THE LUKACS-OLKIN-RUBIN THEOREM ON SYMMETRIC CONES WITHOUT INVARIANCE OF THE “QUOTIENT”

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ABSTRACT. We prove the Lukacs-Olkin-Rubin theorem without invariance of the distribution of the “quotient”, which was the key assumption in the original proof of [Olkin–Rubin, Ann. Math. Stat. 33 (1962), 1272–1280]. Instead we assume existence of strictly positive continuous densities of respective random variables. The main novelty is consideration of the (cone variate) “quotient” for any division algorithm satisfying some natural conditions. For that purpose, the new proof of the Olkin–Baker functional equation on symmetric cones is given.

1. INTRODUCTION

The Lukacs [1955] theorem is one of the most celebrated characterizations of probability distributions. It states that if $X$ and $Y$ are independent, positive, non-degenerate random variables such that their sum and quotient are also independent then $X$ and $Y$ have gamma distributions with the same scale parameter.

This theorem has many generalizations. The most important in the multivariate setting were given by Olkin and Rubin [1962] and Casalis and Letac [1996], where the authors extended characterization to matrix and symmetric cones variate distributions, respectively. There is no unique way of defining the quotient of elements of the cone of positive definite symmetric matrices $\Omega_+$ and in these papers the authors have considered very general form $U = g(X + Y) \cdot X \cdot g^T(X + Y)$, where $g$ is the so-called division algorithm, that is, $g(a) \cdot a \cdot g^T(a) = I$ for any $a \in \Omega_+$, where $I$ is the identity matrix. The drawback of their extension was that the additional strong assumptions of invariance of the distribution of $U$ under a group of automorphisms was imposed. This result was generalized to homogeneous cones in Boutouria et al. [2011].

There were successful attempts in replacing the invariance of the “quotient” assumption with the existence of regular densities of random variables $X$ and $Y$. Bobecka and Wesolowski [2002] assuming existence of strictly positive, twice differentiable densities proved a characterization of Wishart distribution on the cone $\Omega_+$ for division algorithm $g_1(a) = a^{-1/2}$, where $a^{1/2}$ denotes the unique positive definite symmetric root of $a \in \Omega_+$. This result was generalized to all non-octonion symmetric cones of rank greater than 2 and to the Lorentz cone for strictly positive and continuous densities by Kołodziejek [2010, 2013].

Exploiting the same approach, with the same technical assumptions on densities as in Bobecka and Wesolowski [2002], it was proven by Hassairi et al. [2008] that the independence of $X + Y$ and the quotient defined through the Cholesky decomposition, i.e. $g_2(a) = T_a^{-1}$, where $T_a$ is an upper triangular matrix such that $a = T_a \cdot T_a^T \in \Omega_+$, characterizes a wider family of distributions called Riesz (or sometimes called Riesz-Wishart). This fact shows that the invariance property assumed in Olkin and Rubin [1962] and Casalis and Letac [1996] is not of technical nature only. Analogous results for homogeneous cones were obtained by Boutouria [2005, 2009].

In this paper we deal with the density version of Lukacs-Olkin-Rubin theorem on symmetric cones for division algorithm satisfying some natural properties. We assume that the densities of $X$ and $Y$ are strictly positive and continuous. The most important novelty is consideration of quotient $U$ for an arbitrary division algorithm $g$ as in the original paper of Olkin and Rubin [1962], additionally satisfying
some natural conditions. In the known cases \((g = g_1 \text{ and } g = g_2)\) this improves the results obtained in Bobecka and Wesołowski [2002], Hassairi et al. [2008], Kołodziejek [2013a]. In general case, the densities of \(X\) and \(Y\) are given in terms of, so called, \(w\)-multiplicative Cauchy functions, that is functions satisfying
\[
f(x)f(w(I) \cdot y \cdot w^T(I)) = f(w(x) \cdot y \cdot w^T(x)), \quad (x, y) \in \Omega_+^2,
\]
where \(g = w^{-1}\) is a division algorithm. Consistently, we will call \(w\) a multiplication algorithm. Such functions were recently considered in Kołodziejek [2013b].

Unfortunately we can’t answer the question whether there exists division (or equivalently multiplication) algorithm resulting in characterizing other distribution than Riesz or Wishart. Moreover, the simultaneous removal of the assumptions of the invariance of the “quotient” and the existence of densities remains a challenge.

This paper is organized as follows. We start in the next section with basic definitions and theorems regarding analysis on symmetric cones. The statement and proof of the main result are given in Section 4. Section 3 is devoted to consideration of \(w\)-logarithmic Cauchy functions and the Olkin–Baker functional equation. In that section we offer much shorter, simpler and covering more general cones proof of the Olkin–Baker functional equation than given in Bobecka and Wesołowski [2002], Hassairi et al. [2008], Kołodziejek [2013a].

2. Preliminaries

In this section we give a short introduction to the theory of symmetric cones. For further details we refer to Faraut and Korányi [1994].

A Euclidean Jordan algebra is a Euclidean space \(E\) (endowed with scalar product denoted \(\langle x, y \rangle\)) equipped with a bilinear mapping (product)
\[
E \times E \ni (x, y) \mapsto xy \in E
\]
and a neutral element \(e\) in \(E\) such that for all \(x, y, z\) in \(E\):
\begin{itemize}
  \item [(i)] \(xy = yx\),
  \item [(ii)] \(x(x^2y) = x^2(xy)\),
  \item [(iii)] \(xe = x\),
  \item [(iv)] \((x, yz) = \langle xy, z \rangle\).
\end{itemize}

For \(x \in E\) let \(L(x): E \rightarrow E\) be linear map defined by
\[
L(x)y = xy,
\]
and define
\[
P(x) = 2L^2(x) - L(x^2).
\]
The map \(P: E \rightarrow \text{End}(E)\) is called the quadratic representation of \(E\).

An element \(x\) is said to be invertible if there exists an element \(y\) in \(E\) such that \(L(x)y = e\). Then \(y\) is called the inverse of \(x\) and is denoted by \(y = x^{-1}\). Note that the inverse of \(x\) is unique. It can be shown that \(x\) is invertible if and only if \(P(x)\) is invertible and in this case \((P(x))^{-1} = P(x^{-1})\).

Euclidean Jordan algebra \(E\) is said to be simple if it is not a Cartesian product of two Euclidean Jordan algebras of positive dimensions. Up to linear isomorphism there are only five kinds of Euclidean simple Jordan algebras. Let \(K\) denote either the real numbers \(\mathbb{R}\), the complex ones \(\mathbb{C}\), quaternions \(\mathbb{H}\) or the octonions \(\mathbb{O}\), and write \(S_r(K)\) for the space of \(r \times r\) Hermitian matrices with entries valued in \(K\), endowed with the Euclidean structure \(\langle x, y \rangle = \text{Trace}(x \cdot \bar{y})\) and with the Jordan product
\[
xy = \frac{1}{2}(x \cdot y + y \cdot x),
\]
where $\mathbf{x} \cdot \mathbf{y}$ denotes the ordinary product of matrices and $\overline{\mathbf{y}}$ is the conjugate of $\mathbf{y}$. Then $S_r(\mathbb{R})$, $r \geq 1$, $S_r(\mathbb{C})$, $r \geq 2$, $S_r(\mathbb{H})$, $r \geq 2$, and the exceptional $S_3(\mathbb{O})$ are the first four kinds of Euclidean simple Jordan algebras. Note that in this case

\[(2) \quad \mathbb{F}(\mathbf{y})\mathbf{x} = \mathbf{y} \cdot \mathbf{x} \cdot \mathbf{y}.
\]

The fifth kind is the Euclidean space $\mathbb{R}^{n+1}$, $n \geq 2$, with Jordan product

\[(3) \quad (x_0, x_1, \ldots, x_n) (y_0, y_1, \ldots, y_n) = \left( \sum_{i=0}^{n} x_i y_i, x_0 y_1 + y_0 x_1, \ldots, x_0 y_n + y_0 x_n \right).
\]

To each Euclidean simple Jordan algebra one can attach the set of Jordan squares

\[\Omega = \{ \mathbf{x}^2 : \mathbf{x} \in \mathbb{E} \}.
\]

The interior $\Omega$ is a symmetric cone. Moreover $\Omega$ is irreducible, i.e., it is not the Cartesian product of two convex cones. One can prove that an open convex cone is symmetric and irreducible if and only if it is the cone $\Omega$ of some Euclidean simple Jordan algebra. Each simple Jordan algebra corresponds to a symmetric cone, hence there exist up to linear isomorphism also only five kinds of symmetric cones. The cone corresponding to the Euclidean Jordan algebra $\mathbb{R}^{n+1}$ equipped with Jordan product (3) is called the Lorentz cone.

We denote by $G(\mathbb{E})$ the subgroup of the linear group $GL(\mathbb{E})$ of linear automorphisms which preserves $\Omega$, and we denote by $G$ the connected component of $G(\mathbb{E})$ containing the identity. Recall that if $\mathbb{E} = S_r(\mathbb{R})$ and $GL(r, \mathbb{R})$ is the group of invertible $r \times r$ matrices, elements of $G(\mathbb{E})$ are the maps $g : \mathbb{E} \to \mathbb{E}$ such that there exists $a \in GL(r, \mathbb{R})$ with

\[g(\mathbf{x}) = a \cdot \mathbf{x} \cdot a^T.
\]

We define $K = G \cap O(\mathbb{E})$, where $O(\mathbb{E})$ is the orthogonal group of $\mathbb{E}$. It can be shown that

\[K = \{ k \in G : ke = e \}.
\]

A multiplication algorithm is a map $\Omega \to G$; $\mathbf{x} \mapsto w(\mathbf{x})$ such that $w(\mathbf{x})e = \mathbf{x}$ for all $\mathbf{x} \in \Omega$. This concept is consistent with, so called, division algorithm $g$, which was introduced by Olkin and Rubin [1962] and Casalis and Letac [1996], that is a mapping $\Omega \ni \mathbf{x} \mapsto g(\mathbf{x}) \in G$ such that $g(\mathbf{x})\mathbf{x} = e$ for any $\mathbf{x} \in \Omega$. If $w$ is a multiplication algorithm then $g = w^{-1}$ is a division algorithm and vice versa, if $g$ is a division algorithm then $w = g^{-1}$ is a multiplication algorithm. One of two important examples of multiplication algorithms is the map $w_1(\mathbf{x}) = \mathbb{F}(\mathbf{x}^{1/2})$.

We will now introduce a very useful decomposition in $\mathbb{E}$, called spectral decomposition. An element $\mathbf{c} \in \mathbb{E}$ is said to be an idempotent if $\mathbf{cc} = \mathbf{c} \neq 0$. Idempotents $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\mathbf{ab} = 0$. Idempotent $\mathbf{c}$ is primitive if $\mathbf{c}$ is not a sum of two non-null idempotents. A complete system of primitive orthogonal idempotents is a set $(\mathbf{c}_1, \ldots, \mathbf{c}_r)$ such that

\[\sum_{i=1}^{r} \mathbf{c}_i = \mathbf{e} \quad \text{and} \quad \mathbf{c}_i \mathbf{c}_j = \delta_{ij} \mathbf{c}_i \quad \text{for} \quad 1 \leq i < j \leq r.
\]

The size $r$ of such system is a constant called the rank of $\mathbb{E}$. Any element $\mathbf{x}$ of a Euclidean simple Jordan algebra can be written as $\mathbf{x} = \sum_{i=1}^{r} \lambda_i \mathbf{c}_i$ for some complete system of primitive orthogonal idempotents $(\mathbf{c}_1, \ldots, \mathbf{c}_r)$. The real numbers $\lambda_i$, $i = 1, \ldots, r$ are the eigenvalues of $\mathbf{x}$. One can then define trace and determinant of $\mathbf{x}$ by, respectively, $\operatorname{tr} \mathbf{x} = \sum_{i=1}^{r} \lambda_i$ and $\det \mathbf{x} = \prod_{i=1}^{r} \lambda_i$. An element $\mathbf{x} \in \mathbb{E}$ belongs to $\Omega$ if and only if all its eigenvalues are strictly positive.

The rank $r$ and $\dim \Omega$ of simple symmetric cone are connected through relation

\[\dim \Omega = r + \frac{dr(r-1)}{2},
\]

where $d$ is an integer called the Peirce constant.
If \( c \) is a primitive idempotent of \( E \), the only possible eigenvalues of \( L(c) \) are 0, \( \frac{1}{2} \), and 1. We denote by \( E(c,0), E(c, \frac{1}{2}) \) and \( E(c, 1) \) the corresponding eigenspaces. The decomposition

\[ E = E(c,0) \oplus E(c, \frac{1}{2}) \oplus E(c, 1) \]

is called the Peirce decomposition of \( E \) with respect to \( c \). Note that \( P(c) \) is the orthogonal projection of \( E \) onto \( E(c,1) \).

Fix a complete system of orthogonal idempotents \( \{c_i\}_{i=1}^r \). Then for any \( i, j \in \{1,2,\ldots,r\} \) we write

\[ E_{ij} = E(c_i, 1) = Rc_i, \]

\[ E_{ij} = E \left( c_i, \frac{1}{2} \right) \cap E \left( c_j, \frac{1}{2} \right) \text{ if } i \neq j. \]

It can be proved (see Faraut and Korányi [1994, Theorem IV.2.1]) that

\[ E = \bigoplus_{i<j} E_{ij} \]

and

\[ E_{ij} \cdot E_{ij} \subseteq E_{ii} + E_{ij}, \]

\[ E_{ij} \cdot E_{jk} \subseteq E_{ik}, \text{ if } i \neq k, \]

\[ E_{ij} \cdot E_{kl} = \{0\}, \text{ if } \{i,j\} \cap \{k,l\} = \emptyset. \]

Moreover (Faraut and Korányi [1994, Lemma IV.2.2]), if \( x \in E_{ij}, y \in E_{jk}, i \neq k \), then

\begin{equation}
\begin{aligned}
x^2 &= \frac{1}{2}||x||^2(c_i + c_j), \\
||xy||^2 &= \frac{1}{8}||x||^2||y||^2.
\end{aligned}
\end{equation}

The dimension of \( E_{ij} \) is the Peirce constant \( d \) for any \( i \neq j \). When \( E \) is \( S_r(\mathbb{K}) \), if \( (e_1,\ldots,e_r) \) is an orthonormal basis of \( \mathbb{R}^r \), then \( E_{ii} = \mathbb{R}e_i e_i^T \) and \( E_{ij} = \mathbb{K} (e_i e_j^T + e_j e_i^T) \) for \( i < j \) and \( d \) is equal to \( \dim \mathbb{R} \mathbb{K} \).

For \( 1 \leq k \leq r \) let \( P_k \) be the orthogonal projection onto \( E^{(k)} = E(c_1 + \ldots + c_k, 1) \), \( \det^{(k)} \) the determinant in the subalgebra \( E^{(k)} \), and, for \( x \in \Omega \), \( \Delta_k(x) = \det^{(k)}(P_k(x)) \). Then \( \Delta_k \) is called the principal minor of order \( k \) with respect to the Jordan frame \( \{c_k\}^r_{k=1} \). Note that \( \Delta_r(x) = \det x \). For \( s = (s_1,\ldots,s_r) \in \mathbb{R}^r \) and \( x \in \Omega \), we write

\[ \Delta_s(x) = \Delta_1(x)^{s_1} \cdot \Delta_2(x)^{s_2} \cdots \Delta_r(x)^{s_r}. \]

\( \Delta_s \) is called a generalized power function. If \( x = \sum_{i=1}^r s_i c_i \), then \( \Delta_s(x) = \alpha_1^{s_1} \alpha_2^{s_2} \cdots \alpha_r^{s_r} \).

We will now introduce some basic facts about triangular group. For \( x \) and \( y \) in \( \Omega \), let \( x \Box y \) denote the endomorphism of \( E \) defined by

\[ x \Box y = L(xy) + L(x)L(y) - L(y)L(x). \]

If \( c \) is an idempotent and \( z \in E(c, \frac{1}{2}) \) we define the Frobenius transformation \( \tau_c(z) \) in \( G \) by

\[ \tau_c(z) = \exp(2z \Box c). \]

Since \( 2z \Box c \) is nilpotent of degree 3 (see Faraut and Korányi [1994, Lemma VI.3.1]) we get

\begin{equation}
\begin{aligned}
\tau_c(z) &= I + (2z \Box c) + \frac{1}{2}(2z \Box c)^2.
\end{aligned}
\end{equation}

Given a Jordan frame \( \{c_i\}^r_{i=1} \), the subgroup of \( G \),

\[ T = \left\{ \tau_{c_1}(z^{(1)}) \cdots \tau_{c_{r-1}}(z^{(r-1)}) \bigoplus \left( \sum_{i=1}^r \alpha_i c_i \right) : \alpha_i > 0, z^{(j)} \in \bigoplus_{k=j+1}^r E_{jk} \right\} \]
is called the triangular group corresponding to the Jordan frame \((c_i)_{i=1}^r\). For any \(x\) in \(\Omega\) there exists a unique \(t_x\) in \(T\) such that \(x = t_x e\), that is, there exist (see [Faraut and Korányi, 1994, VII.1.1.])

\[
\text{if only } \Delta_{(8)}
\]

\(c\) is called the triangular group corresponding to the Jordan frame \((c_i)_{i=1}^r\). Absolutely continuous Riesz distribution \(R\) is such that \(\tau(c \cdot x) := \int_0^\infty \Delta_s(tx) \, dx\), \(x \in \Omega\) such that \(\int_0^\infty \Delta_s(tx) \, dx = 1\),

\[
\Gamma\]

Let us define for \(1 \leq i \leq j \leq r\) matrix \(\mu_{ij} = (\gamma_{kl})_{i \leq k, l \leq r}\) such that \(\gamma_{ij} = 1\) and all other entries equal 0. Then for Jordan frame \((c_i)_{i=1}^r\), where \(c_k = \mu_{kk}\), \(k = 1, \ldots, r\), we have \(z_{jk} = (\mu_{jk} + \mu_{kj}) \in \mathbb{E}_{jk}\) and \(\|z_{jk}\|^2 = 2\), \(1 \leq j, k \leq r\), \(j \neq k\). If \(z^{(i)} \in \bigoplus_{j=1}^r \mathbb{E}_{ij}\), \(i, 1, \ldots, r - 1\), then there exists \(\alpha^{(i)} = (\alpha_{i+1}, \ldots, \alpha_r) \in \mathbb{R}^{r-1}\) such that \(z^{(i)} = \sum_{j=i+1}^r \alpha_j z_{ij}\). Then the Frobenius transformation reads

\[
\tau(c(z^{(i)}))x = F_i(\alpha^{(i)}) \cdot x \cdot F_i(\alpha^{(i)})^T,
\]

where \(F_i(\alpha^{(i)})\) is so called Frobenius matrix:

\[
F_i(\alpha^{(i)}) = I + \sum_{j=i+1}^r \alpha_j \mu_{ji},
\]

i.e. bellow ith one of identity matrix there is a vector \(\alpha^{(i)}\), particularly

\[
F_2(\alpha^{(2)}) = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & \alpha_3 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \alpha_r & 0 & \cdots & 1
\end{pmatrix}.
\]

It can be shown ([Faraut and Korányi, 1994, Proposition VI.3.10]) that for each \(t \in T\), \(x \in \Omega\) and \(s \in \mathbb{R}^r\),

\[
\Delta_s(tx) = \Delta_s(te) \Delta_s(x)
\]

and for any \(z \in \mathbb{E}(c_i, 1/2), i = 1, \ldots, r,\)

\[
\Delta_s(\tau(z) e) = 1,
\]

if only \(\Delta_s\) and \(T\) are associated with the same Jordan frame \((c_i)_{i=1}^r\).

We will now introduce some necessary basics regarding certain probability distribution on symmetric cones. Absolutely continuous Riesz distribution \(R_{s,a}\) on \(\Omega\) is defined for any \(a \in \Omega\) and \(s = (s_1, \ldots, s_r) \in \mathbb{R}^r\) such that \(s_i > (i - 1)d/2\), \(i = 1, \ldots, r\), though its density

\[
R_{s,a}(dx) = \frac{\Delta_s(a)}{\Gamma_\Omega(s)} \Delta_{s - \dim \Omega/r}(x) e^{-(a,x)} I_\Omega(x) \, dx, \quad x \in \Omega,
\]

where \(\Delta_s\) is the generalized power function with respect to a Jordan frame \((c_i)_{i=1}^r\) and \(\Gamma_\Omega\) is the Gamma function of the symmetric cone \(\Omega\). It can be shown that \(\Gamma_\Omega(s) = (2\pi)^{(\dim \Omega/r - 1)/2} \prod_{j=1}^{r} \Gamma(s_j - (j - 1)d/2)\) (see [Faraut and Korányi, 1994, VII.1.1.]).

Absolutely continuous Wishart distribution \(\gamma_{p,a}\) on \(\Omega\) is a special case of Riesz distribution for \(s_1 = \cdots = s_r = p\). If \(a \in \Omega\) and \(p > \dim \Omega/r - 1\) it has density

\[
\gamma_{p,a}(dx) = \frac{(\det a)^p}{\Gamma_\Omega(p)} (\det x)^{p - \dim \Omega/r} e^{-(a,x)} I_\Omega(x) \, dx, \quad x \in \Omega,
\]

where \(\Gamma_\Omega(p) := \Gamma_\Omega(p, \ldots, p)\). Wishart distribution is a generalization of gamma distribution (case \(r = 1\)).
In generality, Riesz and Wishart distributions do not always have densities, but due to the assumption of existence of densities in Theorem 4.1 we are not interested in other cases.

### 3. FUNCTIONAL EQUATIONS

#### 3.1. Logarithmic Cauchy functions

As will be seen, the densities of respective random variables will be given in terms of \( w \)-logarithmic Cauchy functions, i.e., functions \( f : \Omega \to \mathbb{R} \) that satisfy the following functional equation

\[
f(x) + f(w(e)y) = f(w(x)y), \quad (x, y) \in \Omega^2,
\]

where \( w \) is a multiplication algorithm. If \( f \) is \( w \)-logarithmic, then \( e^f \) is called \( w \)-multiplicative. In the following section we will give the form of \( w \)-logarithmic Cauchy functions for two basic multiplication algorithms, one connected with quadratic representation \( w_1(x) = \mathbb{P}(x^{1/2}) \), and the other related to a triangular group \( T \), \( w_2(x) = t_x \in T \). Such functions were recently considered without any regularity assumptions in Kołodziejek [2013b].

Functional equation (9) for \( w_1 \) were already considered by Bojdecki and Wesołowski [2003] for differentiable functions and by Molnár [2006] for continuous functions of real or complex Hermitian positive definite matrices of rank greater than 2. Without any regularity assumptions it was solved on the Lorentz cone by Wesołowski [2007].

Case of \( w_2(x) = t_x \in T \) for a triangular group \( T \), perhaps a bit surprisingly, leads to a different solution. It was indirectly solved for differentiable functions by Hassairi et al. [2008].

By [Faraut and Korányi, 1994, Proposition III.4.3], for any \( g \) in the group \( G \),

\[
\text{det}(g(x)) = (\text{det}(g))^{r/dim \Omega} \text{det}(x),
\]

where \( \text{Det} \) denotes the determinant in the space of endomorphisms on \( \Omega \). Inserting a multiplication algorithm \( g = w(y) \), \( y \in \Omega \), and \( x = e \) we obtain

\[
\text{det}(w(y)) = (\text{det}(y))^{dim \Omega/r}
\]

and hence

\[
\text{det}(w(y)x) = \text{det}(y) \text{det}(x)
\]

for any \( x, y \in \Omega \). This means that \( f(x) = H(\text{det}(x)) \), where \( H \) is generalized logarithmic function, i.e. \( H(ab) = H(a) + H(b) \) for \( a, b > 0 \), is always a solution to (9), regardless of the choice of multiplication algorithm \( w \). If a \( w \)-logarithmic functions \( f \) is additionally \( K \)-invariant \( (f(x) = f(kx) \) for any \( k \in K \)), then \( H(\text{det}(x)) \) is the only possible solution (Theorem 3.3).

In Kołodziejek [2013b] the following theorems have been proved. They will be useful in the proof of the main theorems in this paper.

**Theorem 3.1** \((w_1\text{-logarithmic Cauchy functional equation})\) Let \( f : \Omega \to \mathbb{R} \) be a function such that

\[
f(x) + f(y) = f \left( \mathbb{P} \left( x^{1/2} \right) y \right), \quad (x, y) \in \Omega^2.
\]

Then there exists a logarithmic function \( H \) such that for any \( x \in \Omega \),

\[
f(x) = H(\text{det}(x)).
\]

**Theorem 3.2** \((w_2\text{-logarithmic Cauchy functional equation})\) Let \( f : \Omega \to \mathbb{R} \) be a function satisfying

\[
f(x) + f(y) = f(t_y x)
\]
for any $x$ and $y$ in the cone $\Omega$ of rank $r$, $\mathfrak{t}_y \in \mathcal{T}$, where $\mathcal{T}$ is the triangular group with respect to the Jordan frame $(c_j)_{j=1}^{\ell}$. Then there exist generalized logarithmic functions $H_1, \ldots, H_r$ such that for any $x \in \Omega$,

$$f(x) = \sum_{k=1}^{r} H_k(\Delta_k(x)),$$

where $\Delta_k$ is the principal minor of order $k$ with respect to $(c_j)_{j=1}^{\ell}$.

**Remark 3.3** If we impose on $f$ in Theorem 3.2 some mild conditions (eg. measurability), then there exists $s \in \mathbb{R}^r$ such that for any $x \in \Omega$,

$$f(x) = \log \Delta_s(x).$$

**Theorem 3.4** Let $f : \Omega \to \mathbb{R}$ be a function satisfying (9). Assume additionally that $f$ is $K$-invariant, ie. $f(kx) = f(x)$ for any $k \in K$ and $x \in \Omega$. Then there exists a logarithmic function $H$ such that for any $x \in \Omega$,

$$f(x) = H(\det x).$$

**Lemma 3.5** ($w$-logarithmic Pexider functional equation) Assume that $a$, $b$, $c$ defined on the cone $\Omega$ satisfy following functional equation

$$a(x) + b(y) = c(w(x)y), \quad (x, y) \in \Omega^2.$$

Then there exist $w$-logarithmic function $f$ and real constants $a_0, b_0$ such that for any $x \in \Omega$,

$$a(x) = f(x) + a_0,$$
$$b(x) = f(w(x)) + b_0,$$
$$c(x) = f(x) + a_0 + b_0.$$

### 3.2. The Olkin–Baker functional equation

In the following section we deal with the Olkin-Baker functional equation on simple symmetric cones, which is related to the Lukacs independence condition (see proof of the Theorem 4.1).

Henceforth we will assume that multiplication algorithm $w$ additionally is homogeneous of degree 1, that is $w(sx) = sw(x)$ for any $s > 0$ and $x \in \Omega$. It is easy to create a multiplication algorithm without this property, for example:

$$w(x) = \begin{cases} 
  w_1(x), & \text{if } \det x > 1, \\
  w_2(x), & \text{if } \det x \leq 1.
\end{cases}$$

The problem of solving

$$f(x)g(y) = p(x + y)q(x/y), \quad (x, y) \in (0, \infty)^2$$

for unknown positive functions $f$, $g$, $p$ and $q$ was first posed in [Olkin 1975]. Its general solution was given in [Baker 1976], and later analyzed in [Lajkó 1979] using a different approach. Recently, in [Mészáros 2010] and [Lajkó and Mészáros 2012] the equation (11) was solved assuming that it is satisfied almost everywhere on $(0, \infty)^2$ for measurable functions which are non-negative on its domain or positive on some sets of positive Lebesgue measure, respectively. Finally, a new derivation of solution to (11), when the equation holds almost everywhere on $(0, \infty)^2$ and no regularity assumptions on unknown positive functions are imposed, was given in [Ger et al. 2013]. The following theorem is concerned with an adaptation of (11) (after taking logarithm) to the symmetric cone case.

**Theorem 3.6** (Olkin–Baker functional equation on symmetric cones) Let $a$, $b$, $c$ and $d$ be real continuous functions on a simple symmetric cone $\Omega$ of rank $r$. Assume

$$a(x) + b(y) = c(x + y) + d(g(x + y)x), \quad (x, y) \in \Omega^2,$$
where $g^{-1} = w$ is a homogeneous of degree 1 multiplication algorithm. Then there exist constants $C_i \in \mathbb{R}$, $i = 1, \ldots, 4$, $\Lambda \in \mathbb{E}$ such that for any $x \in \Omega$ and $u \in D = \{ x \in \Omega : e - x \in \Omega \}$,

$$a(x) = \langle \Lambda, x \rangle + e(x) + C_1,$$

$$b(x) = \langle \Lambda, x \rangle + f(x) + C_2,$$

$$c(x) = \langle \Lambda, x \rangle + e(x) + f(x) + C_3,$$

$$d(u) = e(u)e(u) + f(e - e(u))u + C_4,$$

where $e$ and $f$ are continuous $w$-logarithmic Cauchy functions and $C_1 + C_2 = C_3 + C_4$.

We will need following simple lemma. For the elementary proof we refer to Kołodziejek 2013a, Lemma 3.2.

**Lemma 3.7** (Additive Pexider functional equation on symmetric cones) Let $a$, $b$ and $c$ be measurable functions on a symmetric cone $\Omega$ satisfying

$$a(x) + b(y) = c(x + y), \quad (x, y) \in \Omega^2. \tag{13}$$

Then there exist constants $\alpha, \beta \in \mathbb{R}$ and $\lambda \in \mathbb{E}$ such that for all $x \in \Omega$,

$$a(x) = \langle \lambda, x \rangle + \alpha,$$

$$b(x) = \langle \lambda, x \rangle + \beta,$$

$$c(x) = \langle \lambda, x \rangle + \alpha + \beta. \tag{14}$$

Now we can come back and give a new proof the Olkin–Baker functional equation.

**Proof of Theorem 3.6.** In the first part of the proof we adapt the argument given in Ger et al. 2013, where the analogous result on $(0, \infty)$ was analyzed, to the symmetric cone setting.

For any $s > 0$ and $(x, y) \in \Omega^2$ we get

$$a(sx) + b(sy) = c(s(x + y)) + d(s(sx + sy)zx). \tag{15}$$

Since $w$ is homogeneous of degree 1 we have $g(sx) = \frac{1}{s}g(x)$ and so $g(sx + sy)zx = g(x + y)zx$ for any $s > 0$. Subtracting now (12) from (15) for any $s \in (0, \infty)$ we arrive at the additive Pexider equation on symmetric cone $\Omega$,

$$a_s(x) + b_s(y) = c_s(x + y), \quad (x, y) \in \Omega^2,$$

where $a_s$, $b_s$ and $c_s$ are functions defined by $a_s(x) := a(sx) - a(x)$, $b_s(x) := b(sx) - b(x)$ and $c_s(x) := c(sx) - c(x)$.

Due to continuity of $a$, $b$ and $c$ and Lemma 3.7 it follows that for any $s > 0$ there exist constants $\lambda(s) \in \mathbb{E}$, $\alpha(s) \in \mathbb{R}$ and $\beta(s) \in \mathbb{R}$ such that for any $x \in \Omega$,

$$a_s(x) = \langle \lambda(s), x \rangle + \alpha(s),$$

$$b_s(x) = \langle \lambda(s), x \rangle + \beta(s),$$

$$c_s(x) = \langle \lambda(s), x \rangle + \alpha(s) + \beta(s).$$

By the definition of $a_s$ and the above observation it follows that for any $(s, t) \in (0, \infty)^2$ and $z \in \Omega$

$$a_{st}(z) = a_t(sz) + a_s(z).$$

Hence,

$$\langle \lambda(st), z \rangle + \alpha(st) = \langle \lambda(t), sz \rangle + \alpha(t) + \langle \lambda(s), z \rangle + \alpha(s). \tag{16}$$

Since (16) holds for any $z \in \Omega$ we see that $\alpha(st) = \alpha(s) + \alpha(t)$ for all $(s, t) \in (0, \infty)^2$. That is $\alpha(s) = k_1 \log s$ for $s \in (0, \infty)$, where $k_1$ is a real constant.
Analogous considerations for function $b$ for equation (19) where \( \langle \lambda(t), z \rangle = \langle \lambda(t), z \rangle (t - 1) \) implies that there exist \( w \) for any \( s > 0 \) and \( z \in \Omega \). It then follows that for all \( s \in (0, \infty) \) and \( z \in \Omega \),

\begin{align}
  a_s(z) &= a(sz) - a(z) = \langle \lambda, z \rangle (s - 1) + k_1 \log s.
\end{align}

Let us define function \( \bar{a} \) by formula

\begin{align}
  \bar{a}(x) &= a(x) - \langle \Lambda, x \rangle.
\end{align}

From (18) we get

\begin{align}
  \bar{a}(sx) &= \bar{a}(x) + k_1 \log s
\end{align}

for \( s > 0 \) and \( x \in \Omega \).

Analogous considerations for function \( b_s \) gives existence of constant \( k_2 \) such that \( \bar{b}(sx) = \bar{b}(x) + k_2 \log s \), where

\begin{align}
  \bar{b}(x) &= b(x) - \langle \Lambda, x \rangle,
\end{align}

hence \( \bar{c}(sx) = \bar{c}(x) + (k_1 + k_2) \log s \) and

\begin{align}
  \bar{c}(x) &= c(x) - \langle \Lambda, x \rangle
\end{align}

for any \( s > 0 \) and \( x \in \Omega \).

Functions \( \bar{a}, \bar{b}, \bar{c} \) and \( d \) satisfy original Olkin-Baker functional equation:

\begin{align}
  \bar{a}(x) + \bar{b}(y) = \bar{c}(x + y) + d(g(x + y) x), \quad (x, y) \in \Omega^2.
\end{align}

Taking \( x = y = \mathbf{v} \in \Omega \) in (20), we arrive at

\begin{align}
  \bar{a}(v) + \bar{b}(v) = \bar{c}(2v) + d(\frac{1}{2}v) = \bar{c}(v) + (k_1 + k_2) \log 2 + d(\frac{1}{2}v).
\end{align}

Insert \( x = \alpha w(v) u \) and \( y = w(v)(e - \alpha u) \) into (20) for \( 0 < \alpha < 1 \) and \( (u, v) \in (\mathcal{D}, \Omega) \). Using (19) we obtain

\begin{align}
  \bar{a}(w(v) u) + \bar{b}(w(v)(e - \alpha u)) &= \bar{c}(v) + d(\alpha u) - k_1 \log \alpha, \quad (u, v) \in (\mathcal{D}, \Omega).
\end{align}

Let us observe, that due to continuity of \( \bar{b} \) on \( \Omega \) and \( \lim_{\alpha \to 0} \{ w(v)(e - \alpha u) \} = w(v)e = v \in \Omega \) (convergence in the norm generated by scalar product \( \langle \cdot, \cdot \rangle \)), limit as \( \alpha \to 0 \) of the left hand side of the above equality exists. Hence, the limit of the right hand side also exists and

\begin{align}
  \bar{a}(w(v) u) + \bar{b}(v) = \bar{c}(v) + \lim_{\alpha \to 0} \{ d(\alpha u) - k_1 \log \alpha \}, \quad (u, v) \in (\mathcal{D}, \Omega).
\end{align}

Subtracting (22) from (21) we have

\begin{align}
  \bar{a}(w(v) u) = \bar{a}(v) + g(u)
\end{align}

for \( u \in \mathcal{D}, v \in \Omega \), where \( g(u) = \lim_{\alpha \to 0} \{ d(\alpha u) - k_1 \log \alpha \} - (k_1 + k_2) \log 2 - d(\frac{1}{2}v) \). Due to the property of equation (23) holds for any \( u \in \Omega \), so we arrive at the \( w \)-logarithmic Pexider equation. Lemma 3.3 implies that there exist \( w \)-logarithmic function \( e \) such that

\begin{align}
  \bar{a}(x) = e(x) + C_1
\end{align}

for any \( x \in \Omega \) and a constant \( C_1 \in \mathbb{R} \). Function \( e \) is continuous, because \( \bar{a} \) is continuous. Coming back to the definition of \( \bar{a} \), we obtain

\begin{align}
  a(x) = \langle \Lambda, x \rangle + e(x) + C_1, \quad x \in \Omega.
\end{align}
Analogously for function $b$, considering equation (20) for $x = w(v)(e - au)$ and $y = aw(v)u$ after passing to the limit as $\alpha \to 0$, we show that there exist continuous $w$-logarithmic function $f$ such that

$$b(x) = \langle A, x \rangle + f(x) + C_2, \quad x \in \Omega$$

for a constant $C_2 \in \mathbb{R}$. The form of $c$ follows from (21). Taking $x = w(e)u$ and $y = e - w(e)u$ in (20) for $u \in D$, we obtain the form of $d$.

4. The Lukacs-Olkin-Rubin theorem without invariance of the quotient

In the following section we prove the density version of Lukacs-Olkin-Rubin theorem for any multiplication algorithm $w$ satisfying

(i) $w(sx) = sw(x)$ for $s > 0$ and $x \in \Omega$,

(ii) differentiability of mapping $\Omega \ni x \mapsto w(x) \in G$.

We assume (ii) to ensure that Jacobian of the considered transformation exists.

The following theorem generalizes results obtained in Boecka and Wesolowska [2002], Hassairi et al. [2008], Kołodziejek [2013a]. The most important innovation is considering quotient $U$ for any multiplication algorithm $w$. Respective densities are then expressed in terms of $w$-multiplicative Cauchy functions.

**Theorem 4.1** (The Lukacs-Olkin-Rubin theorem with densities on symmetric cones) Let $X$ and $Y$ be independent rv’s valued in irreducible symmetric cone $\Omega$ with strictly positive and continuous densities. Set $V = X + Y$ and $U = g(X + Y)X$ for any multiplication algorithm $w = g^{-1}$ satisfying conditions (i) and (ii). If $U$ and $V$ are independent then there exist $\Lambda \in E$ and $w$-multiplicative functions $e$, $f$ such that

$$f_X(x) \propto e(x)e^{\langle \Lambda, x \rangle}I_\Omega(x),$$

$$f_Y(x) \propto f(x)e^{\langle \Lambda, x \rangle}I_\Omega(x),$$

where $E$ is the Euclidean Jordan algebra associated with the cone $\Omega$.

In particular,

(1) if $g(x) = g_1(x) = P(x^{-1/2})$, the there exist constants $p_i > \dim \Omega/r - 1$, $i = 1, 2$, and $a \in \Omega$ such that $X \sim \gamma_{p_1,a}$ and $Y \sim \gamma_{p_2,a}$.

(2) if $g(x) = g_2(x) = t_x^{-1}$, the there exist constants $s_i = (s_{i,j})^r_{j=1}$, $s_{i,j} > (j - 1)d/2$, $i = 1, 2$, and $a \in \Omega$ such that $X \sim R_{s_1,a}$ and $Y \sim R_{s_2,a}$.

**Proof.** Let $\psi: \Omega \times \Omega \to D \times \Omega$ be a mapping defined through

$$\psi(x, y) = (u, v) = (g(x + y)x, x + y).$$

Then $(U, V) = \psi(X, Y)$. The inverse mapping $\psi^{-1} : D \times \Omega \to \Omega \times \Omega$ is given by

$$(x, y) = \psi^{-1}(u, v) = (w(v)u, w(v)(e - u)),$$

hence $\psi$ is a bijection. We are looking for the Jacobian $J$ of the map $\psi^{-1}$, that is, the determinant of the linear map

$$\begin{pmatrix} du \\ dv \end{pmatrix} \mapsto \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dx/du & dx/dv \\ dy/du & dy/dv \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

We have

$$J = \begin{vmatrix} w(v) & dx/dv \\ -w(v) & dy/dv \end{vmatrix} = \begin{vmatrix} w(v) & dx/dv \\ 0 & Id_{\Omega} \end{vmatrix} = \text{Det}(w(v)).$$

where Det denotes the determinant in the space of endomorphisms on $\Omega$. By (10) we get

$$\text{Det}(w(v)) = (\det v)^{\dim \Omega/r}.$$
Now we can find the joint density of \((U, V)\). Since \((X, Y)\) and \((U, V)\) have independent components, the following identity holds almost everywhere with respect to Lebesgue measure:
\[
\left(\det(x + y)\right)^{\text{dim} \Omega / r} f_X(x)f_Y(y) = f_U(g(x + y)x) f_V(x + y),
\]
where \(f_X, f_Y, f_U\) and \(f_V\) denote densities of \(X, Y, U\) and \(V\), respectively. Since the respective densities are assumed to be continuous, the above equation holds for every \(x, y \in \Omega\). Taking logarithms of both sides of the above equation (it is permitted since \(f_X, f_Y > 0\) on \(\Omega\)) we get
\[
a(x) + b(y) = c(x + y) + d\left(\mathbb{P}\left((x + y)^{-1/2}\right)\right) x,
\]
where
\[
\begin{align*}
a(x) &= \log f_X(x), \\
b(x) &= \log f_Y(x), \\
c(x) &= \log f_V(x) - \frac{\text{dim} \Omega}{r} \log \det(x), \\
d(u) &= \log f_U(u),
\end{align*}
\]
for \(x \in \Omega\) and \(u \in D\).

The first part of the conclusion follows now directly from Theorem 3.6. Thus there exist constants \(\Lambda \in \mathbb{E}, C_i \in \mathbb{R}, i \in \{1, 2\}\) and \(w\)-logarithmic functions \(e\) and \(f\) such that
\[
\begin{align*}
f_X(x) &= e^{a(x)} = f_X^{C_1} e(x) e^{\left(\Lambda, x\right)}, \\
f_Y(x) &= e^{b(x)} = f_Y^{C_2} f(x) e^{\left(\Lambda, x\right)},
\end{align*}
\]
for any \(x \in \Omega\).

Let us observe that if \(w(x) = w_1(x) = \mathbb{P}(x^{1/2})\), then for Theorem 3.6 there exist constants \(\kappa_i, i = 1, 2\), such that \(e(x) = (\det x)^{\kappa_i}\) and \(f(x) = (\det x)^{\kappa_2}\). Since \(f_X\) and \(f_Y\) are densities it follows that \(a = -\Lambda \in \Omega, k_i = -\kappa_i (\text{dim} \Omega)/r > -1\) and \(e^{C_i} = (\det a)^{\kappa_i}/T_\Omega(p_i), i = 1, 2\).

Analogously, if \(w(x) = w_2(x) = t_x\) then Theorem 3.2 and Remark 3.3 imply that there exist constants \(s_i = (s_{ij})_{j=1}^r, s_{ij} > (j - 1)d/2, i = 1, 2,\) and \(a = -\Lambda \in \Omega\) such that \(X \sim R_{s_1,a} Y \sim R_{s_2,a}\).

With Theorem 4.1 one can easily re-prove original Lukacs-Olkin-Rubin theorem (version of Olkin and Rubin 1964 and Casalis and Letac 1996), when the distribution of \(U\) is invariant under a group of automorphisms:

**Remark 4.2** Let us additionally assume in Theorem 4.1 that the quotient \(U\) has distribution which is invariant under a group of automorphisms, that is \(kU = U\) for any \(k \in K\). By Theorem 3.6 it follows that there exist continuous \(w\)-logarithmic functions \(e\) and \(f\) such that
\[
\log f_U(u) = d(u) = e(w(e)u) - f(e - w(e)u).
\]
The distribution of \(U\) is invariant under \(K\), thus density \(f_U\) is a \(K\)-invariant function, that is \(f_U(u) = f_U(ku)\) for any \(k \in K\). Note that \(w(e) \in K\), thus
\[
e(u) - f(e - u) = e(ku) - f(e - ku), \quad (k, u) \in K \times D.
\]
We will show that both functions \(e\) and \(f\) are \(K\)-invariant. Recall that \(e(x) + e(w(e)y) = e(w(x)y)\), therefore after taking \(y = \alpha e\) we obtain \(e(\alpha x) = e(x) + e(\alpha e)\) for any \(\alpha > 0\) and \(x \in \Omega\). Inserting \(u = \alpha v\) into (25) we arrive at
\[
e(v) + e(\alpha e) - f(e - \alpha v) = e(\alpha v) - f(e - \alpha v) = e(\alpha kv) - f(e - \alpha kv) = e(kv) + e(\alpha e) - f(e - k\alpha v).
\]
Thus \( e(v) + f(e^{-\alpha v}) = e(kv) + f(e^{-k\alpha v}) \) for any \( \alpha \in (0, 1) \), \( v \in D \). This implies that \( e \) is \( K \)-invariant and so is \( f \). By Theorem 3.4 and continuity of \( e \) and \( f \) we get that there exist constants \( \kappa_1, \kappa_2 \) such that \( e(x) = \kappa_1 \log \det x \) and \( f(x) = \kappa_2 \log \det x \), hence \( X \) and \( Y \) have Wishart distributions.

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