Nonlinearity-induced PT-symmetry without material gain

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Abstract

Parity-time symmetry has raised a great deal of attention in optics in recent years, yet its application has been so far hindered by the stringent requirements on coherent gain balanced with loss. In this paper, we show that the conditions to enable parity and time symmetry can be simultaneously satisfied for a pair of modes with mixed frequencies interacting in a nonlinear medium, without requiring the presence of material gain. First, we consider a guided wave structure with second order nonlinearity for a pair of modes with mixed frequencies interacting in a nonlinear medium, without requiring the presence of material gain. Then, we derive the PT-symmetric Hamiltonian that governs the interaction of two waves of mixed frequencies when accompanied by a high intensity pump beam at the sum frequency. We also extend the results to an array of coupled nonlinear waveguide channels. It is shown that the evolution dynamics of the low-frequency waves is associated with a periodic PT-symmetric lattice while the phase of the pump beams can be utilized as a control parameter to modify the gain and loss distribution, thus realizing different PT lattices by design. Our results suggest that nonlinear wave mixing processes can form a rich platform to realize PT-symmetric Hamiltonians of arbitrary dimensions in optical systems, without requiring material gain.

1. Introduction

Non-conservative parity-time (PT-) symmetric systems have recently attracted considerable attention in the context of optical physics. Interest in such systems was raised after the realization that a particular class of non-Hermitian quantum mechanical Hamiltonians, the so-called PT-symmetric Hamiltonians, can exhibit entirely real eigenvalue spectra [1–4]. In general a PT-symmetric operator commutes with the parity-time operator (PT) while it may not commute with parity (P) or time (T) operators solely. Another interesting property of such systems is their phase transition from entirely real to partially or completely complex eigenvalue spectra [1]. This phase transition typically occurs when increasing a non-Hermiticity parameter above the so-called PT-symmetry-breaking threshold, which is associated with an exceptional point singularity in the complex eigenvalue space of the Hamiltonian.

Quite recently it was noted that PT-symmetry can be fruitfully explored and observed within the realm of optics, owing to the fact that optical gain and loss can be properly used, in conjunction with refraction, to realize non-Hermitian PT-symmetric potentials [5–7]. In general, for a scalar optical field, a necessary condition for PT-symmetry requires $n^2(-r) = n(r)$, which means that the real and imaginary parts of the complex refractive index distribution should be even and odd functions of position, respectively [5]. For vectorial fields, similar conditions should be satisfied for the involved electromagnetic permittivity and permeability [8]. As it has been shown by many works, such spatially distributed balanced gain and loss distributions can lead to a wide range of interesting phenomena, which do not necessarily have a counterpart in conservative settings [9–30]. These include unidirectional invisibility [9–12], simultaneous lasing-absorbing [13, 14], negative refraction [15], pseudo-Hermitian Bloch oscillations [16, 17], and optical solitons [18–23] to mention a few. In addition, single mode lasing [24–26] and optical isolation in PT-symmetric structures [27–29] have also been demonstrated.

To some extent, most works in this area have been focused on structures that utilize spatially separated regions of material gain and loss in order to realize PT-symmetric potentials for a spatially distributed...
electromagnetic field. The realization of such structures can be quite demanding due to physical limitation of gain materials, as well as the codependency of the real and imaginary parts of refractive index landscapes as dictated by Kramers–Kronig relations. Therefore, it becomes of great interest to devise alternative platforms for the realization of PT-symmetric Hamiltonians, which do not necessarily have to rely on material gain and loss.

In a different context, optical parametrical amplification has been widely used in commercially available devices [31]. As opposed to other nonlinear processes, parametric amplification has all the properties of a pseudo–Hermitian system [32]. First of all, given that parametric amplification is based on external pumping, the evolution dynamics of the associated system is described via a non-Hermitian operator. In addition, there is phase transition point at which parametric gain can overcome the internal losses, as well as the phase mismatch. Given the similarity with an exceptional phase transition in a PT-symmetric system, it is of great interest to draw a link between these two entirely different phenomena. It is worth noting that notions related to non-Hermitian physics have been previously explored in the context of nonlinear optics. For example, it has been proposed theoretically and justified experimentally that coherent perfect absorption can be achieved in a time-reversed parametric amplifier [34, 35] while non-Hermitian phase matching for parametric amplification has also been proposed [36]. A clear connection between the two fields, however, has not been presented to the best of our knowledge. In this context, it is of interest to explore how the exact conditions for PT symmetry can be fulfilled in a nonlinear optics platform, and whether the prospect of utilizing notions from non-Hermitian Hamiltonians can inspire novel design schemes for nonlinear processes.

Here, we explore these possibilities by showing the connection between a parametrically pumped wave mixing scenario and PT-symmetry. In this regard, we show that nonlinear optics can provide a fertile ground to realize the conditions for PT-symmetry for different harmonics of interest. We first consider a three wave mixing scenario, and we show how the governing evolution operator can be transformed to a PT-symmetric Hamiltonian. The results are then extended to arrays of coupled waveguides with quadratic nonlinearity. While in such segmented arrangements, the pump beam can remain localized to individual channels, the probe can talk to different waveguides while the evolution is governed by a PT-symmetric Hamiltonian. The associated band structure is derived analytically and the condition for PT symmetry breaking is discussed. Finally, we show that the phase of the pump beams applied to different channels play an important role and can be properly designed to realize periodic lattices with local and global PT-symmetric properties.

2. Single nonlinear waveguide

In this section, we briefly discuss the conventional three wave mixing process, and show how the pump beam can realize the condition of PT-symmetry for two mixed lower frequency beams. Consider a general planar waveguide with $\chi^{(2)}$ nonlinearity, with propagation direction along $z$. Under transverse-electric polarization, and assuming no polarization mixing, the propagation of light is governed by the equation:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{n^2(x; \omega)}{c_0^2} \frac{\partial^2}{\partial t^2} \right) E(x, z, t) = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2},$$

Here $n(x; \omega)$ represents the refractive index profile, $c_0$ is the free space speed of light, and the nonlinear polarization can be written as $P_{NL} = d_{eff} E^2$, where $d_{eff}$ is the effective second order nonlinear coefficient. Consider now a non-degenerate wave mixing process where the propagation of two signals with different frequencies $\omega_a \neq \omega_b$, as well as a strong pump beam $\omega_p$, related through the relation $\omega_p = \omega_a + \omega_b$ is considered. Consider solutions of the form $E(t) = \sum_{j=1}^{3} u_j(z) e^{i(\omega_j t - k_j x)} + c.c.,$ where $\ell = a, b, c$, the transverse field profiles $c_r(x)$ represent the waveguide eigenmodes, the functions $u_r(z)$ are the slowly varying envelopes of each wave component, and $k_r = k_0 n_r \omega_r$ is the associated wavevector. By ignoring waveguide losses, and under the slowly varying envelope approximation, the coupled mode equations can be written as [33]:

$$\frac{d u_a}{d z} = - \frac{i \omega_a c_0}{2 n_a^2} d_{eff} u_a u_b^* e^{-i(k_b - k_a)z}, \tag{1a}$$

$$\frac{d u_b}{d z} = - \frac{i \omega_b c_0}{2 n_b^2} d_{eff} u_a u_b^* e^{-i(k_b - k_a)z}, \tag{1b}$$

$$\frac{d u_c}{d z} = - \frac{i \omega_c c_0}{2 n_c^2} d_{eff} u_a u_b e^{i(k_b - k_c)z}. \tag{1c}$$

In the undepleted pump regime $d u_a / dz \approx 0$, by defining $a(z) = \sqrt{n_a / \omega_a} u_a(z)$ and $b(z) = \sqrt{n_b / \omega_b} u_b(z)$, one can write $\frac{d a}{d z} = -i \frac{\beta_2}{2} a e^{-i\Delta k z}$ and $\frac{d b}{d z} = -i \frac{\beta_2}{2} b e^{-i\Delta k z}$, where $\beta_2 = d_{eff} \frac{2 n_b}{\sqrt{n_a n_b}} u_a$ and $\Delta k = k_c - k_b - k_a$. By using the gauge transformation $(a, b) \to (a, b) e^{-i\frac{\Delta k z}{2}}$, the evolution equations can be simplified into
which can be written as

\[
\frac{\mathrm{d}a}{\mathrm{d}z} = +i \frac{\Delta k}{2} a - i \frac{g}{2} b^* ,
\]

(2a)

\[
\frac{\mathrm{d}b}{\mathrm{d}z} = +i \frac{\Delta k}{2} b - i \frac{g}{2} a^* ,
\]

(2b)

The Hamiltonian \( \mathcal{H} \) can be written in terms of parity and time operators:

\[
\mathcal{H} = \frac{\Delta k}{2} \mathcal{J} - \frac{g}{2} \mathcal{P} \mathcal{T} ,
\]

(4)

where \( \mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is the identity operator, \( \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) represents the parity operator, and the time operator \( \mathcal{T} \) is an anti-linear operator that performs a complex conjugation operation. It is straightforward to show that the Hamiltonian (4) commutes with the \( \mathcal{P} \mathcal{T} \) operator since \([ \mathcal{H}, \mathcal{P} \mathcal{T} ] = \frac{\Delta k}{2} \mathcal{I} [ \mathcal{I}, \mathcal{P} \mathcal{T} ] - \frac{g}{2} \mathcal{P} \mathcal{T} \mathcal{P} \mathcal{T} = 0 \). As a result, the necessary condition for PT-symmetry is satisfied for the two wave components at frequency \( \omega_a \) and \( \omega_b \).

Considering the complex conjugate of the second equation, the evolution equation (3) can be written in terms of a linear operator as

\[
\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} a \\ b^* \end{pmatrix} = i \mathcal{L} \begin{pmatrix} a \\ b^* \end{pmatrix} ,
\]

(5)

where the Hamiltonian

\[
\mathcal{L} = \begin{pmatrix} \frac{\Delta k}{2} & - \frac{g}{2} \\ \frac{g}{2} & - \frac{\Delta k}{2} \end{pmatrix}
\]

is non-Hermitian. However, given that the relation \( \mathcal{L}^\dagger = \eta^{-1} \mathcal{L} \eta \) holds for \( \eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \mathcal{L} \) is pseudo-Hermitian and can therefore exhibit real eigenvalues [4]. Interestingly, by defining a unitary operator

\[
\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} ,
\]

via a similarity transformation \( \mathcal{L}' = \mathcal{U} \mathcal{L} \mathcal{U}^{-1} \), the Hamiltonian \( \mathcal{L} \) can be transformed into a linear PT-symmetric Hamiltonian \( \mathcal{L}' \) which is isospectral with the original Hamiltonian. The operator \( \mathcal{L}' \) acts on the new variables \( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathcal{U} \begin{pmatrix} a \\ b^* \end{pmatrix} \), i.e., \( \alpha = \frac{1}{\sqrt{2}} (a + ib^*) \) and \( \beta = \frac{1}{\sqrt{2}} (a - ib^*) \), as:

\[
\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = i \mathcal{L}' \begin{pmatrix} \alpha \\ \beta \end{pmatrix} ,
\]

(7)

while the transformed Hamiltonian is found to be

\[
\mathcal{L}' = \frac{1}{2} \begin{pmatrix} +ig & \Delta k \\ \Delta k & -ig \end{pmatrix} .
\]

(8)

As schematically depicted in figure 1, the propagation of these two waves, with frequencies \( \omega_a \) and \( \omega_b \), in the nonlinear waveguide channel can be viewed as two mixed modes \( \alpha \) and \( \beta \) propagating in two linear channels with balanced loss and gain. One channel involves parametric gain (of strength \( +g/2 \)) and the other channel has the same amount of parametric loss (\( -g/2 \)). Quite interestingly, the phase mismatch \( \Delta k/2 \) plays the role of coupling between the two modes.
The eigenmodes of this system can be simply obtained by assuming eigensolutions of the form
\[ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{iQz}, \]
leading to
\[ Q = \pm \frac{1}{2} \sqrt{(\Delta k)^2 - g^2}. \]

This relation clearly shows the presence of an exceptional point at \( \Delta k = g \), at which the system undergoes a phase transition from unitary propagation to broken symmetry regime. After this point, one mode amplifies and the other mode attenuates parametrically. Figure 2 depicts the eigenvalues of the system as a function of the gain/loss parameter \( g/2 \), while the associated eigenstates are visualized on a Bloch sphere.

It is worth noting that, as opposed to many nonlinear processes in which phase mismatch is undesirable, here its presence is essential to guarantee operation in the PT phase regime. In addition, given that the phase mismatch can in principle be either positive or negative, the effective coupling of the nonlinearity-induced PT coupler can take both signs.

3. Coupled nonlinear waveguides with in-phase pumps

Consider now an array of evanescently coupled waveguides each exhibiting second order nonlinearity (see figure 3(a)). All waveguides are assumed to be identical and coherently excited with in-phase pump beams. In this case, each wave with frequency \( \omega_{a,b} \) can be evanescently coupled with waves of similar frequencies in adjacent channels, and their evolution can be described with coupled mode equations:

\[ \frac{da_n}{dz} = +i\frac{\Delta k}{2} a_n - \frac{i}{2} b_n^* + i\frac{\kappa}{2} (a_{n-1} + a_{n+1}), \]  
\[ \frac{db_n}{dz} = +i\frac{\Delta k}{2} b_n - \frac{i}{2} a_n^* + i\frac{\kappa}{2} (b_{n-1} + b_{n+1}). \]

In these relations, \( a_n \) and \( b_n \) represent the slowly varying envelopes of frequency components \( a \) and \( b \) in the \( n \)th waveguide respectively, and \( \kappa/2 \) is the strength of coupling between two adjacent channels. Given that the pump beams are operating at much higher frequencies, one can ignore the coupling of pump beams in adjacent waveguides.
channels. Thus, we can assume that each pump beam remains in its own channel. In addition, in writing equations (10) we have assumed that the coupling coefficient for coupling between waves of frequency $\omega_n$ is the same as $\omega_k$, which is a valid approximation as long as the two frequencies are close.

As in the case of a single waveguide channel, here we define two mixed modes $\alpha_n = \frac{1}{\sqrt{2}}(a_n + ib_n^*)$ and $\beta_n = \frac{1}{\sqrt{2}}(a_n - ib_n^*)$, which are governed by the dynamical equations:

\[
\frac{d}{dz}\alpha_n = \frac{i}{2} \Delta k \beta_n - \frac{g}{2} \alpha_n + \frac{i}{2} \kappa(\beta_{n-1} + \beta_{n+1}), \quad (11a)
\]

\[
\frac{d}{dz}\beta_n = \frac{i}{2} \Delta k \alpha_n + \frac{g}{2} \beta_n + \frac{i}{2} \kappa(\alpha_{n-1} + \alpha_{n+1}). \quad (11b)
\]

These equations govern the evolution of light in a locally PT-symmetric lattice, as shown in figure 3(b) [17, 23, 37, 38]. As clearly seen in this figure, while each unit cell is PT-symmetric, gain and loss are interchanged in every cell in a way that each gain (loss) channel is evanescently coupled to the loss (gain) channels of the neighboring cells.

In order to find the band structure of an infinite lattice described by equations (11), consider a discrete plane wave ansatz of the form

\[
\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\mu z}, \quad (12)
\]

where $\sigma$ and $\mu$ represent the transverse Bloch momentum and propagation constant respectively. Equations (11) and (12) can be used to find the dispersion diagram as follows:

\[
\mu = \pm \frac{1}{2} \sqrt{(\Delta k + 2\kappa \cos \sigma)^2 - g^2}. \quad (13)
\]

The peculiar band structure of equation (13) exhibits interesting properties, which are completely different from the standard tight-binding PT-symmetric lattices [39]. Before analyzing the band structure of the PT lattice, consider first the corresponding Hermitian system obtained for $g = 0$. In this case, the dispersion diagram involves two bands described by $\mu = \pm (\Delta k/2 + \kappa \cos \sigma)$. Based on the involved parameters, this band diagram can be categorized in two different cases: (a) for $|\Delta k| < 2\kappa$, the two bands intersect and there is no band gap, (b) for $|\Delta k| > 2\kappa$, the two bands are separated by a forbidden band. Note that for $\Delta k > 0$ the maxima and minima occur at the center ($\sigma = 0$), while for $\Delta k < 0$ the maxima and minima happen to be at the band edges ($\sigma = \pm \pi$). The presence or absence of a bandgap, however, is independent from the sign of the phase mismatch $\Delta k$. The photonic bandgap of the Hermitian system is an important property of the band structure, as it is closely related to the PT-symmetry breaking threshold [40, 41]. In fact, in the absence of a bandgap, a small value of gain can break PT-symmetry and thus create a partially complex band. In the presence of the gap, on the other hand, the PT system will have a finite symmetry breaking threshold. As a result, in order to fulfill the requirement of unbroken PT-symmetry regime, one should necessarily impose the condition of $|\Delta k| < 2\kappa$. It follows that, as in the case of a single nonlinear channel, here the presence of the phase mismatch $\Delta k$ is crucial to ensure the condition of unbroken PT-symmetry. Based on relation (13), the band structure is entirely real as long as the following conditions hold simultaneously:

\[
|\Delta k| > 2\kappa, \quad g < \min(|\Delta k| \pm 2\kappa). \quad (14)
\]

By increasing the pump power, the band gap gradually decreases, until it closes completely at the so-called exceptional point, which is located either at the center of the band $\sigma = 0$, or at the edges $\sigma = \pm \pi$, depending on the sign of the phase mismatch $\Delta k$. By further increasing the pump power, imaginary pieces appear in the band structure for certain range of the transverse Bloch momentum $\sigma$, and the band structure becomes partially complex. Figures 4(a)–(d) depict the band structure for different sets of parameters.

4. Coupled nonlinear waveguides with out-of-phase pumps

It should be noted that, in writing equations (11), it is assumed that the pump beams are applied to all channels with the same initial phase. As we discuss in this section, however, even though the initial phase of the pump beam does not play a role in the dynamics of a single waveguide, it can completely alter the behavior of the coupled waveguide scenario when out-of-phase pump beams are applied to different waveguide channels. In general, assuming that the pump beam applied to the $n$th channel is initially modulated with a phase $\phi_n$, the evolution equations can be modified as:

\[
\frac{da_n}{dz} = \frac{i}{2} \Delta k a_n - \frac{g}{2} e^{i\phi_n} b_n^* + \frac{i}{2} \kappa(a_{n-1} + a_{n+1}), \quad (15a)
\]

\[
\frac{db_n}{dz} = \frac{i}{2} \Delta k b_n + \frac{g}{2} a_n + \frac{i}{2} \kappa(a_{n-1} + a_{n+1}). \quad (15b)
\]
Here, we consider a particular case in which the pump phase of each waveguide channel is $\pi$ shifted with respect to its adjacent channels (see figure 5(a)). In this case, the evolution equations for even labeled channels is given by:

$$\frac{db_n}{dz} = +i\Delta k b_n - \frac{g}{2} e^{i\phi_n} a_n^+ + i\frac{\kappa}{2} (b_{n-1} + b_{n+1}).$$

(15b)

Figure 4. Band structure of the diatomic PT lattice of figure 3 for different set of parameters: (a) $\Delta k < 0, \kappa/|\Delta k| = 1/3$, (b) $\Delta k < 0, \kappa/|\Delta k| = 1/3$, (c) $\Delta k > 0, \kappa/|\Delta k| = 2/3$, and (d) $\Delta k < 0, \kappa/|\Delta k| = 2/3$. In all cases, black, blue, red and green colors are associated with $g/|\Delta k| = 0, 0.8/3, 1/3$, and $2/3$ respectively while the solid and dashed lines represent the real and imaginary parts of the band.

Figure 5. (a) A coupled arrangement of nonlinear optical waveguides where the pump beam applied to every other channel is initially modulated by $\pi$. (b) The equivalent PT-symmetric lattice. (c) and (f) The band structure for $g/\Delta k = 0, 0.8, 1$ and 1.5 respectively. In all cases $\Delta k > 0$ and $\kappa/\Delta k = 1/3$ while solid and dash represent the real and imaginary parts of the band.
\[ \frac{da_{2m}}{dz} = +i \frac{\Delta k}{2} a_{2m} - i g \frac{g}{2} b_{2m} + i \frac{\kappa}{2} (a_{2m-1} + a_{2m+1}), \quad (16a) \]

\[ \frac{db_{2m}}{dz} = +i \frac{\Delta k}{2} b_{2m} - i g \frac{g}{2} a_{2m} + i \frac{\kappa}{2} (b_{2m-1} + b_{2m+1}), \quad (16b) \]

while for odd channels:

\[ \frac{da_{2m+1}}{dz} = +i \frac{\Delta k}{2} a_{2m+1} + i g \frac{g}{2} b_{2m+1} + i \frac{\kappa}{2} (a_{2m} + a_{2m+2}), \quad (17a) \]

\[ \frac{db_{2m+1}}{dz} = +i \frac{\Delta k}{2} b_{2m+1} + i g \frac{g}{2} a_{2m+1} + i \frac{\kappa}{2} (b_{2m} + b_{2m+2}). \quad (17b) \]

In this case, by defining \( \alpha_{2m} = \frac{1}{\sqrt{2}}(a_{2m} + ib_{2m}) \), \( \beta_{2m} = \frac{1}{\sqrt{2}}(a_{2m} - ib_{2m}) \), \( \alpha_{2m+1} = \frac{1}{\sqrt{2}}(a_{2m+1} - ib_{2m+1}) \) and \( \beta_{2m+1} = \frac{1}{\sqrt{2}}(a_{2m+1} + ib_{2m+1}) \), equations (16) and (17) can be combined in a single set of coupled equations:

\[ \frac{d}{dz} \alpha_n = i \frac{\Delta k}{2} \beta_n - g \frac{g}{2} \alpha_n + i \frac{\kappa}{2} (\alpha_{n-1} + \alpha_{n+1}), \quad (18a) \]

\[ \frac{d}{dz} \beta_n = i \frac{\Delta k}{2} \alpha_n + g \frac{g}{2} \beta_n + i \frac{\kappa}{2} (\beta_{n-1} + \beta_{n+1}), \quad (18b) \]

which represents a globally PT-symmetric lattice, as shown in figure 5(b). In this case, by using the same ansatz as equation (12), the band diagram is obtained as

\[ \mu = \kappa \cos \sigma \pm \frac{1}{2} \sqrt{(\Delta k)^2 - g^2}. \quad (19) \]

Here, quite interestingly, the PT-symmetry breaking threshold is dictated solely by the wavevector mismatch \( \Delta k \) and the gain/loss coefficient \( g \). The periodicity thus seems to not interplay with PT-symmetry, given that the effective gain/loss distribution and periodicity of the structure are extended along orthogonal directions. As a result, the symmetry-breaking threshold is independent from the transverse Bloch momentum \( \sigma \). The band structure is shown in figures 5(c)–(f) for different sets of parameters. As shown in the figure, below PT-symmetry breaking threshold, this system exhibits two similar real bands separated vertically by \( \sqrt{(\Delta k)^2 - g^2} \). At the symmetry breaking threshold \( g = |\Delta k| \), the two bands overlay on top of each other and the bi-layer lattice thus effectively behaves like a single lattice. Finally for \( g > |\Delta k| \), while the real part of the band remains the same, a constant imaginary part emerges for the entire range of \(-\pi < \sigma < \pi\).

5. Conclusions

Here we have shown that multi-wave mixing in nonlinear optics can provide a fertile ground to realize and explore non-Hermitian, and in particular PT-symmetric Hamiltonians, without the need for material gain. As opposed to most previous works on PT-symmetry in optics, which discuss PT-symmetry in spatially distributed continuous systems, our scheme is based on a PT-symmetric interplay of discrete frequency harmonics. We showed the equivalence of parametric amplification in a three-wave mixing process and PT-symmetry breaking in a PT coupler. The results were also generalized to a coupled nonlinear waveguide scenario, showing that, by modulating the phase of the pump beams, different PT lattices can be realized in a dynamic fashion. In all cases, the phase mismatch plays an important role in fulfilling the requirements for unbroken PT-symmetry.

Although, our results were presented based on a guided wave structure with second order nonlinearity, similar approaches can be utilized to envision PT-symmetric Hamiltonians in other structures and to form PT-symmetric metamaterials. Other types of nonlinear mechanisms, including four-wave mixing, may also be envisioned. In addition, the resulting Hamiltonians offer a high degree of flexibility, owing to the fact that both the effective coupling and gain/loss can be externally controlled via the pump beam power levels, as well as the frequency of the mixing beams, while the pump phases can be modulated to achieve different lattice geometries. Given that the nonlinearity-induced effective gain and loss processes are parametric, the results presented here are expected to provide low noise and a high degree of coherence, opening to the possibility of exploring PT-symmetric scenarios down to the few photon level, as opposed to previous experimental works on PT-symmetry based on bulk gain and loss mechanisms, which are expected to be limited to classical systems. Finally, our scheme can be utilized in nonlinear multi-wave mixing processes in order to engineer the associated parametric mechanisms in a dynamic and reconfigurable fashion.
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