COMBINATORIAL MODELS FOR THE COHOMOLOGY AND \( K \)-THEORY OF SOME LOOP SPACES

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ABSTRACT. Generalized flag varieties \( G/P \) have a Schubert cell decomposition, yielding a canonical \( \mathbb{Z} \)-basis of the cohomology ring \( H^*(G/P) \). The core of Schubert calculus is the study of the structure coefficients of \( H^*(G/P) \) with respect to this standard basis.

We apply the philosophy of Schubert calculus to the loop spaces \( \Omega(\Sigma(G/P)) \), through the homotopy model given by James reduced product \( J(G/P) \). Building on work of Baker and Richter (2008), we describe a canonical Schubert cell decomposition of \( J(G/P) \), yielding a canonical basis of its cohomology. We obtain combinatorial models for the cohomology rings and their distinguished bases.

For concreteness and convenience, we focus primarily on the case \( G/P = \mathbb{C}P^\infty \), where we explicitly identify the Schubert basis of \( H^*(J(\mathbb{C}P^\infty)) \) with monomial quasisymmetric functions. In this context, we also introduce and study a “cellular \( K \)-theory” Schubert basis for \( K(J(\mathbb{C}P^\infty)) \mathbb{Z}_2 \). We characterize this \( K \)-theory ring and develop quasisymmetric representatives with an explicit combinatorial description.

1. INTRODUCTION

Traditionally, Schubert calculus studies the cohomology rings of \emph{generalized flag varieties}, quotients \( G/P \) of a Kac–Moody Lie group \( G \) by a parabolic subgroup \( P \). The key feature enabling a combinatorial analysis of \( H^*(G/P) \) is the existence of a stratification of \( G/P \) by \emph{Schubert varieties}, giving it the structure of a CW-complex with only even-dimensional cells. Such a cell decomposition yields a distinguished linear basis of \( H^*(G/P) \), which one models through combinatorial tools, e.g., \emph{Schubert polynomials}. This approach yields a very precise and fine understanding of \( H^*(G/P) \). For example, when \( G/P \) is a complex Grassmannian, its cohomology is governed by the combinatorics of Schur functions and the structure coefficients of \( H^*(G/P) \) are the \emph{Littlewood–Richardson coefficients}, computed in an explicit positive combinatorial fashion by the theory of \emph{Young tableaux} [LR34] or \emph{puzzles} [KTW04].

In this paper, we take an analogous approach to the cohomology and \( K \)-theory of the loop spaces \( \Omega(\Sigma(G/P)) \). (Here, \( \Sigma \) and \( \Omega \) denote the pointed suspension and loop space functors, where we treat the unique 0-dimensional Schubert cell of \( G/P \) as the basepoint.) Our work is inspired by a paper of A. Baker and B. Richter [BR08] identifying \( H^*(\Omega\Sigma\mathbb{C}P^\infty) \) with the ring \emph{QSym} of quasisymmetric functions. Quasisymmetric functions play a key role in the theory of \emph{combinatorial Hopf algebras} [GR20] and throughout algebraic combinatorics, where they have attracted much study since their introduction in [Sta72, Ges84]; see [Mas19] for a recent survey. Baker and Richter use their topological approach to establish algebraic properties of \emph{QSym}; however, in contrast to our work, they do not identify a geometrically-natural basis, preventing them from extending the analogy with classical Schubert calculus. (Similarly, [Oes19] identifies \emph{QSym} with the Chow ring of an Artin stack, but in a way that does not manifest a geometrically-natural basis.) Our work generalizes the aforementioned papers, establishing a Schubert calculus for \( \Omega(\Sigma(G/P)) \); see Section 6. Furthermore, in the case \( G/P = \mathbb{C}P^\infty \), which is the primary focus of our paper, we make explicit connections to established combinatorics of quasisymmetric polynomials and power series.

Our motivation is two-fold. Firstly, Schubert calculus has been a powerful tool for understanding the combinatorics of \emph{symmetric functions}, while such an approach to quasisymmetric functions has been mostly missing, despite the pioneering work of [BR08, Oes19]. We wish to remedy this situation by providing a geometric context for the combinatorics of \emph{QSym}. Secondly, we are motivated by a desire to understand the geometric content of \emph{elliptic cohomology}. Elliptic cohomology is a complex oriented cohomology theory

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that arises from formal algebraic considerations. A geometric perspective on what is measured by elliptic cohomology would be expected to give important insight into string theory, but is impeded by the difficulty of performing explicit calculations even for simple spaces. Since the elliptic cohomology of a space $X$ is approximated by the $K$-theory of the loop space $\Omega X$, it becomes of interest to identity examples where we can develop a concrete understanding of $K^*(\Omega X)$. We will produce explicit combinatorial models for $K^*(\Omega\Sigma(CP^\infty))$. See, for example, [LZ17, RW20, KRW20] for related work developing the beginnings of an elliptic theory of Schubert calculus and [BT21] for geometric models of elliptic cohomology.

Our approach to studying $\Omega\Sigma X$ (also followed by [BR08]) is through the homotopy model given by the **James reduced product** $J(X)$. For a topological space $X$ with basepoint $e$, its James reduced product [Jan55] (see [Whi78, VII.2] for a textbook treatment) is

$$J(X) = \left( \coprod_{n \geq 1} X^n \right) / \sim$$

where $(x_1, \ldots, x_n) \sim (x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ if $x_i = e$. One can think of $J(X)$ as a topologicalization of the free monoid on the points of $X$. The space $J(X)$ is homotopy equivalent to $\Omega\Sigma X$, so we may freely study $H^*(J(X))$ in place of $H^*(\Omega\Sigma X)$.

Let

$$J_n(X) = X^n / \sim,$$

where $(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, e)$. Then

$$X = J_1(X) \subseteq J_2(X) \subseteq \cdots$$

and

$$J(X) = \bigcup_{n \geq 1} J_n(X).$$

Furthermore, if $X$ has a CW-structure with $e$ being a 0-cell, then the quotient map

$$q_n : X^n \to J_n(X)$$

endows $J_n(X)$ with the structure of a CW-complex, thereby making $J(X)$ a CW-complex as well.

We will find that $J(CP^\infty)$ behaves in many ways like a generalized flag variety. However, unlike generalized flag varieties, which are naturally smooth quasiprojective (Ind)-varieties, $J(CP^\infty)$ cannot be realized as a normal quasiprojective variety; this fact, discussed in Section 5, causes some technical complications when we study the $K$-theory of $J(CP^\infty)$.

A composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence of positive integers; we write $\ell(\alpha) = k$ for the length of $\alpha$. Consider the power series ring $A = \mathbb{Z}[x_1, x_2, \ldots]$ in countably-many variables. A power series $f \in A$ is quasisymmetric if, for each composition $(\alpha_1, \ldots, \alpha_k)$ and each increasing integer sequence $1 \leq j_1 < j_2 < \cdots < j_k$, the coefficients of

$$x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_k^{\alpha_k} \text{ and } x_{j_1}^{\alpha_1}x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$$

in $f$ are equal. Following [LP07], we write $mQSym \subseteq A$ for the ring of quasisymmetric power series and let $QSym$ denote the subring of quasisymmetric power series of bounded degree. For each composition $\alpha$, the **monomial quasisymmetric function** $M_\alpha \in QSym$ is the smallest power series containing the monomial $x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_k^{\alpha_k}$; explicitly,

$$M_\alpha = \sum_{1 \leq j_1 < j_2 < \cdots < j_k} x_1^{\alpha_1}x_{j_1}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}.$$

The monomial quasisymmetric functions are one of the most classical bases of $QSym$ and are fundamental to its combinatorial theory.

Our main cohomological theorem is as follows.

**Theorem 1.1.** The James reduced product $J(CP^\infty)$ is a CW-complex whose cells $\{e_\alpha\}_\alpha$ are indexed by compositions $\alpha$. Treating the variables in $QSym$ as having degree 2, we have

$$H^*(J(CP^\infty); \mathbb{Z}) \cong QSym$$

as graded rings. This isomorphism identifies the monomial quasisymmetric function $M_\alpha$ with the cellular cohomology class of $e_\alpha$. 

We note that the isomorphism of $H^*(J(\mathbb{C}P^\infty))$ with QSym is due originally to [BR08]; we give an alternative proof that is arguably more explicit and avoids consideration of the ring NSym of noncommutative symmetric functions. Our identification of a geometrically natural basis for QSym is new.

In Schubert calculus, one has a basis of $H^*(G/P)$ given by the cellular cohomology classes of the Schubert varieties of $G/P$. We think of the cells $e_\alpha$ from Theorem 1.1 as analogous to Schubert varieties and their classes as analogous to Schubert classes. In combinatorial Schubert calculus, one likes to identify Schubert classes with, for example, Schur functions or Schubert polynomials; for us, the monomial quasisymmetric functions play an analogous role.

A Littlewood–Richardson rule for the multiplication of monomial quasisymmetric functions $M_\alpha$ was given by M. Hazewinkel in [Haz01]. We recall this multiplication rule in Section 6. Using Hazewinkel’s positive combinatorial formula together with Theorem 1.1, one may easily perform very concrete, geometrically-motivated calculations in $H^*(J(\mathbb{C}P^\infty))$, analogous to how one may use a Littlewood–Richardson rule for Schur functions to explicate the intersection theory of Grassmannians. We extend these results and multiplicity rules to the James reduced products of a broader range of spaces in Section 6.

We now turn to a $K$-theoretic analogue of Theorem 1.1. In Schubert calculus, one traditionally (see, for example, [FL94, Buc02, PY17a]) associates a $K$-theory class to a Schubert variety by considering the structure sheaf of the variety (although other related choices are possible, e.g. [TY11]). Here, the structure sheaf naturally lives in $K_0(G/P)$, but one can transfer it to $K^0(G/P)$ by taking a resolution by locally-free sheaves, since $G/P$ is smooth. Such an approach for $J(\mathbb{C}P^\infty)$ is not viable since, as observed in Section 5, $J(\mathbb{C}P^\infty)$ cannot be realized as a normal variety. Nonetheless, for each cell $e_\alpha$, we define an explicit $K$-class $[e_\alpha]$ living in the completed tensor product $K(J(\mathbb{C}P^\infty)) \hat{\otimes}_Z \mathbb{Q}$, in such a way that the set of classes $[e_\alpha]$ span $K(J(\mathbb{C}P^\infty)) \hat{\otimes}_Z \mathbb{Q}$. The completed tensor product is a natural object to work with since it arises as the inverse limit of the rational $K$-rings of $K(J_n(\mathbb{P}^m))_\mathbb{Q}$ and therefore affords us access to the Chern character. The construction of this class $[e_\alpha]$ is somewhat technical; see Definition-Lemma 4.6 for the precise definition. The ideas underlying this construction have already seen applications in [PS22], where the authors established a combinatorial conjecture of K. Mészáros, L. Setiabrat, and A. St. Dizier [MSS22].

We introduce new power series $\overline{M}_\alpha \in \text{QSym}$, indexed by compositions; these power series, which we call quasisymmetric monomial glides, are given by a combinatorial rule (see Section 3) and are closely related to the monomial and fundamental slides of [AS17], as well as to the fundamental glides of [PS19] (see [Sea20, PS20, MPS21] for related discussion). In Theorem 4.7, we show that the set of $\overline{M}_\alpha$ form a Schauder basis for the ring of quasisymmetric power series with rational coefficients (each $\overline{M}_\alpha$ is $M_\alpha$ plus higher degree terms); furthermore, we establish the following $K$-theoretic analogue of Theorem 1.1.

**Theorem 1.2.** We have an isomorphism

$$K(J(\mathbb{C}P^\infty)) \hat{\otimes}_Z \mathbb{Q} \cong \text{mQSym} \hat{\otimes}_Z \mathbb{Q}.$$ 

This isomorphism identifies the $K$-class $[e_\alpha]$ with the quasisymmetric monomial glide $\overline{M}_\alpha$.

**Remark 1.3.** The ring mQSym $\hat{\otimes}_Z \mathbb{Q}$ is nothing more than quasisymmetric power series with rational coefficients.

The ring mQSym has been previously studied, for example by [LP07, Pat16, LM21], in relation to combinatorial constructions considered to be “$K$-theoretic”; to our knowledge, however, mQSym has never before been explicitly identified with a $K$-theory ring, nor have nonsymmetric elements of mQSym been previously identified with $K$-theory classes.

A slightly different “$K$-theoretic” analogue of monomial quasisymmetric functions was proposed briefly in [LP07, Remark 5.14] by T. Lam and P. Pylyavskyy; these were inspired by certain other inhomogeneous power series related to the $K$-theory of Grassmannians, however, no explicit connection with $K$-theoretic geometry was obtained. We had initially imagined that the isomorphism of Theorem 1.2 would identify $[e_\alpha]$ with those Lam–Pylyavskyy multimonomial quasisymmetric functions, and it is interesting to see that this is not the case. See Remark 3.15 for a more precise comparison.

To connect with the topology of Theorem 1.2, we find it useful to define the quasisymmetric monomial glides via a cancellative formula involving the Möbius function on a certain partially-ordered set. However, to calculate explicitly with quasisymmetric monomial glides it is preferable to have an explicit non-cancellative combinatorial formula. In Section 3, we establish such a formula. Our formula uses “sliding” combinatorics
This paper is organized as follows. In Section 2, we establish Theorem 1.1, matching the cellular basis of $H^*(J(\mathbb{C}P^\infty))$ with the combinatorics of monomial quasisymmetric functions. In Section 3, we introduce quasisymmetric monomial glides and prove a combinatorial formula for them. In Section 4, we construct a “cellular” $K$-theory basis of $K(J(\mathbb{C}P^\infty)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and connect it to the combinatorics of $\text{mQSym}$ and quasisymmetric monomial glides, establishing Theorem 1.2. The brief Section 5 explains the geometric obstacles to simplifying the construction of Section 4. Finally, in Section 6, we sketch the extension of Theorem 1.1 to James reduced products of a general class of CW complexes, with special emphasis on finite-dimensional generalized flag varieties and the classifying spaces $BU(k)$; for example, Theorem 6.9 gives an explicit positive combinatorial rule for the cellular structure coefficients of $H^*(J(BU(k)))$.

2. Quasisymmetric functions from the cells of the James of the James reduced product

We begin by fixing notation that will be used throughout the paper. A weak composition is a finite sequence of nonnegative integers; recall from the introduction that a composition is a finite sequence of positive integers. Given a weak composition $\alpha$, its positive part $\alpha^+$ is the composition obtained from $\alpha$ by deleting all 0 terms. For example, the positive part of $(2, 0, 4, 0, 0, 2)$ is $(2, 4, 2)$. As a shorthand, for positive integers $n < n'$, we write $[n] = \{1, 2, \ldots, n\}$ and $[n, n'] = \{n, n + 1, \ldots, n'\}$.

We denote by $e_0, e_2, e_4, \ldots$ the cells of $\mathbb{C}P^\infty$ with dim $e_i = i$. By definition, the CW complex structure on $J_n \mathbb{C}P^\infty$ is induced from the cellular quotient map

$$q_n : (\mathbb{C}P^\infty)^n \to J_n \mathbb{C}P^\infty.$$ 

By construction, $q_n$ identifies the cells $e_{2b_1} \times \cdots \times e_{2b_n}$ and $e_{2c_1} \times \cdots \times e_{2c_n}$ of $(\mathbb{C}P^\infty)^n$ if and only if $(b_1, \ldots, b_n)^+ = (c_1, \ldots, c_n)^+$.

We see then that the cells of $J_n \mathbb{C}P^\infty$ are indexed by compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ with length $k \leq n$. It follows that the cells of $J \mathbb{C}P^\infty$ are indexed by the set of all compositions. For any composition $\alpha$, we let $e_{\alpha} \subset J \mathbb{C}P^\infty$ denote the corresponding cell. Since all cells of $J \mathbb{C}P^\infty$ are even-dimensional, we see that its integral homology is freely generated by the cells; hence, we will not distinguish between cells and their cellular homology classes:

$$H_*(J(\mathbb{C}P^\infty); \mathbb{Z}) \cong \bigoplus_\alpha \mathbb{Z}e_\alpha.$$ 

Similarly, $H^d(J(\mathbb{C}P^\infty); \mathbb{Z}) = \text{Hom}(H_d(J(\mathbb{C}P^\infty); \mathbb{Z}), \mathbb{Z})$, so

$$H^*(J(\mathbb{C}P^\infty); \mathbb{Z}) \cong \bigoplus_\alpha \mathbb{Z}x_\alpha,$$

where $x_\alpha$ denotes the function dual to $e_\alpha$. In particular, we see from this description of the cells of $J_n(\mathbb{C}P^\infty)$ and $J(\mathbb{C}P^\infty)$ that if

$$\iota_n : J_n(\mathbb{C}P^\infty) \to J(\mathbb{C}P^\infty)$$

denotes the inclusion, then

$$\iota_n^* : H^{2d}(J(\mathbb{C}P^\infty); \mathbb{Z}) \to H^{2d}(J_n(\mathbb{C}P^\infty); \mathbb{Z})$$

is an isomorphism for all $n \geq d$. As a result

$$(2.1) \quad H^*(J(\mathbb{C}P^\infty); \mathbb{Z}) = \lim_{n \to \infty} H^*(J_n(\mathbb{C}P^\infty); \mathbb{Z})$$

At this point, we have established the first sentence of Theorem 1.1; the remainder is immediately implied by Theorem 2.1 below. Recall that QSym denotes the graded ring of quasisymmetric functions in variables $x_1, x_2, \ldots$, where each $x_i$ has degree 2.

**Theorem 2.1.** We have an isomorphism of graded rings

$$H^*(J(\mathbb{C}P^\infty); \mathbb{Z}) \cong \text{QSym}$$

sending $x_\alpha$ to the quasisymmetric monomial function $M_\alpha$. 


Proof. First, by the Künneth formula,

\[ H^*((\mathbb{C}P^\infty)^n; \mathbb{Z}) = H^*((\mathbb{C}P^\infty); \mathbb{Z})^\otimes n \cong \mathbb{Z}[x_1, \ldots, x_n]; \]

the isomorphism \( \phi_n \) identifies \( x_i \) with the dual of the cell \( e_0 \times \cdots \times e_0 \times e_2 \times e_0 \times \cdots \times e_0 \), where \( e_2 \) appears in the \( i \)-th position.

We see that if \( k \leq n \), then

\[ q_n^{-1}(e_{(\alpha_1, \ldots, \alpha_k)}) = \prod_i e_i, \]

where

- \( \iota : [k] \to [n] \) is injective;
- \( e_i = e_{i_1} \times \cdots \times e_{i_n} \);
- \( i_j = 0 \) if \( j \) is not in the image of \( \iota \), and otherwise \( i_{\iota(j)} = 2\alpha_j \).

As a result,

\[ q_n^*(x_{(\alpha_1, \ldots, \alpha_k)}) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} =: M_{n,(\alpha_1, \ldots, \alpha_k)}; \]

notice that \( M_{n,(\alpha_1, \ldots, \alpha_k)} \) is the quasisymmetric monomial function \( M_{(\alpha_1, \ldots, \alpha_k)} \), truncated to the finite set of variables \( x_1, \ldots, x_n \).

Let

\[ \text{QSym}_n \subset \mathbb{Z}[x_1, \ldots, x_n] \]

be the subring of quasisymmetric polynomials in \( x_1, \ldots, x_n \). Since \( H^*(J_n(\mathbb{C}P^\infty); \mathbb{Z}) \) is freely generated by all \( x_{(\alpha_1, \ldots, \alpha_k)} \) with \( k \leq n \), we see the image of \( q_n^* \) is contained in \( \text{QSym}_n \). We therefore obtain a graded ring map

\[ q_n^* : H^*(J_n(\mathbb{C}P^\infty); \mathbb{Z}) \rightarrow \text{QSym}_n. \]

This map is surjective since \( q_n^*(x_{(\alpha_1, \ldots, \alpha_k)}) = M_{n,(\alpha_1, \ldots, \alpha_k)} \) and the elements \( M_{n,(\alpha_1, \ldots, \alpha_k)} \) with \( k \leq n \) form a \( \mathbb{Z} \)-basis for \( \text{QSym}_n \). Furthermore, the map \( q_n^* \) is injective since if \( f \in H^*(J_n(\mathbb{C}P^\infty); \mathbb{Z}) \), then the coefficient of \( x_{(\alpha_1, \ldots, \alpha_k)} \) in the expression for \( f \) is equal to the coefficient of \( x_1^{\alpha_1} \cdots x_k^{\alpha_k} \) in the expression for \( q_n^*(f) \). Thus, (2.2) is an isomorphism of graded rings.

To finish the proof, we note that for all \( n \), we have a commutative diagram

\[
\begin{array}{ccc}
(\mathbb{C}P^\infty)^n & \xrightarrow{J_n} & (\mathbb{C}P^\infty)^{n+1} \\
\downarrow{q_n} & & \downarrow{q_{n+1}} \\
J_n(\mathbb{C}P^\infty) & \xrightarrow{J_n} & J_{n+1}(\mathbb{C}P^\infty)
\end{array}
\]

which induces a commutative diagram

\[
\begin{array}{ccc}
H^*(J_{n+1}(\mathbb{C}P^\infty); \mathbb{Z}) & \xrightarrow{\phi_{n+1}} & \text{QSym}_{n+1} \\
\downarrow{\pi_n} & & \downarrow{\pi_n'} \\
H^*(J_n(\mathbb{C}P^\infty); \mathbb{Z}) & \xrightarrow{\phi_n} & \text{QSym}_n
\end{array}
\]

of graded rings; the maps \( \pi_n \) and \( \pi_n' \) kill \( x_{n+1} \) and send \( x_i \) to \( x_i \) for \( i \leq n \). Combining this commutative diagram with equation (2.1), we see that

\[ H^*(J(\mathbb{C}P^\infty); \mathbb{Z}) = \lim_n H^*(J_n(\mathbb{C}P^\infty); \mathbb{Z}) \cong \lim_n \text{QSym}_n = \text{QSym}, \]

where the limit is taken in the category of graded rings.

\[ \square \]

Remark 2.2. For the reader less familiar with limits in the category of graded rings, we give a few more details about why \( \lim_n \text{QSym}_n = \text{QSym} \). The limit in the category of graded rings is constructed by taking
the direct sum of the limits of each graded piece. Letting, \( \text{QSym}_n^d \) denote the quasisymmetric polynomials of degree \( d \), we have then

\[
\lim_{n} \text{QSym}_n = \bigoplus_{d} \lim_{n} \text{QSym}_n^d.
\]

The limit \( \lim_{n} \text{QSym}_n^d \) is now taken in the category of all rings, and therefore consists of all quasisymmetric power series of degree \( d \). As a result,

\[
\lim_{n} \text{QSym}_n = \text{QSym}.
\]

3. Quasisymmetric Monomial Glides

In this section, we first define the quasisymmetric monomial glides \( \overline{M}_\alpha \) via a cancellative formula using a Möbius function on a certain poset. This characterization will be useful to connect with the geometry in Section 4, where we establish that \( \overline{M}_\alpha \) represents the \( K \)-class of the cell \( e_\alpha \) of \( J(\mathbb{CP}^\infty) \). On the other hand, this characterization is inefficient for other uses, due to the cancellations and the complexity of the poset. Moreover, the quasisymmetry of \( \overline{M}_\alpha \) is not easily apparent from this definition. The main task of this section then is to establish a simpler non-cancellative formula for the \( \overline{M}_\alpha \), which additionally manifests their quasisymmetry. We will additionally need this quasisymmetry in Section 4.

Throughout this section, let

\[
(3.1) \quad \alpha = (a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k)
\]

be a composition where \( a_i \neq a_{i+1} \) for \( 1 \leq i < k \). Fix

\[
n \geq N := \sum_{i=1}^{k} N_i.
\]

3.1. Definitions and basic properties. Let \( S_\alpha \) be the \( \binom{n}{N} \)-element set of all length-\( n \) strings obtained from \( \alpha \) by inserting 0s. That is to say,

\[
S_\alpha = \{(b_1, \ldots, b_n) : b \text{ is a weak composition with } b^+ = \alpha\},
\]

where we are conflating tuples with strings. Let \( P_\alpha \) be the closure of the set \( S_\alpha \) under componentwise maximum, i.e., \( (c_1, \ldots, c_n) \in P_\alpha \) if there exist \( (b_1, \ldots, b_n) \in S_\alpha \) such that \( c_j = \max_i b_{ij} \). We give \( P_\alpha \) the structure of a poset by componentwise comparison, so that

\[
(b_1, \ldots, b_n) \leq (b'_1, \ldots, b'_n) \text{ if and only if } b_i \leq b'_i \text{ for all } i.
\]

Example 3.1. Let \( \alpha = (1, 3) \) and let \( n = 4 \). Then

\[
S_\alpha = \{0013, 0103, 0130, 1003, 1030, 1300\},
\]

where we drop commas and parentheses in weak compositions for concision. The reader may check that, as a set,

\[
P_\alpha = S_\alpha \cup \{0113, 1103, 1013, 1130, 1313, 1330, 0133, 1033, 1303, 1333, 1133, 1313, 1333\}.
\]
We may visually illustrate the poset structure on $\mathcal{P}_\alpha$ by the (Hasse) diagram below:

Here, $b \leq b'$ if and only if one can reach $b'$ from $b$ via a sequence of ascending edges.

Let $\hat{\mathcal{P}}_\alpha$ be the poset obtained from $\mathcal{P}_\alpha$ by adjoining a minimum element $\hat{0}$ satisfying $\hat{0} < p$ for all $p \in \mathcal{P}_\alpha$.

For a poset $\mathcal{P}$ and elements $p, q \in \mathcal{P}$, a **lower bound** of $p$ and $q$ is any element $r \in \mathcal{P}$ such that $r \leq p$ and $r \leq q$. Similarly, an **upper bound** is any element $s \in \mathcal{P}$ with $s \geq p$ and $s \geq q$. A **meet** of $p$ and $q$ is an element $p \wedge q \in \mathcal{P}$ such that

- $p \wedge q$ is a lower bound of $p$ and $q$, and
- $p \wedge q \geq r$, for any lower bound $r$ of $p$ and $q$.

Similarly, a **join** of $p$ and $q$ is an element $p \vee q \in \mathcal{P}$ that is an upper bound of $p$ and $q$ and is less than any other such upper bound. Clearly, meets and joins are unique if they exist; indeed, treating $\mathcal{P}$ as a category in the standard way, meet and join coincide with categorical product and coproduct.

**Example 3.2.** In the poset $\mathcal{P}_{13}$ depicted in Example 3.1, we have for example that $0103 \vee 0130 = 0133$ and that $1103 \wedge 1013 = 1003$. On the other hand, the meet $1013 \wedge 1130$ does not exist, as no element of the poset lies below both 1013 and 1130.

A **lattice** is a poset such that every pair of elements has both a meet and a join. By Example 3.2, the poset $\mathcal{P}_{13}$ of Example 3.1 is not a lattice as some meets do not exist. On the other hand, we have the following useful observation.

**Lemma 3.3.** The poset $\hat{\mathcal{P}}_\alpha$ is a lattice.

**Proof.** Consider any two elements $p, q \in \hat{\mathcal{P}}_\alpha$. If $p = \hat{0}$, then $p \vee q = q$, while if $q = \hat{0}$, then $p \vee q = p$. Otherwise, we have $p, q \in \mathcal{P}_\alpha$, so set $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$. By construction, they have a join $s = (s_1, \ldots, s_n) \in \hat{\mathcal{P}}_\alpha$ given by $s_i = \max\{p_i, q_i\}$. Thus, $\hat{\mathcal{P}}_\alpha$ has all joins.

Since $\hat{\mathcal{P}}_\alpha$ is a finite poset with all joins and with a minimum element $\hat{0}$, it is a lattice by [Sta97, Proposition 3.3.1].

Consider the alphabet

$$
\mathbb{B} = \{0, 1, 2, 3, \ldots\} \cup \{\bar{1}, \bar{2}, \bar{3}, \ldots\}.
$$

We refer to elements of the first set in $\mathbb{B}$ as **unbarred** and elements of the second as **barred**. We introduce some local moves that may be performed on strings in the alphabet $\mathbb{B}$:

- **(M.1)** $\ldots 0p\ldots \Rightarrow \ldots p0\ldots$, for $p \in \mathbb{Z}_{>0}$;
- **(M.2)** $\ldots 0p\ldots \Rightarrow \ldots p\bar{p}\ldots$, for $p \in \mathbb{Z}_{>0}$. 




That is, if a string $\sigma$ contains the symbol 0 followed by an unbarred symbol $p$, we may either swap them by (M.1) or else replace them with the substring $pp$ by (M.2). Let $\tilde{C}_\alpha$ denote the set of strings that can be obtained from elements of $S_\alpha$ (thought of as strings in $\mathbb{Z}_{\geq 0}$) by iterated application of the local moves (M.1) and (M.2); here, $C$ stands for “combinatorial” since this set will ultimately be used to construct the aforementioned non-cancellative formula for the quasisymmetric monomial glide $M_\alpha$. Note that $S_\alpha$ is the set of strings obtainable by iterated application of just the local move (M.1), starting from the string $(0^{n-N}) - \alpha$, where $0^{n-N}$ denotes a string of $n - N$ zeros and $\sim$ represents concatenation of strings.

Let $C_\alpha$ be the set of strings obtained from $\tilde{C}_\alpha$ by replacing each barred symbol with the corresponding unbarred symbol. Note that multiple elements of $\tilde{C}_\alpha$ may yield the same element of $C_\alpha$. Equivalently, $C_\alpha$ is the set of strings obtainable from elements of $S_\alpha$ by iterated application of (M.1) and the local move (M.2) $\ldots 0 \ldots \Rightarrow \ldots pp \ldots$, for $p \in \mathbb{Z}_{>0}$.

**Example 3.4.** Continuing with $\alpha = (1, 3)$ and $n = 4$ from Example 3.1, we have

$$\tilde{C}_\alpha = S_\alpha \cup \{01\bar{1}3, 1103, 101\bar{3}, \bar{1}1\bar{3}0, 10\bar{1}3, 13\bar{3}0, 013\bar{3}, 10\bar{3}3, 13\bar{3}3, 113\bar{3}, 133\bar{3}\},$$

while

$$C_\alpha = S_\alpha \cup \{0113, 1103, 1013, 1\bar{1}30, 11\bar{1}3, 13\bar{3}0, 013\bar{3}, 10\bar{3}3, 13\bar{3}3, 113\bar{3}, 133\bar{3}\},$$

so that $|C_\alpha| = |\tilde{C}_\alpha|$ in this case. Comparing with Example 3.1, we see that $P_\alpha = C_\alpha \cup \{1313\}$.

On the other hand, for $\beta = (1, 1)$ and $n = 3$, we have

$$S_\beta = \{011, 101, 110\} \text{ and } \tilde{C}_\beta = S_\beta \cup \{111, 111\},$$

while

$$C_\beta = S_\beta \cup \{111\} = P_\beta.$$

In this case, $C_\beta$ is strictly smaller than $\tilde{C}_\beta$.

For $\tilde{s} \in \tilde{C}_\alpha$, let $\text{Bar}(\tilde{s})$ denote the number of barred symbols in $\tilde{s}$.

**Lemma 3.5.** Viewing $P_\alpha$ as a set (forgetting the poset structure), we have

$$C_\alpha \subseteq P_\alpha.$$

**Proof.** For any $\sigma \in C_\alpha$, let $r(\sigma)$ be the least nonnegative integer $r$ such that $\sigma$ can be obtained from $S_\alpha$ by $r$ applications of the local moves (M.1)/(M.2). We induct on $r$.

Let $\sigma$ be a string of nonnegative integers such that $\sigma \in C_\alpha$. If $\sigma \in S_\alpha$ (i.e., $r(\sigma) = 0$), then $\sigma \in P_\alpha$ by definition. Otherwise, choose $\sigma' \in C_\alpha$ such that $r(\sigma') = r(\sigma) - 1$, assuming inductively that $\sigma' \in P_\alpha$.

Suppose $\sigma$ can be obtained from $\sigma'$ by application of the local move (M,q) in positions $i, i + 1$, where $q \in \{1, 2\}$. Then the $i$th entry $\sigma'(i') = 0$ and the $(i + 1)$th entry $\sigma'(i + 1) = p$ for some positive integer $p$. Since $\sigma' \in P_\alpha$, there exist $\tau_1', \ldots, \tau_k' \in S_\alpha$ such that $\bigvee_{j=1}^{k} \tau_j' = \sigma'$. In particular, we have that $\tau_j'(i) = 0$ and $\tau_j'(i + 1) \leq p$ for all $j$; moreover, $\tau_j'(i + 1) = p$ for at least one $j$. Define $\tau_1, \ldots, \tau_k$ by

$$\tau_j = \begin{cases} \tau_j', & \text{if } \tau_j'(i + 1) = 0; \\ (M.q) \cdot \tau_j', & \text{if } \tau_j'(i + 1) > 0 \end{cases}$$

where $(M.q) \cdot \tau_j'$ is performed in positions $i, i + 1$. Then, each $\tau_j \in S_\alpha$ by construction and $\bigvee_{j=1}^{k} \tau_j = \sigma$, so $\sigma \in P_\alpha$.\qed

Let $\mu_{P_\alpha} : P_\alpha \to \mathbb{Z}$ be defined by the property that for all $p \in P_\alpha$, we have

$$\sum_{q \leq p} \mu_{P_\alpha}(q) = 1. \quad (3.2)$$

We call $\mu_{P_\alpha}$ the **Möbius function** on $P_\alpha$. Note that, traditionally, the Möbius function for $\widehat{P}_\alpha$ is defined by $\mu_{\widehat{P}_\alpha}(0) = 1$ and $\sum_{q \leq p} \mu_{\widehat{P}_\alpha}(q) = 0$ for $q \neq 0$. So, to convert back and forth between our convention and the traditional one, we have $\mu_{P_\alpha}(p) = -\mu_{\widehat{P}_\alpha}(p)$ for all $p \in P_\alpha$. (The reason for this convention is to match a choice from [Knu09] which will be applied in Section 4.) For background on Möbius functions on posets, we refer to [Sta97, Chapter 3].
**Example 3.6.** Consider the poset $P_{13}$ from Example 3.1. The values of the Möbius function $\mu_{P_{13}}$ are recorded below.

These values are computed by labeling each minimal element with 1 and then working upwards by the recurrence Equation (3.2).

On the other hand, the poset $P_{11}$ of Example 3.4 has diagram

and Möbius function $\mu_{P_{11}}$ given by

Let $\pi: \tilde{C}_\alpha \to C_\alpha$ be the projection map that forgets barring on all symbols. For $\sigma \in C_\alpha$, let

$$\mu_{C_\alpha}^\prime (\sigma) := \sum_{\tilde{\sigma} \in \pi^{-1}(\sigma)} (-1)^{\text{Bar}(\tilde{\sigma})}.$$

We introduce the following two polynomials. Our primary goal in this section is to prove that they coincide.

**Definition 3.7.** For any tuple $c = (c_1, \ldots, c_n)$, let $y^c := y_1^{c_1} \cdots y_n^{c_n}$ where the $y_i$ are indeterminates. For any composition $\alpha$, let

$$\overline{M}_\alpha'(y_1, \ldots, y_n) = \sum_{\sigma \in C_\alpha} \mu_{C_\alpha}^\prime (\sigma) y^\sigma$$

and

$$\overline{M}_\alpha(y_1, \ldots, y_n) = \sum_{\sigma \in P_\alpha} \mu_{P_\alpha} (\sigma) y^\sigma.$$

We call $\overline{M}_\alpha(y_1, \ldots, y_n)$ the *quasisymmetric monomial glide* for the composition $\alpha$. 
The following theorem is our main result of this section.

**Theorem 3.8.** For any composition \( \alpha \), we have

\[
\mathcal{M}_\alpha(y_1, \ldots, y_n) = \mathcal{M}_\alpha^\tau(y_1, \ldots, y_n).
\]

Furthermore, these polynomials are both quasisymmetric in the variables \( y_1, \ldots, y_n \).

**Example 3.9.** Let \( \alpha = (1, 3) \). Then by comparing Example 3.1, Example 3.4, and Example 3.6, we find that

\[
\mathcal{M}_{(1,3)}(y_1, y_2, y_3, y_4) = \mathcal{M}_{(1,3)}^\tau(y_1, y_2, y_3, y_4)
\]

\[
= y^{0013} + y^{0103} + y^{0130} + y^{1003} + y^{1030} + y^{1300} - y^{0113} - y^{1103} - y^{1013} - y^{1113} - y^{1130} - y^{1310} - y^{1303} - y^{1033} + y^{1133} + y^{1333}.
\]

Note that these polynomials are equal and quasisymmetric as claimed by Theorem 3.8. \( \Box \)

By Lemma 3.5, we can prove Theorem 3.8 by showing that \( \mu_{\mathcal{P}_\alpha}(\sigma) = \mu_{\mathcal{C}_\alpha}^\tau(\sigma) \) for all \( \sigma \in \mathcal{C}_\alpha \), and that \( \mu_{\mathcal{P}_\alpha}(\sigma) = 0 \) for all \( \sigma \in \mathcal{P}_\alpha \setminus \mathcal{C}_\alpha \). These tasks are accomplished in the next two subsections.

First, we need the following lemma, characterizing \( \mathcal{C}_\alpha \) as a subset of \( \mathcal{P}_\alpha \).

**Lemma 3.10.** Let \( \sigma \in \mathcal{P}_\alpha \). Then \( \sigma \in \mathcal{C}_\alpha \) if and only if \( \sigma \) is of the form \( \sigma = \sigma_1 \bar{\cdots} \bar{\sigma}_k \) where each \( \sigma_i \) is a string containing at least \( N_i \) instances of \( a_i \) with all other symbols being 0.

**Proof.** By Lemma 3.5, we have \( \mathcal{C}_\alpha \subseteq \mathcal{P}_\alpha \).

Suppose \( \sigma \in \mathcal{C}_\alpha \) and consider some \( \bar{\sigma} \in \pi^{-1}(\sigma) \). Then by construction, we must have \( \bar{\sigma} = \bar{\sigma}_1 \bar{\cdots} \bar{\sigma}_k \) where each \( \bar{\sigma}_i \) is obtained from the length-\( N_i \) string \( (a_i, \ldots, a_i) \) by inserting additional 0’s or additional barred symbols \( \bar{a}_i \) (the latter not in first position). Thus, \( \bar{\sigma}_i \) must have \( \ell_i \geq N_i \) elements from \( \{a_i, \bar{a}_i\} \) (for some \( \ell_i \)), exactly \( \ell_i - N_i \) of them must be barred, and the first nonzero entry of \( \bar{\sigma}_i \) must be unbarred. In particular, \( \sigma \) has the form \( \sigma = \sigma_1 \bar{\cdots} \bar{\sigma}_k \) where each entry of each \( \sigma_i \) is either 0 or \( a_i \), and \( \sigma \) has exactly \( \ell_i \geq N_i \) entries equal to \( a_i \). Conversely, it is straightforward to see that every such \( \sigma \) is in \( \mathcal{C}_\alpha \). \( \Box \)

### 3.2. \( \mu_{\mathcal{P}_\alpha} \) vanishes away from \( \mathcal{C}_\alpha \)

In this subsection, we establish the following proposition.

**Proposition 3.11.** If \( \sigma \in \mathcal{P}_\alpha \setminus \mathcal{C}_\alpha \), then \( \mu_{\mathcal{P}_\alpha}(\sigma) = 0 \).

The reader may enjoy confirming directly that Proposition 3.11 holds in the case of Example 3.1 and Example 3.6. We first handle the following special case, which will be the key to proving the result in general.

**Lemma 3.12.** Suppose that \( a_1, \ldots, a_k \) are pairwise distinct. If \( \sigma \in \mathcal{P}_\alpha \setminus \mathcal{C}_\alpha \), then \( \mu_{\mathcal{P}_\alpha}(\sigma) = 0 \).

**Proof.** Fix \( \sigma \in \mathcal{P}_\alpha \setminus \mathcal{C}_\alpha \). The set \( S := S_\alpha \) consists of the atoms of \( \widehat{\mathcal{P}}_\alpha \), i.e., the minimal elements of \( \mathcal{P}_\alpha \). Since \( \widehat{\mathcal{P}}_\alpha \) is a lattice by Lemma 3.3, Rota’s Crosscut Theorem [Rot64] tells us that if

\[
\mathcal{R} = \{ R \subseteq S : \bigvee R = \sigma \},
\]

then

\[
-\mu_{\mathcal{P}_\alpha}(\sigma) = \mu_{\widehat{\mathcal{P}}_\alpha}(\sigma) = \sum_{R \in \mathcal{R}} (-1)^{|R|}.
\]

To prove \( \mu_{\mathcal{P}_\alpha}(\sigma) = 0 \), it suffices to construct a fixed-point-free involution

\[
\iota : \mathcal{R} \to \mathcal{R}
\]

with the property that \(|\iota(R)| = |R| \pm 1\) for all \( R \in \mathcal{R} \). Indeed, then we have

\[
\mu_{\mathcal{P}_\alpha}(\sigma) = \sum_{R \in \mathcal{R}} (-1)^{|\iota(R)|} = -\mu_{\mathcal{P}_\alpha}(\sigma),
\]

so that \( \mu_{\mathcal{P}_\alpha}(\sigma) = 0 \).

Suppose the zero entries of \( \sigma \) occur in positions \( z_1 < \cdots < z_M \). Note that if \( \sigma = \bigvee R \), then every \( \tau \in R \) has \( \tau \leq \sigma \); in particular, \( \tau \) must also have zeros in positions \( z_1 < \cdots < z_M \) (and potentially additional
positions). Therefore, restricting attention to the positions \([n] \setminus \{z_1, \ldots, z_M\}\), we may suppose \(\sigma\) has no 0 entries at all. We may therefore write

\[
\sigma = (a_{i_1}, \ldots, a_{i_\ell_1}, a_{i_2}, \ldots, a_{i_\ell_2}, \ldots, a_{i_K}, \ldots, a_{i_K})
\]

where \(i_j \neq i_{j+1}\). Since the first entry of every atom \(\tau \in S\) is either 0 or \(a_1\), and since \(\sigma\) is assumed to have only non-zero entries, we see the first entry of \(\sigma\) is \(a_1\). Similarly, the final entry of \(\sigma\) must be \(a_k\). That is, we have

\[
i_1 = 1 \quad \text{and} \quad i_K = k.
\]

Every atom \(\tau \in S\) is determined by the positions of its non-zero entries. We record these positions (in increasing order) in a vector \(p(\tau) \in \mathbb{Z}_{>0}^k\) and refer to \(p(\tau)\) as the position vector of \(\tau\). For example, if \(\tau = (0, 1, 1, 0, 3, 0, 3)\), then \(p(\tau) = (2, 3, 5, 7)\).

**Case 1**: There exist \(J\) and \(L\) with \(J < L\) and \(i_L < i_J\): Note that if any such \(J\) and \(L\) exist, we may make a canonical choice of \(J\) and \(L\) (depending only on \(\sigma\)) by requiring that \((J, L)\) is the smallest possible in lexicographical order.

Let \(S_- \subseteq \mathcal{S}\) be the set of atoms \(\tau\) lying below \(\sigma\) with an \(a_{i_j}\) occurring in the same position of \(\tau\) as an \(a_{i_j}\)-entry of \(\sigma\); that is,

\[
S_- = \{ \tau \in \mathcal{S} : \tau \leq \sigma \text{ and there exists } h \in [1 + \sum_{g=1}^{J-1} \ell_g, \sum_{g=1}^{J} \ell_g] \text{ with } \tau(h) = \sigma(h) = a_{i_j} \}.
\]

Similarly, let \(S_+ \subseteq \mathcal{S}\) be the set of atoms \(\tau\) lying under \(\sigma\) where an \(a_{i_L}\) occurs in the same position as an \(a_{i_L}\)-entry of \(\sigma\); that is,

\[
S_+ = \{ \tau \in \mathcal{S} : \tau \leq \sigma \text{ and there exists } h \in [1 + \sum_{g=1}^{L-1} \ell_g, \sum_{g=1}^{L} \ell_g] \text{ with } \tau(h) = \sigma(h) = a_{i_L} \}.
\]

Note that \(S_{\pm}\) are also canonically associated to \(\sigma\).

Let \(R \in \mathcal{R}\). Since the \(a_i\) are pairwise distinct, we have \(R \cap S_- \neq \emptyset\) and \(R \cap S_+ \neq \emptyset\). Let \(u_- \in R \cap S_-\) be the atom whose position vector \(p(u_-)\) is least in lexicographical order. Similarly, let \(u_+ \in R \cap S_+\) be the atom whose position vector \(p(u_+)\) is greatest in lexicographical order.

For example, suppose that \(\alpha = (1, 3)\) and \(\sigma = 113311333\). Then \(a_1 = 1\) and \(a_2 = 3\). Moreover,

\[
i_1 = 1, i_2 = 2, i_3 = 1, \quad \text{and} \quad i_4 = 2.
\]

Hence, we have \(i_3 < i_2\) and can only choose \(J = 2\) and \(L = 3\), so \(a_{i_j} = 3\) and \(a_{i_L} = 1\). An example of \(R \in \mathcal{R}\) is

\[
R = \{103000000, 01030000, 00001030, 00000103, 00000013\}.
\]

In this case, we have

\[
R \cap S_- = \{ \tau \in R : \text{there exists } h \in [3, 4] \text{ with } \tau(h) = \sigma(h) = 3 \} = \{10300000, 01030000\}
\]

and

\[
R \cap S_+ = \{ \tau \in R : \text{there exists } h \in [5, 6] \text{ with } \tau(h) = \sigma(h) = 1 \} = \{00001030, 00000103\}.
\]

Therefore, \(u_- = 103000000\) and \(u_+ = 000001030\).

Notice that \(u_{\pm}\) are canonically associated to \(R\) and that \(u_- \neq u_+\) since the locations of their \(a_{i_L}\)-entries differ. Indeed, \(u_+\) has an \(a_{i_L}\) appearing in the interval \([1 + \sum_{g=1}^{L-1} \ell_g, \sum_{g=1}^{L} \ell_g]\); in contrast, since \(u_-\) is an atom, all of its \(a_{i_L}\) appear left of every \(a_{i_j}\) and in particular left of the one in the interval \([1 + \sum_{g=1}^{J-1} \ell_g, \sum_{g=1}^{J} \ell_g]\).

Let \(u = u_R \in \mathcal{S}\) be the atom defined by "splicing together" \(u_-\) and \(u_+\) as follows. For \(i \leq i_L\), \(u\) has the values \(a_i\) in the same positions as \(u_-\); for \(i > i_L\), \(u\) has the values \(a_i\) in the same positions as \(u_+\). Note that \(u\) is well-defined, since in each position where \(u_-\) has a value \(a_i\) with \(i \leq i_L\) is strictly left of each position where \(u_+\) has a value \(a_j\) with \(j > i_L\); moreover, \(u\) is clearly an atom. Also observe that \(u \neq u_-\) since their \(a_{i_j}\)-positions differ, and \(u \neq u_+\) since their \(a_{i_L}\)-positions differ (e.g., in the running example above, we have \(u_R = 10000030\)).
Since \( u \) is canonically associated to \( R \), we may define

\[
\iota(R) = \begin{cases} 
R \cup \{u\}, & u \notin R; \\
R \setminus u, & u \in R,
\end{cases}
\]

a sort of \textit{toggle} in the sense of [Str18, Definition 2.1]. Furthermore, since

\[
u \vee u_- \vee u_+ = u_- \vee u_+,
\]

we see that

\[
\bigvee \iota(R) = \bigvee R = \sigma.
\]

In other words, \( \iota(R) \in \mathcal{R} \). Lastly, since \( \iota(R) \) differs from \( R \) exactly by the element \( u \), and \( u \) agrees with \( u_-(\text{respectively } u_+) \) for \( i \leq i_L \) (respectively \( i > i_L \)), we see \( u_R = u_{i(R)} \); hence, \( \iota(\iota(R)) = R \) and \( \iota \) is an involution. By construction, \( |\iota(R)| = |R| \pm 1 \).

\textbf{(Case 2: We have } \textit{i}_1 < \textit{i}_2 < \cdots < \textit{i}_K \textbf{):} Since \( \sigma \in \mathcal{P}_\alpha \setminus \mathcal{C}_\alpha \), Lemma 3.10 shows that there exists \( J \in [k - 1] \) such that

\[
i_1 = 1, \ldots , i_J = J, \quad \text{and} \quad m := i_{J+1} > J + 1.
\]

We follow a similar strategy as in Case 1. Let \( \mathcal{S}_- \) be the set of atoms \( \tau \leq \sigma \) such that \( \tau(\ell_1 + \cdots + \ell_J + 1) = a_m \).

In other words, these are the atoms below \( \sigma \) where the value \( a_m \) occurs in the same position as the first \( a_m \)-entry of \( \sigma \). Similarly, let \( \mathcal{S}_+ \) be the atoms lying below \( \sigma \) where \( a_J \) occurs in the same position as the last \( a_J \) of \( \sigma \).

For example, suppose that \( \alpha = (5, 4, 7, 1) \) and \( \sigma = 55771 \). Then, \( J = 1 \) and \( m = 3 \). We have

\[
\mathcal{S}_- = \{54701, 54710\} \quad \text{and} \quad \mathcal{S}_+ = \{05471\}.
\]

Let \( R \in \mathcal{R} \). Again, we have \( R \cap \mathcal{S}_- \neq \emptyset \) and \( R \cap \mathcal{S}_+ \neq \emptyset \). As in Case 1, let \( u_- \in R \cap \mathcal{S}_- \) be the atom whose position vector \( p(u_-) \) is least in lexicographical order and let \( u_+ \in R \cap \mathcal{S}_+ \) be the atom whose position vector \( p(u_+) \) is greatest in lexicographical order. Since \( J + 1 < m \), we see the positions of the \( a_{J+1} \)-entries differ in \( u_+ \), and so \( u_- \neq u_+ \).

Continuing our example, a choice of \( R \in \mathcal{R} \) is

\[
R = \{05471, 54701\}.
\]

With this choice, we have

\[
R \cap \mathcal{S}_- = \{54701\} \quad \text{and} \quad R \cap \mathcal{S}_+ = \{05471\}.
\]

Then, \( u_- = 54701 \) and \( u_+ = 05471 \).

Let \( u := u_R \) be the atom defined as follows. For \( i \leq J \), we let \( u \) have the values \( a_i \) in the same positions as does \( u_- \); for \( i > J \), we let \( u \) have the values \( a_i \) in the same positions as does \( u_+ \). (In our running example, \( u = 50471 \).) Then \( u \neq u_- \) since their \( a_{J+1} \)-positions differ, and \( u \neq u_+ \) since their \( a_J \)-positions differ. We can therefore again define \( \iota(R) \) by Equation (3.3). Once again, \( u \vee u_- \vee u_+ = u_- \vee u_+ \), and so \( \iota(\iota(R)) = R \).

We also have \( u_R = u_{\iota(R)} \), so \( \iota(\iota(R)) = R \). Clearly, \( |\iota(R)| = |R| \pm 1 \). \( \square \)

We will prove Proposition 3.11 by reducing to the special case covered by Lemma 3.12. First, let us rewrite Equation (3.1) in different notation. Let

\[
\alpha = (\alpha_1, \ldots , \alpha_N);
\]

here, we do not assume the \( \alpha_i \) are pairwise distinct, and we even allow the possibility that \( \alpha_i = \alpha_{i+1} \). Let \( \omega \in S_N \) be the unique permutation with the properties that

\[
\alpha_{\omega(1)} \leq \alpha_{\omega(2)} \leq \cdots \leq \alpha_{\omega(N)}
\]

and if \( \alpha_{\omega(i)} = \alpha_{\omega(i+1)} \), then \( \omega(i) < \omega(i + 1) \); that is to say, \( \omega \) is the shortest permutation that sorts \( \alpha \). Let

\[
\beta = (\beta_1, \ldots , \beta_N), \quad \text{where} \quad \beta_i = \omega^{-1}(i).
\]
For example, if \( \alpha = (2, 1, 3, 1) \), then we have \( \omega = 2413 \), where we write a permutation \( \omega \in S_N \) in **one-line notation** \( w = w(1)w(2) \ldots w(N) \), for then \((\alpha_\omega(1), \alpha_\omega(2), \alpha_\omega(3), \alpha_\omega(4)) = (1, 1, 2, 3) \). Therefore, \( \omega^{-1} = 3142 \), and so \( \beta = (\beta_1, \beta_2, \beta_3, \beta_4) = (3, 1, 4, 2) \).

Then we have two posets \( \mathcal{P}_\alpha \) and \( \mathcal{P}_\beta \) of strings of length \( n \). Recall that \( S_\alpha \) (respectively \( S_\beta \)) is the set of atoms of \( \mathcal{P}_\alpha \) (respectively \( \mathcal{P}_\beta \)). Since an atom is determined by the positions of its nonzero entries, we have mutually-inverse bijections

\[
S_\alpha \overset{\text{st}}{\rightarrow} S_\beta
\]

where \( \text{st} \) replaces the \( i \)th nonzero entry with \( \beta_i \), and \( \text{sst} \) replaces the \( i \)th nonzero entry with \( \alpha_i \). We refer to \( \text{st} \) as **standardization** and \( \text{sst} \) as **semistandardization**, since they are analogous to the classical (semi)standardization bijections on Young tableaux (see, e.g., [Sta99, Kir01, PY17b] for discussion of these maps in that setting). These maps induce bijections on the power sets of \( S_\alpha \) and \( S_\beta \), which we continue to call the standardization and semistandardization maps. We use superscripts to denote these functions, e.g., for \( \tau \in S_\alpha \) and \( T \subseteq S_\alpha \), we write \( \tau^{\text{st}} \in S_\beta \) and \( T^{\text{st}} := \{ \theta^{\text{st}} : \theta \in R \} \subseteq S_\beta \) for their standardizations.

**Lemma 3.13.** If \( T \subseteq S_\beta \) and \( \bigvee T \in C_\beta \), then \( \bigvee (T^{\text{st}}) \in C_\alpha \).

**Proof.** Let \( \tau := \bigvee T \). Suppose the zero entries of \( \tau \) occur in positions \( z_1 < \cdots < z_M \). Then every atom in \( T \) has zeros in positions \( z_1 < \cdots < z_M \) (and potentially additional zeros), which implies that every atom in \( T^{\text{st}} \) has zeros in positions \( z_1 < \cdots < z_M \) as well. Therefore, restricting attention to the positions \( [n] \setminus \{z_1, \ldots, z_M\} \), we may suppose \( \tau \) has no zero entries at all.

By Lemma 3.10, we then know

\[
\tau = (\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_2, \ldots, \beta_N, \ldots, \beta_N)
\]

with each \( \ell_i > 0 \). This implies that in the positions of the interval \([1 + \sum_{i=1}^{j-1} \ell_i, \sum_{i=1}^{j} \ell_i]\), the only allowable entries for atoms in \( T \) are \( 1, 2, \ldots, \beta_j \); moreover at every position in this range, there exists some atom \( \theta \in T \) whose value at this position is \( \beta_j \). By semistandardization, we learn that in positions \([1 + \sum_{i=1}^{j-1} \ell_i, \sum_{i=1}^{j} \ell_i]\), the only allowable entries for atoms in \( T^{\text{st}} \) are \( \alpha_{\omega(1)}, \alpha_{\omega(2)}, \ldots, \alpha_{\omega(\omega^{-1}(j))} = \alpha_j \); moreover at every position in this range, there exists some atom \( \theta^{\text{st}} \in T^{\text{st}} \) whose value at this position is \( \alpha_j \). Since

\[
\alpha_{\omega(1)} \leq \alpha_{\omega(2)} \leq \cdots \leq \alpha_j,
\]

it follows that

\[
\bigvee (T^{\text{st}}) = (\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_N, \ldots, \alpha_N)
\]

which is in \( C_\alpha \) by Lemma 3.10.

We now turn finally to proving Proposition 3.11.

**Proof of Proposition 3.11.** Fix \( \sigma \in \mathcal{P}_\alpha \setminus C_\alpha \). Let \( \mathcal{R} \) be the collection of all subsets \( R \subseteq S_\alpha \) such that \( \bigvee R = \sigma \). As in the proof of Lemma 3.12, it suffices to construct a fixed-point-free involution

\[
j : \mathcal{R} \to \mathcal{R}
\]

such that \( |j(R)| = |R| \pm 1 \) for all \( R \in \mathcal{R} \). Note that if \( \bigvee (R^{\text{st}}) \in C_\beta \), then Lemma 3.13 shows \( \bigvee ((R^{\text{st}})^{\text{st}}) = \bigvee R = \sigma \in C_\alpha \), contradicting the hypothesis on \( \sigma \). Therefore

\[
\tau_R := \bigvee (R^{\text{st}}) \in \mathcal{P}_\beta \setminus C_\beta.
\]

For each \( R \in \mathcal{R} \), let \( T_R = \{ T \subseteq S_\beta \mid \bigvee T = \tau_R \} \) and let

\[
i_R : T_R \to T_R
\]

denote the involution constructed in the proof of Lemma 3.12. We define

\[
j(R) := (i_R(R^{\text{st}}))^{\text{st}}.
\]

We first show \( j(R) \in \mathcal{R} \), i.e., \( \bigvee j(R) = \bigvee R \). For this, we recall from the proof of Lemma 3.12 that associated to \( R^{\text{st}} \), we have two distinct atoms \( u_{\pm} \in R^{\text{st}} \). We then form a third atom \( u \) by splicing together
partial data from \( u_- \) and \( u_+ \); specifically, there is an index \( I \) such that \( u \) agrees with \( u_- \) at all positions \( i \leq I \) and agrees with \( u_+ \) at all positions \( i > I \). Then \( \iota_R(R^\text{st}) = R^\text{st} \cup \{ u \} \) if \( u \notin R^\text{st} \), and \( \iota_R(R^\text{st}) = R^\text{st} \setminus \{ u \} \) if \( u \in R^\text{st} \). Therefore, we see that \( j(R) = R \cup \{ u^\text{st} \} \) if \( u^\text{st} \notin R \), and \( j(R) = R \setminus \{ u^\text{st} \} \) if \( u^\text{st} \in R \). By the construction of \( u \), we see \( u^\text{st} \) agrees with \( u^\text{st} \) at all positions \( i \leq I \) and agrees with \( u_+ \) at all positions \( i > I \). So, \( u^\text{st} \cup u^\text{st} \cup u^\text{st} = u^\text{st} \cup u^\text{st} \), which implies \( \bigvee j(R) = \bigvee R \).

Next, since \( \tau \) and \( \tau \) are mutually-inverse bijections of atoms, we have

\[
|j(R)| = |(\tau_R(R^\text{st}))^\text{st}| = |j(R)| = |R^\text{st}| \pm 1 = |R| \pm 1.
\]

Finally, we must show \( j \) is an involution. Directly from the definitions, we see

\[
\mu_R(R^\text{st})^\text{st} = j(R).
\]

where \( R' = (\iota_R(R^\text{st}))^\text{st} \). Since

\[
\tau' := \bigvee (R')^\text{st} = \bigvee \iota_R(R^\text{st}) = \bigvee R^\text{st} =: \tau_R,
\]

we see that \( \tau = \tau' \). Furthermore, since \( \iota_R \) (respectively \( \iota_R \)) depends only on \( \tau_R \) (respectively \( \tau_R' \)), we see

\[
\mu_R(R^\text{st})^\text{st} = (\iota_R(R^\text{st}))^\text{st} = (R^\text{st})^\text{st} = R,
\]

as desired. \( \square \)

3.3. Proof of Theorem 3.8. Having shown in Proposition 3.11 that \( \mu_{R^\alpha} \) vanishes away from \( C^\alpha \), it remains to compute \( \mu_{R^\alpha} \) on the subset \( C^\alpha \).

Lemma 3.14. If \( \sigma \in C^\alpha \), then \( \mu_{R^\alpha}(\sigma) = \mu_{C^\alpha}(\sigma) \). As in Lemma 3.10, let \( \sigma = \sigma_1 \cdots \sigma_k \) where each \( \sigma_i \) is a string containing at least \( N_i \) instances of \( a_i \) with all other symbols being 0. If \( \sigma_i \) has exactly \( \ell_i \) entries equal to \( a_i \), then

\[
\mu_{R^\alpha}(\sigma) = (-1)^{\ell_1 + \cdots + \ell_k - N_i} \prod_{i=1}^{k} \left( \frac{\ell_i - 1}{N_i - 1} \right) = \mu_{C^\alpha}(\sigma).
\]

Proof. As in Lemma 3.10, if \( \sigma \in C^\alpha \) and \( \bar{\sigma} \in \pi^{-1}(\sigma) \), then \( \bar{\sigma} \) has the form \( \bar{\sigma} = \bar{\sigma}_1 \cdots \bar{\sigma}_k \), where each \( \bar{\sigma}_i \) has \( \ell_i \geq N_i \) elements from \( \{a_i, \bar{a}_i\} \) and exactly \( \ell_i - N_i \) must be barred. Moreover, the first nonzero entry of \( \bar{\sigma}_i \) must be unbarred, and all subsets with size \( \ell_i - N_i \) of the remaining \( \ell_i \) nonzero entries can occur barred in \( \bar{\sigma}_i \) for some \( \bar{\sigma} \in \pi^{-1}(\sigma) \). Thus, there are \( \left( \frac{\ell_i - 1}{N_i - 1} \right) \) possibilities for \( \bar{\sigma}_i \) and each of these has \( \text{Bar}(\bar{\sigma}_i) = \ell_i - N_i \). Thus,

\[
\mu_{C^\alpha}(\sigma) = (-1)^{\ell_1 + \cdots + \ell_k - N_i} \prod_{i=1}^{k} \left( \frac{\ell_i - 1}{N_i - 1} \right).
\]

It remains to prove that \( \mu_{R^\alpha}(\sigma) \) is also computed by the same product of binomial coefficients. Extend the domain of the function \( \mu_{C^\alpha} \) from \( C^\alpha \) to \( \alpha \beta \) by declaring that \( \mu_{C^\alpha} \) is zero on \( \alpha \beta \setminus C^\alpha \). We will show \( \mu_{R^\alpha}(\theta) = \mu_{C^\alpha}(\theta) \) for all \( \theta \in \alpha \beta \); by Proposition 3.11, we know this is the case if \( \theta \in \alpha \setminus C^\alpha \). To prove \( \mu_{R^\alpha}(\theta) = \mu_{C^\alpha}(\theta) \) for all \( \theta \), it suffices to show by induction that every \( \theta \in \alpha \beta \) satisfies

\[
\sum_{\tau \leq \theta} \mu'_{C^\alpha}(\tau) = 1,
\]

so that \( \mu_{C^\alpha} \) and \( \mu_{R^\alpha} \) satisfy the same recurrence and initial conditions.

First, suppose the zero entries of \( \theta \) occur in positions \( z_1 < \cdots < z_M \). Then note that if \( \tau \leq \theta \), then \( \tau \) must also have zeros in positions \( z_1 < \cdots < z_M \). Restricting attention to the positions \( \{a_i \} \setminus \{ z_1, \ldots, z_M \} \), we may therefore suppose \( \theta \) has no zero entries at all. Next, in order to prove that Equation (3.4) holds, it suffices to prove that

\[
\sum_{\tau \leq \theta} \mu'_{C^\alpha}(\tau) = 1,
\]

since for \( \tau \notin C^\alpha \), we know \( \mu'_{C^\alpha}(\tau) = 0 \) by Proposition 3.11.

To handle the base case, assume \( \theta \in \mathcal{S} \) is minimal. Then \( \theta \) is obtained from \( \alpha \) by inserting \( n - N \) zero entries. Therefore, \( \theta \) has precisely \( \ell_i = N_i \) entries equal to \( a_i \), so

\[
\mu'_{C^\alpha}(\theta) = (-1)^{N - N} \prod_{i=1}^{k} \left( \frac{N_i - 1}{N_i - 1} \right) = 1 = \mu_{R^\alpha}(\theta).
\]

14
We next handle the induction step. Suppose that \( \theta \) has \( \ell_i \) entries equal to \( a_i \). Then any \( \tau \leq \theta \) with \( \tau \in C_\alpha \) is obtained from \( \theta \) by picking some \( j_i \) with \( N_i \leq j_i \leq \ell_i \), choosing \( j_i \) of the \( a_i \)-entries, and replacing the \( \ell_i - j_i \) remaining \( a_i \)-entries with zeros. Thus, proving Equation (3.4) amounts to showing

\[
\sum_{N_i \leq j_i \leq \ell_i} (-1)^{j_1 + \cdots + j_k - N} \prod_{i=1}^{k} \binom{j_i - 1}{N_i - 1} = 1.
\]

Since

\[
\sum_{N_i \leq j_i \leq \ell_i} (-1)^{j_1 + \cdots + j_k - N} \prod_{i=1}^{k} \binom{j_i - 1}{N_i - 1} = \prod_{i=1}^{k} \sum_{N_i \leq j_i \leq \ell_i} (-1)^{j_i - N_i} \binom{j_i - 1}{j_i},
\]

it suffices to prove Equation (3.5) in the case where \( k = 1 \). That is to say, we will prove the identity

\[
\sum_{j=1}^{\ell} (-1)^{j-N} \binom{j-1}{N-1} = 1
\]

by induction on \( \ell \). When \( \ell = N \), we see that Equation (3.5) holds. For \( \ell > N \), we use the Pascal's triangle identity \( \binom{j}{k} = \binom{j-1}{k} + \binom{j-1}{k-1} \) to rewrite the left side of Equation (3.6) as

\[
\sum_{j=1}^{\ell} (-1)^{j-N} \binom{j-1}{N-1} = \sum_{j=1}^{\ell-1} (-1)^{j-N} \binom{j-1}{N-1} + (-1)^{\ell-N} \binom{\ell-1}{N-1}.
\]

By the inductive hypothesis, the second sum of (3.7) equals 1, so it remains to show that the first sum of (3.7) vanishes. For this, we rewrite it as

\[
\sum_{j=1}^{\ell} (-1)^{j-N} \binom{j-1}{N-1} = \binom{\ell-1}{N-1} \sum_{j=1}^{\ell} (-1)^{j-N} \binom{j-N}{j-1} = 0.
\]

We now prove the main theorem of this section.

**Proof of Theorem 3.8.** Proposition 3.11 shows that if \( \sigma \in P_\alpha \setminus C_\alpha \), then \( \mu_{P_\alpha}(\sigma) = 0 \). On the other hand, by Lemma 3.14, if \( \sigma \in C_\alpha \), then \( \mu_{C_\alpha}(\sigma) = \mu_{P_\alpha}(\sigma) \). So, by definition, we then have

\[
\overline{M}_\alpha(y_1, \ldots, y_n) = \overline{M}_\alpha(y_1, \ldots, y_n),
\]

as desired.

Lastly, suppose \( \sigma, \sigma' \in P_\alpha \) are such that \( \sigma^+ = (\sigma')^+ \), so that they differ only in the location of zeros. Then Lemma 3.10 shows that \( \sigma \in C_\alpha \) if and only if \( \sigma' \in C_\alpha \), and moreover, that \( |\pi^{-1}(\sigma)| = |\pi^{-1}(\sigma')| \). Therefore, the coefficients in \( \overline{M}_\alpha(y_1, \ldots, y_n) \) of \( y^\sigma \) and \( y^{\sigma'} \) are equal. It follows then that \( \overline{M}_\alpha(y_1, \ldots, y_n) \) is a quasisymmetric polynomial. Since \( \overline{M}_\alpha(y_1, \ldots, y_n) = \overline{M}_\alpha(y_1, \ldots, y_n) \), the polynomial \( \overline{M}_\alpha(y_1, \ldots, y_n) \) is also quasisymmetric. \( \square \)

**Remark 3.15.** T. Lam and P. Pylyavskyy introduced inhomogeneous deformations of monomial quasisymmetric functions in [LP07, Remark 5.14]. They refer to these power series as *multinomial quasisymmetric functions* \( \tilde{M}_\alpha \) and consider them to be “\( K \)-theoretic” through analogy with certain other inhomogeneous power series related to the \( K \)-theory of Grassmannians. To our knowledge, multinomial quasisymmetric functions have never been studied outside of that remark; in particular, no explicit connection between multinomial quasisymmetric functions and \( K \)-theoretic geometry has been obtained.

The multinomial quasisymmetric functions \( \tilde{M}_\alpha \) are similar to, but different from our monomial quasisymmetric glides \( \overline{M}_\alpha \). (There appears to be a typo in the definition of [LP07, Remark 5.14]: assuming that the lowest-degree piece of \( \tilde{M}_\alpha \) should be the monomial quasisymmetric function, as the authors suggest, the “\( \subset \)” in their definition should be “\( \supset \)” otherwise, the power series \( \overline{M}_\alpha \) and \( \tilde{M}_\alpha \) are even more different.) For example, suppose that \( \alpha = (1, 3) \). Then, truncating to four variables, we have

\[
\overline{M}_{(1,3)}(y_1, y_2, y_3, y_4) = y_{0013} + y_{0103} + y_{0130} + y_{1003} + y_{1030} + y_{1300} - y_{0113} - y_{1103} - y_{1013}
\]

\[
- y_{1130} + y_{1113} - y_{1330} - y_{0133} - y_{1033} - y_{1303} + y_{1133} + y_{1333}.
\]
(as in Example 3.9), while
\[
\tilde{M}_{(1,3)}(y_1, y_2; y_3, y_4) = y^{0113} + y^{0103} + y^{0130} + y^{1003} + y^{1030} + y^{1300} - y^{1310} - y^{1301} - y^{1220} - y^{1202} - 2y^{1130} - 2y^{1103} - y^{1031} - y^{1022} - 2y^{1013} - y^{0131} - y^{0122} - 2y^{0113} + y^{1311} + y^{1221} + y^{1212} + 2y^{1131} + 2y^{1122} + 3y^{1113}.
\]

**Remark 3.16.** Following [AS17], there has been a flurry of recent work on lifting important bases of \(QSym_n\) to the full polynomial ring \(\mathbb{Z}[y_1, \ldots, y_n]\) (see, e.g., [AS18, PS19, PS20, MS21]).

In this vein, we note that the quasisymmetric monomial glides \(\tilde{M}_a(y_1, \ldots, y_n)\) have a natural such lift. Specifically, for a weak composition \(a = (a_1, \ldots, a_n)\), one may define the monomial glide \(\tilde{M}_a(y_1, \ldots, y_n)\) to be the generating function for strings obtained from \(a\) via the local moves (M.1) and (M.2). That is, letting \(\mathcal{C}_a\) denote the set of such obtainable strings, we define
\[
\tilde{M}_a(y_1, \ldots, y_n) = \sum_{\sigma \in \mathcal{C}_a} \mu'_{\sigma}(\sigma)y^{\sigma}.
\]

Note that this definition recovers the quasisymmetric monomial glides, as we have
\[
\tilde{M}_a(y_1, \ldots, y_n) = \tilde{M}_{(y^{a-\ell(a)})-\alpha}(y_1, \ldots, y_n);
\]
moreover, the monomial glides that are quasisymmetric are exactly those of this form. We do not pursue these generalizations further in this paper.

4. The \(K\)-theory of the James Reduced Product

If \(X\) is a complex algebraic variety, we let \(K^0(X)\) be its Grothendieck group of algebraic vector bundles and \(K_0(X)\) be its Grothendieck group of coherent sheaves. When \(Y\) is a CW complex, we let \(K(Y)\) denote the representable topological complex \(K\)-theory, that is to say homotopy classes of maps from \(Y\) to \(BU \times \mathbb{Z}\). For particularly nice spaces, these three groups can all be identified with each other; however, in general, they are all distinct and we will need to consider some slightly subtle interactions among them.

**Remark 4.1.** Another definition of “topological \(K\)-theory” one might consider here is \(K_{top}(Y)\), the Grothendieck group of topological vector bundles. For compact CW complexes, we have a natural isomorphism \(K_{top}(Y) \cong K(Y)\). However, these two notions may differ even for simple examples of infinite CW complexes, see e.g. [JO96]; in fact, the functor \(K_{top}\) does not even satisfy Bott periodicity. One of the key properties we use about \(K\) is that if \(Y\) is the union of an expanding sequence of compact subspaces \(Y_n\), then the natural map
\[
K(Y) \cong \lim K(Y_n)
\]
is an isomorphism, where the limit is taken in the category of commutative rings; see [BM68, Corollary 1]. Furthermore, the limit is independent of the choice of subspaces \(Y_n\) by the remark after [AS69, Proposition 4.1].

Throughout, we denote the integral cohomology of a CW complex \(Y\) by \(H^*(Y)\), and for any ring \(R\), we use \(R_\mathbb{Q}\) to denote \(R \otimes \mathbb{Z} \mathbb{Q}\). For a finite CW complex, we have a Chern character map
\[
ch: K(Y) \to H^*(Y)_\mathbb{Q}.
\]
Similarly, if \(X\) is a complex variety, we have a Chern character map
\[
ch: K^0(X) \to A^*(X)_\mathbb{Q},
\]
where \(A^*\) is the Chow ring. When \(X\) is a complex projective variety, there is a natural map
\[
K^0(X) \to K(X)
\]
taking an algebraic vector bundle to its underlying topological vector bundle.
4.1. Characterizing $K(J\mathbb{C}P^\infty) \otimes_\mathbb{Z} \mathbb{Q}$ as quasisymmetric functions. In what follows we will need to compute $K(J\mathbb{C}P^\infty)$ in terms of the $K(J_n\mathbb{C}P^m)$. Unfortunately, (4.1) only allows one to compute $K$-theory as an inverse limit of a sequence of CW complexes, as opposed to a doubly infinite sequence. The following result shows that, in our case of interest, (4.1) remains true for doubly infinite sequences as well.

**Lemma 4.2.** Consider the diagram

\[
\begin{array}{ccccccccc}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

where each $Y_{n,m}$ is a finite CW complex with only even-dimensional cells and all subsets are inclusions of subcomplexes. Let $Y = \bigcup_{n,m} Y_{n,m}$ and assume $Y_{n,m} \cap Y_{n',m'} = Y_{\min(n,n'),\min(m,m')}$. Then

$$K(Y) = \lim_{n,m} K(Y_{n,m})$$

where the limit is taken in the category of commutative rings.

**Proof.** We know from (4.1) that

$$K(Y) = \lim_{r} K(Y'_r)$$

where $Y'_r = \bigcup_{n+m \leq r} Y_{n,m}$. So it remains to prove $K(Y'_r) = \lim_{n+m \leq r} K(Y_{n,m})$. In fact, we show the following stronger result. Let $S \subseteq \mathbb{Z}_{\geq 0}$ be a finite set with the property that if $(n,m) \in S$, $n' \leq n$, and $m' \leq m$, then $(n',m') \in S$. Letting $Y_S = \bigcup_{(n,m) \in S} Y_{n,m}$, we prove

$$K(Y_S) = \lim_{(n,m) \in S} K(Y_{n,m})$$

by induction on $|S|$. If $|S| = 1$, then $S = \{(0,0)\}$ and there is nothing to prove. For the induction step, fix $S$ and $(a,b) \notin S$ such that $(a-1,b) \in S$ if $a > 0$ and $(a,b-1) \in S$ if $b > 0$. Let $S' = S \cup \{(a,b)\}$. Since the $Y_{n,m}$ are compact Hausdorff with only even-dimensional cells, the higher $K$-groups are concentrated in even degrees. As a result, long exact sequence on $K$-groups yields

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{K}(Y_{S'}/Y_{a,b}) & \longrightarrow & \tilde{K}(Y_{S'}) & \longrightarrow & \tilde{K}(Y_{a,b}) & \longrightarrow & 0 \\
& & & & \| & & & & \\
0 & \longrightarrow & \tilde{K}(Y_{S'}/(Y_{S} \cap Y_{a,b})) & \longrightarrow & \tilde{K}(Y_{S}) & \longrightarrow & \tilde{K}(Y_{S} \cap Y_{a,b}) & \longrightarrow & 0
\end{array}
\]

where the two rows are exact and $\tilde{K}$ denotes the reduced $K$-groups. It follows that the righthand square of the diagram is cartesian, i.e., $\tilde{K}(Y_{S'})$ is the limit of $K(Y_S)$, $K(Y_{a,b})$, and $K(Y_{S} \cap Y_{a,b})$ in the category of abelian groups. Since all morphisms in the righthand square are the natural pullback maps, they are also ring maps, and hence $K(Y_{S'})$ is the limit of $K(Y_S)$, $K(Y_{a,b})$, and $K(Y_{S} \cap Y_{a,b})$ in the category of rings. Using our hypothesis that $Y_{n,m} \cap Y_{n',m'} = Y_{\min(n,n'),\min(m,m')}$, we see $Y_S \cap Y_{a,b} = Y_T$, where $T$ is the set of $(n,m)$ with $n \leq a-1$ and $m \leq b$, or $n \leq a$ and $m \leq b-1$. Applying the induction statement to $S$ and $T$, we have proved the result for $S'$.

We now apply Lemma 4.2 to our case of interest: the James reduced product of $\mathbb{C}P^\infty$.

**Corollary 4.3.** We have

$$K(J\mathbb{C}P^\infty) \cong \lim_{n,m} K(J_n\mathbb{C}P^m)$$

and

$$K(J\mathbb{C}P^\infty) \otimes_\mathbb{Z} \mathbb{Q} \cong \lim_{n,m} (K(J_n\mathbb{C}P^m))_\mathbb{Q}.$$
**Lemma 4.2**

Upon showing (4.2) and its proof show that, on cohomology,
\[
\text{where the second equality uses flatness of } Z.
\]

We may therefore write the composition \( f \), the induced maps \( \text{K} \) is an isomorphism and that \( \text{K} \) is surjective. the induced maps \( \text{K}(\text{JCP}) \cong \lim_{n,b} \text{K}(\text{J}_{n,b}) \to \text{K}(\text{JCP}) \) are surjective.

We may therefore write \( \text{K}(\text{J}_{n}^{\text{P}}) = \text{K}(\text{JCP})/\text{I}_{n,m} \) where \( \text{I}_{n,m} \) is an ideal. Then, by definition,
\[
\text{K}(\text{JCP}) \otimes_{\mathbb{Z}} \mathbb{Q} = \text{lim}_{n,m} (\text{K}(\text{JCP})/\text{I}_{n,m} \otimes_{\mathbb{Z}} \mathbb{Q})) = \text{lim}_{n,m} (\text{K}(\text{J}_{n}^{\text{P}})/\mathbb{Q}),
\]
where the second equality uses flatness of \( Z \to \mathbb{Q} \).

Next, let \( \pi_{i} : (\mathbb{P}^{\mathbb{N}}) \to \mathbb{P}^{\mathbb{N}} \) denote projection onto the \( i \)-th factor and let
\[
y_{i} = 1 - [\pi_{i} \circ \text{O}_{\mathbb{P}^{\mathbb{N}}}(-1)].
\]

It is well-known (see, e.g., Theorems 2.2 and 4.1 of [Sau08]) that the map
\[
\text{K}^{0}((\mathbb{P}^{\mathbb{N}})^{n}) \cong \text{K}(\mathbb{P})^{n}
\]
is an isomorphism and that
\[
\text{K}^{0}((\mathbb{P}^{\mathbb{N}})^{n}) = \mathbb{Z}[y_{1}, \ldots, y_{n}]/(y_{1}^{m+1}, \ldots, y_{n}^{m+1}).
\]

The following proposition is the main result of this subsection.

**Proposition 4.4.** With notation as in (4.3) and (4.5), the map
\[
q_{n}^{*} : \text{K}(\text{J}_{n}^{\text{P}})/\mathbb{Q} \to \text{K}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} \cong \text{K}^{0}((\mathbb{P}^{\mathbb{N}})^{n})\mathbb{Q} = \mathbb{Q}[y_{1}, \ldots, y_{n}]/(y_{1}^{m+1}, \ldots, y_{n}^{m+1})
\]
is an injection which identifies \( \text{K}(\text{J}_{n}^{\text{P}})/\mathbb{Q} \) with quasisymmetric polynomials in \( y_{1}, \ldots, y_{n} \). These injections yield a natural isomorphism
\[
\psi : \text{K}(\text{JCP}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \text{mQSym} \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

**Proof.** Since the Chern character commutes with pullback, we have a commutative diagram
\[
\begin{array}{ccc}
\text{K}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} & \xrightarrow{\text{ch}} & \text{H}^{*}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} \\
\text{K}(\text{J}_{n}^{\text{P}})/\mathbb{Q} & \xrightarrow{\text{ch}} & \text{H}^{*}(\text{J}_{n}^{\text{P}})/\mathbb{Q} \\
\end{array}
\]

Theorem 2.1 and its proof show that, on cohomology, \( q_{n}^{*} \) is injective and identifies \( \text{H}^{*}(\text{J}_{n}^{\text{P}})/\mathbb{Q} \) with the subring of \( \text{H}^{*}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} = \mathbb{Q}[x_{1}, \ldots, x_{n}]/(x_{1}^{m+1}, \ldots, x_{n}^{m+1}) \) consisting of quasisymmetric polynomials in the \( x_{i} \).

As a result, the K-theoretic pullback \( q_{n}^{*} \) is injective and identifies \( \text{K}(\text{J}_{n}^{\text{P}})/\mathbb{Q} \) with the subring of \( \text{K}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} \) consisting of polynomials \( f(y_{1}, \ldots, y_{n}) \) for which \( \text{ch}(f) \) is quasisymmetric in the \( x_{i} \).

Thus, given \( f(y_{1}, \ldots, y_{n}) \in \text{K}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} \), we must show the following two conditions are equivalent:

(i) \( f \) is quasisymmetric in the \( y_{i} \), and
(ii) \( \text{ch}(f) \) is quasisymmetric in the \( x_{i} \).

To prove (i) implies (ii), first suppose \( f(y_{1}, \ldots, y_{n}) \) is a quasisymmetric power series. Let \( g(x) \in \mathbb{Q}[x] \) be any power series with no constant term. Then it is immediate from the definition of quasisymmetry that the composition \( f(g(x_{1}), \ldots, g(x_{n})) \) is also quasisymmetric.

Now, consider \( f(y_{1}, \ldots, y_{n}) \in \text{K}(\mathbb{P}^{\mathbb{N}})^{n}/\mathbb{Q} \); we show that \( \text{ch}(f) \) is quasisymmetric in the \( x_{i} \). We have
\[
\text{ch}(f) = f(1-e^{-x_{1}}, \ldots, 1-e^{-x_{n}});
\]
this is because $x_i$ is the class of $\mathbb{P}^m \times \cdots \times \mathbb{P}^m \times \mathbb{P}^{m-1} \times \mathbb{P}^m \times \cdots \times \mathbb{P}^m$ with $\mathbb{P}^{m-1}$ appearing in the $i$-th factor, and so $-x_i = c_1(\pi_i^*\mathcal{O}(-1))$. Since $1 - e^{-x}$ is a power series with no constant term, it follows that ch($f$) is quasisymmetric, establishing that (i) implies (ii).

Since (i) implies (ii), we see that the quasisymmetric polynomials in $y_i$ are contained in $q_m^*(K(J_n\mathbb{P}^m)_\mathbb{Q})$, the set of polynomials $f$ such that ch($f$) is quasisymmetric. However, these two sets have the same dimension when viewed as $\mathbb{Q}$-vector spaces, since ch defines an isomorphism between $K(J_n\mathbb{P}^m)_\mathbb{Q}$ and the vector space of quasisymmetric polynomials in the $x_i$. Thus, the sets coincide and (ii) implies (i), as well.

Lastly, having now shown that $K(J_n\mathbb{P}^m)\mathbb{Q}$ agrees with the quasisymmetric polynomials in the ring $\mathbb{Q}[y_1, \ldots, y_n]/(y_1^{m+1}, \ldots, y_n^{m+1})$, we see from Corollary 4.3 that

$$K(J_n\mathbb{CP}^\infty) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \lim_{\substack{n, m}} (K(J_n\mathbb{P}^m)_\mathbb{Q}) \cong m\text{QSym} \otimes_{\mathbb{Z}} \mathbb{Q},$$

where the limit is in the category of commutative rings.

We end this subsection with the following well-known computation, which we make use of later.

**Lemma 4.5. In $K^0(\mathbb{P}^m)$, we have**

$$[\mathcal{O}_{\mathbb{P}^r}] = (1 - [\mathcal{O}_{\mathbb{P}^m}(-1)])^{m-r}.$$

*Proof.* The second statement follows immediately from the first. To prove the first statement, we induct on $m - r$. For $r = m$, there is nothing to show. For $r < m$, we use the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{r+1}}(-1) \to \mathcal{O}_{\mathbb{P}^{r+1}} \to \mathcal{O}_{\mathbb{P}^r} \to 0,$$

which shows

$$[\mathcal{O}_{\mathbb{P}^r}] = [\mathcal{O}_{\mathbb{P}^{r+1}}] - [\mathcal{O}_{\mathbb{P}^{r+1}}(-1)].$$

By induction, we have

$$[\mathcal{O}_{\mathbb{P}^r}] = \sum_{i=0}^{m-r-1} (-1)^i \binom{m-r-1}{i} [\mathcal{O}_{\mathbb{P}^m}(-i)] - \sum_{i=0}^{m-r-1} (-1)^i \binom{m-r-1}{i} [\mathcal{O}_{\mathbb{P}^m}(-(i+1))].$$

A straightforward computation shows that this quantity simplifies to be

$$[\mathcal{O}_{\mathbb{P}^r}] = \sum_{i=0}^{m-r} (-1)^i \binom{m-r}{i} [\mathcal{O}_{\mathbb{P}^m}(-i)] = \sum_{i=0}^{m-r} \binom{m-r}{i} (-[\mathcal{O}_{\mathbb{P}^m}(-1)])^i = (1 - [\mathcal{O}_{\mathbb{P}^m}(-1)])^{m-r}. \quad \Box$$

### 4.2. $K$-classes are represented by quasisymmetric monomial glides.

Our first aim in this section is to associate a $K$-class to each cell $e_{\alpha}$ of $J_n\mathbb{CP}^\infty$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. Then for any $n \geq k$ and $m \geq \max_i \alpha_i$, we see $e_{\alpha}$ is a cell of $J_n\mathbb{P}^m$. If $J_n\mathbb{P}^m$ were a smooth complex variety, we would want to associate $e_{\alpha}$ with the class of its structure sheaf. However, as discussed in Section 5 this approach is not viable, so we must substitute a more delicate construction. Although we cannot use the class of the structure sheaf on $J_n\mathbb{P}^m$ itself, we can instead pull $e_{\alpha}$ back to the smooth variety $(\mathbb{P}^m)^n$ via the quotient map $q_n: (\mathbb{P}^m)^n \to J_n\mathbb{P}^m$, and then consider the associated $K$-class of the closure of $q_n^{-1}(e_{\alpha})$. It turns out that we get better behaviour under the necessary limits if we first take the Poincaré dual of the closure of $q_n^{-1}(e_{\alpha})$ before taking the $K$-class; the reason for this is that we want the codimension of the $K$-class to be independent of the choice of $n$ and $m$.

We can describe $q_n^{-1}(e_{\alpha})$ and its Poincaré dual explicitly as follows. Let $\mathcal{I}_n$ be the set of order-preserving injections $\iota: [k] \to [n]$. For every $\iota \in \mathcal{I}_n$, define $b(\iota) = (b_1, \ldots, b_n)$ by

$$b_i = \begin{cases} \alpha_j, & \text{if } i = \iota(j); \\ 0, & \text{if } i \notin \text{im } \iota. \end{cases}$$
and let $r(i) = (r_1, \ldots, r_n)$ be defined by $r_i = m - b(i)$. We set $B_{\alpha,n} = \{b(i) \mid i \in \mathcal{I}_n\}$ and $R_{\alpha,n,m} = \{r(i) \mid i \in \mathcal{I}_n\}$. Then
\[
q^{-1}_n(e_\alpha) = \bigcup_{(b_1, \ldots, b_n) \in B_{\alpha,n}} \mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_n}
\]
and its Poincaré dual is given by
\[
Z_{\alpha,n,m} = \bigcup_{(r_1, \ldots, r_n) \in R_{\alpha,n,m}} \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}.
\]
We consider the class $[O_{Z_{\alpha,n,m}}] \in K^0((\mathbb{P}^m)^n)_Q$, where we use smoothness to resolve $O_{Z_{\alpha,n,m}}$ by locally-free sheaves to produce a class in $K^0$. Using then the natural isomorphism (4.4), we conflate this algebraic class with the topological class, which we also write as $[O_{Z_{\alpha,n,m}}] \in K((\mathbb{P}^m)^n)_Q$.

**Definition-Lemma 4.6.** We define the $K$-class of $e_\alpha$ viewed as a cell of $J_1$ to be the unique class 
$[e_\alpha]_{n,m} \in K(J_1)_{\mathbb{P}^m}$ that pulls back to $[O_{Z_{\alpha,n,m}}] \in K((\mathbb{P}^m)^n)_Q$ under the quotient map $q_n$.

We define the $K$-class of $e_\alpha$ viewed as a cell of $JC\mathbb{P}^\infty$ to be the unique class
\[
[e_\alpha] \in K(JC\mathbb{P}^\infty) \hat{\otimes} \mathbb{Q}
\]
which pulls back to $[e_\alpha]_{n,m}$ for all $n, m$ sufficiently large.

**Proof.** The uniqueness of the class $[e_\alpha]_{n,m} \in K(J_1)_{\mathbb{P}^m}$ is immediate from the injectivity statement of Proposition 4.4. The existence of $[e_\alpha]_{n,m} \in K(J_1)_{\mathbb{P}^m}$ will be established below in Theorem 4.7, which also then establishes the existence and uniqueness of the class $[e_\alpha] \in K(JC\mathbb{P}^\infty) \hat{\otimes} \mathbb{Q}$. □

The following theorem completes the proof of Definition-Lemma 4.6 and also establishes Theorem 1.2.

**Theorem 4.7.** Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition. Let $n \geq k$ and $m \geq \max_i \alpha_i$ so that $e_\alpha$ is a cell of $J_n$. Then the following hold.

1. $[O_{Z_{\alpha,n,m}}] \in K((\mathbb{P}^m)^n)_Q$ is in the image of $q_n^* : K(J_1)_{\mathbb{P}^m} \to K((\mathbb{P}^m)^n)_Q$, so determines a unique class $[e_\alpha]_{n,m} \in K(J_1)_{\mathbb{P}^m}$.

2. If $N \geq n$ and $M \geq m$, then under the inclusion $\iota : J_{1}\mathbb{P}^m \to J_N\mathbb{P}^M$, we have $[e_\alpha]_{n,m} = \iota^*[e_\alpha]_{N,M}$.

This system of equalities therefore determines a class $[e_\alpha] \in \lim_{N,M} K(JN\mathbb{P}^M)_Q = K(JC\mathbb{P}^\infty) \hat{\otimes} \mathbb{Q}$.

3. The $[e_\alpha]$ form a Schauder basis for $K(JC\mathbb{P}^\infty) \hat{\otimes} \mathbb{Q}$.

4. Under the natural isomorphism
\[
\psi : K(JC\mathbb{P}^\infty) \hat{\otimes} \mathbb{Q} \sim m\text{QSym} \hat{\otimes} \mathbb{Q}
\]
from Proposition 4.4, we have
\[
\psi([e_\alpha]) = M_\alpha(y_1, y_2, \ldots).
\]

**Proof.** Recall that under the isomorphism $\psi$, the element $y_i$ corresponds to $1 - [\pi_i^*O(-1)] \in K((\mathbb{P}^M)^N)_Q$ for any $N \geq i$, where $\pi_i : (\mathbb{P}^M)^N \to \mathbb{P}^M$ is the projection map onto the $i$th factor. Thus, we will henceforth conflate $y_i$ and $1 - [\pi_i^*O(-1)]$. To prove (1), (2), and (4), we claim it is enough to show
\[
[O_{Z_{\alpha,n,m}}] = M_\alpha(y_1, \ldots, y_n, 0, 0, \ldots).
\]
Indeed, since $M_\alpha$ is quasisymmetric by Theorem 3.8, (4.6) implies that $[O_{Z_{\alpha,n,m}}]$ is quasisymmetric, and hence lives in $K(J_1)_{\mathbb{P}^m}$ by Proposition 4.4; this proves the existence part of (1). The uniqueness part of (1) is immediate from the injectivity statement of Proposition 4.4. Next, the map $\iota^* : K(J_N\mathbb{P}^M)_Q \to K(J_1\mathbb{P}^m)_Q$ corresponds to the map
\[
\mathbb{Q}[y_1, \ldots, y_N]/(y_i^{m+1}) \cong K((\mathbb{P}^M)^N)_Q \to K((\mathbb{P}^m)^n)_Q \cong \mathbb{Q}[y_1, \ldots, y_n]/(y_i^{m+1})
\]
sending $y_i$ to 0 for $i > n$. It follows from (4.6) that
\begin{equation}
(4.7) \quad \iota^*[O_{Z_{a,n,m}}] = \iota^*[\mathcal{M}_a(y_1, \ldots, y_N, 0, 0, \ldots) = \mathcal{M}_a(y_1, \ldots, y_n, 0, 0, \ldots) = [O_{Z_{a,n,m}}],
\end{equation}
proving (2). Since $\psi$ is constructed from isomorphisms at each finite level
\[K(J_n \mathbb{P}^m)_{\mathbb{Q}} \cong (\mathfrak{m} \mathfrak{Q} \mathfrak{S} \mathfrak{y} \mathfrak{m} \mathfrak{S} \mathfrak{b} \mathfrak{m} / (y_1^{m+1}, \ldots, y_n^{m+1}, y_{n+1}, y_{n+2}, \ldots)),\]
we see that (4) follows from (4.6) and (4.7).

We turn now to the proof of (4.6). We consider the Chow ring $A^*((\mathbb{P}^m)^n)_{\mathbb{Q}}$. The variety $(\mathbb{P}^m)^n$ is a generalized flag variety with a Schubert decomposition given by the cells $e_{2b_1} \times \cdots \times e_{2b_n}$ for $n$-tuples $(b_1, \ldots, b_n)$ of nonnegative integers. The closures of these cells are the subvarieties of the form $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_n}$, which are the Schubert varieties of $(\mathbb{P}^m)^n$. The Chow classes of the Schubert varieties form a Schubert basis of the Chow ring. (For background on generalized flag varieties and their Schubert varieties in much more generality, see, e.g., [Ful97, BL00].)

The Chow class of $Z_{a,n,m}$ in $A^*((\mathbb{P}^m)^n)_{\mathbb{Q}}$ is the sum of the classes of its irreducible components. By construction, the irreducible components of $Z_{a,n,m}$ are Schubert varieties, so $Z_{a,n,m} \in A^*((\mathbb{P}^m)^n)_{\mathbb{Q}}$ is a multiplicity-free sum of Schubert classes. (By “multiplicity-free,” we mean that the Chow class is a linear combination of Schubert classes where all coefficients are 0 or 1.) M. Brion [Bri03] developed a powerful flat degeneration of any subvariety of a homogeneous space whose Chow class is multiplicity-free in the Schubert basis. Building on this work, A. Knutson [Knu09, Theorem 3] gave a formula for determining $K$-classes from Chow classes in this setting. Specifically, in our context, [Knu09, Theorem 3] tells us that
\[\mathcal{O}_{Z_{a,n,m}} = \sum_{W \in \mathcal{P}'} \mu_{\mathcal{P}'}(W)[\mathcal{O}_W],\]
where $\mathcal{P}'$ is the poset of Schubert varieties of $(\mathbb{P}^m)^n$ that are subvarieties of $Z_{a,n,m}$, partially ordered by inclusion. Here, $\mu_{\mathcal{P}'}$ denotes the M"obius function on the poset $\mathcal{P}'$, defined (in Knutson’s somewhat nonstandard conventions) as the unique function on $\mathcal{P}'$ such that for all $W \in \mathcal{P}'$, we have
\[\sum_{W' \preceq W} \mu_{\mathcal{P}'}(W') = 1.\]
(Technically, Knutson’s formula is stated for $K_0((\mathbb{P}^m)^n)$ rather than $K^0((\mathbb{P}^m)^n)$, but $(\mathbb{P}^m)^n$ is smooth, so we have $K_0((\mathbb{P}^m)^n) \cong K^0((\mathbb{P}^m)^n)$.) Identifying the Schubert variety $\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_n}$ with the tuple $(m - b_1, \ldots, m - b_n)$, we obtain an order-reversing anti-isomorphism between $\mathcal{P}'$ and the poset $\mathcal{P}_\alpha$ from Section 3. As a result,
\[\mathcal{O}_{Z_{a,n,m}} = \sum_{\alpha = (\sigma_1, \ldots, \sigma_n) \in \mathcal{P}_\alpha} \mu_{\mathcal{P}_\alpha}(\alpha)[\mathcal{O}_{\mathbb{P}^{b_1} \times \cdots \times \mathbb{P}^{b_n}}] = \sum_{\alpha = (\sigma_1, \ldots, \sigma_n) \in \mathcal{P}_\alpha} \mu_{\mathcal{P}_\alpha}(\alpha)y_1^{\sigma_1} \cdots y_n^{\sigma_n} = \mathcal{M}_a(y_1, \ldots, y_n);\]
the second equality follows from Lemma 4.5 and the last equality holds by definition. We have therefore proved (4.6).

Lastly, to show (3), we prove that the $\mathcal{M}_a$ form a Schauder basis for $\mathfrak{m} \mathfrak{Q} \mathfrak{S} \mathfrak{y} \mathfrak{m} \mathfrak{S} \mathfrak{b} \mathfrak{m} / \mathbb{Q}$, the ring of quasisymmetric power series with rational coefficients. The key observation here is that the lowest degree terms of $\mathcal{M}_a$ are the monomial quasisymmetric function $M_a$, so linear independence is immediate. Consider $f \in \mathfrak{m} \mathfrak{Q} \mathfrak{S} \mathfrak{y} \mathfrak{m} \mathfrak{S} \mathfrak{b} \mathfrak{m} / \mathbb{Q}$ with lowest degree terms in degree $\ell$; let $f_\ell$ denote the degree $\ell$ homogeneous part of $f$. Since the degree $\ell$ homogeneous part of $\mathfrak{m} \mathfrak{Q} \mathfrak{S} \mathfrak{y} \mathfrak{m} \mathfrak{S} \mathfrak{b} \mathfrak{m} / \mathbb{Q}$ is finite-dimensional, with a basis of monomial quasisymmetric functions $M_\beta$ such that $|\beta| = \ell$, we may write $f_\ell$ as a finite $\mathbb{Q}$-linear combination of monomial quasisymmetric functions:
\[f_\ell = \sum_{i=1}^{k} c_i M_\beta_i,\]
for some compositions $\beta_i$ and coefficients $c_i \in \mathbb{Q}$. Now, consider
\[
f' = f - \sum_{i=1}^{k} c_i \overline{M}_{\beta_i} \in \text{mQSym} \otimes_{\mathbb{Z}} \mathbb{Q}.
\]
The lowest degree terms of $f'$ are in degree $\ell' > \ell$, so by iterating this process we may write $f$ as a countable sum of quasisymmetric monomial glides $\overline{M}_n$ such that there are only finitely-many $\alpha$ with $|\alpha| < p$ appearing for each degree $p$.

5. The James reduced product is not normal

One may wonder whether we cannot endow $J_n \mathbb{P}^m$ with the structure of an algebraic variety and define the $K$-class of a cell $e_n$ as the structure sheaf of the closure of the cell, i.e. $[\mathcal{O}_{e_n}] \in K_0(J_n \mathbb{P}^m)$. Proposition 5.1 shows that if it were possible to give $J_n \mathbb{P}^m$ the structure of an algebraic variety, it would be quite singular. In particular, $K_0(J_n \mathbb{P}^m)$ would not have a natural ring structure under intersection products.

**Proposition 5.1.** Let $n > 1$. Suppose $J_n \mathbb{P}^m$ can be given the structure of a complex variety such that $q_n \colon (\mathbb{P}^m)^n \to J_n \mathbb{P}^m$ is a map of varieties. Then $J_n \mathbb{P}^m$ is non-normal and $q_n$ is the normalization map.

**Proof.** Since $q_n$ is surjective and $(\mathbb{P}^m)^n$ is irreducible, $J_n \mathbb{P}^m$ is irreducible as well. Since $J_n \mathbb{P}^m$ is a variety, it is separated; it follows that $q_n$ is proper as $(\mathbb{P}^m)^n$ is proper. Since, by construction, $q_n$ is also quasi-finite, Zariski’s Main Theorem tells us $q_n$ is finite.

There is a unique top-dimensional cell $e_{(n,\ldots,n)}$ of $J_n \mathbb{P}^m$. Its complement is equal to $q_n(Z)$, where $Z = (\mathbb{P}^m)^n \setminus (\mathbb{A}^m)^n$. Since $Z$ is closed and $q_n$ is a closed map (because it is proper), we see $e_{(n,\ldots,n)}$ is Zariski open in $J_n \mathbb{P}^m$. By generic flatness, there is a non-empty Zariski open subset $U \subset J_n \mathbb{P}^m$ over which $q_n$ is flat. Since $J_n \mathbb{P}^m$ is irreducible, $V := U \cap e_{(n,\ldots,n)}$ is non-empty and dense. Over $V$, the map $q_n$ is flat and bijective, hence an isomorphism. Therefore, $q_n$ is a finite birational map.

Let $\nu \colon J_n \mathbb{P}^m \to J_n \mathbb{P}^m$ be the normalization map. Since $(\mathbb{P}^m)^n$ is normal and $q_n$ is surjective, by the universal property of normalization, there is a unique map $\overline{q}_n$ making the diagram

\[
\begin{array}{ccc}
(\mathbb{P}^m)^n & \xrightarrow{\overline{q}_n} & J_n \mathbb{P}^m \\
\downarrow q_n & & \downarrow \nu \\
J_n \mathbb{P}^m & \xleftarrow{\nu} & J_n \mathbb{P}^m
\end{array}
\]

commute. Since $\nu$ and $q_n$ are finite, $\overline{q}_n$ is as well. Since $q_n$ is an isomorphism over $V \subset J_n \mathbb{P}^m$ and $(\mathbb{P}^m)^n$ is smooth, we see $V$ is smooth, hence normal. As a result, $\nu$ is an isomorphism over $V$ and so $\overline{q}_n$ is a finite birational map to a normal variety. Another application of Zariski’s Main Theorem tells us $\overline{q}_n$ is an isomorphism. In other words, $q_n$ is the normalization map of $J_n \mathbb{P}^m$.

Lastly, if $J_n \mathbb{P}^m$ were normal, then $q_n$ would be an isomorphism. This is not possible since $q_n$ is $n$-to-$1$ over the cell $e_{(1)}$ and $n > 1$.

It would be interesting to find an embedding of $J(\mathbb{CP}^\infty)$ inside a smooth infinite-type scheme $W$ of the same homotopy type. If $W$ had a stratification by complex affine spaces, it would be a sort of “James reduced product analogue” of a thick Kashiwara flag variety [Kas89] and come with another canonical basis of $K$-theory in analogy with [KS09, LSS10].

6. Extending to flag varieties

We end this paper by generalizing Theorem 1.1 to a much larger class of CW complexes, particularly with applications to James reduced products of generalized flag varieties. We suppress some of the details that are identical to those in the $\mathbb{CP}^\infty$ case. Throughout this section, $X$ denotes a CW complex with only even-dimensional cells, only finitely many cells of any given dimension, and a unique 0-cell.

We denote by $\{e_\theta : \theta \in I\}$ the cells of $X$, where $\theta$ runs through some index set $I$; we require that $I$ contains an element 0 such that $e_0$ is the unique 0-cell. We then have
\[
H_*(X; \mathbb{Z}) \cong \bigoplus_{\theta \in I} \mathbb{Z} e_\theta \quad \text{and} \quad H^*(X; \mathbb{Z}) \cong \bigoplus_{\theta \in I} \mathbb{Z} x_\theta,
\]
where \( x_\theta \) denotes the function dual to the cell \( e_\theta \).

A **weak \( I \)-composition** is a finite sequence of elements of \( I \), and an **\( I \)-composition** is a finite sequence of elements of \( I \setminus \{0\} \). Every weak \( I \)-composition \( T \) has an associated **positive part** \( T^+ \) obtained from \( a \) by deleting the 0 terms. For example, if \( \theta_1, \theta_2, \theta_3, \theta_4 \in I \setminus \{0\} \), then the positive part of \( (\theta_1, 0, \theta_2, 0, 0, \theta_4) \) is \( (\theta_1, \theta_2, \theta_4) \). Note that compositions and weak compositions as discussed in Section 2 are, in this more general terminology, \( \mathbb{Z}_{\geq 0} \)-compositions and weak \( \mathbb{Z}_{\geq 0} \)-compositions, respectively.

The CW structure on the James reduced product \( J_n(X) \) is induced from the cellular quotient map

\[
q_n: X^n \to J_n(X),
\]

which, by construction, identifies the cells \( e_{\theta_1} \times \cdots \times e_{\theta_n} \) and \( e_{\kappa_1} \times \cdots \times e_{\kappa_n} \) if and only if

\[
(\theta_1, \ldots, \theta_n)^+ = (\kappa_1, \ldots, \kappa_n)^+.
\]

Hence, the cells of \( J_n(X) \) are indexed by the \( I \)-compositions \( \Theta = (\theta_1, \ldots, \theta_k) \) where \( k \leq n \). We denote the corresponding cell of \( J_n(X) \) by \( e_{(\theta_1, \ldots, \theta_k)} \). We see then that the cells \( e_\Theta \) of \( J(X) \) are indexed by all \( I \)-compositions \( \Theta \) of arbitrary length. As was the case for \( J(\mathbb{C}P^\infty) \), we have

\[
H_*(J(X); \mathbb{Z}) \cong \bigoplus_{\Theta} \mathbb{Z}e_\Theta \quad \text{and} \quad H^*(J(X); \mathbb{Z}) \cong \bigoplus_{\Theta} \mathbb{Z}x_\Theta,
\]

where \( x_\Theta \) denotes the function dual to the cell \( e_\Theta \) and both sums are over all \( I \)-compositions \( \Theta \). We therefore again have

\[
H^*(J(X); \mathbb{Z}) = \lim_{n \to \infty} H^*(J_n(X); \mathbb{Z}).
\]

We prove that \( H^*(J(X); \mathbb{Z}) \) is built out of \( \text{QSym} \), where one substitutes cohomology classes from \( H^*(X; \mathbb{Z}) \) in for the variables \( x_i \). In order to make this precise, we first introduce a new ring as follows.

**Definition 6.1.** Let \( R \) be a commutative graded ring which is free as a graded \( \mathbb{Z} \)-module. Let \( \eta: R \to \mathbb{Z} \) be a map of graded rings where \( \mathbb{Z} \) is placed in degree 0. For every \( n \geq 1 \), \( 1 \leq i \leq n \), and \( r \in R \), let

\[
r^{(i)} := 1 \otimes \cdots \otimes 1 \otimes r \otimes 1 \otimes \cdots \otimes 1 \in R^\otimes n
\]

with \( r \) in the \( i \)th tensor factor. Let

\[
\text{QSym}_{\eta}(R) \subset R^\otimes n
\]

be the graded subring generated by expressions of the form

\[
M_{n, (r_1, \ldots, r_k)} := \sum_{1 \leq i_1 < \cdots < i_k \leq n} r^{(i_1)}_1 r^{(i_2)}_2 \cdots r^{(i_k)}_k
\]

where \( r_j \in R \). The map \( \eta \) yields a graded map \( R^{\otimes (n+1)} \to R^\otimes n \otimes \mathbb{Z} \mathbb{Z} = R^\otimes n \) for each \( n \), which induces a graded map

\[
\text{QSym}_{n+1}(R) \to \text{QSym}_n(R).
\]

We let

\[
\text{QSym}(R) = \lim_n \text{QSym}_n(R)
\]

be the limit in the category of graded rings. Furthermore, for every choice \( (r_1, \ldots, r_k) \), we have an element

\[
M_{(r_1, \ldots, r_k)} \in \text{QSym}(R)
\]

mapping to \( M_{n, (r_1, \ldots, r_k)} \) for each \( n \geq k \).

In our case of interest, we take \( R = H^*(X; \mathbb{Z}) \) and the map \( \eta: H^*(X; \mathbb{Z}) \to H^*(e_0; \mathbb{Z}) = \mathbb{Z} \) in Definition 6.1 is taken to be the pullback map induced by the inclusion \( e_0 \in X \). For notational convenience, for each \( I \)-composition \( \Theta = (\theta_1, \ldots, \theta_k) \), we let

\[
M_{n, \Theta} := M_{n, (x_{\theta_1}, \ldots, x_{\theta_k})} \in \text{QSym}_n(H^*(X; \mathbb{Z}))
\]

and

\[
M_{\Theta} := M_{(x_{\theta_1}, \ldots, x_{\theta_k})} \in \text{QSym}(H^*(X; \mathbb{Z}));
\]

recall that \( x_{\theta_i} \in H^*(X; \mathbb{Z}) \) is the dual function to the cell \( e_{\theta_i} \). Note that for each \( n \), the set

\[
\{M_{n, \Theta}: \Theta = (\theta_1, \ldots, \theta_k), k \leq n\}
\]

is a homogeneous basis for \( \text{QSym}_n(H^*(X; \mathbb{Z})) \).
Theorem 6.2. Let $X$ be a CW complex with only even-dimensional cells, only finitely many cells of any given dimension, and a unique 0-cell $e_0$. Then we have a natural isomorphism of graded rings

$$H^*(J(X); \mathbb{Z}) \cong QSym(H^*(X; \mathbb{Z}))$$

sending $x_\Theta$ to $M_\Theta$ for every $I$-composition $\Theta$.

Proof. Since we know that $H^d(X; \mathbb{Z})$ is free of finite rank for every $d$, the Künneth formula (see, e.g., [Hat02, Theorem 3.16]) tells us that

$$(6.1) \quad H^*(X^n; \mathbb{Z}) \cong H^*(X; \mathbb{Z})^\otimes n.$$ 

For every $I$-composition $\Theta = (\theta_1, \ldots, \theta_k)$, any $n \geq k$, and any injection $\iota : [k] \to [n]$, define a cell $e_\iota$ of $X^n$ by

$$e_\iota := e_{\kappa_1} \times \cdots \times e_{\kappa_n},$$

where

$$\kappa_j := \begin{cases} 0, & \text{if } j \notin \text{im } \iota; \\ \theta_{j-1(i)}, & \text{otherwise}. \end{cases}$$

We see the that

$$q_n^{-1}(e_\Theta) = \coprod_\iota e_\iota,$$

where the disjoint union is over all injections $\iota : [k] \to [n]$. As a result, making use of Equation (6.1) and the notation introduced in Definition 6.1, we have

$$q_n^*(x_\Theta) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{\theta_{i_1}}^{(i_1)} \cdots x_{\theta_k}^{(i_k)} = M_{n, \Theta} \in QSym_n(H^*(X; \mathbb{Z})).$$

For each $n$, the set $\{x_\Theta : \Theta = (\theta_1, \ldots, \theta_k), k \leq n\}$ is a homogeneous basis for $H^*(J_n(X); \mathbb{Z})$ and the set $\{M_{n, \Theta} : \Theta = (\theta_1, \ldots, \theta_k), k \leq n\}$ is a homogeneous basis for $QSym_n(H^*(X; \mathbb{Z}))$. It follows that $q_n^*$ induces a graded isomorphism

$$q_n^* : H^*(J_n X; \mathbb{Z}) \xrightarrow{\cong} QSym_n(H^*(X; \mathbb{Z})).$$

To finish the proof of the theorem, we note that the commutative diagram

$$\begin{array}{ccc} X^n & \xrightarrow{J'_n} & X^{n+1} \\ \downarrow q_n & & \downarrow q_{n+1} \\ J_n X & \xrightarrow{J_n} & J_{n+1} X \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} H^*(J_{n+1} X; \mathbb{Z}) & \xrightarrow{q_{n+1}^*} & QSym_{n+1}(H^*(X; \mathbb{Z})) \xrightarrow{\cong} H^*(X; \mathbb{Z})^\otimes (n+1) \xrightarrow{\cong} H^*(X^{n+1}; \mathbb{Z}) \\ \downarrow J_n^* & & \downarrow \pi_n & & \downarrow (J'_n)^* \\ H^*(J_n X; \mathbb{Z}) & \xrightarrow{q_n^*} & QSym_n(H^*(X; \mathbb{Z})) \xrightarrow{\cong} H^*(X; \mathbb{Z})^\otimes n \xrightarrow{\cong} H^*(X^n; \mathbb{Z}) \end{array}$$

of graded rings. Hence,

$$H^*(J(X); \mathbb{Z}) = \lim_{n \to \infty} H^*(J_n(X); \mathbb{Z}) \cong \lim_{n \to \infty} QSym_n(H^*(X; \mathbb{Z})) = QSym(H^*(X; \mathbb{Z})), $$

where the limits are taken in the category of graded rings. \qed
6.1. James reduced products of generalized flag varieties. We obtain particularly interesting applications of Theorem 6.2 in the case where \( X \) is a generalized flag variety, as in many of these cases we have significant combinatorial understanding of the cohomology ring \( H^*(X; \mathbb{Z}) \) (see, e.g., [Mac91, TY09, CP12, SY16]).

Let \( G \) be a connected complex reductive Lie group with Borel subgroup \( B \) and \( B \subseteq P \subseteq G \) be a parabolic subgroup. Let \( W \) be the Weyl group of \( G \) and let \( W_P \subseteq W \) be the Weyl group of the Levi subgroup of \( P \). The subgroup \( W_P \) is generated by the simple reflections \( \{s_i\} \) indexed by the nodes of \( P \) in the Dynkin diagram from \( G \). The **Coxeter length** of \( w \in W \) is the length \( \ell(w) \) of a shortest expression for \( w \) as a product of simple reflections \( w = s_{i_1}s_{i_2}\cdots s_{i_{\ell(w)}} \). Every coset in \( W/W_P \) has a canonical representative given by the unique element with shortest Coxeter length. We write \( W^P \) for the set of minimum length representatives of the cosets \( W/W_P \). For further expositions of these ideas, see, e.g., [Hum78, Hum90].

The **generalized flag variety** \( G/P \) has a Schubert cell decomposition with cells indexed by the elements of \( W^P \). Here, the cells are the \( B \)-orbits induced from the canonical action of \( B \) on \( G \). We choose the convention such that the cell \( e_w \) has (real) dimension \( 2\ell(w) \). The generalized flag variety has a finite number of cells, all even dimensional. Moreover, there is a unique \( 0 \)-cell labeled by the identity element. Hence, the hypotheses of Theorem 6.2 hold and we obtain an explicit description of the cohomology ring \( H^*(G/P; \mathbb{Z}) \) in terms of \( H^*(G/P; \mathbb{Z}) \). In the cases where we have explicit combinatorial models for \( H^*(G/P; \mathbb{Z}) \), we can leverage this understanding to obtain combinatorial models of \( H^*(G/P; \mathbb{Z}) \).

**Example 6.3.** Let \( G = \text{GL}_n(\mathbb{C}) \) and take \( B = P \) to be the Borel subgroup of upper triangular matrices. Then \( \text{Flags}_n := G/P \) is the **complete flag variety**, parametrizing complete flags

\[
V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n
\]

of nested vector subspaces of \( \mathbb{C}^n \). In this case, \( W = W^P = S_n \) is the symmetric group on \( n \) letters, so we obtain \( n! \) cells in the Schubert cell decomposition of \( \text{Flags}_n \), labeled by permutations. Borel [Bor53] gave a presentation of \( H^*(\text{Flags}_n; \mathbb{Z}) \) as the quotient

\[
H^*(\text{Flags}_n; \mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/I,
\]

where \( I \) is the ideal generated by the \( n \) **elementary symmetric polynomials**

\[
M_{(1)}, M_{(1,1)}, \ldots, M_{(1,\ldots,1)}.
\]

The cohomology classes \( x_w \) dual to the Schubert cells \( e_w \) form a \( \mathbb{Z} \)-basis of \( H^*(\text{Flags}_n; \mathbb{Z}) \), so we identify them with elements of \( \mathbb{Z}[x_1, \ldots, x_n]/I \). Indeed, a remarkable choice of coset representatives for these classes was produced by A. Lascoux and M.-P. Schützenberger [LS82] (see also, [Mac91]), and explained geometrically in [FR03, KM05]. These representatives are the **Schubert polynomials** \( \mathfrak{S}_w \), indexed by the permutations \( w \in S_n \).

Various explicit combinatorial formulas for Schubert polynomials are known, e.g., [BJS93, BB93, WY18].

By Theorem 6.2, the cohomology \( H^*(J(\text{Flags}_n); \mathbb{Z}) \) of the James reduced product has a cellular \( \mathbb{Z} \)-basis indexed by tuples \( \Theta = (w_1, \ldots, w_k) \) of nonidentity permutations in \( S_n \). Considering an infinite array of variables \( x^{(j)}_i \) for \( 1 \leq i \leq n \) and \( j \in \mathbb{Z}_{\geq 0} \), we obtain explicit power series representatives for these classes as follows. For \( j \in \mathbb{Z}_{\geq 0} \), let \( \mathfrak{S}^{(j)} \) denote the Schubert polynomial \( \mathfrak{S}_w \) in the variables \( x^{(j)}_1, \ldots, x^{(j)}_n \). Then, Theorem 6.2 gives that the cellular cohomology class \( x_\Theta \) is represented in

\[
H^*(J(\text{Flags}_n); \mathbb{Z}) \cong \mathcal{QSym}(H^*(\text{Flags}_n; \mathbb{Z}))
\]

by the power series

\[
M_\Theta = \sum_{1 \leq j_1 < j_2 < \cdots < j_k} \mathfrak{S}^{(j_1)}_{w_1} \mathfrak{S}^{(j_2)}_{w_2} \cdots \mathfrak{S}^{(j_k)}_{w_k}.
\]

With this description of \( M_\Theta \) in hand, it is then straightforward to compute any of the structure coefficients of \( H^*(J(\text{Flags}_n)) \) with respect to the cellular basis. \( \diamond \)

6.2. James reduced products of the classifying spaces \( BU(k) \). Slightly extending the above discussion, we will apply Theorem 6.2 to the classifying spaces \( BU(k) \). Let \( G_n = \text{GL}_n(\mathbb{C}) \) and let \( P_n \subset G_n \) denote the maximal parabolic subgroup of block upper triangular matrices with block sizes \( k \) and \( n-k \). Each \( G_n/P_n \) is the **Grassmannian** \( \text{Gr}_k(\mathbb{C}^n) \), parametrizing (complex) \( k \)-dimensional linear subspaces of \( \mathbb{C}^n \). There are natural inclusions \( G_n/P_n \hookrightarrow G_{n+1}/P_{n+1} \) and \( BU(k) \), the classifying space for rank \( k \) complex vector bundles, is obtained as the colimit of this system.
Note that these inclusion maps respect the CW structures, so we obtain a Schubert cell decomposition of \( BU(k) \) induced by those of the various \( Gr_k(\mathbb{C}^n) \). An integer \( 1 \leq k < n \) is a descent of the permutation \( w \in S_n \) if \( w(k) > w(k+1) \). A permutation is \( k \)-Grassmannian if it is the identity permutation or it has \( k \) as its only descent. Writing \( W_n = S_n \) for the Weyl group of \( G_n \), one computes that \( W_n^k \) is the set of \( k \)-Grassmannian permutations in \( S_n \). Hence, these permutations naturally index the cells of \( Gr_k(\mathbb{C}^n) \).

It is more traditional, however, to index the cells of \( Gr_k(\mathbb{C}^n) \) by partitions, i.e., nonincreasing sequences of nonnegative integers. Here, the translation from a \( k \)-Grassmannian permutation \( w \) to a partition \( \lambda \) is given by

\[
w \mapsto (w(k) - k, \ldots, w(2) - 2, w(1) - 1).
\]

For example, the 5-Grassmannian permutation \( w = 124693578 \in S_9 \) corresponds to the partition \( (4, 2, 1, 0, 0) \). We write \( w_\lambda \) for the permutation associated to the partition \( \lambda \). The cell \( c_\lambda = c_{w_\lambda} \) has dimension \( 2|\lambda| \), where \(|\lambda|\) denotes the sum of the elements of \( \lambda \). In this notation, the cells of \( Gr_k(\mathbb{C}^n) \) are indexed by the set of all \( \binom{n}{k} \) partitions of length \( k \) with all elements at most \( n-k \). Taking the colimit, the cells of \( BU(k) \) are indexed by all partitions of length \( k \) (with no upper bound on the elements). Note that there is a unique 0-cell (indexed by the zero partition), all cells are even dimensional, and there are only finitely-many cells in any fixed dimension. Hence, Theorem 6.2 applies to explicate the topology of \( J(BU(k)) \).

Given such a partition \( \lambda \) of length \( k \), the Schur polynomial \( s_\lambda \in \mathbb{Z}[x_1, \ldots, x_k] \) is a classical symmetric polynomial arising historically from the representation theory of \( S_n \) and \( GL_n(\mathbb{C}) \). In our context, the important feature of Schur polynomials is that they represent the cohomology classes \( x_\lambda \in H^*(BU(k)) \). That is, we have

\[
x_\lambda \cdot x_\mu = \sum_{\nu} c_{\lambda,\mu}^\nu x_\nu \quad \iff \quad s_\lambda \cdot s_\mu = \sum_{\nu} c_{\lambda,\mu}^\nu s_\nu;
\]

where the structure coefficients \( c_{\lambda,\mu}^\nu \) are Littlewood–Richardson coefficients, computed combinatorially by the various Littlewood–Richardson rules, e.g., [LR34, KTW04], and both sums are over partitions of length \( k \).

To give a positive combinatorial rule for the structure coefficients of \( H^*(J(BU(k))) \), we must first recall one such Littlewood–Richardson rule for multiplying Schur polynomials, as well as Hazewinkel’s [Haz01] positive combinatorial multiplication rule for monomial quasisymmetric functions \( M_\alpha \).

Given a partition \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of length \( k \), its Young diagram is formed by placing \( \lambda_i \) left justified boxes in row \( i \). For example, the Young diagram of the partition \( (4, 2, 1) \) is:

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & & \\
& & \\
\end{array}
\]

if the corresponding containment holds between their Young diagrams. In such a case, \( \nu \setminus \lambda \) denotes the set-theoretic difference \( \nu \setminus \lambda \) of their Young diagrams. A semistandard tableau of shape \( \nu \setminus \lambda \) is an assignment of positive integers to the boxes of \( \nu \setminus \lambda \) such that the labels are nondecreasing left-to-right along rows and are strictly increasing top-to-bottom down columns. The content of a semistandard tableau \( T \) is the integer vector \( (c_1(T), \ldots) \), where \( c_i(T) \) counts the number of instances of \( i \) as a box label in \( T \). The reading word \( w(T) \) of a semistandard tableau \( T \) is the string obtained by reading the labels of \( T \) by rows right-to-left and then top-to-bottom. A semistandard tableau \( T \) is ballot if, for all \( i \geq 1 \), every initial segment of \( w(T) \) contains at least as many instances of \( i \) as of \( i+1 \). The following is one classical version of a Littlewood–Richardson rule.

**Proposition 6.4.** If \( \lambda, \mu, \nu \) are partitions of length \( k \), then the Littlewood–Richardson coefficient \( c_{\lambda,\mu}^\nu \) equals the number of ballot semistandard tableaux of shape \( \nu \setminus \lambda \) that have content \( \mu \). \qed

**Example 6.5.** Let \( \lambda = \mu = (2, 1, 0) \) and let \( \nu = (3, 2, 1) \). Then by Proposition 6.4, the Littlewood–Richardson coefficient \( c_{\lambda,\mu}^\nu = 2 \), as witnessed by the two ballot semistandard tableaux

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & & \\
& & \\
\end{array} \quad \text{and} \quad \begin{array}{ccc}
1 & 1 & 1 \\
& & \\
& & \\
\end{array}
\]

26
Note that the third filling of this shape with two 1s and one 2 fails the ballotness condition, so does not contribute to $c_{\lambda,\mu}^\nu = 2$.

Let $(m, n)$ be a pair of positive integers. Then, an overlapping shuffle of $(m, n)$ is a surjection

$$\tau : [m + n] \to [k]$$

for some $\max\{m, n\} \leq k \leq m + n$ such that

- $i < j \leq m \Rightarrow \tau(i) < \tau(j)$ and
- $m < i < j \Rightarrow \tau(i) < \tau(j)$.

That is to say, $\tau$ is separately strictly order preserving on the first $m$ elements of $[m + n]$ and on the last $n$ elements, but has no requirements on the relative behaviour of elements from opposite ends of the interval $[m + n]$.

Let $\alpha$ be a composition of length $m$ and $\beta$ a composition of length $n$. Then, for $\tau$ an overlapping shuffle of $(m, n)$, we define a composition $\gamma^\tau$ by

$$\gamma_i^\tau = \sum_{\tau(j) = i} (\alpha \concat \beta)_j,$$

where $\alpha \concat \beta$ denotes concatenation of strings. (Note that here the summation has at most 2 summands.)

The overlapping shuffle product of $\alpha$ and $\beta$ is the formal sum

$$\alpha \shuffle_0 \beta = \sum_{\tau} \gamma^\tau$$

over overlapping shuffles $\tau$ of $(m, n)$. (Here, the binary operator is a modified Cyrillic letter “Sha” for “shuffle,” with a subscript “o” for “overlapping” to distinguish from the more common shuffle product of S. Eilenberg and S. Mac Lane [EML53].) Note that distinct overlapping shuffles $\tau, \tau'$ can yield the same composition $\gamma^\tau = \gamma'^\tau$, so the formal sum (6.3) can have nontrivial coefficients.

**Example 6.6.** The overlapping shuffle product of the compositions $\alpha = (3)$ and $\beta = (1, 3)$ is

$$(3) \shuffle_0 (1, 3) = (3, 1, 3) + 2 \cdot (1, 3, 3) + (4, 3) + (1, 6).$$

The composition $(1, 3, 3)$ appears once from an overlapping shuffle $\tau$ with $\tau(1) = 2$ and once from an overlapping shuffle $\tau'$ with $\tau'(1) = 3$.  

The following is a monomial quasisymmetric function analogue of the Littlewood–Richardson rule. By Theorem 2.1, it yields a positive combinatorial formula for the structure coefficients of $H^*(J(CP^\infty)) = H^*(J(BU(1)))$ with respect to the cellular basis $\{x_\alpha\}$.

**Proposition 6.7 ([Haz01]).** For compositions $\alpha, \beta$, the corresponding monomial quasisymmetric functions multiply as

$$M_\alpha \cdot M_\beta = \sum_\gamma c_{\alpha,\beta}^\gamma M_\gamma,$$

where $c_{\alpha,\beta}^\gamma$ denotes the multiplicity of the composition $\gamma$ in the overlapping shuffle product $\alpha \shuffle_0 \beta$.

**Proof (sketch).** Consider the monomial expansions of $M_\alpha$ and $M_\beta$ from their definitions. Now multiply, distributing term by term. Each monomial appearing from this product corresponds to a $\gamma$ from the sum on the right. It is not hard to see that the multiplicities are also correct.  

**Example 6.8.** From Example 6.6 and Proposition 6.7, we have that

$$M_{(3)} \cdot M_{(1, 3)} = M_{(3, 1, 3)} + 2M_{(1, 3, 3)} + M_{(4, 3)} + M_{(1, 6)}.$$  

The reader may enjoy checking this computation directly after restricting to a small number of variables.  

Finally, we can give a positive combinatorial rule for the structure coefficients of $H^*(J(BU(k)))$. We write $s^{(j)}$ for the Schur polynomial $s_{\lambda}$ in the variables $x_1^{(j)}, \ldots, x_k^{(j)}$. Then Theorem 6.2 says that $J(BU(k))$ has cells indexed by tuples $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of nonzero partitions; moreover, the corresponding cellular cohomology class $x_{\Lambda}$ is represented by the power series

$$M_{\Lambda} = \sum_{1 \leq j_1 < j_2 < \cdots < j_m} s_{\lambda_{j_1}}^{(j_1)} s_{\lambda_{j_2}}^{(j_2)} \cdots s_{\lambda_{j_m}}^{(j_m)}.$$ 

Combining Littlewood–Richardson rule Proposition 6.4 with the Hazewinkel rule Proposition 6.7, we can compute the structure coefficients $c_{\Lambda, M}^{N}$ of $H^*(J(BU(k)))$ in positive combinatorial fashion.

**Theorem 6.9.** Let $\Lambda = (\lambda_1, \ldots, \lambda_r)$, $M = (\mu_1, \ldots, \mu_m)$, and $N = (\nu_1, \ldots, \nu_n)$ be tuples of nonempty partitions, each partition of length $k$. Then the structure coefficient $c_{\Lambda, M}^{N}$ of $H^*(J(BU(k)))$ equals the sum over order preserving injections $i : [\ell] \to [\nu]$ and $j : [\mu] \to [\nu]$ of the number of $n$-tuples $(T_1, \ldots, T_n)$ of ballot semistandard tableaux, where $T_i$ has shape $\nu_i/\lambda_{i^{-1}(i)}$ and content $\mu_{j^{-1}(i)}$. (Here, we take the convention that $\lambda_{i^{-1}(i)}$ is the empty partition if $i \not\in \im i$, and similarly for $\mu_{j^{-1}(i)}$.)

**Proof.** This is straightforward by combining Proposition 6.4 and Proposition 6.7. \qed

**Example 6.10.** Set $k = 3$, let $\Lambda = ((1,0,0), (2,1,0))$, and let $M = ((2,1,0))$. Then the coefficient $c_{\Lambda, M}^{N}$ of the class $x_{N} \in H^*(J(BU(3)))$ for $N = ((2,1,0), (1,0,0), (2,1,0))$ in the product $x_{\Lambda} \cdot x_{M}$ is 1, witnessed by the injections $i : [2] \to [3]$ and $j : [1] \to [3]$ given by $i(1) = 2$, $i(2) = 3$, and $j(1) = 1$, together with the 3-tuple of tableaux

$$\left(\begin{array}{l} 1 \\ 1 \\ 2 \end{array}\right),$$

where $\varnothing$ denotes the empty tableau.

Similarly, for $N' = ((1,0,0), (2,1,0), (2,1,0))$, the coefficient $c_{\Lambda, M}^{N'}$ is easily computed to be 2, witnessed by the 3-tuples of tableaux

$$\left(\begin{array}{l} \varnothing \\ 1 \\ 2 \end{array}\right) \text{ and } \left(\begin{array}{l} \varnothing \\ 1 \\ 2 \end{array}\right).$$

A more interesting calculation is the coefficient $c_{\Lambda, M}^{N''}$ for $N'' = ((1,0,0), (1,0,0), (3,2,1))$. Here, we have

$$c_{\Lambda, M}^{N''} = 4,$$

witnessed by the four 3-tuples

$$\left(\begin{array}{lll} \varnothing & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & \end{array}\right), \left(\begin{array}{lll} \varnothing & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & \end{array}\right), \left(\begin{array}{lll} 1 & \varnothing & 1 \\ 1 & 2 & 1 \\ 2 & 1 & \end{array}\right), \text{ and } \left(\begin{array}{lll} 1 & \varnothing & 1 \\ 1 & 2 & 1 \\ 2 & 1 & \end{array}\right).$$

In the same fashion, an analogous characterization of the structure coefficients of $H^*(J(X))$ can be given in any case with a correspondingly explicit combinatorial description of $H^*(X)$.

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