi^2 \frac{d}{dt} \ln y_\mu &= -c_i^e g_i^2 + 3y_b^2 + y_\tau^2, \\
16\pi^2 \frac{d}{dt} \ln y_\tau &= -c_i^e g_i^2 + 3y_b^2 + 4y_\tau^2,
\end{align*}
\] (2.20)

are obtained by substituting eq. (2.18) into eqs. (2.13), again neglecting first and second generation Yukawa coupling contributions to the running.

Although we assumed \( y_u^2 \ll y_c^2 \ll y_t^2 \) and \( y_d^2 \ll y_s^2 \ll y_b^2 \) in deriving eqs. (2.19) and (2.20), we did not assume that the mixing angles \( \theta_i \) were small. Because the third generation mixes with the first two, third generation quarks induce some running of the mixing angles \( \theta_1 \) and \( \theta_2 \) and the ratios of first and second generation quarks, \( y_u/y_c \) and \( y_d/y_s \). The amount of running of these quantities, however, is typically quite small because of the smallness of \( \theta_3 \sim 0.05 \); they change by less than 0.1% from the GUT scale to the electroweak scale if \( y_b, y_t \lesssim 1.5 \). We will therefore be
justified in the following in neglecting terms proportional to \(\sin^2 \theta_3\) on the r.h.s. of the RG equations (2.19) and (2.20).

3. Approximate Solutions to the RG Equations

In this section, we find explicit solutions to the renormalization group equations for the fermion masses and CKM matrix elements. To do so, we need to make several approximations. In deriving the RG equations (2.19) and (2.20) in the last section, we neglected the running caused by the first and second generations of Yukawa couplings. We now make the further assumption that the mixing angles are small. In this approximation, the RG equations for the CKM matrix elements simplify to

\[
16\pi^2 \frac{d}{dt} \ln V_{\alpha\beta} = \begin{cases} 
-\frac{y_t^2}{g_t^2} - \frac{y_b^2}{g_b^2} & \text{for } \alpha\beta = ub, cb, td, \text{ and } ts, \\
0 & \text{for } \alpha\beta = ud, us, cd, cd, \text{ and } tb,
\end{cases}
\]  

(3.1)

and the Yukawa eigenvalues satisfy

\[
16\pi^2 \frac{d}{dt} \ln y_{u,c} = -c_i^u g_t^2 + 3y_t^2,
\]

\[
16\pi^2 \frac{d}{dt} \ln y_t = -c_i^t g_t^2 + 6y_t^2 + y_b^2,
\]

\[
16\pi^2 \frac{d}{dt} \ln y_{d,s} = -c_i^d g_t^2 + 3y_b^2 + y_{\tau}^2,
\]

\[
16\pi^2 \frac{d}{dt} \ln y_b = -c_i^b g_t^2 + y_t^2 + 6y_b^2 + y_{\tau}^2,
\]

\[
16\pi^2 \frac{d}{dt} \ln y_{e,\mu} = -c_i^e g_t^2 + 3y_b^2 + y_{\tau}^2,
\]

\[
16\pi^2 \frac{d}{dt} \ln y_{\tau} = -c_i^\tau g_t^2 + 3y_b^2 + 4y_{\tau}^2.
\]

(3.2)

If we define the scaling factors

\[
A_\alpha(\mu) = \exp \left[ \frac{1}{16\pi^2} \int_{\ln \mu}^{\ln \mu'} c_i^\alpha g_t^2(\mu')d[\ln \mu'] \right]
\]

(3.3)
\[ B_\alpha(\mu) = \exp \left[ -\frac{1}{16\pi^2} \int_{\ln \mu}^{\ln \mu'} y_\alpha^2(\mu') d(\ln \mu') \right], \]  

(3.4)

then the solutions to eqs. (3.1) and (3.2) are given by

\[ V_{\alpha\beta}(\mu) = \begin{cases} 
V_{\alpha\beta} B_t^{-1} B_b^{-1} & \text{for } \alpha\beta = ub, cb, td, \text{ and } ts, \\
V_{\alpha\beta} & \text{for } \alpha\beta = ud, us, cd, cd, \text{ and } tb,
\end{cases} \]  

(3.5)

and

\[
\begin{align*}
y_u(\mu) &= \overline{y}_u A_u B_t^3, \\
y_d(\mu) &= \overline{y}_d A_d B_b^3 B_\tau, \\
y_e(\mu) &= \overline{y}_e A_e B_b^3 B_\tau, \\
y_c(\mu) &= \overline{y}_c A_u B_t^3, \\
y_s(\mu) &= \overline{y}_s A_d B_b^3 B_\tau, \\
y_\mu(\mu) &= \overline{y}_\mu A_e B_b^3 B_\tau, \\
y_t(\mu) &= \overline{y}_t A_u B_t^6 B_b, \\
y_b(\mu) &= \overline{y}_b A_d B_t B_b^6 B_\tau, \\
y_\tau(\mu) &= \overline{y}_\tau A_e B_b^3 B_\tau^4,
\end{align*}
\]  

(3.6)

where the overline denotes quantities evaluated at the GUT scale. The \( A_\alpha \) factors encapsulate the running induced by the gauge couplings; the \( B_\alpha \) factors that induced by the Yukawa couplings.

In the approximation of small mixing angles, the four off-diagonal CKM matrix elements involving the third generation all run with the same scale factor \( B_t^{-1} B_b^{-1} \), while the remaining five matrix elements do not run at all, as has been observed previously [17, 20, 21]. Also in this approximation, the ratios of first and second generation Yukawa couplings, \( y_u/y_c, y_d/y_s, \text{ and } y_e/y_\mu \), are invariant under running induced by third generation Yukawa couplings, as well as under running induced by gauge couplings [17, 20, 21]. These results hold to all orders in perturbation theory [21].

In order to use eqs. (3.5) and (3.6) to scale fermion mass and mixing angle relations from the GUT scale to the supersymmetry breaking scale, we must know the values of \( A_\alpha(\mu_{\text{susy}}) \) and \( B_\alpha(\mu_{\text{susy}}) \). The scaling factors due to the gauge couplings
\( A_\alpha(\mu) \) obey
\[
16\pi^2 \frac{d}{dt} \ln A_\alpha = -c_i^\alpha g_i^2 ,
\]
and are easily calculated using eqs. (2.1) and (2.2):
\[
A_\alpha(\mu) = \prod_{i=1}^{3} \left[ \frac{g_i(\mu)}{\bar{g}} \right]^{-c_i^\alpha / b_i} = \prod_{i=1}^{3} \left[ 1 - \frac{b_i \pi^2}{8\pi^2} \ln \left( \frac{\mu}{\mu} \right) \right] c_i^\alpha / 2b_i .
\]
These factors are given at the supersymmetry breaking scale by
\[
A_u(\mu_{\text{susy}}) = 3.21, \quad A_d(\mu_{\text{susy}}) = 3.13, \quad A_e(\mu_{\text{susy}}) = 1.48, \quad \text{for } \mu_{\text{susy}} = 170 \text{ GeV},
\]
using eqs. (2.3) and (2.5). The scaling factors due to the Yukawa couplings \( B_\alpha(\mu) \) obey
\[
16\pi^2 \frac{d}{dt} \ln B_t = y_t^2 = \bar{y}_t^2 A_u^2 B_t^{12} \left[ B_u^2 \right],
\]
\[
16\pi^2 \frac{d}{dt} \ln B_b = y_b^2 = \bar{y}_b^2 A_d^2 B_b^{12} \left[ B_t^2 B_b^2 \right],
\]
\[
16\pi^2 \frac{d}{dt} \ln B_\tau = y_\tau^2 = \bar{y}_\tau^2 A_e^2 B_\tau^{12} \left[ (B_b / B_\tau)^4 B_b^2 \right].
\]
The \( B_\alpha \) are equal to 1 at the GUT scale, and decrease monotonically as one lowers the scale. The equations (3.10) do not have an analytic solution. We can obtain an approximate solution by setting the factors in brackets equal to 1. The equations then decouple from one another, and have the solutions
\[
B_t(\mu) \approx \left[ 1 + \bar{y}_t^2 K_u(\mu) \right]^{-1/12},
\]
\[
B_b(\mu) \approx \left[ 1 + \bar{y}_b^2 K_d(\mu) \right]^{-1/12}, \quad K_\alpha(\mu) = \frac{3}{4\pi^2} \int_{\ln \mu}^{\ln \mu'} A_\alpha^2(\mu') d(\ln \mu').
\]
\[
B_\tau(\mu) \approx \left[ 1 + \bar{y}_\tau^2 K_e(\mu) \right]^{-1/12},
\]
We numerically integrate $K_\alpha(\mu)$ to find

$$K_u(\mu_{\text{susy}}) = 8.65, \quad K_d(\mu_{\text{susy}}) = 8.33, \quad K_e(\mu_{\text{susy}}) = 3.77, \quad \text{for } \mu_{\text{susy}} = 170 \text{ GeV}. \quad (3.12)$$

The terms in the brackets in eq. (3.10) are all less than 1, so omitting them tends to increase the running of the factors $B_\alpha$. Consequently, the expressions in eq. (3.11) are smaller than the exact values of $B_\alpha$ at $\mu = \mu_{\text{susy}}$ by about 1 or 2%. As we will see in sect. 4, this approximation tends to exaggerate the effect of the running induced by the Yukawa couplings.

In the limit $y_b, y_\tau \ll y_t$, the approximate solutions (3.11) reduce to the exact result for the running induced by the top quark alone [12, 13]

$$B_t(\mu) = \left[1 + \frac{y_t^2 K_u(\mu)}{12}\right]^{-1/12}, \quad B_b(\mu) = B_\tau(\mu) = 1. \quad (3.13)$$

4. Running Relations

We examine in this section several different sets of fermion mass and mixing angle relations, and the effect of renormalization group running on these relations. We assume that physics at the GUT scale dictates certain forms, or textures, for the matrices of Yukawa couplings. Different physics leads to different textures. In this section, we focus on three different textures: the Georgi-Jarlskog texture, the Giudice texture, and the Fritzsch texture. We will not be concerned so much with the physics behind these textures, but simply take them as given, and examine the relations among fermion masses and mixing angles to which they give rise.

The relations derived from these various textures hold at the GUT scale, and need to be scaled down to low energy to yield predictions for measured parameters. We use the spectrum of the minimal supersymmetric standard model to run the relations from the GUT scale down to the scale at which supersymmetry is broken,
\( \mu_{\text{susy}} = 170 \text{ GeV} \). Below \( \mu_{\text{susy}} \), the CKM matrix elements do not evolve much, but the Yukawa eigenvalues continue to run due to QED and QCD effects. This additional running is incorporated in the factors \( \eta_\alpha \), defined by

\[
\eta_\alpha = \frac{y_\alpha(m_\alpha)}{y_\alpha(\mu_{\text{susy}})}.
\]  

(4.1)

In this paper, \( m_\alpha \) denotes not the physical mass but rather the running mass of the fermion, defined by

\[
m_\alpha = y_\alpha(m_\alpha) \frac{v}{\sqrt{2}} \begin{cases} 
\sin \beta & \text{for } \alpha = u, c, \text{ and } t, \\
\cos \beta & \text{for } \alpha = d, s, e, \mu, \text{ and } \tau.
\end{cases}
\]

(4.2)

The physical (pole) mass of the top quark is then related to its running mass by

\[
m^{\text{phys}}_t = \left[ 1 + \frac{4}{3\pi} \alpha_3(m_t) + O(\alpha_3^2) \right] m_t.
\]

(4.3)

In eqs. (4.1) and (4.2), \( y_\alpha(m_\alpha) \) should be replaced by \( y_\alpha(1 \text{ GeV}) \) for the three lightest quarks, \( \alpha = u, d, \text{ and } s \).

When specific numerical values are required in the following, we will use [22]

\[
m_t = 1.7841^{+0.0027}_{-0.0036} \text{ GeV}, \quad m_c = 1.27 \pm 0.05 \text{ GeV}, \quad m_b = 4.25 \pm 0.10 \text{ GeV},
\]

(4.4)

for the masses, and [13]

\[
\eta_u = 2.17, \quad \eta_s = 2.16, \quad \eta_c = 1.89, \quad \eta_b = 1.47, \quad \eta_\tau = 1.02,
\]

(4.5)

for the QCD/QED scaling factors, corresponding to \( \alpha_3(M_Z) = 0.111 \). We will generally assume that \( m_t \) is close enough to \( \mu_{\text{susy}} = 170 \text{ GeV} \) that running between

\* The \( t, b, \text{ and } \tau \) Yukawa couplings continue to induce running down to the scale of their masses, but the amount of running is much less than that from \( \mu \) to \( \mu_{\text{susy}} \).
the two scales is small,

$$\eta_t \approx 1. \quad (4.6)$$

There is considerable uncertainty in the values of the scaling factors (4.5) due to the uncertainty in $\alpha_3(M_Z)$ [15, 16]. Since our results are analytic, it is easy to determine the effects of choosing other values for $m_\alpha$ and $\eta_\alpha$.

### 4.1. The Georgi-Jarlskog Texture

The first texture for the Yukawa matrices we consider is

$$Y_U = \begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix}, \quad Y_D = \begin{pmatrix} 0 & F e^{i\phi} & 0 \\ F e^{-i\phi} & E & 0 \\ 0 & 0 & D \end{pmatrix}, \quad Y_E = \begin{pmatrix} 0 & F & 0 \\ 0 & -3E & 0 \\ 0 & 0 & D \end{pmatrix}, \quad (4.7)$$

assumed to hold at the grand unification scale. Georgi and Jarlskog [2] originally posited this form for the Yukawa matrices in an SU(5) grand unified theory, and Harvey, Ramond, and Reiss [3] used it in an SO(10) model. Recently, a number of authors [10, 12, 13, 15, 16] have re-examined this texture in a supersymmetric context. The relations between the $Y_D$ and $Y_E$ matrix elements follow if the charged leptons and charge $-\frac{4}{3}$ quarks belong to the same grand unified representation. Entries of the two matrices that are equal in magnitude result from Yukawa couplings to a Higgs field in the 5 of SU(5) or the 10 of SO(10). Entries differing by a factor of $-3$ result from Yukawa couplings to a Higgs field in the 45 of SU(5) or the 126 of SO(10). The zero entries of $Y_U$, $Y_D$, and $Y_E$ are due to discrete symmetries [4, 5] at the grand unified scale.

The Georgi-Jarlskog texture (4.7) leads to six relations among fermion masses and mixing angles. The eigenvalues of the Yukawa matrices (4.7) obey the SU(5) relation [1]

$$\frac{\bar{y}_b}{\bar{y}_\tau} = 1 \quad (4.8)$$
and the Georgi-Jarlskog relations [2]

\[ \frac{\overline{y}_\mu - \overline{y}_e}{\overline{y}_s - \overline{y}_d} = 3, \quad \frac{\overline{y}_e \overline{y}_\mu}{\overline{y}_d \overline{y}_s} = 1. \quad (4.9) \]

The latter two equations can be combined into

\[ \frac{(\overline{y}_d / \overline{y}_s)}{[1 - (\overline{y}_d / \overline{y}_s)]^2} = \frac{9 (\overline{y}_e / \overline{y}_\mu)}{[1 - (\overline{y}_e / \overline{y}_\mu)]^2}. \quad (4.10) \]

The quark Yukawa matrices \( Y_U \) and \( Y_D \) are diagonalized by bi-unitary transformations \( \hat{Y}_F = L_F \dagger Y_F R_F \), with

\[
L_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \overline{c}_3 & -\overline{s}_3 \\ 0 & \overline{s}_3 & \overline{c}_3 \end{pmatrix}, \quad L_D = \begin{pmatrix} \overline{e}^{i\phi} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

and with \( R_U \) and \( R_D \) equal to \( L_U \) and \( L_D \) respectively, modulo diagonal matrices of signs to make the eigenvalues \( \overline{y}_\alpha \) positive. In the unitary matrices (4.11), \( \overline{c}_i = \cos \overline{\theta}_i \) and \( \overline{s}_i = \sin \overline{\theta}_i \), with

\[
\tan^2 \overline{\theta}_1 = \frac{\overline{y}_d}{\overline{y}_s}, \quad \tan^2 \overline{\theta}_2 = \frac{\overline{y}_u}{\overline{y}_c}, \quad \tan^2 \overline{\theta}_3 = \frac{\overline{y}_c - \overline{y}_u}{\overline{y}_t} \approx \frac{\overline{y}_c}{\overline{y}_t}. \quad (4.12)
\]

The unitary transformations (4.11) result in a CKM matrix of exactly the form (2.18)

\[
\overline{V} = L_U \dagger L_D = \begin{pmatrix} \overline{s}_1 \overline{s}_2 \overline{c}_3 + \overline{c}_1 \overline{c}_2 \overline{e}^{i\phi} & \overline{c}_1 \overline{s}_2 \overline{c}_3 - \overline{s}_1 \overline{c}_2 \overline{e}^{i\phi} & \overline{s}_2 \overline{s}_3 \\ \overline{s}_1 \overline{c}_2 \overline{c}_3 - \overline{c}_1 \overline{s}_2 \overline{e}^{i\phi} & \overline{c}_1 \overline{c}_2 \overline{c}_3 + \overline{s}_1 \overline{s}_2 \overline{e}^{i\phi} & \overline{c}_2 \overline{s}_3 \\ -\overline{s}_1 \overline{c}_3 & -\overline{c}_1 \overline{s}_3 & \overline{c}_3 \end{pmatrix}. \quad (4.13)
\]

Therefore, in the approximation that the mixing angles are small, the CKM matrix
elements satisfy the Harvey-Ramond-Reiss (HRR) relation \[3\]

\[ |V_{cb}| \approx \sqrt{\frac{y_c}{y_t}}, \quad (4.14) \]

which depends on the top quark Yukawa coupling, as well as the relations

\[ |V_{us}| \approx \left| \frac{y_d}{y_s} e^{-i\phi} \frac{y_u}{y_c} \right|, \quad \frac{|V_{ub}|}{|V_{cb}|} \approx \sqrt{\frac{y_u}{y_c}}, \quad (4.15) \]

which depend on only the first and second generation Yukawa couplings. The six relations (4.8)–(4.10), (4.14), and (4.15) hold at the GUT scale.

The Yukawa matrices do not retain the form (4.7) below the GUT scale because of RG running induced by large Yukawa couplings. Rather than follow the evolution of the Yukawa matrices, however, we will determine the effect of the running on the relations (4.8)–(4.10), (4.14), and (4.15). We use eqs. (3.5) and (3.6) to scale the relations from the GUT scale \( \mu \) to the supersymmetry breaking scale \( \mu_{\text{susy}} \). (Whenever the scaling factors \( A_\alpha \) and \( B_\alpha \) are written without an explicit scale \( \mu \) throughout this section, \( \mu = \mu_{\text{susy}} \) is understood.) The further running of the Yukawa couplings from \( \mu_{\text{susy}} \) to the scale of the fermion masses is included in the factors \( \eta_\alpha \) defined in eq. (4.1). Thus, the SU(5) relation (4.8) leads to

\[ \frac{m_b}{m_\tau} = \frac{A_d \eta_b B_t B_b^3}{A_e \eta_\tau B_t^3}, \quad (4.16) \]

and the Georgi-Jarlskog relations (4.9) and (4.10) imply

\[ \frac{m_e m_\mu}{m_d m_s} = \frac{A_e^2 \eta_c \eta_\mu}{A_d^2 \eta_d \eta_s}, \quad \frac{(m_d/m_s)}{[1 - (m_d/m_s)]^2} = \frac{9 (\eta_\mu m_e/\eta_\mu m_\mu)}{[1 - (\eta_\mu m_e/\eta_\mu m_\mu)]^2}, \quad (4.17) \]

since \( \eta_d = \eta_s \). The HRR relation (4.14) implies

\[ |V_{cb}| \approx \sqrt{\frac{\eta_m c}{\eta_c m_t}} \sqrt{\frac{B_t}{B_b}}, \quad (4.18) \]
and the CKM relations (4.15) lead to

$$|V_{us}| \approx \sqrt{|m_d/m_s - e^{-i\phi} \eta_c m_u|}, \quad |V_{ub}|/|V_{cb}| \approx \sqrt{\eta_c m_u}. \quad (4.19)$$

The running induced by the third generation of fermions is contained in the factors $B_t$, $B_b$, and $B_\tau$.

Dimopoulos, Hall, and Raby [12] found that four of the six relations implied by the Georgi-Jarlskog texture, viz., the two Georgi-Jarlskog relations (4.9) and the two CKM matrix element relations (4.15), are unaffected by the running induced by the top quark, in the limit that the mixing angles are small. We see from eqs. (4.17) and (4.19) that, not surprisingly, these four relations are also insensitive to the running induced by the bottom quark and $\tau$ lepton. Therefore, the predictions of ref. [12] for $m_s, m_d, \phi$, and $|V_{ub}/V_{cb}|$ remain unchanged even when all three third generation Yukawa couplings contribute significantly to the running.

The SU(5) and HRR relations, however, are modified by the running induced by the third generation of fermions. The scaling factors $B_t$, $B_b$, and $B_\tau$ are determined by the RG equations (3.10), which have the approximate solutions

$$B_t \approx [1 + y_t^2 K_u]^{-1/12}, \quad B_b \approx [1 + y_b^2 K_d]^{-1/12}, \quad B_\tau \approx [1 + y_\tau^2 K_e]^{-1/12}, \quad (4.20)$$

where $K_\alpha = K_\alpha(\mu_{\text{susy}})$ are given in eq. (3.12). The scaling factors $B_\alpha$ depend on the GUT scale Yukawa couplings $\overline{y}_t$, $\overline{y}_b$, and $\overline{y}_\tau$. These couplings are related to the fermion masses by

$$m_\tau = \frac{v_A \eta_\tau}{\sqrt{2}} \overline{y}_\tau B_b^3 B_\tau^4 (\cos \beta), \quad (4.21)$$

$$m_b = \frac{v_A \eta_b}{\sqrt{2}} \overline{y}_b B_t B_b^6 B_\tau (\cos \beta), \quad (4.22)$$

$$m_t = \frac{v_A \eta_t}{\sqrt{2}} \overline{y}_t B_t^6 B_b (\sin \beta) \approx \frac{v_A \eta_t}{\sqrt{2} K_u B_b} \sqrt{1 - B_t^{12} (\sin \beta)}, \quad (4.23)$$

using eqs. (3.6), (4.1), and (4.2), where the last equality in eq. (4.23) depends on the approximation (4.20).
The three GUT scale Yukawa couplings $\bar{y}_t$, $\bar{y}_b$, and $\bar{y}_\tau$ are not independent, however. First, the Georgi-Jarlskog texture dictates that $\bar{y}_\tau = \bar{y}_b$. Second, from the SU(5) relation (4.16), we have

$$B_t = k \left( \frac{B_\tau}{B_b} \right)^3, \quad k \equiv \frac{A_e \eta_{1\tau} m_b}{A_d \eta_{1b} m_\tau} \approx 0.78. \quad (4.24)$$

(The deviation of $k$ from unity shows that significant running must be induced by the Yukawa couplings for the SU(5) relation to be valid.) Equation (4.24), together with eq. (4.20), may be used to determine $\bar{y}_t$ in terms of $\bar{y}_b$. Hence, $B_t$, $B_b$, and $B_\tau$, as well as the fermion masses, may be written in terms of a single GUT scale parameter $\bar{y}_b$. This implies a relation between $\beta$ and $m_t$. The parameter $\beta$ may be expressed in terms of $\bar{y}_b$ as

$$\sec \beta = \frac{v A_e \eta_{1\tau}}{\sqrt{2} m_\tau} (\bar{y}_b B_\tau^3 B_\tau^4), \quad (4.25)$$

using eq. (4.21) and $\bar{y}_\tau = \bar{y}_b$. The top quark mass is given by

$$m_t \approx \frac{v A_u \eta_1}{\sqrt{2} K_u} B_b \sqrt{1 - k^{12} (B_\tau/B_b)^{36} \sin^2 \beta}, \quad (4.26)$$

using eqs. (4.23) and (4.24). Plotting (4.25) and (4.26) parametrically as functions of $\bar{y}_b$, we obtain the relation between $m_t$ and $\tan \beta$ shown by the solid line in fig. 1. To see more explicitly the dependence of $m_t$ on $\beta$, we neglect terms of $O(\bar{y}_b^3)$ on the r.h.s. of eq. (4.25) to obtain

$$\bar{y}_b \approx \frac{\sec \beta}{150}, \quad (4.27)$$

then expand eq. (4.26) in terms of $\bar{y}_b$ to obtain the approximate relation

$$m_t \approx (185 \text{ GeV}) (\sin \beta)[1 - (5 \times 10^{-5}) \sec^2 \beta + \ldots]. \quad (4.28)$$

The Georgi-Jarlskog texture implies an upper bound on the top quark mass, $m_t \lesssim 185$ GeV, which is saturated for $\tan \beta \sim 10$. 

20
If the running induced by the $b$ quark and $\tau$ lepton is neglected, $m_t$ increases monotonically with $\beta$, as shown by the dotted line in fig. 1, and eq. (4.28) reduces to the linear relation between $m_t$ and $\sin \beta$ found in refs. [12, 13]. The inclusion of the effects of all three third generation Yukawa couplings is responsible for the last factor in eq. (4.28), and causes $m_t$ to decrease for large values of $\tan \beta$. Thus, a given value of $m_t$ may correspond to two possible values of $\beta$ [15, 16].

The relation between $m_t$ and $\tan \beta$ obtained by numerically integrating the RG equations (3.10) without approximation is shown by the dashed line in fig. 1, and is similar to plots shown in refs. [10, 11, 15, 16]. The solid line, based on the approximation (4.20), has the same qualitative behavior as the dashed line, but the effects of the Yukawa-induced running for large $\tan \beta$ are exaggerated, as anticipated in sect. 3.

The HRR relation (4.18) determines the dependence of $|V_{cb}|$ on $\beta$. Using eq. (4.18) together with eq. (4.24), we find

$$|V_{cb}| \approx \frac{\sqrt{A_c \eta_\tau \eta_\tau m_b m_c}}{A_d \eta_b \eta_b m_\tau m_t} \sqrt{\frac{B_3^3}{B_b^4}}.$$  \hspace{1cm} (4.29)

(When running induced by the $b$ quark and $\tau$ lepton are neglected, this reduces to eq. (22) of ref. [12].) Using eqs. (4.25) and (4.29), we plot $|V_{cb}|$ as a function of $\tan \beta$ in fig. 2 (solid line). As before, the dashed line indicates the result based on numerical integration of the RG equations (similar to plots in ref. [16]). Expanding eq. (4.29) in terms of $\eta_b$, we obtain the approximate relation

$$|V_{cb}| \approx \frac{0.053}{\sqrt{\sin \beta}}[1 + (7 \times 10^{-5}) \sec^2 \beta + \ldots].$$  \hspace{1cm} (4.30)

The Georgi-Jarlskog texture implies that $|V_{cb}|$ must be $\gtrsim 0.053$.

In principle, given $|V_{cb}|$, we could use eq. (4.29) to determine $\beta$ (up to a two-fold ambiguity) and therefore $m_t$. The uncertainty in $|V_{cb}|$, however, allows us only
to set bounds on $\beta$. For example, requiring $|V_{cb}| < 0.058$ leads to a lower bound on $\tan \beta$ [12, 13]

$$|V_{cb}| < 0.058 \quad \Rightarrow \quad \tan \beta \gtrsim 1.5 \quad \Rightarrow \quad m_t \gtrsim 155 \text{ GeV},$$

(4.31)

as well as an upper bound [15, 16]

$$|V_{cb}| < 0.058 \quad \Rightarrow \quad \tan \beta \lesssim 40 \quad \Rightarrow \quad m_t \gtrsim 170 \text{ GeV}.$$  

(4.32)

It is the running induced by the $b$ quark and $\tau$ lepton that causes $|V_{cb}|$ to increase for large $\tan \beta$ (4.30), and therefore allows us to derive this upper bound on $\tan \beta$; neglecting this running leads to the relation shown by the dotted line in fig. 2, which obeys $|V_{cb}| < 0.058$ for all $\tan \beta \gtrsim 1.5$.

Finally, we plot $|V_{cb}|$ as a function of $m_t$ in fig. 3. Here our analytic approximation (solid line) almost exactly coincides with the numerical solution (dashed line). Each value of $|V_{cb}| \gtrsim 0.053$ corresponds to two different values of $m_t$ [16]. The lower branch of the curve is approximately described by the relation

$$\frac{|V_{cb}|}{0.053} \approx \sqrt{\frac{185 \text{ GeV}}{m_t}},$$

(4.33)

and holds for $\tan \beta \lesssim 8$. The upper branch of the curve holds for $\tan \beta \gtrsim 8$; here the relation between $|V_{cb}|$ and $m_t$ is approximately linear

$$|V_{cb}| \approx 0.053 + 0.075 \left(1 - \frac{m_t}{185 \text{ GeV}}\right),$$

(4.34)

which we obtain by combining eqs. (4.28) and (4.30).
4.2. THE GIUDICE TEXTURE

Next we consider the Giudice texture [14] for the Yukawa matrices,

\[
Y_U = \begin{pmatrix} 0 & 0 & B \\ 0 & B & 0 \\ B & 0 & A \end{pmatrix}, \quad Y_D = \begin{pmatrix} 0 & F e^{i\phi} & 0 \\ F e^{-i\phi} & E & D \\ 0 & D & C \end{pmatrix}, \quad Y_E = \begin{pmatrix} 0 & F & 0 \\ F & -3E & D \\ 0 & D & C \end{pmatrix},
\]

at the GUT scale. (Giudice additionally imposes the ad hoc relation \( D = 2E \), but we leave these matrix elements unrelated.) This texture leads to six relations among fermion masses and mixing angles. The eigenvalues of the Yukawa matrices obey not only the Georgi-Jarlskog and SU(5) relations

\[
\frac{\bar{y}_e}{\bar{y}_d} \approx \frac{1}{3}, \quad \frac{\bar{y}_\mu}{\bar{y}_s} \approx 3, \quad \frac{\bar{y}_b}{\bar{y}_r} = 1,
\]

but also the geometric mean relation

\[
\bar{y}_u \bar{y}_t = \bar{y}_c^2.
\]

The quark Yukawa matrices are diagonalized by

\[
L_U = \begin{pmatrix} \bar{c}_2 & 0 & -\bar{s}_2 \\ 0 & 1 & 0 \\ \bar{s}_2 & 0 & \bar{c}_2 \end{pmatrix}, \quad L_D = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & \bar{c}_3 & \bar{s}_3 \\ 0 & -\bar{s}_3 & \bar{c}_3 \end{pmatrix}, \quad L_E = \begin{pmatrix} \bar{c}_1 & -\bar{s}_1 & 0 \\ \bar{s}_1 & \bar{c}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

with

\[
\tan^2 \bar{\theta}_1 = \frac{\bar{y}_d}{\bar{y}_s}, \quad \tan^2 \bar{\theta}_2 = \frac{\bar{y}_u}{\bar{y}_t}, \quad \bar{\theta}_3 \approx \frac{D}{\bar{y}_b}.
\]

The unitary transformations (4.38) lead to a CKM matrix of the form

\[
\bar{V} = L_U^\dagger L_D = \begin{pmatrix} -\bar{s}_1 \bar{s}_2 \bar{s}_3 + \bar{c}_1 \bar{c}_2 e^{i\phi} & -\bar{c}_1 \bar{s}_2 \bar{s}_3 - \bar{s}_1 \bar{c}_2 e^{i\phi} & \bar{s}_2 \bar{c}_3 \\ \bar{s}_1 \bar{c}_3 & \bar{c}_1 \bar{c}_3 & \bar{s}_3 \\ -\bar{s}_1 \bar{c}_2 \bar{s}_3 - \bar{c}_1 \bar{s}_2 e^{i\phi} & -\bar{c}_1 \bar{c}_2 \bar{s}_3 + \bar{s}_1 \bar{s}_2 e^{i\phi} & \bar{c}_2 \bar{c}_3 \end{pmatrix}.
\]
Thus, for small mixing angles, the CKM matrix elements obey the relations

\[
|V_{us}| \approx \sqrt{\frac{y_d}{y_s}}, \quad |V_{ub}| \approx \sqrt{\frac{y_u}{y_t}}. \quad (4.41)
\]

The matrix element \(|V_{cb}| = \sin \theta_3\) remains arbitrary unless additional constraints \([14]\) are imposed on the parameters of the texture (4.35).

The six GUT scale relations (4.36), (4.37) and (4.41) are modified at low energies by RG running. Using eqs. (3.5), (3.6), and (4.1), we obtain the relations

\[
\frac{m_e}{m_d} \approx \frac{1}{3} \frac{A_e \eta_e}{A_d \eta_d}, \quad \frac{m_\mu}{m_s} \approx \frac{3}{A_e \eta_s} \frac{A_\mu \eta_\mu}{A_d \eta_d}, \quad |V_{us}| \approx \sqrt{\frac{m_d}{m_s}}, \quad (4.42)
\]

which are not affected by the running induced by third generation fermions, and the relations

\[
\frac{m_b}{m_\tau} = \frac{A_d \eta_b}{A_e \eta_\tau} \frac{B_\tau B_b^3}{B_3^3}, \quad (4.43)
\]

\[
m_u = \eta_u \frac{m_c^2}{\eta_c m_t} \frac{B_\tau}{B_b}, \quad (4.44)
\]

\[
|V_{ub}| \approx \sqrt{\frac{\eta_u m_u}{\eta_u m_t}} \sqrt{\frac{B_\tau}{B_b}}, \quad (4.45)
\]

which are. These relations reduce to those given in ref. \([14]\) if we set \(B_b = B_\tau = \eta_\tau = 1\).

The relation between \(m_t\) and \(\tan \beta\) shown in fig. 1 depends only on the SU(5) relation (4.43), and therefore holds for the Giudice texture as well as for the Georgi-Jarlskog texture. For the same reason, eqs. (4.24)–(4.28) also continue to hold.

The up quark mass is determined by \(\beta\) in the Giudice texture. The geometric mean relation (4.44) together with eq. (4.24) yields

\[
m_u = \frac{\eta_u \eta_k m_c^2 k^3}{\eta_c^2 m_t} \left( \frac{B_\tau^9}{B_b^8} \right), \quad (4.46)
\]

which we plot as a function of \(\tan \beta\) in fig. 4. Expanding eq. (4.46) in terms of \(\sqrt{y_b}\),
we obtain

\[ m_u = \frac{2.5 \text{ MeV}}{\sin \beta} [1 + (1.7 \times 10^{-4}) \sec^2 \beta + \ldots], \]

(4.47)

implying a lower bound on the up quark mass, \( m_u \gtrsim 2.5 \text{ MeV} \). We plot the relation between \( m_u \) and \( m_t \) in fig. 5. Using eqs. (4.28) and (4.47), we find that

\[ \frac{m_u}{2.5 \text{ MeV}} \approx \frac{185 \text{ GeV}}{m_t} \]

(4.48)

for \( \tan \beta \lesssim 8 \), and

\[ \left( \frac{m_u}{2.5 \text{ MeV}} - 1 \right) \approx 3.6 \left( 1 - \frac{m_t}{185 \text{ GeV}} \right), \]

(4.49)

for \( \tan \beta \gtrsim 8 \). The central value of the up quark mass given in ref. [22], \( m_u = 5.1 \pm 1.5 \text{ MeV} \), could correspond either to a small value of \( \tan \beta (\sim 0.56) \) with \( m_t \sim 90 \text{ GeV} \) [14], or to a much larger value of \( \tan \beta (\sim 50) \) with \( m_t \sim 130 \text{ GeV} \) [15]. An up quark in the range \( 3.6 \text{ MeV} < m_u < 6.6 \text{ MeV} \) requires either \( 0.4 \lesssim \tan \beta \lesssim 1.0 \) with \( 70 \text{ GeV} \lesssim m_t \lesssim 130 \text{ GeV} \), or \( 50 \lesssim \tan \beta \lesssim 60 \) with \( 100 \text{ GeV} \lesssim m_t \lesssim 160 \text{ GeV} \).

The Giudice texture also determines \( |V_{ub}| \) as a function of \( \beta \). From eqs. (4.24), (4.44), and (4.45), we have

\[ |V_{ub}| \approx \frac{\eta_{K_C b} k^2 B_T^6}{\eta_{C m_t} B_b^6}, \]

(4.50)

which is plotted as a function of \( \tan \beta \) in fig. 6. Expanding eq. (4.50) in terms of \( \eta_b \), we find

\[ |V_{ub}| \approx \frac{0.0022}{\sin \beta} [1 + (1.5 \times 10^{-4}) \sec^2 \beta + \ldots], \]

(4.51)

implying the lower bound \( |V_{ub}| \gtrsim 0.0022 \). The rather weak constraint \( |V_{ub}| < 0.007 \) requires \( \tan \beta \gtrsim 0.3 \).
4.3. The Fritzsch Texture

Finally, we consider the Fritzsch texture [5, 23] for the quark Yukawa matrices,

\[
Y_U = \begin{pmatrix} 0 & C & 0 \\ C & 0 & B \\ 0 & B & A \end{pmatrix}, \quad Y_D = \begin{pmatrix} 0 & F e^{i\phi_1} & 0 \\ F e^{-i\phi_1} & 0 & E e^{i\phi_2} \\ 0 & E e^{-i\phi_2} & D \end{pmatrix},
\]

(4.52)

at the GUT scale. There is no ansatz for the lepton Yukawa matrix. The matrices (4.52) are diagonalized by

\[
L_U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_2} & 0 \end{pmatrix} \begin{pmatrix} \bar{v}_2 & -\bar{s}_2 & 0 \\ \bar{s}_2 & \bar{v}_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(4.53)

\[
L_D = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{-i\phi_2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_1} & 0 \\ 0 & e^{-i\phi_2} & 0 \end{pmatrix} \begin{pmatrix} \bar{v}_1 & -\bar{s}_1 & 0 \\ \bar{s}_1 & \bar{v}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(4.54)

with

\[
\tan^2 \theta_1 = \frac{\bar{y}_d}{\bar{y}_s}, \quad \tan^2 \theta_2 = \frac{\bar{y}_u}{\bar{y}_c}, \quad \tan^2 \theta_3 = \frac{\bar{y}_c}{\bar{y}_t},
\]

(4.55)

When the mixing angles are small, the unitary matrices (4.53) lead to a CKM matrix whose elements are given by

\[
|V_{us}| \approx \frac{\bar{y}_d}{\bar{y}_s} - e^{-i\phi_1} \frac{\bar{y}_u}{\bar{y}_c}, \quad |V_{ub}| \approx \frac{\bar{y}_u}{\bar{y}_c}, \quad |V_{cb}| \approx \frac{\bar{y}_s}{\bar{y}_b} - e^{-i\phi_2} \frac{\bar{y}_c}{\bar{y}_t}
\]

at the GUT scale. There are no relations among the Yukawa couplings.
We scale down the relations (4.55) using eqs. (3.5) and (3.6). The relations involving only the first and second generation Yukawa couplings

\[ |V_{us}| \approx \sqrt{\frac{m_d}{m_s} - e^{-i\phi_1} \frac{\eta_c m_u}{\eta_u m_c}}, \quad |V_{ub}| \approx \sqrt{\frac{\eta_c m_u}{\eta_u m_c}}, \quad (4.56) \]

are the same as in the Georgi-Jarlskog texture, and are not affected by the running induced by large Yukawa couplings. The relation for \( |V_{cb}| \), however, is given by

\[ |V_{cb}| \approx \sqrt{\frac{B_b}{B_t}} \frac{\eta_b m_s - e^{-i\phi_2}}{\eta_s m_b} \sqrt{\frac{B_t}{B_b}} \frac{\eta_t m_c}{\eta_c m_t}, \quad (4.57) \]

in the Fritzsch texture.

The relation (4.57) yields a connection between the top quark mass and \( |V_{cb}| \). The small experimental value for \( |V_{cb}| \), requires a large amount of cancellation between the two terms in eq. (4.57), which seems to imply both \( \phi_2 \approx 0 \) and a relatively light top quark [23]. If we neglect the running induced by Yukawa couplings and require \( m_s > 120 \text{ MeV}, \ m_b < 4.35 \text{ GeV} \) and \( m_c < 1.32 \text{ GeV} \) [22], we obtain the upper bound for the top quark mass

\[ |V_{cb}| < 0.058 \quad \Rightarrow \quad m_t < 110 \text{ GeV}, \quad \text{if } B_t = B_b = 1. \quad (4.58) \]

The inclusion of Yukawa coupling-induced running implies an even lower upper bound for \( m_t \) for small or moderate values of \( \tan \beta \), since typically \( B_t < B_b \). If \( \tan \beta \) is very large, however, then it is possible that \( B_b < B_t \), loosening the bound on the top quark, as Babu and Shafi have pointed out [17]. They demonstrated this numerically for a specific value of \( \tan \beta \). We will derive a modified bound for \( m_t \) using the analytic approximation (4.20). In this approximation, the top quark mass is given by

\[ m_t \approx \frac{v A_u \eta_t}{\sqrt{2K_u}} B_b \sqrt{1 - B_t^{12}} \quad \text{for } \tan \beta \gg 1. \quad (4.59) \]

The largest amount of cancellation will occur in eq. (4.57) when \( B_b/B_t \) is minimized. Holding \( m_t \) fixed in eq. (4.59), we find that \( B_b/B_t \) is minimized when
\[ B_t = \left( \frac{1}{7} \right)^{1/12} \approx 0.85, \text{ and therefore} \]

\[ \left( \frac{B_b}{B_t} \right)_{\text{min}} \approx \frac{m_t}{150 \text{ GeV}}. \] (4.60)

Substituting this into eq. (4.57), we obtain

\[ |V_{cb}| < 0.058 \Rightarrow m_t \lesssim 140 \text{ GeV}, \] (4.61)

a considerably higher upper bound than eq. (4.58). This high value of the top quark mass requires a large value of \( \tan \beta \). From eq. (4.22), and using the approximation (4.20), we have

\[ \sec \beta \approx \frac{v A_d \eta_b}{m_b \sqrt{2 R_d}} B_t B_{\tau} \sqrt{1 - B_b^{12}}, \] (4.62)

so that

\[ m_t = 140 \text{ GeV} \Rightarrow \sec \beta \sim 50 B_{\tau}. \] (4.63)

In the Fritzsch texture, \( \eta_{\tau} \) is unrelated to \( \eta_b \) so \( B_{\tau} \) is undetermined, but \( 0.8 \lesssim B_{\tau} \leq 1 \) for \( 0 \leq \eta_{\tau} < 2 \).

### 5. Conclusions

Relations among fermion masses and mixing angles at the scale of grand unification are modified at lower energies by renormalization group running induced by gauge and Yukawa couplings. In supersymmetric theories, the \( b \) quark and \( \tau \) lepton Yukawa couplings, as well as the \( t \) quark coupling, may cause significant running if \( \tan \beta \) is large.

In this paper, we have analyzed the running of fermion masses and mixing angles caused by the entire third generation of Yukawa couplings. We made several approximations along the way. First, we assumed a hierarchy of masses between generations. In this approximation, we derived explicit RG equations for the mixing
angles (2.19). Next, we assumed that the mixing angles in the RG equations were small; the error from this approximation was shown to be less than 0.1%. Finally, we made the approximation that the RG equations (3.10) for the scaling factors $B_\alpha$ decoupled from one another. This allowed us to obtain the analytic expressions (3.11) for these scaling factors, which differ from the exact values by no more than 1 or 2%.

We then used the approximate analytic expressions for the scaling factors to determine how running induced by the third generation of fermions affects the predictions arising from the GUT scale Yukawa matrices. We summarize here the results for each of the three textures considered in this paper, using the input data (4.4) and (4.5). It is easy to determine the effect of choosing different values for the input parameters because our results are analytic.

The Georgi-Jarlskog texture incorporates the SU(5) relation (4.16), which implies the relation between $m_t$ and $\tan \beta$ displayed in fig. 1 and given approximately by (4.28). The top quark mass is bounded above by $\sim 185$ GeV, the bound being saturated for $\tan \beta \sim 10$. This texture also implies the HRR relation (4.18); this leads to the relation between $|V_{cb}|$ and $\tan \beta$ shown in fig. 2 and given approximately by eq. (4.30). Consequently, $|V_{cb}|$ must be greater than $\sim 0.053$. Requiring $|V_{cb}| < \sim 0.058$ implies that $1.5 < \sim \tan \beta < \sim 40$, and therefore $m_t > \sim 155$ GeV.

The Giudice texture also incorporates the SU(5) relation and therefore the relation between $m_t$ and $\tan \beta$ shown in fig. 1. This texture also implies the geometric mean relation (4.44), which leads to the relation between $m_u$ displayed in fig. 4 and given approximately by eq. (4.47). The up quark mass is bounded below by $\sim 2.5$ MeV. If the up quark lies in the range $3.6$ MeV $< m_u < 6.6$ MeV [22], then the Giudice texture implies that either $0.4 < \sim \tan \beta < \sim 1.0$ with $70$ GeV $< m_t < 130$ GeV, or $50 < \sim \tan \beta < 60$ with $100$ GeV $< m_t < 160$ GeV. Finally the Giudice texture implies the relation between $|V_{ub}|$ and $\tan \beta$ shown in fig. 6 and given approximately by eq. (4.51).

The Fritzsch texture relates $|V_{cb}|$ to $m_t$ through eq. (4.57). For small or mod-
erate values of $\tan \beta$, this relation leads to an upper bound $m_t \lesssim 110$ GeV, but for large $\tan \beta$ ($\sim 50$), the top quark could be as heavy as 140 GeV.

All of these textures seem able to accommodate a top quark mass in the range preferred by the analysis of electroweak radiative corrections, at least for some value of $\tan \beta$. Which one, or whether any of them, accurately describes reality remains an open question. There also remains the more fundamental question: what underlying mechanism determines the form of the Yukawa matrices at the GUT scale?
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FIGURE CAPTIONS

1) The relation between $m_t$ and $\tan \beta$ implied by the SU(5) relation (4.16).
Solid line: approximate analytic solution. Dashed line: numerical solution.
Dotted line: $b$ and $\tau$ induced running neglected.

2) The relation between $|V_{cb}|$ and $\tan \beta$ implied by the Georgi-Jarlskog texture.
Solid line: approximate analytic solution. Dashed line: numerical solution.
Dotted line: $b$ and $\tau$ induced running neglected.

3) The relation between $|V_{cb}|$ and $m_t$ implied by the Georgi-Jarlskog texture.
Solid line: approximate analytic solution. Dashed line: numerical solution.

4) The relation between $m_u$ and $\tan \beta$ implied by the Giudice texture. Solid
line: approximate analytic solution. Dashed line: numerical solution. Dotted
line: $b$ and $\tau$ induced running neglected.

5) The relation between $m_u$ and $m_t$ implied by the Giudice texture. Solid line:
approximate analytic solution. Dashed line: numerical solution.

6) The relation between $|V_{ub}|$ and $\tan \beta$ implied by the Giudice texture. Solid
line: approximate analytic solution. Dashed line: numerical solution. Dotted
line: $b$ and $\tau$ induced running neglected.