Cosmological angular trispectra and non-Gaussian covariance

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Abstract. Angular cosmological correlators are infamously difficult to compute due to the highly oscillatory nature of the projection integrals. Motivated by recent development on analytic approaches to cosmological perturbation theory, in this paper we present an efficient method for computing cosmological four-point correlations in angular space, generalizing previous works on lower-point functions. This builds on the FFTLog algorithm that approximates the matter power spectrum as a sum over power-law functions, which makes certain momentum integrals analytically solvable. The computational complexity is drastically reduced for correlators in a “separable” form — we define a suitable notion of separability for cosmological trispectra, and derive formulas for angular correlators of different separability classes. As an application of our formalism, we compute the angular galaxy trispectrum at tree level, with and without primordial non-Gaussianity. This includes effects of redshift space distortion and bias parameters up to cubic order. We also compute the non-Gaussian covariance of the angular matter power spectrum due to the connected four-point function, beyond the Limber approximation. We demonstrate that, in contrast to the standard lore, the Limber approximation can fail for the non-Gaussian covariance computation even for large multipoles.

Keywords: non-gaussianity, galaxy clustering, cosmological perturbation theory, redshift surveys

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1 Introduction

Our knowledge of the composition of the universe and the physics of its early stages has undergone tremendous advances over the last three decades. So far, the best constraints on the statistics of the primordial fluctuations are provided by detailed measurements of the cosmic microwave background (CMB) \[1\], which will soon be improved with the advent of the next generation CMB experiments, measuring polarization \[2–4\] and temperature fluctuations at small scales \[5, 6\] with higher precision. Complementary to the CMB, upcoming surveys
such as the Large Synoptic Survey Telescope (LSST) [7], SPHEREx [8], Euclid [9], and the Dark Energy Spectroscopic Instrument (DESI) [10] will be refining the measurements of the large-scale structure (LSS) of the universe, using different probes involving galaxy clustering, weak lensing, 21-cm emission line, etc. Containing three-dimensional information, these LSS surveys can ultimately surpass the CMB in providing stronger constraints on the statistics of the scalar fluctuations, and thus offering a better window into primordial non-Gaussianity.

Understanding the LSS is inherently more difficult than the CMB due to the intrinsically nonlinear nature of gravity. The gravitational evolution of dark matter density is most commonly studied in Fourier space, either using the theoretical framework of cosmological perturbation theory (and its extensions) or through numerical $N$-body simulations. In practice, we do not directly observe distributions of dark matter, but rather that of their tracers such as halos and galaxies. The relationship between dark matter and its tracers is called bias, and provides a crucial link between observations and the physics of cosmological perturbations. Amongst the two biased tracers, the fact that halos are nothing but gravitationally bound matter makes it possible for us simulate them in gravity-only simulations, and empirically study their bias. In contrast, galaxy dynamics is vastly more complicated, and currently we do not have first principles understanding of its formation process. This makes it difficult to reliably make theoretical predictions for galaxy distributions or simulating them on cosmological scales.

At present, direct observations provide the best means of studying galaxy distributions. When surveys have limited sky coverage, the sky is effectively flat, which makes it possible to do a Fourier analysis. Near-future spectroscopic redshift surveys will become deeper and wider, which instead requires a full-sky formalism of computing cosmological observables. Conventionally, it is still preferred to do cosmological analyses in Fourier space, in which theoretical calculations can be most naturally done. This, however, involves one extra complication: we must assume a fiducial cosmology to translate redshifts into distances, which, if differs from the true cosmology, results in distortions of data.$^1$ No such assumption is needed when working in redshift space, since observations directly map the positions of galaxies in terms of their redshifts and angles. Angular correlation functions in redshift space are therefore in principle the most natural observable to describe galaxy distributions (see figure 1). It is therefore desirable to be able to directly compare our theoretical models with observations in redshift space.

There are, however, well-known numerical challenges involved in projecting cosmological observables onto two-dimensional redshift surfaces. The main difficulty is that the projection integrals consist of products of spherical Bessel functions that are highly oscillatory, requiring a large number of integration points to reach a desired accuracy. The computational cost quickly becomes infeasible as we go to higher points due to the sheer multi-dimensionality of integrals. Worse, cosmological parameter estimation using Markov chain Monte Carlo methods in upcoming galaxy surveys would require observables to be computed millions of times, summed over many redshift bins as well as their cross-correlations. All these limitations have so far restricted performing cosmological data analyses in redshift space far from ideal.

Over the past few years, there have emerged novel approaches to overcome the numerical obstacles of computing angular observables in cosmology. One of the most insightful ideas has been the use of the so-called FFTLog algorithm, originally proposed in [12]. In this approach,$^1$

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$^1$This simply follows from the fact that the comoving distance between two objects with different redshifts is given by an integral involving the cosmology-dependent Hubble parameter over the redshift difference. Accounting for this requires a modeling of the so-called Alcock-Paczynski effect [11].
one decomposes the matter power spectrum into a sum of power-law functions, in attempts
to analytically solve certain integrals that are otherwise difficult to evaluate numerically. It
turns out that this analytic method not only leads to a dramatic increase in the speed of
the calculations, but also significantly improves the numerical stability of the integrals. The
utility of this method, and similar methods that bypass the spherical Bessel integrals, has been
explored in a number of recent works [13–19]. These theoretical studies on angular correlators,
and codes for computing them (e.g. [16, 20–23]), have so far restricted their analyses to two-
point and three-point statistics, due to yet existing complications of higher-point functions.

However, there are good motivations to go beyond and study four-point statistics. First
of all, shapes of four-point functions can exhibit new qualitative features that provide useful
information for constraining the physics of inflation beyond what is available in three-point
functions. For instance, many inflationary models involving light degrees of freedom produce
four-point functions whose sizes can be larger than that of three-point functions. Also,
the trispectrum — the Fourier counterpart of the four-point function — of biased tracers
from gravitational evolution at tree level has contributions from bias parameters up to third
order, which also enter the power spectrum calculation at two-loop. Having the full shape
information of four-point functions can thus help breaking degeneracy between different bias
parameters. Last but not least, the computation of the trispectrum is required in order to
accurately capture the covariance of the power spectrum, which encodes the statistical error
information. An accurate account of the covariance will be important for realizing the full
promise of the next generation LSS surveys.

The impact of the non-Gaussian contribution to the power spectrum covariance from the
connected four-point function has been studied in Fourier space e.g. in [24–31]. In contrast,
no full computation of the covariance for the angular power spectrum has yet been performed
due to the aforementioned numerical difficulties. In [32–34], the tree-level terms contributing
to the covariance for galaxy clustering were derived and a subset of them were computed
in the context of the halo model [35, 36]. For weak lensing, the non-Gaussian covariance
was computed using the flat-sky or Limber approximations [24, 37, 38], which are valid for
small-angle sky coverage. However, the full validity of these approximations have not been
tested, due to the lack of a stable method for computing the full angular four-point function
without relying on these approximations.
In this work, we present a method that bypasses these numerical difficulties by generalizing the previous FFTLog-based methods to angular four-point functions. In doing so, we revisit the separability condition of cosmological trispectra. Our method has a number of applications. First, it allows us to compute angular four-point functions with Gaussian and non-Gaussian initial conditions, the latter of which will be relevant for constraining primordial physics. Similarly, this can help constraining the cubic bias parameters that enters the galaxy four-point function at tree level. In addition, this FFTLog-based method provides a fast and reliable way of computing the non-Gaussian component of the covariance. We point out that, for the computation of the non-Gaussian covariance, the Limber approximation loses its validity even for high multipoles, in contrast to general expectations. We demonstrate this by computing the tree-level contribution to the covariance in standard cosmological perturbation theory.

Outline. The paper is organized as follows. In section 2, we describe an efficient method to compute cosmological angular four-point functions. In section 3, we describe different trispectrum shapes that we consider in our analyses. We apply the method to compute the angular galaxy trispectrum in section 4 and the non-Gaussian covariance of the angular matter power spectrum in section 5. We conclude in section 6. A number of appendices contain supplementary and technical details. In appendix A, we describe the method of dealing with spurious divergences. In appendix B, we give details of cubic bias operators and their momentum space representation. Finally, in appendix C we present useful properties of spin-weighted functions on a sphere, and use them to write contact separable trispectra.

Notations and convention. We use $k_i = |k_i|$ to denote the magnitudes of spatial momenta. The Fourier convention is $\tilde{O}(k) = \int_{\mathbb{R}^3} d^3x e^{-ik \cdot x} O(x)$. The matter power spectrum is computed with CLASS using the best-fit parameters of the ΛCDM model from Planck 2018 [39]: $h = 0.674$, $\Omega_b h^2 = 0.0224$, $\Omega_c h^2 = 0.120$, $\tau = 0.054$, $A_s = 2.10 \times 10^{-9}$ at $k = 0.05 \text{Mpc}^{-1}$, and $n_s = 0.965$.

2 Methodology

We are interested in computing correlation functions on a two-dimensional sphere, so it is natural to expand them in spherical harmonics. The advantage of this decomposition is that it trivializes the angular integrations for statistically isotropic observables. At the same time, the challenge is that the radial integrals are difficult to evaluate, consisting of highly-oscillatory spherical Bessel functions. Having efficient algorithms for evaluating these integrals will be important for analyzing large sets of observational datasets from upcoming experiments.

In this section, we present an efficient method for evaluating angular correlation functions, mainly focusing on the four-point case. For a self-contained discussion, we first briefly review the basics of angular correlations in cosmology in section 2.1. We then discuss the separability of correlators in section 2.2, reviewing the familiar case of bispectra and introducing a classification of separable trispectra. Lastly, we introduce a method to compute the angular trispectrum based on the FFTLog algorithm in section 2.3.

2.1 Angular correlators

Many cosmological observables are measured in redshift space. These include the CMB, as well as weak lensing shear and galaxy density fields from redshift surveys. Suppose that there
is an object located at some position $x$ and redshift $z$ with respect to an observer sitting at the origin. The actual observable seen in the sky depends on the entire trajectory the light has undertaken from the object to reach the observer at redshift $z = 0$. A cosmological observable $\mathcal{O}$ at redshift $z$ is thus defined as an integration along the line-of-sight direction $\hat{n} \equiv x/|x|$ over some kernel $W_\mathcal{O}$ as

$$
\mathcal{O}(\hat{n}, z) = \int_0^\infty d\chi \, W_\mathcal{O}(\chi) \mathcal{O}(\chi \hat{n}, z) = \sum_{\ell m} \mathcal{O}_{\ell m}(z) Y_{\ell m}(\hat{n}),
$$

(2.1)

where $\chi$ stands for comoving distance, and we have expanded the observable in spherical harmonics $Y_{\ell m}$, with $\sum_{\ell m} \equiv \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell$. The kernel can be either a sharp or broad function depending on the observable under consideration.

At fixed redshift, it is natural to characterize the statistics of the observable $\mathcal{O}$ in terms of the angular variable $\mathcal{O}_{\ell m}$ on the sphere. Cosmological observables are, however, usually first computed in Fourier space. Taking the Fourier transform of $\mathcal{O}$ and using the spherical harmonics expansion of the plane wave\(^2\)

$$
e^{ik \cdot r} = 4\pi \sum_{\ell m} \ell^\ell j_\ell(kr) Y_{\ell m}^*(\hat{k}) Y_{\ell m}(\hat{r}),
$$

(2.2)

where $j_\ell$ is the spherical Bessel function, the projected observable can be expressed as

$$
\mathcal{O}_{\ell m}(z) = 4\pi \ell^\ell \int_0^\infty d\chi \, W_\mathcal{O}(\chi) \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} j_\ell(k\chi) Y_{\ell m}^*(\hat{k}) \mathcal{O}(\hat{k}, z),
$$

(2.3)

where $\mathcal{O}$ denotes the Fourier conjugate of the observable $\mathcal{O}$. The corresponding angular $n$-point function is then (see figure 2)

$$
\langle \mathcal{O}_{\ell_1 m_1} \cdots \mathcal{O}_{\ell_n m_n} \rangle = (4\pi)^n \ell_1^{\ell_1} \cdots \ell_n^{\ell_n} \int \left[ \prod_{i=1}^n \frac{d\chi_i d^3k_i}{(2\pi)^3} W_\mathcal{O}(\chi_i) j_{\ell_i}(k_i\chi_i) Y_{\ell_1 m_1}(\hat{k}_i) \right] \langle \hat{\mathcal{O}}_1 \cdots \hat{\mathcal{O}}_n \rangle,
$$

(2.4)

where we have suppressed the redshift dependence on the left-hand side. We defined $\ell_1 \cdots \ell_n \equiv \ell_1 + \cdots + \ell_n$, and

$$
\langle \hat{\mathcal{O}}_1 \cdots \hat{\mathcal{O}}_n \rangle = \langle \hat{\mathcal{O}}_1 \cdots \hat{\mathcal{O}}_n \rangle' \times (2\pi)^3 \delta_D(k_1 + \cdots + k_n)
$$

(2.5)

denotes an $n$-point function in Fourier space, with $\hat{\mathcal{O}}_i \equiv \hat{\mathcal{O}}_i(k_i, z_i)$, $\delta_D$ is the Dirac delta function that enforces momentum conservation, and $\langle \cdots \rangle'$ is a correlator with the delta function stripped off. The above formula (2.4) tells us how to take a correlator in $k$-space and project it onto $\ell$-space.

In this work, we consider angular correlators of galaxy density field $\delta_g$ at different redshifts, although the method can be similarly applied to other angular observables. These are related to matter density field $\delta$ by means of a bias expansion, which we describe in section 4.1. There are two sources of matter density fields: (i) late-time gravitational nonlinearities and (ii) non-Gaussian initial conditions, with the latter being characterized by correlators of the primordial curvature perturbation $\zeta$. The former contribution can be computed using the

\(^2\)By $Y_{\ell m}(\hat{k})$, we mean the spherical harmonic as a function of the angles of $\hat{k}$ with respect to some fixed reference frame, which becomes irrelevant once we integrate over the angles.
framework of standard perturbation theory, which we briefly describe in section 3.1. The relation between $\delta$ and $\zeta$ is given by

$$
\delta(k, z) = M(k, z) \zeta(k), \quad M(k, z) = -\frac{2 k^2 T(k) D_g(z)}{5 \Omega_{m,0} H_0^2},
$$

(2.6)

where $M$ is the transfer function that evolves $\zeta$ to $\delta$ at redshift $z$, with normalization $T(0) = 1$, and $D_g$ is the linear growth factor normalized to unity at $z = 0$, $D_g(0) = 1$.

We see from (2.4) that the computation of an $n$-point angular correlator in general requires evaluating $4^n$ coupled integrals, which becomes highly intractable as $n$ increases. This is further exacerbated due to the presence of the spherical Bessel functions, which makes the radial integrands highly oscillatory and therefore difficult to numerically integrate. Performing the angular integrations involving spherical harmonics, however, can always be done straightforwardly, which leads to a form restricted by rotational invariance. For the power spectrum and the bispectrum, statistical isotropy implies that they can be written as

$$
\langle O_{\ell m} O_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell},
\langle O_{\ell_1 m_1} O_{\ell_2 m_2} O_{\ell_3 m_3} \rangle = G_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3} b_{\ell_1 \ell_2 \ell_3},
$$

(2.7)

(2.8)

where the geometric factor $G_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3}$ called the Gaunt coefficient is defined as

$$
G_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{array} \right),
$$

(2.9)

$$
g_{m_1 m_2 m_3}^{\ell_1 \ell_2 \ell_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \left( \begin{array}{ccc} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{array} \right),
$$

(2.10)

and round-bracket matrices denote Wigner $3-j$ symbols. The physical degrees of freedom of the angular bispectrum are thus characterized by the reduced bispectrum $b_{\ell_1 \ell_2 \ell_3}$, which is a function of three multipoles.

Unlike for $n = 2$ and $3$, rotational invariance does not uniquely fix the form of angular $n$-point functions for $n \geq 4$. This is simply due to the fact that there are multiple ways of
choosing diagonal multipoles for higher-point functions. For instance, the angular trispectrum can be expressed in a rotationally invariant form as \[30\]

\[
\langle O_{\ell_1 m_1} \cdots O_{\ell_4 m_4} \rangle = \sum_{LM} (-1)^M \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & M \end{pmatrix} \begin{pmatrix} \ell_3 & \ell_4 & L \\ m_3 & m_4 & -M \end{pmatrix} T^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) \tag{2.11}
\]

where in the second line we have decomposed the trispectrum into a sum over different channels.\(^3\) Since we are interested in the connected part of the four-point function, we will assume that the disconnected (Gaussian) part has been subtracted off. Note that \(P^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L)\) is not invariant under \(\ell_1 \leftrightarrow \ell_2\) or \(\ell_3 \leftrightarrow \ell_4\), but instead satisfies \(P^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) = (-1)^{\ell_1 + L} P^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) = (-1)^{\ell_3 + L} P^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L)\) due to the properties of the 3-j symbols. To exhaust all the permutation symmetry, it can be broken apart into four permutations as

\[
P^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) = t^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) + (-1)^{\ell_2 + \ell_3 + \ell_4} t^{\ell_2 \ell_3}_{\ell_4 \ell_1} (L) + (-1)^{\ell_1 + \ell_3 + L} t^{\ell_1 \ell_3}_{\ell_2 \ell_4} (L) + (-1)^{\ell_1 + \ell_2 + L} t^{\ell_2 \ell_3}_{\ell_4 \ell_1} (L), \tag{2.12}
\]

where \(t^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L)\) is called the reduced trispectrum. This is invariant under the exchange of the upper and lower indices, \(t^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) = t^{\ell_3 \ell_4}_{\ell_1 \ell_2} (L)\). Later on, we will see that the geometric factor \(g^{\ell_1 \ell_2 \ell_3}\) repeatedly shows up in the calculation, as in the bispectrum case. It is therefore convenient to further decompose the reduced trispectrum as

\[
t^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) = g^{\ell_1 \ell_2 \ell_3} L g^{\ell_1 \ell_2 \ell_3} L \left( T^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) + \tau^{\ell_1 \ell_2 \ell_3} (L) \right). \tag{2.13}
\]

We will call \(T^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L)\) the super-reduced trispectrum.\(^4\) The full trispectrum can be built out of \(4! = 24\) permutations of this basic building block.

Using various identities of the Wigner symbols, we can express the trispectrum in terms of a single pairing \(\{\ell_1 \ell_2, \ell_3 \ell_4\}\) as

\[
T^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) = P^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) \tag{2.14}
\]

\[
+ (2L + 1) \sum_{L'} (-1)^{\ell_2 + \ell_3} \begin{pmatrix} \ell_1 & \ell_2 & L' \\ \ell_4 & \ell_3 & L \end{pmatrix} P^{\ell_1 \ell_2 \ell_3}_{L} (L') + (-1)^{\ell_1 + \ell_2 + L'} \begin{pmatrix} \ell_1 & \ell_2 & L' \\ \ell_3 & \ell_4 & L \end{pmatrix} P^{\ell_1 \ell_2 \ell_3}_{L} (L'),
\]

where the curly-bracketed matrices denote the Wigner 6-j symbols. We see that the reduced trispectrum has five independent degrees of freedom, as opposed to six for the 4-point function in Fourier space. More generally, the total number of independent degrees of freedom for \(n\)-point functions \(3n - 6\) (for \(n \geq 3\)) in \(k\)-space gets reduced to \(2n - 3\) in \(\ell\)-space, which geometrically is described by an \(n\)-gon (see figure 3).

2.2 Separability

A brute-force evaluation of (2.4) for a general momentum-space correlator is highly intractable due to the sheer number of coupled multi-dimensional integrals. However, these integrals become greatly simplified for certain correlators that are separable, i.e. for those that can be expressed as a product of functions of momenta. Let us first briefly review the separability of the bispectrum in section 2.2.1, and then discuss an analogous criterion for the trispectrum in section 2.2.2.

\(^3\)For implications of statistical isotropy on general angular \(n\)-point functions, see [41].

\(^4\)In [42], the terminology “extra-reduced trispectrum” was used to refer to \(T^{\ell_1 \ell_2}_{\ell_3 \ell_4} (L) + \tau^{\ell_1 \ell_2 \ell_3} (L)\).
Figure 3. Kinematic configurations of four-point functions in $k$-space (left) and $\ell$-space (right). The internal momenta are denoted by $s \equiv k_1 + k_2$ and $t \equiv k_2 + k_3$, while the internal multipole for the pairing $\{\ell_1\ell_2, \ell_3\ell_4\}$ is denoted by $L$.

2.2.1 Bispectrum: review

Due to spatial isotropy, bispectra must be functions of the dot products of momenta $k_i \cdot k_j$, which can be traded with wavenumbers $k_i = |k_i|$ by momentum conservation. A bispectrum in momentum space is then said to be separable if it can be expressed as

$$\langle \mathcal{O}(k_1, z_1) \mathcal{O}(k_2, z_2) \mathcal{O}(k_3, z_3) \rangle' = f_1(k_1, z_1) f_2(k_2, z_2) f_3(k_3, z_3).$$  (2.15)

In general, a full bispectrum will consists of a finite sum over such separable terms as well as other permutations.

It turns out that many physical bispectra can be expressed in the above separable form. To see why this is the case, let us first classify the possible shapes of primordial bispectra, for which there is no redshift dependence. Suppose first that the bispectrum is a rational function of momenta, so that it can be expressed as

$$\langle \zeta \zeta \zeta \rangle' = \frac{G(k_1, k_2, k_3)}{H(k_1, k_2, k_3)},$$  (2.16)

where $G$, $H$ are polynomials. The question of separability then depends on the form of $H$, which is determined in terms of its zeros or the singular behavior of the bispectrum, which can be either of the type $k_i \to 0$ or $k_1 + k_2 + k_3 \to 0$. The former arises because the bispectrum is proportional to the power spectrum, while the latter is due to the kind of time integral involved in computing late-time correlators in inflation. This factor alone can be expressed as

$$\frac{1}{(k_1 + k_2 + k_3)^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\tau \tau^{n-1} e^{-(k_1 + k_2 + k_3)\tau},$$  (2.17)

which is simply the wick-rotated version of the time integral, which is numerically easier to evaluate than the Lorentzian version. A naively non-separable bispectrum containing a factor $(k_1 + k_2 + k_3)^{-n}$ can then be made separable by approximating the above integral by a finite sum [43].

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This is true when the Bunch-Davies initial condition is imposed. Excited initial conditions can lead to singularities of the type $k_1 + k_2 - k_3 \to 0$ that blow up in the folded configuration.
To see what the condition (2.15) implies for the angular bispectrum, we first expand the delta function in plane waves using
\[(2\pi)^3 \delta_D(k_1 + \cdots + k_n) = \int_{\mathbb{R}^3} \mathrm{d}^3r \, e^{ik_1 \cdot r} \cdots e^{ik_n \cdot r}, \quad (2.18)\]
and then project onto the spherical harmonics basis using (2.2). We can then easily perform the angular part of the integrals in (2.4). After stripping off the geometric factor in (2.4), the reduced bispectrum can then be expressed as
\[b_{\ell_1 \ell_2 \ell_3} = \frac{1}{(2\pi)^3} \int_0^\infty \mathrm{d}r \, r^2 I^{(1)}_{\ell_1}(r) I^{(2)}_{\ell_2}(r) I^{(3)}_{\ell_3}(r), \quad (2.19)\]
where we defined
\[I^{(i)}_{\ell_i}(r) \equiv 4\pi \int_0^\infty \, \mathrm{d}\chi \, W_\ell(\chi) \int_0^\infty \, \mathrm{d}k \, k^2 f_i(k, z(\chi)) j_{\ell_i}(kr) j_{\ell_i}(k\chi). \quad (2.20)\]
The expensive part of this calculation is the \(k\)-integral involving highly-oscillating spherical Bessel functions. Once this is done, however, the remaining \(r\)-integral has in general a smooth integrand, which can be replaced by a finite quadrature.

2.2.2 Trispectrum

Having reviewed the well-known separable properties of bispectra, let us now discuss similar separability conditions for trispectra. (See [42, 44] for earlier works on separable trispectra.) Analogous to the bispectrum case, one would be tempted to think that the condition
\[\langle \mathcal{O}(k_1, z_1) \cdots \mathcal{O}(k_4, z_4) \rangle' = f_1(k_1, z_1) \cdots f_4(k_4, z_4), \quad (2.21)\]
is a suitable definition of the separability for the trispectrum. This turns out to be too restrictive in general. To see why, let us briefly review the basic kinematics of trispectra. Spatial isometries imply that the number of independent degrees of freedom for an \(n\)-point function is \(3n - 6\) (for \(n \geq 3\)), or six for the trispectrum. It is natural to choose four of them to be the magnitudes of the external momenta, \(k_i\) for \(i = 1, \cdots, 4\). For our purposes, we will find it convenient to parameterize the remaining two degrees of freedom with the magnitudes of two of the three internal momenta, \(s, t, u\) and \(s, t, u\) and then imposing the momentum conservation at each vertex. It is useful to first decompose the trispectrum into a sum over different channels as (temporarily dropping the \(z\) dependence to avoid clutter)
\[\langle \mathcal{O}(k_1, z_1) \cdots \mathcal{O}(k_4, z_4) \rangle = P_{\mathcal{O}}(k_1, k_2, k_3, k_4) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4), \quad (2.22)\]
where each channel can further be decomposed into
\[P_{\mathcal{O}}(k_1, k_2, k_3, k_4) = \tau_{\mathcal{O}}(k_1, k_2, k_3, k_4) + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (12 \leftrightarrow 34), \quad (2.23)\]
in analogy to (2.12). The trispectrum in the \(s\)-channel can then be written as
\[\tau_{\mathcal{O}}(k_1, k_2, k_3, k_4) = \int \mathrm{d}^3s \int \mathrm{d}^3t \, \tau_{\mathcal{O}}(k_1, k_2, k_3, k_4, s, t) \delta_D(k_{12} - s) \delta_D(k_{23} - t) \delta_D(k_{1234}), \quad (2.24)\]
Scalar-Exchange (scE) 

\[ f(s) \]

Contact (C) 

\[ s^{2J} \]

Spin-Exchange (spE) 

\[ f(s)t^{2J} \]

where \( k_{i_1 \ldots i_n} \equiv k_{i_1} + \cdots + k_{i_n} \). In analogy to the bispectrum case, one can say that the trispectrum is separable if it can be written as

\[
\langle O(k_1, z_1) \cdots O(k_4, z_4) \rangle' = f_1(k_1, z_1) \cdots f_4(k_4, z_4)f(s)g(t),
\]

(2.25)

where we have restored the redshift dependence, and \( \langle \cdots \rangle' \) now indicates that the integrals over \( s \) and \( t \) with the delta functions are also stripped off. The separability condition for other permutations is obtained by a cyclic shift: \( s \rightarrow t, t \rightarrow u \). As we explain in section 3.2, this choice of variables naturally parameterizes the shapes that arise from exchanging a mediator particle. For instance, the intermediate particle in the \( s \)-channel carries the momentum \( k_1 + k_2 \) and the trispectrum becomes a polynomial in \( t^2 \) whose degree reflects the spin of the intermediate particle [45]. The spin-\( J \) factor \( t^{2J} \) can be replaced by the dot product between momenta in the cross channel, \( (\mathbf{k}_2 \cdot \mathbf{k}_3)^J \), or equivalently by a linear combination of the Legendre polynomials \( P_J(\mathbf{k}_2 \cdot \mathbf{k}_3) \). Combining these facts, we consider the following ansatz for a separable trispectrum:

\[
\langle O(k_1, z_1) \cdots O(k_4, z_4) \rangle' = f_1(k_1, z_1) \cdots f_4(k_4, z_4)f(s)t^{2J},
\]

(2.26)

for the \( s \)-channel, where \( J \geq 0 \) is a non-negative integer. For the primordial trispectrum, the redshift dependence can be ignored.

A general separable trispectrum will be a sum over separable pieces obeying (2.26). Given a separable trispectrum in Fourier space, we would like to see what form of the angular trispectrum that this leads to. As previously, the general strategy to derive angular correlators is as follows. We first use (2.18) to express the delta functions as integrals over plane waves, which are then projected onto the spherical harmonic basis using (2.2). We then substitute the Fourier-space trispectrum to the projection formula (2.4) and perform the integrations over the angles. This only leaves radial integrals to be evaluated, weighted by a geometric factor, which can be simplified and put in the standard isotropic form (2.11) using various identities involving the Wigner symbols.

Let us now classify different types of separable trispectra, corresponding to different choices of \( f(s) \) and \( J \). We consider three cases illustrated by the following diagrams:

We have chosen suggestive names that reflect the physical processes that give rise to each shape dependence, as shown above. For example, scalar-exchange diagrams can have nontrivial dependence on the internal momentum, but with \( J = 0 \). Contact diagrams can be viewed as a special case of the scalar-exchange diagram, where the dependence on the internal momentum is given by non-negative integer powers of \( s^2 \). Lastly, the spin-exchange diagrams generalize the scalar-exchange case to nonzero \( J \).

\[ ^{6}\text{A similar classification was used in [44], where scalar-exchange and spin-exchange separability were collectively referred to as “exchange separability”. As we will see shortly, these two cases are qualitatively different, so it will be useful to present formulas for these two cases separately.} \]
In what follows, we present formulas for the angular trispectra belonging to different separability classes in the $s$-channel. Answers for different channels can be obtained by permutations. Derivations involve straightforward algebra but are rather unilluminating, so we mostly just quote the results.

Scalar-Exchange (scE). First, we consider the case with $J = 0$ and a generic function $f(s)$. Although contact separability leads to a simpler structure, we show the result for this case first because the contact-separable trispectrum can be derived as a special case of scE-separable one. We can rewrite the trispectrum (2.24) as

$$
\langle O_1 O_2 O_3 O_4 \rangle = (2\pi)^3 f_1(k_1, z_1) \cdots f_4(k_4, z_4) \int_{\mathbb{R}^3} d^3 s \, f(s) \delta_D(k_{12} - s) \delta_D(k_{34} + s),
$$

(2.27)

where we have absorbed the total-momentum-conserving delta function $\delta_D(k_{1234})$ inside the $s$-integral.\footnote{It is also possible not to absorb $\delta_D(k_{1234})$ in the $s$-integral. However, after projection this leads to integrals involving three spherical Bessel functions, which are trickier to deal with.} Using the plane-wave expansion of the delta function, and then evaluating the angular integrals, and substituting into the projection formula (2.4), we obtain the super-reduced trispectrum

$$(\text{scE}) : \tau_{\ell_1 \ell_2 \ell_3 \ell_4}(L) = \frac{1}{(2\pi)^5} \int_0^\infty dr r^2 I^{(1)}_{\ell_1}(r) I^{(2)}_{\ell_2}(r) \int_0^\infty dr' r'^2 I^{(3)}_{\ell_3}(r') I^{(4)}_{\ell_4}(r') J^{(s)}_L(r, r'),
$$

(2.28)

where the function $I^{(i)}_\ell$ was defined in (2.47). We see that the $s$-dependence of the momentum-space trispectrum leads to two coupled radial integrals, with the coupling integral given by

$$
J^{(s)}_L(r, r') \equiv 4\pi \int_0^\infty dk k^2 f(k) j_L(kr) j_L(kr').
$$

(2.29)

This integral basically has the same structure as the Bessel integral inside $I^{(i)}_\ell$, and is thus difficult to numerically evaluate for generic $L$. In the next section, we will see how this integral can be trivialized with the help of the FFTLog transform. The double integral in (2.28), having smooth integrands, can then be numerically evaluated with a finite quadrature.

Contact (C). The above result was valid for any function $f(s)$. Contact separability corresponds to the special case of scalar-exchange separability, where $f(s) = s^{2n}$ with non-negative integer $n$, which is also equivalent to setting $f(s) = 1$ and $J = n$ in the $t$-channel. There are multiple ways of dealing with this case. Here we present a method that utilizes (2.28), which we find to give the most economical representation.

When $n$ is a non-negative integer the coupling integral (2.29) becomes divergent, but we can treat it as a distribution. For example, when $n = 0$, (2.29) simply becomes proportional to the delta function $2\pi^2 \delta_D(r - r')/r^2$ due to the closure relation for spherical Bessel functions. To deal with the case $n > 0$, note that $j_\ell$ satisfies a differential equation [13]

$$
D_\ell(r) j_\ell(sr) = s^2 j_\ell(sr) \quad \text{with} \quad D_\ell(r) \equiv -\partial^2_r - \frac{2}{r} \partial_r + \frac{\ell(\ell + 1)}{r^2},
$$

(2.30)

so that we can formally express the integral as

$$
J^{(s)}_L(r, r') = \frac{2\pi^2}{r^2} [D_\ell(r)]^n \delta_D(r - r').
$$

(2.31)
We can then integrate by parts to act the $D_\ell$ operator on the $r$ integrand, after which we impose the delta function to collapse the two radial integrals to a single one. Doing so, and stripping off the geometric factor, we find the super-reduced trispectrum to be

$$(C): \quad \tau_{\ell_1 \ell_2 \ell_3 \ell_4}^{l_1 t_2}(L) = \frac{1}{(2\pi)^4} \int_0^\infty dr \left[ \tilde{D}_\ell(r) \right]^n r^2 I_{l_1}^{(1)}(r) I_{l_2}^{(2)}(r) I_{l_3}^{(3)}(r) I_{l_4}^{(4)}(r), \quad (2.32)$$

where

$$\tilde{D}_\ell(r) = -\partial_r^2 + \frac{2}{r} \partial_r + \frac{\ell(\ell + 1) - 2}{r^2}. \quad (2.33)$$

The derivatives of $\tilde{D}_\ell$ can be taken either numerically or analytically as described in appendix A. Since taking many derivatives can lead to numerical instabilities, in practice the derivatives are better taken symmetrically on the 1,2- and 3,4-legs at the same time. Since (2.32) consists of a single radial integral, it has the same degree of computational complexity as the reduced bispectrum in (2.19).

As a remark, let us mention that there is an alternative way of computing the trispectrum of the contact type via the use of spin-weighted spherical harmonics. The idea is to write factors of $s^{2n}$ in terms of the dot product $k_1 \cdot k_2$, and then replace these with the radial derivatives acting on plane waves, e.g., $k_1 \cdot k_2 e^{i k_1 \cdot r} e^{i k_2 \cdot r} = -\partial_r e^{i k_1 \cdot r} \partial_r e^{i k_2 \cdot r}$, which in turn raise the spins of the spherical harmonics after projection. We give details of this method in appendix C (see [44] for the application of this method for $n = 1$). However, this method quickly becomes complicated and we find it to be not easily generalizable for $n > 1$. So in practice, it is better to use the representation (2.32).

**Spin-Exchange (spE).** In this case, we organize the momentum-conserving delta functions in the same way as (2.27), with an extra integral for $t$. As mentioned before, we can always write $t^{2J}$ in terms of a linear combination of the Legendre polynomials $P_m(k_2 \cdot k_3)$ with $m = 0, \cdots, J$. Via the addition theorem, the degree-$J$ Legendre polynomial can be written as

$$P_J(k_2 \cdot \hat{k}_3) = \frac{4\pi}{2J + 1} \sum_{m=-J}^J Y_{Jm}(k_2) Y_{Jm}^*(\hat{k}_3). \quad (2.34)$$

The presence of the extra spherical harmonics leads to a more complicated geometric factors compared to the scE-separable case when we perform the angular integrations. After a bit of algebra, we find that the super-reduced trispectrum takes the form (see also [46])

$$(\text{spE}): \quad \tau_{\ell_1 \ell_2 \ell_3 \ell_4}^{l_1 t_2}(L) = \sum_{L' \ell_1' \ell_2' \ell_3' \ell_4'} \frac{h_{\ell_1' \ell_2' \ell_3' \ell_4'}(L, J, L', \ell_1', \ell_2', \ell_3', \ell_4')}{(2\pi)^5} \times \int_0^\infty dr r^2 I_{l_1}^{(1)}(r) I_{l_2}^{(2)}(r) \int_0^\infty dr' r'^2 I_{l_3}^{(3)}(r') I_{l_4}^{(4)}(r') J_L^J(r, r '), \quad (2.35)$$

where the geometric factor is given by

$$h_{\ell_1' \ell_2' \ell_3' \ell_4'}(L, J, L', \ell_1', \ell_2', \ell_3', \ell_4') \equiv \frac{4\pi (-1)^{l_1 + l_2} \frac{1}{2}(l_1 + l_2 + l_3) + L + L' + J (2L + 1)}{2J + 1} \times \frac{g_{\ell_1' \ell_2' \ell_3' \ell_4'} g_{\ell_1 \ell_2 \ell_3 \ell_4} g_{\ell_1 \ell_2 \ell_3 \ell_4} g_{\ell_1 \ell_2 \ell_3 \ell_4}}{g_{\ell_1 \ell_2 \ell_3 \ell_4} g_{\ell_1 \ell_2 \ell_3 \ell_4}} \left\{ \begin{array}{ccc} \ell_1 & \ell_2 & L \\ J' & J & \ell_1' \end{array} \right\} \left\{ \begin{array}{ccc} \ell_3 & \ell_4 & L \\ J & J & \ell_3' \end{array} \right\}, \quad (2.36)$$
and the curly brackets denote the Wigner 6-$j$ symbol. The $g$-factors in the denominator are due to the way we defined $\tau_{\ell_1 \ell_2 \ell_3 \ell_4}(L)$ in (2.13). In the above, the integral
\begin{equation}
I^{(i)}_{\ell\ell'}(r) \equiv 4\pi \int_0^\infty d\chi W_\ell(\chi) \int_0^\infty dk \frac{k^2 f_i(k, z(\chi)) j_\ell(k\chi) j_{\ell'}(k r)}{f_i(k, z(\chi)) j_\ell(k\chi) j_{\ell'}(k r)},
\end{equation}
slightly generalizes (2.20) by allowing the two multipoles of the spherical Bessel functions to be different. Naively, one might worry about the appearance of the three extra summations in (2.35). However, it turns out most terms vanish in the sums due to the triangle conditions imposed by the 6-$j$ symbols. As a consequence, the computation is not much slower than the scalar-exchange case. The performance will be discussed in detail in section 4.4.

Note that when $f(s) = s^{2n}$ with non-negative integer $n$, (2.35) can be simplified in the same manner as the contact-separable case by writing $J_\ell(s)$ in terms of the delta function and then collapsing one of the radial integrals. In this case, (2.35) simplifies to
\begin{equation}
\tau_{\ell_1 \ell_2 \ell_3 \ell_4}(L) = \sum_{L', \ell'_{13}} \frac{h_{\ell_1 \ell_2}(L, J, L', \ell_1', \ell_3')}{(2\pi)^4} \int_0^\infty dr [\tilde{D}_L(r)] \frac{|r^2 I^{(1)}_{\ell_1 \ell_1'}(r) I^{(2)}_{\ell_2}(r) I^{(3)}_{\ell_3 \ell_3'}(r') I^{(4)}_{\ell_4}(r)}{r^{2n} I^{(4)}_{\ell_4}(r)}.
\end{equation}
As we describe in section 3, this type of trispectrum can arise from higher-derivative self-interactions of $\zeta$, which would be generated by integrating out spinning particles that couple to $\zeta$ during inflation.

### 2.3 Bessel integrals

In the previous section, we saw that a separable trispectrum leads to an integral over a product of factorized momentum integrals. The most challenging part of the computation is evaluating these momentum integrals consisting of an highly-oscillatory integrand. In this section, we describe an efficient method to compute these Bessel integrals based on the FFTLog algorithm, originally introduced in [12], and further developed in [13–16, 47].

#### 2.3.1 FFTLog

The FFTLog is defined to be a discrete Fourier transform with $N_\eta$ logarithmically-spaced sampling points in the $k$-interval $[k_{\text{min}}, k_{\text{max}}]$. Effectively, this decomposes functions into a sum of complex power-laws. For a given function $f(k, z)$, its FFTLog decomposition is given by
\begin{equation}
f(k, z) = \sum_{m = -N_\eta/2}^{N_\eta/2} c_m(z) k^{-b + i\eta_m} \quad \text{with} \quad \eta_m = \frac{2\pi m}{\log(k_{\text{max}}/k_{\text{min}})},
\end{equation}
and the coefficients $c_m$ are given by the inverse transform
\begin{equation}
c_m(z) = \frac{2 - \delta_{|m|, N_\eta/2}}{2N_\eta} \sum_{n=0}^{N_\eta-1} f(k_n, z) k_n^{b - i\eta_m} e^{-2\pi i mn/N_\eta},
\end{equation}
where the Kronecker delta ensures correct weighting factor at the end points $m = \pm N_\eta/2$. The parameter $b \in \mathbb{R}$ is inserted in order to ensure convergence of the FFTLog; see e.g. [13, 47] for more details.

\[\text{--- 13 ---}\]
In our analysis, we consider the FFTLog of two quantities: the matter power spectrum $P(k, z)$ and the transfer function $M(k, z)$. These are relevant for computing angular correlators with Gaussian and non-Gaussian initial conditions, respectively. Let us show the FFTLog of these two quantities at redshift $z = 0$ by taking $f(k, z)$ in (2.39) to be $P(k) \equiv P(k, z = 0)$ and $M(k) \equiv M(k, z = 0)$, in which case the coefficients $c_m$ are $z$-independent. We use the CLASS\footnote{https://class-code.net.} code\footnote{Obviously, we cannot trust the linear approximation for the entire interval. We nevertheless choose a sufficiently large interval to avoid ringing in the FFTLog decomposition and to ensure convergence of momentum integrals. Since the integrals have support effectively on a finite interval of $k$ in the intermediate regime, they are not highly sensitive to the way the high- and low-$k$ limits are regulated; see also [14].} to numerically compute these functions in the interval $k \in [10^{-5}, 10^{2}] h$/Mpc. Their asymptotic behaviors are given by

$$P(k) \propto \begin{cases} k^{n_s} & k \ll k_{\text{eq}} \\ k^{n_s - 4} \log^2 k & k \gg k_{\text{eq}} \end{cases},$$

where $n_s \approx 0.96$ is the spectral index and $k_{\text{eq}} \sim 10^{-2}$ Mpc$^{-1}$ is the scale corresponding to the matter-radiation equality. These behaviors set the allowed range of $b$: we find that the best convergence of the FFTLog is achieved when $b \in [0.5, 2]$ and $b \in [-0.5, -1.5]$ for $P(k)$ and $M(k)$, respectively. In figures 5 and 6, we compare these functions and their FFTLog expansions. We see that we need about $N_{\eta} = 200$ terms to require a percent-level precision for $P(k)$, whereas a less number $N_{\eta} = 100$ is required for the convergence of $M(k)$.

2.3.2 Analytic method

Let us see how the FFTLog can be used to efficiently compute angular galaxy correlators, following the method introduced in the earlier work [13]. We review their method in this section and extend to the cases involving the integral $J_{s}^{L}$ and primordial non-Gaussianity. Galaxies are measured over finite redshift bins, and due to errors in measuring their photometric redshifts, some of them may smear into other bins. This can be modeled with a

\[ \text{Figure 5. Comparison of the linear matter power spectrum and its FFTLog expansion with parameters } k_{\text{min}} = 10^{-4} \text{ h/Mpc, } k_{\text{max}} = 10^2 \text{ h/Mpc, and } b = 0.9. \text{ The bottom panel shows the relative error for } N_{\eta} = 200 \text{ in percentage.} \]
Gaussian window function

\[ W_\delta(\chi; \bar{\chi}, \sigma_\chi) \equiv \frac{1}{\sqrt{2\pi}\sigma_\chi} e^{- (\chi - \bar{\chi})^2 / 2\sigma_\chi^2}, \]

where \( \bar{\chi} \) is the mean redshift and \( \sigma_\chi \) is the width of the bin. When dealing with galaxy correlators, we will frequently encounter separable coefficient functions of the form

\[ f_i(k, z) = k^{p_i} P(k) D_g(z), \quad f(k) = k^{p_s} P(k), \]

Substituting this to (2.20) and (2.29) gives the integrals

\[ I^{(i)}(r) = 4\pi \int_0^\infty d\chi W_\delta(\chi) \int_0^\infty dk k^{2(1+p_i)} P(k) j_\ell(kr) j_\ell(k\chi), \]
\[ J^{(s)}_L(r, r') = 4\pi \int_0^\infty dk k^{2(1+p_s)} P(k) j_L(kr) j_L(kr'), \]

where \( W_\delta(\chi) \equiv D_g(z(\chi))W_\delta(\chi; \bar{\chi}, \sigma_\chi) \). For high multipoles, these integrals can be trivially done using the Limber approximation [49, 50]. The idea is to note that the spherical Bessel function \( j_\ell(x) \) becomes highly oscillatory for high \( x > \ell \) and decays fast for small \( x < \ell \), so that the integrand effectively becomes sharply peaked at \( x \sim \ell \) for high \( \ell \). This allows us to effectively replace the Bessel function as a Dirac delta function as

\[ j_\ell(x) \to \sqrt{\frac{\pi}{2\ell}} \delta_D(\ell - x) \quad \Rightarrow \quad I^{(i)}_\ell(r) \to \frac{2\pi^2}{F^2} W_\delta(r), \]
\[ J^{(s)}_L(r, r') \to \frac{2\pi^2}{L^2} \left( \frac{L}{r} \right)^{2+2p_s} P(L/r) \delta_D(r - r'). \]

\[ ^{11} \text{For spectroscopic surveys, galaxy redshifts can be measured with much greater resolution, which makes a top-hat window function a more appropriate choice.} \]
This relies heavily on the assumption that the rest of the integrand is not highly varying over the integration region, so that the Delta function approximation is valid for high multipoles.

Let us now see how the FFTLog can be used to evaluate the integral (2.44) without the Limber approximation. Substituting the FFTLog of the matter power spectrum \( P(k) = \sum_n c_n k^{-b+i\eta_n} \) to (2.44) gives [13]

\[
I_{\ell}^{(i)}(r) = \sum_n c_n \int_0^\infty d\chi \mathcal{W}_\delta(\chi) \chi^{-\nu_n} - 2p_i I_\ell(\nu_n + 2p_i, \frac{r}{\chi}), \tag{2.47}
\]

where we defined \( \nu_n \equiv 3 - b + i\eta_n \) and

\[
l_\ell(\nu, w) \equiv 4\pi \int_0^\infty dx x^{\nu-1} j_\ell(x) j_\ell(wx)
\]

\[
= \frac{2^{\nu-1}\pi \Gamma(\ell + \frac{\nu}{2})}{\Gamma(\frac{\nu}{2}) \Gamma(\ell + \frac{\nu}{2})} \ {}_2F_1\left[\frac{\nu+1}{2}, \ell + \frac{\nu}{2}, w^2\right] (|w| \leq 1), \tag{2.48}
\]

with \( {}_2F_1 \) being the hypergeometric function. The second line \( \tilde{I}_\ell(\nu, w) \) represents the analytic solution for the integral valid when \( |w| \leq 1 \), and the integral converges for \( -2\ell < \Re[\nu] < 3 \) when \( w \neq 1 \) and \( -2\ell < \Re[\nu] < 2 \) when \( w = 1 \) [51]. The function \( I_\ell(\nu, w) \) is discontinuous at \( w = 1 \), invalidating a naive analytic continuation of the hypergeometric function appearing in (2.48) beyond the unit circle \( |w| = 1 \). To compute the integral for \( w > 1 \), we simply use the scaling property of the integral\(^{12}\)

\[
I_\ell(\nu, w) = w^{-\nu} I_\ell \left( \nu, \frac{1}{w} \right), \tag{2.50}
\]

The analytic representation (2.48) is very useful, since the hypergeometric series converges very fast; we refer the reader to [13, 14, 16] for more details on efficient evaluation of (2.48). The fact that we can convert an integral over highly oscillating Bessel functions to a finite sum over hypergeometric series is what makes the FFTLog decomposition very powerful.

When choosing \( b \in [0.5, 2] \) for the convergence of the FFTLog of \( P(k) \), the spherical Bessel integral is be UV-divergent for \( p_i > 0 \). One way of dealing with the non-convergent case is by treating the integral formally as a distribution and then repeatedly acting with a differential operator on a lower-order integral as [13]

\[
l_\ell(\nu + 2n; w) = [D_\ell(w)]^n l_\ell(\nu, w), \tag{2.51}
\]

where \( D_\ell \) was defined in (2.30). We can then integrate this operator by parts, after which the derivatives act on the window function without affecting the rest of the radial integral; in other words, we have

\[
I_{\ell}^{(i)}(r) = \sum_n c_n \int_0^\infty d\chi \left[ \tilde{D}_\ell^{\nu_i}(\chi) \mathcal{W}_\delta(\chi) \right] \chi^{-\nu_n} I_\ell \left( \nu_n, \frac{r}{\chi} \right), \tag{2.52}
\]

\(^{12}\) A naive analytic continuation of the hypergeometric function in (2.48) would lead to

\[
\tilde{I}_\ell(\nu, w) = e^{-\frac{i}{2} \nu \pi} w^{-\nu} \left[ \cos \left( \frac{\pi \nu}{2} \right) I_\ell \left( \nu, \frac{1}{w} \right) - i \sin \left( \frac{\pi \nu}{2} \right) I_{\ell-1} \left( \nu, \frac{1}{w} \right) \right] (|w| \geq 1), \tag{2.49}
\]

which is the same as the right-hand side of (2.50) for even integer \( \nu \) only. Other analytic properties of the hypergeometric function can still be used as long as we stay within the unit circle.
where we have dropped the boundary terms, which are negligible for the window function \((2.42)\). Similarly, the coupling integral \(J^{(s)}_L(r, r')\) for exchange-separable trispectra (cf. \((2.28)\) and \((2.35)\)) can be expressed as

\[
J^{(s)}_L(r, r') = 4\pi \int_0^{\infty} dk k^{2+2p_s} P(k) j_L(kr) j_L(kr')
= [\mathcal{D}_L(r)]^{p_s} \sum_n c_n r^{\nu_n} \eta_1 (\nu_n, \frac{r}{r'}) .
\]

(2.54)

Notice that the operator \(\mathcal{D}_L(r)\) here depends on \(r\) instead of \(\chi\). Integrating this operator by parts will then hit \(I^{(i)}_L\), similar to \((2.32)\) but without producing a delta function. The numerical computation of the radial integrals then simply reduces to a matrix multiplication for a finite array of \(J^{(s)}_L(r, r')\).

In the case of primordial non-Gaussianity, we will deal with

\[
f_i(k, z) = k^{2p_i+\alpha_i} \mathcal{M}(k) D_g(z) ,
\]

(2.55)

with \(\alpha_i \in \{0, n_s - 4\}\), which follows from the relation \((2.6)\) and the \(\zeta\) power spectrum, \(P_\zeta(k) \propto k^{n_s-4}\). The integral \((2.20)\) then becomes

\[
I^{(i)}_L(r) = 4\pi \int_0^{\infty} d\chi W_\delta(\chi) \int_0^{\infty} dk k^{2(1+\nu_i)+\alpha_i} \mathcal{M}(k) j_L(kr) j_L(k\chi) .
\]

(2.56)

The FFTLog of the transfer function \(\mathcal{M}(k) = \sum_n \tilde{c}_n k^{-\tilde{b}+i\tilde{n}}\) then gives

\[
I^{(i)}_L(r) = \sum_n \tilde{c}_n \int_0^{\infty} d\chi W_\delta^{(1)}(\chi) \chi^{-2\nu_i-2p_i-\alpha_i} \left( \tilde{\nu}_n + 2p_1 + \alpha_i, \frac{r}{\chi} \right) ,
\]

(2.57)

where \(\tilde{\nu}_n \equiv 3 - \tilde{b} + i\tilde{n}\). For the convergence of the FFTLog, we choose \(\tilde{b} \in [-0.5, -1.5]\). This implies that the integral is UV-divergent for \(\alpha_i = 0\), which can be dealt with following the same procedure outlined above.

### 3 Shapes of trispectra

In this section, we present the expressions of the shapes of the trispectra, for both Gaussian and non-Gaussian initial conditions, and identify their separability types according to the classification introduced in the precious section. We first briefly summarize the gravitationally-induced trispectrum in section 3.1. We then describe a few physically-motivated primordial trispectra in section 3.2.

---

\(^{13}\)Note that the relation \((2.51)\) is in fact a valid identity of the hypergeometric function for any \(\nu\), i.e.

\[
\tilde{I}_L(\nu + 2n; w) = [\mathcal{D}_L(w)]^n \tilde{I}_L(\nu, w) ,
\]

(2.53)

which can be shown using the known identities for the hypergeometric function. Naively, this implies that integrating \(\mathcal{D}_L\) by part is in principle not necessary, and we can simply use the left-hand side of \((2.51)\) to deal with the UV divergence. The problem is that the hypergeometric function has a singularity \(I_\nu(\nu, w) \to (1 - w)^{2-\nu}\) as \(w \to 1\) when \(\Re[\nu] > 2\) (or a logarithmic singularity when \(\Re[\nu] = 2\)), making the line-of-sight integral very sensitive near \(\chi = r\). In general, we thus follow \((2.52)\), although it turns out that using \((2.53)\) can still approximately give the correct result when the singularity is somewhat mild, e.g. for \(\Re[\nu] \lesssim 5\).
3.1 Non-Gaussianity from gravitational evolution

Gravitational attraction is a nonlinear process. Statistics of density perturbations in the late-universe thus become non-Gaussian even if they were initially Gaussian distributed. A well-established formalism to compute the cosmological evolution of density perturbations is standard perturbation theory (SPT), see [52] for a review. Here we present the bare minimum of SPT required for describing the gravitationally-induced trispectrum at tree level, i.e. at leading order in perturbation theory.\(^{14}\)

Treating dark matter as a pressureless fluid, the evolution of the matter density field \(\delta\) and its velocity divergence \(\theta \equiv \nabla \cdot \mathbf{v}\) is described by the continuity and the Euler equations. For small \(\delta\) and \(\theta\), the equations of motion can be solved perturbatively as an expansion in the linear solution \(\delta^{(1)}(k, z) = D_g(z)\delta_0(k)\), where \(\delta_0\) is the density field at \(z = 0\). Similarly, we have \(\theta^{(1)}(k, z) = -\mathcal{H}f_g(z)D_g(z)\delta_0(k)\) where \(f_g \equiv d\log D_g / d\log a\) denotes the logarithmic derivative of \(D_g\) with respect to the scale factor \(a\). Under the approximation \(f_g \approx \Omega_m^{1/2}\), the nonlinear solutions can be written as power series in \(\delta_0\), separately for \(\delta\) and \(\theta\) as

\[
\delta(k, z) = \sum_{n=1}^{\infty} D_g(z)^n \delta^{(n)}(k, z), \tag{3.1}
\]

\[
\theta(k, z) = -\mathcal{H}\Omega_m^{1/2} \sum_{n=1}^{\infty} D_g(z)^n \theta^{(n)}(z), \tag{3.2}
\]

with the \(n\)-th order solutions given by

\[
\delta^{(n)}(k) = \int_{q_1, \ldots, q_n} (2\pi)^3 \delta_D(k - q_{1-n}) F_n^{\text{sym}}(q_1, \ldots, q_n)\delta_0(q_1)\cdots\delta_0(q_n), \tag{3.3}
\]

\[
\theta^{(n)}(k) = \int_{q_1, \ldots, q_n} (2\pi)^3 \delta_D(k - q_{1-n}) G_n^{\text{sym}}(q_1, \ldots, q_n)\delta_0(q_1)\cdots\delta_0(q_n), \tag{3.4}
\]

where \(\int_{q_1, \ldots, q_n} \equiv (2\pi)^{-3n} \int d^3q_1 \cdots d^3q_n\) and \(F_n^{\text{sym}}, G_n^{\text{sym}}\) denote symmetrization of the kernels \(F_n, G_n\) that can be computed iteratively using the formulas given in [55, 56]. For example, we have \(F_1 = G_1 = 1\) and

\[
F_2^{\text{sym}}(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{2k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2, \tag{3.5}
\]

\[
G_2^{\text{sym}}(k_1, k_2) = \frac{3}{7} + \frac{k_1 \cdot k_2}{2k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2. \tag{3.6}
\]

The computation of correlation functions of \(\delta\) proceeds in an analogous way as computing Feynman diagrams. At tree-level, there are two contributions to the matter trispectrum assuming Gaussian initial conditions. For \(\delta(k) \equiv \delta(k, z = 0)\), two different contractions yield

\[
\langle \delta(k_1)\delta(k_2)\delta(k_3)\delta(k_4) \rangle = T_{2211}(k_1, k_2, k_3, k_4) + T_{3111}(k_1, k_2, k_3, k_4), \tag{3.7}
\]

where

\[
T_{2211}(k_1, k_2, k_3, k_4) = 4F_2^{\text{sym}}(k_{12}, -k_2)F_2^{\text{sym}}(k_{12}, k_3)P(k_{12})P(k_2)P(k_3) + 11 \text{ perms}, \tag{3.8}
\]

\[
T_{3111}(k_1, k_2, k_3, k_4) = 6F_3^{\text{sym}}(k_1, k_2, k_3)P(k_1)P(k_2)P(k_3) + 3 \text{ perms}, \tag{3.9}
\]

\(^{14}\)It is well-known that SPT fails to be consistent beyond tree level. In order to continue to make sense of perturbation theory at loop level, other formalisms have been developed such as the effective field theory of large-scale structure [53, 54]. As our analysis is restricted to tree level, SPT will be sufficient for our purposes.
and \( F^\text{sym}_3 \) denotes the symmetrized version of the SPT kernel
\[
F_3(k_1, k_2, k_3) = \frac{1}{126} \left( \frac{3k_1 \cdot k_{12}}{k_{12}^2} + \frac{2(k_1 \cdot k_2)^2}{(k_1 k_2)^2} \right) \left( \frac{7k_{12} \cdot k_{123}}{k_{12}^2} + \frac{(k_1 \cdot k_3)^2}{k_{12}^2} \right) + \frac{1}{18} \left[ \frac{k_1 \cdot k_{123}}{k_{123}^2} \left( \frac{5k_2 \cdot k_{23}}{k_{23}^2} + \frac{(k_2 \cdot k_3)^2}{(k_2 k_3)^2} \right) + \frac{(k_1 \cdot k_{23})^2}{7k_{12}^2} \left( \frac{3k_2 \cdot k_{23}}{k_{23}^2} + \frac{(2k_2 \cdot k_3)^2}{(k_2 k_3)^2} \right) \right],
\]
with \( k_{i_1 \cdots i_n} \equiv |k_{i_1} + \cdots + k_{i_n}|. \) To see what separability class this trispectrum falls into, we first convert the dot products \( k_i \cdot k_j \) to diagonal momenta. For \( F^\text{sym}_2(k_{12}, -k_2) \) in (3.8), we can express it as
\[
28 F^\text{sym}_2(k_{12}, -k_2) = \left( 3k_2^2 - 5k_1^2 + \frac{2k_2^4}{k_1^2} \right) \frac{1}{s^2} + 10 + \frac{3k_2^2}{k_1^2} - \frac{5s^2}{k_1^2}.
\]
Similarly, \( F^\text{sym}_3(k_{12}, k_3) = F^\text{sym}_2(-k_{34}, k_3) \) also depends on \( s \) but not \( t \). Combined with \( P(k_{12}) = P(s) \) in (3.8), we see that \( T_{2211} \) is scalar-exchange separable. Now consider \( T_{3111} \). Naively, \( F_3(k_1, k_2, k_3) \) contains products of different permutations of dot products, so that there could be terms that depend on two diagonal momenta at the same time. However, using momentum conservation one can show that each term in \( T_{3111} \) depends no more than one diagonal momentum. We therefore see that the gravitationally-induced matter trispectrum is scalar-exchange separable.

It turns out that the unsymmetrized kernel \( F_3(k_1, k_2, k_3) \) depends both on \( s \) and \( t \), even though there are no products between these factors, so that it is still scalar-exchange separable. For the purpose of computing (reduced) angular trispectra, it will be slightly more convenient to rearrange the \( F_3 \) kernel in a way that makes the symmetry between different channels more manifest. To this end, we define a related kernel \( \hat{F}_3 \), which when symmetrized gives the same result as \( F_3 \), i.e. \( F^\text{sym}_3(k_1, k_2, k_3) = \hat{F}^\text{sym}_3(k_1, k_2, k_3) \), but each permutation of which depends only on a single internal momentum. We give its precise definition in (B.13), and use this basis of kernel henceforth.

### 3.2 Non-Gaussianity from initial conditions

We now consider the shapes of a few primordial trispectra generated during inflation (see [57] for a recent review). When evolved to late times, the \( \zeta \) trispectrum is related to the matter trispectrum by
\[
\langle \delta(k_1, z_1) \cdots \delta(k_1, z_1) \rangle' = M(k_1, z_1) \cdots M(k_4, z_4) \langle \zeta(k_1) \cdots \zeta(k_4) \rangle',
\]
at tree level. Since multiplying by transfer functions does not induce any diagonal momentum dependence, here we discuss separability types of primordial trispectra.

**Local shape.** A simple parameterization of non-Gaussianity is given by a local expansion of Gaussian random fields in real space
\[
\zeta(x) = \zeta_G(x) + \frac{3}{5} f_{NL}(\zeta_G^2(x) - \langle \zeta_G^2(x) \rangle) + \frac{9}{25} g_{NL}(\zeta_G'(x) - \langle \zeta_G'(x) \rangle),
\]
with Gaussian \( \zeta_G \). These terms lead to two contributions to the local trispectrum given by
\[
\langle \zeta^4 \rangle_{fNL} = \tau_{NL} \left[ P(\zeta(k_1)P(\zeta(k_3)P(\zeta(|k_1 + k_2|) + 11 \text{ perms}) \right],
\]
\[
\langle \zeta^4 \rangle_{gNL} = \frac{54}{24} f_{NL} \left[ P(\zeta(k_1)P(\zeta(k_2)P(\zeta(k_3) + 3 \text{ perms}) \right],
\]
where $\langle \zeta^4 \rangle \equiv \langle \zeta(k_1) \cdots \zeta(k_4) \rangle$. The current observational bounds from the CMB\(^{15}\) are $g_{\text{NL}} = (-5.8 \pm 6.5) \times 10^4$ (68% C.L.) \(^{61}\) and $\tau_{\text{NL}} < 2.8 \times 10^3$ (95% C.L.) \(^{62}\). From their momentum dependence, we see that the $\tau_{\text{NL}}$ trispectrum is scalar-exchange separable, while the $g_{\text{NL}}$ trispectrum is contact separable.

In single-field models of inflation, a nonzero bispectrum in the squeezed limit (at which the local shape peaks) necessarily generates the trispectrum of the $\tau_{\text{NL}}$ shape whose amplitude saturates\(^{16}\) the Suyama-Yamaguchi bound $\tau_{\text{NL}} \geq (\frac{\tau_2}{f_{\text{NL}}})^2$ \(^{63}\), while $g_{\text{NL}}$ is an independent variable that characterizes quartic self-interactions of the inflaton field. In contrast, certain non-single-field models — such as multi-field inflation \(^{64}\), quasi-single-field inflation \(^{65}\) or models with higher-spin fields \(^{66}\) — can generate a large trispectrum with $\tau_{\text{NL}} \gg (\frac{\tau_2}{f_{\text{NL}}})^2$.

In the large-scale structure, these models can be constrained by stochasticity in the bias expansion \(^{67,68}\) (see also \(^{69,70}\)).

**Equilateral shape.** Another category of primordial trispectra involves the shapes generated by quartic self-interactions in the inflationary action. At leading order in derivatives, the three quartic interactions that contribute to the trispectrum are $\dot{\sigma}^4$, $\dot{\sigma}^2(\partial_i \sigma)^2$, and $(\partial_i \sigma)^4$ in the effective field theory of inflation \(^{71}\), where $\sigma$ is some additional light scalar \(^{72}\). These lead to the following shapes of the trispectrum \(^{44}\): \(^{17}\)

$$
\begin{align*}
\langle \zeta^4 \rangle_{\text{eq,1}} &= \frac{221184}{25} g_{\text{NL}} \frac{1}{k_1 k_2 k_3 k_4 (k_1 + k_2 + k_3 + k_4)^5}, \\
\langle \zeta^4 \rangle_{\text{eq,2}} &= -\frac{27648}{325} g_{\text{NL}}^2 \frac{k_1^2 + 3(k_3 + k_4)k_4 + 12k_3 k_4}{k_1 k_2 (k_3 k_4)^3 (k_1 + k_2 + k_3 + k_4)^5} (k_3 \cdot k_4) + 5 \text{ perms}, \\
\langle \zeta^4 \rangle_{\text{eq,3}} &= \frac{165888}{2575} g_{\text{NL}}^3 \frac{2k_1^4 - 2k_1^2 \sum_i k_i^2 + k_1 \sum_i k_i^2 + 12k_1 k_2 k_3 k_4}{(k_1 k_2 k_3 k_4)^3 (k_1 + k_2 + k_3 + k_4)^5} ((k_1 \cdot k_2)(k_3 \cdot k_4) + 2 \text{ perms}),
\end{align*}
$$

whose amplitudes are normalized such that $\langle \zeta^4 \rangle = \frac{216}{25} g_{\text{NL}} P_\zeta(k)^3$ in the tetrahedral configuration where $k_i = k$ and $\hat{k}_i \cdot \hat{k}_j = -1/3$ for $i \neq j$. As before, the factors of $(k_1 + k_2 + k_3 + k_4)^5$ in the denominator can be made separable using the trick (2.17), after which the shapes become contact separable.\(^{17}\) In single-field models, a large trispectrum may also be generated from higher-derivative interactions \(^{75–82}\). Trispectra in these models contain higher powers of momentum dot products and generally also fall into the contact-separable type.

\(^{15}\)See \(^{42,58–60}\) for the construction of trispectrum estimators.

\(^{16}\)This holds for models in which the dominant contribution to non-Gaussianity is given by a single, non-gravitational source.

\(^{17}\)In this case, we would have factors such as $f_i(k) = k^\nu e^{-\kappa n}$ appearing in the momentum integrals. It turns out that there exists an analytic formula for the Bessel integral in this case as well, which is given by \(^{73,74}\)

$$
J_{\ell \ell'}(\nu, \alpha, w) \equiv 4\pi \int_0^\infty dx x^{\nu-1} e^{-\alpha x} J_\ell(x) j_{\ell'}(wx) = \frac{w^{\nu}}{2^{\nu+\ell+\ell'}} \frac{\Gamma(\nu + 2m) \Gamma(\ell + \ell' + \nu + 2m)}{\Gamma(\ell + \nu + 2m) \Gamma(\ell' + \nu + 2m)} 2F_1 \left[-\ell - m, -m, \frac{1}{1 + \ell'}, w^2 \right], \quad (|w| \leq 1, \alpha > 1).
$$

For integer $\ell, \ell'$, the hypergeometric function above can be identified as the Jacobi polynomial, i.e. the summand above is a finite polynomial of $w^2$. Unfortunately, this analytic formula is not valid for small $\alpha \lesssim 1/x$, from which the integral actually receives a dominant contribution. In practice, this means that the above formula should be used in conjunction with ordinary numerical integration within a small range. Nevertheless, the formula (3.17) is still useful, since it allows us to circumvent the oscillatory part of the integral.
Exchange shape. When there are extra particles that couple to the inflaton during inflation, they can produce distinct shapes of non-Gaussianity that carry information about the masses and spins of the particles. This allows a model-independent way of constraining the particle spectrum during inflation, akin to collider searches for new particles in particle accelerators. The physics of these non-Gaussian correlators was highlighted in [83] and the resulting phenomenology has been greatly explored, see e.g. [65, 66, 70, 84–99].

An analytic expression for the exchange four-point function in inflation was presented in [45, 100], assuming weak couplings to the inflaton. Schematically, it has the following structure in the s-channel:

\[
\langle \zeta_4 \rangle_{ex}' = \frac{g^2 s^3}{(k_1 k_2 k_3 k_4)^{3/2}} \sum_{m=0}^{J} \Pi^{(J,m)}(\alpha, \beta, \tau) U^{(J,m)}_{12}(\partial_u, u, \alpha) U^{(J,m)}_{34}(\partial_v, v, \beta) \hat{F}(u, v), \tag{3.18}
\]

with the kinematic variables defined by

\[
u \equiv \frac{s}{k_1 + k_2}, \quad v \equiv \frac{s}{k_3 + k_4}, \quad \alpha \equiv k_1 - k_2, \quad \beta \equiv k_3 - k_4, \quad \tau \equiv (k_1 - k_2) \cdot (k_3 - k_4), \tag{3.19}
\]

and a coupling constant \( g \). In the above, \( \Pi^{(J,m)} \) is a polarization structure, which depends on the angular variable \( \tau \) as \( \Pi^{(J,m)} \sim \tau^m \), and \( U^{(J,m)}_{ij} \) are second-order differential operators that acts on the seed function \( \hat{F}(u, v) \) that encodes the scalar-exchange shape. From the relation \( \tau = k_1^2 + k_2^2 + k_3^2 + k_4^2 - s^2 - 2t^2 \), we see that the spin-\( J \) exchange trispectrum in the s-channel is a degree-\( J \) polynomial in \( t^2 \).

The exchange trispectrum (3.18), being a function of \( u \) and \( v \), is not manifestly separable. However, its functional form is dramatically simplified in certain kinematic configurations: in the collapsed limit \( s \to 0 \), the operators \( U^{(J,m)}_{ij} \) become trivial, and the shape dependence reduces to

\[
\langle \zeta_4 \rangle_{ex}' \xrightarrow{s \to 0} \frac{g^2 s^3}{(k_1 k_3 s)^{3/2}} \left( \frac{s^2}{k_1 k_3} \right)^{2+i\mu} \sum_{\lambda=0}^{J} c_{J,\lambda}(\mu) Y_{J,\lambda}(\hat{k}_1) Y_{J,\lambda}^*(\hat{k}_3) + c.c., \tag{3.20}
\]

where \( c.c. \) stands for complex conjugate. This has a clear physical interpretation as indicating particle production during inflation, where the parameter \( \mu \sim M/H \) refers to the mass of the particle in Hubble units during inflation. The mass-dependent coefficient \( c_{J,\lambda}(\mu) \) is fixed by conformal symmetry [45, 83], which goes as \( c_{J,\lambda}(\mu) \sim e^{-\pi\mu} \) for \( \mu \gg 1 \).\(^{18}\) Away from the collapsed limit, the shape becomes dominated by the equilateral shapes in (3.16) for large masses. For data analysis purposes, it is then convenient to approximate the exchange trispectrum with the following template given by a sum over two contributions as

\[
\langle \zeta_4 \rangle_{ex}' \approx \langle \zeta_4 \rangle_{ex}' \bigg|_{s \to 0} + r(\mu) \langle \zeta_4 \rangle'_{eq}, \tag{3.21}
\]

where \( T_{eq} \) is a linear combination of the equilateral trispectra in (3.16) with some relative coefficient \( r(\mu) \) that goes as \( r(\mu) \sim 1/\mu^2 \) for \( \mu \gg 1 \). See [95] for a similar template constructed for the bispectrum. This is a good approximation in the large-mass regime, but breaks down for \( \mu \lesssim 1 \), in which case the collapsed-limit shape receives corrections from a tower of higher-derivative shapes. In this regime, one should instead use the full shape given by (3.18).

\(^{18}\)In the effective field theory of inflation context, these coefficients are fixed in terms of the propagation speeds of individual helicity modes [101].
4 Angular galaxy trispectrum

In this section, we compute the angular galaxy trispectrum using the FFTLog-based method described in section 2. The computation of the angular trispectrum beyond the Limber approximation in this section is new to this paper and have not been computed before.\(^{19}\) We briefly review the cubic bias expansion of galaxy density fields in section 4.1, and describe how to implement redshift space distortion (RSD) in section 4.2. We present the shapes of the angular galaxy trispectrum with and without non-Gaussian initial conditions in section 4.3.

4.1 Cubic bias

We do not observe the distribution of matter density directly, but rather that of its tracers such as galaxies. One way of relating the galaxy density field \(\delta_g\) to that of matter is to express the former as a local functional of operators that characterize the underlying matter density. The \(n\)-th order galaxy density field is then given by an expansion in a set of operators

\[
\delta_g^{(n)}(x, z) = \sum_{\mathcal{O} \in \mathcal{O}_n} b_{\mathcal{O}} \mathcal{O}^{(n)}(x, z), \tag{4.1}
\]

where the superscript of \(\mathcal{O}^{(n)}\) indicates that it is \(n\)-th order in the linear density \(\delta^{(1)}\), \(\mathcal{O}_n\) denotes a set of independent \(n\)-th order operators, and the coefficients \(b_{\mathcal{O}}\) are called bias parameters or simply biases, which are redshift-dependent in general.

The operators that appear in the bias expansion can be classified by the number of fields and their derivatives. A set of independent operators that appear at cubic order is \([104, 105]\)

\[
\mathcal{O}_3 = \{ \delta, \delta^2, \mathcal{G}_2(\Phi_g), \delta^3, \mathcal{G}_2(\Phi_g)\delta, \mathcal{G}_3(\Phi_g), \Gamma_3 \}, \tag{4.2}
\]

where \(\Gamma_3 \equiv \mathcal{G}_2(\Phi_g) - \mathcal{G}_2(\Phi_v)\) and \(\mathcal{G}_i\) are the Galileon operators defined by

\[
\mathcal{G}_2(\Phi_g) \equiv (\nabla_i \nabla_j \Phi_g)^2 - (\nabla^2 \Phi_g)^2, \tag{4.3}
\]

\[
\mathcal{G}_3(\Phi_g) \equiv \frac{3}{2}(\nabla_i \nabla_j \Phi_g)^2 \nabla^2 \Phi_g - (\nabla_i \nabla_j \Phi_g)(\nabla_j \nabla_k \Phi_g)(\nabla_k \nabla_i \Phi_g) - \frac{1}{2}(\nabla^2 \Phi_g)^3, \tag{4.4}
\]

with \(\Phi_g \equiv \nabla^{-2} \delta\) and \(\Phi_v \equiv \nabla^{-2} \theta\) being the gravitational and velocity potentials, respectively. Of course, the choice of this set is not unique, and one could equally choose a set given by a linear combinations of the operators in (4.2). The relation to some other sets of operators that are also used in the literature is described in appendix B.

Using the bias expansion, we find that the tree-level galaxy trispectrum with Gaussian initial conditions has the form

\[
\langle \delta_g(k_1, z_1) \cdots \delta_g(k_4, z_4) \rangle' = T_{2111}^g(k_1, k_2, k_3, k_4) + T_{3111}^g(k_1, k_2, k_3, k_4), \tag{4.5}
\]

\(^{19}\)In addition to the galaxy clustering trispectrum, there are other types of angular trispectra in the large-scale structure that are also sourced by gravitational nonlinearities such as the lensing trispectrum, studied e.g. in \([102, 103]\). Combining the information from clustering and lensing correlation functions (as well as their cross-correlations) is important for extracting optimal cosmological constraints.
with two contributions given by

\[
T_{2211}^g(k_1, \ldots, k_4) = 4b_3^2D_2D_3D_4^2P(k_1)P(k_3)P(s)\left(b_3F_2^{\text{sym}}(k_1, -k_{12}) + b_{12}^2 + b_{23}^2\sigma_{k_1, -k_{12}}^2\right)
\times \left(b_3F_2^{\text{sym}}(k_3, k_{12}) + b_{12}^2 + b_{23}^2\sigma_{k_3, k_{12}}^2\right) + 11 \text{ perms},
\]

\[
T_{3111}^g(k_1, \ldots, k_4) = b_3^3D_1D_2D_3D_4^2P(k_1)P(k_2)P(k_3)
\times \left(6b_3F_3^{\text{sym}}(k_1, k_2, k_3) + b_{12}^3 - b_{23}^3(k_1 \cdot k_2)(k_2 \cdot k_3)(k_3 \cdot k_1)\right)
\]

\[
+ \left[b_{23}^2 + \frac{3}{2}b_{3}\sigma_{k_1,k_2}^2 + 2(b_{12}^2 + b_{3})\sigma_{k_3,k_2}^2F_2^{\text{sym}}(k_1, k_2),
\] (4.7)

\[
+ 2b_{12}^2F_2^{\text{sym}}(k_1, k_2) - 2b_3\sigma_{k_1,k_2}^2G_2^{\text{sym}}(k_1, k_2) + 5 \text{ perms}\right]\right) + 3 \text{ perms},
\]

where \(D_i \equiv \Gamma_g(z_i)\) and \(\sigma_{k,q}^2 \equiv (\hat{k} \cdot \hat{q})^2 - 1\). Let us classify the terms that appear in the trispectrum according to the types of separability we introduced in section 2.2.2. First of all, we see that terms that arise from \(T_{2211}^g\) all have a nontrivial dependence on the internal momentum, \(s = |k_{12}|\) for the particular permutation shown above. In particular, \(T_{2211}^g\) does not depend on \(t\), implying that it is scalar-exchange separable. The only spin-exchange separable term is that due to \(b_{23}^3\) in \(T_{3111}^g\) that depends on two independent angles, with the rest being either contact or scalar-exchange separable depending on their momentum dependence. We summarize the separability classes of the trispectra from different bias parameters (as well as that of primordial non-Gaussianity) in table 1.

### 4.2 Redshift space distortion

What we actually measure in galaxy surveys are the fluctuations \(\Delta(\hat{n}, z)\) in galaxy number counts \(N(\hat{n}, z)\) defined by

\[
\Delta(\hat{n}, z) \equiv \frac{N(\hat{n}, z) - \langle N(\hat{n}, z) \rangle}{\langle N(\hat{n}, z) \rangle}.
\] (4.8)

This is a gauge-invariant quantity that should encapsulate all relativistic effects. At leading order in perturbation theory, this is related to the density fluctuation in Fourier space as

\[
\Delta^{(1)}(\hat{k}, z) = \delta^{(1)}(\hat{k}, z) \left[b_3 + f_g(z)(\hat{k} \cdot \hat{n})^2\right].
\] (4.9)

This correction is the standard redshift-space distortion term that arises from peculiar velocities of galaxies, known as the Kaiser effect [106]. The full inclusion of all relativistic effects can be found in [107–110], but here we consider a simple prescription by just keeping this
The final angular correlator is dimensionless. For \( f_i(k) = 1 \) and \( f_i(k) = P(k) \), the window function is centered at \( z = 1 \) with \( \sigma_z = 0.1 \).

RSD term. The correction in (4.9) leads to a change in the \( I^{(i)}_\ell \) integral (2.20) as [13]

\[
I^{(i)}_\ell (r) \rightarrow 4\pi \int_0^\infty d\chi W_g(\chi) \int_0^\infty dk k^2 \left[b_3 j_\ell(k\chi) - f_3(\chi) j_\ell''(k\chi)\right] j_\ell(kr)f_i(k) \\
= 4\pi \int_0^\infty d\chi \left[b_3 \tilde{D}_\ell(\chi)W_g(\chi) - (f_3 \cdot W_g)''(\chi)\right] \int_0^\infty dk j_\ell(k\chi) j_\ell(kr)f_i(k),
\]

(4.10)

where in the second line we have integrated by parts to trade the derivatives of the spherical Bessel function with that of the window function, and \( (f_3 \cdot W_g)''(\chi) \equiv \frac{d^2}{d\chi^2}(f_3(\chi)W_g(\chi)) \).

When \( f_i(k) = k^{2p_i} \), the integral simplifies. Using the identity

\[
4\pi \int_0^\infty dk j_\ell(k\chi) j_\ell(kr) = \frac{\pi^2}{\ell + \frac{3}{2}} \left(\frac{\chi}{r}\right)^\ell (\chi < r),
\]

(4.11)

we have

\[
I^{(i)}_\ell (r) \rightarrow \begin{cases} 
\frac{2\pi^2}{r^{\ell+1}} \tilde{D}^{p_i-1}_\ell (r) \left[b_3 \tilde{D}_\ell(r)W_g(r) - (f_3 \cdot W_g)''(r)\right] & p_i > 0 \\
\frac{2\pi^2}{\ell^2} b_3 W_g(r) - \frac{\pi^2}{\ell + \frac{3}{2}} \int_0^1 dx x^{\ell} \left[(f_3 \cdot W_g)''(rx) + \frac{(f_3 \cdot W_g)''(r/x)}{x^4}\right] & p_i = 0
\end{cases}
\]

(4.12)

In figure 7, we show the integral \( I^{(i)}_\ell \) with and without RSD for \( f_i(k) = 1 \) and \( f_i(k) = P(k) \). Notice that the two different integrals have different units, but these cancel out in the end so that the final angular correlator is dimensionless. For \( f_i(k) = 1 \), the integral without RSD simply collapses into a window function and is therefore independent of \( \ell \). We see that the RSD effect becomes suppressed at high multipoles, as can also be seen from (4.12). When \( f_i(k) = P(k) \), the overall shape of the integral still looks like a window function. Again, the RSD effect shrinks for high multipoles.
4.3 Shapes of angular trispectra

We now consider the shape of the galaxy trispectrum in angular space. To better illustrate the shape contribution of individual terms, we compute the super-reduced trispectrum in the $s$-channel. For example, the contribution of the bias parameter $b_\delta$ in (4.7) to the reduced trispectrum is

$$\tau^\delta_{2211}(k_1, k_2, k_3, k_4) = 2b_\delta^4 D_1^2 D_2^2 D_3 D_4 P(k_1) P(k_3) P(s) \left[ F^\text{sym}_2(k_1, -k_4) F^\text{sym}_2(k_3, k_4) \right],$$  \hfill (4.13)

$$\tau^\delta_{3111}(k_1, k_2, k_3, k_4) = \frac{1}{6} b_\delta^4 D_1 D_2 D_3^3 P(k_1) P(k_2) P(k_3) \left[ \hat{F}_3(k_1, k_2, k_3) \right].$$  \hfill (4.14)

From (3.11), we see that $\tau_{2211}$ is a linear combination of terms

$$\tau^\delta_{2211}(k_1, k_2, k_3, k_4) \propto \left[ D_1 k_1^{2p_1} P(k_1) \right] \left[ D_2 k_2^{2p_2} P(k_2) \right] \left[ D_3^2 k_3^{2p_3} \right] \left[ D_4^2 k_4^{2p_4} \left[ s^{2p_5} P(s) \right] \right],$$  \hfill (4.15)

with $p_1, p_2 \in \{-1, 0, 1\}$, $p_3, p_4 \in \{0, 1, 2\}$, and $p_5 \in \{-2, -1, 0, 1, 2\}$. Note that we have terms that go as $s^{-4} P(s)$, so that the integral over $s$ is naively IR-divergent for $L = 0$. This is because the integrand goes as $s^2 j_L(r s) j_L(r' s) s^{-4} P(s) \sim s^{-b-2+2L}$ as $s \to 0$; for $b \in [0.5, 2.0]$ that is required for the convergence of the FFTLog decomposition, the integral diverges for $L = 0$. However, this divergence is spurious, since we have not done the radial integrals that impose the momentum conservation between $s$ and other momenta. This suggests that these divergences could be removed by shifting factors of $k_i^2$ and $s^2$ amongst different integrals that would render the final integral finite, similar to the way we used to deal with the UV divergences in section 2.3.2. We verify in appendix A that this IR divergence can indeed be removed in this way.

The expression (B.13) of $F^\text{sym}_3$ implies that $\tau^b_{3111}$ consists of terms as

$$\tau^\delta_{3111}(k_1, k_2, k_3, k_4) \propto \left[ D_1 k_1^{2p_1} P(k_1) \right] \left[ D_2 k_2^{2p_2} P(k_2) \right] \left[ D_3 k_3^{2p_3} P(k_3) \right] \left[ D_4^3 k_4^{2p_4} \right] \left[ s^{2p_5} \right].$$  \hfill (4.16)

Again, the momentum integrals in this case become divergent for certain values of $p_i$, which can be dealt with in the same way as the above case. Note that the dependence on the diagonal momentum simply involves integer powers of $s$. In the special case $p_s = 0$, the $s$ integral just becomes delta function, and the double radial integral collapses to a single integral. When $p_s = -1$, we can use the analytic formula for the two Bessel integral

$$4\pi \int_0^\infty ds j_L(r s) j_L(r' s) = \frac{\pi^2}{L + \frac{1}{2}} \left( \frac{r}{r' \tau} \right)^L \left( \frac{r}{r'} \right) \left( \frac{1}{r' \tau} \right).$$  \hfill (4.17)

For $p_s > 1$, we lower $p_s$ by the use of the operator $D_\ell$. It is straightforward to read off the super-reduced trispectra for other bias parameters, which can be analyzed in the same way as above. A complete list of all of them can be found in appendix B.

Figure 8 shows plots of the angular galaxy trispectrum from all cubic bias parameters in the equilateral configuration, using a Gaussian window function centered at redshift $z = 1$ with $\sigma_z = 0.1$. The trispectrum for $z = 2$ looks very similar, with slightly lower amplitude and shifted scales, and hence we do not show it explicitly. Also not shown are the cross-terms between $b_5, b_4$, and $b_7$. These are given in (B.18)–(B.20) and are straightforward to add. In generating the plots, we assumed constant bias parameters, and normalized the trispectrum by setting $b_O = 1$ for each bias and $b_\delta = 1$ in all cases. The Limber approximation is used for both the external and internal multipoles, replacing $I^{(i)}_\ell$ and $J^{(x)}_L$ as in (2.6). We have chosen sufficiently large sampling points $N_r = 100$ for the radial integrals, so that the
results faithfully represent the true shapes, but not so large that they have converged within 1% accuracy over all multipoles. For instance, we find that the FFTLog method with $N_r = 100$ agrees with a brute-force method of performing the numerical multi-dimensional integrals at the level of 1% for low $\ell$’s, while the convergence is not reached for high $\ell$’s. As a consequence, we find a quantitative difference between the Limber and non-Limber results at high multipoles, which can differ up to a factor of two. However, we emphasize that this is only a numerical artifact, and the discrepancy indeed goes away upon increasing the sampling points up to e.g. $N_r = 400$. We will say more about the precision of the computation in the next section. Apart from this, the Limber approximation works well until it breaks down for small $\ell$, except for $b_G^2 \delta$ for which the approximation fails at almost all scales. The effect of RSD shows up at low $\ell$ as expected, but leads to a rather small amplitude difference.

It is straightforward to compute the trispectrum with non-Gaussian initial conditions. For concreteness and simplicity, let us consider primordial non-Gaussianity of the local type. Figure 9 shows plots of the tree-level angular galaxy trispectrum for two types of local non-Gaussianity — eqs. (3.14) and (3.15) — for different multipole configurations, with $\tau_{NL} = 10^3$, $g_{NL} = 10^4$, and $b_3 = 1$. To show the plot with multipole ranging up to $10^3$, we set the reference multipoles to $\tilde{\ell} = 500$. In the two soft limits, the two trispectra behave differently: while they both grow as $\ell_1 \to 0$ due to the presence of $P_\zeta(k_1)$ in the primordial trispectrum, only

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Angular galaxy trispectrum in the equilateral configuration, $\ell_i = L = \ell$. The trispectrum is evaluated with the window function centered at redshift $z = 1$ with $\sigma_z = 0.1$. We used $N_\eta = 200$ terms in the FFTLog expansion of $P(k)$ with $b = 1.9$, and $N_\chi = 50$, $N_r = 100$ sampling points to numerically evaluate the integrals. In the upper panel, we compare the angular trispectrum computed using the FFTLog method (solid line) and the Limber approximation (dashed line). In the lower panel, we compare the angular trispectrum with (dot-dashed line) and without RSD (solid line), both computed using the FFTLog method. Different colors indicate the trispectra computed with the bias parameter $b_O = 1$ for each $O \in O_3$.}
\end{figure}
Figure 9. Angular galaxy trispectrum (with no RSD) from local primordial non-Gaussianity with $\tau_{NL} = 10^3$ (black) and $g_{NL} = 10^4$ (green). The solid and dashed lines are computed with Gaussian window functions centered at $z = 1, 2$ with $\sigma_z = 0.1, 0.2$, respectively. In the upper panels, the left (right) plot shows the equilateral (collapsed) configurations. In the lower panels, the left (right) plot shows the soft $\ell_1 (\ell_2)$ limit. We used $N_\eta = 100$ terms in the FFTlog expansion of $\mathcal{M}(k)$ with $b = -1.1$.

the $g_{NL}$ shape grows in the $\ell_2 \to 0$ limit. This is somewhat misguiding, since it is an artifact of just looking at a single permutation; the full $\tau_{NL}$ trispectrum should grow in any $\ell_i \to 0$ limit. The difference between the shapes is instead most pronounced in the collapsed limit. We see that the $\tau_{NL}$ shape grows as $L \to 0$, while the $g_{NL}$ shape stays constant. This is easy to see from the primordial trispectrum, since $g_{NL}$ doesn’t depend on the internal momentum. For equilateral configurations, both trispectra take similar shapes.

4.4 Performance and precision

An order of estimate for the computational cost of the angular trispectrum $\tau_{\ell_3 \ell_4}(L)$ in the scE-separable case for each multipole configuration is

$$N = \mathcal{O}(N_r N_\eta N_r^2 N_\chi),$$

where $N_r$ and $N_\chi$ are the number of sampling points for numerically computing the $r$ and $\chi$ integrals, respectively, $N_\eta$ is the number of frequencies in the FFTlog decomposition, and $N_r$ is the number of separable terms in the trispectrum. Typically, $N_\chi \sim \mathcal{O}(50)$, $N_r \sim \mathcal{O}(100)$ and $N_\eta \sim \mathcal{O}(100)$ terms are required for convergence, whereas $N_r$ differs from term to term, and ranges between 1 and $\mathcal{O}(100)$ depending on the number of terms in the bias operator considered. For the contact-separable case, the scaling reduces to $N = \mathcal{O}(N_r N_\eta N_r N_\chi)$. We summarize the runtime and the parameters used for evaluating the angular galaxy trispectrum for a single configuration from different bias operators in table 2.
Table 2. Parameters and performance results for different bias operators. The last column denotes the approximate time for evaluating the trispectrum at a single multipole configuration using our Mathematica code on a laptop with Intel Core i9 CPU @ 2.3 GHz core.

| Bias | Type | $N_r$ | $N_\eta$ | $N_r'$ | $N_\chi$ | $\tau_{\ell_1\ell_2}'(L)$ |
|------|------|------|---------|-------|---------|------------------|
| $b_\delta$ | scE | 76 | 200 | 100 | 100 | 50 | 2.5 min |
| $b_{g_2}$ | scE | 7 | 200 | 100 | 100 | 50 | 15 sec |
| $b_{g_3}$ | C | 1 | 200 | 100 | 0 | 50 | 1 sec |
| $b_{G_2}$ | scE | 72 | 200 | 100 | 100 | 50 | 2.5 min |
| $b_{G_2\delta}$ | C | 6 | 200 | 100 | 0 | 50 | 10 sec |
| $b_{\Gamma_3}$ | scE | 36 | 200 | 100 | 100 | 50 | 1 min |
| $b_{G_3}$ | spE | 31 | 200 | 100 | 100 | 50 | 4 min |

Let us give some further remarks on our parameter choices. For exchange-separable trispectra, we chose $N_r = 100$ to generate the plots, which was enough to for the shapes to have sufficient converged over a wide range of (intermediate) multipoles. However, in order to reach convergence within 1% accuracy over the entire multipole range, we require a much higher number of sampling points of about $N_r \sim 400$. This has to do with our current sampling scheme, in which we sample an $N_r \times N_r$ grid of equally-spaced points from the domain of a two-dimensional integral. In any numerical integration, the sampling scheme should be carefully chosen in order to faithfully represent the integral. In our one-dimensional problem, the $\chi$-integrand of $I^{(i)}_\ell(r)$ is dictated by the Gaussian window function, which is peaked at $\chi = \bar{\chi}$ with a width $\sigma_\chi$. We find that about $N_\chi \sim 50$ is enough to sufficiently sample this integral. The resulting function $I^{(i)}_\ell(r)$ then inherits this shape, which is also peaked around $r = \bar{r}$ with the same width, as shown in figure 7. Naively, this suggests that the $(r,r')$-integrand should follow a bivariate Gaussian shape centered at $r = r' = \bar{\chi}$ with radius $\sigma_\chi$. This would be true if the two integrals are factorized, but is obviously false in our problem due to the presence of the coupling integral $J^{(s)}_L(r,r')$. The integral has no knowledge about $\bar{\chi}$, and is instead peak at $r = r'$. In figure 10, we show the behavior of the coupling integral in the two-dimensional plane around the center $(r,r') = (\bar{r}, \bar{r})$ in units of $\sigma_\chi$, corresponding to our canonical choice $\bar{z} = 1$ and $\sigma_z = 0.1$. We see that the two-dimensional integral is dominated along the line $r = r'$ and quickly diminishes away from it. This behavior becomes more extreme for higher $L$, which implies that sampling for the double radial integral can be done almost one-dimensionally. Indeed, the Limber approximation is the limit in which the domain of integration precisely reduces to the line $r = r'$. We have not optimized the integration scheme, and simply used a square grid of points to sample the integral, implemented as an $N_r \times N_r$ matrix multiplication. This can be rather cost-ineffective for large $N_r$, and a better sampling scheme can be implemented to effectively reduce the computational cost of the radial integration for high $\ell$.

For a given redshift, an angular trispectrum is described by five independent degrees of freedom. Using $N^5 \ell$-bins, there would be a total of $\mathcal{O}(N^5)$ trispectrum configurations to

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20In the statistics context, there is a well-known method to efficiently sample correlated variables with multivariate normal distributions using the Cholesky decomposition of the covariance matrix. Our problem is similar, with the coupling integral playing the role of the covariance matrix with a strong positive correlation.
Figure 10. Contour plots of the coupling integral $J_L^{(s)}(r, r')$ with $p_s = -1$ for $L = 10$ and 50. The two axes represent the distance away from the point $(r, r') = (\bar{\chi}, \bar{\chi})$ in units of $\sigma_{\chi}$, corresponding to $\bar{z} = 1$ and $\sigma_z = 0.1$.

be evaluated, which gets reduced by a factor of 4! due to permutation symmetry. Since it takes about $O(1)$ minutes to evaluate the trispectrum for a single configuration (see table 2), using $N = 10$ we would require about $O(10^2)$ CPU hours to compute the trispectrum for all configurations, for a fixed cosmology and redshift. As discussed above, we have not optimized our integration method in our code, and a better performance can be achieved with parallelization and an improved sampling scheme.

5 Non-Gaussian covariance of angular power spectrum

In order to obtain accurate constraints on cosmological parameters, it is important to have a precise theoretical prediction for the covariance. However, to the best of our knowledge, the full computation of the covariance matrix for the power spectrum (including the connected part) in angular space beyond the Limber approximation has not yet been performed, due to the difficulty associated with computing the trispectrum. In this section, equipped with the formalism for computing the angular trispectrum, we compute the non-Gaussian covariance of the angular power spectrum from the connected part of the trispectrum.\(^{21}\) For simplicity, we set $b_5 = 1$ and all other bias parameters to zero, which is equivalent to computing the matter power spectrum covariance.

\(^{21}\)In the literature, other types of the power spectrum covariance in multipole space have also been studied. For example, one may consider a partial wave expansion of the anisotropic power spectrum with respect to the angle between the momentum vector and a line-of-sight direction, relevant for the galaxy power spectrum with RSD [30, 31]. The resulting covariance depends both on the multipole and the wavenumber $k$, with nonzero monopole, quadrupole and octupole. This is clearly different from the object we are computing, which is the covariance of the angular power spectrum in $\ell m$-space.
5.1 Power spectrum estimator and covariance

At tree level, the theoretical angular matter power spectrum can be computed as
\[ C_\ell = \frac{1}{2\pi^2} \int_0^\infty dr W_s(r) I_\ell(r), \] (5.1)
where \( I_\ell(r) \) is defined as in (2.20) with \( f_i(k, z) = D_g(z) P(k) \). Since the power spectrum only consists of a single radial integral, the sampling points do not need to be as dense as in the trispectrum calculation, and \( N_r = 50 \) is sufficient. The optimal estimator \( \hat{C}_\ell \) for the angular power spectrum is given by summing over all measured multipoles,
\[ \hat{C}_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^\ell |\delta_{\ell m}^{(\text{obs})}|^2, \] (5.2)
which is unbiased as \( \langle \hat{C}_\ell \rangle = C_\ell. \) The covariance matrix of the power spectrum estimator is \[ C_{\ell\ell'} = \frac{\delta_{\ell\ell'}}{\ell + 1} \left[ T_{\ell\ell'}^{(0)}(0) - C_\ell C_{\ell'} \right], \] (5.3)
where in the second line we have subtracted the purely disconnected piece from the trispectrum. The trispectrum in a single channel \( P_{\ell\ell'}^{(L)}(L) \) can be obtained by summing over permutations of reduced trispectra as in (2.12). The symmetries of \( P_{\ell\ell'}^{(L)}(L) \) implies that it vanishes for odd \( L \), so the \( L \)-sum is only over even multipoles. This enters in the Fisher matrix as usual,
\[ F_{\alpha\beta} = \sum_{\ell\ell'} \frac{\partial C_\ell}{\partial \lambda_\alpha} \frac{\partial C_{\ell'}}{\partial \lambda_\beta} , \] (5.4)
for a set of parameters \( \{\lambda_\alpha\} \). The diagonal components of the Fisher matrix give the marginalized 1-\( \sigma \) uncertainties in the parameters \( \lambda_\alpha \). The off-diagonal components from the non-Gaussian part induce correlations between the uncertainties, which in general degrade parameter constraints.

It is instructive to compare (5.3) to the analogous calculation in Fourier space. The power spectrum estimator in Fourier space is
\[ \hat{P}(k) = V_f \int_{V_s(k)} \frac{d^3 q}{V_s(k)} \delta(q) \delta(-q), \] (5.5)
where \( V_f = (2\pi)^3/V \) is the volume of the fundamental shell and the integration is performed over the is the differential volume of the shell of radius \( k \), \( V_s(k) = 4\pi k^2 \delta k \). The covariance is given by [24] (see also [27, 29])
\[ C(k, k') = \frac{P(k)^2}{(V_s(k)/V_f)} \delta_{kk'} + \frac{1}{V} \int_{V_s(k)} \frac{d^3 q}{V_s(k)} \int_{V_s(k')} \frac{d^3 q'}{V_s(k')} \ T(q, -q, q', -q'). \] (5.6)
\[ \text{In practice, the observable is not measured over the entire sky. This means that different multipoles in the spherical harmonic expansion will be correlated, which results in a biased estimator. The effect of partial sky coverage can be accounted for by adding a mode-coupling kernel [111].} \]
We see that the covariance receives contribution from the trispectrum only in the \textit{collapsed} configuration, which corresponds to the limit in which the internal momentum is collapsed to zero length, i.e. \( s \to 0 \) in the \( s \)-channel. Note that the covariance in angular space \((5.3)\), too, is evaluated in the collapsed multipole configuration \( L = 0 \). When projected on the sphere, we are integrating over all momenta, so the covariance in some sense receives contributions from all wavelengths. However, the covariance in angular space is still mostly captured by terms that dominate in the \( s \to 0 \) limit.

5.2 Angular matter power spectrum covariance at tree level

We now turn to the computation of the non-Gaussian covariance of the angular matter power spectrum. There are essentially three most relevant pieces that contribute to the non-Gaussian covariance: the connected four-point function at tree level and one loop, and the super-sample covariance \([112, 113]\). In this section, we consider the contribution from the tree-level trispectrum.

Figure 11 shows the diagonal elements of the covariance from the Gaussian and non-Gaussian parts. We show two results for the latter part, one using the FFTLog method and the other with the Limber approximation. Since we are evaluating the trispectrum at \( L = 0 \), this time we use the Limber approximation only for the external multipoles, and the coupling integral \( J_L^{(s)} \) was computed with the FFTLog method in both cases. One can immediately notice that there is quite a large discrepancy between the FFTLog- and Limber-based calculations, even for large multipoles for which we normally think that the Limber approximation should be valid. We argue that this Limber-based calculation cannot be trusted. This has to do with the subtle issue about the numerical accuracy of the Limber approximation, as we explain further below. The FFTLog calculation shows that the non-Gaussian part from the connected four-point function gives a small contribution to the full covariance in angular space.
Table 3. Comparison of angular matter trispectrum computed using the FFTLog method and the Limber approximation from terms proportional to $s^{-4}$ in $T_{2211}$, evaluated at $\ell = 100$, $L = 0$.

| Method      | $\gamma_{LL}^{\ell \ell}(0) \times 10^{10}$ | $e_1$ | $e_2$ | $e_3$ | $e_1 + e_2 + e_3$ |
|-------------|---------------------------------------------|------|------|------|-------------------|
| FFTLog      | -10.3936                                   | 17.3228 | -6.92884 | 3.80825 $\times 10^{-4}$ |
| Limber      | -10.4896                                   | 17.4829 | -7.01109 | -1.77898 $\times 10^{-2}$ |

FFTLog vs. Limber. First of all, we would like to know which terms in the trispectrum cause the large discrepancy between the FFTLog method and Limber approximation. To understand the root of the problem, we decompose the terms proportional to $s^{-4}$ in $T_{2211}$, which give a large contribution to the covariance. From (4.6), we have (see also appendix A)

$$T_{2211} \supset D_1 D_2 D_3 D_4 P(k_1) P(k_3) P(s) \left( 3k_2^2 - 5k_1^2 + \frac{2k_4^4}{k_4^2} \right) \left( 3k_2^2 - 5k_3^2 + \frac{2k_4^4}{k_4^2} \right) \frac{1}{s^4}. \quad (5.7)$$

This leads to the radial integral of the form

$$\int_0^\infty dr r^2 \left( 3I_{\ell}^{(1,0)}I_{\ell}^{(2,1)} + (-5I_{\ell}^{(1,1)}I_{\ell}^{(2,0)} + 2I_{\ell}^{(1,-1)}I_{\ell}^{(2,2)} \right) J_0(s^{-4}), \quad (5.8)$$

where we have suppressed the arguments and the second radial integral consisting of terms depending on $k_3$, $k_4$. The issue is that the values of all three terms in (5.8) are quite similar, with cancellations occurring at $10^{-5}$ level. We thus need to evaluate $I_{\ell}^{(i)}$ at very high precision in order to account for the correct cancellation between these terms. Note that this cancellation occurs already at the level of the $r$ integrand, so we can see that this is problematic even before computing the full trispectrum, and that this is unrelated to the sampling scheme we choose for the radial integrals. In table 3, we tabulate the values of the terms proportional to $s^{-4}$, computed with and without the Limber approximation. We see that although individual terms computed using the Limber approximation agree with the FFTLog-based result at 1% level, this accuracy is not enough to ensure the full cancellation between the terms, resulting in an orders of magnitude difference in the final trispectrum. We find a similar level of cancellation occurs for the terms proportional to $s^{-2}$, whereas the Limber approximation works well for those proportional to $s^0$, $s^2$, and $s^4$.

A desired level of precision may be achieved with the FFTLog method by choosing a sufficiently large number of sampling points. In contrast, the Limber approximation has an intrinsic level of error, simply due to the fact that it replaces the Bessel function with a Dirac delta function, whose accuracy also depends on the width of the window function used. We find that the level of accuracy of the Limber approximation is roughly at 0.1% level for large multipoles. This can be seen from e.g. figure 12, where we show the comparison between the FFTLog method and the Limber approximation, both for the angular matter power spectrum and its radial integrand. In producing the plots, we used high enough precision to make sure that the results converged for each Limber and non-Limber calculation.

It is rather striking that the Limber approximation can dramatically fail in a range of scales we normally think it can be safely trusted. We find that increasing the precision of numerical integrations does not change this conclusion. As far as we are aware, a similar observation have not been made for lower-point functions, likely because the kinematic
Figure 12. Comparison between the FFTLog method and the Limber approximation for the angular matter power spectrum (left) and its integrand for $\ell = 300$ (right). The bottom panel for each plot shows the relative error in percentage. We used the window function located at $z = 1$ with $\sigma_z = 0.1$, and chose higher number of sampling points $N_r = N_\chi = 200$ than usual.

configurations of two- and three-point functions are simpler than the four-point case. The validity of the Limber approximation should thus be carefully checked whenever it is used, especially when dealing with the non-Gaussian covariance.

6 Conclusions

In the era of high-precision cosmology and large datasets, it is important to build efficient algorithms for calculating and estimating cosmological observables. In this paper, we presented an efficient semi-analytic method to compute cosmological angular trispectra. This generalizes the method of [13] to four-point angular statistics, and we used the method to compute the galaxy angular trispectrum and the non-Gaussian covariance of the angular matter power spectrum. We also defined a suitable separable ansatz for cosmological four-point functions, and classified their separability types based on the physical criteria that correlators ought to satisfy.

There are numerous other applications in cosmology one could explore using the FFTLog algorithm. First of all, it would be interesting to generalize the current formalism to other types of angular observables. Some would be simpler than others: the trivial list involves angular trispectra of the cosmic microwave background, weak lensing, etc., which simply require modifications of the line-of-sight integral kernels. Similarly, tensor observables in angular space, carrying the same Bessel integral structure as spin-0 fields but dressed with more complicated geometric dependence, involve a straightforward generalization.

More nontrivial applications involve applying the formalism to observables that go beyond the linear regime. In perturbation theory, these are systematically captured by loop integrals in Fourier space. A parallel investigation of FFTLog-based methods in Fourier space [15, 17, 114, 115] has revealed that power-law cosmologies have many analytic solutions in this case too, bearing similarities with standard loop integrals in quantum field theory. It is natural to unify the two methods to compute e.g. the one-loop angular power
spectrum. There also exist more phenomenological approaches to nonlinear scales in the large-scale structure such as the halo model [35]. In this setup, one deals with extra layers of integrals over halo profiles to compute correlators, but otherwise whose integrands consist of separable products of functions. It should thus be possible in principle to apply the FFTLog algorithm to the halo model as well, both in angular and in Fourier space.

Another interesting application involves building efficient separable templates for inflationary correlation functions. Many shapes can be made separable using the trick (2.17), and other traditional approaches involving estimating shapes in terms of an orthogonal basis of polynomials [42, 116, 117]. However, sometimes it is possible to exploit the soft limit behavior of the correlators to directly construct separable templates by pure ansatz, see e.g. [99, 118]. In particular, it would be nice if some simple templates for the equilateral and exchange-type four-point functions can be constructed in this way. We have not fully explored the shape dependence of the angular galaxy trispectrum in this work. Having easier way to compute primordial trispectra in angular space would also make it possible to study the correlations between different shapes. For four-point functions, there is a reduction in the number of degrees of freedom of shapes when going from three to two dimensions, so it is not totally obvious whether the shape correlations will remain the same after projection. It would be interesting to systematically study correlations between non-Gaussian shapes from inflation and from bias parameters, and run a detailed forecast to figure out which shapes are easier to detect than others in the large-scale structure.

One of the main utilities of our formalism is a fast and reliable way of computing the non-Gaussian covariance of the power spectrum. Usually, the Limber approximation is believed to work well in the small-scale regime. However, our investigations show a direct counter example. This happens in situations where there are numerical cancellations that are finer than the intrinsic level of precision that the Limber approximation offers, which we find to be about $O(0.1\%)$ at high $\ell$. It would be nice to explore the viability of the Limber approximation more generally, as well as that of the closely related flat-sky approximation, and how this affects cosmological parameter estimation in future surveys (see [18, 119–121] for recent works). Our formalism paves a way towards quantitatively tackling the question about how important the non-Gaussian contribution is to the covariance in angular space. Some initial steps in this direction were taken recently in [32, 33, 122, 123], mainly focusing on a subset of terms that are amenable to the Limber approximation in the halo model. We leave further progress in this interesting direction to future work.

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A Spurious divergence

In computing angular correlations, we convert the momentum-conserving delta functions into integrals over plane waves. These integrals are numerically evaluated after performing the momentum integrals. Because of this, the momentum integrals may give rise to spurious divergences from unphysical momentum configurations in the intermediate steps. For the
exchange-separable trispectrum, this happens when the coupling integral

\[ J_L^{(s)}(r, r') = 4\pi \sum_m c_m \int_0^\infty ds s^{\nu_m + 2p_s - 1} j_L(sr) j_L(sr') \, , \]  

(A.1)

is divergent, where \( \nu_m = 3 - b + i\eta_m \). The (non-)divergent nature of this integral is determined by the integer \( p_s \): for \( 1 < b < 2 \), it is UV divergent for \( p_s > 0 \) and IR divergent for \( p_s < -L \). In this appendix, we describe a procedure to remove these spurious divergences that were encountered in the main text.

**A.1 IR divergence**

Let us first consider the IR divergence, which occurs when \( p_s < -L \). For the matter trispectrum, we saw that the minimum of \( p_s \) was given by \( p_s = -2 \), making it divergent for \( L = 0, 1 \). We will consider the \( L = 0 \) case here, since this is what is relevant for computing the covariance from the connected trispectrum.

To see how this works in practice, we consider the naively divergent term in the matter trispectrum,

\[ T_{2211} \supset D_1 D_2^3 D_3^2 D_4^2 P(k_1) P(k_3) P(s) \left( 3k_2^2 - 5k_1^2 + 2\frac{k_4^2}{k_1^2} \right) \left( 3k_4^2 - 5k_3^2 + 2\frac{k_4^2}{k_3^2} \right) \frac{1}{s^4} \, , \]  

(A.2)

where we dropped a constant prefactor. To remove this apparent singularity in our integral formulas, we will soften the UV behavior of the terms inside the brackets by integrating by parts. As we saw in (2.52), we remove such UV divergence of \( k_i \) by shifting the frequency with the action of \( D_\ell(\chi) \) on \( I_\ell \) and then integrating it by parts to act it acts on the window function. This time, we instead shift the frequency with the operator \( D_\ell (r) \) and then integrate by part to hit the coupling integral \( J_L^{(s)}(r, r') \).

Let us look at the 1,2-leg first. We label each integral with \( J_{i_1}^{(i,p_1)} \) and \( J_{i_2}^{(s,p_2)} \). For \( L = 0 \), we must have \( \ell_1 = \ell_2 \equiv \ell \). Then the \( r \)-integral consists of terms

\[ \int_0^\infty dr r^2 \left( 3I_{\ell_1}^{(1,0)} I_{\ell_1}^{(2,1)} - 5I_{\ell_1}^{(1,1)} I_{\ell_1}^{(2,0)} + 2I_{\ell_1}^{(1,-1)} I_{\ell_1}^{(2,2)} \right) J_0^{(s,-4)} \, , \]  

(A.3)

where we have suppressed the arguments. Let us first shift \( I_{\ell_1}^{(2,1)} = D_\ell I_{\ell_1}^{(1,0)} \) in the first term. We then get

\[ 3r^2 I_{\ell_1}^{(1,0)} I_{\ell_1}^{(2,1)} J_0^{(s,-4)} \rightarrow 3r^2 I_{\ell_1}^{(2,0)} \left( I_{\ell_1}^{(1,1)} J_0^{(s,-4)} + I_{\ell_1}^{(1,0)} J_0^{(s,-2)} + 2I_{\ell_1}^{(1,-1)} \partial_r I_{\ell_1}^{(1,0)} \partial_r J_0^{(s,-4)} \right) \, . \]  

(A.4)

We can disregard the boundary terms in this process, since \( I_{\ell_1}^{(2,p_2)} \) essentially behaves as a window function, e.g. \( I_{\ell_1}^{(2,p_2)} = \frac{2\pi^2}{r^2} D_\ell^{2\ell_1} W^{(2)} \) for \( p_2 \geq 0 \), and thus have zero support on the boundary. Now, we integrate by parts twice the last term in (A.3) to shift \( I_{\ell_1}^{(2,2)} \rightarrow I_{\ell_1}^{(2,0)} \). Doing so, the leading divergent piece precisely cancels with other terms, and the \( r \)-integral becomes

\[ \int_0^\infty dr r^2 \left[ (5I_{\ell_1}^{(1,0)} I_{\ell_1}^{(2,0)} + 2I_{\ell_1}^{(1,-1)} I_{\ell_1}^{(2,1)} ) J_0^{(s,-2)} - 2(2I_{\ell_1}^{(2,0)} \partial_r I_{\ell_1}^{(1,-1)} + 5I_{\ell_1}^{(2,0)} \partial_r I_{\ell_1}^{(1,0)} ) \partial_r J_0^{(s,-4)} \right] \, . \]  

(A.5)

By using the identity \( \partial_r J_\ell (kr) = \frac{\ell}{r} j_\ell (kr) - kj_{\ell+1}(kr) \), the radial derivative of \( I_{\ell_1}^{(i,p_1)} (r) \) can be expressed as

\[ \partial_r I_{\ell_1}^{(i,p_1)}(r) = \sum_n c_n r^{-1-\nu_n} \int_0^\infty d\chi \left[ (1 - \nu) W_\delta(\chi) + \chi W_\delta'(\chi) \right] I_\ell \left( \nu_n + 2p_s, \frac{\chi}{r} \right) \, , \]  

(A.6)
then integrating by parts as
\[ \partial_r J^{(s,-2)}_0(r, r') = - \sum_n c_n \int_0^\infty dk k^{n+2p_s} j_1(kr) j_0(kr'). \]  
(A.7)

Since \( j_1(kr) \sim k \) as \( k \to 0 \), this has the same degree of divergence as \( J^{(s,-2)}_0 \). We can go through the same exercise for the 3,4-legs, with \( \ell_3 = \ell_4 = \ell' \). This will lead to terms such as \( J^{(s,0)}_0, \partial_s J^{(s,-2)}_0, \partial_s J^{(s,-2)}_0, \) and \( \partial_s \partial_s J^{(s,-4)}_0 \), which are manifestly free of IR divergences.

### A.2 UV divergence

Dealing with UV divergences is simpler than the IR divergent case. As we just saw, in the latter case we have to make sure the divergent \( J^{(s)}_L \) terms cancel off each other after integration by parts. In the former case, we simply need to integrate by parts a sufficient number of times to remove the divergence of \( J^{(s)}_L \). The way it works is that integrating by parts shifts UV divergences of \( J^{(s)}_L \) to \( I^{(i)}_{Li} \), which we can handle using the trick introduced in (2.52). For instance, we can lower the frequency \( p_s \) by one unit by writing \( J^{(s,p_s)}_L = D_L J^{(s,p_s-1)}_L \) and then integrating by parts as

\[
\int_0^\infty dr r^2 I^{(1,p_1)}_{\ell_1}(r) I^{(2,p_2)}_{\ell_2}(r) J^{(s,p_s)}_L(r, r') = \int_0^\infty dr J^{(s,p_s-1)}_L(r, r') \tilde{D}_L(r) \left[ r^2 I^{(1,p_1)}_{\ell_1}(r) I^{(2,p_2)}_{\ell_2}(r) \right]
\]

\[
= \int_0^\infty dr J^{(s,p_s-1)}_L(r, r') \left[ (L(L+1) - \ell_1(\ell_1+1) - \ell_2(\ell_2+1)) I^{(1,p_1)}_{\ell_1}(r) I^{(2,p_2)}_{\ell_2}(r) \right]
\]

\[
+ r^2(I^{(1,p_1+1)}_{\ell_1}(r) I^{(2,p_2)}_{\ell_2}(r) + I^{(1,p_1)}_{\ell_1}(r) I^{(2,p_2+1)}_{\ell_2}(r) - 2 \partial_s I^{(1,p_1)}_{\ell_1}(r) \partial_r J^{(2,p_2)}_{\ell_2}(r)), \quad (A.8)
\]

where we have only shown the \( r \) integral and written \( \tilde{D}_L \) in terms of \( D_{\ell_i} \) that in turn shift the frequencies of \( I^{(i,p_i)}_{\ell_i} \). Since the matter trispectrum contains terms up to \( p_s = 2 \), we would need to integrate by parts twice. This can get quickly complicated, giving rise to many terms and thus slowing down the computation. Since the radial integrands are rather smooth, in practice we can also take these derivatives numerically. To avoid numerical instability of taking multiple numerical differentiation, one should apply derivatives on the \( r \) and \( r' \) integrands as symmetrically as possible.

### B Cubic bias

In this appendix, we provide explicit expressions of the galaxy trispectrum in momentum space in section 4.1 in terms of the variables \( \{k_1, k_2, k_3, k_4, s, t\} \) arising from the set \( \mathcal{O}_3 \) of all independent cubic bias operators listed in (4.2). The set of operators we used is of course not a unique choice. Another common choice of bias operators employed in the literature is (see e.g. [69])

\[
\{ \delta, \delta^2, K^2, \delta^3, K^2\delta, K^3, O_{ad} \}. \quad (B.1)
\]

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The two sets of operators are related by
\[
\begin{bmatrix}
\delta \\
\delta^2 \\
\mathcal{G}_2 \\
\delta^3 \\
\mathcal{G}_2 \delta \\
\mathcal{G}_3 \\
\Gamma_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{3} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2}{3} & 1 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{3} & \frac{1}{2} & 0 & -1 \\
0 & 0 & 0 & -\frac{16}{63} & \frac{8}{21} & 0 & 1
\end{bmatrix} \begin{bmatrix}
\delta \\
\delta^2 \\
K^2 \\
\delta^3 \\
K^2 \delta \\
K^3 \delta \\
O_{\alpha \beta}
\end{bmatrix}.
\] (B.2)

More relations between third-order bias parameters can be found in [34, 124, 125].

Let us summarize the momentum dependence of the \(T_{3111}\) part of the galaxy trispectrum in arising from different bias parameters \(b_\Omega\), denoted by \(F_\Omega\). (For the \(T_{2211}\) part, see (3.11).) In the \(s\)-channel, they are given by
\[
\begin{align*}
F_{g_2 \delta}(k_1, k_2) &= \sigma_{k_1, k_2}^2, \\
F_{g_3}(k_1, k_2) &= F_2^{\text{sym}}(k_1, k_2), \\
F_{\delta g}(k_1, k_2, k_3) &= F_3^{\text{sym}}(k_1, k_2, k_3), \\
F_{g_3}(k_1, k_2, k_3) &= 2(k_1 \cdot \hat{k}_2)(k_1 \cdot \hat{k}_3)(\hat{k}_2 \cdot \hat{k}_3) + \sigma_{k_1, k_2}^2, \\
F_{\delta g_3}(k_1, k_2, k_3) &= \sigma_{k_1, k_2, k_3}^2 (F_2^{\text{sym}}(k_1, k_2) - G_2^{\text{sym}}(k_1, k_2)) .
\end{align*}
\] (B.3)-(B.7)

When expressed in terms of the scalar variables, these become rather lengthy expressions, so we introduce a compact notation
\[
K^{(m,n)} = \sum_{i \neq j}^3 k_i^m k_j^n, \quad k_{i \pm}^{(m,n)} = (k_i^m \pm k_j^n)^n,
\] (B.8)
to express the external wavenumber dependence. We have
\[
\begin{align*}
\sigma_{k_1, k_2}^2 &= \frac{k_{12}^{(1,2)} k_{12}^{(1,2)} - 2 k_{12}^{(2,1)} + s^4}{4k_1^2 k_2^2}, \\
F_2^{\text{sym}}(k_1, k_2) &= \frac{-5 k_{12}^{(2,2)} + 3 k_{12}^{(2,1)} s^2 + 2s^4}{28k_1^2 k_2^2}, \\
G_2^{\text{sym}}(k_1, k_2) &= \frac{-3 k_{12}^{(2,2)} - k_{12}^{(2,1)} s^2 + 4s^4}{28k_1^2 k_2^2}.
\end{align*}
\] (B.9)-(B.11)

The dot products in \(F_2\) depend on more than one angle, so it is convenient to express it in terms of \(s^2\) and the angle \(\hat{k}_2 \cdot \hat{k}_3\) so that we can directly apply our spin-exchange separable result (2.35). We have
\[
(k_1 \cdot \hat{k}_2)(k_1 \cdot \hat{k}_3)(\hat{k}_2 \cdot \hat{k}_3) \equiv \frac{(k_1^2 + k_2^2 - s^2)(2k_2 k_3 (\hat{k}_2 \cdot \hat{k}_3) + k_2^2 - k_3^2 + s^2)(\hat{k}_2 \cdot \hat{k}_3)}{4k_1^2 k_2 k_3}.
\] (B.12)

The \(F_3^{\text{sym}}\) term is rather complicated and depend on all of \(s\), \(t\), and \(u\). However, as we described in the main text, it can be written in a manifestly crossing symmetric form in terms of \(\hat{F}_3\) defined by
\[
\hat{F}_3(k_1, k_2, k_3) \equiv \frac{1}{9} + \frac{g_0(k_1, k_2, k_3, k_4) + g_s(k_1, k_2, k_3, k_4, s)}{3024(k_1 k_2 k_3)^2},
\] (B.13)
where
\[
\begin{align*}
g_0(k_1, \cdots, k_4) & = -49 K^{(4,2)} + (24 K^{(2,2)} - 29 K^{(4,0)}) k_4^2 - 2 K^{(2,0)} k_4^4, \\
g_e(k_1, \cdots, k_4, s) & = 3k_{12}^{(1,2)} k_{13}^{(1,1)} k_{34}^{(1,1)} (7k_3^2 + 2k_4^2)s^2 - 3(7k_{12}^{(2,1)} - 14k_3^2 - 2k_4^2)s^4 - 14s^6 + \left[ 7(5k_{12}^{(2,2)} + 2k_{12}^{(2,1)} - 4k_3^4) + (23k_{12}^{(2,1)} + 20k_3^2)k_4^2 + 8k_4^4 \right] s^2.
\end{align*}
\]

This basis is convenient since \( \tilde{F}_3^\text{sym} = \tilde{F}_3^\text{sym} \), but each permutation \( \tilde{F}_3 \) depends on only one diagonal moment, whereas \( F_3 \) depends on two.

Finally, let us list all the super-reduced trispectra associated with all combinations of bias parameters:
\[
\begin{align*}
\tau^2_{2211}(k_1, k_2, k_3, k_4) & = 2b_2^{12} b_2^{32} D_1 D_2 D_3 D_4^3 P(k_1) P(k_3) P(s), \\
\tau^2_{2211}(k_1, k_2, k_3, k_4) & = 2b_2^{12} b_2^{32} D_1 D_2 D_3 D_4^3 P(k_1) P(k_3) P(s) \left[ \sigma_{k_1, -k_12}^2 \sigma_{k_3, k_12}^2 \right], \\
\tau^2_{2211}(k_1, k_2, k_3, k_4) & = 4b_2^{12} b_2^{32} D_1 D_2 D_3 D_4^3 P(k_1) P(k_3) P(s) \left[ F_2^\text{sym}(k_1, -k_12) \right], \\
\tau^2_{2211}(k_1, k_2, k_3, k_4) & = 4b_2^{12} b_2^{32} D_1 D_2 D_3 D_4^3 P(k_1) P(k_3) P(s) \left[ F_2^\text{sym}(k_1, -k_12) \sigma_{k_3, k_12}^2 \right], \\
\tau^2_{2211}(k_1, k_2, k_3, k_4) & = 2b_2^{12} b_2^{32} D_1 D_2 D_3 D_4^3 P(k_1) P(k_3) P(s) \left[ F_2^\text{sym}(k_1, -k_12) \sigma_{k_3, k_12}^2 \right], \\
\end{align*}
\]

where \( \delta \times \delta^2 \), \( \delta \times \bar{G}_2 \), \( \delta^2 \times \bar{G}_2 \) denote the cross-terms, and
\[
\begin{align*}
\tau^2_{3111}(k_1, k_2, k_3, k_4) & = 2b_3^{12} b_3^{32} D_1 D_2 D_3 D_4^3 \left[ F_2^\text{sym}(k_1, k_2) \right], \\
\tau^2_{3111}(k_1, k_2, k_3, k_4) & = 2b_3^{12} b_3^{32} D_1 D_2 D_3 D_4^3 \left[ F_2^\text{sym}(k_1, k_2) \right], \\
\tau^2_{3111}(k_1, k_2, k_3, k_4) & = 2b_3^{12} b_3^{32} D_1 D_2 D_3 D_4^3 \left[ F_2^\text{sym}(k_1, k_2) \right], \\
\tau^2_{3111}(k_1, k_2, k_3, k_4) & = 2b_3^{12} b_3^{32} D_1 D_2 D_3 D_4^3 \left[ F_2^\text{sym}(k_1, k_2) \right], \\
\tau^2_{3111}(k_1, k_2, k_3, k_4) & = 2b_3^{12} b_3^{32} D_1 D_2 D_3 D_4^3 \left[ F_2^\text{sym}(k_1, k_2) \right], \\
\end{align*}
\]

The total s-channel trispectrum can be obtained by summing over 8 permutations of each super-reduced trispectrum. The shapes of the corresponding angular trispectra were shown in section 4.3.

C Spin-weighted functions

As we mentioned in the main text, there are alternative representations of separable trispectra in angular space. One such representation involves spin-weighted harmonics, whose origin we can understand as follows. When expressing correlation functions in three-dimensional space in terms the spherical coordinates, some momentum dependence can be traded with derivatives with respect to the angular coordinates. These derivatives, carrying directional information, in turn will transform scalars into spin-weighted fields on the sphere. For scalar correlators like we are studying, these spin weights necessarily cancel in the end and are thus fake, but still provides a useful way of describing angular observables. In this appendix, we present details on spin-weighted functions on a sphere and use these to express contact separable trispectra.
The first study of spin-weighted functions goes back to \cite{126, 127}, which we first briefly review. The line element of the three-dimensional Euclidean space in the spherical coordinates is

\[
d s^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.
\] (C.1)

It is convenient to work in terms of the orthonormal basis given by

\[
e_r = \partial_r, \quad e_\theta = \frac{1}{r} \partial_\theta, \quad e_\varphi = \frac{1}{r \sin \theta} \partial_\varphi.
\] (C.2)

From these we can form a new, “helicity” basis vectors \(e_\pm\) with their dual 1-forms \(\omega^\pm\) by

\[
e_\pm = \frac{1}{\sqrt{2}} (e_\theta \pm ie_\varphi), \quad \omega^\pm = \frac{1}{\sqrt{2}} (r d\theta \mp ir \sin \theta d\varphi).
\] (C.3)

These are defined such that under a standard rotation of angle \(\gamma\) they transform as \(e_\pm \rightarrow e^{\pm i \gamma} e_\pm\). The natural set of derivative operators we can use in this basis are the \textit{spin-raising} \(\partial\) and its adjoint, \textit{spin-lowering} \(\bar{\partial}\), whose action on a spin-\(J\) function \(J f\) are defined as

\[
\partial (J f) = -(\partial_\theta + i \csc \theta \partial_\varphi + J \cot \theta) J f, \\
\bar{\partial} (J f) = -(\partial_\theta - i \csc \theta \partial_\varphi - J \cot \theta) J f.
\] (C.4)

These operators act very naturally on spherical harmonics, which we review at the end of this section. Standard Cartesian derivatives can be recast in terms of these two operators together with the radial derivative \(\partial_r\). For example, given two scalar fields \(f\) and \(g\), we have

\[
\partial^a f \partial_a g = \partial_r f \partial_r g + \frac{1}{2r^2} (\partial f \bar{\partial} g + \bar{\partial} f \partial g),
\] (C.5)

where \(\partial_\pm f = e_\pm (f)\). Going beyond first derivatives requires us to work out covariant derivatives, which in the orthonormal basis are defined in terms of the connection 1-form \(\omega^a_b\) by the relation \(\nabla_a e_b = (\omega^a_b) e_c\). To obtain these, we first compute the exterior derivatives of the dual 1-forms

\[
d\omega^0 = 0, \quad d\omega^\pm = \frac{1}{r} \omega^0 \wedge \omega^\pm - \frac{\cot \theta}{\sqrt{2}} \omega^0 \wedge \omega^\mp.
\] (C.6)

Comparing these exterior derivatives with Cartan’s first structure formula, we deduce that

\[
d\omega^0 = -\omega^0_a \wedge \omega^a, \quad d\omega^\pm = -\omega^\pm_a \wedge \omega^a \Rightarrow \omega^0_\pm = \frac{1}{r} \omega^\pm, \quad \omega^\pm_\mp = \frac{\cot \theta}{\sqrt{2} r} \omega^\mp.
\] (C.7)

Using these, we can express double covariant derivatives in terms of \(\{\partial, \bar{\partial}, \partial_r\}\) as

\[
\nabla_a \nabla^a f = \partial^2 f, \quad \nabla_+ \nabla^r f = \frac{1}{2r^2} \partial (r \partial_r f - f), \quad \nabla_+ \nabla^+ f = \frac{1}{2r^2} (\overline{\partial} \overline{\partial} + 2r \partial_r) f,
\] (C.8)

and similarly for \(\nabla_-\). For example, some relevant formulas for their action on scalar fields are

\[
\nabla_a \nabla^a f = \frac{1}{r^2} \partial_r (r^2 \partial_r f) + \frac{1}{2r} (\overline{\partial} \overline{\partial} + \overline{\partial}) f,
\] (C.9)

\[
\nabla_a \nabla_b f \nabla^a \nabla^b g = \frac{1}{4r^4} \left[ (\overline{\partial} f) (\overline{\partial} g) + (\overline{\partial} f + 2r \partial_r f) (\overline{\partial} g + 2r \partial_r g) + 4(r \partial_r \overline{\partial} f - \overline{\partial} f) (r \partial_r \overline{\partial} g - \overline{\partial} g) + c.c. \right] + (\partial_r^2 f) (\partial_r^2 g).
\] (C.10)

We show the role played by these derivatives in computing certain angular correlations in section C.2.
Spin-weighted spherical harmonics. Spin-$j$ spherical harmonic $jY_{\ell m}$ is defined by the action of the spin-raising operator on the usual spherical harmonic $Y_{\ell m}$ as

\begin{align}
\partial(jY_{\ell m}) &= \sqrt{(\ell - j)(\ell + j + 1)}j_{j+1}Y_{\ell m}, \\
\bar{\partial}(jY_{\ell m}) &= -\sqrt{(\ell + j)(\ell - j + 1)}j_{j-1}Y_{\ell m},
\end{align}

with $jY_{\ell m} = \sqrt{(\ell - j)(\ell + j)!} \theta Y_{\ell m}$ and $jY_{\ell m} = 0$ if $|j| > \ell$. Their explicit representation in terms of the angles $\theta$ and $\varphi$ is

\begin{equation}
jY_{\ell m}(\theta, \varphi) = \sqrt{(\ell + m)!(\ell - m)!2\ell + 1} \frac{\sin 2\ell \theta}{4\pi} \frac{\ell}{r} \left( \frac{\ell + j}{r + j - m} \right) (-1)^{\ell - r - j} e^{im\varphi} \cos^{2r + j - m} \left( \frac{\theta}{2} \right).
\end{equation}

The tensor spherical harmonics have the useful property that they are orthonormal functions on the sphere

\begin{equation}
\int_{S^2} d\Omega_{\hat{n}} jY_{\ell m}(\hat{n}) jY_{\ell'm'}^*(\hat{n}) = \delta_{\ell\ell'} \delta_{mm'}.
\end{equation}

This together with the identity that relates a product of two spherical harmonics to a sum over spherical harmonics is

\begin{equation}
j_1Y_{\ell_1 m_1 j_2} Y_{\ell_2 m_2} = \sum_{\ell_3 m_3} \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 2)(2\ell_3 + 3)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -j_1 & -j_2 & -j_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} j_3Y_{\ell_3 m_3},
\end{equation}

allows us to easily perform angular integration involving these functions; for example, the integral over three spin-weighted spherical harmonics is

\begin{equation}
\int_{S^2} d\Omega_{\hat{n}} j_1Y_{\ell_1 m_1}\hat{n}) j_2 Y_{\ell_2 m_2}(\hat{n}) j_3 Y_{\ell_3 m_3}(\hat{n})
= \sqrt{\frac{(2\ell_1 + 1)(2\ell_2 + 2)(2\ell_3 + 3)}{4\pi}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -j_1 & -j_2 & -j_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.
\end{equation}

For $j_1 = j_2 = j_3 = 0$, this reduces to the Gaunt integral (2.9).

C.2 Angular trispectrum

Let us now see how a contact separable trispectrum can be represented in angular space in terms of spin-weighted spherical harmonics. We consider a trispectrum that depends on $(k_3 \cdot k_4)^J$ in the $s$-channel, which in momentum space takes the form

\begin{equation}
\langle O_1 \cdots O_4 \rangle = f_1(k_1, z_1) \cdots f_4(k_4, z_4)(k_3 \cdot k_4)^J \times (2\pi)^3 \delta_D(k_1 + k_2 + k_3 + k_4)
= f_1(k_1, z_1) \cdots f_4(k_4, z_4) \int_{\mathbb{R}^3} d^3r e^{i(k_1 \cdot r)} e^{i(k_2 \cdot r)} \nabla a_1 \cdots \nabla a_2 e^{i(k_3 \cdot r)} \nabla a_3 \cdots \nabla a_4 e^{i(k_4 \cdot r)},
\end{equation}

where we have traded the dot products with gradients acting on the appropriate plane waves. Expanding the planes waves in spherical harmonics gives, these gradients turn into radial and angular derivatives acting on the spherical Bessel functions and spherical harmonics,
respectively. To illustrate how this works in practice, let us explicitly work out the case \( J = 1 \). Using the expansion (2.2) and then performing the angular integrations, we get

\[
\langle O_1 \cdots O_4 \rangle = \sum_{\ell_1' m_1'} \cdots \sum_{\ell_4' m_4'} \prod_{i=1}^{4} \left[ f_i(k_i, z_i) Y_{\ell_i' m_i'}(\hat{k}_i) \right] \\
\times \int_0^\infty \mathrm{d}r \ j_{\ell_1'}(k_1 r) j_{\ell_2'}(k_2 r) \int_{S_2} \mathrm{d}\Omega_{n} Y_{\ell_1' m_1'}(\hat{r}) Y_{\ell_2' m_2'}(\hat{r}) \\
\times \left[ r^2 \partial_r j_{\ell_3'}(k_3 r) \partial_r j_{\ell_4'}(k_4 r) Y_{\ell_3' m_3'}(\hat{r}) Y_{\ell_4' m_4'}(\hat{r}) \\
+ \frac{j_{\ell_3'}(k_3 r) j_{\ell_4'}(k_4 r)}{2} (\partial Y_{\ell_3' m_3'}(\hat{r}) \partial Y_{\ell_4' m_4'}(\hat{r}) + \text{c.c.}) \right].
\] (C.18)

Plugging this into the projection formula (2.4) and performing the angular integrations, we obtain the reduced trispectrum (for \( J = 1 \))

\[
t_{\ell_1\ell_2\ell_3\ell_4}(L) = \frac{h_{\ell_1\ell_2 L} h_{\ell_3\ell_4 L}}{(2\pi)^4} \int_0^\infty \mathrm{d}r \ r^2 I_{\ell_1}(r) I_{\ell_2}(r) \partial_r I_{\ell_3}(r) \partial_r I_{\ell_4}(r) \\
+ \frac{h_{\ell_1\ell_2 L} (h_{\ell_3\ell_4 L} h_{\ell_5\ell_6 L})}{2(2\pi)^4} \int_0^\infty \mathrm{d}r \ I_{\ell_1}(r) I_{\ell_2}(r) I_{\ell_3}(r) I_{\ell_4}(r),
\] (C.19)

where we have defined

\[
h_{\ell_1 j_1 j_2 j_3} \equiv h_{\ell_1 \ell_2 \ell_3} \prod_{i=1}^{3} \left[ (-1)^{|j_i| - j_i} (\ell_i - j_i) j_i (\ell_i + 1) j_i \right]^{1/2},
\] (C.20)

and \((a)_n \equiv \Gamma(a + n) / \Gamma(a)\) is the Pochhammer symbol. It is straightforward to use (C.9) and (C.10) to similarly work out the formula for \( J = 2 \).

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