DOUBLE GRAPH COMPLEX AND CHARACTERISTIC CLASSES OF FIBRATIONS

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Abstract. In this paper, we construct a double chain complex generated by certain graphs and a chain map from that to the Chevalley-Eilenberg double complex of the dgl of symplectic derivations on a free dgl. It is known that the target of the map is related to characteristic classes of fibrations. We can describe some characteristic classes of fibrations whose fiber is a 1-punctured even-dimensional manifold by linear combinations of graphs though the cohomology of the dgl of derivations.

1. Introduction

The Chevalley-Eilenberg complex of the limit of the Lie algebra of symplectic derivations on (graded) free Lie algebras is isomorphic to the graph complex defined by the cyclic Lie operad (details in [9, 10, 3, 4]). In this paper, we introduce an extension of (the dual of) the construction to a Lie algebra of symplectic derivations on free dglś. Let \((W, \omega)\) be a graded vector space with symmetric inner product of even degree \(N\) and \(\delta\) a differential of degree \(-1\) on the completed free Lie algebra \(\hat{L}W\) satisfying the symplectic condition \(\delta \omega = 0\). An important example is the case that \((\hat{L}W, \delta)\) is a Chen’s dgl model of an even dimensional manifold and \(\omega\) is its intersection form. We construct a \(W\)-labeled graph complex \(C^\bullet \com(W)\) and a chain map

\[ C^\bullet \com(W) \to C^\bullet \com(\Der^+_\omega(\hat{L}W)) \]

to the Chevalley-Eilenberg (double) complex \(C^\bullet \com(\Der^+_\omega(\hat{L}W))\) of the differential graded Lie algebra \((\Der^+_\omega(\hat{L}W), \text{ad}(\delta))\) of positive symplectic derivations on \(\hat{L}W\). Furthermore the non-labeled part \(C^\bullet \com(N, Z)\) of the graph complex, which depends on only the integer \(N\) and the set \(Z\) of degrees of \(W\), we can obtain a chain map

\[ C^\bullet \com(N, Z) \subset C^\bullet \com(W)^{\text{Sp}(W, \delta)} \to C^\bullet \com(\Der^+_\omega(\hat{L}W))^{\text{Sp}(W, \delta)}, \]

where \(\text{Sp}(W, \delta)\) is the group of graded linear isomorphisms of \(W\) preserving \(\omega\) and \(\delta\). In the case of \(N = 0\) and \(Z = \{0\}\), the map corresponds to the Kontsevich’s one [9, 10].

The construction above gives characteristic classes of fibrations. It is known that characteristic classes of simply-connected fibrations are related to Lie algebras of derivations [17, 18]. In non-simply connected cases, we got relations between characteristic classes and Lie algebras of derivations as in [14, 8]. In this paper, we consider the case that the boundary of a fiber is a sphere. For a simply-connected compact manifold \(X\) with \(\partial X = S^{n-1}\), let \(\text{aut}_\partial(X)\) be the monoid of self-homotopy equivalences of \(X\) fixing the boundary pointwisely and \(\text{aut}_{\partial,0}(X)\) its connected component containing \(\text{id}_X\). According to [1], the isomorphism

\[ H^\bullet(B \text{ aut}_{\partial,0}(X); \mathbb{Q}) \simeq H^\bullet \com(\Der^+_\omega(L_X)) \]
is obtained. Here $L_X$ is a cofibrant dgl model of $X$. The underlying Lie algebra of $L_X$ is generated by the linear dual $W$ of the suspension of the reduced cohomology of $X$. So the graph complex above gives the invariant part of the cohomology $H^*_C (\text{Der}^+_W (L_X))$ with respect to the action of the group $\text{Sp}(W, \delta)$ of automorphisms of $W$ with intersection form preserving the differential $\delta$ of $L_X$. Using the Serre spectral sequence for the fibration

$$B \text{aut} \delta_0 (X) \to B \text{aut} \delta (X) \to B \pi_0 (\text{aut} \delta) (X),$$

the image of the natural map $H^* (B \text{aut} \delta (X); \mathbb{Q}) \to H^* (B \text{aut} \delta_0 (X); \mathbb{Q})$ is included in the invariant part. We give a chain map

$$C^*_{\text{com}} (N, Z) \to C^*_{C \text{E}} (\text{Der}^+_W (L_X))^\text{Sp}(W, \delta),$$

by the construction above. Considering $W$-labeled graphs, we can also obtain a $W$-labeled version $C^*_{\text{com}} (W)$ and a chain map

$$C^*_{\text{com}} (W) \to C^*_{C \text{E}} (\text{Der}^+_W (L_X)).$$

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2. PRELIMINARY

In this paper, all vector spaces are over a field $K$ whose characteristic is zero. A field $K$ is regarded as a $\mathbb{Z}$-graded vector space whose all elements have degree 0.

For a finite set $U$, the number of elements in $U$ is denoted by $\# U$.

All tensor products of linear maps between $\mathbb{Z}$-graded vector spaces contain their signs: for homogeneous linear maps $f : A \to V$, $g : B \to W$ between $\mathbb{Z}$-graded vector spaces, we set

$$(f \otimes g) (a, b) := (-1)^{|a|} f(a) \otimes g(b)$$

for $a \in A$ and $b \in B$. (We often denote by $|a|$ the degree of an element $a$. But we omit the symbol $| \cdot |$ of the degree when it appears in a power of $-1$. For example, $(-1)^{|a|}$ means $(-1)^{|a|}$.)

Let $V$ be a $\mathbb{Z}$-graded vector space. We denote $V^i$ the subspace of elements of $V$ of cohomological degree $i$ and $V_i = V^{-i}$ the subspace of elements of homological degree $i$. Remark that the linear dual $V^* = \text{Hom}(V, \mathbb{R})$ of $V$ is graded by $(V^*)^i = \text{Hom}(V_i, \mathbb{R})$.

The $p$-fold suspension $V[p]$ of $V$ for an integer $p$ is defined by

$$V[p]^i := V^{i+p}$$

and elements of $V[p]^i$ are presented by $x \sigma$ for $x \in V^{i+p}$ using the symbol $\sigma$ of cohomological degree $-p$. The $p$-suspension map $V \to V[p]$ is also denoted by $\sigma$.

Let $V$ be a $\mathbb{Z}$-graded vector space and $\alpha : V \otimes V \to K$ be a non-degenerate bilinear map of (cohomological) degree $n$. Out of the two conditions

(i) $\alpha(x, y) = (-1)^{|x|} \alpha(y, x)$ for homogeneous elements $x, y \in V$, and

(ii) $\alpha(x, y) = (-1)^{|x|} \alpha(y, x)$ for homogeneous elements $x, y \in V$,

the pair $(V, \alpha)$ is called symmetric vector space with degree $n$ if satisfying (i), and symplectic vector space with degree $n$ if satisfying (ii).
2.1. Derivations. Let $W$ be a $\mathbb{Z}$-graded vector space. We denote the completed tensor algebra by

$$\hat{TW} := \prod_{r=0}^{\infty} W^{r \otimes r}.$$  

Its product $\mu$ and coproduct $\Delta$ are defined by

$$\mu(x_1 \otimes \cdots \otimes x_s, x_{s+1} \otimes \cdots \otimes x_r) = x_1 \otimes \cdots \otimes x_r,$$

$$\Delta(x_1 \otimes \cdots \otimes x_r) = \sum_{s=0}^{r} \sum_{\tau \in \text{Ush}(s, r-s)} \epsilon \cdot (x_{\tau(1)} \otimes \cdots \otimes x_{\tau(s)}) \otimes (x_{\tau(s+1)} \otimes \cdots \otimes x_{\tau(r)})$$

for homogeneous elements $x_1, \ldots, x_r \in W$, where $\text{Ush}(s, r-s)$ is the set of $(r, s-r)$-unshuffles and $\epsilon$ is the Koszul sign of the permutation $(x_1, \ldots, x_r) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(r)})$ (Definition 3.2). The primitive part of $\hat{LW}$ is the completed free Lie algebra $\hat{LW}$.

These algebras have the gradings defined by the grading of $W$.

Given a symplectic inner product $\omega$ of degree $N$ on $W$, we define the Lie algebra of symplectic derivations

$$\text{Der}_\omega(\hat{LW}) := \{ D \in \text{Der}(\hat{LW}); \ D(\omega) = 0 \}.$$  

Here $\omega$ is regarded as $\omega \in W^{\otimes 2}$ using non-degeneracy.

Let $\delta$ be an element in $\text{Der}_\omega(\hat{LW})$ of homological degree $-1$ such that $\delta^2 = 0$. Then $\text{ad}(\delta)$ is a differential operator on $\text{Der}_\omega(\hat{LW})$. Using the differential, we can obtain the positive truncation $\text{Der}_\omega^+(\hat{LW})$ of the chain complex $(\text{Der}_\omega(\hat{LW}), \text{ad}(\delta))$. Thus its homological grading is described as

$$\text{Der}_\omega^+(\hat{LW})_i := \begin{cases} 
\text{Der}_\omega(\hat{LW})_1 & (i > 2) \\
\text{Ker}(\text{ad}(\delta))_1 & (i = 1) \\
0 & \text{(otherwise)}.
\end{cases}$$

The differential $\text{ad}(\delta)$ induces the differential operator $L_\delta$ on the Chevalley-Eilenberg complex $C_{CE}^{\bullet \bullet}(\text{Der}_\omega^+(\hat{LW}))$ by the Leibniz rule with respect to the wedge product. Defining the linear map $i_\delta : C_{CE}^{p+1}(\text{Der}_\omega^+(\hat{LW})) \to C_{CE}^{p}(\text{Der}_\omega^+(\hat{LW}))$ by

$$(i_\delta)(D_1, \ldots, D_p) := c(\delta, D_1, \ldots, D_p)$$

for $D_1, \ldots, D_p \in \text{Der}_\omega^+(\hat{LW})$, we can obtain the relation $L_\delta = i_\delta d_{CE} - d_{CE} i_\delta$. The operator $i_\delta$ of homological degree $-1$ is called interior product. One can give cochains in the Chevalley-Eilenberg complex $C_{CE}^{\bullet \bullet}(\text{Der}_\omega^+(\hat{LW}))$ two degree: the first degree is as a Chevalley-Eilenberg cochain and the second degree is derived from the degree of $\text{Der}_\omega^+(\hat{LW})$. Since $d_{CE} L_\delta + L_\delta d_{CE} = 0$ holds, $(C_{CE}^{\bullet \bullet}(\text{Der}_\omega^+(\hat{LW})), d_{CE}, L_\delta)$ is a double complex by these two gradings. We also think the invariant space $C_{CE}^{\bullet \bullet}(\text{Der}_\omega^+(\hat{LW}))^{\text{Sp}(W, \delta)}$, where $\text{Sp}(W, \delta)$ is the group of symplectic linear isomorphisms $W \to W$ preserving $\delta$.

2.2. dgl model with symplectic form of manifolds. In this subsection, we review a Chen’s dgl model of a manifold. Let $X$ be a smooth manifold. Put $A = A^*(X)$ and $H = H_{DR}^*(X)$. Fix a homotopy transfer diagram

$$\begin{array}{ccc} A & \longrightarrow & H, \end{array}$$

e.g. in the case that $X$ is a closed manifold, it is obtained by using the Hodge decomposition of the de Rham complex $A$. Since $A$ is a commutative dga with
symmetric form (intersection form), \( H \) has the structure of minimal cyclic \( C_\infty \)-algebra by the diagram (details in [11, 15, 7, 13, 5] for instance).

Let \( I \) be the intersection form on \( H, m \) the cyclic \( C_\infty \)-algebra structure on \( H \) obtained by the homotopy transfer diagram and \( s : H \to H[1] \) be the suspension map. We denote \( V = H[1]^* \). Defining the suspension of \( m_i \) by \( \tilde{m}_i := s \circ m_i \circ (s^{-1})^\otimes i \) for all \( i \geq 1 \) and of \( I \) by \( \omega := I \circ (s^{-1})^\otimes 2 \), then these dual defines the symplectic inner product \( \omega \) on \( H[1]^* \) of degree \( N = n - 2 \) and the linear map \( \delta_i : V \to V^\otimes n \) of homological degree \(-1\). Thus extending the unique derivation \( \delta_i : \hat{LV} \to \hat{LV} \) by the Leibniz rule, then we have the derivation of homological degree \(-1\)

\[
\delta := \sum_{i=1}^\infty \delta_i \in \text{Der}_\omega(\hat{LV}).
\]

Furthermore we can prove that \( \tilde{\delta} \) is a differential since \( m \) satisfies the \( A_\infty \)-relations and quadratic, i.e. \( \delta(V) \subset \prod_{i \geq 2} V^\otimes i \), since \((H,I,m)\) is minimal.

The Chen’s dgl model is a reduced version of the construction. Suppose \( X \) is connected and put

\[
W := H[1]_{\geq 0}^* = H_+(X; \mathbb{R})[-1].
\]

Then we have the restriction \( \delta : \hat{LW} \to \hat{LW} \) of \( \tilde{\delta} \) and \( \omega : W^\otimes 2 \to \mathbb{R} \). If \( X \) is simply-connected, we can restrict \( \delta \) to the differential on the free Lie algebra \( LW \).

**Theorem 2.1** (Chen[2]). For a simply-connected closed manifold \( X \) with base point \(*\), the dgl \((LW, \delta)\) is a Quillen model of \( X \), i.e., there is a Lie algebra isomorphism

\[
H_\bullet(LW, \delta) \simeq \pi_\bullet(\omega X) \otimes \mathbb{Q}.
\]

### 3. Graph complex

#### 3.1. Orientation and ordering of graded sets.

The set of orderings on a set \( U \) is defined by

\[
\text{Ord}(U) := \{(u_1, \ldots, u_k) \in U^\times k; U = \{u_1, \ldots, u_k\}\},
\]

where \( k := \#U \).

**Definition 3.1.** Let \( U \) be a \( \mathbb{Z} \)-graded set, i.e. a finite set \( U \) given a map \(|-| : U \to \mathbb{Z} \).

- The symmetric algebra \( SU \) generated by \( U \) is the \( \mathbb{Z} \)-graded commutative algebra which is the quotient algebra obtained from the \( \mathbb{Z} \)-graded tensor algebra \( TU \) by introducing the relation
  
  \[
  xy = (-1)^{|x||y|}yx
  \]
  
  for \( x, y \in U \). The image of \( U^\otimes k \) for an integer \( k \) in \( SU \) is denoted by \( S^kU \).

- The exterior algebra \( AU \) generated by \( U \) is the \( \mathbb{Z} \)-graded anti-commutative algebra which is the quotient algebra obtained from the \( \mathbb{Z} \)-graded tensor algebra \( TU \) by introducing the relation
  
  \[
  xy = -(-1)^{|x||y|}yx
  \]
  
  for \( x, y \in U \). The image of \( U^\otimes k \) for an integer \( k \) in \( AU \) is denoted by \( \Lambda^kU \).

For an element \( (u_1, \ldots, u_k) \in \text{Ord}(U) \), we denote the image of \( u_1 \otimes \cdots \otimes u_k \) in \( AU \) by \([u_1, \ldots, u_k]\). The 1-dimensional vector space generated by this element is written by

\[
O(U) := \langle [u_1, \ldots, u_k] \rangle \subset AU.
\]
Definition 3.2. Let $U$ be a $\mathbb{Z}$-graded set. For distinct elements $u_1, \ldots, u_k \in U$ and a permutation $\pi \in \mathfrak{S}_k$, the sign $\epsilon$ defined by the equation on $S^k U$

$$u_1 \cdots u_k = \epsilon \cdot u_{\pi(1)} \cdots u_{\pi(k)}$$

is called the Koszul sign of $(u_1, \ldots, u_k)\mapsto (u_{\pi(1)}, \ldots, u_{\pi(k)})$. Similarly the sign $\bar{\epsilon}$ defined by the same equation in $\Lambda^k U$ is called the anti-Koszul sign. Note that the equation $\bar{\epsilon} = \text{sgn} \cdot \epsilon$.

Definition 3.3. Let $V$ be a $\mathbb{Z}$-graded vector space. We define the subspace $V_{\text{cyc}}^k$ of cyclic tensors in $V^\otimes k$ by the image of the map $[-, \ldots, -]_{\text{cyc}}: V^\otimes k \to V^\otimes k$ obtained by

$$x_1 \otimes \cdots \otimes x_k \mapsto \sum_{\tau \in \mathbb{Z}/k\mathbb{Z}} \epsilon \cdot x_{\tau(1)} \otimes \cdots \otimes x_{\tau(k)},$$

where $\mathbb{Z}/k\mathbb{Z}$ is identified with the group of cyclic permutations and $\epsilon$ is the Koszul sign of $(x_1, \ldots, x_k)\mapsto (x_{\tau(1)}, \ldots, x_{\tau(k)})$. For a $\mathbb{Z}$-graded set $U$, we denote

$$\text{Cyc}(U) := \langle [u_1, \ldots, u_k]_{\text{cyc}} \rangle (u_1, \ldots, u_k) \in \text{Ord}(U) \subset (RU)^k_{\text{cyc}}.$$

3.2. Definition of graph complex. Let $(W, \omega)$ be a symplectic vector space with even degree $N$ and suppose $Z := \{a \in \mathbb{Z}; W_a \neq 0\} \subset \{0, \ldots, N\}$. Our labeled graph complex depends on $(W, \omega)$.

3.2.1. Definition of graphs.

Definition 3.4. An $N$-graded graph $\Gamma$ consists of the following information:

- The set $H(\Gamma)$ of half-edges.
- The set $V(\Gamma)$ of vertices. It is a partition of the set $H(\Gamma)$, i.e.

$$H(\Gamma) = \bigsqcup_{v \in V(\Gamma)} v, \quad v \neq \emptyset (v \in V(\Gamma)).$$

The number of elements of any $v \in V(\Gamma)$ is called the valency of $v$. A vertex with valency $> 1$ is called an internal vertex and one with valency 1 is called an external vertex. The set of internal (resp. external) vertices is denoted by $V_i(\Gamma)$ (resp. $V_e(\Gamma)$).

- The set $E(\Gamma)$ of edges. It is a partition of the set $H(\Gamma)$ such that the number of elements of any $e \in E(\Gamma)$ is two, i.e.

$$H(\Gamma) = \bigsqcup_{e \in E(\Gamma)} e, \quad \#e = 2 (e \in E(\Gamma)).$$

- The cohomological degree of half-edges. It is a map $|\cdot| : H(\Gamma) \to \mathbb{Z}$ such that $|h_1| + |h_2| = N$ for an edge $e = \{h_1, h_2\} \in E(\Gamma)$. Then the cohomological degrees of vertices and edges are defined by

$$|v| := |h_1| + \cdots + |h_r| - N, \quad |e| := N$$

for $v = \{h_1, \ldots, h_r\} \in V(\Gamma)$ and $e \in E(\Gamma)$.

- The division of the set $V_i(\Gamma)$ of internal vertices to two disjoint sets

$$V_i(\Gamma) = V_n(\Gamma) \amalg V_s(\Gamma)$$

such that all elements in $V_s(\Gamma)$ have cohomological degree $-1$ and the valency $\geq 3$. An element of $V_n(\Gamma)$ is called normal vertex, and one of $V_s(\Gamma)$ is called special vertex.
The set of isomorphism classes of such graphs is denoted by $G(N)$. Here an isomorphism between $N$-graded graphs is a bijection between the sets of half-edges preserving all information of $N$-graded graphs.

3.2.2. **Decoration on vertices.** We shall give the relation equivalent to the dual of vertices defined by the cyclic Lie operad as in [3, 4, 12].

**Definition 3.5.** Let $\Gamma$ be an $N$-graded graph.

- We introduce to $\text{Cyc}(v)[N]$ for $v \in V_i(\Gamma)$ the *commutativity relation*

  $$S_{v,h,s}(\sigma) := \sum_{\tau \in \text{Sh}(s,r,s-1)} \sigma^{(v,h,s)} = 0,$$

  $$\sigma^{(v,h,s)} := \epsilon[h_{r(1)}, \ldots, h_{r(r-1)}, h_r] \sigma,$$

  for $r - 1 > s > 0$ and $\sigma = [h_1, \ldots, h_r] \sigma \in \text{Cyc}(v)[N]$, where $\text{Sh}(p,q)$ is the set of $(p,q)$-shuffles, $\sigma$ is the symbol of the $N$-fold suspension, and $\epsilon$ is the Koszul sign. Then we denote the obtained space by $C(v) = \text{Cyc}(v)[N]/(\text{com. rel.})$. (In the case of $r = 3$, it is the AS-relation for Jacobi diagrams.)

![Figure 1. Commutativity ($r = 3, 4$). (Koszul signs are omitted in figures.)](image)

3.2.3. **Decoration on $N$-graded graphs.** Set

$$\hat{\mathcal{O}}_{\text{com}}(W, \Gamma) := \bigoplus_{e \in E(\Gamma)} O(e) \otimes \bigwedge_{u \in V_e(\Gamma)} W[-N]_{|u|} \otimes \bigwedge_{v^* \in V_s(\Gamma)} C(v^*) \otimes \bigwedge_{v \in V_n(\Gamma)} C(v),$$

where

$$\bigoplus_{u \in U} V(u) := \left\{ v_{u_1} \cdots v_{u_k} \in S^k \left( \bigoplus_{u \in U} V(u) \right) : v_{u_i} \in V(u_i), (u_1, \ldots, u_k) \in \text{Ord}(U) \right\},$$

$$\bigwedge_{u \in U} V(u) := \left\{ v_{u_1} \cdots v_{u_k} \in \Lambda^k \left( \bigoplus_{u \in U} V(u) \right) : v_{u_i} \in V(u_i), (u_1, \ldots, u_k) \in \text{Ord}(U) \right\}$$

for a family $(V(u))_{u \in U}$ of $\mathbb{Z}$-graded vector spaces indexed by a finite set $U$. This tensor product consists of four factors: the first factor means directions of edges of $\Gamma$, the second factor $W$-labels of external vertices of $\Gamma$, the third factor (equivalence classes of) cyclic orderings on special vertices of $\Gamma$, and the forth factor the same on normal vertices of $\Gamma$. Note that $W[-N]_{|u|} = W[h]_{[-N]}$ for an external vertex $u = \{h\}$. 


We need to identify elements of $\hat{O}_{\text{com}}(W, \Gamma)$ by the symmetry of $\Gamma$. An automorphism $\alpha$ of an $N$-graded graph $\Gamma \in \mathcal{G}(N)$ induces the linear isomorphism $C(v) \to C(\alpha(v))$ for $v \in V_i(\Gamma)$ described by

$$[h_1, \ldots, h_k] \mapsto [\alpha(h_1), \ldots, \alpha(h_k)],$$

and the identity map $W[-N][u] \to W[-N][\alpha(u)] = W[-N][u]$ for $u \in V_e(\Gamma)$. Therefore the automorphism group of $\Gamma$ acts on the vector space $\hat{O}_{\text{com}}(W, \Gamma)$ by the induced permutation of half-edges. Then the coinvariant vector space of $\hat{O}_{\text{com}}(W, \Gamma)$ by this action is denoted by $\hat{O}_{\text{com}}(W, \Gamma)$. We often consider an element $o$ of $\hat{O}_{\text{com}}(W, \Gamma)$ described by the form

$$o = [o_1, \ldots, o_i; w_1, \ldots, w_k; e_1^*, \ldots, e_i^*; c_1, \ldots, c_k],$$

where $w_i \in W[-N][u_i]$ and

$$o_i = [\hat{o}_i], \quad e_i^* = [\hat{e}_i^*], \quad e_i = [\hat{e}_i],$$

for $\hat{o}_i \in \text{Ord}(e_i), \hat{e}_i \in \text{Ord}(v_i)$ and $\hat{e}_i^* \in \text{Ord}(v_i^*)$. Such element $o$ is called an orientation of $\Gamma$, a pair $(\Gamma, o)$ is an oriented graph, and the information

$$\hat{o} = (\hat{o}_1, \ldots, \hat{o}_i; w_1, \ldots, w_k; e_1^*, \ldots, e_i^*; c_1, \ldots, c_k)$$

is called a lift of an orientation $o = [\hat{o}]$ on $\Gamma$. The vector space $\hat{O}_{\text{com}}(W, \Gamma)$ is generated by orientations.

3.2.4. Definition of the bigraded vector space $\hat{C}^{\bullet, \bullet}_{\text{com}}(W)$. The cohomological bigegree $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ of $\Gamma \in \mathcal{G}(N)$ is defined by

$$p = \#V_o(\Gamma), \quad q = \sum_{v \in V_o(\Gamma)} |v| = \#V_v(\Gamma) + N(\#E(\Gamma) - \#V(\Gamma)) - \sum_{u \in V_e(\Gamma)} |u|,$$

and bigegree of elements in $\hat{O}_{\text{com}}(W, \Gamma)$ is defined by that of $\Gamma$. We define the space of $N$-graded ribbon graphs by

$$\hat{C}^{\bullet, \bullet}_{\text{com}}(W) := \bigoplus_{\Gamma \in \mathcal{G}(N)} \hat{O}_{\text{com}}(W, \Gamma), \quad \hat{C}^{p,q}_{\text{com}}(W) := \bigoplus_{\Gamma \in \mathcal{G}^{p,q}(W)} \hat{O}_{\text{com}}(W, \Gamma),$$

where $\mathcal{G}^{p,q}(W)$ is the subset of $\mathcal{G}(N)$ consisting $N$-graded graphs of degree $(p, q)$. Then $\hat{C}^{\bullet, \bullet}_{\text{com}}(W)$ can be regarded as bigraded vector space. We often denote an element in $\hat{C}^{\bullet, \bullet}_{\text{com}}(W)$ corresponding to $o \in \hat{O}_{\text{com}}(W, \Gamma)$ for $\Gamma \in \mathcal{G}(N)$ by $(\Gamma, o)$.

3.2.5. Definition of the first differential $d$. We define the linear map $\alpha^{a,b}_{v;h^1, h^2} : \hat{O}_{\text{com}}(W, \Gamma) \to \hat{C}^{\bullet, \bullet}_{\text{com}}(W)$ for $\Gamma \in \mathcal{G}(N), v \in V_o(\Gamma), a, b \in \mathbb{Z}$ satisfying $a + b = N$, and $h^1 = h_r \neq h^2 = h_i \in v$ such that

$$\alpha^{a,b}_{v;h^1, h^2}(\Gamma, [\cdot; -; -; [h_1, \ldots, h_r] \sigma, -])$$

$$= (\Gamma^{a,b}_{v;h^1, h^2}, [\cdot, [h', h'']^{-1}; -; -; [h_1, \ldots, h_i, h'] \sigma, [h'', h_{i+1}, \ldots, h_r] \sigma, -]).$$

Here $\sigma$ is the $N$-fold suspension, and the $N$-graded graph $\Gamma^{a,b}_{v;h^1, h^2}$ is defined by

$$H(\Gamma^{a,b}_{v;h^1, h^2}) = H(\Gamma) \sqcup \{h', h''\}, \quad V(\Gamma^{a,b}_{v;h^1, h^2}) = (V(\Gamma) \setminus \{v\}) \sqcup \{v', v''\},$$

$$V_s(\Gamma^{a,b}_{v;h^1, h^2}) = V_s(\Gamma), \quad E(\Gamma^{a,b}_{v;h^1, h^2}) = E(\Gamma) \cup \{e_0\},$$

where $v' = \{h_1, \ldots, h_i, h'\}, v'' = \{h'', h_{i+1}, \ldots, h_r\}, e_0 = \{h', h''\}, |h'| = a$ and $|h''| = b.$
Then we obtain the linear map \( d : \hat{C}_{\text{com}}(W) \to \hat{C}_{\text{com}}(W) \) by
\[
d_v(\Gamma, o) := \frac{1}{2} \sum_{a+b=N} \sum_{h^1 \neq h^2 \in v} d_{v,h^1,h^2}^{a,b}(\Gamma, o),
\]
The map \( d \) can be also described by
\[
d_v(\Gamma, o) = \sum_{a+b=N} \sum_{0 \leq s < t < r} d_{v,h^s,h^t}^{a,b}(\Gamma, o),
\]
where \( o = [-;--; h_1, \ldots, h_r]\sigma, -] \) and \( v = \{h_1, \ldots, h_r\} \). Remark the relation
\[
d_{v,h^1,h^2}^{a,b}(\Gamma, o) = d_{v,h^2,h^1}^{b,a}(\Gamma, o)
\]
for half-edges \( h^1 \neq h^2 \in v \). Here well-definedness of \( d \) is proved by the relation with the commutativity relation:

**Proposition 3.6.** Using the notations above, \( d_v S_{v,h^1,h^2}(\Gamma, o) \) is equal to zero under the commutativity relation.

**Proof.** For integers \( p, q \), we define the linear ordered set \([p, q]\) by \( \{p < p + 1 < \cdots < q - 1 < q\} \). If \( p > q \), put \( [p, q] = \emptyset \).

For partial ordered sets \( P_1, P_2 \), we denote their direct sum by \( P_1 \oplus P_2 \) (in the category of posets), and their ordinal sum by \( P_1 \oplus P_2 \). Then a \((p, q)\)-shuffle is equivalent to the inverse of an order-preserving bijection \([1, p] + [p + 1, p + q] \to [1, p + q]\).

Let \( \tau^{-1} : [1, i] + [i + 1, r - 1] \to [1, r - 1] \) be an \((i, r - i - 1)\)-shuffle and \( 0 \leq s < t < r \) integers. Put \( L = \tau([s + 1, t]) \) and \( l = t - s \).

If \( \tau(s + 1), \ldots, \tau(t) \) are \( \leq i \), then we have \( \tau(s + m) = \tau(s + 1) + (m - 1) \) for \( 1 \leq m \leq t - s \) since \( [1, i] \to \tau^{-1}([1, i]) \) is an isomorphism between posets. Put \( a = \tau(s + 1) - 1 \). Then we obtain the shuffle \( \tau_2 \) by \( \tau \):

\[
\begin{align*}
[1, i - l + 1] + [i - l + 2, r - l] & \xrightarrow{\tau_2^{-1}} [1, r - l] \\
[1, a] \oplus \{s\} & \oplus [a + l, i] + [i + 1, r - 1] & \xrightarrow{\text{bij.}} & [1, s] \oplus \{s\} \oplus [t + 1, r - 1] \\
[1, i] + [i + 1, r - 1] & \xrightarrow{\tau^{-1}} [1, r - 1]
\end{align*}
\]

The shuffle \( \tau \) can recover from a pair \((a, l, \tau_2)\), where \( \{a + 1, \ldots, a + l\} \subseteq [1, i] \) and an \((i - l + 1, r - i - 1)\)-shuffle \( \tau_2 \).

Similarly, if \( \tau(s + 1), \ldots, \tau(t) \) are \( \geq i + 1 \), we can obtain a triple \((a, l, \tau_2)\), where \( \{a + 1, \ldots, a + l\} \subseteq [i + 1, r - 1] \) and an \((i - l + 1, r - i - 1)\)-shuffle \( \tau_2 \).
Otherwise, put \( p = \#(L \cap [1, i]) \). Then we obtain the shuffle \( \tau_1 \) by restricting \( \tau \):

\[
[1, p] + [p + 1, l] \xrightarrow{\tau^{-1}} [1, l]
\]

We consider \( \bar{L} = ([1, i] + [i + 1, r - 1]) \setminus L \) and the order-preserving bijection \( \rho^{-1} : \bar{L} \to [1, s] \oplus [t + 1, r - 1] \) defined by the restriction of \( \tau^{-1} \). The shuffle \( \tau \) recovers from a pair \((\rho, \tau_1)\), where \( \rho^{-1} : \bar{L} \to [1, s] \oplus [t + 1, r - 1] \) is an order-preserving bijection and \( \tau_1 \) is a \((p, l - p)\)-shuffle.

Thus we have

\[
d_v S_{v, h, c}([h_1, \ldots, h_r] \sigma) = \sum_{l=1}^{r-1} \left( \sum_{p=1}^{l-1} \sum_{\rho} o_\rho (\epsilon, h') [h_1, \ldots, h_{u_p}, h'] \sigma, [h_{u_p}, \ldots, h_{u_p(s)}, h', h_{p(t+1)}, \ldots, h_{p(r-1)}, h_r] \sigma, o_a, l \right) + \left( \sum_{a} S_{v, h', c, \tau} (\sigma) \right)
\]

where \( L = \{1, \ldots, r - 1\} \setminus \bar{L} = \{u_1 < \cdots < u_p \} \) as integers,

\[
o_\rho = \epsilon([h_{u_p}, \ldots, h_{u_p(s)}, h', h_{p(t+1)}, \ldots, h_{p(r-1)}, h_r] \sigma, [h_1, \ldots, h_{u_p(s)}, h, h_{p(t+1)}, \ldots, h_{p(r-1)}, h_r] \sigma, o_a, l, c, \ldots]
\]

and \( \epsilon, \epsilon' \) are appropriate Koszul signs.

\[\Box\]

3.2.6. Definition of the second differential \( L \). For \( \Gamma \in \mathcal{G}(N) \), let \( i_v(\Gamma) \) be the \( N \)-graded graph obtained by specializing a normal vertex \( v \) of degree \(-1\). We define the linear map \( i_v : O_{com}(W, \Gamma) \to O_{com}(W, \Gamma) \) for \( o \in O_{com}(W, \Gamma) \) such that

\[
i_v(\Gamma, [-; -; c, \ldots]) = (i_v(\Gamma), [-; -; c, \ldots])
\]

for \( c \in C(v) \) if \( v \) has degree \(-1\), and \( i_v(\Gamma, o) = 0 \) if \( v \) does not have degree \(-1\). Since the relation

\[
i_{v_1} S_{v_2, v_3, k, \ldots} (\Gamma, o) = S_{v_2, \ldots} (i_{v_1} (\Gamma), o)
\]

for \( v_1, v_2 \in V_i(\Gamma) \) holds clearly, the map \( i_v \) is well-defined. Then the linear map

\( L : \hat{C}_{com}^\bullet(W) \to \hat{C}_{com}^\bullet(W) \) is defined by

\[
L := id - di
\]

where the linear map \( i : \hat{C}_{com}^\bullet(W) \to \hat{C}_{com}^\bullet(W) \) is obtained by

\[
i(\Gamma, o) := \sum_{v \in V_v(\Gamma)} i_v(\Gamma, o)
\]

The map \( L \) is also described by

\[
L(\Gamma, o) = \sum_{v \in V_v(\Gamma)} (i_{v'} + i_{v''}) d_v(\Gamma, o)
\]

Then \( d, i, \) and \( L \) have (cohomological) bidegree \((1, 0)\), \((-1, 1)\) and \((0, 1)\) respectively.
3.2.7. **Definition of the underlying bigraded vector space** $C^{\bullet,\bullet}_{com}(W)$. The space $C^{\bullet,\bullet}_{com}(W)$ is the quotient space of $\hat{C}^{\bullet,\bullet}_{com}(W)$ by

- **(positivity)** a graph which has a normal vertex $v$ satisfying $|v| + \#v - 2 \leq 0$ is zero, and $i_{\tilde{v}}d_{\tilde{v}}(\Gamma, o) = 0$ for $\Gamma \in \mathcal{G}(N)$ and a normal vertex $v$ of degree 0,
- **($A_\infty$-relation)**

$$R_v(\Gamma, o) := i_{\tilde{v}}i_{\tilde{v}}d_{\tilde{v}}(\Gamma, o) = 0$$

for $\Gamma \in \mathcal{G}(N)$ and a normal vertex $v$ (of cohomological degree $-2$).

\[\sum \begin{array}{c}
\vdots \\
\vdots
\end{array} \begin{array}{c}
\vdots \\
\vdots
\end{array} \begin{array}{c}
h_{t+1} \\
h_t \\
h_{h+1}
\end{array} \begin{array}{c}
h_1 \\
h_2 \\
h_r
\end{array} = 0 \]

**Figure 3.** $A_\infty$-relation.

- **(Cut-off relation)** For $\Gamma \in \mathcal{G}(N)$ and $e = \{h_1, h_2\} \in E(\Gamma)$, we define the $N$-graded graph $\Gamma_e$ as follows:

$$H(\Gamma_e) = H(\Gamma) \amalg \{\bar{h}_1, \bar{h}_2\},$$
$$E(\Gamma_e) = (E(\Gamma) \setminus \{e\}) \amalg \{\{h_1, \bar{h}_1\}, \{h_2, \bar{h}_2\}\},$$
$$V(\Gamma_e) = V(\Gamma) \amalg \{\{\bar{h}_1\}, \{\bar{h}_2\}\}.$$

Then

$$|\bar{h}_1| = N - |h_1| =: a, \quad |\bar{h}_2| = N - |h_2| =: b.$$

$$\left(\Gamma, [h_1, h_2], \vdots; \vdots; \vdots; \vdots\right) = \sum_{|x^i| = a, |x^j| = b} \omega_{ij}(\Gamma_e, [h_1, \bar{h}_1], [h_2, \bar{h}_2], \vdots; x^i \sigma^{-1}, x^j \sigma^{-1}, \vdots; \vdots),$$

where $\{x^i\}$ is a homogeneous basis of $W$ and $(\omega_{ij})$ is the inverse matrix of $(\omega(x^i, x^j))$.

\[\[
\rightarrow \sum_{|x^i| = a, |x^j| = b} \omega_{ij} \rightarrow x^i \rightarrow x^j \]

**Figure 4.** Cut-off relation.

Remark that $C^{\bullet,\bullet}_{com}(W)$ is generated by $W$-labeled graphs with only one internal vertex by cut-off relation.

3.2.8. **On well-definedness of three operators** $d, i, L$ on $C^{\bullet,\bullet}_{com}(W)$. The endomorphisms $d, i$ and $L$ of $C^{\bullet,\bullet}_{com}(W)$ induce endomorphisms of $C^{\bullet,\bullet}_{com}(W)$ by the equations

$$dR_v(\Gamma, o) = \sum_{u \neq v} R_v d_u(\Gamma, o), \quad iR_v(\Gamma, o) = \sum_{u \neq v} R_v i_u(\Gamma, o)$$

for a normal vertex $v$ of an $N$-graded graph $\Gamma$. 
3.2.9. On two differentials $d, L$ on $C^\bullet_\text{com}(W)$.

**Proposition 3.7.** The bigraded vector space $C^\bullet_\text{com}(W)$ is a double complex with respect to differentials $d$ and $L$. We call $C^\bullet_\text{com}(W)$ **double graph complex**.

**Proof.** First, we show the equation $d^2 = 0$. For a normal vertex $v$ of an $N$-graded graph $(\Gamma, o)$, let $v', v''$ be new vertices obtained by splitting at $v$. Then

$$d_v d_v (\Gamma, o) = -d_v d_v (\Gamma, o) \quad d_u d_v (\Gamma, o) = -d_v d_u (\Gamma, o)$$

for $u \neq v$ holds. So we obtain $d^2 (\Gamma, o) = 0$ by cancellation.

Next, we show $L^2 = 0$. From the equation $(iL - Li)(\Gamma, o) = 2 \sum v R_v (\Gamma, o)$ as element in $\hat{C}^\bullet_\text{com}(W)$, we obtain the relation $iL - Li = 0$ in $C^\bullet_\text{com}(W)$. So the equations

$$L^2 = (id - di)L = idL - diL = idL - dLi = idid - didi,$$

$$L^2 = L(id - di) = Lid - Ldi = LdL - Ldi = -idid + didi$$

hold. Then we obtain $L^2 = 0$. Since $Ld + dL = 0$ holds by definition of $L$, we get the proposition. $\square$

3.3. Construction of the map to Chevalley-Eilenberg complexes. Let $(W, \omega)$ and $Z$ be as Section 3.2 and $\delta$ be a quadratic differential of homological degree $-1$ on $\hat{L}W$. We construct a double chain map

$$C^\bullet_\text{com}(W) \rightarrow C^\bullet_{CE}(\text{Der}_\omega^+(\hat{L}W)).$$

Setting

$$W(r) := (\hat{L}W \otimes W) \cap W^r_{\text{cyc}},$$

we have a graded linear isomorphism

$$\text{Der}_\omega(\hat{L}W) \simeq \prod_{r=2}^\infty W(r)[{-N}]$$

using non-degeneracy of $\omega$.

Put $k = k_e + k_s + k_n$ and

$$(r_1, \ldots, r_k) := (1, \ldots, 1, \# v_{1}^s, \ldots, \# v_{k_s}^s, \# v_1, \ldots, \# v_{k_n}^n).$$

We denote by $\tau(\hat{\delta})$ the linear isomorphism (the permutation of factors of the tensor product)

$$W^{\otimes r_1} \otimes \cdots \otimes W^{\otimes r_k} \rightarrow W^{\otimes 2} \otimes \cdots \otimes W^{\otimes 2} = (W^{\otimes 2})^{\otimes l}$$
corresponding to the permutation of half-edges
\[(h_1, \ldots, h_{k_z}, c_1, \ldots, c_{k_z}, \hat{c}_1, \ldots, \hat{c}_{k_n}) \mapsto (\hat{a}_1, \ldots, \hat{a}_k).\]
Then we define the linear map of cohomological degree \((l - k)N\)
\[\alpha(\Gamma, \hat{o}) : \left( \prod_{r=1}^{\infty} W^{\otimes_r}[-N] \right)^{\otimes k} \to \mathbb{R}\]
by
\[\left( \prod_{r=1}^{\infty} W^{\otimes_r}[-N] \right)^{\otimes k} \xrightarrow{\text{proj}} \bigotimes_{i=1}^{k} (W^{\otimes_{r_i}}[-N]) \xrightarrow{\otimes} \bigotimes_{i=1}^{k} W^{\otimes_{r_i}} \xrightarrow{\tau(\hat{o})} (W^{\otimes_2})^\otimes \to \mathbb{R},\]
where the map \((W^{\otimes_2})^\otimes \to \mathbb{R}\) is the map \(\omega_{e_1} \otimes \cdots \otimes \omega_{e_j}\) obtained by \(\omega_{e_j} := \omega_{(h_1^{e_j}, h_2^{e_j})}\) if \(e_j = \{h_1^{e_j}, h_2^{e_j}\}\). Here we denote by \(\omega_{(d_1, d_2)}\) for integers \(d_1, d_2\) the composition of the projection \(W \otimes W \to W_{d_1} \otimes W_{d_2}\) and the restriction of \(\omega\) to \(W_{d_1} \otimes W_{d_2}\). Restricting \(\alpha(\Gamma, \hat{o})\) by regarding exterior products as subspaces of tensor products, we get the linear map
\[\alpha(\Gamma, o) : \Lambda^k W[-N] \otimes \Lambda^k \text{ Der}_W(\hat{L}W) \otimes \Lambda^k \text{ Der}_W^+(\hat{L}W) \to \mathbb{R}.\]
The map \(\alpha(\Gamma, o)\) is independent of a representation \(\hat{o}\) of \(\hat{L}\) by the definition of an orientation. We define the map \(\psi(\Gamma, o) : \Lambda^k \text{ Der}_W(\hat{L}W) \to \mathbb{R}\) by
\[\psi(\Gamma, o)(D_1, \ldots, D_{k_n}) := \alpha(\Gamma, o)(w_1, \ldots, w_{k_x}, \delta_1, \ldots, \delta, D_1, \ldots, D_{k_n}),\]
for \(D_i \in \text{ Der}_W(\hat{L}W)\). So we obtain the map \(\psi : C^\bullet_n(W) \to C^\bullet_{CE}(\text{ Der}_W^+(\hat{L}W))\). To prove that \(\psi\) is a chain map, we need Lemma 3.8.
Let \(\text{ Der}(\hat{T}W)\) be the Lie algebra of algebra derivations on \(\hat{T}W\), and \(\text{ Der}^r(\hat{T}W)\) be the subspace defined by
\[\text{ Der}^r(\hat{T}W) := \{ D \in \text{ Der}(\hat{T}W); D(W) \subset W^{\otimes(r+1)} \} \simeq \text{ Hom}(W, W^{\otimes(r+1)}).\]
By non-degeneracy of \(\omega\), we have the linear isomorphism of degree 0
\[\text{ Der}^r(\hat{T}W) \simeq W^{\otimes(r+2)}[-N].\]
The Lie algebra structure of \(\text{ Der}(\hat{T}W)\) is described as follows:

**Lemma 3.8.** The Lie bracket \([\ , \ ]\) on \(\text{ Der}(\hat{T}W)\) is equal to the suspension of the map
\[\sum_{d_1 + d_2 = N} (\text{id} \otimes \omega_{(d_1, d_2)}) \left( \sum_{1 \leq t < r_2} \pi_{1;1}^{r_1, r_2} + \sum_{1 \leq s < r_1} \pi_{2;1}^{r_1, r_2} \right) : W^{\otimes r_1} \otimes W^{\otimes r_2} \to W^{\otimes r_1 + r_2 - 2}\]
where \(\pi_{1;1}^{r_1, r_2}, \pi_{2;1}^{r_1, r_2} : W^{\otimes r_1} \otimes W^{\otimes r_2} \to W^{\otimes r_1 + r_2}\) for \(1 \leq i \leq r_1, 1 \leq j \leq r_2\) is defined by
\[\pi_{1;1}^{r_1, r_2}(a_1^{(1)} \cdots a_1^{(r_1)} \otimes a_2^{(1)} \cdots a_2^{(r_2)}) = \epsilon \cdot a_1^{(1)} \cdots a_1^{(1)} \cdots a_1^{(2)} \cdots a_1^{(2)} \cdots a_1^{(2)} \cdots a_1^{(1)}, \]
\[\pi_{2;1}^{r_1, r_2}(a_1^{(1)} \cdots a_1^{(r_1)} \otimes a_2^{(1)} \cdots a_2^{(r_2)}) = \epsilon \cdot a_1^{(1)} \cdots a_1^{(1)} \cdots a_1^{(2)} \cdots a_1^{(2)} \cdots a_1^{(2)} \cdots a_1^{(1)}, \]
for homogeneous elements \(a_1^{(1)}, \ldots, a_1^{(r_1)}, a_2^{(1)}, \ldots, a_2^{(r_2)}\). Here \(\epsilon\) is the Koszul sign of the corresponding permutations.
Proof. Let $x^1, \ldots, x^m$ be a homogeneous basis of $W$. The derivations $x^{i_1} \cdots x^{i_k} \partial/\partial x^{i}$ $(1 \leq i_1, \ldots, i_k, i \leq m)$, which are these elements corresponding to the linear map $x^i \mapsto x^{i_1} \cdots x^{i_k}$, consist a (topological) basis of $\text{Der}(\tilde{T}W)$. The Lie bracket for the basis is described by

$$\left[ x^{j_1} \cdots x^{j_k} \frac{\partial}{\partial x^{j_1}}, x^{j_1} \cdots x^{j_l} \frac{\partial}{\partial x^{j_l}} \right] = \sum_{t} \epsilon \delta^{j_1}_{j_{1-t}} \cdots \delta^{j_k}_{j_{k-t}} x^{i_1} \cdots x^{i_{k-t}} \frac{\partial}{\partial x^{i_t}}$$

where $\epsilon = (-1)^{(x^{i_1} + \cdots + x^{i_k} - x^t)(x^{j_1} + \cdots + x^{j_l} - x^t)}$, $\epsilon' = (-1)^{(x^{j_1} + \cdots + x^{j_l} - x^t)(x^{i_1} + \cdots + x^{i_k} - x^t)}$.

We use the identification by the isomorphism $\text{Der}^{-1}(\tilde{T}W) \simeq W[-N]$ as the $(N)$-suspension which has homological degree $-N$. Then, for $A = x^{i_1} \cdots x^{i_{r_1}}$ and $B = x^{j_1} \cdots x^{j_{r_2}}$, we obtain

$$[A^{\sigma^{-1}}, B^{\sigma^{-1}}] = \sum_{t} \epsilon x^{i_1} \cdots x^{i_{r_1-1}} \cdots x^{i_{r_1-1}} \cdots x^{i_{r_1}} \frac{\partial}{\partial x^{i_t}} + \sum_{s} \epsilon' x^{j_1} \cdots x^{j_{r_2-1}} \cdots x^{j_{r_2-1}} \cdots x^{j_{r_2}} \frac{\partial}{\partial x^{j_{r_1}}}$$

where $\epsilon$ and $\epsilon'$ are the Koszul sign of

$$(x^{i_1}, \ldots, x^{i_{r_1}}, x^{j_1}, \ldots, x^{j_{r_2}}) \mapsto (x^{i_1}, \ldots, x^{i_{r_1-1}}, \ldots, x^{i_{r_1}}, x^{j_1}, \ldots, x^{j_{r_2-1}}),$$

$$(x^{i_1}, \ldots, x^{i_{r_1}}, x^{j_1}, \ldots, x^{j_{r_2}}) \mapsto (x^{i_1}, \ldots, x^{i_{r_1-1}}, \ldots, x^{i_{r_1}}, x^{j_1}, \ldots, x^{j_{r_2-1}})$$

respectively. In the calculus above, note that we use the assumption that $N$ is even.

Well-definedness of $\psi$ is proved by the correspondence through $\psi$ between relations in the graph complex $C^\bullet_\text{com}(W)$ correspond to properties of derivations as the following table:

| graph complex | derivations |
|---------------|-------------|
| cyclicity     | symplectic derivation |
| commutativity | Lie derivation |
| $A_\infty$-relation | $\delta^2 = 0$ |
| positivity    | positive truncated |
| cut-off       | non-degeneracy of symplectic form |

By definition, it is clear except for the $A_\infty$-relation. The correspondence for the $A_\infty$-relation is proved in the end of the proof of the following theorem.

Theorem 3.9. The map $\psi : C^\bullet_\text{com}(W) \to C^\bullet_{CE}(\text{Der}_\omega(\tilde{L}W))$ is a double chain map.
Proof. First, we shall show that $d_{CE} \psi = \psi d$ on $\hat{C}^{\bullet \bullet}_c(W)$. For an oriented graph $(\Gamma, o)$, we define the two lifts $\hat{o}, \hat{o}$ on $\Gamma^{\bullet \bullet}_{\nu; h_v, h_\mu}$ as follows:

\[
\hat{o}^1 = ((h', h''), -; -; -; v_1, \ldots, v'_1, v''_1, \ldots, v_p), \\
\hat{o}^2 = ((h'', h'), -; -; -; v_1, \ldots, v''_1, v'_1, \ldots, v_p),
\]

\[
v_i' = (h_{i+1}', h_\mu', h'), \quad v_i'' = (h_i', h_\mu', h'', h_{i+1}', \ldots, h_{i}''),
\]

where $r_i = \# v_i$. The signs $\epsilon_i$ defined by the equations

\[
o^1 := \epsilon_1[\hat{o}^1], \quad o^2 := \epsilon_2[\hat{o}^2], \quad d_{\nu; h_v, h_\mu} = (-1)^{i-1} o^1 = (-1)^{i-1} o^2.
\]

So we obtain

\[
d(\Gamma, o) = \sum_{i=1}^k \sum_{\nu < \mu} \sum_{\nu + a + b = N} (-1)^{i-1} \left( \Gamma_{a,b}^{\nu; h_v, h_\mu}, o^1 \right)
\]

\[
= \sum_{i=1}^k \sum_{\nu < \mu} \sum_{\nu + a + b = N} (-1)^{i-1} \left( \Gamma_{a,b}^{\nu; h_v, h_\mu}, o^2 \right).
\]

Note that

\[
d_{CE}(\chi \circ \text{Alt}_p) = \frac{1}{2} \sum_{a=1}^p (-1)^{a-1} \chi \circ (1 \otimes \chi \otimes [\; , \; ] \otimes 1^{\otimes p-1}) \circ \text{Alt}_p
\]

for a linear map $\chi : W^{\otimes r_1}[-N] \otimes \cdots \otimes W^{\otimes r_p}[-N] \to \mathbb{R}$ and the anti-symmetrization $\text{Alt}_p$ for $p$-components. So, putting $\psi(\Gamma, \hat{o}) := \alpha(\Gamma, \hat{o})(w_1, \ldots, w_t, \delta, \ldots, \delta, -)$, we should prove

\[
\psi(\Gamma, \hat{o}) \circ (1^{\otimes i-1} \otimes [\; , \; ] \otimes 1^{\otimes p-i-1}) = \sum_{\nu < \mu} \sum_{a+b=N} \left( \epsilon_1 \psi(\Gamma_{a,b}^{\nu; h_v, h_\mu}, \hat{o}^1) + \epsilon_2 \psi(\Gamma_{a,b}^{\nu; h_v, h_\mu}, \hat{o}^2) \right) \circ \tau,
\]

where the map $\tau$ means the permutation

\[
X_1 \otimes \cdots \otimes (x_{\nu+1} \cdots x_\mu x') \otimes (x_1 \cdots x_\mu x'' x_{\mu+1} \cdots x_{r_1}) \otimes \cdots \otimes X_p
\]

\[
\Rightarrow \epsilon \cdot X_1 \otimes \cdots \otimes (x_1 \cdots x_\mu x' x_{\mu+1} \cdots x_{r_1}) \otimes (x_{\nu+1} \cdots x_\mu x'') \otimes \cdots \otimes X_p
\]

and $\epsilon$ is the Koszul sign. It follows the equations

\[
\psi(\Gamma, \hat{o}) \circ (1^{\otimes i-1} \otimes \sigma^{-1}(1 \otimes \omega(a,b))\pi_1^{r_i, r''} \otimes 1) \otimes 1^{\otimes p-i-1}) = \epsilon_1 \psi(\Gamma_{a,b}^{\nu; h_v, h_\mu}, \hat{o}^1),
\]

\[
\psi(\Gamma, \hat{o}) \circ (1^{\otimes i-1} \otimes \sigma^{-1}(1 \otimes \omega(a,b))\pi_2^{r_i, r''} \otimes 1) \otimes 1^{\otimes p-i-1}) = \epsilon_2 \psi(\Gamma_{a,b}^{\nu; h_v, h_\mu}, \hat{o}^2) \circ \tau,
\]

for $r' = \nu + 1$, $r'' = r - \mu + \nu + 1$, and $t = \nu + 1$. The first equation is verified as follows: we have

\[
\omega(x', x'') \psi(\Gamma, \hat{o})(X_1, \ldots, X_p) = \epsilon_1 \psi(\Gamma_{a,b}^{\nu; h_v, h_\mu}, \hat{o}^1)(X_1, \ldots, X_i, X_i', X_i'', X_i, \ldots, X_p)
\]

for $X_k \in W^{\otimes r_k}$, $x' \in W^a$, and $x'' \in W^b$. Here we put $X_i' = x_{\nu+1} \cdots x_\mu x' \sigma^{-1}$ and $X_i'' = x_1 \cdots x_\mu x'' x_{\mu+1} \cdots x_{r_1} \sigma^{-1}$ for $X_i = x_1 \cdots x_\mu \sigma^{-1}$. So we obtain the first equation from

\[
\epsilon_1 X_i \omega(x', x'') = \sigma^{-1}(1 \otimes \omega)\pi_1^{r_i, r''} \otimes 1^{\otimes (X_i' \otimes X_i''})
\]

The second is also verified in the same way.

Next, we shall prove $i_\delta \psi = \psi i$ on $\hat{C}^{\bullet \bullet}_c(W)$. The ordering

\[
\hat{o} := (-; -; -; v_1, v_1, \ldots, v_1, v_p)
\]
is a lift of $\bar{\epsilon}_i \cdot o$, where $\bar{\epsilon}_i$ is the anti-Koszul sign of the permutation 
\[(v_1, \ldots, v_p) \mapsto (v_i, v_1, \ldots, \hat{v}_i, \ldots, v_p).\]

So we have 
\[
\psi_i(\Gamma, o)(X_1, \ldots, X_{p-1}) = \sum_{s=1}^{j} \bar{\epsilon}_s \cdot \alpha(i_v(\Gamma),\hat{\delta})(w_1, \ldots, w_{k_v}, \delta, \ldots, \delta, \text{Alt}_{p-1}(X_1, \ldots, X_{p-1}))
\]
\[
= \sum_{s=1}^{j} \sum_{\pi \in \mathcal{G}_{p-1}} \bar{\epsilon} \cdot \alpha(\Gamma, \hat{o})(w_1, \ldots, w_{k_v}, \delta, \ldots, \delta, X_{\pi(1)}, \ldots, \delta, \ldots, X_{\pi(p-1)})
\]
\[
= \alpha(\Gamma, o)(w_1, \ldots, w_{k_v}, \delta, \ldots, \delta, \text{Alt}_p(\delta, X_1, \ldots, X_{p-1}))
\]
\[
= i_\delta \psi(\Gamma, o)(X_1, \ldots, X_{p-1})
\]
where $\bar{\epsilon}$ is the anti-Koszul sign of 
\[(\delta, X_1, \ldots, X_{p-1}) \mapsto (X_{\pi(1)}, \ldots, \delta, \ldots, X_{\pi(p-1)}).
\]

From the discussion above, the relation $\psi(R_v(\Gamma, o)) = 0$ follows from 
\[
\psi(R_v(\Gamma, o)) = \psi(i_v i_v^\circ d_v(\Gamma, o)) = \psi(\Gamma, o)([\delta, \delta], -) = 0.
\]
Thus $\psi$ induces the map $\psi : C^{\bullet,\bullet}_\text{com}(W) \to C^{\bullet,\bullet}_{CE}(\text{Der}^+_L(\hat{L}W))$. Furthermore, since $\psi$ is commutative with $d$ and $i$, so is $L$. So we complete the proof. \qed

The group $\text{Sp}(W, \delta)$ acts on $C^{\bullet,\bullet}_\text{com}(W)$ by the action on the their labels. Then, the chain map $\psi : C^{\bullet,\bullet}_\text{com}(W) \to C^{\bullet,\bullet}_{CE}(\text{Der}^+_L(\hat{L}W))$ is $\text{Sp}(W, \delta)$-equivariant clearly. Especially we can consider the $\text{Sp}(W, \delta)$-invariant part $C^{\bullet,\bullet}_\text{com}(W)^{\text{Sp}(W,\delta)}$ of the complex $C^{\bullet,\bullet}_\text{com}(W)$. It has the double subcomplex $C^{\bullet,\bullet}_\text{com}(N, Z)$ consisting of $N$-graded graphs which have no external vertex. This complex $C^{\bullet,\bullet}_\text{com}(N, Z)$ does not depend on the symplectic vector space $W$. It depends only a range $Z$ of degrees and a degree $N$ of a symplectic inner product.

**Remark 3.10.** We can define the associative version of $C^{\bullet,\bullet}_\text{com}(W)$ as follows. Set 
\[
\hat{O}_{\text{ass}}(W, \Gamma) := \bigotimes_{v \in V(\Gamma)} O(e) \otimes \bigwedge_{u \in V_+(\Gamma)} W[-N]_{|u|} \otimes \bigwedge_{v^* \in V_+(\Gamma)} \text{Cyc}(v^*)[N] \otimes \bigwedge_{v \in V_-(\Gamma)} \text{Cyc}(v)[N],
\]
\[
C^{\bullet,\bullet}_\text{ass}(W) := \bigoplus_{\Gamma \in \mathcal{G}(N)} O_{\text{ass}}(W, \Gamma), \quad O_{\text{ass}}(W, \Gamma) := \hat{O}_{\text{ass}}(W, \Gamma)_{\text{Aut}(\Gamma)}.
\]
Then $(C^{\bullet,\bullet}_\text{ass}(W), d, L)$ is also a double $\text{Sp}(W, \delta)$-chain complex and the chain map 
\[
C^{\bullet,\bullet}_\text{ass}(W) \to C^{\bullet,\bullet}_{CE}(\text{Der}^+_L(\hat{L}W))
\]
can be defined in the same way. In this case, we can also consider the double subcomplex $C^{\bullet,\bullet}_\text{ass}(N, Z)$ which consists of $N$-graded graphs without external vertices.
4. Applications and Examples

Example 4.1. For a cyclic minimal $A_\infty$-algebra $(H, I, m)$ with even degree, we have the map $C^\bullet_{CE}(W) \to C^\bullet_{CE}(\text{Der}^+_{\omega}(\hat{T}W))$. Here $\hat{T}W$ is the dual of the bar construction of $(H, I, m)$. The map induced by the chain map

$$C^0_{\text{ass}}(N, Z) \to C^0_{CE}(\text{Der}^+_{\omega}(\hat{T}W)) = \mathbb{R}$$

is known as the Kontsevich cocycle ([9, 16, 6]) of a cyclic $A_\infty$-algebra $(H, I, m)$.

Example 4.2. In the case of $Z = \{0\}$ and $\delta = 0$, the chain map

$$C^0_{\text{ass}}(0, \{0\}) \to C^0_{CE}(\text{Der}^+_{\omega}(\hat{T}W))^{Sp(W)}$$

is equal to Kontsevich’s chain map [9, 10].

Example 4.3. Suppose $X = \#_g(S^n \times S^n) \setminus \text{Int} D^{2n}$. Its Quillen model is described by:

$$L_X = L(u_1, v_1, \ldots, u_g, v_g) \ (\deg u_i = \deg v_i = n - 1), \quad \delta = 0,$$

$$\omega(u_i, v_j) = \delta_{ij}, \quad \omega(u_i, u_j) = \omega(v_i, v_j) = 0.$$  

It means $N = n - 1, W = \langle u_1, v_1, \ldots, u_g, v_g \rangle$ and $Z = \{n - 1\}$. Then the dgl $\langle \text{Der}^1_{j}(L_X), 0 \rangle$ is a Quillen model of $B\text{aut}_{\delta,0}(X)$ (which is proved in [1]). In the case, we can forget all special vertices in the graph complex since $\delta = 0$. So we have the chain map

$$C^\bullet_{\text{com}}(n - 1, \{n - 1\})/(\text{special vertices}) \to C^\bullet_{CE}(\text{Der}^+_{\omega}(L_X))^{Sp(W)}.$$

This map is constructed by [1] and it is proved that the map is an isomorphism under the limit $g \to \infty$.

Example 4.4. Suppose $X = \mathbb{C}P^3 \setminus \text{Int} D^6$. Its Quillen model is described by:

$$L_X = L(u_1, u_2) \ (\deg u_i = 2i - 1), \quad \delta = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_2},$$

$$\omega(u_1, u_2) = \omega(u_2, u_1) = 1.$$  

It means $N = 4, W = \langle u_1, u_2 \rangle$ and $Z = \{1, 3\}$. Then the dgl $\langle \text{Der}^1_{j}(L_X), \delta \rangle$ is a Quillen model of $B\text{aut}_{\delta,0}(X)$. Since $Sp(W, \delta) = 1$, we have the chain map

$$C^\bullet_{\text{com}}(W) \to C^\bullet_{CE}(\text{Der}^+_{\omega}(L_X)) = C^\bullet_{CE}(\text{Der}^+_{\omega}(L_X))^{Sp(W, \delta)}.$$  

We shall define a certain sub dgl $\delta$ of $\text{Der}_{\omega}(L_X)$. Put

$$A_1 = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_2}, \quad A_2 = \frac{1}{2}[u_2, u_2] \frac{\partial}{\partial u_1},$$

$$B_1 = \frac{1}{2}[u_1, u_1] \frac{\partial}{\partial u_1} + [u_1, u_2] \frac{\partial}{\partial u_2}, \quad B_2 = [u_1, u_2] \frac{\partial}{\partial u_1} + \frac{1}{2}[u_2, u_2] \frac{\partial}{\partial u_2}.$$  

Then we have

$$\delta(A_1) = \delta(B_1) = \delta(B_2) = 0,$$

$$\delta(A_2) = \frac{1}{2}[[u_1, u_1], u_2] \frac{\partial}{\partial u_2} + \frac{1}{2}[[u_2, u_2], u_1] \frac{\partial}{\partial u_2} = [A_1, A_2] = -[B_1, B_2] =: C,$$

$$[A_i, B_j] = [A_i, A_i] = [B_j, B_j] = 0 \ (i, j = 1, 2),$$

$$\deg A_1 = -1, \quad \deg A_2 = 5, \quad \deg B_1 = 1, \quad \deg B_2 = 3, \quad \deg C = 4.$$  

Here we put $\delta(Z) := [\delta, Z]$ for simplicity. By the relation above,

$$\delta := \langle A_1, A_2, B_1, B_2, C \rangle = \text{Der}^1_{\omega}(L_X) \oplus \text{Der}^2_{\omega}(L_X)$$.
is a sub dgl of Der_ω(L_X). Its positive truncation \( \mathfrak{d}^+ \) is described by
\[
\mathfrak{d}^+ = \langle A_2, B_1, B_2, C \rangle,
\]
\[
\delta(A_2) = -[B_1, B_2] = C, \quad \delta(B_1) = \delta(B_2) = \delta(C) = 0,
\]
\[
[A_2, B_i] = [A_2, A_2] = [B_i, B_i] = [A_2, C] = [B_i, C] = 0 \ (i = 1, 2).
\]
Let \( x, y_1, y_2, z \) be the suspension of the dual basis of \( A_2, B_1, B_2, C \). Then the Chevalley-Eilenberg complex of the dgl \( \mathfrak{d}^+ \) is written by
\[
C_{CE}(\mathfrak{d}^+) = \Lambda(x, y_1, y_2, z) \quad (\text{deg} \ x = 6, \ \text{deg} \ y_1 = 2, \ \text{deg} \ y_2 = 4, \ \text{deg} \ z = 5),
\]
\[
dx = dy_1 = dy_2 = 0, \ dz = x - y_1y_2
\]
and its total cohomology
\[
H^*_CE(\mathfrak{d}^+) = \Lambda(x, y_1, y_2)/(x - y_1y_2).
\]
Since \( \mathfrak{d}^+ \) is the rank \( \leq 2 \) part of Der_ω(L_X), the map \( H^*_CE(\text{Der}_ω(L_X)) \to H^*_CE(\mathfrak{d}^+) \) induced by the inclusion has a section. So non-trivial classes in \( H^*_CE(\mathfrak{d}^+) \) gives non-trivial classes in \( H^*_CE(\text{Der}_ω(L_X)) \).

The relation \( dz = x - y_1y_2 \) in the Chevalley-Eilenberg complex is corresponding to the relation in the graph complex \( C^{••}_\text{com}(W) \) described in Figure 6. Here the classes \( x \) and \( y_1y_2 \) corresponds to the first term and the sum of the second and third terms in the figure. Remark that \( y_1 \) and \( y_2 \) do not correspond to graphs without external vertices.

![Figure 6. the relation of graphs (the orientations are omitted)](image)

**Example 4.5.** Suppose \( X = \mathbb{C}P^4 \setminus \text{Int } D^8 \). Its Quillen model is described by:
\[
L_X = L(u_1, u_2, u_3) \quad (\text{deg} \ u_i = 2i - 1), \quad \delta = \frac{1}{2}[u_1, u_1]_1 \frac{\partial}{\partial u_2} + [u_1, u_2] \frac{\partial}{\partial u_3},
\]
\[
\omega(u_2, u_2) = \omega(u_1, u_3) = 1.
\]
It means \( N = 6, W = \langle u_1, u_2, u_3 \rangle \) and \( Z = \{1, 3, 5\} \). Then the dgl \( (\text{Der}_ω^+(L_X), \delta) \) is a Quillen model of \( B\text{aut}_{\delta,0}(X) \). Defining the linear transformation \( \tau \) by \( \tau(u_1) = -u_1, \ \tau(u_2) = u_2 \) and \( \tau(u_3) = -u_3 \), we have \( \text{Sp}(W, \delta) = \{1, \tau\} \). So \( C^{••}_\text{com}(W)^{\text{Sp}(W, \delta)} \) is generated by graphs labeled by \( u_1, u_2, u_3 \) satisfying \( \#\{u_1, u_3\text{-labeled vertex}\} \) is even. For simplicity, we put
\[
[u_{i_1}, \cdots, u_{i_k}] := [u_{i_1}, \cdots, u_{i_k}]_{\text{cyc}} = \sum_{s=1}^{k} (-1)^{s(k-s)} u_{i_{s+1}} \cdots u_{i_k} u_{i_1} \cdots u_{i_s} \in W^k_{\text{cyc}}.
\]
Using notations in Section 3.3, we can take a basis of \( W(3) \)
\[
\frac{1}{3} [u_1u_1u_1], \ [u_1u_2u_2], \ ([i < j] \in \{1, 2, 3\}), \ [u_1u_2u_3] + [u_1u_3u_2]
\]
and a basis of $W(4)$

$$[u_i u_i u_j u_j] \ (i < j),$$

$$[u_1 u_1 u_2 u_3] + [u_1 u_1 u_3 u_2], \ [u_1 u_2 u_3 u_2] - [u_1 u_3 u_2 u_2], \ [u_1 u_2 u_3 u_3] - [u_1 u_3 u_3 u_2].$$

We put the corresponding rank 1 and rank 2 basis of $\text{Der}_\omega(LX)$

$$A_{iii}, \ A_{iij}, \ A_{123}, \ B_{iijj}, \ B_{1123}, \ B_{1223}, \ B_{1233},$$

and these dual basis $x_{ijk}$ and $y_{ijkl}$ of $A_{ijk}$ and $B_{ijkl}$. Then by direct calculation we have

$$dy_{1122} = x_{222} - 2x_{123} + x_{122}x_{113} - x_{122}x_{122},$$

$$dy_{2233} = x_{333}x_{112} + x_{233}x_{222} - x_{223}x_{223} - 2x_{123}x_{233} + x_{133}x_{223},$$

$$dy_{1133} = x_{233} - x_{133}x_{113} - x_{123}x_{123},$$

$$dy_{1123} = x_{233} - x_{133},$$

$$dy_{1223} = x_{233} - x_{223}x_{113} + x_{223}x_{122} - x_{133}x_{122} - x_{123}x_{222} + x_{123}x_{123},$$

$$dy_{1233} = x_{333} - x_{233}x_{113} + x_{233}x_{122} - x_{123}x_{122} - x_{123}x_{223}.$$  

Here all terms appearing in the right-hand side of the equations are cocycles. The fifth relation is corresponding to the relation in the graph complex $C^{\bullet, \bullet}_\text{com}(W)$ described in Figure 7.

![Figure 7. The relation of graphs (the orientations are omitted)](image)

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