Period Analysis using the Least Absolute Shrinkage and Selection Operator (Lasso)

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Abstract

We introduced least absolute shrinkage and selection operator (lasso) in obtaining periodic signals in unevenly spaced time-series data. A very simple formulation with a combination of a large set of sine and cosine functions has been shown to yield a very robust estimate, and the peaks in the resultant power spectra were very sharp. We studied the response of lasso to low signal-to-noise data, asymmetric signals and very closely separated multiple signals. When the length of the observation is sufficiently long, all of them were not serious obstacles to lasso. We analyzed the 100-year visual observations of \( \delta \) Cep, and obtained a very accurate period of \( 5.366326(16) \) d. The error in period estimation was several times smaller than in Phase Dispersion Minimization. We also modeled the historical data of R Sct, and obtained a reasonable fit to the data. The model, however, lost its predictive ability after the end of the interval used for modeling, which is probably a result of chaotic nature of the pulsations of this star. We also provide a sample R code for making this analysis.

Key words: methods: data analysis — methods: statistical — stars: individual (\( \delta \) Cep, R Sct)

1. Introduction

There has been a wealth of history in analyzing periodicities in time-series data, particularly for variable stars. Determination of periods in variable stars is a very fundamental issue in that they are used in many applications, including classification of variable stars, calibration of the period-luminosity relations, determination of the pulsation modes, detection of stellar rotation, detection of exoplanet transits and determination of the precession mode and frequency in accretion disks. These time-series data in practical astronomy are not usually sampled evenly in the time domain, and it is difficult to apply fast Fourier transform (FFT) directly as in other fields of science. Alternatively, there have been well-established methods based on discrete Fourier transform (Deeming 1975) and on least-squares fit of sinusoids for unevenly spaced data (Lomb-Scargle Periodogram, Scargle 1982; Horne, Baliunas 1986; Zechmeister, Kürster 2009). This type of method has been extended to extract finite numbers of signals by subsequently subtracting the strongest signals (CLEAN; Roberts et al. 1987).

Another class of approach has been taken by evaluating dispersions either in sum of lengths between phase-sorted data (Dworetsky 1983) or sum of dispersions in phased bins against trial periods. This class of approach includes Laffer, Kinman (1965), well-used Phase Dispersion Minimization (PDM, Stellingwerf 1978) and Analysis of Variance (AoV, Schwarzenberg-Czerny 1989).

Yet another class of approach to estimate the power spectrum is by assuming a sum of \( \delta \)-function-like poles on a frequency space extended to the full complex space (Childers 1978, chap. 2; Kay, Marple 1981). This approach is generally referred to as Maximum Entropy Method (MEM) or autoregressive model (AR). Although this approach is potentially very powerful in detecting a small number of sharp signals in the power spectrum, the usage in practical astronomical data has been limited because the well-known quick algorithm (Burg algorithm) using a recursive technique can only be applied to evenly spaced data (Burg 1967).

In recent years, there has been a remarkable progress in the field of Compressed Sensing, which deals with a class of problems in restoring or estimating sparsely scattered, finite number of parameters in a huge dimension. We present an application of this method to period analysis.

2. Methods

2.1. Decomposition to Fourier Components

The following description basically follows the mathematical formulation reviewed in Tanaka (2010). [For a review of Compressed Sensing, see Donoho (2006)]. We assume a time-series data \( Y(t_i) \) with unevenly spaced times of observations at \( t_i \). The data are assumed to be subtracted for their average, and the mean of \( Y \) is zero. The observation can be expressed as a sum of signal (\( Y_s \)) and random errors (\( n \)):

\[
Y_i = Y(t_i) = Y_s(t_i) + n(t_i).
\]
We assume that the signal is composed of a sum of strictly periodic functions. Using sine and cosine Fourier components, \( Y_s \) can be expressed as:
\[
Y_s(t_i) = \sum_j a_j \cos(\omega_j t_i) + \sum_j b_j \sin(\omega_j t_i),
\]
where \( \omega \) are frequencies and \( a \) and \( b \) are amplitudes. The problem can be set to estimate \( a \) and \( b \) from \( Y \). If the number of different \( \omega \) is larger than the half number of observations, the equation cannot be solved uniquely. Rewriting
\[
y = (Y_1, \ldots, Y_N)^T
\]
and
\[
x = (a_1, \ldots, a_M, b_1, \ldots, b_M)^T,
\]
where \( N \) and \( M \) are number of observations and number of different \( \omega \), respectively. We can rewrite a set of equations 1 and 2 as a form of
\[
y = Ax + n
\]
using a \( 2M \times N \) observation matrix \( A \) composed of
\[
A_{i,j} = \begin{cases} 
\cos(\omega_j t_i), & (i \leq M) \\
\sin(\omega_j - M t_i), & (i > M).
\end{cases}
\]
We define \( x_0 \) as the vector to be estimated.

The problem is to estimate sparse \( x_0 \), i.e., with only small number of non-zero elements, under the restriction of \( y = Ax \), if the time-series data is expected to have only a few periodic components, which is often the case with period analysis. We define “0-norm” \( ||x||_0 \) as the number of non-zero elements in \( x_0 \). The problem can be then formalized to choose \( x_0 \):
\[
\hat{x}^{(0)} = \arg\min_x ||x||_0 \text{ subj. to } y = Ax.
\]
This estimate is called \( \ell_0 \) regularization, and is known to be very robust in this problem setting. The problem, however, is that \( \ell_0 \) regularization is computationally difficult, and is known to be an NP-hard problem (Natarajan 1995).

In order to overcome this difficulty, the following alternative \( p \)-norms are usually employed:
\[
||x||_p = \begin{cases} 
(\sum_{i=1}^N |x_i|^p)^{1/p}, & (p > 0) \\
||x||_0, & (p = 0).
\end{cases}
\]
In particular, \( p = 1 \) (\( \ell_1 \)) regularization has an advantage in that it can be solved by conventional ways: by introducing a vector \( u = (u_1, \ldots, u_N)^T \) composed of non-negative elements, the equation becomes
\[
\min_{i=1}^N u_i \text{ subj. to } -u \leq x \leq u, \quad y = Ax,
\]
and it is a well-known linear programming (LP). Most notably, it has been recently demonstrated that this \( \ell_1 \) regularization is also the sparsest solution (\( \ell_0 \) regularization) in most large underdetermined systems (see e.g., Donoho 2006; Candes, Tao 2006).

There have been a number of methods using this \( \ell_1 \) constrained estimations. Among them, Tibshirani (1996) presented a concept of least absolute shrinkage and selection operator (lasso). This estimation has become well-known after the development of a fast algorithm, known as least angle regression (LAR) by Efron et al. (2004). The lasso is defined as:
\[
\hat{x}^{\text{LAR}} = \arg\min_x \{F(x) = \frac{1}{2N}||y - Ax||^2 + \lambda||x||_1\},
\]
where \( \lambda||x||_1 \) is the \( \ell_1 \)-norm penalty function with a parameter \( \lambda \) having a value of \( \lambda \geq 0 \). This estimate becomes identical with a least-squares estimation at \( \lambda = 0 \).

The optimal value of \( \lambda \) may be chosen by using the cross-validation technique: after breaking the data into two groups, use the one group for lasso analysis and use the rest of data for evaluation of residuals. Mean squared errors (MSE) were evaluated using several randomly chosen different partitioning of data. One can then estimate \( \lambda \) giving the smallest MSE. In application to analysis of actual time-series data, it might be helpful to survey the response of lasso using different \( \lambda \) values, rather than fixing at \( \lambda \) giving the smallest MSE, since the derived periods are not very different between different \( \lambda \) values, and since smaller \( \lambda \) gives smaller number of false signals, which is profitable when it is already known that only small number of signals are present in the data. An example of cross-validation diagram is shown in subsection 3.3 (figure 8).

We must note that the present application of \( \ell_1 \) regularization using sinusoidal functions is not unprecedented in astronomy, but that there was at least one in the pre-LAR era in analyzing radial-velocity data (Chen, Donoho 1998) based on the same scheme. We introduce the present method because time-series data in variable stars are commonly met in practical astronomy and are very suitable targets for the present analysis, either in detection of multiple periods or precise determination of the periods. Furthermore, the development of LAR algorithm and its public availability on a popular platform has enabled an application of the Compressed Sensing far easier to reach than in the time of Chen, Donoho (1998).

2.2. Numerical Solution

This LAR algorithm has been implemented as the package lars in R software\(^2\) and we used this package combined with glmnet (generalized linear model via penalized maximum likelihood) as a wrapper, which also provides cross-validation results. See appendix for a sample code.

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1. In actual calculation of lasso estimates, we need to normalize parameters, i.e., \( \sum_j A_{i,j} = 0, \quad \frac{1}{N} \sum_j A_{i,j}^2 = 1, \) for \( j = 1, \ldots, 2M \) (Friedman et al. 2010). We here introduce a conceptual formula and avoided complicated use of normalization factors. The glmnet package in R automatically performs this normalization and we usually do not need to consider the normalization.

2. The R Foundation for Statistical Computing: <http://cran.r-project.org/>.
3. Results

3.1. Response to Artificial Data: Single Frequency

We examined the response of this analysis to the artificial data. In order to reflect typical astronomical time-series observations, we used observations of Cep by Variable Star Observers’ League in Japan (VSOLJ), spanning 1906 through 2001, with a total number of 2955 visual estimates having large gaps (1911–1944, 1946–1955) and inhomogeneous density of observations (figure 1). We presented response to different $\lambda$ values as well as the model at the smallest MSE. We used the times of these observations and replaced the magnitudes with artificial data. The first one (figure 2) is a pure sine wave with a period of 10 d with Gaussian noises whose $\sigma$ is the same as the amplitude of the wave. In this case, the true period was always reproduced within errors of $10^{-4}$ d. Only small noises appear when $\lambda$ is very small. The second case is with Gaussian noises five times larger than the amplitude of the wave (figure 3). In this case, the signal was completely lost when $\lambda$ is small (over-expression of the noises by Fourier components). This result indicates that we need to choose adequate $\lambda$ depending on the signal-to-noise ratio of the data. The model with the smallest MSE is found to be an adequate solution in this case. The third case is the asymmetric waveform. We added second overtone (a period of 5 d) having a half amplitude of the primary wave. The noises were as in the first case (figure 4). The estimates of the period were observed to slightly deviate from the correct value (typically $2 \times 10^{-8}$ d), and the deviation depends on $\lambda$. We allowed two spectral windows 9.9–10.1 d and 4.95–5.05 d and obtained figure 5. After allowing the second window, the main period was more clearly detected. If cycle numbers are sufficient as in this case, asymmetric profiles did not significantly affect the results.

3.2. Separation of Peaks

We then examined the ability of this lasso estimate for identifying very closely separated signals. We analyzed a combination of two periods of 10 d and 10.005 d and they were always detected as separate signals regardless of the different sequence of random numbers used to add noises (figure 6). Assuming an effective baseline of 15000 d, these two periods correspond to a difference of 0.75 cycle between the limits of the baseline. In a case of 10 d and 10.001 d, we could resolve the signals in about half trials. It was impossible to resolve a combination of 10 d and 10.0005 d. This degree of resolution is, naturally, far beyond the reach of conventional techniques, such as PDM and Fourier-type analysis.

3.3. Analysis of Cephei

We performed an analysis of Cephei. The examined periods were 1000 equally spaced bins between 5.36 and 5.37 d. Since the peaks were very sharp, we enlarged the figure to show the profile (figure 7). The cross-validation diagram is shown in figure 8, which indicates that a combination of 5–40 frequencies best describes the data. The resultant (strongest) period was 5.366326 d (in actual calculation, we used ten times smaller bins to obtain this precision). The period in General Catalog of Variable Stars (Kholopov et al. 1985) is 5.366341 d. The period obtained by PDM was 5.36629(4) d (the error was estimated by the methods of Fernie 1989 and Kato et al. 2010). Since lasso estimate is highly nonlinear as in MEM, it is difficult to estimate the error of the obtained period. We...
alternatively performed delete-half jackknifing with random subsampling (Shao, Tu 1995): randomly selected 50% of observations and obtained periods from 10 different sets. The resultant 1-σ standard deviation was 0.000016 d. The estimate of the error appears to be consistent with the difference between our result and the GCVS period, and lasso estimate is several times more accurate than with the PDM in the present case. An addition of random typical errors (0.1 mag) of visual observations did not affect the result, indicating that lasso estimate is very robust in detecting strictly periodic signal under the presence of large errors.

3.4. Application to R Scti

R Sct is a well-known RV Tau-type variable star, characterized by the presence of alternating deep and shallow minima. Many RV Tau-type variables exhibit irregularities and the regular alternations of minima are often disturbed. R Sct is notable in its irregular behavior among RV Tau stars (e.g. Kolláth 1990; Buchler et al. 1996). Buchler et al. (1996) reported that Fourier decomposition reasonably well expresses the observed features of the light curve, but that its predictive power is limited. Since their Fourier decomposition was not based on \( \ell_1 \) regularization, it would be interesting to see whether the periods selected by \( \ell_1 \) regularization equally well describe the light curve and whether the model based on these periods has a better predictive power.

The data used were VSOLJ observations between 1906 and 2001 (16767 points). We used two windows of periods 60–80 d and 120–160 d. The strongest signals were at 143.8 d and 70.9 d, associated with weaker signals (figure 9). We used \( \log \lambda = -0.27 \), which is the most regularized model with a cross-validation error within one standard deviation of the minimum. Although the minimum MSE was reached by \( \log \lambda = -10.18 \), we did not adopt it because it was dominated by false signals and because it is physically less likely that such a large number of pulsation modes coexist in a radially pulsating star. It is noteworthy that the longer period is not the twice of the shorter period, in contrast to Fourier analysis (Kolláth 1990) or our own analysis with the PDM (142.1 d and 71.0 d). These periods may represent fundamental and first-overtone periods close to the 2:1, but slightly different, resonance which is expected to explain the RV Tau-type phenomenon (Takeuti, Petersen 1983).

The lasso model seems to adequately express observations (figure 10), indicating that varying amplitudes are basically a result of a combination of waves with periods close to the 2:1, but slightly different, resonance. It would be noteworthy that interval with very low amplitudes (e.g. JD 2451000–2451600) is well expressed by this model.

Figure 11 presents an example of lasso predictions. The
model soon lost the predictive ability after the end of the epoch (JD 2452240) used for modeling. We also obtained a model to the same data restricted to JD before 2451000. Although the model well describes the observations used in the fit, it again loses the ability after the end of the epoch used for modeling (figure 12). These results are similar to that in Buchler et al. (1996), and it is likely that a simple combination of multiple sinusoidal waves is not an adequate model for expressing the behavior of this object. This is probably caused by the chaotic nature of the pulsations in this star.

3.5. Splitting of Signals for Badly Phased Data

Although equation (6)-type bases are mathematically sufficient for ordinary Fourier analysis, and this formulation for lasso analysis is adequate for many cases (with sufficient number of data and length of time-series), the lasso response to even a simple cyclic function can yield split signals due to the lasso’s characteristics of its norm. This is problematic because it results in erroneous estimation of periods. Whereas we did not observe the problem in the examples presented so far, the problem can arise in certain applications of lasso-based methods to period analysis, and is therefore potentially harmful. We therefore discuss in this subsection what precisely the problem is, and then propose a means to circumvent it.

This situation can be understood by a simple example. If observed data has a form of \( y(t) = \sin(\omega t) \), the lasso result has an expected form of \( \hat{y}^{\text{LAR}}(t) = c \sin(\omega t) \), unless \( \lambda \) is too small. Although the amplitude \( c \) is different from 1, we can get the expected frequency \( \omega \). If the observed data, however, has a form \( y(t) = \sin(\omega t) + \cos(\omega t) \), (11) there are different solutions with equally acceptable squared residuals in a form of

\[
\hat{y}(t) = c_1 \sin(\omega_1 t) + c_2 \cos(\omega_2 t),
\]

and

\[
|c_1| + |c_2| < 2, \quad \omega_1 \neq \omega_2,
\]

which is preferred by lasso, and the result is observed as two signals at \( \omega_1 \) and \( \omega_2 \) around the expected signal \( \omega \). For a numerical example, a lasso application to the generated data for \( t = 1, \ldots, 100 \) and \( \omega = 1/10 \) in equation (11) yields \( \omega_1 = 1/9.818 \) and \( \omega_2 = 1/10.193 \) for any \( \lambda \) in \( -2.58 < \log \lambda < -0.21 \). The coefficients \( c_1 = 0.7632 \) and \( c_2 = 0.7545 \) yield \( \frac{1}{2\pi} \| y - A x \|_2^2 = 0.0039 \) and \( \| x \|_1 = 1.5177 \).\(^4\)

\(^4\) This split occurs regardless of the number of the data and length in time. Although the split is smaller with longer length in time, the deviation from the expected signal is larger than the accuracy of period determination with the PDM.
in equation (10).\(^5\) For \(ω_1 = ω_2 = 1/10\) and \(c_1 = c_2 = 1\), the corresponding values are 0 and 2, respectively. For \(λ\) larger than 0.008, the former set of parameters (with split signals) gives a smaller \(\frac{1}{Nk}||y - Ax||_2^2 + ||x||_1\). Although a very small \(λ\) might improve the situation, such a small \(λ\) will spoil the advantage of \(ℓ_1\) regularization.

Although this difficulty can be avoided by introducing complex coefficients and exp\((-iωt)\)-type basis (Candes et al. 2006; Rauhut 2007), this type of \(ℓ_1\) regularization cannot be achieved by lars package. As an alternative, practical approach to reduce this false splitting of signals, we have introduced an \(N_kM \times N\) matrix in the following form:

\[
A_{kM+i,j} = \cos(ωjt_j + \frac{k}{N_k}π),
\]

(14)

where \(N_k\) is the number of phase bins and \(k = 0, \cdots, N_k - 1\). Although the equation (14)-type basis is not mathematically independent (it is an “over-complete” basis or a “frame”), the non-linear response of lasso, caused by the characteristics of \(ℓ_1\)-norm, enables us to detect an adequate frequency and phase at the same time. The num-

\(^5\) These \(c_1\) and \(c_2\) were chosen to minimize \(\frac{1}{Nk}||y - Ax||_2^2\) in order to best illustrate the problem. The actual values of \(c_1\) and \(c_2\) selected by lasso are dependent on \(λ\). For example, \(λ = 0.095\) yields \(c_1 = 0.5488\) and \(c_2 = 0.5599\), giving \(\frac{1}{Nk}||y - Ax||_2^2 = 0.0399\) and \(||x||_1 = 1.1087\). In this case, \(F(x) = 0.145\) and is indeed smaller than \(F(x) = 0.190\) for \(ω_1 = ω_2 = 1/10\) and \(c_1 = c_2 = 1\).
Fig. 10. A segment of lasso modeling to observations of R Sct. The points represent observations and the curves represent a lasso model.

Fig. 11. Lasso predictions of R Sct. The points represent VSNET (Kato et al. 2004) observations and the curves represent a lasso model. The model soon lost the predictive ability after the end of the interval (JD 2452240) used for modeling.

Fig. 12. A segment of lasso modeling to observations of R Sct. The data are the same as in figure 10. The points represent observations and the curves represent a lasso model fitted for the data before JD 2451000.

The number \( N_k \) can be determined experimentally, and the formulation becomes equivalent to equation (6) in the case of \( N_k = 2 \). In many of randomly sampled and randomly phased actual data, there is no special need to use \( N_k > 2 \). Even in most extreme cases, we have found that \( N_k = 40 \) successfully reproduced the original spectrum.

We used an artificial data \( \mathbf{x} = (1, \ldots, 100)^T \) and used a function of \( y(x) = \sin(2\pi x/10 + 0.63) \) to produce \( y \). The phase offset of 0.63 was chosen to produce the most unfavorable condition. The frequency window corresponds to periods of 9.9–10.1. The effect of different \( N_k \) on this artificial data is shown in figure 13. Since our present aim is to determine the single period precisely, we did not employ cross-validation, which increases the number of detected signals, but used a fixed \( \log \lambda = -2.6 \). In the case of \( N_k = 2 \), the reproduced signals are located on the upper and lower limits of the tested range. In the case of \( N_k = 4 \), the separation between split peaks became smaller, and we could obtain a merged signal with \( N_k = 40 \). With an even larger \( N_k \), false peaks were more prone to appear.

Using \( y(x) = \sin(2\pi x/10 + \pi/4) \), which corresponds to the example in equation 11, we could obtain a reasonable result (single signal) with \( N_k = 4 \). The experiment
suggests that false splitting of signals is expected to be avoided with $N_k$ up to 40, even in the worst cases.

4. Conclusion

We introduced lasso in obtaining periodic signals in unevenly spaced time-series data. A very simple formulation with a combination of a large set of sine and cosine functions has been shown to work very well, and the peaks in the resultant power spectra were very sharp. We studied the response of lasso to low signal-to-noise data, asymmetric signals and very closely separated multiple signals. When the length of the observation is sufficiently long, all of them were not serious obstacles to lasso. We analyzed the 100-year visual observations of $\delta$ Cep, and obtained a very accurate period of 5.366326(16) d. The error in period estimation was several times smaller than in PDM. We also modeled the historical data of R Sct, and obtained a reasonable fit to the data. The main two periods (143.8 d and 70.9 d) were not exactly in a 2:1 ratio. The model, however, lost its predictive ability soon after the end of the data used for modeling, which is probably a result of chaotic nature of the pulsations of this star. We also formulated a scheme by using different set of phases in the basis, and confirmed that this scheme is useful when the phases of observed data are most unfavorable for lasso.

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Appendix. Sample R code

We provide a sample R code to make lasso analysis of time-series data. We assume that the data $t = (t_1, \cdots, t_N)^T$, $y$ are stored as two elements $V1$ and $V2$ in the data frame $d$. The variable $p$ is a vector of assumed periods (for convenience, we provide a function seqfreq to make a series of periods evenly spaced in the frequency space). The lasso estimate can be obtained by $\text{pow} \leftarrow \text{perlasso}(d, p)$. The parameter $\text{ndiv}$ corresponds to $N_k$ in the text, and the parameter $\text{cv}$ is a flag whether to compute cross-validations. The results can be plotted by $\text{plot}(\text{pow}, n)$, where $n$ represents the $\lambda$ bin (the values of $\lambda$ are stored in $\text{pow}$). The result of cross-validation is stored in the list element of $\text{gcv}$ and the element $\text{nmin}$ is the $\lambda$ bin giving the smallest MSE. The function $\text{minper}$ returns the strongest signals and the function $\text{lassofit}$ presents a fit to the data.

```r
library(lars)
library(glmnet)

seqfreq <- function(a,b,...) {
  return(1/seq(1/b,1/a,...))
}

makematlasso <- function(d,p,ndiv) {
  nd <- length(p)
  m <- matrix(0,nrow(d),nd*ndiv)
  for (i in 1:nd) {
    ph <- ((d$V1/p[i]) %% 1)*pi*2
    for (j in 0:(ndiv-1)) {
      m[,i+nd*j] <- sin(ph+pi*j/ndiv)
    }
  }
  return(m)
}

perlasso <- function(d,p,ndiv=2,alpha=1, cv=FALSE) {
  nd <- length(p)
  mat <- makematlasso(d,p,ndiv)
  y <- d$V2 - mean(d$V2)
  m <- glmnet(mat,y,alpha=alpha)
  ndim <- m$dim[2]
  pow <- matrix(0,nd,ndim)
  for (i in 1:ndim) {
    v <- m$beta[,i]
    for (j in 0:(ndiv-1)) {
      pow[,i] <- pow[,i] + v[(nd*+j+1):(nd*(j+1))]^2
    }
  }
  return(pow)
}
```

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```r
library(lars)
library(glmnet)

seqfreq <- function(a,b,...) {
  return(1/seq(1/b,1/a,...))
}

makematlasso <- function(d,p,ndiv) {
  nd <- length(p)
  m <- matrix(0,nrow(d),nd*ndiv)
  for (i in 1:nd) {
    ph <- ((d$V1/p[i]) %% 1)*pi*2
    for (j in 0:(ndiv-1)) {
      m[,i+nd*j] <- sin(ph+pi*j/ndiv)
    }
  }
  return(m)
}

perlasso <- function(d,p,ndiv=2,alpha=1, cv=FALSE) {
  nd <- length(p)
  mat <- makematlasso(d,p,ndiv)
  y <- d$V2 - mean(d$V2)
  m <- glmnet(mat,y,alpha=alpha)
  ndim <- m$dim[2]
  pow <- matrix(0,nd,ndim)
  for (i in 1:ndim) {
    v <- m$beta[,i]
    for (j in 0:(ndiv-1)) {
      pow[,i] <- pow[,i] + v[(nd*+j+1):(nd*(j+1))]^2
    }
  }
  return(pow)
}
```

library(lars)
library(glmnet)

seqfreq <- function(a,b,...) {
  return(1/seq(1/b,1/a,...))
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makematlasso <- function(d,p,ndiv) {
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      m[,i+nd*j] <- sin(ph+pi*j/ndiv)
    }
  }
  return(m)
}

perlasso <- function(d,p,ndiv=2,alpha=1, cv=FALSE) {
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  mat <- makematlasso(d,p,ndiv)
  y <- d$V2 - mean(d$V2)
  m <- glmnet(mat,y,alpha=alpha)
  ndim <- m$dim[2]
  pow <- matrix(0,nd,ndim)
  for (i in 1:ndim) {
    v <- m$beta[,i]
    for (j in 0:(ndiv-1)) {
      pow[,i] <- pow[,i] + v[(nd*+j+1):(nd*(j+1))]^2
    }
  }
  return(pow)
}
Period Analysis using Lasso

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