LIFSHITZ TAILS FOR THE MULTI-PARTICLE CONTINUOUS ANDERSON MODEL

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Abstract. We consider the multi-particle Anderson model in the continuum and show that under some mild assumptions on the inter-particle interaction and the external potential, its lower spectral edge is almost surely constant and is the same with that of the single-particle model. We then obtain the Lifshitz asymptotics for the multi-particle Hamiltonian in the continuum near the bottom of the spectrum.

1. Introduction

The study of multi-particle random Schrödinger operators is a quite recent new direction in the mathematics of random Schrödinger operators. In the single-particle theory, a lot of results on localization, continuity of the integrated density of states and Lifshitz asymptotics have been obtained for different forms of the external potentials see for example [3,7,8,14,20,25,27] for the Anderson localization, [6] for continuity of the integrated density states and [21,24] for the Lifshitz tails. See also the references therein.

For multi-particle systems, we have to distinguish the lattice case with the continuum case. First of all, let us say that today, there is not yet a proof of the existence of the integrated density of states for multi-particle systems on the lattice. In the continuum case, the works by Klopp and Zenk [22,28] for multi-particle homogeneous models establish under some general assumptions, the existence of the integrated density states. Actually, their result is more precised, they showed that the interacting multi-particle integrated density of states exists and is the same with the single-particle one. The proof of Klopp and Zenk uses the Helffer-Sjöstrand formula of the functional calculus.

Although, the object of the paper is not localization, for the reader convenience, we mention some works on Anderson localization for multi-particle systems in both the discrete and the continuum cases [1,2,4,5,9,10,13,15–19,23].

In this paper, we analyze the bottom of the spectrum of the multi-particle continuous Anderson model under fairly general assumptions on the inter-particle interaction and the random external potential. In fact, proving that the multi-particle lower spectral edge is the same with the single-particle one is the heart of the paper. Note, that in our previous works [15,16] for multi-particle systems at low energy on the lattice, we studied the bottom of the spectrum in order to show that it is non-random in absence of ergodicity. We use the general concept and ideas of [16] and adapt them in the continuum case.

Let us now discuss on the results and the structure of the paper. Our main results for multi-particle systems in the continuum are Theorem 1 (the multi-particle lower spectral edges are non-random) and Theorem 2 (multi-particle Lifshitz tails). In the rest of this section, we describe the multi-particle Anderson model in the continuum and the main assumptions. Sections 1.3 and 1.4 are devoted to the statements of the results. Finally, in section 2 we prove the results.

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1.1. **The model.** We fix at the very beginning the number of particles $N \geq 2$. We are concerned with multi-particle random Schrödinger operators of the following forms:

$$H^{(N)}(\omega) := -\Delta + U + V,$$

acting in $L^2((\mathbb{R}^d)^N)$. Sometimes, we will use the identification $(\mathbb{R}^d)^N \cong \mathbb{R}^{Nd}$. Above, $\Delta$ is the multi-particle random external potential also acting as multiplication operator on $L^2(\mathbb{R}^{Nd})$. Additional information on $U$ is given in the assumptions. $V$ is the multi-particle random external potential also acting as multiplication operator on $L^2(\mathbb{R}^{Nd})$. For $x = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$, $V(x) = V(x_1) + \cdots + V(x_N)$ and $\{V(x, \omega), x \in \mathbb{R}^d\}$ is a random i.i.d. stochastic process relative to some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Observe that the non-interacting Hamiltonian $H^{(N)}_0(\omega)$ can be written as a tensor product:

$$H^{(N)}_0(\omega) := -\Delta + V = \sum_{k=1}^{N} 1_{L^2(\mathbb{R}^d)}(x) \otimes H^{(1)}(\omega) \otimes 1_{L^2(\mathbb{R}^d)},$$

where, $H^{(1)}(\omega) = -\Delta + V(x, \omega)$ acting on $L^2(\mathbb{R}^d)$. We will also consider random Hamiltonian $H^{(n)}(\omega)$, $n = 1, \ldots, N$ defined similarly. Denote by $|\cdot|$ the max-norm in $\mathbb{R}^{nd}$.

1.2. **The assumptions.** We now describe our general assumptions on the continuous multi-particle random Hamiltonian.

**(H.1) Short-range interaction.** The global interaction $U$ is of the form:

$$U(x) = \sum_{1 \leq i < j \leq N} U(|x_i - x_j|),$$

where the function $U : \mathbb{R} \rightarrow \mathbb{R}$ is square integrable and non-negative. Further it is plain that $U$ is of finite range, i.e., $\exists r_0 > 0$ such that for $x, y \in \mathbb{R}^d$ with $|x - y| > r_0$, we have $U(|x - y|) = 0$.

For the external potential $V$, we assume:

**(H.2) External potential.** The potential $V$ is square integrable and non-negative.

We have one more assumption

**(H.3) Bounded resolvent.** The operator $U(H^{(N)}_0 - i)^{-1}$ is bounded.

This last assumption was essential in [22] and is valid for example in the particular case of the Coulomb and Yukawa potentials see [26].

1.3. **The results on the bottom of the spectrum.** For any $1 \leq n \leq N$, we denote by $\sigma(H^{(n)}(\omega))$ the spectrum of $H^{(n)}(\omega)$ and by $E^{(n)}_0(\omega)$ the infimum of $\sigma(H^{(n)}(\omega))$. The main result of this subsection is

**Theorem 1** (The multi-particle lower spectral edges are non-random). Let $1 \leq n \leq N$. Assume that the assumptions (H.1), (H.2) and (H.3) hold true. Then with $\mathbb{P}$-probability $1$,

$$0 \in \sigma(H^{(N)}(\omega)) \subset [0; +\infty).$$

Consequently, for $n = 1, \ldots, N$, $E^{(n)}_0 = 0$ almost surely.
1.4. The result on the integrated density of states. Let $1 \leq n \leq N$ and $L > 0$. Denote by $C^0_L(0) = \{x \in (\mathbb{R}^d)^n : |x| < L \}$ the open cube on $(\mathbb{R}^d)^n$. and $H^{(n)}_{C^0_L(0)}$ the restriction of $H^{(n)}(\omega)$ on the cube $C^0_L(0)$ with Dirichlet boundary conditions. We have

**Theorem 2 (Lifshitz tails).** Let $1 \leq n \leq N$. Under assumptions (H.1), (H.2) and (H.3), we have that for any $E \in \mathbb{R}$, the limit

$$\lim_{L \to \infty} L^{-nd} Trace(1_{(-\infty:E]}(H^{(n)}_{C^0_L(0)})), \]$$

exists and is denoted by $N(H^{(n)}, E)$. Further, the quantity $N(H^{(n)}, E)$ (called, the integrated density of states of $H^{(n)}(\omega)$) satisfies the Lifshitz tails: there exist constants $C > 0$ and $\gamma > 0$ such that

$$N(H^{(n)}, E) \sim C \cdot \exp(-\gamma(E - E_0^{(n)})^{-\frac{d}{2}}) \quad \text{as } E \searrow E_0^{(n)}.$$

2. Proof of the results

2.1. **Proof of Theorem 1.** Let $1 \leq n \leq N$. We aim to prove that $0 \in \Sigma(H^{(n)}(\omega)) \subset [0; +\infty)$. Assumption (H.1) implies that $U$ is non-negative and assumption (H.2) also implies that $V$ is non-negative. Since, $-\Delta \geq 0$, we get that almost surely $\sigma(H^{(n)}(\omega)) \subset [0; +\infty)$. It remains to see that $0 \in \sigma(H^{(n)}(\omega))$ almost surely.

Let $k, m \in \mathbb{N}$. Define,

$$B_{k,m} := \{x \in \mathbb{Z}^{nd} : \min_{i \neq j} |x_i - x_j| > r_0 + 2km\},$$

where $r_0 > 0$, is the range of the interaction $U$. We also define the following sequence in $\mathbb{Z}^{nd}$,

$$x^{k,m} := C_{k,m}(1, \ldots, nd),$$

where $C_{k,m} = r_0 + 2km + 1$. Using the identification $\mathbb{Z}^{nd} \cong (\mathbb{Z}^d)^n$, we can also write $x^{k,m} = C_{k,m}(x_1^{k,m}, \ldots, x_n^{k,m})$ with each $x_i^{k,m} \in \mathbb{Z}^d, i = 1, \ldots, n$. Obviously, each term $x^{k,m}$ of the sequence $(x^{k,m})_{k,m}$ belongs to $B_{k,m}$. For $j = 1, \ldots, n$ set

$$H_j^{(1)}(\omega) := -\Delta + V(x_j, \omega).$$

We have that almost surely $\sigma(H_j^{(1)}(\omega)) = [0; +\infty)$, see for example [27]. So, if we set for $j = 1, \ldots, n$,

$$\Omega_j := \{\omega \in \Omega : \sigma(H_j^{(1)}(\omega)) = [0; +\infty)\},$$

$\mathbb{P}\{\Omega_j\} = 1$ for all $j = 1, \ldots, n$. Now, put

$$\Omega_0 := \bigcap_{j=1}^{n} \Omega_j.$$  

We also have that $\mathbb{P}\{\Omega_0\} = 1$. Let $\omega \in \Omega_0$, for this $\omega$, we have that $0 \in \sigma(H_j^{(1)}(\omega))$ for all $j = 1, \ldots, n$ and by the Weyl criterion, there exist $n$ Weyl sequences $\{\phi_j^{(m)} \} : j = 1, \ldots, n$ related to $0$ and each operator $H_j^{(1)}(\omega)$. By the density property of compactly supported functions $C^\infty_c(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, we can directly assume that each $\phi_j^{(m)}$ is of compact support, i.e., $\text{supp}\phi_j^{(m)} \subset C^\infty_{k_j,m}(0)$ for some integer $k_j$ large enough. Set

$$k_0 = \max_{j=1,\ldots,n} k_j.$$
and put, \( x^{k_0,m} = (x_1^{k_0,m}, \ldots, x_n^{k_0,m}) \in B_{k_0,m} \). We translate each function \( \phi_j^m \) to have support contained in the cube \( C_{k_0,m}(x_j^0) \). Next, consider the sequence \((\Phi^m)_m\) defined by the tensor product,

\[ \phi^m := \phi_1^m \otimes \cdots \otimes \phi_n^m. \]

We have that \( \text{supp} \phi^m \subset C_{k_0,m}(x^{k_0,m}) \) and we aim to show that, \((\phi^m)_m\) is a Weyl sequence for \( H^{(n)}(\omega) \) and 0. For any \( y \in \mathbb{R}^{nd} \):

\[ |(H^{(n)}(\omega)\phi^m)(y)| = |(H^{(n)}_0(\omega)\phi^m)(y)|. \]

Indeed, for the values of \( y \) inside the cube \( C_{k_0,m}(x^{k_0,m}) \) the interaction potential \( U \) vanishes and for those values outside that cube \( \phi^m \) equals zero too. Therefore,

\[ \|H^{(n)}(\omega)\phi^m\| \leq \|H^{(n)}_0(\omega)\phi^m\| \leq \prod_{j=1}^n \|H^{(1)}(\omega)\phi_j^m\| \xrightarrow{m \to +\infty} 0, \]

because, for all \( j = 1, \ldots, n \), \( \|H^{(1)}_{j}(\omega)\phi_j^m\| \to 0 \) as \( m \to +\infty \), since \( \phi_j^m \) is a weyl sequence for \( H^{(1)}_{j}(\omega) \) and 0. This complete the proof.

2.2. **Proof of theorem 2** By Theorem 1, we know that all the lower spectral edges of \( H^{(n)}(\omega), n = 1, \ldots, n \) are almost surely equal to 0. Now, by the work of Klopp and Zenk 22, we know that the multi-particle integrated density of states of each \( H^{(n)}(\omega) \) exists and is the same with that of the single-particle. For single-particle Anderson models the integrated density of states exists see 25 and in addition it admits the Lifshitz asymptotics see 27 for example. Finally, since all the lower spectral edges are equal to zero, we conclude that the multi-particle Anderson model in the continuum also admits the Lifshitz tails near its lower spectral edge, i.e., zero. This complete the proof.

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