Additive dimension and the growth of sets

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Dedicated to Konstantin I. Olmezov
(01.12.1995—20.03.2022),
the victim of this heinous war

Annotation

We develop the theory of the additive dimension \( \dim(A) \), i.e. the size of a maximal dissociated subset of a set \( A \). It was shown that the additive dimension is closely connected with the growth of higher sumsets \( nA \) of our set \( A \). We apply this approach to demonstrate that for any small multiplicative subgroup \( \Gamma \) the sequence \( |n\Gamma| \) grows very fast. Also, we obtain a series of applications to the sum–product phenomenon and to the Balog–Wooley decomposition–type results.

1 Introduction

1.1 General results on dimensions and the growth

Let \( G \) be an abelian group and \( k \) be a positive integer. A finite set \( \Lambda \subseteq G \) is called \( k \)-dissociated if any equality of the form

\[
\sum_{\lambda \in \Lambda} \varepsilon_{\lambda} \lambda = 0, \quad \text{where} \quad |\varepsilon_{\lambda}| \leq k, \quad \forall \lambda \in \Lambda
\]

implies \( \varepsilon_{\lambda} = 0 \) for all \( \lambda \). If \( k = 1 \), then \( \Lambda \) is called dissociated. Let \( \dim_k(A) \) be the size of the largest \( k \)-dissociated subset of \( A \) and we call \( \dim(A) := \dim_1(A) \) the additive dimension of \( A \). Clearly, \( \dim_k(A) \geq \log_{2k+1} |A| \) and for \( A = \{1, 2, \ldots, n\} \), say, one has \( \dim_k(A) \ll \log_{2k+1} |A| \).

One of the main ideas of this paper is to treat the ratios (below \( k \) is a large parameter)

\[
\frac{\dim(A)}{\log |A|} \quad \text{and} \quad \frac{\dim_k(A)}{\log_{2k+1} |A|},
\]

as some measures, which control the growth of our set \( A \) see, e.g., inequality (2) of Theorem 1.

The notions of dissociativity and dimensions appeared naturally in analysis, see \[33, 26\], as well as in additive combinatorics, see \[3, 14, 27, 35, 36, 45\]. Previously (see \[14, 26, 38, 45\]), we studied the case when the quantity \( \frac{\dim(A)}{\log |A|} \) is small. Namely, if a set \( A \) is additively rich in a sense (e.g., \( |A + A| \ll |A| \)), then \( \dim(A) \) is small comparable to \( \log |A| \) and thus this case can be treated as the “structural” one. Even the famous Polynomial Freiman–Ruzsa conjecture
from the structural theory of sets addition demands about the possibility to find a large subset \( A^* \) of any set \( A \) with small \( |A + A| \) such that \( A^* \) has small dimension \( \text{dim}(A^*) \).

In this paper we consider the opposite situation, studying the case when \( \text{dim}(A) \) is \textit{large}. Moreover, instead of developing the theory of sets with small sumset \( |A + A| \) we are interested in higher sumsets \( nA \) for large \( n \). We show that the additive dimension controls higher sumsets, as well as higher energies (see rigorous formulation in Section 2). This phenomenon continues the classical line of additive combinatorics which is connected with the Freiman Lemma [19], [46, Lemma 5.13] (also, see recent achievements in [47]). Nevertheless, in the Freiman Lemma we have another dimension, which is defined in terms Freiman’s isomorphisms, see [46, Section 5.3].

Our basic setting is the following. For any set \( A \subseteq G \) consider the growing sequence

\[
|A| \leq |2A| \leq \ldots \leq |nA| \leq \ldots
\]  

For example, if one takes a set \( A \) with small doubling, then sequence \((1)\) is somehow trivial: all sets \( nA \) have comparable sizes with \( A \). In general, sequence \((1)\) can be rather complex. Our first result (it is a combination of Lemmas 18, 20 below) shows that any set \( A \) has three stages of its “life”, i.e. there are three basic lower bounds for the sequence \( |nA| \) in terms of some dimensions (and it is possible that \( A \) does not grow at each stage).

\textbf{Theorem 1} Let \( G \) be an abelian group and \( A \subseteq G \) be a set. Then there is an absolute constant \( C > 0 \) such that for any \( n \) with \( n \leq C^{-1} \log |A| \) one has

\[
|nA| \geq |A| \cdot \left( \frac{\text{dim}(A)}{C \log |A|} \right)^{n-1}.
\]  

Now if \( C^{-1} \log |A| \leq n \leq \text{dim}(A)/4 \), then

\[
|nA| \geq \left( \frac{\text{dim}(A)}{4n} \right)^{n-1}.
\]  

Finally, for \( k = \text{dim}(A) \log \text{dim}(A) \) one has

\[
|\text{dim}_k^2(2) \log \text{dim}_k(A) A| \geq \exp(C^{-1} \text{dim}_k(A) \cdot \log \text{dim}_k(A)).
\]

Thus any set \( A \) grows if there is a nontrivial lower bound for \( \text{dim}(A) \), see stages \((2)\) and \((3)\). After that further growth is possible if for a certain \( k \) another dimension \( \text{dim}_k(A) \) can be estimated non–trivially in a sense, see bound \((4)\), as well as Lemma \((14)\) where the estimate \( \text{dim}_k(A) \cdot \log(k \text{dim}_k(A)) \gg \text{dim}(A) \) was obtained.

Our second structural result allows us to describe all sets such that sequence \((1)\) stops after some steps in a sense that \( |nA| \ll |A|^C \) for a certain constant \( C \geq 1 \). This is an important class of sets and previously the author considered in detail just the case \( n = 2 \), i.e. the family of sets \( A \) with \( |nA| \ll |2A| \) (we mention paper [39] as the first one in this direction). Combining Corollary \((22)\) and Theorem \((28)\) below, we obtain the following result. Given a set \( A = \{a_1, \ldots, a_n\} \subseteq G \), recall that the combinatorial cube \( Q(A) \) is \( \{0,1\} \cdot a_1 + \cdots + \{0,1\} \cdot a_n \).
Theorem 2  Let \( \mathcal{R} \) be a ring, \( A \subseteq \mathcal{R} \) be a set, \( k = \dim(A) \log \dim(A) \), \( K \geq 1 \) be a real number and \( \Lambda_k \) be a maximal \( k \)-dissociated subset of \( A \). Suppose that all numbers \( j \in [k] \) are invertible in \( \mathcal{R} \).

1a. If \( |nA| \leq |A|^K \) for \( n \ll K^2 \log^2 |A| \log(K \log |A|) \), then \( \dim_k(A) \leq K \log |A|/\log(K \log |A|) \).

1b. Now suppose that \( \dim_k(A) \leq K \log |A|/\log(K \log |A|) \). Then \( |nA| = |A|^{O(K)} \) for all \( n = \dim^O(1)(A) \).

2a. Further if \( \dim_k(A) \leq K \dim(A)/\log \dim(A) \), then \( \dim(Q(\Lambda_k)) \leq K \dim(A) \).

2b. If \( \dim(Q(\Lambda_k)) \leq K \dim(A) \), then \( \dim_k(A) \ll K \dim(A)/\log \dim(A) \).

Thus, roughly speaking, we have the following equivalences

\[
|nA| \leq |A|^K, \forall n \leq \dim^O(1)(A) \iff \dim_k(A) \sim_K \frac{\log |A|}{\log \log |A|} \iff \dim(Q(\Lambda_k)) \sim_K \dim(A),
\]

where \( k = \dim(A) \log \dim(A) \), and hence \( A \) has a very strong additive structure if \( \dim_k(A) \sim_K \frac{\log |A|}{\log \log |A|} \), i.e. if \( \dim_k(A) \) is rather small. This description of additively rich sets in terms of \( \dim_k(A) \) seems to be new.

In our next Section 4 and, partially, in Section 5 we develop this approach and obtain several relations between further variants of additive dimensions and another quantity, which is connected with the higher sumsets (namely, the quantity \( T_k(A) \) see formula (6) below or Section 2). These results are applied in the rest of the paper and we formulate just a simple consequence of Theorem 30 below.

Theorem 3  Let \( G \) be an abelian group, \( A \subseteq G \) be a set, \( \dim(A) := M \log |A| \).

1. Then there is \( \mu \) such that \( \mu \ll m \leq \dim(A)/2 \) such that \( T_m(A) \gg |A|^{2m} \exp(-\Omega(\dim(A))) \).

2. On the other hand, for any \( m \) such that \( \log |A|/\log M \ll m \leq \dim(A)/2 \) there exists a set \( A_* \subseteq A \) with \( T_m(A_*) \geq 2^{-m} T_m(A) \) and \( \dim(A_*) \leq \exp(O(M \log M)) \cdot \log |A| \).

1.2 Applications

We now describe some applications, which can be obtained via this method.

Multiplicative subgroups in the prime field is a classical theme of number theory see, e.g., book [26]. Basis properties of subgroups were studied in [8, 25, 26, 39] and in many other papers. For example, in [25] Theorems 2,5 the following result was obtained.

Theorem 4  Let \( p \) be a prime number, \( \varepsilon \in (0,1) \) be a real number, \( \Gamma < \mathbb{F}_p^* \) be a multiplicative subgroup, \( |\Gamma| \geq \frac{\log(p/|\Gamma|)}{\log(\log(p/|\Gamma|)+1)} \). Then

\[
n\Gamma = \mathbb{F}_p, \quad \text{where} \quad n = O_\varepsilon(\log^{2+\varepsilon}(p/|\Gamma|)).
\]

On the other hand, there are infinitely many primes \( p \) and \( \Gamma < \mathbb{F}_p^* \) such that \( n\Gamma \neq \mathbb{F}_p \), provided \( n = O(\log(p/|\Gamma|)) \text{ if } |\Gamma| \geq \log(p/|\Gamma|) \), and

\[
n = O(\log^{1-\varepsilon}(p/|\Gamma|)) \quad \text{if} \quad |\Gamma| \geq \log^{C}(p/|\Gamma|) \quad \text{for any constant } C \geq 1.
\]
Theorem 4 gives an affirmative answer to a question of Heilbronn, see [24]. As Konyagin writes in [25] the conjectured upper bound for \( n \) in (5) is, probably, \( n = O_\varepsilon (\log^{1+\varepsilon} (p/|\Gamma|)) \).

In a natural way, studying multiplicative subgroups \( \Gamma \), the authors of papers [8, 25, 26, 39] applied upper bounds for exponential sums over such subgroups. For example, the upper bound for the Fourier coefficients of \( \Gamma \), which allows to obtain Theorem 4 is sharp in a sense but as we have said before the number \( n \) can be probably decreased. In our approach we do not use this machinery, connected with exponential sums, but rather good lower bounds for some additive dimensions of \( \Gamma \). Let us formulate our result, see Corollaries 40, 41 of Section 5.

**Theorem 5** Let \( p \) be a prime number and \( \Gamma < \mathbb{F}_p^* \) be a multiplicative subgroup.

1. Suppose that \( \varphi(|\Gamma|) \log |\Gamma| \geq \log p \) and \( |\Gamma| \leq (\log p)^C \), where \( C \geq 1 \) is an absolute constant. Then there is \( n = O(\log^2 p/\log \log p) \) such that \( |n\Gamma| \geq p^{\Omega(1/C)} \).

2. Further if \( |\Gamma| \leq \log p \), then for \( n = O(\varphi^2(|\Gamma|) \log |\Gamma|) \) one has \( |n\Gamma| \geq \exp (\log |\Gamma| \cdot \Omega(\min \{ \varphi(|\Gamma|), \log p/\log \log p \})) \).

Thus, say, for \( |\Gamma| \gg \log p \cdot \frac{\log \log p}{\log \log \log p} \) we obtain \( |n\Gamma| \gg p^c \) for a certain absolute constant \( c > 0 \) and \( n = O(\log^2 p/\log \log p) \). Actually, we show that the sequence \( n\Gamma \) grows almost optimally for small \( n \) and moreover, it is possible to estimate the energy

\[
T^+_n(\Gamma) := |\{ (\gamma_1, \ldots, \gamma_k, \gamma'_1, \ldots, \gamma'_k) \in \Gamma^{2k} : \gamma_1 + \cdots + \gamma_k = \gamma'_1 + \cdots + \gamma'_k \}|,
\]

see Corollary 40. For example, if \( |\Gamma| \sim \log p \), then we obtain for all \( n \)

\[
|n\Gamma| = \Omega \left( \frac{|\Gamma|}{n \log^2 |\Gamma|} \right)^n, \quad \text{and} \quad T^+_n(\Gamma) \leq |\Gamma|^n (C_* n \log^3 |\Gamma|)^n,
\]

where \( C_* > 0 \) is an absolute constant.

Any multiplicative subgroup is a set with small product set and in formula (7) or in Theorem 5 we have estimated some additive characteristics of our subgroup. This effect belongs to the wider and well-known sum–product phenomenon see, e.g., [46]. In Section 6 we consider the case when our set \( A \) belongs to a ring \( \mathcal{R} \) and we study dimensions \( \dim^+(A) \), \( \dim^\times(A) \), which are defined for both ring’s operations + and \( \times \). In particular, it is possible to obtain the following sum–product–type result.

**Theorem 6** Let \( A \subset \mathbb{Z} \) be an arbitrary finite set. Then

\[
\max \{ \dim^+(A), \dim^\times(A) \} \gg \log |A| \cdot \frac{\sqrt{\log \log |A|}}{\sqrt{\log \log \log |A|}},
\]

On the other hand, there is a set \( A \subset \mathbb{Z} \) such that

\[
\max \{ \dim^+(A), \dim^\times(A) \} \ll \log |A| \cdot \log \log |A|.
\]
Thus, surprisingly, both dimensions \( \dim^+(A) \), \( \dim^\times(A) \) can be rather small. Further, using some variations of the method, we improve a decomposition result from \cite{30} Corollary 1.3. The first Theorem on decompositions of a set onto two sets with small \( T_s^+ \), \( T_s^\times \) was obtained in \cite{2}.

**Theorem 7** Let \( A \subset \mathbb{Z} \) be a set and \( s \) be an integer parameter,

\[
s \ll \frac{\log |A|}{\sqrt{\log \log |A| \cdot \log \log \log |A|}}.
\]

Then there exist pairwise disjoint sets \( B \) and \( C \) such that \( A = B \cup C \) and

\[
\max\{T_s^+(B), T_s^\times(C)\} \leq |A|^{2s - \frac{c_s \sqrt{\log s}}{\sqrt{\log \log s}}},
\]

where \( c_s > 0 \) is an absolute constant.

Basically, we use just one induction in our proof and that is why we have just one logarithm in estimate (11) in contrast to paper \cite{30}. A similar construction as in Theorem 6 (see \cite{47}, Proposition 1.5) shows that one cannot obtain better saving than \( \Omega(\log s/\log \log s) \) in estimate (11).

Following the argument of the proof from paper \cite{31} Theorem 1.1 one can show that Theorem 7 implies a result on additive/multiplicative Sidon sets in \( \mathbb{Z} \) (the definitions can be found in \cite{31}, also, see preceding paper \cite{12} where the author has to deal with the real case). Recall that a finite set \( A \subset G \) is a \( B^+_h[g] \) set if for any \( x \in G \) the number solutions to the equation \( x = a_1 + \cdots + a_h \) does not exceed \( g \) (and similarly for \( B^\times_h[g] \)). Estimate (12) is a quantitative analogue of the main result from \cite{9}.

**Theorem 8** Let \( h \) be a positive integer, \( A \subset \mathbb{Z} \) be a finite set, and let \( B \) and \( C \) be the largest \( B^+_h[1] \) and \( B^\times_h[1] \) sets in \( A \) respectively. Then

\[
\max\{|B|, |C|\} \gg |A|^{\eta_h/h},
\]

where \( \eta_h \gg (\log h)^{1/2-o(1)} \). In particular, for any \( A \subset \mathbb{Z} \)

\[
|hA| + |A^h| \gg_h |A|(\log h)^{1/2-o(1)}.
\]

Finally, we have obtained a sum–product–type result of another sort. In additive combinatorics we believe that higher sumsets have rich additive structure and this is a classical theme of research. On the other hand, thanks to the sum–product phenomenon it means that such sets must have rather poor multiplicative structure. It turns out that this heuristic can be expressed in terms of some dimensions. For example, we show that for any positive integer \( k \) one has

\[
\dim_k^\times(D) \geq \exp(\Omega(\log |A|/\log K)),
\]

where \( A \) is an arbitrary set with \( |A + A| \leq K|A| \) and \( D := A - A \). This result on dimension of the difference sets is interesting in its own right, especially, due to our crucial inclusion

\[
\{1, 2, \ldots, n\} \subset \{1, 2, \ldots, n\} \leq \frac{D}{D}, \quad \text{where} \quad n = \exp(\Omega(\log |A|/\log K)).
\]

Estimate (13) is exponentially better than the lower bound for the length of the largest arithmetic progression in \( A \pm A \) from \cite{15}.
2 Definitions and preliminaries

By $\mathbf{G}$ we denote an abelian group. Sometimes we underline the group operation writing $+$ or $\times$ in the considered quantities (as energies, the representation function, dimensions and so on). Sometimes it is useful to consider an abelian ring $\mathcal{R}$ instead of our group $\mathbf{G}$. Of course in this case one can apply two ring’s operations simultaneously. We use the same capital letter to denote set $A \subseteq \mathbf{G}$ and its characteristic function $A : \mathbf{G} \to \{0, 1\}$. Given two sets $A, B \subseteq \mathbf{G}$, define the sumset of $A$ and $B$ as

$$A + B := \{a + b : a \in A, b \in B\}.$$  

In a similar way we define the difference sets and higher sumsets, e.g., $2A - A$ is $A + A - A$. Given a positive integer $k$, we put

$$\Sigma_k(A) := \left\{ \sum_{s \in S} s : S \subseteq A, |S| \leq k \right\}.$$  

Clearly, $kA \subseteq \Sigma_k(A)$, and if $0 \in A$, then $\Sigma_k(A) = kA$. Also, trivially, $|\Sigma_k(A)| \leq k|kA|$. For an abelian group $\mathbf{G}$ the Plünnecke–Ruzsa inequality (see, e.g., [46]) holds stating

$$|nA - mA| \leq \left( \frac{|A + A|}{|A|} \right)^{n + m} \cdot |A|,$$  

(14)

where $n, m$ are any positive integers. Further if $|A + B| \leq K|A|$ for some sets $A, B \subseteq \mathbf{G}$, then for any $n$ one has

$$|nB| \leq K^n|A|.$$  

(15)

If $A \subseteq \mathcal{R}$ and $\lambda \in \mathcal{R}$, then we write $\lambda \cdot A := \{\lambda a : a \in A\}$. We use representation function notations like $r_{A+B}(x)$ or $r_{A-B}(x)$ and so on, which counts the number of ways $x \in \mathbf{G}$ can be expressed as a sum $a + b$ or $a - b$ with $a, b \in B$, respectively. For example, $|A| = r_{A-A}(0)$. For any two sets $A, B \subseteq \mathbf{G}$ the additive energy of $A$ and $B$ is defined by

$$E(A, B) = E^+(A, B) = |\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 - b_1 = a_2 - b_2\}|.$$  

If $A = B$, then we simply write $E(A)$ for $E(A, A)$. More generally, for sets (real functions) $A_1, \ldots, A_{2k}$ $(f_1, \ldots, f_{2k})$ belonging to an arbitrary (noncommutative) group $\mathbf{G}$ and $k \geq 2$ define the energy $T_k(A_1, \ldots, A_{2k})$ as

$$T_k(A_1, \ldots, A_{2k}) =$$

$$= |\{(a_1, \ldots, a_{2k}) \in A_1 \times \cdots \times A_{2k} : a_1 a_2^{-1} \cdots a_{2k-1} a_{2k}^{-1} = a_{k+1} a_{k+2}^{-1} \cdots a_{2k-1} a_{2k}^{-1}\}|,$$  

(16)

and

$$T_k(f_1, \ldots, f_{2k}) = \sum_{a_1 a_2^{-1} \cdots a_{2k-1} a_{2k}^{-1} = a_{k+1} a_{k+2}^{-1} \cdots a_{2k-1} a_{2k}^{-1}} f_1(a_1) \cdots f_{2k}(a_{2k}).$$  

One has (see, e.g., [46] Section 2.3)

$$T_k(f_1, \ldots, f_{2k}) \leq \prod_{j=1}^{2k} T_k^{1/2k}(f_j).$$  

(17)
In particular, $T_k^{1/2k}(f)$ defines a norm. For any function $f : \mathbb{G} \to \mathbb{C}$ and $\rho \in \hat{\mathbb{G}}$ define the Fourier transform of $f$ at $\rho$ by the formula

$$\widehat{f}(\rho) = \sum_{g \in \mathbb{G}} f(g)\rho(g).$$  \hspace{1cm} (18)

Given a set $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{G}$, recall that the combinatorial cube $Q(A)$ is $\{0, 1\} \cdot a_1 + \cdots + \{0, 1\} \cdot a_n$. If $|Q(A)| = 2^n$, then the cube is called proper. Similarly, if $P_1, \ldots, P_d \subseteq \mathbb{G}$ are some arithmetic progressions, then the set $S := P_1 + \cdots + P_d$ is called generalized arithmetic progression of dimension $d$ and if $|S| = \prod_{j=1}^d |P_j|$, then $S$ is proper.

The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. When the constants in the signs depend on a parameter $M$, we write $\ll_M$ and $\gg_M$. If $a \ll_M b$ and $b \ll_M a$, then we write $a \sim_M b$. All logarithms are to base 2. By $\mathbb{F}_p$ denote $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$. Let $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$.

If we have a set $A$, then we will write $a \preceq b$ or $b \succeq a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. Let us denote by $\lceil n \rceil$ the set $\{1, 2, \ldots, n\}$.

Given a positive integer $k$ one can define $k$–dissociated set $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ if any equality of the form

$$\sum_{j=1}^n \epsilon_j \lambda_j = 0, \quad \text{where} \quad |\epsilon_j| \leq k$$  \hspace{1cm} (19)

implies that all $\epsilon_j$ are equal to zero. In contrary, if there is a tuple $(\epsilon_1, \ldots, \epsilon_n)$ such that (19) holds and such that not all $|\epsilon_j| \leq k$ are equal to zero, then we say that $(\lambda_1, \ldots, \lambda_n)$ forms an additive $n$–tuple. Thus a $k$–dissociated set $\Lambda$ has no additive $|\Lambda|$–tuples. If $A \subseteq \mathbb{G}$ is a set, then we write $\dim_k(A)$ for the size of the largest $k$–dissociated subset of $A$. In particular, $\dim(A) = \dim_1(A)$. Clearly,

$$\dim_k(A) = \min\{t : \forall a = (a_1, \ldots, a_{t+1}) \in \mathcal{A}^{t+1}, \exists \epsilon = (\epsilon_1, \ldots, \epsilon_{t+1}) \in [-k,k]^{t+1} \text{ s.t. } \langle \epsilon, a \rangle = 0\}. \hspace{1cm} (20)$$

The set $\mathcal{A}^{t+1}$ from formula (20) is the set of all tuples $a_1, \ldots, a_{t+1}$ with different elements. Notice that $\dim_k(A) = \dim_k(A \cup \{0\})$ and $\dim_k(A) \gg \log_k |A|$. Clearly, $\dim_{k_1}(A) \leq \dim_{k_2}(A)$ if $k_1 \geq k_2$ but also almost obviously, that all these dimensions are weakly equivalent for different $k_j$ see, e.g., formula (29) of Lemma 14 below. For any $k$ the dimension is monotone and subadditive, that is, $\dim_k(B) \leq \dim_k(A)$ for any $B \subseteq A$ and for arbitrary $B_1, \ldots, B_t \subseteq \mathbb{G}$ one has

$$\dim_k(\bigcup_{j=1}^t B_j) \leq \sum_{j=1}^t \dim_k(B_j).$$

Similarly, given a set $S \subseteq \mathbb{G}$ and a positive integer $k$ define $\text{Span}_k(S) = \{\sum_{j=1}^n \epsilon_j s_j : |\epsilon_j| \leq k\}$ and let

$$d_k^*(A) := \min\{|S| : A \subseteq \text{Span}_k(S)\} \quad \text{and} \quad d_k(A) := \min\{|S| : S \subseteq A \subseteq \text{Span}_k(S)\}.$$ 

We write $d^*(A)$ for $d_1^*(A)$ and $d(A)$ for $d_1(A)$. Clearly, $d^*(A) \leq d(A) \leq \dim(A)$, $d_k^*(A) \leq \dim_k(A)$ for any $k$ and if $A$ is a dissociated set, then $d(A) = \dim(A) = |A|$ (in contrary, there is a
dissociated set $A$ such that $d^*(A) \sim \dim(A)/\log \dim(A)$, see Example 16 below). Further $d_k^*(A)$ is monotone (but $d_k(A)$ is not, see [38, Example 8.1]) and both $d_k^*(A)$, $d_k(A)$ are subadditive, as well as the dimension $\dim_k(A)$. Again, notice that $d_k^*(A) = d_k^*(A \cup \{0\})$, $d_k(A) = d_k(A \cup \{0\})$ and also $d_k^*(A) \leq k\dim_k(A)$ if $A$ belongs to a ring $R$ such that all elements from $[k]$ are invertible. Clearly, $d_k(A) \leq d_k(A)$ and $d_k^*(A) \leq d_k^*(A)$ if $k_1 \geq k_2$. Unlike $\dim_k(A)$ the dimensions $d_k(A)$, $d_k^*(A)$ enjoy the following properties

$$d_{kt}^*(B_1 + \cdots + B_t) \leq \sum_{j=1}^t d_k^*(B_j) \quad (21)$$

for arbitrary sets $B_j \subseteq G$, $j \in [t]$ and

$$d_k^*(B_1 + \cdots + B_t) \leq \sum_{j=1}^t d_k(B_j) \quad (22)$$

for any disjoint sets $B_j \subseteq G$, $j \in [t]$. For other variants of additive dimensions of a set consult [38] and [13]. We show in the next section that all such dimensions differ by some logarithmic factors, basically, this result is contained in papers [13], [28].

To obtain our applications in Sections 5, 6 we need some results. First of all, we recall the well–known Theorem of Rudin [33] on dissociated sets.

**Theorem 9** Let $\Lambda \subseteq G$ be a dissociated set. Then for any positive integer $k$ one has

$$T_k(\Lambda) \leq (Ck)^k|\Lambda|^k,$$

where $C > 0$ is an absolute constant.

Now let $A \subseteq G$ be a set. Put

$$\beta(A) := \inf_{X,Y \neq \emptyset} \frac{|A + X + Y|}{\sqrt{|X||Y|}} \geq 1.$$

(23)

The quantity $\beta(A)$ is discussed in detail in [29] and in [22]. Basically, $\beta(A)$ measures the additive structure of our set $A$. Notice that by (23) and induction we have for any integer $k \geq 1$

$$|(2^k - 1)A| \geq \beta^{k-1}(A) \cdot |A|.$$

(24)

Thus, applying Proposition 26 below (e.g., consult formula [59]), we see that $\log \beta(A) \ll \dim(A)$ but more precisely by [29, Statement 3.2] one has

$$\beta(A) \leq 2^{\dim(A)}.$$

(25)

It seems like that simple estimate (25) is the only relation between $\beta(A)$ and $\dim(A)$. Indeed, consider the following instructive example: take any set $A \subseteq \mathbb{Z}$ and then (see [29, Statement 3.2]) one has $\beta(A) \leq 2$ but $\dim(A)$ can be arbitrary large.

We need the beautiful result on the connection of $T_k^*(A)$ and the quantity $\beta(A)$ for $A \subset \mathbb{Z}$, see [47, Theorem 1.3].
Theorem 10 Let $A \subset \mathbb{Z}$ be an arbitrary finite set and $\varepsilon \in (0, 1)$, $k \geq 2$ be parameters. Then

$$T_k^+(A) \leq 10^k \beta^{k/\varepsilon} |A| |A|^k + 2\varepsilon k \log k.$$  

We also use a classical result of Bukh [12] that provides an upper bound for a sum of dilates of a set in terms of the additive doubling of that set.

Theorem 11 Let $A$ be a finite subset of an abelian group such that $|A + A| \leq K|A|$. Then for any $\lambda_i \in \mathbb{Z}\{0\}$ we have

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \leq K^O\left(\sum_{i=1}^{k} \log(1+|\lambda_i|)\right) |A|.$$  

Also, we need a relaxation of the well–known sunflower lemma of Erdős and Rao [17], which is due to Füredi [20]. Actually in our regime there is almost no difference between these two results, as well as between a modern relaxation of the sunflower lemma, see [32]. Recall that given a family of sets $A_1, \ldots, A_r$, their common part is the set

$$X := \bigcup_{i \neq j} (A_i \cap A_j).$$

Note that, if $|X| < \min_i |A_i|$, then all the sets $A_1 \setminus X, \ldots, A_r \setminus X$ are nonempty and mutually disjoint and thus it is a relaxation of the notion of the classical sunflower. Now given two positive integers $k$ and $r$, let $f(k, r)$ be the smallest number $n$ such that any collection $C$ of more than $n$ sets, every $C \in C$ has size $k$, then $C$ contains $r$ members with the common part less than $r$. We have the following result, see [20].

Lemma 12 Let $k$ and $r$ be positive integers. Then $f(k, r) \leq (r - 1)^k$.

Finally, we discuss a simple connection between additive dimension and Diophantine approximations, which is, actually, well–known see, e.g., [6], [7], [14], [36], [37] etc.

Let $N$ be a positive integer. Given a positive real number $s$ and a set $A \subseteq \mathbb{Z}/N\mathbb{Z}$ we define

$$D_{s,N}(A) := \min_{q \in [N-1]} \|qa\|_{s,N} \leq s \sum_{a \in A} \|qa\|_{s,N}^{s}.$$  

Similarly, if $A \subset \mathbb{R}$ and $N$ is a positive integer, then we put

$$D_{s,N}(A) := \min_{q \in [N-1]} \|qa\|_{s,N} \leq s \sum_{a \in A} \|qa\|_{s,N}^{s}.$$  

Finally, notice that if $A \subset \mathbb{Z}$ and $A_N := A \pmod{N}$, then always $\dim(A) \geq \dim(A_N)$.

Lemma 13 Let $N$ be a positive integer, $A \subset \mathbb{R}$ or $A \subseteq \mathbb{Z}/N\mathbb{Z}$. Put $d = \dim(A)$ and suppose that for a certain $s > 0$ one has $D_{s,N}(A) \geq |A|/T$, $T \geq 1$. Then

$$d \geq \frac{s \log(N-1)}{\log(dT)}.$$  

Proof. We consider the case $A \subseteq \mathbb{Z}/N\mathbb{Z}$ because if $A \subseteq \mathbb{R}$, then the argument is the same. Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subseteq A$ be a maximal dissociated subset of $A$, $d = \dim(A)$. Using the Dirichlet Theorem, we find $q \in (\mathbb{Z}/N\mathbb{Z}) \setminus \{0\}$ such that $\|q \lambda_j/N\| \leq (N-1)^{-d-1}$ for all $j \in [d]$. Hence

$$\frac{|A|}{T} \leq D_{s,N}(A) \leq \sum_{a \in A} \left\| \frac{qa}{N} \right\|^s \leq |A|(N-1)^{-sd-1}$$

as required.

3 General results on additive dimensions

We start this section with a discussion concerning relations between different types of dimensions. As we have said before all dimensions $\dim_k(A), d_k^*(A), d_k(A)$ differ by some logarithmic factors but we now obtain a more concrete result.

Lemma 14 Let $k, l$ be positive integers, $\mathcal{R}$ be a ring such that all numbers $j \in [k]$ are invertible and $A \subseteq \mathbb{G}$ be a set. Then

$$\dim_l(A) \ll \dim_k(A) \log_{l+1}(k \dim_k(A)). \tag{29}$$

Similarly,

$$d_l(A) \leq \dim(A) \ll \dim_k(A) \log(k \dim_k(A)). \tag{30}$$

Proof. Let $\Lambda_l$ and $\Lambda_k$ be some maximal $l$ and $k$–dissociated subsets of $A$, correspondingly. Thus $|\Lambda_l| = \dim_l(A)$ and $|\Lambda_k| = \dim_k(A)$. By $l$–dissociativity all sums $\sum_{\lambda \in \Lambda_l} n_\lambda \lambda$, where $n_\lambda \in [0, 1, \ldots, l]$ are distinct and hence this set of sums $S$ has size $(l+1)^{\dim_l(A)}$. On the other hand, for any $\lambda \in \Lambda_l$ there is $j \in [k]$ such that $j \lambda \in \text{Span}_l(A_k)$. Splitting all elements of $\Lambda_l$ onto these $k$ sets, we see that any element of the set $S$ belongs to $\sum_{j=1}^k j^{-1} \text{Span}_{kl\dim_l(A)}(A_k)$. Hence we have

$$(l+1)^{\dim_l(A)} \leq \sum_{j=1}^k j^{-1} \text{Span}_{kl\dim_l(A)}(A_k) \leq k(2k \dim_l(A) + 1)^{\dim_k(A)} \tag{31}$$

and thus

$$\dim_l(A) \ll \dim_k(A) \log_{l+1}(k \dim_l(A))$$

as required. To obtain (30) we use inequality (29) and see that

$$d_l(A) \leq d(A) \leq \dim(A) \ll \dim_k(A) \log(k \dim_k(A)). \tag{32}$$

This completes the proof.

Example 15 1) Let $A = [n]$. Then $\dim_k^+(A) \sim \log_k n$ because the set $\{1, (k+1), \ldots, (k+1)^s\} \subset [n], s = [\log_{k+1} n]$ is $k$–dissociated. Also, as in formula (31), we have $(k+1)^{\dim_k^+(A)} \leq 2k \dim_k^+(A)n \leq 2kn^2$ and thus we obtain $\dim_k^+(A) \leq 2 \log_{k+1} n + O(1)$. It is easy to see that for
If each such a set is counted with multiplicity at most \((2^k + 1)\), hence one has \(\|E\| \leq n\). Then for any positive integers \(d, k\) and hence the dimensions \(d(A)\) can differ by a logarithm.

Finally, as in the proof of Lemma 14, see formulae in (30) (or consult [38], Section 8]) for any positive integers \(l, k\) and supposing that all numbers in \([\max\{k, l\}]\) are invertible in \(\mathcal{R}\), we get

\[
A^r(A)/l \leq \dim(A) \leq d_l(A) \cdot \log_t(kd_k^r(A)) \leq k\dim(A) \log_t(k^2\dim(A)).
\]

In particular (see formulae in (30), (32))

\[
dl(A) \leq \dim(A) \leq d_l(A) \cdot \log(kd_k^r(A)).
\]

Thus, dimensions \(d_l(A), d^r(A), d(A)\) differ by some logarithmic factors and this is tight in general, see Example 16. We give another proof of bound (33), which allows us to obtain an upper bound for \(d_k(A)\) in the spirit of [38, Lemma 7.2] but here we had to deal with the dimension \(d(A)\) not \(d_k(A)\).

**Lemma 17** Let \(G\) be an abelian group, \(A \subseteq G\) be a set and \(l, k\) be positive integers. Then \(\dim_1(A) \leq d^r_k(A) \cdot \log_t+1(kd^r_k(A))\).

Also, let \(t, s\) be positive integers. If \(A\) contains \(\exp(\Omega(s \log(k(s + t))))\) additive \(s\)-tuples, then \(d_k(A) \leq |A| - t\).

**Proof.** Let \(d = \dim(A)\) and \(\Lambda = \{\lambda_1, \ldots, \lambda_d\}\) be a maximal \(l\)-dissociated subset of \(A\) and \(S = \{s_1, \ldots, s_{|S|}\}, |S| = d_k^r(A)\) be a set such that \(A \subseteq \text{Span}_k(S)\). It follows that \(\lambda_j = \sum_{i=1}^{|S|} \varepsilon_{ij} s_i\), where \(|\varepsilon_{ij}| \leq k\). Consider the matrix \(E = (\varepsilon_{ij}), i \in |S|\) and \(j \in [d]\). Then for any \(\omega \in [-l, l]^d\) one has \(\|E\omega\|_{\infty} \leq lkd\). Considering \((l + 1)^2\) vectors with elements from \([0, 1, \ldots, l]\), we see that if \((l + 1)^2 > (2kd + 1)^{|S|}\), then there are distinct \(\omega', \omega'' \in [0, 1, \ldots, l]^d\) such that \(E\omega_1 = E\omega_2\) and hence \(E(\omega' - \omega'') = 0\). We see that \(\omega = (\omega_1, \ldots, \omega_l) := \omega' - \omega'' \in [-l, l]^d\) and \(\sum_{j=1}^d \omega_j \lambda_j = 0\), contradicting \(l\)-dissociative of \(\Lambda\). Hence \((l + 1)^2 \leq (2kd + 1)^{|S|}\) and the result follows.

Let us obtain the second part of Lemma 17. Let \(A\) be a maximal \(k\)-dissociated subset of \(A = \{a_1, \ldots, a_{|A|}\}\) and suppose that \(d = |\Lambda| > |A| - t\). Having an additive \(s\)-tuple \(\sum_{j=1}^d \varepsilon_j a_j = 0\), \(|\varepsilon_j| \leq k\) with elements belonging to the set \(A\), we consider a set of \(a_j \in A\) from this tuple. Clearly, each such a set is counted with multiplicity at most \((2k + 1)^s\) and we fix just one set from this
ensemble. Thus we have obtained some (different) sets $S_i \subseteq \Lambda$ of size at most $s$ and by Lemma 12 we can find the relaxation of a sunflower $S$, $|S| = r$, provided there are at least $(2k + 1)^s f(s, r)$ additive $s$–tuples (splitting our family of sets one can suppose that the size of sets is exactly $s$). Adding at most $t$ elements to $S$ and using the argument of the first part of the lemma, we see that the parameter $r$ must be at most $O((s + t) \log (s + t))$ because otherwise we have a contradiction with $k$–dissociativity of $\Lambda$. Applying Lemma 12 we obtain the result. This completes the proof. \qed

Now we are ready to begin to study the connection between our dimensions and higher sumsets. Notice that for any $A, Z \subseteq G$ the following holds

$$|A + Z| \gg |Z| \cdot \frac{\dim(A)}{\log |Z|}. \quad (35)$$

This bound is contained in [33, Theorem 4.2], where the symmetric case was considered or just see the proof of Lemma 18 below, where we propose slightly different argument. In particular, for any integer $n \geq 1$ one has

$$|nA| \gg |A| \cdot \prod_{j=1}^{n-1} \frac{\dim(A)}{\log |jA|}. \quad (36)$$

For the sake of the completeness we give the proof of (35) and improve estimate (36).

**Lemma 18** Let $A, Z \subseteq G$, $k, m$ be positive integers. Suppose that $|m([k] \cdot A) + Z| \leq K|Z|$, $m \leq \dim_k(A)/2$, and $m \ll \log |Z| \leq \dim_k(A)$. Then

$$kd \dim_k(A) \ll K^{1/m} \log |Z|. \quad (37)$$

In particular, there is an absolute constant $C > 0$ such that for any $m$ with $m \ll C^{-1} \log |A| \leq \dim_k(A)/4$ one has

$$|m([k] \cdot A)| \geq |A| \cdot \left(\frac{kd \dim_k(A)}{C \log |A|}\right)^{m-1}. \quad (38)$$

Now if $\log |A| \ll m \leq \dim_k(A)/4$, then

$$|m([k] \cdot A)| \geq \left(\frac{kd \dim_k(A)}{4m}\right)^{m-1}. \quad (39)$$

**Proof.** It is sufficient to obtain (37) because (38) follows if one takes $m \to m - 1$ and $Z = A$ in (37) (notice that for $k = 1$ the condition $\log |A| \ll \dim(A)$ is obviously satisfies). Now let $\Lambda \subseteq A$ be a $k$–dissociated set such that $d := \dim_k(A) = |\Lambda|$. We split $\Lambda$ onto $m$ parts $\Lambda_j$ such that $|\Lambda_j| \geq d/(2m)$. Put $S = [k] \cdot \Lambda_1 + \cdots + [k] \cdot \Lambda_m$. We have $S \subseteq m([k] \cdot A)$ and for an arbitrary positive integer $n \leq d/4m$, considering for any $j$ the sums $k_1 \lambda_1 + \cdots + k_n \lambda_n$, where $\lambda_1, \ldots, \lambda_n \in \Lambda_j$ are distinct elements and $k_i, i \in [n]$ run over $[k]$, we obtain for an arbitrary positive integer $n$

$$|nS| \geq \prod_{j=1}^{m} \frac{k^n |\Lambda_j|^n}{2^n n!} \geq n^{-m} (4m)^{-nm} (kd)^{nm} \quad (40)$$
thanks to $k$–dissociativity of $\Lambda$. Using the Plünnecke inequality \eqref{eq:plunnecke} and our bound \eqref{eq:bound}, we obtain
\[ n^{-nm}(4m)^{-nm}(kd)^{nm} \leq |nS| \leq |nm([k] \cdot A)| \leq K^n|Z| \]
and hence choosing $n$ optimally, that is, $n = c \log |Z|/m$ where $c \in (0, 1]$ is a small constant (the condition $m \ll \log |Z|$ guaranties that $n$ can be chosen as an integer), we get
\[ kd \leq 4nmK^{1/m}|Z|^{1/nm} \ll K^{1/m} \log |Z|. \]
It remains to check the condition $4nm = 4c \log |Z| \leq d$ but our assumption $d = \dim_k(A) \geq \log |Z|$ and sufficiently small $c$ guarantee this. Finally, to get \eqref{eq:condition} just choose in \eqref{eq:bound} the parameters $n = 1$, $Z = A$, $m \to m - 1$ and hence $(kd)^{m-1}(4m)^{-(m-1)} \leq |S| \leq |m([k] \cdot A)|$ as required.
\[ \square \]

Remark 19 Of course we need condition $m \ll \dim(A)$ in \eqref{eq:condition} (for simplicity let $k = 1$) because otherwise the trivial upper bound $|mA| \leq (|A|+m-1)^m$ and any set $A$ with $\dim(A) = |A|$ gives a contradiction. Also, the same example shows that inequality \eqref{eq:condition}, as well as the dependence on $\dim_k(A)$ in the right–hand side of estimate \eqref{eq:dimension} below are tight up to some constants.

Inequality \eqref{eq:condition} works for $m \ll \dim(A)$ (let $k = 1$ for simplicity). We need a result on higher sumsets of $mA$ in terms of some dimensions of $A$, which works for $m \gg \dim(A)$ if for a certain (large) number $k$ the quantity $\dim_k(A) \log \dim_k(A)$ is much greater than $\dim(A)$ (compare with Lemma \eqref{lemma:dimension}). We give a sketch of the proof of estimate \eqref{eq:dimension} below, details are contained in [28, Theorem 1].

Lemma 20 Let $A \subseteq G$ be a set, $l$ be a positive integer parameter, and $\Lambda_l \subseteq A$ be any $l$–dissociated set. Then
\[ |\Lambda_l| \log |\Lambda_l| \gg \dim(\Sigma_{|\Lambda_l|}(\Lambda_l)) \gg \min\{|\Lambda_l| \log |\Lambda_l|, l\}. \tag{41} \]
In particular, for any integer $l$ the following holds
\[ \dim(\Sigma_{\dim(A)}(A)) \gg \min\{\dim(A) \cdot \log \dim(A), l\}. \tag{42} \]
Also, for $k = \dim(A) \log \dim(A)$ and $m \ll \dim_k(A) \cdot \log \dim_k(A)$ one has
\[ |m \Sigma_{\dim_k(A)}(A)| \geq \exp(\Omega(m \log(m^{-1} \dim_k(A) \cdot \log \dim_k(A)))) \tag{43} \]
In particular,
\[ |\dim_k(A) \log \dim_k(A) \Sigma_{\dim_k(A)}(A)| \geq \exp(\Omega(\dim_k(A) \cdot \log \dim_k(A))). \tag{44} \]
Finally, there is an absolute constant $C > 0$ such that for all $m \leq C^{-1} \log |\Sigma_{\dim_k(A)}(A)|$, we have
\[ |m \Sigma_{\dim_k(A)}(A)| \geq |\Sigma_{\dim_k(A)}(A)| \cdot \left(\frac{\log \dim_k(A)}{C \log(e|A|\dim_k^{-1}(A))}\right)^{-1}. \]
Proof. Let us start with the second inequality in \([11]\) (the first one is not so interesting and follows from inequality \([50]\), say). Let \(n = |\Lambda_l|, Q = \Sigma_n(\Lambda_l)\) and \(m = \dim(Q) + 1\). Suppose that \(n > (2\log 3 + o(1))m/\log m\) and \(l \geq m\). Consider \(n \times m\) matrix \(M\) with entries equal 0, 1. In other words, we take a set \(D \subseteq \{0, 1\}^m, |D| = n\) and we construct this set choosing at random and independently of each other \(n\) vectors from the set \(\{0, 1\}^m\). Using the union bound one can show that for any \(\varepsilon \in \{0, \pm 1\}^m \setminus \{0\}\) there is \(d \in D\) such that \(d\) is not orthogonal to \(\varepsilon\) (see [28, Theorem 1]). In other words, using the fact that \(\Lambda_l\) is a \(l\)-dissociated set, \(l \geq m\), we will see that the set of \(m\) columns of \(M\) corresponds to a dissociated set \(S \subseteq Q, |S| = m > \dim(Q)\) by the rule: \(c = (c_1, \ldots, c_n) \in M \rightarrow c_1 + \cdots + c_n := s_c \in S\). It will give a contradiction and hence either \(m \gg n\log n\) or \(m < l\) as required. Indeed, put \(\Lambda_l = \{\lambda_1, \ldots, \lambda_n\}\) and consider an additive \(n\)-tuple with elements \(s_c\). In other words, we take \(\varepsilon \in \{0, \pm 1\}^m \setminus \{0\}\) such that

\[
0 = \sum_{j=1}^{m} \varepsilon_j s_{c(j)} = \sum_{i=1}^{n} \lambda_i \left( \sum_{j=1}^{m} \varepsilon_j c_{i(j)} \right)
\]

and thus by \(l\)-dissociativity of \(\Lambda_l\), we have \(\sum_{j=1}^{m} \varepsilon_j c_{i(j)} = 0\) for any \(i \in [n]\) and this contradicts to our construction of the set \(D\).

Now we are ready to get \([11]\) (the proof of \([13]\) is similar). We know that \(k = \dim(A) \log \dim(A) \geq \dim_k(A) \log \dim_k(A)\) and using \([12]\) with \(l = k\), we obtain \(\dim(\Sigma_{\dim_k(A)}(A)) \gg \dim_k(A) \cdot \log \dim_k(A)\). Putting \(Q = \Sigma_{\dim_k(A)}(A)\) and applying inequality \([39]\) with \(A = Q\) and \(m \gg \dim(Q),\) we derive

\[
|\dim_k(A) \log \dim_k(A) \cdot Q| \geq |mQ| \gg \exp(\Omega(\dim_k(A) \cdot \log \dim_k(A))).
\]

Finally, if we apply inequality \([38]\) instead of \([39]\), we get for \(m \leq C^{-1} \log |Q|\)

\[
|mQ| \geq |Q| \cdot \left( \frac{\dim(Q)}{C \log |Q|} \right)^{m-1} \geq |Q| \cdot \left( \frac{\dim_k(A) \cdot \log \dim_k(A)}{C \log |Q|} \right)^{m-1} \geq |Q| \cdot \left( \frac{\log \dim_k(A)}{C \log(e|A| \dim_k^{-1}(A))} \right)^{m-1}.
\]

Here we have used a trivial upper bound for size of \(Q\), namely,

\[
\log |Q| \leq \log \left( \frac{|A|}{\dim_k(A)} \right) \leq \dim_k(A) \log (e|A| \dim_k^{-1}(A)).
\]

This completes the proof. \(\square\)

To obtain Theorem 1 from the introduction just combine formulae \([38]\), \([39]\) of Lemma 18 (with \(k = 1\)) and inequality \([44]\) of Lemma 20. In the later case it is sufficient to use the trivial bound

\[
|\dim_k(A) \log \dim_k(A) \Sigma_{\dim_k(A)}(A)| \leq \dim_k^2(A) \log \dim_k(A) |\dim_k^2(A) \log \dim_k(A) A|.
\]
Another way to show the same (for small \( k \)) is the following. We know that if \( 0 \in A \), then
\[
\Sigma_{\dim_k(A)}(A) = \dim_k(A) A
\]
but by Theorem 32 below any shift of \( A \) does not change the dimension too much.

The result above allows us to obtain a more delicate connection between \( d^*(A) \) and \( \dim_k(A) \) for large \( k \). It gives in particular, that
\[
\dim_k(A) \ll d^*(A) \leq d(A) \leq \dim(A)
\]
for \( k = \dim(A) \log \dim(A) \).

**Corollary 21** Let \( A \subseteq G \) be a set, \( k = \dim(A) \log \dim(A) \) and \( l \) be a positive integer. Then
\[
\dim_k(A) \ll d_l^*(A) \left(1 + \frac{\log l}{\log \dim_k(A)}\right).
\]

**Proof.** Let \( S \subseteq G \) be a set such that \( A \subseteq \text{Span}_l(S) \) and \( |S| = d_l^*(A) \). Thus for any \( n \) one has
\[
|nA| \leq \text{Span}_n(S) \leq (2nl + 1)^{|S|}.
\]
We choose \( n = \dim_k^2(A) \log \dim_k(A) \). Combining this bound with (44), we get
\[
(2nl + 1)^{|S|} \geq |\dim_k(A) \log \dim_k(A) \Sigma_{\dim_k(A)}(A)| \geq \exp(\Omega(\dim_k(A) \cdot \log \dim_k(A))),
\]
and hence
\[
|S| \log (\log \dim_k(A)) \gg \dim_k(A) \cdot \log \dim_k(A).
\]
This completes the proof. \( \square \)

Lemma 20 allows us to characterize all combinatorial cubes having the property that \( \dim(Q(\Lambda)) \ll \dim(A) \) for a sufficiently dissociated set \( \Lambda \). This characterisation is possible in terms of the dimension \( \dim_k(A) \) for a certain large \( k \). The answer is that \( \dim(Q(\Lambda)) \sim \dim(A) \) iff \( \dim_k(A) \sim \dim(A) \log \dim(A) \).

**Corollary 22** Let \( G \) be a group, and \( A \subseteq G \) be a set. Also, let \( k = \dim(A) \log \dim(A) \), and \( \Lambda_k \) be a maximal \( k \)-dissociated subset of \( A \).
If \( \dim(Q(\Lambda_k)) \leq K \dim(A) \), then \( \dim_k(A) \ll K \dim(A) / \log \dim(A) \).
On the other hand, if \( \dim_k(A) \leq K \dim(A) / \log \dim(A) \), then \( \dim(Q(\Lambda_k)) \ll K \dim(A) \).

**Proof.** From formula (41) of Lemma 20 with \( l = k \), we see that
\[
K \dim(A) \geq \dim(Q(\Lambda_k)) \gg \dim_k(A) \log \dim_k(A) \gg \dim(A)
\]
as required. We have applied Lemma 14 to derive the last inequality in (47) and thus we need to assume that \( G \) is a ring such that all numbers \( j \in [k] \) are invertible in \( G \). Rigorously speaking, we do not need in this implication to obtain Corollary 22.

We now take a dissociated set \( \Lambda^* \subseteq Q(\Lambda_k) \), \( |\Lambda^*| = \dim(Q(\Lambda)) \). Similarly to formula (31) of Lemma 14, we have
\[
2^{|\Lambda^*|} \leq |\text{Span}_{|\Lambda^*|}(\Lambda_k)| \leq (2|\Lambda^*| + 1)^{\dim_k(A)}
\]
or, in other words,
\[ |\Lambda_s| \ll \dim_k(A) \cdot \log |\Lambda_s| \leq \frac{K\dim(A)}{\log \dim(A)} \cdot \log |\Lambda_s|. \]

It gives us \(|\Lambda_s| \ll K\dim(A)| because, trivially, we can assume that \( K \ll \log \dim(A) \). This completes the proof. \( \square \)

We now study the question how to calculate the dimensions of sumsets. The estimate
\[ \dim(A + B) \lesssim \dim(A) + \dim(B) \quad (48) \]
for any sets \( A, B \subseteq \mathbb{G} \) is, basically, contained in [38 Section 8]. Here the sign \( \lesssim \) shows that the factor \( \log \dim(A + B) \) is allowable. For the sake of completeness we give the proof of inequality \((48)\) in Lemma 23 below see, e.g., estimate \((49)\). Let us remark that, in general, formula \((49)\) below is tight. Indeed, take the cube \( Q = \{0, 1\} \cdot a_1 + \cdots + \{0, 1\} \cdot a_d \) from Example 16. Then \( \dim(Q) \sim d\log d \) and it is larger by a logarithm, then \( \sum_{j=1}^{d} \dim(\{0, 1\} \cdot a_j) = d \). Finally, taking a sufficiently sparse set \( A \) with \( |\Lambda| = \lfloor k^2 \rfloor \), we see that inequality \((50)\) is tight as well (take \( l = 1 \) and \( k = |A|^2 \), say, and use that \( \dim(kA) \gg \log |\Lambda| \)).

**Lemma 23** Let \( R \) be a ring, \( k, l \) be positive integers, and \( C_1, \ldots, C_k \subseteq R \) be any sets. Suppose that all numbers \( j \in [l] \) are invertible in \( R \). Then
\[ \dim_l(C_1 + \cdots + C_k) \ll l \sum_{j=1}^{k} \dim_l(C_j) \cdot \log_{l+1}(k \sum_{j=1}^{k} \dim_l(C_j)), \quad (49) \]
and for any \( A \subseteq R \) one has
\[ \dim_l(kA) \leq \dim_l(\Sigma_k(A)) \ll d_l^*(A) \log_{l+1}(k d_l^*(A)) \ll l \dim(A) \log_{l+1}(k \dim(A)). \quad (50) \]

**Proof.** We begin with \((49)\). Let \( d_l^*(C_j) = |S_j| \) such that \( C_j \subseteq \text{Span}_l(S_j), j \in [k] \). Put \( S = \bigcup_{j=1}^{k} S_j \) and let \( C = C_1 + \cdots + C_k \). Then it is easy to see that \( C \subseteq \text{Span}_{kl}(S) \). In other words, thanks to \((21)\) and \((33)\), we have
\[
\frac{\dim_l(C)}{l \log_{l+1}(k \dim_l(C))} \lesssim \frac{d_{kl}^l(C)}{l} \leq \frac{1}{l} \sum_{j=1}^{k} d_l^*(C_j) \leq \sum_{j=1}^{k} \dim_l(C_j)
\]
and hence
\[ \dim_l(C) \ll l \sum_{j=1}^{k} \dim_l(C_j) \cdot \log_{l+1}(k \sum_{j=1}^{k} \dim_l(C_j)) \]
as required.

To obtain \((50)\) just take a minimal set \( \Lambda \subseteq \mathbb{G} \) such that \( A \subseteq \text{Span}_l(\Lambda) \) and notice that \( mA \subseteq \text{Span}_{kl}(\Lambda) \) for all \( m \leq k \). It implies \( d_{kl}^l(\Sigma_k(A)) \leq |\Lambda| = d_l^*(A) \). Applying the second inequality in \((33)\) with \( l = l \) and \( k = kl \), we get
\[ \dim_l(\Sigma_k(A)) \ll d_{kl}^*(\Sigma_k(A)) \log_{l+1}(k d_{kl}^*(\Sigma_k(A))) \ll d_l^*(A) \log_{l+1}(k d_l^*(A)) \]
This completes the proof. □

Upper bounds for dimensions of sets with really small doubling can be found in [38, Theorem 4.2]. Here we give a similar (and slightly sharper) result for \( \dim_k(A) \).

**Theorem 24** Let \( A \subseteq G \) be a set and \( k \) be a positive integer. Suppose that \( |A + A| \leq K|A| \) and \( K \leq \dim_k(A) \). Then

\[
\dim_k(A) - \log_{k+1}|A| \ll \frac{K \log^6(2K) \log \log(4K)}{\log k} \log \left( \frac{K}{\log k} \right). \tag{51}
\]

If \( \mathcal{R} \) is a ring such that all elements of \([k]\) are invertible, \( k \leq |A| \) and \( K \) is an arbitrary number, then

\[
\dim_k(A) \ll (\log_k |A| \log(k \log_k |A|)) + K \log^6(2K) \log \log(4K) \cdot \log_k(K \log_k |A|). \tag{52}
\]

Now let \( k_* = \dim(A) \log \dim(A) \) and \( K \geq 1 \) be an arbitrary number. Then

\[
\dim_{k_*}(A) \ll \frac{\log |A|}{\log \log |A|} + K \log^6(2K) \log \log(4K). \tag{53}
\]

**Proof.** Let \( \Lambda = \{\lambda_1, \ldots, \lambda_d\} \subseteq A \) be a \( k \)-dissociated set, \( d = |\Lambda| = \dim_k(A) \). Consider \((k+1)^d\) distinct sums of the form \( k_1 \lambda_1 + \cdots + k_d \lambda_d \), where \( k_j \in \{0, 1, \ldots, k\} \). Using [38, Lemma 4.1] (which is a consequence of Sander’s result from [37]), we get for \( K \leq d \)

\[
(k+1)^d \leq |dA| \leq \left( \frac{3kd}{K} \right)^{O(K \log^6(2K) \log \log(4K))} |A| \tag{54}
\]

and hence

\[
d \log(k+1) - \log |A| \ll K \log^6(2K) \log \log(4K) \cdot \log(d/K) \ll K \log^6(2K) \log \log(4K) \cdot \log d
\]
as required.

Now let us prove (52). By Chang’s lemma [14] and [38, Lemma 4.1], we have

\[
A \subseteq P - P + (S_1 - S_1) + \cdots + (S_l - S_l), \tag{55}
\]

where \( S_j, P \) are disjoint sets, \( |S_j| \leq 2K, j \in [l], l \ll \log^6(2K) \log \log(4K), \) the sum \( S_1 + \cdots + S_l + P \) is direct and \( |P| \gg |A| \cdot \exp(-\log^6(2K) \log \log(4K)) \) is a proper generalized arithmetic progression of dimension \( d = O(\log^6 K) \).

Clearly, for any \( j \in [l] \) one has \( \dim_k(S_j) \leq 2K \) and thus by formula (22) the following holds

\[
d_k^*(P + S_1 + \cdots + S_l) \leq d_k(P) + \sum_{j=1}^l d_k(S_j) \leq d_k(P) + 2Kl. \tag{56}
\]
Also, writing $P = P_1 + \cdots + P_d$, we have $d^*_k(P) \ll \sum_{j=1}^d \log_{k+1} |P_j| = \log_{k+1} |P|$ and one can see that $d_k(P) \ll d^*_k(A) \cdot \log(kd^*_k(A)) \ll \log_k |P| \log(k \log_k |P|)$ by the second inequality from \((33)\). Using Lemma \(23\) the second inequality from \((33)\) and the obtained bound \((56)\), we get

\[\dim_k(A) \approx (\log_k |P| \log(k \log_k |P|) + K)(\log_k |P| + \log K) \leq (\log_k |A| \log(k \log_k |A|) + K \log^6(2K) \log \log(4K)) \cdot \log_k(K \log_k |A|)\]
as required.

It remains to obtain \((53)\). Put $d_* = \dim_k(A)$ and notice that if $K \geq d^2_* \log d_*$, then the result is trivial. Using formula \((44)\) of Lemma \(20\) as well as \(58\) Lemma 4.1 as in \((54)\), we get

\[
\exp(\Omega(\dim_k(A) \cdot \log \dim_k(A))) \leq |d_* \log d_* \Sigma_{d_*}(A)| \\
\leq d^2_* \log d_* \cdot \left(\frac{3e^2 d^2_* \log d_*}{K}\right)^{O(K \log^6(2K) \log \log(4K))} |A|
\]
or, in other words,

\[d_* \cdot \log d_* \ll K \log^6(2K) \log \log(4K) \cdot \log d_* + \log |A|.
\]
It gives us

\[\dim_k(A) \ll \frac{\log |A|}{\log \log |A|} + K \log^6(2K) \log \log(4K).
\]
This completes the proof. \(\square\)

**Remark 25** As Example 29 (part two) shows one can obtain an analogue of Theorem 24 for sums of sets with small additive doubling.

In the next result we show that polynomial growth and the additive dimension are closely connected to each other (up to some logarithms).

**Proposition 26** Let $k$ be a positive integer, and $G$ be an abelian group.
If $A \subseteq G$ is a set of polynomial growth $|nA| \leq n^d|A|$, then $\dim_k(A) \ll d \log_{k+1} d + \log_{k+1} |A|$. Conversely, let $R$ be a ring such that all numbers $j \in [k]$ are invertible. Then any set $A \subseteq R$ has polynomial growth $|nA| \leq n^d|A|$ with $d \ll \dim_k(A) \log(k+1)$.

**Proof.** Put $L = \log |A|$. Let $A \subseteq A$ be a maximal $k$–dissociated subset of $A$, $|A| = \dim_k(A)$. We obtain even two bounds for $\dim_k(A)$. By Theorem 9 (or simple counting argument), we have for a certain absolute constant $C > 0$

\[\frac{|A|^n}{(Cn)^n} \leq |nA| \leq |nA| \leq n^d|A|.
\]
Taking $n \sim d \log d$, we obtain $|A| \ll d \log d \cdot |A|^{O(1/d \log d)}$. This calculation does not use the fact that $k \geq 1$ and to do this we apply the argument from the proof of Lemma 14. By $k$–dissociativity of $A$ all sums $\sum_{\lambda \in A} n\lambda \lambda$, where $n\lambda \in \{0, 1, \ldots, k\}$ are distinct and hence

\[(k + 1)^{\dim_k(A)} \leq |k \dim_k(A)| \leq (k \dim_k(A))^d |A|\]

\[(58)\]
and the result follows.

Conversely, as in the proof of Lemma 14, we have \( A \subseteq \bigcup_{j \in [k]} j^{-1}\text{Span}_k(\Lambda) := Q \), \(|A| = \dim_k(A)\) and hence (we can assume that \( n \geq 2 \))

\[
|nA| \leq |nQ| \leq k(2nk + 1)^{|A|} \leq n^d \leq n^d|A|,
\]

where \( d = O(\log(k + 1) \cdot \dim_k(A)) \), say. This completes the proof.

Also, we show that any set with small dimension has a rather dense Freiman model (see [46, Section 5.3]). For example, if \( \dim_k(A) \ll \log_k |A| \), then Proposition 27 below gives us a set \( B \subseteq \mathbb{Z}/m\mathbb{Z} \) with \( m = |A|^{O(\log_k(kt))} \) such that \( B \) is an isomorphic image of a large part of \( A \).

**Proposition 27** Let \( \mathcal{R} \) be a ring, \( A \subseteq \mathcal{R} \) be a set, \( k, l \) be positive integers, and \( m \geq k(4kl + 1)^{\dim_k(A)} \). Suppose that all numbers \( j \in [k] \) are invertible. Then there is \( A* \subseteq A \) and \( B \subseteq \mathbb{Z}/m\mathbb{Z} \) such that \( |A*| \geq |A|/l \) and \( A* \) is \( l \)-isomorphic to \( B \).

**Proof.** We follow the standard argument of Ruzsa see, e.g., [46, Lemma 5.26]. In other words, we need to estimate the size of \( |LA - lA| := m \). In terms of \( \dim_k(A) \) it gives us (consult the proof of Lemma 14)

\[ |LA - lA| \leq k(4kl + 1)^{\dim_k(A)}. \]

This completes the proof.

Finally, we consider a rather important case when a set \( A \) stops growing under addition and we give a criterion of this absent of the growth in terms of \( \dim_k(A) \) for a certain \( k \) or, equivalently, in terms of the set \( Q(\Lambda_k) \), thanks to Corollary 22.

**Theorem 28** Let \( A \subseteq G \) be a set, \( k = \dim(A) \log \dim(A) \) and \( K \geq 1 \) be a real number.

If \( |nA| \leq |A|^K \) for \( n \ll K^2 \log^2 |A| \log(K \log |A|) \), then \( \dim_k(A) \ll K \log |A|/\log(K \log |A|) \).

Now suppose that \( \mathcal{R} \) is a ring such that all numbers \( j \in [k] \) are invertible and \( \dim_k(A) \leq K \log |A|/\log(K \log |A|) \). Then \( |nA| = |A|^{O(K)} \) for all \( n = \dim^{O(1)}(A) \).

**Proof.** Define \( N = O(K^2 \log^2 |A| \log(K \log |A|)) \) such that by our assumptions for all \( n \leq N \) the following holds \( |nA| \leq |A|^K \). From \( |nA| \leq |A|^K \), it follows that \( \dim(A) \ll K \log |A| \) (see, e.g., calculations in [57]). We use bound (11) of Lemma 20 (also, it is possible to apply inequality (39) of Lemma 13) to derive

\[
\dim^2(A) \log \dim(A)|A|^K \geq \dim^2(A) \log \dim(A)|NA|
\]

\[
\geq \dim^2(A) \log \dim(A) \log \dim(A) |A|^K \geq |\dim_k(A) \log \dim_k(A) \Sigma_{\dim_k(A)}(A)|
\]

\[
\geq \exp(\Omega(\dim_k(A) \cdot \log \dim_k(A)))
\]

and hence \( \dim_k(A) \ll K \log |A|/\log(K \log |A|) \).

Now suppose that \( \dim_k(A) \ll K \log |A|/\log(K \log |A|) \). Using Lemma 14 we obtain \( \dim(A) \ll K \log |A| \). Now take a \( k \)-dissociated set \( \Lambda_k \) such that \( |\Lambda_k| = \dim_k(A) \). Using the arguments as in (31) of Lemma 13 we get

\[ |nA| \leq k|\text{Span}_{nk}(\Lambda_k)| \leq k(2nk + 1)^{\dim_k(A)} = \exp(\dim_k(A) \log(2nk + 1) + \log k) = |A|^{O(K)}. \]
for all \( n = \dim^{O(1)}(A) \). This completes the proof.

In the proof of Theorem 28 we have considered sets \( A \) with \( \dim(A) \ll K \log |A| \). Clearly, it is a much larger family of sets than just having the property \( |nA| = |A|^{O(K)} \) for all \( n = \dim^{O(1)}(A) \). It is interesting to describe this family and below we give some examples.

**Problem.** Let \( G \) be an abelian group. Characterise all sets \( A \subseteq G \) with \( \dim(A) \ll \log |A| \).

**Example 29** Let \( G \) be an abelian group and \( A \subseteq G \) be set. We write \( L \) for \( \log |A| \).

1) Let \( H_j, j \in [K] \) be some disjoint arithmetic progressions and put \( A(K) = \bigsqcup_j H_j \). Then \( \dim(H_j) < \log |H_j| < \log A(K) \ll KL \). Let us assume, in addition, that the sum \( H_1 + \cdots + H_K \) is direct. Then for all positive integers \( n \) one has \( |nA| < n^K |KA(K)| \ll n^K (|A|/K)^K \).

2) Now let \( A = \sum_{i=1}^t A_i(K_*) \), where \( K_*, t = O(K^{1/2}), K = O(|A|) \), and the sets \( A_i(K_*) \) are constructed as in the previous example. We need to estimate \( \dim(A) \). One has

\[
|nA| \leq \prod_{i=1}^t |nA_i(K_*)| \leq n^{tK*} \prod_{i=1}^t |K_*A_i(K_*)| \ll n^{tK*} (|A|/K_*)^{tK*}.
\]

Taking \( n = \dim(A) \) and using the same argument as in the first part of the proof of Theorem 24 we obtain

\[
\dim(A) \ll tK_* \log(\dim(A)|A|) \ll KL,
\]

thanks to \( K \ll |A| \). It gives us another example of a set with \( \dim(A) \ll KL \).

3) In view of Proposition 20 any set \( A \) with polynomial growth \( d = O(KL/\log L) \) has \( \dim(A) \ll KL \) for small \( K \). As in 2) we can take sums of such sets for small parameters \( t \) and \( d \).

### 4 Additive dimensions and the quantity \( T_k \)

In this section we consider some variants of the dimension \( \dim(A) \), which are convenient for counting \( T_k(A) \). For simplicity we do not have to deal with the dimensions \( \dim_l(A) \) or \( d^*_l(A) \) for \( l > 1 \) because they are connected with the other quantities, namely, with \( T_k(r|A|) \) see, e.g., formulae (39), (40) of Lemma 18.

Let \( \alpha \in (0, 1] \) and \( k \geq 2 \) be an integer. Put

\[
\dim_{\alpha,k}(A) = \min \{ \dim(B) : B \subseteq A, T_k(B) \geq \alpha T_k(A) \},
\]

and

\[
\dim_{\alpha}(A) = \min_{k \geq 2} \dim_{\alpha,k}(A) \quad (61)
\]

Clearly, \( \dim_{\alpha,k}(A) \ll \dim(A) \) for all \( \alpha \) and \( k \). Notice that \( \dim(A) \) is subadditive and monotone but \( \dim_{\alpha,k}(A), \dim_{\alpha}(A) \) are not.

From the proof of Proposition 20 and the Hölder inequality one has for any \( k \) and \( A \subseteq G \)

\[
\dim(A) \geq \log_{2k+1} |kA| \geq \log_{2k+1} \left( \frac{|A|^{2k}}{T_k(A)} \right) \quad (62)
\]
It implies in particular,
\[ T_k(A) \geq \frac{|A|^{2k}}{(2k + 1)^{\dim(A)}} \tag{63} \]
and hence there is a connection between \( \dim(A) \) and \( T_k(A) \) (as well as with the size of the sunset \( kA \)). Below we obtain a stronger result.

**Theorem 30** Let \( \alpha \in (0, 1] \) be a real number and \( k \) be a positive integer. Then
\[ T_k(A) \leq \left( \frac{16Ck}{\dim_{\alpha,k}(A)(1 - \alpha^{1/2k})^2} \right)^k |A|^{2k}, \tag{64} \]
and
\[ T_k(A) \leq \frac{C'|A|^2T_{k-1}(A)}{\dim_{\alpha,k}(A)(1 - \alpha^{1/2k})^2} \cdot \log(|A|^{2k-2}T_{k-1}(A)) \leq \frac{C'k|A|^2T_{k-1}(A)\log |A|}{\dim_{\alpha,k}(A)(1 - \alpha^{1/2k})^2}, \tag{65} \]
where \( C > 0 \) is an absolute constant as in Theorem 38 and \( C' > 0 \) is another absolute constant. Conversely, writing \( d = \dim(A) = M \log |A| \), we find an integer \( m \), \( \log |A|/\log M \ll m \ll \dim(A)/2 \) such that
\[ T_m(A) \geq \frac{|A|^{2m}}{2d^{\alpha}e^{d/(2m)}} \geq \frac{|A|^{2m}}{|A|^{|C_\ast M|}}, \tag{66} \]
where \( C_\ast > 0 \) is an absolute constant. In particular, for any \( B \subseteq A \) with \( |B| = \delta |A| \) and \( d := \dim(B) \) one has
\[ d \log(d/\delta) \gg \log(|A|^{2d+1}T_{d+1}^{-1}(A)). \tag{67} \]

**Proof.** Let \( L = \log |A| \) and \( l \) be a parameter, which we will choose later. Then split \( A \) as \( A = (\bigcup_{j=1}^s A_j) \bigcup A_s \), where \( A_j \) are dissociated, \( |A_j| = l \) and \( \dim(A_s) < l \). Clearly, \( s \ll |A|/l \). By the norm property of \( T_k \) and Rudin’s Theorem 39 we have
\[ T^{1/2k}_k(A) \leq T^{1/2k}_k(A_s) + \sum_{j=1}^s T^{1/2k}_k(A_j) \leq T^{1/2k}_k(A_s) + |A|/l \cdot (Ck)^{1/2}l^{1/2}. \tag{68} \]
Writing \( T_k(A) = \kappa^k|A|^{2k} \) and choosing \( l = Ckk^{-1}(1 - \alpha^{1/2k})^{-2} \), we get \( T_k(A_s) \gg \alpha T_k(A) \). By the definition of the quantity \( \dim_{\alpha,k}(A) \), we see that \( \dim_{\alpha,k}(A) \ll Ckk^{-1}(1 - \alpha^{1/2k})^{-2} \) and hence we derive \([61]\).

The second bound \([65]\) can be obtained similarly to \([38]\) Proposition 5.3. Indeed, put \( \mathcal{E} = \bigcup_{j=1}^s A_j \) and write \( T_k(A) = \omega T_{k-1}(A)|A|^2 \), \( \omega \in (0, 1] \). Let \( p = \log(|A|^{2k-2}T_{k-1}^{-1}(A)) \leq k \log |A| \), \( p \gg \log |A| \) and \( l := C'(1 - \alpha^{1/2k})^{-2}p^{-1} \omega^{-1} \), where \( C' > 0 \) is a sufficiently large absolute constant. Then
\[ T_k(A) = T_k(A, \ldots, A, A_s, A_s) + 2T_k(A, \ldots, A, \mathcal{E}, A_s) + T_k(A, \ldots, A, \mathcal{E}, \mathcal{E}) = \sigma_0 + \sigma_1 + \sigma_2. \]
If \( \sigma_0 \geq \alpha^{1/k}T_k(A) \), then by the Hölder inequality \([17]\) one has \( T_k(A_s) \gg \alpha T_k(A) \) and as before, we obtain \( \dim_{\alpha,k}(A) \ll C'(1 - \alpha^{1/k})^{-2}p^{-1}\omega^{-1} \) as required. Thus we need to estimate \( \sigma_1, \sigma_2 \). Using
the Hölder inequality, Rudin’s Theorem \[\text{5.3, formula (5.11)\]}, we get
\[
\sigma_2^p \leq T_p(\mathcal{E}) \mathcal{T}^{p-1}_{k-1}(A) |A|^{2k-2} \leq (Cp/l)^p |A|^{2p} \mathcal{T}^{p-1}_{k-1}(A) \left( |A|^{2k-2} \mathcal{T}^{-1}_{k-1}(A) \right)
\]
and hence by our choice of the parameters \(p\) and \(l\), we have
\[
\sigma_2 \leq T_k(A) \cdot Cp \omega \omega^{-1} \left( |A|^{2k-2} \mathcal{T}^{-1}_{k-1}(A) \right)^{1/p} \leq T_k(A)(1 - \alpha^{1/k})^2/100,
\]
say. Clearly, by the Cauchy–Schwarz inequality
\[
\sigma_1^2 \leq 4\sigma_0 \sigma_2 \leq T_k^2(A)(1 - \alpha^{1/k})^2/25
\]

hence \(\sigma_1 \leq T_k(A)(1 - \alpha^{1/k})/5\) and thus this sum is also negligible.

It remains to prove \((66), (67)\) and the argument is almost the same. Let us delete zero from \(A\) and with some abuse of the notation we will write \(A\) for the remaining set. Recall that for any \(l\) we denote by \(\overline{A}^{d+1}\) the set of all vectors \(\bar{a} = (a_1, \ldots, a_{l+1}) \in A^{d+1}\) such that all \(a_1, \ldots, a_{l+1}\) are different. Clearly, \(\overline{A}^{d+1} \cap A^{d+1} \geq |A|^{d+1} \exp(-d^2|A|^{-1}) \geq |A|^{d+1} \exp(-d)\). By the definition of the additive dimension for any \(\bar{a} \in \overline{A}^{d+1}\) there is \(\bar{\varepsilon} \in \{0, \pm 1\}^{d+1}\) such that \(\langle \bar{a}, \bar{\varepsilon} \rangle = 0\), see formula \((20)\). Write \(\text{wt}(\bar{\varepsilon}) = \sum_{j=1}^{d+1} |\varepsilon_j|\) and since \(0 \notin A\), it follows that \(\text{wt}(\bar{\varepsilon}) \geq 2\). Let \(1 \leq \Delta \leq (d + 1)/2\) be a parameter. We have

\[
\begin{align*}
|A|^{d+1} \exp(-d) & \leq |A|^{d+1} \sum_{w=2}^{d+1} \exp(-w^2|A|^{-1}) \leq \sum_{w=2}^{d+1} \sum_{\bar{\varepsilon} \in \{0, \pm 1\}^{d+1}, \text{wt}(\bar{\varepsilon})=w} \langle \bar{a}, \bar{\varepsilon} \rangle = 0 \rangle |\bar{a} \in \overline{A}^{d+1} : \langle \bar{a}, \bar{\varepsilon} \rangle = 0\rangle |A|^{d+1-w^2|A|^{-1}} |\bar{a} \in \overline{A}^{d+1} : \langle \bar{a}, \bar{\varepsilon} \rangle = 0\rangle |A|^{d+1-w^2|A|^{-1}} |ar{a} \in \overline{A}^{d+1} : \langle \bar{a}, \bar{\varepsilon} \rangle = 0\rangle |A|^{d+1-w^2|A|^{-1}} = \\
& \leq \sum_{w=2}^{d} |A|^{d+1-w^2|A|^{-1}} \cdot T_{[w/2]}(A)|A|^{-2[w/2]} = \\
& = |A|^{d+1} \sum_{w \leq \Delta} 2^w \binom{d+1}{w} T_{[w/2]}(A)|A|^{-2[w/2]} + |A|^{d+1} \sum_{w > \Delta} 2^w \binom{d+1}{w} T_{[w/2]}(A)|A|^{-2[w/2]} = \\
& = \sigma_1 + \sigma_2 := \sigma .
\end{align*}
\]

(69)

Let us obtain an upper bound for the sum \(\sigma_1\). Trivially estimate \(T_{[w/2]}(A)\) as \(T_{[w/2]}(A) \leq |A|^{2[w/2]-1}\) (below in the paper we will use some better bounds), we see that
\[
\sigma_1 \leq 2|A|^d \left( \frac{2e(d+1)}{\Delta} \right)^{\Delta} \leq |A|^d \left( \frac{4eML}{\Delta} \right)^{\Delta} .
\]

(70)

We choose \(\Delta\) such that
\[
\frac{\Delta^2}{|A|} + \Delta \log \left( \frac{8ed}{\Delta} \right) \ll \Delta \log \left( \frac{8ed}{\Delta} \right) \ll L
\]
or, in other words, we take $\Delta = cL/\log M$, where $c > 0$ is a sufficiently small absolute constant. It gives us $\sigma_2 \geq 2^{-1}|A|^{d+1} \exp(-d)$. Thus (69) implies

$$\exp(-d) \leq 2 \sum_{w > \Delta} 2^w \left( \frac{d+1}{w} \right) T_{[w/2]}(A)|A|^{-|2w/2|} \leq 3^{d+2} \max_{w > \Delta} \{ T_{[w/2]}(A)|A|^{-|2w/2|} \} =$$

$$= |A|^{C_\alpha^* M} \max_{w > \Delta} \{ T_{[w/2]}(A)|A|^{-|2w/2|} \},$$

(71)

where $C_\alpha > 0$ is an absolute constant. Thus we have obtained (66) and to get (67) we repeat the calculations from (69)—(71) with $A = B$. Namely, writing $T_{d+1}(A) = \frac{|A|^{2d+1}}{Q^w}$, $Q \leq |A|$ and using the Hölder inequality

$$T_l(B) \leq T_l(A) \leq \frac{|A|^{2l-1}}{Q^{l-1}}$$

(72)

for all $2 \leq l \leq d+1$, we derive

$$\sigma \leq |B|^{d+1} Q |A|^{-1} \sum_{w=2}^{d+1} Q^{-|w/2|} \left( \frac{2e(d+1)}{w\delta} \right)^w .$$

and hence automatically $d \gg \delta Q^{1/3}$, say. By the definition of the quantity $Q$ we see that

$$d \log(d/\delta) \gg d \log Q = \log(|A|^{2d+1} T_{d+1}^{-1}(A))$$

and we finally, obtain the required bound. This completes the proof.

Corollary 31  Let $\alpha \in (0,1]$ be a real number, $k$ be a positive integer and $\dim_{\alpha}^*(A) := \min_{l\in[2,k]} \dim_{\alpha,l}(A)$. Then for any $2 \leq l \leq k$ one has

$$T_l(A) \leq \left( \frac{C(\alpha)k^3}{\dim_{\alpha}^*(A)} \right)^l \cdot |A|^{2l} .$$

Thus indeed the dimension of a set $A$ is closely connected with the quantity $T_k(A)$: see the lower bound for $\dim(A)$ in (67) and, on the other hand, assuming $\dim_{\alpha,k}(A) \gg \dim(A)$, $\alpha = 2^{-k}$, say, as well as putting $k = c \dim(A)$ (here $c > 0$ is a small absolute constant) in (61), we obtain the upper bound

$$\dim(A) \ll \dim_{\alpha,k}(A) \ll \log(|A|^{2k} T_k^{-1}(A)) .$$

Also, if, say, $T_l(A) \ll |A|^l$ for a certain fixed number $l$, then we get from (69)—(71) and the trivial estimate $T_s(A) \leq |A|^{2s-2l} T_l(A)$, $s \geq l$ that $\dim(A) \gg l \log |A|$ and this is a non–trivial bound. Finally, one can see that if we have the first inequality in (66), that is,

$$T_m(A) \geq \frac{|A|^{2m}}{2dA^{d+1} \exp(d/2m)}$$

for $\log |A|/\log M \ll m \leq \dim(A)/2 = d/2$, then an application of Theorem 30 formula (61) with $\alpha = 2^{-m}$ gives us a set $A_\ast \subseteq A$, $T_m(A_\ast) \geq 2^{-m} T_m(A)$ and $\dim(A_\ast) \leq \exp(O(M \log M)) \cdot \log |A|$.
Thus we obtain the second part of Theorem 3 of the introduction and it shows one more time that the condition of having large $T_k$ is roughly equivalent to the condition of having small dimension.

From (48), it follows that for any $x \in G$ and an arbitrary $A \subseteq G$ one has $\dim(A + x) \lesssim \dim(A)$. We improve the last bound in Theorem 32 below.

**Theorem 32** Let $G$ be an abelian group, $A, X \subseteq G$ be sets. Then

$$\dim(A) \ll \dim(A + X) \ll |X|\dim(A).$$

(73)

In particular, for any $x \in G$ one has

$$\dim(A) \sim \dim(A + x).$$

(74)

**Proof.** We begin with (74). Let $d = \dim(A)$. Consider a dissociated set $\Lambda + x \subseteq A + x$ such that $\dim(A + x) = |\Lambda| = D$. Our task is to show that $d \gg D$. If $\Lambda$ is a dissociated set, then $D \leq d$ and there is nothing to prove. Thus we can suppose that $\Lambda$ is not a dissociated set but nevertheless, we show that $\Lambda$ is rather close to be dissociated. Indeed, for any positive $k$ we have $T_k(\Lambda) = T_k(\Lambda + x)$ and thus by Theorem 9 one has $T_k(\Lambda) \leq (Ck)^k|\Lambda|^k$. We now substitute this bound into the proof of Theorem 30. In the notation of this theorem we have the following restriction on the parameter $\Delta$

$$\Delta \log(8ed/\Delta) + \Delta \log(C\Delta) \ll \Delta \log D.$$  

Hence it is possible to choose $\Delta = cD$, where $c > 0$ is an absolute (small) constant. Using estimate (66) of Theorem 30 we get

$$m \log D - m \log(Cm) \ll \dim(\Lambda) \leq d,$$

Recalling that $m \geq \Delta$, we obtain the required result.

Now let us obtain (73). The first inequality follows from (74) due to

$$\dim(A) \sim \dim(A + x) \ll \dim(A + X),$$

where $x$ is an arbitrary element of $X$. Further by (74) and by subadditivity of $\dim(\cdot)$ one has

$$\dim(A + X) \lesssim \sum_{x \in X} \dim(A + x) \ll |X|\dim(A)$$

as required. \qed

In the next sections we will use the additive dimensions of a set $A$ to estimate $T_k(A)$. It is well–known that the later quantity can be used to estimate the Fourier coefficients of the characteristic function of $A$. Let us make a remark on a simple connection of the Fourier transform of $A$ and $\dim(A)$. Of course Proposition 33 is non–trivial for sets $A$ with $|A| = o(N)$ only.
Proposition 33 Let $N$ be a prime number, $A \subseteq \mathbb{Z}/N\mathbb{Z}$ and for all $r \neq 0$ one has $|\hat{A}(r)| \leq \varepsilon |A|$, $\varepsilon \leq 1/4$. Then $\dim(A) \gg \log N$.

Proof. Let $d = \dim(A)$ and $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subseteq A$ be a maximal dissociated subset of $A$. Suppose that $\dim(A) \leq c \log N$, where $c > 0$ is a sufficiently small absolute constant. Then by the Dirichlet Theorem there is $q \neq 0$ such that $\|q\lambda_j\| \leq (N - 1)^{1-1/d} < N/8$. In other words, $qA$ belongs to the following arithmetic progression $P = (-N/8, N/8)$. Consider another arithmetic progression $Q = P + P$, $|Q| < 2|P|$. Then for any $a \in P$ one has $r_{Q+Q}(a) \geq |P|$. Hence using the Fourier transform and the Parseval identity, we get

$$|A||P| \leq \sum_{a \in qA} r_{Q+Q}(a) < \frac{4|P|^2|A|}{N} + N^{-1} \sum_{r \neq 0} \hat{A}(qr)\hat{Q}(r)^2 < 2^{-1}|A||P| + 2\varepsilon|A||P| \leq |A||P|$$

and this is a contradiction. $\square$

5 On the additive dimensions of multiplicative subgroups

In this section we consider the case of multiplicatively rich sets, i.e. sets $A \subseteq \mathcal{R}$ with $|AA| \ll |A|$, e.g., multiplicative subgroups. The property of having small product set is rather restrictive and implies that all considered dimensions of Sections 3, 4 are essentially the same for $A$ with $|AA| \ll |A|$. In particular, it allows us to estimate $T_k^+(A)$ for such sets $A$. Also, we give rather good lower bounds for the additive dimensions of multiplicative subgroups in the prime field.

Lemma 34 Let $\mathcal{R}$ be a commutative ring without divisors of zero, $A \subseteq \mathcal{R}$ and $|AA| \leq D|A|$. Put $d = \dim(A)$. Then

$$T_k^+(A) \leq |A|^{2k \left(\frac{CkD^6 \log^2 d}{d}\right)^k},$$

where $C > 0$ is an absolute constant.

Now let $N$ be a prime, and $\mathcal{R} = \mathbb{R}$ or $\mathbb{Z}/N\mathbb{Z}$. Suppose that $S \subseteq A$. Then for any $s > 0$ one has

$$D_{s,N}(S) \gg \frac{|S|D_{s,N}(A)}{|A|D^3 \log(D|A|)/D_{s,N}(A))}.$$  \hfill (76)

Proof. Given a set $Z \subseteq \mathcal{R}$ we write $T_k(Z)$ for $T_k^+(Z)$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subseteq A$ be a maximal dissociated subset of $A$. Let us apply the standard probability argument (see, e.g., [46, Exercise 1.1.8]). We take elements of $A\Lambda\Lambda^{-1}$ with probability $p = \frac{C_* \log d}{d}$ (here $C_* > 0$ is an appropriate large constant) uniformly at random and form a set $X$. The probability that a fixed $z \in A\Lambda$ does not belong to $X\Lambda$ is $(1 - p)^d$ and hence the expectation of the cardinality of elements of $A$, which do not in $X\Lambda$ is at most $|AA|(1 - p)^d \leq D|A|(1 - p)^d$. Denoting this set as $\Omega$, we have $|\Omega| \leq D|A|(1 - p)^d$ and the expectation of size of $X$ is $p|A\Lambda\Lambda^{-1}| \leq p|AA^{-1}| \leq D^3|A|$ by the Plünnecke inequality (13). Hence applying Theorem 9 we derive

$$T_k^{1/2k}(A) \leq T_k^{1/2k}(\Omega) + \sum_{x \in X} T_k^{1/2k}(x\Lambda) \leq |\Omega|^{1-1/2k} + p|A|D^3 \sqrt{Ckd} \leq$$
where we have chosen $p$ actually, in this specific case Lemma 34 works better. By the pigeonhole principle there is $t$ we have $X$ we find two sets have $\ast$ This completes the proof.

Let us choose the parameter $p$ as required.

Recalling that $\alpha$ difference between $\dim(\Gamma)$ and $\dim(A)$, we get

$$4^{-k} T_k(A) \leq \frac{D^{2k-1}|A|^{2k-1} \exp(-C_* (2k - 1) \log d) + |A|^{2k} \left( \frac{C' C_*^2 k D^6 \log^2 d}{d} \right)^k}{|A|^{2k} \left( \frac{C k D^6 \log^2 d}{d} \right)^k}$$

as required.

Let us obtain estimate (76). We use the same argument replacing $\Lambda$ with $S$ and $d$ with $|S|$. Let us choose the parameter $p$ (the probability of our random choice) later. With high probability we find two sets $X$ and $\Omega$ such that $|X| \leq D^3|A|$, $|\Omega| \leq D|A|(1 - p)^{|S|}$ and $A \subseteq XS \cup \Omega$. We have

$$D_{s,N}(A) \leq |X| D_{s,N}(S) + |\Omega| \leq |X| D_{s,N}(S) + D|A|(1 - p)^{|S|} \leq 2 |X| D_{s,N}(S),$$

where we have chosen $p = \frac{C_* \log(D|A|/D_{s,N}(A))}{|S|}$ (here $C_* > 0$ is an appropriate absolute constant). Recalling that $|X| \leq D^3|A|$, we obtain the result. This completes the proof.

We now obtain an analogue of Lemma 13 for the dimension $\dim(A)$ and for sets with $|AA| \leq D|A|$.

**Corollary 35** Let $N$ be a prime, $A \subset \mathbb{R}$ or $A \subseteq \mathbb{Z}/N\mathbb{Z}$ be a set with $|AA| \leq D|A|$ and $k$ be a positive integer. Put $d = \dim_k(A)$ and suppose that for a certain $s > 0$ one has $D_{s,N}(A) \geq |A|/T$, $T \geq 1$. Then

$$d \gg \frac{s \log(N - 1)}{\log(kTD^3T\log(DT))}. \quad (77)$$

**Proof.** Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \subseteq A$ be a maximal $k$–dissociated subset of $A$. It means that for any $a \in A$ there is $t \in [k]$ such that $ta = \sum_{j=1}^d l_j \lambda_j$, where $|l_j| \leq k$. In other words, $ta \in \text{Span}_k(A)$. By the pigeonhole principle there is $t \in [k]$ and $S \subseteq A$ such that $|S| \geq |A|/k$ and for any $x \in S$ we have $tx \in \text{Span}_k(A)$. Using Lemma 13 we see that

$$D_{s,N}(S) \gg \frac{|S|}{kTD^3 \log(DT)}. $$

By the Dirichlet Theorem we find $q \in (\mathbb{Z}/N\mathbb{Z}) \setminus \{0\}$ such that $\|q\lambda_j/N\| \leq (N - 1)^{-d^{-1}}$. Since $tx \in \text{Span}_k(A)$, it follows that $\|qtx/N\| \leq dk(N - 1)^{-d^{-1}}$. Thus as in Lemma 13 we obtain

$$\frac{|S|}{kTD^3 \log(DT)} \ll \sum_{x \in S} \left\| \frac{qtx}{N} \right\|^s \leq |S| \frac{dk(N - 1)^{-sd^{-1}}}{}. $$

This completes the proof. 

We now show that in the case of multiplicative subgroups $\Gamma$ of $\mathbb{F}_p^*$ there is almost no difference between $\dim(\Gamma)$ and $\dim_{\alpha}(\Gamma)$. It allows us to use Theorem 30 to estimate $\mathcal{T}_k(\Gamma)$ but, actually, in this specific case Lemma 34 works better.
Lemma 36 Let $\Gamma < \mathbb{F}_p^*$ be a multiplicative subgroup, $k$ be a positive integer, and $\alpha \in (0, 1]$ be a real number. Then
\[
\frac{\alpha \cdot \dim(\Gamma)}{\log |\Gamma|} \ll \dim_{\alpha,k}(\Gamma) \leq \dim(\Gamma).
\] (78)

Proof. Let $t = |\Gamma|$ and consider an arbitrary set $B \subseteq \Gamma$ such that $T_k^+(B) \geq \alpha T_k^+(\Gamma)$. Our task is to estimate $\dim(B)$ from below. We have $T_k^+(\Gamma) = \sum_{x,y \in \Gamma} r_{(k-1)\Gamma - (k-1)\Gamma}(x-y)$. Clearly, the hermitian matrix $M(x,y) = r_{(k-1)\Gamma - (k-1)\Gamma}(x-y)$ is $\Gamma$–invariant in the sense $M(\gamma x, \gamma y) = M(x,y)$ for any $\gamma \in \Gamma$. Hence the eigenfunctions $f_\alpha$, $\alpha \in [t]$ of $M$ are just normalized characters of the subgroup $\Gamma$ (see [39, Proposition 3]). In particular, the main eigenfunction $f_1(x) = \Gamma(x)/t$ where $x$ runs over $\Gamma$ and the correspondent eigenvalue $\mu_1 = \langle M f_1, f_1 \rangle = T_k^+(\Gamma)/t$. It follows that
\[
\alpha T_k^+(\Gamma) \leq T_k^+(B) \leq \sum_{x,y \in B} r_{(k-1)\Gamma - (k-1)\Gamma}(x-y) = \langle MB, B \rangle = \sum_{\alpha \in [t]} \mu_\alpha(B, f_\alpha)^2 \leq T_k^+(\Gamma)|B|/t,
\]
and hence $|B| \geq \alpha t$. Using the random choice as in Lemma 34 we find $X \subseteq \Gamma$ such that $|X| \ll \alpha^{-1} \log t$ and $\Gamma \subseteq XB$. But then by the subadditivity of the dimension $\dim(\cdot)$, we get
\[
\dim(\Gamma) \leq |X| \dim(B) \ll \alpha^{-1} \log t \cdot \dim(B)
\]
as required. \qed

We now obtain some applications to the growth of multiplicative subgroups in $\mathbb{F}_p^*$.

Basis properties of very small subgroups were studied in [8, 25, 26]. In a natural way the authors of these papers were interested in obtaining upper bounds for exponential sums over such subgroups but in our approach we do not want to use this machinery. Nevertheless, both methods rest on lower bounds for the quantity $D_{2,p}$, see Lemmas 37, 38 below. We start with Theorem 4.2 of [26].

Lemma 37 Let $g \in \mathbb{F}_p^*$ has the multiplicative order equals $t$. Then for any $2 \leq r \leq \varphi(t)$ one has
\[
p^2 \cdot D_{2,p}(\{1, g, \ldots, g^{r-1}\}) \geq \left( \frac{p^{2(r-1)} t}{r^{\gamma_{r-1}}} \right)^{1/r},
\]
where $\gamma_{r-1}$ is $(r-1)th$ Hermite constant, $\gamma_{r-1} = O(r)$.

Also, we need [25, Theorem 1, Lemma 6].

Lemma 38 Let $g \in \mathbb{F}_p^*$ be a primitive root, $\varepsilon \in (0, 1)$, $k \geq k(\varepsilon)$ be a sufficiently large positive integer, $\beta = g^k$, $m = \lfloor 6 \ln p (\ln \ln p)^4 \rfloor$, and
\[
p \geq \frac{k \ln k}{\ln^{1-\varepsilon}(\ln k + 1)}.
\] (79)

Then
\[
D_{2,p}(\{\beta, \ldots, \beta^m\}) \geq \frac{1}{(\ln p)^{3e/4}}.
\] (80)
Using the results above we can obtain a good lower bound for $\dim(\Gamma)$ and $\dim_k(\Gamma)$.

**Corollary 39** Let $\Gamma < \mathbb{F}_p^*$ be a multiplicative subgroup, $t = |\Gamma|$. Then

$$\dim(\Gamma) \gg \min \left\{ \log p \log \log p \cdot \frac{\log p}{\log t}, \varphi(t) \right\}. \quad (81)$$

In particular, if $t \gg \frac{\log p}{\log \log p}$ for a certain $\varepsilon \in (0, 1)$, then

$$\dim(\Gamma) \gg \frac{\log p}{\log t}. \quad (82)$$

Similarly, for any positive integer $k$ one has

$$\dim_k(\Gamma) \gg \min \left\{ \log p \log \log p \cdot \frac{\log p}{\log t}, \frac{\log p}{\log k}, \varphi(t) \right\}. \quad (83)$$

**Proof.** Let $d = \dim(\Gamma)$ and $d_k = \dim_k(\Gamma)$. We start with (81). Applying Lemma 37 and Lemma 13 with $T = C p^{2/r^2} t^{1-1/r} r^{-1/r}$, where $C > 0$ is an appropriate constant and $r = \varphi(t)$, we see that

$$d \gg \log p \cdot \min\{r/\log p, \log^{-1} t, \log^{-1} d\}$$

as required. To obtain (82) one can use both Lemmas 37, 38 and we prefer to apply inequality (80) of Lemma 38 with $k = (p - 1)/t$ and $T = m(\ln p)^{3e/4}$. Here we have splitted the sequence $\{g^{(p-1)/t}\}_{j=1}^m$ onto subsequences of length $m$ and also we have assumed that $t \geq m$. One has

$$d \gg \min \left\{ \log p \log \log p \cdot \log m, \log \log \log m \right\} \gg \frac{\log p}{\log t}.$$ 

Also, by the assumption of Lemma 38 we need to check that $k \gg \varepsilon 1$ but if not, then $t \gg \varepsilon p$ and bound (82) is trivial. Similarly, if $t < m$, then by the average arguments $T \sim m(\ln p)^{3e/4}$, $\log m \sim \log t$ and again estimate (82) follows.

To obtain (83), we get by Corollary 35 with $D = 1$ and our choice of the parameter $T = C p^{2/r^2} t^{1-1/r} r^{-1/r}$, $r = \varphi(t)$ that

$$d_k \gg \log p \cdot \min\{r/\log p, \log^{-1} t, \log^{-1} d_k, \log^{-1} k\}$$

as required. 

Using the obtained lower bounds for the dimensions from Corollary 39, we derive rather good upper bounds for the quantity $T_k^+ (\Gamma)$ in the case of small subgroups $\Gamma$.

**Corollary 40** Let $\Gamma < \mathbb{F}_p^*$ be a multiplicative subgroup. If $|\Gamma| \leq \frac{\log p}{\log \log p}$, then for any $k \geq 2$ one has

$$T_k^+ (\Gamma) \leq |\Gamma|^k (C_* k \log^2 |\Gamma| \cdot \log \log |\Gamma|)^k. \quad (84)$$
where $C_* > 0$ is an absolute constant. If $|\Gamma| \geq \log p$, then for any $k \geq 2$ one has

$$T_k^+(\Gamma) \leq |\Gamma|^{2k} \left( \frac{C_* k \log |\Gamma| \cdot \log^2 (\log |\Gamma|) p}{\log p} \right)^k, \quad (85)$$

and if $\frac{\log p}{\log \log p} \leq |\Gamma| \leq \log p$, then

$$T_k^+(\Gamma) \leq |\Gamma|^{2k} \left( \frac{C_* k \cdot \log^2 (\log p)}{\min \{ \phi(t), \frac{\log p}{\log \log p} \}} \right)^k. \quad (86)$$

In particular, if $|\Gamma| \sim \log p$, then

$$|k\Gamma| = \Omega \left( \frac{|\Gamma|}{k \log^2 |\Gamma|} \right)^k. \quad (87)$$

**Proof.** Let $d = \dim(\Gamma)$ and $t = |\Gamma|$. Everything follows from Corollary 39. Indeed, if $t \leq \frac{\log p}{\log \log p}$, then the minimum in (81) is attained at $\phi(t)$ and hence by Lemma 34 and the Cauchy–Schwarz inequality, we get

$$T_k^+(\Gamma) \leq t^{2k} \left( \frac{d}{C_* k \log^2 d} \right)^{-k} \leq t^{2k} \left( \frac{\phi(t)}{C_* k \log^2 t} \right)^{-k} \leq t^{2k} \left( \frac{t}{C_* k \log^2 t \cdot \log \log t} \right)^{-k}. \quad (85)$$

Similarly, if $t \geq \log p$, then the minimum in (81) is attained at $\frac{\log p}{\log t}$ and estimate (85) follows. Finally, if $\frac{\log p}{\log \log p} \leq |\Gamma| \leq \log p$, then $\dim(\Gamma) \gg \min \{ \phi(t), \frac{\log p}{\log \log p} \}$ and we obtain (86). This completes the proof. \qed

An alternative method to obtain bound (87) is to use estimate (35) or formulae (38), (39) of Lemma 18.

Finally, we obtain Theorem 5 from the introduction, exploiting the stronger fact that there is a good lower bound for $\dim_k(\Gamma)$ for rather large $k$.

**Corollary 41** Let $\Gamma < \mathbb{F}_p^*$ be a multiplicative subgroup. Suppose that $\phi(|\Gamma|) \log |\Gamma| \geq \log p$, $|\Gamma| \leq (\log p)^C$, where $C \geq 1$ is an absolute constant. Then there is $n = O(\log^2 p / \log \log p)$ such that $|n\Gamma| \geq p^{O(1/C)}$.

Further if $|\Gamma| \leq \log p$, then for $n = O(\phi^2(t) \log t)$ one has $|n\Gamma| \geq \exp(\log t \cdot \Omega(\min \{ \phi(t), \frac{\log p}{\log \log p} \}))$.

**Proof.** Let $t = |\Gamma|$ and take $k = t \log t \geq \dim(\Gamma) \log \dim(\Gamma)$. Then by Corollary 39 and our assumption $\phi(|\Gamma|) \log |\Gamma| \geq \log p$ we see that $\dim_k(\Gamma) \gg \log p / \log t$. On the other hand, clearly, $\dim_k(\Gamma) \ll \log p / \log t$. Applying formula (44) of Lemma 20 we obtain

$$|n\Gamma| \gg \exp(\Omega(\log p / \log t \cdot \log(\log p / \log t) - \log n)) \gg p^{O(1/C)},$$

where $n = O(\log^2 p / \log \log p)$. The second part of Corollary 41 can be obtained in a similar way, just notice that $\dim_k(\Gamma) \gg \min \{ \phi(t), \frac{\log p}{\log \log p} \}$. This completes the proof. \qed
6 Dimensions and the sum–product phenomenon

We begin this section with estimating multiplicative dimensions of the difference sets $A - A$ for sets $A \subseteq \mathbb{R}$ such that $|A + A| \ll |A|$. Our new inclusion \((88)\) is interesting in its own right.

**Theorem 42** Let $k$ be a positive integer and $A$ be a finite subset of an abelian ring $\mathcal{R}$ such that $|A + A| \leq K|A|$. Put $D = A - A$. Then

$$\frac{n}{|n|} \subseteq D/D, \quad \text{where} \quad n = \exp(\Omega(\log |A|/\log K)). \quad (88)$$

In particular, for $A \subseteq \mathbb{R}$ the following holds

$$\dim_k^x(D) \geq \exp(\Omega(\log |A|/\log K)). \quad (89)$$

Hence for any $m \ll \log |A|$ and $A \subseteq \mathbb{R}$ one has

$$|D^m| \geq |D| \exp(m(\log |A|/\log K - \log \log |A|)), \quad (90)$$

as well as

$$|D^n| \geq \exp(\exp(\Omega(\log |A|/\log K))). \quad (91)$$

Now for $A \subseteq \mathbb{F}_p$ the following holds

$$\dim_k^x(D) \gg \min \left\{ n, \frac{\log p}{k^2(\log \log p)^3} \right\}, \quad (92)$$

and for an arbitrary $m$ one has

$$|(D/D)^m| \geq p^{-o(1)} \min\{n^m, p\}. \quad (93)$$

**Proof.** For any $\lambda_1, \lambda_2 \in \mathbb{Z} \setminus \{0\}$, we have in view of Theorem 11

$$\left|\{(a_1, a_2, a_3, a_4) \in A^4 : \lambda_1(a_1 - a_2) = \lambda_2(a_3 - a_4)\}\right| = E(\lambda_1 \cdot A, \lambda_2 \cdot A) \geq \frac{|A|^4}{|\lambda_1 \cdot A + \lambda_2 \cdot A|} \geq |A|^3 \exp(-O(\log K \cdot \log(1 + |\lambda_1|)(1 + |\lambda_2|))) > |A|^2$$

for any $\lambda_1, \lambda_2 \in [n]$, where $n = \exp(\Omega(\log |A|/\log K))$. Hence $\lambda_2/\lambda_1$ can be expressed as $\frac{a_1 - a_2}{a_3 - a_4}$ or, in other words, $\lambda_2/\lambda_1$ belongs to $D/D$. It gives us inclusion \((88)\).

Now there are $\Omega(n/\log n)$ primes in $[n]$ and thus in view of \((88)\) or inequality \((89)\) of Lemma 23 we get

$$\dim_k^x(D) \gg \frac{n}{\log^2 n} \gg \exp(\Omega(\log |A|/\log K)).$$

To obtain \((90)\) and \((91)\) it remains to use estimates \((88)\), \((89)\) of Lemma 18. To get \((92)\) we take $l \leq n$ such that $l^{kl} < p/2$. Then all primes up to $l$ form a $k$–dissociated set $\Lambda$ modulo $p$. The condition $l^{kl} < p/2$ is equivalent to $l \ll \log p/(k \log \log p)$ and by the prime number theorem we have $|\Lambda| \gg \log p/k(\log \log p)^2$. Hence $\dim_k^x(D/D) \geq |\Lambda|$ and it remains to apply estimate \((89)\) of Lemma 23 with $l = k$ and $k = 2$. Notice that we write $n$ in \((92)\) but not $n/\log n$ because we can just decrease the constant in the symbol $\Omega$ in the definition of the number $n$ from \((88)\).

Finally, to get \((93)\) just use a trivial bound $|(D/D)^m| \geq |[n]^m|$ and apply the standard calculations with the divisor function. This completes the proof. □
Remark 43 If one switches the operations in the result about, then it is easy to obtain that $[n]/[n] \subseteq L/L$, where $L = \log(A/A)$ for any $A \subseteq \mathbb{R}$, say, and hence $\dim^\times(L) \geq \exp(\Omega(\log |A|/\log K))$ for an arbitrary $A$ with $|AA| \leq K|A|$.

The dependence on $m$ in (90) is not logarithmic as in [10], [11] or [23] and the whole bound is much better than [41] Theorem 2. As in [41] bound (90) is a step towards the main conjecture from [1], where authors do not assume that the additional condition of the doubling constant takes place. We now obtain a result in $\mathbb{F}_p$ of the same spirit. It is a byproduct of inclusion (88).

Corollary 44 Let $p$ be a prime number, $\delta \in (0,1)$ be a real number, and $\Gamma \leq \mathbb{F}_p^*$ be a multiplicative subgroup, $|\Gamma| \leq p^{1-\delta}$. Suppose that $A - A \subseteq \Gamma$ and $|A + A| \leq K|A|$. Then

$$|A| \ll \exp(C \log K \cdot \sqrt{\delta^{-1} \log(1/\delta) \log p}),$$

(94)

where $C > 0$ is an absolute constant.

Proof. Let $D = A - A \subseteq \Gamma$. Applying (88) of Theorem 42 we find $n = \exp(\Omega(\log |A|/\log K))$ such that

$$[n] \subset [n]/[n] \subseteq D/D \subseteq \Gamma.$$

By [39] Proposition 7 if $P = a \cdot [k] \subseteq \Gamma$ for an arbitrary $a \neq 0$, then

$$|P| \leq \exp(C \sqrt{\delta^{-1} \log(1/\delta) \log p}),$$

where $C > 0$ is an absolute constant. Putting $a = 1$ and $k = n$, we obtain the required estimate. This completes the proof. \hfill \square

We now obtain the first result on dimensions of sets with small sumset/difference set. In view of forthcoming Theorem 50 these rather simple bounds (95), (96) are not so weak.

Proposition 45 Let $A \subseteq \mathbb{R}$ be a set such that $|A + A| \leq K|A|$ or $|AA| \leq K|A|$. Then

$$\dim^\times(A), \dim^+(A) \gg \log |A| \cdot \log \left(\frac{\log |A|}{\log K}\right),$$

(95)

respectively. Now if $A \subseteq \mathbb{Z}$ and $|AA| \leq K|A|$, then

$$\dim^+(A) \gg \frac{\log^2 |A|}{\log K} \cdot \log \left(\frac{\log |A|}{\log K}\right).$$

(96)

Proof. Let $L = \log |A|$ and $d = \dim(A)$. If $d \gg L \log(L/\log K)$, then there is nothing to prove. To obtain (95) we apply [11] Theorem 1.4 [in the case $|AA| \leq K|A|$ one can alternatively use [40] Theorem 5]. For example, consider the case $|AA| \leq K|A|$. Thanks to [11] Theorem 1.4 with $f(x) = \exp(x)$, we have for all $k$

$$T_{2^k}^+(A) \ll K^{3.2^{k+1}} |A|^{2^{k+1} - k + O(1)}.$$
In other words, $T_k^+(A) \ll K^{6k}|A|^{2k-\log k+O(1)}$. We now substitute this bound in the proof of Theorem 30. In the notation of this theorem we have the following restriction on the parameter $\Delta$

$$\Delta \log(8eKd/\Delta) \ll L \log \Delta.$$  \hspace{1cm} (97)

Hence it is possible to choose $\Delta = c \frac{L}{\log K} \log \left( \frac{L}{\log K} \right)$, where $c > 0$ is an absolute small constant (recall that $d \ll L \log(L/\log K)$). Using estimate (66) of Theorem 30 we get

$$L \log m \ll d + m \log K.$$  \hspace{1cm} (98)

Recalling that $m \geq \Delta$ and using the bound $\log m \gg \log(L/\log K)$, we obtain the required result.

In a similar way, applying Theorem 10, we get for any $\varepsilon \in (0,1)$ and $k \geq 2$

$$T_k^+(A) \leq 10^k \beta^{k/\varepsilon}(A)|A|^{k+2k\log k}.$$  \hspace{1cm} (99)

Clearly, by the Plünnecke inequality (14) one has $\beta(A) \leq K^3$. Inserting bound (98) into the proof of Theorem 30 we need to estimate the parameter $\Delta$ similar to (97). Choosing $\varepsilon \sim 1/\Delta \log \Delta$, we obtain

$$mL \ll d + m \varepsilon^{-1} \log K \ll d + m^2 \log m \log K.$$  \hspace{1cm} (100)

Using the fact that $m \geq \Delta$, we get $d \gg \frac{\log^2 |A|}{\log K} \cdot \log \left( \frac{\log |A|}{\log K} \right)$. This completes the proof. \hspace{1cm} $\square$

Our next step is to obtain a bound similar to (24) for the energy $T_{2k-1}(A)$. It is well–known by the Balog–Szemerédi–Gowers Theorem that the property a set $A$ has small sumset correlates with the largeness of its additive energy. We show that the smallness of $\beta(A)$ is connected with the fact that the energy $T_k(A)$ is large.

**Theorem 46** Let $A \subseteq G$ be a set, $k$ be a positive integer and $K \geq 1$ be a real parameter. Suppose that $T_k(A) = |A|^{2k-1}K^{1-k}$. Then there is $A_* \subseteq A$ such that $|A_*| \geq |A|/M$, $\beta(A_*) \leq M$, \hspace{0.5cm} (100)

where

$$M = \exp(Ck \log kK/\log k),$$  \hspace{1cm} (100)

and $C > 0$ is an absolute constant.

**Proof.** Write $T_j$ for $T_j(A)$. We have $T_k = |A|^{2k-1}K^{1-k}$ and hence there is $j \in [k]$ such that $T_j \geq |A|^2T_{j-1}/K$. We take the largest $j$ with this property. In particular,

$$|A|^{2k-1}K^{1-k} = T_k \leq (|A|^2/K)^{k-j}T_j \leq (|A|^2/K)^{k-j}|A|^2T_{j-1}.$$  \hspace{1cm} (101)
Putting \( L = \log(8K|A|^{2j-3}T_{j=1}^{-1}) \) and using the last formula, we see that \( L \leq \log(8K^j) \leq 16k \log K \). Now by the dyadic Dirichlet principle and the Hölder inequality there is a number \( \Delta > 0 \) and a set \( P = \{x \in G : \Delta < r_{(j-1)}(x) \leq 2\Delta\} \), such that

\[
2L^2 \Delta^2 E(P, A) \geq T_j \geq \frac{|A|^2 T_{j=1}^{-1}}{K} \geq \frac{|A|^2 \Delta^2 |P|}{K}, \tag{102}
\]

and hence \( E(P, A) \geq |P||A|^2/(2L^2K) := |P||A|^2/K_* \). We apply Lemma 52 from the Appendix with \( A = P, B = A \) and \( K = K_* \). According to this Lemma, for any \( l \) we find a set \( H \subseteq G \) such that \( |H + H| \ll M^C|H| \) and for a certain \( x \in G \) one has \( |A \cap (H + x)| \gg |B|/M^C \). Here \( C > 0 \) is an absolute constant and \( M = (|P|/|A|)^{1/2} K_*^{2l/l} \). Put \( A_* = A \cap (H + x) \). Then by the Plünnecke inequality 14 and definition 23 of the quantity \( \beta \) one has

\[
\beta(A_*) \leq \frac{|A_* + H + H|}{|H|} \leq \frac{3H}{|H|} \ll M^{3C}.
\]

It remains to choose \( l \) and obtain a good upper bound for \( M \). Using 102 and 101, we see that

\[
2L^2 |A|^2 \geq 2L^2 |A|^2 (\Delta |P|^2) \geq T_j |P| \geq \frac{|A|^2 T_{j=1}^{-1} |P|}{K} \geq |A|^2 |K^{-j} |P|, \tag{102}
\]

and hence \( |P| \leq 2L^2K \bar{K}|A| \). It follows that

\[
M \leq (2L^2 K^k)^{1/l}(2L^2 K)^{2l/l
\]

Recalling that \( L \leq 16k \log K \) and choosing \( \bar{L} \) optimally as \( \bar{L} = \log k \), we obtain

\[
M \leq \exp(C_* k(\log kK/ \log k)),
\]

where \( C_* > 0 \) is an absolute constant. This completes the proof.

Theorem above implies the first (asymmetric) decomposition result.

**Theorem 47** Let \( A \subseteq \mathbb{Z} \) be a set, and \( q, s \) be some integer parameters, \( sq \log q \ll \log |A| \). Then there exist pairwise disjoint sets \( B \) and \( C \) such that \( A = B \uplus C \) and

\[
T^+_q(B) \leq |B|^{8q/5} \quad \text{and} \quad T^+_s(C) \leq |A|^{2s-c \log s \over \log 2s} \tag{103}
\]

Here \( c_* > 0 \) is an absolute constant.

**Proof.** Let \( L = \log |A| \), and \( \beta = \beta(A) \). Choose a number \( K \) such that \( |A|^{2s-c'(q \log q)^{-1} \log s} := |A|^{2s-1 K^{1-s}} \), where the constant \( c' > 0 \) is sufficiently small. Our proof is a sort of an algorithm. We construct a decreasing sequence of sets \( A = C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k \) and an increasing sequence of sets \( \emptyset = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_{k-1} \subseteq A \) such that for any \( j \in [k] \) the sets \( C_j \) and \( B_{j-1} \) are disjoint and moreover, \( A = C_j \uplus B_{j-1} \). If at some step \( j \) we have \( T^+_s(C_j) \leq |A|^{2s-1 K^{1-s}} \) we stop and set \( C = C_j \), \( B = B_{j-1} \), and \( k = j - 1 \). Else, we have \( T^+_s(C_j) > |A|^{2s-1 K^{1-s}} \). In particular, \( |C_j| \geq |A|K^{2s-1} \geq |A|K^{1/2} \). We apply Theorem 16 to the set \( C_j \), finding \( D_j \subseteq C_j \)

such that $|D_j| > |C_j|/M$, $\beta := \beta(D_j) \leq M$ and $M$ is given by formula (100), that is, $M = \exp(O(s \log s K / \log s))$. We will assume that $|D_j| \geq |A|^{1/2}$, say. Using Theorem 10 with $A = D_j$ and $\varepsilon \sim 1/(q \log q)$, we obtain

$$T_q^+(D_j) \leq 10^q q^{q/\varepsilon} |D_j|^{q+2q \log q} \leq \beta^{100q^2 \log q} |D_j|^{q+q/10} \leq |D_j|^{q+q/5},$$

provided

$$q \log q \cdot \log \beta \ll q \log \beta \cdot s \log s K / \log s \ll L. \quad (104)$$

After that we put $C_{j+1} = C_j \setminus D_j$, $B_j = B_{j-1} \cup D_j$ and repeat the procedure. Clearly, our algorithm stops after at most $K^{1/2} M$ number of steps. Also, it is easy to see that the second estimate in (103) holds with $c_s = c'/2$, say. It remains to check the first inequality from (103).

From the norm property of the energies $T_q^+$ and our condition (104) one has

$$T_q^+(B) \leq \left( \sum_{j=1}^k |D_j|^{3/5} \right)^{2q} \leq |B|^{q+q/5} \cdot (M \sqrt{K})^{2q} \leq |B|^{8q/5}.$$

Similarly, condition (104) gives us $M \sqrt{K} \leq |A|^{1/4}$ and hence $|D_j| \geq \sqrt{|A|}$ as required. Finally, by the choice of the quantity $K$ one has

$$s \log K = \log K^s = c' L (q \log q)^{-1} \log s = 2c_s L (q \log q)^{-1} \log s$$

and thanks to our assumption

$$sq \log q \ll L$$

we see that condition (104) satisfies. This completes the proof. \hfill \Box

We now obtain our new decomposition result in the spirit of [30, Corollary 1.3] and [2].

**Corollary 48** Let $A \subset \Bbb Z$ be a set and $s$ be an integer parameter,

$$s \ll \frac{\log |A|}{\sqrt{\log \log |A| \cdot \log \log \log |A|}}. \quad (105)$$

Then there exist pairwise disjoint sets $B$ and $C$ such that $A = B \cup C$ and

$$\max\{T_s^+(B), T_s^+(C)\} \leq |A|^{2s - \frac{c_s \log q}{\log \log q}}, \quad (106)$$

where $c_s > 0$ is an absolute constant.

**Proof.** We apply Theorem 47 with $q \sim \frac{\log s}{q \log q}$, that is, $q \sim \sqrt{\log s / \log \log s} \ll s$. Then in view of our assumption (105), we see that the condition $sq \log q \ll \log |A|$ takes place. Hence we obtain $T_s^+(C) \leq |A|^{2s - \frac{c_s \log q}{\log \log q}}$ and

$$|B|^{2q - 2s} T_s^+(B) \leq T_q^+(B) \leq |B|^{2q - \frac{c_s \log q}{\log \log q}}.$$

Thus we have obtained the required bound (106). \hfill \Box

Corollary 48 implies a result on additive/multiplicative Sidon sets in $\Bbb Z$, see the details of the proof in [31, Theorem 1.1]. Recall that a finite set $A \subset \Bbb G$ to be a $B^+_h[g]$ set if for any $x \in \Bbb G$ one has $r_{hA}(x) \leq g$ (and similarly for $B^+_h[g]$).
Corollary 49 Let $h$ be a positive integer, $A \subset \mathbb{Z}$ be a finite set, and let $B$ and $C$ be the largest $B_h^+[1]$ and $B_h^x[1]$ sets in $A$ respectively. Then

$$\max\{|B|, |C|\} \gg |A|^\eta_h/h,$$

where $\eta_h \gg (\log h)^{1/2-o(1)}$. In particular, for any $A \subset \mathbb{Z}$

$$|hA| + |A^h| \gg_h |A|^{(\log h)^{1/2-o(1)}}. \quad (107)$$

Now we are ready to obtain a purely sum–product–type result for dimensions.

Theorem 50 Let $A \subset \mathbb{Z}$ be an arbitrary finite set. Then

$$\max\{\dim^+(A), \dim^x(A)\} \gg \log |A| \cdot \frac{\sqrt{\log \log |A|}}{\sqrt{\log \log \log |A|}}. \quad (108)$$

On the other hand, there is $A \subset \mathbb{Z}$ such that

$$\max\{\dim^+(A), \dim^x(A)\} \ll \log |A| \cdot \log \log |A|. \quad (109)$$

Proof. Let $L = \log |A|$. To get (108) we apply Corollary 48 and obtain pairwise disjoint sets $B$ and $C$ such that $A = B \sqcup C$ and estimate (106) takes place. Here $s \ll L/\sqrt{\log L \log \log L}$. Of course either $B$ or $C$ has size at least $|A|/2$ and suppose that this set is $B$. Thus we have very good upper bound (106) for all $T_s^+(B)$ and hence it is possible to apply estimate (66) of Theorem 30. More precisely, as in formulae (97), (99) above, we have

$$\Delta \log(8 \text{dim}^+(B)/\Delta) \ll L \sqrt{\frac{\log \Delta}{\log \log \Delta}}$$

and hence for $m \geq \Delta \sim L/\sqrt{\log L \cdot \log \log L}$ (one can assume that $\dim^+(A) \ll L^2$, say, and whence $\log \dim^+(A) \sim \log L$ because otherwise there is nothing to prove) and further

$$\dim^+(B) \gg L \sqrt{\frac{\log m}{\log \log m}}.$$

Thus we see that

$$\dim^+(A) \gg \dim^+(B) \gg \frac{L \sqrt{\log L}}{\sqrt{\log \log L}}$$

as required.

Now to obtain (109) we use the arguments from [18] and [47, Proposition 1.5]. Namely, let $s, h \geq 2$ be integer parameters, which we will choose later. Put

$$A = \left\{ \prod_{i=1}^s p_i^{l_i} : l_i \in [h] \right\},$$

where $p_i$ are the first $s$ primes.
where \( p_i, i \in [s] \) are the first \( s \) primes. We have \( |A| = h^s \) and

\[
\max A = \exp(h \sum_{i=1}^{s} \log p_i) = \exp(O(h s \log s)).
\]

Trivially, \( A \subseteq [\max A] \) and hence

\[
\dim^+(A) \ll \log(\max A) \ll h s \log s.
\] (110)

To estimate \( \dim^+(A) \), we use the same argument as in formula (57) of Proposition 26. Indeed, thanks to the fact that \( |A^n| \leq (hn)^s \), we obtain

\[
\dim^+(A) \ll n \exp(n^{-1}s \log(hn)) \ll s \log s.
\] (111)

Here we have chosen \( n \sim s \log s \) and \( h = 2 \). Thus both bounds (110), (111) are of the same quality and from \( |A| = h^s = 2^s \), we obtain the required result. \( \square \)

Remark 51 A similar construction as in Theorem 50 (see [7], Proposition 1.5) shows that one cannot obtain something better than \( \Omega(\log s/\log \log s) \) in estimate (110).

7 Appendix

For the convenience of the reader we give a short proof of an asymmetric version of the Balog–Szemerédi–Gowers Theorem with explicit dependence of the parameters on the quantity \( K := |A||B|^2 E^{-1}(A, B) \). Basically, we repeat the argument from the appendix of [34].

Lemma 52 Let \( A, B \subseteq G \) be sets, \( |A| \geq |B| \) and \( E(A, B) \geq |A||B|^2/K \). Also, let \( k \) be a positive integer, \( k \leq \frac{\log |B|}{\log \log |B|} \) and \( M := (|A||B|)^{1/k} K^{2^j/k} \). Then there is a set \( H \subseteq G \) such that \( |H + H| \ll M^C |H| \) and for a certain \( x \in G \) one has \( |B \cap (H + x)| \gg |B|/M^C \). Here \( C > 0 \) is an absolute constant.

Proof. We have

\[
|A||B|^2/K \leq E(A, B) = \sum_{x \in A} \sum_{y} r_{B-B}(y) A(x+y),
\]

and hence using the Hölder inequality several times, we get for any \( j \leq k \)

\[
|A||B|^{2j} K^{-2j-1} \leq \sum_{x \in A} \sum_{y} r_{2^{j-1}B-2^{j-1}B}(y) A(x+y).
\]

Now applying the Cauchy–Schwarz inequality one more time, we see that

\[
|A|^{-1} |B|^{2j+1} K^{-2j} \leq |A|^2 E^{-1}(A) |B|^{2j+1} K^{-2j} \leq T_{2j}(B).
\] (112)

Write \( T_j \) for \( T_j(B) \). Then from the last formula, we have by the pigeonhole principle that there is \( j \in [k] \) with \( T_{2j} \geq |B|^{2j} T_{2j-1}/M \). Put \( L = \log(8|B|^{2j+1-1}T_{2j}^{-1}) \). In view of estimate (112) we see that \( L \leq \log(8K^{2j} |A|/|B|) \leq \log(8K^{2^k} |A|/|B|) \). One can easily see that \( L \ll M^5 \), say.
Indeed, it is sufficient to check that $k \log \log(|A|/|B|) \ll \log(|A|/|B|) + 2^k$ but if not then $k \ll \log \log \log(|A|/|B|)$ and hence $k \log \log(|A|/|B|) \ll \log(|A|/|B|)$ and it gives us a contradiction (one can assume that $|A|$ is much larger than $|B|$, actually, if $A$ and $B$ have comparable sizes, then everything follows immediately from the usual Balog–Szemerédi–Gowers Theorem). Now by the dyadic Dirichlet principle, our choice of the number $L$ and the Hölder inequality there is a number $\Delta > 0$ and a set $P = \{x \in G : \Delta < r_{2j-1}B(x) \leq 2\Delta\}$ such that

$$2L^4\Delta^4E(P) \geq T_{2j} \geq \frac{|B|^{2j}T_{2j-1}}{M} \geq \frac{(\Delta|P|)^2\Delta^2|P|}{M},$$

and hence $E(P) \geq |P|^3/(2L^4M) := |P|^3/Q$. Applying the Balog–Szemerédi–Gowers Theorem [46, Theorem 32], we find a set $H \subseteq P$ such that $|H| \gg |P|/Q^C$ and $|H + H| \ll Q^C|H|$, where $C$ is an absolute constant. Using the definition of the set $P$, we have

$$\Delta|P|Q^{-C} \ll |H| < \sum_{x \in H} r_{2j-1}B(x) \leq |B|^{2j-1-1} \max_{x \in G} |B \cap (H + x)|.$$  (114)

Now applying (113), we get

$$\frac{|B|^{2j}T_{2j-1}}{M} \leq 2L^4\Delta^4|P|^3 \leq 2L^4T_{2j-1}(\Delta|P|)^2,$$

and hence $\Delta|P| \geq 2^{-1}L^{-2}|B|^{2j-1}$. Combining the last bound with (113), we find $x \in G$ such that $|B \cap (H + x)| \gg |B|/Q^{C+1}$. It remains to recall the definitions of $Q$ and $L$. This completes the proof.

\[\blacksquare\]

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