Hybridized Discontinuous Galerkin Method for Elliptic Interface Problems: Error Estimates Under Low Regularity Assumptions of Solutions

Masaru Miyashita · Norikazu Saito

Received: 13 January 2017 / Revised: 19 January 2018 / Accepted: 20 February 2018 / Published online: 27 February 2018
© Springer Science+Business Media, LLC, part of Springer Nature 2018

Abstract New hybridized discontinuous Galerkin (HDG) methods for the interface problem for elliptic equations are proposed. Unknown functions of our schemes are $u_h$ in elements and $\hat{u}_h$ on inter-element edges. That is, we formulate our schemes without introducing the flux variable. We assume that subdomains $\Omega_1$ and $\Omega_2$ are polyhedral domains and that the interface $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is polyhedral surface or polygon. Moreover, $\Gamma$ is assumed to be expressed as the union of edges of some elements. We deal with the case where the interface is transversely connected with the boundary of the whole domain $\overline{\Omega} = \Omega_1 \cap \Omega_2$. Consequently, the solution $u$ of the interface problem may not have a sufficient regularity, say $u \in H^2(\Omega)$ or $u|_{\Omega_1} \in H^2(\Omega_1)$, $u|_{\Omega_2} \in H^2(\Omega_2)$. We succeed in deriving optimal order error estimates in an HDG norm and the $L^2$ norm under low regularity assumptions of solutions, say $u|_{\Omega_1} \in H^{1+s}(\Omega_1)$ and $u|_{\Omega_2} \in H^{1+s}(\Omega_2)$ for some $s \in (1/2, 1]$, where $H^{1+s}$ denotes the fractional order Sobolev space. Numerical examples to validate our results are also presented.

Keywords Discontinuous Galerkin method · Elliptic interface problems · Error estimate · Fractional order Sobolev space

Mathematics Subject Classification 65N30 · 65N15 · 35J25

Norikazu Saito
norikazu@ms.u-tokyo.ac.jp

Masaru Miyashita
masaru.miyashita.z@shi-g.com

1 Technology Research Center, Sumitomo Heavy Industries, Ltd., Natsushima 19, Yokosuka, Kanagawa 237-8555, Japan

2 Graduate School of Mathematical Sciences, The University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan
1 Introduction

Let $\Omega$ be a polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$, with the boundary $\partial \Omega$. We suppose that $\Omega$ is divided into two disjoint polyhedral domains $\Omega_1$ and $\Omega_2$. Then, $\Gamma = \partial \Omega_1 \cap \partial \Omega_2$ implies the interface. See Fig. 1 for example.

Suppose that we are given a matrix-valued function $A = A(x)$ of $\Omega \rightarrow \mathbb{R}^{d \times d}$ such that:

- (smoothness) $A|_{\Omega_i}$ is a $C^1$ function in $\Omega_i$ ($i = 1, 2$);
- (symmetry) $\xi \cdot A(x)\eta = (A(x)\xi) \cdot \eta$, ($\xi, \eta \in \mathbb{R}^d, x \in \Omega$);
- (elliptic condition) $\lambda_{\min} |\xi|^2 \leq \xi \cdot A(x)\xi \leq \lambda_{\max} |\xi|^2$ ($\xi \in \mathbb{R}^d, x \in \Omega$)

with some positive constants $\lambda_{\min}$ and $\lambda_{\max}$. Hereinafter, $|\cdot| = |\cdot|_{\mathbb{R}^d}$ denotes the Euclidean norm in $\mathbb{R}^d$ and $\xi \cdot \eta$ the standard scalar product in $\mathbb{R}^d$.

We consider the following interface problem for second-order elliptic equations for the function $u = u(x)$, $x \in \Omega$,

\begin{align*}
\text{−} \nabla \cdot A \nabla u &= f \quad \text{in } \Omega \setminus \Gamma, \quad \text{(1a)} \\
u &= 0 \quad \text{on } \partial \Omega, \quad \text{(1b)} \\
u|_{\Omega_1} - u|_{\Omega_2} &= g_D \quad \text{on } \Gamma, \quad \text{(1c)} \\
(A \nabla u)|_{\Omega_1} \cdot n_1 + (A \nabla u)|_{\Omega_2} \cdot n_2 &= g_N \quad \text{on } \Gamma, \quad \text{(1d)}
\end{align*}

where $f, g_D, g_N$ are given functions, and $n_1, n_2$ are the unit normal vectors to $\Gamma$ outgoing from $\Omega_1, \Omega_2$, respectively. Moreover, $u|_{\Omega_1}$ stands for the restriction of $u$ to $\Omega_1$ for example.

We note that the gradient $\nabla u$ of the solution may be discontinuous across $\Gamma$, since $A$ may be discontinuous, even if $g_D = 0$ and $g_N = 0$.

Elliptic interface problem of the form (1) arises in many fields of applications such as fluid dynamics and solid mechanics. For instance, the first author has proposed (1) as a convenient model for computing sheath voltage wave form in the radio frequency plasma source within reasonable computational time (see [18]). The model involves the interface where the electronic potential and flux have nontrivial gaps; see also [9]. The case $g_D = 0$, which is sometimes referred to as elliptic problem with discontinuous (diffusion) coefficients, is formulated as the standard elliptic variational problem in $H^1(\Omega)$ and numerical methods are studied by many authors; see [2,5,6,23] for instance. On the other hand, the case $g_D \neq 0$ has further difficulties and a lot of numerical methods have been proposed (see [3,17,19] for example).

The present paper has dual purpose. The first one is to propose new schemes for solving (1) based on the hybridized discontinuous Galerkin (HDG) method. The HDG method is a class of the discontinuous Galerkin (DG) method that is proposed by Cockburn et al. (see [7]; see also [14,15,21] for other pioneering works). In the HDG method, we introduce a new

![Fig. 1 Examples of $\Omega_1, \Omega_2$ and $\Gamma$. Case (I) $\partial \Omega \cap \Gamma \neq \emptyset$. Case (II) $\partial \Omega \cap \Gamma = \emptyset$](image-url)
unknown function $\hat{u}_h$ on inter-element edges in addition to the usual unknown function $u_h$ in elements. We can eliminate $u_h$ from the resulting linear system and obtain the system only for $\hat{u}_h$; consequently, the size of the system becomes smaller than that of the DG method. In this paper, we present another advantage of the HDG method. That is, elliptic interface problem (1) is readily discretized by the HDG method and the resulting schemes (11) and (12) described below naturally satisfy the consistency (see Lemma 5) together with the Galerkin orthogonality (see (22)). The resulting schemes are interpreted as an HDG version of the Nitsche method applied to the interface condition, although we do not intend to apply the Nitsche method directly. It should be kept in mind that Huynh et al. [12] proposed an HDG scheme for (1). They introduced further unknown function $q = A\nabla u$ and rewrote (1) into the system for $(u, q, \hat{u})$ based on the idea of [7], while our unknowns are only $(u, \hat{u})$ by following the idea of [20,21]. Herein, $\hat{u}$ denotes the trace of $u$ into inter-element edges. Moreover, results of numerical experiments were well discussed and no theoretical consideration was undertaken in [12].

The second purpose of this paper is to establish error estimates for the HDG method when a sufficient regularity of solution, say $u \in H^2(\Omega)$, could not be assumed. Actually, if $g_D \neq 0$, the solution cannot be continuous across $\Gamma$. Moreover, we do not always have partial regularities $u|_{\Omega_i} \in H^2(\Omega_i)$, $i = 1, 2$. As a matter of fact, if $\partial \Omega \cap \Gamma \neq \emptyset$, then we know that $u|_{\Omega_i}$ may not belong to $H^2(\Omega_i)$, even when $\Gamma$ and $\partial \Omega$ are sufficiently smooth; see Remark 2. To surmount of this obstacle, we employ the fractional order Sobolev space $H^{1+s}(\Omega_i), s \in (1/2, 1], i = 1, 2$, and are going to attempt to derive an error estimate in an HDG norm $\| \cdot \|_{1+s,h}$ defined in terms of the $H^{1+s}(\Omega_i)$-seminorms (see (19)). One of our final error estimate reads (see Theorem 12)

$$\| u - u_h \|_{1+s,h} \leq C h^5 \left( \| u \|_{H^{1+s}(\Omega_1)} + \| u \|_{H^{1+s}(\Omega_2)} \right),$$

where $u = (u, \hat{u})$ and $u_h = (u_h, \hat{u}_h)$. Moreover, we also derive (see Theorem 13)

$$\| u - u_h \|_{L^2(\Omega)} \leq C h^{2s} \left( \| u \|_{H^{1+s}(\Omega_1)} + \| u \|_{H^{1+s}(\Omega_2)} \right),$$

following the Aubin–Nitsche duality argument. To derive those inequalities, we improve the standard boundedness inequality for the bilinear form (see Lemma 10) and inverse inequality (see Lemma 9) to fit our purpose. We note that those results are actually optimal order estimates, since we assume only $u|_{\Omega_1} \in H^{1+s}(\Omega_1)$ and $u|_{\Omega_2} \in H^{1+s}(\Omega_2)$.

In this paper, we concentrate our consideration on the case where $\Omega_1$ and $\Omega_2$ are polyhedral domains in order to avoid unessential complications about approximation of smooth surfaces/curves. The case of a smooth $\Gamma$ is of great interest; we postpone it for future study. On the other hand, an advantage of this study is to treat the case where the interface $\Gamma$ is transversely connected with $\partial \Omega$, say $\partial \Omega \cap \Gamma \neq \emptyset$, as shown for illustration in case (I) of Fig. 1. In actual applications, the both cases (I) and (II) are of interest; however it seems that little is known for case (I) from the viewpoint of theoretical numerical analysis. Below, we only consider the case $\partial \Omega \cap \Gamma \neq \emptyset$ (case (I)), since the modification to the case $\partial \Omega \cap \Gamma = \emptyset$ (case (II)) is readily and straightforward.

This paper is composed of five sections with an appendix. In Sect. 2, we recall the variational formulation of (1) and state our HDG schemes. The consistency is also proved there. The well-posedness of the schemes is verified in Sect. 3. Section 4 is devoted to error analysis using the fractional order Sobolev space. Finally, we conclude this paper by reporting numerical examples to confirm our error estimates in Sect. 5. In the appendix, we state the proof of a modification of inverse inequality (Lemma 9).
2 Variational Formulation and HDG Schemes

As stated above, \( \Omega, \Omega_1 \) and \( \Omega_2 \) are supposed to be polyhedral domains. Moreover, we assume the following:

\[ \partial \Omega \cap \Gamma \neq \emptyset. \]  

(H1)

That is, we consider only case (I) in Fig. 1. As mentioned in Introduction, the modification to the case \( \partial \Omega \cap \Gamma = \emptyset \) is readily and straightforward, if \( \Omega, \Omega_1 \) and \( \Omega_2 \) are all polyhedral domains.

To state a variational formulation, we need several function spaces. Namely, we use \( L^2(\Omega), H^m(\Omega), m \) being a positive integer, \( H^1_0(\Omega), L^2(\Gamma), H^{1/2}(\Gamma), H^0_{3/2}(\Gamma) \) and so on. We follow the notation of [16] for those Lebesgue and Sobolev spaces and their norms. The standard seminorm of \( H^m(\Omega) \) is denoted by \( |v|_{H^m(\Omega)} \). Supposing that \( S \) is a part of \( \partial \Omega \) or \( \Gamma \), we let \( \gamma(\Omega, S) \) be the trace operator from \( H^1(\Omega) \) into \( L^2(S) \). Set

\[ H^1_2(\Omega_i) = \{ v \in H^1(\Omega_i) | \gamma(\Omega_i, \partial \Omega \cap \partial \Omega_i)v = 0 \}, \quad i = 1, 2. \]

Further set \( \gamma_i = \gamma(\Omega_i, \Gamma), i = 1, 2 \). We introduce

\[ V = \{ v \in L^2(\Omega) | v|_{\Omega_i} \in H^1_2(\Omega_i), \quad i = 1, 2 \} \]

and write \( v_i = v|_{\Omega_i}, i = 1, 2 \), for \( v \in V \).

Variational formulation of (1) is given as follows: Find \( u \in V \) such that

\[ \gamma_1 u_1 - \gamma_2 u_2 = g_D \quad \text{on} \ \Gamma, \]  

\[ a(u, v) = \int_{\Omega} f v \ dx + \int_{\Gamma} g_N v \ dS \quad (\forall v \in H^1_0(\Omega)), \]

where

\[ a(u, v) = \int_{\Omega_1} A \nabla u_1 \cdot \nabla v_1 \ dx + \int_{\Omega_2} A \nabla u_2 \cdot \nabla v_2 \ dx. \]

To state the well-posedness of Problem (2), we have to recall the so-called Lions-Magenes space (see [16, §1.11.5])

\[ H^{1/2}_{00}(\Gamma) = \{ \mu \in H^{1/2}(\Gamma) | \varrho^{-1/2} \mu \in L^2(\Gamma) \} \]

which is a Hilbert space equipped with the norm \( \| \mu \|^2_{H^{1/2}_{00}(\Gamma)} = \| \mu \|^2_{H^{1/2}(\Gamma)} + \| \varrho^{-1/2} \mu \|^2_{L^2(\Gamma)}. \)

Herein, \( \varrho \in C^\infty(\overline{\Gamma}) \) denotes any positive function satisfying \( \varrho|_{\partial \Gamma} = 0 \) and, for \( x_0 \in \partial \Gamma \), \( \lim_{x \to x_0} \varrho(x)/\text{dist}(x, \partial \Gamma) = \varrho_0 > 0 \) with some \( \varrho_0 > 0 \). In particular, \( H^{1/2}_{00}(\Gamma) \) is strictly included in \( H^{1/2}(\Gamma) \). The following result follows directly from [10, Theorem 2.5] and [11, Theorem 1.5.2.3]. (A partial result is also reported in [24, Theorems 1.1 and 5.1].)

**Lemma 1** The trace operator \( v \mapsto \mu = \gamma_1 v \) is a linear and continuous operator of \( H^1_0(\Omega_1) \to H^{1/2}_{00}(\Gamma) \). Conversely, there exists a linear and continuous operator \( E_1 \) of \( H^{1/2}_{00}(\Gamma) \to H^1_0(\Omega_1) \), which is called a lifting operator, such that \( \gamma_1(E_1 \mu) = \mu \) for all \( \mu \in H^{1/2}_{00}(\Gamma) \). The same propositions remain true if \( \gamma_1 \) and \( \Omega_1 \) are replaced by \( \gamma_2 \) and \( \Omega_2 \), respectively.

Suppose that

\[ f \in L^2(\Omega), \quad g_D \in H^{1/2}_{00}(\Gamma) \quad \text{and} \quad g_N \in L^2(\Gamma). \]  

(H2)
In view of Lemma 1, there is \( \tilde{g}_D \in V \) such that \( \gamma \tilde{g}_D = \gamma_2 \tilde{g}_D = g_D \) on \( \Gamma \) and \( \| \tilde{g}_D \|_{H^{1/2}(\Gamma)} \leq C \| g_D \|_{H^{1/2}(\Gamma)} \).

Hereinafter, the symbol \( C \) denotes various generic positive constants depending on \( \Omega \). In particular, it is independent of the discretization parameter \( h \) introduced below. If it is necessary to specify the dependence on other parameters, say \( \mu_1, \mu_2, \ldots \), then we write them as \( C(\mu_1, \mu_2, \ldots) \).

Therefore, we can apply the Lax–Milgram theory to conclude that the problem (2) admits a unique solution \( u \in V \) satisfying

\[
\| u_1 \|_{H^1(\Omega_1)} + \| u_2 \|_{H^1(\Omega_2)} \leq C(\| f \|_{L^2(\Omega)} + \| g_D \|_{H^{1/2}(\Gamma)} + \| g_N \|_{L^2(\Gamma)}),
\]

where \( C = C(A) \).

Next we review the regularity property of the solution \( u \). Suppose further that \( g_D \in H^{3/2}_0(\Gamma) \) and \( g_N \in H^{1/2}(\Gamma) \).

However, in general, we do not expect that \( u_1 \in H^2(\Omega_1) \) and \( u_2 \in H^2(\Omega_2) \), because of the presence of intersection points \( \Gamma \cap \partial \Omega \). (Even if we consider the case \( \Gamma \cap \partial \Omega = \emptyset \), we may have \( u_1 \notin H^2(\Omega_1) \) and \( u_2 \notin H^2(\Omega_2) \).) To state regularity properties of \( u_1 \) and \( u_2 \), it is useful to introduce fractional order Sobolev spaces. We set

\[
|v|^2_{H^{1+\theta}(\omega)} = \sum_{i=1}^d \int_{\omega \times \omega} \frac{|\partial_i v(x) - \partial_i v(y)|^2}{|x-y|^{d+2\theta}} \, dx dy,
\]

where \( \omega \subset \mathbb{R}^d, \theta \in (0,1), \) and \( \partial_i = \partial/\partial x_i \). Then, fractional order Sobolev spaces \( H^{1+\theta}(\Omega_i), i = 1, 2 \), are defined as

\[
H^{1+\theta}(\Omega_i) = \{ v \in H^1(\Omega_i) \mid \| v \|^2_{H^{1+\theta}(\Omega_i)} = \| v \|^2_{H^1(\Omega_i)} + |v|^2_{H^{1+\theta}(\Omega_i)} < \infty \} \quad (3b)
\]

We assume that

\[
g_D \in H_0^{3/2}(\Gamma) \quad \text{and} \quad g_N \in H^{1/2}(\Gamma)
\]

and that the solution \( u \) of (2) has the following regularity property,

\[
\begin{align*}
\left\{ \begin{array}{l}
\ u_1 \in H^{1+s}(\Omega_1), \quad u_2 \in H^{1+s}(\Omega_2) \\
\ N_s(u) \leq C(\| f \|_{L^2(\Omega)} + \| g_D \|_{H^{1/2}_0(\Gamma)} + \| g_N \|_{H^{1/2}(\Gamma)})
\end{array} \right. 
\end{align*}
\]

for some \( s \in (1/2,1) \), where \( N_s(u) = \| u_1 \|_{H^{1+s}(\Omega_1)} + \| u_2 \|_{H^{1+s}(\Omega_2)} \) and \( C = C(A) \).

Remark 2 We can find no explicit reference to (4). Nevertheless, we consider the problem under (4) on the analogy of Poisson interface problem. As an illustration, we consider a smooth bounded domain \( \Omega \) in \( \mathbb{R}^2 \). Suppose that \( x_0 \) is an intersection point of \( \partial \Omega \) and \( \Gamma \). We then set \( U = \partial \Omega \cap \Omega \) and \( U_i = U \cap \Omega_i, i = 1, 2 \), where \( \partial \) is a neighbourhood of \( x_0 \). Assume that \( U \) contains no corners of \( \partial \Omega \cup \Gamma \) and no other intersection points except for \( x_0 \). Consider the unique solution \( w \in H^1_0(\Omega) \) of

\[
\kappa_1 \int_{\Omega_1} \nabla w \cdot \nabla v \, dx + \kappa_2 \int_{\Omega_2} \nabla w \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad (\forall v \in H^1_0(\Omega)),
\]

where \( f \in L^2(\Omega) \) and \( \kappa_1, \kappa_2 \) are positive constants with \( \kappa_1 \neq \kappa_2 \). Then, we have (see [22, Theorem 6.2])

\[
w|_{\Omega_i} \in H^{1+b}(U_i), \quad i = 1, 2, \quad b = \min \left\{ 1, \left. \frac{1}{2\theta} \right\} \in (1/2, 1],
\]

\( \odot \) Springer
Fig. 2 Triangulation satisfying (H4)

where \( \theta \) denotes the maximum interior angle of \( \partial \Omega_1 \) and \( \partial \Omega_2 \) at \( x_0 \). It should be kept in mind that the regularity theory guarantees (5) at worst; the theory does not deny the possibility of \( w_{|\Omega_i} \in H^2(U_i), i = 1, 2 \).

We proceed to the presentation of our HDG schemes. We introduce a family of quasi-uniform triangulations \( \{ T_h \}_h \) of \( \Omega \). That is, \( \{ T_h \}_h \) is a family of shape-regular triangulations that satisfies the inverse assumptions (see [4, (4.4.15)]). Hereinafter, we set \( h = \max \{ h_K \mid K \in T_h \} \), where \( h_K \) denotes the diameter of \( K \). Let \( \mathcal{E}_h = \{ e \subset \partial K \mid K \in T_h \} \) be the set of all faces \( (d = 3) \) edges \( (d = 2) \) of elements, and set \( S_h = \cup_{e \in \mathcal{E}_h} e \). Moreover, we let \( \mathcal{E}_{h, \beta \Omega} = \{ e \in \mathcal{E}_h \mid e \subset \partial \Omega \} \). We assume that there is a positive constant \( \nu_1 \) which is independent of \( h \) such that

\[
\max \left\{ \frac{h_e}{\rho_K}, \frac{h_K}{h_e} \right\} \leq \nu_1 \quad (\forall e \subset \partial K, \forall K \in \mathcal{T}_h \in \{ T_h \}_h), \quad (H3)
\]

where \( h_e \) denotes the diameter of \( e \) and \( \rho_K \) the diameter of the inscribed ball of \( K \).

We use the following function spaces:

\[
H^{1+s}(T_h) = \{ v \in L^2(\Omega) \mid v|_K \in H^{1+s}(K), \ K \in T_h \};
\]

\[
L^2_{\partial \Omega}(S_h) = \{ \tilde{v} \in L^2(S_h) \mid \tilde{v}|_e = 0, \ e \in \mathcal{E}_{h, \beta \Omega} \};
\]

\[
H^{1/2}_{\partial \Omega}(S_h) = \{ \tilde{v} \in H^{1/2}(S_h) \mid \tilde{v}|_e = 0, \ e \in \mathcal{E}_{h, \beta \Omega} \};
\]

\[
V^{1+s}(h) = H^{1+s}(T_h) \times H^{1/2}_{\beta \Omega}(S_h)
\]

for \( s \in (1/2, 1] \).

Further, we assume that

there exists a subset \( \mathcal{E}_{h, \Gamma} \) of \( \mathcal{E}_h \) such that \( \Gamma = \bigcup_{e \in \mathcal{E}_{h, \Gamma}} e \),

as shown for illustration in Fig. 2.

We then set \( \mathcal{E}_{h, 0} = \mathcal{E}_h \setminus (\mathcal{E}_{h, \Gamma} \cup \mathcal{E}_{h, \beta \Omega}) \). Assumption (H4) implies that \( \mathcal{T}_{h,i} = \{ K \in T_h \mid K \subset \Omega_i \} \) is a triangulation of \( \Omega_i \) for \( i = 1, 2 \) and we can write

\[
a(u, v) = \sum_{K \in \mathcal{T}_h} \int_K A \nabla u \cdot \nabla v \ dx. \quad (6)
\]

Throughout this paper, we always assume that (H1), (H2), (H2’), (H3) and (H4) are satisfied.

For derivation of our HDG schemes, we examine a local conservation property of the flux of the solution \( u \). Let \( K \in \mathcal{T}_h \). Recall that, if \( u \) is suitably regular, we have by (1a) and Gauss–Green’s formula.
for any \( w \in H^1(K) \), where \( n_K \) denotes the outer normal vector to \( \partial K \). As mentioned above, the left-hand side of this identity is well-defined, since (4) is assumed for some \( s \in (1/2, 1] \). However, we derive local conservation properties (Lemmas 3 and 4 below) without using the further regularity property (4). That is, based on the identity above, we introduce a functional \( \langle A \nabla u \cdot n_K , \cdot \rangle_{\partial K} \) on \( H^{1/2}(\partial K) \) by

\[
\langle A \nabla u \cdot n_K , \phi \rangle_{\partial K} = \int_K A \nabla u \cdot \nabla (Z\phi) \, dx - \int_K f (Z\phi) \, dx
\]  

for any \( \phi \in H^{1/2}(\partial K) \), where \( Z\phi \in H^1(K) \) denotes a suitable extension of \( \phi \) such that \( \| Z\phi \|_{H^1(K)} \leq C \| \phi \|_{H^{1/2}(\partial K)} \). Actually, the definition of \( \langle A \nabla u \cdot n_K , \cdot \rangle_{\partial K} \) above does not depend on the way of extension of \( \phi \). Below, for the solution \( u \) of (1), we simply write

\[
\int_{\partial K} (A \nabla u \cdot n_K) \phi \, dS = \int_K A \nabla u \cdot \nabla (Z\phi) \, dx - \int_K f (Z\phi) \, dx
\]  

(8) to express (7).

The following lemmas are readily obtainable consequences of (6) and (8).

**Lemma 3** For the solution \( u \) of (2), we have

\[
\sum_{K \in T_h} \int_{\partial K} (A \nabla u \cdot n_K) \hat{v} \, dS = \int_{\Gamma} g_N \hat{v} \, dS \quad (\hat{v} \in H^{1/2}_{\Omega}(S_h)).
\]  

(9)

**Lemma 4** For the solution \( u \) of (2), we have

\[
\sum_{K \in T_h} \int_K A \nabla u \cdot \nabla v \, dx + \sum_{K \in T_h} \int_{\partial K} (A \nabla u \cdot n_K) (\hat{v} - v) \, dS
\]  

\[
= \sum_{K \in T_h} \int_K f v \, dx + \int_{\Gamma} g_N \hat{v} \, dS \quad ((v, \hat{v}) \in V^1(h)).
\]  

(10)

We discretize the expression (10) by the idea of HDG. We use the following finite element spaces:

\[
V_h = \nabla_{h} \times \hat{V}_h;
\]

\[
V_h = V_{h,k} = \{ v \in H^1(T_h) \mid v|_K \in P_k(K), \ K \in T_h \}, \quad k \geq 1; \text{ integer};
\]

\[
\hat{V}_h = \hat{V}_{h,l} = \{ \hat{v} \in L^2_{\Omega}(S_h) \mid \hat{v}|_e \in P_l(e), \ e \in \mathcal{E}_{h,0} \cup \mathcal{E}_{h,1} \}, \quad l \geq 1; \text{ integer};
\]

where \( P_k(K) \) denotes the set of all polynomials defined in \( K \) with degree \( \leq k \).

At this stage, we can state our scheme: Find \( u_h = (u_h, \hat{u}_h) \in V_h \) such that

\[
B_h(u_h, v_h) = L_h(v_h) \quad (\forall v_h = (v_h, \hat{v}_h) \in V_h),
\]  

(11a)

where

\[
B_h(u_h, v_h) = \sum_{K \in T_h} \int_K A \nabla u_h \cdot \nabla v_h \, dx - \sum_{K \in T_h} \int_{\partial K} (A \nabla u_h \cdot n_K)(v_h - \hat{v}_h) \, dS
\]  

\[
= B_1 = B_2
\]
In view of Lemma 4, we know that

\[
\begin{aligned}
- \sum_{K \in T_h} \int_{\partial K} (A \nabla v_h \cdot n_K) (u_h - \hat{u}_h) \, dS + \sum_{K \in T_h} \sum_{e \subset \partial K} \int_e \eta_e (u_h - \hat{u}_h) (v_h - \hat{v}_h) \, dS = \mathcal{B}_3
\end{aligned}
\]

and

\[
\begin{aligned}
L_h(v_h) &= \sum_{K \in T_h} \int_K f v_h \, dx + \int_{\Gamma} g_N \hat{u} \, dS - \sum_{K \in T_h} \sum_{e \subset \partial K} \int_e \sigma_{K,e} \frac{1}{2} g_D (A \nabla v_h \cdot n_K) \, dS \\
&\quad + \sum_{K \in T_h} \sum_{e \subset \partial K} \int_e \sigma_{K,e} \frac{\eta_e}{h_e} g_D (v_h - \hat{v}_h) \, dS. \\
&= \mathcal{B}_4
\end{aligned}
\]

(11b)

Therein, \(\sigma_{K,e}\) is defined by

\[
\sigma_{K,e} = \begin{cases}
1 & (e \in \mathcal{E}_h, e \subset \partial K, K \in \mathcal{T}_h, 1) \\
-1 & (e \in \mathcal{E}_h, e \subset \partial K, K \in \mathcal{T}_h, 2) \\
0 & \text{otherwise}
\end{cases}
\]

(11d)

and \(\eta_e\) denotes the penalty parameter such that

\[
0 < \eta_{\text{min}} = \inf_{T_h \in \mathcal{T}(T_h)} \min_{e \in \mathcal{E}_h} \eta_e, \quad \eta_{\text{max}} = \sup_{T_h \in \mathcal{T}(T_h)} \max_{e \in \mathcal{E}_h} \eta_e < \infty.
\]

(11e)

The main advantage of the scheme (11) is stated as the following lemma.

**Lemma 5** *(Consistency)* Let \(u \in V\) be the solution of (2) and introduce \(\hat{u} \in H^{1/2}_{\sigma\Omega}(S_h)\) by

\[
\hat{u} = \begin{cases}
\frac{1}{2} [\gamma(K_1, e) u + \gamma(K_2, e) u] & (e \in \mathcal{E}_{h,0} \cup \mathcal{E}_{h,1}, e \subset \partial K_1 \cap \partial K_2) \\
\gamma(K, e) u & (e \in \mathcal{E}_{h,\partial\Omega}, e \subset \partial K).
\end{cases}
\]

Then, \(u = (u, \hat{u}) \in V^1(h)\) solves

\[
B_h(u, v_h) = L_h(v_h) \quad (\forall v_h \in V_h).
\]

**Proof** In view of Lemma 4, we know that \(B_1 + B_2 = L_1\). On \(e \in \mathcal{E}_h\), we have by (2a)

\[
u - \hat{u} = \begin{cases}
u_1 - (u_1 + u_2) / 2 = g_D / 2 & (e \in \mathcal{E}_{h,1}, e \subset \partial K, K \in \mathcal{T}_h, 1) \\
u_2 - (u_1 + u_2) / 2 = -g_D / 2 & (e \in \mathcal{E}_{h,1}, e \subset \partial K, K \in \mathcal{T}_h, 2) \\
0 & \text{otherwise},
\end{cases}
\]

where \(u_j = \gamma(\Omega_j, e) u, j = 1, 2\). Therefore, we obtain \(B_3 = L_2\) and \(B_4 = L_3\). \(\square\)

An alternative scheme is given as

\[
B'_h(u_h, v_h) = L'_h(v_h) \quad (\forall v_h \in V_h).
\]

(12a)

where

\[
L'_h(v_h) = L_1 + L_2 + \sum_{K \in T_h} \sum_{e \subset \partial K} \int_e \sigma_{K,e} \frac{\eta_e}{h_e} g_D (v_h - \hat{v}_h) \, dS
\]

(12b)
and
\[
\sigma'_{K,e} = \begin{cases} 
1 & (e \in E_h, K \in T_{h,1}) \\
0 & \text{(otherwise)}.
\end{cases}
\] (12c)

Lemma 5 remains valid for (12) with an obvious modification of the definition of \( \hat{u} \). Therefore, all the following results also remain true for (12). Hence, we explicitly study only (11) below.

## 3 Well-Posedness

In this section, we establish the well-posedness of the scheme (11). First, we recall the following standard results; (13) is the standard inverse inequality (see [4, Lemma 4.5.3]) and (14) follows from the standard trace inequalities (see also “Appendix A”).

**Lemma 6** For \( K \in T_h \), we have following inequalities.

**Inverse inequality**
\[
|v_h|_{H^1(K)} \leq C_{IV} h_K^{-1} \|v_h\|_{L^2(K)} \quad (v_h \in V_h). 
\] (13)

**Trace inequalities**
\[
\|v\|_{L^2(e)}^2 \leq C_{0,T} h_e^{-1} \left( \|v\|_{L^2(K)}^2 + h_K^2 \|v\|_{H^1(K)}^2 \right) \quad (v \in H^1(K)), 
\] (14a)
\[
\|\nabla v\|_{L^2(e)}^2 \leq C_{1,T} h_e^{-1} \left( \|v\|_{H^1(K)}^2 + h_K^2 \|v\|_{H^2(K)}^2 \right) \quad (v \in H^2(K)). 
\] (14b)

Those \( C_{IV}, C_{0,T} \) and \( C_{1,T} \) are absolute positive constants.

We use the following HDG norms:
\[
\|v\|_{1,h}^2 = \sum_{K \in T_h} |v|_{H^1(K)}^2 + \sum_{K \in T_h} \sum_{e \in \partial K} \eta_e \|\hat{v} - v\|_{L^2(e)}^2; 
\] (15a)
\[
\|v\|_{2,h}^2 = \sum_{K \in T_h} |v|_{H^1(K)}^2 + \sum_{K \in T_h} h_K^2 \|v\|_{H^2(K)}^2 + \sum_{K \in T_h} \sum_{e \in \partial K} \eta_e \|\hat{v} - v\|_{L^2(e)}^2. 
\] (15b)

Moreover, set
\[
\alpha = \max \left\{ \sup_{x \in \Omega_1} \frac{|A(x)\xi|}{|\xi|}, \sup_{x \in \Omega_2} \sup_{\xi \in \mathbb{R}^d} \left| \frac{|A(x)\xi|}{|\xi|} \right| \right\}.
\]

**Remark 7** In view of (13), two norms \( \|v\|_{1,h} \) and \( \|v\|_{2,h} \) are equivalent norms in the finite dimensional space \( V_h \). That is, there exists a positive constant \( C_0 \) that depends only on \( C_{IV} \) such that
\[
\|v\|_{1,h} \leq \|v\|_{2,h} \leq C_0 \|v\|_{1,h} \quad (v_h \in V_h). 
\] (16)

**Lemma 8** (Boundedness) For any \( \eta_{\min} > 0 \), there exists a positive constant \( C_b = C_b(\alpha, \eta_{\min}, d, C_{1,T}) \) such that
\[
B_h(w, v) \leq C_b \|w\|_{2,h} \|v\|_{2,h} \quad (w, v \in V^2(h)).
\] (17)

(Coercivity) There exist positive constants \( \eta^* = \eta^*(\alpha, \eta_{\min}, d, C_{1,T}, C_{IV}) \) and \( C_c = C_c(\eta_{\min}, C_{IV}) \) such that, if \( \eta_{\min} \geq \eta^* \), we have
\[
B_h(v_h, v_h) \geq C_c \|v_h\|_{2,h}^2 \quad (v_h \in V_h).
\] (18)
Both inequalities are essentially well-known; however, we briefly state their proofs, since the contribution of parameters on $C_c$ and $C_b$ should be clarified. Moreover, we shall state the extension of (17) below (see Lemma 10) so it is useful to recall the proof of (17) at this stage.

Proof (Proof of Lemma 8) (Boundedness) Let $w = (w, \hat{w}), v = (v, \hat{v}) \in V^2(h)$. For $e \subset \partial K, K \in \mathcal{T}_h$, we have by Schwarz’ inequality
\[
\int_e (A \nabla w \cdot n_K)(v - \hat{v}) \, dS \leq \alpha \left( \frac{h_e}{\eta_e} \right)^{1/2} \| \nabla w \|_{L^2(e)} \cdot \left( \frac{\eta_e}{h_e} \right)^{1/2} \| v - \hat{v} \|_{L^2(e)}.
\]
Hence, using Schwarz’ inequality again,
\[
B_h(w, v) \leq \sum_{K \in \mathcal{T}} \sum_{e \subset \partial K} \sum_{K \in \mathcal{T}_h} \frac{\alpha}{\eta_{\min}} h_e^{1/2} \| \nabla w \|_{L^2(e)} \cdot \left( \frac{\eta_e}{h_e} \right)^{1/2} \| v - \hat{v} \|_{L^2(e)}
\]
\[
+ \sum_{K \in \mathcal{T}} \left( \sum_{e \subset \partial K} \left( \frac{\eta_e}{h_e} \right)^{1/2} \| w - \hat{w} \|_{L^2(e)} \cdot \left( \frac{\eta_e}{h_e} \right)^{1/2} \| v - \hat{v} \|_{L^2(e)} \right) \right) \frac{1}{2}
\]
\[
\leq C \left[ \sum_{K \in \mathcal{T}_h} |v|_{H^1(K)}^2 + \sum_{e \subset \partial K} \frac{\eta_e}{h_e} \| \nabla w \|_{L^2(e)} \cdot \left( \frac{\eta_e}{h_e} \right)^{1/2} \| v - \hat{v} \|_{L^2(e)} \right] \frac{1}{2}
\]
\[
+ \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \left( \frac{\eta_e}{h_e} \| \nabla v \|_{L^2(e)} \cdot \left( \frac{\eta_e}{h_e} \right)^{1/2} \| v - \hat{v} \|_{L^2(e)} \right) \frac{1}{2}
\]
Therefore, using (14b), we obtain (17).
(Coercivity) Let $v_h = (v_h, \hat{v}_h) \in V_h$. Then,
\[
B_h(v_h, v_h) \geq \lambda_{\min} \sum_{K \in \mathcal{T}_h} |v_h|_{H^1(K)}^2
\]
\[
+ \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \frac{\eta_e}{h_e} \| \hat{v} - v \|_{L^2(e)}^2
\]
\[
- 2 \sum_{K \in \mathcal{T}_h} \int_{\partial K} (A \nabla v_h \cdot n_K)(v_h - \hat{v}_h) \, dS.
\]
Letting $e \subset \partial K, K \in \mathcal{T}_h$, we have by (14b), (13), Schwarz’ and Young’s inequalities
\[
\int_e (A \nabla v_h \cdot n_K)(v_h - \hat{v}_h) \, dS \leq \alpha \| \nabla v_h \|_{L^2(e)} \| v_h - \hat{v}_h \|_{L^2(e)}
\]
\[
\leq \alpha C_1 \eta_e^{-1/2} \left( |v_h|_{H^1(K)}^2 + h_e^2 |v_h|_{H^2(K)}^2 \right)^{1/2} \| v_h - \hat{v}_h \|_{L^2(e)}
\]
\[
\leq C_2 (\delta \eta_e)^{-1/2} |v_h|_{H^1(K)} \cdot \left( \frac{\eta_e \delta}{h_e} \right)^{1/2} \| v_h - \hat{v}_h \|_{L^2(e)}
\]
At this stage, choosing $\delta$ using (20) instead of (14b). (Trace inequality) Lemma 9 for $s$. Then, we have the Galerkin orthogonality

$$B_h(w_h, w_h) = 0 \quad (\forall w_h \in V_h).$$

4 Error Analysis

This section is devoted to error analysis of our HDG scheme. We use a new HDG norm:

$$\| v \|_{1+s,h}^2 = \sum_{K \in T_h} |v|_{H^1(K)}^2 + \sum_{K \in T_h} h_K^{2s} |v|_{H^{1+s}(K)}^2 + \sum_{K \in T_h} \sum_{e \in \partial K} \frac{\eta_e}{h_e} \| \hat{\nu} - v \|_{L^2(e)}^2.$$ (19)

for $s \in (1/2, 1)$. We have to improve Lemmas 6 and 8 for our purpose. First, the trace inequality for functions of $H^{1+s}(K)$ is given as follows; the proof will be stated in “Appendix A”.

**Lemma 9** (Trace inequality) Let $s \in (1/2, 1)$. For $K \in T_h$, we have

$$|\nu v|_{L^2(e)}^2 \leq C_{1+s,T} h_e^{-1} \left( |v|_{H^1(K)}^2 + h_K^{2s} |v|_{H^{1+s}(K)}^2 \right) \quad (v \in H^{1+s}(K)).$$ (20)

Moreover, we deduce the following lemma in exactly the same way as the proof of Lemma 8 using (20) instead of (14b).

**Lemma 10** Let $s, t \in (1/2, 1)$. For any $\eta_{\min} > 0$, there exists a positive constant $C_{b,s,t} = C_{b,s,t}(\alpha, \eta_{\min}, d, C_{1+s,T}, C_{1+t,T}, s, t)$ such that

$$B_h(w, v) \leq C_{b,s,t} \| w \|_{1+s,h} \| v \|_{1+t,h} \quad (w \in V^{1+s}(h), \quad v \in V^{1+t}(h)).$$ (21)

**Theorem 11** Let $u \in V$ be the solution of (2) and assume that (4) for some $s \in (1/2, 1]$. Set $u \in V^{1+s}(h)$ as in Lemma 5. Moreover, let $u_h = (u_h, \hat{u}_h) \in V_h$ be the solution of (11). Then, we have the Galerkin orthogonality

$$B_h(u - u_h, v_h) = 0 \quad (\forall v_h \in V_h).$$ (22)
Moreover,
\[ \| u - u_h \|_{1+s,h} \leq C \inf_{v_h \in V_h} \| u - v_h \|_{1+s,h}. \] (23)

**Proof** Let \( v_h \in V_h \) be arbitrarily. Then, (22) is a consequence of (11) and Lemma 5. On the other hand,
\[ C_c \| v_h - u_h \|_{2,h}^2 \leq B_h(v_h - u_h, v_h - u_h) \] (by (18))
\[ \leq B_h(v_h - u, v_h - u_h) + B_h(u - u_h, v_h - u_h) \]
\[ \leq B_h(v_h - u, v_h - u_h) \] (by (22))
\[ \leq C_{b,s,1} \| v_h - u \|_{1+s,h} \| v_h - u_h \|_{2,h} \] (by (21))

This implies
\[ \| v_h - u_h \|_{2,h} \leq \frac{C_{b,s,1}}{C_c} \| v_h - u \|_{1+s,h}. \]

We apply the triangle inequality to obtain
\[ \| u - u_h \|_{1+s,h} \leq \| u - v_h \|_{1+s,h} + C \| v_h - u_h \|_{2,h} \]
\[ \leq \| u - v_h \|_{1+s,h} + C \| v_h - u \|_{1+s,h}, \]
which gives (23). \( \square \)

**Theorem 12** Under the same assumptions of Theorem 11, we have
\[ \| u - u_h \|_{1+s,h} \leq C h^s \left( \| u \|_{H^{1+s}(\Omega_1)} + \| u \|_{H^{1+s}(\Omega_2)} \right). \] (24)

**Proof** It is done by the standard method; see [1, Paragraph 4.3] for example. However, we state the proof, since it is not apparent how to estimate the third term of the left-hand side of (15b). First, we introduce \( u_I \in V_h \) as follows. Let \( K \in T_h \) and let \( u_{I,K} = (u_I)|_K \in P_k(K) \) be the Lagrange interpolation of \( u|_K \). We remark here that \( u_I \) is well-defined, since \( u|_K \in H^{1+s}(K) \).

Further, we introduce \( \hat{u}_I \in V_h \) by setting
\[ \hat{u}_I|_e = (u_{I,K_1}|_e + u_{I,K_2}|_e)/2 \]
for \( e \in E_{h,0} \cup E_{h,\Gamma} \), \( e = \partial K_1 \cap \partial K_2 \) and \( \hat{u}_I|_e = u_{I,K}|_e \) for \( e \in E_{h,\partial \Omega}, e \subset \partial K \). Then, letting \( w_h = (u_I, \hat{u}_I) \in V_h \), we derive an estimation for \( \| u - w_h \|_{1+s,h} \).

For \( e \in E_{h,0} \cup E_{h,\Gamma}, e \subset \partial K \), we have by (14a)
\[ \frac{\eta}{h_e} \| u - u_I \|^2_{L^2(e)} \leq C h_e^{-2} \left( \| u - u_I \|^2_{L^2(K)} + h_e^2 \| u - u_I \|^2_{H^1(K)} \right) \]
Hence, using (H3),
\[ \sum_{K \in T_h} \sum_{e \subset \partial K} \frac{\eta}{h_e} \| u - u_I \|^2_{L^2(e)} \leq C \sum_{K \in T_h} \left( h_K^{-2} \| u - u_I \|^2_{L^2(K)} + |u - u_I|^2_{H^1(K)} \right). \]

On the other hand, for \( e \in E_{h,0} \cup E_{h,\Gamma}, e = \partial K_1 \cap \partial K_2 \),
\[ \| \hat{u} - \hat{u}_I \|^2_{L^2(e)} \leq C \left( \| u |_{K_1} - u |_{K_1} \|^2_{L^2(e)} + \| u |_{K_2} - u |_{K_2} \|^2_{L^2(e)} \right) \]

Therefore, as above, we have
\[ \sum_{K \in T_h} \sum_{e \subset \partial K} \frac{\eta}{h_e} \| \hat{u} - \hat{u}_I \|^2_{L^2(e)} \leq C \sum_{K \in T_h} \left( h_K^{-2} \| u - u_I \|^2_{L^2(K)} + |u - u_I|^2_{H^1(K)} \right). \]
Consequently, we obtain
\[
\| u - w_h \|_{1+s,h}^2 
\leq C \sum_{K \in T_h} \left( h_{K}^{-2} \| u - u_I \|_{L^2(K)}^2 + |u - u_I|_{H^1(K)}^2 + h_{K}^{2s} |u - u_I|_{H^{1+s}(K)}^2 \right).
\]

At this stage, we recall
\[
|u - u_I|_{H_t(K)} \leq C h_s (\| u \|_{H^1(K)^t} + \| u \|_{H^{1+s}(K)^t}) \quad (0 \leq t \leq 2),
\]
where \( \cdot \) is understood as \( \| \cdot \|_{L^2(K)} \). See, for example, [8, Theorems 2.19, 2.22] where the case of integer \( t \) is explicitly mentioned. However, the extension to the case of non-integer \( t \in [0, 1+s] \) is straightforward, since the imbedding \( H^t(K) \subset H^{1+s}(K) \) is continuous.

Combining those inequalities, we deduce
\[
\| u - w_h \|_{1+s,h} \leq C h_s (\| u \|_{H^{1+s}(\Omega_1)} + \| u \|_{H^{1+s}(\Omega_2)}),
\]
which completes the proof. \( \square \)

**Theorem 13** Under the same assumptions of Theorem 11, we have
\[
\| u - u_h \|_{L^2(\Omega)} \leq C h^{2s} (\| u \|_{H^{1+s}(\Omega_1)} + \| u \|_{H^{1+s}(\Omega_2)}).
\]

**Proof** We follow the Aubin–Nitsche duality argument. Set \( e_h = u - u_h \) with \( e_h = u - u_h, \hat{e}_h = \hat{u} - \hat{u}_h \) and consider the adjoint problem: Find \( \psi \in V \) such that
\[
\gamma_1 \psi_1 - \gamma_2 \psi_2 = 0 \quad \text{on} \ \Gamma, \quad a(v, \psi) = \int_{\Omega} v e_h \, dx \quad (\forall v \in V). \quad (25)
\]
(Note that we have taken \( f = e_h, g_D = 0, g_N = 0 \) and used the symmetry of \( a \).) In view of (4), we have \( \psi_1 \in H^{1+s}(\Omega_1), \psi_2 \in H^{1+s}(\Omega_2) \) and
\[
N_s(\psi) \leq C \| e_h \|_{L^2(\Omega)}. \quad (26)
\]
As is verified in Lemma 4, \( \psi = (\psi, \hat{\psi}) \in V^{1+s}(h) \) satisfies
\[
B_h(v, \psi) = \int_{\Omega} v e_h \, dx \quad (\forall v \in V(h)).
\]
HDG scheme for (25) reads as follows: Find \( \psi_h \in V_h \) such that
\[
B_h(v_h, \psi_h) = \int_{\Omega} v e_h \, dx \quad (\forall v_h \in V_h).
\]
Then, we have
\[
\| e_h \|_{L^2(\Omega)}^2 = B_h(e_h, \psi) = B_h(e_h, \psi - \psi_h) \quad (\text{by } (22))
\leq C \| e_h \|_{1+s,h} \| \psi - \psi_h \|_{1+s,h} \quad (\text{by } (21))
\leq C h^s N_s(u) \cdot h^s N_s(\psi) \quad (\text{by } (24))
\leq C h^{2s} N_s(u) \cdot \| e_h \|_{L^2(\Omega)}, \quad (\text{by } (26))
\]
which completes the proof. \( \square \)
5 Numerical Examples

In this section, we confirm the validity of error estimates described in Theorems 12 and 13 using simple numerical examples. We consider polyhedral domains (III), (IV) as portrayed in Fig. 3 and let $A = I$, $f = 0$, $g_N = 0$ and $g_D = \sin(2\pi x_1)$. We use uniform meshes composed of congruent right-angled isosceles triangles for triangulation $\mathcal{T}_h$. The penalty parameter $\eta_e$ is set as $\eta_e = 1000$ for any edge $e$.

Let $u_h = (u_h, \hat{u}_h) \in V_h$ be the solution of (11). Because the exact solutions are not available, we employ the following technique. Set $E_h = |u_h - u|_{H^1(\mathcal{T}_h)}$, $E_{2h} = |u_h - u_{2h}|_{H^1(\mathcal{T}_h)}$, where $u$ denotes the solution of (2). Assume $E_h = Ch^\alpha$ with constants $C > 0$ and $\alpha > 0$. Then, we have $E_h \leq C(1 + 2^\alpha)h^\alpha$. Hence, we observe $E_{2h}$ in order to infer the value of $\alpha$ by numerical experiments. Similarly, we observe $e_h = \|u_h - u_{2h}\|_{L^2(\Omega)}$ to infer the rate of convergence of $\|u - u_h\|_{L^2(\Omega)}$. We also observe the rate of convergence

$$R_h = \frac{\log E_{2h} - \log E_h}{\log 2}, \quad r_h = \frac{\log e_{2h} - \log e_h}{\log 2}.$$

Results for (III) are reported in Table 1. We see that theoretical convergences with $s = 1$ expected from Theorems 12 and 13 actually take place. Since $\Omega$ is not smooth, we cannot apply the regularity result (5) in Remark 2. However, there is a possibility that $u_1$ and $u_2$ belong to $H^2(\Omega_1)$ and $H^2(\Omega_2)$, respectively.

![Fig. 3](image)

\textbf{Case (III)}

![Fig. 3](image)

\textbf{Case (IV)}

\begin{table}[h]
\centering
\begin{tabular}{l|ccc}
\hline
$h$ & $E_h$ & $R_h$ & $e_h$ & $r_h$ \\
\hline
0.06250 & $4.29 \cdot 10^{-2}$ & & $1.35 \cdot 10^{-3}$ & \\
0.03125 & $2.16 \cdot 10^{-2}$ & 0.99 & $3.41 \cdot 10^{-4}$ & 1.98 \\
0.01563 & $1.09 \cdot 10^{-2}$ & 0.99 & $8.62 \cdot 10^{-5}$ & 1.99 \\
0.00781 & $5.44 \cdot 10^{-3}$ & 1.00 & $2.17 \cdot 10^{-5}$ & 1.99 \\
0.00391 & $2.73 \cdot 10^{-3}$ & 1.00 & $5.48 \cdot 10^{-6}$ & 1.99 \\
\hline
\end{tabular}
\end{table}
On the other hand, results for (IV) are reported in Table 2. Convergence at $O(h^{s_1})$ in $|\cdot|_{H^1(T_h)}$ and that at $O(h^{s_0})$ in $\|\cdot\|_{L^2(\Omega)}$ are observed, where $s_1, s_0 \in (1/2, 1)$. Hence, we infer that the solution has the regularity at most $u_1 \in H^{1+s_1}(\Omega_1)$ and $u_2 \in H^{1+s_2}(\Omega_2)$ with some $s \in (1/2, 1)$. Although we do not obtain shape estimates for $s, s_0, s_1$ by our numerical experiments, we succeeded in observing the behavior of the error if the solutions has only low regularity.

**Acknowledgements** We thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper. NS is supported by JST CREST Grant Number JPMJCR15D1, Japan and JSPS KAKENHI Grant Numbers 15H03635, 15K13454 Japan.

### A Proof of Lemma 9

Let $s \in (1/2, 1)$. Let $K \in \mathcal{T}_h$ and $e \subset \partial K$.

The fractional order Sobolev space $H^s(K)$ is defined as

$$H^s(K) = \{ v \in L^2(K) \mid \| v \|^2_{H^s(K)} = \| v \|^2_{L^2(K)} + |v|^2_{H^s(K)} < \infty \},$$

where

$$|v|^2_{H^s(K)} = \int_K \int_K \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} \, dx \, dy.$$

It suffices to prove

$$\| v \|^2_{L^2(e)} \leq C_{s, T} h^{-1} \left( \| v \|^2_{L^2(K)} + h^{2s} \| v \|^2_{H^s(K)} \right) \quad (v \in H^s(K)), \tag{27}$$

since the desired inequality (20) is a direct consequence of (27).

Suppose that $\bar{K}$ is the reference element in $\mathbb{R}^d$ with $\mathrm{diam}(\bar{K}) = 1$. Moreover, let $\bar{e} \subset \partial \bar{K}$ be a face ($d = 3$) or edge ($d = 2$) of $\bar{K}$. Trace theorem implies

$$\| \tilde{v} \|^2_{L^2(e)} \leq \bar{C} \left( \| \tilde{v} \|^2_{L^2(\bar{K})} + |\tilde{v}|^2_{H^s(\bar{K})} \right) \quad (\tilde{v} \in H^1(\bar{K})),
$$

where $\bar{C}$ denotes an absolute positive constant. See [13, Theorem 1, §V.1.1] for example.

Suppose that $\Phi(\xi) = B\xi + c, B \in \mathbb{R}^{d \times d}$, $c \in \mathbb{R}^d$, is the affine mapping which maps $\bar{K}$ onto $K$; $K = \Phi(\bar{K})$. We know

$$\| B \| = \sup_{|\xi| = 1} |B\xi| \leq \frac{h_K}{\rho}, \quad \| B^{-1} \| \leq \frac{\hat{h}}{\rho_K}, \quad d\xi = \frac{\text{meas}_d(\bar{K})}{\text{meas}_d(K)} \, dx,$$
where $\tilde{h} = h_\tilde{K}, \tilde{\rho} = \rho_\tilde{K}$ and $\text{meas}_d(K)$ denotes the $\mathbb{R}^d$-Lebesgue measure of $K$. Moreover,

$$\frac{|x|}{|B^{-1}x|} \leq \sup_{\xi \in \mathbb{R}^d} \frac{|B\xi|}{|\xi|} = \|B\| \quad (x \in \mathbb{R}^d, x \neq 0).$$

We recall that there exists a positive constant $\nu_2$ that independent of $h$ such that $h_K/\rho_K \leq \nu_2$ ($\forall K \in \mathcal{T}_h \in \{T_h\}_h$) by the shape-regularity of the family of triangulations.

Now we can state the proof of (27). By the density, it suffices to consider (27) for $v \in C^1(\tilde{K})$. Set $\tilde{v} = v \circ \Phi \in C^1(\tilde{K})$. Then,

$$\int \tilde{v}^2 d\tilde{\xi} = \frac{\text{meas}_d(\tilde{K})}{\text{meas}_d(K)} \int_K v^2 dx \leq C \rho_K^{-d} \|v\|^2_{L^2(K)}$$

and

$$\int \int_{K \times \tilde{K}} \frac{|\tilde{v}(\xi) - \tilde{v}(\eta)|^2}{|\xi - \eta|^{d+2s}} d\xi d\eta \leq \left(\frac{\text{meas}_d(\tilde{K})}{\text{meas}_d(K)} \right)^2 \int \int_{K \times \tilde{K}} \frac{|v(x) - v(y)|^2}{|B^{-1}x - B^{-1}y|^{d+2s}} dxdy$$

$$\leq C \rho_K^{-2d} \cdot \|B\|^{d+2s} \int \int_{K \times \tilde{K}} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} dxdy$$

$$\leq C h_K^{2d} \frac{v_d^d}{2} \rho_K^{-d} \int \int_{K \times \tilde{K}} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} dxdy.$$

Using those inequalities, we have

$$\|\tilde{v}\|^2_{L^2(\tilde{e})} = \frac{\text{meas}_{d-1}(\tilde{e})}{\text{meas}_{d-1}(\tilde{e})} \int_{\tilde{e}} \tilde{v}(\xi)^2 d\xi$$

$$\leq C h_{\tilde{e}}^{d-1} \cdot \tilde{C} \left( \int_{\tilde{K}} \tilde{v}^2 d\tilde{\xi} + \int \int_{K \times \tilde{K}} \frac{|v(\xi) - v(\eta)|^2}{|\xi - \eta|^{d+2s}} d\xi d\eta \right).$$

$$\leq C v_d^d h_{\tilde{e}}^{-1} \left( \|v\|^2_{L^2(K)} + h_K^{2s} \|v\|^2_{H^s(K)} \right),$$

which completes the proof.

References

1. Arnold, D.N., Brezzi, F., Cockburn, B., Marini, L.D.: Unified analysis of discontinuous Galerkin methods for elliptic problems. SIAM J. Numer. Anal. 39(5), 1749–1779 (2002)
2. Bernardi, C., Verfürth, R.: Adaptive finite element methods for elliptic equations with non-smooth coefficients. Numer. Math. 85(4), 579–608 (2000)
3. Bramble, J.H., King, J.T.: A finite element method for interface problems in domains with smooth boundaries and interfaces. Adv. Comput. Math. 6(2), 109–138 (1997), 1996
4. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics, vol. 15, 3rd edn. Springer, New York (2008)
5. Cai, Z., Ye, X., Zhang, S.: Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations. SIAM J. Numer. Anal. 49(5), 1761–1787 (2011)
6. Chen, Z., Zou, J.: Finite element methods and their convergence for elliptic and parabolic interface problems. Numer. Math. 79(2), 175–202 (1998)
7. Cockburn, B., Gopalakrishnan, J., Lazarov, R.: Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. SIAM J. Numer. Anal. 47(2), 1319–1365 (2009)
8. Feistauer, M.: On the finite element approximation of functions with noninteger derivatives. Numer. Funct. Anal. Optim. 10(1–2), 91–110 (1989)
9. Grapperhaus, M.J., Kushner, M.J.: A semianalytic radio frequency sheath model integrated into a two-dimensional hybrid model for plasma processing reactors. J. Appl. Phys. 81(2), 569–577 (1997)
10. Grisvard.: Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain. In: Numerical Solution of P.D.E’s III, Proc. Third Sympos. (SYNSPADE), pp. 207–274 (1976)
11. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Pitman (Advanced Publishing Program), Boston (1985)
12. Huynh, L.N.T., Nguyen, N.C., Peraire, J., Khoo, B.C.: A high-order hybridizable discontinuous Galerkin method for elliptic interface problems. Int. J. Numer. Methods Eng. 93(2), 183–200 (2013)
13. Jonsson, A., Wallin, H.: Function spaces on subsets of $\mathbb{R}^n$. Math. Rep. 2(1), xiv+221 (1984)
14. Kikuchi, F., Ando, Y.: A new variational functional for the finite-element method and its application to plate and shell problems. Nucl. Eng. Des. 21, 95–113 (1972)
15. Kikuchi, F., Ando, Y.: Some finite element solutions for plate bending problems by simplified hybrid displacement method. Nucl. Eng. Des. 23, 155–178 (1972)
16. Lions, J.L., Magenes, E.: Non-homogeneous Boundary Value Problems and Applications (Translated from the French by Kenneth, P., Die Grundlehren der mathematischen Wissenschaften, Band 181), vol. I. Springer, New York-Heidelberg (1972)
17. Massjung, R.: An unfitted discontinuous Galerkin method applied to elliptic interface problems. SIAM J. Numer. Anal. 50(6), 3134–3162 (2012)
18. Miyashita, M.: Discontinuous model with semi analytical sheath interface for radio frequency plasma. In: The 69th Annual Gaseous Electronics Conference, Session MW6-00077, vol. 61 (2016)
19. Mu, L., Wang, J., Wei, G., Ye, X., Zhao, S.: Weak Galerkin methods for second order elliptic interface problems. J. Comput. Phys. 250, 106–125 (2013)
20. Oikawa, I.: Hybridized discontinuous Galerkin method with lifting operator. JSIAM Lett. 2, 99–102 (2010)
21. Oikawa, I., Kikuchi, F.: Discontinuous Galerkin FEM of hybrid type. JSIAM Lett. 2, 49–52 (2010)
22. Petzoldt, M.: Regularity results for Laplace interface problems in two dimensions. Z. Anal. Anwendungen 20(2), 431–455 (2001)
23. Petzoldt, M.: A posteriori error estimators for elliptic equations with discontinuous coefficients. Adv. Comput. Math. 16(1), 47–75 (2002)
24. Saito, N., Fujita, H.: Remarks on traces of $H^1$-functions defined in a domain with corners. J. Math. Sci. Univ. Tokyo 7, 325–345 (2000)