Methods for a quantitative evaluation of odd-even staggering effects

Alessandro Olmi and Silvia Piantelli

Sezione INFN di Firenze, Via G. Sansone 1, I-50019 Sesto Fiorentino (FI), Italy

Received: date / Revised version:

Abstract. Odd-even effects, also known as staggering effects, are a common feature observed in the yield distributions of fragments produced in different types of nuclear reactions. We review old methods, and we propose new ones, for a quantitative estimation of these effects as a function of proton or neutron number of the reaction products. All methods are compared on the basis of Monte Carlo simulations. We find that some are not well suited for the task, the most reliable ones being those based either on a non-linear fit with a properly oscillating function or on a third (or fourth) finite difference approach. In any case, high statistic is of paramount importance to avoid that spurious structures appear just because of statistical fluctuations in the data and of strong correlations among the yields of neighboring fragments.

PACS. 29.85.Fj Data analysis

1 Introduction

An odd-even effect in the yield distributions of final reaction products has been observed since long time in a variety of nuclear reactions. This effect (often called staggering) usually consists in an enhanced production of even nuclear species with respect to the neighboring odd ones: for example, in the case of charge distributions, even-Z species are produced more abundantly than the neighboring odd-Z ones.

The staggering effect was first investigated long time ago in the fission of actinide nuclei, mainly induced by low-energy neutrons (see e.g. [12,13,14,15,16] and references therein) and it was attributed to the extra energy required to break a pair of protons or neutrons [17]. Experimentally the odd-even effect in fission was found to be less pronounced for neutrons than for protons. Many investigations were devoted to finding a systematic parametrization of the normal shapes, against which experimental results could be compared to extract the odd-even effects. At very low excitation energies, the yield distributions of fission fragments are bell shaped, therefore the normal distributions were assumed to be Gaussians [18]. The odd-even effect appeared as a superimposed saw-tooth modulation.

Having as a reference these semi-empirical shapes, the staggering of low-energy fission was usually quantified by a single value (called global staggering) extracted from the whole fragment distribution. Nowadays staggering effects are studied in a variety of different reactions, therefore no a priori shape of the distribution can be assumed and only local values of the staggering can be extracted from small adjacent regions of the yield distributions. An extensive review of different quantitative methods used for estimating local and global average values of the odd-even effects in fission was given by Gönnenwein [19].

For a quantitative analysis, Amiel and Feldstein [12, 13] parametrized the odd-even effects in fragment distributions by assuming that the yields of even-Z fragments are intensified by a factor \((1 + \Delta)\) with respect to the smooth distribution \(Y_{sm}(Z)\) and those of odd-Z fragments are reduced by a factor \((1 - \Delta)\), with \(|\Delta| < 1\). The normal behavior is therefore represented by the arithmetic mean between odd and even yield values. The generalization, to take into account odd-even effects as a function of both proton and neutron numbers (usually assumed to be independent from each other), leads to the expression

\[ Y(Z, N) = Y_{sm}(Z, N) (1 \pm \Delta_Z) (1 \pm \Delta_N) \] (1)

where \(Y_{sm}(Z, N)\) is the assumed smooth behavior without staggering; the sign ‘+’ is used for even \(Z\) (or \(N\)), the sign ‘−’ for odd ones. \(\Delta = 0\) means no odd-even effects, while \(\Delta > 0\) indicates enhanced yields of even species and reduced yields for odd species (vice versa for \(\Delta < 0\)).

Another approach by Wahl [14] led to a slightly different parametrization

\[ Y(Z, N) = Y_{sm}(Z, N) (F_Z)^{\pm 1} (F_N)^{\pm 1}, \] (2)

with \((F)^{\pm 1} = e^{\pm D}\), which can be cast also in the form

\[ \ln(Y(Z, N)) = \ln(Y_{sm}(Z, N)) \pm D_Z \pm D_N \] (3)

\[^{1}\text{E-mail: olmi@fi.infn.it, piantelli@fi.infn.it}\]

\[^{1}\text{The opposite effect, i.e. enhancement of odd-Z species and reduction of even ones, is rare and is called antistaggering.}\]
where again the sign ‘+’ is used for even \( Z \) (or \( N \)) and the sign ‘−’ for odd ones. In this case \( F=1 \) \( (D=0) \) corresponds to no odd-even effect, while \( F>1 \) \( (D>0) \) indicates enhanced yields for even species and reduced yields for odd species; vice versa for \( F<1 \) \( (D<0) \). In the parameterization by Wahl, the smooth behavior is represented by the geometric mean between odd and even yield values.

Odd-even effects were discovered also in light complex fragments produced in high-energy fragmentation and spallation reactions \[ E/A \leq 50 \text{ MeV/}nucleon \] \[ (15 \leq E/A \leq 50 \text{ MeV/}nucleon) \]. These experiments have stirred renewed interest in staggering phenomena. Indeed, in order to study the symmetry energy \( Y \), one needs to estimate the primary isotopic distributions, which can be reliably reconstructed only if the effects of secondary decays are small or sufficiently well understood.

The staggering has been usually ascribed to nuclear structure effects that manifest in the decay of the excited reaction products or already in the reaction mechanism, if part of the reaction proceeds through low excitation energies \[ (\text{without staggering effects}) \] \[ multiplied by a staggering factor \]. In heavy-ion reaction, where \( Y_{\text{sm}}(Z) \) is unknown, only a local staggering can be determined. The main, essential assumption is that the staggering varies with \( Z \) in a sufficiently gradual way to be considered – to a good approximation – constant over an interval of a few \( Z \) values.

The smoothing procedures used for estimating \( \Delta Z \) or \( D_Z \) from the data are listed below (each with the corresponding abbreviation that will be used in the paper) and some of them are schematically illustrated in figs. 1 and 2.

For more clarity, the procedures that are applied directly to the experimental yields will be distinguished from those that are applied to their logarithms by prefixing “\( Y-\)” in the first case and “\( LY-\)” in the second case. In both figures, the open squares joined by dotted lines indicate the input cross section without staggering \( Y_{\text{sm}}(Z) \), while the open crosses joined by long dashed lines represent what from now on we call the experimental cross section \( Y(Z) \), because it simulates the outcome of an experiment (for illustration purposes no statistical fluctuations are applied in these two figures). The staggering effects are modelled à la Amiel and Feldstein \[ [12,13] \], with a constant value of 0.1 for \( \Delta Z \). The dotted and long dashed lines are just a guide to the eye, joining the points.

### 2 Smoothing methods

When analyzing odd-even effects, the main assumption of eqs. 1-3 is that the experimental yield \( Y(Z) \) can be factorized into the product of the smooth yield \( Y_{\text{sm}}(Z) \) (without staggering effects) multiplied by a staggering factor. In heavy-ion reaction, where \( Y_{\text{sm}}(Z) \) is unknown, only a local staggering can be determined. The main, essential assumption is that the staggering varies with \( Z \) in a sufficiently gradual way to be considered – to a good approximation – constant over an interval of a few \( Z \) values.

In this paper we examine some methods that could be used to put into evidence the presence of the staggering and to quantitatively estimate the magnitude of the phenomenon, without relying on an a priori knowledge of the yield distribution. These methods are briefly described in sect. 2 and the simulation in sect. 3. Section 4 presents the results and a comparison of the different methods.

We note that some of the methods presented in this paper are commonly used for studying odd-even staggering on other nuclear quantities, the most notable of all being nuclear masses and binding energies (see e.g. \[ [36,37] \] and references therein). The distinctive feature of fragment yields is that odd-even effects are here a perturbation superposed on a quantity (the yield) that, depending on the reaction mechanism, may undergo large and rapid changes in magnitude over rather restricted regions of \( Z \).

#### 1. Linear interpolation (LIN or Y-2DI).

If the shape of the charge distribution is sufficiently smooth, the simplest estimate of \( Y_{\text{sm}}(Z) \) is the average between the original experimental distribution and the same distribution shifted, back and forth, by just one \( Z \) unit. This procedure tends to eliminate any periodical oscillation with period 2\( Z \). Practically, the distribution \( Y(Z) \), which is an estimator of the unknown distribution \( Y_{\text{sm}}(Z) \), can be obtained from the experimental one with a linear interpolation:

\[
\mathcal{Y}(Z) = \frac{1}{4}(Y(Z-1) + 2Y(Z) + Y(Z+1)). \tag{4}
\]
In Fig. 1a), $\tilde{Y}$ is estimated by means of the solid line joining the yields $Y_{Z-1/2}$ and $Y_{Z+1/2}$ (open diamonds) obtained by pairwise averaging the experimental data. For each $Z$, only the measured yields of three consecutive elements (from $Z-1$ to $Z+1$) are needed. The staggering parameter $|\Delta Z|$ is given by $|Y(Z)−\tilde{Y}(Z)|/\tilde{Y}(Z)$ and its effect is indicated by the arrows pointing from the experimental $Y(Z)$ (crosses) to $\tilde{Y}(Z)$ (circles). This method is similar to that applied by Zeitlin [18] and later by others [38,39] in high-energy fragmentation studies. Note that in his original version, Zeitlin took a reference for each given $Z$ the line (fine dotted in figure) joining $Y(Z-1)$ and $Y(Z+1)$: in this way he obtained an odd-even effect (distance from cross to triangle, shown only for $Z=7$) exactly twice as large as the present one (distance from cross to full circle).

2. Fit-Interpolation over five points (Y-2DIF). In this paper we propose a possible improvement of the previous method by taking into account also the yields of the two neighboring points $Y(Z-2)$ and $Y(Z+2)$. These five yields are pairwise averaged to produce four intermediate values $Y_{Z-3/2}, Y_{Z-1/2}, Y_{Z+1/2}$ and $Y_{Z+3/2}$ (open diamonds), which are used to perform a parabolic fit (short dashed line). The value of the fit in the central point $Z$ gives the estimated $\tilde{Y}(Z)$ and the difference with respect to the measured value is used to estimate $\Delta Z$.

3. Parabolic fit (Y-PAR). A new smoothing method was recently suggested in Ref. 23, based on a fitting procedure too. For each measured value of the yield $Y(Z)$, the smoothed value $\tilde{Y}(Z)$ is estimated by fitting a parabolic function,

$$\tilde{Y}(Z) = a Z^2 + b Z + c,$$

directly to the measured yields over five consecutive values of $Z$, from $Z-2$ to $Z+2$. Figure 1b) illustrates this method showing three curves, two for the interpolation around even $Z$ (upward parabola, red online) and the other one around an odd $Z$ (downward parabola, blue online). The effect of $\Delta Z$ is again indicated by the arrows pointing from the experimental $Y(Z)$ (crosses) to $\tilde{Y}(Z)$ (full circles). The three parabolas show the peculiarity of this method. Due to the presence of the staggering, the experimental points have an up-and-down behavior that is obviously not well fitted by a parabola. Therefore the estimated effect of staggering (arrows in figure) is enhanced. In fact, with respect to the true yields without staggering $Y_{\text{sm}}(Z)$ (open squares), the estimated values of $\tilde{Y}(Z)$ (full circles) lie systematically below for even $Z$ values, and above for odd ones. This amplifies the staggering effect, but does not make the method more sensitive, because also the statistical errors are enhanced by the same factor. At the same time one has to be aware that this amplification hinders a proper quantitative comparison with the results of other experiments, evaluated with other methods.

4. Parabolic fit with oscillations (Y-COS). In this paper we propose an improvement of the previous method by taking into account the fact that actually, due to staggering, the data oscillate around the assumed smooth behavior. The improvement is accomplished by using a fit function that consists of a parabola multiplied by any properly oscillating function with period $2Z$ and amplitude $d_0$, like for example

$$Y_{\text{fit}}(Z) = (a_0 Z^2 + b_0 Z + c_0) \left(1 + d_0 \cos(\pi Z)\right)$$

$$= Y(Z) \left(1 + d_0 \cos(\pi Z)\right).$$
Of course any other oscillating function (like a rectangular or a triangular one) with period 2Z and proper phase, would have done an equally good job; we choose \( \cos(\pi Z) \) because it is simple and easy to implement in an analysis code. At the cost of a non-linear fitting procedure and by adding a fourth parameter, one obtains a better and more sensible fit (in fact, \( Y_{\text{fit}}(Z) \) does now take into account the oscillation of the data). There is also an additional bonus: the \( \chi^2 \) acquires statistical significance and indeed its distribution agrees very well with that expected for the \( \chi^2 \) function with one degree of freedom. Two variants are possible: (a) deriving \( \Delta Z \) from the difference between the estimated \( Y(Z) \) and the measured \( Y(Z) \), or (b) directly using the fourth parameter of the fit \( \Delta Z \equiv d_0 \). We will call these two variants Y-COSa and Y-COSb, respectively. The result of just one fit in the five-point interval around the charge \( Z=7 \) is shown by the thin solid line in fig. 1(c). The thick dashed line shows the parabolic part \( Y(Z) \) in eq. (9). The estimate of \( \Delta Z \) obtained with variant (a) is indicated once more by the arrow pointing from the experimental \( Y(Z) \) (open cross) to \( Y(Z) \) (full circle). It is apparent that now the circle represents a much better estimate of \( Y_{\text{sm}}(Z) \) than in the previous method.

5. Log-Third difference method (LY-3DI). This is actually the oldest method. First proposed by Tracy \[2\], it has been widely used in the past and also in recent publications (see e.g. [22,19]). It relies on finite difference calculus (see Appendix A) and in particular it uses the third difference of the natural logarithm of the measured yield \( L(Z) = \ln(Y(Z)) \) over four consecutive \( Z \) values:

\[
D_{Z+1/2}^{(3)} = (-1)^{Z} \frac{1}{6} \{L(Z+2) - 3L(Z+1) + 3L(Z) - L(Z-1)\}. \tag{7}
\]

The basic assumption is that, without staggering effects, the shape of the logarithm of the yield, \( L(Z) \), over the considered interval of four \( Z \) values could be described by a parabola

\[
L(Z) = a_T Z^2 + b_T Z + c_T. \tag{8}
\]

The justification by Tracy was that the smooth behavior of \( Y_{\text{sm}}(Z) \) over a small interval of \( Z \) can be well described by a piece of a Gaussian. It is worth noting that this does not necessarily imply that the whole charge distribution needs to be Gaussian-shaped, as it was usually assumed in old-time low-energy fission studies. One can easily verify that with the functional form of eq. (8) (and with no staggering effects) the following identity holds

\[
D_{Z+1/2}^{(3)} \equiv 0. \tag{9}
\]

\[2\] All non-linear fits were performed by means of MINUIT (with successive calls to the routines MIGRAD, HESS and MINOS) and always reached convergence.

\[3\] Actually, as either sign is allowed for the coefficient \( a_T \) in eq. (9), the shape of \( Y(Z) \) is not always a piece of a Gaussian.

If now a fixed quantity \( D_Z \) is alternatively added to \( L(Z) \) for even \( Z \) and subtracted for odd ones, then one obtains \( D_{Z+1/2}^{(3)} \equiv D_Z \). One can easily recognize Wahl’s parametrization [14] of the staggering with \( D_Z \equiv \ln(F_Z) \), see eq. (3). The factor \((-1)^Z\) in eq. (7) takes care of the fact that for staggering the sign of \( D_Z \) is usually positive for even \( Z \) and negative for odd ones. We note also that the result is usually associated with the center of the interval and therefore, being \( D^{(3)} \) an odd finite difference, that center corresponds to a half-integer \( Z \) value. An illustration of this method is shown in fig. 2(a), where the arrows pointing from the four consecutive experimental values of \( Y(Z) \) (crosses) to the estimated smooth yields (open circles). The full lines are the assumed curves that best fit the open circles and the full circle gives the so estimated value of \( Y_{\text{sm}} \); see text for details. Note the expanded vertical scale with suppressed origin.

Fig. 2. (color online) Illustration of the methods (a) LY-3DI and (b) LY-4DI. Open squares and crosses are the same as in fig. 1. The effect of the estimated staggering is shown by the arrows pointing from the experimental data (crosses) to the reconstructed smooth yields (open circles). The full lines are the assumed curves that best fit the open circles and the full circle gives the so estimated value of \( Y_{\text{sm}} \); see text for details. Note the expanded vertical scale with suppressed origin.

The method LY-3DI by Tracy, applied to four consecutive points, is just a particular application of the mathemati-
ternal formalism of Finite Difference Calculus, as briefly illustrated in Appendix A. This formalism has been widely used and discussed in the literature in connection with nuclear structure features, like pairing effects in nuclear masses and binding energies (see e.g. [26,37]), but not so much in connection with the fragment production yields.

Actually, already the calculation for the first method, LIN, produces a 3-point equation that corresponds to the second-order difference in the fragment yields. For this reason we will henceforth address that method as Y-2DI: it represents the simplest useful implementation of this formalism (in fact, a first-difference 2-point formula would probably be too rough).

Other implementations are possible, but – to our knowledge – they were not considered for studying odd-even effects, at least not for what concerns the yields of reaction products. Applying eq. (A4) of Appendix A to the yields \( Y(Z) \) and eq. (A7) to their logarithms \( L(Z) \), one obtains additional methods that are briefly sketched hereafter.

6. Log-Second difference method (LY-2DI).

For sake of completeness, we mention that a linear method, similar to Y-2DI, could be applied also to the logarithm of the yields, leading to the second-difference expression

\[
D^2_Z = (-1)^2 \sum \left\{ -L(Z-1) + 2L(Z) - L(Z+1) \right\}
\]

which is again an estimator of \( D_Z \), with the superscript indicating the order of the finite difference. As in the case of Y-2DI, also this method uses three points.

7. Log-Fourth difference method (LY-4DI).

Much in line with the method of Tracy [2], in the present paper we consider of great interest also the fourth difference of \( L(Z) \) over five consecutive points:

\[
D^4_Z = (-1)^2 \sum \left\{ L(Z-2) - 4L(Z-1) + 6L(Z) - 4L(Z+1) + L(Z+2) \right\}
\]  

Again the assumption is that, without staggering, \( L(Z) \) is well described by eq. (8) and also in this case one obtains as \( D^4_Z \equiv 0 \). However, if a quantity \( D_Z \) is alternatively added to and subtracted from \( L(Z) \), one finds again \( D^4_Z \equiv D_Z \). With respect to the method if Tracy, this one uses five points (like all previous methods except Y-2DI and LY-3DI) and the staggering parameter \( D_Z \) can be more physically attributed to the integer \( Z \) value at the center of the considered interval. An additional advantage of LY-4DI over LY-3DI is that eq. (11) remains valid even if \( L(Z) \) has some cubic component (see Appendix A):

\[
L(Z) = aZ^3 + bZ^2 + cZ + d.
\]

An illustration of LY-4DI is shown in fig. 2(b), with the same meaning of symbols and lines as in fig. 2(a).

8. Third difference method (Y-3DI).

The difference of this method with respect to LY-3DI is that it uses the fragment yields (instead of their logarithms) of four fragments, giving a value of \( \Delta_Z \) that is again attributed to the half-integer intermediate value \( Z + \frac{1}{2} \). From eq. (A4) the staggering parameter \( \Delta_Z \) is obtained as

\[
\Delta^2_{Z+1/2} = (-1)^2 \sum \left\{ Y(Z+2) - 3Y(Z+1) + 3Y(Z) - Y(Z-1) \right\}
\]

\[
\Delta^4_{Z+1/2} = (-1)^2 \sum \left\{ Y(Z+2) - 4Y(Z+1) + 6Y(Z) - 4Y(Z+1) + Y(Z+2) \right\}
\]

9. Fourth difference method (Y-4DI).

The next implementation with five fragment yields, again based on the fourth finite difference of eq. (A4), reads:

\[
\Delta^4_Z = (-1)^2 \sum \left\{ Y(Z-2) - 4Y(Z-1) + 6Y(Z) - 4Y(Z+1) + Y(Z+2) \right\}
\]

This method has been extensively employed in nuclear structure studies of odd-even mass staggering [37].

Some comments are due, before beginning to examine the merits and drawbacks of the various methods that can be used to extract the staggering parameter \( \Delta_Z \) (or \( D_Z \)). First of all, it is not advisable to use still higher-order finite differences because they perform weighted averages over an increasing range of experimental points. One has to find a compromise between using more points to be less sensitive to higher derivatives of the smooth yield function, and using less points to be more sensitive to local variations of the staggering phenomena. We decided not to go beyond five points, which is the number commonly used also by the other methods based on fitting procedures.

We also point out that the methods based on fits and those using finite differences of the fragment yields adopt the staggering parametrization by Amiel-Feldstein [12,13,1], while those using finite differences of the yield logarithms find more natural to adopt that by Wahl [14]. We will see that this has only minor consequences.

A further dissimilarity is that the methods relying on fits do first try to extract, from the data themselves, the smooth yield \( Y(Z) \) (full circles) that best estimates the unknown \( Y_{nm}(Z) \). This in turn allows one to determine \( \Delta_Z \) from the comparison with the experimental yield:

\[
\Delta_Z = (-1)^2 \frac{Y(Z) - Y(Z)}{Y(Z)}
\]

(15)

(usually also Y-COSb gives \( \Delta_Z \) directly from the fit). In eq. (15) the factor \( (-1)^2 \) takes care again of the opposite signs for even and odd \( Z \) values.

On the contrary, in the methods using the finite difference formalism there is generally no need to compute \( Y(Z) \), because they give directly \( \Delta_Z \) (or \( D_Z \)), which is the quantity of physical interest. In figs. 2(a) and (b) the underlying \( Y_{nm}(Z) \) distributions have been reconstructed just for illustrative purposes: by subtracting the staggering effect from the experimental yields \( Y(Z) \) (crosses), one can deduce the smooth yields (open circles) and draw the curves (solid lines) corresponding to eqs. 8 and 12.
The two parametrizations by Amiel-Feldstein and Wahl are approximately equivalent in case of small staggering effects ($\Delta Z \ll 1$), being to first order $D_Z \approx \Delta Z$. In fact, taking the ratio between the yields for even and odd species in both parametrizations, one obtains the relationship

$$(1 + \Delta Z)/(1 - \Delta Z) = (F_Z)^2 \equiv (e^{D_Z})^2$$

and expanding $1 + 2\Delta Z + \mathcal{O}(\Delta Z^2) = 1 + 2D_Z + \mathcal{O}(D_Z^2)$.  

In case of larger staggering, a fair quantitative comparison of the results obtained with the two parametrizations may require corrections of higher order

$$D_Z = \Delta Z + \frac{1}{2} \Delta Z^3 + \frac{1}{4} \Delta Z^5 + \mathcal{O}(\Delta Z^7)$$

and

$$\Delta Z = D_Z - \frac{1}{3} D_Z^3 + \frac{2}{15} D_Z^5 + \mathcal{O}(D_Z^7)$$

3 Simulation

A comparison of the different methods, to find out their possible advantages and disadvantages, is best performed by means of simulations. In this way one can play with the smooth yield distribution $Y_{\text{sm}}(Z)$ and with the parameter $\Delta Z$, in order to produce a final distribution $Y(Z)$ (including statistical fluctuations) that simulates the experimental one. This allows one to test in how far the different methods are able to retrieve the original $Y_{\text{sm}}(Z)$ and the genuine odd-even effect $\Delta Z$. In the simulations for the present paper, the Amiel and Feldstein parametrization ($\Delta Z$) has been adopted and the results of the LY-2DI, LY-3DI and LY-4DI methods (that deliver $D_Z$) have been transformed by means of eq. (18).

The simulation proceeds in the following steps:

1. A smooth analytic function $f(x)$ is assumed and its evaluation for integer $x$ ($x \in \mathbb{N}$) gives the discretized smooth charge distribution $Y_{\text{sm}}(Z)$ without staggering.

2. Following the parametrization by Amiel and Feldstein, the charge distribution with staggering is obtained as

$$Y(Z) = Y_{\text{sm}}(Z) (1 + (-1)^Z \Delta Z),$$

where in principle $\Delta Z$ could even be a slow-varying function of $Z$.

3. A proper normalization converts the yield $Y(Z)$ into the number of counts $N_c(Z)$ and then fluctuations are added to $N_c(Z)$, thus producing an experimental charge distribution that simulates the outcome of an experiment. Statistical fluctuations are produced by means of the random number generator Ranlux [11]. In principle they should be Poisson-distributed, but the normalization is chosen to give values of $N_c(Z)$ large enough, so that the fluctuations can be extracted from a Gaussian distribution with $\sigma^2(Z) = N_c(Z)$.

4. All the described smoothing methods are applied to the experimental distribution to obtain values of $\Delta Z$ as a function of $Z$.

Steps 3 and 4 are repeated for a preset number of times, which in our case is always $10^4$.

5. The propagation of the fluctuations on the individual yields, including all possible correlations and non-linearities of the methods, produces – for each $Z$ – a distribution of $\Delta Z$ that can be well fitted with a Gaussian. The centroids $(\Delta Z(Z))$ give the distribution of the average result (over $10^4$ distributions) for each method.

6. The results obtained with the different methods are finally plotted and examined in detail.

With reference to the above step 5 we emphasize that our procedure is equivalent (as we have easily verified with the simulation) to adding up all $10^4$ replicas and applying the various methods to the summed distribution (which has a huge statistic). The advantage of our procedure is that the widths of the Gaussians, $\sigma_{\Delta Z}$, give the uncertainty on $\Delta Z$ – at the 1σ level – expected in the analysis of one single experimental distribution. This is exactly the meaning that will be given to the dashed lines in the panels (b) to (h) of fig. 3 and to the error bars in the panels (b) to (m) of figs. 3 to 7.

Our final aim is to compare the behavior of the different methods of sect. 2 in relation to both magnitude of the staggering effect and shape of the charge distribution, as well as to assess the robustness of the result with respect to the statistics of the collected data.

For what concerns the last point, to avoid that intrinsic differences among the various analysis methods are obscured by fluctuations, the relative errors on the yields should be kept appropriately small. In the simulation, where the fluctuations are just of statistical nature, it is enough to use a reasonably large number of counts $N_c(Z)$.

One should be aware that the fluctuations on one $Z$ affect the estimated value of the staggering parameter $\Delta Z$ also for neighboring charges, and this happens in quite a complicated and correlated way that is difficult to estimate without a simulation. In fact, while in a real experiment one can analyze just one single realization of the charge distribution, a big advantage of the simulation is that one can generate a large number of replicas of the same charge distribution (differing from each other only in the random fluctuations) and all correlations come naturally out from the simulation itself.

Finally, the simulation allows one to choose freely the shape of $Y_{\text{sm}}(Z)$ in order to see the response of the different methods to the shape of the charge distribution, with special attention to its regions of non-linearity.

4 Results

The first thing that we checked is that indeed all methods are unbiased, i.e. in absence of staggering they give $\Delta Z = 0$ within errors. This is shown in fig. 4 where we consider constant charge distributions, without staggering, with a statistic of 200, 500 and 1000 counts per $Z$ unit (first, second and third columns, respectively).
The main conclusions that can be drawn from fig. 3 are that (a) fluctuations in the charge distribution are amplified by all methods and the values of \( \Delta Z \) for neighboring \( Z \) become strongly correlated; (b) good statistic (and reduction of other sources of fluctuations) is therefore mandatory for any experiment addressing this topic; (c) structures or trends in \( \Delta Z(Z) \) should be interpreted with great caution, after a careful and realistic estimate of their statistical significance.

In the remaining of this paper we will adopt a constant and representative value \( \Delta Z = 0.1 \). In fact we played with possible variations of the staggering magnitude and of its charge dependence, but nothing noteworthy was found. On average (or, equivalently, in a single realization with very small relative errors) almost all methods do an excellent job when the staggering \( \Delta Z = 0.1 \) is applied to a charge distribution that either stays constant, or rises or falls in a linear way, whatever the value of its slope. The only exception concerns the method Y-PAR that, with respect to the other methods, amplifies the staggering by about 40% (see Appendix B for more details).

Apart the trivial (and unrealistic) case of a linear charge distribution, we now have to investigate how the different methods behave in regions where the distribution \( Y_{sm}(Z) \) is highly not linear (namely it presents an appreciable curvature or has an inflection point) on the basis of some representative cases that are shown in figs. 4 to 7.

The figures are organized as follows. The first panel (a) always shows the yield distribution, while the remaining panels, from (b) to (m), present the result \( \langle \Delta Z \rangle \) obtained with the various methods and averaged over \( 10^4 \) replicas of the charge distribution (from now on we will omit the \( \langle \rangle \) for simplicity). The results in the first three rows (panels from (b) to (g)) are based on the second, third and fourth finite differences, applied either directly to the yields (Y-2DI, Y-3DI and Y-4DI in the left columns), or to their logarithms (LY-2DI, LY-3DI and LY-4DI in the right columns). The results in the last two rows (panels

---

![Fig. 3. (color online) Effect of statistical fluctuations for three constant charge distributions with 200, 500 and 1000 counts per Z unit (first, second and third column, respectively), without staggering (\( \Delta Z = 0 \)). Panels in first row (a): relative statistical fluctuations, \( \langle Y(Y) \rangle / \langle Y \rangle \), generated by three different sequences of random numbers (full, red dashed and dotted lines), that applied to the constant distributions produce three experimental realizations. Panels in rows from (b) to (h): average values (full dots) of the parameter \( \Delta Z \) evaluated over \( 10^4 \) distributions with some of the methods illustrated in this paper, compared with \( \Delta Z \) obtained for each of the three realizations (see text). First row, fig. 3(a), the continuous, dashed (red online) and dotted lines show the relative errors \( \Delta Y / Y \) obtained by applying the same three sequences of random errors (step 3 of sect. 3) to the above mentioned constant distributions: in fact the structure is the same, the relative errors being reduced just because of the increasing number of counts. The horizontal dashed lines show the \( \pm 1 \sigma \) error band expected for just one realization and correspond to relative errors \( \Delta Y / Y \) of about \( \pm 7 \% \) for \( N_c = 200 \), down to about \( \pm 3 \% \) for \( N_c = 1000 \). This means that in one single realization of the charge distribution (i.e. in one measurement), for each measured \( Z \) there is still a probability of \( \approx 32 \% \) that its measured yield lies outside this band.

These relative errors may seem rather small, but in the evaluation of \( \Delta Z \) they are amplified by all examined methods. In fact propagating these errors from the yields to \( \Delta Z \) leads to the \( \pm 1 \sigma \) band indicated by the horizontal dashed lines in rows (b) to (h) of fig. 3 its width amounting for all methods to about \( \pm 0.04 \) when \( N_c = 200 \) and \( \pm 0.02 \) when \( N_c = 1000 \).

But this is not the worst part of the story. The impact of these fluctuations on the estimation of \( \Delta Z \) is presented in the lower panels of fig. 3 rows from (b) to (h). One clearly sees that taking the average (step 5 of sect. 3) over \( 10^4 \) generated replicas, differing only in the fluctuations, gives \( \Delta Z = 0 \) with all methods (full dots). However, the correlated propagation of the fluctuations of neighboring charges gives rise to rather large and nonphysical structures (full, dashed and dotted lines), with different and often opposite phases in different realizations: one realization may have a bump where another one has a valley. It is also worth noting that the spurious bumps (or valleys) depend only on the particular set of random fluctuation and appear to be strictly in phase for all analysis methods that can be applied.

---

Similar absolute uncertainties occur also in presence of staggering and cannot be overlooked: for \( \Delta Z = 0.1 \) they correspond to an uncertainty of \( \approx \pm 40 \% \) (20%) for \( N_c = 200 \) (1000).
Fig. 4. (color online) Upper panel (a): charge distributions shaped as vertical parabolas (pairwise with the same curvature). Open dots show a constant staggering $\Delta Z=0.1$ applied to the smooth dashed curves. Panels (b)-(m) present the average $\langle \Delta Z \rangle$ obtained with the methods Y-2DI, LY-2DI, Y-3DI, LY-3DI, Y-4DI, LY-4DI, Y-2DIF, Y-PAR, Y-COSa and Y-COSb, respectively, compared with the nominal input value (dotted lines). The error bars are the $\pm 1\sigma$ widths of the $\Delta Z$ distributions.

from (h) to (m)) always refer to the methods based on various types of fits to the yield distribution (Y-2DIF, Y-PAR, Y-COSa and Y-COSb). Note that the vertical scales for the methods Y-2DI and LY-2DI (panels (b) and (c)) and for Y-2DIF and Y-PAR (panels (h) and (i)) span a range about twice as large as that for the other methods. As already anticipated, the error bars represent the statistical uncertainty (at the $\pm 1\sigma$ level) for the analysis of just one realization of the distributions shown in the panels (a).

Figure 4 shows six examples (three maxima and three minima) of charge distributions shaped as vertical parabolas $y(x) = a(x - x_0)^2 + c$. They have pairwise the same curvature, but with opposite sign ($a = \pm 200$ for the first pair, $\pm 400$ for the second and $\pm 600$ for the third one), while the value in the vertex is always the same, $c = 20000$. With a staggering parameter of $\Delta Z=0.1$, the enhancement (reduction) of even (odd) charges $Z$ near the vertex is always the same, i.e. $+(-)2000$ counts (step 2 in sect. 3).

As expected, the simple Y-2DI and LY-2DI, shown in panels (b) and (c), are not well suited for distributions with an appreciable curvature and indeed they give the worst results. For what concerns the remaining finite-difference methods they all behave rather well. The method LY-4DI in (g) gives some improvement over LY-3DI, the method by Tracy shown in (e), especially for the downward parabolas at the two extremes of the usable $Z$ range (where the statistic is very low). However the linear methods Y-3DI (d) and Y-4DI (f) seem to be slightly better.
For the methods based on fits to the data, the addition of two more points to Y-2DI that is necessary to perform a fit (method Y-2DIF in (h)) certainly improves the results, especially for the smallest curvatures. In the Y-PAR method, panel (i), one sees the already mentioned amplification of about 40% and the intrinsic difference of even and odd Z (already reduced by treating the errors as explained in Appendix B). Y-PAR gives similar results for all curvatures (as expected from its being based on a parabolic fit), but it certainly does not represent a real improvement over Y-2DIF. Much better results are definitely obtained by the two new methods, Y-COSa and Y-COSb (based on the fit with a properly oscillating function).

The situation is in general even worse for inflection points, which make more trouble for nearly all methods. In fig. 5(a) we present six charge distributions that differ either in the slope at the inflection point or in the distance between the minimum and the maximum. The distributions are obtained from cubic functions without quadratic term, \( Y(Z) = a(Z-Z_0)^3 + b(Z-Z_0) + c \), all with the same statistic at the inflection point \( (c = 20000) \). The slope at the inflection point is \( b = 1000, 2000 \) and \( 3000 \) for the first, second and third pair of distributions, respectively, while the cubic coefficient is \( a = -50 \) for the first and \( -150 \) for the second curve of each pair.

The lower panels of fig. 5, from (b) to (m), present the average results for the different methods. The methods Y-2DI and LY-2DI of panels (b) and (c) are certainly not acceptable, because they give systematic deviations up to \( \pm 0.05 \) with respect to the nominal value of \( \Delta_Z = 0.1 \). Of the remaining finite-difference methods, Y-4DI and LY-4DI of panels (f) and (g) are now very remarkable improvements over Y-3DI (d) and LY-3DI by Tracy (e), because they reduce the deviations from the nominal value down to less
Yield a) \( Y(10^3 \text{ counts}) \).

Cross section with staggering smooth input cross section \( D \) averaged over \( 10^4 \) distrib.

Panels (b)-(m) present \( \Delta Z \), as obtained with the methods discussed in this paper, compared with the nominal value of 0.1 (dotted lines). For each \( Z \), the error bar indicates the \( \pm 1\sigma \) width of the \( \Delta Z \) distribution obtained from \( 10^4 \) simulations.

Fig. 6. (color online) Upper panel (a): high-statistic charge distribution based on a natural cubic spline (smooth dashed line); the thin vertical lines indicate the internal knots (see text). Open dots display the effect of a constant staggering \( \Delta Z = 0.1 \). than \( \pm 0.005 \). This does not come totally unexpected, because methods based on the fourth differences should be able to remove smooth terms up to the third order (see Appendix A).

The fit method Y-2DIF of panel (h) represents again an appreciable improvement over the linear method Y-2DI (b), as it cuts the systematic deviations down by about a factor of two, but the method Y-PAR of panel (i), in spite of its amplification and systematic even-odd effect, performs even better than Y-2DIF. What comes rather unexpected is that the methods Y-COSa (l) and especially Y-COSb (m) perform rather well, in a way comparable with Y-4DI (f) and LY-4DI (g). Altogether, the best results among all those displayed in fig. 5 are obtained with the methods Y-4DI and Y-COSb.

The high statistic adopted for the smooth distribution \( Y_{sm}(Z) \) (with relative errors lower than \( \pm 1\% \) in the non-linear regions) simply makes the presentation of the data in fig. 4 and 5 more readable by reducing the error bars, but obviously does not appreciably change the response of the methods, averaged over many distributions. Of course, for a single realization (or experiment) a large statistic of the charge distribution remains of paramount importance.

\[ \text{For example, one sees from eqs. (13), (14) or (15) that a common factor } k \text{ multiplying all yields cancels out in the average } \langle \Delta Z \rangle; \text{ and it cancels out also in } \langle D_Z \rangle \text{, when } \langle \ln(kY) \rangle = \ln(k) + \langle \ln(Y) \rangle \text{ is inserted in eqs. (7) or (11).} \]
A second point worth noting is that the deviations from the nominal value displayed in fig. 4 and 5 are characteristic of each method and of the specific non-linearities of the smooth function \( Y_{sm} \). The simulation also shows that in general the deviations from the nominal value of the staggering parameter are very weakly dependent on the chosen value of \( \Delta Z \). For methods based on the yield logarithms, one can even demonstrate that these deviations do not depend on it. Considering for example the case of LY-3DI, if \( L(Z) \) has the form of eq. (12) with a cubic coefficient \( a \), then eq. (7) gives \( DZ + \frac{3}{4}a \) (instead of \( DZ \)), thus with a constant deviation from the nominal value that amounts to \( \frac{3}{4}a \) and hence does not depend on \( DZ \). Therefore, the choice of a good method is important, especially when the average staggering effects are small.

We tested several other shapes of the charge distribution, finding that in all cases, irrespective of the specific shape, the ranking of the various methods remains similar to that found in figs. 4 and 5.

We show just one case in figs. 6 and 7, where a more realistic charge distribution – open dots in panels (a) – mimics the typical features found in semi-peripheral collisions of intermediate-energy heavy-ion reactions: with increasing \( Z \) there is a very rapid drop of the cross section from the lightest complex fragments with \( Z \approx 4 \) down to a wide valley of intermediate mass fragments (IMF) around \( Z \approx 10 – 15 \), followed by a slow rise toward heavier ones (which may represent either fission-like or projectile-like fragments) until it finally fades away.
This shape was obtained with a natural cubic spline passing through seven equally spaced knots (the five internal ones are indicated by thin vertical lines in the figures): it is a piece-wise interpolation with 3rd degree polynomials that is continuous in the joining knots up to the second derivative. The adjective natural indicates the additional condition that the second derivative is zero in the two external knots.

In fig. 3 the statistic for each of the 10^4 generated replicas is very large: in the IMF valley it corresponds to at least ten thousand counts for each charge Z. The lower panels, from (b) to (m), illustrate again the estimates of \( \Delta Z \) obtained with the various methods already shown in the previous figures. The full dots represent the average obtained over 10^4 simulations, while the error bars, often smaller than the symbol sizes, give the widths of the distributions for each Z. Again dashed lines at \( \Delta Z = 0.1 \) indicate the nominal value.

Also here we observe an evident failure of Y-2DI, LY-2DI and Y-2DIF in the minimum of the IMF valley and in the final tail of the distribution. Of all finite-difference methods, those based on the differences perform better (Y-4DI even slightly better than LY-4DI), but comparatively good results are obtained with the fitting methods Y-COS and particularly with the second one, Y-COSb.

Concerning the finite difference methods, one might think that fragment cross sections that vary over many orders of magnitude are better studied with the logarithmic n-point differences. The present study does not lend much support to this idea, as the results of linear and logarithmic methods presented in figures from 4 to 6 do not differ too much from each other. For the second-difference results of panels (b) and (c), one might have a slight preference for the logarithmic methods, but the choice is hard. The situation for the third and fourth differences (panels (h) vs. (i) and (j) vs. (m), respectively) seems to be slightly in favour of the methods that use directly the yields. In fact, the logarithmic methods seem to suffer more in regions where the yield is lower, see e.g. the IMF valley of fig. 6 or the sides of the downward parabolas in fig. 4.

In order to stress the importance and the effects of the statistic, fig. 7 presents results for the same charge distribution of fig. 3 but in the case of low statistic: about 300 counts per charge Z in the IMF valley. The average behavior is substantially the same as in the previous figure. In fact, for each of the presented methods, the full dots are practically the same in both figures, their differences being much smaller than the symbol sizes. The main difference concerns the ±1σ fluctuations (error bars) expected for a single realization (or measurement), which are of course much larger in fig. 7.

In addition, two examples of the results obtained for two individual random realizations of the charge distribution are shown by the open dots/squares, joined by dashed/dotted lines in the panels (b)-(m) of fig. 7. Here the effects of the statistical fluctuations are clearly visible for all methods and reveal marked deviations from the average behavior, with several points located outside the error bars. What is even worse is the fact that the fluctuations give rise to structures involving several neighboring points, a consequence of their correlation (see next section). The location and the amplitude of these non-physical structures present large random variations from one single realization to the other, being sometimes in-phase and sometimes out-of-phase in the two realizations. This means that in a single experiment, unless the independent relative errors, including the statistical ones, are very small (below the 1% level), the physical significance of structures in the staggering parameter \( \Delta Z \) should be treated with great caution.

5 Correlations

The need of estimating the smooth behavior of the charge distribution from a few data points, naturally introduces short-range autocorrelations among the staggering parameters \( \Delta Z \) pertaining to neighboring charges Z. The degree of autocorrelation is estimated with the usual definition

\[
R(Z', Z'') = \frac{\langle (\Delta Z' - \overline{\Delta Z'}) (\Delta Z'' - \overline{\Delta Z''}) \rangle}{\sigma_{\Delta Z'} \sigma_{\Delta Z''}} ,
\]

(19)

where the overlined \( \overline{\Delta Z} \)'s are the mean values of the staggering parameters for two different charges \( Z' \) and \( Z'' \), the brackets \( \langle \cdot \rangle \) indicate the expectation value of the product of the deviations of the two \( \Delta Z \) from their mean values and the \( \sigma \)'s in the denominator are their standard deviations.

By construction \( R(Z', Z'') \) lies in the range \([-1, 1]\), the two extremes indicating perfect correlation or anticorrelation, respectively (while 0 indicates uncorrelated data).

The correlation factor \( R \) cannot be estimated from a single experiment, but it is easily obtained in a simulation where many replicas of a certain charge distribution are generated, differing only in the fluctuations. As expected, fluctuations introduce a correlation that is positive for two nearby located \( Z' \) and \( Z'' \) values and 0 for distant ones.

The correlation factor is practically independent of the specific shape of the analyzed charge distributions. For a flat distribution with a constant \( \Delta Z \) of 0.1, the region of positively correlated neighboring Z values is obviously at most three charge units wide for the methods Y-2DI and LY-2DI (with FWHM values of 2.37±0.03) and increases to four charge units for Y-3DI and LY-3DI (with FWHM values of about 2.84) and to five charge units for all remaining methods (with a common FWHM of 3.24).

In conclusion, structures possibly found in the staggering analysis that are narrower than the above numbers are highly suspicious and probably they should not be given much physical significance.

6 Conclusions

The odd-even staggering is a widespread phenomenon that usually consists in a relative enhancement of the yields of nuclei with even values of proton (neutron) number \( Z (N) \) with respect to the neighboring nuclei with odd values. All
methods are based on the idea, first proposed by Amiel and Feldstein [12,13], that the measured yield $Y(Z)$ can be factorized into the product of a smooth yield $Y_{sm}(Z)$ times a factor $(1 \pm \Delta Z)$, where the sign `+' holds for even charges and the sign `-' for odd ones.

This is an assumption, widely accepted and probably justifiable on the basis of theoretical arguments, but it remains an assumption and one should definitely remain aware of that.

We have reviewed some old methods and we propose some new methods of extracting the staggering parameter $\Delta Z$ (or $D_Z$) from the experimental charge distributions. Some methods first estimate the underlying smooth behavior of the yield $Y(Z)$ from the data themselves, to derive then the staggering from the difference between the measurement and the smooth yield, while others deliver a direct estimate of the staggering parameter. Some methods use a fit to the data, others take advantage of the finite-difference mathematical formalism.

Almost all methods perform equally well if the charge distribution has a predominant linear behavior, while they greatly differ from each other in the presence of strongly non-linear regions in the charge distribution. So steepness is not an issue, not even for fragment yields. From the present study, using logarithmic n-point differences for fragment yields is just an option, not a necessity.

All methods are extremely sensitive to the fluctuations that may affect the yields of neighboring $Z$ in the experimental data, giving origin to spurious structures without much physical meaning. Therefore for this kind of studies it is of paramount importance to acquire good data with very large statistic and very small independent relative errors, so that the relative random fluctuations for each measured $Z$ are of the order of about 1% or better: only when this condition is fulfilled, one can really appreciate the different quality of the various methods. On the contrary, if the fluctuations are sizably larger, spurious effects appear, and it makes no much sense to choose one method rather than another: the results will be anyhow poor and hard to interpret in a sensible way.

We find that linear interpolations (Y-2DI and LY-2DI) that make use of three neighboring points give the worst results among all examined methods. Extending the linear interpolation with a fit that uses five points (Y-2DIF) brings only some moderate improvement. The recently proposed fit of a simple parabola over five points (Y-PAR) does not represent a real improvement over Y-2DIF. The method Y-PAR presents the peculiarity that it amplifies the signal, but it needs a careful treatment of the errors and even then it does not produce a really smooth estimate of the distribution without staggering, $Y(Z)$.

Very good results are given by the newly proposed method (Y-COS) that uses a five-point fit with a properly oscillating function. As it implies a non-linear fit, a reliable recursive procedure has to be used, such as the well known MINUIT code [40] developed in CERN. Using directly the coefficient of the oscillating term (method Y-COSb) as delivered by the fit seems to be one of the best ways to estimate the staggering parameter $\Delta Z$.

A completely different approach is the oldest one by Tracy, based on the third (LY-3DI) finite differences over four values that are the logarithms of the fragment yields. At variance with the previous methods, it delivers directly an estimate of the staggering parameter, although with a slightly different parametrization first proposed by Wahl [13]. We show that a similar method (Y-3DI) could be applied to the fragment yields, using the parametrization by Amiel [12,13]. In both cases the staggering value, that is usually attributed to the center of the considered interval, corresponds to a half-integer $Z$. These two methods give good results, but not as good as those of Y-COSb.

Further exploiting the Finite Difference formalism, in this paper we propose two more methods, Y-4DI and LY-4DI, that use the fourth finite differences over five points. They give somewhat better results than the original LY-3DI because they are insensitive to possible cubic components in the smooth charge distribution, and probably also because they use one more point and hence they are slightly less sensitive to fluctuations. Moreover, one practical (and aesthetic) advantage is that the results can be attributed to the integer $Z$ values at the center of the five-point intervals.

Concluding, to avoid the appearance in $\Delta Z(Z)$ of structures without much physical significance i) one should collect data with a very large statistic; ii) all other uncertainties that may affect the yields of neighboring $Z$ in an independent way should also be carefully estimated and reduced; iii) once the previous conditions are met, the methods which seem less sensitive to non-linearities of the underlying smooth distribution $Y_{sm}(Z)$ are the methods Y-4DI and Y-COSb, closely followed by LY-4DI and Y-COSa.

Appendix A

The Finite Difference Calculus (see e.g. [42,43,44]) is a widely used mathematical formalism dealing with finite increments of the independent variable(s) of mathematical functions. In particular, here we consider a function $y(x)$ whose values are known for a series of equidistant values of the independent variable $x$: such is the case of the fragment yield $Y(Z)$ that is defined only for integer values of $Z$.

The known values of the function form a sequence $y_k$, where the subscript $k = 0, 1, 2, \ldots$ indicates the number of equal increments with respect to a given reference value $x_0$ (in the general case $k$ can take negative integer values too).From the sequence $y_k$ one can build the first difference
Identifying the fragment yield $Y(Z)$ with $y_k$ (and with a trivial index redefinition) these equations generate all the methods of Sect. 2 that do not make use of a fit to the yield data. For example, from eq. (A7) with $n = 3$, using the definition of eq. (A1), one obtains the method LY-3DI of eq. 3, first proposed long time ago by Tracy [2]; and with $n = 2 (4)$ one obtains the method LY-2DI (LY-4DI). Similarly, from eq. (A4) with $n = 2, 3$ and 4, one obtains the methods Y-2DI, Y-3DI and Y-4DI, respectively.

Appendix B

With respect to the others, the method Y-PAR amplifies the estimated $\Delta Z$ by about 40% above the nominal value. This has to be taken into account when quantitatively comparing the results of this method with those of the others. A second characteristic intrinsic in the Y-PAR method is that this amplification is not the same for even and odd charges. It is somewhat larger for even charges (usually enhanced by staggering) than for odd ones, so that even in case of constant input values of the yield $Y_{sm}(Z)$ and of the parameter $\Delta Z$, one does not obtain a really flat $\Delta Z$ distribution, even for an input value of $\Delta Z=0$ (see full dots in panels of row (d) in fig. 5).

This effect is connected with the weights used in the weighted least squares fit procedure. When the smoothed

![Graph](image-url)

Fig. 8. (color online) Effect of applying a weighted least square fit in the Y-PAR (a) and Y-COS (b) methods, with the weights taken from the statistics of the data. The original distribution (squares) is flat with a statistic of 1000 counts; a $\Delta Z=0.1$ staggering effect (crosses) is superimposed on it. In the Y-PAR method the differing statistical errors for even and odd $Z$ induce an artifact that estimates a larger staggering for even $Z$ than for odd ones (see text).
yield $Y(Z)$ is deduced by means of a fit (like in the Y-PAR and Y-COS methods), some care must be put in the treatment of the errors on the measured yields $Y(Z)$. In fact the weighted least square fit method may introduce a subtle asymmetry between even and odd $Z$ values when the weights mainly depend on the number of counts measured for each $Z$. In this case, even $Z$ yields (which are enhanced) have systematically somewhat larger errors (and hence smaller weights) than odd $Z$ yields.

This causes an artifact in the Y-PAR method because the parabola is not the appropriate function to fit the five ‘zigzagging’ experimental points. This is shown in fig. 8(a), where a constant yield of 1000 counts with $\Delta Z=0.1$ is assumed, leading to statistical weights of about $1/\sqrt{1100}$ and $1/\sqrt{900}$ for the even and odd $Z$, respectively. With their larger weights, the three reduced yields (upper crosses) of an odd $Z$ are more effective in shifting the parabola downward than the three enhanced yields (lower crosses) of an even $Z$ in shifting it upward, due to their smaller weights: as a consequence, even $Z$ values appear to have a larger staggering than odd ones. We stress that this effect cannot be cured by increasing the statistic of the measurement. It is worth noting that, on the contrary, the fits of the Y-COS method shown in fig. 8(b) are not affected by this problem, because this method uses a properly oscillating fit function that well reproduces the data.

The open squares of fig. 9 show the effect on $\Delta Z$ when the errors in Y-PAR are estimated directly from the data, in case of the charge distribution already presented in fig. 7(a). This artifact may be cured either by recurring to the unweighted least square method (which is however unsatisfactory when the yields strongly vary with $Z$) or performing appropriate averages of the weights before fitting the data. In this paper we always used as statistical weights for the Y-PAR method the results of a preceding analysis performed with the Y-2DI linear method. This artifice proved essential to obtain results (full dots in fig. 9) that could be meaningfully compared with the other methods.

We wish to warmly thank Prof. P.R. Maurenzig for continuous support, useful suggestions and very fruitful discussions.
25. M. D’Agostino, M. Bruno, F. Gulminelli, L. Morelli, G. Baiocco, L. Bardelli, S. Barlini, F. Cannata, G. Casini, and E. Geraci et al., Nucl. Phys. A, 861:47, 2011.
26. G. Ademard, J. P. Wieleczko, J. Gomez del Campo, M. La Commara, E. Bonnet, M. Vigilante, A. Chbihi, J. D. Frankland, E. Rosato, and G. Spadaccini et al., Phys. Rev. C, 83:054619, 2011.
27. I. Lombardo, C. Agodi, F. Amorini, A. Anzalone, L. Auditore, I. Berceanu, G. Cardella, S. Cavallaro, M. B. Chatterjee, and E. De Filippo et al., Phys. Rev. C, 84:024613, 2011.
28. M. D’Agostino, M. Bruno, F. Gulminelli, L. Morelli, G. Baiocco, L. Bardelli, S. Barlini, F. Cannata, G. Casini, and E. Geraci et al., Nucl. Phys. A, 875:139, 2012.
29. G. Casini, S. Piantelli, P. R. Maurenzig, A. Olmi, L. Bardelli, S. Barlini, M. Benelli, M. Bini, M. Calviani, and P. Marini et al., Phys. Rev. C, 86:011602(R), 2012.
30. S. Piantelli, G. Casini, P. R. Maurenzig, A. Olmi, S. Barlini, M. Bini, S. Carboni, G. Pasquali, G. Poggi, A. A. Stefanini, and S. Valdré et al., Phys. Rev. C, 88:064607, 2013.
31. M. Colonna and F. Matera. Phys. Rev. C, 71:064605, 2005.
32. R. Raduta and F. Gulminelli. Phys. Rev. C, 75:044605, 2007.
33. M. Huang, Z. Chen, S. Kowalski, Y. G. Ma, R. Wada, T. Keutgen, K. Hagel, M. Barbui, A. Bonasera, and C. Bottosso et al., Phys. Rev. C, 81:044620, 2010.
34. Jun Su, Feng-Shou Zhang, and Bao-An Bian. Phys. Rev. C, 83:014608, 2011.
35. B. Mei, H. S. Xu, X. L. Tu, Y. H. Zhang, Yu. A. Litvinov, K.-H. Schmidt, M. Wang, Z. Y. Sun, X. H. Zhou, Y. J. Yuan, M. V. Ricciardi, and A. Kelic-Heil et al., Phys. Rev. C, 89:054612, 2014.
36. A. S. Jensen, P. G. Hansen, and B. Jonson. Nucl. Phys. A, 431:393, 1984.
37. D. Hove, A. S. Jensen, and K. Riisager. Phys. Rev. C, 88:064329, 2013.
38. G. Iancu, F. Flesch, and W. Heinrich. Radiat. Meas., 39:525, 2005.
39. J. X. Cheng, X. Jiang, S. W. Yan, and D. H. Zhang. J. Phys. G, 39:055104, 2012.
40. F. James. "MINUIT: Function Minimization and Error Analysis - Reference Manual". http://wwwasdoc.web.cern.ch/wwwasdoc/minuit/minmain.html, 2000.
41. M. Luescher. "A portable high-quality random number generator for lattice field theory calculations". Computer Physics Communications, 79, 1994.
42. G. Boole. A treatise on the calculus of finite differences. Cambridge University Press, New York, 2009. Reproduction of the original edition published in 1860.
43. C. Jordan. Calculus of finite differences. Schaum’s Outlines. Chelsea Publishing Comp., New York, 1950.
44. M.R. Spiegel. Calculus of finite differences and difference equations. McGraw-Hill, Inc., 1971.