Toward One-Loop Tunneling Rates of
Near-Extremal Magnetic Black Hole Pair-Production

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ABSTRACT

Pair-production of magnetic Reissner-Nordström black holes (of charges $\pm q$) was recently studied in the leading WKB approximation. Here, we consider generic quantum fluctuations in the corresponding instanton geometry given by the Euclidean Ernst metric, in order to simulate the behaviour of the one-loop tunneling rate. A detailed study of the Ernst metric suggests that for sufficiently weak field $B$, the problem can be reduced to that of quantum fluctuations around a single near-extremal Euclidean black hole in thermal equilibrium with a heat bath of finite size. After appropriate renormalization procedures, typical one-loop contributions to the WKB exponent are shown to be inversely proportional to $B$, as $B \to 0$, indicating that the leading Schwinger term is corrected by a small fraction $\sim \hbar/q^2$. We demonstrate that this correction to the Schwinger term is actually due to a semiclassical shift of the black hole mass-to-charge ratio that persists even in the extremal limit. Finally we discuss a few loose ends.

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1 Motivation

Recently, the pair-production of oppositely charged magnetic black holes in a background magnetic field, has been studied to the leading WKB approximation \cite{1, 2, 3, 4}. The instanton mediating the tunneling process is found to be the Euclidean section of the so-called Ernst metric, and the Euclidean action thereof has been calculated exactly.

One cannot emphasize too much the importance of this process in the context of black hole quantum physics. A crucial issue in the context of the black hole information puzzle \cite{5} is how a potential degeneracy of the black hole configuration would or would not show up in scattering processes \cite{6}. For instance, it is a matter of some controversy whether it is possible for remnants with an infinite degeneracy to be pair-produced only with finite amplitude.

A salient feature of the leading WKB estimate above is that it shows an enhancement factor of $e^{S_{BH}}$ over that of monopole pair-production, where $S_{BH}$ is the Bekenstein-Hawking entropy \cite{7}, seemingly suggesting that the black hole have degeneracy $e^{S_{BH}}$ \cite{2}. But this result is more puzzling than clarifying, for one would normally expect to find the degeneracy, if any, not from the Euclidean action but at the next order WKB where one explores small deviations from the fixed instanton configuration. This naturally leads us to next-to-leading order WKB estimate which we want to explore in this article.

However, our goal here is rather modest. The primary concern here is to determine if there is anything special about the one-loop pair-production rate in the extremal limit which in this context has an alternative description, namely the weak field limit: As emphasized in the recent studies and as will be reiterated in section 2, the temperature of the pair-produced black hole is proportional to the background magnetic field strength $B$. Hence, we can concentrate on small $B$ for the purpose of studying near-extremal cases.

First, let us recall the behaviour of the leading exponent \cite{2}, simply given by the minus of the Euclidean action divided by $\hbar$ \cite{8}

$$-\frac{S_E}{\hbar} = -\frac{\pi q}{\hbar B} + \cdots$$  \hspace{1cm} (1)

$B$ is the background magnetic field strength as above, while $q$ is the absolute value of the magnetic charge of the black hole. The first term is easily recognized as the Schwinger term that also appears in the monopole pair-production \cite{9}, while the ellipsis denotes terms of higher order in $qB$, and includes the entropy term $+S_{BH}$.

In general, since the one-loop correction to this exponent must be independent of $\hbar$ by definition, it has to be a dimensionless function of the parameters of the classical Euclidean instanton, and thus is a function of $qB = qBG$ only \cite{10}. The question is then how strongly it depends on $qB$. One interesting property of Reissner-Nordström black holes is that the distance

\footnote{In this article, we use the geometrized unit $c = G = 1$, unless specified otherwise.}

\footnote{That is, up to the contributions from the zero-mode sector which we shall be forced to disregard in this first attempt.}
to the event horizon diverges along any space-like geodesic in the extremal limit \[10\], implying
that the physical distance to the Euclidean black hole horizon of the instanton also diverges in
the weak field limit. Furthermore, the periodicity of an “asymptotic” Euclidean time coordinate
also diverges owing to the vanishing black hole temperature. Put together, a possible implication
is that increasing number of quantum fluctuations contribute to the tunneling process, and that
the one-loop correction to the exponent here might have a very strong $B$-dependence.

For instance, if the additive one-loop correction to the exponent like $(qB)^{−1−\epsilon}$ with any
positive $\epsilon$, the exponent for near-extremal black hole pair-production would no longer be domi-
nated by $−S_E/\hbar$, signaling a breakdown of the semiclassical method. In this article, we wish to
investigate exactly how strongly this one-loop correction scales as $qB \to 0$.

In the next two sections, we will study the instanton near the horizons, as a preliminary
step. We find through this investigation that, for the purpose of estimating the leading weak
field behaviour of the one-loop correction, it is sufficient to study quantum fields for a single
truncated Euclidean black hole in thermal equilibrium with a heat bath. This happens because
the two lengthscale of the instanton, one associated with the black hole while the other with
the background Melvin universe, are vastly different from each other for the pair-production of
near-extremal black holes. Furthermore, the size of the truncated black hole shall be easily seen
to grow indefinitely in the limit of vanishing $B$. In these two sections, we borrowed heavily, from
references [3] and [4], various results regarding the classical properties of the instanton.

In order to estimate the genuine one-loop contribution to the tunneling rate, one must expand
the action around the instanton background and isolate differential operators $Q_a$’s governing the
gravitational and the electromagnetic fluctuations as well as various Fadeev-Popov-like ghost
fields, determinants of which enter the prefactor.

$$\text{tunneling rate} \simeq \prod_a N_a \exp \left(−\frac{S_E}{\hbar}\right), \quad \log N_a \sim \log \text{Det}Q_a$$

However, since we are interested only in their qualitative behaviour, it is reasonable to consider
generic second-order operators instead, and for the rest of the article, we shall analyze one-
loop contributions from various matter degrees of freedom. (Also, we will consider chargeless
fluctuations only in this article.) Note that by this substitution we completely destroy the
information on the zero-mode sector as well as the negative-mode of the gravitational and
electromagnetic fluctuations. Therefore, we are actually considering the contribution from the
positive mode sectors only.

In section 4, we take on the first examples: those of generic massive quantum fluctuations.
For this case, we estimate the determinant of a general elliptic operator with large positive mass
term by performing the Schwinger-DeWitt expansion [11]. After a renormalization procedure,
we find the one-loop correction to the exponent diverge like $\sim 1/qB$, as $qB \to 0$. Note that this
has the same $B$ dependence as the Schwinger term shown above.

To determine whether a similar effect arises in massless cases, we study two special kinds of massless fluctuations: chargeless Callan-Rubakov modes [12][13][14][15] and four-dimensional conformal fluctuations. Because of the non-local nature of the resulting effective actions, here we need to take into account the heat bath mentioned above. It is shown in section 5 that chargeless Callan-Rubakov modes, just as the massive fluctuations, contribute to the exponent a term $\sim 1/qB$. Furthermore, we estimate the proportionality coefficient exactly. Calculations for the four-dimensional conformal fluctuations are a bit more involved, and we employ an alternate method that is suitable for finding the leading $qB$ dependence. Performing a dimensional analysis of a metric variation of the effective action expressed in terms of the one-loop energy-momentum expectation, we argue that exactly the same $1/qB$ behaviour occurs here as well.

In our conclusion, we demonstrate that this correction actually originates from a semiclassical shift of the near-extremal black hole masses. The correspondingly shifted mass-to-charge ratio of the semiclassically corrected extremal Reissner-Nordström black holes was recently studied and estimated in reference [16], and is due to certain residual quantum radiation that exists despite the vanishing Hawking temperature. The net result is then, we find no large and novel quantum correction associated with near-extremal black hole pair-production as far as contributions from the matter sector are concerned. Also we discuss a few loose ends that need further study.

2 The Geometry of the Euclidean Ernst Metric in the Weak Field Limit

Let us first write down the instanton solution of the Einstein-Maxwell theory [1], various parameters of which are to be identified with those of the pair-produced black holes. We follow the coordinate convention of reference [2].

\[ g^{(4)} = \frac{\Lambda^2}{(Ay - Ax)^2}(-G(y)dt^2 - \frac{dy^2}{G(y)}) + \frac{1}{(Ay - Ax)^2}(\frac{\Lambda^2}{G(x)}dx^2 + \frac{G(x)}{\Lambda^2}d\phi^2) \]  
\[ G(\xi) = (1 + r_- A\xi)(1 - \xi^2 - r_+ A\xi^3) \]  
\[ \Lambda = \Lambda(x,y) = (1 + \frac{1}{2}qBx)^2 + \frac{B^2}{4(Ay - Ax)^2}G(x) \]

The solution comes with two Killing coordinates $T$ and $\phi$, where the latter generates an axial symmetry. The Minkowskian version of this metric has an interpretation as two magnetic black holes of opposite charges, accelerating away from each other in the background Melvin universe. In this section, we are interested in the geometry of this instanton solution in the weak field limit of small $B$. As will be shown later, the only two independent parameters are $q \equiv \sqrt{r_+ r_-}$ and $B$; in the limit $qB \to 0$, $q, A,$ and $B$ are respectively the absolute value of charges, the acceleration of the black holes, and the magnetic field strength on the symmetry axis.
Calling the four roots of $G(\xi)$, $\xi_1, \xi_2, \xi_3, \xi_4$ in the ascending order, $y = \xi_1$, $\xi_2$, $\xi_3$ correspond to the inner and the outer horizons and the acceleration horizon respectively. But along the Euclidean section of real $T$, the inner horizon at $y = \xi_1$ is not part of the instanton, and the $y$-coordinate is restricted to $\xi_2 \leq y \leq \xi_3$, that is between the outer horizon and the acceleration horizon. Similarly, $x$, an angular coordinate, is restricted to $\xi_3 \leq x \leq \xi_4$.

To study the limit $qB \to 0$, it is convenient to expand $\xi_i$ in terms of $r_{\pm}A$ which are also necessarily small in the same limit,

\begin{align*}
\xi_1 &= -\frac{1}{r_-A} \\
\xi_2 &= -\frac{1}{r_+A} + r_+A + \cdots \\
\xi_3 &= -1 - \frac{r_+A}{2} + \cdots \\
\xi_4 &= +1 - \frac{r_+A}{2} + \cdots
\end{align*}

There are four potential conical singularities at the boundaries of the coordinates $x$ and $y$. Naturally, two of them, say at $x = \xi_4$ and $y = \xi_3$, can be removed by adjusting the periods of $\phi$ and $T$, while the removal of the rest forces certain relationships between the parameters of the instanton. For instance, the geometry is smooth at $x = \xi_3$ only if $A$ satisfies the following identity [3],

\[ \frac{(r_+ + r_-)}{2} A = qB + \cdots \] (7)

which is nothing but the Newton’s law, once we identify $(r_+ + r_-)/2$ as the mass of the black hole.

The final conical singularity at the black hole event horizon $y = \xi_2$ is resolved by adjusting $r_+ - r_-$ according to the following constraint [1][3].

\[ \frac{(r_+ - r_-)}{4\pi r_+^2} = \frac{A}{2\pi} + \cdots \] (8)

It is easy to recognize that, up to a factor of $\hbar$ on each side, the right-hand-side is the Unruh temperature [17][18] of an observer with acceleration $A$, and the left is the Hawking temperature $T_{BH}$ of a Reissner-Nordström black hole with horizon radii $r_\pm$.

This suggests that the pair-produced objects are near-extremal magnetic Reissner-Nordström black holes of temperature $\simeq B\hbar/2\pi$, accelerating away from each other. With the present form of the metric that covers the whole Euclidean space, however, it is difficult to see whether the black holes are indeed of Reissner-Nordström type. For instance, the solution (3) lacks the spherical symmetry that is characteristic of such black holes, owing to the background magnetic field that forces the acceleration. But since the background magnetic field and the black holes are described by two separate lengthscales $B^{-1}$ and $q$, the effect of the background magnetic
field on the geometry near black holes may be ignored in the weak field limit \( B^{-1} \gg q \), and a suitable expansion in terms of \( qB \) should reveal the hidden black hole geometry.

Near the Euclidean black hole whose horizon is at \( |y| = -\xi_3 \approx 1/r_+A \approx 1/qB \gg 1 \), then, we might as well expand the metric in terms of \( 1/|y| \). It is particularly convenient to perform the following coordinate transformations \( 3 \).

\[
\tau = T/A, \quad r = -\frac{1}{Ay}
\] (9)

Rewriting the functions in the metric coefficients, in terms of \( r \) and \( x \),

\[
(Ay - Ax)^2 = \frac{1}{r^2}(1 + rAx)^2,
\]

\[
G(y) = (1 - \frac{r_-}{r})(r^2A^2 - 1 + \frac{r_+}{r})\frac{1}{r^2A^2},
\]

\[
G(x) = (1 + r_+Ax)(1 - x^2 - r_+Ax^3),
\]

\[
\Lambda(x,y) = (1 + \frac{qB}{2}x)^2 + \frac{r^2B^2}{4(1 + rAx)^2}G(x),
\]

and retaining the leading nonvanishing term in each expression as \( r_\pm A \to 0 \),

\[
(Ay - Ax)^2 = \frac{1}{r^2} + \cdots, \quad (10)
\]

\[
G(y) = -\frac{1}{r^2A^2}(1 - \frac{r_-}{r})(1 - \frac{r_+}{r}) + \cdots, \quad (11)
\]

\[
G(x) = 1 - x^2 + \cdots, \quad (12)
\]

\[
\Lambda = 1 + \cdots. \quad (13)
\]

Recall that the angular coordinate \( x \) is confined to \( [\xi_3, \xi_4] \simeq [-1,1] \).

Finally inserting these approximate expressions back to the metric \( 3 \), and introducing an angular coordinate \( \cos \theta = x \), we recover the Reissner-Nordström geometry to the leading nonvanishing order, as expected.

\[
g^{(4)} = r^2 \left( \frac{1}{r^2A^2}(1 - \frac{r_-}{r})(1 - \frac{r_+}{r}) A^2 dr^2 + \frac{r^2A^2}{(1 - \frac{r_-}{r})(1 - \frac{r_+}{r})} d(\frac{1}{A}r)^2 \right)
\]

\[+ r^2 \left( \frac{dx^2}{1 - x^2} + (1 - x^2) d\phi^2 \right) + \cdots \]

\[
= \left( 1 - \frac{r_-}{r} \right)(1 - \frac{r_+}{r}) dr^2 + \frac{1}{(1 - \frac{r_-}{r})(1 - \frac{r_+}{r})} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \cdots. \quad (14)
\]

Note that this form of the metric provides a good approximation to the instanton geometry only when the radius \( r \) is much smaller than the scale set by the magnetic field strength, namely when \( r \ll B^{-1} \).

In any case, the discussions above clearly show that, in the weak field limit \( qB \to 0 \), this instanton mediates a quantum tunneling process that pair-produces magnetic Reissner-Nordström black holes of Hawking temperature \( \simeq \hbar B/2\pi \). Since there is only one Euclidean black hole horizon of the instanton, these two black holes must be identified along their bifurcation surfaces, thus forming a single Euclidean black hole when continued to Euclidean space.
3 One-Loop Correction, Vacua and the Accelerating Black Holes

In the previous section we emphasized that, in the weak field limit, the instanton geometry near the Euclidean black hole horizon is that of a single near-extremal Euclidean black hole. In the figure 1, the region described by the approximate metric \( (14) \) corresponds to the bottom “cup” attached to the background Melvin space as depicted by the top “sheet.” Clearly the Euclidean black hole region is not part of the background Euclidean Melvin universe and uniquely associated with the pair-production process.

![Figure 1](image)

**Figure 1**: A schematic diagram for the Euclidean instantons. In the weak field limit, the bottom “cup” is described by a near-extremal Euclidean black hole, while the top “sheet” is the Euclidean Melvin space. The acceleration horizon and the black hole horizon are located at points A and B, respectively. The transitional “mouth” region is denoted by the broken curve where the area of the transverse two-sphere is \( \sim 4\pi q^{2/3} B^{-4/3} \).

Since the tunneling rate can be written as a ratio of two partition functions, one associated with the instanton mediating the tunneling process and the other associated with the background Melvin space, evaluating the contribution from this black hole region should give us a qualitative behaviour of the one-loop correction. We are assuming that there are no large correlations between the bottom “cup” and the top “sheet.”

In principle, therefore, one may perform the necessary mode sum of quantum fluctuations with supports in the bottom “cup” only, in order to extract the one-loop contribution to the exponent, or alternatively, one may start with the off-shell effective action for an arbitrary background geometry and try to evaluate it on the given geometry, as we will do throughout this article. However, if the off-shell effective action to be evaluated is a non-local functional of the metric, the on-shell value of the effective action is a functional not only of the curvatures.
and the connections but also of various Green’s functions, and it is most important to perform the calculations in the right vacuum. And this is where the rest of the instanton is needed.

As illustrated in the figure 1, the instanton geometry possesses two horizons: the black hole horizon B on the bottom “cup” and the acceleration horizon A on the top “sheet.” The surface gravities of these two horizons are identified to ensure the absence of conical singularities [3], which, according to (8), implies that the Hawking radiation of the near-extremal black holes is balanced against the Rindler heat bath [4].

This picture of black holes immersed in Rindler heat baths can also be seen from the fact that the time-coordinate \( T = A \tau \) plays the role of Euclidean Rindler time. Far away from the black hole horizon \( r_+ A |y| \ll 1 \), the Ernst metric (3) can be transformed into the following Rindler-like form [4],

\[
g^{(4)} \simeq \Lambda^2 (\zeta^2 d\zeta^2 + d\phi^2) + \Lambda^{-2} \rho^2 d\phi^2, \quad \Lambda \simeq 1 + \frac{B^2 \rho^2}{4}, \tag{15}
\]

by performing the coordinate transformations

\[
\zeta^2 = \frac{y^2 - 1}{A^2(x - y)^2}, \quad \rho^2 = \frac{1 - x^2}{A^2(x - y)^2}. \tag{16}
\]

Clearly observers at fixed \( \zeta, \phi, \) and \( \rho \) experience acceleration of \( 1/\Lambda \zeta \).

On the other hand, for sufficiently small \( qB \simeq r_+ A \), there is a transitional “mouth” region \( r_+ A |y| \ll 1 \ll |y| \), where the Ernst metric is fairly well approximated by either of the metrics (14) and (13). In terms of the coordinates of (14) and of (13), this transitional region is located at

\[
r_+ \ll r \ll \frac{1}{A}, \quad \text{or} \quad \rho \ll \frac{1}{A} \quad \text{and} \quad \zeta \simeq \frac{1}{A}. \tag{17}
\]

Therefore, the “asymptotic” observers at large fixed \( r \), such that \( r_+ \ll r \ll A^{-1} \), are in fact Rindler observers with the acceleration given by \( \zeta^{-1} \simeq A \), and, to the first nonvanishing order in \( qB \), they must find the Hawking radiation from the black hole in a perfect equilibrium [4] with the Rindler heat bath [17].

To understand another aspect of this vacuum, recall that the local temperature, measured by a “static” observer propagating along \( \partial/\partial r \), is given by the inverse of the proper period of his Euclidean orbit. A “static” observer at large \( r \ll A^{-1} \), for example, has the period \( \simeq \hbar/T_{BH} \), as expected from an approximate black hole geometry (14), and as emphasized above. As we climb out of the bottom “cup” region, however, the geometry is no longer that of a Euclidean black hole, but is described by an approximate Melvin space (15).

\[5\] The fact that the black holes must be in a thermal equilibrium, is certainly consistent with the interpretation of the instanton as a mediator of the tunneling process. A smooth transition from the Euclidean instanton to the Minkowskian Ernst metric, describing pictorially the process of pair-production, is possible because the configuration is at rest at the moment of transition. If we want to extend this picture to the one-loop level, the configuration at the transition must achieve some sort of semiclassical equilibrium. By letting the black holes be in thermal equilibrium with the Rindler heat-baths, we ensure that the instanton solution is reliable even to the one-loop level.
The obvious result is that the local temperature of this vacuum state vanishes rapidly
\((1/2\pi\zeta\Lambda \to 0)\) as \(\zeta \to \infty\), owing to extra gravitational red-shifts, similar to those in the
Rindler space. This is in stark contrast with the usual Hartle-Hawking vacuum around a black
hole, where the local temperature approaches the Hawking temperature \(T_{BH}\) asymptotically.\[^7^\]

The upshot from these observations is that this naturally motivated vacuum looks like the
Hartle-Hawking vacuum inside the bottom “cup,” yet, outside, behaves as if it were an ordinary
Minkowski vacuum as seen by the family of accelerating Rindler observers, up to the modifications
due to the background magnetic fields. This tells us that, among other things, the heat
bath associated with the instanton comes with a natural infrared cut-off given by the size of the
bottom “cup.”

Of course, for the purpose of evaluating the one-loop exponent, we would have had introduced
a similar cut-off in any case, since, as emphasized above, it is only the bottom “cup” region of
the instanton (figure 1) that is associated with the tunneling process and thus contributes to
the tunneling rate. Now what is the size of the bottom “cup”? Note that, according to (\[17\]) and
(\[15\]), the typical curvature of the “mouth” region is roughly \(\sim B^2\) due to the nontrivial factor
\(\Lambda \simeq 1 + B^2 \rho^2 / 4\). On the other hand, the same region is described by the black hole metric (\[14\])
at some large value of \(r\), say \(r_B\), and the corresponding curvature scale is \(\sim q/r_B^3\). Equating
these two scales, we find the approximate value of the \(r\) coordinate along the “mouth” region:
\(r_B \sim (qB^2)^{1/3}\). As a result, whenever \(q \ll r_B \ll B^{-1}\), the bottom “cup” is well approximated
by a truncated Euclidean black hole (\[14\]) with \(r\) restricted to be smaller than \(r_B\), the effective
size of the heat bath. We shall see in the following sections that the precise value of \(r_B\) is
immaterial as far as the leading weak field behaviour is concerned.

To summarize, we argued that a rough estimate of the one-loop prefactor \(N_a\) may be obtained
by evaluating the corresponding effective action on a single near-extremal Euclidean black hole
truncated at \(r = r_B \sim (qB^{-2})^{1/3}\) and in thermal equilibrium with a heat bath.

\[4\] The Leading One-Loop Contribution in the Weak Field Limit:
Massive Fluctuations

Let us first consider the case where the choice of the vacuum is already built in. If the fluct-
uation is massive enough, the calculation of the effective action can be done in a systematic
local expansion in terms of both curvatures and momentum, known as the Schwinger-DeWitt
expansion (\[11\]). Since each term of this expansion is a local expression of the curvatures and
the connections, no ambiguity regarding the boundary condition may arise. Without loss of
generality, take a (bosonic) chargeless fluctuation of mass \(M\) and the kinetic operator \(Q + M^2\).

\[^6^\]If the geometry were that of a Euclidean black hole everywhere, the proper period would approach the finite
value \(\hbar/T_{BH}\) as \(r \to \infty\), for the geometry along the \((\tau, r)\) plane resembles a cylinder asymptotically.

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Then the prefactor $N_{\text{massive}}$ is related to the determinant in the following way,

$$N_{\text{massive}} = \text{Det}^{-1/2} \{ Q + \mathcal{M}^2 \} \equiv e^{-W},$$

(18)

where we evaluate the expressions on the right hand side on the given background geometry. The effective action\(^7\) $W$ is most conveniently expressed in terms of the heat kernel \(^20\).

$$W \equiv \frac{1}{2} \text{Tr} \log(Q + \mathcal{M}^2) = \frac{1}{2} \int_\epsilon^\infty ds \frac{1}{s} \text{Tr} e^{-sQ-s\mathcal{M}^2}$$

(19)

The lower limit $\epsilon$ is an ultraviolet cut-off that has the dimension of length squared. Performing the Schwinger-DeWitt expansion, we find

$$\text{Tr} e^{-sQ-s\mathcal{M}^2} = e^{-s\mathcal{M}^2} \frac{16\pi^2 s}{\sqrt{g}} \sum_{n=0}^\infty s^n \left\{ \int dx^4 \sqrt{g} \mathcal{L}^{(n)} \right\}.$$  

(20)

The integral is over the truncated Euclidean black hole of the previous section. \(\mathcal{L}^{(n)}\) is a polynomial of the curvature tensor $R^{\alpha \beta \gamma \delta}_{\beta \gamma \delta}$ and its derivatives, and has the same scaling dimension as $R^n$. In particular, \(\mathcal{L}^{(0)}\) is always identical to 1, and \(\mathcal{L}^{(1)}\) is proportional to the curvature scalar.

Obviously the first two terms $n = 0$ and $n = 1$ contribute to the renormalization of the cosmological constant and the gravitational constant respectively, while a logarithmically divergent part of the third ($n = 2$) can be absorbed in the renormalization of dimension-four operators.

Then, after the usual renormalization process, the remaining finite contributions from $n \geq 2$ can be integrated over both $s$ and $x$. Note that, since the integrands are completely independent of $\tau$, the integral over the Euclidean time simply produces a universal factor of $\hbar/T_{BH}$ which is the periodicity of $\tau$. On the other hand, the integrals over the radial and angular coordinates as well as over $s$ induces a power series in $1/qM$\(^3\).

$$W = \frac{\hbar}{qT_{BH}} \left\{ \sum_{m=0}^\infty \frac{a_m}{(q\mathcal{M})^{2m}} \right\} + \cdots$$

(21)

Coefficients $a_n$ are constants and the ellipsis denotes terms of higher order in $qB$. Now using the fact that $B/2\pi \simeq T_{BH}/\hbar$,

$$W = \frac{2\pi}{qB} \left\{ \sum_{m=0}^\infty \frac{a_m}{(q\mathcal{M})^{2m}} \right\} + \cdots$$

(22)

Finally denoting by $\sigma/2$ the sum inside the curly bracket, we find the following exponent of one-loop tunneling rate.

$$-S_E - W = -\frac{\pi q}{\hbar B} \left\{ \sum_{m=0}^\infty \frac{a_m}{(q\mathcal{M})^{2m}} \right\} + \cdots = -\frac{\pi q}{\hbar B} \left( 1 + \frac{\sigma}{q^2} \right) + \cdots.$$  

(23)

\(^7\)We shall use the same notation $W$ for various effective actions associated with various quantum fluctuations considered in this article and also for the \textit{total} effective action.

\(^8\)Since the integrands vanish very rapidly, the upper limit of the $r$ integration does not make much difference.
As a consequence, this one-loop effect does not go away in the weak field limit $B \to 0$ and represents a multiplicative correction of the leading Schwinger term by a fraction $\sim \hbar/q^2$, independent of how small $B$ is.

In a sense, this result is what one would expect from the most naive dimensional analysis. Restoring the gravitational constant $G$, the fractional correction above can be rewritten in terms of Planck length $L_{\text{Planck}}$ and the size of the black holes $L_{\text{BH}}$.

$$\frac{\hbar W}{S_E} = \sigma \frac{\hbar}{q^2} + \cdots \sim \frac{hG}{q^2G} \simeq \left( \frac{L_{\text{Planck}}}{L_{\text{BH}}} \right)^2,$$

which is the most natural small parameter characterizing the strength of the quantum fluctuations. But a subtlety arises in our case because there is another small parameter $qB$. 

A priori, the leading one-loop correction can take a more general form.

$$\frac{\hbar W}{S_E} \sim \left( \frac{1}{qB} \right) \frac{\hbar}{q^2}$$

The question is then whether the result (23) indicating $k = 0$ will generalize to other cases. As emphasized in the first section, the bottom “cup” region of figure 1 comes with ever-increasing volume as $qB \to 0$, and it is unclear whether there exists an unknown infrared effect capable of producing such extra factors of $1/qB$. Hence it is imperative to investigate how massless fluctuations contribute.

One crucial property of the expanded effective action above is that the integrands $\mathcal{L}^{(n)}$ are all regular at the Euclidean black hole horizon. This is in turn guaranteed by the term-by-term locality of the effective action. For such integrands, the integrations over the radial and the angular coordinates should be completely determined by the classical length scales of the black hole, and cannot produce extra factors of the inverse temperature $\sim 1/qB$; hence the result (23).

However, the Schwinger-DeWitt expansion is no longer adequate for massless fluctuations, $\mathcal{M} = 0$, since each term with $n > 1$ will cause an infrared divergence upon the $s$ integration. The formula (22) shows this pathology manifestly, for each term of the series inside the curly bracket diverges when $\mathcal{M}$ vanishes. To deal with such massless fluctuations, we need to perform a resummation over the momentum part of the Schwinger-DeWitt expansion, in favor of a manifestly non-local expression of the effective action [21].

In the following two sections, we want to explore the one-loop contributions from massless fluctuations with proper care of the boundary condition. But since the necessary calculations for arbitrary massless fluctuations are hopelessly difficult, we will specialize to two special cases: massless S-waves that are effectively 2-D conformal fields, and then, 4-D conformal fields.
5 The Leading One-Loop Contributions in the Weak Field Limit: Massless S-Wave Fluctuations

For most quantum fluctuations around the instanton solution, there exist potential barriers near the black hole event horizon. For instance, angular momentum $l$ modes of a minimally coupled massless scalar find the following potential barrier $V_l$ in the tortoise coordinate $z$.

$$g = F(z) (-dt^2 + dz^2) + R^2(z) d\Omega^2 \implies V_l(z) = \frac{\partial^2 R}{R} + \frac{l(l+1) F}{R^2}$$  \hspace{1cm} (26)

Such potential barriers are especially inhibitive for low energy excitations which are responsible for the infrared behaviour of the effective action.

One exception to this is the celebrated Callan-Rubakov modes [12][14] in spherically symmetric magnetic field backgrounds, chargeless combinations of which propagate effectively as 2-D conformal fields [13][15]. While the instanton geometry is not spherically symmetric everywhere, we have seen that, in the weak field limit $qB \to 0$, the spherical symmetry is restored near the Euclidean black hole horizon, giving us some hope that the contribution from these uninhibited modes near the Euclidean black hole may capture the essential physics of the next-to-leading WKB.

For 2-D conformal fields, the corresponding effective actions are exactly known in explicitly non-local form. A resummation over the momentum part of the Schwinger-DeWitt expansion turns out to produce a single nonlocal term [21] known as the Polyakov-Liouville action [22], up to a local topological contribution. Accordingly, the prefactor $N_{\text{CR}}$ from $N$ such S-wave modes can be written as following,

$$N_{\text{CR}} = e^{-W} = e^{-NS_{\text{PL}}}, \quad S_{\text{PL}} = \frac{1}{96\pi} \int dx^2 \sqrt{g^{(2)}} R^{(2)} \frac{1}{\nabla^2} R^{(2)} + a \int dx^2 \sqrt{g^{(2)}} R^{(2)}$$  \hspace{1cm} (27)

We included the topological term whose coefficient $a$ depends on the renormalization scheme in two-dimensions.

The form of Polyakov-Liouville action becomes particularly convenient to handle in conformal coordinates, since the scalar curvature $R^{(2)}$ above can be expressed simply by the Laplacian of the conformal mode. Here, let us choose an asymptotically flat conformal coordinate.

$$g^{(2)} = F (d\tau^2 + dz^2) \implies R^{(2)} = -\nabla^2 \log F.$$  \hspace{1cm} (28)

For the case of the Euclidean Reissner-Nordström black hole, in particular, $F$ and $z$ are given by

$$F = F(r) \equiv (1 - \frac{r^-}{r})(1 - \frac{r^+}{r}), \quad z = \int \frac{dr}{F(r)}.$$  \hspace{1cm} (29)

The importance of choosing the conformal gauge can be seen from the following general relationship, where $h$ is any harmonic function on the given space, to be fixed by the boundary condition chosen:

$$\frac{1}{\nabla^2} R^{(2)} = -\frac{1}{\nabla^2} \nabla^2 \log F = -\log F + h.$$  \hspace{1cm} (30)
This way, all the dependence on the choice of vacuum is encoded into a single harmonic function \( h \). On the other hand, as argued in section 3, the black holes created by the instanton [3] are near-extremal Reissner-Nordström in Rindler heat-baths. That is, the Hawking radiation from the event horizon is in thermal equilibrium with the heat bath associated with the acceleration of the ambient region, showing that the proper vacuum to choose is the Hartle-Hawking vacuum.

One important characteristic of the Hartle-Hawking vacuum is that the potential divergence of energy-momentum expectation values on both past and future event horizon disappears. Moreover, we expect \( h \) to be independent of time coordinate, for the state is supposed to be in thermal equilibrium. These two conditions are strong enough to fix \( h \) almost completely.

\[
h = \pm F'(r_+) z + C
\]  

(31)

With this choice of the boundary condition, we find \textit{semiclassical} spacetimes with regular event horizons of non-zero Hawking temperatures (but with infinite ADM masses due to the heat bath), which is exactly what one expects from the Hartle-Hawking vacuum [23][16].

To elaborate the procedure leading to the choice above, it is most convenient to derive the one-loop energy-momentum tensor from a local form of the Polyaev-Liouville action with the aid of an auxiliary scalar field \( \psi \).

\[
S_{PL} = S_{PL}^{local} \bigg|_{\nabla^2 \psi = R^{(2)}}
\]

\[
S_{LP}^{local} \equiv \frac{1}{96\pi} \int dx^2 \sqrt{g^{(2)}} \left( (\nabla \psi)^2 + 2\psi R^{(2)} \right)
\]

\[
\Rightarrow \langle T_{ij} \rangle \sim -2\nabla_i \nabla_j \psi + 2g_{ij} \nabla^2 \psi + \nabla_i \psi \nabla_j \psi - \frac{1}{2} g_{ij} (\nabla \psi)^2.
\]

(32)

(33)

Solving the auxiliary equation with the help of the identity (30), the energy-momentum can be rewritten in the light-cone coordinates \( x^\pm = t \pm z \), after a Wick rotation to the Minkowskian section \( dt^2 = -dx^2 \), as follows,

\[
\langle T_{\pm\pm} \rangle \sim -2\nabla_\pm \nabla_\pm (-\log F + h) + (\nabla_\pm (-\log F + h))^2
\]

\[
= -(\partial_\pm \log F)^2 + 2\partial_\pm \partial_\pm \log F + (\partial_\pm h)^2 - 2\partial_\pm \partial_\pm h.
\]

(34)

For any finite temperature black hole, the divergence (in a local geodesic coordinate) on the event horizons is induced by the first term. Choosing a \( h = h(x^-) \) to cancel this divergence on the future event horizon leads to the familiar Hawking’s radiation, corresponding to Unruh vacuum [16][24]. Choosing a \( h = h(z) \) to cancel this divergence on both past and future event horizons, therefore, must correspond to the Hartle-Hawking vacuum. The resulting choice \( h = h(z) \) can be written most generally as in (34).

This choice of vacuum is not only well-motivated physically but also vital for the validity of the WKB approximation, in that the gravitational backreaction to the quantum fluctuations is now well controlled. Otherwise, the quantum fluctuations around the classical solution can

\[\text{This has been emphasized previously in reference [6].}\]
no longer be regarded as “small” and a systematic expansion based on the Euclidean instanton would be ill-fated.

Note that there are some harmless ambiguities remaining. In particular, an additive shift of the constant $C$ can be translated into an additive shift of the topological term $a \to \tilde{a}$, and our ignorance of $C$ will result in an unknown topological contribution $\sim \chi_{\text{Euler}}$ insensitive to the geometry and thus independent of $qB$.

Evaluating the Polyakov-Liouville action,

$$S_{PL} \rightarrow \frac{1}{96\pi} \int d\tau dz (\partial_z^2 \log F)(\log F - h) + 4\pi \tilde{a} \chi_{\text{Euler}}. \quad (35)$$

Let us first evaluate with $h = F'(r_+)z + \cdots$. Dropping the topological terms $\sim \chi_{\text{Euler}}$ which is finite as $qB \to 0$, we find

$$S_{PL} \bigg|_{\text{on-shell}} \rightarrow -\frac{h}{96\pi T_{BH}} \int_{r_+}^{r_B} dr \frac{F'(r) (F'(r) - F'(r_+))}{F(r)}. \quad (36)$$

If we had chosen $h = -F'(r_+)z + \cdots$ instead, the expression would have changed only by another boundary contribution that is independent of the geometry and again does not enter the leading $qB$ dependence.

The radial integration comes with the upper limit at $r_B$, for the relevant geometry is again the truncated Euclidean black hole, but it matters little owing to the rapidly vanishing behaviour of the integrand. The resulting integral is finite for any $T_{BH}$, and is continuous in the extremal limit.

$$\lim_{r_+ \to r_-} \int_{r_+}^{\infty} \frac{F'(r) (F'(r) - F'(r_+))}{F(r)} = \int_{q}^{\infty} \frac{F'F'}{F} = \frac{4}{3q} \quad (37)$$

Again using the constraint $T_{BH}/h = B/2\pi + \cdots$, we arrive at the following one-loop corrected exponent,

$$-\frac{S_E}{h} - W = -\frac{\pi q}{hB} + \frac{N}{36qB} + \cdots \sim -\frac{\pi q}{hB} \left(1 - \frac{Nh}{36\pi q^2}\right) + \cdots. \quad (38)$$

The ellipsis now denotes terms from quantum fluctuations other than the chargeless Callan-Rubakov modes as well as terms of higher order in $qB = qBG$.

Compared to the massive case (23), we find similar behaviour, for there is no extra factor of $1/qB$ generated. Also we can easily see that the analogue of $\sigma$ in (23) is here given by a negative constant $-N/36\pi$, corresponding to an enhancement of the rate for all small $B$. However, within this large $N$ approximation ($Nh$ finite while $h \to 0$) where one-loop matter contribution is dominant over that of internal gravitons, we need to keep $q^2$ large in order to justify neglecting the higher-loop gravitational contribution as well. Hence, the formula above makes sense only for small $Nh/12\pi q^2$, and the one-loop correction results in only a slight enhancement of the rate.

Before closing this section, we want to mention that the choice of boundary condition actually does not affect the leading behaviour $W \simeq -N/36qB$. Other choices result only in different
boundary terms of order \((qB)^0\). This is closely related to the fact that the functional form of the Polyakov-Liouville action is completely determined by the conformal anomaly, which is a local quantity. Such a highly unusual character of chargeless Callan-Rubakov modes raises a question whether they are “generic” enough to simulate actual gravitational fluctuations that really contribute to the one-loop WKB. In a sense, the derivation in this section should be regarded more as an illustration than anything else.

6 The Leading One-Loop Contributions in the Weak Field Limit: Conformal 4-D Fluctuations

Finding the four-dimensional effective action in its full non-local form is a horrendous task, even for a simple quadratic matter action [21]. On the other hand, what we actually need within the WKB approximation is just values of the effective action on a specific family of metrics. In the case at hand, the instanton has only two independent parameters \(q\) and \(B\), or equivalently \(q\) and \(T_{BH}\). Then, using the standard formula relating a variation of the effective action to the energy-momentum expectation values, we should be able to reduce the problem to that of finding \(\langle T_{\alpha\beta} \rangle\) [23] in a given background.

\[
\frac{\partial W}{\partial q} = \int dx^4 \sqrt{g^{(4)}} \frac{\partial g^{\alpha\beta}}{\partial q} \langle T_{\alpha\beta} \rangle, \tag{39}
\]

where \(W\) is the effective action evaluated on the two-parameter family of the Euclidean metrics. The energy-momentum expectation values are to be evaluated in the Hartle-Hawking vacuum, as was emphasized in section 3.

Now let us consider the approximate 4-D metric (14), rewritten in the tortoise coordinate. It is of practical importance to use the tortoise coordinate \(z\) because the range of \(z\) is independent of the two parameters of the solution.

\[g^{(4)} = F(r(z))(d\tau^2 + dz^2) + r(z)^2(d\theta^2 + \sin^2 \theta d\phi^2) + \cdots\]

Inserting this form of metric to the equation (39) above,

\[- \partial_q W = \frac{4\pi \hbar}{T_{BH}} \int_{-\infty}^{\infty} dz \frac{1}{r^2} \left[ F^r \left( T^r_r + T^z_z \right) + 2 \frac{\partial F}{\partial r} \langle T^\theta_\theta + T^\phi_\phi \rangle \right] + \cdots \tag{40} \]

Inside the integral, the partial derivative \(\partial_q\) should be carried out with fixed \(z\) and also fixed \(B \approx 2\pi T_{BH}/\hbar\). To recover the one-loop contribution \(W\), we just need to recall that the only dimensionless combination of \(q\) and \(B\) is \(qB\), and that any one-loop dimensionless physical quantity, without an explicit cut-off dependence, must be a function of \(qB\).

Regrouping terms above, and using the spherical symmetry, we may simplify the expression a little bit.

\[- \partial_q W = \frac{4\pi \hbar}{T_{BH}} \int_{r_+} dr \frac{1}{r^2} \left[ F^{\alpha} \langle T^\alpha_{\alpha} \rangle + \frac{4\pi \hbar}{T_{BH}} \int_{r_+} dr \frac{1}{r^2} \left[ \frac{\partial F}{\partial q} \langle T^\theta_\theta \rangle - \frac{2\partial F}{\partial q} \langle T^\phi_\phi \rangle \right] \right] + \cdots \tag{41} \]
In general, both of the expectation values \( \langle T^\alpha_\alpha \rangle \) and \( \langle T^\theta_\theta \rangle \) depend on the boundary condition strongly, and their precise behaviour is unknown, especially in the extremal limit.

However, there are special cases where we can exploit a well-known field theoretical fact concerning conformally coupled fluctuations. While the trace of the energy-momentum is classically zero for such matter fields, the one-loop expectation value thereof is nonvanishing due to the conformal anomaly. The conformal anomaly \( \langle T^\alpha_\alpha \rangle \) is explicitly calculable and can be expressed as a polynomial of the curvature tensor and its derivatives, and thus is completely local [26].

To estimate the anomaly contribution to \( W \), first write the coordinate transformation between \( r \) and the tortoise coordinate \( z \), introducing an arbitrary positive constant \( a \).

\[
    z(r) = \int_{(a+1)r_+}^{r} \frac{d\tilde{r}}{\tilde{F}^{\prime}(\tilde{r})}
\]  

(42)

Defining the surface gravity of the horizon by \( \kappa \equiv \tilde{F}^{\prime}(r_+) = 2\pi T_{BH}/\hbar \simeq B \), then, \( \tilde{F}(r(z)) \) is given by the following expansion in \( e^{2\kappa z} \approx (r - r_+)/ar_+ \) as \( z \to -\infty \),

\[ F(r(z)) = 2\kappa ar_+ e^{2\kappa z} + O(e^{4\kappa z}). \]  

(43)

Using the fact that \( \partial_q \kappa \equiv 0 \equiv \partial_q z \), we then find the relative variation \( \partial_q F/F \) approaches \( \partial_q r_+/r_+ \approx 1/r_+ \approx 1/q \) near the black hole horizon provided that \( \kappa \) is positive. In the asymptotic region \( r \to r_B \gg q \), on the other hand, it is not difficult to convince oneself that \( \partial_q F/F \approx 1/r \) at most.

Then, since the value of the trace anomaly is of order \( \sim 1/q^4 \) near the horizon and \( \sim 1/r^5 \) for large \( r \), the first integral of (41) is completely finite and contributes to \( \partial_q W \) a finite leading term of order \( \sim \hbar/q^2 T_{BH} \simeq 2\pi/q^2 B \). Hence, we conclude that the anomaly contribution to the one-loop exponent is given by \( \sim 1/qB \), just as in previous examples.

However, \( \langle T^\theta_\theta \rangle \) strongly depends on the choice of vacuum and shows a substantially different behaviour than the conformal anomaly. For instance, in the Hartle-Hawking vacuum around a black hole, it must approach a constant \( \langle T^\theta_\theta \rangle_\infty \sim T_{BH}^4 \sim B^4 \) at large \( r \), for the quantum state there is that of a heat bath at temperature \( T_{BH} \). Without a suitable infrared cut-off, this could result in a divergence of \( W \) as expressed above, requiring more careful analysis, but as emphasized in section 3, the size of the bottom “cup,” \( r_B \sim (qB^{-2})^{1/3} \), effectively acts as an infrared cut-off, and such an asymptotic heat bath of uniform \( \langle T^\theta_\theta \rangle_\infty \neq 0 \) may contribute to the radial integral higher order terms \( < (qB)^1 \) only.

Then, the only remaining part of (41) that can possibly alter the leading weak field behaviour \( W \sim 1/qB \) from the \( \langle T^\theta_\theta \rangle \) term near the horizon \( r = r_+ \). For this remaining contribution, all we need to do is determine the behaviour of the following quantity \( X(q, \kappa) \) as the extremal limit is approached \( (\kappa \to 0) \).

\[
    X \equiv \int_{r_+}^{2r_+} dr r^2 \left[ \frac{4\partial_q r}{r} - \frac{2\partial_q F}{F} \right] \langle T^\theta_\theta \rangle,
\]  

(44)
The potential divergence of $X$ can only originate at the lower limit of this integral, and it is most important to understand the behaviour of the integrand near the horizon for small $\kappa \simeq B > 0$.

First let us consider the quantity inside the square brackets. Approaching the lower bound, $\partial_q F/F$ converges to $\partial_q r/r$ provided $\kappa > 0$, as shown above, while $\partial_q r/r$ obviously approaches $\partial_q r/r$ for any $\kappa \geq 0$. Henceforth, the summed fractional variation inside the square brackets approaches the finite value $2\partial_q r/r \simeq 2/r \simeq 2/q$.

$$\lim_{r \rightarrow r^+} \left[ \frac{4\partial_q r}{r} - \frac{2\partial_q F}{F} \right] \simeq \frac{2}{q} \quad \text{if } \kappa > 0 \quad (45)$$

From this, we conclude that only rather strong divergence of $\langle T^0_\theta \rangle \geq (r-r^+)^{-1}$ can cause singular behaviour of $X$ as $\kappa \rightarrow 0$.

There exists extensive literature that deals with one-loop energy-momentum tensor outside black holes. Unfortunately, for the cases of Reissner-Nordström black holes, known analytic approximation schemes [27] are unreliable near the event horizon for any nonzero charge of the black hole. But a numerical estimate using a mode sum was carried out by Anderson and collaborators [28], which seems to indicate that $\langle T^0_\theta \rangle$ remains finite at the event horizon for both nonextremal and extremal cases. In fact, they observed that the horizon value of $\langle T^0_\theta \rangle$ achieves a maximum at about $r_-/r^+ \simeq 0.92$ and decreases again as the extremality $r_-/r^+ = 1$ is approached.

If we take this numerical evidence seriously, we may immediately conclude that $\lim_{\kappa \rightarrow 0} X$ is finite and $\partial_q W$ behaves $\sim 1/q^2 B + \cdots$ in the weak field limit $qB \rightarrow 0$. Then, as in previous examples, the one-loop correction of the exponent diverges as $1/qB$ and the net result is again that the Schwinger term is multiplicatively corrected by a fraction $\sim \hbar/q^2$.

Recently Trivedi demonstrated in the same S-wave approximation as in section 5 above that the local energy density, or equivalently $\langle T_{UU} \rangle$ in the null Kruskal coordinates $(U, V)$, is divergent near an extremal Reissner-Nordström horizon [23]. Yet this divergence clearly does not enter the estimate (36), for the latter is more or less determined by $\langle T_{UV} \rangle$ the conformal anomaly. On the other hand, such a divergence in four dimensions could have entered the four-dimensional estimate here, since (41) depends on more than just the traced energy-momentum expectation. The natural question is then how the potential divergence of $\langle T_{UU} \rangle$ affects the behaviour of $\langle T^0_\theta \rangle$ and ultimately that of $\partial_q W$. Is the finite behaviour of $X(q, \kappa)$ as $\kappa \rightarrow 0$ consistent with such a divergence?

In appendix, we present a partial answer by exploiting the conservation of energy-momentum to show that the divergence, if any, of angular components is typically much weaker than that of $\langle T_{UU} \rangle$. Through this analytical study, we shall demonstrate that Trivedi’s divergence is perfectly consistent with finite $\langle T^0_\theta \rangle$ and that the quantity $X$ above is likely to be finite in the

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10See [25], [26], [27], [28], and the references therein.
extremal limit. Naturally, one wishes that more explicit and transparent calculations of the energy-momentum were available. But even without such results, we believe that the arguments given above combined with those in the appendix provide strong evidences for the conclusion above.

7 Conclusion

In summary, we studied the weak field behaviour of the one-loop tunneling rate of near-extremal magnetic black holes pair-production induced by background magnetic fields. In particular, we considered the case of Reissner-Nordström black hole pair-production, mediated by the Euclidean Ernst metric as the instanton.

Roughly, the instanton consists of two components: a single Euclidean black hole without the asymptotic region beyond \( r_B \sim (qB^{-2})^{1/3} \) and a Euclidean Melvin space, glued together along the transitional “mouth” region located at \( \zeta \simeq B^{-1} \) and \( \rho \ll B^{-1} \), in terms of the Rindler-like coordinates (15) of the Melvin space. Furthermore, the Euclidean black hole is a near-extremal Reissner-Nordström black hole with Hawking temperature \( T_{BH} \simeq \hbar B/2\pi \). The natural boundary condition turned out to produce a variant of the Hartle-Hawking vacuum that, outside the black hole region, behaves like an ordinary Minkowski vacuum as seen by Rindler observers.

To simulate the one-loop contributions from quantized gravitational fluctuations, we considered general chargeless quantum fluctuations governed by various elliptic kinetic operators. In all cases considered, we found the following weak field behaviour of the additive one-loop correction \(-W\) to the leading exponent \(-S_E/\hbar\):

\[
W \sim \frac{1}{qB} \quad \Rightarrow \quad -\frac{S_E}{\hbar} - W = -\frac{\pi q}{\hbar B} \left( 1 + \tilde{\sigma} \frac{\hbar}{q^2} \right) + \cdots, \tag{46}
\]

where \( \tilde{\sigma} \) is a proportionality constant.\footnote{What about the recently proposed “geometric entropy” \cite{29} which happens to be proportional to \( q^2 \) and also quadratically divergent? Where would such a divergence appear? As Susskind and Uglum argued eloquently \cite{30}, all the quadratic divergence in the effective action multiplies the curvature scalar (and a possible local boundary term involving the extrinsic curvature) only, and the quadratic divergence of “geometric entropy” can always be traced back to the divergent renormalization of the inverse gravitational constant \( G^{-1} \). This can be easily seen by writing down the effective action in the heat-kernel approach as in section 4. Therefore, once we start with some ultraviolet finite theory of gravity such as superstring theories, and as long as we express all the quantities in terms of the renormalized couplings, no such ultraviolet divergence may occur in the calculation of the tunneling rate. As explicitly stated in section 4 and implicitly assumed in later sections, we discarded all ultraviolet divergences by expressing everything in terms of the renormalized couplings and in particular of the renormalized \( G^{-1} \) which is subsequently set to 1.}

This behaviour is found for both massive and massless fluctuations after appropriate renormalization procedures. Also, we explicitly calculated the additive contribution to \( \tilde{\sigma} \) from \( N \) chargeless Callan-Rubakov modes, and found \(-N/36\pi\).
An immediate consequence of this is that there seems to be nothing special about the weak field limit other than the fact that $T_{BH} > 0$ becomes arbitrarily small. In particular, the divergence of $W \sim 1/qB \sim \hbar/qT_{BH}$ in the limit $T_{BH} \to 0$ results from the ever-increasing periodicity $\hbar/T_{BH}$ of the Euclidean time coordinate rather than from the diverging distance to the Euclidean black hole horizon. Per unit Euclidean time, the contribution of quantum fluctuations at one-loop level remains finite in this limit, in spite of the ever-increasing 3-volume of the bottom “cup” which is in effect a truncated near-extremal Euclidean black hole.

In fact, we believe that this leading one-loop correction can be explained as a direct consequence of the semiclassically shifted mass-to-charge ratio of near-extremal Reissner-Nordström black holes [16]. To see this, it is necessary to restore the classical mass $m \simeq q$ of near-extremal black holes in the expressions (1) and (46),

\[
-SE/\hbar = -\frac{\pi m^2}{\hbar qB} + \cdots, \tag{47}
\]

\[
-SE/\hbar - W = -\frac{\pi m^2}{\hbar qB} \left(1 + \tilde{\sigma} \frac{\hbar}{q^2}\right) + \cdots.
\]

Note that the one-loop corrected expression would look like the original Schwinger term, if we introduce a new parameter $\tilde{m}$ as follows:

\[
-SE/\hbar - W = -\frac{\pi \tilde{m}^2}{\hbar qB} + \cdots \quad \text{with} \quad \tilde{m} \simeq q \left(1 + \tilde{\sigma} \frac{\hbar}{2 q^2}\right). \tag{48}
\]

This leads us to speculate that this particular one-loop correction may represent a semiclassical effect that shifts near-extremal black hole masses.

In order to verify this conjecture, it is necessary to ascertain such a shift $m/q \to \tilde{m}/q$ in some other way. In particular, given a fixed magnetic charge $q$, the lower bound of the Reissner-Nordström black hole masses, or equivalently the mass of the extremal black hole, must be shown to be modified accordingly. Exactly how does such a semiclassical correction arise?

Although the extremal black hole is known to have vanishing Hawking temperature and does not emit the usual late-time thermal radiation, this tells us nothing about transient behaviours before the state settles down to a steady state. In fact, if we imagine an idealized gravitational collapse that produces an extremal black hole [16], it is easy to see that there could be some transient quantum radiations of finite integrated flux. The subsequent correction to the mass-to-charge ratio was first demonstrated and estimated in reference [16], where $N$ chargeless Callan-Rubakov modes are quantized around Reissner-Nordström black holes.

Both analytic and numerical studies, again utilizing the power of Polyakov-Liouville effective action, revealed that the semiclassically corrected extremal black holes obey the following mass-to-charge ratio, when $N\hbar/12\pi q^2$ is small;

\[
\frac{\text{mass}}{\text{charge}} \simeq 1 - \frac{N\hbar}{72\pi q^2}. \tag{49}
\]
Accordingly, the lower bound of black hole masses is reduced in the same fashion, and in particular near-extremal black holes now have the mass approximately of \( q - N h/72\pi q \) rather than of \( q \). This is certainly consistent with the above interpretation (18), since the contribution to \( \tilde{\sigma}/2 \) from \( N \) chargeless Callan-Rubakov modes is indeed \(-N/72\pi\) as demonstrated in section 5. Therefore, the one-loop correction we found merely reflects the fact the mass of the charged object is modified at the one-loop level. The analogous shifts of the mass-to-charge ratio due to general quantum fluctuations are yet to be estimated, but it is reasonable to expect the same interpretation to hold for other cases as well.

Seen the other way around, this observation also provides us with a new and more systematic way of calculating the semiclassical mass-to-charge ratio in the presence of fully four-dimensional quantum fluctuations. In reference [16], the ratio was obtained by evolving an initially empty energy-momentum expectation in accordance with the energy conservation, but the method employed there is not easily generalized beyond the particular case of chargeless Callan-Rubakov modes. On the other hand, the formulae (14) and (18) enable us to convert this complicated evolution problem to a static one, for all we need is a reasonably accurate estimate of the energy-momentum expectations in the Hartle-Hawking vacuum around near-extremal black holes.

Here we discussed only the cases where the pair-produced objects are near-extremal Reissner-Nordström black holes, while there are other known species of instantons [3][4]. In particular, an especially interesting variant exists that pair-produces strictly extremal Reissner-Nordström black holes rather than near-extremal ones. Moreover, this new class of instantons are qualitatively different, in that the matching condition of type (8) is absent: The pair-production of strictly extremal black holes, if any, seems genuinely different from that of the nonextremal ones, however close to the actual extremality the latter might be. A central issue in extending the current analysis to the strictly extremal cases would be how the natural vacuum, analogous to the one we found and utilized here for wormhole type instantons, behaves near the extremal horizon. The mismatch of the two surface gravities seems to suggest a semiclassical instability that involves strongly divergent energy-momentum near the black hole horizon and a large gravitational backreaction thereof. It would be interesting to see if the extremal type instantons allow semiclassical estimates that are reliable at one-loop and beyond.

In any case, the main result (18) is most sensible in that it is exactly what one would expect for pair-production of ordinary charge particles. But at the same time it is rather disappointing. Despite the ever-increasing size of the bottom "cup" that is uniquely associated with the near-extremal black hole pair-production, even the strongest correction in the weak field limit can be explained away and does not lead to new interesting physics. But are we completely sure that we did not leave out some essential physics in the course of carrying out various approximations and simplifications?
It is always a possibility that some nontrivial and large physical effects are hidden in higher order terms, especially in $B$ independent one-loop contributions that must include a correction to the Bekenstein-Hawking entropy. Unlike our calculations here, unfortunately, the estimate of this next-to-leading order in $qB$ appears a lot more sensitive to how one treats the transitional "mouth" region, and thus is much more difficult to carry out. For example, we have been rather cavalier about the boundary at $r = r_B$ of the truncated black hole and pretended that it is infinitely far away, which is justifiable only as far as the leading $qB$ behaviour found above is concerned.

Of course, the most problematic aspect of the Euclidean approach here is the fact that Euclidean path integral is ill-defined with the Einstein-Hilbert action, which we circumvented by substituting general matter fields for actual gravitational fluctuations. In other words, the genuine operators governing small gravitational fluctuations are not even elliptic, and in particular possess infinite number of zero-eigenmodes, not to mention infinite number of negative eigenmodes. Note that by substituting in general matter fluctuations as above, we in effect concentrated on positive eigenmodes-modes only. Although this is not necessarily in conflict with our desire to find the leading infrared behaviour in the weak field limit, we surely neglected the possible degeneracy, if any, of black hole quantum states. Therefore it is a matter of some urgency to understand how to perform a next-to-leading WKB approximation in the presence of gravitational degrees of freedom.

Before closing, we want to bring up another related problem, somewhat tangential to our purpose here but nevertheless rather intriguing on its own. In this article, we needed to consider the vacuum in the Euclidean sector only, which simplified the discussion quite a bit. An interesting question to ask is what does the natural vacuum look like after the pair-production, that is, along the Minkowskian sector where two oppositely charged black holes are uniformly accelerating away from each other with $A \simeq B$. The discussion in section 3 suggests that, as far as the co-accelerating Rindler observers are concerned, the same thermal equilibrium between Hawking radiation and Rindler heat bath must be maintained and these pair-produced black holes are semiclassically stable. But what does an inertial observer see? Does he find Doppler-shifted Hawking radiations from each nonextremal black hole? Or, since the black holes are in equilibrium with Rindler heat baths and do not seem to lose any energy, should he agree that no quantum radiation emerges from the black holes? (This problem of accelerating black hole in the Rindler heat bath was recognized also by Dowker and collaborators in reference [4].)

Although the situation looks superficially similar to the case of accelerating charge, where inertial observers inevitably find the Bremsstrahlung [31][32], the analogy breaks down due to the entirely different nature of the respective radiations. For instance, unlike the case of accelerating charge [33], we cannot expect the interference between the background fields and the radiation to explain the energy conservation at all time, since the Hawking radiation is completely universal.
in its composition while there exist only gravitational and electromagnetic background fields.

While we are tempted by the simplest conclusion that these black holes do not emit any Hawking radiation unless the fine-tuned uniform acceleration is disrupted, it is by no means clear that some subtlety in measuring the mass of such black holes may not explain the two potentially conflicting reports by inertial and co-accelerating observers. It should be a most interesting exercise to obtain the complete semiclassical picture of this system, which probably requires deeper understanding of both classical and semiclassical physics of black holes.

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Appendix: Taming $\langle T^\theta_\theta \rangle$ at the Event Horizon.

In this appendix, we want to study $\langle T^\theta_\theta \rangle$ near the black hole horizon. In the Hartle-Hawking vacuum around a nonextremal black hole, $\langle T^\theta_\theta \rangle$ is expected to remain finite approaching the event horizon. But it is rather uncertain what happens in the extremal limit, in particular in view of the two-dimensional results by Trivedi that $\langle T_{UU} \rangle$ diverges at the extremal horizon \[23\]. The objective of this appendix is to place an upper bound on the possible divergence of $\langle T^\theta_\theta \rangle$ for the purpose of estimating the quantity $X$ of \[44\]. The results here are meant to be complementary to the numerical estimates of reference \[28\].

First, recall that the energy-momentum must satisfy the conservation equations, being Noether currents associated with the diffeomorphism invariance.

$$\nabla_\beta T^\beta_\alpha \equiv 0$$  \hspace{1cm} (50)

Given any static and spherically symmetric expectation values $\langle T_{\alpha\beta} \rangle$, such as expected in the Hartle-Hawking vacuum, these constraints can be reduced to a first order differential equation for $\langle T^z_z \rangle$.\[12\]

Solving it, one finds the following relationship between different components of the energy-momentum expectation.

$$\langle -T^t_t + T^z_z \rangle = \langle -T^r_r + T^z_z \rangle = \left[ -\langle T^0_0 \rangle + \frac{1}{r^2 F} \int_{r_0}^{r} dr \ r^2 \frac{dF}{dr} \langle T^0_0 \rangle \right]$$

\[12\]We are indebted to Jaemo Park for pointing this out.
\[ + \left[ 2 \langle T_\theta^\theta \rangle - \frac{2}{r^2 F} \int_{r_0}^r dr \left( r^2 \frac{dF}{dr} - 2 r F \right) \langle T_\theta^\theta \rangle \right] \]  (51)

Obviously, the lower limit \( r_0 \) for each integral must be chosen in accordance with the expected behaviour of the vacuum.

For instance, in the Boulware vacuum around a nonextremal black hole, \( \langle T_\alpha^\beta \rangle \) must vanish very rapidly as \( r \to \infty \), and the only choice consistent with this is \( r_0 = \infty \). On the other hand, the left-hand-side is related to the energy-density as seen by inertia observers crossing the future event horizon. Calling the in-going Kruskal coordinate \( U \), this energy-density at the horizon scales like

\[ \langle T_{UU} \rangle \sim \frac{\langle -T_t^t + T_z^z \rangle}{F}, \]  (52)

the leading divergence of which at \( F = 0 \) disappears only if the lower limit of the integrals in (51) is at the event horizon. Therefore, in contrast with the Boulware vacuum, the Hartle-Hawking vacuum requires \( r_0 = r_+ \).

In any case, the upshot is that now we can relate the angular components of one-loop energy-momentum at the event horizon to the leading behaviour of the energy-density.

Using this constraint, it is easy to set an upper bound on the potential divergence of \( \langle T_\theta^\theta \rangle \) in the Boulware vacuum by exploiting this constraint, as follows. Assuming that \( \langle T_\theta^\theta \rangle \sim (r - r_+)^{-m} \) near the horizon, we find from (51) with \( r_0 = \infty \) the following leading divergence of the energy-density.

\[ \langle T_{UU} \rangle_{\text{Boulware nonextremal}} \sim \begin{cases} (r - r_+)^{-2} \sim U^{-2} & \text{if } m \leq 1 \\ (r - r_+)^{-1-m} \sim U^{-1-m} & \text{otherwise} \end{cases} \]  (53)

Here we have used the facts that \( \langle T_\alpha^\beta \rangle_{\text{Boulware}} \) vanishes very rapidly in the asymptotic region and that \( F \sim (r - r_+) \) near the nonextremal event horizon. But, the divergence of \( \langle T_{UU} \rangle_{\text{Boulware nonextremal}} \) is closely related to Hawking’s late time thermal radiation. For a realistic black hole, the future event horizon is smooth as seen by freely infalling observers, and the corresponding Unruh vacuum must have an extra contribution to \( \langle T_{UU} \rangle \) that cancels the divergence that appears in the Boulware vacuum. This extra contribution is predominantly a function of \( U \), and thus propagates outward to future null-infinity. The resulting energy flux can be estimated from (53) combined with appropriate coordinate transformation,

\[ \langle T_{uu} \rangle_{\text{Unruh nonextremal}} \bigg|_{u,r \to \infty} \sim \begin{cases} 1/r^2 & \text{if } m \leq 1 \\ e^{(m-1)\kappa u}/r^2 & \text{otherwise} \end{cases} \]  (54)

Here \( u \equiv t - z \) is the asymptotic retarded time, and \( \kappa \) is the surface gravity of the event horizon. The exponentially divergent behaviour of the second \( (m > 1) \) is clearly unacceptable, and in fact, we already know that the radiation flux from the nonextremal black hole is thermal and must be independent of the retarded time \( u \) \[19\]. This requires \( m \) not to be larger than 1, putting an upper bound on the possible divergence of of \( \langle T_\theta^\theta \rangle \) in the Boulware vacuum.

\[ \langle T_\theta^\theta \rangle_{\text{nonextremal}} \leq (r - r_+)^{-1} \sim U^{-1} \quad \iff \quad \langle T_{UU} \rangle_{\text{nonextremal}} \sim (r - r_+)^{-2} \sim U^{-2} \]  (55)
Note that the conservation of energy-momentum restricts the possible divergence of the angular components to be much weaker than that of $\langle T_{UU} \rangle$.

Similar analysis can be carried out for the Hartle-Hawking vacuum, too. In this case, $\langle T_{\theta \theta}^{\theta} \rangle$, in a local geodesic coordinate, are expected to be finite at the nonextremal horizon, and this is certainly consistent with the constraint above. To see this, we perform integrations by part to obtain the following relationship in the Hartle-Hawking vacuum, which is valid provided that both $F \langle T_{\alpha \alpha}^{\alpha} \rangle$ and $F \langle T_{\theta \theta}^{\theta} \rangle$ vanish at $r = r_{+}$,

$$\langle T_{UU} \rangle \sim \frac{1}{r^2 F^2} \int_{r_{+}}^{r} dr \frac{d}{dr} \left[ 2r^2 \langle T_{\theta}^{\theta} \rangle - r^2 \langle T_{\alpha}^{\alpha} \rangle \right] + 4r \langle T_{\theta}^{\theta} \rangle. \quad (56)$$

For regular $\langle T_{\theta}^{\theta} \rangle$, the quantity inside the curly-bracket is finite at $r = r_{+}$, and the definite integral vanishes quadratically $\sim (r - r_{+})^2$ near the nonextremal horizon. Taking into account the fact that $F$ vanishes linearly $\sim (r - r_{+})$, we conclude

$$\langle T_{\theta}^{\theta} \rangle_{\text{Hartle-Hawking nonextremal}} \sim 1 \iff \langle T_{UU} \rangle_{\text{Hartle-Hawking nonextremal}} \sim 1. \quad (57)$$

This gives us more confidence that $X(q, \kappa > 0)$ of (14) is really finite. Now what about the extremal limit $\kappa \to 0$? As mentioned in section 6, the numerical results of Anderson et al., indicate that $\langle T_{\theta}^{\theta} \rangle$ at horizon actually decreases as $\kappa \to 0$ in near-extremal cases $(r_{-}/r_{+} > 0.92)$, which implies finite limit of $X(q, \kappa \to 0)$. But, to be extra cautious, let us try to determine the behaviour of $X$ in the strictly extremal case $\kappa = 0$.

Now we want to extend the above analysis to the extremal limit of the Hartle-Hawking vacuum. Suppose that $\langle T_{\theta}^{\theta} \rangle$ approaches a finite nonzero value near the extremal horizon. Recalling that $F$ now vanishes quadratically $\sim (r - r_{+})^2$, we find from (56) the following divergence of the energy-density.

$$\langle T_{UU} \rangle_{\text{extremal}} \sim (r - r_{+})^{-1} \sim U^{-1}. \quad (58)$$

What if $\langle T_{\theta}^{\theta} \rangle$ is logarithmically divergent $\sim \log(r - r_{+})$? In this case, the differentiation of the square bracket in (56) produces an extra $1/(r - r_{+})$ term that generates the leading term of the integral $\sim (r - r_{+})^2$. As a result, we find

$$\langle T_{UU} \rangle_{\text{extremal}} \sim (r - r_{+})^{-2} \sim U^{-2}. \quad (59)$$

Finally, the case of $\langle T_{\theta}^{\theta} \rangle \sim (r - r_{+})^{-n}$ for general $n$ can be studied similarly from the original equation (51) with $r_{0} = r_{+}$, which yields,

$$\langle T_{UU} \rangle_{\text{extremal}} \sim (r - r_{+})^{-2-n} \sim U^{-2-n}. \quad (60)$$

\[13\] Since the Boulware and the Hartle-Hawking vacua coincide in the extremal limit, we will drop the superscript “Hartle-Hawking” in the following formulae.
The proportionality constant here vanishes if and only if \( n = 0 \). Again we can see that, owing to the conservation of energy-momentum, the divergence of \( \langle T_{UU} \rangle \) is much more severe than that of the angular components. In particular, the divergence shown in (58) is similar to what Trivedi found in two-dimensional context, but the corresponding \( \langle T_{\theta \theta} \rangle \) is perfectly finite.

It is rather difficult to imagine why \( \langle T_{UU} \rangle_{\text{extremal}} = \lim \langle T_{UU} \rangle_{\text{Hartle–Hawking}} \) should be more divergent than \( \langle T_{UU} \rangle_{\text{nonextremal}} \), but, even allowing such strong divergence as \( \langle T_{UU} \rangle_{\text{extremal}} \sim U^{-4} \), we find the quantity \( X(q, \kappa = 0) \) completely finite.

This can be seen from the fact that, in the strict extremal limit (\( \kappa = 0 \)), (42) with extremal \( F \) implies that \( \partial_q F/F \) converges to \( 2\partial_q r_+/r_+ \) instead of \( \partial_q r_+/r_+ \), which translates into

\[
\left[ \frac{4\partial_q r}{r} - \frac{2\partial_q F}{F} \right] \sim \frac{(r - r_+)^2}{r_+^2} \quad \text{if } \kappa = 0.
\]

(61)

The definition in (44), then, shows that \( X(q, 0) \) can be infinite only if the divergence of \( \langle T_{\theta \theta} \rangle \) is at least as strong as \( \sim (r - r_+)^{-3} \). From the general relationships shown above, it is clear that \( X(q, 0) \) remains finite as long as \( \langle T_{UU} \rangle \) is less divergent than \( \sim U^{-5} \).

In this appendix, relying on purely analytical methods, we provided some additional evidence that the pivotal quantity \( X \) remains finite in the extremal limit.

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