Quasilinear wave equations on Schwarzschild–de Sitter

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ABSTRACT
We give an elementary new argument for global existence and exponential decay of solutions of quasilinear wave equations on Schwarzschild–de Sitter black hole backgrounds, for appropriately small initial data. The core of the argument is entirely local, based on time translation invariant energy estimates in spacetime slabs of fixed time length. Global existence then follows simply by iterating this local result in consecutive spacetime slabs. We infer that an appropriate future energy flux decays exponentially with respect to the energy flux of the initial data.

1. Introduction
We revisit the problem of global existence of solutions of quasilinear wave equations, of the form

$$\Box g(\nabla \psi) \psi = \partial \psi \cdot \partial \psi,$$

with $g(\nabla \psi) = g_{M,\Lambda} + h(\nabla \psi)$, where $g_{M,\Lambda}$ is the metric of the Schwarzschild–de Sitter black hole spacetime, also $\partial \psi \cdot \partial \psi = a^i \partial_i \psi \partial_j \psi$, where $a, h$ are sufficiently regular tensors, with $h(0) = 0$. Specifically, we are interested in a spacetime region that is slightly larger than that enclosed by the event and cosmological horizons, respectively $\mathcal{H}^+, \bar{\mathcal{H}}^+$, see the shaded region of Figure 1.

The problem of stability of quasilinear wave equations on such backgrounds has been extensively studied by Hintz and Vasy, see [1–4], where they arrived at global stability results. Their papers appeal to machinery from microlocal analysis and Nash Moser iteration arguments. These results were proceeded by a long list of results on the linear problem, see [5–13]. Moreover, note the recent remarkable global nonlinear stability proof for the slowly rotating Kerr–de Sitter black hole as a solution of the Einstein vacuum equation with $\Lambda > 0$, by Hintz–Vasy [14], based in part on the above works. (For some results on the cosmological region see [15, 16]). Note that the non-linear stability of the pure de Sitter spacetime has been obtained previously by Friedrich [17].

Schwarzschild–de Sitter can be thought of as the $\Lambda > 0$ analogue of the Schwarzschild and Kerr spacetimes, which are celebrated solutions of the vacuum Einstein equation with $\Lambda = 0$. For the study of linear equations on the latter see for instance [18, 19] and the definitive [20]. These results have been used to prove nonlinear stability results for equations of type (1.1),...
Figure 1. The Schwarzschild–de Sitter spacetime.

see [21, 22]. In general, these non-linear problems are more difficult than the $\Lambda > 0$ case, because the expected decay is only polynomial and one has to assume and exploit suitable null structure (see [23]) for the non-linearities. For results on stability of black hole spacetimes with $\Lambda = 0$ see [24–30].

In principle, one approach to the study of (1.1) on Schwarzschild or Kerr–de Sitter backgrounds would be to directly adapt the methods from the $\Lambda = 0$ case. Such an approach, however, would not fully exploit the aspects that make the $\Lambda > 0$ problem easier.

The purpose of this paper is to introduce a physical space approach to (1.1) which is well tailored to this setting. Our approach is based entirely on local in time translation invariant energy estimates (Theorem 1) and an iteration argument in consecutive spacetime regions (Theorem 2). This approach will use the results of our accompanying physical space linear paper [31], where we utilized a physical space commutation with a vector field $G$, see already (1.6), and proved a relatively non-degenerate estimate, using also a Morawetz estimate proved in [10]. Note that Holzegel–Kauffman originally introduced the analogue of the $G$ vector field in the $\Lambda = 0$ case, see [32]. Our physical space commutation with $G$, in the high frequency limit, connects with the work of previous authors on ‘lossless estimates’ and ‘non-trapping estimates’, e.g. see [2, 33–38].

We will present the rough version of our Theorems, for which the reader may wish to refer to Figure 1 for the Schwarzschild–de Sitter spacetime.

We denote as

$$\mathcal{M}_\delta$$

the ‘extended’ exterior region, also see the dark shaded region of Figure 1, where $\delta$ is a smallness parameter that parametrises how far the boundaries of $\mathcal{M}_\delta$ (which we denote as $\mathcal{H}_\delta^+, \overline{\mathcal{H}}_\delta^+$) are from the event horizon $\mathcal{H}^+$ and the cosmological horizon $\overline{\mathcal{H}}^+$, respectively. We choose $\delta$ in the proof of our Theorem 2 in Section 7. For the precise definition of $\mathcal{M}_\delta$ see
already Definition 2.1. The spacetime domain $\mathcal{M}_\delta$, see the dark shaded region of Figure 1, is foliated by the spacelike hypersurfaces

$$\{\bar{t} = \tau\},$$

with $\tau \geq 0$, see Figure 1, where

$$\bar{t}, r, \theta, \phi$$

are appropriate (non-standard) hyperboloidal coordinates in which the metric takes the form (2.6). The coordinate vector field $\partial_{\bar{t}}$ is Killing. We denote by $\Omega_\alpha$, $\alpha = 1, 2, 3$ the generators of the Lie algebra $\text{so}(3)$ associated with the $(\theta, \phi)$ spheres.

We will consider two types of energies.

The first energy is a non-degenerate energy which we define as

$$E_j[\psi](\tau) = \sum_{1 \leq i_1 + i_2 + i_3 \leq j} \sum_{\alpha = 1, 2, 3} \int_{\{\bar{t} = \tau\}} \left( \partial_t^i \Omega^l \partial_r^j \psi \right)^2$$

for all $j \geq 1$, with respect to the induced volume form of $g_{M, \Lambda}$ on the $\{\bar{t} = \tau\}$ hypersurface.

Now, we discuss the second energy that we will consider. The first two terms of the second energy, see (1.7), are associated with the $C^{0,1/2}$ vector field $G$ of [31] (see the previous [32] for $\Lambda = 0$), which is defined as

$$G \doteq \begin{cases} r \sqrt{1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2} \partial_r, & 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 > 0 \\ 0, & 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \leq 0 \end{cases}$$

in appropriate hyperboloidal coordinates $\bar{t}, r, \theta, \phi$, see already Definition 2.4. We define the second energy as

$$E_{G, j}[\psi](\tau) = \int_{\{\bar{t} = \tau\}} \sum_{1 \leq i_1 + i_2 + i_3 \leq j - 1} \sum_{i_3 \geq 1} \left( 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right)^{2i_3 - 1} \left( \partial_t^i \Omega^l \partial_r^j G \psi \right)^2$$

for all $j \geq 2$, with respect to the induced volume form of $g_{M, \Lambda}$ on the $\{\bar{t} = \tau\}$ hypersurface. Note that the highest order integrands of the energies in (1.7) are identically zero where $G \equiv 0$.

Note the inequality

$$E_{j-1}[\psi](\tau) \leq E_{G, j}[\psi](\tau) \lesssim E_j[\psi](\tau),$$

also see already Remark 2.5. For the formal definition of the above energies see Definition 2.5.

The rough version of our main Theorem 1 is the following. (See the dark shaded region of Figure 2 for the local spacetime slab we will consider.)

**Theorem 1** (rough version). Let $k \geq 7$. There exists a constant $C > 0$, depending on $k, M, \Lambda$ and on the tensors $a, h$, see (1.1), such that, for $L > 0$ an arbitrary positive number, the following holds.
There exists a $\tau_{\text{step}}(L) > 0$ sufficiently large and there exist
\[ \delta = \delta(\tau_{\text{step}}) > 0, \quad \epsilon = \epsilon(\tau_{\text{step}}, \delta) > 0 \] (1.9)
sufficiently small such that for all $\tau_1 \geq 0$ and
\[ \tau_2 = \tau_1 + \tau_{\text{step}} \] (1.10)
if we take initial data for (1.1) on $\{\tau = \tau_1\}$ with
\[ E_{k+1}[\psi](\tau_1) \leq \epsilon \] (1.11)
then there exists a unique solution to the quasilinear wave equation (1.1) on $M_\delta \cap \{\tau_1 \leq \tau \leq \tau_2\}$ and the following estimates are satisfied
\[ E_{G,k}[\psi](\tau_2) \leq \frac{1}{L} E_{G,k}[\psi](\tau_1), \] (1.12)
and
\[ E_{G,k}[\psi](\tau') \leq CE_{G,k}[\psi](\tau_1), \]
\[ E_{k+1}[\psi](\tau') \leq CE_{k+1}[\psi](\tau_1), \]
\[ E_{k+2}[\psi](\tau') \leq CE_{k+2}[\psi](\tau_1), \] (1.13)
for all $\tau' \in [\tau_1, \tau_1 + \tau_{\text{step}}]$.

Finally,
\[ E_{k+1}[\psi](\tau_2) \leq Ce^{-\frac{1}{3} \log(L) + \frac{2}{3} \log C} \left(E_{G,k}[\psi](\tau_1)\right)^{1/3} \left(E_{k+2}[\psi](\tau_1)\right)^{2/3}. \] (1.14)

It is instructive to compare the results of Theorem 1 with the main results of our linear theory [31] on a Schwarzschild–de Sitter background. Specifically, the estimate of inequality (1.12) and the first inequality of (1.13) corresponds to the estimate in linear theory that
one obtains from the commutation with the $G$ vector field, see already Theorem 2.1. The remaining inequalities of (1.13) correspond to the uniform boundedness results of [10].

We use the result of Theorem 1, for a well chosen $L$ so that the constant in (1.14) is sufficiently small and, by a completely elementary iteration argument on consecutive spacetime regions, see Figure 2 for such a region, we prove that the solution of the quasilinear wave equation (1.1) exists globally and decays exponentially. The rough version of our main Theorem 2 is the following.

**Theorem 2 (rough version).** Let $k \geq 7$. Then, there exist constants $c_d, c_g > 0$ and there exists a $\delta > 0$ and an $\epsilon > 0$ sufficiently small, such that if the initial energy satisfies

$$E_{k+2}[\psi](0) \leq \epsilon,$$

on $\Sigma = \{ \tilde{t} = 0 \}$ then the solution exists globally on $M_\delta$ and the energy $E_{G,k}[\psi]$ decays exponentially

$$E_{G,k}[\psi](\tau) \lesssim e^{-c_d \tau} E_{G,k}[\psi](0).$$

We note that the classical Sobolev energies $E_{k-1}[\psi](\tau), E_{k+1}[\psi](\tau)$ decay exponentially

$$E_{k-1}[\psi](\tau) \lesssim e^{-c_d \tau} E_{G,k}[\psi](0), \quad E_{k+1}[\psi](\tau) \lesssim e^{-c_d \tau} E_{k+2}[\psi](0)$$

while for the top order energy we only have

$$E_{k+2}[\psi](\tau) \lesssim e^{c_g \tau} E_{k+2}[\psi](0).$$

**Remark 1.1.** Note that the growth constant $c_g$ of (1.18) of Theorem 2 can in fact be made arbitrarily small, restricting to sufficiently small $\epsilon > 0$. See already Remark 4.3.

**Remark 1.2.** In our Theorem 2, we improve slightly on the regularity assumption of initial data of [4]. Specifically, we only require the initial data to be in the Sobolev space $H^9$, see our main Theorems 1, 2 and Remark 4.1.

**Remark 1.3.** In Section 8, we present Theorems 1’ and 2’ which give global non-linear stability in the semilinear case, with the tensor $h \equiv 0$, under weaker assumptions than those of Theorem 2. Specifically, we require the initial data only to be in the Sobolev space $H^7$.

**Remark 1.4.** Note that by using the results of our forthcoming [39] on Kerr–de Sitter, the global stability results of the present paper generalize to the slowly rotating Kerr–de Sitter case.

## 2. Preliminaries and notation

### 2.1. The manifolds, metrics and spacetime domains

The definitions and notation of this Section have already been introduced in our [31].

Fix $M, \Lambda > 0$ such that

$$r_+ < \tilde{r}_+$$

are the two positive real roots of

$$1 - \mu = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2.$$
We need the following definition

**Definition 2.1.** For \( \tau_2 \geq \tau_1 \geq 0 \) we define
\[
D(\tau_1, \tau_2) = [\tau_1, \tau_2]_i \times [r_+ \bar{r}_+]_r \times S^2_{(\theta, \varphi)}.
\] (2.3)

We also define, for \( \delta > 0 \),
\[
D_\delta(\tau_1, \tau_2) = [\tau_1, \tau_2]_i \times [r_+ - \delta \bar{r}_+ + \delta]_r \times S^2_{(\theta, \varphi)}
\] (2.4)
and
\[
\mathcal{M} = D(0, \infty) \equiv [0, \infty]_i \times [r_+ \bar{r}_+]_r \times S^2_{(\theta, \varphi)},
\]
\[
\mathcal{M}_\delta = D_\delta(0, \infty) \equiv [0, \infty]_i \times [r_+ - \delta \bar{r}_+ + \delta]_r \times S^2_{(\theta, \varphi)}.
\] (2.5)

We refer to the coordinates \((\bar{t}, r, \theta, \varphi)\) as regular hyperboloidal coordinates.

We need the following definition

**Definition 2.2.** We denote as \((\mathcal{M}_\delta, g_{M, \Lambda})\) the Schwarzschild–de Sitter spacetime, with metric
\[
g_{M, \Lambda} = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) (d\bar{t})^2 - 2 \frac{1 - \frac{3M}{r}}{\sqrt{1 - 9M^2 \Lambda}} \sqrt{1 + \frac{6M}{r} dr} + \frac{27M^2}{1 - 9M^2 \Lambda} \frac{1}{r^2} (dr)^2 + r^2 d\sigma_{S^2},
\] (2.6)
in regular hyperboloidal coordinates \((\bar{t}, r, \theta, \varphi)\), where \(d\sigma_{S^2} = d\theta^2 + \sin^2 \theta d\varphi^2\) is the standard metric of the unit sphere \(S^2\). We will often denote the metric \(g_{M, \Lambda}\) as
\[
\hat{g}.
\] (2.7)

Note the following remarks

**Remark 2.1.** For a sufficiently small \(\delta > 0\), note that the hypersurfaces
\[
\{\bar{t} = \tau\}
\] (2.8)
are spacelike in \(\mathcal{M}_\delta\), with respect to the metric of the following Definition 2.2.

**Remark 2.2.** Note that \(\delta\) will be fixed in the proof of Theorem 2 in Section 7. In what follows \(\delta\) will always be assumed sufficiently small.

**Remark 2.3.** The inverse metric components of the metric of Definition 2.2 are
\[
\hat{g}^{rr} = (1 - \mu), \quad \hat{g}^{\bar{t}\bar{t}} = -\frac{1 - \xi^2(r)}{1 - \mu}, \quad \hat{g}^{\bar{t}r} = -\xi(r).
\] (2.9)
for
\[
\xi(r) = \frac{1 - \frac{3M}{r}}{\sqrt{1 - 9M^2 \Lambda}} \sqrt{1 + \frac{6M}{r}}.
\] (2.10)

**Remark 2.4.** Note the usual expression of the Schwarzschild–de Sitter metric is in coordinates \((t, r, \theta, \varphi) \in \mathbb{R}_t \times (r_+, \bar{r}_+) \times S^2_{(\theta, \varphi)}\), which reads
\[
\hat{g} = -(1 - \mu) (dt)^2 + (1 - \mu)^{-1} (dr)^2 + r^2 d\sigma_{S^2}.
\] (2.11)
where note the relation
\[ \tilde{t} = t + H(r), \quad H(r) = \int_{3M}^{r} \frac{\xi(r)}{1 - \mu} \, dr. \] (2.12)

We denote as
\[ \nabla \] (2.13)
the covariant derivative of the Riemannian metric \( r^2 \, ds^2 \). Moreover, we denote the standard generators of the \( so(3) \) Lie Algebra
\[ \Omega_\alpha, \quad \alpha = 1, 2, 3, \] (2.14)
associated with the \((\theta, \varphi)\) spheres. These can be thought of as vector fields on \( \mathcal{M}_\delta \).

Note that
\[ |\nabla f|^2 \sim \sum_{\alpha} \frac{1}{r^2} |\Omega_\alpha f|^2 \] (2.15)
for any sufficiently regular \( f \), where the constants in the above only depend on the black hole parameters. (We have included the inessential \( r \) factor above for comparison with the asymptotically flat case.)

### 2.2. The horizons and auxiliary spacelike hypersurfaces

We need the following definition

**Definition 2.3.** The event and cosmological horizons are defined as
\[ \mathcal{H}^+ = \{ r = r_+ \}, \quad \tilde{\mathcal{H}}^+ = \{ \tilde{r} = \tilde{r}_+ \} \] (2.16)
also see [31, 39]. These hypersurfaces are null with respect to the metric \( g_{M, \Lambda} \).

For sufficiently small \( \delta > 0 \), we define the following spacelike hypersurfaces
\[ \mathcal{H}_\delta^+ = \{ r = r_+ - \delta, \tilde{t} \geq 0 \}, \quad \tilde{\mathcal{H}}_\delta^+ = \{ r = \tilde{r}_+ + \delta, \tilde{t} \geq 0 \}, \] (2.17)
in the coordinates \((\tilde{t}, r, \theta, \varphi)\) of Definition 2.2.

### 2.3. Volume forms and normals for the metric \( g_{M, \Lambda} \)

We denote the spacetime volume form of \( \hat{g} \) as
\[ d\hat{g} = r^2 \sin \theta \, d\tilde{t} \, dr \, d\theta \, d\varphi \] (2.18)
with respect to the \((\tilde{t}, r, \theta, \varphi)\) coordinates.

By pulling back the spacetime volume form (2.18) into hypersurfaces of constant \( \tilde{t} \), we obtain that the \( \{ \tilde{t} = \tau \} \) hypersurfaces admit the volume form
\[ d\hat{g}_{[\tilde{t} = \tau]} = r \sqrt{\frac{27M^2}{1 - 9M^2 \Lambda}} \sin \theta \, dr \, d\theta \, d\varphi. \] (2.19)

We note that the normal of \( \{ \tilde{t} = \tau \} \), with respect to \( \hat{g} \), is
\[ \hat{n}_{[\tilde{t} = \tau]} = \sqrt{\frac{27M^2}{1 - 9M^2 \Lambda}} \frac{1}{r^2} \frac{\partial}{\partial \tilde{t}} + \frac{\xi(r)}{\sqrt{\frac{27M^2}{1 - 9M^2 \Lambda}} \frac{1}{r^2}} \frac{\partial}{\partial r}, \] (2.20)
which we also simply denote as
\[ \hat{n}, \]  
(2.21)
for \( \xi(r) \) see Remark 2.3. We denote the normals of the event and cosmological horizons respectively as
\[ \hat{n}_{H^+} = \partial_t, \quad \hat{n}_{\bar{H}^+} = \partial_{\bar{t}}. \]  
(2.22)
With the above choice of normals, the corresponding volume forms of the respective null hypersurfaces take the form
\[ d\hat{g}_{H^+} = r^2 \sin \theta d\sigma_{S^2}, \quad d\hat{g}_{\bar{H}^+} = r^2 \sin \theta d\sigma_{S^2}. \]  
(2.23)
Furthermore, for \( \delta > 0 \) sufficiently small, the vectors
\[ \hat{n}_{H^+}(\delta) = \frac{\nabla r}{\sqrt{\hat{g}(\nabla r, \nabla r)}}, \quad \hat{n}_{\bar{H}^+}(\delta) = -\frac{\nabla r}{\sqrt{\hat{g}(\nabla r, \nabla r)}}, \]  
(2.24)
are the unit outward normals of the spacelike hypersurfaces \( H^+_{\delta}, \bar{H}^+_{\delta} \) respectively, with respect to the metric \( \hat{g} \). Note that there exist smooth functions
\[ c_1 : (r_+ - \delta, r_+) \to \mathbb{R}, \quad \tilde{c}_1 : (\bar{r}_+ + \delta, \bar{r}_+) \to \mathbb{R}, \]  
(2.25)
with
\[ \sqrt{|1 - \mu|} c_1(r) = 1 + O(1 - \mu), \quad \sqrt{|1 - \mu|} \tilde{c}_1(r) = 1 + O(1 - \mu), \]  
(2.26)
as \( r \to r_+ \), \( r \to \bar{r}_+ \) respectively, and \( c_2(r) = -\sqrt{|1 - \mu|}, \tilde{c}_2(r) = \sqrt{|1 - \mu|} \), such that the normals of \( H^+_{\delta}, \bar{H}^+_{\delta} \) can be written respectively as
\[ \hat{n}_{H^+_{\delta}} = c_1(r) \partial_t + c_2(r) \partial_r, \quad \hat{n}_{\bar{H}^+_{\delta}} = \tilde{c}_1(r) \partial_{\bar{t}} + \tilde{c}_2(r) \partial_r. \]  
(2.27)
By pulling back the spacetime volume form (2.18) on hypersurfaces of constant \( r = r_+ - \delta \), or \( r = \bar{r}_+ + \delta \), we obtain the respective volume forms
\[ d\hat{g}_{H^+_{\delta}}, \quad d\hat{g}_{\bar{H}^+_{\delta}} = r^2 \sqrt{|1 - \mu|} \sin \theta d\sigma_{S^2}, \]  
(2.28)
which we both denote simply as \( d\hat{g}_{\mathcal{H}_\delta} \).

### 2.4. Coarea formula

There exists a constant \( C(M, \Lambda) > 0 \) such that
\[ C^{-1} \int_{\tau_1}^{\tau_2} d\tau \int_{\bar{t})^{\tau_1}} f d\bar{g}_{(\bar{t})^{\tau_1}} \leq \int \int_{D_\delta(\tau_1, \tau_2)} f d\hat{g} \leq C \int_{\tau_1}^{\tau_2} d\tau \int_{\bar{t} = \tau_1}^{\bar{t} = \tau_2} f d\hat{g}_{(\bar{t} = \tau_1)} \]  
(2.29)
for any continuous non-negative function \( f \).

### 2.5. The vector fields and the energies

We need the following
Definition 2.4. We define the following vector field
\[ G = r \sqrt{1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \partial_r}, \]  
for \( r \in [r_+, \overline{r}_+] \) with respect to the hyperboloidal coordinates \((\overline{t}, r, \theta, \phi)\).

We extend the vector field (2.30) to a \( C^{0,1/2} \) vector field on \( \mathcal{M}_3 \) by
\[ G \equiv 0 \text{ in } \{ r_+ - \delta \leq r \leq r_+ \} \cup \{ \overline{r}_+ - \delta \leq r \leq \overline{r}_+ + \delta \}. \]  
(2.31)

We need the following definition

Definition 2.5. On the spacelike hypersurface \( \{ \overline{t} = \tau \} \subset D_\delta(0, \infty) \), with respect to the metric \( \hat{g} \), we define the non-degenerate high order energy
\[ E_j[\psi](\tau) = \sum_{1 \leq i_1 + i_2 + i_3 \leq j} \sum_{\alpha=1,2,3} \int_{\{\overline{t} = \tau\}} \left( \partial_{\overline{t}}^{i_1} \partial_r^{i_2} \Omega^{i_3}_\alpha \psi \right)^2 d\hat{g}(\overline{t} = \tau), \]  
with \( j \geq 1 \), where for \( \Omega^\alpha \) see Section 2.1.

We define the following high order energy
\[ E_{G,j}[\psi](\tau) = \int_{\{\overline{t} = \tau\}} \left( \sum_{1 \leq i_1 + i_2 + i_3 \leq j-1} \sum_{\alpha=1,2,3} (1 - \mu)^{2i_1-1} \left( \partial_{\overline{t}}^{i_1} \Omega^{i_2}_\alpha \partial_r^{i_3} G \psi \right)^2 \right. \]  
\[ + \left. \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha=1,2,3} \left( \partial_{\overline{t}}^{i_1} \Omega^{i_2}_\alpha G \psi \right)^2 \right. \]  
\[ + \left. \sum_{1 \leq i_1 + i_2 + i_3 \leq j-1} \sum_{\alpha=1,2,3} \left( \partial_{\overline{t}}^{i_1} \partial_r^{i_2} \Omega^{i_3}_\alpha \psi \right)^2 \right) d\hat{g}(\overline{t} = \tau), \]  
(2.33)

for \( j \geq 2 \).

Remark 2.5. For any \( j \geq 2 \), there exists a constant \( C(j, M, \Lambda) > 0 \), such that for all \( \tau \geq 0 \) the following holds
\[ E_{j-1}[\psi](\tau) \leq E_{G,j}[\psi](\tau) \leq CE_j[\psi](\tau). \]  
(2.34)

Moreover, note the interpolation statement

Lemma 2.1. Let \( k \geq 0 \). Then, there exist a constant
\[ C_{int}(k, M, \Lambda) > 0 \]  
(2.35)

such that for \( \psi \) a sufficiently regular function we obtain
\[ E_{k+1}[\psi](\tau) \leq C_{int} \left( E_{k-1}[\psi](\tau) \right)^{1/3} \left( E_{k+2}[\psi](\tau) \right)^{2/3}, \]  
(2.36)

for any \( \tau \geq 0 \).

2.6. Notation for derivatives

We need the following notations
**Definition 2.6.** Let $X$ be a $C^{0,1/2}$ vector field on the manifold $M_\delta$ and let $\psi$ be a smooth function on the manifold $M_\delta$. Then, for all $j \geq 1$ we define
\[
|\partial \psi| = |\partial_\tau \psi| + |\partial_r \psi| + \sum_{\alpha=1,2,3} |\Omega_\alpha \psi|,
\]
\[
(\partial^j \psi)^2 = \sum_{1 \leq i_1+i_2+i_3 \leq j} \sum_{\alpha} \left( \partial_i^{i_1} \partial_r^{i_2} \Omega_i^{i_3} \psi \right)^2,
\]
\[
(X \partial^j \psi)^2 = \sum_{1 \leq i_1+i_2+i_3 \leq j} \sum_{\alpha} \left( X \partial_i^{i_1} \partial_r^{i_2} \Omega_i^{i_3} \psi \right)^2,
\]
\[
(\partial^j X \psi)^2 = \sum_{1 \leq i_1+i_2+i_3 \leq j} \sum_{\alpha} \left( \partial_i^{i_1} \partial_r^{i_2} X \Omega_i^{i_3} \psi \right)^2.
\] (2.37)

Note that if $X$ is only $C^{0,1/2}$ (for instance $X = G$) then the last expression of (2.37) may not necessarily be finite.

It will be convenient to compare the above expressions with expressions of coordinate derivatives in ambient globally defined Cartesian coordinates. We define the map
\[
M_\delta \to \mathbb{R}^4
\]
\[
(\bar{t}, r, \theta, \phi) \mapsto (x^0, x^1, x^2, x^3)
\] (2.38)
to be a change of coordinates from the coordinates ascribed to the manifold $M_\delta$, see **Definition 2.1**, to Cartesian coordinates, where
\[
x^0 = \bar{t}, \quad x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta.
\] (2.39)

Define the set
\[
\text{Cart}_m = \{ \partial_{x_1} \partial_{x_2} \ldots \partial_{x_1} \partial_{x_2} \ldots \partial_{x_1} \partial_{x_2} \ldots \} \tag{2.40}
\]
to have as elements all the operators that are comprised of any collection of $m$ derivatives of the coordinate vector fields of the Cartesian coordinates $(x^0, x^1, x^2, x^3)$, see (2.38).

For all $j \geq 1$, we note the similarities
\[
|\partial \psi| \sim |\partial_{x^0} \psi| + |\partial_{x^1} \psi| + |\partial_{x^2} \psi| + |\partial_{x^3} \psi|,
\]
\[
(\partial^j \psi)^2 \sim \sum_{1 \leq i \leq j} \sum_{\text{all } D^i \in \text{Cart}_i} \left( D^i \psi \right)^2.
\] (2.41)

Finally, for an arbitrary smooth Lorentzian metric $g$, the wave operator is
\[
\Box_g = g^{ab} \partial_a \partial_b + g^{ab} \Gamma^c_{ab}(g) \partial_c,
\] (2.42)

where $\Gamma^c_{ab}(g)$ are the Christoffel symbols, of the Levi-Civita connection, with respect to the metric $g$.

**2.7. The smooth tensor $a$**

We fix a smooth tensor
\[
a : TM_\delta \times TM_\delta \to \mathbb{R}.
\] (2.43)
Note that for any \( m \in \mathbb{N} \cup \{0\} \) there exist constants \( A_m < \infty \), such that if \( a^{ij} \) are the components of its inverse in Cartesian coordinates (2.38) and \( D^m \) any element of the set \( \text{Cart}_m \), see (2.40), then

\[
|D^m a^{ij}| \leq A_m
\]  

(2.44)

for all \( m, i, j \).

In what follows, if a constant \( C \) depends on \( A_0, A_1, \ldots, A_m \) we denote it as

\[
C = C(A_{[m]}).
\]  

(2.45)

**Remark 2.6.** Recall that in the central quasilinear equation (1.1), namely \( \square_g (\nabla \psi) \psi = \partial \psi \cdot \partial \psi \), the semilinear terms on the RHS are of the form \( \partial \psi \cdot \partial \psi = a^{ij} \partial_i \psi \partial_j \psi \).

### 2.8. Sobolev inequality

Note the following Sobolev inequality.

**Lemma 2.2.** There exists a constant \( C = C(M, \Lambda) > 0 \) such that for \( f \in C^\infty([\bar{t} = c]) \), the following holds

\[
\|f\|^2_{L^\infty([\bar{t} = c])} \leq C(M, \Lambda) \sum_{0 \leq i_1 + i_2 \leq 2} \sum_{\alpha = 1, 2, 3} \int_{[\bar{t} = c]} \left( \partial^{i_1} \Omega^{i_2}_\alpha f \right)^2 d\bar{g}_{[\bar{t} = c]}.
\]  

(2.46)

### 2.9. The \( G \) vector field commutation estimate

In our [31], we proved the following on a Schwarzschild–de Sitter background.

**Theorem 2.1.** (Theorem 3 of [31]) Let \( \psi \) satisfy the inhomogeneous wave equation

\[
\Box_{g_{M,\Lambda}} \psi = F
\]  

(2.47)

on \( D_\delta(\tau_1, \tau_2) \). Then, for any \( k \geq 3 \) there exists a constant \( C(k, M, \Lambda) > 0 \), such that the following estimate holds

\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \\
\leq CE_{G,k}[\psi](\tau_1) \\
+ C \int_{D_\delta(\tau_1, \tau_2)} d\bar{g} \sum_{0 \leq i_1 + i_2 + i_3 \leq k - 2} \sum_{\alpha} \left( \partial^{i_1}_t \partial^{i_2}_r (\Omega^{i_3}_\alpha) F \right)^2 \\
+ (1 - \mu) \sum_{0 \leq i_1 + i_3 \leq k - 2} \sum_{\alpha} \left( \partial^{i_1}_t (\Omega^{i_3}_\alpha) G F \right)^2 \\
+ C \int_{[\bar{t} = \tau_2]} d\bar{g}_{[\bar{t} = \tau]} \sum_{0 \leq i_1 + i_2 + i_3 \leq k - 3} \sum_{\alpha} \left( \partial^{i_1}_t \Omega^{i_2}_\alpha \partial^{i_3}_r F \right)^2 \\
+ (1 - \mu)^{2i_3 + 1} \left( \partial^{i_1}_t \Omega^{i_2}_\alpha \partial^{i_3}_r G F \right)^2
\]  

(2.48)

where the volume forms are with respect to the Schwarzschild–de Sitter metric \( g_{M,\Lambda} \), see Section 2.3.
Proof. This is proved in [31].

Remark 2.7. Note that the highest order term of the error hypersurface terms on the right hand side of (2.48) is of order \( k - 2 \), while the highest order term of the error bulk terms on the right hand side of (2.48) is of order \( k - 1 \). The weights in \((1 - \mu)\) on the error terms on the right hand side of (2.48) ensure that terms on the right hand side of (2.48) are regular for a sufficiently regular function \( F \).

3. Metric close to Schwarzschild–de Sitter and the Local well-posedness result

3.1. Metric close to Schwarzschild–de Sitter

We define the class of metrics close to Schwarzschild–de Sitter.

We fix a sufficiently regular tensor

\[ h \in \Gamma \left( T^*M^*_\delta \times T^*M^*_\delta \times T^*M^*_\delta \right) \]  

with

\[ h(\cdot, \cdot, v = 0) = 0, \]  

and define

\[ g(v) = \tilde{g} + h(v). \]  

We also define \( h^{ij}(v) = \tilde{g}^{ij}(v) - \tilde{g}^{ij} \), where \( \tilde{g}^{ij}(v), \tilde{g}^{ij} \) are the components of the inverses of the respective tensors in Cartesian coordinates.

Note that for every \( m \in \mathbb{N} \cup \{0\} \) there exist constants \( B_m < \infty \), such that, for \( m \geq 0 \), the following hold in Cartesian coordinates

\[ |D^m h_{ijk}|, |D^m h^{ij}_k| \leq B_m, \]  

for all \( m, i, j, k \), where \( h^{ij}_k v^k = h^{ij}(v) \) and \( D^m \) is any element of the set Cart\(_m\), see (2.40).

In what follows, if a constant \( C \) depends on \( B_0, B_1, \ldots, B_m \), we denote it as

\[ C = C(B_{\lfloor m \rfloor}). \]  

3.2. The Cauchy stability type result

We present a Cauchy stability type result for the quasilinear wave equation, in the form that we will use, appropriately tailored to display the energies that are used in our main global stability results, see already Theorems 1 and 2.

Proposition 3.1. Let \( k \geq 7 \) and let the tensors \( a, h \) be as in Sections 2.7 and 3.1, respectively. There exists a constant

\[ C_{wp}(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 1, \]  

where for \( A_{[k+1]}, B_{[k+1]} \) see Sections 2.7 and 3.1, respectively, such that the following holds.

Let \( \tau_{\text{max}} > 0 \) be an arbitrary positive number. There exist

\[ \delta(\tau_{\text{max}}, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}), \quad \epsilon(\tau_{\text{max}}, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0 \]  

both sufficiently small, such that if we take initial data for (1.1) on \( \{\tilde{t} = \tau_1\} \) with

\[ E_{k+1}[\psi](\tau_1) \leq \epsilon, \]  

then...
for some \( \tau_1 \geq 0 \), then there exists a unique \( H^{k+1} \) solution \( \psi \) to the quasilinear wave equation (1.1), on \( D_\delta (\tau_1, \tau_2) \), such that

\[
E_{k+1}[\psi](\tau') \leq C_{wp}E_{k+1}[\psi](\tau_1),
\]

(3.9)

\[
E_{k+2}[\psi](\tau') \leq C_{wp}E_{k+2}[\psi](\tau_1),
\]

(3.10)

for all \( \tau' \in [\tau_1, \tau_1 + \tau_{\max}] \), where (3.10) holds if \( E_{k+2}[\psi](\tau_1) < \infty \), in which case the solution is \( H^{k+2} \).

**Proof** See the Appendix A.

**Remark 3.1.** The result of Proposition 3.1 is a refinement of the usual Cauchy stability and thus we will have to prove it explicitly. The independence of \( C_{wp} \) on \( \tau_{\max} \) is connected to the uniform boundedness result [10] for the linear wave equation.

### 4. The main Theorems

#### 4.1. Theorem on an arbitrary slab of length \( \tau_{\text{step}} > 0 \).

We give the detailed statement of the Theorem, on a fixed large time domain.

**Theorem 1.** Let \( k \geq 7 \) and let the tensors \( a, h \) be as in Sections 2.7 and 3.1, respectively. There exists a constant

\[
C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]

(4.1)

where for \( A_{[k+1]}, B_{[k+1]} \) see Sections 2.7 and 3.1, respectively, such that the following holds.

For all \( L > 0 \) there exists a \( \tau_{\text{step}}(L, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}, C_{\text{int}}, C_{wp}) > 0 \) sufficiently large, where for \( C_{\text{int}}, C_{wp} \) see respectively, Lemma 2.1 and Proposition 3.1, such that there exist

\[
\delta(\tau_{\text{step}}, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}), \quad \epsilon(\tau_{\text{step}}, \delta, k, M, \Lambda, A_{[k+1]}, B_{[k+1]} > 0
\]

(4.2)

sufficiently small such that if we take initial data for (1.1) in \( \{\tau = \tau_1\} \) with

\[
E_{k+1}[\psi](\tau_1) \leq \epsilon, \quad E_{k+2}[\psi](\tau_1) < \infty
\]

(4.3)

and

\[
\tau_2 = \tau_1 + \tau_{\text{step}},
\]

(4.4)

then there exists a unique \( H^{k+2} \) solution in \( D_\delta(\tau_1, \tau_2) \) to the quasilinear wave equation (1.1), and for all \( \tau' \in [\tau_1, \tau_2] \) the following inequalities hold

\[
E_{\tilde{G},k}[\psi](\tau') \leq CE_{\tilde{G},k}[\psi](\tau_1),
\]

(4.5)

\[
E_{k+1}[\psi](\tau') \leq C_{wp}E_{k+1}[\psi](\tau_1),
\]

(4.6)

\[
E_{k+2}[\psi](\tau') \leq C_{wp}E_{k+2}[\psi](\tau_1).
\]

(4.7)

Moreover, the following holds

\[
E_{\tilde{G},k}[\psi](\tau_2) \leq \frac{1}{L}E_{\tilde{G},k}[\psi](\tau_1).
\]

(4.8)

Finally, the following holds

\[
E_{k+1}[\psi](\tau_2) \leq C_{\text{int}}e^{-\frac{1}{4} \log(L) + \frac{3}{2} \log C_{wp}} \left( E_{\tilde{G},k}[\psi](\tau_1) \right)^{1/3} \left( E_{k+2}[\psi](\tau_1) \right)^{2/3}.
\]

(4.9)
Remark 4.1. For the requirement \( k \geq 7 \), see already the computations of inequality (5.19) and Proposition 3.1.

Remark 4.2. Note that if one proves a Morawetz estimate for the linear Klein Gordon equation then the results of the present paper carry over to the non-linear Klein Gordon case immediately.

Moreover, note that without any significant changes in the proof we can treat quasilinear equations of the form \( a^{ij}[\psi] \partial_i \psi \partial_j \psi = \hat{a}^{ij} \cdot (\psi + \partial \psi) \partial_i \psi \partial_j \psi \), where \( \hat{a}^{ij} \in \mathbb{R} \) do not depend on \( \psi \). It would actually be easier to close as one can utilize one extra smallness.

4.2. The global nonlinear stability of the quasilinear wave equation (1.1)

Now, we present the main Theorem of our paper, which is in fact a Corollary of Theorem 1.

**Theorem 2.** Let \( k \geq 7 \), let the tensors \( a, h \) be as in Sections 2.7 and 3.1, respectively. Then, there exist constants

\[
c_d(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}), \quad c_g(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}), \quad C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]

and there exist

\[
\delta = \delta(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0, \quad \epsilon = \epsilon(\delta, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]

sufficiently small, such that for

\[
E_{k+2}[\psi](0) \leq \epsilon,
\]

on \( \Sigma = \{ \tilde{t} = 0 \} \), the arising \( H^{k+2} \) solution of the quasilinear wave equation (1.1) exists globally on \( D_\delta(0, \infty) \) and satisfies the following exponential decay

\[
E_{G,k}[\psi](\tau) \leq C e^{-c_d \tau} E_{G,k}[\psi](0),
\]

for any \( \tau \geq 0 \).

Moreover, we have the following exponential decay of the lower order energies

\[
E_{k-1}[\psi](\tau) \leq C e^{-c_d \tau} E_{G,k}[\psi](0), \quad E_{k+1}[\psi](\tau) \leq C e^{-c_d \tau} E_{k+2}[\psi](0),
\]

while for the top order energy we have the growth estimate

\[
E_{k+2}[\psi](\tau) \leq C e^{c_g \tau} E_{k+2}[\psi](0).
\]

**Remark 4.3.** Note that the growth constant \( c_g \) of inequality (4.15) of Theorem 2 can be made arbitrarily small, if \( \delta > 0 \) and \( \epsilon > 0 \) are sufficiently small. See already the proof of Theorem 2 in Section 7. Although (4.15) allows the top order energy \( E_{k+2}[\psi] \) to a priori grow exponentially, one can in fact prove stronger results. Specifically, with the additional use of a top order uniform boundedness estimate, one can prove Theorem 2 assuming only small \( E_{k+1}[\psi](0) \) initial data, and also uniformly bound the energy \( E_{k+1}[\psi](\tau) \) for all times. (Therefore, we expect to only need the initial energy to lie in the Sobolev space \( H^8 \).) We will not however pursue this here as the weaker estimate provided by Proposition 3.1 leading to (4.15) is sufficient. This improvement is however important in the asymptotically flat case.
4.3. Extension of our results on Kerr–de Sitter

We can extend our results for the quasilinear wave equation to the slowly rotating Kerr–de Sitter case by utilizing the fixed frequency vector field commutation that we introduce in our forthcoming [39]. This will be presented in a future paper.

5. Energy estimates on a spacetime slab of fixed time length

Before turning to the proofs of Theorems 1 and 2, we need a preliminary Proposition.

Proposition 5.1. Let \( k \geq 7 \) and let the tensors \( a, h \) be as in Sections 2.7 and 3.1, respectively. Let \( \psi \) be a solution of the quasilinear wave equation (1.1) on the Schwarzschild–de Sitter domain \( D_\delta(\tau_1, \tau_2) \) for \( \tau_1 \leq \tau_2 \) and any \( \delta > 0 \) sufficiently small. Then, there exists a positive constant

\[
C(k, M, \Lambda, A[k], B[k]) > 0,
\]

where for \( A[k], B[k] \) see Sections 2.7 and 3.1, respectively, and an \( \epsilon > 0 \) sufficiently small, such that if

\[
\sup_{D_\delta(\tau_1, \tau_2)} \sum_{1 \leq j \leq k-1} \sum_{\partial \in \{\bar{\partial}_t, \partial_r, \Omega_1, \Omega_2, \Omega_3\}} |\partial^j \psi| \leq \sqrt{\epsilon}
\]

holds, then the following estimate holds

\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \leq CE_{G,k}[\psi](\tau_1) + C \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) E_{k+1}[\psi](\tau),
\]

for all \( \tau_1 \leq \tau_2 \).

Proof of Proposition 5.1 We will use the derivatives notation of Section 2.6. Moreover, for the semilinear term \( \partial \psi \cdot \partial \psi \) and for the definition of smooth tensors \( a, h \) see Sections 2.7 and 3.1, respectively. In this proof, when we write

\[
h_{ab}
\]

it is to be understood as \( h_{ab}(\nabla \psi) \) in Cartesian coordinates, see (2.38). When we write

\[
h^{ab}
\]

it is to be understood as \( h^{ab}(\nabla \psi) = g^{ab}(\nabla \psi) - \hat{g}^{ab} \), in Cartesian coordinates. Finally, in this proof when we write \( \leq \) it is to be understood that we omit a constant \( C(k, M, \Lambda, A[m], B[m]) \), where for \( A[m], B[m] \) see Sections 2.7 and 3.1, respectively, and \( m \leq k + 1 \).

Note that we may rewrite \( \Box_g(\nabla \psi) \psi = \partial \psi \cdot \partial \psi \) as

\[
\Box_\hat{g} \psi = \left( \Box_\hat{g} - \Box_g(\nabla \psi) \right) \psi + \partial \psi \cdot \partial \psi.
\]

We name, for convenience,

\[
F[\psi] = F_1[\psi] + F_2[\psi] = \left( \Box_\hat{g} - \Box_g(\nabla \psi) \right) \psi + \partial \psi \cdot \partial \psi.
\]
Then, by the integrated energy estimate (2.48) of Theorem 2.1 for the inhomogeneous wave equation (5.6), we obtain the following

\[ E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \]

\[ \lesssim E_{G,k}[\psi](\tau_1) + \int \int_{D(\tau_1,\tau_2)} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_{\alpha} \left( (\partial_t)^i \partial_r^{i_2} (\Omega_\alpha)^i F \right)^2 + (1 - \mu) \sum_{0 \leq i + j \leq k-2} \sum_{\alpha} \left( (\partial_t)^i (\Omega_\alpha)^i \psi \right)^2 \]

\[ \lesssim E_{G,k}[\psi](\tau_1) + \int \int_{D(\tau_1,\tau_2)} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_{\alpha} \left( (\partial_t)^i \partial_r^{i_2} (\Omega_\alpha)^i F_1 \right)^2 + (1 - \mu) \sum_{0 \leq i + j \leq k-2} \sum_{\alpha} \left( (\partial_t)^i (\Omega_\alpha)^i \psi \right)^2 \]

\[ \lesssim E_{G,k}[\psi](\tau_1) + \int \int_{D(\tau_1,\tau_2)} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_{\alpha} \left( (\partial_t)^i \partial_r^{i_2} (\Omega_\alpha)^i F_2 \right)^2 + (1 - \mu) \sum_{0 \leq i + j \leq k-2} \sum_{\alpha} \left( (\partial_t)^i (\Omega_\alpha)^i \psi \right)^2 \]

\[ \lesssim E_{G,k}[\psi](\tau_1) + \int_{[t=\tau_2]} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_{\alpha} (1 - \mu)^{2i+1} \left( (\partial_t)^i \Omega_\alpha^{i_2} \partial_r^{i_3} \psi \right)^2 + \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_{\alpha} \left( (\partial_t)^i \Omega_\alpha^{i_2} \partial_r^{i_3} \psi \right)^2 . \]

We readily bound the term \( \int \int_{D(\tau_1,\tau_2)} (1 - \mu) \sum_{i=0}^{k-2} (\partial_t^i \psi \partial_r^i \psi)^2 \) as follows

\[ \int \int_{D(\tau_1,\tau_2)} (1 - \mu) \sum_{i=0}^{k-2} (\partial_t^i \psi \partial_r^i \psi)^2 = \int \int_{D(\tau_1,\tau_2)} (1 - \mu) \sum_{i=0}^{k-2} (\partial_t^i \psi \partial_r^i \psi)^2 \]

\[ = \int \int_{D(\tau_1,\tau_2)} (1 - \mu) \sum_{i=0}^{k-2} (\partial_t^i ([G, a^{\alpha \beta}] \partial_\alpha \psi \partial_\beta \psi + a^{\alpha \beta} G (\partial_\alpha \psi \partial_\beta \psi)))^2 \]

(5.9)
Therefore, by using the coarea formula (2.29), see Section 2.4, we bound (5.9) from

\[
\frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} d\tau \sum_{1 \leq i+j \leq k-1, j \leq \left\lfloor \frac{k-1}{2} \right\rfloor} \sup_{[i=\tau]} (\partial^i \psi)^2 \int_{[i=\tau]} (\partial^i \psi)^2
\]

\[+ \frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} d\tau \sup_{[i=\tau]} \sum_{1 \leq i+j \leq k-2, j \leq \left\lfloor \frac{k-2}{2} \right\rfloor} (\partial^i \partial^j \psi)^2 \int_{[i=\tau]} (1-\mu)(\partial^i \partial^j G \psi)^2
\]

\[+ \frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} d\tau \sum_{j \geq \left\lfloor \frac{k}{2} \right\rfloor+1, 1 \leq i \leq k-2 - \left\lfloor \frac{k-2}{2} \right\rfloor} \sup_{[i=\tau]} (\partial^{i+2} \psi)^2 \int_{[i=\tau]} (\partial^{i+2} \psi)^2
\]

\[
\lesssim \frac{1}{\tau_2} \int_{\tau_1}^{\tau_2} d\tau \sum_{k-1 \leq \left\lfloor \frac{k}{2} \right\rfloor} E_{k-1}[\psi](\tau) E_{k-1}[\psi](\tau) + \frac{1}{\tau_1} \int_{\tau_1}^{\tau_2} d\tau E_{k-1}[\psi](\tau) E_{G,k}[\psi](\tau)
\]

\[\leq C(k, M, \Lambda, A[k-1], B[k-1]) \int_{\tau_1}^{\tau_2} d\tau E_k[\psi](\tau) E_{G,k}[\psi](\tau)
\]
where in the first inequality we use the Sobolev inequality, see Lemma 2.2, and we also used the requirement $k \geq 6$. The remaining terms
\[
\int \int_{D(\tau_1, \tau_2)} (1 - \mu) \sum_{0 \leq i_1 + i_2 \leq k-2} \sum_{\alpha} \left( (\partial \tau)^{i_1} (\Omega_\alpha)^{i_2} \mathcal{G} F_2 \right)^2
\]  
(5.11)
are similarly bounded by the right hand side of (5.10).

Moreover, we readily bound the term $\int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} (\partial \psi F_2)^2$ on the right hand side of equation (5.8) by using the coarea formula (2.29), see Section 2.4, by the expression below
\[
\int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} (\partial \psi F_2)^2 = \int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} \left( \partial \psi \left( \alpha^{a\beta} \partial_a \psi \partial_\beta \psi \right) \right)^2
\]
\[
\lesssim \int \int_{D(\tau_1, \tau_2)} \sum_{1 \leq i + j \leq k-1} (\partial \psi)^2 (\partial \psi)^2
\]
\[
\lesssim \int_{\tau_1}^{\tau_2} \sum_{1 \leq i + j \leq k-1} \sup_i \int_{[\tau, \tau_1]} (\partial \psi)^2
\]
\[
\lesssim \int_{\tau_1}^{\tau_2} d\tau E_{k-1}[\psi(\tau)] E_{k-1}[\psi(\tau)]
\]
\[
\leq C(k, M, \Lambda, A_{k-1}, B_{k-1}) \int_{\tau_1}^{\tau_2} d\tau E_{k-1}[\psi(\tau)] E_{k}[\psi(\tau)],
\]
where in the second to last inequality we use the Sobolev inequality, see Lemma 2.2, and we also used the requirement $k \geq 4$. The remaining terms
\[
\int \int_{D(\tau_1, \tau_2)} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_{\alpha} \left( (\partial \tau)^{i_1} (\Omega_\alpha)^{i_2} \partial \psi F_2 \right)^2
\]  
(5.13)
are similarly bounded by the right hand side of (5.12). Note that we have not yet appealed to the smallness (5.2), since we have thus far only been discussing semi-linear terms.

We estimate the term $\int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} (\partial \psi \mathcal{G} F_1)^2$, on the right hand side of equation (5.8), by using the coarea formula (2.29), by the following expression
\[
\int_{\tau_1}^{\tau_2} d\tau \int_{[\tau, \tau_1]} \sum_{i=0}^{k-2} \left( \partial \psi \left( \hat{g}^{ab} - g^{ab}(\nabla \psi) \right) \partial_a \partial_b \psi \right)
\]
\[
+ \partial \psi \left( \hat{g}^{ab} \Gamma^c_{ab}(\hat{g}) \partial_c \psi - g^{ab}(\nabla \psi) \Gamma^c_{ab}(g(\nabla \psi)) \partial_c \psi \right)
\]
\[
\leq \int_{\tau_1}^{\tau_2} d\tau \int_{[\tau, \tau_1]} \sum_{i=0}^{k-2} \left( \partial \psi \left( \hat{g}^{ab} - g^{ab}(\nabla \psi) \right) \partial_a \partial_b \psi \right)^2
\]
\[
+ \partial \psi \left( \hat{g}^{ab} \Gamma^c_{ab}(\hat{g}) \partial_c \psi - g^{ab}(\nabla \psi) \Gamma^c_{ab}(g(\nabla \psi)) \partial_c \psi \right)^2
\]  
(5.14)
Now, we note that
\[
\Gamma^c_{ab}(g(\nabla \psi)) = \Gamma^c_{ab}(\hat{g}) + S^c_{ab}(h),
\]  
(5.15)
where
\[
S^c_{ab}(h) \doteq \frac{1}{2} \hat{g}^{ce} \left( \partial_c h_{ab} + \partial_a h_{cb} + \partial_b h_{ca} \right)
\]
\[
+ \frac{1}{2} h^{ce} \left( - \partial_c h_{ab} + \partial_a h_{cb} + \partial_b h_{ca} \right)
\]  
(5.16)
Therefore, we write (5.14) in the following form

\[
\int_{\tau_1}^{\tau_2} d\tau \int_{[\tau_1, \tau]} \sum_{i=0}^{k-2} (\partial_i I'(h^{ab} \partial_a \partial_b \psi) + \partial_i I'(\hat{g}^{ab} S^c_{\alpha \beta} (h) - h^{ab} \Gamma^{c}_{\alpha \beta}(\hat{g}) - h^{ab} S^c_{\alpha \beta}(h)) \partial_c \psi)^2
\]

which, by using the smallness (5.2) for a sufficiently small \( \varepsilon > 0 \), the definition of \( h \), see Section 3.1, and by distributing the derivatives (with the Cartesian coordinates of Section 2.6), we bound the above from

\[
\lesssim \int_{\tau_1}^{\tau_2} d\tau \int_{[\tau_1, \tau]} \sum_{1 \leq i \leq k-1} \left( (\partial^i (h^{ab} \partial_a \partial_b \psi))^2 + (\partial^i (\hat{g}^{ab} S^c_{\alpha \beta}(h) \partial_c \psi))^2 \right)
\]

(5.17)

(5.18)
therefore, by using Sobolev inequalities, see Lemma 2.2, we obtain from (5.18) the following

\[ \lesssim \int_{\tau_1}^{T_2} d\tau \sum_{1 \leq i \leq k-1, i \leq \lfloor \frac{k-1}{2} \rfloor} \sup_{[i=\tau]} (\partial^{i+1} + \psi)^2 \int_{[i=\tau]} (\partial^{i+2} + \psi)^2 \]

+ \sum_{4 \leq i \leq k-1} \sum_{i_1 = \lfloor \frac{k-1}{2} \rfloor + 1}^{\lfloor \frac{k-1}{2} \rfloor - 1} \sup_{[i=\tau]} (\partial^{i+1} + \psi)^2 \int_{[i=\tau]} (\partial^{i+2} + \psi)^2

+ \sum_{5 \leq i \leq k-1} \sum_{i_1 = \lfloor \frac{k-1}{2} \rfloor - 2}^{\lfloor \frac{k-1}{2} \rfloor - 3} \sup_{[i=\tau]} (\partial^{i+2} + \psi)^2 \int_{[i=\tau]} (\partial^{i+1} + \psi)^2

+ \sum_{1 \leq i \leq 4} \sum_{i_1 + i_2 = i} \sup_{[i=\tau]} (\partial^{i+1} + \psi)^2 \int_{[i=\tau]} (\partial^{i+2} + \psi)^2

+ \sum_{1 \leq i_1, i_2 \leq k-1, i_1 \leq \lfloor \frac{k-1}{2} \rfloor} \sup_{[i=\tau]} (\partial^{i+1} + \psi)^2 \int_{[i=\tau]} (\partial^{i+2} + \psi)^2 \]

\[ \lesssim \int_{\tau_1}^{T_2} d\tau E_{k-1}[\psi](\tau)E_{k+1}[\psi](\tau) \]

\[ \leq C(k, M, \Lambda, A_{[k]}, B_{[k]}) \int_{\tau_1}^{T_2} d\tau E_{G,k}[\psi](\tau)E_{k+1}[\psi](\tau) \]

where in the second to last inequality we also used that \( k = 7 \) is the smallest integer such that all of the following hold

\[ \left( \lfloor \frac{k-1}{2} \rfloor + 1 \right) + 2 \leq k - 1, \quad \left( \lfloor \frac{k-1}{2} \rfloor + 1 + 1 \right) + 2 \leq k + 1, \]

\[ \left( \lfloor \frac{k-1}{2} \rfloor + 2 + 1 \right) + 2 \leq k + 1, \quad \left( \lfloor \frac{k-1}{2} \rfloor - 3 + 2 \right) + 2 \leq k - 1 \]

(5.20)

where the addition of +2 on the above inequalities comes from the Sobolev inequality, see Lemma 2.2.

Now, by similar calculations, we bound the following term of the right hand side of (5.8)

\[ \int_{D(\tau_1, \tau_2)} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_{\alpha} (\partial^{i_1} \partial^{i_2} (\Omega_{\alpha})^{i_3} F_1)^2 \]

(5.21)

\[ + \sum_{0 \leq i + j \leq k-2} \sum_{\alpha} (\partial^{i} \partial^{j} (\Omega_{\alpha})^{i} G F_1)^2 \leq C \int_{\tau_1}^{T_2} E_{G,k}[\psi](\tau)E_{k}[\psi](\tau). \]
Finally, by using the bounds \((5.10)\) \((5.12)\), \((5.19)\), \((5.21)\), the integrated energy estimate of equation \((5.8)\) implies the following
\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}(\tau)
\leq C(M, \Lambda)E_{G,k}[\psi](\tau_1) + C(k, M, \Lambda, A_{[k]}, B_{[k]}) \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau)E_{k+1}[\psi](\tau)
+ C(M, \Lambda) \int_{\{i = \tau_2\}} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_{\alpha} (1 - \mu)^{2i_3 + 1} \left( \partial_i^1 \Omega_{\alpha}^1 \partial_i^3 F \right)^2
+ \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_{\alpha} \left( \partial_i^1 \Omega_{\alpha}^1 \partial_i^3 F \right)^2,
\]
where recall that \(F = (\Box \psi - \Box_g(\nabla^2 \psi)) \psi \partial \psi \cdot \partial \psi\).

Finally, we want to absorb the boundary terms of the right hand side at \(\{i = \tau_2\}\) by the relevant boundary term \(E_{G,k}[\psi](\tau_2)\) of the left hand side. By using the smallness assumption \((5.2)\), it is evident that we can appropriately distribute derivatives, in view of the requirement \(k \geq 7\), to conclude that
\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}(\tau)
\leq C(k, M, \Lambda, A_{[k]}, B_{[k]})E_{G,k}[\psi](\tau_1) + C(k, M, \Lambda, A_{[k]}, B_{[k]}) \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau)E_{k+1}[\psi](\tau),
\]
for a constant \(C(k, M, \Lambda, A_{[k]}, B_{[k]})\).

The proof of Proposition 5.1 is thus complete. \(\square\)

6. Proof of Theorem 1

We are ready to prove Theorem 1.

Proof of Theorem 1 Given \(L > 0\), and for a
\[
\tau_{\text{step}}(L, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]
(6.1)
to be determined later (see already \((6.19)\)), we apply Proposition 3.1 with \(\tau_{\text{max}} = \tau_{\text{step}}\) to find
\[
\delta(\tau_{\text{step}}, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0, \quad \epsilon(\tau_{\text{step}}, \delta, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]
(6.2)
sufficiently small, such that if we take
\[
E_{k+1}[\psi](\tau_1) \leq \epsilon
\]
(6.3)
then the solution \(\psi\) of the quasilinear wave equation \((1.1)\) exists in \(D_\delta(\tau_1, \tau_2)\), where
\[
\tau_2 = \tau_1 + \tau_{\text{step}}.
\]
(6.4)

The two inequalities \((4.6)\) and \((4.7)\) are an immediate consequence of the Cauchy stability results \((3.9)\) and \((3.10)\), respectively, of Proposition 3.1.
To obtain the remaining statements, we assume the bootstrap
\[
\sup_{D_k(\tau_1, \tau_2)} \sum_{1 \leq j \leq k-1} \sum_{\partial \in \{\eta, \partial_x, \Omega_1, \Omega_2, \Omega_3\}} |\partial^j \psi| \leq C_b \sqrt{\epsilon},
\] (6.5)
for a constant
\[
C_b(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\] (6.6)
which will be determined later.

First, we prove inequality (4.5), namely
\[
\sup_{\tau \in [\tau_1, \tau_2]} E_{G,k}[\psi](\tau) \leq C(k, M, \Lambda, A_{[k]}, B_{[k]} E_{G,k}[\psi](\tau_1).
\] (6.7)
To prove (6.7), we will introduce an additional bootstrap assumption
\[
\sup_{\tau \in [\tau_1, \tau_2]} E_{G,k}[\psi](\tau) \leq C_{boot} E_{G,k}[\psi](\tau_1).
\] (6.8)
We choose \(\epsilon(C_{boot}) > 0\) sufficiently small so that, in view of the bootstrap (6.5), the energy estimate (5.3) of Proposition 5.1 holds, namely
\[
E_{G,k}[\psi](\tau) + \int_{\tau_1}^{\tau_2} E_{G,k}[\psi](\tau') d\tau' \leq C(k, M, \Lambda, A_{[k]}, B_{[k]} E_{G,k}[\psi](\tau_1)
+ C(k, M, \Lambda, A_{[k]}, B_{[k]} E_{G,k}[\psi](\tau_2)
+ C(k, M, \Lambda, A_{[k]}, B_{[k+1]}) C_{boot} \tau_{step} \epsilon E_{G,k}[\psi](\tau_1)
\leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) E_{G,k}[\psi](\tau_1).
\] (6.9)

Now we use the bootstrap assumption (6.8) and inequality (4.6), which is already proven, to obtain
\[
E_{G,k}[\psi](\tau) + \int_{\tau_1}^{\tau_2} E_{G,k}[\psi](\tau') d\tau' \leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]} E_{G,k}[\psi](\tau_1)
+ C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) C_{boot} \tau_{step} \epsilon E_{G,k}[\psi](\tau_1)
\leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) (1 + C_{boot} \tau_{step} \epsilon) E_{G,k}[\psi](\tau_1).
\] (6.10)
Therefore, by choosing
\[
\frac{C_{boot}(k, M, \Lambda, A_{[k+1]}, B_{[k+1]})}{2} \gg C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]})
\] (6.11)
we take \(\epsilon(\tau_{step}, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0\) sufficiently small in inequality (6.10), and obtain
\[
\sup_{\tau \in [\tau_1, \tau_2]} E_{G,k}[\psi](\tau) \leq \frac{C_{boot}}{2} E_{G,k}[\psi](\tau_1).
\] (6.12)
We improved the bootstrap (6.8) and thus proved (6.7). Therefore, we concluded inequality (4.5). Note that in proving (6.12) we did not use the spacetime integral of the LHS of (6.10).

Now, we proceed to prove inequality (4.8). We use the energy estimate (6.9), in view of the now established bound (6.7), and inequality (4.6), namely \(E_{k+1}[\psi](\tau') \leq C_{wp} E_{k+1}[\psi](\tau_1)\) for all \(\tau' \in [\tau_1, \tau_2]\), to obtain
\[
\int_{\tau_1}^{\tau_2} E_{G,k}[\psi](\tau')d\tau' \leq CE_{G,k}[\psi](\tau_1) + C\tau_{step}E_{G,k}[\psi](\tau_1)E_{k+1}[\psi](\tau_1)
\]
\[
\Rightarrow \int_{\tau_1+\tau_2}^{\tau_2} E_{G,k}[\psi](\tau')d\tau' \leq CE_{G,k}[\psi](\tau_1) + C\tau_{step}E_{G,k}[\psi](\tau_1)E_{k+1}[\psi](\tau_1)
\]
\[
\Rightarrow E_{G,k}[\psi](\tau_{1,2}) \leq \frac{2C(k,M,\Lambda,A_{[k+1]},B_{[k+1]})}{\tau_{step}} \left( E_{G,k}[\psi](\tau_1) + \tau_{step}E_{G,k}[\psi](\tau_1)E_{k+1}[\psi](\tau_1) \right) \tag{6.13}
\]

where we used that there exists a \( \tau_{1,2} \in [\frac{\tau_1+\tau_2}{2},\tau_2] \) such that
\[
\frac{\tau_{step}}{2} E_{G,k}[\psi](\tau_{1,2}) \leq \int_{\tau_1+\tau_2}^{\tau_2} E_{G,k}[\psi](\tau')d\tau', \tag{6.14}
\]
by the mean value theorem. Therefore, from inequality (6.13) we conclude that, if we take
\[
E_{k+1}[\psi](\tau_1) \leq \epsilon, \tag{6.15}
\]
then there exists a \( \tau_{1,2} \in [\frac{\tau_1+\tau_2}{2},\tau_2] \) such that
\[
E_{G,k}[\psi](\tau_{1,2}) \leq \left( \frac{2C}{\tau_{step}} + 2C\epsilon \right) E_{G,k}[\psi](\tau_1). \tag{6.16}
\]

Furthermore, by using the finite in time bound (6.7), for \( E_{G,k}[\psi] \), in the time domain
\[
[\tau_{1,2},\tau_2], \tag{6.17}
\]
in conjunction with (6.16), we find a, potentially different, constant \( C(k,M,\Lambda,A_{[k+1]},B_{[k+1]}) > 0 \) such that
\[
E_{G,k}[\psi](\tau_2) \leq \left( \frac{2C}{\tau_{step}} + 2C\epsilon \right) E_{G,k}[\psi](\tau_1). \tag{6.18}
\]
Finally, we choose \( \tau_{step}(L,k,M,\Lambda,A_{[k+1]},B_{[k+1]}) > 0 \) sufficiently large and conclude
\[
E_{G,k}[\psi](\tau_2) \leq \frac{1}{L} E_{G,k}[\psi](\tau_1), \tag{6.19}
\]
after taking \( \epsilon(\tau_{step}) \) sufficiently small. We have concluded the inequality (4.8).

To conclude (4.9), we use the now established (4.5), (4.6), (4.7), (4.8) and the classical interpolation Lemma 2.1. Namely, by recalling the constant
\[
C_{\text{int}}(k,M,\Lambda) > 0 \tag{6.20}
\]
from the interpolation Lemma 2.1, we obtain
\[
E_{k+1}[\psi](\tau_2) \leq C_{\text{int}} \left( E_{k-1}[\psi](\tau_2) \right)^{1/3} \left( E_{k+2}[\psi](\tau_2) \right)^{2/3}
\]
\[
\leq C_{\text{int}} \left( E_{G,k}[\psi](\tau_2) \right)^{1/3} \left( E_{k+2}[\psi](\tau_2) \right)^{2/3}
\]
\[
\leq C_{\text{int}} \left( \frac{1}{L} \right)^{1/3} \left( C_{wp} \right)^{2/3} \left( E_{G,k}[\psi](\tau_1) \right)^{1/3} \left( E_{k+2}[\psi](\tau_1) \right)^{2/3}
\]
\[
\leq C_{\text{int}} c^{\frac{1}{3}} \log(L) + \frac{1}{2} \log C_{wp} \left( E_{G,k}[\psi](\tau_1) \right)^{1/3} \left( E_{k+2}[\psi](\tau_1) \right)^{2/3}\tag{6.21}
\]
where for the constant
\[
C_{wp}(k,M,\Lambda,A_{[k+1]},B_{[k+1]}) > 0 \tag{6.22}
\]
see Proposition 3.1. We have concluded (4.9).
Of course our inequalities are still conditional on improving the bootstrap assumption (6.5). We see immediately however that by taking \( C_b \gg C_{wp} \) we improve the bootstrap (6.5) by applying a Sobolev inequality on the already proved inequality (4.6).

The proof of Theorem 1 is thus complete.

7. Proof of Theorem 2

We are now ready to prove Theorem 2.

Proof of Theorem 2 We will use the results of Theorem 1 in conjunction with an iteration argument in consecutive spacetime regions.

To apply Theorem 1, we choose \( L > 0 \) sufficiently large so that

\[
L > \left( 2(C_{int} + 1) e^{\frac{2}{3} \log(C_{wp})} \right)^3 = (2(C_{int} + 1))^3 C_{wp}^2
\]

where for the constants \( C_{int}, C_{wp} > 0 \) see equation (4.9). Note that with this choice, the constant on the right hand side of (4.9) satisfies

\[
C_{int} e^{-\frac{1}{3} \log L + \frac{2}{3} \log C_{wp}} < \frac{1}{2}.
\]

This gives us \( \tau_{step}, \delta \) and \( \epsilon \) from the statement of Theorem 1 and our choice of \( L \), namely (7.1).

We obtain

\[
E_{k+1}[\psi](\tau_2) \leq \left( E_{G,k}[\psi](\tau_1) \right)^{1/3} \left( E_{k+2}[\psi](\tau_1) \right)^{2/3},
\]

for \( \tau_2 = \tau_1 + \tau_{step} \).

First, we prove that there exists a strictly increasing sequence of real numbers \( \{t_i\}_{i \in \mathbb{N}}, t_i \to \infty \), such that

\[
t_0 = 0, \quad t_{i+1} - t_i = \tau_{step},
\]

the solution exists in \( D_\delta(0, t_i) \) and the following hold

\[
E_{G,k}[\psi](t_i) \leq \left( \frac{1}{L} \right)^i E_{G,k}[\psi](0),
\]

\[
E_{k+1}[\psi](t_i) \leq C \cdot \epsilon,
\]

\[
E_{k+2}[\psi](t_i) \leq \left( C_{wp} \right)^i E_{k+2}[\psi](0)
\]

where the constant \( C \) in the second inequality of (7.5) depends only on the constant of the trivial inequality \( E_{G,k} \leq C E_{k}[\psi] \).

Note that (7.5) holds for \( t_0 \). For the purpose of using induction, we assume that the solution exists in \( D_\delta(0, t_i) \) and also assume the iterative step that (7.5) holds for \( t_i \), so we want to prove (7.5) for \( t_{i+1} \). Now, in view of Theorem 1 we obtain that the solution exists in \( D_\delta(0, t_{i+1}) \) and moreover in view of inequality (7.3), the condition for \( L \) namely (7.1) and by the iterative
step assumptions we obtain
\[
E_{k+1}[^{\psi}](t_{i+1}) \leq (E_{G,k}[^{\psi}](t_i))^{1/3} (E_{k+2}[^{\psi}](t_i))^{2/3}
\]
\[
\leq \left( \frac{1}{L} \right)^{i/3} e^{\frac{2}{3} \log(c_{wp})} (E_{G,k}[^{\psi}](0))^{1/3} (E_{k+2}[^{\psi}](0))^{2/3}
\]
\[
\leq e^{i \left( \frac{2}{3} \log(c_{wp}) \right)} (E_{G,k}[^{\psi}](0))^{1/3} (E_{k+2}[^{\psi}](0))^{2/3}
\]
\[
\leq C \epsilon.
\] (7.6)

Moreover, since \( E_{k+1}[^{\psi}](0) \) is sufficiently small \( \epsilon > 0 \), we apply (4.8) of Theorem 1 in conjunction with the inductive step and obtain
\[
E_{G,k}[^{\psi}](t_{i+1}) \leq \frac{1}{L} E_{G,k}[^{\psi}](t_i) \leq \left( \frac{1}{L} \right)^{i+1} E_{G,k}[^{\psi}](0).
\] (7.7)

Furthermore, we apply inequality (4.7) of Theorem 1 in conjunction with the inductive step and obtain
\[
E_{k+2}[^{\psi}](t_{i+1}) \leq C_{wp} E_{k+2}[^{\psi}](t_i) \leq (C_{wp})^{i+1} E_{k+2}[^{\psi}](0).
\] (7.8)

Therefore, by (7.6), (7.7), (7.8), it follows that (7.5) holds for \( t_{i+1} \). Therefore, by induction (7.5) hold for all \( t_j \in \mathbb{N} \).

Now we proceed to prove the exponential decay (4.13), of \( E_{G,k}[^{\psi}] \) for all times \( \tau \geq 0 \). Note that for any \( \tau \in \mathbb{R} \) there exists a \( t_l \in \{t_i\}_{i \in \mathbb{N}} \), such that
\[
|\tau - t_l| \leq \tau_{step}, \quad t_l < \tau.
\] (7.9)

We use the finite in time energy estimate (4.5), of Theorem 1, and the decay of the energy of the sequence \( \{t_i\} \), see (7.5), and obtain the following
\[
E_{G,k}[^{\psi}](\tau) \leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) E_{G,k}[^{\psi}](t_l)
\]
\[
\leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) \left( \frac{1}{L} \right)^{l} E_{G,k}[^{\psi}](0).
\] (7.10)

Now, we conclude the desired inequality by noting
\[
E_{G,k}[^{\psi}](\tau) \leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) \left( \frac{1}{L} \right)^{l} E_{G,k}(0)
\]
\[
\leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) e^{-\frac{l}{\log L}} E_{G,k}[^{\psi}](0)
\]
\[
\leq C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) e^{-c_d \tau} E_{G,k}[^{\psi}](0),
\] (7.11)

for \( c_d(K, M, \Lambda, A_{[k+1]}, B_{[k+1]}) = \frac{\log L}{\tau_{step}} \), where in the last inequality we utilized that
\[
t_l = (t_l - t_{l-1}) + (t_{l-1} - t_{l-2}) + (t_{l-2} - t_{l-3}) + \cdots + (t_2 - t_1) + t_1 = \tau_{step} l
\] (7.12)
so
\[
e^{-l \log L} \leq e^{-\frac{L}{\tau_{step}}} \leq e^{\log L - \frac{L}{\tau_{step}}},
\] (7.13)
which concludes inequality (4.13).
The exponential decay of the lower order energy $E_{k+1}[\psi](\tau)$ follows by similar considerations, from inequality (7.6). Specifically, we have the following

$$
E_{k+1}[\psi](t_{i+1}) \leq (E_{G,k}(t_i))^{1/3}(E_{k+2}(t_i))^{2/3} \\
\leq C(k, M, A_{[k+1]}, B_{[k+1]}) \left( \frac{1}{L} \right)^{\frac{i+1}{3}} (C_{wp})^{\frac{i+1}{3}} E_{G,k}[\psi](0)E_{k+1}[\psi](0) 
$$

(7.14)

Therefore, by choosing $L$ larger if needed, see (7.1), we conclude (4.14), for a certain constant $c_d(k, M, A_{[k+1]}, B_{[k+1]}) > 0$.

Note that for the top order energy $E_{k+2}[\psi](\tau)$, we use (4.7) of Theorem 1 and (7.5), and we only obtain

$$
E_{k+2}[\psi](\tau) \leq C(k, M, A_{[k+1]}, B_{[k+1]}) \frac{\log(C_{wp})}{\tau_{step}} \tau E_{k+2}[\psi](0),
$$

which concludes (4.15), for a constant $c_g(k, M, A, A_{[k+1]}, B_{[k+1]}) = \frac{\log(C_{wp})}{\tau_{step}} > 0$, where note that $C_{wp} > 1$, see Proposition 3.1.

The proof of Theorem 2 is thus complete. \hfill \Box

8. The semilinear equation on Schwarzschild–de Sitter

We here present a simplified proof for the global non-linear stability for the semilinear case

$$
\Box_{g_{M,A}} \psi = \partial \psi \cdot \partial \psi,
$$

(8.1)

with $\partial \psi \cdot \partial \psi = a^{ij} \partial_i \psi \partial_j \psi$, where $a^{ij}$ are sufficiently regular components of a smooth tensor

$$
a : T M \times T M \rightarrow \mathbb{R}
$$

(8.2)

in Cartesian coordinates, see Section 2.7. The results here give a more direct proof and a better regularity for the initial data than the relevant Theorems 1 and 2 that treat the more general quasilinear wave equation.

**Remark 8.1.** Note that in this Section we study $D(\tau_1, \tau_2)$ instead of $D_\delta(\tau_1, \tau_2)$, as $\delta = 0$ here. One may easily a posteriori extend the results to $D_\delta(\tau_1, \tau_2)$. See Remark A.1.

We give the statement of our Theorem, the analogue of Theorem 1, on a fixed large time domain.

**Theorem 1’.** Let $k \geq 7$ and let the tensor $a$ be as in Section 2.7.

Then, there exists a $\tau_{step}(k, M, \Lambda, A_{[k-1]}) > 0$ sufficiently large and there exists

$$
\epsilon(\tau_{step}, k, M, \Lambda, A_{[k-1]}) > 0
$$

(8.3)

sufficiently small, where for $A_{[k]}$ see Section 2.7, such that if we take initial data for (8.1) on $\{\tau = \tau_1\}$ with

$$
E_{G,k}[\psi](\tau_1) \leq \epsilon, \quad E_k[\psi](\tau_1) < \infty
$$

(8.4)

and

$$
\tau_2 = \tau_1 + \tau_{step},
$$

(8.5)
then there exists a unique $H^k$ solution in $D_\delta(\tau_1, \tau_2)$ to the semilinear wave equation (8.1), and the following inequality holds

$$E_{G,k}[\psi](\tau') \leq \frac{1}{2} E_{G,k}[\psi](\tau_1),$$

(8.6)

for all $\tau' \in [\tau_1, \tau_2]$.

A Corollary of Theorem 1’, which is also the analogue of Theorem 2, is the following global existence and exponential decay result.

**Theorem 2’**. Let $k \geq 7$ and let the tensor $a$ be as in Section 2.7. Then, there exist positive constants

$$c_d(k, M, \Lambda, A_{[k-1]}), \quad C(k, M, \Lambda, A_{[k-1]}) > 0$$

(8.7)

such that the following holds.

There exists an $\epsilon > 0$ sufficiently small, such that for

$$E_{G,k}[\psi](0) \leq \epsilon, \quad E_k[\psi](0) < \infty$$

(8.8)

the solution of the semilinear wave equation (8.1) exists globally on $D(0, \infty)$ and the following holds

$$E_{G,k}[\psi](\tau) \leq Ce^{-c_d \tau} E_{G,k}[\psi](0),$$

(8.9)

for all $\tau \geq 0$.

Before proving Theorem 1’ we need the following energy estimate on a fixed spacetime slab. This is the analogue of Proposition 3.1.

**Proposition 8.1**. Let $k \geq 5$ and let the tensor $a$ be as in Section 2.7. There exists a positive constant

$$C(k, M, \Lambda, A_{[k-1]}) > 0,$$

(8.10)

where for $A_{[k-1]}$ see Section 2.7, such that the following holds.

Let $\psi$ be a solution of the semilinear wave equation (8.1), on a Schwarzschild–de Sitter background $D(\tau_1, \tau_2)$ for $\tau_1 < \tau_2$. Then, if the following holds

$$\sup_{D(\tau_1, \tau_2)} \sum_{1 \leq i \leq [\frac{k-1}{2}] + 1} \sum_{\partial \in \{\partial_t, \partial_r, \Omega_1, \Omega_2, \Omega_3\}} |\partial^i \psi| \leq \sqrt{\epsilon}$$

(8.11)

for a sufficiently small $\epsilon > 0$, then we obtain

$$E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \leq CE_{G,k}[\psi](\tau_1)$$

$$+ C \left( \sup_{\tau_1 \leq \tau' \leq \tau_2} E_{k-1}[\psi](\tau') \right) \left( \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \right),$$

(8.12)

for all $0 \leq \tau_1 \leq \tau_2$.

**Proof of Proposition 8.1** We will use the derivatives notation of Section 2.6. Moreover, for the semilinear term $\partial \psi \cdot \partial \psi$ and for the definition of the tensor $a$ see Section 2.7.
Let

\[ F = \partial \psi \cdot \partial \psi \]  

(8.13)

then by (2.48) we obtain

\[ \begin{align*}
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \\
\leq C(M, \Lambda)E_{G,k}[\psi](\tau_1) \\
+ C(M, \Lambda) \int \int_{D(\tau_1, \tau_2)} \sum_{i+j=0}^{k-2} \sum_\alpha (1 - \mu) \left( \partial^i (\Omega_\alpha)^j G F \right)^2 \\
+ \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_\alpha \left( \partial_i^1 \partial_r^2 (\Omega_\alpha)^3 F \right)^2 \\
+ C(M, \Lambda) \int \{i=\tau_2\} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_\alpha (1 - \mu)^2 i_3+1 \left( \partial_i^1 \Omega_\alpha \partial_r^3 G F \right)^2 \\
+ \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_\alpha \left( \partial_i^1 \Omega_\alpha \partial_r^3 F \right)^2.
\end{align*} \]  

(8.14)

For the terms

\[ \int \int_{D(\tau_1, \tau_2)} \sum_{i+j=0}^{k-2} \sum_\alpha (1 - \mu) \left( \partial^i (\Omega_\alpha)^j G F \right)^2 \]  

(8.15)

we have already computed in the proof of Proposition 5.1, see inequality (5.10), that

\[ \begin{align*}
\int \int_{D(\tau_1, \tau_2)} \sum_{i+j=0}^{k-2} \sum_\alpha (1 - \mu) \left( \partial^i (\Omega_\alpha)^j G F \right)^2 \\
\leq C(k, M, \Lambda, A_{k-1}) \int_{\tau_1}^{\tau_2} d\tau E_{k-1}[\psi](\tau) \int_{\tau_1}^{\tau_2} E_{G,k}[\psi](\tau) \\
\leq C(k, M, \Lambda, A_{k-1}) \sup_{\tau' \in [\tau_1, \tau_2]} E_{k-1}[\psi](\tau') \int_{\tau_1}^{\tau_2} E_{G,k}[\psi](\tau) d\tau.
\end{align*} \]  

(8.16)

From the terms

\[ \int \int_{D(\tau_1, \tau_2)} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-2} \sum_\alpha \left( \partial_i^1 \partial_r^2 (\Omega_\alpha)^3 F \right)^2 \]  

(8.17)

on the right hand side of (8.14), we only discuss the term

\[ \int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} (\partial_i^1 F)^2 \]  

(8.18)
as the rest will admit the same bound. So, we bound the term (8.18), by using the coarea formula (2.29), as follows

\[
\int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} (\partial_i^2 F)^2 = \int \int_{D(\tau_1, \tau_2)} \sum_{i=0}^{k-2} (\partial_i^2 (a^{\alpha \beta} \partial_{\alpha} \partial_{\beta} \psi))^2 \\
\leq C(k, M, \Lambda, A_{[k-1]}) \int \int_{D(\tau_1, \tau_2)} \sum_{1 \leq i+j \leq k-1} (\partial^i \psi)^2 (\partial^j \psi)^2 \\
\leq C(k, M, \Lambda, A_{[k-1]}) \int_{\tau_1}^{\tau_2} d\tau \sum_{1 \leq i+j \leq k-1, i \leq \lfloor \frac{k-1}{2} \rfloor} \sup_{\bar{t} \in \{\bar{t} = \tau\}} (\partial^i \psi)^2 \int_{\bar{t} = \tau} (\partial^j \psi)^2 \\
\leq C(k, M, \Lambda, A_{[k-1]}) \sup_{\tau_1 \leq \tau' \leq \tau_2} (E_{k-1}[\psi](\tau')) \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau),
\]

where, in the last inequality, we used a Sobolev inequality, see Lemma 2.2, and the assumption that \( k \geq 5 \).

Therefore, by using (8.16), (8.19), inequality (8.14) implies

\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \\
\leq C(k, M, \Lambda) E_{G,k}[\psi](\tau_1) \\
+ C(k, M, \Lambda, A_{[k-1]}) \left( \sup_{\tau_1 \leq \tau' \leq \tau_2} E_{k-1}(\tau') \right) \left( \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \right) \\
+ C(k, M, \Lambda) \int_{\bar{t} = \tau_2} \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} (1 - \mu)^{2i_3+1} \left( \partial_i^1 \Omega^2_{\alpha} \partial_i^1 \Omega^2_{\beta} \partial_i^1 F \right)^2 \\
+ \sum_{0 \leq i_1 + i_2 + i_3 \leq k-3} \sum_{\alpha} \left( \partial^1_i \Omega^2_{\alpha} \partial^1_i F \right)^2,
\]

where recall that \( F = \partial \psi \cdot \partial \psi \).

Finally, we want to absorb the boundary terms of the right hand side at \( \bar{t} = \tau_2 \) by the relevant boundary term \( E_{G,k}[\psi](\tau_2) \) of the left hand side. By using the smallness assumption (8.11), it is evident that we can appropriately distribute derivatives, in view of the restriction \( k \geq 5 \), to conclude that

\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \leq C E_{G,k}[\psi](\tau_1) \\
+ C \left( \sup_{\tau_1 \leq \tau' \leq \tau_2} E_{k-1}[\psi](\tau') \right) \left( \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \right),
\]

for a constant \( C = C(k, M, \Lambda, A_{[k]} \).

The proof of Proposition 8.1 is thus complete. \( \square \)

Now we are ready to prove Theorem 1’.

**Proof of Theorem 1’** Note that the semilinear wave equation (8.1) is well posed in \( H^k, k \geq 7 \), by well-known arguments. The existence of the solution in \( D(\tau_1, \tau_2) \), for \( \tau_2 \) defined below,
will follow easily from the estimates that we will prove, so for convenience we here assume existence.

Moreover, we assume the bootstrap assumption

\[
\sup_{D(\tau_1, \tau_2)} \sum_{1 \leq i \leq \left\lfloor \frac{k}{2} - 1 \right\rfloor + 1} \sum_{\partial \in \{\partial_t, \partial_r, \Omega_1, \Omega_2, \Omega_3\}} |\partial^i \psi| \leq C_b \sqrt{\epsilon}
\]  

(8.22)

for any \( \tau \geq 0 \), for a constant \( C_b(k, M, \Lambda, A_{[k-1]}) > 0 \) to be determined later.

First, we want to prove that for \( \tau_{\text{step}} > 0 \) to be chosen later, there exists an \( \epsilon(\tau_{\text{step}}, C_b) > 0 \) sufficiently small such that if

\[
E_{G,k}[\psi](\tau_1) \leq \epsilon
\]  

(8.23)

for some \( \tau_1 \geq 0 \), then for

\[
\tau_2 = \tau_1 + \tau_{\text{step}}
\]  

(8.24)

we obtain

\[
E_{k-1}[\psi](\tau_2) \leq \epsilon,
\]

\[
E_{G,k}[\psi](\tau_2) \leq \frac{1}{2} E_{G,k}[\psi](\tau_1).
\]  

(8.25)

Let \( \tau_1 \geq 0 \), then in view of the bootstrap assumption (8.22), and for \( \epsilon(\tau_{\text{step}}, C_b) \) we obtain the result of Proposition 8.1, namely

\[
E_{G,k}[\psi](\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \leq CE_{G,k}[\psi](\tau_1)
\]  

\[
+ C \left( \sup_{\tau_1 \leq \tau' \leq \tau_2} E_{k-1}[\psi](\tau') \right) \left( \int_{\tau_1}^{\tau_2} d\tau E_{G,k}[\psi](\tau) \right)
\]  

(8.26)

for some constant \( C = C(k, M, \Lambda, A_{[k-1]}) \). Now, for a sufficiently small \( \epsilon(\tau_{\text{step}}) \) if we assume that

\[
E_{G,k}[\psi](\tau_1) \leq \epsilon,
\]  

(8.27)

we also have that \( E_{k-1}[\psi](\tau_1) \leq \epsilon \), and from (8.26) we conclude that there exists a constant

\[
C(k, M, \Lambda, A_{[k-1]}) > 0
\]  

(8.28)

where for \( A_{[k-1]} \) see Section 2.7, such that

\[
E_{G,k}[\psi](\tau') \leq C(k, M, \Lambda, A_{[k-1]}) E_{G,k}[\psi](\tau_1)
\]  

(8.29)

for all \( \tau' \in [\tau_1, \tau_2] \).

Moreover, by using again the result of Proposition 8.1 (also see the relevant computation in the proof of Theorem 1), there exists a constant \( C(k, M, \Lambda, A_{[k-1]}) > 0 \) such that for any \( \tau_{\text{step}} > 0 \) sufficiently large, there exists an \( \epsilon(\tau_{\text{step}}) > 0 \) sufficiently small, and a value

\[
\tau_{1,2} = \left[ \frac{\tau_1 + \tau_2}{2}, \tau_2 \right]
\]  

(8.30)

such that, after taking

\[
E_{G,k}[\psi](\tau_1) \leq \epsilon
\]  

(8.31)
and recalling $E_{k-1}[\psi](\tau_1) \leq E_{G,k}[\psi](\tau_1)$, we obtain

$$E_{G,k}[\psi](\tau_{1,2}) \leq \frac{1}{\tau_{\text{step}}} C(k, M, \Lambda, A_{[k-1]}) E_{G,k}[\psi](\tau_1),$$

(8.32)

where we used the mean value theorem. By using (8.29) and (8.32) we obtain

$$E_{G,k}[\psi](\tau_2) \leq \frac{1}{\tau_{\text{step}}} C(k, M, \Lambda, A_{[k-1]}) E_{G,k}[\psi](\tau_1)$$

(8.33)

for $\tau_{\text{step}} > 0$ and $\epsilon(\tau_{\text{step}}, C_b) > 0$ sufficiently small, and for a different constant $C(k, M, \Lambda, A_{[k-1]})$. Now, note that

$$E_{k-1}[\psi](\tau_2) \leq E_{G,k}[\psi](\tau_2) \leq \frac{1}{\tau_{\text{step}}} C(k, M, \Lambda, A_{[k-1]}) E_{G,k}[\psi](\tau_1).$$

(8.34)

Therefore, we choose a $\tau_{\text{step}}$ sufficiently large, such that

$$\frac{1}{\tau_{\text{step}}} C(k, M, \Lambda, A_{[k-1]}) < \frac{1}{2}$$

(8.35)

and we obtain

$$E_{k-1}[\psi](\tau_2) \leq E_{G,k}[\psi](\tau_1) \leq \epsilon,$$

$$E_{G,k}[\psi](\tau_2) \leq \frac{1}{2} E_{G,k}[\psi](\tau_1).$$

(8.36)

Of course our inequalities are still conditional on improving the bootstrap assumption (8.22). For the purpose of improving constant $C_b > 0$ of the bootstrap assumption (8.22) we note from (8.29) that

$$E_{k-1}[\psi](\tau') \leq C(k, M, \Lambda, A_{[k-1]}) E_{G,k}[\psi](\tau_1) \leq C(k, M, \Lambda, A_{[k-1]}) \epsilon,$$

(8.37)

for all $\tau' \in [\tau_1, \tau_2]$. Now, since $k = 7$ is the smallest integer such that

$$\left(\left\lfloor \frac{k-1}{2} \right\rfloor + 1 \right) + 2 \leq k - 1$$

(8.38)

we use a Sobolev inequality in (8.37) and prove

$$\sup_{D(0, r)} \sum_{1 \leq i \leq \left\lfloor \frac{k-1}{2} \right\rfloor + 1} \sum_{\hat{\eta} \in \{\partial_1, \partial_2, \partial_3\}} |\hat{\partial}^i \psi| \leq C(k, M, \Lambda, A_{[k-1]}) \sqrt{\epsilon}$$

(8.39)

which improves the bootstrap for a sufficiently large $C_b \gg C(k, M, \Lambda, A_{[k-1]})$.

This concludes the proof of the Theorem. \[ \square \]

Now, we are ready to prove Theorem 2', which is in fact a Corollary of Theorem 1'.

**Proof of Theorem 2’** We want to prove that for sufficiently small initial data

$$E_{G,k}[\psi](0) \leq \epsilon,$$

(8.40)

and for the following sequence of ascending real numbers

$$t_0 = \tau_{\text{step}}, \quad t_{i+1} - t_i = \tau_{\text{step}},$$

(8.41)
the solution exists in $D(0, t_i)$ and we obtain that

$$E_{k-1}[\psi](t_i) \leq C(k, M, \Lambda)\epsilon,$$

$$E_{G,k}[\psi](t_i) \leq \left(\frac{1}{2}\right)^i E_{G,k}[\psi](0), \tag{8.42}$$

for all $i \in \mathbb{N}$, where for the constant $C(k, M, \Lambda)$ see Remark 2.5.

Note that the inequalities (8.42) hold for $t_0$. Therefore, for the purpose of using induction we assume the inductive step that (8.42) hold for $t_i$. Then, from Theorem 1’, specifically from inequality (8.6) and the inductive step we obtain

$$E_{k-1}[\psi](t_{i+1}) \leq C(k, M, \Lambda)\epsilon,$$

$$E_{G,k}[\psi](t_{i+1}) \leq \left(\frac{1}{2}\right)^{i+1} E_{G,k}[\psi](0), \tag{8.43}$$

and moreover conclude that the solution exists in $D(0, t_{i+1})$.

Therefore, we have concluded global existence and the inequalities (8.42) for all $i \in \mathbb{N}$.

Now, we want to conclude the result of the Theorem, namely we want to prove that for

$$E_{G,k}[\psi](0) \leq \epsilon \tag{8.44}$$

we obtain

$$E_{G,k}[\psi](\tau) \leq Ce^{-c_d \tau} E_{G,k}[\psi](0), \tag{8.45}$$

for constants $c_d(k, M, \Lambda, A_{[k-1]})$, $C(k, M, \Lambda, A_{[k-1]}) > 0$. With initial data

$$E_{G,k}[\psi](0) \leq \epsilon \tag{8.46}$$

for a sufficiently small $\epsilon > 0$, we use equation (8.42) and the local in time inequality (8.6) of Theorem 1’, to immediately conclude (8.45).

We conclude the Theorem. \hfill \qed

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### A. Appendix A: Proof of the Cauchy stability result of Proposition 3.1

In this Section we prove the Cauchy stability statement of Proposition 3.1. This proof will require using geometric estimates with respect to the metric $g(\nabla \psi)$, so as to not lose derivatives. We will use some standard definitions and results of the linear theory. Therefore, we first need to discuss some standard notions.

#### A.1. Energy momentum tensor and the divergence theorem of a metric $g$

Let $g$ be a sufficiently regular Lorentzian metric, and $\psi$ a sufficiently regular function. Then, we define the energy momentum tensor with respect to that metric as

$$\mathcal{T}(g)[\psi] = d\psi \otimes d\psi - \frac{1}{2}g|\nabla \psi|^2_g, \quad (A.1)$$

where $|\nabla \psi|^2_g = g^{ab} \partial_a \psi \partial_b \psi$.

Note the following Proposition

**Proposition A.1.** Let $g$ be a sufficiently regular Lorentzian metric on $\mathcal{M}_\delta$ and $\delta(M, \Lambda) \geq 0$ sufficiently small. Also, let $\psi$ satisfy the equation

$$\Box_g \psi = F. \quad (A.2)$$
Then, we obtain the following identity
\[
\int_{[\bar{t} = \tau_2] \cap D_\delta(\tau_1, \tau_2)} \mathbb{T}(g)(X, n)[\psi] + \int_{\mathcal{H}_+^+ \cap D_\delta(\tau_1, \tau_2)} \mathbb{T}(g)(X, n)[\psi] + \int_{\mathcal{H}_+^+ \cap D_\delta(\tau_1, \tau_2)} \mathbb{T}(g)(X, n)[\psi] + \int_{D_\delta(\tau_1, \tau_2)} K^X(g) = \int_{[\bar{t} = \tau_1] \cap D_\delta(\tau_1, \tau_2)} \mathbb{T}(g)(X, n)[\psi] + \int_{D_\delta(\tau_1, \tau_2)} \text{Err}^X[\psi]
\]
\[
(A.3)
\]
for any \(0 \leq \tau_1 \leq \tau_2\) and for any vector field \(X\), where
\[
K^X(g) = \frac{1}{2} (X) \pi_{\mu\nu}(g) \mathbb{T}^{\mu\nu}(g)[\psi], \quad \text{Err}^X[\psi] = -(X\psi) F,
\]
\[
(A.4)
\]
where note that the raised indices are with the metric \(g\).

The volume forms on the spacetime domains are to be understood with respect to the metric \(g\), and the volume forms on the hypersurfaces are to be understood as the ones induced by the volume form of \(g\). Also
\[
n = \partial_{\bar{t}, n}(A.5)
\]
is to be understood as the unit normal of each hypersurface with respect to \(g\).

Note that in the case \(g = g_{M, \Lambda}, \delta \geq 0\), we have explicitly computed the volume forms and the normals, with which the divergence theorem is to be understood, in Section 2.3.

### A.2. The redshift vector field \(N\) on the Schwarzschild background \(\hat{g}\)

We present the following redshift Lemmata, see the Lecture notes [40].

**Lemma A.1.** Let \(q(M, \Lambda) > 0\) be sufficiently small. Then, for all \(\delta(M, \Lambda) > 0\) sufficiently small there exist a timelike vector field
\[
N, \quad \text{(A.6)}
on D_\delta(0, \infty), \text{ such that the following hold}
\]
\[
K^N(\hat{g}) \geq C(M, \Lambda) \mathbb{T}(\hat{g})(N, N)[\psi], \text{ in } \{r_+ - \delta \leq r \leq r_+ + q\} \cup \{\bar{r}_+ - q \leq r \leq \bar{r}_+ + \delta\}
\]
\[
N = \partial_{\bar{t}}, \text{ in } D_\delta(0, \infty) \setminus \{r_+ - \delta \leq r \leq r_+ + q\} \cup \{\bar{r}_+ - q \leq r \leq \bar{r}_+ + \delta\}. \quad \text{(A.7)}
\]
where for \(K^N(\hat{g})\) see the divergence theorem in Proposition A.1.

The following Lemma, is a higher order manifestation of the redshift effect.

**Lemma A.2.** For any \(k \geq 0\) there exist positive constants
\[
\kappa_k > 0, \quad \bar{\kappa}_k > 0 \quad \text{(A.8)}
such that for the equation
\[
\Box_{\hat{g}} \psi = F, \quad \text{(A.9)}
\]
we obtain that the following holds on \(\mathcal{H}^+\)
\[
\Box_{\hat{g}} N^k \psi = \kappa_k N^{k+1} \psi + \sum_{1 \leq i \leq 5} \sum_{0 \leq \sum m_i \leq |m| \leq k+1, m_5 \leq k} \epsilon_m \Omega_1^{m_1} \Omega_2^{m_2} \Omega_3^{m_3} \partial_i^{m_4} N^{m_5} \psi + N^k F, \quad \text{(A.10)}
\]

Proof

The proof of this Lemma is direct by noting that the energy momentum tensor of the metric $g(v)$ is

$$\mathbb{T}(g(v))(X,Y)[\psi] = X\psi Y\psi - \frac{1}{2}g(v)(X,Y)|\nabla\psi|_{g(v)}^2,$$

(A.15)

for any two smooth vector fields $X, Y$.

Specifically, by taking

$$|h_{ij}(v)|, \quad |h^{ij}(v)| \leq \sqrt{c}$$

(A.16)

Note the following lemma

**Lemma A.3.** Let $g(v) = \hat{g} + h(v)$ be a Lorentzian metric that belongs in the class of Section 3.1, where $v \in \Gamma(T, M)$ is considered fixed, and let $(M, \Lambda) > 0$ be sufficiently small.

Then there exists an $\epsilon(\delta) > 0$ such that for

$$|h_{ij}(v)|, \quad |h^{ij}(v)| \leq \sqrt{\epsilon}$$

(A.12)

in Cartesian coordinates, the hypersurface $\{\bar{t} = \tau\}$, is a Cauchy hypersurface, with respect to $g(v)$, in $D(0, \infty)$. Moreover, the unit normal $n$ of $\{\bar{t} = c\}$ is timelike on $D(0, \infty)$ with respect to $g(v)$ and the redshift vector field, see Lemma A.1, is timelike with respect to $g(v)$, as well. Furthermore, the hypersurfaces

$$\mathcal{H}^+ \times \mathbb{R}^+ \times \mathcal{H}^+$$

(A.13)

are spacelike hypersurfaces with respect to the metric $g(v)$ in $D(0, \infty)$.

Finally, we obtain

$$\int_{\{\bar{t} = \tau\}} \mathbb{T}(g(v))(n,n)[\psi] dg(v)_{\{\bar{t} = \tau\}} \sim \int_{\{\bar{t} = \tau\}} \mathbb{T}(\hat{g})(\hat{n}, \hat{n})[\psi] d\hat{g}_{\{\bar{t} = \tau\}},$$

$$\int_{\mathcal{H}^+} \mathbb{T}(g(v))(n,n)[\psi] dg(v)_{\mathcal{H}^+} \sim \int_{\mathcal{H}^+} \frac{dg_{\mathcal{H}^+}}{\sqrt{1 - \mu}} \left(\delta(\partial_r \psi)^2 + (\partial_\tau \psi)^2 + |\nabla\psi|^2\right),$$

(A.14)

$$\int_{\tilde{\mathcal{H}}^+} \mathbb{T}(g(v))(n,n)[\psi] dg(v)_{\tilde{\mathcal{H}}^+} \sim \int_{\tilde{\mathcal{H}}^+} \frac{dg_{\tilde{\mathcal{H}}^+}}{\sqrt{1 - \mu}} \left(\delta(\partial_r \psi)^2 + (\partial_\tau \psi)^2 + |\nabla\psi|^2\right),$$

where the constants only depend on the black hole parameters, and $dg(v)_{\{\bar{t} = \tau\}}, dg(v)_{\mathcal{H}^+}, dg(v)_{\tilde{\mathcal{H}}^+}$ are the induced volume form of the spacetime volume form of the metric $g(v)$ on the hypersurfaces $\{\bar{t} = \tau\}, \mathcal{H}^+, \tilde{\mathcal{H}}^+$, respectively. For the energy momentum tensor $\mathbb{T}$ see Section A.1, for the redshift vector field $N$ see Lemma A.1.

**Proof** The proof of this Lemma is direct by noting that the energy momentum tensor of the metric $g(v)$ is

$$\mathbb{T}(g(v))(X,Y)[\psi] = X\psi Y\psi - \frac{1}{2}g(v)(X,Y)|\nabla\psi|_{g(v)}^2,$$

(A.15)

for any two smooth vector fields $X, Y$.

Specifically, by taking

$$|h_{ij}(v)|, \quad |h^{ij}(v)| \leq \sqrt{c}$$

(A.16)
sufficiently small, we conclude the causal behaviour of the hypersurfaces mentioned, and by pointwise estimates on the integrands we conclude
\[
\int_{[\bar{t} = \tau]} T(g(v))(n, n)[\psi] dg(v)_{[\bar{t} = \tau]}
\]
\[
\sim \int_{[\bar{t} = \tau]} T(\tilde{g})(\bar{n}, \bar{n})[\psi] d\tilde{g}_{[\bar{t} = \tau]},
\]
\[
\int_{\mathcal{H}_3^+} T(g(v))(N, n)[\psi] dg(v)_{\mathcal{H}_3^+}
\]
\[
\sim \int_{\mathcal{H}_3^+} \frac{d\tilde{g}_{\mathcal{H}_3^+}}{\sqrt{1 - \mu}} \left( \delta(\partial_\tau \psi)^2 + (\partial_\tau \psi)^2 + |\bar{\nabla} \psi|^2 + \sqrt{\epsilon(\delta)} \left( (\partial_\tau \psi)^2 + (\partial_\tau \psi)^2 + |\bar{\nabla} \psi|^2 \right) \right),
\]
\[
\int_{\mathcal{H}_3^+} T(g(v))(N, n)[\psi] dg(v)_{\mathcal{H}_3^+}
\]
\[
\sim \int_{\mathcal{H}_3^+} \frac{d\tilde{g}_{\mathcal{H}_3^+}}{\sqrt{1 - \mu}} \left( \delta(\partial_\tau \psi)^2 + (\partial_\tau \psi)^2 + |\bar{\nabla} \psi|^2 + \sqrt{\epsilon(\delta)} \left( (\partial_\tau \psi)^2 + (\partial_\tau \psi)^2 + |\bar{\nabla} \psi|^2 \right) \right).
\]
\[(A.17)\]

Now, for \( \epsilon(\delta) \ll \delta \) we conclude (A.14).

\[\square\]

**A.4. Elliptic estimates and estimates on \( \mathcal{H}_3^+ \), \( \mathcal{H}_3^+ \) for the quasilinear wave equation**

Note the following elliptic estimate

**Lemma A.4.** Let \( k \geq 4 \) and let \( \psi \) satisfy the quasilinear wave equation (1.1) in \( D_3(\tau_1, \tau_2) \) for \( \tau_1 \leq \tau_2 \) and for a sufficiently small \( \delta(M, \Lambda) > 0 \), where for the tensors \( a, h \) see the Sections 2.7, 3.1 respectively. Then, for any \( r_+ < r_0 < r_1 < \bar{r}_+ \), there exist constants

\[
C(r_0, r_1, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}), \quad C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]

(A.18)

independent of \( \delta \), where for \( A_{[k+1]}, B_{[k+1]} \) see Sections 2.7 and 3.1, respectively, and there exists an \( \epsilon = \epsilon(\delta, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) \) sufficiently small such that the following holds.

If

\[
\sup_{D_3(\tau_1, \tau_2)} \sum_{1 \leq j \leq k-1} \sum_{|\partial_j^i \psi|} \leq \sqrt{\epsilon}
\]

(A.19)

holds, we obtain that

\[
\int_{[\bar{t} = \tau']} \sum_{1 \leq i + j_2 + j_3 \leq j} \sum_\alpha \left( \partial_i \partial_{\alpha} \Omega_{i_2}^j \psi \right)^2 d\tilde{g}_{[\bar{t} = \tau']}
\]

\[
\leq C \int_{[\bar{t} = \tau]} \sum_{1 \leq i \leq j-1} T(\tilde{g})(N, n)[N^i \psi] d\tilde{g}_{[\bar{t} = \tau]},
\]

\[
\int_{[\bar{t} = \tau] \cap [r_0, r_1]} \sum_{1 \leq i_2 + j_3 \leq j} \sum_\alpha \left( \partial_i \partial_{\alpha} \Omega_{i_2}^j \psi \right)^2 d\tilde{g}_{[\bar{t} = \tau']}
\]

\[
\leq C(r_0, r_1) \int_{[\bar{t} = \tau] \cap [r_0, r_1]} \sum_{1 \leq i + j_2 + j_3 \leq j} \sum_\alpha \left( \partial_i \Omega_{i_2}^j \psi \right)^2 d\tilde{g}_{[\bar{t} = \tau]},
\]

(A.20)
\[
\int_{\mathcal{H}_{\delta}^+} \frac{d\bar{g}}{\sqrt{1 - \mu}} \left( \delta \sum_{1 \leq i_1 + i_2 + h_2 \leq j, j_2 \geq 1} \sum_{\alpha} \left( \partial_t^{i_1} \partial_r^{i_2} \Omega_{\alpha}^{i_2} \psi \right)^2 + \sum_{1 \leq i_1 + i_2 \leq j} \sum_{\alpha} \left( \partial_r^{i_1} \Omega_{\alpha}^{i_2} \psi \right)^2 \right) \]
\[
\leq C \int_{\mathcal{H}_{\delta}^+} \sum_{1 \leq i \leq j - 1} \mathbb{T}(\bar{g})(N, n)[N^i \psi]d\bar{g}_{\mathcal{H}_{\delta}^+},
\]
\[
\leq C \int_{\mathcal{H}_{\delta}^+} \sum_{1 \leq i \leq j - 1} \mathbb{T}(\bar{g})(N, n)[N^i \psi]d\bar{g}_{\mathcal{H}_{\delta}^+},
\]

for \(1 \leq j \leq k + 2\).

**Proof** The proof of this Lemma comes from the estimates of Lemma A.3.

For the first inequality of (A.20), for \(j = 1\), we use the smallness (A.19), and Lemma A.3. We prove the estimate for all orders \(j\), by first noting that the energy momentum tensor of \(g(\nabla \psi)\) is

\[
\mathbb{T}(g(\nabla \psi))(X, n)[\bar{X}^i \psi] = \mathbb{T}(\bar{g})(X, n)[\bar{X}^i \psi] - \frac{1}{2} h(X, n, \nabla \psi) \left( \nabla \bar{X}^i \psi \right)^2,
\]

for any two smooth vector field \(X, \bar{X}\). Therefore, for \(\epsilon(\delta > 0\) sufficiently small, we use elliptic estimates on the difference \(\Box_{\bar{g}} \psi = (\Box_{\bar{g}} - \Box_{g(\nabla \psi)}) \psi + \partial \psi \cdot \partial \psi\) and conclude. We also used that \(k \geq 4\).

For the second inequality of (A.20), we obtain the estimate by arguing as above and by taking \(\epsilon(\delta) > 0\) small. We also used that \(k \geq 4\).

For the last two inequalities of (A.20), on the hypersurfaces \(\mathcal{H}_{\delta}^+, \bar{\mathcal{H}}_{\delta}^+\), to get the estimate for \(j = 1\), we use the smallness (A.19), and Lemma A.3. We prove the analogous estimate for all orders \(j\), for a sufficiently small \(\epsilon(\delta) > 0\) by noting the form of the energy momentum tensor (A.21), and by using elliptic estimates on the difference \(\Box_{\bar{g}} \psi = (\Box_{\bar{g}} - \Box_{g(\nabla \psi)}) \psi + \partial \psi \cdot \partial \psi\). We also used that \(k \geq 4\).

Finally, note the following estimate

**Lemma A.5.** Let \(k \geq 4\). Let \(\psi\) satisfy the quasilinear wave equation (1.1) in \(D_\delta(\tau_1, \tau_2)\) for a sufficiently small \(\delta(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0\), where for the tensors \(a, h\) see the Sections 2.7 and 3.1, respectively.

Then, there exists a constant

\[
C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0
\]

independent of \(\delta\), and an \(\epsilon = \epsilon(\delta, k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0\) sufficiently small such that the following holds. If

\[
\sup_{D_\delta(\tau_1, \tau_2)} \sum_{1 \leq j \leq k - 1} \sum_{a \in \{\partial_t, \partial_r, \Omega_1, \Omega_2, \Omega_3\}} |\partial^j \psi| \leq \sqrt{\epsilon},
\]

(A.23)
then we obtain that

\[ \left| \int_{\mathcal{H}_{\delta}^+} \sum_{1 \leq i_1 + i_2 \leq j - 1} \sum_{\alpha} \mathbb{T}(\hat{g})(\vartheta_i, n)[a_i^{j_1} \Omega_{ij}^2 \psi] d\hat{g} \mathcal{H}_s \right| - \left| \int_{\mathcal{H}_{\delta}^+} \sum_{0 \leq i_1 + i_2 \leq j - 1} \sum_{\alpha} \left( \delta^{i_1} \Omega_{ij}^2 \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\hat{g} \mathcal{H}_s \right| \leq C\delta \left| \int_{\mathcal{H}_{\delta}^+} \sum_{1 \leq i_1 + i_2 \leq j - 1} \sum_{\alpha} \mathbb{T}(\hat{g})(N, n)[N^j \psi] d\hat{g} \mathcal{H}_s \right|, \]

\[ \left| \int_{\mathcal{H}_{\delta}^+} \sum_{1 \leq i_1 + i_2 \leq j - 1} \sum_{\alpha} \mathbb{T}(\hat{g})(\vartheta_i, n)[a_i^{j_1} \Omega_{ij}^2 \psi] d\hat{g} \mathcal{H}_s \right| - \left| \int_{\mathcal{H}_{\delta}^+} \sum_{0 \leq i_1 + i_2 \leq j - 1} \sum_{\alpha} \left( \delta^{i_1} \Omega_{ij}^2 \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\hat{g} \mathcal{H}_s \right| \leq C\delta \left| \int_{\mathcal{H}_{\delta}^+} \sum_{1 \leq i_1 + i_2 \leq j - 1} \sum_{\alpha} \mathbb{T}(\hat{g})(N, n)[N^j \psi] d\hat{g} \mathcal{H}_s \right|. \]

(A.24)

for \( 1 \leq j \leq k + 2 \).

**Proof** We note that for the metric \( \hat{g} \), we obtain

\[ \mathbb{T}(\hat{g})(\vartheta_i, \mathcal{H}_s)[a_i^{j_1} \psi] = \left( (\delta^{i_1} \psi)^2 \left( c_1(r) - \frac{1}{2} \hat{g}_{II} \hat{g}_{II} - \frac{1}{2} c_2(r) \hat{g}_{rr} \hat{g}\right) \right) \]

\[ + \delta^{i_1} \psi \delta_{i_1} \psi \left( c_2(r) - c_1(r) \right) \hat{g}_{II} \hat{g}_{II} \]

\[ + \left| \vec{\nabla} \delta^{i_1} \psi \right|^2 \left( -\frac{1}{2} c_1(r) \hat{g}_{II} - \frac{1}{2} c_2(r) \hat{g}_{rr} \right) \]

\[ + (\vartheta_i \delta^{i_1} \psi)^2 \left( -\frac{1}{2} c_1(r) \hat{g}_{II} \hat{g}_{II} - \frac{1}{2} c_2(r) \hat{g}_{rr} \hat{g}_{rr} \right), \]

(A.25)

where for \( \hat{g}_{II}, c_1(r), c_2(r) \) see Section 2.3, and specifically \( c_1(r) = \frac{1}{\sqrt{1 - \mu}} + O(\sqrt{1 - \mu}) \) as \( r \to r_+ \) and \( c_2(r) = -\sqrt{1 - \mu} \). At \( r = r_+ - \delta \) the following hold

\[ \left| \frac{1}{2} \hat{g}_{II} \hat{g}_{II} - \frac{1}{2} c_2(r) \hat{g}_{rr} \hat{g} \right| \sim \sqrt{1 - \mu} \sim \delta^{1/2}, \]

\[ \left| c_2(r) - c_1(r) \right| \hat{g}_{II} \hat{g}_{II} \hat{g}_{II} \sim \sqrt{1 - \mu} \sim \delta^{1/2}, \]

\[ \left| -\frac{1}{2} c_1(r) \hat{g}_{II} - \frac{1}{2} c_2(r) \hat{g}_{rr} \right| \sim \sqrt{1 - \mu} \sim \delta^{1/2}, \]

\[ \left| -\frac{1}{2} c_1(r) \hat{g}_{II} \hat{g}_{II} - \frac{1}{2} c_2(r) \hat{g}_{rr} \hat{g}_{rr} \right| \sim 1 - \mu, \quad \delta^{1/2} \sim \delta^{3/2}, \]

(A.26)

see Remark 2.3 for the inverse metric components. A similar estimate holds for the hypersurface \( \mathcal{H}_{\delta}^+ \), since \( \hat{g}_{III} \) has a favorable form, see Section 2.3. We obtain the linear version of the Lemma.
For the quasilinear version we use the smallness \( (A.23) \) to obtain

\[
\left| \int_{H_{r}^{+}} \sum_{1 \leq i_{1} + i_{2} \leq j-1} \sum_{\alpha} T\{(\partial_{i_{1}} \hat{n}_{H_{r}})[a_{i_{1}}^{1} \Omega_{i_{2}}^{2} \psi]\} d\hat{g}_{H_{r}} - \int_{H_{r}^{+}} \sum_{0 \leq i_{1} + i_{2} \leq j-1} \sum_{\alpha} \left(\partial_{i_{1}} a_{i_{2}}^{1} \Omega_{i_{2}}^{2} \psi\right)^{2} \frac{1}{\sqrt{1 - \mu}} d\hat{g}_{H_{r}} \right| \\
\leq C \sqrt{\delta} \int_{H_{r}^{+}} \sum_{1 \leq i_{1} + i_{2} \leq j-1} \sum_{\alpha} T\{(\partial_{i_{1}} \hat{n}_{H_{r}})[N^{i} \psi]\} d\hat{g}_{H_{r}} + \sqrt{\epsilon(\delta)} \int_{H_{r}^{+}} \sum_{0 \leq i_{1} + i_{2} \leq j-1} \sum_{\alpha} \left(\partial_{i_{1}} a_{i_{2}}^{1} \Omega_{i_{2}}^{2} \psi\right)^{2} \frac{1}{\sqrt{1 - \mu}} d\hat{g}_{H_{r}} \\
\leq C \sqrt{\delta} \int_{H_{r}^{+}} \sum_{1 \leq i_{1} + i_{2} \leq j-1} \sum_{\alpha} T\{(\partial_{i_{1}} \hat{n}_{H_{r}})[N^{i} \psi]\} d\hat{g}_{H_{r}} + \sqrt{\epsilon(\delta)} \int_{H_{r}^{+}} \sum_{0 \leq i_{1} + i_{2} \leq j-1} \sum_{\alpha} \left(\partial_{i_{1}} a_{i_{2}}^{1} \Omega_{i_{2}}^{2} \psi\right)^{2} \frac{1}{\sqrt{1 - \mu}} d\hat{g}_{H_{r}} \tag{A.27}
\]

for all \( 1 \leq j \leq k + 2 \), where \( C = C(k, M, \Lambda, A[k+1], B[k+1]) \). Therefore, for a sufficiently small \( \delta \) and \( \epsilon(\delta) \ll \delta \), we use Lemma A.4 and conclude. \( \square \)

### A.5. Proof of Proposition 3.1

Now we proceed to the proof of the Proposition.

**Proof** Let \( \tau_{1} \geq 0 \).

We will use the derivatives notation of Section 2.6. Moreover, for the semilinear term \( \partial \psi \cdot \partial \psi \) and for the definition of smooth tensors \( a, h \), see Sections 2.7 and 3.1, respectively. In this proof, when we write

\[
h_{ab} \tag{A.28}
\]

it is to be understood \( h_{ab}(\mathbf{\nabla} \psi) \) in Cartesian coordinates, see (2.38). When we write

\[
h^{ab} \tag{A.29}
\]

it is to be understood as \( h^{ab}(\mathbf{\nabla} \psi) = g^{ab}(\mathbf{\nabla} \psi) - g^{ab} \), in Cartesian coordinates. Finally, in this proof when we write \( \lesssim \) it is to be understood that we omit a constant \( C(k, M, \Lambda, A[m], B[m]) \), where for \( A[m], B[m] \) see Sections 2.7 and 3.1, respectively, where \( m \leq k + 1 \).

**Existence and uniqueness.**

The existence and uniqueness part of the proof follow readily, by well known arguments, from the energy estimates (3.9) and (3.10). Therefore, we only prove (3.9) and (3.10), assuming existence.

**An auxiliary energy.**

We here introduce an auxiliary energy with which we will estimate the \( \partial_{\bar{t}} \) flux on \( \{ \bar{t} = \tau \} \), see already (A.52). We define

\[
E_{j,q}[\psi](\tau) = \int_{\{\bar{t} = \tau\} \cap [r_{+} + q \leq r \leq \bar{r}_{+} - q]} \sum_{1 \leq i_{1} + i_{2} \leq j} \sum_{\alpha} \left(\partial_{i_{1}} a_{i_{2}}^{1} \Omega_{i_{2}}^{2} \psi\right)^{2} d\hat{g}(\bar{t} = \tau), \tag{A.30}
\]

for \( q > 0 \) as in the redshift Lemma A.1.
The linear wave equation analogue.

Before proving the Cauchy stability estimate (3.9) and (3.10) for the solutions of the quasilinear wave equation, we sketch the proof of (3.9) and (3.10) for the solutions of the linear wave equation

\[ \square_g \psi = 0, \]  

so that the reader can see the similarity between the two proofs. Also see the Lecture notes [40]. The steps of the proof of the relevant quasilinear estimate are similar, but by also treating the non-linearities appropriately.

First, we apply Proposition A.1 with multiplier \( \partial_{\bar{t}} \) and obtain that there exists a constant \( C(M, \Lambda) > 1 \) such that

\[ E_{1,j} \[ \psi \] (\tau_2) \leq C(M, \Lambda) E_{1,j} \[ \psi \] (\tau_1) \]  

for all \( \tau_1 \leq \tau_2 \). By commuting the wave equation appropriately many times with \( \partial_{\bar{t}} \Omega_\alpha \) and by using elliptic estimates, then (A.32) implies

\[ E_{j,q} \[ \psi \] (\tau_2) \leq C(j, M, \Lambda) E_{j,q} \[ \psi \] (\tau_1) \]  

for all \( j \geq 1 \), for some constant \( C(j, M, \Lambda) > 1 \).

Second, we apply the divergence Theorem with multiplier the red-shift vector field \( N \), see Lemma A.1, and obtain

\[ E_{1,j} \[ \psi \] (\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{1,j} \[ \psi \] (\tau) \leq C(j, M, \Lambda) \int_{\tau_1}^{\tau_2} E_{1,j} \[ \psi \] (\tau) + C(j, M, \Lambda) E_{1,j} \[ \psi \] (\tau_1) \]  

for all \( \tau_1 \leq \tau_2 \). By commuting the wave equation \( j - 1 \) times with \( N \), also see the redshift Lemma A.2 and by using elliptic estimates, then (A.34) implies

\[ E_{j,j} \[ \psi \] (\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{j,j} \[ \psi \] (\tau) \leq C(j, M, \Lambda) \int_{\tau_1}^{\tau_2} E_{j,j} \[ \psi \] (\tau) + C(j, M, \Lambda) E_{j,j} \[ \psi \] (\tau_1) \]  

for all \( j \geq 1 \).

Now, by using (A.33) and (A.35) we obtain that there exists a constant \( C(j, M, \Lambda) \) such that

\[ E_{j,j} \[ \psi \] (\tau_2) + \int_{\tau_1}^{\tau_2} E_{j,j} \[ \psi \] (\tau) d\tau \leq C(j, M, \Lambda) E_{j,j} \[ \psi \] (\tau_1)(\tau_2 - \tau_1) + C(j, M, \Lambda) E_{j,j} \[ \psi \] (\tau_1) \]  

for all \( \tau_1 \leq \tau_2 \). Note that (A.34) holds for all \( \tau_1' \geq \tau_1 \) in the place of \( \tau_1 \). We readily obtain that

\[ E_{j} \[ \psi \] (\tau) \leq C(j, M, \Lambda) E_{j} \[ \psi \] (\tau_1) \]  

for all \( \tau \geq \tau_1 \), which concludes the proof of the energy estimates (3.9), (3.10) for (A.31).

The bootstrap.

We now return to the proof of the Proposition 3.1.

We introduce the bootstrap assumption

\[ \sup_{D_b(\tau_1, \tau_1 + \tau_{\text{max}})} \sum_{1 \leq i \leq k - 1} \sum_{\partial \in \{\partial_{\bar{t}}, \partial_r, \Omega_1, \Omega_2, \Omega_3\}} |\partial^j \psi| \leq C_b \sqrt{\epsilon} \]  

for a large constant \( C_b(M, \Lambda) > 0 \) to be determined later.
**Volume form notation.**

In this proof in any integrals over spacetime domains or hypersurfaces without explicit volume forms it is to be understood that the volume forms of the integrals over spacetime domains are with respect to the metric \( g(\nabla \psi) \). The volume forms of the integrals over hypersurfaces are with respect to the induced volume form of the metric \( g(\nabla \psi) \) on those hypersurfaces.

**The multiplier estimates.**

First, we write the general energy identities (multiplier estimates) we need. We will later use them for the vector fields \( \partial_t, N \).

We apply the divergence Theorem, see Proposition A.1, for the quasilinear wave equation (1.1), for the metric \( g(\nabla \psi) \), with multiplier \( X \) and appropriately many commutations with a vector field \( \tilde{X} \), on the extended region

\[
D_\delta(\tau_1, \tau_1 + \tau_{\text{max}}).
\]

We consider any

\[
\tau_2 - \tau_1 \leq \tau_{\text{max}}.
\]

We obtain

\[
\left( \int_{[\bar{t} = \tau_2]} + \int_{H^+_d \cap D_\delta(\tau_1, \tau_2)} + \int_{\tilde{H}^+_d \cap D_\delta(\tau_1, \tau_2)} \right) \mathcal{T}(g(\nabla \psi))(X, n)[\tilde{X}^i \psi]
\]

\[
+ \int \int_{D_\delta(\tau_1, \tau_2)} \frac{1}{2} (\mathcal{L}_X \pi_{\mu \nu}(g(\nabla \psi)) \mathcal{T}_{\mu \nu}(g(\nabla \psi))[\tilde{X}^i \psi]
\]

\[
= \int_{[\bar{t} = \tau_1]} \mathcal{T}(g(\nabla \psi))(X, n)[\tilde{X}^i \psi]
\]

\[
- \int \int_{D_\delta(\tau_1, \tau_2)} X\tilde{X}^i \psi \left( \sum_{l=0}^{j-1} \tilde{X}^l \left[ g(\nabla \psi), \tilde{X} \right] \tilde{X}^{i-1-l} \psi + \tilde{X}^i (a^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi) \right)
\]

for all \( 1 \leq i \leq k + 1 \). Note the deformation tensor of \( g(\nabla \psi) \) is

\[
\frac{1}{2} (\mathcal{L}_X \pi_{\mu \nu}(g(\nabla \psi)) \mathcal{T}_{\mu \nu}(g(\nabla \psi))[\tilde{X}^i \psi]
\]

\[
= \frac{1}{2} \left( \mathcal{L}_X (\tilde{g} + h(\nabla \psi)) \right)_{\mu \nu} \tilde{g}^{\mu \alpha}(\nabla \psi) \tilde{g}^{\nu \beta}(\nabla \psi) \mathcal{T}_{\alpha \beta}(g(\nabla \psi))[\tilde{X}^i \psi]
\]

\[
= \frac{1}{2} \left( \mathcal{L}_X \mathcal{L}_X (\tilde{g} + h(\nabla \psi)) \right)_{\mu \nu} \left( \tilde{g}^{\mu \alpha} + h^{\mu \alpha}(\nabla \psi) \right) \left( \tilde{g}^{\nu \beta} + h^{\nu \beta}(\nabla \psi) \right)
\]

\[
\times \left( \mathcal{T}(\tilde{g})(\partial_\alpha, \partial_\beta)[\tilde{X}^i \psi] - \frac{1}{2} h(\partial_\alpha, \partial_\beta, \nabla \psi)[\nabla \tilde{X}^i \psi]^2_{g(\nabla \psi)} \right)
\]

\[
= \frac{1}{2} (\mathcal{L}_X \pi_{\mu \nu}(\tilde{g}) \mathcal{T}_{\mu \nu}(\tilde{g})[\tilde{X}^i \psi]) + \mathcal{L}_X \pi_{\text{non-lin}}(\tilde{X}^i \psi),
\]
with

\[
(L_{\chi^i} g^\mu\nu) = (L_{\chi^i} g^\mu\nu)_{\mu\nu} \left( g^{\mu\nu} \nabla^2 \nabla \nabla X^i g_{\nabla^2 \nabla X^i} + h^{\mu\nu} T(\mathring{g})(\partial_\alpha, \partial_\beta)[\nabla X^i g_{\nabla^2 \nabla X^i}] + h^{\mu\nu} (\nabla^2 \nabla X^i) (h^{-1}_{\mu\alpha}(\nabla^2 \nabla X^i)) \right)
\]

\[
+ (L_{\chi^i} g^\mu\nu)_{\mu\nu} h^{\mu\nu} T(\mathring{g})(\partial_\alpha, \partial_\beta)[\nabla X^i g_{\nabla^2 \nabla X^i}] + h^{\mu\nu} T(\mathring{g})(\partial_\alpha, \partial_\beta)[\nabla X^i g_{\nabla^2 \nabla X^i}] + h^{\mu\nu} (\nabla^2 \nabla X^i) (h^{-1}_{\mu\alpha}(\nabla^2 \nabla X^i)) \right)
\]

\[
+ (L_{\chi^i} g^\mu\nu)_{\mu\nu} h^{\mu\nu} T(\mathring{g})(\partial_\alpha, \partial_\beta)[\nabla X^i g_{\nabla^2 \nabla X^i}] + h^{\mu\nu} T(\mathring{g})(\partial_\alpha, \partial_\beta)[\nabla X^i g_{\nabla^2 \nabla X^i}] + h^{\mu\nu} (\nabla^2 \nabla X^i) (h^{-1}_{\mu\alpha}(\nabla^2 \nabla X^i)) \right)
\]

where we used that the energy momentum tensor of \( g(\nabla \psi) \) is

\[
T(\mathring{g}(\nabla \psi))(X, Y)[\psi] = T(\mathring{g})(X, Y)[\psi] - \frac{1}{2} h(X, Y, \nabla \psi)[\nabla^2 \nabla X^i g_{\nabla^2 \nabla X^i}] \quad (A.44)
\]

for any two smooth vector field \( X, Y \), with

\[
|\nabla X^i|^2_{\mathring{g}(\nabla \psi)} = g^{\alpha\beta}(\nabla \psi) \partial_\alpha \psi \partial_\beta \psi \quad (A.45)
\]

The \( \partial_\tau \) multiplier estimate.

We apply the energy identity (A.41) with multiplier \( \partial_\tau \) and commutators \( \partial_\tau \Omega_\alpha^i \), in view of (A.42) and of

\[
(\partial_\tau) \pi_{\mu\nu}(\mathring{g}) T^{\mu\nu}(\mathring{g}) = 0. \quad (A.46)
\]

We obtain

\[
\int_{[\tau_1, \tau_2]} T(\mathring{g}(\nabla \psi))(\partial_\tau, n) [\partial_\tau \Omega_\alpha^i \psi]
\]

\[
= \int_{[\tau_1, \tau_2]} T(\mathring{g}(\nabla \psi))(\partial_\tau, n) [\partial_\tau \Omega_\alpha^i \psi] + \int_{[\tau_1, \tau_2]} (\partial_\tau) \pi_{\mu\nu}(\mathring{g}) T^{\mu\nu}(\mathring{g})
\]

\[
- \left( \int_{\mathcal{H}_\tau^1 \cap \mathcal{D}_\delta(\tau_1, \tau_2)} + \int_{\mathcal{H}_\tau^2 \cap \mathcal{D}_\delta(\tau_1, \tau_2)} \right) (T(\mathring{g}(\nabla \psi))(\partial_\tau, n) (\partial_\tau \Omega_\alpha^i \psi)) \quad (A.47)
\]

\[
- \int_{\mathcal{D}_\delta(\tau_1, \tau_2)} (\partial_\tau) \pi_{\mu\nu}(\mathring{g}) (\partial_\tau \Omega_\alpha^i \psi)
\]
\[
- \int \int_{D_\delta(t_1, t_2)} \partial_\tau \partial^a \Omega^i_\alpha \psi \left( \sum_{j_1+j_2+j_3+j_4=i+1} \sum_{j_5+j_6=1} \delta^i_1 \Omega^j_\alpha \left[ \square g(\nabla \psi), \delta^j_0 \Omega^k_\alpha \right] \delta^k_1 \Omega^l_\alpha \psi \right) + \delta^i_1 \Omega^j_\alpha \left( a^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi \right).
\]

Therefore, by summing over \(i_1, i_2, \alpha\) and by adding in the hypersurface terms
\[
\sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \left( \int_{H^+_\delta \cap D_\delta(t_1, t_2)} + \int_{H^+_\delta \cap \partial D_\delta(t_1, t_2)} \right) \left( \delta^i_1 \Omega^j_\alpha \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\bar{\Sigma}_H \quad (A.48)
\]
on both sides of (A.47) and recall the the energy momentum tensor of \(g(\nabla \psi)\), see (A.44), we obtain
\[
\sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \int_{[t = 1]} \mathbb{T}(\bar{g})(\partial_\tau, n)[\delta^i_1 \Omega^j_\alpha \psi]
\]
\[
+ \sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \left( \int_{H^+_\delta \cap D_\delta(t_1, t_2)} + \int_{H^+_\delta \cap \partial D_\delta(t_1, t_2)} \right) \left( \delta^i_1 \Omega^j_\alpha \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\bar{\Sigma}_H
\]
\[
= \sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \int_{[t = 1]} \mathbb{T}(g(\nabla \psi))(\partial_\tau, n)[\delta^i_1 \Omega^j_\alpha \psi]
\]
\[
+ \sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \left( \int_{H^+_\delta \cap D_\delta(t_1, t_2)} + \int_{H^+_\delta \cap \partial D_\delta(t_1, t_2)} \right) \left( \mathbb{T}(g(\nabla \psi))(\partial_\tau, n)[\delta^i_1 \Omega^j_\alpha \psi] \right)
\]
\[
+ \sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \left( \int_{H^+_\delta \cap D_\delta(t_1, t_2)} + \int_{H^+_\delta \cap \partial D_\delta(t_1, t_2)} \right) \left( \delta^i_1 \Omega^j_\alpha \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\bar{\Sigma}_H
\]
\[
- \sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \int \int_{D_\delta(t_1, t_2)} \partial_\tau \pi_{\text{non-lin}}(\delta^i_1 \Omega^j_\alpha \psi)
\]
\[
- \sum_{1 \leq i_1 + i_2 \leq i} \sum_{\alpha} \int \int_{D_\delta(t_1, t_2)} \partial_\tau \delta^i_1 \Omega^j_\alpha \psi \left( \sum_{j_1+j_2+j_3+j_4=i+1} \sum_{j_5+j_6=1} \delta^i_1 \Omega^j_\alpha \left[ \square g(\nabla \psi), \delta^j_0 \Omega^k_\alpha \right] \right)
\]
\[
\times \delta^j_1 \Omega^k_\alpha \psi + \delta^i_1 \Omega^j_\alpha \left( a^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi \right)
\]
(A.49)

Now, note that there exist constants \(c(M, \Lambda), C(M, \Lambda) > 0\), such that
\[
c(M, \Lambda) \left( (\partial_\tau \psi)^2 + |\nabla \psi|^2 \right) \leq \mathbb{T}(\bar{g})(\partial_\tau, \hat{n}_{[t = 1]})[\psi] + C(M, \Lambda)|1 - \mu|(\partial_\tau \psi)^2,
\]
(A.50)
in sufficiently small neighborhoods of the horizons and
\[
c(M, \Lambda, q) \left( (\partial_\tau \psi)^2 + (\partial_\tau \psi)^2 + |\nabla \psi|^2 \right) \leq \mathbb{T}(\bar{g})(\partial_\tau, \hat{n}_{[t = 1]})[\psi]
\]
(A.51)
in \((r_+ + q, \bar{r}_+ - q)\). Therefore, in view of Lemma A.3 and the elliptic estimate of Lemma A.4, we note that there exists a \(\delta > 0\) sufficiently small, and an \(\epsilon(\delta, C_b) > 0\) sufficiently small,
such that
\[ cE_{j,q}(\tau_2) \leq CE_j(\tau_1) \]
\[ + C\delta \int_{[\tau_2]} \sum_{1 \leq i_1 + i_2 + i_3 \leq j} \sum_{\alpha} \left( \partial_i^{i_1} \partial_{i_2}^{i_2} \Omega_i^{\alpha} \psi \right)^2 d\hat{g}_{(i=\tau_1)} \]
\[ - \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \left( \int_{H^+_{\beta} \cap D_\delta(\tau_1, \tau_2)} + \int_{H^+_{\delta} \cap D_\delta(\tau_1, \tau_2)} \right) \left( \mathcal{T}(g(\nabla \psi))(\partial_i, n)[\partial_i^{i_1} \Omega_i^{\alpha} \psi] \right) \]
\[ + \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \left( \int_{H^+_{\beta} \cap D_\delta(\tau_1, \tau_2)} + \int_{H^+_{\delta} \cap D_\delta(\tau_1, \tau_2)} \right) \left( \partial_i^{i_1} \Omega_i^{\alpha} \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\hat{g}_{\mathcal{H}_\delta} \]
\[ + \sum_{1 \leq i_1 + i_2 \leq j} \sum_{\alpha} \int_{[\tau_2]} \frac{1}{2} h(\partial_i, n, \nabla \psi)[\nabla \partial_i^{i_1} \Omega_i^{\alpha} \psi]_g(\nabla \psi) \]
\[ - \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \int_{D_\delta(\tau_1, \tau_2)} (\partial_i^{i_1}) \pi_{\text{non-lin}}(\partial_i^{i_1} \Omega_i^{\alpha} \psi) \]
\[ - \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \int_{D_\delta(\tau_1, \tau_2)} \partial_i \partial_i^{i_1} \Omega_i^{\alpha} \psi \left( \sum_{j_1 + j_2 + j_4 = i_1 + i_2 - 1} \sum_{j_5 + j_6 = 1} \tilde{\partial}_i^{i_1} \Omega_i^{\alpha} [\bar{g}(\nabla \psi), \right. \]
\[ \tilde{\partial}_i^{i_1} \Omega_i^{\alpha} \partial_i^{j_3} \Omega_i^{\alpha} \psi + \tilde{\partial}_i^{i_1} \Omega_i^{\alpha} (a^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi) \right) \]
\[ = CE_j(\tau_1) \]
\[ + C\delta \int_{[\tau_2]} \sum_{1 \leq i_1 + i_2 + i_3 \leq j} \sum_{\alpha} \left( \partial_i^{i_1} \partial_{i_2}^{i_2} \Omega_i^{\alpha} \psi \right)^2 d\hat{g}_{(i=\tau_1)} \]
\[ - \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \left( \int_{H^+_{\beta} \cap D_\delta(\tau_1, \tau_2)} + \int_{H^+_{\delta} \cap D_\delta(\tau_1, \tau_2)} \right) \left( \mathcal{T}(g(\nabla \psi))(\partial_i, n)[\partial_i^{i_1} \Omega_i^{\alpha} \psi] \right) \]
\[ + \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \left( \int_{H^+_{\beta} \cap D_\delta(\tau_1, \tau_2)} + \int_{H^+_{\delta} \cap D_\delta(\tau_1, \tau_2)} \right) \left( \partial_i^{i_1} \Omega_i^{\alpha} \psi \right)^2 \frac{1}{\sqrt{1 - \mu}} d\hat{g}_{\mathcal{H}_\delta} \]
\[ + \sum_{1 \leq i_1 + i_2 \leq j} \sum_{\alpha} \int_{[\tau_2]} \frac{1}{2} h(\partial_i, n, \nabla \psi)[\nabla \partial_i^{i_1} \Omega_i^{\alpha} \psi]_g(\nabla \psi) \]
\[ - \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \int_{D_\delta(\tau_1, \tau_2)} (\partial_i^{i_1}) \pi_{\text{non-lin}}(\partial_i^{i_1} \Omega_i^{\alpha} \psi) \]
\[ - \sum_{1 \leq i_1 + i_2 \leq j-1} \sum_{\alpha} \int_{D_\delta(\tau_1, \tau_2)} \partial_i \partial_i^{i_1} \Omega_i^{\alpha} \psi \left( \sum_{j_1 + j_2 + j_4 = i_1 + i_2 - 1} \sum_{j_5 + j_6 = 1} \tilde{\partial}_i^{i_1} \Omega_i^{\alpha} [\bar{g}(\nabla \psi), \right. \]
\[ \tilde{\partial}_i^{i_1} \Omega_i^{\alpha} \partial_i^{j_3} \Omega_i^{\alpha} \psi + \tilde{\partial}_i^{i_1} \Omega_i^{\alpha} (a^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi) \right) \]
\[ + S_{ab}(h) \partial_c \partial_i^{j_5} \Omega_i^{\alpha} \left( \tilde{\partial}_i^{j_3} \Omega_i^{\alpha} \psi \right) \]
\[ + S_{ab}(h) \partial_c \partial_i^{j_5} \Omega_i^{\alpha} \left( \tilde{\partial}_i^{j_3} \Omega_i^{\alpha} \psi \right) + \tilde{\partial}_i^{j_5} \Omega_i^{\alpha} \left( a^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi \right) \]

(A.52)

for all \( 1 \leq j \leq k + 1 \), for some constants \( c(M, \Lambda), C(k, M, \Lambda, A_{k+1}, B_{k+1}) > 0 \), where for \( S_{ab} \) see (5.16). Note that to get the energies \( E_{j,q}, E_j \) on the left hand side and right hand
side respectively, we used the bootstrap assumption \((A.38)\) and Lemma \(A.3\), for \(\epsilon(\delta, C_b) > 0\) sufficiently small.

We now estimate the hypersurface terms on \(H^+_{\delta}, H^+_{\delta}^c\) on the right hand side of \((A.52)\) (in view of Lemma \(A.5)\) and the two last (non-linear) terms of \((A.52)\). Specifically, in view of the bootstrap \((A.38)\), and the definitions of the smooth tensors \(a, h\), see Sections 2.7 and 3.1, respectively, there exists a \(\delta > 0\) sufficiently small, and an \(\epsilon(\delta, C_b) > 0\) sufficiently small, such that for the two non-linear terms we use the Lemma \(A.3\) and the coarea formula \((2.4)\), to obtain

\[
E_{k+1,g}(\tau_2) \\
\leq CE_{k+1}(\tau_1) + C\delta E_{k+1}(\tau_2) \\
+ C\sqrt{\delta} \left( \int \mathcal{H}^+_{\delta} \cap D_\delta(\tau_1, \tau_2) + \int \mathcal{H}^+_{\delta}^c \cap D_\delta(\tau_1, \tau_2) \right) \sum_{1 \leq i \leq k} \mathbb{T}(N, n)[N^i\psi] d\mathcal{H}_\delta \\
+ \sqrt{\epsilon(\delta)} C \int_{\tau_1}^{\tau_2} d\tau E_j(\tau),
\]

\[(A.53)\]

for \(k \geq 7\), where \(C = C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0\). We appropriately distributed derivatives and used a Sobolev estimate, see Lemma 2.2, in view of the fact that \(k = 7\) is the smallest integer such that

\[\lfloor \frac{k + 1}{2} \rfloor + 2 \leq k - 1. \quad (A.54)\]

**The \(N\) multiplier estimate.**

We apply the energy identity \((A.41)\) with multiplier \(N\) and commutators \(N^l\), in view of \((A.42)\).

We obtain

\[
\left( \int_{[t=\tau_2]} D_\delta(\tau_1, \tau_2) + \int \mathcal{H}^+_{\delta} \cap D_\delta(\tau_1, \tau_2) + \int \mathcal{H}^+_{\delta}^c \cap D_\delta(\tau_1, \tau_2) \right) \mathbb{T}(g(\nabla \psi))(N, n)[N^i\psi] \\
+ \int \int_{D_\delta(\tau_1, \tau_2)} \frac{1}{2} (N) \pi_{\mu\nu}(\hat{g}) \mathbb{T}^{\mu\nu}(\hat{g})[N^i\psi] \\
= \int_{[t=\tau_1]} D_\delta(\tau_1, \tau_2) \mathbb{T}(g(\nabla \psi))(N, n)[N^i\psi] \\
- \int \int_{D_\delta(\tau_1, \tau_2)} (N) \pi_{\text{non-lin}}(N^i\psi) \\
- \int \int_{D_\delta(\tau_1, \tau_2)} N^{i+1}\psi \left( \sum_{l=0}^{i-1} N^l \left[ \square_g(\nabla \psi), N \right] N^{i-1-l}\psi + N^i \left( a^\alpha\beta \partial_\alpha \psi \partial_\beta \psi \right) \right)
\]

\[(A.55)\]

for all \(1 \leq i \leq k\).
Therefore, for a sufficiently small $\delta$ we obtain

\[
\left( \int_{\{i=\tau\}} \cap D_3(\tau_1, \tau_2) + \int \mathcal{H}_1^+ \cap D_3(\tau_1, \tau_2) + \int \mathcal{H}_1^+ \cap D_3(\tau_1, \tau_2) \right) \mathbb{T}(g(\nabla \psi))(N, n)\lfloor N^i \psi \rfloor \\
+ \int \int \int_{D_3(\tau_1, \tau_2)} \left( \{r_+ - q \leq r \leq r_+ + q\} \cup \{r_+ - q \leq r_+ + q\} \right) \frac{1}{2} (N) \mu \nu (\hat{g}) \mathbb{T}^{\mu \nu} (\hat{g})\lfloor N^i \psi \rfloor \\
= \int \int_{\{i=\tau\}} \cap D_3(\tau_1, \tau_2) \mathbb{T}(g(\nabla \psi))(N, n)\lfloor N^i \psi \rfloor \\
- \int \int \int_{D_3(\tau_1, \tau_2)} \left( \{r_+ - q \leq r \leq r_+ + q\} \cup \{r_+ - q \leq r_+ + q\} \right) \frac{1}{2} (N) \mu \nu (\hat{g}) \mathbb{T}^{\mu \nu} (g(\nabla \psi))\lfloor N^i \psi \rfloor \\
- \int \int \int_{D_3(\tau_1, \tau_2)} (N) \pi_{non-lin}(N^i \psi) \\
- \int \int \int_{D_3(\tau_1, \tau_2)} N^{i+1} \psi \left( \sum_{l=0}^{i-1} N^l \left[ \Box g(\nabla \psi), N \right] N^{i-l} \psi + N^i \left( a^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi \right) \right),
\]

for all $1 \leq i \leq k$, where for $q$ see Lemma A.1. Now, we estimate the second term on the right hand side of (A.56), and obtain

\[
\left( \int_{\{i=\tau\}} \cap D_3(\tau_1, \tau_2) + \int \mathcal{H}_1^+ \cap D_3(\tau_1, \tau_2) + \int \mathcal{H}_1^+ \cap D_3(\tau_1, \tau_2) \right) \mathbb{T}(g(\nabla \psi))(N, n)\lfloor N^i \psi \rfloor \\
+ \int \int \int_{D_3(\tau_1, \tau_2)} \left( \{r_+ - q \leq r \leq r_+ + q\} \cup \{r_+ - q \leq r_+ + q\} \right) \frac{1}{2} (N) \mu \nu (\hat{g}) \mathbb{T}^{\mu \nu} (\hat{g})\lfloor N^i \psi \rfloor \\
\leq \int \int_{\{i=\tau\}} \cap D_3(\tau_1, \tau_2) \mathbb{T}(g(\nabla \psi))(N, n)\lfloor N^i \psi \rfloor \\
+ C_{\pi_{\text{max}}} \sup_{\tau \in [\tau_1, \tau_2]} \int_{\{i=\tau\} \cap \{r_+ + q \leq r \leq r_+ - q\}} \sum_{1 \leq i_1 + i_2 + i_3 = i + 1} \sum_{\alpha} \left( \hat{a}^{i_1} \partial_i \hat{a}^{i_3} \Omega_i^\alpha \psi \right)^2 \\
- \int \int \int_{D_3(\tau_1, \tau_2)} (N) \pi_{non-lin}(N^i \psi) \\
- \int \int \int_{D_3(\tau_1, \tau_2)} N^{i+1} \psi \left( \sum_{l=0}^{i-1} N^l \left[ \Box g(\nabla \psi), N \right] N^{i-l} \psi + N^i \left( a^{\alpha \beta} \partial_\alpha \psi \partial_\beta \psi \right) \right),
\]

for all $1 \leq i \leq k$, where the constant $C > 0$ depends only on the black hole parameters.

Now, in view of Lemma A.3 and the elliptic estimates of Lemma A.4, there exist $\delta, \epsilon (\delta, C_b)$, sufficiently small, such that the estimate (A.57) implies
\[ cE_j(\tau_2) + c \int_{\tau_1}^{\tau_2} d\tau E_j(\tau) \]

\[ + c \left( \int_{\mathcal{H}_\delta^{+} \cap D_\delta(\tau_1, \tau_2)} + \int_{\mathcal{H}_\delta^{-} \cap D_\delta(\tau_1, \tau_2)} \right) \sum_{1 \leq i \leq j-1} \mathbb{T}(N, n)[N^i \psi] d\mathbb{g}_{\mathcal{H}_\delta} \]

\[ \leq CE_j(\tau_1) + C\tau_{\text{max}} \sup_{\tau \in [\tau_1, \tau_2]} E_{j,q}(\tau) \]

\[ - \int \int_{D_\delta(\tau_1, \tau_2)} \sum_{1 \leq i \leq j-1} \pounds^{(N)}_{\text{non-lin}}(N^i \psi) \]

\[ - \int \int_{D_\delta(\tau_1, \tau_2)} \sum_{1 \leq i \leq j-1} N^{i+1} \psi \left( \sum_{l=0}^{i-1} N^l \left[ \Box_{g(\psi)} N \right] N^{i-l-1} \psi + N^i \left( a^{\alpha \beta} \partial_{\alpha} \psi \partial_{\beta} \psi \right) \right) \]

\[ = CE_j(\tau_1) + C\tau_{\text{max}} \sup_{\tau \in [\tau_1, \tau_2]} E_{j,q}(\tau) \]

\[ - \int \int_{D_\delta(\tau_1, \tau_2)} \sum_{1 \leq i \leq j-1} \pounds^{(N)}_{\text{non-lin}}(N^i \psi) \]

\[ - \int \int_{D_\delta(\tau_1, \tau_2)} \sum_{1 \leq i \leq j-1} N^{i+1} \psi \left( \sum_{l=0}^{i-1} N^l \left[ \Box_{g} N \right] N^{i-l-1} \psi \right) \]

\[ - \int_{D_\delta(\tau_1, \tau_2)} \int_{1 \leq i \leq j-1} N^{i+1} \psi \left( \sum_{l=0}^{i-1} N^l \left[ H^{ab} \partial_{a} \partial_{b} + S^{c}_{ab} \partial_{c}, N \right] N^{i-l-1} \psi + N^i \left( a^{\alpha \beta} \partial_{\alpha} \psi \partial_{\beta} \psi \right) \right) \]

\[ \text{(A.58)} \]

for all \( 1 \leq j \leq k + 1 \), for some constants \( c(k, M, \Lambda, A_{k+1}, B_{k+1}) \), \( C(k, M, \Lambda, A_{k+1}, B_{k+1}) \), where for \( S^{c}_{ab} \) see (5.16).

Similarly to (A.52), we estimate the two last (non-linear) terms of (A.58). Note that we estimate the third to last bulk term of (A.58), in view on the linear \( N \)-commutation Lemma A.2. Specifically, in view of the bootstrap (A.38), and the definitions of the smooth tensors \( a, h \), see Sections 2.7 and 3.1, respectively, there exists a \( \delta > 0 \) sufficiently small, and an \( \epsilon(\delta, C_b) > 0 \) sufficiently small, such that for the two non-linear terms we use the Lemma A.3 and the coarea formula (2.4), to obtain

\[ E_{k+1}(\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{k+1}(\tau) + \left( \int_{\mathcal{H}_\delta^{+} \cap D_\delta(\tau_1, \tau_2)} + \int_{\mathcal{H}_\delta^{-} \cap D_\delta(\tau_1, \tau_2)} \right) \sum_{1 \leq i \leq k} \mathbb{T}(N, n)[N^i \psi] d\mathbb{g}_{\mathcal{H}_\delta} \]

\[ \leq CE_{k+1}(\tau_1) + C\tau_{\text{max}} \sup_{\tau \in [\tau_1, \tau_2]} E_{j,q}(\tau) \]

\[ \text{(A.59)} \]

for \( k \geq 7 \), where \( 0 < C = C(k, M, \Lambda, A_{k+1}, B_{k+1}) \). We appropriately distributed derivatives and used a Sobolev injection, see Lemma 2.2, in view of that \( k = 7 \) is the smallest integer such that

\[ \left[ \frac{k + 1}{2} \right] + 2 \leq k - 1. \]

We obtain

\[ \text{(A.60)} \]
The combination of the $\partial$ and $N$ multiplier estimates.

Now, we combine the $\partial$-estimate, namely (A.53), and the $N$-estimate, namely (A.59). We obtain

$$E_{k+1}(\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{k+1}(\tau) + \left( \iint_{\mathcal{H}_{\delta}^+ \cap D_{\delta}(\tau_1, \tau_2)} + \iint_{\mathcal{H}_{\delta}^+ \cap D_{\delta}(\tau_1, \tau_2)} \right) \sum_{1 \leq i \leq N} T(N, n)[N^i\psi] d\bar{g}_{\mathcal{H}_{\delta}} \leq CE_{k+1}(\tau_1) + C\delta E_{k+1}(\tau_2) + C\sqrt{\delta} \left( \iint_{\mathcal{H}_{\delta}^+ \cap D_{\delta}(\tau_1, \tau_2)} + \iint_{\mathcal{H}_{\delta}^+ \cap D_{\delta}(\tau_1, \tau_2)} \right) \sum_{1 \leq i \leq k} T(N, n)[N^i\psi] d\bar{g}_{\mathcal{H}_{\delta}} + \sqrt{\epsilon(\delta)} C \int_{\tau_1}^{\tau_2} d\tau E_{k+1}(\tau),$$

(A.61)

for $k \geq 7$.

The integral inequality. Therefore, for $\delta(\tau_{\text{max}}), \epsilon(\tau_{\text{max}}, \delta) > 0$ sufficiently small, then from inequality (A.61) we obtain

$$E_{k+1}(\tau_2) + \int_{\tau_1}^{\tau_2} d\tau E_{k+1}(\tau) \leq CE_{k+1}(\tau_1) + C(\tau_2 - \tau_1) E_{k+1}(\tau_1),$$

(A.62)

for $k \geq 7$, and for a constant $C(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 0$. Note that (A.62) is the quasilinear analogue of the integral inequality (A.36).

Finishing the proof, improving the bootstrap (A.38).

Note that the estimate (A.62) holds for any $\tau'_1 \geq \tau_1$ in the place of $\tau_1$.

Therefore, arguing as in the linear case, see the integral inequality (A.36), we obtain from (A.62) that there exists a constant

$$C_{wp}(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 1$$

(A.63)

independent of $\tau_{\text{max}}$, such that

$$E_{k+1}[\psi](\tau') \leq C_{wp}E_{k+1}[\psi](\tau_1),$$

(A.64)

for all $\tau' \in [\tau_1, \tau_1 + \tau_{\text{max}}]$. Moreover, by using (A.64) and the smallness $E_{k+1}[\psi](\tau_1) \leq \epsilon$, we also prove that

$$E_{k+2}[\psi](\tau') \leq C_{wp}E_{k+2}[\psi](\tau_1),$$

(A.65)

by repeating the arguments of the above proof, and after redefining $C_{wp}(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) > 1$ appropriately, provided that $E_{k+2}[\psi](\tau_1) < \infty$.

Finally, for a sufficiently large $C_{b}(k, M, \Lambda, A_{[k+1]}, B_{[k+1]}) \gg C_{wp}$ and a sufficiently small $\epsilon > 0$, we improve the bootstrap assumption (A.38) by Sobolev inequalities, see Lemma 2.2, on the left hand side of inequality (A.64).

We conclude the energy estimates (3.9) and (3.10) and therefore the Proposition. \[\square\]

Note also the following Remark

Remark A.1. A posteriori, one can remove the dependence of $\delta$ on the $\tau_{\text{step}}$ parameter. To do so, note that we dropped the terms on the hypersurfaces $\mathcal{H}_{\delta}^+, \mathcal{H}_{\delta}^-$ from (A.61). By using the
$N$ redshift vector field in the spacetime regions

$$\{r_+ - \tilde{\delta} \leq r \leq r_+ - \delta\}, \quad \{\tilde{r}_+ + \delta \leq r \leq \tilde{r}_+ + \tilde{\delta}\}$$  \hspace{1cm} (A.66)

for $\delta > 0$ as in Theorem 1, and for a sufficiently small $\tilde{\delta}$ independent of $\tau_{\text{step}}$, we can absorb the contributions at $\mathcal{H}_\delta^+, \tilde{\mathcal{H}}_\delta^+$ by (A.61), and conclude the Cauchy stability result of Proposition 3.1 for a $\tilde{\delta} > 0$ in the place of $\delta$. 