STRONGLY CONTINUOUS SEMIGROUPS ON SOME FRÉCHET SPACES

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ABSTRACT. We prove that for a strongly continuous semigroup $T$ on the Fréchet space $\omega$ of all scalar sequences, its generator is a continuous linear operator $A \in L(\omega)$ and that, for all $x \in \omega$ and $t \geq 0$, the series $\exp(tA)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x)$ converges to $T_t(x)$. This solves a problem posed by Conejero.

Moreover, we improve recent results of Albanese, Bonet, and Ricker about semigroups on strict projective limits of Banach spaces.

1. Introduction

In [Con07] Conejero asked whether on $\omega = \ell^N$ every strongly continuous semigroup $T$ is of the form $T_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x)$ for a continuous linear operator $A$ on $\omega$. This question arose in the context of hypercyclicity: It is shown in [Con07, Theorem 2.7] that no such semigroup on $\omega$ can be hypercyclic. Although Shkarin [Shk11] proved that there are not even supercyclic strongly continuous semigroups on $\omega$, the question of Conejero remained open. Only a partial answer is contained in [ABR10].

The definitions and most basic results for semigroups on locally convex spaces $X$ are the same as for Banach spaces, we refer to [Kom64, Kom68, Ouc73, Yos65]. A strongly continuous semigroup $T$ on $X$ is thus a morphism from the semigroup $([0, \infty), +)$ to that of continuous linear operators $(L(X), o)$ such that all orbits $t \mapsto T_t(x)$ are continuous. If the convergence $T_t \rightarrow id_X$ for $t \rightarrow 0$ is uniform on bounded subsets of $X$, the semigroup is called uniformly continuous.

$T$ is called locally equicontinuous if $\{T(s) : 0 \leq s \leq t\}$ is equicontinuous for every $t > 0$, i.e., for every continuous seminorm $p$ on $X$ there is another continuous seminorm $q$ on $X$ such that $p(T_s(x)) \leq q(x)$ for all $x \in X$ and $0 \leq s \leq t$. On barrelled spaces, every strongly continuous semigroup is already locally equicontinuous because of the Banach-Steinhaus theorem. The (infinitesimal) generator $(A, D(A))$ of a strongly continuous semigroup on a locally convex space is defined as in the Banach space setting as the derivative of the orbit at 0.

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Although under rather weak assumptions (sequential completeness to have a vector valued integral and barrelledness to apply the uniform boundedness principle) many results from the Banach space setting carry over to strongly continuous semigroups on locally convex spaces there are some crucial differences because the exponential series \( \exp(A)(x) = \sum_{k=0}^{\infty} \frac{A^k(x)}{k!} \) need not converge for continuous linear operators. Therefore, one does not always have the familiar representation \( T_t = \exp(tA) \) for semigroups with continuous generators:

**Example 1.** Consider \( C^\infty(\mathbb{R}) \) with its usual topology and the strongly continuous semigroup defined by \( T_t(f)(x) = f(x+t) \). Then \( D(A) = C^\infty(\mathbb{R}) \) and \( A(f) = f' \). For any \( f \) which is flat at the origin but does not vanish on \((0, \infty)\) the series \( \exp(tA)(f) = \sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(k)}(0) \) cannot converge to \( T_t(f)(0) = 0 \). Using E. Borel’s theorem, that a smooth function can have any given sequence of derivatives at 0, one also gets \( f \in C^\infty(\mathbb{R}) \) such that the exponential series diverges.

Answering Conjero’s question, we will prove that such phenomena do not occur on the Fréchet space \( \omega \).

### 2. Semigroups on strict projective limits

Let \( X_n \) be Banach spaces with norms \( \| \cdot \|_n \), \( \pi^m_n : X_m \to X_n \) norm decreasing operators for \( n \leq m \in \mathbb{N} \) with \( \pi^m_n \circ \pi^n_m = \pi^n_k \) as well as \( \pi^n_n = id_{X_n} \), and

\[
X = \{(x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : \pi^m_n(x_m) = x_n \text{ for all } n \leq m \}
\]

its projective limit. Every Fréchet space has such a representation, and \( X \) is called a *quojection* if there is a representation with surjective (hence open) \( \pi^m_n \). Countable products of Banach spaces are of this form, in particular spaces like \( L^p_{loc}(\Omega) \) of locally \( p \)-integrable functions or \( C^n(\Omega) \) for open sets \( \Omega \subseteq \mathbb{R}^d \) and \( n \in \mathbb{N}_0 \), but there are quojections which are not isomorphic to a product.

If \( X \) has a strict representation as above then, by a simple induction, \( \pi_m : X \to X_n, (x_n)_{n \in \mathbb{N}} \mapsto x_m \) are also surjective. Applying this observation to the spaces \( \ell^\infty(I)(X_n) \) of bounded functions \( I \to X_n \) one obtains that \( \pi_m \) lifts bounded sets, that is, there are bounded sets \( D_n \subseteq X \) such that \( \pi_n(D_n) \) contains the unit ball \( B_n \) of the Banach space \( X_n \). This lifting property was first proved in [DZ84].

The following theorem improves results of Albanese, Bonet, and Ricker [ABR10] who showed its first part under restrictive additional assumptions. Moreover, their proposition 3.2 shows that the strong assumption on the space \( X \) in our theorem cannot be relaxed too much: On every nuclear Köthe space (and even every locally convex space containing a complemented copy of such a space) there are equicontinuous, uniformly continuous semigroups whose generator is not everywhere defined and discontinuous.
Theorem 2. Let $X$ be a quojection.

(1) If $T$ is a uniformly continuous semigroup on $X$ then its generator is continuous and everywhere defined, and for all $x \in X$ and $t \geq 0$ we have

$$T_t(x) = \exp(tA)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x).$$

(2) $A \in L(X)$ generates a strongly continuous semigroup if and only if

$$\forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall k \in \mathbb{N}_0 : \pi_m(x) = 0 \implies \pi_n(A^k(x)) = 0.$$

Then the generated semigroup is even uniformly continuous.

Proof. As $X$ is barrelled the semigroup is locally equicontinuous, so that, for every $t_0 > 0$ and $n \in \mathbb{N}$, there are $m \geq n$ and $c > 0$ such that for all $x \in X$ and $t \in [0, t_0]$,

$$\|\pi_m(T_t(x))\|_n \leq c\|\pi_m(x)\|_m.$$

In particular, we have

$$(*) \quad \pi_m(x) = 0 \implies \pi_n(T_t(x)) = 0 \text{ for all } t \leq t_0.$$

As in the case of semigroups on Banach spaces it is easily seen that the Cesaro means

$$C_t(x) = \frac{1}{t} \int_0^t T_s(y)ds$$

belong to $D(A)$, $A(C_t(y)) = \frac{1}{t}(T_t(y) - y)$, and $C_t(y) \to y$ holds uniformly on bounded sets since $T$ is uniformly continuous. Moreover, $C_t$ satisfy the same continuity estimates as $T_t$. Therefore, the following operators are well-defined and continuous for $t \in (0, t_0]$:

$$\tilde{C}_t : X_m \to X_n \pi_m(y) \mapsto \pi_n(C_t(y)).$$

Since $\pi_m$ lifts bounded sets, every $z$ in the unit ball $B_m$ of $X_m$ can be represented as $z = \pi_m(y)$ with $y \in D_m$ and, as $C_t \to \text{id}$ uniformly on $B_m$, we obtain that $\tilde{C}_t$ converges uniformly on $B_m$ to $\pi_m$. Since the set of surjective operators is open in $L(X_m, X_n)$, see e.g. [Lan93, chapter XV, theorem 3.4], we conclude that $\tilde{C}_t$ is surjective for some sufficiently small $t > 0$.

If now $n_0 \in \mathbb{N}$ is given we take $n \geq n_0$ with $\pi_n(z) = 0 \Rightarrow \pi_{n_0}(T_t(z)) = 0$ and then $m \geq n$ again with $(*)$ for $t_0 = 1$, say. Given $x \in X$ we choose $y \in X$ with $\pi_n(x) = \tilde{C}_t(\pi_n(y)) = \pi_n(C_t(y))$ so that $\pi_{n_0}(T_h(x)) = \pi_{n_0}(T_h(C_t(y)))$ for small $h$.

Therefore,

$$\pi_{n_0} \left( \frac{1}{h}(T_h(x) - x) \right) = \pi_{n_0} \left( \frac{1}{h}(T_h(C_t(y)) - C_t(y)) \right)$$

converges to $\pi_{n_0}(\frac{1}{h}(T_t(y) - y))$.

This shows that the difference quotients satisfy the Cauchy condition so that the completeness of $X$ implies that $A(x)$ is defined for every $x \in X$. Moreover, $A$ is continuous either because of the closed graph theorem or because of $\pi_{n_0}(A(x)) = \pi_{n_0}(\frac{1}{h}(T_t(y) - y))$ together with the fact that $y$ can be chosen with $\|\pi_m(y)\| \leq c\|\pi_m(x)\|_n$ using that $\tilde{C}_t$ is open.

Since $A^k(x)$ is the $k$th derivative of the orbit $t \mapsto T_t(x)$ at $0$ we obtain from $(*)$ that $A$ satisfies the condition in the second part of the theorem (this argument uses only local equicontinuity and is thus true for strongly continuous semigroups).
We will now show that under the condition of (2) the exponential series
\[
\exp(tA)(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k(x)
\]
converges absolutely and uniformly on bounded sets. Since then \(\exp(tA)\) is a uniformly continuous semigroup with generator \(A\) we have proved both parts of the theorem because a locally equicontinuous semigroup is uniquely determined by its generator.

We define linear maps \(\tilde{A}_0 = \pi_m^n\) and
\[
\tilde{A}_k : X_m \to X_n, \quad \pi_m(x) \mapsto \pi_n(A^k(x))
\]
so that \(\tilde{A}_k \circ \pi_m = \pi_n \circ A^k\) and \(\tilde{A}_{k+1} \circ \pi_m = \tilde{A}_k \circ \pi_m \circ A\).

As above we take a bounded set \(D_m \subseteq X\) with \(B_m \subseteq \pi_m(D_m)\). The continuity of \(A\) then implies \(\pi_m \circ A(D_m) \subseteq \lambda B_m\) for some \(\lambda > 0\). We now claim that
\[
\|\tilde{A}_k\|_{L(X_m, X_n)} \leq \lambda^k
\]
for all \(k \in \mathbb{N}_0\). For \(k = 0\) this holds because \(\tilde{A}_0 = \pi_m^n\) is norm decreasing. If the claim is true for some \(k \in \mathbb{N}_0\) and \(y \in B_m\) is given we choose \(x \in D_m\) with \(\pi_m(x) = y\) and obtain
\[
\|\tilde{A}_{k+1}(y)n = \|\tilde{A}_{k+1} \circ \pi_m(x)n = \|\tilde{A}_k(\pi_m(A(x)))n \leq \lambda^k\|\pi_m(A(x))\| \leq \lambda^{k+1}.
\]
Finally, we obtain \(\|\pi_n(A^k(x))\|_n = \|\tilde{A}_k(\pi_m(x))\|_n \leq \lambda^k\|\pi_m(x)\|_m\) which clearly implies the absolute convergence of the exponential series.

The very strong form of the convergence of the exponential series shown at the end of the proof need not hold in arbitrary Fréchet spaces. The shift \(T_t(f)(z) = f(z + t)\) on the space of entire functions is generated by \(A(f) = f'\) and is of the form \(\exp(tA)\) because of the Taylor representation of entire functions. However, one cannot estimate \(\|A^k(f)\|_n \leq \lambda^k\|f\|_m\).

Let us remark that part of the implication in (2), namely that a strongly continuous semigroup whose generator is everywhere defined is uniformly continuous, holds for all barrelled spaces. This follows easily from the Banach-Steinhaus theorem for \(\{(T_h - \text{id})/h : h \in [0, 1]\}\). Moreover, the equicontinuity of this family characterizes semigroups with a continuous generator.

The countability of the projective spectrum was only used to obtain barrelledness and that \(\pi_m : X \to X_m\) lifts bounded sets. The theorem is thus true for non-countable strict projective limits of Banach spaces which are barrelled and satisfy this lifting property. In particular, it holds for arbitrary products of Banach spaces. Moreover, as in [ABR10 proposition 3.4] it is easy to extend the results to so-called prequotions \(X\) where only the bidual \(X''\) has a strict representation.

Since strongly continuous, locally equicontinuous semigroups on Montel spaces are easily seen to be uniformly continuous we immediately obtain the answer to Conejero’s question:

**Corollary 3.** Every strongly continuous semigroup on an arbitrary product \(\mathbb{K}^I\) has a continuous generator \(A\) and is of the form \(\exp(tA)\).
Example 4. Consider the backward shift $A : \omega \to \omega$, $x \mapsto (x_2, x_3, \ldots)$. Theorem 2 implies that $A$ does not generate a strongly continuous semigroup on $\omega$. On the other hand, the operator $B(x) = (x_2, 0, x_4, 0, x_6, 0, \ldots)$ satisfies $B^2 = 0$ as well as $(A - B)^2 = 0$ so that both operators do generate strongly continuous semigroups on $\omega$, given by $T(t) = \text{id} + tB$ and $S(t) = \text{id} + t(A - B)$, respectively. It is thus not only the “shape” of the matrix determined by the operator which decides whether it generates a semigroup.

Let us finally remark that the second part of theorem 2 implies that, if a continuous operator $A$ on a quocjection generates a uniformly continuous semigroup, then the same is true for $A^2$. This does not hold for the Fréchet space $\mathcal{C}^\infty(\mathbb{R})$: $A(f) = f'$ generates the shift semigroup but $B(f) = A^2(f) = f''$ does not generate a strongly continuous semigroup on $\mathcal{C}^\infty(\mathbb{R})$.

The Gauß-Weierstraß semigroup $\bar{T}$ on $\mathcal{S}(\mathbb{R})$ given by the convolution with the Gauß kernel $k(t, x) = (\pi t)^{-1/2} \exp(-x^2/t)$ is uniformly continuous (which one can check by Fourier transformation) and has the generator $\bar{B} = B|_{\mathcal{S}(\mathbb{R})}$. If $B$ would generate a semigroup $T$ on $\mathcal{C}^\infty(\mathbb{R})$ we obtain, for fixed $f \in \mathcal{S}(\mathbb{R})$ and $t > 0$, that $\varphi(s) = T_s \circ \bar{T}_{t-s}(f)$ has vanishing derivative so that $T_t(f) = \varphi(t) = \varphi(0) = \bar{T}_t(f)$. But this means that the convolution with the Gauß kernel can be continuously extended from $\mathcal{S}(\mathbb{R})$ to $\mathcal{C}^\infty(\mathbb{R})$ which is not true.

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