Optimal quantum data classification

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Abstract

The problem of simultaneously distinguishing a pure quantum state, generated by some quantum mechanical process, from its constituent observable states optimally is introduced and solved using a non-cooperative game model and the affiliated solution concept of Nash equilibrium. The notion of Nash equilibrium quantum states is introduced and used to classify quantum data optimally.

1 Introduction

Recall that the mathematical model for quantum objects with finitely many observable states is a (projective) complex, finite $d$-dimensional Hilbert space $\mathbf{H}_d$. The elements of an orthogonal basis $B$ of the quantum system $\mathbf{H}_d$ represent the physically observable states of the quantum object. General linear combinations of the elements of $B$, known as quantum superpositions, while equally meaningful physically, cannot be observed directly; instead quantum superpositions can be observed to be in one of their constituent observable states with a certain probability via the physical process of (quantum) measurement. To be more precise, measurement is orthogonal projection of a quantum superposition $q$ onto its constituent observable states in $B$, with the length of each projection representing the probability with which $q$ is observed to be in the corresponding observable state. The following constrained optimization question arises naturally here: is there an optimal measurement of $q$?

The constraints in this problem arise from the orthogonality of the observable states, that is, increasing or decreasing the probability of observing $q$ in any one observable state necessarily effects the probability of observing $q$ in at least another observable state. In particular, decreasing the probability of observing $q$ in a given observable state $b$ so as to distinguish $q$ from $b$ necessarily makes $q$ less distinguishable from some other observable states. Hence, the question of optimal measurement for $q$ can be cast as a problem of optimal distinguishably of $q$ from its constituents observable states $b_i \in B$, given the constraints of the orthogonality of the $b_i$. This is the problem of optimal simultaneous distinguishability of $q$ from the $b_i$. 

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To elaborate further, note that the probability of observing $q$ in any one of $b_i$ can be expressed as a function of the inner-product $\langle , \rangle$ of $H_d$ and therefore as a function of the angle $\theta_{(q, b_i)}$ between $q$ and $b$. Further note that this functional relationship has a diametrically opposite nature so that whenever $\theta_{(q, b_i)}$ is small, the probability that $q$ will be observed to be in the state $b_i$, $|\langle q, b_i \rangle|^2$, is large, and vice versa. Now, the problem of optimal simultaneous distinguishability of $q$ from the $b_i$ is reduced to simultaneous minimization of the quantities $\theta_{(q, b_i)}$ for all $i$, given the orthogonality constraints $\langle b_i, b_j \rangle = 0$ for all $i \neq j$.

This problem begins to take on a game-theoretic flavor when one considers the basis elements $b_i$ to be the payoffs for some notion of “players,” and that these players have non-identical preferences over the various $b_i$. The constraints $\langle b_i, b_j \rangle = 0$ for all $i \neq j$ now simply mean that if two players entertain non-identical preferences over $b_i$ and $b_j$, then these players will compete over distinguishing $q$ from these two basis states respectively. This notion of competition is one contribution of the game-theoretic flavoring just added, together with the exact mechanism for this competition which is a non-cooperative game. Yet another contribution that comes from game theory here is the notion of strategic choices made by the players in the game to realize an outcome that optimizes the distinguishability of $q$ from $b_i$ and $b_j$. Summarizing, game theory offers the correct formalism for controlling the quantum physical problem at hand towards an optimal solution. Details appear in the following sections.

2 Gaming the quantum system

An $n$ player, non-cooperative quantum game [2] is a function $G$ with a finite-dimensional complex (projective) Hilbert space $H_d$ of quantum superpositions as its co-domain, combined with the additional feature of “players” who entertain non-identical preferences over the elements of the co-domain. In symbols

$$G : \Pi_i D_i \rightarrow H_d, \quad i = 1, 2, \ldots, n.$$  (1)

The factor $D_i$ in the domain of $G$ is the set of strategies of player $i$, and a play of the game $G$ is a tuple of strategies in $\Pi_i D_i$ producing a payoff to each player in the form of an outcome, that is, an element of $H_d$.

A Nash equilibrium is a play of $G$ in which every player employs a strategy that is a best reply, with respects to his preferences over the outcomes, to the strategic choice of every other player. In other words, unilateral deviation from a Nash equilibrium by a player in the form of a different choice of strategy will produce an outcome which is less than or equal to in preference to that player than before. The quantum state that is the image of a Nash equilibrium play of $G$ is called a Nash equilibrium quantum state.

The problem of optimal simultaneous distinguishability of $q$ from the $b_i$ now has a game-theoretic solution; that is, $q$ is optimally distinguishable from all the $b_i$ if $q$ is a Nash equilibrium quantum state. Note that gaming the quantum system $H_d$ gives rise to the capability of controlling this problem towards an optimal solution in a very general sense. That is, the quantum game $G$ and the strategy sets $D_i$ can be very general mechanisms and
therefore solutions to the problem can be constructed in several context of both mathematical and physical interests. However, the exact nature of the control mechanism (quantum game) determines the exact nature of the optimal solution to the problem. This game model dependency of the solution is discussed in the following section with the aid of a particular game model.

2.1 Nash equilibrium quantum states

Let

\[ B = \{b_1, \ldots, b_d\} \]

be an orthonormal basis (set of observable states) in \( \mathbf{H}_d \) and let

\[ \text{Pref}_k : b_{j_1}^k \succ b_{j_2}^k \succ \cdots \succ b_{j_d}^k, \quad k = 1, 2, \ldots, n \]

be the preference profile of Player \( k \) over the elements of \( B \), with the symbol \( \succ \) representing the notion of “prefers over”. Hence, player \( k \) prefers the element \( b_{j_1}^k \) over the element \( b_{j_2}^k \) of \( B \), and so forth in ascending order of the lowest subscript, until all the elements of \( B \) are exhausted. The equation

\[ \cos \theta_{(p,q)} = |\langle p, q \rangle|^2 \]

for arbitrary elements \( p \) and \( q \) of \( \mathbf{H}_d \) allows defining the players’ preferences over arbitrary elements of \( \mathbf{H}_d \) as follows. Player \( k \) will prefer an arbitrary element \( p \) of \( \mathbf{H}_d \), that is, a quantum superposition of the elements of \( B \), over another \( q \), whenever \( p \) is “closer” to all the elements of \( B \) with respect to \( \text{Pref}_k \) than \( q \) is. The notion of closeness utilized in the preceding sentence is the one described by the quantity \( ||p - q|| \) and which is an increasing function of \( \theta_{(p,q)} \). The notion of players’ preference over arbitrary quantum states in \( \mathbf{H}_d \) is expressed symbolically as

\[ p \succ q \quad \text{whenever} \quad \theta_{(p,b_{j_1}^k)} < \theta_{(q,b_{j_1}^k)}, \forall j_i. \]

and can be used to characterize the notion of Nash equilibrium in the non-cooperative quantum game \( G \).

Suppose \( E^* = (e_1^*, e_2^*, \ldots, e_k^*, \ldots, e_n^*) \) is a Nash equilibrium play in the non-cooperative quantum game \( G \), producing the Nash equilibrium quantum state \( G(E^*) \in \mathbf{H}_d \). Then a unilateral deviation from \( E^* \) by Player \( k \) to some other play \( E = (e_1^*, e_2^*, \ldots, e_k, \ldots, e_n^*) \) will produce a quantum state \( G(E) \) that will be less than or equal to in preference to Player \( k \), with respect to \( \text{Pref}_k \), than before. Stated explicitly in terms of the quantity \( \theta_{(\cdot)} \), a Nash equilibrium play \( E^* \) of the quantum game \( G \) will satisfy the following inequalities:

\[ \theta_{(G(E),b_{j_1}^k)} \geq \theta_{(G(E^*),b_{j_1}^k)}, \forall j_i \text{ and } \forall k. \]

The existence question of \( E^* \) in \( G \) can addressed by referring to the theory of Hilbert space \([6]\) if \( G \) is a unitary map into \( \mathbf{H}_d \), for in this case its image, \( \text{Im}(G) \), is a sub-Hilbert
space of $H_d$. Therefore, for every element $h \in H_d$, there exist an element $k \in \text{Im}(G)$ such that for all other $k' \in \text{Im}(G)$

$$\theta_{(k', h)} \geq \theta_{(k, h)}.$$  

Setting

$$k' = G(E); \quad k = G(E^*); \quad h = b^k_{ji}$$

in (6) and adding the universal quantifiers $\forall j_i$ and $\forall k$ recovers the Nash equilibrium condition in (5), thus establishing the necessary and sufficient conditions for the existence of a Nash equilibrium in a unitary non-cooperative quantum game $G$. In the more general setting where $G$ is not a unitary operation, the theory of Hilbert space requires that $\text{Im}(G)$ at least be a complete convex subset of $H_d$ in order to meet the existence conditions for Nash equilibrium quantum states characterized above.

Further generalization can see the finite-dimensional Hilbert space $H_d$ replaced with a Hilbert space that entertains a more general notion of observables, and appropriate mathematical conditions imposed to define the notion of Nash equilibrium in quantum states and explore their existence.

3 Quantum data classification

Quantum data are elements of $H_d$. Since observable states in $H_d$ are the states that are ultimately “real” in any practically meaningful sense, all quantum data can be characterized with respect to these states. More precisely, given a quantum superposition $q$, which of the observables $b_i$ is $q$ most like? Or, upon measurement, which observable state will $q$ be in with the highest probability? Further yet, which of the $b_i$ is $q$ least distinguishable from? Viewing quantum data as generated by functions mapping into $H_d$, the game-theoretic model developed here provides an optimal answer to these equivalent question. This model can also be used to design quantum mechanisms that produce Nash equilibrium quantum states with respect to pre-defined preferences over the observable states. This has been done for two qubit quantum computations under a strictly competitive game model in [3] and under the model of the popular game Prisoners’ Dilemma in [4].

Other game models, with different notions of preferences over the elements of $H_d$ will produce different solutions to the problem of optimal quantum data classification.

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References

[1] K. Binmore, Fun and games, D.C. Heath, 1992.
of Computing, 1996.

[2] F.S.Khan, S.J.D. Phoenix, *Gaming the quantum*, Quantum Information & Computation, Volume 13, Numbers 3&4, 2013.

[3] F.S.Khan, S.J.D. Phoenix, *Mini-maximizing two qubit quantum computations*, Quantum Information Processing, Volume 12, Numbers 12, 2013.

[4] F.S.Khan, *Dominant strategies in two qubit quantum computations*, Quantum Information Processing, 10.1007/s11128-015-0945-9, 2015.

[5] D. Meyer, *Quantum games and quantum algorithms*, In Quantum Computation and Quantum Information Science, Contemporary Mathematics Series. American Mathematical Society, 2002.

[6] S. Roman, *Advanced linear algebra*, Graduate Text in Mathematics, Springer, 2008.