Density results and trace operator in weighted Sobolev Spaces defined on the half line equipped with power weights

Radosław Kaczmarek
Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland; radekk@amu.edu.pl, ORCID: 0000-0002-3536-9102

Agnieszka Kałamajska
Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland; A.Kalamajska@mimuw.edu.pl, ORCID: 0000-0001-5674-8059

April 26, 2022

Abstract

We study properties of $W^{1,p}_0(\mathbb{R}^+, t^\beta)$ - the completion of $C_0^\infty(\mathbb{R}^+)$ in the power-weighted Sobolev spaces $W^{1,p}(\mathbb{R}^+, t^\beta)$, where $\beta \in \mathbb{R}$. Among other results, we obtain the analytic characterization of $W^{1,p}_0(\mathbb{R}^+, t^\beta)$ for all $\beta \in \mathbb{R}$. Our analysis is based on the precise study the two trace operators: $\text{Tr}^0(u) := \lim_{t \to 0} u(t)$ and $\text{Tr}^\infty(u) := \lim_{t \to \infty} u(t)$, which leads to the analysis of the asymptotic behavior of functions from $W^{1,p}_0(\mathbb{R}^+, t^\beta)$ near zero or infinity. The obtained statements can contribute to the proper formulation of Boundary Value Problems in ODE’s, or PDE’s with the radial symmetries. We can also apply our results to some questions in the complex interpolation theory, raised by M. Cwikel and A. Einav in 2019, which we discuss within the particular case of Sobolev spaces $W^{1,p}(\mathbb{R}^+, t^\beta)$.

Mathematics subject classification (2020): Primary 46E35; Secondary 46B70.
Keywords and phrases: Sobolev spaces, density, asymptotics, interpolation.

1 Introduction

Results about densities of smooth functions are one of central problems in the theory of Sobolev spaces, see e.g. [12, 13]. Classically, having the domain $U \subseteq \mathbb{R}^n$, one defines
$W_0^{m,p}(U)$ - the subspace of Sobolev space $W^{m,p}(U)$, as the completion of the set of smooth compactly supported functions $C_0^\infty(U)$, in the norm of $W^{m,p}(U)$. Similarly, having given the weights function $\omega$ defined on $U$, one defines $W_0^{m,p}(U, \omega)$ as the completion of $C_0^\infty(U, \omega)$ in the weighted Sobolev space $W^{m,p}(U, \omega)$. Then the following problem arises: How can we recognize in the easy way, if the given function from the space $W^{m,p}(U, \omega)$ belongs to its subspace $W_0^{m,p}(U, \omega)$?

Here we are interested in spaces $W_1^{1,p}(\mathbb{R}^+, t^\beta)$ and $W_0^{1,p}(\mathbb{R}^+, t^\beta)$, where $\beta \in \mathbb{R}$ is given number and we ask about the analytic characterization of $W_0^{1,p}(\mathbb{R}^+, t^\beta)$. Although the above question looks very simple at first glance, we could not find results about such a characterization in the literature. On the other hand, they might be present in the hidden way, as it is in the classical formulation of Hardy inequality, see Remark 4.1.

In the non-weighted, classical approach on bounded and sufficiently regular domains, characterization is known. Namely, having given bounded domain with the sufficiently regular boundary, say Lipschitz, one has well defined the trace operator $Tr : W_1^{1,p}(U) \to L^1(\partial U)$ (see e.g. [1, 11, 15]) for any $p \in [1, +\infty)$. Then the following characterization can be prescribed:

$$W_0^{1,p}(U) = \{u \in W_1^{1,p}(U) : Tr u|_{\partial U} \equiv 0 \text{ a.e. on } \partial U\}.$$  

Our analysis is based on a similar approach, which however requires to have introduced and analyzed the trace operator defined at the endpoint of the interval $(0, \infty)$. It appears, that in our setting, in all cases except at $\beta = p - 1$, such trace operator can be defined at one endpoint $p = 0$ or $p = \infty$ only. For example, as shown in Theorems: 4.1, 4.2 and 4.3 for any $p > 1$ and $u \in W_1^{1,p}(\mathbb{R}^+, t^{\beta})$, we have:

- $Tr^0(u) := \lim_{t \to 0} u(t)$ is well prescribed for $\beta < p - 1$,

- $Tr^\infty(u) := \lim_{t \to \infty} u(t)$ is well prescribed for $\beta > p - 1$,

- when $\beta = p - 1$, then $Tr^0(u)$ cannot be prescribed in general on $W_1^{1,p}(\mathbb{R}^+, t^{p-1})$, while $Tr^\infty(u)$ annihilates all that space.

Moreover, having trace operator precisely defined, when $u \in W_0^{1,p}(\mathbb{R}^+, t^{\beta})$, then one has

$$\lim_{t \to 0} u(t) = 0 \iff \lim_{t \to 0} t^{(\beta - (p-1))/p} u(t), \quad \text{when } \beta < p - 1,$$

$$\lim_{t \to \infty} u(t) = 0 \iff \lim_{t \to \infty} t^{(\beta - (p-1))/p} u(t), \quad \text{when } \beta > p - 1,$$

which contributes to the study of the asymptotic behavior of functions from $W_0^{1,p}(\mathbb{R}^+, t^{\beta})$, see Theorems 4.1 and 4.2. The above facts can be applied to proper formulation of Boundary Value Problems in ODE’s and radial solutions to PDE’s with radial symmetries, as we mention in Remark 4.2. The case of $\beta = p - 1$ is the borderline one between that of $\beta < p - 1$ and of $\beta > p - 1$. It appears that in that case
\(W_{0}^{1,p}(\mathbb{R}^+, t^{p-1}) = W^{1,p}(\mathbb{R}^+, t^{p-1})\), which is shown in Theorem 4.3. As just mentioned, the trace operators do not apply reasonably to that space.

New information about asymptotic behavior can be also applied to Hardy type inequalities. Namely, the set \(W = W(\alpha, \beta)\) such that \(C_{0}^{\infty}(\mathbb{R}^+) \subseteq W \subseteq W^{1,p}(\mathbb{R}^+, t^{\beta})\) and \(W\) is admitted to the inequality:

\[
\int_{\mathbb{R}^+} |x^\alpha |u(t)|^{p}\, dt \leq C_1 \int_{\mathbb{R}^+} |u(t)|^{p}\, dt + C_2 \int_{\mathbb{R}^+} |u'(t)|^{p}\, dt,
\]

has been precisely described in the paper [8] in the purely analytic way. Now we can say that this set is precisely \(W^{1,p}(\mathbb{R}^+, t^{\beta})\) in the case when \(\beta < 1\), \(\alpha \in (-1, 0)\), which is not visible without our analysis. See Remark 4.3 for more precise information.

Other problems of our interest, are density results. Having traces well defined, we can now say that:

- \(W^{1,p}(\mathbb{R}^+, t^{\beta}) = \{ u \in W^{1,p}(\mathbb{R}^+, t^{\beta}) : \lim_{t \to 0} u(t) = 0 \}\) when \(\beta < p - 1\),
- \(W^{1,p}(\mathbb{R}^+, t^{\beta}) = \{ u \in W^{1,p}(\mathbb{R}^+, t^{\beta}) : \lim_{t \to \infty} u(t) = 0 \}\) when \(\beta > p - 1\),
- \(W^{1,p}(\mathbb{R}^+, t^{\beta}) = W^{1,p}(\mathbb{R}^+, t^{\beta})\), when \(\beta = p - 1\),

see Theorems: 4.1, 4.2, 4.3.

As next example application of our results, we focus on the application to complex interpolation theory. Namely, our density results allow us to conclude that:

\[
[W^{1,p}(\mathbb{R}^+, t^{\beta_0}, \mathbb{C}), W^{1,p}(\mathbb{R}^+, \omega_1, \mathbb{C})]_{\theta} = W^{1,p}(\mathbb{R}^+, t^{\beta_\theta}, \mathbb{C}),
\]

whenever, \(\beta_\theta := (1 - \theta)\beta_0 + \theta \beta_1 \in (-\infty, -1) \cup (p - 1, \infty)\) and \([X, Y]_{\theta}\) is complex interpolation pair between Banach spaces \(X, Y\), with \(\theta \in (0, 1)\), see Theorem 4.3. Such result seems new and contributes to the problem posed in 1982 by Lofstrom in [14] and to the discussion undertaken recently by M. Cwikel and A. Einav in [3] about interpolation spaces between weighted Sobolev spaces. In particular, we built some examples partially solving questions posed in [3], when restricting the analysis to the particular cases of weighted Sobolev spaces \(W^{1,p}(\mathbb{R}^+, t^{\beta})\). See Remark 4.4 for the precise formulations and arguments.

The methods we use are rather elementary. In most situations they are based on the appriori estimates, first order integral Taylor’s formulae, and on Hardy inequality (Theorem 2.1). Similar tools were used e.g. in [6] on page 9, as well as in [2], in the proof of Lemma 5.9. More abstract tools were presented in the proof of Theorem 4.3 showing that \(W^{1,p}(\mathbb{R}^+, t^{\beta}) = W^{1,p}_{0}(\mathbb{R}^+, t^{\beta})\), where we used Mazur’s Lemma.

As all of them: density results, asymptotic behavior near endpoints as well as complex interpolation theory are important tools in functional analysis and PDE’s/ODE’s, we hope to contribute in this way for their understanding.
2 Notation and Preliminaries

Basic notation. We will deal with functions defined on $\mathbb{R}_+$ and use the standard notations $C^\infty(\mathbb{R}_+)$, $C_0^\infty(\mathbb{R}_+)$, $\text{Lip}(\mathbb{R}_+)$, $L^p(\mathbb{R}_+, \omega)$ for smooth functions, smooth compactly supported functions, lipschitz functions and for functions in $L^p$ equipped with $\omega$ weight, respectively. If $A$ is a subset in some Banach space $X = (X, \|\cdot\|_X)$, then by $\overline{A}(\|\cdot\|_X)$ we denote the completion of $A$ in $X$ in the norm $\|\cdot\|_X$.

When estimating expressions, notation $A \lesssim B$ will be used when the estimate $A \leq CB$ holds with some universal constant $C$, whose precise value is not important for final conclusion, while $A \sim B$ will mean that $A \preceq B$ and $B \preceq A$ (denoted also as $A \succeq B$).

The special Sobolev spaces. Let $1 \leq p < \infty$ and $\beta \in \mathbb{R}$. We will deal with the following one and two weighted Sobolev spaces $W^{1,p}(\mathbb{R}_+, t^\beta)$ and $Y^{1,p}_{t^\beta}$:

\[
W^{1,p}(\mathbb{R}_+, t^\beta) := \{ u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+) : \|u\|_{W^{1,p}(\mathbb{R}_+, t^\beta)} < \infty \} \quad \text{where} \quad (2.1)
\]

\[
\|u\|_{W^{1,p}(\mathbb{R}_+, t^\beta)} := \left( \int_{\mathbb{R}_+} |u(t)|^p t^\beta dt \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}_+} |u'(t)|^p t^\beta dt \right)^{\frac{1}{p}},
\]

\[
Y^{1,p}_{t^\beta} := \{ u \in W^{1,1}_{\text{loc}}(\mathbb{R}_+) : \|u\|_{Y^{1,p}_{t^\beta}} < \infty \} \quad \text{where} \quad (2.2)
\]

\[
\|u\|_{Y^{1,p}_{t^\beta}} := \left( \int_{\mathbb{R}_+} |u(t)|^p t^\beta (1 + \frac{1}{t^p}) dt \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}_+} |u'(t)|^p t^\beta dt \right)^{\frac{1}{p}}.
\]

Those spaces are complete and they form Banach spaces, see e.g. [13], Theorem 1.11.

By $W^{1,p}_0(\mathbb{R}_+, t^\beta)$, $Y^{1,p}_{t^\beta}0$, we denote the completion of $C_0^\infty(\mathbb{R}_+)$ in $W^{1,p}(\mathbb{R}_+, t^\beta)$ and $Y^{1,p}_{t^\beta}$, respectively. We will also use notation $\text{Lip}_c(\mathbb{R}_+)$, $W^{1,p}_c(\mathbb{R}_+, t^\beta)$, $Y^{1,p}_{t^\beta,c}$ for the sets of functions in a given function space, which have compact support in $\mathbb{R}_+$.

Let us recall that $W^{1,1}_{\text{loc}}(\mathbb{R}_+) \subseteq C(\mathbb{R}_+)$, see e.g. [15], therefore all the functions which belong to weighted Sobolev spaces as in (2.1) and (2.2), are continuous on $\mathbb{R}_+$.

Hardy inequality. We recall the classical Hardy inequality (see e.g. [4], Theorem 330 or [12]), which will be useful for further considerations.

Theorem 2.1 (Hardy inequality) Let $1 < p < \infty$, $\beta \neq p - 1$. Suppose that $u = u(t)$ is an absolutely continuous function in $(0, \infty)$ such that $\int_0^\infty |u'(t)|^p t^\beta dt < \infty$, and let

\[
u(0) := \lim_{t \to 0} u(t) = 0 \quad \text{for} \quad \beta < p - 1,
\]

\[
u(\infty) := \lim_{t \to \infty} u(t) = 0 \quad \text{for} \quad \beta > p - 1.
\]

Then the following inequality holds with sharp constant:

\[
\int_0^\infty |u(t)|^p t^{\beta-p} dt \leq \left( \frac{p}{\beta - p + 1} \right)^p \int_0^\infty |u'(t)|^p t^\beta dt. \quad (2.3)
\]
3 First density result

We start with the following preliminary density result, which deals with Sobolev spaces as in (2.1) and (2.2).

**Lemma 3.1** Let $1 < p < \infty$ and $\beta \in \mathbb{R}$. Then

$$W_0^{1,p}(\mathbb{R}_+, t^\beta) = \overline{\text{Lip}_c(\mathbb{R}_+)(\|\cdot\|_{W^{1,p}(\mathbb{R}_+, t^\beta)})} = \overline{W_c^{1,p}(\mathbb{R}_+, t^\beta)(\|\cdot\|_{W^{1,p}(\mathbb{R}_+, t^\beta)})},$$

(3.1)

$$Y_{t^\beta,0}^{1,p} = \overline{\text{Lip}_c(\mathbb{R}_+)(\|\cdot\|_{t^\beta})} = \overline{Y_{c,t^\beta}^{1,p}(\|\cdot\|_{t^\beta})}$$

(3.2)

and

$$Y_{t^\beta,0}^{1,p} = Y_{t^\beta}^{1,p} \subseteq W_0^{1,p}(\mathbb{R}_+, t^\beta).$$

(3.3)

**Remark 3.1** We will show in the next section that the embedding $Y_{t^\beta}^{1,p} \subseteq W_0^{1,p}(\mathbb{R}_+, t^\beta)$ in (3.3) can be either strict or non-strict, depending on the exponent $\beta$. For example, if $\beta \neq p - 1$, then $Y_{t^\beta}^{1,p} = W_0^{1,p}(\mathbb{R}_+, t^\beta)$, while for $\beta = p - 1$, we have $Y_{t^\beta}^{1,p} \not\subseteq W_0^{1,p}(\mathbb{R}_+, t^\beta)$. See Theorems 4.1, 4.2 and 4.3.

**Proof.** (3.1): Clearly,

$$C^\infty(\mathbb{R}_+) \subseteq \text{Lip}_c(\mathbb{R}_+) \subseteq W_c^{1,p}(\mathbb{R}_+, t^\beta).$$

It suffices to show that $W_c^{1,p}(\mathbb{R}_+, t^\beta) \subseteq W_0^{1,p}(\mathbb{R}_+, t^\beta)$. For that, take $u \in W_c^{1,p}(\mathbb{R}_+, t^\beta)$ with the support $[a, b] \subseteq \mathbb{R}_+$. As on compactly supported sets $t^\beta \sim 1$, therefore $u \in W_1^1(\mathbb{R}_+)$, and is compactly supported. By the standard convolution arguments, the convolutions $u_\varepsilon(x) := \phi_\varepsilon * u$ with the classical mollifier functions $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$ where $\phi \in C^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, supp $u \subseteq [-1, 1]$ and $\int_\mathbb{R} \phi dx = 1$, converge to $u$ in $W_1^1(\mathbb{R}_+)$. Their supports are the subsets of $J := [a - \varepsilon_0, b + \varepsilon_0]$ for $\varepsilon_0 := a/2$ and $\varepsilon \in (0, a/2)$. Again, as $t^\beta \sim 1$ on $J$, therefore $u_\varepsilon$ converge to $u$ also in $W_0^{1,p}(\mathbb{R}_+, t^\beta)$, as $\varepsilon \to 0$. This shows that $u \in W_0^{1,p}(\mathbb{R}_+, t^\beta)$.

(3.2): The proof of (3.2) reduces to the proof of the fact that $Y_{t^\beta,0}^{1,p} \subseteq Y_{t^\beta}^{1,p}$, which follows by the same convolution argument as just before.

(3.3): To prove (3.3), we first show that $Y_{t^\beta,0}^{1,p} \subseteq Y_{t^\beta}^{1,p}(\|\cdot\|_{t^\beta})$. For this, let $u \in Y_{t^\beta}^{1,p}$ and consider

$$u_n(t) := u(t)\phi_n(t)$$

(3.4)

where

$$\phi_n(t) := \begin{cases} 0 & \text{for } t \leq \frac{1}{2n} \\ 2nt - 1 & \text{for } t \in \left(\frac{1}{2n}, \frac{1}{n}\right) \\ 1 & \text{for } t \in \left(\frac{1}{n}, n\right) \\ -\frac{1}{n}t + 2 & \text{for } t \in (n, 2n] \\ 0 & \text{for } t > 2n. \end{cases}$$
Clearly, \( u_n(t) \in Y^{1,p}_{t^\beta} \). We will show that \( u_n \to u \) in \( Y^{1,p}_{t^\beta} \) as \( n \to \infty \). This follows from the following computations:

\[
\int_{\mathbb{R}_+} |u_n(t) - u(t)|^{p t^\beta} \left( 1 + \frac{1}{t^p} \right) dt = \int_{\mathbb{R}_+} |u(t)(\phi_n(t) - 1)|^{p t^\beta} \left( 1 + \frac{1}{t^p} \right) dt \\
\leq \int_{\{0, \infty\} \cup (n, \infty)} |u(t)|^{p t^\beta} \left( 1 + \frac{1}{t^p} \right) dt \xrightarrow{n \to \infty} 0, \quad (3.5)
\]

\[
\int_{\mathbb{R}_+} |(u_n(t) - u(t))'|^{p t^\beta} dt = \int_{\mathbb{R}_+} |(u(t)(\phi_n(t) - 1))'|^{p t^\beta} dt \leq \int_0^\infty |u'(t)|^{p t^\beta} dt \\
+ \int_n^\infty |u'(t)|^{p t^\beta} dt + (2n)^p \int_0^{\frac{1}{2n}} |u(t)|^{p t^\beta} dt + \frac{1}{n^p} \int_n^{2n} |u(t)|^{p t^\beta} dt \\
=: I_n + II_n + III_n + IV_n. \quad (3.6)
\]

Obviously, \( I_n, II_n, IV_n \to 0 \) as \( n \to \infty \). The same holds for \( III_n \) because \( III_n \sim \int_{\mathbb{R}_+} |u(t)|^{p t^\beta - p} dt \leq \int_{\mathbb{R}_+} |u(t)|^{p t^\beta} \left( 1 + \frac{1}{t^p} \right) n \to \infty 0. \)

It follows that

\[ Y^{1,p}_{t^\beta} \subseteq Y^{1,p}_{t^\beta,0} \subseteq Y^{1,p}_{t^\beta} \quad \text{and consequently,} \quad Y^{1,p}_{t^\beta,0} = Y^{1,p}_{t^\beta}. \]

Moreover,

\[ Y^{1,p}_{t^\beta} \subseteq Y^{1,p}_{t^\beta,0} \subseteq W^{1,p}_{t^\beta} \subseteq W^{1,p}_{t^\beta} \quad \text{(3.1)} \]

\[ \equiv W^{1,p}_0(\mathbb{R}_+, t^\beta), \]

where the embedding (a) holds because the norm \( \| \cdot \|_{W^{1,p}(\mathbb{R}_+, t^\beta)} \) is stronger than the norm \( \| \cdot \|_{Y^{1,p}_{t^\beta}} \). \( \square \)

4 Main results: analytic characterization of \( W^{1,p}_0(\mathbb{R}_+, t^\beta) \)

4.1 Auxiliary sets and trace operators

Let us define the following sets:

\[ A^0_{p,t^\beta} := \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \exists u_n : \lim_{n \to \infty} u(t_n) = 0 \}; \]
\[ B^0_{p,t^\beta} := \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \lim_{t \to 0} u(t) = 0 \}; \]
\[ C^0_{p,t^\beta} := \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \lim_{t \to 0} u(t)t^{(\beta-p+1)/p} = 0 \}; \]
\[ A^\infty_{p,t^\beta} := \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \exists t_n : \lim_{n \to \infty} u(t_n) = 0 \}; \]
\[ B^\infty_{p,t^\beta} := \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \lim_{t \to \infty} u(t) = 0 \}; \]
\[ C^\infty_{p,t^\beta} := \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \lim_{t \to \infty} u(t)t^{(\beta-p+1)/p} = 0 \}. \]

6
We will consider the following two trace-endpoints operators, which are well defined on some subspaces $X, Z \subseteq W^{1,p}(\mathbb{R}^+, t^{\beta})$, possibly depending on $p$ and $\beta$:

$$Tr^0 : X \rightarrow \mathbb{R}, \text{ defined as } u \mapsto \lim_{t \rightarrow 0} u(t) \quad (4.1)$$

and

$$Tr^\infty : Z \rightarrow \mathbb{R}, \text{ defined as } u \mapsto \lim_{t \rightarrow \infty} u(t). \quad (4.2)$$

In our further analysis we will indicate on situations when $X, Z$ are the whole spaces $W^{1,p}(\mathbb{R}^+, t^{\beta})$. That observations will be useful to derive density results.

### 4.2 Presentation of main results

Our main goal is to prove the following three statements, which give analytic characterization of $W^{1,p}_0(\mathbb{R}_+, t^{\beta})$ for all possible range of $\beta$.

**Theorem 4.1** (characterization of $W^{1,p}_0(\mathbb{R}_+, t^{\beta})$, $\beta < p - 1$) Let $1 < p < \infty$ and $\beta < p - 1$. Then we have:

i) $W^{1,p}_0(\mathbb{R}_+, t^{\beta}) = A^0_{p,t^{\beta}} = B^0_{p,t^{\beta}} = C^0_{p,t^{\beta}} = Y^{1,p}_{t^{\beta}}$. 

ii) The mapping $Tr^0$ as in (4.1) is well defined as the functional on $W^{1,p}(\mathbb{R}_+, t^{\beta})$. Moreover, 

$$\lim_{t \rightarrow 0} t^{\frac{\beta-(p-1)}{p}} |u(t) - Tr^0 u| = 0, \text{ for any } u \in W^{1,p}(\mathbb{R}_+, t^{\beta}). \quad (4.3)$$

iii) When $\beta \leq -1$, then $W^{1,p}(\mathbb{R}_+, t^{\beta}) = W^{1,p}_0(\mathbb{R}_+, t^{\beta}) = B^0_{p,t^{\beta}}$. In particular the trace operator $Tr^0$, defined by (4.1), annihilates the whole space $W^{1,p}(\mathbb{R}_+, t^{\beta})$ and

$$W^{1,p}(\mathbb{R}_+, t^{\beta}) = W^{1,p}_0(\mathbb{R}_+, t^{\beta}) = A^0_{p,t^{\beta}} = B^0_{p,t^{\beta}} = C^0_{p,t^{\beta}} = Y^{1,p}_{t^{\beta}}. \quad (4.4)$$

iv) When $\beta \in (-1, p - 1)$, then the trace operator $Tr^0 : W^{1,p}(\mathbb{R}_+, t^{\beta}) \rightarrow \mathbb{R}$ is an epimorphism. Moreover, 

$$W^{1,p}(\mathbb{R}_+, t^{\beta}) \ni W^{1,p}_0(\mathbb{R}_+, t^{\beta}) = A^0_{p,t^{\beta}} = B^0_{p,t^{\beta}} = C^0_{p,t^{\beta}} = Y^{1,p}_{t^{\beta}}. \quad (4.5)$$

**Theorem 4.2** (characterization of $W^{1,p}_0(\mathbb{R}_+, t^{\beta})$, $\beta > p - 1$) Let $1 < p < \infty$. When $\beta > p - 1$, then we have:

$$W^{1,p}(\mathbb{R}_+, t^{\beta}) = W^{1,p}_0(\mathbb{R}_+, t^{\beta}) = A^\infty_{p,t^{\beta}} = B^\infty_{p,t^{\beta}} = C^\infty_{p,t^{\beta}} = Y^{1,p}_{t^{\beta}}. \quad (4.6)$$

In particular the trace operator $Tr^\infty$, defined by formula (4.2), annihilates the whole space $W^{1,p}(\mathbb{R}_+, t^{\beta})$, moreover

$$\lim_{t \rightarrow \infty} t^{\frac{\beta-(p-1)}{p}} |u(t)| = 0, \text{ for any } u \in W^{1,p}(\mathbb{R}_+, t^{\beta}). \quad (4.6)$$
Theorem 4.3 (characterization of $W_0^{1,p}(\mathbb{R}^+, t^\beta)$, $\beta = p - 1$) Let $1 < p < \infty$. Then we have:
i) $W_0^{1,p}(\mathbb{R}^+, t^{p-1}) = W^{1,p}(\mathbb{R}^+, t^{p-1})$.
ii) $Y_{t^{p-1}} \subset W^{1,p}(\mathbb{R}^+, t^{p-1})$.
iii) The trace operator $Tr^0$ as in (4.1) is not well defined on $W^{1,p}(\mathbb{R}^+, t^{p-1})$.
iv) We have $W^{1,p}(\mathbb{R}^+, t^{p-1}) = \mathcal{A}_{t^{p-1}}^\infty = B_{t^{p-1}}^\infty$.

We have the following remark about the classical formulation of Hardy inequality.

Remark 4.1 In the classical formulation of Hardy inequality (see Theorem 2.1), the sets of functions admitted to Hardy inequality with right hand sides finite are

$$\{ u \in W^{1,1}_{loc}(\mathbb{R}^+) : \lim_{t \to 0} u(t) = 0, \int_0^\infty |u'(t)|^p t^\beta dt < \infty, \} \text{ when } \beta < p - 1,$$

$$\{ u \in W^{1,1}_{loc}(\mathbb{R}^+) : \lim_{t \to \infty} u(t) = 0, \int_0^\infty |u'(t)|^p t^\beta dt < \infty, \} \text{ when } \beta > p - 1.$$

We know from Theorems 4.1 and 4.2 that

- $W_0^{1,p}(\mathbb{R}^+, t^\beta) = \{ u \in W^{1,p}(\mathbb{R}^+, t^\beta) : \lim_{t \to 0} u(t) = 0 \}$ when $\beta < p - 1$,
- $W_0^{1,p}(\mathbb{R}^+, t^\beta) = \{ u \in W^{1,p}(\mathbb{R}^+, t^\beta) : \lim_{t \to \infty} u(t) = 0 \}$ when $\beta > p - 1$.

Therefore the classical Hardy inequality, if restricted to $W^{1,p}(\mathbb{R}^+, t^\beta)$, holds on the space $W_0^{1,p}(\mathbb{R}^+, t^\beta)$. There is no larger subspace $Z \subseteq W^{1,p}(\mathbb{R}^+, t^\beta)$ admitted to the Hardy inequality with any finite constant. Indeed, for any $u \in W^{1,p}(\mathbb{R}^+, t^\beta) \setminus W_0^{1,p}(\mathbb{R}^+, t^\beta)$ we have either $\lim \inf_{t \to 0} |u(t)| = C > 0$ or $\lim \inf_{t \to \infty} |u(t)| = C > 0$ by Theorems 4.1 or 4.2. But then $(|u|/t)^{p-1}$ is not integrable either near zero or near infinity, so that left hand side in Hardy inequality is infinite. For recent improvement of Hardy inequality:

$$\int_0^\infty \max \left\{ \sup_{0 < s \leq t} \frac{|u(s)|^p}{sp}, \sup_{t \leq s \leq \infty} \frac{|u(s)|^p}{sp} \right\} dt \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty |u'(t)|^p dt,$$

with the admissibility condition $\lim \inf_{t \to 0} |u(t)| = 0$, we refer to the paper [10]. As the weight on the right hand side above is 1, i.e., according to Theorem 4.1 if we restrict such inequality to the subspace of $W^{1,p}(\mathbb{R}^+)$, it must hold precisely on $W_0^{1,p}(\mathbb{R}^+)$. Hardy inequalities:

$$\left( \int_a^b |f(x)|^q u(x) dx \right)^{1/q} \leq C \left( \int_a^b |f'(x)|^p v(x) dx \right)^{1/p},$$

where $-\infty \leq a < b \leq \infty$, with the analytical description of the admitted sets of functions given in the form of the vanishment conditions at the endpoints of the interval $(a, b)$ can also be found in [10].

In the proceeding section we will prove the above statements.
4.3 The proofs

Proof of Theorem 4.1.

i) and ii): The proof follows by steps.

Step 1. We show that $C^0_{p,t^\beta} = B^0_{p,t^\beta} = A^0_{p,t^\beta}$. We obviously have $C^0_{p,t^\beta} \subseteq B^0_{p,t^\beta} \subseteq A^0_{p,t^\beta}$. We will show that the converse inclusions hold. Let $u \in A^0_{p,t^\beta}$. Then there exists $t_n \to 0$ such that $u(t_n) \to 0$. As for $0 < t_n < t$

$$|u(t) - u(t_n)| \leq \int_{t_n}^{t} |u'(\tau)| \, d\tau \leq \int_{0}^{t} |u'(\tau)| \, d\tau,$$

therefore,

$$|u(t)| \leq \int_{0}^{t} |u'(\tau)| \, d\tau = \int_{0}^{t} |u'(\tau)|^{\beta/p} \tau^{\beta/p} \, d\tau \leq \left( \int_{0}^{t} |u'(\tau)|^{p\beta} \, d\tau \right)^{\frac{1}{p}} \left( \int_{0}^{t} \tau^{-(p-1)/p} \, d\tau \right)^{\frac{1}{p}} = a(t)t^{\frac{\beta-(p-1)}{p}},$$

where $a(t) := \left( \frac{\beta-(p-1)}{p} \right)^{\frac{1}{p}} \left( \int_{0}^{t} |u'(\tau)|^{p\beta} \, d\tau \right)^{\frac{1}{p}} \to 0$, as $t \to 0$. This implies

$$t^{\frac{\beta-(p-1)}{p}} |u(t)| \to 0.$$ 

Consequently, $A^0_{p,t^\beta} \subseteq C^0_{p,t^\beta}$, which ends the proof of Step 1.

Step 2. We prove ii).

Let us prove first that the operator $T r^0$ as in (4.1) is well defined on $W^{1,p}(\mathbb{R}_+, t^\beta)$.

For this, let $u \in W^{1,p}(\mathbb{R}_+, t^\beta)$ and let $t_n \to 0$, as $n \to \infty$. By the same arguments as in (4.7) and (4.8), we conclude that

$$|u(t) - u(t_n)| \lesssim t^{\frac{\beta-(p-1)}{p}} \left( \int_{0}^{t} |u'(\tau)|^{p\beta} \, d\tau \right)^{\frac{1}{p}}, \text{ for } t_n \leq t.$$

Therefore, the sequence $\{u(t_n)\}$ is bounded. By the Bolzano-Weierstrass theorem it possesses a convergent subsequence which we denote by the same expression. We can assume that $u(t_n) \to \bar{u} \in \mathbb{R}$ as $n \to \infty$. After passing to the limit in the above expression first with $n$, then with $t$, we get

$$|u(t) - \bar{u}| \lesssim t^{\frac{\beta-(p-1)}{p}} \left( \int_{0}^{t} |u'(\tau)|^{p\beta} \, d\tau \right)^{\frac{1}{p}}, \text{ where } \bar{u} = \lim_{t_n \to 0} u(t_n),$$

$$\lim_{t \to 0} |u(t) - \bar{u}| = 0 \text{ and } \lim_{t \to 0} \frac{t^{\frac{\beta-(p-1)}{p}}}{|u(t) - \bar{u}|} = 0.$$
In particular, \( u \mapsto Tr^0(u) =: u(0) \) is the well defined mapping on \( W^{1,p}(\mathbb{R}_+, t^\beta) \). Such mapping is linear, so its continuity is equivalent to its boundedness. In order to show the boundedness, we use Hölder inequality in the estimates below

\[
|u(0)| = \int_0^1 |u(0)| ds \leq \int_0^1 |u(s) - u(0)| ds + \int_0^1 |u(t)| t^\beta_1 t^{-\beta_2} dt \\
\leq \int_0^1 s^{-\beta_2/(p-1)} \left( \int_0^1 |u'(t)|^{p\beta} dt \right)^{\frac{1}{p}} ds + \left( \int_0^1 |u(t)|^{p\beta} dt \right) \left( \int_0^1 t^{-\beta_2/(p-1)} dt \right)^{1-\frac{1}{p}} \\
\leq \left( \int_0^1 |u'(t)|^{p\beta} dt \right)^{\frac{n}{p}} + \left( \int_0^1 |u(t)|^{p\beta} dt \right)^{\frac{n}{p}} \leq \|u\|_{W^{1,p}(\mathbb{R}_+, t^\beta)}.
\]

This implies the continuity of the operator \( Tr^0 \) and allows to conclude that \( Tr^0 \) is the well defined functional on \( W^{1,p}(\mathbb{R}_+, t^\beta) \). Property \( 4.3 \) follows directly from \( 4.9 \). This ends the proof of Step 2.

**STEP 3.** We show that \( C_{p,t^\beta}^0 = W^{1,p}_0(\mathbb{R}_+, t^\beta) \).

It follows from Step 2 that \( B_{p,t^\beta}^0 \) can also be defined as \( (Tr^0)^{-1}(0) \), so \( B_{p,t^\beta}^0 (= C_{p,t^\beta}^0 = A_{p,t^\beta}^0) \) is a closed subspace in \( W^{1,p}(\mathbb{R}_+, t^\beta) \). Let \( u \in B_{p,t^\beta}^0 \).

By similar arguments as in the proof of formula \( 3.3 \) in Lemma 3.1 defining \( u_n \ (n \in \mathbb{N}) \) by the formula \( 3.4 \), we will show that \( B_{p,t^\beta}^0 \subseteq W^{1,p}_c(\mathbb{R}_+, t^\beta) \) \( (\|u\|_{W^{1,p}(\mathbb{R}_+, t^\beta)} \text{ and allows to conclude that } I_{III_n} \to 0 \text{ as } n \to \infty \) as a result of the following estimates:

\[
III_n \sim \int_{\frac{1}{2n}}^{\frac{1}{n}} |u(t)|^{p\beta_p} dt = \int_{\frac{1}{2n}}^{\frac{1}{n}} \left( |u(t)|^{p\beta_p - 1} \right) t^{-1} dt \\
\leq \sup\{|u(t)|^{p\beta_p - 1} : t \leq \frac{1}{n} \} \ln 2 \xrightarrow{n \to \infty} 0.
\]

This shows that \( C^\infty_{p,t^\beta}(= B_{p,t^\beta}^0) \) Having shown that

\[
C^\infty_{p,t^\beta}(\mathbb{R}_+) \subseteq B_{p,t^\beta}^0 \subseteq W_{c}^{1,p}(\mathbb{R}_+, t^\beta) \ (\|\|_{W^{1,p}(\mathbb{R}_+, t^\beta)}) \text{ Lemma } 3.1 \beta W^{1,p}_0(\mathbb{R}_+, t^\beta),
\]

and remembering \( B_{p,t^\beta}^0 \) is the closed subset in \( W^{1,p}(\mathbb{R}_+, t^\beta) \), we deduce that \( C_{p,t^\beta}^0 = W^{1,p}_0(\mathbb{R}_+, t^\beta) \), when passing to the respective closures. This finishes the proof of Step 3.

**STEP 4.** We show that \( B_{p,t^\beta}^0 = Y_{p,t^\beta}^{1,p} \).

At first we observe that \( Y_{p,t^\beta}^{1,p} \subseteq A_{p,t^\beta}^0 \). Indeed, otherwise we would have \( |u(t)| \geq C > 0 \) for the sufficiently small \( t \), which implies the impossible inequality:

\[
\infty > \int_0^\epsilon |u(t)|^{p\beta_p} (1 + \frac{1}{t^p}) dt \geq \int_0^\epsilon |C|^p t^{\beta_p - p} dt = \infty.
\]
Therefore \( Y_{t^\beta}^{1,p} \subseteq W_0^{1,p}(\mathbb{R}_+, t^\beta) \) by Steps 1 and 3. Hardy inequality \([2,3]\) implies that \( W_0^{1,p}(\mathbb{R}_+, t^\beta) = \mathcal{B}_{p,t^\beta}^0 \subseteq Y_{t^\beta}^{1,p} \). This ends the proof of Step 4 and part i).

**iii):** When \( \beta \leq -1 \) and \( u \in L^p(\mathbb{R}_+, t^\beta) \), we necessarily have \( \lim \inf_{t \to 0} |u(t)| = 0 \), that is \( u \in A_{p,t^\beta}^0 \). Indeed, otherwise we would have \( |u(t)| \geq C > 0 \) for the sufficiently small \( t \). Thus implies

\[
\infty > \int_0^\varepsilon |u(t)|^p t^\beta dt \geq \int_0^\varepsilon |C|^p t^\beta dt = \infty \quad \text{with some } \varepsilon > 0. \tag{4.11}
\]

Thus, the condition \( u \in W_1^1(\mathbb{R}_+, t^\beta) \), together with the already proven part i), gives that \( u \in W_{0}^{1,p}(\mathbb{R}_+, t^\beta) = \mathcal{B}_{p,t^\beta}^0 \) and consequently \( Tr^0 u = 0 \). Therefore, we get (4.4).

**iv):** Let \( \phi \in Lip(\mathbb{R}_+) \) be defined as

\[
\phi(t) = \begin{cases} 
1 & \text{for } t \in (0, 1) \\
-t+2 & \text{for } t \in [1, 2) \\
0 & \text{for } t \geq 2.
\end{cases}
\]

Obviously \( \phi \in W_1^1(\mathbb{R}_+, t^\beta) \) and the extension operator

\[
\mathbb{R} \ni s \mapsto Ext^0(s) := s\phi(\cdot) \in W_1^1(\mathbb{R}_+, t^\beta)
\]

is the right inverse to the operator of trace \( Tr^0 \). In particular the elements of \( W_1^1(\mathbb{R}_+, t^\beta) \) might have the nonzero limits at zero, so \( W_{0}^{1,p}(\mathbb{R}_+, t^\beta) = \mathcal{B}^0_{p,t^\beta} \) is the essential subspace of \( W_1^1(\mathbb{R}_+, t^\beta) \). The identities \( (4.5) \) follow now from \( i) \).

This finishes the proof of the statement. \( \square \)

**Proof of Theorem 4.2.**

The equality of sets \( \mathcal{A}^\infty_{p,t^\beta} = \mathcal{B}^\infty_{p,t^\beta} = \mathcal{C}^\infty_{p,t^\beta} \) can be deduced by almost the same arguments as in the proof of subcase i) Step 1 in Theorem 4.1. The main difference is that in the initial step we choose \( T_n \to \infty \) instead of \( t_n \to 0 \) and we deal with integrals \( \int_{T_n}^t |u'(\tau)| d\tau \) and \( \int_{T_n}^\infty |u'(\tau)| d\tau \) \( (t > 0, t < T_n < \infty) \) instead of \( \int_{t_n}^t |u'(\tau)| d\tau \) and \( \int_0^t |u'(\tau)| d\tau \), respectively.

Next, we observe that when \( u \in L^p(\mathbb{R}_+, t^\beta) \), then we necessarily have

\[
\lim \inf_{t \to \infty} |u(t)| = 0,
\]

as in the other case we would have: \( \infty > \int_0^\varepsilon |u(t)|^p t^\beta dt = \infty \) where \( \varepsilon > 0 \), by similar arguments as in (4.11). Therefore,

\[
W_1^1(\mathbb{R}_+, t^\beta) = \mathcal{A}^\infty_{p,t^\beta} = \mathcal{B}^\infty_{p,t^\beta} = \mathcal{C}^\infty_{p,t^\beta}.
\]

The fact that \( W_1^1(\mathbb{R}_+, t^\beta) = W_{0}^{1,p}(\mathbb{R}_+, t^\beta) \) follows from the similar computations as in (3.5) (with the weight \( t^\beta \)) and (3.6). When estimating \( III_n \), we can use the fact that
any \( u \in W^{1,p}(\mathbb{R}^+, t^\beta) \) converges to zero at \( \infty \), so by the classical Hardy inequality (2.3), we have \( \int_{\mathbb{R}_+} \left( \frac{|u(t)|}{t} \right)^p t^\beta dt < \infty \). Consequently,

\[
III_n \sim \int_{\mathbb{R}_+} \left( \frac{|u(t)|}{t} \right)^p t^\beta dt \xrightarrow{n \to \infty} 0.
\]

Note also that obviously \( Y_{t^\beta}^{1,p} \subseteq W^{1,p}(\mathbb{R}^+, t^\beta) \), while Hardy inequality (2.3), together with the fact that \( W^{1,p}(\mathbb{R}^+, t^\beta) = \mathcal{B}_p^{t^\beta} \), implies that \( Y_{t^\beta}^{1,p} \supseteq W^{1,p}(\mathbb{R}^+, t^\beta) \).

(4.6): The property (4.6) follows from modification of (4.8) (now we choose \( T_n \to \infty \) and the integral over the integral \((t, \infty)\) instead of \((0, t)\)), which reads as:

\[
|u(t)| \leq t^{-\frac{\beta - (p-1)}{p}} \bar{a}(t)
\]

where

\[
\bar{a}(t) := \left( \frac{p-1}{\beta - (p-1)} \right)^{\frac{p-1}{p}} \left( \int_t^\infty |u'(\tau)|^{p} \tau^{\beta} d\tau \right)^{\frac{1}{p}} \to 0, \text{ as } t \to \infty,
\]

which finishes the proof of the statement. \( \square \)

**Proof of Theorem 4.3**

i): The proof follows by steps.

**Step 1.** We show that \( \mathcal{B}_p^{0, t^{-1}} \subseteq W_0^{1,p}(\mathbb{R}^+, t^{-1}) \).

For this, let \( u \in \mathcal{B}_p^{0, t^{-1}} \) and consider the sequence \( \{u_n\}_{n \in \mathbb{N}} \) as in (3.4). Similar estimates as in (3.5) (with the weight \( t^{-1} \)) and (4.10) show that \( u \in W_0^{1,p}(\mathbb{R}^+, t^{-1}) \).

**Step 2.** We show that the set \( L \subseteq W_0^{1,p}(\mathbb{R}^+, t^{-1}) \) where

\[
L := \{ u \in W^{1,p}(\mathbb{R}^+, t^{-1}) : u \text{ is bounded near zero} \} \supseteq \mathcal{B}_p^{0, t^{-1}}.
\]

For this, let us we consider \( u \in L \). By the same computations as in Step 1, we deduce that the sequence \( \{u_n\} \) as in (3.4), is bounded in \( W^{1,p}(\mathbb{R}^+, t^{-1}) \). This is because by (4.10), we have

\[
III_n \lesssim \| u \|_{L^\infty(0, 1)} \ln 2.
\]

The argument as in (3.5) with the weight \( t^{-1} \) gives that \( u_n \to u \) strongly in \( L^p(\mathbb{R}^+, t^{-1}) \). We will prove that

\[
\lim_{n \to \infty} u_n' = u' \quad \text{(weakly in } L^p(\mathbb{R}^+, t^{-1}) \text{).}
\]

To verify (4.12) at first we note that the space \( L^{p/(p-1)}(\mathbb{R}^+, t^{-1}) \) is dual to \( L^p(\mathbb{R}^+, t^{-1}) \), where the duality is expressed by the formula

\[
L^p(\mathbb{R}^+, t^{-1}) 	imes L^{p/(p-1)}(\mathbb{R}^+, t^{-1}) \ni (w, v) \mapsto \int_{\mathbb{R}_+} w(t)v(t)dt = \int_{\mathbb{R}_+} (w \cdot t^{-\frac{1}{p}}) \cdot (v \cdot t^{\frac{1}{p}-1})dt.
\]
Therefore it suffices to test the weak convergence \([4.12]\) on any dense subset in \(L^{p/(p-1)}([0, t^{-1}])\), such as the set of smooth compactly supported functions. Hence, it suffices to verify that for any \(v \in C_0^\infty([0, t])\) supported in \([a, b]\), where \(0 < a < b < \infty\), we have

\[
\int_{[0, t]} (u'_n(t) - u'(t))v(t)dt \xrightarrow{n \to \infty} 0.
\]

This follows from the following simple argument:

\[
\int_{[0, t]} (u'_n(t) - u'(t))v(t)dt = \int_{[a, b]} (u'_n(t) - u'(t))v(t)dt
\]

\[
= \int_{[a, b]} u'(t)(\phi(t) - 1)v(t)dt + \int_{[a, b]} u(t)\phi(t)v(t)dt =: I_n + I_{II}.
\]

As \(\phi_n \equiv 1\) on sets \([\frac{1}{n}, n]\), which exhaust \([a, b]\) for the sufficiently large \(n\), therefore for such \(n\) we have \(I_n = I_{II} = 0\).

Mazur’s theorem (see, e.g. \([1]\), p. 6) says that there exists the sequence \(\{v_n\}\) of convex combinations of the \(u_n\)’s such that \(v_n \to u'\) in \(L^p([0, t^{-1}])\). Simple verification shows that also \(v_n \to u\) strongly in \(L^p([0, t^{-1}])\). In particular \(C_0^\infty([0, t]) \ni v_n \to u\) in \(W^{1, p}([0, t^{-1}])\), as \(n \to \infty\) and consequently, \(u \in W^{1, p}([0, t^{-1}]).\)

**Step 3.** We show that \(W^{1, p}([0, t^{-1}]) = W^{1, p}_0([0, t^{-1}])\), which will finish the proof of part i).

For this, we note that

\[
W^{1, p}_0([0, t^{-1}]) \overset{\text{step 2}}{\supset} L \supset L^\infty([0, t]) \cap W^{1, p}([0, t^{-1}]).
\]

Moreover, \(L^\infty([0, t]) \cap W^{1, p}([0, t^{-1}])\) is a dense subset in \(W^{1, p}([0, t^{-1}]).\) This is because the truncation of any \(u \in W^{1, p}([0, t^{-1}]):

\[
T_\lambda(u)(t) := \begin{cases} 
  u(x) & \text{if } |u(x)| \leq \lambda \\
  -\lambda & \text{if } u(x) \leq -\lambda \\
  \lambda & \text{if } u(x) \geq \lambda
\end{cases}
\]

converge to \(u\), as \(\lambda \to \infty\), in \(W^{1, p}([0, t^{-1}])\) because

\[
\int_{[0, t]} |u - T_\lambda(u)|^{p-1}dt = \int_{|u| > \lambda} |u|^{p-1}dt + \int_{|u| < \lambda} |u + \lambda|^{p-1}dt
\]

\[
= \int_{|u| > \lambda} ||u| - \lambda|^{p-1}dt \leq \int_{|u| > \lambda} |u|^{p-1}dt \xrightarrow{\lambda \to \infty} 0
\]

and similarly

\[
\int_{[0, t]} |u' - (T_\lambda(u))'|^{p-1}dt = \int_{|u| > \lambda} |u'|^{p-1}dt \xrightarrow{\lambda \to \infty} 0.
\]


Therefore, $W^{1,p}(\mathbb{R}^+, t^{p-1}) \subseteq W^{1,p}_0(\mathbb{R}^+, t^{p-1})$, which ends the proof of Step 3.

ii): As $u(t) = \ln(-\ln t) \chi_{(0, \frac{1}{2})} \notin Y^{1,p}_{t^{p-1}}$, therefore $Y^{1,p}_{t^{p-1}} \subseteq W^{1,p}(\mathbb{R}^+, t^{p-1})$.

iii): The function $u(t) = \ln(-\ln t) \chi_{(0, \frac{1}{2})}$ belongs to $W^{1,p}(\mathbb{R}^+, t^{p-1})$ but converges to $\infty$ at zero. Therefore, the trace operator as in (4.1) cannot be well defined on the space $W^{1,p}(\mathbb{R}^+, t^{p-1})$.

iv): The fact that $W^{1,p}(\mathbb{R}^+, t^{p-1}) \subseteq \mathcal{A}_{p, t^{p-1}}^\infty$ follows from integrability of $|u|^{p^{p-1}}$ near infinity and the argument as in (4.11). We are left with the proof that

$$
\mathcal{A}_{p, t^{p-1}}^\infty = \mathcal{B}_{p, t^{p-1}}^\infty.
$$

Suppose on the contrary, that the above is not true, that is there exists the function $u \in W^{1,p}(\mathbb{R}^+, t^{p-1})$, which is oscillating near infinity, that is $\limsup_{t \to \infty} |u(t)| = c > 0 = \liminf_{t \to \infty} |u(t)|$. Using the Darboux property and changing the function $u$ to $Au$ with the suitable chosen constant $A$, we could construct the function $v \in W^{1,p}(\mathbb{R}^+, t^{p-1})$, and sequences $t_n \not\to \infty$, $T_n \not\to \infty$ such that

$$
0 < t_{n-1} < T_n < t_n \text{ and } v(t_n) = \frac{1}{2}, \quad v(T_n) = 1, \quad \frac{1}{2} < v(t) < 1 \text{ on } (t_{n-1}, T_n).
$$

This would imply:

$$
\begin{align*}
\infty &> \int_0^\infty |v(t)|^{p^{p-1}} dt \geq \sum_{n=1}^\infty \int_{t_{n-1}}^{T_n} \left(\frac{1}{2}\right)^p t^{p-1} dt = \frac{1}{p^2} \sum_{n=1}^\infty (T_n^p - t_n^p) \\
&= \frac{1}{p^2} \sum_{n=1}^\infty T_n^p \left(1 - \left(\frac{t_{n-1}}{T_n}\right)^p\right) =: \frac{1}{p^2} \sum_{n=1}^\infty T_n^p (1 - (x_n)^p).
\end{align*}
$$

At the same time

$$
\frac{1}{2} = |v(T_n) - v(t_n)| \leq \int_{t_{n-1}}^{T_n} |v'(t)| dt = \int_{t_{n-1}}^{T_n} |v'(t)| t^{\frac{p-1}{p}} t^{-\frac{p-1}{p}} dt
\leq \left(\int_{t_{n-1}}^{T_n} |v'(t)|^{p^{p-1}} dt\right)^{\frac{1}{p}} \left(\int_{t_{n-1}}^{T_n} t^{-1} dt\right)^{1-\frac{1}{p}} =: a_n b_n.
$$

As $a_n \to 0$ when $n \to \infty$, therefore $b_n \to \infty$ when $n \to \infty$. Consequently, $b_n = \ln\left(\frac{T_n}{t_{n-1}}\right) \to \infty$ as $n \to \infty$ and so $\{x_n\}$ in (4.13) satisfies $x_n = \frac{t_{n-1}}{T_n} \to 0$ as $n \to \infty$, in particular $(1 - x_n^p) > \frac{1}{2}$ for the sufficiently large $n$.

Further estimates in (4.13) would lead to the impossible inequality, with some number $N \in \mathbb{N}$:

$$
\infty > \int_0^\infty |v(t)|^{p^{p-1}} dt \geq \sum_{n=N}^\infty T_n^p = \infty.
$$

We arrive at contradiction, which confirms that $\mathcal{A}_{p, t^{p-1}}^\infty = \mathcal{B}_{p, t^{p-1}}^\infty$ and ends the proof of the statement.
4.4 Applications

We propose three example applications of our results.

Remark 4.2 (application to BVP) Our results can be applied to the analysis of Boundary Value Problems in O.D.E’s. For example, when dealing with the concrete ODE with the solution in $W^{1,p}(\mathbb{R}_+, t^\beta)$, where $p > 1$, one can formulate the boundary decay condition as:

$$
\lim_{t \to 0} u(t) = 0, \quad \text{equivalently} \quad \lim_{t \to 0} t^{(\beta-(p-1))/p} u(t), \quad \text{when } \beta < p - 1,
$$

$$
\lim_{t \to \infty} u(t) = 0, \quad \text{equivalently} \quad \lim_{t \to \infty} t^{(\beta-(p-1))/p} u(t), \quad \text{when } \beta > p - 1,
$$

for $u \in W^{1,p}(\mathbb{R}_+, t^\beta)$. In particular, all the above limits are equivalent to the limits of $t^s u(t)$, for any $s \in [0, (\beta - (p - 1))/p]$. We have shown in Theorems 4.1 and 4.2 that such limits are well defined in the respective Sobolev spaces. Such limits can be also prescribed to the radial solutions to the Boundary Value Problems in PDE’s defined on $\mathbb{R}^n$, where we expect radial solutions $v(x) = u(\|x\|)$ in the weighted Sobolev Spaces $W^{1,p}(\mathbb{R}^n, \|x\|_n^{n-1})$, where $\beta = \alpha + n - 1$.

Remark 4.3 (application to Hardy-type inequality) In [3] K. Pietruska-Pałuba and second author have studied the inequalities:

$$
\int_{\mathbb{R}_+} M(t^\alpha |u(t)|) t^\beta dt \leq C_1 \int_{\mathbb{R}_+} M(|u(t)|) t^\beta dt + C_2 \int_{\mathbb{R}_+} M(|u'(t)|) t^\beta dt, \quad (4.14)
$$

holding on supersets $W$ of $C_0^\infty(\mathbb{R}_+)$, where $M(\cdot)$ is the given Orlicz function, as the special cases of larger class of inequalities from [7], which deal with general weights $\omega, \rho$ in place of $t^\alpha, t^\beta$, respectively. Let us focus on (4.14) restricted to $M(s) = s^p, p > 1$:

$$
\int_{\mathbb{R}_+} |t^\alpha |u(t)||^p t^\beta dt \leq C_1 \int_{\mathbb{R}_+} |u(t)|^p t^\beta dt + C_2 \int_{\mathbb{R}_+} |u'(t)|^p t^\beta dt. \quad (4.15)
$$

The following statement is the special case of Theorem 4.2 from [3], where we put $M(s) = s^p$.

Theorem 4.4 Suppose that $\alpha \in (-1, 0)$. Then we have:

i) Inequality (4.15) with $C_1 = 0$ cannot hold for all $u \in C_0^\infty(\mathbb{R}_+)$ with any finite constant independent on $u$.

ii) There exist positive constants $C_1, C_2$ such that inequality (4.15) holds for all $u \in W(\alpha, \beta)$, where

a) $W = W^{1,p}(\mathbb{R}_+, t^\beta)$ when $\beta > |\alpha|p - 1$,

b) $W = \{ u \in W^{1,p}(\mathbb{R}_+, t^\beta) : \liminf_{t \to 0^+} |u(t)| t^s = 0, \quad s := \frac{\beta+1-|\alpha|p}{p} \}$ when $\beta < |\alpha|p - 1$.

15
iii) The sets $W(\alpha, \beta)$ from ii) are maximal subsets of $W^{1,p}(\mathbb{R}^+, t^\beta)$ for which (4.15) holds.

We can now show that when $\beta < |\alpha|p - 1$ then $W = W^{1,p}_0(\mathbb{R}^+, t^\beta)$ and in particular it is independent on $\alpha$. Indeed, for any $u \in W = W(\alpha, \beta)$, we necessarily have $\liminf_{t \to 0} |u(t)| = 0$, because $s < 0$. Therefore $u \in A_{p,t^\beta}^0$. As $\beta < p - 1$, therefore by Theorem 4.1 we get $u \in W^{1,p}_0(\mathbb{R}^+, t^\beta)$. This gives $W \subseteq W^{1,p}_0(\mathbb{R}^+, t^\beta)$. On the other hand, when $u \in W^{1,p}_0(\mathbb{R}^+, t^\beta)$ then by Theorem 4.1 $u \in C_{p,t^\beta}$ and consequently

$$
\lim_{t \to 0} |u(t)| t^{\frac{\beta + 1 - |\alpha|p}{p}} = 0 \implies \lim_{t \to 0} |u(t)| t^{\frac{\beta + 1 - |\alpha|p}{p}} = \lim_{t \to 0} \left(|u(t)| t^{\frac{\beta + 1 - |\alpha|p}{p}}\right) t^{1-|\alpha|} = 0.
$$

Hence $u \in W$. We have shown that $W = W^{1,p}_0(\mathbb{R}^+, t^\beta)$.

Remark 4.4 (application to complex interpolation theory) It is known by Stein - Weiss Theorem (see i.e. [2], Section 5.5.3, p. 120) that having two weight functions $\omega_0, \omega_1$ defined on some domain $U \subseteq \mathbb{R}^n$, one has

$$
[L^{p\theta}(U, \omega_0, \mathbb{C}), L^{p_1}(U, \omega_1, \mathbb{C})]_\theta = L^{p\theta}(U, \omega_\theta, \mathbb{C}), \text{ where } \theta \in (0, 1),
$$

where $[X,Y]_\theta$ denotes the complex interpolation pair,

$$
\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \omega_\theta^{\frac{\theta}{p_0}} = \omega_0^{\frac{1-\theta}{p_0}} \omega_1^{\frac{\theta}{p_1}} \quad (4.16)
$$

and $L^{p}(U, \omega_0, \mathbb{C}) = L^{p}(U, \omega_0) \times iL^{p}(U, \omega_0)$ denotes complex valued functions $f_1 + if_2$, with coordinate functions $f_1, f_2$ from $L^{p}(U, \omega_0)$.

Let us use the analogous notation for complex-valued Sobolev functions $W^{s,p}(U, \omega, \mathbb{C}) = W^{s,p}(U, \omega) \times iW^{s,p}(U, \omega)$. As proven by Jörgen Löfström [14], under some special assumptions on weight functions, one has

$$
[W^{s_0,p_0}(\mathbb{R}^n, \omega_0, \mathbb{C}), W^{s_1,p_1}(\mathbb{R}^n, \omega_1, \mathbb{C})]_\theta = W^{s_\theta,p_\theta}(\mathbb{R}^n, \omega_\theta, \mathbb{C}),
$$

where $\theta \in (0, 1)$ and $s_\theta = (1 - \theta)s_0 + \theta s_1$, $p_\theta$ and $\omega_\theta$ are given by (4.16) and weighted function Sobolev spaces are defined with the help of Fourier transform. The authors of [3] have raised interesting questions, when one has

$$
[W^{1,p}(U, \omega_0, \mathbb{C}), W^{1,p}(U, \omega_1, \mathbb{C})]_\theta = W^{1,p}(U, \omega_\theta, \mathbb{C}),
$$

or

$$
[W^{1,p}_0(U, \omega_0, \mathbb{C}), W^{1,p}_0(U, \omega_1, \mathbb{C})]_\theta = W^{1,p}_0(U, \omega_\theta, \mathbb{C}),
$$

where $\theta \in (0, 1)$, with the same $\omega_\theta$ as in (4.16) and $p \in [1, \infty)$?
It has been shown in [3] in Theorem 1.16 that under the assumption (1.6) on weights, which are satisfied for positive continuous functions, and local Lipschitzism on \( r_\omega \) one has
\[
[W^{1,p}(U, \omega_0, \mathbb{C}), W^{1,p}(U, \omega_1, \mathbb{C})]_\theta \subseteq W^{1,p}(U, \omega_\theta, \mathbb{C}), \tag{4.17}
\]
\[
\mathcal{W}_0^p(U, \theta, r_\omega) \subseteq [W^{1,p}_0(U, \omega_0, \mathbb{C}), W^{1,p}_0(U, \omega_1, \mathbb{C})]_\theta, \tag{4.18}
\]
\[
\mathcal{W}^p(U, \theta, r_\omega) := \{ \phi : U \to \mathbb{C} \mid \phi \in W^{1,p}(U, w_\theta, \mathbb{C}), \phi \cdot |\nabla \log(r_\omega)| \in L^p(U, w_\theta, \mathbb{C}) \},
\]
\[
\|\phi\|_{\mathcal{W}^p(U, \theta, r_\omega)} := \left( \|\phi\|^p_{W^{1,p}(U, w_\theta, \mathbb{C})} + \int_U |\phi(x)|^p |\nabla \log(r_\omega(x))|^p w_\theta(x) dx \right)^{\frac{1}{p}}, \quad r_\omega := \frac{\omega_0}{\omega_1},
\]
\[
\mathcal{W}_0^p(U, \theta, r_\omega) \text{ is the completion of Lipschitz compactly supported complex-valued functions in } \mathcal{W}^p(U, \theta, r_\omega). \text{ As results from (4.17) and (4.18), we have:}
\]
\[
\mathcal{W}_0^p(U, \theta, r_\omega) \subseteq [W^{1,p}_0(U, \omega_0, \mathbb{C}), W^{1,p}_0(U, \omega_1, \mathbb{C})] \supseteq [W^{1,p}(U, \omega_0, \mathbb{C}), W^{1,p}(U, \omega_1, \mathbb{C})]_\theta \subseteq W^{1,p}(U, \omega_\theta, \mathbb{C}),
\]
where the inclusion (a) follows from the very definition of interpolation pair, see Remark 1.17 in [3]. Therefore if one has
\[
\mathcal{W}_0^p(U, \theta, r_\omega) = W^{1,p}(U, \omega_\theta, \mathbb{C}),
\]
then
\[
[W^{1,p}_0(U, \omega_0, \mathbb{C}), W^{1,p}_0(U, \omega_1, \mathbb{C})]_\theta = [W^{1,p}(U, \omega_0, \mathbb{C}), W^{1,p}(U, \omega_1, \mathbb{C})]_\theta = W^{1,p}(U, \omega_\theta, \mathbb{C})
\]

Our main results cannot be concluded, as a special case, from Theorem 4 in [14] because our weights \( t^\beta \) do not satisfy the "polynomially regularity" condition, an essential and important assumption for the weight function, considered among others in Theorem 4 of [14]. Even if our functions could be extended to those on the whole \( \mathbb{R} \), it is not possible to provide further analysis based on paper [14] because the polynomial components of the weight constrain the fulfillment of condition (3.4) from the definition of polynomially regular weight function in [14] on page 197, which requires in our case the finiteness of \( \sup_{t \in \mathbb{R}^+} \frac{\beta}{t} \). However, in our case this supremum is infinite.

Let us consider \( U = \mathbb{R}_+, w_0 = t^\beta_0, \omega_1 = t^\beta_1 \). Then we have
\[
r_\omega(t) = t^{\beta_0 - \beta_1}, \quad \omega_\theta = t^{\beta_0(1-\theta) + \beta_1} =: t^{\beta_\theta}, \quad |\nabla \log(r_\omega)| \sim \frac{1}{t}
\]
and
\[
\mathcal{W}^p(\mathbb{R}_+, \theta, t^{\beta_0 - \beta_1}) = \{ \phi : \mathbb{R} \to \mathbb{C} \mid \phi \sim (\phi_1, \phi_2), \phi_i \in W^{1,p}(\mathbb{R}_+, t^{\beta_i}) \text{ for } i = 1, 2, \phi/i \in L^p(\mathbb{R}_+, t^{\beta_\theta}) \} = Y^{1,p}_{t^{\beta_\theta}} \times iY^{1,p}_{t^{\beta_\theta}}.
\]
It follows from Theorem 4.1 that
\[ W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}, C) = W^{1,p}(\mathbb{R}^+, t^{\beta_0}, C), \text{ equivalently } W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}) = W^{1,p}(\mathbb{R}^+, t^{\beta_0}), \]
\[ \iff \beta_0 \in (-\infty, -1] \cup [p - 1, +\infty), \]
while from (3.2) in Lemma 3.1 we know that
\[ W_{0}^{p}(\mathbb{R}^+, \theta, t^{\beta_0 - \beta_1}) = W_{0}^{p}(\mathbb{R}^+, \theta, t^{\beta_0}) \]
\[ \iff Y_{t^{\beta_0 \theta}}^{1,p} = Y_{t^{\beta_0 \theta}}^{1,p}, \text{ which holds for any } \beta_0 \in \mathbb{R}, p > 1. \]
Moreover, when only \( p > 1, \beta_0 \neq p - 1 \), then by Theorem 4.1
\[ W_{0}^{p}(\mathbb{R}^+, \theta, t^{\beta_0 - \beta_1}) = W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}, C), \text{ equivalently } Y_{t^{\beta_0 \theta}}^{1,p} = W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}). \]
Altogether leads to the following interpolation result, which seems to us new.

**Theorem 4.5** When \( \theta \in (0, 1) \) \( \beta_0, \beta_1 \in \mathbb{R} \) are such that for \( \beta_0 = (1 - \theta)\beta_0 + \theta \beta_1 \in (-\infty, -1] \cup (p - 1, +\infty) \), then
\[ [W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}, C), W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_1}, C)]_{\theta} = [W^{1,p}(\mathbb{R}^+, t^{\beta_0}, C), W^{1,p}(\mathbb{R}^+, t^{\beta_1}, C)]_{\theta} \]
\[ = W^{1,p}(\mathbb{R}^+, t^{\beta_0}, C) = Y_{t^{\beta_0 \theta}}^{1,p} \times iY_{t^{\beta_0 \theta}}^{1,p}. \]
The related statement is Theorem 1.20 in [3], which gives the condition for
\[ W_{0}^{p}(U, \theta, r_w) \subseteq [W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}, C), W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_1}, C)]_{\theta} \subseteq W_{0}^{1,p}(\mathbb{R}^+, t^{\beta_0}, C) \]
for all \( \theta \in (0, 1) \), in the abstract approach. For that inclusions one has to verify the following assumption:

"whenever \( \phi \) is an element of \( W_{0}^{1,p}(U, \omega_0) \cap W_{0}^{1,p}(U, \omega_1) \) there exists a sequence \( \{\phi_n\} \) of functions in \( \text{Lip}_c(U) \) which converges to \( \phi \) in \( W^{1,p}(U, \omega_0) \) or in \( W^{1,p}(U, \omega_1) \) and is bounded in the other space."

It shows that the knowledge about density results and the techniques of approximation can be the crucial tool for the analysis of interpolation spaces in the complex interpolation theory.

**References**

[1] R.A. ADAMS, J.J.F. FOURNIER, *Sobolev spaces*. Second edition. Pure and Applied Mathematics (Amsterdam) 140 Elsevier/Academic Press, Amsterdam, 2003.

[2] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces, an Introduction*. Grundlehren der Mathematischen Wissenschaften vol. 223, Springer-Verlag, Berlin-New York, 1976.
[3] M. CWIKEL, A. EINAV, Interpolation of weighted Sobolev spaces. J. Funct. Anal. 277(7) (2019), 2381–2441.

[4] I. EKELAND, R. TÉMAM, Convex analysis and variational problems. North-Holland, Amsterdam, 1976.

[5] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, Inequalities. University Press, Cambridge, 1952.

[6] A. KALAMAJSKA, K. PIETRUSKA-PALUBA, On a variant of the Hardy inequality between weighted Orlicz spaces. Studia Math. 193(1) (2009), 1–28.

[7] A. KALAMAJSKA, K. PIETRUSKA-PALUBA, Weighted Hardy-type inequalities in Orlicz spaces. Math. Inequal. Appl. 15(4) (2012), 745–766.

[8] A. KALAMAJSKA, K. PIETRUSKA-PALUBA, New Orlicz variants of Hardy type inequalities with power, power-logarithmic, and power-exponential weights. Cent. Eur. J. Math. 10(6) (2012), 2033–2050.

[9] A. KUFNER, Weighted Sobolev Spaces. Translated from the Czech. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. 116 pp..

[10] A. Kufner, The Hardy inequality with boundary or intermediate conditions. Eurasian Math. J. 8(2) (2017), 105–109.

[11] A. KUFNER, O. JOHN, S. FUČÍK, Function spaces. Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis. Noordhoff International Publishing, Leyden; Academia, Prague, 1977.

[12] A. KUFNER, L. MALIGRANDA, L.-E. PERSSON, The Hardy inequality. About its history and some related results. Vydavatelský Servis, Plzeň, 2007.

[13] A. KUFNER, B. OPIC, How to define reasonably weighted sobolev spaces. Commentationes Mathematicae Universitatis Carolinae 25(3) (1984), 537-554.

[14] J. LÖFSTRÖM, Interpolation of weighted spaces of differentiable functions on \( \mathbb{R}^d \). Ann. Mat. Pura Appl. 132 (1982), 189–214.

[15] V. MAZ'JA, Sobolev spaces. Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985.

[16] R.L. FRANK, A. LAPTEV, T. WEIDL, An improved one-dimensional Hardy inequality, https://arxiv.org/pdf/2204.00877.pdf