Elliptic Algebra $U_{q,p}(\hat{g})$ and Quantum $Z$-algebras

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Abstract A new definition of the elliptic algebra $U_{q,p}(\hat{g})$ associated with an untwisted affine Lie algebra $\hat{g}$ is given as a topological algebra over the ring of formal power series in $p$. We also introduce a quantum dynamical analogue of Lepowsky-Wilson’s $Z$-algebras. The $Z$-algebra governs the irreducibility of the infinite dimensional $U_{q,p}(\hat{g})$-modules. Some level-1 examples indicate a direct connection of the irreducible $U_{q,p}(\hat{g})$-modules to those of the $W$-algebras associated with the coset $\hat{g} \oplus \hat{g} \supset (\hat{g})_{\text{diag}}$ with level $(r - g - 1, 1)$ ($g$: the dual Coxeter number), which includes Fateev-Lukyanov’s $WB_l$-algebra.

Keywords Quantum group · Affine Lie algebra · Virasoro algebra · $W$-algebra · $Z$-algebra

Mathematics Subject Classification (2010) 17B37 · 20G42 · 81R10 · 81R50

1 Introduction

The algebra $U_{q,p}(\hat{g})$ is an elliptic analogue [1, 2] of the quantum affine algebra $U_q(\hat{g})$ in the Drinfeld realization [3]. There are two types of the elliptic quantum groups, the vertex type and the face type [4, 5]. Deriving the $L$-operators [2, 6, 7] and introducing the
Hopf-algebroid structure [8–10] $U_{q,p}(\hat{\mathfrak{g}})$ is now recognized as a face type elliptic quantum group.

Originally $U_{q,p}(\hat{\mathfrak{g}})$ with $p = q^{2r}$ was derived for $\hat{\mathfrak{s}l}_2 = \widehat{\mathfrak{s}l}(2, \mathbb{C})$ [1] as a deformation of the screening currents of the coset conformal field theory (CFT) $\mathfrak{s}l_2 \oplus \hat{\mathfrak{s}l}_2 \supset (\hat{\mathfrak{s}l}_2)_{\text{diag}}$ with level $(r - k - 2, k)$ [11–16] instead of considering a deformation of $U_q(\mathfrak{s}l_2)$ itself. Such coset CFT is known to be realized in terms of the level-$k$ free boson and the $\mathbb{Z}_k$-parafermion [17], or the $Z$-algebra [18] associated with the level-$k$ standard representation of $\mathfrak{s}l_2$. It was then crucial in [1] to realize that the level-$k$ boson should be deformed both $q$- and elliptically [19] whereas the $\mathbb{Z}_k$-parafermion gets only a $q$-deformation to obtain consistent relations for the generators in $U_{q,p}(\hat{\mathfrak{s}l}_2)$.

In [2], a realization of $U_{q,p}(\hat{\mathfrak{g}})$ for general untwisted affine Lie algebra $\hat{\mathfrak{g}}$ was given by modifying the Drinfeld realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$. However its structure associated with the quantum $Z$-algebras has not yet been discussed so far. The purpose of this paper is to address this subject. The general theory of the $Z$-algebra was studied by Lepowsky and Wilson [18] and by Gepner [20] in the representation theory of affine Lie algebras and in CFT, respectively. Its quantum deformation and application to the representations of $U_q(\hat{\mathfrak{g}})$ was partially investigated in [1, 21–24]. A construction of the coset CFT associated with the general $\hat{\mathfrak{g}}$ was also given [25] in terms of the generalized parafermions. We extend these studies to the elliptic algebras $U_{q,p}(\hat{\mathfrak{g}})$. In particular, we define a dynamical analogue $\hat{\mathcal{Z}}_k$ of the quantum $Z$-algebras and show that the level-$k$ highest weight representations of $U_{q,p}(\hat{\mathfrak{g}})$ are realized in terms of $\hat{\mathcal{Z}}_k$ and the level-$k$ elliptic bosons. It is then shown that the irreducibility of the infinite dimensional $U_{q,p}(\hat{\mathfrak{g}})$-modules is governed by the $\hat{\mathcal{Z}}_k$-modules as in the affine Lie algebra cases [18].

On the other hand, it was conjectured [1, 2] that the $U_{q,p}(\hat{\mathfrak{g}})$ provides an algebra of the screening currents of the deformation of the $W$-algebras associated with the coset $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}} \supset (\hat{\mathfrak{g}})_{\text{diag}}$ with level $(r - g - 1, 1)$. For the simply-laced $\hat{\mathfrak{g}}$, such deformed $W$-algebras have been realized in [26–28], and in particular for the $\hat{\mathfrak{s}l}_N$ case the conjecture has been established by an explicit comparison of the free field realizations [7, 26, 27, 29]. However for the non-simply laced $\hat{\mathfrak{g}}$, deformation of the coset type $W$-algebras has not yet been studied at all. One should note that the coset type $W$-algebras associated with the non-simply laced $\hat{\mathfrak{g}}$ are different from those obtained by the quantum Hamiltonian reduction. See for example [30]. We investigate this issue further by giving an explicit realization of the level-1 irreducible highest weight representations of $U_{q,p}(\hat{\mathfrak{g}})$ for $\hat{\mathfrak{g}} = A_1^{(1)}, B_1^{(1)}, D_1^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$. We show that at least for $A_1^{(1)}$ and $D_1^{(1)}$ the level-1 elliptic currents $e_j(z)$ and $f_j(z)$ coincide with the screening currents of the deformed $W$-algebras obtained in [26–28]. We also show that the irreducible representations of $U_{q,p}(\hat{\mathfrak{g}})$ is naturally decomposed into a direct sum of the irreducible $W$-algebras of the coset type for $\hat{\mathfrak{g}} = A_1^{(1)}, B_1^{(1)}, D_1^{(1)}$. This suggests in particular an existence of a deformation of Fateev-Lukyanov’s $WB_l$-algebra [31] as the commutant of the screening operators provided by the level-1 elliptic currents $e_j(z)$ and $f_j(z)$ of $U_{q,p}(B_l^{(1)})$.

It is also worth to mention that the coset type $W$-algebras describe a critical behavior of the face type elliptic solvable lattice models [32, 33]. Correspondingly the $U_{q,p}(\hat{\mathfrak{g}})$ provides an algebraic framework to formulate the lattice model itself in the spirit of Jimbo and Miwa [34]. This has been established for $\hat{\mathfrak{s}l}_N$ in [1, 2, 7, 35, 36] by constructing the $L$-operator.

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\(^1\)The difference between $Z$-algebra and Parafermion is whether one adds zero-modes of the bosons to it or not.
and introducing the Hopf algebroid structure. In order to construct the $L$-operator of $U_{q,p}(\hat{g})$ and also to get a realization of a generating function of the deformation of the $W$-algebras, it is crucial to introduce new types of elliptic bosons, which we call the fundamental weight type $A^j_m$ and the orthonormal basis type $E^\pm_m^j$ distinguishing from the usual ones $\alpha_{j,m}$ ($\alpha_{j,m}^\vee$) corresponding to the simple (co-)root and appearing as generators of $U_{q,p}(\hat{g})$. An idea of such bosons has already appeared in [26–28]. We give an explicit construction of them for $\hat{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$. As a check we calculate the commutation relations among $E^\pm_m^j$ as well as among the elliptic currents $k_{\pm j}(z)$, the generating functions of $E^\pm_m^j$, and show that they have a universal form. See Theorem 5.3 and 5.7.

This paper is organized as follows. In section 2, we define the elliptic algebra $U_{q,p}(\hat{g})$ as a topological algebra generated by the elliptic Drinfeld generators. This is a new definition of $U_{q,p}(\hat{g})$ given independently of $U_q(\hat{g})$ unlike the previous one in Appendix A in [2]. In section 3, we define a quantum dynamical analogue $Z_\gamma$ of Lepowsky and Wilson’s $Z$-algebra associated with the level-$k$ $U_{q,p}(\hat{g})$-module $\gamma$ and its universal counterpart $Z_k$. The irreducibility of the level-$k$ highest weight representation of $U_{q,p}(\hat{g})$ is shown to be governed by the $Z_k$-module. In section 4, we give a simple realization of $Z_k$ in terms of the quantum (non-dynamical) $Z$-algebra associated with the level-$k$ $U_{q,p}(\hat{g})$-module and define a standard representation of $U_{q,p}(\hat{g})$. We provide some level-1 examples of the standard representations and discuss their relation to the deformation of the $W$-algebras. In section 5, we give a construction of the new elliptic bosons of the fundamental weight type and the orthonormal basis type and derive various commutation relations.

## 2 Elliptic Algebra $U_{q,p}(\hat{g})$

### 2.1 Definition

Let $\hat{g} = X_l^{(1)}$ be an untwisted affine Lie algebra associated with the generalized Cartan matrix $A = (a_{ij})$, $i,j \in \{0\} \cup I$, $I = \{1, \cdots, l\}$. We denote by $B = (b_{ij})$, $b_{ij} = d_i a_{ij}$ the symmetrization of $A$. We take $d_i = 1$ ($i \in I$) for the simply laced cases, $d_i = 1$ ($1 \leq i \leq l - 1$), $d_i = 1/2$ for $B_l^{(1)}$ and $d_i = 1$ ($1 \leq i \leq l - 1$), $d_i = 2$ for $C_l^{(1)}$. Let $q = e^{i\pi} \in \mathbb{C}[[h]]$ and set $q_1 = g^{d_i}$. Let $\rho$ be an indeterminate.

Let $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}d$, $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}c$, $\mathfrak{h} = \oplus_{i \in I} \mathbb{C}h_i$ be the Cartan subalgebra of $\hat{g}$. Define $\delta, \Lambda_0, \alpha_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$<\alpha_i, h_j> = a_{j,i}, <\delta, d> = 1 = <\Lambda_0, c>, \tag{2.1}$$

the other pairings are 0. We also define $\tilde{\Lambda}_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$<\tilde{\Lambda}_i, h_j> = \delta_i, j.$$ We set $\tilde{\mathfrak{h}}^* = \oplus_{i \in I} \mathbb{C}\tilde{\Lambda}_i$, $\tilde{\mathfrak{h}}^* = \tilde{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$, $Q = \oplus_{i \in I} \mathbb{Z}\alpha_i$ and $P = \oplus_{i \in I} \mathbb{Z}\tilde{\Lambda}_i$. Let $N = l + 1$ for $X_l = A_l = l$ for $B_l, C_l, D_l = 7$ for $E_6, = 8$ for $E_7, E_8, = 3$ for $G_2 = 4$ for $F_4$ and consider the orthonormal basis $\{\xi_j (1 \leq j \leq N)\}$ in $\mathbb{R}^N$ with the inner product $(\xi_j, \xi_k) = \delta_{j,k}$. For $A_l$, we also set

$$\tilde{\xi}_j = \xi_j - \frac{1}{l + 1} \sum_{j=1}^{l+1} \xi_j. \tag{2.2}$$

We define $\epsilon_j = \tilde{\xi}_j$ for $A_l$ and $= \xi_j$ for other $X_l$. The simple roots $\alpha_j$ and the fundamental weights $\Lambda_j$ ($1 \leq j \leq l$) can be expressed as a linear sum of $\epsilon_j$ [37, 38]. We follow
Kac’s conventions. We define $h_{\epsilon_j} \in \mathfrak{h}$ ($j \in I$) by $\langle \epsilon_i, h_{\epsilon_j} \rangle = (\epsilon_i, \epsilon_j)$ and $h_\alpha \in \mathfrak{h}$ for $\alpha = \sum_j c_j \epsilon_j$, $c_j \in \mathbb{C}$ by $h_\alpha = \sum_j c_j h_{\epsilon_j}$. We regard $\mathfrak{h} \oplus \mathfrak{h}^*$ as the Heisenberg algebra by

$$[h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k]. \tag{2.3}$$

In particular, we have $[h_j, \alpha_k] = a_{jk}$. We also set $h_j^j = h_{\tilde{\alpha}_j}$.

In order to treat the dynamical shifts in the face type elliptic algebra systematically, we introduce another Heisenberg algebra generated by $P_\alpha$ and $Q_\beta$ ($\alpha, \beta \in \mathfrak{h}^*$) satisfying the commutation relations

$$[P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}]. \tag{2.4}$$

We also set

$$[P_{\epsilon_j}, \alpha] = [Q_{\epsilon_j}, \alpha] = 0, \quad [P_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = [Q_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = 0 \tag{2.5}$$

where $P_\alpha = \sum_j c_j P_{\epsilon_j}$ for $\alpha = \sum_j c_j \epsilon_j$. We set $P_\mathfrak{h} = \oplus_{\epsilon \in I} \mathbb{C} P_{\epsilon}$, $Q_\mathfrak{h} = \oplus_{\epsilon \in I} \mathbb{C} Q_{\epsilon}$, $P_j = P_{\alpha_j^j}$, $P^j = P_{\tilde{\alpha}_j^j}$ and $Q_j = Q_{\alpha_j}$, $Q^j = Q_{\tilde{\alpha}_j}$. Here $\alpha_j^j = 2\alpha_j/(\alpha_j, \alpha_j)$.

For the abelian group $\mathcal{R}_\mathcal{Q} = \sum_{j=1}^N \mathbb{Z} Q_{\alpha_j}$, we denote by $\mathbb{C}[\mathcal{R}_\mathcal{Q}]$ the group algebra over $\mathbb{C}$ of $\mathcal{R}_\mathcal{Q}$. We denote by $e^\alpha$ the element of $\mathbb{C}[\mathcal{R}_\mathcal{Q}]$ corresponding to $\alpha \in \mathcal{R}_\mathcal{Q}$. These $e^\alpha$ satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us set $H = \mathfrak{h} \oplus P_\mathfrak{h} = \sum_j \mathbb{C}(P_{\epsilon_j} + h_{\epsilon_j}) + \sum_j \mathbb{C} P_{\epsilon_j} + \mathbb{C} c$ and denote its dual space by $H^* = \tilde{\mathfrak{h}}^* \oplus Q_\mathfrak{h}$. We define the paring by Eq. (2.1), $\langle Q_\alpha, P_\beta \rangle = (\alpha, \beta)$ and $\langle Q_\alpha, h_{\beta} \rangle = \langle Q_\alpha, c \rangle = \langle Q_\alpha, d \rangle = 0 = \langle \alpha, P_\beta \rangle = \langle \delta, P_\beta \rangle = \langle \Lambda_0, P_\beta \rangle$. We define $\mathbb{F} = \mathcal{M}_H$ to be the field of meromorphic functions on $H^*$. We regard a function of $P + h = \sum_j a_j(P_{\epsilon_j} + h_{\epsilon_j})$, $P = \sum_j b_j P_{\epsilon_j}$ and $c$, $\widehat{f} = f(P + h, P, c)$, as an element in $\mathbb{F}$ by $\widehat{f}(\mu) = f(\mu, P + h, \mu, \mu, c)$ for $\mu \in H^*$.

We use the following notations.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_i = \frac{q^n_i - q^{-n}_i}{q_i - q_i^{-1}}, \quad [n]_j = \frac{q^n_j - q^{-n}_j}{q_j - q_j^{-1}},$$

$$[n]_i! = [n]_i[n-1]_i\cdots[1]_i, \quad \left[ \begin{array}{c} m \\ n \end{array} \right] = \frac{[m]_i!}{[n]_i! [m-n]_i!},$$

$$\left( x; q \right)_\infty = \prod_{n=0}^{\infty} (1 - xq^n), \quad \left( x; q, t \right)_\infty = \prod_{n,m=0}^{\infty} (1 - xq^n t^m),$$

$$\Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty.$$
Definition 2.1 The elliptic algebra $U_{q,p}(\mathfrak{g})$ is a topological algebra over $\mathbb{F}[[p]]$ generated by $\mathcal{M}_{H^+}$, $e_{j,m}, f_{j,m}, \alpha_{j,n}^\vee, K_j^\pm, (j \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0})$, $d$ and the central element $c$. We assume $K_j^\pm$ are invertible and set

$$
e j(z) = \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m},$$

$$\psi_j^+(q^{-\frac{1}{p}}z) = K_j^+ \exp \left( -(q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,-n}^\vee}{1 - p^n z^n} \right) \exp \left( (q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1 - p^n z^n} \right),$$

$$\psi_j^-(q^\frac{1}{p}z) = K_j^- \exp \left( -(q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,-n}^\vee}{1 - p^n z^n} \right) \exp \left( (q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1 - p^n z^n} \right).$$

Note that $\psi_j^\pm(z)$ are formal Laurent series in $z$, whose coefficients are well defined in the $p$-adic topology. We call $e_j(z), f_j(z), \psi_j^\pm(z)$ the elliptic currents. The defining relations are as follows. For $g(P), g(P+h) \in \mathcal{M}_{H^+}$,

$$g(P + h) e_j(z) = e_j(z) g(P + h), \quad g(P) e_j(z) = e_j(z) g(P - < Q_{\alpha_j}, P >),$$

$$g(P + h) f_j(z) = f_j(z) g(P + h - < \alpha_j, P + h >), \quad g(P) f_j(z) = f_j(z) g(P),$$

$$[g(P), \alpha_{i,m}^\vee] = [g(P + h), \alpha_{i,n}^\vee] = 0,$$

$$g(P) K_j^\pm = K_j^\pm g(P - < Q_{\alpha_j}, P >),$$

$$g(P + h) K_j^\pm = K_j^\pm g(P + h - < Q_{\alpha_j}, P >),$$

$$[d, g(P)] = [d, g(P + h)] = 0,$$

$$[d, \alpha_{j,n}^\vee] = n \alpha_{j,n}^\vee, \quad [d, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [d, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z),$$

$$K_j^\pm e_j(z) = q_i^{\pm a_{ij}} e_j(z) K_i^\pm, \quad K_j^\pm f_j(z) = q_i^{\pm a_{ij}} f_j(z) K_i^\pm,$$

$$[\alpha_{i,m}^\vee, \alpha_{j,n}^\vee] = \delta_{m+n,0} \frac{[a_{i,m}]_k [cm]_j}{m} \frac{1 - p^{-m}}{1 - p^{-m} q^{-cm}},$$

$$[\alpha_{i,m}^\vee, e_j(z)] = \left[ \frac{[a_{i,m}]_k [cm]_j}{m} \frac{1 - p^{-m}}{1 - p^{-m} q^{-cm}} \right] z^m e_j(z),$$

$$[\alpha_{i,m}^\vee, f_j(z)] = -\left[ \frac{[a_{i,m}]_k [cm]_j}{m} \frac{1 - p^{-m}}{1 - p^{-m} q^{-cm}} \right] z^m f_j(z),$$

$$z_1^{(q_{bij} z_1z_2; p^\infty)} e_i(z_1) e_j(z_2) = -z_2^{(q_{bij} z_1z_2; p^\infty)} e_j(z_2) e_i(z_1),$$

$$z_1^{(q^{-b_{ij}} z_1z_2; p^\infty)} f_i(z_1) f_j(z_2) = -z_2^{(q^{-b_{ij}} z_1z_2; p^\infty)} f_j(z_2) f_i(z_1),$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left( \delta(q^{-c} z_1z_2) \psi_j^-(q^\frac{1}{p}z_2) - \delta(q^c z_1z_2) \psi_j^+(q^{-\frac{1}{p}}z_2) \right).$$

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\[
\sum_{\sigma \in S_n} \prod_{1 \leq m < k \leq a} \frac{(p^*q^2z_{\sigma(k)}/z_{\sigma(m)}; p^*)_{\infty}}{(p^*q^{-2}z_{\sigma(k)}/z_{\sigma(m)}; p^*)_{\infty}} 
\times \sum_{s=0}^{a} (-1)^s \left[ \sum_{1 \leq m \leq s} \left( \frac{(p^*q^2h/w/z_{\sigma(m)}; p^*)_{\infty}}{(p^*q^{-2}h/w/z_{\sigma(m)}; p^*)_{\infty}} \right) \prod_{s+1 \leq m \leq a} \left( \frac{(p^*q^{-b}z_{\sigma(m)}/w; p^*)_{\infty}}{(p^*q^{b}z_{\sigma(m)}/w; p^*)_{\infty}} \right) \right] 
\times e_1(z_{\sigma(1)}) \cdots e_1(z_{\sigma(s)}) e_j(w) e_1(z_{\sigma(s+1)}) \cdots e_1(z_{\sigma(a)}) = 0, \quad (i \neq j, a = 1 - a_i), \quad (2.20)
\]

where \( p^* = pq^{-2c} \) and \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \). We also denote by \( U'_{q,p}(\widehat{\mathfrak{g}}) \) the subalgebra obtained by removing \( d \).

We treat the relations (2.12), (2.15)–(2.21) as formal Laurent series in \( z, w \) and \( z_j \)'s. In each term of Eqs. (2.17)–(2.21), the expansion direction of the structure function given by a ratio of infinite products is chosen according to the order of the accompanied product of the elliptic currents. For example, in the l.h.s of Eq. (2.17), \( \frac{(q^{b}z_{2}/z_{1}; p^*)_{\infty}}{(p^*q^{-b}z_{2}/z_{1}; p^*)_{\infty}} \) should be expanded in \( z_2/z_1 \), whereas in the r.h.s \( \frac{(q^{b}z_{1}/z_{2}; p^*)_{\infty}}{(p^*q^{-b}z_{1}/z_{2}; p^*)_{\infty}} \) should be expanded in \( z_1/z_2 \). In each term in Eq. (2.20), the coefficient function is expanded in \( z_{\sigma(k)}/z_{\sigma(m)} \) (\( m < k \)), \( w/z_{\sigma(m)} \) (\( m \leq s \)) and \( z_{\sigma(m)}/w \) (\( m \geq s + 1 \)). All the coefficients in \( z_j \)'s are well defined in the \( p \)-adic topology.

Remark. In [1, 2, 35, 36], assuming that \( q \) is a transcendental complex number satisfying \( |q| < 1 \), we wrote (2.17), (2.18) as
\[
\begin{align*}
&z_1 \Theta_{p^*}(q^{b}z_{2}/z_{1}) e_1(z_1) e_j(z_2) = -z_2 \Theta_{p^*}(q^{b}z_{1}/z_{2}) e_1(z_2) e_1(z_1), \\
&z_1 \Theta_{p}(q^{-b}z_{2}/z_{1}) f_i(z_1) f_j(z_2) = -z_2 \Theta_{p}(q^{-b}z_{1}/z_{2}) f_i(z_2) f_i(z_1),
\end{align*}
\]
in the sense of analytic continuation.

Let \( U_{q}(\widehat{\mathfrak{g}}) \) be the quantum affine algebra associated with \( \widehat{\mathfrak{g}} \) in the Drinfeld realization [3]. See Appendix A. \( U_{q,p}(\widehat{\mathfrak{g}}) \) is a natural face type (i.e. dynamical) elliptic deformation of \( U_{q}(\widehat{\mathfrak{g}}) \) in the following sense.

Theorem 2.2

\[ U_{q,p}(\widehat{\mathfrak{g}})/pU_{q,p}(\widehat{\mathfrak{g}}) \cong (\mathbb{F} \otimes \mathbb{C} U_{q}(\widehat{\mathfrak{g}})) \otimes \mathbb{C}[\mathcal{R}_{Q}]. \]

Here the smash product \( \boxtimes \) is defined as follows:
\[
g(P, P + h)x \otimes e^a \cdot f(P, P + h)y \otimes e^b = g(P, P + h) f(P - < \alpha, P >, P + h - < \alpha + wt(x), P + h >) x y \otimes e^{a+\beta}
\]
where \( wt(x) \in \widehat{\mathfrak{h}}^* \) s.t. \( q^h x q^{-h} = q^{-wt(x)} x h \) for \( x, y \in U_{q}(\widehat{\mathfrak{g}}), f(P), g(P) \in \mathbb{F}, e^a, e^b \in \mathbb{C}[\mathcal{R}_{Q}] \).
Proof At \( p = 0 \), the relations for \( \alpha_{j,m}^\vee, e_j(z), f_j(z) \) (2.12)–(2.21) coincide with those for \( a_{j,m}^\vee, x_j^+(z), x_j^-(z) \) (6)–(11) of \( U_q(\hat{g}) \). Therefore from Eqs. (2.6)–(2.10), one has the isomorphism

\[
e_j(z) \mapsto x_j^+(z)e^{-Q_{a_j}}, \quad f_j(z) \mapsto x_j^-(z), \quad K_j^\pm \mapsto q^ {\mp h_j} e^{-Q_{a_j}}, \quad \alpha_{j,m}^\vee \mapsto a_{j,m}^\vee \mod pU_{q,p}(\hat{g}).
\]

2.2 \( H \)-algebra \( U_{q,p}(\hat{g}) \)

Let \( A \) be a complex associative algebra, \( H \) be a finite dimensional commutative subalgebra of \( A \), and \( \mathcal{M}_H^* \) be the field of meromorphic functions on \( H^* \) the dual space of \( H \).

**Definition 2.3** (\( H \)-algebra) An \( H \)-algebra is an associative algebra \( A \) with 1, which is bigraded over \( H^* \),

\[
A = \bigoplus_{\alpha, \beta \in H^*} A_{\alpha \beta},
\]

and equipped with two algebra embeddings \( \mu_l, \mu_r : \mathcal{M}_H^* \to A_{00} \) (the left and right moment maps), such that

\[
\mu_l(\hat{f})a = a \mu_l(T_{\alpha} \hat{f}), \quad \mu_r(\hat{f})a = a \mu_r(T_\beta \hat{f}),
\]

\( a \in A_{\alpha \beta}, \hat{f} \in \mathcal{M}_H^* \), where \( T_\alpha \) denotes the automorphism \( T_\alpha \hat{f}(\lambda) = \hat{f}(\lambda + \alpha) \) of \( \mathcal{M}_H^* \).

**Proposition 2.4** \( U = U_{q,p}(\hat{g}) \) is an \( H \)-algebra by

\[
U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha \beta},
\]

\[
U_{\alpha \beta} = \left\{ x \in U \mid q^{P+h}xq^{-(P+h)} = q^{<\alpha,p+h>}x, \quad q^P \cdot x = q^{<\beta,p>}x \ \forall P + h, P \in H \right\}
\]

and \( \mu_l, \mu_r : \mathbb{F} \to U_{00} \) defined by

\[
\mu_l(\hat{f}) = f(P + h, p) \in \mathbb{F}[[p]], \quad \mu_r(\hat{f}) = f(P, p^*) \in \mathbb{F}[[p]].
\]

2.3 Dynamical Representations

Let us consider a vector space \( \mathcal{V} \) over \( \mathbb{F} \), which is \( H \)-diagonalizable, i.e.

\[
\mathcal{V} = \bigoplus_{\lambda, \mu \in H^*} \mathcal{V}_{\lambda, \mu}, \quad \mathcal{V}_{\lambda, \mu} = \{ v \in \mathcal{V} \mid q^{P+h} \cdot v = q^{<\lambda, p+h>}v, \quad q^P \cdot v = q^{<\mu, p>}v \ \forall P + h, P \in H \}.
\]

Let us define the \( H \)-algebra \( \mathcal{D}_{H, \mathcal{V}} \) of the \( \mathbb{C} \)-linear operators on \( \mathcal{V} \) by

\[
\mathcal{D}_{H, \mathcal{V}} = \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H, \mathcal{V}})_{\alpha \beta},
\]

\[
(\mathcal{D}_{H, \mathcal{V}})_{\alpha \beta} = \left\{ X \in \text{End}_\mathbb{C}\mathcal{V} \mid f(P + h)X = Xf(P + h + <\alpha, P + h>), \quad f(P)X = Xf(P + <\beta, P>), \quad f(P), f(P + h) \in \mathbb{F}, X \cdot \mathcal{V}_{\lambda, \mu} \subseteq \mathcal{V}_{\lambda + \alpha, \mu + \beta} \right\},
\]

\[
\mu_l^{\mathcal{D}_{H, \mathcal{V}}}(\hat{f})v = f(<\lambda, P + h>, p)v, \quad \mu_r^{\mathcal{D}_{H, \mathcal{V}}}(\hat{f})v = f(<\mu, P>, p^*)v,
\]

\( \hat{f} \in \mathcal{M}_H^*, v \in \mathcal{V}_{\lambda, \mu} \).
Definition 2.5 We define a dynamical representation of \( U_{q,p}(\hat{g}) \) on \( \mathcal{V} \) to be an \( H \)-algebra homomorphism \( \pi : U_{q,p}(\hat{g}) \to \mathcal{D}_{H,\mathcal{V}} \). By the action \( \pi \) of \( U_{q,p}(\hat{g}) \) we regard \( \mathcal{V} \) as a \( U_{q,p}(\hat{g}) \)-module.

Definition 2.6 For \( k \in \mathbb{C} \), we say that a \( U_{q,p}(\hat{g}) \)-module has level \( k \) if \( c \) acts as the scalar \( k \) on it.

Remark For the level-0 representations, Definition 2.5 is essentially the same as in [8], by identifying \( P \) and \( P + h \) with \( \lambda \) and \( \lambda - \gamma h \), respectively. This definition is valid also for the non-zero level cases [10].

Definition 2.7 For \( \omega \in \mathbb{C} \), we set
\[
\mathcal{V}_\omega = \{ v \in \mathcal{V} \mid -d \cdot v = \omega v \}
\]
and we call \( \mathcal{V}_\omega \) the space of elements homogeneous of degree \( \omega \). We also say that \( X \in \mathcal{D}_{H,\mathcal{V}} \) is homogeneous of degree \( \omega \in \mathbb{C} \) if
\[
[-d, X] = \omega X
\]
and denote by \( (\mathcal{D}_{H,\mathcal{V}})_\omega \) the space of all endomorphisms homogeneous of degree \( \omega \).

Definition 2.8 Let \( \mathcal{H}, \mathcal{N}_+, \mathcal{N}_- \) be the subalgebras of \( U_{q,p}(\hat{g}) \) generated by \( c, d, K_i^\pm \) (\( i \in I \)), by \( \alpha_{i,n}^\vee \) (\( i \in I, n \in \mathbb{Z}_{\geq 0} \)), \( e_{i,n} \) (\( i \in I, n \in \mathbb{Z}_{\geq 0} \)) \( f_{i,n} \) (\( i \in I, n \in \mathbb{Z}_{\geq 0} \)) and by \( \alpha_{i,-n} \) (\( i \in I, n \in \mathbb{Z}_{\geq 0} \)), \( e_{i,-n} \) (\( i \in I, n \in \mathbb{Z}_{\geq 0} \)), \( f_{i,-n} \) (\( i \in I, n \in \mathbb{Z}_{\geq 0} \)), respectively.

Definition 2.9 For \( k \in \mathbb{C}, \lambda \in \mathfrak{h}^* \) and \( \mu \in H^* \), a (dynamical) \( U_{q,p}(\hat{g}) \)-module \( \mathcal{V}(\lambda, \mu) \) is called the level-\( k \) highest weight module with the highest weight \( (\lambda, \mu) \), if there exists a vector \( v \in \mathcal{V}(\lambda, \mu) \) such that
\[
\mathcal{V}(\lambda, \mu) = U_{q,p}(\hat{g}) \cdot v, \quad N_+ \cdot v = 0, \quad c \cdot v = \omega v, \quad f(P) \cdot v = f(<\mu, P>) v, \quad f(P + h) \cdot v = f(<\lambda, P + h>) v.
\]

We define the category \( \mathcal{C}_k \) in the analogous way to the classical affine Lie algebra case [18].

Definition 2.10 For \( k \in \mathbb{C} \), \( \mathcal{C}_k \) is the full subcategory of the category of \( U_{q,p}(\hat{g}) \)-modules consisting of those modules \( \mathcal{V} \) such that
\[
\begin{align*}
\text{(i)} & \quad \mathcal{V} \text{ has level } k \\
\text{(ii)} & \quad \mathcal{V} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{V}_\omega \\
\text{(iii)} & \quad \text{For every } \omega \in \mathbb{C}, \text{ there exists } n_0 \in \mathbb{N} \text{ such that for all } n > n_0, \mathcal{V}_{\omega+n} = 0.
\end{align*}
\]
Since \( \pi \mathcal{N}_+ \subset \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{D}_{H,\mathcal{V}})_n \), any level-\( k \) highest weight \( U_{q,p}(\hat{g}) \)-modules belong to \( \mathcal{C}_k \).

3 The Dynamical Quantum \( Z \)-Algebras

In this section we introduce a quantum and dynamical analogue \( \mathcal{Z}_k \) of Lepowsky-Wilson’s \( Z \)-algebra associated with the level-\( k \) \( U_{q,p}(\hat{g}) \)-modules and define a category \( \mathcal{D}_k \) of the
\( \mathcal{Z}_k \)-modules. Each representation of \( \mathcal{Z}_k \) in \( \mathcal{D}_k \) turns out to be a dynamical analogue of the quantum \( \mathcal{Z} \)-algebra derived by Jing [23] from the level-\( k \) representation in the \( U_q(\hat{\mathfrak{g}}) \) counterpart of \( \mathcal{D}_k \). See sec.4.1. We also provide the Serre relations (3.23) which are not written in [23] explicitly.

3.1 The Heisenberg algebra \( U_{q,p}(\mathcal{H}) \)

Let \( U_{q,p}(\mathcal{H}) \) be the subalgebra of \( U_{q,p}(\hat{\mathfrak{g}}) \) generated by \( \alpha_{i,n}^\vee \) (\( i \in I, n \in \mathbb{Z}_{\neq 0} \)) and \( c \). It is convenient to introduce the simple root type generators \( \alpha_{j,m} \) and \( \alpha'_{j,m} \) defined by

\[
\alpha_{j,m} = [d_j] \alpha_{j,m}^\vee \quad \text{and} \quad \alpha'_{j,m} = \frac{1 - p^m}{1 - p^m q^{cm}} \alpha_{j,m}, \quad (j \in I, m \neq 0).
\]

From Eqs. (2.14), (2.15), (2.16), we have

\[
[a_{i,m} \cdot \alpha_{j,n}] = \frac{[b_{ijm}][cm]}{m} \left( 1 - \frac{p^m}{1 - p^m q^{cm}} \delta_{m+n,0} \right), \quad (3.1)
\]

\[
[a'_{i,m} \cdot \alpha'_{j,n}] = \frac{[b_{ijm}][cm]}{m} \left( 1 - \frac{p^m}{1 - p^m q^{cm}} \delta_{m+n,0} \right), \quad (3.2)
\]

\[
[a_{i,m} \cdot \alpha'_{j,n}] = \frac{[b_{ijm}][cm]}{m} \delta_{m+n,0}, \quad (3.3)
\]

\[
[a_{i,m} \cdot e_j(z)] = \frac{[b_{ijm}][cm]}{m} \left( 1 - \frac{p^m}{1 - p^m q^{cm}} z^m e_j(z) \right), \quad (3.4)
\]

\[
[a'_{i,m} \cdot f_j(z)] = - \frac{[b_{ijm}][cm]}{m} \left( 1 - \frac{p^m}{1 - p^m q^{cm}} z^m f_j(z) \right). \quad (3.5)
\]

Let \( U_{q,p}(\mathcal{H}_+) \) (resp. \( U_{q,p}(\mathcal{H}_-) \)) be the commutative subalgebras of \( U_{q,p}(\mathcal{H}) \) generated by \( \{ c, \alpha_{i,n} (i \in I, n \in \mathbb{Z}_{>0}) \} \) (resp. \( \{ \alpha_{i,-n} (i \in I, n \in \mathbb{Z}_{>0}) \} \)). We have

\[
U_{q,p}(\mathcal{H}) = U_{q,p}(\mathcal{H}_-) U_{q,p}(\mathcal{H}_+).
\]

Let \( \mathbb{C} 1_k \) be the one-dimensional \( U_{q,p}(\mathcal{H}_+) \)-module generated by the vacuum vector \( 1_k \) defined by

\[
c \cdot 1_k = k 1_k, \quad \alpha_{i,n} \cdot 1_k = 0 \quad (n > 0).
\]

Then we have the induced \( U_{q,p}(\mathcal{H}) \)-module

\[
\mathcal{F}_{\alpha,k} = U_{q,p}(\mathcal{H}) \otimes U_{q,p}(\mathcal{H}_+) \mathbb{C} 1_k.
\]

We identify \( \mathcal{F}_{\alpha,k} \) with a polynomial ring \( \mathbb{C} [\alpha_{i,-m} (i \in I, m > 0)] \) by

\[
c \cdot u = ku, \quad \alpha_{i,-n} \cdot u = \alpha_{i,-n} u,
\]

\[
\alpha_{i,n} \cdot u = \sum_j \frac{[b_{ijm}][kn]}{n} \left( 1 - \frac{p^n}{1 - p^n q^{kn}} \frac{\partial}{\partial \alpha_{j,-n}} \right) u \quad (n > 0)
\]

for \( u \in \mathbb{C} [\alpha_{i,-m} (i \in I, m > 0)] \).
3.2 The dynamical quantum $Z$-algebra $\mathcal{Z}_V$

Let $k \in \mathbb{C}^\times$ and $(\mathcal{V}, \pi) \in \mathfrak{C}_k$. We call $\pi U_{q,p}(H) \subset (D_{H,V})_{00}$ the level-$k$ Heisenberg algebra. We define the following vertex operators in $(D_{H,V})_{00}[[z, z^{-1}]]$.

$$E^\pm(\alpha_j, z) = \exp \left( \pm \sum_{n>0} \frac{\pi(\alpha_j, \pm n)}{[kn]} z^{\pm n} \right), \quad E^\pm(\alpha'_j, z) = \exp \left( \mp \sum_{n>0} \frac{\pi(\alpha'_j, \pm n)}{[kn]} z^{\pm n} \right).$$

These satisfy the following relations.

**Proposition 3.1**

$$E^+(\alpha_i, z) E^-(\alpha_j, w) = \frac{(q^{-b_{ij}+2k} w/z; q^{2k})_\infty (q^{-b_{ij}} w/z; p^*)_\infty}{(q^{b_{ij}+2k} w/z; q^{2k})_\infty (q^{b_{ij}} w/z; p^*)_\infty} E^-(\alpha_j, w) E^+(\alpha_i, z),$$  

(3.6)

$$E^+(\alpha'_i, z) E^-(\alpha'_j, w) = \frac{(q^{b_{ij}} w/z; q^{2k})_\infty (q^{b_{ij}+k} w/z; p^*)_\infty}{(q^{-b_{ij}+k} w/z; q^{2k})_\infty (q^{-b_{ij}} w/z; p^*)_\infty} E^-(\alpha'_j, w) E^+(\alpha'_i, z),$$  

(3.7)

$$E^+(\alpha_i, z) E^-(\alpha_j, w) = \frac{(q^{b_{ij}+k} w/z; q^{2k})_\infty}{(q^{-b_{ij}+k} w/z; q^{2k})_\infty} E^-(\alpha'_j, w) E^+(\alpha_i, z),$$  

(3.8)

$$E^+(\alpha'_i, z) E^-(\alpha_j, w) = \frac{(q^{b_{ij}}+2k) w/z; q^{2k})_\infty}{(q^{b_{ij}+2k} w/z; q^{2k})_\infty} E^+(\alpha'_i, z),$$  

(3.9)

$$E^\pm(\alpha_i, z) e_j(w) = \frac{(q^{b_{ij}} (w/z)^{\pm 1}; q^{2k})_\infty (q^{b_{ij}} (w/z)^{\pm 1}; p^*)_\infty}{(q^{b_{ij}} (w/z)^{\pm 1}; q^{2k})_\infty (q^{b_{ij}} (w/z)^{\pm 1}; p^*)_\infty} e_j(w) E^\pm(\alpha_i, z),$$  

(3.10)

$$E^\pm(\alpha'_i, z) f_j(w) = \frac{(q^{b_{ij}+2k} (w/z)^{\pm 1}; q^{2k})_\infty}{(q^{b_{ij}+2k} (w/z)^{\pm 1}; q^{2k})_\infty} f_j(w) E^\pm(\alpha'_i, z),$$  

(3.11)

$$E^\pm(\alpha'_i, z) e_j(w) = \frac{(q^{b_{ij}+k} (w/z)^{\pm 1}; q^{2k})_\infty}{(q^{b_{ij}+k} (w/z)^{\pm 1}; q^{2k})_\infty} e_j(w) E^\pm(\alpha'_i, z),$$  

(3.12)

$$E^\pm(\alpha_i, z) f_j(w) = \frac{(q^{b_{ij}+k} (w/z)^{\pm 1}; q^{2k})_\infty}{(q^{b_{ij}+k} (w/z)^{\pm 1}; q^{2k})_\infty} f_j(w) E^\pm(\alpha_i, z).$$  

(3.13)

**Definition 3.2** We define $\mathcal{Z}_j^\pm(z; \mathcal{V}) \in D_{H,V}[[z, z^{-1}]]$ by

$$\mathcal{Z}_j^+(z; \mathcal{V}) := E^-(\alpha_j, z) \pi(e_j(z)) E^+(\alpha_j, z),$$  

(3.14)

$$\mathcal{Z}_j^-(z; \mathcal{V}) := E^-(\alpha'_j, z) \pi(f_j(z)) E^+(\alpha'_j, z).$$  

(3.15)

for $j \in I$ and call them the dynamical quantum $Z$ operators associated with $(\mathcal{V}, \pi) \in \mathfrak{C}_k$.

Note that due to the truncation property of the grading of $\mathcal{V} \in \mathfrak{C}_k$ w.r.t $-d$, $\mathcal{Z}_j^\pm(z; \mathcal{V})$ are well defined i.e. the coefficients $\mathcal{Z}_j^\pm(\mathcal{V})$ of $\mathcal{Z}_j^\pm(z; \mathcal{V}) = \sum_{n \in \mathbb{Z}} \mathcal{Z}_j^\pm(\mathcal{V}) z^{-n}$ in $z$ are well defined elements in $(D_{H,V})_0$ for all $n \in \mathbb{Z}$. For the sake of simplicity of the presentation, we often drop $\pi$ to denote the elements in $D_{H,V}$.

From the defining relations of $U_{q,p}((\widehat{\mathfrak{g}}))$, we obtain the following relations of the dynamical quantum $Z$ operators.
Theorem 3.3

\[ g(P + h)Z^+_i(z; V) = Z^+_i(z; V)g(P + h), \]
\[ g(P)Z^+_i(z; V) = Z^+_i(z; V)g(P - <Q_{a_i}, P>), \]
\[ g(P + h)Z^-_i(z; V) = Z^-_i(z; V)g(P + h - <a_i, P + h>), \]
\[ g(P)Z^-_i(z; V) = Z^-_i(z; V)g(P), \]  
(3.16)

\[ [d, Z^+_i(z; V)] = -\frac{d}{dz}Z^+_i(z; V), \]
(3.17)

\[ [\alpha_{i,m}, \widehat{Z}^+_j(w; V)] = 0, \]
(3.18)

\[ K^+_i Z^+_j(z; V) = q^{\pm b_{ij}} Z^+_j(z; V) K^+_i, \quad K^-_i Z^-_j(z; V) = q^{\pm b_{ij}} Z^-_j(z; V) K^-_i, \]  
(3.19)

\[ \sum_{\sigma} \prod_{1 \leq m < l \leq a} \frac{(q^{2+2k} z_{\sigma(i)}; q^{2k})_\infty}{(q^{2-k} z_{\sigma(i)}; q^{2k})_\infty} \]
\[ \times \prod_{s=0}^a (-1)^s \left[ \frac{a}{s} \right] \prod_{1 \leq m \leq s} \frac{(q^{-b_{ij}+2k} w/z_{\sigma(m)}; q^{2k})_\infty}{(q^{b_{ij}+2k} w/z_{\sigma(m)}; q^{2k})_\infty} \prod_{s+1 \leq m \leq a} \frac{(q^{-b_{ij}+2k} z_{\sigma(m)}/w; q^{2k})_\infty}{(q^{b_{ij}+2k} z_{\sigma(m)}/w; q^{2k})_\infty} \]
\[ \times Z^+_i(z_{\sigma(1)}; V) \cdots Z^+_i(z_{\sigma(s)}; V) Z^-_j(w; V) Z^+_i(z_{\sigma(s+1)}; V) \cdots Z^+_i(z_{\sigma(a)}; V) = 0 \quad (i \neq j, a = 1 - a_{ij}). \]  
(3.20)

**Proof** The relations (3.17) and (3.18) follow from Eqs. (2.6)–(2.10) and (2.12), respectively. Let us show the relation (3.19). For \( m > 0 \), we have

\[ [\alpha_{i,m}, \widehat{Z}^+_j(z; V)] = [\alpha_{i,m}, E^-(\alpha_j, z)] e_j(z) E^+(\alpha_j, z) + E^-(\alpha_j, z) [\alpha_{i,m}, e_j(z)] E^+(\alpha_j, z). \]

This vanishes due to Eq. (3.4) and

\[ [\alpha_{i,m}, E^-(\alpha_j, z)] = -\frac{[b_{ij} m]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-km} z^m, \]

where \( p^* = pq^{-2k} \). Similarly, \([\alpha_{i,m}, \widehat{Z}^-_j(z; V)] = 0\) follows from Eq. (3.5) and

\[ \left[ \alpha'_{i,m}, E^-(\alpha'_j, z) \right] = \frac{[b_{ij} m]}{m} \frac{1 - p^{*m}}{1 - p^m} q^{km} z^m. \]

The case \( m < 0 \) can be proved in a similar way.
The relation (3.21) follows from

\[
Z_i^+(z; \mathcal{V})Z_j^+(w; \mathcal{V})
= E^-(\alpha_i, z)e_i(z)E^+(\alpha_i, z)E^-(\alpha_j, w)e_j(w)E^+(\alpha_j, w)
= (q^{b_{ij}}+2k w/z; q^{2k})_\infty(q^{b_{ij}} w/z; p^*)_\infty E^- (\alpha_i, z)e_i(z)E^- (\alpha_j, w)e_j(w)E^+(\alpha_j, w)
= (q^{b_{ij}}+2k w/z; q^{2k})_\infty(q^{b_{ij}} w/z; p^*)_\infty E^- (\alpha_i, z)E^- (\alpha_j, w)e_j(z)e_j(w)E^+(\alpha_i, z)E^+(\alpha_j, w)
\]

We also derive (3.22) as follows.

\[
\frac{(q^{b_{ij}}+k w/z; q^{2k})_\infty}{(q^{b_{ij}}+2k w/z; q^{2k})_\infty} Z_i^+(z; \mathcal{V})Z_j^-(w; \mathcal{V})
= \frac{(q^{b_{ij}}+k w/z; q^{2k})_\infty}{(q^{b_{ij}}+2k w/z; q^{2k})_\infty} E^- (\alpha_i, z)e_i(z)E^+(\alpha_i, z)E^- (\alpha_j', w)f_j(w)E^+(\alpha_j', w)
= E^- (\alpha_i, z)E^- (\alpha_j', w)e_i(z)f_j(w)E^+(\alpha_i, z)E^+(\alpha_j', w)
\]

Then use

\[
\psi_i^\pm(q^{\mp k/2} w) = K_i^\pm E^- (\alpha_i, q^{\mp k} w)^{-1} E^- (\alpha_i', w)^{-1} E^+(\alpha_i, q^{\mp k} w)^{-1} E^+(\alpha_i', w)^{-1}
\]

and the property of the delta function.

To prove the Serre relation (3.23) for \(Z_j^+(z)\) we use Eqs. (3.14) and (3.19) and obtain

\[
e_i(z) = E(\alpha_i, z)Z_i^+(z; \mathcal{V}) \quad (3.24)
\]

where we set

\[
E(\alpha_i, z) = E^- (\alpha_i, z)^{-1} E^+ (\alpha_i, z)^{-1}.
\]

From Eq. (3.6), we have

\[
E(\alpha_i, z) E(\alpha_i, w)
= \frac{(q^{-2} w/z; q^{2k})_\infty(q^2 z/w; q^{2k})_\infty (p^* q^{-2} w/z; p^*)_\infty(p^* q^2 z/w; p^*)_\infty E(\alpha_i, w) E(\alpha_i, z)}{(q^2 w/z; q^{2k})_\infty(q^{-2} z/w; q^{2k})_\infty (p^* q^{-2} w/z; p^*)_\infty(p^* q^2 z/w; p^*)_\infty} \quad (3.25)
\]
Next note that (2.20) is equivalent to
\[
0 = \prod_{1 \leq m < l \leq a} (p^* q^2 z_l z_m; p^*)_\infty \prod_{1 \leq i \leq a} (p^* q^{b_i} z_i / w; p^*)_\infty \\
\times \sum_{\sigma \in S_a} \prod_{1 \leq m \leq s} \left( \prod_{1 \leq i \leq a} (p^* q^{b_i} z_i / w; p^*)_\infty \right) \\
\times \sum_{s=0}^{a} (-1)^s \left[ a \choose s \right] \prod_{1 \leq m \leq s} \left( p^* q^{b_i} w / z_\sigma(m); p^* \right)_\infty (p^* q^{b_i} z_\sigma(m) / w; p^*)_\infty \\
\times e_i(z_\sigma(1)) \cdots e_i(z_\sigma(s)) e_j(w) e_i(z_\sigma(s+1)) \cdots e_i(z_\sigma(a)).
\]
Substitute (3.24) into this, and move all \(E(\alpha_i, z_j)\) and \(E(\alpha_j, w)\) to the left. Then we get
\[
0 = \prod_{1 \leq m < l \leq a} (p^* q^2 z_l z_m; p^*)_\infty \prod_{1 \leq i \leq a} (p^* q^{b_i} z_i / w; p^*)_\infty \\
\times \sum_{\sigma \in S_a} \prod_{1 \leq m \leq s} \left( \prod_{1 \leq i \leq a} (p^* q^{b_i} z_i / w; p^*)_\infty \right) \\
\times \sum_{s=0}^{a} (-1)^s \left[ a \choose s \right] \prod_{1 \leq m \leq s} \left( p^* q^{b_i} w / z_\sigma(m); p^* \right)_\infty (p^* q^{b_i} z_\sigma(m) / w; p^*)_\infty \\
\times e(z_\sigma(1), \cdots, z_\sigma(s), w, z_\sigma(s+1), \cdots, z_\sigma(a)) \\
\times Z^+_i(z_\sigma(1); \mathcal{V}) \cdots Z^+_i(z_\sigma(s); \mathcal{V}) Z^+_j(w; \mathcal{V}) Z^+_i(z_\sigma(s+1); \mathcal{V}) \cdots Z^+_i(z_\sigma(a); \mathcal{V}),
\]
where we set
\[
\epsilon(z_\sigma(1), \cdots, z_\sigma(s), w, z_\sigma(s+1), \cdots, z_\sigma(a)) \\
= E(\alpha_i, z_\sigma(1)) \cdots E(\alpha_i, z_\sigma(s)) E(\alpha_j, w) E(\alpha_i, z_\sigma(s+1)) \cdots E(\alpha_i, z_\sigma(a)).
\]
Then moving \(E(\alpha_j, w)\) to the left end by Eq. (3.25), we have
\[
\epsilon(z_\sigma(1), \cdots, z_\sigma(s), w, z_\sigma(s+1), \cdots, z_\sigma(a)) \\
= \prod_{1 \leq i \leq s} \left( q^{b_i} w / z_\sigma(i); q^{2k} \right)_\infty \left( q^{b_i} z_\sigma(i) / w; q^{2k} \right)_\infty (p^* q^{b_i} w / z_\sigma(i); p^*)_\infty (p^* q^{b_i} z_\sigma(i) / w; p^*)_\infty \\
\times \epsilon(w, z_\sigma(1), \cdots, z_\sigma(a)).
\]
Substituting this into Eq. (3.26), we can factor out \(\epsilon(w, z_\sigma(1), \cdots, z_\sigma(a))\) from \(\sum_{s=0}^{a} \). Then exchanging the order of \(E(\alpha_i, z_i)\)'s by Eq. (3.25), we have
\[
\epsilon(w, z_\sigma(1), \cdots, z_\sigma(a)) \\
= \prod_{1 \leq m \leq l \leq a} \left( q^{2} z_l z_m; q^{2k} \right)_\infty \left( q^{2} z_\sigma(l) z_\sigma(m); q^{2k} \right)_\infty (p^* q^2 z_l z_m; p^*)_\infty (q^2 p^* z_\sigma(l) / z_\sigma(m); p^*)_\infty \\
\times e(w, z_1, \cdots, z_a).
Substituting this into Eq. (3.26), we can factor out $\varepsilon(w, z_1, \cdots, z_a)$ completely from $\sum_{\sigma \in S_a}$. Multiply
\[
\prod_{1 \leq m < l \leq a} \frac{(q^2 z_l/z_m; q^{2k})_\infty}{(q^{-2} z_l/z_m; q^{2k})_\infty} \prod_{1 \leq m \leq a} \frac{(q^{-h_{ij}} z_m/w; q^{2k})_\infty}{(q^{h_{ij}} z_m/w; q^{2k})_\infty},
\]
and drop the overall factor depending on $p^*$, one gets the desired relation. One can prove the $Z_j^-(z; V)$ case in the same way. \hfill \Box

**Definition 3.4** For $k \in \mathbb{C}^\times$ and $(V, \pi) \in \mathfrak{c}_k$, we call the $H$-subalgebra of $\mathcal{D}_{H,V}$ generated by $Z_{i,m}^\pm(V), K_i^\pm (i \in I, m \in \mathbb{Z}), \mathcal{M}_{H^*}$ and $d$ the dynamical quantum $Z$-algebra $Z_{V}$ associated with $(V, \pi)$.

### 3.3 The universal algebra $Z_k$

Using the relations in Theorem 3.3, we define the universal dynamical quantum $Z$-algebra as follows.

**Definition 3.5** Let $Z_{i,m}^\pm (i \in I, m \in \mathbb{Z})$ be abstract symbols. We set $Z_i^\pm(z) = \sum_{m \in \mathbb{Z}} Z_{i,m}^\pm z^{-m}$. We define the universal dynamical quantum $Z$-algebra $Z_k$ to be a topological algebra over $\mathbb{F}[[q^{2k}]]$ generated by $Z_{i,m}^\pm, K_i^\pm (i \in I, m \in \mathbb{Z}), d, \mathcal{M}_{H^*}$ subject to the relations obtained by replacing $Z_{i}^\pm(z; V)$ by $Z_i^\pm(z)$ in Theorem 3.3.

We treat the relations as formal Laurent series in $z, w$ and $z_j$'s in a similar way to those of $U_{q,p}(\hat{g})$ in section 2.1. The defining relations are well-defined in the $q^{2k}$-adic topology.

**Proposition 3.6** $Z_k$ is an $H$-algebra with the same $\mu_l, \mu_r$ as in $U_{q,p}(\hat{g})$.

Note that for $(V, \pi) \in \mathfrak{c}_k$ we extend $\pi$ to the map $\pi : Z_k \rightarrow \mathcal{D}_{H,V}$ by $\pi(Z_{i,m}^\pm) = Z_{i,m}^\pm(V)$. Then $V$ is a $Z_k$-module by $\pi$.

**Definition 3.7** For $k \in \mathbb{C}^\times$, we denote by $\mathcal{D}_k$ the full subcategory of the category of $Z_k$-modules consisting of those modules $(\mathcal{W}, \sigma)$ such that

(i) $\mathcal{W}$ has level $k$.
(ii) $\mathcal{W} = \bigsqcup_{\omega \in \mathcal{C}} \mathcal{W}_\omega$, where $\mathcal{W}_\omega = \{w \in \mathcal{W} | -\sigma(d)w = \omega w \}$
(iii) For every $\omega \in \mathcal{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathcal{W}_{\omega+n} = 0$.

Let us consider $(V, \pi) \in \mathfrak{c}_k$. Following Lepowsky and Wilson [18], we define the vacuum space $\Omega_V$ by
\[
\Omega_V = \{v \in V | \pi(\alpha_{i,n})v = 0 \ \forall i \in I, \ n \in \mathbb{Z}_{>0} \}.
\]

From Theorem 3.3, $\Omega_{V'}$ is stable under the action of $Z_{V'}$. For a morphism $f : V \rightarrow V'$ in $\mathfrak{c}_k$, we have
\[
f(\Omega_V) \subset \Omega_{V'}.
\]
Proposition 3.8  For \((V, \pi) \in \mathcal{C}_k\), there is a unique representation \(\sigma\) of \(\mathcal{Z}_k\) on \(\Omega_V\) such that 
\[
\sigma(K^\pm_i) = \pi(K^\pm_i), \quad \sigma(Z^\pm_{i,m}) = Z^\pm_{i,m}(V) \quad \forall i \in I, m \in \mathbb{Z}.
\]

We hence define a functor \(\Omega : \mathcal{C}_k \to \mathcal{D}_k\) by 
\[
\Omega(V, \pi) = (\Omega_V, \sigma), \quad \Omega(f) = f|_{\Omega_V} : \Omega_V \to \Omega_{V'}.
\]

3.4 The functor \(\Lambda\)

We define a reverse functor \(\Lambda : \mathcal{D}_k \to \mathcal{C}_k\) as follows. Let \((W, \sigma) \in \mathcal{D}_k\) be a \(\mathcal{Z}_k\)-module. We define \(U_{q,p}(\mathcal{H})\)-module \(\text{Ind } W\) by requiring \(\alpha_i \cdot W = 0\) and 
\[
\text{Ind } W = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H})} W.
\]

Let \(\mathcal{F}_{a,k}\) be the level-\(k\) Fock module defined in section 3.1. We have a natural isomorphism 
\[
\mathcal{F}_{a,k} \otimes \mathcal{C} W \cong \text{Ind } W \text{ by } (u \otimes 1_k) \otimes w \mapsto u \otimes w [18].
\]
We thus identify the \(U_{q,p}(\mathcal{H})\)-module \(\text{Ind } W\) with \(\mathcal{F}_{a,k} \otimes \mathcal{C} W\), with the action \(\pi\) of \(U_{q,p}(\mathcal{H})\)
\[
\pi(c) = 1 \otimes c, \quad \pi(K^\pm_i) = 1 \otimes \sigma(K^\pm_i), \quad \pi(\alpha_i) = \alpha_i \otimes 1.
\]

For \((V, \sigma) \in \mathcal{D}_k\) and \(\text{Ind } W = \mathcal{F}_{a,k} \otimes \mathcal{C} W\), we define 
\[
e'_j(z), f'_j(z) \in \mathcal{D}_{H, \text{Ind } W}[z, z^{-1}]
\]
by 
\[
e'_j(z) = E^- (\alpha_j, z)^{-1} E^+(\alpha_j, z)^{-1} \otimes \sigma(Z^+_j(z)),
\]
\[
f'_j(z) = E^- (\alpha'_j, z)^{-1} E^+(\alpha'_j, z)^{-1} \otimes \sigma(Z^-_j(z)).
\]

These are well-defined elements of \(\mathcal{D}_{H, \text{Ind } W}[z, z^{-1}]\). By a similar argument to the proof of Theorem 3.3 one can show that \(e'_j(z)\) and \(f'_j(z)\) satisfy the defining relations of \(U_{q,p}(\mathfrak{g})\) with \(c = k\). We hence extend \(\pi : U_{q,p}(\mathcal{H}) \to \mathcal{D}_{H, \text{Ind } W}\) to \(\pi : U_{q,p}(\mathfrak{g}) \to \mathcal{D}_{H, \text{Ind } W}\) as an \(H\)-algebra homomorphism by 
\[
\pi(e_j(z)) = e'_j(z), \quad \pi(f_j(z)) = f'_j(z),
\]
\[
\pi(d) = d \otimes 1 + 1 \otimes \sigma(d).
\]

By construction, the latter map is uniquely determined.

Proposition 3.9  For \((V, \sigma) \in \mathcal{D}_k\), there is a unique level-\(k\) \(U_{q,p}(\mathfrak{g})\)-module \((\text{Ind } W, \pi) \in \mathcal{C}_k\).

We thus reach the following definition.

Definition 3.10  We define a functor \(\Lambda : \mathcal{D}_k \to \mathcal{C}_k\) by

(i) \(\Lambda(V, \sigma) = (\text{Ind } W, \pi)\)

(ii) For a morphism \(f : W \to W'\) in \(\mathcal{D}_k\), define \(\Lambda(f) : \text{Ind } W \to \text{Ind } W'\) to be the induced \(U_{q,p}(\mathcal{H})\)-module map. Then \(\Lambda(f)\) is a \(U_{q,p}(\mathfrak{g})\)-module map.

We obtain the following theorem analogously to the case of the affine Lie algebras [18].

Theorem 3.11  For \(k \in \mathbb{C}^*,\) the two categories \(\mathcal{C}_k\) and \(\mathcal{D}_k\) are equivalent by the functors 
\(\Omega : \mathcal{C}_k \to \mathcal{D}_k\) and \(\Lambda : \mathcal{D}_k \to \mathcal{C}_k\). In particular, the level-\(k\) \(U_{q,p}(\mathfrak{g})\)-module \(\text{Ind } W = \mathcal{F}_{a,k} \otimes \mathcal{C} W \in \mathcal{C}_k\) is irreducible if and only if \(W \in \mathcal{D}_k\) is an irreducible \(\mathcal{Z}_k\)-module.
4 The Induced $U_{q,p}(\hat{g})$-Modules

In this section we give a simple realization of the dynamical quantum $Z$-algebra $Z_k$ in terms of the quantum $Z$-algebra $Z_k$ associated with $U_q(\hat{g})$ and construct the level-$k$ induced $U_{q,p}(\hat{g})$-modules. We also give some examples of the level-1 irreducible representations.

4.1 The quantum $Z_k$-algebra associated with $U_q(\hat{g})$

One can apply the arguments similar to those in sections 3.1–3.3 to the quantum affine algebra $U_q(\hat{g})$ in the Drinfeld realization and define the corresponding quantum $Z$-algebras $Z_k$ associated with the level-$k U_q(\hat{g})$-module $V$ and the universal one $Z_k^\times$.

See Appendix A. We also denote by $C_k$ and $D_k$ the $U_q(\hat{g})$ counterparts of the categories $C_k$ and $D_k$.

Comparing the defining relations of $Z_k$ with those of $Z_k^\times$, we obtain the following isomorphism.

**Proposition 4.1** We have the isomorphism

$$Z_k \cong (F \otimes C_k)\sharp \mathbb{C}[\mathcal{R}_Q]$$

as an $H$-algebra by

$$Z_{j,m}^\pm \mapsto Z_{j,m}^\pm e^{-Q_{a_j}}, \quad Z_{j,m}^- \mapsto Z_{j,m}^\pm,$$

$$K_i^\pm \mapsto q_i^\pm h_i e^{-Q_a_j} \quad (i \in I, m \in \mathbb{Z}), \quad d \mapsto \tilde{d},$$

where $Z_{j,m}^\pm$ denotes the generators in $Z_k$ (Definition A.3).

**Theorem 4.2** For $(W, \tilde{\sigma}) \in D_k$ and generic $\mu \in \mathfrak{h}^*$, there is a dynamical representation $\sigma$ of $Z_k$ on $W_{H,Q}(\mu) := (F \otimes C W) \otimes \mathbb{C} e^{Q_{\tilde{\mu}}} C[\mathcal{R}_Q]$ such that $(W_{H,Q}(\mu), \sigma) \in C_k$ and

$$\sigma(Z_{j,m}^\pm) = \tilde{\sigma}(Z_{j,m}^\pm) \otimes e^{-Q_{a_j}}, \quad \sigma(Z_{j,m}^-) = \tilde{\sigma}(Z_{j,m}^-) \otimes 1,$$

$$\sigma(K_j^\pm) = \tilde{\sigma}(q_j^\pm h_i) e^{-Q_a_j}, \quad \sigma(d) = \tilde{\sigma}(\tilde{d}) \otimes 1 + 1 \otimes P_d,$$

where $P_d$ denotes a $\mathbb{C}$-linear operator on $1 \otimes e^{Q_{\tilde{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ such that

$$[1 \otimes P_d, \sigma(Z_{j,m}^\pm)] = 0.$$

**Proposition 4.3** The representation $(W_{H,Q}(\mu), \sigma)$ of $Z_k$ is irreducible if and only if $W$ is an irreducible $Z_k$-module.

From this and Theorem 3.11, we obtain:

**Proposition 4.4** For a $Z_k$-module $(W, \tilde{\sigma}) \in D_k$ and generic $\mu \in \mathfrak{h}^*$, let $(W_{H,Q}(\mu), \sigma)$ be the $Z_k$-module constructed in Theorem 4.2 and $\text{Ind} W_{H,Q}(\mu) = \mathcal{F}_{a,k} \otimes \mathbb{C} W_{H,Q}(\mu)$ be the level-$k$ induced $U_{q,p}(\hat{g})$-module given in Proposition 3.9. Then $(\text{Ind} W_{H,Q}(\mu), \pi)$ is irreducible if and only if $(W, \tilde{\sigma})$ is irreducible.

4.2 Examples of the irreducible representations

We here give some examples of the level-1 irreducible induced representations of $U_{q,p}(\hat{g})$ of types $\hat{g} = A_i^{(1)}, D_i^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_i^{(1)}$. 

\[\square\]
4.2.1 The simply laced case:

Let $\mathbb{C}[Q]$ be the group algebra of the root lattice $Q = \bigoplus_i \mathbb{Z}\alpha_i$ with the central extension:

$$e^{a_i} e^{a_j} = (-1)^{[\alpha_i, \alpha_j]} e^{a_j} e^{a_i} \quad (i, j \in I).$$

Let us consider the fundamental weight $\Lambda_a$ of $\widehat{\mathfrak{g}}$ with $0 \leq a \leq l$ for $A_l^{(1)}$, $a = 0, 1, l - 1, l$ for $D_l^{(1)}$, $a = 0, 1, 2$ for $E_6^{(1)}$, $a = 0, 1$ for $E_7^{(1)}$, $a = 0$ for $E_8^{(1)}$.

Theorem 4.5 [23, 39] An inequivalent set of the level-1 irreducible $Z_1(\widehat{\mathfrak{g}})$-modules is given by $W(\Lambda_a) = e^{\hat{\Lambda}_a} \mathbb{C}[Q]$, on which the actions of $Z_1^\pm(z)$ are given by

$$Z_j^\pm(z) = e^{\pm\alpha_j z \pm h_j + 1}$$

with

$$z^{\pm h_i} e^{\pm\alpha_j} e^{\hat{\Lambda}_a} = z^{\pm(\alpha_i^\vee \alpha_j + \hat{\Lambda}_a)} e^{\pm\alpha_j} e^{\hat{\Lambda}_a} \quad (i, j \in I).$$

Then for generic $\mu \in \mathfrak{h}^*$, we have from Theorem 4.2 a level-1 irreducible $Z_1(\widehat{\mathfrak{g}})$ module $\mathcal{W}_{H,Q}(\Lambda_a, \mu) := (\mathbb{F} \otimes \mathcal{W}(\Lambda_a)) \otimes e^{Q_{\mu}} \mathbb{C}[R_Q]$ with the action given by

$$Z_j^+(z) = Z_j^-(z) \otimes e^{-Q_{\mu}}, \quad Z_j^-(z) = Z_j^+(z) \otimes 1. \quad (4.2)$$

Then from Proposition 4.4 we obtain:

Theorem 4.6 A level-1 irreducible highest weight representations of $U_{q,p}(\mathfrak{g})$ is given by $\mathcal{V}(\Lambda_a + \mu, \mu) := \text{Ind} \mathcal{W}_{H,Q}(\Lambda_a, \mu)$ with the highest weight $(\Lambda_a + \mu, \mu)$:

$$\mathcal{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{a,1} \otimes \mathcal{W}_{H,Q}(\Lambda_a, \mu) = \bigoplus_{\gamma, \kappa \in Q} \mathcal{F}_{\gamma,\kappa}(\Lambda_a, \mu),$$

where

$$\mathcal{F}_{\gamma,\kappa}(\Lambda_a, \mu) = \mathbb{F} \otimes \mathcal{F}_{a,1} \otimes e^{\hat{\Lambda}_a + \gamma} \otimes e^{Q_{\mu + \kappa}}.$$ 

The highest weight vector is $1_1 \otimes e^{\hat{\Lambda}_a} \otimes e^{Q_{\mu}}$. The derivation operator $d$ is realized as

$$d = -\frac{1}{2} \sum_{j=1}^{l} h_j h_j - N^\alpha \frac{1}{2r^*} \sum_{j=1}^{l} (P_j + 2) P_j - \frac{1}{2r} \sum_{j=1}^{l} ((P + h) j + 2)(P + h) j,$$

$$N^\alpha = \sum_{j=1}^{l} \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{m} \frac{1 - p^{sm}}{1 - p^m} q^m \alpha_{j,-m} \Lambda_{j}^m,$$

where $r, r^* \in \mathbb{C}^\times$, and $\Lambda_{j}^m$ are the fundamental weight type elliptic bosons given in Sec.5.1.

One can easily calculate the character of $\mathcal{V}(\Lambda_a, \mu)$:

$$\text{ch} \mathcal{V}(\Lambda_a + \mu, \mu) = \text{tr} \mathcal{V}(\Lambda_a + \mu, \mu) q^{-d - e(W(\mathfrak{g}))} = \sum_{\gamma, \kappa \in Q} \text{ch} \mathcal{F}_{\gamma,\kappa}(\Lambda_a, \mu),$$

$$\text{ch} \mathcal{F}_{\gamma,\kappa}(\Lambda_a, \mu) = \frac{1}{\eta(q)} q^{\frac{1}{2rr^*} [r(\mu + \kappa + \bar{\rho}) - r^*(\Lambda_a + \mu + \gamma + \bar{\rho})]^2}.$$

Here $c(W(\mathfrak{g})) = l \left(1 - \frac{g(g+1)}{2rr^*}\right)$, and $\eta(q)$ denotes Dedekind’s $\eta$-function given by

$$\eta(q) = q^{\frac{1}{24}} (q; q)_\infty.$$
One should note that the character \( ch_{\mathcal{F}_{\gamma,k}(\Lambda_a, \mu)} \) coincides with the one of the Verma module of the \( W(\mathfrak{g}) \)-algebras for \( \mathfrak{g} = \mathfrak{A}_1, \mathfrak{D}_1, \mathfrak{E}_6, \mathfrak{E}_7, \mathfrak{E}_8 \) with the highest weight \( h = \frac{1}{2r^*}r(\tilde{\mu} + \kappa + \rho) - r^*(\tilde{\Lambda}_a + \tilde{\mu} + \gamma + \kappa + \rho) \) and the central charge \( c(W(\mathfrak{g})) \). In fact, for \( \hat{\mathfrak{g}} = \mathfrak{A}_1^{(1)} \) case, for example, one can construct an action of the deformed \( W(\mathfrak{A}_1) \) algebra on \( \mathcal{F}_{\gamma,k}(\Lambda_a, \mu) \) explicitly.

**Theorem 4.7** [26, 27] For \( p = q^{2r} \) and \( p^* = pq^{-2} = q^{2r^*} \), i.e. \( r^* = r - 1 \), the deformed \( W(\mathfrak{A}_1) \)-algebra acts on \( \mathcal{F}_{\gamma,k}(\Lambda_a + \mu, \mu) \) by

\[
\Lambda_j(z) := \exp \left\{ \sum_{m \neq 0} (q^m - q^{-m})(1 - p^{sm}) \mathcal{E}_m^{+j}(q^j z)^{-m} \right\} : p^{sh_j} (1 \leq j \leq l),
\]

\[
T_n(z) = \sum_{1 \leq j_1 < \cdots < j_n \leq l} : \Lambda_{j_1}(z) \Lambda_{j_2}(zq^{-2}) \cdots \Lambda_{j_n}(zq^{-2(n-1)}) : (1 \leq n \leq l).
\]

Here \( \mathcal{E}_m^{+j} \) denotes the orthonormal basis type elliptic boson given in Eq. (5.3), and \( : \) denotes the normal ordering of the enclosed expression such that the operators \( \mathcal{E}_m^{\pm j} \) for \( m < 0 \) are to be placed to the left of the operators \( \mathcal{E}_m^{\pm j} \) for \( m > 0 \). In addition, the level-1 elliptic currents \( e_j(w) \) and \( f_j(w) \) of \( U_{q,p}(\mathfrak{A}_1^{(1)}) \) obtained from Proposition 3.9, Eqs. (4.1) and (4.2) are the screening currents of the deformed \( W(\mathfrak{A}_1) \)-algebra, i.e. they commute with \( T_n(z) \) up to a total difference.

See also [2, 7, 29]. A similar statement is valid also for the deformed \( W(\mathfrak{D}_1) \) [28] and \( U_{q,p}(\mathfrak{D}_1^{(1)}) \). We also expect that for \( r \in \mathbb{Z}_{>0} \) satisfying \( r > g + 1 \) and for a level-\((r-g-1)\) dominant integral weight \( \mu \), the space \( \mathcal{F}_{\gamma,k}(\Lambda_a + \mu, \mu) \) becomes completely degenerate with respect to the action of the corresponding deformed \( W(\mathfrak{g}) \)-algebra [26–28], although the \( E_{6,7,8}^4 \)-type deformed \( W \) algebras have not yet been constructed explicitly. In order to get the irreducible module one should make the BRST-resolution in terms of the BRST-charge constructed from the half currents of \( U_{q,p}(\hat{\mathfrak{g}}) \). An explicit demonstration for the \( \mathfrak{A}_1^{(1)} \) case has been discussed in [35].

**Remark.** In Theorem 4.7, we assumed \( p = q^{2r} \) in order to make a connection to the deformed \( W(\mathfrak{A}_1) \)-algebra. The same relation arises naturally when one considers the finite dimensional representations of the universal elliptic dynamical \( \mathcal{R} \) matrices [5, 40].

### 4.2.2 The \( \mathfrak{B}_l^{(1)} \) Case

We follow the work [41] and its quantum analogues [42, 43] with a slight modification in the Ramond sector according to [44]. Let \( e^{a_i} \) \( (i \in I) \) be the generators of the group algebra \( \mathbb{C}[Q] \) with the following central extension.

\[ e^{a_i} e^{a_j} = (-1)^{(a_i, a_j) + (a_i, a_i)(a_j, a_j)} e^{a_j} e^{a_i} \]

As before we regard \( h_i \) \( (i \in I) \) as an operator such that

\[ z^{\pm h_i} e^{a_j} = z^{\pm (a_j, a_i)} e^{a_j} z^{\pm h_i} \]

We also need the Neveu-Schwartz (NS) fermion \( \{ \Psi_n | n \in \mathbb{Z} + \frac{1}{2} \} \) and the Ramond (R) fermion \( \{ \Psi_n | n \in \mathbb{Z} \} \) satisfying the following anti-commutation relations.

\[ \{ \Psi_m, \Psi_n \} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m}) \]
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with $\mathcal{N} = 1/(q^{1/2} + q^{-1/2})$. We define

$$\mathcal{F}^{NS} = \mathbb{C}[\Psi_{-1}, \Psi_{-1/2}, \ldots], \quad \mathcal{F}^{R} = \mathbb{C}[\Psi_{-1}, \Psi_{-2}, \ldots]$$

and their submodules $\mathcal{F}^{NS,R}_{even}$ (reps. $\mathcal{F}^{NS,R}_{odd}$) generated by the even (reps. odd) number of $\Psi_{-m}$’s. One should note that for the $R$ fermion $\Psi_{0}^2 = \mathcal{N}$ and $\{\Psi_{m}, \Psi_{0}\} = 0$ for $m \neq 0$. So we have two degenerate vacuum states 1 and $\Psi_{0} 1$. We hence consider the extended space

$$\hat{\mathcal{F}}^{R} = \mathcal{F}^{R} \otimes \mathbb{C}^2$$

and realize the $R$-fermions by

$$\hat{\Psi}_{m} = \Psi_{m} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (m \in \mathbb{Z}_{\neq 0}), \quad \hat{\Psi}_{0} = \mathcal{N}^{1/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $\{\hat{\Psi}_{m}, \hat{\Psi}_{n}\} = \delta_{m+n,0} \mathcal{N}(q^{m} + q^{-m})$. We set

$$\mathcal{F}^{R} = \mathcal{F}^{R}_{even} \otimes \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}^{R}_{odd} \otimes \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of $\Psi_{m}$ on $\mathcal{F}^{NS}$ is given by

$$\Psi_{-m} \cdot u = \Psi_{-m} u, \quad \Psi_{m} \cdot u = \{\Psi_{m}, u\} \quad (m \in \mathbb{Z}_{>0}),$$

where $u \in \mathcal{F}^{NS}$, whereas $\hat{\Psi}_{m}$ acts on $\mathcal{F}^{R}$ as

$$\hat{\Psi}_{-m} \cdot u \otimes v = \Psi_{-m} u \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \quad \hat{\Psi}_{0} \cdot u \otimes v = u \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v,$$

$$\hat{\Psi}_{m} \cdot u \otimes v = \{\Psi_{m}, u\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}),$$

where $u \in \hat{\mathcal{F}}^{R}$, $v \in \mathbb{C}^2$.

Let us define the fermion fields $\Psi^{NS}(z)$ and $\Psi^{R}(z)$ by

$$\Psi^{NS}(z) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \Psi_{n} z^{-n}, \quad \Psi^{R}(z) = \sum_{n \in \mathbb{Z}} \hat{\Psi}_{n} z^{-n}.$$

One can derive the following operator product expansions.

$$\Psi(z) \Psi(w) =: \Psi(z) \Psi(w) : + < \Psi(z) \Psi(w) >,$$

where

$$< \Psi(z) \Psi(w) > = \begin{cases} \frac{(zw)^{1/2}(z-w)}{(z-qw)(z-q^{-1}w)} & \text{for NS} \\ \frac{\mathcal{N}(z-w)(z+w)}{(z-qw)(z-q^{-1}w)} & \text{for R.} \end{cases}$$

Then the quantum $Z$-algebra $Z_{1}(B_{l}^{(1)})$ is realized as follows [23].

$$Z_{i}^{\pm}(z) = e^{\pm\alpha_{i}z^{\pm h_{i}+1} (1 \leq i \leq l-1)},$$

$$Z_{l}^{\pm}(z) = \frac{1}{\mathcal{N}^{1/2}} \Psi(z) e^{\pm\alpha_{l}z^{\pm h_{l}+d_{l}}}.$$
There are three irreducible $Z_1(B_1^{(1)})$-modules given by

\[
W(\Lambda_0) = \mathcal{F}_{e}^{NS} \otimes \mathbb{C}[Q_0] \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[Q_0] e^{\tilde{\Lambda}_1},
\]
\[
W(\Lambda_1) = \mathcal{F}_{e}^{NS} \otimes \mathbb{C}[Q_0] e^{\tilde{\Lambda}_1} \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[Q_0],
\]
\[
W(\Lambda_l) = \mathcal{F}^R \otimes \mathbb{C}[Q] e^{\tilde{\Lambda}_l} \cong \mathcal{F}^R \otimes \mathbb{C}[Q_0] e^{\tilde{\Lambda}_1 + \tilde{\Lambda}_l},
\]

where $Q_0$ denotes the sublattice of $Q$ generated by the long roots. For generic $\mu \in \mathfrak{h}^*$ and $a = 0, 1, l$, we set $\mathcal{V}_{H,Q}(\Lambda_a, \mu) = (\overline{\mathbb{F}} \otimes \mathbb{C}) W(\Lambda_a) \otimes e^{\bar{Q}_\overline{\mathbb{F}}[R]}$. From Proposition 3.9 we have the following three level-1 irreducible $U_{q,p}(\hat{B}_1^{(1)})$-modules with the highest weight $(\Lambda_a + \mu, \mu)$:

\[
\mathcal{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{\alpha,1} \otimes \mathcal{V}_{H,Q}(\Lambda_a, \mu)
\]
\[
= \bigoplus_{\gamma \in Q_0, k \in Q} \bigoplus_{\lambda \in \text{max}(\Lambda_a) \mod (Q_0 + C)} \mathcal{F}_{\lambda,\gamma,k}(\Lambda_a, \mu),
\]

where

\[
\mathcal{F}_{\lambda,\gamma,k}(\Lambda_0, \mu) = \mathbb{F} \otimes \mathbb{C} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{NS}^{even} \otimes \gamma) \otimes e^{\bar{Q}_\overline{\mathbb{F}}},
\]
\[
\mathcal{F}_{\lambda,\gamma,k}(\Lambda_1, \mu) = \mathbb{F} \otimes \mathbb{C} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{NS}^{odd} \otimes e^{\tilde{\Lambda}_1+\gamma}) \otimes e^{Q_\overline{\mathbb{F}}},
\]
\[
\mathcal{F}_{\lambda,\gamma,k}(\Lambda_l, \mu) = \mathbb{F} \otimes \mathbb{C} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{R} \otimes e^{\tilde{\Lambda}_l+\gamma}) \otimes e^{Q_\overline{\mathbb{F}}},
\]
\[
\mathcal{F}_{\lambda,\gamma,k}(\Lambda_l, \mu) = \mathbb{F} \otimes \mathbb{C} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{R} \otimes e^{\tilde{\Lambda}_l+\gamma}) \otimes e^{Q_\overline{\mathbb{F}}},
\]

The highest weight vectors are given by $1 \otimes 1 \otimes 1 \otimes e^{Q_\overline{\mathbb{F}}}$ for $\mathcal{V}(\Lambda_0 + \mu, \mu)$, $1 \otimes 1 \otimes e^{\tilde{\Lambda}_1} \otimes e^{Q_\overline{\mathbb{F}}}$ for $\mathcal{V}(\Lambda_1 + \mu, \mu)$ and $1 \otimes 1 \otimes \left(\frac{1}{l}\right) \otimes e^{\tilde{\Lambda}_l} \otimes e^{Q_\overline{\mathbb{F}}}$ for $\mathcal{V}(\Lambda_l + \mu, \mu)$, respectively.

It is also easy to calculate the characters of these modules:

\[
ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} = tr_{\mathcal{V}(\Lambda_a + \mu, \mu)} q^{-d - \frac{c_W}{2}} = \sum_{\lambda \in \text{max}(\Lambda_a) \mod (Q_0 + C)} \sum_{\gamma \in Q_0, k \in Q} ch_{\mathcal{F}_{\lambda,\gamma,k}(\Lambda_a, \mu)},
\]

where $c_W = \left(1 + \frac{1}{2}\right) \left(1 - \frac{2(l-1)}{rl}\right)$ is the central charge of the $WB_l$ algebra by Fateev and Lukyanov [31], and the derivation operator $d$ is realized as

\[
d = -\frac{1}{2} \sum_{j=1}^{l} h_j h^j - N^\alpha - N^\Psi + \frac{1}{2r^*} \sum_{j=1}^{l} (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^{l} ((P + h) j + 2) (P + h)^j,
\]
\[
N^\alpha = \sum_{j=1}^{l} \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{|m|} \left(1 - \frac{p^{2m}}{1 - p^m}\right)^2 q^m \alpha_j, -m \Lambda_i^j,
\]
\[
N^\Psi = \sum_{m > 0} \frac{m(q^{\frac{1}{2}} - q^{-\frac{1}{2}})}{q^m + q^{-m}} \Psi_m \Psi_m.
\]
where \( r, r^* \in \mathbb{C}^\times \), and \( A^j_m \) are the fundamental weight type elliptic bosons of the type \( B_1 \) given in Sec.5.1. \( \Psi_m \) denotes \( \Psi_m \) on \( \mathcal{F}^{NS} \) and \( \Psi_m \) on \( \mathcal{F}^R \). We obtain:

\[
ch_{\mathcal{V}(\Lambda_+ + \mu, \mu)} = \sum_{\lambda, \mu \in \mathcal{P}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_+ , \mu)} ,
\]

\[
ch_{\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu)} = c_{\Lambda_0} A_0 \frac{1}{2rr^*} \left| r(\bar{\mu} + \bar{\kappa} + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \bar{\kappa} + \bar{\rho}) \right|^2 ,
\]

\[
ch_{\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu)} = c_{\Lambda_1} A_1 \frac{1}{2rr^*} \left| r(\bar{\mu} + \bar{\kappa} + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \bar{\kappa} + \bar{\rho}) \right|^2 ,
\]

\[
ch_{\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu)} = c_{\Lambda_0} A_0 \frac{1}{2rr^*} \left| r(\bar{\mu} + \bar{\kappa} + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \bar{\kappa} + \bar{\rho}) \right|^2 ,
\]

\[
ch_{\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu)} = c_{\Lambda_1} A_1 \frac{1}{2rr^*} \left| r(\bar{\mu} + \bar{\kappa} + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \bar{\kappa} + \bar{\rho}) \right|^2 ,
\]

\[
ch_{\mathcal{F}_{\Lambda_1 - q_1, \gamma, \kappa}(\Lambda_1, \mu)} = c_{\Lambda_1 - q_1} A_1 \frac{1}{2rr^*} \left| r(\bar{\mu} + \bar{\kappa} + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \bar{\kappa} + \bar{\rho}) \right|^2 ,
\]

where

\[
c_{\Lambda_0} = c_{\Lambda_1} = \frac{q^{-1/8}}{2\eta(q)^{1/4}} \left( ( - q^{\frac{1}{2}} ; q)_{\infty} + ( q^{\frac{1}{2}} ; q)_{\infty} \right) ,
\]

\[
c_{\Lambda_0} = c_{\Lambda_1} = \frac{q^{-1/8}}{2\eta(q)^{1/4}} \left( ( - q^{\frac{1}{2}} ; q)_{\infty} - ( q^{\frac{1}{2}} ; q)_{\infty} \right) ,
\]

\[
c_{\Lambda_1} = c_{\Lambda_1 - q_1} = \frac{q^{\frac{1}{2}}} {2\eta(q)^{1/4}} (-q ; q)_{\infty} .
\]

\[
\sum_{\lambda, \mu, \nu} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_+, \mu)} \text{ coincides with the character of the Verma modules of the } WB_1 \text{-algebra with the highest weight } h = \frac{1}{2rr^*} | r(\bar{\mu} + \bar{\kappa} + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \bar{\kappa} + \bar{\rho}) |^2 \text{ and the central charge } c_W \text{ with } r, r^* = r - 1 \in \mathbb{C} \text{ being generic.}
\]

**Conjecture 4.8** There exists a deformation of the \( WB_1 \)-algebra such that

i) its generating functions commute with the level-1 elliptic currents \( e_j(z) \) and \( f_j(z) \) of \( U_{q, \rho}(B_1^{(1)}) \) modulo a total difference, i.e. \( e_j(z) \) and \( f_j(z) \) at \( c = 1 \) are the screening currents of the deformation of the \( WB_1 \)-algebra,

ii) for generic \( r \) and \( \mu \in \mathfrak{h^*} \), \( \mathcal{F}_{\lambda, \xi, \kappa}(\Lambda + \mu, \mu) \) is an irreducible module of the deformation of the \( WB_1 \)-algebra.

**Remark** All the algebras \( W(g) \) appearing in section 4.2.1 and \( WB_1 \) in this subsection are the \( W \)-algebras associated with the coset \( X_1^{(1)} \oplus X_1^{(1)} \to (X_1^{(1)} \text{)}_{\text{diag}} \) with level \( r - g - 1, 1 \). In particular, the \( WB_1 \) is different from the one obtained from the quantum Hamiltonian reduction of the affine Lie algebra \( B_1^{(1)} \). The \( W \)-algebras associated with such coset describe the critical behavior of the face type solvable lattice models introduced by Jimbo, Miwa and Okado [33].
5 Elliptic Bosons of Various Types

In this section we introduce elliptic bosons of the fundamental weight type $A^j_m$ and the orthogonal basis type $E^{\pm j}_m$ for $U_{q,p}(\hat{\mathfrak{g}})$, $\hat{\mathfrak{g}} = A^{(1)}_l, B^{(1)}_l, C^{(1)}_l, D^{(1)}_l$. The level-1 bosons $A^j_m$ and $E^{\pm j}_m$ are used to realize the derivation operator $d$ and the generating function of the deformed $W(A^l_l)$-algebra, respectively, in section 4.2.

5.1 Definition

Let us set $\eta = -t g/2$ ($t = (\text{long root})^2/2$).

Let $\alpha_{i,m}$ be the elliptic bosons of the simple root type as in section 2. We define the fundamental weight type elliptic bosons $A^j_m$ ($1 \leq j \leq l, m \in \mathbb{Z}_{\neq 0}$) by

$$[\alpha_{i,m}, A^j_n] = -\delta_{i,j} \delta_m n + \left[ cm \right] \frac{1 - p^m}{1 - p^{2m}} q^{-cm} (1 \leq i, j \leq l). \quad (5.1)$$

Note that using the matrix $B(m) = (b_{i,j,m})_{1 \leq i,j \leq l}$, we have [28]

$$A^j_m = \sum_{k=1}^l (B(m)^{-1})_{kj} \alpha_{k,m}.$$ 

Solving Eq. (5.1) we obtain the following.

For $A^{(1)}_l$,

$$A^j_m = C_m \left( [(2\eta + j)m] \sum_{k=1}^j [km\alpha_{k,m}] + [jm] \sum_{k=j+1}^l [(2\eta + k)m]\alpha_{k,m} \right) \quad (1 \leq j \leq l).$$

For $B^{(1)}_l$,

$$A^j_m = C_m \left( (q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km\alpha_{k,m}] + [jm] \sum_{k=j+1}^l (q^{(\eta+k)m} + q^{-(\eta+k)m})\alpha_{k,m} \right) \quad (1 \leq j \leq l).$$

For $C^{(1)}_l$,

$$A^j_m = C_m \left( (q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km\alpha_{k,m}] + [jm] \sum_{k=j+1}^{l-1} (q^{(\eta+k)m} + q^{-(\eta+k)m})\alpha_{k,m} + [jm]\alpha_{l,m} \right), \quad (1 \leq j \leq l - 1),$$

$$A^l_m = C_m \left( \sum_{k=1}^{l-1} [km\alpha_{k,m}] + \left[ \frac{m}{2m} \right] [lm]\alpha_{l,m} \right).$$
For $D_l^{(1)}$

$$A_m^j = C_m \left( q^{(\eta+j)m} + q^{-(\eta+j)m} \sum_{k=1}^l [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^{l-2} (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} \right) + [jm] (\alpha_{l-1,m} + a_{l,m}) \quad (1 \leq j \leq l - 2),$$

$$A_{m-1}^j = C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} + \frac{[m]}{[2m]} ([lm] \alpha_{l-1,m} + [(l-2)m] a_{l,m}) \right),$$

$$A_m^l = C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} + \frac{[m]}{[2m]} ([l-2m] \alpha_{l-1,m} + [lm] a_{l,m}) \right).$$

Here

$$C_m = \frac{1}{[m]^2 [2\eta m]} \quad \text{for } A_l^{(1)}$$

$$= \frac{[\eta m]}{[m]^2 [2\eta m]} \quad \text{for } B_l^{(1)}, C_l^{(1)}, D_l^{(1)}.$$

We then devide $A_m^j$ into two terms and define the elliptic bosons $\mathcal{E}_m^{\pm j}$ of the orthogonal basis type as follows.

For $A_l^{(1)}$

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \quad (5.2)$$

$$\mathcal{E}_m^{\pm j} = \pm q^{\pm jm} C_m \left( q^{\pm 2\eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} + \sum_{k=j}^{l-1} [(\eta + k)m] \alpha_{k,m} \right) \quad (5.3)$$

for $1 \leq j \leq l$. It is convenient to define $\mathcal{E}_m^{\pm(l+1)}$ by

$$\mathcal{E}_m^{\pm(l+1)} = \mp \frac{C_m}{q - q^{-1}} \sum_{k=1}^{l} [km] \alpha_{k,m}. \quad (5.4)$$

For $B_l^{(1)}$

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \quad (5.5)$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left( q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l} [(\eta + k)m] \alpha_{k,m} \right) \quad (5.6)$$

for $1 \leq j \leq l$. Here we set

$$[m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

We also define

$$\mathcal{E}_m^0 = \frac{[\eta m]}{[m]} (\mathcal{E}_m^{+l} + \mathcal{E}_m^{-l}). \quad (5.7)$$
For $C^{(1)}_l$,

\[ A^j_m = \mathcal{E}^+_m + \mathcal{E}^-_m, \quad (5.8) \]

\[ \mathcal{E}^\pm_j = q^{\pm j \text{m}} C_m \left( q^{\pm \text{m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-1} [(\eta + k)m] \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right) \quad (1 \leq j \leq l - 1), \quad (5.9) \]

\[ A^l_m = \frac{1}{q^m + q^{-m}} (\mathcal{E}^+_m + \mathcal{E}^-_m), \quad (5.10) \]

\[ \mathcal{E}^\pm_l = q^{\pm \text{m}} C_m \left( q^{\pm \text{m} \sum_{k=1}^{l-1} [km] \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right). \quad (5.11) \]

For $D^{(1)}_l$,

\[ A^j_m = \mathcal{E}^+_m + \mathcal{E}^-_m, \quad (5.12) \]

\[ \mathcal{E}^\pm_j = q^{\pm j \text{m}} C_m \left( q^{\pm \text{m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-2} [(\eta + k)m] \alpha_{k,m} \pm \frac{1}{q - q^{-1}} (\alpha_{l-1,m} + \alpha_{l,m}) \right) \quad (1 \leq j \leq l - 2), \quad (5.13) \]

\[ \mathcal{E}^\pm_{l-1} = C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} \pm \frac{q^{\mp \text{m}}}{q - q^{-1}} (\alpha_{l-1,m} + \alpha_{l,m}) \right), \quad (5.14) \]

\[ \mathcal{E}^\pm_l = q^{\pm \text{m}} C_m \left( \sum_{k=1}^{l-2} [km] \alpha_{k,m} \pm \frac{1}{q - q^{-1}} (q^{\pm \text{m}} \alpha_{l-1,m} - q^{\mp \text{m}} \alpha_{l,m}) \right). \quad (5.15) \]

**Proposition 5.1**

\[ \alpha_{j,m} = \pm [m]^2 (q - q^{-1}) (\mathcal{E}^\pm_j - q^{\mp \text{m}} \mathcal{E}^\pm_{j+1}), \quad (5.16) \]

where $1 \leq j \leq l$ for $A^{(1)}_l$, $1 \leq j \leq l - 1$ for $B^{(1)}_l$, $C^{(1)}_l$, $D^{(1)}_l$, and

\[ \alpha_{l,m} = [m] \left( q^{m/2} - q^{-m/2} \right) \left( q^{-m/2} \mathcal{E}^+_m - q^{m/2} \mathcal{E}^-_m \right) \quad \text{for} \quad B^{(1)}_l, \quad (5.17) \]

\[ \alpha_{l,m} = [m]^2 (q - q^{-1}) \left( q^m \mathcal{E}^+_m - q^{-m} \mathcal{E}^-_m \right) \quad \text{for} \quad C^{(1)}_l, \quad (5.18) \]

\[ \alpha_{l,m} = \pm [m]^2 (q - q^{-1}) \left( \mathcal{E}^{\pm l-1}_m - q^{\pm \text{m}} \mathcal{E}^{\mp l}_m \right) \quad \text{for} \quad D^{(1)}_l. \quad (5.19) \]

**Proposition 5.2** The following relations hold.

\[ \mathcal{E}^\pm_m = \pm \frac{q^{\pm \text{m}}}{q^m - q^{-m}} A^1_m, \quad \mathcal{E}^\pm = \pm \frac{1}{q^m - q^{-m}} \left( q^{\pm \text{m}} A^j_m - A^{j-1}_m \right), \quad (5.20) \]

where $2 \leq j \leq l$ for $A^{(1)}_l$, $2 \leq j \leq l$ for $B^{(1)}_l$, $2 \leq j \leq l - 1$ for $C^{(1)}_l$ and $2 \leq j \leq l - 2$ for $D^{(1)}_l$. In addition, we have

\[ \mathcal{E}^\pm_{l+1} = \pm \frac{1}{q^m - q^{-m}} A^1_m, \quad \sum_{j=1}^{l+1} q^{\pm (j-1)m} \mathcal{E}^\pm_m = 0 \quad \text{for} \quad A^{(1)}_l, \quad (5.21) \]

\[ \mathcal{E}^\pm_l = \pm \frac{1}{q^m - q^{-m}} \left( q^m + q^{-m} \right) q^{\pm \text{m}} A^1_m - A^{l-1}_m \quad \text{for} \quad C^{(1)}_l, \quad (5.22) \]
and
\[
\mathcal{E}_m^{\pm(l-1)} = \pm \frac{1}{q^m - q^{-m}} \left( q^{\pm m} A_{m}^{l-1} + q^{\mp m} A_{m}^{l-2} \right),
\]
(5.23)
\[
\mathcal{E}_m^{\pm l} = \pm \frac{1}{q^m - q^{-m}} \left( q^{\pm 2m} A_{m}^{l-1} - A_{m}^{l-2} \right) \text{ for } D_l^{(l)}.
\]
(5.24)

**Remark** The level-1 case i.e. \( c = 1 \), the \( A_l^{(1)} \) type relation was given in [26, 27] and the \( D_l^{(1)} \) type was essentially given in [28], where parameters \( q \) and \( t \) should be identified with our \( p^{\pm\frac{1}{2}} = p^{\frac{1}{2}} q^{-1} \) and \( p^{\frac{1}{2}} \), respectively. However the \( B_l^{(1)} \) and \( C_l^{(1)} \) cases are different from those given in [28]. At least the formulas for \( B_l^{(1)} \) and \( C_l^{(1)} \) seem to be reversed. Our definitions and relations are valid for arbitrary level \( c \).

Although the expressions of \( \mathcal{E}_m^{\pm j} \) are complicated depending on the types of the affine Lie algebras, their commutation relations are rather universal:

**Theorem 5.3** For \( 1 \leq j, k \leq l \), the following commutation relations hold. For \( A_l^{(1)} \),
\[
[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \delta_{m+n,0} \frac{[cm][2(\eta + 1)m]}{m(q - q^{-1})^2[2\eta m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}},
\]
(5.25)
\[
[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \delta_{m+n,0} q^{\mp (\text{sgn}(k-j)2\eta + k - j)m} \frac{[cm]}{m(q - q^{-1})[2\eta m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}},
\]
(5.26)
\[
[\mathcal{E}_m^{\mp j}, \mathcal{E}_n^{\mp k}] = -\delta_{m+n,0} q^{\mp (\text{sgn}(j-k)2\eta + j - k)m} \frac{[cm]}{m(q - q^{-1})[2\eta m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}}.
\]
(5.27)

For \( B_l^{(1)}, C_l^{(1)}, D_l^{(1)} \),
\[
[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = \delta_{m+n,0} \frac{[cm][\eta m][2(\eta + 1)m]}{m(q - q^{-1})^2[2\eta m][\eta (\eta + 1)m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}},
\]
(5.28)
\[
[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \mp \delta_{m+n,0} q^{\mp (\text{sgn}(k-j)\eta + k - j)m} \frac{[cm][\eta m]}{m[\eta m]^3(q - q^{-1})^2[2\eta m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}},
\]
(5.29)
\[
[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \mp sgn(k - j) \delta_{m+n,0} q^{\mp (\text{sgn}(k-j)\eta + k - j)m} \frac{[cm][\eta m]}{m(q - q^{-1})[2\eta m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}},
\]
(5.30)
\[
[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = \mp \delta_{m+n,0} q^{\mp (\text{sgn}(j-k)\eta + j - k)m} \frac{[cm][\eta m]}{m(q - q^{-1})[2\eta m]} \frac{1 - p^m}{1 - p^{*m} q^{-cm}}.
\]
(5.31)

Here
\[
\text{sgn}(l - j) = \begin{cases} + & (l > j), \\ - & (l < j). \end{cases}
\]

**Proof** Straightforward calculation using Proposition 5.2 and Eq. (5.1). \( \square \)

**Proposition 5.4** For \( 1 \leq i \leq l \), the following commutation relations hold.
\[
[\alpha_i, \mathcal{E}_n^{\pm j}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m} q^{-cm}} (q^{\mp m} \delta_{i,j} - \delta_{i,j-1})
\]
(5.32)
where \( 1 \leq j \leq l \) for \( A_l^{(1)}, B_l^{(1)} \), \( 1 \leq j \leq l - 1 \) for \( C_l^{(1)} \), \( 1 \leq j \leq l - 2 \) for \( D_l^{(1)} \). In addition,

\[
[\alpha_{i,m}, \mathcal{E}_{n}^{\pm l}] = \pm \frac{[cm]}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} q^{-cm} (q^{\mp m} (q^n - q^{-m}) \delta_{i,l} - \delta_{i,l-1}) \text{ for } C_l^{(1)},
\]

(5.33)

and

\[
[\alpha_{i,m}, \mathcal{E}_{n}^{\pm (l-1)}] = \pm \frac{[cm]}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} q^{-cm} (q^{\mp m} \delta_{i,l-1} + q^{\mp m} \delta_{i,l} - \delta_{i,l-2}).
\]

(5.34)

\[
[\alpha_{i,m}, \mathcal{E}_{n}^{\pm l}] = \pm \frac{[cm]}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} q^{-cm} (q^{\mp 2m} \delta_{i,l} - \delta_{i,l-1}) \text{ for } D_l^{(1)}.
\]

(5.35)

From Eqs. (2.15) and (2.16) we also obtain the following relations.

**Proposition 5.5** For \( 1 \leq j \leq l \),

\[
[\mathcal{E}_{m}^{\pm i}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} e_j(z) (q^{\mp m} \delta_{i,j} - \delta_{i-1,j}),
\]

(5.36)

\[
[\mathcal{E}_{m}^{\pm i}, f_j(z)] = \mp \frac{z^m}{m(q^n - q^{-m})} f_j(z) (q^{\mp m} \delta_{i,j} - \delta_{i-1,j})
\]

(5.37)

where \( 1 \leq i \leq l \) for \( A_l^{(1)}, B_l^{(1)} \), \( 1 \leq i \leq l - 1 \) for \( C_l^{(1)} \), \( 1 \leq i \leq l - 2 \) for \( D_l^{(1)} \). In addition,

\[
[\mathcal{E}_{m}^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} e_j(z) (q^{\mp m} (q^n + q^{-m}) \delta_{i,j} - \delta_{i-1,j}),
\]

(5.38)

\[
[\mathcal{E}_{m}^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^n - q^{-m})} f_j(z) (q^{\mp m} (q^n + q^{-m}) \delta_{i,j} - \delta_{i-1,j}) \text{ for } C_l^{(1)}.
\]

(5.39)

and

\[
[\mathcal{E}_{m}^{\pm (l-1)}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} e_j(z) (q^{\mp m} \delta_{i-1,j} + q^{\mp m} \delta_{i,j} - \delta_{i-2,j}),
\]

(5.40)

\[
[\mathcal{E}_{m}^{\pm (l-1)}, f_j(z)] = \mp \frac{z^m}{m(q^n - q^{-m})} f_j(z) (q^{\mp m} \delta_{i-1,j} + q^{\mp m} \delta_{i,j} - \delta_{i-2,j}),
\]

(5.41)

\[
[\mathcal{E}_{m}^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^n - q^{-m})} \frac{1 - p^m}{1 - p^{sm}} e_j(z) (q^{\mp 2m} \delta_{i,j} - \delta_{i-1,j}),
\]

(5.42)

\[
[\mathcal{E}_{m}^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^n - q^{-m})} f_j(z) (q^{\mp 2m} \delta_{i,j} - \delta_{i-1,j}) \text{ for } D_l^{(1)}.
\]

(5.43)
5.2 The Elliptic Currents $k_{\pm j}(z)$

Let us set

$$\psi_j(z) = \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m}}{1 - p^m} p^m z^{-m} \right\} : .$$  \hspace{1cm} (5.44)

Then the elliptic currents $\psi_j^\pm(z)$ in Definition 2.1 can be written as

$$\psi_j^+(q^{-\frac{c}{2}}z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}}z) = K_j^- \psi_j(pq^{-c}z) .$$  \hspace{1cm} (5.45)

Let us introduce the new currents $k_{\pm j}(z) (1 \leq j \leq l)$ associated with $\mathcal{E}_m^{\pm j}$ by

$$k_{\pm j}(z) = \exp \left\{ \sum_{m \neq 0} \frac{[m]_2(q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m^{\pm j} z^{-m} \right\} : .$$  \hspace{1cm} (5.46)

and in addition we define $k_0(z)$ for $B_l^{(1)}$ by

$$k_0(z) = : k_{-l}(q^{-1/2}z) \psi_l(q^{-1/2}z) := k_{+l}(q^{1/2}z) \psi_l(q^{1/2}z)^{-1} : .$$  \hspace{1cm} (5.47)

Then from Proposition 5.1 we have the following decompositions.

**Proposition 5.6**

$$\psi_j(z) = : k_{+j}(z) k_{+(j+1)}(qz)^{-1} := k_{-j}(z)^{-1} k_{-(j+1)}(q^{-1}z) : .$$  \hspace{1cm} (5.48)

where $1 \leq j \leq l - 1$ for $A_l^{(1)}, 1 \leq j \leq l - 1$ for $B_l^{(1)}, C_l^{(1)}$ and $D_l^{(1)}$. In addition,

$$\psi_l(z) = : k_{+l}(z) k_0(q^{-1/2}z)^{-1} := k_{-l}(z)^{-1} k_0(q^{1/2}z) : \quad \text{for } B_l^{(1)},$$  \hspace{1cm} (5.49)

$$= : k_{+l}(q^{-1}z) k_{-l}(q^{-1}z)^{-1} : \quad \text{for } C_l^{(1)},$$  \hspace{1cm} (5.50)

$$= : k_{+(l-1)}(z) k_{-(l-1)}(q^{-1}z)^{-1} := k_{-(l-1)}(q^{-1}z)^{-1} k_{+(l-1)}(z) : \quad \text{for } D_l^{(1)}. \hspace{1cm} (5.51)$$

Now let us introduce the functions $\tilde{\rho}^+(z)$, which appear associated with the elliptic dynamical $R$-matrices [40]:

$$\tilde{\rho}^+(z) = \frac{q^2 z \{ q^2 z^{-2} \} \{ p \bar{x}^2 z \} \{ p / z \}}{\{ q \bar{x}^2 z \} \{ z \} \{ q \bar{x}^2 z \} \{ q^{-2} z \}} \quad \text{for } A_l^{(1)} , \hspace{1cm} (5.52)$$

$$= \frac{\{ q \bar{x}^2 z \} \{ q \bar{x}^2 z \} \{ q^2 z \} \{ p / z \} \{ p \bar{x}^2 z \} \{ p \bar{x}^2 q^{-2} z \} \{ p \bar{x}^2 q^{-2} z \} \{ p q^2 z \}}{\{ q \bar{x}^2 z \} \{ q \bar{x}^2 z \} \{ q \bar{x}^2 z \} \{ q^{-2} z \} \{ p \bar{x}^2 z \} \{ p \bar{x}^2 q^{-2} z \} \{ p q^2 z \}} \quad \text{for } B_l^{(1)}, C_l^{(1)}, D_l^{(1)}. \hspace{1cm} (5.53)$$
where $\xi = q^{-2n}$, $\{z\} = (z; p, \xi^2)_{\infty}$. The following Theorem indicates a deep relationship between $k_{\pm j}(z)$'s and elliptic dynamical $R$-matrices.

**Theorem 5.7**

\[
k_{\pm j}(z_1)k_{\pm j}(z_2) = \frac{\tilde{\rho}^{+*}(z)}{\rho^{+}(z)}k_{\pm j}(z_2)k_{\pm j}(z_1), \quad (1 \leq j \leq l),
\]

\[
k_{+j}(q^j z_1)k_{+j}(q^j z_2) = \frac{\tilde{\rho}^{+*}(z) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(z) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_{+j}(q^j z_2)k_{+j}(q^j z_1) \quad (1 \leq j < k \leq l),
\]

\[
k_{-j}(q^{-j} z_1)k_{-j}(q^{-j} z_2) = \frac{\tilde{\rho}^{+*}(z) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(z) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_{-j}(q^{-j} z_2)k_{-j}(q^{-j} z_1) \quad (1 \leq k < j \leq l),
\]

\[
k_{+j}(q^j z_1)k_{-j}(q^{-j} z_2) = \frac{\tilde{\rho}^{+*}(z) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(z) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_{+j}(q^j z_2)k_{-j}(q^{-j} z_1) \quad (j \neq k),
\]

\[
k_{-j}(q^{-j} z_1)k_{+j}(q^j z_2) = \frac{\tilde{\rho}^{+*}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_{-j}(q^{-j} z_2)k_{+j}(q^j z_1),
\]

where $z = z_1/z_2$ and $\tilde{\rho}^{+*}(z) = \rho^{+}(z)|_{p \to p^*}$. In addition, for $B_1^{(1)}$ we have

\[
k_0(z_1)k_0(z_2) = \frac{\tilde{\rho}^{+*}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_0(z_2)k_0(z_1),
\]

\[
k_{+j}(q^j z_1)k_0(q^{-j+1/2} z_2) = \frac{\tilde{\rho}^{+*}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_0(q^{-j+1/2} z_2)k_{+j}(q^j z_1) \quad (1 \leq j \leq l),
\]

\[
k_{-j}(q^{-j} z_1)k_0(q^{j-1/2} z_2) = \frac{\tilde{\rho}^{+*}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)}{\rho^{+}(u) \Theta_{p^*}(q^{-2} z)\Theta_p(q z)} k_0(q^{j-1/2} z_2)k_{-j}(q^{-j} z_1) \quad (1 \leq j \leq l).
\]

**Proof** Straightforward calculation using Theorem 5.3. \qed

In addition from Proposition 5.5, we obtain:

**Proposition 5.8**

\[
k_{\pm j}(z_1)e_j(z_2) = \frac{\Theta_{p^*}(q^{-c} z)}{\Theta_{p^*}(q^{-c+2} z)} e_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq l),
\]

\[
k_{\pm j}(z_1)e_{j-1}(z_2) = \frac{\Theta_{p^*}(q^{-c+1} z)}{\Theta_{p^*}(q^{-c+1} z)} e_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq l),
\]

\[
k_{\pm j}(z_1)e_k(z_2) = e_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j - 1),
\]

\[
k_{\pm j}(z_1)f_j(z_2) = \frac{\Theta_p(q^{c+2} z)}{\Theta_p(z)} f_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq l),
\]

\[
k_{\pm j}(z_1)f_{j-1}(z_2) = \frac{\Theta_p(q^{c+1} z)}{\Theta_p(z)} f_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq l),
\]

\[
k_{\pm j}(z_1)f_k(z_2) = f_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j - 1).
\]
for $A^{(1)}_l$, $B^{(1)}_l$ with $1 \leq i \leq l$, $C^{(1)}_l$ with $1 \leq i \leq l - 1$, $D^{(1)}_l$ with $1 \leq i \leq l - 2$. In addition, we have

\[
\begin{align*}
  k_0(q^{l-1/2}z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c+l}z_2)\Theta_{p^*}(q^{-c+l-1}z_2)}{\Theta_{p^*}(q^{-c+l-2}z_2)\Theta_{p^*}(q^{-c+l+1}z_2)}e_l(z_2)k_0(q^{l-1/2}z_1), \\
  k_0(q^{l-1/2}z_1)e_j(z_2) &= e_j(z_2)k_0(q^{l-1/2}z_1) \quad (1 \leq j \leq l - 1), \\
  k_0(q^{l-1/2}z_1)f_j(z_2) &= \frac{\Theta_{p}(q^{l-2}z_2)\Theta_{p}(q^{l+1}z_2)}{\Theta_{p}(q^{l}z_2)\Theta_{p}(q^{l-1}z_2)}f_j(z_2)k_0(q^{l-1/2}z_1), \\
  k_0(q^{l-1/2}z_1)f_j(z_2) &= f_j(z_2)k_0(q^{l-1/2}z_1) \quad (1 \leq j \leq l - 1) \quad \text{for } B^{(1)}_l, \\
  k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z_2)}{\Theta_{p^*}(q^{-c\pm 3}z_2)}e_l(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z_2)}{\Theta_{p^*}(q^{-c\pm 3}z_2)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1), \\
  k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_{p}(q^{\mp 3}z_2)}{\Theta_{p}(q^{\mp 1}z_2)}f_l(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_{p}(q^{\mp 1}z_2)}{\Theta_{p}(q^{\mp 1}z_2)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1) \quad \text{for } C^{(1)}_l, \\
  k_{\pm(l-1)}(z_1)e_j(z_2) &= \frac{\Theta_{p^*}(q^{-c-l}z_2)}{\Theta_{p^*}(q^{-c+l+2}z_2)}e_j(z_2)k_{\pm(l-1)}(z_1) \quad (j = l, l - 1), \\
  k_{\pm(l-1)}(z_1)e_{l-2}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z_2)}{\Theta_{p^*}(q^{-c\pm 1}z_2)}e_{l-2}(z_2)k_{\pm(l-1)}(z_1), \\
  k_{\pm(l-1)}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm(l-1)}(z_1) \quad (j \neq l, l - 1, l - 2), \\
  k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z_2)}{\Theta_{p^*}(q^{-c\pm 3}z_2)}e_l(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z_2)}{\Theta_{p^*}(q^{-c\pm 3}z_2)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1), \\
  k_{\pm(l-1)}(z_1)f_j(z_2) &= \frac{\Theta_{p}(q^{\pm 2}z_2)}{\Theta_{p}(z_2)}f_j(z_2)k_{\pm(l-1)}(z_1) \quad (j = l, l - 1), \\
  k_{\pm(l-1)}(z_1)f_{l-2}(z_2) &= \frac{\Theta_{p}(q^{\pm 1}z_2)}{\Theta_{p}(q^{\mp 1}z_2)}f_{l-2}(z_2)k_{\pm(l-1)}(z_1), \\
  k_{\pm(l-1)}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm(l-1)}(z_1) \quad (j \neq l, l - 1, l - 2), \\
  k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_{p}(q^{\mp 3}z_2)}{\Theta_{p}(q^{\pm 1}z_2)}f_l(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_{p}(q^{\mp 1}z_2)}{\Theta_{p}(q^{\mp 1}z_2)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
  k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1) \quad \text{for } D^{(1)}_l.
\end{align*}
\]
The elliptic bosons $\mathcal{E}_m^{\pm j}$ and their elliptic currents $k_{\pm j}(z)$ are useful to realize the $L$-operators and the vertex operators for $U_{q,p}(\hat{\mathfrak{g}})$ as well as deformation of the $W$-algebras. We will discuss this subject in separate papers.

Acknowledgments H.K is supported by the Grant-in-Aid for Scientific Research (C) 22540022 JSPS, Japan. R.M.F is grateful to the Egyptian government for a scholarship.

Appendix A: The Drinfeld Realization of $U_q(\hat{\mathfrak{g}})$

Let $\hat{\mathfrak{g}}$ be an untwisted affine Lie algebra.

Definition A.1 The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ in the Drinfeld realization is a unital $\mathbb{C}$-algebra generated by $q^h$ ($h \in \mathfrak{h}$), $a_{i,n}^+$, $x_{i,m}^\pm$ ($i \in I$, $n \in \mathbb{Z}_{\neq 0}$, $m \in \mathbb{Z}$) $\hat{\mathfrak{d}}$ and the central element $c$. We set

\[ x_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m}^\pm z^{-m}, \]
\[ \psi_i(z) = q_i^{h_i} \exp \left( (q_i - q_i^{-1}) \sum_{n > 0} a_{i,n}^+ z^{-n} \right), \]
\[ \varphi_i(z) = q_i^{-h_i} \exp \left( -(q_i - q_i^{-1}) \sum_{n > 0} a_{i,-n}^- z^n \right). \]

The defining relations are as follows.

\[ [q_i^{\pm h_i} , \hat{\mathfrak{d}}] = 0, \quad [\hat{\mathfrak{d}}, a_{i,n}] = n a_{i,n}, \quad [\hat{\mathfrak{d}}, x_{i,n}^\pm] = nx_{i,n}^\pm, \]
\[ [q_i^{\pm h_i}, a_{j,n}] = 0, \quad q_i^{h_i} x_{j}^\pm(z) = q_i^{\pm a_{ij}} x_j^\pm(z) q_i^{h_i}, \]
\[ [a_{i,n}^+, a_{j,n}^-] = \frac{[a_{ij}n][cn]_j}{n} q^{-c|n|} \delta_{n+m,0}, \]
\[ [a_{i,n}^+, x_{j}^+(z)] = \frac{[a_{ij}n]_j}{n} q^{-c|n|} z^n x_j^+(z), \]
\[ [a_{i,n}^-, x_{j}^-(z)] = -\frac{[a_{ij}n]_j}{n} z^n x_j^-(z), \]
\[ (z - q^{h_j} w)x_i^+(z)x_j^+(w) = (q^{h_j} z - w)x_j^+(w)x_i^+(z), \]
\[ [x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q_i - q_j} \left( \delta(\frac{q^{k} z}{w}) \psi_i(q^{k} w) - \delta(\frac{q^{k} z}{w}) \varphi_i(q^{k} w) \right), \]
\[ \sum_{\sigma \in S_a} \sum_{s=0}^a (-1)^s \left[ \begin{array}{c} a \\ s \end{array} \right] x_i^\pm(z_{\sigma(1)}) \cdots x_i^\pm(z_{\sigma(s)}) x_j^\pm(w) x_i^\pm(z_{\sigma(s+1)}) \cdots x_i^\pm(z_{\sigma(a)}) = 0, \]
\[ (i \neq j, \quad a = 1 - a_{ij}). \]

For $k \in \mathbb{C}$, we define the category $C_k$ of the level-$k$ $U_q(\hat{\mathfrak{g}})$-modules in the same way as $C$ of $U_{q,p}(\hat{\mathfrak{g}})$ in section 2. Let $a_{i,n} = [d_i] a_{i,n}^+$ ($i \in I$, $n \in \mathbb{Z}_{\neq 0}$) be the simple root type level-$k$ Drinfeld bosons. They satisfy

\[ [a_{i,n}, a_{j,m}] = \frac{[b_{ij}n][kn]}{n} q^{-k|n|} \delta_{n+m,0}. \]
For \((V, \pi) \in C_k\), we define the \(Z\)-operators associated with the level-\(k\) \(U_{q,p}(\hat{g})\)-module \(V\) by

\[
Z^\pm_i(z; V) = \exp \left( \mp \sum_{n \geq 1} \frac{\pi(a_i, -n)}{[k^n]} q^{\frac{i+1}{2}k n} z^n \right) \pi(x_i^\pm(z)) \exp \left( \pm \sum_{n \geq 1} \frac{\pi(a_i, n)}{[k^n]} q^{\frac{i-1}{2}k n} z^{-n} \right).
\]

The coefficients \(Z^\pm_{i,n}(V)\) of \(Z^\pm_i(z; V) = \sum_{n \in \mathbb{Z}} Z^\pm_{i,n}(V) z^{-n}\) in \(z\) are well defined elements in \(\text{End}_C V\).

**Theorem A.2** The \(Z\)-operators \(Z^\pm_i(z; V)\) satisfy the same relations in Theorem 3.3 except for Eqs. (3.16), (3.17) with replacement \(Z^\pm_j(z; V), \alpha_j,m, d\) and \(K^\pm\) by \(Z^\pm_i(z; V), a_j,m, \bar{d}\) and \(q_j^{\mp k_i}\), respectively.

**Remark** This theorem is essentially due to Jing [23]. However, in [23] no Serre relations are written explicitly. There are also some misprints in Theorem 2.2 in [23]:

- \((1 - q^{\mp w/z}) q^{\mp k_i} / q^{2k_i}\) should be read as \((1 - q^{\mp w/z}) q^{\mp k_i} / q^{2k_i}\)
- \((1 - q^{\mp z/w}) q^{\mp k_i} / q^{2k_i}\) should be read as \((1 - q^{\mp z/w}) q^{\mp k_i} / q^{2k_i}\)
- \((1 - w/z) q^{\mp k_i} / q^{2k_i}\) should be read as \((1 - w/z) q^{\mp k_i} / q^{2k_i}\)
- \((1 - z/w) q^{\mp k_i} / q^{2k_i}\) should be read as \((1 - z/w) q^{\mp k_i} / q^{2k_i}\)

**Definition A.3** For \(k \in \mathbb{C}^\times\) and \((V, \pi) \in C_k\), we call the subalgebra of \(\text{End}_{C,V}\) generated by \(Z^\pm_{i,m}(V), q^{\pm k_i}_i (i \in I, m \in \mathbb{Z})\) and \(\bar{d}\) the quantum \(Z\)-algebra \(Z_k\) associated with \((V, \pi)\). We also define the universal quantum \(Z\) algebra \(Z_k\) as a topological algebra over \(\mathbb{C}[[q^{2k_i}]]\) in the same way as \(Z_k\) in Definition 3.5. We denote the generators in \(Z_k\) by \(Z^\pm_{j,m}(V)\)\( (j \in I)\).

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