CONSTANT ANGLE RULED SURFACES IN EUCLIDEAN SPACES

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Abstract. In this paper, we study the special curves and ruled surfaces on helix hypersurface whose tangent planes make a constant angle with a fixed direction in Euclidean n-space $E^n$. Besides, we observe some special ruled surfaces in $R^n$ and give requirement of being developable of the ruled surface. Also, we investigate the helix surface generated by a plane curve in Euclidean 3-space $E^3$.

Keywords: Helix surfaces; Constant angle surfaces; Ruled surfaces

Mathematics Subject Classification 2000: 53A04, 53A05, 53A07, 53A10, 53B25, 53C40

1. INTRODUCTION

Ruled surfaces are one of the most important topics of differential geometry. The surfaces were found by Gaspard Monge, who was a French mathematician and inventor of descriptive geometry. And, many geometers have investigated the many properties of these surfaces in [4,6,7].

Constant angle surfaces are considerable subject of geometry. There are so many types of these surfaces. Helix hypersurface is a kind of constant angle surfaces. An helix
A hypersurface in Euclidean n-space is a surface whose tangent planes make a constant angle with a fixed direction. The helix surfaces have been studied by Di Scala and Ruiz-Hernández in [2]. And, A.I. Nistor investigated certain constant angle surfaces constructed on curves in Euclidean 3-space $E^3$ [1].

One of the main purposes of this study is to observe the special curves and ruled surfaces on a helix hypersurface in Euclidean n-space $E^n$. Another purpose of this study is to obtain a helix surface by generated a plane curve in Euclidean 3-space $E^3$.

2. PRELIMINARIES

Definition 2.1 Let $\alpha : I \subset \mathbb{R} \rightarrow E^n$ be an arbitrary curve in $E^n$. Recall that the curve $\alpha$ is said to be of unit speed (or parametrized by the arc-length function $s$) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product in the Euclidean space $E^n$ given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i,$$

for each $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in E^n$.

Let $\{V_1(s), V_2(s), \ldots, V_n(s)\}$ be the moving frame along $\alpha$, where the vectors $V_i$ are mutually orthogonal vectors satisfying $\langle V_i, V_j \rangle = 1$. The Frenet equations for $\alpha$ are given by

$$
\begin{bmatrix}
V'_1 \\
V'_2 \\
V'_3 \\
\vdots \\
V'_{n-1} \\
V'_n
\end{bmatrix}
=
\begin{bmatrix}
0 & k_1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 0 & k_2 & 0 & \ldots & 0 & 0 \\
0 & -1 & 0 & k_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & k_{n-1} \\
0 & 0 & 0 & 0 & \ldots & -1 & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_{n-1} \\
V_n
\end{bmatrix}.
$$

Recall that the functions $k_i(s)$ are called the $i$-th curvatures of $\alpha$ [3].
**Definition 2.2** Given a hypersurface $M \subset \mathbb{R}^n$ and an unitary vector $d \neq 0$ in $\mathbb{R}^n$, we say that $M$ is a helix with respect to the fixed direction $d$ if for each $q \in M$ the angle between $d$ and $T_qM$ is constant. Note that the above definition is equivalent to the fact that $\langle d, \xi \rangle$ is constant function along $M$, where $\xi$ is a normal vector field on $M$ [2].

**Theorem 2.1** Let $H \subset \mathbb{R}^{n-1}$ be an orientable hypersurface in $\mathbb{R}^{n-1}$ and let $N$ be an unitary normal vector field of $H$. Then,

$$f_\theta(x,s) = x + s(\sin(\theta)N(x) + \cos(\theta)d), f_\theta : H \times \mathbb{R} \to \mathbb{R}^n, \theta = \text{constant}$$

is a helix with respect to the fixed direction $d$ in $\mathbb{R}^n$, where $x \in H$ and $s \in \mathbb{R}$. Here $d$ is the vector $(0,0,\ldots,1) \in \mathbb{R}^n$ such that $d$ is orthogonal to $N$ and $H$ [2].

**Definition 2.3** Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^n$ be a unit speed curve with nonzero curvatures $k_i (i = 1,2,\ldots,n)$ in $\mathbb{E}^n$ and let $\{V_1,V_2,\ldots,V_n\}$ denote the Frenet frame of the curve of $\alpha$. We call $\alpha$ a $V_n$-slant helix, if the $n$-th unit vector field $V_n$ makes a constant angle $\varphi$ with a fixed direction $X$, that is,

$$\langle V_n, X \rangle = \cos(\varphi), \varphi \neq \frac{\pi}{2}, \varphi = \text{constant}$$

along the curve, where $X$ is unit vector field in $\mathbb{E}^n$ [3].

3. THE SPECIAL CURVES ON THE HELIX HYPERSURFACES IN EUCLIDEAN n-SPACE $\mathbb{E}^n$

**Theorem 3.1** Let $M$ be a helix hypersurface with the direction $d$ in $\mathbb{E}^n$ and let $\alpha : I \subset \mathbb{R} \to M$ be a unit speed geodesic curve on $M$. Then, the curve $\alpha$ is a $V_2$-slant helix with the direction $d$ in $\mathbb{E}^n$.

**Proof:** Let $\xi$ be a normal vector field on $M$. Since $M$ is a helix hypersurface with respect to $d$, $\langle d, \xi \rangle = \text{constant}$. That is, the angle between $d$ and $\xi$ is constant on every point of the surface $M$. And, $\alpha'(s) = \lambda \xi |_{\alpha(s)}$ along the curve $\alpha$ since $\alpha$ is a geodesic curve on $M$. Moreover, by using the Frenet equation $\alpha'(s) = V'_1 = k_1V_2$, we
obtain $\lambda \xi |_{\alpha(t)} = k_1 V_2$, where $k_1$ is first curvature of $\alpha$. Thus, from the last equation, by taking norms on both sides, we obtain $\xi = V_2$ or $\xi = -V_2$. So, $\langle d, V_2 \rangle$ is constant along the curve $\alpha$ since $\langle d, \xi \rangle$ is constant. In other words, the angle between $d$ and $V_2$ is constant along the curve $\alpha$. Consequently, the curve $\alpha$ is a $V_2$-slant helix with the direction $d$ in $E^n$.

**Corollary 3.1** For $n = 3$, the following Theorem obtained.

Theorem: Let $\alpha : I \subset \mathbb{R} \to M$ be a curve on a constant angle surface $M$ with unit normal $N$ and the fixed direction $k$. If a curve $\alpha$ on $M$ is a geodesic, then $\alpha$ is a slant helix with the axis $k$ in $E^3$ (see [5]).

**Theorem 3.2** Let $M$ be a helix hypersurface in $E^n$ and let $\alpha : I \subset \mathbb{R} \to M$ a unit speed curve on $M$. If the $n$-th unit vector field $V_n$ of $\alpha$ equals to $\xi$ or $-\xi$, where $\xi$ is a normal vector field on $M$, then $\alpha$ is a $V_n$-slant helix with the direction $d$ in $E^n$.

**Proof:** Let $d \neq 0 \in E^n$ be a fixed direction of the helix hypersurface $M$. Since $M$ is a helix hypersurface with respect to $d$, $\langle d, \xi \rangle$ is constant. That is, the angle between $d$ and $\xi$ is constant on every point of the surface $M$. Let the $n$-th unit vector field $V_n$ of $\alpha$ be equals to $\xi$ or $-\xi$. Then $\langle d, V_n \rangle$ is constant along the curve $\alpha$ since $\langle d, \xi \rangle$ is constant. That is, the angle between $d$ and $V_n$ is constant along the curve $\alpha$. Finally, the curve $\alpha$ is a $V_n$-slant helix in $E^n$.

**Theorem 3.3** Let $M$ be a helix hypersurface with the direction $d$ in $E^n$ and let $\alpha : I \subset \mathbb{R} \to M$ ($\alpha(t) \in M$, $t \in I$) be a curve on the surface $M$. If $\alpha$ is a line of curvature on $M$, then $d \in Sp\{T\}$ along the curve $\alpha$, where $T$ is tangent vector field of $\alpha$.

**Proof:** Since $M$ is a helix hypersurface with the direction $d$,
\[ \langle N \circ \alpha, d \rangle = \text{constant} \]
along the curve \( \alpha \), where \( N \) is the normal vector field of \( M \). If we are taking the derivative in each part of the equality with respect to \( t \), we obtain:
\[ \langle (N \circ \alpha)', d \rangle = 0. \]
Since \( \alpha \) is a line of curvature on \( M \), \( (N \circ \alpha)' = S(T) = \lambda T \), where \( S \) is the shape operator of the surface \( M \). So, we have
\[ \langle T, d \rangle = 0. \]
Finally, \( d \in Sp\{T\} \) along the curve \( \alpha \).

4. THE RULED SURFACES IN \( \IR^n \)

**Definition 4.1** Let \( H \subset \IR^{n-1} \) be a orientable hypersurface in \( \IR^{n-1} \) and let \( \beta \) be a curve on \( H \), where
\[ \beta : I \subset \IR \to H \subset \IR^{n-1}. \]
Then,
\[ \Phi(t, s) = \beta(t) + s(\sin(\theta)N(\beta(t)) + \cos(\theta)d) \]
is a ruled surface with dimension 2 on \( f_\theta \) in \( \IR^n \) (\( f_\theta \) was defined in Theorem 2.1), where \( \theta \) constant, \( N \) is a unitary normal vector field of \( H \) and \( d \) is constant vector as defined in Theorem 2.1. The surface \( \Phi \) will be called the ruled surface generated by the curve \( \beta \).

**Theorem 4.1** The ruled surface \( \Phi(t, s) \) defined above is developable if and only if the curve \( \beta \) is a line of curvature on the surface \( H \).

**Proof:** We assume that \( \beta \) is a line of curvature on \( H \).
Let consider the surface \( \Phi(t, s) = \beta(t) + s(\sin(\theta)N(\beta(t)) + \cos(\theta)d) \) with rulings \( X(t) = \sin(\theta)N(\beta(t)) + \cos(\theta)d \) and directrix \( \beta \). If we are taking the partial derivative in each part of the equality with respect to \( t \), we obtain:
\[ \frac{d\Phi}{dt} = \beta' + (s \sin(\theta))(N \circ \beta)' \]

(1)

And, \( S(T) = \lambda T \) since \( \beta \) is a line of curvature on \( H \), where \( T \) is tangent vector field of \( \beta \) and \( S \) is the shape operator of the surface \( H \). Besides, \( (N \circ \beta)' = \frac{dN}{dt} = S(T) \).

Therefore, \( (N \circ \beta)' = \lambda T \) and by using (1), we obtain the equality

\[ \Phi_i = (1 + \lambda s \sin(\theta)) \beta' = (1 + \lambda s \sin(\theta))T. \]

Hence, the system \( \{\Phi_i, T\} \) is linear dependent. And, we know that a tangent plane along a ruling is spanned by \( \Phi_i \) and \( T \). Finally, the tangent planes are parallel along the ruling \( \sin(\theta)N(\beta(t)) + \cos(\theta)d \) passing from the point \( \beta(t) \). That is, the surface \( \Phi(t, s) \) is developable.

We assume that the ruled surface \( \Phi(t, s) \) is developable. Then, the system \( \{\Phi_i, T\} \) is linear dependent. So, from the equality

\[ \Phi_i = \frac{d\Phi}{dt} = \beta' + (s \sin(\theta))(N \circ \beta)' = T + (s \sin(\theta))(N \circ \beta)', \]

we get \( (N \circ \beta)' = \lambda T \). Therefore \( (N \circ \beta)' = S(T) = \lambda T \), where \( S \) is the shape operator of the surface \( H \). That is, \( \beta \) is a line of curvature on \( H \).

**Corollary 4.1** Let \( H \) be the hypersphere

\[ S^{n-2} = \left\{ x = (x_1, x_2, \ldots, x_{n-1}) : f(x) = \sum_{i=1}^{n-1} x_i^2 = 1, \nabla f \neq 0 \right\} \subset IR^{n-1}. \]

Let \( \beta \) be a curve on \( H = S^{n-2} \) where

\[ \beta : I \subset IR \rightarrow H \subset IR^{n-2} \subset IR^{n-1}, \quad t \rightarrow \beta(t) \]

Then, the ruled surface \( \Phi(t, s) = \beta(t) + s(\sin(\theta)\beta(t) + \cos(\theta) d) \subset IR^n \) is always developable from Theorem 4.1. Because, each curve on the hypersphere \( S^{n-2} \) is a line of curvature.

5. **Helix Surfaces Generated by a Plane Curve in Euclidean 3-Space \( E^3 \)**
Let
\[ \alpha : I \subset \mathbb{R} \to \mathbb{R}^3 \]
\[ u \to \alpha(u) \]
be a plane curve in Euclidean 3-space \( \mathbb{R}^3 \). And, we denote the tangent, principal normal, the binormal of \( \alpha \) by \( V_1 = T \), \( V_2 \) and \( V_3 = B \), respectively. Note that binormal of a plane curve in \( \mathbb{R}^3 \) is constant.

**Definition 5.1** We can obtain a ruled surface by using the plane curve \( \alpha \) such that
\[ \phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \]
\[ (u,v) \to \phi(u,v) = \alpha(u) + v(\sin(\theta)V_2(u) + \cos(\theta)B) \]
The ruled surface will be called as the surface generated by the curve \( \alpha \).

**Theorem 5.1** The ruled surface
\[ \phi : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \]
\[ (u,v) \to \phi(u,v) = \alpha(u) + v(\sin(\theta)V_2(u) + \cos(\theta)B) \]
is a helix surface with the direction \( B \) in \( \mathbb{R}^3 \), where \( \theta \) is constant, \( \alpha \) is a plane curve and \( B \) is a constant vector which is perpendicular to the plane of the curve \( \alpha \).

**Proof:** We want to show that \( \langle Z, B \rangle \) is a constant function along \( \phi \), where \( Z \) is a normal vector field of \( \phi \).

First, we are going to find a normal vector field \( Z \). To do this, we will compute the partial derivatives of \( \phi \) with respect to \( u \) and \( v \). Note that \( k_2 = 0 \) for the curve \( \alpha \) since \( \alpha \) is a planer curve in \( \mathbb{R}^3 \).

\[ \phi_u(u,v) = (1-k_1,v \sin(\theta))T \quad \text{and} \quad \phi_v(u,v) = \sin(\theta)V_2 + \cos(\theta)B. \quad (2) \]

Using the equalities in (2), a normal to the surface \( \phi \) is given by
\[ Z = \frac{\phi_u \times \phi_v}{\| \phi_u \times \phi_v \|} = -\cos(\theta)V_2 + \sin(\theta)B. \]
So, we have \( \langle Z, B \rangle = \sin(\theta) = \text{const} \). Finally, \( \phi \) is a helix surface with the direction \( B \) in \( \mathbb{R}^3 \).
This completes the proof.

**Corollary 5.1** The helix surface

\[ \phi: U \subset E^2 \to E^3 \]

\[(u,v) \to \phi(u,v) = \alpha(u) + v(\sin(\theta)V_2(u) + \cos(\theta)B) \]

is always developable.

**Proof:** We know that if \( \det(T, X, X') = 0 \), where \( X = \sin(\theta)V_2 + \cos(\theta)B \) and \( T \) tangent of \( \alpha \), then \( \phi \) is developable.

So, we will compute \( \det(T, X, X') \):

\[ T = \alpha' \]

\[ X = \sin(\theta)V_2 + \cos(\theta)B \]

\[ X' = -k_1(\sin(\theta)T) \]

and so, we have \( \det(T, X, X') = 0 \).

This completes the proof.

**Theorem 5.2** Let

\[ \alpha: I \subset IR \to E^3 \]

\[ s \to \alpha(s) \]

be a plane curve (not a straight line) with unit speed in Euclidean 3-space \( E^3 \). We consider the helix surface (generated by the curve \( \alpha(s) \))

\[ \phi: U \subset E^2 \to E^3 \]

\[(s,v) \to \phi(s,v) = \alpha(s) + v(\sin(\theta)V_2(s) + \cos(\theta)B) \]

Then, the Gauss curvature of \( \phi \) is zero, and the mean curvature of \( \phi \):

\[ H = \frac{1}{2} \frac{k_1 \cos(\theta)}{1 - k_1 \cdot v \cdot \sin(\theta)} \]

where \( k_1 \) is the first curvature of the curve \( \alpha \).

**Proof:** From corollary 5.1, the surface \( \phi \) is developable. So, the Gauss curvature of \( \phi \) is zero.
Now, we are going to prove that \[ H = \frac{1}{2} \frac{k_1 \cos(\theta)}{1-k_1 v \cdot \sin(\theta)}. \]

The system \( \{ x_1, x_2 \} \) is an orthonormal basis for the tangent space of \( \phi \) at the point \( \phi(s, v) \), where \( x_1 = \frac{\phi_s}{\|\phi_s\|} \) and \( x_2 = \phi_v \) (\( \phi_s \) is partial derivative of \( \phi \) with respect to \( s \) and \( \phi_v \) is partial derivative of \( \phi \) with respect to \( v \)). Recall that a normal vector field of \( \phi \) is \( Z(s, v) = -\cos(\theta) V_2(s) + \sin(\theta) B \) by Theorem 5.1. And, we know that the mean curvature of \( \phi \) at a point \( \phi(s, v) \):

\[ H(\phi(s, v)) = \frac{1}{2} \sum_{i=1}^{2} \langle S(x_i), x_i \rangle, \]

where \( S \) is the shape operator of \( \phi \).

So firstly, we will compute \( S(x_1) \) and \( S(x_2) \):

\[ S(x_1) = D_{\phi_s} Z = \frac{1}{\|\phi_s\|} D_{\phi_s} Z = \frac{1}{\|\phi_s\|} \frac{dZ}{ds} = \left( \frac{k_1 \cos(\theta)}{1-k_1 v \cdot \sin(\theta)} \right)^T, \]

and

\[ S(x_2) = D_{\phi_v} Z = \frac{dZ}{dv} = 0, \]

where \( D \) is standard covariant derivative in \( E^3 \). Therefore, we have

\[ \langle S(x_1), x_1 \rangle = \frac{k_1 \cos(\theta)}{1-k_1 v \cdot \sin(\theta)} \quad \text{and} \quad \langle S(x_2), x_2 \rangle = 0. \]

Finally,

\[ H = \frac{1}{2} \sum_{i=1}^{2} \langle S(x_i), x_i \rangle = \frac{1}{2} \frac{k_1 \cos(\theta)}{1-k_1 v \cdot \sin(\theta)}, \]

where \( 1-k_1 v \cdot \sin(\theta) \neq 0 \).

This completes the proof.

**Corollary 5.2** The surface \( \phi \) defined above is minimal if and only if \( \theta = \pi/2 \) where \( 1-k_1 v \cdot \sin(\theta) \neq 0 \). In that case (whenever \( \theta = \pi/2 \)), the surface \( \phi \) is a plane.

**Example 5.1** Let the curve \( \alpha(u) \) be a plane curve parametrized by the vector function
\[ \alpha(u) = \left( \frac{3}{5}\sin(u), 1 + \cos(u), \frac{4}{5}\sin(u) \right), \quad u \in [0, 5\pi]. \]

Then,

\[ V_2 = \left( -\frac{3}{5}\sin(u), -\cos(u), -\frac{4}{5}\sin(u) \right) \]
\[ B = \left( \frac{4}{5}, 0, -\frac{3}{5} \right) \]

where \( V_2 \) is the principal normal and \( B \) is the binormal of \( \alpha \), respectively.

So, If we choose \( \theta = \pi/6 \) and \( v \in [0, \pi] \), the helix surface generated by the curve \( \alpha(u) \) has the parametric representation:

\[ x = \left( \frac{3}{5} - \frac{3v}{10} \right)\sin(u) + \frac{2\sqrt{3}}{5}v \]
\[ y = (1 - \frac{v}{2})\cos(u) + 1 \]
\[ z = \left( \frac{4}{5} - \frac{2v}{5} \right)\sin(u) - \frac{3\sqrt{3}v}{10}. \]

And, the surface generated by the curve \( \alpha \) is shown the following Figure.
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