On decompositions of trigonometric polynomials

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Abstract

Let $\mathbb{R}_t[\theta]$ be the ring generated over $\mathbb{R}$ by $\cos \theta$ and $\sin \theta$, and $\mathbb{R}_t(\theta)$ be its quotient field. In this paper we study the ways in which an element $p$ of $\mathbb{R}_t[\theta]$ can be decomposed into a composition of functions of the form $p = R(q)$, where $R \in \mathbb{R}(x)$ and $q \in \mathbb{R}_t(\theta)$. In particular, we describe all possible solutions of the functional equation $R_1(q_1) = R_2(q_2)$, where $R_1, R_2 \in \mathbb{R}[x]$ and $q_1, q_2 \in \mathbb{R}_t[\theta]$.

1 Introduction

Let $L$ be a rational function with complex coefficients. Any representation of $L$ in the form $L = P \circ W$, where $P$ and $W$ are rational functions of degree greater than one and the symbol $\circ$ denotes the superposition of functions, that is $P \circ W = P(W)$, is called a decomposition of $L$. Two decompositions $L = P_1 \circ W_1$ and $L = P_2 \circ W_2$ of the same function $L$ are called equivalent if there exists a rational function $\mu$ of degree one such that

$$ P_2 = P_1 \circ \mu, \quad W_2 = \mu^{-1} \circ W_1. $$

One of the main problems of the decomposition theory of rational functions is to describe possible solutions of the equation

$$ L = P_1 \circ W_1 = P_2 \circ W_2 \quad (1) $$

in the case where decompositions $L = P_1 \circ W_1$ and $L = P_2 \circ W_2$ are not equivalent. In the case where $L$ is a polynomial, a description of solutions of (1) was given by Ritt in his paper [18] which was a starting point of the decomposition theory of rational functions. Roughly speaking, in this case solutions of (1) up to equivalency reduce either to the solutions

$$ z^n \circ z^r R(z^n) = z^r R^n(z) \circ z^n, $$

where $R$ is a polynomial, and $r \geq 0$, $n \geq 1$, or to the solutions

$$ T_n \circ T_m = T_m \circ T_n, \quad (2) $$

where $T_n, T_m$ are the Chebyshev polynomial, which can be defined by the equality $T_n(\cos \theta) = \cos n\theta$. 

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A description of solutions of (1) in the case where $L$ is a Laurent polynomial, or more generally any rational function with at most two poles, was obtained in the papers [12], [22]. For arbitrary rational functions, a description of solutions of (1) is known only in particular cases. Namely, in the classical papers of Julia, Fatou, and Ritt [6], [8], [19] was given a description of commuting rational functions (that is of solutions of (1) with $P_1 = W_2$ and $P_2 = W_1$), and recently a description of semi-conjugate rational functions (that is of solutions of (1) with $P_1 = W_2$) was given in [10].

The decomposition theory of polynomials turned out to be closely related with the following so called “polynomial moment problem”. Let $P, Q$ be complex polynomial; what are conditions implying that the equalities

$$\int_0^1 P^i dQ = 0, \quad i \geq 0,$$

hold? Indeed, it is easy to see using the change $z \to W(z)$ that (3) is satisfied whenever there exist polynomials $\tilde{P}, \tilde{Q}$, and $W$ such that

$$P = \tilde{P} \circ W, \quad Q = \tilde{Q} \circ W, \quad W(0) = W(1). \quad (4)$$

Furthermore, it was shown in [11] that if polynomials $P, Q$ satisfy (3), then there exist polynomials $Q_j$ such that $Q = \sum_j Q_j$ and the equalities

$$P = \tilde{P}_j \circ W_j, \quad Q_j = \tilde{Q}_j \circ W_j, \quad W_j(0) = W_j(1) \quad (5)$$

hold for some polynomials $\tilde{P}_j, \tilde{Q}_j, W_j$. Thus, the most interesting solutions of the polynomial moment problem arise from polynomials having “multiple” decompositions

$$P = \tilde{P}_1 \circ W_1 = \tilde{P}_2 \circ W_2 = \cdots = \tilde{P}_s \circ W_s. \quad (6)$$

Polynomial solutions of (6) were described in the paper [13], where the corresponding generalization of the result of Ritt about solutions of (1) was obtained. Notice that in the study of the polynomial moment problem one can restrict oneself by the case where considered polynomials have real coefficients. However, the results of [11], [13] imply that in the real case a description of solutions of (3) is only a bit easier than in the complex one.

The polynomial moment problem naturally appears in the study of the center problem for the Abel differential equation with polynomial coefficients (see e. g. the recent papers [3], [2] and the bibliography therein) which is believed to be a simplified analog of the center problem for the Abel differential equation whose coefficients are trigonometric polynomials over $\mathbb{R}$. In its turn the last problem is closely related to the classical center-focus problem of Poincaré ([4]).

In the same way as the center problem for the Abel equation with polynomial coefficients leads to the polynomial moment problem, the center problem for the Abel equation with trigonometric coefficients leads to the following “trigonometric moment problem”. Let

$$p = p(\cos \theta, \sin \theta), \quad q = q(\cos \theta, \sin \theta)$$
be trigonometric polynomials over $\mathbb{R}$, that is elements of the ring $\mathbb{R}_t[\theta]$ generated over $\mathbb{R}$ by the functions $\cos \theta$, $\sin \theta$. What are conditions implying that the equalities

$$\int_0^{2\pi} p^i dq = 0, \quad i \geq 0, \quad (7)$$

hold? Like to the case of the polynomial moment problem one can consider a complexified version of this problem (see [14], [15], [1]). However, examples constructed in [15], [1] suggest that in the trigonometric case the complex version of the problem may be much more complicated than the real one.

Again, a natural sufficient condition for (7) to be satisfied is related with compositional properties of $p$ and $q$. Namely, it is easy to see that if there exist $P, Q \in \mathbb{R}[x]$ and $w \in \mathbb{R}_t[\theta]$ such that

$$p = P \circ w, \quad q = Q \circ w, \quad (8)$$

then (7) hold. Furthermore, if for given $p$ there exist several such $q$ (with different $w$), then (7) obviously holds for their sum. Thus, the trigonometric moment problem leads to the problem of description of solutions of the equation

$$p = P_1 \circ w_1 = P_2 \circ w_2, \quad (9)$$

where $p, w_1, w_2 \in \mathbb{R}_t[\theta]$ and $P_1, P_2 \in \mathbb{R}[x]$, and the main goal of this paper is to provide such a description. Notice that, besides of its relation with the trigonometric moment problem, functional equation (9) or its shortened version

$$P_1 \circ w_1 = P_2 \circ w_2, \quad (10)$$

where as above $w_1, w_2 \in \mathbb{R}_t[\theta]$ and $P_1, P_2 \in \mathbb{R}[x]$, seems to be interesting by itself. In particular, it contains among its solutions the most known trigonometric identity $\sin^2 \theta = 1 - \cos^2 \theta$.

Observe that the problem of description of solutions of (10) absorbs the problem of description of polynomial solutions of (1) over $\mathbb{R}$ since for any polynomial solution of (1) and any $w \in \mathbb{R}_t[\theta]$ we obtain a solution of (10) setting

$$w_1 = W_1 \circ w, \quad w_2 = W_2 \circ w.$$ 

Further, observe that if $P_1, P_2, w_1, w_2$ is a solution of (10), then for any $k \in \mathbb{N}$ and $b \in \mathbb{R}$ we obtain another solution $P_1, P_2, \tilde{w}_1, \tilde{w}_2$ setting

$$\tilde{w}_1(\theta) = w_1(k\theta + b), \quad \tilde{w}_2(\theta) = w_2(k\theta + b).$$

Finally, if $P_1, P_2, w_1, w_2$ is a solution of (10), then for any $U \in \mathbb{R}[t]$ we obtain another solution $\bar{P}_1, \bar{P}_2, w_1, w_2$ setting

$$\bar{P}_1 = U \circ P_1, \quad \bar{P}_2 = U \circ P_2.$$ 

Let $p$ be an element of $\mathbb{R}_t[\theta]$, and $p = P_1 \circ w_1$ and $p = \bar{P}_1 \circ \tilde{w}_1$ be two decompositions of $p$, where $w_1, \tilde{w}_1 \in \mathbb{R}_t[\theta]$ and $P_1, \bar{P}_1 \in \mathbb{R}[x]$. We will call these
decompositions equivalent, and will use the notation $P_1 \circ w_1 \sim \tilde{P}_1 \circ \tilde{w}_1$, if there exists $\mu \in \mathbb{R}[x]$ of degree one such that
\[
\tilde{P}_1 = P_1 \circ \mu, \quad \tilde{w}_1 = \mu^{-1} \circ w_1.
\]
We also will use the symbol $\sim$ for equivalent decompositions of rational functions defined earlier.

Under the above notation our main result about solutions of (10) may be formulated as follows.

**Theorem 1.1.** Assume that $P_1, P_2 \in \mathbb{R}[x]$ and $w_1, w_2 \in \mathbb{R}_t[\theta]$ are not constant and satisfy the equality
\[
P_1 \circ w_1 = P_2 \circ w_2.
\]
Then, up to a possible replacement of $P_1$ by $P_2$ and $w_1$ by $w_2$, one of the following conditions holds:

1) There exist $U, \tilde{P}_1, \tilde{P}_2, W_1, W_2 \in \mathbb{R}[x]$ and $\tilde{w} \in \mathbb{R}_t[\theta]$ such that
\[
P_1 = U \circ \tilde{P}_1, \quad P_2 = U \circ \tilde{P}_2, \quad w_1 = W_1 \circ \tilde{w}, \quad w_2 = W_2 \circ \tilde{w}, \quad \tilde{P}_1 \circ W_1 = \tilde{P}_2 \circ W_2,
\]
and either
\[
a) \quad \tilde{P}_1 \circ W_1 \sim z^n \circ z^r R(z^n), \quad \tilde{P}_2 \circ W_2 \sim z^r R^n(z) \circ z^n,
\]
where $R \in \mathbb{R}[x], r \geq 0, n \geq 1$, or
\[
b) \quad \tilde{P}_1 \circ W_1 \sim T_n \circ T_m, \quad \tilde{P}_2 \circ W_2 \sim T_m \circ T_n,
\]
where $T_n, T_m$ are the Chebyshev polynomials, $m, n \geq 1$, $\text{GCD}(n, m) = 1$.

2) There exist $U, \tilde{P}_1, \tilde{P}_2 \in \mathbb{R}[x], \tilde{w}_1, \tilde{w}_2 \in \mathbb{R}_t[\theta]$, and a polynomial $W(\theta) = k\theta + b$, where $k \in \mathbb{N}, b \in \mathbb{R}$, such that
\[
P_1 = U \circ \tilde{P}_1, \quad P_2 = U \circ \tilde{P}_2, \quad w_1 = \tilde{w}_1 \circ W, \quad w_2 = \tilde{w}_2 \circ W, \quad \tilde{P}_1 \circ \tilde{w}_1 = \tilde{P}_2 \circ \tilde{w}_2,
\]
and either
\[
c) \quad \tilde{P}_1 \circ \tilde{w}_1 \sim z^2 \circ \cos \theta S(\sin \theta), \quad \tilde{P}_2 \circ \tilde{w}_2 \sim (1 - z^2) S^2(z) \circ \sin \theta,
\]
where $S \in \mathbb{R}[x]$, or
\[
d) \quad \tilde{P}_1 \circ \tilde{w}_1 \sim -T_{nl} \circ \cos \left(\frac{(2s + 1)\pi}{nl} + m\theta\right), \quad \tilde{P}_2 \circ \tilde{w}_2 \sim T_{ml} \circ \cos (m\theta),
\]
where $T_{nl}, T_{ml}$ are the Chebyshev polynomials, $m, n \geq 1, l > 1, 0 \leq s < nl$, and $\text{GCD}(n, m) = 1$. 

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Notice that solutions of types a) and b) reduce to polynomial solutions of (1), while solutions of type c) generalize the identity \( \sin^2 \theta = 1 - \cos^2 \theta \). Further, solutions of type d) can be considered as a generalization of the identity

\[
T_n \circ \cos m \theta = T_m \circ \cos n \theta,
\]

although this identity itself is an example of a solution of type b) since

\[
\cos m \theta = T_m \circ \cos \theta, \quad \cos n \theta = T_n \circ \cos \theta.
\]

Our approach to functional equation (10) relies on the isomorphism

\[
\varphi: \cos \theta \rightarrow \left( \frac{z + 1/z}{2} \right), \quad \sin \theta \rightarrow \left( \frac{z - 1/z}{2i} \right),
\]

between the ring \( \mathbb{R}_t[\theta] \) and a subring of the ring \( \mathbb{C}[z, 1/z] \) of complex Laurent polynomials. Clearly, any decomposition \( p = P \circ w \) of \( p \in \mathbb{R}_t[\theta] \), where \( P \in \mathbb{R}(x) \) and \( w \in \mathbb{R}_t[\theta] \), or more generally where \( P \in \mathbb{R}(x) \) and \( w \) is contained in the quotient field \( \mathbb{R}_t(\theta) \) of \( \mathbb{R}_t[\theta] \), descends to a decomposition \( \varphi(p) = P \circ \varphi(w) \) of \( \varphi(p) \), making it possible to use results about decompositions of Laurent polynomials into compositions of rational functions for the study of decompositions of trigonometric polynomials.

The paper is organized as follows. In the second section we recall some basic facts about decompositions of Laurent polynomials and prove their analogues for decompositions in \( \mathbb{R}_t[\theta] \). We also show (Corollary 2.2) that if \( p \in \mathbb{R}_t[\theta] \), then any equivalence class of decompositions of \( \varphi(p) \in \mathbb{C}[z, 1/z] \) into a composition of rational functions over \( \mathbb{C} \) contains a representative which lifts to a decomposition \( p = P \circ w \), where \( P \in \mathbb{R}(x) \) and \( w \in \mathbb{R}_t(\theta) \). This result shows that the decomposition theory for \( \mathbb{R}_t[\theta] \) is “isomorphic” to the decomposition theory for a certain subclass of complex Laurent polynomials, and permits to deduce results about decompositions in \( \mathbb{R}_t[\theta] \) from the ones in \( \mathbb{C}[z, 1/z] \).

Finally, in the third section of the paper, basing on the results of the second section and results about decompositions of Laurent polynomials, we prove Theorem 1.1.

2 **Decompositions in** \( \mathbb{R}_t[\theta] \) **and in** \( \mathbb{C}[z, 1/z] \)**

The goal of this section is to show that decomposition theory for \( \mathbb{R}_t[\theta] \) can be considered as a “part” of the decomposition theory of complex Laurent polynomials.

It is well known that \( \mathbb{R}_t[\theta] \) is isomorphic to a subring of the field \( \mathbb{R}(x) \), where the isomorphism \( \psi: \mathbb{R}_t[\theta] \rightarrow \mathbb{R}(x) \) is defined by the formulas

\[
\psi(\sin \theta) = \frac{2x}{1 + x^2}, \quad \psi(\cos \theta) = \frac{1 - x^2}{1 + x^2}.
\]
Furthermore, the isomorphism $\psi$ extends to an isomorphism between $\mathbb{R}_t[\theta]$ and $\mathbb{R}(x)$, where
\[
x = \psi(\tan(\theta/2)) = \psi\left(\frac{\sin \theta}{1 + \cos \theta}\right).
\]
In particular, this implies by the Lüroth theorem that any subfield $k$ of $\mathbb{R}_t[\theta]$ has the form $k = \mathbb{R}(b)$ for some $b \in \mathbb{R}_t[\theta]$. In this paper however we will use the isomorphism $\varphi$, defined by the formulas
\[
\varphi(\cos \theta) = \frac{z + 1/z}{2}, \quad \varphi(\sin \theta) = \frac{z - 1/z}{2i},
\]
(12) between the ring $\mathbb{R}_t[\theta]$ and a subring of the ring $\mathbb{C}[z, 1/z]$ of complex Laurent polynomials, which seems to be more convenient for the study of compositional properties of $\mathbb{R}_t[\theta]$. For brevity, we will denote the ring $\mathbb{C}[z, 1/z]$ by $L[z]$ and the image of $\mathbb{R}_t[\theta]$ in $L[z]$ under the isomorphism $\varphi$ by $L_\mathbb{R}[z]$. It is easy to see that $L_\mathbb{R}[z]$ consists of Laurent polynomials $L$ such that $L(1/z) = L(z)$, where $\bar{L}$ denotes the Laurent polynomial obtained from $L$ by complex conjugation of all its coefficients. Clearly, the isomorphism $\varphi$ extends to an isomorphism between $\mathbb{R}_t[\theta]$ and $L_\mathbb{R}(z)$, where $L_\mathbb{R}(z)$ consists of rational functions $R$ satisfying the equality $\bar{R}(1/z) = R(z)$.

Any decomposition $p = P \circ w$, where $p \in \mathbb{R}_t[\theta]$, $P \in \mathbb{R}(x)$, and $w \in \mathbb{R}_t(\theta)$, obviously descends to a decomposition $\varphi(p) = P \circ \varphi(w)$, where $\varphi(p) \in L_\mathbb{R}[z]$ and $\varphi(w) \in L_\mathbb{R}(z)$. However, it is clear that $L = \varphi(p)$ may have decompositions $L = A \circ B$, where $A, B \in \mathbb{C}(z)$, such that coefficients of $A$ are not real and $B$ is not contained in $L_\mathbb{R}(z)$. In this context the following simple lemma is useful.

Lemma 2.1. Let $L \in L_\mathbb{R}(z)$ and $L = A \circ B$ be a decomposition of $L$ into a composition of rational functions $A, B \in \mathbb{C}(z)$. Then the inclusion $B \in L_\mathbb{R}(z)$ implies the inclusion $A \in \mathbb{R}(x)$.

Proof. Indeed, since $L, B \in L_\mathbb{R}(z)$, we have:
\[
A \circ B = \bar{A} \circ \bar{B} \circ 1/z = \bar{A} \circ B,
\]
implying that $\bar{A} = A$.

We will call a Laurent polynomial $L$ proper if $L$ is neither a polynomial in $z$, nor a polynomial in $1/z$, or in other words if $L$ has exactly two poles. The lemma below is a starting point of the decomposition theory of Laurent polynomials (see [12],[22]).

Lemma 2.2. Let $L = P \circ W$ be a decomposition of $L \in L[z]$ into a composition of rational functions $P, W \in \mathbb{C}(z)$. Then there exists $\mu \in \mathbb{C}(z)$ of degree one such that either $P \circ \mu$ is a polynomial and $\mu^{-1} \circ W$ is a Laurent polynomial, or $P \circ \mu$ is a Laurent polynomial and $\mu^{-1} \circ W = z^d$, $d \geq 1$.

Proof. Indeed, it follows easily from
\[
L^{-1}\{\infty\} = W^{-1}\{P^{-1}\{\infty\}\} \subseteq \{0, \infty\}
\]
that either $P^{-1}\{\infty\}$ consists of a single point $a \in \mathbb{C}P^1$ and $W^{-1}\{a\} \subseteq \{0,\infty\}$, or $P^{-1}\{\infty\}$ consists of two points $a, b \in \mathbb{C}P^1$ and $W^{-1}\{a, b\} = \{0, \infty\}$. In the first case there exists a rational function $\mu \in \mathbb{C}(z)$ of degree one such that $P \circ \mu$ is a polynomial and $\mu^{-1} \circ W$ is a Laurent polynomial (which is proper if and only if $L$ is proper). In the second case there exists $\mu \in \mathbb{C}(z)$ of degree one such that $P \circ \mu$ is a proper Laurent polynomial and $\mu^{-1} \circ W = z^d$, $d \geq 1$. \hfill \square

The following statement is a “trigonometric” analogue of Lemma 2.2 and essentially is equivalent to Proposition 21 of [7] and to Theorem 5 of [5]. Notice however that the proofs given in [7], [5] are much more complicated than the proof given below. The idea to relate decompositions in $\mathbb{R}_t[\theta]$ with decompositions in $L[z]$ was proposed in [17].

**Lemma 2.3.** Let $p = P \circ w$ be a decomposition of $p \in \mathbb{R}_t[\theta]$ into a composition of $P \in \mathbb{R}(x)$ and $w \in \mathbb{R}_t(\theta)$. Then there exists a rational function $\mu \in \mathbb{R}(x)$ of degree one such that either $P \circ \mu \in \mathbb{R}[x]$ and $\mu^{-1} \circ w \in \mathbb{R}_t[\theta]$, or $P \circ \mu \in \mathbb{R}(x)$ and $\mu^{-1} \circ w = \tan(\theta/2)$, $d \geq 1$.

**Proof.** Setting

$$L = \varphi(p), \quad W = \varphi(w)$$

and considering the equality $L = P \circ W$, we conclude as above that either

$$P^{-1}\{\infty\} = \{a\} \quad \text{and} \quad W^{-1}\{a\} = \{0, \infty\}$$

(13)

for some $a \in \mathbb{C}P^1$, or

$$P^{-1}\{\infty\} = \{a, b\} \quad \text{and} \quad W^{-1}\{a, b\} = \{0, \infty\}$$

(14)

for some $a, b \in \mathbb{C}P^1$.

Since any polynomial with real coefficients is a product of linear and quadratic polynomials with real coefficients, if (13) holds, then the first equality in (13) implies that either $P \in \mathbb{R}[x]$ and $W \in L_{\mathbb{R}}[z]$, or $a \in \mathbb{R}$. In the first case, since $\varphi$ is an isomorphism between $\mathbb{R}_t[\theta]$ and $L_{\mathbb{R}}[z]$, we conclude that $w \in \mathbb{R}_t[\theta]$. On the other hand, if $a \in \mathbb{R}$, then setting $\mu = a + 1/z$ we see that $P \circ \mu \in \mathbb{R}[x]$ and $\mu^{-1} \circ W \in L[z]$. Furthermore, since $W \in L_{\mathbb{R}}(z)$ and $\mu$ has real coefficients, the function $\mu^{-1} \circ W$ is contained in $L_{\mathbb{R}}[z]$ implying that $\mu^{-1} \circ w \in \mathbb{R}_t[\theta]$.

If (14) holds, then we can modify $\mu \in \mathbb{C}(z)$ from Lemma 2.2 so that

$$\mu^{-1} \circ W = \frac{1}{i} \frac{z^d - 1}{z^d + 1} = \frac{1}{i} \left( \frac{z^{d/2} - z^{-d/2}}{z^{d/2} + z^{-d/2}} \right) = \varphi(\tan(d\theta/2)), \quad d \geq 1.$$  

(15)

Furthermore, since the functions $\varphi(\tan(d\theta/2))$ and $W$ are contained in $L_{\mathbb{R}}(z)$, it follows from Lemma 2.1 that $\mu^{-1} \in \mathbb{R}(x)$. Finally, it is clear that $P \circ \mu \in \mathbb{R}(x)$ and $\mu^{-1} \circ w = \tan(d\theta/2)$.

\hfill \square

Notice that if $p = P \circ w$ is a decomposition of $p \in \mathbb{R}_t[\theta]$ such that $P \in \mathbb{R}(x)$ and $w = \tan(d\theta/2)$, $d \geq 1$, then $P$ has the form $P = A/(x^2 + 1)^k$, where $A \in \mathbb{R}[x]$, $k \geq 1$, and deg $A \leq 2k$. This can be proved by arguments similar to
the ones used in the proof of Lemma 2.3. Alternatively, we can observe that \( \tan(\frac{d\theta}{2}) \) considered as a function of complex variable takes all the values in \( \mathbb{C}P^1 \) distinct from \( \pm i \). Therefore, the function \( P \) may have poles only at points \( \pm i \), since otherwise the composition \( p = P \circ w \) would not be an entire function.

Two different types of decompositions of Laurent polynomials appearing in Lemma 2.2 correspond to two different types of imprimitivity systems in their monodromy groups (for more details concerning decompositions of rational functions with two poles we refer the reader to [9]). Namely, if \( L \) is a Laurent polynomial of degree \( n \) we may assume that its monodromy group \( G \) contains the permutation

\[
h = (1 \, 2 \, \ldots \, n_1)(n_1 + 1 \, n_1 + 2 \, \ldots \, n_1 + n_2),
\]

where \( 1 \leq n_1 \leq n, 0 \leq n_2 < n, n_1 + n_2 = n \). Furthermore, the equalities \( n_1 = n, n_2 = 0 \) hold if and only if \( L \) is not proper.

Denote by \( W_{i,d}^1 \) (resp. by \( W_{i,d}^2 \)) a union of numbers from the segment \( [1,n_1] \) (resp. \([n_1+1,n_1+n_2]\)) equal to \( i \) by modulo \( d \). Since \( h \) must permute blocks of any imprimitivity system of \( G \), it is easy to see that if \( E \) is such a system, then either there exists a number \( d|n \) such that any block of \( E \) is equal to \( W_{i_1,d_1} \cup W_{i_2,d_2} \) for some \( i_1, i_2, 1 \leq i_1, i_2 \leq d \), or there exist numbers \( d_1|n, d_2|n \) such that

\[
n_1/d_1 = n_2/d_2
\]

and any block of \( E \) is equal either to \( W_{i_1,d_1}^1 \) for some \( i_1, 1 \leq i_1 \leq d_1 \), or to \( W_{i_2,d_2}^2 \) for some \( i_2, 1 \leq i_2 \leq d_2 \). The imprimitivity systems of the first type correspond to decompositions \( L = A(B) \), where \( A \) is a polynomial and \( B \) is a Laurent polynomial, while imprimitivity systems of the second type correspond to decompositions \( L = A(B) \), where \( A \) is a proper Laurent polynomial and \( B = z^d \).

The following result coincides with Lemma 6.3 of [12]. For the reader convenience we provide below a self-contained proof.

**Lemma 2.4.** Let \( A, B \in \mathbb{C}[z] \setminus \mathbb{C} \) and \( L_1, L_2 \in \mathcal{L}[z] \setminus \mathbb{C} \) satisfy

\[
A \circ L_1 = B \circ L_2.
\]

Assume additionally that \( \deg A = \deg B \). Then either

\[
B = A \circ w^{-1}, \quad L_2 = w \circ L_1
\]

for some polynomial \( w \in \mathbb{C}[z] \) of degree one, or there exist \( r \in \mathbb{N}, a \in \mathbb{C}, \) and a root of unity \( \nu \) such that

\[
w_1 \circ L_1 = \left( z^r + \frac{1}{z^r} \right) \circ (az), \quad w_2 \circ L_2 = \left( z^r + \frac{1}{z^r} \right) \circ (a\nu z)
\]

for some polynomials \( w_1, w_2 \in \mathbb{C}[z] \) of degree one.
Proof. Let \( G \) be the monodromy group of a Laurent polynomial \( L \) defined by any of the parts of equality (17). Then equality (17) implies that \( G \) has two imprimitivity systems \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) of the first type corresponding to the decompositions in (17). Furthermore, since \( \deg A = \deg B \), the blocks of these systems have the same cardinality \( l = \deg L/\deg A \).

If these systems coincide, then equalities (18) hold for some rational function \( w \in \mathbb{C}(z) \) of degree one which obviously is a polynomial. On the other hand, if they are different, then the imprimitivity system \( \mathcal{E}_1 \cap \mathcal{E}_2 \) necessary belongs to the second type, and has blocks consisting of \( r \) elements, where \( 2r = l \). In particular, \( L \) and \( L_1, L_2 \) are proper, and the equalities

\[
L_1 = \tilde{L}_1 \circ W, \quad L_2 = \tilde{L}_2 \circ W, \tag{20}
\]

hold for some rational functions \( \tilde{L}_1, \tilde{L}_2, W \), where \( \deg \tilde{L}_1 = \deg \tilde{L}_2 = 2 \). Applying now Lemma 2.2 to equalities (20) we conclude that

\[
L_1 = \left( \alpha_0 + \alpha_1 z + \frac{\alpha_2}{z} \right) \circ z^r, \quad L_2 = \left( \beta_0 + \beta_1 z + \frac{\beta_2}{z} \right) \circ z^r,
\]

for some \( \alpha_0, \beta_0 \in \mathbb{C} \), and \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C} \setminus \{0\} \). Furthermore, equality (17) implies that

\[
L_1 = \left( \alpha_0 + \alpha_1 z + \frac{\alpha_2}{z} \right) \circ z^r, \quad L_2 = \left( \beta_0 + \alpha_1 \nu_1 z + \frac{\alpha_2 \nu_2}{z} \right) \circ z^r,
\]

for some roots of unity \( \nu_1, \nu_2 \). The lemma follows now from the equalities

\[
\alpha_0 + \alpha_1 z^r + \frac{\alpha_2}{z^r} = \left( \alpha_0 + \frac{\alpha_1 z}{az} \right) \circ \left( z^r + \frac{1}{z^r} \right) \circ (az),
\]

\[
\beta_0 + \alpha_1 \nu_1 z^r + \frac{\alpha_2 \nu_2}{z^r} = \left( \beta_0 + \frac{\alpha_1 \nu_1 z}{a \nu z} \right) \circ \left( z^r + \frac{1}{z^r} \right) \circ (a \nu z),
\]

where \( a \) and \( \nu \) are complex numbers satisfying \( a^{2r} = \alpha_1/\alpha_2 \) and \( \nu^{2r} = \nu_1/\nu_2 \). \( \square \)

Corollary 2.1. Let \( P \in \mathbb{R}[z] \) and \( A, B \in \mathbb{C}[z] \) satisfy the equality \( P = A \circ B \). Then \( A, B \in \mathbb{R}[z] \) whenever the leading coefficient of \( B \) is real.

Proof. Applying Lemma 2.4 to the equality

\[
A \circ B = \tilde{A} \circ \tilde{B} \tag{21}
\]

we conclude that \( \tilde{B} = \alpha B + \beta \), where \( \alpha, \beta \in \mathbb{C} \). Comparing the leading coefficients of the polynomials in the last equality we see that \( \alpha = 1 \). It follows now from equality (21) that \( A(z) = A(z + \beta) \) implying easily that \( \beta = 0 \). Finally, it follows from \( \tilde{B} = B \) and (21) that \( \tilde{A} = A \). \( \square \)

Lemma 2.5. Let \( L = A \circ L_1 \) be a decomposition of \( L \in \mathcal{L}_\mathbb{R}[z] \) into a composition of \( A \in \mathbb{C}[z] \) and \( L_1 = \sum c_i z^i \in \mathcal{L}[z] \). Assume additionally that \( c_{-n} = 1/c_n \). Then the leading coefficient of \( A \) is real and \( |c_n| = |c_{-n}| = 1 \).
Proof. Let $\alpha$ be the leading coefficient of $A$. Then $L \in \mathcal{L}_R[z]$ implies that
\begin{equation}
\alpha c_n = \bar{\alpha}/c_n.
\end{equation}
Multiplying this equality by its conjugated we obtain the equality $(\bar{c}_n c_n)^2 = 1$ implying that $c_n = 1/c_n$ or equivalently that $|c_n| = 1$. Now (22) implies that $\bar{\alpha} = \alpha$.

**Theorem 2.1.** Let $L = A \circ L_1$ be a decomposition of $L \in \mathcal{L}_R[z]$ into a composition of $A \in \mathbb{C}[z]$ and $L_1 \in \mathcal{L}[z]$. Then there exists a polynomial $v \in \mathbb{C}[z]$ of degree one such that $A \circ v^{-1} \in \mathbb{R}[z]$ and $v \circ L_1 \in \mathcal{L}_R[z]$.

**Proof.** Since $L$ belongs to $\in \mathcal{L}_R[z]$, the equality $A \circ L_1 = \bar{A} \circ \bar{L}_1 \circ 1/z$ holds. Applying to this equality Lemma 2.4 we conclude that either
\begin{equation}
\bar{L}_1 \circ 1/z = w \circ L_1,
\end{equation}
for some polynomial $w = az + b$, $a, b \in \mathbb{C}$, or
\begin{equation}
v \circ L_1 = cz^r + \frac{1}{cz^r}
\end{equation}
for some polynomial $v \in \mathbb{C}[z]$ of degree one and $c \in \mathbb{C}$.

In the first case, setting $L_1 = \sum_{-n}^n c_i z^i$, we see that (23) implies the equalities
\[c_{-i} = ac_i, \quad 0 < |i| \leq n.
\]
Taking $c_{-i} \neq 0$, we obtain
\[c_{-i} = \frac{\bar{a}c_i}{c_i} = \bar{\alpha} c_{-i}
\]
implies that $a\bar{a} = 1$ or equivalently that $|a| = 1$.

Set $v = \lambda z + \mu$, where $\lambda$ satisfies $\lambda^2 = a$, and $\mu = \lambda c_0$. Since $\lambda \bar{\lambda} = 1$, we have:
\[\lambda c_{-i} = \bar{\lambda} a c_i = \bar{\lambda} \lambda^2 c_i = \lambda c_i, \quad 0 < |i| \leq n.
\]
Furthermore,
\[\lambda c_0 + \bar{\lambda} c_0 = \lambda c_0 + \lambda c_0,
\]
and hence $v \circ L_1 \in \mathcal{L}_R[z]$. It follows now from the equality
\begin{equation}
L = (A \circ v^{-1}) \circ (v \circ L_1)
\end{equation}
by Lemma 2.1 that $A \circ v^{-1} \in \mathbb{R}[z]$.

In the second case, it follows from equalities (24) and (25) by Lemma 2.5 that $|c| = 1$ implying that $v \circ L_1 \in \mathcal{L}_R[z]$. Finally, Lemma 2.1 implies as above that $A \circ v^{-1} \in \mathbb{R}[z]$.
Corollary 2.2. Let $L = P \circ W$ be a decomposition of $L \in L_R[z]$ into a composition of $P, W \in \mathbb{C}(z)$. Then there exists a rational function $v \in \mathbb{C}(z)$ of degree one such that $P \circ v^{-1} \in \mathbb{R}(x)$ and $v \circ W \in L_R(z)$.

Proof. Arguing as in the proofs of Lemma 2.2 and Lemma 2.3 we see that there exists a rational function $\mu \in \mathbb{C}(z)$ of degree one such that either $P \circ \mu$ is a polynomial and $\mu^{-1} \circ W$ is a Laurent polynomial, or equality (15) holds and $P \circ \mu \in \mathbb{R}(x)$. In the second case the statement of the corollary is obvious, while in the first one it follows from Theorem 2.1.

3 Double decompositions in $\mathbb{R}_t[\theta]$ and in $\mathbb{C}[z, 1/z]$

Recall, that two decompositions $P = A \circ B$ and $P = \tilde{A} \circ \tilde{B}$ of a function $P \in \mathbb{C}(z)$ into compositions of functions $A,B,\tilde{A},\tilde{B} \in \mathbb{C}(z)$ are called equivalent if there exists a function $\mu \in \mathbb{C}(z)$ of degree one such that

$$\tilde{A} = A \circ \mu, \quad \tilde{B} = \mu^{-1} \circ B.$$  

Notice that if both $\tilde{A}$ and $A$ (or $\tilde{B}$ and $B$) are polynomials, then $\mu$ also is a polynomial. In particular, this is the case for most of the equivalences considered below. In case if we consider rational functions defined over an arbitrary field the definition above is modified in an obvious way (in fact, below we are only interested in the cases where the ground field is $\mathbb{C}$ or $\mathbb{R}$).

We start from recalling some basic facts about polynomial solutions of the equation

$$A \circ C = B \circ D. \quad (26)$$

The proposition below reduces a description of solutions of (26) to the case where degrees of $A$ and $B$ as well as of $C$ and $D$ are coprime (see e.g. [9]).

Proposition 3.1. Suppose $A,B,C,D \in \mathbb{C}[z] \setminus \mathbb{C}$ satisfy (26). Then there exist $U, V, \tilde{A}, \tilde{C}, \tilde{B}, \tilde{D} \in \mathbb{C}[z], \mathbb{C}$, where

$$\deg U = \text{GCD}(\deg A, \deg B), \quad \deg V = \text{GCD}(\deg C, \deg D),$$

such that

$$A = U \circ \tilde{A}, \quad B = U \circ \tilde{B}, \quad C = \tilde{C} \circ V, \quad D = \tilde{D} \circ V,$$

and

$$\tilde{A} \circ \tilde{C} = \tilde{B} \circ \tilde{D}. \quad \square$$

In fact, under an appropriate restriction, Proposition (3.1) remains true if to assume that coefficients of polynomials $A, B, C, D$ as well as of $U, V, \tilde{A}, \tilde{C}, \tilde{B}, \tilde{D}$ belong to an arbitrary field (see Theorem 5, Chapter 1 of [20]). In particular, Proposition 3.1 remains true if the ground field is $\mathbb{R}$.

The following result obtained by Ritt [18] describes solutions of (26) in the case where the equalities

$$\text{GCD}(\deg A, \deg B) = 1, \quad \text{GCD}(\deg C, \deg D) = 1 \quad (27)$$

hold, and is know under the name “the second Ritt theorem”.

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Theorem 3.1. Suppose $A,B,C,D \in \mathbb{C}[z] \setminus \mathbb{C}$ satisfy (27) and (28). Then there exist $U, A, B, C, D, W \in \mathbb{C}[z]$, where $\deg U = \deg W = 1$, such that

$$A = U \circ \tilde{A}, \quad B = U \circ \tilde{B}, \quad C = \tilde{C} \circ W, \quad D = \tilde{D} \circ W,$$

and, up to a possible replacement of $A$ by $B$ and $C$ by $D$, one of the following conditions holds:

1) $\tilde{A} \circ \tilde{C} \sim z^n \circ z^r R(z^n), \quad \tilde{B} \circ \tilde{D} \sim z^r R^n(z) \circ z^n,$

where $R \in \mathbb{C}[z]$, $r \geq 0$, $n \geq 1$, and $\gcd(n,r) = 1$;

2) $\tilde{A} \circ \tilde{C} \sim T_n \circ T_m, \quad \tilde{B} \circ \tilde{D} \sim T_m \circ T_n,$

where $T_n, T_m$ are the Chebyshev polynomials, $m, n \geq 1$, and $\gcd(n,m) = 1$. \qed

Again, this theorem remains true if to assume that coefficients of all polynomials involved are real (see Lemma 2, Chapter 1 of [20]), and, under an appropriate modification, even belong to an arbitrary field (see [21] and Theorem 5, Chapter 1 of [20]).

Recall now the main result of the decomposition theory of Laurent polynomials (see [12],[22]) concerning solutions of the equation

$$P_1 \circ W_1 = P_2 \circ W_2, \quad (28)$$

where $P_1, P_2 \in \mathbb{C}[z]$ and $W_1, W_2 \in \mathbb{C}[z, 1/z]$. We will use the notation of [16] (Theorem 3.1). Notice that the main result of [16] (Theorem A) also may be used for a proof of Theorem 1.1. However, the approach using the results of Section 2 is more general and may be used for a solution of another problems related to decompositions of trigonometric polynomials.

Set

$$U_n(z) = \frac{1}{2} \left( z^n + \frac{1}{z^n} \right), \quad V_n(z) = \frac{1}{2i} \left( z^n - \frac{1}{z^n} \right).$$

Observe that

$$U_n = \varphi(\cos n\theta), \quad V_n = \varphi(\sin n\theta). \quad (29)$$

Indeed, the first formula in (29) follows from the equality

$$T_n \circ \frac{1}{2} \left( x + \frac{1}{x} \right) = \frac{1}{2} \left( x^n + \frac{1}{x^n} \right), \quad (30)$$

which in its turn is obtained from the definition of the Chebychev polynomials by the substitution $x = e^{i\theta}$. The second one follows from the formulas

$$T_n'(\cos \theta) \sin \theta = n \sin n\theta$$

and (30). Furthermore, it is easy to see that if $c = \cos a + i \sin a$, where $a \in \mathbb{R}$, then

$$U_n \circ (cz) = \varphi(\cos (n\theta + na)), \quad V_n \circ (cz) = \varphi(\sin(n\theta + na)). \quad (31)$$
Theorem 3.2. Let \( P_1, P_2 \in \mathbb{C}[z] \setminus \mathbb{C} \) and \( W_1, W_2 \in \mathbb{C}[z, 1/z] \setminus \mathbb{C} \) satisfy (28). Then there exist \( U, \tilde{P}_1, \tilde{P}_2 \in \mathbb{C}[z] \) and \( W, \tilde{W}_1, \tilde{W}_2 \in \mathbb{C}[z, 1/z] \) such that

\[
P_1 = U \circ \tilde{P}_1, \quad P_2 = U \circ \tilde{P}_2, \quad W_1 = \tilde{W}_1 \circ W, \quad W_2 = \tilde{W}_2 \circ W, \quad \tilde{P}_1 \circ \tilde{W}_1 = \tilde{P}_2 \circ \tilde{W}_2
\]

and, up to a possible replacement of \( P_1 \) by \( P_2 \) and \( W_1 \) by \( W_2 \), one of the following conditions holds:

1) \( \tilde{P}_1 \circ \tilde{W}_1 \sim z^n \circ z^r R(z^n), \quad \tilde{P}_2 \circ \tilde{W}_2 \sim z^r R^n(z) \circ z^n \), where \( R \in \mathbb{C}[z], \, r \geq 0, \, n \geq 1, \) and \( \text{GCD}(n, r) = 1; \)

2) \( \tilde{P}_1 \circ \tilde{W}_1 \sim T_n \circ T_m, \quad \tilde{P}_2 \circ \tilde{W}_2 \sim T_m \circ T_n \), where \( T_n, T_m \) are the Chebyshev polynomials, \( m, n \geq 1, \) and \( \text{GCD}(n, m) = 1; \)

3) \( \tilde{P}_1 \circ \tilde{W}_1 \sim z^2 \circ U_1 S(V_1), \quad \tilde{P}_2 \circ \tilde{W}_2 \sim (1 - z^2) S^2 \circ V_1 \), where \( S \in \mathbb{C}[z]; \)

4) \( \tilde{P}_1 \circ \tilde{W}_1 \sim -T_{nl} \circ U_m(\varepsilon z), \quad \tilde{P}_2 \circ \tilde{W}_2 \sim T_{ml} \circ U_n \), where \( T_{nl}, T_{ml} \) are the Chebyshev polynomials, \( m, n \geq 1, \, l > 1, \, \varepsilon^{nlm} = -1, \) and \( \text{GCD}(n, m) = 1; \)

5) \[
\tilde{P}_1 \circ \tilde{W}_1 \sim (z^2 - 1)^3 \circ \left( \frac{i}{\sqrt{3}} V_2 + \frac{2\sqrt{2}}{\sqrt{3}} U_1 \right),
\]

\[
\tilde{P}_2 \circ \tilde{W}_2 \sim (3z^4 - 4z^3) \circ \left( \frac{i}{3\sqrt{2}} V_3 + U_2 + \frac{i}{\sqrt{2}} V_1 + \frac{2}{3} \right). \]

Notice that if \( L_1, L_2 \) are polynomials, then \( W \) also is a polynomial and either 1) or 2) holds, in correspondence with Proposition 3.1 and Theorem 3.1.

Proof of Theorem 1.1. Let \( P_1, P_2 \in \mathbb{R}[x] \) and \( w_1, w_2 \in \mathbb{R}_t[\theta] \) satisfy equation (10). Assume first that there exist \( w \in \mathbb{R}_t[\theta] \) and \( \tilde{W}_1, \tilde{W}_2 \in \mathbb{R}[x] \) such that the equalities

\[
w_1 = \tilde{W}_1 \circ w, \quad w_2 = \tilde{W}_2 \circ w
\]

hold. Then equality (10) implies the equality

\[
P_1 \circ \tilde{W}_1 = P_1 \circ \tilde{W}_1,
\]

and it is easy to see using the real versions of Proposition 3.1 and Theorem 3.1 that either the case a) or the case b) has the place.
Assume now that such \( w \) and \( \hat{W}_1, \hat{W}_2 \) do not exist. Set

\[
p = P_1 \circ w_1 = P_2 \circ w_2, \quad L = \varphi(p), \quad W_1 = \varphi(w_1), \quad W_2 = \varphi(w_2),
\]

and apply Theorem 3.2 to equality (28). Observe that our assumption implies that neither the first nor the second case provided by Theorem 3.2 may have the place. Indeed, since \( L \) is a proper Laurent polynomial, if one of these cases holds, then the function \( W \) is also a proper Laurent polynomial. Therefore, applying Theorem 2.1 to the equality \( W_1 = \hat{W}_1 \circ W \), we conclude that there exists a polynomial \( v \in \mathbb{C}[z] \) of degree one such that \( \hat{W}_1 \circ v^{-1} \in \mathbb{R}[x] \) and \( v \circ W \in \mathcal{L}_R[z] \). Furthermore, applying Lemma 2.1 to the equality

\[
W_2 = (\hat{W}_2 \circ v^{-1}) \circ (v \circ W),
\]

we conclude that \( \hat{W}_2 \circ v^{-1} \in \mathbb{R}[x] \) implying that (32) holds for

\[
\hat{W}_1 = \hat{W}_1 \circ v^{-1}, \quad \hat{W}_2 = \hat{W}_2 \circ v^{-1}, \quad w = \varphi^{-1}(v \circ W).
\]

Consider now one by one all the other cases possible by Theorem 3.2. If holds 3), then there exist \( \mu_1, \mu_2 \in \mathbb{C}[z] \) of degree one and \( S \in \mathbb{C}[z] \) such that

\[
P_1 = U \circ z^2 \circ \mu_1^{-1}, \quad W_1 = \mu_1 \circ U_1 S(V_1) \circ W,
\]

and

\[
P_2 = U \circ (1 - z^2) S^2 \circ \mu_2^{-1}, \quad W_2 = \mu_2 \circ V_1 \circ W,
\]

for some \( U \in \mathbb{C}[z] \) and \( W \in \mathcal{L}_R[z] \). Notice that changing if necessary \( \mu_1 \) to \( \mu_1 \circ (\gamma z) \) and \( U \) to \( U \circ (\gamma^2 z) \), where \( \gamma \in \mathbb{C} \) is the leading coefficient of \( S \), without loss of generality we may assume that the polynomial \( S \) is monic. Furthermore, it follows from Lemma 2.2 that \( W \) necessary has the form \( cz^k \), \( c \in \mathbb{C} \setminus \{0\} \).

Let \( \mu_2 = \alpha z + \beta \), where \( \alpha, \beta \in \mathbb{C} \). Since \( W_2 \) is contained in \( \mathcal{L}_R[z] \), the second equality in (34) implies that \( \beta = \beta \) and, by Lemma 2.5, that \( \alpha \in \mathbb{R} \) and \( \bar{c} = 1/c \). Therefore, \( \mu_2 \in \mathbb{R}[x] \) and there exists \( a \in \mathbb{R} \) such that \( c = \cos a + i \sin a \), implying by (31) that \( w_1 = \mu_2 \circ \sin(n\theta + b) \), where \( b = na \).

Since \( \mu_2 \in \mathbb{R}[x] \), applying now Corollary 2.1 to the first equality in (34), we conclude that \( U \in \mathbb{R}[x] \) and \( S^2 \in \mathbb{R}[x] \). Moreover, since \( S \) monic, the last equality implies that \( S \in \mathbb{R}[x] \). Further, since \( S \in \mathbb{R}[x] \) and \( \bar{c} = 1/c \), the Laurent polynomial \( U_1 S(V_1) \circ W \) is contained in \( \mathcal{L}_R[z] \) implying by Lemma (2.1) that

\[
W_1 = \mu_1 \circ \cos(n\theta + b) S(\sin(n\theta + b)).
\]

Therefore, if (33) and (34) hold we arrive to the case c) of Theorem 1.1.

Consider now the case 4). In this case there exist \( \mu_1, \mu_2 \in \mathbb{C}[z] \) of degree one and \( U \in \mathbb{C}[z] \) such that

\[
P_1 = U \circ -T_m \circ \mu_1^{-1}, \quad W_1 = \mu_1 \circ U_m(\varepsilon z) \circ W,
\]

(35)
and

\[ P_2 = U \circ T \circ \mu_2^{-1}, \quad W_2 = \mu_2 \circ U_n \circ W, \quad (36) \]

where \( W = cz^k, \, c \in \mathbb{C} \setminus \{0\} \). As above the second equality in (36) implies that \( \bar{c} = 1/c \) and \( \mu_2 \in \mathbb{R}[x] \). Further, the first equality in (36) implies that \( U \in \mathbb{R}[x] \), and the second equality in (35) implies that \( \mu_1 \in \mathbb{R}[x] \). Therefore, taking into account formulas (31), we conclude that equalities (35) and (36) lead to the case d) of Theorem 1.1.

Let us show finally that the case 5) can not have a place. Assume the inverse. Then

\[ W_1 = \mu \circ \left( \frac{i}{\sqrt{3}} V_2 + \frac{2\sqrt{2}}{\sqrt{3}} U_1 \right) \circ (cz^k), \quad (37) \]

where \( \mu = \alpha + \beta, \, \alpha, \beta, c \in \mathbb{C} \). Since \( W_1 \in \mathcal{L}[z] \), equality (37) implies the equalities

\[ \bar{\alpha} \bar{c} = -\alpha/c, \quad \bar{\alpha} \bar{c} = \alpha/c \]

which are possible only if \( \alpha = 0 \) and \( w_1 \) is a constant.

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