Algebraic entropy for face-centered quad equations

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Abstract

In this paper we define the algebraic entropy test for face-centered quad equations, which are equations defined on vertices of a quadrilateral plus an additional interior vertex. This notion of algebraic entropy is applied to a recently introduced class of these equations that satisfy a new form of multidimensional consistency called consistency-around-a-face-centered-cube (CAFCC), whereby the system of equations is consistent on a face-centered cubic unit cell. It is found that for certain arrangements of equations (or pairs of equations) in the square lattice, all known CAFCC equations pass the algebraic entropy test possessing either quadratic or linear growth.

1 Introduction

Recently the second author has introduced a new set of five-point lattice equations, called face-centered quad equations, which satisfy a new form of multidimensional consistency, called consistency-around-a-face-centered-cube (CAFCC) [38]. The latter property may be regarded as an analogue of the consistency-around-a-cube (CAC) property [14,39] that is satisfied by regular integrable quad equations [10]. For such quad equations there is also the concept of algebraic entropy [9], which provides an algorithmic tool for determining the integrability of the equations based on examining the growth of their degrees under lattice evolution. The main goals of this paper are to introduce an analogous concept of algebraic entropy that is applicable to face-centered quad equations, and to apply this concept to equations that satisfy CAFCC to quantify the nature of their integrability (or non-integrability) in terms of their degree growth.

The concept of algebraic entropy is a method to measure the complexity of birational maps, analogous to the one introduced by Arnold for diffeomorphisms [4]. For finite dimensional maps this concept was first introduced in [16,32,43] and finally formalised in [9]. The concept
of algebraic entropy has been generalised to various different kinds of infinite-dimensional systems, most notably to quad-equations in [42,44,45]. A small generalisation of the concept of algebraic entropy for quad equations, considering the possibility of multiple growths coexisting in the same quad equation, was then given in [20].

Integrability for both continuous and discrete systems is associated with regularity and predictability of the solutions of a given system. Intuitively, this means that the complexity of an integrable system is expected to be low. For birational maps defined over complex projective spaces, this condition translates to the vanishing of the algebraic entropy. Positive algebraic entropy implies that the complexity of the iterates of a birational map grow exponentially. This makes the evolution of the motion almost impossible to predict and very sensitive to the initial conditions: an indication of what is usually referred as chaos. For a complete account on the history and the definition of algebraic entropy in the mentioned cases we refer to the reviews [18,19,29].

Face-centered quad equations are generalisations of quad equations. While quad equations are equations defined on the four vertices of a quadrilateral polygon, face-centered quad equations also involve an additional interior vertex. Thus a face-centered quad equation can be expressed as a sum of powers of the variable at the interior vertex with coefficients given by regular quad equations. In [38] the second author introduced CAFCC as a form of multidimensional consistency for face-centered quad equations. The latter property involves satisfying an over-determined system of fourteen equations on the face-centered cube (or face-centered cubic unit cell) for eight unknown variables, and may be regarded as an analogue of the CAC property defined for regular quad equations. The algorithmic nature of CAC has been exploited to obtain several different classifications of quad equations [1,2,11,28,30]. The quad equations that satisfy CAC have also been shown to arise in the asymptotics of more general equations for hypergeometric functions associated to the star-triangle relations [6–8,36], and this type of connection was used by the second author to derive the set of CAFCC equations. Similarly to the case of integrable quad equations, it was subsequently shown that the property of CAFCC allows for an algorithmic derivation of Lax pairs for face-centered quad equations [37].

The goals of this paper are to define and apply the concept of algebraic entropy to face-centered quad equations. Since the CAFCC equations satisfy a form of multidimensional consistency and have associated Lax pairs, it is expected that they will have a polynomial degree growth. Indeed, for specific arrangements of equations in the lattice it is found that the type-A and type-C equations each have quadratic growths of degrees, and the type-B equations each have linear growths of degrees. The formulation of algebraic entropy for face-centered quad equations and the computation of the generating functions describing the polynomial degree growths for equations satisfying CAFCC are the main results of this paper.

In the process of applying algebraic entropy we also solve the non-trivial problem of finding integrable combinations of CAFCC equations in a $\mathbb{Z}^2$ lattice. This problem is non-trivial because an arbitrarily chosen arrangement of equations is typically not sufficient to obtain a sub-exponential growth of degrees. The reason for this is that type-B and -C CAFCC equations do not satisfy the full symmetries of the square, so a general arrangement of these equations will not give rise to a well-defined lattice of consistent face-centered cubes in three dimensions. This situation is similar to the case of the $H4$- and $H6$-type quad equations [20].
Our constructions of lattices for type-B and -C CAFCC equations may be regarded as the analogue of the black-and-white lattices for the $H_4$ equations \cite{2, 47} and the four-colours lattice for the $H_6$ equations \cite{11, 20}. This means that such type-B and type-C equations are non-autonomous face-centered quad equations alternating on the lattice between different CAFCC equations (see \cite{20} for the analogous case on quad equations).

This paper is organised as follows. In Section 2 we give an overview of the CAFCC equations, and then define the concept of algebraic entropy for face-centered quad equations. We discuss the main properties of the algebraic entropy, and we present a few different examples of growth of degrees. In Section 3 we give the arrangements of type-A, type-B, and type-C CAFCC equations in the square lattice, and give the results from the computations of the algebraic entropy for the corresponding systems of equations. Finally, in Section 4 we give some conclusion and outlook for future works.

2 Face-centered quad equations and algebraic entropy

In Section 2.1 an overview of the concept of the face-centered quad equations will be given. Then in Section 2.2 a definition of algebraic entropy for the face-centered quad equations will be given. The concept of algebraic entropy requires the definition of a birational evolution, which in turn leads to the problem of defining proper initial conditions. These topics will be treated in Section 2.2.

2.1 Face-centered quad equations

The face-centered quad equations are defined on a five-point stencil of the square lattice shown in Figure 1. Figure 1 may be regarded as a face of the face-centered cube, which was the central idea in the formulation of the property of multidimensional consistency for these equations \cite{38}. In Figure 1 the four variables $x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}$, are associated to the four corner vertices, and the variable $x_{m,n}$ is associated to the face vertex. The two-component parameters

\[
\alpha = (\alpha_1, \alpha_2), \quad \beta = (\beta_1, \beta_2),
\]

are associated to the edges of the face, where two opposite edges that are parallel are assigned the same components of the parameters.

The face-centered quad equations that we are interested in, are always linear in the four corner variables $x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}$, but not the face variable $x_{m,n}$ (typically they are quadratic in $x_{m,n}$, except for the more complicated elliptic case). Such face-centered quad equations can be written in a general polynomial form

\[
A = \kappa_1 x_a x_b x_c x_d + \kappa_2 x_a x_b x_c + \kappa_3 x_a x_b x_d + \kappa_4 x_a x_c x_d + \kappa_5 x_b x_c x_d + \kappa_6 x_a x_b + \kappa_7 x_a x_c + \kappa_8 x_a x_d + \kappa_9 x_b x_c + \kappa_10 x_b x_d + \kappa_11 x_c x_d + \kappa_12 x_a + \kappa_13 x_b + \kappa_14 x_c + \kappa_15 x_d + \kappa_16 = 0,
\]

where $x_a = x_{m-1,n+1}, x_b = x_{m+1,n+1}, x_c = x_{m-1,n-1}, x_d = x_{m+1,n-1}$, and the coefficients $\kappa_i(x_{m,n}; \alpha, \beta)$ ($i = 1, \ldots, 16$) depend on the face variable $x_{m,n}$ and the four components...
Figure 1: Variables and parameters associated to the vertices and edges of a face of the face-centered cube. In terms of the graphical representation of CAFCC this is a type-A equation.

\( \alpha, \alpha_2, \beta_1, \beta_2 \) of the parameters \( \alpha, \beta \). The multi-linear expression \((2.2)\) resembles the expression for regular quad equations, except in the latter case the coefficients should only depend on two parameters and no variables.

The face-centered quad equations that were found to satisfy CAFCC \([38]\) are grouped into three different types. Type-A equations \( A(x_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}; \alpha, \beta) \) are invariant under the following exchanges of variables and parameters

\[
\begin{align*}
\{ x_{m-1,n+1} &\leftrightarrow x_{m+1,n+1}, x_{m-1,n-1} \leftrightarrow x_{m+1,n-1}, \beta_1 \leftrightarrow \beta_2 \}, \\
\{ x_{m-1,n+1} &\leftrightarrow x_{m-1,n-1}, x_{m+1,n+1} \leftrightarrow x_{m+1,n-1}, \alpha_1 \leftrightarrow \alpha_2 \}, \\
\{ x_{m-1,n+1} &\leftrightarrow x_{m+1,n-1}, \alpha \leftrightarrow \beta \}.
\end{align*}
\]

(2.3a) (2.3b) (2.3c)

Type-B equations are only invariant under \((2.3a)\) and \((2.3b)\), while type-C equations are only invariant under \((2.3a)\). It is also useful to distinguish these three different types of equations by using three different graphical representations, where the type-A equation is shown in Figure 1 and the type-B and type-C equations are shown in Figure 2. Type-A equations are drawn with single-line edges and type-B equations are drawn with double-line edges. Type-C equation are drawn with both single- and double-line edges, and the reason for this is that they arise on the face-centered cube as an intermediate equation that connects a type-A equation and a type-B equation.

Figure 2: Graphical representations of type-B (left) and type-C (right) face-centered quad equations.

The three types of face-centered quad equations associated to Figures 1 and 2 can typically
also be written in the equivalent forms

\begin{align}
\text{Type-A:} & \quad \frac{a(x_{m,n}; x_{m-1,n+1}; \alpha_2, \beta_1)}{a(x_{m,n}; x_{m+1,n-1}; \alpha_1, \beta_2)} \frac{a(x_{m,n}; x_{m+1,n+1}; \alpha_2, \beta_1)}{a(x_{m,n}; x_{m-1,n-1}; \alpha_1, \beta_2)} = 1, \\
\text{Type-B:} & \quad \frac{b(x_{m,n}; x_{m-1,n+1}; \alpha_2, \beta_1)}{b(x_{m,n}; x_{m+1,n-1}; \alpha_1, \beta_2)} \frac{b(x_{m,n}; x_{m+1,n+1}; \alpha_2, \beta_1)}{b(x_{m,n}; x_{m-1,n-1}; \alpha_1, \beta_2)} = 1, \\
\text{Type-C:} & \quad \frac{a(x_{m,n}; x_{m-1,n+1}; \alpha_2, \beta_1)}{a(x_{m,n}; x_{m+1,n-1}; \alpha_1, \beta_1)} \frac{c(x_{m,n}; x_{m+1,n+1}; \alpha_2, \beta_1)}{c(x_{m,n}; x_{m-1,n-1}; \alpha_1, \beta_1)} = 1.
\end{align}

in terms of three types of “leg” functions \(a(x; y; \alpha, \beta)\), \(b(x; y; \alpha, \beta)\), and \(c(x; y; \alpha, \beta)\). The “leg” function \(a(x_i; x_j; \alpha, \beta)\) is associated to the solid edges of both type-A and type-C equations, and the leg functions \(b(x_i; x_j; \alpha, \beta)\) and \(c(x_i; x_j; \alpha, \beta)\) are associated to the solid double edges of type-B and type-C equations respectively. The leg function \(a(x_i; x_j; \alpha, \beta)\) satisfies a symmetry \(a(x_i; x_j; \alpha, \beta) = 1\). The leg functions \(a(x_i; x_j; \alpha, \beta)\), \(b(x_i; x_j; \alpha, \beta)\), and \(c(x_i; x_j; \alpha, \beta)\), are each rational linear functions of a corner variable \(y\), and depend also on the face variable \(x\) and one component from each of the parameters \(\alpha, \beta\). The expressions (2.4)–(2.6) may be regarded as analogues of the three-leg forms for the regular quad equations [1]. For the cases of type-A equations, the expressions of the form (2.4) turn out to be equivalent to expressions for discrete Laplace-type (or Toda-type) equations associated to type-Q ABS equations [10].

Explicit expressions for the different equations and combinations that have been found to satisfy CAFCC are given in Appendix A. The property of CAFCC itself will not be considered here as it is not essential for this paper.

### 2.2 Algebraic entropy for face-centered quad equations

In this section we introduce the concept of algebraic entropy for face-centered quad equations. This may be regarded as an extension of the definition of algebraic entropy for quad equations [19, 20, 42, 44, 45]. In what follows we will define the forms of the standard initial conditions and describe the essential tools that are involved in the entropy computations.

#### 2.2.1 Double staircases of initial conditions

Like quad equations, face-centered quad equations are multilinear in the extremal variables \(x_{m\pm, n\pm}\). This implies that face-centered quad equations satisfy a fundamental condition required to apply the algebraic entropy criterion: in every direction of evolution the associated map is rational and invertible with a rational inverse, that is they are birational. There are four possible directions of evolution in the square lattice, corresponding to solving the equations with respect to one of the four different corner variables.

In general, initial conditions can be given along straight half-lines in the four directions in
the square lattice. To be more precise, in terms of the following half-lines

\[ M^{(+)}_{m_0,n_0} = \{(m,n_0) \in \mathbb{Z}^2 \mid m_0 \leq m < +\infty \}, \quad (2.7a) \]
\[ M^{(-)}_{m_0,n_0} = \{(m,n_0) \in \mathbb{Z}^2 \mid -\infty \leq m < m_0 \}, \quad (2.7b) \]
\[ N^{(+)}_{m_0,n_0} = \{(m_0,n) \in \mathbb{Z}^2 \mid n_0 \leq n < +\infty \}, \quad (2.7c) \]
\[ N^{(-)}_{m_0,n_0} = \{(m_0,n) \in \mathbb{Z}^2 \mid -\infty \leq n < n_0 \}, \quad (2.7d) \]

the standard initial conditions for face-centered quad equations are defined on the following sets of indices:

\[ I^{(+,+)}_{m_0,n_0} = M^{(+)}_{m_0,n_0} \cup M^{(+)}_{m_0+1,n_0+1} \cup N^{(+)}_{m_0,n_0} \cup N^{(+)}_{m_0+1,n_0+1}, \]
\[ I^{(-,+)}_{m_0,n_0} = M^{(-)}_{m_0,n_0} \cup M^{(-)}_{m_0+1,n_0-1} \cup N^{(+)}_{m_0,n_0} \cup N^{(+)}_{m_0+1,n_0-1}, \]
\[ I^{(+,-)}_{m_0,n_0} = M^{(+)}_{m_0,n_0} \cup M^{(+)}_{m_0-1,n_0+1} \cup N^{(-)}_{m_0,n_0} \cup N^{(-)}_{m_0-1,n_0+1}, \]
\[ I^{(-,-)}_{m_0,n_0} = M^{(-)}_{m_0,n_0} \cup M^{(-)}_{m_0+1,n_0+1} \cup N^{(-)}_{m_0,n_0} \cup N^{(-)}_{m_0+1,n_0+1}. \quad (2.8) \]

However, to compute the algebraic entropy of face-centered quad equations it is more convenient to give initial conditions on double staircase configurations. We discuss later the main rationale behind this choice. An evolution from any staircase-like arrangement of initial values is possible in the lattices of face-centered quad equations. Some prototypical examples of double staircases of points in the lattice are shown in Figure 3. In general, a double staircase configuration might include hook-like configurations as in example (d) of Figure 3. However, a hook-like configuration requires compatibility conditions on the initial data since there will be more than one way to calculate the same value for the dependent variable. For this reason we exclude these kinds of configurations from our consideration.

In general, double staircases go from \((-\infty,-\infty)\) to \((\infty,\infty)\), or from \((-\infty,\infty)\) to \((\infty,-\infty)\), implying that the space of initial conditions is infinite. Like in the case of quad equations, we decide to restrict ourselves to a specific kind of double staircases that we call regular double staircases. We define a double staircase to be regular when the external points (the black ones in Figure 3) lie on a regular single staircase, that is a staircase with steps of constant horizontal length, and constant vertical height. This will leave some freedom in the choice of the interior points (that is the grey ones in Figure 3). We will make a specific choice later. For example, in Figure 3 the double staircases (a) and (b) would be regular, (c) would be irregular, and (d) is excluded since it may lead to incompatibilities. We remark that for the case of quad equations, the problem of computing the entropy using irregular (single) staircases was considered in [31].

Given a positive integer \(N\), a pair of coprime integers \(\mu, \nu\), and a starting point \(p_0 \in \mathbb{C}^2\), we define a restricted regular single staircase \(\Delta^{(N)}_{[\mu,\nu]}(p_0)\) to be the set of points:

\[ \Delta^{(N)}_{[\mu,\nu]}(p_0) = p_0 + \{(k\mu + \text{sgn} \, \mu j_1, k\nu), ((k+1)\mu, k\nu + \text{sgn} \, \nu j_2)\} \quad \text{for} \quad \begin{array}{l}
\sum_{j_1=0,\ldots,|\mu|} \sum_{j_2=0,\ldots,|\nu|} (j_1, j_2) \leq N-1
\end{array} \quad \text{or} \quad \begin{array}{l}
\sum_{j_1=0,\ldots,|\mu|} \sum_{j_2=0,\ldots,|\nu|} (j_1, j_2) = N-1
\end{array}. \quad (2.9) \]

Note that for either \(\mu = 0\) or \(\nu = 0\), \(\Delta^{(1)}_{[\mu,\nu]}(p_0)\) collapses to a set of points on a straight line, and that \(\Delta^{(1)}_{[0,0]}(p_0) = \{p_0\} \).
Then a restricted regular double staircase denoted by $\Gamma_{[\mu,\nu]}^{(N)} (p_0)$, can be defined in terms of regular single staircases as follows:

$$
\Gamma_{[\mu,\nu]}^{(N)} (p_0) = 2\Delta_{[\mu,\nu]}^{(N)} \left( \frac{p_0}{2} \right) \cup 2\Delta_{[\mu,\nu]}^{(N-1)} \left( \frac{p_1}{2} \right) \cup 2\Delta_{[\mu,\nu-\text{sgn} \nu]}^{(1)} \left( \frac{p_2}{2} \right),
$$

(2.10)

where:

$$
p_1 = p_0 + \left( \text{sgn} \mu, \text{sgn} \nu \right),
$$

(2.11a)

$$
p_2 = p_1 + 2(N-1)(\mu, \nu).
$$

(2.11b)

Some examples of restricted regular double staircases are shown in Figure 4. In (2.10), $2\Delta_{[\mu,\nu]}^{(N)} \left( \frac{p_0}{2} \right)$ is a staircase corresponding to black points in Figure 4, and $2\Delta_{[\mu,\nu]}^{(N-1)} \left( \frac{p_1}{2} \right) \cup$
$2\Delta_{[\mu, sgn \mu, \nu, sgn \nu]}^{(1)} \left( \frac{p_2}{2} \right)$ is a staircase corresponding to grey points in Figure 4. The notation $\Gamma_{[\mu, \nu]}(p_0)$ will be used when considering infinite (unrestricted) regular double staircases.

The total number of initial points in a double staircase is $M = 2N(\vert \mu \vert + \vert \nu \vert)$. We consider the associated initial values in an appropriate compactification of $C^M$. In this paper we will consider two different compactifications of $C^M$, namely $CP^M$ and $CP_{\frac{M}{2}+1} \times CP_{\frac{M}{2} - 1}$. In the first case we consider the initial conditions to be parametrised by a line in $CP^M$ in the following form:

$$x_{m,n} = \frac{\alpha_{m,n} t + \beta_{m,n}}{\alpha t + \beta}, \quad (m,n) \in \Gamma^{(N)}_{[\mu, \nu]}.$$  

(2.12)

where $\alpha_{m,n}, \beta_{m,n}, \alpha, \beta \in \mathbb{C}$ are fixed parameters. In the second case we differentiate between the corner vertices and the face vertex of an equation, and parametrise the initial conditions.
with respective lines in $\mathbb{CP}^{M+1}$ and $\mathbb{CP}^{M-1}$:

\[
x_{m,n} = \frac{\alpha_{m,n} t + \beta_{m,n}}{\alpha_{1} t + \beta_{1}}, \quad (m,n) \in 2\Delta_{[\mu,\nu]}^{(N)} \left( \frac{p_{\mu}}{p_{\nu}} \right),
\]

\[
x_{m,n} = \frac{\alpha_{m,n} t + \beta_{m,n}}{\alpha_{2} t + \beta_{2}}, \quad (m,n) \in 2\Delta_{[\mu,\nu]}^{(N-1)} \left( \frac{p_{\mu}}{p_{\nu}} \right) \cup 2\Delta_{[\mu,\nu]}^{(1)},
\]

where $\alpha_{m,n}, \beta_{m,n}, \alpha_{i}, \beta_{i} \in \mathbb{C}$ are fixed parameters. We emphasise that these two are not the only possible compactifications of the space $C^{M}$. However, it is expected that the final result for algebraic entropy will be independent of the choice of the compactification, and thus our two choices of initial conditions should be sufficient for our investigations of the algebraic entropy for face-centered quad equations.

2.2.2 Growth of degrees and algebraic entropy

Now we may calculate the iterates by considering the initial values as given in (2.12) or (2.13). The simplest choice is to apply this construction to the restricted double staircases $\Gamma_{[\pm 1, \pm 1]}^{(N)}$. We will call these the fundamental double staircases (the upper index $(N)$ will be omitted for infinite lines). Furthermore, we denote by $\Gamma_{NE}^{(N)}, \Gamma_{NW}^{(N)}, \Gamma_{SW}^{(N)}, \Gamma_{SE}^{(N)}$, the fundamental double staircases where the evolution is directed in the north-east, north-west, south-west, and south-east directions, respectively. These fundamental double staircases have relatively simple expressions given by

\[
\Gamma_{NE}^{(N)} = \{ (-n,n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N \} \cup \{ (-n-2,n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N-2 \},
\]

\[
\Gamma_{NW}^{(N)} = \{ (n,n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N \} \cup \{ (n+2,n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N-2 \},
\]

\[
\Gamma_{SW}^{(N)} = \{ (n,-n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N \} \cup \{ (n+2,-n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N-2 \},
\]

\[
\Gamma_{SE}^{(N)} = \{ (-n,-n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N \} \cup \{ (-n-2,-n) \in \mathbb{Z}^{2} | 0 \leq n \leq 2N-2 \}.
\]

The above fundamental double staircases are shown in Figure 5. Note that in the case of fundamental regular staircase, there is no ambiguity in the definition of the inner sequence of points.

The number of points in the fundamental double staircases (2.14) is $4N$. Thus the initial condition (2.12) on the fundamental double staircases is taken in $\mathbb{CP}^{4N}$. Furthermore, according to (2.10) the fundamental double staircases decompose into two regular single staircases which possess $2N+1$ and $2N-1$ points respectively. Thus the initial condition (2.13) on the fundamental double staircases is taken in $\mathbb{CP}^{2N+1} \times \mathbb{CP}^{2N-1}$. For convenience, in the remainder of the paper a special notation $X_{1}$ and $X_{2}$ will be used to denote the respective spaces of initial conditions for fundamental double staircases as

\[
X_{1} = \mathbb{CP}^{4N}, \quad X_{2} = \mathbb{CP}^{2N+1} \times \mathbb{CP}^{2N-1}.
\]

Consider the evolution from the fundamental double staircase in the NE direction. From this evolution, sequences of degrees $d_{k}^{(i)}, k = 1, \ldots, N, i = 1, \ldots, N-k$, will be generated that take the shape shown in Figure 6. The degree $d_{k}^{(i)}$ is computed as the maximum of the degree
of numerator and the denominator of the dependent variable on the lattice. In this example of the NE direction, we will call the sequence

$$1, d^{(1)}_1, d^{(1)}_2, d^{(1)}_3, d^{(1)}_4, \ldots$$

(2.16)

the primary sequence of growth, while sequence as

$$1, d^{(m)}_1, d^{(m)}_2, d^{(m)}_3, d^{(m)}_4, \ldots$$

(2.17)

with $m = 2, 3, \ldots$ will be secondary sequence of growth, tertiary sequence, and so on. The sequences in the other directions of fundamental double staircases are defined similarly. In Section 3, it will be seen that CAFCC equations can have up to four different growth patterns in each direction of evolution.

In general, we take as principal sequence the longest one, and then we move towards the direction with less elements. That is, in the NE and SE direction we move left, while in the NW and SW direction we move right. Periodicity along the columns is found looking at repeating patterns on different columns.

The fundamental algebraic entropies of a face-centered quad equation are given in terms
of the generated degrees by:
\[ S^{(i)} = \lim_{k \to \infty} \frac{1}{k} \log d_k^{(i)}. \]  

Furthermore, we define a maximal sequence of growth:
\[ 1, D_1, D_2, D_3, D_4, D_5, \ldots, \]  
where \( D_k = \max_{m=1, \ldots, N-k} d_m^{(m)} \). Associated to (2.19), we define the maximal algebraic entropy of a face-centred quad equation:
\[ S_{\text{Max}} = \lim_{k \to \infty} \frac{1}{k} \log D_k. \]  

The existence of the limits (2.18) and (2.20) may be proven in an analogous way as for finite dimensional systems [9].

We have the following general result on the growth of degrees for an autonomous face-centered quad equation:

**Proposition 2.1.** Assume we are given an autonomous face-centered quad equation, that is an equation of the form (2.2) where the polynomials \( \kappa_i \) are independent of the variables \( (m,n) \). Then its maximal growth is asymptotically given by the solution of the linear difference equation:
\[ D_{k+1} = (K + 2) D_k + D_{k-1}, \]  
that is:
\[ D_k \sim \left[ \frac{1}{2} \left( K + 2 + \sqrt{K^2 + 4K + 8} \right) \right]^k, \quad k \to \infty, \]  
where \( K = \max_{i=1, \ldots, 16} \text{deg} \xi \kappa_i (\xi) \). So, for a fixed \( K \) the maximal value of the algebraic entropy is:
\[ S_K = \log \left[ \frac{1}{2} \left( K + 2 + \sqrt{K^2 + 4K + 8} \right) \right]. \]
Proof. Consider equation (2.2) solved with respect to $x_{m+1,n+1}$. Assuming that no factorisation occurs, we can take the degrees over the one-dimensional plaquette in the $(k + 1)$-th iteration, given in Figure 1. On the right-hand side we have $x_{m,n}$ at most at the $K$th power, while the equation is linear in $x_{m+1,n-1}$, $x_{m-1,n+1}$, and $x_{m-1,n-1}$. This means that at most we have:

$$\deg x_{m+1,n+1} = K \deg x_{m,n} + \deg x_{m+1,n-1} + \deg x_{m+1,n-1} + \deg x_{m-1,n-1}. \quad (2.24)$$

To obtain the estimate (2.21) it is enough to note that we can assume that the maximal sequence $D_k$ is such that

$$\deg x_{m+1,n+1} = D_{k+1},$$

$$\deg x_{m,n} = \deg x_{m+1,n-1} = \deg x_{m+1,n-1} = D_k,$$

$$\deg x_{m-1,n-1} = D_{k-1}. \quad (2.25a)$$

Substituting (2.25) into (2.24), we readily obtain formula (2.21).

We note that for $K = 0$, the formula (2.21) reduces to:

$$D_k \sim \left(1 + \sqrt{2}\right)^k. \quad (2.26)$$

As expected, this is the default growth for regular quad equations [29, 33]. This result implies that from the point of view of algebraic entropy the structure of non-integrable face-centered quad equations is much richer than that of regular quad equations.

Analogously to the case of regular quad equations [33], according to the algebraic entropy and degree of iterates growth for the face-centered quad equations they may be classified as follows:

**Linear growth:** The equation is linearisable.

**Polynomial growth:** The equation is integrable.

**Exponential growth:** The equation is non-integrable.

2.2.3 Generating functions of degree growths

To compute the degree of iterates we adopt some simplification techniques that are used for the case of regular quad equations [18, 19]. First, we choose all of the parameters involved in the equations to be integers. Furthermore, to avoid potential factorisations that may inadvertently affect the results we choose these integers to be prime numbers. A final simplification to speed up the computations is given by considering the factorisation of the iterates in some finite field $\mathbb{K}_r$, with $r$ a given prime number. Using these rules we are able to avoid accidental cancellations and produce a finite sequence of degrees:

$$1, d_1, d_2, d_3, d_4, d_5, \ldots, d_N. \quad (2.27)$$
A standard way to extract the asymptotic behaviour from a finite sequence like (2.27) is to compute its generating function. A generating function is a function \( g = g(z) \) such that the coefficients of its Taylor series

\[
g(z) = \sum_{k=0}^{\infty} d_k z^k
\]

up to order \( N \) coincides with the finite sequence (2.27). The adoption of this heuristic method is justified by the strong algebraic geometry structure behind the definition of algebraic entropy, see [9]. Up to present, in all known examples the method is known to work except, in some pathological cases, see [25].

If a generating function is rational, as it is safe to assume, it can be computed exactly using a finite number of iterates using the method of Padé approximants [5, 40]. An important property of rational generating functions is given in the following theorem:

**Theorem 2.2 ([15]).** A sequence \( \{d_k\}_{k \in \mathbb{N}_0} \) admits a rational generating function \( g \in \mathbb{C}(z) \) if and only if it solves a linear difference equation with constant coefficients. In such case if \( g = P/Q \), with \( P, Q \in \mathbb{C}[z] \) and the polynomial \( Q \) is monic, then the difference equation is given by \( Q(T_k^{-1})(d_k) = 0 \), where \( T_k d_k = d_{k+1} \) is the translation operator.

From Theorem 2.2 we have that the order of the difference equation solved by the sequence \( \{d_k\}_{k \in \mathbb{N}_0} \) is equal to the degree of the denominator of the generating function. Moreover, we see that the numerator of the generating function acts as a sort of “noise” linked to the initial conditions of the sequence. For instance, this implies that if two sequence have the same denominator but different numerators, they will satisfy the same difference equation with different initial conditions. Several examples of this kind of occurrence will appear in Section 3.

Once obtained a generating function is a predictive tool. Indeed one can readily compute the successive terms in the Taylor expansion for (2.28) and confront them with the degrees calculated with the iterations. This means that the assumption that the value of the algebraic entropy is given by the approximate method is in fact very strong and it is very unlikely that the real value will differ from it.

Having a rational generating function will also yield the value of the algebraic entropy from the modulus of the smallest pole of the generating function:

\[
S = \log \min \left\{ |z| \in \mathbb{R}^+ \left| \frac{1}{g(z)} = 0 \right. \right\}^{-1}.
\]

(2.29)

Let us underline that while the value of the algebraic entropy obtained through (2.29) is canonical as it depends only on the equation itself, the sequence of degrees and its generating function are not canonical quantities and might change under appropriate invertible transformations.

From the generating function one can also find an asymptotic fit for the degrees (2.27). This can be done by using the inverse \( \mathcal{Z} \)-transform [15, 35]. Assume we are given a function \( f = f(\zeta) \) of a complex variable \( \zeta \in \mathbb{C} \) analytic in a region \( |\zeta| > r \) for some \( r \in \mathbb{R}^+ \). We define the inverse \( \mathcal{Z} \)-transform of such a function \( f \) to be the sequence:

\[
\mathcal{Z}^{-1}[f(\zeta)]_k \equiv \frac{1}{2\pi i} \oint_C f(\zeta) \zeta^{k-1} \, d\zeta, \quad k \in \mathbb{N}.
\]

(2.30)
In equation (2.30) the contour $C \subset \mathbb{C}$ is a counterclockwise closed path enclosing the origin and entirely in the region of convergence of $f$. From the definition of inverse $Z$-transform (2.30) it can be readily proved that the sequence $\{d_k\}_{k \in \mathbb{N}}$ corresponding to the generating function (2.28) is given by:

$$d_k = Z^{-1} \left[ g \left( \frac{1}{\zeta} \right) \right]_k. \quad (2.31)$$

We note that the general asymptotic behaviour of the sequence $\{d_k\}_{k \in \mathbb{N}_0}$ can be obtained even without computing the inverse $Z$-transform. This is the content of the following proposition:

**Proposition 2.3 ([17]).** Assume that a sequence $\{d_k\}_{k \in \mathbb{N}_0}$ possesses a generating function of radius of convergence $\rho > 0$ and of the following form:

$$g = A(z) + B(z) \left( 1 - \frac{z}{\rho} \right)^{-\beta}, \quad \beta \in \mathbb{R} \setminus \{ -n \}_{n \in \mathbb{N}}. \quad (2.32)$$

where $A$ and $B$ are analytic functions for $|z| < r$ such that $B(\rho) \neq 0$. Then the asymptotic behaviour of the sequence $\{d_k\}_{k \in \mathbb{N}_0}$ as $k \to \infty$ is given by:

$$d_k \sim \frac{B(\rho)}{\Gamma(\beta)} k^{\beta - 1} \rho^{-k}, \quad k \to \infty, \quad (2.33)$$

where $\Gamma(z)$ is the Euler Gamma function. If additionally $\rho \equiv 1$, then

$$d_k \sim k^{\beta - 1}, \quad k \to \infty, \quad (2.34)$$

i.e. the growth is asymptotically polynomial of degree $\beta - 1$.

Notice that when the generating function is polynomial and the radius of convergence is $\rho = 1$, usually the denominator can be factorised as follows:

$$Q(z) = (1 - z)^{\beta_0} \prod_{i=1}^{K} \left( 1 - z^{\beta_i} \right). \quad (2.35)$$

That is, the polynomial $Q$ is the product of the term $(1 - z)^{\beta_0}$ with some cyclotomic polynomials $p_i(z) = 1 - z^{\beta_i}$. From Proposition (2.3), using the factorisation properties of cyclotomic polynomials, we get that $d_k \sim k^{\beta}$ as $k \to \infty$, where $\beta = \beta_0 + K$. We will make constant use of this observation to estimate the asymptotic growth of degrees of the face-centered quad equations in Section 3.

### 2.2.4 Examples

We conclude this section with some illustrative examples of the computation of algebraic entropy that has been formulated above for face-centered quad equations.

**Example 1.** Consider the following face-centered quad equation:

$$Ax_{m+1,n+1}x_{m-1,n-1} + Bx_{m-1,n+1}x_{m+1,n-1} = Cx_{m,n}^2, \quad (2.36)$$
where $A$, $B$, and $C$ are arbitrary constants.

Considering initial conditions in $X_1$ we have a single sequence of degrees of iterates:

$$1, 2, 6, 14, 26, 42, 62, 86, 114, 146, 182, 222, 266 \ldots$$

(2.37)

The corresponding generating function is:

$$g_{X_1}(z) = \frac{z^3 + 3z^2 - z + 1}{(1-z)^3}.$$ 

(2.38)

From this, we infer that the algebraic entropy is positive. Indeed, the smallest singularity of $g_{X_1}$ in absolute value is:

$$z_0 = \frac{1}{3} + \frac{1}{6} \left( 8 + 6\sqrt{78} \right)^{1/3} - \frac{7}{3 \left( 8 + 6\sqrt{78} \right)^{1/3}} \simeq 0.3966082528 \ldots < 1$$

(2.44)

The value of the entropy is then $S = \log z_0^{-1} \simeq \log (2.521379706 \ldots )$. Notice that despite the growth being exponential, it is not maximal. Indeed, since (2.41) has degree 2 in $x_{l,m}$, from equation (2.23) the maximal entropy is $S_2 = \log (2 + \sqrt{5})$, which is clearly greater than $S$.

If we fix our initial conditions in $X_2$ we have the single sequence of degrees of iterates:

$$1, 4, 12, 30, 74, 186, 470, 1186, 2990, 7538, 19006 \ldots ,$$

(2.45)
fitted by the generating function:

\[ g_{X_2}(z) = \frac{-2z^2 + z + 1}{2z^3 - 2z^2 + 3z - 1}. \]  

(2.46)

Since the denominator of \( g_{X_2} \) is the same as \( g_{X_1} \), we obtain, as expected, the same value of the algebraic entropy. \( \square \)

**Example 3.** Consider the following face-centered quad equation:

\[
P_0(x_{m,n}) x_{m-1,n+1} x_{m+1,n+1} x_{m-1,n-1} x_{m+1,n-1}
+ P_1(x_{m,n}) (x_{m-1,n+1} - x_{m+1,n+1}) (x_{m-1,n-1} - x_{m+1,n-1}) = P_2(x_{m,n}).
\]  

(2.47)

where \( P_i(\xi) \in \mathbb{C}[\xi] \) and \( \deg P_1 = 2 \).

Considering initial conditions either in \( X_1 \) or in \( X_2 \) we have a single sequence of degrees of iterates:

\[ 1, 5, 21, 89, 377, 1597, 6765, 28657 \ldots \]  

(2.48)

The corresponding generating function is:

\[ g(z) = \frac{1 + z}{z^3 + 4z - 1}. \]  

(2.49)

So, we infer that the algebraic entropy is positive and maximal. Indeed, the smallest singularity of \( g \) in absolute value is \( z_0 = -2 + \sqrt{2} \). This gives value of the entropy as \( S_2 = \log z_0^{-1} = \log (2 + \sqrt{5}) \), that is the maximal value of the entropy for \( M = 2 \) in formula (2.23). \( \square \)

### 3 Arrangements of CAFCC equations in the lattice and algebraic entropy

In this section we present the results of the heuristic computations of the algebraic entropy on the three types of CAFCC equations. The results of these computations are summarised as follows:

- **Type-A CAFCC equations** possess quadratic growth.
- **Type-B CAFCC equations** possess linear growth.
- **Type-C CAFCC equations** possess quadratic growth.

In the following subsections we will give the details of arrangements of equations in the lattice and the explicit growth patterns for each CAFCC equation found from [38] under the two types of initial conditions (2.15). In particular, it is sometimes the case that individual type-B or type-C equations have exponential growth patterns, but certain pairs of such equations together will give sub-exponential growth patterns. These cases of pairs of equations possess multiple growth patterns, each with the same asymptotic behaviour. Furthermore, for one particular type-C equation a new type-C CAFCC equation was needed to achieve quadratic degree growth \( (C1(\delta_1=1) \text{ in (A.9)}) \), and this equation was not found from the original method used to obtain CAFCC equations [38].
To describe the arrangements of equations and initial conditions, it is useful to colour the vertices of the rotated bipartite square lattice such that black vertices are connected only to white vertices (and vice versa). The convention will be that black vertices are associated to variables \( x_{m,n} \) and the white vertices are associated to variables \( y_{m,n} \). Such a black and white colouring of the lattice is shown in Figure 7. In the example of Figure 7, the initial conditions are indicated by the square vertices, and the evolution of the system of face-centered quad equations would be in the north-east direction. The equations in the lattice can be distinguished according to the numbers of single- and double-line edges that are connected to a vertex. For example, according to the graphical representation of face-centered quad equations shown in Figures 1 and 2, Figure 7 involves all type-A equations, Figure 8 involves all type-B equations, and Figure 9 involves all type-C equations. There are more complicated arrangements of three or more different equations in the lattice that are also expected to give a polynomial degree growth, but for this paper only the minimal number of different equations (either individual or pairs) to achieve polynomial degree growth are considered.

In the following, \( \hat{\alpha} \) and \( \hat{\beta} \) are used to denote

\[
\hat{\alpha} = (\alpha_2, \alpha_1), \quad \hat{\beta} = (\beta_2, \beta_1).
\]
3.1 Type-A

3.1.1 Arrangement of equations in the lattice

The arrangement of type-A equations in the lattice is the most straightforward and is shown in the example of Figure 7. The type-A equations centered at white vertices are

\[ A(y_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}; \alpha, \beta) = 0, \]

and the type-A equations centered at black vertices are

\[ A(x_{m,n}; y_{m-1,n+1}, y_{m+1,n+1}, y_{m-1,n-1}, y_{m+1,n-1}; \hat{\alpha}, \hat{\beta}) = 0. \]

3.1.2 Degree growths for type-A equations

The two type-A equations \( A_3(\delta) \) and \( A_2(\delta_1; \delta_2) \) are given in equations (A.1) and (A.5), respectively. For values \( \delta = 0, 1 \), and \( (\delta_1, \delta_2) = (0, 0), (1, 0), (1, 1) \), these equations each share the same growth patterns when iterated in the lattice of Figure 7. In \( \mathcal{X}_1 \) the growth pattern in all directions is:

\[ 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157 \ldots, \]

while in \( \mathcal{X}_2 \) the growth in all directions is:

\[ 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313 \ldots. \]

The generating functions for these patterns are respectively given by

\[ g_{\mathcal{X}_1}(z) = \frac{z^2 + 1}{(1 - z)^3}, \quad g_{\mathcal{X}_2}(z) = \frac{(z + 1)^2}{(1 - z)^3}. \]

Since all the zeroes of the generating functions lie on the unit circle for each type-A equation the algebraic entropy vanishes. Moreover, due to the presence of \((1 - z)^3\) in the denominator we have from Proposition 2.3 that the growth is quadratic.

We emphasise two important facts on the growth pattern of the type-A equations. The first one is that the growth is the same in all directions because type-A equations on the lattice are symmetric with respect to the exchange \((m, n) \leftrightarrow (n, m)\). The second is that the growth stays the same regardless of the value of the parameter \( \delta \) in \( A_3(\delta) \), or the value of the parameter \( \delta_1 \) in \( A_2(\delta_1, 0) \). For \( \delta \neq 0 \) or \( \delta_1 \neq 0 \), this follows from the fact that these parameters can be scaled to 1. On the other hand, when \( \delta = 0 \) or \( \delta_1 = 0 \) there is no difference in the growth of the degrees for the respective equations, so it can be seen that their behaviour does not depend on these parameters.

As observed in [38], the type-A CAFCC equations are equivalent to discrete Laplace-type equations that are associated to type-Q ABS equations [1]. Thus it is not surprising to find that their growth pattern in \( \mathcal{X}_1 \) is the same as the most general type-Q equation: the \( Q_V \) equation [45]. The growth pattern of \( Q_V \) has been proven rigorously in independent works [41, 46], where the proof of [41] used the gcd-factorisation method that also worked for the case of a two-periodic extension of the \( Q_V \) equation [22].
3.2 Type-B (individual)

In the following the algebraic entropy for individual type-B equations in the lattice will be considered. However, some examples of type-B equations have exponential growth when considered individually in the lattice. Only the cases of individual type-B equations that achieve sub-exponential growth will be considered here, while the remaining type-B equations will be treated in Section 3.4.

3.2.1 Arrangement of equations in the lattice

The arrangement of type-B equations in the lattice is shown in Figure 8. The type-B equations centered at white vertices are

\[ B(y_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}; \alpha, \beta) = 0, \]  

(3.7)

and type-B equations centered at black vertices are

\[ B(x_{m,n}; y_{m-1,n+1}, y_{m+1,n+1}, y_{m-1,n-1}, y_{m+1,n-1}; \hat{\alpha}, \hat{\beta}) = 0. \]  

(3.8)

The arrangement of equations in Figure 8 is equivalent to that of Figure 7, but with double edges instead of single edges that are used to represent type-B equations rather than type-A equations. Because the same type-B equation is centered at both the black vertices and white vertices, these systems of equations will be referred to as \( B(x; y) \) systems.

![Figure 8: Arrangement of type-B equations in the lattice. This is equivalent to the arrangement of Figure 7 but with double edges to indicate type-B equations instead of type-A equations.](image)

Note that there are two degenerate type-B equations given by

\[ B_2(0,0;0) = B_3(0,0;0) = y_{l+1,m+1}y_{l-1,m-1} - y_{l-1,m+1}y_{l+1,m-1} = 0. \]  

(3.9)
This equation (3.9) is not a true face-centered quad equation because it has no dependence on the central face variable. Considering the double pass lattice \((m, n) \to (2M, 2N)\) equation (3.9) can be reduced to the quad equation

\[
y_{M+1,N+1}y_{M,N} - y_{M+1,N}y_{M,N+1} = 0.
\]

(3.10)

This equation is a well-known Darboux integrable quad equation [3] with isotropic linear growth. It is also a special case of the \(D4\) quad equation that appeared in [11].

### 3.2.2 \(B2_{(1;0;0)}(x; y)\) system

The equation \(B2(\delta_1; \delta_2; \delta_3)\) is given in (A.6). For the case of \((\delta_1, \delta_2, \delta_3) = (1, 0, 0)\), this equation exhibits a single growth pattern in \(X_1\) and in \(X_2\), respectively.

In \(X_1\) the growth pattern is the following:

\[
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 \ldots
\]

(3.11)

This growth is clearly linear and is fitted by the following generating function:

\[
g_{X_1}(z) = \frac{1}{(1-z)^2}.
\]

(3.12)

The linearity of the growth (3.11) could also be confirmed using Proposition 2.3, because of the presence of the term \((1-z)^2\) in the denominator of (3.12).

In \(X_2\) the growth pattern is:

\[
1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25 \ldots
\]

(3.13)

This growth is clearly linear and is fitted by the following generating function:

\[
g_{X_2}(z) = \frac{1+z}{(1-z)^2}.
\]

(3.14)

The linearity of the growth (3.13) could also be inferred from Proposition 2.3, because of the presence of the term \((1-z)^2\) in the denominator of (3.14).

As expected, the asymptotic behaviour in \(X_1\) and \(X_2\) is the same. The above findings imply that the \(B2_{(1;0;0)}\) equation is expected to be linearisable.

### 3.2.3 \(B3_{(1;0;0)}(x; y)\) system

The equation \(B3(\delta_1; \delta_2; \delta_3)\) is given in (A.2). For the case of \((\delta_1, \delta_2, \delta_3) = (1, 0, 0)\), this equation exhibits a single growth pattern both in \(X_1\) and in \(X_2\). These two growth patterns coincide and are given by (3.13) with generating function (3.14). This implies that the \(B3_{(1;0;0)}\) equation is expected to be linearisable.

### 3.3 Type-C (individual)

In the following the algebraic entropy for individual type-C equations in the lattice will be considered. However, similarly to the previous case for type-B equations, some examples of type-C equations only have exponential growth when considered individually in the lattice. Only the cases of type-C equations that have sub-exponential growth will be considered here, and the remaining type-C equations will be treated in Section 3.5.

In the following, the integers \(n \bmod 2\) are taken to be elements of \(\{1, 2\}\).
3.3.1 Arrangement of equations in the lattice

The arrangement of type-C equations in the lattice is shown in Figure 9. The type-C equations centered at white vertices are

\[ C(y_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}; \alpha, \beta) = 0. \]  \hspace{1cm} (3.15)

According to Figure 2, the type-C equations centered at black vertices of Figure 9 are rotated by 180°, which is given by

\[ C(x_{m,n}; y_{m+1,n-1}, y_{m-1,n-1}, y_{m+1,n+1}, y_{m-1,n+1}; \alpha, \beta) = 0. \]  \hspace{1cm} (3.16)

Because the same type-C equation is centered at both the black vertices and white vertices, these systems of equations will be referred to as \( C(x; y) \) systems.

![Figure 9: Arrangement of type-C equations in the lattice. Equations centered at black vertices are rotated 180° relative to equations centered at white vertices.](image)

Type-C equations have the most varying behaviour in comparison to both type-A and type-B equations. All type-C equations are found to possess more than one growth pattern, and these patterns may differ greatly depending on the different directions of evolution. Despite this behaviour, it is interesting to note that the maximal degree growth of type-C equations is isotropic.

3.3.2 \( C_{1(0)}(x; y) \) system

The equation \( C_{1(0)}(\delta) \) is given in (A.9). For the case of \( \delta = 0 \), this equation exhibits two different growth patterns in both \( X_1 \) and in \( X_2 \), respectively. The patterns for the different
sets of initial conditions are different from each other. In both cases the patterns appear in reverse order in the evolution matrix in the SW and SE directions.

In $X_1$ the two different degree patterns are given in (B.1) and (B.2). These two degree patterns are fitted by the two different generating functions defined by

$$g_{X_1}^{(\ell)}(z) = \frac{h^{(\ell)}(z)}{(1-z^3)(1-z)^2}, \quad \ell = 1, 2, \quad (3.17)$$

where

$$h^{(\ell)}(z) = \begin{cases} z^4 - z^3 + 2z^2 + z + 1, & \ell = 1, \\ (z^2 - z + 1)(z + 1), & \ell = 2. \end{cases} \quad (3.18)$$

By Proposition 2.3, the growth of degree patterns associated to each of $g_{X_1}^{(\ell)}(z) \ (\ell = 1, 2)$ is quadratic. The maximal growth coincides with the pattern (B.1) fitted by $g_{X_1}^{(\ell=1)}(z)$.

In $X_2$ the two different degree patterns are given in (B.3) and (B.4). These two degree patterns are fitted by the two different generating functions defined by

$$g_{X_2}^{(\ell)}(z) = \frac{h^{(\ell)}(z)}{(1-z^3)(1-z^3)(1-z)}, \quad \ell = 1, 2, \quad (3.19)$$

where

$$h^{(\ell)}(z) = \begin{cases} z^{14} - z^{11} + z^{10} + 4z^9 + 5z^7 + 8z^6 + 8z^5 + 7z^4 + 7z^3 + 8z^2 + 3z + 1, & \ell = 1, \\ (z + 1)(z^{11} - z^{10} + 2z^9 + 4z^7 + 3z^6 + 3z^5 + 4z^4 + 3z^3 + 2z^2 + 3z + 1), & \ell = 2. \end{cases} \quad (3.20)$$

By Proposition 2.3, the growth of degree patterns associated to each of $g_{X_2}^{(\ell)}(z) \ (\ell = 1, 2)$ is quadratic. The maximal degree growth is given by (B.5), and is fitted by the following generating function:

$$G_{X_2}(z) = \frac{(z + 1)(z^{13} - z^{12} + z^9 + 3z^7 + 2z^6 + 6z^5 + 3z^4 + 3z^3 + 4z^2 + 3z + 1)}{(1-z^3)(1-z^3)(1-z)}. \quad (3.21)$$

Its asymptotic behaviour is quadratic.

As expected, the asymptotic behaviour in $X_1$ and $X_2$ is the same. The above findings imply that the $C1_{(0)}$ equation is expected to be integrable.

### 3.3.3 $C2_{(1;0;0)}(x;y)$ system

The equation $C2_{(\delta_1;\delta_2;\delta_3)}$ is given in (A.7). For the case of $(\delta_1, \delta_2, \delta_3) = (1, 0, 0)$, this equation exhibits two different growth patterns in both $X_1$ and in $X_2$. The patterns for the different sets of initial conditions are different from each other. In both cases the patterns appear in reverse order in the evolution matrix in the SW and SE directions.

The two different degree patterns in $X_1$ are given in (B.6) and (B.7). These two degree patterns are fitted by the two different generating functions defined by

$$g_{X_1}^{(\ell)}(z) = \frac{h^{(\ell)}(z)}{(1-z^3)(1-z)^2}, \quad \ell = 1, 2, \quad (3.22)$$
where
\[ h^{(\ell)}(z) = \begin{cases} 
- (z^2 + 1)(z^3 - z^2 - z - 1), & \ell = 1, \\
(z + 1)(z^2 + 1), & \ell = 2.
\end{cases} \] (3.23)

By Proposition 2.3, the growth of degree patterns associated to each of \( g_{X_1}^{(\ell)}(z) \) \((\ell = 1, 2)\) is quadratic. The maximal growth coincides with the pattern (B.6) fitted by \( g_{X_1}^{(\ell=1)}(z) \).

The two different degree patterns in \( X_2 \) are given in (B.8) and (B.9). These two degree patterns fitted by the two different generating functions defined by
\[ g_{X_2}^{(\ell)}(z) = \frac{h^{(\ell)}(z)}{(1 + z^2)(1 - z)^3}, \quad \ell = 1, 2, \] (3.24)
where
\[ h^{(\ell)}(z) = \begin{cases} 
z^{12} - 2z^{11} + 2z^{10} - 2z^9 + z^8 - 2z^5 + 3z^4 - 2z^3 + 4z^2 + z + 1, & \ell = 1, \\
z^{10} - 2z^9 + 2z^8 - 2z^7 + z^6 - z^4 + 3z^3 + 2z + 1, & \ell = 2.
\end{cases} \] (3.25)

By Proposition 2.3, the growth of degree patterns associated to each of \( g_{X_2}^{(\ell)}(z) \) \((\ell = 1, 2)\) is quadratic. The maximal degree growth is given by (B.10), and is fitted by the following generating function
\[ G_{X_2}(z) = \frac{z^{12} - 2z^{11} + 2z^{10} - 2z^9 + z^8 - z^6 - z^4 + 2z^3 + z^2 + 2z + 1}{(1 + z^2)(1 - z)^3}. \] (3.26)

Its asymptotic behaviour is quadratic.

As expected, the asymptotic behaviour in \( X_1 \) and \( X_2 \) is the same. The above findings imply that the \( C2_{(1;0;0)} \) equation is expected to be integrable.

### 3.3.4 \( C3_{(0;0;0)}(x; y) \) system

The equation \( C3_{(\delta_1; \delta_2; \delta_3)} \) is given in (A.3). For the case of \( (\delta_1, \delta_2, \delta_3) = (0, 0, 0) \), this equation exhibits two different growth patterns in both \( X_1 \) and in \( X_2 \). The patterns for the different sets of initial conditions are different from each other. In both cases the patterns appear in reverse order in the evolution matrix in the SW and SE directions.

The two different degree patterns in \( X_1 \) are given in (B.11) and (B.12). These two degree patterns are fitted by the two different generating functions defined by
\[ g_{X_1}^{(\ell)}(z) = \frac{h^{(\ell)}(z)}{(1 - z^4)(1 - z^2)}, \quad \ell = 1, 2, \] (3.27)
where
\[ h^{(\ell)}(z) = \begin{cases} 
2z^2 + z + 1, & \ell = 1, \\
z^4 + z^2 + z + 1, & \ell = 2.
\end{cases} \] (3.28)

By Proposition 2.3, the growth of degree patterns associated to each of \( g_{X_1}^{(\ell)}(z) \) \((\ell = 1, 2)\) is quadratic. The maximal growth coincides with the pattern (B.11) fitted by \( g_{X_1}^{(\ell=1)}(z) \).
The different degree patterns in $X_2$ are given in (B.13) and (B.14). These two degree
patterns are fitted by the two different generating functions defined by
\[ g_{X_2}^{(\ell)}(z) = \frac{h_{X_2}^{(\ell)}(z)}{(1-z^4)(1-z^3)(1-z)}, \quad \ell = 1, 2, \] (3.29)
where
\[ h_{X_2}^{(\ell)}(z) = \begin{cases} 
-(z^8 - 5z^5 - 6z^4 - 8z^3 - 8z^2 - 3z - 1), & \ell = 1, \\
2z^7 + 4z^5 + 7z^4 + 7z^3 + 5z^2 + 4z + 1, & \ell = 2.
\end{cases} \] (3.30)
By Proposition 2.3, the growth of degree patterns associated to each of $g_{X_2}^{(\ell)}(z) \ (\ell = 1, 2)$
is quadratic. The maximal degree growth is given by (B.15), and is fitted by the following generating function
\[ G_{X_2}(z) = \frac{z^9 - z^6 - 5z^5 - 5z^4 - 8z^3 - 7z^2 - 4z - 1}{(z^4 - 1)(z^3 - 1)(z - 1)}. \] (3.31)
Its asymptotic behaviour is quadratic.

As expected, the asymptotic behaviour in $X_1$ and $X_2$ is the same. The above findings imply that the $C3_{(0,0,0)}$ equation is expected to be integrable.

3.3.5 $C3_{(1,0,0)}(x; y)$ system

The equation $C3_{(\delta_1, \delta_2, \delta_3)}$ is given in (A.3). For the case of $(\delta_1, \delta_2, \delta_3) = (1, 0, 0)$, this equation exhibits two different growth pattern both in $X_1$ and in $X_2$. The patterns for the different sets of initial conditions are different from each other. In both cases the patterns appear in reverse order in the evolution matrix in the SW and SE directions.

The different degree patterns in $X_1$ are given in (B.16) and (B.17). These two degree
patterns are fitted by the two different generating functions defined by
\[ g_{X_1}^{(\ell)}(z) = \frac{h_{X_1}^{(\ell)}(z)}{(1-z^3)(1-z)^2}, \quad \ell = 1, 2, \] (3.32)
where
\[ h_{X_1}^{(\ell)}(z) = \begin{cases} 
-(z^5 - z^3 - 2z^2 - z - 1), & \ell = 1, \\
2z^2 + z + 1, & \ell = 2.
\end{cases} \] (3.33)
By Proposition 2.3, the growth of degree patterns associated to each of $g_{X_1}^{(\ell)}(z) \ (\ell = 1, 2)$
is quadratic. The maximal growth coincides with the pattern (B.16) fitted by $g_{X_1}^{(\ell=1)}(z)$.

The different degree patterns in $X_2$ are given in (B.18) and (B.19). These two degree
patterns are fitted by the two different generating functions defined by
\[ g_{X_2}^{(\ell)}(z) = \frac{h_{X_2}^{(\ell)}(z)}{(z^2 + 1)(z - 1)^3}, \quad \ell = 1, 2, \] (3.34)
where
\[ h_{X_2}^{(\ell)}(z) = \begin{cases} 
z^8 - 2z^7 + 2z^6 + z^5 - 3z^4 + 2z^3 - 4z^2 - 1, & \ell = 1, \\
z^6 - 2z^5 + 3z^4 - 4z^3 - 2z - 1, & \ell = 2.
\end{cases} \] (3.35)
By Proposition 2.3 the growth of degree patterns associated to each of \( g_{\mathcal{X}_\ell}(z) \) \( (\ell = 1, 2) \) is quadratic. The maximal degree growth is given by (B.20), and is fitted by the following generating function

\[
G_{\mathcal{X}_2}(z) = \frac{z^8 - 2z^7 + 3z^6 - 2z^5 + 4z^4 - 2z^3 - z^2 - 2z - 1}{(z^2 + 1)(z - 1)^3}.
\] (3.36)

Its asymptotic behaviour is quadratic.

As expected, the asymptotic behaviour in \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) is the same. The above findings imply that the \( C_{3(1;0;0)} \) equation is expected to be integrable.

3.4 Type-B (pairs)

There remains four type-B equations to be treated that give exponential degree growth in the lattice arrangement of Figure 8. These four type-B equations are \( B2_{1;0;1} \), \( B2_{1;1;0} \), \( B3_{(\frac{1}{2};0;\frac{1}{2})} \), and \( B3_{(\frac{1}{2};\frac{1}{2};0)} \). Using different type-B equations centered at black and white vertices respectively there can be found linear degree growths for certain pairs of type-B equations. These pairs of type-B equations are investigated below.

3.4.1 Arrangement of equations in the lattice

This case involves a specific pair of type-B equations which will be denoted as \( \overline{B} \) and \( \overline{\overline{B}} \). The two different type-B equations will be distinguished graphically by different orientations of directed edges, as shown in Figure 10. Only the relative orientation matters, so if \( B \) is associated to the left of Figure 12, then \( \overline{B} \) is associated to the right of Figure 10, and vice versa.

![Type-B Equations](image)

Figure 10: Two different type-B equations which are distinguished graphically by the orientations of double-line edges.

The arrangement of the pair of type-B equations of Figure 10 in the lattice is indicated in Figure 11. The type-B equations centered at white vertices are

\[
\overline{B}(y_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m-1,n-1}, x_{m+1,n-1}; \alpha, \beta) = 0,
\] (3.37)

and the type-B equations centered at black vertices are

\[
B(x_{m,n}; y_{m-1,n+1}, y_{m+1,n+1}, y_{m-1,n-1}, y_{m+1,n-1}; \hat{\alpha}, \hat{\beta}) = 0.
\] (3.38)
Since the two different type-B equations $B$ and $\overline{B}$ are respectively centered at black and white vertices of Figure 11, these systems of equations will be referred to as $B(x) + \overline{B}(y)$ systems.

Figure 11: Arrangement of a pair of type-B equations in the lattice. This is equivalent to the type-B lattice arrangement of Figure 8, but with directed edges used to distinguish the two different type-B equations indicated in Figure 10.

3.4.2 $B_2(1;0;1)(x) + B_2(1;1;0)(y)$ system

The equation $B_2(\delta_1; \delta_2; \delta_3)$ is given in (A.6). The $B_2(1;0;1)(x) + B_2(1;1;0)(y)$ system exhibits two different growth pattern both in $X_1$ and in $X_2$. The patterns for the different sets of initial conditions are different from each other.

In $X_1$ the first growth pattern (of equations centered at $x$ vertices) is the same as given in (3.13). Thus the growth is linear with generating function given by (3.14). The second growth pattern (of equations centered at $y$ vertices) is instead:

$$1, 3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43\ldots$$

(3.39)

This growth is clearly linear and is fitted by the following generating function:

$$g_{X_1}(z) = \frac{2z^2 + z + 1}{(1 - z)^2}.$$  

(3.40)

The linearity of the growth (3.39) could also be confirmed using Proposition 2.3. The maximal growth coincides with the pattern (3.39).

In $X_2$ the first growth pattern (of equations centered in $x$ vertices) is:

$$1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37\ldots$$

(3.41)
This growth is clearly linear and is fitted by the following generating function:

\[ g_{X_2}(z) = \frac{1 + 2z}{(1 - z)^2}. \]  

(3.42)

The linearity of the growth (3.41) follows from Proposition 2.3. The second growth pattern (of equations centered in \( y \) vertices) is instead:

\[ 1, 4, 10, 16, 22, 28, 34, 40, 46, 52, 58, 64 \ldots \]  

(3.43)

This growth is clearly linear and is fitted by the following generating function:

\[ g_{X_2}(z) = \frac{3z^2 + 2z + 1}{(1 - z)^2}. \]  

(3.44)

The linearity of the growth (3.43) also follows from Proposition 2.3. The maximal growth coincides with the pattern (3.43).

As expected, the asymptotic behaviour in \( X_1 \) and \( X_2 \) is the same. The above findings imply that the \( B_{2(1; 1; 0)}(x) + B_{2(1; 0; 1)}(y) \) system is expected to be linearisable.

### 3.5.3 \( B_{3(\frac{1}{2}; 0; \frac{1}{2})}(x) + B_{3(\frac{1}{2}; \frac{1}{2}; 0)}(y) \) system

The equation \( B_3(h_1, h_2; h_3) \) is given in (A.2). The \( B_{3(\frac{1}{2}; 0; \frac{1}{2})}(x) + B_{3(\frac{1}{2}; \frac{1}{2}; 0)}(y) \) system exhibits two different growth pattern both in \( X_1 \) and in \( X_2 \). The patterns for the different sets of initial conditions are different from each other, and coincide with those of the \( B_{2(1; 1; 0)}(x) + B_{2(1; 0; 1)}(y) \) system. Thus we refer back to subsection 3.4.2 for a complete description of each of these patterns. The above findings imply that the \( B_{3(\frac{1}{2}; 0; \frac{1}{2})}(x) + B_{3(\frac{1}{2}; \frac{1}{2}; 0)}(y) \) system is expected to be linearisable.

### 3.5 Type-C (pairs)

There remains six type-C equations to be treated that give exponential degree growth in the lattice arrangement of Figure 9. These six type-C equations are \( C_{1(1)} \), \( C_{2(0; 0; 0)} \), \( C_{2(1; 0; 1)} \), \( C_{2(1; 1; 0)} \), \( C_{3(\frac{1}{2}; 0; \frac{1}{2})} \), and \( C_{3(\frac{1}{2}; \frac{1}{2}; 0)} \). Similarly to the type-B equations that were treated in Section 3.4, the remaining type-C equations can have a quadratic degree growth when combining certain pairs of the equations. In fact, the equation \( C_{1(1)} \) in (A.9) is a new equation that was found in this paper, motivated by the expectation there should be some equation to pair with \( C_{2(0; 0; 0)} \) to achieve the quadratic degree growth. The different pairs of type-C equations are investigated below.

In the following, the integers \( n \mod 4 \) are taken to be elements of \( \{1, 2, 3, 4\} \).

#### 3.5.1 Arrangement of equations in the lattice

This case involves a specific pair of type-C equations which will be denoted as \( C \) and \( \overline{C} \). The two different type-C equations are distinguished graphically by different orientations of directed edges, as shown in Figure 12. Only the relative orientation matters, so if \( C \) is associated to the left of Figure 12, then \( \overline{C} \) is associated to the right of Figure 12, and vice versa.
The arrangement of the pair of type-C equations of Figure 12 in the lattice is indicated in Figure 13. First, there are two types of white vertices in Figure 13 which are distinguished by the orientation of directed edges they are connected to. The following two type-C equations are centered at the two respective types of white vertices

\[ C(y_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m+1,n-1}, x_{m-1,n-1}; \alpha, \beta) = 0, \quad (3.45) \]
\[ \overline{C}(y_{m,n}; x_{m-1,n+1}, x_{m+1,n+1}, x_{m+1,n-1}, x_{m-1,n-1}; \alpha, \beta) = 0. \quad (3.46) \]

Similarly, there are two types of black vertices in Figure 13 which are distinguished by the orientation of directed edges they are connected to. The following two type-C equations are centered at the two respective types of black vertices

\[ \overline{C}(x_{m,n}; y_{m+1,n-1}, y_{m-1,n-1}, y_{m+1,n+1}, y_{m-1,n+1}; \alpha, \beta) = 0, \quad (3.47) \]
\[ C(x_{m,n}; y_{m+1,n-1}, y_{m-1,n-1}, y_{m+1,n+1}, y_{m-1,n+1}; \alpha, \beta) = 0. \quad (3.48) \]

The choice of which equation is centered at which type of black vertex, should be consistent with the orientation of directed edges from the choice made for equations centered at the white vertices. Since the two different type-C equations \( C \) and \( \overline{C} \) each appear centered at both black and white vertices of Figure 11, but with a different ordering, these systems of equations will be referred to as \( C(x; y) + \overline{C}(y; x) \) systems.

There is a general typical behaviour for the growth patterns for pairs of type-C equations in the different directions of the lattice. It turns out that the generating functions for the growth patterns in the NE, SW, NW, SE directions may always be written as

\[
NE : \quad g_X^{(\ell, \ell)}, \quad \ell = 1, 2, 3, 4, \\
SW : \quad g_X^{(\ell-1, \ell-1)}, \quad \ell = 1, 2, 3, 4, \\
NW : \quad g_X^{(2-\ell, -\ell)}, \quad \ell = 1, 2, 3, 4, \\
SE : \quad g_X^{(1-\ell, 1-\ell)}, \quad \ell = 1, 2, 3, 4, \\
\]

where \( X \) is either \( X_1 \) or \( X_2 \), and \( g_X^{(\ell_1, \ell_2)} = zh_1^{(\ell_1)}(z^2) + h_2^{(\ell_2)}(z^2) \) is a generating function that is split into odd and even components, \( h_1^{(\ell_1)}(z) \) and \( h_2^{(\ell_2)}(z) \) respectively. In the following
the generating function $g_{\mathcal{X}}^{(\ell_1, \ell_2)}$ is given for each case, which was found by fitting the relevant degree growths listed in Appendix B.

### 3.5.2 $C1_{(1)}(x; y) + C2_{(0; 0; 0)}(y; x)$ system

The equation $C1_{(1)}$ is given in (A.9), and the equation $C2_{(\delta_1; \delta_2; \delta_3)}$ is given in (A.7). The $C1_{(1)}(x; y) + C2_{(0; 0; 0)}(y; x)$ system of equations exhibits four different growth patterns both in $\mathcal{X}_1$ and in $\mathcal{X}_2$. The patterns for the different sets of initial conditions are different from each other. The maximal growth is isotropic.

#### $\mathcal{X}_1$.

The generating function $g_{\mathcal{X}_1}^{(\ell_1, \ell_2)}(z)$ is

$$
g_{\mathcal{X}_1}^{(\ell_1, \ell_2)}(z) = \frac{zh_1^{(\ell_1)}(z^2) + h_2^{(\ell_2)}(z^2)}{(1 - z^6)(1 - z^2)^2}, \quad \ell_1, \ell_2 = 1, 3, 4 \pmod{4}, \quad (3.50)
$$

where

$$
h_1^{(\ell)}(z) = \begin{cases} 
    z^4 + z^3 + 6z^2 + 5z + 3, & \ell = 1 \pmod{4}, \\
    z^4 + 3z^3 + 6z^2 + 3z + 3, & \ell = 2 \pmod{4}, \\
    2z^3 + 5z^2 + 6z + 3, & \ell = 3 \pmod{4}, \\
    z^4 + 2z^3 + 6z^2 + 4z + 3, & \ell = 4 \pmod{4},
\end{cases} \quad (3.51)
$$
and

$$h_2^{(\ell)}(z) = \begin{cases} 
  z^4 + 5z^3 + 4z^2 + 5z + 1, & \ell = 1 \pmod{4}, \\
  3z^4 + 3z^3 + 6z^2 + 3z + 1, & \ell = 2 \pmod{4}, \\
  5z^3 + 5z^2 + 5z + 1, & \ell = 3 \pmod{4}, \\
  2z^4 + 4z^3 + 5z^2 + 4z + 1, & \ell = 4 \pmod{4}.
\end{cases} \tag{3.52}$$

The four different degree patterns for each of the NE/SW and NW/SE directions are given in (B.21)–(B.24) and (B.25)–(B.28), respectively. The generating functions for the growth patterns in the NE, SW, NW, and SE directions are respectively given in terms of (3.50) as indicated in (3.49). By Proposition 2.3 the growth of degree patterns associated to each of these generating functions is quadratic.

The maximal growth with initial conditions in $X_1$ is isotropic and given by the quadratic degree pattern (B.23) fitted by $g_{X_1}^{(3,3)}(z)$.

$X_2$. The generating function $g_{X_2}^{(\ell_1,\ell_2)}(z)$ is

$$g_{X_2}^{(\ell_1,\ell_2)}(z) = \frac{zh_1^{(\ell_1)}(z^2) + h_2^{(\ell_2)}(z^2)}{(1 - z^2)(1 - z^6)(1 - z^{22})}, \quad \ell_1, \ell_2 = 1, 2, 3, 4 \pmod{4}, \tag{3.53}$$

where

$$h_1^{(\ell)}(z) = \begin{cases} 
  z^{16} + z^{14} + 3z^{13} + 15z^{12} + 24z^{11} + 32z^{10} + 28z^9 + 32z^8 \\
  + 28z^7 + 31z^6 + 28z^5 + 30z^4 + 29z^3 + 26z^2 + 16z + 4, & \ell = 1 \pmod{4}, \\
  z^{15} + z^{14} + 7z^{13} + 16z^{12} + 27z^{11} + 28z^{10} + 32z^9 + 28z^8 \\
  + 31z^7 + 29z^6 + 30z^5 + 29z^4 + 28z^3 + 24z^2 + 12z + 5, & \ell = 2 \pmod{4}, \\
  -z^{14} + 4z^{13} + 13z^{12} + 25z^{11} + 31z^{10} + 28z^9 + 32z^8 \\
  + 28z^7 + 32z^6 + 28z^5 + 31z^4 + 30z^3 + 26z^2 + 17z + 4, & \ell = 3 \pmod{4}, \\
  z^{14} + 5z^{13} + 16z^{12} + 26z^{11} + 28z^{10} + 32z^9 + 28z^8 \\
  + 32z^7 + 28z^6 + 31z^5 + 29z^4 + 29z^3 + 25z^2 + 13z + 5, & \ell = 4 \pmod{4}, 
\end{cases} \tag{3.54}$$

and

$$h_2^{(\ell)}(z) = \begin{cases} 
  z^{16} + z^{14} + 10z^{13} + 20z^{12} + 28z^{11} + 30z^{10} + 30z^9 + 29z^8 \\
  + 31z^7 + 28z^6 + 31z^5 + 28z^4 + 31z^3 + 18z^2 + 11z + 1, & \ell = 1 \pmod{4}, \\
  z^{15} + 4z^{14} + 12z^{13} + 20z^{12} + 29z^{11} + 30z^{10} + 29z^9 + 31z^8 \\
  + 28z^7 + 32z^6 + 28z^5 + 31z^4 + 24z^3 + 19z^2 + 9z + 1, & \ell = 2 \pmod{4}, \\
  -z^{15} + 9z^{13} + 21z^{12} + 27z^{11} + 31z^{10} + 29z^9 + 30z^8 \\
  + 30z^7 + 29z^6 + 31z^5 + 29z^4 + 32z^3 + 19z^2 + 11z + 1, & \ell = 3 \pmod{4}, \\
  2z^{14} + 12z^{13} + 19z^{12} + 30z^{11} + 29z^{10} + 30z^9 + 30z^8 \\
  + 29z^7 + 31z^6 + 28z^5 + 32z^4 + 26z^3 + 20z^2 + 9z + 1, & \ell = 4 \pmod{4}. \tag{3.55}
\end{cases}$$
The four different degree patterns for each of the NE/SW and NW/SE directions are given in (B.29)–(B.32) and (B.33)–(B.36), respectively. The generating functions for the growth patterns in the NE,SW,NW,SE directions are respectively given in terms of (3.53) as indicated in (3.49). By Proposition 2.3 the growth of degree patterns associated to each of these generating functions is quadratic.

The maximal growth with initial conditions in $X_2$ is isotropic and given by (B.37). The maximal degree growth is fitted by the following generating function:

$$g_{X_2}(z) = \frac{z^{16} - z^{13} - 4z^{12} - 5z^{11} - 8z^{10} - 8z^9 - 7z^8 - 7z^7 - 8z^6 - 9z^5 - 6z^4 - 8z^3 - 7z^2 - 4z - 1}{(z^{11} - 1)(z^3 - 1)(z - 1)}.$$  

(3.56)

By Proposition 2.3 the growth of degree patterns associated to $G_{X_2}(z)$ is quadratic.

As expected, the asymptotic behaviour in $X_1$ and $X_2$ is the same. The above findings imply that the $C1_{(1)}(x;y) + C2_{(0;0;0)}(y;x)$ system is expected to be integrable.

### 3.5.3 $C2_{(1;0;1)}(x;y) + C2_{(1;1;0)}(y;x)$ system

The equation $C2_{(\delta_1;\delta_2;\delta_3)}$ is given in (A.7). The $C2_{(1;0;1)}(x;y) + C2_{(1;1;0)}(y;x)$ system of equations exhibits four different growth patterns both in $X_1$ and in $X_2$, respectively. The patterns for the different sets of initial conditions are different from each other. The maximal growth is isotropic.

$X_1$. The generating function $g_{X_1}^{(\ell_1,\ell_2)}(z)$ is

$$g_{X_1}^{(\ell_1,\ell_2)}(z) = \frac{zh_1^{(\ell_1)}(z^2) + h_2^{(\ell_2)}(z^2)}{(z^6 - 1)(z^2 - 1)^2}, \quad \ell_1, \ell_2 = 1, 2, 3, 4 \mod 4, \quad (3.57)$$

where

$$h_1^{(\ell)}(z) = \begin{cases} 3z^5 + z^4 - 2z^3 - 8z^2 - 7z - 3, & \ell = 1 \mod 4, \\
3z^5 + 2z^4 - 2z^3 - 7z^2 - 7z - 3, & \ell = 2 \mod 4, \\
2z^4 - 3z^3 - 5z^2 - 7z - 3, & \ell = 3 \mod 4, \\
-2z^3 - 5z^2 - 6z - 3, & \ell = 4 \mod 4, \end{cases} \quad (3.58)$$

and

$$h_2^{(\ell)}(z) = \begin{cases} z^6 + 3z^5 - 6z^3 - 8z^2 - 5z - 1, & \ell = 1 \mod 4, \\
3z^5 - z^4 - 4z^3 - 8z^2 - 5z - 1, & \ell = 2 \mod 4, \\
z^5 - 4z^4 - 7z^2 - 5z - 1, & \ell = 3 \mod 4, \\
-5z^3 - 5z^2 - 5z - 1, & \ell = 4 \mod 4. \end{cases} \quad (3.59)$$

The four different degree patterns for each of the NE/SW and NW/SE directions are given in (B.38)–(B.41) and (B.42)–(B.45), respectively. The generating functions for the growth patterns in the NE,SW,NW,SE directions are respectively given in terms of (3.57) as indicated in (3.49). By Proposition 2.3 the growth of degree patterns associated to each of these generating functions is quadratic.

The maximal growth with initial conditions in $X_1$ is isotropic and given by the quadratic degree pattern (B.38) fitted by $g_{X_1}^{(1,1)}(z)$.
$X_2$. The generating function $g_{X_2}^{(\ell_1, \ell_2)}(z)$ is

$$g_{X_2}^{(\ell_1, \ell_2)}(z) = \frac{zh_1^{(\ell_1)}(z^2) + h_2^{(\ell_2)}(z^2)}{(1-z^8)(1-z^6)(1-z^2)}, \quad \ell_1, \ell_2 = 1, 2, 3, 4 \pmod{4}, \quad (3.60)$$

where

$$h_1^{(\ell)}(z) = \begin{cases} 
    z^{12} - z^9 - 7z^8 - 7z^7 - 3z^6 + 14z^5 \\
    +31z^4 + 39z^3 + 34z^2 + 18z + 5, & \ell = 1 \pmod{4}, \\
    z^{12} - z^9 - 3z^8 - 6z^7 - z^6 + 13z^5 \\
    +29z^4 + 37z^3 + 32z^2 + 18z + 5, & \ell = 2 \pmod{4}, \\
    z^{11} - z^9 - 5z^7 + 3z^6 + 13z^5 \\
    +27z^4 + 36z^3 + 28z^2 + 19z + 4, & \ell = 3 \pmod{4}, \\
    z^{10} - z^8 - 2z^7 + 3z^6 + 16z^5 \\
    +27z^4 + 33z^3 + 26z^2 + 16z + 5, & \ell = 4 \pmod{4},
\end{cases} \quad (3.61)$$

and

$$h_2^{(\ell)}(z) = \begin{cases} 
    z^{13} - z^{10} - 3z^9 - 7z^8 - 7z^7 + 5z^6 \\
    +22z^5 + 37z^4 + 39z^3 + 26z^2 + 11z + 1, & \ell = 1 \pmod{4}, \\
    z^{12} - z^9 - 6z^8 - 5z^7 + 5z^6 + 21z^5 \\
    +36z^4 + 35z^3 + 26z^2 + 11z + 1, & \ell = 2 \pmod{4}, \\
    z^{12} - 2z^9 - 3z^8 - 2z^7 + 8z^6 + 21z^5 \\
    +32z^4 + 33z^3 + 24z^2 + 11z + 1, & \ell = 3 \pmod{4}, \\
    z^{10} - z^9 - 2z^7 + 10z^6 + 21z^5 \\
    +31z^4 + 32z^3 + 20z^2 + 11z + 1, & \ell = 4 \pmod{4}.
\end{cases} \quad (3.62)$$

The four different degree patterns for each of the NE/SW and NW/SE directions are given in (B.46)–(B.49) and (B.50)–(B.53), respectively. The generating functions for the growth patterns in the NE,SW,NW,SE directions are respectively given in terms of $X_{C}$ indicated in (3.54). By Proposition 2.3 the growth of degree patterns associated to each of these generating functions is quadratic.

The maximal growth with initial conditions in $X_2$ is isotropic and given by the quadratic degree pattern (B.46) fitted by $g_{X_2}^{(1,1)}(z)$.

As expected, the asymptotic behaviour in $X_1$ and $X_2$ is the same. The above findings imply that the $C^2_{2(1;0;1)}(x; y) + C^2_{2(1;1;0)}(y; x)$ system is expected to be integrable.

**3.5.4 $C^3_{2(\frac{1}{2};0;\frac{1}{2})}(x; y) + C^3_{2(\frac{1}{2};\frac{1}{2};0)}(y; x)$ system**

The equation $C^3_{2(\delta_1; \delta_2; \delta_3)}$ is given in (A.3). The $C^3_{2(\frac{1}{2};0;\frac{1}{2})}(x; y) + C^3_{2(\frac{1}{2};\frac{1}{2};0)}(y; x)$ system of equations exhibits four different growth patterns, both in $X_1$ and in $X_2$, respectively. The patterns for the different sets of initial conditions are different from each other. The maximal growth is isotropic.
\( X_1 \). In \( X_1 \) the growth patterns exactly coincide with those found for the \( C2(1;0,1)(x;y) + C3(1;1,0)(y;x) \) system given in Section 3.5.3.

\( X_2 \). The generating function \( g_{X_2}^{(\ell_1,\ell_2)}(z) \) is

\[
g_{X_2}^{(\ell_1,\ell_2)}(z) = \frac{zh_1^{(\ell_1)}(z^2) + h_2^{(\ell_2)}(z^2)}{(z^8 - 1)(z^6 - 1)(z^2 - 1)}, \quad \ell_1, \ell_2 = 1, 2, 3, 4 \pmod{4}, \tag{3.63}
\]

where

\[
h_1^{(\ell)}(z) = \begin{cases} 
7z^8 + 8z^7 + 3z^6 - 14z^5 - 32z^4 - 39z^3 - 34z^2 - 18z - 5, & \ell = 1 \pmod{4}, \\
3z^8 + 6z^7 + 2z^6 - 13z^5 - 29z^4 - 38z^3 - 32z^2 - 18z - 5, & \ell = 2 \pmod{4}, \\
z^9 + 5z^7 - 12z^5 - 27z^4 - 36z^3 - 29z^2 - 19z - 4, & \ell = 3 \pmod{4}, \\
z^8 + z^7 - 3z^6 - 15z^5 - 27z^4 - 33z^3 - 27z^2 - 16z - 5, & \ell = 4 \pmod{4}, 
\end{cases}
\tag{3.64}
\]

and

\[
h_2^{(\ell)}(z) = \begin{cases} 
3z^9 + 8z^8 + 7z^7 - 5z^6 - 23z^5 - 37z^4 - 39z^3 - 26z^2 - 11z - 1, & \ell = 1 \pmod{4}, \\
6z^8 + 5z^7 - 4z^6 - 21z^5 - 36z^4 - 36z^3 - 26z^2 - 11z - 1, & \ell = 2 \pmod{4}, \\
z^9 + 3z^8 + 2z^7 - 7z^6 - 21z^5 - 32z^4 - 34z^3 - 24z^2 - 11z - 1, & \ell = 3 \pmod{4}, \\
z^9 + z^7 - 10z^6 - 20z^5 - 31z^4 - 32z^3 - 21z^2 - 11z - 1, & \ell = 4 \pmod{4}.
\end{cases}
\tag{3.65}
\]

The four different degree patterns for each of the NE/SW and NW/SE directions are given in (B.54)—(B.57) and (B.58)—(B.61), respectively. The generating functions for the growth patterns in the NE,SW,NW,SE directions are respectively given in terms of (3.63) as indicated in (3.49). By Proposition 2.3 the growth of degree patterns associated to each of these generating functions is quadratic.

The maximal growth with initial conditions in \( X_2 \) is isotropic and is given by the quadratic degree pattern (B.54) fitted by \( g_{X_2}^{(1,1)}(z) \).

As expected, the asymptotic behaviour in \( X_1 \) and \( X_2 \) is the same. The above findings imply that the \( C3(\frac{1}{2},0,\frac{1}{2})(x;y) + C3(\frac{1}{2},\frac{1}{2},0)(y;x) \) system is expected to be integrable.

4 Summary and outlook

In this paper we introduced the concept of algebraic entropy for face-centered quad equations, which we have developed in analogy with the concept of algebraic entropy for regular quad equations [42, 44, 45]. A key step was the identification of standard sets of initial conditions, which we call the fundamental double staircases, which allows us to build the sequence of iterates. In particular, we use this concept to analyse the growth of each of the known equations that satisfies the property of consistency-around-a-face-centered-cube (CAFCC) [38]. The arrangement of CAFCC equations in the \( Z^2 \) lattice is non-trivial as they are non-autonomous equations. For all cases the appropriate arrangements in the lattice were found which gave a quadratic growth of degrees for type-A and type-C equations, and a linear growth of degrees for type-B equations. Degree sequences where computed for two different
sets of initial conditions, namely the $X_1$ and the $X_2$ spaces. Analysing the obtained sequences of degrees, we note that, in general, using initial conditions in the space $X_1$ the obtained sequences are less noisy. This suggests that using initial conditions in the space $X_1$ is more suitable from the computational point of view.

CAFCC equations are multidimensionally consistent, and this property was used to obtain the Lax pairs of the equations [37]. Multidimensional consistency is a property usually associated to Bianchi-like identities and Bäcklund transformations. It is well known that Lax pairs and Bäcklund transformations are associated to both linearisable and integrable equations. While in the integrable case the solution of the Lax pair provides genuine nontrivial solutions [12], in the linearisable case the Lax pair is fake [13,21,26,27]. Our growth analysis implies that type-A and type-C equations must be genuinely integrable due to the quadratic growth. On the other hand, since type-B equations have linear growth, this suggests that these equations are linearisable and that their associated Lax pairs are fake. This is analogous to the results presented in [20] about the trapezoidal $H4$ and the $H6$ equations [2,11].

The linear behaviour of the trapezoidal $H4$ and the $H6$ equations was explained in [23,24] through the concept of Darboux integrability for quad equations [3]. We conjecture that type-B equations satisfy an analogous notion for face-centered quad equations. Such a notion has not yet been discussed, but it is an stimulating open issue for future research.

In this paper only the minimal number of different equations (either individual equations or pairs) were used to achieve the vanishing algebraic entropy. However, there are also more complicated arrangements of equations which are also expected to lead to polynomial degree growth of the equations. For example, the type-C equations can be used as a link between type-A and type-B equations (similarly to their use in CAFCC) to construct a system consisting of each of type-A, -B, and -C equations in the lattice. This arrangement of equations could also be reasonably expected to pass the algebraic entropy test and it would be interesting to explore further. This could be considered as an analogue for the case of algebraic entropy for non-standard lattice arrangements of $H4$ and $H6$ quad equations [34].

Acknowledgements

The authors thank Prof. C-M. Viallet for reading the manuscript and providing helpful comments. We also thank the anonymous referee for their helpful comments in correcting some errors in the first version of the paper.

GG has been supported by the Australian Research Council through grant DP200100210 (Prof. N. Joshi and A/Prof. M. Radnović) and by Fondo Sociale Europeo del Friuli Venezia Giulia, Programma operativo regionale 2014-2020 FP195673001 (A/Prof. T. Grava and A/Prof. D. Guzzetti).
Appendix A  CAFCC equations

\[ A3(\delta), B3(\delta_1; \delta_2; \delta_3), C3(\delta_1; \delta_2; \delta_3) \]

\[ A3(\delta)(x; x_a, x_b, x_c, x_d; \alpha, \beta) = \]
\[ x \left( \frac{\beta_1}{\beta_2} - \frac{\beta_2}{\alpha_1} \right) (x_a x_b - x_c x_d) + \frac{\alpha_1}{\alpha_2} (x_a x_c - x_b x_d) - \frac{\alpha_1}{\alpha_2} \frac{\beta_2}{\beta_1} (x_a x_d - x_b x_c) \]
\[ + \left( \frac{\alpha_1}{\beta_2} - \frac{\beta_1}{\alpha_2} \right) (x_a x^2 - x_b x_c x_d) - \left( \frac{\alpha_1}{\beta_2} - \frac{\beta_1}{\alpha_2} \right) (x_b x^2 - x_a x_c x_d) - \frac{\alpha_1}{\alpha_2} (x_c x^2 - x_a x_b x_d) \]
\[ + \left( \frac{\alpha_1}{\beta_2} - \frac{\beta_1}{\alpha_2} \right) (x_d x^2 - x_a x_b x_c) - \delta \left( \frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2} \right) \left( \frac{\alpha_1}{\beta_1} - \frac{\beta_1}{\alpha_2} \right) (x_c x^2 - x_a x_b x_d) \]
\[ + \delta \left( \frac{\alpha_1}{\beta_1} + \frac{\beta_1}{\alpha_2} \right) \left( \frac{\alpha_1}{\beta_1} - \frac{\beta_1}{\alpha_2} \right) \left( \frac{\alpha_1}{\beta_1} - \frac{\beta_1}{\alpha_2} \right) \left( \frac{\alpha_1}{\beta_1} - \frac{\beta_1}{\alpha_2} \right) x \]
\[ = 0. \]  \hspace{1cm} (A.1)

\[ B3(\delta_1; \delta_2; \delta_3)(x; x_a, x_b, x_c, x_d; \alpha, \beta) = \]
\[ x (x_a x_c - x_b x_d) + \delta_1 \left( \frac{\alpha_1}{\beta_1} x_a - \frac{\beta_1}{\alpha_2} x_b + \frac{\beta_1}{\alpha_2} x_c \right) + \delta_2 \left( \frac{\alpha_1}{\beta_1} x_a - \frac{\beta_1}{\alpha_2} x_b + \frac{\beta_1}{\alpha_2} x_c \right) \]
\[ + \delta_3 \left( x_a x_b x_c x_d \left( \frac{x_a x_c}{\beta_1} - \frac{x_b x_d}{\alpha_2} \right) + x_a x_d x_c \left( \frac{x_a x_c}{\beta_1} - \frac{x_b x_d}{\alpha_2} \right) \right) = 0. \]  \hspace{1cm} (A.2)

\[ C3(\delta_1; \delta_2; \delta_3)(x; x_a, x_b, x_c, x_d; \alpha, \beta) = \]
\[ \left( \frac{\alpha_1}{\beta_1} (x_a x_b - x_c x_d) + \beta_1 \beta_2 (x_a x_c - x_b x_d) + \alpha_2 \left( \frac{\beta_1}{\beta_1} - \frac{\beta_1}{\alpha_2} \right) \left( \delta_1 x_a - \delta_2 x_b x_c \right) \right) \]
\[ + \delta_2 \left( \frac{\alpha_1}{\beta_1} x_c x_d \right) + \alpha_2 x_a x_b \left( \beta_2 x_d - \beta_2 x_d \right) + \delta_1 x_a \left( \beta_1 x_b - \beta_2 x_a \right) + \delta_2 \left( \frac{\alpha_1}{\beta_1} x_c x_d \right) \]
\[ + \left( \frac{\alpha_1}{\beta_1} x_d - \delta_3 \left( \frac{\alpha_1}{\beta_1} x_b - \beta_2 x_a \right) + \beta_1 x_b \right) \right) \left( \frac{\alpha_1}{\beta_1} x_d - \delta_3 \left( \frac{\alpha_1}{\beta_1} x_b - \beta_2 x_a \right) + \beta_1 x_b \right) \]
\[ = 0. \]  \hspace{1cm} (A.3)

The equation (A.1) satisfies CAFCC for the two values \( \delta = 0, 1. \) The equations (A.1), (A.2), (A.3), collectively satisfy CAFCC for the four values of \( \delta_1, \delta_2, \delta_3 = (0, 0, 0), (1, 0, 0), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0), \) where the parameter for (A.1) is \( \delta = 2 \delta_2. \)

\[ A2(\delta_1; \delta_2), B2(\delta_1; \delta_2; \delta_3), C2(\delta_1; \delta_2; \delta_3) \]

Define \( \theta_{ij} \) by
\[ \theta_{ij} = \theta_i - \theta_j, \quad i, j \in \{1, 2, 3, 4\}, \quad (\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha_1, \alpha_2, \beta_1, \beta_2). \]  \hspace{1cm} (A.4)
\[ A_{2}(\delta_1; \delta_2)(x; x_a, x_b, x_c, x_d; \alpha, \beta) = \theta_{23}(x_a x_c - x_b x_d) - \theta_{24}(x_a x_d - x_b x_c) - \theta_{13}(x_a x_c^2 - x_b x_d) + \theta_{14}(x_a x_d^2 - x_b x_c) + \left(\theta_{34}(x_a x_c - x_b x_d) + \theta_{12}(x_a x_c - x_b x_d) + (\theta_{13} + \theta_{24})(x_a x_d - x_b x_c)\right)x + \delta_1 \left(\theta_{14} \theta_{23}(\theta_{13} x_b + \theta_{24} x_c)(2x - \theta_{12} \theta_{34}) - \theta_{13} \theta_{24}(\theta_{13} x_a + \theta_{23} x_d)(2x + \theta_{12} \theta_{34})\right) + \delta_1 x \theta_{12} \theta_{34}(\theta_{13} + \theta_{24})(x + x_a + x_b + x_c + x_d - \theta_{12} - \theta_{13} \theta_{23} - \theta_{14} \theta_{24})^\delta_2 + \delta_2 \left(x_a \theta_{13} \theta_{14}(\theta_{24} \theta_{14} - \theta_{34} x_b) - x_3 \theta_{13} \theta_{23}(\theta_{14} \theta_{13} - \theta_{12} x_a) - x_3 \theta_{14} \theta_{24}(\theta_{23} \theta_{24} + \theta_{12} x_a) + x_2 \theta_{23} \theta_{24}(\theta_{13} \theta_{23} + \theta_{34} x_c) + x_3 \theta_{13} \theta_{34} + x_6 \theta_{23} \theta_{24} + \prod_{1 \leq i < j \leq 4} \theta_{ij}(\theta_{13} + \theta_{24})\right) = 0. \]  

\[ B_{2}(\delta_1; \delta_2; \delta_3)(x; x_a, x_b, x_c, x_d; \alpha, \beta) = \delta_1 \left(\theta_{12} \theta_{34}(\theta_{12} x_a - \theta_{13} \theta_{14} - \theta_{23} \theta_{24}) - \theta_{23} x_a - \theta_{24} x_b\right) + \left(x_a(x + \theta_{14} \theta_{13} + \theta_{31} + \theta_{12} \theta_{24} + \theta_{23} \theta_{24}) + \left(x_a x_c - x_b x_d\right)\theta_{12} + (x_a x_b - x_a x_d)\theta_{34} - x_b x_c(x_a + x_d) + x_a x_d(x_b + x_c)\right) + \delta_2 \left(x_2 \theta_{13} \theta_{14}(\theta_{24} \theta_{14} - \theta_{34} x_b) - x_3 \theta_{13} \theta_{23}(\theta_{14} \theta_{13} - \theta_{12} x_a) - x_3 \theta_{14} \theta_{24}(\theta_{23} \theta_{24} + \theta_{12} x_a) + x_2 \theta_{23} \theta_{24}(\theta_{13} \theta_{23} + \theta_{34} x_c) + x_3 \theta_{13} \theta_{34} + x_6 \theta_{23} \theta_{24} + \prod_{1 \leq i < j \leq 4} \theta_{ij}(\theta_{13} + \theta_{24})\right) = 0. \]  

\[ C_{2}(\delta_1; \delta_2; \delta_3)(x; x_a, x_b, x_c, x_d; \alpha, \beta) = (x_d - x_c)(x_c^2 + x_a x_b) + \theta_{34}(x_c^2 - x_a x_b)(\theta_{13} + \theta_{14}) + 2 \delta_3 \left(x_3 \theta_{23} x_a - \theta_{24} x_b\right) + \left(x_a + x_b + 2 \delta_2 \theta_{23} \theta_{24}(x_c - x_d) - (x_a - x_b)(x_3 + \theta_{24}) + \theta_{13} \theta_{23} \theta_{24} + 2 \delta_3 \theta_{34} \theta_{23} x_a x_b\right) + \delta_1 \left(x_2 \theta_{24} x_b - x_3 \theta_{23} \theta_{24} + \theta_{34}(\theta_{23} \theta_{24} - x)\right) + \delta_1 \theta_{23} \theta_{24}(x_c - x_d + \theta_{34}(\theta_{13} + \theta_{14} - 2x) + (x_a + x_b - \theta_{34} - \theta_{23} \theta_{24})^\delta_2 = 0. \]  

The equation (A.5) satisfies CAFCC for the three values \((\delta_1, \delta_2) = (0, 0), (1, 0), (1, 1)\). The equations (A.5), (A.6), (A.7), collectively satisfy CAFCC with the four values \((\delta_1, \delta_2, \delta_3) = (0, 0, 0), (1, 0, 0), (1, 0, 1), (1, 1, 0)\).

\[ A_{2}(\delta_1; \delta_2), \, D_{1}, \, C_{1}(\delta_1) \]

\[ D_1(x_a, x_b, x_c, x_d) = x_a - x_b - x_c + x_d = 0. \]  

\[ C_{1}(x; x_a, x_b, x_c, x_d; \alpha, \beta) = (x_c - x_d)x^2 + 2(\beta_1 - \beta_2)(-\frac{x_c + x_d}{2})x + 2((\beta_2 - \alpha_2)x_a + (\alpha_2 - \beta_1)x_b)(-\frac{x_c + x_d}{2}) + (x_a x_b - \delta_1(\alpha_2 - \beta_1)(\alpha_2 - \beta_2)) = 0. \]  

The equations (A.5), (A.8), (A.9), collectively satisfy CAFCC for the values \((\delta_1, \delta_2) = (0, 0), (1, 0)\).
Four-leg expressions

The above equations have equivalent four-leg type expressions that are given respectively in (2.4)–(2.6), with the functions given in Tables 1 and 2 below. The abbreviation add., indicates an additive form of one of the equations (2.4), (2.5), (2.6), given respectively by

\[ a(x; x_{a}; \alpha_{2}, \beta_{1}) + a(x; x_{d}; \alpha_{1}, \beta_{2}) - a(x; x_{c}; \alpha_{2}, \beta_{2}) - a(x; x_{c}; \alpha_{1}, \beta_{1}) = 0, \quad (A.10) \]
\[ b(x; x_{a}; \alpha_{2}, \beta_{1}) + b(x; x_{d}; \alpha_{1}, \beta_{2}) - b(x; x_{c}; \alpha_{2}, \beta_{2}) - b(x; x_{c}; \alpha_{1}, \beta_{1}) = 0, \quad (A.11) \]
\[ a(x; x_{a}; \alpha_{2}, \beta_{1}) + c(x; x_{d}; \alpha_{1}, \beta_{2}) - a(x; x_{b}; \alpha_{2}, \beta_{2}) - c(x; x_{c}; \alpha_{1}, \beta_{1}) = 0. \quad (A.12) \]

| Type-A          | \(a(x; y; \alpha, \beta)\)                           |
|-----------------|------------------------------------------------------|
| \(A3(\delta=1)\) | \(\frac{\alpha^{2} + \beta^{2}x^{2} - 2\alpha \beta xy}{\beta^{2} + \alpha^{2}y^{2} - 2\alpha \beta y}\) |
| \(A3(\delta=0)\) | \(\frac{\beta x - \alpha y}{\alpha x - \beta y}\)       |
| \(A2(\delta_{1}=1; \delta_{2}=0)\) | \(\frac{(\sqrt{x} + \alpha - \beta)^{2} - y}{(\sqrt{x} - \alpha + \beta)^{2} - y}\) |
| \(A2(\delta_{1}=1; \delta_{2}=0)\) | \(\frac{-x + y + \alpha - \beta}{x - y + \alpha - \beta}\) |
| \(A2(\delta_{1}=0; \delta_{2}=0)\) | \(\frac{\alpha - \beta}{x - y}\) (add.)               |

Table 1: A list of \(a(x; y; \alpha, \beta)\) in (2.4) for type-A equations (A.1), (A.5). Here \(x = \sqrt{x^{2} - 1}\).

| Type-B          | \(b(x; y; \alpha, \beta)\)                           |
|-----------------|------------------------------------------------------|
| \(B3(\delta_{1}=\frac{1}{2}; \delta_{2}=\frac{1}{2}; \delta_{3}=0)\) | \(\beta^{2} + \alpha^{2}x^{2} - 2\alpha \beta xy\) |
| \(B3(\delta_{1}=\frac{1}{2}; \delta_{2}=\frac{1}{2}; \delta_{3}=\frac{1}{2})\) | \(\frac{\alpha y - \beta x}{\alpha x y - \beta}\)       |
| \(B3(\delta_{1}=1; \delta_{2}=0; \delta_{3}=0)\) | \(\beta - \alpha xy\)                                 |
| \(B3(\delta_{1}=0; \delta_{2}=0; \delta_{3}=0)\) | \(y\)                                               |
| \(B2(\delta_{1}=1; \delta_{2}=1; \delta_{3}=0)\) | \(x + \alpha - \beta)^{2} - y\)                     |
| \(B2(\delta_{1}=1; \delta_{2}=0; \delta_{3}=1)\) | \(\sqrt{x + y + \alpha - \beta}\)                   |
| \(B2(\delta_{1}=1; \delta_{2}=0; \delta_{3}=0)\) | \(-\sqrt{x + y + \alpha - \beta}\)                  |
| \(B2(\delta_{1}=0; \delta_{2}=0; \delta_{3}=0)\) | \(y\) (add.)                                        |
| \(D1\)          | \(y\) (add.)                                        |

| Type-C          | \(c(x; y; \alpha, \beta)\)                           |
|-----------------|------------------------------------------------------|
| \(C3(\delta_{1}=\frac{1}{2}; \delta_{2}=\frac{1}{2}; \delta_{3}=0)\) | \(\frac{\alpha - \beta x y}{\alpha x y - \beta}\)       |
| \(C3(\delta_{1}=\frac{1}{2}; \delta_{2}=\frac{1}{2}; \delta_{3}=\frac{1}{2})\) | \(\frac{\alpha^{2} + \beta x^{2} - 2\alpha xy}{\alpha x y - \beta}\) |
| \(C3(\delta_{1}=1; \delta_{2}=0; \delta_{3}=0)\) | \(xy - \frac{\alpha}{\beta}\)                          |
| \(C3(\delta_{1}=0; \delta_{2}=0; \delta_{3}=0)\) | \(y\)                                               |
| \(C2(\delta_{1}=1; \delta_{2}=1; \delta_{3}=0)\) | \(-\sqrt{x + y - \alpha + \beta}\)                   |
| \(C2(\delta_{1}=1; \delta_{2}=0; \delta_{3}=1)\) | \(\sqrt{x + y - \alpha + \beta}\)                   |
| \(C2(\delta_{1}=1; \delta_{2}=0; \delta_{3}=0)\) | \((x - \alpha + \beta)^{2} - y\)                     |
| \(C2(\delta_{1}=0; \delta_{2}=0; \delta_{3}=0)\) | \(x + y - \alpha + \beta\)                           |
| \(C1(\delta_{1}=1)\) | \(-\frac{\alpha + \beta}{\sqrt{x}}\) (add.)          |
| \(C1(\delta_{1}=0)\) | \(y\) (add.)                                        |

Table 2: Left: A list of \(b(x; y; \alpha, \beta)\) in (2.5) for type-B equations (A.2), (A.6). Right: A list of \(c(x; y; \alpha, \beta)\) in (2.6) for type-C equations (A.3), (A.7). For \(C3(\delta_{1}; \delta_{2}; \delta_{3})\), \(C2(\delta_{1}; \delta_{2}; \delta_{3})\), and \(C1(\delta_{1})\), the \(a(x; y; \alpha, \beta)\) are respectively given by \(A3(\delta_{1}); A2(\delta_{1}; \delta_{2})\), and \(A2(\delta_{1}; \delta_{2}=0)\). Here \(x = \sqrt{x^{2} - 1}\).
Appendix B  Lists of degree growths for type-C equations

This Appendix gives the lists of degree growths that have been obtained for type-C equations with the two types of initial conditions (2.15).

\( C_{1(0)}(x; y) \) system

\( \mathcal{X}_1 \). The two different growth patterns are
\[
1, 3, 7, 11, 17, 25, 33, 43, 55, 67, 81, 97, 113 \ldots \\
1, 3, 5, 9, 15, 21, 29, 39, 49, 61, 75, 89 \ldots
\]  
\( \mathcal{X}_2 \). The two different growth patterns are
\[
1, 4, 12, 20, 30, 46, 62, 78, 101, 125, 149, 178, 210, 242, 278, 317, 357, 401, 446, 494, 546, 598, 653, 713, 773, 834, 902, 970, 1038, 1113, 1189 \ldots
\]  
\( \mathcal{X}_2 \). The two different growth patterns are
\[
1, 4, 12, 20, 30, 46, 62, 78, 101, 125, 149, 178, 210, 242, 278, 317, 357, 401, 446, 494, 546, 598, 653, 713, 773, 834, 902, 970, 1038, 1113 \ldots
\]

\( C_{2(1;0,0)}(x; y) \) system

\( \mathcal{X}_1 \). The two different growth patterns are
\[
1, 3, 7, 12, 19, 27, 36, 47, 59, 72, 87, 103, 120 \ldots \\
1, 3, 6, 11, 17, 24, 33, 43, 54, 67, 81, 96 \ldots
\]  
\( \mathcal{X}_2 \). The two different growth patterns are
\[
1, 4, 12, 22, 34, 49, 67, 87, 110, 135, 163, 193, 225, 260, 298, 338, 380, 425, 473, 523, 575, 630, 688, 748, 810 \ldots
\]  
\( \mathcal{X}_2 \). The two different growth patterns are
\[
1, 4, 12, 22, 34, 49, 67, 87, 110, 135, 163, 193, 225, 260, 298, 338, 380, 425, 473, 523, 575, 630, 688, 748 \ldots
\]

\( \text{Maximal.} \) The maximal growth in \( \mathcal{X}_1 \) is given by (B.1). The maximal growth in \( \mathcal{X}_2 \) is:
\[
1, 5, 12, 22, 34, 49, 67, 87, 110, 135, 163, 193, 225, 260, 298, 338, 380, 425, 473, 523, 575, 630, 688, 748 \ldots
\]

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\( C3_{(0,0,0)}(x; y) \) system

\( \mathcal{X}_1 \). The two different growth patterns are

\[
1, 3, 7, 12, 18, 26, 35, 45, 57, 70, 84, 100, 117 \ldots \\
1, 3, 6, 10, 16, 23, 31, 41, 52, 64, 78, 93 \ldots
\] (B.11)

\( \mathcal{X}_2 \). The two different growth patterns are

\[
1, 4, 12, 21, 31, 47, 64, 82, 104, 129, 155, 185, 216, 250, 288, 327, 367, \\
413, 460, 508, 560, 615, 671, 731, 792, 856, 924, 993, 1063, 1139, 1216 \ldots \\
1, 5, 10, 18, 30, 43, 57, 77, 98, 120, 146, 175, 205, 239, 274, 312, 354, \\
397, 441, 491, 542, 594, 650, 709, 769, 833, 898, 966, 1038, 1111 \ldots
\] (B.13)

Maximal. The maximal growth in \( \mathcal{X}_1 \) is given by (B.11). The maximal growth in \( \mathcal{X}_2 \) is:

\[
1, 5, 12, 21, 31, 47, 64, 82, 104, 129, 155, 185, 216, 250, 288, 327, 367, \\
413, 460, 508, 560, 615, 671, 731, 792, 856, 924, 993, 1063, 1139 \ldots
\] (B.15)

\( C3_{(1;0,0)}(x; y) \) system

\( \mathcal{X}_1 \). The two different growth patterns are

\[
1, 3, 7, 13, 20, 28, 38, 49, 61, 75, 90, 106, 124 \ldots \\
1, 3, 7, 12, 18, 26, 35, 45, 57, 70, 84, 100 \ldots
\] (B.16)

\( \mathcal{X}_2 \). The two different growth patterns are

\[
1, 4, 12, 22, 34, 50, 68, 88, 111, 137, 165, 195, 228, 264, \\
302, 342, 385, 431, 479, 529, 582, 638, 696, 756, 819 \ldots \\
1, 5, 11, 21, 33, 47, 64, 84, 106, 130, 157, 187, 219, 253, \\
290, 330, 372, 416, 463, 513, 565, 619, 676, 736 \ldots
\] (B.18)

Maximal. The maximal growth in \( \mathcal{X}_1 \) is given by (B.16). The maximal growth in \( \mathcal{X}_2 \) is:

\[
1, 5, 12, 22, 34, 50, 68, 88, 111, 137, 165, 195, 228, 264, \\
302, 342, 385, 431, 479, 529, 582, 638, 696, 756 \ldots
\] (B.20)

\( C1_{(1)}(x; y) + C2_{(0,0,0)}(y; x) \) system

\( \mathcal{X}_1 – \text{NE & SW directions} \). The four different growth patterns are:

\[
1, 3, 7, 11, 17, 25, 33, 43, 55, 67, 81, 97, 113 \ldots \\
1, 3, 5, 9, 15, 21, 29, 39, 49, 61, 75, 89 \ldots
\] (B.21)

\[
1, 3, 7, 12, 18, 26, 35, 45, 57, 70, 84, 100 \ldots
\] (B.23)

\[
1, 3, 6, 10, 16, 23, 31, 41, 52, 64, 78 \ldots
\] (B.24)
\(x_1 - \text{NW \& SE directions.}\) The four different growth patterns are:

1, 3, 7, 11, 18, 25, 35, 43, 57, 67, 84, 97, 117, 131, 155, 171, 198, 217, 247, 267, 301 \ldots \tag{B.25}

1, 3, 5, 10, 15, 23, 29, 41, 49, 64, 75, 93, 105, 127, 141, 166, 183, 211, 229, 261 \ldots \tag{B.26}

1, 3, 7, 12, 17, 26, 33, 45, 55, 70, 81, 100, 113, 135, 151, 176, 193, 222, 241, 273 \ldots \tag{B.27}

1, 3, 6, 9, 16, 21, 31, 39, 52, 61, 78, 89, 109, 123, 146, 161, 188, 205, 235 \ldots \tag{B.28}

\(x_2 - \text{NE \& SW directions.}\) The four different growth patterns are:

1, 4, 12, 20, 30, 46, 62, 79, 101, 125, 150, 179, 210, 243, 280, 317, 358, 403, 448, 495, 548, 601, 655, 715, 776, 838, 904, 973, 1043, 1117, 1192, 1270, 1352, 1434, 1519, 1609, 1699 \ldots \tag{B.29}

1, 5, 10, 17, 29, 41, 54, 74, 94, 115, 141, 169, 198, 231, 266, 303, 344, 385, 430, 479, 528, 579, 636, 693, 751, 815, 880, 946, 1016, 1089, 1163, 1241, 1320, 1402, 1488, 1574 \ldots \tag{B.30}

1, 4, 12, 21, 31, 47, 64, 81, 104, 129, 154, 183, 216, 249, 286, 325, 366, 411, 457, 505, 558, 612, 666, 727, 789, 851, 918, 988, 1058, 1132, 1209, 1287, 1369, 1452, 1538, 1628 \ldots \tag{B.31}

1, 5, 10, 18, 30, 43, 57, 77, 98, 119, 146, 175, 204, 237, 274, 311, 352, 395, 440, 489, 539, 591, 648, 706, 764, 829, 895, 961, 1032, 1106, 1180, 1258, 1339, 1421, 1507 \ldots \tag{B.32}

\(x_2 - \text{NW \& SE directions.}\) The four different growth patterns are:

1, 4, 12, 20, 31, 46, 64, 79, 104, 125, 154, 179, 216, 243, 286, 317, 366, 403, 457, 495, 558, 601, 666, 715, 789, 838, 918, 973, 1058, 1117, 1209, 1270, 1369, 1434, 1538, 1609, 1719, 1790, 1909, 1986, 2107, 2189, 2319, 2402, 2537, 2626, 2767, 2860, 3007, 3102, 3257, 3356, 3515, 3620, 3786, 3891, 4065, 4176, 4353, 4468, 4654, 4771, 4962, 5084, 5281 \ldots \tag{B.33}

1, 5, 10, 18, 29, 43, 54, 77, 94, 119, 141, 175, 198, 237, 266, 311, 344, 395, 430, 489, 528, 591, 636, 706, 751, 829, 880, 961, 1016, 1106, 1163, 1258, 1320, 1421, 1488, 1594, 1663, 1778, 1851, 1969, 2048, 2174, 2253, 2386, 2471, 2608, 2697, 2842, 2933, 3084, 3179, 3336, 3437, 3599, 3701, 3872, 3979, 4152, 4265, 4447, 4560, 4748, 4867, 5060 \ldots \tag{B.34}

1, 4, 12, 21, 30, 47, 62, 81, 101, 129, 150, 183, 210, 249, 280, 325, 358, 411, 448, 505, 548, 612, 655, 727, 776, 851, 904, 988, 1043, 1132, 1192, 1287, 1352, 1452, 1519, 1628, 1699, 1811, 1888, 2008, 2085, 2212, 2295, 2426, 2513, 2652, 2741, 2886, 2979, 3130, 3229, 3385, 3485, 3650, 3755, 3922, 4033, 4209, 4320, 4502, 4619, 4806, 4927, 5121 \ldots \tag{B.35}

1, 5, 10, 17, 30, 41, 57, 74, 98, 115, 146, 169, 204, 231, 274, 303, 352, 385, 440, 479, 539, 579, 648, 693, 764, 815, 895, 946, 1032, 1089, 1180, 1241, 1339, 1402, 1507, 1574, 1684, 1757, 1873, 1946, 2071, 2150, 2277, 2361, 2497, 2582, 2723, 2814, 2961, 3056, 3209, 3306, 3467, 3568, 3733, 3840, 4012, 4119, 4299, 4412, 4595, 4712, 4904 \ldots \tag{B.36}

**Maximal.** The maximal growth with initial conditions in \(x_1\) is isotropic and given by (B.23).
The maximal growth with initial conditions in $\mathcal{X}_2$ is isotropic and given by

$$1, 5, 12, 21, 31, 47, 64, 81, 104, 129, 154, 183, 216, 249, 286, 325, 366, 411, 457,$$
$$505, 558, 612, 666, 727, 789, 851, 918, 988, 1058, 1132, 1209, 1287, 1369, 1452, 1538 \ldots$$

(B.37)

$C_{2(1; 0; 1)}(x; y) + C_{2(1; 1; 0)}(y; x)$ **system**

$\mathcal{X}_1$ – **NE & SW directions.** The four different growth patterns are:

$$1, 3, 7, 13, 21, 31, 42, 54, 68, 83, 99, 117, 136, 156, 178, 201, 225, 251, 278, 306, 336 \ldots$$

(B.38)

$$1, 3, 7, 13, 21, 30, 40, 52, 65, 79, 95, 112, 130, 150, 171, 193, 217, 242, 268, 296, 325, 355 \ldots$$

(B.39)

$$1, 3, 7, 13, 20, 28, 38, 49, 61, 75, 90, 106, 124, 143, 163, 185, 208, 232, 258, 285, 313, 343 \ldots$$

(B.40)

$$1, 3, 7, 12, 18, 26, 35, 45, 57, 70, 84, 100, 117, 135, 155, 176, 198, 222, 247, 273, 301, 330 \ldots$$

(B.41)

$\mathcal{X}_1$ – **NW & SE directions.** The four different growth patterns are:

$$1, 3, 7, 13, 20, 31, 38, 54, 61, 83, 90, 117, 124, 156, 163,$$
$$201, 208, 251, 258, 306, 313, 367, 374, 433, 440 \ldots$$

(B.42)

$$1, 3, 7, 12, 21, 26, 40, 45, 65, 70, 95, 100, 130, 135, 171,$$
$$176, 217, 222, 268, 273, 325, 330, 387, 392 \ldots$$

(B.43)

$$1, 3, 7, 13, 21, 28, 42, 49, 68, 75, 99, 106, 136, 143,$$
$$178, 185, 225, 232, 278, 285, 336, 343, 399, 406 \ldots$$

(B.44)

$$1, 3, 7, 13, 18, 30, 35, 52, 57, 79, 84, 112, 117, 150,$$
$$155, 193, 198, 242, 247, 296, 301, 355, 360 \ldots$$

(B.45)

$\mathcal{X}_2$ – **NE & SW directions.** The four different growth patterns are:

$$1, 5, 12, 23, 38, 57, 78, 101, 127, 155, 186, 221, 257, 296, 338, 382,$$
$$428, 477, 529, 583, 640, 700, 762, 827, 894, 963, 1036, 1111, 1188,$$
$$1269, 1352, 1437, 1525, 1615, 1708, 1804, 1902, 2003, 2107, 2213, 2321 \ldots$$

(B.46)

$$1, 5, 12, 23, 38, 55, 74, 97, 122, 149, 180, 212, 247, 285, 325, 368,$$
$$414, 462, 512, 565, 621, 679, 740, 804, 870, 939, 1010, 1083, 1160,$$
$$1239, 1320, 1405, 1492, 1581, 1673, 1767, 1864, 1964, 2066, 2171 \ldots$$

(B.47)

$$1, 4, 12, 23, 36, 51, 70, 91, 114, 141, 170, 201, 236, 272, 311, 353, 397, 443, 493, 545,$$
$$600, 657, 717, 779, 844, 911, 981, 1054, 1130, 1207, 1288, 1371, 1456, 1544, 1635, 1728 \ldots$$

(B.48)

$$1, 5, 12, 21, 32, 47, 65, 85, 108, 133, 160, 191, 223, 258, 296, 337, 380, 426, 474, 525,$$
$$578, 634, 692, 754, 818, 884, 953, 1025, 1098, 1175, 1254, 1336, 1421, 1508, 1597 \ldots$$

(B.49)
\( \mathcal{X}_2 \) -- NW & SE directions. The four different growth patterns are:

\[
1, 5, 12, 23, 36, 57, 70, 101, 114, 155, 170, 221, 236, 296, 311, 382, 397, 477, 493, 583, 600, 700, 717, 827, 844, 963, 981, 1111, 1130, 1269, 1288, 1437, 1456, 1615, 1635, 1804, 1825, 2003, 2024, 2213, 2234, 2432, 2454, 2662, 2685, 2903, 2926, 3154, 3177 \ldots
\]

\[
1, 5, 12, 21, 38, 47, 74, 85, 122, 133, 180, 191, 247, 258, 325, 337, 414, 426, 512, 525, 621, 634, 740, 754, 870, 884, 1010, 1025, 1160, 1175, 1320, 1336, 1492, 1508, 1673, 1690, 1864, 1881, 2066, 2084, 2279, 2297, 2501, 2520, 2734, 2753, 2977, 2997 \ldots
\]

\[
1, 4, 12, 23, 38, 51, 78, 91, 127, 141, 186, 201, 257, 272, 338, 353, 428, 443, 529, 545, 640, 657, 762, 779, 894, 911, 1036, 1054, 1188, 1207, 1352, 1371, 1525, 1544, 1708, 1728, 1902, 1923, 2107, 2128, 2321, 2342, 2546, 2568, 2781, 2804, 3027, 3050 \ldots
\]

\[
1, 5, 12, 23, 32, 55, 65, 97, 108, 149, 160, 212, 223, 285, 296, 368, 380, 462, 474, 565, 578, 679, 692, 804, 818, 939, 953, 1083, 1098, 1239, 1254, 1405, 1421, 1581, 1597, 1767, 1784, 1964, 1981, 2171, 2189, 2389, 2407, 2616, 2635, 2854, 2873 \ldots
\]

Maximal. The maximal growth with initial conditions in \( \mathcal{X}_1 \) is isotropic and given by (B.38). The maximal growth with initial conditions in \( \mathcal{X}_2 \) is isotropic and given by (B.46).

\[
C_3^{\left(\frac{1}{2}; 0, \frac{1}{2}\right)}(x; y) + C_3^{\left(\frac{1}{2}; \frac{1}{2}, 0\right)}(y; x) \text{ system}
\]

\( \mathcal{X}_1 \) -- NE & SW directions. The same growth patterns as were given in (B.38)--(B.41).

\( \mathcal{X}_2 \) -- NE & SW directions. The four different growth patterns are:

\[
1, 5, 12, 23, 36, 57, 70, 101, 114, 155, 170, 221, 236, 296, 311, 382, 397, 477, 493, 583, 600, 700, 717, 827, 844, 963, 981, 1111, 1130, 1269, 1288, 1437, 1456, 1615, 1635, 1804, 1825, 2003, 2024, 2213, 2234, 2432, 2454, 2662, 2685, 2903, 2926, 3154, 3177 \ldots
\]

\[
1, 5, 12, 23, 38, 47, 74, 85, 122, 133, 180, 191, 247, 258, 325, 337, 414, 426, 512, 525, 621, 634, 740, 754, 870, 884, 1010, 1025, 1160, 1175, 1320, 1336, 1492, 1508, 1673, 1690, 1864, 1881, 2066, 2084, 2279, 2297, 2501, 2520, 2734, 2753, 2977, 2997 \ldots
\]

\[
1, 4, 12, 23, 38, 51, 78, 91, 127, 141, 186, 201, 257, 272, 338, 353, 428, 443, 529, 545, 640, 657, 762, 779, 894, 911, 1036, 1054, 1188, 1207, 1352, 1371, 1525, 1544, 1708, 1728, 1902, 1923, 2107, 2128, 2321, 2342, 2546, 2568, 2781, 2804, 3027, 3050 \ldots
\]

\[
1, 5, 12, 23, 32, 55, 65, 97, 108, 149, 160, 212, 223, 285, 296, 368, 380, 462, 474, 565, 578, 679, 692, 804, 818, 939, 953, 1083, 1098, 1239, 1254, 1405, 1421, 1581, 1597, 1767, 1784, 1964, 1981, 2171, 2189, 2389, 2407, 2616, 2635, 2854, 2873 \ldots
\]

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\( \mathcal{X}_2 - \text{NW & SE directions.} \) The four different growth patterns are:

\[
1, 5, 12, 23, 36, 57, 71, 101, 115, 156, 171, 222, 237, 297, 313, 383, 399, \\
479, 496, 586, 603, 703, 721, 830, 848, 967, 986, 1116, 1135, 1274, 1294, \\
1442, 1462, 1621, 1642, 1811, 1832, 2010, 2032, 2220, 2242, 2440, 2463 \ldots \quad (B.58)
\]

\[
1, 5, 12, 21, 38, 48, 75, 123, 134, 181, 192, 248, 260, 327, 340, \\
416, 429, 515, 528, 624, 638, 744, 759, 874, 889, 1015, 1030, 1165, 1181, \\
1326, 1343, 1498, 1515, 1680, 1697, 1871, 1889, 2074, 2093, 2287, 2306 \ldots \quad (B.59)
\]

\[
1, 4, 12, 23, 38, 52, 78, 92, 127, 142, 187, 202, 258, 274, 339, 355, \\
429, 446, 531, 548, 643, 661, 765, 783, 897, 916, 1040, 1059, 1193, 1213, \\
1357, 1377, 1530, 1551, 1714, 1735, 1909, 1931, 2114, 2136, 2328, 2351 \ldots \quad (B.60)
\]

\[
1, 5, 12, 23, 33, 55, 66, 98, 109, 150, 161, 213, 225, 286, 299, 370, \\
383, 464, 477, 568, 582, 682, 697, 808, 823, 943, 958, 1088, 1104, 1244, \\
1261, 1411, 1428, 1587, 1604, 1774, 1792, 1971, 1990, 2179, 2198 \ldots \quad (B.61)
\]

**Maximal.** The maximal growth with initial conditions in \( \mathcal{X}_2 \) is isotropic and given by \( (B.54) \).

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