Singular solutions for the constant $Q$–curvature problem

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Abstract

This paper is devoted to the construction of weak solutions to the singular constant $Q$–curvature problem. We build on several tools developed in the last years. This is the first construction of singular metrics on closed manifolds of sufficiently large dimension with constant (positive) $Q$–curvature.

Contents

1 Introduction 2

2 Preliminaries 5

2.1 Function spaces . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5

2.2 Fermi coordinates . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5

2.3 The singular solution . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

2.4 Approximate solutions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

2.4.1 Isolated singularities . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

2.4.2 Higher dimensional singularities . . . . . . . . . . . . . . . . . . . . . . 7

3 Linearized operator in $\mathbb{R}^N \setminus \{0\}$ 8

3.1 Indicial roots . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

3.2 Study of the linearized operator . . . . . . . . . . . . . . . . . . . . . . . . 10

4 Linearized operator on $\mathbb{R}^n \setminus \mathbb{R}^k$ 13

5 Injectivity of $L_\varepsilon$ on $C^{4,\alpha}_{\mu,\beta}(\Omega \setminus \Sigma)$ 15

6 Uniform surjectivity of $L_\varepsilon$ on $C^{4,\alpha}_{\mu,\beta}(\Omega \setminus \Sigma)$ 21

7 Fixed point arguments 25

8 Proofs of Theorems 1.1 and 1.2 28
1 Introduction

The last years have seen several important works on the $Q$–curvature problem in dimensions bigger or equal to five, since the discovery by Gursky and Malchiodi [GM15] of a natural geometric maximum principle associated to the Paneitz operator. Building on this work, Hang and Yang [HY16] realized that one could give some conformally covariant conditions under which such a maximum principle holds and provided an Aubin-type result for existence of constant $Q$–curvature metrics on closed manifolds.

The present paper is devoted to the construction of singular solutions to the constant $Q$–curvature problem for dimensions bigger or equal to five. Before explaining our results, we review the by-now classical setting of the Yamabe problem. Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold, $n \geq 3$. If $\Sigma \subset M$ is any closed set, then the ‘standard’ singular Yamabe problem concerns the existence and geometric properties of complete metrics of the form $\bar{g} = u^{4/(n-2)}g$ with constant scalar curvature. This corresponds to solving the partial differential equation

$$-\Delta_g u + \frac{n-2}{4(n-1)} R^g u = \frac{n-2}{4(n-1)} R^\bar{g} u^{\frac{n+2}{n-2}}, \quad u > 0,$$

where $R^\bar{g}$ is constant and with a ‘boundary condition’ that $u \to \infty$ sufficiently quickly at $\Sigma$ so that $\bar{g}$ is complete. It is known that solutions with $R^\bar{g} < 0$ exist quite generally if $\Sigma$ is large in a capacitary sense [Lab03], whereas for $R^\bar{g} > 0$ existence is only known when $\Sigma$ is a smooth submanifold (possibly with boundary) of dimension $k \leq (n-2)/2$, see [MP96], [Fak03].

There are both analytic and geometric motivations for studying this problem. For example, in the positive case ($R > 0$), solutions to this problem are actually weak solutions across the singular set, so these results fit into the broader investigation of possibly singular sets of weak solutions of semilinear elliptic equations. On the geometric side is a well-known theorem by Schoen and Yau [SY88] stating that if $(M, g)$ is a compact manifold with a locally conformally flat metric $g$ of positive scalar curvature, then the developing map $D$ from the universal cover $\tilde{M}$ to $S^n$, which by definition is conformal, is injective, and moreover, $\Sigma := S^n \setminus D(M)$ has Hausdorff dimension less than or equal to $(n-2)/2$. Regarding the lifted metric $\tilde{g}$ on $\tilde{M}$, this provides an interesting class of solutions of the singular Yamabe problem which are periodic with respect to a Kleinian group, and for which the singular set $\Sigma$ is typically nonrectifiable. More generally, that paper also shows that if $g$ is the standard round metric on the sphere and if $\tilde{g} = u^{\frac{2}{n-2}}g$ is a complete metric with positive scalar curvature and bounded Ricci curvature on a domain $S^n \setminus \Sigma$, then $\dim \Sigma \leq (n-2)/2$.

In this work, we address the same type of question for the $Q$–curvature equation. The equation involves a fourth order operator, the so-called Paneitz operator, is therefore
significantly more challenging to investigate. However, in the recent years, there has been several new insights on this difficult problem, thanks to the work of Gursky and Malchiodi [GM15]. A major problem for considering higher order equations is the lack of maximum principle. In particular, in general, one cannot ensure that any reasonable approximation yielding to a weak solution of the equation is actually positive. However, the breakthrough of Gursky and Malchiodi ensures that under some geometric assumptions on the manifold, one can ensure that one can obtain a positive solution. Their maximum principle led to an existence result similar to the Yamabe problem using a flow approach. A more variational point of view was later implemented by Hang and Yang in [HY16], weakening also the geometric conditions to have a maximum principle. Keeping in mind the importance of the $Q$-curvature problem both analytically and geometrically, it is then a natural question to ask whether one can construct singular solutions as for the second order case. This is the main result of this paper.

We first describe the setting of our contribution: let $(M, g)$ be a smooth closed $n$-dimensional Riemannian manifold with $n \geq 5$. The $Q$-curvature $Q_g$ is given by

$$Q_g = -\frac{1}{2(n-1)}\Delta R - \frac{2}{(n-2)^2} |\text{Ric}|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R^2$$

$$= -\Delta J - 2|A|^2 + \frac{n}{2} J^2,$$

(2)

where $R$ is the scalar curvature, $\text{Ric}$ is the Ricci curvature tensor, and

$$J = \frac{R}{2(n-1)}, \quad A = \frac{1}{n-2} (\text{Ric} - J g).$$

The Paneitz operator is given by

$$P\varphi = \Delta^2 \varphi + \frac{4}{n-2} \text{div}(\text{Ric}(\nabla \varphi, e_i) e_i) - \frac{n^2 - 4n + 8}{2(n-1)(n-2)} \text{div}(R \nabla \varphi) + \frac{n-4}{2} Q \varphi$$

$$= \Delta^2 \varphi + \text{div}(4A(\nabla \varphi, e_i) e_i - (n-2) J \nabla \varphi) + \frac{n-4}{2} Q \varphi.$$  

(3)

Here $e_1, \ldots, e_n$ is an orthonormal frame with respect to $g$.

For a given closed sub-manifold $\Sigma$ of $M$, we are interested in finding weak solutions to

$$Pu = u^{\frac{n+4}{n-4}} \text{ in } M \setminus \Sigma,$$  

(4)

such that $u$ goes to infinity as one approaches $\Sigma$, that is, for every $p \in \Sigma$ and $x_k \to p$ with $x_k \in M \setminus \Sigma$, $u(x_k) \to \infty$.

Our main result is the following.

**Theorem 1.1** Let $\Sigma$ be a connected smooth closed (compact without boundary) submanifold of $M$. Assume that $Q \geq 0$, $Q \neq 0$ and $R \geq 0$. If $0 < \dim(\Sigma) := k < \frac{n-4}{2}$ then there exists an infinite dimensional family of complete metrics on $M \setminus \Sigma$ with constant $Q$-curvature.
Actually, Theorem 1.1 is a corollary of the following result.

**Theorem 1.2** Let $\Omega \subset \mathbb{R}^n$ be a smooth open set and $\Sigma = \cup_{i=1}^K \Sigma_i$ a disjoint union of smooth, closed submanifolds of dimensions $k_i$ in $\Omega$. Assume that $n$ and $k_i$ for $i = 1, \ldots, K$ satisfy

$$\frac{n-k_i}{n-k_i-4} < p < \frac{n-k_i+4}{n-k_i-4}$$

or equivalently

$$n - \frac{4p+4}{p-1} < k_i < n - \frac{4p}{p-1}.$$

Then there exists a positive weak solution to

$$\begin{cases}
\Delta^2 u = u^p & \text{in } \Omega \setminus \Sigma \\
u = \Delta u = 0 & \text{on } \partial \Omega \\
u > 0 & \text{in } \Omega.
\end{cases}\quad (5)$$

that blows up exactly at $\Sigma$. Furthermore if at least one of the $k_i > 0$, there is an infinite dimensional solution space for (5).

**Remark 1** Notice that in the previous theorem if $k_i = 0$ for all $i = 1, \ldots, K$, then the exponent $p$ has to be subcritical with respect to the Sobolev exponent and supercritical with respect to the Serrin’s exponent provided $n \geq 5$. On the other hand, if one of the $k_i$ is positive, then the critical exponent $\frac{n+4}{n-4}$ is allowed for $p$, but the dimension $n$ has to be large enough.

The proof of Theorem 1.1 uses several tools ranging from geometric theory of edge operators (as in [MP96]) to a more general viewpoint on this type of problem provided in [ACD+]. Since we are dealing with a fourth order equation, even the ODE analysis, which is instrumental in [MP96], is rather involved. On the other hand, the authors in [ACD+] had to develop ODE-free method to deal with their quite general operators. Using the model $\mathbb{R}^n \setminus \mathbb{R}^k$ which is conformally equivalent to the product $S^{n-k-1} \times \mathbb{H}^{k+1}$ with the canonical metric, a straightforward computation of the $Q-$curvature on this model provides the condition $0 < k < \frac{n-4}{2}$ for positive $Q-$curvature metrics (see e.g. [BPS]). This model plays a crucial role in our theory since it allows to by-pass some tricky ODE arguments by having an “explicit” form of the solution using the Fourier-Helgason transform on hyperbolic space. See [CHY04] for much deeper results related to the dimension restriction and [BPS] for multiplicity results on the Q-curvature problem. Finally, as in the second order case (Yamabe problem), we use Delaunay-type solutions as building blocks of the approximate solution (see [GWZ17] and the Appendix for some existence results on these solutions). Note also that [AB08] provides also singular solutions using the trivial profile $|x|^{-\frac{4}{n-4}}$ but those allow to build only local solutions (see e.g. [MS91] for the Yamabe case).
2 Preliminaries

2.1 Function spaces

Let $\Sigma$ be a smooth $k$ dimensional submanifold of $\Omega \subset \mathbb{R}^n$ (or a union of submanifolds with different dimensions). For $\sigma > 0$ small we let $N_\sigma$ to be the geodesic tubular neighborhood of radius $\sigma$ around $\Sigma$. For $\alpha \in (0,1)$, $s \in (0,\sigma)$, $k \in \mathbb{N} \cup \{0\}$ and $\nu \in \mathbb{R}$ we define the seminorms

$$|w|_{k,\alpha,s} := \sum_{j=0}^{k} s^j \sup_{N_\sigma \setminus N_2} |\nabla^j w| + s^{k+\alpha} \sup_{x,x' \in N_\sigma \setminus N_2} \frac{|\nabla^k w(x) - \nabla^k w(x')|}{|x-x'|^\alpha},$$

and the weighted Hölder norm

$$\|w\|_{C^k,\alpha} := |w|_{C^k,\alpha}(\Omega \setminus \Sigma) + \sup_{0<s<\sigma} s^{-\nu}|w|_{k,\alpha,s}.$$

The weighted Hölder space $C^k_{\nu,\alpha}(\Omega \setminus \Sigma)$ is defined by

$$C^k_{\nu,\alpha}(\Omega \setminus \Sigma) := \left\{ w \in C^k_{\mathrm{loc}}(\Omega \setminus \Sigma) : \|w\|_{C^k,\alpha} < \infty \right\}.$$

The subspace of $C^k_{\nu,\alpha}(\Omega \setminus \Sigma)$ with Navier boundary conditions will be denoted by

$$C^k_{\nu,n}(\Omega \setminus \Sigma) := \{ w \in C^k_{\nu,\alpha}(\Omega \setminus \Sigma) : w = \Delta w = 0 \text{ on } \partial \Omega \}.$$

The space $C^k_{\nu,\alpha}(\mathbb{R}^n \setminus \{0\})$ is defined by

$$\|w\|_{C^k,\alpha}(\mathbb{R}^n \setminus \{0\}) := \|w\|_{C^k,\alpha}(B_2 \setminus \{0\}) + \sup_{r \geq 1} (r^{-\nu'} \|w(r)\|_{C^k,\alpha}(B_2 \setminus B_1)).$$

We also set

$$\|w\|_{C^k,\alpha}(\mathbb{R}^m \setminus \mathbb{R}^n) := \|w\|_{C^k,\alpha}(B_2 \setminus \mathbb{R}^n) + \sup_{r \geq 1} (r^{-\nu'} \|w(r)\|_{C^k,\alpha}(B_2 \setminus B_1)),$$

where $B_r$ denotes the tubular neighborhood of radius $r$ of $\mathbb{R}^m$ in $\mathbb{R}^n$.

We now list some useful properties of the space $C^k_{\nu,\alpha}(\Omega \setminus \Sigma)$, see e.g. [MP96] and the book [PR00].

**Lemma 2.1** The following properties hold.

i) If $w \in C^{k+1,\alpha}_{\gamma}(\Omega \setminus \Sigma)$ then $\nabla w \in C^{k,\alpha}_{\gamma-1}(\Omega \setminus \Sigma)$.

ii) If $w \in C^{k+1,0}_{\gamma}(\Omega \setminus \Sigma)$ then $w \in C^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma)$ for every $\alpha \in [0,1)$.

iii) For every $w_i \in C^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma)$, $i=1,2$, we have

$$\|w_1 w_2\|_{k,\gamma_1+\gamma_2,\alpha} \leq C\|w_1\|_{k,\gamma_1,\alpha}\|w_2\|_{k,\gamma_2,\alpha},$$

for some $C > 0$ independent of $w_1, w_2$.

iv) There exists $C > 0$ such that for every $w \in C^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma)$ with $w > 0$ in $\Omega \setminus \Sigma$ we have

$$\|w^p\|_{k,\gamma,\alpha} \leq C\|w\|_{k,\gamma,\alpha}^p$$

for some $p > 0$. 


2.2 Fermi coordinates

We now compute the Fermi coordinates for our problem. For \( \sigma > 0 \) small we can choose fermi coordinates in \( N_\sigma \) as follows: First we fix any local coordinate system \( y = (y_1, \ldots, y_k) \) on \( \Sigma \) (\( k \) is the dimension of \( \Sigma \)). For every \( y_0 \in \Sigma \) there exists an orthonormal frame field \( E_1, \ldots, E_{n-k} \), basis of the normal bundle of \( \Sigma \). Then we consider the coordinate system

\[
\Sigma \times \mathbb{R}^{n-k} \ni (y, z) \rightarrow y + \sum z_i E_i(y).
\]

For \( |z| < \sigma \) with \( \sigma \) small, these are well defined coordinate system in a neighborhood of \( y_0 \).

In this coordinate system the Euclidean metric has the following expansion

\[
g_{\mathbb{R}^n} = g_{\mathbb{R}^{n-k}} + g_\Sigma + O(|z|)dzdy + O(|z|)dy^2.
\]

Therefore,

\[
\Delta_{\mathbb{R}^n} = \Delta_{\mathbb{R}^{n-k}} + \Delta_\Sigma + e_1 \nabla + e_2 \nabla^2,
\]

where \( e_i, i = 1, 2 \) satisfy

\[
\|e_1\|_{C^{0,\alpha}_0} + \|e_2\|_{C^{0,\alpha}_1} \leq c.
\]

Using this we also have

\[
\Delta_{\mathbb{R}^n}^2 = \Delta_{\mathbb{R}^{n-k}}^2 + \Delta_\Sigma^2 + 2\Delta_{\mathbb{R}^{n-k}} \Delta_\Sigma + \sum_{i=1}^4 e_i \nabla^i,
\]

where

\[
\|e_1\|_{C^{0,\alpha}_0} + \|e_2\|_{C^{0,\alpha}_1} + \|e_3\|_{C^{0,\alpha}_0} + \|e_4\|_{C^{0,\alpha}_1} \leq c.
\]

2.3 The singular solution

The building block for our theory is the existence of a singular solution with different behaviour at the origin and at infinity. The following theorem provides such a solution. We refer the reader to the appendix for a proof of this result.

**Theorem 2.2** Let \( N \geq 5 \). Suppose that \( \frac{N}{N-4} < p < \frac{N+4}{N-4} \). Then for every \( \beta > 0 \) there exists a unique radial solution \( u \) to

\[
\begin{cases}
\Delta^2 u = u^p & \text{in } \mathbb{R}^N \setminus \{0\} \\
u > 0 & \text{in } \mathbb{R}^N \setminus \{0\} \\
\lim_{|x| \to 0} u(x) = \infty,
\end{cases}
\]

such that

\[
\lim_{r \to 0^+} r^{N-4} u(r) = \beta, \quad \lim_{r \to \infty} r^{\frac{4}{p-1}} u(r) = c_p := \left[ k(p, N) \right]^{p-1},
\]

where

\[
k(p, N) = \frac{8(p + 1)}{(p - 1)^4} \left[ N^2(p - 1)^2 + 8p(p + 1) + N(2 + 4p - 6p^2) \right].
\]
2.4 Approximate solutions

Let $u$ be a singular radial solution to (5). Then $u_{\varepsilon}(x) := \varepsilon^{-\frac{4}{p-1}} u(\frac{x}{\varepsilon})$ is also a solution to (5). Note that

$$u_{\varepsilon}(x) \leq C(\delta, u) \varepsilon^{N-4-\frac{4}{p-1}}$$

for $|x| \geq \delta$, which shows that $u_{\varepsilon} \to 0$ locally uniformly in $\mathbb{R}^N \setminus \{0\}$. Due to this scaling and the asymptotic behavior of $u$ at infinity, for a given $\alpha > 0$, we can find a solution $u_1$ such that

$$r^4 u_1^{p-1}(r) \leq \alpha$$

on $1 < r < \infty$.

2.4.1 Isolated singularities

Let $\Sigma = \{x_1, x_2, \ldots, x_K\}$ be a set of finite points in $\Omega$. To construct a solution to (5) which is singular precisely at the points of $\Sigma$, we start by constructing an approximate solution to (5) which is singular exactly on $\Sigma$. Let us first fix a smooth cut-off function $\chi$ such that $\chi = 1$ on $B_1$ and $\chi = 0$ on $B_2^c$. Also fix $R > 0$ such that $B_{2R}(x_i) \subset \Omega$ and $B_{2R}(x_i) \cap B_{2R}(x_j) = \emptyset$ for every $i \neq j$, $i, j \in \{1, 2, \ldots, K\}$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_K)$ be a $K$-tuple of dilation parameter. An approximate solution $\bar{u}_\varepsilon$ is defined by

$$\bar{u}_\varepsilon(x) = \sum_{i=1}^K \chi_R(x - x_i) u_{\varepsilon_i}(x - x_i) = \sum_{i=1}^K \varepsilon_i^{-\frac{4}{p-1}} \chi(\frac{x - x_i}{R}) u_1(\frac{x - x_i}{\varepsilon_i}).$$

The asymptotic behavior of $u_1$ at infinity leads to the following error of approximation:

**Lemma 2.3** The error $f_{\varepsilon} := \Delta^2 \bar{u}_\varepsilon - \bar{u}_\varepsilon^p$ satisfies

$$\|f_{\varepsilon}\|_{C^{\alpha}_{\gamma-4}} \leq C\gamma \varepsilon_0^{N-\frac{4\gamma}{p-1}}$$

for $0 < \varepsilon_i \leq \varepsilon_0 \leq 1$, for every $\gamma \in \mathbb{R}$.

2.4.2 Higher dimensional singularities

Let $\Sigma_i \subset \Omega$ be a $k_i$-dimensional submanifold in $\Omega$ for $i = 1, 2, \ldots, K$. We fix $\sigma > 0$ small such that Fermi coordinates are well-defined on the Tubular neighborhood $N_{i, \sigma}$ of $\Sigma_i$ for every $i = 1, \ldots, K$, and $N_{i, 2\sigma} \cap N_{j, 2\sigma} = \emptyset$ for $i \neq j$. Fix a smooth radially symmetric cut-off function $\chi$ such that $\chi = 1$ on $B_1$ and $\chi = 0$ on $B_2^c$. Then for $0 < \varepsilon_i < 1$ and $0 < R < \frac{\sigma}{2}$ we set

$$\bar{u}_{\varepsilon_i}(x, y) = \varepsilon_i^{-\frac{4}{p-1}} u_1(\frac{x}{\varepsilon_i}) \chi(\frac{y}{R}) =: u_{\varepsilon_i}(x) \chi_R(y).$$

An approximate solution which is singular only on $\bigcup_{i=1}^K \Sigma_i$ is defined by

$$\bar{u}_\varepsilon = \sum_{i=1}^K \bar{u}_{\varepsilon_i}.$$
Using the expansion (6) and the estimate (7) we see that the error $f_{\bar{\varepsilon}} := \Delta^2 \bar{u}_{\varepsilon} - \bar{u}_p$ satisfies

$$\|f_{\bar{\varepsilon}}\|_{C^{0,\alpha}_{\gamma - 4}} \leq C\varepsilon_0^q, \quad 0 < \varepsilon_1 \leq \varepsilon_0 \leq 1, \quad q := \frac{p - 5}{p - 1} - \gamma > 0,$$

for $\gamma < \frac{p - 5}{p - 1}$. In our applications, $\gamma$ will be bigger than $-\frac{4}{p - 1}$.

**Remark 2** To prove existence of solution to (5) with singular set $\Sigma$, we shall look for solutions of the form $u = \bar{u}_{\varepsilon} + v$ (in both cases, that is, $\Sigma$ is finite and higher dimensional). Then, $v$ has to satisfy

$$L_{\varepsilon}v + f_{\varepsilon} + Q(v) = 0, \quad L_{\varepsilon} := \Delta^2 - pu_{\varepsilon}^{p-1}, \quad (9)$$

where

$$Q(v) = -(\bar{u}_{\varepsilon} + v)^p + \bar{u}_p^{p-1} + pu_{\varepsilon}^{p-1}v. \quad (10)$$

### 3 Linearized operator in $\mathbb{R}^N \setminus \{0\}$

Since our purpose is to use an implicit function theorem, it is crucial to understand the linearized problem. For this, we invoke the analytic theory of edge operators as in [Maz91, MV14] but also some more general arguments in [ACD+] as we mentioned in the introduction.

We consider the linearized operator

$$L_1 = \Delta^2 - pu_1^{p-1}$$

where in polar coordinates we denote

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\theta.$$  

#### 3.1 Indicial roots

Next we compute indicial roots of the linearized operator $L_1$. We recall that $\gamma_j$ is a indicial root of $L_1$ at 0 if $L_1(|x|^{\gamma_j} \varphi_j) = o(|x|^{-4})$, where $\varphi_j$ is the $j$-th eigenfunction of $-\Delta_\theta$, that is $-\Delta_\theta \varphi_j = \lambda_j \varphi_j$, $\lambda_0 = 0$, $\lambda_j = N - 1$, for $j = 1, \ldots, N$, and so on. One shows that $\gamma_j$ is a solution to

$$[\gamma(\gamma - 1) + (N - 1)\gamma - \lambda_j][\gamma - 2(\gamma - 3) + (N - 1)(\gamma - 2) - \lambda_j] - A_p = 0,$$
where

\[ A_p := p \lim_{r \to 0} r^4 u_1^{p-1}(r) = pk(p, N). \]  

(11)

The solutions can be given by

\[ \gamma_j^{\pm} = \frac{1}{2} \left[ 4 - N \pm \sqrt{(N - 2)^2 + 4 + 4\lambda_j \pm 4\sqrt{(N - 2)^2 + 4\lambda_j + A_p}} \right]. \]

We have that (\( \Re \) denotes the real part)

\[ \gamma_0^+ < 4 - N < -\frac{4}{p - 1} < \Re(\gamma_0^-) \leq \frac{4 - N}{2} \leq \Re(\gamma_0^+) < 0 < 2 < \gamma_0^+, \]  

(12)

and

\[ \gamma_j^{-} < -\frac{4}{p - 1}, \quad \Re(\gamma_0^-) < \gamma_j^{\pm} \quad \text{for } j \geq 1. \]  

(13)

To prove the above relations one uses that \( A_p \) is monotone. Indeed, for any fixed \( N \geq 5 \), \( \frac{\partial}{\partial p} A_p \) vanishes at the following points

\[ p_0 := \frac{N + 2}{N - 6}, \quad p_1^+ = \frac{N + 4 \pm 2\sqrt{N^2 + 4}}{3N - 8}. \]

Using this one would get that \( A_p \) is monotone increasing on \( \left( \frac{N}{N-4}, \frac{N+4}{N-4} \right) \).

Since \( \lim_{r \to \infty} r^4 u_1^{p-1}(r) = 0 \), the indicial roots of \( L_1 \) at infinity are the same as for the \( \Delta^2 \) itself. These values are given by

\[ \tilde{\gamma}_j^{\pm} = \frac{1}{2} \left[ 4 - N \pm \sqrt{(N - 2)^2 + 4 + 4\lambda_j \pm 4\sqrt{(N - 2)^2 + 4\lambda_j}} \right]. \]

In particular,

\[ \tilde{\gamma}_0^{\pm} \in \{2 - N, 4 - N, 0, 2\}, \quad \tilde{\gamma}_1^{\pm} \in \{1, 3\}, \quad \tilde{\gamma}_j^{\pm} \geq 1 \quad \text{and} \quad \tilde{\gamma}_j^{\pm} < 4 - N \quad \text{for } j \geq 1. \]

We shall choose \( \mu, \nu \) in the region

\[ -\frac{4}{p - 1} < \nu < \Re(\gamma_0^-) \leq \frac{4 - N}{2} \leq \Re(\gamma_0^+) < \mu, \]  

(14)

so that \( \mu + \nu = 4 - N \).
3.2 Study of the linearized operator

For a function \( w = w(r, \theta) \) we decompose it as

\[
w(r, \theta) = \sum_{j=0}^{\infty} w_j(r) \varphi_j(\theta).
\]

Then

\[
\Delta^2(w_j \varphi_j) = \left( w_j^{iv} + \frac{2(N-1)}{r} w_j''' + \frac{N^2 - 4N + 3 - 2\lambda_j}{r^2} w_j'' - \frac{(N-3)(N-1 + 2\lambda_j)}{r^3} w_j' + \frac{2(N - 4)\lambda_j + \lambda_j^2}{r^4} w_j \right) \varphi_j
\]

\[= (w_j^{iv} + \frac{a_{1,j}}{r} w_j''' + \frac{a_{2,j}}{r^2} w_j'' - \frac{a_{3,j}}{r^3} w_j' + \frac{a_{4,j}}{r^4} w_j) \varphi_j.
\]

Thus, \( L_1 w = 0 \) if and only if, for every \( j = 0, 1, 2, \ldots \)

\[
w_j^{iv} + \frac{a_{1,j}}{r} w_j''' + \frac{a_{2,j}}{r^2} w_j'' - \frac{a_{3,j}}{r^3} w_j' + \frac{a_{4,j}}{r^4} w_j = 0, \quad V_p(r) := pr^4 w_1^{p-1}(r).
\]

One obtains

\[
r^{N-1} w_j(w_j^{iv} + \frac{a_{1,j}}{r} w_j''' + \frac{a_{2,j}}{r^2} w_j'' - \frac{a_{3,j}}{r^3} w_j' + \frac{a_{4,j}}{r^4} w_j)
\]

\[= (r^{N-1} w_j w_j'''')' - (r^{N-1} w_j w_j''')' + (N - 1)(r^{N-2} w_j w_j''')' - (N - 1 + 2\lambda_j)(r^{N-3} w_j w_j''')'
\]

\[+ (N - 1 + 2\lambda_j)r^{N-3}(w_j')^2 + r^{N-1}(w_j')^2 + a_{4,j} r^{N-5} w_j^2.
\]

Proposition 3.1 Let \( w \in C_{\mu,0}^4(\mathbb{R}^N \setminus \{0\}) \) be a solution to \( L_1 w = 0 \). Then \( w \equiv 0 \).

Proof. From the definition of the space \( C_{\mu,0}^4(\mathbb{R}^N \setminus \{0\}) \) we have that

\[
\begin{cases}
|w(x)| \leq C \log(2 + |x|), & |x|^k |\nabla^k w(x)| \leq C & \text{for } |x| \geq 1, \quad k = 1, \ldots, 4 \\
|x|^{-\mu+k} |\nabla^k w(x)| \leq C & \text{for } 0 < |x| \leq 1, \quad k = 0, \ldots, 4.
\end{cases}
\]

Now we decompose \( w \) as in (15).

First we show that \( w_0 = 0 \). From the choice of \( \mu \) we see that if \( w_0 \neq 0 \) then \( w_0 \) should behave like \( r^{\gamma_0^+} \) around the origin. Therefore, without loss of any generality, we can assume that \( w_0 \geq 0 \) in a small neighborhood of the origin. Using the crucial fact \( q := \gamma_0^+ > 2 \), thanks to (12), we shall show that \( w \) is actually \( C^2 \) and \( \Delta w_0(0) = 0 \). Indeed, as \( w_0 \) satisfies \( \Delta^2 w_0 = pu_1^{p-1} w_0 =: f \) in \( \mathbb{R}^N \setminus \{0\} \), for \( 0 < \varepsilon < r \) we have

\[
(\Delta w_0)'(r) r^{N-1} = (\Delta w_0)'(\varepsilon) \varepsilon^{N-1} + \omega_1 \int_{\varepsilon < |x| < r} f \, dx, \quad \omega_1 := |S^{N-1}|^{-1}.
\]

10
As \( \mu > (4 - N)/2 \) we see that

\[
|(\Delta w_0)'(\varepsilon)| \varepsilon^{N-1} \leq C \varepsilon^{\mu-3} \varepsilon^{N-1} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

Hence, for \( 0 < r_1 < r_2 \) we get

\[
\Delta w_0(r_2) = \Delta w_0(r_1) + \omega_N \int_{r_1}^{r_2} \frac{1}{t^{N-1}} \int_{|x|<t} f(x) dx dt.
\]

As \( (\Delta w_0)' > 0 \) on \((0, \varepsilon_0)\), that is \( \Delta w_0 \) is monotone increasing, we see from the above relation that \( \lim_{r_1 \to 0^+} \) exists and finite. Here we used that \( f(x) \approx |x|^{q-4}, q > 2 \). Thus, \( w_0 \) is \( C^2 \), and again as \( q > 2 \), we must have \( \Delta w_0(0) = 0 \). In conclusion, \( \Delta w_0(r) \geq \Delta w_0(1) > 0 \) for \( r > 1 \), which leads to \( w_0(r) \geq r^2 \), a contradiction to (18).

For \( j \geq 1 \), \( w_j \) behaves like \( r^{4-N-\frac{1}{N}} \) as \( r \to \infty \). Since \( j^- \frac{1}{N} < 4 - N \) and \( w_j = O(\log r) \) at infinity, we must have \( w_j \approx r^{4-N-\frac{1}{N} \pm} \).

Now multiplying the equation (16) by \( r^{-N}w_j \), and using (17) we get (this is justified thanks to (18), and the asymptotic behavior of \( w_j \) at infinity)

\[
(N - 1 + 2\lambda_j) \int_0^\infty r^{-N-3} w_j^2 dr + \int_0^\infty r^{-N-1} w_j'^2 dr = \int_0^\infty [V_p(r) - 2(N - 4)\lambda_j - \lambda_j^2] r^{N-5} w_j^2 dr
\]

\[
\leq \left[ \frac{p^p + 1}{2} k(p, N) - 2(N - 4)\lambda_j - \lambda_j^2 \right] \int_0^\infty r^{N-5} w_j^2 dr
\]

\[
\leq \left[ \frac{1}{16} N^3(N + 4) - 2(N - 4)\lambda_j - \lambda_j^2 \right] \int_0^\infty r^{N-5} w_j^2 dr
\]

\[
=: C(N, j) \int_0^\infty r^{N-5} w_j^2 dr.
\]

An integration by parts gives

\[
\int_0^\infty r^{-N-5} w_j^2 dr \leq \frac{4}{(N - 4)^2} \int_0^\infty r^{-N-3} w_j^2 dr
\]

\[
\int_0^\infty r^{-N-3} w_j'^2 dr \leq \frac{4}{(N - 2)^2} \int_0^\infty r^{-N-1} w_j'^2 dr.
\]

This leads to

\[
\int_0^\infty r^{-N-1} w_j'^2 dr \leq \left[ \frac{4C(N, j)}{(N - 4)^2} - (N - 1 + 2\lambda_j) \right] \int_0^\infty r^{-N-3} w_j'^2 dr
\]

\[
\leq \left[ \frac{4C(N, j)}{(N - 4)^2} - (N - 1 + 2\lambda_j) \right] \frac{4}{(N - 2)^2} \int_0^\infty r^{-N-1} w_j'^2 dr
\]

\[
=: \tilde{C}(N, j) \int_0^\infty r^{-N-1} w_j'^2 dr.
\]
One can show that $\tilde{C}(N,j) < 1$ for $j \geq N + 1$, and hence $w_j \equiv 0$ for $j \geq N + 1$.

Finally, we consider the case $j = 1, \ldots, N$. For $\lambda_1 = n - 1$ we have that $\tilde{\gamma}_1^{\pm} \in \{1, 3\}$, and $\tilde{\gamma}_1^{-} < 4 - N$. Therefore, $w_1$ should behave like $r^{1-N}$ or $r^{3-N}$ at infinity.

Let us first show that if $w_1 = O(r^{1-N})$ at infinity and $w_1 \varphi_1 \in C^{4,\alpha}_{\mu,0}$ then $w_1 \equiv 0$.

We set

$$W(r) = - \int_r^\infty w_1(t)dt, \quad r > 0,$$

so that $W' = w_1$. For $\varepsilon > 0$ small let $\Omega_{\varepsilon}$ be the domain

$$\Omega_{\varepsilon} := \{x \in \mathbb{R}^N : x_1 > 0, |x| > \varepsilon\}.$$

We know that $u'_1 < 0$, $\Delta u_1 < 0$, $(\Delta u_1)' > 0$ on $(0, \infty)$,

$$u_1(r) \approx r^{4-N}, \quad u'_1(r) \approx -r^{3-N}, \quad \Delta u_1(r) \approx -r^{2-N}, \quad (\Delta u_1)'(r) \approx r^{1-n} \quad \text{as } r \to \infty,$$

and

$$u_1(r) \approx r^{-\frac{1}{p-1}}, \quad u'_1(r) \approx -r^{-\frac{1}{p-1}} - 1, \quad \Delta u_1(r) \approx -r^{-\frac{1}{p-1}} - 2, \quad (\Delta u_1)'(r) \approx r^{-\frac{1}{p-1} - 3} \quad \text{as } r \to 0.$$

Therefore,

$$W(r) = o(u_1(r)), \quad W'(r) = o(u'_1(r)), \quad (\Delta W)'(r) = o((\Delta u_1)'(r)) \quad \text{as } r \to 0 \text{ or } \infty.$$

Setting

$$\tilde{w}_\rho(x) = (W(|x|) - \rho u'_1(|x|)) \frac{x_1}{|x|} = \frac{\partial}{\partial x_1}(W - \rho u_1)(|x|),$$

we see that for $\rho \gg 1$ we have

$$\tilde{w}_\rho \geq 0 \quad \text{and} \quad \Delta \tilde{w}_\rho = (\Delta W - \rho \Delta u_1)'(x) \frac{x_1}{|x|} \leq 0 \quad \text{in } \Omega_{\varepsilon},$$

equivalently

$$W' - \rho u_1' \geq 0 \quad \text{and} \quad (\Delta W)' - \rho (\Delta u_1)' \leq 0 \quad \text{in } \Omega_{\varepsilon}. \quad (19)$$

Now we set

$$\rho_{\varepsilon} := \inf \{\rho > 0 : (19) \text{ holds}\}.$$

We claim that $\rho_{\varepsilon} \to 0$ as $\varepsilon \to 0$. Indeed, as $\tilde{w}_{\rho_{\varepsilon}}$ satisfies (recall that $\varphi_1 = \frac{x_1}{|x|}$)

$$\Delta^2 \tilde{w}_{\rho_{\varepsilon}} = p u^{p-1} \tilde{w}_{\rho_{\varepsilon}} \geq 0, \quad \Delta \tilde{w}_{\rho_{\varepsilon}} \leq 0 \quad \text{in } \Omega_{\varepsilon},$$

by maximum principle

$$\tilde{w}_{\rho_{\varepsilon}} > 0 \quad \text{and} \quad \Delta \tilde{w}_{\rho_{\varepsilon}} < 0 \quad \text{in } \Omega_{\varepsilon}. \quad (20)$$
On the other hand, if $\rho_\varepsilon > 0$ then there exists $x_\varepsilon \in \bar{\Omega}_\varepsilon$ such that
\[
W'(x_\varepsilon) - \rho_\varepsilon u_1'(x_\varepsilon) = 0 \quad \text{or} \quad (\Delta W)'(x_\varepsilon) - \rho_\varepsilon (\Delta u_1)'(x_\varepsilon) = 0,
\]
thanks to the definition of $\rho_\varepsilon$ and the asymptotic behavior of $W'$, $(\Delta W)'$, $u_1'$, $(\Delta u_1)'$. Since $W$ and $u_1$ are radially symmetric, (20) implies that $|x_\varepsilon| = \varepsilon$. Hence, from the behavior of $W'$, $(\Delta W)'$, $u_1'$, $(\Delta u_1)'$ around the origin, we conclude that $\rho_\varepsilon \to 0$.

Thus we have shown that $W' \geq 0$ on $(0, \infty)$. In a similar way, taking $\tilde{w}_\rho(x) = (-W'(|x|) - \rho u_1'(|x|)) x_1 / |x|$ we would get that $W' \leq 0$ on $(0, \infty)$. This completes the roof.

The same proof shows that there is no solution $w_1$ such that $w_1(r) = u_1'(r)(1 + o(1))$ around the origin, and $w_1(r) = o(u_1'(r))$ at infinity.

Now we show that if $w_1 = O(r^{3-N})$ at infinity and $w_1 \varphi_1 \in C^{4,\alpha}_{\mu,0}$ then $w_1 \equiv 0$. Indeed, if $w_1 \not\equiv 0$, then $\tilde{w}_1 := u_1' + aw_1$ would satisfy $\tilde{w}_1 = u_1'(1 + o(1))$ around the origin and $\tilde{w}_1 = o(u_1')$ at infinity for some non-zero constant $a$, which is a contradiction.

This finishes the lemma.

**Proposition 3.2** Let $w \in C^{4,\alpha}_{\mu,\mu}(\mathbb{R}^N \setminus \{0\})$ be a solution to
\[
\Delta^2 w - \frac{A_p}{r^4} w = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]
where $A_p$ is given by (11). Then $w \equiv 0$.

**Proof.** Note that in this case also we have same indicial roots as before. Therefore, $w$ is given by
\[
w = \sum_{j \geq 0} \left( (c_{1,j} r^{\tau_j^+} + c_{2,j} r^{\tau_j^-} + c_{3,j} r^{\tau_j^+} + c_{4,j} r^{\tau_j^-}) \varphi_j \right),
\]
for some $c_{i,j} \in \mathbb{R}$. Since, $r^{\tau_j^\pm}$ is not bounded by $r^\mu$ simultaneously at the origin and at infinity, we have that $c_{i,j} = 0$ for every $(i,j)$. This finishes the lemma.

**4 Linearized operator on $\mathbb{R}^n \setminus \mathbb{R}^k$**

The main goal of this section is to prove the following injectivity of the linearized operator on $\mathbb{R}^n \setminus \mathbb{R}^k$. We closely follow the approach in [ACD$^+$.]

**Proposition 4.1** The linearized operator
\[
L_1 = \Delta^2_{x,y} - pu_1^{p-1}
\]
is injective in $C^{4,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^k)$.
In order to prove the above lemma we shall use our previous injectivity results on $\mathbb{R}^N \setminus \{0\}$. The idea is to show that both operators have same symbol. To be more precise, we write the Euclidean metric in $\mathbb{R}^N$ as

$$|dx|^2 = dr^2 + r^2 g_{S^{N-1}}.$$  

We consider the conformal change

$$g_0 := \frac{1}{r^2} |dx|^2 = dt^2 + g_{S^{N-1}}, \quad r = e^{-t},$$

which is a complete metric on the cylinder $\mathbb{R} \times S^{N-1}$. Then the conformal Laplacian $P^g_{\gamma}$ of order $2\gamma$ with $0 < \gamma < \frac{N}{2}$ is given by

$$P^g_{\gamma} w = r^{\gamma/2} (-\Delta)^\gamma u, \quad u := r^{-\frac{N-2}{2}} w.$$ 

In the following we shall use the following normalization on the definition of Fourier transformation on $\mathbb{R}$:

$$\hat{w} (\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} w(t) dt.$$ 

The following lemma can be found in [DG18].

**Lemma 4.2** Let $P^g_{\gamma}$ be the projection of the operator $P^g_{\gamma}$ on the eigenspace $\langle \varphi_j \rangle$. Then, writing $w(t, \theta) = \sum_{j=0}^{\infty} w_j(t) \varphi_j(\theta)$ we have

$$\widehat{P^g_{\gamma} w_j} = \Theta^j_\gamma(\xi) \hat{w}_j,$$

where the Fourier symbol is given by

$$\Theta^j_\gamma(\xi) = 2^{2\gamma} \frac{|\Gamma(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2} \sqrt{(\frac{N}{2} - 1)^2 + \lambda_j + \xi i})|^2}{|\Gamma(\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2} \sqrt{(\frac{N}{2} - 1)^2 + \lambda_j + \xi i})|^2}. \quad (21)$$

Now we move on to the case when the singularity is along $\mathbb{R}^k$. For a point $z = (x, y) \in \mathbb{R}^n \setminus \mathbb{R}^k$ we shall use the following notations: $x \in \mathbb{R}^N$, $y \in \mathbb{R}^k$ where $\mathbb{R}^n = \mathbb{R}^N \times \mathbb{R}^k$. We shall also write $z = (x, y) = (r, \theta, y)$ where $r = |x|$ and $\theta \in S^{N-1}$. Then the Euclidean metric on $\mathbb{R}^n$ can be written as

$$|dz|^2 = |dx|^2 + |dy|^2 = dr^2 + r^2 g_{S^{N-1}} + dy^2.$$ 

Now we consider the conformal metric

$$g_k := \frac{1}{r^2} |dz|^2 = g_{S^{N-1}} + \frac{dr^2 + dy^2}{r^2} = g_{S^{N-1}} + g_{H^{k+1}},$$
where $\mathbb{H}^{k+1}$ is the Hyperbolic space. The conformal Laplacian is given by

$$P_{g_k}^w = r^{-\frac{n+2}{2}}(-\Delta)^\gamma u, \quad u = r^{-\frac{n-2}{2}}w.$$  

For a function $w$ on $S^{N-1} \times \mathbb{H}^{k+1}$ we decompose it as $w(\theta, \zeta) = \sum_{j=0}^\infty w_j(\zeta)\phi_j(\theta)$, with $\zeta \in \mathbb{H}^{k+1}$.

The next lemma can be found in [ACD+].

**Lemma 4.3** Let $P_{j}^\gamma$ be the projection of the operator $P_{g_k}^w$ on the eigenspace $\langle \phi_j \rangle$. Then

$$\hat{P_{j}^\gamma}w_j = \Theta_j^\gamma(\lambda)\hat{w}_j, \quad \Theta_j^\gamma(\lambda) = 2^{2\gamma}|\Gamma(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2}\sqrt{(\frac{N}{2} - 1)^2 + \lambda + \frac{\lambda}{\gamma}})|^2$$

where $\hat{\cdot}$ denotes the Fourier-Helgason transform on $\mathbb{H}^{k+1}$.

**Proof of Proposition 4.1** As we mentioned before, we shall use Proposition 3.1. Let $\phi$ be a solution to

$$\Delta^2 \phi - pu_{p-1} \phi = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{R}^k.$$  

We set $w = r^{-\frac{n+2}{2}}\phi$. Let $w_j$ be the projection of $w$ on the eigenspace $\langle \phi_j \rangle$. Let $\hat{w}_j(\lambda, \omega)$ be the Fourier-Helgason transform of $w_j$, $(\lambda, \omega) \in \mathbb{R} \times S^{N-1}$. As the symbol (22) coincides with the symbol (21) for every $\omega \in S^{N-1}$, our problem is equivalent to that of Proposition 3.1. This concludes the proof. \hfill \Box

In a similar way, using Proposition 3.2 one can prove the following Proposition:

**Proposition 4.4** Solutions to

$$\Delta^2 w - \frac{A_p}{r^4}w = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{R}^k,$$  

are trivial in the space $w \in C^{4,\alpha}_{\mu,\nu}(\mathbb{R}^n \setminus \mathbb{R}^k)$.

**5 Injectivity of $L_\varepsilon$ on $C^{4,\alpha}_{\mu,\nu}(\Omega \setminus \Sigma)$**

In this section we study injectivity of the linearized operator

$$L_\varepsilon w := \Delta^2 w - pu_{\varepsilon}^{-1}w.$$  

We shall use the following notations:

$$\Omega_\varepsilon := \Omega \setminus \cup_{i=1}^K B_i(\Sigma_i), \quad f^+ := \max\{f,0\}, \quad f^- := \min\{f,0\}.$$
Lemma 5.1 There exists $\varepsilon_0 > 0$ such that if $\varepsilon_i < \varepsilon_0$ for every $i$, then after a suitable normalization of $u_1$, the operator $L_{\varepsilon}$ satisfies maximum principle in $\Omega_{\varepsilon}$, that is

$$\begin{cases}
L_{\varepsilon} w \geq 0 & \text{in } \Omega_{\varepsilon} \\
w \geq 0 & \text{on } \partial\Omega_{\varepsilon} \\
\Delta w \leq 0 & \text{on } \partial\Omega_{\varepsilon}
\end{cases} \Rightarrow w \geq 0 \text{ and } \Delta w \leq 0 \text{ in } \Omega_{\varepsilon}.$$

Proof. Let $v$ and $\tilde{v}$ be given by

$$\begin{cases}
-\Delta v = -(\Delta w)^- & \text{in } \Omega_{\varepsilon} \\
v = w & \text{on } \partial\Omega_{\varepsilon}, \\
-\Delta \tilde{v} = (\Delta w)^+ & \text{in } \Omega_{\varepsilon} \\
\tilde{v} = 0 & \text{on } \partial\Omega_{\varepsilon}.
\end{cases}$$

Then $v \geq 0$ and $\tilde{v} \geq 0$ in $\Omega_{\varepsilon}$. We shall show that $\tilde{v} = 0$, and hence $v = w$.

It follows that

$$\begin{cases}
-\Delta (v - w) = (\Delta w)^+ \geq 0 & \text{in } \Omega_{\varepsilon} \\
v - w = 0 & \text{on } \partial\Omega_{\varepsilon}, \\
-\Delta (\tilde{v} + w) = -(\Delta w)^- \geq 0 & \text{in } \Omega_{\varepsilon} \\
\tilde{v} + w \geq 0 & \text{on } \partial\Omega_{\varepsilon}.
\end{cases}$$

Therefore,

$$v - w \geq 0, \quad -w \leq \tilde{v} \quad \text{in } \Omega_{\varepsilon} \quad \text{and} \quad \frac{\partial(v - w)}{\partial\nu} \leq 0 \quad \text{on } \partial\Omega_{\varepsilon},$$

where $\nu$ is the outward unit normal vector. We compute

$$\int_{\Omega_{\varepsilon}} (v - w) \Delta^2 w dx = \int_{\Omega_{\varepsilon}} \Delta (v - w) \Delta w dx + \int_{\partial\Omega_{\varepsilon}} (v - w) \frac{\partial \Delta w}{\partial\nu} - \Delta w \frac{\partial (v - w)}{\partial\nu} d\sigma$$

$$= -\int_{\Omega_{\varepsilon}} [(\Delta w)^+]^2 dx - \int_{\partial\Omega_{\varepsilon}} \Delta w \frac{\partial (v - w)}{\partial\nu} d\sigma$$

$$\leq -\int_{\Omega_{\varepsilon}} [(\Delta w)^+]^2 dx.$$
where $\delta > 0$ can be chosen arbitrarily small by normalizing $u_1$ so that $pr^4u_1^{n-1}(r) \leq \delta$ for $r \geq 1$. Since $\tilde{v} = 0$ on $\partial \Omega_\varepsilon$ (same arguments for $v - w$), integrating by parts we obtain

$$\int_{\Omega_\varepsilon} \frac{\tilde{v}(x)^2}{|x - x_i|^4} dx = \int_{\Omega_\varepsilon} \frac{\tilde{v}(x)^2}{|x - x_i|^4} dx$$

$$= -\frac{1}{2(n-4)} \int_{\Omega_\varepsilon} \tilde{v}(x)^2 \Delta_\varepsilon \frac{1}{|x - x_i|^2} dx$$

$$= -\frac{1}{n-4} \int_{\Omega_\varepsilon} \tilde{v} \Delta \tilde{v} + |\nabla \tilde{v}|^2 dx$$

$$\leq -\frac{1}{n-4} \int_{\Omega_\varepsilon} \tilde{v} \Delta \tilde{v} dx$$

$$\leq \frac{1}{n-4} \left( \int_{\Omega_\varepsilon} \frac{\tilde{v}(x)^2}{|x - x_i|^4} dx \right)^{\frac{1}{2}} \left( \int_{\Omega_\varepsilon} (\Delta \tilde{v}(x))^2 dx \right)^{\frac{1}{2}},$$

which gives

$$\int_{\Omega_\varepsilon} \frac{\tilde{v}(x)^2}{|x - x_i|^4} dx \leq \frac{1}{(n-4)^2} \int_{\Omega_\varepsilon} (\Delta \tilde{v}(x))^2 dx = \frac{1}{(n-4)^2} \int_{\Omega_\varepsilon} [(\Delta w(x))^+]^2 dx.$$

Going back to (23)

$$\int_{\Omega_\varepsilon} [(\Delta w)^+]^2 dx \leq \delta K \frac{1}{(n-4)^2} \int_{\Omega_\varepsilon} [(\Delta w)^+]^2 dx,$$

and hence $(\Delta w)^+ = 0$.

We conclude the lemma.

□

**Remark 3** $L_\varepsilon$ satisfies maximum principle on $\cup_{i=1}^{K} (B_\sigma(\Sigma_i) - B_\varepsilon(\Sigma_i))$ for $0 < \varepsilon_i < \varepsilon_0$.

**Lemma 5.2** Fix $\varepsilon_0 > 0$ such that $L_\varepsilon$ satisfies maximum principle on $\Omega_\varepsilon$. Let $4 - N < \gamma < 0$ be fixed. Let $w_\varepsilon$ be a solution to $L_\varepsilon w_\varepsilon = f_\varepsilon$ on $\Omega_\varepsilon$ for some $f_\varepsilon \in C^{0,\alpha}_\varepsilon(\Omega_\varepsilon)$, and $0 < \varepsilon_i \leq \varepsilon_0$. Assume that $w_\varepsilon = \Delta w_\varepsilon = 0$ on $\partial \Omega$. Then there exists $C > 0$ such that

$$\|w_\varepsilon\|_{4,\alpha,\gamma} \leq C \left( \|f_\varepsilon\|_{0,\alpha,\gamma-4} + \sum_{i=1}^{K} \left( \varepsilon_i^{-\gamma}\|w_\varepsilon\|_{C^0(\partial B_\varepsilon(\Sigma_i))} + \varepsilon_i^{2-\gamma}\|\Delta w_\varepsilon\|_{C^0(\partial B_\varepsilon(\Sigma_i))} \right) \right). \tag{24}$$

**Proof.** Let $\sigma > 0$ be as in Section 2.4.2 so that $\tilde{u}_\varepsilon$ is supported in $\cup_{i=1}^{K} B_\sigma(\Sigma_i)$. We fix a smooth positive function $\phi$ on $\Omega \setminus \cup \Sigma_i$ such that $\phi(x) = d(x, \Sigma_i)\gamma$ in each $B_\sigma(\Sigma_i)$. For simplicity we assume that $\Sigma_i$ is a point $x_i$. Then $\phi(x) = |x - x_i|\gamma$ on $B_\sigma(\Sigma_i)$. We compute

$$\Delta^2 \varphi(x) = c_{N,\gamma}|x - x_i|^{-4}, \quad c_{N,\gamma} := \gamma(\gamma - 2)(N^2 + \gamma^2 + 2N\gamma - 6N - 6\gamma + 8) > 0,$$
and
\[ \Delta \phi(x) = \tilde{c}_{N, \gamma}|x - x_i|^{-2}, \quad \tilde{c}_{N, \gamma} := \gamma(N + 2 - \gamma) < 0. \]

This shows that for a suitable choice of \( u_i \), we have for some \( \delta > 0 \)
\[ L_\varepsilon \phi(x) = \Delta^2 \phi - p\tilde{u}_\varepsilon^{p-1} \phi \geq \delta |x - x_i|^{-4} \quad \text{on } \Omega := \bigcup_{i=1}^K B_\varepsilon(\Sigma_i) \setminus B_{2\varepsilon}(\Sigma_i). \]

Therefore, we can choose \( c_{1, \varepsilon} = \| f_\varepsilon \|_{0, \alpha, \gamma - 4} \) so that
\[ L_\varepsilon(w_\varepsilon + c_{1, \varepsilon} \phi) \geq 0 \quad \text{on } \Omega. \]

We can also choose
\[
c_{2, \varepsilon} \approx \sum_{i=1}^K \left( \varepsilon_i^{-\gamma} \| w_\varepsilon \|_{C^0(\partial B_{\varepsilon}(\Sigma_i))} + \varepsilon_i^{2-\gamma} \| \Delta w_\varepsilon \|_{C^0(\partial B_{\varepsilon}(\Sigma_i))} \right)
+ \sum_{i=1}^K \left( \| w_\varepsilon \|_{C^0(\partial B_{\varepsilon}(\Sigma_i))} + \| \Delta w_\varepsilon \|_{C^0(\partial B_{\varepsilon}(\Sigma_i))} \right)
=: c_{3, \varepsilon} + c_{4, \varepsilon},
\]
so that
\[ w_\varepsilon + (c_{1, \varepsilon} + c_{2, \varepsilon}) \phi \geq 0 \quad \text{and } \Delta w_\varepsilon + (c_{1, \varepsilon} + c_{2, \varepsilon}) \Delta \phi \leq 0 \quad \text{on } \Omega. \]

Then by Maximum principle we have that (to get the other inequality use \(-\phi\))
\[ |w_\varepsilon| \leq (c_{1, \varepsilon} + c_{2, \varepsilon}) \phi \quad \text{and } |\Delta w_\varepsilon| \leq -(c_{1, \varepsilon} + c_{2, \varepsilon}) \Delta \phi \quad \text{in } \Omega. \]

Since, \( \Delta^2 w_\varepsilon = f_\varepsilon \) in \( \Omega_\varepsilon \setminus \Omega \), we get that
\[ |w_\varepsilon(x)| + |\Delta w_\varepsilon(x)| \lesssim (c_{1, \varepsilon} + c_{2, \varepsilon}) \quad x \in \Omega_\varepsilon \setminus \Omega. \]

We claim that
\[ c_{4, \varepsilon} \lesssim c_{3, \varepsilon} + \| f_\varepsilon \|_{0, \alpha, \gamma - 4}. \tag{25} \]

We assume by contradiction that the above claim is false. Then there exists a family of solutions \( w_\ell = w_{\varepsilon_\ell} \) to \( L_{\varepsilon_\ell} w_\ell = f_\ell \) with \( 0 < \varepsilon_{\ell, \varepsilon} < \varepsilon_0 \), \( f_\ell \in C^{0, \alpha}_{\gamma - 4}(\Omega_{\varepsilon_\ell}) \), \( \Delta w_\ell = 0 \) on \( \partial \Omega \) such that
\[ c_{4, \varepsilon_\ell} = 1 \quad \text{and } c_{3, \varepsilon_\ell} + \| f_\ell \| \to 0. \tag{26} \]

Then, up to a subsequence, \( \Omega_{\varepsilon_\ell} \to \tilde{\Omega} \), where \( \tilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{i=1}^K B_{\varepsilon_i}(\Sigma_i) \) for some \( 0 \leq \varepsilon_i \leq \varepsilon_0 \).

Here \( B_{\varepsilon_i}(\Sigma_i) = \Sigma_i \) if \( \varepsilon_i = 0 \) for some \( i \).

From the estimates on \( w_\ell \) we see that \( w_\ell \to w \) in \( \tilde{\Omega} \setminus \bigcup_{i=1}^K B_{\varepsilon_i}(\Sigma_i) \). Moreover, \( w \) satisfies
\[ L_\varepsilon w = 0 \quad \text{in } \tilde{\Omega}_\varepsilon. \]
where \( L_\varepsilon = \Delta^2 - p\bar{u}_\varepsilon^{p-1} \), with the understanding that if \( \varepsilon_i = 0 \) for some \( i \) then \( \bar{u}_\varepsilon = 0 \) on \( B_\sigma(\Sigma_i) \). Notice that \( w \) satisfies

\[
w = \Delta w = 0 \quad \text{on } \partial \Omega \cup \varepsilon_i \neq 0 \partial B_\varepsilon(\Sigma_i).
\]

If \( \varepsilon_i = 0 \) for some \( i \), then \( w \) is bi-harmonic in \( B_\sigma(\Sigma) \setminus \Sigma_i \), and as \( w(x) = O(d(x,\Sigma_i)\gamma) \) with \( 4 - N < \gamma < 0 \), we see that the singularity on \( \Sigma_i \) is removable. Thus, we can use maximum principle to conclude that \( w = 0 \) in \( \tilde{\Omega}_\varepsilon \). This contradicts the first condition in (26).

In this way we have that there exists \( C > 0 \) independent of \( \varepsilon \), but depending only on the right hand side of (24) such that

\[
|w_\varepsilon| \leq C \phi \quad \text{and} \quad |\Delta w_\varepsilon| \leq C(1 + |\Delta \phi|) \quad \text{in } \Omega_\varepsilon.
\]

The desired estimate follows from Schauder theory.

\[\square\]

**Lemma 5.3** Let \( (w_\ell) \subset C^{4,\alpha}_\mu(B_\sigma(\Sigma_i)) \) be a sequence of solutions to \( L_1 w_\ell = 0 \) in \( B_\sigma(\Sigma_i) \), for some fixed \( i \in \{1, 2, \ldots, K\} \). If \( |w_\ell| + |\Delta w_\ell| \leq C \) on \( B_\sigma(\Sigma_i) \setminus B_\sigma^2(\Sigma_i) \) then \( \|w_\ell\|_{C^{4,\alpha}_\mu(\Sigma_i)} \) is uniformly bounded.

**Proof.** It suffices to show that

\[
S_\ell := \sup \left( r^{-\mu}|w_\ell| + r^{2-\mu}|\Delta w_\ell| \right) \leq C.
\]

We assume by contradiction that the above supremum is not uniformly bounded. Let \( x_\ell = (r_\ell, \theta_\ell, y_\ell) \in B_\sigma(\Sigma_i) \) be such that

\[
S_\ell \approx r_\ell^{-\mu}|w_\ell(x_\ell)| + r_\ell^{2-\mu}|\Delta w_\ell(x_\ell)|.
\]

We claim that \( r_\ell \to 0 \). On the contrary, if \( r_\ell \to r_\infty \neq 0 \), then setting \( \bar{w}_\ell = \frac{w_\ell}{S_\ell} \) we see that \( \bar{w}_\ell \to \bar{w}_\infty \), where

\[
L_1 \bar{w}_\infty = 0 \quad \text{in } B_\sigma(\Sigma_i), \quad \bar{w}_\infty \equiv 0 \quad \text{in } B_\sigma(\Sigma_i) \setminus B_\sigma^2(\Sigma_i).
\]

Therefore, \( \bar{w}_\infty \equiv 0 \) in \( B_\sigma(\Sigma_i) \), which contradicts to

\[
r_\infty^{-\mu}|\bar{w}_\infty(x_\infty)| + r_\infty^{2-\mu}|\Delta \bar{w}_\infty(x_\infty)| \approx 1.
\]

If \( \Sigma_i = \{x_i\} \), we set

\[
\bar{w}_\ell(r, \theta) = \frac{r_\ell^{-\mu}w_\ell(rr_\ell, \theta)}{S_\ell}, \quad 0 < r < \frac{\sigma}{r_\ell}.
\]

Then

\[
r^{-\mu}|\bar{w}_\ell(r, \theta)| + r^{2-\mu}|\Delta \bar{w}_\ell(r, \theta)| \lesssim 1 \approx \sup_{\partial B_i}(|\bar{w}_\ell| + |\Delta \bar{w}_\ell|),
\]

19
and \( \tilde{w}_\ell \) satisfies

\[
L_{r_\ell} \tilde{w}_\ell = 0 \quad \text{in } B_{\frac{\sigma}{r_\ell}}.
\]

If \( \Sigma \) is higher dimensional, then \( y_\ell \to y_\infty \), and we choose Fermi coordinates around \( y_\infty \) so that \( y_\infty = 0 \), and the coordinates are defined for \( |y| < \tau \) for some \( \tau > 0 \). Then we set

\[
\tilde{w}_\ell(r, \theta, y) := \frac{r_\ell^{-\mu} w_\ell(r_{r_\ell}, \theta, r_{r_\ell}(y + \tilde{y}_\ell))}{S_\ell} \quad \text{with } \tilde{y}_\ell := \frac{y_\ell}{r_\ell} \quad 0 < r < \frac{\sigma}{r_\ell}, \quad |y| < \frac{\tau}{2r_\ell}.
\]

In this case \( \tilde{w}_\ell \) satisfies the equation \( L_{r_\ell} \tilde{w}_\ell = o(1) \).

In both cases we have that \( \tilde{w}_\ell \to \tilde{w}_\infty \neq 0 \), \( \tilde{w}_\infty \) satisfies \( r^{-\mu} |\tilde{w}_\infty| \leq C \). For the point singularity case, the limit function satisfies

\[
\Delta^2 \tilde{w}_\infty = \frac{A_p}{r^4} \tilde{w}_\infty \quad \text{in } \mathbb{R}^n \setminus \{0\},
\]

where \( A_p \) is given by (11). By proposition 3.2 we get that \( \tilde{w}_\infty \equiv 0 \), a contradiction.

For the higher dimensional case

\[
\Delta^2 \tilde{w}_\infty = \frac{A_p}{r^4} \tilde{w}_\infty \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^k,
\]

and we get a contradiction by Proposition 4.4. \( \square \)

**Lemma 5.4** There exists \( \varepsilon_0 > 0 \) sufficiently small such that if each \( \varepsilon_i < \varepsilon_0 \) then

\[
L_{\varepsilon} : C^{4,\alpha}_{\mu,A}(\Omega \setminus \Sigma) \to C^{0,\alpha}_{\mu-4}(\Omega \setminus \Sigma)
\]

is injective.

**Proof.** We assume by contradiction that for every \( \varepsilon_i^\ell := \frac{1}{i} \) there exists \( \varepsilon_i^\ell \) with each \( \varepsilon_i^\ell < \varepsilon_0 \) such that \( L_{\varepsilon^\ell} \) is not injective. Let \( w_\ell \in C^{4,\alpha}_{\mu,A}(\Omega \setminus \Sigma) \) be a non-trivial solution to \( L_{\varepsilon^\ell} w_\ell = 0 \). We normalize \( w_\ell \) so that

\[
\max_{\partial \Omega_{\varepsilon^\ell}} \left( \rho(x)^{-\mu} |w_\ell(x)| + \rho(x)^2 |\Delta w_\ell(x)| \right) = 1.
\]

Then by Lemma 5.2 we get that

\[
\sup_{\Omega_{\varepsilon^\ell}} \left( \rho(x)^{-\mu} |w_\ell(x)| + \rho(x)^2 |\Delta w_\ell(x)| \right) \leq C. \tag{27}
\]

First consider the case when \( \Sigma \) is a set of finite points. Assume that the above maximum is achieved on \( \partial B_{\varepsilon_j^\ell}(x_j) \) for some \( j \), and up to a translation, assume that \( x_j = 0 \). We set

\[
\tilde{w}_\ell(x) = (\varepsilon_j^\ell)^{-\mu} w_\ell(\varepsilon_j^\ell x), \quad |x| < \frac{\sigma}{\varepsilon_j^\ell}.
\]
Then $L_1 \hat{w}_\ell = 0$ on $B_{R_\ell}$ for some $R_\ell \to \infty$, and $r^{-\mu}|\hat{w}_\ell| + r^{2-\mu} |\Delta \hat{w}_\ell| \leq C$ for $1 \leq r \leq \frac{\sigma}{\varepsilon_j}$. Therefore, by Lemma 5.3

$$r^{-\mu}|\hat{w}_\ell| + r^{2-\mu} |\Delta \hat{w}_\ell| \leq C \quad \text{for } r \leq \frac{\sigma}{\varepsilon_j}.$$ 

Hence $\hat{w}_\ell \to \hat{w}_\infty$, where $\hat{w}_\infty$ satisfies

$$L_1 \hat{w}_\infty = 0 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad r^{-\mu}|\hat{w}_\infty| + r^{2-\mu} |\Delta \hat{w}_\infty| \leq C.$$ 

Hence, by Proposition 3.1 we have $\hat{w}_\infty \equiv 0$, a contradiction to $\max_{\partial B_1} |\hat{w}_\infty| + |\Delta \hat{w}_\infty| = 1$.

Next we consider the case of higher dimensional singularity. Let $x_\ell = (r_\ell, \theta_\ell, y_\ell)$ be a point around $\Sigma_j$ for some $j$ such that

$$r_\ell^{-\mu}|w_\ell(x_\ell)| + r_\ell^{2-\mu} |\Delta w_\ell(x_\ell)| \approx \sup_{\Omega} \rho(x)^{-\mu}|w_\ell| + \rho(x)^{2-\mu} |\Delta w_\ell| =: S_\ell.$$

We can also assume that $r_\ell \leq \varepsilon_j^\ell$, thanks to (27). We shall take $r_\ell = \varepsilon_j^\ell$ if they are of the same order so that either $r_\ell = o(\varepsilon_j^\ell)$ or $r_\ell = \varepsilon_j^\ell$. We choose Fermi coordinates around $y_\ell$ so that $y_\ell = 0$, and the coordinates are defined for $|y| < \tau$ for some $\tau > 0$. We set

$$\tilde{w}_\ell(r, \theta, y) := \frac{r_\ell^{-\mu}w_\ell(r_\ell, \theta, y_\ell(y + \tilde{y}_\ell))}{S_\ell}, \quad \tilde{y}_\ell := \frac{y_\ell}{r_\ell}, \quad 0 < r < \frac{\sigma}{r_\ell}, \quad |y| < \frac{\tau}{2r_\ell}.$$ 

As before one gets a contradiction, thanks to Propositions 4.1 and 4.4. □

6 Uniform surjectivity of $L_{\bar{\varepsilon}}$ on $C^{4,\alpha}_{\mu,\gamma}(\Omega \setminus \Sigma)$

Let $\rho$ be a smooth function on $\Omega \setminus \Sigma$ with positive lower bound such that $\rho(\cdot) = \text{dist}(\cdot, \Sigma_i)$ in $B_\sigma(\Sigma_i)$, for every $i$. The weighted space $L_2^\delta(\Omega \setminus \Sigma)$ is defined by

$$L_2^\delta(\Omega \setminus \Sigma) := \left\{ w \in L_{\text{loc}}^2(\Omega \setminus \Sigma) : \int_{\Omega} \rho^{-4-2\delta} |w|^2 dx < \infty \right\}.$$ 

Let $L_2^{-\delta}(\Omega \setminus \Sigma)$ be the dual of $L_2^\delta(\Omega \setminus \Sigma)$ with respect to the pairing

$$L_2^\delta(\Omega \setminus \Sigma) \times L_2^{-\delta}(\Omega \setminus \Sigma) \ni (w_1, w_2) \mapsto \int_{\Omega} w_1 w_2 \rho^{-4} dx.$$ 

We note that the following embedding is continuous

$$C^{k,\alpha}_{\gamma}(\Omega \setminus \Sigma) \hookrightarrow L_2^\delta(\Omega \setminus \Sigma) \quad \text{for } \delta < \gamma + \frac{N-4}{2}.$$ 

21
The domain $D(L_\varepsilon)$ of the operator $L_\varepsilon$ is the set of functions $w \in L^2_\delta$ (for simplicity we drop the domain $\Omega \setminus \Sigma$ from the notation) such that $L_\varepsilon w = h \in L^2_{\delta-4}$ in the sense of distributions. One can show that the following elliptic estimate holds:

$$\sum_{\ell=1}^{4} \| \nabla^{\ell} w \|_{L^2_{\delta-\ell}(\Omega_\sigma)} \leq C(\|h\|_{L^2_{\delta-4}} + \|w\|_{L^2_\delta}),$$

where $\Omega_\sigma := \bigcup_{i=1}^{K} \left( B_{\sigma}(\Sigma_i) \setminus \Sigma_i \right)$. In particular, $L_\varepsilon : L^2_\delta \to L^2_{\delta-4}$ is densely defined, and it has closed graph. Moreover, if $\delta - \frac{N-4}{2} \not\in \{ \Re \gamma_{j}^{\pm} : j = 0, 1, \ldots \}$, then $L_\varepsilon$ is Fredholm (see [Maz91]).

We shall fix $\delta > 0$ slightly bigger than $\mu + \frac{N-4}{2}$, where $\mu$ is fixed according to (14). The adjoint of the operator $L_\varepsilon : L^2_{\delta-4} \to L^2_{\delta}$ is given by

$$L^*_\varepsilon : L^2_{\delta} \to L^2_{\delta+4}, \quad w \mapsto \rho^{4} L_\varepsilon(w \rho^{-4}). \quad (28)$$

Then the adjoint operator (28) is injective, and $L_\varepsilon$ in (28) is surjective. Using the isomorphism $\rho^{2\delta} : L^2_{\delta} \to L^2_{2\delta+\delta}, \quad w \mapsto \rho^{2\delta} w,$

we identify the adjoint operator as

$$L^*_\varepsilon : L^2_{\delta+4} \to L^2_{\delta}, \quad w \mapsto \rho^{4-2\delta} L_\varepsilon(w \rho^{2\delta-4}).$$

Now we consider the composition

$$\mathcal{L} = L_\varepsilon \circ L^*_\varepsilon : L^2_{\delta+4} \to L^2_{\delta-4}, \quad w \mapsto L_\varepsilon[\rho^{4-2\delta} L_\varepsilon(w \rho^{2\delta-4})].$$

Then $\mathcal{L}$ is an isomorphism, and hence there exists a two sided inverse

$$G_\varepsilon : L^2_{\delta-4} \to L^2_{\delta+4}.$$ 

Consequently, the right inverse of $L_\varepsilon$ is given by $G_\varepsilon := L^*_\varepsilon G_\varepsilon$. It follows from [MP96] that

$$G_\varepsilon : C^{0,\alpha}_{\nu-4}(\Omega \setminus \Sigma) \to C^{4,\alpha}_{\nu,N}(\Omega \setminus \Sigma)$$

is bounded.

**Lemma 6.1** Let $\varepsilon_0 > 0$ be as in Lemma 5.4. Then the system $L_\varepsilon w_1 = 0, w_1 = L^*_\varepsilon w_2$ with $w_1 \in C^{4,\alpha}_{\nu,N}(\Omega \setminus \Sigma)$ and $w_2 \in C^{8,\alpha}_{\nu+4,N}(\Omega \setminus \Sigma)$ has only trivial solution.
Proof. We set \( w = \rho^{2\delta-4}w_2 \). Then \( L_\varepsilon[\rho^{4-2\delta}L_\varepsilon w] = 0 \). Multiplying the equation by \( w \) and then integrating by parts we get
\[
0 = \int_\Omega \rho^{4-2\delta}|L_\varepsilon w|^2\,dx.
\]
Since \( \mu + 2\delta > \mu \), we have \( w \in C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma) \subset C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma) \). Then by Lemma 5.3 we get that \( w = 0 \), equivalently \( w_1 = w_2 = 0 \).

Lemma 6.2 There exists \( \varepsilon_0 > 0 \) small such that if \( 0 < \varepsilon_i < \varepsilon_0 \) for every \( i \), then the sequence of solutions \( (w_{1,\varepsilon_i}) \subset C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma) \cap L_\varepsilon^*[C_{4,\alpha}^{8,\alpha}(\Omega \setminus \Sigma)] \) to \( L_\varepsilon w_{1,\varepsilon_i} = f_\varepsilon \) is uniformly bounded in \( C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma) \), provided \( (f_\varepsilon) \) is uniformly bounded in \( C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma) \).

Proof. Assume by contradiction that the lemma is false. Then there exists a sequence of \( K \)-tuples \( (\varepsilon_i) \) with \( \varepsilon_i \to 0 \) for each \( i = 1, \ldots, K \), and \( w_{1,\varepsilon_i} \in C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma) \cap L_\varepsilon^*[C_{4,\alpha}^{8,\alpha}(\Omega \setminus \Sigma)] \) with \( L_\varepsilon w_{1,\varepsilon_i} = f_\varepsilon \) such that \( \|f_\varepsilon\|_{C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma)} \leq C \). By Lemma 5.2
\[
\|w_{1,\varepsilon_i}\|_{C_{4,\mu}^{4,\alpha}(\Omega \setminus \Sigma)} \leq C + C \max_{\Omega \setminus B_{1,\varepsilon_i}(\Sigma)} \left( \rho^{-\nu}|w_{1,\varepsilon_i}| + \rho^{2-\nu}|\Delta w_{1,\varepsilon_i}| \right) =: C + CS_{\varepsilon_i}.
\]

First we consider the case when \( \Sigma \) is a set of finitely many points. We distinguish the following two cases.

Case 1 \( S_{\varepsilon_i} \leq C \).

In this case we proceed as in the proof of Lemma 5.3. Let \( x_\varepsilon = (r_\varepsilon, \theta_\varepsilon) \) be such that
\[
\sup_{\Omega \setminus B_{1,\varepsilon_i}(\Sigma)} \left( \rho^{-\nu}|w_{1,\varepsilon_i}| + \rho^{2-\nu}|\Delta w_{1,\varepsilon_i}| \right) \approx \rho^{-\nu}(x_\varepsilon)|w_{1,\varepsilon_i}(x_\varepsilon)| + \rho^{2-\nu}(x_\varepsilon)|\Delta w_{1,\varepsilon_i}(x_\varepsilon)| =: S_\varepsilon \to \infty.
\]

Up to a subsequence, \( x_\varepsilon \in B_{1,\varepsilon_i}(\Sigma) \) for some fixed \( i \). Then necessarily \( r_\varepsilon = o(\varepsilon_i) \). Setting
\[
\tilde{w}_{1,\varepsilon_i}(r, \theta) := \frac{r^{-\nu}w_{1,\varepsilon_i}(rr_\varepsilon, \theta)}{S_\varepsilon}
\]
one would get that \( \tilde{w}_{1,\varepsilon_i} \to \tilde{w}_1 \not\equiv 0 \) where
\[
\tilde{L}_1 \tilde{w}_1 = 0 \quad \text{in} \ \mathbb{R}^n \setminus \{0\}, \quad r^{-\nu}|\tilde{w}_1| \leq C, \quad \tilde{L}_1 := \Delta^2 - \frac{A_p}{r^4},
\]
where \( A_p \) as in (11). Since \( \nu \) does not coincide with indicial roots of \( \tilde{L}_1 \), as in the proof of Proposition 5.2 one obtains that \( \tilde{w}_1 \equiv 0 \), a contradiction.

Case 2 \( S_{\varepsilon_i} \to \infty \).
In this case we first divide the function \( \tilde{w}_{1,\varepsilon} \) by \( S_{\varepsilon} \). Then consider the scaling with respect to \( \varepsilon^{\ell} \) instead of \( r_{\ell} \) (see Lemma 5.4) and proceeding as before we would get that \( w_{1,\varepsilon} \to \tilde{w}_1 \neq 0 \) where

\[
L_1 \tilde{w}_1 = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad r^{-\nu}\| \tilde{w}_1 \| \leq C.
\]

Since \( \tilde{w}_1 \) decays at infinity, its decay rate is determined by the indicial roots of \( L_1 \) (which are exactly the same as \( \Delta^2 \)) at infinity. In fact, \( \tilde{w}_1 \) would be bounded by \( r^{4-N} \) at infinity, see e.g. the proof of Proposition 5.1.

Since \( \tilde{w}_{1,\varepsilon} \in L^s_{\varepsilon} \mathcal{C}_{\nu+4,N}(\Omega \setminus \Sigma) \), we have \( w_{1,\varepsilon} = \rho^{4-2\delta} L_{\varepsilon} w_{2,\varepsilon} \) for some \( w_{2,\varepsilon} \in \mathcal{C}_{\nu+2\delta,N}(\Omega \setminus \Sigma) \). As \( 2\delta + \nu > \mu \), applying Lemmas 5.2 and 5.3 we can show that the scaled functions

\[
\tilde{w}_{2,\varepsilon}(r, \theta) := \frac{\varepsilon^{-\nu - 2\delta} w_{2,\varepsilon}(r \varepsilon^{i,\ell}, \theta)}{S_{\ell}},
\]

converges to a limit function \( \tilde{w}_2 \), where

\[
L_1 \tilde{w}_2 = r^{2\delta-4} \tilde{w}_1 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad r^{-\nu-2\delta}| \tilde{w}_2 | \leq C.
\]

Thus, \( L_1[r^{4-2\delta} L_{\varepsilon} \tilde{w}_2] = 0 \). We multiply this equation by \( \tilde{w}_2 \) and integrate it on \( \mathbb{R}^N \). Then an integration by parts leads \( L_1 \tilde{w}_2 = 0 \) (this is justified because of the decay of \( \tilde{w}_1 \) at infinity, provided we choose \( \delta > 0 \) sufficiently close to \( \mu + \frac{N}{2} \)). Again, as \( 2\delta + \nu > \mu \), by Proposition 5.1 we have \( \tilde{w}_2 = \tilde{w}_1 = 0 \), a contradiction.

When \( \Sigma \) is of positive dimension, we need to do the scaling as in Lemmas 5.4 and 5.3. We now prove that the limit is independent of the variable \( y \). The argument is based on the theory of edge operators and their parametrices.

As in the previous section, we set

\[
\tilde{w}_{1,\varepsilon}(r, \theta, y) := \frac{r^{-\nu} w_{1,\varepsilon}(rr_{\ell}, \theta, r_{\ell}(y+\tilde{y}_{\ell})))}{S_{\ell}}, \quad \tilde{y}_{\ell} := \frac{y_{\ell}}{r_{\ell}}, \quad 0 < r < \sigma, \quad |y| < \frac{\tau}{2r_{\ell}}.
\]

We now proceed as before to show, because of the normalization, that \( \tilde{w}_{1,\varepsilon} \to \tilde{w}_1 \neq 0 \) and

\[
\tilde{L}_1 \tilde{w}_1 = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{R}^k, \quad r^{-\nu}\| \tilde{w}_1 \| \leq C, \quad \tilde{L}_1 := \Delta^2 - \frac{A_{\ell}}{r^4}.
\]

Claim: The function \( \tilde{w}_1 \) does not depend on \( y \in \mathbb{R}^k \). By standard theory in edge calculus (see Maz91), each operator \( L_{\varepsilon} \) has a left parametrix \( G_{\varepsilon} \) since the solutions are normalized. In other words, there exists a compact (in the sense of pseudo-differential operators) \( R_{\varepsilon} \) such that

\[
G_{\varepsilon} L_{\varepsilon} = Id + R_{\varepsilon}
\]

along every sequence \( \varepsilon^{\ell} \). Furthermore, since \( R_{\varepsilon} \) is compact, it maps polyhomogeneous functions into functions with fast decay. Applying the previous identity to \( \tilde{w}_{1,\varepsilon} \), one sees

\[\text{We are very grateful to Rafe Mazzeo for explaining us the argument, already mentioned in MP96.}\]
right away that $\tilde{w}_{1,\epsilon}$ is itself polyhomogeneous. Consider now any derivative $\partial^\alpha y w_{1,\epsilon}$, denoted for simplicity $w_{1,\epsilon}^{(\alpha)}$. By appropriately normalizing the latter function and using the fact that the compact operator $R_{\epsilon \ell}$ is itself polyhomogeneous in $y$, one gets passing to the limit in the previous equation that the limiting function has to be in kernel of $\tilde{L}$ with faster decay, hence its identically zero. Hence the function $\tilde{w}_1$ is independent of $y$.

Therefore, we are back to the case of a point singularity. This proves the lemma.

\[ \square \]

7 Fixed point arguments

To prove existence of solution to (9) we use a fixed point argument on the space $C^{4,\alpha}(\Omega \setminus \Sigma)$. Since we need to find $v$ such that $\bar{u}_\epsilon + v$ is positive in $\Omega$, we shall solve the equation

\[
\begin{align*}
\Delta^2 (\bar{u}_\epsilon + v) &= |\bar{u}_\epsilon + v|^p \quad \text{in } \Omega \\
v &= \Delta v = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Equivalently,

\[ L_\epsilon v + f_\epsilon + Q(v) = 0, \quad Q(v) := -|\bar{u}_\epsilon + v|^p + \bar{u}_\epsilon^p + p\tilde{u}_\epsilon^{p-1}v, \]

where $f_\epsilon = \Delta^2 \bar{u}_\epsilon - \bar{u}_\epsilon^p$ as before. Applying the inverse of $L_\epsilon$, that is $G_\epsilon$, we rewrite the above equation as

\[ v + G_\epsilon f_\epsilon + G_\epsilon Q(v) = 0. \]

The crucial fact we shall use is that the norm of $G_\epsilon$ is uniformly bounded if $\epsilon_0$ is sufficiently small.

We note that if $v \in C^{4,\alpha}(\Omega \setminus \Sigma)$ is a weak solution to the above equation then by maximum principle we have that $\bar{u}_\epsilon + v > 0$ in $\Omega$. This is a simple consequence of the fact that $\nu > -\frac{1}{p-1}$, and therefore, $\Delta(\bar{u}_\epsilon + v) < 0$ and $\bar{u}_\epsilon + v > 0$ in a small neighborhood of $\Sigma$, thanks to the asymptotic behavior of $\bar{u}_\epsilon$, $\Delta \bar{u}_\epsilon$ around the origin.

We recall that the error $f_\epsilon = \Delta^2 \bar{u}_\epsilon - \bar{u}_\epsilon^p$ satisfies the estimate $\|f_\epsilon\|_{0,\alpha,\nu-4} \leq C\epsilon_0^{-\frac{4\nu}{p-1}}$ if $\Sigma$ is a discrete set, and $\|f_\epsilon\|_{0,\alpha,\nu-4} \leq C\epsilon_0^{q-1}$, $q = \frac{p^2}{p-1} - \nu$ otherwise. Let us first consider the case when $\Sigma$ is a set of finitely many points. Then, there exists $C_0 > 0$ such that $\|G_\epsilon f_\epsilon\|_{4,\alpha,\nu} \leq C_0 \epsilon_0^{-\frac{4\nu}{p-1}}$. This suggests to work on the ball

\[ \mathcal{B}_{\epsilon_0, M} = \left\{ v \in C^{4,\alpha}_\nu : \|v\|_{4,\alpha,\nu} \leq M\epsilon_0^{-\frac{4\nu}{p-1}} \right\}, \]

for some $M > 2C_0$ large. We shall show that the map $v \mapsto G_\epsilon[f_\epsilon + Q(v)]$ is a contraction on the ball $\mathcal{B}_{\epsilon_0, M}$. To this end we shall assume that $\epsilon_i \in [\alpha \epsilon_0, \epsilon_0]$ for every $i = 1, \ldots, K$ for some fixed $\alpha \in (0, 1)$. 

25
Lemma 7.1 Let $M_1 > 1$ be fixed. Then for $\varepsilon_0 << 1$ we have

$$\|Q(v_1) - Q(v_2)\|_{0,\alpha,-4} \leq \frac{1}{M_1} \|v_1 - v_2\|_{4,\alpha,\nu} \quad \text{for every } v_1, v_2 \in \mathcal{B}_{\varepsilon_0,M}.$$ 

Proof. We start by showing that there exists $0 < \tau < \sigma$ small, independent of $\varepsilon_0 << 1$, such that

$$|v(x)| \leq \frac{1}{10} \bar{u}_\varepsilon(x) \quad \text{for every } x \in \bigcup_{i=1}^{K} B_\varepsilon(x_i), \, v \in \mathcal{B}_{\varepsilon_0,M}. \quad (31)$$

To prove this we recall that for any fixed $R > 1$ there exists $c_1, c_2 > 1$ such that

$$\frac{1}{c_1} \leq |x|^{\frac{4}{p+1}} u_{\varepsilon_i}(x) \leq c_1 \quad \text{for } |x| \leq R\varepsilon_0,$$

$$\frac{1}{c_2} \leq \varepsilon_0^{-N+\frac{4p}{p-1}} |x|^{-4} u_{\varepsilon_i}(x) \leq c_2 \quad \text{for } R\varepsilon_0 \leq |x| \leq \tau.$$

On the other hand,

$$\varepsilon_0^{-N+\frac{4p}{p-1}} \rho(x)^{-\nu} |v(x)| \leq M.$$

As $\nu > 4 - N$, we have (31) for some $\tau > 0$ small.

We have

$$Q(v_1) - Q(v_2) = \int_0^1 \frac{d}{dt}|\bar{u}_\varepsilon + v_1 + t(v_2 - v_1)|^p dt + p\bar{u}_\varepsilon^{p-1}(v_1 - v_2)$$

$$= p(v_2 - v_1) \int_0^1 (|\bar{u}_\varepsilon + v_1 + t(v_2 - v_1)|^{p-1} - \bar{u}_\varepsilon^{p-1}) \, dt$$

$$=: p(v_2 - v_1) \int_0^1 Q(v_1, v_2) \, dt.$$

Next, using that

$$(1 + a)^{p-1} = 1 + O(|a|) \quad \text{for } |a| \leq \frac{1}{2},$$

we estimate for $x \in \bigcup_{i=1}^{K} B_{R\varepsilon_0}(x_i)$

$$|Q(v_1, v_2)|(x) \leq C\bar{u}_\varepsilon(x)^{p-2}(|v_1|(x) + |v_2|(x))$$

$$\leq CM \varepsilon_0 \varepsilon_{\frac{4p}{p-1}} \varepsilon_0^{N-4} \rho^{-4}(x),$$

and for $x \in \bigcup_{i=1}^{K} (B_\varepsilon(x_i) \setminus B_{R\varepsilon_0}(x_i))$

$$|Q(v_1, v_2)|(x) \leq C_{\tau,R} \max \{\varepsilon_0^{N-4-p}, \varepsilon_0^{(N-4-p)(p-1)}\} \rho^{-4}(x),$$

For $x \in \Omega \setminus \bigcup_{i=1}^{K} B_\varepsilon(x_i)$

$$|Q(v_1, v_2)|(x) \leq C(\bar{u}_\varepsilon^{p-1} + |v_1|^{p-1} + |v_2|^{p-1})(x)$$

$$\leq C_{\tau,M} \varepsilon_0^{(N-4)p-N} \rho^{-4}(x),$$

26
where in the last inequality we have used that, in this region,
\[ \tilde{u}_\varepsilon(x) + |v_1(x)| + |v_2(x)| \leq C_\tau,M \varepsilon_0^{N-4p}. \]

Combining these estimates we get for \( \varepsilon_0 \ll 1 \)
\[ \|Q(v_1) - Q(v_2)\|_{0,0,\nu-4} \leq c_\varepsilon_0 \|v_1 - v_2\|_{4,\alpha,\nu}, \]
where \( c_\varepsilon_0 \to 0 \) as \( \varepsilon_0 \to 0 \).

In order to estimate the weighted Hölder norm of \( Q(v_1) - Q(v_2) \) we note that the function \( |\tilde{u}_\varepsilon + v|^{p-1} \) is only \( C^{0,p-1} \) for \( 1 < p < 2 \), which in turn implies that \( Q(v_1, v_2) \) is only \( C^{0,p-1} \). This suggests that we need to take the Hölder exponent \( \alpha \leq p - 1 \).

For \( 0 < s < \sigma \) we write
\[
\begin{align*}
&\sup_{x,x' \in N_s \setminus N_{\varepsilon}} \frac{|[Q(v_1) - Q(v_2)](x) - [Q(v_1) - Q(v_2)](x')|}{|x - x'|^\alpha} \\
&\leq 4\|Q(v_1) - Q(v_2)\|_{0,0,\nu-4} + 4^{4-\nu+\alpha} \sup_{x,x' \in N_s \setminus N_{\varepsilon}, |x - x'| \leq \frac{\varepsilon}{4}} \frac{|[Q(v_1) - Q(v_2)](x) - [Q(v_1) - Q(v_2)](x')|}{|x - x'|^\alpha}.
\end{align*}
\]

Notice that for \( x, x' \in N_s \setminus N_{\varepsilon} \) with \( |x - x'| \leq \frac{\varepsilon}{4} \), the line segment \([x, y]\) joining \( x \) and \( y \) lies in \( N_{2s} \setminus N_{\varepsilon} \). The desired estimate follows on the region \( \cup_{i=1}^{K} B_r(x_i) \) by estimating \( Q(v_1, v_2)(x) - Q(v_1, v_2)(x') \) using the following gradient bound (we are using that \( |\tilde{u}_\varepsilon + v|^{p-1} \) is \( C^1 \) in this region)
\[
\begin{align*}
\nabla Q(v_1, v_2) &= (p-1) \left[ (\tilde{u}_\varepsilon + v_1 + t(v_2 - v_1)^{p-2} - \tilde{u}_\varepsilon^{p-2}) \right] \nabla \tilde{u}_\varepsilon \\
&\quad + (p-1)(\tilde{u}_\varepsilon + v_1 + t(v_2 - v_1)^{p-2}) \nabla v_1 + (t(v_2 - v_1)) \\
&= O(1) \tilde{u}_\varepsilon^{p-3}(|v_1| + |v_2|) |\nabla u_\varepsilon| + O(1) \tilde{u}_\varepsilon^{p-2}(|\nabla v_1| + |\nabla v_2|).
\end{align*}
\]

In fact, gradient bounds can also be used for the region \( \Omega \setminus \cup_{i=1}^{K} B_r(x_i) \) if \( p \geq 2 \). For \( 1 < p \leq 2 \), one can use the following inequality
\[
||\phi||_{p-1}(x) - |\phi||_{p-1}(x')| \leq |\phi(x) - \phi(x')| \leq ||\nabla \phi||_{C^0([x,x'])}|x - x'|^{p-1},
\]
with \( \phi = \tilde{u}_\varepsilon \) and \( \phi = \tilde{u}_\varepsilon + v_1 + t(v_2 - v_1) \).

We conclude the lemma.

\[ \square \]

From the above lemma we see that for a suitable choice of \( M \) and \( M_1 \), the map \( v \mapsto G_v[f_\varepsilon + Q(v)] \) is a contraction on the ball \( B_{\varepsilon_0,M} \) onto itself, for \( \varepsilon_0 \ll 1 \). Hence, we get a solution to (30) as desired.
When $\Sigma$ is not discrete, one shows in a similar way that the map $v \mapsto G_\varepsilon[f_\varepsilon + Q(v)]$ is a contraction on the ball

$$B_{\varepsilon_0,M} = \{ v \in C^4_{\nu} : \| v \|_{4,\alpha,\nu} \leq M\varepsilon_0^2 \},$$

for some suitable $M >> 1$. Here $q = \frac{p-5}{p-1} - \nu$, and the parameter $\nu$ satisfies $-\frac{4}{p-1} < \nu < \min\{\frac{p-5}{p-1}, \Re(\gamma_0^-)\}$.

8 Proofs of Theorems 1.1 and 1.2

This section is devoted to the completion of the proofs of the theorems stated in the introduction. In the previous section, we constructed for a fixed $\bar{\varepsilon}$ a solution of (5).

**Proof of Theorem 1.1** As noticed already in [MP96], the modifications are very minor. Recall the equation

$$P_{g_0}u = u^{\frac{n+4}{n-4}} \text{ in } M \setminus \Sigma,$$

where $g_0$ is a fixed metric. Using Fermi coordinates and the rescaled Delaunay-type solutions shows that, since the linearized operator is the bilaplacian with lower order terms, those terms disappear in the rescaling/blow-up and one can prove in an exactly parallel way that the linearization is uniformly surjective provided $\varepsilon$ is small enough. The geometric assumptions in the theorem ensures then that the constructed solution in the fixed point, is positive.

**Proof of Theorem 1.2** The statement follows from a combination of the solution constructed in the previous section and the application of the implicit function theorem as described in [MP96]. To get the infinite dimensionality of the solution space, we invoke as in [MP96], the edge calculus in [Maz91].

9 Appendix: Singular radial solutions in $\mathbb{R}^N \setminus \{0\}$

In this appendix, we collect several results related to the ODE analysis of Delaunay-type solutions for our problem (see [GG06, GWZ17]). For sake of completeness, we provide the proofs. Furthermore, since we need rather fine properties of these solutions, we also straighten some of the arguments in the above-mentioned papers.

**Lemma 9.1** Let $u$ be a radial solution to (8) with $\frac{N}{N-4} < p < \frac{N+4}{N-4}$ as given by Theorem 2.2. Then

$$r^4u^{p-1}(r) \leq \frac{p+1}{2}k(p,N).$$
Proof. Set
\[ \tilde{u}(y) = |x|^{N-4} u(x), \quad x = \frac{y}{|y|^2}. \]

Then \( \tilde{u} \) satisfies
\[ \Delta^2 \tilde{u} = |y|^\alpha \tilde{u} \quad \text{in} \quad \mathbb{R}^N, \quad \alpha := (N-4)p - (N+4) \in (-4, 0), \quad (32) \]
and \( \tilde{u} \) does not have any singularity at the origin. Now we set
\[ \bar{u}(t) = r^{\frac{N+4}{p-1}} \tilde{u}(r) = r^{-\frac{4}{p-1}} u\left(\frac{1}{r}\right), \quad t = \log r. \]

One checks that
\[ \bar{u}(t) \to 0, \quad \bar{u}'(t) \to 0, \quad \bar{u}''(t) \to 0, \quad \bar{u}'''(t) \to 0 \quad \text{as} \quad t \to -\infty. \]

Moreover,
\[ \bar{u}'''(t) + K_3 \bar{u}''(t) + K_2 \bar{u}'(t) + K_1 \bar{u}(t) + K_0 \bar{u}(t) = \bar{u}^p(t), \]
where (see e.g. \cite{GWZ17, GG06})
\[ K_0 := \frac{4 + \alpha}{(p-1)^4} \left[ 2(N-2)(N-4)(p-1)^3 + (4 + \alpha)(N^2 - 10N + 20)(p-1)^2 \right. \]
\[ \left. -2(4 + \alpha)^2(N-4)(p-1) + (4 + \alpha)^3 \right] = k(p, N) \]
\[ K_1 := -\frac{2}{(p-1)^3} \left[ (N-2)(N-4)(p-1)^3 + (4 + \alpha)(N^2 - 10N + 20)(p-1)^2 \right. \]
\[ \left. -3(\alpha^2 + 8\alpha + 16)(N-4)(p-1) + 2\alpha(\alpha^2 + 12\alpha + 48) + 128 \right] \]
\[ = -\frac{2}{(p-1)^3} \left[ (6N - N^2 - 8)p^3 + (22N - N^2 - 56)p^2 + (5N^2 - 14N - 56)p - 3N^2 - 8 - 14N \right], \]
\[ K_2 := \frac{1}{(p-1)^2} \left[ (N^2 - 10N + 20)(p-1)^2 - 6(4 + \alpha)(N-4)(p-1) + 6\alpha(\alpha + 8) + 96 \right], \]
\[ K_3 := \frac{2}{p-1} \left[ (N-4)(p-1) - 2(4 + \alpha) \right] = \frac{2}{p-1} [N + 4 - p(N-4)]. \]

It follows that \( K_3 > 0 \) for \( \frac{N}{N-4} < p < \frac{N+4}{N-4} \), and \( K_1 \) vanishes at the following points
\[ p_1 := \frac{N + 4}{N - 4}, \quad p_2^+ := \frac{6 - N \pm 2\sqrt{N^2 - 4N + 8}}{N - 2}. \]

We also have that \( p_2^- < 0 < p_2^+ < \frac{N}{N-4} \). In particular, as \( K_1(\infty) > 0 \), we have that \( K_1 < 0 \) for \( \frac{N}{N-4} < p < \frac{N+4}{N-4} \).
Let us now define the energy

\[ E(t) := \frac{1}{p+1} \bar{u}^{p+1}(t) - \frac{K_0}{2} \bar{u}(t)^2 - \frac{K_2}{2} |\bar{u}'(t)|^2 + \frac{1}{2} |\bar{u}''(t)|^2. \]

If \( \bar{u}'(t_1) = 0 \) for some \( t_1 \in \mathbb{R} \), then following the proof of [GG06, Lemma 6] we get

\[ E(t_1) - E(-\infty) = K_1 \int_{-\infty}^{t_1} |\bar{u}'(t)| \, dt - K_3 \int_{-\infty}^{t_1} |\bar{u}''(t)| \, dt \leq 0. \]

Thus, \( \bar{u}'(t_1) = 0 \) implies that

\[ \bar{u}^{p-1}(t_1) \leq \frac{p+1}{2} K_0. \]

The proof follows from this, and the asymptotic behavior of \( u \) at the origin. \( \square \)

The next lemma provides uniqueness of solutions to (32). We start with the following lemma:

**Lemma 9.2** Let \( u \) be a non-negative bounded radial solution to (32) on \( B_1 \setminus \{0\} \). Then \( u \) is Hölder continuous for every \( \alpha \in (-4, 0) \), and it is \( C^2 \) for \( \alpha \in (-2, 0) \). Moreover,

\[ \lim_{r \to 0^+} r^{-\alpha-2} \Delta u(r) = c_{N,\alpha} u(0)^p, \quad \alpha \in (-4, -2) \tag{33} \]

and

\[ \lim_{r \to 0^+} \frac{\Delta u(r)}{\log r} = c_{N,\alpha} u(0)^p, \quad \alpha = 2, \tag{34} \]

for some constant (independent of \( u \)) \( c_{N,\alpha} < 0 \).

**Proof.** We set

\[ v(x) = \frac{1}{\gamma_N} \int_{B_1} \frac{1}{|x-y|^{N-1}} |y|^\alpha u^p(y) \, dy, \quad h(x) := u - v(x), \]

where \( \frac{1}{\gamma_N} \frac{1}{|x-y|^{N-1}} \) is a fundamental solution of \( \Delta^2 \) in \( \mathbb{R}^N \). Since \( u \) is bounded, one easily gets that \( v \) is Hölder continuous for \( \alpha \in (-4, 0) \), and differentiating under the integral sign, \( v \in C^2 \) for \( \alpha \in (-2, 0) \). Thus, \( h \) is a bounded biharmonic function on \( B_1 \setminus \{0\} \). Therefore, the singularity at zero is removable, and \( h \) is smooth in \( B_1 \). This completes the proof of regularity of \( u \).

Now we prove (33). We fix \( 0 < \delta < 1 \) such that \( \alpha \delta - \alpha - 2 > 0 \). Using that \( u \) is continuous, we estimate for \( r = |x| \neq 0 \)

\[
\Delta v(x) = c_{N,\alpha} u(0)^p \int_{|y| < r} \frac{|y|^\alpha}{|x-y|^{N-2}} (1 + o(1)) \, dy + O(1) \int_{1 < |y| < r^\delta} \frac{|y|^\alpha}{|x-y|^{N-2}} \, dy \\
= c_{N,\alpha} u(0)^p \int_{|y| < r} \frac{|y|^\alpha}{|x-y|^{N-2}} (1 + o(1)) \, dy + O(1) r^{\delta \alpha},
\]

30
where $o(1) \to 0$ uniformly in $y \in B_r$ as $r \to 0$. Using a change of variable $y \mapsto |x| y$ we obtain

$$\int_{|y|<r^\delta} \frac{|y|^\alpha}{|x-y|^{N-2}} dy = r^{2+\alpha} \int_{|y|<r^{\delta-1}} \frac{|y|^\alpha}{\left|\frac{x}{r} - y\right|^{N-2}} dy = r^{2+\alpha} \int_{\mathbb{R}^N} \frac{|y|^\alpha}{\left|\frac{x}{r} - y\right|^{N-2}} dy + o(1)r^{2+\alpha},$$

where the last integral is finite as $-4 < \alpha < -2$, and $o(1) \to 0$ as $r \to \infty$. Combining these estimates, and as $h$ is smooth, we get (33).

To prove (34) we fix $0 < \varepsilon << 1 << R < \infty$. As before we would get

$$\Delta v(x) = c_N u(0)^p \int_{|y|<\varepsilon} \frac{|y|^{-2}}{|x-y|^{N-2}} (1 + o(1)) dy + O_{\varepsilon}(1),$$

and after a change of variable

$$\int_{|y|<\varepsilon} \frac{|y|^{-2}}{|x-y|^{N-2}} dy = \int_{|y|<\varepsilon} \frac{|y|^{-2}}{|x-y|^{N-2}} dy = \int_{R<|y|<\varepsilon} \frac{|y|^{-2}}{|x-y|^{N-2}} dy + O_{R}(1).$$

Since

$$\frac{1}{|\frac{x}{r} - y|^{N-2}} = \frac{1}{|y|^{N-2}(1 + o_R(1))} \quad \text{for} \quad |y| \geq R >> 1,$$

we have

$$\int_{|y|<\varepsilon} \frac{|y|^{-2}}{|x-y|^{N-2}} dy = |S^{N-1}|(1 + o_R(1)) \log \frac{1}{r} + O_{\varepsilon}(1) + O_{R}(1).$$

Combining these estimates and first taking $r \to 0^+$, and then taking $\varepsilon \to 0^+$, $R \to \infty$ we obtain (34).

Next we prove uniqueness of radial solutions to (32). We shall use the following identity:

$$w(r_2) - w(r_1) = \int_{r_1}^{r_2} \frac{1}{|S^{N-1}| |t^{N-1}|} \int_{B_t} \Delta w(x) dx dt, \quad w \text{ is radial}. \quad (35)$$

**Lemma 9.3** Let $u_1, u_2$ be two non-negative bounded radial solutions to (32) on $\mathbb{R}^N \setminus \{0\}$ with $\alpha \in (-4, 0)$. If $u_1(0) = u_2(0)$ then $u_1 = u_2$ on $\mathbb{R}^N$.

**Proof.** Let us first assume that

$$\lim_{|x| \to 0^+} \Delta \bar{u}(x) = 0, \quad \bar{u} := u_1 - u_2. \quad (36)$$

Then using (35) we obtain

$$\bar{u}(x) = o(1)|x|^2 \quad \text{as} \quad |x| \to 0. \quad (37)$$
By (35)-(36)
\[
\Delta \bar{u}(r) = \int_0^r \frac{1}{|S^{N-1}|t^{N-1}} \int_{B_t} |x|^\alpha (u^p_1(x) - u^p_2(x)) dx dt
\]
\[
= o(1)|\bar{u}|_r \int_0^r \frac{1}{t^{N-1}} \int_{B_t} |x|^{2+\alpha} dx dt
\]
\[
= o(1)|\bar{u}|_r r^{4+\alpha},
\]
where we have set $|\bar{u}|_r := \sup_{0 < t < r} t^{-2} |\bar{u}(t)|$. This leads to
\[
\bar{u}(r) = o(1)|\bar{u}|_r \int_0^r \frac{1}{t^{N-1}} \int_{B_t} |x|^{4+\alpha} dx dt = o(1)|\bar{u}|_r r^{6+\alpha},
\]
which gives
\[
r^{-2} |\bar{u}(r)| \leq \frac{1}{2} \sup_{0 < t < r} t^{-2} |\bar{u}(t)| \quad \text{for every } 0 < r \leq r_0,
\]
for some $r_0 > 0$ sufficiently small. From this and (37) we get that $\bar{u} \equiv 0$ in a small neighborhood of the origin, and consequently we have $\bar{u} \equiv 0$ in $\mathbb{R}^N$.

It remains to prove (36), and we do that in few steps.

**Step 1** Assume that $\bar{u}(x) = O(1)|x|^\gamma$ for some $\gamma \geq 0$. Then setting $\hat{\gamma} := \alpha + \gamma + 2$ we have
\[
\Delta \bar{u}(x) = O(1) \begin{cases} 
|x|^\hat{\gamma} & \text{if } \hat{\gamma} < 0 \\
\log |x| & \text{if } \hat{\gamma} = 0 \\
1 & \text{if } \hat{\gamma} > 0,
\end{cases}
\hat{u}(x) = O(1) \begin{cases} 
|x|^\hat{\gamma} + 2 & \text{if } \hat{\gamma} < 0 \\
|x|^2 \log |x| & \text{if } \hat{\gamma} = 0 \\
|x|^2 & \text{if } \hat{\gamma} > 0.
\end{cases}
\]

We set $\bar{v} := v_1 - v_2$, $\bar{h} := h_1 - h_2$ where
\[
v_i(x) := \frac{1}{\gamma N} \int_{B_1} \frac{1}{|x - y|^{N-4}} |y|^\alpha u^p_i(y) dy, \quad h_i := u_i - v_i, \quad i = 1, 2.
\]
Then using that $|u^p_1(x) - u^p_2(x)| \leq C|\bar{u}(x)| \leq C|x|^\gamma$
\[
\Delta \bar{v}(x) = c_\alpha \int_{B_1} \frac{|y|^\alpha}{|x - y|^{N-2}} (u^p_1(y) - u^p_2(y)) dy
\]
\[
= O(1) \int_{B_1} \frac{|y|^{\alpha + \gamma}}{|x - y|^{N-2}} dy
\]
\[
= O(1) \begin{cases} 
|x|^{\alpha + \gamma + 2} & \text{if } \alpha + \gamma + 2 < 0 \\
\log |x| & \text{if } \alpha + \gamma + 2 = 0 \\
1 & \text{if } \alpha + \gamma + 2 > 0,
\end{cases}
\]
thanks to Lemma 9.4. First part of Step 1 follows as $\bar{h}$ is smooth in $B_1$. The second part follows immediately by the first part and the identity (35).
Step 2 The function $\bar{u}$ is $C^2$.

Since $\bar{u}(x) = O(1)$, we can use Step 1 with $\gamma = 0$, and deduce that $\bar{u}(x) = O(|x|^{4+\alpha})$ (or the other growths at 0). In fact, we can repeat this process finitely many times to eventually get that $\bar{u}(x) = O(|x|^2)$. Then, as $\alpha > -4$, from the integral representation of $\bar{v}$ it is easy to see that $\bar{v}$ is $C^2$, and consequently $\bar{u}$ is $C^2$.

Step 3 (36) holds.

Since $\bar{u}$ is $C^2$, $a := \lim_{|x| \to 0} \Delta \bar{u}(x)$ exists. If $a > 0$ then a repeated use of (35) yields that $\Delta \bar{u} \geq a$ on $\mathbb{R}^n$. In particular, $\bar{u}(x) \geq 2n a |x|^2$ on $\mathbb{R}^n$, a contradiction as $\bar{u}$ is bounded. The case $a < 0$ is similar. □

Proof of the following lemma is straight forward.

Lemma 9.4 For $q_1, q_2 \in (0, N)$ there exists $c = c(N, q_1, q_2) > 0$ such that for any $R > 0$ we have

$$\lim_{|x| \to 0^+} |x|^{q_1+q_2-N} \int_{B_R} \frac{dy}{|x-y|^{q_1}|y|^{q_2}} = c$$

if $q_1 + q_2 > N$,

and

$$\lim_{|x| \to 0^+} -(\log |x|)^{-1} \int_{B_R} \frac{dy}{|x-y|^{q_1}|y|^{q_2}} = c$$

if $q_1 + q_2 = N$,

Theorem 9.5 There exists a positive radial solution $u \in C^0(\mathbb{R}^N) \cap C^4(\mathbb{R}^N \setminus \{0\})$ to (32) such that $u$ is monotone decreasing and $u$ vanishes at infinity. In fact, $u(r) \leq C r^{-\frac{4}{N-2}}$ at infinity.

To prove the theorem we consider the auxiliary equation

$$\begin{cases}
\Delta^2 u = \lambda |x|^\alpha (1 + u)^p & \text{in } B_1 \\
u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1.
\end{cases}$$

(38)

Next we prove existence of a positive radial solution to (38) for some $\lambda > 0$. This will be done by Schauder fixed point theorem on the space

$$X := C^0_{rad}(\bar{B}_1), \quad \|u\| := \|u\|_{C^0(B_1)}.$$ 

We define $T : X \to X$, $u \mapsto \bar{u}$ where we have set

$$\bar{u}(x) := \int_{B_1} G(x, y) |y|^\alpha (1 + |u(y)|)^p dy,$$

(39)

where $G = G(x, y)$ is the Green function for $\Delta^2$ on $B_1$ with Dirichlet boundary conditions. It is easy to see that $T$ is well-defined, and in fact, $T$ is compact. Therefore, there exists $0 < t_0 \leq 1$ and $u_0 \in X$ such that $t T u_0 = u_0$. Then $u_0$ is positive, monotone decreasing, and it satisfies (35) with $\lambda = t_0$.

33
As \( u_0 \) is a super solution to (38) for \( 0 < \lambda \leq t_0 \), one can prove existence of positive, radially symmetric, monotone decreasing, minimal solution \( u = u_\lambda \) to (38) for every \( 0 < \lambda \leq t_0 \).

Next we prove uniqueness of the minimal solutions for \( \lambda > 0 \) small. In order to do that let us recall the following Pohozaev identity from [GGS10, Theorem 7.27].

**Lemma 9.6** Let \( u \) be a solution to

\[
\begin{cases}
\Delta^2 u = f(x, u) & \text{in } \Omega \subset \mathbb{R}^N \\
u_0 = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

Then setting \( F(x, t) = \int_0^t f(x, s) ds \) we have

\[
\int_\Omega \left[ F(x, u) + \frac{1}{N} x \cdot F_x(x, u) - \frac{N-4}{2N} (\Delta u)^2 \right] dx = \frac{1}{2N} \int_{\partial \Omega} (\Delta u)^2 (x \cdot \nu) d\sigma.
\]

Since \( \int_\Omega uf(x, u) dx = \int_\Omega |\Delta u|^2 dx \), and \( x \cdot \nu = 1 \) on \( \partial \Omega \) for \( \Omega = B_1 \), the above Pohozaev identity leads to

\[
\int_{B_1} \left[ F(x, u) + \frac{1}{N} x \cdot F_x(x, u) - \sigma uf(x, u) \right] dx \geq \left( \frac{N-4}{2N} - \sigma \right) \int_{B_1} (\Delta u)^2 dx.
\]

We also need the following Hardy-Sobolev inequality:

\[
\left( \int_{B_1} \frac{|u|^{2(N-\beta)}}{|x|^\beta} dx \right)^{\frac{N-\beta}{N-4}} \leq c_0 \int_{B_1} |\Delta u|^2 dx \quad \text{for } u \in H_0^2(B_1),
\]

where \( B_1 \subset \mathbb{R}^N \), \( N \geq 5 \) and \( 0 < \beta < 4 \). This can be derived from the Sobolev inequality \( \|u\|_{L^2^*} \leq C\|\Delta u\|_{L^2} \) \( (2^* := \frac{2N}{N-4}) \), Hardy inequality \( \|\frac{u}{|x|^\beta}\|_{L^2} \leq C\|\Delta u\|_{L^2} \) and Hölder inequality.

Using (41) and (42) one can prove the following lemma, see e.g. [ACD+ Proposition 2.2].

**Lemma 9.7** There exists \( \lambda_0 > 0 \) such that for every \( \lambda \in (0, \lambda_0) \) the minimal solution \( u = u_\lambda \) to (38) is the unique solution on the space \( C^0(\overline{B_1}) \).

From [Rab73, Theorem 6.2] we know that the closure of the set of solutions \( \{(\lambda, u)\} \subset \mathbb{R} \times X \) to (38) is unbounded in \( (0, \infty) \times X \). Therefore, there exists an unbounded sequence \( (\lambda_k, u_{\lambda_k}) \in (0, \infty) \times X \) of solutions to (38). Then necessarily \( u_{\lambda_k}(0) = \max u_{\lambda_k} \to \infty \), and by Lemma 9.7, \( \lambda_k \not\to 0 \). We set

\[
v_k(x) := \frac{u_{\lambda_k}(r_k x)}{u_{\lambda_k}(0)}, \quad r_k^{4+a} \lambda_k u_{\lambda_k}(0)^{p-1} := 1.
\]

34
Then \( r_k \to 0 \) and \( v_k \) satisfies
\[
\Delta^2 v_k = |x|^\alpha \left( \frac{1}{u_{x_k}(0)} + v_k \right)^p \quad \text{in } B_{\frac{1}{r_k}}, \quad 0 \leq v_k \leq 1, \quad v_k(0) = 1, \quad \Delta v_k \leq 0.
\]

By elliptic estimates, up to a subsequence, \( v_k \to v \) locally uniformly in \( \mathbb{R}^N \), where \( v \) is a non-trivial bounded positive radial solution to
\[
\Delta^2 v = |x|^\alpha v^p \quad \text{in } \mathbb{R}^N.
\]

Now to prove the decay estimate of \( v \) at infinity, we use that \( v \) is monotone decreasing, and \( \Delta v < 0 \) on \( \mathbb{R}^N \). For \( r > 0 \), by (35), we get
\[
\Delta v(2r) = \Delta v(r) + \int_r^{2r} \frac{1}{|S^{N-1}|t^{N-1}} \int_{B_t} |x|^\alpha v^p(x) dx dt
\geq \Delta v(r) + \int_r^{2r} \frac{1}{|S^{N-1}|t^{N-1}} \int_{B_t} |x|^\alpha v^p(x) dx dt
\geq \Delta v(r) + c_1 r^{\alpha+2} v^p(r),
\]
for some constant \( c_1 > 0 \). Thus
\[
\Delta v(r) + c_1 r^{\alpha+2} v^p(r) < 0,
\]
which leads to
\[
v(2r) = v(r) + \int_r^{2r} \frac{1}{|S^{N-1}|t^{N-1}} \int_{B_t} \Delta v(x) dx dt
\leq v(r) + \int_r^{2r} \frac{1}{|S^{N-1}|t^{N-1}} \int_{B_t} \Delta v(x) dx dt
\leq v(r) + c_2 \Delta v(r) v^2
\leq v(r) - c_1 c_2 r^{4+\alpha} v^p(r),
\]
for some constant \( c_2 > 0 \).

This finishes the proof of Theorem 9.5.

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