Gauge Symmetry of the Heat-Kernel and Anomaly Formulas

Shoichi ICHINOSE and Noriaki IKEDA

DAMTP, University of Cambridge,
Silver Street, Cambridge CB3 9EW, UK
†Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-01, Japan

Abstract

We consider a gauge symmetry in a quantum Hilbert space. The symmetry leads to that of the heat-kernel and of the anomaly formulae which were previously obtained by the authors. This greatly simplifies and clarifies the structure of the formulae. We explicitly obtain the anomaly formulae in two and four dimensions, which "unify" all kinds of anomaly. The symmetry corresponds to that of the counterterm formulae in the background field method. As an example, the non-abelian anomaly is considered.

1 On leave of absence, until Jan. 31,1997, from Department of Physics, University of Shizuoka, Yada 52-1, Shizuoka 422, Japan. E-mail address: s.ichinose@damtp.cam.ac.uk ; ichinose@momo1.u-shizuoka-ken.ac.jp
2 E-mail address: nori@kurims.kyoto-u.ac.jp
1 Introduction

Like renormalization, anomalies are an important aspect of the quantum field theory (QFT) and have been provided us with important insights. There are many kinds of anomalies (Weyl, U(1) chiral, Non-abelian, gravitational U(1) chiral, pure gravitational (Einstein), etc.) in different theories and in different dimensions. The applications are diverse and range from a phenomenological one to a purely theoretical or mathematical one. In this circumstance, it is useful, from both formal and practical viewpoints, to look at all anomalies from one framework. With this in mind, we present anomaly formulas, which are valid for all kinds of anomalies of all theories in 2 and 4 spacetime dimensions.

In the case of renormalization, ’t Hooft\cite{1} succeeded in determining the 1-loop counter-term formula, which is valid for all 4-dim theories. It was exploited, from mid to late 70’s, for the (un)renormalizability check of various gravity and supergravity theories. On the other hand, the anomaly, which has the same origin (ultraviolet divergences of local interaction), has not been explored as much. Important progress in a “unified treatment” of anomalies was made by Fujikawa\cite{5} who noted that all anomalies can be interpreted in the path-integral formalism as a change in the measure due to a symmetry transformation. We take this standpoint and realize the “unified” approach in a more manifest way as ’t Hooft did with renormalization\cite{3}.

Anomalies are grouped into two types: those which are related to the scale transformation (i.e. the Weyl anomaly), and those related to the chiral transformation (that is the chiral anomaly). The former is caused by the regularization of the unit operator $1 \cdot \delta^n(x - y)$, while the latter is by the regularization of the chiral operator $\gamma_5 \cdot \delta^n(x - y)$. We know of many regularization methods in QFT and choose an appropriate one depending on the specific problem. In the case of anomalies, popular choices are regularizations of Pauli-Villars, heat-kernel and higher-derivative method. (Dimensional regularization is not appropriate for chiral anomalies due to the difficulty in treating $\gamma_5$). In this paper we use heat-kernel regularization. Its characteristics are as follows: 1) we can formulate the theory explicitly in a covariant way even though a dimensional parameter, $t$, called proper-time is introduced\cite{7}. This is due to the fact that the parameter is introduced not as a mass but as an additional component of the spacetime (time or temperature). This must be compared with the case of Pauli-Villars; 2) A key quantity $G(x,y;t)$, called the heat-kernel, is defined by a heat equation with an elliptic differential operator. It has been closely examined and is well-defined mathematically\cite{8, 9, 10}. This point can be compared with the higher-derivative regularization. 3) This regularization gives a similar result to the dimensional regularization. Dimensional regularization essentially picks up only logarithmic divergences, which encapsulate the most important information of the QFT. The same information appears, in the heat-kernel, as a $t^0$-order term which is the remaining finite term in the limit $t \to +0$.

The symmetry of the heat kernel is derived from a gauge symmetry of the quantum Hilbert space on which the elliptic differential operator acts. The formulae of anomalies are shown

\footnote{It was generalized to higher-loops\cite{2, 3} and to higher-dimensions\cite{4}.}
to satisfy the gauge symmetry. The symmetry clarifies the anomaly formulas and simplifies the calculation. The formula is very powerful as in the case of the counterterm formula of 'tHooft[1]. The 4-dim anomaly formula coincides with the 't Hooft's 1-loop counterterm formula up to total derivative terms. However, contrary to the case of counterterms, the total derivative terms have significance in the anomaly calculation.

In this paper, we take the non-abelian anomaly in a 4-dim chiral gauge model as a concrete application of the anomaly formulas. We calculate both the consistent and covariant anomalies and compare them.

2 Gauge symmetry of the heat-kernel

We consider the bosonic and fermionic cases separately. Focussing initially on the bosonic case. The 1-loop action in n-dim space is generally represented as

\[ S[\phi, D_b] = \int d^nx \frac{1}{2} \phi^\dagger D_b \phi, \]  

where \( \phi \) is a bosonic quantum field and \( D_b \) is written in terms of background fields. (Here we use the matrix notation: \( \phi^\dagger D_b \phi = \phi^i D^j_b \phi_j \) where \( i,j \) are field-suffixes.) Let \( S[\phi, D_b] \) be invariant for a general local transformation.

\[ \phi' = e^{-\Sigma(x)} \phi, \]
\[ \phi'^\dagger = \phi^\dagger e^{-\Sigma(x)^\dagger}, \]
\[ D'_b = e^{\Sigma(x)\dagger} D_b e^{\Sigma(x)}, \]  

where \( \Sigma(x) \) is a local parameter (local gauge, general coordinate transformation, local Weyl). (In the case of the general coordinate transformation, \( \Sigma(x) \) has derivatives). We analyze the anomaly for the above general local symmetry. The partition function,

\[ Z = \int D\phi e^{-S[\phi, D_b]}, \]

generally changes under the above symmetry as

\[ Z' = \int D\phi' e^{-S[\phi', D_b]} = \int D\phi(x) \det \frac{\partial \phi'(y)}{\partial \phi(x)} e^{-S[\phi, D_b]}, \]  

due to a change in the measure which is identified with the anomaly[3]. The Jacobian is formally written as

\[ J \equiv \det \frac{\partial \phi'(y)}{\partial \phi(x)} = \det \left( e^{-\Sigma(x)} \delta^n(x-y) \right) = \exp(-\text{Tr} \left[ \Sigma(x) \delta^n(x-y) \right] + O(\Sigma^2)), \]

where the infinitesimal \( \Sigma(x) \) is taken in the last expression. The anomaly is represented by (3).
Next, we consider the fermion case. The 1-loop action in n-dim space is written as

\[ S[\psi, \bar{\psi}; D_f] = \int d^n x \ i \bar{\psi} D_f \psi, \]  

where \( \psi \) is a (Dirac) fermionic quantum field and \( \bar{\psi} = \psi^\dagger \gamma^4 \). \( D_f \) is written in terms of background fields and is generally not hermitian. Let this action be invariant for a local transformation.

\[ \psi' = e^{-\Sigma(x)} \psi, \quad \bar{\psi}' = \bar{\psi} \gamma^4 e^{-\Sigma(x)^\dagger} \gamma^4, \quad D'_f = \gamma^4 e^{\Sigma(x)^\dagger} \gamma^4 D_f e^{\Sigma(x)}. \]  

The anomaly for the above symmetry is given, as in the bosonic case, by the following Jacobian of the above local transformation:

\[ J \equiv \begin{vmatrix} \det \left( \frac{\partial \psi'(y)}{\partial \psi(x)}, \frac{\partial \bar{\psi}'(y)}{\partial \bar{\psi}(x)} \right) \end{vmatrix}^{-1} = \det (e^{\Sigma(x)} \delta^n(x-y)) \det (\gamma^4 e^{\Sigma(x)^\dagger} \gamma^4 \delta^n(x-y)') \]

\[ = \exp(\text{Tr} [\Sigma(x) \delta^n(x-y) + \gamma^4 \Sigma(x)^\dagger \gamma^4 \delta^n(x-y)'] + O(\Sigma^2)) , \]

where \( \delta^n(x-y)' \) is used to note that it can be different from \( \delta^n(x-y) \) at the regularization level.

In order to regularize the delta function \( \delta^n(x-y) \) appearing both in (5) and in (8), we introduce the heat-kernel,

\[ G(x, y; t) \equiv <x| e^{-t \bar{D}} |y>, \quad t > 0, \]

using a properly chosen elliptic differential operator \( \bar{D} \). The heat-kernel \( G(x, y; t) \) , introduced “symbolically” above, is more precisely defined by

\[ \left( \frac{\partial}{\partial t} + \bar{D} \right) G(x, y; t) = 0, \quad G(x, y; t)(\frac{\partial}{\partial t} + \bar{D}^\dagger v) = 0, \quad t > 0, \]

\[ \lim_{t \to +0} G(x, y; t) = \delta^n(x-y), \]

where \( G(x, y; t) \) is defined symmetrically with respect to \( x \) and \( y \). The last equation of (10) guarantees that we can use \( G(x, y; t) \) to regularize \( \delta^n(x-y) \). Then the anomaly is obtained as [3],

\[ \ln J = - \lim_{t \to +0} \text{Tr}[\Sigma(x)G(x, y; t)], \quad \text{for the bosonic case}, \]

\[ \ln J = \lim_{t \to +0} \text{Tr} [\Sigma(x)G(x, y; t) + \gamma^4 \Sigma(x)^\dagger \gamma^4 G(x, y; t)'], \quad \text{for the fermionic case}. \]

where \( G(x, y; t)' \equiv <x| e^{-t \bar{D}'} |y> \) is generally a heat-kernel with some another operator \( \bar{D}' \).

In order for the heat kernel to be well-defined, and to be defined by the action of (1) or (6) we require the following conditions for \( \bar{D} \): \( \bar{D} \) is a second order elliptic differential operator;

\[ \gamma^4 \] is understood as the n-dim generalization of \( \gamma^4 \) in 4 dimension.
$\vec{D}$ is made of the differential operators appearing in the field equation. We do not impose hermiticity on $\vec{D}$. (See Sect.6 for further discussion). For example, in the bosonic case, we can take

$$
\vec{D} = D_b = D_b^\dagger.
$$

In the fermionic case, we have two cases.

- $D_f$ is hermitian
  $$\vec{D} = D_f D_f$$

- $D_f$ is not hermitian
  $$\vec{D} = \begin{cases} 
  D_f D_f, & D_f^\dagger D_f, \\
  \frac{1}{2}(D_f^\dagger D_f + D_f D_f^\dagger), & etc.
  \end{cases}$$

In the non-hermitian case, the operators $D_f^2$ and $(D_f^\dagger)^2$ are not hermitian. The different choices of the heat kernel operator correspond to the different choices of regularization. It causes the different anomalies, in appearance, which are examined in a later section.

We denote the basis of the Hilbert space, on which $D_i$ ($i = b$ or $f$) operates, as $\{f_n(x)\}$. The Hermite conjugate of $D_i$, appearing above, is defined precisely as follows:

$$
(f_m, D_i f_n) = (D_i^\dagger f_m, f_n),
$$

where $(\Phi, \Psi) \equiv \int d^n x \Phi^\dagger(x)\Psi(x)$ is an inner product on the Hilbert space. On this space, let us introduce the following local gauge transformation:

$$
D_{\xi}' = e^{-\Lambda(x)} D_{\xi} e^{\Lambda(x)},
$$

(16)

where $\Lambda(x)$ is the anti-hermitian gauge parameter: $\Lambda^{\dagger} = -\Lambda(x)$. For the above transformation (16), the matrix element of $D_i$, defined by the inner product (15), does not change.

In both boson and fermion cases, $\vec{D}$ transforms as

$$
\vec{D}'_x = e^{-\Lambda(x)} \vec{D}_x e^{\Lambda(x)}, \quad \vec{D}'_y = e^{-\Lambda(x)} \vec{D}_y^\dagger e^{\Lambda(x)}.
$$

(17)

Because the heat kernel is defined by the operator $\vec{D}$, through (10), and $\vec{D}$ changes as above, the heat kernel must itself change. The heat kernel equation transforms as follows:

$$
\frac{\partial}{\partial t} G'(x, y; t) = 0, \quad G'(x, y; t)\left(\frac{\partial}{\partial t} + \vec{D}_y^\dagger\right) = 0,
$$

(18)

which implies

$$
e^{-\Lambda(x)}\left(\frac{\partial}{\partial t} + \vec{D}_x\right) e^{\Lambda(x)} G'(x, y; t) = 0, \quad G'(x, y; t) e^{-\Lambda(y)}\left(\frac{\partial}{\partial t} + \vec{D}_y\right) e^{\Lambda(y)} = 0,
$$

(19)

where $G'(x, y; t)$ is the transform of $G(x, y; t)$. Therefore, by (10) and (19), the solution $G(x, y; t)$ has to transform covariantly as follows:

$$
G'(x, y; t) = e^{-\Lambda(x)} G(x, y; t) e^{\Lambda(y)}.
$$

(20)
3 Covariants and Covariant Derivative

We now move on to the gauge symmetry of the anomaly formulae. The formulae are obtained by calculating \( x = y \) part (trace part). So we consider the case of \( x = y \) in (20). If we take the infinitesimal form, (17) and (20) become

\[
\delta \bar{D}_x = -[\Lambda(x), \bar{D}_x], \quad \delta G(x, x; t) = -[\Lambda(x), G(x, x; t)].
\]  

(21)

Generally, \( \bar{D}_x \) is a second order differential operator, so its most general form is written as

\[
\bar{D}_x = - (\delta_{\mu\nu} + W_{\mu\nu}(x)) \partial_\mu \partial_\nu - N_\mu(x) \partial_\mu - M(x),
\]  

(22)

where \( W_{\mu\nu}(x) = W_{\nu\mu}(x) \), \( N_\mu(x) \) and \( M(x) \) are represented by background fields\(^5\) (The above equation is written, in component form, as \( \bar{D}^i_\mu = - (\delta_{\mu\nu} + W_{\mu\nu}(x)) \partial_\mu \partial_\nu - N_\mu(x) \partial_\mu - M(x) \), where \( i \) and \( j \) are all suffixes of fields; \( i, j = 1, 2, \cdots, N \)). From (21), we can derive the transformation of \( W_{\mu\nu}(x) \), \( N_\mu(x) \) and \( M(x) \) as

\[
\delta W_{\mu\nu} = -[\Lambda, W_{\mu\nu}], \quad \delta N_\mu = 2 (\delta_{\mu\nu} + W_{\mu\nu}) \partial_\nu \Lambda - [\Lambda, N_\mu],
\]

\[
\delta M = (\delta_{\mu\nu} + W_{\mu\nu}) \partial_\mu \partial_\nu \Lambda + N_\mu \partial_\mu \Lambda - [\Lambda, M].
\]  

(23)

We assume that the 'background coefficient' of the second order differential terms \( (\delta_{\mu\nu} + W_{\mu\nu}(x)) \) is non-degenerate. It is necessary for the propagator of \( \phi(x) \) or \( \psi(x) \) to be defined. We can define \( Z_{\mu\nu} \) as

\[
(\delta_{\mu\nu} + W_{\mu\nu})(\delta_{\nu\lambda} + Z_{\nu\lambda}) = \delta_{\mu\lambda},
\]  

(24)

where \( Z_{\mu\nu} = Z_{\nu\mu} \). It is expressed as an infinite series in \( W_{\mu\nu} \) by

\[
Z_{\mu\nu} = - W_{\mu\nu} + W_{\mu\lambda} W_{\lambda\nu} - W_{\mu\lambda} W_{\lambda\rho} W_{\rho\nu} + O(W^4).
\]  

(25)

\( Z_{\mu\nu} \) transforms covariantly, as derived from (24),

\[
\delta Z_{\mu\nu} = -[\Lambda, Z_{\mu\nu}].
\]  

(26)

We can find the generalized gauge field \( A_\mu \) of the present general gauge transformation as,

\[
A_\mu \equiv \frac{1}{2} (\delta_{\mu\nu} + Z_{\mu\nu}) N_\nu,
\]  

(27)

which surely transforms, from (23) and (27), as a gauge field.

\[
\delta A_\mu = \partial_\mu \Lambda - [\Lambda, A_\mu].
\]  

(28)

\(^5\) \( \delta_{\mu\nu} + W_{\mu\nu}(x) \) is the metric of the present general system.
We want to obtain the solution $G(x, x; t)$, expressed by $W_{\mu \nu}(x)$, $N_{\mu}(x)$ and $M(x)$. Since $G(x, x; t)$ is covariant with respect to the gauge transformation (20) or (23), it must be constructed by covariant quantities. We find them, by (23), (26) and (28), as

$$X \equiv M - (\delta_{\mu \nu} + W_{\mu \nu}) \partial_{\mu} A_{\nu} - (\delta_{\mu \nu} + W_{\mu \nu}) A_{\mu} A_{\nu}, \quad Y_{\mu \nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}], \quad W_{\mu \nu}.$$ 

Indeed, $X$ and $Y_{\mu \nu}$ transform covariantly,

$$\delta X = -[\Lambda, X], \quad \delta Y_{\mu \nu} = -[\Lambda, Y_{\mu \nu}].$$ 

Those covariants were examined in the counterterm formula, for the special case of $W_{\mu \nu} = 0$ by [1] and for the general case by [11]. Moreover we can define a covariant derivative $D_{\mu}$ on a covariant $\Phi$, as

$$D_{\mu} \Phi \equiv \partial_{\mu} \Phi + [A_{\mu}, \Phi].$$ 

Their transformations are

$$\delta \Phi = -[\Lambda, \Phi], \quad \delta (D_{\mu} \Phi) = -[\Lambda, D_{\mu} \Phi].$$ 

Therefore we can conclude that $G(x, x; t)$ must be written by the covariant quantities, $X(x)$, $Y_{\mu \nu}(x)$, $W_{\mu \nu}(x)$ and $D_{\mu}(x)$.

$$G(x, x; t) = F(X(x), Y_{\mu \nu}(x), W_{\mu \nu}(x), D_{\mu}(x)),$$ 

where $F(X, Y, W, D)$ is some function which will be determined in the next section. For later use, we note mass-dimension of the covariants: $[X] = [Y_{\mu \nu}] = (\text{Mass})^{2}$, $[D_{\mu}] = \text{Mass}$, $[W_{\mu \nu}] = (\text{Mass})^{0}$.

### 4 Anomaly Formulae

Decomposing the operator $\bar{D}_x$ as $
\bar{D}_x = -\delta_{\mu \nu} \partial_\mu \partial_\nu - \bar{V}(x),$ 
$\bar{V}(x) \equiv W_{\mu \nu}(x) \partial_\mu \partial_\nu + N_\mu \partial_\mu + M,$
the heat kernel equation (10) can be written as

$$\left(\frac{\partial}{\partial t} - \partial^2\right) G(x, y; t) = \bar{V}(x) G(x, y; t),$$ 

We solve this equation by the propagator approach [12, 3]. The solution of the heat equation is given by

$$\left(\frac{\partial}{\partial t} - \partial^2\right) G_0(x, y; t) = 0 \quad \Rightarrow \quad G_0(x, y; t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{(x-y)^2}{4t}} I_N \quad \text{for } t > 0,$$

$$G_0(x, y; t) \equiv 0 \quad \text{for } t \leq 0.$$ 


where \( I_N \) is the \( N \times N \) identity matrix and \( G_0 \) satisfies the initial condition: \( \lim_{t \to +0} G_0(x - y; t) = \delta^n(x - y) I_N \). The heat equation with the delta-function source defines the heat propagator and which is given by

\[
(\frac{\partial}{\partial t} - \nabla^2) S(x, y; t - s) = \delta(t - s) \delta^n(x - y) I_N ,
\]

\[
S(x, y; t) = \theta(t) G_0(x - y; t) ,
\]

where \( \theta(t) \) is the step function defined by \( \theta(t) = 1 \) for \( t > 0 \), \( \theta(t) = 0 \) for \( t < 0 \). Using the above quantities, the formal solution of (34) (or (10)) is given by

\[
G(x, y; t) = G_0(x - y; t) + \int d^n z \int_{-\infty}^{\infty} ds \ S(x - z; t - s) \tilde{V}(z) G(z, y ; s) .
\]

The iterative solution of (37) with respect to the order of \( \tilde{V} \) is calculated from the diagonal-part of the solution.

\[
G(x, x; t) = G_0(0; t) + G_1(x, x; t) + G_2(x, x; t) + \cdots ,
\]

where the suffix numbers show the iteration order. Generally, in \( n \) dimensions, the terms up to \( G_{n/2} \) are practically sufficient for the anomaly calculation.

Further analysis will be done for each dimension. For simplicity, we assume that \( W_{\mu \nu} \) commutes with \( W_{\mu \nu}, N_\mu \) and \( M \). This assumption is valid for most field theories in flat and curved spacetime.\(^6\) Only the term of \( G(x, x; t)|_\phi \) is necessary for the anomaly calculation, because we take the limit \( t \to +0 \). Divergent parts of \( O(t^{-m}) \) \( (m = 1, 2, \cdots > 0) \) are considered to be renormalized. By using the gauge symmetry found in the previous section, we can express the anomaly formulas obtained in \( \ref{33} \) in a covariant way. Let us consider the 2-dim and 4-dim cases.

(i) Two Dimensions \((n=2)\)

First for the special case of \( W_{\mu \nu} = 0 \),

\[
G(x, x; t)|_\phi = \frac{1}{4\pi} X , \quad W_{\mu \nu} = 0 , \quad X = M - \frac{1}{2} \partial_\mu N_\mu - \frac{1}{4} N_\mu N_\mu .
\]

For this case, \( G(x, x; t)|_\phi \) is completely fixed. For the general case of \( W_{\mu \nu} \), we obtain

\[
G(x, x; t)|_\phi = \frac{1}{4\pi} \left[ X - \frac{1}{12} D^2 W_{\mu \nu} + \frac{1}{3} D_\mu D_\nu W_{\mu \nu} + O((W, X, Y)^2) \right] ,
\]

where \( O((W, X, Y)^3) \) are, from the dimensional reason, restricted to the following forms.

\[
X \cdot O_1(W) , \quad Y \cdot O_2(W) , \quad DDW \cdot O_3(W) , \quad DW \cdot DW \cdot O_4(W^0)
\]

where \( O_i(W^s) \) \( (i = 1 \sim 4; s = 0, 1) \) are some functions made only of \( W \) with respective orders of \( W^s \).

\(^6\) This is the case \( W_{\mu \nu} \propto \delta^{ij} \). The case that \( W \) does not commute with \( W, N, M \) takes place, for example, when we consider Weyl anomaly in the gravity-photon coupling system with a non-Feynman-like gage: \( \mathcal{L}_{int} = \frac{1}{2} \sqrt{g} (A_\lambda (g^{\alpha \nu} \nabla^\alpha - \alpha \nabla^\lambda \nabla^\nu) A_\nu ) , \alpha \neq 0 \).
(ii) Four Dimensions (n=4)

In the four dimension, we obtain the following formulas. For the case of $W_{\mu\nu} = 0$, we obtain a complete expression.

$$G(x, x; t)|_\nu = \frac{1}{(4\pi)^2} \left[ \frac{1}{16} D^2 X + \frac{1}{2} X^2 + \frac{1}{12} Y_{\mu\nu} Y_{\mu\nu} \right], \quad D_\mu X = \partial_\mu X + \frac{1}{2} [N_\mu, X],$$

$$X = M - \frac{1}{4} \partial_\mu N_\mu - \frac{1}{4} N_\mu N_\mu, \quad Y_{\mu\nu} = \frac{1}{2} (\partial_\mu N_\nu - \partial_\nu N_\mu) + \frac{1}{4} [N_\mu, N_\nu]. \quad (42)$$

The first total derivative term is the difference from the 1-loop counter-term formula\(\footnote{1}\). This formula will be used in the next section. For the general case of $W_{\mu\nu}$, we have

$$G(x, x; t)|_\nu = \frac{1}{(4\pi)^2} \left[ \frac{1}{16} D^2 X - \frac{1}{120} D^2 D^2 W_{\mu\nu} + \frac{1}{20} D^2 D_\mu D_\nu W_{\mu\nu} 
+ \frac{1}{2} X^2 + \frac{1}{12} Y_{\mu\nu} Y_{\mu\nu} - \frac{1}{12} D^2 W_{\mu\nu} \cdot X + \frac{1}{3} D_\mu D_\nu W_{\mu\nu} \cdot X - \frac{1}{6} D_\lambda W_{\mu\nu} \cdot Y_{\mu\nu} 
- \frac{1}{18} D_\lambda W_{\mu\nu} \cdot D_\lambda X - \frac{1}{12} W_{\mu\nu} D^2 X + \frac{1}{6} D_\mu W_{\mu\nu} \cdot D_\nu X + \frac{1}{6} W_{\mu\nu} D_\mu D_\nu X 
+ \frac{1}{6} D_\lambda W_{\lambda\nu} \cdot D_\mu Y_{\mu\nu} + 0 \times D_\mu W_{\lambda\nu} \cdot D_\lambda Y_{\mu\nu} + 0 \times W_{\lambda\nu} D_\lambda D_\mu Y_{\mu\nu} 
+ (DDW \cdot DDW-terms, see Table 1) + (DW \cdot DDDDW-terms) 
+ (W \cdot DDDDW-terms) + O((W, X, Y)^3) \right]. \quad (43)$$

Suppressed parts above are not necessary for the practical use. Terms of $O((W, X, Y)^3)$ are restricted to some types, which can be determined as in $n=2$ case. Graph names, used in Table 1, are defined in \footnote{3} where each expression is shown graphically.

| Graph | Expression | Coeff | Graph | Expression | Coeff |
|-------|------------|-------|-------|------------|-------|
| A1    | $D_\sigma D_\lambda W_{\mu\nu} \cdot D_\sigma D_\nu W_{\mu\lambda}$ | 1/45  | $\bar{Q}^2$ | $(D_\mu D_\nu W_{\mu\nu})^2$ | 1/18  |
| A2    | $D_\sigma D_\lambda W_{\lambda\mu} \cdot D_\sigma D_\nu W_{\mu\nu}$ | -2/45 | $\bar{C}1$ | $D_\mu D_\nu W_{\lambda\lambda} \cdot D_\mu D_\nu W_{\sigma\sigma}$ | 1/360 |
| A3    | $D_\sigma D_\lambda W_{\lambda\mu} \cdot D_\mu D_\nu W_{\nu\sigma}$ | -2/45 | $\bar{C}2$ | $D^2 W_{\mu\nu} \cdot D^2 W_{\mu\nu}$ | 1/144 |
| B1    | $D_\nu D_\lambda W_{\sigma\sigma} \cdot D_\lambda D_\mu W_{\mu\nu}$ | -1/90 | $\bar{C}3$ | $D_\mu D_\nu W_{\lambda\lambda} \cdot D^2 W_{\mu\nu}$ | -1/90 |
| B2    | $D^2 W_{\lambda\mu} \cdot D_\lambda D_\mu W_{\mu\nu}$ | 1/180 | $P \bar{Q}$ | $D^2 W_{\lambda\lambda} \cdot D_\mu D_\nu W_{\mu\nu}$ | -1/36 |
| B3    | $D_\mu D_\nu W_{\lambda\sigma} \cdot D_\mu D_\sigma W_{\nu\lambda}$ | 1/180 | $P^2$ | $(D^2 W_{\lambda\lambda})^2$ | 1/288 |
| B4    | $D_\mu D_\nu W_{\lambda\sigma} \cdot D_\lambda D_\sigma W_{\mu\nu}$ | 1/180 |

\footnote{3}{Table 1 Anomaly Formula for $(DDW)^2$-part of $G_2(x, x; t)|_\nu$}

The overall factor is $1/(4\pi)^2$. Graph names are defined in \footnote{3}
When we consider an anomaly in a gravitational model, the general case of $W_{\mu\nu}$ must be taken. Even in a flat theory, the case appears when we evaluate a gauge-loop effect to the (Weyl) anomaly in a gauge different from Feynman gauge.

5 The Non-abelian Anomaly in Four Dimensions

As a typical example, we consider the non-abelian anomaly of the Dirac fields coupled to the V-A current in 4 dim space \[13, 14, 15, 16\]. The Lagrangian is written as

$$L = -\bar{\psi} \gamma^\mu (\partial_\mu - iV_\mu - iA_\mu \gamma_5) \psi,$$

where $V_\mu \equiv T^a V_\mu^a$ and $A_\mu \equiv T^a A_\mu^a$ are gauge fields and the convention: $(\gamma^\mu)^\dagger = \gamma^\mu$, $\gamma_5^\dagger = \gamma_5$, is taken. We take the left and right decomposition: $R_\mu = V_\mu + A_\mu$, $L_\mu = V_\mu - A_\mu$. For simplicity, we set $L_\mu = 0$ and analyze only the $R_\mu$ part. The Lagrangian is then represented as

$$L_R = -\bar{\psi} \gamma^\mu (\partial_\mu - iR_\mu \gamma_5) \psi = -\bar{\psi} \partial_R^+ \psi,$$

where we define the following differential operators,

$$\partial_R^+ = \gamma^\mu (\partial_\mu - iR_\mu \gamma_5), \quad \partial_R^- = \gamma^\mu (\partial_\mu - i\gamma_5 R_\mu).$$

We consider the case: $D_\mu = i\partial_R^+ \psi$ in (44). We note here $i\partial_R^+$ is hermitian, whereas $(i\partial_R^+)^\dagger = i\partial_R^-$. The Lagrangian (45) is invariant under the following local gauge symmetry,

$$\psi \rightarrow \psi' = e^{-i\lambda(x)\frac{1+\gamma_5}{2}} \psi, \quad \bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi} e^{i\lambda(x)\frac{1-\gamma_5}{2}},$$

$$R_\mu \rightarrow R'_\mu = U(\lambda)^{-1} R_\mu U(\lambda) - i\partial_\mu U(\lambda)^{-1} U(\lambda), \quad U(\lambda) = e^{i\lambda(x)}.$$
These operators are not hermitian. We find
\[
\exp(tD_{R^+}D_{R^+}) = \exp\{t\left[\partial\left(\frac{1-\gamma_5^2}{2}\right)D_R\left(\frac{1+\gamma_5^2}{2}\right) + D_R\left(\frac{1+\gamma_5^2}{2}\right)\partial\left(\frac{1-\gamma_5^2}{2}\right)\right]\}
\]
\[
= \frac{1+\gamma_5^2}{2} \exp(t\partial D_R) + \frac{1-\gamma_5^2}{2} \exp(tD_R\partial) ,
\]
\[
\exp(tD_{R^-}D_{R^-}) = \frac{1+\gamma_5^2}{2} \exp(tD_R\partial) + \frac{1-\gamma_5^2}{2} \exp(t\partial D_R) ,
\]
\[
\ln J_{\text{con}} = i \lim_{t \to +0} \text{Tr} \lambda(x)[\gamma_5 < x] \exp(t\partial D_R)|y >] .
\] (50)

The quantities \(M, N_\mu\) and \(W_{\mu\nu}\) for the operator \(-\partial D_R\) are
\[
M = -i\gamma^\mu\gamma^\nu\partial_\mu R_\nu , \quad N_\mu = -i\gamma^\mu\gamma^\nu R_\nu , \quad W_{\mu\nu} = 0 .
\] (51)

The corresponding covariants \(X(x)\) and \(Y_{\mu\nu}(x)\) are found to be
\[
X = -i\frac{1}{2} \partial_\mu R_\mu - \frac{1}{2} R_\mu R_\mu - \frac{1}{8} i[\gamma^\mu, \gamma^\nu] F_{\mu\nu}^R ,
\]
\[
Y_{\mu\nu} = -\frac{1}{2} i\gamma^\nu\partial_\mu R_\lambda - \frac{1}{4} \gamma^\mu\gamma^\nu R_\lambda R_\rho - (\mu \leftrightarrow \nu) ,
\]
\[
F_{\mu\nu}^R = \partial_\mu R_\nu - \partial_\nu R_\mu - i[R_\mu, R_\nu] .
\] (52)

Here \(X\) and \(Y_{\mu\nu}\) are not expressed covariantly in terms of the gauge field \(R_\mu\). (This is because the general gauge symmetry (23), which is introduced irrespective of the explicit forms of \((W_{\mu\nu}, N_\mu, M)\), cannot be realized by the gauge transformation of \(R_\mu\)). This must be compared with the covariant anomaly of the next item. Finally the consistent anomaly is determined to be
\[
\ln J_{\text{con}} = i \frac{1}{(4\pi)^2} \frac{2}{3} \epsilon^{\mu\nu\lambda\rho} \text{Tr} \lambda(x) \left[\partial_\mu (R_\rho \partial_\lambda R_\rho - \frac{i}{2} R_\rho R_\lambda R_\rho)\right] ,
\] (53)

where the convention \(\text{tr} \gamma_5^\mu \gamma^\nu \gamma^\lambda \gamma^\rho = -4 \epsilon^{\mu\nu\lambda\rho}, \epsilon^{1234} = 1\), is used. This coincides with the result of the conventional calculation [14].

If we take \(\tilde{D}\) and \(\tilde{D}'\) in (49) in the exchanged way: \(\tilde{D} = -\partial D_R- D_{R^-}, \tilde{D}' = -\partial D_R+ D_{R^-}\), we get \(\ln J_{\text{con}} = i \lim_{t \to +0} \text{Tr} \lambda(x)[\gamma_5 < x] \exp(t\partial D_R)|y >]\). The \(-\partial D_R\partial\) operator is written as
\[
M = 0 , \quad N_\mu = -i\gamma^\nu\gamma^\mu R_\nu , \quad W_{\mu\nu} = 0 .
\] (54)

The corresponding covariants, \(X(x)\) and \(Y_{\mu\nu}(x)\), are written as
\[
X = \frac{1}{2} i\partial_\mu R_\mu - \frac{1}{2} R_\mu R_\mu - \frac{1}{8} i[\gamma^\mu, \gamma^\nu] F_{\mu\nu}^R ,
\]
\[
Y_{\mu\nu} = -\frac{1}{2} i\gamma^\nu\gamma^\mu \partial_\mu R_\lambda - \frac{1}{4} \gamma^\lambda\partial_\mu R_\lambda R_\rho - (\mu \leftrightarrow \nu) .
\] (55)

The final anomaly is the same as (53).
(ii) Covariant anomaly

The covariant anomaly is obtained by taking the heat kernel operators as

\[ \vec{D} = \partial_{R}D_{R}^{\dagger} + \partial_{R}D_{R}, \quad \vec{D}' = \partial_{R}D_{R}^{\dagger} + \partial_{R}D_{R}, \]

which are hermitian and invariant for the \( R_{\mu} \) gauge transformation. The anomaly is evaluated as:

\[ \exp(-t\partial_{R}D_{R}^{\dagger} + \partial_{R}D_{R}) = \frac{1}{2}\exp(t\partial_{R}D_{R}) + \frac{1}{2}\exp(t\partial_{R}D_{R}) \]

\[ \exp(-t\partial_{R}D_{R}^{\dagger} + \partial_{R}D_{R}) = \frac{1}{2}\exp(t\partial_{R}D_{R}) + \frac{1}{2}\exp(t\partial_{R}D_{R}) \]

\[ \ln J_{\text{cov}} = i \lim_{t \to +0} \text{Tr} [\lambda(x)\gamma_{5} < x | \exp(t\partial_{R}D_{R})| y >] . \]

The operator \(- (\partial_{R})^{2}\) is expressed by the following \( M, N_{\mu} \) and \( W_{\mu\nu} \).

\[ M = -i\partial_{\mu}R_{\mu} - R_{\mu}R_{\mu} - \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]F_{\mu\nu}^{R}, \quad N_{\mu} = -2iR_{\mu}, \quad W_{\mu\nu} = 0 \]

where \( F_{\mu\nu}^{R} \) is defined in (52). So we have covariant quantities as follows,

\[ X = -\frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}]F_{\mu\nu}^{R}, \quad Y_{\mu\nu} = -iF_{\mu\nu}^{R} \]

In the covariant anomaly, all covariants \((X, Y_{\mu\nu})\) under the general gauge transformation (23) are also covariant under the \( R_{\mu} \)-gauge transformation (17). Substituting the above expressions into the anomaly formula (42), the anomaly is

\[ \ln J_{\text{cov}} = i \frac{1}{(4\pi)^{2}} \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \text{Tr} [\lambda(x)F_{\mu\nu}^{R}F_{\lambda\rho}^{R}] \]

which is the covariant anomaly [5, 18].

6 Conclusion

We have explained the general gauge symmetry which appears in the general heat-kernel \( G(x, y; t) \). The symmetry simplifies and clarifies the structure of the anomaly formulae. They are expressed by the covariants. The general gauge symmetry is attributed to the local gauge freedom of the Hilbert space on which the operator \( \vec{D} \) acts. This analysis is in contrast with the case of the counterterm formula [1] where the symmetry is that of the action. In fact the 4-dim anomaly formula of (12) is different from the ’t Hooft’s by a total derivative term.
We have obtained the new anomaly formulas for two and four dimensional theories. These are valid for all anomalies in most quantum field theories both on flat and on curved space-times. Messy integral-calculation of the anomaly has been eliminated. Using $W_{\mu \nu}, N_\mu, M$ in $\vec{D}$, (22), (which corresponds to ‘which theory’ we take) and $\Sigma$ in (2) or (4) (which corresponds to ‘which symmetry’ we consider anomalous), its anomaly is expressed by (11) for the bosonic case and by (12) for the fermionic case. The heat-kernel $G(x,x;0)$ in (11) or (12) is given by (39)-(43) depending on the relevant situation. The generalization to higher dimensional anomaly formulae is straightforward.

In the initial condition of the heat-kernel $\lim_{t \to +0} G(x,y;t) = \lim_{t \to +0} <x|e^{-t\vec{D}}|y> = \delta^n(x-y)$, $\delta^n(x-y)$ is a real quantity (distribution). This fact, however, does not necessarily imply the hermiticity condition on $\vec{D}$. In fact we have proved, in subsec.IIB of [6], the existence of a (perturbative) solution of $G(x,y;t)$, which satisfies the above initial condition, for a general $\vec{D}$ which includes the non-hermitian case. The assumption used in the proof is that the weak-field perturbation is valid and the 0-th order of $\vec{D}_{ij}$ is $-\delta_{\mu \nu} \delta^{ij} \partial_\mu \partial_\nu$.

As examples, we have calculated both consistent and covariant non-abelian anomalies. The different appearance of the same anomaly in the same theory is caused by the freedom in the choice of the heat kernel operators $\vec{D}$. The (covariantly-expressed) anomaly formulae most efficiently work in a covariant approach as shown in the covariant nonabelian anomaly. Other examples are shown in [6].

Much more needs to be explored in the anomaly. Its most popular usage is, at present, the anomaly cancellation for model building in various theories, including the (super)string theories (critical string). It could have, however, a more positive meaning as demonstrated in 2-dim models for the case of the chiral anomaly by Polyakov and Wiegmann[19] and by Witten[20] (WZNW-model) and in the case of Weyl anomaly by Polyakov[21] and by Knizhnik et al[22] (non-critical string). For clarification, we point out the importance of the gauge or gravitational quantum effects to anomalies, especially to the Weyl anomaly. The Weyl anomaly is directly related to the renormalization-group $\beta$-function and contains rich dynamical information of the system. Compared with the chiral anomaly, the geometrical description of the Weyl anomaly is not so clear [23, 24, 25, 26]. Furthermore the higher loop (higher than 1-loop) effect has not been examined. We believe the present results are useful for the analysis to such a direction.

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\footnote{In [6], we insufficiently explained this point.}
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