PLANAR NON-FORMALITY OF THE LITTLE DISCS OPERAD
IN CHARACTERISTIC TWO

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Abstract. We show that the little discs operad $D_2$ is not formal over $\mathbb{F}_2$ as a planar (or non-symmetric) operad. We compute explicitly the homological obstruction using as chain model the cells of the spineless cacti operad.

1. Introduction

The little discs operads $D_n$ have been heavily studied since their introduction by Boardman-Vogt in the seventies. A major result states that they are formal over a field of characteristic 0 as symmetric operads. This was proved by Kontsevich [6] and Lambrechts-Volic [7] over $\mathbb{R}$, by Tamarkin over $\mathbb{Q}$ for $n = 2$ [15], and later by Fresse-Willwacher over $\mathbb{Q}$ for all $n$ [4]. We recall that a topological operad $O$ is formal over a ring $R$ if its homology operad $H_\ast(O, R)$ and its singular chain operad $C_\ast(O, R)$ are connected by a zig-zag of quasi-isomorphisms, i.e. chain operad maps inducing an isomorphism in homology. It is not difficult to see that $D_n$ cannot be formal over a field $F$ of positive characteristic as a symmetric operad (Remark 6.9 in [3]). However if we forget the action of the symmetric groups, and we consider $D_n$ as a planar (or non-symmetric) operad, then the weaker notion of planar formality over $F$ is much harder to check. A reason to study it is the relation to knot theory: there is a vast literature relating the space $K_n$ of long knots in $\mathbb{R}^n$ to $D_n$. [14] For example if the pair $(D_n, D_1)$ was a formal pair of planar operads, then the Sinha-Vassiliev spectral sequence computing the homology of $K_n$ would collapse. This is a 20 years old conjecture by Vassiliev that has been verified in characteristic 0 using the rational formality of $D_n$ [8]. In this paper we show:

Theorem 1.1. The little 2-discs operad $D_2$ is not formal as a planar operad over $\mathbb{F}_2$.

An immediate consequence is that $D_2$ is not formal over any field of characteristic 2, and also over the integers. We prove Theorem 1.1 by obstruction theory, adapting the work of Halperin-Stasheff to the framework of planar operads. We use a finite dimensional model for the chain operad of the little 2-discs, that is basically the cell complex of the spineless cacti operad ([12], sect. 4), or equivalently the second filtration of the surjection operad (1.2 in [2]). The non-formality of $D_2$ as a planar operad, together with the non-formality of the single spaces $D_2(k)$ in characteristic 2 for $k > 3$, that we proved in [13], show that the behaviour in characteristic 2 is opposite to that in characteristic 0, where instead Hopf operad formality holds [4] (both at the level of spaces and operads). I am grateful to Joana Cirici and Benoît Fresse for some fruitful discussions.
2. Obstruction theory for planar operads

The obstruction theory by Halperin-Stasheff [5] for rational commutative algebras can be translated almost verbatim to the framework of planar operads. The rational version for symmetric operads was worked out by Markl [9]. We recall the definition of a planar operad in a concrete symmetric monoidal category, that in practice will be either the category $\text{Top}$ of topological spaces with the cartesian product, or the category $\text{Ch}_F$ of $\mathbb{N}$-graded chain complexes over a field $F$ with the tensor product.

**Definition 2.1.** A planar operad in a concrete symmetric monoidal category $(\mathcal{C}, \otimes)$ is a sequence of objects $O(k)$ of $\mathcal{C}$, with $k \in \mathbb{N}$ ( $k$ is called the arity),

- together with an element $\iota \in O(1)$ called the unit, and composition maps $\circ_i : O(k) \otimes O(l) \to O(k + l - 1)$ for $1 \leq i \leq k$

such that for $a \in O(n)$, $b \in O(p)$, $c \in O(q)$

$$a \circ_i b \circ_{j+p-1} c = (a \circ_j c) \circ_i b \quad \text{for} \quad 1 \leq i < j \leq n$$

$$a \circ_i (b \circ_j c) = (a \circ_i b) \circ_{i+j-1} c \quad \text{for} \quad 1 \leq i \leq n, 1 \leq j \leq p$$

and $\iota$ is a bi-sided unit for the $\circ_i$-operations.

A symmetric operad $O$ is a planar operad equipped with the action of the symmetric group $\Sigma_k$ on $O(k)$ for each $k$, so that the actions and the composition maps are compatible in an appropriate sense. A planar operad in $\text{Top}$ is called a planar topological operad, and an operad in $\text{Ch}_F$ is called a differential graded planar operad (DGPO) over $F$. A DGPO equipped with a trivial differential is called a graded planar operad (GPO). A GPO $O$ is reduced if $O(0) = 0$ and $O(1) = F\{\iota\}$.

We sketch the construction of the bigraded model of a GPO. Let $\mathcal{F}$ be the free functor from the category of sequences of graded vector spaces $V = \{V(i)\}_{i \geq 2}$ to the category of reduced graded planar operads, that is left adjoint to the forgetful functor. Roughly speaking the free functor is constructed by means of directed planar trees with vertices labelled by elements of $V$ (compare Def. 6 in [11]).

There is a splitting $\mathcal{F}(W) = \mathcal{F}^+(V) \oplus V \oplus F\{\iota\}$, where $\mathcal{F}^+(V)$ contains the decomposable elements.

**Definition 2.2.** A bigraded DGPO is one of the form $(\mathcal{F}(W), d)$, where

$$W = (W(k))_{k \geq 2} = (\oplus_{i,j} W_j^i(k))_{k \geq 2}$$

is a sequence of bigraded vector spaces, We say that $W_j^i(k)$ contains the elements of arity $k$, dimension $j$, and level $i$. The dimension is the grading of the underlying chain complex, and the level is an additional grading. We denote $W^i = \oplus_{k} W_j^i(k)$, $W^{\leq i} = \oplus_{i', \leq i} W^i$, and $W(\leq k) = \oplus_{k' \leq k} W(k')$. By adding up degrees $\mathcal{F}(W)(k)$ inherits a bigrading for each $k$. We require $d$ to be homogeneous of bidegree $(-1, -1)$ with respect to the dimension and the level grading.

We can consider the homology with respect to the level grading obtaining for each $i$ a GPO $H^i(\mathcal{F}(W), d)$. 
Proposition 2.5. Let $O$ be a DGPO such that $H = H_*(O)$ is a reduced GPO. Let $\rho : (\mathcal{F}(W), d) \to (H, 0)$ be the inclusion of DGPO such that

- $W^0 = V$,
- $\rho|_V : V \to H$ is the inclusion
- $\rho|_W : W^i \to H$ is trivial for $i > 0$
- $H^i(\mathcal{F}(W), d) = 0$ for $i > 0$ and $H^0(\mathcal{F}(W), d) \cong H$
- $d(x) \in \mathcal{F}^+(W)$ for $x \in W$ (the differential is decomposable)

Under these conditions $(\mathcal{F}(W), d)$ is uniquely defined up to isomorphism, and is called the bigraded model, or the minimal model of $H$. In particular $W^1 = sR$ is the suspended sequence of relations, where $s$ raises the dimension degree by one. In general

$$W^i = s(H^{i-1}(\mathcal{F}(W^{\leq(i-1)}, d)))/H$$

is the suspended module of $H$-bimodule-indecomposables and

$$d : W^i \to \mathcal{F}(W^{\leq(i-1)})$$

is a splitting of the projection.

Definition 2.4. Let $O(k) = \bigoplus_{i=1}^k O^i(k)$ be a sequence of bigraded vector spaces. We say that a linear map $f : O \to O$ lowers the filtration level by $k$ if $f(O^i) \subseteq O^{\leq(i-k)}$ for each $i$.

Proposition 2.7. Let $C_1$ and $C_2$ be DGPO with $H = H_*(C_1) = H_*(C_2)$ reduced, and with respective filtered models $(\mathcal{F}(W), D_1)$, $(\mathcal{F}(W), D_2)$. Then $C_1$ and $C_2$ are weakly equivalent by a zig-zag inducing the identity in homology if and only if there is an isomorphism $\Phi : (\mathcal{F}(W), D_1) \cong (\mathcal{F}(W), D_2)$ with $\Phi - \text{id}$ lowering the filtration level by 1.

Definition 2.8. A DGPO $C$ is formal if it is weakly equivalent to its homology $H_*(C)$ equipped with the trivial differential. A topological planar operad $O$ is formal over $\mathbb{F}$ if the singular chain DGPO $C_*(O, \mathbb{F})$ is formal.
From Proposition 2.7 we obtain the following corollary.

**Corollary 2.9.** A DGPO \( C \) is formal if and only if there is an isomorphism

\[ \Phi : (\mathcal{F}(W), d) \cong (\mathcal{F}(W), D) \]

between the bigraded model of \( H_\ast(C) \) and a filtered model of \( C \), with \( \Phi - \text{id} \) lowering the filtration level by 1.

Let \( C \) be a DGPO such that \( H_\ast(C) \) is reduced, and has the bigraded model \((\mathcal{F}(W), d)\). We perform a partial construction of the filtered model of \( C \) up to level 2 and define the first obstruction to the formality of \( C \).

We start by defining up to level 1

\[ \pi : \mathcal{F}(W^{\leq 1}, d) \to C \]

by sending linear generators \( x \in W^0 \) to cycles \( \pi(x) \in Z(C) \) representing

\[ \rho(x) \in H_\ast(C), \text{ and relations } y \in W^1 \text{ to elements } \pi(y) \text{ such that} \]

\[ \pi(d(y)) = d_C(\pi(y)) \in B(C) \text{ is a boundary representing the relation } y \text{ in the homology } H_\ast(C) = \mathcal{F}(W^0)/(dW^1). \]

Now for \( z \in W^2 \) consider \( \alpha(z) := \pi(dz) \). Since \( \pi \) commutes with the differentials, \( d_C(\pi(dz)) = \pi(ddz) = 0 \), and we can consider the homology class \( \alpha(z) = a(z) \in H_\ast(C) \).

We have that \( \alpha \in \text{Hom}_{-1}(W^2, H_\ast(C)) \), where the latter is the vector space of arity preserving linear maps that lower the dimension by 1. Clearly \( \pi \) can be extended to a DGPO-map up to level 2

\[ \pi : (\mathcal{F}(W^{\leq 2}), d) \to (C, d_C) \]

if and only if \( \alpha = 0 \). In that case \( D = d \) on \( W^{\leq 2} \). Otherwise let \( \eta : H_\ast(C) \to \mathcal{F}(W^0) \) be a linear splitting of the projection. We define, on \( W^2, D = d - \eta \circ \alpha \).

**Proposition 2.10.** There exists a DGPO-map \( \pi : (\mathcal{F}(W^{\leq 2}), D) \to C \) commuting with the differentials.

**Proof.** For any linear generator \( z \in W^2 \) by construction the cycle

\[ \pi(D(z)) = \pi(d(z)) - \pi(\eta(\alpha(z))) \]

represents the homology class \( \alpha(z) - \alpha(z) = 0 \) and so there exists \( c \in C \) such that \( d_C(c) = \pi(D(z)) \). Set \( \pi(z) = c \).

\( \square \)

This procedure can be continued as in the proof of Theorem 4.4 in [5] to obtain the filtered model of \( C \). We are now concerned about the formality of \( C \).

**Definition 2.11.** Let \( \text{Hom}_0(W^1, H_\ast(C)) \) be the vector space of linear maps preserving both the arity and the dimension. There is a homomorphism

\[ \partial : \text{Hom}_0(W^1, H_\ast(C)) \to \text{Hom}_{-1}(W^2, H_\ast(C)) \]

defined as follows: for \( f : W^1 \to H_\ast(C) \), extend it to a linear map \( f : W^{\leq 1} = W^0 \oplus W^1 \to H_\ast(C) \) by \( f|_{W^0} = \rho \). By the universal property \( f \) defines an operad map \( f : \mathcal{F}(W^{\leq 1}) \to H_\ast(C) \). Then \( \partial(f) \) is the composition

\[ \partial(f) : W^2 \xrightarrow{d} \mathcal{F}(W^{\leq 1}) \xrightarrow{f} H_\ast(C) \]

**Proposition 2.12.** There exists an isomorphism \( \Phi : (\mathcal{F}(W^{\leq 2}), d) \to (\mathcal{F}(W^{\leq 2}), D) \)

such that \( \Phi - \text{id} \) lowers the filtration by 1 if and only if \( \alpha \in \text{Im}(\partial) \).

The proof is similar to that of Lemma 6.7 in [13].
Corollary 2.13. If $\alpha \notin \text{Im}(\partial)$ then $C$ is not formal.

Thus $\alpha$ is the first obstruction to the formality of $C$.

3. The bigraded model of the homology of the little discs operad

Let us describe the first generators of the bigraded model of the homology of the little 2-discs $H = H_*(D_2, F_2)$, as planar operad, over $F_2$. We will consider the non-unitary version of the little discs $D_2$ with $D_2(0) = \emptyset$.

It is well known that $H$ is the Gerstenhaber operad, and has the following presentation as a symmetric operad.

Proposition 3.1. The Gerstenhaber operad $H$, as a symmetric operad, is generated by

- the product $m = m(x_1, x_2) = x_1 x_2$ in degree 0 and arity 2,
- the bracket $b = b(x_1, x_2) = [x_1, x_2]$ in degree 1 and arity 2

modulo the following relations expressed using the module structure of $H_*$ over the group ring of the symmetric group $F_2[\Sigma_3]$. On the right we express the corresponding relations holding in Gerstenhaber algebras.

- the associativity relation

$$m \circ_1 m = m \circ_2 m \quad x_1(x_2 x_3) = (x_1 x_2)x_3$$

- the commutativity relation for the product

$$(21)m = m \quad x_1 x_2 = x_2 x_1$$

- the (anti)commutativity relation for the bracket

$$(21)b = b \quad [x_1, x_2] = [x_2, x_1]$$

- the Jacobi relation

$$(123) + (231) + (312))(b \circ_1 b) = 0 \quad [x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0$$

- The Poisson relation

$$b \circ_1 m = ((123) + (213))(m \circ_2 b) \quad [x_1 x_2, x_3] = x_1[x_2, x_3] + x_2[x_1, x_3]$$

The Poincare polynomials $P_k(t) = \sum_i \dim H_i(k)t^i$ in arity $k = 2, 3, 4$ are

$$P_2(t) = 1 + t, \quad P_3(t) = 1 + 3t + 2t^2, \quad P_4(t) = 1 + 6t + 11t^2 + 6t^3$$

We list basis generators

$H_0(2) = F_2\{x_1 x_2\}$

$H_1(2) = F_2\{[x_1, x_2]\}$

$H_0(3) = F_2\{x_1 x_2 x_3\}$

$H_1(3) = F_2\{[x_1, x_2]x_3, [x_1, x_3]x_2, [x_2, x_3]x_1\}$

$H_2(3) = F_2\{[[x_1, x_2], x_3], [x_1, [x_2, x_3]]\}$

$H_0(4) = F_2\{x_1 x_2 x_3 x_4\}$

$H_1(4) = F_2\{[x_1, x_2]x_3 x_4, [x_1, x_3]x_2 x_4, [x_1, x_4]x_2 x_3, [x_2, x_3]x_1 x_4, [x_2, x_4]x_1 x_3, [x_3, x_4]x_1 x_2\}$

$H_2(4) = F_2\{[[x_1, x_2], x_3]x_4, [[x_1, x_2], x_4]x_3, [x_2, [x_3, x_4]]x_1, [x_2, [x_3, x_4]]x_1, [x_1, [x_3, x_4]]x_2, [x_1, [x_3, x_4]]x_2, [x_1, x_2]x_3 x_4, [x_1, x_2]x_4 x_3, [x_1, x_3]x_2 x_4, [x_1, x_4]x_2 x_3\}$

$H_3(4) = F_2\{[x_1, [x_2, x_3]]x_4, [x_1, [x_2, x_4]]x_3, [[x_1, x_4], x_2, x_3]], [[x_1, x_4], x_2, x_3], [[x_1, x_3], x_2, x_4]\}$
A geometric description of the cycles is given in [10].

The presentation of $H_*(D_2)$ as a planar operad is very different from the symmetric presentation. For example consider the symmetric sub-operad of $H$ generated by the bracket $b$: up to one operadic suspension this is the Lie operad $\text{Lie}$, that we studied as a planar operad in [11].

**Theorem 3.2.** [11] The operad $\text{Lie}$ is a free planar operad on an infinite number of generators growing exponentially in the arity.

The next planar generator of $\text{Lie}$ after the bracket is $l = (1324)((b \circ_2 b) \circ_1 b)$ corresponding to the arity 4 operation $[[x_1, x_3], [x_2, x_4]]$.

We compute all generators of the bigraded model of $H_*(D_2)$ up to arity 4.

**Theorem 3.3.** The bigraded model $(\mathcal{F}(W), d) \rightarrow H = H_*(D_2, \mathbb{F}_2)$ of the Gerstenhaber operad over $\mathbb{F}_2$ satisfies

\[
W^0(\leq 4) = \mathbb{F}_2\{m, b, u, l\}, W^1(\leq 4) = \mathbb{F}_2\{A, B\}, W^2(\leq 4) = \mathbb{F}_2\{P, C\}
\]

In arity 2, level 0, we have.

- the product $m$ in dimension 0
- the bracket $b$ in dimension 1.

In arity 3, level 1, we have

- a 1-dimensional generator $A$ resolving associativity such that
  \[
  d(A) = m \circ_1 m + m \circ_2 m,
  \]
- a 2-dimensional generator $B$ resolving the Poisson relation such that
  \[
  d(B) = b \circ_1 m + b \circ_2 m + m \circ_1 b + m \circ_2 b.
  \]
- In arity 4, level 0, dimension 2, there is a generator
  \[
  u = (1324)((m \circ_2 b) \circ_1 b) = [x_1, x_3][x_2, x_4]
  \]
- In arity 4, level 0, dimension 3, we have the generator presented earlier
  \[
  l = (1324)((b \circ_2 b) \circ_1 b) = [[x_1, x_3], [x_2, x_4]]
  \]
- In arity 4, level 2, dimension 2, we have a generator $P$ resolving the Pentagon relation
  \[
  d(P) = m \circ_1 A + m \circ_2 A + A \circ_1 m + A \circ_2 m + A \circ_3 m.
  \]
- Finally in arity 4, level 2, dimension 3, there is a generator $C$ with
  \[
  d(C) = A \circ_1 b + A \circ_2 b + A \circ_3 b + b \circ_1 A + b \circ_2 A + B \circ_1 m + B \circ_2 m + B \circ_3 m + m \circ_1 B + m \circ_2 B.
  \]

**Proof.** The product and the bracket generate under operad composition all of $H(3)$, all of $H_i(4)$ for $i \leq 1$, a subspace of $H_2(4)$ of codimension 1 not containing $u$, and a subspace of codimension 1 of $H_3(4)$ not containing $l$. In arity 3 and dimension 0

\[
\mathcal{F}(W^0)_0(3) = \{m \circ_1 m, m \circ_2 m\}
\]

and the kernel of $\rho : W^0_0(3) \rightarrow H_0(3) \cong \mathbb{F}_2$ is generated by $dA$. In arity 3 and dimension 1

\[
\mathcal{F}(W^0)_1(3) = \{m \circ_1 b, m \circ_2 b, b \circ_1 m, b \circ_2 m\}
\]
and the kernel of $\rho : W^0(3) \to H_1(3) \cong (\mathbb{F}_2)^3$ is generated by $dB$. In arity 3 and dimension 2

$$\mathcal{F}(W^0)_2(3) = \{b \circ_1 b, b \circ_2 b \} \cong H_2(3).$$

In arity 4 and dimension 1

$$\mathcal{F}(W^1)_1(4) \cong (\mathbb{F}_2)^5$$

and

$$d : \mathcal{F}(W^1)_1(4) \to \mathcal{F}(W^0)_0(4) \cong (\mathbb{F}_2)^5$$

has rank 4, with cokernel $H_0(4)$ and kernel generated by $dP$. In arity 4 and dimension 2

$$\mathcal{F}(W^1)_2(4) \cong (\mathbb{F}_2)^{10}$$

and

$$d : \mathcal{F}(W^1)_2(4) \to \mathcal{F}(W^0)_1(4) \cong (\mathbb{F}_2)^{15}$$

has rank 9 with cokernel $H_1(4)$ and kernel generated by $dC$. In arity 4 and dimension 3

$$\mathcal{F}(W^1)_3(4) \cong (\mathbb{F}_2)^5$$

and

$$d : \mathcal{F}(W^1)_3(4) \to \mathcal{F}(W^0)_2(4) \cong (\mathbb{F}_2)^{15}$$

has rank 5 with cokernel the codimension 1 subspace of $H_3(4)$ generated by $m$ and $b$, and trivial kernel.

4. The cacti model for the little discs

We consider the 2nd filtration $S$ of the surjection operad as a model for the chain operad of the little 2-discs operad. The operad $S$ is the cellular chain complex of the spineless cacti operad as explained in [12]. For any $k S(k)$ is a free $\mathbb{F}_2[\Sigma_k]$-module.

**Definition 4.1.** The linear generators of $S(k)$ of dimension $i$ are sequences of length $i + k$ containing all integers $1, 2, \ldots, k$ such that

- No adjacent numbers are equal
- No ordered sub-sequence of the form $aabb$ with $a \neq b$ occurs.

The symmetric group $\Sigma_k$ acts on $S(k)$ by acting on the values of sequences.

**Example 4.2.**

- $S_0(2)$ has the $\mathbb{F}_2$-basis $\{12, 21\}$
- $S_1(2)$ has the $\mathbb{F}_2$-basis $\{121, 212\}$
- $S_0(3)$ is the free $\mathbb{F}_2[\Sigma_3]$-module on $\{123\}$
- $S_1(3)$ is the free $\mathbb{F}_2[\Sigma_3]$-module on $\{1231, 1213, 1232\}$
- $S_2(3)$ is the free $\mathbb{F}_2[\Sigma_3]$-module on $\{12321, 12131\}$

The top dimensional generators of $S(k)$ are in dimension $k - 1$.

The differential $\delta$ of $S$ is obtained by removing an element from a sequence in all possible ways, and adding the results, deleting those sequences that do not satisfy the conditions of definition 4.1.

For example

$$\delta(12321) = 2321 + 1321 + 1231 + 1232$$

since 1221 is not allowed.

In order to define the operad composition of $S$ we need the following definition.
Definition 4.3. An interval decomposition of a sequence \( y \) into \( m \) subsequences is the overlapping partition of \( y \) into \( m \) ordered non-empty subsequences \( y_1, \ldots, y_m \) of \( y \) such that two adjacent subsequences \( y_i, y_{i+1} \) have in common exactly the last element of \( y_i \) and the first element of \( y_{i+1} \).

For example the interval decompositions of 123 into three subsequences are
\[
\{1, 12, 23\}, \{1, 123\}, \{123, 3\}, \{12, 2, 3\}, \{12, 23\}\]

Definition 4.4. A composition of linear generators of \( S \) of the form \( x \circ_i y \in S(p + q - 1) \), with \( x \in S(p) \) and \( y \in S(q) \) is obtained by the following procedure: let \( n \) be the number of occurrences of the value \( i \) in \( x \).

for each interval decomposition of \( y \) into \( n \) subsequences \( y_1, \ldots, y_n \) let \( y'_j \) be the sequence obtained by adding \( i - 1 \) to each value of \( y_j \).

Add \( q - 1 \) to each value in \( x \) larger than \( i \).

Replace the \( j \)-th occurrence of \( i \) in \( x \) by the sequence \( y'_j \) for each \( j = 1, \ldots, n \).

Take the sum of the resulting sequences over all interval decompositions of \( y \) into \( n \) subsequences.

For example
\[
2132 \circ_2 121 = (2)1(232)4 + (23)1(32)4 + (232)1(2)4
\]
where we added some parenthesis to emphasize the subsequences \( y'_j \).

Theorem 4.5. (Berger) \([1, 2]\). The operad \( S \) is weakly equivalent to the chain operad of the little 2-discs operad \( C_*(D_2, F_2) \).

By Berger’s theorem the formality of \( S \) is equivalent to the formality of \( D_2 \) over \( F_2 \). Under the identification \( H_*(S) \cong H_*(D_2) \cong H \)

- the product \( m \) is represented by the cycle \( 12 \in S^0(2) \)
- the bracket \( b \) is represented by the cycle \( 121 + 212 \in S^1(2) \)

5. Computing the obstruction

We construct a DGPO homomorphism as in section 2
\[
\pi : \mathcal{F}(W^{\leq 1}(\leq 4)) \to S
\]
in arity \( \leq 4 \) and level \( \leq 1 \). The choice
\[
\pi(m) = 12
\]
\[
\pi(b) = 121 + 212
\]
\[
\pi(A) = 0
\]
\[
\pi(B) = 21312 + 23132 + 12131 + 31323
\]
is compatible with the differential on \( A \) because the multiplication is strictly associative in \( S \), i.e. \( (12) \circ_1 (12) = 123 = (12) \circ_2 (12) \), and so \( \pi(dA) = \pi(m \circ_1 + m \circ_2 m) = (12) \circ_1 (12) + (12) \circ_2 (12) = 0 = \delta(0) = \delta(\pi(A)) \).

It is also compatible with the differential on \( B \) since
\[
\delta(\pi(B)) = (1312 + 2312 + 2132 + 2131) + (3132 + 2132 + 2312 + 2313) +
(2131 + 1231 + 1213) + (1323 + 3123 + 3132) =
(1232 + 1312 + 3123) + (1231 + 2313 + 2123) + (1213 + 2123) + (1232 + 1323) =
\]
\begin{align*}
&(121 + 212) \circ_1 (12) + (121 + 212) \circ_2 (12) + (12) \circ_1 (121 + 212) + 12 \circ_2 (121 + 212) = \\
&\pi(b \circ_1 m + b \circ_2 m + m \circ_1 b + m \circ_2 b) = \\
&\pi(d(B)).
\end{align*}

There is a non-trivial obstruction to extend \( \pi \) to arity 4 and level 2, where we have two generators \( P \) and \( C \):
\begin{align*}
\pi(d(P)) &= 0 \text{ (no obstruction for this generator), but } \pi(d(C)) \text{ is the sum of } \\
\pi(B \circ_1 m) &= 312423 + 314123 + 341243 + 123242 + 131242 + 131412 + 412434 \\
\pi(B \circ_2 m) &= 231413 + 214123 + 234143 + 241423 + 123141 + 414234 \\
\pi(B \circ_3 m) &= 213412 + 234142 + 213412 + 121341 + 314243 + 313243 \\
\pi(m \circ_1 B) &= 213124 + 231324 + 121314 + 313234 \\
\pi(m \circ_2 B) &= 132423 + 134243 + 123242 + 142434,
\end{align*}
that is the sum of 24 generators, as two copies of 313234 and 123242 cancel out.

\begin{lemma}
The homology class of the 2-cycle \( \pi d(C) \in S_2(4) \) is
\( \alpha(C) = (243)(m \circ_2 b \circ_1 b = [x_1, x_4][x_2, x_3] \in H_2(S(4)) \cong H_2(D_2(4)) \cong (\mathbb{F}_2)^{11} \)
\end{lemma}

\begin{proof}
The class \([x_1, x_4][x_2, x_3]\) is represented by the 2-cycle
\( y = 141232 + 414232 + 141323 + 414323 \in S^2(4) \).
Consider in \( S^3(4) \) the element
\( \gamma = 2314132 + 2341432 + 2131412 + 3414243 + 2141242 + 2141232 + 2313242 + 2414232 + 3141323 + 3413423 + 3132432 + 3134232 + 1213141 + 4142434 \).
Then \( \delta(\gamma) = y - \pi d(C) \) and this proves the claim. \( \square \)

By lemma [5.1]
\( \alpha : W^2(4) = \mathbb{F}_2\{C, P\} \rightarrow H_4(S(4)) \)
is defined by \( \alpha(C) = [x_1, x_4][x_2, x_3] \) and \( \alpha(P) = 0 \).

\begin{theorem}
The class \( \alpha \) is not in the image of
\( \partial : Hom_0(W^1, H) \rightarrow Hom_{-1}(W^2, H) \)
and therefore it represents a non-trivial obstruction to the formality of \( S \).
\end{theorem}

\begin{proof}
The only generators in \( W^1 \) that can contribute non-trivially to \( Im(\partial) \) in arity \( \leq 4 \) are \( A \in W^1_1(3) \) and \( B \in W^1_2(3) \) that are both in arity 3. Since
\begin{itemize}
\item \( H_1(S(3)) \) has a basis \([[[x_1, x_2], x_3], [x_1, x_3], [x_1, x_2]x_3] \)
\item \( H_2(S(3)) \) has a basis \([[[x_1, x_2], x_3], [x_1, [x_2, x_3]] \}
\end{itemize}
we have that \( Hom_0(W^1(3), H(S(3))) \) is 5-dimensional, spanned by
\begin{align*}
f_1 : A &\mapsto m \circ_1 b = [x_1, x_2]x_3; B \mapsto 0 \\
f_2 : A &\mapsto [x_1, x_3]x_2; B \mapsto 0 \\
f_3 : A &\mapsto m \circ_2 b = x_1[x_2, x_3]; B \mapsto 0 \\
f_4 : A &\mapsto 0; B \mapsto b \circ_1 b = [x_1, x_2], x_3] \\
f_5 : A &\mapsto 0; B \mapsto b \circ_2 b = [x_1, [x_2, x_3]]
\end{align*}
On the other hand

\[ \text{Hom}_{-1}(W^2(4), H(S(4))) = \text{Hom}(\mathbb{F}_2\{P\}, H_1(S(4))) \oplus \text{Hom}(\mathbb{F}_2\{C\}, H_2(S(4))) \cong H_1(S(4)) \oplus H_2(S(4)) \]

has dimension 6+11=17. By definition 2.11

\[
\partial(f_j) : P \mapsto \sum_{i=1}^{2} \hat{f}_j(m) \circ_i \hat{f}_j(A) + \sum_{i=1}^{3} \hat{f}_j(A) \circ_i \hat{f}_j(m) = \\
\sum_{i=1}^{2} (x_1 x_2) \circ_i \hat{f}_j(A) + \sum_{i=1}^{3} \hat{f}_j(A) \circ_i (x_1 x_2).
\]

\[
\partial(f_j) : C \mapsto \sum_{i=1}^{2} \hat{f}_j(b) \circ_i \hat{f}_j(A) + \sum_{i=1}^{3} \hat{f}_j(A) \circ_i \hat{f}_j(b) + \\
\sum_{i=1}^{2} \hat{f}_j(m) \circ_i \hat{f}_j(B) + \sum_{i=1}^{3} \hat{f}_j(B) \circ_i \hat{f}_j(m) = \\
\sum_{i=1}^{2} [x_1, x_2] \circ_i f_j(A) + \sum_{i=1}^{3} f_j(A) \circ_i [x_1, x_2] + \sum_{i=1}^{2} (x_1 x_2) \circ_i f_j(B) + \sum_{i=1}^{3} f_j(B) \circ_i (x_1 x_2)
\]

By substituting the values for \( f_j(A) \) and \( f_j(B) \) we find that

\[
\partial(f_1) : P \mapsto [x_1, x_2]x_3 x_4; C \mapsto . . . \\
\partial(f_2) : P \mapsto [x_1, x_4]x_2 x_3; C \mapsto . . . \\
\partial(f_3) : P \mapsto x_1 x_2 [x_3, x_4]; C \mapsto . . . \\
\partial(f_4) : P \mapsto 0; C \mapsto [x_1, x_4][x_2, x_3] + [x_1, x_2][x_3, x_4] \\
\partial(f_5) : P \mapsto 0; C \mapsto [x_1, x_4][x_2, x_3] + [x_1, x_2][x_3, x_4]
\]

The image of \( C \) in the first three cases is not important since \( \partial(f_j)(P) \) are linearly independent for \( j = 1, 2, 3 \) but \( \alpha(P) = 0 \) and so if \( \alpha \in \text{Im}(\partial) \) then it should be a linear combination of \( \partial(f_4) \) and \( \partial(f_5) \). However \( \alpha(C) \) is not a multiple of \( \partial(f_4)(C) = \partial(f_5)(C) \) and this proves that \( \alpha \notin \text{Im}(\partial) \).

Theorem 5.2 together with corollary 2.13 prove that \( S \) is a non-formal planar operad over \( \mathbb{F}_2 \), and this proves the main theorem 1.1.

What happens in characteristic \( p \) for \( p \) odd? The work by Cirici-Horel 3 indicates that the obstruction defined by \( \alpha \) vanishes in that case, and we should look for a higher obstruction. In fact we should construct the filtered model up to level \( p \) in order to find an obstruction mod \( p \). There is a striking similarity between this problem and the open problem of the formality over \( \mathbb{F}_p \) of the little disc spaces \( D_2(k) \), homotopy equivalent to the ordered configuration spaces \( F_k(\mathbb{R}^2) \) of \( k \) points in the plane, that we discuss briefly at the end of the paper 1.3.

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