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SOME REMARKS ON THE HEAT FLOW FOR FUNCTIONS AND FORMS

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Abstract
This note is concerned with the differentiation of heat semigroups on Riemannian manifolds. In particular, the relation $dP_t f = P_t df$ is investigated for the semigroup generated by the Laplacian with Dirichlet boundary conditions. By means of elementary martingale arguments it is shown that well-known properties which hold on complete Riemannian manifolds fail if the manifold is only BM-complete. In general, even if $M$ is flat and $f$ smooth of compact support, $\|dP_t f\|_\infty$ cannot be estimated on compact time intervals in terms of $f$ or $df$.

1 Introduction

Let $(M, g)$ be a Riemannian manifold and $\Delta$ its Laplacian. Consider the minimal heat semigroup associated to $\frac{1}{2}\Delta$ on functions given by

$$ (P_t f)(x) = E\left[(f \circ X_t(x)) 1_{\{t < \zeta(x)\}}\right] \tag{1.1} $$

where $X_t(x)$ is Brownian motion on $M$ starting at $x$, with (maximal) lifetime $\zeta(x)$. Denote by $W_{0, r}: T_x M \to T_{X_t(x)} M$ the linear transport on $M$ along $X_t(x)$ determined by the following pathwise covariant equation:

$$ \begin{cases} \frac{d}{dr} W_{0, r} v = \frac{1}{2} \text{Ric}(W_{0, r} v, \cdot)^\# \\ W_{0, 0} v = v. \end{cases} \tag{1.2} $$

By definition, $\frac{d}{dr} = /_{0, r} \frac{d}{dr} /_{0, r}^{-1}$ where $/_{0, r}$ denotes parallel transport along $X_t(x)$. For 1-forms $\alpha \in \Gamma(T^* M)$ let

$$ (P_t^{(1)} \alpha) v = E\left[\alpha_{X_t(x)} W_{0, t} v 1_{\{t < \zeta(x)\}}\right], \quad v \in T_x M. \tag{1.3} $$

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It is a well-known consequence of the spectral theorem that on a complete Riemannian manifold $M$

$$
  dP_t f = P_t^{(1)} df
$$

holds for all $f \in C_c^\infty(M)$ (compactly supported $C^\infty$ functions on $M$) if, for instance,

$$
  \mathbb{E} \left[ \| W_{0,t} \| 1_{\{ X_t(x) \in K \}} 1_{\{ t < \zeta(x) \}} \right] < \infty
$$

for any $x \in M$ and any compact subset $K \subset M$. Indeed, (1.4) holds true for the semigroups

associated to the self-adjoint extensions of the Laplacian on functions, resp. 1-forms. These

semigroups defined by the spectral theorem can be identified with the stochastic versions

(1.1) and (1.3) as soon as (1.3) is well-defined. The identification can be done, for instance,

with straightforward martingale arguments by exhausting the manifold through a sequence of

regular domains.

Note that from the defining equation (1.2) one gets

$$
  \| W_{0,t} \| \leq \exp \left\{ -\frac{1}{2} \int_0^t \text{Ric}(X_s(x)) \, ds \right\}
$$

where $\text{Ric}(x)$ is the smallest eigenvalue of the Ricci tensor $\text{Ric}_x$ at $x$. Thus (1.5) reads as a

condition imposing lower bounds on the Ricci curvature of $M$.

The Brownian motions $X(x)$ may be constructed as solutions of a globally defined (non-
intrinsic) Stratonovich SDE on $M$ of the form

$$
  dX = A(X) \circ dZ + A_0(X) \, dt
$$

with $A \in \Gamma(\mathbb{R}^r \otimes TM)$, $A_0 \in \Gamma(TM)$ and $Z$ an $\mathbb{R}^r$-valued Brownian motion on some filtered

probability space satisfying the usual completeness conditions. For $x \in M$, let

$$
  \mathcal{F}_t(x) := \mathcal{F}_t^{X(x)} = \sigma\{ X_s(x) : 0 \leq s \leq t \}
$$

be the filtration generated by $X(x)$ starting at $x$. Then, by [4], $A$ and $A_0$ in the SDE (1.6)

can be chosen in such a way that

$$
  W_{0,t} v 1_{\{ X_t(x) \in K \}} = \mathbb{E}^{\mathcal{F}_t(x)}[ /\!\!/_{t\downarrow 0} 1_{\{ T_x X_t \} v 1_{\{ X_t(x) \in K \}}}].
$$

Suppose that, instead of (1.5), we have

$$
  \mathbb{E} \left[ \| T_x X_t \| 1_{\{ X_t(x) \in K \}} 1_{\{ t < \zeta(x) \}} \right] < \infty
$$

for any $x \in M$ and any compact subset $K \subset M$. Then

$$
  (P_t^{(1)} df)v = \mathbb{E}[(df)_{X_t(x)} T_x X_t v 1_{\{ t < \zeta(x) \}}], \quad v \in T_x M.
$$

Thus, supposing for simplicity that $(M,g)$ is BM-complete, i.e., $\zeta(x) \equiv \infty$ a.s. for all $x \in M$,

relation (1.4) comes down to a matter of differentiation under the integral.

This point of view rises the question whether completeness of $M$ is an essential ingredient

for (1.4) to hold. However, we show that (1.4) may fail on metrically incomplete manifolds,

even if the manifold is flat and BM-complete. Even then, $\limsup_{t \to 0^+} \| dP_t f \|_\infty$ may be infinite for

compactly supported $f \in C^\infty(M)$. 


2 Differentiation of semigroups

We follow the methods of [7]. In the sequel we write occasionally $T_x f$ instead of $df_x$ for the differential of a function $f$ to avoid mix-up with stochastic differentials. Finally, we denote by $B(M)$ the bounded measurable functions on $M$ and by $bC^1(M)$ the bounded $C^1$-functions on $M$ with bounded derivative.

**Lemma 2.1** Let $(M,g)$ be a Riemannian manifold and $f \in B(M)$. Fix $t > 0$, $x \in M$, and $v \in T_x M$. Then

\[
N_s := T_{X_s(x)}(P_t-s)f) T_x X_s v, \quad 0 \leq s < t \wedge \zeta(x), \\
\bar{N}_s := T_{X_s(x)}(P_t-s)f) W_{0,s}v, \quad 0 \leq s < t \wedge \zeta(x),
\]

are local martingales (with respect to the underlying filtration).

**Proof** To see the first claim, note that $(P_t-sf)(X_s(x))$ is a local martingale depending on $x$ in a differentiable way. Thus, the derivative with respect to $x$ is again a local martingale, see [1]. The second claim is reduced to the first one by conditioning with respect to $\mathcal{F}(x)$ to filter out redundant noise. The second part may also be checked directly using the Weitzenböck formula

\[
d\Delta f \equiv \Delta^{(1)}df = \Delta^{\text{hor}}df - \text{Ric}(df^\#, \cdot)
\]

(2.1)

where $\Delta^{(1)}$ is the Laplacian on 1-forms and $\Delta^{\text{hor}}df$ the horizontal Laplacian on $O(M)$ acting on $df$ when considered as equivariant function on $O(M)$. Indeed, by lifting things up to the orthonormal frame bundle $O(M)$ over $M$, we can write

\[
\bar{N}_s = F(s,U_s) \cdot U^{-1}_s W_{0,s}v
\]

where $U$ is a horizontal lift of the BM $X_s(x)$ to $O(M)$ (i.e., a horizontal BM on $O(M)$ with generator $\frac{1}{2}\Delta^{\text{hor}}$) and

\[
F: [0,t] \times O(M) \to \mathbb{R}^d, \quad F_t(s,u) := (dP_t-sf)_{\pi(u)}(ue_i), \quad i = 1, \ldots, d = \dim M.
\]

Then $d\bar{N}_s \equiv 0$ (equality modulo differentials of local martingales) follows by means of Itô’s formula. $\square$

**Notation** For the Brownian motion $X_s(x)$ on $M$, let

\[
B = \int_0^t ||_{\mathbb{R}^d} \circ dX_r(x)
\]

denote the anti-development of $X_s(x)$ taking values in $T_x M$. By definition, $B$ is a BM on $T_x M$ satisfying

\[
A(X(x)) dZ = ||_{\mathbb{R}^d} dB.
\]
Lemma 2.2 Let \((M, g)\) be a Riemannian manifold, \(f \in B(M)\), \(x \in M\) and \(t > 0\). Let 
\[ \Theta(x) : T_x M \to T_{x(t)} M \]
be linear maps such that 
\[ T_{x(t)}(P_{t-s} f) \Theta_{0,s} v, \quad 0 \leq s < t \wedge \zeta(x), \]
is a continuous local martingale. Then 
\[ T_{x(t)}(P_{t-s} f) \Theta_{0,s} h_s - \int_0^s \langle T_{x(t)}(P_{t-r} f) \Theta_{0,r} \rangle dh_r, \quad 0 \leq s < t \wedge \zeta(x), \tag{2.2} \]
is again a continuous local martingale for any adapted \(T_M\)-valued process \(h\) of locally bounded variation. In particular, 
\[ T_{x(t)}(P_{t-s} f) \Theta_{0,s} h_s - (P_{t-s} f)(X_s(x)) \int_0^s \langle \Theta_{0,r} \dot{h}_r, \|/_{0,r} dB_r \rangle, \quad 0 \leq s < t \wedge \zeta(x), \]
is a local martingale for any adapted process \(h\) with paths in the Cameron-Martin space \(H([0, t], T_M)\), i.e., \(h(\omega) \in H([0, t], T_M)\) for almost all \(\omega\).

Proof Indeed, by Itô’s lemma, 
\[ d(T_{x(t)}(P_{t-s} f) \Theta_{0,s} h_s) = (T_{x(t)}(P_{t-s} f) \Theta_{0,s}) d h_s + d(T_{x(t)}(P_{t-s} f) \Theta_{0,s}) \cdot h_s \]
\[ \equiv (T_{x(t)}(P_{t-s} f) \Theta_{0,s}) d h_s \]
where \(\equiv\) stands for equality modulo local martingales. The second part can be seen using the formula 
\[ (P_{t-s} f)(X_s(x)) = \int_0^s T_{x(t)}(P_{t-r} f) \|/_{0,r} dB_r. \]
This proves the Lemma. \(\square\)

Lemma 2.2 leads to explicit formulae for \(dp_t f\) by means of appropriate choices for \(h\) which make the local martingales in Lemma 2.2 to uniformly integrable martingales. This can be done as in [7].

Theorem 2.3 [7] Let \(f : M \to \mathbb{R}\) be bounded measurable, \(x \in M\) and \(v \in T_M\). Then, for any bounded adapted process \(h\) with paths in \(H(\mathbb{R}^+, T_M)\) such that \(\left( \int_0^{\tau_D \wedge t} |h_s|^2 \, ds \right)^{1/2} \in L^1\), and the property that \(h_0 = v\), \(h_s = 0\) for all \(s \geq \tau_D \wedge t\), the following formula holds:
\[ d(P_t f)_x v = -\mathbb{E} \left[ f(X_t(x)) 1_{\{t < \zeta(x)\}} \int_0^{\tau_D \wedge t} \langle W_{0,s} (h_s), \|/_{0,s} dB_s \rangle \right] \tag{2.3} \]
where \(\tau_D\) is the first exit time of \(X(x)\) from some relatively compact open neighbourhood \(D\) of \(x\).
Theorem 2.4 Let \((M, g)\) be a BM-complete Riemannian manifold such that \(\text{Ric} \geq \alpha\) for some constant \(\alpha\).

(i) For \(f \in bC^1(M)\) the relation \(dP_s f = P_s^{(1)} df\) holds for \(0 \leq s \leq t\) if and only if
\[
\sup_{0 \leq s \leq t} \|dP_s f\|_\infty < \infty. \tag{2.4}
\]

(ii) Let \(f \in C^1(M)\) be bounded such that (2.4) is satisfied. Then, for \(t > 0\),
\[
\|dP_t f\|_\infty \leq \left( \left( \frac{1 - e^{-\alpha t}}{\alpha} \right)^{1/2} \frac{1}{t} \|f\|_\infty \right) \wedge \left( e^{-\alpha t/2} \|df\|_\infty \right) \tag{2.5}
\]
with the convention \((1 - e^{-\alpha t})/\alpha = t\) for \(\alpha = 0\).

Proof (i) Of course, \(dP_t f = P_t^{(1)} df\) implies (2.4) in case \(df\) is bounded. On the other hand, let \(f \in C^1(M)\) such that (2.4) holds. Condition (2.4) ensures the local martingale
\[
\mathcal{K}_s = (dP_{t-s} f)_{X_s(x)} W_{0,s} v, \quad v \in T_x M,
\]
of Lemma 2.1 to be a martingale for \(0 \leq s \leq t\), which gives by taking expectations
\[
(dP_t f)_v = E[(df)_{X_t(x)} W_{0,t} v] = P_t^{(1)} df(v).
\]

(ii) As in (i), condition (2.4) for \(f \in C^1(M)\) implies \(dP_t f)_x v = E[(df)_{X_t(x)} W_{0,t} v]\) which shows \(|d(P_t f)_x| \leq e^{-\alpha t/2} \|df\|_\infty\). On the other hand, by Lemma 2.2,
\[
T_{X_t(x)} (P_{t-.} f) W_{0,.}, h, (P_{t-.} f)(X_t(x)) \int_0^t \langle W_{0,r} h_r, //_{0,r} dB_r \rangle \tag{2.6}
\]
is a local martingale for any adapted process \(h\) with \(h_r(\omega) \in \mathbb{H}([0,t], T_x M)\). If we take \(h_s := (1-s/t)v\) where \(v \in T_x M\), then by means of assumption (2.4) and the bound on the Ricci curvature, (2.6) is seen to be a uniformly integrable martingale, hence
\[
d(P_t f)_x v = -\frac{1}{t} E \left[ f \circ X_t(x) \int_0^t \langle W_{0,r} v, //_{0,r} dB_r \rangle \right].
\]
Thus
\[
|d(P_t f)_x| \leq \frac{1}{t} \|f\|_\infty \left( E \int_0^t \|W_{0,r}\|^2 dr \right)^{1/2} \leq \frac{1}{t} \|f\|_\infty \left( \int_0^t e^{-\alpha r} dr \right)^{1/2} \leq \frac{1}{t} \left( \frac{1 - e^{-\alpha t}}{\alpha} \right)^{1/2} \|f\|_\infty
\]
which shows part (ii).

Remark 2.5 [8] Let \(M\) be an arbitrary Riemannian manifold and \(D \subset M\) an open set with compact closure and nonempty smooth boundary. Let \(f \in B(M)\). Then, for \(x \in D\) and \(t > 0\),
\[
|d(P_t f)_x| \leq c \|f\|_\infty
\]
with a constant \(c\) depending on \(t\), \(\text{dim } M\), \(\text{dist}(x, \partial D)\) and the lower bound of \(\text{Ric}\) on \(D\). This follows from Theorem 2.3 with an explicit choice for \(h\). See [8] for the details.
Remark 2.6 In the abstract framework of the $\Gamma_2$-theory of Bakry and Emery (e.g. [2]) lower bounds on the Ricci curvature $\text{Ric} \geq \alpha$ (i.e. $\Gamma_2 \geq \alpha \Gamma$) may be expressed equivalently in terms of the semigroup as

$$|dP_tf|^2 \leq e^{-\alpha t} P_t |df|^2, \quad t \geq 0,$$

for $f$ in a sufficiently large algebra of bounded functions on $M$. However, in general, the setting does not include the Laplacian on metrically incomplete manifolds. On such spaces, we may have $\limsup_{t \to 0^+} \|dP_tf\|_\infty = \infty$ for $f \in C^\infty_c(M)$, as can be seen from the examples below.

3 An example

Let $\mathbb{R}^2 \setminus \{0\}$ be the plane with origin removed. For $n \geq 2$, let $M_n$ be an $n$-fold covering of $\mathbb{R}^2 \setminus \{0\}$ equipped with the flat Riemannian metric. See [6] for a detailed analysis of the heat kernel on such BM-complete spaces. In terms of polar coordinates $x = (r, \vartheta)$ on $M_n$ with $0 < r < \infty$, $0 \leq \vartheta < 2n\pi$, $h(x) = \cos(\vartheta/n) J_{1/n}(r)$ (3.1) is a bounded eigenfunction of $\Delta$ on $M_n$ (with eigenvalue $-1$); here $J_{1/n}(r)$ denotes the Bessel function of order $1/n$. Note that $J_{1/n}(r) = O(r^{1/n})$ as $r \to 0$, consequently $dh$ is unbounded on $M_n$. The martingale property of

$$m_t = e^{t/2} (h \circ X_t(x)), \quad t \geq 0,$$

implies $P_t h = e^{-t/2} h$ which means that $dP_t h$ is unbounded on $M_n$ as well.

Example 3.1 On $M_n$ the relation $dP_t f = P_t^{(1)} df$ fails in general for compactly supported $f \in C^\infty(M_n)$. If this happens, then by Theorem 2.4 (i),

$$\sup_{0 \leq s \leq t} \|dP_sf\|_\infty = \infty$$

(3.2)

for $f \in C^\infty(M_n)$ of compact support.

Proof Otherwise (3.2) holds true for all compactly supported $f \in C^\infty(M_n)$. Fix $t > 0$. Then by Theorem 2.4 (ii)

$$\|dP_t f\|_\infty \leq \frac{1}{\sqrt{t}} \|f\|_\infty$$

(3.3)

for any compactly supported $f \in C^\infty(M_n)$. On the other hand, we may choose a sequence $(f_\ell)$ of nonnegative, compactly supported elements in $C^\infty(M_n)$ such that $f_\ell \not\to h^c := h + c$ with $h$ given by (3.1) and $c$ a constant such that $h + c \geq 0$. But then (see Chavel [3] p. 187 Lemma 3; note that this is a local argument which can be applied on any open relatively compact subset of $M$)

$$P_t f_\ell \not\to P_t h^c$$

and

$$dP_t f_\ell \to dP_t h^c \quad \text{as} \quad \ell \to \infty.$$

By (3.3) we would have

$$\|dP_t h^c\|_\infty \leq \frac{1}{\sqrt{t}} \|h^c\|_\infty,$$

in contradiction to $\|dP_t h^c\|_\infty = e^{-t/2} \|dh\|_\infty = \infty$. □
Remark 3.2 In [5] it is shown that if a stochastic dynamical system of the type (1.6) is strongly 1-complete, and if for each compact set $K$ there is a $\delta > 0$ such that
\[
\sup_{x \in K} E \| T_x X_s \|^{1+\delta} < \infty,
\]
then $dP_t f = P_t^{(1)} df$ holds true for functions $f \in bC^1(M)$. Example 3.1 shows that the strong 1-completeness is necessary and cannot be replaced by completeness.

On $M_n$ consider the heat equation for 1-forms
\[
\begin{aligned}
\frac{\partial}{\partial t} \alpha &= \frac{1}{2} \Delta^{(1)} \alpha \\
\alpha|_{t=0} &= df
\end{aligned}
\tag{3.4}
\]
where $f \in C^\infty(M_n)$. Take $f \in C^\infty(M_n)$ of compact support with $dP_t f \neq P_t^{(1)} df$. Then
\[
\alpha_1^t := P_t^{(1)} df \quad \text{and} \quad \alpha_2^t := dP_t f
\]
define two different smooth solutions to (3.4). Note that $\|\alpha_i^t\| \in L^2$, $i = 1, 2$.

Corollary 3.3 On the $n$-fold cover $M_n$ of the punctured plane ($n \geq 2$) there are infinitely many nontrivial classical solutions to
\[
\begin{aligned}
\frac{\partial}{\partial t} \alpha &= \frac{1}{2} \Delta^{(1)} \alpha \\
\alpha|_{t=0} &= 0
\end{aligned}
\]
of the form $\alpha_t = P_t^{(1)} df - dP_t f$ with $f \in C^\infty(M_n)$ of compact support.

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