Entanglement of Accelerating Particles

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Abstract

We study how the entanglement of a maximally entangled pair of particles is affected when one or both of the pair are uniformly accelerated, while the detector remains in an inertial frame. We find that the entanglement is unchanged if all degrees of freedom are considered. However, particle pairs are produced, and the entanglements of different bipartite systems may change with the acceleration. In particular, the entanglement between accelerating fermions is transferred preferentially to the produced antiparticles when the acceleration is large, and the entanglement transfer is complete when the acceleration approaches infinity. However, for scalar particles, no entanglement transfer to the antiparticles is observed.

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Entanglement is an important property of quantum mechanical systems. It is useful in the field of quantum information and quantum computing, such as in quantum teleportation [1]. It also finds many applications in quantum control [2] and quantum simulations [3]. Studying quantum entanglement in relativistic systems may give us insights on the relationship between quantum mechanics and general relativity. It has been shown that entanglement is Lorentz invariant [4, 5]. However, an accelerating observer measures less entanglement than an inertial observer in both the scalar [6, 7] and fermion [8] cases. This degradation in entanglement is due to the splitting of space-time, as a result of which the vacuum observed in one frame can become excited in another frame - the case of Unruh effect [9]. Classically, the trajectory of a uniformly accelerating particle observed by an inertial observer is the same as that of an inertial particle measured by a uniformly accelerating observer with appropriate acceleration, and it will be interesting to study how the acceleration of particles affects the entanglement of the originally entangled states. Pair production occurs when a relativistic particle accelerates. We will show that the entanglement remains unchanged if all degrees of freedom are considered. However, the entanglements between different bipartite systems may change with the acceleration. In particular, for fermions the entanglement is preferentially transferred to the antiparticles when the acceleration is large, and the entanglement transfer is complete when the acceleration approaches infinity. However, for scalar particles, no entanglement transfer to the antiparticles is observed. Entanglement transfer to produced particles is a unique quantum mechanical and relativistic phenomenon, and it may play a role in many problems such as the black hole information paradox.

We add an electric field in the Dirac and Klein-Gordon equations that would lead to uniform acceleration in the classical limit. A strong electric field makes the vacuum unstable and leads to pair production [10, 11, 12], which has been studied in the time dependent gauge [13, 14], in Rindler coordinates [15] and in a finite region [16, 17, 18]. We quantize the fields and use the in/out formalism to calculate the pair production, and we calculate the entanglements in different bipartite systems. The results are compared with those observed by an accelerating observer. Entanglements of different degrees of freedom will be shown.

The entanglement of a bipartite system can be quantified by the logarithmic negativity [19, 20, 21]. For a density operator $\rho_{A,B}$ corresponding to a bipartite system $A$ and $B$, we define the trace norm $||\rho_{A,B}|| \equiv tr|\rho_{A,B}| = tr\sqrt{\rho^\dagger_{A,B}\rho_{A,B}}$, and the negativity, $N_e \equiv (||\rho^T_A|| - 1)/2$, where $\rho^T_A$ is the partial transpose of $\rho_{A,B}$ with respect to the party A.
$N_e$ can be calculated from the absolute value of the sum of the negative eigenvalues of $\rho^T_A$. Then the logarithmic negativity of the bipartite system $A$ and $B$ is defined by,

$$LN(\rho_{A,B}) \equiv \log_2||2N_e + 1||.$$  \hfill (1)

For a product state, $LN(\rho_{A,B}) = 0$, and for entangled states, $LN(\rho_{A,B}) > 0$.

The Klein-Gordon equation ($\hbar = c = 1$) for a unit-charged particle with mass $m$ in a uniform electric field $E$ \cite{22, 23} is

$$(D_\mu D^\mu + m^2)\phi = 0,$$ \hfill (2)

where $D_\mu = \partial_\mu + iA_\mu$ and the gauge is chosen to be $A_0 = -Ex$ and $A_x = 0$. By assuming the form of solution as, $\phi_\omega(t,x) = C\exp(i\omega t)\chi_\omega(x)$, where $C$ is a normalization constant, we obtain from Eq. (2)

$$\left[\frac{\partial^2}{\partial x^2} + E^2(x - \omega/E)^2\right]\chi_\omega(x) = m^2\chi_\omega(x).$$ \hfill (3)

The solutions of Eq. (3) can be found in \cite{24}, and they are parabolic cylinder functions, $D_{-\frac{1}{2}}(x)$. We can classify the solutions in the in/out basis \cite{22, 25} and have the in-basis functions,

$$\phi_{\omega,p}^{\text{in}}(x,t) = \frac{e^{-3\pi\mu^2/4}}{(2E)^{1/4}}e^{i\omega t}D_{\mu^2-1/2}[e^{-3i\pi/4}\sqrt{2E}(x - \omega/E)],$$ \hfill (4)

$$\phi_{\omega,a}^{\text{in}}(x,t) = \phi_{-\omega,p}^{\text{in}}(-x,t),$$ \hfill (5)

where $\mu^2 = m^2/2E$. The subscripts $p$ and $a$ stand for particles and antiparticles respectively. We also obtain the out-basis solutions,

$$\phi_{\omega,p}^{\text{out}}(x,t) = \phi_{\omega,p}^{\text{in}}(x,-t),$$ \hfill (6)

$$\phi_{\omega,a}^{\text{out}}(x,t) = \phi_{-\omega,p}^{\text{in}}(-x,-t).$$ \hfill (7)

As there are two different complete bases, we can quantize the field in two ways,

$$\phi = \sum_\omega (a^{\text{in}}_\omega \phi_{\omega,p}^{\text{in}} + b^{\text{in}}_\omega \phi_{\omega,a}^{\text{in}}),$$ \hfill (8)
\[
\phi = \sum_\omega \left( a^\text{out}_\omega \phi^\text{out}_{\omega,p} + b^\text{out}_\omega \phi^\text{out*}_{\omega,a} \right).
\]

(9)

The operators \(a^\text{in}_\omega(b^\text{in}_\omega), a^\text{out}_\omega(b^\text{out}_\omega)\) are the annihilation (creation) operators in the in-basis and out-basis, and they are related by the Bogoliubov transformation,

\[
a^\text{out}_\omega = \alpha^* a^\text{in}_\omega - \beta^* b^\text{in}_\omega^\dagger,
\]

(10)

\[
b^\text{out}_\omega = \alpha^* b^\text{in}_\omega - \beta^* a^\text{in}_\omega^\dagger,
\]

where

\[
\alpha = \frac{\sqrt{2\pi} e^{-\pi/4} e^{-\pi \mu^2/2}}{\Gamma(1/2 + i\mu^2)},
\]

(11)

\[
\beta = e^{i\pi/2} e^{-\pi \mu^2},
\]

which have the relation

\[
|\alpha|^2 - |\beta|^2 = 1.
\]

(12)

We can express the in-vacuum state as a linear combination of out states,

\[
|0\rangle_\text{in} = \prod_\omega \frac{1}{\alpha} \exp \left( \frac{\beta^*}{\alpha} \sum_\omega a^\text{out}_\omega b^\text{out}_\omega^\dagger \right) |0\rangle_\text{out}.
\]

(13)

We let \(\alpha = e^{i\phi_1} \cosh r\) and \(\beta = e^{i\phi_2} \sinh r\), where \(r\) is a parameter related to the acceleration (via Eq. 11), and we ignore the phase factors, which do not affect the following calculations of entanglement. Taking the single-mode approximation, we get the in-vacuum state in terms of the out states,

\[
|0_p\rangle_\text{in} = \frac{1}{\cosh r} \sum_{n=0}^{\infty} \tanh^n r |n_p\rangle_\text{out} |n_a\rangle_\text{out}.
\]

(14)

Similarly the one-particle state is

\[
|1_p\rangle_\text{in} = \frac{1}{\cosh^2 r} \sum_{n=0}^{\infty} \tanh^n r \sqrt{n + 1} |(n + 1)_p\rangle_\text{out} |n_a\rangle_\text{out}.
\]

(15)
For a unit-charged fermions with mass $m$ coupled to an uniform electric field,

$$[\gamma^\mu \pi_\mu - m] \psi = 0, \quad (16)$$

where

$$\pi_\mu \equiv i \partial_\mu - A_\mu. \quad (17)$$

Here, $A_\mu$ is the vector potential, and $\gamma_\mu$ is the gamma matrix. Then we let

$$\psi = (\gamma^\nu \pi_\nu + m) \phi \quad (18)$$

to obtain

$$\left[ \pi^2 - m^2 - \frac{i}{2} \gamma^\mu \gamma^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \right] \phi = 0. \quad (19)$$

We choose the gauge to be $A_0 = 0$, $A_3 = -Et$ and get back a Klein-Gordon equation from Eq. (19). The solution is still the parabolic cylinder function. The in/out basis solution of the second order ODE is still the in/out basis solution of the Dirac equation Eq. (16). Therefore, we obtain the Bogoliubov coefficients, which have been calculated in Ref. [26],

$$a_{n}^{\text{out}} = \alpha_f a_{n}^{\text{in}} - \beta_f^{*} b_{n}^{\text{in}} \dagger, \quad (20)$$

$$b_{n}^{\text{out}} \dagger = \beta_f a_{n}^{\text{in}} + \alpha_f^{*} b_{n}^{\text{in}} \dagger,$$

where

$$\beta_f = e^{-\pi \mu^2}, \quad (21)$$

$$\alpha_f^{*} = -i \sqrt{\frac{2\pi}{\mu^2} e^{-\pi \mu^2/2}} \frac{1}{\Gamma(i\mu^2)},$$

with $\alpha_f$ and $\beta_f$ having the relation,

$$|\alpha_f|^2 + |\beta_f|^2 = 1. \quad (22)$$

We let $\alpha_f = \cos rf e^{i\phi}$ and $\beta_f = \sin rf$, $rf$ being a parameter with values between 0 and $\pi/2$ and related to the acceleration (via Eq. [21]). Also, we can relate the incoming states with the outgoing states as in the case of an accelerating detector [8],
\[ |0_p\rangle_{in} = \cos r_f e^{-i\phi} |0_p\rangle_{out} |0_a\rangle_{out} - \sin r_f |1_p\rangle_{out} |1_a\rangle_{out}, \]
\[ |1_p\rangle_{in} = |1_p\rangle_{out} |0_a\rangle_{out}. \]

Initially, we have the incoming entangled state,
\[ \Psi_i = \frac{1}{\sqrt{2}} [ |0_{s,p}\rangle_{in} |0_{\omega,p}\rangle_{in} + |1_{s,p}\rangle_{in} |1_{\omega,p}\rangle_{in} ]. \]

Then either one or both of the particles in \( \omega \) and \( s \) modes are accelerated by the electric field, and the in states in Eq. (24) are replaced by the out states in Eq. (23). If only the \( \omega \) mode is accelerated, we have
\[ \Psi_f = \frac{1}{\sqrt{2}} \left\{ |0_{s,p}\rangle_{out} \otimes \left[ \cos r_f e^{-i\phi} |0_{\omega,p}\rangle_{out} |0_{\omega,a}\rangle_{out} - \sin r_f |1_{\omega,p}\rangle_{out} |1_{\omega,a}\rangle_{out} \right] + |1_{s,p}\rangle_{out} \otimes \left( |1_{\omega,p}\rangle_{out} |0_{\omega,a}\rangle_{out} \right) \right\}. \]

If both the \( s \) and \( \omega \) modes are accelerated with the same \( r_f \), we have
\[ \Psi_f = \frac{1}{\sqrt{2}} \left\{ \left[ \cos r_f e^{-i\phi} |0_{s,p}\rangle_{out} |0_{s,a}\rangle_{out} - \sin r_f |1_{s,p}\rangle_{out} |1_{s,a}\rangle_{out} \right] \otimes \left[ \cos r_f e^{-i\phi} |0_{\omega,p}\rangle_{out} |0_{\omega,a}\rangle_{out} - \sin r_f |1_{\omega,p}\rangle_{out} |1_{\omega,a}\rangle_{out} \right] \right\}. \]

The degradation of entanglement in the case of an accelerating detector is due to the fact that some degrees of freedom have been traced out. An accelerating detector 'sees' the space-time being split into two causally disconnected regions, and it cannot access information in one of them. We have verified explicitly that the entanglement between the particles in \( s \) mode and \( \omega \) mode is unchanged if there is no tracing out of any space-time region. On the other hand, in the case of accelerating particles, the detector, which is in an inertial frame, can access all degrees of freedom and the orthogonality of the states is unchanged; therefore, the entanglement of accelerating particles is unchanged.

However, more degrees of freedom are produced and we can calculate the entanglements between different bipartite systems. In Ref. [4], it was shown that entanglement is Lorentz
invariant. If one traces out the momentum, the entanglement decreases, and the entanglement is transferred between the momentum and the spin degrees of freedom. We will show that entanglement transfer also occurs in accelerating fermions. Detailed calculations for both non-relativistic and relativistic particles are shown in [27].

If only the particle in the $\omega$ mode is accelerated, we can study the three bipartite systems: $A =$ the $s$ mode, $B =$ the particles in the $w$ mode, the antiparticles in $w$ mode, or the entire $w$ mode including both the particles and antiparticles. The density matrices are called $\rho_{s,p}$, $\rho_{s,a}$, and $\rho_{s,(p,a)}$ respectively. The entanglements are

$$
\begin{align*}
LN(\rho_{s,(p,a)}) &= 1, \\
LN(\rho_{s,p}) &= \log_2(1 + \cos^2 r_f), \\
LN(\rho_{s,a}) &= \log_2(1 + \sin^2 r_f),
\end{align*}
$$

which are plotted in Fig. 1. It is obvious that the entanglement of $\rho_{s,p}$ is transferred to $\rho_{s,a}$.

When both the $s$ and $\omega$ modes are accelerated with the same $r_f$, we can calculate the entanglements between the five bipartite systems: particles in $s$ mode and particles in $\omega$ mode ($\rho_{p,p}$), antiparticles in $s$ and antiparticles in $\omega$ ($\rho_{a,a}$), antiparticles in $s$ and particles in $\omega$ ($\rho_{p,a}$), particles in $s$ and antiparticles in $\omega$ ($\rho_{p,a}$), and the entire $s$ and $\omega$ modes ($\rho_{(p,a),(p,a)}$). The logarithmic negativities are

$$
\begin{align*}
LN(\rho_{(p,a),(p,a)}) &= 1, \\
LN(\rho_{p,p}) &= \log_2 \left[ 1 + \cos^4 r_f \right], \\
LN(\rho_{a,a}) &= \log_2 \left[ 1 + \sin^4 r_f \right], \\
LN(\rho_{p,a}) &= \log_2 \left[ 1 + \cos^2 r_f \sin^2 r_f \right].
\end{align*}
$$

By symmetry, $LN(\rho_{a,p}) = LN(\rho_{p,a})$. The results are shown in Fig. 1. The entanglement is transferred from $\rho_{p,p}$ not only to $\rho_{p,a}$, but also to $\rho_{a,a}$. In fact, when the acceleration of the particles tends to infinity, the entanglement is completely transferred to between the antiparticles $\rho_{a,a}$.

Similarly, for scalar particles, we calculate the entanglements of $\rho_{s,p}$, $\rho_{s,a}$, $\rho_{p,p}$, $\rho_{a,a}$ and $\rho_{p,a}$. The results are shown in Fig. 2. Again, the entanglement between the entire $s$ and $\omega$ modes remains unchanged, such that $LN(\rho_{s,\omega}) = LN(\rho_{(p,a),(p,a)}) = 1$ for all $r$. In contrast to fermions, there is no entanglement transfer to the antiparticles, and $LN(\rho_{s,a}) = LN(\rho_{p,a}) = LN(\rho_{a,a}) = 0$ for all $r$, even though the entanglement between the particles in the $s$ and $\omega$ modes decreases as $r$ increases.
FIG. 1: Logarithmic negativities of several bipartite systems when one or both fermions are accelerated, the magnitude of which is parameterized by $r_f$. In both cases, the entanglement between the entire $s$ mode and $\omega$ mode is unchanged (dot-dashed line). The solid lines show the results when both particles are accelerated together, for three bipartite systems: particles in $s$ mode and particles in $\omega$ mode ($\rho_{p,p}$), particles in $s$ mode and antiparticles in $\omega$ mode ($\rho_{p,a}$), and antiparticles in $s$ mode and antiparticles in $\omega$ mode ($\rho_{a,a}$). For comparison, the dashed lines show the results when only the particle in the $\omega$ mode is accelerated, in which case the two bipartite systems are particle in $s$ mode and particles in $\omega$ ($\rho_{s,p}$), and particle in $s$ mode and antiparticles in $\omega$ ($\rho_{s,a}$).

FIG. 2: Same as Fig. 1, but for scalar particles. Again, the entanglement of the entire $s$ mode and $\omega$ mode (both particles and antiparticles), indicated by the dot-dashed line, remains unchanged. Note that $LN(\rho_{s,a}) = LN(\rho_{p,a}) = LN(\rho_{a,a}) = 0$ for all $r$.

We have studied how the entanglement of a pair of maximally entangled particles is affected when one or both of the pair is uniformly accelerated, as measured by an inertial detector, and compared it with that of inertial particles observed by a uniformly accelerating detector. While there is a degradation of entanglement in the latter case due to the splitting
of the space-time, the entanglement in the former case is unchanged by the acceleration when all degrees of freedom are considered. However, particle pairs are produced, and the entanglements of different bipartite systems may change as the acceleration. In particular, for fermions, the entanglement is preferentially transferred to the produced antiparticles and when the acceleration approaches infinity, the entanglement is completely transferred to the antiparticles. However, for scalar particles, no entanglement transfer to the antiparticles is observed.

Our results raise the possibility that when an entangled pair falls into a black hole, their entanglement may be partially transferred to the produced particles, which should not be ignored in considering the black hole information paradox. Studying quantum entanglement in curved space-time may therefore give us insights on the relation between quantum mechanics and general relativity.

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