Anisotropic Yield Criteria

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Abstract. The most widely used isotropic yield criteria are defined in terms of the principal values of the stress deviator. In this paper, it is demonstrated that these yield criteria are expressible as polynomials in terms of the second and third invariants of the stress deviator. Using these new mathematical results, new, simple and explicit expression of Yld91 and Karafillis and Boyce [1] orthotropic yield criteria in terms of stress components are obtained. Moreover, it is shown that for orthotropic BCC materials, Yld 91 and Karafillis and Boyce [1] criteria are particular forms of Cazacu and Barlat [2] orthotropic yield criterion.

1. Introduction
For metallic materials, the plastic response is independent of the hydrostatic pressure. Therefore, for an isotropic material, the yield function depends on the stress deviator $s$ only through its invariants $J_2$ and $J_3$, or the principal stresses $s_1, s_2, s_3$. The most widely used criterion was proposed by von Mises. A yield criterion that involves both invariants was proposed by Drucker [3]:

$$ (J_2)^3 - c(J_3)^2 = \tau_y^6, $$

with $c$ being a material constant and $\tau_y$ being the yield limit in simple shear.

Based on crystal plasticity calculations of randomly oriented face-centered cubic (FCC) and body-centered cubic (BCC) polycrystals the following criterion, known as Hershey-Hosford criterion was proposed:

$$ \phi(\sigma_1, \sigma_2, \sigma_3) = |\sigma_1 - \sigma_2|^m + |\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m = 2\sigma_y^m $$

with $m$ being an integer, $\sigma_y$ being the yield limit in uniaxial tension. The recommended value is $m = 6$ for BCC materials and $m = 8$, for FCC materials, respectively. Another widely used isotropic criterion is:

$$ \phi_2(\sigma_1, \sigma_2, \sigma_3) = |\sigma_1|^m + |\sigma_2|^m + |\sigma_3|^m = \left( \frac{2m + 2}{3m} \right) \sigma_y^m. $$

Karafillis and Boyce [1] proposed a generic isotropic yield function, which is obtained by linear interpolation of the above functions $\phi_1$ and $\phi_2$, i.e.:

$$ \alpha \frac{3^m}{2m-1} \left( |\sigma_1|^m + |\sigma_2|^m + |\sigma_3|^m \right) + (1 - \alpha) (|\sigma_1 - \sigma_2|^m + |\sigma_2 - \sigma_3|^m + |\sigma_3 - \sigma_1|^m) = 2\sigma_y^m $$

where $0 \leq \alpha \leq 1$. 

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There are two rigorous methodologies for extending isotropic yield functions such as to account for orthotropy; one is based on the use of orthotropic generalizations of the stress invariants $J_2$ and $J_4$ (e.g. [1]), the other is based on a linear transformations applied to the stress tensor or its deviator (e.g. [3]).

Using representation theorems for tensor functions, Cazacu and Barlat [1] demonstrated that relative to the coordinate system $(x,y,z)$ associated with the orthotropy axes, the orthotropic generalizations of the second-invariant and third-invariant of the stress deviator should be expressed as follows:

$$J_2^o = \frac{a_1}{6}(\sigma_{xx} - \sigma_{yy})^2 + \frac{a_2}{6}(\sigma_{yy} - \sigma_{zz})^2 + \frac{a_3}{6}(\sigma_{zz} - \sigma_{xx})^2 + a_4\sigma_{xy}^2 + a_5\sigma_{xz}^2 + a_6\sigma_{yz}^2,$$

and

$$J_3^o = \frac{1}{27}(b_1 + b_2)\sigma_{xx}^3 + \frac{1}{27}(b_1 + b_3)\sigma_{yy}^3 + \frac{1}{27}(b_1 + b_4)\sigma_{zz}^3 + \frac{1}{9}(b_2 - b_3)\sigma_{xy}^2 + \frac{1}{9}(b_3 - b_4)\sigma_{xz}^2 + \frac{1}{9}(b_4 - b_5)\sigma_{yz}^2$$

$$+ \frac{2}{9}(b_1 + b_5)\sigma_{xx}\sigma_{yy}\sigma_{zz} - \frac{\sigma_{xx}^2}{3}[2b_6\sigma_{xx} - b_7\sigma_{zz} - (2b_8 - b_9)\sigma_{yy}]$$

$$- \frac{\sigma_{yy}^2}{3}[2b_9\sigma_{zz} - b_8\sigma_{xx} - (2b_10 - b_11)\sigma_{xx}] - \frac{\sigma_{zz}^2}{3}[2b_1\sigma_{xx} - b_2\sigma_{yy} - (2b_3 - b_4)\sigma_{zz}]$$

$$+ 2b_1\sigma_{xy}\sigma_{xz}\sigma_{yz}$$

(5)

In the above expressions $a_i$, $i=1...6$ and $b_k$ ($k=1...11$) are constants. Specifically, using these orthotropic invariants, it is possible to extend any isotropic yield criterion simply by substituting $J_2^o$ for $J_2$ and $J_3^o$ for $J_3$ in the respective isotropic expression. For example, Cazacu and Barlat [2] proposed the following orthotropic criterion:

$$F(J_2^o, J_3^o) = (J_2^o)^3 - c(J_3^o)^2.$$  

(6)

The other methodology for accounting for anisotropy consists in replacing in the expression of the isotropic criterion the stress deviator $\mathbf{s}$ with a transformed stress tensor $\mathbf{S}$. It was first used by Barlat et al. [4] in conjunction with the yield function $\phi_1$, and later in Karafillis and Boyce [3] with the isotropic yield function given by Eq. (4). These 3-D yield criteria, called Yld91 and KB93, respectively are very versatile. However, to use these orthotropic yield criteria one needs to determine the principal values of the transformed tensor $\mathbf{S}$, which amounts to solving an algebraic third-degree characteristic equation. Moreover, calculation of the derivatives of these orthotropic yield functions requires consideration of the different possible ordering of the principal values of the transformed tensors and singular cases. Since no explicit expressions of these yield functions in terms of stress components are known, parameter sensitivity studies are rather complex. This contributes to the limited use of these yield functions as compared to 2-D orthotropic yield functions for which explicit expressions in stresses exist.

In this paper, it is shown that the widely used isotropic yield criteria given by Eq. (2)-(4) are expressible as simple polynomials in $J_2$ and $J_3$. Moreover, it is shown that $\phi_1$ and $\phi_2$ are identical in form. Next, new closed-form polynomial expressions in terms of the stress components are deduced.
for Yld 91 and KB93 criteria for BCC metals. Finally, it is shown that both Yld 91 and KB93 are in fact particular forms of Cazacu and Barlat [2] yield criterion.

2. Explicit expression of Yld91 in terms of stresses

Hershey-Hosford criterion for isotropic BCC materials can be expressed in terms of the invariants \( J_2 \) and \( J_3 \) as follows (for more details, see [5]):

\[
\phi_{\text{Yld91}}(s_1,s_2,s_3) = (s_1 - s_2)^6 + (s_2 - s_3)^6 + (s_1 - s_3)^6 = 66J_2^3 - 81J_3^2 \tag{7}
\]

Therefore, Yld91 for orthotropic BCC materials takes the simple form:

\[
\text{Yld91}_{\text{BCC}} = (\tilde{S}_1 - \tilde{S}_2)^6 + (\tilde{S}_2 - \tilde{S}_3)^6 + (\tilde{S}_1 - \tilde{S}_3)^6 = 66J_2^3 - 81J_3^2 \tag{8}
\]

In the above equation, \( \tilde{S}_i \) are the principal values of the transformed stress tensor \( \tilde{S} = \mathbf{C}s \), with the fourth-order tensor \( \mathbf{C} \) being orthotropic, symmetric, and deviatoric; \( J_2 = tr(\tilde{S}^2)/2 \) and \( J_3 = tr(\tilde{S}^3)/3 \), being the the second and third-invariant of the transformed tensor \( \tilde{S} \). The associated equivalent stress for Yld91\(_{\text{BCC}}\):

\[
\bar{\sigma} = 1.746\left(\tilde{J}_2^3 - 1.227\tilde{J}_3^3\right)^{1/6} \tag{9}
\]

If the fourth-order orthotropic tensor \( \mathbf{C} \) is represented by the 6x6 matrix:

\[
\mathbf{C} = \frac{1}{3}
\begin{bmatrix}
 b + \bar{c} & -\bar{c} & -b & 0 & 0 & 0 \\
 -\bar{c} & \bar{c} + a & -a & 0 & 0 & 0 \\
 -b & -a & a + b & 0 & 0 & 0 \\
 0 & 0 & 0 & 3f & 0 & 0 \\
 0 & 0 & 0 & 0 & 3g & 0 \\
 0 & 0 & 0 & 0 & 0 & 3h
\end{bmatrix}
\]

where, \( a \), \( b \), \( \bar{c}, f, g, h \) are independent parameters. Note that the invariants \( \tilde{J}_2 \) and \( \tilde{J}_3 \) of the transformed tensor \( \tilde{S} \) can be easily expressed in terms of the components of the stress deviator \( s \) as:

\[
\tilde{J}_2 = f^2s_{xx}^2 + g^2s_{yy}^2 + h^2s_{zz}^2 + \left[ \left( b + \bar{c} \right)s_{xx} - \bar{c}s_{yy} - bs_{zz} \right]^2 / 18 + \left[ -\bar{c}s_{xx} + \left( \bar{c} + a \right)s_{yy} - as_{zz} \right]^2 / 18 + \left[ -bs_{xx} - as_{yy} + \left( a + b \right)s_{zz} \right]^2 / 18
\]

\[
\tilde{J}_3 = 2(fgh)s_{xy}s_{xz}s_{yz} + \frac{1}{27} \left[ \left( b + \bar{c} \right)s_{xx} - \bar{c}s_{yy} - bs_{zz} \right] \left[ -\bar{c}s_{xx} + \left( \bar{c} + a \right)s_{yy} - as_{zz} \right] - \frac{f^2s_{xy}^2}{3} \left[ \left( b + \bar{c} \right)s_{xx} - \bar{c}s_{yy} - bs_{zz} \right] - \frac{g^2s_{yz}^2}{3} \left[ -\bar{c}s_{xx} + \left( \bar{c} + a \right)s_{yy} - as_{zz} \right] - \frac{h^2s_{xz}^2}{3} \left[ -bs_{xx} - as_{yy} + (a + b)s_{zz} \right] \tag{10}
\]

Remark: It is worth noting that Yld91 yield criterion for BCC materials is a particular case of the orthotropic yield criterion of Cazacu and Barlat [2] corresponding to \( c = 81/66 \). Indeed, by comparing...
the expressions of $J_2$ and $J_3$ (see Eq.(10)) with those of the orthotropic invariants given by Eq. (5) it is evident that $J_2$ is a particular case of $J_2^o$ which involves only 6 anisotropy coefficients, the expressions of the coefficient $a_i$, $i = 1...6$ in terms of $a$, $b$, $\overline{c}$, $f$, $g$, $h$ being:

$$a_i = \frac{\overline{c}(2\overline{c} + a + b) - ab}{3}; a_2 = \frac{a(2a + b + \overline{c}) - b\overline{c}}{3}; a_3 = \frac{b(2b + a + \overline{c}) - a\overline{c}}{3};$$

$$a_4 = h^2; a_5 = g^2; a_6 = f^2$$  \hspace{1cm} (11)

As demonstrated in [1], $\tilde{J}_3$ is a particular form of $J_3^o$ containing only six independent coefficients, the expressions of the parameters $b_k$, $k = 1...11$ in terms of $a$, $b$, $\overline{c}$, $f$, $g$, $h$ being:

$$b_1 = a\left(b^i - \overline{c}^i\right) / 3 + b\overline{c}(2\overline{c} + b) / 3; b_2 = b\overline{c}(b + \overline{c}) - b;$$

$$b_3 = b\overline{c}(a + \overline{c}) - b;$$

$$b_4 = ah^i; b_5 = \overline{c}f^i; b_6 = bf^i; b_7 = ag^i; b_8 = g^i(a + b) / 2;$$

$$b_{10} = h^i(a + b) / 2; b_{11} = fh$$  \hspace{1cm} (12)

Comparison between the expression of Yld91\textsubscript{BCC} in terms of $\tilde{J}_2$ and $\tilde{J}_3$ and the expression of the Cazacu and Barlat (2001) criterion (see Eq.(6)) shows that Yld91\textsubscript{BCC} is a particular form of the latter corresponding to $c = 81/66$, and with parameters $a_i$, $i = 1...6$ and $b_k$, $k = 1...11$ expressible in terms of the six independent coefficients $a$, $b$, $\overline{c}$, $f$, $g$, $h$ (see Eq.(11)-(12)).

3. Explicit expression of KB93 in terms of stresses

For isotropic BCC materials, the yield criterion given by Eq. (3) can be expressed in terms of the invariants $J_2$ and $J_3$ as follows (for more details, see [5]):

$$\phi_{2,\text{BCC}}(s_1, s_2, s_3) = s_1^6 + s_2^6 + s_3^6 = 3J_3^2 + 2J_2^2.$$  \hspace{1cm} (13)

Therefore, for $m = 6$, both $\phi_1(s_1, s_2, s_3)$ (i.e. Hershey-Hosford criterion) and $\phi_2(s_1, s_2, s_3)$ reduce to the Drucker [2] yield function with $c = 1.227$ and $c = -3/2$, respectively.

As mentioned, KB93 is the orthotropic extension of the isotropic yield function of Eq.(4), for BCC materials the recommended values of the parameters $m$ and $\alpha$ being: $m = 6$ and $\alpha = 0.17$, respectively. It follows that the effective stress according to the KB93 criterion for orthotropic BCC materials has the following simple expression:

$$\overline{\sigma} = 2.235\left(\tilde{J}_2^3 - 1.209\tilde{J}_3^2\right)^{1/6}.$$  \hspace{1cm} (14)

The expressions of the invariants $\tilde{J}_2$ and $\tilde{J}_3$ in terms of the stress components and the anisotropy coefficients $a$, $b$, $\overline{c}$, $f$, $g$, $h$ involved in the linear transformation are given by Eq.(10).

Remark1: It is worth noting that the explicit expression of KB93 given by Eq. (14) allows us to recognize that KB93 is in fact Cazacu and Barlat [2] criterion corresponding to $c = 1.209$ (see also Eq. (6) and the discussion in the preceding section concerning the invariants $\tilde{J}_2$ and $\tilde{J}_3$ as particular expressions of the orthotropic ones given by Eq. (5)).
Remark 2: Comparison between the explicit expressions of Yld 91 and KB93 for orthotropic BCC materials shows that the corresponding yield surfaces are very close. Indeed, Yld91 corresponds to $c = 1.227$ while KB93 corresponds to $c = 1.209$ (see Eq. (9) and Eq. (14), respectively) in the Cazacu and Barlat [2] criterion.

4. Application to DC06 steel

As an example, in Figure 1(a)-(b) are shown the predicted variation of the yield stresses and $r$-values in the $(x,y)$ plane or (RD, TD) according to the Cazacu and Barlat [2] and Yld 91 yield criterion, respectively, in comparison with measured yield stresses for a DC06 steel sheet (data reported in [6]-[7]). The coefficients involved in the Cazacu and Barlat [2] criterion were determined from the experimental yield stresses and $r$-values for $\theta = 0^\circ$, $30^\circ$, $45^\circ$, $60^\circ$, $90^\circ$ and the equibaxial tensile yield stress $\sigma_T^b$ and their numerical values are: $a_1 = 1.303$, $a_2 = 0.968$, $a_3 = 0.9822$, $a_4 = 1.0471$, $b_1 = 1.531$, $b_2 = 2.247$, $b_3 = 2.399$, $b_4 = 0.6070$, $b_5 = 1.6540$, $b_{10} = 1.248$, and $c = 1.30$ (for more details on the identification procedure, see Cazacu and Barlat [2]). The numerical values of the Yld91 anisotropy coefficients for this material are: $a = 0.8446$, $b = 0.8853$, $\bar{c} = 1.246$, and $h = 1.03$.

![Graph](image)

**Figure 1.** Predicted anisotropy according to Cazacu and Barlat [2] and Yld 91 [4] criteria for a DC06 steel sheet: (a) Uniaxial yield stresses; (b) Lankford coefficients. Data after [6]-[7].

5. Conclusions

In this paper, it was shown that the most widely used isotropic yield criteria expressible in terms of the principal values of the stress deviator are in fact identical in form. Moreover, explicit expressions of these isotropic yield criteria in terms of stress invariants have been presented. These expressions allowed recognizing that for BCC materials the orthotropic yield criteria Yld91 and KB93 coincide with Cazacu and Barlat [2] corresponding to specific values of the parameter $c$ and of the anisotropy coefficients, respectively. While in this paper, discussion was devoted to BCC materials, it is possible to obtain the explicit expressions in terms of $J_2$ and $J_3$ for isotropic FCC materials, and moreover deduce the explicit expressions of the Yld91 and KB93 orthotropic yield criteria in terms of stresses.

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References

[1] Karafillis A, Boyce M. 1993. *J Mech Phys Solids* 41:1859–1886.
[2] Cazacu O, Barlat F. 2001. *Math Mech Solids* 6:613–630.
[3] Drucker D.C. 1949. *ASME J Appl Mech* 16:349–35.
[4] Barlat F, Lege DJ, Brem JC. 1991. *Int J Plast* 7:693–712.
[5] Cazacu O, Revil-Baudard, B., Chandola, N. 2018 *Plasticity-Damage Couplings:From Single Crystal to Polycrystalline Materials* (New York: Springer).
[6] Haddadi H, Bouvier S, Banu M, Maier C, Teodosiu C. 2006. *Int J Plast* 22:2226–2271.
[7] Mattiasson K, Sigvant M. 2008. *Int J Mech Sci* 50:774–787.