Research Article

Local Exact Controllability to the Trajectories of Burgers–Fisher Equation

Xiaofeng Shi1,2

1School of Mathematical Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China
2Nonlinear Scientific Research Center, Jiangsu University, Zhenjiang, Jiangsu 212013, China

Correspondence should be addressed to Xiaofeng Shi; sxf961@ujs.edu.cn

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This paper is addressed to study local exact controllability to the trajectories of the Burgers–Fisher (BF) equation. By using the global Carleman estimate for the second-order parabolic operator, we establish the observable inequality and obtain the exact controllability to the trajectories of the linear system. Then, by local inverse theory, we consider the controllability result for the Burgers–Fisher equation.

1. Introduction

Burgers–Fisher-type equations describe the interaction between reaction mechanism, convection effect and diffusion process. Due to this, these equations have a wide range of applications in plasma physics, fluid physics, capillary-gravity waves, nonlinear optics, chemical physics and population dynamics [1–3].

Burgers–Fisher-type equation is as follows:

\[ y_t + a y y_x = d y_{xx} + b y (1 - y), \]  

where \( a, b \) and \( d \) are the nonnegative numbers.

As we know, the approximation numerical methods for the Burgers-type equations have been developed by many researchers, including the moving mesh PDE method [4], the Adomian decomposition method [5, 6], the direct discontinuous Galerkin method [7], and the B-spline quasi-interpolation method [8].

Due to the quadratic nonlinearity of the Burgers–Fisher equation, the nonlinear phenomena results in very complex and unfavorable behaviors (e.g., blowing up, shock waves, and chaos). Control by using internal or external actuation has been expected as an effective method to reduce or totally avoid those undesired phenomena.

Although the controllability of infinite dimensional systems has been studied extensively, e.g., null controllability [9, 10], local controllability to trajectories [11–14], approximate controllability [15, 16], exact controllability [17, 18], and boundary controllability [19], control of the Burgers–Fisher equation is still in its infancy and remains open. In this paper, we will deal with the local exact controllability to the trajectories of the Burgers–Fisher equation.

We consider the controlled system described by the Burgers–Fisher equation:

\[
\begin{align*}
    y_t - y_{xx} + a y y_x - b y (1 - y) &= u(x,t) \\
    y(t,0) &= y(t,1) \\
    y_x(t,0) &= y_x(t,1) \\
    y(0,x) &= y_0(x)
\end{align*}
\]

where \( u(t,x) \) is an internal control and \( \omega \) is a nonempty open interval of \([0,1]\).

Our problem is to guide the solution of (2) to a given trajectory. More accurately, for any given time \( T > 0 \) and a
suitable space $X, Y$ (seen below), if with any initial value $y_0 \in X$ and the given $\gamma \in Y$ satisfying

\[
\begin{align*}
\dot{y}_t - \dot{y}_{xx} + a \dot{y} y_x - \beta \dot{y} (1 - \dot{y}) &= 0, \\
\gamma(t, 0) &= 0 = \gamma(t, 1), \\
\dot{y}_x (t, 0) &= 0 = \dot{y}_x (t, 1), \\
\gamma(0, x) &= \gamma_0 (x).
\end{align*}
\]

Then, there exists a control $u(t, x)$ such that the solution $y$ of (2) satisfies $y(0, x) = y_0$ and can touch $y(T, x) = \gamma(T, x)$.

Letting $q = y - \gamma$, we get a new controlled system:

\[
\begin{align*}
q_t - q_{xx} + (2 \beta \gamma_x + a \gamma y_x - \beta)q + a \gamma q_x + \alpha |q|^2 = u x, \\
q(t, 0) &= 0 = q(t, 1), \\
q_x (t, 0) &= 0 = q_x (t, 1), \\
q(0, x) &= q_0 (x).
\end{align*}
\]

It is easy to find that the exact controllability to the trajectories of system (2) is equivalent to the null controllability of system (4).

For the convenience of narration, we firstly introduce some notations as a preliminary:

(i) $L_T y = y_t - y_{xx} + (2 \beta \gamma_x + a \gamma y_x - \beta) y + a \gamma y_x$ is a linear operator.

(ii) Let $\psi(x) \in C^\infty [0, 1]$, which satisfies $\forall x \in [0, 1], 0 \leq \psi \leq 1; \psi(0) = \psi(1) = 0; \forall x \in [0, 1] \phi(x, \gamma, \psi_x) > 0$.

For given positive constants $\lambda$ and $\mu$, we construct two weight functions:

\[
\phi(x, t) = \frac{e^{\lambda t}}{t(T - t)},
\]

\[
a(x, t) = \frac{e^{\mu t} - e^{\lambda t}}{t(T - t)},
\]

\[
\eta = \lambda a,
\]

\[
\theta = \phi^0.
\]

(iii) For simplicity, we abbreviate $\int_0^1 \int_0^T (\cdot)$ as either $\iint (\cdot)$ or $\int_Q (\cdot)$, where $Q = (0, 1) \times (0, T)$.

(iv) Let

\[
\begin{align*}
\chi(t) &= \begin{cases}
1, & t \in \left[0, \frac{T}{2}\right], \\
0, & t \in \left[\frac{3T}{4}, T\right],
\end{cases}
\]

\[
\phi(t) = \begin{cases}
\frac{4}{3T^2}, & t \in \left(0, \frac{T}{2} \right), \\
\frac{1}{t(T - t)}, & t \in (\frac{T}{2}, T),
\end{cases}
\]

\[
m = \frac{T^2}{4} \min_{x \in (0, 1)} a \left(\frac{x}{T}\right)
\]

\[
M = \frac{T^2}{4} \max_{x \in (0, 1)} a \left(\frac{x}{T}\right)
\]

and

\[
F = \left\{ f : e^{-\lambda \phi} \phi^{(1/2)} f \in \mathcal{L}^1 \left(0, T ; \mathcal{L}^2 (0, 1) \right) \right\},
\]

\[
U = \left\{ u : e^{-\lambda \phi} \phi^{(3/2)} u \in \mathcal{L}^2 \left(0, T ; \mathcal{L}^2 (\omega) \right) \right\},
\]

\[
Y = \left\{ y : e^{-\lambda \phi} \phi^{(3/2)} y \in \mathcal{C} \left(0, T ; \mathcal{L}^2 (0, T ; H^1) \right) \right\}.
\]

(6)

To end this introductory, let us mention how this work is organized. Sections 2 and 3, respectively, establish the well posedness of the linear Burgers–Fisher equation and the nonlinear one. In Section 4, we establish the Carleman estimate for the second parabolic operator (similar estimates can be found in [20–22]). Section 5 is contributed to the null controllability of system (4).

2. Well Posedness of Linear BF Equation

Definition 1. If for any $\psi \in H^1 (0, 1) \cap \mathcal{L}^2 (0, 1)$, there exists $y \in \mathcal{C} ([0, T]; \mathcal{L}^2 (0, 1)) \cap \mathcal{L}^2 (0, T; H^1)$, such that

\[
(y(t), \psi) + \int_0^t \left( (y, \psi_x) + (a y + b y_x, \psi) \right) dt = (y_0, \psi) + \int_0^t (f, \psi) dt,
\]

then $y$ is called a weak solution of the following system:
We consider the existence of this solution through a series of mathematical estimates. Firstly, we establish a Galerkin approximate solution for the linear BF equation; secondly, we prove the existence of this solution through a series of mathematical estimates; finally, the uniqueness of the solution is proved.

2.1. Galerkin Approximate Solutions. We consider the system

\[
Ay = y_{xx}, \quad \forall y \in D(A) = \left\{ y \in H^2(0, 1) \mid y(t, 0) = 0 = y(t, 1), y_x(t, 0) = 0 = y_x(t, 1) \right\}. \tag{11}
\]

Obviously, \( A \) is the second-order operator defined on \( L^2(0, 1) \). We can find an orthogonal basis \( \{ \varphi_k \}_{k=1}^\infty \) of \( L^2(0, 1) \) as the eigenfunctions corresponding to the eigenvalues \( \lambda_k \) of operator \( A \).

Let

\[
y_m = \sum_{k=1}^m C_m^k(t) \varphi_k, \quad f_m = \sum_{k=1}^m (f, \varphi_k) \varphi_k, \tag{12}
\]

where \( C_m^k(t) (k = 1, 2, \ldots, m) \) is the solution of the following ordinary differential equation

\[
\begin{aligned}
&\left( y_m - y_{xxx}, \varphi_k \right) + (ay_m + by_{xx}, \varphi_k) = (f_m, \varphi_k), \\
&C_m^k(0) = (y_0, \varphi_k).
\end{aligned} \tag{13}
\]

According to the classical theory of the ordinary differential equation, equation (13) has a unique solution in the interval \([0, T_m]\).

2.2. Energy Estimates. Next, we prove that the solution \( y_m \) mentioned above is bounded when \( T_m \to T \).

Multiplying the first formula of equation (13) by \( C_m^k(t) \) on both sides and taking summation about \( k \) from 1 to \( m \), the following is obtained:

\[
\sum_{k=1}^m \int_0^T \left( \frac{\partial y_m}{\partial t}, \varphi_k \right) + \left( ay_m, \varphi_k \right) + \left( by_{xx}, \varphi_k \right) = \left( f_m, \varphi_k \right),
\]

The main result for the well posedness of linear system (8) is as follows:

**Proposition 1.** Let \( a, b \in L^2(0, T; L^\infty(0, 1)) \) if \( y_0 \in L^2(0, 1) \) and \( f \in L^1(0, T; L^2) \), then there exists a unique solution of (8): \( y \in C([0, T]; L^2(0, T; H^1)) \) and there is a constant \( C \) independent of \( y_0, f, a, b, \) and \( T \), such that

\[
\left\| y \right\|_{C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1)} \leq C \left( \left\| \psi \right\|_{L^2(0, 1)} + \left\| f \right\|_{L^1(0, T; L^2)} \right) \times \left[ \int_0^T \left( \left\| y \right\|_{L^2(0, 1)} + \left\| f \right\|_{L^2(0, T; L^2)} \right)^{1/2} dt \right]. \tag{9}
\]

Let

\[
\begin{aligned}
y(t, 0) &= 0 = y(t, 1), \\
y_x(t, 0) &= 0 = y_x(t, 1), \\
y(0, x) &= y_0(x).
\end{aligned} \tag{10}
\]

(Proposition 1 will be proved in the following four sections step-by-step. Firstly, we establish a Galerkin approximate solution for the linear BF equation; secondly, we prove the existence of this solution through a series of mathematical estimates; finally, the uniqueness of the solution is proved.)

Let

\[
\begin{aligned}
&\left( y_m - y_{xxx}, y_m \right) + (ay_m + by_{xx}, y_m) = (f_m, y_m), \tag{14}
\end{aligned}
\]

That is to say

\[
\frac{1}{2} \frac{d}{dt} \left\| y_m \right\|_{L^2(0, 1)}^2 + \left\| y_{xx} \right\|_{L^2(0, 1)}^2 = - \int_0^1 ay_m^2 dx - \int_0^1 by_m y_{xx} dx + (f_m, y_m). \tag{15}
\]

Noticing that

\[
\begin{aligned}
&\int_0^1 ay_m^2 dx \leq \|a\|_{L^\infty(0, 1)} \int_0^1 y_m^2 dx, \\
&\int_0^1 by_m y_{xx} dx \leq \|b\|_{L^\infty(0, 1)} \int_0^1 y_m y_{xx} dx, \\
&\frac{1}{2} \|b\|_{L^\infty(0, 1)} \int_0^1 y_{xx}^2 dx + \frac{1}{2} \int_0^1 y_m^2 dx, \\
&\left( f_m, y_m \right) \leq \|f_m\|_{L^2(0, 1)} \|y_m\|_{L^2(0, 1)},
\end{aligned} \tag{16}
\]

Therefore

\[
\left\| y \right\|_{C([0, T]; L^2(0, 1)) \cap L^2(0, T; H^1)} \leq C \left( \left\| \psi \right\|_{L^2(0, 1)} + \left\| f \right\|_{L^1(0, T; L^2)} \right) \times \left[ \int_0^T \left( \left\| y \right\|_{L^2(0, 1)} + \left\| f \right\|_{L^2(0, T; L^2)} \right)^{1/2} dt \right].
\]
\[
\frac{1}{2} \frac{d}{dt} \| y_m \|_{L^2(0,1)}^2 + \| y_{mx} \|_{L^2(0,1)}^2 \leq \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) \| y_m \|_{L^2(0,1)}^2 + 2 \| f_m \|_{L^2(0,1)} \| y_m \|_{L^2(0,1)}
\]

According to Gronwall’s inequality, we have

\[
\| y_m \|_{C[0,T;L^2(0,1)]} \leq C e^{C \int_0^T \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) dt} \left( \| y_m(0) \|_{L^2(0,1)} + \| f_m \|_{L^2(0,1)} \right).
\]

Integrating both sides of (17) about \( t \) on \([0,T]\) and combining with (18), we can get

\[
\| y_m(T) \|_{L^2(0,1)}^2 + \int_0^T \| y_{mx} \|_{L^2(0,1)}^2 dt \\
\leq \| y_m(0) \|_{L^2(0,1)}^2 + \int_0^T \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) \| y_m \|_{L^2(0,1)}^2 dt + 2 \int_0^T \| f_m \|_{L^2(0,1)} \| y_m \|_{L^2(0,1)} dt \\
\leq \| y_m(0) \|_{L^2(0,1)}^2 + \| y_m \|_{C[0,T;L^2(0,1)]} \int_0^T \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) dt + 2 \| f_m \|_{L^2(0,T;L^2(0,1))} \| y_m \|_{C[0,T;L^2(0,1)]} \\
\leq \| y_m(0) \|_{L^2(0,1)}^2 + \| y_m \|_{C[0,T;L^2(0,1)]} \int_0^T \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) dt + \| f_m \|_{L^2(0,T;L^2(0,1))} \| y_m \|_{C[0,T;L^2(0,1)]} \\
\leq C e^{C \int_0^T \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) dt} \left( \| y_m(0) \|_{L^2(0,1)} + \| f_m \|_{L^2(0,1)} \right)^2.
\]

Combining (18) and (19), we have

\[
\| y_m \|_{C[0,T;L^2(0,1)] \cap L^2(0,T;H^1(0,1))} \\
\leq C e^{C \int_0^T \left( 2 \| a \|_{L^\infty(0,1)} + \| b \|_{L^\infty(0,1)} \right) dt} \left( \| y_m(0) \|_{L^2(0,1)} + \| f_m \|_{L^2(0,T;L^2(0,1))} \right)^{1/2}
\]

Let us denote \( H^{1*}(0,1) \) as the dual space of \( H^1(0,1) \) and \( H^{1*} \) for convenience. Similarly, \( L^\infty(0,1) \), \( L^2(0,1) \), and so on; also, omit \( (0,1) \) for convenience.

Noticing that

\[
\int_0^T \| ay_m \|_{H^1}, dt = \int_0^T \sup_{\| \psi \|_{H^1(0,1)}} < ay_m, \psi >_{H^1, H^1}, dt = \int_0^T \sup_{\| \psi \|_{H^1(0,1)}} (ay_m, \psi) dt \\
\leq \int_0^T \sup_{\| \psi \|_{H^1(0,1)}} \| a \|_{L^\infty} \| y_m \|_{L^2} \| \psi \|_{L^2}, dt \leq C \int_0^T \sup_{\| \psi \|_{H^1(0,1)}} \| a \|_{L^\infty} \| y_m \|_{L^2} \| \psi \|_{H^1}, dt \\
\leq C \int_0^T \| a \|_{L^\infty} \| y_m \|_{L^2}, dt \leq C \| a \|_{L^1(0,T;L^\infty)} \| y_m \|_{C[0,T;L^2(0,1)]}
\]

Similarly
\[
\int_0^T \| b y_{mx} \|_{L^2} \, dt = \int_0^T \sup_{\| \varphi \|_{H^1}} < b y_{mx}, \varphi >_{H^1, H^1} \, dt = \int_0^T \sup_{\| \varphi \|_{H^1}} (b y_{mx}, \varphi) \, dt \\
\leq \int_0^T \sup_{\| \varphi \|_{H^1}} \| b \|_{L^2} \| y_m \|_{H^1} \| \varphi \|_{L^2} \, dt \\ 
\leq C \int_0^T \| a \|_{L^\infty} \| y_m \|_{H^1} \, dt \leq C \| b \|_{L^1(0,T;L^2)} \| y_m \|_{C([0,T];H^1)} \leq C \| b \|_{L^1(0,T;L^\infty)} \| y_m \|_{L^2(0,T;H^1)}
\]

(22)

According to the above two estimates and (20), we can obtain

\[
\int_0^T \| y_{mt} \|_{H^1} \, dt = \int_0^T \sup_{\| \varphi \|_{H^1}} < y_{mx}, \varphi >_{H^1, H^1} \, dt \\
= \int_0^T \sup_{\| \varphi \|_{H^1}} < y_{mx} - ay_m - by_{mx} + f, \varphi >_{H^1, H^1} \, dt \\
\leq C(T, y_m(0), f, a, b),
\]

which implies \( \{ y_{mt} \}_{m \geq 0} \) is bounded in \( L^1(0,T; H^1) \).

2.3. Existence of Weak Solutions. We are in position to use the energy estimates to gain weak solutions. According to (20) and (23), combined with the Lions–Aubin theorem, there exists \( y \in C([0,T]; L^2) \cap L^2(0,T; H^1) \) such that

\[
y_m \rightarrow y, \text{ weakly } * \text{ in } L^\infty(0,T; L^2), \\
y_m \rightarrow y, \text{ weakly } \text{ in } L^2(0,T; H^1), \\
y_{mt} \rightarrow y_t, \text{ weakly } \text{ in } L^2(0,T; H^1), \\
y_{mt} \rightarrow y_t, \text{ strongly } \text{ in } L^2(0,T; H^1).
\]

(24)

According to the first formula in (13), for \( \varphi \in H^1(0,1) \), we have

\[
\int_0^T (a y_m - ay, \varphi_X(0,t)) \, dt = \int_0^T (a y_m - ay, \varphi(0,t)) \, dt \leq \int_0^T \| a \|_{L^\infty} \| y_m - \varphi \|_{L^2} \, dt \\
\leq \| a \|_{L^\infty} \int_0^T \| y_m - \varphi \|_{L^2} \, dt \leq \| a \|_{L^\infty} \| y_m - \varphi \|_{L^2(0,T; L^\infty)} \| y_m - \varphi \|_{L^2(0,T; L^1)}.
\]

(27)

\[
\int_0^T (b y_{mx} - b y_x, \varphi_X(0,t)) \, dt = \int_0^T (b y_{mx} - b y_x, \varphi(0,t)) \, dt \leq \int_0^T \| b \|_{L^\infty} \| y_{mx} - y_x \|_{L^2} \, dt \\
\leq \| b \|_{L^\infty} \| y_{mx} - y_x \|_{L^2} \| y_m - \varphi \|_{L^2(0,T; L^\infty)} \| y_m - \varphi \|_{L^2(0,T; H^1)}.
\]

Therefore

\[
\int_0^T (a y_m, \varphi_X(0,t)) \, dt \rightarrow \int_0^T (a y, \varphi_X(0,t)) \, dt, \\
\int_0^T (b y_{mx}, \varphi_X(0,t)) \, dt \rightarrow \int_0^T (b y_x, \varphi_X(0,t)) \, dt.
\]

(28)

Taking the limit on both sides in (26) and combining with (24) and (28), the following can be obtained:

\[
\int_0^T \left[ (y_t, \varphi_X(0,t)) + (ay + by_x, \varphi_X(0,t)) \right] \, dt = \int_0^T (f, \varphi_X(0,t)) \, dt.
\]

(29)
\begin{equation}
\int_0^t [(y_r \phi) + (y_s \phi)] + (ay + by_x \phi) dt = \int_0^t (f, \phi) dt,
\end{equation}
which shows that \( y \) satisfies (7).

2.4. Uniqueness of Weak Solutions. If \( y_1 \) and \( y_2 \) are two solutions of (8), letting \( y = y_1 - y_2 \), then \( y \) satisfies
\begin{equation}
\begin{aligned}
y - y_x + a y + b y_x &= 0, \\
y(t, 0) &= y(t, 1), \\
y_x(t, 0) &= y_x(t, 1), \\
y(0, x) &= 0.
\end{aligned}
\end{equation}

When (20) is used for (31) and according to the convergence of solution, we obtain
\begin{equation}
\|y\|_{C[0,T;L^2_1(0,1)]} + \|v\|_{L^2(0,T;H^1(0,1))} = 0.
\end{equation}
Thus
\begin{equation}
y = 0,
\end{equation}
that is
\begin{equation}
y_1 = y_2.
\end{equation}

3. Well Posedness of Nonlinear BF Equation

**Theorem 1.** For \( y_0 \in L^2(0, 1), a, b \in L^2(0, T; L^{\infty}(0, 1)), \) and \( f \in L^1(0, T; W^{-1,1}(0, 1)), \) if there exists a positive number \( r, \) such that
\begin{equation}
\|y_0\|_{L^2(0,1)} + \|f\|_{L^1(0,T;W^{-1,1}(0,1))} \leq r,
\end{equation}
then the nonlinear system
\begin{equation}
\begin{aligned}
y - y_x + a y + b y_x &= f, \\
y(t, 0) &= y(t, 1), \\
y_x(t, 0) &= y_x(t, 1), \\
y(0, x) &= y_0(x),
\end{aligned}
\end{equation}
has a unique solution: \( y \in C([0,T];H^{-1}(0,1)) \cap L^2(0,T; L^2(0,1)). \)

**Proof.** Inspired by [12]. Let \( y_0 \in L^2(0, 1) \) and \( f \in L^1(0, T; W^{-1,1}(0, 1)), \) satisfying (35).
We define a mapping:
\begin{equation}
\Pi: l \in L^2(0, T; L^2) \longrightarrow y \in L^2(0, T; L^2),
\end{equation}
where \( y \) is the solution of the following system:
\begin{equation}
\begin{aligned}
y - y_x + a y + b y_x &= f - a l_x + \beta l^2, \\
y(t, 0) &= y(t, 1), \\
y_x(t, 0) &= y_x(t, 1), \\
y(0, x) &= y_0(x).
\end{aligned}
\end{equation}
Comparing (36) and (38), we know that \( y \) is the solution of (36) if and only if \( y \) is a fixed point of the mapping \( \Pi. \)
Similar to formula (9), we have
\begin{equation}
\Pi(l)^2 \leq C\left(\int \|y(0)\|_{L^2_1(0,1)} + \|f\|_{L^1(0,T;W^{-1,1}(0,1))} \right) + \left(\|y(0)\|_{L^2(0,1)} + \|f\|_{L^1(0,T;W^{-1,1}(0,1))}\right)^{(1/2)}.
\end{equation}
We have
\begin{equation}
\Pi(l)^2 \leq \frac{1}{2} \|y(0)\|_{L^2_1(0,1)}^2 + \|f\|_{L^1(0,T;W^{-1,1}(0,1))}^2.
\end{equation}
For any \( R, \) we note
\begin{equation}
B(0,R) = \left\{ l \in L^2(0, T, L^2) \right\} \|l\|_{L^2(0, T; L^2)} \leq R \right\}.
\end{equation}
It is easy to see that the appropriate \( r, R > 0 \) can be selected to satisfy
\begin{equation}
C\left( r + \left(\frac{\alpha}{2} + \beta \right) R^2 \right) < R,
\end{equation}
then \( \Pi_{B(0,R)} \subset B(0, R). \)

Now we are in the position to prove that \( \Pi \) is a contractive mapping.
\[ \|\Pi (l) - \Pi (\tilde{l})\|_{L^2 (0, T; L^2 (0, 1))} \leq C \left[ \|a(\Pi_x - \Pi_{\tilde{x}}) + \beta \left( T_2 - T_1 \right)\|_{L^2 (0, T; W^{-1, 1} (0, 1))} \right]. \]  

(45)

Noticing that
\[ \|\Pi_x - \Pi_{\tilde{x}}\|_{L^2 (0, T; W^{-1, 1} (0, 1))} \leq \frac{1}{2} \|T_2 - T_1\|_{L^2 (0, T; L^2 (0, 1))} \]
\[ \leq \frac{1}{2} \|T - \|l\|_{L^2 (0, T; L^2 (0, 1))} - \|l\|_{L^2 (0, T; L^2 (0, 1))}\|_{L^2 (0, T; L^2 (0, 1))}. \]

(46)

We have
\[ \|\Pi (l) - \Pi (\tilde{l})\|_{L^2 (0, T; L^2 (0, 1))} \leq C (\alpha + 2\beta) R (\|\tilde{T} - l\|_{L^2 (0, T; L^2 (0, 1))}). \]

(47)

Thus, the mapping \( \Pi \) is contractive when \( C (\alpha + 2\beta) R \leq 1 \).

By the Banach fixed-point theorem, \( \Pi \) has a unique fixed point, i.e., \( \Pi (y) = y \), which is a solution of (36). \( \square \)

### 4. Carleman Estimate

The main result in this section is given as follows:

**Theorem 2.** There exist constants \( C_0 \) and \( C > 0 \) such that for 
\[ \lambda > C_0 (T + T^2), \forall y \in D (A), \] 
we have
\[ \int_Q \left( \frac{1}{\lambda^2} \phi (y_x^2 + y_{xx}^2) + \lambda^3 \phi^3 \theta^2 y^2 + \lambda \phi \theta^2 y_x^2 \right) dx \]  
\[ \leq C \left( \int_Q \theta^2 f^2 dx dt + \int_{Q^*} \lambda^3 \phi^3 \theta^2 y^2 dx dt \right). \]

(48)

In order to prove Theorem 2, we need the following conclusion.

**Proposition 2.** Let \( y_t - y_{xx} = f \) and \( u = \theta y = e^{\theta} y \), then

\[ \theta f = u_t + \left( -\eta_x^2 + \eta + \eta_{xx} \right) u + 2\eta_x u_x - u_{xx} = I_1 + I_2 + Ru, \]

(49)

where
\[ I_1 = u_t + 2\eta_x u_x + \eta_{xx} u, I_2 = -\eta_x^2 u - u_{xx}, \]
\[ Ru = -\eta_x u. \]

(50)

**Proof**

\[ u = \theta y, \]
\[ y = e^{-\eta} u, \]
\[ y_t = -e^{-\eta} \eta_t u + e^{-\eta} u_t, \]
\[ y_x = -e^{-\eta} \eta_x u + e^{-\eta} u_x, \]
\[ y_{xx} = -e^{-\eta} \eta_x^2 u - e^{-\eta} \eta_{xx} u - 2e^{-\eta} \eta_x u_x + e^{-\eta} u_{xx}. \]

Substituting the above formulas into \( y_t - y_{xx} = f \), we achieve (49) and (50).

The proof of Theorem 2 will be completed in the following four steps:

**Step 1:** We will get the following estimate:
\[ \int_Q \left( I_1^2 + I_2^2 + 2\lambda \mu^2 \phi^3 \psi_2^2 \psi_x^2 + \lambda \mu^2 \phi \psi_x^2 u_x^2 \right) dx dt \leq C \left( \int_Q \theta^2 f^2 dx dt + \int Q \lambda^3 \mu^3 \phi^3 u^2 dx dt \right). \]

(52)

Let \( (I_1 u, I_2 u) = \sum I_{i,j} (i = 1, 2, 3; j = 1, 2) \), where \( I_{i,j} \) denotes the \( L^2 \) inner product between the \( i \)th term of \( I_1 u \) and the \( j \)th term of \( I_2 u \).

For all \( I_{i,j} \), integrating by part and using the boundary conditions, we have

\[ I_{1,1} = \iint u_x^2 u - \int_0^1 \int_0^T u_x^2 u dt u = \iint u (2u_x \eta u_x u + \eta_x^2 u) = \iint \eta_x u_x u u^2, \]
\[ I_{1,2} = -\iint u_x u_x u - \int_0^1 \int_0^T u_x u_x u_x dt u = \iint \iint u_x u_x u dx = \frac{1}{2} \int_0^1 \int_0^1 \frac{d}{dt} (u_x^2) dt = 0, \]
\[ I_{2,1} = -\iint u_x^2 u = -\iint \int_0^1 \int_0^T u_x^2 u dt u = -\iint \int_0^1 \int_0^T u_x^2 u_x^2 = \iint \int_0^1 \int_0^1 u_x u_x u_x^2, \]
\[ I_{2,2} = -\iint \eta_x u_x u_x u = -\iint \int_0^1 \int_0^T \eta_x u_x u_x dt u = -\iint \int_0^1 \int_0^T \eta_x u_x^2 u_x = \iint \int_0^1 \int_0^1 \eta_x u_x u_x^2, \]
\[ I_{3,1} = -\iint \iint u_x^2 u_x^2 u_x^2, I_{3,2} = -\iint \iint \iint u_x u_x u_x u_x = \iint \iint (\frac{1}{2}) u_x u_x u_x u_x^2. \]

(53)
Adding the above equations all together, we have

\[
(I_1u, I_2u)_2 = \sum_{i,j=1}^{2} I_{ij} = I(u) + I(u_x),
\]

where

\[
I(u) = \int \left( \eta_x \eta_{xt} + 2 \eta_x^2 \eta_{xx} - \frac{1}{2} \eta^{xxxx} \right) u^2, \quad (54)
\]

\[
I(u_x) = \int 2 \eta_{xx} u_x^2. \quad (55)
\]

According to the definition of \( \eta(x,t) \), we note that

\[
\left( \eta_x \eta_{xt} + 2 \eta_x^2 \eta_{xx} - \frac{1}{2} \eta^{xxxx} \right) u^2 = (2\lambda \mu^2 \varphi^3 \psi_x^1 + R_0)u^2,
\]

\[
2\eta_{xx} u_x^2 = (\lambda \mu^2 \varphi^2 \psi_x^2 + R_1)u_x^2,
\]

where \(|R_0| \leq \lambda^3 \mu^2 \varphi^3\) and \(|R_1| \leq C\lambda \mu \varphi\).

Noticing that \( I_1 + I_2 = \theta f + \eta u \),

we have

\[
\left\| I_1 + I_2 \right\|_{L^2_Q}^2 = \int_Q (I_1^2 + I_2^2 + 2I_1 I_2) \, dx \, dt = \left\| \theta f + \eta u \right\|_{L^2_Q}^2 
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_Q \lambda \varphi^3 u^2 \, dx \, dt \right].
\]

Combining (54), (56) and (57), we obtain (52).

Step 2: we prove the following estimate:

\[
\int_Q \left( I_1^2 + I_2^2 + \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt 
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt \right].
\]

In fact, from (52) we get

\[
\left[ \int_Q \left( I_1^2 + I_2^2 + \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt \right] 
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt \right].
\]

Noticing that \( |\psi_x| > 0 \), \( \forall x \in [0,1] \setminus \omega_0 \),

we obtain

\[
\int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt 
\leq C(\psi) \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q=0} \lambda^3 \mu^2 \varphi^3 u^2 \, dx \, dt \right]
= C(\psi) \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q=0} \lambda^3 \mu^2 \varphi^3 u^2 \, dx \, dt \right] + \int_{Q=0} \lambda^3 \mu^2 \varphi^3 u^2 \, dx \, dt.
\]

In the formula above, if we take \( \mu_0 = C(\psi) + 1 \), then

\[
\int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt 
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt \right],
\]

that is

\[
\int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt 
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q=0} \left( \lambda^3 \mu^2 \varphi^3 u^2 + \lambda \mu \varphi u_x^2 \right) \, dx \, dt \right].
\]

holds for \( \mu \geq \mu_0 \),

From (59) and (62), we can infer that
\[
\int_Q (I_1^2 + I_2^2) \, dx \, dt \\
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q^o} (\lambda^3 \mu^3 \phi^3 u^2 + \lambda \mu \phi u_x^2) \, dx \, dt \right]. 
\]
\quad (63)

Combining (62) with (63), (58) is easy to be achieved.

Step 3: the term \( \int_{Q^o} \lambda \mu \phi u_x^2 \, dx \, dt \) can be absorbed in

\[
\int_Q (I_1^2 + I_2^2 + \lambda^3 \mu^3 \phi^3 u^2 + \lambda \mu \phi u_x^2) \, dx \, dt \\
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q^o} (\lambda^3 \mu^3 \phi^3 u^2 + \lambda \mu \phi u_x^2) \, dx \, dt \right].
\]
\quad (64)

such that the following estimate can be obtained:

\[
\int_Q (I_1^2 + I_2^2 + \lambda^3 \mu^3 \phi^3 u^2) \, dx \, dt \\
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q^o} \lambda^3 \mu^3 \phi^3 u^2 \, dx \, dt \right].
\]
\quad (65)

In the following discussion, we consider two nonempty open intervals \( \omega_2, \omega \), satisfying: \( \omega_2 \subset \omega_1 \subset \omega \) and select a nonnegative function \( \chi \in C_0^\infty (\omega_1) \), which satisfies \( \chi \equiv 1 \) in \( \omega_0 \).

Thus

\[
\int_{Q_0} \lambda \mu \phi u_x^2 \, dx \, dt \leq C \int_{Q^o} \lambda^3 \mu^3 \phi^3 u^2 \, dx \, dt + \epsilon \int_{Q^o} \frac{1}{\lambda \mu \phi} u_x^2 \, dx \, dt \\
+ C(\epsilon) \int_{Q^o} \lambda^3 \mu^3 \phi^3 u^2 \, dx \, dt.
\]
\quad (66)

According to the definition of \( I_2 = -\eta_2^2 u - u_{xx} \), then

\[
\int_{Q_0} \frac{1}{\lambda \mu \phi} u_x^2 \, dx \, dt = \int_{Q^o} \frac{1}{\lambda \mu \phi} (I_2 + \eta_2^2 u)^2 \, dx \, dt \\
\leq C \int_Q (I_2^2 + \lambda^3 \mu^3 \phi^3 u^2) \, dx \, dt.
\]
\quad (67)

Substituting (67) into (66) and taking enough small \( \epsilon \) and sufficient large \( \lambda \), it is easy to see that (65) holds.

Step 4: let us prove the Carleman estimate (48).

Similar to formula (67), according to the definition of \( I_1 = u_t + 2\eta_1 u_x + \eta_{xx} u \), we have

\[
\int_{Q_0} \frac{1}{\lambda \mu \phi} u_t^2 \, dx \, dt = \int_{Q^o} \frac{1}{\lambda \mu \phi} (I_1 - 2\eta_1 u_x - \eta_{xx} u)^2 \, dx \, dt \\
\leq C \int_Q (I_1^2 + \lambda^3 \mu^3 \phi^3 u^2 + \lambda \mu \phi u_x^2) \, dx \, dt.
\]
\quad (68)

Combining (67) with (68), we have

\[
\int_{Q_0} \frac{1}{\lambda \mu \phi} (u_t^2 + u_x^2) \, dx \, dt \\
\leq C \int_Q \left( I_1^2 + I_2^2 + \lambda^3 \mu^3 \phi^3 u^2 + \lambda \mu \phi u_x^2 \right) \, dx \, dt.
\]
\quad (69)

According to (65) and (68), we can get

\[
\int_{Q_0} \left[ \int_{Q^o} \frac{1}{\lambda \mu \phi} (u_t^2 + u_x^2) + \lambda^3 \mu^3 \phi^3 u^2 + \lambda \mu \phi u_x^2 \right] \, dx \, dt \\
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q^o} \lambda^3 \mu^3 \phi^3 u^2 \, dx \, dt \right].
\]
\quad (70)

By substituting \( u \) with \( \theta y \), we obtain

\[
\int_{Q_0} \left[ \int_{Q^o} \frac{1}{\lambda \mu \phi} \theta^2 (y_t^2 + y_{xx}^2) + \lambda^3 \mu^3 \phi^3 \theta^2 y^2 + \lambda \mu \phi \theta^2 y_x^2 \right] \, dx \, dt \\
\leq C \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q^o} \lambda^3 \mu^3 \phi^3 \theta^2 y^2 \, dx \, dt \right].
\]
\quad (71)

Taking a suitable \( \mu_0 \), when \( \mu \geq \mu_0 \), the estimate can be reduced to

\[
\int_{Q_0} \left[ \int_{Q^o} \frac{1}{\lambda \phi} \theta^2 (y_t^2 + y_{xx}^2) + \lambda^3 \phi^3 \theta^2 y^2 + \lambda \phi \theta^2 y_x^2 \right] \, dx \, dt \\
\leq C(\mu_0) \left[ \int_Q \theta^2 f^2 \, dx \, dt + \int_{Q^o} \lambda^3 \phi^3 \theta^2 y^2 \, dx \, dt \right],
\]
\quad (72)

which implies (48).

\[ \square \]

5. Controllability to the Trajectory

In this section, the controllability of linear system (73) is obtained by the duality of observability-controllability, and the result for the nonlinear system is obtained by means of a local inverse theorem, where the idea can be referred to [13].

Firstly, the conclusion for the linear system is as follows.

Proposition 3. \( \forall y_0 \in L^2 (0, 1) \) and \( f \in F \), then there exists a control \( u \in U \), such that
\[
\begin{align*}
L_T y &= f + u \chi_\omega, \\
y(t, 0) &= 0 = y(t, 1), \\
y_x(t, 0) &= 0 = y_x(t, 1), \\
y(0, x) &= y_0(x),
\end{align*}
\] (73)

has a solution \( y \in Y \), satisfying \( y(\cdot, T) = 0 \). Furthermore, there exists \( C > 0 \) such that
\[
\begin{align*}
\|u\|_U &\leq C(\|f\|_F + \|y_0(x)\|_{L^2(0,1)}), \\
\|y\|_Y &\leq C(\|f\|_F + \|u\|_U + \|y_0(x)\|_{L^2(0,1)}).
\end{align*}
\] (74)

Proposition 3 will be proved in three steps. According to the dual theory, in order to get the controllability of system (73), we need an observable inequality of dual operator (76) (for more details, refer [14]).

Step 1: we firstly prove the following estimate:
\[
\int_Q e^{2\lambda \Phi} q^2 \, dx \, dt + \int_0^1 q^2(x, 0) \, dx \\
\leq C \left( \int_Q e^{2\lambda \Phi} \|L_T q\|^2 \, dx \, dt + \int_Q e^{2\lambda \Phi} \|q^2 \, dx \, dt \right)
\] (75)

Consider the dual operator:
\[
\begin{align*}
L_T^* q &= -q_t - q_{xx} + (2\beta \bar{y} + \alpha \bar{y}_x - \beta)q - \alpha \bar{y} q_x, \\
qu(t, 0) &= 0 = qu(t, 1), \\
q_{x}(t, 0) &= 0 = q_{x}(t, 1), \\
qu(0, x) &= q_0(x).
\end{align*}
\] (76)

Similar to (48), the following estimate can be shown:
\[
\int_Q \left( \frac{1}{\lambda \Phi} \beta^2 (q_t^2 + q_{x}^2) + \lambda^3 \phi^3 \theta^2 q^2 + \lambda \Phi \theta^2 q_{x}^2 \right) \, dx \, dt \\
\leq C \left( \int_Q \frac{1}{\lambda \Phi} \beta^2 \|q_t\|^2 \, dx \, dt + \int_Q \lambda^3 \phi^3 \theta^2 q^2 \, dx \, dt \right).
\] (77)

According to (77), we have
\[
\begin{align*}
&\int_0^{T/2} \int_0^1 e^{2\lambda \Phi} \|q^2 \, dx \, dt + \int_0^1 q^2(x, 0) \, dx \\
&\leq C \left( \int_0^{T/2} \int_0^1 q^2 \, dx \, dt + \int_0^1 q^2(x, 0) \, dx \right)
\end{align*}
\] (78)
Similarly

\[
\int_0^1 e^{2\lambda\phi} q^3 \, dx \, dt + \int_0^1 q^2(x, 0) \, dx \leq C \left( \int_Q e^{21\lambda\phi} |L_T q|^2 \, dx \, dt + \int_Q \lambda^3 q^3 \, dt \right)
\]

\[
= C \left( \int_Q e^{21\lambda\phi} |L_T q|^2 \, dx \, dt + \int_0^{t/2} \int_0^1 \lambda^3 q^3 \, dx \, dt + \int_{t/2}^T \int_0^1 \lambda^3 q^3 \, dx \, dt \right)
\]

\[
\leq C \left( \int_Q e^{21\lambda\phi} |L_T q|^2 \, dx \, dt + \int_Q e^{21\lambda\phi} \psi^2 q^2 \, dx \, dt \right).
\]

By adding the left and right sides of the two formulas separately, we have

\[
\int_Q e^{2\lambda\phi} q^3 \, dx \, dt + \int_Q q^2(x, 0) \, dx \leq C \left( \int_Q e^{21\lambda\phi} |L_T q|^2 \, dx \, dt + \int_Q \lambda^3 q^3 \, dx \, dt \right)
\]

\[
= C \left( \int_Q e^{21\lambda\phi} |L_T q|^2 \, dx \, dt + \int_0^{t/2} \int_0^1 \lambda^3 q^3 \, dx \, dt + \int_{t/2}^T \int_0^1 \lambda^3 q^3 \, dx \, dt \right)
\]

Step 2: we can obtain the following observable inequality:

\[
\| e^{\lambda m\phi} q^{(1/2)} \|_{L^\infty(L^2)} + \| q(x, 0) \|_{L^2(0, 1)} \leq C \left( \| e^{\lambda m\phi} L_T q^* \|_{L^2(L^2)} + \| e^{\lambda m\phi} \psi^{3/2} q^3 \|_{L^2(L^2)} \right).
\]

Step 3: we will prove Proposition 3.

Note the right side of (81) as
\[ \|q\| = \left\| e^{\lambda M \phi} L^*_\gamma p \right\|_{L^2((0,T);L^2)} + \left\| e^{\lambda M \phi} f^{(3/2)} \right\|_{L^2((0,T);L^2)} \]  

(83)

\[ Q = \left\{ q \in C([0, T] ; L^2) \cap L^2(Q) \right\} \bigg| q(0, t) = q(1, t) = q_x(0, t) = q_x(1, t) = 0, \ \forall t \in [0, T] \bigg\}. \]  

(84)

Noticing that \( Q \) is a complete space of \( Q \) with respect to the above norm, \( Q \) is obviously a Hilbert space, on which the inner product is defined as

\[ \langle p, q \rangle = \int_Q e^{2 \lambda M \phi} L^*_\gamma p L^*_\gamma q \ dx \ dt + \int_Q e^{2 \lambda M \phi} p q \ dx \ dt. \]  

(85)

Recalling the first formula in (73)

\[ L_\gamma y = f + u \chi_\omega. \]  

(86)

Multiply both sides by \( q \) and integrating over \( Q \)

\[ \int_Q q L_\gamma y = \int_Q q f + \int_Q q u \chi_\omega, \]  

(87)

that is

\[ \int_Q q [y_t - y_{xx} + (2 \beta \gamma + \alpha \gamma_x - \beta) y + \alpha \gamma y_x] \ dx \ dt \]

\[ = \int_Q q f \ dx \ dt + \int_Q q u \chi_\omega \ dx \ dt. \]  

(88)

Integrating by parts, we have

\[ \int_Q y(x, T)q(x, T) T_1 \ dx - \int_Q y_0(x) q(x, 0) \ dx + \int_Q y L^*_\gamma q \ dx \ dt \]

\[ = \int_Q q f \ dx \ dt + \int_Q q u \chi_\omega \ dx \ dt, \]  

(89)

that is

\[ \int_Q q f \ dx \ dt + \int_Q y_0(x) q(x, 0) \ dx \]

\[ = \int_Q y(x, T)q(x, T) T_1 \ dx + \int_Q y L^*_\gamma q \ dx \ dt - \int_Q q u \chi_\omega \ dx \ dt. \]  

(90)

The left of (90) defines a linear function: \( \mathscr{L} : Q \rightarrow R: \)

\[ \mathscr{L} = \int_Q q f \ dx \ dt + \int_Q y_0(x) q(x, 0) \ dx, \]  

(91)

with

\[ |\mathscr{L} (q)| \leq \left\| e^{-\lambda M \phi} f^{(1/2)} \right\|_{L^2((0,T);L^2)} \| q_0(x) \|_{L^2((0,T);L^2)} \]  

\[ \leq \left( \| e^{-\lambda M \phi} f^{(1/2)} \|_{L^2((0,T);L^2)} + \| y_0(x) \|_{L^2((0,T);L^2)} \right) \]  

\[ \leq C \left( \| e^{-\lambda M \phi} f^{(1/2)} \|_{L^2((0,T);L^2)} + \| y_0(x) \|_{L^2((0,T);L^2)} \right) \]  

(92)

Therefore, \( \mathscr{L} \) is continuous. Then, there exists a unique \( p \) on \( Q \), such that

\[ \mathscr{L} (q) = \langle p, q \rangle, \ \forall q \in Q. \]  

(93)

Thus

\[ \int_Q q f \ dx \ dt + \int_Q y_0(x) q(x, 0) \ dx \]

\[ = \int_Q e^{2 \lambda M \phi} L^*_\gamma p L^*_\gamma q \ dx \ dt + \int_Q e^{2 \lambda M \phi} p q \ dx \ dt. \]  

(94)

Let \( y = e^{2 \lambda M \phi} L^*_\gamma p \) and \( u = e^{2 \lambda M \phi} p \), then (94) can also be written as follows:

\[ \mathscr{L} (q) = \langle p, q \rangle, \ \forall q \in Q. \]  

(95)

Combining (90) and (95), we obtain

\[ \int_Q y(x, T)q(x, T) T_1 \ dx = 0, \ \forall q \in Q, \]  

(96)

which implies \( y(x, T) = 0. \)

Furthermore, in the right side of (94), letting \( q = p \) and taking

\[ L^*_\gamma p = e^{-2 \lambda M \phi} y, \]  

\[ p = -e^{-2 \lambda M \phi} u, \]  

(97)
we have

\[
\begin{align*}
\|e^{-\lambda M\phi}y\|^2_{L^2(0,T;L^2)} + \|e^{-\lambda M\phi}(-\lambda/2)u\|^2_{L^2(0,T;L^2)} &= \mathcal{L}(p) \leq C\left(\|e^{-\lambda M\phi}(-1/2)f\|^2_{L^1(0,T;L^2)} + \|y_0(x)\|^2_{L^2(0,1)}\right)
\end{align*}
\]

Therefore

\[
\begin{align*}
\|e^{-\lambda M\phi}y\|^2_{L^2(L^1)} + \|e^{-\lambda M\phi}(-3/2)u\|^2_{L^2(L^1)} &\leq C\left(\|e^{-\lambda M\phi}(-1/2)f\|^2_{L^1(0,T;L^2)} + \|y_0(x)\|^2_{L^2(0,1)}\right),
\end{align*}
\]

that is

\[
\|u\|_U \leq C\left(\|f\|_F + \|y_0(x)\|_L^2(0,1)\right).
\]

In order to obtain \(\|y\|_Y \leq C\left(\|f\|_F + \|u\|_U + \|y_0(x)\|_L^2(0,1)\right)\), we construct a new system.

Let \(\xi = e^{-\lambda M\phi} - \lambda/2\), then \(\xi y\) satisfies the following equations:

\[
\begin{align*}
L\phi(\xi y) &= \xi f + \xi u_m \chi y, \\
(\xi y)(t, 0) &= 0 = (\xi y)(t, 1), \\
(\xi y)_x(t, 0) &= 0 = (\xi y)_x(t, 1), \\
(\xi y)_x(0, x) &= \xi(0)y_0(x).
\end{align*}
\]

According to the definition of \(\phi(t)\) and by direct calculation, we obtain

\[
\begin{align*}
\xi' &= e^{-\lambda M\phi}(-\lambda M\phi'(-3/2) - \lambda/2), \\
|\xi'| &= e^{-\lambda M\phi}(\lambda M\phi'(-3/2) - \lambda/2) \leq Ce^{-\lambda M\phi}.
\end{align*}
\]

In the following, we estimate \(\|\xi f\|_{L^1(0,T;L^2)}\), \(\|\xi u_m\|_{L^1(0,T;L^2)}\), and \(\|\xi y\|_{L^1(0,T;L^2)}\):

\[
\begin{align*}
\|\xi f\|_{L^1(0,T;L^2)} &= \|e^{-\lambda M\phi}(-3/2)f\|_{L^1(0,T;L^2)} = \|e^{(\lambda - M)\phi}(\lambda M\phi'(-1/2) - 1)\|_{L^1(0,T;L^2)} \leq C\|e^{-\lambda M\phi}(-1/2)f\|^2_{L^1(0,T;L^2)} = C\|f\|_F, \\
\|\xi u_m\|_{L^1(0,T;L^2)} &= \|e^{-\lambda M\phi}(-3/2)u_m\|_{L^1(0,T;L^2)} = \|e^{-\lambda M\phi}(-3/2)u\|_{L^1(0,T;L^2(0,1))} \leq C\|e^{-\lambda M\phi}(-1/2)\|_{L^1(0,T;L^2(0,1))} = C\|u\|_U, \\
\|\xi y\|_{L^1(0,T;L^2)} &\leq C\|e^{-\lambda M\phi}y\|_{L^1(0,T;L^2)} \leq C\|e^{-\lambda M\phi}y\|_{L^2(0,T;L^2)} \leq C\left(\|f\|_F + \|y_0(x)\|_{L^2(0,1)}\right).
\end{align*}
\]

According to the results of well posedness, we have

\[
\begin{align*}
\|e^{-\lambda M\phi}(-3/2)y\|_{C(0,T;L^2(0,1);L^2(0,T;H^1(0,1)))} &\leq C\left(\|f\|_F + \|u\|_U + \|y_0(x)\|_{L^2(0,1)}\right),
\end{align*}
\]

which implies

\[
\|y\|_Y \leq C\left(\|f\|_F + \|u\|_U + \|y_0(x)\|_{L^2(0,1)}\right).
\]

The proof of Proposition 3 is completed.

Next, we consider the controllability of the nonlinear system. We have the following result.
Proposition 4. Let $\forall \varepsilon > 0$ and $y_0 \in L^2(0, 1)$, satisfying $\|y_0\|_{L^2(0,1)} < \varepsilon$, then there exists a control $u \in U$, such that

$$||y(t)||_{L^2(0,1)} \leq c $$

has a solution $y \in Y$, satisfying $y(\cdot, T) = 0$. Before the proof, we decompose the nonlinear operator:

$$Ny = \alpha y x + \beta y^2$$

Proof.

$$\|N_{Pro}(y_1, y_2)\|_F = \left\| e^{-\lambda \Phi \Phi_1^{1/2}} y_1, y_2 \right\|_{L^2(0, T; L^2)} = \alpha \int_0^T e^{-\lambda \Phi \Phi_1^{1/2}} \|y_1, y_2\|_{L^2(0,1)} dt$$

$$\leq \alpha C(\lambda, M) \int_0^T e^{-2\lambda M \Phi \Phi^3} \|y_1, y_2\|_{L^2(0,1)} dt \leq \alpha C(\lambda, M) \int_0^T e^{-2\lambda M \Phi \Phi^3} \|y_1, y_2\|_{L^2(0,1)} dt$$

Lemma 1. For $N_{Pro}: Y \times Y \rightarrow F$ and $N_{Bil}: Y \times Y \rightarrow F$, there exists $C > 0$, such that

$$\|N_{Pro}(y_1, y_2)\|_F \leq C \|y_1, y_2\|_Y$$

$$\|N_{Bil}(y_1, y_2)\|_F \leq C \|y_1, y_2\|_Y$$

Lemma 2 (local inverse theorem). Let $E$ and $G$ be two Banach spaces, $\Lambda: E \rightarrow G$ satisfying $\Lambda \in C^1(E, G)$. If for $\varepsilon \in E, \Lambda'(\varepsilon) = \hat{g}$ and $\Lambda'(\varepsilon): E \rightarrow G$ is a surjection. Then, for any $g$ satisfying $\|g - \hat{g}\| < \varepsilon$, the equation $\Lambda(e) = g$ must have a solution $e$.

With Lemmas 1 and 2, we can prove Proposition 4.

Proof. Taking $E = \{(u, y) \in U \times Y | L_T y - u \chi_y \in F\}$, with the norm defined as

$$\|y\|_F = \left( \|u\|_U^2 + \|y\|_Y^2 + \|L_T y - u \chi_y\|_F^2 \right)^{1/2}$$

and $G = F \times L^2(0, 1)$. We define the set-valued mapping:

$$\Lambda: E \rightarrow G$$

$$e = (u, y) \mapsto \left( L_T y - u \chi_y + N y, y_0(x) \right)$$

Obviously, $\Lambda(0, 0) = (0, 0)$ and $\Lambda \in C^1(E, G)$. Let $\bar{e} = (0, 0)$ and $\bar{y} = (0, 0)$, then

$$\|y\|_F = \left( \|u\|_U^2 + \|y\|_Y^2 + \|L_T y - u \chi_y\|_F^2 \right)^{1/2}$$
$\Lambda'(\tilde{e}): E \rightarrow G,$

$$e = (u, y) \mapsto \left( L_\gamma y - u \chi_\omega, y_0(x) \right). \quad (114)$$

The conclusion of Proposition 3 shows the fact that $\Lambda'(\tilde{e}): E \rightarrow G$ is a surjection.

According to Lemma 2, system (106) has a solution $y \in Y$ and

$$y(\cdot, T) = 0. \quad (115)$$

In this way, null controllability of system (4) is obtained. Equivalently, local exact controllability to the trajectory of system (2) is achieved. \hfill \Box

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares no conflicts of interest.

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