Domain Structures and Zig-Zag Patterns
Modeled by a Fourth-Order Ginzburg-Landau Equation

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1 Introduction

The formation of steady spatial structures in systems far from equilibrium has been studied in great detail over the past years, the classical examples being Rayleigh-Bénard convection and Taylor vortex flow [7]. In quasi-one-dimensional geometries they usually share a common feature: the stable structures that arise after the decay of transients are strictly periodic in space and if they are weakly perturbed they relax diffusively back to the periodic state. This relaxation has been investigated experimentally in various systems [24] and the results agree with theoretical results based on the phase diffusion equation [21, 13, 15]. It describes the slow dynamics of the local phase, the gradient of which is the local wave number. The structures are generally stable over a range of wave numbers which is limited (at least) by the Eckhaus instability which is characterized by a vanishing of the diffusion coefficient. It leads through the destruction (or creation) of one or more unit cells, i.e. a roll pair in convection or a vortex pair in Taylor vortex flow, to a new stable periodic state with a different wave number.

Recently it has been pointed out that a vanishing of the diffusion coefficient does not necessarily invoke the Eckhaus instability [4, 22]. Instead, under certain conditions, the system can go to a state consisting of distinct domains in which the wave number has different values and which does not relax to a strictly periodic structure. The domains are separated by domain walls in which the wave number changes rapidly. This situation has been treated successfully using a higher-order phase equation [4, 22].

Experimentally, inhomogeneous structures have been found in a variety of systems. Most of them involve at least one time-dependent structure, e.g. counterpropagating traveling waves (or spirals) [4], (turbulent) twist vortices amidst regular Taylor vortices [4], localized traveling-wave pulses in binary-mixture convection [3] and steady Turing patterns amidst chemical traveling waves [16]. Recently, however, domain structures involving only steady convection rolls of two different sizes have been observed in Rayleigh-Bénard convection in a very narrow channel [3]. It has been speculated that these structures may be related to the phase-diffusion mechanism discussed above [4].

Within the framework of the phase equation domain structures consisting of an array of domain walls are not stable due to the attractive interaction between the walls. This leads to a coarsening of the structures during which domain walls annihilate each other. This dynamics is closely related to that observed in spinodal decomposition of binary mixtures after quenches into the miscibility gap [10]. In general, the coarsening will eventually lead to periodic structures with a constant wave number. Only if the boundary conditions conserve the total phase, i.e. require the total number of convection rolls, say, to be constant, will the final state consist of a
(single) pair of domain walls.

Here we investigate the possibility of stable domain structures in the absence of phase conservation. To do so we study systems in which two periodic patterns differing only in their wave numbers are equally likely to arise. In certain cases the competition between the two wave numbers can be described by an extended Ginzburg-Landau equation. We study this equation with spatially ramped control parameter in order to allow the total phase to change and find that even in this general case domain structures of varying sizes can be stable. This Ginzburg-Landau equation can also be viewed as a one-dimensional version of the Ginzburg-Landau equation for two-dimensional patterns in isotropic and anisotropic systems. The domain structures correspond then to ‘zig-zag’ patterns. In two dimensions phase conservation is less common than in one dimension due to the possibility of focus singularities in the pattern which often arise at the boundaries [6]. Our results may therefore also shed some additional light on the behavior of the ‘zig-zag’ patterns studied previously [17, 3].

2 Numerical Simulation of Fourth-Order Ginzburg-Landau Equation

A situation in which the competition between two wave numbers can be investigated with relative ease is obtained if the neutral curve which marks the stability boundary of the basic state has two almost equal minima at (slightly) different wave numbers. For values of the control parameter \(\Sigma\) slightly above these minima the pattern can be described by a Ginzburg-Landau equation for the amplitude \(A\) [18],

\[
\partial_T A = D_2 \partial_X^2 A + iD_3 \partial_X^3 A - \partial_X A + \Sigma A - |A|^2 A,
\]

which gives for instance the vertical fluid velocity in convection via

\[
v_z(x, z, t) = e^{i q c x} A(X, T) f(z) + h.o.t. + c.c.. \tag{2}
\]

For simplicity we assume in the following that the neutral curve has reflection symmetry with respect to \(q_c\) and set \(D_3 = 0\). In this case the neutral curve has two minima if \(D_2\) is negative.

The control parameter \(\Sigma\) is proportional to the temperature difference across the fluid layer, say.

In order to allow the number of rolls to change freely without any pinning by the boundaries we apply a subcritical spatial ramping to \(\Sigma\). Thus \(A\) goes to zero before the boundary of the system and the wave number selected by the ramp [11, 21] corresponds to one of the minima of the neutral curve. For the numerical simulation a Crank-Nicholson scheme with \(dx \approx 0.034\) is used. Such a small grid spacing is required to reduce the pinning by the grid below the small attractive force between domain walls.

For slowly varying wave numbers \(Q(\tau, \xi) \equiv \partial_\xi \phi\), the Ginzburg-Landau equation [1] can be reduced to an equation for the phase of the amplitude \(A = Re^{i\phi}\),

\[
\partial_\tau \phi = (D + E \partial_\xi \phi + F(\partial_\xi \phi)^2) \partial^2_\xi \phi - G \partial^4_\xi \phi. \tag{3}
\]

Domain structures arise for negative values of \(D\). For large values of \(\Sigma\) wave-number gradients across domain walls are sufficiently small to be described by [3], and within this framework the interaction between domain walls is purely attractive. Thus, in the presence of a subcritical ramp domain structures are expected to evolve to a strictly periodic pattern. This is confirmed
for $\Sigma = 100$ and $D_2 = -1$, using initial conditions with three domains, i.e. a region of low wave number between two regions of high wave number.

For $\Sigma = 1$, on the other hand, the behavior is quite different, as shown in fig. 1. It gives the total phase $\int Q \, dX$ in the system (between the ramps) as a function of time for different initial conditions which are obtained by changing the width of the central (low wave number) domain. While for large $\Sigma$ the system evolves to the same periodic state independent of the initial total phase, it converges here to one of several final states with different total phase. These states are pictured in fig. 2. They differ in the width of the central domain and, most strikingly, their local wave number exhibits oscillatory behavior in space.

The possibility for spatial oscillations in the wave number can be seen in a linear stability analysis of the periodic state. It shows that solutions which approach a periodic state for $X \to \pm \infty$ can do so in an oscillatory manner if the fourth derivative is present. The purely attractive interaction between domain walls within the phase equation, and therefore also for large $\Sigma$, is due to the monotonic behavior of the wave number across the wall. The non-monotonic behavior found for smaller $\Sigma$ therefore strongly suggests an oscillatory contribution to the interaction which would explain the stability of bound pairs of domain walls.

To study the transition between the two regimes one could investigate the persistence of the domain structures when increasing $\Sigma$. Since the scale for $\Sigma$ is set by $D_2$, which determines the depth of the wells in the neutral curve, increasing $\Sigma$ is equivalent to making $D_2$ less negative. In fact, one of the 2 parameters could be scaled away ($D_2 \to D_2/|\Sigma|, \Sigma \to \pm 1$). We therefore investigate the persistence of bound pairs by changing $D_2$ rather than $\Sigma$ since this also sheds some light on the stability of the two-dimensional patterns discussed below. The result is shown in fig. 3. It gives the total phase within the unramped region of various states as a function of time.
Figure 2: Coexisting stable final states obtained from the time evolution shown in fig.1. The thick line gives the local wave number in the bulk (800 < i < 2400, dx = 0.034). Note that the amplitude goes to zero toward the boundaries due to the subcritical ramping.

For these simulations the system has been chosen much larger in order to avoid interactions between the domain walls and the ramps. The periodic state is indicated by a dashed line. The different symbols denote where the bound pairs with 1 to 5 minima in the local wave number, respectively, disappear. While the states with 1 to 3 minima jump to the periodic solution above that value of $D_2$, the 4-state jumps to the 3-state and the 5-state merges with the 4-state. The 5-state is relatively hard to track due to the rapid decay of the oscillations away from the domain walls. For $D_2 < -0.8$ all six states coexist stably. In this regime arrays of domain walls should be possible in which the widths of successive domains alternate chaotically. Clearly, none of the bound pairs investigated persists all the way to $D_2 = 0$. This corresponds to the previous result that for large $\Sigma$ the phase equation becomes valid. Strikingly, the value of $D_2$ at which the solutions cease to exist is not a monotonic function of the number of minima in the wave number. Rather, the solution branch corresponding to the state with 3 minima is the last non-periodic state to disappear. If one were to start with a chaotic array of domain walls and to increase $D_2$, fig.(3) suggests that domains of various widths would successively be eliminated beginning with the longest and the shortest ones until only domains with 3 minima in the wave number remain. Note that these simulations only address the interaction between two domain
walls. In a general array with many domains the regime of existence may be therefore somewhat different.

![Figure 3: Regime of existence of bound pairs of domain walls for Σ = 1. The calculation was performed with 12800 points and a grid spacing of 0.034. The ramped part is 400 points wide on each side.](image)

3 Application to Two Dimensional Systems: Zig-Zags

The above results are expected to capture also certain aspects of patterns extended in two dimensions. A particularly clear example is given by electro-hydrodynamic convection (EHC) in nematic liquid crystals. In the nematic phase the rod-like molecules are predominantly oriented in one direction rendering the fluid anisotropic. Depending on parameters, convection arises in the form of rolls perpendicular or oblique to that preferred direction. Due to reflection symmetry, the oblique rolls can have wave number \((Q, P)\) or \((Q, -P)\). Often domains of oblique rolls with opposite orientation are observed to coexist, leading to ‘zig-zag’ patterns. They are characterized by sharp transitions between the domains with the two orientations.

Quite generally, in isotropic systems, as for instance in Rayleigh-Bénard convection (RBC), straight roll patterns exhibit a secondary instability which leads to an undulatory deformation of the rolls. It arises for small wave numbers and tends to increase the local wave number. In the course of the nonlinear evolution the undulations can grow sharper and form zig-zag-type patterns which may in fact be stable, as recently shown in simulations of chemical Turing patterns [8].

Near onset both systems can be described by suitable Ginzburg-Landau equations which
differ in their scaling of the spatial variables. In the isotropic case one obtains
\[ \partial_T A = -(i\partial_X + \partial_Y^2)^2 A + \lambda A - |A|^2 A, \] (4)
whereas the anisotropic case yields
\[ \partial_T A = (\partial_Y^2 - iZ\partial_X \partial_Y^2 + W \partial_Y^2 - \partial_Y^4)A + \lambda A - |A|^2 A. \] (5)

In the anisotropic case the normal/oblique transition occurs at \( W = -ZQ \).

Focussing on solutions which are strictly periodic in \( X \), \( A = A_1(Y,T)e^{iQX} \), one obtains eq.(1) for \( A_1 \),
\[ \partial_T A_1 = D_2 \partial_Y^2 A_1 - \partial_Y^4 A_1 + \Sigma A - |A_1|^2 A_1, \] (6)
with
\[ D_2 = Q, \quad \Sigma = \lambda - Q^2 \] for (4) \( (7) \)
\[ D_2 = W + ZQ, \quad \Sigma = \lambda - Q^2 \] for (5). \( (8) \)

Note that \( Y \) and \( P \) in the two-dimensional systems correspond to \( X \) and \( Q \) in eq.(1). Thus, the domain structures discussed in sec.2 are one-dimensional analogs of zig-zag patterns. The general stability of zig-zag patterns in anisotropic systems has been studied by Pesch and Kramer \( [17] \) and by Bodenschütz et al. \( [3] \) who find islands of stability for the zig-zag states. They are presumably related to the discrete set of domains with different widths discussed above. Of course, the stability of the zig-zags may be limited by two-dimensional instabilities which may preempt some of the transitions discussed here. The interaction of domain walls separating the two kinds of oblique rolls has, however, not been discussed in detail.

The straight (normal) roll state corresponds to a solution with constant \( A_1 \). Reducing its wave number is equivalent to decreasing \( D_2 \), which goes through zero at the onset of the zig-zag instability. It leads to the growth of undulatory deformations of the rolls which correspond to periodic variations of the local wave number in eq.(1). Based on the simulations of that equation one would expect that \( D_2 \) has to be sufficiently negative for the resulting zig-zag structure to persist. Therefore, close to the onset of the zig-zag instability the coarsening dynamics predicted by the phase equation (3) will prevail, and the expected final state would be a pattern of straight (oblique) rolls of either sign if the phase is not conserved. For deeper quenches into the unstable regime\( ^4 \), however, the undulations are expected to lock into each other due to the oscillatory interaction and form an array of domain walls, i.e. an array of zigs and zags, which again could be spatially chaotic.

4 Conclusion

The competition between patterns differing only in their wave number can lead to complex spatial patterns in which the interaction between the walls separating the domains with different wave number plays an important roll. Previously, we investigated them for conditions under which the total phase, i.e. the number of convection rolls in the system, is conserved \( [19] \). There we found a very rich bifurcation structure which originates from interactions with other modes arising from the Eckhaus instability.

\[^1\]Note \( |D_2/\Sigma| \rightarrow \infty \) at the neutral curve \( \Sigma = \lambda - Q^2 \)
In experiments the phase is not always conserved. In the thin layers used in electro-convection of liquid crystals, for instance, inhomogeneities in the layer thickness are often strong enough that convection arises in large patches which are separated by non-convecting regions. In isotropic two-dimensional systems focus singularities are expected to arise in the corners of the container which also allow rolls to disappear smoothly without strong pinning \[\text{[6]}\]. Here we therefore studied domain structures in the presence of spatial ramps. We found that they can exist even without phase conservation and we attribute this to the oscillatory behavior of the local wave number which should lead to an oscillatory interaction between adjacent domain walls. The distance between the walls is discretized accordingly. Close to the onset of the zig-zag instability in two-dimensional systems, these oscillations are small; to stabilize zig-zag patterns one therefore has to quench deeper into the unstable regime.

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References

[1] Baxter, B.W., Andereck, C.D., “Formation of Dynamical Domains in a Circular Couette System”, Phys. Rev. Lett. 57 (1986) 3046.

[2] Bensimon, D., Kolodner, P., Surko, C.M., “Competing and Coexisting Dynamical States of Traveling-Wave Convection in an Annulus”, J. Fluid Mech. 217 (1990) 441.

[3] Bodenschatz, E., Kaiser, M., Kramer, L., Pesch, W., Weber, A., Zimmermann, W., “Patterns and Defects in Liquid Crystals” in The Geometry of Non-Equilibrium, eds. Coullet, P., Huerre, P., p. 111, Plenum 1990.

[4] Brand, H.R., Deissler, R.J., “Confined States in Phase Dynamics”, Phys. Rev. Lett. 63 (1989) 508; Brand, H.R., Deissler, R.J., “Properties of Confined States in Phase Dynamics”, Phys. Rev. A 41 (1990) 5478; Deissler, R.J., Lee, Y.C., Brand, H.R., “Confined States in Phase Dynamics: the Influence of Boundary Conditions and Transient Behavior”, Phys. Rev. A 42 (1990) 2101.

[5] Couillet, P., Elphick, C., Repaux, D., “Nature of Spatial Chaos”, Phys. Rev. Lett. 58 (1987) 431.

[6] Cross, M.C., Newell, A.C., “Convection Patterns in Large Aspect Ratio Systems”, Physica 10D (1984) 299.

[7] Cross, M.C., Hohenberg, P.C., “Pattern Formation Outside of Equilibrium”, Rev. Mod. Phys. (to appear).

[8] Dufiet, V., Boissonade, J., “Conventional and Unconventional Turing Patterns”, J. Chem. Phys. 96 (1992) 664.

[9] Hegseth, J., Vince, J.M., Dubois, M., Bergé, P., “Pattern Domains in Rayleigh-Bénard Slot Convection”, Europhys. Lett. 17 (1992) 413.

[10] Kawasaki, K., Ohta, T., “Kink Dynamics in One-Dimensional Nonlinear Systems”, Physica 116A (1982) 573; Kawasaki, K., this volume.
[11] Kramer, L., Ben-Jacob, E., Brand, H., Cross M., “Wavelength Selection in Systems Far from Equilibrium”, Phys. Rev. Lett. 49 (1982) 1891.

[12] Kramer, L., Zimmermann, W., “On the Eckhaus Instability for Spatially Periodic Pattern”, Physica 16D (1985) 221.

[13] Lücke, M., Roth, D., “Structure and Dynamics of Taylor Vortex Flow and the Effect of Subcritical Driving Ramps”, Z. Phys. B 78 (1990) 147.

[14] Mutabazi, I., Hegseth, J.J., Andereck, C.D., Wesfreid, J.E., “Spatiotemporal Pattern Modulations in the Taylor-Dean System” Phys. Rev. Lett. 64 (1990) 1729.

[15] Paap, H.-G., Riecke, H., “Drifting Vortices in Ramped Taylor Vortex Flow: Quantitative Results from Phase Equation”, Phys. Fluids A3 (1991) 1519.

[16] Perraud, J.-J., Dulos, E., De Kepper P., De Wit, A., Dewel, G., Borckmans, P., “Turing-Hopf Localized Structures”, in Spatio-Temporal Organization in Nonequilibrium Systems, eds. Müller, S.C., Plesser, Th., 205, Projekt Verlag, 1992.

[17] Pesch, W., Kramer, L., “Nonlinear Analysis of Spatial Structures in Two-Dimensional Anisotropic Pattern Forming Systems”, Z. Phys. B63 (1986) 121.

[18] Proctor, M.R.E., “Instabilities of Roll-Like Patterns for Degenerate Marginal Curves”, Phys. Fluids A 3 (1991) 299.

[19] Raitt, D., Riecke, H., “Domain Structures: Existence and Stability in a Fourth-Order Ginzburg-Landau Equation”, preprint.

[20] Riecke, H. Paap, H.-G., “Stability and Wave-Vector Restriction of Axisymmetric Taylor Vortex Flow”, Phys. Rev. A 33 (1986) 547.

[21] Riecke, H., Paap, H.-G., “Perfect Wave-Number Selection and Drifting Patterns in Ramped Taylor Vortex Flow”, Phys. Rev. Lett. 59 (1987) 2570.

[22] Riecke, H., “Stable Wave-Number Kinks in Parametrically Excited Standing Waves”, Europhys. Lett. 11 (1990) 213; Riecke, H., “On the Stability of Parametrically Excited Standing Waves”, in Nonlinear Evolution of Spatio-Temporal Structures in Dissipative Continuous Systems, eds. Busse, F.H., Kramer, L., p. 437, Plenum 1990.

[23] Segel, L.A., “Slow Amplitude Modulation of Cellular Convection”, J. Fluid Mech. 38 (1969) 203; Newell, A.C., Whitehead, J.A., “Finite Bandwidth Finite Amplitude Convection”, J. Fluid Mech. 38 (1969) 279.

[24] Wu, M., Andereck, C.D., “Phase Modulation of Taylor vortex flow”, Phys. Rev. A 43 (1991) 2074.