DUAL CONSTRUCTIONS FOR PARTIAL ACTIONS OF HOPF ALGEBRAS

ELIEZER BATISTA AND JOOST VERCRUYSSE

Abstract. The duality between partial actions and co-actions of a Hopf algebra are fully explored in this work. The good properties of Hopf algebras with respect to duality are enlightened, giving rise to new constructions, like partial $H$ module coalgebras and partial $H$ comodule coalgebras. The inter relation between partial coactions of commutative Hopf algebras and Hopf algebroids is analysed. The construction of partial co-smash coproducts, which are coalgebras, points out to a deeper inter relation between partial actions and partial coactions.

Contents

1. Introduction 1
2. Preliminaries: Partial actions and partial representations 3
   2.1. Algebraic structures 3
   2.2. Partial actions 5
   2.3. Partial representations 6
3. Partial actions of co-commutative Hopf algebras 7
4. Partial comodule algebras 11
   4.1. Symmetric partial comodule algebras 11
   4.2. Partial coactions of commutative Hopf algebras 14
5. Partial module coalgebras 27
   5.1. Definition and examples 27
   5.2. Partial module coalgebras as coalgebra objects 31
   5.3. The $C$-ring associated to a module coalgebra 32
   5.4. Dualities 35
6. Partial comodule coalgebras and partial co-smash coproducts 36
   6.1. Definition and examples 36
   6.2. The partial cosmash coproduct 37
   6.3. Dualities 38
7. Conclusions and outlook 39
Acknowledgments 40
References 40

1. Introduction

The theory of Hopf algebras traditionally has been strongly inspired by group theory. This is due to two dual constructions leading to mayor classes of examples of Hopf algebras and their (co)modules upon which they (co)act. First of all, given any group $G$ and a commutative base ring $k$, the associated group algebra $H = kG$ is a Hopf $k$-algebra. Actions of the group $G$ are in correspondence with module coalgebras over $H$ and representations of $G$ are exactly modules over $H$. In a dual way, supposing now that $G$ is moreover finite, the algebra of $k$-valued functions $\mathbb{1}$
on $G$ is a again a Hopf algebra, say $K$. Actions of $G$ now correspond to comodule algebras over $K$ and representations of $G$ are precisely the $K$ comodules. If $G$ is more generally an algebraic group, one could take $K$ to be the algebra of rational functions on $G$ and obtain the same results. The Hopf algebras $H$ and $K$, as well as their respective actions and coactions, are related by dual pairings.

In recent years, a rich theory on partial group actions has been developed \[8, 7\]. In particular, the Galois theory for partial group actions \[10\] motivated the introduction of partial actions for Hopf algebras \[5\], which on its turn triggered a new branch of research in Hopf algebra theory. Partial actions and coactions of Hopf algebras were verified to have nice properties with relation to globalization, that is, every partial action (resp. coaction) of a Hopf algebra $H$ on an algebra $A$ is coming from a restriction of an action (resp. coaction) of $H$ on a bigger algebra $B$ such that $A$ is isomorphic to a unital ideal of $B$ \[1, 2\]. In the case of partial actions of groups on algebras the existence of globalization is restricted to partial actions in which the ideals defined by the partial action are unital ideals \[8\]. this can be better understood by the fact that partial actions of the group algebra $kG$ are in correspondence with partial actions of the group $G$ such that the ideals defined by the partial action are unital \[5\].

As in the global case, the category of partial modules over a Hopf algebra forms a monoidal category, in which the algebra objects are exactly the partial actions \[3\]. The notion of a partial module of a Hopf algebra $H$ is closely related to that of a partial representation of $H$, and in case where $H$ equals the group algebra $kG$, these coincide with the partial representations of the group $G$.

The first aim of this paper is to complete the picture of partial actions of Hopf algebras, with respect to the known existing dualities for usual actions. In geometric terms, partial group actions describe no global symmetries of a space, but only “local” symmetries of certain subspaces. We make this viewpoint more explicit in Section \[4\] where we introduce the notion of a partial action of an algebraic group on an affine space. As it is the case for usual actions, such a partial action allows in a natural way to construct examples of partial comodule algebras. As partial actions of groups can be viewed as a theory of partially defined symmetries in classical geometry, in the same way, partial actions fo Hopf algebras can be viewed as partial symmetries in noncommutative geometry.

This observation is in fact the starting point and motivation to develop a complete theory of dual constructions for partial actions of Hopf algebras, together with their mutual dualities. Moreover, we show that the internal algebraic structures associated to partial (co)actions are sometimes richer than first expected.

In particular, given a partial action of a Hopf algebra $H$ on an algebra $A$, one can construct the partial smash product $A\#H$, which is an $A$-ring. In case where $H$ is cocommutative and $A$ is commutative, we show in Section \[5\] that $A\#H$ has even the structure of a $A$-Hopf algebroid.

Second, given a commutative Hopf algebra $H$ and right partial coaction on a commutative algebra $A$, one can construct a commutative $A$-Hopf algebroid out of these data, called the partial split Hopf algebroid, denoted by $A\otimes H$. The interconnection between partial coactions of commutative Hopf algebras on commutative algebras and Hopf algebroids also works in the opposite direction. More precisely, given a commutative $A$ Hopf algebroid $\mathcal{H}$ and an algebra morphism $F : H \to \mathcal{H}$ such that $\mathcal{H}$ can be totally defined from the image of $F$ and the image of the source map $s : A \to \mathcal{H}$, then there is a natural partial coaction of $H$ on the base algebra $A$ constructed from these data. This is a dual version of a theorem due to Kellendonk and Lawson \[11\], which gives an equivalence, in the categorical sense, between partial actions of a group $G$ and star injective functors between groupoids $\mathcal{G}$ and the group $G$.

These previous examples of Hopf algebroids, the partial smash product and the partial split Hopf algebroid are in duality. More precisely, consider a co-commutative Hopf algebra $H$, and a commutative Hopf algebra $K$ such that there exists a pairing between them. Let $A$ be a commutative algebra which is a right partial $K$ comodule algebra. Then $A$ is also a left partial $H$ module algebra and there exists a skew pairing, in the sense of Schauenburg \[12\], between $A\#H$ and $A\otimes H$, considering them as $\times_A$ bialgebras. All this is done in Section \[4\].
In Section 3 we study partial $H$ module coalgebras. These are related to coalgebra objects in the category of partial $H$ modules, but unlike the case of algebra objects the correspondence is not one to one. It can be proved that for every partial $H$ module coalgebra, we can associate a coalgebra object in the category of partial $H$ modules, but the converse is not always possible. For the Hopf algebra being the group algebra $kG$, for some group $G$, then the partial $kG$ module coalgebras correspond to partial actions of the group $G$ on coalgebras. We show that if $A$ is a left $H$ module coalgebra over a bialgebra $H$ give rise to a $C$ ring structure over the subspace $H \otimes C \subseteq H \otimes C$.

Partial $H$ comodule coalgebras, on its turn, were first defined in [14]. In Section 3 we use this notion to construct partial comodule coproducts, which are naturally coalgebras over the base field. Given two Hopf algebras $H$ and $K$ such that there is a dual pair between them, there is again a duality between partial $H$ module algebras and partial $K$ comodule algebras. Moreover, if $A$ is a left partial $H$ module algebra and $C$ is a left partial $K$ comodule algebra such that there is a dual pairing between them, then there exists a dual pairing between the partial smash product $A \# H$ and the partial co smash coproduct $C \rhd K$.

Finally, in Section 4 the global picture of all dualities involved is pointed out and some further developments of the theory are outlined.

2. Preliminaries: Partial actions and partial representations

2.1. Algebraic structures.

Hopf algebroids. Throughout this note, $k$ denotes a field, however, often a commutative ring is sufficient. An algebra, coalgebra or Hopf algebra means the respective structure with respect to the base $k$, unadorned tensor products are tensor products over $k$.

Let $A$ be a $k$-algebra. An $A$-coring is a coalgebra object in the category of $A$-bimodules, i.e. it is a triple $(C, \Delta, \epsilon)$ where $C$ is an $A$-bimodule, $\Delta : C \rightarrow C \otimes A C$ and $\epsilon : C \rightarrow A$ are $A$-bimodule maps satisfying the usual coassociativity and counit conditions.

Given a $k$ algebra $A$, a left (resp. right) bialgebroid over $A$ is given by the data $(H, s, t, \Delta, \psi)$ (resp. $(H, A, s, t, \Delta, \psi)$) such that:

1. $H$ is a $k$ algebra.
2. The map $s_i$ (resp. $s_r$) is a morphism of algebras between $A$ and $H$, and the map $t_i$ (resp. $t_r$) is an anti-morphism of algebras between $A$ and $H$. Their images commute, that is, for every $a, b \in A$ we have $s_i(a)t_i(b) = t_i(b)s_i(a)$ (resp. $s_r(a)t_r(b) = t_r(b)s_r(a)$).
3. The triple $(H, \Delta, \psi)$ (resp. $(H, \Delta, \psi)$) is an $A$ coring relative to the structure of $A$ bimodule defined by $s$ and $t$ (resp. $s_r$ and $t_r$).
4. The image of $\Delta$ (resp. $\Delta$) lies on the Takeuchi subalgebra $H \times_A H = \{ \sum_i h_i \otimes k_i \in H \otimes_A H : \sum_i h_i t_i(a) \otimes k_i = \sum_i h_i \otimes k_i s(a) \ \forall a \in A \}$, resp. $H \times A H = \{ \sum_i h_i \otimes k_i \in H \otimes_A H : \sum_i s_r(a) h_i \otimes k_i = \sum_i h_i \otimes t_r(a) k_i \ \forall a \in A \}$, and it is an algebra morphism.
5. For every $h, k \in H$, we have $\xi_t(hk) = \xi_t(hs_t(\xi_t(k))) = \xi_t(ht_t(\xi_t(k)))$, $\xi_r(hk) = \xi_r(s_r(\xi_r(h))k) = \xi_r(t_r(\xi_r(h)))k$.

Given two anti-isomorphic algebras $A$ and $\tilde{A}$ (ie, $A \cong \tilde{A}^{\text{op}}$) and an algebra $H$ that is endowed with at the same time a left $A$ bialgebroid structure $(H, A, s, t, \Delta, \psi)$ and a right $\tilde{A}$ bialgebroid
structure $(\mathcal{H}, \tilde{A}, s_r, t_r, \Delta, \Delta_r)$, we say that $\mathcal{H}$ is a Hopf algebroid if it is equipped with an antipode, that is, an anti algebra homomorphism $S : \mathcal{H} \to \mathcal{H}$ such that

(i) $s_1 \circ \Delta \circ t_r = t_r$, $t_1 \circ \Delta \circ s_r = s_r$, $s_r \circ \Delta \circ t_l = t_l$ and $t_l \circ \Delta \circ s_1 = s_l$.

(ii) $(\Delta_l \otimes \tilde{A}) \circ \Delta_r = (I \otimes A \Delta_r) \circ \Delta_l$ and $(I \otimes \tilde{A}) \circ \Delta_r = (\Delta_r \otimes A) \circ \Delta_l$.

(iii) $S(t_l(a) h_S(b)) = s_r(b') S(h) s_l(a)$, for all $a \in A$, $b' \in \tilde{A}$ and $h \in \mathcal{H}$.

(iv) $\mu_H \circ (S \otimes I) \circ \Delta_r = s_r \circ \Delta_l$ and $\mu_H \circ (I \otimes S) \circ \Delta_r = s_l \circ \Delta_l$.

An example of a Hopf algebroid is the so-called split Hopf algebroid: Let $H$ be a cocommutative Hopf algebra and $A$ a cocommutative $H$-comodule algebra. Then $A \otimes H$ is an $A$-Hopf algebroid with structure maps $s_r(a) = s_1(a) = s(a) = a \otimes 1_H$, $t_r(a) = t_1(a) = a_{[0]} \otimes a_{[1]}$, $\Delta_r(a \otimes h) = \Delta_l(a \otimes h) = a \otimes h_{(1)} \otimes 1 \otimes h_{(2)}$, $\epsilon_r(a \otimes h) = \epsilon_l(a \otimes h) = a \epsilon_H(h)$, $S(a \otimes h) = a \otimes S_H(h)$.

**Dual pairings.** A dual pairing between a $k$-algebra $A$ and a $k$-coalgebra $C$ is a $k$-linear map

$$\langle -, - \rangle : A \otimes C \to k$$

satisfying the following conditions

$$\langle ab, c \rangle = \langle a, c_{(1)} \rangle \langle b, c_{(2)} \rangle, \quad \langle 1_A, c \rangle = \epsilon_C(c),$$

for all $a \in A$, $c \in C$. A dual pairing induces two $k$-linear morphisms

$$\phi : A \to C^*, \quad \phi(a)(c) = \langle a, c \rangle$$

$$\psi : C \to A^*, \quad \phi(c)(a) = \langle a, c \rangle$$

for all $a \in A$, $c \in C$. Clearly, a linear map $\langle -, - \rangle$ is a dual pairing if and only if the induced map $\phi$ is an algebra morphism. Hence, a dual pairing induces a functor

$$\Phi : \mathcal{M}^C \to A \mathcal{M}$$

that sends a $C$-comodule $M$ to an $A$-module with the same underlying $k$-module $M$ and $A$-action given by

$$a \cdot m = m^{(0)} \langle a, m^{(1)} \rangle.$$
2.2. Partial actions. The following definition, of partial actions of Hopf algebras, first appeared in [5] and was motivated by examples of partial actions of groups on algebras.

**Definition 2.1.** A left partial action of a Hopf algebra $H$ over a unital algebra $A$ is a linear map

$$\cdot_1 : \quad H \otimes A \rightarrow A$$

$$h \otimes a \mapsto h \cdot_1 a$$

such that

(PLA1) For every $a \in A$, $1_h \cdot_1 a = a$.

(PLA2) For every $h \in H$ and $a, b \in A$, $h \cdot_1 (ab) = (h^{(1)} \cdot_1 a)(h^{(2)} \cdot_1 b)$.

(PLA3) For every $h, k \in H$ and $a \in A$, $h \cdot_1 (k \cdot_1 a) = (h^{(1)} \cdot_1 1_A)(h^{(2)} k \cdot_1 a)$

The partial action is **symmetric** if in addition

(PLA3') For every $h, k \in H$ and $a \in A$, $h \cdot_1 (k \cdot_1 a) = (h^{(1)} k \cdot_1 a)(h^{(2)} \cdot_1 1_A)$

The algebra $A$ is said to be a **partial left $H$ module algebra**. Given two partial left $H$ module algebras $A$ and $B$, a morphism of partial actions is an algebra morphism $f : A \rightarrow B$ such that, for every $h \in H$ and $a \in A$ we have $h \cdot_1 B f(a) = f(h \cdot_1_A a)$. The category of left partial actions of $H$ will be denoted by $\mathbb{H} \text{ParAct}$

**Remark 2.2.** Throughout this paper, all partial actions will be considered to be symmetric. It is easy to see that $H$ module algebras are partial $H$ module algebras, in fact, one can prove that a partial action is global if, and only if, for every $h \in H$ we have $h \cdot_1 1_A = \epsilon(h) 1_A$.

Given a partial action of a Hopf algebra $H$ on a unital algebra $A$, one can define an associative product on $A \otimes H$, given by

$$(a \otimes h)(b \otimes k) = a(h^{(1)} \cdot_1 b) \otimes h^{(2)} k,$$

for all $a, b \in A$ and $h, k \in H$. Then, a new unital algebra is constructed as

$$A\# L H = (A \otimes H)(1_A \otimes 1_H).$$

This unital algebra is called **partial smash product** [5]. This algebra is generated by typical elements of the form

$$a \# L h = a(h^{(1)} \cdot_1 1_A) \otimes h^{(2)}.$$ 

One then proofs that

$$a \# L h = a(h^{(1)} \cdot_1 1_A) \# L h^{(2)};$$

and that

$$(a \# L h)(b \# L k) = a(h^{(1)} \cdot_1 1_A) \# L h^{(2)} k.$$ (1)  

**Examples 2.3.**

1. A group $G$ acts partially on a set $X$, if there exists a family of subsets $\{X_g\}_{g \in G}$ and a family of bijections $\alpha_g : X_{g^{-1}} \rightarrow X_g$ such that
   (a) $X_e = X$, and $\alpha_e = \text{Id}_X$;
   (b) $\alpha_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh}$;
   (c) If $x \in X_{h^{-1}} \cap X_{(gh)^{-1}}$, then $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$.

One says moreover that a group $G$ acts partially on an algebra $A$, if $G$ acts partially on the underlying set of the algebra $A$, such that each $A_g$ is an ideal of $A$ and each $\alpha_g$ is multiplicative. If $G$ acts partially by $\alpha$ on the set $X$, then $G$ acts partially by $\theta$ on the algebra $A = \text{Fun}(X, k)$, where $A_g = \text{Fun}(X_g, k)$ and $\theta_g(f)(x) = f(\alpha_g^{-1}(x))$, where $f \in A_{g^{-1}}$ and $x \in X_g$. Remark that in this example, the ideals $A_g = \text{Fun}(X_g, k)$ are unital algebras. Partial actions of the group algebra $kG$ over any unital algebra $A$ are one-to-one correspondence with partial actions of the group $G$ on $A$ in which the domains $A_g$ are unital ideals, that is, there exists a central idempotent $1_g \in A$ such that $A_g = 1_g A$, and the partially defined isomorphisms $\alpha_g : A_{g^{-1}} \rightarrow A_g$ are unital isomorphisms for each $g \in G$, see [5].
(2) Recall from [1] that if $B$ is an $H$ module algebra and $e$ is a central idempotent in $B$, then there exist a partial action of $H$ on the ideal $A = eB$, given by

$$h \cdot a = e(h \triangleright a),$$

where $h \in H$, $a \in A$ and $\triangleright : H \otimes A \to A$ is the (global) action of $H$ on $A$.

Of course, one can introduce in a symmetric way the category $\text{ParAct}_H$ of right partial actions of $H$, also called right partial $H$-module algebras, and morphisms between them. Furthermore, if $A$ is a right partial $H$-module algebra, then one can construct the right partial smash product $H \#_R A = (1_H \otimes 1_A)(H \otimes A)$.

2.3. Partial representations. Closely related to partial actions, there is the concept of a partial representation. Partial representations were first defined for groups are related to partial actions by means of a pair of adjoint functors, see [9], [3].

Definition 2.4. [3] Let $H$ be a Hopf $k$-algebra, and let $B$ be a unital $k$-algebra. A partial representation of $H$ on $B$ is a linear map $\pi : H \to B$ such that

(PR1) $\pi(1_H) = 1_B$
(PR2) $\pi(h)\pi(k(1))\pi(S(k(2))) = \pi(hk(1))\pi(S(k(2)))$
(PR3) $\pi(k(1))\pi(S(k(2)))\pi(k) = \pi(h(1))\pi(S(h(2)))k$
(PR4) $\pi(h)\pi(S(k(1)))\pi(k(2)) = \pi(hS(k(1)))\pi(k(2))$
(PR5) $\pi(S(h(1)))\pi(h(2))\pi(k) = \pi(S(h(1)))\pi(h(2))k$

A left (resp. right) partial $H$-module is a pair $(M, \pi)$ in which $M$ is a $k$ vector space and $\pi$ is a partial representation of $H$ on the algebra $\text{End}_k(M)$ (resp. $\text{End}_k(M)^{op}$). A morphism between two partial $H$ modules $(M, \pi)$ and $(N, \phi)$ is a linear function $f : M \to N$ such that for each $h \in H$ one has, $f \circ \pi(h) = \phi(h) \circ f$. The category of left (resp. right) partial $H$ modules is denoted by $\mathcal{H}M^\text{par}$ (resp. $\mathcal{H}M^\text{par}_H$).

Let left $(M, \pi)$ be a partial $H$-module. For $h \in H$, we will denote $h \bullet_m = \pi(h)$ in the algebra $\text{End}_k(M)$.

It was shown in [3] that the category of partial modules is equivalent to a category of modules over a Hopf algebroid $H_{\text{par}}$, which can be explicitly constructed out of $H$ by means of a universal property (hence it is a quotient algebra of the free tensor algebra over $H$), in particular, there is a partial representation $[-] : H \to H_{\text{par}}$. We denote the action of $H_{\text{par}}$ on a partial $H$-module $M$ as $x \triangleright m$. In particular, for $h \in H$ and $x = [h] \in H_{\text{par}}$, we have $[h] \triangleright m = h \bullet_m$.

As a consequence of the Hopf algebroid structure, the category of partial $H$-modules is closed monoidal and allows a monoidal forgetful functor to the category of $A$-bimodules that preserves internal Homs. Here the base algebra $A$ of the Hopf algebroid $H_{\text{par}}$ can be identified with the subalgebra of $H_{\text{par}}$ generated by elements of the form $\epsilon_h = [h(1)](S(h(2)))$. For details we refer to [3]. Using this monoidal structure, it was shown in [3] that the category of (left) $H$-module algebras $\mathcal{H}\text{ParAct}$ coincides with the category $\text{Alg}(\mathcal{H}M^\text{par})$ of algebra objects in $\mathcal{H}M^\text{par}$, what justifies the name “partial module algebra”. Let us now state explicitly the internal Hom structure of the category of partial modules.

Proposition 2.5. Consider the objects $M, N, P \in \mathcal{H}M^\text{par}$, then

(i) $\text{Hom}_A(M, N)$ is an object in $\mathcal{H}M^\text{par}$, where

$$\langle x \triangleright F \rangle(m) = x(1) \triangleright_N F(S(x(2)) \triangleright_M m)$$

for all $F \in \text{Hom}_A(M, N)$ and $x \in H_{\text{par}}$;

(ii) the $k$-linear map

$$\hat{\text{Hom}}(M \otimes_A N, P) \to \mathcal{H}\text{par}, \text{Hom}(M, \text{Hom}_A(N, P)), \ F \mapsto \hat{F},$$

with $\hat{F}(m)(n) = F(m \otimes_A n)$, for all $m \in M$, $n \in N$ is an isomorphism, natural in $M$ and $P$;

(iii) $\text{End}_A(M)$ is a left partial $H$-module algebra.
We have also another duality theorem involving the dual vector space of a partial $H$ module. This reveals that the algebra $H_{\text{par}}$ should posses a richer structure than just that of an $A$-Hopf algebroid, which is a subject of future investigation.

**Proposition 2.6.** Given an object $M \in \mathcal{H}M_{\text{par}}$, its dual vector space $M^* = \text{Hom}_k(M,k)$ has both left and right partial $H$ module structures.

*Proof.* Given $\phi \in M^*$ and $h \in H$, we define the functionals $h \bullet_l \phi$ and $\phi \bullet_r h$ by

$$(|h| \triangleright \phi)(m) = \phi(S(h)) \triangleright_M m), \quad (\phi \triangleright |h|)(m) = \phi(|h| \triangleright_M m).$$

It is easy to see that these are, respectively, a left and a right partial module structures. In the generality of a Hopf algebroid, however, in this commutative case several axioms from the general definition of a Hopf algebroid simplify drastically.

**3. Partial actions of co-commutative Hopf algebras.**

In [3] it was shown that the universal partial Hopf algebra $H_{\text{par}}$ (see Section 2.3), which has a structure of a Hopf algebroid, is moreover isomorphic to a partial smash product $A\#H$, where $A$ is a particular subalgebra of $H_{\text{par}}$. In this section, we will show that a general partial smash product $A\#H$ of a co-commutative Hopf algebra with a commutative left partial $H$ module algebra $A$ also has the structure of a Hopf algebroid, however, in this commutative case several axioms from the general definition of a Hopf algebroid simplify drastically.

First, we define the left and right source and target maps all to be the same map

$$s_l = t_l = s_r = t_r : A \rightarrow \frac{A\#H}{a\#1_H},$$

which might be denoted as $s$ or $t$ below. Clearly, $s$ is an algebra morphism and as $A$ is commutative, then the target map $t = s$ can be viewed as an anti morphism as well. Again by the commutativity of $A$, and the images of $s$ and $t$ obviously commute in $\mathcal{H} = A\#H$. Nevertheless, the images of $s$ and $t$ lie not necessarily in the center of $A\#H$, hence the pairs $(s_l, t_l)$ and $(s_r, t_r)$ induce different $A$-(bi)module structures on $A\#H$. Explicitly, the “left handed” $A$ bimodule structure in $\mathcal{H}$ is given by

$$a \triangleright (b\#h) \triangleleft c = s_l(a)t_l(c)(b\#h) = abc\#h,$$

and the “right handed” $A$ bimodule structure by

$$a \bowtie (b\#h) \bowtie c = (b\#h)s_r(c)t_r(a) = b(h_{(1)} \cdot (ac))\#h_{(2)}.$$  

By the commutativity of $A$, both bimodule structures are in fact central, i.e. they are just $A$-module structures or

$$a \triangleright (b\#h) = (b\#h) \triangleleft a, \quad a \bowtie (b\#h) = (b\#h) \bowtie a.$$ 

Hence, different from what is done in a general Hopf algebroid, it is in our situation not needed to keep in mind the two bimodule structures on $\mathcal{H}$, but just the two module structures. In particular, we will show in lemma’s 3.1 and 3.2 below that $\mathcal{H}$ endowed with one of the $A$-module structures described above can moreover be endowed with respective coalgebra structures. Of course these coalgebra structures can be viewed as coring structures to fit the definition of a Hopf algebroid.

In what follows, we will denote the $A$ tensor product with respect to the bimodule structure [3] as $\otimes_A^l$ and the tensor product with respect to [4] as $\otimes_A^r$.

**Lemma 3.1.** Let $H$ be a cocommutative Hopf algebra and $A$ be a commutative left partial $H$ module algebra. Then the partial smash product $\mathcal{H} = A\#LH$ has the structure of an $A$-coalgebra with the $A$-module structure [3] and comultiplication and counit given by

$$\Delta(a\#h) = (a\#h_{(1)}) \otimes_A^l (1_A\#h_{(2)}),$$

$$\varepsilon(a\#h) = a(h \cdot 1_A).$$
Proof. By the definitions of $\Delta$ and of $\epsilon$, one can verify immediately that the comultiplication and the counit are $A$-linear maps and that $\Delta$ is coassociative. The right counit axiom reads

\[
(I \otimes \epsilon) \circ \Delta \circ (a \# h) = (a \# h_{(1)}) \triangleleft (h_{(2)} \cdot 1_A) = a(h_{(2)} \cdot 1_A) \# h_{(1)} = a(h_{(1)} \cdot 1_A) \# h_{(2)} = a \# h
\]

where we used the co-commutativity of $H$ and $\mathbb{1}$. Similarly, we check the left counit axiom

\[
(\epsilon \otimes I) \circ \Delta \circ (a \# h) = (a(h_{(1)} \cdot 1_A)) \triangleright (1_A \# h_{(2)}) = a(h_{(1)} \cdot 1_A) \# h_{(2)} = a \# h
\]

Therefore, $(A \# L H, \Delta, \epsilon)$ is an $A$ coalgebra. \hfill \Box

Lemma 3.2. Let $A$, $H$ be as in Lemma 3.1. Then the partial smash product $\mathcal{H} = A \# L H$ has the structure of an $A$ coalgebra with $A$ bimodule structure given by $\mathbb{1}$, and comultiplication and counit given by

\[
\Delta_r(a \# h) = (a \# h_{(1)}) \otimes_A^r (1_A \# h_{(2)}), \quad \epsilon_r(a \# h) = S(h) \cdot a.
\]

Proof. Let us begin with the $A$-linearity of the comultiplication.

\[
\Delta_r(a \rhd (b \# h)) = \Delta_r(b(h_{(1)} \cdot a) \# h_{(2)}) = b(h_{(1)} \cdot a) \# h_{(2)} \otimes_A^r 1_A \# h_{(3)} = a \rhd (b \# h_{(1)}) \otimes_A^r (1_A \# h_{(2)}) = a \rhd \Delta_r(b \# h)
\]

For the $A$-linearity of the counit, we have

\[
\epsilon_r(a \rhd (b \# h)) = \epsilon_r(b(h_{(1)} \cdot a) \# h_{(2)}) = S(h_{(2)}) \cdot (b(h_{(1)} \cdot a)) = (S(h_{(3)}) \cdot b)(S(h_{(2)}) \cdot (h_{(1)} \cdot a)) = (S(h_{(3)}) \cdot b)(S(h_{(1)})) \cdot (h_{(2)} \cdot a) = (S(h) \cdot b)a = a \epsilon_r(b \# h).
\]

The cocommutativity is straightforward, then it remains to verify the counit axiom:

\[
(I \otimes \epsilon) \circ \Delta_r \circ (a \# h) = (a(h_{(1)})) \lhd (S(h_{(2)}) \cdot 1_A) = a(h_{(1)} \cdot (S(h_{(3)})) \cdot 1_A) \# h_{(2)} = a(h_{(1)} \cdot 1_A)(h_{(2)} S(h_{(4)}) \cdot 1_A) \# h_{(3)} = a(h_{(1)} \cdot 1_A)(h_{(2)} S(h_{(3)})) \cdot 1_A) \# h_{(4)} = a(h_{(1)} \cdot 1_A) \# h_{(2)} = a \# h
\]

and

\[
(\epsilon \otimes I) \circ \Delta_r \circ (a \# h) = (S(h_{(1)})) \rhd (1_A \# h_{(2)}) = (h_{(2)} \cdot (S(h_{(3)}))) \# h_{(3)} = (h_{(2)} S(h_{(1)} \cdot a)(h_{(3)} \cdot 1_A) \# h_{(4)} = (h_{(1)} S(h_{(2)}) \cdot a)(h_{(3)} \cdot 1_A) \# h_{(4)} = a(h_{(1)} \cdot 1_A) \# h_{(2)} = a \# h.
\]

Therefore, $A \# H$ is an $A$ coalgebra with respect to the module structure $\mathbb{1}$. \hfill \Box

Although in the previous Lemma’s we found coalgebra structures rather than coring structures on $\mathcal{H}$, there are still two different module structures on $\mathcal{H}$, so it still makes sense to consider the Takeuchi product. We obtain the following result.

Lemma 3.3. With notation as in Lemma 3.1 and Lemma 3.2,

1. The image of the map $\Delta_l$ lies in the left-Takeuchi tensor product

\[
A \# H \times_A A \# H = \left\{ \sum x_i \otimes_A^l y_i \in A \# H \otimes_A^l A \# H \mid \sum x_i \otimes_A^l y_i = \sum a \rhd x_i \otimes_A^l y_i \lhd a, \forall a \in A \right\}.
\]

2. The image of the map $\Delta_r$ lies in the right-Takeuchi tensor product

\[
A \# H \times_A A \# H = \left\{ \sum x_i \otimes_A^r y_i \in A \# H \otimes_A^r A \# H \mid \sum a \rhd x_i \otimes_A^r y_i = \sum x_i \otimes_A^r y_i \lhd a, \forall a \in A \right\}.
\]

Moreover, in both cases, the co-restriction of the co-multiplication is an algebra morphism.
Consider \(a \# h \in A \# H\) and \(b \in A\), then

\[
b \triangleright (a \# h_{(1)}) \otimes_A' (1_A \# h_{(2)}) = (a(h_{(1)} \cdot b) \# h_{(2)}) \otimes_A' (1_A \# h_{(3)})
\]
\[
= (a \# h_{(2)}) \triangleright (h_{(1)} \cdot b) \otimes_A' (1_A \# h_{(3)})
\]
\[
= (a \# h_{(1)}) \triangleright (h_{(2)} \cdot b) \triangleright (1_A \# h_{(3)})
\]
\[
= (a \# h_{(1)}) \otimes_A' (h_{(2)} \cdot b) \triangleright (1_A \# h_{(3)})
\]
\[
= (a \# h_{(1)}) \otimes_A' ((h_{(2)} \cdot b) \# h_{(3)})
\]
\[
= (a \# h_{(1)}) \otimes_A' (1_A \# h_{(2)}) \triangleright b.
\]

(2). Consider, again, \(a \# h \in A \# H\) and \(b \in A\), then

\[
b \triangleright (a \# h_{(1)}) \otimes_A' 1_A \# h_{(2)} = (ba \# h_{(1)}) \otimes_A' 1_A \# h_{(2)}
\]
\[
= ab(h_{(1)} \cdot 1_A \# h_{(2)}) \otimes_A' 1_A \# h_{(3)}
\]
\[
= a(h_{(1)} \cdot 1_A)(h_{(2)}S(h_{(3)}) \cdot b) \# h_{(4)} \otimes_A' 1_A \# h_{(5)}
\]
\[
= (a(h_{(1)} \cdot (S(h_{(2)}) \cdot b))) \# h_{(3)} \otimes_A' 1_A \# h_{(4)}
\]
\[
= (a \# h_{(1)}) \triangleright (S(h_{(2)}) \cdot b) \triangleright (1_A \# h_{(3)})
\]
\[
= a \# h_{(1)} \otimes_A' (S(h_{(2)}) \cdot b) \triangleright (1_A \# h_{(3)})
\]
\[
= a \# h_{(1)} \otimes_A' (h_{(3)} \cdot (S(h_{(2)}) \cdot b)) \# h_{(4)}
\]
\[
= a \# h_{(1)} \otimes_A' (h_{(2)} \cdot 1_A)(h_{(3)}S(h_{(4)}) \cdot b) \# h_{(5)}
\]
\[
= a \# h_{(1)} \otimes_A' (h_{(2)} \cdot 1_A)b \# b \# h_{(3)}
\]
\[
= a \# h_{(1)} \otimes_A' b \# b \# h_{(2)}
\]
\[
= a \# h_{(1)} \otimes_A' (1_A \# h_{(2)}) \triangleleft b.
\]

Finally, let us verify that the co-restriction \(\Delta_r : H \to H \times_A H\) is an algebra morphism (the proof for \(\Delta_l\) is similar). On one hand we have

\[
\Delta_r((a \# h)(b \# k)) = \Delta_r(a(h_{(1)} \cdot b) \# h_{(2)k})
\]
\[
= (a(h_{(1)} \cdot b) \# h_{(2)k_{(1)}}) \otimes_A' (1_A \# h_{(3)k_{(2)}})
\]

on the other hand, we have

\[
\Delta_r(a \# h)\Delta_r(b \# k) = ((a \# h_{(1)}) \otimes_A' (1_A \# h_{(2)}))[[(b \# k_{(1)}) \otimes_A' (1_A \# k_{(2)})]]
\]
\[
= (a \# h_{(1)}) (b \# k_{(1)}) \otimes_A' (1_A \# h_{(2)}) (1_A \# k_{(2)})
\]
\[
= (a(h_{(1)} \cdot b) \# h_{(2)k_{(1)}}) \otimes_A' (1_A \# h_{(3)k_{(2)}}).
\]

Therefore, \(\Delta_r\) is an algebra morphism. \(\square\)

**Lemma 3.4.** With notation as in Lemma 3.21 and Lemma 3.22, the following identities hold for all \(x, y \in A \# H\).

(1) \(\xi(1_H) = 1_A = \xi(1_H)\);
(2) \(\xi(xy) = \xi(x \triangleright \xi(y)) (= \xi(xs(\xi(y))))\)
(3) \(\xi(xy) = \xi(\xi(x) \triangleright y) (= \xi(s(\xi(x))y)\)

**Proof.** The identity (1) is straightforward. For (2), take \(x = a \# h\) and \(y = b \# k\). Then, on one hand, we have

\[
\xi((a \# h)(b \# k)) = \xi(a(h_{(1)} \cdot b) \# h_{(2)k})
\]
\[
= a(h_{(1)} \cdot b)(h_{(2)k} \cdot 1_A)
\]
\[
= a(h_{(1)} \cdot b)(h_{(2)} \cdot 1_A)(h(x)_k \cdot 1_A)
\]
\[
= a(h_{(1)} \cdot b)(h_{(2)} \cdot (k \cdot 1_A))
\]
\[
= a(h_{(1)} \cdot b)(h(2) \cdot (k \cdot 1_A))
\]
On the other hand,
\[
\varepsilon((a\#h) \triangleright \varepsilon(b\#k)) = \varepsilon((a\#h) \triangleright (b\cdot 1_A)) \\
= \varepsilon(a(h_{(1)} \cdot (b\cdot 1_A))) \# h_{(2)} \\
= a(h_{(1)} \cdot (b\cdot 1_A))(h_{(2)} \cdot 1_A) \\
= a(h \cdot (b\cdot 1_A)).
\]

For (3) take again \(x = a\#h\) and \(y = b\#k\). Then,
\[
\varepsilon((a\#h)(b\#k)) = \varepsilon(a(h_{(1)} \cdot b)\# h_{(2)} k) \\
= S(h_{(2)} k) \cdot (a(h_{(1)} \cdot b)) \\
= (S(h_{(3)} k_{(3)}) \cdot a)(S(h_{(2)} k_{(2)}) \cdot (h_{(1)} \cdot b)) \\
= (S(h_{(3)} k_{(2)}) \cdot a)S(k_{(1)})S(h_{(1)})h_{(2)} \cdot b) \\
= (S(k_{(2)})S(h) \cdot a)(S(k_{(1)}) \cdot b) \\
= S(k) \cdot ((S(h) \cdot a)b) \\
= \varepsilon((S(h) \cdot a)b\#k) \\
= \varepsilon((S(h) \cdot a)) \triangleright (b\#k) \\
= \varepsilon(\varepsilon(a\#h) \triangleright (b\#k)).
\]

Combining the results of Lemmas 3.1, 3.2, 3.3 and 3.4 so far we have proved that \(\mathcal{H} = A\#H\) has the structures of left and right \(A\)-bialgebroid. In order to prove that it is a Hopf algebroid we need to define the antipode map
\[
\mathcal{S} : \mathcal{H} \to \mathcal{H}, \quad \mathcal{S}(a\#h) = (S(h_{(2)}) \cdot a)\#S(h_{(1)}), \forall a\#h \in \mathcal{H}.
\]

**Theorem 3.5.** Using notation as in Lemma’s 3.1 and 3.2 the data \((A\#H, A, s_1, t_1, s_r, t_r, \Delta, \varepsilon, \varepsilon)\) define a structure of a Hopf algebroid over the base algebra \(A\).

**Proof.** First, we need to show that \(\mathcal{S}\) is an anti-algebra morphism. Indeed, taking \(a\#h\) and \(b\#k\) in \(A\#H\), we have
\[
\mathcal{S}((a\#h)(b\#k)) = S(a(h_{(1)} \cdot b) \# h_{(2)} k) \\
= (S(h_{(3)} k_{(3)}) \cdot a)(S(h_{(2)} k_{(2)}) \cdot (h_{(1)} \cdot b)) \# S(h_{(2)} k_{(1)}) \\
= (S(h_{(3)} k_{(2)}) \cdot a)(S(h_{(2)})(h_{(1)} \cdot b)) \# S(h_{(2)} k_{(1)}) \\
= (S(h_{(3)} k_{(3)}) \cdot a)(S(k_{(3)})S(h_{(2)})h_{(3)} \cdot b) \# S(h_{(1)} k_{(1)}) \\
= (S(h_{(2)} k_{(3)}) \cdot a)(S(k_{(2)}) \cdot b) \# S(h_{(1)} k_{(1)}).
\]
on the other hand
\[
\mathcal{S}(b\#k)\mathcal{S}(a\#h) = ((S(k_{(2)}) \cdot b) \# S(k_{(1)}))((S(h_{(2)}) \cdot a) \# S(h_{(1)})) \\
= (S(k_{(3)}) \cdot b)(S(k_{(2)}) \cdot (S(h_{(2)}) \cdot a)) \# S(k_{(1)})S(h_{(1)}) \\
= (S(k_{(3)}) \cdot b)(S(k_{(2)})S(h_{(2)}) \cdot a) \# S(h_{(1)} k_{(1)}) \\
= (S(h_{(2)} k_{(3)}) \cdot a)(S(k_{(2)}) \cdot b) \# S(h_{(1)} k_{(1)}).
\]

As \((A\#H, A, s_1, t_1, \Delta, \varepsilon)\) is a left \(A\) bialgebroid, and \((A\#H, A, s_r, t_r, \Delta, \varepsilon)\) is a right \(A\) bialgebroid, we need only to verify the compatibility identities and the identities relative to the antipode in the definition of a Hopf algebroid as recalled on page 4.
For the item (i) consider \( a \in A \). Since \( s_l = t_l = s_r = t_r \) in our case, we only have to verify two identities:

\[
\begin{align*}
    s_l \circ \underline{\sigma} \circ t_r(a) &= s_l(\underline{\sigma}(a \# 1_H)) \\
    &= s_l(a(1_H \cdot 1_A)) = s_l(a) = t_r(a),
\end{align*}
\]

and

\[
\begin{align*}
    s_r \circ \underline{\sigma} \circ t_l(a) &= s_r(\underline{\sigma}(a \# 1_H)) \\
    &= s_r(S(1_H) \cdot a) = s_r(a) = t_l(a).
\end{align*}
\]

The item (ii) is straightforward.

For the item (iii), take \( a, c \in A \) and \( b \# h \in A \# H \), then

\[
\begin{align*}
    S(t_l(a)(b \# h)t_r(c)) &= S((a \# 1_H)(b \# h)(c \# 1_H)) \\
    &= S(ab(h(1) \cdot c) \# h(2)) \\
    &= (S(h(3)) \cdot (a(b(h(1) \cdot c))) \# S(h(2)) \\
    &= (S(h(3)) \cdot a(S(h(4)) \cdot b)(S(h(3)) \cdot (h(1) \cdot c)) \# S(h(2)) \\
    &= (S(h(3)) \cdot a(S(h(4)) \cdot b)(S(h(2))h(3) \cdot c) \# S(h(1)) \\
    &= (S(h(3)) \cdot a(S(h(2)) \cdot b)c \# S(h(1)) \\
    &= (c \# 1_H)((S(h(3)) \cdot b)(S(h(2)) \cdot a) \# S(h(1))) \\
    &= (c \# 1_H)((S(h(2)) \cdot b) \# S(h(1)))(a \# 1_H) \\
    &= s_r(c)S(b \# h)s_l(a).
\end{align*}
\]

Finally, for the item (iv), take \( a \# h \in A \# H \), then

\[
\begin{align*}
    \mu \circ (S \otimes I) \circ \Delta_{\mathcal{A}}(a \# h) &= S(a \# h(1))(1_A \# h(2)) \\
    &= ((S(h(2)) \cdot a) \# S(h(1))) \cdot (1_A \# h(3)) \\
    &= (S(h(3)) \cdot a)(S(h(4)) \cdot 1_A \# S(h(1))h(4)) \\
    &= (S(h(3)) \cdot a) \# S(h(1))h(2) \\
    &= (S(h) \cdot a) \# 1_H \\
    &= s_r \circ \underline{\sigma}(a \# h),
\end{align*}
\]

and

\[
\begin{align*}
    \mu \circ (I \otimes S) \circ \Delta_{\mathcal{A}}(a \# h) &= (a \# h(1))S(1_A \# h(2)) \\
    &= (a \# h(1))(S(h(3)) \cdot 1_A) \# S(h(2)) \\
    &= a(h(1) \cdot (S(h(4)) \cdot 1_A)) \# h(2)S(h(3)) \\
    &= a(h(1) \cdot (S(h(2)) \cdot 1_A)) \# 1_H \\
    &= a(h(1) \cdot 1_A)(h(2)S(h(3)) \cdot 1_A) \# 1_H \\
    &= a(h \cdot 1_A) \# 1_H \\
    &= s_l \circ \underline{\sigma}(a \# h).
\end{align*}
\]

Therefore, \( A \# H \) is a Hopf algebroid. \( \square \)

4. Partial comodule algebras

4.1. Symmetric partial comodule algebras.

4.1.1. Definitions and examples. In [5] the notion of a partial comodule algebra was introduced. We recall this notion here and add a symmetry axiom to it.

**Definition 4.1.** Let \( H \) be a Hopf algebra. A unital algebra \( A \) is said to be a right partial \( H \) comodule algebra, or is said to possess a partial coaction of \( H \), if there exists a linear map

\[
\begin{align*}
    \overline{\sigma} : A &\rightarrow A \otimes H \\
    a &\mapsto \overline{\sigma}(a) = a^{[0]} \otimes a^{[1]}
\end{align*}
\]

...
that satisfies the following identities:

(PRHCA1) For every $a, b \in A$, $\overline{\rho}(ab) = \rho(a)\rho(b)$.
(PRHCA2) For every $a \in A$, $(I \otimes \epsilon)\overline{\rho}(a) = a$.
(PRHCA3) For every $a \in A$, $(\overline{\rho} \otimes I)\rho(a) = [(I \otimes \Delta)\overline{\rho}(a)](\overline{\rho}(1_A) \otimes 1_H)$. 

A partial coaction is moreover called symmetric if

(PRHCA4) For every $a \in A$, $(\overline{\rho} \otimes I)\rho(a) = (\overline{\rho}(1_A) \otimes 1_H)\rho(a)]$.

Let $A$ and $B$ be two right partial $H$ comodule algebras. We say that $f : A \to B$ is a morphism of partial comodule algebras if it is an algebra morphism such that $f(a^{[0]} \otimes a^{[1]} = f(a)^{[0]} \otimes f(a)^{[1]}$.

The category of right partial $H$ comodule algebras and their morphisms is denoted as $\text{ParCoAct}_H$.

Denoting the partial coaction in a Sweedler like notation,

\[ \overline{\rho}(a) = a^{[0]} \otimes a^{[1]}, \]

we can rewrite the axioms for partial coactions in the following manner:

(PRHCA1) \((ab)^{[0]} \otimes (ab)^{[1]} = a^{[0]}b^{[0]} \otimes a^{[1]b^{[1]}},\)
(PRHCA2) \(a^{[0]}c(a^{[1]}) = a,\)
(PRHCA3) \(a^{[0]}a^{[0]} \otimes a^{[0]}a^{[1]} \otimes a^{[1]} \otimes 1_A^{[1]} \otimes 1_A^{[2]} \otimes a^{[1]} = 1_A^{[0]}a^{[0]} \otimes a^{[1]} \otimes 1_A^{[1]} \otimes 1_A^{[2]} \otimes a^{[1]}\)
(PRHCA4) \(a^{[0]}a^{[0]} \otimes a^{[0]}a^{[1]} \otimes a^{[1]} = 1_A^{[0]}a^{[0]} \otimes a^{[1]} \otimes 1_A^{[1]} \otimes 1_A^{[2]} \otimes a^{[1]}\).

Symmetrically, one can define also the notion of a left partial $H$ comodule algebra, but throughout this text, we deal basically with right partial comodule algebras. It is important to note that, as the partial coaction $\overline{\rho} : A \to A \otimes H$ is a morphism of algebras, then we have that $\overline{\rho}(1_A)$ is an idempotent in the algebra $A \otimes H$, and for any $a \in A$ we have $\overline{\rho}(a) = \overline{\rho}(a)\overline{\rho}(1_A) = \overline{\rho}(1_A)\overline{\rho}(a)$. Remark however, that $\overline{\rho}(1_A)$ is only central in the image of $\overline{\rho}$ and not in the whole of $A \otimes H$. We obtain that the image of the coaction is contained in the unitary ideal $A_\Delta H = (A \otimes H)\overline{\rho}(1_A)$ and the projection $\pi : A \otimes H \to A_\Delta H$ is given simply by the multiplication by $\overline{\rho}(1_A) = 1^{[0]} \otimes 1^{[1]}$. A typical element in $A_\Delta H$ can be written as

\[ x = \sum_i a^i 1^{[0]} \otimes h^i 1^{[1]}, \quad \text{for } a^i \in A, \text{ and } h^i \in H. \]

Let us recall the following basic example from [1].

Example 4.2. Let $H$ be a Hopf algebra and $B$ a right $H$ comodule algebra with coaction $\rho : B \to B \otimes H$. Suppose that $A \subset B$ is a unital ideal in $B$, then $A$ is right partial $H$ comodule algebra, with coaction $\overline{\rho} : A \to A \otimes H$, $\overline{\rho}(a) = (1_A \otimes 1_H)\rho(a)$.

4.1.2. Duality between partial actions and partial coactions. There is a natural duality between partial actions and partial coactions, as was observed in [1, 5], which hold in the symmetric case as we will point out now.

Proposition 4.3. Consider a dual pairing of Hopf algebras $\langle -, - \rangle : H \otimes K \to k$ and let $A$ be a symmetric right partial $K$-comodule algebra. Then the map

\[ \cdot : H \otimes A \to A \]
\[ f \otimes a \mapsto \sum a^{[0]}\langle f, a^{[1]} \rangle \]

is a symmetric left partial action of $H$ on $A$. This construction yields a functor

\[ \Phi : \text{ParCoAct}_K \to H\text{ParAct} \]

Proof. Let us just check the duality between the symmetry of the coaction and the action. For the remaining part, we refer to [1] where the statement was proven in case $k$ is a field, but the proof generalizes without any problem to any commutative base ring $k$. Consider $a \in A$ and
As the coaction is symmetric, then we can write
\[ a^{[0]} \otimes a^{[1]} = 1^{[0]} \otimes 1^{[1]}(1) \otimes a^{[1]}(2), \]
as well, then, the same calculation above gives us
\[ h \cdot (k \cdot a) = (h(1) \cdot 1_A)(h(2) \cdot k \cdot a) \]
Therefore, the partial action of $A$ on $A$ is symmetric. \hfill \Box

Let us use the same notation as in the statement of Proposition 3. Let $a \in A$ and write $a^{[0]} \otimes a^{[1]} = \sum_{i=1}^{n} a_i \otimes x_i$ for certain $a_i \in A$ and $x_i \in K$. Then for all $h \in H$, we find that
\[ a \cdot h = \sum_{i=1}^{n} a_i \langle h_i, x_i \rangle. \]
This leads us to the following definition.

**Definition 4.4.** Let $H$ be a Hopf algebra and $A$ be a left partial $H$-module algebra. We say that the partial action of $H$ on $A$ is *rational* if for every $a \in A$ there exists $n = n(a) \in \mathbb{N}$ and a finite set $\{a_i, \varphi^i\}_{i=1}^{n}$, with $a_i \in A$ and $\varphi^i \in H^*$ such that, for every $h \in H$ we have
\[ h \cdot a = \sum_{i=1}^{n} \varphi^i(h)a_i. \]

The full subcategory of $\text{ParAct}_H$ consisting of all rational left partial actions is denoted by $\text{ParAct}^r_H$.

Recall that a $k$-module $M$ is said to be locally projective (in the sense of Zimmermann-Huisgen) if for all $m \in M$, there exists a finite dual base $\{e_i, f_i\}_{i=1}^{n} \in M \otimes M^*$ i.e. $m = \sum_{i=1}^{n} e_i f_i(m)$. Moreover the fact that $M$ is locally projective over $R$ and a subspace $D \subset M^*$ is dense with respect to the finite topology is equivalent with $M$ satisfying the $D$-relative $\alpha$-condition, which states that for any $k$-module $N$ the canonical map
\[ \alpha_{N,D} : N \otimes M \rightarrow \text{Hom}(D, N), \quad \alpha_{N,D}(n \otimes m)(d) = d(m)n \]
is injective, for details see e.g. [13].

**Theorem 4.5.** Consider a non-degenerate dual pairing of Hopf algebras $\langle -, - \rangle : H \otimes K \rightarrow k$ where $H$ is locally projective over $k$ and let $A$ be a rational symmetric left partial $H$-module algebra. Then $A$ can be endowed with the structure of a symmetric right partial $K$ comodule algebra such that $\Phi(A)$ is the initial left partial $H$-module algebra. This construction, to together with Proposition 3.3 yields an isomorphism of categories
\[ \text{ParAct}^r_H \cong \text{ParCoAct}_K. \]

**Proof.** Consider the following diagram.
\[
\begin{array}{ccc}
A & \longrightarrow & A \otimes K \\
\beta \downarrow & & \alpha_{A,K} \downarrow \\
\text{Hom}(H, K) & & \\
\end{array}
\]
where we define $\beta(a)(h) = h \cdot a$. Then, since $A$ is rational and $K$ is dense in $H^*$ by the non-degeneracy of the pairing, the image of $\beta$ is contained in the image of $\alpha_{A,K}$. Moreover, since $H$ is locally projective over $k$ and $H$ is dense in $K^*$, the map $\alpha_{A,K}$ is injective. Hence there exists a well-defined map $\overline{\beta} : A \rightarrow A \otimes K$ that renders the diagram commutative, i.e.
\[ h \cdot a = (I \otimes \langle h, \cdot \rangle)\overline{\beta}(a). \]
Similarly to [1], where the proof was made considering $k$ a field, one can now show the stated equivalence. Let us just check the duality between the symmetry of the coaction and the action.

We have to verify the axiom (PRHCA3), to this, consider $h, k \in H$ then

$$(I \otimes \langle h, - \rangle \otimes \langle k, - \rangle)(\rho \otimes I)(a) = (I \otimes \langle h, - \rangle)(\sum_{i=1}^{n} \rho(a_i) \langle k, x^i \rangle) = (I \otimes \langle h, - \rangle)\rho(k \cdot a)$$

$$h \cdot (k \cdot a) = (h_{(1)} k \cdot a_{(2)} \cdot 1_A) = [(I \otimes \langle h_{(1)}, - \rangle)(\rho(a)]((I \otimes \langle h_{(2)}, - \rangle)\rho(1_A))$$

$$= [(I \otimes \langle h_{(1)}, - \rangle \otimes \langle k, - \rangle)(I \otimes \Delta)(\rho(a)]((I \otimes \langle h_{(2)}, - \rangle)\rho(1_A)) \otimes 1_H)$$

By the nondegeneracy of the pairing, one can conclude that

$$(\rho \otimes I)\rho(a) = ((I \otimes \Delta)(\rho(a)))(\rho(1_A) \otimes 1_H).$$

The following result can implicitly be found in [5], hence we omit an explicit proof and give only the structure maps.

**Lemma 4.6.** Let $K$ be a Hopf algebra and $A$ be a right partial $K$ comodule algebra, then the reduced tensor product $A \bar{\otimes} K = (A \otimes K)\rho(1_A)$ has a structure of an $A$-coring with bimodule structure, comultiplication and counit given by

$$b \cdot (a^0 \otimes x^0) \cdot b' = bab^0 \otimes xb'$$

$$\Delta(a^0 \otimes x^1) = a^0 \otimes x_{(1)}^1 \otimes A 1^0 \otimes x_{(2)}^1,$$

$$\tau(a^0 \otimes x^1) = ae(x).$$

Moreover, if there is a dual pairing $\langle -, - \rangle : H \otimes K \to k$ between $K$ and a second Hopf algebra $H$, then there is a dual pairing between the $A$-coring $A \bar{\otimes} K$ and the smash product $(A^{op} \otimes H^{cop})^{op}$ given by

$$\langle a \hat{\otimes} h, b \otimes x \rangle = ba(h_{(2)} \cdot 1_A) \langle h_{(1)}, x \rangle$$

Remark that if $A$ is a left partial $H$-module algebra, then $A^{op}$ is also a left partial $H^{cop}$-algebra and hence the smash product $A^{op} \otimes H^{cop}$ makes sense. The dual paring between the coring and smash product above means that there is a morphism of $A$-rings

$$\alpha : (A^{op} \otimes H^{cop})^{op} \to (* (A \bar{\otimes} K), \alpha(a \hat{\otimes} h)(b \otimes x) = \langle a \hat{\otimes} h, b \otimes x \rangle$$

**4.2. Partial coactions of commutative Hopf algebras.** When we restrict the study of partial coactions to the case of commutative Hopf algebras on commutative algebras we find a richer structure. First of all, these turn out to be closely related to partial actions of affine algebraic groups on affine algebraic varieties. Furthermore, we can construct a Hopf algebroid out of such a commutative partial coaction. Conversely, given a morphism of bialgebroids between a commutative Hopf $H$ and a commutative $A$-Hopf algebroid $H$ such that the map induced by this morphism $\Pi : A \otimes H \to H$ splits as a morphism of algebras, then there is a partial coaction of $H$ on $A$.

**4.2.1. Partial coactions from algebraic geometry.** In this section we extend the well-known correspondence between actions of affine algebraic groups on affine varieties and comodule algebras over affine Hopf algebras to the partial setting.

**Definition 4.7.** Let $G$ be an affine algebraic group and $M$ an affine algebraic variety. A partial action $\{(M_g)_{g \in G}, \{a_g\}_{g \in G}\}$ of $G$ on the underlying set $M$ is said to be algebraic if

1. For all $g \in G$, $M_g$ is a subvariety.
2. For all $g \in G$, the maps $a_g : M_{g^{-1}} \to M_g$ are algebraic.
3. For all $g \in G$ there exists a subvariety $M'_g$ such that $M = M_g \sqcup M'_g$.

**Remark 4.8.** If a partial action of an affine algebraic group $G$ on an affine variety $M$ is algebraic, then each domain $M_g$ is a disjoint union of a finite number of connected components.
**Example 4.9.** Take the variety $M$ which is the union of two horizontal circles of radius 1, one centered at $(0, 0, 1)$ and the other at $(0, 0, 0)$. This is an algebraic variety, whose algebra of coordinate functions is given by

$$A = k[x, y, z]/(x^2 + y^2 - 1, z^2 - z)$$

There is a partial action of the affine algebraic group $G = S^1 \rtimes \mathbb{Z}_2$, whose algebra of coordinate functions can be written as

$$H = k[x_1, x_2, x_3]/(x_1^2 + x_2^2 - 1, z^2 - 1)$$

Geometrically, the group is the union of two disjoint circles, $G_1$ whose elements are of the form $g = (x_1, x_2, 1)$ and $G_2$ whose elements are of the form $g = (x_1, x_2, -1)$. For $g \in G_1$, we have $M_g = M$ and the action is given by

$$\alpha_{(x_1, x_2, 1)}(x, y, z) = (x, x_1 - y, x_2 + y, x_1, z) \quad \text{for } z = 0, 1,$$

For $g \in G_2$, the domain $M_g$ is only the unit circle centered at $(0, 0, 0)$, and the action is given by

$$\alpha_{(x_1, x_2, -1)}(x, y, z) = (-x, x_1 + y, x_2 + y, x_1, -z) \quad \text{for } z = 0.$$

This partial action is clearly algebraic.

Algebraic partial group actions give rise to partial coactions of commutative Hopf algebras.

**Proposition 4.10.** Let $G$ be an affine algebraic group and $M$ be an affine algebraic variety. Then each algebraic partial action of $G$ on $M$ defines a symmetric partial coaction of the commutative Hopf algebra $H = \mathcal{O}(G)$ over the commutative algebra $A = \mathcal{O}(M)$.

**Proof.** Suppose that $M$ is a subvariety of the affine space $k^n$, therefore, all coordinate algebras, $A = \mathcal{O}(M)$ and $A_g = \mathcal{O}(M_g)$ are quotients of the polynomial algebra $k[x_1, \ldots, x_n]$. As the partial action of $G$ on $M$ is algebraic, then, for every $g \in G$, we have $M = M_g \sqcup M_g'$. Denote the ideals relative to $M_g$ and $M_g'$, respectively by $I_g$ and $J_g$, then

$$\mathcal{O}(M) = k[x_1, \ldots, x_n]/I_g.J_g.$$

Moreover, as $M_g \cap M_g' = \emptyset$, then $I_g + J_g = k[x_1, \ldots, x_n]$, that is, there exists $u_g \in I_g$ and $v_g \in J_g$ such that $u_g + v_g = 1$. Denote by $e_g$ and $f_g$, respectively, the classes of $v_g$ and $u_g$ modulo $I_g.J_g$, it is easy to see that $e_g$ is an idempotent, indeed,

$$e_g + f_g = (v_g + u_g) + I_g.J_g = 1 + I_g.J_g,$$

and

$$e_g.f_g = (v_g + I_g.J_g)(u_g + I_g.J_g) = u_g v_g + I_g.J_g = 0 + I_g.J_g,$$

then

$$e_g = e_g(1 + I_g.J_g) = e_g(e_g + f_g) = e_g^2 + e_g.f_g = e_g^2.$$  

The algebra $A_g = \mathcal{O}(M_g)$ can be seen as the unital ideal $e_g.\mathcal{O}(M) \subseteq \mathcal{O}(M)$. Define the map

$$\overline{\alpha} : \mathcal{O}(M) \rightarrow \mathcal{O}(M) \otimes \mathcal{O}(G) \cong \mathcal{O}(M \times G)$$

by

$$\overline{\alpha}(f)(p, g) = \left\{ \begin{array}{ll} f(\alpha_{g^{-1}}(p)) & \text{if } p \in M_g \\ 0 & \text{otherwise} \end{array} \right.$$  

This is in fact a polynomial function because the coaction can also be written as

$$\overline{\alpha}(f)(p, g) = e_g.(f \circ \alpha_{g^{-1}})(p)$$

If the partial action was not be algebraic, then the projections over the subvarieties $M_g$ would not be polynomial and the right hand side would not be well defined.

It is easy to see that $\overline{\alpha}$ above defined is multiplicative, then it satisfies (PRHCA1). Now, consider $f \in \mathcal{O}(M)$ and $p \in M$ then

$$(I \otimes e)\overline{\alpha}(f)(p) = \overline{\alpha}(f)(p, e) = f(\alpha_e(p)) = f(p), \quad \text{for } p \in M_e = M.$$
Therefore, $\mathcal{P}$ satisfies (PRHCA2). Finally, for $f \in \mathcal{O}(M)$, $p \in M$ and $g, h \in G$,

$$(\mathcal{P} \otimes I)(f)(p, g, h) = (\mathcal{P} \otimes I)(f^0 \otimes f^1)(p, g, h)$$

$$= \begin{cases} f^0(\alpha_{g^{-1}}(p))f^1(h) & \text{for } p \in M_g \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f(\alpha_{h^{-1}}(\alpha_{g^{-1}}(p))) & \text{for } p \in M_g \text{ and } \alpha_{g^{-1}}(p) \in M_h \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f(\alpha_{g^{-1}}(\alpha_{g^{-1}}(p))) & \text{for } p \in \alpha_g(M_g^{-1} \cap M_h) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f(\alpha_{(gh)^{-1}}(p)) & \text{for } p \in M_g \cap M_{gh} \\ 0 & \text{otherwise} \end{cases}$$

on the other hand,

$$[(I \otimes \Delta)\mathcal{P}(f))(p, g, h)(\mathcal{P}(1_{\mathcal{O}(M)}) \otimes 1_{\mathcal{O}(G)}) = (I \otimes \Delta)\mathcal{P}(f)(p, g, h)(\mathcal{P}(1_{\mathcal{O}(M)}) \otimes 1_{\mathcal{O}(G)})(p, g, h)$$

$$= \mathcal{P}(f)(p, gh)\mathcal{P}(1_{\mathcal{O}(M)})(p, g)$$

$$= \begin{cases} \mathcal{P}(f)(p, gh) & \text{for } p \in M_g \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f(\alpha_{(gh)^{-1}}(p)) & \text{for } p \in M_g \cap M_{gh} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, $\mathcal{P}$ satisfies (PRHCA3), which proves that this is a partial coaction of the commutative Hopf algebra $H = \mathcal{O}(G)$ over the commutative algebra $A = \mathcal{O}(M)$. \hfill \Box

Conversely, given a partial coaction of a commutative Hopf algebra on a commutative algebra, one can construct a partial action of the affine algebraic group $\text{Hom}_{\mathcal{O}(H)}(H, k)$ the spectrum of the algebra $A$.

**Proposition 4.11.** Let $H$ be a commutative Hopf algebra and $A$ be a commutative right partial $H$ comodule algebra. Then there is a partial action of the affine algebraic group $G = \text{Hom}_{\mathcal{O}(H)}(H, k)$ on the affine algebraic variety $M = \text{Hom}_{\mathcal{O}(H)}(A, k)$.

**Proof.** Denoting $\mathcal{P}(1_A) = 1^0 \otimes 1^1$, define, for each $g \in G$, the element

$$1_g = 1^0g(1^1) \in A.$$

It is easy to see that $1_g$ is an idempotent,

$$1_g1_g = 1^01^0g(1^1)g(1^1) = 1^01^01^01^1 = 1^0g(1^1) = 1_g.$$

Define, for each $g \in G$, the ideals $A_g = 1_gA$, a typical element in $A_g$ is of the form $a1^0g(1^1)$, for $a \in A$. Note that, as $\mathcal{P}(a) = \mathcal{P}(a1_A)$ then the elements of the form $a^0g(a^1)$ are also in the ideal $A_g$. Define also the linear maps $\theta_g : A_{g^{-1}} \rightarrow A$, by $\theta_g = (I \otimes g(\cdot)) \circ \mathcal{P}|_{A_{g^{-1}}}$. The map $\theta_g$ is an algebra isomorphism between $A_{g^{-1}}$ and $A_g$. Indeed, take $a \in A_{g^{-1}}$, then we have $a = a1^0g^{-1}(1^1)$ and we find

$$\theta_g(a) = (I \otimes g) \circ \mathcal{P}(a1^0g^{-1}(1^1)) = a1^01^0g(1^1)1^01^01^1 = a^0g(a^1).$$

Also, we have, for $a \in A_{g^{-1}}$,

$$\theta_g \circ \theta_g(a) = \theta_{g^{-1}}(a^0g(a^1)) = a1^0g^{-1}(a^0g(a^1)) = a1^0g^{-1}(a^0g(a^1)) = a1^0g^{-1}(a^0g(a^1)) = a1^0g^{-1}(a^0g(a^1)) = a.$$
Then \( \theta_{g^{-1}} \circ \theta_g = \text{Id}_{A_g} \). Analogously, we have \( \theta_g \circ \theta_{g^{-1}} = \text{Id}_{A_g} \). Then \( \theta_g \) is bijective. Finally, for \( a, b \in A_{g^{-1}} \),
\[
\theta_g(ab) = (ab)^{(0)}g((ab)^{(1)}) = a^{(0)}b^{(0)}g(a^{(1)}b^{(1)}) = a^{(0)}g(a^{(1)})b^{(0)}g(b^{(1)}) = \theta_g(a)\theta_g(b).
\]
Therefore, \( \theta_g \) is an algebra isomorphism.

The data \( \{ A_g \}_{g \in G}, \{ \theta_g \}_{g \in G} \) defines a partial action of the group \( G \) on the algebra \( A \). Indeed, it is easy to see that \( A_e = A \) and \( \theta_e = \text{Id}_A \). Take now an element \( y \in A_h \cap A_{g^{-1}} \), then
\[
\theta_h^{-1}(y) = \theta_h^{-1}(y^{(0)}1^{(0)}h(1^{(1)})g^{-1}(1^{(1)'})) = y^{(0)}h^{-1}(y^{(1)}1^{(0)}1^{(0)'}h^{-1}(y^{(1)}1^{(1)}1^{(1)'})) = y^{(0)}h^{-1}(y^{(1)}1^{(0)}h^{-1}(1^{(1)}1^{(1)'}))g^{-1}(1^{(1)'}2).
\]
Then \( \theta_h^{-1}(A_h \cap A_{g^{-1}}) \subseteq A_{h^{-1}} \cap A_{(gh)^{-1}} \). Finally, for any \( x \in \theta_{h^{-1}}(A_h \cap A_{g^{-1}}) \) we have
\[
\theta_g\theta_h(x) = \theta_g(x^{(0)}h(x^{(1)})1^{(0)}g^{-1}(1^{(1)})) = x^{(0)}1^{(0)}g(x^{(0)}1^{(0)}1^{(0)}h(x^{(1)})) = x^{(0)}g(x^{(1)}1^{(0)}1^{(0)}h(x^{(1)})) = x^{(0)}gh(x^{(1)}1^{(0)}1^{(0)}g(x^{(1)})),
\]
while, on the other hand, we have
\[
\theta_{gh}(x) = \theta_{gh}(x^{(0)}1^{(0)}h^{-1}(1^{(1)})) = x^{(0)}1^{(0)}g(gh(x^{(0)}1^{(0)}1^{(0)})) = x^{(0)}gh(x^{(1)}1^{(0)}h^{-1}(1^{(1)})) = x^{(0)}gh(x^{(1)}1^{(0)}1^{(0)}g(x^{(1)})),
\]
Then, \( \theta_g \circ \theta_h(x) = \theta_{gh}(x) \), making the data \( \{ A_g \}_{g \in G}, \{ \theta_g \}_{g \in G} \) a partial action of the group \( G \) on the algebra \( A \).

The last step is to make a partial action of \( G \) on the set \( M = \text{Hom}_{Alg}(A, k) \), defining for each \( g \in G \) the subsets \( M_g = \text{Hom}_{Alg}(A_g, k) \), and the maps \( \alpha_g : M_{g^{-1}} \to M_g \) given by \( \alpha_g(P) = P \circ \theta_{g^{-1}} \), for \( P \in M_{g^{-1}} \). It is easy to see that this data defines a partial action of \( G \) on \( M \). This concludes our proof.

The above constructions are clearly functorial. Hence we obtain the following result, generalizing the classical result for global actions

**Corollary 4.12.** The constructions above provide an equivalence of categories between the category of affine Hopf algebras and the category of affine algebraic groups. Let \( H \) be an affine Hopf algebra and \( G = \text{Hom}_{Alg}(H, k) \) the corresponding algebraic group, then there is an equivalence between the category of affine partial \( H \)-comodule algebras and the category of algebraic partial actions of \( G \).

As finite groups provide (trivial) examples of algebraic groups, the above result applies in particular to this situation. Hence we obtain a bijective correspondence between partial actions of a finite group \( G \) on a finite set \( X \) and partial coactions of the dual group algebra \((kG)^*\) on
the algebra \( A = \text{Fun}(X, k) \) of \( k \) valued functions on \( X \). Explicitly, given a partial action \( \alpha \) of \( G \) on \( X \), the partial coaction of \( \rho \) is given by
\[
\rho(f) = \sum_{g \in G} (f \circ \alpha_{g^{-1}})(1) \otimes p_g.
\]
where \( f \in \text{Fun}(X, k) \) and \( p_g \in kG^* \) is given by \( p_g(h) = \delta_{g, h} \) for all \( g, h \in G \).

4.2.2. The Hopf algebroid associated to a partial coaction. As shown previously, from a left partial action of a co-commutative Hopf algebra \( H \) over a commutative algebra \( A \), one can endow the partial smash product \( A \#_L H \) with a structure of an \( A \) Hopf algebroid. The aim of this section is to show that in the dual case, of a right partial coaction of a commutative Hopf algebra \( H \) over a commutative algebra \( A \), one can construct a Hopf algebroid as well.

From now on, let \( H \) denote a commutative Hopf algebra \( H \) and \( A \) a commutative right partial \( H \) module algebra, with partial coaction \( \overline{\rho} : A \to A \otimes H \). We can define the left source and target maps \( s = s_l, t = t_l : A \to A \otimes H \) as
\[
s(a) = a^{[0]} \otimes 1^{[1]}, \quad t(a) = \overline{\rho}(a) = a^{[0]} \otimes a^{[1]}.
\]
The right source and target maps are defined as \( s_r = t_l \) and \( t_r = s_l \). It is easy to see that both maps are algebra morphisms, as \( A \) and \( H \) are commutative algebras, then \( t \) is an algebra anti-morphism and the images of \( s_l \) and \( t_l \) mutually commute. The \( A \) bimodule structure on \( A \otimes H \), is given by
\[
a \cdot (b^{[0]} \otimes h^{[1]}) \cdot c = s_l(a)t_l(c)(b^{[0]} \otimes h^{[1]}) = abc^{[0]}1^{[0]} \otimes c^{[1]}1^{[1]} = (b^{[0]} \otimes h^{[1]})s_r(c)t_r(a).
\]
Remark 4.13. It is important to remark that these coring operations will be the same for both, the left and the right bialgebroid structures.

There is another useful way to express the comultiplication:
\[
\tilde{\Delta}(a^{[0]} \otimes h^{[1]}) = a^{[0]}1^{[0]} \otimes h_{(1)}1^{[1]}1^{[1]} \otimes_A 1^{[0]} \otimes h_{(2)}1^{[1]}1^{[1]}.
\]
Due to the axiom (PRHCA3) of partial coactions, we can write the above expression as
\[
\begin{align*}
&= a^{[0]}1^{[0]} \otimes h_{(1)}1^{[1]}1^{[1]} \otimes_A 1^{[0]} \otimes h_{(2)}1^{[1]}1^{[1]} \\
&= a^{[0]}1^{[0]} \otimes h_{(1)}1^{[1]}1^{[1]} \otimes_A 1^{[0]} \otimes h_{(2)}1^{[1]}1^{[1]} \\
&= a^{[0]}1^{[0]} \otimes h_{(1)}1^{[1]}1^{[1]} \otimes_A 1^{[0]} \otimes h_{(2)}1^{[1]}1^{[1]} \\
&= a^{[0]}1^{[0]} \otimes h_{(1)}1^{[1]}1^{[1]} \otimes_A 1^{[0]} \otimes h_{(2)}1^{[1]}1^{[1]} \\
&= a^{[0]}1^{[0]} \otimes h_{(1)}1^{[1]}1^{[1]} \otimes_A 1^{[0]} \otimes h_{(2)}1^{[1]}1^{[1]},
\end{align*}
\]
which is the definition of the comultiplication in \( A \otimes H \).

**Lemma 4.14.** The data \((A \otimes H, A, s_l, t_l, \tilde{\Delta}, \tilde{\varepsilon})\) define a structure of a left \( A \) bialgebroid.

**Proof.** We have already seen that \( A \otimes H \) is an \( A \) coring and because of the commutativity of \( A \), we have that the \( A \) bimodule structure of this coring is compatible with both source and target above defined. Again, as \( A \) and \( A \otimes H \) are commutative, then the image of the comultiplication is automatically into the Takeuchi’s product
\[
\begin{align*}
(A \otimes H) \times_A (A \otimes H) &= \\
&= \{ \sum_i X_i \otimes Y_i \in (A \otimes H) \otimes (A \otimes H) \mid \sum_i X_i t(a) \otimes Y_i = \sum_i X_i \otimes Y_i s(a), \forall a \in A \}
\end{align*}
\]
The comultiplication is also multiplicative,
\[
\Delta(a^{[0]} \otimes h^{[1]})\Delta(b^{[0]} \otimes k^{[1]})
= (a^{[0]} \otimes h(1)1^{[1]} \otimes A 1^{[0']} \otimes h(2)1^{[1']})(b1^{[0']} \otimes k(1)1^{[1']} \otimes A 1^{[0'']} \otimes k(2)1^{[1'']})
= (ab1^{[0]} \otimes h(1)1^{[1]} \otimes A 1^{[0']} \otimes h(2)k(2)1^{[1']})
\]
\[
\Delta(ab1^{[0]} \otimes hk1^{[1]})
= \Delta((a^{[0]} \otimes h1^{[1]})(b1^{[0']} \otimes k1^{[1']})).
\]

Let us verify the axiom of the counit in a bialgebroid:
\[
\tilde{e}(XY) = \tilde{e}(Xs(\tilde{e}(Y))) = \tilde{e}(Xt(\tilde{e}(Y))), \quad \forall X, Y \in A \otimes H.
\]

For the first equality, we have
\[
\tilde{e}((a^{[0]} \otimes h1^{[1]})s(\tilde{e}(b1^{[0']} \otimes k1^{[1']}))) = \tilde{e}((a^{[0]} \otimes h1^{[1]})(bc1^{[0]} \otimes 1^{[1']})) = \tilde{e}(abc1^{[0]} \otimes h1^{[1]}) = abc(h)e(k) = \tilde{e}(ab1^{[0]} \otimes hk1^{[1]}) = \tilde{e}((a^{[0]} \otimes h1^{[1]})(b1^{[0']} \otimes k1^{[1']})).
\]

And for the second equality,
\[
\tilde{e}((a^{[0]} \otimes h1^{[1]})t(\tilde{e}(b1^{[0']} \otimes k1^{[1']}))) = \tilde{e}((a^{[0]} \otimes h1^{[1]})(b1^{[0]}e(k) \otimes b1^{[1]})) = \tilde{e}(ab1^{[0]}e(k)1^{[0]} \otimes hb1^{[1]}1^{[1]}) = ab1^{[0]}e(b1^{[1]})e(h)e(k) = abc(h)e(k) = \tilde{e}(ab1^{[0]} \otimes hk1^{[1]}) = \tilde{e}((a^{[0]} \otimes h1^{[1]})(b1^{[0']} \otimes k1^{[1']})).
\]

Therefore, we have that \((A \otimes H, s, t, \Delta, \tilde{e})\) is an \(A\) bialgebroid.

The structure of a right \(A\) bialgebroid is completely analogous, due to the fact that \(A\) is a commutative algebra. Then we have the following result.

**Lemma 4.15.** The data \((A \otimes H, A, s, t, \Delta, \tilde{e})\) define a structure of a right \(A\) bialgebroid.

Finally, we have the structure of a Hopf algebroid on \(A \otimes H\).

**Theorem 4.16.** Let \(H\) be a commutative Hopf algebra and \(A\) be a commutative right \(H\) comodule algebra with partial coaction \(\mathcal{P}\), as above. Then the algebra \(A \otimes H = \mathcal{P}(1_{A})(A \otimes H)\) is a commutative Hopf algebroid over the base algebra \(A\), called the partial split Hopf algebroid. The partial coaction \(prho\) is global if, and only if, the Hopf algebroid \(A \otimes H\) coincides with the split Hopf algebroid \(A \otimes H\).

**Proof.** The Hopf algebroid structure is given by the antipode
\[
\tilde{S}(a^{[0]} \otimes h1^{[1]}) = a^{[0]}1^{[0]} \otimes a^{[1]}S(h)1^{[1]}.
\]

The map \(\tilde{S}\) is an anti algebra morphism. Indeed, take \(a^{[0]} \otimes h1^{[1]}\) and \(b1^{[0]} \otimes k1^{[1]}\) in \(A \otimes H\), then
\[
\tilde{S}((a^{[0]} \otimes h1^{[1]})(b1^{[0]} \otimes k1^{[1]})) = \tilde{S}(ab1^{[0]} \otimes hk1^{[1]}) = a^{[0]}b^{[0]}1^{[0]} \otimes a^{[1]}b^{[1]}S(k)S(h)1^{[1]} = (b^{[0]}1^{[0]} \otimes b^{[1]}S(k)1^{[1]})(a^{[0]}1^{[0]} \otimes a^{[1]}S(h)1^{[1]}) = \tilde{S}(b^{[0]} \otimes k1^{[1]})(a^{[0]} \otimes h1^{[1]}).
The item (i) of the definition of Hopf algebroid (see Section 2.1) can be easily deduced, for example

\[ t_l \circ \varrho \circ s_r(a) = t_l(\varrho(a^{[0]} \otimes a^{[1]})) = t_l(a^{[0]} \epsilon(a^{[1]})) = t_l(a) = s_r(a). \]

the other identities are completely analogous.

The item (ii) is consequence of the fact that the left and right coring structure coincide, and then the compatibility simply means the coassociativity.

The item (iii), take \( a, c \in A \) and \( b1^{[0]} \otimes h1^{[1]} \in A \otimes H \), then

\[
\tilde{S}(t_l(a)(b1^{[0]} \otimes h1^{[1]}))_{r(c)} = \tilde{S}(a^{[0]}bc1^{[0]} \otimes a^{[1]}h1^{[1]}) \\
= a^{[0]}b^{[0]}c^{[0]}1^{[0]} \otimes a^{[0]}b^{[1]}c^{[1]}1^{[1]}S(h)S(a^{[1]}) \\
= (c^{[0]} \otimes c^{[1]}) (b^{[0]}a^{[0]}b^{[0]}1^{[0]} \otimes b^{[1]}a^{[1]}c^{[1]}1^{[1]}S(a^{[1]})S(h)) \\
= \epsilon_r(c) \left( b^{[0]}1^{[0]}a^{[0]} \otimes b^{[1]}S(h)1^{[1]}a^{[1]}(1)S(a^{[1]}(2)) \right) \\
= \epsilon_r(c) \left( b^{[0]}1^{[0]}a^{[0]}c^{[1]}(a^{[1]}) \otimes b^{[1]}S(h)1^{[1]} \right) \\
= \epsilon_r(c) \left( b^{[0]}1^{[0]}a \otimes b^{[1]}S(h)1^{[1]} \right) \\
= \epsilon_r(c) \left( b^{[0]}1^{[0]} \otimes b^{[1]}S(h)1^{[1]}a \right) \\
= \epsilon_r(c) \tilde{S}(b1^{[0]} \otimes h1^{[1]})s_1(a).
\]

Finally, for the item (iv), take \( a1^{[0]} \otimes h1^{[1]} \in A \otimes H \), then

\[
\mu \circ (\tilde{S} \otimes I) \circ \tilde{\Delta}(a1^{[0]} \otimes h1^{[1]}) = \tilde{S}(a1^{[0]} \otimes h(1)1^{[1]})(1^{[0]} \otimes h(2)1^{[1]}) \\
= (a^{[0]}1^{[0]} \otimes a^{[1]}S(h(1))1^{[1]})(1^{[0]} \otimes h(2)1^{[1]}) \\
= a^{[0]}1^{[0]} \otimes a^{[1]}S(h(1))h(2)1^{[1]} \\
= a^{[0]} \epsilon(h)1^{[0]} \otimes a^{[1]}1^{[1]} \\
= \epsilon_r(ac(h)) \\
= \epsilon_r(\varrho(a1^{[0]} \otimes h1^{[1]})),
\]

and

\[
\mu \circ (I \otimes \tilde{S}) \circ \tilde{\Delta}(a1^{[0]} \otimes h1^{[1]}) = (a1^{[0]} \otimes h(1)1^{[1]}), \tilde{S}(1^{[0]} \otimes h(2)1^{[1]}) \\
= (a1^{[0]} \otimes h(1)1^{[1]}), \tilde{S}(1^{[0]} \otimes S(h(2))1^{[1]}) \\
= a1^{[0]} \otimes h(1), S(h(2))1^{[1]} \\
= a1^{[0]} \epsilon(h) \otimes 1^{[1]} \\
= s_1(ac(h)) \\
= s_1(\varrho(a1^{[0]} \otimes h1^{[1]})).
\]

Therefore, \( A \otimes H \) is a Hopf algebroid.

Now consider a global coaction \( \overline{\varpi} : A \to A \otimes H \), then \( \overline{\varpi} \) is a unital morphism, that is, \( \overline{\varpi}(1_A) = 1_A \otimes 1_H \). Therefore

\[ A \otimes H = \overline{\varpi}(1_A)(A \otimes H) = A \otimes H. \]
The usual Hopf algebroid structure on $A \otimes H$, named split Hopf algebroid, is given by

\[ s(a) = t_r(a) = a \otimes 1, \]
\[ t_l(a) = s_r(a) = \overline{\pi}(a), \]
\[ \tilde{\Delta}(a \otimes h) = a \otimes h^{(1)} \otimes_A 1_A \otimes h^{(2)}, \]
\[ \tilde{\epsilon}(a \otimes h) = a \epsilon(h), \]
\[ \tilde{S}(a \otimes h) = a^{[0]} \otimes a^{[1]} S(h), \]

Which are the same expressions for the Hopf algebroid structure on $A \otimes H$ with the simplification $1^{[0]} \otimes 1^{[1]} = 1_A \otimes 1_H$.

On the other hand, if $A \otimes H = A \otimes H$, then $\overline{\pi}(1_A) = 1_A \otimes 1_H$ which makes $\overline{\pi}$ into a global coaction. \hfill \Box

4.2.3. A dual version of a theorem by Kellendonk and Lawson. Note that we have a morphism of algebras

\[ F : H \to A \otimes H \]
\[ h \to 1^{[0]} \otimes h^{[1]} \]

It is easy to show that this map also satisfies

1. $\tilde{\epsilon} \circ F = \eta_A \circ \epsilon$.
2. $\tilde{\Delta} \circ F = \pi \circ (F \otimes F) \circ \Delta$.

where the map $\pi : (A \otimes H) \otimes (A \otimes H) \to (A \otimes H) \otimes_A (A \otimes H)$ is the natural projection. That is equivalent to say that the map $F$ is a morphism of Hopf algebroids with different base algebras, considering the Hopf algebra $H$ as a Hopf algebroid having as base algebra the base field $k$.

Finally, note that, the Hopf algebroid, $A \otimes H$ is totally defined by the image of the source map and the image of the map $F$. Indeed, a general element of $A \otimes H$ can be written as

\[ \sum_i a^i 1^{[0]} \otimes h^i^{[1]} = \sum_i s(a^i) F(h^i). \]

Given a commutative Hopf algebra $H$ and a commutative $A$ Hopf algebroid $\mathcal{H}$ such that there exists an algebra morphism $F : H \to \mathcal{H}$, one can construct out of these data an algebra morphism

\[ \Pi : A \otimes H \to \mathcal{H} \]
\[ a \otimes h \to s(a) F(h) \]

In the case of the Hopf algebroid $A \otimes H$, the morphism $\Pi$ is surjective and splits by the canonical inclusion. This motivates the following definition:

**Definition 4.17.** Given a commutative Hopf algebra $H$ and a commutative $A$ Hopf algebroid $\mathcal{H}$, an algebra morphism $F : H \to \mathcal{H}$ is said to be right dual star injective, if

(DSI1) The map $F$ is a morphism of Hopf algebroids, considering $H$ as a Hopf algebroid over the base field $k$.

(DSI2) The Hopf algebroid $\mathcal{H}$ is generated, as algebra, by the image of the source map and the image of $F$. That is $\mathcal{H} = s(A) F(H)$.

(DSI3) The surjective algebra map $\Pi : A \otimes H \to \mathcal{H}$ given by $\Pi(a \otimes h) = s(a) F(h)$ splits as an algebra morphism.

Obviously, there is a left version, by modifying the item (DSI3) by taking the algebra map $\Pi' : H \otimes A \to \mathcal{H}$. The origin of the name “dual star injective” will be explained later with the example. The next result shows a deeper connection between partial coactions of commutative Hopf algebras and commutative Hopf algebroids.

**Theorem 4.18.** Let $H$ be a commutative Hopf algebra, $\mathcal{H}$ be a commutative $A$ Hopf algebroid and $F : H \to \mathcal{H}$ be a right dual star injective algebra morphism. Then $A$ is a right partial $H$ comodule algebra. The partial coaction defined on $A$ is global if, and only if, the morphism $\Pi : A \otimes H \to \mathcal{H}$ is an isomorphism.
The projection $\sigma$ image of $\sigma$ we have the following expression for $\sigma$ morphisms of $A - H$ bimodules. Then for any element

$$x = \sum_i s(a^i)F(h^i) \in \mathcal{H},$$

we have the following expression for $\sigma(x)$,

$$\sigma(x) = \sum_i \sigma(s(a^i)F(h^i)) = \sum_i \sigma(a^i \cdot 1_H \cdot h^i) = \sum_i a^i \cdot \sigma(1_H) \cdot h^i.$$

Writing $\sigma(1_H) = 1^{[0]} \otimes 1^{[1]}$, which is an idempotent in $A \otimes H$, we have that any element in the image of $\sigma$ is of the form

$$\sigma(x) = \sum_i a^i 1^{[0]} \otimes h^i 1^{[1]}.$$

The projection $\sigma \circ \Pi : A \otimes H \to \sigma(\mathcal{H})$ is implemented by the multiplication by the idempotent $\sigma(1_H) = 1^{[0]} \otimes 1^{[1]}$.

For any $a \in A$ we have, obviously, $\sigma(s(a)) = a 1^{[0]} \otimes 1^{[1]}$ and let us denote

$$\bar{\rho}(a) = \sigma(t(a)) = a^{[0]} \otimes a^{[1]}.$$

The morphism $\bar{\rho}$ is automatically an algebra morphism, because $\sigma$ and $t$ are morphisms of algebras. We shall prove that this morphism $\bar{\rho}$ is a right partial coaction of the Hopf algebra $H$ on the algebra $A$.

Let $\theta : \sigma(\mathcal{H}) \otimes_A \sigma(\mathcal{H}) \to \sigma(\mathcal{H}) \otimes H$ be the linear map given by

$$\theta(a 1^{[0]} \otimes h 1^{[1]} \otimes A h 1^{[0]} \otimes k 1^{[1]}) = ak 1^{[0]} 1^{[0]} \otimes h 1^{[1]} 1^{[1]} \otimes k.$$

One can easily prove that $\theta$ is also an algebra map.

In order to prove that $\bar{\rho}$ is a partial coaction, we need to verify two identities:

(I) $\epsilon_H \circ \Pi = I \otimes \epsilon_H$.

(II) $\theta \circ (\sigma \otimes \sigma) \circ \Delta_H \circ \Pi = ((\sigma \circ \Pi) \otimes I) \circ (I \otimes \Delta_H)$

For the identity (I), take $a \otimes h \in A \otimes H$, then

$$\epsilon_H \circ \Pi(a \otimes h) = \epsilon_H(s(a)F(h)) = \epsilon_H(a \cdot F(h)) = a(\epsilon_H(F(h))) = a(\eta_A(\epsilon_H(h))) = a \epsilon_H(h) = (I \otimes \epsilon_H)(a \otimes h).$$

And for the identity (II),

$$\theta \circ (\sigma \otimes \sigma) \circ \Delta_H \circ \Pi(a \otimes h) = \theta \circ (\sigma \otimes \sigma) \circ \Delta_H(s(a)F(h)) = \theta \circ (\sigma \otimes \sigma) \circ \Delta_H(a \cdot F(h)) = a \cdot (\theta \circ (\sigma \otimes \sigma) \circ \Delta_H(F(h))) = a \cdot (\theta \circ (\sigma \otimes \sigma)(F(h_{(1)}) \otimes_A F(h_{(2)}))) = a \cdot \theta(1^{[0]} 1^{[1]} h_{(1)} \otimes_A 1^{[0]} 1^{[1]} h_{(2)}) = a \cdot (1^{[0]} 1^{[1]} h_{(1)} \otimes h_{(2)}) = a 1^{[0]} 1^{[1]} h_{(1)} \otimes h_{(2)} = \sigma \circ \Pi(a \otimes h_{(1)}) \otimes h_{(2)} = ((\sigma \circ \Pi) \otimes I) \circ (I \otimes \Delta_H)(a \otimes h).$$
Then we are in position to verify the axioms of a right partial coaction for the map $\overline{\rho} = \sigma \circ t : A \to \sigma(\mathcal{H}) \subseteq A \otimes H$. The axiom (PRHCA1) is automatically satisfied because both $\sigma$ and $t$ are algebra maps, therefore $\overline{\rho}$ is an algebra map.

For the axiom (PRHCA2) take $a \in A$, then

$$(I \otimes \epsilon_H) \circ \overline{\rho}(a) = (I \otimes \epsilon_H) \circ \sigma(t(a))$$

$$= \epsilon_H \circ \Pi \circ \sigma(t(a))$$

$$= \epsilon_H(t(a))$$

$$= \epsilon_H(1_H \cdot a)$$

$$= \epsilon_H(1_H) a = a.$$ 

For the axiom (PRHCA3) we first note that $$(\overline{\rho} \otimes I) \circ \overline{\rho} = \theta \circ (\sigma \otimes \sigma) \circ \Delta_H \circ t.$$ Indeed, for $a \in A$

$$\theta \circ (\sigma \otimes \sigma) \circ \Delta_H(t(a)) = \theta \circ (\sigma \otimes \sigma)(\Delta_H(1_H \cdot a))$$

$$= \theta \circ (\sigma \otimes \sigma)((1_H \otimes_A 1_H) \cdot a)$$

$$= \theta(1^0 \otimes 1^{[1]} \otimes_A a^{[0]} \otimes a^{[1]})$$

$$= 1^0 a^{[0][0]} \otimes 1^{[1]} a^{[0][1]} \otimes a^{[1]}$$

$$= a^{[0][0]} \otimes a^{[0][1]} \otimes a^{[1]}$$

$$= (\overline{\rho} \otimes I)(a^{[0]} \otimes a^{[1]}).$$

On the other hand,

$$\theta \circ (\sigma \otimes \sigma) \circ \Delta_H(t(a)) = \theta \circ (\sigma \otimes \sigma)(\Delta_H(1_H \cdot a))$$

$$= ((\sigma \otimes \Pi) \otimes I) \circ (I \otimes \Delta_H) \circ \sigma(t(a))$$

$$= ((\sigma \otimes \Pi) \otimes I) \circ (I \otimes \Delta_H) \circ \overline{\rho}(a)$$

$$= (\sigma(1_H) \otimes 1_H)(I \otimes \Delta_H) \circ \overline{\rho}(a)$$

$$= (\overline{\rho}(1_A) \otimes 1_H)(I \otimes \Delta_H) \circ \overline{\rho}(a).$$

Therefore, $$(\overline{\rho} \otimes I) \circ \overline{\rho}(a) = (\overline{\rho}(1_A) \otimes 1_H)(I \otimes \Delta_H) \circ \overline{\rho}(a)$$ for any $a \in A$. This concludes the proof that $\overline{\rho}$ is a right partial coaction.

If the partial coaction $\overline{\rho} = \sigma \circ t$ is global, then $\overline{\rho}(1_A) = 1^0 \otimes 1^{[1]} = 1_A \otimes 1_H$. Then, we have for any $a \otimes h \in A \otimes H$

$$\sigma \circ \pi(a \otimes h) = a1^0 \otimes h1^{[1]} = a \otimes h$$

Therefore, $\sigma = \Pi^{-1}$, which implies that $\mathcal{H} \cong A \otimes H$.

On the other hand, if $\Pi : A \otimes H \to \mathcal{H}$ is an isomorphism, then the projection $\sigma \circ \Pi = \text{Id}_{A \otimes H}$, on the other hand

$$(\overline{\rho} \otimes I) \circ \overline{\rho}(a) = \theta \circ (\sigma \otimes \sigma) \circ \Delta_H(t(a)) = ((\sigma \otimes \Pi) \otimes I) \circ (I \otimes \Delta_H) \circ \overline{\rho}(a) = (I \otimes \Delta_H) \circ \overline{\rho}(a).$$

Therefore, the partial coaction $\overline{\rho}$ is global. \qed

The results above can be interpreted as a dual version of the classical result which establishes a correspondence between partial actions of a given group $G$ on sets, and groupoids which admits a star injective functor to the group, viewed as a one object groupoid $\mathbb{I}$. 

**Definition 4.19.** [11] 1) Let $C$ be a small category and $x$ be an object in $C$. The star over $x$ is the set $S(x) = \{ f \in \text{Hom}_C(x, y) \mid y \in C \}$.

2) A functor $F : C \to D$ between two small categories is said to be star injective (surjective) if for every $x \in C$, the function $F|_{S(x)}$ is injective (surjective).
To make the statement more precise, consider a partial action of a group \( G \) on a set \( X \), given by the data \( \{(X_g \subseteq X)_{g \in G}, \{\alpha_g : X_{g^{-1}} \to X_g\}_{g \in G}\} \). From these data it is possible to construct a groupoid
\[
\mathcal{G}(G, X, \alpha) = \{(x, g) \in X \times G \mid x \in X_g\}
\]
The objects in this groupoid are the elements of \( X \). The groupoid structure is given by the source and target maps,
\[
s(x, g) = \alpha_{g^{-1}}(x), \quad t(x, g) = x,
\]
the product
\[
(x, g)(y, h) = \begin{cases} (x, gh) & \text{for } y = \alpha_{g^{-1}}(x) \\ - & \text{otherwise}, \end{cases}
\]
and the inverse
\[
(x, g)^{-1} = (\alpha_{g^{-1}}(x), g^{-1}).
\]

It is easy to see that the projection map
\[
\pi_2 : \mathcal{G}(G, X, \alpha) \to G \quad (x, g) \mapsto g
\]
is a functor, considering \( G \) as a groupoid with one single element. It can be easily verified that this functor is star injective, which in this context means that given \( x \in X \) and \( g \in G \) there exists at most one element \( \gamma \in \mathcal{G}(G, X, \alpha) \) such that \( s(\gamma) = x \) and \( \pi_2(\gamma) = g \) (by the way, this element exists only when \( x \in X_{g^{-1}} \), and it is written explicitly as \( \gamma = (\alpha_g(x), g) \)). The functor \( \pi_2 \) is star surjective if, and only if, the action of \( G \) on \( X \) is global \([11]\).

On the other hand, given a groupoid \( \mathcal{G} \) and a star injective functor \( F : \mathcal{G} \to G \), one can construct a partial action of the group \( G \) on the set \( X \) of objects of \( \mathcal{G} \). For this, we define for each \( g \in G \) the subset
\[
X_g = \{x \in X \mid \exists \gamma \in \mathcal{G}, s(\gamma) = x, \ F(\gamma) = g^{-1}\}.
\]
And the partially defined bijections \( \alpha_g : X_{g^{-1}} \to X_g \) given by
\[
\alpha_g(x) = t(\gamma), \quad \text{such that } s(\gamma) = x, \text{ and } F(\gamma) = g.
\]
Because of the star injectivity of \( F \), it is easy to prove that \( \alpha_g \) is well defined as a function and it is bijective, the proof that this is indeed a partial action can be found in the reference \([11]\).

This defined partial action is global if, and only if, the functor \( F \) is star surjective \([11]\).

**Example 4.20.** In order to relate our results with partial actions of groups, consider a partial action \( \alpha \) of a finite group \( G \) on a finite set \( X \). Consider the right partial coaction of the Hopf algebra \( H = (kG)^* \) on the algebra \( A = \text{Fun}(X, k) \) as defined by the formula
\[
\overline{\rho}(f)(p, g) = \begin{cases} f(\alpha_{g^{-1}}(p)) & \text{for } p \in X_g \\ 0 & \text{otherwise} \end{cases}
\]
denoting the basis of \( (kG)^* \) by \( \{p_g\}_{g \in G} \), where \( p_g(h) = \delta_{g,h} \) and denoting the characteristic functions of \( X_g \) by \( 1_g \), then the partial coaction can be written as
\[
\overline{\rho}(f) = \sum_{g \in G} 1_g \cdot (f \circ \alpha_{g^{-1}}) \otimes p_g
\]
The Hopf algebroid \( A \otimes H \) is exactly the Hopf algebroid of functions on the groupoid \( \mathcal{G}(G, X, \alpha) \).

Indeed, first note that
\[
\overline{\rho}(1_A) = \sum_{g \in G} 1_g \otimes p_g,
\]
then, an element \( (a \otimes h)(\overline{\rho}(1_A)) \in A \otimes H \) is a function defined on \( \mathcal{G}(G, X, \alpha) \), because if evaluated on a pair \( (x, g) \) it will be nonzero only if \( x \in X_g \). On the other hand,
\[
\text{Fun}(\mathcal{G}(G, X, \alpha), k) \subseteq \text{Fun}(X \times G, k) \cong A \otimes H.
\]
denoting by $\chi_x \in \text{Fun}(X, k)$ the characteristic function of the element $x \in X$, then any $\varphi \in \text{Fun}(G(G, X, \alpha), k)$ can be written as

$$\varphi = \sum_{x \in X, g \in G} a_{x,g} \chi_x \otimes p_g,$$

with $a_{x,g} \in k$, for every $x \in X$, $g \in G$. But, as $\varphi$ is only defined on the groupoid $G(G, X, \alpha)$ then $a_{x,g} = 0$ if $x \notin X_g$, therefore

$$\varphi = \sum_{g \in G} \left( \sum_{x \in X_g} a_{x,g} \chi_x \right) 1_g \otimes p_g \in (A \otimes H)\mathfrak{p}(1_A) = A \otimes H.$$

On the other hand, considering a star injective functor $F : G \to G$, where $G$ is a finite groupoid and $G$ is a finite group. Then the functor $F$ induces an algebra morphism $\hat{F}$ between the Hopf algebra $H = (kG)^*$ and the Hopf algebroid $\mathcal{H} = \text{Fun}(G, k)$ satisfying $\hat{F}(f)(\gamma) = f(F(\gamma))$, for all $f \in (kG)^*$ and $\gamma \in G$. The functoriality of $F$ implies directly the item DSI1 of the definition of a dual star injective algebra morphism.

For the item DSI2, note that, for a finite groupoid $G$ its algebra of functions has a basis of characteristic functions $\{\chi_{\gamma} | \gamma \in G\}$. Let $\gamma \in G$ denote by $x = s(\gamma)$ and $g = F(\gamma)$. Then, it is easy to see that the characteristic function $\chi_{\gamma}$ can be written as

$$\chi_{\gamma} = s(\chi_x)\hat{F}(p_g) = \Pi(\chi_x \otimes p_g).$$

Finally, for the item DSI3, consider the map $\sigma : \mathcal{H} \to A \otimes H$ given by

$$\sigma(f) = \sum_{\gamma \in G} f(\gamma) \chi_{t(\gamma)} \otimes p_F(\gamma).$$

It is easy to see that $\Pi \circ \sigma = \text{Id}_{\mathcal{H}}$. Therefore, the morphism $\hat{F}$ is dual star injective.

4.2.4. A duality result. Given two dually paired Hopf algebras, $H$ and $K$, a right partial comodule algebra $A$ is automatically a left partial $H$ module algebra. In the special case when $H$ is a co-commutative Hopf algebra, and consequently $K$ is commutative, and $A$ is a commutative algebra, then the duality between partial actions and partial coactions can be viewed as a duality between bialgebroids. In order to make this statement more precise, let us define what is a skew pairing between left $A$ bialgebroids (for $A$ not necessarily commutative).

**Definition 4.21.** (See reference [12]) Let $A$ be a $k$ algebra and $(\Lambda, s, t, \Delta, \varepsilon)$ and $(L, s, t, \Delta, \varepsilon)$ be two left $A$ bialgebroids. A skew pairing between $\Lambda$ and $L$ is a $k$ linear map $\langle \langle \cdot, \cdot \rangle \rangle : \Lambda \otimes_k L \to A$ satisfying

(SP1) $\langle \langle s(a)t(b)\xi s(c)t(d)\ell|e \rangle \rangle = a\langle \langle \xi|s(c)t(e)\ell s(d)t(b) \rangle \rangle$ for every $\xi \in \Lambda$, $\ell \in L$, and $a, b, c, d, e \in A$.

(SP2) $\langle \langle \xi|m \ell \rangle \rangle = \langle \langle \xi|l(\ell) (m) \ell \rangle \rangle$, for every $\xi \in \Lambda$ and $\ell, m \in L$.

(SP3) $\langle \langle \xi|\ell \rangle \rangle = \langle \langle \xi|l(\ell(1)) \ell(2) \rangle \rangle$, for every $\xi, \ell, \ell \in \Lambda$ and $\ell \in L$.

(SP4) $\langle \langle 1|L \rangle \rangle = \varepsilon(\ell)$ for every $\xi \in \Lambda$.

(SP5) $\langle \langle 1A|\ell \rangle \rangle = \ell(\ell)$ for every $\ell \in L$.

We know are ready to state the announced duality between our constructed Hopf algebroids. Remark that this duality is exactly the pairing between the $A$-coring and $A$-ring of Lemma 4.6 in the non-commutative case (where there is no Hopf algebroid structure).

**Theorem 4.22.** Let $H$ be a co-commutative Hopf algebra, $K$ be a commutative Hopf algebra with a dual pairing $\langle \cdot, \cdot \rangle : H \otimes K \to k$, and let $A$ be a commutative algebra which is, at the same time, a left partial $H$ module algebra and a right partial comodule algebra with the compatibility condition $h \cdot a = a^{[0]}(h, a^{[1]})$. Then the map

$$\langle \langle \cdot, \cdot \rangle : A \otimes_k H \otimes_k A \to A$$

$$b^{[0]} \otimes \xi^{[1]}(a \# h) \otimes_k (a \# h) \mapsto ab(h^{[1]} \cdot 1_A)(h^{[2]}, \xi)$$

is a skew pairing between the $A$ bialgebroids $A \otimes H$ and $A \# H$. 
Proof. First, note that the same map can be written as

\[ \langle a_{1}^{[0]} \otimes \xi_{1}^{[1]}|b \# k \rangle = ab_{1}^{[0]}(h, \xi_{1}^{[1]}) \]

that is because the compatibility condition between the left partial H-module structure and the right partial H-comodule structure on \( A \). It is easy to see that the map \( \langle \cdot, \cdot \rangle \) is a well defined, indeed, as we know, in the partial smash product the equality \( b \# h = b(h_{(1)} \cdot 1_{A}) \# h_{(2)} \) holds, then, for any \( a_{1}^{[0]} \otimes \xi_{1}^{[1]} \in A \otimes H \) we have

\[ \langle a_{1}^{[0]} \otimes \xi_{1}^{[1]}|b(h_{(1)} \cdot 1_{A}) \# h_{(2)} \rangle = ab(h_{(1)} \cdot 1_{A})\langle h_{(2)}, \xi_{1}^{[1]} \rangle = \langle a_{1}^{[0]} \otimes \xi_{1}^{[1]}|a \# h \rangle. \]

Let us verify the axiom (SP1). As the bialgebroid \( A \otimes H \) is commutative, this enables a simplification, because, for each \( a_{1}, a_{2} \in A \) and \( x \in A \otimes H \) we have \( s(a_{1})t(a_{2})x = xs(a_{1})t(a_{2}) \). Then, taking \( a_{1}, a_{2}, a_{3} \in A, b_{1}^{[0]} \otimes \xi_{1}^{[1]} \in A \otimes H \) and \( c \# h \in A \# H \), we have

\[ \langle s(a_{1})t(a_{2})(b_{1}^{[0]} \otimes \xi_{1}^{[1]})|c \# h \rangle a_{3} = \langle (a_{1}b_{2}^{[0]} a_{1}^{[0]} \otimes a_{2}^{[1]} \xi_{1}^{[1]})|c \# h \rangle a_{3} = a_{1}bca_{3}(h_{(1)} \cdot 1_{A})a_{2}^{[0]} \langle h_{(2)}, a_{2}^{[1]} \xi \rangle = a_{1}bca_{3}(h_{(1)} \cdot 1_{A})a_{2}^{[0]} \langle h_{(2)}, a_{2}^{[1]} \rangle \langle h_{(3)}, \xi \rangle = a_{1}bca_{3}(h_{(1)} \cdot 1_{A})a_{2} \langle h_{(2)}, \xi \rangle = a_{1}bca_{3}(h_{(1)} \cdot a_{2}) \langle h_{(2)}, \xi \rangle = a_{1}\langle b_{1}^{[0]} \otimes \xi_{1}^{[1]}|a_{3}c(h_{(1)} \cdot a_{2}) \# h_{(2)} \rangle = a_{1}\langle b_{1}^{[0]} \otimes \xi_{1}^{[1]}|s(a_{3})(c \# h)(a_{2} \# 1_{H}) \rangle = a_{1}\langle b_{1}^{[0]} \otimes \xi_{1}^{[1]}|s(a_{3})(c \# h)(t(a_{2})) \rangle \]

The last expression is also equal to \( \langle b_{1}^{[0]} \otimes \xi_{1}^{[1]}|s(a_{1})s(a_{3})(c \# h)t(a_{2}) \rangle \).

For the axiom (SP2), take \( a_{1}^{[0]} \otimes \xi_{1}^{[1]} \in A \otimes H \), and \( b \# h, c \# k \in A \# H \), then

\[ \langle a_{1}^{[0]} \otimes \xi_{1}^{[1]}|(b \# h)(c \# k) \rangle = \langle a_{1}^{[0]} \otimes \xi_{1}^{[1]}|(b(h_{(1)} \cdot c) \# (h_{(2)}k_{(1)} \cdot 1_{A})) \rangle \langle h_{(2)}k_{(1)}, \xi \rangle \]

while, on the other hand

\[ \langle a_{1}^{[0]} \otimes \xi_{1}^{[1]}|(b \# h)(c(k_{(1)} \cdot 1_{A}) \# 1_{H}) \rangle \rangle h_{(2)}k_{(1)}, \xi \rangle \]

For the axiom (SP3), take \( a_{1}^{[0]} \otimes \xi_{1}^{[1]}, b_{1}^{[0]} \otimes \xi_{1}^{[1]} \in A \otimes H \), and \( c \# h \in A \# H \), then

\[ \langle (a_{1}^{[0]} \otimes \xi_{1}^{[1]})(b_{1}^{[0]} \otimes \xi_{1}^{[1]})|c \# h \rangle = \langle (ab_{1}^{[0]} \otimes \xi_{1}^{[1]})(c \# h) \rangle = abc(h_{(1)} \cdot 1_{A}) \langle h_{(2)}, \xi \rangle, \]
while, on the other hand
\[ \langle a1^0 \otimes \xi^1 | 1_A \# 1_H \rangle = a(1_H, \xi) = \epsilon(a) = \langle a1^0 \otimes \xi^1 | 1_A \# 1_H \rangle. \]

Finally, for (SP5), take \( a \# h \in A \# H \), then
\[ \langle 1^0 \otimes 1^1 | a \# h \rangle = a(h(1) \cdot 1_A)(h(2), 1_K) = a(h(1) \cdot 1_A)(h(2)) = a(h \cdot 1_A) = \epsilon(a \# h). \]

Therefore, the map \( \langle , \rangle \) is a skew pairing between these two A bialgebroids.

\[ \square \]

5. Partial module coalgebras

5.1. Definition and examples.

**Definition 5.1.** A \( k \) coalgebra \( C \) is said to be a left partial \( H \) module coalgebra if there is a linear map
\[ \gamma : \ H \otimes C \rightarrow C \]

\[ H \otimes c \rightarrow h \cdot c \]

satisfying the following conditions:

(P LHMC1) For all \( h \in H \) and \( c \in C \), \( \Delta(h \cdot c) = (h(1) \cdot c(1)) \otimes (h(2) \cdot c(2)). \)

(P LHMC2) \( 1_H \cdot c = c \), for all \( c \in C \).

(P LHMC3) For all \( h, k \in H \) and \( c \in C \),
\[ h \cdot (k \cdot c) = (hk(1) \cdot c(1))\epsilon(k(2) \cdot c(2)) \]

It is called symmetric, if, in addition, it satisfies

(P LHMC3') For all \( h, k \in H \) and \( c \in C \),
\[ h \cdot (k \cdot c) = \epsilon(hk(1) \cdot c(1))(hk(2) \cdot c(2)) \]

In a symmetric way, it is possible to define the notion of a right partial \( H \) module coalgebra. We have the following immediate results.

**Proposition 5.2.** Let \( H \) be a bialgebra and \( C \) be a left partial \( H \) module coalgebra, then,

(i) for every \( h \in H \) and \( c \in C \), we have
\[ h \cdot c = \epsilon(h(1) \cdot c(1))(h(2) \cdot c(2)) = (h(1) \cdot c(1))\epsilon(h(2) \cdot c(2)). \]

(ii) for every \( h \in H \) and \( c \in C \), we have
\[ \epsilon(h \cdot c) = \epsilon(h(1) \cdot c(1))\epsilon(h(2) \cdot c(2)) \]

(iii) \( C \) is a left \( H \) module coalgebra if, and only if, for every \( h \in H \) and \( c \in C \), we have
\[ \epsilon(h \cdot c) = \epsilon(h)\epsilon(c). \]

**Proof.** (i). This follows immediately from the axiom (PLHMC1). By applying \( \epsilon \otimes I \), we obtain the first equality, and by applying \( I \otimes \epsilon \) we obtain the second equality.

(ii) follows from (i).

(iii). If \( C \) is a left \( H \) module coalgebra, then, by definition, \( \epsilon(h \cdot c) = \epsilon(h)\epsilon(c) \). Now suppose that this equality happens for every \( h \in H \) and \( c \in C \), then, by (PLHMC3) we have
\[ h \cdot k \cdot c = (hk(1) \cdot c(1))\epsilon(k(2) \cdot c(2)) \]
\[ = (hk(1) \cdot c(1))\epsilon(k(2))\epsilon(c(2)) \]
\[ = hk \cdot c, \]
Lemma 5.4. For a coalgebra \( C \) and a coalgebra morphism \( C \to H \), there exists a subcoalgebra \( D \) of \( C \) such that the universal \( \epsilon: \) is an isomorphism of coalgebras.

Remark 5.3. It is important to notice that a different notion of “partial module coalgebra” was introduced earlier in the arXiv-version of [5]. There, a coalgebra \( C \) was said to be a (right) \( H \)-module coalgebra if and only if there exists a map \( C \to H \), \( c \mapsto h \cdot c \), such that the map \( H \to C \to C \to H \), \( c \otimes h \mapsto h(1) \otimes c \cdot h(2) \), is a partial entwining structure. Clearly, this definition, as does ours, generalizes usual module coalgebras. However, the definition introduced in this note is motivated by the examples of partial actions of groups on coalgebras and by duality results with partial module algebras as illustrated below. Moreover, partial module coalgebras in the sense defined here are closely related to the coalgebra objects in the monoidal category of partial modules.

Our next aim is to define a partial action of a group on a coalgebra. Recall that in the case of partial actions of a group on an algebra, we considered ideals \( A \) that are generated by central idempotents \( 1 \). The existence of a central idempotent ensures that \( A \cong A \times A' \), for some other unitary algebra \( A' \). We will now prove a similar result for partial module coalgebras.

Recall (see e.g. [6, Proposition 1.4.19]) that the category of coalgebras has coproducts. In particular, if \( D \) and \( D' \) are two coalgebras, then their coproduct \( D \coprod D' \) in the category of coalgebras is computed by taking the direct sum of the underlying \( k \)-modules \( D \oplus D' \) and endowing it with the following comultiplication and counit

\[
\Delta(d, d') = (d(1), 0) \otimes (d(2), 0) + (0, d'(1)) \otimes (0, d'(2))
\]

\[
\epsilon(d, d') = \epsilon_D(d)\epsilon_{D'}(d')
\]

for all \( d, d' \in D \).

Lemma 5.4. For a coalgebra \( C \) and a coalgebra map \( \iota : D \to C \), the following are equivalent:

(i) there exists a coalgebra \( D' \) and a coalgebra map \( \iota' : D' \to C \) such that the universal morphism \( J : D \coprod D' \to C \) is an isomorphism of coalgebras.

(ii) there is a \( k \)-linear map \( P : C \to D \) that satisfies

- \( P \circ \iota = \iota_D \)
- \( \Delta \circ P = (P \otimes P) \circ \Delta \)
- \( \iota(P(c)) = c(1)\epsilon_D(P(c(1))) = \epsilon_D(P(c(1)))c(2), \) all \( c \in C \)

(iii) There is a \( k \)-linear map \( P : C \to D \) such that

- \( P \circ \iota = \iota_D \)
- \( \Delta(\iota(P(c))) = c(1) \otimes \iota(P(c(2))) = \iota(P(c(1))) \otimes c(2) = \iota(P(c(1))) \otimes \iota(P(c(2))), \) all \( c \in C \)

Proof. (i) \( \Rightarrow \) (ii). It is easy to verify that the canonical projection \( D \oplus D' \to D \) satisfies all stated conditions.

(ii) \( \Rightarrow \) (iii). We only have to prove the third condition. Take any \( c \in C \) and use the fact that \( \iota \) is a coalgebra morphism, then we find

\[
\Delta(\iota(P(c))) = \iota(P(c(1))) \otimes \iota(P(c(2))) = c(1)\epsilon_D(P(c(2))) \otimes \epsilon(D(P(c(3)) = c(1) \otimes \iota\circ P(c(2))
\]

(iii) \( \Rightarrow \) (i). Define \( D' = \ker P \). Then clearly \( C \cong D \oplus D' \) as \( k \)-modules. Moreover, using the second and third condition one can easily verify that \( D' \) is a subcoalgebra of \( C \). For any \( c \in C \), we can write \( c = \iota \circ P(c) + c - \iota \circ P(c) = \iota(d) + \iota'(d') \) where \( d = P(c), d' = c - \iota \circ P(c) \) and \( \iota' : D' \to C \) is the canonical injection. We conclude that \( C \cong D \coprod D' \).

Definition 5.5. (Partial group actions on coalgebras) A partial action of a group \( G \) on a coalgebra \( C \) consists of a family of subcoalgebras \( \{C_g\}_{g \in G} \) of \( C \) and coalgebra isomorphisms \( \theta_g : C \to C_g \), \( g \in G \), such that

(i) for every \( g \in G \), the coalgebra \( C_g \) is a subcoalgebra-direct summand of \( C \), i.e. there exists a projection \( P_g : C \to C_g \) satisfying the (equivalent) conditions of Lemma 5.4.
(ii) \( C_g = C \) and \( \theta_g = P_g = \text{id}_C \), where \( e \) is the unit of \( C \);

(iii) For all \( g, h \in G \), we the following equations hold

\[
F_h \circ F_g = F_g \circ F_h \tag{9}
\]

\[
\theta_{h^{-1}} \circ F_{g^{-1}} \circ P_h = F_{(gh)^{-1}} \circ \theta_{h^{-1}} \circ P_h \tag{10}
\]

\[
\theta_g \circ \theta_h \circ P_{h^{-1}} \circ F_{(gh)^{-1}} = \theta_{gh} \circ P_{h^{-1}} \circ F_{(gh)^{-1}} \tag{11}
\]

Remark 5.6. The three equalities involving the projections in Definition 5.3 were motivated by dualizing a partial action of \( G \) over an algebra \( A \) in which every ideal \( A_g \) is generated by a central idempotent \( e_g \) dualized by suppressing the existence of the projection \( P_g \). The fact that the idempotents mutually commute (i.e. they are central) is now expressed in the equality \( \tag{9} \). The equation \( \tag{10} \) is basically saying that \( \theta_{h^{-1}}(C_h \cap C_{g^{-1}}) = C_{(gh)^{-1}} \cap C_{h^{-1}} \) which is the second axiom of partial action. The equality \( \tag{11} \) reflects the fact that for any \( x \in C_{(gh)^{-1}} \cap C_{h^{-1}} \), we have \( \theta_g \circ \theta_h(x) = \theta_{gh}(x) \). We remark as well that thanks to conditions \( \tag{9} \) and \( \tag{10} \), the equality \( \tag{11} \) makes sense, in particular, one checks that the image of \( \theta_h \circ P_{h^{-1}} \circ F_{(gh)^{-1}} \) is included in the image of \( F_{g^{-1}} \).

Theorem 5.7. Let \( C \) be a \( k \)-coalgebra and \( G \) a group. Then there is a bijective correspondence between partial actions of \( G \) on the coalgebra \( C \) and maps \( kG \otimes C \to C \) that turn \( C \) into a symmetric partial \( kG \) module coalgebra.

Proof. Suppose that \( C \) is a partial \( kG \) module coalgebra. For any \( g \in G \) we denote \( \delta_g \in kG \) the corresponding base element. Then for any \( c \in C \), we find

\[
\delta_g \cdot (\delta_{g^{-1}} \cdot c) = \epsilon(\delta_{g^{-1}} \cdot c(1))\delta_g\delta_{g^{-1}} \cdot c(2) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) = c(1)\epsilon(\delta_{g^{-1}} \cdot c(2))
\]

where we used (PLHMC3) and (PLHMC3’). We then define \( P_g : C \to C \), as

\[
P_g(c) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2), \tag{12}
\]

and because of the symmetry of the partial action, we can also write

\[
P_g(c) = c(1)\epsilon(\delta_{g^{-1}} \cdot c(2)). \tag{13}
\]

It is easy to see that these linear operators are projections, and upon these projections we can define \( C_g = \text{Im}P_g \). Let us observe that \( C_g \) is a subcoalgebra of \( C \).

\[
\Delta(P_g(c)) = \Delta(\epsilon(\delta_{g^{-1}} \cdot c(1))c(2)) = \epsilon(\delta_{g^{-1}} \cdot c(1))\epsilon(\delta_{g^{-1}} \cdot c(2))c(3) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) \otimes c(3) \otimes c(4)
\]

\[
= \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) \otimes c(3) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) \otimes c(3) = P_g(c(1)) \otimes P_g(c(2)) \in C_g \otimes C_g
\]

For the counit condition, we find

\[
P_g(c(1))\epsilon(P_g(c(2))) = \epsilon(\delta_{g^{-1}} \cdot c(1))\epsilon(c(2))c(3) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) = P_g(c)
\]

and by symmetry we also have \( \epsilon(P_g(c(1)))P_g(c(2)) = P_g(c) \).

Let us show that \( P_g \) satisfies the conditions of Lemma 5.4\( \text{(ii)} \). In order to see that \( \Delta \circ P_g = (P_g \otimes P_g) \circ \Delta \) we use the both expressions \( \tag{12} \) and \( \tag{13} \), indeed,

\[
(P_g \otimes P_g) \circ \Delta(c) = P_g(c(1)) \otimes P_g(c(2)) = c(1)\epsilon(\delta_{g^{-1}} \cdot c(2)) \otimes c(3) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) \otimes c(3)
\]

\[
= \Delta(\epsilon(\delta_{g^{-1}} \cdot c(1))c(2)) = \Delta(P_g(c)).
\]

Finally, for all \( g \in G \) and \( c \in C \) we have

\[
\epsilon(P_g(c(1)))c(2) = \epsilon(\delta_{g^{-1}} \cdot c(1))\epsilon(c(2))c(3) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2) = P_g(c).
\]

With the expression \( \tag{13} \) we can prove also that \( P_g(c) = c(1)\epsilon(P_g(c(2))) \).

Let us check that the projections mutually commute, i.e. \( \text{(ii)} \) is satisfied. Take \( c \in C \), then for all \( g, h \in C \) we find, using \( \tag{12} \) and \( \tag{13} \)

\[
P_g \circ P_h(c) = P_g(c(1)\epsilon(\delta_{h^{-1}} \cdot c(2))) = \epsilon(\delta_{g^{-1}} \cdot c(1))c(2)\epsilon(\delta_{h^{-1}} \cdot c(2)) = P_h \circ P_g(c).
\]
Now, define for any \( g \in G \) a map \( \theta_g : C_{g^{-1}} \to C_g \) by \( \theta_g(P_{g^{-1}}(c)) = \delta_g \cdot P_{g^{-1}}(c) \). Then we find for all \( g \in G \) and \( c \in C \) that
\[
\theta_g \circ P_{g^{-1}}(c) = \delta_g \cdot c_{(1)} \epsilon(\delta_g \cdot c_{(2)}) = \delta_g \cdot c.
\]
One can easily observe that \( C_c = C \) and \( \theta_c = id_C \).

Take \( x = P_h(c) \in C_h \), then we find
\[
P_{(gh)^{-1}} \circ \theta_{h^{-1}}(x) = P_{(gh)^{-1}}(\delta_{h^{-1}} \cdot x) = \delta_{h^{-1}} \cdot \epsilon(x_{(1)})(\delta_{gh} \cdot x_{(2)})
= \delta_{h^{-1}} \cdot \epsilon(x_{(1)})(\delta_{gh} \cdot x_{(2)}) = \delta_{h^{-1}} \cdot \epsilon(x_{(1)})(\delta_{gh} \cdot x_{(2)})
= \theta_{h^{-1}} \circ P_{g^{-1}}(x)
\]

So (10) is verified. Finally take any \( c \in C \) and let us check (11) (recall from Remark 5.6 that both expressions are well-defined), using (14)
\[
\theta_g \circ \theta_h \circ P_{h^{-1}} \circ P_{(gh)^{-1}}(c) = \theta_g \circ \theta_h \circ P_{h^{-1}}(c_{(1)}) \epsilon(\delta_{gh} \cdot c_{(2)}) = \theta_g(\delta_h \cdot c_{(1)}) \epsilon(\delta_{gh} \cdot c_{(2)})
= \epsilon(\delta_h \cdot c_{(1)})(\delta_{gh} \cdot c_{(2)}) = \theta_{gh} \circ P_{gh^{-1}} \circ P_{h^{-1}}(c)
\]

Conversely, suppose that we have a partial action \( \{C_g\}_{g \in G}, \{\theta_g : C_{g^{-1}} \to C_g \}_{g \in G} \) of \( G \) on \( C \). Then, we define a linear map \( \cdot : kG \otimes C \to C \) by \( \delta_g \cdot c = \theta_g(P_{g^{-1}}(c)) \) (and linear extension).

Thanks to axiom (ii) in Definition 5.5, we find for any \( c \in C \)
\[
1_{kG} \cdot c = \delta_c \cdot c = c.
\]

As both \( \theta_g \) and \( P_{g^{-1}} \) are comultiplicative, then it follows directly that
\[
\Delta(\delta_g \cdot c) = \Delta(\theta_g(P_{g^{-1}}(c))) = \theta_g(\epsilon(\delta_{1_{kG}} \cdot c_{(1)})) \otimes \theta_g(\epsilon(\delta_{1_{kG}} \cdot c_{(2)})) = \delta_g \cdot c_{(1)} \otimes \delta_g \cdot c_{(2)}.
\]

Next, let us first remark that
\[
\epsilon(\delta_g \cdot c_{(1)})(c_{(2)}) = \epsilon(\theta_g \circ P_{g^{-1}}(c_{(1)}))c_{(2)} = \epsilon(P_{g^{-1}}(c_{(1)}))c_{(2)} = P_{g^{-1}}(c),
\]
where we used that \( \theta_g \) is a coalgebra morphism in the second equality and the properties of the projection \( P_{g^{-1}} \), (see Lemma 5.4) in the third equality.

Then, for any \( g, h \in G \) and \( c \in C \) we have
\[
\delta_g \cdot (\delta_h \cdot c) = \theta_g \circ P_{g^{-1}} \circ \theta_h \circ P_{h^{-1}}(c) = \theta_g \circ \theta_h \circ P_{(gh)^{-1}} \circ P_{h^{-1}}(c)
= \theta_{gh} \circ P_{(gh)^{-1}} \circ P_{h^{-1}}(c) = \delta_{gh} \cdot (P_{h^{-1}}(c))
= \delta_{gh} \cdot c_{(1)}(\delta_g \cdot c_{(2)}).
\]

Here we used (10) in the second equality, (11) in the third equality and (15) in the last equality.

Along the lines, we have already verified that both constructions are mutual inverses. \( \square \)

**Example 5.8.** (Induced partial actions) Let \( C \) be a left \( H \) module coalgebra with the action \( \triangleright : H \otimes C \to C \). Consider \( D \subseteq C \) a subcoalgebra with a \( k \)-linear projection \( P : C \to D \) that is comultiplicative and satisfies
\[
P(c) = c_{(1)} \epsilon(P(c_{(2)})) = \epsilon(P(c_{(1)}))c_{(2)},
\]
for all \( c \in C \). Then the linear map defined by
\[
\cdot : H \otimes D \to D
h \otimes d \mapsto h \cdot d = P(h \triangleright d)
\]
is a symmetric partial action, turning \( D \) into a symmetric left partial \( H \) module coalgebra.

Indeed, for proving the item (PLHMC1), consider \( d \in D \), then
\[
1_H \cdot c = P(1_H \triangleright d) = P(d) = d.
\]

Now, for the item (PLHMC2), consider \( h \in H \) and \( d \in D \), then
\[
\Delta(h \cdot d) = \Delta(P(h \triangleright d)) = (P \otimes P) \circ \Delta(h \triangleright d)
= (P \otimes P)(h_{(1)} \triangleright c_{(1)}) \otimes (h_{(2)} \triangleright c_{(2)})
= (h_{(1)} \cdot c_{(1)}) \otimes (h_{(2)} \cdot c_{(2)}).
\]
Finally, for the item (PLHMC3), consider \( h, k \in H \) and \( d \in D \), then
\[
\begin{align*}
\Delta h \cdot \iota (k \cdot d) &= P(h \triangleright (P(k \triangleright d))) \\
&= P(h \triangleright (k(1) \triangleright c(1))) \epsilon (P(k(2) \triangleright c(2))) \\
&= P(hk(1) \triangleright c(1)) \epsilon (P(k(2) \triangleright c(2))) \\
&= (hk(1) \cdot c(1)) \epsilon (k(2) \cdot c(2)).
\end{align*}
\]
where we used the definition of \( \cdot \) in the first and last equality, \([10]\) in the second equality and the fact that \( C \) is an \( H \) module coalgebra in the third equality.

The symmetry of the action follows easily from the symmetry in the identity \([10]\).

5.2. Partial module coalgebras as coalgebra objects. In the case of partial actions of Hopf algebras, we have seen that there is a categorical equivalence between the category of left (resp. right) partial actions (partial \( H \) module algebras) and the category of algebras in \( H \mathcal{M} \). For the case of partial \( H \) module coalgebras, there is still a correspondence in one direction, that is, for all left (resp. right) partial \( H \) module coalgebra, we can associate a coalgebra object in the category \( H \mathcal{M} \) (resp. \( \mathcal{M}_H \)). But we cannot make a categorical equivalence as in the case of algebras because there is no canonical way to associate to each coalgebra object in the monoidal category \( H \mathcal{M} \) (resp. \( \mathcal{M}_H \)) a \( k \) coalgebra with a structure of left (resp. right) partial \( H \) module coalgebra. Nevertheless, we still have the duality between coalgebra and algebra objects in \( H \mathcal{M} \) as we shall see later.

**Proposition 5.9.** Let \( H \) be a Hopf algebra with invertible antipode and \( C \) be a symmetric left (resp. right) partial \( H \) module coalgebra. Then there is a coalgebra object in the category of left (resp. right) partial \( H \) modules canonically associated to the coalgebra \( C \).

**Proof.** Let us show for the left sided case, the right sided case is analogous. Let \( C \) be a left partial \( H \) module coalgebra, define the structure of \( A \) bimodule on \( C \) by
\[
\varepsilon_h \triangleright c = h(1) \triangleright_l (S(h(2)) \triangleright_l c) = \epsilon(S(h(3)) \triangleright_l c(1)) (h(1) S(h(2)) \triangleright_l c(2)) = \epsilon(S(h) \triangleright_l c(1)c(2)),
\]
and
\[
c \triangleright \varepsilon_h = h(2) \triangleright_l (S^{-1}(h(1)) \triangleright_l c) = (h(3) S^{-1}(h(2)) \triangleright_l c(1)) \epsilon(S^{-1}(h(1)) \triangleright_l c(2)) = c(1) \epsilon(S^{-1}(h) \triangleright_l c(2)).
\]
With this \( A \) bimodule structure, we can define the left \( H \mathcal{M} \) structure in the following way: for \( x = [h^1] \ldots [h^n] = \varepsilon_{h(1)} \ldots \varepsilon_{h_{n-1}} \cdot h_n [h_{n-1}] \ldots [h^1] \), define
\[
x \triangleright c = \varepsilon_{h(1)} \ldots \varepsilon_{h_{n-1}} \cdot h_n \triangleright_l c.
\]
It is easy to see that, with this definition, we have \([h] \triangleright c = h \triangleright_l c\). In order to verify the composition, let \( h, k \in H \) and \( c \in C \)
\[
[h][k] \triangleright c = \varepsilon_{h(1)} \triangleright (h(2) k \triangleright_l c) = \epsilon(S(h(1)) \triangleright_l (h(2)k(1) \triangleright_l c(1)) (h(2)k(2) \triangleright_l c(2))
\]
\[
= \epsilon(S(h(1)) \triangleright_l (h(2)k(1) \triangleright_l c(1)) (h(3)k(2) \triangleright_l c(2))
\]
\[
= \epsilon(S(h(1))h(2)k(1) \triangleright_l c(1)) \epsilon(h(3)k(2) \triangleright_l c(2)) (h(4)k(3) \triangleright_l c(3))
\]
\[
= \epsilon(k(1) \triangleright_l c(1)) (hk(2) \triangleright_l c(2)) = h \triangleright_l (k \triangleright_l c) = [h] \triangleright (([k] \triangleright c)
\]
Using induction, we conclude that \( x \triangleright (y \triangleright c) = xy \triangleright c \) for every \( x, y \in H \mathcal{M} \) and \( c \in C \).

Now, we introduce a new comultiplication \( \Delta_C : C \rightarrow C \otimes_A C \) and a new counit \( \varepsilon_C : C \rightarrow A \) given respectively by
\[
\Delta_C = \Pi \circ \Delta, \quad \text{and} \quad \varepsilon_C(x \triangleright c) = \epsilon(x) \epsilon(c), \forall x \in H \mathcal{M}, \forall c \in C
\]
where \( \Pi : C \otimes C \rightarrow C \otimes_A C \) is the canonical projection. These two maps are in fact morphisms of left \( H \mathcal{M} \) modules. The counit was defined to be a morphism, while, for the comultiplication, take \( h \in H \) and \( c \in C \), then
\[
\Delta_C([h] \triangleright c) = \Pi \circ \Delta(h \triangleright c) = \Pi((h(1) \triangleright c(1)) \otimes (h(2) \triangleright c(2))) = ([h(1)] \triangleright c(1)) ⊗ ([h(2)] \triangleright c(2)) = [h] \triangleright \Delta_C(c)
\]
It is easy to see that the comultiplication is coassociative and that the axiom of counit is satisfied. Therefore, \((C, \Delta_C, \varepsilon_C)\) is a coalgebra object in \(H\mathcal{M}_{par}\).

**Proposition 5.10.** Let \((C, \Delta_C, \varepsilon_C)\) be a coalgebra object in \(H\mathcal{M}_{par}\), then the right dual space \(C^* = \text{Hom}_A(C, A)\) is an algebra object in the same category. In particular, if \(C\) is a partial \(H\)-module coalgebra, then \(C^*\) is a partial \(H\)-module algebra.

**Proof.** This result follows easily from the fact that \(C^* = \text{Hom}_A(C, A)\) is an internal Hom in the category of left partial \(H\) modules and because of the Hom-tensor relation in Proposition 2.5, we see automatically that the dual of a coalgebra object is an algebra object. Explicitly, the algebra structure on \(C^*\) is given by the convolution product

\[(f \ast g)(c) = f(g(c)) \circ c, \quad \forall f, g \in C^*, \forall c \in C,\]

and the unit

\[1^*(c) = \varepsilon_C(c).\]

\[\square\]

### 5.3. The \(C\)-ring associated to a module coalgebra.

As we have seen so far, if \(A\) is a right \(H\) comodule algebra, then the space \(A \otimes_H H\) has a structure of an \(A\) coring. In the same way if \(C\) is a left partial \(H\)-module coalgebra, it is possible to define an algebraic structure on a subspace of \(H \otimes C\), namely a \(C\)-ring structure.

**Definition 5.11.** Let \(C\) be a coalgebra, a \(C\)-ring is a monoid in the monoidal category \((C\mathcal{M}^C, \square^C, C)\).

**Proposition 5.12.** Let \(H\) be a bialgebra and \(C\) be a left symmetric partial \(H\) module coalgebra, then the subspace

\[H \otimes C = \{h \otimes c = \epsilon(h_{(1)} \cdot c_{(1)})h_{(2)} \otimes c_{(2)} \in H \otimes C\},\]

has a structure of a \(C\) ring.

**Proof.** First it is important to note that, for every \(h \in H\) and \(c \in C\) we have

\[h \otimes c = \epsilon(h_{(1)} \cdot c_{(1)})h_{(2)} \otimes c_{(2)},\]

this follows easily from (8).

The left \(C\) comodule structure in \(H \otimes C\) is given by

\[\lambda(h \otimes c) = h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \otimes c_{(2)};\]

By the axiom (PLHMC1), we can see that \((I \otimes \lambda) \circ \lambda = (\Delta \otimes I) \circ \lambda\), and because of (17), we obtain \((\epsilon \otimes I) \circ \lambda = I\).

The right comodule structure, in its turn, is given by \(\rho(h \otimes c) = h \otimes c_{(1)} \otimes c_{(2)}\), which satisfies trivially the equalities \((\rho \otimes I) \circ \rho = (I \otimes \Delta) \circ \rho\) and \((I \otimes \epsilon) \circ \rho = I\). Therefore \(H \otimes C \in C\mathcal{M}^C\).

The cotensor product \(H \otimes C \square^C H \otimes C\), as the equalizer between \(\rho \otimes I\) and \(I \otimes \lambda\), can be characterized as the subspace of \(H \otimes C \otimes H \otimes C\) spanned by elements

\[\sum_i h^i \otimes c^{i1} \otimes k^i \otimes d^i,\]

such that

\[\sum_i h^i \otimes c^{i1} \otimes c^{i2} \otimes k^i \otimes d^i = \sum_i h^i \otimes c^{i1} \otimes k^i \otimes d^{i1} \otimes k^i_{(2)} \otimes d^{i2}_{(2)},\]

\[\text{Equation 18}\]

The multiplication map, \(\mu : H \otimes C \square^C H \otimes C \to H \otimes C\) is defined as

\[\sum_i (h^i \otimes c^i)(k^i \otimes d^i) = \sum_i (\epsilon(h^i) \cdot c^i)\epsilon(k^i) \cdot d^{i1}_{(1)} \otimes k^i_{(2)} \otimes d^{i2}_{(2)}\]
It is a morphism of $C$ bicomodules: For the left coaction, we have

\[
\lambda \circ \mu \left( \sum_i h^i \otimes c^i \otimes k^i \otimes d^i \right) \\
= \lambda \left( \sum_i \epsilon(h^i(1) \cdot c^i) \epsilon(k^i(1) \cdot d^i(1)) h^i(2) k^i(2) \otimes d^i(2) \right) \\
= \sum_i \epsilon(k^i(1) \cdot d^i(1)) h^i(2) \otimes \epsilon(k^i(1) \cdot c^i) h^i(3) k^i(3) \otimes d^i(3) \\
= \sum_i \epsilon(h^i(1) \cdot c^i(1)) (h^i(2) \cdot d^i(1)) \otimes \epsilon(h^i(1) \cdot c^i(2)) h^i(3) k^i(3) \otimes d^i(3) \\
= \sum_i h^i(1) \cdot c^i(1) \otimes \mu \left( h^i(2) \cdot c^i(2) \otimes k^i \otimes d^i \right) \\
= (I \otimes \mu) \circ (\lambda \otimes I) \left( \sum_i h^i \otimes c^i \otimes k^i \otimes d^i \right).
\]

And for the right coaction,

\[
\rho \circ \mu \left( \sum_i h^i \otimes c^i \otimes k^i \otimes d^i \right) \\
= \rho \left( \sum_i \epsilon(h^i(1) \cdot c^i) \epsilon(k^i(1) \cdot d^i(1)) h^i(2) k^i(2) \otimes d^i(2) \right) \\
= \sum_i \epsilon(h^i(1) \cdot c^i) \epsilon(k^i(1) \cdot d^i(1)) h^i(2) k^i(2) \otimes d^i(2) \otimes d^i(3) \\
= \sum_i \mu \left( h^i \otimes c^i \otimes k^i \otimes d^i(1) \right) \otimes d^i(3) \\
= (\mu \otimes I) \circ (I \otimes \rho) \left( \sum_i h^i \otimes c^i \otimes k^i \otimes d^i \right).
\]
We can prove also that this multiplication is associative. Indeed, on one hand we have,

\[
\mu \circ (\mu \otimes I) \left( \sum h^i \otimes c^j \otimes k^i \otimes d^i \otimes l^i \otimes e^j \right)
\]

\[
= \mu \left( \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(k_{1(1)} \cdot \epsilon^j) h_{1(1)} h_{2(2)} \otimes d^i \right) \otimes l^i \otimes e^j
\]

\[
= \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(k_{1(1)} \cdot \epsilon^j) \epsilon(h_{1(2)} k_{2(2)} \cdot \epsilon^i) \epsilon(l_{1(1)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j
\]

\[
= \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(h_{1(2)} \cdot (k_{1(1)} \cdot \epsilon^j)) \epsilon(l_{1(1)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j
\]

\[
= \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(h_{1(2)} \cdot \epsilon^j) \epsilon(k_{2(2)} \cdot \epsilon^j) \epsilon(l_{1(2)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j.
\]

On the other hand,

\[
\mu \circ (I \otimes \mu) \left( \sum_i h^i \otimes c^j \otimes k^i \otimes d^i \otimes l^i \otimes e^j \right)
\]

\[
= \mu \left( \sum_i h^i \otimes c^j \otimes \epsilon(k_{1(1)} \cdot \epsilon^j) \epsilon(l_{1(1)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j \right)
\]

\[
= \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(k_{1(1)} \cdot \epsilon^j) \epsilon(l_{1(1)} \cdot \epsilon^j) \epsilon(k_{2(2)} \cdot \epsilon^j) \epsilon(l_{2(2)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j
\]

\[
= \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(k_{1(1)} \cdot \epsilon^j) \epsilon(h_{1(1)} \cdot \epsilon^j) \epsilon(l_{1(2)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j
\]

\[
= \sum_i \epsilon(h_{1(1)} \cdot \epsilon^i) \epsilon(k_{1(1)} \cdot \epsilon^j) \epsilon(h_{1(2)} \cdot \epsilon^j) \epsilon(l_{1(2)} \cdot \epsilon^j) h_{1(3)} k_{3(3)} l^i \otimes e^j.
\]

The unit map \( \eta : C \to H \otimes C \) is given by \( \eta(c) = 1_H \otimes c \). We can see that it is a bi-comodule map. For the left side, we have

\[
\lambda \circ \eta(c) = \lambda(1_H \otimes c) = 1_H \cdot c_{(1)} \otimes 1_H \otimes c_{(2)}
\]

\[
= c_{(1)} \otimes 1_H \otimes c_{(2)} = c_{(1)} \otimes \eta(c_{(2)})
\]

\[
= (I \otimes \eta)(c_{(1)} \otimes c_{(2)}) = (I \otimes \eta) \circ \Delta(c),
\]

and for the right side,

\[
\rho \circ \eta(c) = \rho(1_H \otimes c) = 1_H \otimes c_{(1)} \otimes c_{(2)}
\]

\[
= \eta(c_{(1)}) \otimes c_{(2)} = (\eta \otimes I)(c_{(1)} \otimes c_{(2)})
\]

\[
= (\eta \otimes I) \circ \Delta(c).
\]
For any $h \otimes c \in H \otimes C$ the images of $(\eta \otimes I) \circ \lambda$ and $(I \otimes \eta) \circ \rho$ are in the co-tensor product $H \otimes C \overset{C}{\otimes} H \otimes C$. Indeed, taking $h \otimes c \in H \otimes C$,

$$(\rho \otimes I) \circ (\eta \otimes I) \circ \lambda (h \otimes c) = (\rho \otimes I)(1_H \otimes h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)})$$

$$= (1_H \otimes h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \cdot c_{(2)} \otimes h_{(3)} \otimes c_{(3)})$$

$$= (I \otimes \lambda)(1_H \otimes h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \otimes c_{(2)})$$

$$= (I \otimes \lambda)(\eta \otimes I) \circ \lambda (h \otimes c),$$

with a similar proof for $(I \otimes \eta) \circ \rho (h \otimes c)$. The unit axioms for a $C$ ring can be written as

(i) $\mu \circ (\eta \otimes I) \circ \lambda = 1_H$.

(ii) $\mu \circ (I \otimes \eta) \circ \rho = 1_H$.

For the identity (i), take $h \otimes c \in H \otimes C$,

$$\mu \circ (\eta \otimes I) \circ \lambda (h \otimes c) = \mu((1_H \otimes h_{(1)} \cdot c_{(1)} \otimes h_{(2)} \otimes c_{(2)})$$

$$\mu = \epsilon(1_H \cdot h_{(1)} \cdot c_{(1)})\epsilon(h_{(2)} \cdot c_{(2)})h_{(3)} \otimes c_{(3)}$$

$$= \epsilon(h_{(1)} \cdot c_{(1)})h_{(2)} \otimes c_{(2)} = h \otimes c.$$ 

and for the identity (ii),

$$\mu \circ (I \otimes \eta) \circ \rho (h \otimes c) = \mu(h \otimes c_{(1)} \otimes 1_H \otimes c_{(2)})$$

$$= \epsilon(h_{(1)} \cdot c_{(1)})\epsilon(1_H \cdot c_{(2)})h_{(2)} \otimes c_{(3)}$$

$$= \epsilon(h_{(1)} \cdot c_{(1)})\epsilon(c_{(2)})h_{(2)} \otimes c_{(3)}$$

$$= \epsilon(h_{(1)} \cdot c_{(1)})h_{(2)} \otimes c_{(2)} = h \otimes c.$$ 

Therefore, $H \otimes C$ is a $C$-ring.

Finally, there exist a duality between the $C$-rings obtained from partial module coalgebras and $A$-corings defined from partial comodule algebras. More precisely, we have,

**Proposition 5.13.** Let $H$ and $K$ be two bialgebras with a dual pairing $(\cdot, \cdot) : K \otimes H \to k$. Consider $A$ be a left $H$ comodule algebra, with left partial coaction $\lambda : A \to H \otimes A$, and $C$ a left partial $K$ module coalgebra. Consider also that there is a dual pairing $(\cdot, \cdot) : C \otimes A \to k$, such that for every $x \in C$, $\xi \in K$ and $a \in A$ we have $(\xi \cdot x, a) = (\xi, a^{-1}) (x, a^{[0]})$. Then there is a dual pairing between the $A$-coring $H \otimes A = \lambda(1_A)(H \otimes A)$ and the $C$-ring $K \otimes C$.

**5.4. Dualities.**

**Module coalgebras versus module algebras.**

**Theorem 5.14.** Let $H$ be a Hopf algebra and $(\cdot, \cdot) : A \otimes C \to k$ be a non-degenerate dual pairing between the coalgebra $C$ and the algebra $A$. Then there is a bijective correspondence between the structures of right partial module algebra on $A$ and left partial module coalgebra on $C$. Explicitly, for $a \in A$, $c \in C$ and $h \in H$, this correspondence of the partial actions is given by

$$(a \cdot h \cdot c) = (a, h \cdot c)$$

**Proof.** As the pairing is non-degenerate, is suffices to check that, taking arbitrary $a \in A$ and $c \in C$, the (right handed versions of) axioms (PLA1)-(PLA4) correspond to axioms (PLHMC1)-(PLHMC4). Let us check this for one axiom, leaving the others for the reader. Suppose that $A$ is right partial $H$ module algebra, then

$$(a, h \cdot (k \cdot c)) = ((a \cdot h) \cdot k, c) = ((a \cdot h k_{(1)}) (1_A \cdot k_{(2)}) , c)$$

$$= ((a \cdot h k_{(1)}), c_{(1)}) (1_A \cdot k_{(2)} , c_{(2)}) = (a, h k_{(1)} \cdot c_{(1)})(1_A, k_{(2)} \cdot c_{(2)})$$

$$= (a, h k_{(1)} \cdot c_{(1)})(c_{(2)} \cdot k_{(2)} ) = (a, (h k_{(1)} \cdot c_{(1)})c_{(2)} )$$

for any $a \in A$ then $(k \cdot c) = (h k_{(1)} \cdot c_{(1)})c_{(2)}$, therefore, $C$ is a left partial $H$ module coalgebra. \[\Box\]
Module coalgebras versus comodule algebras.

**Theorem 5.15.** Let \((-,-): H \otimes K \to k\) be a dual pairing between the Hopf algebras \(H\) and \(K\) and \((-,-): A \otimes C \to k\) be a non-degenerate dual pairing between the coalgebra \(C\) and the algebra \(A\). If \(A\) is a symmetric left \(K\)-comodule algebra with coaction \(\rho: A \to K \otimes A\), \(\rho(a) = a^{-1}[1] \otimes a^{[0]}\), then \(C\) is a symmetric partial left \(H\)-module coalgebra, with action given by

\[ (a, h \cdot c) = \langle h, a^{-1}[1] \rangle (a^{[0]} \cdot c) \]

for all \(a \in A\), \(c \in C\) and \(h \in H\).

**Proof.** Let us check (PHMC3’). Consider \(a \in A\), \(c \in C\) and \(h, k \in H\), then

\[
(a, h \cdot (k \cdot c)) = \langle h, a^{-1}[1] \rangle (a^{[0]} \cdot k \cdot c) = \langle h, a^{-1}[1] \rangle \langle k, a^{[0]} \rangle \langle a^{[0][0]} \rangle (a^{[0][0]} \cdot c)
\]

\[
= \langle h, a^{-1}[1] \rangle \langle k, 1^{-1}[1] a^{-1}[1][2] \rangle (1^{[0]} a^{[0]} c)
\]

\[
= \langle h, a^{-1}[1] \rangle \langle k, 1^{-1}[1] \rangle \langle k[2], a^{-1}[2] \rangle \langle 1^{[0]} c(1) \rangle \langle a^{[0]} c(2) \rangle
\]

\[
= \langle 1, k(1) \cdot c(1) \rangle \langle h k(2), a^{-1} \rangle \langle a^{[0]} c(2) \rangle = \epsilon(k(1) \cdot c(1)) \langle a, h k(2) \cdot c(2) \rangle
\]

\[
= \langle a, c(k(1) \cdot c(1)) (h k(2) \cdot c(2)) \rangle,
\]

for every \(a \in A\), then we have \(h \cdot (k \cdot c) = \epsilon(k(1) \cdot c(1))(h k(2) \cdot c(2))\). The other axioms are verified similarly. Therefore, \(C\) is a symmetric left partial \(H\) module coalgebra. \(\square\)

### 6. Partial Comodule Coalgebras and Partial Co-smash Coproducts

#### 6.1. Definition and examples.

**Definition 6.1.** A left partial coaction of a Hopf algebra \(H\) on a \(k\)-coalgebra \(C\) is a linear map

\[
\lambda: C \to H \otimes C
\]

\[
c \mapsto c^{[-1]} \otimes c^{[0]}
\]

such that

(PLHCC1) \((I \otimes \Delta) \circ \lambda(c) = c^{[-1]}(c^{[-1]} \otimes c^{[0]}) \otimes c^{[0]}, \) for all \(c \in C\).

(PLHCC2) \((c \otimes I) \circ \lambda(c) = c\), for all \(c \in C\).

(PLHCC3) For all \(c \in C\) we have

\[
(I \otimes \lambda) \circ \lambda(c) = c^{[-1]}(\epsilon(c^{[0]})) c^{[-1]}(1) \otimes c^{[-1]}(2) \otimes c^{[0]}(2)\]

The coalgebra \(C\) is called a left partial \(H\) comodule coalgebra. If, in addition, the partial coaction satisfies the condition,

(PLHCC3’) For all \(c \in C\) we have

\[
(I \otimes \lambda) \circ \lambda(c) = c^{[-1]}(c^{[0]}(1)) c^{[-1]}(2) \otimes c^{[-1]}(1) \otimes c^{[0]}(2),
\]

then the partial coaction is said to be symmetric.

It is easy to see that any left \(H\) comodule coalgebra is a left partial \(H\) comodule coalgebra.

Indeed, the axioms (PLHCC1) and (PLHCC2) are the same as the classic case. For the axiom (PLHCC3), if

\[
c^{[-1]} \epsilon(c^{[0]}) = \epsilon(c)1_H
\]

for any \(c \in C\) then

\[
c^{[-1]} \epsilon(c^{[0]}) = \epsilon(c^{[0]}) (1) \otimes c^{[-1]}(1) \otimes c^{[-1]}(2) \otimes c^{[0]}(2)\]

\[
= \epsilon(c^{[0]}(1)) c^{[-1]}(1) \otimes c^{[-1]}(2) \otimes c^{[0]}(2)\]

\[
= c^{[-1]}(1) \otimes c^{[-1]}(2) \otimes c^{[0]}(2)\]

\[
= (\Delta \otimes I) \circ \lambda(c)
\]

\[
= (I \otimes \lambda) \circ \lambda(c).
\]
This also shows that a left partial $H$ comodule coalgebra satisfying (19) is in fact a left $H$ comodule coalgebra.

In [14], the author gave a definition of left partial $H$ comodule coalgebra, but in the last axiom, the author considered only the nonsymmetric version. As we have learned so far, the most interesting properties of partial actions can be found when we consider more symmetric versions of the actions and coactions.

An immediate example of a left partial $H$ comodule coalgebra is $C = D/I$ where $D$ is a left $H$ comodule coalgebra with coaction $\delta : D \to H \otimes D$, denoted by $\delta(d) = d^{(-1)} \otimes d^{(0)}$, and $I$ is a right coideal of $D$ such that $C$ is a coalgebra. The partial coaction is given by $\lambda(d) = d_{(2)}^{(-1)} \otimes \epsilon_C(d_{(1)})d_{(2)}^{(0)}$ as shown in [14].

**Lemma 6.2.** Let $H$ be a bialgebra and $C$ be a left partial $H$ comodule coalgebra, with coaction $\lambda : C \to H \otimes C$ denoted by $\lambda(c) = c^{[-1]} \otimes c^{[0]}$. Then, for every $c \in C$ we have

$$c^{[-1]} \otimes c^{[0]} = c_{(1)}^{[-1]}c_{(2)}^{[-1]} \otimes c_{(1)}^{[0]} \otimes c_{(2)}^{[0]}.$$  

(20)

Applying $(I \otimes \epsilon \otimes I)$ on both sides of (20), we have

$$c^{[-1]} \otimes \epsilon(c_{[1]}^{[0]})c_{[2]}^{[0]} = c_{(1)}^{[-1]}c_{(2)}^{[-1]} \otimes \epsilon(c_{(1)}^{[0]})c_{(2)}^{[0]},$$

denoting by $\lambda(c) = c^{[-1]}e(c^{[0]})$ is an idempotent in the convolution algebra $\text{Hom}_H(C,H)$.\[\Box\]

**Corollary 6.3.** Let $H$ be a bialgebra and $C$ be a left partial $H$ comodule coalgebra, with coaction $\lambda : C \to H \otimes C$ denoted by $\lambda(c) = c^{[-1]} \otimes c^{[0]}$. Then the map $\psi : C \to H$ given by $\psi(c) = c^{[-1]}e(c^{[0]})$ is an idempotent with relation to the convolution product.

**Proof.** If one applies $(I \otimes \epsilon)$ to any of the equalities obtained in the previous lemma, then it would end up with

$$\psi(c) = c^{[-1]}e(c^{[0]}) = c_{(1)}^{[-1]}e(c_{(1)}^{[0]})c_{(2)}^{[-1]}e(c_{(2)}^{[0]}) = \psi(c_{(1)})\psi(c_{(2)}) = \psi \circ \psi(c),$$

showing that $\psi$ is idempotent with relation to the convolution product.\[\Box\]

### 6.2. The partial cosmash coproduct

From the definition of a left partial $H$ comodule coalgebra, define on the tensor product $C \otimes H$ a subspace $C \bowtie H$ spanned by elements of the form

$$c \bowtie h = c_{(1)} \otimes c_{(2)}^{[-1]}e(c_{(2)}^{[0]})h.$$  

By Corollary 6.3 it is easy to see that

$$c \bowtie h = c_{(1)} \bowtie c_{(2)}^{[-1]}e(c_{(2)}^{[0]})h.$$  

**Proposition 6.4.** Let $H$ be a Hopf algebra and $C$ be a left partial $H$ comodule coalgebra. Then the space $C \bowtie H$ is a coalegebra with the comultiplication given by

$$\hat{\Delta}(c \bowtie h) = c_{(1)} \bowtie c_{(2)}^{[-1]}h_{(1)} \otimes c_{(2)}^{[0]} \bowtie h_{(2)},$$

and counit given by

$$\hat{\epsilon}(c \bowtie h) = \epsilon_C(c)e_H(h).$$

This coalegebra is will be called partial co-smash coproduct.
Proof. Let us check the counit axioms. First, we have
\[(I \otimes \widehat{\epsilon})(c \triangleright h) = (I \otimes \widehat{\epsilon})(c_{(1)} \triangleright c_{(2)}^{-1}h_{(1)} \otimes c_{(2)}[0] \triangleright h_{(2)}) \]
\[= c_{(1)} \triangleright c_{(2)}^{-1} \epsilon(c_{(2)}[0])h_{(1)} \epsilon(h_{(2)}) \]
\[= c_{(1)} \triangleright c_{(2)}^{-1} \epsilon(c_{(2)}[0])h = c \triangleright h.\]
And on the other hand
\[(\widehat{\epsilon} \otimes I)(c \triangleright h) = (\widehat{\epsilon} \otimes I)(c_{(1)} \triangleright c_{(2)}^{-1}h_{(1)} \otimes c_{(2)}[0] \triangleright h_{(2)}) \]
\[= \epsilon(c_{(1)})\epsilon(c_{(2)}^{-1})c_{(2)}[0] \triangleright \epsilon(h_{(1)})h_{(2)} \]
\[= \epsilon(c_{(1)})c_{(2)} \triangleright h = c \triangleright h.\]
For the coassociativity, on one hand, we have
\[(\widehat{\Delta} \otimes I)(c \triangleright h) = (\widehat{\Delta} \otimes I)(c_{(1)} \triangleright c_{(2)}^{-1}h_{(1)} \otimes c_{(2)}[0] \triangleright h_{(2)}) \]
\[= \epsilon(c_{(1)}) \epsilon(c_{(2)}^{-1}) c_{(2)}[0] \triangleright \epsilon(h_{(1)})h_{(2)} \]
\[= \epsilon(c_{(1)})c_{(2)} \triangleright h = c \triangleright h.\]
On the other hand
\[(I \otimes \widehat{\Delta})(c \triangleright h) = (I \otimes \widehat{\Delta})(c_{(1)} \triangleright c_{(2)}^{-1}h_{(1)} \otimes c_{(2)}[0] \triangleright h_{(2)}) \]
\[= c_{(1)} \triangleright c_{(2)}^{-1} h_{(1)} \otimes c_{(2)}[0]_{(1)} \triangleright c_{(2)}[0]_{(2)}^{-1} h_{(2)} \otimes c_{(2)}[0]_{(2)} \triangleright h_{(3)} \]
\[= c_{(1)} \triangleright c_{(2)}^{-1}c_{(3)}^{-1} h_{(1)} \otimes c_{(2)}[0]_{(1)} \triangleright c_{(3)}[0]_{(2)} \triangleright h_{(2)} \otimes c_{(3)}[0]_{(2)} \triangleright h_{(3)} \]
\[= c_{(1)} \triangleright c_{(2)}^{-1}c_{(3)}^{-1} \epsilon(c_{(3)}[0]) c_{(4)}^{-1} h_{(1)} \otimes c_{(2)}[0]_{(1)} \triangleright c_{(4)}[0] \triangleright h_{(2)} \otimes c_{(4)}[0] \triangleright h_{(3)} \]
\[= c_{(1)} \triangleright c_{(2)}^{-1} \epsilon(c_{(2)}[0]) c_{(3)}^{-1} h_{(1)} \otimes c_{(2)}[0]_{(1)} \triangleright c_{(3)}[0] \triangleright h_{(2)} \otimes c_{(3)}[0] \triangleright h_{(3)} \]
\[= c_{(1)} \triangleright c_{(2)}^{-1}c_{(3)}^{-1} h_{(1)} \otimes c_{(2)}[0]_{(1)} \triangleright c_{(3)}[0] \triangleright h_{(2)} \otimes c_{(3)}[0] \triangleright h_{(3)},\]
where in the third equality we used the axiom (i), in the fourth equality, the axiom (iv), in the fifth equality, we used the axiom (i) again and in the sixth equality, only the counit axiom.

Therefore, \(C \triangleright H\) is a coalgebra with the above defined comultiplication and counit axiom. \(\square\)

6.3. Dualities.

Comodule coalgebras versus comodule algebras.

Theorem 6.5. Let \(H\) be a Hopf algebra and \((-,-) : A \otimes C \rightarrow k\) be a non-degenerate dual pairing between the coalgebra \(C\) and the algebra \(A\). Then there is a bijective correspondence between the structures of symmetric right partial comodule algebra on \(A\) and symmetric left partial comodule algebra on \(C\) given by the relation
\[\left(a^0, c\right) a^1 = c^{-1} \left(a, c^0\right).\]

Proof. Suppose that \(A\) is a symmetric right \(H\) comodule algebra, and consider \(a \in A\) and \(c \in C\), then
\[c^{-1} \otimes c^0[-1] \left(a, c^0[0]\right) = c^{-1} \otimes \left(a^0, c^0[0]\right) a^1 = \left(a^0, c[0]\right) a^1 \otimes a^1[1] \]
\[= \left(a^0, 1, c\right) a^1 [1] \otimes a^1[2] = \left(a^0, c(1), 1^0[0], c(2)\right) a^1[1] (1^0, a^1[1] \otimes a^1[2]) \]
\[= \left(a^0, c(1) \right) a^1 [1] (c(2) [-1], 1^0 \otimes a^1[2]) \]
\[= c^{-1} a^1 (1, c) \left(a, c^0[0]\right) a^1 [1] (c(2) [-1], c(1) [-1]) \]
\[= c^{-1} a^1 [1] (c(2) [-1], c(1) [-1]) \left(a, c(1) [0]\right),\]
for any \(a \in A\), by the nondegeneracy of the pairing, we have
\[c^{-1} \otimes c^0[-1] \otimes c^0[0] = c^{-1} \left(a^1, c(1) \right) c^{-1} c(2) [-1] \otimes c(1) [-1] \otimes c(1) [0],\]
Therefore, \( C \) is a symmetric left partial \( H \) comodule coalgebra. \( \square \)

Comodule coalgebras versus module coalgebras.

**Theorem 6.6.** Consider a dual pairing of Hopf algebras \( (-, -) : H \otimes K \rightarrow k \) and let \( C \) be a left partial \( K \)-comodule coalgebra. Then the map

\[
\cdot : C \otimes H \rightarrow C \quad c \otimes h \mapsto \sum \langle h, c^{-1} \rangle c^{[0]}
\]

turns \( C \) into a right partial \( H \)-module coalgebra. This construction yields a functor from the category of left partial \( K \)-comodule coalgebras to the category of right partial \( H \)-module coalgebras. If the dual pairing \( (-, -) \) is moreover non-degenerate, then the above functor corestricts to an isomorphism of categories between the category of left partial \( K \)-comodule coalgebras to the category of rational right partial \( H \)-module coalgebras, which are those for which \( c \cdot H \) is a finitely generated \( k \)-module for all \( c \in C \).

Comodule coalgebras versus module algebras.

**Theorem 6.7.** Let \( (-, -) : H \otimes K \rightarrow k \) be a dual pairing between the Hopf algebras \( H \) and \( K \) and \( (-, -) : A \otimes C \rightarrow k \) be a non-degenerate dual pairing between the coalgebra \( C \) and the algebra \( A \). If \( C \) is a left \( K \)-comodule coalgebra with coaction \( \rho : C \rightarrow K \otimes C \), \( \rho(c) = c^{-1} \otimes c^{[0]} \), then \( A \) is a partial left \( H \)-module algebra, with action given by

\[
(h \cdot a, c) = \langle h, c^{-1} \rangle (a, c^{[0]})
\]

for all \( a \in A, c \in C \) and \( h \in H \). Furthermore

- under these conditions, there is a pairing \( (-, -) : A \# H \otimes C \rightarrow K \rightarrow k \) between the algebra \( A \# H \) and the coalgebra \( C \rightarrow K \) given by \( \langle a \# h, x \rightarrow \xi \rangle = a(h_{(1)}, 1_A), x)\langle h_{(2)}, \xi \rangle \);
- if the pairing between \( H \) and \( K \) is moreover non-degenerate then there is a bijective correspondence between the structures of rational left \( H \)-module algebra on \( A \) and the structures of left \( K \)-comodule coalgebra on \( C \).

**Proof.** The first and last statement follows from Proposition \( 4.3 \), Theorem \( 4.5 \) and Theorem \( 6.6 \). The details to verify that \( (-, -) \) is a dual pairing are left to the reader, we warn however that computations become very technical. \( \square \)

# 7. Conclusions and outlook

In this paper, introduced the dual notions of module algebras and made several algebraic constructions with them. Let us summarize this in the following table

| \( H \)-module | \( H \)-module algebra \( A \) | \( H \)-module coalgebra \( C \) |
|----------------|-------------------------------|-------------------------------|
| \( K \)-comodule | \( K \)-comodule algebra \( A \) | \( K \)-comodule coalgebra \( C \) |

If there is a non-degenerate pairing between the algebra \( A \) and the coalgebra \( C \), then one can use this duality to move from the right column to the left column and visa versa. If the Hopf algebras \( H \) and \( K \) are in duality, then one an use the associated pairing to move from the lower row to the upper row and visa versa if the pairing is non-degenerate and one restricts to rational modules.

In the general case, when no commutativity or cocommutativity conditions are imposed, each of these notions leads to an algebraic construction as follows (we keep same rows and columns)

| \( H \)-module | smash product algebra \( A \# H \) | \( C \)-ring \( H \otimes C \) |
|----------------|--------------------------------|--------------------------|
| \( K \)-comodule | \( A \)-coring \( A \otimes K \) | cosmash coproduct coalgebra \( C \otimes K \) |

Furthermore, if one adds commutativity conditions, then the obtained structure is even richer. In particular, we found that if \( A \) is a commutative algebra, \( H \) is a cocommutative Hopf algebra and \( K \) is a commutative Hopf algebra, then \( A \# H \) and \( A \otimes K \) are Hopf algebroids and the dualities from the first table are lead to a skew pairing between these Hopf algebroids.
In fact, we believe that the structure is even richer than what we obtained so far. First, we would like to see all dualities that appear in the first table, also to be apparent in the second table. Indications that this is true are given by the duality (without commutativity constraints) between the $A$-coring structure of $A \otimes K$ and the $A$-ring structure of $A \# H$ (see Lemma 4.10) and the coalgebra structure of $C \triangleright K$ and the ring structure of $A \# H$, (see Theorem 6.7). Furthermore, we expect that if $C$ is supposed to be a cocommutative coalgebra, $H$ is cocommutative and $K$ is commutative then $H \otimes C$ and $C \triangleright K$ are Hopf coalgebroids, and all dualities also be brought to this level. All this is subject for future investigations.

Acknowledgments

One of the authors, E. Batista, is supported by CNPq (Ciência sem Fronteiras), proc no 236440/2012-8, and he would like to thanks the Département des Mathématiques de l’Université Libre de Bruxelles for their kind hospitality. E. Batista is also partially supported by Fundação Araucária, project n. 490/16032. The last author would like to thank the FNRS for the CDR “Symmetries of non-compact non-commutative spaces: Coactions of generalized Hopf algebras” that partially supported this collaboration. We both would like to thank to Marcelo Muniz S. Alves for fruitful discussions along the elaboration of this paper.

References

[1] M.M.S. Alves and E. Batista, Enveloping Actions for Partial Hopf Actions, Comm. Algebra 38 (2010), 2872–2902.
[2] M. M. S. Alves, E. Batista, Globalization theorems for partial Hopf (co)actions and some of their applications, Contemp. Math. 537 (2011), 13-30.
[3] M.M.S. Alves, E. Batista, J. Vercruysse, Partial representations of Hopf algebras, arXiv:1309.1659 (2013).
[4] G. Böhm, Hopf algebroids, In: “Handbook of algebra”. Vol. 6, 173–235, Handb. Algebr., 6, Elsevier/North-Holland, Amsterdam, 2009.
[5] S. Caenepeel, K. Janssen, Partial (co)actions of Hopf algebras and partial Hopf-Galois theory, Comm. Algebra 36 (2008), 2923-2946.
[6] S. Dăscălescu, C. Năstăscu and Ş. Raianu, “Hopf Algebras”, Marcel Dekker Inc., New York, 2001.
[7] M. Dokuchaev, Partial actions: a survey, Contemp. Math. V.537 (2011) 51-63.
[8] M. Dokuchaev, R. Exel, Associativity of Crossed Products by Partial Actions, Enveloping Actions and Partial Representations, Trans. Amer. Math. Soc. 357 (2005), 1931–1952.
[9] M. Dokuchaev, R. Exel, P. Piccione, Partial Representations and Partial Group Algebras, J. Algebra 226 (2000), 505–532.
[10] M. Dokuchaev, M. Ferrero, A. Paques, Partial Actions and Galois Theory, J. Pure Appl. Algebra 208 (1) (2007), 77-87.
[11] J. Kellendonk, M. Lawson, Partial Actions of Groups, Int. J. Alg. Comp. 14 (2004), 87–114.
[12] P. Schauenburg, Duals and Doubles of Quantum Groupoids ($\times_R$-Hopf Algebras), Contemp. Math. 267 (2000), 273–299.
[13] J. Vercruysse, Local units versus local projectivity. Dualisations : Corings with local structure maps, Comm. Algebra 34 (2006), 2079–2103.
[14] C. Wang, Enveloping Coactions for Partial Hopf Coactions, Multimedia Technology (ICMT), 2011 IEEE International Conference on (2011), 2096–2098.

Departamento de Matemática, Universidade Federal de Santa Catarina, Brazil
E-mail address: ebatista@mtm.ufsc.br

Département de Mathématiques, Université Libre de Bruxelles, Belgium
E-mail address: jvercruy@ulb.ac.be