THE EFFECT OF GRAVITATION ON THE POLARIZATION STATE OF A LIGHT RAY

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ABSTRACT

In the present work, detailed calculations have been carried out on the rotation of the polarization vector of an electromagnetic wave due to the presence of a gravitational field of a rotating body. This has been done using the general expression of Maxwell’s equation in curved spacetime. Considering the far-field approximation (i.e., the impact parameter is greater than the Schwarzschild radius and rotation parameter), the amount of rotation of the polarization vector as a function of impact parameter has been obtained for a rotating body (considering Kerr geometry). The present work shows that the rotation of the polarization vector cannot be observed in the case of Schwarzschild geometry. This work also calculates the rotational effect when considering prograde and retrograde orbits for the light ray. Although the present work demonstrates the effect of rotation of the polarization vector, it confirms that there would be no net polarization of an electromagnetic wave due to the curved spacetime geometry in a Kerr field.

Key words: gravitation – polarization – stars: rotation

1. INTRODUCTION

It was way back in 1958 when Balazs (1958), for the first time, showed that the gravitational field of a rotating body can actually rotate the direction of the polarization vector of an electromagnetic wave passing in its field. The author gave an idea on how the gravitational field affects the orientation of the plane of polarization of the electromagnetic wave. The basic concept behind the idea of Balazs (1958) was to derive the four Maxwell’s equations (Jackson 1962; Born & Wolf 1999; Stephani 2004; Landau & Lifshitz 1998) and express the electric displacement vector in terms of the electric and magnetic fields. He tried to figure out whether or not the gravitational field would have any effect on polarization. It should be noted here that the work by Balazs (1958) was carried out before the establishment of the Kerr metric, so he described the metric as

\[ g_{\alpha\beta} = \begin{pmatrix} 1 - \frac{2\phi}{c^2} & \frac{\phi}{c^2} \\ \frac{\phi}{c^2} & 1 + \frac{2\phi}{c^2} \end{pmatrix}, \]

\[ \gamma = -i(g_{00}) = \left( \frac{2G}{c^3R^3} \right) L \times R \]

where \( L \) represents the angular momentum of a rotating mass and \( R \) represents the radius vector of a rotating body. He investigated the problem further by considering a rotating frame instead of a rotating body.

In the very next year, another investigator, Plebanski (1960), put forward his investigation into the propagation of an electromagnetic wave affected by gravitation. He tried to understand the similar physical phenomenon of polarization vector rotation due to gravitational effect in an isolated physical system or macroscopic medium. To resolve the physical problem, he used the geometrical optics approximation to estimate the displacement vector and considered the case of very far-field approximation. In that work, from the change of the energy momentum tensor, he proved that energy conservation remains intact for the modified electric displacement vector. He described the electric displacement vector as

\[ D_{i} = E_{i} + \varepsilon_{ik}E_{k} + \varepsilon_{ikl}R_{l}H_{i}. \]

He showed that when space becomes flat at a very far distance, then \( D_{i} \sim E_{i} \). This work also focused on the effect of the factors \( \varepsilon_{ik} \) and \( g_{kl} \) on the rotation of the polarization vector. The lowest order approximation for \( \varepsilon_{ik} \) and \( g_{kl} \) was expressed in terms of matter tensor (as described in Plebanski 1960), and he concluded that the effect is very small. Finally, the work gave the total rotation of the polarization vector for a rotating frame, and the author applied the result to some selected cases. This work clearly established the rotation of the polarization vector when light is deflected by a gravitational field and also estimated the amount of rotation of the polarization vector. In 1977, Pineault & Roeder (1977) estimated the rotation of the polarization vector with respect to a distant observer and calculated the image distortion due to this rotation. In 1980, Su & Mallett (1980) gave a comprehensive review of the work done to that date in order to calculate the amount of rotation that the polarization vector of a light ray will undergo when passing close to a massive rotating body (viz., a black hole). In their review, they discussed the cases where some authors got non-vanishing results and others got vanishing results, with the reasons for such differences.

In more recent work in the last decade, Sereno (2004) investigated the rotation of the polarization vector due to a gravitational field and gave the approximate result for gravitational Faraday rotation. He took the weak-field approximation limit and analytically presented the result for gravitational Faraday rotation. The author claimed the model to be suitable for estimating Faraday-like rotation of the polarization vector up to third-order correction to gravitational potential. He also briefly discussed the case for a gravitational lens moving with constant velocity. Prior to that, a very systematic treatment of the image formation due to a moving gravitational lens was presented by Paczynski (1986).

As discussed above, it has been already well established that the direction of the polarization vector gets rotated due to the
effect of gravity (Carroll 2004; Balazs 1958; Vilenkin 1984; Fernández & Davis 2011). In general, the spacetime curvature created by the presence of a rotating massive body can cause the rotation of the polarization vector of an electromagnetic wave or light. Section 2 of this work gives the general expression for the rotation of the polarization vector for a linearly polarized light ray considering the far-field approximation, which means the impact parameter is much greater than the Schwarzschild radius and rotation parameter. The expression for the electric displacement vector in the present work has been written with knowledge of the Kerr metric (Kerr 1963), which was not possible in the time of Balazs (1958) and Plebanski (1960), as the Kerr metric expression was published later. As discussed by Sereno (2004), the model is significant in micro-lensing phenomena, where massive bodies act as moving micro-lenses in the case of galactic gravitational lensing of a light ray. However, this situation clearly differs from our case of interest, where we studied the effect of a rotating gravitational body on the direction of polarization vector of a single ray. Su & Mallett (1980) developed their technique to calculate the rotation angle for two types of observers: in the first case, the observer is in a global inertial frame (GIF), i.e., in an asymptotic flat space, and in the second case, the observer is in a non-inertial frame, at rest in a locally non-inertial frame. Only the result they obtained under the first case (viz. GIF) can be compared to the findings from our present study, as we limit our work to an observer in an asymptotic flat space only (i.e., at infinite distance from the massive rotating body). In Section 5, the degree of polarization that may be expected from this phenomena has been obtained with a null result. To derive the effect of the gravitational field, here in the present work we derive the relation between the electric displacement vector and the electric field. A light source placed at an asymptotically flat space emits an electromagnetic wave, which travels through the curved spacetime and is received by an observer at another asymptotically flat space. The effect of the gravitational field on the electromagnetic wave has been calculated in the present work from knowledge of the metric tensor representing the curved spacetime. For simplicity, the emitted light from the source has been considered to be linearly (or plane) polarized for our convenience.

2. GENERAL EXPRESSION FOR THE ROTATION OF THE POLARIZATION VECTOR OF AN ELECTROMAGNETIC WAVE IN CURVED SPACETIME

To derive the expression for the desired rotation of the polarization vector of an electromagnetic wave, the expressions for the electromagnetic displacement vector have to be calculated in generalized form from the co-variant form of Maxwell’s equation. These expressions can be written as (Landau & Lifshitz 1998; Stephani 2004; Misner et al. 1973):

\[
\frac{\partial F_{ik}}{\partial x^j} + \frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{kj}}{\partial x^i} = 0 \quad (1a)
\]

\[
F_{ik} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} (\sqrt{-g} F^{ik}) \quad (1b)
\]

After the general expression of the displacement vector and magnetic induction have been established, the general expression for the rotation of the polarization vector in curved spacetime for a known gravitational field can be obtained.

2.1. Expression for the Electromagnetic Displacement Vector in Curved Spacetime

After writing Maxwell’s equations in co-variant form, we can write down the expressions for the electric field and displacement vector in the corresponding three-dimensional form as below (Landau & Lifshitz 1998; Frolov & Shoom 2011):

\[
E_{\alpha} = F_{0\alpha} \quad B_{\alpha\beta} = F_{0\beta} \quad D^{\alpha} = -\frac{\partial E^\alpha}{\partial x^0} \quad H^{\alpha\beta} = \sqrt{\frac{\det g}{g}} F^{\alpha\beta} \quad (2)
\]

where \(E\) is the electric component, \(B\) is the magnetic component, \(D\) is the electric displacement vector, and \(H\) is the magnetic induction of the electromagnetic wave. The \(\alpha\) and \(\beta\) indices run from 1 to 3, representing the spatial axis of the four-dimensional spacetime geometry. In this work, \(D\) and \(E\) are also represented in vectorial form in three-dimensional geometry. From Equation (2), as outlined by Landau & Lifshitz (1998) and Frolov & Shoom (2011), we may re-write the following expression in a more detailed manner.

\[
E_{\alpha} = F_{0\alpha} \Rightarrow E_{\alpha} = g_{00} g_{\alpha\beta} F^{\beta} \\
E_{\alpha} = g_{00} (g_{0\alpha} F^{\alpha} + g_{\alpha\beta} F^{\beta}) \\
E_{\alpha} = g_{00} (g_{0\alpha} F^{\alpha} + g_{\alpha\beta} F^{\beta} + g_{\alpha\beta} g_{\gamma\delta} F^{\gamma\delta}) \\
E_{\alpha} = \frac{g_{0\alpha}}{g_{00}} = \frac{g_{\alpha\beta}}{g_{00}} F^{\beta} + \frac{g_{\alpha\beta} g_{\gamma\delta}}{g_{00}} F^{\gamma\delta} \\
E_{\alpha} = \frac{-g_{\alpha\beta}}{g_{00}} (-g_{00} F^{\beta}) + \frac{g_{\alpha\beta}}{g_{00}} F^{\beta} = \frac{g_{\alpha\beta}}{g_{00}} H^{\beta} \\
E_{\alpha} = \frac{g_{\alpha\beta}}{g_{00}} + \frac{g_{\alpha\beta}}{g_{00}} D^{\beta} + \frac{g_{\alpha\beta}}{g_{00}} H^{\beta} \\
E_{\alpha} = \frac{g_{\alpha\beta}}{g_{00}} (g_{\alpha\beta} g_{\gamma\delta}) D^{\beta} + \frac{g_{\alpha\beta}}{g_{00}} H^{\alpha\beta} \quad (3)
\]

With the help of the identities of the unit tensor (Landau & Lifshitz 1998; Weinberg 1972; Frolov & Shoom 2011) in a three-dimensional coordinate system, Equation (3) can be rewritten in vector form as follows:

\[
D = \frac{E}{\sqrt{g}} + H \times g \quad (4)
\]

So, from Equation (4), one can determine the components of the displacement vector for a given gravitational field and
subsequently the rotation of the polarization vector can be obtained for a linearly polarized electromagnetic light ray.

2.2. Rotation of the Polarization Vector of an Electromagnetic Wave in the Gravitational Field of a Rotating Object

2.2.1. Light Traveling Along the Equatorial Plane

Let us assume that a linearly (or plane) polarized light ray is coming from an asymptotic flat space (represented by $r = -\infty$) and passing through the gravitational field of a rotating object represented by the metric $g_{\mu\nu}$. After that, the ray is received by an observer at $r = \infty$. The displacement vector defined by Equation (4) for the three-dimensional coordinate system in the present case has to be derived. For our convenience, we define the geometry of the system as follows: the rotating gravitational mass is placed at the origin of the three-dimensional Cartesian coordinate system, with the rotation axis defining the $Z$-axis. The incoming light ray is traveling along a plane that is perpendicular to the $Z$-axis. The direction of the incoming light ray and the origin of the coordinate system together define the $X$-$Y$ plane. The light ray approaches the origin with impact parameter “b.” The original direction of the light ray marks the direction parallel to the $X$-axis (as illustrated by Figure 1). The $Y$-axis is now defined as normal to the $X$- and $Z$-axes.

In this geometry, the light ray is contained within the equatorial plane of the rotating body. The incoming light ray is assumed to be plane polarized, and it will have its electric vector projected into the directions $(Y, Z)$ with components $E_y$ and $E_z$. Thus, the position angle of the polarization ($\chi_{\text{eq}}$) can be expressed by the relation

$$\chi_{\text{eq}} = \arctan \left( \frac{E_y}{E_z} \right). \tag{5}$$

As the light propagates through the gravitational field, the effect of gravitation can be equated with the change in the refractive index of the medium (Balazs 1958; Plebanski 1960; Sen 2010; Roy & Sen 2014). As a result, in the new material medium, the strength of the electric field vectors will be now estimated by the displacement vectors $D_y$ and $D_z$. Thus, the position angle of the polarization vector will be now estimated by the equation

$$\tilde{\chi}_{\text{eq}} = \arctan \left( \frac{D_y}{D_z} \right). \tag{6}$$

Therefore, the rotation of the polarization vector at each location of space would be some value over the total change from the position angle defined by Equation (5). Now Equation (6) only gives the position angle of the polarization vector at any point under the influence of the gravitational field. To find out the total rotation of the polarization vector at any point, we have to calculate the rate of change of the displacement vector at a particular point. One can find the values of the components of the displacement vector along the $Y$- and $Z$-axes with the help of Equations (4) and (5). We note that the electromagnetic wave is propagating parallel to the $X$-axis and there will be no component of the displacement vector or the magnetic induction vector along the direction of propagation (which is the $X$-axis). One can write:

$$D_y = \frac{E_y}{\sqrt{800}} + H_z g_x, \tag{7a}$$

$$D_z = \frac{E_z}{\sqrt{800}} - H_y g_x. \tag{7b}$$

Combining with Equation (6):

$$\tilde{\chi}_{\text{eq}} = \arctan \left( \frac{D_y}{D_z} \right) = \arctan \left( \frac{\frac{E_y}{\sqrt{800}} + H_z g_x}{\frac{E_z}{\sqrt{800}} - H_y g_x} \right) \tag{9}$$

where $\tilde{\chi}_{\text{eq}}$ is the position angle of the polarization vector at a given point, under the influence of the gravitational field. To find the change in the electric displacement vector, we must know the components of electric vector $E_y$ and $E_z$ along the $Y$- and $Z$-axes at the point $r = -\infty$. Now, if we assume that the polarization vector makes an angle $\chi_{\text{eq}}$ with the $Y$-axis at the point $r = -\infty$, then the components of the electric vector will be related by the equation $E_y = \xi_{\text{eq}} E_z$ (say), and $\xi_{\text{eq}}$ is given by

$$\chi_{\text{eq}} = \arctan \left( \frac{E_y}{E_z} \right) = \arctan(\xi_{\text{eq}}). \tag{10}$$

The different components of the magnetic field ($H$) at the starting point of the incoming ray can be written in terms of the electric field using the following relation $H_y = \hat{n} \times E_y$ and $H_z = \hat{n} \times E_z$ (p. 112 in Landau & Lifshitz 1998), where $\hat{n}$ is the normal unit vector along the direction of propagation (in this case parallel to the $X$-axis). Taking only the magnitude of the magnetic field from Equations ?(a) and (b) and taking note of our earlier assumption that $E_y = \xi E_z$, we can write

$$D_y = E_z \left\{ \frac{\xi}{\sqrt{800}} + g_x \right\}, \tag{10a}$$

$$D_z = E_z \left\{ \frac{1}{\sqrt{800}} - \xi g_x \right\}. \tag{10b}$$
As a result we may finally write

\[ \bar{\chi}_{eq} = \arctan \left( \frac{D_y}{D_z} \right) \]

\[ = \arctan \left( \frac{\xi_{eq} + \sqrt{800 \xi_{eq}^2}}{1 - \xi_{eq} \sqrt{800 \xi_{eq}^2}} \right) \]  

(11)

Equation (11) gives the position angle of the rotated polarization vector of the electromagnetic wave. Now, at the initial point \((r = -\infty)\) in our case, \(D_y = E_z\) and \(D_z = E_x\). The general equation for \(D_y, D_z\) at any point \(r\) (say) will be,

\[ D_y = \frac{\xi_{eq} E_z}{\sqrt{800(r)}} + E_z \xi_{eq} g_x(r) \]  

(12a)

\[ D_z = \frac{E_z}{\sqrt{800(r)}} - E_z \xi_{eq} g_x(r) \]  

(12b)

From Equations 12(a) and 12(b) it is clear that both \(D_y\) and \(D_z\) are functions of position only.

### 2.2.2. Light Propagation Along the Rotation Axis

If the light ray propagates along the rotation axis, i.e., along the defined Z-axis, then it will have its electric vector projected into the directions (X, Y) with components \(E_x\) and \(E_y\). Thus, the position angle of the polarization (\(\chi_{ax}\)) can be expressed as

\[ \chi_{ax} = \arctan \left( \frac{E_x}{E_y} \right) \]  

(13)

After the light ray propagates out through the gravitational field, the strength of the electric field vectors will be estimated by the displacement vectors \(D_x, D_y\) (Balazs 1958; Plebanski 1960; Sen 2010; Roy & Sen 2014). Thus, the position angle of the polarization vector will be estimated by the formula

\[ \bar{\chi}_{ax} = \arctan \left( \frac{D_x}{D_y} \right) \]  

(14)

As shown in Section 2.2.1, one can find the values of \(D_x\) and \(D_y\) by assuming that \(E_x = \xi_{ax} E_y\) at any point \(r\) (say) using the following relations:

\[ D_x = \frac{\xi_{ax} E_y}{\sqrt{800(r)}} + E_y \xi_{ax} g_x(r) \]  

(15a)

\[ D_y = \frac{E_y}{\sqrt{800(r)}} - E_y \xi_{ax} g_x(r) \]  

(15b)

Here again from Equations 15(a) and (b) it is clear that both \(D_x\) and \(D_y\) are functions of position \((r)\) only.

### 3. Metric Element for a Light Ray Traveling Along and Perpendicular to the Equatorial Plane

In this section the effect of the gravitational field of a rotating object on the total amount of rotation of the polarization vector will be discussed. In our case, the electric displacement vector as in Equations 10(a) and (b) (and Equations 15(a) and (b)) have been defined in the Cartesian coordinate system, so the situation demands that we must express the metric \(g\) or the corresponding line element in Cartesian form. Su & Mallett (1980) had suitably parameterized the Kerr line element to look for symmetry and to find a solution to the problem. They considered two geometries, the light ray passing parallel to the rotation axis and the ray passing along the equatorial plane. The Kerr field line element is generally expressed in spherical coordinates as given below (Kerr 1963; Visser 2007),

\[ ds^2 = \sum \frac{r^2}{\Delta} \left[ dt^2 - a \sin^2 \theta d\phi^2 \right] \]

\[ + \frac{\sin^2 \theta}{\Delta} \left[ (r^2 + a^2) d\phi - a dt \right]^2 + \Delta d\theta^2 \]  

(16)

where \(\Delta = r^2 - r_g r + a^2\) and \(\Sigma = r^2 + a^2 \cos^2 \theta\). Below we shall consider the above two cases of geometry and evaluate the metric in Cartesian form for the two cases separately.

#### 3.1. Light Ray Traveling Along the Equatorial Plane

In this case, we have taken the light ray to be contained in the equatorial plane of the system, so \(\sin \theta = 1\) and \(\cos \theta = 0\). Taking this into account, we have \(\Sigma = r^2\). Thus the above Equation (16) can be rewritten as

\[ ds^2 = \left( 1 - \frac{r_g}{r^2 + y^2} \right) dt^2 \]

\[ + \left[ \frac{x^2}{(r^2 - r_g r + a^2)} + \frac{y^2 (2a^2 + 2r_g r + r^4 - r^2)}{r^6} \right] dx^2 \]

\[ + \left[ \frac{y^2}{(r^2 - r_g r + a^2)} + \frac{x^2 (2a^2 + 2r_g r + r^4 - r^2)}{r^6} \right] dy^2 \]

\[ + 2 \frac{r_g a y}{(x^2 + y^2)^2} dx dy - 2 \frac{r_g a x}{(x^2 + y^2)^2} dx dy + 2 xy \]

\[ \times \left( \frac{1}{r^2 - r_g r + a^2} - \frac{2a^2 + 2r_g r + r^4 - r^2}{r^6} \right) dx dy. \]

(17)

So the metric for the Kerr line element will be,

\[
\begin{pmatrix}
-1 & \frac{r_g a y}{(x^2 + y^2)^2} & \frac{y^2}{r^2 - r_g r + a^2} \\
\frac{r_g a y}{(x^2 + y^2)^2} & -1 & \frac{x^2}{r^2 - r_g r + a^2} \\
\frac{y^2}{r^2 - r_g r + a^2} & \frac{x^2}{r^2 - r_g r + a^2} & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]  

(18)
From Equation (18), $g_{00}$ and $g_\ell$ (or $g_{0\ell}$) can be easily obtained as

$$g_{00} = \left(1 - \frac{r_g}{\sqrt{x^2 + y^2}}\right)$$  \hspace{1cm} (19)

$$g_\ell = \frac{r_g a y}{(x^2 + y^2)^{3/2}}$$  \hspace{1cm} (20)

The spatial component $\gamma_{0\beta}$ (Landau & Lifshitz 1998) could be extracted from Equation (17) and written as

$$\gamma_{0\beta} = \begin{pmatrix}
\frac{x^2}{r^2 - r_g r + a^2} + \frac{y^2 (2a^2 + r_g r + r^4 - r^2)}{r^6} \\
\frac{1}{r^2 - r_g r + a^2} - \frac{2a^2 + r_g r + r^4 - r^2}{r^6} \\
0
\end{pmatrix}
$$

For the geometry given in Figure 1 for the impact parameter $b$, we can write (approximately)

$$r^2 = x^2 + b^2.$$  \hspace{1cm} (22)

As the amount of deflection is small, we have used $y = b$ for the entire trajectory of light. We also note $r \gg b$. Applying the result for sufficiently large $r$ in Equations 12(a) and (b), we can say that the displacement vector ($D$) is a function of $x$ only. So the displacement vector at point ($x + \Delta x$) will be

$$D_x = E_z \left(\frac{\xi}{\sqrt{1 - \frac{r_g}{\sqrt{(x+\Delta x)^2 + b^2}}}} + \frac{r_g ab}{(\sqrt{(x+\Delta x)^2 + b^2})^3}\right)$$  \hspace{1cm} (23a)

$$D_z = E_z \left(\frac{1}{\sqrt{1 - \frac{r_g}{\sqrt{(x+\Delta x)^2 + b^2}}}} + \frac{\xi r_g ab}{(\sqrt{(x+\Delta x)^2 + b^2})^3}\right)$$  \hspace{1cm} (23b)

Thus the change in $D_x$ is $D_x(x + \Delta x) - D_x(x)$ for the change in the position coordinate $\Delta x$. Therefore, the rate of change would be given as

$$\lim_{\Delta x \to 0} \frac{D_x(x + \Delta x) - D_x(x)}{\Delta x} = E_z \lim_{\Delta x \to 0} \frac{\xi}{\sqrt{1 - \frac{r_g}{\sqrt{(x+\Delta x)^2 + b^2}}}} + \frac{r_g ab}{(\sqrt{(x+\Delta x)^2 + b^2})^3} - \frac{\xi}{\sqrt{1 - \frac{r_g}{\sqrt{(x)^2 + b^2}}}} + \frac{r_g ab}{(\sqrt{(x)^2 + b^2})^3}.$$  \hspace{1cm} (24)

We note here that $E_x$ and $E_z$ are independent of $x$. After simplifying, the rate of change of the electric displacement vector along the Y-axis can be calculated as

$$\frac{dD_y}{dx} = E_z \left[\frac{\xi}{2 \sqrt{x^2 + b^2}} \frac{x}{\sqrt{x^2 + b^2} - r_g} \left(\frac{1}{\sqrt{x^2 + b^2}} - \frac{1}{\sqrt{x^2 + b^2} - r_g}\right) - 3r_g ab \frac{x}{(x^2 + b^2)^{3/2}}\right]$$  \hspace{1cm} (25)

It is easy to observe that the function $\frac{dD_y}{dx}$ is an even function so the total change of the electric displacement vector from the initial point ($r = -\infty$) to the final point ($r = \infty$) will be

$$\Delta D_y \bigg|_{r=-\infty}^{r=\infty} = 2 \int_{0}^{\infty} E_z \left[\frac{\xi}{2 \sqrt{x^2 + b^2}} \frac{x}{\sqrt{x^2 + b^2} - r_g} \times \left(\frac{1}{\sqrt{x^2 + b^2} - r_g}\right) - 3r_g ab \frac{x}{(x^2 + b^2)^{3/2}}\right] dx$$

$$= 2 \int_{0}^{\infty} E_z \frac{r_g}{2} \frac{x}{(x^2 + b^2)^{3/2}} \frac{x}{\sqrt{x^2 + b^2} - r_g} dx$$

$$= 2E_z \xi \left[-\sqrt{\frac{x^2 + b^2}{\sqrt{x^2 + b^2} - r_g}} + \frac{r_g ab}{(x^2 + b^2)^{3/2}}\right]_0^{\infty}$$

$$= 2E_z \xi \left[-1 + \frac{\sqrt{b}}{\sqrt{b} - r_g} - \frac{r_g a}{b^2}\right]$$  \hspace{1cm} (26)
From Equation (26), we can get the electric vector at the final point \( r = +\infty \) by adding the total change \( \Delta D_y \) to the initial value of the electric vector as
\[
D_y(x = +\infty) = E_y + \Delta D_y \bigg|_{x=-\infty} = E_y \xi \left[ 1 + 2 \left( -1 + \frac{\sqrt{b^2 - r_y^2}}{b^2} - \frac{r_y a}{b^2} \right) \right].
\]

Similarly, from Equation 23(b), we can find the expression for the electric vector along the Z-axis as
\[
D_z(x = +\infty) = E_z + \Delta D_z \bigg|_{x=-\infty} = E_z \left[ 1 + 2 \left( -1 + \frac{\sqrt{b^2 - r_y^2}}{b^2} + \frac{r_y a}{b^2} \right) \right].
\]

It is also clear from the definition of the problem that \( D_y(x = \infty) \) and \( D_z(x = \infty) \) are nothing but the components of the electric field at an asymptotically flat space \( (x = \infty) \). So the total rotation of the polarization vector for the light ray traveling along the equatorial plane would be (combining Equations (10), (11), (27), and (28)),
\[
\Delta \chi_{eq} = \arctan(\xi) - \arctan \left( \frac{D_y(at \ r = \infty)}{D_z(at \ r = \infty)} \right)
= \arctan(\xi) - \arctan \left( \frac{1 + 2 \left( -1 + \frac{\sqrt{b^2 - r_y^2}}{b^2} - \frac{r_y a}{b^2} \right)}{1 + 2 \left( -1 + \frac{\sqrt{b^2 - r_y^2}}{b^2} + \frac{r_y a}{b^2} \right)} \right)
= \arctan(\xi) - \arctan \left( \frac{1 - \frac{2\xi r_y^2}{2b^2 - r_y^2}}{1 + \frac{2\xi r_y^2}{2b^2 - r_y^2}} \right)
= \arctan \left( \frac{1 + \frac{\sqrt{b^2 - r_y^2}}{2b^2 - r_y^2}}{1 + \frac{\sqrt{b^2 - r_y^2}}{2b^2 - r_y^2}} \right) + \tan^2(\chi_{eq}) \left( 1 - \frac{2\xi r_y^2}{2b^2 - r_y^2} \right).
\]

Equation (29) gives the general expression for the rotation of the polarization vector of a light ray traveling along the equatorial plane of a Kerr mass, where \( \chi_{eq} \) is the initial value of the position angle of the polarization (electric vector) when the light leaves its source at \( r = -\infty \). We can clearly see that when we substitute \( a = 0 \) (representing a Schwarzschild field), the value of \( \Delta \chi_{eq} \) becomes zero from Equation (29). This confirms that for Schwarzschild geometry, there will be no rotation of the polarization vector.

### 3.2. Light Ray Traveling Parallel to the Rotation Axis

Now, we consider the case when the light ray propagates parallel to the rotation axis, i.e., the Z-axis according to our definition. In this case, we could consider that the light ray is in the \( Y-Z \) plane and the choice of the X-axis becomes arbitrary. Looking back to the Boyer–Lindquist equation, Equation (16), one would have \( d\phi = 0 \) and Equation (16) will be reduced to
\[
ds^2 = \frac{\Delta}{\Sigma} dr^2 - \frac{2}{\Sigma} \frac{\Delta}{\Sigma} r dr dz + \frac{d\theta^2}{\sin^2\theta} + \frac{d\psi^2}{(d\tau)^2} + \frac{d\phi^2}{(d\psi)^2}.
\]

where \( \Sigma \) and \( \Delta \) are previously defined in Equation (16). Following the same steps as shown in Section 3.1, one could simplify Equation (30) as
\[
ds^2 = -\left( \frac{\Delta}{\Sigma} + \frac{\gamma^2 a^2}{r^2\Sigma^2} \right) dt^2 + \left( \frac{\Sigma y^2}{\Sigma} + \frac{\Sigma (1 - \frac{\gamma^2}{r^2})}{\Sigma} \right) dy^2 + \left( \frac{\Sigma y^2}{\Delta r} - \frac{\gamma^2}{r} \right) dz^2 + 2 \left( \frac{\Sigma y^2}{\Delta r} + \frac{\Sigma (1 - \frac{\gamma^2}{r})}{\Sigma} \right) dydz.
\]

Following the same assumption as in Section 3.1, for the large value of \( r(\gg b) \), impact parameter) we can write
\[r^2 = z^2 + b^2.\]

But from Equation (31), we see that here \( g_{\alpha\beta} = g_{\alpha\alpha} = 0 \), so from Equation (14), with the help of Equations 15(a) and (b), we can write
\[
\tilde{\chi}_{\alpha\psi} = \arctan \left( \frac{E_z}{E_y} \right) = \chi_{\alpha\psi}.
\]

Therefore, there will be no change in the position of the polarization angle \( \chi \) when the electromagnetic radiation propagates parallel to the rotational axis of a rotating mass.

### 4. SOME SAMPLE CALCULATIONS FOR THE ROTATION OF THE POLARIZATION VECTOR WHEN THE LIGHT RAY TRAVELS ALONG THE EQUATORIAL PLANE

In order to study the variation of \( \Delta \chi_{eq} \) with respect to \( \chi_{eq} \), we take its first-order derivative from Equation (29). Thus we see \( \Delta \chi_{eq} \) becomes an extremum when \( \frac{d\Delta \chi_{eq}}{d\chi_{eq}} = 0 \). Then we...
have
\[
\frac{d\chi_{eq}}{d\chi_{eq}} = \sec^2 \chi \left[ \frac{4r_g}{b^2 \sqrt{\sqrt{b} - r_g}} \left( a + \frac{4r_g}{b^2 \sqrt{\sqrt{b} - r_g}} \right) \right] - \frac{4r_g}{b^2 \sqrt{\sqrt{b} - r_g}} a \tan^2(\chi_{eq}) \left( 1 - \frac{4r_g}{b^2 \sqrt{\sqrt{b} - r_g}} \right) = 0.
\]
\]
(33)

From Equation (33), we can see that \(\sec \chi_{eq}\) cannot be zero, and setting the other part of the equation to zero we have
\[
\chi_{eq} = \sqrt{\frac{1 - \frac{4r_g}{b^2 \sqrt{\sqrt{b} - r_g}}}{1 + \frac{4r_g}{b^2 \sqrt{\sqrt{b} - r_g}}} a}.
\]
\]
(34)

From Equation (34) we can determine the value of \(\chi_{eq}\) for which the rotation in position angle will be an extremum. Taking the second derivative, we can show that this value represented by Equation (34) gives the condition for a maximum.

4.1. Case I: \(E_y = 0\) i.e., \(\xi = 0\) or \(\chi_{eq} = 0\)

If we consider \(E_y = 0\), then \(\xi\) will be zero. Now from Equation (29) the total rotation of the polarization vector will be zero.

4.2. Case II: \(E_y = E_x\) i.e., \(\xi = 1\) or \(\chi_{eq} = 45^\circ\)

Assuming that both components are equal, the initial position angle of the polarization vector will be \(45^\circ\). In this case, from Equation (29) we could write
\[
\Delta \chi_{eq} = \arctan \frac{2 \frac{c a}{b^2}}{2 \sqrt[4]{\sqrt{b} - r_g} - 1}.
\]
\]
(35)

The Sun is rotating with an angular speed of \(1.664 \times 10^{-40}\) s\(^{-1}\) (degrees per second) and it has a moment of inertia \(5.8 \times 10^{36}\) kg m\(^2\). Accordingly, we can calculate the rotational parameter \(a\) for the Sun to be 1.69749 km. Considering the impact parameter \(b\) of the Sun to be the physical radius of the Sun \((0.7 \times 10^6\) km), from Equation (35) we can estimate the value of \(\Delta \chi_{eq}\) for the Sun as 4.2873 \(\mu\)as. One can say that there will be no observable change for the rotation of the polarization vector in the case of the Sun. However, when we apply Equation (35) to other fast rotating compact objects, we find the result to be quite fascinating. For these compact rotating objects, the light ray passes closer to the center of mass and stronger field is experienced by light ray. So we obtain much higher values for the total rotation of the polarization vector. We choose the cases of two millisecond pulsars, PSR J1748-2446ad (Hessels et al. 2006; Nunez & Nowakowski 2010; Dubey & Sen 2014) and PSR B1937+21 (Ashworth et al. 1983; Nunez & Nowakowski 2010; Dubey & Sen 2014), for which the \(r_g\) and \(a\) values are \((4.050\) km, \(2.42325\) km\) and \((4.050\) km, \(2.19438\) km\), respectively. These two pulsars have been chosen arbitrarily, but we tried our best to choose pulsars whose Schwarzschild radii \((r_g)\) and rotation parameters \((a)\) values are very close to that of the Sun, so that the comparison will be easy. We take their impact parameters to be the physical radii, which are \(20.1\) km (Hessels et al. 2006) and \(20.2\) km (Ashworth et al. 1983), respectively. Using Equation (35), we find the values of \(\Delta \chi_{eq}\) to be \(2^\circ.2471\) and \(2^\circ.0172\), for the two pulsars PSR J1748-2446ad and PSR B1937+21, respectively. So the results are quite appreciable. These results have been listed in Table 1. In Figure 2, we plot \(\Delta \chi_{eq}\) against \(\chi_{eq}\) for the pulsar PSR J1748-2446ad for various values of impact parameters from the center of mass. The impact parameter values were taken as \(1 \times, 1.5 \times, \) and \(2 \times\) of the physical radius. We can clearly see the dependence of the rotation of the polarization vector \(\Delta \chi_{eq}\) on the initial value of the polarization vector \(\chi_{eq}\). We observe this from Figure 2, where the rotation value attains a maximum when the initial position angle is approximately \(47^\circ\). From Equation (34) also, we get the same value of \(47^\circ.25\) for the rotation value to be maximum.

As a further illustration of our calculation, we now consider prograde and retrograde orbits of the light ray with respect to the rotation axis of the gravitational (or Kerr) mass. In the case of prograde orbit, the equation would be the same as Equation (29). But for the reverse or retrograde orbit, the rotational parameter \((\alpha)\) would be negative in Equation (29). Thus, the expressions for rotation of the polarization vector would be written as
\[
\Delta \chi_{eq}(\text{pro}) = \arctan \frac{2 \frac{c a}{b^2}}{2 \sqrt[4]{\sqrt{b} - r_g} - 1},
\]
\]
(36)

and
\[
\Delta \chi_{eq}(\text{retro}) = - \arctan \frac{2 \frac{c a}{b^2}}{2 \sqrt[4]{\sqrt{b} - r_g} - 1}.
\]
\]
(37)

| Name of Stars | \(r_g\) (km) | \(R\) (km) | \(a\) (km) | \(\Delta \chi_{eq}\) |
|---------------|--------------|------------|-----------|----------------|
| Sun           | 3.000        | 0.7 \times 10^6 | 1.69749   | 4^\circ 2873 \times 10^{-6} |
| PSR J1748-2446ad | 4.050        | 20.1       | 2.42325   | 2^\circ 2471 |
| Hessels et al. (2006) |          |           |          |                |
| PSR B1937+21 | 4.050        | 20.2       | 2.19438   | 2^\circ 0172 |
| Ashworth et al. (1983) | | | | |

**Note.** Original \(\chi_{eq} = 45^\circ, \xi = 1\).
We note that Equation (36) is the same as Equation (35) obtained earlier, for prograde orbit. Calculated values for prograde and retrograde orbits differ only in sign or direction.

4.3. Case III: $E_x = 0$, i.e., $\xi = \infty$

Consider that only the $Z$ component is present between two electric field components ($\chi_{eq} = 90^\circ$). Going back to Equation (29), we can show that $\lim_{\xi \to \infty} \Delta \chi_{eq} = 0$ by applying L’Hospital’s rule. This is also evident from Figure 2. So in this case there will be no rotation of the polarization vector. At this stage, it will be worth to make a comparison with the work of Su & Mallett (1980). They found that for an asymptotic flat observer (GIF case), the rotation of the polarization vector is solely due to the frame dragging effect. This is in perfect agreement with our observation in the present work. Further, Su & Mallett (1980) considered distances beyond $r_g$ and obtained the amount of rotation to be up to a few degrees. This is also grossly in agreement with what we have observed. An exact comparison between the approaches and results obtained by Su & Mallett (1980) with those we obtained is not possible due to several reasons. The former authors obtained numerical results by running some specific codes and sometimes a very high degree of approximation was applied. Also it appears that they limited their rotation parameter values (in the Kerr line element) to be lower than the Schwarzschild impact parameter. In addition, they obtained solutions for the cases where the impact parameter (distance of closest approach of the light ray to the massive body) is at least six times the Schwarzschild radius. Further, the authors also limited all their calculations to an initial condition where the polarization of the light ray is $45^\circ$ with respect to the equatorial plane.

In our approach, all the above three restrictions are relaxed and the solutions that we obtained are more general in nature. However, the underlying physics in the two approaches appear to be the same. This is obvious from Equation (4.27) of Su & Mallett (1980) and our Equation (11), where the rotation angle (of the polarization vector) depends on the same set of metric elements and the mathematical nature of the dependence is the same. In spite of the finer differences between the two approaches, we can make an attempt to compare the results obtained by the two methods as follows. As can be seen from Figure 3 of Su & Mallett (1980), the rotation angle is about $2^\circ.5$ for GIF when the rotation parameter $= r_g$ and impact parameter $= 6r_g$. With these same set of parameters, we can calculate from our Equation (35) the rotation of the polarization vector to be about $2^\circ.69$. Since our work has fewer assumptions and approximations, we claim that the results we obtained are more accurate. In addition, the present work is more general in nature, direct, and is more flexible in accommodating different geometries.

5. POLARIZATION INTRODUCED IN THE LIGHT RAY DUE TO CURVED SPACETIME

In this section, we explore whether or not the effect of curved spacetime introduces any polarization to the light ray propagating through it. Now from the definition of Stokes parameter (Stokes 1852; Born & Wolf 1999) for a linearly polarized electromagnetic wave, one can write the expression for polarization ($P$) in terms of the Stokes vectors (Stokes 1852) as

\[ I = \langle D_x^2 \rangle + \langle D_z^2 \rangle \]  
\[ Q = \langle D_x^2 \rangle - \langle D_z^2 \rangle \]  
\[ U = 2 \langle D_x D_z \cos \delta \rangle \]

where $\delta$ is the phase difference between $D_x$ and $D_z$ and $\langle . . \rangle$ indicates the time average. In the present work, it is clear that
both components $D_\nu$ and $D_\lambda$ are in the same phase. From Equations (10a) and (b) we can calculate $D_\nu$ and $D_\lambda$, where it has been assumed that the light is linearly polarized. So from the above definition of polarization, we can write (omitting the (..) sign everywhere for convenience)

$$P = \frac{\sqrt{Q^2 + U^2}}{I} = \frac{\sqrt{(D_\nu^2 - D_\lambda^2)^2 + (2D_\nu D_\lambda)^2}}{D_\nu^2 + D_\lambda^2} = 1. \quad (39)$$

Thus we find that the light remains 100% plane polarized, as it was before entering the gravitational field. As a result, one can conclude here that the curved spacetime geometry (of the Kerr field) does not introduce any new polarization to the electromagnetic wave (or light). However, the gravitational field just rotates the direction of the polarization vector (or electric vector).

6. RESULT AND DISCUSSIONS

The above work clearly demonstrated and systematically calculated the effect of curved spacetime on the polarization states of a light ray. From this work, one can easily come to the conclusion that there is no effect on polarization due to the gravitational field of a static non-rotating body. For example, the Schwarzschild field will have no effect on the polarization of the electromagnetic field, which is very obvious and supports previous works by several authors.

The present work clearly indicates, however, that the gravitational field generated by a rotating body would affect the direction of the polarization vector of an electromagnetic wave. The work presented here provides an analytical expression to determine the amount of rotation of the polarization vector. This expression is much simpler and more straightforward compared to the ones obtained by previous authors, viz. Su & Mallett (1980), which requires many approximations and numerical techniques to obtain the exact value of rotation. Calculations carried out here, for several pulsars as test cases, demonstrate that the result is appreciable and it is on the order of a few arcdegrees, which can be measured by present day telescopes. However, for the Sun the calculated rotation is on the order of a few micro-arcseconds, and this can be measured by new generation telescopes and polarimeters. Also, based on the calculations presented here, conclusions can be drawn that the gravitational field would not produce or introduce any polarization in the light, and it will only rotate the direction of the existing polarization vector.

Our present study may also open new avenues for indirect detection of gravitational waves (Abbott et al. 2016). This may be possible in two different ways: (i) the electromagnetic counterpart of a gravitational wave may be emitted in a way involving axial symmetry (viz., binary black hole coalescence; Abbott et al. 2016), which will cause the rotation of the polarization vector of an electromagnetic wave passing close by; and/or (ii) the electromagnetic wave passing through a region of gravitational wave will be lensed (Sbytov 1973), resulting in a good possibility of the rotation of its polarization vector, especially if the gravitational wave is plane polarized. However, these are only possibilities, which we would like to explore in our future work.

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REFERENCES

Abbott, B., Abbott, R., Abbott, T., et al. 2016, PhRvL, 116, 061102
Ashworth, M., Lyne, A. G., & Smith, F. G. 1983, Natur, 301, 313
Balazs, N. L. 1958, PhRv, 110, 236
Born, M., & Wolf, E. 1999, Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light (Cambridge: Cambridge Univ. Press)
Carroll, S. M. 2004, Spacetime and Geometry. An Introduction to General Relativity, Vol. 1 (San Francisco, CA: Pearson)
Dubey, A. K., & Sen, A. K. 2014, IJTP, 54, 2398
Fernández, R., & Davis, S. W. 2011, ApJ, 730, 131
Frolov, V. P., & Shoun, A. A. 2011, PPhRvD, 84, 044026
Hessels, J. W., Ransom, S. M., Stairs, I. H., et al. 2006, Sci, 311, 1901
Jackson, J. D. 1962, Classical Electrodynamics, Vol. 3 (New York: Wiley)
Kerr, R. 1963, PhRvL, 11, 237
Landau, L. D., & Lifshitz, E. M. 1998, The Classical Theory of Fields, Vol. 2 (Portsmouth, NH: Butterworth Heinemann)
Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973, Gravitation (San Francisco: Freeman)
Nunez, P. D., & Nowakowski, M. 2010, JApA, 31, 105
Paczynski, B. 1986, ApJ, 304, 1
Pineault, S., & Roeder, R. 1977, ApJ, 213, 548
Plebanski, J. 1960, PhRv, 118, 1396
Roy, S., & Sen, A. K. 2015, Ap&SS, 360, 23
Sbytov, Y. G. 1973, JETP, 36, 387
Sen, A. K. 2010, Ap, 53, 560
Sereno, M. 2004, PhRvD, 69, 087501
Stephani, H. 2004, Relativity: An Introduction to Special and General Relativity (Cambridge: Cambridge Univ. Press)
Stokes, G. G. 1852, TCaPS, 9, 399
Su, F., & Mallett, R. 1980, ApJ, 238, 1111
Vilenkin, A. 1984, ApJL, 282, L51
Visser, M. 2007, arXiv:0706.0622
Weinberg, S. 1972, Gravitation and Cosmology: Principle and Applications of General Theory of Relativity (New York: Wiley)