Symplectic Structures and Self-dual Fields in
(4k+2) Dimensions

Ioannis Giannakis\textsuperscript{a} and V. P. Nair\textsuperscript{b}

\textsuperscript{(a)} Physics Department, Rockefeller University
1230 York Avenue, New York, NY 10021-6399
e-mail: giannak@theory.rockefeller.edu

\textsuperscript{(b)} Physics Department, City College of the City University of New York
New York, NY 10031
e-mail: vpn@ajanta.sci.ccny.cuny.edu

Abstract

We discuss symplectic structures for the chiral boson in (1 + 1) dimensions and the self-dual field in (4k + 2) dimensions. Dimensional reduction of the six-dimensional field on a torus is also considered.
1. Introduction.

Self-dual fields, which are described by differential $p$-forms with the corresponding field strength being self-dual in $(2p + 2)$ dimensions, play an important role in many theories of current interest such as the six-dimensional and type IIB ten-dimensional supergravities, M-theory and heterotic strings [1]. It is also well known that there are difficulties in the construction of manifestly Lorentz-invariant actions for such fields [2]. Possible ways of circumventing these difficulties have also been recently explored [3]. The absence of a manifestly covariant description can be awkward but does not necessarily hinder the construction of the quantum theory. The latter requires a canonical realization of the Poincaré algebra with the equations of motion, viz., the selfduality conditions, being the Heisenberg equations of motion. With this motivation, in this letter, we shall explore the canonical structure for the self-dual fields, directly defining it from the equations of motion. Such a description can obviously lead to a first-order action. However, in our approach the action is more of a derived quantity. It can also depend on the choice of field variables. One may be able to choose different fields (e.g., related by duality transformation) which give different actions.

A simple situation which illustrates these ideas is the free Maxwell theory with the equations of motion

$$
\partial_0 E_i = \epsilon_{ijk} \partial_j B_k, \quad \partial_0 B_i = -\epsilon_{ijk} \partial_j E_k
$$

(1)

with the constraints $\nabla \cdot E = 0$, $\nabla \cdot B = 0$. The evolution equations involve first derivatives with respect to time and hence the field variables $(E_i, B_i)$ at a fixed time are sufficient to label the classical solutions or in other words they describe the phase space $P$ of the theory. The canonical structure $\Omega$ is thus a differential two-form on the space of all fields $(E_i, B_i)$ (at a fixed time). The vector fields which describe the time-evolution and Poincaré transformations of the fields should have the Hamiltonian property with respect to $\Omega$, i.e., the Lie derivative of $\Omega$ with respect to the vector field must be zero. In situations where the first cohomology of the phase space is trivial this is equivalent to the contraction of the vector field $\xi$ with $\Omega$, denoted by $i_\xi \Omega$, being the differential of a function on the phase space

$$
i_\xi \Omega = -\delta f.
$$

(2)

(Here $\delta$ denotes the exterior derivative on $P$.) The function $f$ is the generator of the transformation. For many theories, these conditions suffice to identify $\Omega$ and the various generators. (In what follows we shall impose the requirement (2) rather than the weaker Lie derivative condition. If the first cohomology of the phase space is nontrivial, one can have more general solutions to the Lie derivative condition with associated $\theta$-angles. We shall not consider such situations.)

As an example, for the Maxwell theory, time-evolution is described by the vector field

$$
\xi = \int d^3 x \ \epsilon_{ijk} \left( \partial_j B_k \frac{\delta}{\delta E_i} - \partial_j E_k \frac{\delta}{\delta B_i} \right).
$$

(3)

By writing $\Omega$ in the general form

$$
\Omega = \int_{x,y} \delta E_i(x) \wedge \delta B_j(y) \ M_{ij}(x,y) + \frac{1}{2} \delta E_i(x) \wedge \delta E_j(y) \sigma_{ij}(x,y) + \frac{1}{2} \delta B_i(x) \wedge \delta B_j(y) \tau_{ij}(x,y)
$$

(4)

one can easily see that a solution to the requirement (2) is

$$
\Omega = \int_{x,y} \delta E_i(x) \wedge \delta B_j(y) \epsilon_{ijk} \partial_k g(x, y)
$$

$$
H = \frac{1}{2} \int_x E^2_T + B^2_T
$$

(5)
where \( g(x,y) \) is the Coulomb Green’s function, \(-\partial^2 g(x,y) = \delta(x-y)\), and \((E_T, B_T)\) denote the tranverse components of the fields. Defining the potential \( A_i = \int_y \epsilon_{ijk} \partial_k g(x,y) \; B_j(y) \), we can write \( \Omega = \delta \alpha \) with the symplectic potential \( \alpha = \int E_i \delta A_i \). This leads to the standard first-order action

\[
S = \int E_i \partial_0 A_i - H
\]

Alternatively, one can define the dual potential \( \tilde{A}_i = -\int \epsilon_{ijk} \partial_k g(x,y) \; E_j(y) \) with the symplectic potential \( \tilde{\alpha} = \int B_i \delta \tilde{A}_i \); this gives the dual description. (See in this connection [4].)

In what follows, we explore a similar approach for self-dual fields in \((4k+2)\) dimensions. Our approach, then, would be as follows. We start with the equations of motion of the underlying theory. Some of these describe the time-evolution of fields, some are constraints on the fields at a given time. By writing the equations of motion in first-order form one can identify a suitable set of variables as coordinates for the phase space of the theory. Subsequently we introduce a symplectic structure on the phase space which should be a closed two-form. The time-evolution equations of motion must correspond to a Hamiltonian vector field for this symplectic two-form. (More generally all Poincaré transformations must correspond to Hamiltonian vector fields.) Strictly speaking, the symplectic two-form should be nondegenerate but very often it would be more convenient to start with a two-form which has degenerate directions. The constraints on the fields must correspond precisely to the degenerate directions or to gauge symmetries. Starting with the equations of motion and using these requirements, in many situations one is able to obtain a canonical formulation and a first-order action for the theory. It is also possible to construct functions on the phase space that generate the spacetime symmetries of the theory. Quantization is then fairly straightforward. (A similar approach has been used for analyzing the interactions of anyons in [5].)

Most of our discussion will be for the chiral boson in \((1 + 1)\) dimensions and the self-dual field in six dimensions. We shall also discuss the dimensional reduction of latter to four dimensions with the resulting dual symmetric electrodynamics [6]. The supersymmetric version of the six-dimensional case and the general \((4k+2)\)-dimensional case are commented upon.

2. Chiral boson in \((1 + 1)\) dimensions.

This is described by a scalar field \( \phi(x,t) \) which obeys the equation of motion

\[
\partial_0 \phi(x,t) - \partial_1 \phi(x,t) = 0
\]

where \( x^0 = t \) and \( x^1 = x \). The equation of motion being first-order in time derivatives, the classical trajectories can be labelled by the initial value of \( \phi(x,t) \). Thus the phase space can be described by field configurations \( \phi(x,t) \) at a fixed time \( t \). The symplectic two-form is thus given by

\[
\Omega = \frac{1}{2} \int_{x,y} M(x,y,\phi) \delta \phi(x) \wedge \delta \phi(y)
\]

where we denote exterior differentiation on the space of \( \phi \)’s by \( \delta \). Evidently from the antisymmetry of the exterior product, \( M(x,y) = -M(y,x) \).

The time-evolution of \( \phi(x,t) \) as given in (7) corresponds to the vector field

\[
\xi = \int_x \partial_1 \phi \frac{\delta}{\delta \phi}
\]
The requirement that $\xi$ be a Hamiltonian vector field for $\Omega$ leads to

$$i_\xi \Omega = \int \partial_x \phi(x) M(x, y, \phi) \delta \phi(y) = - \int \phi(x) \partial_x M(x, y, \phi) \delta \phi(y) = - \delta H. \quad (10)$$

The simplest solution to this is given by $M(x, y, \phi)$ which is independent of $\phi$. In this case we can satisfy (10) by the choice

$$\partial_x M(x, y) = \delta(x, y). \quad (11)$$

This gives

$$H = \frac{1}{2} \int_x \phi^2(x). \quad (12)$$

Given (11), we find $i_V \Omega = - \delta \phi$ where $V(x)$, the vector field corresponding to $\phi$, is given by $- \partial_x \frac{\delta}{\delta \phi}$. The fundamental Poisson bracket is thus

$$[\phi(x), \phi(y)] = i_{V(x)} i_{V(y)} \Omega = \partial_x \delta(x - y). \quad (13)$$

The symplectic potential $\alpha$ defined by $\Omega = \delta \alpha$ is given by

$$\alpha = \frac{1}{2} \int_{x,y} \phi(x) M(x, y) \delta \phi(y). \quad (14)$$

Once we have the symplectic potential, a first-order action for the theory can be constructed. In the present case it is given by

$$S = \frac{1}{2} \int_{x,y,t} \phi(x) M(x, y) \dot{\phi}(y) - \frac{1}{2} \int_{x,t} \phi^2(x). \quad (15)$$

(The overdot on the field $\phi$ denotes the derivative with respect to time.)

Linear momentum $P$ corresponds to the vector field $\zeta = \int \partial_x \phi \frac{\delta}{\delta \phi}$; evidently $H = P$. Consider Lorentz transformations. If $\phi$ is a scalar, we have, infinitesimally,

$$\delta \phi(x, t) = v(t \partial_x \phi + x \partial_t \phi) = v(t + x) \partial_x \phi \quad (16)$$

which corresponds to a vector field

$$\tilde{v}_L = \int (t + x) \partial_x \phi \frac{\delta}{\delta \phi}. \quad (17)$$

The contraction of this vector field with $\Omega$ is given by

$$i_{\tilde{v}_L} \Omega = - t \delta H - \int_{x} x \delta T^{00} - \int_{x,y} \phi(x) M(x, y) \delta \phi(y) = - \delta (t H + \int_{x} x T^{00}) - 2 \alpha \quad (18)$$

where $T^{00} = \frac{1}{2} \phi^2$ is the energy density. We cannot write $\alpha$ as $\delta f$ since $\delta \alpha = \Omega \neq 0$. Thus $\tilde{v}_L$ is not a Hamiltonian vector field. In order to achieve this we must modify the transformation property of $\phi$, Eq.(16). A suitable modification is given by

$$v_L = \int (t + x) \partial_x \phi \frac{\delta}{\delta \phi} + \int \phi \frac{\delta}{\delta \phi}, \quad i_{v_L} \Omega = - \delta (t H + \int_{x} x T^{00}). \quad (19)$$

The generator of Lorentz transformations is thus

$$L = t H + \frac{1}{2} \int_x x \phi^2(x). \quad (20)$$
The full Poincaré algebra is easily checked for \( H, P, L \). The transformation property of \( \phi \)

\[
[L, \phi(x)] = -(t + x)\partial_x \phi(x) - \phi(x)
\]

shows that \( \phi \) has nontrivial spin.

As far as time-evolution is concerned, it is possible to have more general solutions to (10). For example we can take \( \partial_x M(x, y) = G(x, y) \), \( H = \frac{1}{2} \int_{x,y} \phi(x)G(x, y)\phi(y) \) where \( G(x, y) \) is some symmetric function of \( x, y \). For the Lorentz transformation we then find

\[
i_v \Omega = -\delta(tH + \int_x xT^{00}) - \frac{1}{2} \int_{x,y} (x - y)\phi(x)G(x, y)\delta\phi(y).
\]

For the Poincaré algebra to be satisfied, the last term must vanish. Since this must be true for any field we must have \( (x - y)G(x, y) = 0 \), with the solution that \( G(x, y) \) is proportional to \( \delta(x - y) \). For the chiral boson in \((1 + 1)\)-dimensions this basically leads to the action (15) and the Poisson bracket (13) with \( M(x, y) \) given by (11). What we have obtained is the Floreanini-Jackiw description of the chiral boson [7].

3. Self-dual field in six dimensions.

In order to study the characteristics of self-dual theories where the antisymmetric tensors are gauge fields, it is sufficient to consider the six-dimensional case. Generalization to higher dimensions is then straightforward. The gauge field in six dimensions is given by a two-form potential \( C \). Its field strength \( H = dC \) is self-dual and it is invariant under the gauge transformations \( C \to C + dN \), where \( N \) is a one-form. The equations of motion, viz., the selfduality of \( H \), are given by \( H = * H \), * denoting the Hodge dual. In components this reads

\[
\partial_\mu C_{\nu\rho} + \partial_\nu C_{\rho\mu} + \partial_\rho C_{\mu\nu} = \frac{1}{3!} \epsilon_{\mu\nu\alpha\beta\gamma} \left( \partial_\alpha C_{\beta\gamma} + \partial_\beta C_{\alpha\gamma} + \partial_\gamma C_{\alpha\beta} \right)
\]

with \((\mu, \nu, \ldots = 1, 2, \ldots, 5)\). We can use the gauge freedom to set \( C_{0\mu} = 0 \) and simplify the equations of motion. This will be assumed in what follows. Although we have imposed a gauge condition, there is still the residual freedom of time-independent gauge transformations, \( C_{\mu\nu} \to C_{\mu\nu} + \partial_\mu N_\nu - \partial_\nu N_\mu \). The equations of motion are first order in time and hence solutions of (23) may be labelled by the set of configurations \( C_{\mu\nu}(x) \) at a fixed time, denoted by \( \mathcal{P} \). There is still redundancy in the parametrization because of the freedom of time-independent gauge transformations. The true phase space, \( \mathcal{P} \), is obtained from \( \mathcal{P} \) by the identification of configurations which only differ by a gauge transformation. We can however start with a symplectic two-form \( \Omega \) on \( \mathcal{P} \). Such an \( \Omega \) should have zero components along the gauge directions of the phase space. Indeed the vector field \( V_\mu \) generating the gauge transformation, viz.,

\[
V_\mu = \int (\partial_\mu N_\nu - \partial_\nu N_\mu) \frac{\delta}{\delta C_{\mu\nu}}
\]

should be a zero mode of \( \Omega \), i.e., \( i_{V_\mu} \Omega = 0 \). For \( \Omega \), we can assume the general form

\[
\Omega = \frac{i}{2} \int_{x,y} M^{\mu\nu\lambda}(x, y, C) \delta C_{\mu\nu}(x) \wedge \delta C_{\nu\lambda}(y)
\]

where \( M^{\mu\nu\lambda}(x, y) = -M^{\lambda\mu\nu}(y, x) \). The requirement of \( i_{V_\mu} \Omega = 0 \) gives \( \partial_\mu M^{\mu\nu\lambda}(x, y) = 0 \), a solution to which may be taken as

\[
M^{\mu\nu\lambda}(x, y) = \epsilon_{\mu\nu\lambda\rho} \partial^\rho F(x, y).
\]

Since \( \Omega \) has zero modes, it is not invertible. In order to have an invertible \( \Omega \) we have to impose a gauge-fixing condition. Upon gauge-fixing and inverting we get the Dirac brackets. These have the property that
\[ \phi, f \]_D = 0 for any observable \( \phi \) where \( f \) is the gauge-fixing constraint. In our case we can construct the basic bracket relations without going through this formal procedure. With \( M^{\mu \nu \lambda}(x, y) \) as given by (26) we can easily check that the vector field \( P^{\mu \nu} = \frac{\delta}{\delta C_{\mu \nu}} \) is a Hamiltonian vector field if \( F(x, y) \) is independent of \( C_{\mu \nu} \). The corresponding function on phase space is given by

\[
P^{\mu \nu}(x) = -\int_z \epsilon_{\mu \nu \alpha \beta} \partial^\alpha C_{\kappa \lambda}(z) \partial^\beta F(x, z) = \int_z \epsilon_{\mu \nu \alpha \beta} \partial^\alpha C_{\kappa \lambda}(z) F(x, z).
\] (27)

The fundamental bracket relations are thus

\[
[P^{\mu \nu}(x), P^{\alpha \lambda}(y)] = (i_{P^{\mu \nu}} i_{P^{\alpha \lambda}}) \Omega = -\epsilon_{\mu \nu \alpha \beta} \partial^\beta F(x, y).
\] (28)

Notice that \( P^{\mu \nu}(x) \) is gauge-invariant. In solving (28) to obtain the brackets for \( C_{\mu \nu} \)'s, one has to choose a gauge-fixing condition. From (23) we see that time-evolution is generated by

\[
\xi = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} \partial^\alpha C_{\kappa \lambda} \frac{\delta}{\delta C_{\mu \nu}}.
\] (29)

We then find

\[
i_\xi \Omega = -\int_{x,y} C_{\beta \gamma}(x) \delta C_{\rho \sigma}(y) L^{\beta \gamma, \rho \sigma} F(x, y)
\]

\[
L^{\beta \gamma, \rho \sigma} = \left[ \delta^{\beta \rho} \gamma^2 - \delta^{\beta \rho} \sigma^\alpha \partial_\alpha \partial_\gamma + \delta^{\beta \sigma} \partial_\alpha \partial_\gamma \right]
\] (30)

As it stands, \( \xi \) is not a Hamiltonian vector field, due to the \( \partial_\alpha \partial_\rho \) and \( \partial_\alpha \partial_\gamma \)-terms in the above expression. Also the operator \( L^{\beta \gamma, \rho \sigma} \) has zero modes and is not invertible; this is a reflection of the residual gauge invariance. A convenient gauge-fixing condition is given by \( \partial^\rho C_{\mu \nu}(x) = 0 \). As can be easily seen, this condition is preserved in time by the evolution equations (23) (with \( C_{0 \mu}(x) = 0 \) of course). In this case (30) simplifies as

\[
i_\xi \Omega = -2 \int_{x,y} C_{\rho \sigma}(x) \delta C_{\rho \sigma}(y) \partial^2 F(x, y) = -\delta H
\] (31)

with \( H = \int_{x,y} C_{\rho \sigma}(x) \partial^2 F(x, y) C_{\rho \sigma}(y) \). Thus \( \xi \) is indeed a Hamiltonian vector field and \( H \) is the Hamiltonian. With the gauge-fixing condition \( \partial^\rho C_{\mu \nu}(x) = 0 \) we can solve (28) to obtain the bracket relations for \( C_{\mu \nu}(x) \) as

\[
[C_{\mu \nu}(x), C_{\alpha \beta}(y)] = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} \partial^\gamma \Delta(x, y)
\] (32)

where \( \Delta(x, y) \) is the inverse of \( \partial^2 F(x, y) \). A simple choice which is consistent with Lorentz symmetry is to take

\[
\partial^2 F(x, y) = \delta(x - y), \quad \Delta(x, y) = \delta(x - y).
\] (33)

In this case we have for the energy density \( T^{00}(x) = C_{\alpha \beta}(x) C_{\alpha \beta}(x) \) with \( H = \int_x T^{00}(x) \) and also (32) becomes

\[
[C_{\mu \nu}(x), C_{\alpha \beta}(y)] = \frac{1}{2} \epsilon_{\mu \nu \alpha \beta} \partial^\gamma \delta(x, y)
\] (34)

The relation (34) is identical to the bracket relation given by Henneaux and Teitelboim in [8]. Notice also that (34) gives \( [\partial^\mu C_{\mu \nu}(x), C_{\alpha \beta}(y)] = 0 \).

Using the expression for \( T^{00}(x) \) we can obtain some of the other components of the energy-momentum tensor. For example, the bracket relation

\[
[T^{00}(x), T^{00}(y)] = (T^{00}(x) + T^{00}(y)) \partial_\alpha \delta(x - y)
\] (35)

which holds for a Poincaré-invariant theory can be used, along with eqs. (33) and (34), to identify \( T^{00}(x) \) as

\[
T^{00}(x) = \frac{1}{2} \epsilon^{\alpha \mu \nu \beta \gamma} C_{\mu \nu}(x) C_{\beta \gamma}(x).
\] (36)
The linear momentum is given by $P^\alpha = \int_x T^{0\alpha}(x)$ and obeys the relation

$$[P^\alpha, C_{\mu\nu}(x)] = -\partial^\alpha C_{\mu\nu}(x)$$  \hspace{1cm} (37)

up to the condition $\partial^\mu C_{\mu\nu} = 0$.

We now consider Lorentz transformations. Naively the orbital part of the Lorentz boosts corresponds to a vector field

$$\tilde{V}_L = \int_x \left[ x^0 v^j \partial_j C_{\mu\nu}(x) + \frac{1}{2} v^j x^l \epsilon_{\mu\nu\alpha\beta} \partial_\alpha C_{\beta\gamma}(x) \right] \frac{\delta}{\delta C_{\mu\nu}}.$$  \hspace{1cm} (38)

It is easily seen that $\tilde{V}_L$ is not a Hamiltonian vector field, i.e., $i_{\tilde{V}_L} \Omega \neq -\delta f$ for some $f$. One has to modify the vector field to

$$V_L = \int_x \left[ x^0 v^j \partial_j C_{\mu\nu}(x) + \frac{1}{2} v^j x^l \epsilon_{\mu\nu\alpha\beta} \partial_\alpha C_{\beta\gamma}(x) + v^\kappa \epsilon_{\mu\nu\rho\sigma} C_{\rho\sigma}(x) \right] \frac{\delta}{\delta C_{\mu\nu}}.$$  \hspace{1cm} (39)

In this case $i_{V_L} = -\delta(v^\mu L_\mu)$ with

$$L^\mu = x^0 P^\mu - \int_x x^\mu T^{00}(x).$$  \hspace{1cm} (40)

The anomalous term, viz., $v^\kappa \epsilon_{\mu\nu\rho\sigma} C_{\rho\sigma}(x)$ in (39) is as expected. This is the “operator gauge transformation” part which arises because we have used the non-invariant gauge condition $C_{0\mu}(x) = 0$ (For a discussion on the operator gauge transformation which reestablishes the Coulomb gauge in the new Lorentz frame in electrodynamics see [9]). From the relation $[L^\mu, L^\nu] = M^{\mu\nu}$, we identify the spatial angular momentum as

$$M^{\mu\nu} = \int_x \left( x^\mu T^{0\nu}(x) - x^\nu T^{0\mu}(x) \right).$$  \hspace{1cm} (41)

This has the bracket relation

$$[M^{\mu\nu}, C^{\alpha\beta}(x)] = -\left( x^\mu \partial^\nu - x^\nu \partial^\mu \right) C^{\alpha\beta}(x) + \left( \delta^{\nu\alpha} C^{\mu\beta}(x) - \delta^{\nu\beta} C^{\mu\alpha}(x) \right) \frac{\delta}{\delta C_{\mu\nu}}.$$  \hspace{1cm} (42)

which shows the infinitesimal change of $C^{\alpha\beta}$ corresponding to the orbital and spin transformations. Similarly we can construct the generators of dilatations and special conformal transformations. Dilatations correspond to a Hamiltonian vector field

$$\zeta = \int_x \left[ \partial_\alpha \left( x^\mu C^{\gamma\nu} + x^\nu C^{\mu\gamma} + x^\gamma C^{\mu\nu} \right) + \frac{1}{2} x^0 \epsilon_{0\mu\alpha\beta\gamma} \partial_\alpha C_{\beta\gamma} \right] \frac{\delta}{\delta C_{\mu\nu}}.$$  \hspace{1cm} (43)

such that $i_\zeta \Omega = -\delta D$, with the generator of dilatations given by

$$D = \int_x \left( x^0 T^{00}(x) + x^\alpha T^{0\alpha}(x) \right).$$  \hspace{1cm} (44)

The bracket relation

$$[D, C_{\alpha\beta}(x)] = -3 C_{\alpha\beta}(x) - x^\kappa \partial_\kappa C_{\alpha\beta}(x) - x^0 \partial_0 C_{\alpha\beta}(x).$$  \hspace{1cm} (45)

gives the transformation of $C^{\alpha\beta}$ under dilatations. Finally we consider special conformal transformations. Naively they correspond to a vector field $\tilde{\zeta}$

$$\tilde{\zeta} = \int_x \left( 2k^\lambda x^\nu C_{\mu\lambda} - 2k^\nu x^\lambda C_{\mu\lambda} + 2k^\lambda x^\mu C_{\lambda\nu} - 2k^\mu x^\lambda C_{\lambda\nu} - 6k^\lambda x^\nu C_{\mu\nu} + \delta x^A \partial_A C_{\mu\nu} \right) \frac{\delta}{\delta C_{\mu\nu}}.$$  \hspace{1cm} (46)
Self-Dual Fields

with \( \delta x^A = k^B (2x^B x^A - x^2 \delta^{AB}) \) and \( A, B = 0, \ldots, 5 \). As with Lorentz transformations, for this vector field to have the Hamiltonian property, we need to modify the transformation properties of \( C_{\mu\nu} \) under special conformal transformations. More specifically we need to add to \( \zeta \) the term

\[
\zeta' = \int_x \left( \epsilon^{\mu\nu\lambda\sigma} k^\lambda x^\nu C_{\rho\sigma} - \epsilon^{\mu\nu\lambda\sigma} k^\sigma x^\lambda C_{\rho\mu} \right) \frac{\delta}{\delta C_{\mu\nu}}
\]

The vector field \( \zeta = \zeta + \zeta' \) is Hamiltonian, i.e., \( i_\zeta \zeta \Omega = -\delta (k^A K^A) \) with

\[
K^A = \int_x \left( x^2 \delta^{AB} - 2x^A x^B \right) T^{0B}(x)
\]

The transformation properties of the gauge field \( C_{\mu\nu} \) under special conformal transformations also reveal an anomalous term, viz., \( \epsilon^{\mu\nu\lambda\rho\sigma} k^\lambda x^\nu C_{\rho\sigma} - \epsilon^{\mu\nu\lambda\rho\sigma} k^\rho x^\lambda C_{\mu\nu} \) which reestablishes the gauge condition \( \partial_\mu C_{\mu\nu}(x) = 0 \) after the transformation.

The discussion upto this point shows that the symplectic structure \( \Omega \) of eqs. (25, 26) with \( F \) given by eq. (33) gives a Poincaré-invariant (actually conformally invariant) description of the selfdual field. We can now consider an action for this field. The symplectic potential \( \alpha \) for the two-form \( \Omega \) is given by

\[
\alpha = \frac{1}{2} \int_{x,y} C_{\mu\nu}(x) \delta C_{\alpha\beta}(y) \epsilon^{\mu\alpha\beta\gamma} \partial_\gamma F(x,y).
\]

A first order action for the self-dual field is thus

\[
S = \frac{1}{2} \int_{x,y} C_{\mu\nu}(x) \dot{C}_{\alpha\beta}(y) \epsilon^{\mu\alpha\beta\gamma} \partial_\gamma F(x,y) - \int_{x} C_{\mu\nu}(x) C_{\mu\nu}(x).
\]

Our analysis goes over in a straightforward way to \((4k + 2)\) dimensions. The fields in this case are described by a 2k-form with components \( C_{\mu_1\mu_2...\mu_{2k}} \), \( \mu_i = 1, 2, \ldots, (2k+1) \) and \( C_{0\mu_1...} = 0 \) as before. \( \Omega \) is given by

\[
\Omega = \frac{1}{2} \int \epsilon_{\mu_1\mu_2...\mu_{2k}\nu_1\nu_2...\nu_{2k}\alpha} \delta C_{\mu_1\mu_2...\mu_{2k}}(x) \wedge \delta C_{\nu_1\nu_2...\nu_{2k}}(y) \partial_\alpha F(x,y)
\]

with \( \partial^2 F(x,y) = \delta(x-y) \). We also have

\[
T^{00} = \frac{1}{2} (2k)! C_{\mu_1\mu_2...\mu_{2k}}(x) C_{\mu_1\mu_2...\mu_{2k}}(x)
\]

The basic Poisson brackets are

\[
[C_{\mu_1\mu_2...\mu_{2k}}(x), C_{\nu_1\nu_2...\nu_{2k}}(y)] = \frac{1}{((2k)!)^2} \epsilon_{\mu_1\mu_2...\mu_{2k}\nu_1\nu_2...\nu_{2k}\alpha} \partial_\alpha \delta(x-y)
\]

4. The supersymmetric self-dual field.

In many situations, the supersymmetric version of the self-dual field is of interest. This has been studied before and here we simply point out that supersymmetrization does not lead to new features within our approach. For supersymmetrization of the six-dimensional case with Euclidean signature, we need complex \( C_{\mu\nu} \) since self-duality requires complex fields [10]. Further we have two Weyl spinors \( \psi_i^A \), \( A = 1, 2, i = 1, 2, 3, 4 \) which transform on the \( i \)-index as the fundamental representation of \( SU(4) \) (i.e., spinor of \( SO(6) \)). We also have a complex scalar \( \phi \). With Minkowski signature, one can impose a symplectic Majorana condition on
the $\psi$'s and make $C_{\mu\nu}, \phi$ real. The supersymmetry transformations (with Euclidean signature) are of the form
\[
\delta C_{\mu\nu} = \bar{\psi}^A \gamma_{\mu\nu} \eta_A + \eta_A^* \gamma_{\mu\nu} \psi^A
\]
\[
\delta \psi^A = -i \epsilon^{AB} \eta_B \gamma_{\mu} \partial^\mu \phi - \frac{i}{12} \epsilon^{AB} \eta_B \gamma_{\rho\nu\lambda} H^{\rho\nu\lambda}
\]
(54)
\[
\delta \phi = \eta_A \bar{\psi}^A + \eta_A^* \psi^A
\]
In eq.(54) and elsewhere in this section $\mu, \nu = 1, \ldots, 6$. The parameter $\eta_A, A = 1, 2$ of the transformations are spinors of $SO(6)$ and $\epsilon^{AB} = -\epsilon^{BA}, \epsilon^{12} = 1$. $\gamma_{\mu}$ are $\gamma$-matrices for six dimensions and $\gamma_{\mu\nu} = \frac{1}{2} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu}), \gamma_{\rho\nu\lambda} = \frac{1}{6!} (\gamma_{\rho} \gamma_{\nu} \gamma_{\lambda} - \gamma_{\nu} \gamma_{\rho} \gamma_{\lambda} + \text{cyclic}).$

As has been known for sometime, the closure of the supersymmetry algebra requires the fields to obey the equations of motion
\[
H(x) - \ast H(x) = 0, \quad \gamma^\alpha \partial_\alpha \psi(x) = 0, \quad \partial^2 \phi(x) = 0.
\]
(55)
An action and canonical formulation of the first of these equations is what we have obtained. The other two equations, viz., the Dirac equation for $\psi$ and the d’Alembertian for $\phi$, can be derived from the standard actions for these fields. The symplectic structures for them are also standard.

5. Dimensional reduction.

The dimensional reduction of the self-dual field in six dimensions where two of the dimensions form a torus is known to give a dual symmetric version of electrodynamics [6]. The modular transformations of the torus act as duality transformations on the Maxwell field. It is interesting to see how this works out in terms of the action (50).

The five-dimensional spatial manifold is taken as $R^3 \times T^2$. The torus $T^2$ is described as usual by the complex coordinate $z = \sigma_1 + \tau \sigma_2, 0 \leq \sigma_1, \sigma_2 \leq 1$, $\tau$ being the modular parameter. The Coulomb Green's function $-F(x,y)$ is given by
\[
-F(x,y) = \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{e^{2\pi i n \cdot (\sigma - \sigma')}}{(Im\tau)[k^2 + 4\pi^2 g_{ij} \bar{n}_i \bar{n}_j]}
\]
(56)
where
\[
g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{Re\tau}{|\tau|^2} \end{pmatrix}.
\]
(57)
The ansatz for dimensional reduction is $C = \beta d\sigma_2 - B d\sigma_1$, where $\beta, B$ are one-forms on $R^3$ and depend on the coordinates $x_1, x_2, x_3$ of $R^3$. The action (50) can now be worked out. We find
\[
S = -i \int d^3x d^3y \frac{(W_i(x) \bar{W}_j(y) - \bar{W}_i(x) W_j(y))}{Im\tau} \epsilon^{ijk} \partial_\kappa g(x,y) + \frac{2}{Im\tau} \int d^3x \ W_i(x) \bar{W}_i(x)
\]
(58)
where
\[
W = \beta + \tau B, \quad g(x,y) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\bar{k} \cdot (\bar{x} - x')}}{k^2}.
\]
(59)
(The first term in eq.(58) has an $\epsilon$-tensor. The integration measure for this term is thus $d^3x \ d^3y \sqrt{g}$ where $g = (det g_{ij})$. Apart from this observation, the integration over the coordinates of $T^2$ is straightforward.)

It is easily checked that the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ leaves $S$ unchanged apart from the replacement $\beta \rightarrow B, B \rightarrow \beta$. Integrating out $\beta$ in the functional integral corresponding to $S$ of (58) will lead
to the standard Maxwell action, integrating out \( B \) will give the dual action. Consider the integration over \( \beta \). Because of the condition \( \partial_\mu C_{\mu\nu}(x) = 0 \), we must have \( \nabla \cdot \beta = 0, \nabla \cdot B = 0 \). The integration measure must take account of this. Up to irrelevant constants, the measure for \( \beta \)-integration is thus

\[
d\mu(\beta) = [d\beta] \delta(\nabla \beta) = [d\beta d\varphi] e^{-i\int \nabla \varphi \beta}. \tag{60}
\]

We also introduce the potential \( A_i(x) \) by

\[
A_i(x) = -\int_y \epsilon_{ijk} \partial_k g(x, y) \ B_j(y). \tag{61}
\]

Integration of \( e^S \) over \( \beta, \varphi \) then leads to a reduced action

\[
S = 2 \int_x Im \tau (E^2 - B^2) - 4 \int_x Re \tau \ E \cdot B \tag{62}
\]

where \( E_i = A_i \). Writing \( Im \tau = \frac{g^2}{2\pi} \) and \( Re \tau = \frac{\theta}{2\pi} \) and rescaling the fields \( (E, B) \rightarrow \sqrt{\frac{1}{16\pi}} (E, B) \), the above equation becomes

\[
S = \frac{1}{2g^2} \int_x (E^2 - B^2) - \frac{\theta}{8\pi^2} \int_x E \cdot B \tag{63}
\]

which is the standard Maxwell action with a \( \theta \)-term. Similarly integration over \( B \) will give a dual description.

6. Acknowledgments.

VPN thanks R. Jackiw for discussions and N. Khuri for hospitality at Rockefeller University. This work was supported in part by the Department of Energy Contract Number DE-FG02-91ER40651-TASKB and by the National Science Foundation Grant number PHY-9322591.

References.

[1] E. Witten, IASSNS-HEP-96-101 preprint, [hep-th/9610234]; J. Schwarz, CALT-68-2046 preprint, [hep-th/9604171]; M. Perry and J. Schwarz, DAMTP-R-96-49 preprint, [hep-th/9611063]; P. Argyres and K. Dienes, Phys. Lett. 387B (1996) 727; J. Schwarz, CALT-68-2091 preprint, [hep-th/9701008].

[2] N. Marcus and J. Schwarz, Phys. Lett. 115B (1982) 111; A. Sen and J. Schwarz, Nucl. Phys. B411 (1994) 35.

[3] W. Siegel, Nucl. Phys. B238 (1984) 307; N. Berkovits, Phys. Lett. 388B (1996) 743; IFT-P-039-96 preprint, [hep-th/9610134]; IFT-P-042-96 preprint, [hep-th/9610226]; P. Pasti, D. Sorokin and M. Tonin, Phys. Rev. D52 (1995) 4277; Phys. Lett. 352B (1995) 59.

[4] R. Jackiw in Diverse Topics in Theoretical and Mathematical Physics, World Scientific, Singapore, (1995).

[5] C. Chou, V. P. Nair and A. Polychronakos, Phys. Lett. 304B (1993) 105.

[6] E. Verlinde, Nucl. Phys. B455 (1995) 211; D. Berman, CERN-TH-96-366 preprint, [hep-th/9612191].

[7] R. Floreanini and R. Jackiw, Phys. Rev. Lett. 59 (1987) 1873.

[8] M. Henneaux and C. Teitelboim, Phys. Lett. 206B (1988) 650.

[9] B. Zumino, Jour. Math. Phys. 1 (1960) 1.

[10] L. Romans, Nucl. Phys. B276 (1986) 71; A. Sagnotti, Phys. Lett. 294B (1992) 196; E. Bergshoeff, E. Sezgin and E. Sokatchev, Class. Quant. Grav. 13 (1996) 2875.