Adapted Exponential Type Estimator in the Presence of Non-response

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Received 15 August 2020; Accepted 19 March 2021; Publication 22 June 2021

Abstract
In this article, we propose an estimator using the exponential function for the population mean in the case of non-response on both the study and the auxiliary variables. The equations for the Bias and Mean Square Error (MSE) are derived to the first order of approximation and theoretical comparisons are made with existing estimators in literature. Besides, we examine the efficiency of the proposed estimator according to the classical ratio and regression estimator, Hansen-Hurwitz unbiased estimator, and the estimator of Singh et al. (2009). Following theoretical comparisons, we infer that the proposed estimator is more efficient than compared estimators under the obtained conditions in theory. Moreover, these theoretical results are supported numerically by providing an empirical study on five different data sets.

Keywords: Exponential type estimator, auxiliary variable, non-response, population mean, efficiency.

1 Introduction
In sample surveys, it is well known that while estimating the population parameters, the information of the auxiliary variable is usually used in order
to improve the efficiency of the estimators. In other words, the main aim of studies is to find out efficient estimators by using the auxiliary information. For this reason, many authors have proposed type of ratio, product, regression and exponential estimators using the auxiliary information in recent years. However, required information on different variables may not be obtained correctly and completely. This situation is named as non-response and this decreases the efficiency. Hansen and Hurwitz [1] considered a method in order to deal with this problem and introduced a new technique of sub-sampling the non-respondents. In this method, suppose that \( S = (S_1, S_2, \ldots, S_N) \) consists of \( N \) units \((N_1 + N_2 = N)\) is composed of \( N_1 \) and \( N_2 \) belonging to responding units and non-responding units, respectively, and sample of size \( n \) is drawn without replacement (SRSWOR) which is divided into two groups as \( n_1 \) units are “responding group” and \( n_2 \) units are “not responding group”. Here, \((y_i, x_i)\) are the values of the study and auxiliary variables for the \( i \)th unit \((i = 1, 2, \ldots, 10)\) of the population, respectively. A sub-sample of size \( r = n_2/h \) \((h > 1)\) units is randomly drawn from \( n_2 \) non-responding units where \( h \) is the inverse sampling rate at the second phase sample of size \( n \). \( W_1 = N_1/N \) and \( W_2 = N_2/N \) are proportions for responding and non-responding for the population, respectively.

Hansen and Hurwitz [1] were the first to propose the unbiased estimator for the population mean in the presence of non-response. The unbiased estimator is given as

\[
t_1 = w_1 \bar{y}_1 + w_2 \bar{y}_{2(r)},
\]

where \( w_1 = n_1/n \) and \( w_2 = n_2/n \) denote responding and non-responding proportions, respectively, for the sample. In addition, \( \bar{y}_1 \) and \( \bar{y}_{2(r)} \) symbolize the sample means of the study variable \( y \) depending on \( n_1 \) and \( r \) units, respectively. The variance of the estimator in (1) is given by

\[
V(t_1) = \bar{Y}^2 \left( \lambda C_y + \frac{W_2(h-1)}{n} C_{y(2)} \right),
\]

Here, \( f = n/N \), \( \lambda = \frac{1-f}{n} \), \( C_y^2 = S_y^2/\bar{Y}^2 \). Also, \( C_{y(2)}^2 = S_{y(2)}^2/\bar{Y}^2 \) is the coefficient of variation of the study variable for \( N_2 \) non-responding units of the population.

When non-response exists on both the study and the auxiliary variables and when the population mean of the auxiliary variable \( (\bar{X}) \) is known, some of important estimators in the presence of non-response in literature may be considered as follows:
Cochran [2] suggests a ratio estimator for the population mean as
\[ t_2 = \frac{\bar{y}^*}{\bar{x}^*} \bar{X}, \tag{3} \]
and its MSE, up to the first order of approximation, is given by
\[ MSE(t_2) = \bar{Y}^2 \left( \lambda (C^2_y + C^2_x - 2C_{yx}) \right. \]
\[ + \left. \frac{W_2(h - 1)}{n} \left( C^2_{y(2)} + C^2_{x(2)} - 2\rho_{yx(2)}C_{y(2)}C_{x(2)} \right) \right), \tag{4} \]
where \( C^2_x = \frac{S^2_x}{\bar{X}^2}, \ C_{xy} = \rho_{xy}C_xC_y, \ C^2_{x(2)} = \frac{S^2_{x(2)}}{\bar{X}^2} \) and \( C_{yx(2)} = \rho_{yx(2)}C_{y(2)}C_{x(2)} \). Note that \( \rho_{xy} \) and \( \rho_{yx(2)} \) are the population correlation coefficient of the response and non-response group between the study and auxiliary variables, respectively.

Using the technique of Hansen and Hurwitz, an exponential estimator, which is introduced by Singh et al. [3], is
\[ t_3 = \bar{y}^* \exp \left( \frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*} \right), \tag{5} \]
when there are non-response units on both the study as well as the auxiliary variables. The MSE of the estimator in (5) is given by
\[ MSE(t_3) = \bar{Y}^2 \left( \lambda C^2_y + \lambda \frac{C^2_x}{4} - \lambda C_{yx} + \frac{W_2(h - 1)}{n} \right. \]
\[ \times \left( C^2_{y(2)} + \frac{C^2_{x(2)}}{4} - \rho_{yx(2)}C_{y(2)}C_{x(2)} \right) \right). \tag{6} \]

Cochran [2] also adapted classical regression estimator to the case of the incomplete information on the study and auxiliary variables as
\[ t_4 = \bar{y}^* + b^* (\bar{X} - \bar{x}^*), \tag{7} \]
where \( b^* = \frac{S_{xy}^*}{S_x^*} \). The equation of MSE, up to the first order of approximation, for the regression estimator in (7) is given by
\[ MSE(t_4) = \bar{Y}^2 \left( \lambda C^2_y (1 - \rho_{xy}^2) + \frac{W_2(h - 1)}{n} \right. \]
\[ \times \left( C^2_{y(2)} + \rho_{xy}^2 \frac{C^2_y}{C^2_x} C_{x(2)} - 2\rho_{xy} \frac{C_y}{C_x} C_{yx(2)} \right) \right). \tag{8} \]
Main motivation of this study is to find out more efficient estimator than existing estimators in literature considering the non-response. For this reason, we adapt the estimator which is proposed by Vishwakarma et al. [4] to a new estimator considering the case of non-response units on both the study and the auxiliary variables for the estimation of the population mean. In Section 2, the expressions of the Bias and MSE for the adapted estimator are also obtained. Theoretical and numerical comparisons between the adapted estimator and existing estimators, such as Hansen-Hurwitz unbiased estimator, adapted classical ratio and regression estimators, Singh et al. [3] exponential estimator, are made in Sections 3 and 4, respectively.

2 The Adapted Estimator

We adapt the exponential type estimator which is proposed by Vishwakarma et al. [4] to a new estimator considering the case of non-response occurs on both the study and the auxiliary variables as follows. The adapted estimator is given as follows:

\[ t_5 = \alpha \bar{y}^* + (1 - \alpha) \bar{y}^* \exp \left( \frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*} \right). \]  

(9)

To obtain the Bias and MSE of the proposed estimator in (9), we can write that

\[ \bar{y}^* = \bar{Y}(1 + e_y^*), \quad \bar{x}^* = \bar{X}(1 + e_x^*), \]

then, we have

\[ E(e_x^2) = E(e_y^2) = 0, \quad E(e_x^2) = \lambda C_x^2 + \frac{W(h-1)}{n} C_x^2, \]

\[ E(e_y^2) = \lambda C_y^2 + \frac{W(h-1)}{n} C_y^2 \text{ and } E(e_x e_y^*) = \lambda \rho_{xy} C_x C_y + \frac{W(h-1)}{n} \rho_{xy} \]

\[ C_x(2) C_y(2). \]

Now, expressing the estimator in (9), in terms of \( e_i^* (i = x, y) \), we have

\[ t_5 = \alpha \bar{Y}(1 + e_0^*) + (1 - \alpha) \bar{Y}(1 + e_0^*) \exp \left( \frac{\bar{X} - \bar{X} - X e_0^*}{\bar{X} + X + X e_0^*} \right), \]

(10)

\[ = \bar{Y} (\alpha + \alpha e_0^*) + \bar{Y} (1 + e_0^* - \alpha - \alpha e_0^*) \exp \left( \frac{-e_1^*}{2} \left( 1 + \frac{e_1^*}{2} \right)^{-1} \right), \]

(11)

\[ = \bar{Y} \left( 1 + e_0^* - \frac{e_1^*}{2} + \frac{e_1^*}{2} + \frac{3 e_1^*}{8} - \frac{3 \alpha e_1^*}{8} - \frac{e_0^* e_1^*}{2} \right). \]

(12)
Expanding the right hand side of (12), to the first degree of approximation, we have

\[(t_5 - \bar{Y}) = \bar{Y} \left( e_0^* + e_1^* \left( \frac{\alpha}{2} - \frac{1}{2} \right) + e_1^2 \left( \frac{3}{8} - \frac{3\alpha}{8} \right) + e_0^* e_1^* \left( \frac{\alpha}{2} - \frac{1}{2} \right) \right).\]

(13)

Taking the expectation of both sides of (13), we get the bias as

\[BIAS(t_5) = \bar{Y} \left( \frac{3}{8} (1 - \alpha) \left( \lambda C_x^2 + \frac{W_2(h-1)}{n} C_{x(2)}^2 \right) \right.
\[\left. + \left( \frac{\alpha - 1}{2} \right) \left( \lambda C_y + \frac{W_2(h-1)}{n} \rho_{yx(2)} C_{y(2)} C_{x(2)} \right) \right).\]

(14)

Squaring both sides of (13), we have

\[(t_5 - \bar{Y})^2 = \bar{Y}^2 \left( e_0^{*2} + e_1^{*2} \left( \frac{\alpha^2}{4} - \frac{\alpha}{2} + \frac{1}{4} \right) + e_0^* e_1^* (\alpha - 1) \right),\]

then taking expectation on both sides, we get the MSE as

\[MSE(t_5) = \bar{Y}^2 \left( \lambda C_y^2 + \frac{W_2(h-1)}{n} C_{y(2)}^2 \right.
\[\left. + \left( \lambda C_x^2 + \frac{W_2(h-1)}{n} C_{x(2)}^2 \right) \left( \frac{\alpha^2}{4} - \frac{\alpha}{2} + \frac{1}{4} \right) \right.
\[\left. + \left( \lambda \rho_{xy} C_x C_y + \frac{W_2(h-1)}{n} \rho_{yx(2)} C_{y(2)} C_{x(2)} \right) (\alpha - 1) \right).\]

(15)

The optimal value of \(\alpha\) is obtained by minimizing the MSE in (15) as

\[\alpha^* = \frac{A_1 - 2A_2}{A_1},\]

(16)

where \(A_1 = E(e_1^{*2}) = \lambda C_x^2 + \frac{W_2(h-1)}{n} C_{x(2)}^2\) and \(A_2 = E(e_0^* e_1^*) = \lambda C_{yx} + \frac{W_2(h-1)}{n} \rho_{yx(2)} C_{y(2)} C_{x(2)}\).
Replacing $\alpha$ in (15) with the optimal value of $\alpha$, given in (16), we have the minimum MSE of the estimator as

$$\text{MSE}_{\text{min}}(t_5) = \bar{Y}^2 \left( \lambda \left( C_y^2 + \frac{A_2}{A_1} \left( \frac{A_2}{A_1} - 2\rho_{xy} \frac{C_y}{C_x} \right) C_x^2 \right) + \frac{W_2(h-1)}{n} \left( C_y^{(2)} + \frac{A_2}{A_1} \left( \frac{A_2}{A_1} - 2\rho_{xy}^{(2)} \frac{C_y^{(2)}}{C_x^{(2)}} \right) C_x^{(2)} \right) \right).$$

(17)

Also, by substituting the obtained $A_1$ and $A_2$ equations, the minimum MSE of the estimator can be rewritten as follows:

$$\text{MSE}_{\text{min}}(t_5) = \bar{Y}^2 \left[ \left( \lambda C_y^2 + \frac{W_2(h-1)}{n} C_y^{(2)} \right) \frac{\left( \lambda C_{xy} + \frac{W_2(h-1)}{n} \rho_{yx}^{(2)} C_y^{(2)} C_{x}^{(2)} \right)^2}{\left( \lambda C_x^2 + \frac{W_2(h-1)}{n} C_x^{(2)} \right)} \right].$$

3 Efficiency Comparisons

In this section, we compare the MSE equation of the adapted estimator ($t_5$) with the MSE equations of the mentioned estimators, such as Hansen and Hurwitz [1] unbiased estimator, Cochran [2] classical ratio and regression estimators, Singh et al. [3] exponential estimator, mentioned in Section 1.

We find the efficiency conditions of the proposed estimator as follows:

(i) $\text{MSE}_{\text{min}}(t_5) < \text{MSE}(t_1)$

$$\left( \lambda C_{xy} + \frac{W_2(h-1)}{n} C_{yx}^{(2)} \right)^2 > 0$$

(18)

(ii) $\text{MSE}_{\text{min}}(t_5) < \text{MSE}(t_2)$

$$\left( \left( \lambda C_x^2 + \frac{W_2(h-1)}{n} C_x^{(2)} \right) - \left( \lambda C_{yx} + \frac{W_2(h-1)}{n} C_{yx}^{(2)} \right) \right)^2 > 0$$

(19)
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(iii) $\text{MSE}_{\text{min}}(t_5) < \text{MSE}(t_3)$

\[
\left( \lambda C_{xy} + \frac{W_2(h-1)}{n} \rho_{yx(2)} C_y C_{x(2)} \right) - \frac{1}{2} \left( \lambda C_x^2 + \frac{W_2(h-1)}{n} C_x^2 \right)^2 > 0
\]

(iv) $\text{MSE}_{\text{min}}(t_5) < \text{MSE}(t_4)$

\[
\left( \frac{W_2(h-1)}{n} C_x^2 \rho_{xy} C_y C_x \right) - \left( \frac{W_2(h-1)}{n} C_y C_{x(2)} \right)^2 > 0
\]

The conditions between (18)–(21) are always satisfied, we infer that the proposed estimator, $t_5$, is more efficient than the compared estimators $t_1, t_2, t_3$ and $t_4$.

4 Numerical Illustration

To examine the appropriateness of the proposed estimator, we have used five popular different data sets considered by many researchers in literature. The descriptive statistics and results for each data set are given as follows:

The PREs of $t_1, t_2, t_3, t_4$ and $t_5$ for various values of $h$ are presented in Tables 2–6 based on five populations, respectively.

We would like to remark that the PRE of the adapted estimator is more efficient than the other compared estimators, $t_1, t_2, t_3$ and $t_4$ in the presence

| Table 1 Descriptive statistics for each data set |
|-----------------------------------------------|
| Parameters | Population | $N$ | $n$ | $W_2$ | $\bar{X}$ | $\bar{Y}$ | $C_y$ | $C_x$ | $\rho_{yx}$ | $C_y(2)$ | $C_x(2)$ | $\rho_{yx(2)}$ |
|-------------|------------|-----|-----|-------|---------|---------|-------|-------|-------------|-----------|-----------|----------------|
| 1 Khare and Kumar [5] | 96 | 25 | 0.25 | 1807.23 | 185.22 | 1.053 | 1.0633 | 0.904 | 0.528 | 0.853 | 0.895 |
| 2 Khare and Sinha [6] | 96 | 40 | 0.25 | 144.87 | 137.92 | 1.32 | 0.81 | 0.77 | 2.08 | 0.94 | 0.72 |
| 3 Khare and Srivastava [7] | 70 | 35 | 0.2 | 1755.53 | 981.29 | 0.6254 | 0.801 | 0.778 | 0.4087 | 0.574 | 0.445 |
| 4 Sinha and Kumar [8] | 109 | 35 | 0.25 | 255.97 | 485.92 | 0.6559 | 0.6037 | 0.857 | 0.7335 | 0.6897 | 0.834 |
| 5 Sinha and Kumar [9] | 109 | 35 | 0.25 | 41.24 | 485.92 | 0.6559 | 1.126 | 0.451 | 0.4785 | 1.166 | 0.714 |
Table 2  PREs of the Proposed Estimator \((t_5)\) and Other Estimators for Population 1

|       | \(h = 2\) | \(h = 3\) | \(h = 4\) | \(h = 5\) | \(h = 6\) |
|-------|------------|------------|------------|------------|------------|
| \(t_1\) | 100.0000  | 100.0000  | 100.0000  | 100.0000  | 100.0000  |
| \(t_2\) | 425.4729  | 370.1973  | 332.8156  | 305.8494  | 285.4779  |
| \(t_3\) | 301.6963  | 310.1851  | 317.9189  | 324.9939  | 331.4910  |
| \(t_4\) | 419.9153  | 350.2990  | 306.3844  | 276.1563  | 254.0789  |
| \(t_5\) | **491.1458** | **463.1675** | **447.5542** | **438.3615** | **432.8506** |

Table 3  PREs of the Proposed Estimator \((t_5)\) and Other Estimators for Population 2

|       | \(h = 2\) | \(h = 3\) | \(h = 4\) | \(h = 5\) | \(h = 6\) |
|-------|------------|------------|------------|------------|------------|
| \(t_1\) | 100.0000  | 100.0000  | 100.0000  | 100.0000  | 100.0000  |
| \(t_2\) | 202.2646  | 194.3660  | 190.6994  | 188.5823  | 187.2039  |
| \(t_3\) | 148.0884  | 144.4216  | 142.6821  | 141.6667  | 141.0011  |
| \(t_4\) | 219.9692  | 212.4751  | 208.9698  | 206.9382  | 205.6122  |
| \(t_5\) | **220.6768** | **214.9752** | **212.6081** | **211.3493** | **210.5799** |

Table 4  PREs of the Proposed Estimator \((t_5)\) and Other Estimators for Population 3

|       | \(h = 2\) | \(h = 3\) | \(h = 4\) | \(h = 5\) | \(h = 6\) |
|-------|------------|------------|------------|------------|------------|
| \(t_1\) | 100.0000  | 100.0000  | 100.0000  | 100.0000  | 100.0000  |
| \(t_2\) | 124.3555  | 108.5754  | 98.8639   | 92.2849   | 87.5332   |
| \(t_3\) | 208.3451  | 188.8973  | 176.1676  | 167.1876  | 160.5133  |
| \(t_4\) | 209.0156  | 184.9004  | 169.7404  | 159.3284  | 151.7359  |
| \(t_5\) | **210.8401** | **189.2205** | **176.1734** | **167.4633** | **161.2454** |

Table 5  PREs of the Proposed Estimator \((t_5)\) and Other Estimators for Population 4

|       | \(h = 2\) | \(h = 3\) | \(h = 4\) | \(h = 5\) | \(h = 6\) |
|-------|------------|------------|------------|------------|------------|
| \(t_1\) | 100.0000  | 100.0000  | 100.0000  | 100.0000  | 100.0000  |
| \(t_2\) | 351.9514  | 342.8085  | 337.4328  | 333.8937  | 331.3874  |
| \(t_3\) | 233.9950  | 232.7577  | 232.0054  | 231.4997  | 231.1363  |
| \(t_4\) | 359.2028  | 350.8473  | 345.8426  | 342.5690  | 340.2464  |
| \(t_5\) | **359.4776** | **351.3462** | **346.5934** | **343.4756** | **341.2729** |

of non-response. Furthermore, the PREs of the adapted estimator stands out for Population 1, 2 and 3, especially, according to the results in Tables 2–6. We also see that the PRE of the \(t_5\) estimator decrease with the increasing values of \(h\) except the Population 5.
Table 6: PREs of the Proposed Estimator ($t_5$) and Other Estimators for Population 5

|      | $h = 2$                  | $h = 3$                  | $h = 4$                  | $h = 5$                  | $h = 6$                  |
|------|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $t_1$ | 100.0000                 | 100.0000                 | 100.0000                 | 100.0000                 | 100.0000                 |
| $t_2$ | 38.8760                  | 37.0780                  | 35.8300                  | 34.9130                  | 34.2109                  |
| $t_3$ | 107.8945                 | 110.9662                 | 113.3976                 | 115.3701                 | 117.0024                 |
| $t_4$ | 133.8007                 | 140.4609                 | 145.9319                 | 150.5061                 | 154.3873                 |
| $t_5$ | 133.8633                 | 140.6118                 | 146.1823                 | 150.8558                 | 154.8319                 |

5 Conclusion

In this study, we propose a new exponential type estimator for the estimation of the population mean using the information of the auxiliary variable in the presence of non-response. Equations for the bias and minimum MSE of the proposed estimator are obtained. In theoretical comparisons, the proposed estimator is found more efficient than the estimators in literature, such as Hansen and Hurwitz [1] unbiased estimator, Cochran [2] adapted ratio and regression estimators, and Singh et al. [3] exponential estimator, under the obtained conditions. We use five data sets with the aim of supporting the results in theory and we show that the proposed estimator is quite efficient than other compared estimators as seen in Tables 2–6. Hence, the proposed estimator is recommended based on the theoretical and numerical results and can be used in applications.

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