The inverse scattering problem for perturbed Kadomtsev-Petviashvili multi-line solitons I: solvability of the Cauchy integral equation

Derchyi Wu
Institute of Mathematics, Academia Sinica, Taipei, Taiwan
e-mail: mawudc@gate.sinica.edu.tw
May 17, 2022

ABSTRACT

The stability problem of the Kadomtsev-Petviashvili Grassmannian solitons is investigated using the inverse scattering theory. Building upon previous studies that constructed forward scattering transform via non holomorphic data of an eigenfunction of the Lax equation with a perturbed Kadomtsev-Petviashvili Grassmannian soliton potential and with a Sato eigenfunction boundary data, this paper focuses on solving the Cauchy integral equation prescribing these scattering data and boundary condition. It is found topology, near singularities of the scattering data, of the solution can be characterized by rescaled Hölder spaces and multi-valued functions. It is shown that the Cauchy integral equation is well-posed and the solution can be constructed through an iteration scheme starting with the Sato eigenfunction. Finally, a representation formula for the Kadomtsev-Petviashvili solution is derived formally.

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1. Introduction

The Kadomtsev-Petviashvili (KP) equation \[19\]

\[-4u_x + u_{x_1x_1} + 6uu_{x_1} + 3u_{x_2x_2} = 0, \quad (1.1)\]

where \(u(x) = u(x_1, x_2, x_3)\) represents the wave amplitude at the point \((x_1, x_2)\) in the plane for fixed time \(x_3\), can approximate quasi-two dimensional dispersive systems in the weakly nonlinear, long wave regime \([4], [23]\). It is a fundamentally important integrable system in the sense that many known integrable systems, including the KdV equation, can be derived as special reductions of the KP hierarchy which consists of the KP equation together with its infinitely many symmetries \([20]\). The KP equation is related to various areas of mathematics and physics, such as algebraic and enumerative geometry, representation theory, random matrix theory, and quantum field theory \([5]\).

One major breakthrough in the KP theory was given by Sato \([29]\). He realized that solutions of the KP equation could be written in terms of points of an infinite-dimensional Grassmannian. In particular, a regular real finite dimensional version of the Sato theory concerns \(\text{Gr}(N, M)_{\geq 0}\) KP solitons constructed by \([8], [9], [22]\)

\[u_0(x) = 2\partial_{x_1}^2 \ln \tau(x), \quad (1.2)\]

where the \(\tau\)-function is the Wronskian determinant

\[
\tau(x) = \left| \begin{array}{cccc}
    a_{11} & a_{12} & \cdots & a_{1M} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{N1} & a_{N2} & \cdots & a_{NM}
\end{array} \right|
\begin{array}{cccc}
    E_1 & \cdots & \kappa_1^{N-1} E_1 \\
    E_2 & \cdots & \kappa_2^{N-1} E_2 \\
    \vdots & \ddots & \vdots \\
    E_M & \cdots & \kappa_M^{N-1} E_M
\end{array}
\]

\[
= \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1, \ldots, j_N}(A) E_{j_1, \ldots, j_N}(x), \quad (1.3)
\]

with \(\kappa_1 < \cdots < \kappa_M, A = (a_{ij}) \in \text{Gr}(N, M)_{\geq 0}\) (full rank \(N \times M\) real matrices with non negative minors), \(E_j(x) = \exp \theta_j(x) = \exp(\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3), \Delta_{j_1, \ldots, j_N}(A)\) is the \(N \times N\) minor of the matrix \(A\) whose columns are labelled by the index set \(J = \{j_1 < \cdots < j_N\}\), and \(E_{j_1, \ldots, j_N}(x) = \Pi_{1 \leq m < N} (\kappa_{j_m} - \kappa_{j_m}) \exp \left( \sum_{n=1}^{N} \theta_{j_n}(x) \right)\). Important progress in combinatoric properties, classification theories, and wave resonant theory of \(\text{Gr}(N, M)_{\geq 0}\) KP solitons have been developed recently \([17], [20], [21]\). Connection of Grassmannian KP solitons to real finite gap KP solutions and applications are investigated in \([1], [2], [3]\).
For the simplest $\text{Gr}(1, 2)_{>0}$ KP solitons
\[
u_0(x) = \frac{(\kappa_1 - \kappa_2)^2}{2} \text{sech}^2 \frac{\theta_1(x) - \theta_2(x) - \ln a}{2}, \tag{1.4}
\]

through PDE approaches, $H^s$-global well posedness has been solved by Molinet-Saut-Tzvetkov \[24\], and $L^2$ orbital stability and $L^2$ instability theories have been justified by Mizumachi \[25\], \[26\]. So far none global well posedness or stability results for $\text{Gr}(N, M)_{>0}$ KP multiline solitons are derived through the PDE approach.

Our interest is to study the stability problem of $\text{Gr}(N, M)_{>0}$ KP solitons via the inverse scattering theory (IST). Based on the Lax pair:
\[
\begin{align*}
(-\partial_{x_2} + \partial_{x_1}^2 + u)\Phi(x, \lambda) &= 0, \\
(-\partial_{x_3} + \partial_{x_1}^3 + \frac{3}{2} u \partial_{x_1} + \frac{3}{4} u_x + \frac{3}{4} \partial_{x_1}^{-1} u_{x_2} - \lambda^3)\Phi(x, \lambda) &= 0
\end{align*}
\tag{1.5}
\]
pioneering research on the IST for perturbed $\text{Gr}(1, 2)_{>0}$ KP solitons \[31\], \[10\]. Substantial contributions on the IST for perturbed $\text{Gr}(N, M)_{>0}$ KP solitons, in particular, introducing the Sato theory to construct the explicit formula for the Green function, to derive boundedness of the discrete part of the Green function, as well as to discover the whole symmetries, namely, the $D^\flat$-symmetry of the Sato eigenfunction and $D^\sharp$-symmetry of the eigenfunction (see (1.18)) \[28\], \[11\], \[12\], \[13\], \[14\], \[15\], \[16\], have been established by Boiti, Pempinelli, Pogrebkov, and Prinari in the passed two decades. But rigorous analysis is not completed even for the direct scattering problem for perturbed $\text{Gr}(1, 2)_{>0}$ KP solitons.

We complete rigorous analysis of the direct scattering problem for perturbed $\text{Gr}(N, M)_{>0}$ KP solitons in \[34\], \[35\], \[36\]. Precisely, given
\[
u(x_1, x_2) = \nu_0(x_1, x_2, 0) + v_0(x_1, x_2),
\]
\[
u_0 \in \text{Gr}(N, M)_{>0}, \quad (1 + |x_1| + |x_2|)^2 \partial_x^k v_0 \in L^1 \cap L^\infty, |k| \leq 4, \quad |v_0|_{L^1 \cap L^\infty} \ll 1,
\tag{1.6}
\]
and $z_1 = 0, \{z_n, \kappa_j\}, 1 \leq n \leq N, 1 \leq j \leq M$, distinct real numbers, we prove the unique existence of the eigenfunction
\[
(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x_1, x_2))m(x_1, x_2, \lambda) = 0,
\]
\[
m(x_1, x_2, \lambda) \to \tilde{\chi}(x_1, x_2, 0, \lambda) = \frac{(\lambda - z_1)^{N-1}}{\Pi_{2 \leq n \leq N}(\lambda - z_n)} \chi(x_1, x_2, 0, \lambda),
\tag{1.7}
\]
for $\lambda \notin \{z_n, \kappa_j\}$, construct the forward scattering transform
\[
\mathcal{S}(\nu(x_1, x_2)) = \{z_n, \kappa_j, \mathcal{D}, s_e(\lambda)\},
\tag{1.8}
\]
and establish the Cauchy integral equation

\[ m(x_1, x_2, \lambda) = 1 + \sum_{n=1}^{N} \frac{m_{\text{res}}(x_1, x_2)}{\lambda - z_n} + \mathcal{C}T_0 m, \ \forall \lambda \neq 0, \]

(1.9)

Here \( \chi(x, \lambda) = \tilde{\chi}_{\lambda} R, \) is the normalized Sato eigenfunction; \( z_n \) and \( \kappa_j \) are blowing up and discontinuous points of \( m \), \( \mathcal{D} \) are norming constants between discontinuities of \( m \) at \( \kappa_j \),

\( e^{\kappa_1 x_1 + \kappa_2 x_2} m(x_1, x_2, \lambda_1), \ldots, e^{\kappa_M x_1 + \kappa_2 x_2} m(x_1, x_2, \lambda_M) \mathcal{D} = 0; \)

\( s_c(\lambda) \) is the continuous scattering data

\[ s_c(\lambda) = \frac{\Pi_{2 \leq n \leq N} (\lambda - z_n) \text{sgn}(\lambda I)}{\lambda - z_1} \int \int e^{-[(\lambda - \lambda) x_1 + (\lambda - \lambda) x_2]} \times \tilde{\xi}(x_1, x_2, 0, \lambda) v_0(x_1, x_2) m(x_1, x_2, \lambda) dx_1 dx_2, \]

(1.11)

with \( \tilde{\xi}(x, \lambda) = \frac{1}{\gamma(x)} \sum_{1 \leq j_1 < \cdots < j_N \leq M} \Delta_{j_1 \cdots j_N} (A) \frac{E_{j_1 \cdots j_N}(x)}{1 - \frac{\lambda}{\lambda_1} - \cdots - \frac{\lambda}{\lambda_N}}, \) and satisfies

\[ \partial^\gamma m(x_1, x_2, \lambda) = s_c(\lambda) e^{(\lambda - \lambda) x_1 + (\lambda - \lambda) x_2} m(x_1, x_2, \lambda), \ \lambda \notin \mathbb{R}. \]

Algebraic and analytic constraints for scattering data are

- if \( u(x_1, x_2) \) is a perturbed \( \text{Gr}(1, 2) > 0 \) KP soliton, then

\[ \mathcal{D} = \begin{pmatrix} \kappa_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_N \\ \mathcal{D}_{N+1} & \cdots & \mathcal{D}_{N+1} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{M+1} & \cdots & \mathcal{D}_{M+1} \end{pmatrix}, \quad \mathcal{D} = \mathcal{D}(0, \kappa_1, \kappa_2, A, 0), \ A = (1, a_{12}), \]

(1.13)

\[ s_c(\lambda) = \begin{cases} \frac{\text{sgn}(\lambda)}{\lambda - \kappa_1} + \text{sgn}(\lambda) h_1(\lambda), & \lambda \in D_{\kappa_1}^x, \\ \frac{\text{sgn}(\lambda)}{\lambda - \kappa_2} + \text{sgn}(\lambda) h_2(\lambda), & \lambda \in D_{\kappa_2}^x, \\ \text{sgn}(\lambda) h_3(\lambda), & \lambda \in D_{\kappa_3}^x. \end{cases} \]

(1.14)

- if \( u(x_1, x_2) \) is a perturbed \( \text{Gr}(N, M) > 0 \) KP soliton, then

\[ \mathcal{D} = \begin{pmatrix} \kappa_1^N & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_M^N \\ \mathcal{D}_{N+1} & \cdots & \mathcal{D}_{N+1} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{M+1} & \cdots & \mathcal{D}_{M+1} \end{pmatrix}, \quad \mathcal{D} = \mathcal{D}(z_n, \kappa_j, A, v_0), \ A \in \text{Gr}(N, M) > 0, \]

(1.15)

\[ s_c(\lambda) = \begin{cases} \frac{\text{sgn}(\lambda)}{\lambda - \kappa_1} + \text{sgn}(\lambda) h_1(\lambda), & \lambda \in D_{\kappa_1}^x, \\ \frac{\text{sgn}(\lambda)}{\lambda - \kappa_2} + \text{sgn}(\lambda) h_2(\lambda), & \lambda \in D_{\kappa_2}^x, \\ \text{sgn}(\lambda) h_3(\lambda), & \lambda \in D_{\kappa_3}^x. \end{cases} \]

(1.16)
and
\begin{equation}
(1 - \sum_{j=1}^{M} E_{kj}) \left| \sum_{l=0}^{1} (\lambda - \lambda)^l s_c(\lambda) \right| + \left| (\overline{\lambda}^2 - \lambda^2) s_c(\lambda) \right| \leq L^2(\lambda) \leq L^\infty \tag{1.17}
\end{equation}
\begin{align*}
&+ \sum_{j=1}^{M} \left| \gamma_j + h_j \right|_{C^1(D_{kj})} + \sum_{n=1}^{N} \left| h_n \right|_{C^1(D_{zn})} + \left| \mathcal{D} - \mathcal{D}(z_n, \kappa_j, A, 0) \right|_{L^\infty} \\
&\leq C \sum_{l \leq 2} \left| (1 + |x_1|^2 + |x_2|^2) \partial_{x_1}^l \partial_{x_2}^l v_0 \right|_{L^1 \cap L^\infty}, \\
&s_c(\lambda) = s_c(\overline{\lambda}), h_j(\lambda) = -h_j(\overline{\lambda}), h_n(\lambda) = -h_n(\overline{\lambda}).
\end{align*}

Here \( E_z(\lambda) \equiv 1 \) on \( D_z \), \( E_z(\lambda) \equiv 0 \) elsewhere, \( D_z = \{ \lambda = z + re^{i\alpha} : 0 \leq r \leq \delta, |\alpha| \leq \pi \} \), \( D_z^*=D_z/\{z\}, \delta = \frac{1}{2} \inf |z - z'|, z \neq z' \in \mathcal{Z} = \{ z_n, \kappa_j \} \) and, here and throughout the paper,

\( C \) denotes a uniform constant which is independent of \( x, \lambda \),

and \( \mathcal{D} = \mathcal{D}(z_n, \kappa_j, A, v_0) \) in (1.13), (1.15) can be expressed as
\begin{equation}
\begin{align*}
\mathcal{D} &= \tilde{\mathcal{D}} \times \left( \begin{array}{ccc}
\tilde{D}_{11} & \cdots & \tilde{D}_{1N} \\
\vdots & \ddots & \vdots \\
\tilde{D}_{N1} & \cdots & \tilde{D}_{NN}
\end{array} \right)^{-1} \quad \text{diag}(\kappa_1^N, \ldots, \kappa_N^N), \\
\tilde{\mathcal{D}} &= \text{diag}(\Pi_{2 \leq n \leq N}(\kappa_1 - z_n)^{-1}, \ldots, \Pi_{2 \leq n \leq N}(\kappa_M - z_n)^{-1}) \mathcal{D}^i \\
\mathcal{D}^i &= \left( \mathcal{D}_{ji}^b \right) = \left( \mathcal{D}_{ji} + \sum_{l=0}^{M} c_{jl} \mathcal{D}_{li}^b 1 - c_{jj} \right), \quad c_{jl} = -\int v_1(x) \psi_j(x) \varphi_l(x) dx, \\
\mathcal{D}^b &= \text{diag}(\kappa_1^N, \ldots, \kappa_M^N) A^T,
\end{align*}
\end{equation}

where \( \psi_j, \varphi_l \) are the residue adjoint eigenfunction at \( \kappa_j \) and Sato eigenfunction evaluating at \( \kappa_i \) \cite{36} Theorem 2).

Moreover, \( C \) is the Cauchy integral operator, \( T_0 \) is defined by
\begin{equation}
C \phi = -\frac{1}{2\pi i} \int \frac{\phi(x, \zeta)}{\zeta - \lambda} d\zeta \wedge d\zeta,
\end{equation}
\( T_0 \phi = s_c(\lambda) e^{(\overline{\lambda} - \lambda)x_1 + (\overline{\lambda}^2 - \lambda^2)x_2} \phi(x_1, x_2, \overline{\lambda}), \)
and \( T_0 m = \partial_x m \) for \( \lambda \notin \mathbb{R}; W^0 \) is the space of eigenfunctions satisfying
\begin{enumerate}
\item \( \phi(x_1, x_2, \lambda) = \overline{\phi(x_1, x_2, \overline{\lambda})} \); \\
\item \( (1 - \sum_{n=1}^{N} E_{zn}) \phi(x_1, x_2, \lambda) \in L^\infty \); \\
\item \( \phi(x, \lambda) = \frac{\phi_{zn, \text{res}}(x_1, x_2)}{\lambda - z_n} + \phi_{zn, r}(x_1, x_2, \lambda), \)
\end{enumerate}
where \( \phi_{z_n, \text{res}}(x_1, x_2), \phi_{z_n, r}(x_1, x_2, \lambda) \in L^\infty(D_{z_n}); \)

(d) For \( \lambda \in D_{\kappa_j}^\times \),

\[
\frac{\phi(x_1, x_2, \lambda) - \phi^\lambda(x_1, x_2, \lambda)}{1 + |x_1| + |x_2|} \in C^4(D_{\kappa_j}),
\]

with \( \phi^\lambda(x_1, x_2, \lambda) \equiv \phi(x_1, x_2, \kappa_j + 0^+ e^{i\alpha}) \),

\[
\phi(x_1, x_2, \lambda) = \frac{\Theta_1(x_1, x_2)}{1 - \gamma|\alpha|}, \quad \lambda \in D_{\kappa_1}^\times,
\]

\[
\phi(x_1, x_2, \lambda) = \frac{\Theta_2(x_1, x_2)}{1 - \gamma|\pi - \alpha|}, \quad \lambda \in D_{\kappa_2}^\times,
\]

for perturbed \( \text{Gr}(1, 2)_{>0} \) KP solitons;

\[
\phi(x_1, x_2, \lambda) = \frac{\Theta_j(x_1, x_2)}{1 - \gamma|\alpha|}, \quad \lambda \in D_{\kappa_j}^\times,
\]

for perturbed \( \text{Gr}(N, M)_{>0} \) KP solitons and

\[
(e^{\kappa_1 x_1 + \kappa_2 x_2} \phi(x_1, x_2, \lambda_1), \ldots, e^{\kappa_M x_1 + \kappa_M x_2} \phi(x_1, x_2, \lambda_M))D = 0
\]

where \( D, \lambda_j \) are defined by (1.13) and (1.15).

Finally, a linearization theorem for the scattering data (1.8) is proved in [35, Lemma 4.1], [36 Theorem 5]. Namely, if \( \Phi(x, \lambda) = e^{\lambda x_1 + \lambda x_2 m(x, \lambda)} \) satisfies the Lax pair, then

\[
s_c(\lambda, x_3) = e^{(\lambda - \lambda^3) x_3} s_c(\lambda), \\
D_{mn}(x_3) = e^{(\kappa_n^3 - \kappa_m^3) x_3} D_{mn}.
\]

Consequently, we define the continuous scattering operator \( T \)

\[
T \phi = s_c(\lambda) e^{(\lambda - \lambda^3) x_1 + (\lambda^2 - \lambda^3) x_2 + (\lambda^4 - \lambda^3) x_3} \phi(x, \lambda),
\]

and the \( D \)-symmetry

\[
(e^{\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3} \phi(x, \lambda_1), \ldots, e^{\kappa_M x_1 + \kappa_M x_2 + \kappa_M x_3} \phi(x, \lambda_M))D = 0.
\]

**Definition 1.** A scattering data \( S = \{0, \kappa_j, D, s_c(\lambda)\} \) is called an admissible KdV scattering data if (1.13), (1.14) with \( \kappa_1 < \kappa_2, \kappa_j ≠ 0, a_{12} > 0, \) and

\[
\epsilon_0 \equiv ||(1 - \sum_{j=1}^2 E_{\kappa_j}) \sum_{l=0}^1 |(\lambda - \lambda)^l s_c(\lambda)| + |(\lambda^2 - \lambda^3) s_c(\lambda)| ||_{L^2(\omega_b \cup \omega_\lambda \cap \omega_b)} (1.25)
\]

\[
+ \sum_{j=1}^2 (|\gamma_j| + |h_j| c^1(D_{a_j})) + |h_0| c^1(D_{\alpha_0}), \quad \epsilon_0 \ll 1,
\]
are satisfied. A scattering data $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$ is called an admissible BPP scattering data if\n(1.15), (1.16),
\[
D = \tilde{D} \times \begin{pmatrix}
\tilde{d}_{11} & \cdots & \tilde{d}_{1N} \\
\vdots & \ddots & \vdots \\
\tilde{d}_{N1} & \cdots & \tilde{d}_{NN}
\end{pmatrix}^{-1}
diag(k_1^N, \cdots, k_N^N),
\]
(1.26)

\[
\tilde{D} = \text{diag}(\prod_{2 \leq n \leq N}(k_1 - z_n)^{N-1}, \cdots, \prod_{2 \leq n \leq N}(k_M - z_n)^{N-1})D^2
\]
(1.27)

\[
D^b = \text{diag}(k_1^N, \cdots, k_M^N)A^T, \quad A \in \text{Gr}(N, M),
\]
(1.28)

\[
\det \begin{pmatrix} 1 \frac{1}{k_1 - z_1} & \cdots & \frac{1}{k_1 - z_N} \\
\vdots & \ddots & \vdots \\
\frac{1}{k_N - z_1} & \cdots & \frac{1}{k_N - z_N}
\end{pmatrix} \neq 0, \ z_1 = 0, \ \{z_n, \kappa_j\} \ \text{distinct real},
\]
(1.29)

and
\[
\epsilon_0 \equiv \left| (1 - \sum_{j=1}^M E_{\gamma_j}) \sum_{l=0}^1 \left| (\lambda - \lambda) s_c(\lambda) \right| + \left| (\lambda^2 - \lambda^2) s_c(\lambda) \right| \right|_{L^2(\lambda_{\lambda_{\Lambda_{\alpha_{\Lambda}}} \cap L^\infty}} + \sum_{j=1}^M (|\gamma_j| + |h_j|_{C^1(D_{\gamma_j})}) + \sum_{n=1}^N |h_n|_{C^1(D_{z_n})} + |D^2 - D^b|_{L^\infty}, \ \epsilon_0 \ll 1,
\]
(1.30)

are fulfilled.

We turn to the inverse problem in this paper. Main results are stated as follows:

**Theorem A.** For an admissible KdV or BPP scattering data $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$, there exists $W = W_x$, consisting of functions satisfying the $D$-symmetry, such that $W^0 \subset W_{x_1, x_2, 0}$ and the Cauchy integral equation (CIE)
\[
m(x, \lambda) = 1 + \sum_{n=1}^N \frac{m_{z_n, \text{res}}(x)}{\lambda - z_n} + CTm, \quad \forall \lambda \notin \{z_1, \cdots, z_N\},
\]
(1.31)

\[
m(x, \lambda) \in W, \ |m(x, \lambda) - \tilde{\chi}(x, \lambda)|_W \leq C\epsilon_0
\]
is uniquely solved, where $\tilde{\chi}(x, \lambda) = \tilde{\chi}_{x_1, \kappa_j, A}(x, \lambda) = \frac{(\lambda - z_n)^{N-1}}{\prod_{2 \leq n \leq N}(\lambda - z_n)}\chi_{x_1, \kappa_j, A}(x, \lambda)$.

**Theorem B.** Suppose $u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2)$, $u_0$ is a Grassmannian KP soliton with data $\kappa_j \neq 0$, $A \in \text{Gr}(N, M), \epsilon_0 \ll 1$, $\det(\frac{1}{\kappa_j - z_n})_{1 \leq j, n \leq N} \neq 0$, $z_1 = 0, \ {\{z_n, \kappa_j\}}_{1 \leq j \leq M, 1 \leq n \leq N} \ 	ext{distinct real}$. Then there exist $W$, consisting of functions satisfying the $D$-symmetry, and a unique $m \in W$ satisfying
\[
m(x, \lambda) = 1 + \sum_{n=1}^N \frac{m_{z_n, \text{res}}(x)}{\lambda - z_n} + CTm, \quad \forall \lambda \notin \{z_1, \cdots, z_N\},
\]
(1.31)

\[
m(x_1, x_2, 0, \lambda) = m(x_1, x_2, \lambda), \quad |m(x, \lambda) - \tilde{\chi}(x, \lambda)|_W \leq C\epsilon_0.
\]
where \( \tilde{\chi}(x, \lambda) = \tilde{\chi}_{z_n, \kappa_j, A}(x, \lambda) = \frac{(\lambda - z_1)^{N-1}}{\prod_{2 \leq n \leq N} (\lambda - z_n)} \chi_{\kappa_j, A}(x, \lambda) \).

We are going to find \( W \), construct a recursive sequence

\[
\phi^{(k)}(x, \lambda) \equiv 1 + \sum_{n=1}^{N} \frac{\psi^{(k)}_{z_n, \text{res}}(x)}{\lambda - z_n} + C T \phi^{(k-1)}(x, \lambda),
\]

(1.32)

and prove \( \{\phi^{(k)}\} \subset W \) converge to the unique solution of the CIE. Building upon \[33\], \( D \) symmetries, and a Sato theory, it reduces to deriving \( x \)-uniform estimates of the Cauchy integral operator \( C T \) (CIO) near \( \kappa_j \). Since pointwise structure near \( \kappa_j \) is required to define \( D \)-symmetry. If Sobolev type estimates are adopted then \( H^k(D_{\kappa_j}) \) estimates have to be discussed at least for \( k \geq 1 \). Due to singular structures of \( s_c \) at \( \kappa_j \), we cannot use Sobolev type estimates.

A scaling invariant property of the leading terms of the CIO at \( \kappa_j \) suggests that rescaled Hölder type estimates could work possibly. Indeed, we are going to introduce miscellaneous analytical techniques to tangle the inverse problem on spaces which are direct sums of multi-valued functions \( f^\ast \in L^\infty(D_{\kappa_j}) \) and rescaled Hölder continuous functions \( f^\ast \in C^\mu(D_{\kappa_j}) \). Here the rescaling parameter \( \tilde{\sigma} = \max\{|X_1|, \sqrt{|X_2|}, \sqrt[3]{|X_3|}\} \) and \( X_k \) are the coefficients of the phase function of the forward scattering transformation \( T \) in polar coordinates \( \lambda = z + re^{i\alpha} \),

\[
(\lambda - \lambda)x_1 + (\lambda^2 - \lambda^2)x_2 + (\lambda^3 - \lambda^3)x_3 = -i(X_1r \sin \alpha + X_2r^2 \sin 2\alpha + X_3r^3 \sin 3\alpha).
\]

(1.33)

**Theorem C.** Given an admissible KdV or BPP scattering data \( S = \{z_n, \kappa_j, \mathcal{D}, s_c(\lambda)\} \), then formally

\[
(-\partial_{x_2} + \partial^2_{x_1} + 2\lambda \partial_{x_1} + u(x)) m(x, \lambda) = 0,
\]

with

\[
u(x) \equiv \frac{i}{\pi} \partial_{x_1} \int\int T m \, d\zeta \wedge d\zeta - 2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, \text{res}}(x).
\]

(1.34)

The paper is organized as: In Section 2 we present main estimates of this paper, i.e., estimates of the CIO near \( \kappa_j \). Due to characteristic manifestation of the solitonic content, including highly oscillatory, non homogeneous, asymmetric, and singular structures of the CIO at \( \kappa_j \), natural topologies of the space of eigenfunction \( W \) near \( \kappa_j \) are direct sums of the span of power series in \( \ln(1 - \gamma_j|\beta|) \) or \( \ln(1 - \gamma_j|\pi - \beta|) \) and rescaled Hölder continuous spaces. Hence, proper decomposition of the CIO, principal integration, dilating techniques, Hölder estimates, affine transformations, deformation methods, and stationary point analysis are utilized to make the breakthrough.
In Section 3, we provide estimates of the CIO near \( \infty \) and \( z_n \). Thus a complete characterization of the eigenfunction space \( W \) and estimates of the CIO on \( W \) are obtained.

In Section 4, using Sato theories and the admissible condition, in particular, \( D \) symmetries, we construct the iteration scheme (1.32) in \( W \). Theorem A and B are proved via Sato theories, the admissible condition, and estimates of the CIO. The paper is concluded by a derivation of a formal representation formula of the KPII solution \( u(x) \), (1.34), also called the inverse scattering transform \( S^{-1}\{z_n, \kappa_j, D, s_c(\lambda)\} \). Rigorous analysis for describing the topology space of \( u \) and proving \( u \) solves the KPII equation, which will give an answer to the stability problem of the \( \text{Gr}(N,M) > 0 \) KP solitons, are the future research plan.

Acknowledgments. This research project was partially supported by NSC 109-2115-M-001-003-. Special thanks need to paid to A. K. Pogrebkov and Y. Kodama, this work cannot be possible without their kind and genius input.

2. The Cauchy integral operator \( CT \) near \( \kappa_j \)

Given an admissible scattering data \( \mathcal{S} = \{z_n, \kappa_j, D, s_c(\lambda)\} \), we shall derive \( L^\infty \) estimates of the Cauchy integral operator \( CT \) (CIO) near \( \kappa_j \) in this section. Major difficulties arise from singularities at \( \lambda, \kappa_j \), the highly oscillatory non homogeneous phase function

\[
\varphi(x, \zeta) = i[ (\zeta - \zeta_0)x_1 + (\zeta^2 - \zeta_0^2)x_2 + (\zeta^3 - \zeta_0^3)x_3]
\]

(2.1)
of the continuous scattering transform \( T \), and no good symmetries for a cancellation of the leading term of CIO at \( \kappa_j \). Hence the CIO is a singular integral which is worse than the standard Calderón Zygmund operator.

2.1. Preliminaries. We first investigate leading terms of the CIO at \( \kappa_j \). From (1.14), (1.16), and (1.23), in terms of the polar coordinates at \( \kappa_j \),

\[
\lambda = \kappa_j + re^{i\alpha}, \quad \zeta = \kappa_j + se^{i\beta}, \quad 0 \leq r, s \leq \delta, \quad |\alpha|, |\beta| \leq \pi,
\]

(2.2)
the principal parts of \( T \) are expressed by

\[
\tilde{\gamma}_1(\zeta) = \frac{i \text{sgn}(\zeta)}{\zeta - \kappa_1} \frac{\gamma}{1 - \gamma|\beta|} = -\frac{i}{2}\partial_{\beta} \ln(1 - \gamma|\beta|),
\]

\[
\tilde{\gamma}_2(\zeta) = \frac{i \text{sgn}(\zeta)}{\zeta - \kappa_2} \frac{-\gamma}{1 - \gamma|\pi - \beta|} = -\frac{i}{2}\partial_{\beta} \ln(1 - \gamma|\pi - \beta|)
\]

(2.3)
for admissible KdV scattering data, or

\[
\tilde{\varrho}_j(\zeta) = \frac{i \text{sgn}(\zeta)}{\zeta - \kappa_j} \frac{\gamma_j}{1 - \gamma_j|\beta|} = -\frac{i}{2}\partial_{\beta} \ln(1 - \gamma_j|\beta|)
\]

for admissible BPP scattering data.
The following lemma shows that leading terms of the CIO at \( \kappa_j \), can be integrated and the outcomes are multi-valued functions. Moreover, iterating leading term integration, we have closed formula too.

**Lemma 2.1.** Fixed \( \lambda = \kappa_j + re^{\alpha} \in D_{\kappa_j}^\alpha \), for any non negative integer \( l \),

\[
C\tilde{g}_1 E_{\kappa_j} \left[ -\ln(1 - \gamma|\beta|) \right]^l \quad \frac{[-\ln(1 - \gamma|\alpha|)]^{l+1}}{l + 1} - \frac{1}{2\pi i} \int_{[\kappa - \kappa_1] = \delta} \frac{\frac{1}{l+1}[-\ln(1 - \gamma|\beta|)]^{l+1}}{\zeta - \lambda} d\zeta,
\]

\[
C\tilde{g}_2 E_{\kappa_j} [-\ln(1 - \gamma|\pi - \beta|)]^l \quad \frac{[-\ln(1 - \gamma|\pi - \alpha|)]^{l+1}}{l + 1} - \frac{1}{2\pi i} \int_{[\kappa - \kappa_2] = \delta} \frac{\frac{1}{l+1}[-\ln(1 - \gamma|\pi - \beta|)]^{l+1}}{\zeta - \lambda} d\zeta,
\]

\[
C\tilde{g}_j E_{\kappa_j} [-\ln(1 - \gamma_j|\beta|)]^l \quad \frac{[-\ln(1 - \gamma_j|\alpha|)]^{l+1}}{l + 1} - \frac{1}{2\pi i} \int_{[\kappa - \kappa_j] = \delta} \frac{\frac{1}{l+1}[-\ln(1 - \gamma_j|\beta|)]^{l+1}}{\zeta - \lambda} d\zeta.
\]

**Proof.** In view of

\[
\tilde{g}_1(\zeta) = -\partial_\zeta \ln(1 - \gamma|\beta|),
\]

and applying Stokes’ theorem,

\[
C\tilde{g}_1 E_{\kappa_1} 1 = -\frac{1}{2\pi i} \lim_{\epsilon \to 0} \left\{ \int_{D_{\kappa_1}/D_{\kappa_1,\epsilon} \cup D_{\lambda,\epsilon}} -\partial_\zeta \ln(1 - \gamma|\beta|) d\zeta \wedge d\zeta 
\right.
+
\left. \frac{1}{2} \text{sgn}(\zeta, t) \frac{\gamma}{1 - \gamma|\beta|} d\zeta \wedge d\zeta + \int_{D_{\lambda,\epsilon}} \frac{1}{2} \text{sgn}(\zeta, t) \frac{\gamma}{1 - \gamma|\beta|} d\zeta \wedge d\zeta \right\}

= -\ln(1 - \gamma|\alpha|) + \frac{1}{2\pi i} \int_{[\kappa - \kappa_1] = \delta} \frac{\ln(1 - \gamma|\beta|)}{\zeta - \lambda} d\zeta.
\]

With the help of \( \tilde{g}_2(\zeta) = -\partial_\zeta \ln(1 - \gamma|\pi - \beta|) \) and \( \tilde{g}_j(\zeta) = -\partial_\zeta \ln(1 - \gamma_j|\beta|) \), others can be proved by analogy. \( \square \)

To study the lower order term estimates, suggested by Lemma 2.1 and the scaling invariant property of the leading terms

\[
C\tilde{g}_j E_{\kappa_j} f(\lambda) \quad -\frac{\frac{1}{2}\text{sgn}(\zeta) + \gamma_j}{\zeta - \kappa_j} e^{(\zeta - \kappa_j) x_1 + (\zeta^2 - \zeta^3) x_2 + (\zeta^2 - \zeta^3) x_3} f(\zeta)
\]

\[
= -\frac{1}{2\pi i} \int_{D_{\kappa_j,\delta}} \frac{\frac{1}{2}\text{sgn}(\eta) + \gamma_j}{\eta - \kappa_j} e^{(\eta - \kappa_j) \frac{x_1}{a} + (\eta^2 - \eta^3) \frac{x_2}{a} + (\eta^2 - \eta^3) \frac{x_3}{a}} f \left( \kappa_j + \frac{\eta - \kappa_j}{a} \right) d\eta \wedge d\eta,
\]

\[
= -\frac{1}{2\pi i} \int_{D_{\kappa_j,\delta}} \frac{\frac{1}{2}\text{sgn}(\eta) + \gamma_j}{\eta - \kappa_j} e^{(\eta - \kappa_j) \frac{x_1}{2a} + (\eta^2 - \eta^3) \frac{x_2}{2a} + (\eta^2 - \eta^3) \frac{x_3}{2a}} f \left( \kappa_j + \frac{\eta - \kappa_j}{2a} \right) d\eta \wedge d\eta.
\]
which can tame the highly oscillatory properties of the phase function, we introduce topology of our eigenfunction spaces at $\kappa_j$ as follows.

**Definition 2.** Let

\[
\varphi(s, \beta, X) = X_1 s \sin \beta + X_2 s^2 \sin 2\beta + X_3 s^3 \sin 3\beta,
\]

\[
X_1(z, x) = 2(x_1 + 2x_2z + 3x_3z^2), \quad X_2(z, x) = 2(x_2 + 3x_3z), \quad X_3(z, x) = 2x_3,
\]

and

\[
\bar{\sigma} = \max\{|X_1|, \sqrt{|X_2|}, \sqrt{|X_3|}\}.
\]

and dilate the polar coordinates (2.2) by

\[
\lambda = z + re^{i\alpha} = z + \frac{\tilde{r}}{\sigma}e^{i\alpha}, \quad r \leq \delta;
\]

\[
\tilde{\lambda} = z + \tilde{r}e^{i\alpha}, \quad \tilde{\zeta} = z + \tilde{s}e^{i\beta}; \quad \tilde{r}, \tilde{s} \leq \bar{\sigma}\delta.
\]

Let $0 < \mu < 1$, $H^\mu_\varphi(D_z)$ be the sets of dilated Hölder functions

\[
|f|_{H^\mu_\varphi(D_z)} \equiv \sup_{0 < \tilde{s}_j < \bar{\sigma}\delta, |\tilde{s}_1 - \tilde{s}_2| \leq 1} \left| \frac{f(\tilde{s}_1, \beta_1, X) - f(\tilde{s}_2, \beta_2, X)}{|\tilde{s}_1 e^{i\beta_1} - \tilde{s}_2 e^{i\beta_2}|^\mu} \right|,
\]

and $C^\mu_\varphi(D_z) \equiv \{ f : f \in L^\infty(D_z) \cap H^\mu_\varphi(D_z) \}$, $C^{\mu}_\varphi(D_z) \equiv \{ f : f \in L^\infty(D_z) \cap H^\mu_\varphi(D_z) \}$.

Define $f = f^\sharp + f^\flat$ with $f^\sharp(\zeta) \equiv f(z + 0^+e^{i\beta})$ and $f^\flat(\zeta) \equiv f(\zeta) - f^\sharp(\zeta)$. Let

\[
\mathfrak{M}(D_{\kappa_1}) \equiv \{ f : f^\sharp \in C^\mu_\varphi(D_{\kappa_1}), f^\flat = \sum_{l=0}^{\infty} f_l(X)(-\ln(1 - \gamma|\beta|))^l \in L^\infty(D_{\kappa_1}) \},
\]

\[
\mathfrak{M}(D_{\kappa_2}) \equiv \{ f : f^\sharp \in C^\mu_\varphi(D_{\kappa_2}), f^\flat = \sum_{l=0}^{\infty} f_l(X)(-\ln(1 - \gamma|\pi - \beta|))^l \in L^\infty(D_{\kappa_2}) \},
\]

for admissible KdV scattering data;

\[
\mathfrak{M}(D_{\kappa_j}) \equiv \{ f : f^\sharp \in C^\mu_\varphi(D_{\kappa_j}), f^\flat = \sum_{l=0}^{\infty} f_l(X)(-\ln(1 - \gamma_j|\beta|))^l \in L^\infty(D_{\kappa_j}) \},
\]

with $1 \leq j \leq M$ for admissible BPP scattering data, and

\[
|f|_{\mathfrak{M}(D_z)} \equiv |f^\sharp|_{C^{\mu}_\varphi(D_z)} + |f^\flat|_{L^\infty(D_z)}.
\]

**Lemma 2.2.** If $\phi \in L^p(D_z)$, $p > 2$, then for $\nu = \frac{p - 2}{p}$,

\[
|CE_\varphi\phi|_{L^\infty} \leq C|\phi|_{L^p(D_z)},
\]

\[
|CE_\varphi\phi|_{H^\nu(D_z)} \leq C|\phi|_{L^p(D_z)}.
\]

**Proof.** Please see [30, Theorem 1.19] for the details.
The above lemma shows that, estimates of \( |C T E_{\kappa_1} e^{-i\mathcal{F}(s,\beta,X)} f(s,-\beta,X)|_{\text{B}(D_{\kappa_1})} \), for \( f \in \mathcal{M}(D_{\kappa_1}) \), can be reduced to the leading term \( |C \gamma_{\kappa_1} E_{\kappa_1} e^{-i\mathcal{F}(s,\beta,X)} f(s,-\beta,X)|_{\text{B}(D_{\kappa_1})} \) estimates. To this aim, using the scaling invariant property and the \( \tilde{\sigma} \) rescaled coordinates, decompose

\[
C \gamma_{\kappa_1} E_{\kappa_1} e^{-i\mathcal{F}(s,\beta,X)} f(s,-\beta,X) = - \frac{1}{2\pi i} \int_{D_{\kappa_1,\delta}} \frac{\gamma_{\kappa_1}(\tilde{s},\beta) e^{-i\mathcal{F}(\tilde{s},\beta,\delta,X)} f(\tilde{s},\beta,\delta,X)}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta}
\]

\( \equiv I_1 + I_2 + I_3 + I_4 + I_5 \),

where \( I_j = I_j(\tilde{\lambda}, X) \),

\[
I_1 = - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\tilde{s} < 2} \frac{\gamma_{\kappa_1}(\tilde{s},\beta) f^0(\tilde{s},-\beta,X)}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta},
\]

\[
I_2 = - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\tilde{s} < 2} \frac{\gamma_{\kappa_1}(\tilde{s},\beta) [e^{-i\mathcal{F}(\tilde{s},-\beta,X)} - 1] f^0(\tilde{s},-\beta,X)}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta},
\]

\[
I_3 = - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\tilde{s} < 2} \frac{\gamma_{\kappa_1}(\tilde{s},\beta) e^{-i\mathcal{F}(\tilde{s},\beta,\delta,X)} f^0(\tilde{s},-\beta,X)}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta},
\]

\[
I_4 = - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{2 < \tilde{s} < \tilde{\delta}} \frac{\gamma_{\kappa_1}(\tilde{s},\beta) e^{-i\mathcal{F}(\tilde{s},-\delta,X)} f(\tilde{s},-\beta,\delta,X)}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta},
\]

\[
I_5 = - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{\tilde{s} < \tilde{\delta}} \frac{\gamma_{\kappa_1}(\tilde{s},\beta) e^{-i\mathcal{F}(\tilde{s},\beta,\delta,X)} f(\tilde{s},-\beta,\delta,X)}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta}.
\]

In the following subsections, we shall apply Stokes’ theorem and Hölder interior estimates to derive estimates for \( I_1, I_2, \) and \( I_3 \).

For \( I_4 \) and \( I_5 \), integrals on non uniformly compact domains, we shall take advantage of the oscillatory factors. An efficient way to use oscillatory factors is the deformation method which relies on meromorphic or holomorphic properties in \( \tilde{s} \). To carry out these estimates step by step, we introduce the following definition.

**Definition 3.** Let \( X \) be defined by (2.1). \( X \) is called \( z \)-homogeneous if \( X_2 = X_3 = 0 \), or \( X_1 = X_3 = 0 \), or \( X_1 = X_2 = 0 \); \( z \)-degenerated non homogeneous by \( X_1 = 0 \), or \( X_2 = 0 \), or \( X_3 = 0 \); and \( z \)-fully non homogeneous by \( X_1X_2X_3 \neq 0 \).

### 2.2. Homogeneous cases.

**Lemma 2.3.** Suppose \( S = \{ z_n, \kappa_j, D, s_c(\lambda) \} \) is an admissible scattering data and \( E_{\kappa_j} f \) is \( \lambda \)-holomorphic. For the quadratic or cubic homogeneous cases,

\[
|C T E_{\kappa_j} f|_{\text{B}(D_{\kappa_j})} \leq C \epsilon_0 |f|_{\text{B}(D_{\kappa_j})}.
\]
Proof. Since proofs are identical. Along with Lemma 2.2, for simplicity, we shall assume
\[ \kappa_j = \kappa_1, \quad |\lambda - \kappa_1| \leq \frac{\delta}{2}, \quad X_i \geq 0, \quad |X| > 1, \]
and reduce the proof to estimating principal parts. Introduce the scaled coordinates (2.6)
and use the decomposition (2.11)-(2.15).

Step 1 (Estimates for \( I_1, I_2, \) and \( I_3 \): Results for \( I_1-I_3 \) in this step can be applied to
linear homogeneous or non homogeneous cases. The proof for the compact parts remini-
ses estimates of the Beltrami’s equation (cf. [30, §8, Chapter I]).

Firstly, from Lemma 2.2 and \( |\tilde{v}_1(\tilde{s}, \beta)[e^{-i\tilde{v}(\tilde{x}, \beta, X)} - 1]| < C \) for \( \tilde{s} < 2 \),
\[ |I_2|_{C_0^u(D_{\kappa_1})} \leq C\epsilon_0|f^\delta|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f|_{\mathcal{M}(D_{\kappa_1})}. \]

Besides, applying Lemma 2.1
\[ I_1 = \theta(1 - \tilde{r})F^\delta(\tilde{\lambda}, X) - \frac{\theta(1 - \tilde{r})}{2\pi i} \oint_{|\zeta - \kappa_1| = 2} \frac{F^\delta(\tilde{\zeta}, X)}{\tilde{\zeta} - \tilde{\lambda}} d\tilde{\zeta}, \]
where
\[ \hat{f}^\delta(\zeta, X) = \sum_{l=0}^{\infty} f_l(X)[-\ln(1 - \gamma |\beta|)]^l, \]
\[ F^\delta(\zeta, X) = \sum_{l=0}^{\infty} f_l(X)[-\ln(1 - \gamma |\beta|)]^{l+1}. \]

Thus
\[ I_1' = \theta(1 - \tilde{r})F^\delta(\tilde{\lambda}, X) - \frac{\theta(1 - \tilde{r})}{2\pi i} \oint_{|\zeta - \kappa_1| = 2} \frac{F^\delta(\tilde{\zeta}, X)}{\tilde{\zeta} - \tilde{\lambda}} d\tilde{\zeta} \in \mathcal{M}(D_{\kappa_1}), \]
\[ |I_1'|_{L^\infty(D_{\kappa_1})} + |I_1|_{C_0^u(D_{\kappa_1})} \leq C\epsilon_0|f^\delta|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f|_{\mathcal{M}(D_{\kappa_1})}. \]

For \( I_3 \), one has \( |\tilde{v}_1(\tilde{s}, \beta)[f^\delta(\tilde{x}, \beta, X)|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f^\delta|_{H_\delta^u(D_{\kappa_1})} \tilde{s}^{\mu-1} \) from \( f^\delta \in C_0^u(D_{\kappa_1}) \)
and \( f^\delta(\kappa_1, X) = 0 \). Therefore, an improper integral yields
\[ |I_3|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0|f^\delta|_{H_\delta^u(D_{\kappa_1})}. \]

To derive the \( H_\delta^u \)-estimate of \( I_3 \), from Lemma 2.2 let \( \tilde{\lambda}_j = \kappa_1 + \tilde{r}_j e^{i\alpha_j}, \tilde{r}_j \leq 1, j = 1, 2, \)
and decompose
\[ I_3(\lambda_1, X) - I_3(\lambda_2, X) \]
\[ = -\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \oint_{|\zeta| \leq 2} \tilde{v}_1(\tilde{\zeta}) \frac{\varphi_{\tilde{f}^\delta(\tilde{x}, \beta, X)} - \varphi_{\tilde{f}^\delta(\tilde{x}, \alpha_1, X)}}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\tilde{\zeta} \wedge d\tilde{\zeta} \]
\[ -\frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \oint_{|\zeta| \leq 2} \tilde{v}_1(\tilde{\zeta}) \frac{\varphi_{\tilde{f}^\delta(\tilde{x}, \beta, X)} - \varphi_{\tilde{f}^\delta(\tilde{x}, \alpha_2, X)}}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\tilde{\zeta} \wedge d\tilde{\zeta} \]
+ \frac{\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)}{4\pi i} \iint_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta}) \left[ \frac{1}{\tilde{\zeta} - \tilde{\lambda}_2} - \frac{1}{\tilde{\zeta} - \tilde{\lambda}_1} \right] d\tilde{\zeta} \wedge d\tilde{\zeta},

where \( \varphi_{f^2}(x, \zeta) = e^{-i\varphi(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)} f^2(x, \tilde{\zeta}) \).

In view of \( f^2 \in C^\mu_\sigma(D_{\kappa_1}) \) and \( f^2(\kappa_1, X) = 0 \), we have

\[ |\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha, X)|_{L^\infty(D_{\kappa_1})} \leq C |f^2|_{H^\mu_\sigma(D_{\kappa_1})} \tilde{r}^\mu. \quad (2.21) \]

Along with Lemma \[2.1\] yields

\[ |\frac{\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)}{4\pi i}| \iint_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta}) \left[ \frac{1}{\tilde{\zeta} - \tilde{\lambda}_2} - \frac{1}{\tilde{\zeta} - \tilde{\lambda}_1} \right] d\tilde{\zeta} \wedge d\tilde{\zeta} | \leq C \varepsilon_0 |f^2|_{H^\mu_\sigma(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^{\mu_1}, \tilde{r} = \tilde{r}_2, \]

\[ |\frac{\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)}{4\pi i}| \iint_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta}) \left[ \frac{1}{\tilde{\zeta} - \tilde{\lambda}_2} - \frac{1}{\tilde{\zeta} - \tilde{\lambda}_1} \right] d\tilde{\zeta} \wedge d\tilde{\zeta} | \leq C \varepsilon_0 |f^2|_{L^\infty(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^{\mu_1}, \alpha_1 = \alpha_2 \]

respectively. Therefore,

\[ |\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)| \iint_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta}) \left[ \frac{1}{\tilde{\zeta} - \tilde{\lambda}_2} - \frac{1}{\tilde{\zeta} - \tilde{\lambda}_1} \right] d\tilde{\zeta} \wedge d\tilde{\zeta} | \leq C \varepsilon_0 |f^2|_{C^\mu_\sigma(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^{\mu_1}. \quad (2.22) \]

In an entirely similar way,

\[ |\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_2, X)| \iint_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta}) \left[ \frac{1}{\tilde{\zeta} - \tilde{\lambda}_2} - \frac{1}{\tilde{\zeta} - \tilde{\lambda}_1} \right] d\tilde{\zeta} \wedge d\tilde{\zeta} | \leq C \varepsilon_0 |f^2|_{C^\mu_\sigma(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2|^{\mu_1}. \quad (2.23) \]

Let us now investigate the first term on the right hand side of (2.20). Applying Lemma \[2.2\] it suffices to derive the estimate for all \( \lambda_1, \lambda_2, |\lambda_1| \leq 1 \) with \( \bar{D} \subset \{ \tilde{s} \leq 2 \} \) being a disk centred at \( \tilde{\lambda}_1 \) with radius \( l \) and \( l = 2|\tilde{\lambda}_2 - \tilde{\lambda}_1| \) (cf. [18 5.1]). Write

\[ - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{s} \leq 2} \tilde{\gamma}_1(\tilde{\zeta}) \varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \beta, X) \varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X) d\tilde{\zeta} \wedge d\tilde{\zeta}. \quad (2.24) \]

\[ = - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\tilde{D}} \tilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \beta, X) - \varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\tilde{\zeta} \wedge d\tilde{\zeta}, \]

\[ - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\{\tilde{s} \leq 2\} / \tilde{D}} \tilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \beta, X) - \varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\tilde{\zeta} \wedge d\tilde{\zeta}. \]

Let \( \tilde{D}_0 = \{ \tilde{\zeta} : |\tilde{\zeta} - \tilde{\lambda}_1| < l \} \cap \{ \tilde{\zeta} : |\tilde{\zeta} - \tilde{\lambda}_2| < l \} \).

- If \( \tilde{\zeta} \in \{ \tilde{s} \leq 2 \} / \tilde{D} \) and \( \kappa_1 \in \tilde{D}_0 \), then

\[ \frac{1}{C} \leq \frac{|\tilde{\zeta} - \tilde{\lambda}_1|}{|\tilde{\zeta} - \tilde{\lambda}_2|} \leq C. \]

In this case, using \( f^2 \in C^\mu_\sigma(D_{\kappa_1}) \) and [30] Chapter 1,§6.1,

\[ \left| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\{\tilde{s} \leq 2\} / \tilde{D}} \tilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \beta, X) - \varphi_{f^2}(\frac{\tilde{r}_2}{\sigma}, \alpha_1, X)}{(\tilde{\zeta} - \tilde{\lambda}_1)(\tilde{\zeta} - \tilde{\lambda}_2)} d\tilde{\zeta} \wedge d\tilde{\zeta} \right| \quad (2.25) \]
Therefore the second term on the RHS of (2.24) is done.

If \( \tilde{\zeta} \in \{ \tilde{s} \leq 2 \} / \tilde{D} \) and \( \kappa_1 \notin \tilde{D}_0 \) then

\[
\frac{1}{C} \leq \left| \frac{\tilde{\zeta} - \tilde{\lambda}_1}{\zeta - \lambda_2} \right| \leq C, \quad |\tilde{\lambda}_1 - \tilde{\lambda}_2| \leq \min\{|\tilde{\lambda}_1 - \kappa_1|, |\tilde{\lambda}_2 - \kappa_1|\}.
\]

In this case, using \( f^\sharp \in C^\mu_\rho(D_{\kappa_1}) \) and [30] Chapter 1,§6.1,

\[
| - \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \iint_{\{ \tilde{s} \leq 2 \} / \tilde{D}} \tilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\tilde{\xi}, \beta, X) - \varphi_{f^\sharp}(\tilde{\xi}, \alpha_1, X)}{(\zeta - \lambda_1)(\zeta - \lambda_2)} d\tilde{\zeta} \wedge d\tilde{\zeta} | \leq C \epsilon_0 |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} |\tilde{\lambda}_1 - \tilde{\lambda}_2| \mu.
\]

Therefore the second term on the RHS of (2.24) is done.

Let \( \tilde{D}(\zeta) = 0 \) be the line perpendicular to \( \tilde{\lambda}_1 \tilde{\lambda}_2 \) and passing through \( \frac{1}{2} (\lambda_1 + \lambda_2) \). Set

\[
\tilde{D}_{\tilde{\lambda}_1, \pm} = \tilde{D} \cap \{ \zeta : L(\zeta) L(\lambda_1) \geq 0 \}.
\]

Therefore, thanks to \( f^\sharp \in C^\mu_\rho(D_{\kappa_1}) \), and setting \( \eta = \frac{\tilde{\zeta} - \lambda_1}{|\lambda_1 - \lambda_2|}; \frac{\tilde{\zeta} - \kappa_1}{|\lambda_1 - \lambda_2|} = \eta - r_0 e^{i\alpha_0} \),

\[
\frac{1}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, \pm}} \tilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\tilde{\xi}, \beta, X) - \varphi_{f^\sharp}(\tilde{\xi}, \alpha_1, X)}{(\zeta - \lambda_1)(\zeta - \lambda_2)} d\tilde{\zeta} \wedge d\tilde{\zeta} \]

\[
\leq C \epsilon_0 |\lambda_1 - \lambda_2||f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} \iint_{\tilde{D}_{\tilde{\lambda}_1, \pm}} |\zeta - \kappa_1| |\zeta - \lambda_1|^{1-\mu} |\zeta - \lambda_2| d\zeta \wedge d\tilde{\zeta} \]

\[
\leq C \epsilon_0 |\lambda_1 - \lambda_2|^\mu |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} \iint_{\{|\eta| \leq 2\} \cap \tilde{D}_{\tilde{\lambda}_1, \pm}} \frac{1}{|\eta - r_0 e^{i\alpha_0}| |\eta|^{1-\mu} e^{|\eta - e^{i\alpha_0}|}} d\eta d\eta \]

\[
\leq C \epsilon_0 |\lambda_1 - \lambda_2|^\mu |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} \iint_{\{|\eta| \leq 2\} \cap \tilde{D}_{\tilde{\lambda}_1, \pm}} \frac{1}{|\eta - r_0 e^{i\alpha_0}| |\eta|^{1-\mu} d\eta d\eta} \]

\[
\leq C \epsilon_0 |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} |\lambda_1 - \lambda_2|^\mu.
\]

By analogy,

\[
\frac{1}{4\pi i} \iint_{\tilde{D}_{\tilde{\lambda}_1, \pm}} \tilde{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^\sharp}(\tilde{\xi}, \beta, X) - \varphi_{f^\sharp}(\tilde{\xi}, \alpha_1, X)}{(\zeta - \lambda_1)(\zeta - \lambda_2)} d\tilde{\zeta} \wedge d\tilde{\zeta} \]

\[
\leq |\lambda_1 - \lambda_2| |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} \iint_{\tilde{D}_{\tilde{\lambda}_1, \pm}} |\zeta - \kappa_1| |\zeta - \lambda_1|^{1-\mu} |\zeta - \lambda_2| d\zeta \wedge d\tilde{\zeta} \]

\[
+ |\lambda_1 - \lambda_2| |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} \iint_{\tilde{D}_{\tilde{\lambda}_1, \pm}} |\zeta - \kappa_1| |\zeta - \lambda_1|^{1-\mu} |\zeta - \lambda_2| d\zeta \wedge d\tilde{\zeta} \]

\[
\leq C \epsilon_0 |\lambda_1 - \lambda_2|^\mu |f^\sharp|_{C^\mu_\rho(D_{\kappa_1})} |\lambda_1 - \lambda_2|^\mu.
\]
\[
\leq C\epsilon_0 |f^2|_{C^\mu_\varphi (D_{\kappa_1})}|\tilde{\lambda}_1 - \tilde{\lambda}_2|^\mu
\]
\[
+ \left| \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{D_{\bar{\lambda}_1,-}} \bar{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2} \left( \frac{\tilde{\zeta}}{\varphi}, \alpha_2, X \right) - \varphi_{f^1} \left( \frac{\tilde{\zeta}}{\varphi}, \alpha_1, X \right)}{(\tilde{\zeta} - \bar{\lambda}_1)(\tilde{\zeta} - \bar{\lambda}_2)} d\tilde{\zeta} d\bar{\zeta} \right|.
\]

Applying \( f^2 \in C^\mu_\varphi (D_{\kappa_1}) \), Stokes’ theorem, and \( |\tilde{\zeta} - \bar{\lambda}_1|, |\tilde{\zeta} - \bar{\lambda}_2| \sim |\bar{\lambda}_1 - \bar{\lambda}_2| \) on the boundary of \( D_{\bar{\lambda}_1,-} \) (assured by \( |\tilde{\lambda}| \leq 1, \tilde{D} \subset \{ \tilde{s} < 2 \} \)),
\[
\left| \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{D_{\bar{\lambda}_1,-}} \bar{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2} \left( \frac{\tilde{\zeta}}{\varphi}, \alpha_2, X \right) - \varphi_{f^1} \left( \frac{\tilde{\zeta}}{\varphi}, \alpha_1, X \right)}{(\tilde{\zeta} - \bar{\lambda}_1)(\tilde{\zeta} - \bar{\lambda}_2)} d\tilde{\zeta} d\bar{\zeta} \right|
\leq C |f^2|_{C^\mu_\varphi (D_{\kappa_1})} |\bar{\lambda}_1 - \bar{\lambda}_2|^{1+\mu} \int_{D_{\bar{\lambda}_1,-}} \bar{\gamma}_1(\tilde{\zeta}) \frac{1}{(\zeta - \bar{\lambda}_1)(\zeta - \bar{\lambda}_2)} d\zeta d\bar{\zeta}
\]
\[
= C |f^2|_{C^\mu_\varphi (D_{\kappa_1})} |\bar{\lambda}_1 - \bar{\lambda}_2|^{1+\mu} \int_{D_{\bar{\lambda}_1,-}} \frac{\partial_\zeta \left[ \ln(1 - |\gamma|/|\beta|) \frac{1}{\zeta - \bar{\lambda}_2} \right]}{\zeta - \bar{\lambda}_2} d\zeta d\bar{\zeta}
\]
\[
\leq C\epsilon_0 |f^2|_{C^\mu_\varphi (D_{\kappa_1})} |\bar{\lambda}_1 - \bar{\lambda}_2|^\mu.
\]
Therefore the first term on the RHS of (2.24) is done. Thus
\[
\left| \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{s} \leq 2} \bar{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2} \left( \frac{\tilde{\zeta}}{\varphi}, \beta, X \right) - \varphi_{f^1} \left( \frac{\tilde{\zeta}}{\varphi}, \alpha_1, X \right)}{(\tilde{\zeta} - \bar{\lambda}_1)(\tilde{\zeta} - \bar{\lambda}_2)} d\tilde{\zeta} d\bar{\zeta} \right|
\leq C\epsilon_0 |f^2|_{C^\mu_\varphi (D_{\kappa_1})} |\bar{\lambda}_1 - \bar{\lambda}_2|^\mu,
\text{for } |\tilde{\lambda}_j - \kappa_1| \leq 1, j = 1, 2. \quad (2.28)
\]

In an entirely similar way,
\[
\left| \frac{\tilde{\lambda}_1 - \tilde{\lambda}_2}{4\pi i} \int_{\tilde{s} \leq 2} \bar{\gamma}_1(\tilde{\zeta}) \frac{\varphi_{f^2} \left( \frac{\tilde{\zeta}}{\varphi}, \beta, X \right) - \varphi_{f^1} \left( \frac{\tilde{\zeta}}{\varphi}, \alpha_2, X \right)}{(\tilde{\zeta} - \bar{\lambda}_1)(\tilde{\zeta} - \bar{\lambda}_2)} d\tilde{\zeta} d\bar{\zeta} \right|
\leq C\epsilon_0 |f^2|_{C^\mu_\varphi (D_{\kappa_1})} |\bar{\lambda}_1 - \bar{\lambda}_2|^\mu,
\text{for } |\tilde{\lambda}_j - \kappa_1| \leq 1, j = 1, 2. \quad (2.29)
\]

Plugging (2.22), (2.28), (2.28), and (2.29) into (2.20), we obtain
\[
|I_3(\lambda_1, X) - I_3(\lambda_2, X)| \leq C\epsilon_0 |f^2|_{C^\mu_\varphi (D_{\kappa_1})} |\bar{\lambda}_1 - \bar{\lambda}_2|^\mu \quad (2.30)
\]

for \( |\tilde{\lambda}_j - \kappa_1| \leq 1, j = 1, 2. \)

Hence
\[
|I_3|_{C^\mu_\varphi (D_{\kappa_1})} \leq C\epsilon_0 |f^2|_{C^\mu_\varphi (D_{\kappa_1})} \leq C\epsilon_0 |f|_{\mathfrak{m}(D_{\kappa_1})}. \quad (2.31)
\]

\underline{Step 2 (Proof for \( I_4 \))} : Since the domains are non uniformly compact for \( I_4 \) and \( I_5 \), we shall take advantage of the oscillatory factors. An efficient way to use oscillatory factors is the deformation method which relies on holomorphic properties in \( \tilde{s} \).

Consider the deformation
\[
\tilde{s} \mapsto \tilde{s}e^{ir}, \quad 0 < \tilde{s} < \tilde{\sigma} \delta = X_k^{1/k} \tilde{\delta}, \quad (2.32)
\]
with
\[\tau \in \mathfrak{d} = \begin{cases} 
[-\epsilon_1, 0], & \sin k\beta > 0, \ |\alpha - \beta| \leq \frac{\epsilon_1}{2}, \\
[-\frac{\epsilon_1}{4}, 0], & \sin k\beta > 0, \ |\alpha - \beta| \geq \frac{\epsilon_1}{2}, \\
[0, +\epsilon_1], & \sin k\beta < 0, \ |\alpha - \beta| \leq \frac{\epsilon_1}{2}, \\
[0, +\frac{\epsilon_1}{4}], & \sin k\beta < 0, \ |\alpha - \beta| \geq \frac{\epsilon_1}{2},
\end{cases}\]
and
\[\tau_\dagger = \begin{cases} 
\mp \epsilon_1, & \sin k\beta \gtrless 0, \ |\alpha - \beta| \leq \frac{\epsilon_1}{2}, \\
\mp \frac{\epsilon_1}{4}, & \sin k\beta \gtrless 0, \ |\alpha - \beta| \geq \frac{\epsilon_1}{2},
\end{cases}\]
for \(\epsilon_1 < \frac{\pi}{2k}\). Then
\[|\tilde{s}e^{i\tau_\dagger} - \tilde{r}e^{i(\alpha - \beta)}| \geq \frac{1}{C} \max\{\tilde{r}, \tilde{s}\}, \quad (2.33)\]
\[\Re (-i\phi(\tilde{s}e^{i\tau}, \beta, X)) \leq -|\sin k\tau \sin k\beta|\tilde{s}^k. \quad (2.34)\]

If \(f\) is holomorphic in \(\lambda\), from a residue theorem,
\[I_4 = -\frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \]
\[\times \left( \int_{S_<} + \int_{\Gamma_4} + \int_{S_>} \right) \frac{e^{i\nu\epsilon(\tilde{s}e^{i\tau}, \beta, X)} f(\tilde{s}e^{i\tau}, -\beta, X)}{\tilde{s}e^{i\tau} - \tilde{r}e^{i(\alpha - \beta)}} d\tilde{s}e^{i\tau}\]
where \(S_\geq = S_\geq(\beta, X, \lambda), \Gamma_4 = \Gamma_4(\beta, X, \lambda), \)
\[S_< = \{2e^{i\tau} : \tau \text{ is defined by (2.32)}\}, \]
\[\Gamma_4 = \{\tilde{s}e^{i\tau} : 2 \leq \tilde{s} \leq \tilde{\sigma}\delta, \ \tau_\dagger \text{ is defined by (2.32)}\}, \quad (2.36)\]
\[S_> = \{\tilde{\sigma}\delta e^{i\tau} : \tau \text{ is defined by (2.32)}\}.
\]

In view of (2.16), \(\tilde{r} < 1\), and (2.36),
\[|\frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \]
\[\times \left( \int_{S_<} + \int_{S_>} \right) \frac{e^{i\nu\epsilon(\tilde{s}e^{i\tau}, \beta, X)} f(\tilde{s}e^{i\tau}, -\beta, X)}{\tilde{s}e^{i\tau} - \tilde{r}e^{i(\alpha - \beta)}} d\tilde{s}e^{i\tau}|_{C^\mu(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \quad (2.37)\]
Applying (2.33), (2.34), and (2.36),
\[
\left|\frac{\theta(1 - \tilde{r})}{2\pi i}\int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_4} \frac{e^{-i\psi(s)^i e^{i\tau}, \beta, X} f(s^i e^{i\tau}, -\beta, X)}{s e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)}} d\bar{s} e^{i\tau} |c_\beta^\mu(D_{c_1})\right| (2.38)
\]
\[
\leq C \sum_{n=1,2} \left|\frac{\theta(1 - \tilde{r})}{2\pi i}\int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] e^{-i(n-1)\beta} \times \int_{\Gamma_4} \frac{e^{-i\psi(s)^i e^{i\tau}, \beta, X} f(s^i e^{i\tau}, -\beta, X)}{(s e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)})^n} d\bar{s} e^{i\tau} |L^\infty(D_{c_1})\right|
\]
\[
\leq C\epsilon_0 |f|_{L^\infty(D_{c_1})} \int_{-\pi}^{\pi} d\beta \int_{\mathbb{R}} e^{-s^k |\sin k\tau_1 \sin k\beta|} d\bar{s}.
\]

We have to pay attention to the stationary point \( \sin k\beta = 0 \) of \( \phi \). Precisely, if \( \text{deg} \phi = k = 2 \) or \( 3 \), then by
\[
\tilde{s} \mapsto t = \tilde{s} k^3 |\sin k\beta| = \frac{\sin k\beta}{\sin k\beta},
\]
and improper integrals, (2.38) turns into
\[
\left|\frac{\theta(1 - \tilde{r})}{2\pi i}\int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_4} \frac{e^{-i\psi(s)^i e^{i\tau}, \beta, X} f(s^i e^{i\tau}, -\beta, X)}{s e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)}} d\bar{s} e^{i\tau} |c_\beta^\mu(D_{c_1})\right| (2.40)
\]
\[
\leq C\epsilon_0 |f|_{L^\infty(D_{c_1})} \int_{-\pi}^{\pi} d\beta \frac{1}{\sqrt{|\sin k\beta|}} \int_0^\infty e^{-t^k |\sin k\tau_1|} dt
\]
\[
\leq C\epsilon_0 |f|_{L^\infty(D_{c_1})}.
\]

Combining (2.35), (2.37), and (2.40), we derive
\[
|I_4|_{2\mathbb{R}(D_{c_1})} = |I_4|_{c_\beta^\mu(D_{c_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{c_1})}.
\]

\textit{Step 3 (Proof for } I_5 \): Using the deformation (2.32), holomorphic property of \( f \), and a residue theorem, one has
\[
I_5 = -\frac{\theta(\tilde{\rho} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_5} e^{-i\psi(s)^i e^{i\tau}, \beta, X} f(s^i e^{i\tau}, -\beta, X) \frac{d\bar{s} e^{i\tau}}{s e^{i\tau} - \tilde{r} e^{i(\alpha - \beta)}}
\]
\[
-\theta(\tilde{\rho} - 1) \int_{\alpha + \beta \in \partial} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \text{sgn}(\tau_1(\beta)) e^{-i\psi(s)^i e^{i\tau}, \beta, X} f(s^i e^{i\tau}, -\beta, X) (2.42)
\]
where \( \partial = \partial(\beta, X, \lambda) \) is defined by (2.32), \( S_\gamma \) is defined by (2.36), \( \Gamma_5 = \Gamma_5(\beta, X, \lambda) \), and
\[
\Gamma_5 = \{ s e^{i\tau_1} : 0 \leq \tilde{s} \leq \tilde{\sigma} \delta, \tau_1 \text{ is defined by } (2.32) \}. \]

From (2.33), (2.34), and (2.43), we can apply the same method as that for \( I_4 \) in \textit{Step 3} to derive
\[
|I_5|_{L^\infty(D_{c_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{c_1})}.
\]
Suppose Proposition 2.1. It is sufficient to enhance the proof of Lemma 2.3, it is sufficient to enhance the proof of I

To derive $|I_5|_{\mathcal{M}(D_{s_1})}$, it reduces to studying, $\tilde{\lambda}_j = \kappa_1 + \tilde{r}_j \rho_{ij}$, $\tilde{r}_1 \geq 1$, $|\lambda_1 - \lambda_2| \leq 1/4$,

$$
\left| \frac{1}{2\pi i} \int \frac{\bar{\gamma}_1(\tilde{s}, \beta) e^{-i\varphi(\bar{s}, \beta, X)} f(\tilde{s}, -\beta, X)}{\tilde{s} - \lambda_1} d\tilde{\zeta} \right| - \frac{1}{2\pi i} \int \frac{\bar{\gamma}_1(\tilde{s}, \beta) e^{-i\varphi(\bar{s}, \beta, X)} f(\tilde{s}, -\beta, X)}{\tilde{s} - \lambda_2} d\tilde{\zeta}
$$

$$
\leq \frac{1}{2\pi i} \int \theta(1/2 - |\tilde{s} - \tilde{r}_1|) \bar{\gamma}_1(\tilde{s}, \beta) e^{-i\varphi(\bar{s}, \beta, X)} f(\tilde{s}, -\beta, X) \left[ \frac{1}{\tilde{s} - \lambda_1} - \frac{1}{\tilde{s} - \lambda_2} \right] d\tilde{\zeta}
$$

Applying $f \in \mathcal{M}(D_{s_1})$ and Lemma 2.3 for the first term on the right hand side of (2.45), $\tilde{r}_1 \geq 1$, $|\lambda_1 - \lambda_2| \leq 1/4$,

$$
|\theta(1/2 - |\tilde{s} - \tilde{r}_1|) - 1/2| \leq C \frac{1}{1 + \tilde{s}^2} |\lambda_1 - \lambda_2|
$$

for the second term, and combining (2.45),

$$
|I_5|_{\mathcal{M}(D_{s_1})} \leq C|I_5|_{L^\infty(D_{s_1})} \leq C\varepsilon_0 |f|_{L^\infty(D_{s_1})}. \quad (2.46)
$$

The lemma is proved from (2.17), (2.18), (2.31), (2.41), and (2.46). \qed

Proposition 2.1. Suppose $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$ is an admissible scattering data and $E_{\kappa_j}f$ is holomorphic. For the quadratic or cubic homogeneous cases and $n \geq 2$,

$$
||(CTE_{\kappa_j})_n f||_{\mathcal{M}(D_{s_1})} \leq C\varepsilon_0 |(CTE_{\kappa_j})^{n-1} f|_{\mathcal{M}(D_{s_1})} + C\varepsilon_0^2 |(CTE_{\kappa_j})^{n-2} f|_{\mathcal{M}(D_{s_1})}. \quad (2.47)
$$

Consequently, for $n \geq 0$,

$$
||(CTE_{\kappa_j})_n f||_{\mathcal{M}(D_{s_1})} \leq (C\varepsilon_0)^n |f|_{\mathcal{M}(D_{s_1})}. \quad (2.48)
$$

Proof. Without loss of generality, we assume (2.16) holds. Define $\tilde{\sigma}$ and scaled coordinates by Definition 2 and decompose the principal part of the CIO by (2.10)–(2.15). From the proof of Lemma 2.3, it is sufficient to enhance the proof of $I_4$ and $I_5$. To this aim, denote

$$
f^{[k]} = (CTE_{\kappa_j})^k f = CTE_{\kappa_j} f^{[k-1]}, \quad f^{[0]} = f, \quad (2.49)
$$
From Fubini's theorem, the meromorphic property of the Cauchy kernel, and a residue theorem, \( I_4^{[k]} \) can be expressed as:

\[
I_4^{[k]} = -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{2 < \tilde{s} < \delta} \frac{\bar{\gamma}_1(\tilde{s}, \beta) e^{-i\phi(\frac{d}{d\alpha}, \beta, X) f^{[k-1]}(\frac{\tilde{s}}{\alpha}, -\beta, X)} \bar{\zeta}}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta},
\]

\[
I_5^{[k]} = -\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{\tilde{s} < \delta} \frac{\bar{\gamma}_1(\tilde{s}, \beta) e^{-i\phi(\frac{d}{d\alpha}, \beta, X) f^{[k-1]}(\frac{\tilde{s}}{\alpha}, -\beta, X)} \bar{\zeta}}{\zeta - \lambda} d\zeta \wedge d\bar{\zeta}.
\]

**Step 1 (Proof for \( I_4^{[k]} \)):** From Fubini’s theorem, the meromorphic property of the Cauchy kernel, and a residue theorem,

\[
I_4^{[k]} = \frac{\theta(1 - \tilde{r})}{(2\pi i)^2} \int_{-\pi}^{\pi} d\beta e^{+i\beta [\partial_\beta \ln(1 - \gamma |\beta|)]} \int_{2 < \tilde{s} < \delta} \frac{e^{-i\phi(\frac{d}{d\alpha}, \beta, X) f^{[k-1]}(\frac{\tilde{s}}{\alpha}, -\beta, X)}}{\bar{\zeta}} d\zeta
\]

\[
\times \left( \int_{S_<} + \int_{\Gamma_4} + \int_{S_>} \frac{\bar{\gamma}_1(\tilde{s}', \beta') e^{-i\phi(\frac{d}{d\alpha}, \beta', X) f^{[k-2]}(\frac{\tilde{s}}{\alpha}, -\beta', X)} d\zeta}{\tilde{s}' - \bar{s} e^{-i(\beta + \beta')}} \right)
\]

\[
\times \left( \int_{S_<} + \int_{\Gamma_4} + \int_{S_>} \frac{\bar{\gamma}_1(\tilde{s}', \beta') e^{-i\phi(\frac{d}{d\alpha}, \beta', X) f^{[k-2]}(\frac{\tilde{s}}{\alpha}, -\beta', X)} d\zeta}{\tilde{s}' e^{i(\beta + \beta')} - \bar{\tau} e^{i(\alpha - \beta)}} \right)
\]

where \( \tilde{s}' = k_1 + \tilde{s}' e^{i\beta}', \quad S_< = S_\beta(\beta, X, \lambda), \quad \Gamma_4 = \Gamma_\alpha(\beta, X, \lambda) \) are defined by (2.30) and \( \partial = \partial(\beta, X, \lambda) \) is defined by (2.32).

Observe that the first term on the RHS of (2.50) is the integration of \( f^{[k-1]} \) on the deformed contour \( S_<, \Gamma_4, \) and \( S_> \). Hence, by induction and applying (2.16), (2.39),

\[
| -\frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma |\beta|)] | 
\]

\[
\times \left( \int_{S_<} + \int_{\Gamma_4} + \int_{S_>} \frac{\bar{\gamma}_1(\tilde{s}', \beta') e^{-i\phi(\frac{d}{d\alpha}, \beta', X) f^{[k-1]}(\frac{\tilde{s}}{\alpha}, -\beta', X)} d\zeta}{\tilde{s}' e^{i(\beta + \beta')} - \bar{\tau} e^{i(\alpha - \beta)}} \right) \leq C \epsilon_0 |f^{[k-1]}|_{\mathcal{B}(D_{\alpha})}.
\]

For the second term, if we deform \( \tilde{s}' \) then the real part of the sum of phase functions are

\[
\Re(-\tilde{s}'^k e^{i(k + \beta' + \gamma)} \sin k\beta + \tilde{s}'^k e^{ik\tau} \sin k\beta')
\]

\[
= \tilde{s}'^k \left( \sin k(\beta + \beta') \sin k\beta + \sin k(\beta' + \sin k\beta) + l.o.t. \right),
\]

and, thanks to \( \beta' + \beta \in \partial \),

\[
\sin k\beta < 0, \quad k(\beta + \beta') > 0, \quad \sin k\beta + \sin k\beta' > 0,
\]

\[
\sin k\beta > 0, \quad k(\beta + \beta') < 0, \quad \sin k\beta + \sin k\beta' < 0.
\]
Thus introduce the deformation 
\[ \tau \in \begin{cases} 
[\epsilon_1, 0], & \beta + \beta' > 0, \ |\alpha - 2\beta - \beta'| \leq \frac{\epsilon_1}{2}, \\
[-\frac{\epsilon_1}{4}, 0], & \beta + \beta' > 0, \ |\alpha - 2\beta - \beta'| \geq \frac{\epsilon_1}{4}, \\
[0, +\epsilon_1], & \beta + \beta' < 0, \ |\alpha - 2\beta - \beta'| \leq \frac{\epsilon_1}{2}, \\
[0, +\frac{\epsilon_1}{4}], & \beta + \beta' < 0, \ |\alpha - 2\beta - \beta'| \geq \frac{\epsilon_1}{2}, 
\end{cases} \]
we have
\[ |\tilde{s}' e^{i(\tau_1 + \beta + \beta')} - \tilde{r} e^{i(\alpha - \beta)}| \geq \frac{1}{C} \max \{\tilde{r}, \tilde{s}'\}. \]
\[ \Re \left( -i\left( \tilde{s}' e^{i(\beta + \beta' + \tau)} \right), \beta, X \right) + \Re \left( \tilde{s}' e^{i(\beta' + \tau)}, \beta', X \right) \right) \leq \epsilon_0 |\tilde{r} e^{i(\alpha - \beta)}| \].

Consequently, by the same approach for \( I_4 \) in Lemma 2.3 and induction,
\[ \left| \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta \ e^{i\beta} \left[ \partial_\beta \ln(1 - \gamma |\beta|) \right] \int_{\Re(D_{\nu I})} d\tilde{s} \ e^{-i\tilde{s} e^{i(\beta + \beta')}} \right| \]
\[ \times \frac{-\tilde{s}' e^{i(\beta + \beta')}}{\tilde{s}' - \tilde{s} e^{-i(\beta + \beta')}} \leq C\epsilon_0 |f^{k-2}|_{\Re(D_{\nu I})} + C\epsilon_0^2 |f^{k-2}|_{\Re(D_{\nu I})}. \]

**Step 1 (Proof for \( I_4^{[k]} \))**: From Fubini’s theorem, the meromorphic property of the Cauchy kernel, and a residue theorem,
\[ I_5^{[k]} = \frac{\theta(\tilde{r} - 1)}{(2\pi i)^2} \int_{-\pi}^{\pi} d\beta \ e^{i\beta} \left[ \partial_\beta \ln(1 - \gamma |\beta|) \right] \int_{0 < s < \delta} d\tilde{s} \ e^{-i\tilde{s} e^{i(\beta + \beta')}} \]
\[ \times \int_{D_{\nu I}} \frac{e^{-i\tilde{s}' \gamma_1(s', \beta') e^{-i\tilde{s} e^{i(\beta + \beta')}} f^{k-2}(\tilde{s}', \beta, X) \tilde{s}'} - \tilde{s} e^{-i(\beta + \beta')}}{\tilde{s}' e^{i(\beta + \beta')}} \]
Following similar argument as above,

\[ |I_5^{[k]}|_{L_\infty(D_{n_1})} \leq C \varepsilon_0 |f|^{k-1}_{\text{N}(D_{n_1})} + C \varepsilon_0^2 |f|^{k-2}_{\text{N}(D_{n_1})}. \]

Along with estimates for \( I_1, I_2, I_3 \), yields

\[ |(CTE_{\kappa_1})^k f|_{\text{N}(D_{n_1})} \leq C \varepsilon_0 |f|^{k-1}_{\text{N}(D_{n_1})} + C \varepsilon_0^2 |f|^{k-2}_{\text{N}(D_{n_1})}. \]

In an entirely similar way and applying Lemma 2.3, one has

\[ |(CTE_{\kappa_1})^n f|_{\text{N}(D_{n_1})} \leq \sum_{\nu=0}^n (C \varepsilon_0)^n |(CTE_{\kappa_1})^{n-\nu} f|_{\text{N}(D_{n_1})} \leq (C + 1)^{n-1} C \varepsilon_0^n |f|_{\text{N}(D_{n_1})}. \]

Estimating for the linear homogeneous cases is more involved. We have to take advantage of symmetries of integrands as well as \( \lambda \)-holomorphic properties of \( E_{\kappa_1} f \) to deal with the stationary points \( \sin \beta = 0 \).

**Lemma 2.4.** Suppose \( \mathcal{S} = \{z_0, \kappa_j, D, s_c(\lambda)\} \) is an admissible scattering data and \( E_{\kappa_1} f \) is holomorphic. For linear homogeneous cases,

\[ |(CTE_{\kappa_1})^n f|_{\text{N}(D_{n_1})} \leq C \varepsilon_0 |f|_{\text{N}(D_{n_1})}. \]

**Proof.** Without loss of generality, we assume (2.16) holds. Define \( \tilde{\sigma} \) and scaled coordinates by Definition 2 and decompose the principal part of the CIO by (2.10)-(2.15). From the proof of Lemma 2.3, it is sufficient to enhance the proof of \( I_4 \) and \( I_5 \).

**Step 1 (Proof for \( I_4 \))**: A finer decomposition can squeeze out extra \( |\sin \beta| \)-decay on \( \Gamma_4 \) or \( \Gamma_5 \) by the symmetry. Namely,

\[ I_4 = - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_2^{X_1 \delta} (J_1 + J_2 + J_3 + J_4 + J_5), \]

\[ J_1 = \theta \left( \frac{1}{|\sin \beta|} - |\tilde{s} - \tilde{r}| \right) \left( e^{-i \tilde{s} \sin \beta} - 1 \right) \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} d\tilde{s}, \]

\[ J_2 = - \theta \left( \frac{1}{|\sin \beta|} - |\tilde{s} - \tilde{r}| \right) \left( e^{+i \tilde{s} \sin \beta} - 1 \right) \frac{f(\tilde{s}, -\beta, X)}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} d\tilde{s}, \]

\[ J_3 = \theta \left( \frac{1}{|\sin \beta|} - |\tilde{s} - \tilde{r}| \right) e^{+i \tilde{s} \sin \beta} f \left( \frac{\tilde{s}}{X_1}, -\beta, X \right) \left[ \frac{1}{\tilde{s} - \tilde{r} e^{i(\alpha - \beta)}} - \frac{1}{\tilde{s} - \tilde{r} e^{i(\alpha + \beta)}} \right] d\tilde{s}, \]

\[ J_4 = \theta \left( \frac{1}{|\sin \beta|} - |\tilde{s} - \tilde{r}| \right) e^{+i \tilde{s} \sin \beta} \frac{f \left( \frac{\tilde{s}}{X_1}, -\beta, X \right) - f \left( \frac{\tilde{s}}{X_1}, +\beta, X \right)}{\tilde{s} - \tilde{r} e^{i(\alpha + \beta)}} d\tilde{s}, \]

\[ J_5 = \theta \left( |\tilde{s} - \tilde{r}| \right) \left( e^{-i \tilde{s} \sin \beta} f \left( \frac{\tilde{s}}{X_1}, -\beta, X \right) - e^{i \tilde{s} \sin \beta} \frac{f \left( \frac{\tilde{s}}{X_1}, +\beta, X \right)}{\tilde{s} - \tilde{r} e^{i(\alpha + \beta)}} \right) d\tilde{s}. \]
According to the signatures of the real parts of the exponential factors in $J_j$, we deform

$$|I_4|c_2(D_{n_1}) \leq C_0 |f|_{L^\infty(D_{n_1})} + | - \frac{\theta(1 - \bar{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{S_\chi} \mathfrak{J}_1 \mathfrak{J}_2 \mathfrak{J}_3 \mathfrak{J}_4 \mathfrak{J}_5 |_{L^\infty(D_{n_1})},$$

where $S_\chi$, $\Gamma_4$ are defined by (2.31) and

$$\mathfrak{J}_1 = \theta(\frac{1}{|\sin \beta|} - |\bar{s} - \bar{r}|) \frac{[e^{-i\bar{s}e^{i\tau} \sin \beta} - 1] f(\bar{s}e^{i\tau} \chi_1, -\beta, \chi) - \bar{s}e^{i\tau} - \bar{r} e^{i(\alpha-\beta)}}{ \bar{s}e^{i\tau} - \bar{r}e^{i(\alpha-\beta)}} d\bar{s}e^{i\tau},$$

$$\mathfrak{J}_2 = \theta(\frac{1}{|\sin \beta|} - |\bar{s} - \bar{r}|) \frac{[1 - e^{-i\bar{s}e^{i\tau} \sin \beta}] f(\bar{s}e^{i\tau} \chi_1, +\beta, \chi) - \bar{s}e^{i\tau} - \bar{r} e^{i(\alpha+\beta)}}{ \bar{s}e^{i\tau} - \bar{r}e^{i(\alpha-\beta)}} d\bar{s}e^{i\tau},$$

$$\mathfrak{J}_3 = \theta(\frac{1}{|\sin \beta|} - |\bar{s} - \bar{r}|) \frac{e^{-i\bar{s}e^{i\tau} \sin \beta} f(\bar{s}e^{i\tau} \chi_1, -\beta, \chi) - f(\bar{s}e^{i\tau} \chi_1, +\beta, \chi)}{ \bar{s}e^{i\tau} - \bar{r}e^{i(\alpha-\beta)}} d\bar{s}e^{i\tau},$$

$$\mathfrak{J}_4 = \theta(\frac{1}{|\sin \beta|} - |\bar{s} - \bar{r}|) \frac{e^{i\bar{s}e^{i\tau} \sin \beta} f(\bar{s}e^{i\tau} \chi_1, -\beta, \chi) - f(\bar{s}e^{i\tau} \chi_1, +\beta, \chi)}{ \bar{s}e^{i\tau} - \bar{r}e^{i(\alpha+\beta)}} d\bar{s}e^{i\tau},$$

$$\mathfrak{J}_5 = \theta(|\bar{s} - \bar{r}| - \frac{1}{|\sin \beta|}) (e^{-i\bar{s}e^{i\tau} \sin \beta} f(\bar{s}e^{i\tau} \chi_1, -\beta, \chi) - f(\bar{s}e^{i\tau} \chi_1, +\beta, \chi) + e^{-i\bar{s}e^{i\tau} \sin \beta} f(\bar{s}e^{i\tau} \chi_1, +\beta, \chi)) d\bar{s}e^{i\tau}.$$

with $\tau$ (and $\tau_1$) defined by (2.32) for $\beta \in [0, \pi]$.

(J1,J2) From the mean value theorem, (2.33), and (2.34),

$$\frac{|e^{\pm i\bar{s}e^{i\tau_1} \sin \beta} - 1|}{|\bar{s}e^{\pm i\tau_1} - \bar{r} e^{i(\alpha-\beta)}|} \leq \frac{|(e^{\bar{s} \sin \tau_1 \sin \beta} - 1) e^{\mp i\bar{s} \cos \tau_1 \sin \beta} + e^{\mp i\bar{s} \cos \tau_1 \sin \beta} - 1|}{|\bar{s}e^{\pm i\tau_1} - \bar{r} e^{i(\alpha-\beta)}|} \leq C |\sin \beta|.$$

Along with (2.16), (2.32), (2.34), (2.59), (2.60), and the change of variables

$$\bar{s} \mapsto t = \bar{s} |\sin \beta|,$$

yields

$$\sum_{j=1}^{2} \left| - \frac{\theta(1 - \bar{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] (\int_{S_\chi} + \int_{\Gamma_4} + \int_{S_\chi}) \mathfrak{J}_j |_{L^\infty(D_{n_1})}$$

(2.66)
\[ \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} + C|f|_{L^\infty(D_{\kappa_1})} \left( \int_0^\pi d\beta [\partial_\beta \ln (1 - \gamma |\beta|)] \right) \]

\[ \times \int_{2|\sin \beta|}^{X_1|\sin \beta|} \theta(1 - |t - \tilde{r}| \sin \beta |) \text{d}t |_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \]

(J3) Thanks to $|\tilde{r}| < 1$ and $\tilde{s} > 2$,

\[ \frac{1}{\tilde{s} e^{-i\tilde{r} \tau} - \tilde{r} e^{i(\alpha - \beta)}} - \frac{1}{\tilde{s} e^{-i\tilde{r} \tau} - \tilde{r} e^{i(\alpha + \beta)}} \leq \frac{1}{\tilde{s}^2} \]  

Together with (2.61), (2.16), (2.32), and (2.34), yields

\[ | - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi \text{d}[\partial_\beta \ln (1 - \gamma |\beta|)](\int_{S_\beta} + \int_{\Gamma_4} + \int_{S_\kappa}) \text{d}\beta |_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \]

(J4) From holomorphic properties of $E_\kappa, f$ and (2.33),

\[ \left| f(\tilde{s} e^{-i\tilde{r} \tau}, -\beta, X) - f(\tilde{s} e^{-i\tilde{r} \tau}, +\beta, X) \right| \leq C \frac{\tilde{s} |\sin \beta|}{|\tilde{s} e^{-i\tilde{r} \tau} - \tilde{r} e^{i(\alpha + \beta)}|} \leq C |\sin \beta|. \]  

Along with (2.16), (2.32), (2.34), (2.62), and the change of variable $s$ (2.65), yields

\[ | - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi \text{d}[\partial_\beta \ln (1 - \gamma |\beta|)](\int_{S_\beta} + \int_{\Gamma_4} + \int_{S_\kappa}) \text{d}\beta |_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \]

(J5) Using (2.16), (2.32), (2.34), (2.63), $\epsilon_1 > 0$, and rescaling (2.65), one obtains

\[ | - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi \text{d}[\partial_\beta \ln (1 - \gamma |\beta|)](\int_{S_\beta} + \int_{\Gamma_4} + \int_{S_\kappa}) \text{d}\beta |_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} + C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} \int_{2|\sin \beta|}^{X_1|\sin \beta|} \theta(|t - \tilde{r}| \sin \beta |) \left( e^{-t \sin \beta |} \right) \text{d}t \]

\[ \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \]

As a result,

\[ |I_4|_{\mathcal{M}(D_{\kappa_1})} = |I_4|_{C_0^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}. \]  

\textbf{Step 2 (Proof for $I_5$):} Applying Lemma 2.3 if $|\beta| > \epsilon_1/8$, and following a similar argument as that in the previous step,

\[ |I_5|_{\mathcal{M}(D_{\kappa_1})} = |I_5|_{L^\infty(D_{\kappa_1})} \]

\[ \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} + | - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi \text{d}[\partial_\beta \ln (1 - \gamma |\beta|)](\frac{\epsilon_1}{8} - |\beta|) \]

\[ \times (\int_{\Gamma_5} + \int_{S_\kappa}) (\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5) |_{L^\infty(D_{\kappa_1})}. \]
Lemma 2.5. For \( \tilde{s} > 1 \) and \( \tau_1 \) defined by (2.32) for \( \beta \in [0, \pi] \),

\[
|C_{\kappa_1 + \frac{i}{\tilde{s}} e^{i(\gamma \beta)}} T E_{\kappa_1} g - C_{\kappa_1 + \frac{i}{\tilde{s}} e^{-i(\gamma \beta)}} T E_{\kappa_1} g| \\
\leq C\epsilon_0 |g|_{L^\infty(D_{\kappa_1})} \max\{\tilde{s}|\sin \beta|, |\tilde{s}| \sin \beta|^\mu\}.
\]  

Moreover,

\[
| - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta(\frac{\epsilon_1}{8} - |\beta|) \text{sgn}(\tau_1(\beta)) \times e^{-i\tilde{s} e^{i(\gamma - \beta)}} f(\frac{\tilde{r}}{s} e^{i(\alpha - \beta)}, -\beta, X)|_{L^\infty(D_{\kappa_1})}
\]

where \( S_\gamma \) is defined by (2.36), \( \Gamma_5 \) defined by (2.43), and \( \tilde{J}_j \), \( 1 \leq j \leq 5 \), defined by (2.59)-(2.63).

From (2.16), (2.33), (2.34), and (2.43), for \( 1 \leq j \leq 5 \),

\[
|\theta(\tilde{r} - 1) \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta(\frac{\epsilon_1}{8} - |\beta|) \int_{S_{\gamma}} \tilde{J}_j|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]  

From \( |\beta| < \epsilon_1/8 \) and (2.33),

\[
\left| \frac{1}{\tilde{s} e^{-i\tau_1} - \tilde{r} e^{i(\alpha - \beta)}} \frac{1}{\tilde{s} e^{-i\tau_1} - \tilde{r} e^{i(\alpha + \beta)}} \right| \\
= \left| \frac{\tilde{r} e^{i\alpha} 2 \sin \beta}{(\tilde{s} e^{-i\tau_1} - \tilde{r} e^{i(\alpha - \beta))}(\tilde{s} e^{-i\tau_1} - \tilde{r} e^{i(\alpha + \beta))} \right| \\
\leq C|\sin \beta|.
\]

Along with (2.64), (2.69), the change of variables (2.65), and argument in previous step for \( \tilde{J}_j \) on \( \Gamma_4 \), \( 1 \leq j \leq 5 \), yields

\[
| - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta(\frac{\epsilon_1}{8} - |\beta|) \int_{\Gamma_4} \tilde{J}_j|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]  

Thus \( L^\infty \) estimates for \( I_5 \) is complete by (2.73), (2.45), and (2.76). One can adopt the same argument as that from (2.45) to (2.46) to derive

\[
|I_5|_{W(D_{\kappa_1})} \leq C|I_5|_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]  

\[\square\]

In order to generalize Lemma 2.4 to \((C T E_{\kappa_1})^n f\), we need to improve

(J4) Estimate (2.69), (2.76) for \( \tilde{J}_4 \) on \( \Gamma_4 \) or \( \Gamma_{5,\text{out}} \),

where \( \lambda \)-holomorphic \( E_{\kappa_1} f(\lambda) \) is used. The following lemma is a generalized version of J4.

Lemma 2.5. For \( \tilde{s} > 1 \) and \( \tau_1 \) defined by (2.32) for \( \beta \in [0, \pi] \),

\[
|C_{\kappa_1 + \frac{i}{\tilde{s}} e^{i(\gamma \beta)}} T E_{\kappa_1} g - C_{\kappa_1 + \frac{i}{\tilde{s}} e^{-i(\gamma \beta)}} T E_{\kappa_1} g| \\
\leq C\epsilon_0 |g|_{L^\infty(D_{\kappa_1})} \max\{\tilde{s}|\sin \beta|, |\tilde{s}| \sin \beta|^\mu\}.
\]  

Moreover,

\[
| - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta(\frac{\epsilon_1}{8} - |\beta|) \int_{\Gamma_4} \tilde{J}_4|_{L^\infty(D_{\kappa_1})}
\]
\[ + | - \frac{\theta(\bar{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - |\beta|)] \int_{\Gamma_4} \mathcal{J}_4 |L^{\infty}(D_{\kappa_1}) \]

where \( \Gamma_4, \Gamma_5 \) are defined by \((2.36), (2.43)\), and \( \mathcal{J}_4 \) defined by \((2.62)\) with \( f = CTE_\kappa g \).

Proof. Step 1 (Proof for \((2.78)\)) : Decompose \( CTE_\kappa g = \sum_{j=1}^5 I_j \) via \((2.10)-(2.13)\). To prove \((2.78)\), it amounts to the investigation of \(|I_5(\bar{s}e^{-i\tau}, -\beta, X) - I_5(\bar{s}e^{-i\tau}, +\beta, X)|\).

Write

\[
\theta(1/4 - |\bar{s}\sin \beta|)|I_5(\bar{s}e^{-i\tau}, -\beta, X) - I_5(\bar{s}e^{-i\tau}, +\beta, X)|
\]

\[
= - \frac{\theta(\bar{s} - 1)\theta(1/4 - |\bar{s}\sin \beta|)}{2\pi i} \int_{-\pi}^{\pi} d\beta' |\partial_{\beta'} \ln(1 - |\beta'|)|
\]

\[
\times \int_0^{\tilde{s}\delta} \theta(1/2 - |\bar{s}' - \bar{s}e^{i(-\gamma_1 + \beta - \beta')}|) e^{-iv\theta'_{\beta'} X} \left[ g(\frac{\bar{s}}{\bar{s}'}, -\beta', X) - g(\frac{\bar{s}}{\bar{s}'}, +\beta', X) \right] d\bar{s}'
\]

\[
\times \int_0^{\tilde{s}\delta} \theta(|\bar{s}' - \bar{s}e^{i(-\gamma_1 + \beta - \beta')}| - 1/2) e^{-iv\theta'_{\beta'} X} \left[ \frac{-2i\bar{s}e^{i(-\gamma_1 - \beta') \sin \beta} g(\frac{\bar{s}}{\bar{s}'}, -\beta', X)}{(\bar{s}' - \bar{s}e^{i(-\gamma_1 - \beta')})(\bar{s}' - \bar{s}e^{i(-\gamma_1 + \beta - \beta')})} d\bar{s}'
\]

\[ \equiv A_1 + A_2. \]

In view of \(|\bar{s}\sin \beta| < 1/4, \]

\[ \left| \frac{\theta(|\bar{s}' - \bar{s}e^{i(-\gamma_1 + \beta - \beta')}| - 1/2)}{(\bar{s}' - \bar{s}e^{i(-\gamma_1 - \beta')})(\bar{s}' - \bar{s}e^{i(-\gamma_1 + \beta - \beta')})} \right| \leq \frac{C}{(\bar{s}')^2}, \]

one obtains

\[ |A_2|_{L^{\infty}(D_{\kappa_1})} \leq C\epsilon_0 |\bar{s}| \sin \beta |g|_{L^{\infty}(D_{\kappa_1})}. \]

From \( \bar{s} > 1, |\bar{s}\sin \beta| < 1/4, \) and \( |\bar{s}' - \bar{s}e^{i(-\gamma_1 + \beta - \beta')}| < 1/2, \) one can apply Lemma \((2.2)\) to derive

\[ |A_1|_{L^{\infty}(D_{\kappa_1})} \leq C\epsilon_0 |g|_{L^{\infty}(D_{\kappa_1})} |\bar{s}\sin \beta|^\mu. \]

Plugging the above estimates into \((2.80)\), we obtain

\[ \theta(1/4 - |\bar{s}\sin \beta|)|I_5(\bar{s}e^{-i\tau}, -\beta, X) - I_5(\bar{s}e^{-i\tau}, +\beta, X)| \]

\[ \leq C\epsilon_0 |g|_{L^{\infty}(D_{\kappa_1})} |\bar{s}\sin \beta|^\mu. \] (2.81)

On the other hand, write

\[ \theta(|\bar{s}\sin \beta| - 1/4)(I_5(\bar{s}e^{-i\tau}, -\beta, X) - I_5(\bar{s}e^{-i\tau}, +\beta, X)) \]

\[ = 2i\bar{s} \sin \beta \left( \frac{-1}{2\pi i} \int_{-\pi}^{\pi} d\beta' |\partial_{\beta'} \ln(1 - |\beta'|)| \int_0^{\tilde{s}\delta} e^{-iv\theta'_{\beta'} X} g(\frac{\bar{s}}{\bar{s}'}, -\beta', X) \right) \] (2.82)
\[ \times \theta(1/8 - |\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')}|) \left( \frac{e^{i(-\tau_1 - \beta')}}{(\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')})(\tilde{s}' - \tilde{s}e^{i(-\tau_1 + \beta - \beta')})} \right) ds' \]

\[ + 2i\tilde{s} \sin \beta \frac{\theta(\tilde{s} - 1)(\tilde{s} \sin \beta) - 1}{2\pi i} \int_{-\pi}^{\pi} d\beta' \theta(1 - \gamma |\beta'|) \int_{0}^{\bar{\alpha}} e^{-i\tilde{p}(\tilde{s}',\beta',X)} g \left( \frac{\tilde{s}'}{\sigma}, -\beta', X \right) \]

\[ \times \theta(1/8 - |\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')}|) \left( \frac{e^{i(-\tau_1 - \beta')}}{(\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')})(\tilde{s}' - \tilde{s}e^{i(-\tau_1 + \beta - \beta')})} \right) ds' \]

\[ + 2i\tilde{s} \sin \beta \frac{\theta(\tilde{s} - 1)(\tilde{s} \sin \beta) - 1}{2\pi i} \int_{-\pi}^{\pi} d\beta' \theta(1 - \gamma |\beta'|) \int_{0}^{\bar{\alpha}} e^{-i\tilde{p}(\tilde{s}',\beta',X)} g \left( \frac{\tilde{s}'}{\sigma}, -\beta', X \right) \]

\[ \times \left[ 1 - \theta(1/8 - |\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')}|) - \theta(1/8 - |\tilde{s}' - \tilde{s}e^{i(-\tau_1 + \beta - \beta')}|) \right] \]

\[ \times \left( \frac{e^{i(-\tau_1 - \beta')}}{(\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')})(\tilde{s}' - \tilde{s}e^{i(-\tau_1 + \beta - \beta')})} \right) ds' \]

\[ \equiv A'_1 + A'_2 + A'_3. \]

Thanks to $\tilde{s} > 1$, $|\tilde{s} \sin \beta| > 1/4$, $|\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')}| < 1/8$, and Lemma 2.2, one has

\[ |A'_1| \leq C\varepsilon_0|g|_{L^\infty(D_{\kappa_1})}|\tilde{s} \sin \beta|. \]  \hspace{1cm} (2.83)

By analogy,

\[ |A'_2| \leq C\varepsilon_0|g|_{L^\infty(D_{\kappa_1})}|\tilde{s} \sin \beta|. \]  \hspace{1cm} (2.84)

Since

\[ \left| 1 - \theta(1/8 - |\tilde{s}' - \tilde{s}e^{i(-\tau_1 - \beta - \beta')}|) - \theta(1/8 - |\tilde{s}' - \tilde{s}e^{i(-\tau_1 + \beta - \beta')}|) \right| \leq \frac{C}{(\tilde{s}')^2}. \]

We have

\[ |A'_3| \leq C\varepsilon_0|g|_{L^\infty(D_{\kappa_1})}|\tilde{s} \sin \beta|. \]  \hspace{1cm} (2.85)

Therefore, (2.78) is justified by (2.52)–(2.85).

**Step 2 (Proof for (2.79))**: From (2.78), for $\tilde{s} > 1$, $\tau_1$ defined by (2.32) for $\beta \in [0, \pi]$,

\[ \left| \frac{C_{\kappa_1 + \tilde{s}e^{-i(\tau_1 + \beta)}} T E_{\kappa_1} g - C_{\kappa_1 + \tilde{s}e^{-i(\tau_1 - \beta)}} T E_{\kappa_1} g}{\tilde{s}e^{-i\tau_1} - \tilde{r}e^{i(\alpha + \beta)}} \right| \leq C\varepsilon_0|g|_{L^\infty(D_{\kappa_1})}|\sin \beta|^\mu. \]  \hspace{1cm} (2.86)

Along with an improper integral and the change of variables (2.65), yields

\[ \left| - \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{\Gamma_4} d\beta \theta(1 - \gamma |\beta|) \int_{0}^{\tilde{s}'} 3_4|_{L^\infty(D_{\kappa_1})} \right| \leq (C\varepsilon_0)^2 |g|_{L^\infty(D_{\kappa_1})} \int_{\Gamma_4} d\beta \int_{0}^{\pi} e^{-\tilde{s}'\sin \tau_1 \sin \beta}|\sin \beta|^\mu d\tilde{s}'. \]

Adapting the above argument,

\[ \left| - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{\Gamma_5} d\beta \theta(1 - \gamma |\beta|) \int_{0}^{\tilde{s}'} 3_4|_{L^\infty(D_{\kappa_1})} \right| \leq (C\varepsilon_0)^2 |g|_{L^\infty(D_{\kappa_1})}. \]  \hspace{1cm} (2.88)
\[ \partial_{\tilde{s}} \varphi(\frac{\tilde{s}}{\beta}, \beta, X) \]

| Case | \( \partial_{\tilde{s}} \varphi(\frac{\tilde{s}}{\beta}, \beta, X) \) | \( \tilde{s}_*(\beta, X) \) |
|------|------------------------------------------------|-----------------|
| (21) | \( 2 \sin 2\beta(\tilde{s} - \tilde{s}_*) \) | \(- \frac{X_1}{2X_2} \sin \beta \) |
| (31) | \( 3 \sin 3\beta(\tilde{s} + \tilde{s}_*)(\tilde{s} - \tilde{s}_*) \) | \( \sqrt{- \frac{X_1}{3X_3} \sin \beta} \) |
| (32) | \( 3 \sin 3\beta\tilde{s}(\tilde{s} - \tilde{s}_*) \) | \(- \frac{2X_2}{3X_3} \sin 2\beta \) |
| (23) | \( \frac{3\sqrt{X_2}}{X_3} \sin 3\beta(\tilde{s} - \tilde{s}_*) \) | \(- \frac{2X_2^2}{3X_3^2} \sin 2\beta \) |
| (12) | \( \frac{2\sqrt{X_2}}{X_3} \sin 2\beta(\tilde{s} - \tilde{s}_*) \) | \( \frac{X_1}{2X_2} \sin \beta \) |
| (13) | \( \frac{3\sqrt{X_3}}{X_1} \sin 3\beta(\tilde{s} + \tilde{s}_*)(\tilde{s} - \tilde{s}_*) \) | \( \sqrt{- \frac{X_1}{3X_3} \sin \beta} \) |

Table 2.1. Stationary points \( \tilde{s}_* \) for degenerated homogeneous cases

Proposition 2.2. Suppose \( S = \{z_n, \kappa_j, D, s_c(\lambda)\} \) is an admissible scattering data and \( E_{\kappa_j}f \) is holomorphic. For the linear homogeneous cases and \( n \geq 2 \),

\[
|(CTE_{\kappa_j})^n f|_{\mathfrak{M}(D_{\kappa_j})} \leq C\epsilon_0 |(CTE_{\kappa_j})^{n-1} f|_{\mathfrak{M}(D_{\kappa_j})} + C\epsilon_0^2 |(CTE_{\kappa_j})^{n-2} f|_{\mathfrak{M}(D_{\kappa_j})}. \tag{2.89}
\]

Consequently, for \( n \geq 0 \),

\[
|(CTE_{\kappa_j})^n f|_{\mathfrak{M}(D_{\kappa_j})} \leq (C\epsilon_0)^n |f|_{\mathfrak{M}(D_{\kappa_j})}. \tag{2.90}
\]

**Proof.** With the help of Lemma 2.5, we can adapt the approach in the proof of Proposition 2.1 to derive estimates for this proposition.

\[ \square \]

2.3. Degenerated non homogeneous cases. We first have the classification

\[
\begin{align*}
(12) & \quad \sqrt{|X_2|} \leq |X_1|, & X_3 &= 0; \\
(13) & \quad \sqrt{|X_3|} \leq |X_1|, & X_2 &= 0; \\
(23) & \quad \sqrt{|X_3|} \leq \sqrt{|X_2|}, & X_1 &= 0;
\end{align*}
\]

Define \( \tilde{\sigma} \) by Definition 2 and scaling coordinates by (2.6). One has Table 2.1. Hence beside the fixed critical point \( \tilde{s} = 0 \) (i.e. \( \tilde{\zeta} = \kappa_1 \)), extra techniques are demanded near movable stationary points \( \tilde{s}_* > 0 \) such that \( \partial_{\tilde{s}} \varphi(\frac{\tilde{s}}{\beta}, \beta, X) = 0 \). We shall sort (2.91) into two categories and search for an approach which is consistent with that for homogeneous cases and can be applied to all degenerated non homogeneous cases and fully non homogeneous cases.

- Case (21), (31), (32), (23): In this category, we shall introduce new coordinates near \( \tilde{s}_* \) and find contours such that (2.33), (2.34) type conditions hold. Argument
in Lemma 2.2 can be adopted to derive estimates outside compact neighborhood

• at $\tilde{s}_*$.  

- Case (12), (13): Following the above approach won’t achieve uniform estimates, up to $\sin k\beta$ factors, such as (2.33), (2.34) type conditions. We shall introduce new dilated affine coordinates near $\tilde{s}_*$ to overcome the difficulty.

**Lemma 2.6.** Given an admissible scattering data $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$, for the degenerate non homogeneous Case (21), (31), (32), (23) and $E_{\kappa_j}f$ holomorphic,

$$|(CT E_{\kappa_j})_n f|_{\mathfrak{M}(D_{\kappa_j})} \leq (C\epsilon_0)^n |f|_{\mathfrak{M}(D_{\kappa_j})}, \quad \forall n \geq 0.$$  

**Proof.** For simplicity and WLOG, we only give a proof assuming (2.16) holds and reduce the proof to estimating principal parts. Define $\tilde{\sigma}$ and scaled coordinates by Definition 2, and decompose the principal part of the CIO by (2.10)-(2.15). Thanks to

$$|\partial_{\tilde{\sigma}} \varphi(\frac{\tilde{s}}{\tilde{\sigma}}, \beta, X)| \leq C, \quad \forall \tilde{s} < 2,$$

estimates on compact domains $I_1$-$I_3$ can be derived via the same approach in the proof of Lemma 2.3. As for $I_4$ and $I_5$, we need to dwell more on the deformation method.

**Step 1 (Deformation):** Let the stationary points $\tilde{s}_* = \tilde{s}_*(\beta, X)$ defined by Table 2.1.

Define the essential critical points $\tilde{s}_{j,*} = \tilde{s}_{j,*}(\beta, X), j = 0, 1$, defined by

$$\tilde{s}_{0,*} \equiv 0; \quad \tilde{s}_{1,*} \equiv \begin{cases} - \tilde{s}_* < 0, \\
\tilde{s}_* > 0 \end{cases},$$

where $-$ means no definition. Given $\epsilon_1 < \frac{\pi}{2k} \ll 1$, define $\bar{\mathcal{U}}_j^\sharp(\beta, X) \supset \mathcal{U}_j(\beta, X)$ by

$$\mathcal{U}_0 \equiv \begin{cases} \{0 \leq \tilde{s} \leq \tilde{\sigma}\delta\}, \\
\{0 \leq \tilde{s} \leq \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}\}, \quad \tilde{s}_* < 0, \\
\{\frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*} \leq \tilde{s} \leq \tilde{\sigma}\delta\} \equiv \mathcal{U}_{1<} \cup \mathcal{U}_{1>}, \quad \tilde{s}_* > 0,
\end{cases}$$

$$\mathcal{U}_1 \equiv \begin{cases} \tilde{\sigma}, \\
\{1 - \frac{1}{2\cos \epsilon_1}\tilde{s}_{1,*} \leq \tilde{s} \leq \tilde{s}_{1,*}\} \cup \{\tilde{s}_{1,*} \leq \tilde{s} \leq \tilde{\sigma}\delta\} \equiv \mathcal{U}_{1<} \cup \mathcal{U}_{1>}, \quad \tilde{s}_* < 0, \\
\{0 \leq \tilde{s} \leq \tilde{s}_{1,*}\}\setminus \mathcal{U}_{1<} \equiv \mathcal{U}_0^\circ, \\
\{0 \leq \tilde{s} \leq \tilde{s}_{1,*}\} \cup \{\tilde{s}_{1,*} \leq \tilde{s} \leq \tilde{\sigma}\delta\} = \mathcal{U}_1^\circ \cup \mathcal{U}_1^\circ.
\end{cases}$$

Write

$$\lambda = \kappa_1 + \frac{\tilde{r}_j e^{i\alpha}}{\tilde{\sigma}} = \kappa_1 + \frac{\tilde{s}_{j,*} e^{i\beta} + \tilde{r}_j e^{i\alpha_j}}{\tilde{\sigma}},$$

$$\tilde{r}_j = \tilde{r}_j(\beta, X, \lambda), \quad \alpha_j = \alpha_j(\beta, X, \lambda), j = 0, 1.$$
Due to Table 2.2 and Figure 3, we define the deformation on $\mathcal{U}_j$: 

\[
\tilde{s} \mapsto \xi_j \equiv \tilde{s}_j e^{i\tau_j} + \tilde{s}_{j,*}, \quad \tilde{s} \equiv \pm \tilde{s}_j + \tilde{s}_{j,*} \in \mathcal{U}_j^z \text{ if } |\tau_j| \leq \frac{\pi}{2}, \quad \tilde{s}_j \geq 0,
\]

where if $\tilde{s}_* < 0$, 

\[
\tau_0 = \begin{cases} 
[-\epsilon_1, 0], & \text{for } \sin k\beta > 0, \ |\alpha - \beta| \leq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
[-\frac{\alpha}{4}, 0], & \text{for } \sin k\beta > 0, \ |\alpha - \beta| \geq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
[0, +\epsilon_1], & \text{for } \sin k\beta < 0, \ |\alpha - \beta| \leq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
[0, +\frac{\alpha}{4}], & \text{for } \sin k\beta < 0, \ |\alpha - \beta| \geq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, 
\end{cases}
\]

and 

\[
\tau_0,^+ = \begin{cases} 
\mp \epsilon_1, & \text{for } \sin k\beta \geq 0, \ |\alpha - \beta| \leq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
\mp \frac{\alpha}{4}, & \text{for } \sin k\beta \geq 0, \ |\alpha - \beta| \geq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, 
\end{cases}
\]

with $k = \deg \varphi$; if $\tilde{s}_* > 0$, 

\[
\tau_0 \in \begin{cases} 
[0, +\epsilon_1], & \text{for } \sin k\beta > 0, \ |\alpha_0 - \beta| \leq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
[0, +\frac{\alpha}{4}], & \text{for } \sin k\beta > 0, \ |\alpha_0 - \beta| \geq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
[-\epsilon_1, 0], & \text{for } \sin k\beta < 0, \ |\alpha_0 - \beta| \leq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, \\
[-\frac{\alpha}{4}, 0], & \text{for } \sin k\beta < 0, \ |\alpha_0 - \beta| \geq \frac{\alpha}{2}, \ \tilde{s} \in \mathcal{U}_0^z, 
\end{cases}
\]
Therefore, for $j = 0, 1$,

$$|\tilde{s}_j e^{i\tau_j} - \tilde{r}_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max \{\tilde{r}_j, \tilde{s}_j\},$$

(2.98)

and

$$\Re(-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)) \leq -\frac{1}{C}|\sin k\tau_j \sin k\beta|\tilde{s}^2_j, \quad \tilde{s} \in \mathcal{V}_j \text{ for Case (21), (31), (32)},$$

$$\Re(-i\varphi(\frac{\xi_j}{\sigma}, \beta, X)) \leq -\frac{1}{C}|\sin 2\tau_j \sin 2\beta|\tilde{s}^2_j, \quad \tilde{s} \in \mathcal{V}_j \text{ for Case (23)}.$$  

(2.99)
Step 2 (Proof for $I_4$, $n = 1$) : From (2.97), the holomorphic property of $E_{\kappa_j}f$, and a residue theorem,

\[
|I_4|c_{\kappa_1}^{\bullet}(D_{\kappa_1}) \leq \sum_{\nu=1,2} \left| \frac{\theta(1-\tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1-\gamma|\beta|)] \left\{ \left( \int_{S_\prec} + \int_{\Gamma_{40}} \right) \frac{e^{-i\phi(\xi_0,\beta, X)} f(\xi_0, -\beta, X) e^{i(\alpha_0-\beta)f})}{\nu} d\xi_0 \right\} \right.
\]

\[
+ \int_{\Gamma_{41}} \frac{e^{-i\phi(\xi_0,\beta, X)} f(\xi_0, -\beta, X) e^{i(\alpha_0-\beta)f})}{\nu} d\xi_0 + \int_{S_\succ} \frac{e^{-i\phi(\xi_0,\beta, X)} f(\xi_0, -\beta, X) e^{i(\alpha_0-\beta)f})}{\nu} d\xi_0 \right\} \]

with

\[
S_\prec(\beta, X, \lambda) = \{ \xi_0 : \tilde{s} = 2, \tau_0 \text{ defined by (2.97)} \},
\]

\[
\Gamma_{40}(\beta, X, \lambda) = \{ \xi_0 : \tilde{s} \in (2, \tilde{s} \delta) \cap \bar{U}_0(\beta, X), \tau_0 = \tau_{0,\dagger} \},
\]

\[
\Gamma_{41}(\beta, X, \lambda) = \{ \xi_1 : \tilde{s} \in (2, \tilde{s} \delta) \cap \bar{U}_1(\beta, X), \tau_1 = \tau_{1,\dagger} \},
\]

\[
S_\succ(\beta, X, \lambda) = \{ \xi_h : h = \sup_{\tilde{\omega}_j \neq \phi} \tilde{s}_j, \tilde{s} = \tilde{s}_\delta, \tau_h \text{ defined by (2.97)} \},
\]

with $\xi_j$, $\tau_{j,\dagger}$, $\bar{U}_j = U_j(\beta, X)$ defined by (2.97), (2.98).

From $\tilde{r} < 1$, $\epsilon_1 > 0$, $f \in \mathcal{M}(D_{\kappa_1})$, (2.16), (2.98)-(2.101), changes of variables

\[
\tilde{s}_j \mapsto \tilde{t}_j = s_j |\sin k|^{1/k} \quad \text{for } \Gamma_{4j}, \text{ Case (21), (31), (32)},
\]

\[
\tilde{s}_j \mapsto \tilde{t}_j = s_j |\sin 2\beta|^{1/2} \quad \text{for } \Gamma_{4j}, \text{ Case (23)},
\]

and improper integrals, one obtains

\[
|I_{4j}|_{\mathcal{M}(D_{\kappa_1})} = |I_4|c_{\kappa_1}^{\bullet}(D_{\kappa_1}) \leq C\epsilon_0 |f|_{\mathcal{M}(D_{\kappa_1})}
\]

\[
+ C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})} \int_{-\pi}^{\pi} d\beta \int_0^{\infty} \frac{1}{k |\sin k\beta|} e^{-t|\sin k\tau_1|} dt |_{L^\infty(D_{\kappa_1})} \leq C\epsilon_0 |f|_{\mathcal{M}(D_{\kappa_1})}, \quad (2.103)
\]
From (2.97), the holomorphic property of $S_I$.

**Step 3 (Proof for $I_5$, $n = 1$):** From (2.97), the holomorphic property of $f$, and the residue theorem,

$$I_5 = -\frac{\theta(\bar{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \left\{ \int_{\Gamma_{50}} \frac{e^{-i\varphi(\xi_0, \beta, X)} f(\xi_0, -\beta, X)}{s_0 e^{i\tau_0} - \bar{r}_0 e^{i(\alpha - \beta)}} d\xi_0 \right. \right.$$

$$+ \left. \int_{\Gamma_5} \frac{e^{-i\varphi(\xi_1, \beta, X)} f(\xi_1, -\beta, X)}{s_1 e^{i\tau_1} - \bar{r}_1 e^{i(\alpha - \beta)}} d\xi_1 \right. + \left. \int_{S_o} \frac{e^{-i\varphi(\xi_h, \beta, X)} f(\xi_h, -\beta, X)}{s_h e^{i\tau_h} - \bar{r}_h e^{i(\alpha - \beta)}} d\xi_h \right. \right.$$

$$- \theta(\bar{r} - 1)\theta(1 - \frac{1}{4}) \int_{\alpha - \beta \in \Phi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \text{sgn}(\tau_1(\beta))$$

$$\times e^{-i\varphi(\xi, \beta, X)} f(\bar{r} e^{i(\alpha - \beta)}, -\beta, X),$$

where $S_o$ defined by (2.101), and $\Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda), j = 0, 1$, defined by

$$\Gamma_{50} = \{ \xi_0 : \tilde{s} \in \mathcal{U}_0, \tau_0 = \tau_{0,\dagger}\};$$

$$\Gamma_{51} = \Gamma_{51,\text{out}} \cup S_{51} \cup \Gamma_{51,\text{in}},$$

with

$$\Gamma_{51,\text{out}} = \left\{ \begin{array}{ll}
\{ \xi_1 : \tilde{s} \in \mathcal{U}_1, \tau_1 = \tau_{1,\dagger}\}, & \bar{r}_1 > \frac{1}{4}, \\
\{ \xi_1 : \tilde{s} \in \mathcal{U}_1, \tau_1 = \tau_{1,\dagger}, |\tilde{s}_1 - \bar{r}_1| > 1/2\}, & \bar{r}_1 < \frac{1}{4}, \\
\phi, & \bar{r}_1 > \frac{1}{4}, \\
\phi, & \bar{r}_1 < \frac{1}{4}, \\
\tau_1 = \pi \text{ on } \mathcal{U}_{1d}, |\tilde{s}_1 - \bar{r}_1| < 1/2, & \bar{r}_1 > \frac{1}{4}, \\
\{ \xi_1 : \tau_1 \text{ defined by (2.97), } |\tilde{s}_1 - \bar{r}_1| = 1/2\}, & \bar{r}_1 < \frac{1}{4}, \\
\phi, & \bar{r}_1 > \frac{1}{4}, \\
\phi, & \bar{r}_1 < \frac{1}{4}, \\
\end{array} \right.$$}

and $\alpha_j, \xi_j, \tau_{j,\dagger}, \mathcal{U}_j = \mathcal{U}_j(\beta, X)$ defined by (2.94) - (2.97).

Using $\bar{r} > 1$, $\epsilon_1 > 0$, $f \in \mathcal{M}(D_{\kappa_1})$, (2.16), (2.98), (2.99), (2.104), (2.105), changes of variables (2.102), and improper integrals,

$$|I_5|_{\mathcal{M}(D_{\kappa_1})} \leq C\epsilon_0 |f|_{\mathcal{M}(D_{\kappa_1})} + \int_{\Gamma_{51,\text{in}} \cup S_{51}} \frac{e^{-i\varphi(\xi_1, \beta, X)} f(\xi_1, -\beta, X)}{s_1 e^{i\tau_1} - \bar{r}_1 e^{i(\alpha - \beta)}} d\xi_1 |_{\mathcal{M}(D_{\kappa_1})}. \quad (2.106)$$
By $\tilde{r} > 1$, compactness, $f \in \mathcal{M}(D_{\kappa_1})$, (2.98), (2.99), (2.105),
\[
| - \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - |\beta|)] \int_{\mathfrak{M}} e^{-ip(\frac{\xi}{\tilde{r}}, \beta, X)} f(\frac{\xi}{\tilde{r}}, \beta, X) \frac{\tilde{s}_1 e^{i\tau_1} - \tilde{r}_1 e^{i(\alpha_1 - \beta)}}{d\xi_1|_{\mathfrak{M}(D_{\kappa_1})}} \leq C \epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]
Step 2 (Proof of Proposition 2.1) by using the deformation in (2.92), estimates on compact domains.
Step 3 (Step 2 and Step 3). Thanks to (2.106)-(2.108), estimates on compact domains.
Step 4 (Proof for $I_4$, $I_5$, $n > 1$): Estimates can be derived via the approach of the proof of Proposition 2.1 by using the deformation in Step 2 and Step 3. We skip details for simplicity.

Lemma 2.7. Given an admissible scattering data $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$, for the degenerated non homogeneous Case (12), (13), and holomorphic $E_{\kappa_j} f$,
\[
|(CTE_{\kappa_j})^n f|_{\mathcal{M}(D_{\kappa_j})} \leq (C \epsilon_0)^n |f|_{\mathcal{M}(D_{\kappa_j})}, \quad \forall n \geq 0.
\]
Proof. For simplicity and without loss of generality, we only give a proof assuming (2.16) and reduce the proof to estimating principal parts. Define the parameter $\sigma_0 = \tilde{\sigma}$ by Definition 2 and decompose the principal part of the CIO into (2.10)-(2.15). Thanks to (2.92), estimates on compact domains $I_1-I_3$ can be derived by the approach in Lemma 2.3.
Step 1 (Deformation): For $I_4$ and $I_5$, let $\tilde{s}_j, \tilde{\sigma}_j(\beta, X), \tilde{\sigma}_j^2(\beta, X), j = 0, 1, \tilde{\sigma}_{1, \geq}(\beta, X), \tilde{\sigma}_{1, \geq}(\beta, X)$ defined by (2.93), (2.94), (2.95). Let $\sigma_j$ defined by
\[
\sigma_0 = \tilde{\sigma}, \quad \sigma_1 = k |X|, \quad k = \deg(\phi).
\]
Write
\[
\lambda = \kappa_1 + \frac{\tilde{r}}{\tilde{\sigma}} e^{i\alpha} = \kappa_1 + \frac{s_j e^{i\beta} + r_j e^{i\alpha_j}}{\sigma_j},
\]
\[
s_j = \frac{\sigma_j}{\sigma_0} \quad r_j = r_j(\beta, X, \lambda) \geq 0, \quad \alpha_j = \alpha_j(\beta, X, \lambda),
\]
with $\tilde{s}_{j,*}$ defined by (2.93). Thus

$$\inf_{\beta} s_{1,*} \geq \frac{1}{4} \tag{2.112}$$

for Case (12) and (13) by Table 2.1. 

In view of Table 2.3 and Figure 3, define the deformation

$$\frac{\tilde{s}}{\sigma} \rightarrow \frac{\vartheta_j}{\sigma_j} \equiv \frac{s_j e^{i\tau_j} + s_{j,*}}{\sigma_j},$$

$$\tilde{s} \equiv (\pm s_j + s_{j,*}) \frac{\sigma_0}{\sigma_j} \in \mathfrak{U}_2(\beta, X), \text{ if } |\tau_j| \leq \frac{\pi}{2}, \ s_j \geq 0$$

where $\tau_0$, $\tau_{0,\dagger}$ are defined as in (2.97) for $k\epsilon_1 \leq \frac{\pi}{2}$, $\deg \varphi = k$. Therefore,

$$|s_j e^{i\tau_{j,\dagger}} - r_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max \{r_j, s_j\}, \tag{2.114}$$

and

$$\Re(-i\varphi(\frac{\vartheta_0}{\sigma_0}, \beta, X)) \leq -|\sin k\tau_0 \sin \beta|s_0, \ \tilde{s} \in \mathfrak{U}_0(\beta, X), \tag{2.115}$$

$$\Re(-i\varphi(\frac{\vartheta_1}{\sigma_1}, \beta, X)) \leq -|\sin k\tau_1 \sin k\beta|s_1^k, \ \tilde{s} \in \mathfrak{U}_1(\beta, X). \tag{2.116}$$

**Step 2 (Proof for $I_4$, $n = 1$)**: From (2.113), the holomorphic property of $E_{\kappa_j}f$, and a residue theorem,

$$|I_4|_{\mathfrak{N}(D_{s_1})} \leq \frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^{\pi} d\beta |\partial_\beta \ln(1 - \gamma|\beta|)| \left( \int_{S_<} + \int_{\Gamma_{40}} \right) \mathfrak{I}_<|_{\mathfrak{N}(D_{s_1})}$$

$$\quad + \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta |\partial_\beta \ln(1 - \gamma|\beta|)| \int_{\Gamma_{41}} e^{-i\varphi(\frac{\vartheta_1}{\sigma_1}, \beta, X)} \frac{f(\frac{\vartheta_1}{\sigma_1}, -\beta, X)}{s_1 e^{i\tau_1} - r_1 e^{i(\alpha_1 - \beta)}} d\vartheta_1|_{\mathfrak{N}(D_{s_1})}$$

$$\quad + \frac{\theta(1 - \tilde{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta |\partial_\beta \ln(1 - \gamma|\beta|)| \int_{S_> \mathfrak{J}} \mathfrak{J}>|_{\mathfrak{N}(D_{s_1})},$$

where

$$S_<(\beta, X, \lambda) = \{\vartheta_0 : \tilde{s} = 2, \tau_0 \text{ defined by } (2.113)\}.$$

| Case | $\tilde{s} \in \mathfrak{U}_0$ | $\tilde{s} \in \mathfrak{U}_1$ |
|------|-----------------|-----------------|
| (12) | $\Re(-i\varphi(\frac{\vartheta_0}{\sigma_0}, \beta, X))$ | $\Re(-i\varphi(\frac{\vartheta_0}{\sigma_0}, \beta, X))$ |
|      | $= \frac{\vartheta_0}{\vartheta_1} \sin 2\tau_0 \sin 2\beta \tilde{s}(\tilde{s} - \frac{\sin \tau_0}{\sin 2\tau_0} \tilde{s}_{1,*})$ | $= \sin 2\tau_1 \sin 2\beta \tilde{s}^2_1$ |
| (13) | $\Re(-i\varphi(\frac{\vartheta_1}{\sigma_1}, \beta, X))$ | $\Re(-i\varphi(\frac{\vartheta_1}{\sigma_1}, \beta, X))$ |
|      | $= \frac{\vartheta_1}{\vartheta_0} \sin 3\tau_0 \sin 3\beta \tilde{s}(\tilde{s}^2 - \frac{3\sin \tau_0}{\sin 3\tau_0} \tilde{s}_{1,*}^2)$ | $= \sin 3\tau_1 \sin 3\beta \tilde{s}_1^2(s_1 + \frac{3\sin 2\tau_1}{\sin 3\tau_1} \tilde{s}_{1,*})$ |
\[ \Gamma_{40}(\beta, X, \lambda) = \{ \vartheta_0 : \bar{s} \in (2, \bar{s}_0) \cap \mathcal{U}_0(\beta, X), \quad \tau_0 = \tau_{0,1} \}, \]
\[ \Gamma_{41}(\beta, X, \lambda) = \{ \vartheta_j : \bar{s} \in (2, \bar{s}_0) \cap \mathcal{U}_1(\beta, X), \quad \tau_1 = \tau_{1,1} \}, \]
\[ S_>(\beta, X, \lambda) = \{ \vartheta_h : h = \sup_j, \bar{s} = \bar{s}_0, \quad \tau_h \text{ defined by (2.97)} \}, \]

with \( \vartheta_j(\beta, X), \tau_1, \mathcal{U}_j = \mathcal{U}_j(\beta, X) \) defined by (2.113), (2.94); 

\[ J_< = \begin{cases} \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5, & \text{if } 0 < \beta < \epsilon_1/8, \\ 0, & \text{if } -\epsilon_1/8 < \beta < 0, \\ e^{-i\pi(\frac{\pi}{2}, \beta, X)} f(\frac{\pi}{2}, -\beta, X) \frac{s}{s_0 e^{i\tau_0 - r_0 e^{i(\alpha_0 - \beta)}}} d\vartheta_0, & \text{if } |\beta| > \epsilon_1/8; \end{cases} \]
\[ J_> = \begin{cases} \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5, & \text{if } h = 0, 0 < \beta < \epsilon_1/8, \\ 0, & \text{if } h = 0, -\epsilon_1/8 < \beta < 0, \\ e^{-i\pi(\frac{\pi}{2}, \beta, X)} f(\frac{\pi}{2}, -\beta, X) \frac{s_1}{s_1 e^{i\tau_1 - r_1 e^{i(\alpha_1 - \beta)}}} d\vartheta_1, & \text{if } h = 1, \end{cases} \]

From (2.112), if \( \bar{r} < 1 \), then \( \tau_1 > 1 \). Along with (2.16), (2.112), (2.114), (2.116), similar analysis as that in Lemma 2.3 yields 

\[ \frac{\theta(1 - \bar{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{40}} e^{-i\pi(\frac{\pi}{2}, \beta, X)} f(\frac{\pi}{2}, -\beta, X) \frac{s}{s_0 e^{i\tau_0 - r_0 e^{i(\alpha_0 - \beta)}}} d\vartheta_0 |_{\gamma(\mathcal{D}_{\epsilon_1})} \]
\[ + \frac{\theta(1 - \bar{r})}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{S_>} \mathcal{J}_{>_{\gamma(\mathcal{D}_{\epsilon_1})}} \leq C \epsilon_0 |f|_{L^\infty(\mathcal{D}_{\epsilon_1})}. \quad (2.120) \]
On $\Gamma_{40}$, it suffices to consider $|\beta| \leq \epsilon_1/8$. Note that (J3), (J4), (J5) for linear homogeneous cases in Lemma 2.4 can be adapted to derive estimates for Case (12) and (13) on $\mathcal{U}_0$. As for (J1,J2), in view of Table 2.1:

$$|\varphi(\frac{\eta_0}{\sigma_0}, \beta, X)| \leq C|\sin \beta| s_0, \quad \text{for } \tilde{s} \in \mathcal{U}_0(\beta, X) \text{ and } |\beta| < \frac{\epsilon_1}{8}. \quad (2.121)$$

Consequently, (2.64) can be generalized to

$$\left|\frac{e^{-i\rho(\frac{\eta_0}{\sigma_0}, \pm \beta, X)} - 1}{s e^{i\pi t} - r e^{i(\alpha - \beta)}}\right| \leq C \left|\frac{\tilde{s} \sin \beta}{s e^{i\pi t} - r e^{i(\alpha - \beta)}}\right| \leq C|\sin \beta| \quad (2.122)$$

for $\tilde{s} \in \mathcal{U}_0(\beta, X) \text{ and } |\beta| < \frac{\epsilon_1}{8}$.

Applying (2.114), (2.115), (2.122), and similar analysis as that in Lemma 2.4 one obtains

$$\left|\frac{\theta(1 - \tilde{r})}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta(\frac{\epsilon_1}{8} - |\beta|) \int_{\Gamma_{40}} (\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4 + \mathcal{J}_5)\right|_{\Re(D_{s_1})}$$

$$\leq C\epsilon_0 |f|_{L^\infty(D_{s_1})}. \quad (2.123)$$

As a result,

$$|I_4|_{\Re(D_{s_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{s_1})}. \quad (2.124)$$

**Step 3 (Proof for $I_5$, $n = 1$):** From (2.113), the holomorphic property of $E_{\kappa,j} f$, and the residue theorem,

$$I_5 = -\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_0^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{50}} \mathcal{J}_<$$

$$-\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{51}} e^{\frac{i\eta_0}{\sigma_0} \beta, X} f(\frac{\eta_0}{\sigma_0}, -\beta, X) \frac{\partial_1}{\partial_1} - r_1 e^{i(\alpha_1 - \beta)} d\partial_1$$

$$-\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^\pi d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{S>} \mathcal{J}_>$$

$$-\theta(\tilde{r} - 1) \theta(1 - \frac{1}{4}) \int_{\alpha_\beta \in \Theta} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \text{sgn}(\tau_1(\beta))$$

$$\times e^{-i\rho(\frac{\eta_0}{\sigma_0}, \beta, X)} f(\frac{\eta_0}{\sigma_0}, -\beta, X)\}.$$

where $\mathcal{J}_<$, $\mathcal{J}_>$ are defined by (2.119), $S_>$ defined by (2.118), and $\Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda)$, $j = 0, 1$, defined by

$$\Gamma_{50} = \{\theta_0 : \tilde{s} \in \mathcal{U}_0, \tau_0 = \tau_{0,1}\};$$

$$\Gamma_{51} = \Gamma_{51,\text{out}} \cup S_{51} \cup \Gamma_{51,\text{in}}; \quad (2.126)$$

with
\[\Gamma_{51,\text{out}} = \begin{cases} \{ \vartheta_1 : \tilde{s} \in \mathcal{O}_1, \tau_1 = \tau_{1,\dagger} \}, & r_1 > \frac{1}{16}, \\ \{ \vartheta_1 : \tilde{s} \in \mathcal{O}_1, \tau_1 = \tau_{1,\dagger}, |s_1 - r_1| > 1/8 \}, & r_1 < \frac{1}{16}, \\ \phi, & r_1 > \frac{1}{16}, \end{cases}\]

\[\Gamma_{51,\text{in}} = \begin{cases} \{ \vartheta_1 : \tilde{s} \in \mathcal{O}_1, \tau_1 = 0 \text{ on } \mathcal{O}_{1,>}, \tau_1 = \pi \text{ on } \mathcal{O}_{1,<}, |s_1 - r_1| < 1/8 \}, & r_1 < \frac{1}{16}, \\ \phi, & r_1 > \frac{1}{16}, \end{cases}\]

\[S_{51} = \begin{cases} \{ \vartheta_1 : \tau_1 \text{ defined by (2.97)}, |s_1 - r_1| = 1/8 \} & r_1 > \frac{1}{16}, \\ \phi, & r_1 < \frac{1}{16}, \end{cases}\]

and \(\alpha_j, \vartheta_j, \tau_{j,\dagger}, \mathcal{O}_j = \mathcal{O}_j(\beta, X)\) defined by (2.94)-(2.97).

Applying (2.16), (2.114), (2.116), (2.112) (used in estimating integrals on \(\Gamma_{51,\text{in}}\) in particular), (2.126), similar analysis as that in Lemma 2.3, one obtains

\[
\begin{align*}
\left| \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta \partial_{\beta} \ln(1 - |\beta|) \right| \int_{\mathcal{G}_{51}} e^{-i\varphi(\frac{\alpha}{\sigma}, \beta, X)} f\left( \frac{\alpha}{\sigma}, -\beta, X \right) d\vartheta_1 & |_{\mathfrak{M}(D_{\kappa_1})} \\
+ \left| \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta \partial_{\beta} \ln(1 - |\beta|) \int_{S_>} J_1 |_{\mathfrak{M}(D_{\kappa_1})} \right. \\
+ \left| \frac{\theta(\tilde{r} - 1)}{2\pi i} \frac{\theta}{16} \int_{\alpha - \beta \in \Omega} d\beta \partial_{\beta} \ln(1 - |\beta|) \right| \text{sgn}(\tau_1(\beta)) \times e^{-i\varphi(\frac{\alpha}{\sigma}, \beta, X)} f\left( \frac{\alpha}{\sigma}, -\beta, X \right) \big|_{\mathfrak{M}(D_{\kappa_1})} \\
\leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\end{align*}
\] (2.127)

On \(\Gamma_{50}\), it suffices to consider \(|\beta| \leq \epsilon_1/8\). Applying (2.114), (2.115), (2.122), similar analysis as that in Lemma 2.4, one obtains

\[
\begin{align*}
\left| \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{0}^{\pi} d\beta \partial_{\beta} \ln(1 - |\beta|) \right| \theta \left( \frac{\epsilon_1}{8} - |\beta| \right) \int_{\Gamma_{50}} (J_1 + J_2 + J_3 + J_4 + J_5) |_{\mathfrak{M}(D_{\kappa_1})} \\
\leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\end{align*}
\] (2.128)

Therefore,

\[
|I_5|_{\mathfrak{M}(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\] (2.129)

**Step 4 (Proof for \(I_4, I_5, n > 1\))**: Estimates can be derived via the approach of the proof of Proposition 2.1, 2.2 by using the deformation in **Step 2** and **Step 3**. We skip details for simplicity.

\[\square\]
| Case | $\partial_x \varphi(\frac{x}{\sigma}, \beta, X)$ | Stationary points $\tilde{s}_\pm(\beta, X)$ |
|------|---------------------------------|---------------------------------|
| (F1) | $\sin \beta + 2 \frac{X_3}{X_1} \tilde{s} \sin 2\beta + 3 \frac{X_1}{X_3} \tilde{s}^2 \sin 3\beta$ | $-\frac{1+\sqrt{1-\frac{\beta^2}{\sin^2 \beta}}}{3 \frac{X_1}{X_3} \tilde{s} \sin^2 \beta}$ |
| (F2) | $\frac{X_1}{X_2} \sin \beta + 2 \tilde{s} \sin 2\beta + 3 \frac{X_1}{X_2} \tilde{s}^2 \sin 3\beta$ | $-\frac{1+\sqrt{1-\frac{\beta^2}{\sin^2 \beta}}}{3 \frac{X_1}{X_2} \tilde{s} \sin^2 \beta}$ |
| (F3) | $\frac{X_1}{X_3} \sin \beta + 2 \frac{X_3}{X_2} \tilde{s} \sin 2\beta + 3 \tilde{s}^2 \sin 3\beta$ | $-\frac{1+\sqrt{1-\frac{\beta^2}{\sin^2 \beta}}}{3 \frac{X_1}{X_3} \tilde{s} \sin^2 \beta}$ |

**Table 2.4.** Stationary points for fully non homogeneous cases

2.4. **Fully non homogeneous cases.** Fully non homogeneous cases can be classified as

(F1) $\sqrt{|X_3|} \leq \sqrt{|X_2|} \leq |X_1|$ or $\sqrt{|X_2|} \leq \sqrt{|X_3|} \leq |X_1|$;

(F2) $\sqrt{|X_3|} \leq |X_1| \leq \sqrt{|X_2|}$ or $|X_1| \leq \sqrt{|X_3|} \leq \sqrt{|X_2|}$;

(F3) $\sqrt{|X_2|} \leq |X_1| \leq \sqrt{|X_3|}$ or $|X_1| \leq \sqrt{|X_2|} \leq \sqrt{|X_3|}$.

(2.130)

Define the scaling parameter $\tilde{\sigma}$ by Definition 2 and scaling coordinates by (2.6). Thus we have Table 2.4 where

$$\Delta = 3 \frac{X_1 X_3 \sin \beta \sin 3\beta}{X_2^2 \sin^2 2\beta}. \quad (2.131)$$

For simplicity and without loss of generality, we only give a proof assuming (2.16) and reduce the proof to estimating principal parts. Let

$$\Omega_1 = \{0 \leq |\beta| \leq \frac{\pi}{3}\}, \quad \Omega_2 = \{\frac{\pi}{3} \leq |\beta| \leq \frac{\pi}{2}\}, \quad \Omega_3 = \{\frac{\pi}{2} \leq |\beta| \leq \frac{2\pi}{3}\}, \quad (2.132)$$

as is shown in Figure 3. The dynamics of the stationary points $\tilde{s}_\pm$, $\Delta$ can be further characterized by Table 2.5. Namely, for any fully non homogeneous case, we classify $(\beta, X)$ into $\mathfrak{A}, \ldots, \mathfrak{C}$ according to the determinant $\Delta$. Then properties of corresponding stationary points $\tilde{s}_\pm$ and $\beta$-domains are listed.

**Lemma 2.8.** Given an admissible scattering data $S = \{z_n, \kappa_j, \mathcal{D}, s_c(\lambda)\}$, for fully non homogeneous cases (F3), and holomorphic $E_{\kappa_j} f$,

$$|(CT E_{\kappa_j})^n f|_{\mathfrak{M}(D_{\kappa_j})} \leq (C\epsilon_0)^n |f|_{\mathfrak{M}(D_{\kappa_j})}.$$  

**Proof.** For simplicity and without loss of generality, we only give a proof assuming (2.16) and reduce the proof to estimating the principal part. Define the scaling parameter $\tilde{\sigma}$ by Definition 2, scaled coordinates by (2.6), and decompose the principal part into (2.10)–(2.15). Thanks to (2.92), estimates on compact domains, i.e., $I_j$, $j = 1, 2, 3$ can be derived as in Lemma 2.3.
Step 1 (Deformation): Let the generalized critical points \( \tilde{s}_{j,*} = \tilde{s}_{j,*}(\beta, X) \), \( j = 0, 1, 2 \) defined by

\[
\tilde{s}_{0,*} = 0, \quad \tilde{s}_{1,*} = \begin{cases} 
\frac{\tilde{s}_+ + \tilde{s}_-}{2} > 0, & \text{Type } \mathfrak{A}'' \text{,} \\
\inf \tilde{s}_+ > 0, & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
\sup \tilde{s}_+ > 0, & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
- & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}' ,
\end{cases}
\]

\[
\tilde{s}_{2,*} = \begin{cases} 
- & \text{Type } \mathfrak{A}'', \\
\sup \tilde{s}_+, & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
- & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
- & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}' ,
\end{cases}
\]

\[
(2.133)
\]

where \(-\) means no definition. Since \( \partial \tilde{s}_{j,*}(\frac{\tilde{s}_+}{\sigma^2}, \beta, X) \neq 0 \) for Type \( \mathfrak{A}'' \), \( \tilde{s}_{j,*} \) are called generalized critical points. Given \( 0 < \epsilon_1 < \frac{\pi}{2\kappa} \ll 1 \), define \( \mathcal{U}_j^\beta(\beta, X) \supset \mathcal{U}_j(\beta, X) \), \( j = 0, 1, 2 \), by

\[
\mathcal{U}_0 = \begin{cases} 
[0, \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}], & \text{Type } \mathfrak{A}'', \\
[0, \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}], & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
[0, \frac{1}{2\cos \epsilon_1} \tilde{s}_{1,*}], & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
[0, \tilde{\sigma}], & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}' ,
\end{cases}
\]

\[
(2.134)
\]

\[
\mathcal{U}_1 = \begin{cases} 
[(1 - \frac{1}{2\cos \epsilon_1}) \tilde{s}_{1,*}, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{s}_{1,*}] \equiv \mathcal{U}_{1,<} \cup \mathcal{U}_{1,>}, & \text{Type } \mathfrak{A}'', \\
[(1 - \frac{1}{2\cos \epsilon_1}) \tilde{s}_{1,*}, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{s}_{1,*} + \tilde{s}_{2,*} - \tilde{s}_{1,*}] \equiv \mathcal{U}_{1,<} \cup \mathcal{U}_{1,>}, & \text{Type } \mathfrak{B}'', \mathfrak{C}'', \\
(1 - \frac{1}{2\cos \epsilon_1}) \tilde{s}_{1,*}, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{s}_{1,*}] \equiv \mathcal{U}_{1,<} \cup \mathcal{U}_{1,>}, & \text{Type } \mathfrak{D}, \mathfrak{E}, \\
\phi, & \text{Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}' ,
\end{cases}
\]
\[ \mathcal{U}_2 = \left\{ \begin{array}{ll} \phi, & T \overline{\mathbb{A}''}, \\
\left[ \tilde{s}_{2,*} - \frac{\tilde{s}_{2,*} - \tilde{s}_{1,*}}{2 \cos \xi}, \tilde{s}_{2,*} \right] \cup \left[ \tilde{s}_{2,*}, \tilde{\sigma} \right] \equiv \mathcal{U}_{2,<} \cup \mathcal{U}_{2,>}, & T \overline{\mathbb{B}''}, \mathbb{C}'', \\
\phi, & T \overline{\mathbb{D}}, \mathbb{E}, \\
\phi, & T \overline{\mathbb{A}'}, \mathbb{B}', \mathbb{C}', \end{array} \right. \]

\[ \mathcal{U}_0^0 = \left\{ \begin{array}{ll} [0, \tilde{s}_{1,*}], & T \overline{\mathbb{A}''}, \mathbb{B}'', \mathbb{C}'', \mathbb{D}, \mathbb{E}, \\
[0, \tilde{\sigma}], & T \overline{\mathbb{A}'}, \mathbb{B}', \mathbb{C}', \end{array} \right. \]

\[ \mathcal{U}_1^1 = \left\{ \begin{array}{ll} [0, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{\sigma} \tilde{\delta}] \equiv \mathcal{U}_{1,<} \cup \mathcal{U}_{1,>}, & T \overline{\mathbb{A}'}, \mathbb{D}, \mathbb{E}, \\
[0, \tilde{s}_{1,*}] \cup [\tilde{s}_{1,*}, \tilde{s}_{2,*}] \equiv \mathcal{U}_{1,<} \cup \mathcal{U}_{1,>}, & T \overline{\mathbb{B}'}, \mathbb{C}'', \\
\phi, & T \overline{\mathbb{A}'}, \mathbb{B}'', \mathbb{C}', \end{array} \right. \]

\[ \mathcal{U}_2^2 = \left\{ \begin{array}{ll} \phi, & T \overline{\mathbb{A}'}, \mathbb{B}', \mathbb{C}', \mathbb{D}, \mathbb{E}, \\
[\tilde{s}_{1,*}, \tilde{s}_{2,*}] \cup [\tilde{s}_{2,*}, \tilde{\sigma} \tilde{\delta}] \equiv \mathcal{U}_{2,<} \cup \mathcal{U}_{2,>}, & T \overline{\mathbb{B}'}, \mathbb{C}'. \end{array} \right. \]

Write
\[
\lambda = \kappa_j + \frac{\tilde{r} e^{i \alpha}}{\tilde{\sigma}} = \kappa_j + \frac{\tilde{s}_{j,*} e^{i \beta} + \tilde{r}_{j} e^{i \alpha_j}}{\tilde{\sigma}},
\]
\[
(2.135)
\]
\[ \tilde{r}_j = \tilde{r}_j(\beta, X, \lambda), \quad \alpha_j = \alpha_j(\beta, X, \lambda), \quad j = 0, 1, 2. \]

Due to Figure 3, Table 2.5, 2.6, the deformation defined by
\[
\tilde{s} \mapsto \xi_j \equiv \tilde{s}_j e^{i \tau_j} + \tilde{s}_{j,*},
\]
\[
\tilde{s} \equiv \pm \tilde{s}_j + \tilde{s}_{j,*} \in \mathcal{U}_j^j, \quad |\tau_j| \leq \frac{\pi}{2}, \quad \tilde{s}_j \geq 0, \quad j = 0, 1, 2,
\]
with
\[
\left\{ \begin{array}{ll}
\pm \pi \leq \tau_1 \pm \pi \pm \epsilon_1, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{A}''}, \\
\pm \epsilon_1 \leq \tau_1 \leq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{A}''}, \\
\pm \pi \leq \tau_1 \leq \mp \pi \pm \epsilon_1, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{B}''}, \mathbb{C}'', \\
\mp \epsilon_1 \leq \tau_1 \leq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \leq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{B}''}, \mathbb{C}'', \\
\pm \pi \leq \tau_1 \pm \mp \pi \leq \pm \epsilon_1, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{D}}, \mathbb{E}, \\
\pm \epsilon_1 \leq \tau_1 \leq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{D}}, \mathbb{E}, \\
\mp \pi \leq \tau_1 \leq \mp \pi \pm \frac{\pi}{2}, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{A}'}, \mathbb{B}'', \mathbb{C}'', \\
\pm \frac{\pi}{2} \leq \tau_1 \leq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{A}'}, \mathbb{B}'', \mathbb{C}'', \\
\pm \pi \leq \tau_1 \pm \mp \frac{\pi}{2}, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{D}}, \mathbb{E}, \\
\pm \frac{\pi}{2} \geq \tau_1 \geq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{D}}, \mathbb{E}, \\
\pm \pi \leq \tau_1 \pm \pi \pm \frac{\pi}{2}, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{B}'}, \mathbb{C}', \\
\pm \frac{\pi}{2} \geq \tau_1 \geq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{B}'}, \mathbb{C}', \\
\pm \pi \leq \tau_1 \leq \mp \pi \pm \frac{\pi}{2}, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,<}, \ T \overline{\mathbb{D}}, \mathbb{E}, \\
\pm \frac{\pi}{2} \geq \tau_1 \leq 0, & \text{for } \sin 3 \beta \geq 0, \quad || \alpha_1 - \beta | - \pi | \geq \frac{\pi}{2}, \quad \tilde{s} \in \mathcal{U}_{1,>}, \ T \overline{\mathbb{D}}, \mathbb{E},
\end{array} \right. \]
\begin{align*}
\pm \varepsilon_1 \geq \tau_0 \geq 0, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ Type \ \mathcal{D}, \ \mathcal{E}, \\
\pm \frac{\alpha}{4} \geq \tau_0 \geq 0, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ Type \ \mathcal{D}, \ \mathcal{E}, \\
\mp \varepsilon_1 \leq \tau_0 \leq 0, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ otherwise, \\
\mp \frac{\alpha}{4} \leq \tau_0 \leq 0, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ otherwise, \\
\pm \pi \leq \tau_2 \leq \pm \pi \mp \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_2 - \beta| - \pi| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \varepsilon_1 \leq \tau_2 \leq 0, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_2 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\pm \frac{\alpha}{4} \leq \tau_2 \leq 0, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_2 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\end{align*}

and

\begin{align*}
\tau_{0,\uparrow} = \\
\pm \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ Type \ \mathcal{D}, \ \mathcal{E}, \\
\pm \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ Type \ \mathcal{D}, \ \mathcal{E}, \\
\mp \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ otherwise, \\
\mp \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_0 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_0, \ otherwise, \\
\mp \pi \pm \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_1 - \beta| - \pi| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{A}'', \\
\mp \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_1 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{A}'', \\
\mp \pi \pm \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_1 - \beta| - \pi| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_1 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \pi \pm \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_1 - \beta| - \pi| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{A}'', \\
\mp \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_1 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{A}'', \\
\mp \pi \pm \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_1 - \beta| - \pi| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_1 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \pi \pm \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_1 - \beta| - \pi| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{A}'', \\
\mp \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_1 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_1, \ Type \ \mathcal{A}'', \\
\mp \pi \pm \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_2 - \beta| - \pi| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \varepsilon_1, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_2 - \beta| \leq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \pi \pm \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad ||\alpha_2 - \beta| - \pi| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\mp \frac{\alpha}{4}, & \quad \text{for } \sin 3\beta \geq 0, \quad |\alpha_2 - \beta| \geq \frac{\alpha}{2}, \quad \tilde{s} \in \mathcal{U}_2, \ Type \ \mathcal{B}'', \ \mathcal{C}'', \\
\end{align*}

satisfies, for \( j = 0, 1, 2, \)

\begin{align*}
|\bar{s}_j e^{i\tau_{j,\uparrow}} - \tilde{r}_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max\{\tilde{r}_j, \bar{s}_j\}, \quad (2.137)
\end{align*}

and

\begin{align*}
\Re(-i\varphi(\frac{\xi}{\sigma}, \beta, X)) \leq -\frac{1}{C} \sin 3\tau_j \sin 3\beta |\bar{s}_j^3|, \quad \tilde{s} \in \mathcal{U}_j(\beta, X). \quad (2.138)
\end{align*}
Step 2 (The estimates for $I_4$, $I_5$, $n = 1$): With (2.137)-(2.138), one can adapt argument in Lemma 2.6 to derive estimates for $I_4$ and $I_5$. For illustration, we give the proof for $I_5$.

A residue theorem implies

$$I_5 = \frac{-\theta(\bar{r} - 1)}{2\pi i} \sum_{j=0}^{n-1} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{\beta_j}} e^{-i\beta \xi_j(\hat{\beta}, X)} f(\xi_j, -\beta, X) \frac{e^{-i\beta \xi_j(\hat{\beta}, X)}}{\xi_j e^{\nu_j} - \bar{r}_j e^{\nu_j(\alpha_\gamma - \beta)}} d\xi_j$$

(2.139)

and

$$-\frac{\theta(\bar{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{S^*_\beta} e^{-i\beta \xi_j(\hat{\beta}, X)} f(\xi_j, -\beta, X) \frac{e^{-i\beta \xi_j(\hat{\beta}, X)}}{\xi_j e^{\nu_j} - \bar{r}_j e^{\nu_j(\alpha_\gamma - \beta)}} d\xi_j$$

Table 2.6. Deformation for Case $(F1)$, $(F2)$, $(F3)$

| Case | Type $A''$ |
|------|-------------|
| $\tilde{s} \in \mathcal{U}_0$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_1$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_2$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |

| Case | Type $B''$, $C''$ |
|------|------------------|
| $\tilde{s} \in \mathcal{U}_0$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_1$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_2$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |

| Case | Type $D$, $E$ |
|------|----------------|
| $\tilde{s} \in \mathcal{U}_0$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_1$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_2$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |

| Case | Type $A'$, $B'$, $C'$ |
|------|------------------------|
| $\tilde{s} \in \mathcal{U}_0$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_1$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
| $\tilde{s} \in \mathcal{U}_2$ | $\Re(-i\phi(\frac{3\rho r}{\sigma}, \beta, X))$ |
\[ -\theta(\bar{r} - 1)\theta(\bar{r}_1 - \frac{1}{4})\theta(\bar{r}_2 - \frac{1}{4}) \int_{\alpha - \beta < 0} d\beta [\partial_{\beta} \ln(1 - \gamma|\beta|)] \text{sgn}(\tau_1(\beta)) \times e^{-i\psi(\bar{r}_j\alpha - \beta, X)} f(\frac{e^{i(\alpha - \beta)}}{\bar{\sigma}}, -\beta, X), \]

where \( S_\gamma = S_\gamma(\alpha, X, \lambda) \), and \( \Gamma_{5j} = \Gamma_{5j}(\alpha, X, \lambda) \), \( j = 0, 1, 2 \), \( n = 1, 2 \), defined by

\[
\begin{align*}
\Gamma_{50} &= \{ \xi_0 : \hat{s} \in \mathcal{U}, \tau_0 = \tau_{0,\dagger} \}, \\
\Gamma_{5n} &= \Gamma_{5n,\text{out}} \cup S_{5n} \cup \Gamma_{5n,\text{in}},
\end{align*}
\]

with

\[
\begin{align*}
\Gamma_{5n,\text{out}} &= \left\{ \begin{array}{ll}
\{ \xi_n : \hat{s} \in \mathcal{U}_n, \tau_n = \tau_{n,\dagger} \}, & \bar{r}_n > \frac{1}{4}, \\
\{ \xi_n : \hat{s} \in \mathcal{U}_n, \tau_n = \tau_{n,\dagger}, |\bar{s}_n - \bar{r}_n| > 1/2 \}, & \bar{r}_n < \frac{1}{4}, \\
\phi, & \bar{r}_n > \frac{1}{4}, \\
\phi, & \bar{r}_n < \frac{1}{4}, \\
\end{array} \right. \\
\Gamma_{5n,\text{in}} &= \left\{ \begin{array}{ll}
\{ \xi_n : \hat{s} \in \mathcal{U}_n, \tau_n = 0 \text{ on } \mathcal{U}_{n,\gamma} \}, & \bar{r}_n < \frac{1}{4}, \\
\{ \xi_n : \tau_n = \pi \text{ on } \mathcal{U}_{n,\gamma} <, |\bar{s}_n - \bar{r}_n| < 1/2 \}, & \bar{r}_n > \frac{1}{4}, \\
\end{array} \right. \\
S_{5n} &= \left\{ \begin{array}{ll}
\phi, & \bar{r}_n > \frac{1}{4}, \\
\{ \xi_n : \tau_n \text{ defined by (2.136)}, |\bar{s}_n - \bar{r}_n| = 1/2 \}, & \bar{r}_n < \frac{1}{4}, \\
\end{array} \right. \\
S_{\gamma}(\alpha, X, \lambda) &= \{ \xi_h : h = \sup_{i,j\neq\phi} j, \bar{s} = \bar{s}_\delta \tau_h \text{ defined by (2.136)} \},
\end{align*}
\]

and \( \alpha_j, \tau_j, \tau_{i,\dagger}, \mathcal{U}_j \) are defined by (2.134)-(2.136).

Using (2.137)-(2.140), applying argument as that for \( I_5 \) in Step 3 for \( I_5 \) in Lemma 2.6 to the above deformation, we obtain

\[
|I_5|_{\mathcal{M}(D_{\kappa_1})} \leq C\delta_0 |f|_{L^\infty(D_{\kappa_1})},
\]

Step 3 (The estimates for \( I_4, I_5, n > 1 \)): Estimates can be derived via the approach of the proof of Proposition 2.7. We skip details for simplicity.
Lemma 2.9. Given an admissible scattering data \( S = \{ z_n, \kappa_j, D, sc(\lambda) \} \), for fully non homogeneous cases (F2), and holomorphic \( E_{\kappa_j}f \),

\[
| (CE_{\kappa_j})^n f |_{\mathfrak{M}(D_{\kappa_j})} \leq (C \epsilon_0)^n | f |_{\mathfrak{M}(D_{\kappa_j})}.
\]

Proof. For simplicity and without loss of generality, we only give a proof assuming (2.16) and reduce the proof to estimating the principal part. Define the scaling parameter \( \tilde{\sigma} \) by Definition 2, scaled coordinates by (2.6), and decompose the principal part into (2.10)-(2.15). Thanks to (2.92), estimates for \( I_j, j = 1, 2, 3 \) can be derived as in Lemma 2.3.

Step 1 (Deformation): We shall justify the deformation (2.136) is good for Case (F2) in this step.

Note that

\[
3 \cdot \frac{X_3}{X_3^{3/2}} \sin 3\beta = \frac{X_2^{1/2}}{X_1} \sin \beta \Delta \sin 2\beta.
\]

Together with Figure 3, Table 2.4-2.6, \( \Delta > \frac{1}{2} \), and (2.136), yields

\[
\Re (-i\phi(\tilde{s}_j e^{i\tau_j \tilde{\sigma}} + \tilde{s}_j^*, \beta, X)) \leq -\frac{1}{C} | \sin 2\beta | | \tilde{s}_j^3 |,
\]

\( \tilde{s} \in \mathfrak{M}_j, j = 0, 1, 2, \) for Type \( \mathfrak{A}'' \), \( \mathfrak{B}'' \).

Moreover, Figure 3, Table 2.4-2.6 and (2.136) imply

\[
| \tilde{s}_j + (-1)^{j+1} \frac{\sqrt{1 - \Delta}}{3 \cdot \frac{X_3}{\sigma X_2} \sin 3\beta \sin 3\tau_j} | \geq \frac{1}{C} \left| \frac{1}{X_3 \sin 3\beta} \right|, \text{ on } \mathfrak{M}_j, j = 1, 2, \text{ for Type } \mathfrak{C}'',
\]

\[
| \tilde{s}_1 \pm \frac{\sqrt{1 - \Delta}}{3 \cdot \frac{X_3}{\sigma X_2} \sin 3\beta \sin 3\tau_1} | \geq \frac{1}{C} \left| \frac{1}{X_3 \sin 3\beta} \right|, \text{ on } \mathfrak{M}_1, 0 < \tilde{s}_\pm \text{ for Type } \mathfrak{D}, \mathfrak{E},
\]

\[
| \tilde{s}_{1,*} - \tilde{s}_{2,*} | \geq \frac{1}{C} \left| \frac{1}{X_3 \sin 3\beta} \right|, \text{ for Type } \mathfrak{C}'' , \mathfrak{D}, \mathfrak{E}.
\]

As a result,

\[
\Re (-i\phi(\tilde{s}_j e^{i\tau_j \tilde{\sigma}} + \tilde{s}_j^*, \beta, X)) \leq -\frac{1}{C} | \sin 2\beta | | \tilde{s}_j^2 |
\]

\( \text{for } \tilde{s} \in \mathfrak{M}_j, j = 0, 1, 2, \text{ for Type } \mathfrak{C}'' , \mathfrak{D}, \mathfrak{E} \).

Finally,

\[
\Re (-i\phi(\tilde{s}_0 \sigma, \beta, X)) \leq -\frac{1}{C} | \sin 3\tau_0 \sin 3\beta | \tilde{s}_0^3, \text{ } \tilde{s} \in \mathfrak{M}_0, \text{ for Type } \mathfrak{A}', \mathfrak{B}', \mathfrak{C}'.
\]
Lemma 2.10. Given an admissible scattering data $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$, for fully nonhomogeneous cases (F1), and holomorphic $E_{\kappa_j}f$,

$$|(CTE_{\kappa_j})^n f|_{\mathfrak{M}(D_{\kappa_j})} \leq (C\epsilon_0)^n |f|_{\mathfrak{M}(D_{\kappa_j})}.$$

Proof. For simplicity and without loss of generality, we only give a proof assuming (2.16) and reduce the proof to estimating the principal part. Define the scaling parameter $\bar{\sigma}$ by Definition 2, scaled coordinates by (2.6), and decompose the principal part into (2.10)-(2.15). Thanks to (2.92), estimates on compact domains, i.e., $I_j, j = 1, 2, 3$ can be derived as in Lemma 2.3.

Step 1 (The deformation): To estimate $I_4$ and $I_5$ for the fully non homogeneous cases (F1), introduce $\sigma_j, j = 0, 1, 2$,

$$\begin{cases} 
  \sigma_0 = \bar{\sigma}, & \sigma_1 = \sigma_2 = \sqrt{|X_3|}, \text{ for Type } \mathfrak{A}, \mathfrak{B}, \mathfrak{E}, \\
  \sigma_0 = \bar{\sigma}, & \sigma_1 = \sigma_2 = \sqrt{|X_2|}, \text{ for Type } \mathfrak{C}, \mathfrak{D},
\end{cases} \tag{2.146}$$

and $s_{j,*}$,

$$s_{j,*} = \tilde{s}_{j,*} \frac{\sigma_j}{\sigma_0}, \quad r_j = \tilde{r}_j \frac{\sigma_j}{\sigma_0}, \tag{2.147}$$

where $\tilde{s}_{j,*}$, and $\tilde{r}_j$ are defined by (2.133) and (2.135). Define the deformation

$$\frac{\tilde{s}}{\bar{\sigma}} \mapsto \frac{\tilde{\vartheta}_j}{\sigma_j} \equiv \frac{s_j e^{i\tau_j} + s_{j,*}}{\sigma_j},$$

$$\tilde{s} \equiv (\pm s_j + s_{j,*}) \frac{\sigma_0}{\sigma_j} \in \mathcal{O}_j, \quad s_j \geq 0, \quad \text{for } |\tau_j| \leq \frac{\pi}{2}, \tag{2.148}$$

and $\mathcal{O}_j, \epsilon_1, \tau_j$, are defined by (2.134), (2.136). Therefore,

$$|s_j e^{i\tau_j} - r_j e^{i(\alpha_j - \beta)}| \geq \frac{1}{C} \max \{r_j, s_j\}. \tag{2.149}$$

In order to derive estimates, we have to justify

$$\inf_{\beta} s_{1,*} = c_0, \quad 0 < c_0 < 1 \tag{2.150}$$

first (cf. (2.112)). To this aim, for Type $\mathfrak{A}, \mathfrak{B}, \mathfrak{E}$, from Table 2.5 (2.146), and

$$|s_+s_-| = \left| \frac{-X_1}{X_3^{1/3}} \sin \beta \sin 3\beta \right| \geq 1, \quad s_{\pm} = \frac{-1 \pm \sqrt{1 - \Delta}}{3X_3^{2/3} \sin 3\beta},$$

we derive $|s_+| \sim |s_-|$ and then (2.150) for Type $\mathfrak{A}, \mathfrak{B}, \mathfrak{E}$.

On the other hand, from (2.131), Table 2.5, and (2.146),

$$|s_+| = \left| \frac{\Delta}{6X_3 \sin 3\beta} + \text{l.o.t.} \right| = \left| \frac{3X_1 X_3 \sin \beta \sin 3\beta}{X_3^2 \sin^2 2\beta} + \text{l.o.t.} \right| \geq \frac{1}{C},$$

$$|s_-| = \left| \frac{-2}{3X_3 \sin 3\beta} + \text{l.o.t.} \right| \geq \frac{1}{C} |s_+| \geq \frac{1}{C}. $$
Hence (2.150) is proved for Type $\mathfrak{C}$ and $\mathfrak{D}$.

Next, from (2.146)-(2.148), Table 2.6 and results of Case (F3),

$$\Re(-i\varphi(\frac{s_j e^{i\tau_j \tau}}{\sigma_j}, \beta, X)) \leq -\frac{1}{C} |\sin 3\beta| s_j^3, \ s \in \bar{U}_j, j = 1, 2,$$

for Type $\mathfrak{A''}$, $\mathfrak{B''}$, $\mathfrak{C}$;  

and from (2.146)-(2.148), Table 2.6 and results of Case (F2),

$$\Re(-i\varphi(\frac{s_j e^{i\tau_j \tau}}{\sigma_j}, \beta, X)) \leq -\frac{1}{C} |\sin 2\beta| s_j^2, \ s \in \bar{U}_j, j = 1, 2,$$

for Type $\mathfrak{C''}$, $\mathfrak{D}$.

Besides, note

$$3\frac{X_3}{X_1} \sin 3\beta = \left( \frac{X_2}{X_1} \sin \beta \right)^2 \sin \beta \Delta, \ 3\frac{X_3}{X_1X_2} \sin 3\beta = \frac{X_2}{X_1^2} \sin \beta \Delta. \quad (2.153)$$

Therefore, using

\[ \begin{cases} 
\text{the second term of } \Re(-i\varphi) \text{ in Table 2.6}, & \text{for } \mathfrak{A}, \Delta \geq \frac{3}{2}, \\
\text{the first term of } \Re(-i\varphi) \text{ in Table 2.6}, (2.136), (2.153), & \text{for } \mathfrak{A}, 1 \leq \Delta \leq \frac{3}{2}, \ \\
(2.136), (2.153), \frac{1}{2} \leq \Delta \leq 1, \text{ Table 2.4}, (2.6) & \text{for } \mathfrak{B}, \ \\
\text{Table 2.4, (2.6), (2.153), } |(s - \bar{s}_+)(s - \bar{s}_-)| \geq \frac{1}{C} \left( \frac{X_3}{X_1^2} \sin \beta \Delta \right)^2 |\Delta|, & \text{for } \mathfrak{C}, \mathfrak{D}, \\
\text{Table 2.4, (2.6), (2.153), } |(s - \bar{s}_+)(s - \bar{s}_-)| \geq \frac{1}{C} \left( \frac{X_3}{X_1^2} \sin \beta \Delta \right)^2 |\Delta|, & \text{for } \mathfrak{E}, \\
\end{cases} \quad (2.154) \]
we obtain

$$\Re(-i\psi(s_0^{\frac{\pi}{\sigma_0}}, \beta, X)) \leq -\frac{1}{C}|\sin \beta|s_0, \quad s \in U_0, \quad \text{for Type } \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}. \tag{2.155}$$

Finally, in view of Table 2.3, 2.5, 2.6, 2.7, and adapting argument as in deriving (2.154), we have

$$|\psi(s_0^{\frac{\pi}{\sigma_0}}, \beta, X)| \leq C|\sin \beta|s_0, \quad \psi \in U_0, \quad \text{for Type } \mathcal{A}, \mathcal{B''}, \mathcal{C''}. \tag{2.156}$$

Consequently, (2.64) can be generalized to

$$\left|\frac{e^{-i\psi(s, \beta, X)}}{|\psi - \tilde{\psi}(\alpha - \beta)|} - 1\right| \leq C|\sin \beta|, \quad \psi \in U_0, \quad \text{for Type } \mathcal{A}, \mathcal{B''}, \mathcal{C''}. \tag{2.157}$$

**Step 2 (The estimates for I₄, I₅, n = 1):** With (2.149), (2.151), (2.152), (2.155), and (2.157), we can adapt argument in Lemma 2.3 and 2.7 to derive estimates for I₄ and I₅. For illustration, we give the proof for I₅.

A residue theorem implies

$$I_5 = -\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{\psi_0}} J_< + \sum_{n=1,2} \frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{\psi_n}} e^{-i\psi(s_n^{\frac{\pi}{\sigma_n}}, \beta, X)} f(s_n^{\frac{\pi}{\sigma_n}}, -\beta, X) \frac{d\psi_n}{s_n e^{i\tau_n} - r e^{i(\alpha_n - \beta)}} d\psi_n$$

$$-\frac{\theta(\tilde{r} - 1)}{2\pi i} \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\psi_n} J_> - \theta(\tilde{r} - 1) \theta(r_1 - \frac{c_0}{4}) \theta(r_2 - \frac{c_0}{4}) \int_{\alpha - \beta < \psi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \text{sgn}(\tau_1(\beta)) \times e^{-i\psi(s, \beta, X)} f(s, -\beta, X),$$

where c₀ is defined by (2.150), J₉, J₈ are defined by

$$J_< = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5, & \text{if } 0 < \beta < \epsilon_1/8, \\ 0, & \text{if } -\epsilon_1/8 < \beta < 0, \\ e^{-i\psi(s_0^{\frac{\pi}{\sigma_0}}, \beta, X)} f(s_0^{\frac{\pi}{\sigma_0}}, -\beta, X) \frac{d\psi_0}{s_0 e^{i\tau_0} - r e^{i(\alpha_0 - \beta)}} d\psi_0, & \text{if } |\beta| > \epsilon_1/8, \end{cases} \tag{2.159}$$

$$J_> = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5, & \text{if } h = 0, 0 < \beta < \epsilon_1/8, \\ 0, & \text{if } h = 0, -\epsilon_1/8 < \beta < 0, \\ e^{-i\psi(s_0^{\frac{\pi}{\sigma_0}}, \beta, X)} f(s_0^{\frac{\pi}{\sigma_0}}, -\beta, X) \frac{d\psi_0}{s_0 e^{i\tau_0} - r e^{i(\alpha_0 - \beta)}} d\psi_0, & \text{if } h = 0, |\beta| > \epsilon_1/8, \end{cases} \quad h = \sup_{\psi_j \neq \phi} j;$$

$$\text{with } J_<, J_> \text{ as defined.}$$
with \( \mathfrak{J}_j, 1 \leq j \leq 5 \), defined by (2.119); \( S_> = S_>(\beta, X, \lambda) \), and \( \Gamma_{5j} = \Gamma_{5j}(\beta, X, \lambda), j = 0, 1, 2, n = 1, 2, \) by

\[
\Gamma_{50} = \{ \vartheta_0 : \tilde{s} \in \mathfrak{U}_0, \tau_0 = \tau_{0,1} \},
\]

\[
\Gamma_{5n} = \Gamma_{5n,\text{out}} \cup S_{5n} \cup \Gamma_{5n,\text{in}},
\]

with

\[
\Gamma_{5n,\text{out}} = \begin{cases} 
\{ \vartheta_n : \tilde{s} \in \mathfrak{U}_n, \tau_n = \tau_{n,1} \}, & r_n > \frac{c_0}{4}, \\
\{ \vartheta_n : \tilde{s} \in \mathfrak{U}_n, \tau_n = \tau_{n,1}, |s_n - r_n| > c_0/2 \}, & r_n < \frac{c_0}{4},
\end{cases}
\]

\[
\Gamma_{5n,\text{in}} = \begin{cases} 
\{ \vartheta_n : \tilde{s} \in \mathfrak{U}_n, \tau_n = 0 \text{ on } \mathfrak{U}_{n,>}, \tau_n = \pi \text{ on } \mathfrak{U}_{n,<}, |s_n - r_n| < c_0/2 \}, & r_n < \frac{c_0}{4},
\end{cases}
\]

\[
S_{5n} = \{ \vartheta_n : \tau_n \text{ defined by (2.136)}, |s_n - r_n| = c_0/2 \}
\]

\[
S_>(\beta, X, \lambda) = \{ \vartheta_h : h = \sup_{\mathfrak{J}_j \neq \varnothing} j, \tilde{s} = \tilde{\sigma} \delta, \tau_h \text{ defined by (2.136)} \},
\]

and \( \alpha_j, \tau_j, \tau_{j,1}, \mathfrak{U}_j \), defined by (2.134)-(2.135).

Applying (2.16), (2.149), (2.150) (in particular for estimating \( \Gamma_{5n,\text{in}} \)), (2.151), (2.152), (2.155), (2.160), similar analysis as that in Lemma 2.3 one obtains

\[
\sum_{n=1,2} |\theta(\tilde{r} - 1)| \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{\Gamma_{5n}} e^{-ip(\vartheta_n, \beta, X) f(\vartheta_n, \beta, X)} s_n e^{ir_n} - T_n e^{i(\alpha - \beta)} d\vartheta_n |_{\mathfrak{M}(D_{\kappa_1})}
\]

\[
+ |\theta(\tilde{r} - 1)| \int_{-\pi}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \int_{S_>} |_{\mathfrak{M}(D_{\kappa_1})}
\]

\[
+ |\theta(\tilde{r} - 1)\theta(r_1 - \frac{c_0}{4})\theta(r_2 - \frac{c_0}{4}) \int_{\alpha - \beta \in \varnothing} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \text{sgn}(\tau_1(\beta))
\]

\[
\times e^{-ip(\varphi^{(\alpha - \beta)}), \beta, X) f(\varphi^{(\alpha - \beta)}), \beta, X)} |_{\mathfrak{M}(D_{\kappa_1})}
\]

\[
\leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]

On \( \Gamma_{50} \), it suffices to consider \( |\beta| \leq \epsilon_1/8 \). Applying (2.149), (2.155), (2.157), and (2.160), we can adapt argument in Lemma 2.3 and 2.7 one obtains

\[
|\theta(\tilde{r} - 1)| \int_{0}^{\pi} d\beta [\partial_\beta \ln(1 - \gamma|\beta|)] \theta(\epsilon_1/8 - |\beta|)
\]

\[
\times \int_{\Gamma_{50}} (\mathfrak{J}_1 + \mathfrak{J}_2 + \mathfrak{J}_3 + \mathfrak{J}_4 + \mathfrak{J}_5) |_{\mathfrak{M}(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]

Thus

\[
|I_5|_{\mathfrak{M}(D_{\kappa_1})} \leq C\epsilon_0 |f|_{L^\infty(D_{\kappa_1})}.
\]
Step 3 (The estimates for \(I_4, I_5, n > 1\)): Estimates can be derived via the approach of the proof of Proposition 2.1, 2.2. We skip details for simplicity.

\[\square\]

2.5. Summaries. From Lemma 2.2, Proposition 2.1, 2.2, and Lemma 2.6-2.10 we have

**Theorem 1.** Suppose \(S = \{z_n, \kappa_j, D, s_c(\lambda)\}\) is an admissible KdV or BPP scattering data and \(E_{\kappa_j} f\) is holomorphic for \(1 \leq j \leq M\). Then

\[\sum_{j=1}^{M} |(CTE_{\kappa_j})^n f|_{\mathcal{M}(D_{z_n})} + \sum_{j=1}^{M} |(CTE_{\kappa_j})^n f|_{L^\infty(D_{z_n})} \leq (C\epsilon_0)^n \sum_{j=1}^{M} |f|_{\mathcal{M}(D_{z_n})}.\]

3. Estimates for the Cauchy integral operator CT

Given an admissible KdV or BPP scattering data \(S = \{z_n, \kappa_j, D, s_c(\lambda)\}\), one can adapt the approach in Section 2 to derive estimates of the CIO near \(z_n\).

**Lemma 3.1. (Estimates near \(z_n\))** Given an admissible KdV or BPP scattering data \(S = \{z_n, \kappa_j, D, s_c(\lambda)\}\), if \(\phi(x, \lambda) = \frac{\phi_{z_n, r}(x) + \phi_{z_n, r}(x, \lambda)}{\lambda - z_n}\) then

\[|CTE_{z_n}\phi|_{L^\infty} + \sum_{j=1}^{M} |CTE_{z_n}\phi|_{C^0(D_{z_n})} \leq C\epsilon_0 (|\phi_{z_n, r}|_{L^\infty} + |\phi_{z_n, r}|_{L^\infty(D_{z_n})}).\]

**Proof.** From (1.14), (1.16), (2.5), and (2.6), decompose

\[C_\lambda E_{z_n} T \phi = -\frac{1}{2\pi i} \int_{D_{z_n, \delta}} \frac{\text{sgn}(\beta)h_n(x, \beta)e^{-ip(x, \beta, X)}\phi_{z_n, \text{res}}(X)}{(\zeta - \lambda)(\zeta - z_n)} d\zeta \wedge d\bar{\zeta} + C_\lambda E_{z_n} T \phi_{z_n, r} = II_1 + II_2 + II_3 + II_4 + II_5,\]

where

\[II_1 = -\frac{\theta(1 - \bar{\tau})}{2\pi i} \int_{\bar{\delta} < 1} \frac{\text{sgn}(\beta)h_n(x, \beta)\phi_{z_n, \text{res}}(X)}{(\zeta - \lambda)(\zeta - z_n)} d\zeta \wedge d\bar{\zeta},\]

\[II_2 = -\frac{\theta(1 - \bar{\tau})}{2\pi i} \int_{\bar{\delta} < 1} \frac{\text{sgn}(\beta)h_n(x, \beta)[e^{-ip(x, \beta, X)} - 1]\phi_{z_n, \text{res}}(X)}{(\zeta - \lambda)(\zeta - z_n)} d\zeta \wedge d\bar{\zeta},\]

\[II_3 = C_\lambda E_{z_n} T \phi_{z_n, r},\]

\[II_4 = -\frac{\theta(1 - \bar{\tau})}{2\pi i} \int_{2<\delta<\bar{\delta}} \frac{\text{sgn}(\beta)h_n(x, \beta)e^{-ip(x, \beta, X)}\phi_{z_n, \text{res}}(X)}{(\zeta - \lambda)(\zeta - z_n)} d\zeta \wedge d\bar{\zeta},\]

\[II_5 = -\frac{\theta(1 - \bar{\tau})}{2\pi i} \int_{\bar{\delta} < 1} \frac{\text{sgn}(\beta)h_n(x, \beta)e^{-ip(x, \beta, X)}\phi_{z_n, \text{res}}(X)}{(\zeta - \lambda)(\zeta - z_n)} d\zeta \wedge d\bar{\zeta}.\]
From the admissible condition and a standard Hilbert transform theory \[18\],
\[ |II_1|_{L^\infty}, |II_2|_{L^\infty} \leq C \epsilon_0 |\phi_{z_n, res}|_{L^\infty}. \]

Similar argument as that for \( I_4 \) and \( I_5 \) in Section 2 can be adapted to derive
\[ |II_4|_{L^\infty(D_{z_n})}, |II_5|_{L^\infty(D_{z_n})} \leq C \epsilon_0 |\phi_{z_n, res}|_{L^\infty}. \]

Thanks to Lemma 2.2,
\[ |II_3|_{L^\infty}, |II_3|_{C^\mu(D_{\kappa_j})} \leq C \epsilon_0 |\phi_{z_n, r}|_{L^\infty(D_{z_n})}; \]
\[ |II_5|_{L^\infty(D_{z_n})}, |II_5|_{C^\mu(D_{\kappa_j})} \leq C \epsilon_0 |\phi_{z_n, res}|_{L^\infty}. \]

\[ \square \]

**Lemma 3.2. (Estimates near \( \infty \))** Given an admissible KdV or BPP scattering data \( S = \{z_n, \kappa_j, D, s_c(\lambda)\} \), we have
\[ |CT(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi|_{L^\infty} + \sum_{j=1}^{M} |CT(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi|_{C^\mu(D_{\kappa_j})} \leq C \epsilon_0 |(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi|_{L^\infty}. \]

Proof. Estimates for \( |CT(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi|_{L^\infty} \) can be proved by \[33\] Lemma 6.III] and the admissible condition. Besides, since
\[ |CT(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi|_{C^\mu(D_{\kappa_j})} \]
\[ \leq C |\int_{|\zeta - \kappa_j| \leq 2\delta} \frac{T(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi}{\zeta - \lambda} d\zeta \land d\zeta|_{C^\mu(D_{\kappa_j})} \]
\[ + C |\int_{|\zeta - \kappa_j| \geq 2\delta} \frac{T(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi}{\zeta - \lambda} d\zeta \land d\zeta|_{L^\infty(D_{\kappa_j})} \]
\[ + C |\int_{|\zeta - \kappa_j| \geq 2\delta} \frac{T(1 - \sum_{n=1}^{N} E_{z_n} - \sum_{j=1}^{M} E_{\kappa_j})\phi}{(\zeta - \lambda)^2} d\zeta \land d\zeta|_{L^\infty(D_{\kappa_j})}. \]

Applying Lemma 2.2 to the first term and \[33\] Lemma 6.III to the remaining terms on the right hand side of the above decomposition, we can derive the estimates.

\[ \square \]

**Definition 4.** Given an admissible scattering data \( S = \{z_n, \kappa_j, D, s_c(\lambda)\} \), the eigenfunction space \( W \equiv W_x = W_{x_1, x_2, x_3} \) is the set of functions satisfying
\[ (a) \ \phi(x, \lambda) = \phi(x, \vec{\lambda}); \]
(b) \((1 - \sum_{n=1}^{N} E_{z_n}) \phi(x, \lambda) \in L^\infty;\)

(c) For \(\lambda \in D_{z_n};\)
\[
\phi(x, \lambda) = \frac{\phi_{z_n, \text{res}}(x)}{\lambda - z_n} + \phi_{z_n, r}(x, \lambda),
\]
where \(\phi_{z_n, \text{res}}, E_{z_n, \phi_{z_n, r}} \in L^\infty(D_{z_n});\)

(d) For \(\lambda \in D_{\kappa_j}, \) the D-symmetry
\[
(e^{\kappa_1 x_1 + \kappa_2^2 x_2 + \kappa_3^2 x_3} \phi(x, \lambda_1), \ldots, e^{\kappa M x_1 + \kappa_2 x_2 + \kappa_3^2 x_3} \phi(x, \lambda_M)) D = 0
\]
is valid and \(E_{\kappa_j} \phi \in M(D_{\kappa_j})\) (see Definition 2).

For any function \(\phi,\) not necessarily obeys the D-symmetry, define
\[
|\phi|_W \equiv |(1 - \sum_{n=1}^{N} E_{z_n}) \phi|_{L^\infty} + \sum_{n=1}^{N} |\phi_{z_n, \text{res}}|_{L^\infty} + \sum_{n=1}^{N} |\phi_{z_n, r}|_{L^\infty(D_{z_n})} + \sum_{j=1}^{M} |\phi|_{\mathfrak{M}(D_{\kappa_j})}, \quad (3.1)
\]
\[
|\phi|_{W'} \equiv |(1 - \sum_{n=1}^{N} E_{z_n}) \phi|_{L^\infty} + \sum_{n=1}^{N} |\phi_{z_n, r}|_{L^\infty(D_{z_n})} + \sum_{j=1}^{M} |\phi|_{\mathfrak{M}(D_{\kappa_j})}. \quad (3.2)
\]

We conclude this section by the estimate of the Cauchy integral operator on \(W.\)

**Theorem 2.** Given an admissible scattering data \(S = \{z_n, \kappa_j, D, s_c(\lambda)\}\) and \(E_{\kappa_j} \phi\) is holomorphic for \(1 \leq j \leq M,\) one has
\[
|(CT)^n \phi|_{W'} \leq (C\epsilon_0)^n |\phi|_W.
\]

4. The inverse problem

4.1. Admissible KdV scattering data. Suppose \(S = \{0, \kappa_1, \kappa_2, D, s_c(\lambda)\}\) is an admissible KdV scattering data. Via the D-symmetry and the normalized Sato eigenfunction
\[
\chi(x, \lambda) = \tilde{\chi}_{\kappa_j, A}(x, \lambda) = \frac{(1 - \kappa_1 \lambda) e^{3}\theta_1 + (1 - \kappa_2 \lambda) a e^{3}\theta_2}{e^{3}\theta_1 + a e^{3}\theta_2}, \quad (4.1)
\]
we shall construct a recursive sequence
\[
\phi^{(k)}(x, \lambda) \equiv 1 + \frac{\psi^{(k)}(x)}{\lambda} + C T \phi^{(k-1)} \in W \quad (4.2)
\]
and prove the sequence converge to the solution of the Cauchy integral equation which satisfies constraints in (1.31). Indeed, let us observe: if \(\phi^{(k)} \in W\) then the D-symmetry yields
\[
\psi^{(k)}(x) = -\frac{\kappa_1 e^{3}\theta_1 + \kappa_2 a e^{3}\theta_2}{e^{3}\theta_1 + a e^{3}\theta_2} - \frac{\kappa_1 e^{3}\theta_1 \mathcal{C}_1 T \phi^{(k-1)} + \kappa_2 a e^{3}\theta_2 \mathcal{C}_2 T \phi^{(k-1)}}{e^{3}\theta_1 + a e^{3}\theta_2}, \quad (4.3)
\]
where $\theta_j = \kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3$, $\lambda_1 = \kappa_1 + 0^+ e^{i\alpha}$, and $\lambda_2 = \kappa_2 + 0^+ e^{i(\pi + \alpha)}$. Consequently, to use (4.2) to define the recursive sequence, we need to justify the right hand side of (4.3) is univalent. Secondly, combining (4.2) and (4.3), we obtain

$$\phi^{(k)}(x, \lambda) = \chi(x, \lambda) - \frac{\kappa_1 e^{\theta_1} C_{\lambda_1} T \phi^{(k-1)}}{e^{\theta_1} + a e^{\theta_2}} + \frac{\kappa_2 a e^{\theta_2} C_{\lambda_2} T \phi^{(k-1)}}{e^{\theta_1} + a e^{\theta_2}} + C T \phi^{(k-1)}. \quad (4.4)$$

**Lemma 4.1.** Let $S = \{0, \kappa_1, \kappa_2, D, s_c(\lambda)\}$ be an admissible KdV scattering data. Then $\chi(x, \lambda) \in W$ and $E_{\kappa_1} \chi$ is holomorphic.

**Proof.** By a direct computation

$$\kappa_1 e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_3} \chi(\kappa_1) + \kappa_2 a e^{\kappa_2 x_1 + \kappa_2^2 x_2 + \kappa_2^3 x_3} \chi(\kappa_2) = 0. \quad (4.5)$$

Hence $\chi \in W$. \hfill \Box

We now inducse. Let

$$\phi^{(0)}(x, \lambda) = \chi(x, \lambda),$$

$$\phi^{(k)}(x, \lambda) = 1 + \frac{\psi^{(k)}(x, \lambda)}{\lambda} + C T \phi^{(k-1)}, \quad (4.6)$$

$$\psi^{(k)}(x, \lambda) = - \frac{\kappa_1 e^{\theta_1} + \kappa_2 a e^{\theta_2}}{e^{\theta_1} + a e^{\theta_2}} - \frac{\kappa_1 e^{\theta_1} C_{\lambda_1} T \phi^{(k-1)} + \kappa_2 a e^{\theta_2} C_{\lambda_2} T \phi^{(k-1)}}{e^{\theta_1} + a e^{\theta_2}}.$$

**Lemma 4.2.** Let $S = \{0, \kappa_1, \kappa_2, D, s_c(\lambda)\}$ be an admissible KdV scattering data. Then, for $\forall k > 0$, $\psi^{(k)}(x, \lambda) \equiv \psi^{(k)}(x)$, $\phi^{(k)} \in W$, and

$$|\phi^{(k)} - \phi^{(k-1)}|_W \leq (C e_0)^k. \quad (4.7)$$

**Proof.** From Theorem 2, Lemma 4.1 and (4.6), it is sufficient to establish, for $\forall k$, $\lambda_1 = \kappa_1 + 0^+ e^{i\alpha}$, and $\lambda_2 = \kappa_2 + 0^+ e^{i(\pi + \alpha)}$,

$$\kappa_1 e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_3} C_{\lambda_1} T \phi^{(k-1)} + a \kappa_2 e^{\kappa_2 x_1 + \kappa_2^2 x_2 + \kappa_2^3 x_3} C_{\lambda_2} T \phi^{(k-1)} \quad (4.8)$$

is independent of $\alpha$ and

$$\kappa_1 e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_3} \phi^{(k)}(x, \lambda_1) + a \kappa_2 e^{\kappa_2 x_1 + \kappa_2^2 x_2 + \kappa_2^3 x_3} \phi^{(k)}(x, \lambda_2) = 0. \quad (4.9)$$

**Step 1 (Proof for (4.8)):** From Lemma 3.1 and 3.2 we have only to prove

$$\kappa_1 e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_3} C_{\lambda_1} T E_{\kappa_1} \phi^{(k-1)} + a \kappa_2 e^{\kappa_2 x_1 + \kappa_2^2 x_2 + \kappa_2^3 x_3} C_{\lambda_2} T E_{\kappa_2} \phi^{(k-1)}$$

is independent of $\alpha$. From the proof of Lemma 2.3.2.10 it is sufficient to show

$$\kappa_1 e^{\kappa_1 x_1 + \kappa_1^2 x_2 + \kappa_1^3 x_3} C_{\lambda_1} T E_{\kappa_1} [\phi^{(k-1)}]^b + \kappa_2 a e^{\kappa_2 x_1 + \kappa_2^2 x_2 + \kappa_2^3 x_3} C_{\lambda_2} T E_{\kappa_2} [\phi^{(k-1)}]^b = 0. \quad (4.10)$$
By induction, $\phi^{(k-1)} \in W$. Therefore,

$$
\left[ \phi^{(k-1)} \right]^{(b)} (\zeta, x) = \sum_{l=0}^{k} f_l(x) \left[ -\ln(1 - \gamma|\beta|) \right]^l, \quad \zeta \in D_{\kappa_1}
$$

(4.11)

$$
\left[ \phi^{(k-1)} \right]^{(b)} (\zeta, x) = \sum_{l=0}^{k} \tilde{f}_l(x) \left[ -\ln(1 - \gamma|\pi - \beta|) \right]^l, \quad \zeta \in D_{\kappa_2},
$$

with

$$
\kappa_1 e^{\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3} f_l(x) + a \kappa_2 e^{\kappa_2 x_1 + \kappa_2 x_2 + \kappa_3 x_3} \tilde{f}_l(x) = 0, \quad \forall l.
$$

(4.12)

Applying Lemma 2.1 (Stokes’ theorem) and (4.11),

$$
C_{\lambda_1} T E_{\kappa_1} \left[ \phi^{(k-1)} \right]^{(b)} = \sum_{l=0}^{k} f_l(x) \frac{[-\ln(1 - \gamma|\alpha|)]^{l+1}}{l+1},
$$

(4.13)

$$
C_{\lambda_2} T E_{\kappa_2} \left[ \phi^{(k-1)} \right]^{(b)} = \sum_{l=0}^{k} \tilde{f}_l(x) \frac{[-\ln(1 - \gamma|\pi - (\pi + \alpha)|)]^{l+1}}{l+1}.
$$

So (4.10) follows from (4.12) and (4.13).

**Step 2 (Proof for (4.9))**: From (4.6),

$$
\phi^{(k)} (x, \lambda_1) = 1 + \frac{\psi^{(k)}(x)}{\kappa_1} + C_{\lambda_1} T \phi^{(k-1)}
$$

$$
= 1 - \frac{1}{\kappa_1} \kappa_1 e^{\theta_1} + \kappa_2 a e^{\theta_2} - \frac{1}{\kappa_1} \kappa_1 e^{\theta_1} C_{\lambda_1} T \phi^{(k-1)} + \kappa_2 a e^{\theta_2} C_{\lambda_2} T \phi^{(k-1)} + C_{\lambda_1} T \phi^{(k-1)}
$$

$$
= \left[ (1 - \frac{\kappa_2}{\kappa_1}) + C_{\lambda_1} T \phi^{(k-1)} - \frac{\kappa_2}{\kappa_1} C_{\lambda_2} T \phi^{(k-1)} \right] a e^{\theta_2},
$$

and

$$
\phi^{(k)} (x, \lambda_2) = 1 + \frac{\psi^{(k)}(x)}{\kappa_2} + C_{\lambda_2} T \phi^{(k-1)}
$$

$$
= 1 - \frac{1}{\kappa_2} \kappa_1 e^{\theta_1} + \kappa_2 a e^{\theta_2} - \frac{1}{\kappa_2} \kappa_1 e^{\theta_1} C_{\lambda_1} T \phi^{(k-1)} + \kappa_2 a e^{\theta_2} C_{\lambda_2} T \phi^{(k-1)} + C_{\lambda_2} T \phi^{(k-1)}
$$

$$
= \left[ (1 - \frac{\kappa_1}{\kappa_2}) + C_{\lambda_2} T \phi^{(k-1)} - \frac{\kappa_1}{\kappa_2} C_{\lambda_1} T \phi^{(k-1)} \right] e^{\theta_1}.
$$

So (4.9) holds for $\phi^{(k)}$.

We prove Theorem A and B provided the scattering data is KdV type.
**Theorem 3.** Given an admissible KdV scattering data \( S = \{0; \kappa_1, \kappa_2, D, s_c(\lambda)\} \), there exists uniquely \( m \in W \) satisfying

\[
m(x, \lambda) = 1 + \frac{m_{\text{res}}(x)}{\lambda} + CTm, \quad \forall \lambda \neq 0,
\]

\[
|m - \chi|_W \leq C\epsilon_0.
\]  (4.14) (4.15)

**Proof.** From (4.6) and Lemma 4.2 the sequence \( \phi^{(k)} \) is Cauchy. Thus the iteration sequence \( \phi^{(k)} \) converge to \( m \in W \) satisfying the CIE (4.14). Moreover,

\[
|\phi^{(k)} - \chi|_W \leq \sum_{n=1}^{k} |\phi^{(n)} - \phi^{(n-1)}|_W \leq \sum_{n=1}^{k} (C\epsilon_0)^n \leq C\epsilon_0,
\]  (4.16)

so (4.15) is justified.

Suppose \( m_1, m_2 \in W \) and satisfy (4.14) and (4.15). Hence

\[
m_1(x, \lambda) - m_2(x, \lambda) = \frac{m_{1,\text{res}}(x) - m_{2,\text{res}}(x)}{\lambda} + CT(m_1 - m_2),
\]

\[
m_{j,\text{res}}(x) = -\frac{\kappa_1 e^{\theta_1} + \kappa_2 a e^{\theta_2}}{e^{\theta_1} + a e^{\theta_2}} - \frac{\kappa_1 e^{\theta_1} C_{\lambda_j} T m_j + \kappa_2 a e^{\theta_2} C_{\lambda_j} T m_j}{e^{\theta_1} + a e^{\theta_2}}.
\]  (4.17)

Along with Theorem 2 yields

\[
|m_1 - m_2|_W \leq C\epsilon_0|m_1 - m_2|_W.
\]

Thanks to \( |m_j - \chi|_W \leq C\epsilon_0 \) for \( \epsilon_0 \ll 1 \), \( m_1(x, \lambda) \equiv m_2(x, \lambda) \).

\[\square\]

**Theorem 4.** Given \( u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2) \), if \( u_0 \) is a Gr(1, 2) \( KP \) soliton with data \( \kappa_j \neq 0 \), \( A \in \text{Gr}(1, 2)_{>0} \), \( \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} v_0 \in L^1(dx_1dx_2) \cap L^\infty(dx_1dx_2) \) for \( |k| \leq 4 \), \( v_0(x_1, x_2) \in \mathbb{R} \), and \( |v_0|_{L^1 \cap L^\infty(dx_1dx_2)} \leq \epsilon_0 \ll 1 \), then there exists uniquely \( m \in W \) satisfying

\[
m(x, \lambda) = 1 + \frac{m_{\text{res}}(x)}{\lambda} + CTm, \quad \forall \lambda \neq 0,
\]

\[
m(x_1, x_2, 0, \lambda) = m(x_1, x_2, \lambda), \quad |m(x, \lambda) - \chi(x, \lambda)|_W \leq C\epsilon_0,
\]

with

\[
(-\partial_{x_2} + \partial_{x_1}^{2} + 2\lambda \partial_{x_1} + u(x_1, x_2))m(x_1, x_2, \lambda) = 0,
\]

\[
m(x_1, x_2, \lambda) \to \chi(x_1, x_2, 0, \lambda).
\]  (4.18)

**Proof.** Applying results in the direct scattering problem [34], [35], [36], (4.18) is uniquely solved and \( S(u(x_1, x_2)) = \{0, \kappa_1, \kappa_2, D, s_c(\lambda)\} \) is admissible KdV type. From (1.9) and Theorem 3 it suffices to justify the initial data \( m(x_1, x_2, \lambda) \in W_{(x_1, x_2, 0)} \).

Applying definition of \( W^0 \) on page 4-5 and Definition 4, it reduces to proving \( m^3(x_1, x_2, \lambda) \) at \( \kappa_j \) can be expressed by \( \sum_{i=0}^\infty f_i(X)(-\ln(1 - \gamma|\beta|))^i \) and \( \sum_{i=0}^\infty \tilde{f}_i(X)(-\ln(1 - \gamma|\pi - \beta|))^i \) respectively. But these properties follows from the multi-valued properties (1.19) at \( \kappa_j \).
4.2. Admissible BPP scattering data. Given an admissible BPP scattering data $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$, via admissible condition, the $D$-symmetry, a Sato theory, and Theorem 2, we shall construct a recursive sequence

$$
\phi^{(k)}(x, \lambda) \equiv 1 + \sum_{n=1}^{N} \frac{\psi^{(k)}_{z_n, \text{res}}(x)}{\lambda - z_n} + CT\phi^{(k-1)}(x, \lambda),
$$

(4.19)

$$
\phi^{(0)}(x, \lambda) \equiv 1,
$$

such that $\phi^{(k)} \in W$ for $k > 0$ and $\phi^{(k)}$ converge to the unique solution of the CIE (1.31).

Lemma 4.3. Suppose $S = \{z_n, \kappa_j, D, s_c(\lambda)\}$ is an admissible BPP scattering data, and $\phi^{(k)}$ are defined by (4.19). Then for $k > 0$, $\phi^{(k)}$ satisfies the $D$-symmetry iff

$$
\begin{pmatrix}
\psi^{(k)}_{z_1, \text{res}} \\
\vdots \\
\psi^{(k)}_{z_N, \text{res}}
\end{pmatrix}
= -B^{-1}A
\begin{pmatrix}
1 + C_{\kappa_1} T\phi^{(k-1)} \\
\vdots \\
1 + C_{\kappa_M} T\phi^{(k-1)}
\end{pmatrix},
$$

(4.20)

with

$$
A =
\begin{pmatrix}
\kappa_1^N e^{\theta_1} & \cdots & 0 & D_{N+1,1} e^{\theta_{N+1}} & \cdots & D_{M,1} e^{\theta_M} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \kappa_N^N e^{\theta_N} & D_{N+1,N} e^{\theta_{N+1}} & \cdots & D_{M,N} e^{\theta_M}
\end{pmatrix},
$$

(4.21)

$$
B = A
\begin{pmatrix}
\frac{1}{\kappa_1-\kappa_1} & \cdots & \frac{1}{\kappa_1-\kappa_N} \\
\vdots & \ddots & \vdots \\
\frac{1}{\kappa_M-\kappa_1} & \cdots & \frac{1}{\kappa_M-\kappa_N}
\end{pmatrix},
$$

$$
e^{\theta_j} = e^{\kappa_j x_1 + \kappa_j^2 x_2 + \kappa_j^3 x_3}.
$$

Moreover,

$$
|\psi^{(k)}_{z_n, \text{res}}|_{L^\infty} \leq C(1 + (C\epsilon_0)^k |\chi|_{W})
$$

(4.22)

$$
|\psi^{(k)}_{z_n, \text{res}} - \psi^{(k-1)}_{z_n, \text{res}}|_{L^\infty} \leq (C\epsilon_0)^k,
$$

for $\phi^{(k)}$ and $\psi^{(k)}_{z_n, \text{res}}$ defined by (4.19) and (4.20),
Proof. Using the $\mathcal{D}$-symmetry and evaluating (4.19) at $\kappa_j^+ = \kappa_j + 0^+$, we obtain a linear system for $M + N$ variables \( \{ \phi^{(k)}(x, \kappa_j^+), \psi_{z_n, \text{res}}^{(k)}(x) \} \),

\[
\begin{pmatrix}
\kappa_1^N e^\theta_1 & \cdots & 0 & D_{N+1,1} e^\theta_{N+1} & \cdots & D_{M,1} e^\theta_M & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \kappa_N^N e^\theta_N & D_{N+1,N} e^\theta_{N+1} & \cdots & D_{M,N} e^\theta_M & 0 & \cdots & 0 \\
-1 & \cdots & 0 & 0 & \cdots & 0 & 1 & \kappa_{11^{-1}} & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & \cdots & -1 & 1 & \kappa_{M1^{-1}} & \cdots & 1 \\
\end{pmatrix}
\begin{pmatrix}
\phi^{(k)}(x, \kappa_1^+) \\
\vdots \\
\phi^{(k)}(x, \kappa_M^+) \\
\psi_{z_1, \text{res}}^{(k)}(x) \\
\vdots \\
\psi_{z_N, \text{res}}^{(k)}(x)
\end{pmatrix}
\]

(4.23)

Solving $\phi^{(k)}(x, \kappa_j^+)$ in terms of $\psi_{z_n, \text{res}}^{(k)}$ and plugging the outcomes into (4.23) again yield

\[
B \begin{pmatrix}
\psi_{z_1, \text{res}}^{(k)}(x) \\
\vdots \\
\psi_{z_N, \text{res}}^{(k)}(x)
\end{pmatrix} = -\tilde{A} \begin{pmatrix}
1 + C_{\kappa_1^+} T \phi^{(k-1)} \\
\vdots \\
1 + C_{\kappa_M^+} T \phi^{(k-1)}
\end{pmatrix},
\]

(4.24)

with $B$ and $\tilde{A}$ defined by (4.21). By the admissible condition, the system (4.23) is just determined and is equivalent to (4.20).

To prove (4.22), firstly, the admissible condition implies that

\[
\mathcal{D} = \mathcal{D}(z_n, \kappa_j, A', 0), \quad \mathcal{D}^z = \text{diag}(\kappa_1^N, \ldots, \kappa_M^N) (A')^T, \\
A' \in Gr(N, M)_{>0}, \quad |A'| < C.
\]

(4.25)
Corresponding to the $Gr(N, M)_{> 0}$ KP soliton with data $\kappa_1, \cdots, \kappa_M, A'$, the normalized Sato eigenfunction, denoted as

$$\tilde{\chi}'(x, \lambda) = \tilde{\chi}_{z_n, \kappa_j, A'}(x, \lambda),$$

satisfies the $\mathcal{D}$-symmetry. Therefore, following the above argument, namely, using $\mathcal{D}$-symmetry and evaluating $\tilde{\chi}'$ at $\kappa_j$, yield

$$
\begin{pmatrix}
\tilde{\chi}'_{z_1, \text{res}}(x) \\
\vdots \\
\tilde{\chi}'_{z_N, \text{res}}(x)
\end{pmatrix} = -B^{-1} \tilde{A}
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix},
$$

(4.26)

with $B$ and $\tilde{A}$ defined by (4.21), in particular, with $E_j = e^{\theta_j}$,

$$\tilde{A} = \begin{pmatrix}
\kappa_1^N e^{\theta_1} & \cdots & 0 & D_{N+1,1} e^{\theta_{N+1}} & \cdots & D_{M,1} e^{\theta_M} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \kappa_N^N e^{\theta_N} & D_{N+1,N} e^{\theta_{N+1}} & \cdots & D_{M,N} e^{\theta_M}
\end{pmatrix},
$$

(4.27)

$$B = \text{diag} \left( \frac{\kappa_1^{N-1}}{\Pi_{n \neq 1}(\kappa_1 - z_n)}, \cdots, \frac{\kappa_N^{N-1}}{\Pi_{n \neq 1}(\kappa_N - z_n)} \right) A' \text{ diag} (E_1, \cdots, E_M)
$$

$$
\times \begin{pmatrix}
\frac{\Pi_{n \neq 1}(\kappa_1 - z_n)\kappa_1^N}{\kappa_1} & \cdots & \frac{\Pi_{n \neq 1}(\kappa_1 - z_n)\kappa_N^N}{(\kappa_1 - z_N)\kappa_1^N} \\
\vdots & \ddots & \vdots \\
\frac{\Pi_{n \neq 1}(\kappa_M - z_n)\kappa_M^N}{\kappa_M} & \cdots & \frac{\Pi_{n \neq 1}(\kappa_M - z_n)\kappa_N^N}{(\kappa_M - z_N)\kappa_M^N}
\end{pmatrix}.
$$

From the Sato theory (cf. (1.3), (1.10), (4.26)-(4.27), matching the coefficients of $E_1 \times \cdots \times E_N$, and (1.29)), we conclude that

$$B^{-1} = \frac{1}{\tau'(x)} \begin{pmatrix}
b_{11} & \cdots & b_{1N} \\
\vdots & \ddots & \vdots \\
b_{N1} & \cdots & b_{NN}
\end{pmatrix},$$

(4.28)

$$b_{kl} = \sum_{|J(kl)|=N-1} \Delta_{J(kl)} E_{J(kl)}(x),$$

$$\tau'(x) \text{ is the tau function with data } \kappa_j, A',$$

$$|\Delta_{J(kl)}| = |\Delta_{J(kl)}(z_n, \kappa_j, A')| < C.$$
Moreover,
\[
\tau'(x)\tilde{\chi}_{\bar{z}_h,\text{res}}(x)
\]
\[
= \text{the h-row of } 
\begin{pmatrix}
  b_{11} & \cdots & b_{1N} \\
  \vdots & \ddots & \vdots \\
  b_{N1} & \cdots & b_{NN}
\end{pmatrix}
\begin{pmatrix}
  \kappa_N^N E_1 + \cdots + \mathcal{D}_{N+1,1} E_{N+1} + \cdots + \mathcal{D}_{M,1} E_M \\
  \vdots \\
  \vdots \\
  \vdots \\
  \kappa_N^N E_N + \cdots + \mathcal{D}_{N+1,N} E_{N+1} + \cdots + \mathcal{D}_{M,N} E_M
\end{pmatrix}
\]
\[
= (\kappa_1^N E_1 + \cdots + \mathcal{D}_{N+1,1} E_{N+1} + \cdots + \mathcal{D}_{M,1} E_M) \sum_{|J(h)|=N-1} \Delta_{J(h_1)} E_{J(h_1)}(x)
\]
\[
+ \cdots + (\kappa_N^N E_N + \cdots + \mathcal{D}_{N+1,N} E_{N+1} + \cdots + \mathcal{D}_{M,N} E_M) \sum_{|J(h)|=N-1} \Delta_{J(h_N)} E_{J(h_N)}(x)
\]
\[
\equiv (\tilde{a}_{11} E_1 + \cdots + \tilde{a}_{1M} E_M) \sum_{|J(h)|<N} \Delta_{J(h_1)} E_{J(h_1)}(x)
\]
\[
+ \cdots + (\tilde{a}_{N1} E_1 + \cdots + \tilde{a}_{NM} E_M) \sum_{|J(h)|=N-1} \Delta_{J(h_N)} E_{J(h_N)}(x),
\]
and
\[
0 = \tilde{a}_{1k} E_k \sum_{k \in J(h_1), |J(h_1)|=N-1} \Delta_{J(h_1)} E_{J(h_1)}(x) + \cdots
\]
\[
+ \tilde{a}_{Nk} E_k \sum_{k \in J(h_N), |J(h_N)|=N-1} \Delta_{J(h_N)} E_{J(h_N)}(x)
\]
(4.30)
by the Sato theory.

Using (4.20), (4.27), (4.28), (4.30), and multi linearity,
\[
\tau'(x)\psi^{(k)}_{\bar{z}_h,\text{res}}(x)
\]
\[
= \tau'(x)\tilde{\chi}_{\bar{z}_h,\text{res}}(x)
\]
\[
+ \text{the h-row of } 
\begin{pmatrix}
  b_{11} & \cdots & b_{1N} \\
  \vdots & \ddots & \vdots \\
  b_{N1} & \cdots & b_{NN}
\end{pmatrix}
\begin{pmatrix}
  \tilde{a}_{11} E_1 C_{\kappa_1^k} T_{\phi^{(k-1)}} + \cdots + \tilde{a}_{1M} E_M C_{\kappa_M^k} T_{\phi^{(k-1)}} \\
  \vdots \\
  \vdots \\
  \vdots \\
  \tilde{a}_{N1} E_1 C_{\kappa_1^k} T_{\phi^{(k-1)}} + \cdots + \tilde{a}_{NM} E_M C_{\kappa_M^k} T_{\phi^{(k-1)}}
\end{pmatrix}
\]
\[
= \sum_{|J(h)|=N} \Delta_{J(h)} E_{J(h)}(x),
\]
with

\(J(h)\) consisting of \(N\) distinct numbers in \(\{1, \cdots, M\}\).
\[|\Delta J(h)| = |\Delta J(h)(z_n, \kappa_j, A')| < C(1 + \sum_{j=1}^{M} |C_{\kappa_j}T\phi^{(k-1)}|).\]

Along with the totally positive condition of \(A'\), yield

\[|\psi_{z_n, \text{res}}^{(k)}(x)| \leq C(1 + \sum_{j=1}^{M} |C_{\kappa_j}T\phi^{(k-1)}|),\]

and (4.22) follows from Theorem 2.

\[\square\]

Stipulating residues \(\psi_{z_n, \text{res}}^{(k)}(x)\) in (1.19) by (1.20), we construct the iteration sequence \(\{\phi^{(k)}\}\). Applying Theorem 2 and Lemma 4.3, we have \(\phi^{(k)} \in W\) for \(k > 0\).

Note that the choice of the initial map \(\phi^{(0)}\) is not unique, it can be replaced by any function satisfying (a), (b), (c) in \(W\), \(E_{\kappa_j} \phi \in \mathfrak{M}(D_{\kappa_j})\), and \(E_{\kappa_j} \phi\) holomorphic in \(\tilde{s}\). For instance, \(\tilde{x}_{z_n, \kappa_j, A}\) and constant functions fulfill these criteria.

Adapting the approach in previous subsection, we have

**Theorem 5.** Given an admissible BPP scattering data \(S = \{z_n, \kappa_j, D, s_c(\lambda)\}\), there exists uniquely \(m \in W\) satisfying

\[m(x, \lambda) = 1 + \sum_{n=1}^{N} \frac{m_{z_n, \text{res}}(x)}{\lambda - z_n} + CTm, \quad \forall \lambda \notin \{z_1, \ldots, z_N\}\]

with

\[|m(x, \lambda) - \tilde{x}_{z_n, \kappa_j, A}(x, \lambda)|_{W} \leq C\epsilon_0.\]

**Theorem 6.** Given \(u(x_1, x_2) = u_0(x_1, x_2, 0) + v_0(x_1, x_2)\), if \(u_0\) is a \(\text{Gr}(N, M)\) KP soliton with data \(\kappa_j, A \in \text{Gr}(N, M)\), \(\partial_{x_1}^k \partial_{x_2}^k v_0 \in L^1(dx_1dx_2) \cap L^\infty(dx_1dx_2)\) for \(|k| \leq 4\), \(v_0(x_1, x_2) \in \mathbb{R}\), and \(|v_0|_{L^1 \cap L^\infty(dx_1dx_2)} \leq \epsilon_0 \ll 1\), \(\det(\frac{1}{\kappa_j - z_n})_{1 \leq j, n \leq N} \neq 0\), with \(z_1 = 0\), \(\{z_n, \kappa_j\}\) distinct real, then there exists a unique \(m(x, \lambda) \in W\) satisfying that

\[m(x, x_2, 0, \lambda) = m(x_1, x_2, 0), \quad |m(x, \lambda) - \tilde{x}_{z_n, \kappa_j, A}(x, \lambda)|_{W} \leq C\epsilon_0,\]

with

\[\begin{align*}
(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x_1, x_2))m(x_1, x_2, \lambda) = 0, \\
\end{align*}\]

\[m(x_1, x_2, \lambda) \rightarrow \tilde{x}_{z_n, \kappa_j, A}(x_1, x_2, 0, \lambda).\]
4.3. A representation formula for $u(x)$. We formally derive a representation formula of $u(x)$ in this section.

Let

$$-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} = -\nabla_2 + \nabla_1^2, \quad \nabla_1 = \partial_{x_1} + \lambda, \quad \nabla_2 = \partial_{x_2} + \lambda^2,$$

and

$$m = 1 + Jm + CTm, \quad J \phi = \sum_{n=1}^{N} \frac{\phi_{z_n, res}(x)}{\lambda - z_n},$$

(4.33)

Hence

$$(-\nabla_2 + \nabla_1^2)m = (-\nabla_2 + \nabla_1^2)(1 + Jm + CTm)$$

$$= \left[-\nabla_2 + \nabla_1^2, J + CT \right] m + (J + CT) (-\nabla_2 + \nabla_1^2)m.$$  

(4.34)

Note

$$[\nabla_j, T] = 0, \quad j = 1, 2,$$

hence

$$[-\nabla_2 + \nabla_1^2, CT] m = \left[-\nabla_2 + \nabla_1^2, C \right] Tm = 2 [\lambda, C] \partial_{x_1} (Tm)$$

$$= -\frac{i}{\pi} \partial_{x_1} \iint Tm \, d\zeta \wedge d\zeta.$$ 

Moreover,

$$[\nabla_2 + \nabla_1^2, J] m = \left[-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1}, J \right] m$$

$$= (-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1}) \sum_{n=1}^{N} \frac{m_{z_n, res}(x)}{\lambda - z_n} - \sum_{n=1}^{N} \frac{((-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1})m)_{z_n, res}(x)}{\lambda - z_n}$$

$$= 2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, res}(x).$$

Consequently,

$$[-\nabla_2 + \nabla_1^2, J + CT] m = -\frac{i}{\pi} \partial_{x_1} \iint Tm \, d\zeta \wedge d\zeta + 2\partial_{x_1} \sum_{n=1}^{N} m_{z_n, res}(x)$$

$$\equiv -u(x)$$

(4.35)

which is a function of $x$ alone. Combining (4.33), (4.34), and (4.35), we obtain

$$(-\nabla_2 + \nabla_1^2)m = -(1 - J - CT)^{-1}u(x)1 = u(x)(1 - J - CT)^{-1}1 = -u(x)m(x, \lambda).$$

(4.36)
So we have

**Theorem 7.** Given an admissible KdV or BPP scattering data $S = \{z_n, \kappa_j, \mathcal{D}, s_c(\lambda)\}$, let $m$ be the unique eigenfunction derived from Theorem 3 or 5. Formally,

\[
\left(-\partial_{x_2} + \partial_{x_1}^2 + 2\lambda \partial_{x_1} + u(x)\right) m(x, \lambda) = 0, \quad \lambda \neq z_n, \kappa_j,
\]

with

\[
u(x) = \frac{i}{\pi} \partial_{x_1} \int \int Tm \; d\zeta \wedge d\zeta - 2\partial_{x_1} \sum_{n=1}^{N} m_{zn, \text{res}}(x).
\]

Fourier and complex function theories provide an efficient approach to derive a rigorous proof and prove $u(x)$ satisfies the KP equation (1.1) provided the background is vacuum \[33], \[32]. We leave the investigation in our case, which gives an answer to the stability problem of $Gr(N, M) > 0$ KP solitons, for the future.

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