Exercise: ±1 bug and center of an array problem

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Abstract. A problem that is constantly cropping up in designing even the simplest algorithm or a program is dealing with ±1 bug, or one-off bug, when we calculate positions within an array, very noticeably while splitting it in half. This bug is often found in buffer overflow type of bugs. While designing one complicated algorithm, we needed various ways of splitting an array, and we found the lack of general guidance for this apparently minor problem. We present an exercise that tracks the cause of the problem and leads to the solution. This problem looks trivial because it seems obvious or insignificant, however, treating it without outmost precision can lead to subtle bugs, unbalanced solution, not transparent expressions for various languages. Basically, the exercise is about dealing with \( \leq \) as well as \( \frac{n}{2}, \frac{n}{2} - 1, \frac{n+1}{2}, n-1 \) and similar expressions when they are rounded down to the nearest integer and used to define a range.

Mathematics never crashes - program does.

Wobbling center of array and other stories

An array is a continuous buffer in the memory. It starts at position \( b \) and ends at position \( e \). All positions between \( b \) and \( e \) (including \( b \) and \( e \)) belong to the array. The size or length of the array is \( n=e-b+1 \). The problem is apparently very simple: where is the center of the array and how does it relate to the size, start and end position? No more than that. There are several causes of the problem:

1. There is confusion in various languages about the initial index of an array. For example, by default, in PHP, C, C++ it is 0, in COBOL, Fortran, Smalltalk it is 1, other languages use the lowest value of index type like Ada, Pascal. The quick rules for one language are not transparent.
2. Expressions like \( \frac{n}{2} \) and similar with integer division are rounded down to the nearest integer which is not obvious from the syntax.
3. Definition of center is vague. An array with even size does not have a central position.
4. A wrong assumption that the center of an array is obviously \( \frac{n}{2} \) which you need to adjust because it does not work all the time.
5. The existing code in the literature or elsewhere did not follow any particular guidance. Reading such a code is a source of confusion.
6. The index that is used in the algorithm or program could have several hidden meanings throughout the code.
7. Foggy understanding of expression like \( \frac{n-1}{2}+1, \frac{n+1}{2}, \frac{n}{2} - 1 \) when they are in integers rounded down to the lowest integer with a constant attempt to reason in \( \text{mod}\ 2 \).
8. Because the problem looks trivial, it is not given sufficient attention.
9. The treatment of indices does look somewhat magical even in very standard and praised programming literature.
10. Special cases like \( n=0 \) and \( n=1 \) are merged into the general solution.
11. There is additional confusion about using \( < \) or \( \leq \)
12. Formula from mathematics is applied directly, rather than carefully reconstructed.
13. There is confusion about the usage of positional value as length value or the other way around. For example, 0 can be first position in an array, but as length it represents an empty array.
14. A headache about counting a number of elements in an array with unclear specification for boundaries. How many numbers do we have from 0 to 5? 6, 5, 4?

In the exercise, we will use two simple examples: 01234 and 012345 and two values for finding a center: \( \frac{n}{2} \) and \( \frac{e-b}{2}=s/2 \). These two examples are all what we need. We will explain what happens when the initial index is not 0. We explain everything using 0-index case because all other cases can be easily derived from it. All values are rounded down to the nearest integer, i.e. we are dealing with integer division. We use two
ways of trying to reach the center: through \((n\pm 1)/2\) and \(n/2\pm 1\). We will find in our analysis that the following are the best expressions for various programming languages. The goal was to find as few as possible items to memorize, and to find an easy way of extending them to various situations. In the work, we give a very detailed explanation and potential pitfalls for other expressions. We deliberately dig down to the last detail just to explain how even the most correct mathematical formula cannot prevent bugs.

| Final solution that handles central position | left half | right half |
|-------------------------------------------|-----------|------------|
| Natural division                          | \(0 \leq i < n/2\) | \((n+1)/2 \leq i < n\) |
| Left+ division                            | \(0 \leq i < (n+1)/2\) | \((n+1)/2 \leq i < n\) |
| Right+ division                           | \(0 \leq i < n/2\) | \(n/2 \leq i < n\) |
| Cut out center division                   | left cut  | \(0 \leq i < (n+1)/2-1\) | \((n+1)/2 \leq i < n\) |
|                                          | right cut | \(0 \leq i < n/2\) | \(n/2 + 1 \leq i < n\) |

For this entire table, you need to remember only one range \(0 \leq i < n/2\), \((n+1)/2 \leq i < n\), which is obvious if you know the formula
\[
\left[ \frac{n}{2} \right] + \left[ \frac{n + 1}{2} \right] = n
\]

**List of the common rules**

Let us start with few examples to illustrate the problem. If we have value \(r\), \(0 \leq r \leq n\), then

- \(b \leq i \leq e\) has length \(n = e - b + 1\)
- \(b < i \leq e\) has length \(n - 1\)
- \(b \leq i < e\) has length \(n - 1\)
- \(b < i < e\) has length \(n - 2\)
- \(b \leq i < b + r\) has length \(r\)

\(\times\) \(b \leq i \leq b + r\) has length \(r + 1\), which means that we **must** have \(r < n\)

- \(b + r \leq i \leq e\) has length \(n - r\)
- \(b + r < i \leq e\) has length \(n - r - 1\)
- \(b \leq i \leq e - r\) has length \(n - r\)
- \(b \leq i < e - r\) has length \(n - r - 1\)

\(i \leq e\) can be replaced with \(i < n\) only if \(e = n - 1\), which means that we are using default value for 0-index language or \(b\) has constant value 0 through the program, \(b = 0\)

If we have the size of an array, in order to cover all possible combinations, similar to those above, we will use the general expression for ranges mostly in this form

\[
u \leq i < w \quad n = w - u
\]

This expression is connecting the length \(n\) and the position \([8]\) in a simple and obvious way: the number of positions is simply \(n = w - u\). Additionally, it suits well for many programming languages that require a strict initial bound. If we have any other expression we compare with this expression and:

- switching \(\leq\) to \(<\) is a **reduction by one** and we have to subtract 1 from the difference
- switching \(<\) to \(\leq\) is a **promotion by one** and we have to add 1 to the difference

Another advantage is that excluding elements from the beginning or end of an array becomes automatic. To exclude \(g\) elements from the beginning, we simply write \(g \leq i < n\) (\(n\) is not the size of an array any longer, if we need to work with the new size it will be \(g \leq i < g + (n - g)\), \(n - g\) is a new size) or if the initial position is \(b\) then \(b + g \leq i < b + n\) (or \(b + g \leq i < b + g + (n - g)\), where \(n - g\) is a new size). Excluding \(h\) elements from the end becomes \(0 \leq i < n - h\) or with other value for \(b\), \(b \leq i < b + (n - h)\) (\(n - h\) is a new size of an array in this case).
Remember that the left side of $u \leq i < w$ is inclusive and right side excluding.
If we need $k$ elements to the left of the element at $p$, excluding $p$, we have $p-k \leq i < p$. If we need $k$ elements including $p$, it becomes $p-k+1 \leq i < p+1$ which becomes, after the promotion on the right side, and reduction on the left $p-k < i \leq p$.
If we need $k$ elements to the right of $p$, including $p$, the expression becomes $p \leq i < p+k$, and excluding $p$ it is then $p+1 \leq i < p+k+1$ or after reduction on the left side and promotion on the right it becomes $p < i \leq p+k$.
Both expressions are easy to understand. The expression $p-k < i \leq p$ is easy to understand because we say including $p$, this is the reason we have $\leq p$. Equally $p < i \leq p+k$ is clear since we say excluding $p$, which is why we have $p <$. However, either way, they both have $k$ elements.
Overall this means that even the expression $u < i \leq w$ is keeping the rule $n=w-u$.
We show shortly how to use the rules for $b \leq i \leq b+r$. We notice that from expected $... \leq ... < ...$ we have changed one $<$ to $\leq$ which is a promotion, thus adding 1 to the result, so the number of positions in this expression is $(b+r)-b+1=r+1$.
If we have only right and left bounds then we express the range as $b \leq i \leq e$ with $n=e-b+1$ since we have a promotion on the right side. Overall we have this table:

| $u \leq i \leq w$ | Number of positions | Used expressions | Adjustment to $w-u$ |
|-------------------|---------------------|-----------------|-------------------|
| $u \leq i < w$   | $n=w-u+1$           | $\leq\leq$     | $+1$              |
| $u < i \leq w$   | $n=w-u$             | $\leq\leq$     | $0$               |
| $u < i < w$      | $n=w-u-1$           | $\leq\leq\leq$ | $-1$              |

You could make a mnemonic rule that using $<$ has a hidden penalty of -1/2, and on the other hand, $\leq$ has a cost of +1/2, if they are used together the penalties cancel each other, but if we use two $<$ we have a total cost of -1, while two $\leq$ need +1 adjustment. To complete the summary we are adding:

- switching from $u \leq i < w$ to the form $b \leq i < b+n$ $u \leq i < u+(w-u)$
- switching from $u \leq i < w$ to the form $b \leq i \leq e$ $u \leq i \leq w$-1

Now, back to the division problem. Let us see what happens if we try to use the expression $n/2$ directly to split the array into half assuming $b=0$.

**Ranges with central position included**

| number of elements | $n=1$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ |
|--------------------|-------|-------|-------|-------|-------|-------|
| 1                  | 0     | 0     | 0     | 1     | 1     | 2     |
| 2                  | 0     | 0     | 0     | 1     | 1     | 2     |
| 3                  | 1     | 1     | 2     | 2     | 3     | 3     |
| 4                  | 0     | 0     | 1     | 2     | 2     | 2     |

*includes central position

**Ranges that exclude some element around center if not the center itself**

| number of elements | $n=1$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ |
|--------------------|-------|-------|-------|-------|-------|-------|
| 1                  | 0     | 0     | 0     | 1     | 1     | 2     |
| 3                  | 0     | 0     | 0     | 1     | 1     | 1     |
Problems:

- $0 \leq i \leq n/2$ for $n=-1$ could mean $0 \leq i \leq 0$ or $0 \leq i \leq -1$, because some languages have different implementation for integer division for negative values.
- For $n=0$, the expression $0 \leq i \leq n/2$ is incorrect.
- Pair of expressions 1 and 4 has different distribution of halves than 2 and 3. With 2 and 3 the left half is always larger.
- The only useful pair of expressions that attempts to exclude the central position is $0 \leq i < n/2$ and $n/2 < i < n$. However, with these two, one element is always excluded regardless of the number of elements. If we do not want to extract any element at all when $n$ is even, because the center is not well defined for $n$ even, using only $n/2$ cannot help.

Let us dig deeper into the problem.

**01234 and 012345 and (n±1)/2 case**

We read and calculate first what we need from the array **01234** $b=0$, $e=4$, $n=e-b+1=5$, $n$ is odd

Then, we will calculate $(n-1)/2$, $n/2$, $(n+1)/2$ and $(s-1)/2$, $s/2$, $(s+1)/2$, $s=(e-b)=n-1=4$

| (n-1)/2 | n/2 | (n+1)/2 |
|---------|-----|---------|
| 2       | 2   | 3       |

| (s-1)/2 | s/2 | (s+1)/2 |
|---------|-----|---------|
| 1       | 2   | 2       |

We read and calculate next the array **012345** $b=0$, $e=5$, $n=e-b+1=6$, $n$ is even

We will calculate $(n-1)/2$, $n/2$, $(n+1)/2$ and $(s-1)/2$, $s/2$, $(s+1)/2$, $s=(e-b)=n-1=5$

| (n-1)/2 | n/2 | (n+1)/2 |
|---------|-----|---------|
| 2       | 3   | 3       |

| (s-1)/2 | s/2 | (s+1)/2 |
|---------|-----|---------|
| 2       | 2   | 3       |

We immediately notice that there is no common rule for $n/2$. It points to the center only if $n$ is odd. If $n$ is even it points only to the closest element to the right, closest to the physical center of the array, 2.5, rounded **up**. On the other hand, $s/2$ does point to the center if $n$ is odd, but it points to the closest element to the left of the center, closest to the physical center of the array, 2.5, rounded **down**.

Because there is no central position of an array in every situation, it is better to split an array into **left** and **right half**, with the center in case $n$ is odd, or without a center, if $n$ is even. We mark two special elements, end of left half, $el$, and start of right half, $rs$. In case $n=1$, these two are not well defined.

| Array in case $n$ is even |
|---------------------------|
| $el$ | $rs$ |
| left part | right part |

| Array in case $n$ is odd |
|--------------------------|
| $el$ | $c$ | $rs$ |
| left part | center | right part |
Summary for two cases

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| el | c | rs |   |   |   |
|   | (n-1)/2 | (n+1)/2 |   |   |   |
|   | n/2 |   |   |   |   |
|   | (s-1)/2 | s/2 |   |   |   |
|   | (s+1)/2 |   |   |   |   |

Table 1 The summary for 01234

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| el | rs |   |   |   |   |   |
|   | (n-1)/2 | n/2 |   |   |   |   |
|   | (n+1)/2 |   |   |   |   |   |
|   | (s-1)/2 | (s+1)/2 |   |   |   |   |
|   | s/2 |   |   |   |   |   |

Table 2 The summary for 012345

Is there any rule? Regarding n/2 and s/2, there is none. The only rule that can be extracted is given by the next proposition

**Proposition (n±1)/2 case**

Given all the above definitions, and the operation / taken as integer division, for \( n>1 \), \((n+1)/2\) always points to the first position of the right half of an array, and \((s-1)/2\) always points to the last position of the left half of an array. For \( n=1 \), \((s-1)/2\) points to the only element in the array and \((n+1)/2\) points outside an array. For \( n=0 \), \( s \) is not defined, (unless we define it indirectly as \( n-1 \)), and \((n+1)/2\) points outside an array. For \( n=0 \), if we define \( s \) indirectly as \( n-1 \), then \((s-1)/2\) is outside the array as well.

**Proof.** We can easily extend the above cases 01234 and 012345 inductively two by two elements and observe the same position for all mentioned values: \((n+1)/2, n/2, (n-1)/2, (s+1)/2, s/2, (s-1)/2\). We have started from \( n=4 \) and \( n=5 \), but there is no difference in reasoning for \( n=2,3,4 \). For \( n=1 \), we have \((n+1)/2=1 \) and \((s-1)/2=0 \), which is as in the proposition. For \( n=0 \), we have \((n+1)/2=0 \) and \( s \) is not defined because there are no elements in the array, both as in the proposition as well. If we define \( s \) indirectly, not as \( e-b \), but as \((n-1)/2\), then for \( n=0 \) we have \((n-1)/2=0 \), which is defined but outside of an array.

Overall, think about an array with three sections. Check if \( n=0 \) and \( n=1 \) are two special cases. Because in general, there is no center of an array, you have to decide what type of division you need. Then, you apply a specific range to it:

- **Natural division** divides an array in two equal halves. The central positions, if exists, is not included.
- **Left+ division** is splitting the array so that the center, if exists, is included in the left half of an array, otherwise it is the same as natural division.
- **Right+ division** is splitting the array so that the center, if exists, is included in the right half of an array, otherwise it is the same as natural division.
- **Cut-out center division** is splitting the array so that one element at the center is always excluded from both halves. In case we have no center, we exclude the element closest to the physical center of the array, either left or right.
Cut-out center division is excluding the center, if that one exists, or it is excluding \( el \) or \( rs \) respectively.

We have one expression for Cut-out center already: we simply exclude \( s/2 \) or \( n/2 \). Cut-out center cannot be balanced; it always uses one of the halves more.

However, there are two problems. First, instead of \( n/2 < i < n \) or \( s/2 < i < n \) we would prefer having a version \( x \leq i < n \), with \( \leq \) on the left side, because many programming languages require specifying the lower bound. Second, we would like to have other versions of division somehow connected. Because of that, we will develop expressions for Cut-out division from other expressions.

The rules for left and right half of an array, using \( s \) and \( n \) where the scope of \( i \) is over integers follows.

If we use \( s=n-1 \) we have

If the index does not start with 0, but rather with \( b\neq0 \), we just add \( b \), and then we end the other expression with \( e \), observe the change \( < \) to \( \leq \) for \( e \). everything else is the same.
Let us check the cases $n=0$ and $n=1$ for Left+ division. Left side $0 \leq i < \frac{(n+1)}{2}$, right side $\frac{(n+1)}{2} \leq i < n$.

If Left+ division is used then

- $n=0$ gives left side $0 \leq i < 0$, right side $0 \leq i < 0$, so array is not accessed at all
- $n=1$ gives left side $0 \leq i < 1$, right side $1 \leq i < 1$, so array is accessed only for $i=0$

Both results are bug-free, i.e. we access each element in the array respecting its bounds.

For $n=0$ and $n=1$, although apparently correct results, Right and Natural division may have a subtle bug in calculating the range. For $n=1$, we have $(n-2)/2=\frac{-1}{2}$, which can be $-1$ if integer division is implemented as literally rounding down integer, like in Python, or $0$ otherwise.

$n=0$ this is correct

- Natural division: left side $0 \leq i \leq -1$, right side $0 \leq i < 0$
- Right+ division: left side $0 \leq i \leq -1$, right side $-1 < i < 0$

$n=1$ may have a problem with both divisions

- Natural division: left side $0 \leq i \leq \frac{(-1)}{2}$, right side $1 \leq i < 1$
- Right+ division: left side $0 \leq i \leq \frac{(-1)}{2}$, right side $\frac{-1}{2} < i < 1$

The problem with Natural division is obvious; we may not be able to access the array at all. The problem with Right+ division is not so apparent at first. If $(-1)/2=0$ then we will access one existing element through the condition on the left, $0 \leq i \leq 0$, because the other expression $0 < i < 1$ does not have a solution. However, if $(-1)/2=-1$ then it is the other way around, $0 \leq i \leq -1$ has no solution and $-1 < i < 1$ will have one solution, $i=0$. This, unfortunately, may be two very different situations, because we do not have to execute the same code under both constraints, and testing might not reveal the problem immediately.

In case Natural or Right+ division is used as it is above, it would be probably best to treat $n=0$ and $n=1$ as two special cases. Fortunately, we do not have to deal with it.

01234 and 012345 and n/2±1 case

We read and calculate first what we need from the array 01234 $b=0$, $e=4$, $n=e-b+1=5$, $n$ is odd
Then we will calculate $n/2-1$, $n/2$, $n/2+1$ and $s/2-1$, $s/2$, $s/2+1$, $s=(e-b)=n-1=4$

| 01234 | 01234 |
|---|---|
| $n/2-1$ | $s/2-1$ |
| $n/2$ | $s/2$ |
| $n/2+1$ | $s/2+1$ |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |

We read and calculate next the array 012345 $b=0$, $e=5$, $n=e-b+1=6$, $n$ is even
And we will calculate $n/2-1$, $n/2$, $n/2+1$ and $s/2-1$, $s/2$, $s/2+1$, $s=(e-b)=n-1=5$

| 012345 | 012345 |
|---|---|
| $n/2-1$ | $s/2-1$ |
| $n/2$ | $s/2$ |
| $n/2+1$ | $s/2+1$ |
| 2 | 1 |
| 3 | 2 |
| 4 | 3 |
Summary for two cases

|   | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
|   | \(el\) | \(c\) | \(rs\) |   |
| \(n/2-1\) | \(n/2\) | \(n/2+1\) |   |
| \(s/2-1\) | \(s/2\) | \(s/2+1\) |   |

Table 3 The summary for 01234

|   | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
|   | \(el\) | \(rs\) |   |
| \(n/2-1\) | \(n/2\) | \(n/2+1\) |   |
| \(s/2-1\) | \(s/2\) | \(s/2+1\) |   |

Table 4 The summary for 012345

Again, is there any rule? The rule we can extract is given by the next proposition

**Proposition \(n/2\pm1\) case**

Given all the above definitions, and the operation / taken as integer division, for \(n>1\), \(s/2+1\) points to the first position of the right half of an array, and \(n/2-1\) to the last position of the left half of the array. For \(n=1\), \(s/2+1\) points to the only element in the array and \(n/2-1\) points outside the array. For \(n=0\), \(s\) is not defined, (unless we define it indirectly as \(n-1\)) and \(s/2+1\) points outside the array. For \(n=0\), if we define \(s\) indirectly as \(n-1\), then \(s/2+1\) is outside the array as well. •

**Proof.** The same reasoning as for proposition \((n\pm1)/2\). However \(n=2\) must be treated as a special case.

We read and calculate everything for the array \(01\) \(b=0\), \(e=1\), \(n=e-b+1=2\), \(n\) is even.

We calculate \(n/2-1\), \(n/2\), \(n/2+1\) and \(s/2-1\), \(s/2\), \(s/2+1\), \(s=(e-b)=n-1=1\)

|   | 01 | 01 |
|---|----|----|
| \(n/2-1\) | \(n/2\) | \(n/2+1\) |
| 0 | 1 | 2 |

|   | -1 | 0 | 1 | 2 |
|---|----|---|---|---|
| \(s/2-1\) | \(s/2\) | \(s/2+1\) |
| -1 | 0 | 1 |

From the table it is obvious that the proposition is correct for \(n=2\) as well.

\(\Box\)

The summary for \(n/2\pm1\) case follows

|               | left half | right half |
|---------------|-----------|------------|
| Natural division | \(0 \leq i \leq n/2-1\) | \(s/2+1 \leq i < n\) |
| Left+ division | \(0 \leq i < s/2+1\) | \(s/2+1 \leq i < n\) |
| Right+ division | \(0 \leq i \leq n/2-1\) | \(n/2-1 \leq i < n\) |
The same table in case we replace \( s=n-1 \)

|                      | left half                  | right half                |
|----------------------|----------------------------|----------------------------|
| Natural division     | \( 0 \leq i \leq n/2-1 \) | \( (n-1)/2+1 \leq i < n \) |
| Left+ division       | \( 0 \leq i < (n-1)/2+1 \) | \( (n-1)/2+1 \leq i < n \) |
| Right+ division      | \( 0 \leq i \leq n/2-1 \)  | \( n/2-1 \leq i < n \)    |

Although correct, the combination \( n/2-1 \) and \( (n-1)/2+1 \) is somewhat difficult to memorize and quickly justify. With this version of \( n/2\pm 1 \) expressions, we do not have a problem with \( -1)/2 \) for Natural and Left+ division, because for \( n=0 \) \( (n-1)/2+1 \leq i < n \) becomes \( -1)/2+1 \leq i < 0 \) which has no solution for \( i \) even if \( -1)/2=-1 \). We have no problem for \( n=0, n=1 \), with Right+ division either.

- \( n=0 \) gives left side \( 0 \leq i \leq -1 \), right side \(-1 < i < 0 \), which means no access - correct
- \( n=1 \) gives left side \( 0 \leq i \leq -1 \), right side \(-1 < i < 1 \), gives access for \( i=0 \), which is correct

Apart from tricky mnemonic rules, \( n/2\pm 1 \) approach is giving the right results for all divisions even for special cases \( n=0 \) and \( n=1 \).

Before we continue, we will mention one important connection between Left+ and Right+ divisions:

- Left+ division with first element removed from an array becomes Right+ division of the new array
- Right+ division with last element removed from an array becomes Left+ division of the new array

For example, removing the first element is done by \( b=b+1 \), and the last by \( n=n-1 \).

### Equivalent expressions

The expressions in \( (n\pm 1)/2 \) and \( n/2\pm 1 \) for ranges in Normal, Left and Right+ divisions are somewhat equivalent. We will make a table to display this clearly. They are not equivalent if integer division is implemented in a different way, neither for \( n=0 \) and \( n=1 \), so we need to check this all additionally - at least for some small negative numbers.

| Equivalent expressions for integer division except for \( n=0 \) and \( n=1 \) |
|-------------------------------|-------------------------------|
| \( n/2-1 \)                   | \( (n-2)/2 \)                 |
| \( (n-1)/2+1 \)               | \( (n+1)/2 \)                 |

If we try to replace \( (n-2)/2 \) with \( n/2-1 \) we might resolve the problem with \( -1)/2 \) for \( n=1 \)

|                      | left half                  | right half                |
|----------------------|----------------------------|----------------------------|
| Natural division     | \( 0 \leq i \leq n/2-1 \)  | \( (n+1)/2 \leq i < n \)  |
| Left+ division       | \( 0 \leq i < (n+1)/2 \)   | \( (n+1)/2 \leq i < n \)  |
| Right+ division      | \( 0 \leq i \leq n/2-1 \)  | \( n/2-1 \leq i < n \)    |

Let us check.

- \( n=0 \) is correct
  - Natural division: left side \( 0 \leq i \leq -1 \), right side \( 0 \leq i < 0 \)
  - Right+ division: left side \( 0 \leq i \leq -1 \), right side \(-1 < i < n \)

- \( n=1 \) is correct as well
  - Natural division: left side \( 0 \leq i \leq -1 \), right side \( l \leq i < l \)
  - Right+ division: left side \( 0 \leq i \leq -1 \), right side \(-1 < i < l \)

Besides all above transformations, we have few options more. For example, if we are already in the loop that handles the left half of an array, and we need to access the right half, we can always use symmetry \( (i, n-1-i) \) between them. Additionally, we can use as well the fact that \( i < t+1 \) is equivalent to \( i \leq t \) for integers as mentioned in promotion/reduction rules before. We will exploit this last in the next chapter.
Regarding $b$ and $e$, we mention here a nice expression that connects $s$ and $b+e$: $(b+e)/2 = b+(e-b)/2 = b+s/2$

Selected expressions

We will make a selected solution in form of another proposition. Additionally, we will make an adjustment for expected positions of $\leq$ and $<.$

**Proposition** *All correct divisions* Given all the above definition, the following table represents correct boundaries for Left+, Right+ and Natural division and additionally all of them cover $n=0$ and $n=1$ cases correctly.

|                | left half                  | right half                |
|----------------|----------------------------|----------------------------|
| Natural division | $0 \leq i \leq n-1-(n+1)/2$ | $n-n/2 \leq i < n$        |
| Left+ division      | $0 \leq i < n-n/2$         | $n-n/2 \leq i < n$        |
| Right+ division     | $0 \leq i \leq n-1-(n+1)/2$ | $n-1-(n+1)/2 < i < n$     |

Proof

We have already proven that all expressions are giving the correct results for $n>1$, either by previous propositions or by exploiting equivalent expressions. We shortly summarize $n=0$ and $n=1$ cases.

| n=0                | left half                  | right half                |
|-------------------|----------------------------|----------------------------|
| Natural division  | $0 \leq i \leq 0/2-1=-1$  | $(0+1)/2=0 \leq i < 0$    |
| Left+ division    | $0 \leq i < (0+1)/2=0$    | $(0+1)/2=0 \leq i < 0$    |
| Right+ division   | $0 \leq i \leq 0/2-1=-1$  | $0/2-1=-1 < i < 0$        |

| n=1                | left half                  | right half                |
|-------------------|----------------------------|----------------------------|
| Natural division  | $0 \leq i \leq 1/2-1=-1$  | $(1+1)/2=1 \leq i < 1$    |
| Left+ division    | $0 \leq i < (1+1)/2=1$    | $(1+1)/2=1 \leq i < 1$    |
| Right+ division   | $0 \leq i \leq 1/2-1=-1$  | $1/2-1=-1 < i < 1$        |

It is obvious that all cases are correct. For $n=0$ there is no access at all anywhere. For $n=1$, Natural division has no access and Left and Right have one access with correct index 0 on the correct side of the branching. 'left half' for Left+ and 'right half' for Right+. This last has to be so because the center always belongs to left half for Left+ and to right half for Right+ division.

**Proposition** *Cut-out center division* The expression for right Cut-out center division is the same as Right+ division with first element removed from the right side, and for left Cut-out center divisions as Left+ division with last element removed from the left side.

Proof

This follows from the definition of the Right+ and Left+ divisions. □
Finally, in case we are dividing an array in two, and we want to have cases \( n=0 \) and \( n=1 \) included, bullet-proof expressions for ranges shall use only \( n/2-1 \) and \( (n+1)/2 \). Although these expressions include cases \( n=0 \) and \( n=1 \), we can guarantee that the program will access the array correctly, nothing more. If these cases are special, they should be treated so.

|                      | left half                  | right half                  |
|----------------------|---------------------------|----------------------------|
| Natural division     | \( 0 \leq i \leq n/2-1 \) | \( (n+1)/2 \leq i < n \)  |
| Left+ division       | \( 0 \leq i < (n+1)/2 \)  | \( (n+1)/2 \leq i < n \)  |
| Right+ division      | \( 0 \leq i \leq n/2-1 \) | \( n/2 \leq i < n \)     |
| Cut-out center division | left cut | \( 0 \leq i < (n+1)/2-1 \) | \( (n+1)/2 \leq i < n \) |
|                      | right cut                 | \( 0 \leq i \leq n/2-1 \) | \( n/2 < i < n \) |

Since Right+ division is the only one that uses \(< \) for right half, we give the expression with \( \leq \), because it is normal to define an initial index in many programming languages, for example, for various loop expressions. We include Cut-out center division as well to have it all in one place.

|                      | left half                  | right half                  |
|----------------------|---------------------------|----------------------------|
| Natural division     | \( 0 \leq i < n/2 \)      | \( (n+1)/2 \leq i < n \)  |
| Left+ division       | \( 0 \leq i < (n+1)/2 \)  | \( (n+1)/2 \leq i < n \)  |
| Right+ division      | \( 0 \leq i \leq n/2 \)   | \( n/2 \leq i < n \)     |
| Cut-out center division | left cut | \( 0 \leq i < (n+1)/2-1 \) | \( (n+1)/2 \leq i < n \) |
|                      | right cut                 | \( 0 \leq i \leq n/2 \)   | \( n/2 + 1 \leq i < n \) |

Observe that both sides in Natural division have length \( n/2 \). We repeat the same expressions with \( b \) and \( e \) and \( n \), or just \( b \) and \( n \)

|                      | left half                  | right half                  |
|----------------------|---------------------------|----------------------------|
| Natural division     | \( b \leq i < b+n/2 \)    | \( b+(n+1)/2 \leq i \leq e \) \((or < b+n)\) |
| Left+ division       | \( b \leq i < b+(n+1)/2 \) | \( b+(n+1)/2 \leq i \leq e \) |
| Right+ division      | \( b \leq i < b+n/2 \)    | \( b+n/2 \leq i \leq e \)  |
| Cut-out center division | left cut | \( b \leq i < b+(n+1)/2-1 \) | \( b+(n+1)/2 \leq i \leq e \) |
|                      | right cut                 | \( b \leq i < b+n/2 \)    | \( b+n/2+1 \leq i \leq e \) |

We revisit the symmetrical expressions as well.

|                      | left half                  | right half                  |
|----------------------|---------------------------|----------------------------|
| Natural division     | \( 0 \leq i < n-(n+1)/2 \)| \( n-n/2 \leq i < n \)   |
| Left+ division       | \( 0 \leq i < n-n/2 \)    | \( n-n/2 \leq i < n \)   |
| Right+ division      | \( 0 \leq i < n-(n+1)/2 \)| \( n-(n+1)/2 \leq i < n \) |

Nice to remember a simple formula that speaks everything
\[
n = \frac{n}{2} + \left\lfloor \frac{n+1}{2} \right\rfloor
\]

**Corollary** Calculating the length of the region \( (n+1)/2 \leq i < n \) using the above formula
\[
n - \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor
\]
More generally, if we need to split up an array into several close to equal sections, this formula can help
\[\sum_{r=0}^{k-1} \left\lfloor \frac{n+r}{k} \right\rfloor = n \quad k > 1\]

We could start from this formula, but the purpose of this exercise is to show the nuances in implementing an apparently correct mathematical formula within any programming language. However, to illustrate its usage, we will write the range expressions using only \(b\) and \(e\), without employing \(n\). For the range \(b \leq i \leq e\) which is \(b \leq i < e + 1\) we have:

\[b + e + 1 = \left\lfloor \frac{b + e + 1}{2} \right\rfloor + \left\lfloor \frac{b + e + 2}{2} \right\rfloor = \left\lfloor \frac{b + e + 1}{2} \right\rfloor + \left( \left\lfloor \frac{b + e}{2} \right\rfloor + 1 \right)\]

(We could, of course, repeat the entire analysis.) If we use only \(b\) and \(e\), and replace \(b+e = m\) from the above formula, we have, more or less directly, nice expressions given in the table below. They work even for \(n=1\), providing we agree that in that situation \(e=b, m=2b\).

If we need to cover \(n=0\) using the same formulas, we must have \(e \leq b-1, m \leq 2b-1\). Obviously, \(n=0\) and \(n=1\) are not natural cases if we want to express the ranges using \(b\) and \(e\) only. Observe that for \(n=2\), case \(b=0\) and \(e=1\) will give \((e+b)/2=0\) which is going back to the value of \(b\), making a dead loop a very possible outcome if you apply a division incorrectly.

For these expressions, it is obvious that left Cut-out center division looks more natural. Observe that similar expressions \(b \leq i < m/2\) and \(b \leq i < b+n/2\) do not behave the same way: \(b \leq i < m/2, m/2 < i \leq e\) cuts out element at \(s/2\), \(b \leq i < b+n/2, b+n/2 < i \leq e\) cuts out element at \(n/2\).

It is not difficult to find a pattern in all equations: for expressions with \(n\) we use only \(n/2, (n+1)/2\) and for \(b\) and \(e\) we use \((m+1)/2, (m+2)/2=m/2+1\), with the rule about excluding last left or first right element.

You could notice that we have avoided all expressions with \((n-1)/2\), since they do not work for \(n=0\). We introduce this connecting formula instead, with the constraint \(n>0\).

\[\frac{n + 1}{2} - \frac{n - 1}{2} = 1 \quad n > 0\]

An interesting derivative of the above Natural and right divisions is the one using \(b\) and \(e+1\) only.

### Using \(b\) and \(e+1\) with desirable \(\leq\) and \(<\) arrangement

|                | left half                               | right half                              |
|----------------|-----------------------------------------|-----------------------------------------|
| Natural division | \(b \leq i < (b+(e+1))/2\) \((b+(e+1)-1)/2+1 \leq i < e+1\) | \(b+(e+1)/2 \leq i < e+1\)             |
| Right+ division | \(b \leq i < (b+(e+1))/2\) \((b+(e+1))/2 \leq i < e+1\) | \(b+(e+1)/2+1 \leq i < e+1\)           |
| Cut-out center division | **right cut** \(b \leq i < (b+(e+1))/2\) \((b+(e+1)+1)/2 \leq i < e+1\) | \(b+(e+1)+1 \leq i < e+1\)             |

If we stay within the same division, the expressions for ranges with \((b, e), (b, n), (b, e+1)\) are all equivalent, so we can combine them the way we like. In the next example, in \((b, e)\) expressions, we use \(b+n/2\) for the center.

### An example of deriving a combined expression which is using \(b\) and \(e+1\) (but not \(b+e\)) and still has \(\leq\) and \(<\)

|                | left half                               | right half                              |
|----------------|-----------------------------------------|-----------------------------------------|
| Cut-out center | **right cut** \(b \leq i < b+n/2\) \ we use \(n=e-b+1\) \(b+n/2+1 \leq i < e+1\) | \(b+(e+1)/2 \leq i < e+1\)             |
|                | \(b \leq i < ((e+1)+b)/2\)              | \(b+(e+1)+b \leq i < e+1\)             |
|                | \(b \leq i < b+(ex-b)/2\)              | \(b+(ex-b)/2+1 \leq i < ex\)           |
Reaching the center

If we want to reach strictly the center of an array, or nothing in case $n$ is even, reading Table 1, 2, 3, 4 gives several options

1. $\frac{n-2}{2} < i < \frac{(n+1)}{2}$
2. $\frac{n}{2} < i < \frac{(n-1)}{2} + 1$
3. $\frac{n}{2} \leq i \leq \frac{(n-1)}{2}$

Option 3. is not a typo, we just exploit the fact that $n/2 = (n-1)/2$ if $n$ is odd and $(n-1)/2 < n/2$ otherwise. Unfortunately, we cannot avoid the problem with negative values like $n=-1$, if integer division is implemented in various ways, since we do not have any test against $n$ or 0, and both sides are linear functions. Option 2. and option 3. are the problem because they do not work for $n=0$ at all. The only option is to combine some of the above, and the most optimal choice seems to be

$$\frac{n}{2} \leq i < \frac{n+1}{2} \quad n \geq 0$$

This expression is actually an excellent fit with other expressions, including our agreement on $\leq$ and $<$ usage. It is not difficult to memorize it because it points to the section between right and left half of Natural division. Overall, we are not interested just in central position, and this expression becomes an indispensable tool when we want to manipulate a couple of elements symmetrically placed around the center. For example, this is a central subarray with $k$ elements to the left and $k$ elements to the right from the center (including the central position if exists) $\frac{n}{2} - k \leq i < \frac{(n+1)}{2} + k$. If we use the conversion rules like $(n+1)/2 = n-n/2$, we have maybe more familiar formulation $\frac{n}{2} - k \leq i < n-n/2 + k$. However, if we write this directly, the part to the right $< n-n/2 + k$ would require a small mental testing routine on its own to realize why we need $<$ and not $\leq$ and why we do not have $+1$ and so on. This is what the text is about, pointing out a strict and unmistakable set of rules.

Example says it all

All above sounds useful but we have to show one clear example or practical usage. We will write a simple binary search. Why binary search? It seems so obvious and nice elementary problem and solution that nobody ever agreed on its final specification. Do we exclude element in the center if it does not match? If we do how do we take care of bounds? And so on. What does a programmer need more but his naked imagination?

Binary search:

Find an element in a sorted array using division of the array based on comparison between central elements and lookup element, as well as the fact that all elements in a sorted array to the left of any element are smaller than or equal to, and on the right are greater than or equal to the observed element.

Solution:

We will first use Right+ division. We will ask if the first element of the right half is greater than the lookup element. If so, we continue using left half of Right+ division, otherwise we use its right half. As a reminder: Right+ division is $b \leq i < b+n/2$ and $b+n/2 \leq i < b+n$. We use C/C++. Observe that right half of Right+ division always has at least one element, so it must have the first element. The code is straightforward because all basic elements are cleared, no special effort is needed any longer[9].
int binarySearch (int a[], int n, int t)
{
  if(n<=0) return -1;   // special treatment, we need at least one element to start our game
  int b=0;
  while(n > 1) { // first position b=0, and for size we have n, so it is full array, for n>1, b is in the left half
    if(a[b+n/2] > t) { // n > 1 condition above and constant dividing n/2 will force the loop out eventually
      n = n/2;     // if it is greater, target left half next: so b remains left, while n shrinks to b+n/2-b=n/2
    } else {        // if it is not greater than t, we switch to the first position of right half
      b = b + n/2;
      n = n - n/2;        // if this is the correct size of right half, it is as it is written n-n/2, not(() n/2
      // the rule is indeed: if you move the start position up - shrink the length by equal amount
      // n is incrementing here (or staying the same for n==1)
    }
  }
  if(a[b] == t) return b;    // either we found it since n=1 and we return the position ...
  return -1;                  // ... or we did not find it so we return the signal back
}

\( n-n/2 \) could be written as \((n+1)/2\) which allows further optimization. Equally simple is the code if we have decided to keep excluding the central element that does or does not match the lookup element. Classical solution, seen in most books, which is using \( e, b \) and \((e+b)/2\), with exclusion, can be derived from left Cut-out center division and made strict, because it may not be obvious from the classical code that \( b \) and \( e \) values will remain within the array bounds. Didactically, the classical \( b, e \) solution may be a wrong first choice, intuitive as it seems. Precise programming almost never matches our intuitive feeling about the solution.

Let us try to use right Cut-out division: \( b \leq i < b+n/2 \) and \( b+n/2+1 \leq i < b+n \) with some typical optimization this time.

int binarySearch (int a[], int n, int t)
{
  int r;
  int b=0;
  while(n > 0) { // if n=0 we did not find it
    r=b+n/2;
    // save and reuse the reference position
    if(a[r] == t) return r;    // we found it, return the position, otherwise we can cut out the position at r
    if(a[r] < t) { // since we deal with right cut, this is a check for staying on the right side
      b = r + 1;
      n = (n-1)/2;
      // check this above: n= \((b+n) - (b+n/2+1) = n-(n/2)-1=(n+1)/2-1, \)
      // here we have n>0, and we know that \((n+1)/2-(n-1)/2=1\) so \((n+1)/2-1=(n-1)/2\)
    } else { // otherwise we stay to the left
      n = n/2;
    }
  }
  return -1;
}
Well, that last *else* we can really squeeze into one statement:

```c
int binarySearch (int a[], int n, int t)
{
    int r;
    int b=0;
    while(n > 0)
    {
        r=b+ n/2;
        if(a[r] == t) return r;
        if(a[r] < t)
        {
            b = r + 1;
            n = n - 1; // we will divide by 2 below anyway
        }
        n = n/2; // we have incremented n maybe, and now we divide it by 2 so that together will exit the loop eventually
    }
    return -1;
}
```

We have the same construction steps for any other algorithms that use integer bounds: choose proper bound expression, *b* and *n*, *b* and *e* and *n*, *b* and *e*, proper conditions ≤ and <, then proper cut if that one is used, and only then using acceptable transformation switch to more optimal expressions. Writing a code this way becomes similar to writing a mathematical formula correctly. You do not have to go back and understand why your attempts have failed, looking for the bug, spending additional time fixing your reasoning in iterations. All you need to do is to rewrite the formula correctly. The bug, if there, can guide you to the error you made in deriving the code, not to the construction error that needs to be fixed.

Programming is not full of tricks. The tricks fail.

**Conclusion**

This work clearly displays the problems of using quick ad hoc rules for boundaries of an array, in case we want to, for example, divide an array. A bug can be very subtle and noticed only when some later refactoring is attempted.

The final solution is an example of concise and justified expressions for dividing an array under various assumptions. The key point is in observing an array as a 3-part object. When we decide how to treat the array and what program or algorithm really needs, we can decide which expression to apply. It is important to notice that with Natural division, the start of the right half is not a clear increment of the end of the left half. In memorizing this, it is best to memorize Left+ division first, then a rule to exclude the center for Natural division, and finally, Right+ division will come up by itself.

Example of *for* loop in C/C++ for all divisions.

|               | left half                      | right half                    |
|---------------|-------------------------------|-------------------------------|
| Natural division | `for (int i = 0; i < n/2; i++)` | `for (int i = (n+1)/2; i < n; i++)` |
| Left+ division   | `for (int i = 0; i < (n+1)/2; i++)` | `for (int i = (n+1)/2; i < n; i++)` |
| Right+ division   | `for (int i = 0; i < n/2; i++)` | `for (int i = n/2; i < n; i++)` |
| Right cut-out    | `for (int i = 0; i < n/2; i++)` | `for (int i = n/2+1; i < n; i++)` |
Appendix

To illustrate the problem of switching from positional value to length (dimensional value) look at the sort algorithm below.

```c
void swap(int a[], int i, int j)
{
    int t = a[i]; a[i] = a[j]; a[j] = t;
}

void quicks(int a[], int n)
{
    if (n<=0) return;

    int t=a[0], k=0, m;
    for(int i=1; i<n; i++)
    {
        if(a[i]<t)
        {
            swap(a,k,i);
            k++;
            swap(a,k,i);
        }
        a[k]=t;
    }
    quicks(a,k); // there is an implicit conversion from positional to dimensional system through k
    k++;
    quicks(a+k,n-k) ; // there is an implicit formula for conversion through n-k and a+k
}
```

The definition of quicks(int a[], int n) means quicks(array, size) which is quicks(object, dimension). In the first above highlighted code line quicks(a,k); k is a positional argument so there is a silent conversion from position to length, positional argument to dimensional argument. There is no syntax help regarding this conversion. Another subtle swindle is that by passing k in quicks(a,k); we are silently removing kth element. It is even worse with the second highlighted expression, because a+k is hiding (object+position → object) conversion and in the second parameter (length+position → length) conversion. If a developer keeps reading the code in this manner, it is not strange that he or she starts mixing position and length.

This problem, which has its origin in shorthand syntax-optimization of code for hardware specialization and speed, remains even with many object-oriented languages. This can make code hard to read, understand and refactor.
Problems

1. Are the above ranges correct if we have an array with negative indices?

2. We have assumed that the index is increasing. If an array starts at 10 and ends at 2, what adjustment do we need to make in the expressions?

3. Write merge sort using natural division. How do you treat the central position?

4. Write binary search using other divisions that exclude (or not) central element if it does not match. Check the literature. Did they use the same expressions? What is a potential problem if they did not?

5. What happens if \( n, b, e \ldots \) is close to the maximum integer value?

6. The central position exists if \( n \) is odd. Imagine a string, an array of characters. A string contains words separated by spaces. A word is crossing the center if some of its characters belong to the left half, right half as well as the center of the array, if the center exists. For example, cat in ‘One cat above’ is not crossing the center, but word cats in ‘Two cats above’ is. Single letter word cannot cross the center. What is the shortest expression for checking if any word crosses the center? Do not search for words in the string.

7. If division and used variables \( n, i, p, b, e \ldots \) suddenly become real numbers, not integers, although they are still used to keep the indices of an array, what will happen with the given expressions? Do you need to fix anything?

8. Check if all expressions are correct even if \( n<0 \). Is there a problem if \((-1)/2=-1\)?

9. Recreate the simple expressions which use only \( b \) and \( e \), without \( n \) involved. What is going on in case \( n=0, n=1 \)? Why or when would you prefer using \( b \) and \( e \) only instead of \( n, b \) and \( e \)? (We recommend that you repeat the entire analysis for \( b \) and \( e \) case.)

10. We have mentioned that \((n-1)/2=n/2\) is the test for the central position. Can you write the test using \( b \) and \( e \)? If a language supports what do you think about simply checking the parity of the lowest bit of \( n \)? Is there anything a developer should think about before using that test?

11. Left and right cut-out center divisions are not balanced. Is there a way to make them balanced? Can we create a way to split an array that would always extract one element around the center but in a balanced way, not always only one side?

12. Write the missing symmetrical expressions for left and right cut-out divisions.

13. Quickly write a range for substring that starts at position \( p \) and has length \( s \). Quickly write a range that includes 5 characters from the middle of the string for \( n \) odd, and 4 for \( n \) even. Write a range for substring where \( g \) is a number of elements excluded from the beginning of the string and \( h \) from the end of the string. Write the correct expressions for 1-index and 0-index language.

14. Prove the formula

\[
\sum_{i=0}^{k-1} \left\lceil \frac{n+i}{k} \right\rceil = n
\]

What happens if \( k>n \)? Try using the formula to write three ranges that divide \( 0 \leq i < n \) into three equal or almost equal parts. Do it for Natural division, Right+ division and right Cut-out center division.

15. Chop-sorted array is an array \( a_0, a_1, a_2, \ldots, a_k \ldots \) that has several in-sorted elements \( b_j \) where all elements to the left are less than or equal to \( b_j \) and all elements to the right are greater than or equal to \( b_j \), but elements between two \( b_j \) and \( b_{j+1} \) do not have to be sorted (example in: 1,4,3,6,13,11,15,18; 1, 6, 15, 18 are in-sorted). Test if your solution can find in-sorted elements \( b_j \) in chop-sorted array.
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