Provably robust verification of dissipativity properties from data

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Abstract—Dissipativity properties have proven to be very valuable for systems analysis and controller design. With the rising amount of available data, there has therefore been an increasing interest in determining dissipativity properties from (measured) trajectories directly, while an explicit model of the system remains undisclosed. Most existing approaches for data-driven dissipativity, however, guarantee the dissipativity condition only over a finite time horizon and provide weak or no guarantees on robustness in the presence of noise. In this paper, we present a framework for verifying dissipativity properties from measured data with desirable guarantees. We first consider the case of input-state measurements, where we provide non-conservative and computationally attractive conditions even in the presence of noise. We then extend this approach to input-output data, where similar results hold in the noise-free case.

Index Terms—Data-based systems analysis, Dissipativity, Learning, Optimization, Linear Systems, Identification for Control

I. INTRODUCTION

With the rising complexity of systems, obtaining a suitable mathematical model for a yet unknown system becomes more and more cumbersome. At the same time, data is becoming ubiquitous and cheap. Therefore, there has been a rising interest in establishing a data-driven framework that allows for systems analysis and control from data with the same guarantees as obtained through the well-known and established model-based approach. Especially for linear time-invariant (LTI) systems, there has recently been considerable progress in setting up such a data-driven framework. The basis for this line of work can be attributed to the seminal work in [1], in which the authors prove in the behavioral framework that the behavior of an LTI system can be described by suitable data-dependent matrices under the condition that the input is persistently exciting. This representation in state-space as stated and discussed in [2] and proven in [3], provides a basis that allows for systems analysis and controller design with rigorous guarantees on the basis of (measured) trajectories. Recent developments in this direction include state-feedback design from input-state trajectories [4], robust controller synthesis from noisy input-state trajectories [5], data-driven model predictive control [6, 7], data informativity [8], dissipativity properties from input-output trajectories [9, 10, 11] and from input-state trajectories [12].

As in [9, 10, 11, 12], we are interested in dissipativity properties from data. Dissipativity properties cannot only be used for systems analysis giving insights to an unknown system, but knowledge of dissipativity properties allows for direct application of well-known feedback theorems with guaranteed stability of the closed loop. For examples of such feedback theorems and stabilizing, robust or distributed controller design on the basis of dissipativity properties, the reader is referred to the standard literature with respect to dissipativity properties, which includes [13, 14, 15]. Due to the well-established literature on dissipativity-based controller design, there has been a considerable number of approaches to determine such dissipativity properties from data.

Very generally, the literature on data-driven dissipativity can be roughly categorized into three inherently different setups. Firstly, a large number of approaches consider online sampling schemes for LTI systems, where it is assumed that it is possible to choose the input and measure the output in an iterative fashion. This line of works includes [16, 17, 18, 19, 20]. While there are some distinct advantages and disadvantages to each of these methods, the joint limitation is that iterative experiments are needed, which requires access to the plant and is potentially a more time-consuming task than purely computational and offline approaches. We define the second category as approaches that apply for rather general classes of nonlinear systems, but require large or even huge amounts of input-output trajectories (e.g. [21], [22], [23], [24], [25]). While these works consider more general nonlinear systems, the sheer amount of required data hampers their application.

Finally, the third category includes all offline computational approaches from one input-state or input-output trajectory for LTI systems, which includes [9, 10, 11, 12, 26]. The approaches in [9, 10, 11, 26] do not provide guarantees from noisy trajectories. Therefore, we extend in this work the idea presented in [11], where rigorous and quantitative guarantees from noise-corrupted input-state trajectories can be given. However, the herein presented result are generally not tight and also the computational complexity grows with increasing amounts of data. In this paper, we employ ideas similar to recent results on data-driven controller design in [27] in order to derive both non-conservative and computationally attractive conditions for data-driven dissipativity.

The remainder of the paper is structured as follows. In Sec. II we introduce the problem formulation and present some related results that will be used throughout the paper. We then introduce an equivalent dissipativity characterization
purely on the basis of input-state data in Sec. III followed by a noisy consideration thereof in Sec. IV where we provide a tight robust verification framework for dissipativity properties. Finally, we extend the results to input-output trajectories in the noise-free case in Sec. V.

II. PROBLEM SETUP

We consider multiple-input multiple-output discrete-time LTI systems for which there exists a (controllable) minimal realization of the form

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k, \quad x_0 = \bar{x}, \\
    y_k &= Cx_k + Du_k,
\end{align*}
\]

with \( x_k \in \mathbb{R}^n \), \( u_k \in \mathbb{R}^m \) and \( y_k \in \mathbb{R}^p \).

In this paper, we develop a framework for verifying dissipativity properties of (1) directly from measured data, without identifying a model of the system. We consider two problem setups:

- **Input-state data** (Sec. 3 & 5): We assume that \( A \) and \( B \) are unknown, but one input-state trajectory \( \{x_k\}_{k=0}^N \), \( \{u_k\}_{k=0}^{N-1} \) is available. Further, we assume that \( C, D \) are known.

- **Input-output data** (Sec. 5): We assume that \( A, B, C \) and \( D \) are unknown, but one input-output trajectory \( \{u_k\}_{k=0}^{N-1} \), \( \{y_k\}_{k=0}^{N-1} \) is available as well as an upper bound on the lag of the system \( l \geq \ell \) (cf. Def. 15).

Furthermore, we distinguish between noise-free measurements of (1) (Sec. 3 & 5) and the noisy case (Sec. 4). We collect the respective data sequences \( \{u_k\}_{k=0}^{N-1} \), \( \{x_k\}_{k=0}^N \) or \( \{y_k\}_{k=0}^{N-1} \) in the following matrices

\[
\begin{align*}
    X &:= \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}, \\
    X_k &:= \begin{pmatrix} x_k \end{pmatrix}, \\
    U &:= \begin{pmatrix} u_0 & u_1 & \cdots & u_{N-1} \end{pmatrix}, \\
    Y &:= \begin{pmatrix} y_0 & y_1 & \cdots & y_{N-1} \end{pmatrix}.
\end{align*}
\]

Our approach is based on data, i.e., on measured trajectories of the system (1), with the only assumption that this measured trajectory is informative enough. One condition in this respect, which will play an important role in the following section, is the rank condition

\[
\text{rank} \begin{pmatrix} X \\ U \end{pmatrix} = n + m. \tag{2}
\]

Generally speaking, this condition can be ensured by requiring that the input of the measured trajectory is sufficiently persistently exciting. Given a finite sequence \( \{u_k\}_{k=0}^{N-1} \), we first define the corresponding Hankel matrix

\[
H_L(u) := \begin{bmatrix}
    u_0 & u_1 & \cdots & u_{N-L} \\
    u_1 & u_2 & \cdots & u_{N-L+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{L-1} & u_L & \cdots & u_{N-1}
\end{bmatrix}.
\]

We can now recall the notion of persistency of excitation.

\[\text{Definition 1.} \quad \text{We say that a sequence} \ \{u_k\}_{k=0}^{N-1} \ \text{with} \ u_k \in \mathbb{R}^m \ \text{is persistently exciting of order} \ L, \ \text{if} \ \text{rank}(H_L(u)) = mL.\]

With this definition of persistency of excitation, we can find a sufficient condition to ensure (2).

\[\text{Lemma 2} \ (\text{Corollary 2}). \quad \text{If the sequence} \ \{u_k\}_{k=0}^{N-1} \ \text{with} \ u_k \in \mathbb{R}^m \ \text{is persistently exciting of order} \ n+1, \ \text{then condition} \ (2) \ \text{holds.}\]

Since their introduction in [28], dissipativity properties have become increasingly relevant in systems analysis and control. Usually, these properties can be verified using a full mathematical model of the system. In this paper, we are interested in determining dissipativity properties directly from (noisy) data with guarantees. While the notion of dissipativity was introduced in [28] for general (nonlinear) systems, we make use of equivalent formulations for LTI systems with quadratic supply rates as, e.g., presented in [29]. Quadratic supply rates are functions \( s : \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R} \) defined by

\[
s(u, y) = \begin{pmatrix} u \end{pmatrix}^\top \Pi \begin{pmatrix} u \\ y \end{pmatrix}, \tag{3}
\]

The matrix \( \Pi \in \mathbb{R}^{(m+p) \times (m+p)} \) will be partitioned throughout this paper as

\[
\Pi = \begin{pmatrix} R & S^\top \\ S & Q \end{pmatrix}.
\]

with \( Q = Q^\top \in \mathbb{R}^{p \times p}, \ S \in \mathbb{R}^{p \times m} \) and \( R = R^\top \in \mathbb{R}^{m \times m} \).

\[\text{Definition 3.} \quad \text{A system (1) is said to be dissipative w.r.t. the supply rate} \ s \ \text{if there exists a function} \ V : \mathbb{R}^n \rightarrow \mathbb{R} \ \text{which is bounded from below such that} \]

\[
V(x_{k'}) - V(x_{k''}) \leq \sum_{k=k'}^{k''-1} s(u_k, y_k) \tag{4}
\]

for all \( 0 \leq k' < k'' \) and all signals \( (u, x, y) \) which satisfy (1). It is said to be strictly dissipative if instead of (4)

\[
V(x_{k'}) - V(x_{k''}) \leq \sum_{k=k'}^{k''-1} s(u_k, y_k) - \epsilon^2 \sum_{k=k'}^{k''-1} \|u_k\|^2_2
\]

holds for all \( 0 \leq k' < k'' \), all signals \( (u, x, y) \) which satisfy (1) and some \( \epsilon > 0 \).

Hereby, the matrices \( (Q, S, R) \) in the supply rate define the system property at hand. With the supply rates defined by

\[
\Pi_\gamma = \begin{pmatrix} \gamma I & 0 \\ 0 & -I \end{pmatrix}, \quad \Pi_p = \begin{pmatrix} 0 & I \\ I & \rho I \end{pmatrix},
\]

to name two well-known examples, we retrieve the operator gain \( \gamma \) and the output-feedback passivity parameter \( \rho \), respectively. The general dissipativity property specified by \( (Q, S, R) \) will in the following also be referred to as \( (Q, S, R) \)-dissipativity.

In the remainder of the paper, we make use of different equivalent conditions on dissipativity of an LTI system. The following standard result together with explanations and the proofs can be found, e.g. in [29], [30] and references therein.
Theorem 4. Suppose that the system \((1)\) is controllable and let \(s\) be a quadratic supply rate of the form \((3)\). Then the following statements are equivalent.

(a) The system is dissipative with respect to the supply rate \(s\).

(b) There exists a quadratic storage function \(V(x) := x^TPx\) with \(P = P^T \geq 0\) such that
\[
V(x_{k+1}) - V(x_k) \leq s(u_k, y_k)
\]
for all \(k\) and all \((u, x, y)\) satisfying \((1)\).

(c) There exists a matrix \(P = P^T \geq 0\) such that
\[
\begin{pmatrix} (A^TPA - P - \hat{Q}) & (A^TPB - \hat{S}) \\ (A^TPB - \hat{S})^T & -\hat{R} + B^TPB \end{pmatrix} \leq 0
\]
with \(\hat{Q} = C^TQC\), \(\hat{S} = C^TS + C^TQD\) and \(\hat{R} = D^TQD + (D^TS + S^TD) + R\).

Remark 5. The attentive reader might have noticed that, unlike in [29], we require the storage function \(V\) to be lower bounded, which yields the condition \(P = P^T \geq 0\). Generally speaking, dissipativity as in [29] can be defined without requiring a lower bounded storage function. However, a key motivation of inferring dissipativity properties from data (and hence a key motivation of the present paper) is to use such dissipativity properties in order to design controllers, e.g., for closed-loop stability. In this case, it is meaningful to only consider lower bounded storage functions, similar to much of the related literature (cf. e.g. [37]). All results in this paper can be directly extended to using data to verify "cyclo-dissipativity", compare [31], in which case the storage function does not need to be bounded from below (i.e., \(P \not\geq 0\) in Theorem 4). Similarly, if a positive definite storage function is desired, one can simply substitute \(P \geq 0\) by \(P > 0\) in Theorem 4.

Other approaches to determine dissipativity from data rely on an input-output formulation of dissipativity (e.g. [10], [11], [9], [12]). To put this into perspective, the following result shows that this input-output definition is equivalent to Definition 3.

Theorem 6 (12). A system \((1)\) is dissipative w.r.t. the supply rate \(s\) in \((3)\) according to Definition 3 if and only if
\[
\sum_{k=0}^{r} s(u_k, y_k) \geq 0, \quad \forall r \geq 0,
\]
for all trajectories \(\{u_k, y_k\}_{k=0}^{\infty}\) of \(G\) with initial condition \(x_0 = 0\).

While this result shows the equivalence of the state-space definition of dissipativity and the input-output definition, in many other works where dissipativity is determined from data, dissipativity is only considered over a finite horizon. In these works (e.g. [9], [11], [16], [17], [19]), the condition \((6)\) is only verified over the horizon \(r\) for \(r \leq L\), which is also called \(L\)-dissipativity. Throughout this paper, we consider the classical definition of dissipativity as provided in Definition 3.

Furthermore, Thm. 6 also allows to infer dissipativity of systems which are not given in a minimal realizations by investigating dissipativity of a minimal realization with the same input-output behavior. Whenever two systems have the same input-output behavior (i.e., same span of input-output trajectories with zero initial condition), they satisfy the same condition \((6)\). This insight will be especially important in Sec. V when considering data-driven dissipativity from input-output data.

In the remainder of this paper, we use the equivalences stated in Theorem 4 and Theorem 6 to verify or find dissipativity properties from data. We start in the following section by considering noise-free input and state trajectories.

III. DATA-DRIVEN DISSIPATIVITY FROM INPUT-STATE TRAJECTORIES

With the definitions and analysis in the previous section, we can directly state an equivalent formulation for dissipativity from noise-free input and state trajectories. The necessary and sufficient condition is a simple LMI that can be solved using standard solvers.

Theorem 7 (12). Given input and state trajectories \(\{u_k\}_{k=0}^{N-1}, \{x_k\}_{k=0}^{N}\) of a controllable LTI system \(G\) and the feasibility problem to find \(P = P^T \geq 0\) such that
\[
X^T P X + X^T P X - \begin{pmatrix} U \\ CX + DU \end{pmatrix}^T \begin{pmatrix} R & S^T \\ S & Q \end{pmatrix} \begin{pmatrix} U \\ CX + DU \end{pmatrix} \leq 0.
\]

1) If there exists a \(P = P^T \geq 0\) such that \((1)\) holds and, additionally, the rank condition \((2)\) is satisfied, then \(G\) is \((Q, S, R)\)-dissipative.

2) If there exists no \(P = P^T \geq 0\) such that \((7)\) holds, then \((G, Q, S, R)\) is dissipative.

Proof. Substituting \(X_+ = AX + BU\), the semidefiniteness condition in \((7)\) can be equivalently written as
\[
X^T \begin{pmatrix} A^TPA - P - \hat{Q} \\ A^TPB - \hat{S} \end{pmatrix} (A^TPB - \hat{S})^T \leq -\hat{R} + B^TPB,
\]
with \(\hat{Q} = C^TQC\), \(\hat{S} = C^TS + C^TQD\) and \(\hat{R} = D^TQD + (D^TS + S^TD) + R\).

1) With \((2)\), the semidefiniteness condition \((8)\) in turn implies that \((5)\) holds, which implies dissipativity by Theorem 4.

2) If problem \((7)\) is infeasible, this directly implies that \((5)\) is not negative semidefinite for any \(P\), i.e. \(G\) is not dissipative by Theorem 4.

Remark 8. The condition \((2)\) can easily be checked for the available data. With Lemma 2, this rank condition can also be enforced by requiring or choosing the input \(\{u_k\}_{k=0}^{N-1}\) to be persistently exciting of order \(n+1\). Note that \(\{u_k\}_{k=0}^{N-1}\) being persistently exciting of order \(n+1\) requires a minimum length of the input trajectory, namely \(N \geq (n+1)n + m\).

The result stated in Theorem 7 is conceptionally similar to the approach in [38] where methods for data-based system analysis (e.g., controllability, stability) are provided by verifying such properties for all systems which are consistent with
the data. The data-based formulation of dissipativity given by Thm. 7 is particularly simple and only requires solving a single semidefinite program. The proof relies on the fact that, if the matrix \( \begin{pmatrix} X \\ U \end{pmatrix} \) has full row rank, then it spans all possible system trajectories. Multiplying (5) from both sides by this matrix and exploiting the system dynamics \( X_+ = AX + BU \), we obtain the stated result.

In comparison to other input-output approaches (e.g. [9], [11], [16]), we exploited the here the state-space definition of dissipativity which can be verified by looking at a difference viewpoint, i.e. looking at the difference at two time points (cf. Def. 5). This yields the advantage with respect to many other data-driven dissipativity approaches that guarantees on the infinite horizon can be given and also rigorous guarantees in the noisy case, as will be discussed in the next section.

IV. DISSIPATIVITY PROPERTIES FROM NOISY INPUT-STATE TRAJECTORIES

While Section III provides a simple, computationally attractive condition to verify dissipativity properties of unknown systems, it also assumes that exact measurements of input and state variables are available. This assumption does rarely hold in practice. Therefore, in this section, we extend the results to the case that the measured data are affected by noise. More precisely, we consider in this section LTI systems that are disturbed by process noise of the form

\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + B_w w_k, \\
y_k &= Cx_k + Du_k,
\end{align*}
\]

(9)

where \( w_k \in \mathbb{R}^{m_w} \) denotes the noise and \( B_w \) is some known matrix describing the influence of the noise on the system dynamics. We denote the actual noise sequence which yields the available input-state trajectory \( \{u_k\}_{k=0}^{N-1}, \{x_k\}_{k=0}^{N-1} \) by \( \{\bar{w}_k\}_{k=0}^{N-1} \). While this noise sequence \( \{\bar{w}_k\}_{k=0}^{N-1} \) is unknown, we assume that some information on the noise is available in form of a bound on the stacked matrix

\[
\bar{W} = (\bar{w}_0 \ \bar{w}_1 \ \cdots \ \bar{w}_{N-1})
\]

as specified in the following assumption

Assumption 9. The matrix \( \bar{W} \) denoting the stacked process noise \( \{\bar{w}_k\}_{k=0}^{N-1} \) is an element of the set

\[
W = \{ W \in \mathbb{R}^{m_w \times N} \mid \begin{pmatrix} W^T \\ I \end{pmatrix}^T \begin{pmatrix} Q_w & S_w \\ S_w^T & R_w \end{pmatrix} \begin{pmatrix} W^T \\ I \end{pmatrix} > 0 \} \tag{10}
\]

with \( Q_w \in \mathbb{R}^{N \times N}, \ S_w \in \mathbb{R}^{N \times m_w} \) and \( R_w \in \mathbb{R}^{m_w \times m_w} \) with \( Q_w \prec 0 \).

This quadratic bound on the noise matrix \( \bar{W} \) is a flexible noise or disturbance description. Similar bounds on the noise were also used, for example, in [5], [12], [27]. This quadratic matrix bound can incorporate bounds on sequences (\( \|\bar{w}\|_2 < \bar{w} \)) and bounds on separate components (\( \|w_k\|_2 < \bar{w} \) for all \( k \)), to name a few exemplary cases.

Due to the presence of noise, there generally exist multiple matrix pairs \( (A_d, B_d) \) which are consistent with the data for some noise sequence \( W \in W \). The set of all such matrix pairs consistent with the input-state data and the noise bound is in the following denoted by

\[
\Sigma_{X,U} = \{(A_d, B_d) \mid X_+ = A_d X + B_d U + B_w W, W \in W \}.
\]

By assumption, this set includes the system of interest \( (A, B) \) which generated the data.

To verify that a system (1) is indeed \((Q, S, R)\)-dissipative from noisy data, it is necessary to verify that all systems that are consistent with the data are \((Q, S, R)\)-dissipative. In the language of [8], we verify whether the data are informative for dissipativity. For this, we make use of an equivalent representation of the set \( \Sigma_{X,U} \) provided in [27]. This new equivalent representation of \( \Sigma_{X,U} \) is a key step for retrieving a tight condition on dissipativity and decreasing the conservatism with respect to the results in [12].

Lemma 10. It holds that

\[
\Sigma_{X,U} = \{(A_d, B_d) \mid \begin{pmatrix} A_d^T \\ B_d \\ I \end{pmatrix}^T \begin{pmatrix} \bar{Q}_w & \bar{S}_w & \bar{R}_w \end{pmatrix} \begin{pmatrix} A_d^T \\ B_d \\ I \end{pmatrix} > 0 \} \tag{11}
\]

with

\[
\bar{Q}_w = \begin{pmatrix} X \\ U \end{pmatrix}^T Q_w \begin{pmatrix} X \\ U \end{pmatrix}, \quad \bar{S}_w = \begin{pmatrix} X \\ U \end{pmatrix}^T (Q_w X_+^T + S_w B_w^T), \quad \bar{R}_w = X^T Q_w X_+^T + X^T S_w B_w^T + B_w S_w X_+ + B_w R_w B_w^T.
\]

Proof. This statement follows from [27] Lemma 4 and Remark 2.

Remark 11. Note that including a known matrix \( B_w \) into the analysis offers the possibility to include additional knowledge on the influence of the process noise on the system into the optimization problem. If no additional information on the effect of the noise on the different states is available, one can simply choose the identity matrix \( B_w = I \).

The equivalent formulation of \( \Sigma_{X,U} \) in Lemma 10, which is only based on data and the noise bound, allows us to rewrite the problem in a form such that we can directly apply robust analysis tools from the literature [29]. While a similar idea was already exploited in [12] to verify and find dissipativity properties from input-state data, only a superset of \( \Sigma_{X,U} \) could be considered, hence introducing conservatism. On the contrary, we improve this result in the following by providing tight conditions on dissipativity.

It follows from Lemma 10 that the set of all LTI systems consistent with the data can be written as

\[
x_{k+1} = \begin{pmatrix} A_d & B_d \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}
\]

for some \( (A_d, B_d) \in \Sigma_{X,U} \). We can equivalently reformulate this uncertain system as a linear fractional transformation
This concludes the proof.

Remark 13. Thm. 12 provides a powerful tool to verify dissipativity properties from noisy input-state measurements, based on a simple LMI (15). Compared to the approach presented in (12), Thm. 12 verifies dissipativity for a tight description of the systems consistent with the data and is thus considerably less conservative. Further reducing conservatism, e.g., by considering parameter-dependent storage functions, is an interesting issue for future research.

Remark 14. Since the definiteness condition in (15) is linear in the matrices \((Q, S, R)\), optimizing for specific dissipativity properties or over the matrices \(Q, S\) or \(R\) can be done via a simple SDP. This includes for example finding the operator gain \(\gamma\) via the SDP \(\min_{\gamma} \gamma^2\) such that (7) holds for \(R = \gamma^2 I\), \(S = 0\) and \(Q = -I\).

Theorem 12 can be seen as a counterpart to the data-driven controller design presented in (27), focusing on data-driven system analysis instead. As we discuss above, the result is powerful, being non-conservative and computationally simple. However, it also requires the availability of state measurements, which can be restrictive in practice, where often only input-output data are available. In the next section, we extend the results of Section III to an input-output setting.

V. DISSIPATIVITY FROM INPUT-OUTPUT TRAJECTORIES

Instead of input and state measurements, we consider in this section the case where only an input-output sequence \(\{u_k, y_k\}_{k=0}^{N-1}\) of (1) is available, and we use this sequence to verify dissipativity properties. We start by defining the lag of a system.

Definition 15. The lag \(l\) of system (1) is the smallest integer \(l \in \mathbb{N}_+\) such that the observability matrix of \(G\) given by

\[
O_l := \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{l-1} \end{pmatrix}
\]

has rank \(n\).

In this section, we use an extended state, based on \(l\) consecutive inputs and outputs, in order to verify dissipativity properties. The following lemma shows that this is in principle possible since the corresponding stacked system has the same input-output behavior as (1).

Lemma 16. Let \(l \geq l\). Then there exists a system \(\tilde{G}\) with matrices \(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\) which can explain the data \(\{u_k\}_{k=0}^{N-1}, \{y_k\}_{k=0}^{N-1}\), i.e., there exists \(\xi_0\) such that for \(k = 0, \ldots, N-1\),

\[
\xi_{k+1} = \tilde{A}\xi_k + \tilde{B}u_k, \quad y_k = \tilde{C}\xi_k + \tilde{D}u_k,
\]

where the extended state is defined by

\[
\xi_k := \begin{pmatrix} u_{k-l} & u_{k-l+1} & \cdots & u_{k-1} & y_{k-l} & y_{k-l+1} & \cdots & y_{k-1} \end{pmatrix}^T.
\]

We believe that this is a well-known fact, but we nevertheless add a proof in the appendix for completeness and to provide some intuition.

The converse of Lemma 16 follows trivially from its proof and the constructed extended system in (24): All input-output trajectories of the extended system (24) with zero initial condition \(\xi_0 = 0\) (or \(\xi_0 \in \mathcal{X}_\xi\), where \(\mathcal{X}_\xi\) denotes the set of reachable states) are also input-output trajectories of the system (1).

Similar to the results of this section, (4) uses such an extended state to design data-driven controllers but, instead of the lag \(l\), the system order \(n\) is used, which can lead to a significantly larger state dimension for MIMO systems. Furthermore, (4) assumes controllability of the extended state, which is generally not the case, unless \(l = n\) and SISO systems are considered.

From the proof of Lemma 16 we conclude that the extended system (24) has the same input-output behavior as \(G\) if the initial condition \(\xi_0\) is restricted to the set of reachable states. This implies that both systems have the same input-output behavior for zero initial condition \(\xi_0 = 0, x_0 = 0\). Together with Thm. 6 this in turn implies that dissipativity of the extended system (24) (if the initial condition is reachable) is equivalent to dissipativity of a minimal realization of \(G\), compare (1). Hence, using Lemma 16 we can reduce the problem of verifying dissipativity from input-output trajectories to the problem of verifying dissipativity from input-state trajectories of the potentially uncontrollable system (16).
Remark 17. In a purely data-driven setup, knowledge on the lag \( l \) is often not available. However, as shown above, it is sufficient for the purpose of this section to have an upper bound on \( l \) or even an upper bound an \( n \) since \( l \leq n \).

Before stating our main result of this section, we recall the Fundamental Lemma introduced in [1]:

Lemma 18 (Fundamental Lemma). Suppose \( \{u_k, y_k\}_{k=0}^{N-1} \) is a trajectory of a controllable LTI system \( G \), where \( u \) is persistently exciting of order \( l + n \). Then, \( \{\tilde{u}_k, \tilde{y}_k\}_{k=0}^{N-1} \) is a trajectory of \( G \) if and only if there exists \( \alpha \in \mathbb{R}^{N-l+1} \) such that

\[
\begin{bmatrix}
H_l(u) \\
H_l(y)
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\tilde{u} \\
\tilde{y}
\end{bmatrix} = 0.
\]

We now use the result of Lemma 16 together with the Fundamental Lemma above to determine dissipativity properties from input-output trajectories. For this, we collect the extended state data analogously to Sec. III in the following form

\[
\Xi := (\xi_l \quad \xi_{l+1} \quad \cdots \quad \xi_{N-1}),
\]

\[
\Xi_+ := (\xi_{l+1} \quad \xi_{l+2} \quad \cdots \quad \xi_N),
\]

which directly leads us to the main result of this section.

Theorem 19. Given an input-output trajectory \( \{u_k, y_k\}_{k=0}^{N-1} \) of a controllable LTI system \( G \) of the form (1) with lag \( l \). Let \( l \geq 1 \) and consider the feasibility problem to find \( P = P^T \geq 0 \) such that

\[
\Xi_+^T P \Xi_+ - \Xi^T P \Xi Y^T QY - Y^T SU - (SU)^T Y - U^T RU \leq 0. \tag{17}
\]

1) If there exists a \( P = P^T \geq 0 \) such that (17) holds and additionally \( \{u_k\}_{k=0}^{N-1} \) is persistently exciting of order \( n + l + 1 \), then \( G \) is \((Q, S, R)\)-dissipative.

2) If there exists no \( P = P^T \geq 0 \) such that (17) holds, then \( G \) is not \((Q, S, R)\)-dissipative.

Proof. 1) First, we notice that the data matrix \( \Xi \) can be written as a Hankel matrix of the form

\[
\Xi = \begin{pmatrix}
H_l(\{u_k\}_{k=0}^{N-2}) \\
H_l(\{y_k\}_{k=0}^{N-2})
\end{pmatrix}
\begin{pmatrix}
u_0 & u_1 & \cdots & u_{N-l-1} \\
u_1 & u_2 & \cdots & u_{N-l} \\
\vdots & \vdots & \ddots & \vdots \\
y_0 & y_1 & \cdots & y_{N-l-1} \\
y_1 & y_2 & \cdots & y_{N-l} \\
\vdots & \vdots & \ddots & \vdots \\
y_{l-1} & y_{l} & \cdots & y_{N-2}
\end{pmatrix}
\]

Since there exists a controllable realization (of order \( n \)) with the same input-output behavior as our extended system, the Fundamental Lemma implies that the image of \( \Xi \) spans the whole reachable state space of the extended system \( \chi_ξ \). More specifically, if \( \{u_k\}_{k=0}^{N-1} \) is persistently exciting of order \( n + l \), then Lemma 18 guarantees that the columns in \( \Xi \) span all possible input-output trajectories of the system \( G \), and hence the whole reachable state space of the extended system (16).

If \( \{u_k\}_{k=0}^{N-1} \) is persistently exciting of order \( n + l + 1 \), then it additionally holds that \( \begin{pmatrix} \Xi \\ U \end{pmatrix} \) spans the space of all input-state trajectories of (16).

Using \( \Xi_k = A \Xi + B U \) and rearranging (17), we obtain

\[
\begin{pmatrix} \Xi \\ U \end{pmatrix}^T \begin{pmatrix}
-\hat{P} A \hat{P} - \hat{Q} \\
-\hat{B} \hat{P} \hat{B} - \hat{S}
\end{pmatrix} \begin{pmatrix} \Xi \\ U \end{pmatrix} \leq 0. \tag{18}
\]

Since \( \begin{pmatrix} \Xi \\ U \end{pmatrix} \) spans the space of all input-state trajectories, this implies

\[
\begin{pmatrix} \xi_k \\ u_k \end{pmatrix}^T \begin{pmatrix}
-\hat{P} A \hat{P} - \hat{Q} \\
-\hat{B} \hat{P} \hat{B} - \hat{S}
\end{pmatrix} \begin{pmatrix} \xi_k \\ u_k \end{pmatrix} \leq 0 \tag{19}
\]

for all \( k \) and all trajectories \( (u, \xi) \) of the extended system (16) (with \( \xi_0 \in \chi_ξ \)). Hence, there exists a quadratically lower bounded storage function for the extended system (16). This implies that the system (16) is \((Q, S, R)\)-dissipative which in turn, using Theorem 6 and Lemma 16 implies that the system (1) is \((Q, S, R)\)-dissipative.

2) If system (1) is \((Q, S, R)\)-dissipative, then, according to Thm. 4, there exists a quadratic storage function \( V(x_k) = x_k^T P x_k \) such that

\[
x_{k+1}^T P x_{k+1} - x_k^T P x_k \leq s(u_k, y_k)
\]

holds for all \( k \) and all \((u, x, y)\) satisfying (1). From the proof of Lemma 16 we know that there exists a transformation matrix \( T \) such that \( x_{k-1} = T \xi_k \) holds for all reachable states \( \xi_k \) and all \( k \). Hence, the matrix \( P = T^T P T \geq 0 \) satisfies (19) for all \( k \) and all \((u, \xi)\) of the extended system (16). Using the Fundamental Lemma, this implies that (18) holds for the matrix \( \begin{pmatrix} \Xi \\ U \end{pmatrix} \) and thus there exists a \( P \geq 0 \) s.t. (17) holds.

Theorem 19 provides an equivalent formulation of dissipativity based on input-output data. The result itself and its proof are conceptually similar to the state measurements case in Theorem 7. A key challenge is that, in contrast to the matrix
the matrix \( \begin{pmatrix} X \\ U \end{pmatrix} \) does usually not have full row rank, even if the input is persistently exciting, since the system (16) is usually not controllable. However, the Fundamental Lemma implies that, assuming persistence of excitation of order \( l+n \), the matrix \( \Xi \) spans the space of all state trajectories of the extended system (16). Under the stronger assumption of persistence of excitation of order \( l+n+1 \), which we assume in Theorem 19 it even holds that \( \begin{pmatrix} \Xi \\ U \end{pmatrix} \) spans the space of all input-state trajectories of (16). Using this fact, it is then straightforward to derive (17), which provides an equivalent data-driven characterization of dissipativity.

VI. CONCLUSION AND OUTLOOK

In this work, we introduced simple verification methods of dissipativity properties with guarantees from (noisy) input-state and input-output data based on LMIs. While in [11] the general ideas were presented to determine dissipativity properties that hold over the infinite horizon from finite input and state trajectories, the guarantees given in this paper for noisy input-state data are non-conservative. Furthermore, we introduced an equivalent data-based dissipativity condition from noisy-free input-output data.

Ongoing work includes the problem of deriving conditions for dissipativity properties from noisy input-output data. Furthermore, we plan to investigate in numerical as well as experimental examples the applicability of the approach in practice. Finally, for future work, it might be interesting to investigate how existing works on data-driven descriptions of nonlinear systems (Hammerstein and Wiener systems, second-order Volterra systems, bilinear systems, and polynomial systems) as presented in [5], [34], [35], respectively, can be applied to verify and find dissipativity properties of unknown nonlinear systems from data.

APPENDIX

PROOF OF LEMMA [16]

Proof. The input-output behavior of the system \( G \) in (1) over \( l \) steps can be written as (20), which yields with the introduced matrix notation

\[
\begin{pmatrix} -R & I \end{pmatrix} \xi_k = O_l x_k.
\]

Using the definition of the lag, we know that \( O_l \) has full column rank, and hence, there exists a left-inverse \( O_l^{-1} \) (which has full row rank) such that

\[
\begin{pmatrix} -R & I \end{pmatrix} \xi_k = x_k.
\]

In general \( T \) not unique, but there exists a set of matrices \( T \in T \).

The general system description from (1) yields

\[
y_k = CA^l x_{k-l} + (CA^{l-1} B \ldots CB) \begin{pmatrix} u_{k-1} \\ \vdots \\ u_{k-l} \end{pmatrix}.
\]

Together with any linear transformation matrix \( T \in T \) from (22), this leads to (21). This proves that \( (A, B, C, D) \) can explain the input-output trajectory (for all \( T \in T \)).

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\[
\begin{pmatrix}
  y_{k-1} \\
y_{k-l+1} \\
  \vdots \\
y_{k-2} \\
y_{k-1}
\end{pmatrix} = \begin{pmatrix}
  C \\
  CA \\
  \vdots \\
  CA^{l-1}
\end{pmatrix} x_{k-l} + \begin{pmatrix}
  D & 0 & \cdots & 0 \\
  CB & D & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  C A^{l-2} B & C A^{l-3} B & \cdots & C A B & C B & D
\end{pmatrix} \begin{pmatrix}
  u_{k-l} \\
u_{k-l+1} \\
  \vdots \\
u_{k-2} \\
u_{k-1}
\end{pmatrix} 
\]

\[
\begin{pmatrix}
  u_{k-l+1} \\
u_{k-l+2} \\
  \vdots \\
u_{k} \\
y_{k-l+1} \\
y_{k-l+2} \\
  \vdots \\
y_{k}
\end{pmatrix} = \begin{pmatrix}
  0 & I & \cdots & 0 & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & \cdots & I & 0 & 0 & \cdots & 0 \\
  0 & 0 & \cdots & 0 & I & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots \\
  0 & 0 & \cdots & \cdots & \cdots & \cdots & I & \cdots \\
  C A^{l-1} B & \cdots & \cdots & C B & 0 & \cdots & 0 & 0
\end{pmatrix} \begin{pmatrix}
  0 \\
  \vdots \\
  0 \\
  0 \\
  \vdots \\
  0 \\
  0 \\
  0
\end{pmatrix}
\]

\[
y_k = \begin{pmatrix}
  0 & \cdots & 0 & I
\end{pmatrix} \dot{\xi}_k + \begin{pmatrix}
  D
\end{pmatrix} u_k
\]

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