This work investigates dense packings of congruent hard infinitesimally–thin circular arcs in the two–dimensional Euclidean space. It focuses on those denotable as major whose subtended angle $\theta \in (\pi, 2\pi]$. Differently than those denotable as minor whose subtended angle $\theta \in [0, \pi]$, it is impossible for two hard infinitesimally–thin circular arcs with $\theta \in (\pi, 2\pi]$ to arbitrarily closely approach once they are arranged in a configuration, e.g. on top of one another, replicable ad infinitum without introducing any overlap. This makes these hard concave particles, in spite of being infinitesimally thin, most densely pack with a finite number density. This raises the question as to what are these densest packings and what is the number density that they achieve. Supported by Monte Carlo numerical simulations, this work shows that one can analytically construct compact closed circular groups of hard major circular arcs in which a specific, $\theta$–dependent, number of them (anti–)clockwise intertwine. These compact closed circular groups then arrange on a triangular lattice. These analytically constructed densest–known packings are compared to corresponding results of Monte Carlo numerical simulations to assess whether they can spontaneously turn up.

I. INTRODUCTION

Systems of hard (i.e. non–interacting except for non–intersecting) particles are elementary model systems with which to investigate (condensed) states of matter [1, 2]. Out of the many aspects of this investigation [3–8], one is the determination of those packings (i.e. single configurations) that exhibit the maximal density [4, 8]. In the two–dimensional Euclidean space ($\mathbb{R}^2$), it has long been mathematically proven that hard circles, the simplest hard particles, most densely pack in a triangular lattice [9]. Equally mathematically proven has been that hard convex particles, if centro–symmetric, most densely pack in a lattice [10] while, if non–centro–symmetric, pack in a double lattice that covers, as a minimum, a fraction of $\mathbb{R}^2$ equal to $\frac{\sqrt{3}}{2}$ [11]. For hard concave particles, the sole conjecturing what may be the densest packings may not be immediate, leaving aside devising a convincing mathematical proof that these packings are such indeed. In this context, numerical methods, that originated in physics, such as the Monte Carlo and molecular dynamics methods [12–16], become increasingly useful. In two recent works, event–driven molecular dynamics was used to determine the densest–known packings of hard convex and concave superdiscs [17] while a method of the Monte Carlo type [18] was used to determine the densest–known packings of an ample variety of hard convex and concave particles [19].

Out of the many generalisations of a circle, one views it as a special circular arc. It is then natural to consider, as hard, generally non–circular, particles, hard circular arcs in the entire interval of the subtended angle $\theta$, from the linear segment limit, corresponding to $\theta = 0$, to the circle, corresponding to $\theta = 2\pi$ [Fig. 1 (a–e)]. These hard particles are deceptively simple: they are generally concave and, in spite of being infinitesimally thin, they generally exclude an area to one another. The class of circular arcs can be divided into two sub–classes: (i) those...
circular arcs with $\theta \in [0, \pi]$, i.e. minor circular arcs [Fig. 1(f)], from the linear segment limit to the semi–circular arc; (ii) those circular arcs with $\theta \in [\pi, 2\pi]$, i.e. major circular arcs [Fig. 1(g)], from the semi–circular arc to the circle. Two hard particles of the former sub–class can arbitrarily closely approach: e.g. once they are arranged on top of one another. Since this arrangement is replicable ad infinitum, without introducing any overlap, there exist infinitely dense packings of these hard particles. For two hard particles of the latter sub–class, this two joint conditions do not hold any more. It is then natural to inquire what are their densest packings and what is the number density that they achieve.

This work attempts to answer this question. Supported by Monte Carlo numerical simulations, one can peculiarly interlace a specific number, depending on $\theta$, of hard major circular arcs to analytically construct compact closed circular groups; one can then arrange these compact closed circular groups on a triangular lattice. To appreciate the mechanism for constructing these compact closed circular groups, it is useful to examine the characteristics of the excluded area of two hard (major) circular arcs (section II A). Once this has been accomplished, the mechanism for constructing compact closed circular groups of a specific number of them is devised; it immediately leads to the densest–known packings by arranging these compact closed circular groups on a triangular lattice (section II B). To asses whether these compact closed circular groups can spontaneously either make up or undo, specific Monte Carlo calculations are then carried out. In general, the closed circular groups that do form on compression in these Monte Carlo calculations have a number of constituent hard major circular arcs smaller than that achievable by analytic construction; the analytically constructed compact closed circular groups are nevertheless able to unfasten on decompression in these Monte Carlo calculations (section II). This suggests that the spontaneous replication of the complete mechanism responsible for the formation of these optimal packings of hard major circular arcs, although possible in theory, will be extremely improbable in practice, but similar sub–optimal packings of hard (colloidal, granular) major circular arcs will nevertheless be readily accessible (section IV).

II. EXCLUDED AREA AND CONSTRUCTION OF THE DENSEST–KNOWN PACKINGS

A. excluded area

In a two–dimensional space, the excluded area of two hard particles is that surface whose constituent points the centroid of one particle cannot occupy due to the presence of the other particle otherwise the two hard particles would overlap. For two hard circular particles, the excluded area is the area of a circle whose centre coincides with the centre of one particle and whose radius is the sum of the radii of the two particles. For two hard non–circular particles, the excluded area depends on their fixed relative orientation. One of them, particle 1, can be held fixed, e.g. with its centroid at the origin of the Cartesian reference frame and an axis along one of the two Cartesian axes, while the other, particle 2, whose axis is rotated by a certain angle $\psi$ with respect to the particle 1 axis, displaces around, freely except for the constraint that it cannot overlap particle 1. While displacing, the particle 2 centroid is as if it effectively generate a region delimited by a boundary; all internal points are prohibited positions for the particle 2 centroid, since, if it occupied one of them, particle 2 would overlap particle 1, while all external points are permitted positions for the particle 2 centroid, since, if it occupies one of them, particle 2 does not overlap particle 1.

Their concavity and infinitesimal thinness make the area that two hard circular arcs exclude to one another peculiar: it deserves an examination.

Since a circular arc is axis–symmetric, it suffices to consider the angle $\psi \in [0, \pi]$. Once $\psi$ has been fixed, a pair configuration of hard circular arcs can be completely defined by $\vec{r}$, the distance vector separating the centres of the parent circles, whose radius is $R$, with $r = |\vec{r}|$ the modulus of this vector, and $\alpha$ the angle that it forms with the $x$–axis of the Cartesian reference frame (Fig. 2). One can define a function $f(\vec{r}; \psi)$ that takes on the value 1 if the two hard circular arcs overlap and the value -1 if they do not, the overlap condition being established according to an exact and suitable overlap criterion. The excluded–area boundary points can be considered as the zeros of $f(\vec{r}; \psi)$.

Without much loss of generality, one can focus on two hard major circular arcs. In this case, it is very useful to introduce the angle

$$\delta = (\theta - \pi) \in (0, \pi].$$

For each value of $\delta$, one should distinguish between the
case $0 \leq \psi \leq \delta$ and the case $\delta < \psi \leq \pi$.

In the case $0 \leq \psi \leq \delta$, one can distinguish between two situations: either the two respective contact points at which the two hard circular arcs touch are both internal [Fig. 3(a)] or at least one of these contact points is external [Fig. 3(b, c)]. In the former situation, the two hard major circular arcs behave as two hard circles, with the boundary points that follow a circumference of radius $2R$ [Fig. 3(a)]. In the latter situation, the two hard major circular arcs behave distinctively, with the boundary points that follow a circumference of radius $R$ [Fig. 3(b, c)]. This latter situation occurs for $\alpha \in (\alpha_1, \alpha_2)$ and $\alpha \in (\alpha_3, \alpha_4)$ with $\alpha_1 = \frac{\delta}{2} + \psi$, $\alpha_2 = \pi - \frac{\delta}{2} + \psi$, $\alpha_3 = \pi + \frac{\delta}{2}$ and $\alpha_4 = 2\pi - \frac{\delta}{2}$ [Fig. 3(d, e)]. For values of $\alpha$ equal to the mid-values $\alpha_{12} = \frac{\alpha_1 + \alpha_2}{2}$ and $\alpha_{34} = \frac{\alpha_3 + \alpha_4}{2}$, the two hard major circular arcs touch as peculiarly as in Fig. 3(c): each extremal point of one of them respectively touches an internal point of the other. On increasing the angle $\psi$, both extremes of the interval $(\alpha_1, \alpha_2)$ increase of the same quantity $\psi$ with respect to the values that they have at $\psi = 0$ [cf. panel (d) with panel (e) of Fig. 3]. While the shape of the boundary changes, the area that it encloses does not: the value of the excluded area stays constant throughout the interval $0 \leq \psi \leq \delta$ [cf. panel (d) and panel (e) of Fig. 3] the difference between the area of a circle of radius $2R$ and the excluded area consists of the sum of the two arched triangular regions comprised between $\alpha_1$ and $\alpha_2$ and between $\alpha_3$ and $\alpha_4$: on going from panel (d), corresponding to $\psi = 0$, to panel (e), corresponding to $\psi = 0.4\pi$, the former of these two arched triangular regions rotates round an axis passing through the point $(0,0)$ and perpendicular to the plane of an angle equal to $0.4\pi$ while the latter of these two arched triangular regions does not change: hence, that difference does not vary and consequently neither does the excluded area. It is pertinent to observe that, for $0 \leq \psi \leq \delta$, the excluded area of two hard major circular arcs coincides with that that two hard lunate particles, formed by the juxtaposition of the same major circular arc with the minor circular arc that subtends the ex- }
}
FIG. 3. Examples of three types of contact configuration between two hard circular arcs: (a) internal point – internal point; (b) internal point – extremal point; (c) each extremal point of one hard circular arc respectively touches an internal point of the other hard circular arc. In any of these panels, the extremal points of the hard circular arc held fixed are labelled as A and B while those of the displacing hard circular arc as C and D. The excluded area, the shaded region enclosed by the black continuous line, of two hard major circular arcs with \( \theta = 1.6\pi \), i.e. \( \delta = 0.6\pi \), for the angle of relative orientation \( \psi = 0 \) (d) and \( \psi = 0.4\pi \) (e). Discontinuous round lines either correspond to the circumference with centre \((0,0)\) and radius \(2R\) or to circumferences with the centres that are labelled as the extremal points in panels (a), (b) and (c) and radius \(R\). The special angles are: \( \alpha_1 = \frac{\delta}{2} + \psi \); \( \alpha_2 = \pi - \frac{\delta}{2} + \psi \); \( \alpha_3 = \pi + \frac{\delta}{2} \); \( \alpha_4 = 2\pi - \frac{\delta}{2} \). Lengths in panels (d) and (e) are in units of \(R\).

their capability of intertwining, without overlapping, so closely that the centres of their parent circles essentially coincide.

B. construction of the densest-known packings

That it is possible to closely intertwine two hard major circular arcs (section [IIA]) is exploited to demonstrate that it is actually possible to closely intertwine more hard major circular arcs thus arriving at constructing compact closed circular groups of them. Specifically, it is demonstrated that, for a given angle \( \delta = \theta - \pi \), one can arrange \( n = \left\lfloor \frac{2\pi}{\delta} \right\rfloor \), with \( \lfloor x \rfloor \) the strict floor function [20], hard major circular arcs so that the centres of their parent circles essentially coincide without making these hard particles overlap; i.e., \( n \) hard major circular arcs essentially arrange on the same circumference, thus maximally exploiting their concavity and infinitesimal thinness.

First, one observes that it is impossible to arrange more than \( n \) hard major circular arcs on the same circumference. If that were possible, there would exist a pair of hard major circular arcs whose angle of relative orientation \( \psi \) would be smaller than \( \delta \). However, as shown in section [IIA], two such hard major circular arcs would be incapable of approaching so closely to allow the centres of their parent circles to essentially coincide; i.e., they would be incapable of arranging on the same circumference.

Then, one demonstrates that it is exactly \( n \) the number of hard major circular arcs that can be arranged so that the centres of their parent circles essentially coincide; i.e., that can be arranged on the same circumference.

One begins by arranging the centres of the parent cir-
FIG. 5. Examples of three types of contact configuration between two hard major circular arcs in the case $\delta < \psi \leq \pi$: (a) the contact involves the concave side of one of the hard major circular arcs; such contact pair configurations correspond to the dark gray sector in panels (d, e); (b) the two hard major circular arcs are intertwined; such contact pair configurations correspond to the light gray sector in panels (d, e); (c) the two hard major circular arcs are so closely intertwined that the centres of their parent circles essentially coincide: such contact pair configurations correspond to the close neighbourhood of the point (0,0) in panels (d, e). In any of these panels, the extremal points of the hard major circular arc held fixed have been labelled as A and B while those of the displacing hard major circular arcs as C and D. The excluded area, the shaded region enclosed by the union of the black line, the dark gray line and the light gray line, of two hard circular arcs with $\theta = 1.67\pi$, i.e. $\delta = 0.67\pi$, for the angle of relative orientation $\psi = 0.65\pi$ (d) and $\psi = \pi$ (e). Discontinuous round lines either correspond to the circumference with centre (0,0) and radius 2R or to circumferences with centres that are labelled as the extremal points in panels (a, b) and radius R. The special angles are: $\beta_1 = \frac{\pi}{2} + \frac{\delta}{2}$, $\beta_2 = \frac{\pi}{2} + \frac{\delta}{2} + \frac{\psi}{2}$, $\beta_3 = \frac{3\pi}{2} - \frac{\delta}{2} + \frac{\psi}{2}$ and $\beta_4 = \frac{3\pi}{2} - \frac{\delta}{2} + \psi$. The angles indicated as $\Delta \psi$ are those formed by the segments starting off the point (0,0) and tangential to the dark gray and light gray lines; those segments that are tangent to the dark gray lines correspond to the special angles $\beta_1$ and $\beta_4$; the amplitude of these angles is $\psi - \delta$. Lengths in panels (d) and (e) are in units of R.

FIG. 6. For hard major circular arcs with $\theta = 1.3\pi$, i.e. $\delta = 0.3\pi$, $n = 6$. These hard major circular arcs are arranged with the centres of their parent circles on the vertices of a regular convex hexagon and their symmetry axes tangential to the circumscribing circumference whose radius is $R_{\text{crf}}$ and (anti-)clockwise rotating.

relative orientations with angles $\psi_k$ and $\psi_{n-k} = 2\pi - k\delta^*$ are equivalent, it suffices to consider the angles of rela-
that joins the centre of its parent circle with the centre one can appreciate that the angle aligned along the y–axis of the Cartesian reference frame, reference hard major circular arc has its symmetry axis relative orientation is convex hexagon while, in part (b), the corresponding ex-
circular arcs are positioned at the vertices of a regular the centres of the six parent circles of the hard major
exemplificative and explicative Fig. 7 where, in part (a), orientation
ψ

The conditions that define the centres of the parent circles are azimuthally arranged such that if the

θ
= 2 hard major circular arcs. Out of these, consider one hard major circular arcc as a reference with its symmetry axis forms an angle ψ
k
with the symmetry axis of the reference hard major circular arc. (b) The corresponding excluded area between the reference hard major circular arc and the k = 2 hard major circular arc in (a). The black filled circle marks the intersection of the segment, that starts off the point (0,0) and lies mid–way the segments tangential to the dark gray curve and light gray curve, with this dark gray curve. The angles

θ

and

Δψ

in (b) replicate those in (a). Lengths in panel (b) are in units of R.

that joins the centre of its parent circle with the centre one can appreciate that the angle aligned along the y–axis of the Cartesian reference frame, reference hard major circular arc has its symmetry axis relative orientation is convex hexagon while, in part (b), the corresponding ex-
circular arcs are positioned at the vertices of a regular the centres of the six parent circles of the hard major
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The conditions that define the centres of the parent circles are azimuthally arranged such that if the

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θ

and

Δψ

in (b) replicate those in (a). Lengths in panel (b) are in units of R.

ψ
k
≤ π. It is now useful to refer to the exemplificative and explicative Fig. 4 where, in part (a), the centres of the six parent circles of the hard major circular arcs are positioned at the vertices of a regular convex hexagon while, in part (b), the corresponding excluded area of two such hard major circular arcs, whose relative orientation is ψ
k
, is depicted for k = 2. If the reference hard major circular arc has its symmetry axis aligned along the y–axis of the Cartesian reference frame, one can appreciate that the angle α of the distance vector that joins the centre of its parent circle with the centre of the parent circle of the k–th other hard major cir-
cular arc lies mid–way the special angular interval comprised between the segments that start off the point (0,0) and are tangential to the dark gray and light gray curves [Fig. 5 (d, e) as well as Fig. 7 (b)]. One recalls that the amplitude of this angular interval is

Δψ
k
= ψ
k
− δ
k
(section II A). If the angle α is within such an angular interval, the k–th other hard major circular arc can arbitrarily closely approach the reference hard major circular arc. The modulus of the distance that separates the centres of the parent circle of these two hard major circular arcs is equal to 2R

crf
sin

(ψ
k
2
− δ
k
2
).

Provided

R

crf

is such that

R

crf

< \frac{\sin \left(\frac{k\delta^* - 2\pi}{2}\right)}{\sin \left(\frac{2\pi}{2}\right)},

the k–th other hard major circular arc does not overlap with the reference hard major circular arc. One can observe that

R

crf

is an increasing function of kδ*. Thus, it suffices that the condition is verified for k = 1. The condition

R

crf

< \frac{\sin \left(\frac{\delta^* - 2\pi}{2}\right)}{\sin \left(\frac{2\pi}{2}\right)}

allows one to progressively shrink the circumscribing circumference on which the centres of the parent circles are positioned without this leading to the n hard major circular arcs overlapping. Thus, as

R

crf
→ 0,

the centres of the parent circles ultimately end up to essentially coincide, i.e. the n hard major circular arcs ultimately end up to essentially arrange on the same circumference (Fig. 8).

Having succeeded to construct compact closed circular groups of n hard major circular arcs, it is then natural to arrange these circular groups on a triangular lattice [Fig. 9 (a)]. These generally non–lattice and non–periodic infinitely degenerate packings have a dimensionless number density

ρR^2(\theta) = \frac{1}{2\sqrt{3}} \left[\frac{2\pi}{\theta - \pi}\right] = \frac{n}{2\sqrt{3}}.

Observe that the strictness of the floor function that defines n is necessary to recover the densest packing of hard circles and its number density [Fig. 9 (c)]]. These generally non–lattice and non–periodic infinitely degenerate packings constitute the densest–known packings for hard major circular arcs.

One different packing strategy could have been to arrange the hard major circular arcs in one of the (non–)lattice infinitely degenerate packings where these hard particles are placed on top of one another as in Fig. 9 (b) [22]. One can appreciate that the dimensionless number density of
FIG. 8. Examples of the compact closed circular groups that can be analytically constructed: (a, b) θ = 1.05π, i.e. δ = 0.05π, leading to n = 39, and (a) R_{cnf} = 0.025 and (b) R_{cnf} = 2 \times 10^{-12}; (c, d) θ = 1.3π, i.e. δ = 0.3π, leading to n = 6, and (c) R_{cnf} = 0.1 and (d) R_{cnf} = 4 \times 10^{-5}; (e, f) θ = 1.6π, i.e. δ = 0.6π, leading to n = 3 and (e) R_{cnf} = 0.11 and (f) R_{cnf} = 0.03. Lengths are in units of R.

all these (non-)lattice packings [22] is given by

\[ \rho R^2(\theta) = \frac{1}{2 \cos \left( \pi - \frac{\theta}{2} \right) \sqrt{4 - \cos^2 \left( \frac{\theta}{2} \right)}}. \]

However, it is smaller than that of the packings formed by triangularly arranging the compact closed circular groups of n hard major circular arcs [Fig. 9(c)].

One could have attempted to combine the two packing strategies. First, one essentially arranges a number m of hard major circular arcs on the same circumference by successively rotating them of an angle infinitesimally larger than δ. The number m = 1, ..., p, with p = \left\lfloor \frac{\pi}{\delta} \right\rfloor [20], is sufficiently small that the group thus constructed is generally arched and open with an effective subtended angle equal to \( \theta^* = \theta + (m-1)\delta = \pi + m\delta \). Then, one arranges these arched open groups as it has been done with a single hard major circular arc in Fig. 9(b). The dimensionless number density of such packings is:

\[ \rho R^2(\theta) = \frac{m}{2 \cos \left( \pi - \frac{\theta}{2} \right) \sqrt{4 - \cos^2 \left( \frac{\theta}{2} \right)}}. \]

Irrespective of the value of δ, this function monotonically increases with m, attaining its maximum at m = p, i.e. once the arched and open groups have actually become closed and circular and ended up to arranging on a triangular lattice. However, p < n: the circular and closed groups thus constructed are not as numerous as those constructed by the method based on positioning the centres of their parent circles at the vertices of a regular convex polygon: consequently, the dimensionless number density achieved by these packings is smaller than that of the packings formed by triangularly arranging the compact closed circular groups of n hard major circular arcs.

III. MONTE CARLO CALCULATIONS

One may inquire whether closed circular groups similar to those described in section II B that constitute the structural units of the densest-known packings of hard major circular arcs may spontaneously form and how compact and numerous they result to be.

To address this point, specific Monte Carlo (MC) [12, 14, 16] calculations were carried out. In these calcula-
tions, systems of $N$ hard major circular arcs in a de-
formable parallelogrammatic container with hard walls
were compressed from a low to a high pressure. The ini-
tial value of $N$ was 2. These two hard major circular
arcs were initially placed in a large square container with
hard walls. The initial value for the dimensionless
pressure $P^\ast = \frac{P \sigma^2}{k_B T}$, with $P$ the pressure, $\sigma^2$ the area of that
part of a spherical surface subtended by the angle $\theta$, $k_B$
the Boltzmann constant and $T$ the absolute temperature,
was 1. Successively, $P^\ast$ was gradually increased in log-
arithmic steps up to a value equal to 100. For each of
these successive values of $P^\ast$, 1 million of MC cycles were
usually carried out, with a MC cycle defined as a set of
$N$ random translations of a randomly selected particle,
$N$ random rotations of a randomly selected particle and
one random change of the length and/or orientation of a
randomly selected side of the container. In the course of
these MC calculations, the maximal sizes for a random
translation, a random rotation and a random change of a
side of the container were progressively adjusted to en-
sure that a fraction comprised between 0.2 and 0.3 of
each of these trial moves was accepted. The acceptance
of a new shape and a new size for the container was
subject to the usual Metropolis criterion of a constant–
pressure MC calculation [13,16]. On completion of this
sequence of MC calculations, it was observed whether a
closed circular group was formed. The number of hard
major circular arcs $N$ was then increased by one and the
same sequence of MC calculations was repeated for that
new value of $N$. This incremental addition of hard major
circular arcs and successive MC–method–based compres-
sion of the system were carried on until it was observed
that the formation of a closed circular group of $N$ hard
major circular arcs did not occur. The largest value of
$N$ that led to the formation of a closed circular group
was then registered. For a number of values of $\theta$, this
entire process was repeated a number of times, usually
seven, changing the initial configurations of the $N$ hard
major circular arcs and the seed of the random num-
ber generator $\text{mt19937}[23]$. By operating in this way,
Table I was constructed. For a number of values of $\theta$, it
reports the seven largest values of $N$ that led to the
formation of a closed circular group. These numerical
results prove that the spontaneous formation of closed
circular groups is indeed possible, even for a value of $\theta$
as large as $359/180\pi$. Yet, one has to also observe that
these moderately fluctuating largest values of $N$ are usu-
ally incapable of equalling the larger number $n$ of hard
major circular arcs that is possible to closely intertwine
by analytic construction (section II.B).

Based on these results, one may then inquire whether
the compact closed circular groups that are analytically
constructed can spontaneously unfasten when a system
of them is decompressed in analogous MC calculations.
For values of the angle $\theta \geq 1.2\pi$, starting from a di-

tensionless pressure equal to 100 and decreasing it by
logarithmic steps until a value of 0.01 was reached, it
proved relatively easy to completely unfasten the analy-
tically constructed compact close circular groups. For
values of the angle $\theta < 1.2\pi$, this procedure was insuffi-
cient: these more compact and numerous analytically
constructed closed circular groups survived down to that
very small value of $P^\ast$. Nevertheless, they ultimately
succeeded to unfasten in relatively painful MC calcula-
tions that protracted up to $10^8$ million of MC cycles once
they were suitably modified so that the random change
of the container shape and size was accepted only if it
led to a larger container area.

| $\theta$ | $N_1$ | $N_2$ | $N_3$ | $N_4$ | $N_5$ | $N_6$ | $N_7$ | $n$ |
|---------|-------|-------|-------|-------|-------|-------|-------|-----|
| $\pi$   | 16    | 17    | 17    | 17    | 16    | 16    | 16    | $\infty$ |
| 1.05$\pi$ | 13    | 11    | 11    | 11    | 12    | 12    | 12    | 39   |
| 1.17$\pi$ | 8     | 8     | 9     | 9     | 9     | 9     | 9     | 19   |
| 1.15$\pi$ | 8     | 7     | 7     | 7     | 7     | 7     | 7     | 13   |
| 1.2$\pi$ | 6     | 6     | 6     | 6     | 7     | 6     | 6     | 9    |
| 1.25$\pi$ | 6     | 5     | 5     | 5     | 5     | 5     | 5     | 7    |
| 1.3$\pi$ | 5     | 5     | 5     | 4     | 4     | 5     | 5     | 6    |
| 1.21/90$\pi$ | 4    | 4     | 4     | 4     | 4     | 4     | 4     | 5    |
| 1.4$\pi$ | 4     | 4     | 4     | 4     | 4     | 4     | 4     | 4    |
| 1.45$\pi$ | 3     | 3     | 3     | 3     | 3     | 3     | 3     | 4    |
| 1.5$\pi$ | 3     | 3     | 3     | 2     | 3     | 3     | 3     | 3    |
| 1.55$\pi$ | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 3    |
| 1.6$\pi$ | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 3    |
| 1.95$\pi$ | 2     | 2     | 2     | 2     | 2     | 2     | 2     | 2    |
| 359/180$\pi$ | 2 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |

TABLE I. Largest value of the number of hard major circular arcs per closed circular group, $N_i$, attained in the $i$–th sequence of Monte Carlo calculations on compression, as well as the corresponding $n$, as a function of the subtended angle $\theta$.

### IV. CONCLUSION

In this work, densest–known packings of congruent
hard infinitesimally–thin circular arcs are analytically
constructed. The interest is actually in those hard
infinitesimally–thin circular arcs denotable as major
whose subtended angle $\theta \in (\pi,2\pi)$. In spite of being
infinitesimally thin, there exists no pair configuration of
them where they arbitrarily closely approach and which
is replicable ad infinitum, without introducing any over-
lap, as it occurs for hard infinitesimally–thin minor circu-
lar arcs whose subtended angle $\theta \in [0,\pi)$. Nevertheless,
it is shown that hard infinitesimally–thin major circular
arcs can be carefully arranged in compact closed circular
groups formed by $n = \left\lfloor \frac{2\pi}{\theta - \pi} \right\rfloor [20]$ of them. These
compact closed circular groups can then be arranged with
the communal centres at the sites of a triangular lattice thus
leading to generally non–lattice non–periodic infinitely
degenerate packings with dimensionless number density
equal to $\frac{n}{2\sqrt{3}}$. These densest–known packings are classi-
fiable as purely entropy-driven cluster (porous) crystals. It is notable as very simple particles, such as these hard infinitesimally-thin concave particles, devoid of any attractive or complicated interactions between them, can first cluster and then these clusters act as structural units to form a (porous) crystal. Monte Carlo calculations confirm that these clusters, albeit not as numerous as those analytically constructed, can spontaneously form on compression.

On these premises, the investigation of the complete phase diagram of systems of hard infinitesimally-thin circular arcs will prove interesting. Once the latter phase diagram will have been mapped, one will possibly move on to considering dense packings and phase behaviour of hard finitely-thin circular arcs. In actuality, one real example of these systems has been recently investigated [24] [25]: hard colloidal C-shaped particles, corresponding to hard finitely-thin circular arcs with \( \theta = \frac{3}{2} \pi \), have been prepared and their phase behaviour first experimentally investigated in a tilted two-dimensional gravitational column [24] and then theoretically rationalised [25]: dimerisation has been observed. It is presumable that progressive hard-(colloidal, granular)-particle thickening will increasingly impede reaching a number of intertwined hard (colloidal, granular) major circular arcs as large as in the infinitesimally-thin case but it should still prove interesting to generally examine up to which extent.

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20. The strict floor function is \( \lfloor x \rfloor = -1 - \lceil -x \rceil \), with \( \lfloor x \rfloor \) the ordinary ceiling function.
21. These packings are generally non-lattice [3] [8] as their fundamental cell generally contains more than one hard particle. They are also generally non-periodic [4] [8] as the hard major circular arcs that form a compact closed circular group can be arranged either anti- or clockwise and the compact closed circular groups can be rotated of an arbitrary angle with respect to one another. They are infinitely degenerate as these arbitrary rotations do not affect the value of the number density.
22. Irrespective as to whether the hard major circular arcs orient the unit vectors that join the centre of their parent circle with the respective vertex either parallel, i.e. they form a lattice packing, or anti-parallel, i.e. they form a non-lattice packing such as the one in Fig. 9 (b), the number density is the same. Each of these (non-)lattice packings of hard major circular arcs coincides with the corresponding packing of the corresponding hard lunate particles such as in Fig. 4.
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