Links between generalized Montréal-functors

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Abstract

Let $\mathcal{O}$ be the ring of integers in a finite extension $K/\mathbb{Q}_p$ and $G = G(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split reductive group $G$ defined over $\mathbb{Z}_p$ with connected centre and split Borel $B = TN$. We show that Breuil’s pseudocompact $(\varphi, \Gamma)$-module $D^\vee_{\xi}(\pi)$ attached to a smooth $\mathcal{O}$-torsion representation $\pi$ of $B = B(\mathcal{O}_p)$ is isomorphic to the pseudocompact completion of the basechange $O_V \otimes_{\Lambda(N_0), \ell} \widetilde{D_{SV}}(\pi)$ to Fontaine’s ring (via a Whittaker functional $\ell : N_0 = N(\mathbb{Z}_p) \to \mathbb{Z}_p$) of the étale hull $\widetilde{D_{SV}}(\pi)$ of $D_{SV}(\pi)$ defined by Schneider and Vigneras [8]. Moreover, we construct a $G$-equivariant map from the Pontryagin dual $\pi^\vee$ to the global sections $\mathcal{Y}(G/B)$ of the $G$-equivariant sheaf $\mathcal{Y}$ on $G/B$ attached to a noncommutative multivariable version $D^\vee_{\xi, \ell, \infty}(\pi)$ of Breuil’s $D^\vee_{\xi, \ell}(\pi)$ whenever $\pi$ comes as the restriction to $B$ of a smooth, admissible representation of $G$ of finite length.

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1 Introduction

1.1 Notations

Let $G = \mathbb{G}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a $\mathbb{Q}_p$-split connected reductive group $\mathbb{G}$ defined over $\mathbb{Z}_p$ with connected centre and a fixed split Borel subgroup $\mathbb{B} = T \mathbb{N}$. Put $B := \mathbb{B}(\mathbb{Q}_p)$, $T := T(\mathbb{Q}_p)$, and $N := \mathbb{N}(\mathbb{Q}_p)$. We denote by $\Phi_+$ the set of roots of $T$ in $N$, by $\Delta \subset \Phi_+$ the set of simple roots, and by $u_\alpha : \mathbb{G}_a \to N_\alpha$, for $\alpha \in \Phi_+$, a $\mathbb{Q}_p$-homomorphism onto the root subgroup $N_\alpha$ of $N$ such that $t u_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in \mathbb{Q}_p$ and $t \in T(\mathbb{Q}_p)$, and $N_0 = \prod_{\alpha \in \Phi_+} u_\alpha(\mathbb{Z}_p)$ is a subgroup of $N(\mathbb{Q}_p)$. We put $N_{\alpha,0} := u_\alpha(\mathbb{Z}_p)$ for the image of $u_\alpha$ on $\mathbb{Z}_p$. We denote by $T_+$ the monoid of dominant elements $t$ in $T(\mathbb{Q}_p)$ such that $\text{val}_p(\alpha(t)) \geq 0$ for all $\alpha \in \Phi_+$, by $T_0 \subset T_+$ the maximal subgroup, by $T_{++}$ the subset of strictly dominant elements, i.e. $\text{val}_p(\alpha(t)) > 0$ for all $\alpha \in \Phi_+$, and we put $B_{++} = N_0 T_+, B_0 = N_0 T_0$. The natural conjugation action of $T_+$ on $N_0$ extends to an action on the Iwasawa $\mathbb{o}$-algebra $\Lambda(N_0) = \mathcal{O}[N_0]$. For $t \in T_+$ we denote this action of $t$ on $\Lambda(N_0)$ by $\varphi_t$. The map $\varphi_t : \Lambda(N_0) \to \Lambda(N_0)$ is an injective ring homomorphism with a distinguished left inverse $\psi_t : \Lambda(N_0) \to \Lambda(N_0)$ satisfying $\psi_t \circ \varphi_t = \text{id}_{\Lambda(N_0)}$ and $\psi_t(\varphi_t(\lambda)u) = \psi_t(\varphi_t(\lambda)u) = 0$ for all $u \in N_0 \setminus t N_0 t^{-1}$ and $\lambda \in \Lambda(N_0)$.

Each simple root $\alpha$ gives a $\mathbb{Q}_p$-homomorphism $x_\alpha : N \to \mathbb{G}_a$ with section $u_\alpha$. We denote by $\ell_\alpha : N_0 \to \mathbb{Z}_p$, resp. $\iota_\alpha : \mathbb{Z}_p \to N_0$, the restriction of $x_\alpha$, resp. $u_\alpha$, to $N_0$, resp. $\mathbb{Z}_p$.

Since the centre of $G$ is assumed to be connected, there exists a cocharacter $\xi : \mathbb{Q}_p^\times \to T$ such that $\alpha \circ \xi$ is the identity on $\mathbb{Q}_p^\times$ for each $\alpha \in \Delta$. We put $\Gamma := \xi(\mathbb{Z}_p^\times) \leq T$ and often denote the action of $s := \xi(p)$ by $\varphi = \varphi_s$.

By a smooth $\mathcal{O}_\mathcal{E}$-torsion representation $\pi$ of $G$ (resp. of $B = \mathbb{B}(\mathbb{Q}_p)$) we mean a torsion $\mathcal{O}_\mathcal{E}$-module $\pi$ together with a smooth (ie. stabilizers are open) and linear action of the group $G$ (resp. of $B$).

For example, $\mathbb{G} = \text{GL}_n$, $B$ is the subgroup of upper triangular matrices, $N$ consists of the strictly upper triangular matrices (1 on the diagonal), $T$ is the diagonal subgroup, $N_0 = \mathbb{N}(\mathbb{Z}_p)$, the simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ where $\alpha_i(\text{diag}(t_1, \ldots, t_n)) = t_i t_{i+1}^{-1}$, $x_{\alpha_i}$ sends a matrix to its $(i, i+1)$-coefficient, $u_{\alpha_i}(\cdot)$ is the strictly upper triangular matrix, with $(i, i+1)$-coefficient $1$ and $0$ everywhere else.

Let $\ell : N_0 \to \mathbb{Z}_p$ (for now) any surjective group homomorphism and denote by $H_0 \triangleleft N_0$ the kernel of $\ell$. The ring $\Lambda_{\ell}(N_0)$, denoted by $\Lambda_{H_0}(N_0)$ in [3], is a generalisation of the ring $\mathcal{O}_\mathcal{E}$, which corresponds to $\Lambda_{\text{id}}(N_0^{(2)})$ where $N_0^{(2)}$ is the $\mathbb{Z}_p$-points of the unipotent radical of a split Borel subgroup in $\text{GL}_2$. We refer the reader to [3] for the proofs of some of the following claims.

The maximal ideal $\mathcal{M}(H_0)$ of the completed group $\mathcal{O}_\mathcal{E}$-algebra $\Lambda(H_0) = \mathcal{O}[H_0]$ is generated by $\varpi$ and by the kernel of the augmentation map $\mathcal{O}[H_0] \to \mathcal{O}$.

The ring $\Lambda_{\ell}(N_0)$ is the $\mathcal{M}(H_0)$-adic completion of the localisation of $\Lambda(N_0)$ with respect to the Ore subset $S_\ell(N_0)$ of elements which are not in $\mathcal{M}(H_0)\Lambda(N_0)$. The ring $\Lambda(N_0)$ can be viewed as the ring $\Lambda(H_0)[[X]]$ of skew Taylor series over $\Lambda(H_0)$ in the variable $X = [u] - 1$ where $u \in N_0$ and $\ell(u)$ is a topological generator of $\ell(N_0) = \mathbb{Z}_p$. Then $\Lambda_{\ell}(N_0)$ is viewed as the ring of infinite skew Laurent series $\sum_{n \in \mathbb{Z}} a_n X^n$ over $\Lambda(H_0)$ in the variable $X$ with $\text{lim}_{n \to -\infty} a_n = 0$ for the compact topology of $\Lambda(H_0)$. For a different characterization of this ring in terms of a projective limit $\Lambda_{\ell}(N_0) \cong \varprojlim_{k} \Lambda(N_0/H_k)[1/X]/\varpi^n$ for $H_k \triangleleft N_0$ normal subgroups contained and open in $H_0$ satisfying $\bigcap_{k \geq 0} H_k = \{1\}$ see also [11].
For a finite index subgroup \( G_2 \) in a group \( G_1 \) we denote by \( J(G_1/G_2) \subset G_1 \) a (fixed) set of representatives of the left cosets in \( G_1/G_2 \).

### 1.2 General overview

By now the \( p \)-adic Langlands correspondence for \( GL_2(\mathbb{Q}_p) \) is very well understood through the work of Colmez [3], [4] and others (see [1] for an overview). To review Colmez’s work let \( K/\mathbb{Q}_p \) be a finite extension with ring of integers \( o \), uniformizer \( \varpi \) and residue field \( k \). The starting point is Fontaine’s [7] theorem that the category of \( o \)-torsion Galois representations of \( \mathbb{Q}_p \) is equivalent to the category of torsion \((\varphi, \Gamma)\)-modules over \( \mathcal{O}_\mathcal{E} = \varprojlim_k o/\varpi^h((X)) \). One of Colmez’s breakthroughs was that he managed to relate \( p \)-adic (and mod \( p \)) representations of \( GL_2(\mathbb{Q}_p) \) to \((\varphi, \Gamma)\)-modules, too. The so-called “Montréal-functor” associates to a smooth \( o \)-torsion representation \( \pi \) of the standard Borel subgroup \( B_2(\mathbb{Q}_p) \) of \( GL_2(\mathbb{Q}_p) \) a torsion \((\varphi, \Gamma)\)-module over \( \mathcal{O}_\mathcal{E} \). There are two different approaches to generalize this functor to reductive groups \( G \) other than \( GL_2(\mathbb{Q}_p) \). We briefly recall these “generalized Montréal functors” here.

The approach by Schneider and Vigneras [5] starts with the set \( B_+(\pi) \) of generating \( B_+ \)-subrepresentations \( W \leq \pi \). The Pontryagin dual \( W^\vee = \text{Hom}_o(W, K/o) \) of each \( W \) admits a natural action of the inverse monoid \( B_+^{-1} \). Moreover, the action of \( N_0 \leq B_+^{-1} \) on \( W^\vee \) extends to an action of the Iwasawa algebra \( \Lambda(N_0) = o[[N_0]] \). For \( W_1, W_2 \in B_+(\pi) \) we also have \( W_1 \cap W_2 \in B_+(\pi) \) (Lemma 2.2 in [5]) therefore we may take the inductive limit \( D_{SV}(\pi) := \varinjlim_{W \in B_+(\pi)} W^\vee \). In general, \( D_{SV}(\pi) \) does not have good properties: for instance it may not admit a canonical right inverse of the \( T_\pi \)-action making \( D_{SV}(\pi) \) an \( \mathcal{T}_\pi \)-module over \( \Lambda(N_0) \). However, by taking a resolution of \( \pi \) by compactly induced representations of \( B \), one may consider the derived functors \( D_{SV}^i \) of \( D_{SV} \) for \( i \geq 0 \) producing étale \( T_\pi \)-modules \( D_{SV}^i(\pi) \) over \( \Lambda(N_0) \). Note that the functor \( D_{SV} \) is neither left- nor right exact, but exact in the middle. The fundamental open question of [5] whether the topological localizations \( \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}^i(\pi) \) are finitely generated over \( \Lambda_\ell(N_0) \) in case when \( \pi \) comes as a restriction of a smooth admissible representation of \( G \) of finite length. One can pass to usual \( 1 \)-variable étale \((\varphi, \Gamma)\)-modules—still not necessarily finitely generated—over \( \mathcal{O}_\mathcal{E} \) via the map \( \ell : \Lambda_\ell(N_0) \to \mathcal{O}_\mathcal{E} \) which step is an equivalence of categories for finitely generated étale \((\varphi, \Gamma)\)-modules (Thm. 8.20 in [5]).

More recently, Breuil [2] managed to find a different approach, producing a pseudocompact (i.e. projective limit of finitely generated) \((\varphi, \Gamma)\)-module \( D_{\hat{\xi}}^i(\pi) \) over \( \mathcal{O}_\mathcal{E} \) when \( \pi \) is killed by a power \( \varpi^h \) of the uniformizer \( \varpi \). In [2] (and also in [5]) \( \ell \) is a generic Whittaker functional, namely \( \ell \) is chosen to be the composite map

\[
\ell : N_0 \to N_0 / (N_0 \cap [N, N]) \cong \prod_{\alpha \in \Delta} N_{\alpha,0} / o = 1 - \alpha_{\Delta} \to \mathbb{Z}_p.
\]

Breuil passes right away to the space of \( H_0 \)-invariants \( \pi^{H_0} \) of \( \pi \) where \( H_0 \) is the kernel of the group homomorphism \( \ell : N_0 \to \mathbb{Z}_p \). By the assumption that \( \pi \) is smooth, the invariant subspace \( \pi^{H_0} \) has the structure of a module over the Iwasawa algebra \( \Lambda(N_0/H_0)/\varpi^h \cong o/\varpi^h[[X]] \). Moreover, it admits a semilinear action of \( F \) which is the Hecke action of \( s := \xi(p) \): For any \( m \in \pi^{H_0} \) we define

\[
F(m) := \text{Tr}_{H_0/sH_0s^{-1}}(sm) = \sum_{u \in J(H_0/sH_0s^{-1})} usm.
\]
So $\pi^{H_0}$ is a module over the skew polynomial ring $\Lambda(N_0/H_0)/\varpi^h[F]$ (defined by the identity $FX = (sX)^{-1}F = ((X+1)^{p-1})F$). We consider those (i) finitely generated $\Lambda(N_0/H_0)/\varpi^h[F]$-submodules $M \subset \pi^{H_0}$ that are (ii) invariant under the action of $\Gamma$ and are (iii) admissible as a $\Lambda(N_0/H_0)/\varpi^h$-module, i.e. the Pontryagin dual $M^\vee = \text{Hom}_o(M, o/\varpi^h)$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h$. Note that this admissibility condition (iii) is equivalent to the usual admissibility condition in smooth representation theory, i.e. that for any (or equivalently for a single) open subgroup $N' \leq N_0/H_0$ the fixed points $M^{N'}$ form a finitely generated module over $o$. We denote by $\mathcal{M}(\pi^{H_0})$ the—via inclusion partially ordered—set of those submodules $M \leq \pi^{H_0}$ satisfying (i), (ii), (iii). Note that whenever $M_1, M_2$ are in $\mathcal{M}(\pi^{H_0})$ then so is $M_1 + M_2$. It is shown in [4] (see also [5] and Lemma 2.6 in [2]) that for $M \in \mathcal{M}(\pi^{H_0})$ the localized Pontryagin dual $M^\vee/1/X$ naturally admits a structure of an étale $(\varphi, \Gamma)$-module over $o/\varpi^h((X))$. Therefore Breuil [2] defines

$$D^\vee_\xi(\pi) := \lim_{M \in \mathcal{M}(\pi^{H_0})} M^\vee[1/X].$$

By construction this is a projective limit of usual $(\varphi, \Gamma)$-modules. Moreover, $D^\vee_\xi$ is right exact and compatible with parabolic induction [2]. It can be characterized by the following universal property: For any (finitely generated) étale $(\varphi, \Gamma)$-module over $o/\varpi((X) \cong o/\varpi[\mathbb{Z}_p][([1] - 1)^{-1}]$ (here $[1]$ is the image of the topological generator of $\mathbb{Z}_p$ in the Iwasawa algebra $o/\varpi^h[\mathbb{Z}_p]$) we may consider continuous $\Lambda(N_0)$-homomorphisms $\pi^\vee \rightarrow D$ via the map $\ell : N_0 \rightarrow \mathbb{Z}_p$ (in the weak topology of $D$ and the compact topology of $\pi^\vee$). These all factor through $(\pi^\vee)^{H_0} \cong (\pi^{H_0})^\vee$. So we may require these maps be $\psi_s$- and $\Gamma$-equivariant where $\Gamma = \xi(\mathbb{Z}_p \setminus \{0\})$ acts naturally on $(\pi^{H_0})^\vee$ and $\psi_s : (\pi^{H_0})^\vee \rightarrow (\pi^{H_0})^\vee$ is the dual of the Hecke-action $F : \pi^{H_0} \rightarrow \pi^{H_0}$ of $s$ on $\pi^{H_0}$. Any such continuous $\psi_s$- and $\Gamma$-equivariant map $f$ factors uniquely through $D^\vee_\xi(\pi)$. However, it is not known in general whether $D^\vee_\xi(\pi)$ is nonzero for smooth irreducible representations $\pi$ of $G$ (restricted to $B$).

The way Colmez goes back in construction of representations of $GL_2(\mathbb{Q}_p)$ requires the following construction. From any $(\varphi, \Gamma)$-module over $\mathcal{E} = \mathcal{O}_\xi[1/p]$ and character $\delta : \mathbb{Q}_p^\times \rightarrow o^\times$ Colmez constructs a $GL_2(\mathbb{Q}_p)$-equivariant sheaf $\mathcal{F} : U \mapsto D \boxtimes_U (U \subseteq \mathbb{P}^1)$ of $K$-vectorspaces on the projective space $\mathbb{P}^1(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)/B_2(\mathbb{Q}_p)$. This sheaf has the following properties: (i) the centre of $GL_2(\mathbb{Q}_p)$ acts via $\delta$ on $D \boxtimes \mathbb{P}^1$; (ii) we have $D \boxtimes \mathbb{Z}_p \cong D$ as a module over the monoid $\left(\begin{array}{cc} Z_p \setminus \{0\} & Z_p \\ 0 & 1 \end{array}\right)$ (where we regard $\mathbb{Z}_p$ as an open subspace in $\mathbb{P}^1 = \mathbb{Q}_p \cup \{\infty\}$). Moreover, whenever $D$ is 2-dimensional and $\delta$ is the character corresponding to the Galois representation of $\Lambda^2 D$ via local class field theory then the $\Gamma$-representation of global sections $D \boxtimes \mathbb{P}^1$ admits a short exact sequence

$$0 \rightarrow \Pi(\tilde{D})^\vee \rightarrow D \boxtimes \mathbb{P}^1 \rightarrow \Pi(D) \rightarrow 0$$

where $\Pi(\cdot)$ denotes the $p$-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$ and $\tilde{D} = \text{Hom}(D, \mathcal{E})$ is the dual $(\varphi, \Gamma)$-module.

In [9] the functor $D \mapsto \mathcal{F}$ is generalized to arbitrary $\mathbb{Q}_p$-split reductive groups $G$ with connected centre. Assume that $\ell = \ell_\alpha : N_0 \rightarrow N_{\alpha,0} \cong \mathbb{Z}_p$ is the projection onto the root subgroup corresponding to a fixed simple root $\alpha \in \Delta$. Then we have an action of the monoid $T_+$ on the ring $\Lambda_\ell(N_0)$ as we have $tH_0 \ell_{\alpha}^{-1} \leq H_0$ for any $t \in T_+$. Let $D$ be an étale $(\varphi, \Gamma)$-module finitely generated over $\mathcal{O}_\xi$ and choose a character $\delta : \text{Ker}(\alpha) \rightarrow o^\times$. Then we may let the monoid $\xi(\mathbb{Z}_p \setminus \{0\})\text{Ker}(\alpha) \leq T$ (containing $T_+$) act on $D$ via the character $\delta$ of $\text{Ker}(\alpha)$.
and via the natural action of \( \mathbb{Z}_p \setminus \{0\} \cong \varphi^{\mathbb{N}_0} \times \Gamma \) on \( D \). This way we also obtain a \( T_+ \)-action on \( \Lambda(T) \otimes_{u_0} D \) making \( \Lambda(T) \otimes_{u_0} D \) an étale \( T_+ \)-module over \( \Lambda(T) \). In [9] a \( G \)-equivariant sheaf \( \mathcal{F} \) on \( G/B \) is attached to \( D \) such that its sections on \( C_0 := N_0 w_0 B/B \subset G/B \) is \( B_+ \)-equivariantly isomorphic to the étale \( T_+ \)-module \( (\Lambda(T) \otimes_{u_0} D)^{bd} \) over \( \Lambda(T) \) consisting of bounded elements in \( \Lambda(T) \otimes_{u_0} D \) (for a more detailed overview see section 6).

### 1.3 Summary of our results

Our first result is the construction of a noncommutative multivariable version of \( D_{\xi}^\vee(\pi) \). Let \( \pi \) be a smooth \( o \)-torsion representation of \( B \) such that \( \varpi^h \pi = 0 \). The idea here is to take the invariants \( \pi^H \) for a family of open normal subgroups \( H_k \leq H_0 \) with \( \bigcap_{k \geq 0} H_k = \{1\} \). Now \( \Gamma \) and the quotient group \( N_0/H_k \) act on \( \pi^H \) (we choose \( H_k \) so that it is normalized by both \( \Gamma \) and \( N_0 \)). Further, we have a Hecke-action of \( s \) given by \( F_k := \text{Tr}_{H_k/s H_k} \circ (s \cdot) \). As in [2] we consider the set \( M_k(\pi^H) \) of finitely generated \( \Lambda(N_0/H_k)[F_k]-\text{submodules of} \ \pi^H \) that are stable under the action of \( \Gamma \) and admissible as a representation of \( N_0/H_k \). In section 2 we show that for any \( M_k \in M_k(\pi^H) \) there is an étale \( (\varphi, \Gamma) \)-module structure on \( M_k^{\varphi}[1/X] \) over the ring \( \Lambda(N_0/H_k)/\varpi^h[1/X] \). So the projective limit

\[
D_{\xi,\ell,\infty}(\pi) := \lim_{\kappa \geq 0} \lim_{M_k \in M_k(\pi^H)} M_k^{\varphi}[1/X]
\]

is an étale \( (\varphi, \Gamma) \)-module over \( \Lambda(T)/\varpi^h = \lim_{\kappa \geq 0} \Lambda(N_0/H_k)/\varpi^h[1/X] \). Moreover, we also give a natural isomorphism \( D_{\xi,\ell,\infty}(\pi)_{H_0} \cong D_{\xi}(\pi) \) showing that \( D_{\xi,\ell,\infty}(\pi) \) corresponds to \( D_{\xi}(\pi) \) via (the projective limit of) the equivalence of categories in Thm. 8.20 in [9]. Note that this shows that \( D_{\xi,\ell,\infty}(\pi) \) is naturally attached to \( \pi \)—not just via the equivalence of categories (loc. cit.)—in the sense that any \( \psi \)- and \( \Gamma \)-equivariant map from \( \pi^\vee \) to an étale \( (\varphi, \Gamma) \)-module over \( o/\varpi^h((X)) \) factors uniquely through the corresponding multivariable \( (\varphi, \Gamma) \)-module. This fact is used crucially in the subsequent sections of this paper.

In section 3 we develop these ideas further and show that the natural map \( \pi^\vee \to D_{\xi,\ell,\infty}(\pi) \) factors through the map \( \pi^\vee \to D_{SV}(\pi) \). In fact, we show (Prop. 3.1) that \( D_{\xi,\ell,\infty}(\pi) \) has the following universal property: Any continuous \( \psi \)- and \( \Gamma \)-equivariant map \( f: D_{SV} \to D \) into a finitely generated étale \( (\varphi, \Gamma) \)-module \( D \) over \( \Lambda(T) \) factors uniquely through \( \pi \mapsto \text{pr}_{\xi}: D_{SV}(\pi) \to D_{\xi,\ell,\infty}(\pi) \). The association \( \pi \mapsto \text{pr}_{\xi} \) is a natural transformation between the functors \( D_{SV} \) and \( D_{\xi,\ell,\infty} \).

In order to be able to compute \( D_{\xi,\ell,\infty}(\pi) \) (hence also \( D_{\xi}(\pi) \)) from \( D_{SV}(\pi) \) we introduce the notion of the étale hull of a \( \Lambda(T) \)-module with a \( \psi \)-action of \( T_+ \) (or of a submonoid \( T_* \leq T_+ \)). Here a \( \Lambda(T) \)-module \( D \) with a \( \psi \)-action of \( T_+ \) is the analogue of a \( (\psi, \Gamma) \)-module over \( o[[X]] \) in this multivariable noncommutative setting. The étale hull \( \tilde{D} \) of \( D \) (together with a canonical map \( \iota: D \to \tilde{D} \)) is characterized by the universal property that any \( \psi \)-equivariant map \( f: D \to D' \) into an étale \( T_+ \)-module \( D' \) over \( \Lambda(T) \) factors uniquely through \( \iota \). It can be constructed as a direct limit \( \varphi_\iota^n D \) where \( \varphi_\iota^n D = \Lambda(T) \otimes_{\Lambda(T)} \Lambda(T) \) (Prop. 4.5). We show (Thm. 4.12) that the pseudocompact completion of \( \Lambda(T) \otimes_{\Lambda(T)} \tilde{D}_{SV}(\pi) \) is canonically isomorphic to \( D_{\xi,\ell,\infty}(\pi) \) as they have the same universal property.

In order to go back to representations of \( G \) we need an étale action of \( T_+ \) on \( D_{\xi,\ell,\infty}(\pi) \), not just of \( \xi(\mathbb{Z}_p \setminus \{0\}) \). This is only possible if \( tH_0 t^{-1} \leq H_0 \) for all \( t \in T_+ \) which is not the case for generic \( \ell \). So in section 4 we equip \( D_{\xi,\ell,\infty}(\pi) \) with an étale action of \( T_+ \) (extending
that of \( \xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+ \) in case \( \ell = \ell_\alpha \) is the projection of \( N_0 \) onto a root subgroup \( N_{\alpha,0} \cong \mathbb{Z}_p^* \) for some simple root \( \alpha \) in \( \Delta \). Moreover, we show (Prop. 6.5) that the map \( \text{pr}: D_{SV}(\pi) \to D_{\xi,\ell,\infty}(\pi) \) is \( \psi \)-equivariant for this extended action, too. Note that \( D_{\xi,\ell,\infty}(\pi) \) may not be the projective limit of finitely generated étale \( T_+ \)-modules over \( \Lambda(\ell N_0) \) as we do not necessarily have an action of \( T_+ \) on \( M_{\infty}[1/X] \) for \( M \in \mathcal{M}(\pi H_0) \), only on the projective limit. So the construction of a \( G \)-equivariant sheaf on \( G/B \) with sections on \( \mathcal{C}_0 = N_0 w_0 B/B \subset G/B \) isomorphic to a dense \( B_+ \)-stable \( \Lambda(\ell N_0) \)-submodule \( D_{\xi,\ell,\infty}(\pi)^{bd} \) of \( D_{\xi,\ell,\infty}(\pi) \) is not immediate from the work \([9]\) as only the case of finitely generated modules over \( \Lambda(\ell N_0) \) is treated in there. However, as we point out in section \([4]\) the most natural definition of bounded elements in \( D_{\xi,\ell,\infty}(\pi) \) works: The \( \Lambda(\ell N_0) \)-submodule \( D_{\xi,\ell,\infty}(\pi)^{bd} \) is defined as the union of \( \psi \)-invariant compact \( \Lambda(\ell N_0) \)-submodules of \( D_{\xi,\ell,\infty}(\pi) \). This section is devoted to showing that the image of \( \tilde{\text{pr}}: D_{SV}(\pi) \to D_{\xi,\ell,\infty}(\pi) \) is contained in \( D_{\xi,\ell,\infty}(\pi)^{bd} \) (Cor. 6.4) and that the constructions of \([9]\) can be carried over to this situation (Prop. 6.7). We denote the resulting \( G \)-equivariant sheaf on \( G/B \) by \( \mathfrak{Y} = \mathfrak{Y}_{\alpha,\pi} \).

Now consider the functors \((\cdot)^\vee: \pi \mapsto \pi^\vee\) and the composite \( \mathfrak{Y}_{\alpha,\pi}(G/B): \pi \mapsto D_{\xi,\ell,\infty}(\pi) \mapsto \mathfrak{Y}_{\alpha,\pi}(G/B) \) both sending smooth, admissible \( o/\varpi^h \)-representations of \( G \) of finite length to topological representations of \( G \) over \( o/\varpi^h \). The main result of our paper (Thm. 7.8) is a natural transformation \( \beta_{G/B} \) from \((\cdot)^\vee\) to \( \mathfrak{Y}_{\alpha,\pi} \). This generalizes Thm. IV.4.7 in \([4]\). The proof of this relies on the observation that the maps \( \mathcal{H}_g: D_{\xi,\ell,\infty}(\pi)^{bd} \to D_{\xi,\ell,\infty}(\pi)^{bd} \) in fact come from the \( G \)-action on \( \pi^\vee \). More precisely, for any \( g \in G \) and \( W \in \mathcal{B}_+(\pi) \) we have maps

\[
(g^\cdot): (g^{-1} W \cap W)^\vee \to (W \cap gW)^\vee
\]

where both \((g^{-1} W \cap W)^\vee\) and \((W \cap gW)^\vee\) are naturally quotients of \( W^\vee \). We show in (the proof of) Prop. 7.7 that these maps fit into a commutative diagram

\[
\begin{array}{ccccccccc}
W^\vee & \longrightarrow & (g^{-1} W \cap W)^\vee & \stackrel{g^\cdot}{\longrightarrow} & (W \cap gW)^\vee \\
\text{pr}_W \downarrow & & \downarrow & & \downarrow \\
D_{\xi,\ell,\infty}(\pi)^{bd} & \stackrel{\text{res}_{g^{-1} \mathcal{C}_0}^{\mathcal{C}_0}}{\longrightarrow} & D_{\xi,\ell,\infty}(\pi)^{bd} & \stackrel{g^\cdot}{\longrightarrow} & \text{res}_{g \mathcal{C}_0 \cap \mathcal{G}_0}^{\mathcal{C}_0}(D_{\xi,\ell,\infty}(\pi)^{bd})
\end{array}
\]

allowing us to construct the map \( \beta_{G/B} \). The proof of Thm. 7.8 is similar to that of Thm. IV.4.7 in \([4]\). However, unlike that proof we do not need the full machinery of “standard presentations” in Ch. III.1 of \([4]\) which is not available at the moment for groups other than \( \text{GL}_2(\mathbb{Q}_p) \).

**Acknowledgements**

Our debt to the works of Christophe Breuil \([2]\), Pierre Colmez \([3]\) \([4]\), Peter Schneider, and Marie-France Vigneras \([8]\) \([9]\) will be obvious to the reader. We would especially like to thank Breuil for discussions on the exactness properties of his functor and its dependence on the choice of \( \ell \).
2 A $\Lambda_{\ell}(N_0)$-variant of Breul's functor

Our first goal is to associate a $(\phi, \Gamma)$-module over $\Lambda_{\ell}(N_0)$ (not just over $O_E$) to a smooth o-
torsion representation $\pi$ of $G$ in the spirit of [2] that corresponds to $D_{\ell}^\Gamma(\pi)$ via the equivalence of categories of [2] between $(\phi, \Gamma)$-modules over $O_E$ and over $\Lambda_{\ell}(N_0)$.

Let $H_k$ be the normal subgroup of $N_0$ generated by $s^kH_0s^{-k}$, i.e. we put

$$H_k = \langle n_0 s^k H_0 s^{-k} n_0^{-1} \mid n_0 \in N_0 \rangle.$$ 

$H_k$ is an open subgroup of $H_0$ normal in $N_0$ and we have $\bigcap_{k \geq 0} H_k = \{1\}$. Denote by $F_k$ the operator $\text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s \cdot)$ on $\pi$ and consider the skew polynomial ring $\Lambda(N_0/H_k)/\mathfrak{h}[F_k]$ where $F_k \lambda = (s^k \lambda s^{-k})F_k$ for any $\lambda \in \Lambda(N_0/H_k)/\mathfrak{h}$. We denote by $\mathcal{M}_k(\pi^H_k)$ the set of finitely generated $\Lambda(N_0/H_k)[F_k]$-submodules of $\pi^H_k$ that are stable under the action of $\Gamma$ and admissible as a representation of $N_0/H_k$.

**Lemma 2.1.** We have $F = F_0$ and $F_k \circ \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \cdot) = \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \cdot) \circ F_0$ as maps on $\pi^H_0$.

**Proof.** We compute

$$F_k \circ \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \cdot) = \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s \cdot) \circ \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \cdot) =$$

$$\text{Tr}_{H_k/s^kH_0s^{-k}} \circ \text{Tr}_{s^kH_0s^{-k}/s^{k+1}H_0s^{-k-1}} \circ (s^{k+1}) = \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \cdot) \circ \text{Tr}_{H_0/s^kH_0s^{-k-1}} \circ (s \cdot) =$$

$$\text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \cdot) \circ F_0 .$$

\[\square\]

Note that if $M \in \mathcal{M}(\pi^H_0)$ then $\text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k M)$ is a $s^k N_0 s^{-k} H_k$-subrepresentation of $\pi^H_k$. So in view of the above Lemma we define $M_k$ to be the $N_0$-subrepresentation of $\pi^H_k$ generated by $\text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k M)$, i.e. $M_k := N_0 \text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k M)$. By Lemma [2.1] $M_k$ is a $\Lambda(N_0/H_k)/\mathfrak{h}[F_k]$-submodule of $\pi^H_k$.

**Lemma 2.2.** For any $M \in \mathcal{M}(\pi^H_0)$ the $N_0$-subrepresentation $M_k := N_0 \text{Tr}_{H_k/s^kH_0s^{-k}} (s^k M) \leq \pi^H_k$ lies in $\mathcal{M}_k(\pi^H_k)$.

**Proof.** Let $\{m_1, \ldots, m_r\}$ be a set of generators of $M$ as a $\Lambda(N_0/H_0)/\mathfrak{h}[F_k]$-module. We claim that the elements $\text{Tr}_{H_k/s^kH_0s^{-k}} (s^k m_i)$ ($i = 1, \ldots, r$) generate $M_k$ as a module over $\Lambda(N_0/H_k)/\mathfrak{h}[F_k]$. Since both $H_k$ and $s^k H_0 s^{-k}$ are normalized by $s^k N_0 s^{-k}$, for any $u \in N_0$ we have

$$\text{Tr}_{H_k/s^kH_0s^{-k}} (s^k us^{-k} \cdot) = (s^k us^{-k} \cdot) \circ \text{Tr}_{H_k/s^kH_0s^{-k}} .$$

Therefore by continuity we also have

$$\text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \lambda s^{-k} \cdot) = (s^k \lambda s^{-k} \cdot) \circ \text{Tr}_{H_k/s^kH_0s^{-k}}$$

for any $\lambda \in \Lambda(N_0/H_0)/\mathfrak{h}$. Now writing any $m \in M$ in the form $\sum_{j=1}^{r} \lambda_j F^{i_j} m_j$ we compute

$$\text{Tr}_{H_k/s^kH_0s^{-k}} \circ (s^k \sum_{j=1}^{r} \lambda_j F^{i_j} m_j) = \sum_{j=1}^{r} (s^k \lambda s^{-k}) F^{i_j} \text{Tr}_{H_k/s^kH_0s^{-k}} (s^k m_j) \in \sum_{j=1}^{r} \Lambda(N_0/H_k)/\mathfrak{h}[F_k] \text{Tr}_{H_k/s^kH_0s^{-k}} (s^k m_j) .$$

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For the stability under the action of $\Gamma$ note that $\Gamma$ normalizes both $H_k$ and $s^kH_0s^{-k}$ and the elements in $\Gamma$ commute with $s$.

Since $M$ is admissible as an $N_0$-representation, $s^kM$ is admissible as a representation of $s^kN_0s^{-k}$. Further, by Lemma 2.3 the map $\text{Tr}_{H_k/s^kH_0s^{-k}}$ is $s^kN_0s^{-k}$-equivariant therefore its image is also admissible. Finally, $M_k$ can be written as a finite sum

$$\sum_{u \in I(N_0/s^kN_0s^{-k}H_k)} u\text{Tr}_{H_k/s^kH_0s^{-k}}(s^kM)$$

of admissible representations of $s^kN_0s^{-k}$ therefore the statement. \hfill $\Box$

**Lemma 2.3.** Fix a simple root $\alpha \in \Delta$ such that $\ell(N_0,0) = \mathbb{Z}_p$. Then for any $M \in \mathcal{M}(\pi^H_0)$ the kernel of the trace map

$$\text{Tr}_{H_0/H_k} : Y_k := \sum_{u \in I(N_0/s^kN_0s^{-k})} u\text{Tr}_{H_k/s^kH_0s^{-k}}(s^kM) \to N_0F^k(M) \quad (2)$$

is finitely generated over $o$. In particular, the length of $Y_k^{\vee}[1/X]$ as a module over $o/\wp^h((X))$ equals the length of $M^{\vee}[1/X]$.

**Proof.** Since any $u \in N_0,0 \leq N_0$ normalizes both $H_0$ and $H_k$ and we have $N_0,0H_0 = N_0$ by the assumption that $\ell(N_0,0) = \mathbb{Z}_p$, the image of the map (2) is indeed $N_0F^k(M)$. Moreover, by the proof of Lemma 2.6 in [2] the quotient $M/N_0F^k(M)$ is finitely generated over $o$. Therefore we have $M^{\vee}[1/X] \cong (N_0F^k(M))^{\vee}[1/X]$ as a module over $o/\wp^h((X))$. In particular, their length are equal:

$$l := \text{length}_{o/\wp^h((X))}M^{\vee}[1/X] = \text{length}_{o/\wp^h((X))}(N_0F^k(M))^{\vee}[1/X].$$

We compute

$$l = \text{length}_{o/\wp^h((X))}M^{\vee}[1/X] = \text{length}_{o/\wp^h((X))}(s^kM)^{\vee}[1/X] \geq \text{length}_{o/\wp^h((X))}(\text{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^{\vee}[1/X] = \text{length}_{o/\wp^h((X))}(o/\wp^h[[X]] \otimes_{o/\wp^h[[X]]} \text{Tr}_{H_k/s^kH_0s^{-k}}(s^kM))^{\vee}[1/X] \geq \text{length}_{o/\wp^h((X))}Y_k^{\vee}[1/X].$$

By the existence of a surjective map (2) we must have equality in the above inequality everywhere. Therefore we have $\text{Ker}(\text{Tr}_{H_0/H_k})^{\vee}[1/X] = 0$, which shows that $\text{Ker}(\text{Tr}_{H_0/H_k})$ is finitely generated over $o$, because $M$ is admissible, and so is $\text{Ker}(\text{Tr}_{H_0/H_k}) \leq M$. \hfill $\Box$

The kernel of the natural homomorphism $\Lambda(N_0/H_k)/\wp^h \to \Lambda(N_0/H_0)/\wp \cong k[[X]]$ is a nilpotent prime ideal in the ring $\Lambda(N_0/H_k)/\wp^h$. We denote by $\Lambda(N_0/H_k)/\wp^h[1/X]$ the localization at this ideal. For the justification of this notation note that any element in $\Lambda(N_0/H_k)/\wp^h[1/X]$ can uniquely be written as a formal Laurent-series $\sum_{n \geq -\infty} a_nX^n$ with coefficients $a_n$ in the finite group ring $o/\wp^h[H_0/H_k]$. Here $X$—by an abuse of notation—denotes the element $[u_0] - 1$ for an element $u_0 \in N_0,0 \leq N_0$ with $\ell(u_0) = 1 \in \mathbb{Z}_p$. The ring $\Lambda(N_0/H_k)/\wp^h[1/X]$ admits a conjugation action of the group $\Gamma$ that commutes with the operator $\varphi$ defined by $\varphi(\lambda) := s\lambda s^{-1}$ (for $\lambda \in \Lambda(N_0/H_k)/\wp^h[1/X]$). A $(\varphi, \Gamma)$-module
over $\Lambda(N_0/H_k) / \varpi^h[1/X]$ is a finitely generated module over $\Lambda(N_0/H_k) / \varpi^h[1/X]$ together with a semilinear commuting action of $\varphi$ and $\Gamma$. Note that $\varphi$ is no longer injective on $\Lambda(N_0/H_k) / \varpi^h[1/X]$ for $k \geq 1$, in particular it is not flat either. However, we still call a $(\varphi, \Gamma)$-module $D_k$ over $\Lambda(N_0/H_k) / \varpi^h[1/X]$ étale if the natural map

$$1 \otimes \varphi: \Lambda(N_0/H_k) / \varpi^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k) / \varpi^h[1/X]} D_k \rightarrow D_k$$

is an isomorphism of $\Lambda(N_0/H_k) / \varpi^h[1/X]$-modules. For an object $M \in \mathcal{M}(\pi^H)$ we put

$$M^\vee_k[1/X] := \Lambda(N_0/H_k) / \varpi^h[1/X] \otimes_{\Lambda(N_0/H_k) / \varpi^h} M^\vee_k$$

where $(\cdot)^\vee$ denotes the Pontryagin dual $\text{Hom}_o(\cdot, K/o)$.

The group $N_0/H_k$ acts by conjugation on its finite normal subgroup $H_0/H_k$. Therefore the kernel of this action has finite index. In particular, there exists a positive integer $r$ such that $s^r N_0 H_k \leq N_0/H_k$ commutes with $H_0/H_k$. Therefore the group ring $o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k]$ is a subring of $\Lambda(N_0/H_k) / \varpi^h[1/X]$.

**Lemma 2.4.** As modules over the group ring $o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k]$ we have an isomorphism

$$M^\vee_k[1/X] \rightarrow o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k] \otimes_{o/\varpi^h \langle \varphi^r(X) \rangle} Y^\vee_k[1/X] .$$

In particular, as a representation of the finite group $H_0/H_k$ the module $M^\vee_k[1/X]$ is induced, so the reduced (Tate-) cohomology groups $\widehat{H}^i(H', M^\vee_k[1/X])$ vanish for all subgroups $H' \leq H_0/H_k$ and $i \in \mathbb{Z}$.

**Proof.** By the definition of $M_k$ we have a surjective $o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k]$-linear map

$$f: o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k] \otimes_{o/\varpi^h \langle \varphi^r(X) \rangle} Y_k \rightarrow M_k$$

sending $\lambda \otimes y$ to $\lambda y$ for $\lambda \in o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k]$ and $y \in Y_k$. Further, by Lemma 2.3 the kernel of the restriction of $f$ to the $H_0/H_k$-invariants

$$(o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k] \otimes_{o/\varpi^h \langle \varphi^r(X) \rangle} Y_k)_{H_0/H_k} = (\sum_{h \in H_0/H_k} h) \otimes Y_k$$

is finitely generated over $o$. By taking the Pontryagin dual of $f$ and inverting $X$ we obtain an injective $o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k]$-homomorphism

$$f^\vee[1/X]: M^\vee_k[1/X] \rightarrow (o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k] \otimes_{o/\varpi^h \langle \varphi^r(X) \rangle} Y_k)^\vee[1/X] \cong$$

$$\cong o/\varpi^h(\langle \varphi^r(X) \rangle)[H_0/H_k] \otimes_{o/\varpi^h \langle \varphi^r(X) \rangle} (Y^\vee_k[1/X])$$

that becomes surjective after taking $H_0/H_k$-coinvariants. Since $M^\vee_k[1/X]$ is a finite dimensional representation of the finite $p$-group $H_0/H_k$ over the local artinian ring $o/\varpi^h(\langle X \rangle)$ with residual characteristic $p$, the map $f^\vee[1/X]$ is in fact an isomorphism as its cokernel has trivial $H_0/H_k$-coinvariants.

Denote by $H_{k,-}/H_k$ the kernel of the group homomorphism $s(\cdot)s^{-1}: N_0/H_k \rightarrow N_0/H_k$. It is a finite normal subgroup contained in $H_0/H_k \leq N_0/H_k$. If $k$ is big enough so that $H_k$ is contained in $sH_0s^{-1}$ then we have $H_{k,-} = s^{-1}H_k s$, otherwise we always have $H_{k,-} = H_0 \cap$
The ring homomorphism $\varphi: \Lambda(N_0/H_k)/\varpi^h \to \Lambda(N_0/H_k)/\varpi^h$ factors through the quotient $\Lambda(N_0/H_{k,-})/\varpi^h$ of $\Lambda(N_0/H_k)/\varpi^h$. We denote by $\tilde{\varphi}$ the induced ring homomorphism $\tilde{\varphi}: \Lambda(N_0/H_{k,-})/\varpi^h \to \Lambda(N_0/H_k)/\varpi^h$. Note that $\tilde{\varphi}$ is injective and makes $\Lambda(N_0/H_k)/\varpi^h$ a free module of rank

$$\nu := |\text{Coker}(s\cdot)s^{-1}: N_0/H_k \to N_0/H_k| = p|\text{Coker}(s\cdot)s^{-1}: H_0/H_k \to H_0/H_k| = p|\text{Ker}(s\cdot)s^{-1}: H_0/H_k \to H_0/H_k| = p|H_{k,-}/H_k|$$

over $\Lambda(N_0/H_{k,-})/\varpi^h$.

**Lemma 2.5.** We have a series of isomorphisms of $\Lambda(N_0/H_k)/\varpi^h[1/X]$-modules

$$\begin{align*}
\text{Tr}^{-1} = \text{Tr}^{-1}_{H_{k,-}/H_k}: (\Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h} M_k)^{\vee}[1/X] & \overset{(1)}{\to} \\
& \overset{(1)}{\to} \text{Hom}_{\Lambda(N_0/H_k)}(\Lambda(N_0/H_k), M_k'[1/X]) \\
& \overset{(2)}{\to} \text{Hom}_{\Lambda(N_0/H_{k,-})}(\Lambda(N_0/H_k), (M_k'[1/X])_{H_{k,-}}) \\
& \overset{(3)}{\to} \Lambda(N_0/H_k) \otimes_{\Lambda(N_0/H_{k,-})} \tilde{\varphi}(M_k'[1/X])_{H_{k,-}} \\
& \overset{(4)}{\to} \Lambda(N_0/H_k) \otimes_{\Lambda(N_0/H_{k,-})} \tilde{\varphi}(M_k'[1/X])_{H_{k,-}} \\
& \overset{(5)}{\to} \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h, \tilde{\varphi}} M_k'[1/X].
\end{align*}$$

**Proof.** (1) follows from the adjoint property of $\otimes$ and $\text{Hom}$. The second isomorphism follows from noting that the action of the ring $\Lambda(N_0/H_k)$ over itself via $\varphi$ factors through the quotient $\Lambda(N_0/H_{k,-})$ therefore $H_{k,-}$ acts trivially on $\Lambda(N_0/H_k)$ via this map. So any module-homomorphism $\Lambda(N_0/H_k) \to M_k'[1/X]$ lands in the $H_{k,-}$-invariant part $M_k'[1/X]_{H_{k,-}}$ of $M_k'[1/X]$. The third isomorphism follows from the fact that $\Lambda(N_0/H_k)$ is a free module over $\Lambda(N_0/H_{k,-})$ via $\tilde{\varphi}$. The fourth isomorphism is given by (the inverse of) the trace map $\text{Tr}_{H_{k,-}/H_k}: (M_k'[1/X])_{H_{k,-}} \to M_k'[1/X]_{H_{k,-}}$ which is an isomorphism by Lemma 2.4. The last isomorphism follows from the isomorphism $(M_k'[1/X]_{H_{k,-}})_{H_{k,-}} \cong \Lambda(N_0/H_{k,-}) \otimes_{\Lambda(N_0/H_k)} M_k'[1/X]$. \hfill $\Box$

**Remark.** Here $\varphi$ always acted only on the ring $\Lambda(N_0/H_k)$, hence denoting $\varphi_t$ the action $n \mapsto tnt^{-1}$ for a fixed $t \in T_+$ and choosing $k$ large enough such that $tH_0t^{-1} \geq H_k$ we get analogously an isomorphism

$$\begin{align*}
\text{Tr}^{-1}_{t^{-1}H_k/tH_k}: (\Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h} M_k)^{\vee}[1/X] & \to \\
& \to \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h, \varphi_t} M_k'[1/X].
\end{align*}$$

We denote the composite of the five isomorphisms in Lemma 2.5 by $\text{Tr}^{-1}$ emphasising that all but (4) are tautologies. Our main result in this section is the following generalization of Lemma 2.6 in [2].

**Proposition 2.6.** The map

$$\begin{align*}
\text{Tr}^{-1} \circ (1 \otimes F_k)^{\vee}[1/X]: M_k'[1/X] \to \Lambda(N_0/H_k)/\varpi^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/\varpi^h[1/X]} M_k'[1/X]
\end{align*}$$

(3)
is an isomorphism of $\Lambda(N_0/H_k)/ω^h[1/X]$-modules. Therefore the natural action of $Γ$ and the operator

$$\varphi : M_k^\nu[1/X] \to M_k^\nu[1/X]$$

$$f \mapsto (\text{Tr}^{-1} \circ (1 \otimes F_k)^\nu[1/X])^{-1}(1 \otimes f)$$

make $M_k^\nu[1/X]$ into an étale $(\varphi, Γ)$-module over the ring $\Lambda(N_0/H_k)/ω^h[1/X]$.

**Proof.** Since $M_k$ is finitely generated over $\Lambda(N_0/H_k)/ω^h[F_k]$ by Lemma 2.2, the cokernel $C$ of the map

$$1 \otimes F_k : \Lambda(N_0/H_k)/ω^h \otimes_{\varphi, \Lambda(N_0/H_k)/ω^h} M_k \to M_k \quad (4)$$

is finitely generated as a module over $\Lambda(N_0/H_k)/ω^h$. Further, it is admissible as a representation of $N_0$ (again by Lemma 2.2), therefore $C$ is finitely generated over $o$. In particular, we have $C^\nu[1/X] = 0$ showing that (3) is injective.

For the surjectivity put $Y_k := \sum_{u \in \mathcal{J}(N_0,o/sN_0,s^{-k})} u \text{Tr}_{H_k/s^kH_k/s^k}(s^kM)$. This is an $o/ω^h[1/X]$-submodule of $M_k$. By Lemma 2.3 we have

$$\text{length}_{o/ω^h(φ^r(X))}(Y_k^\nu[1/X]) = |N_{0,0} : s^rN_{0,0}s^{-r}|\text{length}_{o/ω^h(\{\})}(Y_k^\nu[1/X]) = p^r l.$$

By Lemma 2.4 we obtain

$$\text{length}_{o/ω^h(φ^r(X))}M_k^\nu[1/X] = |H_0 : H_k| \cdot \text{length}_{o/ω^h(φ^r(X))}Y_k^\nu[1/X] = |H_0 : H_k|p^r l.$$

Consider the ring homomorphism

$$\varphi : \Lambda(N_0/H_k)/ω^h[1/X] \to \Lambda(N_0/H_k)/ω^h[1/X]. \quad (5)$$

Its image is the subring $\Lambda(sN_0s^{-1}H_k/H_k)/ω^h[1/φ(X)]$ over which $\Lambda(N_0/H_k)/ω^h[1/X]$ is a free module of rank $ν = |N_0 : sN_0s^{-1}H_k| = p|H_k, \ldots : H_k|$. So we obtain

$$\text{plength}_{o(φ^r(X))} \Lambda(N_0/H_k)/ω^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/ω^h[1/X]} M_k^\nu[1/X] = = \text{length}_{o(φ^r+1(X))} \Lambda(N_0/H_k)/ω^h[1/X] \otimes_{\varphi, \Lambda(N_0/H_k)/ω^h[1/X]} M_k^\nu[1/X] = = ν\text{length}_{o(φ^r(X))} \Lambda(sN_0s^{-1}H_k/H_k)/ω^h[1/φ(X)] \otimes_{\varphi, \Lambda(N_0/H_k)/ω^h[1/X]} M_k^\nu[1/X] \overset{(\ast)}{=} = ν\text{length}_{o(φ^r(X))} M_k^\nu[1/X]_{H_k, \ldots} = ν\text{length}_{o(φ^r(X))}(o/ω^h[H_0/H_{k, \ldots} \otimes_o ω^h Y_k^\nu[1/X]]) = = ν|H_0 : H_k|p^r l = p|H_0 : H_k|p^r l = p\text{length}_{o/ω^h(φ^r(X))} M_k^\nu[1/X].$$

Here the equality $(\ast)$ follows from the fact that the map $φ$ induces an isomorphism between $\Lambda(N_0/H_{k, \ldots})/ω^h[1/X]$ and $\Lambda(sN_0s^{-1}H_k/H_k)/ω^h[1/φ(X)]$ sending the subring $o((φ^r(\{\}))$ isomorphically onto $o((φ^{r+1}(\{\))))$.

This shows that (5) is an isomorphism as it is injective and the two sides have equal length as modules over the artinian ring $o/ω^h(\{\})$. □

**Remark.** We also obtain in particular that the map (4) has finite kernel and cokernel. Hence there exists a finite $\Lambda(N_0/H_k)/ω^h$-submodule $M_{k,*}$ of $M_k$ such that the kernel of $1 \otimes F_k$ is contained in the image of $\Lambda(N_0/H_k)/ω^h \otimes_{φ} M_{k,*}$ in $\Lambda(N_0/H_k)/ω^h \otimes_{φ} M_k$. We denote by $M_k^*$ the image of $1 \otimes F_k$. 11
Note that for $k = 0$ we have $M_0 = M$. Let now $0 \leq j \leq k$ be two integers. By Lemma 2.4 the space of $H_j$-invariants of $M_k$ is equal to $\text{Tr}_{H_j/H_k}(M_k)$ up to finitely generated modules over $\sigma$. On the other hand, we compute

$$N_0F_j^{k-j}(M_j) = N_0\text{Tr}_{H_j/sH_0s^{-k}}(s^{k-j}) \circ \text{Tr}_{H_j/sH_0s^{-j}}(s^j M) =$$

$$= N_0\text{Tr}_{H_j/sH_0s^{-k}}(s^k M) = N_0\text{Tr}_{H_j/H_k} \circ \text{Tr}_{H_k/sH_0s^{-k}}(s^k M) =$$

$$= \text{Tr}_{H_j/H_k}(N_0\text{Tr}_{H_k/sH_0s^{-k}}(s^k M)) = \text{Tr}_{H_j/H_k}(M_k)$$

since both $H_k$ and $H_j$ are normal in $N_0$ whence we have $(u \cdot) \circ \text{Tr}_{H_j/H_k} = \text{Tr}_{H_j/H_k} \circ (u \cdot)$ for all $u \in N_0$. So taking $H_j/H_k$-coinvariants of $M_k^\vee[1/X]$, we have a natural identification

$$M_k^\vee[1/X]_{H_j/H_k} \cong (M_k^{H_j/H_k})^\vee[1/X] \cong (\text{Tr}_{H_j/H_k}(M_k))^\vee[1/X] = (N_0F_j^{k-j}(M_j))^\vee[1/X] \cong M_k^\vee[1/X]$$

(6)

induced by the inclusion $N_0F_j^{k-j}(M_j) \subseteq M_k^{H_j} \subseteq M_k$.

**Lemma 2.7.** We have $\text{Tr}_{H_j/H_k} \circ F_k = F_j \circ \text{Tr}_{H_j/H_k}$.

**Proof.** We compute

$$\text{Tr}_{H_j/H_k} \circ F_k = \text{Tr}_{H_j/H_k} \circ \text{Tr}_{H_k/sH_0s^{-1}} \circ (s \cdot) = \text{Tr}_{H_j/sH_0s^{-1}} \circ (s \cdot) =$$

$$= \text{Tr}_{H_j/sH_0s^{-1}} \circ \text{Tr}_{sH_0s^{-1}/sH_0s^{-1}}(s \cdot) = \text{Tr}_{H_j/sH_0s^{-1}}(s \cdot) \text{Tr}_{H_j/H_k} = F_j \circ \text{Tr}_{H_j/H_k}.$$

\[\square\]

**Proposition 2.8.** The identification (6) is $\varphi$ and $\Gamma$-equivariant.

**Proof.** It suffices to treat the case when $k$ is large enough so that we have $H_{k-} = s^{-1}H_k s$. So from now on we assume $H_k \leq sH_0s^{-1} \leq sN_0s^{-1}$. As $\Gamma$ acts both on $M_k$ and $M_j$ by multiplication coming from the action of $\Gamma$ on $\pi$, the map (6) is clearly $\Gamma$-equivariant. In order to avoid confusion we are going to denote the map $\varphi$ on $M_k^\vee[1/X]$ (resp. on $M_j^\vee[1/X]$) temporarily by $\varphi_k$ (resp. by $\varphi_j$). Let $f$ be in $M_k^\vee$ such that its restriction to $M_{k^*}$ is zero (see the Remark after Prop. 2.6). We regard $f$ as an element in $(M_k^{s}/M_{k^*})^\vee \leq (M_k^\vee)^\vee$. We are going to compute $\varphi_k(f)$ and $\varphi_j(f|_{\text{Tr}_{H_j/H_k}(M_k^s)})$ explicitly and find that the restriction of $\varphi_k(f)$ to $\text{Tr}_{H_j/H_k}(M_k^s)$ is equal to $\varphi_j(f|_{\text{Tr}_{H_j/H_k}(M_k^s)})$. Note that we have an isomorphism $M_k^\vee[1/X] \cong M_k^{\vee\vee}[1/X] \cong (M_k^{s}/M_{k^*})^\vee[1/X]$ (resp. $M_j^\vee[1/X] \cong \text{Tr}_{H_j/H_k}(M_k^s)^\vee[1/X]$).

Let $m \in M_k^{s} \leq M_k$ be in the form

$$m = \sum_{u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)$$

with elements $m_u \in M_k$ for $u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})$. By the remark after Proposition 2.6, $M_k^{s}$ is a finite index submodule of $M_k$. Note that the elements $m_u$ are unique up to $M_{k^*} + \text{Ker}(F_k)$. Therefore $\varphi_k(f) \in (M_k^{\vee})^\vee$ is well-defined by our assumption that $f|_{M_{k^*}} = 0$.
noting that the kernel of $F_k$ equals the kernel of $\text{Tr}_{H_k,-/H_k}$ since the multiplication by $s$ is injective and we have $F_k = s \circ \text{Tr}_{H_k,-/H_k}$. So we compute

$$\varphi_k(f)(m) = ((1 \otimes F_k)^{-1}(\text{Tr}_{H_k,-/H_k}(1 \otimes f))(m) =$$

$$= ((1 \otimes F_k)^{-1}(1 \otimes \text{Tr}_{H_k,-/H_k}(f)) \sum_{u \in J((N_0/H_k)/s(N_0/H_k)s^{-1})} uF_k(m_u)) =$$

$$= \text{Tr}_{H_k,-/H_k}(f)(F_k^{-1}(u_0F_k(m_u))) = f(\text{Tr}_{H_k,-/H_k}((s^{-1}u_0s)m_u))$$

(7)

where $u_0$ is the single element in $J(N_0/sN_0s^{-1})$ corresponding to the coset of 1. In order to simplify notation put $f_s$ for the restriction of $f$ to $\text{Tr}_{H_j/H_k}(M_k)$ and

$$U := J(N_0/sN_0s^{-1}) \cap H_j sN_0s^{-1}.$$ 

Note that we have $0 = \varphi_j(f_s)(uF_j(m'))$ for all $m' \in M_j$ and $u \in J(N_0/sN_0s^{-1}) \setminus U$. Therefore using Lemma 2.7 we obtain

$$\varphi_j(f_s)(\text{Tr}_{H_j/H_k} m) = \varphi_j(f_s)(\text{Tr}_{H_j/H_k} \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u)) =$$

$$= \varphi_j(f_s)(\sum_{u \in J(N_0/sN_0s^{-1})} uF_j \circ \text{Tr}_{H_j/H_k}(m_u)) =$$

$$= \sum_{u \in U} f(\text{Tr}_{H_j,-/H_k}(s^{-1}us\text{Tr}_{H_j/H_k}(m_u))) =$$

$$= \sum_{u \in U} f(s^{-1}us\text{Tr}_{H_j,-/H_k}(m_u))$$

(8)

where for each $u \in U$ we choose a fixed $\overline{u}$ in $sN_0s^{-1} \cap H_j u$. Note that $f(s^{-1}\overline{u}s\text{Tr}_{H_j,-/H_k}(m_u))$ does not depend on this choice: If $\overline{u} \overline{t} \in sN_0s^{-1} \cap H_j u$ is another choice then we have $(\overline{u} \overline{t})^{-1}\overline{u} \overline{s} \in sN_0s^{-1} \cap H_j$ whence $s^{-1}(\overline{u} \overline{t})^{-1}\overline{u}s$ lies in $H_j,$ $= N_0 \cap s^{-1}H_j s$ so we have

$$s^{-1}\overline{u}s\text{Tr}_{H_j,-/H_k}(m_u) = s^{-1}\overline{u}ss^{-1}(\overline{t}^{-1})^{-1}\overline{s}\text{Tr}_{H_j,-/H_k}(m_u) = s^{-1}\overline{u}t\text{Tr}_{H_j,-/H_k}(m_u).$$

Moreover, the equation 8 also shows that $\varphi_j(f_s)$ is a well-defined element in $(\text{Tr}_{H_j/H_k}(M_k^*))^\lor$. On the other hand, for the restriction of $\varphi_k(f)$ to $\text{Tr}_{H_j/H_k}(M_k)$ we compute

$$\varphi_k(f)(\text{Tr}_{H_j/H_k} m) = \varphi_k(f)(\sum_{w \in J(H_j/H_k)} w \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u)) =$$

$$= \sum_{w \in J(H_j/H_k)} \sum_{u \in J(N_0/sN_0s^{-1})} \varphi_k(f)(wuF_k(m_u)) =$$

$$= \sum_{u \in U} f(\text{Tr}_{H_k,-/H_k}((s^{-1}us)m_u)) =$$

$$= f(\sum_{\overline{u} : s^{-1}\overline{u}us \in J(H_j,-/H_k,-)} \text{Tr}_{H_k,-/H_k} us^{-1}\overline{u}sm_u) =$$

$$= \sum_{u \in U} f(s^{-1}\overline{u}s\text{Tr}_{H_j,-/H_k}(m_u))$$
that equals $\varphi_j(f_*)(\text{Tr}_{H_j/H_k} m)$ by \[(5). Finally, let now $f \in M_k^{\vee}$ be arbitrary. Since $M_{k,s}$ is finite, there exists an integer $r \geq 0$ such that $X^r f$ vanishes on $M_{k,s}$. By the above discussion we have $\varphi_k(X^r f)(\text{Tr}_{H_j/H_k} m) = \varphi_j(X^r f_*)(\text{Tr}_{H_j/H_k} m)$. The statement follows noting that $\varphi(X^r)$ is invertible in the ring $\Lambda(N_0/H_j)/\varpi^h[1/X]$.

So we may take the projective limit $M_k^{\vee}[1/X] := \varprojlim_k M_k^{\vee}[1/X]$ with respect to these quotient maps. The resulting object is an étale $(\varphi, \Gamma)$-module over the ring

$$\varprojlim_k \Lambda(N_0/H_k)/\varpi^h[1/X] \cong \Lambda(N_0)/\varpi^h.$$ Moreover, by taking the projective limit of \[(6) with respect to $k$ we obtain a $\varphi$- and $\Gamma$-equivariant isomorphism $(M_k^{\vee}[1/X])_{H_j} \cong M_j^{\vee}[1/X]$. So we just proved

**Corollary 2.9.** For any object $M \in \mathcal{M}(\pi^{H_0})$ the $(\varphi, \Gamma)$-module $M^{\vee}[1/X]$ over $\mathfrak{o}/\varpi^h((X))$ corresponds to $M_k^{\vee}[1/X]$ via the equivalence of categories in Theorem 8.20 in \[9\].

Note that whenever $M \subseteq M'$ are two objects in $\mathcal{M}(\pi^{H_0})$ then we have a natural surjective map $M'^{\vee}[1/X] \rightarrow M_k^{\vee}[1/X]$. So in view of the above corollary we define

$$D_{\xi, \Gamma, \infty}(\pi) := \varprojlim_{k \geq 0, M \in \mathcal{M}(\pi^{H_0})} M_k^{\vee}[1/X] = \varprojlim_{M \in \mathcal{M}(\pi^{H_0})} M_k^{\vee}[1/X].$$

We call two elements $M, M' \in \mathcal{M}(\pi^{H_0})$ equivalent ($M \sim M'$) if the inclusions $M \subseteq M + M'$ and $M' \subseteq M + M'$ induce isomorphisms $M^{\vee}[1/X] \cong (M + M')^{\vee}[1/X] \cong M'^{\vee}[1/X]$. This is equivalent to the condition that $M_k$ equals $M'$ up to finitely generated $\mathfrak{o}$-modules. In particular, this is an equivalence relation on the set $\mathcal{M}(\pi^{H_0})$. Similarly, we say that $M_k, M'_k \in \mathcal{M}(\pi^{H_k})$ are equivalent if the inclusions $M_k \subseteq M_k + M'_k$ and $M'_k \subseteq M_k + M'_k$ induce isomorphisms $M^{\vee}_k[1/X] \cong (M_k + M'_k)^{\vee}[1/X] \cong M^{\vee'}_k[1/X]$.\[\]

**Proposition 2.10.** The maps

$$M \mapsto N_0 \text{Tr}_{H_0/s^k H_0 s^{-k}}(s^k M)$$

$$\text{Tr}_{H_0/H_k}(M_k) \mapsto M_k$$

induce a bijection between the sets $\mathcal{M}(\pi^{H_0})/\sim$ and $\mathcal{M}(\pi^{H_0})/\sim$. In particular, we have

$$D_{\xi, \Gamma, \infty}(\pi) = \varprojlim_{k \geq 0, M_k \in \mathcal{M}(\pi^{H_k})} M_k^{\vee}[1/X].$$

**Proof.** We have $\text{Tr}_{H_0/H_k}(N_0 \text{Tr}_{H_0/s^k H_0 s^{-k}}(s^k M)) = N_0 \text{Tr}_{H_0/s^k H_0 s^{-k}}(s^k M) = N_0 F^k(M)$ which is equivalent to $M$. Conversely, $N_0 \text{Tr}_{H_k/s^k H_k s^{-k}}(s^k \text{Tr}_{H_0/H_k}(M_k)) = N_0 \text{Tr}_{H_k/s^k H_k s^{-k}}(s^k M_k) = N_0 F^k(M_k)$ is equivalent to $M_k$ as it is the image of the map

$$1 \otimes F^k_k : \Lambda(N_0/H_k)/\varpi^h \otimes_{\mathfrak{o}^k, \Lambda(N_0/H_k)/\varpi^h} M_k$$

having finite cokernel.
We equip the pseudocompact $\Lambda_\ell(N_0)$-module $D^\vee_{\xi,\ell,\infty}(\pi)$ with the weak topology, i.e., with the projective limit topology of the weak topologies of $M^\vee_\infty[1/X]$. (The weak topology on $\Lambda_\ell(N_0)$ is defined in section 8 of [3].) Recall that the sets
\[
O(M, l, l') := f_{M,l}^{-1}(\Lambda(N_0/H_l) \otimes_{u_\alpha} X^\prime \, M^\vee[1/X]^{++})
\]
for $l, l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ form a system of neighbourhoods of 0 in the weak topology of $D^\vee_{\xi,\ell,\infty}(\pi)$. Here $f_{M,l}$ is the natural projection map $f_{M,l} : D^\vee_{\xi,\ell,\infty}(\pi) \to M^\vee[1/X]$ and $M^\vee[1/X]^{++}$ denotes the set of elements $d \in M^\vee[1/X]$ with $\varphi^n(d) \to 0$ in the weak topology of $M^\vee[1/X]$ as $n \to \infty$.

3 A natural transformation from the Schneider–Vigneras $D$-functor to $D^\vee_{\xi,\ell,\infty}$

In order to avoid confusion we denote by $D_{SV}(\pi)$ the $\Lambda(N_0)$-module with an action of $B_{\pi}^{-1}$ associated to the smooth $o$-torsion representation $\pi$ defined as $D(\pi)$ in [5] (note that in [5] the notation $V$ is used for the $o$-torsion representation that we denote by $\pi$). For a brief review of this functor see section 1.2.

**Lemma 3.1.** Let $W$ be in $\mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$. There exists a positive integer $k_0 > 0$ such that for all $k \geq k_0$ we have $s^k \, M \subseteq W$. In particular, both $M_k = N_0 \text{Tr}_{H_k/s^{k}H_0} (s^k \, M)$ and $N_0 F^k(M)$ are contained in $W$ for all $k \geq k_0$.

**Proof.** By the assumption that $M$ is finitely generated over $\Lambda(N_0/H_0)/\varpi^h[F]$ and $W$ is a $B_+\text{-subrepresentation}$ it suffices to find an integer $s^{k_0}$ such that we have $s^{k_0} \, m_i$ lies in $W$ for all the generators $m_1, \ldots, m_r$ of $M$. This, however, follows from Lemma 2.1 in [5] noting that the powers of $s$ are cofinal in $T_+$. \hfill \Box

In particular, we have a homomorphism $W^\vee \to M^\vee_k$ of $\Lambda(N_0)$-modules induced by this inclusion. We compose this with the localisation map $M^\vee_k \to M^\vee_k[1/X]$ and take projective limits with respect to $k$ in order to obtain a $\Lambda(N_0)$-homomorphism
\[
\text{pr}_{W,M} : W^\vee \to M^\vee_\infty[1/X].
\]

**Lemma 3.2.** The map $\text{pr}_{W,M}$ is $\psi_s$- and $\Gamma$-equivariant.

**Proof.** The $\Gamma$-equivariance is clear as it is given by the multiplication by elements of $\Gamma$ on both sides. For the $\psi_s$-equivariance let $k > 0$ be large enough so that $H_k$ is contained in $sH_0s^{-1} \leq sN_0s^{-1} \text{(ie. } H_{k_r} = s^{-1} H_k s \text{)}$ and $M_k$ is contained in $W$. Let $f$ be in $W^\vee = \text{Hom}_\alpha(W, o/\varpi^h)$ such that $f|_{N_0 s M_k} = 0$. By definition we have $\psi_s(f)(w) = f(sw)$ for any $w \in W$. Denote the restriction of $f$ to $M_k$ by $f|_{M_k}$ and choose an element $m \in M_k^\vee \leq M_k$ written in the form
\[
m = \sum_{u \in J(N_0/sN_0s^{-1})} uF_k(m_u) = \sum_{u \in J(N_0/sN_0s^{-1})} uF_{H_{k_r}/H_k}(m_u).
\]
Then we compute
\[
 f|_{M_k}(m) = \sum_{u \in J(N_0/sN_0s^{-1})} f(usTr_{H_k,\gamma/H_k}(m_u)) = \\
 = \sum_{u \in J(N_0/sN_0s^{-1})} (u^{-1}f)(sTr_{H_k,\gamma/H_k}(m_u)) = \\
 = \sum_{u \in J(N_0/sN_0s^{-1})} \psi_s(u^{-1}f)(Tr_{H_k,\gamma/H_k}(m_u)) = \\
 \phi \sum_{u \in J(N_0/sN_0s^{-1})} \varphi(\psi_s(u^{-1}f)|_{M_k})(F_k(m_u)) = \\
 = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)|_{M_k})(uF_k(m_u)) = \\
 = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi_s(u^{-1}f)|_{M_k})(m)
\]
as for distinct \( u, v \in J(N_0/sN_0s^{-1}) \) we have \( u\varphi(f_0)(vF_k(m_v)) = 0 \) for any \( f_0 \in (M_k^\vee) \). So by inverting \( X \) and taking projective limits with respect to \( k \) we obtain
\[
 pr_{W,M}(f) = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(pr_{W,M}(\psi_s(u^{-1}f)))
\]
as we have \((M_k^\vee)[1/X] \cong M_k[1/X]\). However, since \( M_k^\vee[1/X] \) is an étale \((\varphi, \Gamma)\)-module over \( \Lambda_\ell(N_0)/\varpi^h \) we have a unique decomposition of \( pr_{W,M}(f) \) as
\[
 pr_{W,M}(f) = \sum_{u \in J(N_0/sN_0s^{-1})} u\varphi(\psi(u^{-1}pr_{W,M}(f)))
\]
so we must have \( \psi(pr_{W,M}(f)) = pr_{W,M}(\psi_s(f)) \). For general \( f \in W^\vee \) note that \( N_0sM_{k,s} \) is killed by \( \varphi(X^r) \) for \( r \geq 0 \) big enough, so we have \( X^r\psi(pr_{W,M}(f)) = \psi(pr_{W,M}(\varphi(X^r)f)) = pr_{W,M}(\psi_s(\varphi(X^r)f)) = X^rpr_{W,M}(\psi_s(f)) \). The statement follows since \( X^r \) is invertible in \( \Lambda_\ell(N_0) \).

By taking the projective limit with respect to \( M \in \mathcal{M}(\pi^{H_0}) \) and the injective limit with respect to \( W \in B_+(\pi) \) we obtain a \( \psi_-\) and \( \Gamma\)-equivariant \( \Lambda(N_0)\)-homomorphism
\[
 pr := \lim_{W} \lim_{M} pr_{W,M} : D_{SV}(\pi) \to D_{SV,\ell,\infty}(\pi).
\]

**Remarks.**

1. The natural maps \( \pi^\vee \to D_{SV,\ell,\infty}^\vee(\pi) \) and \( \pi^\vee \to D_{SV,\ell,\infty}^\vee(\pi) \) both factor through the map \( \pi^\vee \to D_{SV}(\pi) \).

2. The natural topology on \( D_{SV} \) obtained as the quotient topology from the compact topology on \( \pi^\vee \) via the surjective map \( \pi^\vee \to D_{SV}(\pi) \) is compact, but may not be Hausdorff in general. However, if \( B_+(\pi) \) contains a minimal element (as in the case of the principal series [6]) then it is also Hausdorff. However, the map \( pr \) factors through the maximal Hausdorff quotient of \( D_{SV}(\pi) \), namely \( \overline{D_{SV}(\pi)} := (\bigcap_{W \in B_+(\pi)} W)^\vee \). Indeed, \( pr \) is continuous and \( D_{SV,\ell,\infty}^\vee(\pi) \) is Hausdorff, so the kernel of \( pr \) is closed in \( D_{SV}(\pi) \) (and contains 0).
3. Assume that \( h = 1 \), i.e. \( \pi \) is a smooth representation in characteristic \( p \). Then \( D_{\xi,\ell,\infty}^\prime(\pi) \) has no nonzero \( \Lambda(0)/\varpi \)-torsion. Hence the \( \Lambda(0)/\varpi \)-torsion part of \( D_{SV}(\pi) \) is contained in the kernel of \( \text{pr} \).

4. If \( D_{SV}(\pi) \) has finite rank and its torsion free part is étale over \( \Lambda(0) \) then \( \Lambda(0) \otimes_{\Lambda(0)} D_{SV}(\pi) \) is also étale and of finite rank \( r \) over \( \Lambda(0) \). Moreover, the map \( \Lambda(0) \otimes_{\Lambda(0)} \text{pr} : \Lambda(0) \otimes D_{SV}(\pi) \to D_{\xi,\ell,\infty}(\pi) \) has dense image by Lemma 3.1. Thus \( D_{\xi,\ell,\infty}(\pi) \) has rank at most \( r \) over \( \Lambda(0) \). In particular, for \( \pi \) being the principal series \( D_{SV}(\pi) \) has rank 1 and its torsion free part is étale over \( \Lambda(0) \) (cf. Example 7.6 of [2]), hence we obtained that \( D_{\xi,\ell,\infty}(\pi) \) has rank 1 over \( \Lambda(0) \) (cf. Example 7.6 of [2]).

One can show the above Remark 2 algebraically, too. Let \( M \in \mathcal{M}(\pi^{H_0}) \) be arbitrary. Then the map \( 1 \otimes \text{id}_{M'} : M' \to M'[1/X] \) has finite kernel, so the image \( (1 \otimes \text{id}_{M'})(M') \) is isomorphic to \( M_0' \) for some finite index submodule \( M_0 \leq M \). Moreover, \( M_0' \) is a \( \psi \)- and \( \Gamma \)-invariant trellis in \( D := M'[1/X] = M_0'[1/X] \). Therefore the map \( 1 \otimes (F)^\prime \) is injective on \( M_0' \) since it is injective after inverting \( X \) and \( M_0' \) has no \( X \)-torsion. This means that \( 1 \otimes F : \mathcal{O}/\varpi^h[X] \otimes_{\mathcal{O}/\varpi^h[X],\varphi} M_0 \to M_0 \) is surjective, i.e. we have \( M_0 = N_0 F^k(M_0) \) for all \( k \geq 0 \). However, for any \( W \in \mathcal{B}_+(\pi) \) and \( k \) large enough (depending a priori on \( W \)) we have \( N_0 F^k(M_0) \subseteq W \), so we deduce \( M_0 \subset \cap_{W \in \mathcal{B}_+} W \).

**Corollary 3.3.** If \( \pi = \text{ind}_{B_0}^{B \varphi} \pi_0 \) is a compactly induced representation of \( B \) for some smooth \( o/\varpi^h \)-representation \( \pi_0 \) of \( B_0 \) then we have \( D_{\xi}^\prime(\pi) = 0 \). In particular, \( D_{\xi}^\prime \) is not exact on the category of smooth \( o/\varpi^h \)-representations of \( B \). (However, it may still be exact on a smaller subcategory with additional finiteness conditions.)

**Proof.** By the 2nd remark above the map \( \pi^\prime \to D_{\xi}^\prime(\pi) \) factors through the maximal Hausdorff quotient \( D_{SV}(\pi) \) of \( D_{SV}(\pi) \). By Lemma 3.2 in [8], we have \( D_{SV}(\pi) = (\cap_{\sigma} W_{\sigma})^\prime \) where the \( B_+ \)-subrepresentations \( W_{\sigma} \) are indexed by order-preserving maps \( \sigma : T_+ / T_0 \to \text{Sub}(\pi_0) \) where \( \text{Sub}(\pi_0) \) is the partially order set of \( B_0 \)-subrepresentations of \( \pi_0 \). The explicit description of the \( B_+ \)-subrepresentations \( W_{\sigma} \) (there denoted by \( M_{\sigma} \)) before Lemma 3.2 in [8] shows that we have in fact \( \cap_{\sigma} W_{\sigma} = \{0\} \) whence the natural map \( \pi^\prime \to D_{\xi}^\prime(\pi) \) is zero. However, by the construction of this map this can only be zero if \( D_{\xi}^\prime(\pi) = 0 \).

Since the principal series arises as a quotient of a compactly induced representation, the exactness of \( D_{\xi}^\prime \) would imply the vanishing of \( D_{\xi}^\prime \) on the principal series, too—which is not the case by Ex. 7.6 in [2].

**Proposition 3.4.** Let \( D \) be an étale \( (\varphi, \Gamma) \)-module over \( \Lambda(0)/\varpi^h \), and \( f : D_{SV}(\pi) \to D \) be a continuous \( \psi \)- and \( \Gamma \)-equivariant \( \Lambda(0) \)-homomorphism. Then \( f \) factors uniquely through \( \text{pr} \), i.e. there exists a unique \( \psi \)- and \( \Gamma \)-equivariant \( \Lambda(0) \)-homomorphism \( \hat{f} : D_{\xi,\ell,\infty}(\pi) \to D \) such that \( f = \hat{f} \circ \text{pr} \).

**Proof.** Note that the uniqueness of \( \hat{f} \) follows from Lemma 3.1 since any continuous \( \Lambda(0) \)-homomorphism of \( D_{\xi,\ell,\infty}(\pi) \) factors through \( M_{\infty}[1/X] \) for some \( M \in \mathcal{M}(\pi^{H_0}) \). Indeed, if \( f' \) is another lift then the image of \( \text{pr} \) is contained in the kernel of \( \hat{f} - f' \).

At first we construct a homomorphism \( \hat{f}_{H_0} : D_{\xi}^\prime = (D_{\xi,\ell,\infty})_{H_0} \to D_{H_0} \) such that the
Consider the composite map $f' : \pi^\vee \to D_{SV}(\pi) \xrightarrow{f} D \to D_{H_0}$. Note that $f'$ is continuous and $D_{H_0}$ is Hausdorff, so Ker($f'$) is closed in $\pi^\vee$. Therefore $M_0 = (\pi^\vee / \text{Ker}(f'))^\vee$ is naturally a subspace in $\pi$. We claim that $M_0$ lies in $\mathcal{M}(\pi^{H_0})$. Indeed, $M_0^\vee$ is a quotient of $\pi^{H_0}_0$, hence $M_0 \leq \pi^{H_0}$ and it is $\Gamma$-invariant since $f'$ is $\Gamma$-equivariant. $M_0$ is admissible because it is discrete, hence $M_0^\vee$ is compact, equivalently finitely generated over $0/\varpi^h[[X]]$, because $M_0^\vee$ can be identified with a $0/\varpi^h[[X]]$-submodule of $D_{H_0}$ which is finitely generated over $0/\varpi^h((X))$. The last thing to verify is that $M$ is finitely generated over $0/\varpi^h[[X]][F]$, which follows from the following

**Lemma 3.5.** Let $D$ be an étale $(\varphi, \Gamma)$-module over $0/\varpi^h((X))$ and $D_0 \subset D$ be a $\psi$ and $\Gamma$-invariant compact (or, equivalently, finitely generated) $0/\varpi^h[[X]]$ submodule. Then $D_0^\vee$ is finitely generated as a module over $0/\varpi^h[[X]][F]$ where for any $m \in D_0^\vee = \text{Hom}_0(D_0, 0/\varpi^h)$ we put $F(m)(f) := m(\psi(f))$ (for all $f \in D_0$).

**Proof.** As the extension of finitely generated modules over a ring is again finitely generated, we may assume without loss of generality that $h = 1$ and $D$ is irreducible, i.e. $D$ has no nontrivial étale $(\varphi, \Gamma)$-submodule over $0/\varpi((X))$.

If $D_0 = \{0\}$ then there is nothing to prove. Otherwise $D_0$ contains the smallest $\psi$ and $\Gamma$ stable $0[[X]]$-submodule $D^2$ of $D$. So let $0 \neq m \in D_0^\vee$ be arbitrary such that the restriction of $m$ to $D^2$ is nonzero and consider the $0/\varpi[[X]][F]$-submodule $M := 0/\varpi[[X]][F]m$ of $D_0^\vee$ generated by $m$. We claim that $M$ is not finitely generated over $0$. Suppose for contradiction that the elements $F^nm$ are not linearly independent over $0/\varpi$. Then we have a polynomial $P(x) = \sum_{i=0}^n a_ix^i \in 0/\varpi[x]$ such that $0 = P(F)m(f) = m(\sum a_i\psi^i(f)) = m(P(\psi)f)$ for any $f \in D^2 \subset D_0$. However, $P(\psi) : D^2 \to D^2$ is surjective by Prop. II.5.15. in [2], so we obtain $m|_{D^2} = 0$ which is a contradiction. In particular, we obtain that $M^\vee[1/X] \neq 0$. However, note that $M^\vee[1/X]$ has the structure of an étale $(\varphi, \Gamma)$-module over $0/\varpi((X))$ by Lemma 2.6 in [2]. Indeed, $M$ is admissible, $\Gamma$-invariant, and finitely generated over $0/\varpi[[X]][F]$ by construction. Moreover, we have a natural surjective homomorphism $D = D_0[1/X] = (D_0^\vee)^\vee[1/X] \to M^\vee[1/X]$ which is an isomorphism as $D$ is assumed to be irreducible. Therefore we have $(D_0^\vee/M)^\vee[1/X] = 0$ showing that $D_0^\vee/M$ is finitely generated over $0$. In particular, both $M$ and $D_0^\vee/M$ are finitely generated over $0/\varpi[[X]][F]$ therefore so is $D_0^\vee$.

Now $D_0 = M_0^\vee$ is a $\psi$- and $\Gamma$-invariant $0/\varpi[[X]]$-submodule of $D$ therefore we have an injection $f_0 : M_0^\vee[1/X] \hookrightarrow D$ of étale $(\varphi, \Gamma)$-modules. The map $\hat{f}_{H_0} : D_0^\vee \to D_{H_0}$ is the composite map $D_0^\vee \to M_0^\vee[1/X] \hookrightarrow D$. It is well defined and makes the above diagram commutative, because the map

$$\pi^\vee \to D_{SV}(\pi) \xrightarrow{pr} D_{\xi, \ell, \infty}^\vee(\pi) \xrightarrow{(\cdot)_{H_0}} D_{H_0}(\pi) \xrightarrow{f} D \xrightarrow{(\cdot)_{H_0}} D_{H_0}$$

is the same as $\pi^\vee \to M_0^\vee \to M_0^\vee[1/X]$.
Finally, by Corollary 2.9 \( M'v[1/X] \) (resp. \( D_{H_0} \)) corresponds to \( M''[1/X] \) (resp. to \( D \)) via the equivalence of categories in Theorem 8.20 in [9] therefore \( f_0 \) can uniquely be lifted to a \( \varphi^- \) and \( \Gamma \)-equivariant \( \Lambda(\varnothing(0))-\text{homomorphism} \) \( f : M'v[1/X] \hookrightarrow D \). The map \( \tilde{f} \) is defined as the composite \( D_{\xi,\ell,\infty}^{\ell} \hookrightarrow M''[1/X] \hookrightarrow D \). Now the image of \( f - \tilde{f} \circ \text{pr} \) is a \( \psi^- \)-invariant \( \Lambda(N_0) \)-submodule in \( (H_0 - 1)D \) therefore it is zero by Lemma 8.17 and the proof of Lemma 8.18 in [9]. Indeed, for any \( x \in D_{SV}(\pi) \) and \( k \geq 0 \) we may write \((f - \tilde{f} \circ \text{pr})(x)\) in the form \( \sum_{u \in J(N_0/s^kN_0s^{-k})} u \varphi^k((f - \tilde{f} \circ \text{pr})(\psi^k(u^{-1}x))) \) that lies in \( (H_k - 1)D \).

\[ \square \]

4 Étale hull

In this section we construct the étale hull of \( D_{SV}(\pi) \): an étale \( T^- \)-module \( \tilde{D}_{SV}(\pi) \) over \( \Lambda(N_0) \) with an injection \( \iota : D_{SV}(\pi) \hookrightarrow \tilde{D}_{SV}(\pi) \) with the following universal property: For any étale \( (\varphi, \Gamma) \)-module \( D' \) over \( \Lambda(N_0) \), and \( \psi^- \) and \( \Gamma \)-equivariant map \( f : D_{SV}(\pi) \rightarrow D' \), \( f \) factors through \( \tilde{D}_{SV}(\pi) \), i.e. there exists a unique \( \psi^- \) and \( \Gamma \)-equivariant \( \Lambda(N_0) \)-homomorphism \( \tilde{f} : \tilde{D}_{SV}(\pi) \rightarrow D' \) making the diagram

\[
\begin{array}{ccc}
D_{SV}(\pi) & \xrightarrow{\iota} & \tilde{D}_{SV}(\pi) \\
\downarrow f & & \downarrow \tilde{f} \\
D' & & D'
\end{array}
\]

commutative. Moreover, if we assume further that \( D' \) is an étale \( T^- \)-module over \( \Lambda(N_0) \) and the map \( f \) is \( \psi^- \)-equivariant for all \( t \in T^- \) then the map \( \tilde{f} \) is \( T^- \)-equivariant.

**Definition 4.1.** Let \( D \) be a \( \Lambda(N_0) \)-module and \( T^- \leq T^- \) be a submonoid. Assume moreover that the monoid \( T^- \) (or in the case of \( \psi^- \) actions the inverse monoid \( T^- \)) acts \( \sigma \)-linearly on \( D \), as well.

We call the action of \( T^- \) a \( \varphi^- \)-action (relative to the \( \Lambda(N_0) \)-action) and denote the action of \( t \) by \( d \mapsto \varphi_t(d) \), if for any \( \lambda \in \Lambda(N_0) \), \( t \in T^- \) and \( d \in D \) we have \( \varphi_t(\lambda d) = \lambda \varphi_t(d) \). Moreover, we say that the \( \varphi^- \)-action is injective if for all \( t \in T^- \) the map \( \varphi_t \) is injective. The \( \varphi^- \)-action of \( T^- \) is nondegenerate if for all \( t \in T^- \) we have

\[
D = \sum_{u \in J(N_0/tN_0t^{-1})} \text{Im}(u \circ \varphi_t) = \sum_{u \in J(N_0/tN_0t^{-1})} u(\varphi_t(D)).
\]

We call the action of \( T^- \) a \( \psi^- \)-action of \( T^- \) (relative to the \( \Lambda(N_0) \)-action) and denote the action of \( t^{-1} \in T^- \) by \( d \mapsto \psi_t(d) \), if for any \( \lambda \in \Lambda(N_0) \), \( t \in T^- \) and \( d \in D \) we have \( \psi_t(\lambda d) = \lambda \psi_t(d) \). Moreover, we say that the \( \psi^- \)-action of \( T^- \) is surjective if for all \( t \in T^- \) the map \( \psi_t \) is surjective. The \( \psi^- \)-action of \( T^- \) is nondegenerate if for all \( t \in T^- \) we have

\[
\{0\} = \bigcap_{u \in J(N_0/tN_0t^{-1})} \text{Ker}(\psi_t \circ u^{-1}).
\]

The nondegeneracy is equivalent to the condition that for any \( t \in T^- \) \( \text{Ker}(\psi_t) \) does not contain any nonzero \( \Lambda(N_0) \)-submodule of \( D \).

We say that a \( \varphi^- \) and a \( \psi^- \)-action of \( T^- \) are compatible on \( D \), if
Lemma 4.2. For any $t \in T_\ast$ we have $\psi_t \circ \varphi_t = \text{id}_D$.

(\varphi \psi 1) for any $t \in T_\ast$ we have $\psi_t \circ \varphi_t = \text{id}_D$.

(\varphi \psi 2) for any $t \in T_\ast$, $\lambda \in \Lambda(N_0)$, and $d \in D$ we have $\psi_t(\lambda \varphi_t(d)) = \psi_t(\lambda) d$.

We also consider $\varphi$- and $\psi$-actions of the monoid $\mathbb{Z}_p \setminus \{0\}$ on $\Lambda(N_0)$-modules via the embedding $\xi: \mathbb{Z}_p \setminus \{0\} \to T_\ast$. Modules with a $\varphi$-action (resp. $\psi$-action) of $\mathbb{Z}_p \setminus \{0\}$ are called $(\varphi, \Gamma)$-modules (resp. $(\psi, \Gamma)$-modules).

For example, the natural $\varphi$- and $\psi$-actions of $T_\ast$ on $\Lambda(N_0)$ are compatible.

Remarks. 1. Note that the $\psi$-action of the monoid $T_\ast$ is in fact an action of the inverse monoid $T_\ast^{-1}$. However, we assume $T_\ast$ to be commutative so it may also be viewed as an action of $T_\ast$.

2. Pontryagin duality provides an equivalence of categories between compact $\Lambda(N_0)$-modules with a continuous $\psi$-action of $T_\ast$ and discrete $\Lambda(N_0)$-modules with a continuous $\varphi$-action of $T_\ast$. The surjectivity of the $\psi$-action corresponds to the injectivity of $\varphi$-action. Moreover, the $\psi$-action is nondegenerate if and only if so is the corresponding $\varphi$-action on the Pontryagin dual.

If $D$ is a $\Lambda(N_0)$-module with a $\varphi$-action of $T_\ast$ then there exists a homomorphism

$$\Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_t} D \to D, \lambda \otimes d \mapsto \lambda \varphi_t(d) \quad (10)$$

of $\Lambda(N_0)$-modules. We say that the $T_\ast$-action on $D$ is étale if the above map is an isomorphism. The $\varphi$-action of $T_\ast$ on $D$ is étale if and only if it is injective and for any $t \in T_\ast$ we have

$$D = \bigoplus_{u \in J(N_0/\langle tN_0 \rangle \setminus \{t\})} u \varphi(D) \quad (11)$$

Similarly, we call a $\Lambda(N_0)$-module together with a $\varphi$-action of the monoid $\mathbb{Z}_p \setminus \{0\}$ an étale $(\varphi, \Gamma)$-module over $\Lambda(N_0)$ if the action of $\varphi = \varphi_\ast$ is étale.

If $D$ is an étale $T_\ast$-module over $\Lambda(N_0)$ then there exists a $\psi$-action of $T_\ast$ compatible with the étale $\varphi$-action (see $\S$ Section 6).

Dually, if $D$ is a $\Lambda(N_0)$-module with a $\psi$-action of $T_\ast$ then there exists a map

$$t_\ast: D \to \Lambda(N_0) \otimes_{\Lambda(N_0), \psi_t} D$$

$$d \to \sum_{u \in J(N_0/\langle tN_0 \rangle \setminus \{t\})} u \otimes \psi_t(u^{-1}d).$$

Lemma 4.2. For any $t \in T_\ast$ the map $t_\ast$ is a homomorphism of $\Lambda(N_0)$-modules. It is injective for all $t \in T_\ast$ if and only if the $\psi$-action of $T_\ast$ on $D$ is nondegenerate.

Proof. Fix $t \in T_\ast$. For any $\lambda \in \Lambda(N_0)$ and $u, v \in N_0$ we put $\lambda_{u,v} := \psi_t(u^{-1} \lambda v)$. Note that for any fixed $v \in N_0$ we have

$$\lambda v = \sum_{u \in J(N_0/\langle tN_0 \rangle \setminus \{t\})} u \varphi_t(\lambda_{u,v})$$

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and for any fixed $u \in N_0$ we have
\[ u^{-1} \lambda = \sum_{v \in J(N_0/tN_0 t^{-1})} \varphi_t(\lambda_{u,v}) v^{-1}. \]

So we compute
\[
\iota_t(\lambda x) = \sum_{u, v \in J(N_0/tN_0 t^{-1})} u \otimes \psi_t(u^{-1} \lambda x) = \sum_{u, v \in J(N_0/tN_0 t^{-1})} u \otimes \psi_t(\varphi_t(\lambda_{u,v}) v^{-1} x) = \]
\[
= \sum_{u, v \in J(N_0/tN_0 t^{-1})} u \otimes \lambda_{u,v} \psi_t(v^{-1} x) = \sum_{u, v \in J(N_0/tN_0 t^{-1})} u \psi_t(\lambda_{u,v}) \otimes \psi_t(v^{-1} x) = \]
\[
= \sum_{v \in J(N_0/tN_0 t^{-1})} \lambda v \otimes \psi_t(v^{-1} x) = \lambda \iota_t(x). \]

The second statement follows from noting that $\Lambda(N_0)$ is a free right module over itself via the map $\varphi_t$ with free generators $u \in J(N_0/tN_0 t^{-1})$. \hfill \Box

\textbf{Lemma 4.3.} Let $D$ be a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$ and $t \in T_*$. Then there exists a $\psi$-action of $T_*$ on $\varphi^*_t D := \Lambda(N_0) \otimes_{\Lambda(N_0), \varphi_t} D$ making the homomorphism $\iota_t$ $\psi$-equivariant.

Moreover, if we assume in addition that the $\psi$-action on $D$ is nondegenerate then so is the $\psi$-action on $\varphi^*_t D$.

\textbf{Proof.} Let $t' \in T_*$ be arbitrary and define the action of $\psi_{t'}$ on $\varphi^*_t D$ by putting
\[
\psi_{t'}(\lambda \otimes d) := \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi\left(\lambda \varphi_t(u')\right) \otimes \psi_{t'}(u'^{-1} d) \text{ for } \lambda \in \Lambda(N_0), d \in D,
\]
and extending $\psi_{t'}$ to $\varphi^*_t D$ $o$-linearly. Note that we have
\[
\psi_{t'}(\varphi_{t'}(\mu) \lambda \otimes d) = \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\varphi_{t'}(\mu) \lambda \varphi_t(u')) \otimes \psi_{t'}(u'^{-1} d) = \mu \psi_{t'}(\lambda \otimes d).
\]

Moreover, the map $\psi_{t'}$ is well-defined since we have
\[
\psi_{t'}(\lambda \varphi_t(\mu) \otimes d) = \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(\mu) \varphi_t(u')) \otimes \psi_{t'}(u'^{-1} d) = \]
\[
= \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(\mu u')) \otimes \psi_{t'}(u'^{-1} d) = \]
\[
= \sum_{u', v' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(u')) \varphi_t(\mu u', v') \otimes \psi_{t'}(v'^{-1} d) = \]
\[
= \sum_{u', v' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(u')) \varphi_t(\mu u', v') \psi_{t'}(v'^{-1} d) = \]
\[
= \sum_{u', v' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(u')) \varphi_t(\mu u', v') v'^{-1} d = \]
\[
= \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(u')) \otimes \psi_{t'}(u'^{-1} \mu d) = \psi_{t'}(\lambda \otimes \mu d),
\]
where $\mu_{u',u''} = \psi_{t'}(u'^{-1}\mu u'')$. We further compute
\[
\psi_{t'}(\psi_{t'}(\lambda \otimes d)) = \psi_{t'}\left( \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\lambda \varphi_t(u')) \otimes \psi_{t'}(u'^{-1}d) \right) = \\
\sum_{u'' \in J(N_0/t''N_0 t'^{-1})} \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\psi_{t'}(\lambda \varphi_t(u')) \varphi_t(u'')) \otimes \psi_{t'}(u''^{-1}\psi_{t'}(u'^{-1}d)) = \\
\sum_{u'' \in J(N_0/t''N_0 t'^{-1})} \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \psi_{t'}(\psi_{t'}(\lambda \varphi_t(u'') \varphi_t(u')) \otimes \psi_{t'}(\varphi_t(u''^{-1}u'^{-1}d)) = \\
= \psi_{t'}(\lambda \otimes d)
\]
showing that it is indeed a $\psi$-action of the monoid $T_s$.

For the second statement of the Lemma we compute
\[
\psi_{t'}(t_{x}(x)) = \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \sum_{u \in J(N_0/tN_0 t^{-1})} \psi_{t'}(u \varphi_t(u')) \otimes \psi_{t'}(u'^{-1}\psi_{t'}(u^{-1}x)) = \\
= \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \sum_{u \in J(N_0/tN_0 t^{-1})} \psi_{t'}(u \varphi_t(u')) \otimes \psi_{t'}(\varphi_t(u'^{-1}u^{-1}x)) .
\]

Note that in the above sum $u \varphi_t(u')$ runs through a set of representatives for the cosets $N_0/t't'N_0 t^{-1}$ Moreover, $v := \psi_{t'}(u \varphi_t(u'))$ is nonzero if and only if $u \varphi_t(u')$ lies in $t'N_0 t^{-1}$ and the nonzero values of $v$ run through a set $J(N_0/tN_0 t^{-1})$ of representatives of the cosets $N_0/tN_0 t^{-1}$. In case $v \neq 0$ we have $\psi_{t'}(\varphi_t(u'^{-1}u^{-1}x)) = \psi_{t'}(\varphi_t(u'^{-1}u^{-1}x) \psi_{t'}(x))$. So we obtain
\[
\psi_{t'}(t_{x}(x)) = \sum_{v \in J(N_0/tN_0 t^{-1})} v \otimes \psi_{t'}(\varphi_t(v)x) = \\
= \sum_{v \in J(N_0/tN_0 t^{-1})} v \otimes \psi_{t'}(v^{-1}\psi_{t'}(x)) = t_{x}(\psi_{t'}(x)) .
\]

Assume now that the $\psi$-action of $T_s$ on $D$ is nondegenerate. Any element in $x \in \varphi_t^*D$ can be uniquely written in the form $\sum_{u \in J(N_0/tN_0 t^{-1})} u \otimes x_u$. Assume that for a fixed $t' \in T_s$ we have $\psi_{t'}(u'^{-1}x) = 0$ for all $u' \in N_0$. Then we compute
\[
0 = \psi_{t'}(u'^{-1}x) = \sum_{u' \in J(N_0/t'N_0 t'^{-1})} \sum_{u \in J(N_0/tN_0 t^{-1})} \psi_{t'}(u'^{-1}u \varphi_t(u')) \otimes \psi_{t'}(u'^{-1}x_u) .
\]

Put $y = u'^{-1}u \varphi_t(u')$. For any fixed $u'_0$ the set $\{ y \mid u \in J(N_0/t'N_0 t'^{-1}), u' \in J(N_0/t'N_0 t'^{-1}) \}$ forms a set of representatives of $N_0/t't'N_0 (tt')^{-1}$, and we have $\psi_{t'}(y) \neq 0$ if and only if $y$ lies in $t'N_0 t'^{-1}$ in which case we have $\psi_{t'}(y) = t'^{-1}y t'$. So the nonzero values of $\psi_{t'}(y)$ run through a set of representatives of $N_0/tN_0 t^{-1}$. Since we have the direct sum decomposition $\varphi_t^*D = \bigoplus_{v \in J(N_0/tN_0 t^{-1})} v \otimes D$ we obtain $\psi_{t'}(u'^{-1}x_u) = 0$ for all $u' \in J(N_0/t'N_0 t'^{-1})$ and $u \in J(N_0/tN_0 t^{-1})$ such that $y = u'^{-1}u \varphi_t(u')$ is in $t'N_0 t'^{-1}$. However, for any choice of $u'$ and $u$ there exists such a $u'_0$, so we deduce $x = 0$.

**Proposition 4.4.** Let $D$ be a $\Lambda(N_0)$-module with a $\psi$-action of $T_s$. The following are equivalent:
1. There exists a unique $\varphi$-action on $D$, which is compatible with $\psi$ and which makes $D$ an étale $T_*$-module.

2. The $\psi$-action is surjective and for any $t \in T_*$ we have

$$D = \bigoplus_{u_0 \in J(N_0/tN_0t^{-1})} \bigcap_{u \in J(N_0/tN_0t^{-1}) \atop u \neq u_0} \ker(\psi_t \circ u^{-1}) . \quad (12)$$

In particular, the action of $\psi$ is nondegenerate.

3. The map $\iota_t$ is bijective for all $t \in T_*$.

**Proof.** 1 $\implies$ 3 In this case the map $\iota_t$ is the inverse of the isomorphism (11) so it is bijective by the étale property.

3 $\implies$ 2: The injectivity of $\iota_t$ shows the nondegeneracy of the $\psi$-action. Further if $1 \otimes d = \iota_t(x)$ then we have $\psi_t(x) = d$ so the $\psi$-action is surjective. Moreover, $\iota_t^{-1}(u_0 \otimes D)$ equals $\bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \ker(\psi_t \circ u^{-1})$ therefore $D$ can be written as a direct sum (12).

2 $\implies$ 1: Fix $t \in T_*$. For any $d \in D$ we have to choose $\varphi_t(d)$ such that $\psi_t(\varphi_t(d)) = d$. By the surjectivity of $\psi_t$ we can choose $x \in D$ such that $\psi_t(x) = d$. Using the assumption we can write $x = \sum_{u_0 \in J(N_0/tN_0t^{-1})} x_{u_0}$, with

$$x_{u_0} \in \bigcap_{u \in J(N_0/tN_0t^{-1}) \atop u \neq u_0} \ker(\psi_t \circ u^{-1}) .$$

By the compatibility we should have

$$\varphi_t(d) \in \bigcap_{u \in J(N_0/tN_0t^{-1}) \atop u \neq 1} \ker(\psi_t \circ u^{-1}) .$$

A convenient choice is $\varphi_t(d) = x_1$, and there exists exactly one such element in $D$: if $x'$ would be an other, then

$$x_1 - x' \in \bigcap_{u \in J(N_0/tN_0t^{-1})} \ker(\psi_t \circ u^{-1}) = \{0\} .$$

This shows the uniqueness of the $\varphi$-action and the property $(\varphi \psi 1)$ is also clear. Further, $x_1 = \varphi_t(d) = 0$ would mean that $x$ lies in $\ker(\psi_t)$ whence $d = \psi_t(x) = 0$ – therefore the injectivity. Similarly, by definition we also have $x_{u_0} = u_0 \varphi_t \circ \psi_t(u_0^{-1}x)$ for all $u_0 \in J(N_0/sN_0s^{-1})$. By the surjectivity of the $\psi$-action any element in $D$ can be written of the form $\psi_t(u_0^{-1}x)$ for any fixed $u_0 \in J(N_0/tN_0t^{-1})$ so we obtain

$$u_0 \varphi_t(D) = \bigcap_{u_0 \neq u \in J(N_0/tN_0t^{-1})} \ker(\psi_t \circ u^{-1}) .$$

The étale property (11) follows from this using our assumption 2. Moreover, this also shows $\psi_t(u_0 \varphi_t(d)) = 0$ for all $u \in N_0 \setminus tN_0t^{-1}$ which implies $(\varphi \psi 2)$ using $(\varphi \psi 1)$. Finally, $\varphi_t(\lambda) \varphi_t(d) - \varphi_t(\lambda d)$ lies in the kernel of $\psi_t \circ u_0^{-1}$ for any $u_0 \in J(N_0/tN_0t^{-1})$, $\lambda \in \Lambda(N_0)$ and $d \in D$, so it is zero.
From now on if we have an étale $T_*$-module over $\Lambda(N_0)$ we a priori equip it with the compatible $\psi$-action, and if we have a $\Lambda(N_0)$-module with a $\psi$-action, which satisfies the above property 2, we equip it with the compatible $\varphi$-action, which makes it étale. The construction of the étale hull and its universal property is given in the following

**Proposition 4.5.** For any $\Lambda(N_0)$-module $D$, with a $\psi$-action of $T_*$ there exists an étale $T_*$-module $\widetilde{D}$ over $\Lambda(N_0)$ and a $\psi$-equivariant $\Lambda(N_0)$-homomorphism $\iota: D \to \widetilde{D}$ with the following universal property: For any $\psi$-equivariant $\Lambda(N_0)$-homomorphism $f: D \to D'$ into an étale $T_*$-module $D'$ we have a unique morphism $\tilde{f}: \widetilde{D} \to D'$ of étale $T_*$-modules over $\Lambda(N_0)$ making the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\iota} & \widetilde{D} \\
\downarrow f & & \downarrow \tilde{f} \\
D' & & 
\end{array}
$$

commutative. $\widetilde{D}$ is unique up to a unique isomorphism. If we assume the $\psi$-action on $D$ to be nondegenerate then $\iota$ is injective.

**Proof.** We will construct $\widetilde{D}$ as the injective limit of $\varphi_t^*D$ for $t \in T_*$. Consider the following partial order on the set $T_*$: we put $t_1 \leq t_2$ whenever we have $t_2t_1^{-1} \in T_*$. Note that by Lemma 4.3 we obtain a $\psi$-equivariant isomorphism $\varphi_{t_2t_1}^*\varphi_{t_1}^*D \cong \varphi_{t_2}^*D$ for any pair $t_1 \leq t_2$ in $T_*$. In particular, we obtain a $\psi$-equivariant map $\iota_{t_1,t_2}: \varphi_{t_1}^*D \to \varphi_{t_2}^*D$. Applying this observation to $\varphi_{t_1}^*D$ for a sequence $t_1 \leq t_2 \leq t_3$ we see that the $\Lambda(N_0)$-modules $\varphi_t^*D$ ($t \in T_*$) with the $\psi$-action of $T_*$ form a direct system with respect to the connecting maps $\iota_{t_1,t_2}$. We put

$$
\widetilde{D} := \lim_{t \in T_*} \varphi_t^*D
$$

as a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$. For any fixed $t' \in T_*$ we have

$$
\varphi_{t'}^*\widetilde{D} = \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_{t'}} \lim_{t \in T_*} \varphi_t^*D \cong \lim_{t \in T_*} \Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_t} \varphi_t^*D \cong \lim_{t' \in T_*} \varphi_{t'}^*D \cong \widetilde{D}
$$

showing that there exists a unique $\varphi$-action of $T_*$ on $\widetilde{D}$ making $\widetilde{D}$ an étale $T_*$-module over $\Lambda(N_0)$ by Proposition 4.4.

For the universal property, let $f: D \to D'$ be an $\psi$-equivariant map into an étale $T_*$-module $D'$ over $\Lambda(N_0)$. By construction of the map $\varphi_t$ on $\widetilde{D}$ ($t \in T_*$) we have $\varphi_t(\iota(x)) = (1 \otimes x)_t$ where $(1 \otimes x)_t$ denotes the image of $1 \otimes x \in \varphi_t^*D$ in $\widetilde{D}$. So we put

$$
\tilde{f}((\lambda \otimes x)_t) := \lambda \varphi_t(f(x)) \in D'
$$

and extend it $\mathcal{o}$-linearly to $\widetilde{D}$. Note right away that $\tilde{f}$ is unique as it is $\varphi_t$-equivariant. The
map $\tilde{f}: \tilde{D} \to D'$ is well-defined as we have

$$\tilde{f}(\iota_{t,t'}) (1 \otimes_{t} x) = \tilde{f}(\sum_{u' \in N_0/tN_0^{t-1}} u' \otimes_{t'} \psi_{t'}(u'^{-1} \otimes_{t} x)) =$$

$$= \sum_{u',v' \in N_0/t'N_0^{t'v-1}} \tilde{f}(u' \otimes_{t'} \psi_{t'}(u'^{-1} \varphi_{t}(v')) \otimes_{t} \psi_{t'}(v'^{-1} x)) =$$

$$= \sum_{u',v' \in N_0/t'N_0^{t'v-1}} \tilde{f}(u' \varphi_{t'} \circ \psi_{t'}(u'^{-1} \varphi_{t}(v')) \otimes_{t'\psi_{t'}} \psi_{t'}(v'^{-1} x)) =$$

$$= \sum_{v' \in N_0/t'N_0^{t'v-1}} \varphi_{t}(v' \varphi_{t'} \circ \psi_{t'}(v'^{-1} f(x))) = \varphi_{t}(f(x)) = \tilde{f}(1 \otimes_{t} x)$$

noting that $\iota_{t,t'}$ is a $\Lambda(N_0)$-homomorphism. Here the notation $\otimes_{t}$ indicates that the tensor product is via the map $\varphi_{t}$. By construction $\tilde{f}$ is a homomorphism of étale $T_{\ast}$-modules over $\Lambda(N_0)$ satisfying $\tilde{f} \circ \iota = f$.

The injectivity of $\iota$ in case the $\psi$-action on $D$ is nondegenerate follows from Lemmata 4.12 and 4.13.

Example 4.6. If $D$ itself is étale then we have $\tilde{D} = D$.

Corollary 4.7. The functor $\tilde{D}$ from the category of $\Lambda(N_0)$-modules with a $\psi$-action of $T_{\ast}$ to the category of étale $T_{\ast}$-modules over $\Lambda(N_0)$ is exact.

Proof. $\Lambda(N_0)$ is a free $\varphi_{t}(\Lambda(N_0))$-module, so $\Lambda(N_0) \otimes_{\Lambda(N_0),\varphi_{t}} -$ is exact, and so is the direct limit functor.

Corollary 4.8. Assume that $D$ is a $\Lambda(N_0)$-module with a nondegenerate $\psi$-action of $T_{\ast}$ and $f: D \to D'$ is an injective $\psi$-equivariant $\Lambda(N_0)$-homomorphism into the étale $T_{\ast}$-module $D'$ over $\Lambda(N_0)$. Then $\tilde{f}$ is also injective.

Proof. Since $D$ is nondegenerate we may identify $\varphi_{t}^{\ast} D$ with a $\Lambda(N_0)$-submodule of $\tilde{D}$. Assume that $x = \sum_{u \in J(N_0/tN_0^{t-1})} u \otimes_{t} x_{u} \in \varphi_{t}^{\ast} D$ lies in the kernel of $\tilde{f}$. Then $x_{u} = \psi_{t}(u^{-1} x) \in D \subseteq \varphi_{t}^{\ast} D \subseteq \tilde{D}$ ($u \in J(N_0/tN_0^{t-1})$) also lies in the kernel of $\tilde{f}$. However, we have $\tilde{f}(x_{u}) = f(x_{u})$ showing that $x_{u} = 0$ for all $u \in J(N_0/tN_0^{t-1})$ as $f$ is injective.

Example 4.9. Let $D$ be a (classical) irreducible étale $(\varphi, \Gamma)$-module over $k[[X]]$ and $D_{0} \subseteq D$ a $\psi$- and $\Gamma$-invariant trellis in $D$. Then we have $\tilde{D}_{0} \cong D$ unless $D$ is 1-dimensional and $D_{0} = D$ in which case we have $\tilde{D}_{0} = D_{0}$.

Proof. If $D$ is 1-dimensional then $D$ is an étale $(\varphi, \Gamma)$-module over $k[[X]]$ (Prop. II.5.14 in [3]) therefore it is equal to its étale hull. If $\dim D > 1$ then we have $D = D^{\#} \subseteq D_{0}$ by Cor. II.5.12 and II.5.21 in [3]. By Corollary 4.8 $D^{\#} \subseteq D_{0}$ injects into $D$ and it is $\varphi$- and $\psi$-invariant. Since $D^{\#}$ is not $\varphi$-invariant (Prop. II.5.14 in [3]) and it is the maximal compact $o[[X]]$-submodule of $D$ on which $\psi$ acts surjectively (Prop. II.4.2 in [3]) we obtain that $\tilde{D}_{0}$ is not compact. In particular, its $X$-divisible part is nonzero therefore equals $D$ as the $X$-divisible part of $\tilde{D}_{0}$ is an étale $(\varphi, \Gamma)$-submodule of the irreducible $D$. 

\[25\]
Proposition 4.10. The $T_+^{-1}$ action on $D_{SV}(\pi)$ is a surjective nondegenerate $\psi$-action of $T_+$.

Proof. Let $d \in D_{SV}(\pi)$ and $t \in T_+$. Since the action of both $t$ and $\Lambda(N_0)$ on $D_{SV}(\pi)$ comes from that on $\pi^\vee$ we have $t^{-1} \varphi_t(\lambda) d = t^{-1} t \lambda t^{-1} d = \lambda t^{-1} d$, so this is indeed a $\psi$-action. The surjectivity of each $\psi_t$ follows from the injectivity of the multiplication by $t$ on each $W \in B_+(\pi)$. Finally, if $W$ is in $B_+(\pi)$ then so is $t^* W := \sum_{u \in J(N_0/tN_0 t^{-1})} u t W$ for any $t \in T_+$. Take an element $d \in D_{SV}(\pi)$ lying in the kernel of $\psi_t(u^{-1})$ for all $u \in J(N_0/tN_0 t^{-1})$. Then there exists a generating $B_+$-subrepresentation $W$ of $\pi$ such that the restriction of $t^{-1} u^{-1} d$ to $W$ is zero for all $u \in J(N_0/tN_0 t^{-1})$. Then the restriction of $d$ to $t^* W$ is zero showing that $d$ is zero in $D_{SV}(\pi)$ therefore the nondegeneracy. Alternatively, the nondegeneracy of the $\psi$-action also follows from the existence of a $\psi$-equivariant injective map $D_{SV}(\pi) \hookrightarrow D_{SV}^0(\pi)$ into an étale $T_+$-module $D_{SV}^0(\pi)$ (\cite{3} Proposition 3.5 and Remark 6.1).\hfill \Box

Question 1. Let $D_{SV}^{(0)}(\pi)$ as in \cite{3}. We have that $D_{SV}^{(0)}(\pi)$ is an étale $T_+$-module over $\Lambda(N_0)$ \cite{3} Proposition 3.5) and $f : D_{SV}(\pi) \hookrightarrow D_{SV}^{(0)}(\pi)$ is a $\psi$-equivariant map (\cite{3} Remark 6.1). By the universal property of the étale hull and Corollary 4.10 $\widetilde{D}_{SV}(\pi)$ also injects into $D_{SV}^{(0)}(\pi)$. Whether or not this injection is always an isomorphism is an open question. In case of the Steinberg representation this is true by Proposition 11 in \cite{7}.

We call the submonoid $T'_* \leq T_* \leq T_+$ cofinal in $T_*$ if for any $t \in T_*$ there exists a $t' \in T'_*$ such that $t \leq t'$. For example $\xi(\mathbb{Z}_p \setminus \{0\})$ is cofinal in $T_+$.

Corollary 4.11. Let $D$ be a $\Lambda(N_0)$-module with a $\psi$-action of $T_*$ and denote by $\widetilde{D}$ (resp. by $\widetilde{D}'$) the étale hull of $D$ for the $\psi$-action of $T_*$ (resp. of $T'_*$). Then we have a natural isomorphism $\widetilde{D}' \sim_{\psi} \widetilde{D}$ of étale $T'_*$-modules over $\Lambda(N_0)$. More precisely, if $f : D \to D_1$ is a $\psi$-equivariant $\Lambda(N_0)$-homomorphism into an étale $T'_*$-module $D_1$ then $f$ factors uniquely through $\iota : D \to \widetilde{D}$.

Proof. Since $T'_* \leq T_*$ is cofinal in $T_*$ we have $\varprojlim_{t' \in T'_*} \varphi_{t'} D \cong \varprojlim_{t \in T_*} \varphi_t D = \widetilde{D}$.

By Corollary 4.11 there exists a homomorphism $\widetilde{pr} : \widetilde{D}_{SV}(\pi) \to D^{\psi}_{\xi, \ell, \infty}(\pi)$ of étale $(\varphi, \Gamma)$-modules over $\Lambda(N_0)$ such that $\text{pr} = \widetilde{pr} \circ \iota$. Our main result in this section is the following

Theorem 4.12. $D^{\psi}_{\xi, \ell, \infty}(\pi)$ is the pseudocompact completion of $\Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi))$ in the category of étale $(\varphi, \Gamma)$-modules over $\Lambda(\ell(N_0))$, i.e. we have $D^{\psi}_{\xi, \ell, \infty}(\pi) \cong \varprojlim_{\widetilde{D}} D$ where $D$ runs through the finitely generated étale $(\varphi, \Gamma)$-modules over $\Lambda(\ell(N_0))$ arising as a quotient of $\Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi))$ by a closed submodule. This holds in any topology on $\Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi))$ making both the maps $1 \otimes \iota : D_{SV}(\pi) \to \Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi))$, $d \mapsto 1 \otimes \iota(d)$ and $1 \otimes \widetilde{pr} : \Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi)) \to D^{\psi}_{\xi, \ell, \infty}(\pi)$ continuous.

Remark. Since the map $\text{pr} : D_{SV}(\pi) \to D^{\psi}_{\xi, \ell, \infty}(\pi)$ is continuous, there exists such a topology on $\Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi))$. For instance we could take either the final topology of the map $D_{SV}(\pi) \to \Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi))$ or the initial topology of the map $\Lambda(\ell(N_0) \otimes_{\Lambda(N_0)} \widetilde{D}_{SV}(\pi)) \to D^{\psi}_{\xi, \ell, \infty}(\pi)$. 26
Proof. The homomorphism \( \tilde{\varphi} \) factors through the map \( 1 \otimes \text{id}: \tilde{D}_{SV}(\pi) \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \tilde{D}_{SV}(\pi) \) since \( D^\vee_{\xi,\ell,\infty}(\pi) \) is a module over \( \Lambda_\ell(N_0) \), so we obtain a homomorphism

\[
1 \otimes \tilde{\varphi}: \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \tilde{D}_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)
\]

of étale \((\varphi, \Gamma)\)-modules over \( \Lambda_\ell(N_0) \). At first we claim that \( 1 \otimes \tilde{\varphi} \) has dense image. Let \( M \in \mathcal{M}(\pi^{H_0}) \) and \( W \in \mathcal{B}_+(\pi) \) be arbitrary. Then by Lemma 3.1 the map \( \text{pr}_{W,M,k}: W^\vee \to M_k^\vee \) is surjective for \( k \geq 0 \) large enough. This shows that the natural map

\[
1 \otimes \text{pr}_{W,M,k}: \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} W^\vee \to \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} M_k^\vee = M_k^\vee[1/X]
\]

is surjective. However, \( 1 \otimes \text{pr}_{W,M,k} \) factors through \( \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \) by the Remarks after Lemma 3.2. In particular, the natural map

\[
1 \otimes \text{pr}_{M,k}: \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to M_k^\vee[1/X]
\]

is surjective for all \( M \in \mathcal{M}(\pi^{H_0}) \) and \( k \geq 0 \) large enough (whence in fact for all \( k \geq 0 \)). This shows that the image of the map

\[
1 \otimes \text{pr}: \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to D^\vee_{\xi,\ell,\infty}(\pi)
\]

is dense whence so is the image of \( 1 \otimes \tilde{\varphi} \). By the assumption that \( 1 \otimes \tilde{\varphi} \) is continuous we obtain a surjective homomorphism

\[
\tilde{1} \otimes \tilde{\varphi}: \varprojlim_\xi D \to D^\vee_{\xi,\ell,\infty}(\pi)
\]

of pseudocompact \((\varphi, \Gamma)\)-modules over \( \Lambda_\ell(N_0) \) where \( D \) runs through the finitely generated étale \((\varphi, \Gamma)\)-modules over \( \Lambda_\ell(N_0) \) arising as a quotient of \( \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} \tilde{D}_{SV}(\pi) \).

Let \( 0 \neq (x_D)_D \) be in the kernel of \( \tilde{1} \otimes \tilde{\varphi} \). Then there exists a finitely generated étale \((\varphi, \Gamma)\)-module \( D \) over \( \Lambda_\ell(N_0) \) with a surjective continuous homomorphism \( \Lambda_\ell(N_0) \otimes_{\Lambda(N_0)} D_{SV}(\pi) \to D \) such that \( x_D \neq 0 \). By Proposition 3.4 this map factors through \( D^\vee_{\xi,\ell,\infty}(\pi) \) contradicting to the assumption \( \tilde{1} \otimes \tilde{\varphi}((x_D)_D) = 0 \).

\[\square\]

5 Nongeneric \( \ell \) and the action of \( T_+ \)

Assume from now on that \( \ell = \ell_\alpha \) is a nongeneric Whittaker functional defined by the projection of \( N_0 \) onto \( N_{\alpha,0} \cong \mathbb{Z}_p \) for some simple root \( \alpha \in \Delta \). Our goal in this section is to define a \( \varphi \)-action of \( T_+ \) on \( D^\vee_{\xi,\ell,\infty}(\pi) \) or, equivalently, on \( D^\vee_{\xi}(\pi) \) extending the action of \( \mathbb{Z}_p \setminus \{0\} \) and making \( D^\vee_{\xi,\ell,\infty}(\pi) \) an étale \( T_+ \)-module over \( \Lambda_\ell(N_0) \).

Remark. In [2] the Whittaker functional \( \ell \) is assumed to be generic. However, even if \( \ell \) is not generic, the functor \( D^\vee_{\xi} \) (hence also \( D^\vee_{\xi,\ell,\infty} \)) is right exact. Moreover, Thm. 6.1 (the compatibility with parabolic induction) in [2] also extends to this case. In particular, the value of \( D^\vee_{\xi} \) at the principal series is the same \((\varphi, \Gamma)\)-module of rank 1 regardless of the choice of \( \ell \). However, the restriction of \( D^\vee_{\xi} \) to the category \( SP_{\mathfrak{o}/\omega} \) may not be exact in general.
Proof. The particular shape of \( \ell \) is only used in Lemma 6.5 and section 8 in [2]. Note that even though the statement of Lemma 6.5 (loc. cit.) is not true for non-generic \( \ell \), the argument using it in the proof of Prop. 6.2 can be replaced by the following (we use the notations of [2] where \( H_0 = \text{Ker}(\ell: N_0 \to \mathbb{Z}_p) \) is denoted by \( N_1 \)): For an element \( w \neq 1 \) in the Weyl group we have \((w^{-1}N_p - w \cap N_0) \setminus N_0/N_1 \not= \{1\} \) if and only if \( N_1 \) does not contain \( w^{-1}N_p - w \cap N_0 \). In case we have \( \{1\} \neq w^{-1}N_p - w \cap N_0 \subseteq N_1 \), the operator \( F \) acts on the space \( C((w^{-1}N_p - w \cap N_0) \setminus N_0, \pi^w) \) nilpotently. Indeed, the trace map

\[
\text{Tr}_{N_1/sN_1s^{-1}}: C((w^{-1}N_p - w \cap N_0) \setminus N_0, \pi^w) sN_1s^{-1} \to C((w^{-1}N_p - w \cap N_0) \setminus N_0, \pi^w) N_1
\]
is zero as each double coset \((w^{-1}N_p - w \cap N_1) \setminus N_1/sN_1s^{-1}\) has size divisible by \( p \) and any function in \( C((w^{-1}N_p - w \cap N_0) \setminus N_0, \pi^w) sN_1s^{-1}\) is constant on these double cosets. \( \square \)

Let \( t \in T_+ \) be arbitrary. Note that by the choice of this \( \ell \) we have \( tH_0t^{-1} \subseteq H_0 \). In particular, \( T_+ \) acts via conjugation on the ring \( \Lambda(N_0/H_0) \cong o[[X]] \); we denote the action of \( t \in T_+ \) by \( \varphi_t \). This action is via the character \( \alpha \) mapping \( T_+ \) onto \( \mathbb{Z}_p \setminus \{0\} \). In particular, \( o[[X]] \) is a free module of finite rank over itself via \( \varphi_t \). Moreover, we define the Hecke action of \( t \in T_+ \) on \( \pi^{H_0} \) by the formula \( F_t(m) := \text{Tr}_{H_0/tH_0t^{-1}}(tm) \) for any \( m \in \pi^{H_0} \). For \( t, t' \in T_+ \) we have

\[
F_{t'} \circ F_t = \text{Tr}_{H_0/t'H_0t'^{-1}} \circ (t') \circ \text{Tr}_{H_0/tH_0t^{-1}} \circ (t) = \text{Tr}_{H_0/t'H_0t'^{-1}} \circ \text{Tr}_{t'H_0t'^{-1}/t'H_0t^{-1}} \circ (t't') = F_{t't}.
\]

For any \( M \in \mathcal{M}(\pi^{H_0}) \) we put \( F_t^* M := N_0 F_t(M) \).

**Lemma 5.1.** For any \( M \in \mathcal{M}(\pi^{H_0}) \) we have \( F_t^* M \in \mathcal{M}(\pi^{H_0}) \).

**Proof.** We have \( F(F_t^* M) = F(N_0 F_t(M)) \subset N_0 F^t(M) = N_0 F_{t'}(F(M)) \subseteq F_t^* M \). So \( F_t^* M \) is a module over \( \Lambda(N_0/H_0)/(\pi^h(F]]) \). Moreover, if \( m_1, \ldots, m_r \) generates \( M \), then the elements \( F_t(m_i) \) generate \( F_t^* M \), so it is finitely generated. The admissibility is clear as \( F_t^* M = \sum_{u \in J(N_0/H_0)} u F_t(M) \) is the sum of finitely many admissible submodules. Finally, \( F_t^* M \) is stable under the action of \( \Gamma \) as \( F_t \) commutes with the action of \( \Gamma \). \( \square \)

By the definition of \( F_t^* M \) we have a surjective \( o/\pi^h[[X]] \)-homomorphism

\[
1 \otimes F_t: o/\pi^h[[X]] \otimes o/\pi^h[[X]] \otimes \varphi_t M \to F_t^* M
\]
which gives rise to an injective \( o/\pi^h((X)) \)-homomorphism

\[
(1 \otimes F_t)[[1/X]]: (F_t^* M)^[[1/X]] \hookrightarrow o/\pi^h((X)) \otimes o/\pi^h((X)) \otimes \varphi_t M^[[1/X]]. \tag{13}
\]

Moreover, there is a structure of an \( o/\pi^h[[X]][F] \)-module on \( o/\pi^h[[X]] \otimes o/\pi^h[[X]] \otimes F(M) \) by putting \( F(\lambda \otimes m) := \varphi_t(\lambda) \otimes F(m) \). Similarly, the group \( \Gamma \) also acts on \( o/\pi^h[[X]] \otimes o/\pi^h[[X]] \otimes \varphi_t M \) semilinearly. The map \( 1 \otimes F_t \) is \( \Gamma \)-equivariant as \( F_t \), \( F_t \), and the action of \( \Gamma \) all commute. We deduce that \( (1 \otimes F_t)^[[1/X]] \) is a \( \varphi_t \) and \( \Gamma \)-equivariant map of étale (\( \varphi_t \), \( \Gamma \))-modules.

Note that for any \( t \in T_+ \) there exists a positive integer \( k \geq 0 \) such that \( t \leq s^k \), i.e., \( t' := t^{-1}s^k \) lies in \( T_+ \). So we have \( F_t^*(F_{t'}^* M) = F_t^* M = N_0 F^k(M) \subseteq M \). So we obtain an isomorphism \( M^[[1/X]] \cong (F_{s^k}^* M)^[[1/X]] = (F_t^*(F_{t'}^* M))^[[1/X]] \) as \( M/N_0 F^k(M) \) is finitely generated over \( o \).

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Theorem 8.20 in [9] we have a $\varphi$ with injective ring homomorphisms. On the other hand, by the equivalence of categories in Lemma 5.2.

The composite $(1 \otimes F_t)^\vee[1/X] \circ (1 \otimes F_t)^\vee[1/X] = (1 \otimes F^k)^\vee[1/X]$ is an isomorphism by Lemma 2.6 in [2]. So $(1 \otimes F_t)^\vee[1/X]$ is also an isomorphism as both $(1 \otimes F_t)^\vee[1/X]$ and $(1 \otimes F_t)^\vee[1/X]$ are injective.

Now taking projective limits we obtain an isomorphism of pseudocompact étale $(\varphi, \Gamma)$-modules

$$(1 \otimes F_t)^\vee[1/X] : D^\vee_\xi(\pi) \rightarrow \lim_{M \in \mathcal{M}(\pi^H_0)} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X])$$

$$(m)(F_t M)^\vee[1/X] \mapsto ((1 \otimes F_t)^\vee[1/X](m))_{M^\vee[1/X]}.$$ Moreover, since $o((X))$ is finite free over itself via $\varphi_t$, we have an identification

$$\lim_{M \in \mathcal{M}(\pi^H_0)} (o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} M^\vee[1/X]) \cong o/\varpi^h((X)) \otimes_{o/\varpi^h((X)), \varphi_t} D^\vee_\xi(\pi).$$

Using the maps $(1 \otimes F_t)^\vee[1/X]$ we define a $\varphi$-action of $T_+$ on $D^\vee_\xi(\pi)$ by putting $\varphi_t(d) := ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d)$ for $d \in D^\vee_\xi(\pi)$.

Proposition 5.3. The above action of $T_+$ extends the action of $\xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+$ and makes $D^\vee_\xi(\pi)$ into an étale $T_+$-module over $o/\varpi^h[[X]]$.

Proof. By the definition of the $T_+$-action it is indeed an extension of the action of the monoid $\mathbb{Z}_p \setminus \{0\}$. For $t, t' \in T_+$ we compute

$$\varphi_{t'} \circ \varphi_t(d) = ((1 \otimes F_t)^\vee[1/X])^{-1} \circ ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) =$$

$$= ((1 \otimes F_t)^\vee[1/X] \circ (1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = ((1 \otimes F_{t'})^\vee[1/X])^{-1}(1 \otimes d) =$$

$$\varphi_{t'}(d) = \varphi_{t'}(d).$$

Further, we have

$$\varphi_t(\lambda d) = ((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes \lambda d) = ((1 \otimes F_t)^\vee[1/X])^{-1}(\varphi_t(\lambda) \otimes d) =$$

$$= \varphi_t(\lambda)((1 \otimes F_t)^\vee[1/X])^{-1}(1 \otimes d) = \varphi_t(\lambda) \varphi_t(d)$$

showing that this is indeed a $\varphi$-action of $T_+$. The étale property follows from the fact that $(1 \otimes F_t)^\vee[1/X]$ is an isomorphism for each $t \in T_+$.

The inclusion $u_\alpha : \mathbb{Z}_p \rightarrow N_{a, 0} \leq N_0$ induces an injective ring homomorphism—still denoted by $u_\alpha$ by a certain abuse of notation—$u_\alpha : o((X)) \hookrightarrow \Lambda_{\ell}(N_0)$. For each $t \in T_+$ this gives rise to a commutative diagram

$$o((X)) \xrightarrow{u_\alpha} \Lambda_{\ell}(N_0)$$

$$\varphi_t \downarrow \quad \downarrow \varphi_t$$

$$o((X)) \xrightarrow{u_\alpha} \Lambda_{\ell}(N_0)$$

with injective ring homomorphisms. On the other hand, by the equivalence of categories in Thm. 8.20 in [9] we have a $\varphi$- and $\Gamma$-equivariant identification $M^\vee_{\infty}[1/X] \cong \Lambda_{\ell}(N_0) \otimes_{o((X)), u_\alpha}$.
\[ (1 \otimes F_t)^\vee_\infty[1/X] : (F_t^*M)^\vee_\infty[1/X] \cong \Lambda_\ell(N_\ell) \otimes_{u_\alpha} (F_t^*M)^\vee[1/X] \rightarrow \Lambda_\ell(N_\ell) \otimes_{u_\alpha} o/\varpi^h((X)) \otimes_o/\varpi^h(X),\varphi_t M^\vee[1/X] \cong \Lambda_\ell(N_\ell) \otimes_{o,\varphi_t} \Lambda_t(N_\ell) \otimes_{\Lambda_t(N_\ell),\varphi_t} M^\vee[1/X]. \] (14)

Taking projective limits again we deduce an isomorphism

\[ (1 \otimes F_t)^\vee_\infty[1/X] : D^\vee_{\xi,\ell,\infty}(\pi) \rightarrow \Lambda_\ell(N_\ell) \otimes_{\Lambda_t(N_\ell),\varphi_t} D^\vee_{\xi,\ell,\infty}(\pi) \]

for all \( t \in T_+ \) using the identification

\[ \lim_{M \in \mathcal{M}(\pi_{u_\alpha})} (\Lambda_\ell(N_\ell) \otimes_{\Lambda_t(N_\ell),\varphi_t} M^\vee[1/X]) \cong \Lambda_\ell(N_\ell) \otimes_{\Lambda_t(N_\ell),\varphi_t} D^\vee_{\xi,\ell,\infty}(\pi). \]

Using the maps \( (1 \otimes F_t)^\vee_\infty[1/X] \) we define a \( \varphi \)-action of \( T_+ \) on \( D^\vee_{\xi,\ell,\infty}(\pi) \) by putting \( \varphi_t(d) := ((1 \otimes F_t)^\vee_\infty[1/X])^{-1}(1 \otimes d) \) for \( d \in D^\vee_{\xi,\ell,\infty}(\pi). \)

**Corollary 5.4.** The above action of \( T_+ \) extends the action of \( \xi(\mathbb{Z}_p \setminus \{0\}) \leq T_+ \) and makes \( D^\vee_{\xi,\ell,\infty}(\pi) \) into an étale \( T_+ \)-module over \( \Lambda_\ell(N_\ell) \). The reduction map \( D^\vee_{\xi,\ell,\infty}(\pi) \rightarrow D^\vee_{\xi}(\pi) \) is \( T_+ \)-equivariant for the \( \varphi \)-action.

We can view this \( \varphi \)-action of \( T_+ \) in a different way: Let us define \( F_{t,k} := \text{Tr}_{H_k/\ell H_k} \circ (t^\cdot). \) Then we have a map

\[ 1 \otimes F_{t,k} : \Lambda(N_0/H_k)/\varpi^h \otimes_{\Lambda(N_0/H_k)/\varpi^h,\varphi_t} M_k \rightarrow F_{t,k}^* M_k := N_0 F_{t,k}(M_k), \]

(15)

where we have \( F_{t,k}^* M \in \mathcal{M}_k(\pi_{H_k}). \) Let \( k \) be large enough such that we have \( tH_k t^{-1} \supseteq H_k. \) After taking Pontryagin duals, inverting \( X, \) taking projective limit and using the remark after Lemma 2.35 we obtain a homomorphism of étale \( (\varphi, \Gamma) \)-modules

\[ \lim_{k \rightarrow k} \text{Tr}_{t^{-1}H_k} \circ (1 \otimes F_{t,k})^\vee[1/X] : (F_{t,k}^* M)^\vee[1/X] \rightarrow \Lambda_\ell(N_\ell) \otimes_{\Lambda_t(N_\ell),\varphi_t} M^\vee[1/X]. \] (16)

This map is indeed \( \Gamma \)- and \( \varphi \)-equivariant because we compute

\[ F_k \circ F_{t,k} = \text{Tr}_{H_k/sH_k} \circ (s^\cdot) \circ \text{Tr}_{H_k/\ell H_k} \circ (t^\cdot) = \text{Tr}_{H_k/\ell H_k} \circ (s^k\cdot) = \text{Tr}_{H_k/\ell H_k} \circ (s^\cdot) \circ \text{Tr}_{H_k/sH_k} \circ (s^\cdot) = F_{t,k} \circ F_k. \]

Now we have two maps (13) and (16) between the \( (\varphi, \Gamma) \)-modules \( (F_{t,k}^* M)^\vee[1/X] \) and \( \Lambda_\ell(N_\ell) \otimes_{\Lambda_t(N_\ell),\varphi_t} M^\vee[1/X] \) that agree after taking \( H_0 \)-coinvariants by definition. Hence they are equal by the equivalence of categories in Thm. 8.20 in [9].

We obtain in particular that the map (15) has finite kernel and cokernel as it becomes an isomorphism after taking Pontryagin duals and inverting \( X. \) Hence there exists a finite \( \Lambda(N_0/H_k)/\varpi^h \)-submodule \( M_{k,i} \) of \( M_k \) such that the kernel of \( 1 \otimes F_{t,k} \) is contained in the image of \( \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi} M_{k,i} \) in \( \Lambda(N_0/H_k)/\varpi^h \otimes_{\varphi} M_k. \) We denote by \( M^\ast_{t,k} \leq F_{t,k}^* M_k \) the
image of $1 \otimes F_{t,k}$. We conclude that as in Proposition 2.6, we can describe the $\varphi_t$-action in the following way:

$$
\varphi_t: M_k^\vee[1/X] \to (F_{t,k}^* M_k)^\vee[1/X]
$$

$$
f \mapsto (\text{Tr}_{t^{-1}H_k/tH_k} \circ (1 \otimes F_{t,k})^\vee[1/X])^{-1}(1 \otimes f)
$$  \hspace{1cm} (17)

Being an étale $T_+$-module over $\Lambda_t(N_0)$ we equip $D_{\xi,t,\infty}^\vee(\pi)$ with the $\psi$-action of $T_+$: $\psi_t$ is the canonical left inverse of $\varphi_t$ for all $t \in T_+$.

**Proposition 5.5.** The map $\text{pr}: D_{SV}(\pi) \to D_{\xi,t,\infty}(\pi)$ is $\psi$-equivariant for the $\psi$-actions of $T_+$ on both sides.

**Proof.** We proceed as in the proofs of Proposition 2.8 and Lemma 3.2. We fix $t \in T_+$, $W \in \mathcal{B}_+(\pi)$ and $M \in \mathcal{M}(\pi^{H_0})$ and show that $\text{pr}_{W,M}$ is $\psi_t$-equivariant. Fix $k$ such that $F_{t,k}^* M_k \leq W$ and $tH_0t^{-1} \geq H_k$.

At first we compute the formula analogous to (7). Let $f$ be in $M_k^\vee$ such that its restriction to $M_{t,k,*}$ is zero and $m \in M_{t,k,*} \leq F_{t,k}^* M_k$ be in the form

$$
m = \sum_{u \in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u)
$$

with elements $m_u \in M_k$ for $u \in J(N_0/tN_0t^{-1})$. $M_{t,k,*}^*$ is a finite index submodule of $F_{t,k}^* M_k$.

Note that the elements $m_u$ are unique up to $M_{t,k,*} + \text{Ker}(F_{t,k})$. Therefore $\varphi_t(f) \in (M_{t,k}^*)^\vee$ is well-defined by our assumption that $f|_{M_{t,k,*}} = 0$ noting that the kernel of $F_{t,k}$ equals the kernel of $\text{Tr}_{t^{-1}H_k/tH_k}$ since the multiplication by $t$ is injective and we have $F_{t,k} = t \circ \text{Tr}_{t^{-1}H_k/tH_k}$. So we compute

$$
\varphi_t(f)(m) = ((1 \otimes F_{t,k})^\vee)^{-1}(\text{Tr}_{t^{-1}H_k/tH_k}((1 \otimes f)))(m) = 
$$

$$
= ((1 \otimes F_{t,k})^\vee)^{-1}(1 \otimes \text{Tr}_{t^{-1}H_k/tH_k}(f))\left(\sum_{u \in J((N_0/H_k)/t(N_0/H_k)t^{-1})} uF_{t,k}(m_u)\right) = 
$$

$$
\text{Tr}_{t^{-1}H_k/tH_k}(f)(F_{t,k}^{-1}(u_0 F_{t,k}(m_{u_0}))) = f(\text{Tr}_{t^{-1}H_k/tH_k}((t^{-1}u_0 t)m_{u_0}))
$$  \hspace{1cm} (18)

where $u_0$ is the single element in $J(N_0/tN_0t^{-1})$ corresponding to the coset of 1.

Now let $f$ be in $W^\vee$ such that the restriction $f|_{N_0 t M_{t,k,*}} = 0$. By definition we have $\psi_t(f)(w) = f(tw)$ for any $w \in W$. Choose an element $m \in M_{t,k,*}^* \leq F_{t,k}^* M_k$ written in the form

$$
m = \sum_{u \in J(N_0/tN_0t^{-1})} uF_{t,k}(m_u) = \sum_{u \in J(N_0/tN_0t^{-1})} uf\text{Tr}_{t^{-1}H_k/tH_k}(m_u) .
$$
Then we compute

\[ f|_{F_{t,k}^* M_k}(m) = \sum_{u \in J(N_0/tN_0 t^{-1})} f(utT_{t-1} H_k t/H_k(m_u)) = \]

\[ = \sum_{u \in J(N_0/tN_0 t^{-1})} \psi_t(u^{-1} f)(T_{t-1} H_k t/H_k(m_u)) = \]

\[ = \sum_{u \in J(N_0/tN_0 t^{-1})} \varphi_t(\psi_t(u^{-1} f)|_{F_{t,k}^* M_k})(F_{t,k}(m_u)) = \]

\[ = \sum_{u \in J(N_0/tN_0 t^{-1})} u \varphi_t(\psi_t(u^{-1} f)|_{M_k})(uF_{t,k}(m_u)) = \]

\[ = \sum_{u \in J(N_0/tN_0 t^{-1})} u \varphi_t(\psi_t(u^{-1} f)|_{M_k})(m) \]

as for distinct \( u, v \in J(N_0/tN_0 t^{-1}) \) we have \( u \varphi_t(f_0)(vF_{t,k}(m_v)) = 0 \) for any \( f_0 \in (M_{t,k}^*)^\vee \). So by inverting \( X \) and taking projective limits with respect to \( k \) we obtain

\[ \text{pr}_{W,F_{t}^* M}(f) = \sum_{u \in J(N_0/tN_0 t^{-1})} u \varphi_t(\text{pr}_{W,M}(\psi_t(u^{-1} f))) \]

as we have \((M_{t,k}^*)^\vee[1/X] \cong (F_{t,k}^* M)^\vee[1/X] \). Since the map (14) is an isomorphism we may decompose \( \text{pr}_{W,F_{t}^* M}(f) \) uniquely as

\[ \text{pr}_{W,F_{t}^* M}(f) = \sum_{u \in J(N_0/tN_0 t^{-1})} u \varphi_t(\psi_t(u^{-1} \text{pr}_{W,F_{t}^* M}(f))) \]

so we must have \( \psi_t(\text{pr}_{W,F_{t}^* M}(f)) = \text{pr}_{W,M}(\psi_t(f)) \). For general \( f \in W^\vee \) note that \( N_0 s M_{t,k,s} \) is killed by \( \varphi_t(X^r) \) for \( r \geq 0 \) big enough, so we have \( X^r \psi_t(\text{pr}_{W,F_{t}^* M}(f)) = \psi_t(\text{pr}_{W,F_{t}^* M}(\varphi_t(X^r)f)) = \text{pr}_{W,M}(\psi_t(\varphi_t(X^r)f)) = X^r \text{pr}_{W,M}(\psi_t(f)) \). Since \( X^r \) is invertible in \( \Lambda_t(N_0) \), we obtain

\[ \psi_t(\text{pr}_{W,F_{t}^* M}(f)) = \text{pr}_{W,M}(\psi_t(f)) \]

for any \( f \in W^\vee \). The statement follows taking the projective limit with respect to \( M \in \mathcal{M}(\pi^{l_0}) \) and the inductive limit with respect to \( W \in \mathcal{B}_+(\pi) \).

We end this section by proving a Lemma that will be needed several times later on.

**Lemma 5.6.** For any \( M \in \mathcal{M}(\pi^{l_0}) \) there exists an open subgroup \( T' = T'(M) \leq T \) such that \( M \) is \( T' \)-stable.

**Proof.** Choose \( m_1, \ldots, m_a \in M \) \((a \geq 1)\) generating \( M \) as a module over \( o/\omega h[X][F] \). Since \( \pi \) is smooth, there exists an open subgroup \( T' \leq T_0 \) stabilizing all \( m_1, \ldots, m_a \). Now \( T' \) normalizes \( N_0 \) and all the elements \( t \in T' \) commute with \( F \) we deduce that \( T' \) acts on \( M \).
6 A $G$-equivariant sheaf $\mathcal{Y}$ on $G/B$ attached to $D_{\xi,\ell,\infty}^{\vee}(\pi)$

Assume in this section that $\ell = \ell_{\alpha}$ for some simple root $\alpha \in \Delta$.

Let $D$ be an étale $(\varphi, \Gamma)$-module over the ring $\Lambda_{\ell}(N_0)/\mathcal{O}$. Recall that the $\Lambda(N_0)$-submodule $D^{bd}$ of bounded elements in $D$ is defined [9] as

$$D^{bd} = \{ x \in D \mid \ell_D(\psi^k_s(u^{-1}x)) \mid k \geq 0, u \in N_0 \} \subseteq D_{H_0} \text{ is bounded} \} .$$

where $\ell_D$ denotes the natural map $D \to D_{H_0}$. Note that $D_{H_0}$ is an étale $(\varphi, \Gamma)$-module over $o/\mathcal{O}((X))$, so the bounded subsets of $D_{H_0}$ are exactly those contained in a compact $o/\mathcal{O}[[X]]$-submodule of $D_{H_0}$.

**Lemma 6.1.** Assume that $D$ is a finitely generated étale $(\varphi, \Gamma)$-module over $\Lambda_{\ell}(N_0)/\mathcal{O}$. Then $d \in D$ lies in $D^{bd}$ if and only if $d$ is contained in a compact $\psi_s$-invariant $\Lambda(N_0)$-submodule of $D$.

**Proof.** If $d$ is in $D^{bd}$ then it is contained in $D^{bd}(D_0) = \{ x \in D \mid \ell_D(\psi^k_s(u^{-1}x)) \subseteq D_0 \}$ for some treillis $D_0 \subset D_{H_0}$ where $D^{bd}(D_0)$ is a compact $\psi_s$-stable $\Lambda(N_0)$-submodule of $D$ by Prop. 9.10 in [9]. On the other hand if $x \in D_1$ for some compact $\psi_s$-invariant $\Lambda(N_0)$-submodule $D_1 \subset D$ then we have $\{ \ell_D(\psi^k_s(u^{-1}x)) \mid k \geq 0, u \in N_0 \} \subseteq \ell(D(D_1))$ is bounded as $D_1$ is compact and $\ell_D$ is continuous. \hfill \Box

We call a pseudocompact $\Lambda_{\ell}(N_0)$-module together with a $\varphi$-action of the monoid $T_+$ (resp. $\mathbb{Z}_p \setminus \{0\}$) a pseudocompact étale $T_+$-module (resp. $(\varphi, \Gamma)$-module) over $\Lambda_{\ell}(N_0)$ if it is a topologically étale $o[B_+]$-module in the sense of section 4.1 in [9]. Recall that a pseudocompact module over the pseudocompact ring $\Lambda_{\ell}(N_0)$ is the projective limit of finitely generated $\Lambda_{\ell}(N_0)$-modules. As for $D = D_{\xi,\ell,\infty}^{\vee}(\pi)$ in section 2 we equip the pseudocompact $\Lambda_{\ell}(N_0)$-modules $D$ with the weak topology, i.e. with the projective limit topology of the weak topologies of these finitely generated quotients of $D$. Recall from section 4.1 in [9] that the condition for $D$ to be topologically étale means in this case that the map

$$B_+ \times D \rightarrow D, \quad (b, x) \mapsto \varphi_b(x) \quad (19)$$

is continuous and $\psi = \psi_s : D \to D$ is continuous (Lemma 4.1 in [9]).

**Lemma 6.2.** $D_{\xi,\ell,\infty}^{\vee}(\pi)$ is a pseudocompact étale $T_+$-module over $\Lambda_{\ell}(N_0)$ (under the assumption that $\ell = \ell_{\alpha}$).

**Proof.** At first we show that the map (19) is continuous in the weak topology of $D = D_{\xi,\ell,\infty}^{\vee}(\pi)$. Let $b = ut \in B_+ \quad (u \in N_0, t \in T_+), x, y \in D_{\xi,\ell,\infty}^{\vee}(\pi)$ be such that $u_1 \varphi_t(y) = x$ and let $M \in \mathcal{M}(\pi H_0), l, l' \geq 0$ be arbitrary. We need to verify that the preimage of $x + O(M, l, l')$ under (19) contains a neighbourhood of $(b, y)$. By Lemma 5.6 there exists an open subgroup $T' \leq T_0 \leq T$ acting on $M$ therefore also on $M_{\ell,1}[1/X]$ as $T_0$ normalizes $H_l$ for all $l \geq 0$ by the assumption $\ell = \ell_{\alpha}$. Moreover, this action is continuous in the weak topology of $M_{\ell,1}[1/X]$, so there exists an open subgroup $T_1 \leq T'$ such that we have $(T_1 - 1)x \subset O(M, l, l')$. Moreover, since we have $D_{\xi,\ell,\infty}^{\vee}(\pi)/O(M, l, l') \cong M_{\ell,1}[1/X]/(\Lambda(N_0/H_1) \otimes u_0 X_1M_{\ell,1}[1/X]^{++})$ is a smooth representation of $N_0$, we have an open subgroup $N_1 \leq N_0$ with $(N_1 - 1)x \subset O(M, l, l')$. Moreover, we may
assume that $T_1$ normalizes $N_1$ so that $B_1 := N_1T_1$ is an open subgroup in $B_0 \leq B_+$ for which we have $(B_1 - 1)x \subset O(M, l, l')$ as $O(M, l, l')$ is $N_0$-invariant. Choose an element $t' \in T_+$ such that $tt' = s^r$ for some $r \geq 0$. Note that the composite map $D^\vee_{\ell, \infty}(\pi) \xrightarrow{\psi} D^\vee_{\ell, \infty} \rightarrow M^\vee[1/X]$ factors through the $\varphi_s$-equivariant map

$$( (1 \otimes F_i)^\vee[1/X] )^{-1} : (F_i^* M)^\vee[1/X] \rightarrow M^\vee[1/X]$$

mapping $X^\ell_i (F_i^* M)^\vee[1/X]^{++}$ into $X^\ell_i M^\vee[1/X]^{++}$. Since $X^\ell_i M^\vee[1/X]^{++}$ is $B_1$-invariant (as each $\varphi_i$ for $t_1 \in T_1$ commutes with $\varphi_s$), so is $O(M, l, l')$. We deduce that

$$B_1 b \times (y + O(F_i^* M, l, l')) \subset B_+ \times D^\vee_{\ell, \infty}(\pi)$$

maps into $x + O(M, l, l')$ via (19).

The continuity of $\psi_s$ follows from Proposition 8.22 in [9] since $\psi_s : D^\vee_{\ell, \infty}(\pi)$ is the projective limit of the maps $\psi_s : M^\vee_X[1/X] \rightarrow M^\vee[1/X]$ for $M \in \mathcal{M}(\pi^H)$. □

In view of the above Lemmata we define $D^{bd}$ for a pseudocompact étale $(\varphi, \Gamma)$-module $D$ over $\Lambda_\ell(N_0)$ as

$$D^{bd} = \bigcup_{D_c \in \mathfrak{C}_0(D)} D_c$$

where we denote the set of $\psi_s$-invariant compact $\Lambda(N_0)$-submodules $D_c \subset D$ by $\mathfrak{C}_0 = \mathfrak{C}_0(D)$.

The following is a generalization of Prop. 9.5 in [9].

**Proposition 6.3.** Let $D$ be a pseudocompact étale $(\varphi, \Gamma)$-module over $\Lambda_\ell(N_0)$. Then $D^{bd}$ is an étale $(\varphi, \Gamma)$-module over $\Lambda(N_0)$. If we assume in addition that $D$ is an étale $T_+$-module over $\Lambda_\ell(N_0)$ (for a $\varphi$-action of the monoid $T_+$ extending that of $\xi(\mathbb{Z}_p \setminus \{0\})$) then $D^{bd}$ is an étale $T_+$-module over $\Lambda(N_0)$ (with respect to the action of $T_+$ restricted from $D$).

**Proof.** We prove the second statement assuming that $D$ is an étale $T_+$-module. The first statement follows easily the same way.

At first note that $D^{bd}$ is $\psi_s$-invariant for all $t \in T_+$ as for $D_c \in \mathfrak{C}_0$ we also have $\psi_t(D_c) \in \mathfrak{C}_0$. So it suffices to show that it is also stable under the $\varphi$-action of $T_+$ since these two actions are clearly compatible (as they are compatible on $D$). At first we show that we have $\varphi_s(D^{bd}) \subset D^{bd}$. Let $D_c \in \mathfrak{C}_0$ be arbitrary. Then the $\psi$-action of the monoid $p\mathbb{Z}$ (ie. the action of $\psi_s$) is nondegenerate on $D_c$ as $D_c$ is a $\psi_s$-invariant submodule of a étale module $D$. So by the remark after Proposition 1.3 and by Corollary 4.8 we obtain an injective $\psi_s$ and $\varphi_s$-equivariant homomorphism $\iota : \tilde{D}_c \hookrightarrow D$. However, each $\varphi_s \iota(D_c) \subseteq \tilde{D}_c$ is compact and $\psi_s$-equivariant therefore the image of $\tilde{D}_c$ is contained in $D^{bd}$ showing that $\varphi_s(D_c) \subset N_0 \varphi_s(D_c) = \iota(\varphi_s(D_c)) \subseteq D^{bd}$. However, for each $t \in T_+$ there exists a $t' \in T_+$ with $tt' = s^k$ for some $k \geq 0$, so $\varphi_t(D_c) = \psi_t(\varphi_s(D_c)) \subseteq D^{bd}$ showing that $D^{bd}$ is $\varphi_t$-invariant for all $t \in T_+$. □

**Corollary 6.4.** The image of the map $\tilde{\text{pr}} : \tilde{D}^\vee_{SV}(\pi) \rightarrow D^\vee_{\ell, \infty}(\pi)$ is contained in $D^\vee_{\ell, \infty}(\pi)^{bd}$.

**Proof.** By Propositions 1.3 and 6.3 it suffices to show that the image of $\text{pr} : D^\vee_{SV}(\pi) \rightarrow D^\vee_{\ell, \infty}(\pi)$ lies in $D^\vee_{\ell, \infty}(\pi)^{bd}$. However, this is clear since $\text{pr}(D^\vee_{SV}(\pi))$ is a $\psi_s$-invariant compact $\Lambda(N_0)$-submodule of $D^\vee_{\ell, \infty}(\pi)$. □
Let $\mathfrak{C}$ be the set of all compact subsets $C$ of $D_{\xi,\ell,\infty}(\pi)$ contained in one of the compact subsets $D_c \in \mathfrak{C}_0 = \mathfrak{C}_0(D_{\xi,\ell,\infty}(\pi))$. Recall from Definition 6.1 in [9] that the family $\mathfrak{C}$ is said to be special if it satisfies the following axioms:

$\mathfrak{C}(1)$ Any compact subset of a compact set in $\mathfrak{C}$ also lies in $\mathfrak{C}$.

$\mathfrak{C}(2)$ If $C_1, C_2, \ldots, C_n \in \mathfrak{C}$ then $\bigcup_{i=1}^n C_i$ is in $\mathfrak{C}$, as well.

$\mathfrak{C}(3)$ For all $C \in \mathfrak{C}$ we have $N_0 C \in \mathfrak{C}$.

$\mathfrak{C}(4)$ $D(\mathfrak{C}) := \bigcup_{C \in \mathfrak{C}} C$ is an étale $T_+\text{-submodule of } D$.

**Lemma 6.5.** The set $\mathfrak{C}$ is a special family of compact sets in $D_{\xi,\ell,\infty}(\pi)$ in the sense of Definition 6.1 in [7].

**Proof.** $\mathfrak{C}(1)$ is satisfied by construction. So is $\mathfrak{C}(3)$ by noting that any $C \in \mathfrak{C}$ is contained in a $D_c \in \mathfrak{C}_0$ which is $N_0\text{-stable}$. For $\mathfrak{C}(2)$ note that for any $D_{c,1}, \ldots, D_{c,r} \in \mathfrak{C}_0$ we have $\bigcup_{i=1}^r D_{c,i} \in \mathfrak{C}_0$. Finally, $\mathfrak{C}(4)$ is just Proposition 6.3.

Our next goal is to construct a $G\text{-equivariant}$ sheaf $\mathfrak{Y} = \mathfrak{Y}_{A,\pi}$ on $G/B$ in [9] with sections $\mathfrak{Y}(C_0)$ on $C_0 := N_0 w_0 B/B$ isomorphic to $D_{\xi,\ell,\infty}(\pi)^{bd}$ as a $B_+\text{-module}$. Here $w_0 \in N_G(T)$ is a representative of an element in the Weyl group $N_G(T)/C_G(T)$ of maximal length. For this we identify $D_{\xi,\ell,\infty}(\pi)^{bd}$ with the global sections of a $B_+\text{-equivariant}$ sheaf on $N_0$ as in [9]. The restriction maps $\text{res}_{us^k N_0 s^{-k}}^N$ are defined as $u \circ \varphi^k_s \circ \psi^k_s \circ u^{-1}$. The open sets $us^k N_0 s^{-k}$ form a basis of the topology on $N_0$, so it suffices to give these restriction maps. Indeed, any open compact subset $\mathcal{U} \subseteq N_0$ is the disjoint union of cosets of the form $us^k N_0 s^{-k}$ for $k \geq k'(\mathcal{U})$ large enough. For a fixed $k \geq k'(\mathcal{U})$ we put

$$\text{res}_\mathcal{U} = \text{res}_{us^k N_0 s^{-k}}^N := \sum_{u \in \mathcal{U}(N_0/s^k N_0 s^{-k})} u \varphi^k_s \circ \psi^k_s \circ u^{-1}.$$  

This is independent of the choice of $k \geq k'(\mathcal{U})$ by Prop. 3.16 in [9]. Note that the map

$$u \mapsto x_u := uw_0 B/B \in C_0$$

is a $B_+\text{-equivariant}$ homeomorphism from $N_0$ to $C_0$ therefore we may view $D_{\xi,\ell,\infty}(\pi)^{bd}$ as the global sections of a sheaf on $C_0$. For an open subset $U \subseteq N_0$ we denote the image of $U$ by $x_U \subseteq C_0$ under the above map $u \mapsto x_u$. Moreover, we regard res as an $\text{End}_{\mathfrak{C}_0}(D_{\xi,\ell,\infty}(\pi))\text{-valued measure on } C_0$, i.e. a ring homomorphism $\text{res} : C^\infty(C_0, o) \rightarrow \text{End}_{\mathfrak{C}_0}(D_{\xi,\ell,\infty}(\pi))$. We restrict res to a map res: $C^\infty(C_0, o) \rightarrow \text{Hom}_{\mathfrak{C}_0}(D_{\xi,\ell,\infty}(\pi)^{bd}, D_{\xi,\ell,\infty}(\pi))$. Put $C := Nw_0 B/B \supset C_0$. By the discussion in section 5 of [9] in order to construct a $G\text{-equivariant}$ sheaf on $G/B$ with the required properties we need to integrate the map

$$\alpha_g : C_0 \rightarrow \text{Hom}_{\mathfrak{C}_0}^\text{cont}(D_{\xi,\ell,\infty}(\pi)^{bd}, D_{\xi,\ell,\infty}(\pi))$$

$$x_u \mapsto \alpha(g, u) \circ \text{res}(1_{\alpha(g, u)^{-1}C_0 \cap C_0})$$

with respect to the measure res where for $x_u \in g^{-1}C_0 \cap C_0 \subset g^{-1}C \cap C$ we take $\alpha(g, u)$ to be the unique element in $B$ with the property

$$guw_0 N = \alpha(g, u)uw_0 N.$$
Note that since \( x_u \) lies in \( g^{-1}C_0 \cap C_0 \) we also have \( x_u \in \alpha(g, u)^{-1}C_0 \cap C_0 \) so the latter set is nonempty and open in \( G/B \). Recall from section 6.1 in [9] that a map \( F : C_0 \to \text{Hom}_c(D_{\xi, \ell, \infty}(\pi)^{bd}, D_{\xi, \ell, \infty}(\pi)) \) is called integrable with respect to \( (s, \text{res}, \mathcal{C}) \) if the limit

\[
\int_{C_0} F \text{res} := \lim_{k \to \infty} \sum_{u \in J(N_0/s^k N s^{-k})} F(x_u) \circ \text{res}(1_{x_u s^k N s^{-k}})
\]

exists in \( \text{Hom}_c(D_{\xi, \ell, \infty}(\pi)^{bd}, D_{\xi, \ell, \infty}(\pi)) \) and does not depend on the choice of the sets of representatives \( J(N_0/s^k N s^{-k}) \).

**Proposition 6.6.** The map \( \alpha_g \) is \( (s, \text{res}, \mathcal{C}) \)-integrable for any \( g \in G \).

**Proof.** By Proposition 6.8 in [9] it suffices to show that \( \mathcal{C} \) satisfies:

\( \mathfrak{C}(5) \) For any \( C \in \mathcal{C} \) the compact subset \( \psi_s(C) \subseteq D_{\xi, \ell, \infty}(\pi) \) also lies in \( \mathcal{C} \).

\( \mathfrak{T}(1) \) For any \( C \in \mathcal{C} \) such that \( C = N_0 C \), any open \( o[N_0] \)-submodule \( D \) of \( D_{\xi, \ell, \infty}(\pi) \), and any compact subset \( C_+ \subseteq T_+ \) there exists a compact open subgroup \( B_1 = B_1(C, D, C_+) \subseteq B_0 \) and an integer \( k(C, D, C_+) \geq 0 \) such that

\[
\varphi_s^k \circ (1 - B_1) C_+ \psi_s^k(C) \subseteq D \quad \text{for any } k \geq k(C, D, C_+).
\]

Here the multiplication by \( C_+ \) is via the \( \varphi \)-action of \( T_+ \) on \( D_{\xi, \ell, \infty}(\pi) \).

The condition \( \mathfrak{C}(5) \) is clearly satisfied as for any \( D \in \mathcal{C} \) we have \( \psi_s^k(D) \in \mathcal{C} \), as well. For the condition \( \mathfrak{T}(1) \) choose a \( C \in \mathcal{C} \) with \( C = N_0 C \), a compact subset \( C_+ \subseteq T_+ \), and an open \( o[N_0] \)-submodule \( D \subseteq D_{\xi, \ell, \infty}(\pi) \). As \( D_{\xi, \ell, \infty}(\pi) \) is the topological projective limit \( \lim_{\leftarrow \text{M} \in \mathcal{M}(H_0)} M_n[1/X] \) we may assume without loss of generality that \( D \) is the preimage of a compact \( \Lambda(N_0) \)-submodule \( D_n \leq M_n[1/X] \) with \( D_n[1/X] = M_n[1/X] \) under the natural surjective map \( \varphi_{M,n} : D_{\xi, \ell, \infty}(\pi) \to M_n[1/X] \) for some \( M \in \mathcal{M}(H_0) \) and \( n \geq 0 \). Moreover, since \( B_0 \) is compact and normalizes \( H_0 \), the \( T_0 \)-orbit of any element \( m \in M \leq \pi H_0 \) is finite and contained in \( \pi H_0 \). Therefore we also have \( B_0 M = T_0 M \in \mathcal{M}(H_0) \). So we may assume without loss of generality that \( M \) is \( B_0 \)-invariant whence we have an action of \( B_0 \) on \( M_n[1/X] \). Choose a \( D \in \mathcal{C} \) with \( C \subseteq D \). Since \( D \) is \( \psi_s \)-invariant, we have \( C_+ \psi_s^k(C) \subseteq C_+ \psi_s^k(D) \subseteq C_+ D \). Moreover, \( C_+ D \) is compact as both \( C_+ \) and \( D \) are compact, so \( f_{M,n}(C_+ \psi_s^k(C)) \subseteq M_n[1/X] \) is bounded. In particular, we have a compact \( \Lambda(N_0) \)-submodule \( D' \) of \( M_n[1/X] \) containing \( f_{M,n}(C_+ \psi_s^k(C)) \). So by the continuity of the action of \( B_0 \) on \( M_n[1/X] \) there exists an open subgroup \( B_1 \subseteq B_0 \) such that we have

\[
(1 - B_1) f_{M,n}(C_+ \psi_s^k(C)) \subseteq \Lambda(N_0/H_n) \otimes_{\Lambda(N_0, o)} (M^+[1/X]) \leq \Lambda(N_0/H_n) \otimes_{\Lambda(N_0, o)} M^+[1/X] \approx M_n[1/X]
\]

for any \( k \geq 0 \). Here \( M^+[1/X] \) denotes the trellis in \( M^+[1/X] \) consisting of those elements \( d \in M^+[1/X] \) such that \( \varphi_s^n(d) \to 0 \) in \( M^+[1/X] \) as \( n \to \infty \) (cf. section I.3.2 in [1]). Finally, since \( D_n \) is open and \( M^+[1/X] \) is finitely generated over \( \Lambda(N_0, o) \equiv o[1/X] \) there exists an integer \( k_1 \geq 0 \) such that \( \varphi_s^k(\Lambda(N_0/H_n) \otimes_{\Lambda(N_0, o)} (M^+[1/X]^{++})) \) is contained in \( D_n \) for all \( k \geq k_1 \). In particular, we have

\[
f_{M,n}(\varphi_s^k \circ (1 - B_1) C_+ \psi_s^k(C)) = \varphi_s^k \circ (1 - B_1)(f_{M,n}(C_+ \psi_s^k(C))) \subseteq \varphi_s^k \circ (1 - B_1)(M^+[1/X]^{++}) \subseteq D_n
\]

showing that \( \varphi_s^k \circ (1 - B_1) C_+ \psi_s^k(C) \) is contained in \( D \).
For all \( g \in G \) we denote by \( \mathcal{H}_g \in \text{Hom}_{\text{cont}}(D^\vee_{\xi,\ell,\infty}(\pi)^{bd}, D^\vee_{\xi,\ell,\infty}(\pi)) \) the integral
\[
\mathcal{H}_g := \int_{C_0} \alpha_g \, d\text{res} = \lim_{k \to \infty} \sum_{u \in J(N_0/s^k N_0s^{-k})} \alpha_g(x_u) u \circ \varphi^k_s \circ \psi^k_s \circ u^{-1}
\]
we have just proven to converge. We denote the \( k \)th term of the above sequence by
\[
\mathcal{H}_g^{(k)} = \mathcal{H}_{g,J(N_0/s^k N_0s^{-k})} := \sum_{u \in J(N_0/s^k N_0s^{-k})} \alpha_g(x_u) u \circ \varphi^k_s \circ \psi^k_s \circ u^{-1}.
\]

Our main result in this section is the following

**Proposition 6.7.** The image of the map \( \mathcal{H}_g : D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \to D^\vee_{\xi,\ell,\infty}(\pi) \) is contained in \( D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \).

There exists a \( G \)-equivariant sheaf \( \mathcal{Y} = \mathcal{Y}_{g,\pi} \) on \( G/B \) with sections \( \mathcal{Y}(C_0) \) on \( C_0 \) isomorphic \( B_+ \)-equivariantly to \( D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \) such that we have \( \mathcal{H}_g = \text{res}^G_{C_0} \circ (g \cdot) \circ \text{res}^G_{C_0} \) as maps on \( D^\vee_{\xi,\ell,\infty}(\pi)^{bd} = \mathcal{Y}(C_0) \).

**Proof.** By Prop. 5.14 and 6.9 in [9] it suffices to check the following conditions:

\( \mathcal{C}(6) \) For any \( C \in \mathcal{C} \) the compact subset \( \varphi_s(C) \subseteq M \) also lies in \( \mathcal{C} \).

\( \mathcal{C}(2) \) Given a set \( J(N_0/s^k N_0s^{-k}) \subseteq N_0 \) of representatives for cosets in \( N_0/s^k N_0s^{-k} \) for all \( k \geq 1 \), for any \( x \in D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \) and \( g \in G \) there exists a compact \( \psi_s \)-invariant \( \Lambda(N_0) \)-submodule \( D_{x,g} \in \mathcal{C} \) and a positive integer \( k_{x,g} \) such that \( \mathcal{H}_g^{(k)}(x) \subseteq D_{x,g} \) for any \( k \geq k_{x,g} \).

The condition \( \mathcal{C}(6) \) follows from (the proof of) Prop. 6.3 as for \( C \subseteq D_c \in \mathcal{C}_0 \) we have \( \varphi_s(C) \subseteq \varphi_s(D) \subseteq i(\varphi^s_s D_c) \in \mathcal{C}_0 \).

The proof of \( \mathcal{C}(2) \) is very similar to the proof of Corollary 9.15 in [9]. However, it is not a direct consequence of that as \( D^\vee_{\xi,\ell,\infty} \) is not necessarily finitely generated over \( \Lambda_{f}(N_0) \), so we recall the details. Since \( \mathcal{H}_g^{(k)}(x) \) lies in \( D^{bd} \) for any fixed \( k \), we only need to show that for \( k \) large enough the difference
\[
s_g^{(k)}(x) := \mathcal{H}_g^{(k)}(x) - \mathcal{H}_{g,J(N_0/s^{k+1} N_0s^{-k-1})}(x)
\]
lies in a compact submodule \( D_{x,g} \leq D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \) in \( \mathcal{C}_0 \) independent of \( k \). Equation (43) in [9] shows that for any compact open subgroup \( B_1 \leq B_0 \) there exist integers \( 0 \leq k_g^{(1)} \leq k_g^{(2)}(B_1) \) and a compact subset \( \Lambda_g \subseteq T_+ \) such that for \( k \geq k_g^{(2)}(B_1) \) we have
\[
s_g^{(k)} \in \langle N_0 s^{k-k_g^{(1)}}(1 - B_1) \Lambda_g s \psi_s^{k+1} N_0 \rangle_0 ,
\]
where we denote by \( \langle \cdot \rangle_0 \) the generated \( \mathfrak{g} \)-submodule. Here \( k_g^{(1)} \) is chosen so that \( \{ \alpha(g, u)us^{k_g^{(1)}} | x_u \in g^{-1} C_0 \cap C_0 \} \) is contained in \( B_+ = N_0 T_+ \). There exists such an integer \( k_g^{(2)} \) since \( \{ \alpha(g, u)u | x_u \in g^{-1} C_0 \cap C_0 \} \) is a compact subset in \( N_0 T_+ \). Choose a compact \( \psi_s \)-invariant \( \Lambda(N_0) \)-submodule \( D_c \in \mathcal{C}_0 \) containing the element \( x \in D^\vee_{\xi,\ell,\infty}(\pi)^{bd} \) and an \( M \) in \( \mathcal{M}(\pi^{H_0}) \). Applying \( \mathfrak{S}(1) \) in the situation \( C = D_c, C_+ = \Lambda_g s \), and \( D = f_{M,0}^{-1}(M^{\vee}[1/X]^{++}) \) we find an integer \( k_1 \geq 0 \) and a compact open subgroup \( B_1 \leq B_0 \) such that \( \varphi_s \circ (1 - B_1) \Lambda_g s D_c \subseteq D \) for all \( k \geq k_1 \).

Noting that \( D_c \) is \( \psi_s \)-stable and \( D \) is a \( \Lambda(N_0) \)-submodule we obtain \( s_g^{(k)}(D_c) \subseteq N_0 \varphi_s(D) \) for

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\[ k \geq r + k_1 + k_g^{(2)}(B_1). \]

Applying \( \psi_s^r \) to this using (20) and putting \( k_g(M) := k_1 + k_g^{(2)}(B_1) \) we deduce

\[
\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq \mathcal{D} \quad \text{for all } k \geq k_g(M) \text{ and } r \leq k - k_g(M). \tag{21}
\]

Note that the subgroup \( B_1 \) depends on \( M \) therefore so do \( k_g^{(2)}(B_1) \) and \( k_g(M) \), but not \( k_g^{(1)} \).

On the other hand, we are going to find another treillis \( D_1 \leq M^\vee[1/X] \) such that for all \( k \geq k_g(M) \) and \( r \geq k - k_g(M) \) we have

\[
\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq D_1 := f_{M,0}^{-1}(D_1). \tag{22}
\]

For \( x_u \in g^{-1}C_0 \cap C_0 \) write \( \alpha(g,u)u \) in the form \( \alpha(g,u)u = n(g,u)t(g,u) \) with \( n(g,u) \in N_0 \) and \( t(g,u) \in T \). Since \( g^{-1}C_0 \cap C_0 \) is compact, \( t(g,\cdot) \) is continuous, and \( k_g(M) \geq k_g^{(1)} \) the set \( C'_+ := \{ t(g,u)s_{k_g(M)} | x_u \in g^{-1}C_0 \cap C_0 \} \subseteq T \) is compact and contained in \( T_+ \). So we compute

\[
\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) = \psi_s^r(\Lambda(N_0) \sum_{u \in J(N_0/s_{k_g(M}-k})} n(g,u)\varphi_{t(g,u)s_{k_g(M)}} \circ \psi_s^r(u^{-1}D_c)) \subseteq \\
\psi_s^r(\Lambda(N_0)\varphi_k \circ \psi_s^r(u^{-1}D_c)) \subseteq \psi_s^r(\Lambda(N_0)\varphi_k \circ \varphi_{t(g,u)s_{k_g(M)}}(D_c)) \subseteq \psi_s^r(\Lambda(N_0)C'_+(D_c)).
\]

Since \( C'_+ \subseteq T_+ \) is compact, there exists an integer \( k(C'_+) \) such that \( s^kt^{-1} \) lies in \( T_+ \) for all \( t \in C'_+ \). So we have \( C'_+(D_c) \subseteq i(\varphi_{k(C'_+)}\mathcal{D}_{\xi,t,\infty}(\pi)^{bd}) \in C_0 \) showing that

\[
D_1 := f_{M,0}(i(\varphi_{k(C'_+)}\mathcal{D}_{\xi,t,\infty}(\pi)^{bd}))
\]

is a good choice as \( i(\varphi_{k(C'_+)}\mathcal{D}_{\xi,t,\infty}(\pi)^{bd}) \) is a \( \psi_s \)-stable \( \Lambda(N_0) \) submodule. Finally, for each fixed \( k \geq k_g^{(1)} \) there exists a compact \( \psi_s \)-invariant \( \Lambda(N_0) \)-submodule \( D_{c,k} \in C_0 \) containing \( \mathcal{H}_g^{(k)}(D_c) \). In particular, we may choose a treillis \( D_2 \leq M^\vee[1/X] \) containing

\[
\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c))
\]

for all \( k \geq k_g^{(1)} \leq k \leq k_g(M) \) and \( r \geq 0 \). Putting \( D_2 := f_{M,0}^{-1}(D_2) \) and combining this with (21) and (22) we obtain

\[
\psi_s^r(\Lambda(N_0)\mathcal{H}_g^{(k)}(D_c)) \subseteq \mathcal{D} + D_1 + D_2 \tag{23}
\]

for all \( k \geq k_{x,g} := k_g^{(1)} \) and \( r \geq 0 \). Denote by \( f_{M,\infty} \) the natural surjective map \( f_{M,\infty} : D_{\xi,t,\infty}^\vee \rightarrow M_{\infty}^\vee[1/X] \). Note that \( f_{M,0} \) factors through \( f_{M,\infty} \). The equation (23) implies (in fact, is equivalent to) that

\[
f_{M,\infty} \left( \bigcup_{k \geq k_{x,g}} \mathcal{H}_g^{(k)}(D_c) \right) \subseteq M_{\infty}^\vee[1/X]^{bd}(M_{\infty}^\vee[1/X]^{++} + D_1 + D_2)
\]

where

\[
M_{\infty}^\vee[1/X]^{bd}(M_{\infty}^\vee[1/X]^{++} + D_1 + D_2) = \\
\{ m \in M_{\infty}^\vee[1/X] | \ell_M(\psi_s^r(u^{-1}m)) \text{ is in } M_{\infty}^\vee[1/X]^{++} + D_1 + D_2 \text{ for all } r \geq 0, u \in N_0 \}
\]

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is a compact $\psi_\ell$-invariant $\Lambda(N_0)$-submodule in $M^\nu_\infty[1/X]$ (Prop. 9.10 in [9]). So we put $D_{x,g}(M) := \bigcap \mathcal{D}$ where $\mathcal{D}$ runs through all the $\psi_\ell$-invariant compact $\Lambda(N_0)$-submodules of $M^\nu_\infty[1/X]$ containing $f_{M,\infty}(\bigcup_{k \geq k_{x,g}} \mathcal{H}^{(k)}_g(D_c))$. Therefore

$$D_{x,g} := \lim_{M \in \mathcal{M}(M^\nu_\infty)} D_{x,g}(M)$$

is a $\psi_\ell$-invariant compact $\Lambda(N_0)$-submodule of $D_{\xi,\ell,\infty}^\nu(\pi)$ (ie. we have $D_{x,g} \in \mathcal{C}_0$) containing $\bigcup_{k \geq k_{x,g}} \mathcal{H}^{(k)}_g(D_c)$.

We end this section by putting a natural topology (called the weak topology) on the global sections $\mathcal{Y}(G/B)$ that will be needed in the next section. At first we equip $Y$ the weak topology on $D_{x,g}(M)$ where $\mathcal{M}$ is a $\psi_\ell$-submodule of $D_{\xi,\ell,\infty}(\pi)$ isometrically inherited from $D_{\xi,\ell,\infty}(\pi)$ therefore it is Hausdorff. So the topology on $\mathcal{Y}(G/B)$ is also Hausdorff as for any two different global sections $x \neq y \in \mathcal{Y}(G/B)$ there exists an element $g \in G$ such that $\text{res}_{g \mathcal{C}_0}^G(x) \neq \text{res}_{g \mathcal{C}_0}^G(y)$.

**Lemma 6.8.** The operators $\mathcal{H}_g$ and $\text{res}_U$ on $D_{\xi,\ell,\infty}(\pi)^{bd}$ are continuous in the weak topology of $D_{\xi,\ell,\infty}(\pi)^{bd}$ for all $g \in G$ and $U \subseteq N_0$ compact open. In particular, $D_{\xi,\ell,\infty}(\pi)^{bd}$ is the topological direct sum of $\text{res}_U(D_{\xi,\ell,\infty}(\pi)^{bd})$ and $\text{res}_N \mathcal{H}(D_{\xi,\ell,\infty}(\pi)^{bd})$.

**Proof.** By the property $\mathcal{T}(2)$ the restriction of $\mathcal{H}^{(k)}_g$ to a compact subset $D_c$ in $\mathcal{C}_0$ has image in a compact set $D_{c,g} \in \mathcal{C}_0$ for all large enough $k$. Moreover, each $\mathcal{H}^{(k)}_g$ is continuous by Lemma 6.2. On the other hand, the limit $\mathcal{H} = \lim_{k \to \infty} \mathcal{H}^{(k)}_g$ is uniform on each compact subset $D_c \in \mathcal{C}_0$ by Proposition 6.3 in [9], so the limit $\mathcal{H}_g : D_c \to D_{c,g}$ is also continuous. Taking the inductive limit on both sides we deduce that $\mathcal{H}_g : D_{\xi,\ell,\infty}(\pi) \to D_{\xi,\ell,\infty}(\pi)$ is also continuous. The continuity of $\text{res}_U$ follows in a similar but easier way.

So far we have put a topology on $D_{\xi,\ell,\infty}(\pi)^{bd} = \mathcal{Y}(\mathcal{C}_0)$. The multiplication by an element $g \in G$ gives a $\sigma$-linear bijection $g : \mathcal{Y}(\mathcal{C}_0) \to \mathcal{Y}(g \mathcal{C}_0)$. We define the weak topology on $\mathcal{Y}(g \mathcal{C}_0)$ so that this is a homeomorphism. Now we equip $\mathcal{Y}(G/B)$ with the coarsest topology such that the restriction maps $\text{res}_{g \mathcal{C}_0}^G : \mathcal{Y}(G/B) \to \mathcal{Y}(g \mathcal{C}_0)$ are continuous for all $g \in G$. We call this the weak topology on $\mathcal{Y}(G/B)$ making $\mathcal{Y}(G/B)$ a linear-topological $\sigma$-module.

**Lemma 6.9.** a) The multiplication by $g$ on $\mathcal{Y}(G/B)$ is continuous (in fact a homeomorphism) for each $g \in G$.

b) The weak topology on $\mathcal{Y}(G/B)$ is Hausdorff.

**Proof.** For a) we need to check that the composite of the function $(g \cdot)_{G/B} : \mathcal{Y}(G/B)$ with the projections $\text{res}_{h \mathcal{C}_0}^G$ is continuous for all $h \in G$. However, $\text{res}_{h \mathcal{C}_0}^G \circ (g \cdot)_{G/B} = (g \cdot)_{g^{-1}h \mathcal{C}_0} \circ \text{res}_{g^{-1}h \mathcal{C}_0}^G$ is the composite of two continuous maps hence also continuous.

For b) note that the weak topology on $D_{\xi,\ell,\infty}(\pi)^{bd}$ is finer than the subspace topology inherited from $D_{\xi,\ell,\infty}(\pi)$ therefore it is Hausdorff. So the topology on $\mathcal{Y}(G/B)$ is also Hausdorff as for any two different global sections $x \neq y \in \mathcal{Y}(G/B)$ there exists an element $g \in G$ such that $\text{res}_{g \mathcal{C}_0}^G(x) \neq \text{res}_{g \mathcal{C}_0}^G(y)$.
7  A $G$-equivariant map $\pi^\vee \to \mathcal{Y}(G/B)$

Here we generalize Thm. IV.4.7 in [4] to $\mathbb{Q}_p$-split reductive groups $G$ over $\mathbb{Q}_p$ with connected centre. Assume in this section that $\ell = \ell_\alpha$ for some simple root $\alpha \in \Delta$ and that $\pi$ is an admissible smooth $o/\varpi^h$-representation of $G$ of finite length.

By Corollary 6.4 we have the composite maps

$$\beta_{oC_0} : \pi^\vee \xrightarrow{g^{-1}} \pi^\vee \xrightarrow{BD_{SV}(\pi)} D_{SV}(\pi) \xrightarrow{D_{SV}(\pi)(\xi, t, \infty)} \mathcal{Y}(C_0) \xrightarrow{p} \mathcal{Y}(gC_0)$$

for each $g \in G$. By definition we have $\beta_{oC_0}(\mu) = g\beta_{o}(g^{-1}\mu)$ for all $\mu \in \pi^\vee$ and $g \in G$. Our goal is to show that these maps glue together to a $G$-equivariant map $\beta_{G/B} : \pi^\vee \to \mathcal{Y}(G/B)$.

Let $n_0 = n_0(G) \in \mathbb{N}$ be the maximum of the degrees of the algebraic characters $\beta \circ \xi : \mathbb{G}_m \to \mathbb{G}_m$ for all $\beta$ in $\Phi^+$ and put $U^{(k)} := \text{Ker}(G_0 \to G(\mathbb{Z}_p/p^k\mathbb{Z}_p))$ where $G_0 = G(\mathbb{Z}_p)$.

**Lemma 7.1.** For any fixed $r_0 \geq 1$ we have $t^{-1}U^{(k)}t \leq U^{(k-r_0)}$ for all $t \leq s^{r_0}$ in $T_+$ and $k \geq r_0 n_0$.

**Proof.** The condition $t \leq s^{r_0}$ implies that $v_p(\beta(t)) \leq v_p(\beta(s^{r_0})) = v_p(\beta(\xi(p^n))) \leq r_0 n_0$ for all $\beta \in \Phi^+$. On the other hand, by the Iwahori factorization we have $U^{(k)} = (U^{(k)} \cap T)(U^{(k)} \cap N)$. Since $t$ is in $T_+$ we deduce

$$t^{-1}(U^{(k)} \cap N)t \leq (U^{(k)} \cap N) \leq (U^{(k-r_0)} \cap N)$$

$$t^{-1}(U^{(k)} \cap T)t = (U^{(k)} \cap T) \leq (U^{(k-r_0)} \cap T)$$

$$t^{-1}(U^{(k)} \cap N)t = \prod_{\beta \in \Phi^+} t^{-1}(U^{(k)} \cap N_\beta)t \leq \prod_{\beta \in \Phi^+} (U^{(k-r_0)} \cap N_\beta) = (U^{(k-r_0)} \cap N) .$$

\[\square\]

**Lemma 7.2.** Assume that $\pi$ is an admissible representation of $G$ of finite length. Then there exists a finitely generated $o$-submodule $W_0 \leq \pi$ such that $\pi = BW_0$.

**Proof.** Since $\pi$ has finite length, by induction we may assume it is irreducible (hence killed by $\varpi$). In this case we may take $W_0 = \pi^{U^{(1)}}$ which is $G_0$-stable as $U^{(1)}$ is normal in $G_0$. It is nonzero since $\pi$ is, and finitely generated over $o$ as $\pi$ is admissible. By the Iwasawa decomposition we have $\pi = GW_0 = BG_0W_0 = BW_0$.

Let $W_0$ be as in Lemma 7.2 and put $W := B_+W_0$. Put $W_r := \bigcup_{t \leq s^r} N_0 t W_0$ so we have

$$W = \lim_{r \to \infty} W_r = \bigcup_{r \geq 0} W_r$$

(24)

where $W_r$ is finitely generated over $o$ for all $r \geq 0$. By construction $W$ is a generating $B_+$-subrepresentation of $\pi$. So the map $pr_{SV}$ factors through the natural projection map $pr_W : \pi^\vee \to W^\vee$. Here the Pontryagin dual $W^\vee$ is a compact $\Lambda(N_0)$-module with a $\psi$-action of $T_+$ coming from the multiplication by $T_+$ on $W$. By Proposition 4.50 we may form the étale hull $W^\vee$ of $W^\vee$ which is an étale $T_+$-module over $\Lambda(N_0)$. Since $D^\vee_{SV}(\pi, t, \infty)$ is an étale
The operators $\mathcal{H}_g^{(k)}$ make sense as maps $\overline{W^v} \rightarrow \overline{W^v}$ and the map $\overline{W^v} \rightarrow D_{\xi,\ell,\infty}(\pi)$ is $\mathcal{H}_g^{(k)}$-equivariant as it is a morphism of étale $T_+$-modules over $\Lambda(N_0)$. More precisely, let $g$ be in $G$ and put $\mathcal{U}_g := \{u \in N_0 \mid x_u \in g^{-1}C_0 \cap C_0\}$, $\mathcal{U}_g^{(k)} := J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}_g$. For any $u \in \mathcal{U}_g$ we write $gu$ in the form $g = n(g,u)t(g,u)\overline{u}(g,u)$ for some unique $n(g,u) \in N_0$, $t(g,u) \in T$, $\overline{u}(g,u) \in \overline{N}$.

**Lemma 7.3.** There exists an integer $k_0 = k_0(g)$ such that for all $k \geq k_0$ and $u \in \mathcal{U}_g$ we have $us^kN_0s^{-k} \subseteq \mathcal{U}_g$, $s^kt(g,u) \in T_+$, and $s^{-k}\overline{u}(g,u)s^k \in J_0 = G_0 \cap \overline{N}$. In particular, for any set $J(N_0/s^kN_0s^{-k})$ of representatives of the cosets in $N_0/s^kN_0s^{-k}$ we have $\mathcal{U}_g = \bigcup_{u \in \mathcal{U}_g^{(k)}} us^kN_0s^{-k}$.

*Proof.* Since $\mathcal{U}_g$ is compact and open in $N_0$, it is a union of finitely many cosets of the form $us^kN_0s^{-k}$ for $k$ large enough. Moreover, the maps $t(g,\cdot)$ and $\overline{u}(g,\cdot)$ are continuous in the $p$-adic topology. So the image of $t(g,\cdot)$ is contained in finitely many cosets of $T/T_0$ as $T_0$ is open. For the statement regarding $\overline{u}(g,u)$ note that we have $\overline{N} = \bigcup_{k \geq 0} s^k\overline{N}s^{-k}$.

For $k \geq k_0$ let $J(N_0/s^kN_0s^{-k}) \subseteq N_0$ be an arbitrary set of representatives of $N_0/s^kN_0s^{-k}$. Recall (cf. [9]) that we defined

$$\mathcal{H}_g^{(k)} = \mathcal{H}_{g,J(N_0/s^kN_0s^{-k})} := \sum_{u \in \mathcal{U}_g^{(k)}} n(g,u)\varphi_{t(g,u)s^k} \circ \psi_s^k \circ (u^{-1}) .$$

Further, any open compact subset $\mathcal{U} \subseteq N_0$ is the disjoint union of cosets of the form $us^kN_0s^{-k}$ for $k \geq k'(\mathcal{U})$ large enough. For a fixed $k \geq k'(\mathcal{U})$ we put

$$\text{res}_\mathcal{U} := \sum_{u \in J(N_0/s^kN_0s^{-k}) \cap \mathcal{U}} u\varphi_s^k \circ \psi_s^k \circ (u^{-1}) .$$

The operators $\mathcal{H}_g^{(k)}$ and $\text{res}_\mathcal{U}$ make sense in any étale $T_+$-module over $\Lambda(N_0)$, in particular also in $\overline{W^v}$ and $D_{\xi,\ell,\infty}(\pi)$. Moreover, $\text{res}_\mathcal{U}$ is independent of the choice of $k \geq k'(\mathcal{U})$. Further, any morphism between étale $T_+$-modules over $\Lambda(N_0)$ is $\mathcal{H}_g^{(k)}$- and $\text{res}_\mathcal{U}$-equivariant.

**Lemma 7.4.** Let $g$ be in $G$, $u$ be in $\mathcal{U}_g$, and $k \geq k_0 + 1$ be an integer. Then the map

$$n(g,\cdot): us^kN_0s^{-k} \rightarrow n(g,u)t(g,u)s^kN_0s^{-k}t(g,u)^{-1}$$

is a bijection. In particular, for any set $J(N_0/s^kN_0s^{-k})$ of representatives of the cosets in $N_0/s^kN_0s^{-k}$ the set $\mathcal{U}_{g-1}$ is the disjoint union of the cosets $n(g,u)t(g,u)s^kN_0s^{-k}t(g,u)^{-1}$ for $u \in \mathcal{U}_g^{(k)}$.
Proof. By our assumption \( k \geq k_0 + 1 \), \( s^{-k} \mathfrak{m}(g, u) s^k \) lies in \( s^{-1} N_0 s \subseteq U^{(1)} \). So for any \( v \in N_0 \) we have \( s^{-k} \mathfrak{m}(g, u) s^k v = v v_1 \) for some \( v_1 \) in \( v^{-1} U^{(1)} v = U^{(1)} \). Further, by the Iwahori factorization we have \( U^{(1)} = (N \cap U^{(1)})(T \cap U^{(1)}) \cap (N \cap U^{(1)}) \). So we obtain that \( s^{-k} \mathfrak{m}(g, u) s^k v w_0 B \subseteq C_0 \) for all \( v \in N_0 \), whence we deduce \( s^{-k} \mathfrak{m}(g, u) s^k C_0 \subseteq C_0 \). Similarly we have \( s^{-k} \mathfrak{m}(g, u) s^k C_0 \subseteq C_0 \) showing that in fact \( s^{-k} \mathfrak{m}(g, u) s^k C_0 = C_0 \). We compute

\[
g(u s^k N_0 s^{-k}) w_0 B = g u s^k N_0 w_0 B = n(g, u) t(g, u) s^k (s^{-k} \mathfrak{m}(g, u) s^k) C_0 = n(g, u) t(g, u) s^k C_0 = n(g, u) (t(g, u) s^k N_0 s^{-k} t(g, u)^{-1}) w_0 B.
\]

Since the map \( n(g, \cdot) \) is induced by the multiplication by \( g \) on \( g^{-1} C_0 \cap C_0 \) (identified with \( U_g \)), we deduce that the map (25) is a bijection. The second statement follows as \( n(g, \cdot) : U_g \to U_{g^{-1}} \) is a bijection and we have a partition of \( U_g \) into cosets \( u s^k N_0 s^{-k} \) for \( u \in U_g^{(k)} \) by Lemma 7.3.

Lemma 7.5. Let \( M \) be arbitrary in \( \mathcal{M}(\pi_{H_0}) \) and \( l, l' \geq 0 \) be integers. There exists an integer \( k_1 = k_1(M, W_0, l, l') \geq 0 \) such that for all \( r \geq k_1 \) the image of the natural composite map

\[
(W/W_r)^L \hookrightarrow W^L \to D_{\xi, l, \infty}(\pi) \twoheadrightarrow M_l^L[1/X]
\]

lies in \( \Lambda(N_0/H_1) \otimes_{\Lambda} X^L M^L[1/X]^{++} \subset \Lambda(N_0/H_1) \otimes_{\Lambda} M_l^L[1/X] \cong M_l^L[1/X] \). Here \( M^L[1/X]^{++} \) denotes the \( \mathfrak{o}/\mathfrak{w}^L[X] \)-submodule of the \( (\varphi, \Gamma) \)-module \( M^L[1/X] \) consisting of elements \( d \in M^L[1/X] \) with \( \varphi_n(d) \to 0 \) as \( n \to \infty \).

Proof. By (24) the \( \Lambda(N_0) \)-submodules \( (W/W_r)^L \) form a system of neighbourhoods of 0 in \( W^L \). On the other hand, \( X^L M^L[1/X]^{++} \) being a treillis in \( M^L[1/X] \) (Prop. II.2.2 in [3]), \( \Lambda(N_0/H_1) \otimes_{\Lambda} X^L M^L[1/X]^{++} \) is open in the weak topology of \( M_l^L[1/X] \). Therefore its preimage in \( W^L \) contains \( (W/W_r)^L \) for \( r \) large enough.

Since \( t(g, \cdot) \) is continuous and \( U_g \) is compact, there exists an integer \( c \geq 0 \) such that for all \( u \in U_g \) there is an element \( t'(g, u) \in T_+ \) such that \( t(g, u) s^k t'(g, u) = s^c \).

Lemma 7.6. For any fixed \( M \in \mathcal{M}(\pi_{H_0}) \) there are finitely many different values of \( F_{t'(g, u)}^* M \) where \( g \in G \) is fixed and \( u \) runs on \( U_g \).

Proof. By Lemma 7.3 there exists an open subgroup \( T' \leq T_+ \) acting on \( M \). In particular, \( F_{t'(g, u)}^* M \) only depends on the coset \( t'(g, u) T' \). Now \( t'(g, \cdot) = s^{-k_0} t(g, \cdot)^{-1} \) is continuous and \( U_g \) is compact therefore there are only finitely many cosets of the form \( t'(g, u) T' \).

Our key proposition is the following:

Proposition 7.7. For all \( g \in G \) we have \( \text{res}_{g C_0 \cap C_0} \circ \beta_{C_0} = \text{res}_{g C_0 \cap C_0} \circ \beta_{C_0} \).

Proof. Note that since \( G/B \) is totally disconnected in the \( p \)-adic topology, in particular \( g C_0 \cap C_0 \) is both open and closed in \( C_0 \), we have \( \mathcal{P}(C_0) = \mathcal{P}(g C_0 \cap C_0) \setminus \mathcal{P}(C_0 \setminus g C_0) \). By Prop. 6.7 \( H_g \) is the composite map

\[
D_{\xi, l, \infty}(\pi)^{bd} = \mathcal{P}(C_0) \to \mathcal{P}(g C_0) \to \mathcal{P}(g C_0 \cap C_0) \to \mathcal{P}(C_0) = D_{\xi, l, \infty}(\pi)^{bd}.
\]

So we are bound to show that for any \( g \in G \) and \( \mu \in \pi^L \) we have

\[
\mathcal{H}_g(\text{pr} \circ \text{pr}_{SV}(g^{-1} \mu)) = \text{res}_{U_{s^{-1}}} \circ \text{pr} \circ \text{pr}_{SV}(\mu).
\]
Putting $\mathcal{U}_g^{(k)} := \{ u \in J(N_0/s^kN_0s^{-k}) \mid x_u \in g^{-1}C_0 \cap C_0 \}$ we compute

$$\mathcal{H}_g^{(k)} \circ \overline{\text{pr}_W(g^{-1}\mu)} = \sum_{u \in \mathcal{U}_g^{(k)}} n(g, u) \varphi_{t(g,u)s^k} \circ \psi_{s^k} (u^{-1}\overline{\text{pr}_W(g^{-1}\mu)}) = \sum_{u \in \mathcal{U}_g^{(k)}} n(g, u) \varphi_{t(g,u)s^k} \circ \overline{\text{pr}_W(s^{-k}u^{-1}g^{-1}\mu)} = \sum_{u \in \mathcal{U}_g^{(k)}} \iota_{t(g,u)s^k,\infty}(n(g, u) \otimes_s s^k \text{pr}_W(s^{-k}u^{-1}g^{-1}\mu)) = \sum_{u \in \mathcal{U}_g^{(k)}} \iota_{t(g,u)s^k,\infty}(n(g, u) \otimes_s s^k \text{pr}_W((s^{-k}u^{-1}g^{-1})t(g, u)^{-1}s^{-k}n(g, u)^{-1}\mu)) = \sum_{u \in \mathcal{U}_g^{(k)}} \iota_{t(g,u)s^k,\infty}(n(g, u) \otimes_s \text{pr}_W((s^{-k}u^{-1}g^{-1})t(g, u)^{-1}s^{-k}n(g, u)^{-1}\mu))$$

(26)

where $\iota_{t(g,u)s^k,\infty}: \varphi_{t(g,u)s^k} W^\vee \to \lim_{t \to t} \varphi_t W^\vee = \overline{W^\vee}$ is the natural map. By Lemma 7.3 we have

$$s^{-k}u^{-1}g^{-1}n(g, u)^{-1}\mu \in s^{-k+k_0}(G_0 \cap N)s^{k-k_0} \leq U^{(k-k_0)}.$$

As $\pi$ is a smooth representation of $G$ and $W_0$ is finite, there exists an integer $k_2 = k_2(W_0)$ such that for all $k' \geq k_2$ the subgroup $U^{(k')} \subset U^{(k-k_0)}$. Therefore by Lemma 7.3 and (26) we obtain

$$\mathcal{H}_g^{(k)} \circ \overline{\text{pr}_W(g^{-1}\mu)} - \text{res}_{U^{(k'-k_0)}} \circ \overline{\text{pr}_W(\mu)} = \mathcal{H}_g^{(k)} \circ \overline{\text{pr}_W(g^{-1}\mu)} - \sum_{u \in \mathcal{U}_g^{(k)}} n(g, u) \varphi_{t(g,u)s^k} \circ \psi_{s^k} (n(g, u)^{-1}\overline{\text{pr}_W(\mu)}) = \sum_{u \in \mathcal{U}_g^{(k)}} \iota_{t(g,u)s^k,\infty}(n(g, u) \otimes_{t(g,u)s^k} \text{pr}_W((s^{-k}u^{-1}g^{-1})t(g, u)^{-1}s^{-k}n(g, u)^{-1}\mu)) \in \sum_{u \in \mathcal{U}_g^{(k)}} \iota_{t(g,u)s^k,\infty}(\Lambda(N_0) \otimes \Lambda(N_0)^{t(g,u)s^k} (W/W_r)^\vee).$$

Finally, the sets $O(M, l, l') \subset D_{\xi,\ell,\infty}^\vee(\pi)$ in (3) form a system of open neighbourhoods of $0$ in $D_{\xi,\ell,\infty}^\vee(\pi)$. Moreover, for any fixed choice $l, l' \geq 0$ and $M \in \mathcal{M}(\pi^{H_0})$ there exists an integer $k_1 \geq 0$ such that for all $r \geq k_1$ and $u \in \mathcal{U}_g$ we have

$$\text{pr}_{W,F_{\ell'}(g,u)} M((W/W_r)^\vee) \subseteq \Lambda(N_0/H_1) \otimes_{u_0} X^\vee(F_{\ell'}(g,u)M)^\vee[1/X]^{++}$$

(see Lemmata 7.5 and 7.6). Note that the composite map $D_{\xi,\ell,\infty}^\vee(\pi) \xrightarrow{\varphi_{t(g,u)s^k}} D_{\xi,\ell,\infty}^\vee(\pi) \xrightarrow{f_{M,0}} M^\vee[1/X]$ factors through the $\varphi_s$-equivariant map

$$((1 \otimes F_{\ell}(g,u)s^k)^\vee[1/X])^{-1}: (F_{\ell}(g,u)M)^\vee[1/X] \to M^\vee[1/X]$$

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mapping $X^\nu(F_{t'(g,u)}M)^\nu[1/X]^{++}$ into $X^\nu M^\nu[1/X]^{++}$. So we deduce that
\[ \mathcal{H}_g^{(k)} \circ \text{pr} \circ \text{pr}_{SV}(g^{-1}\mu) = \text{res}_{\nu-1} \circ \text{pr} \circ \text{pr}_{SV}(\mu) \]
lies in $O(M, l, l')$ for all $k \geq k_0 + k_2 + n_0 k_1$ and any choice of $J(N_0/s^k N_0 s^{-k})$. The result follows by taking the limit $\mathcal{H}_g = \lim_{k \to \infty} \mathcal{H}_g^{(k)}$.

Now for any fixed $\mu \in \pi^\vee$ consider the the elements $\beta_{gC_0}(\mu) \in \mathfrak{J}(gC_0)$ for $g \in G$. By Proposition 7.7 we also deduce
\[ \res_{gC_0 \cap hC_0}^{gC_0} \circ \beta_{gC_0}(\mu) = \res_{gC_0 \cap hC_0}^{gC_0} (g\beta_{C_0}(g^{-1}\mu)) = g \res_{C_0 \cap hC_0}^{g_0 \cap hC_0} \circ \beta_{C_0}(g^{-1}\mu) = \]
\[ \text{res}_{gC_0 \cap hC_0}^{g^{-1}hC_0} \circ \beta_{g^{-1}hC_0}(g^{-1}\mu) = \text{res}_{gC_0 \cap hC_0}^{g^{-1}hC_0} (g(g^{-1}h)\beta_{C_0}((g^{-1}h)^{-1}g^{-1}\mu)) = \]
\[ = \text{res}_{gC_0 \cap hC_0}^{hC_0} (h\beta_{C_0}(h^{-1}\mu)) = \text{res}_{gC_0 \cap hC_0}^{hC_0} (h\beta_{C_0}(h^{-1}\mu)) \]
for all $g, h \in G$. Since $\mathfrak{J}$ is a sheaf and we have $\bigcup_{g \in G} gC_0 = G/B$, there exists a unique element $\beta_{G/B}(\mu)$ in the global sections $\mathfrak{J}(G/B)$ with $\res_{gC_0}^{G/B}(\beta_{G/B}(\mu)) = \beta_{gC_0}(\mu)$ for all $g \in G_0$. So we obtained a map $\beta_{G/B} : \pi^\vee \to \mathfrak{J}(G/B)$. Our main result is the following

**Theorem 7.8.** The family of morphisms $\beta_{G/B, \pi}$ for smooth, admissible o-torsion representations $\pi$ of $G$ of finite length form a natural transformation between the functors $(\cdot)^\vee$ and $\mathfrak{J}_{\alpha, \cdot}(G/B)$.

**Proof.** At first we need to check that $\beta_{G/B, \pi} : \pi^\vee \to \mathfrak{J}_{\alpha, \pi}(G/B)$ is $G$-equivariant and continuous for all $\pi$. For $g, h \in G$ and $\mu \in \pi^\vee$ we compute
\[ \res_{gC_0}^{G/B} (\beta_{G/B}(h\mu)) = \beta_{gC_0}(h\mu) = g \beta_{C_0}(g^{-1}h\mu) = \]
\[ = h \beta_{h^{-1}gC_0}(\mu) = h \res_{h^{-1}gC_0}^{G/B} \circ \beta_{G/B}(\mu) = \res_{gC_0}^{G/B} (h\beta_{G/B}(\mu)) \]
showing that $\beta_{G/B}(h\mu)$ and $h\beta_{G/B}(\mu)$ are equal locally everywhere, so they are equal globally, too. The continuity follows from the fact that $\beta_{gC_0}$ is continuous for each $g \in G$.

By Thm. 9.24 in [9] the assignment $\pi \mapsto \mathfrak{J}_{\alpha, \pi}$ is functorial. Moreover, by definition we have $\beta_{gC_0, \pi} = (g \cdot) \circ \beta_{C_0, \pi} \circ (g^{-1} \cdot)$ so we are reduced to showing the naturality of $\beta_{C_0, \cdot}$. This follows from the fact that for any morphism $f : \pi \to \pi'$ of smooth, admissible o-torsion representations of $G$ of finite length and $M_k \in \mathcal{M}_k(\pi^H_k)$ for any $k \geq 0$ we have $f(M_k) \in \mathcal{M}_k(\pi'^H_k)$.

**Remark.** Whenever $D^\vee(\pi)$ is nonzero, the map $\beta_{G/B}$ is nonzero either. In particular, if we further assume that $\pi$ is irreducible then $\beta_{G/B}$ is injective.

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