A Strict Inequality for a Minimal Degree of a Direct Product

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Abstract

The minimal faithful permutation degree $\mu(G)$ of a finite group $G$ is the least non-negative integer $n$ such that $G$ embeds in the symmetric group $\text{Sym}(n)$. Work of Johnson and Wright in the 1970’s established conditions for when $\mu(H \times K) = \mu(H) + \mu(K)$, for finite groups $H$ and $K$. Wright asked whether this is true for all finite groups. A counter-example of degree 15 was provided by the referee and was added as an addendum in Wright’s paper. Here we provide a counter-example of degree 12.

1 Introduction

The minimal faithful permutation degree $\mu(G)$ of a finite group $G$ is the least non-negative integer $n$ such that $G$ embeds in the symmetric group $\text{Sym}(n)$. It is well known that $\mu(G)$ is the smallest value of $\sum_{i=1}^{n} \lvert G : G_i \rvert$ for a collection of subgroups $\{G_1, \ldots, G_n\}$ satisfying $\bigcap_{i=1}^{n} \text{core}(G_i) = \{1\}$, where $\text{core}(G_i) = \bigcap_{g \in G} G_i^g$.

We first give a theorem due to Karpilovsky [3] which will be needed later. The proof of it can be found in [2] or [6].

**Theorem 1.1.** Let $A$ be a non-trivial finite abelian group and let $A \cong A_1 \times \ldots \times A_n$ be its direct product decomposition into non-trivial cyclic groups of prime power order. Then

$$\mu(A) = a_1 + \ldots + a_n,$$
where $|A_i| = a_i$ for each $i$.

One of the themes of Johnson and Wright’s work was to establish conditions for when

$$\mu(H \times K) = \mu(H) + \mu(K)$$

(1)

for finite groups $H$ and $K$. The next result is due to Wright [8].

**Theorem 1.2.** Let $G$ and $H$ be non-trivial nilpotent groups. Then $\mu(G \times H) = \mu(G) + \mu(H)$.

Further in [8], Wright constructed a class of groups $\mathcal{C}$ with the property that for all $G \in \mathcal{C}$, there exists a nilpotent subgroup $G_1$ of $G$ such that $\mu(G_1) = \mu(G)$. It is a consequence of Theorem (1.2) that $\mathcal{C}$ is closed under direct products and so (1) holds for any two groups $H, K \in \mathcal{C}$. Wright proved that $\mathcal{C}$ contains all nilpotent, symmetric, alternating and dihedral groups, however the extent of it is still an open problem. In [1], Easdown and Praeger showed that (1) holds for all finite simple groups.

The counter-example to (1) was provided by the referee in Wright’s paper [8] and involved subgroups of the standard wreath product $C_5 \wr \text{Sym}(3)$, specifically the group $G(5,5,3)$ which is a member of a class of unitary reflection groups. We give a brief exposition on these groups now.

Let $m$ and $n$ be positive integers, let $C_m$ be the cyclic group of order $m$ and $B = C_m \times \ldots \times C_m$ be the product of $n$ copies of $C_m$. For each divisor $p$ of $m$ define the group $A(m, p, n)$ by

$$A(m, p, n) = \{ (\theta_1, \theta_2, \ldots, \theta_n) \in B \mid (\theta_1 \theta_2 \ldots \theta_n)^{m/p} = 1 \}.$$ 

It follows that $A(m, p, n)$ is a subgroup of index $p$ in $B$ and the symmetric group $\text{Sym}(n)$ acts naturally on $A(m, p, n)$ by permuting the coordinates.

$G(m, p, n)$ is defined to be the semidirect product of $A(m, p, n)$ by $\text{Sym}(n)$. It follows that $G(m, p, n)$ is a normal subgroup of index $p$ in $C_m \wr \text{Sym}(n)$ and thus has order $m^n n!/p$.

It is well known that these groups can be realized as finite subgroups of $GL_n(\mathbb{C})$, specifically as $n \times n$ matrices with exactly one non-zero entry, which is a complex $m$th root of unity, in each row and column such that the product of the entries is a complex $(m/p)$th root of unity. Thus the groups $G(m, p, n)$
are sometimes referred to as monomial reflection groups. For more details on the groups $G(m, p, n)$, see [4].

2 Calculation of $\mu(G(4, 4, 3))$

Recall that $G(4, 4, 3) = A(4, 4, 3) \rtimes Sym(3)$, where

$$A(4, 4, 3) = \{(\theta_1, \theta_2, \theta_3) \in C_4 \times C_4 \times C_4 \mid \theta_1 \theta_2 \theta_3 = 1\}$$

which is isomorphic to a product of two copies of the cyclic group of order 4. Hence

$$G(4, 4, 3) \cong (C_4 \times C_4) \rtimes Sym(3).$$

From now on, we will let $G$ denote $G(4, 4, 3)$. A presentation for this group can be given thus

$$G = \langle x, y, a, b \mid x^4 = y^4 = b^3 = a^2 = 1, xy = yx, x^a = y, x^b = y, y^b = x^{-1}y^{-1}, b^a = b^{-1} \rangle.$$ 

Since $\langle x, y \rangle \cong C_4 \times C_4$ is a proper subgroup of $G$ we have by Theorem 1.1, that $8 = \mu(\langle x, y \rangle) \leq \mu(G)$. Moreover since $G$ is a proper subgroup of the wreath product $W := C_4 \wr Sym(3)$, for which $\mu(W) = 12$, we have the inequalities

$$8 \leq \mu(G) \leq 12.$$ 

We will prove that in fact $\mu(G) = 12$ by a sequence of lemmas.

**Lemma 2.1.** $\langle x^2, y^2 \rangle$ is the unique minimal normal subgroup of $G$.

**Proof.** Observe by the conjugation action of $a$ and $b$ on $x^2$ and $y^2$ that $M = \langle x^2, y^2 \rangle$ is indeed normal in $G$. Let $N$ be a non-trivial normal subgroup of $G$ so there exists an

$$\alpha = x^i y^j b^k a^l$$

in $N$ where $i, j \in \{0, 1, 2, 3\}, k \in \{0, 1, 2\}, l \in \{0, 1\}$ are not all zero. It remains to show that $M$ is contained in $N$.

**Case (a):** $k = l = 0$.

**Subcase (i):** $i = j$ so $\alpha = x^i y^i$.

Then $\alpha \alpha^b = x^i y^i y^{-1} x^{-1} y^{-1} = y^i \in N$, so $y^{-i} \alpha = x^i \in N$. But $i \neq 0$, so $M \subseteq \langle x^i, y^i \rangle$. Hence $M \subseteq N$, as required.
Subcase (ii): \( i + j \not\equiv 0 \mod 4 \).
Then \( \alpha \alpha = x^{i+j} y^{i+j} \) and we are back in Subcase (i).

Subcase (iii): \( i + j \equiv 0 \mod 4 \).
Then \( \alpha \alpha = x^{i-j} y^j \).
If \( 2i - j \not\equiv 0 \mod 4 \), then we are back in Subcase (ii), so suppose \( 2i \equiv j \mod 4 \). Then together with \( i + j \equiv 0 \mod 4 \) it follows that \( i = 0 \). Therefore \( j \) is zero and \( \alpha \) is trivial. This completes case (a).

Case (b): \( k \neq 0 \) or \( l \neq 0 \).

Subcase (i): \( l = 0 \) so \( k \neq 0 \)

Then \( \alpha \alpha^{-l} = x^i y^j b^k (x^{-j} y^{i-j} b^k)^{-1} = x^{i+j} y^{2j-i} \). If \( i + j \not\equiv 0 \) or \( 2j - i \not\equiv 0 \mod 4 \), then we are back in Case (a) so suppose \( i + j \equiv 2j - i \equiv 0 \mod 4 \). Solving gives \( i = j = 0 \) and so \( \alpha = b^k \), whence \( \langle b \rangle \in N \). Hence

\[
b^{-1} b^x = b^{-1} x^{-1} b x = y^{-1} x \in N
\]

and we are back in Case (a).

Subcase (ii): \( l \neq 0 \) and \( k \neq 0 \).

Then \( \alpha \alpha^{-a} = x^i y^j b^k a^{(x^j y^{i-j} a^l)^{-1}} = x^i y^j b^k a^{l} b^k x^{-j} y^{-i} = x^p y^q b^{2k} \)
where \( p, q \in \{0, 1, 2, 3\} \) and we are back in Subcase (i), replacing \( k \) by \( 2k \).

Subcase (iii): \( k = 0 \) so \( l = 1 \)

Then

\[
\alpha \alpha^{-b} = x^i y^j a (x^j y^i a)^{-b} = x^p y^q b^2
\]

for some \( p, q \in \{0, 1, 2, 3\} \) and again we are back in Subcase (i).

This completes the proof. \( \square \)

It is worth observing at this point that Lemma 2.1 tells us that any minimal faithful representation of \( G \) is necessarily transitive. That is, any minimal faithful collection of subgroups \( \{G_1, \ldots, G_n\} \) is just a single core-free subgroup.

**Lemma 2.2.** Elements of \( \langle x, y \rangle b \) and \( \langle x, y \rangle b^2 \) have order 3. All other elements of \( G \) have order dividing by 8.
Proof. It is a routine calculation to show that any element of the form \( \alpha = x^i y^j b^k \) for \( k \) nonzero has order three. Now suppose \( \alpha = x^i y^j b^k a^l \) where \( l \) is nonzero. Then \( l = 1 \) and we have

\[
\alpha^2 = x^p y^q (b^k a)^2 = x^p y^q,
\]

for some \( p, q \), which has order dividing 4. Therefore \( \alpha \) has order dividing 8. \( \square \)

It is an immediate consequence that \( G \) does not contain any element of order 6.

**Lemma 2.3.** If \( L \) is a core-free subgroup of \( G \) then \( |G : L| \geq 12 \).

**Proof.** Suppose for a contradiction that \( \text{core}(L) = \{1\} \) and \( |G : L| < 12 \). Since \( |G| = 96 \), \( |L| > 8 \). However, if \( |L| > 12 \) then \( |G : L| < 8 \) and so \( \mu(G) < 8 \) contradicting that \( \mu(G) \geq 8 \). Therefore \( |L| = 12 \) and so by the classification of groups of order 12, see [5], \( L \) is isomorphic to one of the following groups

\[
L \cong \begin{cases} 
C_{12} \\
C_6 \times C_2 \\
A_4 \\
D_6 \\
T = \langle s, t \mid s^6 = 1, s^3 = t^2, st = ts = s \rangle
\end{cases}
\]

Notice that the groups \( C_{12}, C_6 \times C_2, D_6 \) and \( T \) each contain an element of order 6 and so cannot be isomorphic to \( L \) by Lemma 2.2.

Hence \( L \) is isomorphic to \( A_4 \) and so we can find two non-commuting elements \( \alpha = x^i y^j b^k \) and \( \beta = x^s y^t b^r \) of order three that generate it such that \( \alpha \beta \) has order two. Now

\[
\alpha \beta = x^p y^q b^{k+r}
\]

for some \( p, q \in \{0, 1, 2, 3\} \) and so \( k + r \equiv 0 \mod 3 \) by Lemma 2.2. Without loss of generality let \( k = 1 \). Now

\[
\alpha \beta = \begin{cases} 
x^2 \\
y^2 \\
x^2 y^2
\end{cases}
\]
and upon conjugation by $\alpha = x^i y^j b$, we get respectively,

$$(\alpha \beta)^\alpha = \begin{cases} y^2 \\ x^2 y^2 \\ x^2. \end{cases}$$

So in each case we get $\langle x^2, y^2 \rangle \subseteq L$, contradicting that $L$ is core-free. \qed

Combining the above lemmas we find that any minimal faithful representation of $G$ is necessarily transitive and that any faithful transitive representation has degree at least 12. Therefore we have $12 \leq \mu(G)$. But $\mu(G) \leq 12$. Therefore we have proved the following:

**Theorem 2.4.** The minimal faithful permutation degree of $G(4, 4, 3)$ is 12.

## 3 $G(4, 4, 3)$ forms a Counter-Example of Degree 12

Let $W = C_4 \wr \text{Sym}(3)$ be the wreath product of the cyclic group of order 4 by the symmetric group on 3 letters. Observe at this point that since the base group of $W$ is $C_4 \times C_4 \times C_4$, and $\mu(C_4 \times C_4 \times C_4) = 12$ by Theorem 1.1, $\mu(W) = 12$. Let $\gamma_1, \gamma_2, \gamma_3$ be generators for the base group of $W$ and let $a = (23), b = (123)$ be generators for $\text{Sym}(3)$ acting coordinate-wise on the base group. It follows that $\gamma := \gamma_1 \gamma_2 \gamma_3$ commutes with $a$ and $b$ and thus lies in the centre of $W$. Let $H = \langle \gamma \rangle$, so $\mu(H) = 4$.

Set $x = \gamma_1^{-1} \gamma_2 \gamma_3^{-1}$ and $y = \gamma_1^{-1} \gamma_2^{-1} \gamma_3^2$. Then it readily follows that

$$x^a = x^b = y, \ y^a = x, \ y^b = x^{-1}y^{-1},$$

so that $G = \langle x, y, a, b \rangle$ is isomorphic to $G(4, 4, 3)$. Moreover with a little calculation, it can be shown that $G \cap H = \{1\}$.

It now follows that $W$ is an internal direct product of $G$ and $H$. Therefore by Theorem 2.4 we have

$$12 = \mu(G \times H) < \mu(G) + \mu(H) = 16$$

and so $G$ and $H$ form a counter-example to (1) of degree 12.
Finally, we remark that using the result from [7] that $\mu(G(p, p, p)) = p^2$ for $p$ a prime, it follows that $\mu(G(3, 3, 3)) = 9$. However the centralizer, $C_{Sym(9)}(G(3, 3, 3))$ in $Sym(9)$ is a proper subgroup of $G(3, 3, 3)$. So it is not possible to get a counter-example to (1) of degree 9 in this case, by this method.

Similarly by realizing $G(2, 2, 3)$ as $Sym(4)$, it is immediate that $\mu(G(2, 2, 3)) = 4$ and again a counter-example to (1) of degree 4 is impossible by this method.
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