Abstract

We investigate the AdS$_3$/CFT$_2$ correspondence for the Euclidean AdS$_3$ space compacted on a solid torus with the CFT field on the regularizing boundary surface in the bulk. Correlation functions corresponding to the bulk theory at finite temperature tend to the standard CFT correlation functions in the limit of renormalized regularization. In both regular and $Z_N$ orbifold cases, in the sum over geometries, the two-point correlation function for massless modes factors, up to divergent terms proportional to the volume of the $SL(2;Z) = Z$ group, into the finite sum of products of the conformal/anticontformal CFT Green’s functions.

1 Introduction

The AdS/CFT correspondence [1, 2, 3] has been verified for interacting field cases [4, 5] (three- and four-graviton scattering, etc.), and it is interesting to check it also in cases where the space-time geometry is more involved than the sphere. Various approaches to this problem were proposed [6, 7, 8].

In [7, 8], we considered the massless scalar field theory on AdS$_3$ space compacted on a solid torus (toroidal handlebody). We considered both the case of a homogeneously compacted AdS$_3$ manifold without (topological) singularity in the interior and the $Z_N$-orbifold case. The classical scalar field theory on the AdS$_3$ manifold in the bulk then provided the appropriate quantum correlation functions on the boundary.

Recall that the compactification in the Euclidean case corresponds to considering the finite temperature theory in the case of 2 + 1 dimensions, so we actually calculate correlation functions of boundary fields for the AdS$_3$ space at finite temperature [3]. Following [3], one must take into account all possible solutions of the Einstein gravity that have the anti-de Sitter metric at infinity. In the AdS$_3$ case, the black-hole solutions of Hawking and Page [9, 10, 11] turn out [11, 12] to be T-dual to the case of a pure AdS$_3$ space without internal singularities; this holds for the corresponding correlation functions as well [3]. Developing these ideas, authors of [13] proposed that the total CFT free energy must be developed into the sum over all possible AdS$_3$ geometries having the same two-dimensional boundary surface. This necessarily incorporates the sum over possible modular group SL(2;Z) factorized over the parabolic subgroup $Z$; we must therefore include the AdS geometry, the BTZ black-hole geometry, etc., but each such geometry is equivalent to a unique AdS$_3$ geometry without a singularity in the bulk, and we can as well take the sum only over such smooth geometries.

The thermal correlation functions for 1 + 1-dimensional boundary theory in the presence of the Lorentzian BTZ black hole were obtained using the image technique in [14]; however, our approach differs from the one used there because we do not imply that the boundary theory is conformally invariant ad hoc. Instead, as in [7, 8], we introduce the regularizing surface inside the AdS$_3$ space;
this surface must be invariant under the action of the discrete symmetry group. This approach goes in parallel with the approach of Krasnov [13] to the 3D gravity. In the latter, for arbitrary genus handlebody, one must take a bounding "surface invariant w.r.t. the action of the corresponding Fuchsian subgroup and having the induced constant two-dimensional curvature. This surface is determined by the height function $e^{-r}$ with the aid of the Liouville equation.

In Sec. 2, we recall the general structure of $\text{AdS}_3$ manifolds and introduce the "cone regularization" in order to make the volume and the boundary area finite in the solid torus case. In Sec. 3, we derive the Green's function for the points located at the torus boundary in the bulk in the limit $\epsilon$. In Sec. 4, we take the sum over geometries for the smooth $\text{AdS}_3$ space case with the necessary cancellations performed and demonstrate that, in the massless case, we retain the holomorphic factorization of the corresponding two-point correlation function. In Sec. 5, we investigate the corresponding sum over geometries for the massless scalar field correlation function in the case of $Z_N$ orbifold geometry and demonstrate its splitting into a finite sum of conformal blocks. A brief discussion on mass case and perspectives is in Sec. 6.

2 Geometric of $\text{AdS}_3$ Manifolds

The group $\text{SL}(2;C)$ of conformal transformations of the complex plane admits the continuation to the upper half-space $\mathbb{H}^3$ endowed with the constant negative curvature ($\text{AdS}_3$ space). In the Schottky uniformization picture, $\mathbb{H}^3$ ann surfaces of higher genera can be obtained from $C$ by factoring it over a nicely generated free-acting discrete (Fuchsian) subgroup $\text{SL}(2;C)$. Therefore, we can continue the action of this subgroup to the whole $\text{AdS}_3$ and, after factorization, obtain a three-dimensional manifold of constant negative curvature (an $\text{AdS}_3$ manifold) whose boundary is (topologically) a two-dimensional $\text{SL}(2;C)$ ann surface $\mathbb{R}$. The simplest, genus one, $\text{AdS}_3$ manifold (a handlebody with the torus boundary) can be obtained upon the identification

$$(-;t) \rightarrow (t^{-1};q;ijj);$$

(2.1)

where $q = e^{2\pi i}$ is the modular parameter, $\Im t > 0$, $\bar{z} = x + iy$; $x$ and $y$ are the coordinates on $C$, and $t > 0$ is the third coordinate in $\mathbb{H}^3$.

Adopting the $\text{AdS/CFT}$ correspondence principle, we should first regularize expressions in order to make them finite (see [13]). For this, we set the boundary data on a two-dimensional submanifold of the $\text{AdS}_3$ manifold that is invariant under the Fuchsian group action in the bulk. Such a submanifold in the torus case is the "cone" set of points $(t;j)$ such that

$$t = "j ij;"$$

This cone is obviously invariant w.r.t. the action of the Fuchsian element [23] and becomes torus upon the identification. The part of the $\mathbb{H}^3$ bounded from below by this cone becomes the interior of the toroidal handlebody upon factorization. Given the boundary data on this cone, we x the problem setting the Laplace equation then has a unique solution (the Dirichlet problem on a compact manifold).

Geometrically, performing the "cone" regularization and factoring over the group of transformations [23], we obtain the solid torus on whose boundary (the two-dimensional torus) the CFT exile dwell. The "center" of the cone is a unique closed geodesic, which has the length $\log j$ (the image of the vertical half-line $= 0$, while the $\text{AdS}$-invariant (proper) distance $r$ from this geodesic to the image of the "cone" is constant, $\cosh r = 1$.

Following the summation over geometries, we must x the two-dimensional submanifold and consider all possible $\text{AdS}_3$ metrics with this submanifold being the boundary. We restrict the possible class of metrics to be handlebody metrics characterized by a unique geodesic line on the boundary homomorphic to a unique contractible circle (a cycle) in the handlebody. The choice of the complex entry b-cycle homomorphic to the closed geodesic inside the solid torus is not unique, and the freedom is exactly the Abelian (parabolic) subgroup $Z$ allowing adding a-cycle windings to a given b-cycle. We therefore re-derive the Green's function for the solid torus in a way simpler than in [8] and, then, perform the summation over the geometries. We are especially interested in
the case of m massless scalar eld on the A dS 3 space. Insertions of this e 1d m must correspond in the A dS/CFT dictionary to insertions of the c = 1 CFT energy (m om entum tensor @X @X for the free scalar e 1d X (z; @) of the boundary theory. For this e 1d m, as the result, we do reconstruct (up to some e divergences) the conformal block structure of the CFT correlation function on the torus.

3 Green’s function in A dS 3

As is well known, A dS spaces are uniform, that is, we can introduce the interval in terms of the proper distance to the reference point,

\[ ds^2 = dr^2 + \sinh^2 r [d^2 + \sin^2 d^2] \]  

(3.1)

For the integrity reasons, we present the action of the scalar e 1d m assim on A dS 3 in coordinates \( z \),

\[ S = \int dz \sinh r \sin (\theta \sinh r \sin (\theta ))^2 + \frac{d^2}{\sinh^2 r} + \frac{d^2}{\sinh^2 r} + m^2 \]  

(3.2)

which, upon segregating angular degrees of freedom, results in the equation of motion for the radial part \( r \) (if the total angular momentum is \( l \)):

\[ \theta_r \sinh^2 r \theta_r (r) - l(l + 1) (r) \sinh^2 r (r) = 0 \]  

(3.3)

The Green’s function for the e 1d m assim in the bulk of the A dS 3 space must therefore satisfy the equation

\[ \sinh (d + 1) \theta_r \sinh \theta_r G (r \theta) \sinh \theta_r G (r \theta) = (d) (r) \]  

(3.4)

with obvious conditions of decreasing at infinity. We are interested in \( G (r \theta) \) at the regime of large \( r \) only. Then

\[ G (r \theta) \frac{1}{\sinh r} \quad r \to 2 \quad d \quad m = 0 \]  

(3.5)

Recall that the mass spectrum in the A dS 3 is governed by the eigenvalues of the total angular momentum operator in the complementary sphere \( S_d \), which produces the discrete mass spectrum \( m^2 = l(l + 1) \), whereas the corresponding values of \( l \) are integers,

\[ d + 1 \]  

(3.6)

From now on, we restrict the consideration only to the A dS 3 case. Then,

\[ G (r \theta) = \frac{1}{4} \frac{\sinh \theta r}{\sinh r} = \frac{1}{4} \frac{\sinh \theta r}{\sinh r} \]  

(3.7)

Choosing two points, \( \theta_1 \) and \( \theta_2 \), on the complex plane and considering their images on the "cone, i.e., the points \( \theta_1 \) and \( \theta_2 \) in \( \theta \)-plane, the distance between these two points in the bulk. For two points in the upper half-space with heights \( R_1 \) and \( R_2 \) and the distance \( d \) in the plane coordinates, the exact proper distance is

\[ r = \log^4 \frac{R_2}{2R_1} \frac{R_1}{2R_2} + 1 \left( \frac{R_1}{R_2} + \frac{R_2}{R_1} \right) \frac{d^2}{R_1 R_2} \]  

(3.8)

with \( R_1 = \theta_1 \) and \( d = \theta_2 \) in our case. We can however take into account that \( d \geq R_1 \) and \( R_2 \) for any two points on the "cone and then

\[ r = \log^4 \frac{R_2}{2R_1} \frac{R_1}{2R_2} + \frac{d^2}{R_1 R_2} \]  

(3.9)
and, at large distances, we may have
\[ r' = \log \frac{1}{2} \sum_{j} \left( \frac{z_j^2}{j} + O(m^2) \right) : \]  

Passing from the Schottky uniformization to the standard complex structure on two-dimensional torus using the exponential mapping
\[ e^{i \omega z}; \quad e^{i \omega w} ; \quad (3.11)\]
and using (3.5), we obtain the formula for the Green's function for two points on the "cone at large proper distances:
\[ G(z,w; n) = \frac{1}{4} \sum_{j} e^{i \omega z} e^{i \omega w} j^2 = \frac{m^2}{8} \sinh (z \cdot w) j^2 ; \quad (3.12)\]

We keep here the scaling factor $m^2$, which will be removed only at the very end of calculations, after the summation over geometries. The reason for it will be clarified in the succeeding section. The function in (3.12) is periodic under shifts $z = z + m$ and $w = w + k$ for $k, m \in \mathbb{Z}$. In order to obtain the Green's function for the solid torus boundary, we must now take the sum over all images $z_i$ of the point $z$, $z_n = z + n$. Thus, the final answer for the Green's function on the toroidal handlebody is
\[ G_{\text{torus}}(z,w; n) = \frac{m^2}{8} \sum_{n=1}^{\infty} \sinh (z \cdot w + n) j^2 ; \quad (3.13)\]

This formula (up to scaling factors) exactly reproduces the answer obtained in [8] in the massless case ($\kappa = 2$) and in [13] for the general massive case.

4 Summation over geometries

The most difficult problem when evaluating the sum over geometries is the choice of the proper summation measure. For this, several proposals had been made [13, 18]. Using ideas of [18], we assume that one must consider the same two-dimensional boundary surface, the proper continuation to the $\text{AdS}_3$ being then completely determined by the choice of (contractible) a-cycles; in the torus case, we have just one a-cycle determined by two relatively prime integers $(c,d)$ with the identification $(c,d) \mod (c,d)$. The proper modular transformation from the $\text{SL}(2;\mathbb{Z}) = \mathbb{Z}$ is then
\[ \begin{pmatrix} 0 & a + b \\ c + d \end{pmatrix} z ; \quad \begin{pmatrix} 0 & z \\ c + d \end{pmatrix} w ; \quad \begin{pmatrix} a & b \\ c + d \end{pmatrix} = 1 ; \quad (4.1)\]
where the pair $(a;b)$ must be taken modulo the parabolic group transformation $(a;b) \mod (a + c;b + d)$. This is because the choice of the contractible cycle is completely determined by the pair $(c,d)$ as the image of the straight line along the vector $c + d$ upon the identification $z' = z + 1'$. But the choice of the complex entry $b$-cycle (the image of the straight line along the vector $a + b$) is fixed only up to the freedom of adding the a-cycle vector. The condition of the unit determinant then just expresses that these two cycles have exactly one intersection point at the torus.

We take the summation measure to be just the hyperbolic volume of the toroidal handlebody bounded by the torus in the $\text{AdS}$ space with the a-cycle selected. A Graham and Lee theorem claims that choosing a conformal metric on the boundary of a ball, the smooth $\text{AdS}$ continuation of the metric inside the ball is unique. The AdS metric inside the handlebody is then completely determined by the conformal metric on the boundary Riemann surface together with the choice of a set of contractible a-cycles. In order to calculate both the corresponding metric and the volume, we use the scheme depicted in Fig.1.

In the geometry of Fig.1, we consider the action of the Schottky group on the complex plane $\mathbb{C}$ with the modular parameter \( \frac{c}{(c,d)} = \frac{c}{c + d} \). It acts by identifying the circle $\mathbb{R} \equiv 1$ and the circle.
\[ j^j = \text{Im}_{(c;d)} \]. Note that for the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with the unit determinant,

\[ \text{Im}_{(c;d)} = \frac{\text{Im}_{(0;1)}}{j^j + d^j}; \tag{4.2} \]

Next, we continue this identification using the formula \( \text{Im}_{(c;d)} \) into the whole AdS\(_3\). The natural coordinates in Fig. 1 are the spherical coordinates \((\sigma;\eta')\), where \( \sigma = \frac{1}{2} \log(-+t) \) is the logarithm of the Euclidean distance to the origin and \( \sigma \) and \( \eta' \) are the corresponding angles; the interval \([0,1]\) in this coordinates is

\[ ds^2 = \frac{d\sigma^2 + d\eta'^2 + \cos^2\eta' d\sigma^2}{\sin^2\eta'}; \tag{4.3} \]

We take such the bounding surface for which the proper AdS\(_3\) circumferences \( l_{a}^{[\text{c,d}]} \) of the a-circle is exactly proportional to the length of the corresponding geodesic in the plane metric on \( C \), i.e., we set

\[ l_{a}^{[\text{c,d}]} = 2 \frac{\cos\eta_{(\text{c,d})}}{\sin\eta_{(\text{c,d})}} = j_c + d_j^{[0;1]} = j_c + d_j \cos\eta_{(0;1)} / \sin\eta_{(0;1)}; \tag{4.4} \]

that is

\[ \frac{\cos\eta_{(\text{c,d})}}{\sin\eta_{(\text{c,d})}} = j_c + d_j \frac{\cos\eta_{(0;1)}}{\sin\eta_{(0;1)}}; \tag{4.5} \]

This choice is justified by that it leads to the proper scaling behavior of Green's functions (see below). It then follows from the hyperbolic geometry that

\[ l_{b}^{[\text{c,d}]} = \frac{\text{Im}_{(\text{c,d})}}{\sin\eta_{(\text{c,d})}} = \frac{\text{Im}_{(0;1)}}{j_c + d_j \sin\eta_{(\text{c,d})}}; \tag{4.6} \]

\[ \text{Fig. 1. The regularizing surface for the torus. Here } l_{a}^{[\text{c,d}]} \text{ is the proper (AdS) circumferences of the regularizing cone (which is actually a cylinder in the proper distance geometry) and } l_{b}^{[\text{c,d}]} \text{ is the proper (AdS) distance between identified circles on the cone.} \]

For the \( \eta' \)-cone, the volume of the toroidal handlebody bounded by a torus of the induced area \( S_{(\text{c,d})} \) lying at the distance \( \arcsinh \eta' \) from a unique closed geodesic is

\[ V_{(\text{c,d})} = \frac{1}{2} S_{(\text{c,d})} \cos\eta_{(\text{c,d})} = \frac{1}{2} l_{a}^{[\text{c,d}]} l_{b}^{[\text{c,d}]} \cos\eta_{(\text{c,d})}; \tag{4.7} \]
and using formulas (4.10) (4.13), we obtain the simple relation:

$$V_{(c;d)} = V_{(0;1)};$$

(4.8)

ie., under our prescription, the hyperbolic volume of the regularizing manifold becomes exactly modular invariant, and we have the same weight factors standing by the Green's functions of the form (3.13). Then, due to (4.5), the factor $n^2_{(c;d)}$ in (3.12) becomes in the limit $n \to 0$ just the proper scaling factor $j^2 + d j^2 = 2$ standing by the relevant Green's function: upon the transformation (4.11), we have

$$G_{\text{torus}}(z;w) = G_{\text{torus}}(z,w);$$

(4.9)

where we have eventually omitted the irrelevant overall factor $n!_{(0;1)}=(0!)$, and it only remains to take the sum over all $(c;d)$-pairs.

Because the small-$z$ behavior of the function (4.9) is obviously

$$G_{\text{torus}}(z) \sim \frac{1}{z};$$

(4.10)

and it is independent on the $(c;d)$-pair choice, in order to obtain the finite answer when performing the summation over $(c;d)$-pairs, we must subtract from each (except one) term of this sum the corresponding (doubly periodically continued) function

$$F(z) = \sum_{n,k \neq 0} \frac{1}{z+n+k^2};$$

(4.11)

The thus regularized sum becomes especially instructive in the massless case $m=2$. Using the representation for the function $\sin^2(z)$,

$$\sin^2(z) = \sum_{k=1}^{\infty} \frac{1}{(z+k)^2};$$

(4.12)

we can present the sum over geometries in the following form (omitting irrelevant factors):

$$\sum_{(c;d)} G_{\text{torus}}(z) = 2 = F(z) + \sum_{(c;d)} \frac{1}{z+k_1(c+d) + n(a+b)}(z+k_2(c+d) + n(a+b));$$

(4.13)

(the primed sum denotes that we must count pairs $(c;d)$ and $(c;\bar{d})$ just once). In this sum, we must take for each pair $(c;d)$ a single complementary pair $(a;b)$ such that the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the unity.

Collecting the coefficients standing by and unit factors in (4.13), we obtain

$$F(z) = \frac{1}{(z+s_1+t)^2(z+s_2+t)^2} \sum_{sp, 2 \neq 0} \frac{1}{z+n+k^2} = \frac{1}{(z+s+t^2)^2} \sum_{sp2 \neq 0} \frac{1}{z+n+k^2};$$

(4.14)
where \( s_{1,2} \) and \( t_{1,2} \) are solutions of the system of equations

\[
\begin{align*}
na + k_1 c &= s_1; \\
nb + k_2 c &= s_2; \\
nb + k_2 d &= t_1; \\
nb + k_2 d &= t_2;
\end{align*}
\]

Subtracting the second equation from the first and the fourth from the third, we have

\[
k_c = s; \quad k_d = t; \quad (4.15)
\]

where \( k, k_1, k_2, s, s_1, s_2, t, t_1, t_2 \). Because \( \text{GCD}(c; d) = 1 \) and we identify \( (c; d) \) \( (c; d) \) equations \( (4.15) \) admit a unique solution

\[
k = \text{GCD}(s; t); \quad c = s = k; \quad d = t = k
\]

unless \( s = t = 0 \). Recall that \( c \) and \( d \) may vanish, but not simultaneously, and if \( c = 0 \), then \( d = 1 \) and vice versa. Given \( c \) and \( d \) (and, correspondingly, \( a \) and \( b \)), the solution of, say, the matrix equation w.r.t. \( n \) and \( k_1 \),

\[
\begin{pmatrix}
s_1; k_1
\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} s_2; t_1 \end{pmatrix};
\]

exists and is unique in \( Z \) because the determinant of the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is the unity. The contribution at \( s = t = 0 \) implies \( k = 0 \), and this contribution is then exactly cancelled by the contribution from the function \( F(z; j) \).

We therefore obtain that the regularized sum in \( (4.15) \),

\[
X \overset{0 \text{(reg)}}{\text{G}}_{\text{torus}}(z, j) = \frac{1}{\text{GCD}(s; t)} \sum_{n, m} \frac{1}{(z + s + t)^2} = 1 \quad (z; j); \quad (j; j); \quad (4.16)
\]

becomes just the squared modulus of the \( \text{W} \) Eisenstein \( j \)-function

\[
\begin{align*}
\text{W}(z; j) &= \frac{X}{(z + s + t)^2};
\end{align*}
\]

We may expect that this term provides the relevant contribution to the CFT correlation function in the sum over the \( AdS_5 \) metrics because it exhibits the structure of CFT conformal blocks. Say, the function \( (4.16) \) exactly coincides with the correlation function \( \Theta X (z; \bar{z}) \Theta X (z; \bar{z}) \Theta X (0; 0) \Theta X (0; 0) \) for the \( c = 1 \) free scalar \( X (z; \bar{z}) \) on torus.

5 Summing over geometries in \( Z_N \)-orbifold case

Using our technique, we can find a sum over geometries for the correlation functions of the massless \( \text{W} \) also in the case where a conical singularity with solid angle 2 \( = N \) is located at the closed geodesic line (the vertical axis in Fig. 1) of the toroidal handlebody. We can resolve this singularity by considering the \( N \) -sheet covering of the corresponding space; this covering is just the \( \text{old} \) toroidal handlebody.

This corresponds to imposing the following double periodic conditions on admitted functions of the scaled variable

\[
\begin{align*}
z &= N \quad (5.1)
\end{align*}
\]

in the torus case:

\[
\begin{align*}
f(z) &= f(z + 1); \quad f(z) = f(z + 1).
\end{align*}
\]

The corresponding correlation functions in the massless case are [3]

\[
G_{\text{torus}}(z; j) = \frac{X}{(z + s + t)^2} \sin \left( (z + p + n)N \right); \quad (5.2)
\]

\[
G_{\text{torus}}(z; j) = \frac{X}{(z + s + t)^2} \sin \left( (z + p + n)N \right); \quad (5.3)
\]
and in the sum over geometries, we must take \( 0 \) \((\text{cf. (4.13)})\)

\[
X \sum_{(c,d)} G_{\text{torus}}^{(c,d)}(z; \mathcal{F}) = F(z; \mathcal{F}) + \sum_{(c,d)} G_{\text{torus}}^{(c,d)}(z; \mathcal{F}) F(z; \mathcal{F}); \tag{5.4}
\]

where

\[
G_{\text{torus}}^{(c,d)}(z; \mathcal{F}) = \prod_{n=1}^{\infty} \prod_{p=1}^{N} \frac{1}{(c + d + n(\mathcal{F} + p + n)^N)^2}; \tag{5.5}
\]

with \( z^0 \) from \((4.1)\). The function \( F(z; \mathcal{F}) \) is given by the same formula \((4.11)\). Using the same trick as in the smooth torus case, we represent the expression in \((5.4)\) as the series

\[
N^4 F(z; \mathcal{F}) + N^4 \sum_{(c,d)} \sum_{k_1, k_2, p_1, p_2} \prod_{j=1}^{2} \frac{1}{(z + s_j + t_j)^2 (\mathcal{F} + s_j + t_j)^2}; \tag{5.6}
\]

where \( s_{1,2} \) and \( t_{1,2} \) are solutions of the system of equations

\[
na + (N k_1 + p)c = s_1; \\
na + (N k_2 + p)c = s_2; \\
nb + (N k_1 + p)d = t_1; \\
nb + (N k_2 + p)d = t_2;
\]

which again has a unique solution if \( s = N x, t = N y \) and \( x, y \notin \mathbb{Z} \) do not vanish simultaneously; \(^2\) the contribution when \( x = y = 0 \) is again exactly cancelled by the function \( F(z; \mathcal{F}) \).

Eventually, we have

\[
X \sum_{(c,d)} G_{\text{torus}}^{(c,d)}(z; \mathcal{F}) = \prod_{s, t \in \mathbb{Z}} X^{(s + t)^2} = \prod_{s, t \in \mathbb{Z}} \frac{1}{(z + p + q + N s + N t)^2 (\mathcal{F} + p + q + N s + N t)^2}; \tag{5.7}
\]

where

\[
\prod_{s, t \in \mathbb{Z}} \frac{1}{(z + p + q + N s + N t)^2 (\mathcal{F} + p + q + N s + N t)^2}; \tag{5.7}
\]

is the Weierstrass \( \wp \)-function with characteristics and the sum in \((5.7)\) exhibits the properties of the sum over conformal blocks of the CFT with the twisted boundary conditions corresponding to the \( Z_N \)-orbifold case.

\(^2\) We conveniently express this condition as \( x^2 + y^2 > 0 \).
6 Discussion. Massive modes

We have demonstrated that correlation functions in the AdS space in the sum over geometries exhibit properties of sums over conformal blocks of the underlying CFT. Although we have considered only the massless case in details, it seems plausible that the same procedure (with slight modifications) can be applied to the whole spectrum of mass appearing in the AdS/CFT correspondence pattern. The generalization seems to be rather straightforward. The factorization property must be nevertheless corrected; to see this, let us consider an example of the correlation function for fields at the second mass level. Using (4.6) and the formula for $l=\sin^2(z)$,

$$\frac{1}{\sin^4(z)} = \sum_{n=1}^{\infty} \frac{1}{(z + n)^2} + \frac{2}{3(z + n)^2};$$

and performing the summation over $(c;d)$-pairs, we obtain

$$\sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) = \sum_{(c;d)} F(z^2) + \sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) |^{(c;d)} = \sum_{(c;d)} F(z^2)$$

Besides the factored term $\frac{1}{z^2} \mathcal{G}_{\text{torus}}(z | \mathbf{n})$, appearing when summing over $(c;d)$-pairs, the products of leading $(1st)$ terms in parentheses in (6.1) with the singularity cancelled by the term $F(z^2)$, we have the contributions

$$\sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) = \sum_{(c;d)} F(z^2) + \sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) |^{(c;d)} = \sum_{(c;d)} F(z^2)$$

which converge due to the presence of the $(c + d)$ factors in denominators. But, again, just because of these factors, we cannot perform the resummed procedure as in Secs. 4 and 5, and the possible modular behavior of these sums become somewhat involved. For instance, we can perform the resummation in the $1st$ contribution passing to the summation variables $s_1, t_1, s_2, t_2$, which yields

$$\sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) = \sum_{(c;d)} F(z^2) + \sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) |^{(c;d)} = \sum_{(c;d)} F(z^2)$$

where the coefficient by the second term is

$$\sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) = \sum_{(c;d)} F(z^2) + \sum_{(c;d)} \mathcal{G}_{\text{torus}}(z | \mathbf{n}) |^{(c;d)} = \sum_{(c;d)} F(z^2)$$
and \( E_2 \) is the Eisenstein series.

Even more difficult (in the ideological sense) problem is the problem of determining the proper sum measure. Under our bulk regularization for the Green’s function, it seems plausible that in order to reproduce the proper scaling behavior of the Green’s functions for different geometries, this measure (or, the corresponding hyperbolic volume) must be constant for all solid torus geometries with the given boundary surface metric; this result however in the necessity to introduce regularizing factors; nevertheless, it turns out that these regularizing factors are independent on the additional structure (on the choice of the \( a \)-cycles) and can be therefore determined unambiguously as soon as we have the two-dimensional metric. Still, it is an important question whether it is possible to obtain the relevant regularizing factors from the field-theory considerations related to \( D1(D5 \) brane system) (see [20] and references therein). It would be interesting to check whether the Hamiltonian prescription of [21] concerning local contributions may help in constructing such a regularization. We however hope that the very appearance of the conformal block structure in the sum over geometries in our approach justifies further studies of these, relatively simple, system. Another rather straightforward generalization can be considering the supersymmetricization of the whole pattern.

Another, really challenging, problem is to consider generalizations of this technique to handle bodies of higher genus. There, we must use the Schottky uniformization picture on the complex \( \mathbb{C} \) plane while the regularizing surface must be determined by the equation

\[
t = \frac{1}{e^t \pi}
\]

with \( \left( \frac{1}{\pi} \right) \) satisfying the Liouville equation. That is, we must be able to work with expressions of the form (cf. [3.12])

\[
G_{\text{higher genus}}(\cdot ; j) = \sum_{k} \exp \left( \frac{j}{2\pi^2} \frac{k^2}{e^t \pi} - \frac{j}{e^t \pi} \right)
\]

where the sum ranges all images of the point under the Schottky group action. And it then still remains the problem of performing the summation over all \( \text{AdS}_3 \) geometries determined by all possible choices of the \( a \)-cycle structures on the relevant \( \text{Riemann} \) surface. This may lead to a progress in studying Liouville system as well.

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