Abstract Through Borel summation methods, we analyze two different variations of the Navier-Stokes equation—the Boussinesq equation for fluid motion and temperature field and the magnetic Bénard equation which approximates electro-magnetic effects on fluid flow under some simplifying assumptions. In the Boussinesq equation,

\begin{equation}
\begin{aligned}
    u_t - \nu \Delta u & = -P[u \cdot \nabla u - a\xi^2 \Theta] + f \\
    \Theta_t - \mu \Delta \Theta & = -u \cdot \nabla \Theta,
\end{aligned}
\end{equation}

where $d = 2$ or $3$ is the dimension, $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$, and $\Theta : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$. For the magnetic Bénard equation,

\begin{equation}
\begin{aligned}
    v_t - \nu \Delta v & = -P[v \cdot \nabla v - \frac{1}{\mu \rho} B \cdot \nabla B] + f \\
    B_t - \frac{1}{\mu \sigma} \Delta B & = -P[v \cdot \nabla B - B \cdot \nabla v],
\end{aligned}
\end{equation}

where $v, B : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$.

This method has previously been applied to the Navier-Stokes equation in [5], [7], and [8]. We show that this approach can be used to show local existence for the Boussinesq and magnetic Bénard equation, either for $d = 2$. 

H. Rosenblatt
Department of Mathematics: The Ohio State University
222 Math Tower 231 West 18th Avenue Columbus, OH 43210-1174
Tel.: 614-292-7648
Fax: 614-292-1479
E-mail: rosenblatt@math.ohio-state.edu

S. Tanveer
402 Math Tower 231 West 18th Avenue Columbus, OH 43210-1174
Tel.: 614-292-5710
E-mail: tanveer@math.ohio-state.edu
or $d = 3$. We prove that an equivalent system of integral equations in each case has a unique solution, which is exponentially bounded for $p \in \mathbb{R}^+$, $p$ being the Laplace dual variable of $1/t$. This implies the local existence of a classical solution to (1) and (2) in a complex $t$-region that includes a real positive time ($t$)-axis segment. Further, it is shown that within this real time interval, for analytic initial data and forcing, the solution remains analytic and has the same analyticity strip width. Further, under these conditions, the solution is Borel summable, implying that that formal series in time is Gevrey-1 asymptotic for small $t$. We also determine conditions on the integral equation solution in each case over a finite interval $[0, p_0]$ that result in a better estimate for existence time of the PDE solution.

1 Introduction

We consider two variations of the incompressible Navier-Stokes equation. In the first case, we consider the coupling of temperature field with fluid flow under the assumption that the temperature induced changes in density have negligible effects on momentum, but cause a significant buoyant force. The corresponding Boussinesq equation for $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and $\Theta : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ with $d = 2, 3$ are

$$u_t - \nu \Delta u = -P[u \cdot \nabla u - ae_2 \Theta] + f, \quad u(x, 0) = u_0(x) \quad (3)$$

$$\Theta_t - \mu \Delta \Theta = -u \cdot \nabla \Theta, \quad \Theta(x, 0) = \Theta_0(x)$$

where $P = I - \nabla \Delta^{-1}(\nabla \cdot)$ is the Hodge projection operator to the space of divergence free vector fields and $e_2$ is the unit vector aligned opposite to gravity and parameter $a$ is proportional to gravity. Here $(u, \Theta)$ corresponds to the fluid velocity and temperature field. Using standard energy methods, see for instance [17], existence of Leray type solutions in $L^\infty(0, T, L^2(\mathbb{R}^d)) \cap L^2(0, T, H^1(\mathbb{R}^d))$ follows easily for any $T > 0$. In $\mathbb{R}^2$ a unique classical global solution can be shown to exist for all time. Further, in $\mathbb{R}^3$ there is a unique solution under the additional assumption that the solution lies in $L^\infty(0, T, H^1(\mathbb{R}^3))$. In [2], local existence and uniqueness for Boussinesq equation are shown in $L^p(0, T, L^q(\mathbb{R}^d))$ for $d < p < \infty$ and $\frac{2d}{p} + \frac{d}{q} \leq 1$.

For the second problem, we study the the viscous magnetic Bénard equation, or MHD equation, which arises in the motion of a magnetic fluid in situations where displacement current and charge density variations are negligible [4]. The equations for $v, B : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ are

$$v_t - \nu \Delta v = -P[v \cdot \nabla v - \frac{1}{\mu \rho} B \cdot \nabla B] + f, \quad v(x, 0) = v_0(x) \quad (4)$$

$$B_t - \frac{1}{\mu \sigma} \Delta B = -P[v \cdot \nabla B - B \cdot \nabla v], \quad B(x, 0) = B_0(x)$$

where $d = 2, 3$ as before, $v$ is the fluid velocity, $B$ is the magnetic field, while $\nu$, $\rho$, $\mu$ and $\sigma$ are constants related to fluid viscosity, density, magnetic permeability and electric conductivity respectively. The question of regularity of solutions to the MHD equation in two and three dimensions has been well
studied. Duraut and Lions [9] constructed a class of global weak solutions and a local class of strong solutions using energy methods in both two and three dimensions. In the two dimensional case, uniqueness and smoothness were established for all time. More generally, Sermange and Temam [16] showed existence in three dimensions in the class $L^\infty(0,T;L^2(\mathbb{R}^d)) \cap L^2(0,T,H^1(\mathbb{R}^d))$ and uniqueness assuming the solution lies in $L^\infty(0,T,H^1(\mathbb{R}^d))$. Many others [10], [11], [18], and [3] have a variety of results improving regularity.

In both the problems above, the existence of classical solutions, globally in time, remains an open problem as it is for the 3-D Navier-Stokes equation. Control of a higher order energy norm (like the $H^1$ norm of velocity) has remained a serious impediment for a long time. This motivates one to look at other formulations of the existence problem that do not rely on energy bounds.

The primary purpose of this paper is to show that the Borel transform methods, developed earlier in [5] and [8] in the context of Navier-Stokes equations, can be extended to determine classical PDE solutions for the Boussinesq and magnetic Bénard equations. This provides an alternate existence and uniqueness theory for a class of nonlinear PDEs for which the question of global existence of solution to the PDE becomes one of asymptotics for known solution to the associated nonlinear integral equations. While this asymptotics problem is difficult and yet to be resolved, it is shown (Thm 24) how information about solution on a finite interval in the dual variable for specific initial condition and forcing may be used for obtaining better exponential bounds in the Borel plane and therefore better existence time for classical solutions to the PDEs.

Further, many analyticity properties readily follow from this representation. Time analyticity for $\Re s \frac{1}{t} > \alpha$ follow from the solution representation. We also prove that the classical $H^2(\mathbb{R}^d)$ solution, which is unique, has the Laplace transform representation given here, provided initial data and forcing are in $L^1 \cap L^\infty$ in Fourier space. Furthermore, for analytic initial data and forcing, we prove that the formal expansion in powers of $t$ is Borel summable and hence Gevrey asymptotic for small $t$. In the latter case, it is also shown that the associated power series in the Borel plane has a radius of convergence independent of size of initial data and forcing when initial data and forcing have a fixed number of Fourier modes, this is useful in computing the solution in the Borel plane.

2 Main Results

We first write the equations as integral equations in Fourier space. We denote by $\hat{f}$ the Fourier transform of $f$ and $\hat{f} * \hat{g}$ the Fourier convolution. The Fourier transform operator is denoted by $\mathcal{F}$. As usual, a repeated index $j$ denotes the sum over $j$ from 1 to $d$. $P_k$ is the Fourier transform of the Hodge projection and has the representation

$$ P_k \equiv \left( 1 - \frac{k(k\cdot)}{|k|^2} \right). $$
Moreover, $u$, $v$, and $B$ are divergence free. Formal derivation based on inversion of the heat operator in Fourier space in (3) leads to the following integral equations:

\[ \hat{u}(k,t) = -\int_0^t e^{-\nu|k|^2(t-\tau)} \left( ik_j P_k [\hat{u}_j \ast \hat{u} - a e_2 \hat{\theta}] (k,\tau) - \hat{f}(k) \right) d\tau + e^{-\nu|k|^2t} \hat{u}_0(k) \]  
\[ \hat{\theta}(k,t) = -\int_0^t e^{-\mu|k|^2(t-\tau)} \left( ik_j [\hat{u}_j \ast \hat{\theta}] (k,\tau) \right) d\tau + e^{-\mu|k|^2t} \hat{\theta}_0(k) \]

and, for the magnetic Bénard equation (4), one obtains

\[ \hat{v}(k,t) = -\int_0^t e^{-\nu|k|^2(t-\tau)} \left( ik_j P_k \left[ \hat{v}_j \ast \hat{v} - \frac{1}{\mu \rho} \hat{B}_j \ast \hat{B} \right] (k,\tau) - \hat{\tilde{f}}(k) \right) d\tau + e^{-\nu|k|^2t} \hat{v}_0(k) \]
\[ \hat{B}(k,t) = -\int_0^t e^{-\frac{\nu|k|^2(t-\tau)}{\sigma}} \left( ik_j P_k \left[ \hat{v}_j \ast \hat{B} - \hat{B}_j \ast \hat{v} \right] (k,\tau) \right) d\tau + e^{-\frac{\nu|k|^2t}{\sigma}} \hat{B}_0(k). \]

Remark 21 We may assume the initial conditions $u_0$ in the Boussinesq equation and $v_0$, $B_0$ for the Bénard equation, as well as the forcing $f$ are divergence free, since any non-zero divergence part of $f$ can be included in a gradient term, which has been projected away. We assume $f = f(x)$ to be time independent for simplicity although a time dependent $f$ with some restrictions may be treated in a similar manner. Additional forcing terms on the temperature and magnetic equations can be accommodated in the formalism here.

Definition 22 We introduce the norm $|| \cdot ||_{\gamma,\beta}$ for some $\beta \geq 0$ and $\gamma > d$ by

\[ ||\hat{f}||_{\gamma,\beta} = \sup_{k \in \mathbb{R}^d} (1 + |k|)^{\gamma} e^{\beta|k|} ||\hat{f}(k)||, \text{ where } \hat{f}(k) = \mathcal{F}[f(\cdot)](k). \]

Definition 23 We also use the space $L^1 \cap L^\infty$ with the norm defined by

\[ ||\hat{f}||_{L^1 \cap L^\infty} = \max \left\{ \int_{\mathbb{R}^d} |\hat{f}(k)| dk, \sup_{k \in \mathbb{R}^d} |\hat{f}(k)| \right\}. \]

In the case when results hold either for $|| \cdot ||_{\gamma,\beta}$ or $|| \cdot ||_{L^1 \cap L^\infty}$ norm, we will use $|| \cdot ||_{N}$ for brevity of notation.

We assume $||(1 + |k|)^2(\hat{u}_0, \hat{\theta}_0)||_N < \infty$, $||(1 + |k|)^2(\hat{v}_0, \hat{B}_0)||_N < \infty$, and $||\hat{f}||_N < \infty$ in what follows. If $|| \cdot ||_N = || \cdot ||_{\gamma,\beta}$ and $\beta > 0$ then the initial condition and forcing are real analytic in $x$ in a strip of width at least $\beta$. 

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1. While derivation is formal, in the space of functions where existence is proved, it will be clear the integral and differential formulations are equivalent.
Theorem 21 (Boussinesq Existence and Uniqueness)

If \(|1+\cdot|^2(\hat{u}_0, \hat{\Theta}_0)|_N < \infty\) and \(|f|_N < \infty\), then the following hold.

i) The Boussinesq equation (5) has a solution \((\hat{u}, \hat{\Theta})(k,t)\) such that \(||(\hat{u}, \hat{\Theta})(\cdot, t)||_N < \infty\) for \(\Re \alpha > \omega\) for \(\omega\) sufficiently large. Specifically, (43) holds, where \((\hat{u}_1, \hat{\Theta}_1)\), defined in (44), depends on the initial data and forcing.

ii) The solution has the Laplace transform representation

\[
(\hat{u}, \hat{\Theta})(k,t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k,p)e^{-p/t}dp
\]  

(7)

where \((\hat{H}, \hat{S})\) satisfies a set of integral equations that has a unique solution for \(|(\hat{H}, \hat{S})(\cdot, p)||_N e^{-\omega p} \in L^1(0, \infty)\). From this representation \((u, \Theta)(x,t) = \mathcal{F}^{-1}[(\hat{u}, \hat{\Theta})(k)](x,t)\) is analytic in \(t\) for \(\Re \alpha > \omega\). This implies that if \(\beta > 0\) then \((u, \Theta)\) is analytic in \(x\) in a strip of the same width as the analyticity strip for the initial data and forcing for any \(t \in [0, \omega^{-1})\).

iii) Further for this solution, \(|(1+\cdot|^2(\hat{u}, \hat{\Theta})(\cdot, t)||_N < \infty\) for \(t \in (0, \omega^{-1})\). Moreover, \((u, \Theta)(x,t)\) solves (4) and is the unique solution in \(L^\infty(0, T; H^2(\mathbb{R}^d))\).

In other words, given any solution in \(H^2(\mathbb{R}^d)\) to the Boussinesq equation for which the initial data and forcing satisfy the given assumption then the solution has the representation (7).

iv) A sufficient condition for global existence of smooth solution to the Boussinesq equation is that \(e^{-\omega p}||[(\hat{H}, \hat{S})(\cdot, p)||_N \in L^1(0, \infty)\) for any \(\omega > 0\).

Theorem 22 (MHD Existence and Uniqueness) If \(|(1+\cdot|^2(\hat{v}_0, \hat{B}_0)|_N < \infty\) and \(|f|_N < \infty\), then the following hold.

i) The magnetic Bénard equation (6) has a solution \((\hat{v}, \hat{B})(k,t)\) such that \(||(\hat{v}, \hat{B})(\cdot, t)||_N < \infty\) for \(\Re \alpha > \alpha\) for \(\alpha\) sufficiently large. Specifically, (44) holds, where \((\hat{v}_1, \hat{B}_1)\), defined in (45), depends on the initial data and forcing.

ii) The solution has the Laplace transform representation

\[
(\hat{v}, \hat{B})(k,t) = (\hat{v}_0, \hat{B}_0)(k) + \int_0^\infty (\hat{W}, \hat{Q})(k,p)e^{-p/t}dp
\]  

(8)

where \((\hat{W}, \hat{Q})\) satisfies a set of integral equations that has a unique solution for \(||(\hat{W}, \hat{Q})(\cdot, p)||_N e^{-\omega p} \in L^1(0, \infty)\). From this representation \((v, B)(x,t) = \mathcal{F}^{-1}[(\hat{v}, \hat{B})(k)](x,t)\) is analytic in \(t\) for \(\Re \alpha > \alpha\). This implies that if \(\beta > 0\) then \((v, B)\) is analytic in \(x\) in a strip of the same width as the analyticity strip for the initial data and forcing for any \(t \in [0, \alpha^{-1})\).

iii) Further for this solution, \(|(1+\cdot|^2(\hat{v}, \hat{B})(\cdot, t)||_N < \infty\) for \(t \in (0, \alpha^{-1})\). Moreover, \((v, B)(x,t)\) is the unique solution to (4) in \(L^\infty(0, T; H^2(\mathbb{R}^d))\). In other words, given any solution in \(H^2(\mathbb{R}^d)\) to the MHD equation for which the initial data and forcing satisfy the given assumption then the solution has the representation (8).

iv) A sufficient condition for global existence of smooth solution to the magnetic Bénard equation is that \(e^{-\omega p}||[(\hat{W}, \hat{Q})(\cdot, p)||_N \in L^1(0, \infty)\) for any \(\alpha > 0\).
Remark 24 If the initial condition and forcing are known to be in \( L^1 \) in Fourier space but not necessarily to be in \( L^\infty \), then we have a unique solution to (5) or (6) for which \( \| \hat{u}(0, \Theta) \|_{L^1(\mathbb{R}^d)} < \infty \) and \( \| \hat{v}(0, B) \|_{L^1(\mathbb{R}^d)} < \infty \) for \( t \in (0, \omega^{-1}) \), respectively. This solution is smooth pointwise and solves the corresponding equation (3) or (4).

Remark 25 The guaranteed existence time \( T = \omega^{-1} \) or \( \alpha^{-1} \), depending on the equation being considered, depends on \( \| (1 + | \cdot |)^2 (\hat{u}_0, \Theta_0)() \|_N \) or \( \| (1 + | \cdot |)^2 (\hat{v}_0, B_0)() \|_N \). This condition can be weakened using an accelerated version of the Borel transform as in [8], i.e. using an alternate representation for \( n > 1 \):

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k, q)e^{-q/|\alpha|^2}dq
\]

Remark 26 Using an accelerated variable instead of \( p \), as in [4] for \( n \) sufficiently large, we expect to be able to prove that in the case without forcing for the periodic case \( x \in \mathbb{T}^d \), global solutions of the PDEs implies that the growth rate \( \alpha \) for associated integral equation solution is arbitrarily small, a result already shown for 3-D Navier-Stokes [8].

Theorem 23 (Borel Summability) i) For analytic initial data and forcing and \( \beta > 0 \) the solution to the Boussinesq equation, \( (u, \Theta) \), and the solution to the magnetic Bénard equation, \( (v, B) \), are Borel summable in \( t \). That is there exists \( (H, S)(x, p) \) and \( (W, Q)(x, p) \) analytic in a neighborhood of \( \{0\} \cup \mathbb{R}^+ \), exponentially bounded and analytic in \( x \) for \( |\text{Im}(x)| < \beta \) such that

\[
(u, \Theta)(x, t) = (u_0, \Theta_0)(x) + \int_0^\infty (H, S)(x, p)e^{-p/\beta}dp
\]

and

\[
(v, B)(x, t) = (v_0, B_0)(x) + \int_0^\infty (W, Q)(x, p)e^{-p/\beta}dp.
\]

In particularly by Watson’s Lemma, as \( t \to 0^+ \)

\[
(u, \Theta)(x, t) \sim (u_0, \Theta_0)(x) + \sum_{m=1}^\infty (u_m, \Theta_m)(x)t^m
\]

and

\[
(v, B)(x, t) \sim (v_0, B_0)(x) + \sum_{m=1}^\infty (v_m, B_m)(x)t^m,
\]

where \( |(u_m, \Theta_m)(x)| \leq m!A_0D_0^m \) and \( |(v_m, B_m)(x)| \leq m!\tilde{A}_0\tilde{D}_0^m \) with constants \( A_0, \tilde{A}_0, D_0 \) and \( \tilde{D}_0 \) generally dependent on the initial condition and forcing through Lemma [7].

ii) Further, if analytic initial data and forcing have only a finite number of Fourier modes and \( \beta > 0 \), the solutions \( (\hat{H}, \hat{S})(k, p) \) and \( (\hat{W}, \hat{Q})(k, p) \) have
radii of convergence independent of the size of the initial data and forcing. In particular, constants $A_0$, $\hat{A}_0$ depend on the initial condition and forcing and constants $D_0$, and $\hat{D}_0$ depend on the number of Fourier modes of the initial condition and forcing but are independent of the size of initial data and forcing.

Remark 27 In the case $\beta > 0$, we do not need the restriction $\gamma > d$. If $||\hat{u}||_{\gamma,\beta} < \infty$, then for $\beta' \in (0, \beta)$ we have for any $n \in \mathbb{N}$, $||\hat{u}||_{\gamma+n,\beta'} < \infty$.

Remark 28 Besides the nature of early time asymptotics, the finite radius of convergence of the series in $p$ being independent of size of initial condition, at least for data with finite Fourier modes, helps determine the solution in $[0, p_0]$. Knowledge of the solution on $[0, p_0]$ can be exploited (as in the following Theorem 24) to compute a revised estimate on $\omega$ and $\alpha$ for specific initial data and forcing.

Let $(\hat{H}, \hat{S})(k, p)$ be the solution to (23) provided by Lemma 57. Define

$$(\hat{H}, \hat{S})(a)(k, p) = \begin{cases} (\hat{H}, \hat{S})(k, p) & \text{for } p \in (0, p_0) \subset \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{H}^{(s)}(k, p) = \frac{ik_j \pi}{2|k|\sqrt{c^p}} \int_0^{\min(p, 2p_0)} G(z, z') \hat{G}_j^{(1), (a)}(k, p') dp' + 2\hat{u}_1(k) \frac{J_1(2|k|\sqrt{c^p})}{2|k|\sqrt{c^p}}$$

$$+ \frac{a \pi}{2|k|\sqrt{c^p}} \int_0^{\min(p, p_0)} G(z, z') P_k[e_2 \hat{S}^{(a)}(k, p')] dp'$$

$$\hat{S}^{(s)}(k, p) = \frac{ik_j \pi}{2|k|\sqrt{c^p}} \int_0^{\min(p, 2p_0)} G(z, z') \hat{G}_j^{(2), (a)}(k, p') dp' + 2\hat{\Theta}_1(k) \frac{J_1(2|k|\sqrt{c^p})}{2|k|\sqrt{c^p}}$$

where

$$\hat{G}_j^{(1), (a)}(k, p) = -P_k[\hat{u}_0, j * \hat{H}^{(a)}(k)] + \hat{H}_j^{(a)} * \hat{u}_0 + \hat{H}_j^{(a)} * \hat{S}^{(a)}$$

$$\hat{G}_j^{(2), (a)}(k, p) = -[\hat{u}_0, j * \hat{S}^{(a)}] + \hat{H}_j^{(a)} * \hat{\Theta}_0 + \hat{S}_j^{(a)} * \hat{S}^{(a)}.$$
where

\[ B_0(k) = C_0 \sup_{p_0 \leq p} \left| G(z, z')/z \right|, \quad B_1 = 2 \sup_{k \in \mathbb{R}^d} |k| |B_0(k)||\hat{u}_0, \hat{\Theta}_0||_N, \]

\[ B_2 = 2 \sup_{k \in \mathbb{R}^d} |k| B_0(k)||\hat{G}(z, z')/z||, \quad B_3 = \sup_{k \in \mathbb{R}^d} |k| B_0(k), \quad B_4 = a \sup_{k \in \mathbb{R}^d} B_0(k). \]

Then, over an extended interval, the solution satisfies the relation

\[ \| (\hat{H}(\cdot, p), \hat{S}(\cdot, p)) \|_N \in L^1 \left( e^{-\omega p} dp \right) \]

for any \( \omega \geq \omega_0 \) satisfying

\[ \omega > \epsilon_1 + 2 \sqrt{B_3} \epsilon, \]

where \( f \in L^1(e^{-\omega p} dp) \) means \( \int_0^\infty |f(p)| e^{-\omega p} dp < \infty. \)

**Remark 29** This means that if the solution \((\hat{H}, \hat{S})\), restricted to \([0, p_0]\), to the integral equation equivalent to the Boussinesq equation is known, through computation of power series in \( p \) or otherwise, and the corresponding functionals \( \epsilon \) and \( B_3 \epsilon \) are small, as is the case for sufficiently rapidly decaying \((\hat{H}, \hat{S})\) over a large enough interval \([0, p_0]\), then a long time interval of existence \((0, \omega^{-1})\) for classical solutions to Boussinesq equation is guaranteed. A specific choice of \( \omega_0 \) may be made to optimize the lower bound on \( \omega \) in the above calculations. The point of Theorem 24 is that solutions to the integral equation over a finite interval in \( p \) (either in the form of a Taylor series in \( p \), as appropriate for analytic data and initial conditions, or in the form of numerical calculations, where rigorous error control are expected similar to 3-D Navier-Stokes [8]) can lead to a revised asymptotic bounds on \( \omega \) which translates into a longer existence time for the PDE.

**Remark 210** A similar result holds for the magnetic Bénard equation with the obvious changes.

### 3 Formulation of Integral Equation: Borel Transform

Our goal is to take the Borel transform and create equivalent integral equations. To ensure decay in \( 1/t \) and avoid dealing with delta distribution when applying the Borel transform in \( 1/t \), it is convenient to define \( \hat{h}, \hat{w}, \hat{s}, \) and \( \hat{q} \) so that

\[ \hat{u}(k, t) = \hat{u}_0(k) + \hat{h}(k, t) \]

\[ \hat{\Theta}(k, t) = \hat{\Theta}_0(k) + \hat{s}(k, t) \]

\[ \hat{v}(k, t) = \hat{v}_0(k) + \hat{w}(k, t) \]

\[ \hat{B}(k, t) = \hat{B}_0(k) + \hat{q}(k, t). \]

For \( \hat{\delta}^{[1]} \), we define

\[ \hat{\delta}^{[1]} = -P_h[\hat{h}_j \hat{s} + \hat{h}_j \hat{s} \hat{u}_0 + \hat{u}_0 \hat{s} \hat{h}] \]
\[
g_{j}^{[2]} := -[\hat{h}_j \hat{s} + \hat{h}_j \hat{\Theta}_0 + \hat{u}_{0,j} \hat{s}]
\]

and

\[
\dot{u}_1(k) := -\nu|k|^2 \hat{u}_0 - i k_j P_k[\hat{u}_{0,j} \hat{u}_0] + a P_k[e_2 \hat{\Theta}_0] + \dot{f} \tag{14}
\]

\[
\hat{\Theta}_1(k) := -\mu|k|^2 \hat{\Theta}_0 - i k_j (\hat{u}_{0,j} \hat{\Theta}_0).
\]

Similarly, for (15), we define

\[
g_{j}^{[3]} := -P_k[\hat{v}_{0,j} \hat{w} + \hat{v}_j \hat{v}_0 + \hat{w}_j \hat{w}] + \frac{1}{\mu \rho} P_k[\dot{B}_{0,j} \hat{q} + \dot{q}_j \hat{B}_0 + \dot{q}_j \hat{q}] \tag{15}
\]

\[
g_{j}^{[4]} := -P_k[\hat{v}_{0,j} \hat{q} + \hat{v}_j \hat{B}_0 + \hat{w}_j \hat{q}] + P_k[\dot{B}_{0,j} \hat{w} + \dot{q}_j \hat{v}_0 + \dot{q}_j \hat{w}]
\]

and

\[
\dot{v}_1(k) := -\nu|k|^2 \hat{v}_0 - i k_j P_k[\hat{v}_{0,j} \hat{v}_0] - \frac{1}{\mu \rho} \hat{B}_{0,j} \hat{B}_0] + \dot{f} \tag{16}
\]

\[
\hat{B}_1(k) := -\frac{1}{\mu \sigma} |k|^2 \hat{B}_0 - i k_j P_k[\hat{v}_{0,j} \hat{B}_0 - \dot{B}_{0,j} \hat{v}_0].
\]

Using these definitions in (15) and (16) and integrating terms whose \(\tau\) dependence appears only in the exponential, we obtain the integral equations

\[
\hat{h}(k, t) = -ik_j \int_0^t e^{-\nu|k|^2(t-s)} \left(g_{j}^{[1]}(k, s') - P_k[ae_2 \hat{s}](k, s')\right) ds' + \left(1 - \frac{e^{-\nu|k|^2 t}}{\nu|k|^2}\right) \hat{u}_1 \tag{17}
\]

\[
\hat{s}(k, t) = -ik_j \int_0^t e^{-\nu|k|^2(t-s)} \left(g_{j}^{[2]}(k, s)\right) ds + \left(1 - \frac{e^{-\mu|k|^2 t}}{\mu|k|^2}\right) \hat{\Theta}_1
\]

and

\[
\hat{w}(k, t) = -ik_j \int_0^t e^{-\nu|k|^2(t-s)} \left(g_{j}^{[3]}(k, s)\right) ds + \left(1 - \frac{e^{-\mu|k|^2 t}}{\mu|k|^2}\right) \hat{v}_1 \tag{18}
\]

\[
\hat{q}(k, t) = -ik_j \int_0^t e^{-\mu \sigma^{-1}|k|^2(t-s)} \left(g_{j}^{[4]}(k, s)\right) ds + \left(1 - \frac{e^{-\mu \sigma^{-1}|k|^2 t}}{(\mu \sigma)^{-1}|k|^2}\right) \hat{B}_1.
\]

In both systems, we seek a solution as a Laplace transform,

\[
\hat{(h, s)}(k, t) = \int_0^\infty \left(\hat{H}, \hat{S}\right)(k, p)e^{-p/t} dp
\]

\[
\hat{(w, q)}(k, t) = \int_0^\infty \left(\hat{W}, \hat{Q}\right)(k, p)e^{-p/t} dp.
\]
With this goal, we take the formal inverse Laplace transform in $1/t$ of our two equations. The inverse Laplace transform of $f$ is given as usual by

$$[\mathcal{L}^{-1} f](p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s)e^{sp}ds,$$

where $c$ is chosen so that for $\text{Re } s \geq c$, $f$ is analytic and has suitable asymptotic decay. We define

$$\mathcal{H}^{(\nu)}(p,p',k) := \int_0^1 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^{-1} e^{\nu|k|^2\tau^{-1}(1-s)+(p-p's^{-1})\tau}d\tau ds.

Then (17) becomes

$$\dot{H}(k,p) = -ik_j \int_0^p \mathcal{H}^{(\nu)}(p,p',k)\dot{G}^{[1]}_j(k,p')dp' + \int_0^p \mathcal{H}^{(\nu)}(p,p',k)P_k[\mu_2\dot{S}](k,p)dp$$

$$+ \dot{u}_1(k)\mathcal{L}^{-1}\left(\frac{1-e^{-\nu|k|^2t}}{\nu|k|^2}\right)(p)$$

and (18) becomes

$$\dot{W}(k,p) = -ik_j \int_0^p \mathcal{H}^{(\nu)}(p,p',k)\dot{G}^{[3]}_j(k,p')dp' + \dot{v}_1(k)\mathcal{L}^{-1}\left(\frac{1-e^{-\nu|k|^2t}}{\nu|k|^2}\right)(p)$$

In the above, $\dot{G}^{[1,2,3,4]}_j = \mathcal{L}^{-1}[g^{[1,2,3,4]}_j]$. Specifically,

$$\dot{G}^{[1]}_j = P_k[\dot{u}_{0,j}\dot{H} + \dot{H}_j\dot{u}_0 + \dot{H}_j\dot{H}],$$

$$\dot{G}^{[2]}_j = [\ddot{u}_{0,j}\dot{S} + \dot{H}_j\dot{\dot{S}} + \dot{H}_j\dot{\dot{S}}],$$

$$\dot{G}^{[3]}_j = P_k[\dot{v}_{0,j}\dot{W} + \dot{W}_j\dot{v}_0 + \dot{W}_j\dot{W}] - \frac{1}{\mu \rho} P_k[\dot{\dot{B}}_0,j\dot{\dot{Q}} + \dot{\dot{Q}}_j\dot{\dot{B}}_0 + \dot{Q}_j\dot{\dot{Q}}],$$

$$\dot{G}^{[4]}_j = P_k[\dot{v}_{0,j}\dot{\dot{Q}} + \dot{\dot{W}}_j\dot{\dot{B}}_0 + \dot{W}_j\dot{\dot{Q}} - P_k[\dot{\dot{B}}_0,j\dot{\dot{W}} + \dot{\dot{Q}}_j\dot{\dot{v}}_0 + \dot{\dot{Q}}_j\dot{\dot{W}}].$$

While the derivation of the integral equation is formal, we prove later (Lemma 1) that the unique solution to the integral equation in the Borel plane generates a solution to the Boussinesq/magnetic Bénard equation through Laplace transform.
where \( \ast \) denotes the Laplace convolution followed by Fourier convolution (order is unimportant). We now make the observation that our kernel \( H(\nu)(p,p',k) \) has a representation in terms of Bessel functions. Namely,

\[
H(\nu)(p,p',k) = \frac{\pi}{z} G(z,z') := \frac{\pi z'}{z} \{-J_1(z)Y_1(z') + Y_1(z)J_1(z')\}
\]

where \( J_1 \) and \( Y_1 \) are the Bessel functions of order 1, \( z = 2|k|\sqrt{\nu}p \), and \( z' = 2|k|\sqrt{\nu}p' \). In similar spirit, we have

\[
\frac{2J_1(z)}{z} = L^{-1} \left( \frac{1 - e^{-\nu|k|^2\tau}}{\nu|k|^2} \right)(p).
\]

These assertions are proved in the appendix in Lemma 91 and Lemma 92.

Thus, our integral Boussinesq equation becomes

\[
\hat{H}(k,p) = \frac{ik_1\pi}{2|k|\sqrt{\nu}p} \int_0^p \dot{G}(z,z')\hat{G}_j^{[1]}(k,p')dp' + a\pi \int_0^p \frac{\dot{G}(z,z')}{z} P_k[e_2\hat{T}(k,p')]dp' + 2\hat{u}_1(k)J_1(z) \]

and

\[
\hat{S}(k,p) = \frac{ik_1\pi}{2|k|\sqrt{\nu}p} \int_0^p \dot{G}(\zeta,\zeta')\hat{G}_j^{[2]}(k,p')dp' + 2\hat{\Theta}_1(k)J_1(\zeta) \]

where \( \zeta = 2|k|\sqrt{\nu}p \), and \( \zeta' = 2|k|\sqrt{\nu}p' \). Abstractly, we may write the set of equations (23) as

\[
(\hat{H}, \hat{S})(k,p) = \mathcal{N}[(\hat{H}, \hat{S})](k,p).
\]

Similarly, our integral MHD equation becomes

\[
\hat{W}(k,p) = \frac{ik_1\pi}{2|k|\sqrt{\nu}p} \int_0^p \dot{G}(\hat{z},\hat{z}')\hat{G}_j^{[3]}(k,p')dp' + 2\hat{u}_1(k)J_1(\hat{z}) 
\]

and

\[
\hat{Q}(k,p) = \frac{ik_1\pi}{2|k|\sqrt{\nu}p} \int_0^p \dot{G}(\hat{\zeta},\hat{\zeta}')\hat{G}_j^{[4]}(k,p')dp' + 2\hat{B}_1(k)J_1(\hat{\zeta}) \]

where \( \hat{z} = 2|k|\sqrt{\nu}p, \hat{z}' = 2|k|\sqrt{\nu}p' \), \( \hat{\zeta} = 2|k|\sqrt{\mu\sigma} \), and \( \hat{\zeta}' = 2|k|\sqrt{\mu\sigma} \). Abstractly, we will denote the set of integral equations in (25) as

\[
(\hat{W}, \hat{Q})(k,p) = \mathcal{M}[(\hat{W}, \hat{Q})](k,p).
\]

Remark 31 By properties of Bessel functions \( |\hat{G}(z,z')| \) is bounded for all real nonnegative \( z' \leq z \). (The approximate bound is 0.6).

Remark 32 By properties of Bessel functions \( |\hat{G}(z,z')/z| \) is bounded for all real nonnegative \( z' \leq z \).
To prove Theorem 21 and 22, we will show \(N\) and \(M\) are contractive in a suitable space, so \((\hat{H}, \hat{S})\) and \((\hat{W}, \hat{Q})\) are Laplace transformable in \(1/t\). Then Lemma 91 tells us that \((\hat{h}, \hat{s})\) and \((\hat{w}, \hat{q})\) the Laplace transforms satisfy (17) and (18) for \(\Re(1/t)\) large enough. This means that at least for small enough \(t\),

\[
(\hat{u}, \hat{\Theta})(k,t) = (\hat{u}_0, \hat{\Theta}_0) + \int_0^\infty (\hat{H}, \hat{S})(k,p)e^{-p/t}dp
\]
solves the Boussinesq equation (5) in the Fourier space with given initial condition and

\[
(\hat{v}, \hat{B})(k,t) = (\hat{v}_0, \hat{B}_0) + \int_0^\infty (\hat{W}, \hat{Q})(k,p)e^{-p/t}dp
\]
solves the magnetic Bénard equation (6) in the Fourier space with given initial condition. Furthermore, we show \((u, \Theta)(x,t) = F^{-1}[(\hat{u}, \hat{\Theta})(\cdot, t)](x)\) (respectively, \((v, B)(x,t) = F^{-1}[(\hat{v}, \hat{B})(\cdot, t)](x)\)) is a classical solution to the Boussinesq (magnetic Bernard) problem.

4 Norms in \(p\)

Recall the norm \(\|\cdot\|_N\) in \(k\) is either the \((\gamma, \beta)\) norm given in Definition 22 for some \(\beta \geq 0\) and \(\gamma > d\) or the \(L^1 \cap L^\infty\) norm.

Definition 41: For \(\alpha \geq 1\), we define

\[
\|\hat{f}\|^{(\alpha)} = \sup_{p \geq 0} (1 + p^2)e^{-\alpha p}\|\hat{f}(\cdot, p)\|_N.
\]

Definition 42: We define \(A^\alpha\) to be the Banach space of continuous function of \((k, p)\) for \(k \in \mathbb{R}^d\) and \(p \in \mathbb{R}^+\) for which \(\|\cdot\|^{(\alpha)}\) is finite. In similar spirit, we define the space \(A^\alpha_1\) of locally integrable functions for \(p \in [0, L]\), and continuous in \(k\) such that

\[
\|\hat{f}\|^{(\alpha)}_1 = \int_0^L e^{-\alpha p}\|\hat{f}(\cdot, p)\|_N dp < \infty.
\]

Definition 43: Finally, we also define \(A^\alpha_L\) to be the Banach space of continuous functions in \((k, p)\) for \(k \in \mathbb{R}^d\) and \(p \in [0, L]\) such that

\[
\|\hat{f}\|^{(\alpha)}_L = \sup_{p \in [0, L]} \|\hat{f}(\cdot, p)\|_N < \infty.
\]

These norms are used in the analysis of the solutions to (24) and (25). The norms are used to guarantee the solutions have the properties necessary to insure their Laplace transforms satisfy the corresponding integral equations, (5) and (6). Furthermore, to show Borel summability for analytic data and forcing, more regularity in \(p\) is required than provided by \(\|\cdot\|^{(\alpha)}\). By proving the solution is unique in the spaces \(A^\alpha_1\) and \(A^\alpha_L\), where one clearly contains the other for finite \(L\), we are assured of regularity in \(p\).
5 Existence of a Solution to (23) and (25)

We need some preliminary lemmas. Recall, $d = 2$ or $d = 3$ denotes the dimension in $x$ or its dual $k$. Often constants appearing in subalgebra bounds will depend on dimension. We will explicitly state the dependence when defining them and suppress the dependence elsewhere.

**Lemma 51** If $||\hat{v}||_{\gamma,\beta}$ and $||\hat{w}||_{\gamma,\beta} < \infty$ for $\gamma > d$ and $k \in \mathbb{R}^d$, then

$$||\hat{v} \hat{w}^*||_{\gamma,\beta} \leq \tilde{C}_0(d)||\hat{v}||_{\gamma,\beta}||\hat{w}||_{\gamma,\beta},$$

where

$$\tilde{C}_0(2) = 2^{\gamma+1} \int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)^\gamma} dk' = \frac{\pi 2^{\gamma+2}}{(\gamma - 1)(\gamma - 2)}$$

and

$$\tilde{C}_0(3) = 2^{\gamma+1} \int_{k' \in \mathbb{R}^3} \frac{1}{(1 + |k'|)^\gamma} dk' = \frac{\pi 2^{\gamma+4}}{(\gamma - 1)(\gamma - 2)(\gamma - 3)}.$$

**Proof** The $d = 3$ case can be found in [5] and the $d = 2$ case is basically the same. For a detailed proof see [15]. From the definition of $|| \cdot ||_{\gamma,\beta}$ and the fact that $e^{-\beta(|k| + |k'|)} \leq e^{-\beta|k|}$, we have

$$|\hat{v} \hat{w}^*| \leq e^{-\beta|k|}||\hat{v}||_{\gamma,\beta}||\hat{w}||_{\gamma,\beta} \int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)^\gamma(1 + |k - k'|)^\gamma} dk'.$$

Split the integral into two domains $|k'| \leq |k|/2$ and its compliment to show

$$\int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)^\gamma(1 + |k - k'|)^\gamma} dk' \leq 2^{\gamma+1} \int_{k' \in \mathbb{R}^2} \frac{1}{(1 + |k'|)^\gamma(1 + |k - k'|)^\gamma} dk'$$

$$= \frac{\pi 2^{\gamma+2}}{(\gamma - 1)(\gamma - 2)},$$

where polar coordinates and integration by parts are used to evaluate the last integral.

**Corollary 52** If $||\hat{v}||_N$, $||\hat{w}||_N < \infty$, then for $C_0 = C_0(d)$ chosen such that $C_0 = C_0$ for $N = (\gamma, \beta)$, $\gamma > d$ and $C_0 = 1$ for $N = L^1 \cap L^\infty$, we have

$$||\hat{v} \hat{w}^*||_N \leq C_0 ||\hat{v}||_N ||\hat{w}||_N.$$

**Lemma 53** Also, notice that

$$\left\| \left( P_k(\hat{f}), P_k(\hat{g}) \right) \right\|_N \leq ||(\hat{f}, \hat{g})||_N$$

**Proof** $P_k$ is the projection of a vector onto $k^\perp$. 
Lemma 54 With $C_0$ as defined in Corollary 52 appropriately modified for $d = 2$ or 3, and constants

\[
C_2 = \frac{\pi C_0}{\min(\sqrt{p'}, \sqrt{\mu})} \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} |G(z, z')|,
\]

\[
C_4 = 2\pi \max\left(\frac{1}{\sqrt{p'}}, \frac{1}{\sqrt{\mu}}\right) \max(1, \frac{1}{\mu P}) C_0 \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} |G(z, z')|,
\]

\[
C_3 = \pi a \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} \frac{|G(z, z')|}{z},
\]

we have the following bounds on the norm in $k$, for operators $N$ and $M$ defined in (27) and (28) respectively:

\[
\|N[(\hat{H}, \hat{S})(\cdot, p)]\|_N \leq C_2 \int_0^p \left(\|[(\hat{H}, \hat{S})(\cdot, p')]|_N + \|[(\hat{H}, \hat{S})(\cdot, p')]|_N + C_3 \int_0^p \|S(p', p')\|_N dp'\right)
\]

(27)

\[
\|M[(\hat{W}, \hat{Q})(\cdot, p)]\|_N \leq C_4 \int_0^p \left(\|[(\hat{W}, \hat{Q})(\cdot, p')]|_N + \|[(\hat{W}, \hat{Q})(\cdot, p')]|_N + C_3 \int_0^p \|\hat{S}(p', p')\|_N dp'\right)
\]

(28)

and

\[
\|N[(\hat{H}^{[1]}, \hat{S}^{[1]})(\cdot, p) - N[(\hat{H}^{[2]}, \hat{S}^{[2]})(\cdot, p)]\|_N \leq \frac{C_2}{\sqrt{p}} \int_0^p \left(\|[(\hat{H}^{[1]}, \hat{S}^{[1]})(\cdot, p')]|_N + \|[(\hat{H}^{[2]}, \hat{S}^{[2]})(\cdot, p')]|_N + \|[(\hat{H}^{[1]}, \hat{S}^{[1]})(\cdot, p')]|_N\right)\right. + C_3 \int_0^p \|\hat{S}^{[1]}(p', p')\|_N dp' + C_3 \int_0^p \|\hat{S}^{[2]}(p', p')\|_N dp'
\]

(29)

\[
\|M[(\hat{W}^{[1]}, \hat{Q}^{[1]})(\cdot, p)] - M[(\hat{W}^{[2]}, \hat{Q}^{[2]})(\cdot, p)]\|_N \leq \frac{C_4}{\sqrt{p}} \int_0^p \left(\|[(\hat{W}^{[1]}, \hat{Q}^{[1]})(\cdot, p')]|_N + \|[(\hat{W}^{[2]}, \hat{Q}^{[2]})(\cdot, p')]|_N + \|[(\hat{W}^{[1]}, \hat{Q}^{[1]})(\cdot, p')]|_N\right)\right. + C_3 \int_0^p \|\hat{S}^{[1]}(p', p')\|_N dp' + C_3 \int_0^p \|\hat{S}^{[2]}(p', p')\|_N dp'.
\]

(30)

Proof We will give the proof for (28) and (30). The two inequalities for $(\hat{H}, \hat{S})$ are very similar. From (1), $|J_1(z)/z| \leq 1/2$ for $z \in \mathbb{R}^+$ and

\[
\left\|2 \left(\hat{\nu}_1(k) \frac{J_1(\hat{k})}{\hat{k}}, \hat{B}_1(k) \frac{J_1(\hat{k})}{\hat{k}}\right)\right\|_N \leq \|[(\hat{\nu}_1, \hat{B}_1)]\|_N.
\]

(31)
From Corollary 52, we have
\[
||\hat{v}_0| |(\hat{W}, \hat{Q})| |+| |\hat{W}_0| |+| |\hat{W}|^2(\hat{W}, \hat{Q})||_N \leq
2C_0|||v_0, B_0|||_N|||(\hat{W}, \hat{Q})| |_N + C_0|||\hat{W}(\cdot, p)|||_N * |||(\hat{W}, \hat{Q})(\cdot, p)|||_N .
\]

Similarly,
\[
||\hat{B}_0| |\left( \frac{\hat{\mathcal{G}}}{\mu \rho} \hat{W} \right) + | |\check{\hat{Q}}| |\left( \frac{\hat{B}_0}{\mu \rho} \check{v}_0 \right) + | |\hat{Q}| |\left( \frac{\hat{\mathcal{G}}}{\mu \rho} \hat{W} \right) \right| |_N \leq \max \left( 1, \frac{1}{\mu \rho} \right).
\]

Then using Lemma 53, the two inequalities above, and Schwartz inequality we obtain
\[
||k_j(\hat{G}_j^{[2]}, \check{\hat{G}}_j^{[4]})||_N \leq 4C_0|k| \max \left( 1, \frac{1}{\mu \rho} \right) \left( ||(\hat{W}, \hat{Q})(\cdot, p')||_N * ||(\hat{W}, \hat{Q})(\cdot, p)||_N \right.
+ \left. ||(\check{v}_0, B_0)||_N ||(\hat{W}, \hat{Q})(\cdot, p')||_N .
\]

Now, noticing that
\[
\left| k_j \left( \frac{\hat{\mathcal{G}}(z, z')}{\sqrt{\rho}} \hat{G}_j^{[2]}, \sqrt{\mu \sigma} \mathcal{\check{G}}(\zeta, \zeta') \check{G}_j^{[4]} \right) \right| \leq \max \left( \frac{1}{\sqrt{\rho}}, \sqrt{\mu \sigma} \right) |k_j(\hat{G}_j^{[2]}, \check{\hat{G}}_j^{[4]})| \sup_{z \in \mathbb{R}^+, 0 \leq z' \leq z} |\mathcal{G}(z, z')|
\]
(28) follows directly. To obtain (30) notice that
\[
\hat{W}_j^{[1]}(\hat{W}^{[1]}, \hat{Q}^{[1]}) - \hat{W}_j^{[2]}(\hat{W}^{[2]}, \hat{Q}^{[2]}) = \hat{W}_j^{[1]}(\hat{W}^{[1]}, \hat{Q}^{[1]} - (\hat{W}^{[2]}, \hat{Q}^{[2]})) + (\hat{W}_j^{[1]} - \hat{W}_j^{[2]}) (\hat{W}^{[2]}, \hat{Q}^{[2]}).
\]

From which we get
\[
\left\| \hat{W}_j^{[1]}(\hat{W}^{[1]}, \hat{Q}^{[1]} - (\hat{W}^{[2]}, \hat{Q}^{[2]})) \right\|_N \leq C_0 \left( ||(\hat{W}^{[1]}, \hat{Q}^{[1]} - (\hat{W}^{[2]}, \hat{Q}^{[2]}))||_N \right.
+ \left. ||(\hat{W}^{[1]}, \hat{Q}^{[1]} - (\hat{W}^{[2]}, \hat{Q}^{[2]}))||_N .
\]

Similarly,
\[
\left\| \hat{Q}_j^{[1]}(\hat{Q}^{[1]}, \hat{W}^{[1]} - (\hat{Q}^{[2]}, \hat{W}^{[2]})) \right\|_N \leq C_0 \left( ||(\hat{W}^{[1]}, \hat{Q}^{[1]} - (\hat{W}^{[2]}, \hat{Q}^{[2]}))||_N \right.
+ \left. ||(\hat{W}^{[1]}, \hat{Q}^{[1]} - (\hat{W}^{[2]}, \hat{Q}^{[2]}))||_N .
\]

Combining this bound and bounds using Lemma 53 as in the first part of the proof, we get (30).
Lemma 55  For \( \hat{f}, \hat{g} \in \mathcal{A}_p, \mathcal{A}_q, \text{ or } \mathcal{A}_s^\infty \)

\[
\| \hat{f} \ast \hat{g} \|^{(\alpha)} \leq M_0 C_0 \| \hat{f} \|^{(\alpha)} \| \hat{g} \|^{(\alpha)}
\]

\[
\| \hat{f} \ast \hat{g} \|^{(1)}_1 \leq C_0 \| \hat{f} \|^{(1)}_1 \| \hat{g} \|^{(1)}_1
\]

\[
\| \hat{f} \ast \hat{g} \|^{(\infty)}_L \leq L C_0 \| \hat{f} \|^{(\infty)}_L \| \hat{g} \|^{(\infty)}_L,
\]

where \( M_0 \approx 3.76 \ldots \) is large enough so

\[
\int_0^p \frac{(1 + p^2) ds}{(1 + s^2)(1 + (p - s)^2)} \leq M_0.
\]

This means the Banach spaces listed in the norms section form subalgebras under the operation \( \ast \). The properties listed are independent of dimension except for a change in \( C_0 \) showing up due to the Fourier convolution. The proof is in [5]. The basic idea is that \( k \) and \( p \) act separately in the norm. So, we need only consider how the \( p \) portion of the norm effects \( \int_0^p u(p)v(p - s) ds \).

Lemma 56  (This lemma expands the bounds in Lemma 55 to bounds in \( p \) in some of our other norms). On \( \mathcal{A}_p^r \), the operators \( \mathcal{M} \) and \( \mathcal{N} \) satisfy the following inequalities

\[
\| \mathcal{N}(\hat{H}, \hat{S}) \|^{(1)}_1 \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|(\hat{H}, \hat{S})\|^{(1)}_1 \right)^2 + \|(\hat{u}_0, \hat{\Theta}_0)\|_N \|(\hat{H}, \hat{S})\|^{(1)}_1 \right\}
\]

\[
+ \alpha^{-1} \|(\hat{u}_1, \hat{\Theta}_1)\|_N \alpha^{-1} C_0 \|(\hat{H}, \hat{S})\|^{(1)}_1, \quad (32)
\]

\[
\| \mathcal{M}(\hat{W}, \hat{Q}) \|^{(1)}_1 \leq C_4 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|(\hat{W}, \hat{Q})\|^{(1)}_1 \right)^2 + \|(\hat{u}_0, \hat{B}_0)\|_N \|(\hat{W}, \hat{Q})\|^{(1)}_1 \right\}
\]

\[
+ \alpha^{-1} \|(\hat{v}_1, \hat{B}_1)\|_N, \quad (33)
\]

and

\[
\| \mathcal{N}(\hat{H}^{[1]}, \hat{S}^{[1]}) - \mathcal{N}(\hat{H}^{[2]}, \hat{S}^{[2]}) \|^{(1)}_1 \leq
\]

\[
C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|(\hat{H}^{[1]}, \hat{S}^{[1]})\|^{(1)}_1 + \|(\hat{H}^{[2]}, \hat{S}^{[2]})\|^{(1)}_1 \right) \left( \|(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})\|^{(1)}_1 \right) \right\}
\]

\[
+ \|(\hat{u}_0, \hat{\Theta}_0)\|_N \|(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})\|^{(1)}_1 \right\} + \alpha^{-1} C_5 \|(\hat{H}^{[1]} - \hat{S}^{[2]}\|^{(1)}_1, \quad (34)
\]

\[
\| \mathcal{M}(\hat{W}^{[1]}, \hat{Q}^{[1]}) - \mathcal{M}(\hat{W}^{[2]}, \hat{Q}^{[2]}) \|^{(1)}_1 \leq
\]

\[
C_4 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \|(\hat{W}^{[1]}, \hat{Q}^{[1]})\|^{(1)}_1 + \|(\hat{W}^{[2]}, \hat{Q}^{[2]})\|^{(1)}_1 \right) \left( \|(\hat{W}^{[1]} - \hat{Q}^{[2]}\|^{(1)}_1 \right) \right\}
\]

\[
+ \|(\hat{v}_0, \hat{B}_0)\|_N \|(\hat{W}^{[1]}, \hat{Q}^{[1]}) - (\hat{W}^{[2]}, \hat{Q}^{[2]})\|^{(1)}_1 \right\} \right\}. \quad (35)
\]

Similarly, for \( \mathcal{A}_s^\infty \), we have

\[
\| \mathcal{N}(\hat{H}, \hat{S}) \|^{(\infty)}_E \leq C_2 \sqrt{L} \left\{ L \left( \|(\hat{H}, \hat{S})\|^{(\infty)}_E \right)^2 + \|(\hat{u}_0, \hat{\Theta}_0)\|_N \|(\hat{H}, \hat{S})\|^{(\infty)}_E \right\}
\]
Similarly,

\[ ||\mathcal{M}(\mathcal{W}, \mathcal{Q})||^2_L \leq C_4 \sqrt{L} \left\{ L(||(\mathcal{W}, \mathcal{Q})||^2_L)^2 + ||(\mathcal{W}, \mathcal{Q})||_N \right\} + ||(\mathcal{W}, \mathcal{Q})||_N, \quad (37) \]

\[ ||\mathcal{N}(\hat{\mathcal{H}}^{[1]}, \hat{\mathcal{S}}^{[1]}) - \mathcal{N}(\hat{\mathcal{H}}^{[2]}, \hat{\mathcal{S}}^{[2]})||^2_L \leq C_2 \sqrt{L} \left\{ L \left(||(\hat{\mathcal{H}}^{[1]}, \hat{\mathcal{S}}^{[1]})||^2_L + ||(\hat{\mathcal{H}}^{[2]}, \hat{\mathcal{S}}^{[2]})||^2_L \right) \left(||(\hat{\mathcal{H}}^{[1]}, \hat{\mathcal{S}}^{[1]}) - (\hat{\mathcal{H}}^{[2]}, \hat{\mathcal{S}}^{[2]})||^2_L \right) + ||(\mathcal{W}, \mathcal{Q})||_N \left(||(\hat{\mathcal{H}}^{[1]}, \hat{\mathcal{S}}^{[1]}) - (\hat{\mathcal{H}}^{[2]}, \hat{\mathcal{S}}^{[2]})||^2_L \right) \right\} + L C_3 ||\hat{\mathcal{S}}^{[1]} - \hat{\mathcal{S}}^{[2]}||^2_L \quad (38) \]

\[ ||\mathcal{M}(\hat{\mathcal{W}}^{[1]}, \hat{\mathcal{Q}}^{[1]}) - \mathcal{M}(\hat{\mathcal{W}}^{[2]}, \hat{\mathcal{Q}}^{[2]})||^2_L \leq C_4 \sqrt{L} \left\{ L \left(||(\hat{\mathcal{W}}^{[1]}, \hat{\mathcal{Q}}^{[1]})||^2_L + ||(\hat{\mathcal{W}}^{[2]}, \hat{\mathcal{Q}}^{[2]})||^2_L \right) \left(||(\hat{\mathcal{W}}^{[1]}, \hat{\mathcal{Q}}^{[1]}) - (\hat{\mathcal{W}}^{[2]}, \hat{\mathcal{Q}}^{[2]})||^2_L \right) + ||(\mathcal{W}, \mathcal{Q})||_N \left(||(\hat{\mathcal{W}}^{[1]}, \hat{\mathcal{Q}}^{[1]}) - (\hat{\mathcal{W}}^{[2]}, \hat{\mathcal{Q}}^{[2]})||^2_L \right) \right\} \quad (39) \]

**Proof** For the space \( \mathcal{A}_p \) and any \( L > 0 \), we note that

\[ \int_0^L e^{-\alpha p} ||(\mathcal{W}, \mathcal{Q})||_N dp \leq \alpha^{-1} ||(\mathcal{W}, \mathcal{Q})||_N \]

and

\[ \int_0^L e^{-\alpha p} \alpha^{-1/2} dp = \Gamma \left( \frac{1}{2} \right) \alpha^{-1/2} = \sqrt{\pi} \alpha^{-1/2}. \]

We further notice that for \( y(p') \geq 0 \), we have

\[ \int_0^L e^{-\alpha p} \alpha^{-1/2} \left( \int_0^{y(p')} d\alpha \right) dp = \int_0^L y(p') e^{-\alpha p} \left( \int_0^L e^{-\alpha (p-p') \alpha^{-1/2} dp} \right) dp' \]

\[ \leq \int_0^L y(p') e^{-\alpha p} \left( \int_0^L e^{-\alpha s} e^{-1/2 ds} \right) dp' \leq \int_0^L y(p') e^{-\alpha p} \sqrt{\pi} \alpha^{-1/2} dp'. \quad (40) \]

Similarly,

\[ \int_0^L e^{-\alpha p} \left( \int_0^{y(p')} d\alpha \right) dp = \int_0^L ||\mathcal{S}(\cdot, p')||_N e^{-\alpha p'} \left( \int_0^L e^{-\alpha (p-p') \alpha^{-1/2} dp} \right) dp' \]

\[ = \int_0^L ||\mathcal{S}(\cdot, p')||_N e^{-\alpha p'} \left( \int_0^L e^{-\alpha s} ds \right) dp' \leq \alpha^{-1} ||\mathcal{S}||_N^2. \]
Then, using (40) in (27) and the idea in Lemma 55 that $\int_0^L e^{-\alpha p}||h||_N \leq ||y||_H$, we have

$$
\int_0^L e^{-\alpha p}||N(\hat{H}, \hat{S})||_N dp \leq C_2 \sqrt{\pi} \alpha^{-1/2} \left\{ \left( \left< ||(\hat{H}, \hat{S})||^2 \right> \right)^2 
+ ||(\hat{u}_0, \hat{\Theta}_0)||_N \left< ||(\hat{H}, \hat{S})||_1 \right> \right\} + \alpha^{-1} ||(\hat{u}_1, \hat{\Theta}_1)||_N + \alpha^{-1} C_4 ||\hat{S}||_N^2. \quad (41)
$$

This proves (32). Further, from (29), it also follows that

$$
\int_0^L e^{-\alpha p}||N(\hat{H}^{[1]}, \hat{S}^{[1]})(\cdot, p) - N(\hat{H}^{[2]}, \hat{S}^{[2]})(\cdot, p)||_N dp \leq \frac{C_2 \sqrt{\pi} \alpha^{-1/2}}{2} \left\{ \left( \left< ||(\hat{H}^{[1]}, \hat{S}^{[1]})||_N^2 + ||(\hat{H}^{[2]}, \hat{S}^{[2]})||_N^2 \right> \right)^{1/2} \right\} ||(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})||_1^2
+ \left[ ||(\hat{u}_0, \hat{\Theta}_0)||_N \left< ||(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})||_1^2 \right> \right] + \alpha^{-1} C_4 ||\hat{S}^{[1]} - \hat{S}^{[2]}||_N^2. \quad (42)
$$

This proves (33). The inequalities for (\hat{W}, \hat{Q}) similarly follow from (28) and (40).

Now, we consider $A_L^\infty$. We note that for $p \in [0, L]$, we have

$$
\left| p^{-1/2} \int_0^p y(p') dp' \right| \leq \sup_{p \in [0, L]} |y(p)| \sqrt{L}.
$$

We recall from Lemma 54 that

$$
\left| \int_0^p y_1(s)y_2(p-s) ds \right| \leq L \left( \sup_{p \in [0, L]} |y_1(p)| \right) \left( \sup_{p \in [0, L]} |y_2(p)| \right).
$$

Taking

$$
y(p) = ||(\hat{H}, \hat{S})(\cdot, p)||_N * ||(\hat{H}, \hat{S})(\cdot, p)||_N + ||(\hat{u}_0, \hat{\Theta}_0)||_N \left< ||(\hat{H}, \hat{S})(\cdot, p)||_N \right|
$$

and $y_1(p) = y_2(p) = ||(\hat{H}, \hat{S})(\cdot, p)||_N$,

(36) follows from (27). To get the bound in (33) we will choose,

$$
y(p) = \left[ ||(\hat{H}^{[1]}, \hat{S}^{[1]})||_N + ||(\hat{H}^{[2]}, \hat{S}^{[2]})||_N \right] * \left( ||(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})||_N \right)_N
+ \left[ ||(\hat{u}_0, \hat{\Theta}_0)||_N \right] \left< ||(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})||_N \right>_N
$$

$$
y_1(p) = ||(\hat{H}^{[1]}, \hat{S}^{[1]})||_N + ||(\hat{H}^{[2]}, \hat{S}^{[2]})||_N
$$

$$
y_2(p) = \left< ||(\hat{H}^{[1]}, \hat{S}^{[1]}) - (\hat{H}^{[2]}, \hat{S}^{[2]})||_N \right>_N
$$

now using (29) the proof follows. The bounds on (\hat{W}, \hat{Q}), (37) and (39), are proved in similar spirit.
Lemma 57  Equation (23) has a unique solution in $A_1^\omega$ for any $L > 0$ in a ball of size $2\omega^{-1}||\tilde{u}_1, \tilde{\Theta}_1||_N$ for $\omega$ large enough to guarantee

$$2C_2\sqrt{\pi} \omega^{-1/2} \left( 2\omega^{-1}||\tilde{u}_1, \tilde{\Theta}_1||_N + ||\tilde{u}_0, \tilde{\Theta}_0||_N + \frac{C_3}{C_2 \sqrt{\pi}} \omega^{-1/2} \right) < 1$$

where $(\tilde{u}_1, \tilde{\Theta}_1)$ is given in (13). Similarly, equation (25) has a unique solution in $A_1^\omega$ for any $L > 0$ in a ball of size $2\alpha^{-1}||\tilde{v}_1, \tilde{B}_1||_N$ for $\alpha$ large enough to guarantee

$$2C_2L^{1/2} \left( 2L||\tilde{v}_1, \tilde{B}_1||_N + ||\tilde{v}_0, \tilde{B}_0||_N \right) < 1$$

where $(\tilde{v}_1, \tilde{B}_1)$ is given in (16). Furthermore, the solutions also belong to $A_2^\omega$ for $L$ small enough to ensure either

$$2C_2L^{1/2} \left( 2L||\tilde{v}_1, \tilde{B}_1||_N + ||\tilde{v}_0, \tilde{B}_0||_N \right) < 1$$

or

$$2C_4L^{1/2} \left( 2L||\tilde{v}_1, \tilde{B}_1||_N + ||\tilde{v}_0, \tilde{B}_0||_N \right) < 1$$

depending on the equation being considered. Moreover, $\lim_{p \to 0^+} (\tilde{H}, \tilde{S})(k, p) = (\tilde{u}_1, \tilde{\Theta}_1)(k)$ and $\lim_{p \to 0^+} (\tilde{W}, \tilde{Q})(k, p) = (\tilde{v}_1, \tilde{B}_1)(k)$.

Proof The estimates in Lemma 56 imply that $\mathcal{M}$ maps a ball of radius $2\omega^{-1}||\tilde{u}_1, \tilde{B}_1||_N$ in $A_2^\omega$ into itself and is contractive when $\alpha$ is large enough to satisfy (44). Similarly, $\mathcal{M}$ maps a ball of size $2\alpha^{-1}||\tilde{v}_1, \tilde{B}_1||_N$ in $A_2^\omega$ into itself and is contractive when $L$ is small enough to satisfy (45). Therefore, there is a unique solution to the Bénard integral system of equations in the ball. Furthermore, $A_2^\omega \subseteq A_1^\omega$, so the solutions are in fact one and the same. Similarly, $\mathcal{N}$ is contractive on a ball of radius $2\omega^{-1}||\tilde{u}_1, \tilde{\Theta}_1||_N$ in $A_2^\omega$ for $\omega$ large enough to satisfy (44), and a ball of size $2\alpha^{-1}||\tilde{v}_1, \tilde{\Theta}_1||_N$ in $A_2^\omega$ for $L$ small enough to satisfy (45). So, the Boussinesq integral system has a unique solution in each of these spaces. Since $A_2^\omega \subseteq A_1^\omega$, the solutions are in fact one and the same.

Moreover, applying (38) (respectively, (39)) with $(\tilde{H}^{[1]}, \tilde{S}^{[1]}) = (\tilde{H}, \tilde{S})$ (respectively, $(\tilde{W}^{[1]}, \tilde{Q}^{[1]}) = (\tilde{W}, \tilde{Q})$) and $(\tilde{H}^{[2]}, \tilde{S}^{[2]}) = (\tilde{W}^{[2]}, \tilde{Q}^{[2]}) = 0$, we obtain

$$\left\| \left( \tilde{H}, \tilde{S} \right)(k, p) - \left( \tilde{u}_1(k) \frac{2J_1(z)}{z}, \tilde{\Theta}_1(k) \frac{2J_1(\zeta)}{\zeta} \right) \right\|_L \leq C_2L^{1/2} \left( L(||\tilde{H}, \tilde{S}||^2_N + ||\tilde{u}_0, \tilde{\Theta}_0||_N(||\tilde{H}, \tilde{S}||^2_L) \right) + LC_3\|\tilde{S}\|_N$$

and

$$\left\| \left( \tilde{W}, \tilde{Q} \right)(k, p) - \left( \tilde{v}_1(k) \frac{2J_1(\tilde{z})}{\tilde{z}}, \tilde{B}_1(k) \frac{2J_1(\tilde{\zeta})}{\tilde{\zeta}} \right) \right\|_L \leq$$
\[ C_4 L^{1/2} \{ L(\|\hat{W}, \hat{Q}\|_\infty^2) + \|\hat{t}_0, \hat{B}_0\|_N(\|\hat{W}, \hat{Q}\|_\infty^2) \}. \]

Since \( \|\hat{H}, \hat{S}\|_\infty^2 \) and \( \|\hat{W}, \hat{Q}\|_\infty^2 \) are bounded for small \( L \), letting \( L \to 0 \),

\[
\left\| (\hat{H}, \hat{S})(k, p) - \left( \hat{u}_1(k) \frac{2J_1(z)}{z}, \hat{\Theta}_1(k) \frac{2J_1(\zeta)}{\zeta} \right) \right\|_L \to 0
\]

and

\[
\left\| (\hat{W}, \hat{Q})(k, p) - \left( \hat{v}_1(k) \frac{2J_1(z)}{z}, \hat{B}_1(k) \frac{2J_1(\zeta)}{\zeta} \right) \right\|_L \to 0.
\]

As \( \lim_{z \to 0} 2J_1(z)/z = 1 \), for fixed \( k \), \( \lim_{p \to 0} (\hat{H}, \hat{S})(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k) \). Similarly, for fixed \( k \), \( \lim_{p \to 0} (\hat{W}, \hat{Q})(k, p) = (\hat{v}_1, \hat{B}_1)(k) \).

### 6 Properties of the solutions

We have unique solutions to our two integral equations, (19) and (20). We show in the following Lemma 61 that these solutions Laplace transforms give solutions to (19) and (20), which are analytic in \( t \) for \( \Re(1/t) > \alpha \) (resp. \( \alpha \)). Lemma 64 below shows that any solution of (3) with \( \|J_1(z)/z\|_N < \infty \) or respectively (4) with \( \|J_1(\zeta)/\zeta\|_N < \infty \) is inverse Fourier transformable with \( (u, \Theta) \) solving (3) and \( (v, B) \) solving (4). Lemma 62 below insures that \( \|J_1(z)/z\|_N < \infty \) and \( \|J_1(\zeta)/\zeta\|_N < \infty \). Thus, combining these two results, we have \( (u, \Theta)(x, t) = \mathcal{F}^{-1}(\hat{u}, \hat{\Theta})(k, t) \) and \( (v, B)(x, t) = \mathcal{F}^{-1}(\hat{v}, \hat{B})(k, t) \) are classical solutions to (3) and (4) respectively.

**Lemma 61** For any solutions \( (\hat{H}, \hat{S}) \) and \( (\hat{W}, \hat{Q}) \) of (12) and (22) such that \( \|\hat{H}, \hat{S}\|_N \in L^1(e^{-\omega p} dp) \) and \( \|\hat{W}, \hat{Q}\|_N \in L^1(e^{-\alpha p} dp) \) the Laplace transform

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty (\hat{H}, \hat{S})(k, p)e^{-p/t} dp \tag{47}
\]

and

\[
(\hat{v}, \hat{B})(k, t) = (\hat{v}_0, \hat{B}_0)(k) + \int_0^\infty (\hat{W}, \hat{Q})(k, p)e^{-p/t} dp \tag{48}
\]

solve (3) for \( \Re(1/t) > \omega \) and (4) for \( \Re(1/t) > \alpha \) respectively. Moreover, \( (\hat{u}, \hat{\Theta})(k, t) \) is analytic for \( t \in (0, \omega^{-1}) \) and \( (\hat{v}, \hat{B})(k, t) \) is analytic for \( t \in (0, \alpha^{-1}) \).

**Proof** We may write

\[
\mathcal{H}(p, p', k) = \int_0^1 \left\{ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tau^{-1}e^{\nu|\tau|} e^{(1-s)\tau} (p - p's^{-1}) \tau \right\} \frac{d\tau}{d\tau} ds
\]

since by contour deformation the integral with respect to \( \tau \) can be pushed to \( +\infty \) and is therefore zero for \( s \in (0, p'/p) \). Let \( \hat{G}_i = -ik_j \hat{G}^{[i]}_{j} + P_k(ae_2 \hat{S}) \).
and $\hat{G}_l = -ik_j \hat{G}_j^{(l)}$ for $l = 2, 3, 4$. Changing variable $p'/s \to p'$ and applying Fubini's theorem gives

$$\int_0^p \left( H^{(r)}(p, p', k) \hat{G}_1(k, p'), H^{(r)}(p, p', k) \hat{G}_2(k, p') \right) dp'$$

$$= \int_0^1 s \left\{ \int_0^p \left( \hat{G}_1(k, p's) I^{(r)}(p - p', s, k), \hat{G}_2(k, p's) I^{(r)}(p - p', s, k) \right) dp' \right\} ds$$

and

$$\int_0^p \left( H^{(r)}(p, p', k) \hat{G}_3(k, p'), H^{(r)}(p, p', k) \hat{G}_4(k, p') \right) dp'$$

$$= \int_0^1 s \left\{ \int_0^p \left( \hat{G}_3(k, p's) I^{(r)}(p - p', s, k), \hat{G}_4(k, p's) I^{(r)}(p - p', s, k) \right) dp' \right\} ds,$$

where for $p > 0$

$$I^{(r)}(p, s, k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tau^{-1} \exp[-\nu |k|^2 \tau^{-1}(1 - s) + pr] d\tau.$$
Recalling the integral equations for \( \dot{H}, \dot{S} \) and \( \dot{W}, \dot{Q} \) given in \([19]\) and \([20]\), we have
\[
(\dot{h}, \dot{s})(k,t) - \left( \hat{u}_1(k) \left( \frac{1 - e^{-\nu|k|^2 t}}{\nu|k|^2} \right), \hat{\Theta}_1(k) \left( \frac{1 - e^{-\mu|k|^2 t}}{\mu|k|^2} \right) \right)
= t \int_0^1 \left( e^{-\nu|k|^2 t(1-s)} \hat{g}_1(k, st), e^{-\nu|k|^2 t(1-s)} \hat{g}_2(k, st) \right) ds
- \int_0^t \left( e^{-\nu|k|^2 (t-s)} \hat{g}_1(k, s), e^{-\nu|k|^2 (t-s)} \hat{g}_2(k, s) \right) ds
\]
and
\[
(\dot{w}, \dot{q})(k,t) = \int_0^t \left( e^{-\nu|k|^2 (t-s)} \hat{g}_3(k, s), e^{-\nu|k|^2 (t-s)} \hat{g}_4(k, s) \right) ds
+ \left( \hat{v}_1(k) \left( \frac{1 - e^{-\nu|k|^2 t}}{\nu|k|^2} \right), \hat{B}_1(k) \left( \frac{1 - e^{-(\mu \sigma)^{-1} |k|^2 t}}{(\mu \sigma)^{-1} |k|^2} \right) \right).
\]
Therefore, we directly verify \((\hat{u}, \hat{\Theta})(k,t) = (\hat{u}_0, \hat{\Theta}_0)(k) + (\dot{h}, \dot{s})(k,t)\) satisfies \([13]\) and \((\hat{v}, \hat{B})(k,t) = (\hat{v}_0, \hat{B}_0)(k) + (\dot{w}, \dot{q})(k,t)\) satisfies \([10]\). Moreover, analyticity in \(t\) follows from the representations
\[
(\hat{u}, \hat{\Theta})(k,t) = (\hat{u}_0, \hat{\Theta}_0)(k) + \int_0^\infty \left( \hat{H}, \hat{S} \right)(k,p)e^{-\rho t} dp
= (\hat{v}_0, \hat{B}_0)(k) + \int_0^\infty \left( \hat{W}, \hat{Q} \right)(k,p)e^{-\rho t} dp.
\]

**Lemma 62** (Instantaneous smoothing) Assume \(\| (\hat{u}_0, \hat{\Theta}_0) \|_N < \infty \), \(\| (\hat{v}_0, \hat{B}_0) \|_N < \infty \), and \(\| f \|_N < \infty \) with \(N\) either \(L^1 \cap L^\infty(\mathbb{R}^d)\) or \((\gamma, \beta)\) with \(\gamma > d, \beta \geq 0\). For the solution \((\hat{v}, \hat{B})\) known to exist by Lemma \([37]\) for \(t \in (0,T]\) with \(T < \alpha^{-1}\), we have \(\| (1 + | \cdot |^2)(\hat{v}, \hat{B})(\cdot,t) \|_N < \infty \) for \(t \in (0,T]\). Respectively, \(\| (1 + | \cdot |^2)(\hat{u}, \hat{\Theta})(\cdot,t) \|_N < \infty \) for \(t \in (0,T]\) with \(T < \omega^{-1}\).

**Proof** The two cases are similar, we present the Bénard case. Our goal is to bootstrap up using derivatives of \((\hat{v}, \hat{B})\). Consider the time interval \([\epsilon, T]\) for \(\epsilon > 0\) and \(T < \alpha^{-1}\). Define
\[
\hat{V}_\epsilon(k) = \sup_{\epsilon \leq t \leq T} \| (\hat{v}, \hat{B})(k,t) \|.
\]
Since \(\| (\hat{v}, \hat{B})(k,t) \| \leq \| (\hat{v}_0, \hat{B}_0)(k) \| + \int_0^\infty \| (\hat{W}, \hat{Q})(k,p) \| e^{-\alpha p} dp\),
\[
\| \hat{V}_\epsilon(k) \|_N \leq \| (\hat{v}_0, \hat{B}_0)(k) \|_N + \| (\hat{W}, \hat{Q})(k,p) \|_1 \epsilon < \infty.
\]
On \([\epsilon, T]\) for \(\epsilon > 0\),
\[
\hat{v}(k,t) = e^{-\nu|k|^2 t} \hat{v}_0(k) - \int_0^t e^{-\nu|k|^2 (t-\tau)} \left( ik_j P_k \hat{v}_j \hat{\Phi} - \frac{1}{\mu \rho} \hat{B}_j \hat{\Phi} \right)(k, \tau) - \hat{f}(k) d\tau
\]
\[ \dot{B}(k,t) = e^{-\frac{ik^2}{\nu}} \dot{B}_0(k) - ik \int_0^t e^{-\frac{ik^2(t-\tau)}{\nu}} \left\{ P_k[\dot{v}_j \dot{B} + \dot{B}_j \dot{v}](k,\tau) \right\} d\tau. \]

Therefore,

\[ |k||(\dot{v}, \dot{B})(k,t)| \leq \left| (\dot{v}_0, \dot{B}_0)(k) \right| \sqrt{\frac{1}{e \min(\nu, \frac{1}{\nu})} \sup_{z \geq 0} z e^{-z^2}} + |f| \int_0^t |k|e^{-\min(\nu, \frac{1}{\mu\sigma})} |k|^2(t-\tau) d\tau + 2\dot{V}_0 \dot{V}_0 \int_0^t |k|e^{-\min(\nu, \frac{1}{\mu\sigma})} |k|^2(t-\tau) d\tau. \]

Noticing that

\[ \int_0^t |k|^2 e^{-\min(\nu, \frac{1}{\mu\sigma})} |k|^2(t-\tau) d\tau \leq \frac{1}{\min(\nu, \frac{1}{\mu\sigma})}, \]

and

\[ \int_0^t |k|e^{-\min(\nu, \frac{1}{\mu\sigma})} |k|^2(t-\tau) d\tau \leq \sup_{z \geq 0} \frac{1 - e^{-z}}{\sqrt{2z}} \frac{\sqrt{T}}{\min(\nu, \frac{1}{\mu\sigma})}, \]

it follows that

\[ \left\| |k|\dot{V}_{t/2} \right\|_N \leq \frac{C}{\epsilon e^{1/2}} \left\| (\dot{v}_0, \dot{B}_0) \right\|_N + \frac{1}{\min(\nu, \frac{1}{\mu\sigma})} \left( 2C_0 \left\| \dot{V}_0 \right\|_N^2 + C\sqrt{T} \left\| f \right\|_N \right) < \infty. \]

In the same spirit, for \( t \in [\frac{\epsilon}{2}, T] \), we have

\[ \dot{v}(k,t) = e^{-\nu|k|^2} \dot{v}(k, \epsilon/2) - \int_{\epsilon/2}^t e^{-\nu|k|^2(t-\tau)} \left( iP_k(\dot{v}_j \dot{B} + \dot{B}_j \dot{v})(k,\tau) - f(k) \right) d\tau \]

\[ \dot{B}(k,t) = e^{-\frac{ik^2}{\nu}} \dot{B}(k, \epsilon/2) - i \int_{\epsilon/2}^t e^{-\frac{ik^2(t-\tau)}{\nu}} \left\{ P_k(\dot{v}_j \dot{B} + \dot{B}_j \dot{v})(k,\tau) \right\} d\tau, \]

where we used the divergence free conditions \( k \cdot \dot{v} = 0 \) and \( k \cdot \dot{B} = 0 \). Multipling by \( |k|^2 \) and using our previous bounds, we have for \( t \in [\epsilon, T] \)

\[ |k|^2 |(\dot{v}, \dot{B})(k,t)| \leq |(\dot{v}, \dot{B})(k, \epsilon/2)| \left( \sqrt{\frac{1}{(t-\epsilon/2) \min(\nu, \frac{1}{\nu})}} \sup_{z \geq 0} z e^{-z^2} \right. \]

\[ + \left. (2\dot{V}_{t/2} |k| \dot{V}_{t/2} + |f|) \int_{\epsilon/2}^t |k|^2 e^{-\min(\nu, \frac{1}{\mu\sigma})} |k|^2(t-\tau) d\tau \right. \]

Hence,

\[ \left\| |k|^2 \dot{V}_{t/2} \right\|_N \leq \frac{C}{\epsilon} \left\| (\dot{v}_0, B_0) \right\|_N + \frac{1}{\min(\nu, \frac{1}{\mu\sigma})} \left( 2C_0 \left\| \dot{V}_{t/2} \right\|_N^2 + \left\| \dot{V}_{t/2} \right\|_N + \left\| f \right\|_N \right). \]

All the terms on the right hand side are bounded, which gives \( \left\| (1+|k|^2) \dot{V}_{t/2} \right\|_N < \infty \). Further, as \( \epsilon > 0 \) is arbitrary, it follows that \( \left\| (1+|\cdot|^2)(\dot{v}, B)(\cdot, t) \right\|_N < \infty \)

for \( t \in (0, T] \).
Remark 63 We note that the smoothness argument in $x$ of the previous Lemma can be easily extended further to show $\| (1 + |k|^2) \hat{f} \|_N$, is finite provided $\| (1 + |k|^2) \hat{f} \|_N$, is finite. Since $\epsilon > 0$ is arbitrary this implies instantaneous smoothing two orders more than the forcing.

Lemma 64 Given $(\hat{u}, \hat{\Theta})$ a solution to (3) such that $\| (1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty$ for $t \in (0, \omega^{-1})$, then $(u, \Theta) \in L^\infty[0, \omega^{-1}, H^2(\mathbb{R}^d)]$ solves (3). Respectively, given $(\hat{v}, \hat{B})$ a solution to (8) such that $\| (1 + |\cdot|^2)(\hat{v}, \hat{B})(\cdot, t) \|_N < \infty$ for $t \in (0, \alpha^{-1})$, then $(v, B) \in L^\infty[0, \alpha^{-1}, H^2(\mathbb{R}^d)]$ solves (8).

Proof The two cases are similar, we show the Boussinesq case. Suppose $(\hat{u}, \hat{\Theta})$ a solution to (3) such that $\| (1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t) \|_N < \infty$ for $t \in (0, \omega^{-1})$. We notice that by our choice of norms, $(1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t) \in L^2(\mathbb{R}^d)$ for any $t \in (0, \omega^{-1})$. Indeed for $N = (\gamma, \beta)$, we have

$$\left( \int (1 + |k|^4)(\hat{u}, \hat{\Theta})(k, t)^2 \, dk \right)^{1/2} \leq \| (1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t) \|_{\gamma, \beta} \left( \int \frac{e^{-2|\beta|k}}{(1 + |k|^2)^{\gamma}} \, dk \right)^{1/2}.$$ 

As $\gamma > d$, $\int \frac{1}{(1 + |k|^2)^\gamma} e^{-2|\beta|k} \, dk < \infty$. For $N = L^1 \cap L^\infty$ we have,

$$\int (1 + |k|^4)(\hat{u}, \hat{\Theta})(k, t)^2 \, dk \leq \int (1 + |k|^2)(\hat{u}, \hat{\Theta})(k, t) \, dk \sup_{k \in \mathbb{R}^d} (1 + |k|^2)(\hat{u}, \hat{\Theta})(k, t).$$

So, $\| (1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t) \|_{L^2(\mathbb{R}^d)} \leq \| (1 + |\cdot|^2)(\hat{u}, \hat{\Theta})(\cdot, t) \|_{L^1 \cap L^\infty(\mathbb{R}^d)}$. Thus, by well known properties of the Fourier transform $(u, \Theta) = \mathcal{F}^{-1}(\hat{u}, \hat{\Theta})(x, t) \in L^\infty(0, \omega^{-1}, H^2(\mathbb{R}^d))$. As $(\hat{u}, \hat{\Theta})$ solves (3), $(\hat{u}, \hat{\Theta})$ is differentiable almost everywhere and

$$\begin{align*}
\hat{u}_t + \nu |k|^2 \hat{u} &= -ik_j P_k[\hat{u}_{j\hat{c}} + \hat{a} P_k[e_j \hat{\Theta}]] + \hat{f}, \\
\hat{\Theta}_t + \mu |k|^2 \hat{\Theta} &= -ik_j[\hat{u}_{j\hat{c}} + \hat{\Theta}], \quad k \in \mathbb{R}^d \quad t \in \mathbb{R}^+.
\end{align*}$$

Further, $(\hat{u}, \hat{\Theta})(k, t) \in L^\infty(0, \omega^{-1}, L^2(\mathbb{R}^d))$ since $(1 + |k|^2)(\hat{u}, \hat{\Theta})(k, t) \in L^\infty(0, \omega^{-1}, L^2(\mathbb{R}^d))$. Hence, $(u, \Theta)(x, t) = \mathcal{F}^{-1}(\hat{u}, \hat{\Theta})(x, t)$ solves

$$\begin{align*}
u \Delta u &= -P[\hat{u} \cdot \nabla \hat{u} - \hat{c} \hat{\Theta}] + f(x) \\
\Theta &= -\mu \Delta \hat{\Theta} = -u \cdot \nabla \Theta.
\end{align*}$$

Proof of Theorems 21 and 22 Suppose $\| (1 + |\cdot|^2)(\hat{u}_0, \hat{\Theta}_0) \|_N < \infty$ and $\| \hat{f} \|_N < \infty$. Then from the definition of $(\hat{u}_1, \hat{\Theta}_1)$ in (13) we see $\| (\hat{u}_1, \hat{\Theta}_1) \|_N < \infty$, since

$$\| (\hat{u}_1, \hat{\Theta}_1) \|_N \leq \max(\nu, \mu) \| |k|^2(\hat{u}_0, \hat{\Theta}_0) \|_N + C_0 \| \hat{u}_0 \|_N \| \hat{f} \|_N + a \| \hat{\Theta}_0 \|_N + \| \hat{f} \|_N.$$
Therefore, when \( \omega \) is large enough to ensure (43), Lemma \( \text{[57]} \) gives \( (\hat{H}, \hat{S})(k, \cdot) \) is in \( L^1(e^{-\gamma_2 t}dp) \). Applying Lemma \( \text{[61]} \) we know for \( t \) such that \( \mathbb{R}_T > \omega \), \( (\hat{H}, \hat{S})(k, p) \) is Laplace transformable in \( 1/t \) with \( (\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + (\hat{h}, \hat{s})(k, t) \) satisfying Boussinesq equation in the Fourier space. (5). Since \( ||(\hat{H}, \hat{S})\cdot p ||_N < \infty \), we have \( \|(\hat{u}, \hat{\Theta})\cdot t \|_N < \infty \) if \( \mathbb{R}_T > \omega \), and i) is proved. Moreover, Lemma \( \text{[61]} \) shows that \( (\hat{u}, \hat{\Theta}) \) is analytic for \( \mathbb{R}_T > \omega \) and has the representation

\[
(\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(x) + \int_0^\infty (\hat{H}, \hat{S})(k, p)e^{-p/t}dp
\]

proving ii). For iii), Lemma \( \text{[62]} \) shows that \( \|[(1 + \cdot)^2(\hat{u}, \hat{\Theta})(\cdot, t)]\|_X < \infty \) for \( t \in [0, \omega^{-1}] \) while Lemma \( \text{[64]} \) shows that \( (u, \Theta)(x, t) \in L^\infty(0, T, H^2(\mathbb{R}^d)) \) solves \( \text{[3]} \). Moreover, \( (u, \Theta)(x, t) \) is the unique solution to \( \text{[3]} \) in \( L^\infty(0, T, H^2(\mathbb{R}^d)) \) as classical solutions are known to be unique. [17]. Finally, suppose \( (\hat{H}, \hat{S})(k, \cdot) \) is in \( L^1(e^{-\gamma_2 t}dp) \) for any \( \omega > 0 \). By Lemma \( \text{[61]} \) we know for any \( t > 0 \), \( (\hat{H}, \hat{S})(k, p) \) is Laplace transformable with \( (\hat{u}, \hat{\Theta})(k, t) = (\hat{u}_0, \hat{\Theta}_0)(k) + (\hat{h}, \hat{s})(k, t) \) satisfying Boussinesq equation in the Fourier space. (5). Further, appealing to instantaneous smoothing Lemma \( \text{[62]} \) the solution is smooth. Thus, if \( (\hat{H}, \hat{S})(k, \cdot) \) is in \( L^1(e^{-\gamma_2 t}dp) \) for any \( \omega > 0 \), then a smooth global solution exists and iv) is proved. This shows the Boussinesq existence theorem. The proof of Theorem \( \text{[22]} \) is very similar.

### 7 Borel-Summability

We now show Borel-summability of the solutions guaranteed by Theorem \( \text{[21]} \) and Theorem \( \text{[22]} \) for \( \beta > 0 \). This requires us to show that the solutions \( (\hat{H}, \hat{S})(k, p) \) and \( (\hat{W}, \hat{Q})(k, p) \) to the Boussinesq and MHD equations, respectively, are analytic in \( p \) for \( p \in \{0\} \cup \mathbb{R}^+ \). First, we will seek a solution which is a power series

\[
(\hat{H}, \hat{S})(k, p) - (\hat{u}_1, \hat{\Theta}_1)(k) = \sum_{i=1}^{\infty} (\hat{H}^{[i]}, \hat{S}^{[i]})(k)p^i \tag{55}
\]

\[
(\hat{W}, \hat{Q})(k, p) - (\hat{v}_1, \hat{B}_1)(k) = \sum_{i=1}^{\infty} (\hat{W}^{[i]}, \hat{Q}^{[i]})(k)p^i. \tag{56}
\]

**Remark 71** We will use induction to bound the successive terms of the power series. Many of these bounds have constants depending on the dimension in \( k \) as before. For brevity of notation the dependence on dimension is suppressed after introducing the constants.

For the purpose of finding power series solutions, \( \text{[23]} \) and \( \text{[25]} \) are not good representations of the equations. By construction, \( \sum \mathcal{G}(z, z') \) satisfies...
We will also use the fact that
\[ l \]
For
\[ \frac{\partial_p}{l} \]
\[ \partial_{p'} \left( \frac{x}{z} \right) \rightarrow 0 \] and \[ \partial_{p} \left( \frac{x}{z} \right) \rightarrow \frac{1}{p} \] as \( p' \) approaches \( p \) from below. Hence, we have the equivalent equations
\[
[p \partial_{pp} + 2 \partial_{p} + \nu|k|^2] \tilde{H} = i k_j \tilde{G}^{[1]}_j + a \tilde{P}_k [\hat{e}_2 \hat{S}] \\
[p \partial_{pp} + 2 \partial_{p} + \nu|k|^2] \tilde{S} = i k_j \tilde{G}^{[2]}_j
\]
and
\[
[p \partial_{pp} + 2 \partial_{p} + \nu|k|^2] \tilde{W} = i k_j \tilde{G}^{[3]}_j \\
[p \partial_{pp} + 2 \partial_{p} + \nu|k|^2] \tilde{Q} = i k_j \tilde{G}^{[4]}_j.
\]
We substitute (55) into (57) and (56) into (58) and identify powers of \( p' \) to get a relationship for the coefficients. We will use that \( 1 \ast p' = p'^{l+1}/(l+1) \).
We will also use the fact that
\[
p' \ast p^n = \frac{ln!}{(l + n + 1)!} p'^{l+n+1}.
\]
For \( l = 0 \), we have
\[
2 \hat{H}^{[1]} = -ik_j \tilde{P}_k [\hat{u}_{1,j} \hat{\tilde{u}}_0 + \hat{u}_{0,j} \hat{\tilde{u}}_1] - \nu|k|^2 \hat{\tilde{u}}_1 + \tilde{P}_k [ae_2 \hat{\Theta}_1] \\
2 \hat{S}^{[1]} = -ik_j [\hat{u}_{1,j} \hat{\tilde{\Theta}}_0 + \hat{u}_{0,j} \hat{\tilde{\Theta}}_1] - \mu|k|^2 \hat{\tilde{\Theta}}_1
\]
and
\[
2 \hat{W}^{[1]} = -ik_j \tilde{P}_k \left[ \hat{\tilde{v}}_{1,j} \hat{\tilde{v}}_0 + \hat{\tilde{v}}_{0,j} \hat{\tilde{v}}_1 - \left( \frac{\hat{B}_{0,j} \hat{B} + \hat{B}_{1,j} \hat{B} + \hat{B}_{0,j} \hat{B}}{\mu \rho} \right) \right] - \nu|k|^2 \hat{\tilde{v}}_1
\]
\[
2 \hat{Q}^{[1]} = -ik_j \tilde{P}_k \left[ \hat{\tilde{v}}_{1,j} \hat{\tilde{B}}_0 + \hat{\tilde{v}}_{0,j} \hat{\tilde{B}}_1 - (\hat{B}_{1,j} \hat{\tilde{v}}_0 + \hat{B}_{0,j} \hat{\tilde{v}}_1) \right] - \frac{1}{\mu \sigma} |k|^2 \hat{\tilde{B}}_1.
\]
For \( l = 1 \), we have
\[
6 \hat{H}^{[2]} + \nu|k|^2 \hat{H}^{[1]} = -ik_j \tilde{P}_k [\hat{H}^{[1]}_j \hat{\tilde{u}}_0 + \hat{u}_{0,j} \hat{\tilde{H}}^{[1]}_j + \hat{u}_{1,j} \hat{\tilde{u}}_1] + \tilde{P}_k [ae_2 \hat{S}^{[1]}_j] \\
6 \hat{S}^{[2]} + \mu|k|^2 \hat{S}^{[1]} = -ik_j [\hat{S}^{[1]}_j \hat{\tilde{\Theta}}_0 + \hat{u}_{0,j} \hat{\tilde{S}}^{[1]}_j + \hat{u}_{1,j} \hat{\tilde{\Theta}}_1]
\]
and
\[
6 \hat{W}^{[2]} + \nu|k|^2 \hat{W}^{[1]} = -ik_j \tilde{P}_k [\hat{W}^{[1]}_j \hat{\tilde{v}}_0 + \hat{v}_{0,j} \hat{\tilde{W}}^{[1]}_j + \hat{v}_{1,j} \hat{\tilde{v}}_1] \\
+ \frac{ik}{\mu \rho} \tilde{P}_k [\hat{Q}^{[1]}_j \hat{\tilde{B}}_0 + \hat{B}_{0,j} \hat{\tilde{Q}}^{[1]}_j + \hat{B}_{1,j} \hat{\tilde{B}}_1]
\]
\[
6 \hat{Q}^{[2]} + \frac{1}{\mu \sigma} |k|^2 \hat{Q}^{[1]} = -ik_j \tilde{P}_k [\hat{W}^{[1]}_j \hat{\tilde{B}}_0 + \hat{v}_{0,j} \hat{\tilde{Q}}^{[1]}_j + \hat{v}_{1,j} \hat{\tilde{B}}_1] \\
+ \frac{ik}{\mu \rho} \tilde{P}_k [\hat{Q}^{[1]}_j \hat{\tilde{v}}_0 + \hat{B}_{0,j} \hat{\tilde{Q}}^{[1]}_j + \hat{B}_{1,j} \hat{\tilde{v}}_1].
\]
In the MHD case, we have

\[
(l + 1)(l + 2)\hat{H}^{[l+1]} = -\nu|k|^2\hat{H}^{[l]} - ik_P \sum_{l_1=1}^{l-2} \frac{l_1!(l-l_1-1)!}{l!} \hat{H}^{[l_1]} \ast \hat{H}^{[l-l_1-1]} - ik_P [\tilde{u}_{0,j} \ast \hat{H}^{[l]} + \hat{H}^{[l]} \ast \tilde{u}_0 + \frac{1}{l} \tilde{u}_{1,j} \ast \hat{H}^{[l]} + \frac{1}{l} \hat{H}^{[l]} \ast \tilde{u}_1 + P_k [\eta_2 \tilde{S}^{[l]}]]
\]

(63)

More generally, for \(l \geq 2\) in the Boussinesq case, we have

\[
(l + 1)\tilde{S}^{[l+1]} = -\mu|k|^2\tilde{S}^{[l]} - ik_P \sum_{l_1=1}^{l-2} \frac{l_1!(l-l_1-1)!}{l!} \hat{S}^{[l_1]} \ast \hat{S}^{[l-l_1-1]} - ik_P [\tilde{u}_{0,j} \ast \tilde{S}^{[l]} + \hat{H}^{[l]} \ast \tilde{S}_0 + \frac{1}{l} \tilde{u}_{1,j} \ast \tilde{S}^{[l]} + \frac{1}{l} \hat{H}^{[l]} \ast \tilde{S}_1].
\]

(64)

In the MHD case, we have

\[
(l + 1)\hat{W}^{[l+1]} = -\nu|k|^2\hat{W}^{[l]} - ik_P \sum_{l_1=1}^{l-2} \frac{l_1!(l-l_1-1)!}{l!} \hat{W}^{[l_1]} \ast \hat{W}^{[l-l_1-1]} + \tilde{v}_{0,j} \ast \hat{W}^{[l]} + \hat{W}^{[l]} \ast \tilde{v}_0 + \frac{1}{l} \tilde{v}_{1,j} \ast \hat{W}^{[l]} + \frac{1}{l} \hat{W}^{[l]} \ast \tilde{v}_1 + ik_P [\tilde{\theta}_{0,j} \ast \hat{Q}^{[l]} + \hat{Q}^{[l]} \ast \tilde{\theta}_0] + \frac{ik_P}{\mu \rho} P_k [\tilde{B}_{0,j} \ast \hat{Q}^{[l]} + \hat{Q}^{[l]} \ast \tilde{B}_0] + \sum_{l_1=1}^{l-2} \frac{l_1!(l-l_1-1)!}{l!} \hat{Q}^{[l_1]} \ast \hat{Q}^{[l-l_1-1]} + \frac{1}{l} \tilde{B}_{1,j} \ast \hat{Q}^{[l]} + \frac{1}{l} \hat{Q}^{[l]} \ast \tilde{B}_1]
\]

(65)

\[
(l + 1)\hat{Q}^{[l+1]} = -\frac{1}{\mu \sigma} \hat{Q}^{[l]} - ik_P \sum_{j_1=1}^{l-2} \frac{l_1!(l-l_1-1)!}{l!} \hat{W}^{[l_1]} \ast \hat{Q}^{[l-l_1-1]} + \tilde{v}_{0,j} \ast \hat{Q}^{[l]} + \hat{Q}^{[l]} \ast \tilde{v}_0 + \frac{1}{l} \tilde{v}_{1,j} \ast \hat{Q}^{[l]} + \frac{1}{l} \hat{Q}^{[l]} \ast \tilde{v}_1 + ik_P [\tilde{\theta}_{0,j} \ast \hat{Q}^{[l]} + \hat{Q}^{[l]} \ast \tilde{\theta}_0] + \frac{1}{l} \tilde{B}_{1,j} \ast \hat{Q}^{[l]} + \frac{1}{l} \hat{Q}^{[l]} \ast \tilde{B}_1]
\]

(66)

\[
\hat{Q}_n(y) = \sum_{j=0}^{n} 2^{n-j} \frac{y^j}{j!}
\]

Definition 72 It is useful to define a n-th order polynomial, call it \(Q_n\),

Definition 73 It is also useful to define constants

\[
M_1 = \max(\nu, \mu)
\]

\[
M_2 = \max \left( \nu, \frac{1}{\mu \sigma} \right)
\]

\[
M_3 = \max \left( 1, \frac{1}{\mu \rho} \right)
\]
Lemma 74 If \( \| \langle \dot{u}_0, \dot{\Theta}_0 \rangle \|_{\gamma+2, \beta} < \infty \) and \( \| \langle \dot{v}_0, \dot{B}_0 \rangle \|_{\gamma+2, \beta} < \infty \) for \( \gamma > d \) and \( \beta > 0 \), then there are constants \( A_0, D_0, \tilde{D}_0 > 0 \) not depending on \( l \) or \( k \) such that

\[
|\langle \hat{H}^{[l]}, \hat{S}^{[l]} \rangle| \leq e^{-\beta |k|} A_0 D_0 (1 + |k|)^{-\gamma} Q_2(\beta |k|) (2l + 1)^2 \tag{67}
\]

\[
|\langle \hat{W}^{[l]}, \hat{Q}^{[l]} \rangle| \leq e^{-\beta |k|} \tilde{A}_0 \tilde{D}_0 (1 + |k|)^{-\gamma} Q_2(\beta |k|) (2l + 1)^2. \tag{68}
\]

Furthermore, the solutions guaranteed to exist in Lemma 53 have convergent power series representations in \( p \), and for \( \| p \| < (4D_0)^{-1} \),

\[
\langle \hat{H}, \hat{S} \rangle(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k) + \sum_{l=1}^{\infty} (\hat{H}^{[l]}, \hat{S}^{[l]}) (k)p^l
\]

and for \( \| p \| < (4\tilde{D}_0)^{-1} \),

\[
\langle \hat{W}, \hat{Q} \rangle(k, p) = (\hat{v}_1, \hat{B}_1)(k) + \sum_{l=1}^{\infty} (\hat{W}^{[l]}, \hat{Q}^{[l]}) (k)p^l.
\]

To prove this lemma we will establish bounds for \( \langle \hat{H}^{[l]}, \hat{S}^{[l]} \rangle \) and \( \langle \hat{W}^{[l]}, \hat{Q}^{[l]} \rangle \) using induction.

Lemma 75 For the base case, we have

\[
|\langle \hat{H}^{[1]}, \hat{S}^{[1]} \rangle(k, p)| \leq \frac{e^{-\beta |k|} Q_2(\beta |k|) A_0 D_0}{(1 + |k|)^{\gamma} 9} \tag{69}
\]

\[
|\langle \hat{W}^{[1]}, \hat{Q}^{[1]} \rangle(k, p)| \leq \frac{e^{-\beta |k|} Q_2(\beta |k|) \tilde{A}_0 \tilde{D}_0}{(1 + |k|)^{\gamma} 9} \tag{70}
\]

for

\[
A_0 D_0 \geq 9 \beta^2 \| (\hat{u}_1, \hat{\Theta}_1) \|_{\gamma, \beta} \left( C_0 \beta \| (\hat{u}_0, \hat{\Theta}_0) \|_{\gamma, \beta} + M_1 + 4\beta^2 \right)
\]

\[
\tilde{A}_0 \tilde{D}_0 \geq 9 \beta^2 \| (\hat{v}_1, \hat{B}_1) \|_{\gamma, \beta} M_2 M_3 \left( 1 + C_0 \beta \| (\hat{v}_0, \hat{B}_0) \|_{\gamma, \beta} \right)
\]

Proof From \( \| \langle \dot{u}_0, \dot{\Theta}_0 \rangle \|_{\gamma+2, \beta} < \infty \) and Lemma 53, we get

\[
|\langle \hat{H}^{[1]}, \hat{S}^{[1]} \rangle(k, p)| \leq \frac{e^{-\beta |k|}}{2(1 + |k|)^{\gamma}} \left( |k|^2 \| (\hat{u}_1, \hat{\Theta}_1) \|_{\gamma, \beta} M_1 \right. \tag{71}
\]

\[
\left. + 2 C_0 |k| \left\| (\hat{u}_0, \hat{\Theta}_0) \right\|_{\gamma, \beta} \| (\hat{u}_1, \hat{\Theta}_1) \|_{\gamma, \beta} + a \| \hat{\Theta}_1 \|_{\gamma, \beta} \right)
\]

\[
|\langle \hat{W}^{[1]}, \hat{Q}^{[1]} \rangle(k, p)| \leq \frac{e^{-\beta |k|} M_2 M_3}{2(1 + |k|)^{\gamma}} \left( |k|^2 + 4 C_0 |k| \left\| (\hat{v}_0, \hat{B}_0) \right\|_{\gamma, \beta} \right) \tag{72}
\]

The result now follows from \( \| \langle \dot{u}_0, \dot{\Theta}_0 \rangle \|_{\gamma+2, \beta} < \infty \) and \( \| \langle \dot{v}_0, \dot{B}_0 \rangle \|_{\gamma+2, \beta} < \infty \) after noting that \( Q_2(\beta |k|) = 4 + 2\beta |k| + 1/2(\beta |k|)^2 \).
For the general terms we will need a series of lemmas, which depend heavily on the lemmas developed in the Fourier inequalities section, bounding the terms that appear on the right side of (65) and (65).

**Lemma 76** Assume that \((\hat{H}^l, \hat{S}^l)\) satisfies \((67)\) and \((\hat{W}^l, \hat{Q}^l)\) satisfies \((68)\) for \(l \geq 1\). Then we have,

\[
\frac{|k|^2|\hat{H}^l, \hat{S}^l|}{(l+1)(l+2)} \leq \frac{6A_0D_0^l e^{-\beta|k|}Q_{2l+2}(\beta|k|)}{\beta^2(1+|k|)^\gamma(2l+3)^2},
\]

\[
\frac{|k|^2|\hat{W}^l, \hat{Q}^l|}{(l+1)(l+2)} \leq \frac{6\tilde{A}_0\tilde{D}_0^l e^{-\beta|k|}Q_{2l+2}(\beta|k|)}{\beta^2(1+|k|)^\gamma(2l+3)^2}.
\]

**Proof** The proof follows from \((67)\) or \((68)\) directly by noting that for \(y \geq 0\)

\[
\frac{y^2Q_{2l}(y)}{(2l+2)(2l+1)} \leq Q_{2l+2}(y) \quad \text{and} \quad \frac{(2l+2)(2l+3)^2}{(l+1)(l+2)(2l+1)} \leq 6.
\]

**Lemma 77** Suppose \((\hat{H}^l, \hat{S}^l)\) satisfies \((67)\) and \((\hat{W}^l, \hat{Q}^l)\) satisfies \((68)\) for \(l \geq 1\). Then both

\[
\frac{1}{(l+1)(l+2)} \left| k_j \left( P_k(\hat{u}_{0,j} \ast \hat{H}^l), \hat{u}_{0,j} \ast \hat{S}^l \right) \right| \quad \text{and} \quad \frac{1}{(l+1)(l+2)} \left| k_j \left( P_k(\hat{H}^l_j \ast \hat{u}_0), \hat{H}^l_j \ast \hat{\vartheta}_0 \right) \right|
\]

are bounded by

\[
2^\gamma|||(\tilde{u}_0, \tilde{\vartheta}_0)|||_{\gamma, \beta} \frac{9C\pi A_0D_0^l e^{-\beta|k|}}{2^\beta \cdot (2l+3)^2 (1+|k|)^\gamma Q_{2l+2}(|\beta k|)}.
\]

Similarly,

\[
\frac{1}{(l+1)(l+2)} \left| k_j \left( P_k(\hat{v}_{0,j} \ast \hat{W}^l), P_k(\hat{v}_{0,j} \ast \hat{Q}^l) \right) \right|, \quad \frac{1}{(l+1)(l+2)} \left| k_j \left( P_k(\hat{W}^l_j \ast \hat{v}_0), P_k(\hat{W}^l_j \ast \hat{B}_0) \right) \right|, \quad \frac{1}{(l+1)(l+2)} \left| k_j \left( P_k(\hat{S}^l_j \ast \hat{B}_0), P_k(\hat{Q}^l_j \ast \hat{v}_0) \right) \right|
\]

are bounded by

\[
2^\gamma|||(\tilde{v}_0, \tilde{B}_0)|||_{\gamma, \beta} \frac{9C\pi \tilde{A}_0\tilde{D}_0^l e^{-\beta|k|}}{2^\beta \cdot (2l+3)^2 (1+|k|)^\gamma Q_{2l+2}(|\beta k|)}.
\]

**Proof** We use the estimate \((67)\) on \((\hat{H}^l, \hat{S}^l)\) and Lemma \((91)\) in \(\mathbb{R}^d\) with \(n = 0\) to get

\[
|k_j\hat{v}_{0,j} \ast (\hat{H}^l, \hat{S}^l)| \leq ||\hat{u}_0||_{\gamma, \beta} A_0D_0^l \left( \frac{2l+1}{(2l+1)^2} \right) \left( \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta|k'|+|k-k'|}}{(1+|k'|)^\gamma(1+|k-k'|)^\gamma} Q_{2l}(|\beta k'|)dk' \right)
\]

\[
\leq \frac{||\hat{u}_0||_{\gamma, \beta} A_0D_0^l}{(2l+1)^2} \sum_{m=0}^{2l-m} \frac{2^{2l-m}}{m!} \left| k \right| \left( \int_{k' \in \mathbb{R}^d} \frac{e^{-\beta|k'|+|k-k'|}}{(1+|k'|)^\gamma(1+|k-k'|)^\gamma} |\beta k'|^mdk' \right).
\]
\[ \leq \frac{C_7 \pi ||\hat{u}_0||_{\gamma, \beta} A_0 D_0^2 e^{-\beta|k|}}{(2l + 1)^2 \beta^d (1 + |k|)^\gamma} \sum_{m=0}^{2l} 2^{2l-m} (m+2) Q_{m+2}(\beta|k|) \]
\[ \leq 2^\gamma C_7 \pi ||\hat{u}_0||_{\gamma, \beta} A_0 D_0 e^{-\beta|k|} \left( l + 2 \right) Q_{2l+2}(\beta|k|). \]

The first part of the lemma now follows noting that \( \frac{2(2l+3)^2}{(2l+1)(l+1)} \leq 9 \) for \( l \geq 1 \). For the other four terms, we use the estimate \( \frac{9}{2l+1} \) on \( \hat{W}^{(l)} \), \( \hat{Q}^{(l)} \) and Lemma \ref{lem:9} in \( \mathbb{R}^d \) with \( n = 0 \). Hence, the proof is the same as that given above with \( A_0 \) in place of \( A_0 \) and \( D_0 \) in place of \( D_0 \).

**Lemma 78** Suppose \( \hat{H}^{(l-1)}, \hat{S}^{(l-1)} \) satisfies \( \ref{eq:67} \) and \( \hat{W}^{(l-1)}, \hat{Q}^{(l-1)} \) satisfies \( \ref{eq:68} \) for \( l \geq 2 \). Then both
\[ \frac{1}{l(l+1)} \left| k_j \left( P_k(\hat{u}_{1,j} \hat{H}^{(l-1)}), \hat{u}_{1,j} \hat{S}^{(l-1)} \right) \right| \text{ and } \frac{1}{l(l+1)} \left| k_j \left( P_k(\hat{H}_j^{(l-1)} \hat{u}_{1}), \hat{H}_j^{(l-1)} \hat{\Theta}_1 \right) \right| \]
are bounded by
\[ 2^{\gamma} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \frac{9C_7 \pi A_0 D_0 e^{-\beta|k|} Q_{2l}(\beta|k|)}{2^{3d} (2l + 1)^2 (1 + |k|)^\gamma}. \]

Similarly,
\[ \frac{1}{l(l+1)} \left| k_j \left( P_k(\hat{u}_{1,j} \hat{W}^{(l-1)}), P_k(\hat{v}_{1,j} \hat{Q}^{(l-1)}) \right) \right|, \frac{1}{l(l+1)} \left| k_j \left( P_k(\hat{W}_j^{(l-1)} \hat{v}_{1}), P_k(\hat{W}_j^{(l-1)} \hat{B}_1) \right) \right|, \]
\[ \frac{1}{l(l+1)} \left| k_j \left( P_k(\hat{B}_1,j \hat{Q}^{(l-1)}), P_k(\hat{B}_{1,j} \hat{W}^{(l-1)}) \right) \right|, \text{ and } \frac{1}{l(l+1)} \left| k_j \left( P_k(\hat{Q}_j^{(l-1)} \hat{B}_1), P_k(\hat{Q}_j^{(l-1)} \hat{v}_{1}) \right) \right| \]
are bounded by
\[ 2^{\gamma} ||(\hat{u}_1, \hat{B}_1)||_{\gamma, \beta} \frac{9C_7 \pi \hat{A}_0 \hat{D}_0 e^{-\beta|k|} Q_{2l}(\beta|k|)}{2^{3d} (2l + 1)^2 (1 + |k|)^\gamma}. \]

The proof is the same as that for Lemma \ref{lem:7} with \( l \) replaced by \( l - 1 \) and \( (\hat{u}_0, \hat{\Theta}_0) \) replaced by \( (\hat{u}_j, \hat{\Theta}_1) \) or \( (\hat{v}_0, \hat{B}_0) \) replaced by \( (\hat{v}_j, \hat{B}_1) \).

**Lemma 79** Let \( l \geq 3 \). Suppose \( \hat{H}^{(l)}, \hat{S}^{(l)} \) and \( \hat{H}^{(l-1)}, \hat{S}^{(l-1)} \) satisfy \( \ref{eq:69} \) and \( \hat{W}^{(l)}, \hat{Q}^{(l)} \) and \( \hat{W}^{(l-1)}, \hat{Q}^{(l-1)} \) satisfy \( \ref{eq:69} \) for \( l_1 = 1, \ldots, l - 2 \). Then
\[ \left| \sum_{l_1=1}^{l-3} \frac{l_1!(l - 1 - l_1)!}{l!} \left( P_k(\hat{H}_j^{(l)} \hat{H}^{(l-1)}), \hat{H}_j^{(l)} \hat{S}^{(l-1)} \right) \right| \]
is bounded by
\[ 2^{\gamma+3} C_7 A_0 D_0 e^{-\beta|k|} Q_{2l}(\beta|k|) \frac{\beta^d (2l + 3)^2}{\beta^d (2l + 3)^2}. \]
Similarly, both

\[
\left| \frac{k_j}{(l+1)(l+2)} \sum_{l_1=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l!} \left( P_k(\tilde{W}_j^{[l_1]} \ast \tilde{W}^{[l-1-l_1]}), P_k(\tilde{Q}_j^{[l_1]} \ast \tilde{Q}^{[l-1-l_1]}) \right) \right|
\]

and

\[
\left| \frac{k_j}{(l+1)(l+2)} \sum_{l_1=1}^{l-2} \frac{l_1!(l-1-l_1)!}{l!} \left( P_k(\tilde{Q}_j^{[l_1]} \ast \tilde{Q}^{[l-1-l_1]}), P_k(\tilde{Q}_j^{[l_1]} \ast \tilde{W}^{[l-1-l_1]}) \right) \right|
\]

are bounded by

\[
2^{\gamma+3}C^2 \tilde{A}_0^2 \tilde{D}_0^{-1} (1 + |k|)^{-\gamma} e^{-\beta |k|} \frac{Q_{2l}(\beta |k|)}{\beta^d (2l + 3)^2}.
\]

**Proof** The proof is similar to that in [5] with \(\tilde{W}^{[l_1]}\) replaced by \((\tilde{W}^{[l_2]}, \tilde{Q}^{[l_2]}))\). If \(l \geq 3\) then \(l_2 = l - l_1 - 1 \geq 0\) for \(l_1 = 1, \ldots, l - 2\) and we apply Lemma \[12\] in \(\mathbb{R}^d\) giving,

\[
l_1! l_2! (k_j \tilde{W}_j^{[l_1]} \ast (\tilde{W}^{[l_2]}, \tilde{Q}^{[l_2]})) \leq \frac{l_1! l_2!}{l!(2l_1 + 1)^2(2l_2 + 1)^2} \tilde{A}_0^2 \tilde{D}_0^{-1} |k|.
\]

\[
\int_{k' \in \mathbb{R}^d} e^{-\beta (|k'| + |k - k'|)} (1 + |k'|)^{-\gamma} (1 + |k - k'|)^{-\gamma} \frac{Q_{2l_1}(\beta |k'|) Q_{2l_2}(\beta |k - k'|) dk'}{(2l_1 + 1)^2(2l_2 + 1)^2} \leq \frac{C_l l_1! l_2! \tilde{A}_0^2 \tilde{D}_0^{-1} \beta^{2l}}{3\beta^d (1 + |k|)^{\gamma}} Q_{2l}(\beta |k|).
\]

Thus,

\[
\sum_{l_1=1}^{l-2} \frac{l_1! l_2!}{l!(l+1)(l+2)} \left| k_j \tilde{W}_j^{[l_1]} \ast (\tilde{W}^{[l_2]}, \tilde{Q}^{[l_2]})) \right| \leq \frac{C_l l_1! l_2! \tilde{A}_0^2 \tilde{D}_0^{-1} \beta^{2l}}{3\beta^d (1 + |k|)^{\gamma}} Q_{2l}(\beta |k|) \sum_{l_1=1}^{l-2} \frac{l_1! l_2!}{(l-1)!(2l_1 + 1)^2(2l_2 + 1)^2}.
\]

After noting that \(\frac{l_1! l_2!}{(l-1)!} \leq 1, \frac{(2l_1 + 1)(2l_2 + 1)}{(l+1)(l+2)} \leq 4\), and

\[
\sum_{l_1=1}^{l-2} \frac{1}{(2l_1 + 1)^2(2l_2 + 1)^2} \leq \frac{C}{(2l + 3)^2},
\]

where \(C = 1.0755 \ldots \leq 3\), the second inequality is proved. The others are done in the same manner.

**Lemma 710** For \(l = 2\) we have,

\[
|\tilde{H}^{[2]}, \tilde{S}^{[2]}| \leq e^{-\beta |k|} \frac{Q_4(\beta |k|)}{5^2(1 + |k|)^{\gamma}} \left( \frac{6A_0 D_0 M_1}{\beta^2} + \frac{2^9 9C \pi A_0 D_0 \{ |\tilde{u}_0, \tilde{Q}_0| \} \gamma}{\beta^d} \right)
\]
\[ + A_0 D_0 a + \frac{C_0}{\beta} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta}^2 \]
\[ ||(\hat{W}^{[2]}, \hat{Q}^{[2]})|| \leq \frac{e^{-\beta|k|} Q_4(\beta|k|)}{5^2(1 + |k|)^{\gamma}} \left( 6 \tilde{A}_0 \tilde{D}_0 M_2 \right) \frac{+ \frac{2^{\gamma+1}9C_7 \pi M_3 \tilde{A}_0 \tilde{D}_0 ||(\hat{v}_0, \hat{B}_0)||_{\gamma, \beta}}{\beta^d} \right) \]
\[ + M_2 \frac{2C_0}{\beta} ||(\hat{v}_1, \hat{B}_1)||_{\gamma, \beta}^2 \]

Thus, \((\hat{H}^{[2]}, \hat{S}^{[2]})\) satisfies (67) and \((\hat{W}^{[2]}, \hat{Q}^{[2]})\) satisfies (68) for
\[ D_0^2 \geq \frac{6D_0 M_1}{\beta^2} + D_0 a + \frac{2^{\gamma+1}9C_7 \pi D_0}{\beta^d} ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} + \frac{C_0}{A_0 \beta} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta}^2 \] (73)
and
\[ \tilde{D}_0^2 \geq \frac{6 \tilde{D}_0 M_2}{\beta^2} + \frac{2^{\gamma+1}9C_7 \pi \tilde{M}_3 \tilde{D}_0 ||(\hat{v}_0, \hat{B}_0)||_{\gamma, \beta}}{\beta^d} + \frac{2C_0 \tilde{M}_3 ||(\hat{v}_1, \hat{B}_1)||_{\gamma, \beta}^2}{A_0 \beta} \] (74)

**Proof** We start from (61) or (62). For the first term we use Lemma 76. For the second term, appearing in (73), the Boussinesq case only, we use our induction assumption and \( \frac{Q_4(\beta|k|)}{4^3} \leq \frac{Q_4(\beta|k|)}{2^5} \). For the next term, we use Lemma 77. For the last term, apply Corollary 52 and use \(|k|/6 \leq \frac{Q_4(\beta|k|)}{2^5 \beta^3} \).

**Proof of Lemma 74** The base case is proved picking \( D_0 \) and \( \tilde{D}_0 \) large enough so (73), (74), (69), and (71) hold. For general \( l \geq 2 \) suppose \((\hat{H}^{[m]}, \hat{S}^{[m]})\) satisfies (67) and \((\hat{W}^{[m]}, \hat{Q}^{[m]})\) satisfies (68) for \( m = 1, \ldots, l \). We estimate terms on the right of (63), (64), (65), and (66), using Lemma 76, 77, 78 and 79 and the fact that \( Q_2(y) \leq 1/4Q_{2l+2}(y) \), to get
\[ ||(\hat{H}^{[l+1]}, \hat{S}^{[l+1]})|| \leq \frac{A_0 \tilde{D}_0^{l-1} Q_{2l+2}(\beta|k|)}{(2l + 3)^2(1 + |k|)^{\gamma}} \left( \frac{6D_0 M_1}{\beta^2} + \frac{aD_0}{2} + \frac{2^{\gamma+1}9C_7 \pi D_0}{\beta^d} ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} \right) \]
\[ + \frac{2^{\gamma+1}9C_7 \pi (2l + 3)^2}{4(l + 2)(2l + 1)^2 \beta^d} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} + \frac{2^{\gamma+1}9C_7 \pi A_0}{4\beta^d} \]
\[ \leq \frac{A_0 \tilde{D}_0^{l-1} e^{-\beta|k|}}{(1 + |k|)^{\gamma}(2l + 3)^2} Q_{2l+2}(\beta|k|) \]

and
\[ ||(\hat{W}^{[l+1]}, \hat{Q}^{[l+1]})|| \leq \frac{\tilde{A}_0 \tilde{D}_0^{l-1} Q_{2l+2}(\beta|k|)}{(2l + 3)^2(1 + |k|)^{\gamma}} \left( \frac{6 \tilde{D}_0 M_2}{\beta^2} + M_3 \left[ \frac{2^{\gamma+1}9C_7 \pi \tilde{D}_0}{\beta^d} ||(\hat{v}_0, \hat{B}_0)||_{\gamma, \beta} \right] \right) \]
\[ + \frac{2^{\gamma+1}9C_7 \pi (2l + 3)^2}{4(l + 2)(2l + 1)^2 \beta^d} ||(\hat{v}_1, \hat{B}_1)||_{\gamma, \beta} + \frac{2^{\gamma+1}9C_7 \pi \tilde{D}_0}{4\beta^d} \]
\[ \leq \frac{\tilde{A}_0 \tilde{D}_0^{l-1} e^{-\beta|k|}}{(1 + |k|)^{\gamma}(2l + 3)^2} Q_{2l+2}(\beta|k|), \]

where \( D_0 \) has been chosen large enough so
\[ \left\{ \frac{6D_0 M_1}{\beta^2} + \frac{aD_0}{2} + \frac{2^{\gamma+1}9C_7 \pi D_0}{\beta^d} ||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma, \beta} + \frac{2^{\gamma+1}9C_7 \pi D_0}{4\beta^d} ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \right\} \]
\[ + \frac{2^{\gamma+1} C_7 A_0}{\beta^d} \right\} \leq D_0^2 \]

and \( \bar{D}_0 \) large enough so

\[
\left\{ \frac{6\bar{D}_0 M_2}{\beta^2} + M_3 \left[ \frac{2^{\gamma+1} 9 C_7 \pi \bar{D}_0}{\beta^d} \right][|\hat{v}_0, \hat{B}_0|]_{\gamma, \beta} + \frac{2^{\gamma+1} 9 C_7 \pi}{4 \beta^d} [||\hat{v}_1, \hat{B}_1||]_{\gamma, \beta} + \frac{2^{\gamma+2} C_7 A_0}{\beta^d} \right\} \leq \bar{D}_0^2. \]

We also used \((2\gamma+3)^2 \leq 1\) in the above. Thus, by induction, we have (67) and (68) satisfied for any \( l \geq 1 \). So, \( \sum_{l=1}^{\infty} (\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l \) is convergent for \( |p| \leq \frac{\gamma, \beta}{2} \) and \( \sum_{l=1}^{\infty} (\hat{W}^{[l]}, \hat{Q}^{[l]})(k)p^l \) is convergent for \( |p| \leq \frac{1}{4\bar{D}_0} \) since \( Q_{9\gamma} (\beta |k|) \leq 4^{1/\beta} |k|/2 \). By construction of the iteration, \((\hat{H}, \hat{S}) - (\hat{u}_1, \hat{\Theta}_1) = \sum_{l=1}^{\infty} (\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l \) is a solution to (57) which is zero at \( p = 0 \). Similarly, \((\hat{W}, \hat{Q}) - (\hat{v}_1, \hat{B}_1) = \sum_{l=1}^{\infty} (\hat{W}^{[l]}, \hat{Q}^{[l]})(k)p^l \) is a solution to (58) which is zero at \( p = 0 \). However, we know there are unique solutions to (57) and (58) which are zero and \( p = 0 \) in the space \( \mathcal{A}_C^\infty \), which includes analytic functions at the origin for \( L \) sufficiently small. Thus, for \((\hat{H}, \hat{S}) \) and \((\hat{W}, \hat{Q}) \) the solutions guaranteed by Lemma 57 we have,

\[
(\hat{H}, \hat{S})(k, p) = (\hat{u}_1, \hat{\Theta}_1)(k) + \sum_{l=1}^{\infty} (\hat{H}^{[l]}, \hat{S}^{[l]})(k)p^l
\]

\[
(\hat{W}, \hat{Q})(k, p) = (\hat{v}_1, \hat{B}_1)(k) + \sum_{l=1}^{\infty} (\hat{W}^{[l]}, \hat{Q}^{[l]})(k)p^l.
\]

Estimates on \( \partial_p^l (\hat{H}, \hat{S})(k, p) \) and \( \partial_p^l (\hat{W}, \hat{Q})(k, p) \)

We now want to develop estimates on \( \partial_p^l (\hat{H}, \hat{S})(k, p) \) and \( \partial_p^l (\hat{W}, \hat{Q})(k, p) \) in order to show that the series about any \( p = p_0 \in \mathbb{R}^+ \) is convergent. We will proceed in the same spirit as above. That is to use induction to bound the successive derivatives. Our goal is to show that we can analytically extend our solutions along \( \mathbb{R}^+ \) with a radius of convergence independent of center \( p_0 \) along \( \mathbb{R}^+ \). Combining this with the fact that the solutions are exponentially bounded will give Borel Summability.

**Definition 711** For \( l \geq 1 \) we define,

\[
(\hat{H}^{[l]}, \hat{S}^{[l]})(k, p) = \frac{1}{l!} \partial_p^l (\hat{H}, \hat{S})(k, p)
\]

\[
(\hat{H}^{[0]}, \hat{S}^{[0]})(k, p) = (\hat{H}, \hat{S})(k, p) - (\hat{u}_1, \hat{\Theta}_1)
\]

\[
(\hat{W}^{[l]}, \hat{Q}^{[l]})(k, p) = \frac{1}{l!} \partial_p^l (\hat{W}, \hat{Q})(k, p)
\]

\[
(\hat{W}^{[0]}, \hat{Q}^{[0]})(k, p) = (\hat{W}, \hat{Q})(k, p) - (\hat{v}_1, \hat{B}_1).
\]
Lemma 7.12 If \(||(\hat{u}_0, \hat{\Theta}_0)||_{\gamma,2,\beta} < \infty \) and \(||(\hat{v}_0, \hat{B}_0)||_{\gamma+2,\beta} < \infty\) for and \(\beta > 0\), then there are constants \(A, D, \bar{A}, \bar{D} > 0\) not depending on \(l, k, p\) such that

\[
|\langle H^{[l]}, \tilde{S}^{[l]} \rangle(k, p)| \leq \frac{e^{\omega p e^{-\beta |k|}} A D^l Q_2(|\beta|)}{(1 + p^2)(1 + |k|)^{\gamma}(2l + 1)^2} \tag{75}
\]

\[
|\langle \hat{W}^{[l]}, \hat{Q}^{[l]} \rangle(k, p)| \leq \frac{e^{\alpha p e^{-\beta |k|}} \bar{A} \bar{D}^l Q_2(|\beta|)}{(1 + p^2)(1 + |k|)^{\gamma}(2l + 1)^2} \tag{76}
\]

where \(\omega' = \omega + 1\) and \(\alpha' = \alpha + 1\) for \(\omega\) and \(\alpha\) chosen as in Lemma 57. We will prove the lemma by induction, and as before we will develop several lemmas to establish the bound.

For \(l = 0\), we use Lemma 61 which says that for \(\omega\) and \(\alpha\) sufficiently large

\[
|\langle \hat{H}, \tilde{S} \rangle(k, p)| \leq \frac{2e^{\omega p e^{-\beta |k|}}||\hat{u}_1, \hat{\Theta}_1||_{\gamma,\beta}}{(1 + |k|)^{\gamma}}
\]

\[
|\langle \hat{W}, \hat{Q} \rangle(k, p)| \leq \frac{2e^{\alpha p e^{-\beta |k|}}||\hat{v}_1, \hat{B}_1||_{\gamma,\beta}}{(1 + |k|)^{\gamma}}.
\]

We chose \(\omega' = \omega + 1\) and \(\alpha' = \alpha + 1\) and recall Definition 711 to get

\[
|\langle \hat{H}^{[0]}, \tilde{S}^{[0]} \rangle(k, p)| \leq \frac{3e^{\omega p e^{-\beta |k|}}||\hat{u}_1, \hat{\Theta}_1||_{\gamma,\beta}}{(1 + p^2)(1 + |k|)^{\gamma}}
\]

\[
|\langle \hat{W}^{[0]}, \hat{Q}^{[0]} \rangle(k, p)| \leq \frac{3e^{\alpha p e^{-\beta |k|}}||\hat{v}_1, \hat{B}_1||_{\gamma,\beta}}{(1 + p^2)(1 + |k|)^{\gamma}}.
\]

and the base cases of (75) and (76) are proved for \(A = 3||\hat{u}_1, \hat{\Theta}_1||_{\gamma,\beta}\) and \(\bar{A} = 3||\hat{v}_1, \hat{B}_1||_{\gamma,\beta}\).

For the general case \((l \geq 1)\) we take \(\partial_p^l\) in (57) or (58) and divide by \(l!\), to obtain

\[
pH_{pp}^{(l+2)} + \nu |k|^{2} \hat{H}^{(l)} = \left(-ik_{j} P_{k}[\hat{u}_{0,j} \hat{\Theta} + \hat{u}_{1,j} \hat{\Theta} - \nu |k|^{2} \hat{u}_{1}] \hat{\Theta}_{0} \right)
\]

\[
-ik_{j} P_{k} \left[ \int_{0}^{p} \mathcal{H}_{j}^{(l)}(\nu, p, s) \hat{H}^{(0)}(\nu, s) ds + \sum_{l_{1}=1}^{l-1} \frac{l_{1}! (l - l_{1})!}{l!} \mathcal{H}_{j}^{(l_{1})}(\nu, 0) \hat{H}^{(l-l_{1}-1)}(\nu, p) \right]
\]

\[
-ik_{j} P_{k} \left[ \frac{1}{l!} \left( \hat{u}_{1,j} \hat{\Theta} \right) \hat{H}^{(l-1)} + \hat{H}^{(l-1)} \hat{\Theta} \hat{u}_{1,j} + \hat{H}^{(l)} \hat{\Theta} \hat{u}_{0,j} \hat{\Theta} + \hat{H}^{(l)} \hat{\Theta} \hat{u}_{1,j} \hat{\Theta} \right) \right]
\]

\[
pS_{pp}^{(l+2)} + \nu |k|^{2} \hat{S}^{(l)} = \left(-ik_{j} [\hat{u}_{0,j} \hat{\Theta} + \hat{u}_{1,j} \hat{\Theta} - \mu |k|^{2} \hat{\Theta}] \hat{\Theta}_{0} \right)
\]

\[
-ik_{j} \left[ \int_{0}^{p} \mathcal{H}_{j}^{(l)}(\nu, p, s) \hat{S}^{(0)}(\nu, s) ds + \sum_{l_{1}=1}^{l-1} \frac{l_{1}! (l - l_{1} - 1)!}{l!} \mathcal{H}_{j}^{(l_{1})}(\nu, 0) \hat{S}^{(l-l_{1}-1)}(\nu, p) \right]
\]

(78)
\[ -i k_j \frac{1}{l} (u_{1,j} \hat{s}^{(l-1)} + \hat{H}_j^{(l-1)} \hat{\Theta}_1) + \hat{H}_j^{(l)} \hat{\Theta}_0 + \dot{u}_{0,j} \hat{s}^{(l)} + \delta_{l=1} \dot{u}_{1,j} \hat{\Theta}_1 \]

(79)

and

\[
p\hat{W}_p^{[l]} + (l+2)\hat{W}_p^{[0]} + \nu |k|^2 \hat{W}^{[l]} = \\
-ik_j P_k [\tilde{v}_{0,j} \hat{s} \hat{v}_1 + \hat{B}_{1,j} \hat{\hat{v}}_0] + ik_j P_k [\hat{B}_{0,j} \hat{s} \hat{B}_1 + \hat{B}_{1,j} \hat{B}_0] - \nu |k|^2 \hat{v}_1 \delta_{l,0} \\
-ik_j P_k \left( \int_0^\infty \hat{W}_j^{[l]} (\cdot, p-s) \hat{W}_j^{[0]} (\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1! (l - l_1 - 1)!}{l!} \hat{W}_j^{[l_1]} (\cdot, s) \hat{W}_j^{[l-l_1-1]} (\cdot, p) \right) \\
+ \frac{ik_j P_k}{\mu \rho} \left( \int_0^p \hat{Q}_j^{[l]} (\cdot, p-s) \hat{Q}_j^{[0]} (\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1! (l - l_1 - 1)!}{l!} \hat{Q}_j^{[l_1]} (\cdot, s) \hat{Q}_j^{[l-l_1-1]} (\cdot, p) \right) \\
- ik_j P_k [\tilde{v}_{0,j} \hat{s} \hat{w}_{l-1} + \hat{W}_j^{[l-1]} \hat{\hat{v}}_1] + \hat{W}_j^{[l]} \hat{\hat{v}}_0 + \dot{v}_{0,j} \hat{W}_j^{[l]} + \delta_{l,1} \dot{v}_{1,j} \hat{\hat{v}}_1 \\
+ \frac{ik_j P_k}{\mu \rho} [\tilde{B}_{1,j} \hat{s} \hat{Q}_{l-1} + \hat{Q}_j^{[l-1]} \hat{B}_1] + \hat{Q}_j^{[l]} \hat{\hat{B}}_0 + \hat{B}_{0,j} \hat{s} \hat{Q}_l + \delta_{l,1} \hat{B}_{1,j} \hat{B}_1 \hat{B}_1 \\
\]

(80)

\[
p\hat{Q}_p^{[l]} + (l+2)\hat{Q}_p^{[0]} + \frac{1}{\mu \sigma} |k|^2 \hat{Q}^{[l]} = \\
-ik_j P_k [\tilde{v}_{0,j} \hat{s} \hat{B}_1 + \hat{v}_{1,j} \hat{\hat{B}}_0] + ik_j P_k [\hat{B}_{0,j} \hat{s} \hat{v}_1 + \hat{B}_{1,j} \hat{\hat{v}}_0] - \frac{1}{\mu \sigma} |k|^2 \hat{B}_1 \delta_{l,0} \\
-ik_j P_k \left( \int_0^\infty \hat{W}_j^{[l]} (\cdot, p-s) \hat{Q}_j^{[0]} (\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1! (l - l_1 - 1)!}{l!} \hat{W}_j^{[l_1]} (\cdot, s) \hat{Q}_j^{[l-l_1-1]} (\cdot, p) \right) \\
+ \frac{ik_j P_k}{\mu \rho} \left( \int_0^p \hat{Q}_j^{[l]} (\cdot, p-s) \hat{W}_j^{[0]} (\cdot, s) ds + \sum_{l_1=1}^{l-1} \frac{l_1! (l - l_1 - 1)!}{l!} \hat{Q}_j^{[l_1]} (\cdot, s) \hat{W}_j^{[l-l_1-1]} (\cdot, p) \right) \\
- ik_j P_k [\tilde{v}_{1,j} \hat{s} \hat{Q}_{l-1} + \hat{W}_j^{[l-1]} \hat{\hat{B}}_1] + \hat{W}_j^{[l]} \hat{\hat{B}}_0 + \hat{v}_{0,j} \hat{Q}_l + \delta_{l,1} \hat{v}_{1,j} \hat{\hat{B}}_1 \\
+ ik_j P_k [\tilde{B}_{1,j} \hat{s} \hat{W}_{l-1} + \hat{W}_j^{[l-1]} \hat{\hat{v}}_1] + \hat{W}_j^{[l]} \hat{\hat{v}}_0 + \hat{B}_{0,j} \hat{s} \hat{W}_l + \delta_{l,1} \hat{B}_{1,j} \hat{\hat{v}}_1 \hat{\hat{B}}_1 \\
\]

(81)

Identify the right hand side of these four equations by $R_m^{[l]}$ for $m = 1, \ldots, 4$ respectively.

**Lemma 713** For any $l \geq 0$ and for some absolute constant $C_0$, if $(\hat{H}_j^{[l]}, \hat{S}_j^{[l]})$ satisfies (73), $(\hat{W}_j^{[l]}, \hat{Q}_j^{[l]})$ satisfies (76), and both are bounded at $p = 0$ then

\[
| (\hat{H}_j^{[l+1]}, \hat{S}_j^{[l+1]}) (k, p) | \leq \frac{C_0}{(l+1)^{5/3}} \sup_{p' \in [0, p]} | (\hat{R}_1^{[l]}, \hat{R}_2^{[l]} ) | + \frac{M_4 |k|^2 |(\hat{H}_j^{[l]}, \hat{S}_j^{[l]}) (k, 0)|}{(l+1)(l+2)}
\]
\begin{align*}
&\frac{(\hat{W}^{[l+1]}, \hat{Q}^{[l+1]})(k,p)}{\left(\int_{l+1}^{[l+1]}\right)} \leq \frac{C_0}{(l+1)^{\beta/3}} \sup_{p' \in [0,p]} |(\hat{R}_0^{[l]}, \hat{R}_1^{[l]})| + \frac{M_2|k|^2|\tilde{W}^{[l]}, \tilde{Q}^{[l]}|}{(l+1)(l+2)}.
\end{align*}

\textbf{Proof} The proof is in [5] under Lemma 4.4. The lemma is dependent only on the operator \( D \) which is the same in our case. The idea of the proof is as follows. We invert the operator on the left of (78) with the requirement that \( \hat{H} \) is bounded at \( p = 0 \), obtaining
\[
\hat{H}^{[l]}(k,p) = \int_0^p \mathcal{L}(2|k|\sqrt{\nu p}, 2|k|\sqrt{\nu p'}) \hat{R}_1^{(l)}(k,p') dp' + 2l+1(\frac{J_{l+1}(z)}{z^{l+1}} \hat{H}^{[l]}(k,0)),
\]
where
\[
\mathcal{L}(z, z') = \pi z^{-(l+1)} \left[-J_{l+1}(z)z'^{(l+1)}Y_{l+1}(z') + z'^{(l+1)}J_{l+1}(z')Y_{l+1}(z) \right].
\]
Then, we take a derivative with respect to \( p \) yielding
\[
(\hat{H}^{[l+1]}(k,p) = \frac{|k|}{\nu p} \int_0^p \mathcal{L}(2|k|\sqrt{\nu p}, 2|k|\sqrt{\nu p'}) \hat{R}_1^{(l)}(k,p') dp' - 2l+2(\frac{J_{l+2}(z)}{z^{l+2}} \hat{H}^{[l]}(k,0)).
\]
Using properties of Bessel functions, it is know that
\[
2^{l+2}(l+1)! \left| \frac{J_{l+2}(z)}{z^{l+2}} \right| \leq \frac{1}{l+2}
\]
and that
\[
\int_0^z \frac{z'}{z} \mathcal{L}_k(z, z') \, dz' \leq \frac{C}{(l+1)^{2/3}},
\]
where the constant is independent of \( l \). Thus, after a change of variables,
\[
(l+1)|\hat{H}^{[l+1]}(k,p)| \leq \sup_{p' \in [0,p]} |\hat{R}_1^{(l)}| \frac{C}{(l+1)^{2/3}} + \frac{|k|^2|\hat{H}(k,0)|}{l+2},
\]
and the claim follows.

\textbf{Lemma 714} \textit{Sup}
\[
\left| k_j \left( P_k(\hat{u}_{0,j}^{[l]} + \hat{\Theta}_0), \hat{u}_{0,j}^{[l]} + \hat{\Theta}_0 \right) \right| \quad \text{and} \quad \left| k_j \left( P_k(\hat{\Theta}_j^{[l]} + \hat{\Theta}_0), \hat{\Theta}_j^{[l]} + \hat{\Theta}_0 \right) \right|
\]
are bounded by
\[
C_1 |(\hat{u}_0, \hat{\Theta}_0)|_{\gamma, \beta} (l+1)^{2/3} AD e^{-\beta |k| + \omega p} \frac{(1+2l+1)(1+p^2)(1+|k|)^2}{\gamma} Q_{2l+2}(|\beta|).
\]

Similarly,
\[
\left| k_j \left( P_k(\hat{v}_{0,j}^{[l]} + \hat{\Theta}_0), \hat{v}_{0,j}^{[l]} + \hat{\Theta}_0 \right) \right|, \left| k_j \left( P_k(\hat{\Theta}_j^{[l]} + \hat{\Theta}_0), \hat{\Theta}_j^{[l]} + \hat{\Theta}_0 \right) \right|
\]
are bounded by
\[
C_1 |(\hat{v}_0, \hat{\Theta}_0)|_{\gamma, \beta} (l+1)^{2/3} AD e^{-\beta |k| + \omega p} \frac{(1+2l+1)(1+p^2)(1+|k|)^2}{\gamma} Q_{2l+2}(|\beta|).
\]
Proof For the first inequality, we use (75) and then apply Lemma 913 to get

\[ C_1 ||(\hat{v}_0, \hat{B}_0)||_{\gamma, \beta} \frac{(1 + 1)^{2/3} \tilde{A}D^{l-1}e^{-\beta|k|} + \alpha'}{(2l + 1)(1 + p^3)(1 + |k|)^\gamma} Q_{2l+2}(|\beta k|). \]

We also have

\[ |P_k(ae_2 S[t])| \leq \alpha \frac{e^{\omega p}e^{-\beta|k|}AD^l}{(1 + p^2)(1 + |k|)} Q_{2l}(|\beta k|). \]

In the above, \( C_1 = C_1(d) \) is defined in Lemma 913.

Proof For the first inequality, we use (75) and then apply Lemma 913 to get

\[
(1 + p^3) e^{-\omega p}|k_j \hat{u}_{0,j} + \hat{S}[l]| \leq ||\hat{u}_0||_{\gamma, \beta} \frac{AD^l}{(2l + 1)^2} |k| \int_{k' \in \mathbb{R}^d} e^{-\beta (|k'| + |k-k'|)} Q_{2l}(|\beta k'|) dk'.
\]

The other inequalities are similar except for the last which is simply the statement of the assumed bound.

Lemma 715 Suppose \((\hat{H}[l], \hat{S}[l])\) satisfies (75) and \((\hat{W}[l], \hat{Q}[l])\) satisfies (76) for \( l \geq 1 \). Then both

\[
\left| \frac{k_j}{l} \left( P_k(\hat{u}_{1,j} + \hat{H}[l-1], \hat{u}_{1,j} + \hat{S}[l-1]) \right) \right| \quad \text{and} \quad \left| \frac{k_j}{l} \left( P_k(\hat{H}[l-1] + \hat{u}_1), \hat{H}[l-1] + \hat{\Theta}_1 \right) \right|
\]

are bounded by

\[ C_1 ||(\hat{u}_1, \hat{\Theta}_1)||_{\gamma, \beta} \frac{P^l AD^{l-1} e^{-\beta|k|} + \alpha'}{l(2l - 1)(1 + p^2)(1 + |k|)^\gamma} Q_{2l}(|\beta k|). \]

Similarly,

\[
\left| \frac{k_j}{l} \left( P_k(\hat{v}_{1,j} + \hat{W}[l-1]), P_k(\hat{v}_{1,j} + \hat{Q}[l-1]) \right) \right|, \quad \left| \frac{k_j}{l} \left( P_k(\hat{W}[l-1] + \hat{v}_1), P_k(\hat{W}[l-1] + \hat{B}_1) \right) \right|,
\]

\[
\left| \frac{k_j}{l} \left( P_k(\hat{B}_{1,j} + \hat{Q}[l-1]), P_k(\hat{B}_{1,j} + \hat{W}[l-1]) \right) \right|, \quad \text{and} \quad \left| \frac{k_j}{l} \left( P_k(\hat{Q}[l-1] + \hat{B}_1), P_k(\hat{Q}[l-1] + \hat{v}_1) \right) \right|
\]

are bounded by

\[ C_1 ||(\hat{v}_1, \hat{B}_1)||_{\gamma, \beta} \frac{P^l AD^{l-1} e^{-\beta|k|} + \alpha'}{l(2l - 1)(1 + p^2)(1 + |k|)^\gamma} Q_{2l}(|\beta k|). \]

The proof is the same as Lemma 914 with \( l \) replacing \( l \), \((\hat{u}_1, \hat{\Theta}_1)\) replacing \((\hat{u}_0, \hat{\Theta}_0)\), and \((\hat{v}_1, \hat{B}_1)\) replacing \((\hat{v}_0, \hat{B}_0)\).
Lemma 716  Suppose \((\hat{H}^{[l]}, \hat{S}^{[l]})\) satisfies \((75)\) and \((\hat{W}^{[l]}, \hat{Q}^{[l]})\) satisfies \((76)\) for \(l \geq 1\). Then
\[
\left| \frac{k_j}{l} \left( P_k(\hat{H}_j^{[l-1]}(.0) \hat{H}^{[0]}(., p)), \hat{H}_j^{[l-1]}(.0) \hat{S}^{[0]}(., p) \right) \right|
\]
is bounded by
\[
C_1 \frac{(l + 1)^{2/3} A^2 D_{l-1} e^{-\beta|k| + \alpha^p}}{l(2l - 1)(1 + |k|)^\gamma (1 + p^2)} Q_{2l}(\beta|k|).
\]
We also have
\[
\left| \frac{k_j}{l} \left( P_k(\hat{W}_j^{[l-1]}(.0) \hat{W}^{[0]}(., p)), P_k(\hat{W}_j^{[l-1]}(.0) \hat{Q}^{[0]}(., p)) \right) \right|
\]
and
\[
\left| \frac{k_j}{l} \left( P_k(\hat{Q}_j^{[l-1]}(.0) \hat{Q}^{[0]}(., p)), P_k(\hat{Q}_j^{[l-1]}(.0) \hat{W}^{[0]}(., p)) \right) \right|
\]
bounded by
\[
C_1 \frac{(l + 1)^{2/3} A^2 D_{l-1} e^{-\beta|k| + \alpha^p}}{l(2l - 1)(1 + |k|)^\gamma (1 + p^2)} Q_{2l}(\beta|k|).
\]
Proof  We give the proof of one of the magnetic Bénard cases the others are similar. Using \((76)\) with \(p = 0\) and \((77)\) with \(A = 3\left(|\hat{r}_1, \hat{B}_1|\right|_{\gamma, \beta} + l/1\), we get
\[
(1 + p^2) e^{-\alpha^p} \left|\frac{k_j}{l} \left( \hat{W}_j^{[l-1]}(.0) \hat{W}^{[0]}(., p) \right) \right|
\]
\[
\leq \frac{A^2 D_{l-1}}{l(2l - 1)^2 |k|} \int_{k' \in \mathbb{R}^d} e^{-\beta(|k'| + |k - k'|)} Q_{2l-2}(|k'|) dk'
\]
\[
\leq C_1 \frac{1^{2/3} A^2 D_{l-1} e^{-\beta|k|}}{l(2l - 1)(1 + |k|)^\gamma} Q_{2l}(\beta|k|)
\]
From this the lemma follows after noting \((1 + l)^{2/3} \geq l^{2/3}\) and using Lemma 53.

Lemma 717  Suppose \((\hat{H}^{[l_1]}, \hat{S}^{[l_1]})\) and \((\hat{W}^{[l_1]}, \hat{Q}^{[l_1]})\) satisfies \((75)\) and \((\hat{W}^{[l_1-1]}, \hat{Q}^{[l_1]})\) satisfies \((76)\) for \(l_1 = 1, \ldots, l - 2\) where \(l \geq 2\). Then for \(C_8 = 82\) and \(C_7 = C_7(d)\) given in Lemma 712 we have
\[
\left| \sum_{l_1 = 1}^{l-2} \frac{l_1!(l - l_1 - 1)!}{l!} \left( P_k(\hat{H}_j^{[l_1]}(.0) \hat{W}^{[l_1-1]}(., p)), \hat{H}_j^{[l_1]}(.0) \hat{S}^{[l_1-1]}(., p) \right) \right|
\]
is bounded by
\[
C_8 C_7 \pi A^2 D_{l_1-1} \frac{e^{-\beta|k| + \omega^p}}{3 \beta^d (1 + p^2)(1 + |k|)^\gamma (2l + 3)^2}.
\]
Similarly, for the second, we have
\begin{align*}
&\left| k_j \sum_{l_1=1}^{l-2} \frac{l_1!(l - l_1 - 1)!}{l!} \left( P_k(\hat{W}_j^{[l_1]}(\cdot, 0) \hat{S}^{[l_1-1]}(\cdot, p)), P_k(\hat{Q}_j^{[l_1]}(\cdot, 0) \hat{Q}^{[l_1-1]}(\cdot, p)) \right) \right| \\
&\quad \text{and} \\
&\left| k_j \sum_{l_1=1}^{l-2} \frac{l_1!(l - l_1 - 1)!}{l!} \left( P_k(\hat{Q}_j^{[l_1]}(\cdot, 0) \hat{Q}^{[l_1-1]}(\cdot, p)), P_k(\hat{Q}_j^{[l_1]}(\cdot, 0) \hat{W}^{[l_1-1]}(\cdot, p)) \right) \right|
\end{align*}
are bounded by
\[ C_0 C_7 2^7 \pi A^2 D_{l-1} \frac{e^{-\beta|k| + \alpha'}}{3\beta^2(1 + p^2)(1 + |k|)^\gamma (2l + 3)^2} Q_{2l+2}(\beta|k|). \]
The proof is the same as in [5] the only difference is a change in the constants arising when Lemma [512] in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) is applied.

**Lemma 718** Suppose \((\hat{H}^{[l]}, \hat{S}^{[l]})\) satisfies (75) and \((\hat{W}^{[l]}, \hat{Q}^{[l]})\) satisfies (76) for \( l \geq 0 \). Then
\[ k_j \int_0^p \left( P_k(\hat{H}_j^{[l]}(\cdot, p - s) \hat{S}^{[l]}(\cdot, s)), \hat{H}_j^{[l]}(\cdot, p - s) \hat{S}^{[l]}(\cdot, s) \right) ds \]
\[ \leq C_1 M_0 A^2 D^l \frac{(l + 1)^2/3 e^{-\beta|k| + \omega'}}{(2l + 1)(1 + |k|)^\gamma (1 + p^2)} Q_{2l+2}(\beta|k|). \]
Similarly,
\[ k_j \int_0^p \left( P_k(\hat{W}_j^{[l]}(\cdot, p - s) \hat{W}^{[l]}(\cdot, s)), P_k(\hat{W}_j^{[l]}(\cdot, p - s) \hat{W}^{[l]}(\cdot, s)) \right) ds \]
\[ k_j \int_0^p \left( P_k(\hat{Q}_j^{[l]}(\cdot, p - s) \hat{Q}^{[l]}(\cdot, s)), P_k(\hat{Q}_j^{[l]}(\cdot, p - s) \hat{Q}^{[l]}(\cdot, s)) \right) ds \]
are bounded by
\[ C_0 M_0 A^2 D^l \frac{(l + 1)^2/3 e^{-\beta|k| + \omega'}}{(2l + 1)(1 + |k|)^\gamma (1 + p^2)} Q_{2l+2}(\beta|k|). \]
In the above, \( M_0 \), defined in Lemma [55], is such that
\[ \int_0^p \frac{1}{(1 + (p - s)^2)(1 + s^2)} ds \leq \frac{M_0}{1 + p^2}. \]

**Proof** Using (76) for the first inequality and Lemma [513] and Lemma [55] for the second, we have
\[ k_j \int_0^p \left( P_k(\hat{W}_j^{[l]}(\cdot, p - s) \hat{Q}^{[l]}(\cdot, s)), P_k(\hat{W}_j^{[l]}(\cdot, p - s) \hat{W}^{[l]}(\cdot, s)) \right) ds \leq \]
The rest are computed in the same way.

**Lemma 719** We have

\[
|k_j \left( P_k(\tilde{u}_{0,j} + \tilde{\gamma_1}, \tilde{u}_{0,j} + \tilde{\gamma_0}) + k_j \left( P_k(\tilde{u}_{1,j} + \tilde{\gamma_0}), \tilde{u}_{1,j} + \tilde{\gamma_0} \right) \right) | \leq \frac{2C_0|k|e^{-|k|}}{(1 + |k|)^{\gamma}} ||(\tilde{u}_0, \tilde{\gamma_0})||_{\gamma, \beta} ||(\tilde{u}_1, \tilde{\gamma_0})||_{\gamma, \beta}
\]

Similarly, we have

\[
k_j \left( P_k(\tilde{v}_{0,j} + \tilde{\gamma_1}), P_k(\tilde{v}_{0,j} + \tilde{\gamma_0}) + k_j \left( P_k(\tilde{v}_{1,j} + \tilde{\gamma_0}), P_k(\tilde{v}_{1,j} + \tilde{\gamma_0}) \right) \right)
\]

bounded by

\[
\frac{2C_0|k|e^{-|k|}}{(1 + |k|)^{\gamma}} ||(\tilde{v}_0, \tilde{\gamma_0})||_{\gamma, \beta} ||(\tilde{v}_1, \tilde{\gamma_0})||_{\gamma, \beta}.
\]

Finally, we have

\[
|k_j \left( P_k(\tilde{v}_{1,j} + \tilde{\gamma_1}), P_k(\tilde{v}_{1,j} + \tilde{\gamma_0}) \right) | \text{ and } |k_j \left( P_k(\tilde{v}_{1,j} + \tilde{\gamma_0}), P_k(\tilde{v}_{1,j} + \tilde{\gamma_0}) \right) |
\]

bounded by

\[
\frac{Ae^{-|k| + |k'|}}{(1 + |k|)^{\gamma}(1 + |k'|)^{\gamma}} ||(\tilde{v}_1, \tilde{\gamma_1})||_{\gamma, \beta}.
\]

**Proof** The first two claims follow directly from Corollary 52 and Lemma 53.

The last uses the additional fact that

\[
\frac{4Q_4(|\beta|)|C_0}{25\beta} \geq \frac{32\beta|k|C_0}{25\beta} \geq C_0|k|.
\]

Thus,

\[
\frac{|k|e^{-|k|}C_0}{(1 + |k|)^{\gamma}} ||(\tilde{v}_1, \tilde{\gamma_1})||_{\gamma, \beta} \leq \frac{Ae^{-|k| + |k'| + |k|}}{(1 + |k'|)^{\gamma}(1 + |k'|)^{\gamma}} ||(\tilde{v}_1, \tilde{\gamma_1})||_{\gamma, \beta}
\]

and the last claim follows.
Lemma 720 For the case $l = 1$, we have
\begin{align*}
|\langle \hat{H}^{[1]} , \hat{S}^{[1]} \rangle (k, p) | & \leq \frac{e^{\alpha'_p e^{-\beta |k|}} AD}{(1 + p^2)(1 + |k|)^\gamma} Q_2 (|\beta k|), \\
|\langle \hat{W}^{[1]} , \hat{Q}^{[1]} \rangle (k, p) | & \leq \frac{e^{\alpha'_p e^{-\beta |k|}} \tilde{A}D}{(1 + p^2)(1 + |k|)^\gamma} Q_2 (|\beta k|),
\end{align*}
where
\begin{align*}
A & \geq C_6 \left( \frac{C_0}{\beta} \right)^2 ||| (\hat{u}_0, \hat{\Theta}_0) |||_{\gamma, \beta} ||| (\hat{v}_1, \hat{\Theta}_1) |||_{\gamma, \beta} + M_1 \frac{2}{\beta^2} ||| (\hat{u}_1, \hat{\Theta}_1) |||_{\gamma, \beta} \\
& \quad + C_1 M_0 A^2 + 2 C_1 A ||| (\hat{u}_0, \hat{\Theta}_0) |||_{\gamma, \beta} + \frac{aA}{4}, \\
\tilde{A} & \geq C_6 \left( \frac{2 C_0 M_1}{\beta} \right)^2 ||| (\hat{v}_0, \hat{B}_0) |||_{\gamma, \beta} ||| (\hat{v}_1, \hat{B}_1) |||_{\gamma, \beta} + 2 M_2 \frac{2}{\beta^2} ||| (\hat{v}_1, \hat{B}_1) |||_{\gamma, \beta} \\
& \quad + 2 C_1 M_3 M_0 A^2 + 4 C_1 M_3 A ||| (\hat{v}_0, \hat{B}_0) |||_{\gamma, \beta}).
\end{align*}
Proof Lemma 713 with $l = 0$ tells us that
\begin{align*}
|\langle \hat{H}^{[1]} , \hat{S}^{[1]} \rangle (k, p) | & \leq C_6 \sup_{p' \in [0, p]} |\langle \hat{R}^{[0]}_1 , \hat{R}^{[0]}_2 \rangle (k, p') | \\
|\langle \hat{W}^{[1]} , \hat{Q}^{[1]} \rangle (k, p) | & \leq C_6 \sup_{p' \in [0, p]} |\langle \hat{R}^{[0]}_3 , \hat{R}^{[0]}_4 \rangle (k, p') |
\end{align*}
since $(\hat{H}^{[0]} , \hat{S}^{[0]}) (k, 0) = 0$ and $(\hat{W}^{[0]} , \hat{Q}^{[0]}) (k, 0) = 0$. In both cases, we use Lemma 714, Lemma 715, and Lemma 719 to bound the terms appearing in $R_m s$. The terms are kept in the same order as they appear in $R_m s$ as much as possible to help with organization.
\begin{align*}
|\langle \hat{R}^{[0]}_1 , \hat{R}^{[0]}_2 \rangle (k, p) | & \leq 2 C_0 \frac{|e^{-\beta |k|}|}{(1 + |k|)^\gamma} ||| (\hat{u}_0, \hat{\Theta}_0) |||_{\gamma, \beta} ||| (\hat{u}_1, \hat{\Theta}_1) |||_{\gamma, \beta} \\
& \quad + M_1 \frac{|e^{-\beta |k|}|}{(1 + |k|)^\gamma} ||| (\hat{v}_1, \hat{\Theta}_1) |||_{\gamma, \beta} + C_1 M_0 A^2 \frac{e^{-\beta |k| + \omega' p}}{(1 + |k|)^\gamma (1 + p^2)} Q_2 (|\beta k|) \\
& \quad + 2 C_1 ||| (\hat{u}_0, \hat{\Theta}_0) |||_{\gamma, \beta} \frac{A e^{-\beta |k| + \omega' p}}{(1 + p^2)(1 + |k|)^\gamma} Q_2 (|\beta k|) + \frac{e^{\alpha'_p e^{-\beta |k|}} A}{(1 + p^2)(1 + |k|)^\gamma}
\end{align*}
and
\begin{align*}
|\langle \hat{R}^{[0]}_3 , \hat{R}^{[0]}_4 \rangle (k, p) | & \leq 4 C_6 M_3 |e^{-\beta |k|}| \frac{|e^{-\beta |k|}|}{(1 + |k|)^\gamma} ||| (\hat{v}_0, \hat{B}_0) |||_{\gamma, \beta} ||| (\hat{v}_1, \hat{B}_1) |||_{\gamma, \beta} \\
& \quad + M_2 \frac{|e^{-\beta |k|}|}{(1 + |k|)^\gamma} ||| (\hat{v}_1, \hat{B}_1) |||_{\gamma, \beta} + 2 C_1 M_3 M_0 \tilde{A}^2 \frac{e^{-\beta |k| + \alpha' p}}{(1 + |k|)^\gamma (1 + p^2)} Q_2 (|\beta k|) \\
& \quad + 4 C_1 M_3 ||| (\hat{v}_0, \hat{B}_0) |||_{\gamma, \beta} \frac{A e^{-\beta |k| + \alpha' p}}{(1 + p^2)(1 + |k|)^\gamma} Q_2 (|\beta k|).
\end{align*}
The lemma now follows since $4 |k| \leq \frac{2 \Theta_4}{\beta}$ and $|k|^2 \leq \frac{2 \Theta_4^2}{\beta^2}$. 
Proof of Lemma 7.12: Lemma 7.20 and (7.4) prove the base case. Suppose, for the purpose of induction, that for \( l \geq 1 \) (75) and (76) hold. Then by Lemma 7.13 we need only prove a bound for \( |(\hat{R}'_1, \hat{R}'_2)| \) and \( |(\hat{R}'_3, \hat{R}'_4)| \) whose terms we bounded in the previous lemmas.

\[
|(\hat{R}'_1, \hat{R}'_2)| \leq \frac{AD^2 e^{-\beta|k|}}{(2l + 3)^2(1 + p^2)(1 + |k|)} Q_{2l+2}(\beta|k|) \left\{ \frac{C_1 M_0 AD(l + 1)^{2/3}(2l + 3)^2}{(2l + 1)} + \frac{C_1 \tilde{A} \tilde{D}(l + 1)^{2/3}(2l + 3)^2}{12 \beta^3} + \frac{C_1 \tilde{A} \tilde{D}(1 + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} \right\}
\]

and

\[
|(\hat{R}'_3, \hat{R}'_4)| \leq \frac{\tilde{A} \tilde{D}^2 e^{-\beta|k|}}{(2l + 3)^2(1 + p^2)(1 + |k|)} Q_{2l+2}(\beta|k|) \left\{ \frac{M_3}{(2l + 1)^{1/3}} \left( \frac{2C_1 C_2 \tilde{A} \tilde{D}(l + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} + \frac{2C_2 G \pi A l}{12 \beta^3} + \frac{C_1 \tilde{A} \tilde{D}(1 + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} \right) + \frac{4C_1 \tilde{A} \tilde{D}(1 + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} \right\}.
\]

We also note that as \( (\hat{H}'[0], \hat{S}'[0]) \) satisfies (75) and \( (\hat{W}'[0], \hat{Q}'[0]) \) satisfies (76),

\[
\frac{|k|^2 |\hat{H}'[0], \hat{S}'[0]|(k, 0)}{(l + 1)(l + 2)} \leq \frac{|k|^2 e^{-\beta|k|} AD^2 Q_{2l}(\beta|k|)}{(l + 1)(l + 2)(1 + |k|)^7(2l + 1)^2} \leq \frac{AD^2 e^{-\beta|k|}}{(2l + 3)^2(1 + p^2)(1 + |k|)} Q_{2l+2}(\beta|k|) \frac{6}{\beta^2}
\]

and

\[
\frac{|k|^2 |\hat{W}'[0], \hat{Q}'[0]|(k, 0)}{(l + 1)(l + 2)} \leq \frac{AD \tilde{A} \tilde{D} e^{-\beta|k|}}{(2l + 3)^2(1 + p^2)(1 + |k|)} Q_{2l+2}(\beta|k|) \frac{6}{\beta^2}.
\]

Here, we used the following two facts

\[
\frac{y^2 Q_{2l}(y)}{(2l + 2)(2l + 1)} \leq Q_{2l+2}(y) \text{ and } \frac{(2l + 2)(2l + 3)^2}{(l + 1)(l + 2)(2l + 1)} \leq 6.
\]

Thus, for \( D \) and \( \tilde{D} \) chosen, independently of \( l, k, \) and \( p, \) large enough so

\[
D^2 \geq C_0 \left\{ \frac{C_1 M_0 AD(l + 1)^{2/3}(2l + 3)^2}{(l + 1)(l + 2)} + \frac{C_1 A(2l + 3)^2}{4(l + 1)(l + 2)} + \frac{C_5 G \pi A l}{12 \beta^3(l + 1)^{5/3}} \right\}
\]

and

\[
\frac{C_1 |(\hat{u}_0, \hat{e}_1)|_{\gamma, \beta}(2l + 3)^2}{2(l + 1)^{5/3}(2l + 1)^{5/3}} + \frac{2C_1 \tilde{A} \tilde{D}(1 + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} + \frac{4C_1 \tilde{A} \tilde{D}(1 + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} + \frac{25\delta_1 \tilde{A} \tilde{D}(1 + 1)^{2/3}(2l + 3)^2}{(2l + 1)^{1/3}} + \frac{M_1 6D}{\beta^2}
\]
\[
\hat{D}^2 \geq C_6 M_3 \left( \frac{2C_1 M_0 \hat{A} \hat{D}(2l + 3)^2}{(l + 1)(2l + 1)} + \frac{C_1 \hat{A}(2l + 3)^2}{2(l + 1)(l(l - 1))} + \frac{C_8 \hat{C}_2^2 \pi \hat{A} l}{12\beta^l(l + 1)^{5/3}} \right. \\
+ \frac{C_l \|\hat{v}_l, \hat{B}_l\|_{\gamma, \beta}(2l + 3)^2}{(l + 1)^{5/3} l^{l/3}(2l - 1)} + 4C_1 \hat{D} \|\hat{v}_0, \hat{B}_0\|_{\gamma, \beta} (2l + 3)^2 \\
\left. + 25\delta_{l,1} C_0 \sum_{l=0}^{\infty} \|\hat{v}_l, \hat{B}_l\|_{\gamma, \beta} \right) + M_2 \frac{6\hat{D}}{\beta^2},
\]

and \( (75) \) hold and the lemma is proved.

As \( Q_{2l}(\beta|k|) \leq 4^l e^{\|\beta k\|/2} \),

\[
(\tilde{H}, \tilde{S})(k, p; p_0) = \sum_{l=0}^{\infty} (\hat{H}^{[l]}, \hat{S}^{[l]})(k, p_0)(p - p_0)^l \\
(\tilde{W}, \tilde{Q})(k, p; p_0) = \sum_{l=0}^{\infty} (\hat{W}^{[l]}, \hat{Q}^{[l]})(k, p_0)(p - p_0)^l
\]

are convergent for \(|p - p_0| \leq \frac{1}{4\beta} \) (or respectively \(|p - p_0| \leq \frac{1}{4\beta} \)) where \( D \) is independent of \( p_0 \). Moreover, the following lemma proved in [5] says that these series are indeed local representations of the solution \((\tilde{H}, \tilde{S})(k, p)\) or respectively \((\tilde{W}, \tilde{Q})(k, p)\).

**Lemma 721** The unique solution to \((57)\) satisfying \((\tilde{H}, \tilde{S})(k, 0) = 0\) guaranteed in Lemma \( 57 \) has a local representation given by \((\tilde{H}, \tilde{S})(k, p; p_0)\) for \( p_0 \in \mathbb{R}^+ \). So, the solution is analytic on \( \mathbb{R}^+ \cup \{0\} \). Similarly, the unique solution to \((58)\) satisfying \((\tilde{W}, \tilde{Q})(k, 0) = 0\) again guaranteed in Lemma \( 57 \) has a local representation given by \((\tilde{W}, \tilde{Q})(k, p; p_0)\) for \( p_0 \in \mathbb{R}^+ \) and is therefore analytic on \( \mathbb{R}^+ \cup \{0\} \).

**Proof** The proof is in [5].

**Proof of Theorem 23 i)** We prove the Boussinesq case. The MHD case is the same with the obvious changes. Using Lemma \( 721 \) and the fact that \( \|g\|_{L^\infty} \leq \|\hat{g}\|_{L^1} \), we know that

\[
\|H^{[l]}, S^{[l]}(x, p_0)\| \leq \frac{8\pi A(4\beta)^l e^{\omega p_0}}{\beta(2l + 1)^2(1 + p_0^2)}
\]

\[
|D(H^{[l]}, S^{[l]}(x, p_0))| \leq \frac{8\pi A(4\beta)^l e^{\omega p_0}}{\beta(2l + 1)^2(1 + p_0^2)}
\]

\[
|D^2(H^{[l]}, S^{[l]}(x, p_0))| \leq \frac{16\pi A(4\beta)^l e^{\omega p_0}}{\beta^2(2l + 1)^2(1 + p_0^2)}
\]

and the series \((82)\) converges for \(|p - p_0| \leq \frac{1}{4\beta} \). By Lemma \( 721 \) the series is the local representation of the solution guaranteed to exist by Lemma \( 57 \) which is zero at \( p = 0 \). Combining this with the facts that the solution is analytic in a neighborhood of zero and exponentially bounded for large \( p \),
recall \((\hat{H}, \hat{S}) \in \mathcal{A}^c\), implies Borel Summability in \(1/t\). Watson’s Lemma then implies as \(t \to 0^+\)

\[(u, \Theta)(x, t) \sim (u_0, \Theta_0)(x) + \sum_{m=1}^{\infty} (u_m, \Theta_m)(x)t^m\]

where \(|(u_m, \Theta_m)(x)| \leq m! A_0 D_0^m\) with constants \(A_0\) and \(D_0\) generally dependent on the initial condition and forcing through Lemma 74.

8 Extension of existence time

We have shown by Theorem 21 and Theorem 22 that there is a unique solution to (23) and (25) within the class of locally integrable functions, which are exponentially bounded in \(p\), uniformly in \(x\). Further, the solutions \((\hat{H}, \hat{S})(k, p)\) and \((\hat{W}, \hat{Q})(k, p)\) generate, in each case, a smooth solution to the Boussinesq and magnetic Bénard equation for \(t \in [0, \omega^{-1}]\) where \(\omega\) is the exponential growth rate of the integral equation (23) respectively, for \(t \in [0, \alpha^{-1}]\) where \(\alpha\) is the exponential growth rate of the integral equation (25). By Theorem (23), we know that the solution is Borel Summable. The question of global existence in either problem can then be reduced to a question of exponential growth for the integral equation solution. If \((\hat{H}, \hat{S})(k, p)\) or \((\hat{W}, \hat{Q})(k, p)\) grow subexponentially, then global existence will follow. The exponential growth rate \(\omega\) or \(\alpha\) previously found is suboptimal and ignores possible cancellations in the integrals. If we improve the estimates, we get a longer interval of existence. One example of improvement is given in the second part of Theorem 23, in the special case when the initial condition and forcing have a finite number of Fourier modes, then the radius of convergence in the Borel plane is independent of the size of initial data and forcing. We then prove Theorem 24 which says that based on detailed knowledge of the solution to the integral equation in \([0, p_0]\) given either by the power series at \(p = 0\) or by numerical calculation, if the solution is small for \(p\) towards the right of this interval then \(\omega\) or \(\alpha\) can be shown to be small.

8.1 Improved Radius of Convergence

When the initial data and forcing are analytic Borel summability given in Theorem 23 implies that

\[
(\hat{H}, \hat{S})(k, p) = \sum_{m=1}^{\infty} (\hat{u}[m], \hat{\Theta}_m)(k) \frac{p^{m-1}}{(m-1)!} = \sum_{m=0}^{\infty} (\hat{u}[m+1], \hat{\Theta}[m+1])(k) \frac{p^m}{m!}
\]

\[
(\hat{W}, \hat{Q})(k, p) = \sum_{m=1}^{\infty} (\hat{v}[m], \hat{B}_m)(k) \frac{p^{m-1}}{(m-1)!} = \sum_{m=0}^{\infty} (\hat{v}[m+1], \hat{B}[m+1])(k) \frac{p^m}{m!}
\]

has a finite radius of convergence depending on the size of the initial data and forcing. However, in the special case when the initial data and forcing
have only a finite number of Fourier modes the radius of convergence is in fact independent of the size of the initial data or \( f \). The argument allows forcing to be time dependent.

**Proof of Theorem 23 ii)** We show the Boussinesq case the other begin similar. For small time 

\[
(u, \Theta)(x, t) = (u^{[0]}, \Theta^{[0]})(x) + \sum_{m=1}^{\infty} (u^{[m]}, \Theta^{[m]})(x)t^m
\]

\[
\hat{f}(k, t) = \hat{f}^{[0]} + \sum_{m=1}^{\infty} \hat{f}^{[m]}(k)t^m,
\]

where by (5) for \( m \geq 0 \)

\[
\hat{u}^{[m+1]} = \frac{1}{m+1} \left[ \hat{f}^{[m]} - \nu|k|^2 \hat{u}^{[m]} - ik_j P_k \left( \sum_{l=0}^{m} \hat{u}^{[l]} \ast \hat{u}^{[m-l]} \right) + aP_k(e_2 \hat{\Theta}^{[m]}) \right]
\]

(84)

\[
\hat{\Theta}^{[m+1]} = \frac{1}{m+1} \left[ -\mu|k|^2 \hat{\Theta}^{[m]} - ik_j \left( \sum_{l=0}^{m} \hat{u}^{[l]} \ast \hat{\Theta}^{[m-l]} \right) \right].
\]

Suppose the initial data and forcing have a finite number of Fourier m odes. Let

\[
K_1 = \max(\sup_{k \in \text{supp}(\hat{u}^{[0]}, \hat{\Theta}^{[0]})} |k|, \sup_{k \in \text{supp}(\hat{f})} |k|). Then by induction on \( k \) we have

\[
\sup_{k \in \text{supp}(\hat{u}^{[m]}, \hat{\Theta}^{[m]})} |k| \leq (m + 1)K_1.
\]

Taking the \( \| \cdot \|_{\gamma, \beta} \) norm of both sides of (84) with respect to \( k \) and writing

\[
a_m = \|(\hat{u}^{[m]}, \hat{\Theta}^{[m]})\|_{\gamma, \beta}, \quad b_m = \|\hat{f}^{[m]}\|_{\gamma, \beta},
\]

we obtain

\[
a_{m+1} \leq \frac{1}{m+1} \left[ b_m + \max(\nu, \mu) \left( \|k|^2 \|(\hat{u}^{[m]}, \hat{\Theta}^{[m]})\|_{\gamma, \beta} \right) + aP_k(e_2 \hat{\Theta}^{[m]}) \right] + \sum_{l=0}^{m} \|k| \hat{u}^{[l]}\|_{\gamma, \beta} \left( \hat{u}^{[m-l]} \ast \hat{\Theta}^{[m-l]} \right) + aa_m
\]

(85)

\[
\leq \frac{1}{m+1} \left[ b_m + \max(\nu, \mu)K_1^2(m + 1)^2 a_m + K_1C_0(m + 2) \sum_{l=0}^{m} a_l a_{m-l} + aa_m \right]
\]

\[
\leq \frac{b_m}{m+1} + \frac{aa_m}{m+1} + K_1^2 \max(\nu, \mu)(m + 1)a_m + 2K_1C_0 \sum_{l=0}^{m} a_l a_{m-l}.
\]

Now, consider the formal power series

\[
y_0(t) := \sum_{m=1}^{\infty} \tilde{a}_m t^m,
\]

where

\[
\tilde{a}_0 = a_0
\]
\[ \tilde{a}_{m+1} = \frac{b_m}{m+1} + \frac{\bar{a} \tilde{a}_m}{m+1} + K_1^2 \max(\nu, \mu) (m+1) \tilde{a}_m + 2K_1C_0 \sum_{l=0}^{m} \tilde{a}_l \tilde{a}_{m-l}. \]

Clearly, \( a_m \leq \tilde{a}_m \), so \( y_0(t) \) majorizes \( \| (\hat{u}, \hat{\Theta})(\cdot, t) \|_{\gamma, \beta} \). If we multiply both sides of (85) by \( t^m \) and sum over \( m \), then

\[
\sum_{m=0}^{\infty} \tilde{a}_{m+1} t^m = \sum_{m=0}^{\infty} \frac{b_m + \bar{a} \tilde{a}_m}{m+1} t^m + K_1^2 \max(\nu, \mu) \sum_{m=0}^{\infty} (m+1) \tilde{a}_m t^m + 2K_1C_0 \sum_{m=0}^{\infty} \sum_{l=0}^{m} \tilde{a}_l \tilde{a}_{m-l} t^m.
\]

In other words, \( y_0(t) \) is a formal power series solution to

\[
\frac{1}{t} (y - \bar{a}_0) = w + \frac{a}{t} \int_0^t y(\tau) d\tau + K_1^2 \max(\nu, \mu) (ty)' + 2K_1C_0y^2,
\]

where \( w(t) = \sum_{m=0}^{\infty} \frac{b_m + \bar{a} \tilde{a}_m}{m+1} t^m \). With the change of variables \( s = 1/t \), we have

\[
-K_1^2 \max(\nu, \mu) y' + 2K_1C_0s^{-1}y^2 + (K_1^2 \max(\nu, \mu) s^{-1} - 1) y + (s^{-1} w + \bar{a}_0) + as \int_0^{1/s} y(\tau) d\tau = 0.
\]

A singularity of \( B(y(s)) \) in the Borel plane exhibits itself as an exponential small correction to \( y_0 \). So, we let \( y = y_0 + \delta \) and construct the equation for \( \delta \):

\[
-K_1^2 \max(\nu, \mu) \delta' + 2K_1C_0s^{-1}(\delta^2 + 2y_0 \delta) + (K_1^2 \max(\nu, \mu) s^{-1} - 1) \delta + as \int_0^{1/s} \delta(\tau) d\tau = 0.
\]

If we assume \( \delta \) is exponentially small, then to leading order the equation is

\[
-K_1^2 \max(\nu, \mu) \delta' + [(4K_1C_0s^{-1} \bar{a}_0 + (K_1^2 \max(\nu, \mu) s^{-1} - 1) \delta = 0,
\]

which yields

\[
\delta \sim e^{-K_1^{-2} \max(\nu, \mu)^{-1} s} \bar{a}_0 C_0 K_1^{-1} \max(\nu, \mu)^{-1} + 1.
\]

So, the radius of convergence of \( B(y) \) is at least \( K_1^{-2} \max(\nu, \mu)^{-1} \) which is independent of the size of initial data as claimed. As \( y \) majorizes our solution \( (\hat{u}, \hat{\Theta})(k, t) \) the radius of convergence of (83) is independent of the size of initial data or forcing as well.
8.2 Improved growth estimates based on knowledge of the solution to \(23\) in \([0, p_0]\).

Let \(( \hat{H}, \hat{S})(k, p)\) be the solution to \(23\) provided by Theorem \(21\). Define

\[
( \hat{H}, \hat{S})^{(a)}(k, p) = \begin{cases} 
( \hat{H}, \hat{S})(k, p) & \text{for } p \in (0, p_0) \subset \mathbb{R}^+ \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\hat{H}^{(s)}(k, p) = \frac{ik_j \pi}{2|k|\sqrt{\nu p}} \int_0^{|\min(p, 2p_0)|} G(z, z') \hat{G}_j^{(b)}(k, p') dp' + 2 \hat{u}_1(k) \frac{J_1(2|k|\sqrt{\nu p}p)}{2|k|\sqrt{\nu p}}
\]

\[
+ \frac{a \pi}{2|k|\sqrt{\nu p}} \int_0^{|\min(p, p_0)|} G(z, z') P_k [e_2 \hat{S}^{(a)}(k, p')] dp'
\]

\[
\hat{S}^{(s)}(k, p) = \frac{ik_j \pi}{2|k|\sqrt{\nu p}} \int_0^{|\min(p, 2p_0)|} G(\zeta, \zeta') \hat{G}_j^{(a)}(k, p') dp' + 2 \hat{\Theta}_1(k) \frac{J_1(2|k|\sqrt{\nu p}p)}{2|k|\sqrt{\nu p}},
\]

where

\[
\hat{G}_j^{[1],(a)}(k, p) = - P_k [\hat{u}_0 j^* \hat{H}^{(a)} + \hat{H}_j^{(a)} j^* \hat{u}_0 + \hat{H}_j^{(a)} j^* \hat{H}^{(a)} - \hat{u}_0 j^* \hat{H}^{(a)}]
\]

\[
\hat{G}_j^{[2],(a)}(k, p) = - [\hat{u}_0 j^* \hat{S}^{(a)} + \hat{H}_j^{(a)} j^* \hat{\Theta}_0 + \hat{H}_j^{(a)} j^* \hat{S}^{(a)} - \hat{u}_0 j^* \hat{S}^{(a)}]
\]

are known functions depending on \(( \hat{H}, \hat{S})^{(a)}(k, p)\). Using these definitions, we introduce the following functionals dependent on the initial condition, forcing, and \(( \hat{H}, \hat{S})^{(a)}\). Further, for any chosen \(\omega_0 \geq 0\), define

\[
b = \omega_0 \int_{p_0}^{\infty} e^{-\omega_0 p} ||( \hat{H}, \hat{S})^{(s)}(\cdot, p)||_{\gamma, \sigma} dp
\]

\[
\epsilon_1 = B_1 + B_4 + \int_{0}^{p_0} e^{-\omega_0 p} B_2(p) dp,
\]

where

\[
B_0(k) = C_0 \sup_{p_0 \leq p' \leq p} |G(z, z')|/|z|, \quad B_1 = 2 \sup_{k \in \mathbb{R}^d} |k| B_0(k) ||( \hat{u}_0, \hat{\Theta}_0)||_N,
\]

\[
B_2 = 2 \sup_{k \in \mathbb{R}^d} |k| B_0(k) ||( \hat{H}, \hat{S})^{(a)}(\cdot, p)||_N, \quad B_3 = \sup_{k \in \mathbb{R}^d} |k| B_0(k), \quad B_4 = a \sup_{k \in \mathbb{R}^d} B_0(k).
\]

Now, let \(( \hat{H}, \hat{S})^{(b)} = ( \hat{H}, \hat{S}) - ( \hat{H}, \hat{S})^{(a)}\). It is convenient to write the integral equation for \(( \hat{H}, \hat{S})^{(b)}\) for \(p > p_0\),

\[
\hat{H}^{(b)}(k, p) = \frac{\pi}{2|k|\sqrt{\nu p}} \int_{p_0}^{p} G(z, z') (ik_j \hat{G}_j^{[1],(b)}(k, p') + P_k [e_2 \hat{S}^{(b)}(k, p')]) dp' + \hat{H}^{(s)}(k, p)
\]
\[ \hat{S}^{(b)}(k, p) = \frac{ik_0 \pi}{2|k| \sqrt{pp}} \int_{p_0}^{p} G(\zeta, \zeta') \hat{G}^{[2],(b)}(k, p') dp' + \hat{S}^{(a)}(k, p), \]

where
\[
\hat{G}^{[1],(b)}(k, p) = -P_k [\hat{u}_0 + \hat{H}^{(b)} + \hat{\Theta}_0 + \hat{\Theta}^{(a)} + \hat{H}^{(b)} + \hat{H}^{(a)} + \hat{H}^{(b)} + \hat{H}^{(a)}] \\
\hat{G}^{[2],(b)}(k, p) = -[\hat{u}_0, \hat{S}^{(b)} + \hat{H}^{(b)} + \hat{\Theta}_0 + \hat{\Theta}^{(a)} + \hat{H}^{(b)} + \hat{H}^{(a)} + \hat{H}^{(b)} + \hat{H}^{(a)}].
\]

We also define
\[ R^{(b)}(k, p) = ik_j (\hat{G}^{[1]}(k, p) + aP_k [2 \hat{S}^{(b)}(k, p)], \]

**Proof of Theorem 24** We note that
\[
|R^{(b)}(k, p)| \leq \left( |k| \left[ |\hat{u}_0| \text{ } |(\hat{H}, \hat{S})| + |H^{(b)}| \hat{u}_0, \Theta_0) + 2|H, \hat{S}^{(a)}| \hat{S}^{(b)}| + a|H^{(b)}| \right](k, p),
\]

where \(| \cdot |\) is the usual euclidean norm. Let \( \psi(p) = ||(\hat{H}, \hat{S})^{(b)}(\cdot, p)||_{\gamma, \beta} \). Then
\[
\left( |k| \left[ |\hat{u}_0| \text{ } |(\hat{u}_0, \Theta_0)||_{\gamma, \beta} + 2|H, \hat{S}^{(a)}| \hat{S}^{(b)}| + a|H^{(b)}| \right](k, p)
\]

= \( (B_1 \psi + B_2 \psi + B_3 \psi \psi + B_4 \psi) (p) \).

Taking the \((\gamma, \beta)\) norm in \( k \) on both sides of (\ref{eq:90}) and multiplying by \( e^{-\omega p} \) for \( \omega \geq \omega_0 \geq 0 \) and integrating from \( p_0 \) to \( M \) gives
\[
L_{p_0, M} := \int_{p_0}^{M} e^{-\omega p} \psi^{(s)}(p) dp \leq \int_{p_0}^{M} \int_{p_0}^{p} (B_1 \psi + B_2 \psi + B_3 \psi \psi + B_4 \psi) (p') dp' dp
\]

where \( \psi^{(s)} = ||(\hat{H}, \hat{S})^{(s)}(\cdot, p)||_{\gamma, \beta} \). Recalling that \( \psi = 0 \) on \([0, p_0] \), we note that for any \( u \)
\[
\int_{p_0}^{M} e^{-\omega p} (\psi \ast u)(p) dp = \int_{p_0}^{M} \int_{p_0}^{p} e^{-\omega p} \psi^{(s)} u(p-s) ds dp
\]

= \( \int_{p_0}^{M} \psi^{(s)}(s) e^{-\omega s} \int_{0}^{M-s} e^{-\omega p} u(dp) ds. \)
Using this, we obtain

\[
L_{p_0,M} \leq \frac{1}{\omega} \left\{ (B_1 + \int_0^{M-p_0} e^{-\omega p} B_2(p) dp) L_{p_0,M} + B_3 L_{p_0,M}^2 + B_4 L_{p_0,M} \right\} + b \omega^{-1}
\leq \omega^{-1} \left\{ \epsilon_1 L_{p_0,M} + B_3 L_{p_0,M}^2 + b \omega^{-1} \right\}.
\]

For

\[
\epsilon_1 < \omega \quad \text{and} \quad (\epsilon_1 - \omega)^2 > 4B_3b,
\]

we get an estimate for \( L_{p_0,M} \) that is independent of \( M \). Namely,

\[
L_{p_0,M} \leq \frac{1}{2B_3} \left[ \omega - \epsilon_1 - \sqrt{(\epsilon_1 - \omega)^2 - 4B_3b} \right].
\]

So, \( ||\hat{H}, \hat{S}(\cdot, p)||_{\gamma_3} \in L^1(e^{-\nu p} dp) \), and the solution to the Boussinesq equation exists for \( t \in (0, \omega^{-1}) \) for \( \omega \) sufficiently large so that

\[
\omega \geq \omega_0 \quad \text{and} \quad \omega > \epsilon_1 + 2\sqrt{B_3b}.
\]

Equivalently, we could choose our original \( \omega_0 \) large enough so that \( \omega_0 > \epsilon_1 + 2\sqrt{B_3b} \). This completes the proof of Theorem 24.

9 Appendix

Lemma 91 The kernel \( G(z, z') \) given by

\[
G(z, z') = z'(-J_1(z)Y_1(z') + Y_1(z)J_1(z'))\text{, where } z = 2|k|\sqrt{\nu p} \text{ and } z' = 2|k|\sqrt{\nu p'}
\]

satisfies \( \frac{1}{2}G(z, z') = \mathcal{H}^{(\nu)}(p, p', k) \) with

\[
\mathcal{H}^{(\nu)}(p, p', k) = \int_1^{1/p'} \left\{ \int_{p'/p}^{e^{-\nu i}} \int_{e^{-\nu i}}^{e^{\nu i}} \tau^{-1} \exp[-\nu |k|^2 \tau^{-1}(1-s) + (p - p's^{-1})\tau] d\tau \right\} ds
\]

\[
= \frac{p'}{p} \int_1^{p/p'} F(\eta) d\eta,
\]

where

\[
\eta = \nu |k|^2 p \left( 1 - \frac{sp'}{p} \right) \left( 1 - \frac{1}{s} \right), \quad F(\eta) = \frac{1}{2\pi i} \int_C \zeta^{-1} e^{\zeta - \eta \zeta^{-1}} d\zeta,
\]

and \( C \) is the contour starting and ending at \( \infty e^{-\pi i} \) turning around the origin in counterclockwise direction and ending at \( \infty e^{\pi i} \).
Proof. We will show that $\mathcal{H}^{(\nu)}(p, p', k)$ solves \((p\partial_{pp} + 2\partial_p + \nu|k|^2)\mathcal{H}^{(\nu)} = 0\) for \(0 < p' < p\) with the condition that $\mathcal{H}^{(\nu)}(p, p', k) \to 0$ and $\mathcal{H}^{(\nu)}_p(p, p', k) \to \frac{1}{p}$ as $p'$ approaches $p$ from below.

In the appendix of [8], it is shown that $F$ is entire, $\mathcal{F}(0) = 1$, and $F$ satisfies $\eta F''(\eta) + F'(\eta) + F(\eta) = 0$. We will use these facts as given. As $F$ is continuous and the interval of integration shrinks to length zero, $\mathcal{H}^{(\nu)}(p, p', k) \to 0$ as $p'$ tends to $p$ from below. For $p > p'$, $\mathcal{H}^{(\nu)}$ is twice differentiable in $p$ as $F$ is twice continuously differentiable. Moreover, we have

$$\mathcal{H}^{(\nu)}_p(p, p', k) = -\frac{1}{p} \mathcal{H}^{(\nu)}(p, p', k) + \frac{1}{p} F(0) + p' \int_1^{p/p'} F''(\eta) \frac{d\eta}{dp} ds,$$

$$(p\mathcal{H}^{(\nu)}_p)_p = -\mathcal{H}^{(\nu)} + F''(0) \nu|k|^2(1 - \frac{p'}{p}) + p' \int_1^{p/p'} F''(\eta) \left( \frac{d\eta}{dp} \right)^2 ds,$$

where the second equality uses that $\frac{d\eta}{dp} = \nu|k|^2 (1 - \frac{1}{s})$ is $p$ independent.

Thus, as $F(0) = 1$, we have $\mathcal{H}^{(\nu)}_p(p, p', k) \to \frac{1}{p}$ as $p'$ tends to $p$ from below. We notice that

$$\frac{d\eta}{dp} = \nu|k|^2 \left( 1 - \frac{1}{s} \right),$$

and

$$\left( \frac{d\eta}{dp} \right)^2 = \frac{\nu|k|^2}{p} \left( 1 + \frac{p' s^2 - p}{s(p - sp')} \right) = \frac{\nu s}{p(p - sp')} \frac{d\eta}{dp} = \frac{\nu s}{p} \frac{d\eta}{dp} = \frac{\nu|k|^2(s - 1)}{p} \frac{d\eta}{dp}.$$

So, integrating by parts and using $\eta F''(\eta) + F'(\eta) + F(\eta) = 0$, we have

$$(p\mathcal{H}^{(\nu)}_p)_p + \mathcal{H}^{(\nu)}_p = F''(0) \nu|k|^2(1 - \frac{p'}{p}) + p' \int_1^{p/p'} F''(\eta) \left( \frac{\nu|k|^2}{p} \right) ds$$

$$- p' \int_1^{p/p'} \frac{d}{ds} \left( F''(\eta) \right) \frac{\nu|k|^2(s - 1)}{p} ds$$

$$= \frac{\nu|k|^2p'}{p} \int_1^{p/p'} \eta F''(\eta) ds + \frac{p' \nu|k|^2}{p} \int_1^{p/p'} F''(\eta) ds = -\nu|k|^2 \mathcal{H}^{(\nu)}.$$

In other words, $p\mathcal{H}^{(\nu)}_p + 2\mathcal{H}^{(\nu)} + \nu|k|^2 \mathcal{H}^{(\nu)} = 0$, and the Lemma is proved.

**Lemma 92** We also have the representation in terms of Bessel functions

$$\mathcal{L}^{-1} \left( \frac{1 - e^{-\nu|k|^2 r^{-1}}}{\nu|k|^2} \right)(p) = \frac{2 J_1(z)}{z}.$$

**Proof** Notice that by contour deformation the integral of $\frac{1}{|k|^2} e^{-\nu|k|^2 r^{-1}}$ is zero. Factorizing out $|k|\sqrt{p}$ in the exponent and using the change of variables $\frac{r \sqrt{z}}{|k|^2} \to \tau$, we have

$$\mathcal{L}^{-1} \left( \frac{1 - e^{-\nu|k|^2 r^{-1}}}{\nu|k|^2} \right)(p) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{-\nu|k|^2 r^{-1} + pr} \frac{d\tau}{\nu|k|^2}.$$
\[
\frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{[k\sqrt{\nu_p} (w - w^{-1})]} \frac{dw}{|k|\sqrt{\nu_p}} = 2J_1(z).
\]

Fourier Inequalities in two dimensions

In the appendix of [5], Fourier inequalities are developed in \( \mathbb{R}^3 \). We present the counterparts to those inequalities in \( \mathbb{R}^2 \) here. Where a Lemma is referenced from this section, we use either the \( \mathbb{R}^2 \) version or \( \mathbb{R}^3 \) version as appropriate for our two problems. The basic idea is that in 2-d Lemma 99 below differs by a constant from 3-d case. All other lemmas are basically the same for \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \) once the change in Lemma 99 is taken into account.

**Definition 93** Define the polynomial

\[ P_n(z) = \sum_{j=0}^{n} \frac{n!}{j!} z^j. \]

**Remark 94** Integration by parts gives

\[ \int_{0}^{z} e^{-\tau^2} \tau^n d\tau = -e^{-z} P_n(z) + n!. \]

**Lemma 95** For all \( y \geq 0 \) and nonnegative integers \( m, n \) we have

\[ y^{m+1} \int_{0}^{1} \rho^m P_n(y(1 - \rho)) d\rho = m! n! \sum_{j=0}^{n} \frac{y^{m+j+1}}{(m+j+1)!}. \]

**Proof** Integration by parts gives

\[ \int_{0}^{1} (1 - \rho)^j \rho^m d\rho = \frac{m! j!}{(m+j+1)!}. \]

The result now follows by a direct calculation using the definition of \( P_n \) given by Definition 93.

**Lemma 96** For all \( y \geq 0 \) and integers \( n \geq m \geq 0 \), we have

\[ y^{m+1} \int_{0}^{\infty} e^{-2y(\rho-1)^2} \rho^m P_n(y(\rho - 1)) d\rho \leq \frac{2^{-m} n!}{m!} \sum_{j=0}^{m} \frac{y^{j}}{j!}. \]

**Proof** This again follows from direct calculation and is the same as in [5].

**Lemma 97** For all \( y \geq 0 \) and integers \( n \geq m \geq 0 \), we have

\[ y^{m+1} \int_{0}^{\infty} e^{-y(\rho-1)^2 + \text{sgn}(\rho-1)} \rho^m P_n(y(1 - \rho)) d\rho \leq m! n! Q_{m+n+1}(y). \]

**Proof** This is a combination of the previous two lemmas after splitting the integral at 1.
Proposition 98 Let \( n \) be an integer no less than 0 and \( r \geq 0 \) and \( \rho \geq 0 \) fixed. Then

\[
\int_0^{2\pi} e^{-|\rho-re^{i\theta}|} |\rho - re^{i\theta}|^{n} d\theta \leq 6\pi e^{-|\rho-r|} P_n(|\rho - r|).
\]

Proof Let \( f(\theta) = e^{-|\rho-re^{i\theta}|} |\rho - re^{i\theta}|^{n} \). Then notice that \( f'(\theta) = e^{-|\rho-re^{i\theta}|} |\rho - re^{i\theta}|^{n-2} \rho \sin(\theta) (-|\rho - re^{i\theta}| + n) \). We want to maximize \( f(\theta) \), so we split it into two cases.

Case 1. Suppose \( r \leq \rho - n \) or \( r \geq \rho + n \). As \( |\rho - re^{i\theta}| \geq n \), \( f(\theta) \) reaches its maximum at \( \theta = 0 \). Thus, \( |f(\theta)| \leq e^{-|\rho-r|} |\rho - r|^n \leq e^{-|\rho-r|} P_n(|\rho - r|) \). If \( n = 0 \) this is the only case to consider. For \( n \geq 1 \) we have a second case.

Case 2. Suppose \( \rho - n < r < \rho + n \). Then \( f(\theta) \) is maximized for \( \theta \) such that \( |\rho - re^{i\theta}| = n \). Hence, \( |f(\theta)| \leq e^{-n} n^n \). Now, we use the fact that for \( r \in (\rho - n, \rho + n) \)

\[
e^{-|\rho-r|} \leq \sum_{j=0}^{n} \frac{|\rho-r|^j}{j!} + \frac{e^n}{(n+1)!} = \frac{P_n(|\rho-r|)}{n!} + \frac{e^n}{(n+1)!}.
\]

So,

\[
|f(\theta)| \leq e^{-n} n^n \leq e^{-|\rho-r|} \left( \frac{P_n(|\rho-r|) e^{-n} n^n}{n!} + \frac{e^n}{(n+1)!} \right) \leq 3e^{-|\rho-r|} P_n(|\rho-r|),
\]

where the last inequality uses \( e^{-n} n^n \leq n! \) and \( \frac{e^n}{(n+1)!} \leq 2P_n(|\rho-r|) \). Putting these two cases together bounds the integrand by \( 3e^{-|\rho-r|} P_n(|\rho-r|) \) and the proposition follows.

Lemma 99 If \( m \) and \( n \) are integers no less than \(-1\), then

\[
|q| \int_{q' \in \mathbb{R}^d} e^{-|q' - q|^2 - |q|^2} |q'|^m |q - q'|^n dq' \leq C_7(d) \pi(m+1)! (n+1)! Q_{m+n+3}(|q|),
\]

where \( C_7(2) = 18 \) and \( C_7(3) = 2 \).

Proof We note that we may assume without loss of generality that \( m \leq n \) since a change of variables \( q' \rightarrow q - q' \) switches the roles of \( m \) and \( n \). Write \( q = pe^{i\phi}, q' = re^{i\psi} \) and \( \phi = \phi - \psi \). Let \( I \) be the integral on the left hand side. Then switching to polar coordinates gives

\[
I = \rho \int_0^\infty \int_0^{2\pi} e^{\rho - r |p - e^{i\phi}|} |p - e^{i\phi}|^n r dr d\theta.
\]

For \( n \geq 0 \), using Proposition 98 above gives,

\[
I \leq 6\pi \rho \int_0^\infty e^{-\rho r (m+1)} |p - e^{i\phi}|^n e^{\rho - r} P_n(|p - r|) dr.
\]

Now, we let \( \tilde{\rho} = \frac{1}{\rho} \). Then \( d\tilde{\rho} = \frac{dr}{\rho} \) and \(-|\rho - r| = -\rho(\tilde{\rho} - 1)\text{sgn}(\tilde{\rho} - 1)\), so

\[
I \leq 6\pi \rho^{m+3} \int_0^\infty e^{-\rho(\tilde{\rho} - 1)(1 + \text{sgn}(\tilde{\rho} - 1))} \rho^{m+1} P_n(\rho(\tilde{\rho} - 1)) d\tilde{\rho}.
\]
Applying Lemma 97 with $m = m + 1$ and $n = n$ gives

\[ I \leq 6\pi \rho (m + 1)! n! Q_{m+n+2}(\rho) \leq 18\pi (m + 1)! (n + 1)! Q_{m+n+3}(\rho), \]

where the last inequality follows as $m \leq n$, so

\[ \rho^{m+n+2} \sum_{j=0}^{2m+n+2-j} \frac{\rho^j}{j!} \leq \sum_{j=1}^{m+n+3} \frac{2^{m+n+3-j} \rho^j}{(j-1)!} \]

\[ \leq Q_{m+n+3}(\rho)(m + n + 3) \leq 3(n + 1) Q_{m+n+3}(\rho). \]

For $n = m = -1$, we use a slightly different approach. Assuming $q$ is not zero, we split the integral over two regions, a ball of radius $3|q|/2$ centered at zero and its compliment. For the compliment region we have $|q - q'| \geq |q|/2$, so

\[ |q| \int_{|q'| \geq 3|q|/2} e^{|q|-|q'| - |q-q'|} \frac{1}{|q||q-q'|} dq' \leq 2e^{|q|/2} \int_0^{2\pi} \int_{3|q|/2} \pi e^{-r} dr d\theta = 4\pi e^{-|q|} \leq 4\pi. \]

For the interior region we have

\[ |q| \int_{|q'| \leq 3|q|/2} e^{|q|-|q'| - |q-q'|} \frac{1}{|q||q-q'|} dq' \leq |q| \int_{|q'| \leq 3|q|/2} \frac{1}{|q||q-q'|} dq'. \]

We now note that $\int_{|q'| \leq 3|q|/2} \frac{1}{|q||q-q'|} dq'$ is bounded. Without trying to be precise we can bound the integral by $13\pi$ by spitting the region into two disks of radius $|q|/2$ centered at $0$ and $q$ and the compliment, call the compliment $D$. We have

\[ \int_{|q'| \leq |q|/2} \frac{1}{|q'||q-q'|} dq' \leq \frac{2}{|q|} \int_{|q'| \leq |q|/2} \frac{1}{|q'|} dq' \leq 2\pi. \]

Similarly,

\[ \int_{|q-q'| \leq |q|/2} \frac{1}{|q'||q-q'|} dq' \leq 2\pi. \]

Finally,

\[ \int_{D} \frac{1}{|q'||q-q'|} dq' \leq \frac{4}{|q|^2} \int_{D} dq' \leq \frac{4}{|q|^2} \int_{|q'| \leq 3|q|/2} dq' \leq 9\pi. \]

Thus,

\[ |q| \int_{|q'-q'| \geq |q'-q'|} \frac{1}{|q'||q-q'|} dq' \leq 13\pi |q| + 4\pi \leq 18(|q| + 2) = 18Q_1(|q|) \]

for all nonzero $q$. Hence, the lemma is proved with $C_7(2) = 18$. 

Lemma 910  For any $\gamma \geq 1$ and nonnegative integers $m$ and $n$, we have
\[
|k| \int_{k' \in \mathbb{R}^d} \frac{e^{-|k'|^2 + |k-k'|^2}}{(1 + |k'|)^\gamma (1 + |k-k'|)^\gamma} |\beta|^{m+n} \beta |k-k'|^{n} \, dk' \\
\leq \frac{C_7 \pi 2^\gamma e^{-|k|} m! n!}{\beta^d (1 + |k|)^\gamma} (m + n + 2) Q_{m+n+2}(\beta |k|).
\]

Proof  The proof is exactly as in [5] using our new bound in Lemma 99. The idea is to split into two regions $|k'| \leq |k|/2$ and its compliment. In the ball, we have
\[
\frac{1}{(1 + |k-k'|)^\gamma (1 + |k'|)^\gamma} \leq \frac{\beta}{(1 + |k|)^\gamma |\beta k'|},
\]
and we use Lemma 99 with $m$ replaced by $m - 1$. In the compliment, we have
\[
\frac{1}{(1 + |k-k'|)^\gamma (1 + |k'|)^\gamma} \leq \frac{\beta}{(1 + |k|)^\gamma |\beta (k-k')|},
\]
and we use Lemma 99 with $n$ replaced by $n - 1$.

Lemma 911  For any $\gamma \geq 2$ and $n \in \mathbb{N} - 0$, we have
\[
|k| \int_{k' \in \mathbb{R}^d} \frac{e^{-|k'|^2 + |k-k'|^2}}{(1 + |k'|)^\gamma (1 + |k-k'|)^\gamma} |\beta|^{n} \beta |k-k'|^{n} \, dk' \\
\leq \frac{C_7 \pi 2^\gamma e^{-|k|} |k|!}{\beta^d (1 + |k|)^\gamma} \left\{ (n - 1)! Q_{n+1}(|q|) + \frac{3(n+1)! |q|^{2/3} n! \sum_{j=0}^{n+1} |q|^{j}}{2^{2/3}} \right\}.
\]

Proof  We again break into two integrals $\int_{|k'| \leq |k|/2} + \int_{|k'| \geq |k|/2}$. In the outer region, we have $(1 + |k'|)^{-\gamma} \leq 2^\gamma (1 + |k|)^{-\gamma}$, and in the inner, we have $(1 + |k-k'|)^{-\gamma} \leq 2^\gamma (1 + |k|)^{-\gamma}$. We use this and $\gamma \geq 2$ for the first inequality and Lemma 99 for the second to get a bound for the outer region
\[
|k| \int_{|k'| \geq |k|/2} \frac{e^{-|k'|^2 + |k-k'|^2}}{(1 + |k'|)^\gamma (1 + |k-k'|)^\gamma} |\beta|^{n} \beta |k-k'|^{n} \, dk' \\
\leq \frac{2^\gamma e^{-|k|} |k|!}{\beta^d (1 + |k|)^\gamma} \int_{q' \in \mathbb{R}^d} e^{-|q'|^2} (|a - q'| - |q - q'|) \, dq' \\
\leq \frac{C_7 \pi 2^\gamma e^{-|k|} |k|!}{\beta^d (1 + |k|)^\gamma} (n - 1)! Q_{n+1}(|q|).
\]

In the inner region, we also use $(1 + |k'|)^{-\gamma} \leq (|k'|)^{-2+2/3}$, a change to polar coordinates as in the proof of Lemma 99, and integration by parts to get
\[
|k| \int_{|k'| \leq |k|/2} \frac{e^{-|k'|^2 + |k-k'|^2}}{(1 + |k'|)^\gamma (1 + |k-k'|)^\gamma} |\beta|^{n} \beta |k-k'|^{n} \, dk' \\
\leq \frac{2^\gamma e^{-|k|} |k|!}{\beta^{d+2/3} (1 + |k|)^\gamma} \int_{|q'| \leq |q|/2} e^{-|q'|^2} (|q'| - |q - q'|) \, dq' + \frac{q' \, dq'}{(|q'|-|q-q'|)^{2+2/3}} |q - q'|^n \, dq'
\]
\[
\begin{align*}
\int_{0}^{2\pi} e^{\rho - \rho \cdot r e^{i\eta}} |r - r e^{i\theta}|^{n} r^{-2+2/3} r d\theta d r & \\
& = \frac{2^{\gamma} e^{-\beta |k|}}{\beta^{d-1+2/3} (1 + |k|)^{\gamma}} \rho^{\rho/2} \int_{0}^{2\pi} e^{\rho - \rho \cdot r e^{i\eta}} |r - r e^{i\theta}|^{n} r^{-2+2/3} r d\theta d r \\
& \leq \frac{2^{\gamma} e^{-\beta |k|}}{\beta^{d-1+2/3} (1 + |k|)^{\gamma}} 6\pi \rho^{\rho/2} r^{-1+2/3} P_{n}(|\rho - r|) d r \\
& \leq \frac{2^{\gamma} e^{-\beta |k|}}{\beta^{d-1+2/3} (1 + |k|)^{\gamma}} 6\pi n! \rho^{1+2/3} \sum_{j=0}^{n} \rho^{j} \int_{0}^{1} \tilde{r}^{-1+2/3} (1 - \tilde{r})^{j} d \tilde{r} \\
& \leq \frac{2^{\gamma} e^{-\beta |k|}}{\beta^{d-1+2/3} (1 + |k|)^{\gamma}} \frac{18}{2} \pi n! \rho^{2/3} \sum_{j=0}^{n} \frac{\rho^{j+1}}{j!}.
\end{align*}
\]

**Lemma 912.** For any \( \gamma \geq 1 \) and nonnegative integers \( l_{1}, l_{2} \geq 0 \), we have

\[
|k| \int_{k' \in \mathbb{R}^{d}} e^{\beta (|k| - |k'|-|k-k'|)} Q_{2l_{1}}(\beta |k'|) Q_{2l_{2}}(\beta |k-k'|) d k' \\
\leq \frac{C_{\gamma} \pi 2^{\gamma} e^{-\beta |k|}}{3 \beta^{d} (1 + |k|)^{\gamma}} (2l_{1} + 2l_{2} + 1) (2l_{1} + 2l_{2} + 2) (2l_{1} + 2l_{2} + 3) Q_{2l_{1}+2l_{2}+2}(|\beta |k|).
\]

The proof is exactly the same as in [5] with \( K = \frac{C_{\gamma} \pi 2^{\gamma} e^{-\beta |k|}}{3 \beta^{d} (1 + |k|)^{\gamma}} \). The idea of the proof is to use the definition of \( Q_{2l_{1}} \) and \( Q_{2l_{2}} \) with Lemma 910 to bound the left hand side by

\[
K \sum_{j=0}^{2l_{1}+2l_{2}} 2^{2l_{1}+2l_{2}+2-(j+2)} (j + 2) (j + 1) Q_{j+2}(|q|) \\
\leq K Q_{2l_{1}+2l_{2}+2}(|q|) \sum_{j=0}^{2l_{1}+2l_{2}} (j + 1) (j + 2)
\]

from which the result follows.

**Lemma 913.** If \( \gamma \geq 2 \) and \( l \geq 0 \), then

\[
\begin{align*}
\frac{|k|}{(l + 1)^{2/3}} \int_{k' \in \mathbb{R}^{d}} e^{-\beta (|k'| + |k-k'|)} |Q_{2l}(|\beta (k-k')|) d k' \\
& \leq \frac{C_{1} e^{-\beta |k|}}{(1 + |k|)^{\gamma}} (2l + 1) Q_{2l+2}(|\beta |k|),
\end{align*}
\]

where

\[
C_{1} = C_{1}(d) = 6C_{\gamma} \pi 2^{\gamma} \beta^{-d+1/3} + C_{\gamma} \pi 2^{\gamma} \beta^{-d+1} + \frac{1}{2} C_{0} \beta^{-1}.
\]

**Proof** The proof is again the same as in [5] except when Lemma 6.8. is invoked in [5] we use our Lemma 911. The idea is to split into a few cases. When \( l = 0 \), the claim holds with \( C_{1} = \frac{1}{2} C_{0} \beta^{-1} \). For \( l \geq 1 \), we separate the constant term

\[
\begin{align*}
|k| \int_{k' \in \mathbb{R}^{d}} e^{-\beta (|k'| + |k-k'|)} 2^{2l} d k' & \leq \frac{C_{0} e^{-\beta |k|}}{2\beta (1 + |k|)^{\gamma}} Q_{2l+2}(|\beta |k|).
\end{align*}
\]
Then, we use Lemma 9.11 to bound the terms of
\[
|k| \int_{k' \in \mathbb{R}^d} e^{-\beta(|k'|+|k-k'|)} \left((1 + |k'|)^\gamma (1 + |k-k'|)^\gamma (Q_{2t}(\beta |k|) - 2^{2t})ight) dk'
\]
and over bound the remaining sums to get the rest of the terms appearing in \( C_1(d) \).

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