MOTIVIC MEASURES AND STABLE BIRATIONAL GEOMETRY

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Abstract. We study the motivic Grothendieck group of algebraic varieties from the point of view of stable birational geometry. In particular, we obtain a counter-example to a conjecture of M. Kapranov on the rationality of motivic zeta-function.

1. Introduction

1.1. Grothendieck ring of varieties. Fix a field $k$. Let $V_k$ denote the category of $k$-varieties. By a $k$-variety we mean a reduced separated scheme of finite type over $k$. Consider the Grothendieck ring $K_0[V_k]$: this is the abelian group generated by isomorphism classes of $k$-varieties, subject to the relations $[X - Y] = [X] - [Y]$, where $Y$ is a closed in $X$. The product over $k$ turns it into a commutative ring with 1. It appears that very little is known about this interesting ring. For example, one would like to know the answer to the following basic question.

Question 1.2. Let $X,Y$ be $k$-varieties such that $[X] = [Y]$. Is it possible to partition $X$ and $Y$ by a finite number of locally closed subvarieties which are pairwise isomorphic?

1.3. Motivic measures. Let $A$ be a commutative ring. An $A$-valued motivic measure on $V_k$ is a ring homomorphism $\mu : K_0[V_k] \to A$.

1.4. Motivic zeta-function. For a $k$-variety $X$ denote by $X^{(n)}$ the $n$-fold symmetric product of $X$. Given a motivic measure $\mu : K_0[V_k] \to A$
we define (following [Ka]) the motivic zeta-function of $X$ as the formal power series

$$\zeta_\mu(X, t) = 1 + \sum_{n=1}^{\infty} \mu([X^{(n)}]) t^n \in A[[t]].$$

For example,

$$\zeta_{id}(\mathbb{P}^1, t) = \sum_{n=0}^{\infty} [\mathbb{P}^n] t^n = \frac{1}{(1-t)(1-[\mathbb{A}^1]t)},$$

$$\zeta_{id}(E, t) = 1 + \sum_{n=1}^{\infty} [E][\mathbb{P}^{n-1}] t^n = \frac{1 + ([E] - [\mathbb{P}^1]) t + [\mathbb{A}^1] t^2}{(1-t)(1-[\mathbb{A}^1]t)}$$

for any elliptic curve $E$, and

$$\zeta_{id}(\mathbb{P}^2, t) = \frac{1}{(1-t)(1-[\mathbb{A}^1]t)(1-[\mathbb{A}^2]t)}.$$

Checking these formulas requires a certain amount of cutting and pasting, using the relation $[X] = [X - Y] + [Y]$ systematically.

Kapranov proves the following theorem in [Ka]:

**Theorem 1.5.** Let $A$ be a field and $\mu : K_0[V_k] \to A$ be a motivic measure. Let $X \in V_k$ be a curve. Then $\zeta_\mu(X, t)$ is a rational function.

In the same paper Kapranov remarks that it is natural to expect a similar rationality result for varieties of higher dimension. We give a negative solution to this problem.

**Theorem 1.6.** Assume that $k = \mathbb{C}$. There exists a field $\mathcal{H}$ and a motivic measure $\mu : K_0[V_\mathbb{C}] \to \mathcal{H}$ with the following property: if $X$ is a smooth complex projective surface such that $P_g(X) = h^{2,0}(X) \geq 2$, then the zeta-function $\zeta_\mu(X, t)$ is not rational.

We expect that a similar result holds for any smooth projective variety of even dimension and positive Kodaira dimension.

1.7. **Stable birational geometry.** The above theorem on the irrationality of the zeta-function follows easily from our analysis of the Grothendieck ring $K_0[V_\mathbb{C}]$ from the point of view of stable birational equivalence of varieties. Namely, recall that (irreducible) varieties $X,Y$ are stably birational if $X \times \mathbb{P}^k$ is birational to $Y \times \mathbb{P}^l$ for some $k, l \geq 0$. Let $SB$ denote the multiplicative monoid of classes of stable birational equivalence of varieties; let $\mathbb{Z}[SB]$ be the corresponding monoid ring. Consider the principal ideal $I \subset K_0[V_\mathbb{C}]$ generated by the class of the
affine line $\mathbb{A}^1$. The main result of the next section is the following isomorphism of rings.

$$K_0[\mathcal{V}_C]/I \simeq \mathbb{Z}[SB].$$

This isomorphism is interesting in its own right. It implies, in particular, that classes of stable birational equivalence from a $\mathbb{Z}$-basis of the group $K_0[\mathcal{V}_C]/I$. The hard part in establishing this isomorphism is to show that the map from the LHS to the RHS is well defined. This is essentially the content of Theorem 2.3.

We also show that for a ring homomorphism $\phi : K_0[\mathcal{V}_C] \to B$ the following conditions are equivalent

i) $I \subset \text{Ker}(\phi),$

ii) if $X, Y$ are smooth complete varieties which are birational, then $\phi([X]) = \phi([Y]).$

2. Grothendieck ring of varieties and stable birational equivalence

2.1. The Grothendieck ring of varieties revisited. We make a few remarks about the Grothendieck ring $K_0[\mathcal{V}_k]$ which will be used later.

A. If a variety $X$ is partitioned by locally closed subvarieties $X_1, ... X_n$, then

$$[X] = \sum [X_i].$$

B. Every variety can be partitioned by a finite number of smooth varieties. Hence classes of smooth varieties generate the group $K_0[\mathcal{V}_k]$.

C. Let $E \to X$ be a vector bundle of dimension $d$. Then it is locally trivial in the Zariski topology. Hence the same is true about its projectivization $\overline{E} \to X$. Thus $[\overline{E}] = [X][\mathbb{P}^{d-1}]$ in the ring $K_0[\mathcal{V}_k]$.

D. Let $f : X \to Y$ be a proper morphism of smooth varieties, which is a blowup with a smooth center $Z \subset Y$ of codimension $d$. Then the projective bundle $f^{-1}(Z) \to Z$ is the projectivization of the normal bundle of $Z$ in $Y$. Hence, by the previous remark $[f^{-1}(Z)] = [Z][\mathbb{P}^{d-1}]$ in the ring $K_0[\mathcal{V}_k]$.

E. Assume that $\text{char}(k) = 0$. By Hironaka’s theorem every smooth variety $X$ is isomorphic to a dense open subset of a smooth complete variety. Hence the group $K_0[\mathcal{V}_k]$ is generated by smooth complete varieties.

Let $k = \mathbb{C}$. We will be using the following theorem of Wlodarczyk (see [W], [AKMW]).
Theorem 2.2. Let $\phi : X_1 \to X_2$ be a rational birational map of smooth complete varieties. Let $U \subset X_1$ be an open subset where $\phi$ is an isomorphism. Then $\phi = \phi_1 \cdots \phi_1$, where each $\phi_i$ is a rational birational map of smooth complete varieties and either $\phi_i$ or $\phi_i^{-1}$ is a regular map which is a blowup with a smooth center disjoint from $U$.

The basic result of this paper is the following:

Theorem 2.3. Put $k = \mathbb{C}$. Let $G$ be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. Denote by $\mathcal{M}$ the multiplicative monoid of isomorphism classes of smooth complete irreducible varieties. Let $\Psi : \mathcal{M} \to G$ be a homomorphism of monoids such that

(i) $\Psi([X]) = \Psi([Y])$ if $X$ and $Y$ are birational;
(ii) $\Psi([\mathbb{P}^n]) = 1$ for all $n \geq 0$.

Then there exists a unique ring homomorphism

$$\Phi : K_0[V_\mathbb{C}] \to \mathbb{Z}[G]$$

such that $\Phi([X]) = \Psi([X])$ for $[X] \in \mathcal{M}$.

Proof. To simplify notation we will write $\Psi(X)$ and $\Phi(X)$ for $\Psi([X])$ and $\Phi([X])$ respectively.

We will define the elements $\Phi(X) \in \mathbb{Z}[G]$ by induction on the dimension of the variety $X$. The induction step will require checking that

i) $\Phi(X)$ is well defined for each variety $X$;
ii) $\Phi$ preserves the defining relations of the abelian group $K_0[V_\mathbb{C}]$;
iii) $\Phi$ is multiplicative.

Let us formulate a series of constructions and assertions which depend on $n$.

Construction $A_n$. If $X$ is an irreducible smooth complete variety of dimension $\leq n$ then $\Phi(X) := \Psi(X)$.

Construction $B_n$. Let $X$ be a smooth variety of dimension $\leq n$ with connected components $X_1, ..., X_k$. For each $i$ choose an open embedding $X_i \hookrightarrow \overline{X_i}$ where $\overline{X_i}$ is smooth, complete, and irreducible. Then put

$$\Phi(X) := \sum \Phi(\overline{X_i}) - \sum \Phi(\overline{X_i} - X_i).$$

Construction $C_n$. Let $X$ be an arbitrary variety of dimension $\leq n$. Then put

$$\Phi(X) := \Phi(X - X^{\text{sing}}) + \Phi(X^{\text{sing}}).$$

Assertion $D_n$. Let $X, Y$ be varieties of dimension $\leq n$, and $f : X \to Y$ be a morphism with the following property: there exists a stratification
of $Y$ by locally closed subvarieties $Y_i$, so that $f^{-1}(Y_i) \simeq \mathbb{P}^{n_i} \times Y_i$ for some $n_i \geq 0$ and $f : f^{-1}(Y_i) \to Y_i$ is the projection. Then

$$\Phi(X) = \Phi(Y).$$

**Assertion** $E_n$. Let $X$ be a variety of dimension $\leq n$ and $Y \subset X$ a closed subvariety. Then

$$\Phi(X) = \Phi(Y) + \Phi(X - Y).$$

**Assertion** $F_n$. Let $X, Y$ be varieties such that $\dim X + \dim Y \leq n$, then

$$\Phi(X \times Y) = \Phi(X) \cdot \Phi(Y).$$

We will use the following logic in proving the theorem. Assume that $\Phi$ is constructed according to Constructions $A_{n-1}, B_{n-1}, C_{n-1}$, Construction $B_{n-1}$ is unambiguous, and Assertions $D_{n-1}, E_{n-1}, F_{n-1}$ are proved. Then, in particular, $\Phi$ is defined on all classes $[X]$ of varieties $X$ of dimension $\leq n - 1$ in such a way that

1) $\Phi(X) = \Psi(X)$ if $X$ is irreducible, smooth, and complete.
2) $\Phi(X) = \Phi(Y) + \Phi(X - Y)$ if $Y \subset X$ is a closed.
3) $\Phi(X \times Y) = \Phi(X) \cdot \Phi(Y)$ if $\dim X + \dim Y \leq n - 1$.

We extend $\Phi$ by linearity to linear combinations of such classes. Then we use Definitions $A_n, B_n, C_n$ to extend $\Phi$ to classes of varieties of dimension $\leq n$ and prove Assertions $D_n, E_n, F_n$ which are needed to ensure that Definition $B_n$ is unambiguous and that this extension of $\Phi$ satisfies the properties 1),2),3) above. This will prove the existence statement of the theorem. The uniqueness is clear since classes of smooth complete irreducible varieties generate the group $K_0[\mathcal{V}_C]$.

**Base case.** We can use Definitions $A_0, B_0, C_0$ without ambiguity, and Assertions $D_0, E_0, F_0$ obviously hold.

**Induction step** ($n - 1) \to n$. Assume that the map $\Phi$ has been defined using Constructions $A_{n-1}, B_{n-1}, C_{n-1}$, Construction $B_{n-1}$ is unambiguous, and Assertions $D_{n-1}, E_{n-1}, F_{n-1}$ are true.

Let $X$ be an irreducible smooth complete variety of dimension $n$. Then define $\Phi(X)$ according to $A_n$.

Let $X$ be a smooth $n$-dimensional variety. Use $B_n$ to define $\Phi(X)$. We prove that it is independent of the choice of smooth compactifications (such compactifications exist by Hironaka’s theorem). We may assume that $X$ is irreducible. Let $X \leftrightarrow \overline{X}, X \leftrightarrow \overline{X}'$ be two open embeddings with $\overline{X}, \overline{X}'$ smooth and complete. Put $Y := \overline{X} - X$, $Y' := \overline{X}'$. We need to show that $\Phi(\overline{X}) - \Phi(Y) = \Phi(\overline{X}') - \Phi(Y')$. Since $\Psi$ is a birational invariant, Construction $A_n$ implies $\Phi(\overline{X}) = \Phi(\overline{X}')$. So
it remains to show that $\Phi(Y) = \Phi(Y')$. By the Theorem 2.2 we may reduce to the case that there exists a morphism $f : X' \to X$ which is a blowup with a smooth center $Z \subset Y$. Then by Remark C above the map $f : f^{-1}(Z) \to Z$ is a Zariski locally trivial fibration with fibre $\mathbb{P}^k$. So by $D_{n-1}$, $\Phi(Y) = \Phi(Y')$. This justifies the definition $B_n$.

Now for a general $n$-dimensional variety $X$ we define $\Phi(X)$ according to $C_n$.

It remains to prove the Assertions $D_n$, $E_n$, $F_n$. For a variety $W$ put $W^{ns} := W - W^{sing}$.

Proof of $E_n$. Let $X$ be a variety of dimension $n$ and $Y \subset X$ be a closed subvariety. Put $U := X - Y$. We need to prove that $\Phi(X) = \Phi(U) + \Phi(Y)$.

Assume first that $X$ is smooth. If $W$ is any variety and $W_1, ..., W_s$ are the connected components of $W$, then by Constructions $B_n$ and $C_n$, $\Phi(W) = \sum \Phi(W_i)$. So we may assume that $X$ is connected. If $\dim Y = n$, then $U = \emptyset$ and we are done. So assume that $\dim Y < n$.

Let $\overline{X} \to \overline{X}$ be a smooth compactification of $X$, $Z := \overline{X} - X$, $S := \overline{X} - U = \bigcup \overline{Y}$. Then by Construction $B_n$,

$$\Phi(X) = \Phi(\overline{X}) - \Phi(Z), \quad \Phi(U) = \Phi(\overline{X}) - \Phi(S).$$

By $E_{n-1}$ $\Phi(S) = \Phi(Z) + \Phi(Y)$. So

$$\Phi(X) = \Phi(U) + \Phi(Y).$$

Now let $X$ be any variety. We have

$$U^{sing} = X^{sing} \cap U, \quad U^{ns} = X^{ns} \cap U.$$

By $C_n$,

$$\Phi(X) = \Phi(X^{ns}) + \Phi(X^{sing}),$$

$$\Phi(U) = \Phi(U^{ns}) + \Phi(U^{sing}).$$

By the argument above,

$$\Phi(X^{ns}) = \Phi(U^{ns}) + \Phi(Y \cap X^{ns}),$$

and therefore

$$\Phi(X) = \Phi(U) - \Phi(U^{sing}) + \Phi(Y \cap X^{ns}) + \Phi(X^{sing}).$$

So it remains to prove that

$$\Phi(Y) = \Phi(X^{sing}) + \Phi(Y \cap X^{ns}) - \Phi(U^{sing}).$$

Since $X^{sing} = U^{sing} \bigcup (Y \cap X^{sing})$, by $E_{n-1}$,

$$\Phi(X^{sing}) = \Phi(U^{sing}) + \Phi(Y \cap X^{sing}).$$
So it suffices to prove that \( \Phi(Y) = \Phi(Y \cap X^{ns}) + \Phi(Y \cap X^{sing}) \). The last equality is \( E_n \), with \( X, Y, U \) replaced by \( Y, Y \cap X^{sing}, Y \cap X^{ns} \). Since we may assume that \( Y \) is a proper subset of \( X \) the proof is finished by Noetherian induction on \( X \).

**Proof of \( F_n \).** Let \( X, Y \) be varieties such that \( \dim X + \dim Y = n \). We need to prove that \( \Phi(X \times Y) = \Phi(X) \cdot \Phi(Y) \). If \( \dim Y = 0 \), then we are done. So may assume that \( \dim X, \dim Y \geq 1 \). By \( E_n \) we have

\[
\Phi(X \times Y) = \Phi(X^{ns} \times Y^{ns}) + \Phi(X^{ns} \times Y^{sing}) + \Phi(X^{sing} \times Y^{ns}) + \Phi(X^{sing} \times Y^{sing}),
\]

\[
\Phi(X) \cdot \Phi(Y) = \Phi(X^{ns}) \cdot \Phi(Y^{ns}) + \Phi(X^{ns}) \cdot \Phi(Y^{sing}) + \Phi(X^{sing}) \cdot \Phi(Y^{ns}) + \Phi(X^{sing}) \cdot \Phi(Y^{sing}).
\]

Hence by \( F_{n-1} \) we may assume that \( X, Y \) are smooth (and connected). Let \( X \leftrightarrow \overline{X}, Y \leftrightarrow \overline{Y} \) be smooth compactifications of \( X, Y \), \( X' := \overline{X} - X, Y' := \overline{Y} - Y \). Again by \( E_n \),

\[
\Phi(\overline{X}) \cdot \Phi(\overline{Y}) = \Phi(X) \cdot \Phi(Y) + \Phi(X') \cdot \Phi(Y) + \Phi(X') \cdot \Phi(Y'),
\]

\[
\Phi(\overline{X} \times \overline{Y}) = \Phi(X \times Y') + \Phi(X' \times Y) + \Phi(X' \times Y').
\]

We have \( \Phi(\overline{X} \times \overline{Y}) = \Phi(\overline{X}) \cdot \Phi(\overline{Y}) \), since \( \Psi \) is multiplicative. Now using \( F_{n-1} \) we conclude that \( \Phi(X \times Y) = \Phi(X) \cdot \Phi(Y) \).

**Proof of \( D_n \).** Using \( E_n \) repeatedly we may assume that \( X = Y \times \mathbb{P}^k \) and \( f : X \to Y \) is the projection. Then by \( F_n \), \( \Phi(X) = \Phi(Y) \cdot \Phi(\mathbb{P}^k) \), and \( \Phi(\mathbb{P}^k) = \Psi(\mathbb{P}^k) = 1 \).

**Remark 2.4.** The Grothendieck group \( K_0[V_C] \) is generated by classes of smooth complete varieties. In [L] Looijenga asserts that it suffices to consider the following relations: let \( X, Y \) be smooth complete and \( f : X \to Y \) be a morphism which is a blowup with a smooth center \( Z \subset Y \); then

\[
[X] - [f^{-1}(Z)] = [Y] - [Z].
\]

This is a strong result, which immediately implies our Theorem 2.3. Indeed, \( f^{-1}(Z) \) is birational to \( Z \times \mathbb{P}^k \). So by the hypotheses of Theorem 2.3, \( \Psi(X) = \Psi(Y), \Psi(f^{-1}(Z)) = \Psi(Z \times \mathbb{P}^k) = \Psi(Z) \). Therefore the desired ring homomorphism \( \Phi \) exists. However, since we could not produce a proof of Looijenga’s result, we chose to give an argument that does not depend on it. (Looijenga informs us that a proof is being written up by one of his students.)
2.5. **The universal homomorphism** $\Psi : \mathcal{M} \to G$. There exists a universal homomorphism of monoids $\Psi$ which satisfies the hypotheses of Theorem 2.3. Namely, recall that varieties $X, Y$ are *stably birational* if $X \times \mathbb{P}^k$ and $Y \times \mathbb{P}^l$ are birational for some $k, l \geq 0$. Denote by $SB$ the multiplicative monoid of stable birational equivalence classes of varieties. We have a tautological (surjective) homomorphism $\Psi_{SB} : \mathcal{M} \to SB$ which satisfies the hypotheses (i),(ii) of Theorem 2.3 (with $\Psi = \Psi_{SB}, G = SB$). By definition any homomorphism $\Psi$ as in the theorem factors through $\Psi_{SB}$. Denote by $\Phi_{SB} : K_0[\mathcal{V}_C] \to \mathbb{Z}[SB]$ the ring homomorphism corresponding to $\Psi_{SB}$ by the theorem. We obtain the following immediate corollary.

**Corollary 2.6.** Let $X_1, ... X_k, Y_1, ... Y_m$ be smooth complete varieties. Let $m_i, n_j \in \mathbb{Z}$ be such that

$$\sum m_i[X_i] = \sum n_j[Y_j]$$

in $K_0[\mathcal{V}_C]$. Then $k = m$ and after renumbering the varieties $X_i$ and $Y_i$ are stably birational and $m_i = n_i$.

**Proof.** Applying the ring homomorphism $\Phi_{SB}$ to the above equality we obtain the equality in the monoid ring $\mathbb{Z}[SB]$: 

$$\sum m_i \Psi_{SB}(X_i) = \sum n_j \Psi_{SB}(Y_j)$$

and the proposition follows.

The above corollary means that any variety is a *unique* (up to a stable birational equivalence) linear combination (in $K_0[\mathcal{V}_C]$) of smooth complete varieties. This is in the spirit of the basic Question formulated in the introduction. The difference is, of course, that instead of cutting varieties in pieces we complete and resolve singularities.

The next proposition clarifies the relation between the Grothendieck ring $K_0[\mathcal{V}_C]$ and the monoid ring $\mathbb{Z}[SB]$.

**Proposition 2.7.** The kernel of the (surjective) homomorphism $\Phi_{SB} : K_0[\mathcal{V}_C] \to \mathbb{Z}[SB]$ is the principal ideal generated by the class $[\mathbb{A}^1]$ of the affine line $\mathbb{A}^1$.

**Proof.** Since $\Phi_{SB}([\mathbb{P}^1]) = \Phi_{SB}([1] + [\mathbb{A}^1]) = 1$, we have $\Phi_{SB}([\mathbb{A}^1]) = 0$.

Let $a \in \text{Ker}(\Phi_{SB})$. Express $a$ as a linear combination

$$a = [X_1] + ... + [X_k] - [Y_1] - ... - [Y_l],$$

where $X_i, Y_j$ are smooth and complete. Since

$$\Phi_{SB}(a) = \sum \Psi_{SB}(X_i) - \sum \Psi_{SB}(Y_j) = 0,$$
we get \( k = l \) and, after renumbering, \( X_i \) is stably birational to \( Y_j \). Thus it suffices to show that if \( X, Y \) are smooth, complete, and stably birational, then \([X] - [Y] \in K_0[V_C] \cdot [A^1] \). Note that
\[
[X \times \mathbb{P}^k] - [X] = [X] \cdot [A^1 + A^2 + \cdots + A^k],
\]
so we may assume that \([X] \) and \([Y] \) are birational. Moreover by Theorem 2.2 we may assume that \( X \) is a blowup of \( Y \) with a smooth center \( Z \subset Y \) and exceptional divisor \( E \subset X \). Then \([E] = [P^t] \cdot [Z] \) for some \( t \) and
\[
[X] - [Y] = [E] - [Z] = ([A^1] + [A^2] + \cdots + [A^t]) \cdot [Z].
\]

The next proposition “explains” the role of \( A^1 \) in birational geometry.

**Proposition 2.8.** Let \( \alpha : K_0[V_C] \to B \) be a ring homomorphism (i.e. \( \alpha \) is a motivic measure). The following conditions are equivalent:

1. \( \alpha([A^1]) = 0 \),
2. if smooth complete varieties \( X, Y \) are birational, then \( \alpha([X]) = \alpha([Y]) \).

If these conditions hold, then \( \alpha([Z]) = \alpha([W]) \) for any smooth complete varieties \( Z, W \) which are stably birationally equivalent.

**Proof.** Assume i). Then \( \alpha \) factors through the homomorphism \( \Phi_{SB} \) and ii) follows. Assume ii). Let \( \tilde{X} \to X \) be a blowup of a smooth complete surface at a point. Then \([\tilde{X}] = [X] + [A^1] \) and hence \( \alpha([A^1]) = 0 \).

To prove the last assertion note that \( \alpha([A^1]) = 0 \) implies that \( \alpha([\mathbb{P}^n]) = 1 \).

The last assertion of the proposition means that birational motivic measures are automatically stably birational.

3. Irrationality of the zeta-function

3.1. The motivic measure \( \mu_h : K_0[V_C] \to \mathcal{H} \). Let \( C \subset \mathbb{Z}[t] \) be the multiplicative monoid of polynomials with a positive leading coefficient. Consider the corresponding monoid ring \( \mathbb{Z}[C] \).

**Lemma 3.2.** The ring \( \mathbb{Z}[C] \) is an integral domain.

**Proof.** The ring \( \mathbb{Z}[t] \) is factorial and any element of \( C \) is a unique product of elements of \( C \), which are prime in \( \mathbb{Z}[t] \) (the only unit in \( C \) is 1). Thus \( C \) is isomorphic to the group \( \oplus \mathbb{N} \), where the summation is over all prime elements of \( \mathbb{Z}[t] \). Hence \( \mathbb{Z}[C] \) is a polynomial ring, so it is an integral domain. \( \square \)
Definition 3.3. Let $\mathcal{H}$ be the field of fractions of $\mathbb{Z}[C]$.

Definition 3.4. For a smooth projective irreducible complex variety $X$ of dimension $d$ define

$$\Psi_h(X) := 1 + h^1(X)t + \ldots + h^d(X)t^d \in C.$$ 

For any smooth complete complex irreducible variety $Z$ put $\Psi_h(Z) = \Psi_h(X)$, where $X$ is a smooth projective variety which is birational to $Z$. It is well known that if smooth projective varieties $X$ and $Y$ are birational then $\Psi_h(X) = \Psi_h(Y)$ ([Ha], Ch. 2, Exercise 8.8). Therefore $\Psi_h$ is well defined. The Künneth formula implies that $\Psi_h$ is a homomorphism from the multiplicative monoid $\mathcal{M}$ of isomorphism classes of smooth complete irreducible varieties to $C$. It satisfies the hypotheses of Theorem 2.3. Thus it extends to a motivic measure $\mu_h : K_0[V_C] \to \mathcal{H}$.

Definition 3.5. For a smooth projective irreducible variety $X$ of dimension $d$ denote as usual

$$P_g(X) = h^d(X) = h^0(X, \omega_X),$$

where $\omega_X$ is the canonical line bundle on $X$. For an arbitrary irreducible variety $Z$ of dimension $d$ put

$$P_g(Z) := P_g(X),$$

where $X$ is any smooth projective variety in the birational class of $Z$. Thus $P_g$ becomes a multiplicative function from the collection of isomorphism classes of irreducible varieties to natural numbers.

Lemma 3.6. Let $Y_1, \ldots, Y_s, Z$ be irreducible varieties of dimension $d$. Assume that $\mu_h([Z]) = \sum n_i \mu_h([Y_i])$ for some $n_i \in \mathbb{Z}$. If $P_g(Z) \neq 0$, then $P_g(Z) = P_g(Y_i)$ for some $i$.

Proof. Note that for any irreducible variety $W$ of dimension $d$ we have the equality in $K_0[V_C]$

$$[W] = [\overline{W}] + \sum m_j[W_j],$$

where $\overline{W}$ is a smooth projective variety in the birational class of $W$ and $W_j$'s are smooth projective varieties of dimension $< d$. Thus replacing $Y_1, \ldots, Y_s, Z$ by smooth projective varieties $\overline{Y}_1, \ldots, \overline{Y}_s, \overline{Z}$ from the same birational class we obtain an equality

$$\mu_h([\overline{Z}]) = \sum n_i \mu_h([\overline{Y}_i]) + \sum l_\beta \mu_h([X_\beta]).$$
for some $l_\beta \in \mathbb{Z}$ and some smooth projective irreducible varieties $X_\beta$ of dimension $< d$. By definition this means

$$\Psi_h(Z) = \sum n_i \Psi_h(Y_i) + \sum l_\beta \Psi_h(X_\beta),$$

and the lemma follows.

**Theorem 3.7.** Consider the motivic measure $\mu_h : K_0[V_C] \to \mathcal{H}$. Let $X$ be a smooth complex projective surface. Assume that $P_g(X) \geq 2$. Then the zeta function $\zeta_{\mu_h}(X, t)$ is not rational.

**Proof.** For a variety $Y$ we will write for short $\mu(Y) = \mu_h([Y])$. Let $X$ be as in the theorem and assume that the zeta-function

$$\zeta_{\mu_h}(X, t) = 1 + \sum_{n=1}^{\infty} \mu(X^{(n)}) \in \mathcal{H}[[t]]$$

is rational. Recall the following characterization of rational power series (with coefficients in a field). A power series $\sum a_i t^i$ is a rational function if and only if there exist $n > 0, n_0 > 0$ such that for each $m > n_0$ the determinant of the matrix

$$\begin{pmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m+n} & a_{m+n+1} & \cdots & a_{m+2n} \end{pmatrix}$$

is zero.

In case $\sum a_i t^i = \zeta_{\mu_h}(X, t)$ this equality implies

$$\sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \mu(X^{(m-1+\sigma(1))} \times X^{(m+\sigma(2))} \times \cdots \times X^{(m+n-1+\sigma(n+1))}) = 0. \quad (*)$$

Note that the summand $\mu(X^{(m)} \times X^{(m+2)} \times \cdots \times X^{(m+2n)})$ appears exactly once.

**Claim.** There exists $m > 0$ such that

$$P_g(X^{(m)} \times \cdots \times X^{(m+2n)}) = P_g(X^{(m-1+\sigma(1))} \times \cdots \times X^{(m+n-1+\sigma(n+1))})$$

implies that $\sigma$ is the identity permutation.

Assuming the claim we apply Lemma 3.6 with $Z = X^{(m)} \times X^{(m+2)} \times \cdots \times X^{(m+2n)}$ and $Y_j$'s being the other summands in $(*)$. We get a contradiction and the theorem follows. So it remains to prove the claim.
Lemma 3.8. Let \( r = P_g(X) \). Then
\[
P_g(X^{(n)}) = \binom{r + n - 1}{r - 1}.
\]

Assume the lemma and let us prove the claim. By our assumption \( r = P_g(X) \geq 2 \), so
\[
P_g(X^{(m+1)} \times X^{(m+2)} \times \cdots \times X^{(m+n-1+\sigma(n+1))})
\]
\[
= \prod_{j=0}^{n} P_g(X^{(m+j-1+\sigma(j+1))})
\]
\[
= \frac{1}{(r-1)! n!} \prod_{j=0}^{n} (m + r + j - 2 + \sigma(j + 1)) \cdots (m + j + \sigma(j + 1)).
\]

Therefore the multiset \( \{ i + \sigma(i) \mid i = 1, \ldots, n + 1 \} \) is determined by the expression for geometric genus, regarded as a polynomial in \( m \). This proves the claim and the theorem.

Proof of the lemma. Consider the quotient morphism \( \pi : X^n \to X^{(n)} \). Then \( H^i(X^{(n)}, \mathbb{C}) = H^i(X^n, \mathbb{C})^{S_n} \), where the action of \( S_n \) on \( H^i(X^n, \mathbb{C}) \) is twisted by the sign if \( i \) is odd. Clearly, the \( S_n \)-action preserves the subspaces \( H^{p,q}(X, \mathbb{C}) \). The embedding \( H^*(X^{(n)}, \mathbb{C}) \hookrightarrow H^*(X^n, \mathbb{C}) \) is a morphism of mixed Hodge structures; thus the Hodge structure on \( H^*(X^{(n)}, \mathbb{C}) \) is, in fact, pure and \( H^{p,q}(X^{(n)}, \mathbb{C}) = H^{p,q}(X^n, \mathbb{C})^{S_n} \).

In particular,
\[
H^{2n,0}(X^{(n)}, \mathbb{C}) = \text{Sym}^n H^{2,0}(X, \mathbb{C}),
\]
and
\[
h^{2n,0}(X^{(n)}) = \binom{r + n - 1}{r - 1}.
\]

It remains to prove that \( P_g(X^{(n)}) = h^{2n,0}(X^{(n)}) \), i.e. \( h^{2n,0}(\bar{X}) = h^{2n,0}(X^{(n)}) \) for a resolution of singularities \( \bar{X} \to X \).

Denote by \( X^{[n]} \) the nth Hilbert scheme parametrizing closed zero-dimensional subschemes of length \( n \) of \( X \). It is known that \( X^{[n]} \) is a smooth projective variety of dimension \( 2n \) and there exists a natural map \( \pi : X^{[n]} \to X^{(n)} \) which is a resolution of singularities ([Na]). Let us show that \( h^{2n,0}(X^{[n]}) = h^{2n,0}(X^{(n)}) \). This follows immediately from the result of L. Gottsche and W. Soergel which computes the Hodge structure on the cohomology \( H^*(X^{[n]}, \mathbb{C}) \). Let \( P(n) \) be the set
of partitions of $n$. We can write $\alpha \in P(n)$ as $n = \alpha_1 \cdot 1 + \ldots + \alpha_r \cdot r$. Put $|\alpha| = \sum \alpha_i$, $X^{(\alpha)} = X^{(\alpha_1)} \times \ldots \times X^{(\alpha_r)}$.

**Theorem 3.9.** ([GS]). There exists a canonical isomorphism of mixed Hodge structures

$$H^{i+2n}(X^{[n]}, \mathbb{Q}) \otimes \mathbb{Q}(n) = \bigoplus_{\alpha \in P(n)} H^{i+2|\alpha|}(X^{(\alpha)}, \mathbb{Q}) \otimes \mathbb{Q}(|\alpha|).$$

Here $\mathbb{Q}(1)$ is the one-dimensional Hodge structure of weight $-2$ and bidegree $(-1, -1)$. The proof of this theorem is based on the decomposition theorem for mixed Hodge modules.

Note that $\dim X^{(\alpha)} = 2|\alpha|$. Since the Hodge structure $H^\bullet(X^{(\alpha)}, \mathbb{C})$ is a substructure of $H^\bullet(X^{\alpha_1} \times \ldots \times X^{\alpha_r}, \mathbb{C})$, it does not contain a summand $H^{p,q}$, unless $|p - q| \leq 2|\alpha|$. The same remains true after a twist by a tensor power of $\mathbb{Q}(1)$. Therefore the only summand on the RHS which contributes to $H^{2n,0}(X^{[n]}, \mathbb{Q}) \otimes \mathbb{Q}(n)$ is $H^{0+2n}(X^{[n]}, \mathbb{Q}) \otimes \mathbb{Q}(n)$. This proves the lemma.

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