The Cauchy problem
for a semilinear ordinary differential equation
in the homogeneous and isotropic spacetime

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Abstract

A semilinear ordinary differential equation is derived from a semilinear Schrödinger equation in the homogeneous and isotropic spacetime by the Ehrenfest theorem. The Cauchy problem for the equation is considered. Exact solutions and nonexistence of global weak solutions of the equation are also considered in the de Sitter spacetime. The effects of spatial expansion and contraction are studied.

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1 Introduction

The Cauchy problems for some semilinear Klein-Gordon equations have been considered in the de Sitter spacetime in [3, 12, 17]. In some cases, the spatial expansion [resp. contraction] can be characterized as a dissipative [resp. anti-dissipative] term in the equations (for example, see the energy estimate (2.3) in [12]), by which the existence-time of the solution is deeply affected. A semilinear Schrödinger equation was derived from the semilinear Klein-Gordon equation in [13] by the nonrelativistic limit, and its Cauchy problem was considered. In this paper, we derive a semilinear ordinary differential equation from the semilinear Schrödinger equation based on the Ehrenfest theorem, and we consider its Cauchy problem in more general spacetime known in Cosmology.

We consider the Cauchy problem for the semilinear ordinary differential equation given by (1.4), below, in the homogeneous and isotropic spacetime which is described by a scale-function given by (1.2), below. We start from the introduction of the scale-function. Let $n \geq 1$, $\sigma \in \mathbb{R}$. For $a_0 > 0$ and $a_1 \in \mathbb{R}$, put $H := a_1/a_0$ which is called the Hubble constant. When $\sigma \neq -1$ and $a_1 \neq 0$, we put

$$T_0 := -\frac{2a_0}{n(1+\sigma)a_1}.$$ 

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We define
\[ T_1 := \begin{cases} T_0 & \text{if } (1 + \sigma)a_1 < 0 \text{ (i.e., } T_0 > 0), \\ \infty & \text{if } (1 + \sigma)a_1 \geq 0. \end{cases} \]

Under the assumption of the cosmological principle, namely, that the space is homogeneous and isotropic, the solution of the Einstein equations with the spatial flat curvature is given by the Friedmann-Lemaître-Robertson-Walker metric (the FLRW metric) given by
\[ -c^2(d\tau)^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -c^2(dx^0)^2 + a(x^0)^2 \sum_{j=1}^{n}(dx^j)^2, \quad (1.1) \]
where \( \tau \) is the proper time, and \( a(\cdot) \) is the scale-function of the spacetime defined by
\[ a(t) := \begin{cases} a_0 \left(1 + \frac{n(1+\sigma)a_1}{2a_0} \right)^{2/\sigma} & \text{if } \sigma \neq -1, \\ a_0e^{a_1t/a_0} & \text{if } \sigma = -1. \end{cases} \quad (1.2) \]
for \( 0 \leq t < T_1 \) and \( \sigma \in \mathbb{R} \) (see [4, 5] for the references of the Einstein equations and the FLRW metric). We denote the first order derivative by the variable \( t \) by
\[ D_t := \frac{d}{dt}. \]

We note that \( a(0) = a_0 \) and \( D_t a(0) = a_1 \) hold.

The Minkowski spacetime is obtained when \( a_1 = 0 \) (i.e., \( a(\cdot) = a_0 \) is a constant), and the de Sitter spacetime is obtained when \( \sigma = -1 \). The scale-function \( a(\cdot) \) defined by (1.2) blows up [resp. vanishes] at finite time \( T_0 > 0 \) when \( a_1 > 0 \) and \( \sigma < -1 \) [resp. \( a_1 < 0 \) and \( \sigma > -1 \)], which is called Big-Rip [resp. Big-Crunch] in cosmology. The case \( \sigma = -1 \) shows the exponential expansion [resp. contraction] of the space when \( a_1 > 0 \) [resp. \( a_1 < 0 \)]. The case \( a_1 > 0 \) and \( \sigma > -1 \) [resp. \( a_1 < 0 \) and \( \sigma < -1 \)] shows the polynomial expansion [resp. contraction] of the space. These models have been studied as some models of the universe.

We define a function \( q_0(\cdot) \) and a weight function \( A(\cdot) \) by
\[ q_0(t) := \frac{D_t a^2(t)}{a^2(t)}, \quad A(t) := -\frac{1}{4}q_0(t)^2 - \frac{1}{2}D_t q_0(t) \quad (1.3) \]
for \( 0 \leq t < T_1 \). For \( T \) with \( 0 < T \leq T_1 \), we consider the Cauchy problem for a semilinear ordinary differential equation given by
\[ \begin{cases} D_t^2 Y(t) + A(t)Y(t) + f(Y)(t) = 0, \\ Y(0) = Y_0, \quad D_t Y(0) = Y_1 \end{cases} \quad (1.4) \]
for \( 0 \leq t < T \), where \( Y = (Y^1, \ldots, Y^n), Y_0 = (Y_0^1, \ldots, Y_0^n) \in \mathbb{R}^n, Y_1 = (Y_1^1, \ldots, Y_1^n) \in \mathbb{R}^n, \)
\[ |Y| := \left( \sum_{j=1}^{n}(Y^j)^2 \right)^{1/2}, \quad (1.5) \]
\[ \lambda \in \mathbb{R}, \ 1 < p < \infty \text{ and} \]
\[ f(Y) := \lambda |Y|^{p-1}Y \text{ or } f(Y) := \lambda |Y|^p. \]  

The differential equation in (1.4) is the Newtonian equation of motion and is derived from the semilinear Schrödinger equation in the homogeneous and isotropic space-time with the scale-function \( a(\cdot) \) in (1.2) by the Ehrenfest theorem (see Section 2 below). We regard the solution of the Cauchy problem (1.4) as the fixed point of the integral operator \( \Psi \) defined by (3.2), below. The differential equation in (1.4) is the Newtonian equation of motion and is derived from the semilinear Schrödinger equation in the homogeneous and isotropic space-time with the scale-function \( a(\cdot) \) in (1.2) by the Ehrenfest theorem (see Section 2 below). We regard the solution of the Cauchy problem (1.4) as the fixed point of the integral operator \( \Psi \) defined by (3.2), below.

The differential equation in (1.4) with \( f(Y) := \lambda |Y|^{p-1}Y \) is rewritten as
\[ D_s^2Z + \frac{2\beta + 1}{s} D_sZ + \frac{\beta^2 + A(t(s))}{s^2} Z + \lambda s^{\beta(p-1)-2} |Z|^{p-1}Z = 0 \]  

by the change of variable \( s := e^{t} \) and \( Z(s) = Y(t(s))s^{-\beta} \) for \( \beta \in \mathbb{R} \). The equation \( D_s^2Z + B(s)|Z|^{p-1}Z = 0 \) is known as the Emden-Fowler type equation for some function \( B(\cdot) \). We refer to [9] for the nonoscillation, [10] for the blow-up rate, [2] [6] [11] for the asymptotic behavior, and [7] [8] for the uniqueness of the solution of the Emden-Fowler type equation. The equation \( D_s^2Z + BZ/s^2 - |Z|^{p-1}Z = 0 \) is considered in [1] for a constant \( B \). Our equation (1.7) has a dissipative or anti-dissipative term \((2\beta+1)D_sZ/s\), and the coefficient of \( Z/s^2 \) is a function \( \beta^2 + A(t(s)) \) dependent on the scale-function in (1.2).

We say that \( Y \) is a global solution of (1.4) if \( Y \) exists on the time-interval \([0, T_1)\) since the spacetime does not exist after \( T_1 \). Let us consider the four cases (i), (ii), (iii) and (iv) given by
\[ \begin{align*}
(1.8) \quad & (i) \quad a_1 > 0, \quad \sigma > -1 + \frac{2}{n}, \\
& (ii) \quad a_1 = 0, \quad \sigma \in \mathbb{R}, \\
& (iii) \quad a_1 \neq 0, \quad \sigma = -1 + \frac{2}{n}, \\
& (iv) \quad a_1 < 0, \quad \sigma > -1 + \frac{2}{n}.
\end{align*} \]

The scale-function \( a(t) = a_0 + a_1t \) given in the case (iii) corresponds to the scale-function for the Milne universe. In the cases (i), (ii) and (iii), we have \( A \geq 0 \) and \( D_tA \leq 0 \) on \([0, T_1)\) (see Lemma 2.6 below). We put
\[ D := \sqrt{A(0)}|Y_0| + |Y_1|. \]

Let us consider the case (i). For \( T > 0 \) and \( R > 0 \), we define the norm \( \| \cdot \|_{X(T)} \) by
\[ \|Y\|_{X(T)} := \|D_tY\|_{L^\infty(0,T)} + \|\sqrt{A}Y\|_{L^\infty((0,T))} + \|\sqrt{-D_tA}Y\|_{L^2((0,T))}, \]
and the function spaces \( X(T) \) and \( X(T, R) \) by
\[ X(T) := \{Y; \|Y\|_{X(T)} < \infty\} \text{ and } X(T, R) := \{Y; \|Y\|_{X(T)} \leq R\}. \]

We obtain the local and global solutions of the problem (1.4) as follows.

**Theorem 1.1** (Cauchy problem for the case (i)). Let \( a_1 > 0 \) and \( \sigma > -1 + 2/n \) (so that, \( T_0 < 0, T_1 = \infty \)). Let \( \lambda \in \mathbb{R} \) and \( 1 < p < \infty \). Let \( q_* \) be an arbitrary number which satisfies \( 1 \leq q_* < \infty \) and \( 1 - p/2 \leq 1/q_* \).
Here, the constants $C > 0$ which is independent of $Y_0$ and $Y_1$

(2) The solution $Y$ in (1) satisfies $Y \in C^1((0, T))$.

(3) The solution $Y$ in (1) is unique in $C^1([0, T]) \cap X(T)$.

(4) If $\lambda \geq 0$ and $f(Y) = \lambda |Y|^{p-1}Y$, then the solution $Y$ in (1) is a global solution. Namely, $T$ can be taken as $T = T_1$.

Let us consider the cases (ii) and (iii). For $T > 0$, $R_0 > 0$ and $R_1 > 0$, we put

$$
X(T, R_0, R_1) := \{Y; \|Y\|_{L^\infty((0, T))} \leq R_0, \|D_t Y\|_{L^\infty((0, T))} \leq R_1\}.
$$

**Theorem 1.2** (Cauchy problem for the cases (ii) and (iii)). Let $a_1 = 0$ and $\sigma \in \mathbb{R}$ (so that, $T_1 = \infty$), or $a_1 \neq 0$ and $\sigma = -1 + 2/n$ (so that, $T_1 = \infty$ when $a_1 > 0$, and $T_1 = T_0 > 0$ when $a_1 < 0$). Let $\lambda \in \mathbb{R}$ and $1 < p < \infty$. Then the following results hold. Moreover, the results from (2) to (4) in Theorem 1.1 hold with $X(T)$ replaced by

$$
\{Y; \|Y\|_{L^\infty((0, T))} < \infty, \|D_t Y\|_{L^\infty((0, T))} < \infty\}.
$$

(1) For any $Y_0 \in \mathbb{R}^n$ and $Y_1 \in \mathbb{R}^n$, there exists $T > 0$, constants $C_0 > 0$ and $C > 0$ such that the Cauchy problem (1.4) has a unique solution $Y \in X(T, R_0, R_1)$ for any $R_0$ and $R_1$ with $R_0 \geq 2|Y_0|$ and $R_1 \geq C_0|Y_1|$, where $T$ can be arbitrarily taken as

$$
0 < T \leq \min \left\{ \frac{C}{R_0^{p-1}}, \frac{C R_1}{R_0^{p-1/2}}, \frac{C R_1}{R_0}, \frac{R_0}{2 R_1} \right\}.
$$

**Theorem 1.2** holds with $X(T)$ replaced by

$$
\{Y; \|Y\|_{L^\infty((0, T))} < \infty, \|D_t Y\|_{L^\infty((0, T))} < \infty\}.
$$

Here, the constants $C_0$ and $C$ are independent of $Y_0$ and $Y_1$.

(2) When $\sigma = -1 + 2/n$ and $a_1 < 0$ (so that, $T_1 = T_0 > 0$), if $|Y_0|$ and $|Y_1|$ are sufficiently small, then the solution $Y$ in (1) is a global solution.

In (2) in Theorem 1.2 we obtain global solutions for small data when the space is contracting (i.e., $a_1 < 0$), while we have only local solutions in (1) in the Minkowski spacetime (i.e., $a_1 = 0$).

Let us consider the case (iv). We note $T_1 = T_0 > 0$, $A > 0$ and $D_t A > 0$ on $[0, T_1)$ in this case (see (7) in Lemma 2.6 below). We put

$$
\begin{align*}
\|Y\|_{X'(T)} &:= \|A^{-1/2} D_t Y\|_{L^\infty((0, T))} + \|Y\|_{L^\infty((0, T))} + \|A^{-1} \sqrt{D_t AD_t Y}\|_{L^2((0, T))}, \\
X'(T) &:= \{Y; \|Y\|_{X'(T)} < \infty\}, \\
X'(T, R) &:= \{Y; \|Y\|_{X'(T)} \leq R\}, \\
D' &:= |Y_0| + A(0)^{-1/2}|Y_1|,
\end{align*}
$$

for $0 < T \leq T_1$ and $R > 0$. 

Theorem 1.3 (Cauchy problem for the case (iv)). Let $a_1 < 0$ and $\sigma > -1 + 2/n$ (so that, $T_1 = T_0 > 0$). Let $\lambda \in \mathbb{R}$ and $1 < p < \infty$. Then there exists constants $C_0 > 0$ and $C > 0$ such that the following results hold.

1. For any $Y_0 \in \mathbb{R}^n$ and $Y_1 \in \mathbb{R}^n$, there exists $T$ with $0 < T < T_1$, constants $C_0 > 0$ and $C > 0$ such that the Cauchy problem (1.4) has a unique solution $Y \in X'(T, C_0 D')$, where $T$ can be arbitrarily taken under the condition

$$ C|\lambda|a_0 T \left|\frac{1 - T}{2T_0}\right| (C_0 D')^{p-1} \leq 1. \quad (1.13) $$

Here, the constants $C_0$ and $C$ are independent of $Y_0$ and $Y_1$.

2. If $D'$ is sufficiently small such that

$$ \frac{2C|\lambda|a_0^2}{n(1 + \sigma)|a_1|^2} (C_0 D')^{p-1} \leq 1, \quad (1.14) $$

then the solution $Y$ in (1) is a global solution. Namely, $T$ can be taken as $T = T_1$.

3. The solution $Y$ in (1) satisfies $Y \in C^1([0, T))$.

4. The solution $Y$ in (1) is unique in $C^1([0, T)) \cap X'(T)$.

5. If $\lambda \geq 0$ and $f(Y) = \lambda |Y|^{p-1}Y$, then the solution $Y$ in (1) is a global solution.

In (2) in Theorem 1.3, we obtain global solutions when the space is contracting (i.e., $a_1 < 0$) under the condition (1.14). We note that global solutions are obtained even for large data (i.e., $D'$ is large) if the spatial contraction is sufficiently large (i.e., $|a_1|$ is sufficiently large).

In the above theorems, the case $\sigma \geq -1 + 2/n$ has been considered when $a_1 \neq 0$, which is required for our energy estimates (see Lemmas 2.7 and 2.8 below). When $a_1 = 0$, we note that $a(\cdot) = a_0$ in (1.2) is independent of $\sigma$. The case $\sigma = -1$ in (1.2) corresponds to the de Sitter spacetime with the flat spatial curvature which is one of important models of the expanding or contracting universe. In this case, we have the following two theorems. The first theorem shows the exact solutions of the Cauchy problem (1.4).

Theorem 1.4 (Exact solutions in the de Sitter spacetime). Let $n \geq 1$, $H \in \mathbb{R}$, $a(t) = e^{Ht}$, $\lambda \in \mathbb{R}$, $p \in \mathbb{R}$, $f(Y) = \lambda |Y|^{p-1}Y$.

1. Let $p = 1$. Then the solution of (1.4) is given by

$$ Y(t) = \begin{cases} Y_0 \cos \sqrt{\lambda - H^2}t + \frac{Y_1}{\sqrt{\lambda - H^2}} \sin \sqrt{\lambda - H^2}t & \text{if } \lambda > H^2, \\
Y_0 + Y_1 t & \text{if } \lambda = H^2, \\
Be^{\sqrt{H^2 - \lambda} t} + Ce^{-\sqrt{H^2 - \lambda} t} & \text{if } \lambda < H^2, 
\end{cases} \quad (1.15) $$

where $B$ and $C$ are constants defined by

$$ B := \frac{1}{2} \left( Y_0 + \frac{Y_1}{\sqrt{H^2 - \lambda}} \right) \quad \text{and} \quad C := \frac{1}{2} \left( Y_0 - \frac{Y_1}{\sqrt{H^2 - \lambda}} \right). \quad (1.16) $$

2. Let $p \neq 1$. Let $Y^3 = \cdots = Y^n = 0$. Put $R := \sqrt{(Y_0^1)^2 + (Y_0^2)^2}$. Then the solution $Y$ of (1.4) is given as follows.
(i) If $\lambda R^{p-1} > H^2$ (namely, $\lambda > 0$ and $R > (H^2/\lambda)^{1/(p-1)}$), then
\[ Y^1(t) = R \cos(\omega t + \delta), \quad Y^2(t) = R \sin(\omega t + \delta), \quad (1.17) \]
where we have put $\omega := \sqrt{\lambda R^{p-1} - H^2}$ and $\delta$ is a number which satisfies $Y^1_0 = R \cos \delta$, $Y^2_0 = R \sin \delta$, $Y^1_1 = -R \omega \sin \delta$ and $Y^2_1 = R \omega \cos \delta$.

(ii) If $\lambda R^{p-1} = H^2$ (namely, $\lambda = H = 0$, or $\lambda > 0$ and $R = (H^2/\lambda)^{1/(p-1)}$), then
\[ Y^1(t) = Y^1_0, \quad Y^2(t) = Y^2_0 \]
and $Y^1_1 = Y^2_1 = 0$. Namely, $Y^1$ and $Y^2$ are constants.

(iii) If $\lambda R^{p-1} < H^2$ (namely, $\lambda < 0$ and $H \in \mathbb{R}$, or $\lambda = 0$ and $H \neq 0$, or $\lambda > 0$ and $R < (H^2/\lambda)^{1/(p-1)}$), then the solution $Y^1 = Y^2 = 0$ is only allowed.

In Theorem 1.4, $\lambda = H^2$ and $\lambda R^{p-1} = H^2$ are thresholds for the results in (1) and (2), respectively. Especially, we have the new solutions for the positive $\lambda > 0$ with $0 < \lambda < H^2$ in (1) and $0 < \lambda < H^2/R^{p-1}$ for (iii) in (2) when $H \neq 0$.

**Remark 1.5** (Rotational motion by the gravity). Let $M > 0$ be a mass constant. Let $G$ be the Newton gravitational constant. The first equation in the Cauchy problem (1.4) for $\sigma = -1$, $f(Y) = \lambda |Y|^{p-1}Y$, $p = -2$ and $\lambda = GM$ is written as
\[ D_t^2 Y(t) - H^2 Y(t) + \frac{GM}{|Y(t)|^2} Y(t) = 0 \quad (1.18) \]
for $0 \leq t < \infty$, which is the Newton equation for a star which goes around another star with the mass $M$ in the de Sitter spacetime. When we consider a star moving on the plane by $(Y^1, Y^2)$ coordinates (i.e., $Y^3 = \cdots = Y^n = 0$), the result (i) in (2) in Theorem 1.4 gives the solution
\[ Y^1(t) = R \cos (\omega t + \delta), \quad Y^2(t) = R \sin (\omega t + \delta) \quad (1.19) \]
for some constant $\delta$ if $H$ satisfies $H^2 < GM/R^3$, where $\omega = \sqrt{GM/R^3 - H^2}$. We note that the original variable $X$ is given by $X(t) = Y(t)/a(t) = Y(t)/e^{Ht}$ in (2.3), below. The solution (1.19) shows that the angular velocity $\omega$ of the star in the de Sitter spacetime ($H \neq 0$) is smaller than the angular velocity $\sqrt{GM/R^3}$ in the Minkowski spacetime ($H = 0$). Let us calculate its difference by an example of the sun and the earth. We use the values
\[ T = T = 3.1556925 \times 10^7 \text{[s]}, \quad M = M = 1.9884 \times 10^{30} \text{[kg]}, \quad \dot{M} = 3.085677581 \times 10^{19} \text{[km] with } \dot{P} = 3.085677581. \]

where $s$ denotes the second, and Mpc denotes Mega parsec which is $1 \text{Mpc} = \dot{P} \times 10^{19}$ [km] with $\dot{P} = 3.085677581$. When $H = 0$, we must have $\omega T = \sqrt{GMR^{-3} \cdot T} = 2\pi$, while we have $\sqrt{GMR^{-3}} \cdot T = 2 \times 3.141004674$ from the above values. Now, we
calculate the angular velocity in the de Sitter spacetime as
\[
\sqrt{\frac{GM}{R^5} - H^2} = \sqrt{\frac{GM}{R^5} \left( 1 - \frac{\dot{H}^2 \dot{R}^3}{GMP^2} \times 10^{-22} \right)}^{1/2},
\]
where
\[
\frac{\dot{H}^2 \dot{R}^3}{GMP^2} \equiv 1.298358447.
\]

Secondly, we consider the case \( \sigma = -1, n = 1, Y_0 \in \mathbb{R}, Y_1 \in \mathbb{R}, H \geq 0, \lambda > 0, 1 < p < \infty, a(t) = e^{Ht}, f(Y) = \lambda |Y|^p \) in \( (1.20) \). Namely,
\[
\begin{aligned}
&\left\{ \begin{array}{ll}
D_t^2 Y(t) - H^2 Y(t) + \lambda |Y(t)|^p \phi(t) &= 0 \quad \text{for } t \geq 0, \\
Y(0) = Y_0, \quad D_t Y(0) = Y_1.
\end{array} \right.
\end{aligned}
\tag{1.20}
\]
We say that \( Y \) is the global weak solution if \( Y \) satisfies
\[
-Y_1 \phi(0) + Y_0 D_t \phi(0) + \int_0^\infty Y(t) D_t^2 \phi(t) - H^2 Y(t) \phi(t) + \lambda |Y(t)|^p \phi(t) dt = 0 \tag{1.21}
\]
for any \( \phi \in C_0^2([0, \infty)) \). The definition of the global weak solution follows from the differential equation in \( (1.20) \). Namely,
\[
0 = (D_t^2 Y - H^2 Y + \lambda |Y|^p) \phi = D_t (D_t Y \phi - Y D_t \phi) + Y D_t^2 \phi - H^2 Y \phi + \lambda |Y|^p \phi,
\]
which yields \( (1.21) \) by the integration. The next theorem shows the nonexistence of the nontrivial global weak solution of \( (1.20) \), which implies that nontrivial local solutions of \( (1.20) \) must blow up in finite time.

**Theorem 1.6** (Nonexistence of nontrivial global weak solution). \( \) Let \( n = 1, H \geq 0, a(t) = e^{Ht}, \lambda > 0, 1 < p < \infty \). If \( HY_0 + Y_1 \leq 0 \), then the global weak solution \( Y \) of \( (1.20) \) must satisfy \( Y = 0 \).

On the condition \( HY_0 + Y_1 \leq 0 \) in Theorem 1.6, we note that \( Y_1 \) must be non-positive for the Minkowski spacetime (i.e., \( H = 0 \)), while \( Y_1 \) can be positive if \( Y_0 < 0 \) and the spatial expansion is sufficiently large (i.e., \( H \) is sufficiently large).

We use the following notations. We denote the Lebesgue space by \( L^q(I) \) for an interval \( I \subset \mathbb{R} \) and \( 1 \leq q \leq \infty \) with the norm
\[
\|Y\|_{L^q(I)} := \left\{ \frac{1}{q} \int_I |Y(t)|^q dt \right\}^{1/q} \quad \text{if } 1 \leq q < \infty,
\]
\[
\text{ess. sup}_{t \in I} |Y(t)| \quad \text{if } q = \infty.
\]
We denote the inequality \( A \leq CB \) for some constant \( C > 0 \) which is not essential for the argument by \( A \lesssim B \). Put \( \nabla := (\partial_1, \ldots, \partial_n) \) and \( \Delta := \sum_{j=1}^n \partial_j^2 \).

This paper is organized as follows. We show the derivation of the differential equation in \( (1.4) \) based on the Ehrenfest theorem for the Schrödinger equation in the homogeneous and isotropic spacetime in Section 2. We also show some estimates for the solution of the linear equation, and the energy estimates for the inhomogeneous equation. We prove the above theorems in Sections from 3 to 7.
2 Derivation of the equation

In this section, we show the derivation of the first equation in (1.4). We derive it from the Schrödinger equation by the Ehrenfest theorem, where the Schrödinger equation is derived from the Klein-Gordon equation by the nonrelativistic limit.

We use the following convention. The Greek letters $\alpha, \beta, \gamma, \cdots$ run from 0 to $n$, and the Latin letters $j, k, \ell, \cdots$ run from 1 to $n$. We use the Einstein rule for the sum of indices, namely, the sum is taken for same upper and lower repeated indices, for example, $\partial_j \partial_j := \sum_{j=1}^{n} \partial_j \partial_j$, $T_{\alpha \alpha} := \sum_{\alpha=0}^{n} T_{\alpha \alpha}$, and $T_{jj} := \sum_{j=1}^{n} T_{jj}$ for any tensor $T_{\alpha \beta}$.

We put $x := (x^0, x^1, \cdots, x^n) \in \mathbb{R}^{1+n}$, $t := x^0$, $x' := (x^1, \cdots, x^n)$.

Firstly, let us consider the Klein-Gordon equation

\begin{equation}
\frac{1}{\sqrt{-g}} \partial_\alpha \left( \sqrt{-g} g^{\alpha \beta} \partial_\beta \phi \right) - \frac{m^2 c^2}{\hbar^2} \phi - \frac{2m}{\hbar^2} U(x') \phi = 0 \tag{2.1}
\end{equation}

in the spacetime with the metric $(g_{\alpha \beta}(t)) = \text{diag} (-c^2, a(t)^2, \cdots, a(t)^2)$, where $g$ is the determinant of the matrix $(g_{\alpha \beta})$, the matrix $(g^{\alpha \beta})$ denotes the inverse matrix of $(g_{\alpha \beta})$, $x = (x^0, x^1, \cdots, x^n)$ with $x^0 = t$ and $U = U(x')$ for $x' = (x^1, \cdots, x^n)$ is a real-valued potential.

Put $w(t) := b_0 \left( \frac{a_0}{a(t)} \right)^{n/2}$, $b(t) := w(t) e^{-imc^2 t/\hbar}$, $u(x) := \frac{\phi(x)}{b(t)}$ for the constant $a_0 > 0$ in (1.2), and any constant $b_0 (= b(0)) \in \mathbb{R}$.

We obtain the Schrödinger equation by the nonrelativistic limit as follows.

**Lemma 2.1.** The Schrödinger equation

\begin{equation}
i \frac{2m}{\hbar} \partial_t u + \frac{1}{a^2} \Delta u - \frac{2m}{\hbar^2} U u = 0 \tag{2.2}
\end{equation}

is obtained from (2.1) by the nonrelativistic limit (i.e., $c \to \infty$). The equation (2.2) is rewritten as

\begin{equation}
i \partial_t u - \frac{1}{2m} p^j p_j u - U u = 0, \tag{2.3}
\end{equation}

where $x_\alpha := g_{\alpha \beta} x^\beta$, $p_\alpha := -i\hbar \partial_\alpha$, $p^\alpha := g^{\alpha \beta} p_\beta$ for $0 \leq \alpha \leq n$.

**Proof.** Since we have

\begin{equation}\frac{1}{\sqrt{-g}} \partial_\alpha \left( \sqrt{-g} g^{\alpha \beta} \partial_\beta \phi \right) = -\frac{1}{c^2} \left( \frac{n \partial_a a}{a} \partial_t \phi + \partial_t^2 \phi \right) + \frac{1}{a^2} \Delta \phi,
\end{equation}

the equation (2.1) is rewritten as

\begin{equation}
-\frac{1}{c^2} \cdot I + II = 0, \tag{2.4}
\end{equation}

where we have put

\begin{align*}
I := & \frac{n \partial_a a}{a} \partial_t \phi + \partial_t^2 \phi + \frac{m^2 c^4}{\hbar^2} \phi, \\
II := & \frac{1}{a^2} \Delta \phi - \frac{2m}{\hbar^2} U(x') \phi.
\end{align*}
Put $\alpha := -im/h$. Since we have

$$I = b(\partial_t^2 u + 2\alpha c^2 \partial_t u + u \tilde{M}), \quad \tilde{M} = b \left( \frac{1}{a^2} \Delta u - \frac{2m}{h^2} U u \right),$$

where we have put

$$\tilde{M} := - \left( \frac{n \partial_t a}{2a} \right)^2 - \frac{n}{2} \partial_t \left( \frac{\partial_t a}{a} \right),$$

the equation (2.4) is rewritten as

$$0 = -\frac{1}{c^2}(\partial_t^2 u + u \tilde{M}) - 2\alpha \partial_t u + \frac{1}{a^2} \Delta u - \frac{2m}{h^2} U u.$$

By the nonrelativistic limit ($c \to \infty$) of this equation, we obtain the Schrödinger equation (2.2), which is rewritten as (2.3) by $p_j p_j = -\hbar^2 a^{-2} \Delta$.

For the solution $u$ of the Schrödinger equation, we define the expectation values for $x^j, x_j, p_j$ and any function $h = h(x')$ for $x' \in \mathbb{R}^n$ by

$$\langle x^j \rangle(t) := \int_{\mathbb{R}^n} u(t, x') x^j u(t, x') dx', \quad \langle x_j \rangle(t) := \int_{\mathbb{R}^n} u(t, x') x_j u(t, x') dx',$$

$$\langle p_j \rangle(t) := \int_{\mathbb{R}^n} u(t, x') p_j u(t, x') dx', \quad \langle h \rangle(t) := \int_{\mathbb{R}^n} u(t, x') h(x') u(t, x') dx'.$$

We have the equation of motion as follows.

**Lemma 2.2.** Let $u$ be the solution of (2.2). The equations

1. $m D_t \langle x^j \rangle(t) = \langle p_j \rangle(t), \quad 2. m D_t^2 \langle x^j \rangle(t) = -\frac{D_t a^2(t)}{a^2(t)} \langle p_j \rangle(t) - \langle \partial^j U \rangle(t)$

hold for $t \geq 0$ and $1 \leq j \leq n$. Especially, the equation of motion

$$m D_t^2 \langle x^j \rangle(t) + m \frac{D_t a^2(t)}{a^2(t)} D_t \langle x^j \rangle(t) + \langle \partial^j U \rangle(t) = 0$$

holds for $0 \leq t < T_1$.

**Proof.** (1) Put $I_j^j := \bar{u} x^j i \hbar \partial_t u$ and $\ni^j := \bar{u} x^j \Delta u$. By the equation (2.2), we have

$$\text{Im} I^j = -\frac{\hbar^2}{2ma^2} \text{Im} \i^j.$$

Since we have $\nabla(\bar{u} u) = 2\text{Re}(\bar{u} \nabla u)$ and $\int_{\mathbb{R}^n} \nabla(\bar{u} u) dx' = 0$ by the divergence theorem, we have

$$\int_{\mathbb{R}^n} \bar{u} \nabla u dx' = i \text{Im} \int_{\mathbb{R}^n} \bar{u} \nabla u dx'.$$

Since $\i^j$ is rewritten as

$$\i^j = \nabla \cdot (\bar{u} x^j \nabla u) - x^j |\nabla u|^2 - \bar{u} \partial^j u,$$
we have \( \text{Im} \, I^j = \nabla \cdot \text{Im} (\overline{u} x^j \nabla u) - \text{Im} (\overline{\partial_j u}) \), which yields
\[
\int_{\mathbb{R}^n} \text{Im} \, I^j \, dx' = -\text{Im} \int_{\mathbb{R}^n} \overline{\partial_j u} \, dx' = i \int_{\mathbb{R}^n} \overline{\partial_j u} \, dx'.
\]
(2.7)
by (2.6). By the definition of \( \langle x^j \rangle \) and (2.7), we have

\[
\begin{align*}
i \hbar D_t \langle x^j \rangle & = 2i \int_{\mathbb{R}^n} \text{Im} \, I^j \, dx' \\
& = -\frac{i \hbar^2}{ma^2} \int_{\mathbb{R}^n} \text{Im} \, I^j \, dx' \\
& = \frac{\hbar^2}{ma^2} \int_{\mathbb{R}^n} \overline{\partial_j u} \, dx'.
\end{align*}
\]
We obtain the required equation by \( p^j u = a^{-2} (p^j u) = -i \hbar a^{-2} \partial_j u \).

(2) Put \( III^j := 2 \text{Re} (\partial_j \overline{u} \cdot i \hbar \partial_t u) \). By (1) and \( p^j = -i \hbar a^{-2} \partial_j \), we have

\[
\begin{align*}
m D_t^2 \langle x^j \rangle & = D_t \langle p^j \rangle \\
& = -\frac{D_t a^2}{a^2} \int_{\mathbb{R}^n} \overline{u} p^j \, dx' + \frac{1}{a^2} \int_{\mathbb{R}^n} III^j \, dx' - \frac{1}{a^2} \int_{\mathbb{R}^n} \partial_j (\overline{\partial_j u} \, dx') \\
& = -\frac{D_t a^2}{a^2} \langle p^j \rangle + \frac{1}{a^2} \int_{\mathbb{R}^n} III^j \, dx',
\end{align*}
\]
(2.8)
where we have used the definition of \( \langle p^j \rangle \) and the divergence theorem. Let us estimate the term \( III^j \). By the equation (2.2), we have

\[
\begin{align*}
III^j & = -\frac{\hbar^2}{ma^2} \text{Re} (\partial_j \overline{\partial_j u}) = 2U \text{Re} (u \partial_j \overline{\partial_j u}) =: III^j_1 + III^j_2.
\end{align*}
\]
By simple calculations, we have

\[
\begin{align*}
III^j_1 & = -\nabla \cdot \left( \frac{\hbar^2}{ma^2} \text{Re} (\partial_j \overline{\partial_j u}) \right) + \partial_j \left( \frac{\hbar^2}{2ma^2} |\overline{\partial_j u}|^2 \right), \\
III^j_2 & = \partial_j (|u|^2) - \partial_j U |u|^2,
\end{align*}
\]
which yield \( \int_{\mathbb{R}^n} III^j_1 \, dx' = 0 \) and \( \int_{\mathbb{R}^n} III^j_2 \, dx' = -\langle \partial_j U \rangle \) by the divergence theorem and the definition of the expectation value \( \langle \cdot \rangle \). So that, we obtain the required equation from (2.8). We also obtain (2.5) inserting the result (1) into (2).

Now, we consider the potential \( U \) defined by

\[
U(x') := \frac{m}{2} U_*(\langle x^j \rangle \langle x_j \rangle) x^k x_k
\]
for a function \( U_*(\cdot) \). Put

\[
X^j := \langle x^j \rangle, \quad X_j := \langle x_j \rangle = a^2 X^j, \quad \text{and} \quad r := (X^j X_j)^{1/2}.
\]
Put
\[
Y^j(t) := a(t) X^j(t) \quad \text{for} \quad t \geq 0,
\]
(2.9)
and

\[
A := -\frac{1}{4} \left( \frac{D_t a^2}{a^2} \right)^2 - \frac{1}{2} D_t \left( \frac{D_t a^2}{a^2} \right).
\] (2.10)

The equation (2.5) is rewritten as follows.

**Lemma 2.3.** The following results hold for \(1 \leq j \leq n\).

1. \(\langle \partial^j U \rangle = mU_*(r^2)X^j\).
2. The equation (2.5) is rewritten as

\[
D_t^2 X^j + \frac{D_t a^2}{a^2} D_t X^j + U_*(r^2)X^j = 0.
\] (2.11)

3. The equation (2.11) is rewritten as

\[
D_t^2 Y^j + AY^j + U_*|Y|^2 Y^j = 0.
\] (2.12)

**Proof.**
(1) By \(\partial_j (x^k x_k) = 2x_j\), we have

\[
\partial_j U(x') = \frac{m}{2} U_*(r^2) \partial_j (x^k x_k) = mU_*(r^2)x_j.
\]

Thus, we obtain

\[
\langle \partial^j U(x') \rangle = mU_*(r^2)\langle x^j \rangle = mU_*(r^2)X^j.
\]

(2) The equation (2.11) follows from (2.5) directly by \(X^j = \langle x^j \rangle\) and (1).

(3) We have \(X^j = a^{-1}Y^j\) and \(r^2 = X^j X_j = |Y|^2\) by (1.3). By

\[
D_t X^j = a^{-1} \left( D_t - \frac{D_t a^2}{2a^2} \right) Y^j,
\] (2.13)

we have

\[
D_t^2 X^j = a^{-1} \left[ D_t^2 Y^j - \frac{D_t a^2}{a^2} D_t Y^j + \frac{1}{4} \left\{ - \left( \frac{D_t a^2}{a^2} \right)^2 - 2D_t \left( \frac{D_t a^2}{a^2} \right) \right\} Y^j \right].
\] (2.14)

By \(\partial^j U(x') = mU_*(r^2)x_j\), we have \(\langle \partial^j U \rangle = mU_*(r^2)X^j\). We have the required equation (2.12) by (2.11), (2.13) and (2.14).

To rewrite the Cauchy problem (1.4) as the integral equation, we prepare some fundamental results for ordinary differential equations.

**Lemma 2.4.** For any fixed nonnegative function \(\tilde{a} \in C([0, T])\) for \(T > 0\), let \(\rho_0\) and \(\rho_1\) be the solutions of the Cauchy problem

\[
\begin{cases}
(D_t^2 + \tilde{a}(t)) \rho_j(t) = 0 & \text{for } t \in [0, T), \\
\rho_j(0) = \delta_{0j}, & D_t \rho_j(0) = \delta_{1j},
\end{cases}
\] (2.15)

where \(\delta_{ij} = 1\) for \(0 \leq i = j \leq 1\), \(\delta_{ij} = 0\) for \(0 \leq i \neq j \leq 1\). Put \(\bar{A} := \begin{pmatrix} 0 & 1 \\ -\tilde{a} & 0 \end{pmatrix}\),

\[
\Phi_m(t) := \begin{cases}
E & \text{if } m = 0, \\
\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} \bar{A}(t_1) \bar{A}(t_2) \cdots \bar{A}(t_m) dt_m \cdots dt_2 dt_1 & \text{if } m \geq 1,
\end{cases}
\]
Thus, the solution \( \rho \) of the equation

\[
(D_t^2 + \tilde{a}(t))\rho(t) = b(t)
\]  

(2.16)

for \( 0 \leq t < T \). Then the following results hold.

1. \( \Phi = \begin{pmatrix} \rho_0 & \rho_1 \\ D_t\rho_0 & D_t\rho_1 \end{pmatrix} \).
2. \( \det \Phi = 1 \).
3. The solution \( \rho \) is given by

\[
\begin{pmatrix} \rho(t) \\ D_t\rho(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} \rho(0) \\ D_t\rho(0) \end{pmatrix} + \int_0^t \Phi(t)\Phi(s)^{-1} \begin{pmatrix} 0 \\ b(s) \end{pmatrix} \, ds,
\]

which is rewritten as

\[
\rho(t) = \rho_0(t)\rho(0) + \rho_1(t)D_t\rho(0) + \int_0^t \rho_{12}(t,s)b(s)ds,
\]  

(2.17)

\[
D_t\rho(t) = D_t\rho_0(t)\rho(0) + D_t\rho_1(t)D_t\rho(0) + \int_0^t \rho_{22}(t,s)b(s)ds,
\]  

(2.18)

where \( \rho_{12} \) and \( \rho_{22} \) are defined by

\[
\rho_{12}(t,s) := -\rho_0(t)\rho_1(s) + \rho_1(t)\rho_0(s),
\]  

(2.19)

\[
\rho_{22}(t,s) := -D_t\rho_0(t)\rho_1(s) + D_t\rho_1(t)\rho_0(s).
\]  

(2.20)

4. If \( \tilde{a} \geq 0 \) and \( D_t\tilde{a} \leq 0 \) on \([0,T)\), then

\[
|\rho_0(t)| \leq \sqrt{\tilde{a}(t)} \quad |D_t\rho_0(t)| \leq \sqrt{\tilde{a}(0)} \quad |\rho_1(t)| \leq \sqrt{\frac{1}{\tilde{a}(t)}} \quad |D_t\rho_1(t)| \leq 1.
\]

5. If \( \tilde{a} \geq 0 \) and \( D_t\tilde{a} \geq 0 \) on \([0,T)\), then

\[
|\rho_0(t)| \leq 1 \quad |D_t\rho_0(t)| \leq \sqrt{\tilde{a}(t)} \quad |\rho_1(t)| \leq \sqrt{\frac{1}{\tilde{a}(0)}} \quad |D_t\rho_1(t)| \leq \sqrt{\frac{\tilde{a}(t)}{\tilde{a}(0)}}.
\]

6. \( \rho \in C([0,T)) \). Moreover, if \( b \in C([0,T)) \), then \( \rho \in C^1([0,T)) \).

Proof. (1) We note that the solution \( \rho \) in (2.16) with \( b = 0 \) satisfies

\[
D_t \left( \begin{pmatrix} \rho(t) \\ D_t\rho(t) \end{pmatrix} \right) = \tilde{A}(t) \left( \begin{pmatrix} \rho(t) \\ D_t\rho(t) \end{pmatrix} \right).
\]

We have \( D_t\Phi = \sum_{m=0}^{\infty} \tilde{A}\Phi_{m-1} = \tilde{A}\Phi \) by \( D_t\Phi_0 = 0 \) and \( D_t\Phi_m = \tilde{A}\Phi_{m-1} \) for \( m \geq 1 \). Thus, the solution \( \rho \) in (2.16) with \( b = 0 \) satisfies

\[
\left( \begin{pmatrix} \rho(t) \\ D_t\rho(t) \end{pmatrix} \right) = \Phi(t) \left( \begin{pmatrix} \rho(0) \\ D_t\rho(0) \end{pmatrix} \right)
\]
since it satisfies
\[
D_t \left( \frac{\rho(t)}{D_t \rho(t)} \right) = D_t \Phi(t) \left( \frac{\rho(0)}{D_t \rho(0)} \right) = \bar{A}(t) \Phi(t) \left( \frac{\rho(0)}{D_t \rho(0)} \right) = \bar{A}(t) \left( \frac{\rho(t)}{D_t \rho(t)} \right) .
\]

So that, the solutions \( \rho_0 \) and \( \rho_1 \) satisfy \( \begin{pmatrix} \rho_0 \\ D_t \rho_0 \end{pmatrix} = \Phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} \rho_1 \\ D_t \rho_1 \end{pmatrix} = \Phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), which yields \( \begin{pmatrix} \rho_0 \\ \rho_1 \end{pmatrix} = \Phi E = \Phi \), namely, the required result.

(2) Since \( \det \Phi = \rho_0 D_t \rho_1 - D_t \rho_0 \rho_1 \) by (1), we have
\[
D_t \det \Phi = D_t \rho_0 D_t \rho_1 + \rho_0 D_t^2 \rho_1 - D_t^2 \rho_0 \rho_1 - D_t \rho_0 D_t \rho_1 = 0
\]
by \( D_t^2 \rho_0 = -\bar{a} \rho_0 \) and \( D_t^2 \rho_1 = -\bar{a} \rho_1 \). So that, we have \( \det \Phi(t) = \det \Phi(0) = 1 \) by \( \Phi(0) = E \).

(3) Since the solution \( \rho \) in (2.16) satisfies
\[
D_t \left( \frac{\rho(t)}{D_t \rho(t)} \right) = \bar{A}(t) \left( \frac{\rho(0)}{D_t \rho(0)} \right) + \begin{pmatrix} 0 \\ b(t) \end{pmatrix} ,
\]
we have
\[
\begin{pmatrix} \rho(t) \\ D_t \rho(t) \end{pmatrix} = \Phi(t) \left( \begin{pmatrix} \rho(0) \\ D_t \rho(0) \end{pmatrix} \right) + \Phi(t) \int_0^t \Phi(s)^{-1} \begin{pmatrix} 0 \\ b(s) \end{pmatrix} ds \quad (2.22)
\]
since it satisfies (2.21). Since we have \( \Phi^{-1} := \begin{pmatrix} D_t \rho_1 & -\rho_1 \\ -D_t \rho_0 & \rho_0 \end{pmatrix} \) by (2), we obtain
\[
\Phi(t) \Phi(s)^{-1} = \begin{pmatrix} \rho_0(t) D_t \rho_1(s) - \rho_1(t) D_t \rho_0(s) & -\rho_0(t) \rho_1(s) + \rho_1(t) \rho_0(s) \\ D_t \rho_0(t) D_t \rho_1(s) - D_t \rho_1(t) D_t \rho_0(s) & -D_t \rho_0(t) \rho_1(s) + D_t \rho_1(t) \rho_0(s) \end{pmatrix} \quad (2.23)
\]
where we have defined \( \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22} \) by the right hand side. We obtain (2.17) and (2.18) by (1), (2.22) and (2.23).

(4) Since \( \rho_0 \) and \( \rho_1 \) are the solutions of (2.15), we have
\[
D_t \rho_j (D_t^2 + \bar{a}) \rho_j = 0
\]
for \( j = 0, 1 \). By \( D_t \rho_j D_t^2 \rho_j = D_t (D_t \rho_j)^2 / 2 \) and \( D_t \rho_j \rho_j = D_t (\rho_j^2) / 2 \), we have
\[
D_t (D_t \rho_j)^2 + \bar{a} D_t \rho_j^2 = 0, \quad (2.24)
\]
which is rewritten as
\[
D_t \left( (D_t \rho_j)^2 + \bar{a} \rho_j^2 \right) = D_t \bar{a} \cdot \rho_j^2
\]
by \( \bar{a} D_t \rho_j^2 = D_t (\bar{a} \rho_j^2) = D_t \bar{a} \cdot \rho_j^2 \). Under the condition \( D_t \bar{a} \leq 0 \), we have
\[
(D_t \rho_j)^2(t) + \bar{a}(t) \rho_j^2(t) \leq (D_t \rho_j)^2(0) + \bar{a}(0) \rho_j^2(0),
\]

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namely,
\[(D_t \rho_0)^2(t) + \tilde{a}(t)\rho_0^2(t) \leq \tilde{a}(0), \quad (D_t \rho_1)^2(t) + \tilde{a}(t)\rho_1^2(t) \leq 1,\]
which yield the required inequalities.

(5) We rewrite (2.21) as
\[\tilde{a}^{-1}D_t(D_t \rho_j)^2 + D_t\rho_j^2 = 0.\]
By \(\tilde{a}^{-1}D_t(D_t \rho_j)^2 = D_t(\tilde{a}^{-1}(D_t \rho_j)^2) + \tilde{a}^{-2}D_t\tilde{a}(D_t \rho_j)^2\), we have
\[D_t\{\tilde{a}^{-1}(D_t \rho_j)^2 + \rho_j^2\} = -\tilde{a}^{-2}D_t\tilde{a} \cdot (D_t \rho_j)^2.\]
Under the conditions \(\tilde{a} \geq 0\) and \(D_t\tilde{a} \geq 0\), we have
\[\rho_j^2(t) + \tilde{a}^{-1}(t)(D_t \rho_j)^2(t) \leq \rho_j^2(0) + \left(\tilde{a}^{-1}(D_t \rho_j)^2\right)(0),\]
namely,
\[\rho_0^2(t) + \tilde{a}^{-1}(t)(D_t \rho_0)^2(t) \leq 1, \quad \rho_1^2(t) + \tilde{a}^{-1}(t)(D_t \rho_1)^2(t) \leq \tilde{a}^{-1}(0),\]
which yield the required inequalities.

(6) We note that \(\rho\) is given by (2.17). Since we have \(\rho_0, \rho_1 \in C^2([0, T])\) by \(\tilde{a} \in C([0, T])\) and \(D_t^2 \rho_j = -\tilde{a}\rho_j \in C([0, T])\), we have \(\rho_0(\cdot)\rho(0) + \rho_1(\cdot)D_t\rho(0) \in C^2([0, T])\) and \(\rho_{12} \in C^2([0, T] \times [0, T])\). Put \(\phi(t) := \int_0^t \rho_{12}(t, s) b(s) ds\). For any \(t \in [0, T]\) and \(\epsilon \in \mathbb{R}\) with \(|\epsilon|\) sufficiently small, we have
\[\phi(t + \epsilon) - \phi(t) = \int_0^{t+\epsilon} \rho_{12}(t + \epsilon, s) b(s) ds - \int_0^t \rho_{12}(t, s) b(s) ds = \int_t^{t+\epsilon} \rho_{12}(t + \epsilon, s) b(s) ds + \int_0^t (\rho_{12}(t + \epsilon, s) - \rho_{12}(t, s)) b(s) ds =: I + II,\]
where we take \(\epsilon > 0\) when \(t = 0\). Let \(T_*\) satisfy \(t + |\epsilon| < T_* < T\). We have
\[|I| \leq \|\rho\|_{L^\infty([0, T_*])} \cdot \left|\int_t^{t+\epsilon} b(s) ds\right| \rightarrow 0\]
as \(\epsilon \rightarrow 0\) by \(b \in L^1((0, T))\). We also have
\[|II| \leq \int_0^t |\rho_{12}(t + \epsilon, s) - \rho_{12}(t, s)| \cdot |b(s)| ds,\]
and \(|II| \leq 2\|\rho_{12}\|_{L^\infty([0, T_*])} \cdot \|b\|_{L^1([0, T_*])},\) by which we have \(|II| \rightarrow 0\) as \(\epsilon \rightarrow 0\) by the Lebesgue convergence theorem. Thus, we have \(\phi \in C([0, T])\). So that, we obtain
\(\rho \in C([0, T])\) by (2.17).

Let us consider the case \(b \in C([0, T])\). We have
\[\frac{1}{\epsilon} (\phi(t + \epsilon) - \phi(t)) = III + IV,\]
where we have put

\[ III := \frac{1}{\varepsilon} \int_0^t \rho_{12}(t + \varepsilon, s) b(s) ds, \quad IV := \int_0^t \rho_{12}(t + \varepsilon, s) - \rho_{12}(t, s) \cdot b(s) ds. \]

We have \( III \to \rho_{12}(t, t)b(t) = 0 \) by \( \rho_{12}(t, t) = 0 \) and \( b \in C([0, T]) \). We also have \( IV \to \int_0^t D_t \rho_{12}(t, s) b(s) ds \) by \( \rho_{12} \in C^1([0, T] \times [0, T]) \). Thus, we have

\[ D_t \phi(t) = \int_0^t D_t \rho_{12}(t, s) b(s) ds. \]

We have \( D_t \phi \in C([0, T]) \) by \( D_t \rho_{12} \in C([0, T] \times [0, T]) \) and \( b \in L^1([0, T]) \). Thus, we have \( \phi \in C^1([0, T]) \). So that, we obtain \( \rho \in C^1([0, T]) \) as required if \( b \in C([0, T]) \). \( \square \)

**Lemma 2.5.** The scale-function \( a(\cdot) \) and \( q_0(\cdot) \) defined by (1.2) and (1.3) satisfy the followings.

1. \( q_0 = \frac{2a_1}{a_0} \left( 1 + \frac{n(1 + \sigma)a_1 t}{2a_0} \right)^{-1} \)
2. \( D_t q_0 = -\frac{n(1 + \sigma)}{4} q_0^2 \)

**Proof.** (1) Since we have

\[ D_t a(t) = a(t) \frac{a_1}{a_0} \left( 1 + \frac{n(1 + \sigma)a_1 t}{2a_0} \right)^{-1} \]

and \( q_0 = 2D_t a/a \), we obtain the required result.

(2) By (1), we have

\[ D_t q_0 = -\frac{n(1 + \sigma)}{a_0} a_1^2 \left( 1 + \frac{n(1 + \sigma)a_1 t}{2a_0} \right)^{-2} = -\frac{n(1 + \sigma)}{4} q_0^2 \]

as required. \( \square \)

Let us classify the cases \( A > 0, D_t A > 0, D_t A < 0 \) as follows which are needed when we consider the energy estimates for the Cauchy problem (1.4).

**Lemma 2.6.** For the function \( A \) defined by (1.3), the following results hold.

1. \( A = \frac{n^2}{8} \left( \sigma + 1 - \frac{2}{n} \right) q_0^2 \)
2. \( A = 0 \) holds if and only if \( \sigma = -1 + 2/n \) or \( a_1 = 0 \). \( A > 0 \) holds if and only if \( \sigma > -1 + 2/n \) and \( a_1 \neq 0 \).
3. \( D_t A = -\frac{n^2}{16} \left( \sigma + 1 - \frac{2}{n} \right) (\sigma + 1) q_0^3 \)
4. \( D_t A = 0 \) holds if and only if \( \sigma = -1 + 2/n \) or \( \sigma = -1 \) or \( a_1 = 0 \). \( D_t A > 0 \) holds if and only if \( (\sigma + 1 - 2/n)(\sigma + 1)a_1 < 0 \). \( D_t A < 0 \) holds if and only if \( (\sigma + 1 - 2/n)(\sigma + 1)a_1 > 0 \).
5. If \( a_1 = 0 \) or \( \sigma = -1 + 2/n \), then \( A = D_t A = 0 \).
6. If \( a_1 > 0 \) and \( \sigma > -1 + 2/n \), then \( A > 0 \) and \( D_t A < 0 \).
7. If \( a_1 < 0 \) and \( \sigma > -1 + 2/n \), then \( A > 0 \) and \( D_t A > 0 \).
Proof. The result (1) follows from the definition of \( q_0 \) and (2) in Lemma 2.5 as
\[
A = -\frac{1}{4}q_0^2 - \frac{Dvq_0}{2} = \frac{-q_0^2}{4} + \frac{n(1 + \sigma)q_0^2}{8} = \frac{nq_0^2}{8} \left( \sigma + 1 - \frac{2}{n} \right).
\]
The result (2) follows from (1) directly, where \( q_0 = 0 \) is equivalent to \( a_1 = 0 \) by (1) in Lemma 2.5. The result (3) follows from \( D_tA = n(\sigma + 1 - 2/n)q_0Dvq_0/4 \) and (2) in Lemma 2.5. The result (4) follows from (3), and (1) in Lemma 2.5. The results (5), (6) and (7) follow from (2) and (4).

**Lemma 2.7** (Energy estimates when \( D_tA \leq 0 \)). Let \( 0 < T \leq T_1 \). For any function \( h = (h^1, \cdots, h^n) \), let \( Y = (Y^1, \cdots, Y^n) \) be the solution of the equation
\[
D_t^2Y^j(t) + A(t)Y^j(t) + h^j(t) = 0 \tag{2.25}
\]
for \( 1 \leq j \leq n \) and \( 0 \leq t < T \). Let \( A \geq 0 \) and \( D_tA \leq 0 \) on \([0, T)\).

1. The following estimate holds;
\[
\|D_tY\|_{L^\infty ([0, T])} + \|\sqrt{A}Y\|_{L^\infty ([0, T])} + \|\sqrt{-D_tA}Y\|_{L^2 ([0, T])} \lesssim |D_tY(0)| + \sqrt{A(0)}|Y(0)| + \|h\|_{L^1 ([0, T])}.
\]

2. Let \( h = \lambda|Y|^{p-1}Y \) for \( \lambda \in \mathbb{C} \) and \( 1 < p < \infty \). Put
\[
e^0 := \frac{1}{2}|D_tY|^2 + \frac{1}{2}A|Y|^2 + \frac{\lambda}{p+1}|Y|^{p+1}, \quad e^1 := -\frac{1}{2}D_tA|Y|^2.
\]
Then the following estimate holds;
\[
e^0(t) + \int_0^t e^1(s)ds = e^0(0).
\]

Proof. Multiplying \( D_tY^j \) to the both sides in the equation (2.25), we have
\[
D_t \left\{ \frac{1}{2}|D_tY|^2 + \frac{1}{2}A|Y|^2 \right\} - \frac{1}{2}D_tA|Y|^2 + \sum_{j=1}^n h^jD_tY^j = 0. \tag{2.26}
\]
Integrating the both sides in this equation on the interval \([0, t]\), we have
\[
\frac{1}{2}|D_tY(t)|^2 + \frac{1}{2}A(t)|Y(t)|^2 + \frac{1}{2}\left\|\sqrt{-D_tA}Y\right\|_{L^2 ([0, t])}^2 + \sum_{j=1}^n \int_0^t h^j(s)D_tY^j(s)ds = \frac{1}{2}|D_tY(0)|^2 + \frac{1}{2}A(0)|Y(0)|^2.
\]
So that, we have
\[
\|D_tY\|_{L^\infty ([0, T])} + \|\sqrt{A}Y\|_{L^\infty ([0, T])} + \left\|\sqrt{-D_tA}Y\right\|_{L^2 ([0, T])}^2 \lesssim |D_tY(0)|^2 + A(0)|Y(0)|^2 + \sum_{j=1}^n \int_0^T |h^j(s)D_tY^j(s)| ds.
\]

Since we have
\[
\sum_{j=1}^{n} \int_{0}^{T} |h^j(s)D_t Y^j(s)|ds \leq \varepsilon \sum_{j=1}^{n} \|D_t Y^j\|_{L^\infty((0,T))}^2 + \frac{1}{4\varepsilon} \sum_{j=1}^{n} \left( \int_{0}^{T} |h^j(s)|ds \right)^2
\]
for any \( \varepsilon > 0 \), we obtain the required result taking \( \varepsilon \) sufficiently small.

(2) Since we have
\[
\sum_{j=1}^{n} h^j D_t Y^j = \lambda |Y|^{p-1} \sum_{j=1}^{n} Y^j D_t Y^j = \frac{\lambda}{p+1} D_t |Y|^{p+1},
\]
we obtain the required result by (2.26). \( \Box \)

**Lemma 2.8** (Energy estimates when \( D_t A \geq 0 \)). For any function \( h = (h^1, \ldots, h^n) \), let \( Y = (Y^1, \ldots, Y^n) \) be the solution of the equation
\[
D_t^2 Y^j(t) + A(t) Y^j(t) + h^j(t) = 0 \tag{2.27}
\]
for \( 1 \leq j \leq n \) and \( 0 \leq t < T \). Let \( A > 0 \) and \( D_t A \geq 0 \) on \([0,T)\).

1. The following estimate holds;
\[
\|A^{-1/2} D_t Y\|_{L^\infty((0,T))} + \|Y\|_{L^\infty((0,T))} + \|A^{-1/2} \sqrt{D_t A D_t Y}\|_{L^2((0,T))} \leq |A(0)^{-1/2} D_t Y(0)| + |Y(0)| + \|A^{-1/2} h\|_{L^1((0,T))}.
\]

2. Let \( h := \lambda |Y|^{p-1} Y \) for \( \lambda \in \mathbb{C} \) and \( 1 < p < \infty \). Put
\[
e^0 := \frac{1}{2} |A^{-1/2} D_t Y|^2 + \frac{1}{2} |Y|^2 + \frac{\lambda}{p+1} A^{-1} |Y|^{p+1},
\]
\[
e^1 := \frac{1}{2} A^{-2} D_t A |D_t Y|^2 + \frac{\lambda}{p+1} A^{-2} D_t A |Y|^{p+1}.
\]
Then the following estimate holds;
\[
e^0(t) + \int_{0}^{t} e^1(s)ds = e^0(0).
\]

**Proof.** (1) Multiplying \( A^{-1} D_t Y^j \) to the both sides in the equation (2.27), we have
\[
D_t \left\{ \frac{1}{2} A^{-1} |D_t Y|^2 + \frac{1}{2} |Y|^2 \right\} + \frac{1}{2} A^{-2} D_t A |D_t Y|^2 + A^{-1} \sum_{j=1}^{n} h^j D_t Y^j = 0. \tag{2.28}
\]
Integrating the both sides in this equation on the interval \([0,t]\), we have
\[
\frac{1}{2} A^{-1}(t) |D_t Y(t)|^2 + \frac{1}{2} |Y(t)|^2 + \frac{1}{2} \left\| A^{-1} \sqrt{D_t A D_t Y} \right\|_{L^2((0,t))}^2 + \sum_{j=1}^{n} \int_{0}^{t} A^{-1}(s) h^j(s) D_t Y^j(s)ds = \frac{1}{2} A^{-1}(0) |D_t Y(0)|^2 + \frac{1}{2} |Y(0)|^2.
\]
Thus, we have
\[ \|A^{-1/2}D_t Y\|_{L^\infty([0,T])}^2 + \|Y\|_{L^\infty([0,T])}^2 + \left\|A^{-1}\sqrt{D_tA} D_t Y\right\|_{L^2([0,T])}^2 \]
\[ \lesssim A^{-1}(0) |D_t Y(0)|^2 + |Y(0)|^2 + \sum_{j=1}^n \int_0^T \left| A^{-1}(s) h^j(s) D_t Y^j(s) \right| ds. \]
Since we have
\[ \|Y\|_{L^p([0,T])} = \left\|\left(\sum_{j=1}^n h^j(s) D_t Y^j(s)\right)\right\|_{L^p([0,T])} \]
\[ \lesssim \|A^{-1/2}D_t Y\|_{L^\infty([0,T])}^2 \left\|A^{-1/2}h\right\|_{L^1([0,T])} \]
for any \( \varepsilon > 0 \), we obtain the required result taking \( \varepsilon \) sufficiently small.

(2) Since we have
\[ A^{-1} \sum_{j=1}^n h^j D_t Y^j = \lambda A^{-1} |Y|^{p-1} \sum_{j=1}^n Y^j D_t Y^j \]
\[ = \frac{\lambda}{p+1} A^{-1} |Y|^{p+1} \]
\[ = D_t \left( \frac{\lambda}{p+1} A^{-1} |Y|^{p+1} \right) + \frac{\lambda}{p+1} A^{-2} D_t A |Y|^{p+1}, \]
we obtain the required result by (2.28). \( \square \)

We prepare some estimates for the semilinear term in (1.6) as follows.

**Lemma 2.9.** For any \( Y, Z \in \mathbb{R}^n \) and \( 1 < p < \infty \), the following inequalities hold.

1. \( \|Y|^{p-1} Y - |Z|^{p-1} Z\| \leq p \left( \|Y|^{p-1} + |Z|^{p-1} \right) |Y - Z| \)
2. \( \|Y|^{p} - |Z|^{p} \| \leq p \left( \max\{ |Y|, |Z| \} \right) |Y - Z| \)

**Proof.** When \( Y = 0 \) or \( Z = 0 \) or \( Y = Z \), the results are trivial. We assume \( Y \neq 0 \), \( Z \neq 0 \) and \( Y \neq Z \).

1. (1) If there is not \( \theta \) such that \( 0 < \theta < 1 \) and \( Z + \theta (Y - Z) = 0 \), then we have
\[ |Y|^{p-1} Y - |Z|^{p-1} Z \]
\[ = \int_0^1 \frac{d}{d\theta} \left( |Z + \theta (Y - Z)|^{p-1} \cdot (Z + \theta (Y - Z)) \right) d\theta \]
\[ = (p-1) \int_0^1 |Z + \theta (Y - Z)|^{p-3} (Z + \theta (Y - Z)) \cdot (Y - Z) (Z + \theta (Y - Z)) d\theta \]
\[ + \int_0^1 |Z + \theta (Y - Z)|^{p-1} (Y - Z) d\theta. \]
Thus, we have
\[ \|Y|^{p-1} Y - |Z|^{p-1} Z\| \leq p \int_0^1 |Z + \theta (Y - Z)|^{p-1} |Y - Z| d\theta \]
\[ \leq p \max\{ |Y|^{p-1}, |Z|^{p-1} \} |Y - Z|. \]

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Thus, we have $\theta > (1 - \theta) > 0$. Since we have $Z = \alpha Y$, we have $\theta = \alpha Y$. We prove Theorem 1.1 only for the case $f \alpha$. Since we have $1 + \alpha \leq (1 + \alpha^p)(1 + \alpha)$, we have $||Y||^{1-p} - |Z|^{1-p}Z \leq |Y||^{p-1}(1 + \alpha^p)\max\{\max\{|Y|, |Z|\}\}^{p-1}|Y - Z|$. So that, we obtain the required result.

(2) If there is not $\theta$ such that $0 < \theta < 1$ and $Z = \theta(Y - Z) = 0$, then we have $|Y|^{p-1}Y - |Z|^{p-1}Z = |Y|^{p-1}Y(1 + \alpha^p) = |Y|^{p-1}\frac{1 + \alpha^p}{1 + \alpha}(Y - Z)$. Thus, we have $||Y||^{p-1}Y - |Z|^{p-1}Z \leq |Y|^{p-1}(1 + \alpha^p)|Y - Z| = (|Y| + |Z|)^{p-1}|Y - Z|$. Let us consider the case that there is $\theta$ such that $0 < \theta < 1$ and $Z = \theta(Y - Z) = 0$. For this $\theta$, we put $\alpha := \theta/(1 - \theta) > 0$. We have $Z = \alpha Y$ and $|Y|^{p-1}Y - |Z|^{p-1}Z = |Y|^{p-1}(1 - \alpha^p)$ and $1 - \alpha^p = \int_0^1 d\tau(\alpha + \tau(1 - \alpha))^p d\tau = \int_0^1 (\alpha + \tau(1 - \alpha))^p d\tau \cdot (1 - \alpha)$. Thus, we have $||Y||^{p-1}Y - |Z|^{p-1}Z \leq \int_0^1 p(\max\{|Y|, |Z|\})^{p-1}|Y - Z|$. Since we have $|Y|\max\{\alpha, 0\} = \max\{|Y|, |Z|\}$ and $|Y| \cdot |1 - \alpha| = ||Y|| - |Z| \leq |Y - Z|$, we obtain the required result.

\section{Proof of Theorem 1.1}

We have $A > 0$, $D^2_1 A < 0$ and $T_0 < 0$ by the assumption $a_1 > 0$ and $\sigma > -1 + 2/n$. We prove Theorem 1.1 only for the case $f(Y) := \lambda|Y|^{p-1}Y$ since the case $f(Y) := \lambda|Y|^p$ is proved analogously by the use of (2) instead of (1) in Lemma 2.9. Put $U_*|Y|^p := \lambda|Y|^{p-1}$. We regard the solution of the Cauchy problem (1.4) as the fixed point of the operator $\Psi$ defined by $\Psi(Y)_j := \rho_0(t)Y_0^j + \rho_1(t)Y_1^j - \int_0^t \rho_{12}(t, s)f(Y)_j^*(s)ds$ (3.2)
for \(1 \leq j \leq n\), where \(\rho_0, \rho_1\) and \(\rho_{12}\) are the functions in Lemma 2.4 with \(\tilde{a}(t) := A(t)\).

(1) Put \(A_q := A^{1/2-1/q}(-D_t A)^{1/q}\) for \(2 \leq q \leq \infty\). By the H"older inequality, we have
\[
\|A_q Y\|_{L^q((0,T))} \leq \|\sqrt{A} Y\|_{L^\infty((0,T))}^{1-2/q} \|\sqrt{-D_t A Y}\|_{L^2((0,T))}^{2/q}.
\] (3.3)
Since we have \(A = n(\sigma + 1 - 2/n)q_0^2/8\) and \(-D_t A = n^2(\sigma + 1 - 2/n)(\sigma + 1)q_0^3/16\) by (1) and (3) in Lemma 2.6 we have
\[
A_q = C q_0^{1+1/q}
\] (3.4)
for some constant \(C > 0\), where \(q_0 > 0\) by (1) in Lemma 2.5. Let \(q_*\) be a real number which satisfies
\[
\max\left\{0, 1 - \frac{p}{2}\right\} \leq \frac{1}{q_*} \leq 1.
\] (3.5)
Then there exist \(q_1\) and \(q_2\) with \(2 \leq q_1, q_2 \leq \infty\) and
\[
\frac{1}{q_*} = 1 - \frac{p - 1}{q_1} - \frac{1}{q_2}.
\] (3.6)
Especially, (3.5) is satisfied for \(q_* = \infty\) and \(p \geq 2\). By \(f(Y) = \lambda |Y|^{p-1} Y\), we have
\[
\|f(Y)\|_{L^1((0,T))} \leq |\lambda| \|A_{q_1} Y\|_{L^1((0,T))}^{p-1} \|A_{q_2} Y\|_{L^{q_2}((0,T))} \leq |\lambda| \|Y\|_X^p
\] (3.7)
if \(q_*\) satisfies \(1 \leq q_* \leq \infty\) and (3.6), where we have put
\[
I := \|A_{q_1}^{-p+1} A_{q_2}^{-1}\|_{L^{q_*}((0,T))}.
\] (3.8)
By (3.4), we have
\[
I \lesssim \|q_0^{-p+1/q_*}\|_{L^{q_*}((0,T))} =: I II.
\] (3.9)
By Lemma 2.7 we have
\[
\|\Psi(Y)\|_X \lesssim |D_t Y(0)| + \sqrt{A(0)}|Y(0)| + \|f(Y)\|_{L^1((0,T))},
\]
which yields
\[
\|\Psi(Y)\|_X \leq C_0 D + C II R^p
\]
for any \(Y \in X(T, R)\) by (3.7) and (3.9), where \(C_0 > 0\) and \(C > 0\) are constants independent of \(Y\). Thus, the operator \(\Psi\) maps \(X(T, R)\) into itself if \(R > 0\) satisfies
\[
R \geq 2C_0 D, \quad 2C II R^{p-1} \leq 1.
\] (3.10)
Put \(C' := (2a_1/a_0)^{-p+1/q_*} \{1/(p+1)q_*\}^{1/q_*}\). We have \(q_0 = 2a_1/a_0 (1 - t/T_0)\) by (1) in Lemma 2.5. Since we have
\[
II = C' \cdot \left[-T_0 \left\{ \left(1 - \frac{T}{T_0}\right)^{(p+1)q_*} - 1 \right\} \right]^{1/q_*}
\]
when \( q_* \neq \infty \), the condition \( 2CIR R^{p-1} \leq 1 \) is rewritten as
\[
T \leq -T_0 \left\{ 1 - \frac{1}{T_0(2CC'R^{p-1})^{q_*}} \right\}^{1/(p+1)q_*} - 1 \].
(3.11)
Since we have \( II = C'(1 - T/T_0)^{p+1} \) when \( q_* = \infty \), the condition \( 2CIR R^{p-1} \leq 1 \) is rewritten as
\[
T \leq -T_0 \left\{ (2CC'R^{p-1})^{-1/(p+1)} - 1 \right\} \text{ and } R \leq \left( \frac{1}{2CC'} \right)^{1/(p-1)}
(3.12)
when \( q_* = \infty \).

Since we have \( f(Y) - f(Z) = \lambda(|Y|^{p-1}Y - |Z|^{p-1}Z) \), we have
\[
|f(Y) - f(Z)| \leq \lambda|p|(|Y|^{p-1} + |Z|^{p-1})|Y - Z|
by Lemma 2.9. We have
\[
\|f(Y) - f(Z)\|_{L^1((0,T))} \lesssim |\lambda|I \left( \|A_{q_1}Y\|_{L^{q_1}((0,T))} + \|A_{q_2}Z\|_{L^{q_2}((0,T))} \right)^{p-1} \|A_{q_2}(Y - Z)\|_{L^{q_2}((0,T))}
\lesssim |\lambda|I \max\{|Y|,|Z|\}^{p-1} \|Y - Z\|_X
(3.13)
by the similar argument to derive (3.7). Since we have
\[
\|\Psi(Y) - \Psi(Z)\|_X \lesssim \|f(Y) - f(Z)\|_{L^1((0,T))}
by Lemma 2.7, there exist a constant \( C > 0 \) such that
\[
\|\Psi(Y) - \Psi(Z)\|_X \leq CIR^{p-1} \|Y - Z\|_X
(3.14)
holds for any \( Y, Z \in X(T,R) \). Thus, \( \Psi \) is a contraction mapping on \( X(T,R) \) under the conditions (3.10). By the Banach fixed point theorem, \( \Psi \) has a unique fixed point \( Y \in X(T,R) \).

(2) Let \( Y \) be the fixed point of \( \Psi \) obtained in (1). Since \( A \in C([0,T)) \), and \( f(Y) \in L^1((0,T)) \) by (6.7) and \( Y \in X(T) \), we have \( \Psi(Y) \in C([0,T)) \) by (6) in Lemma 2.4. Thus, we have \( f(Y) = \Psi(Y) \) which follows. Assume \( T_* < T \). Let \( \varepsilon > 0 \) be a small number such that \( T_* + \varepsilon < T \). Put the interval \( J := (T_*, T_* + \varepsilon) \). Let \( \|Y\|_{X(J)} \) be the norm defined by (1.9) with the interval \((0,T)\) replaced by \( J \). By the analogous argument on (3.13) and (3.14), we have
\[
\|Y - Z\|_{X(J)} \leq C_0 \left\{ |D_t(Y - Z)(T_*)| + \sqrt{A(T_*)}|(Y - Z)(T_*)| \right\}
+ C|\lambda|I(J) \max\{|Y|_{X(J)},|Z|_{X(J)}\}^{p-1} \|Y - Z\|_{X(J)},
\]
where $I(J)$ is defined by \(3.8\) with the interval \((0, T)\) replaced by $J$. Since $|D_t(Y - Z)(T_*)| = |(Y - Z)(T_*)|$ = 0 and $C|\lambda|I(J) \max\{\|Y\|_{X(J)}, \|Z\|_{X(J)}\}^{p-1} < 1$ for sufficiently small $\varepsilon > 0$ by the continuity of $Y, Z \in C^1([0, T]),$ \(3.10\), $\|Y\|_{X(J)} \leq R$ and $\|Z\|_{X(J)} \to \|Y\|_{X(J)}$ as $\varepsilon \searrow 0$, we obtain $\|Y - Z\|_{X(J)} = 0$ for sufficiently small $\varepsilon > 0$. Thus, we have $Y = Z$ on $[T_*, T_* + \varepsilon)$, which contradicts to the definition of $T_*$. So that, we obtain $T_* = T$ as required.

4 Proof of Theorem 1.2

We note $A(\cdot) = 0$ holds when $a_1 = 0$ and $\sigma \in \mathbb{R}$, or $a_1 \in \mathbb{R}$ and $\sigma = -1 + 2/n$ by Lemma 2.6.

(1) Let us consider the operator $\Psi$ defined by \(\mathbb{3.2}\). By the elementary equation $\Psi(Y)(t) = Y_0 + \int_0^t D_t \Psi(Y)(\tau) d\tau$, we have

$$\|\Psi(Y)\|_{L^\infty([0, T]))} \leq \|Y_0\| + T \|D_t \Psi(Y)\|_{L^\infty([0, T]))} \leq \|Y_0\| + TR_1$$

(4.1)

for $Y \in X(T, R_0, R_1)$. We have

$$\|f(Y)\|_{L^1([0, T]))} \leq T \|f(Y)\|_{L^\infty([0, T]))} \leq |\lambda|T\|Y\|_{L^\infty([0, T]))}^{p} \leq |\lambda|TR_0^p$$

(4.2)

and

$$\|f(Y) - f(Z)\|_{L^1([0, T]))} \leq T \|f(Y) - f(Z)\|_{L^\infty([0, T]))} \leq |\lambda|pT \left(\|Y\|_{L^\infty([0, T]))}^{p-1} + \|Z\|_{L^\infty([0, T]))}^{p-1}\right) \|Y - Z\|_{L^\infty([0, T]))} \leq |\lambda|pTR_0^{p-1}\|Y - Z\|_X$$

(4.3)

by Lemma 2.9. We have

$$\|D_t \Psi(Y)\|_{L^\infty([0, T]))} \lesssim \|Y_1\| + \|f(Y)\|_{L^1([0, T]))} \lesssim \|Y_1\| + |\lambda|TR_0^p$$

(4.4)

by (1) in Lemma 2.7 $A(\cdot) = 0$ and \(4.2\). Since $\Psi(Y)$ and $\Psi(Z)$ satisfy

$$D_t^2 (\Psi(Y) - \Psi(Z)) + f(Y) - f(Z) = 0,$$
we have
\[ \|D_t(\Psi(Y) - \Psi(Z))\|_{L^\infty(0,T)} \lesssim \|f(Y) - f(Z)\|_{L^1((0,T))} \leq |\lambda|TR_0^{p-1}\|Y - Z\|_X \quad (4.5) \]
similarly to (4.4) by (4.3). Moreover, by (Ψ(Y) - Ψ(Z))(t) = \int_0^t D_t(\Psi(Y) - \Psi(Z))(\tau)d\tau, we have
\[ \|\Psi(Y) - \Psi(Z)\|_{L^\infty((0,T))} \leq T\|D_t(\Psi(Y) - \Psi(Z))\|_{L^\infty((0,T))} \lesssim |\lambda|T^2R_0^{p-1}\|Y - Z\|_X. \quad (4.6) \]
By (4.1), (4.4), (4.5) and (4.6), we have
\[ \|\Psi(Y)\|_{L^\infty((0,T))} \leq |Y_0| + TR_1 \leq R_0, \]
\[ \|D_t\Psi(Y)\|_{L^\infty((0,T))} \leq C_0|Y_1| + CTR_0^p \leq R_1, \]
\[ \|D_t(\Psi(Y) - \Psi(Z))\|_{L^\infty((0,T))} \leq CTR_0^{p-1}\|Y - Z\|_X \leq \frac{1}{2}\|Y - Z\|_X, \]
and
\[ \|\Psi(Y) - \Psi(Z)\|_{L^\infty((0,T))} \leq C^2T^2R_0^{p-1}\|Y - Z\|_X \leq \frac{1}{2}\|Y - Z\|_X \]
for some constants C_0 > 0 and C > 0 if
\[ R_0 \geq 2|Y_0|, \quad 2TR_1 \leq R_0, \quad R_1 \geq 2C_0|Y_1|, \]
\[ 2CTR_0^p \leq R_1, \quad 2CTR_0^{p-1} \leq 1, \quad 2CT^2R_0^{p-1} \leq 1. \]
Since these conditions are satisfied under the condition (1.11), Ψ is a contraction mapping on X(T, R_0, R_1). The solution is obtained as the fixed point of Ψ.

(2) When a_1 < 0 and \sigma = -1 + 2/n, we have T_1 = T_0 > 0. By the argument in (1), we obtain the global solution if (1.11) holds with T replaced by T_0, which is satisfied if R_0 = 2T_0R_1 and R_0 is sufficiently small. So that, we obtain the global solution if |Y_0| and |Y_1| are sufficiently small.

The results of (2), (3) and (4) in Theorem 1.1 follow from the similar proofs for Theorem 1.1. Especially for (4), the energy estimate
\[ \frac{1}{2}|D_tY(t)|^2 + \frac{\lambda}{p+1}|Y(t)|^{p+1} = \frac{1}{2}|Y_1|^2 + \frac{\lambda}{p+1}|Y_0|^p + 1 \]
by (2) in Lemma 2.7 shows the boundedness of Y(t) and D_tY(t) for 0 \leq t < T_1 when \lambda > 0, by which we obtain the global solution. When \lambda = 0, we also have the global solution since the differential equation in (1.4) is linear for Y.

5 Proof of Theorem 1.3

We note A > 0, D_tA > 0 and T_0 > 0 holds when a_1 < 0 and \sigma > -1 + 2/n by Lemma 2.6.

(1) Let us consider the operator Ψ defined by (3.3). We have
\[ \|A^{-1/2}f(Y)\|_{L^1((0,T))} \leq |\lambda|\|A^{-1/2}\|_{L^1((0,T))}\|Y\|^p_{L^\infty((0,T))}. \]
Since $A = n(\sigma + 1 - 2/n)q_0^2/8$ by (1) in Lemma 2.6, we have
\[
\|A^{-1/2}\|_{L^1((0,T))} \lesssim \|q_0^{-1}\|_{L^1((0,T))} = \frac{a_0}{2|a_1|} \left\| 1 - \frac{t}{T_0} \right\|_{L^1((0,T))} = \frac{a_0}{2|a_1|} T \left(1 - \frac{T}{2T_0}\right)
\]
by Lemma 2.6. Since we have
\[
\|\Psi(Y)\|_{X'(T)} \lesssim D' + \|A^{-1/2}f(Y)\|_{L^1((0,T))}
\]
by Lemma 2.8, there exist constants $C_0 > 0$ and $C > 0$ such that
\[
\|\Psi(Y)\|_{X'(T)} \leq C_0D' + \frac{C|\lambda|a_0}{|a_1|} T \left(1 - \frac{T}{2T_0}\right) R^p
\]
holds for any $Y \in X'(T, R)$, where $D'$ is defined by (1.12). Thus, we have $\|\Psi(Y)\|_{X'(T)} \leq R$ if $T$ and $R$ satisfy
\[
R \geq 2C_0D', \quad \frac{2C|\lambda|a_0}{|a_1|} T \left(1 - \frac{T}{2T_0}\right) R^{p-1} \leq 1. 
\tag{5.1}
\]
Similarly to (3.13) and (3.14), we are able to show
\[
\|\Psi(Y) - \Psi(Z)\|_{X'(T)} \leq \frac{1}{2}\|Y - Z\|_{X'(T)}
\]
for any $Y, Z \in X'(T, R)$ under the conditions (5.1). So that, $\Psi$ is a contraction mapping on $X'(T, R)$, and the solution is obtained as its fixed point.

(2) Since $T(1 - T/2T_0) \leq T \leq T_0$ by $T_0 > 0$, the second condition in (5.1) is satisfied if
\[
\frac{2C|\lambda|a_0}{|a_1|} T_0 R^{p-1} \left(= \frac{4C|\lambda|}{n(1 + \sigma)H^2} R^{p-1}\right) \leq 1.
\]
Thus, the conditions (5.1) are satisfied with $T = T_0$ if $R > 0$ and $D' > 0$ are sufficiently small. Namely, we obtain the global solution for small data.

(3) For the solution $Y \in X'(T)$, since $f(Y) \in L^1((0,T))$ by
\[
\|f(Y)\|_{L^1((0,T))} \leq T\|f(Y)\|_{L^\infty((0,T))}
\]
and
\[
\|f(Y)\|_{L^\infty((0,T))} \leq |\lambda|\|Y\|_{L^\infty((0,T))} \leq |\lambda|\|Y\|_{X'(T)}^p,
\]
we have $Y \in C([0,T))$ by (6) in Lemma 2.4. Thus, we have $f(Y) \in C([0,T))$, and moreover $Y \in C^1([0,T))$ again by (6) in Lemma 2.4.

(4) The result follows from the analogous argument in the proof of (3) in Theorem 1.3.

(5) By the energy estimate Lemma 2.8, we have $e^0(t) + \int_0^t e^1(s)ds = e^0(0)$ for $0 \leq t < T \leq T_0$. Since $D_tA > 0$ and $\lambda \geq 0$, we have $\int_0^t e^1(s)ds \geq 0$ and
\[
\frac{1}{2}\|A^{-1/2}(t)D_t Y(t)\|^2 + \frac{1}{2}\|Y(t)\|^2 \leq e^0(t) \leq e^0(0).
\]
Since $D'(t) := |Y(t)| + A(t)^{-1/2}|D_t Y(t)|$ is uniformly bounded on $[0, T)$, the existence time $T$ for the local solution can be taken uniformly under the condition (1.13) with $a_0, a_1$ and $D'$ replaced by $a(t)$, $D_t a(t)$ and $D'(t)$. So that, we can obtain the global solution on $[0,T_0)$ connecting the local solutions on short intervals in $[0,T_0)$.
6 Proof of Theorem 1.4

We have $A = -H^2$ by $a(t) = e^{Ht}$ and (1.3).

(1) Since the differential equation in (1.4) is rewritten as $D_t^2 Y + (\lambda - H^2) Y = 0$ when $p = 1$, the solution $Y$ is given by (1.15) with (1.16).

(2) The differential equation in (1.4) is rewritten as

\[ D_t^2 Y + (\lambda R^{p-1} - H^2) Y = 0 \]  

by $R = |Y|$. Let $Y^3 = \cdots = Y^n = 0$.

(i) When $R$ satisfies $\lambda R^{p-1} - H^2 > 0$, then $Y$ is given by (1.17). Here, $\lambda R^{p-1} - H^2 > 0$ holds if and only if $\lambda > 0$ and $R > (H^2/\lambda)^{1/(p-1)}$.

(ii) Let $R$ satisfy $\lambda R^{p-1} - H^2 = 0$, which holds if and only if $\lambda = H = 0$, or $\lambda > 0$ and $R = (H^2/\lambda)^{1/(p-1)}$. Since (6.1) is rewritten as $D_t^2 Y = 0$, $Y$ is given by $Y^j = B^j + C^j t$ for some constants $B^j$ and $C^j$ for $j = 1, 2$. Since $(B^1)^2 + (B^2)^2 = R^2$ and $C^1 = C^2 = 0$ by $|Y| = R$, we obtain $Y = (B^1, B^2, 0, \cdots, 0)$ with $B^1 = Y_0^1$, $B^2 = Y_0^2$, $Y_1^1 = C^1 = 0$, $Y_1^2 = C^2 = 0$ and $(Y_0^1)^2 + (Y_0^2)^2 = R^2$.

(iii) Let $R$ satisfy $\lambda R^{p-1} - H^2 < 0$. Since (6.1) is rewritten as $D_t^2 Y - (H^2 - \lambda R^{p-1}) Y = 0$, $Y$ is given by

\[ Y^j = B^j e^{\sqrt{H^2 - \lambda R^{p-1}} t} + C^j e^{-\sqrt{H^2 - \lambda R^{p-1}} t} \]

for some constants $B^j$ and $C^j$ for $j = 1, 2$. We have $B^1 = B^2 = 0$ by $|Y| \to \infty$ as $t \to \infty$ if $B^1 \neq 0$ or $B^2 \neq 0$ which contradicts to $|Y| = R$. Moreover, $R = 0$ must hold by $|Y| = \sqrt{(C^1)^2 + (C^2)^2} e^{-\sqrt{H^2 - \lambda R^{p-1}} t} \to 0$ as $t \to \infty$. Thus, $Y = 0$ is only allowed.

7 Proof of Theorem 1.6

Since we have

\[ |Y(t)| = e^{H|X(t)|} \text{ and } A = -H^2 \]

by (2.3) and (2.10), where $|X| := \left\{ \sum_{j=1}^{n} (X^j)^2 \right\}^{1/2}$ by (1.5), the Cauchy problem (1.20) is rewritten as

\[
\begin{aligned}
D_t^2 X(t) + 2 HD_t X + \lambda e^{(p-1)H t}|X(t)|^p & = 0 \quad \text{for } t \geq 0, \\
X(0) = X_0, \quad D_t X(0) = X_1
\end{aligned}
\]  

(7.1)

when $a(t) = e^{Ht}$ by Lemma 2.3. We say that $X$ is a global weak solution if $X$ satisfies

\[
-X_1 \phi(0) + X_0 D_t \phi(0) - 2H X_0 \phi(0) + \int_0^\infty X(t) D_t^2 \phi - 2H X(t) D_t \phi(t) + \lambda e^{(p-1)H t}|X(t)|^p \phi(t) dt = 0
\]  

(7.2)

for any $\phi \in C^2_0([0, \infty))$ which is equivalent to (1.21). So that, Theorem 1.6 is equivalent to the following Theorem 7.1 since $X_0 = Y_0$ and $X_1 = -HY_0 + Y_1$ when $a(t) = e^{Ht}$ under (2.3). It suffices to show Theorem 7.1 to prove Theorem 1.6.
Theorem 7.1. Let $n = 1$, $\sigma = -1$ in (7.2), $H \geq 0$, $\lambda > 0$, $1 < p < \infty$. If $X_1 + 2H X_0 \leq 0$, then any global weak solution $X$ of (7.1) must satisfy $X = 0$.

Proof. Let $\eta \in C_0^\infty([0, \infty))$ be a non-negative function with $\eta(t) = 1$ for $0 \leq t \leq 1/2$, $\eta(t) = 0$ for $t \geq 1$, and $|D_t \eta(t)|^2/\eta(t) \lesssim 1$ for $1/2 \leq t \leq 1$. An example of $\eta$ is given by

$$
\eta(t) := \begin{cases} 
1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\
1 - \eta_0 \int_{1/2}^t e^{-(s-1/2)^{-1}(1-s)^{-1}} ds & \text{if } \frac{1}{2} < t < 1, \\
0 & \text{if } t \geq 1,
\end{cases}
$$

where $\eta_0 := \left( \int_{1/2}^{1} e^{-(s-1/2)^{-1}(1-s)^{-1}} ds \right)^{-1}$. For $R > 0$, put $\eta_R(t) := \eta(t/R)$ and $\phi := \eta_R^p$, where $p'$ is defined by $1/p + 1/p' = 1$. We note $\phi \in C^2([0, \infty))$ by

$$
\begin{align*}
D_t \eta_R^p(t) &= \frac{p'}{R} \eta_R^{p-1}(t) D_t \eta \left( \frac{t}{R} \right), \\
D_t \eta_R^p(t) &= \frac{p'(p'-1)}{R^2} \eta_R^{p-1}(t) \cdot \frac{(D_t \eta(t/R))^2}{\eta(t/R)} + \frac{p'}{R^2} \eta_R^{p-1}(t) D_t \eta \left( \frac{t}{R} \right),
\end{align*}
$$

and $|D_t \eta(t)|^2/\eta(t) \lesssim 1$ for $1/2 \leq t \leq 1$. The equation (7.2) is rewritten as

$$
\lambda I = X_1 + 2H X_0 - J + L
$$

by $\phi(0) = 1$ and $D_t \phi(0) = 0$, where we have put

$$
I := \int_0^\infty e^{(p-1)Ht} |X(t)|^p \phi(t) dt, \quad J := \int_0^\infty X(t) D_t^2 \phi(t) dt, \\
L := 2H \int_0^\infty X(t) D_t \phi(t) dt.
$$

By $|D_t \eta_R^p| \lesssim R^{-2} \eta_R^{p-1} \chi_{[R/2, R]}$ and the Hölder inequality, we have

$$
|J| \lesssim \frac{1}{R^2} \int_{R/2}^R |X(t)| \eta_R^{p-1}(t) dt
$$

$$
\lesssim \frac{1}{R^2} \left\{ \int_{R/2}^R e^{-Ht} dt \right\}^{1/p'} \left\{ \int_{R/2}^R e^{(p-1)Ht} |X(t)|^p \eta_R^{p'}(t) dt \right\}^{1/p}
\lesssim \frac{J_* I^{1/p}}{R^2},
$$

where $\chi_{[R/2, R]}$ is the characteristic function on the interval $[R/2, R]$, and we have put

$$
J_* := \left\{ \int_{R/2}^R e^{-Ht} dt \right\}^{1/p'} \left( \lesssim R^{1/p'} \max \left\{ e^{-HR/2p'}, e^{-HR/p'} \right\} \right). \quad (7.4)
$$

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By $|D_t \eta_R^{p'}| \lesssim R^{-1} \eta_R^{p'-1} \chi_{[R/2, R]}$ and the Hölder inequality, we have

$$
|L| \lesssim \frac{H}{R} \int_{R/2}^{R} |X(t)| \eta_R^{p'-1}(t) dt
$$

$$
\lesssim \frac{H}{R} \left\{ \int_{R/2}^{R} e^{-Ht} dt \right\}^{1/p'} \left\{ \int_{R/2}^{R} e^{(p-1)Ht} |X(t)| \eta_R^{p'}(t) dt \right\}^{1/p}
$$

$$
\lesssim \frac{H J_* I^{1/p}}{R}.
$$

Since we have $X_1 + 2HX_0 \leq 0$ in (7.3) by the assumption, we obtain

$$
\lambda I \leq -J + L \leq |J| + |L| \lesssim \frac{J_* I^{1/p}}{R^2} + \frac{H J_* I^{1/p}}{R}.
$$

Dividing the both sides by $I^{1/p}$, we have

$$
\lambda I^{1/p'} \lesssim \frac{J_*}{R^2} + \frac{H J_*}{R} \leq \left( R^{1/p'-2} + \frac{HR^{1/p'-1}}{2} \right) \max \left\{ e^{-HR/2p'}, e^{-HR/p'} \right\}
$$

by (7.3). Since $\lambda > 0$ and the right hand side tends to $0$ as $R$ tends to infinity by $H \geq 0$, we have $\lim_{R \to \infty} I = 0$, which yields

$$
\int_{0}^{\infty} e^{(p-1)Ht} |X(t)|^p dt = 0
$$

by the definition of $I$. So that, we obtain $X = 0$ as required. \qed

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