On the multiplicity of $A_\alpha$-eigenvalues and the rank of complex unit gain graphs

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Abstract

Let $\Phi = (G, \varphi)$ be a connected complex unit gain graph ($T$-gain graph) on a simple graph $G$ with $n$ vertices and maximum vertex degree $\Delta$. The associated adjacency matrix and degree matrix are denoted by $A(\Phi)$ and $D(\Phi)$, respectively. Let $m_\alpha(\Phi, \lambda)$ be the multiplicity of $\lambda$ as an eigenvalue of $A_\alpha(\Phi) := \alpha D(\Phi) + (1 - \alpha) A(\Phi)$, for $\alpha \in [0, 1)$. In this article, we establish that $m_\alpha(\Phi, \lambda) \leq (\Delta - 2)n + 2\Delta - 1$, and characterize the classes of graphs for which the equality hold. Furthermore, we establish a couple of bounds for the rank of $A(\Phi)$ in terms of the maximum vertex degree and the number of vertices. One of the main results extends a result known for unweighted graphs and simplifies the proof in \cite{15}, and other results provide better bounds for $r(\Phi)$ than the bounds known in \cite{8}.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G)$ are the vertex set and the edge set of $G$, respectively. If two vertices $v_s$ and $v_t$ are connected by an edge, we write $v_s \sim v_t$. If $v_s \sim v_t$, then the edge between them is denoted by $e_{s,t}$. The degree of a vertex $v_s$ is denoted by $d(v_s)$ and is defined as the number of vertices adjacent to $v_s$. Then the maximum vertex degree of $G$ is denoted by $\Delta(G)$ (or, simply $\Delta$). The degree matrix of a graph $G$ is a diagonal matrix, denoted by $D(G)$, is defined

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by $D(G) := \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$. The adjacency matrix $A(G)$ of a graph $G$ is a symmetric matrix whose $(s,t)$th entry is 1 if $v_s \sim v_t$, and zero otherwise. The nullity of $A(G)$ is the multiplicity of zero eigenvalue of $A(G)$, and is called the nullity of $G$, denoted by $\eta(G)$. The rank of $G$ is the rank of $A(G)$, and is denoted by $r(G)$. Thus $\eta(G) = n - r(G)$.

Let $G$ be a simple undirected graph. An oriented edge from the vertex $v_s$ to the vertex $v_t$ is denoted by $\overrightarrow{e}_{s,t}$. For each undirected edge $e_{s,t} \in E(G)$, there is a pair of oriented edges $\overrightarrow{e}_{s,t}$ and $\overrightarrow{e}_{t,s}$. The collection $\overrightarrow{E(G)} := \{\overrightarrow{e}_{s,t}, \overrightarrow{e}_{t,s} : e_{s,t} \in E(G)\}$ is the oriented edge set associated with $G$. Let $T = \{z \in \mathbb{C} : |z| = 1\}$. A complex unit gain graph (or $T$-gain graph) on a simple graph $G$ is an ordered pair $(G, \varphi)$, where the gain function $\varphi : \overrightarrow{E(G)} \to T$ is a mapping such that $\varphi(\overrightarrow{e}_{s,t}) = \varphi(\overrightarrow{e}_{t,s})^{-1}$, for every $e_{s,t} \in E(G)$. A $T$-gain graph $(G, \varphi)$ is denoted by $\Phi$. The adjacency matrix of a $T$-gain graph $\Phi = (G, \varphi)$ is a Hermitian matrix, denoted by $A(\Phi)$ and its $(s,t)$th entry is defined as follows:

$$A(\Phi)_{st} = \begin{cases} \varphi(\overrightarrow{e}_{s,t}) & \text{if } v_s \sim v_t, \\ 0 & \text{otherwise.} \end{cases}$$

The rank and the nullity of $A(\Phi)$ are the rank and nullity of $\Phi$, denoted by $r(\Phi)$ and $\eta(\Phi)$, respectively. The degree matrix $D(\Phi)$ and maximum vertex degree $\Delta(\Phi)$ of $\Phi$ are same as $D(G)$ and $\Delta(G)$, respectively. The notion of adjacency matrix of $T$-gain graphs generalize the notion of adjacency matrix of undirected graphs, adjacency matrix of signed graphs and the Hermitian adjacency matrix of a digraph. The notion of gain graph was introduced in [2]. For more information about the properties of gain graphs and $T$-gain graphs, we refer to [9, 11, 12, 13, 19, 20].

Let $G$ be a simple graph with adjacency matrix $A(G)$ and degree matrix $D(G)$. In [10], the author introduced the following new family of matrices, denoted by $A_\alpha(G)$, associated with an undirected graph:

$$A_\alpha(G) = \alpha D(G) + (1 - \alpha) A(G), \quad \alpha \in [0, 1].$$

Then $A_0(G) = A(G)$, $A_{\frac{1}{2}}(G) = \frac{1}{2} Q(G)$ and $A_1(G) = D(G)$, where $Q(G)$ is the signless Laplacian of $G$. The eigenvalues of $A_\alpha(G)$ are known as $A_\alpha$-eigenvalues of $G$. For more details about $A_\alpha$-eigenvalues, we refer to [10, 15]. Let $\Phi = (G, \varphi)$ be a $T$-gain graph on underlying graph $G$. In [6], the authors introduced $A_\alpha$-matrix for a complex unit gain graph $\Phi$, denoted by $A_\alpha(\Phi)$, defined as follows:

$$A_\alpha(\Phi) = \alpha D(\Phi) + (1 - \alpha) A(\Phi), \quad \alpha \in [0, 1].$$
The eigenvalues of $A_{\alpha}(\Phi)$ are called $A_{\alpha}$-eigenvalues of $\Phi$. Note that $A_{\alpha}(\Phi)$ is a generalization of $A_{\alpha}(G)$ and hence the matrices $A_{\alpha}(G)$, $A(\Phi)$, $A(G)$ are particular cases of $A_{\alpha}(\Phi)$. Let $\lambda$ be an eigenvalue of $A_{\alpha}(\Phi)$. Then the multiplicity of $\lambda$ is denoted by $m_{\alpha}(\Phi, \lambda)$.

In [15], authors proved the following upper bound for the multiplicity of $A_{\alpha}$-eigenvalues of a connected graph $G$.

**Theorem 1.1.** [15, Theorem 3.1, Theorem 3.3] Let $G$ be a connected graph of $n$ vertices with maximum vertex degree $\Delta \geq 2$. If $m_{\alpha}(G, \lambda)$ is the multiplicity of $\lambda$ as an eigenvalue of $A_{\alpha}(G)$, then for $\alpha \in [0, 1)$,

$$m_{\alpha}(G, \lambda) \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}$$

Equality occur if and only if $G$ and $\lambda$ satisfy one of the following:

(i) $G = K_n$ and $\lambda = \alpha n - 1$,

(ii) $G = C_n$ with even order and $\lambda \in \{2\alpha + 2(1 - \alpha) \cos\left(\frac{2\pi j}{n}\right) : j = 1, 2, \ldots, \frac{n}{2}\}$,

(iii) $G = C_n$ with odd order and $\lambda \in \{2\alpha + 2(1 - \alpha) \cos\left(\frac{2\pi j}{n}\right) : j = 1, 2, \ldots, \frac{n-1}{2}\}$,

(iv) $G = K_{\frac{n}{2}, \frac{n}{2}}$ and $\lambda = \frac{\alpha n}{2}$.

One of the main objectives of this article is to extend Theorem 1.1 for $A_{\alpha}$-matrices of $T$-gain graphs, and provide an alternate simple proof (Theorem 3.1).

Establishing bounds for the rank and the nullity of a graph, in terms of $\Delta$ and $n$, is an interesting problem considered in literature [1, 16, 17, 21]. The following bounds for the rank of $T$-gain graphs are known.

**Theorem 1.2** (8, Theorem 3.2]. Let $\Phi = (G, \varphi)$ be any $T$-gain graph of $n$ vertices with maximum vertex degree $\Delta$. Then

$$r(\Phi) \geq \frac{n}{\Delta}.$$

**Theorem 1.3** (8, Theorem 3.4]. Let $\Phi = (G, \varphi)$ be a $T$-gain graph with $n$ vertices and maximum vertex degree $\Delta$ such that $2\Delta \nmid n$. Then

$$r(\Phi) \geq \frac{n + 1}{\Delta}.$$ 

Equality occur if and only if $\Phi = \frac{n-2\Delta+1}{2\Delta}K_{\Delta, \Delta}^\xi \cup K_{(\Delta-1), \Delta}^\xi$ and each $C_4^\xi$ (if any) in $K_{\Delta, \Delta}^\xi$ and $K_{(\Delta-1), \Delta}^\xi$ is of Type A.
The second objective of this article is to improve these bounds for T-gain graphs (Theorem 4.1 and Theorem 4.2). The proofs of the main results use some known results from the field of zero-forcing sets, and in [14] the authors used the same techniques to derive bounds for the nullity of the adjacency matrices of graphs.

2 Definitions, notation and preliminary results

The notion of a zero-forcing set of a simple graph $G$ is introduced in [5].

**Definition 2.1** ([5, Definition 2.1]). [Color-change rule] Let $G$ be a simple graph such that each vertex of $G$ is colored either black or white. Suppose vertex $v$ is a black vertex and exactly one neighbor $w$ of $v$ is white among all other neighbors. Then change the color of $w$ to black.

The derived coloring of a given coloring of $G$ is the resulting coloring after applying the color-change rule such that no more changes are possible. A subset $Z$ of the vertex set of $G$ is called a zero forcing set of $G$, if initially the vertices of $Z$ are all colored black and the remaining vertices are colored white, the derived coloring of $G$ are all black. The zero forcing number of $G$ is defined as $Z(G) = \min_{Z} |Z|$, $Z$ is a zero-forcing set of $G$.

The cycle and the complete graph on $n$ vertices are denoted by $C_n$ and $K_n$, respectively. A complete bipartite graph, of partition size $m$ and $n$, is denoted by $K_{m,n}$. In [3] and [4], the authors established upper bounds for the zero forcing number of a graph in terms of the maximum vertex degree.

**Theorem 2.1** ([3, Theorem 1(ii)]). If $G$ is a connected graph of $n$ vertices with maximum vertex degree $\Delta \geq 2$. Then

$$Z(G) \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}.$$

Equality occurs if and only if $G$ is either $C_n$, or $K_n$, or $K_{\frac{n}{2}, \frac{n}{2}}$.

**Theorem 2.2** ([4, Theorem 4]). Let $G$ be any connected graph of $n$ vertices and the maximum vertex degree $\Delta \geq 3$. Then

$$Z(G) \leq \frac{(\Delta - 2)n}{\Delta - 1}$$

holds if and only if $G \notin \{G_1, G_2, K_n, K_{\frac{n}{2}, \frac{n}{2}}, K_{\frac{n+1}{2}, \frac{n+1}{2}}\}$, where $G_1$ and $G_2$ are given in Figure 7.
Lemma 2.1 ([5 Proposition 2.2]). Let $B$ be any square matrix on some field with $\eta(B) > s$. Then there exists a nonzero vector $y \in \ker(B)$ vanishing at $s$ specified positions.

For a graph $G$, any function $\zeta : V(G) \to \mathbb{T}$ is called a switching function. Let $\Phi_1 = (G, \varphi_1)$ and $\Phi_2 = (G, \varphi_2)$ be two $\mathbb{T}$-gain graphs. Then $\Phi_1$ and $\Phi_2$ are switching equivalent, denoted by $\Phi_1 \sim \Phi_2$, if there exists a switching function $\zeta$ such that $\varphi_1(e_{i,j}) = \zeta(v_i)^{-1}\varphi_2(e_{i,j})\zeta(v_j)$, for all $e_{i,j} \in E(G)$. If $\Phi_1 \sim \Phi_2$, then $A(\Phi_1)$ and $A(\Phi_2)$ are diagonally similar and hence have the same spectra. Let $\overrightarrow{C_n}$ be an oriented cycle in a $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ with oriented edges $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$, then $\varphi(\overrightarrow{C_n}) = \prod_{i=1}^{n} \varphi(\overrightarrow{e_i})$. If $\varphi(\overrightarrow{C_n})$ is a real number, then we simply write $\varphi(C_n)$. A $\mathbb{T}$-gain graph $\Phi = (G, \varphi)$ is balanced if $\varphi(C) = 1$ for all cycle $C$ in $\Phi$. If $\Phi$ is balanced then we write $\Phi \sim (G, 1)$. It is known that $(C_n, \varphi_1) \sim (C_n, \varphi_2)$ if and only if $\varphi_1(\overrightarrow{C_n}) = \varphi_2(\overrightarrow{C_n})$. Therefore, if $\varphi_1(\overrightarrow{C_n}) = \varphi_2(\overrightarrow{C_n})$, then $(C_n, \varphi_1)$ and $(C_n, \varphi_2)$ have the same spectra.

Definition 2.2 ([7 Definition 2]). Let $\Phi = (C_n, \varphi)$ be any $\mathbb{T}$-gain graph on a cycle $C_n$. Then $\Phi$ is called

\[
\begin{cases}
\text{Type A, if } n \text{ is even and } \varphi(C_n) = (-1)^{\frac{n}{2}} \\
\text{Type B, if } n \text{ is even and } \varphi(C_n) \neq (-1)^{\frac{n}{2}} \\
\text{Type C, if } n \text{ is odd and } \text{Re}((-1)^{\frac{n-1}{2}} \varphi(C_n)) > 0 \\
\text{Type D, if } n \text{ is odd and } \text{Re}((-1)^{\frac{n-1}{2}} \varphi(C_n)) < 0 \\
\text{Type E, if } n \text{ is odd and } \text{Re}((-1)^{\frac{n-1}{2}} \varphi(C_n)) = 0.
\end{cases}
\]

The rank of the gain adjacency matrices of cycles are known.

Theorem 2.3 ([18 Theorem 7]). Let $\Phi = (C_n, \varphi)$ be any $\mathbb{T}$-gain graph on $C_n$ with $n$ vertices. Then
\[ r(\Phi) = \begin{cases} 
  n - 2, & \text{if } \Phi \text{ is Type A} \\
  n, & \text{if } \Phi \text{ is Type B} \\
  n, & \text{if } \Phi \text{ is Type C} \\
  n, & \text{if } \Phi \text{ is Type D} \\
  n - 1, & \text{if } \Phi \text{ is Type E}. 
\end{cases} \]

The following results will be used in the proofs of the main theorems.

**Lemma 2.2** ([18 Lemma 3(iii)]). Let \( \Phi = (G, \varphi) \) be a \( T \)-gain graph and \( \Phi_1 \) be an induced subgraph of \( \Phi \). Then \( r(\Phi_1) \leq r(\Phi) \).

**Theorem 2.4** ([9 Theorem 4.1]). Let \( \Phi = (G, \varphi) \) be a \( T \)-gain graph on a bipartite graph \( G \). Then the eigenvalues of \( A(\Phi) \) are symmetric about origin.

Let us recall a couple of results about the \( A_\alpha \)-eigenvalues of a \( T \)-gain graph \( \Phi \).

**Lemma 2.3** ([6 Lemma 2.9]). Let \( \Phi = (C_n, \varphi) \) be a \( T \)-gain graph such that \( \varphi(C_n^\rightarrow) = e^{i\theta} \). Then the \( A_\alpha \)-eigenvalues of \( \Phi \) are
\[
\left\{ 2\alpha + 2(1-\alpha) \cos \left( \frac{\theta + 2\pi j}{n} \right) : j = 0, 1, \ldots, (n-1) \right\}.
\]

**Corollary 2.1** ([6 Corollary 2.12]). Let \( \Phi = (C_n, \varphi) \) be a \( T \)-gain graph such that \( \varphi(C_n^\rightarrow) = e^{i\theta} \). If \( m_\alpha(\Phi, \lambda) \) is the multiplicity of \( \lambda \) as an eigenvalue of \( A_\alpha(\Phi) \), then \( m_\alpha(\Phi, \lambda) \leq 2 \), for \( \lambda \in \mathbb{R} \) and \( \alpha \in [0, 1) \). Equality occur if and only if any one of the following holds:

(i) \( \theta = 0 \) and \( \lambda \in \left\{ 2\alpha + 2(1-\alpha) \cos \left( \frac{2\pi j}{n} \right) : j = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right\} \),

(ii) \( \theta = \pi \) and \( \lambda \in \left\{ 2\alpha + 2(1-\alpha) \cos \left( \frac{(2j+1)\pi}{n} \right) : j = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right\} \).

### 3 Multiplicity of an \( A_\alpha \)-eigenvalue of \( T \)-gain graph

Let \( H_n \) denote the set of all Hermitian matrices of order \( n \). For \( B \in H_n \), \( G(B) \) is the matrix with \((i,j)\)th entry defined as follows:
\[
G(B)_{ij} = \begin{cases} 
 B_{ij} & \text{if } B_{ij} \neq 0, \\
 0 & \text{otherwise.} 
\end{cases}
\]

where \( B_{ij} \) denote the \((i,j)\)th entry of the matrix \( B \). Let \( \Phi = (G, \varphi) \) be any \( T \)-gain graph of \( n \) vertices. A matrix \( B = (B_{ij}) \in H_n \) is a *matrix of type \( \Phi \) if \( G(B)_{ij} = A(\Phi)_{ij} \) for all \( i \neq j \). Define \( \mathcal{H}(\Phi) = \{B \in H_n : B \text{ is of type } \Phi\} \). Let \( \eta(B) \) be the nullity of the matrix \( B \).
Define $M(\Phi) := \max\{\eta(B) : B \in \mathcal{H}(\Phi)\}$. Then $\eta(A(\Phi)) \leq M(\Phi)$. For any $T$-gain graph $\Phi$, the underlying graph is denoted by $\Gamma(\Phi)$. For $y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n$, the support of $y$ is the set of indices $j$ such that $y_j \neq 0$, and is denoted by $\text{supp}(y)$. A zero forcing set of a $T$-gain graph $\Phi$ is the zero forcing set of its underlying graph $\Gamma(\Phi)$. The following lemma is an extension of [5, Proposition 2.3] for the complex matrices. For the sake of completeness we include a proof here.

**Lemma 3.1.** Let $\Phi$ be any $T$-gain graph, and $Z$ be a zero forcing set of $\Phi$. Let $B \in \mathcal{H}(\Phi)$ and $y \in \text{Ker}(B)$ with $\text{supp}(y) \cap Z = \emptyset$. Then $y = 0$.

**Proof.** Let $V(\Phi)$ be the vertex set of $\Phi$. If $Z = V(\Phi)$, then $y = 0$. Suppose $Z \subset V(\Phi)$. Since $Z$ is a zero forcing set, so all the white vertices in $V(\Phi) \setminus Z$ can be colored black by color change rule. Let $v_i \in Z$ be such that it has exactly one white neighbor vertex $v_t$. Then the $i$-th entry $(By)_i = B_{ii}y_i + \sum_{v_i \sim v_j} B_{ij}y_j = B_{it}y_t = 0$. Thus $y_t = 0$. As $Z$ is a zero forcing set, so all the components of $y$ associated with white vertices are zero. Hence $y = 0$. \hfill $\square$

The following lemma is an extension of [5, Proposition 2.4] for the complex matrices. For the sake of completeness we include a proof here.

**Lemma 3.2.** Let $\Phi$ be any $T$-gain graph. Then $M(\Phi) \leq Z(\Gamma(\Phi))$.

**Proof.** Let $Z$ be a zero forcing set of $\Phi$. Suppose that $M(\Phi) > |Z|$. Then there exists a matrix $B \in \mathcal{H}(\Phi)$ such that $\eta(B) > |Z|$. Therefore, by Lemma 2.1, there exist a nonzero $y \in \text{Ker}(B)$ such that $\text{supp}(y) \cap Z = \emptyset$. By Lemma 3.1, we get $y = 0$, a contradiction. Thus $M(\Phi) \leq |Z|$, and hence $M(\Phi) \leq Z(\Gamma(\Phi))$. \hfill $\square$

**Lemma 3.3.** Let $\Phi = (K_n, \varphi)$ be a $T$-gain graph. If $\mu$ is an eigenvalue of $\Phi$ with multiplicity $(n-1)$ if and only if $\{\alpha(n-1) + (1-\alpha)\mu\}$ is an $A_\alpha$-eigenvalue of $\Phi$ with multiplicity $(n-1)$, for $\alpha \in [0,1)$.

Now we are ready to establish one of our main results.

**Theorem 3.1.** Let $\Phi = (G, \varphi)$ be a connected $T$-gain graph of $n$ vertices with maximum vertex degree $\Delta \geq 2$. If $m_\alpha(\Phi, \lambda)$ is the multiplicity of $\lambda$ as an $A_\alpha$-eigenvalue of $\Phi$, where $\alpha \in [0,1)$, then

$$m_\alpha(\Phi, \lambda) \leq \frac{(\Delta - 2)n + 2}{(\Delta - 1)}.$$

Equality occurs if and only if one of the following holds:
\( (i) \) \( \Phi = (K_n, \varphi) \) with \( \mu \in \text{spec}(\Phi) \) has multiplicity \((n - 1)\) and \( \lambda = \alpha(n - 1) + (1 - \alpha)\mu. \)

\( (ii) \) \( \Phi = (C_n, \varphi) \) with \( \varphi(C_n) = 1 \) and \( \lambda \in \{2\alpha + 2(1 - \alpha) \cos \left(\frac{2\pi j}{n}\right) : j = 0, 1, \ldots, \frac{n}{2} - 1\}. \)

\( (iii) \) \( \Phi = (C_n, \varphi) \) with \( \varphi(C_n) = -1 \) and \( \lambda \in \{2\alpha + 2(1 - \alpha) \cos \left(\frac{(2j+1)\pi}{n}\right) : j = 0, 1, \ldots, \frac{n}{2} - 1\}. \)

\( (iv) \) \( \Phi \sim (K_{\frac{n}{2}, \frac{n}{2}}, 1) \) and \( \lambda = \frac{an}{2}. \)

**Proof.** For \( \alpha \in [0, 1] \), we have \( A_\alpha(\Phi) = \alpha D(\Phi) + (1 - \alpha)A(\Phi) \), where \( D(\Phi) \) is the degree matrix of \( \Phi \). Let \( \lambda \) be an eigenvalue of \( A_\alpha(\Phi) \). Then the matrix \( B := (A_\alpha(\Phi) - \lambda I) \) is Hermitian, where \( I \) is the identity matrix of order \( n \), and hence the multiplicity of the eigenvalue \( \lambda \) is same as the nullity of \( B \). That is, \( \eta(B) = m_\alpha(\Phi, \lambda) \). Let \( B_{ij} \) be the \((i,j)\)th-entry of the matrix \( B \). If \( B_{ij} \neq 0 \), then \( B_{ij} = \frac{(1 - \alpha)A(\Phi)_{ij}}{(1 - \alpha)A(\Phi)_{ij}} = A(\Phi)_{ij}, \) for \( i \neq j \) and \( \alpha \in [0, 1] \).

Thus \( B \in \mathcal{H}(\Phi) \). Therefore, \( \eta(B) \leq M(\Phi) \). Now, by combining Lemma 3.2 Theorem 2.1 and the fact that \( \eta(B) \leq M(\Phi) \), we have

\[
m_\alpha(\Phi, \lambda) = \eta(B) \leq M(\Phi) \leq Z(G) \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}.
\]

If \( m_\alpha(\Phi, \lambda) = \frac{(\Delta - 2)n + 2}{\Delta - 1} \), then \( Z(G) = \frac{(\Delta - 2)n + 2}{\Delta - 1} \). Therefore, by Theorem 2.1, \( G \) is either \( K_n \) or \( C_n \) or \( K_{\frac{n}{2}, \frac{n}{2}} \).

**Case 1:** Suppose \( \Phi = (K_n, \varphi) \) and \( m_\alpha(\Phi, \lambda) = \frac{(\Delta - 2)n + 2}{\Delta - 1} \). Then \( m_\alpha(\Phi, \lambda) = n - 1 \). Therefore, \( \lambda \) is an eigenvalue of \( A_\alpha(\Phi) \) with multiplicity \((n - 1)\). By Lemma 3.3, statement (i) holds.

**Case 2:** Suppose \( \Phi = (C_n, \varphi) \) and \( m_\alpha(\Phi, \lambda) = \frac{(\Delta - 2)n + 2}{\Delta - 1} \). Then \( m_\alpha(\Phi, \lambda) = 2 \). Therefore, by Corollary 2.1, either statement (ii) or statement (iii) holds.

**Case 3:** Suppose \( \Phi = (K_{\frac{n}{2}, \frac{n}{2}}, \varphi) \) and \( m_\alpha(\Phi, \lambda) = \frac{(\Delta - 2)n + 2}{\Delta - 1} \). Then \( m_\alpha(\Phi, \lambda) = n - 2 \). Then there is an eigenvalue \( \mu \) of \( A(\Phi) \) with multiplicity \((n - 2)\) such that \( \lambda = \frac{an}{2} + (1 - \alpha)\mu. \)

Since \( \Phi \) is bipartite, so by Theorem 2.4 the eigenvalues are symmetric about origin. Then \( \mu = 0 \). Therefore \( r(\Phi) = 2 \). Let \( (C_4, \varphi) \) be an induced 4-cycle in \( \Phi \). Using Lemma 2.2, \( 2 \leq r(C_4, \varphi) \leq r(\Phi) = 2 \). Since \( r(C_4, \varphi) = 2 \), so by Lemma 2.3, \( (C_4, \varphi) \) is of type A and hence \( \varphi(C_4) = 1 \). Therefore, any 4-cycle in \( \Phi \) is neutral. Let us take an arbitrary cycle.
$C_{2k} \equiv v_1 - v_2 - \cdots - v_{2k}$. Then
\[
\varphi(C_{2k}) = \varphi(e_1,2)\varphi(e_{2,3})\cdots\varphi(e_{(2k-1),2k}) = \\{\varphi(e_{1,2})\varphi(e_{2,3})\varphi(e_{3,4})\varphi(e_{4,1})\} \\
\{\varphi(e_{1,4})\varphi(e_{4,5})\varphi(e_{5,6})\varphi(e_{6,1})\} \\
\vdots \\
\{\varphi(e_{1,(2k-2)})\varphi(e_{(2k-2),(2k-1)})\varphi(e_{(2k-1),2k})\varphi(e_{2k,1})\}
\]
\[= 1.
\]
Therefore $\Phi \sim (K_{\frac{n}{2}, \frac{n}{2}}, 1)$ and $\lambda = \frac{\alpha n}{2}$. Thus statement (iv) holds.

The converse is easy to verify.

Remark 3.1. It is easy to see that Theorem 3.1 is an extension of Theorem 1.1. Also, the above proof simplifies the proof of Theorem 1.1.

4 Lower bounds of rank for connected $\mathbb{T}$-gain graph

In this section, we establish two lower bounds for the rank of a $\mathbb{T}$-gain graph $\Phi$ in terms of the number of vertices $n$ and the maximum vertex degree $\Delta$. The first bound is a consequence of Theorem 3.1

Theorem 4.1. Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph of $n$ vertices with rank $r(\Phi)$ and the maximum vertex degree $\Delta \geq 2$. Then

\[r(\Phi) \geq \frac{n - 2}{\Delta - 1}.
\]

Equality holds if and only if $\Phi$ is either $(K_{\frac{n}{2}, \frac{n}{2}}, 1)$ or $(C_n, \varphi)$, where $n$ is even and $\varphi(C_n) = \pm 1$.

Proof. Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph. Then for $\alpha \in [0, 1)$, we have $A_\alpha(\Phi) = \alpha D(\Phi) + (1 - \alpha)A(\Phi)$ and $m_\alpha(\Phi, \lambda)$ is the multiplicity of $\lambda$ as an eigenvalue of $A_\alpha(\Phi)$. Consider $\alpha = 0$ and $\lambda = 0$. Then $A_0(\Phi) = A(\Phi)$ and $m_0(\Phi, 0)$ is the nullity of $\Phi$. That is, $m_0(\Phi, 0) = \eta(\Phi)$. Therefore, by Theorem 3.1

\[\eta(\Phi) \leq \frac{(\Delta - 2)n + 2}{(\Delta - 1)}.
\]

Also

\[\eta(\Phi) = n - r(\Phi),
\]

so

\[r(\Phi) \geq \frac{n - 2}{\Delta - 1}.
\]

(1)

Since rank of the adjacency matrix of $\Phi = (K_n, \varphi)$ is at least 2, so by Theorem 3.1 equality occurs in (1) if and only if $\Phi$ is either $(K_{\frac{n}{2}, \frac{n}{2}}, 1)$ or $(C_n, \varphi)$, where $n$ is even and $\varphi(C_n) = \pm 1$.
Let $Φ = (G, ν)$ be a connected $T$-gain graph with $n$ vertices and the maximum vertex degree $Δ$. Then either $n < 2Δ$ or $n ≥ 2Δ$. If $n < 2Δ$, then $r(Φ) ≥ 2 > \frac{n}{Δ} ≥ \frac{n-2}{Δ-1}$. If $n ≥ 2Δ$, then $r(Φ) ≥ \frac{n-2}{Δ-1} ≥ \frac{n}{Δ}$. Therefore, $\frac{n-2}{Δ-1}$ is better than $\frac{n}{Δ}$. Thus the bound derived in the above theorem improves the lower bound of $r(Φ)$ given in Theorem 1.2.

Now we establish a bound for $r(Φ)$ in terms of $n$ and $Δ$.

**Theorem 4.2.** Let $Φ$ be any connected $T$-gain graph with $n$ vertices and the maximum vertex degree $Δ(Φ) ≥ 3$. Then

$$r(Φ) ≥ \frac{n}{Δ-1}$$

equality holds if and only if $Φ ∉ \{(K_2, \frac{n}{2}, 1), (K_{\frac{n+1}{2}, \frac{n+1}{2}}, 1)\}$.

**Proof.** Let $Φ$ be any connected $T$-gain graph. Then, by Lemma 3.2 and Theorem 2.2,

$$η(Φ) ≤ M(Φ) ≤ Z(Γ(Φ)) ≤ \frac{(Δ-2)n}{Δ-1}.$$  (2)

Now the right most inequality in (2) holds if and only if $Γ(Φ) ∉ \{G_1, G_2, K_n, K_{\frac{n}{2}, \frac{n}{2}}, K_{\frac{n+1}{2}, \frac{n+1}{2}}\}$. Therefore, if $Γ(Φ) ∉ \{G_1, G_2, K_n, K_{\frac{n}{2}, \frac{n}{2}}, K_{\frac{n+1}{2}, \frac{n+1}{2}}\}$, then $r(Φ) ≥ \frac{n}{Δ-1}$ holds. From (2), $η(Φ) ≤ \frac{(Δ-2)n}{Δ-1}$ is possible, even if $Z(Γ(Φ)) > \frac{(Δ-2)n}{Δ-1}$. So, for some of the $T$-gain graphs whose underlying graphs in $\{G_1, G_2, K_n, K_{\frac{n}{2}, \frac{n}{2}}, K_{\frac{n+1}{2}, \frac{n+1}{2}}\}$ may have rank greater or equal to $\frac{n}{Δ-1}$. Let $L = \frac{n}{Δ-1}$.

**Case 1:** Let $Φ$ be a gain graph with $Γ(Φ) = G_1$. Then $L = 2.5$. Suppose $r(Φ) = 2$. Then by Lemma 2.2 all induced subgraphs of $Φ$ have rank at most 2. Since the rank of any non empty graph is at least 2. So all non empty induced subgraphs of $Φ$ have rank 2. Consider two cyclic subgraphs of length 4, namely $Φ_1 = (C_1, ν)$ and $Φ_2 = (C_2, ν)$, where $C_1 ≡ v_1-v_2-v_3-v_4-v_1$ and $C_2 ≡ v_1-v_2-v_3-v_5-v_1$. Now $r(Φ_1) = r(Φ_2) = 2$. Therefore, by Theorem 2.3, $Φ_1$ and $Φ_2$ are balanced. Hence both the 3-cycles in $Φ$ are balanced. Now the rank of any balanced 3-cycles is 3, a contradiction. Thus $r(Φ) ≥ \frac{n}{Δ-1}$ holds for any $Φ$.

**Case 2:** Let $Φ$ be a gain graph with $Γ(Φ) = G_2$. Then $L = \frac{7}{3} = 2.333$. Since $Φ$ can not have rank 2, so $r(Φ) ≥ \frac{n}{Δ-1}$ holds for any $Φ$.

**Case 3:** Let $Φ$ be a gain graph with $Γ(Φ) = K_n$. Then $L = 1 + \frac{2}{n-2}$. Therefore, $r(Φ) ≥ 2 ≥ \frac{n}{Δ-1}$ holds for any $Φ$.

**Case 4:** Let $Φ$ be a gain graph with $Γ(Φ) = K_{\frac{n}{2}, \frac{n}{2}}$. Then $L > 2$. Therefore $r(Φ) ≥ 3$. Using Case 3 of Theorem 3.1, the $T$-gain graph $Φ$ is of rank 2 if and only if $Φ$ is balanced. Therefore, inequality holds for any $T$-gain graph $Φ = (K_{\frac{n}{2}, \frac{n}{2}}, ν)$ except $(K_{\frac{3}{2}, \frac{3}{2}, 1})$.

**Case 5:** Let $Φ$ be a gain graph with $Γ(Φ) = K_{\frac{n+1}{2}, \frac{n+1}{2}}$. Then $L > 2$. Therefore, similar to case 4, $r(Φ) ≥ \frac{n}{Δ-1}$ holds for any $Φ$ except $(K_{\frac{3}{2}, \frac{3}{2}, 1})$. 

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Remark 4.1. It is easy to see that $\frac{n+1}{\Delta} \leq \frac{n}{\Delta-1}$ holds. The bound for $r(\Phi)$ in Theorem 4.2 is better than that of Theorem 1.3.

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