GENUS 2 CURVES AND GENERALIZED THETA DIVISORS

SONIA BRIVIO, FILIPPO F. FAVALE

Abstract. In this paper we investigate generalized theta divisors $\Theta_r$ in the moduli spaces $\mathcal{U}_C(r, r)$ of semistable vector bundles on a curve $C$ of genus 2. We provide a desingularization $\Phi$ of $\Theta_r$ in terms of a projective bundle $\pi : \mathbb{P}(V) \to \mathcal{U}_C(r - 1, r)$ which parametrizes extensions of stable vector bundles on the base by $\mathcal{O}_C$. Then, we study the composition of $\Phi$ with the well known theta map $\theta$. We prove that, when it is restricted to the general fiber of $\pi$, we obtain a linear embedding.

Introduction

Theta divisors play a fundamental role in the study of moduli spaces of semistable vector bundles on curves. First of all, the classical notion of theta divisor of the Jacobian variety of a curve can be generalized to higher rank. Let $C$ be a smooth, irreducible, complex, projective curve of genus $g$. The study of isomorphism classes of stable vector of fixed rank $r$ and degree $n$ goes back to Mumford. The compactification of this moduli space is denoted by $\mathcal{U}_C(r, n)$ and has been introduced by Seshadri. In the particular case when the degree is equal to $r(g - 1)$ it admits a natural Brill-Noether locus $\Theta_{r,L}$, which is said theta divisor of $\mathcal{U}_C(r, r(g - 1))$. Riemann’s singularity Theorem extends to $\Theta_r$, see [Las91].

When we restrict our attention to semistable vector bundles of rank $r$ and fixed determinant $L \in \text{Pic}^r_{g-1}(C)$, we have the moduli space $\mathcal{SU}_C(r, L)$ and a Brill-Noether locus $\Theta_{r,L}$ which is said theta divisor of $\mathcal{SU}_C(r, L)$. The line bundle associated to $\Theta_{r,L}$ is the ample generator $\mathcal{L}$ of the Picard variety of $\mathcal{SU}_C(r, L)$, which is said the determinant line bundle, see [DN89].

For semistable vector bundles with integer slope, one can also introduce the notion of associated theta divisor. In particular for a stable $E \in \mathcal{SU}_C(r, L)$ with $L \in \text{Pic}^r_{g-1}(C)$ we have that the set

$$\{N \in \text{Pic}^0(C) \mid h^0(E \otimes N) \geq 1\}$$

is either all $\text{Pic}^0(C)$ or an effective divisor $\Theta_E$ which is said the theta divisor of $E$. Moreover the map which associates to each bundle $E$ its theta divisor $\Theta_E$ defines a rational map

$$\theta : \mathcal{SU}_C(r, L) \dashrightarrow |r\Theta_M|,$$

where $\Theta_M$ is a translate of the canonical theta divisor of $\text{Pic}^{r-1}(C)$ and $M$ is a line bundle such that $M^{\otimes r} = L$. Note that the indeterminacy locus of $\theta$ is given by the vector bundles which does not admit theta divisor.

Actually, this map is defined by the determinant line bundle $\mathcal{L}$, see [BNR89] and it has been studied by many authors. It has been completely described for $r = 2$ with the contributions of many authors. On the other hand, when $r \geq 3$, very little is known. In particular, the genus 2 case seems to be interesting. First of all, in this case we have that $\dim \mathcal{SU}_C(r, L) = \dim |r\Theta_M|$. For $r = 2$ it is proved in [NR69] that $\theta$ is an isomorphism, whereas, for $r = 3$ it is a double covering.
ramified along a sextic hypersurface (see [Ort05]). For \( r \geq 4 \) this is no longer a morphism, and it is generically finite and dominant, see [Bea06] and [BV07].

In this paper, using the theory of extensions of vector bundles, we give a birational description of \( \Theta_r \) as a projective bundle over the moduli space \( \mathcal{U}_C(r-1, r) \). Our first result is Theorem 2.5 which can be stated as follows:

**Theorem.** There exists a vector bundle \( \mathcal{V} \) on \( \mathcal{U}_C(r-1, r) \) of rank \( 2r-1 \) whose fiber at the point \([F]\) is \( \text{Ext}^1(F, \mathcal{O}_C) \). Let \( \mathbb{P}(\mathcal{V}) \) be the associated projective bundle and \( \pi : \mathbb{P}(\mathcal{V}) \to \mathcal{U}_C(r-1, r) \) the natural projection. Then the map

\[
\Phi : \mathbb{P}(\mathcal{V}) \to \Theta_r
\]

sending \([v]\) to the vector bundle which is extension of \( \pi([v]) \) by \( \mathcal{O}_C \), is a birational morphism.

In particular, notice that this theorem gives a desingularization of \( \Theta_r \) as \( \mathbb{P}(\mathcal{V}) \) is smooth.

As a corollary of the above Theorem we have, (see 2.7), that \( \Theta_{r,L} \) is birational to a projective bundle over the moduli space \( \mathcal{SU}_C(r-1, L) \) for any \( r \geq 3 \). This has an interesting consequence (see Corollary 2.8):

**Corollary.** \( \Theta_{r,L} \) is a rational subvariety of \( \mathcal{SU}_C(r, L) \).

The proof of the Theorem and its corollaries can be found in Section 2.

The second result of this paper is contained in Section 3 and it involves the study of the restriction of \( \Phi \) to the general fiber \( \mathbb{P}_F = \pi^{-1}([F]) \) of \( \pi \) and its composition with the theta map. The main result of this section is Theorem 3.4 which can be stated as follows:

**Theorem.** For a general stable bundle \( F \in \mathcal{SU}_C(r-1, L) \) the map

\[
\theta \circ \Phi|_{\mathbb{P}_F} : \mathbb{P}_F \to |r\Theta_M|
\]

is a linear embedding.

In the proof we are actually more precise about the generality of \( F \): we describe explicitly a open subset of the moduli space \( \mathcal{SU}_C(r-1, L) \) where the above theorem holds. Let us stress that one of the key argument in the proof involves the very recent result about the stability of secant bundles [BD18].

1. **Background and known results**

In this section we recall some definitions and useful results about generalized Theta divisors, secant bundles and 2-symmetric product of curves that we will use in the following sections.

1.1. **Theta divisors.**

Let \( C \) be a smooth, irreducible, complex, projective curve of genus \( g = 2 \). For any \( r \geq 2 \) and for any \( n \in \mathbb{Z} \), let \( \mathcal{U}_C(r, n) \) denote the moduli space of semistable vector bundles on the curve \( C \) with rank \( r \) and degree \( n \). It is a normal, irreducible, projective variety of dimension \( r^2 + 1 \), whose points are \( S \)-equivalence classes of semistable vector bundles of rank \( r \) and degree \( n \); we recall that two vector bundles are said \( S \)-equivalent if they have isomorphic graduates, where the graduate \( gr(E) \) of \( E \) is the polystable bundle defined by a Jordan-Holder filtration of \( E \), see [Ses82] and [LeP97].

We denote by \( \mathcal{U}_C(r, n)^\ast \) the open subset corresponding to isomorphism classes of stable bundles. For \( r = 2 \) one has that \( \mathcal{U}_C(r, n) \) is smooth, whereas, for \( r \geq 3 \) one has

\[
\text{Sing}(\mathcal{U}_C(r, n)) = \mathcal{U}_C(r, n) \setminus \mathcal{U}_C(r, n)^\ast.
\]
Moreover, \( \mathcal{U}_C(r, n) \cong \mathcal{U}_C(r, n') \) whenever \( n' - n = kr \), with \( k \in \mathbb{Z} \), and \( \mathcal{U}_C(r, n) \) is a fine moduli space if and only if \( (r, n) = 1 \).

For any line bundle \( L \in \text{Pic}^0(C) \), let \( SU_C(r, L) \) denote the moduli space of semistable vector bundles on \( C \) with rank \( r \) and fixed determinant \( L \). These moduli spaces are the fibres of the natural map \( \mathcal{U}_C(r, n) \to \text{Pic}^0(C) \) which associates to each vector bundle its determinant.

When \( n = r \), we consider the following Brill-Noether loci:

\[
\Theta_r = \{ [E] \in \mathcal{U}_C(r, r) \mid h^0(\text{gr}(E)) \geq 1 \},
\]

\[
\Theta_{r, L} = \{ [E] \in SU_C(r, L) \mid h^0(\text{gr}(E)) \geq 1 \},
\]

where \([E]\) denotes \( S\)-equivalence class of \( E \). Actually, \( \Theta_r \) (resp. \( \Theta_{r, L} \)) is an integral Cartier divisor which is said the theta divisor of \( \mathcal{U}_C(r, r) \) (resp. \( SU(r, L) \)), see [DN89]. The line bundle \( \mathcal{L} \) associated to \( \Theta_{r, L} \) is called the determinant bundle of \( SU_C(r, L) \) and it is the generator of its Picard variety. We denote by \( \Theta_{r}^* \subset \Theta_r \) the open subset of stable points. Let \( [E] \in \Theta_{r}^* \), then the multiplicity of \( \Theta_r \) at the point \([E]\) is \( h^0(E) \), see [Las91]. This implies:

\[
\text{Sing}(\Theta_r^*) = \{ [E] \in \Theta_r^* | h^0(E) \geq 2 \}.
\]

For semistable vector bundles with integer slope we can introduce the notion of theta divisors as follows. Let \( E \) be a semistable vector bundle on \( C \) with integer slope \( m = \frac{\deg E}{r} \). The tensor product defines a morphism

\[
\mu: \mathcal{U}_C(r, rm) \times \text{Pic}^{1-m}(C) \to \mathcal{U}_C(r, r)
\]

sending \([E], N\) \( \to [E \otimes N] \).

The intersection \( \mu^* \Theta_r \cdot ([E] \times \text{Pic}^{1-m}(C)) \) is either an effective divisor \( \Theta_E \) on \( \text{Pic}\^{1-m}(C) \) which is called the theta divisor of \( E \), or all \((E) \times \text{Pic}^{1-m}(C)) \), and in this case we will say that \( E \) does not admit theta divisor. For more details see [Real03].

Set theoretically we have

\[
\Theta_E = \{ N \in \text{Pic}^{1-m}(C) \mid h^0(\text{gr}(E) \otimes N) \geq 1 \}.
\]

For all \( L \in \text{Pic}^r(C) \) fixed we can choose a line bundle \( M \in \text{Pic}^m(C) \) such that \( L = M^\otimes r \). If \([E] \in SU_C(r, L)\), then \( \Theta_E \in [r \Theta_M] \) where

\[
\Theta_M = \{ N \in \text{Pic}^{1-m}(C) \mid h^0(M \otimes N) \geq 1 \}
\]

is a translate of the canonical theta divisor \( \Theta \subset \text{Pic}^{g-1}(C) \). This defines a rational map, which is said theta map of \( SU_C(r, L) \)

\[
(1) \quad SU_C(r, L) \dashrightarrow [r \Theta_M].
\]

As previously recalled \( \theta \) is the map induced by the determinant bundle \( \mathcal{L} \) and the points \([E]\) which do not admit theta divisor give the indeterminacy locus of \( \theta \). Moreover \( \theta \) is an isomorphism for \( r = 2 \), it is a double covering ramified along a sextic hypersurface for \( r = 3 \). For \( r \geq 4 \) it is no longer a morphism: it is generically finite and dominant.
1.2. 2-symmetric product of curves.

Let $C^{(2)}$ denote the 2-symmetric product of $C$, parametrizing effective divisors $d$ of degree 2 on the curve $C$. It is well known that $C^{(2)}$ is a smooth projective surface, see [ACGH85]. It is the quotient of the product $C \times C$ by the action of the symmetric group $S_2$; we denote by

\[ \pi: C \times C \to C^{(2)}, \quad \pi(x, y) = x + y, \]

the quotient map, which is a double covering of $C^{(2)}$, ramified along the diagonal $\Delta \subset C \times C$.

Let $N^1(C^{(2)})\mathbb{Z}$ be the Neron-Severi group of $C^{(2)}$, i.e. the quotient group of numerical equivalence classes of divisors on $C^{(2)}$. For any $p \in C$, let’s consider the embedding

\[ i_p: C \to C^{(2)} \]

sending $q \to q + p$, we denote the image by $C + p$ and we denote by $x$ its numerical class in $N^1(C^{(2)})\mathbb{Z}$. Let $d_2$ be the diagonal map

\[ d_2: C \to C^{(2)} \]

sending $q \to 2q$. Then $d_2(C) = \pi(\Delta) \simeq C$, we denote by $\delta$ its numerical class in $N^1(C^{(2)})\mathbb{Z}$. Finally, let’s consider the Abel map

\[ A: C^{(2)} \to \text{Pic}^2(C) \simeq J(C) \]

sending $p + q \to O_C(p + q)$. Since $g(C) = 2$, it is well known that actually $C^{(2)}$ is the blow up of $\text{Pic}^2(C)$ at $\omega_C$ with expectional divisor

\[ E = \{ d \in C^{(2)} | O_C(d) \simeq \omega_C \} \simeq \mathbb{P}^1. \]

This implies that:

\[ K_{C^{(2)}} = A^*(K_{\text{Pic}^2(C)}) + \mathcal{E} = \mathcal{E}, \]

since $K_{\text{Pic}^2(C)}$ is trivial.

Let $\Theta \subset J(C)$ be the theta divisor, its pull back $A^*(\Theta)$ is an effective divisor on $C^{(2)}$, we denote by $\theta$ its numerical class in $N^1(C^{(2)})\mathbb{Z}$. It is well known that $\delta = 2(3x - \theta)$, or, equivalently,

\[ (2) \quad \theta = 3x - \frac{\delta}{2} \]

If $C$ is a general curve of genus 2 then $N^1(C^{(2)})\mathbb{Z}$ is generated by the classes $x$ and $\frac{\delta}{2}$ (see [ACGH85]). The Neron-Severi lattice is identified by the relations

\[ x \cdot x = 1, \quad x \cdot \frac{\delta}{2} = 1, \quad \frac{\delta}{2} \cdot \frac{\delta}{2} = -1. \]

1.3. Secant bundles on 2-symmetric product of curves.

Let’s consider the universal effective divisor of degree 2 of $C$:

\[ \mathcal{I}_2 = \{ (d, y) \in C^{(2)} \times C \mid y \in \text{Supp}(d) \}, \]

it is a smooth irreducible divisor on $C^{(2)} \times C$. Let $\iota$ be the embedding of $\mathcal{I}_2$ in $C^{(2)} \times C$, $r_1$ and $r_2$ be the natural projections of $C^{(2)} \times C$ onto factors and $q_i = r_i \circ \iota$ the restriction to $\mathcal{I}_2$ of $r_i$. Then $q_1$ is a surjective map of degree 2. Denote also with $p_1$ and $p_2$ the natural projections of $C \times C$ onto factors.

We have a natural isomorphism

\[ \nu: C \times C \to \mathcal{I}_2, \quad (x, y) \to (x + y, y) \]
and, under this isomorphism, the map \( q_1: \mathcal{I}_2 \to C^{(2)} \) can be identified with the map \( \pi: C \times C \to C^{(2)} \). It is also easy to see that the map \( q_2 \), under the isomorphism \( \nu \), can be identified with the projection \( p_2 \). We have then a commutative diagram

![Diagram](image)

Now we will introduce the secant bundle \( \mathcal{F}_2(E) \) associated to a vector bundle \( E \) on \( C \) as well as some properties which will be useful in the sequel. For details one can see [Sch64] or the Ph.D. thesis of E. Mistretta for the secant bundles of line bundles. The general case is studied in [BD18].

Let \( E \) be a vector bundle of rank \( r \) on \( C \), we can associate to \( E \) a sheaf on \( C^{(2)} \) which is defined as

\[
\mathcal{F}_2(E) = q_1^*(q_2^*(E)).
\]

\( \mathcal{F}_2(E) \) is a vector bundles of rank \( 2r \) which is called secant bundle associated to \( E \) on \( C^{(2)} \).

Let ’s consider the pull back of the secant bundle on \( C \times C \): \( \pi^*\mathcal{F}_2(E) \). Outside the diagonal \( \Delta \subset C \times C \) we have:

\[
\pi^*\mathcal{F}_2(E) \simeq p_1^*E \oplus p_2^*E.
\]

Actually, these bundles are related by the following exact sequence:

\[
0 \to \mathcal{F}_2(E) \to p_1^*E \oplus p_2^*E \to p_1^*(E)|_\Delta = p_2^*(E)|_\Delta \simeq E \to 0,
\]

where the last map sends \((u,v) \to u|_\Delta - v|_\Delta\).

Finally, from the exact sequence on \( C^{(2)} \times C \):

\[
0 \to \mathcal{O}_{C^{(2)} \times C}(-\mathcal{I}_2) \to \mathcal{O}_{C^{(2)} \times C} \to \iota_*\mathcal{O}_{\mathcal{I}_2} \to 0,
\]

tensoring with \( r_2^*(E) \) we get:

\[
0 \to r_2^*(E)(-\mathcal{I}_2) \to r_2^*(E) \to \iota_*(q_2^*(E)) \to 0,
\]

where, to simplify notations, we set \( r_2^*(E) \otimes \mathcal{O}_{C^{(2)} \times C}(-\mathcal{I}_2) = r_2^*(E)(-\mathcal{I}_2) \) and we have used the projection formula

\[
r_2^*(E) \otimes \iota_*\mathcal{O}_{\mathcal{I}_2} = \iota_*(\iota^*(r_2^*E) \otimes \mathcal{O}_{\mathcal{I}_2}) = \iota_*(q_2^*E).
\]

By applying \( r_1^* \), we get

\[
0 \to r_1^*(r_2^*(E)(-\mathcal{I}_2)) \to H^0(E) \otimes \mathcal{O}_{C^{(2)}} \to \mathcal{F}_2(E) \to
\]

\[
\to R^1r_1^*(r_2^*(E)(-\mathcal{I}_2)) \to H^1(E) \otimes \mathcal{O}_{C^{(2)}} \to \cdots
\]

since we have: \( r_1^*(\iota_*(q_2^*E)) = q_1^*q_2^*E = \mathcal{F}_2(E) \) and

\[
R^0r_1^*r_2^*E = H^0(E) \otimes \mathcal{O}_{C^{(2)}}.
\]
Moreover, by projection formula $H^0(C^{(2)}, \mathcal{F}_2(E)) \simeq H^0(C, E)$ and the map
\[ H^0(E) \otimes O_{C^{(2)}} \to \mathcal{F}_2(E) \]
appearing in (5) is actually the evaluation map of global sections of the secant bundle; we will it denoted by $ev$. Notice that, if we have $h^1(E) = 0$, the exact sequence (5) becomes
\[ 0 \to r_1^*(r_2^*(E)(-\mathcal{I}_2)) \to H^0(E) \otimes O_{C^{(2)}} \overset{ev}{\to} \mathcal{F}_2(E) \to R^1r_1^*(r_2^*(E)(-\mathcal{I}_2)) \to 0 \]
We will call the exact sequence (5) (and its particular case (6)) the exact sequence induced by the evaluation map of the secant bundle.

If $degE = n$, then the Chern character of $\mathcal{F}_2(E)$ is given by the following formula:
\[ ch(\mathcal{F}_2(E)) = n(1 - e^{-x}) - r + r(3 + \theta)e^{-x}, \]
where $x$ and $\theta$ are the numerical classes defined above. From this we can deduce the Chern classes of $\mathcal{F}_2(E)$:
\[ c_1(\mathcal{F}_2(E)) = (n - 3r)x + r\theta, \]
\[ c_2(\mathcal{F}_2(E)) = \frac{1}{2}(n - 3r)(n + r + 1) + r^2 + 2r. \]

We recall the following definition:

**Definition 1.1.** Let $X$ be a smooth, irreducible, complex projective surface and let $H$ be an ample divisor on $X$. For a torsion free sheaf $E$ on $X$ we define the slope of $E$ with respect to $H$:
\[ \mu_H(E) = \frac{c_1(E) \cdot H}{rk(E)}. \]
$E$ is said semistable with respect to $H$ if for any non zero proper subsheaf $F$ of $E$ we have $\mu_H(F) \leq \mu_H(E)$, it is said stable with respect to $H$ if for any proper subsheaf $F$ with $0 < rk(F) < rk(E)$ we have $\mu_H(F) < \mu_H(E)$.

For stability of secant bundles, we have the following result, see [BD18]:

**Proposition 1.1.** Let $E$ be a semistable vector bundle on $C$ with rank $r$ and $deg(E) \geq r$, then $\mathcal{F}_2(E)$ is semistable with respect to the ample class $x$; if $deg(E) > r$ and $E$ is stable, then $\mathcal{F}_2(E)$ is stable too with respect to the ample class $x$.

2. Description of $\Theta_r$ and $\Theta_{r,L}$.

In this section we will give a description of $\Theta_r$ (resp. $\Theta_{r,L}$) which gives a natural desingularization. Fix $r \geq 3$.

**Lemma 2.1.** Let $E$ be a stable vector bundle with $[E] \in \Theta_r$, then there exists a vector bundle $F$ such that $E$ fit into the following exact sequence:
\[ 0 \to O_C \to E \to F \to 0, \]
with $[F] \in U_C(r - 1, r)$. 

6
Proof. Since $E$ is stable, $E \cong gr(E)$ and, as $[E] \in \Theta_r$, $h^0(E) \geq 1$. Let $s \in H^0(E)$ be a non zero global section, since $E$ is stable of slope 1, $s$ cannot be zero in any point of $C$, so it defines an injective map of sheaves
\[ i_s : \mathcal{O}_C \rightarrow E \]
which induces the following exact sequence of vector bundles:
\[ 0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow F \rightarrow 0, \]
where the quotient $F$ is a vector bundle of rank $r - 1$ and degree $r$. We will prove that $F$ is semistable, hence $[F] \in \mathcal{U}_C(r - 1, r)$, which implies that it is also stable.

Let $G$ be a non trivial destabilizing quotient of $F$ of degree $k$ and rank $s$ with $1 \leq s \leq r - 2$. Since $G$ is also a quotient of $E$, by stability of $E$ we have
\[ 1 = \mu(E) < \mu(G) \leq \mu(F) = \frac{r}{r - 1}, \]
i.e.
\[ 1 < \frac{k}{s} \leq 1 + \frac{1}{r - 1}. \]
Hence we have
\[ s < k \leq s + \frac{s}{r - 1} \]
which is impossible since $s < r - 1$.

A short exact sequence of vector bundles
\[ 0 \rightarrow G \rightarrow E \rightarrow F \rightarrow 0, \]
is say to be an extension of $F$ by $G$, see [Ati57]. Recall that equivalence classes of extensions of $F$ by $G$ are parametrized by
\[ H^1(\text{Hom}(F, G)) \simeq \text{Ext}^1(F, G); \]
where the extension corresponding to $0 \in \text{Ext}^1(F, G)$ is $G \oplus F$ and is called the trivial extension. Given $v \in \text{Ext}^1(F, G)$ we will denote by $E_v$ the vector bundle which is the extension of $F$ by $G$ in the exact sequence corresponding to $v$. Moreover, if $v_2 = \lambda v_1$ for some $\lambda \in \mathbb{C}^*$, we have $E_{v_1} \simeq E_{v_2}$.
Lastly, recall that $\text{Ext}^1$ is a functorial construction so are well defined on isomorphism classes of vector bundles.

Lemma 2.2. Let $[F] \in \mathcal{U}_C(r - 1, r)$, then $\dim \text{Ext}^1(F, \mathcal{O}_C) = 2r - 1$.

Proof. We have: $\text{Ext}^1(F, \mathcal{O}_C) \simeq H^1(F^\vee) \simeq H^0(F \otimes \omega_C)^\vee$, so by Riemann-Roch theorem:
\[ \chi_C(F \otimes \omega_C) = \deg(F \otimes \omega_C) + \text{rk}(F \otimes \omega_C)(1 - g(C)) = 2r - 1. \]
Finally, since $\mu(F \otimes \omega_C) = 3 + \frac{1}{r - 1} \geq 2g - 1 = 3$, then $h^1(F \otimes \omega_C) = 0$. \qed

Let $F$ be a stable bundle, with $[F] \in \mathcal{U}_C(r - 1, r)$. The trivial extension $E_0 = \mathcal{O}_C \oplus F$ gives an unstable vector bundle. However, this is the only unstable extension of $F$ by $\mathcal{O}_C$ as proven in the following Lemma.

Lemma 2.3. Let $[F] \in \mathcal{U}_C(r - 1, r)$ and $v \in \text{Ext}^1(F, \mathcal{O}_C)$ be a non zero vector. Then $E_v$ is a semistable vector bundle of rank $r$ and degree $r$, moreover $[E_v] \in \Theta_r$.  

Proof. By lemma 2.2 \( \dim \text{Ext}^1(F, \mathcal{O}_C) = 2r - 1 > 0 \), let \( v \in \text{Ext}^1(F, \mathcal{O}_C) \) be a non zero vector and denote by \( E_v \) the corresponding vector bundle. By construction we have an exact sequence of vector bundles

\[
0 \to \mathcal{O}_C \to E_v \to F \to 0
\]

from which we deduce that \( E_v \) has rank \( r \) and degree \( r \).

Assume that \( E_v \) is not semistable. Then there exists a proper subbundle \( G \) of \( E_v \) with \( \mu(G) > \mu(E_v) = 1 \). Denote with \( s \) and \( k \) respectively the rank and the degree of \( G \). Hence we have

\[
1 \leq s \leq r - 2 \quad k > s.
\]

Let \( \alpha \) be the composition of the inclusion \( G \hookrightarrow E_v \) with the surjection \( \varphi : E_v \to F \), let \( K = \ker \alpha \).

Then we have a commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
K & \xrightarrow{\alpha} & \text{Im}(\alpha) \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\quad
\begin{array}{ccc}
0 & \to & E_v \\
\downarrow & & \downarrow \\
\mathcal{O}_C & \xrightarrow{\varphi} & F \\
\end{array}
\quad
\begin{array}{cc}
0 & \to \\
\end{array}
\]

If \( K = 0 \) then \( G \) is a subsheaf of \( F \), which is stable, so

\[
\mu(G) = \frac{k}{s} < \mu(F) = 1 + \frac{1}{r - 1}
\]

and

\[
s < k < s + \frac{s}{r - 1},
\]

which is impossible as \( 1 \leq s \leq r - 2 \). Hence we have that \( \alpha \) has non trivial kernel \( K \), which is a subsheaf of \( \mathcal{O}_C \), so \( K = \mathcal{O}_C(-A) \) for some divisor \( A \geq 0 \) with degree \( a \geq 0 \). Then \( \text{Im}(\alpha) \) is a subsheaf of \( F \), which is stable so:

\[
\frac{k + a}{s - 1} < 1 + \frac{1}{r - 1},
\]

hence we have

\[
s + a < k + a < s - 1 + \frac{s - 1}{r - 1}
\]

and

\[
a < -1 + \frac{s - 1}{r - 1}
\]

which is impossible as \( a \geq 0 \). This proves that \( E_v \) is semistable. Finally, note that \( h^0(E_v) \geq h^0(\mathcal{O}_C) = 1 \), so \([E] \in \Theta_r\). \( \square \)

We would like to study extensions of \( F \in \mathcal{U}_C(r - 1, r) \) by \( \mathcal{O}_C \) which give vector bundles of \( \Theta_r \setminus \Theta^*_r \). Note that if \( E_v \) is not stable, then there exists a proper subbundle \( S \) of \( E_v \) with slope 1. We will prove that any such \( S \) actually comes from a subsheaf of \( F \) of slope 1.

Let \([F] \in \mathcal{U}_C(r - 1, r)\), observe that any proper subsheaf \( S \) of \( F \) has slope \( \mu(S) \leq 1 \). Indeed, let \( s = \text{rk}(S) \leq r - 1 \), by stability of \( F \) we have

\[
\frac{\deg(S)}{s} < 1 + \frac{1}{r - 1}
\]

which implies \( \deg(S) < s + \frac{s}{r - 1} \), hence \( \deg(S) \leq s \). Assume that \( S \) is a subsheaf of slope 1. Then we are in one of the following cases:
A subsheaf $S$ of $F$ with slope 1 and rank $s \leq r - 2$ is a subbundle of $F$ and it is said a maximal subbundle of $F$ of rank $s$, see [LN83]. Note that any maximal subbundle $S$ is semistable hence $[S] \in \mathcal{U}_C(s,s)$. We will denote by $\mathcal{M}_s(F)$ the set of maximal subbundles of $F$ of rank $s$.

A subsheaf $S$ of $F$ of slope 1 and rank $r - 1$ is obtained by an elementary transformation of $F$ at a point $p \in C$, i.e. it fits into an exact sequence as follows:

$$0 \to S \to F \to \mathbb{C}_p \to 0.$$ 

More precisely, let’s denote with $F_p$ the fiber of $F$ in $p$, all the elementary transformations of $F$ at $p$ are parametrized by $\mathbb{P}(\text{Hom}(F_p, \mathbb{C}))$. In fact, for any non zero form $\gamma \in \text{Hom}(F_p, \mathbb{C})$, by composing it with the restriction map $F \to F_p$, we obtain a surjective morphism $F \to \mathbb{C}_p$ and then an exact sequence

$$0 \to G_\gamma \to F \to \mathbb{C}_p \to 0,$$

where $G_\gamma$ is actually a vector bundle which is obtained by the elementary transformation of $F$ at $p$ defined by $\gamma$. Finally, $G_{\gamma_1} \simeq G_{\gamma_2}$ if and only if $[\gamma_1] = [\gamma_2]$ in $\mathbb{P}(\text{Hom}(F_p, \mathbb{C}))$, see [Mar82] and [Bri17].

We have the following result:

**Proposition 2.1.** Let $[F] \in \mathcal{U}_C(r-1,r)$, $v \in \text{Ext}^1(F, \mathcal{O}_C)$ a non zero vector and $E_v$ the extension of $F$ defined by $v$. If $G$ is a proper subbundle of $E_v$ of slope 1, then $G$ is semistable and satisfies one of the following conditions:

- $G$ is a maximal subbundle of $F$ and $1 \leq \text{rk}(G) \leq r - 2$;
- $G$ has rank $r - 1$ and it is obtained by an elementary transformation of $F$.

**Proof.** Let $s = \text{rk}(G) = \text{deg}(G)$. As in the proof of Lemma 2.1 we can construct a commutative diagram

$$0 \to K \to G \to \text{Im}(\alpha) \to 0$$

$$0 \to \mathcal{O}_C \to E_v \to F \to 0$$

form which we obtain that either $K = 0$ of $K = \mathcal{O}_C(-A)$ with $A \geq 0$. In the second case, let $a$ be the degree of $A$. As in the proof of Lemma 2.1 we have that the slope of $\text{Im}(\alpha)$ satisfies

$$\mu(\text{Im}(\alpha)) = \frac{s + a}{s - 1} < 1 + \frac{1}{r - 1}$$

which gives a contradiction

$$0 \leq a < -1 + \frac{s - 1}{r - 1}.$$ 

So can assume that $K = 0$, so $\alpha: G \to F$ is an injective map of sheaves, we denote by $Q$ the quotient.

If $s = r - 1$ we have that $Q$ is a torsion sheaf of degree 1, i.e. a skyscraper sheaf over a point with the only non trivial fiber of dimension 1. Hence $G$ is obtained by an elementary transformation of $F$ at a point $p \in C$. 


If $s \leq r - 2$, we claim that $\alpha$ is an injective map of vector bundles. On the contrary, if $G$ is not a subbundle, then $Q$ is not locally free, so there exists a subbundle $G_f \subset F$ containing $\alpha(G)$, with $\text{rk}(G_f) = \text{rk}(G)$ and $\text{deg}(G_f) = \text{deg}G + b$, $b \geq 0$:

\[
\begin{array}{cccc}
0 & 0 & 0 & \\
& G & G_f & \\
& & \alpha & \\
& O_C & E & F & 0 \\
& & & & Q_f & Q \\
& & & & 0 & 0
\end{array}
\]

Then, as $F$ is stable, we have:

$$
\mu(G_f) = \frac{s + b}{s} = 1 + \frac{b}{s} < 1 + \frac{1}{r - 1} = \mu(F),
$$

hence

$$
0 \leq b < \frac{s}{r - 1}
$$

which implies $b = 0$.

Finally, note that $G$ is semistable. In fact, since $\mu(G) = \mu(E)$, a subsheaf of $G$ destabilizing $G$ would be a subsheaf destabilizing $E$.

For the proof see [NR69].

The above lemma allows us to prove the following result:

**Proposition 2.2.** Let $[F] \in \mathcal{U}_C(r - 1, r)$. Then:
Let $G_\gamma$ be the elementary transformation of $F$ at $p \in C$ defined by $[\gamma] \in \mathbb{P}(F_p^\vee)$, there exists a unique $[v] \in \mathbb{P}(\text{Ext}^1(F, \mathcal{O}_C))$ such that the inclusion $G_\gamma \hookrightarrow F$ can be lifted to $E_v$.

Let $S$ be a maximal subbundle of $F$ of rank $s$ and $\iota : S \hookrightarrow F$ the inclusion, then the set of classes $[v]$ which extend $\iota$ is a linear subspace of $\mathbb{P}(\text{Ext}^1(F, \mathcal{O}_C))$ of dimension $2r - 2s - 2$.

In particular, for any maximal subbundle of $F$ and for any elementary transformation, we obtain at least an extension of $F$ which is in $\Theta_r \setminus \Theta^\ast_r$.

Proof. Let’s start with the case of elementary transformation. We are looking for the extensions of $F$ by $\mathcal{O}_C$ such that there exists a lift $\iota : G_\gamma \to E_v$ such that the diagram

\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_C \otimes_{\mathbb{P}^1} E_v \to F \to 0 \\
\downarrow & & \downarrow \phi_v \\
0 & \to & \mathcal{O}_C \to \mathcal{O}_C(p) \to \mathbb{C}_p \to 0
\end{array}
\]

commutes. By Lemma 2.4, there exists $\tilde{\iota}$ if and only if the class of the extension $E_v$ lives in the image of $H^1(\iota^\ast)$ in the diagram

\[
\begin{array}{cccc}
\text{Hom}(F, \mathcal{O}_C) & \to & \text{Hom}(F, E_v) & \to \text{Hom}(F, F) \to \text{Ext}^1(F, \mathcal{O}_C) \\
\phi_v^\ast & \downarrow \iota^\ast & \delta_v & \downarrow H^1(\iota^\ast) \\
\text{Hom}(G_\gamma, \mathcal{O}_C) & \to & \text{Hom}(G_\gamma, E_v) & \to \text{Hom}(G_\gamma, F) \to \text{Ext}^1(G_\gamma, \mathcal{O}_C)
\end{array}
\]

If we apply the functor $\text{Hom}(-, \mathcal{O}_C)$ to the vertical exact sequence we obtain the exact sequence

\[0 \to F^\vee \to G_\gamma^\vee \to \mathbb{C}_p \to 0\]

from which we obtain

\[\cdots \to H^1(F^\vee) \to H^1(G_\gamma^\vee) \to 0.\]

In particular, the map $H^1(\iota^\ast)$ is surjective so its kernel has dimension

\[
\dim(\ker(H^1(\iota^\ast))) = \text{ext}^1(F, \mathcal{O}_C) - \text{ext}^1(G_\gamma, \mathcal{O}_C) =
\]

\[
h^0(F \otimes \omega_C) - h^0(G_\gamma \otimes \omega_C) = 2r - 1 - 2(r - 1) = 1.
\]

Hence there exist only one possible extension which extend $\iota$.

Let $S$ be a maximal subbundle of $F$ of rank $s, 1 \leq s \leq r - 2$, and let $\iota : S \to F$ the inclusion. By Lemma 2.4, we have that the set of $[v]$ which extends $\iota$ lifts is $\mathbb{P}(\ker(H^1(\iota^\ast)))$. As in the previous case, one can verify that $H^1(\iota^\ast)$ is surjective and

\[
\dim(\ker(H^1(\iota^\ast))) = \text{ext}^1(F, \mathcal{O}_C) - \text{ext}^1(S, \mathcal{O}_C) =
\]

\[
h^0(F \otimes \omega_C) - h^0(S \otimes \omega_C) = 2r - 1 - 2(s) = 2r - 2s - 1.
\]

\[\square\]
The above properties of extensions allow us to give the following description of theta divisor of $U_C(r, r)$:

**Theorem 2.5.** There exists a vector bundle $V$ on $U_C(r-1, r)$ of rank $2r - 1$ whose fiber at the point $[F] \in U_C(r-1, r)$ is $\text{Ext}^1(F, O_C)$. Let $\mathbb{P}(V)$ be the associated projective bundle and $\pi : \mathbb{P}(V) \to U_C(r-1, r)$ the natural projection. Then, the map

$$\Phi : \mathbb{P}(V) \to \Theta_r$$

sending $[v]$ to $[E_v]$, where $E_v$ is the extension of $\pi([v])$ by $O_C$ defined by $v$, is a birational morphism.

**Proof.** Let $\mathcal{P}$ be a universal bundle on $U_C(r-1, r)$, i.e. $\mathcal{P}$ is a vector bundle on $C \times U_C(r-1, r)$ such that $\mathcal{P}|_{C \times [F]} \simeq F$ for any $[F] \in U_C(r-1, r)$. Let $p_1$ and $p_2$ denote the projections of $C \times U_C(r-1, r)$ onto factors. Consider on $C \times U_C(r-1, r)$ the vector bundle $p_1^*(O_C)$, note that $p_1^*(O_C)|_{C \times [F]} \simeq O_C$, for any $[F] \in U_C(r-1, r)$. Let consider on $U_C(r-1, r)$ the first direct image of the sheaf $\mathcal{H}om(\mathcal{P}, p_1^*(O_C))$, i.e. the sheaf

$$(12) \quad V = R^1 p_2_* \mathcal{H}om(\mathcal{P}, p_1^*(O_C)).$$

For any $[F] \in U_C(r-1, r)$ we have

$$V|_{[F]} = H^1(C, \mathcal{H}om(\mathcal{P}, p_1^*(O_C))|_{C \times [F]}) = H^1(C, \mathcal{H}om(F, O_C)) = \text{Ext}^1(F, O_C)$$

which, by lemma 2.2 has dimension $2r - 1$. Hence we can conclude that $V$ is a vector bundle on $U_C(r-1, r)$ of rank $2r - 1$ whose fibre at $[F]$ is actually $\text{Ext}^1(F, O_C)$. Let’s consider the projective bundle associated to $V$ and the natural projection map

$$\pi : \mathbb{P}(V) \to U_C(r-1, r).$$

Note that for any $[F] \in U_C(r-1, r)$ we have:

$$H^0(C, \mathcal{H}om(\mathcal{P}, p_1^*(O_C))|_{C \times [F]}) = H^0(C, \mathcal{H}om(F, O_C)) = H^0(C, F^*) = 0,$$

since $F$ is stable with positive slope. Then by [NR69 Proposition 3.1], there exists a vector bundle $E$ on $C \times V$ such that for any point $v \in V$ the restriction $E|_{C \times v}$ is naturally identified with the extension $E_v$ of $F$ by $O_C$ defined by $v \in \text{Ext}^1(F, O_C)$ which, by lemma 2.3 is semistable and has sections, unless $v = 0$. Denote by $V_0$ the zero section of the vector bundle $V$, i.e. the locus parametrizing trivial extensions by $O_C$. Then $V \setminus V_0$ parametrize a family of semistable extensions of elements in $U_C(r-1, r)$ by $O_C$. This implies that the map sending $v \in V \setminus V_0$ to $[E_v]$ is a morphism. Moreover this induces a morphism

$$\Phi : \mathbb{P}(V) \to \Theta_r$$

sending $[v] \in \mathbb{P}(\text{Ext}^1(F, O_C))$ to $[E_v]$.

Note that we have:

$$\dim \mathbb{P}(V) = \dim U_C(r-1, r) + 2r - 2 = (r-1)^2 + 1 + 2r - 2 = r^2 = \dim \Theta_r.$$
Theorem 2.5 we have seen that the fiber of \( \Phi \) over a stable point \([E]\) with \( h^0(E) = 1 \) is a single point. For stable points it is possible to say something similar:

**Lemma 2.6.** Let \([E] \in \Theta_r^s\), there is a bijective morphism

\[
\nu: \mathbb{P}(H^0(E)) \to \Phi^{-1}(E).
\]

**Proof.** Let \( s \in H^0(E) \) be a non zero global section of \( E \). As in the proof of lemma 2.1, \( s \) induces an exact sequence of vector bundles:

\[
0 \to \mathcal{O}_C \to E \to F_s \to 0,
\]

where \( F_s \) is stable, \([F_s] \in \mathcal{U}_C(r-1, r)\) and \( E \) is the extension of \( F_s \) by a non zero vector \( v_s \in \text{Ext}^1(F_s, \mathcal{O}_C) \). By tensoring 13 with \( F_s^* \) and taking cohomology, since \( h^0(F_s^*) = h^0(F_s^* \otimes E) = 0 \), we get:

\[
0 \to H^0(F_s^* \otimes F_s) \to H^1(F_s^*) \to H^1(F_s^* \otimes E) \to H^1(F_s^* \otimes F_s) \to 0,
\]

from which we see that \( \langle v_s \rangle \) is the kernel of \( \lambda_s \).

So we have a natural map:

\[
H^0(E) \setminus \{0\} \to \mathbb{P}(V)
\]

sending a non zero global section \( s \in H^0(E) \) to \([v_s]\). Let \( s \) and \( s' \) be non zero global sections such that \( s' = \lambda s \), with \( \lambda \in C^* \). As in the proof of Theorem 2.5 it turns out that \( v_{s'} = \lambda v_s \) in \( \text{Ext}^1(F, \mathcal{O}_C) \). So we have a map:

\[
\nu: \mathbb{P}(H^0(E)) \to \mathbb{P}(V)
\]

sending \([s] \to [v_s]\), whose image is actually \( \Phi^{-1}(E) \).

We claim that this map is a morphism. Let \( \mathbb{P}^n = \mathbb{P}(H^0(E)) \), with \( n \geq 1 \), one can prove that there exists a vector bundle \( Q \) on \( \mathbb{P}^n \times C \) of rank \( r-1 \) such that \( Q_{[s]} \simeq F_s \). Hence we have a morphism \( \sigma: \mathbb{P}^n \to \mathcal{U}_C(r-1, r) \), sending \([s] \to [F_s] \), and a vector bundle \( \sigma^* V \) on \( \mathbb{P}^n \). Finally, there exists a vector bundle \( G \) on \( \mathbb{P}^n \) with \( G_{[s]} = H^1(F_s^* \otimes E) \) and a map of vector bundles:

\[
\lambda: \sigma^*(V) \to G,
\]

where \( \lambda_{[s]} \) is the map appearing in 13. Since \( \langle v_s \rangle = \ker \lambda_s \), this implies the claim.

To conclude the proof, we show that \( \nu \) is injective. Let \( s_1 \) and \( s_2 \) be global sections and assume that \([v_{s_1}] = [v_{s_2}]\). Then \( s_1 \) and \( s_2 \) defines two exact sequences which gives two extensions which multiples one of the other. Then, there exists an isomorphism \( \sigma \) of \( E \) such that the diagram

\[
\begin{array}{cccccc}
0 & \to & \mathcal{O}_C & \xrightarrow{s_1} & E & \xrightarrow{\sigma} & F & \to & 0 \\
\downarrow{id} & & \downarrow{\lambda id} & & \downarrow{\lambda id} & & & & \\
0 & \to & \mathcal{O}_C & \xrightarrow{s_2} & E & \xrightarrow{\sigma} & F & \to & 0
\end{array}
\]
is commutative. But $E$ is stable, so $\sigma = \lambda \text{id}$. Then, clearly, $\sigma_1 = \lambda \sigma_2$.

Let $L \in \text{Pic}^r(C)$ and $SU_C(r - 1, L)$ be the moduli space of stable vector bundles with determinant $L$. As we have seen, $SU_C(r - 1, L)$ can be seen as a subvariety of $U_C(r - 1, r)$. Let $V$ be the vector bundle on $U_C(r - 1, r)$ defined in the proof of Theorem 2.5. Let $V_L$ denote the restriction of $V$ to $SU_C(r - 1, L)$. We will denote with $\pi : \mathbb{P}(V_L) \to SU_C(r - 1, L)$ the projection map. Then, with the same arguments of the proof of Theorem 2.5 we have the following:

**Corollary 2.7.** Fix $L \in \text{Pic}^r(C)$. The map

$$\Phi_L : \mathbb{P}(V_L) \to \Theta_{r,L}$$

sending $[v]$ to the extension $[E_v]$ of $\pi([v])$ by $O_C$ defined by $v$, is a birational morphism.

As $\gcd(r, r - 1) = 1$ we have that $SU_C(r - 1, L)$ is a rational variety (see [New75, KS99]). Hence, as a consequence of our theorem we have also this interesting corollary:

**Corollary 2.8.** For any $L \in \text{Pic}^r(C)$, $\Theta_{r,L}$ is a rational subvariety of $SU_C(r,l)$.

### 3. General fibers of $\pi$ and $\theta$ map

In this section, we would restrict the morphism $\Phi$ to extensions of a general vector bundle $[F] \in U_C(r - 1, r)$. First of all we will deduce some properties of general elements of $U_C(r - 1, r)$.

For any vector bundle $F$, let $M_1(F^*)$ be the set of maximal line subbundles of $F^*$. Note that, if $[F] \in U_C(r - 1, r)$, then maximal line subbundles of $F^*$ are exactly the line subbundles of degree $r - 2$.

**Proposition 3.1.** Let $r \geq 3$, a general $[F] \in U_C(r - 1, r)$ satisfies the following properties:

1. if $r \geq 4$, $F$ does not admit maximal subbundles of rank $s \leq r - 3$;
2. $F$ admits finitely many maximal subbundles of rank $r - 2$;
3. we have $M_{r-2}(F) \simeq M_1(F^*)$.

**Proof.** For any $1 \leq s \leq r - 2$ let’s consider the following locus:

$$T_s = \{ [F] \in U_C(r - 1, r) \mid \exists S \hookrightarrow F \text{ with } \text{deg}(S) = \text{rk}(S) = s \}.$$

The set $T_s$ is locally closed, irreducible of dimension

$$\dim T_s = (r - 1)^2 + 1 + s(r - s + 2),$$

see [LN02, RT99]. If $r \geq 4$ and $s \leq r - 3$, then $\dim T_s < \dim U_C(r - 1, r)$, which proves (1).

2. Let $r \geq 3$ and $s = r - 2$. Then actually $T_{r-2} = U_C(r - 1, r)$ and a general $[F] \in U_C(r - 1, r)$ has finitely many maximal subbundles of rank $r - 2$, see [LN02, RT99] for the general case and [LN83] for $r = 3$, where actually the property holds for any $[F] \in U_C(2,3)$.

3. Let $[F] \in U_C(r - 1, r)$ be a general element and $[S] \in M_{r-2}(F)$, then $S$ is semistable and we have an exact sequence

$$0 \to S \to F \to Q \to 0$$

with $Q \in \text{Pic}^2(C)$. Moreover $S$ and $Q$ are general in their moduli spaces as in [LN02]. This implies that $\text{Hom}(F,Q) \simeq \mathbb{C}$. In fact, by taking the dual of the above sequence and tensoring with $Q$ we obtain

$$0 \to Q^* \otimes Q \to F^* \otimes Q \to S^* \otimes Q \to 0.$$
and, passing to cohomology we get
\[ 0 \rightarrow H^0(O_C) \rightarrow H^0(F^* \otimes Q) \rightarrow H^0(S^* \otimes Q) \rightarrow \cdots. \]

Since \( S \) and \( Q \) are general \( h^0(S^* \otimes Q) = 0 \) and we can conclude
\[ \text{Hom}(F,Q) \simeq H^0(F^* \otimes Q) \simeq H^0(O_C) = \mathbb{C}. \]

We have a natural map \( q: M_{r-2}(F) \rightarrow M_1(F^*) \) sending \( S \) to \( Q^* \). The map \( q \) is surjective as any maximal line subbundle \( Q^* \rightarrow F^* \) gives a surjective map \( \phi: F \rightarrow Q \) whose kernel is a maximal subbundle \( S \) of \( F \). The map is also injective. Indeed, assume that \([S_1]\) and \([S_2]\) are maximal subbundles such that \( q(S_1) = q(S_2) = Q^* \). Then \( S_1 = \ker \phi_1 \) and \( S_2 = \ker \phi_2 \), with \( \phi_1 \in \text{Hom}(F,Q) \simeq \mathbb{C} \). This implies that \( \phi_2 = \rho \phi_1 \), \( \rho \in \mathbb{C}^* \), hence \( S_1 \simeq S_2 \).

**Lemma 3.2.** For any \( r \geq 3 \) and \( |F| \in \mathcal{U}_C(r-1,r) \), let \( \text{ev} \) be the evaluation map of the secant bundle \( F_2(F \otimes \omega_C) \). If \( M_1(F^*) \) is finite, then \( \text{ev} \) is generically surjective and its degeneracy locus \( Z \) is the following:
\[ Z = \{ d \in C^{(2)} \mid O_C(-d) \in M_1(F^*) \}. \]

Moreover, \( Z \simeq M_1(F^*) \) if and only if \( h^0(F) = 1 \); if \( h^0(F) \geq 2 \) then \( Z = \mathcal{E} \cup Z' \), where \( \mathcal{E} = \{ \omega_C \} \) (see Section 7) and \( Z' \) is a finite set.

**Proof.** As we have seen in section 11, \( F_2(F \otimes \omega_C) \) is a vector bundle of rank \( 2r-2 \) on \( C^{(2)} \) and \( H^0(C^{(2)}, F_2(F \otimes \omega_C)) \simeq H^0(C,F \otimes \omega_C) \). Recall that the evaluation map of the secant bundle of \( F \otimes \omega_C \) is the map
\[ \text{ev}: H^0(F \otimes \omega_C) \otimes O_{C^{(2)}} \rightarrow F_2(F \otimes \omega_C) \]
and is such that, for any \( d \in C^{(2)} \), \( \text{ev}_d \) can be identified with the restriction map
\[ H^0(F \otimes \omega_C) \rightarrow (F \otimes \omega_C)_d, \quad s \mapsto s|_d. \]

Observe that
\[ H^1(F \otimes \omega_C(-d)) \simeq H^0(F^* \otimes O_C(d))^* \simeq \text{Hom}(F, O_C(d))^*. \]

Note that for any \( d \in C^{(2)} \) we have:
\[ \text{coker}(\text{ev}_d) \simeq H^1(F \otimes \omega_C(-d)), \]
hence \( \text{ev}_d \) is not surjective if and only if \( \text{Hom}(F, O_C(d)) \neq 0 \), that is \( O_C(-d) \) is a maximal line subbundle of \( F^* \). If \( F \) has finitely many maximal line subbundles we can conclude that \( \text{ev} \) is generically surjective and its degeneracy locus is the following:
\[ Z = \{ d \in C^{(2)} \mid \text{rk}(\text{ev}_d) < 2r-2 \} = \{ d \in C^{(2)} \mid O_C(-d) \in M_1(F^*) \}. \]

Let \( a: C^{(2)} \rightarrow \text{Pic}^{-2}(C) \) be the map sending \( d \rightarrow O_C(-d) \), \( a \) is the composition of \( A: C^{(2)} \rightarrow \text{Pic}^2(C) \) sending \( d \) to \( O_C(d) \) with the isomorphism \( \sigma: \text{Pic}^2(C) \rightarrow \text{Pic}^{-2}(C) \) sending \( Q \rightarrow Q^* \). Then \( Z = a^{-1}(M_1(F^*)) \). Note that
\[ Z \simeq M_1(F^*) \iff \omega_C^{-1} \notin M_1(F^*) \iff h^0(F) = 1 \]
If \( h^0(F) \geq 2 \), then \( \mathcal{E} = \{ \omega_C \} \subset Z \) and this concludes the proof.

**Remark 3.1.** Under the hypothesis of Lemma 3.2, the evaluation map fit into an exact sequence
\[ 0 \rightarrow M \rightarrow H^0(F \otimes \omega_C) \otimes O_{C^{(2)}} \rightarrow F_2(F \otimes \omega_C) \rightarrow T \rightarrow 0, \]
where \( M \) is a line bundle and \( \text{Supp}(T) = Z \).
Let \([F] \in \mathcal{U}_C(r - 1, r)\) be a general vector bundle, by proposition 3.1, \(\mathcal{M}_1(F^*) \simeq \mathcal{M}_{r-2}(F)\) is a finite set, moreover \(\text{Hom}(F, O_C(d)) \simeq \mathbb{C}\) when \(O_C(-d) \in \mathcal{M}_1(F^*)\). Finally, being \([F]\) general, we have \(h^0(F) = 1\) and this implies

\[ Z \simeq \mathcal{M}_1(F^*). \]

Taking the dual sequence of \([\mathcal{L}]\) we have:

\[ 0 \to \mathcal{F}_2(F \otimes \omega_C)^* \to H^0(F \otimes \omega_C)^* \otimes O_{C(2)} \to M^* \otimes J_Z \to 0, \]

and computing Chern classes we obtain:

\[ c_1(M^*) = c_1(\mathcal{F}_2(F \otimes \omega_C)), \]

\[ c_1(\mathcal{F}_2(F \otimes \omega_C)^*)c_1(M^*) + c_2(\mathcal{F}_2(F \otimes \omega_C)^*) + l(Z) = 0, \]

from which we deduce:

\[ l(Z) = c_1(\mathcal{F}_2(F \otimes \omega_C))^2 - c_2(\mathcal{F}_2(F \otimes \omega_C)). \]

We have:

\[ c_1(\mathcal{F}_2(F \otimes \omega_C)) = x + (r - 1)\theta, \quad c_2(\mathcal{F}_2(F \otimes \omega_C)) = r^2 + 2r - 2, \]

so we obtain:

\[ l(Z) = (r - 1)^2. \]

This gives the cardinality of \(\mathcal{M}_{r-2}(F)\) and of \(\mathcal{M}_1(F^*)\). This formula actually holds also for \(F \in \mathcal{U}_C(r, d)\), see \([\text{Ghi}81, \text{Lan}85]\) for \(r = 3\) and \([\text{OT}02, \text{Oxb}00]\) for \(r \geq 4\).

The stability properties of the secant bundles allow us to prove the following.

**Proposition 3.3.** Let \(r \geq 3\) and \([F] \in \mathcal{U}_C(r - 1, r)\) with \(h^0(F) \leq 2\). If \(\mathcal{M}_1(F^*)\) is finite, then every non trivial extension of \(F\) by \(O_C\) gives a vector bundle which admits theta divisor.

**Proof.** Let \(E\) be an extension of \(F\) by \(O_C\) which does not admit theta divisor. Hence

\[ 0 \to O_C \to E \to F \to 0, \]

and, by tensoring with \(\omega_C\) we obtain

\[ 0 \to \omega_C \to \tilde{E} \xrightarrow{\psi} \tilde{F} \to 0, \]

where, to simplify the notations, we have set \(\tilde{E} = E \otimes \omega_C\) and \(\tilde{F} = F \otimes \omega_C\). Note that \(\tilde{E}\) does not admit theta divisor too, hence

\[ \{l \in \text{Pic}^{-2}(C) | h^0(\tilde{E} \otimes l) \geq 1\} = \text{Pic}^{-2}(C). \]

This implies that \(\forall d \in C(2)\) we have \(h^0(\tilde{E} \otimes O_C(-d)) \geq 1\) too. Let’s consider the cohomology exact sequence induced by the exact sequence \([\mathcal{L}]\)

\[ 0 \to H^0(\omega_C) \to H^0(\tilde{E}) \xrightarrow{\psi_0} H^0(\tilde{F}) \to H^1(\omega_C) \to 0, \]

where we have used \(h^1(\tilde{E}) = 0\) as \(\mu(\tilde{E}) = 3 \geq 2\). Let’s consider the subspace of \(H^0(\tilde{F})\) given by the image of \(\psi_0\), i.e.

\[ V = \psi_0(H^0(\tilde{E})). \]

In particular \(\dim V = h^0(\tilde{F}) - 1 = 2r - 2\) so \(V\) is an hypersplane.

**Claim:** For any \(d \in C(2) \setminus \mathcal{E}\) we have \(V \cap H^0(\tilde{F} \otimes O_C(-d)) \neq 0\).

In fact, by tensoring the exact sequence \([\mathcal{L}]\) with \(O_C(-d)\) we have:

\[ 0 \to \omega_C \otimes O_C(-d) \to \tilde{E} \otimes O_C(-d) \to \tilde{F} \otimes O_C(-d) \to 0, \]

for a general \(d \in C(2)\), then passing to cohomology we obtain the inclusion:

\[ 0 \to H^0(\tilde{E} \otimes O_C(-d)) \to H^0(\tilde{F} \otimes O_C(-d)). \]
which implies the claim since \( h^0(\tilde{E} \otimes O_C(-d)) \neq 0 \).

Let \( ev: H^0(\tilde{F}) \otimes O_{C^{(2)}} \rightarrow \mathcal{F}_2(\tilde{F}) \) be the evaluation map of the secant bundle associated to \( \tilde{F} \) and consider its restriction to \( V \otimes O_{C^{(2)}} \). We have a diagramm as follows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker(ev_V)^c & \rightarrow & V \otimes O_{C^{(2)}} & \xrightarrow{ev_V} & \text{im}(ev_V) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M & \rightarrow & H^0(\tilde{F}) \otimes O_{C^{(2)}} & \xrightarrow{ev} & \mathcal{F}_2(\tilde{F}) & \rightarrow & T & \rightarrow & 0 \\
\end{array}
\]

where \( M \) is a line bundle, \( T \) has support on \( Z \) as in Lemma \[3.2\]. For any \( d \in C^{(2)} \) we have that the stalk of \( \ker(ev_V) \) at \( d \) is

\[
\ker(ev_V)_d = \ker \left( (ev_V)_d : V \otimes O_d \rightarrow \mathcal{F}_2(\tilde{F})_d \right) = H^0(\tilde{F} \otimes O_C(-d)) \cap V.
\]

Notice that, as a consequence of the claim,

\[
\dim \left( H^0(\tilde{F} \otimes O_C(-d)) \cap V \right) \geq 1
\]

for any non canonical divisor \( d \). Hence \( \ker(ev_V) \) is a torsion free sheaf of rank 1. For all \( d \in C^{(2)} \setminus Z \) we have \( h^0(\tilde{F} \otimes O_C(-d)) = 1 \), hence, for these points, we have

\[
\ker(ev_V)_d = H^0(\tilde{F} \otimes O_C(-d))
\]

In particular, as \( M \) and \( \ker(ev_V) \) coincide outside \( Z \), we have that the support of \( Q \) is contained in \( Z \).

In order to conclude the proof we will use the stability property of the secant bundle. With this aim, recall that, as seen in \[3.2\], \( c_1(\mathcal{F}(\tilde{F})) = x + (r - 1)\theta \) and thus, \( c_1(\mathcal{F}(\tilde{F})) \cdot x = 2r - 1 \). In particular, if \( H \) is an ample divisor with numerical class \( x \) we have

\[
(20) \quad \mu_H(\mathcal{F}(\tilde{F})) = \frac{2r - 1}{2r - 2}.
\]

We will distinguish two cases depending on the value of \( h^1(F) \).

**Assume that** \( h^0(F) = 1 \). In this case \( Z \simeq \mathcal{M}_1(F^*) \) is a finite set (see Lemma \[3.2\]). The support of \( T \) is finite too so we have

\[
c_1(\text{im}(ev_V)) = -c_1(\ker(ev_V)) = -c_1(M) = c_1(\mathcal{F}_2(\tilde{F})).
\]

Hence, we can conclude that \( \text{im}(ev_V) \) is a proper subsheaf of the secant bundle with rank \( 2r - 3 \) and with the same first Chern class. Hence

\[
(21) \quad \mu_H(\text{im}(ev_V)) = \frac{c_1(\text{im}(ev_V)) \cdot x}{2r - 3} = \frac{x \cdot (x + (r - 1)\theta)}{2r - 3} = \frac{2r - 1}{2r - 3},
\]

but this contradicts Proposition \[1.1\]. This conclude this case.

**Assume that** \( h^0(F) = 2 \). In this case \( Z = \mathcal{E} \cup Z' \) with \( Z' \) of dimension 0 as proven in Lemma \[3.2\]. Recall that the numerical class of \( \mathcal{E} \) in \( C^{(2)} \) is \( \theta - x \) (see Section \[1\]). Observe that \( \text{Supp}(T) = \mathcal{E} \cup Z' \) and for any \( d \in \mathcal{E} \) we have: \( \dim T_d = 1 \). From the exact sequence of the evaluation map of the secant bundle we obtain:

\[
c_1(M) = \mathcal{E} - c_1(\mathcal{F}_2(\tilde{F})).
\]

Since \( \text{Supp}(Q) \subset Z \), we distinguish two cases depending to its dimension.
(a) If \( \dim \text{Supp}(Q) = 0 \), then we have
\[
\chi_1(\text{im}(ev_V)) = -\chi_1(\ker(ev_V)) = -\chi_1(M),
\]
hence \( \chi_1(\text{im}(ev_V)) = \chi_1(\ker(ev_V)) = -\chi_1(M) \).

(22) \[
\mu_H(\text{im}(ev_V)) = \frac{x \cdot (2x + (r - 2)\theta)}{2r - 3} = \frac{2r - 2}{2r - 3}.
\]
But this is impossible since the secant bundle is semistable by Proposition 1.1.

(b) If \( \dim \text{Supp}(Q) = 1 \), since \( \text{Supp}(Q) \subset Z \) and \( \mathfrak{E} \) is irreducible, then \( \text{Supp}(Q) = \mathfrak{E} \cup Z' \), with \( Z' \) finite or empty. Observe that for any \( d \in \mathfrak{E} \) we have: \( \dim Q_d = 1 \). So we have
\[
\chi_1(\text{im}(ev_V)) = -\chi_1(\ker(ev_V)) = -\chi_1(M) + \mathfrak{E},
\]
hence \( \chi_1(\text{im}(ev_V)) = \chi_1(F_2(\tilde{F})) \) and we can conclude as above. □

Fix a line bundle \( L = M \otimes r \), with \( M \in \text{Pic}^1(C) \). Let \( [F] \in \mathcal{SU}_C(r - 1, L) \), we consider the fibre of the projective bundle \( \pi: P(V) \to \mathcal{U}_C(r - 1, r) \) at \( [F] \):
\[
P_F = \mathbb{P}(\text{Ext}^1(F, \mathcal{O}_C)) = \pi^{-1}([F]) \cong \mathbb{P}^{2r-2},
\]
and the restriction of the morphism \( \Phi \) to \( P_F \):
(23) \[
\Phi_F = \Phi|_{P_F}: P_F \to \Theta_{r,L}.
\]
By Corollary 2.7 the map \( \Phi_L: P(V_L) \to \Theta_{r,L} \) is a birational morphism. Then, there exists a non empty open subset \( U \subset \Theta_{r,L} \) such that
\[
\Phi_L|_{\Phi_L^{-1}(U)}: \Phi_L^{-1}(U) \to U
\]
is an isomorphism. Hence, for general \( F \in \mathcal{SU}_C(r - 1, L) \) the intersection \( \Phi^{-1}(U) \cap P_F \) is a non empty open subset of \( P_F \) and \( \Phi_F: P_F \to \Theta_{r,L} \) is a birational morphism onto its image.

Recall that
(24) \[
\mathcal{SU}_C(r, L) \dashrightarrow \theta \to |r\Theta_M|.
\]
is the rational map which sends \([E]\) to \( \Theta_E \). Note that if \( F \) is generic then, by Proposition 3.3, we have that \( \theta \) is defined in each element of \( \text{im}(\Phi_F) \) so it make sense to study the composition of \( \Phi_F \) with \( \theta \) which is then a morphism:

\[
\begin{array}{ccc}
P_F & \xrightarrow{\Phi_F} & \Theta_{r,L} \\
\downarrow & & \downarrow \theta \\
\theta \circ \Phi_F & \to & |r\Theta_M|
\end{array}
\]

We have the following result:

**Theorem 3.4.** For a general stable bundle \( F \in \mathcal{SU}_C(r - 1, L) \) the map
\[
\theta \circ \Phi_F: P_F \to |r\Theta_M|
\]
is a linear embedding.
Proof. As previously noted, as $F$ is generic we have that

$$
\Phi_F: \mathbb{P}_F \rightarrow \Theta_{r,L}
$$

is a birational morphism onto its image and that the composition $\theta \circ \Phi_F$ is a morphism by proposition 3.3. We recall that $\theta$ is defined by the determinat line bundle $\mathcal{L} \in \text{Pic}^0(SU_C(r, L))$. For simplicity, we set $\mathbb{P}^N = |r\Theta_M|$.

In order to prove that, for $F$ general, $\theta \circ \Phi_F$ is a linear embedding, first of all we will prove that $(\theta \circ \Phi_F)^*(\mathcal{O}_{\mathbb{P}^N}(1)) \simeq \mathcal{O}_{\mathbb{P}_F}(1)$.

For any $\xi \in \text{Pic}^0(C)$ the locus

$$
D_\xi = \{[E] \in SU_C(r, L)^s : h^0(E \otimes \xi) \geq 1\}
$$

is an effective divisor in $SU_C(r, L)$ and $\mathcal{O}_{SU_C(r, L)}(D_\xi) \simeq \mathcal{L}$, see [DN89].

Note that

$$
(\theta \circ \Phi_F)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \Phi_F^*(\mathcal{O}_{\mathbb{P}_F}(1)) = \Phi_F^*(\mathcal{L}|_{\Theta_{r,L}}) = \Phi_F^*(\mathcal{O}_{\Theta_{r,L}}(D_\xi)).
$$

Moreover, one can verify that for general $E \in \Theta_{r,L}$ there exists an irreducible reduced divisor $D_\xi$ passing through $E$ such that $E$ is a smooth point of the intersection $D_\xi \cap \Theta_{r,L}$. This implies that for general $F$ the pull back $\Phi_F^*(D_\xi)$ is a reduced divisor.

Observe that if $\xi$ is such that if $h^1(F \otimes \xi) \geq 1$ (this happens, for example, if $\xi = 0$), then any extension $E_v$ of $F$ has sections:

$$
h^0(E_v \otimes \xi) = h^1(F \otimes \xi) \geq 1.
$$

In particular this implies that $\Phi_F(\mathbb{P}_F) \subset D_\xi$. On the other hand this does not happen for $\xi$ general and we are also able to be more precise about this. Indeed, let $\xi \in \text{Pic}^0(C)$, then there exists an effective divisor $d \in C^{(2)}$ such that $\xi = \omega_C(-d)$. We have that $h^1(F \otimes \xi) \geq 1$ if and only $d \in Z$, where $Z$ is defined in Lemma 3.2. Moreover, we can assume that $Z$ is finite by Proposition 3.2 as $F$ is generic. From now on we will assume that $d \notin |\omega_C|$ and $d \notin Z$. We can consider the locus

$$
H_\xi = \{[v] \in \mathbb{P}_F | h^0(E_v \otimes \xi) \geq 1\}.
$$

We will prove that $H_\xi$ is an hyperplane in $\mathbb{P}_F$ and $\Phi_F^*(D_\xi) = H_\xi$.

From the exact sequence

$$
0 \rightarrow \xi \rightarrow E_v \otimes \xi \rightarrow F \otimes \xi \rightarrow 0,
$$

passing to cohomology, since $h^0(\xi) = 0$ we have

$$
0 \rightarrow h^0(E_v \otimes \xi) \rightarrow h^0(F \otimes \xi) \rightarrow \cdots
$$

from which we deduce that $[v] \in H_\xi$ if and only if there exists a non zero global section of $H^0(F \otimes \xi)$ which is in the image of $h^0(E_v \otimes \xi)$. Since $d \notin Z$, then $h^0(F \otimes \xi) = 1$, let’s denote by $s$ a generator of $H^0(F \otimes \xi)$.

Claim: if $\xi$ is general, we can assume that the zero locus $Z(s)$ of $s$ is actually empty. This can be seen as follows. By stability of $F \otimes \xi$ we have that $Z(s)$ has degree at most 1. Suppose that $Z(s) = x$, with $x \in C$. Then we would have an injective map $O_C(x) \hookrightarrow F \otimes \xi$ of vector bundles which gives us $\xi^{-1}(x) \in \mathcal{M}_1(F)$. Since $F$ is general, if $r \geq 4$ then $\mathcal{M}_1(F)$ is empty by Proposition 3.1 so the zero locus of $s$ is indeed empty. If $r = 3$, then

$$
\mathcal{M}_1(F) = \{T_1, \ldots, T_m\}
$$

is finite. For each $i \in \{1, \ldots, m\}$ consider the locus

$$
T_{F,i} = \{\xi \in \text{Pic}^0(C) | \exists x \in C : \xi^{-1}(x) = T_i\}.
$$
This is a closed subset of $\text{Pic}^0(C)$ of dimension 1. Indeed, $T_{F,i}$ is the image, under the embedding $\mu_i : C \to \text{Pic}^0(C)$ which send $x$ to $T_i(-x)$. Hence the claim follows by choosing $\xi$ outside the divisor $\bigcup_{i=1}^m T_{F,i}$.

As consequence of the claim, we have that $s$ induces an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\iota_*} F \otimes \xi \longrightarrow Q \longrightarrow 0.$$  

Observe that $[v] \in H_\xi$ if and only if $\iota_* s$ can be lifted to a map $\tilde{\iota}_s : \mathcal{O}_C \to E \otimes \xi$. Then, by Lemma 2.4 we have that $H_\xi$ is actually the projectivization of the kernel of the following map:

$$H^1(\iota_*^s) : H^1(\text{Hom}(F \otimes \xi, \xi)) \to H^1(\text{Hom}(\mathcal{O}_C, \xi))$$

which proves that $H_\xi$ is an hyperplane as $H^1(\iota_*^s)$ is surjective and

$$H^1(\text{Hom}(\mathcal{O}_C, \xi)) \simeq H^1(\xi) \simeq \mathbb{C}.$$  

Note that we have the inclusion $\Phi_F^*(D_\xi) \subseteq H_\xi$. Since both are effective divisors and $H_\xi$ is irreducible we can conclude that they have the same support. Finally, since $\Phi_F^*(D_\xi)$ is reduced, then they are the same divisor. In particular, as claimed, we have

$$\Phi_F^*(\mathcal{O}_{\Theta}(\mathcal{L})) = \mathcal{O}_{\mathbb{P}(\mathcal{L})}(1).$$

In order to conclude we simply need to observe that the map is induced by the full linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{L})}(1)|$. But this easily follows from the fact that $\theta \circ \Phi_F$ is a morphism. Hence $\theta \circ \Phi_F$ is a linear embedding and the Theorem is proved. 

**Remark 3.3.** The above Theorem implies that $\Phi_F^*(\mathcal{L})$ is a unisecant line bundle on the projective bundle $\mathbb{P}(\mathcal{V}_L)$.

**References**

[Ati57] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85**, (1957), 181–207.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic curves, I*, Springer verlag, Berlin (1985).

[Bea03] A. Beauville, *Some stable vector bundles with reducible theta divisors*, Man. Math. **110** (2003), 343-349.

[Bea06] A. Beauville, *Vector bundles and the theta functions on curves of genus 2 and 3*, Amer. J. of Math. **128**(n3), (2006), 607–618.

[BD18] S. Basu, K. Dan, *Stability of secant bundles on the second symmetric power of curves*, Arch. Math. (Basel) **110**, (2018), 245–249.

[BNR89] A. Beauville, M. S. Narasimhan, S. Ramanan, *Spectral curves and the generalised theta divisor*, J. Reine angew. Math. **398**(1989), 169–178.

[Bri17] S. Brivio, *Families of vector bundles and linear systems of theta divisors*, Inter. J. Math. **28**, n 6, (2017), 1750039 (16 pages).

[BV07] S. Brivio, A. Verra, *The Brill Noether curve of a stable vector bundle on a genus two curve*, in ”Algebraic Cycles and Motives”, London Math. Soc. LNS **344**, v 2, (2007), ed. J. Nagel, C. Peters, Cambridge Univ. Press.

[DN89] I. M. Drezet, M. S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stable sur les courbes algébriques*, Invent. Math. **97**(1989), 53–94.

[Ghi81] F. Ghione, *Quelques résultats de Corrado Segre sur les surfaces réglées*, Math. Ann. **255**, (1981), 77–96.

[KS99] A. King, A. Schofield, *Rationality of moduli of vector bundles on curves*, Indag. Math. (N.S.) **10**. 4, (1999), 519–535.

[Las91] Y. Laszlo, *Un théorème de Riemann pour les diviseurs thêta sur les espaces de modules de fibrés stables sur une courbe*, Duke Math. J. **64**, (1991), pp. 333-347.
[LeP97] J. Le Potier, *Lectures on vector bundles*, Cambridge Univ. Press, (1997).

[Lan85] H. Lange, *Hohere Sekantenvarietaten und Vektorbündel auf Kurven*, Manuscripta Math. **52** (1985), 63–80.

[LN83] H. Lange and M.S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves*, Math. Ann. **266**, (1983) 55–72

[LN02] H. Lange and P.E. Newstead, *Maximal subbundles and Gromov-Witten invariants*, A tribute to C. S. Seshadri (Chennai, 2002), Trends Math., Birkhäuser Basel, (2003) 310–322

[Mar82] M. Maruyama, *Elementary transformations in the theory of algebraic vector bundles*, Lecture Notes Math. **961**, (1982) 241–266.

[NR69] M.S. Narasimhan, S. Ramanan, *Moduli of vector bundles on a compact Riemann Surface*, Ann. of Math. **89**(2), (1969), 14–51.

[New75] P.E. Newstead, *Rationality of moduli spaces of stable bundles*, Math. Ann., **215**, (1975) 251–268

[OT02] C. Okonek and A. Teleman, *Gauge theoretical equivariant Gromov-Witten invariants and the full Seiberg-Witten invariants of ruled surfaces*, Comm. Math. Phys. **227** 3, (2002) 551–585

[Ort05] A. Ortega, *On the moduli space of rank 3 vector bundles on a genus 2 curve and the Cable cubic*, J. Alg. Geom. **14**, (2005), 327–356.

[Oxb00] W. M. Oxbury, *Varieties of maximal line subbundles*, Math. Proc. Cambridge Phil. Soc. **129** (2000), 9–18.

[RT99] B. Russo and M. Teixidor i Bigas, *On a Conjecture of Lange*, J. Alg. Geom. **8** (1999), 483-496.

[Sch64] R. L. E. Schwarzenberger, *The secant bundle of a projective variety*, Proc. London Math. Soc. (3), **14** (1964), 369–384.

[Seg89] C. Segre, *Recherches générales sur les courbes et les surfaces réglées algébriques II*, Math. Ann. **34** (1889), 1–25.

[Ses82] C.S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque, **96** (1992).

(Sonia Brivio) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MILANO-BICOCCA, VIA ROBERTO COZZI, 55, 20125 MILANO (MI)

E-mail address: sonia.brivio@unimib.it

(Filippo F. Favale) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MILANO-BICOCCA, VIA ROBERTO COZZI, 55, 20125 MILANO (MI)

E-mail address: filippo.favale@unimib.it