Betti Numbers of GIT quotients of products of projective planes

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Abstract

We study the GIT quotients for the diagonal action of the algebraic group $SL_3(\mathbb{C})$ on the $n$-fold product of $\mathbb{P}^2(\mathbb{C})$: in particular we determine a strategy in order to determine the (intersection) Poincaré polynomial of any quotient variety. In the special case $n=6$ we determine an explicit formula for the (intersection) Betti numbers of a quotient variety, depending only on the combinatorics of the weights of the polarization $m \in \mathbb{Z}_{>0}^6$.

Introduction

Geometric Invariant Theory gives a method for constructing “quotients” varieties for any linear action of a complex reductive algebraic group $G$ on a projective variety $X$ (see [6] and [1] for the general setting). The quotients depend on the choice of an ample linearized line bundle $L$: in particular Dolgachev-Hu [2] and Thaddeus [7] proved that only a finite number of GIT quotients can be obtained when $L$ varies and gave a general description of the maps relating the various quotients.

In this paper we restrict our attention to the case when $G = PSL_3(\mathbb{C})$ acts diagonally on $X$, the $n$-fold product of $\mathbb{P}^2(\mathbb{C})$, $\mathbb{P}^2(\mathbb{C})^n$ (that we have already studied in [5]) and describe the topology of an arbitrary quotient variety, both geometric and categorical: as a main result we describe an algorithm to compute the (intersection) Poincaré polynomial of any quotient variety. Moreover in the special case $n=6$ we obtain an explicit formula for the (intersection) Betti numbers of a quotient variety, depending only on the combinatorics of the weights of the polarization $m \in \mathbb{Z}_{>0}^6$.

The contents of the paper are more precisely as follows.

Section 1 is concerned with the main features of the quotients of $G = PSL_3(\mathbb{C})$ acting on $X = \mathbb{P}^2(\mathbb{C})^n$: we remind the main features of a categorical quotient $X^{SS}///G$ and we describe the $G$ ample cone $C^G(X)$. At the end we discuss the birational maps $\theta_{\pm k}$ that can relate different quotients.

Section 2 discusses small resolutions: we give conditions for a birational map
\[ \theta_{+k} \] to be a small map, and we discuss the problem of the existence of a small resolution.

In Section 3 we give an algorithm that permits to compute the (intersection) Poincaré polynomial of any quotient. The main key is the decomposition theorem of Beilinson-Bernstein-Deligne.

Section 4 contains the special case \( n = 6 \): Theorem 4.2 contains an explicit formula for the (intersection) Betti numbers of any quotient: this formula depends only on the combinatorics of the polarization \( m \in \mathbb{Z}_{>0} \).

## 1 \( G = \mathbb{P}SL_3(\mathbb{C}) \) acting on \( X = \mathbb{P}^2(\mathbb{C})^n \)

First of all remind that \( \text{Pic}^G(X) \cong \mathbb{Z}^n \): an ample line bundle \( L \) over \( X \) is determined by \( L = L(m) := L(m_1, \ldots, m_n) \), where \( m_i \in \mathbb{Z}_{>0}, \forall i \) (supp. \( m_i \geq m_{i+1} \) for all \( i : 1, \ldots, n-1 \)). Then the set of semi-stable points \( X^{SS}(m) \) is described by the Hilbert-Mumford numerical criterion; let \( x = (x_1, \ldots, x_n) \in X \) and \( |m| := \sum_{i=1}^n m_i \):

\[
x \in X^{SS}(m) \Leftrightarrow \begin{cases} \sum_{k,x_k=y} m_k \leq \frac{|m|}{3} \\ \sum_{j,x_j \in r} m_j \leq 2\frac{|m|}{3} \end{cases}
\]

for every point \( y \in \mathbb{P}^2(\mathbb{C}) \) and for every line \( r \subset \mathbb{P}^2(\mathbb{C}) \). Moreover \( x \) is a stable point, \( x \in X^{S}(m) \), iff the numerical criterion (1) is verified with strict inequalities.

The numerical criterion can be restated as follows: if \( K \) and \( J \) are subsets of \( \{1, \ldots, n\} \), then we can associate them with the numbers:

\[
\gamma^C_K(m) = |m| - 3 \sum_{k \in K} m_k, \quad \gamma^L_J(m) = 2|m| - 3 \sum_{j \in J} m_j.
\]

In particular we have: \( \gamma^J_J(m) = -\gamma^{J'}_J(m) \) where \( J' = \{1, \ldots, n\} \setminus J \).

Now consider \( K \subset \{1, \ldots, n\} \) with \( |K| \geq 2 \) and the set \( V^C_K \) of configurations \( (x_1, \ldots, x_n) \in X \) where the points indexed by \( K \) are coincident and there are no further coincidences and no non-implied collinearities; in the same way, if \( J \subset \{1, \ldots, n\}, |J| > 2 \) consider the set \( V^J_J \) of configurations \( (x_1, \ldots, x_n) \in X \) where the points indexed by \( J \) are collinear and there are no further collinearities.

In order to study \( X^{SS}(m) \) and \( X^{S}(m) \), it is sufficient to consider the subsets \( V^C_K \) and \( V^J_J \): in fact

\[
V^C_K \subseteq X^{SS}(m) \Leftrightarrow m_i \leq \frac{|m|}{3} \text{ for all } i \text{ and } \gamma^C_K(m) \geq 0,
\]

and

\[
V^J_J \subseteq X^{SS}(m) \Leftrightarrow m_i \leq \frac{|m|}{3} \text{ for all } i \text{ and } \gamma^J_J(m) \geq 0,
\]

with similar statements for \( X^{S}(m) \) (for more details, see [5]).

Moreover we observe that if \( m_i < |m|/3 \) for all \( i \), then the number of coincident
points is at most $n - 3$.

Now consider a point

$$\xi \in Z(m) := (X^{SS}(m)/G) \setminus (X^S(m)/G);$$

this is the image in $X^{SS}(m)/G$ of different, strictly semi-stable orbits, that all have in their closure a closed, minimal orbit $Gx$, for a certain configuration $x$ that has $|K|$ coincident points, and the others $n - |K|$ collinear; by the numerical criterion, we get $\gamma^G_K(m) = 0$ and $\gamma^L_K(m) = 0$, where $K$ indicates the coincident points, while $K' = \{1, \ldots\} \setminus K$ indicates the collinear ones. In this case $\phi(\overline{V^C_K \cup V^L_{K'}})$ has dimension $n - |K| - 3$, where $\phi$ is the projection map to the quotient: $\phi : X^{SS}(m) \to X^{SS}(m)/G$.

Some quotients are particularly easy to compute: let’s consider two examples.

**Example 1.1.** Consider a polarization $m = (s, s, s, 1, 1, \ldots, 1, 1)$ such that

$$\begin{align*}
0 & < s < \frac{1}{3}(4s + n - 4) \\
\frac{1}{3}(4s + n - 4) & < 2s < \frac{2}{3}(4s + n - 4) \quad \Rightarrow \quad s > 2(n - 4). \\
3s & > \frac{2}{3}(4s + n - 4)
\end{align*}$$

In this case the quotient $X^{SS}(m)/G$ is isomorphic to the product of $(n - 4)$ copies of $\mathbb{P}^2(\mathbb{C})$.

**Example 1.2.** Consider the polarization $m = (s, s, 1, 1, \ldots, 1)$ such that

$$\begin{align*}
0 & < s < \frac{1}{3}(2s + n - 2) \\
\frac{1}{3}(2s + n - 2) & < 2s < \frac{2}{3}(2s + n - 2) \quad \Rightarrow \quad n - \frac{7}{2} < s < n - 2. \\
n - 3 & < \frac{1}{3}(2s + n - 2)
\end{align*}$$

In this case the quotient $X^{SS}(m)/G$ is isomorphic to the product of $\mathbb{P}^{n-4}(\mathbb{C}) \times \mathbb{P}^{n-4}(\mathbb{C})$.

Dolgachev and Hu proved in [2] that varying the line bundle $L$ only a finite number of different quotients can be obtained: the space that parametrizes these quotients is the $G$-ample cone $C^G(X)$, the convex cone in $NS^G(X) \otimes \mathbb{R}$ spanned by ample $G$-linearized line bundles $L$ with $X^{SS}_L \neq \emptyset$, where $NS^G(X)$ is the (Néron-Severi) group of $G$-line bundles modulo homological equivalence. This cone is subdivided in walls and chambers: a polarization $L$ lies on a wall if and only if $X^S_L \subsetneq X^{SS}_L$. A chamber is a connected component of the complement of the union of walls: all the polarizations in a chamber define the same set of stable points.
In our case

\[ C^G(X) = \{(m_1, \ldots, m_n; \lambda) \in \mathbb{R}^n \times \mathbb{R}_+ : \sum_{i=1}^n m_i = 3 \lambda, 0 \leq m_i \leq \lambda, i = 1, \ldots, n\}. \]

Moreover \( X^S(m) \subsetneq X^{SS}(m) \) if and only if there exists a subset \( K \subset \{1, \ldots, n\} \) such that

\[ \sum_{i \in K} m_i = \frac{|m|}{3}; \]

this is equivalent to the condition that \( L(m) \) belongs to the hyperplane

\[ H_{C^K, L_{K'}} = \left\{ \left( m; \frac{|m|}{3} \right) \in C^G(X) : \sum_{i \in K} m_i = \frac{|m|}{3}, \sum_{j \in K'} m_j = \frac{2}{3}|m|, K \subset \{1, \ldots, n\} \right\}. \]

This is a codimension-1 wall.

Now we want to introduce the birational maps that may be constructed between two quotients: they will be crucial in determining the Betti numbers of the quotients.

Let \( m \) be a polarization such that 3 divides \( |m| \) and \( X^S(m) \neq \emptyset, X^S(m) \subsetneq X^{SS}(m) \); let us consider “variations” of \( m \) as follows:

\[ \hat{m} = m \pm (0, \ldots, 0, 1 \overbrace{0, \ldots, 0}^{k}, 0, \ldots, 0). \]

Then applying the numerical criterion we get \( X^{SS}(m) \subset X^S(\hat{m}) = X^{SS}(\hat{m}) \subset X^{SS}(m) \): these inclusions induce a morphism

\[ \theta_{\pm k} : X^S(\hat{m})/G \rightarrow X^{SS}(m)/G, \quad (2) \]

which is an isomorphism over \( X^S(m)/G \), while over \( Z(m) \) it is a contraction of subvarieties. Moreover if the quotient \( X^{SS}(m)/G \) is singular (possible for \( n \geq 6 \)), then \( \theta_{\pm k} \) is a resolution of singularities: from now on we assume \( n \geq 6 \).

Consider a point \( \xi \in Z(m) \): for what we have already observed, \( \xi \) is determined by a closed, minimal orbit \( Gx \) for a certain configuration \( x \) that has \( |K| \) coincident points and the other \( n - |K| \) collinear (\( \gamma_{C^K}^L(m) = \gamma_{K'}^L(m) = 0 \)).

Now we want to calculate \( d \), the dimension of \( \theta_{\pm k}^{-1}(\xi) \): by the numerical criterion, only one between \( V_{C^K}^L \) and \( V_{K'}^L \) is included in \( X^S(\hat{m}) \).

Dealing with an elementary transformation of “plus” type \((\hat{m} \xrightarrow{+1k} m)\), then

- if \( k \in K \) \( \Rightarrow \)

\[ \theta^{-1}_{+k}(\xi) \cong \mathbb{P}^{n-|K|-3} (\mathbb{C}), \quad d = n - |K| - 3; \quad (3) \]

- if \( k \in K' \) \( \Rightarrow \)

\[ \theta^{-1}_{+k}(\xi) \cong \mathbb{P}^{2|K|-3} (\mathbb{C}), \quad d = 2|K| - 3. \quad (4) \]
Similarly, with an elementary transformation of “minus” type \((\hat{m} - 1)k\), we have

\[
k \in K \Rightarrow d = 2|K| - 3; \quad k \in K' \Rightarrow d = n - |K| - 3.
\]

From now on we will consider only elementary transformation of “plus” type, \(\theta_k\); in particular we want to study the properties of \(\theta_k\): when \(\theta_k\) is a blow-up map? When a small map?

## 2 Small resolutions

**Definition 2.1.** A proper surjective algebraic map \(f : Y_1 \to Y_2\) between irreducible complex \(N\)-dimensional algebraic varieties is small if \(Y_1\) is nonsingular and, for all \(r > 0\),

\[
\text{codim}_C \{ y \in Y_2 \mid \dim_C f^{-1}(y) \geq r \} > 2r.
\]

A small resolution is a resolution of singularities which is a small map.

Small maps are particularly relevant, because they preserve intersection homology.

**Proposition 2.2.** Let \(X^{SS}(m)/G\) be a quotient such that \(m\) lies on a \(1\) codimension wall of \(C^G(X)\) and consider a variation of the weights \(\hat{m}\); then the birational map

\[
\theta_k : X^S(\hat{m})/G \to X^{SS}(m)/G,
\]

is a small resolution if for each \(\xi \in Z(m)\), determined by a minimal closed orbit in \(\overline{V^L_K} \cup \overline{V^L_{K'}}\) (\(2 \leq |K| \leq n - 3\)),

\[
\dim(\theta_k^{-1}(\xi)) = \begin{cases} 2|K| - 3 & \text{if } 2 \leq |K| < \frac{n+1}{3}, \\ n - |K| - 3 & \text{if } \frac{n+1}{3} < |K| \leq n - 3. \end{cases}
\]

**Remark 2.3.** Studying the subdivision in chambers and walls of the \(G\)-ample cone \(C^G(X)\), the existence of a small resolution appears quite natural: consider a polarization \(m\) that determines a singular quotient \(X^{SS}(m)/G\) and lies on a codimension \(-1\) wall: each singularity of the quotient, determined by a certain \(\phi(\overline{V^L_K} \cup \overline{V^L_{K'}})\), can be solved by the birational morphisms

\[
\theta_k : X^S(m^+)/G \to X^{SS}(m)/G \quad \text{and} \quad \theta_k : X^S(m^-)/G \to X^{SS}(m)/G
\]

where the polarizations \(m^+\) and \(m^-\) lie on opposite chambers. In order to have a small resolution we have to choose the “right side” of each wall.

**Proof.** The demonstration is based on the definition of small map: in order to verify the definition in our case, we have to consider the codimension of the sets

\[
\{ \xi \in Z(m) \mid \dim(\theta_k^{-1}(\xi)) \geq r \}
\]

for \(r > 0\). We have to distinguish two different cases:
1. if \( \dim(\theta_{+k}^{-1}(\xi)) = 2|K| - 3 \), then

\[
\operatorname{codim} \{ \xi \in Z(m) \mid \dim(\theta_{+k}^{-1}(\xi)) \geq 2|K| - 3 \} \geq 2(2|K| - 3) \\
2(n - 4) - (n - |K| - 3) > 4|K| - 6 \\
3|K| < n + 1.
\]

2. if \( \dim(\theta_{+k}^{-1}(\xi)) = n - |K| - 3 \), then

\[
\operatorname{codim} \{ \xi \in Z(m) \mid \dim(\theta_{+k}^{-1}(\xi)) \geq n - |K| - 3 \} \geq 2(n - |K| - 3) \\
2(n - 4) - (n - |K| - 3) > 2n - 2|K| - 6 \\
3|K| > n - 1.
\]

In other words, \( \theta_{+k} : X^S(\hat{m}) \to X^{SS}(m)/G \) is a small map if for every \( \xi \in Z(m) \), \( \theta_{+k}^{-1}(\xi) \) has minimum dimension between \( 2|K| - 3 \) and \( n - |K| - 3 \).

**Remark 2.4.** Combining the previous result and formulas \( 2 \) and \( 4 \), we can restate Proposition 2.2 as follows: in the same hypothesis of \( 2 \), the birational map \( \theta_{+k} \) is a small resolution if for every \( \xi \in Z(m) \), \( \theta_{+k}^{-1}(\xi) \) has minimum dimension between \( 2|K| - 3 \) and \( n - |K| - 3 \).

**Example 2.5.** Let us consider an example: \( n = 7, m = (9, 4, 4, 4, 4, 4, 1), |m| = 30 \); this polarization lies on the codimension-1 wall \( H_{17,1234567} \).

The map \( \theta_{+6} : X^S(9, 4, 4, 4, 4, 4, 3, 1)/G \to X^{SS}(m)/G \) is a small map, while \( \theta_{+1} : X^S(8, 4, 4, 4, 4, 4, 4, 1)/G \to X^{SS}(m)/G \) is not a small map.

As an immediate consequence we have \( IH_{\bullet}(X^{SS}(m)/G) \) is isomorphic to \( H_{\bullet}(X^{SS}(\hat{m})) \) if \( \theta_{+k} \) is a small map.

In general we have the following stronger result regarding the existence of small resolutions whose proof is the same of \( 4 \) Theorem 2.5, in the case of a torus action; in fact using the Gelfand-MacPherson correspondence, the moduli space \( (\mathbb{F}^2(\mathbb{C})^{SS}(m))/\text{SL}_3(\mathbb{C}) \) can be identified with a quotient in the Grassmannian \( \text{Gr}(3, \mathbb{C}^n) \) acted on by the torus \( (\mathbb{C}^*)^{n-1} \).

**Theorem 2.6.** For every singular quotient \( X^{SS}(m)/G \), there exists a polarization \( \hat{m} \) such that \( \hat{m} \) lies on a chamber close enough to \( m \) and \( \theta : X^S(\hat{m})/G \to X^{SS}(m)/G \) is a small resolution.

**Remark 2.7.** We have already proved the result for \( m \) lying on a codimension-1 wall; for higher codimension cases, the proof is based on the observation that a codimension-\( N \) wall is the intersection of \( N \) codimension-1 walls \( H_i \) and for each wall \( H_i (i = 1, \ldots, N) \), \( C^{G}(X) \setminus H_i \) has two connected components lying in two sides of \( H_i \); one of these components, named \( C_{i,s} \), defines a small resolution according to the previous result. Then the proof examines each wall, detects the right component \( C_{i,s} \) and studies their intersection for \( i = 1, \ldots, N \).
Example 2.8. Consider \( n = 8 \) and the polarization \( m = (19, 16, 7, 6, 2, 2, 1) \): it lies on a codimension–3 wall, given by

\[
H_{C_{18},L_{234567}} \cap H_{C_{13458},L_{13458}} \cap H_{C_{345},L_{12678}}.
\]

A polarization \( \hat{m} \) such that \( \theta : X^S(\hat{m})/G \to X^{SS}(m)/G \) is a small map is \( \hat{m} = (19, 16, 7, 6, 2, 1, 1) \): in fact \( X^S(m) \subset X^S(\hat{m}) \subset X^{SS}(m) \) and \( \theta^{-1}(\xi) \) has the correct dimension for each \( \xi \in Z(m) \). The polarization \( \hat{m} \) is obtained studying the three codimension–1 walls in order to detect the right components and their intersection.

3 Poincaré polynomial

Let us consider a quotient \( Y = Y(m) = X^{SS}(m)/G \) and define the intersection Betti numbers of \( X^{SS}(m)/G \) as

\[
ib_i(Y) := \dim IH_i(Y, \mathbb{Q}), \quad i : 0, \ldots, 4(n - 4).
\]

If \( Y \) is non-singular, this definition coincides with the classical one. In what follows we will use

\[
P(Y) = \sum_i t^i \dim(H_i(Y))
\]

to denote the ordinary Poincaré polynomial of a variety \( Y \) and use

\[
IP(Y) = \sum_i t^i \cdot ib_i(Y)
\]

to denote the intersection Poincaré polynomial of \( Y \).

We have already stressed that small maps preserve intersection homology: if \( \theta_{+k} : X^S(\hat{m})/G \to X^{SS}(m)/G \) is a small map, then \( H_\bullet(X^S(\hat{m})/G) \cong IH_\bullet(X^{SS}(m)/G) \), and as a consequence the Betti numbers and the Poincaré polynomial are preserved.

In the previous section we have determined the conditions for a \( \theta_{+k} \) map to be small, but in the general case \( \theta_{+k} \) is not small. In this way in order to compute the Betti numbers for a general quotient it is necessary to use the decomposition theorem of Beilinson-Bernstein-Deligne (in particular we will consider the simplified version of [4]):

Theorem 3.1. Let \( f : X \to Y \) be a projective algebraic map and \( X \) be a nonsingular variety. Then there exists

1. a stratification \( Y = \bigcup_{\alpha} Y_{\alpha} \) of \( Y \)

2. a list of enriched strata \( E_\beta = (Y_\beta, L_\beta) \) where \( Y_\beta \) is a stratum of \( Y \) and \( L_\beta \) is a local system over \( Y_\beta \); moreover assume that every local system \( L_\beta \) is trivial.
Then there exists a collection of polynomials \( \psi_\beta \) for all strata such that,

\[
P(X) = \sum_{\beta} IP(Y_\beta) \cdot \psi_\beta,
\]

and for a point \( y \in Y \)

\[
IP(f^{-1}(y)) = \sum_{\beta} IP_Y(Y_\beta) \cdot \psi_\beta.
\]

Consider a polarization \( m \) on a codimension\(-1\) wall \( H_{\mathcal{K},L_{\mathcal{K}'},} \), and \( \hat{m}, \overline{m} \) two polarizations close enough to \( m \) lying in different sides of the wall \( H_{\mathcal{K},L_{\mathcal{K}'},} \):

\[
X^S(\hat{m})/G \quad \xrightarrow{\hat{\theta}} \quad X^S(m)/G \quad \xleftarrow{\overline{\theta}} \quad X^S(\overline{m})/G
\]

Consider the map \( \hat{\theta} : X^S(\hat{m})/G \to X^{SS}(m)//G \): suppose that for \( \xi \in Z(m) \), \( \hat{\theta}^{-1}(\xi) \cong \mathbb{P}^{n-|\mathcal{K}|-3}(\mathbb{C}) \). Now apply the decomposition theorem to the map \( \hat{\theta} \), where the stratification of \( X^{SS}(m)//G \) is given by

\[
X^{SS}(m)//G = X^S(m)/G \cup Z(m).
\]

Then there exist two polynomials \( \psi_0 \) and \( \varphi_1 \) such that

\[
P(X^S(\hat{m})/G) = \psi_0 IP(X^{SS}(m)//G) + \psi_1 IP(Z(m)).
\]

These two polynomials may be determined:

\[
\psi_0 = 1, \quad \psi_1 = 1 + t^2 + \ldots + t^{2(n-|\mathcal{K}|-3)} - IP_\xi(X^{SS}(m)//G),
\]

where \( \xi \) is a point in \( Z(m) \).

Then if \( \xi \) is any point in \( Z(m) \), we have that \( P(X^S(\hat{m})/G) \) is equal to

\[
IP(X^{SS}(m)//G) + \left( 1 + \ldots + t^{2(n-|\mathcal{K}|-3)} - IP_\xi(X^{SS}(m)//G) \right) IP(Z(m)). \tag{8}
\]

In the same way, study the map \( \overline{\theta} : X^S(\overline{m})/G \to X^{SS}(m)//G \): suppose that for \( \xi \in Z(m) \), \( \overline{\theta}^{-1}(\xi) \cong \mathbb{P}^{2|\mathcal{K}|-3}(\mathbb{C}) \).

Then if \( \xi \) is any point in \( Z(m) \), we have that \( P(X^S(\overline{m})/G) \) is equal to

\[
IP(X^{SS}(m)//G) + \left( 1 + \ldots + t^{2|\mathcal{K}|-3} - IP_\xi(X^{SS}(m)//G) \right) IP(Z(m)). \tag{9}
\]

Subtracting (8) and (9), \( P(X^S(\hat{m})/G) - P(X^S(\overline{m})/G) \) is equal to

\[
P(X^S(\hat{m})/G) - P(X^S(\overline{m})/G) = \varepsilon(H_{\mathcal{K},L_{\mathcal{K}'},})Q_\xi(H_{\mathcal{K},L_{\mathcal{K}'},}) IP(Z(m)),
\]

\[8\]
where \( \varepsilon(H_{C,K,L'}) \) and \( Q_t(H_{C,K,L'}) \) are defined by

\[
\varepsilon(H_{C,K,L'}) = \begin{cases} 
1 & \text{if } n - |K| - 3 > 2|K| - 3 \Rightarrow |K| < \frac{4}{3} \\
-1 & \text{if } n - |K| - 3 < 2|K| - 3 \Rightarrow |K| > \frac{4}{3} \\
0 & \text{if } n - |K| - 3 = 2|K| - 3 \Rightarrow |K| = \frac{4}{3} 
\end{cases}
\]

\[
Q_t(H_{C,K,L'}) = \begin{cases} 
\frac{t^{2(|K| - 3) + 2} + \ldots + t^{2(n-|K|-3)}}{t^{2(n-|K|-3) + 2} + \ldots + t^{2|K|-3}} & \text{if } |K| < \frac{4}{3} \\
0 & \text{if } |K| > \frac{4}{3}
\end{cases}
\]

We can also write

\[
P(X^S(m)/G) = P(X^S(m)/G) + \varepsilon(H_{C,K,L'})Q_t(H_{C,K,L'})IP(Z(m)) \tag{10}
\]

The previous formula tells us how the Poincaré polynomial and the Betti numbers vary, when we cross a codimension \(-1\) wall \(H_{C,K,L'}\); in particular let study \(IP(Z(m))\).

The locus \(Z(m)\) is equal to \(X^{SS}(m)/G \setminus X^S(m)/G\) and it is determined by

\[
\phi(V_K' \cup V_{K}^c) = \left( \mathbb{P}^1(\mathbb{C})^{n - |K|}(m') \right)^{SS} / \text{SL}_2(\mathbb{C}),
\]

where \(m'\) is obtained from \(m\), by deleting all those weights \(m_i\) with \(i \in K\). For the Hilbert-Mumford numerical criterion, the open set of stable points for \(m'\) is equal to the open set of semi-stable points for \(m'\), i.e. the categorical quotient \((\mathbb{P}^1(\mathbb{C})^{n - |K|}(m'))^{SS} / \text{SL}_2(\mathbb{C})\) is also geometric.

Now the Poincaré polynomials for this kind of quotient are well known (see [3] for details):

\[
P(Z(m)) = \frac{1}{1 - t^2} \sum_{J \in S_{n-|K|}} \left( t^{2|J| - t^{2(n-|K|-|J|-2)}} \right) \tag{11}
\]

where \(S_{n-|K|} = \{ J \subset \{1, 2, \ldots, n - |K|\} : m'_{n-|K|} + \sum_{j \in J} m'_j < \sum_{i \notin J} m_i \} \).

**Example 3.2.** In Example 2.3 we have introduced the polarization \(m = (9, 4, 4, 4, 4, 4, 1)\) that lies on the codimension \(-1\) wall \(H_{C,7,L_{23456}}\).

We have \(Z(m) \cong (\mathbb{P}^1(\mathbb{C})^{5}(4, 4, 4, 4, 4))^{SS} / \text{SL}_2(\mathbb{C})\) and

\[
S_5 = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{4\} \}.
\]

\[
\Rightarrow P(Z(m)) = \frac{1}{1 - t^2} (1 - t^6 + 4t^2 - 4t^4) = 1 + 5t^2 + t^4.
\]

In fact \(Z(m)\) is isomorphic to the blow-up of \(\mathbb{P}^2(\mathbb{C})\) in four points.

Now collecting all the results, we can calculate the Poincaré polynomial of any quotient \(X^{SS}(m)/G\); this is the strategy:

1. If \(m\) lies on a face of \(C^G(X)\) and there exists one or more weights \(m_i\) such that \(m_i = 0\), then the quotient has lower dimension than expected: study the polarization \(m'\) obtained from \(m\) by deleting all the zero weights, and follow the next steps;
2) If \( m \) lies on a face of \( C^G(X) \) and there exists one weight \( m_i \) such that \( m_i = \frac{|m|}{3} \), then the quotient degenerates to \( (\mathbb{P}^1(\mathbb{C})^{n-1}(m)')^{SS} / SL_2(\mathbb{C}) \), where \( m' \) is obtained from \( m \) by deleting \( m_i \); if \( m' \) determines a categorical quotient that is also geometric use the Hausmann-Knutson formula, otherwise apply the decomposition theorem to this case.

3) If \( m \) lies on a face of \( C^G(X) \) and there exists two weights \( m_i, m_j \) such that \( m_i = m_j = \frac{|m|}{3} \), then the quotient degenerates to a point.

4) If \( m \) lies on a wall (its codimension is not relevant), then determine a small resolution \( X_S(m)/G \) of \( X^{SS}(m)/G \): the Poincaré polynomial does not change; now follow 5) for the polarization \( \tilde{m} \);

5) If \( m \) lies in a chamber, then construct a path \( \gamma \) in \( C^G(X) \) that goes from a polarization \( \tilde{m} \) whose quotient is well-known (see Examples 1.1 and 1.2) to \( m \), and \( \gamma \) meets only codimension-1 walls. Then if \( \gamma \) crosses the codimension-1 walls \( H_{C_{K_1,L_{K_1}'},...}, H_{C_{K_N,L_{K_N}'}} \),

\[
\gamma : m = \tilde{m} = m_0 \xrightarrow{H_{C_{K_1,L_{K_1}'}}} m_1 \xrightarrow{H_{C_{K_2,L_{K_2}'}}} m_2 \rightarrow ... \xrightarrow{H_{C_{K_N,L_{K_N}'}}} m_N = m
\]

study each crossing and apply result (10):

\[
P(X^S(m_{j+1}))/G = P(X^S(m_j))/G + \varepsilon(H_{C_{K_j,L_{K_j}'}})Q_I(H_{C_{K_j,L_{K_j}'}})IP(Z(m_{j_{j+1}})),
\]

where \( m_{j_{j+1}} \) indicates the polarization that lies on the wall \( H_{C_{K_{j+1},L_{K_{j+1}'}}} \) and “connects” \( m_j \) and \( m_{j+1} \). The Poincaré polynomial of each \( Z(m_{j_{j+1}}) \) can be computed using formula (11).

4 A special case: \( n = 6 \)

Let’s study the particular case \( n = 6 \): it is really interesting because in this case every map \( \theta_{+k} \) is the eventual composition of a small map and a blow-up map.

First of all remind that the number of chambers in which the \( G \)-ample cone \( C^G(X) \) is divided is less than or equal to 38 (see [5] for details).

For quotients \( X^{SS}(m)//G \) such that \( X^S(m) \subseteq X^{SS}(m) \), it is not so easy to give an upper bound to the number of quotients, but we can get some important informations about their structure: the following result (5) classifies the different types of points that may appear in \( Z(m) \):

**Theorem 4.1.** Let \( X = \mathbb{P}^2(\mathbb{C})^6 \) and \( m \in \mathbb{Z}_{\geq 0}^6 \) a polarization with \( 3 \mid |m| \) and \( m_i < |m|/3 \forall i \); if

1. there are two different indexes \( i, j \) s.t. \( m_i + m_j = |m|/3 \), then the quotient includes a curve \( C_{ij} \cong \mathbb{P}^1(\mathbb{C}) \), that corresponds to strictly semi-stable
orbits s.t. \( x_i = x_j \) or \( x_h, x_k, x_l, x_n \) collinear. In particular points \( \xi \) of \( C_{ij} \) are singular: locally, the variety \( (X^{SS}(m)/G, \xi) \) is isomorphic to the toric variety
\[
\mathbb{C}[T_1, T_2, T_3, T_4, T_5]/(T_1 T_4 - T_2 T_3).
\]

2. there is a “partition” of \( m \) such that \( m_i + m_j = m_h + m_l = m_k + m_n \), then the quotient includes three curves \( C_{ij}, C_{hl}, C_{kn} \cong \mathbb{P}^1(\mathbb{C}) \), that have a common point \( O_{ij,hl,kn} \).
In particular \( O_{ij,hl,kn} \) is singular: locally the variety \( (X^S(m)/G, O_{ij,hl,kn}) \) is isomorphic to the toric variety
\[
\mathbb{C}[T_1, T_2, T_3, T_4, T_5]/(T_1 T_2 T_3 - T_4 T_5).
\]

3. there are three indexes \( h, i, j \) s.t. \( m_h + m_i + m_j = |m|/3 \), then the quotient includes a point \( O_{hij} \) that correspond to the minimal, closed, strictly semi-stable orbit \( Gx \) such that \( x_h = x_i = x_j \) and \( x_k, x_l, x_n \) are collinear. The point \( O_{hij} \) is non singular.

If \( \xi \) is of type 3 in Theorem 4.1 then \( \theta_{\pm k}^{-1}(\xi) \) may be a point or \( \mathbb{P}^3(\mathbb{C}) \); in the last case \( \theta_{\pm k} \) blows down \( \mathbb{P}^3(\mathbb{C}) \) to the point \( \xi \):
\[
\theta_{\pm k}^{-1}(\xi) \cong \begin{cases} \text{point} & k \notin K \\ \mathbb{P}^3(\mathbb{C}) & k \in K \end{cases} \quad \theta_{\mp k}^{-1}(\xi) \cong \begin{cases} \text{point} & k \notin K \\ \mathbb{P}^3(\mathbb{C}) & k \in K \end{cases} \quad (12)
\]

If \( \xi \) is of type 1, then \( \xi \in C_{ij} \cong \mathbb{P}^1(\mathbb{C}) \) and \( \theta_{\pm k}^{-1}(C_{ij}) \) has dimension two (in particular it is one of the two dimensional quotients \( \mathbb{P}^3(\mathbb{C})^5(m')/\text{PSL}_3(\mathbb{C}) \)): it is a small contraction.
As the end we can consider \( \theta_{\pm k} : X^S(m)/G \to X^S(m)/G \) as the eventual composition of a blow-down map with a small map; moreover if \( X^S(m)/G \) is singular, then \( \theta_{\pm k} \) is a resolution of singularities.

The main result of this section is the following formula for the Poincaré polynomial of an arbitrary categorical quotient \( Y = X^{SS}(m)/G \):

**Theorem 4.2.** Let \( m = (m_1, \ldots, m_6) \) be a polarization such that \( 0 < m_i < \frac{1}{3} \sum_m, m_i \geq m_{i+1} \), then there may be five different cases:

1. if
\[
m_1 + m_2 + m_3 < \frac{2}{3} m,
\]
then \( IP(Y) = 1 + 6t^2 + 7t^4 + 6t^6 + t^8 \);

2. if
\[
\begin{cases} m_1 + m_2 + m_3 \geq \frac{2}{3} m \\ m_1 + m_2 + m_4 < \frac{2}{3} m \end{cases}
\]
then \( IP(Y) = 1 + 5t^2 + 6t^4 + 5t^6 + t^8 \).
3. if
\[
\begin{cases}
m_1 + m_2 + m_4 \geq \frac{2}{3}|m| \\
m_1 + m_2 + m_5 < \frac{2}{3}|m| \\
m_1 + m_3 + m_4 < \frac{2}{3}|m|
\end{cases}
\]
then \(IP(Y) = 1 + 4t^2 + 5t^4 + 4t^6 + t^8;\)

4. if
\[
\begin{cases}
m_1 + m_2 + m_5 \geq \frac{2}{3}|m| \\
m_1 + m_2 + m_6 < \frac{2}{3}|m|
\end{cases}
or \begin{cases}
m_1 + m_3 + m_4 \geq \frac{2}{3}|m| \\
m_2 + m_3 + m_4 < \frac{2}{3}|m|
\end{cases}
\]
then \(IP(Y) = 1 + 3t^2 + 4t^4 + 3t^6 + t^8;\)

5. if
\[
m_1 + m_2 + m_6 \geq \frac{2}{3}|m| \quad \mathrm{or} \quad m_2 + m_3 + m_4 \geq \frac{2}{3}|m|,
\]
then \(IP(Y) = 1 + 2t^2 + 3t^4 + 2t^6 + t^8.\)

**Proof.** First of all let us observe that given two quotients \(Y(m)\) and \(Y(m')\) (geometric or categorical) it is always possible to find a finite sequence of algebraic maps \(\theta_{\pm k}\) such that \(Y(m) \xrightarrow{\theta_{\pm 1}} \ldots \xrightarrow{\theta_{\pm k}} Y(m')\), where each \(\theta_{\pm k}\) is a composition of a blow-down (or blow-up) map with a small map (or only one of these).

If \(\theta_{\pm k}\) is a small map, then it induces an isomorphism from the (intersection) homology of \(Y(\hat{m})\) to the intersection homology of \(Y(m)\). It means that when we cross a wall of the \(G\)-ample cone, the intersection Betti numbers vary if the maps \(\theta_{\pm k}\) involve some blow-ups and blow-downs.

As we have seen the map \(\theta_{\pm k} : Y(\hat{m}) \longrightarrow Y(m)\) is a blow-down map if there is at least a point \(\xi = O_{hij} \in Y(m)\) of type 3 (Theorem 4.1), that satisfies (12):
\[
\theta_{\pm k}^{-1}(O_{hij}) = \theta_{\pm k}^{-1}\left(\phi(\overline{V_{hij}^{\theta_{\pm k}}} \cup \overline{V_{[6]\setminus\{h,i,j\}}^L})\right) = \tilde{\phi}(\overline{V_{[6]\setminus\{h,i,j\}}}) \cong \mathbb{P}^3(\mathbb{C}),
\]
where \(\phi\) and \(\tilde{\phi}\) are the projection to the quotients \(Y(m)\) and \(Y(\hat{m})\). In other words, \(\theta_{\pm k}\) is a blow-down if \(X_{SS}(\hat{m})\) contains \(V_{[6]\setminus\{h,i,j\}}^L\) and not \(V_{hij}^C\).

How many points \(O_{hij}\) there may be in a categorical quotient \(Y(m)\)? The answer is at most four and there are two possible sets of points:
\[
O_{456}, \quad O_{356}, \quad O_{256}, \quad O_{156}, \quad \mathrm{or} \quad O_{456}, \quad O_{356}, \quad O_{346}, \quad O_{345}.
\]

For example if \(m = (666621)\) then \(O_{156}, O_{256}, O_{356}, O_{456} \in X_{SS}(m) // G\) (this quotient is particularly easy to compute: it is \(\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\)).

In this way, in order to compute the intersection Betti numbers of a quotient
\(Y(m)\) (both geometric and categorical) it is sufficient to know the intersection Betti numbers for a quotient \(Y(m')\) and then check how many \(V_K^L\) sets (with \(|K| = 3\) are NOT included in \(X^S(m)\): in fact for what we have just observed, we can always “connect” two quotients by a finite sequence of maps that change the Betti numbers if and only if they are blow-ups or blow-down of at most four copies of \(\mathbb{P}^3(\mathbb{C})\). These copies of \(\mathbb{P}^3(\mathbb{C})\) are detected by the sets \(V_K^L\) that are contained in \(X^S(m)\), where \(K\) is one of the following:

\[
456, 356, 256, 156 \quad \text{or} \quad 456, 356, 346, 345.
\]

Consider now the chamber of \(C^G(X)\) that contains the polarization \((5, 5, 5, 1, 1)\); then for every polarization \(m\) in this chamber we have \(Y(m) = \mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) and moreover there are four sets \(V_K^L\) that are not included in \(X^S(m)\). Its intersection Betti numbers are well-known and now we are able to compute all the Betti numbers:

- if \(m'\) is such that \(X^S(m')\) does NOT contain four sets \(V_K^L\) then its intersection Betti numbers are the same of \(\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\),

- if \(m'\) is such that \(X^S(m')\) does NOT contain three sets \(V_K^L\) then its intersection Betti numbers are the same of \(\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) blown-up in one point,

- if \(m'\) is such that \(X^S(m')\) does NOT contain two sets \(V_K^L\) then its intersection Betti numbers are the same of \(\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) blown-up in two points,

- if \(m'\) is such that \(X^S(m')\) does NOT contain one sets \(V_K^L\) then its intersection Betti numbers are the same of \(\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) blown-up in three points,

- if \(m'\) is such that \(X^S(m')\) contains all sets \(V_K^L\) then its intersection Betti numbers are the same of \(\mathbb{P}^2(\mathbb{C}) \times \mathbb{P}^2(\mathbb{C})\) blown-up in four points.

The relations of Theorem 4.2 compute the number of \(V_K^L\) sets that are included in \(X^S(m)\). \(\blacksquare\)

References

[1] I.V. Dolgachev, Lectures on Invariant Theory, Cambridge University Press, Lecture Note Series 296, 2003.

[2] I.V. Dolgachev, Y. Hu, Variations of geometric invariant theory quotients, Duke Math. Jour, 68, 1992, 151-184.

[3] J.-C. Hausmann, A. Knutson, The cohomology ring of polygon spaces, Ann. Inst. Fourier (Grenoble), 48, 1998, no. 1, 281-321.
[4] Y. Hu, *The geometry and topology of quotient varieties of torus actions*, Publ. Math. IHES, **87**, 1998, 5-51.

[5] F. Incensi, *GIT quotients of products of projective planes*, Accepted by Rendiconti Seminari Matematici dell’Università di Padova, 2009.

[6] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, Springer-Verlag, third edition, 1994.

[7] M. Thaddeus, *Geometric invariant theory and flips*, Jour. Amer. Math. Soc., **9**, 1996, 691-723.

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