A NOTE ON RESOLUTION QUIVERS

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Abstract. Recently, Ringel introduced the resolution quiver for a connected Nakayama algebra. It is known that each connected component of the resolution quiver has a unique cycle. We prove that all cycles in the resolution quiver are of the same size. We introduce the notion of weight for a cycle in the resolution quiver. It turns out that all cycles have the same weight.

1. Introduction

Let $A$ be a connected Nakayama algebra without simple projective modules. All modules are left modules of finite length. We denote the number of simple $A$-modules by $n(A)$. Let $\gamma(S) = \tau \text{soc} P(S)$ for a simple $A$-module $S$ [5], where $P(S)$ is the projective cover of $S$ and $\tau = DTr$ is the Auslander-Reiten translation [1].

Ringel [5] defined the resolution quiver $R(A)$ of $A$ as follows: the vertices correspond to simple $A$-modules and there is an arrow from $S$ to $\gamma(S)$ for each simple $A$-module $S$. The resolution quiver gives a fast algorithm to decide whether $A$ is a Gorenstein algebra or not, and whether it is CM-free or not; see [5].

Using the map $f$ introduced in [3], the notion of resolution quiver applies to any connected Nakayama algebra. It is known that each connected component of $R(A)$ has a unique cycle.

Let $A$ be a connected Nakayama algebra and $C$ be a cycle in $R(A)$. Assume that the vertices of $C$ are $S_1, S_2, \cdots, S_m$. We define the weight of $C$ to be $\sum_{k=1}^{m} c_k n(A)$, where $c_k$ is the length of the projective cover of $S_k$. The aim of this note is to prove the following result.

Proposition 1.1. Let $A$ be a connected Nakayama algebra. Then all cycles in its resolution quiver are of the same size and of the same weight.

As a consequence of Proposition 1.1, if the resolution quiver has a loop, then all cycles are loops; this result is obtained by Ringel [5, 6]. The proof of Proposition 1.1 uses left retractions of Nakayama algebras studied in [2].

2. The proof of Proposition 1.1

Let $A$ be a connected Nakayama algebra. Recall that $n = n(A)$ is the number of simple $A$-modules. Let $S_1, S_2, \cdots, S_n$ be a complete set of pairwise non-isomorphic simple $A$-modules and $P_i$ be the projective cover of $S_i$. We require that $\text{rad} P_i$ is a factor module of $P_{i+1}$. Here, we identify $n + 1$ with 1.

Recall that $c(A) = (c_1, c_2, \cdots, c_n)$ is an admissible sequence for $A$, where $c_i$ is the length of $P_i$; see [1, Chapter IV. 2]. We denote $p(A) = \min\{c_1, c_2, \cdots, c_n\}$. The algebra $A$ is called a line algebra if $c_n = 1$ or, equivalently, the valued quiver of $A$ is a line; otherwise, $A$ is called a cycle algebra or, equivalently, the valued quiver of...
A is a cycle. Then \( A \) is a cycle algebra if and only if \( A \) has no simple projective modules.

Following [3], we introduce a map \( f_A : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\} \) such that \( n \) divides \( f_A(i) - (c_i + i) \) for \( 1 \leq i \leq n \). The resolution quiver \( R(A) \) of \( A \) is defined as follows: its vertices are \( 1, 2, \cdots, n \) and there is an arrow from \( i \) to \( f_A(i) \). Observe that for a cycle algebra \( A \) we have \( \gamma(S_i) = S_{f_A(i)} \). Then by identifying \( i \) with \( S_i \), the resolution quiver \( R(A) \) coincides with that in [5].

Assume that \( A \) is a cycle algebra which is not self-injective. After possible cyclic permutations, we may assume that its admissible sequence \( c(A) = (c_1, c_2, \cdots, c_n) \) is normalized [2], that is, \( p(A) = c_1 = c_n - 1 \). Recall from [2] that there is an algebra homomorphism \( \eta : A \to L(A) \) with \( L(A) \) a connected Nakayama algebra such that its admissible sequence \( c(L(A)) = (c'_1, c'_2, \cdots, c'_{n-1}) \) is given by \( c'_i = c_i - \left\lceil \frac{c_i + i - 1}{n} \right\rceil \) for \( 1 \leq i \leq n - 1 \); in particular, \( n(L(A)) = n(A) - 1 \). Here, for a real number \( x \), \( [x] \) denotes the largest integer not greater than \( x \). The algebra homomorphism \( \eta \) is called the left retraction [2] of \( A \) with respect to \( S_n \).

We introduce a map \( \pi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n - 1\} \) such that \( \pi(i) = i \) for \( i < n \) and \( \pi(n) = 1 \). The following result is contained in the proof of [2, Lemma 3.7].

**Lemma 2.1.** Let \( A \) be a cycle algebra which is not self-injective. Then \( \pi f_A(i) = f_{L(A)} \pi(i) \) for \( 1 \leq i \leq n \).

**Proof.** Let \( c_i + i = kn + j \) with \( k \in \mathbb{N} \) and \( 1 \leq j \leq n \). In particular, \( f_A(i) = j \).

For \( i < n \), we have

\[
(1) \quad c'_i = c_i + i - \left\lfloor \frac{c_i + i - 1}{n} \right\rfloor = kn + j - \left\lfloor \frac{kn + j - 1}{n} \right\rfloor = k(n-1) + j.
\]

Then \( \pi f_A(i) = \pi(j) \) and \( f_{L(A)} \pi(i) = f_{L(A)}(i) = \pi(j) \).

For \( i = n \), we have

\[
(2) \quad c'_n = c_n - 1 + n - \left\lfloor \frac{c_n - 1}{n} \right\rfloor = kn + j - 1 - \left\lfloor \frac{kn + j - n - 1}{n} \right\rfloor = k(n-1) + j.
\]

Then \( \pi f_A(n) = \pi(j) \) and \( f_{L(A)} \pi(n) = f_{L(A)}(1) = \pi(j) \). \( \square \)

The previous lemma gives rise to a unique morphism of resolution quivers

\[
\tilde{\pi} : R(A) \longrightarrow R(L(A))
\]

such that \( \tilde{\pi}(i) = \pi(i) \). Then \( \tilde{\pi} \) sends the unique arrow from \( i \) to \( f_A(i) \) to the unique arrow in \( R(L(A)) \) from \( \pi(i) \) to \( f_{L(A)} \pi(i) = f_{L(A)}(i) \). The morphism \( \tilde{\pi} \) identifies the vertices \( 1 \) and \( n \) as well as the arrows starting from \( 1 \) and \( n \). Because \( 1 \) and \( n \) are in the same connected component of \( R(A) \), we infer that \( R(A) \) and \( R(L(A)) \) have the same number of connected components.

Let \( A \) be a connected Nakayama algebra and \( C \) be a cycle in \( R(A) \). The size of \( C \) is the number of vertices in \( C \). We recall that the weight of \( C \) is given by \( w(C) = \sum_{v \in V(C)} \frac{1}{\pi(v)} \), where \( k \) runs over all vertices in \( C \). We mention that \( w(C) \) is an integer; see [3]. A vertex \( x \) in \( R(A) \) is said to be cyclic provided that \( x \) belongs to a cycle.

**Lemma 2.2.** Let \( A \) be a cycle algebra which is not self-injective. Then \( \tilde{\pi} \) induces a bijection between the set of cycles in \( R(A) \) and the set of cycles in \( R(L(A)) \), which preserves sizes and weights.

**Proof.** We observe that for two vertices \( x \) and \( y \) in \( R(A) \), \( \pi(x) = \pi(y) \) if and only if \( x = y \) or \( \{x, y\} = \{1, n\} \). Note that \( f_A(1) = f_A(n) \). So the vertices \( 1 \) and \( n \) are in the same connected component of \( R(A) \) and they are not cyclic at the same time.
Let $C$ be a cycle in $R(A)$ with vertices $x_1, x_2, \cdots, x_s$ such that $x_{i+1} = f_A(x_i)$. Here, we identify $s + 1$ with 1. Since the vertices 1 and $n$ are not cyclic at the same time, we have that $\pi(x_1), \pi(x_2), \cdots, \pi(x_s)$ are pairwise distinct and $\pi(C)$ is a cycle in $R(L(A))$. Hence $\pi$ induces a map from the set of cycles in $R(A)$ to the set of cycles in $R(L(A))$. Obviously the map is injective. On the other hand, recall that $R(L(A))$ and $R(A)$ have the same number of connected components, thus they have the same number of cycles. Hence $\pi$ induces a bijection between the set of cycles in $R(A)$ and the set of cycles in $R(L(A))$ which preserves sizes.

It remains to prove that $w(C) = w(\pi(C))$. We assume that $e_{x_i} + x_i = k_i n + x_{i+1}$ with $k_i \in \mathbb{N}$. Then we have

$$w(C) = \sum_{i=1}^s \frac{c_{x_i}}{n} = \sum_{i=1}^s k_i.$$ 

Recall that $\eta(L(A)) = n - 1$. We note that $c'_{\pi(x_i)} + x_i = k_i(n - 1) + x_{i+1}$; see (1) and (2). Hence $\sum_{i=1}^s c'_{\pi(x_i)} = (n - 1) \sum_{i=1}^s k_i$ and the assertion follows. □

Recall from [2, Theorem 3.8] that there exists a sequence of algebra homomorphisms

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \rightarrow \cdots \rightarrow A_{r-1} \xrightarrow{\eta_{r-1}} A_r,$$

such that each $A_i$ is a connected Nakayama algebra, $\eta_i : A_i \rightarrow A_{i+1}$ is a left retraction and $A_r$ is self-injective.

We now prove Proposition 1.1.

**Proof of Proposition 1.1** Assume that $A$ is a connected self-injective Nakayama algebra with $n(A) = n$ and admissible sequence $c(A) = (c, c, \cdots, c)$. Then a direct calculation shows that $R(A)$ consists entirely of cycles and each cycle is of size $\frac{n}{(n,c)}$ and of weight $\frac{c}{(n,c)}$, where $(n,c)$ is the greatest common divisor of $n$ and $c$. In particular, all cycles in $R(A)$ are of the same size and of the same weight.

In general, let $A$ be a connected Nakayama algebra whose admissible sequence is $c(A) = (c_1, c_2, \cdots, c_n)$. Take $A'$ to be a connected Nakayama algebra with admissible sequence $c(A') = (c_1 + n, c_2 + n, \cdots, c_n + n)$. Then $R(A) = R(A')$ and for any cycle $C$ in $R(A)$, the corresponding cycle $C'$ in $R(A')$ satisfies $w(C') = w(C) + s(C)$, where $s(C)$ denotes the size of $C$. The statement for $A$ holds if and only if it holds for $A'$.

We now assume that $A$ is a connected Nakayama algebra with $p(A) > n(A)$. One proves by induction that each $A_i$ in the sequence (4) satisfies $p(A_i) > n(A_i)$. In particular, each $A_i$ is a cycle algebra. We can apply Lemma 2.2 repeatedly. Then the statement for $A$ follows from the statement for the self-injective Nakayama algebra $A_r$, which is already proved above. □

We conclude this note with a consequence of the above proof.

**Corollary 2.3.** Let $A$ be a connected Nakayama algebra of infinite global dimension. Then we have the following statements.

1. The number of cyclic vertices of the resolution quiver $R(A)$ equals the number of simple $A$-modules of infinite projective dimension.

2. The number of simple $A$-modules of infinite projective dimension equals the number of simple $A$-modules of infinite injective dimension.

**Proof.** (1) All the algebras $A_i$ in the sequence (4) have infinite global dimension; see [2, Lemma 2.4]. In particular, they are cycle algebras. We apply Lemma 2.2 repeatedly and obtain a bijection between the set of cyclic vertices of $R(A)$ and the set of cyclic vertices of $R(A_r)$. Recall that all vertices of $R(A_r)$ are cyclic,
and \( n(A_r) \) equals \( n(A) \) minus the number of simple \( A \)-modules of finite projective dimension; see [2, Theorem 3.8]. Then the statement follows immediately.

(2) Recall from [3, Corollary 3.6] that a simple \( A \)-module \( S \) is cyclic in \( R(A) \) if and only if \( S \) has infinite injective dimension. Then (2) follows from (1). \( \square \)

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