Abstract

With appropriate boundary conditions the anisotropic XY chain in a magnetic field is known to be invariant under quantum group transformations. We generalize this model defining a class of integrable chains with several fermionic degrees of freedom per site. In order to maintain the quantum group symmetry a general condition on the parameters of these systems is derived. It is shown that the corresponding quantum algebra is a multi-parameter deformation of the Clifford algebra. Discussing a special physical example we observe a new type of zero modes.
1 Introduction

In statistical mechanics the anisotropic XY chain is one of the simplest exactly solvable models. Its $L$-site Hamiltonian with periodic boundary conditions

$$H_{\text{per}}^{XY}(\eta, h) = -\frac{1}{2} \sum_{j=1}^{L} \left( \eta \sigma_j^x \sigma_{j+1}^x + \eta^{-1} \sigma_j^y \sigma_{j+1}^y \right) - h \sum_{j=1}^{L} \sigma_j^z$$

depends on two parameters, namely the anisotropy parameter $\eta$ and the magnetic field $h$ ($\sigma_j^{x,y,z}$ are Pauli matrices acting on site $j$). This Hamiltonian has a long history [1, 2] and provides a good model for Helium adsorbed on metallic surfaces. It also describes the master equation of the kinetic Ising model [3] and plays a role in one-dimensional reaction-diffusion processes [4].

The present work is based on the investigation of the anisotropic XY chain with a special kind of boundary conditions defined by the Hamiltonian

$$H^{XY}(\eta, q) = -\frac{1}{2} \sum_{j=1}^{L-1} \left( \eta \sigma_j^x \sigma_{j+1}^x + \eta^{-1} \sigma_j^y \sigma_{j+1}^y + q \sigma_j^z + q^{-1} \sigma_{j+1}^z \right)$$

where $q$ is related to the magnetization by $2h = q + q^{-1}$ (notice that compared to Eq. (1.1) there are additional surface fields at the ends of the chain). These boundary conditions make the system to be invariant under quantum group transformations [5, 6]. Beside their mathematical relevance these boundary conditions are also of physical interest since they appear naturally in a special one-dimensional reaction-diffusion process with open ends [4].

The attempts to introduce diagonalizable generalizations of the XY model go back to Suzuki [7]. Following these ideas we consider generalized quantum chains with a higher number of degrees of freedom per site. We maintain the quantum group symmetry by choosing special boundary conditions and imposing appropriate restrictions on the parameters. We are interested in both the physical properties of these generalized chains (like their spectra) and the mathematical structure of the corresponding quantum algebra.

Let us briefly summarize the results of Ref. [6]. The XY chain Hamiltonian (1.2) is invariant under a two-parameter quantum Clifford algebra which is defined by the generators $T^1, T^2$ and the central element $E$ with the commutation relations

$$\{T^1, T^1\} = 2 [E]_{\alpha_1}, \quad \{T^2, T^2\} = 2 [E]_{\alpha_2}, \quad \{T^1, T^2\} = [E, T^1] = [E, T^2] = 0,$$

where $\alpha_1$ and $\alpha_2$ are deformation parameters and

$$[E]_{\alpha_\mu} = \frac{\alpha_\mu^E - \alpha^{-E}_\mu}{\alpha_\mu - \alpha^{-1}_\mu}. \quad (1.4)$$

The coproducts of these generators read

$$\Delta(T^1) = \alpha_1^{E/2} \otimes T^1 + T^1 \otimes \alpha_1^{-E/2}$$
$$\Delta(T^2) = \alpha_2^{E/2} \otimes T^2 + T^2 \otimes \alpha_2^{-E/2}$$
$$\Delta(E) = E \otimes 1 + 1 \otimes E. \quad (1.5)$$

For $\alpha_1 = \alpha_2 = 1$ the quantum algebra (1.3) reduces to the (classical) Clifford algebra

$$\{T^\mu, T^\nu\} = 2 E \delta^{\mu\nu}, \quad [E, T^\mu] = 0. \quad (\mu, \nu = 1, 2) \quad (1.6)$$
Apart from the trivial one-dimensional representation the algebra (1.3) has only two-di-

dimensional irreducible representations, in particular the fermionic representation corresponds to
taking $T^1 = \sigma^x$, $T^2 = \sigma^y$, and $E = 1$.

The explicit expressions for the generators in the case of the $XY$ chain can be obtained
from the fermionic (one-site) representation by a multiple application of the coproducts (1.3).
In order to do so, let us introduce local fermionic operators $\tau^1_j$ and $\tau^2_j$ by a Jordan-Wigner
transformation

$$\tau^1_j = \left( \prod_{i=1}^{j-1} \sigma^z_i \right) \sigma^x_j, \quad \tau^2_j = \left( \prod_{i=1}^{j-1} \sigma^z_i \right) \sigma^y_j$$

which obey the Clifford algebra

$$\{ \tau^\mu_i, \tau^\nu_j \} = 2 \delta_{ij} \delta^{\mu\nu} . \quad (i, j = 1, \ldots, L; \ \mu, \nu = 1, 2)$$

In terms of these operators the Hamiltonian (1.2) can be written as

$$H^{XY}(\eta, q) = \frac{i}{2} \sum_{j=1}^{L-1} \left( \eta \tau^2_j \tau^1_{j+1} - \eta^{-1} \tau^1_j \tau^2_{j+1} + q \tau^1_j \tau^2_j + q^{-1} \tau^1_{j+1} \tau^2_{j+1} \right) .$$

The explicit expressions for the generators $T^1$, $T^2$ and $E$ read

$$T^1 = \alpha_1 \frac{L+1}{2} \sum_{j=1}^{L} \alpha_1^j \tau^1_j, \quad T^2 = \alpha_2 \frac{L+1}{2} \sum_{j=1}^{L} \alpha_2^j \tau^2_j, \quad E = L ,$$

where

$$\alpha_1 = \frac{q}{\eta}, \quad \alpha_2 = q \eta$$

are the deformation parameters. Both of them are essential, i.e. it is impossible to remove
one of the parameters by similarity transformation. Notice furthermore that the generator $E$
simply counts the number of sites. Therefore if one of the deformation parameters is a
root of unity, the irreducible representations of the algebra (1.3) depend on the length of the
chain which requires to define a special thermodynamical limit in this case [6].

The generators (1.10) commute with the Hamiltonian and appear physically as a fermionic
zero mode. This zero mode is present for arbitrary parameters $q$ and $\eta$ and causes all levels
of the spectrum to be at least two-fold degenerated. We want to emphasize that such a
zero mode cannot be observed in the case of periodic or free boundary conditions. In other
words, the quantum group symmetry is directly related to the special boundary conditions
in Eq. (1.2).

If both deformation parameters $\alpha_1$ and $\alpha_2$ coincide, the total magnetization

$$S^z = \sum_{j=1}^{L} \sigma^z_j$$

also commutes with the Hamiltonian and generates an additional $U(1)$ symmetry. In this
case the quantum algebra (1.3) can be enlarged by adding the commutation relations

$$[T^1, N] = 2iT^2, \quad [T^2, N] = -2iT^1, \quad [E, N] = 0$$

and the coproduct

$$\Delta(N) = N \otimes 1 + 1 \otimes N ,$$

where $N = \frac{1}{2}(S^z + L)$. The resulting algebra is the $U_q[SU(1/1)]$ superalgebra [3].

A first attempt to generalize the quantum group invariant $XY$ chain has been made in...
Hamiltonian depends up to normalization on ten parameters: 

\[ H = -\sum_{k=1}^{K} \sum_{j=1}^{L'} (\tilde{J}_{x,k} \sigma^x_j \sigma^x_{j+k} + \tilde{J}_{y,k} \sigma^y_j \sigma^y_{j+k}) \sigma^z_{j+1} \cdots \sigma^z_{j+k-1} + \hbar \sum_{j=1}^{L'} \sigma^z_j \]  

(1.15)

possesses a quantum group symmetry provided that

\[ \tilde{J}_{x,k} = -J_x \delta_{m,k}, \quad \tilde{J}_{y,k} = -J_y \delta_{m,k} \]  

(1.16)

This case is trivial for the following reason. If one performs the transformation

\[ \sigma^{x,y}_{mr+s} \rightarrow (\prod_{i=0}^{r-1} \prod_{j=s+1}^{m} \sigma^z_{mi+j}) (\prod_{i=r+1}^{L'/m} \prod_{j=1}^{s-1} \sigma^z_{mi+j}) \sigma^{x,y}_{mr+s} \]  

(1.17)

\[ \sigma^z_{mr+s} \rightarrow \sigma^z_{mr+s} \]  

(1.18)

one obtains the Hamiltonian

\[ \tilde{H}' = \sum_{r=0}^{L'/m-1} \sum_{s=1}^{m} (\tilde{J}_x \sigma^x_{mr+s} \sigma^x_{m(r+1)+s} + \tilde{J}_y \sigma^y_{mr+s} \sigma^y_{m(r+1)+s}) + \hbar \sum_{r=0}^{L'/m-1} \sum_{s=1}^{m} \sigma^z_{mr+s} \]  

which is a sum of \( m \) identical anisotropic XY chains. Since the transformation (1.17) does not change the algebra of the Pauli matrices, \( \tilde{H} \) and \( \tilde{H}' \) differ only by a similarity transformation. Therefore the physical properties of the \( \tilde{H} \) are already known. In this paper we show that it is possible to define nontrivial generalizations of the XY chain maintaining both the quantum group symmetry and the integrability in terms of free fermions. In contrast to Ref. [8] we start from the physical point of view generalizing the XY chain directly in its fermionic formulation (1.9). In order to implement the quantum group symmetry we then derive a general condition for the existence of zero modes. As an example we consider a four-state quantum chain defined on two commuting copies of Pauli matrices \( \sigma^x,y,z \) and \( \gamma^x,y,z \). Its Hamiltonian depends up to normalization on ten parameters:

\[ H (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \omega_{12}, \omega_{34}, \omega_{14}, \omega_{23}, \omega_{13}, \omega_{24}) = \]  

\[ -\frac{1}{2} \sum_{j=1}^{L-1} \left[ \omega_{12} (\gamma_1^{-1} \gamma_2 \sigma^x_j \sigma^x_{j+1} + \gamma_1 \gamma_2^{-1} \sigma^y_j \sigma^y_{j+1} + \gamma_1 \gamma_2 \sigma^z_j + \gamma_1^{-1} \gamma_2^{-1} \sigma^z_{j+1}) \right. \]

\[ + \omega_{34} (\gamma_3^{-1} \gamma_4 \sigma^x_j \sigma^x_{j+1} + \gamma_3 \gamma_4^{-1} \sigma^y_j \sigma^y_{j+1} + \gamma_3 \gamma_4 \sigma^z_j + \gamma_3^{-1} \gamma_4^{-1} \sigma^z_{j+1}) \]

\[ + \omega_{14} (\gamma_1^{-1} \gamma_4 \sigma^x_j \sigma^x_{j+1} + \gamma_1 \gamma_4^{-1} \sigma^y_j \sigma^y_{j+1} + \gamma_1 \gamma_4 \sigma^z_j + \gamma_1^{-1} \gamma_4^{-1} \sigma^z_{j+1}) \]

\[ + \omega_{23} (\gamma_2^{-1} \gamma_3 \sigma^x_j \sigma^x_{j+1} + \gamma_2 \gamma_3^{-1} \sigma^y_j \sigma^y_{j+1} + \gamma_2 \gamma_3 \sigma^z_j + \gamma_2^{-1} \gamma_3^{-1} \sigma^z_{j+1}) \]

\[ - \omega_{13} (\gamma_1^{-1} \gamma_3 \sigma^x_j \sigma^x_{j+1} + \gamma_1 \gamma_3^{-1} \sigma^y_j \sigma^y_{j+1} + \gamma_1 \gamma_3 \sigma^z_j + \gamma_1^{-1} \gamma_3^{-1} \sigma^z_{j+1}) \]

\[ - \omega_{24} (\gamma_2^{-1} \gamma_4 \sigma^x_j \sigma^x_{j+1} + \gamma_2 \gamma_4^{-1} \sigma^y_j \sigma^y_{j+1} + \gamma_2 \gamma_4 \sigma^z_j + \gamma_2^{-1} \gamma_4^{-1} \sigma^z_{j+1}) \]  

(1.19)

Although at first sight this Hamiltonian seems to be rather artificial we will see that it is indeed a natural generalization of the XY chain Hamiltonian (1.2). We will show that this chain is invariant under a four-parameter deformation of the Clifford algebra. This quantum algebra is defined by the generators \( T^1, T^2, T^3, T^4 \) and \( E \) with the commutation relations

\[ \{T^\mu, T^\nu\} = 2 \delta^{\mu\nu} [E]_{\alpha\nu}, \quad [E, T^\mu] = 0, \quad (\mu, \nu = 1, \ldots, 4) \]  

(1.20)

where \( \alpha_\mu = \gamma_\mu^2 (\mu = 1, \ldots, 4) \) are four deformation parameters. As in the case of the XY chain, we observe additional symmetries if some of these parameters coincide.

The model defined in Eq. (1.19) can be understood as a system of two interacting XY chains:
In opposition to a single $XY$ chain (which is completely described by the deformation parameters) its Hamiltonian depends on further six parameters $\omega_{ij}$ which do not occur in the quantum algebra and in cannot generally be eliminated by similarity transformation. Since these parameters allow to implement nontrivial couplings between the $XY$ chains without breaking the quantum group symmetry, we expect a richer structure than in the case of decoupled $XY$ chains as in Eq. (1.15). Switching these couplings off by taking $\omega_{13} = \omega_{14} = \omega_{23} = \omega_{24} = 0$ and performing the following automorphism on the Pauli matrices

$$
\sigma_j^{x,y} \rightarrow (\prod_{i=1}^{j-1} \sigma_i^z) \sigma_j^{x,y}, \quad \sigma_j^z \rightarrow \sigma_j^z, \quad  \sigma_j^x \rightarrow \sigma_j^x \quad (j = 1 \ldots L) \quad (1.21)
$$

the Hamiltonian (1.19) decouples into a sum of two independent anisotropic $XY$ chains:

$$
H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \omega_{12}, \omega_{34}) = 
-\frac{1}{2} \sum_{j=1}^{L-1} \left[ \omega_{12} (\gamma_1^{-1} \gamma_2 \sigma_j^x \sigma_{j+1}^x + \gamma_1 \gamma_2^{-1} \sigma_j^y \sigma_{j+1}^y + \gamma_1 \gamma_2 \sigma_j^z + \gamma_1^{-1} \gamma_2^{-1} \sigma_{j+1}^z) 
+ \omega_{34} (\gamma_3^{-1} \gamma_4 \sigma_j^x \sigma_{j+1}^x + \gamma_3 \gamma_4^{-1} \sigma_j^y \sigma_{j+1}^y + \gamma_3 \gamma_4 \sigma_j^z + \gamma_3^{-1} \gamma_4^{-1} \sigma_{j+1}^z) \right],
$$

where $\omega_{12}$ and $\omega_{34}$ appear as normalization constants.

As an application we finally consider the Hamiltonian (1.19) for a particular choice of the parameters $\omega_{ij}$ so that the strength of the couplings between the two $XY$ chains is controlled by a single parameter $\xi$. Computing the corresponding spectrum we observe that for a special value of $\xi$ the interaction becomes singular so that one obtains $2^{L-1}$-fold (instead of 4-fold) degenerations. The supplementary symmetry is caused by $L - 1$ additional zero modes. Normally zero modes are known to be exponential modes acting globally on the whole chain. Contrarily the additional zero modes turn out to act only in a specific part of the chain. We thereby find a new type of zero modes which cannot be observed in the case of two-state models.

The paper is organized as follows. In Sect. 2 we define the class of quantum chains to be investigated and outline the diagonalization method. In Sect. 3 we derive a general condition for the existence of fermionic zero modes. Sect. 4 discusses the structure of the corresponding quantum algebra which is a multi-parameter deformation of the Clifford algebra. It is shown that if some of the deformation parameters coincide, additional algebra automorphisms allow the number of free parameters to be reduced. In Sect. 5 we turn our attention to a particular physical four-state model. We discuss our results and consider a special case where additional zero modes occur. Finally we summarize our conclusions in Sect. 6. Appendix A explains how the zero mode condition has been derived. In Appendix B we show that a recently discovered duality property of the anisotropic $XY$ chain [9] also exists in the generalized case.
2 Multifermionic chains and their diagonalization

In order to define a natural generalization of the XY chain, we first rewrite the fermionic version of the Hamiltonian \([1,2]\) in the general bilinear form

\[
H(A, B, C) = \frac{i}{2} \sum_{j=1}^{L-1} \sum_{\mu, \nu=1}^{2n} \left(A^{\mu \nu} \tau_j^\mu \tau_{j+1}^\nu + \frac{1}{2} B^{\mu \nu} \tau_j^\mu \tau_j^\nu + \frac{1}{2} C^{\mu \nu} \tau_j^\mu \tau_{j+1}^\nu \right),
\]

where \(n = 1\) and

\[
A = \begin{pmatrix} 0 & -\eta^{-1} \\ \eta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & q^{-1} \\ -q^{-1} & 0 \end{pmatrix}.
\]

It is near at hand to consider the Hamiltonian \((2.1)\) for \(n > 1\) pairs of local fermionic operators \(\tau_j^\mu\) per site. These operators are supposed to obey the Clifford algebra

\[
\{\tau_j^\mu, \tau_j^\nu\} = 2 \delta_{ij} \delta^{\mu \nu} \quad (i, j = 1 \ldots L; \ \mu, \nu = 1, \ldots, 2n)
\]

and therefore the \(2n \times 2n\) matrices \(B\) and \(C\) can be assumed to be antisymmetric (since symmetric contributions would result in an irrelevant constant). This defines a \(2n\)-state model which is a natural generalization of the XY chain in the fermionic language. As we will see below, the four-state Hamiltonian \((1.19)\) just corresponds to the case \(n = 2\).

We are now going to diagonalize these generalized quantum chains. Since the Hamiltonian \((2.1)\) is bilinear in the operators \(\tau_j^\mu\), it is possible to apply standard methods described in Ref. \([1]\). Accordingly \(H\) can be written in the diagonal form

\[
H(A, B, C) = \sum_{k=0}^{L-1} \sum_{\gamma=1}^{n} \Lambda_k^\gamma i T_k^{2\gamma} \tau_k^{2\gamma-1}
\]

where \(\Lambda_k^\gamma\) are fermionic excitation energies and \(T_k^{\mu}\) are diagonal Clifford operators:

\[
\{T_k^{\mu}, T_l^{\nu}\} = 2 \delta_{kl} \delta^{\mu \nu}. \quad (k, l = 0, \ldots, L - 1; \ \mu, \nu = 1, \ldots, 2n)
\]

They are related to the local fermionic operators \(\tau_j^\mu\) by an orthogonal transformation:

\[
T_k^{\nu} = \sum_{j=1}^{L} \sum_{\mu=1}^{2n} \psi_{k,j}^{\nu \mu} \tau_j^\mu, \quad \sum_{j=1}^{L} \sum_{\mu=1}^{2n} \psi_{k,j}^{\gamma \mu} \psi_{l,j}^{\delta \mu} = \delta_{kl} \delta^{\gamma \delta}
\]

Thus the expressions \(i T_k^{2\gamma} \tau_k^{2\gamma-1}\) in Eq. \((2.4)\) have the eigenvalues \(\pm 1\) so that the knowledge of all excitation energies \(\Lambda_k^\gamma\) allows the spectrum of the Hamiltonian \((2.1)\) to be constructed by taking all combinations into account. As shown in Ref. \([1]\) the excitation energies \(\Lambda_k^\gamma\) and the transformation coefficients \(\psi_{k,j}^{\gamma \mu}\) are solutions of the eigenvalue problem

\[
\sum_{j,\nu} M_{ij}^{\gamma \nu} \Phi_{k,j}^{\gamma \nu} = \pm 2i \Lambda_k^\gamma \Phi_{k,i}^{\gamma \mu}, \quad (\gamma = 1, \ldots, n; \ \mu = 1, \ldots, 2n)
\]

where \(\Phi_{k,i}^{\gamma \mu} = \psi_{k,i}^{2\gamma-1 \mu} \pm \psi_{k,i}^{2\gamma \mu}\) and \(M\) is the following \(2nL \times 2nL\) matrix:

\[
M = \begin{pmatrix}
B & A & A & A \\
-A^T & B + C & A & A \\
& -A^T & B + C & A & A \\
& & & -A^T & B + C & A \\
& & & & -A^T & B + C \\
& & & & & -A^T \\
& & & & & & & & \vdots
\end{pmatrix}.
\]
Since $M$ is antisymmetric we expect its eigenvalues to occur in pairs with different signs.
We thus are free to choose the sign of $A_k^\gamma$. However, the spectrum does not depend on this choice.

Let us summarize our results at this stage. We have constructed a class of Hamiltonians of the form (2.1) defined on $n$ pairs of fermionic operators per site. These chains depend up to normalization on $2n^2 - n$ parameters which are arranged in $2n \times 2n$ matrices $A$, $B$ and $C$. Their spectra can be determined by solving the reduced eigenvalue problem (2.7). In the following section we are going to derive an additional condition on the matrices $A$, $B$ and $C$ in order to implement the quantum group symmetry and to eliminate unessential parameters.

### 3 A condition for the existence of zero modes

In case of the $XY$ chain the quantum group symmetry appears as a fermionic zero mode $\Lambda_0 = 0$ for arbitrary parameters $\eta$ and $q$. Our aim is to implement a similar structure in the case of generalized chains (2.1). We therefore search for a general condition which implies the existence of fermionic zero modes. According to Eq. (2.7) such zero modes are solutions of the system of equations

$$
\sum_{j=1}^L \sum_{\nu=1}^{2n} M_{ij}^{\mu\nu} \tilde{\psi}_{0,j}^{\gamma,\nu} = 0. 
\quad (i = 1, \ldots, L; \mu = 1, \ldots, 2n; \gamma = 1, \ldots, n)
$$

As a necessary condition the determinant of $M$ therefore has to vanish. Since it is quite difficult to solve this problem in general we are only interested in a special set of solutions. As shown in Appendix A the systems of equations (3.1) simplifies essentially if the matrices $A$, $B$, and $C$ satisfy the condition

$$
A^T + CA^{-1}B = 0. 
$$

This zero mode condition is assumed to be valid throughout the rest of this paper. It implies that the components of the zero mode eigenvectors $\tilde{\psi}_{0,j}^{\gamma,\nu} = (\psi_{0,j}^{\mu,1}, \ldots, \psi_{0,j}^{\mu,2n})$ obey a simple power-law:

$$
\tilde{\psi}_{0,j}^{\gamma,\nu} = (-A^{-1}B)^{j-1} \tilde{\psi}_{0,1}^{\gamma,\nu}. 
$$

It is easy to check that in the $XY$ chain case the matrices (2.2) satisfy the zero mode condition (3.2). We now consider the generalized case $n > 1$. In order to simplify the eigenvectors (3.1) and remove unessential parameters one can use the invariance of the Clifford algebra (1.8) under orthogonal transformations $O(2n)$:

$$
\tau_i^{\mu} \rightarrow \tau_i^{\mu'} = \sum_{\nu=1}^{2n} O^{\mu\nu} \tau_i^{\nu}. 
\quad (OO^T = O^TO = 1)
$$

Therefore a change of basis

$$
A \rightarrow A' = OAO^T 
B \rightarrow B' = OBO^T 
C \rightarrow C' = CCO^T
$$

(3.5)

corresponds to a similarity transformation of the Hamiltonian (2.1):

$$
H(A', B', C') = U H(A, B, C) U^{-1}. 
$$

(3.6)
This allows us to choose a basis where the matrix $-A^{-1}B$ in Eq. (3.3) is already diagonal:

$$(-A^{-1}B)^{\mu\nu} = \alpha_{\mu} \delta^{\mu\nu}. \quad (3.7)$$

According to Eq. (2.6) the zero mode operators then read

$$T_0^\mu = (\alpha_{\mu})^{-\frac{1}{4\pi}} \sum_{j=1}^{L} (\alpha_{\mu})^{j} \tau_j^{\mu} \quad (\mu = 1, \ldots, 2n) \quad (3.8)$$

Because of $\Lambda_0^\gamma = 0$ (c.f. Eq. (2.4)) these operators commute with $H(A, B, C)$ and therefore all levels of the spectrum are at least $2^n$-fold degenerated. As will be seen in the next section, they appear as the generators of the corresponding quantum algebra.

Another very useful advantage of the zero mode condition (3.2) is a further simplification of the eigenvalue problem (2.7). It turns out that the eigenvalues of $M$ (beside the zero modes $\Lambda_0^\mu = 0$) are the solutions of the polynomial

$$\det\left(-A^T e^{-i\pi k/L} + (B + C - 2i\Lambda_k^\gamma) + A e^{i\pi k/L}\right) = 0, \quad (3.9)$$

where $k$ runs from 1 to $L - 1$. This polynomial contains only even powers of $\Lambda_k^\gamma$ (due to the freedom of choosing its sign) and constitutes the dispersion relation of the chain.

Notice that the zero modes are always related to exponential wave functions and cannot be derived from Eq. (2.9). Here it is useful to give some comment. It is a well-known property of integrable quantum chains with open boundary conditions that beside the excitations with trigonometric wave functions there is always a set of exceptional excitations with exponential behaviour. In the thermodynamic limit $L \to \infty$ these wave functions are located at the ends of the chain and have a vanishing energy. In our models a special choice of the boundary conditions causes these excitation energies to vanish exactly for finite $L$ giving the exponential wave functions the physical meaning of zero modes.

Hamiltonians of the form (2.1) obeying the zero mode condition (3.2) in the basis (3.7) can be constructed as follows. Let us choose an arbitrary diagonal $2n \times 2n$ matrix $\Gamma$ and an arbitrary antisymmetric $2n \times 2n$ matrix $\Omega$. Then the matrices

$$A = -\Gamma \Omega \Gamma^{-1}, \quad B = \Gamma \Omega \Gamma, \quad C = \Gamma^{-1} \Omega \Gamma^{-1} \quad (3.10)$$

satisfy the zero mode condition (3.3) and $-A^{-1}B = \Gamma^2$ is already diagonal. The corresponding Hamiltonian $H(\Omega, \Gamma)$ therefore depends up to normalization on $2n^2 + n$ parameters.

Let us illustrate this construction for the case $n = 2$. Choosing

$$\Omega = \begin{pmatrix}
0 & \omega_{12} & \omega_{13} & \omega_{14} \\
-\omega_{12} & 0 & \omega_{23} & \omega_{24} \\
-\omega_{13} & -\omega_{23} & 0 & \omega_{34} \\
-\omega_{14} & -\omega_{24} & -\omega_{34} & 0
\end{pmatrix}, \quad \Gamma = \begin{pmatrix}
\gamma_1 & 0 & 0 & 0 \\
0 & \gamma_2 & 0 & 0 \\
0 & 0 & \gamma_3 & 0 \\
0 & 0 & 0 & \gamma_4
\end{pmatrix} \quad (3.11)$$

and inserting the matrices (3.10) into Eq. (2.1) we obtain a ten-parameter Hamiltonian with two fermionic zero modes. Their deformation parameters $\alpha_{\mu}$ in Eq. (3.8) are simply given by $\alpha_{\mu} = \gamma_{\mu}^2$. Then performing a generalized Jordan-Wigner transformation

$$\tau_j^1 = \prod_{i=1}^{j-1} \sigma_i^z \rho_i^z \quad \tau_j^2 = \prod_{i=1}^{j-1} \sigma_i^z \rho_i^z \sigma_j^y \\
\tau_j^3 = \prod_{i=1}^{j-1} \sigma_i^z \rho_i^z \sigma_j^z \rho_j^z \\
\tau_j^4 = \prod_{i=1}^{j-1} \sigma_i^z \rho_i^z \sigma_j^z \rho_j^y \quad (3.12)$$
one obtains directly the ten-parameter Hamiltonian (2.1). It is now clear that the somewhat artificial appearance of this Hamiltonian is nothing but a simple consequence of Jordan-Wigner factors while in the fermionic formulation the generalization is a quite natural one.

We now apply Eq. (3.9) in order to compute the spectrum of the Hamiltonian (2.1). One obtains the fermionic excitation energies

\[
\Lambda_k^1 = \sqrt{p_k + \sqrt{p_k^2 - q_k}}, \quad (k = 1, \ldots, L - 1) \quad (3.13)
\]

\[
\Lambda_k^2 = \sqrt{p_k - \sqrt{p_k^2 - q_k}}, \quad (3.14)
\]

where

\[
p_k = \frac{1}{2} \sum_{1 \leq \mu < \nu \leq 4} \left( \cos \frac{\pi k}{L} - \alpha_{\mu} + \frac{\alpha_{\mu}^{-1}}{2} \right) \left( \cos \frac{\pi k}{L} - \alpha_{\nu} + \frac{\alpha_{\nu}^{-1}}{2} \right) \omega_{\mu \nu}^2 \quad (3.15)
\]

\[
q_k = (\omega_{12} \omega_{34} - \omega_{13} \omega_{24} + \omega_{14} \omega_{23})^2 \prod_{\mu=1}^{4} \left( \cos \frac{\pi k}{L} - \frac{\alpha_{\mu} + \alpha_{\mu}^{-1}}{2} \right). \quad (3.16)
\]

The levels of the spectrum can be computed by taking all fermionic combinations into account (see Eq. (2.4)). Because of the zero modes \(\Lambda_0^1 = \Lambda_0^2 = 0\) each level is at least four-fold degenerated. Obviously the spectrum is massless if at least one of the deformation parameters is on the unit circle. Moreover we observe that the spectrum is invariant under discrete transformations \(\alpha_{\mu} \to \alpha_{\mu}^{-1}\). This symmetry is related to a generalized duality property which will be discussed in Appendix B.

### 4 The Clifford quantum algebra

If the Hamiltonian (2.1) satisfies the zero mode condition (3.2), it is invariant under a \(2n\)-parameter deformation of the Clifford algebra. This quantum algebra is defined by the commutation relations

\[
\{T^\mu, T^\nu\} = 2 \delta^{\mu \nu} [E]_{\alpha_{\mu}}, \quad \quad [E, T^\mu] = 0 \quad (\mu, \nu = 1, \ldots, 2n) \quad (4.1)
\]

and the coproducts

\[
\Delta(T^\mu) = \alpha_{\mu}^{E/2} \otimes T^\mu + T^\mu \otimes \alpha_{\mu}^{-E/2} \quad (\mu = 1, \ldots, 2n) \quad (4.2)
\]

\[
\Delta(E) = E \otimes 1 + 1 \otimes E \quad (\mu, \nu = 1, \ldots, 2n) \quad (4.3)
\]

with the co-unit

\[
\epsilon(T^\mu) = \epsilon(E) = 0 \quad (4.4)
\]

and the antipode

\[
S(T^\mu) = T^\mu, \quad S(E) = -E. \quad (4.5)
\]

It has been given in a similar form in Ref. [11], where \(2n\) distinct central elements and one deformation parameter have been used (instead of \(2n\) deformation parameters and one central element \(E\) in our case which leads to a different representation theory). Notice that by construction of our model the dimension of the algebra (4.1) is always even (the odd case however is also possible but not of interest in this paper). If all deformation parameters \(\alpha_1, \ldots, \alpha_{2n}\) are equal to one, the algebra reduces to the (classical) Clifford algebra

\[
\{T^\mu, T^\nu\} = 2 \delta^{\mu \nu}. \quad (4.6)
\]
Beside the trivial one-dimensional representation \( T^\mu = E = 0 \) the algebra \([4.1]\) possesses only \( 2n \)-dimensional irreducible representations of the form
\[
T^\mu = \sqrt{e} \alpha_\mu t^\mu, \quad E = e \mathbf{1},
\]
(4.7)
where \( e \) is a number and the \( t^\mu \) denote a canonical representation of the \( 2n \)-dimensional classical Clifford algebra \( \{ t^\mu, t^\nu \} = 2 \delta^{\mu\nu} \). For \( n = 2 \) a possible choice is
\[
t^{+1} = \sigma^x \otimes \mathbf{1} \quad t^{-1} = \sigma^y \otimes \mathbf{1} \quad t^{+2} = \sigma^z \otimes \sigma^x \quad t^{-2} = \sigma^z \otimes \sigma^y.
\]
(4.8)
In particular the fermionic representation corresponds to taking \( e = 1 \). The coproduct (4.2) then explicitly reads
\[
\Delta(t^\mu) = \alpha_\mu^{1/2} t^{2n+1} \otimes t^\mu + \alpha_\mu^{-1/2} t^\mu \otimes \mathbf{1},
\]
(4.9)
where \( t^{2n+1} = \sigma^z \otimes \sigma^z \) plays the role of a grading operator. By a multiple application of this coproduct we obtain the \( L \)-site representation
\[
T^\mu = \sum_{j=1}^L (\alpha_\mu)^{j-\frac{L+1}{2}} t^\mu_j, \quad E = L. \quad (\mu = 1, \ldots, 2n)
\]
(4.10)
These generators are nothing but the zero mode operators defined in Eq. \([3.8]\). They commute with the Hamiltonian \([2.1]\) and therefore the chain is invariant under the deformed Clifford algebra \([1.1]\).

If one of the deformation parameters \( \alpha_1, \ldots, \alpha_{2n} \) is a root of unity (i.e. the parameters are non-generic) the r.h.s. of Eq. \([1.1]\) may vanish. In this case the two-dimensional irreducible representations \([4.7]\) break down and only the trivial one survives. In the spectrum non-generic cases appear as level crossings. Here the Hamiltonian possesses zero-norm eigenvectors and one has to consider an appropriate subspace and a redefined scalar product. We do not want to discuss this case in detail here and therefore we will assume the deformation parameters to be generic.

If some of the deformation parameters \( \alpha_1, \ldots, \alpha_{2n} \) coincide, it is possible to perform orthogonal transformations \([3.4]\) in the corresponding subspace \textit{without} altering the commutation relations \([1.1]\). This allows further parameters in the matrix \( \Omega \) (see Eq. \([3.11]\)) to be eliminated. Let us first consider the case where all deformation parameters coincide. Then by means of \( O(2n) \) rotations we can always transform the matrix \( \Omega \) to the block-diagonal form
\[
\left(\begin{array}{ccccc}
0 & \omega_{1,2} & & & \\
-\omega_{1,2} & 0 & & & \\
& \cdots & & & \\
& & \cdots & & \\
& & & 0 & \omega_{2n-1,2n} \\
& & & -\omega_{2n-1,2n} & 0
\end{array}\right).
\]
(4.11)
Hence for equal deformation parameters the generalized Hamiltonian \([2.1]\) (together with the zero mode condition \([3.3]\)) always decomposes into a sum of \( n \) isotropic \( XY \) chains of the form \([1.2]\). However, the normalizations and deformation parameters \( q \) of each copy may be different.

If only \( m < 2n \) deformation parameters coincide (while the others are pairwise distinct), similar considerations show that one can remove \( \frac{1}{2}(m^2 - m) \) parameters in the matrix \( \Omega \) by means of \( O(m) \) transformations.
5 A special physical example

In this section we want to illustrate our results in the example of the chain (1.19). We only consider a special choice of the coupling constants \( \omega_{ij} \) to be defined below. This choice is motivated by physical reasons as follows. Thinking of two coupled \( XY \) chains and neglecting the influence of the deformation parameters we suppose the internal interactions of each chain to have the same strength (\( \omega_{12} = \omega_{34} = 1 \)). On the other hand there are couplings between both chains which can be controlled by a single parameter \( \xi \). Since within each \( XY \) chain only \( X-X \) and \( Y-Y \) interactions are present, we assume for physical reasons the same to be true for the interactions between both chains, i.e., we exclude \( X-Y \) interactions by setting \( \omega_{13} = \omega_{24} = 0 \). Therefore taking care of the signs in Eq. (1.19) a physically reasonable choice of the coupling constant is

\[
\omega_{12} = \omega_{34} = 1, \quad \omega_{14} = -\omega_{23} = \xi, \quad \omega_{13} = \omega_{24} = 0.
\] (5.12)

The Hamiltonian (1.19) then reads

\[
H (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \xi) = \frac{1}{2} \sum_{j=1}^{L-1} \left[ \gamma_1^{-1} \gamma_2 \sigma_j^x \sigma_{j+1}^x + \gamma_1 \gamma_2^{-1} \sigma_j^y \sigma_{j+1}^y + \gamma_1 \gamma_2 \sigma_j^z + \frac{1}{2} (\gamma_1^{-1} \gamma_2^{-1} + \gamma_1 \gamma_2 - \xi) \right] (5.13)
\]

\[
+ \gamma_3 \gamma_4 \sigma_j^x \sigma_{j+1}^x + \gamma_3 \gamma_4^{-1} \sigma_j^y \sigma_{j+1}^y + \gamma_3 \gamma_4 \sigma_j^z + \frac{1}{2} (\gamma_3 \gamma_4^{-1} - \gamma_3^{-1} \gamma_4) \sigma_j^x \sigma_{j+1}^x + (\gamma_3^{-1} \gamma_4^{-1} - \gamma_3 \gamma_4) \sigma_j^y \sigma_{j+1}^y + \frac{1}{2} (\gamma_3^{-1} \gamma_4^{-1} - \gamma_3 \gamma_4) \sigma_j^z \sigma_{j+1}^z + \frac{1}{2} (\gamma_3^{-1} \gamma_4^{-1} - \gamma_3 \gamma_4) \sigma_j^x \sigma_{j+1}^x + (\gamma_3^{-1} \gamma_4^{-1} - \gamma_3 \gamma_4) \sigma_j^y \sigma_{j+1}^y + \frac{1}{2} (\gamma_3^{-1} \gamma_4^{-1} - \gamma_3 \gamma_4) \sigma_j^z \sigma_{j+1}^z.
\]

Let us first consider the case where all deformations \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma \) are equal. As shown in Fig. 1, the spectrum of \( H \) depends linearly on \( \xi \) in this case. For \( \xi = 0 \) the spectrum is just the sum of two identical isotropic \( XY \) chains

\[
H (\gamma, \gamma, \gamma, \gamma, 0) = H^{XY} (1, \gamma^2) \otimes 1 + 1 \otimes H^{XY} (1, \gamma^2),
\] (5.14)

where \( H^{XY} (\eta, q) \) is given in Eq. (1.2) and ‘\( \otimes \)’ denotes equality up to similarity transformation. Looking at Fig. 1 it is obvious that for arbitrary \( \xi \) only the normalizations of the two \( XY \) chains vary linearly:

\[
H (\gamma, \gamma, \gamma, \gamma, \xi) = (1 + \xi) H^{XY} (1, \gamma^2) \otimes 1 + (1 - \xi) 1 \otimes H^{XY} (1, \gamma^2)
\] (5.15)

Thus for equal deformation parameters the generalized chain (5.13) always decouples into a sum of two isotropic \( XY \) chains in agreement with the observation in Eq. (1.14). In particular for \( \xi = 1 \) only one of them survives so that each level of the spectrum is at least \( 2^{L+1} \)-fold degenerated. Here the Hamiltonian (5.13) is invariant under local rotations generated by the anticommuting operators

\[
S_j^1 = \tau_j^1 - \tau_j^3, \quad \quad \quad S_j^2 = \tau_j^2 - \tau_j^4, \quad (j = 1, \ldots, L)
\] (5.16)

We will come back to this case below.

For arbitrary deformation parameters the spectrum cannot be decomposed into a sum of \( XY \) chain spectra (see Fig. 2). The only exceptions are \( \xi = 0 \) and \( \xi = 1 \). In the first case we have

\[
H (\gamma_1, \gamma_2, \gamma_3, \gamma_4, 0) = H^{XY} (\gamma_1^{-1} \gamma_2, \gamma_1 \gamma_2) \otimes 1 + 1 \otimes H^{XY} (\gamma_3^{-1} \gamma_4, \gamma_3 \gamma_4)
\] (5.17)
in agreement with Eq. (1.22). For $\xi = 1$ we claim that the spectrum of $H$ coincides up to degenerations with the spectrum of a single anisotropic $XY$ chain:

$$H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, 1) \cong 2 H^{XY}(\eta, q) \otimes 1.$$ (5.18)

Here the parameters $\eta$ and $q$ are solutions of the following trigonometric equations

$$\nu = \left( \frac{\eta + \eta^{-1}}{2} \right)^2 + \left( \frac{q + q^{-1}}{2} \right)^2 - 1 = \frac{1}{4}(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_4),$$ (5.19)

$$\mu = 4 \left( \frac{\eta - \eta^{-1}}{2} \right)^2 \left( \frac{q - q^{-1}}{2} \right)^2 = \frac{1}{4}(\Delta_1 + \Delta_3 - \Delta_2 - \Delta_4)^2,$$ (5.20)

where

$$\Delta_i = \frac{\alpha_i + \alpha_i^{-1}}{2} = \frac{\gamma_i^2 + \gamma_i^{-2}}{2}.$$ (5.21)

In order to prove Eq. (5.18) we checked that the Hamiltonian

$$H(\gamma_1, \gamma_2, \gamma_3, \gamma_4, 1) = 2 \sum_{j=1}^{L-1} h_j$$ (5.22)

satisfies indeed the algebraic relations of the anisotropic $XY$ chain [3]:

$$[h_j h_{j\pm 1} h_j - h_{j\pm 1} h_j h_{j\pm 1} + (\nu - 1)(h_j - h_{j\pm 1})](h_j - h_{j\pm 1}) = \mu$$ (5.23)

$$h_j^2 = \nu.$$ (5.24)

Hence for arbitrary deformation parameters a special tuning of the coupling constants (due to the choice $\xi = 1$) provides a strong symmetry. The spectrum is equivalent to that of a single anisotropic $XY$ chain and each level is at least $2^{L+1}$-fold degenerated. The corresponding symmetry operators read

$$L_1^j = \sum_{k=1}^{j} (\alpha_1^{k-j-1/2} \tau_j^{-1} - \alpha_3^{k-j-1/2} \tau_j^{-3})$$ \hspace{1cm} (j = 1, \ldots, L - 1) (5.25)

$$L_2^j = \sum_{k=1}^{j} (\alpha_2^{k-j-1/2} \tau_j^{-2} - \alpha_4^{k-j-1/2} \tau_j^{-4})$$

$$[L_i^\mu, H] = 0.$$ (5.26)

These generators may be understood as $L - 1$ additional zero modes. Together with the four zero mode generators $T^\mu_0$ they cause $2^{L+1}$-fold degenerations.

The special property of the zero modes (5.25) is that they act only in a part of the chain extending from the left boundary to a certain position $j$. Similarly there are zero mode operators acting from position $j+1$ to the right boundary:

$$R_1^j = \sum_{k=j+1}^{L} (\alpha_1^{k-j-1/2} \tau_j^{1} - \alpha_3^{k-j-1/2} \tau_j^{3})$$ \hspace{1cm} (j = 1, \ldots, L - 1) (5.26)

$$R_2^j = \sum_{k=j+1}^{L} (\alpha_2^{k-j-1/2} \tau_j^{2} - \alpha_4^{k-j-1/2} \tau_j^{4})$$
Because of
\[ L_j^\mu + R_j^\mu = \alpha_{\mu}^{L/2-j} T_0^\mu - \alpha_{\mu+2}^{L/2-j} T_0^{\mu+2} \quad (\mu = 1, 2) \] (5.27)
only one set of operators (e.g. \{L_j^\mu\}) is independent. It is easy to check that for all deformation parameters being equal one retrieves the local symmetry operators (5.16) by taking appropriate linear combinations.

As already mentioned in Sect. 3, the existence of exponential modes is a well-known property of integrable chains with non-periodic boundary conditions. Normally there are only four exponential modes in our model, namely the zero modes (3.8). These modes always act globally. In contrast for \( \xi = 1 \) the additional zero modes (5.25) and (5.26) act only to the left and to the right of a certain position, respectively. To our knowledge this phenomenon has not been observed before. It has its origin in a singularity of the interaction \((\text{det}(\Omega) = 0)\) for \( \xi = 1 \). Roughly speaking certain modifications of the states at site \( j \) do not affect the situation at site \( j + 1 \). Therefore if one combines exponential modes in an appropriate way they ‘trickle away’ at a certain position.

6 Conclusions

The present work is based on previous investigations of the anisotropic XY chain in a magnetic field with a special kind of boundary conditions. These boundary conditions imply the existence of a fermionic zero mode which is related to a quantum group symmetry.

In this article we found a class of integrable quantum chains which can be understood as generalizations of the XY chain. These \( 2n \)-state models are defined on \( n \) fermionic degrees of freedom per site and can be diagonalized in terms of free fermions as well. In analogy to the XY chain case we found a general condition for the existence of fermionic zero modes. This condition implies in turn that the Hamiltonian is invariant under a \( 2n \)-parameter deformation of the \( 2n \)-dimensional Clifford algebra and causes \( 2^n \)-fold degenerations of each energy level. Discussing the structure of this algebra we observed that if some of the deformation parameters coincide, the symmetry of the chain is increased by means of orthogonal algebra automorphisms leading to higher degenerations of the spectrum.

The structure of the quantum group allows complicated internal couplings to be implemented. These couplings are nontrivial in the sense that the spectrum of such a chain cannot be decomposed into a sum of XY chain spectra. As an example we discussed a four-state model and computed the corresponding spectrum. In this case one can think of two XY chains with nearest-neighbour couplings between them. The corresponding Hamiltonian depends on ten parameters, four of them being deformation parameters of the Clifford algebra. We restricted our attention to a special choice of the other six parameters which is supposedly the most physical one (we allow only XX and YY couplings between the chains) and illustrated our results. For a special tuning of the coupling constants the interaction matrix becomes singular and the spectrum coincides with that of a single XY chain. However, the degenerations are much larger due to the existence of \( L - 1 \) additional zero modes. In contrast to usual exponential modes zero modes of this kind act only in a particular part of the chain extending from the left boundary to a certain position.

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Appendix A  The structure of fermionic zero modes

The quantum group symmetry of the XY chain Hamiltonian (1.2) and its generalization (1.19) relies on the existence of fermionic zero modes for arbitrary parameters. In this appendix we show that zero modes of this kind possess a very simple and general structure. For this we derive the zero mode condition (3.2) on the level of the diagonalization of the matrix \( M \) in Eq. (2.7).

Since the matrix \( M \) in Eq. (2.8) possesses a tridiagonal block structure it is near at hand to solve the system of equations \( M\vec{\psi} = 0 \) 'line by line'. Introducing the vector notation \( \vec{\psi}_{\mu,0} = (\psi_{\mu,0}, \ldots, \psi_{\mu,2n}) \) and fixing \( \vec{\psi}_{\mu,0} \) on the leftmost site, the first line of the matrix \( M \) allows \( \vec{\psi}_{\mu,0,2} \) to be computed by

\[
\vec{\psi}_{\mu,0,2} = -A^{-1}B\vec{\psi}_{\mu,0,1}.
\]

This equation can be understood as a left boundary condition. In the same way the last line

\[
\vec{\psi}_{\mu,0,L-1} = (A^T)^{-1}C\vec{\psi}_{\mu,0,L}
\]

acts as a right boundary condition. Between the boundaries the lines \( j = 2, \ldots, L - 1 \)

\[
\vec{\psi}_{\mu,j+1} = -A^{-1}(B + C)\vec{\psi}_{\mu,1} + A^{-1}A^T\vec{\psi}_{\mu,j+1}
\]

(A.3)

describe the bulk behaviour of the zero mode. These equations allow the zero mode eigenvector to be constructed as follows. Once if \( \vec{\psi}_{\mu,0,1} \) is fixed, Eq. (A.1) yields \( \vec{\psi}_{\mu,0,2} \). Then by Eq. (A.3) which can be understood as a two-stage recurrency relation the vectors \( \vec{\psi}_{\mu,0,3}, \ldots, \vec{\psi}_{\mu,0,L} \) (and therewith the entire zero mode vector) can be computed recursively. Finally the last two vectors \( \vec{\psi}_{0,0,L-1} \) and \( \vec{\psi}_{0,0,L} \) have to satisfy the right boundary condition Eq. (A.2).

Normally it is rather difficult to determine \( \vec{\psi}_{\mu,0,1} \) so that the right boundary condition (A.2) is satisfied since the parameters of the Hamiltonian \( H(A,B,C) \) are involved in a highly non-linear way.

The situation becomes much simpler if we impose the additional condition

\[
A^T + CA^{-1}B = 0
\]

on the matrices \( A, B \) and \( C \). The two-stage recurrency relation (A.3) then reduces to a one-stage relation \( \vec{\psi}_{\mu,j} = -A^{-1}B\vec{\psi}_{\mu,j+1} \) yielding solution in Eq. (3.3). It is easy to see that these eigenvectors satisfy the boundary conditions (A.1) and (A.2) automatically.

Appendix B  Discrete Symmetry Transformations

Beside the quantum group invariance the anisotropic XY chain (1.2) possesses a further important symmetry. Diagonalizing the Hamiltonian (1.2) one observes that the exchange of the parameters \( \eta \) and \( q \) does not modify the spectrum:

\[
H^{XY}(\eta,q) = H^{XY}(q,\eta)
\]

(B.1)

In Ref. [9] we derived the corresponding similarity transformation

\[
H^{XY}(\eta,q) = U H^{XY}(q,\eta) U^{-1}
\]

(B.2)
and showed that $U$ reduces in a special limit to the Ising duality transformation. For this reason we denoted the transformation (B.2) as ‘generalized duality transformation’ (although the XY chain is not self-dual in the usual sense). In this section we show that a similar symmetry exists in the case of generalized chains of the form (2.1) obeying the zero mode condition (3.2).

We first notice that in the XY chain case the transformation (B.1) just inverts the deformation parameter $\alpha_1 \leftrightarrow \alpha_1^{-1}$ while $\alpha_2$ is not changed (in the same way it is possible to construct a similarity transformation which inverts $\alpha_2$ and keeps $\alpha_1$ fixed). Then looking at the fermionic excitation energies (3.13) of the generalized Hamiltonian (1.19) with two fermionic degrees of freedom per site we recognize that the inversion of any deformation parameter $\alpha_\mu \leftrightarrow \alpha_\mu^{-1}$ ($\mu = 1, \ldots, 4$) does not alter the spectrum. We therefore expect this observation to hold for arbitrary $n$, i.e. using the notation of Eq. (3.10) we assume that for every $\mu = 1, \ldots, 2n$ we have

$$H(\alpha_1, \ldots, \alpha_\mu^{-1}, \ldots, \alpha_{2n}, \Omega) = H(\alpha_1, \ldots, \alpha_\mu, \ldots, \alpha_{2n}, \Omega).$$

(B.3)

In analogy to the results of Ref. [9] it turns out that the corresponding similarity transformation depends exclusively on the deformation parameter it is inverting:

$$H(\alpha_1, \ldots, \alpha_\mu^{-1}, \ldots, \alpha_{2n}, \Omega) = U(\alpha_\mu) H(\alpha_1, \ldots, \alpha_\mu, \ldots, \alpha_{2n}, \Omega) U^{-1}(\alpha_\mu).$$

(B.4)

Denoting

$$N_\mu = 2^{L-1} \frac{1 + \alpha_\mu^L}{(1 + \alpha_\mu)^L}, \quad \omega_\mu = \frac{\alpha_\mu^{1/2} - \alpha_\mu^{-1/2}}{\alpha_\mu^{1/2} + \alpha_\mu^{-1/2}}$$

(B.5)

this transformation can be written in terms of a ‘time-ordered’ exponential

$$U(\alpha_\mu) = \frac{1}{\sqrt{N_\mu}} T \exp(\omega_\mu G_\mu),$$

(B.6)

where $G_\mu$ is a nonlocal generator

$$G_\mu = \sum_{1 \leq j_1 < j_2 \leq L} \tau_{j_1}^{\mu} \tau_{j_2}^{\mu}.$$

(B.7)

$T$ is an ordering operator defined by

$$T \tau_i^{\mu} \tau_j^{\mu} = \begin{cases} 
\tau_i^{\mu} \tau_j^{\mu} & i < j \\
-\tau_j^{\mu} \tau_i^{\mu} & i > j \\
0 & i = j
\end{cases}.$$ 

(B.8)

Explicitely the transformation $U(\alpha_\mu)$ is given by the polynomial

$$U(\alpha_\mu) = \frac{1}{\sqrt{N_\mu}} \left( 1 + \sum_{k=0}^{[L/2]} \omega_\mu^k \sum_{1 \leq j_1 < j_2 < \ldots < j_{2k} \leq L} \tau_{j_1}^{\mu} \tau_{j_2}^{\mu} \ldots \tau_{j_{2k}}^{\mu} \right)$$

(B.9)

where $[L/2]$ denotes the truncation of $L/2$ to an integer number. It is an orthogonal transformation

$$U(\alpha_\mu)^T = U(\alpha_\mu)^{-1}$$

(B.10)

and its inverse is given by

$$U(\alpha_\mu)^{-1} = U(\alpha_\mu^{-1}).$$

(B.11)
Therefore $U(\alpha_\mu)$ reduces to the identity if the deformation parameter in question is equal to one. Because of $[G_\mu, G_\nu] = 0$ the transformations $U(\alpha_\mu)$ commute for different $\mu$ and can be combined freely. Notice that for non-generic deformation parameters ($\alpha^L_\mu = \pm 1$) the transformation $U(\alpha_\mu)$ does not exist since the normalization $N_\mu$ diverges.

It is well known in the theory of quantum groups that the inversion of a deformation parameter corresponds to an algebra homomorphism. The transformation (B.6) shows that the same is true for the whole physical system. It should be emphasized that different physical situations are related (e.g. disordered and frozen states). If the deformation parameter in question is equal to one, the system undergoes a massless phase transition. Two types of transitions are possible. If the dispersion of the massless excitations is linear in $k$, we have a critical Ising transition, otherwise if the dispersion is quadratic in $k$, we observe a Pokrovsky-Talapov phase transition.
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Figure Captions:

Fig. 1: Spectrum of $H\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \xi\right)$ for 3 sites.

Fig. 2: Spectrum of $H\left(\frac{2}{7}, \frac{2}{7}, \frac{5}{7}, \frac{6}{7}, \xi\right)$ for 3 sites.
Fig. 1
Fig. 2