ON GENERALIZED POWERS-STØRMER’S INEQUALITY

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Abstract. A generalization of Powers-Størmer’s inequality for operator monotone functions on \([0, +\infty)\) and for positive linear functional on general \(C^*\)-algebras will be proved. It also will be shown that the generalized Powers-Størmer inequality characterizes the tracial functionals on \(C^*\)-algebras.

1. Introduction

Powers-Størmer’s inequality (see, for example, \cite{12}) asserts that for \(s \in [0, 1]\) the following inequality
\begin{equation}
2 \text{Tr}(A^s B^{1-s}) \geq \text{Tr}(A + B - |A - B|)
\end{equation}
holds for any pair of positive matrices \(A, B\). This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory \cite{1}. This inequality was first proven in \cite{1}, using an integral representation of the function \(t^s\). After that, M. Ozawa gave a much simpler proof for the same inequality, using fact that for \(s \in [0, 1]\) function \(f(t) = t^s\) \((t \in [0, +\infty))\) is an operator monotone. Recently, Y. Ogata in \cite{10} extended this inequality to standard von Neumann algebras. The motivation of this paper is that if the function \(f(t) = t^s\) is replaced by another operator monotone function (this class is intensively studied, see \cite{7} \cite{11}), then \(\text{Tr}(A + B - |A - B|)\) may get smaller upper bound that is used in quantum hypothesis testing. Based on M. Ozawa’s proof we formulate Powers-Størmer’s inequality for an arbitrary operator monotone function on \([0, +\infty)\) in the context of general \(C^*\)-algebras.

Finally, we will show that the Powers-Størmer’s inequality characterizes the trace property for a normal linear positive functional on a von Neumann algebras and for a linear positive functional on a \(C^*\)-algebra.

Recall that a positive linear functional \(\varphi\) on a von Neumann algebra \(\mathcal{M}\) is said to be normal if \(\varphi(\sup A_i) = \sup \varphi(A_i)\) for every bounded increasing net \(A_i\) of positive elements in \(\mathcal{M}\). A linear functional \(\varphi\) on a \(C^*\)-algebra \(\mathcal{A}\) is said to be tracial if \(\varphi(AB) = \varphi(BA)\) for all \(A, B \in \mathcal{A}\).

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For all other notions used in the paper, we refer the reader to the monograph [8].

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2. Main results

Let \( n \in \mathbb{N} \) and \( M_n \) be the algebra of \( n \times n \) matrices. Let \( I \) be an interval in \( \mathbb{R} \) and \( f : I \to \mathbb{R} \) be a continuous function. We call a function \( f \) matrix monotone of order \( n \) or \( n \)-monotone in short whenever the inequality

\[
A \leq B \implies f(A) \leq f(B)
\]

for an arbitrary selfadjoint matrices \( A, B \in M_n \) such that \( A \leq B \) and all eigenvalues of \( A \) and \( B \) are contained in \( I \).

Let \( H \) be a separable infinite dimensional Hilbert space and \( B(H) \) be the set of all bounded linear operators on \( H \). We call a function \( f \) operator monotone whenever the inequality

\[
A \leq B \implies f(A) \leq f(B)
\]

for an arbitrary selfadjoint matrices \( A, B \in B(H) \) such that \( A \leq B \) and all eigenvalues of \( A \) and \( B \) are contained in \( I \).

We denote the spaces of operator monotone functions by \( P_\infty(I) \). The spaces for \( n \)-monotone functions are written as \( P_n(I) \). We have then

\[
P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I).
\]

Here we note that \( \bigcap_{n=1}^\infty P_n(I) = P_\infty(I) \) and each inclusion is proper [7][11].

The following result is well-known. For example see the proof in [5] Theorem 2.5].

**Lemma 2.1.** Let \( f \) be a strictly positive, continuous function on \([0, \infty)\). If the function \( f \) is \( 2n \)-monotone, then for any positive semidefinite \( A \) and a contraction \( C \) in \( M_n \) we have

\[
C^* f(A) C \leq f(C^* A C).
\]

**Lemma 2.2.** Let \( f \) be a continuous function on \((0, \infty)\) such that \( 0 \notin f((0, \infty)) \). Then, \( f \) is \( n \)-monotone if and only if the function \( -\frac{1}{f'(t)} \) is \( n \)-monotone.
Proof. For any \( t_1, t_2, \cdots, t_n \in (0, \infty) \) we have

\[
\frac{1}{f(t_i)} - \frac{1}{f(t_j)} = \frac{f(t_i) - f(t_j)}{f(t_i)f(t_j)}
\]

\[
= -\frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j}.
\]

Since \( f \) is \( n \)-monotone, the matrix \( \frac{f(t_i) - f(t_j)}{t_i - t_j} \) is positive semidefinite by \([9]\), hence, we have

\[
\left[ -\frac{1}{f(t_i)} - \frac{1}{f(t_j)} \right] = -\left[ \frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j} \right]
\]

\[
= -\left[ \frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j} \right] \circ \left[ \frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j} \right]
\]

\[
\geq 0,
\]

where \( \circ \) means the Hadamard product.

Therefore, the function \(-\frac{1}{f(t)}\) is \( n \)-monotone by \([9]\).

Conversely, if \(-\frac{1}{f}\) is \( n \)-monotone, we have

\[
\left[ \frac{f(t_i) - f(t_j)}{t_i - t_j} \right] = \left[ f(t_i)f(t_j) \right] \circ \left[ -\frac{1}{f(t_i)} - \frac{1}{f(t_j)} \right]
\]

\[
\geq 0,
\]

hence \( f \) is \( n \)-monotone. \( \square \)

**Proposition 2.1.** Let \( f \) be a strictly positive, continuous function on \([0, \infty)\). If \( f \) is \( 2n \)-monotone, the function \( g(t) = \frac{1}{f(t)} \) is \( n \)-monotone on \([0, \infty)\).

Proof. Let \( A, B \) be positive matrices in \( M_n \) such that \( 0 < A \leq B \).

Let \( C = B^{-\frac{1}{2}}A^\frac{1}{2} \). Then \( \|C\| \leq 1 \). Since \( f \) is \( 2n \)-monotone, \(-f\) satisfies the Jensen type inequality from Lemma 2.1 that is,

\[
-f(A) = -f(C^*BC) \leq -C^*f(B)C
\]

\[
-f(A) \leq -A^\frac{1}{2}B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}A^\frac{1}{2}
\]

\[
-A^{-\frac{1}{2}}f(A)A^{-\frac{1}{2}} \leq -B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}
\]

\[
-A^{-1}f(A) \leq -B^{-1}f(B)
\]
Hence, the function \( -\frac{f(t)}{t} \) is \( n \)-monotone. Therefore, from Lemma 2.2 we conclude that

\[
-\frac{1}{f(t)} = \frac{t}{f(t)}
\]

is \( n \)-monotone. \( \square \)

Remark 1. The condition of \( 2n \)-monotonicity of \( f \) is needed to guarantee the \( n \)-monotonicity of \( g \). Indeed, it is well-known that \( t^3 \) is monotone, but not \( 2 \)-monotone. In this case the function \( g(t) = \frac{t}{t^3} = \frac{1}{t^2} \) is obviously not \( 1 \)-monotone.

**Proposition 2.2.** Let \( h: [0, \infty) \to [0, \infty) \) be a Borel function such that \( h \) is a continuous, \( n \)-monotone on \( (0, \infty) \), and \( h(0) = 0 \). Then for any \( A, B \in M_n^+ \) with \( A \leq B \) we have

\[
h(A) \leq h(B).
\]

**Proof.** Let \( B = \sum s \mu_s q_s \) be a spectral decomposition. Set \( 1 - q \) as a projection on \( \text{Ker}(B) \). Then \( B = Bq = qB = \sum s' \mu_s' q_s' \) and \( q = \sum s' q_s' \).

Similarly, let \( A = \sum \lambda p \) be a spectral projection and \( (1 - p) \) be a projection on \( \text{Ker}(A) \). Since \( A \leq B \), \( p \leq q \) and \( A = \sum v \lambda p_v \) and \( p = \sum v' p_v \). Note that since \( h(0) = 0 \), by the function calculus we have

\[
h(A) = \sum_v h(\lambda_v) p_v \quad \text{and} \quad h(B) = \sum_{s'} h(\mu_{s'}) q_{s'}.
\]

For any \( \varepsilon > 0 \) since

\[
0 < \sum v' \lambda_v p_v + \varepsilon 1 \leq \sum s' \mu_{s'} q_{s'} + \varepsilon 1
\]

and \( h \) is \( n \)-monotone on \( (0, \infty) \), we have

\[
h(\sum v' (\lambda_v + \varepsilon) p_v + \varepsilon (1 - p)) \leq h(\sum_{s'} (\mu_{s'} + \varepsilon) q_{s'} + \varepsilon (1 - q)).
\]

Since

\[
\sum v' h(\lambda_v + \varepsilon) p_v + h(\varepsilon)(1 - p) = h(\sum v' (\lambda_v + \varepsilon) p_v + \varepsilon (1 - p)) \leq h(\sum_{s'} (\mu_{s'} + \varepsilon) q_{s'} + \varepsilon (1 - q)) = \sum_{s'} h(\mu_{s'} + \varepsilon) q_{s'} + h(\varepsilon)(1 - q)
\]


and } p \leq q, \text{ it follows that} \sum_{t'} h(\lambda_t + \varepsilon)p_t \leq \sum_{t'} h(\lambda_t + \varepsilon)p_t + h(\varepsilon)q(1 - p)q 
\leq \sum_{s'} h(\mu_{s'} + \varepsilon)q_{s'}.

Therefore, since } h \text{ is continuous on } (0, \infty), \text{ as } \varepsilon \to 0 \text{ we have}

\begin{align*}
h(A) &= \sum_{t'} h(\lambda_{t'}) p_{t'} 
&\leq \sum_{s'} h(\mu_{s'}) q_{s'}
&= h(B).
\end{align*}

\hfill \square

**Corollary 2.1.** Let } f \text{ be a } 2n\text{-monotone, continuous function on } [0, \infty) \text{ such that } f((0, \infty)) \subset (0, \infty), \text{ and let } g \text{ be a Borel function on } [0, \infty) \text{ defined by } g(t) = \begin{cases} \frac{t}{1+t} & (t \in (0, \infty)) \\
0 & (t = 0) \end{cases}. \text{ Then for any pair of positive matrices } A, B \in M_n \text{ with } A \preceq B, \text{ } g(A) \preceq g(B).

**Proof.** Since } f \text{ is } 2n\text{-monotone, continuous function on } [0, \infty) \text{ such that } f((0, \infty)) \subset (0, \infty), \text{ from Proposition 2.1 } g \text{ is } n\text{-monotone on } (0, \infty).

Hence, since } g \text{ is a Borel function on } [0, \infty) \text{ with } g(0) = 0, \text{ from Proposition 2.2 it follows that } g(A) \leq (B). \hfill \square

**Theorem 2.1.** Let } Tr \text{ be a canonical trace on } M_n \text{ and } f \text{ be a } 2n\text{-monotone function on } [0, \infty) \text{ such that } f((0, \infty)) \subset (0, \infty). \text{ Then for any pair of positive matrices } A, B \in M_n

\begin{align*}
(2) \quad Tr(A) + Tr(B) - Tr(|A - B|) &\leq 2 Tr(f(A)^{1/2} g(B) f(A)^{1/2}),
\end{align*}

where } g(t) = \begin{cases} \frac{t}{1+t} & (t \in (0, \infty)) \\
0 & (t = 0) \end{cases}.

**Proof.** Let } A, B \text{ be any positive matrices in } M_n.

For operator } (A - B) \text{ let us denote by } P = (A - B)^+ \text{ and } Q = (A - B)^- \text{ its positive and negative part, respectively. Then we have}

\begin{align*}
(3) \quad A - B &= P - Q \quad \text{and} \quad |A - B| = P + Q,
\end{align*}

from that it follows that

\begin{align*}
(4) \quad A + Q &= B + P.
\end{align*}
On account of (1) the inequality (2) is equivalent to the following
\[ \text{Tr}(A) - \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}) \leq \text{Tr}(P). \]
Since \( B + P \geq B \geq 0 \) and \( B + P = A + Q \geq A \geq 0 \), we have \( g(A) \leq g(B + P) \) by Corollary 2.1 and
\[ \text{Tr}(A) - \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}) \leq \text{Tr}(P). \]
Hence, we have the conclusion. 

\[ \square \]

**Remark 2.**

(i) When given positive matrices \( A, B \) in \( M_n \) satisfies the condition \( A \leq B \), the inequality (2) becomes
\[ \text{Tr}(A) \leq \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}). \]

(ii) As pointed in Proposition 2.1, \( 2 \)-monotonicity of \( f \) is needed to guarantee the inequality (2). Indeed, let \( f(t) = t^3 \) and \( n = 1 \). Then, for any \( a, b \in (0, \infty) \), the inequality (2) would imply
\[ a \leq f(a)^{1/2}g(b)f(a)^{1/2}, \]
that is,
\[ \frac{a}{f(a)} \leq \frac{b}{f(b)}. \]
Since \( \frac{1}{f(t)} \) is, however, not 1-monotone, the latter inequality is impossible.

As an application we get Powers-Størmer’s inequality.

**Corollary 2.2.** (Theorem 1) Let \( A \) and \( B \) be positive matrices, then for all \( s \in [0,1] \)
\[ \text{Tr}(A + B - |A - B|) \leq \text{Tr}(A^sB^{1-s}). \]
Proof. Let \( f(t) = t^s \) (\( s \in [0, 1] \)). Then \( f \) is operator monotone with \( f(0, \infty) \subset (0, \infty) \) and \( g(t) = t^{1-s} \). Hence, we have the conclusion from Theorem 2.1. \( \square \)

Since any \( C^* \)-algebra can be realized as a closed selfadjoint \(*\)-algebra of \( B(H) \) for some Hilbert space \( H \). We can generalize Theorem 2.1 in the framework of \( C^* \)-algebras.

**Theorem 2.2.** Let \( \tau \) be a tracial functional on a \( C^* \)-algebra \( \mathcal{A} \), \( f \) be a strictly positive, operator monotone function on \([0, \infty)\). Then for any pair of positive elements \( A, B \in \mathcal{A} \)

\[
\tau(A) + \tau(B) - \tau(|A - B|) \leq 2\tau(f(A)^{1/2}g(B)f(A)^{1/2}),
\]

where \( g(t) = tf(t)^{-1} \).

**Proof.** Since the function \( \frac{1}{f(0)} \) is operator monotone on \((0, \infty)\) by [5, Corollary 6], we can get the conclusion through the same steps in the proof of Theorem 2.1. \( \square \)

**Remark 3.** For matrices \( A, B \in M_n^+ \) let us denote

\[
Q(A, B) = \min_{s \in [0, 1]} \text{Tr}(A^{(1-s)/2}B^sA^{(1-s)/2})
\]

and

\[
Q_{F_{2n}}(A, B) = \inf_{f \in F_{2n}} \text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}),
\]

where \( F_{2n} \) is the set of all \( 2n \)-monotone functions on \([0, +\infty)\) satisfy condition of the Theorem 2.1 and \( g(t) = tf(t)^{-1} \) \((t \in [0, +\infty))\). Note that the function \( f(t) = t^s \) \((t \in [0, +\infty))\) satisfies the conditions of Theorem 2.1. Since the class of \( 2n \)-monotone functions is large enough [11], we know that \( Q_{F_{2n}}(A, B) \leq Q(A, B) \). Hence, we hope on finding another \( 2n \)-monotone function \( f \) on \([0, +\infty)\) such that

\[
\text{Tr}(f(A)^{1/2}g(B)f(A)^{1/2}) < Q(A, B).
\]

If we can find such a function, then we can refine the quantum Chernoff bound used in quantum hypothesis testing [1].

### 3. Characterizations of the trace property

In this section the generalized Powers-Størmer inequality in the previous section implies the trace property for a positive linear functional on operator algebras.

**Lemma 3.1.** Let \( \varphi \) be a positive linear functional on \( M_n \) and \( f \) be a continuous function on \([0, \infty)\) such that \( f(0) = 0 \) and \( f((0, \infty)) \subset (0, \infty) \). If the following inequality

\[
\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{1/2}g(B)f(A)^{1/2})
\]

...
holds true for all \( A, B \in M_n^+ \), then \( \varphi \) should be a positive scalar multiple of the canonical trace \( \text{Tr} \) on \( M_n \), where \( g(t) = \begin{cases} \frac{1}{t} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases} \).

Proof. As is well known, every positive linear functional \( \varphi \) on \( M_n \) can be represented in the form \( \varphi(\cdot) = \text{Tr}(S_{\varphi}\cdot) \) for some \( S_{\varphi} \in M_n^+ \). It is easily seen that without loss of generality we can assume that \( S_{\varphi} = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \), and we have to prove that \( \alpha_i = \alpha_j \) for all \( i, j = 1, \ldots, n \). Clearly, it is sufficient to prove that \( \alpha_1 = \alpha_2 \). By assumption, the inequality (9) holds true, in particular, for any positive matrices \( X = [x_{ij}]_{i,j=1}^n, Y = [y_{ij}]_{i,j=1}^n \) from \( M_n^+ \) such that \( 0 = x_{ij} = y_{ij} \) if \( 3 \leq i \leq n \) or \( 3 \leq j \leq n \). Thus, it suffices to consider the case \( n = 2 \).

Assume that \( S_{\varphi} = \text{diag}(d, 1) \) \((d \in [0, 1])\) and \( \varphi(D) = \text{Tr}(S_{\varphi}D), \forall D \in M_2 \). We show that \( d = 1 \). For arbitrary positive numbers \( \lambda, \mu \) such that \( \lambda < \mu \) we consider the following matrices

\[
A = \begin{pmatrix} \frac{\lambda}{\sqrt{\lambda\mu}} & \sqrt{\frac{\lambda\mu}{\lambda}} \\ \sqrt{\frac{\lambda\mu}{\mu}} & \frac{\mu}{\sqrt{\lambda\mu}} \end{pmatrix}
\]

and

\[
B = \begin{pmatrix} \frac{\lambda}{\sqrt{\lambda\mu}} & -\sqrt{\frac{\lambda\mu}{\mu}} \\ -\sqrt{\frac{\lambda\mu}{\lambda}} & \frac{\mu}{\sqrt{\lambda\mu}} \end{pmatrix}.
\]

It is clear that these are positive scalar multiple of projections of rank one. In addition,

\[
f(A)^\frac{1}{2}g(B)f(A)^\frac{1}{2} = \left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 A.
\]

We have

\[
2\varphi(f(A)^\frac{1}{2}g(B)f(A)^\frac{1}{2}) = 2 \left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 \text{Tr}(S_{\varphi}A)
\]

\[
= 2 \left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 (d\lambda + \mu).
\]

By direct calculation,

\[
|A - B| = \begin{pmatrix} 2\sqrt{\lambda\mu} & 0 \\ 0 & 2\sqrt{\lambda\mu} \end{pmatrix}.
\]

Consequently,

\[
\varphi(A + B) - \varphi(|A - B|) = d(2\lambda - 2\sqrt{\lambda\mu}) + 2\mu - 2\sqrt{\lambda\mu}.
\]

Then the inequality (9) becomes

\[
\left( \frac{\mu - \lambda}{\mu + \lambda} \right)^2 (d\lambda + \mu) \geq d(\lambda - \sqrt{\lambda\mu}) + \mu - \sqrt{\lambda\mu}.
\]

Dividing two side by \( \sqrt{\lambda}(\sqrt{\mu} - \sqrt{\lambda}) \), we get

\[
d + \frac{(\sqrt{\mu} - \sqrt{\lambda})(\sqrt{\mu} + \sqrt{\lambda})^2}{\sqrt{\lambda}^2(\mu + \lambda)^2}(d\lambda + \mu) \geq \sqrt{\frac{\mu}{\lambda}}.
\]
Tending $\lambda$ to $\mu$ from the left we obtain

$$d \geq 1.$$ 

Since $d \in [0, 1]$, $d = 1$. This means that $\varphi$ is a positive scalar multiple of the canonical trace $\text{Tr}$ on $M_n$. □

**Remark 4.** Let $\varphi$ be a positive linear functional on $M_n$ and $s \in [0,1]$. From Lemma 3.1 it is clear that if the following inequality

$$\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(A^{\frac{1}{2-s}}B^{\frac{1}{1-s}}A^{\frac{1}{2-s}})$$

holds true for any $A, B \in M_n^+$, then $\varphi$ is a tracial. In particular, when $s = 0$ the following inequality characterizes the trace property

$$\varphi(B) - \varphi(A) \leq \varphi(|A - B|) \quad (A, B \in M_n^+).$$

**Corollary 3.1 ([14]).** Let $\varphi$ be a positive linear functional on $M_n$ and the following inequality

$$\varphi(|A + B|) \leq \varphi(|A|) + \varphi(|B|)$$

holds true for any self-adjoint matrices $A, B \in M_n$. Then $\varphi$ is a tracial.

**Proof.** From the assumption, we have

$$\varphi(|B - A|) \geq \varphi(|B|) - \varphi(|A|)$$

for any pair of self-adjoint matrices $A, B \in M_n$. Moreover, for any pair of positive matrices $A, B \in M_n$ we have

$$\varphi(|B - A|) \geq \varphi(B) - \varphi(A).$$

On account of Remark [4] it follows that $\varphi$ should be a tracial. □

**Corollary 3.2 ([4]).** Let $\varphi$ be a positive linear functional on $M_n$ and the following inequality

$$|\varphi(A)| \leq \varphi(|A|)$$

holds true for any self-adjoint matrix $A \in M_n$. Then $\varphi$ is a tracial.

**Proof.** Let $A, B \in M_n$ be arbitrary positive matrices. Then $C = B - A$ is a self-adjoint matrix. Since $A, B \geq 0$, the values $\varphi(A)$ and $\varphi(B)$ are real. From the assumption, we have

$$\varphi(B) - \varphi(A) \leq |\varphi(B) - \varphi(A)| = |\varphi(B - A)| \leq \varphi(|B - A|).$$

On account of Remark [4] it follows that $\varphi$ should be a tracial. □

By analogy with a number of other similar cases (see [4] or [14]), the proof for the trace property of a positive normal functional satisfying the inequality (9) on a von Neumann algebra can be reduced to the case of the algebra $M_2$ of all matrices of order $2 \times 2$. But for self-contained we will give a sketch of its proof.
Theorem 3.1. Let $\phi$ be a positive normal linear functional on a von Neumann algebra $\mathcal{M}$ and $f$ be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality
\begin{equation}
\phi(A) + \phi(B) - \phi(|A - B|) \leq 2\phi(f(A)^{1/2}g(B)f(A)^{1/2})
\end{equation}
holds true for any pair $A, B \in \mathcal{M}^+$, then $\phi$ is a trace, where $g(t) = \begin{cases} \frac{1}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$.

Proof. Let $P_1, P_2$ be a pair of nonzero mutually orthogonal equivalent projections in $\mathcal{M}$, that means $P_1 = V^*V$ and $P_2 = VV^*$ for some nonzero partial isometry $V \in \mathcal{M}$. Consider the $*$-algebra $\mathcal{N}$ in $(P_1 + P_2)\mathcal{M}(P_1 + P_2)$ generated by the partial isometry $V$. Then $\mathcal{N}$ is isomorphic to $\mathcal{M}_2$ and inequality (14) still holds true for the operators in $\mathcal{N}$ and for the restriction of the functional $\phi$ to $\mathcal{N}$. According to Lemma 3.1 this restriction is a tracial functional on $\mathcal{N}$, and hence $\phi(P_1) = \phi(P_2)$. By [5, Vol2, Proposition 8.1.1] it follows that $\phi$ is a trace. \qed

Corollary 3.3. Let $\phi$ be a positive linear functional on a $C^*$-algebra $\mathcal{A}$ and $f$ be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality
\begin{equation}
\phi(A) + \phi(B) - \phi(|A - B|) \leq 2\phi(f(A)^{1/2}g(B)f(A)^{1/2})
\end{equation}
holds true for any pair $A, B \in \mathcal{A}^+$, then $\phi$ is a tracial functional, where $g(t) = \begin{cases} \frac{1}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$.

Proof. Let $\pi$ be the universal representation of $C^*$-algebra $\mathcal{A}$ and $\mathcal{M} = \pi(\mathcal{A})''$. Let $\hat{\phi}$ be the positive normal functional on $\mathcal{M}$ such that $\hat{\phi}(\pi(A)) = \phi(A)$ for $A \in \mathcal{A}$. By the Kaplansky density theorem, for any pair $\hat{A}, \hat{B} \in \mathcal{M}^+$ there exist bounded nets $\{A_\alpha\}$ and $\{B_\alpha\}$ in $\mathcal{A}^+$ such that $\pi(A_\alpha) \to \hat{A}$ and $\pi(B_\alpha) \to \hat{B}$ in the strong operator topology. Using (15) and the continuity of the corresponding operations in the strong operator topology, we have
\[ \hat{\phi}(\hat{A}) + \hat{\phi}(\hat{B}) - \hat{\phi}(|\hat{A} - \hat{B}|) \leq 2\hat{\phi}(f(\hat{A})^{1/2}g(\hat{B})f(\hat{A})^{1/2}). \]

By Theorem 3.1 $\hat{\phi}$ is a tracial functional $\mathcal{M}$, and hence $\phi$ is a tracial functional on $\mathcal{A}$. \qed

Remark 5. Let $\mathcal{A}$ be a von Neumann algebra and $\phi$ be a positive linear functional on $\mathcal{A}$. The set $P(\mathcal{A})$ of all orthogonal projections from $\mathcal{A}$ is enough as a testing space for some inequality to characterize the trace property of $\phi$ (see [3]). But, in the case of the inequality (14) the set $P(\mathcal{A})$ is not enough as a testing set.
Indeed, let $p, q$ be arbitrary orthogonal projections from a von Neumann algebra $\mathcal{M}$. Since $q \geq p \wedge q$ it follows that $pqp \geq p(p \wedge q)p = p \wedge q$. So $pqp \geq p \wedge q$ holds for any pair of projections. From that it follows

$$\varphi(p + q - |p - q|) = 2\varphi(p \wedge q) \leq 2\varphi(pqp) = 2\varphi\left(f(p)^{\frac{1}{2}}g(q)f(p)^{\frac{1}{2}}\right)$$

whenever $\varphi$ is an arbitrary positive linear functional on $\mathcal{M}$.

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