PARTIALLY HYPERBOLIC DIFFEOMORPHISMS WITH ONE-DIMENSIONAL NEUTRAL CENTER ON 3-MANIFOLDS

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Abstract. We prove that for any partially hyperbolic diffeomorphism with one dimensional neutral center on a 3-manifold, the center stable and center unstable foliations are complete; moreover, each leaf of center stable and center unstable foliations is a cylinder, a Möbius band or a plane.

Further properties of the Bonatti-Parwani-Potrie type of partially hyperbolic diffeomorphisms are studied. Such examples are obtained by composing the time $m$-map (for $m > 0$ large) of a non-transitive Anosov flow $\phi_t$ on an orientable 3-manifold with Dehn twists along some transverse tori, and the examples are partially hyperbolic with one-dimensional neutral center. We prove that the center foliation gives a topologically Anosov flow which is topologically equivalent to $\phi_t$. We also prove that for the precise example constructed by Bonatti-Parwani-Potrie, the center stable and center unstable foliations are robustly complete.

1. Introduction

1.1. Our setting. In 1970s, M. Brin and Y. Pesin [BP] proposed a notion called partial hyperbolicity. A diffeomorphism is called partially hyperbolic if the tangent bundle of the manifold splits into three invariant bundles: one of which is uniformly contracting under the dynamics, another is uniformly expanding, and the center is intermediate.

One of the main topics on partially hyperbolic systems is the classification according to different properties. This field is the intersection of topology and dynamical systems, and many important projects are proposed by F. Rodriguez Hertz, J. Rodriguez Hertz and R. Ures in a series of papers and talks. Extending a conjecture by E. Pujals (formalized in [BW]), F. Rodriguez Hertz, J. Rodriguez Hertz and R. Ures [HHU6] proposed the following:

Conjecture 1. Any dynamically coherent partially hyperbolic diffeomorphism on a 3-manifold is, up to finite iterations and finite lifts, leaf conjugated to one of the following three models:

- linear Anosov diffeomorphism on $\mathbb{T}^3$;
- time one map of an Anosov flow;
- skew products over linear Anosov diffeomorphisms on torus.

We remark that it is necessary to consider finite iterations and lifts, see the examples in [BW, Section 4].

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For this conjecture, some partial results are obtained, see for instance [Bo], [BW], [Ca], [HaPo1], [HaPo2], [G]. In [BW], a notation called completeness was proposed for the structure of invariant foliations of a dynamically coherent partially hyperbolic diffeomorphism and the completeness of invariant foliations played an important role in attacking this conjecture. We remark that for all the three models in Conjecture I, their invariant foliations are complete.

Recently, C. Bonatti, K. Parwani and R. Potrie [BPP] gave a mechanism to build new partially hyperbolic diffeomorphisms which are counter examples to the conjecture above. The new partially hyperbolic diffeomorphism is obtained by composing the time $n$-map of a specific non-transitive Anosov flow with a Dehn twist along a transverse torus. Then in [BZ], it is shown that such construction can be made to any non-transitive Anosov flow. One can ask: to what extent the properties of the three models in the conjecture are preserved by the new partially hyperbolic diffeomorphisms? To be precise:

**Question.** For the new partially hyperbolic diffeomorphisms in [BPP, BZ], is every center stable (resp. center unstable) leaf either a cylinder or a plane? Are the center stable and center unstable foliations complete? Furthermore, what is the relation between the new partially hyperbolic diffeomorphism and the Anosov flow used for building it?

In this paper, we give answers to the questions above.

1.2. Statement of the results. Let $M$ be a closed 3-manifold. We denote by $\mathcal{PH}(M)$ the set of partially hyperbolic diffeomorphisms with the splitting of the form

$$TM = E^s \oplus E^c \oplus E^u,$$

where $\dim(E^s) = \dim(E^c) = 1$.

Given $f \in \mathcal{PH}(M)$, one says that $f$ has neutral behavior along $E^c$ or neutral center if there exists a constant $K > 1$ such that

$$\frac{1}{K} \leq \| Df^n|_{E^c(x)} \| \leq K,$$

for any $x \in M$ and any $n \in \mathbb{Z}$.

By Theorem 7.5 of [HHU1], one has that $f$ is dynamically coherent, that is, there exist $f$-invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively. Moreover, one has that the invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are plaque expansive. It is well known that $E^s$ and $E^u$ are uniquely integrated into two invariant foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{uu}$. By Remark 3.7 of [HHU2], the neutral behavior along the center implies that the center distribution $E^c$ is also uniquely integrable.

Given a partially hyperbolic diffeomorphism $f$, assume that $f$ is dynamically coherent, then one can get an $f$-invariant foliation $\mathcal{F}^c$ tangent to $E^c$. We denote $\mathcal{F}^{ss}(\mathcal{F}^c(x)) := \cup_{y \in \mathcal{F}^c(x)} \mathcal{F}^{ss}(y)$. We say that the center stable foliation is complete if for any $x \in M$, we have that

$$\mathcal{F}^{ss}(\mathcal{F}^c(x)) = \mathcal{F}^{cs}(x).$$

Although we don’t have the examples of dynamically coherent partially hyperbolic diffeomorphisms whose invariant foliations are not complete, we cannot rule out this possibility.
Our first result is the following:

**Theorem A.** Let $M$ be a closed 3-manifold and $f \in \mathcal{PH}(M)$. Assume that $f$ has neutral behavior along the center, then we have the followings:

- the center stable and center unstable foliations are complete;
- every center stable (resp. center unstable) leaf is a plane, a Möbius band or a cylinder. Moreover, a center stable (resp. center unstable) leaf is a cylinder or a Möbius band if and only if this leaf contains a compact center leaf.

**Remark 1.1.** By [HHU1, Theorem 7.5], one has that both the center stable and center unstable foliations are plaque expansive. Hence, there exists a small neighborhood $U$ of $f$ such that each $g \in U$ is dynamically coherent and every center stable (center unstable) leaf is a plane, a Möbius band or a cylinder.

By Remark 1.1, the second item of Theorem A is a robust property. However, we don’t know if the first property is robust.

**Question 1.** Does there exist a small neighborhood $V$ of $f$ such that for any $g \in V$, the center stable and center unstable foliations of $g$ are complete?

Let $\phi_t$ be a smooth non-transitive Anosov flow on an orientable 3-manifold $M$. Consider a smooth Lyapunov function $L : M \to \mathbb{R}$ of the flow $\phi_t$ (for definition see Section 2.3). Let $\{c_1, \ldots, c_m\}$ be the values of $L$ on the hyperbolic basic sets. We say that $L^{-1}(c)$ is a wandering regular level of $L$ if $c$ is a regular value of $L$ and $c$ is in $L(M) \setminus \{c_1, \ldots, c_m\}$. Then the wandering regular level $L^{-1}(c)$ consists of finite pairwise disjoint tori transverse to the Anosov flow, see for instance [Br].

Now, we define the set of partially hyperbolic diffeomorphisms that we consider. Given an orientable 3-manifold $M$ and let $\phi_t$ be a non-transitive Anosov flow on $M$, we denote by $\mathcal{PH}_{\phi_t}(M) \subset \mathcal{PH}(M)$ the set of diffeomorphisms such that for each $f \in \mathcal{PH}_{\phi_t}(M)$, one has that

- $f$ is partially hyperbolic with one dimensional center;
- there exist $\tau > 0$ and a family of tori $\{T_1, \ldots, T_k\}$ contained in a wandering regular level of a smooth Lyapunov function of $\phi_t$ such that

$$f = \psi_1 \circ \cdots \circ \psi_k \circ \phi_\tau,$$

where $\psi_i$ is a Dehn twist along $T_i$ and is supported in $\{\phi_t(T_i)\}_{t \in (0, \tau)}$.

**Remark 1.2.** It is shown in [BZ] that for each smooth non-transitive Anosov flow $\phi_t$ on $M$, one always has that $\mathcal{PH}_{\phi_t}(M)$ is non-empty, see [BZ, Theorem 7.1 and Lemma 7.4].

**Theorem B.** Let $\phi_t$ be a non-transitive Anosov flow on an orientable 3-manifold $M$. For any $f \in \mathcal{PH}_{\phi_t}(M)$, one has that

- the diffeomorphism $f$ has neutral center;
- We denote by $\mathcal{F}^c$ the center foliation of $f$, then there exist a continuous flow $\{\theta_t\}_{t \in \mathbb{R}} : M \to M$ and a homeomorphism $h : M \to M$ such that for any $x \in M$, one has

$$\text{Orb}(x, \theta_t) = \mathcal{F}^c(x);$$
\[ h(\text{Orb}(x, \theta_t)) = \text{Orb}(h(x), \phi_t) \text{ and } h(\text{Orb}^+(x, \theta_t)) = \text{Orb}^+(h(x), \phi_t); \]

Now, we discuss the particular example \( f_b \) built in [BPP]. In order to state the further properties of \( f_b \) that we get, we need to recall some terminology to give the statement of our result. In [BPP Section 4], the authors firstly build a smooth non-transitive Anosov flow \( \psi_t \) on a 3-manifold \( N \) such that

- the non-wandering set consists of one attractor and one repeller.
- there exist two transverse tori and each orbit has no return on the union of these two tori.
- the foliations induced by the stable and unstable foliations of the Anosov flow on each transverse torus consist of two Reeb components;
- the union of these two tori separates the manifold into two connected components which contain the attractor and the repeller respectively.

The partially hyperbolic diffeomorphism \( f_b \) in [BPP] is obtained by composing a Dehn twist along one transverse torus with \( \psi_n \) (for \( n > 0 \) large), and \( f_b \) has one dimensional neutral center (see [BPP Lemma 9.1]).

By Proposition 1.9 in [BZ], the two transverse tori for \( \psi_t \) are contained in a wandering regular level of a smooth Lyapunov function of \( \psi_t \). Hence Theorem [B] can be applied to \( f_b \). This shows that the action of \( f_b \) on the space of center leaves just “permutes” the center leaves of the Anosov flow on each center stable (center unstable) leaf without changing the structure of the invariant foliations of the Anosov flow \( \psi_t \). Combining with [BPP Theorem 9.6], one has the following corollary:

**Corollary 1.3.** There exist a 3-manifold \( M \) and a dynamically coherent partially hyperbolic diffeomorphism \( f \) on \( M \) such that

- the manifold \( M \) supports Anosov flows;
- the center foliation of \( f \) defines a continuous flow which is topologically equivalent to an Anosov flow;
- there is no lifts or iterations of \( f \) that is leaf conjugate to the time one map of an Anosov flow.

By Theorem 7.1 and Lemma 7.4 in [BZ], one can apply Theorems [A] and [B] to the new examples in [BZ], and one gets that their center stable (resp. center unstable) leaves are planes, cylinders, or Möbius bands; moreover, their center foliations are topologically Anosov. One can ask a more ambitious question:

**Question 2.** Given a dynamically coherent partially hyperbolic diffeomorphism \( f \in \mathcal{PH}(M) \). Up to finite lifts and finite iterations, does one of the followings hold:

- \( f \) is leaf conjugate to a linear Anosov diffeomorphism on \( \mathbb{T}^3 \);  
- or \( f \) is leaf conjugate to a skew product over a linear Anosov diffeomorphism on \( \mathbb{T}^2 \);
- or the center foliation of \( f \) defines a continuous flow which is topologically equivalent to an Anosov flow?

By Theorem [A], one also has that for the examples in [BZ], the center stable and center unstable foliations are complete. Then one can ask:
Question 3. For the partially hyperbolic diffeomorphisms with neutral center built in [BZ], are the center stable and center unstable foliations robustly complete?

We don’t have the answer for the question above, but one can get the robust completeness of the center stable and center unstable foliations for the particular example $f_b$ in [BPP].

Proposition 1.4. There exists a $C^1$ neighborhood $\mathcal{U}$ of $f_b$ such that for any $g \in \mathcal{U}$, one has that the center stable and center unstable foliations of $g$ are complete.

Remark 1.5. The proof of Proposition 1.4 depends on the fact that the non-wandering set of $\phi_t$ consists of one attractor and one repeller.

2. Preliminary

In this section, we collect the notations and results that we need.

Let $f$ be a diffeomorphism on a compact manifold $M$, and recall that $f$ is partially hyperbolic, if there exist a $Df$ invariant splitting $TM = E^s \oplus E^c \oplus E^u$ and a positive integer $N$ such that for any $x \in M$, one has

$$\|Df^N|_{E^s(x)}\| < \min\{m(Df^N|_{E^c(x)}), 1\} \leq \max\{\|Df^N|_{E^c(x)}\|, 1\} < m(Df^N|_{E^u(x)}).$$

2.1. Dynamical coherence, plaque expansiveness and completeness.

Definition 2.1. Let $f$ be a partially hyperbolic diffeomorphism. We say that $f$ is cs (resp. cu)-dynamically coherent, if there exists an $f$-invariant foliation $\mathcal{F}^{cs}$ (resp. $\mathcal{F}^{cu}$) tangent to $E^s \oplus E^c$ (resp. $E^c \oplus E^u$). In particular, $f$ is dynamically coherent, if $f$ is both cs-dynamically coherent and cu-dynamically coherent.

A partially hyperbolic diffeomorphism might not be dynamical coherent even if the center dimension is one (see for instance [HHU4]).

Let $\mathcal{F}$ be an $f$-invariant foliation. We denote by $\mathcal{F}(x)$ the $\mathcal{F}$-leaf through the point $x$ and by $\mathcal{F}_\epsilon(x)$ the $\epsilon$-neighborhood of $x$ in the leaf $\mathcal{F}(x)$. A sequence of points $\{x_n\}_{n \in \mathbb{Z}}$ is called an $\epsilon$ pseudo orbit with respect to $\mathcal{F}$, if $f(x_n)$ belongs to $\mathcal{F}_\epsilon(x_{n+1})$ for any $n \in \mathbb{Z}$.

Definition 2.2. Given an $f$-invariant foliation $\mathcal{F}$. We say that $\mathcal{F}$ is plaque expansive, if there exists $\epsilon > 0$ satisfying the following: if $\{x_n\}_{n \in \mathbb{Z}}$ and $\{y_n\}_{n \in \mathbb{Z}}$ are two $\epsilon$-pseudo orbits with respect to $\mathcal{F}$ and if one has $d(x_n, y_n) < \epsilon$ for any $n \in \mathbb{Z}$, then $x_n$ and $y_n$ belong to a common $\mathcal{F}$-leaf for any $n \in \mathbb{Z}$.

By Theorem 7.5 and Corollary 7.6 in [HHU1], a partially hyperbolic diffeomorphism with neutral center is dynamically coherent; moreover, the center, center stable and center unstable foliations are plaque expansive. By Theorem 7.1 in [HPS] one has that the plaque expansiveness in this setting is a robust property and implies the structure stability of the invariant foliation (ie. leaf conjugacy). In [PS], the authors prove that if the center foliation is plaque expansive, then the leaf conjugacy for the center foliation is also the leaf conjugacy for the center stable and center unstable foliations. To summarize, one has the following result:
Theorem 2.3. Let $f$ be a partially hyperbolic diffeomorphism. If $f$ has one dimensional neutral center, there exists a $C^1$ small neighborhood $U$ of $f$ such that for any $g \in U$, one has the following properties:

- (dynamical coherence) $g$ is dynamically coherent;
- (plaque expansive) the center, center stable and center unstable foliations $g$ are plaque expansive;
- (leaf conjugacy) there exists a homeomorphism $h_g : M \mapsto M$ such that for any point $x \in M$ and $i = c, cs, cu$, one has that
  $$h_g(F^i_g(x)) = F^i_f(h_g(x)) \text{ and } h_g(g(F^i_g(x))) = f(h_g(F^i_g(x))).$$

Remark 2.4. The homeomorphism $h_g$ tends to identity as $g$ tends to $f$.

Let $f$ be a dynamically coherent partially hyperbolic diffeomorphism. A point $y$ is called an accessible boundary point with respect to $F^c(F^c(x))$ for some $x \in M$, if there exists a $C^1$ curve $\sigma : [-1, 0] \mapsto M$ tangent to the center bundle such that

$$\sigma([-1, 0]) \subset F^{ss}(F^c(x)) \text{ and } \sigma(0) = y \notin F^{ss}(F^c(x)).$$

The set of accessible boundary points with respect to $F^{ss}(F^c(x))$ is called accessible boundary with respect to $F^{ss}(F^c(x))$.

With the notations above, one has the following result due to [BW]:

Proposition 2.5. The accessible boundary with respect to $F^{ss}(F^c(x))$ is saturated by strong stable leaves.

We will call each strong stable leaf in the accessible boundary with respect to $F^{ss}(F^c(x))$ as a boundary leaf with respect to $F^{ss}(F^c(x))$ or a boundary leaf for simplicity.

2.2. Existence of compact leaves. In 1965, S. Novikov gave a criterion for the existence of compact leaves of codimension one foliations on closed 3-manifolds.

Theorem 2.6. [N] Let $F$ be a codimension one foliation on a 3-manifold $M$. $F$ has a compact leaf, if one of the followings is satisfied:

- there exists a null-homotopy closed transversal for $F$;
- there exists a non-null homotopic closed path in a $F$-leaf which is null homotopy in $M$.

With the help of Novikov’s theorem, [HHU5] proves the non-existence of compact leaf for center stable (center unstable) foliation. More precisely,

Theorem 2.7. [HHU5, Theorem 1.1] Let $f$ be a partially hyperbolic diffeomorphism on a 3-manifold $M$ with the splitting $TM = E^s \oplus E^c \oplus E^u$. Assume that $f$ is $cs$-dynamically coherent, then the center stable foliation $F^{cs}$ has no compact leaves.

In our context, instead of Theorem 2.7, we can also use the following result:

Theorem 2.8. [HHU3] Assume that $f$ is partially hyperbolic diffeomorphism on a 3-manifold $M$ and admits a torus tangent to $E^s \oplus E^c$, then $M$ fibers over $S^1$ with torus fiber.

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1I thank Rafael Potrie for pointing out this fact.
In fact, the dynamical coherent partially hyperbolic diffeomorphisms on 3-manifolds fibering $S^1$ over torus do not admit compact center stable or center unstable leaves, see [HaPo1, HaPo2].

2.3. Anosov flow. A non-singular $C^1$ vector field $X$ on a closed manifold $M$ is called an Anosov vector field, if there exists a splitting $TM = E^s \oplus <X> \oplus E^u$ which is invariant under the tangent flow $\Phi^X_t$ of flow $X_t$ such that $E^s$ is uniformly contracting and $E^u$ is uniformly expanding under $\Phi^X_t$. The flow generated by an Anosov vector field is called as Anosov flow.

It is well known that the Anosov flow is structurally stable, that is, the flows generated by the vector fields in a $C^1$ small neighborhood of an Anosov vector field are topologically equivalent to each other. The topological equivalence of two flows is defined as follow:

**Definition 2.9.** Let $\varphi_t : M \mapsto M$ and $\theta_t : N \mapsto N$ be two continuous flows. We say that $\varphi_t$ is topologically equivalent to $\theta_t$, if there exists a homeomorphism $h : M \mapsto N$ preserving the orientation of the flows and sending the orbits of the flow $\varphi_t$ to the orbits of the flow $\theta_t$, that is, for any $x \in M$, one has

$$h(\text{Orb}(x, \varphi_t)) = \text{Orb}(h(x), \theta_t) \text{ and } h(\text{Orb}^+(x, \varphi_t)) = \text{Orb}^+(h(x), \theta_t).$$

Different from the Anosov diffeomorphisms on surfaces, the Anosov flows on 3-manifolds might be non-transitive, see for instance [FW]. Given an Anosov flow $\phi_t$ on $M$, a smooth Lyapunov function for $\phi_t$ is a smooth function $L : M \mapsto \mathbb{R}$ such that

- $L$ is not increasing along every orbit;
- $L$ is strictly decreasing along an orbit if and only if this orbit is in the wandering domain.

For a smooth Anosov flow, there always exist smooth Lyapunov functions, see [S, Page 18].

Recall that an embedded torus defined by $e : T^2 \mapsto M$ is incompressible, if the induced map $e_* : \pi_1(T) \mapsto \pi_1(M)$ is injective. With the help of Lyapunov functions, for non-transitive Anosov flows, one can separate the hyperbolic basic sets by finite incompressible transverse tori, see [Br].

2.4. Dehn twists. Now, we give the definition of Dehn twists on 3-manifolds.

**Definition 2.10.** Let $T$ be an embedded torus on a 3-manifold $M$. We say that a diffeomorphism $\Psi : M \mapsto M$ is a Dehn twist along the torus $T$, if there exists an orientation preserving diffeomorphism $\varphi : T^2 \times [0, 1] \mapsto M$ such that

- $\varphi(T^2 \times \{0\}) = T$,
- $\Psi$ is identity in the complement of $\varphi(T^2 \times [0, 1])$;
- under the coordinate of $\varphi$, one has that

  1. $\varphi^{-1} \circ \Psi \circ \varphi : T^2 \times [0, 1] \mapsto T^2 \times [0, 1]$ is of the form $\varphi^{-1} \circ \Psi \circ \varphi(x, t) = (\phi_t(x), t)$, for any $(x, t) \in T^2 \times [0, 1]$;
  2. $\phi_t$ equals to identity when $t$ is close to $0$ or $1$;
  3. for each $x \in T^2$, the closed path $\{\phi_t(x)\}_{t \in [0, 1]}$ is non-null homotopy.
3. The topological structure of the center stable and center unstable foliations: Proof of Theorem A

Let $f$ be a partially hyperbolic diffeomorphism on a 3-manifold $M$, exhibiting neutral behavior along one dimensional center. Since it has already been proven that $f$ is dynamically coherent, we denote the center stable and center unstable foliations as $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ respectively. We will first show that $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are complete, then we give the description of their leaves.

Proof of Theorem A. By the uniform transversality of strong stable direction and center direction restricted to every center stable leaf, there exists $\delta > 0$ such that for any point $x \in M$, the $\delta$ neighborhood of the leaf $\mathcal{F}^c(x)$ in the center stable leaf $\mathcal{F}^{cs}(x)$ is contained in $\mathcal{F}^{ss}(\mathcal{F}^c(x))$.

We will prove the completeness of $\mathcal{F}^{cs}$ by contradiction. Assume that $\mathcal{F}^{cs}$ is not complete, then by Proposition 2.5, there exists a point $x \in M$ such that $\mathcal{F}^{ss}(\mathcal{F}^c(x))$ has a boundary leaf $\mathcal{F}^{ss}(y)$ for some $y \in M$ (there might be infinitely many boundary leaves). By the invariant property of the center and strong stable foliations, we have that $\mathcal{F}^{ss}(f^n(y))$ is a boundary leaf with respect to $\mathcal{F}^{ss}(\mathcal{F}^c(f^n(x)))$, for any integer $n \in \mathbb{Z}$. By the choice of $\delta$, one has that when it is restricted to the center stable leaf $\mathcal{F}^{cs}(f^n(x))$, the strong stable leaf $\mathcal{F}^{ss}(f^n(y))$ is $\delta$ away from the center leaf $\mathcal{F}^c(f^n(x))$.

By the definition of boundary leaves, there exists a $C^1$-curve $\sigma : [0, 1] \to M$ such that

$$\sigma(t) \subset \mathcal{F}^c(y), \quad \sigma(0) = y \quad \text{and} \quad \sigma((0, 1]) \subset \mathcal{F}^{ss}(\mathcal{F}^c(x)).$$

Up to shrinking $\sigma$, we can assume that the length $\ell(\sigma)$ of $\sigma$ is strictly less than $\frac{\delta}{4K^2}$, where $K > 1$ is the number satisfying

$$\frac{1}{K} \leq \| Df^n|_{E^c(p)} \| \leq K, \quad \text{for any } n \in \mathbb{Z} \text{ and any point } p \in M.$$

Since $\sigma(1)$ is on the strong stable manifold of a point $z \in \mathcal{F}^c(x)$, there exists an integer $m$ large enough such that $f^m(\sigma(1))$ is in the $\frac{\delta}{2}$ neighborhood of $f^m(z)$ with respect to the distance on the center stable leaf $\mathcal{F}^{cs}(f^m(x))$. Since $f^m(y)$ is still on the boundary leaf $\mathcal{F}^{ss}(f^m(y))$, we have that the length

$$\ell(f^m(\sigma)) > \frac{\delta}{2}.$$

On the other hand, we have the estimate

$$\ell(f^m(\sigma)) \leq \max_{p \in M} \| Df^m|_{E^c(p)} \| \cdot \ell(\sigma) < \frac{\delta}{4K} < \frac{\delta}{2},$$

a contradiction. This proves the first item of Theorem A.

Let $\tilde{M}$ be the universal cover of $M$. The metric on $\tilde{M}$ is the pull back of the metric on $M$ by the covering map. We denote by $\tilde{\mathcal{F}}^i$ the lift of $\mathcal{F}^i$ on the universal cover $\tilde{M}$, for $i = ss, cs, c, cu, uu$.

To prove the second item, we need the following lemma:

Lemma 3.1. For any $x \in \tilde{M}$, the lifted leaves $\tilde{\mathcal{F}}^{cs}(x)$ and $\tilde{\mathcal{F}}^{cu}(x)$ are planes.
Proof. For any \( x \in \tilde{M} \), the lifted leaf \( \tilde{F}_{cs}(x) \) is a two dimensional manifold without boundary; to prove that it is a plane, we only need to show that its fundamental group is trivial. Assume that there exists a closed curve \( \gamma \) in \( \tilde{F}_{cs}(x) \) which is non-null homotopy in the leaf. Since \( \gamma \) is null homotopy in \( \tilde{M} \), then the projection of \( \gamma \) on \( M \) is null homotopy in \( M \) and is non-null homotopy in a \( F_{cs} \)-leaf. By Theorem \([2.6]\), the foliation \( F_{cs} \) has a compact leaf, which contradicts to Theorem \([2.7]\).

Analogously, one can show that \( \tilde{F}_{cu}(x) \) is also a plane. \( \Box \)

Claim 3.2. The lifted foliations \( \tilde{F}_{cs} \) and \( \tilde{F}_{cu} \) are complete, that is, they are trivially bi-foliated by \( \tilde{F}_{ss} \) and \( \tilde{F}_{c} \).

Proof. Let \( \tilde{f} \) be a lift of \( f \), then \( \tilde{f} \) is a partially hyperbolic diffeomorphism with one dimensional neutral center and whose invariant foliations are the lifts of the invariant foliations of \( f \). Hence, the argument for \( f \) applies for \( \tilde{f} \). \( \Box \)

For any compact center leaf \( \gamma \) of \( f \), we prove the following:

Lemma 3.3. The center stable leaf contains \( \gamma \) is either a cylinder or a Möbius band.

Proof. Up to taking a double cover of the manifold, we can assume that the strong stable bundle is orientable and we give it an orientation. By Theorem \([2.7]\) the leaf \( F_{cs}(\gamma) \) is not compact.

If every strong stable leaf through a point on \( \gamma \) intersects \( \gamma \) only once, then one claims that the center stable leaf of \( \gamma \) is a cylinder. On the universal cover, a lift \( \tilde{F}_{cs}(\gamma) \) of \( F_{cs}(\gamma) \) is a plane. By completeness and the fact that each strong stable leaf through \( \gamma \) intersects \( \gamma \) only once, one has that the lift of \( \gamma \) in \( \tilde{F}_{cs}(\gamma) \) has only one connected component. Let \( \Gamma \subset \pi_1(M) \) be the subgroup which keeps \( \tilde{F}_{cs}(\gamma) \) invariant, then \( F_{cs}(\gamma) \) is the quotient of \( \tilde{F}_{cs}(\gamma) \) by the action of \( \Gamma \) and \( \Gamma \) is the fundamental group of \( F_{cs}(\gamma) \). Let \( \tilde{\gamma} \) be the lift of \( \gamma \) in \( \tilde{F}_{cs}(\gamma) \), then \( \tilde{\gamma} \) is homeomorphic to \( \mathbb{R} \) and each element of \( \Gamma \) keeps \( \tilde{\gamma} \) invariant. Since each non-trivial element of \( \Gamma \) has no fixed points, each non-trivial element acting on \( \tilde{\gamma} \) has no fixed points. To summarize, one has that \( \Gamma \) is isomorphic to a subgroup of \( \text{Homeo}(\mathbb{R}) \), whose non-trivial element has no fixed points. Hölder theorem \([10]\) asserts that any group acting freely on \( \mathbb{R} \) is Abelian. Hence \( \Gamma \) is Abelian, then \( F_{cs}(\gamma) \) can only be a cylinder.

If not, there exists a strong stable leaf intersects \( \gamma \) at least twice. Consider the universal cover \( \tilde{M} \) and the lift \( \mathcal{P} \) of the leaf \( F_{cs}(\gamma) \) which is a plane, then there exist two center leaves which are the lifts of \( \gamma \) on \( \mathcal{P} \); by the completeness of the center stable foliation on the universal cover, one has that for every point on \( \gamma \), the strong stable curve through this point positively goes back to \( \gamma \) after some uniform finite length, which implies that the center stable leaf \( F_{cs}(\gamma) \) is a closed surface, a contradiction. \( \Box \)

The following lemma ends the proof of Theorem \([A]\)

Lemma 3.4. Any center stable leaf which contains no compact center leaves is a plane.
Proof. Let $\mathcal{F}^{cs}(x)$ be a center stable leaf which contains no compact center leaves. We will first use the argument from [BW] to prove that $\mathcal{F}^{cs}(x)$ is either a plane or a cylinder, then we show that $\mathcal{F}^{cs}(x)$ can only be a plane.

Consider a lift $\tilde{\mathcal{F}}^{cs}(y)$ of $\mathcal{F}^{cs}(x)$ and let $\Gamma$ be the subgroup of $\pi_1(M)$ which keeps the leaf $\tilde{\mathcal{F}}^{cs}(y)$ invariant, then $\mathcal{F}^{cs}(x)$ is the quotient of $\tilde{\mathcal{F}}^{cs}(y)$ by the action of $\Gamma$. By Claim 3.2 we have that the space of center leaves in $\tilde{\mathcal{F}}^{cs}(y)$ is a real line, as well as the space of strong stable leaves in $\tilde{\mathcal{F}}^{cs}(y)$. Hence, $\Gamma$ induces two actions on these two spaces respectively and the action of $\Gamma$ is a sub-action of Cartesian product of non-trivial element of $\Gamma$ keeping a center leaf invariant which implies that there exists a leaf without fixed points and preserving the orientation, otherwise there exists a non-trivial element of $\Gamma$ acting on the space of center leaves. Similar argument applies to the action of $\Gamma$ on the space of strong stable leaves, proving that this action is orientation preserving and has a compact center leaf. Similar argument applies to the action of $\Gamma$ on the space of center leaves, which is a leaf different from $\mathcal{F}^{cs}(y)$, hence these two actions are Abelian actions. As a consequence, the action of $\Gamma$ is Abelian and is orientation preserving, which implies that $\mathcal{F}^{cs}(x)$ is an orientable two dimensional manifold whose fundamental group is Abelian. Once again, by Theorem 2.7, we have that $\mathcal{F}^{cs}(x)$ is either a cylinder or a plane.

By assumption, one has that $\mathcal{F}^{cs}(f^n(x))$ contains no compact center leaves for any integer $n \in \mathbb{Z}$. We will prove, by contradiction, that $\mathcal{F}^{cs}(x)$ is a plane. Assume that $\mathcal{F}^{cs}(x)$ is a cylinder, then one has the following result:

Claim 3.5. The center leaf $\mathcal{F}^{c}(x)$ intersects $\mathcal{F}^{ss}(x)$ at least twice.

Proof. Let $y$ be a lift of $x$. Since leaf $\tilde{\mathcal{F}}^{cs}(y)$ is complete and the group $\Gamma$ is non-trivial, given a non-trivial element $\varphi \in \Gamma$, then $\varphi(\tilde{\mathcal{F}}^{c}(y))$ is a leaf different from $\tilde{\mathcal{F}}^{c}(y)$ (otherwise, $\mathcal{F}^{c}(x)$ is compact). As a consequence, $\varphi(\tilde{\mathcal{F}}^{c}(y))$ intersects $\tilde{\mathcal{F}}^{ss}(y)$ in a point different from $y$ and $\varphi(y)$, hence $\mathcal{F}^{c}(x)$ intersects $\mathcal{F}^{ss}(x)$ at least twice. □

One can take a point $z \in \mathcal{F}^{c}(x) \cap \mathcal{F}^{ss}(x) \setminus \{x\}$ such that the interior of the center curve $L$ with endpoints $\{x, z\}$ does not intersect $\mathcal{F}^{ss}(x)$. By the transversality and completeness of the foliation $\tilde{\mathcal{F}}^{cs}$, one has that for any point $w \in \mathcal{F}^{cs}(x) \setminus \mathcal{F}^{ss}(x)$, the strong stable leaf $\mathcal{F}^{ss}(w)$ intersects $L$ in a unique point. Since $z \in \mathcal{F}^{ss}(x)$ and $f$ has neutral behavior along $E^{c}$, we have that a subsequence of $f^n(L)$ tends to a closed center leaf $C$ in the $C^1$-topology. By the uniform transversality between $E^c$ and $E^c \oplus E^u$, and the compactness of $M$, there exist two small positive numbers $\epsilon$ and $\delta$ such that for any two points $x_1, x_2$ satisfying $d(x_1, x_2) < \delta$, one has that $\mathcal{F}^{ss}_{\epsilon}(x_1)$ intersects $\mathcal{F}^{cu}_{\epsilon}(x_2)$ in a unique point which is strictly contained in $\mathcal{F}^{ss}_{\epsilon/2}(x_1) \cap \mathcal{F}^{cu}_{\epsilon/2}(x_2)$.

We take $n$ large enough such that $f^n(L)$ is $\delta$ close to $C$, then $\mathcal{F}^{ss}(f^n(L))$ intersects the annulus or Möbius band $\mathcal{F}^{cu}_{\epsilon}(C)$ in a compact center curve without boundary in the interior of $\mathcal{F}^{cs}_{\epsilon}(C)$ (see Figure 1 below), which, therefore, is a compact center leaf in $f^n(\mathcal{F}^{cs}(x))$, a contradiction. This ends the proof of Lemma 3.4. □
Now, the proof of Theorem A is completed. □

At the end, we prove a property for the lifted foliations of partially hyperbolic diffeomorphisms, which will be used in the next section.

**Lemma 3.6.** Let $f$ be a partially hyperbolic diffeomorphism on a 3-manifold $M$ such that $f$ has neutral behavior along the center. We denote by $\tilde{M}$ the universal cover of $M$ and by $\tilde{F}^i$ the lift of $F^i$ for $i = ss, cs, c, cu, uu$.

Then for any $x, y \in \tilde{M}$, the leaf $\tilde{F}^{cs}(x)$ intersects $\tilde{F}^{cu}(y)$ in at most one center leaf.

**Proof.** Assume that there exist two points $x, y \in \tilde{M}$ such that the intersection of $\tilde{F}^{cs}(x)$ and $\tilde{F}^{cu}(y)$ contains two different center leaves $L_1, L_2$.

By the completeness, there exists a strong stable segment $\sigma$ whose endpoints are contained in $L_1, L_2$ respectively. Since $L_1, L_2$ are contained in the same center unstable leaf, using a classical argument, one has that $\tilde{F}^{cu}$ admits a closed transversal which implies that $F^{cu}$ admits a null-homotopy closed transversal. By Theorem 2.6, one gets that center unstable foliation $F^{cu}$ has compact leaves, contradicting to Theorem 2.7. □

4. CENTER FLOW CARRIED BY THE PARTIALLY HYPERBOLIC DIFFEOMORPHISMS DERIVED FROM DEHN SURGERY: PROOF OF THEOREM B

In this section, we first study the properties of the partially hyperbolic diffeomorphisms in the assumption of Theorem B, then we give the proof of Theorem B. At last, we recall the precise example in [BPP] and give the proof of Proposition 1.4.

4.1. Partially hyperbolic diffeomorphisms from the Dehn surgery of Anosov flows. Let $\phi_t$ be a non-transitive Anosov flow on an orientable 3-manifold $M$ and $\mathcal{L} : M \to \mathbb{R}$ be a smooth Lyapunov function of $\phi_t$. 
Consider a family of transverse tori \( \{T_1, \ldots, T_k\} \) which is contained in a wandering regular level \( \mathcal{L}^{-1}(c) \). By the definition of a smooth Lyapunov function, one has that the regions \( M^+ = \mathcal{L}^{-1}([c, +\infty]) \) and \( M^- = \mathcal{L}^{-1}(-\infty, c) \) are the repelling and attracting regions for the flow \( \phi_t \) respectively. We denote by \( \mathcal{R} \) and \( \mathcal{A} \) the maximal invariant sets in \( M^+ \) and \( M^- \) respectively. By definition, one has that \( \partial M^+ = \partial M^- = \mathcal{L}^{-1}(c) \). We denote by \( TM = E^s_X \oplus < X > \oplus E^u_X \) the hyperbolic splitting for the Anosov flow \( \phi_t \). In the following text, we denote by \( W^{ss}, W^{cs}, W^c, W^{cu} \) and \( W^{uu} \) the invariant foliations of \( \phi_t \) tangent to \( E^s_X, E^c_X \oplus < X >, < X >, < X > \oplus E^u_X \) respectively.

Proposition 4.1. Assume that there exist \( \tau > 0 \) and Dehn twist \( \psi_i \) whose support is in \( \{\phi_t(T_i)\}_{t \in (0, \tau)} \) such that the diffeomorphism \( f = \psi_1 \circ \cdots \psi_k \circ \phi_{\tau} \) is partially hyperbolic with one-dimensional center. Let \( TM = E^s \oplus E^c \oplus E^u \) be the partially hyperbolic splitting for \( f \).

Then one has that

- the regions \( M^+ \) and \( M^- \) are the repelling and attracting regions for \( f \) respectively;
- the maximal invariant sets of \( f \) in \( M^+ \) and \( M^- \) are \( \mathcal{R} \) and \( \mathcal{A} \) respectively;
- in the region \( M^+ \), the bundles \( < X > \oplus E^u_X \) and \( E^u_X \) coincide with \( E^c \oplus E^u \) and \( E^u \) respectively;
- in the region \( M^- \), the bundles \( E^c_X \oplus < X > \) and \( E^c_X \) coincide with \( E^s \oplus E^c \) and \( E^s \) respectively;
- the splitting \( E^s_X \oplus < X > \oplus E^u_X \) coincides with \( E^s \oplus E^c \oplus E^u \) restricted to \( \mathcal{A} \cup \mathcal{R} \).

Proof. Since \( \psi_i \) is supported on \( \{\phi_t(T_i)\}_{t \in [0, \tau]} \), one has that \( f|_{M^-} = \phi_{-\tau} \) and \( f^{-1}|_{M^+} = \phi_{\tau} \). This proves the first and the second items.

Remember that \( TM = E^s_X \oplus < X > \oplus E^u_X \) is also the partially hyperbolic splitting for \( \phi_{\tau} \). Since \( \dim(M) = 3 \), by the uniqueness of partially hyperbolic splitting, one has that restricted to \( \mathcal{A} \cup \mathcal{R} \), the splitting \( E^s_X \oplus < X > \oplus E^u_X \) coincides with \( E^s \oplus E^c \oplus E^u \). This proves the last item.

The following classical result ends the proof of Proposition 4.1.

Lemma 4.2. Let \( U \) be an attracting region for a diffeomorphism \( g \). Assume that the maximal invariant set \( \Lambda \) of \( g \) in \( U \) admits a dominated splitting \( T_{\Lambda}M = E \oplus F \), where \( \dim(E(x)) \) is a constant. Then there exists a unique \( Dg \)-invariant continuous bundle \( \tilde{E} \) defined on \( U \) such that \( \tilde{E}|_{\Lambda} = E \).

□

As a corollary, one has that

Corollary 4.3. Let \( f \) be a partially hyperbolic diffeomorphism as in the assumption of Proposition 4.1, then \( f \) has one dimensional neutral center.

The proof of Corollary 4.3 would be same as the proof of [BPP, Lemma 9.1]. As it is short, we add the proof.
Proof. By the forth item in Proposition \ref{prop:4.1} on the set \(M^-\), the bundles \(E^s\) and \(E^s \oplus E^c\) coincide with the bundles \(E^s_X\) and \(E^s_X \oplus <X>\). As \(E^c\) is uniformly transverse to \(E^s\), in the set \(M^-\), each unit vector \(v\) in \(E^c\) has uniform component in the bundle \(<X>\) which is bounded from above and below. Since \(f\) coincides with \(\phi_r\) on \(M^-\) the diffeomorphism \(f\), the forward iterations of \(v\) are uniformly bounded from above and below.

On the set \(M^+\), for the backward iterations of each unit vector, one has the analogous property. Since the one dimensional center bundle is \(f\) invariant and each orbit intersects the interior of \(\bigcup_{i=1}^{k} \{\phi_t(T_i)\}_{t \in [0,\tau]}\) at most once, one gets neutral property for the center bundle. □

Remark 4.4. (1) As the center direction of \(f\) is the intersection of the center stable and center unstable bundles, one has that the center bundle of \(f\) coincides with \(<X>\) in a neighborhood of the boundary of \(\{\phi_t(T_i)\}_{t \in [0,\tau]}\);

(2) Since each Dehn twist \(\psi_i\) coincides with identity map in a neighborhood of the boundary of \(\{\phi_t(T_i)\}_{t \in [0,\tau]}\), by the third and fourth items of Proposition \ref{prop:4.1}, one has that the center stable and center unstable leaves of \(f\) are invariant under \(f\).

Similar to the situation of Anosov flow \(\phi_t\), one has the following corollary:

Corollary 4.5. For any \(z \in M \setminus (\mathcal{A} \cup \mathcal{R})\), the center leaf \(\mathcal{F}^c(z)\) intersects \(\mathcal{L}^{-1}(c)\) in a unique point.

Proof. Since the orbits of the transverse tori in \(\mathcal{L}^{-1}(c)\) are pairwise disjoint and the center foliation of \(f\) is obtained as the intersection of the center stable and center unstable foliations, one has that each center leaf \(\mathcal{F}^c(z)\) intersects at most one connected component of \(\mathcal{L}^{-1}(c)\).

For \(z \in M \setminus (\mathcal{A} \cup \mathcal{R})\), one that there exists an integer \(n\) such that \(f^n(z) \in \{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [0,\tau]}\). The center foliation restricted in \(\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [0,\tau]}\) is obtained as the intersection of the foliations \(\mathcal{F}^{cs}\) and \(\mathcal{F}^{cu}\). By Proposition \ref{prop:4.1}, one has that
\[
\mathcal{F}^{cu}|_{\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [\tau,0]}} = f(W^{cu}|_{\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [\tau,0]}})
\]
and
\[
\mathcal{F}^{cs}|_{\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [\tau,0]}} = W^{cs}|_{\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [\tau,0]}}.
\]
Hence, for any \(w \in \{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [0,\tau]}\), the restricted center leaf \(\mathcal{F}^c(w)|_{\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [0,\tau]}}\) has two endpoints belonging to \(T_j\) and \(\phi_r(T_j)\) respectively, where \(T_j\) is a connected component of \(\mathcal{L}^{-1}(c)\). Combining with the fact that \(f\) equals \(\phi_r\) in a neighborhood of \(\{\phi_t(\mathcal{L}^{-1}(c))\}_{t \in [0,\tau]}\), one has that the center leaf \(\mathcal{F}^c(z)\) intersects \(T_i\) and \(\phi_r(T_i)\) into a unique point respectively, where \(T_i\) is a connected component of \(\mathcal{L}^{-1}(c)\); since \(\phi_t(T_i) \cap T_i = \emptyset\) for \(t \neq 0\), one has that the intersection of \(\mathcal{L}^{-1}(c)\) and \(\mathcal{F}^c(z)\) is a unique point. □

For the partially hyperbolic diffeomorphisms satisfying the hypothesis of Proposition \ref{prop:4.1}, we will show that the center foliations of such diffeomorphisms are orientable and therefore give continuous flows (recall that the center bundle is uniquely integrable).
Lemma 4.6. Let $f$ be a diffeomorphism satisfying the assumption of Proposition 4.1. We denote by $F^c$ the center foliation of $f$. Then there exists a continuous flow $\theta_t$ on $M$ such that

- for any $x \in M$, one has $\text{Orb}(x, \theta_t) = F^c(x)$.
- the direction of flow $\theta_t$ gives the same transverse orientation to $L^{-1}(c)$ as the direction of flow $\phi_t$ does.
- restricted to $A \cup R$, the orientation of $E^c$ given by the direction of flow $\theta_t$ coincides with the orientation given by the direction of the flow $\phi_t$.

Proof. By Remark 4.4, one has that the center foliation $F^c$ coincides with the center foliation $W^c$ in a neighborhood of $L^{-1}(c)$.

For each center leaf intersecting $L^{-1}(c)$, we give it the same orientation as the one of $W^c$ given by the flow direction. Since in the region $M^-$, the center stable and strong stable foliations of $f$ coincide with the ones of the Anosov flow $\phi_t$, by the fact that $F^c$ is transverse to the strong stable direction in each center stable leaf, one has that in the region $M^\perp \setminus A$, restricted to each center stable leaf, each leaf of $F^c$ cuts the strong stable leaf with the same orientation as $W^c$ does; another way to observe this is to lift them to the universal cover, and one still has that the lift of $F^c$ coincides with the lift of $W^c$ in a neighborhood of the lift of $L^{-1}(c)$ and the lift of strong stable foliations $F^{ss}, W^{ss}$ restricted to the lift of $M^\perp \setminus A$ also coincide. Combining with Corollary 4.5 one has that the orientation of the $F^c$-leaves intersecting $L^{-1}(c)$ induces the same orientation of the center leaves in $A$ as the the one given by the flow. One can apply the same argument to the region $M^+$. Finally, one gets that the center foliation $F^c$ is orientable, and one can give it an orientation such that

- it gives the same orientation on the set $A \cup R$ as the one given by the flow $\phi_t$;
- it gives the same transverse orientation to $L^{-1}(c)$ as the one given by the flow $\phi_t$.

Hence, the center foliation $F^c$ can give a continuous flow, denoted as $\theta_t$ and satisfying the posited properties. This ends the proof of Lemma 4.6. \qed

Now, we are ready to give the proof of Theorem B.

Proof of Theorem B. By Corollary 4.3, the partially hyperbolic diffeomorphism $f$ has neutral center.

Let $\theta_t$ be the continuous flow given by Lemma 4.6 with respect to the center foliation $F^c$ of $f$.

We shall build the conjugation between the center flow $\theta_t$ and the original Anosov flow $\phi_t$. We shall first define the conjugation on the set $M^\perp \setminus A$ which is $\text{Id}$ restricted to the boundary of $M^\perp$, then we extend it to be a homeomorphism of $M^\perp$ which equals $\text{Id}$ on $A$. Similarly, we define the conjugation on the set $M^+$. In the end, we get the conjugation between $\theta_t$ and $\phi_t$.

We denote by $F^{ss}, F^{cs}, F^{cu}$ and $F^{ uu}$ the strong stable, center stable, center unstable and strong unstable foliations of $f$ respectively.
Let $\pi : \tilde{M} \rightarrow M$ be the universal cover of $M$. We denote by $\tilde{F}^l$ and $\tilde{W}^l$ the lifts of the foliations $F^l$ and $W^l$ respectively, for any $l = ss, cs, c, cu, uu$. Given a foliation $F$ on $M$ and a submanifold $M' \subset M$, the leaf of $F|_{M'}$ through a point $x \in M'$ is the connected component of $F(x) \cap M'$ containing $x$.

**Proposition 4.7.** There exists a homeomorphism $h^* : M^- \rightarrow M^-$ such that
- the map $h^*$ preserves every leaf of the foliation $W^{cs}|_{M^-} = F^{cs}|_{M^-}$;
- the map $h^*$ takes the orbits of the flow $\phi_t|_{M^-}$ to the ones of $\theta_t|_{M^-}$;
- the map $h^*$ coincides with $\text{Id}$ on the set $\mathcal{A} \cup \mathcal{L}^{-1}(c)$.

From now on, for each set $A \subset M$, we denote by $\tilde{A} = \pi^{-1}(A)$. To prove Proposition 4.7, we need the following lemma:

**Lemma 4.8.** For any points $x \in \tilde{M}^+$ and $y \in \tilde{M}^-$, the center unstable leaf $\tilde{W}^{cu}(x)$ intersects the center stable $\tilde{W}^{cs}(y)$ non-empty if and only if the center unstable leaf $F^{cu}(x)$ intersects the center stable $F^{cs}(y)$ non-empty. More precisely, we have the following:

$$z \in \tilde{W}^{cu}(x) \cap \tilde{W}^{cs}(y) \cap \mathcal{L}^{-1}(c) \iff z \in F^{cu}(x) \cap F^{cs}(y) \cap \mathcal{L}^{-1}(c).$$

**Proof of Lemma 4.8** First, we need the following result:

**Claim 4.9.** Given two points $p, q \in M^-$, if $p \in F^{cs}(q)$, then $p, q$ are in the same connected component of $F^{cs}(q) \cap M^-$.

**Proof.** By the fourth item of Proposition 4.1, one has that $\text{Orb}^+(p, \phi_t) \cup \text{Orb}^+(q, \phi_t)$ is contained in $F^{cs}(q)$. Since the orbits of $p, q$ under the flow $\phi_t$ converge to an orbit in $\mathcal{A}$, once again by the fourth item of Proposition 4.1, one has that $p, q$ are in the same connected component of $F^{cs}(q) \cap M^-$. \hfill $\Box$

Given $x \in \tilde{M}^-$ and $y \in \tilde{M}^+$, by Lemma 3.6 the intersection of $\tilde{W}^{cs}(x)$ and $\tilde{W}^{cu}(y)$ consists of at most one leaf of $\tilde{W}^c$, and the same property holds for the foliations $\tilde{F}^{cs}$ and $\tilde{F}^{cu}$. If the intersection of $\tilde{W}^{cs}(x)$ and $\tilde{W}^{cu}(y)$ is not empty, by the choices of $x, y$, there exists a unique point $z \in \mathcal{L}^{-1}(c)$ contained in $\tilde{W}^{cu}(x)$ and $\tilde{W}^{cs}(y)$, which is equivalent to that $z \in \mathcal{L}^{-1}(c)$ is contained in $\tilde{F}^{cu}(x)$ and $\tilde{F}^{cs}(y)$; this is due to Claim 4.9 and the fact that these foliations coincide on the corresponding region. This ends the proof of Lemma 4.8. \hfill $\Box$

Now, we are ready to give the proof of Proposition 4.7.

**Proof of Proposition 4.7.** Given $x \in \tilde{M}^+ \setminus \tilde{A}$, there exists a point $y \in \tilde{M}^+ \setminus \tilde{R}$ such that $x$ is on the positive orbit of $y$ for the lifted flow $\tilde{\phi}_t$; by Corollary 4.5, one has the analogous property for the flow $\tilde{\theta}_t$. Let $z$ be the unique intersection between $\text{Orb}(x, \tilde{\phi}_t) = \tilde{W}^c(x)$ and the transverse section $\pi^{-1}(\mathcal{L}^{-1}(c))$, then one has that

$$z \in \tilde{W}^{cu}(y) \cap \tilde{W}^{cs}(x) \cap \mathcal{L}^{-1}(c).$$

By Lemma 4.8 one has that

$$z \in \tilde{F}^{cu}(y) \cap \tilde{F}^{cs}(x) \cap \mathcal{L}^{-1}(c).$$

\hfill $\Box$
Since each leaf of the foliation $\tilde{F}^{cs}$ is a plane, by completeness and transversality, the center leaf $\tilde{F}^{c}(z)$ intersects the strong stable leaf $\tilde{F}^{ss}(x)$ in a unique point and we denote it as $h^{s}(x)$ (as it is shown in Figure 2). Similarly, we can define a map $\tau^{s}$ by exchanging the roles of $(\tilde{\phi}_{t}, \tilde{W}^{ss})$ and $(\tilde{\theta}_{t}, \tilde{F}^{ss})$ in the definition of $h^{s}(x)$.

![Figure 2.](image)

By definition, one can check that the maps $h^{s}, \tau^{s} : \tilde{M}^{-}\setminus \tilde{A} \mapsto \tilde{M} \setminus (\tilde{A} \cup \tilde{R})$ are continuous and commutative with the automorphisms of $\tilde{M}$ induced by $\pi_{1}(M)$. Moreover, one has that

- the maps $h^{s}, \tau^{s}$ coincide with $\text{Id}$ in a neighborhood of $\tilde{L}^{-1}(c)$ restricted in $\tilde{M}^{-}$;
- the maps $h^{s}, \tau^{s}$ preserve the orientation of the foliations $\tilde{F}^{cs}|_{\tilde{M}^{-}}, \tilde{F}^{ss}|_{\tilde{M}^{-}}$ and send the positive orbits of one center flow to the positive orbits of the other.

**Claim 4.10.** The maps $h^{s}$ and $\tau^{s}$ are injective.

**Proof.** Assume that $h^{s}$ is not injective, then there exist two different points $x, y \in \tilde{M}^{-}\setminus \tilde{A}$ such that $h^{s}(x) = h^{s}(y)$. By definition, one has that $x \in \tilde{F}^{ss}(y)$. Moreover, $x$ and $y$ are on the same orbit of the Anosov flow $\tilde{\phi}_{t}$. By the forth item in Proposition 4.11, the orbit segment from $x$ to $y$ for the flow $\tilde{\phi}_{t}$ is contained in the leaf $\tilde{F}^{cs}(x)$ and is transverse to $\tilde{F}^{ss}$. Since the leaf $\tilde{F}^{cs}(x)$ is a plane and the leaf $\tilde{F}^{ss}(x)$ intersects $L(x, y)$ twice, we get the contradiction.

The same argument applies for $\tau^{s}$. \qed

**Claim 4.11.** The images of $h^{s}$ and $\tau^{s}$ are contained in $\tilde{M}^{-}\setminus \tilde{A}$.

**Proof.** Let us first recall that

$$\tilde{F}^{cs}|_{\tilde{M}^{-}} = \tilde{W}^{cs}|_{\tilde{M}^{-}} \quad \text{and} \quad \tilde{F}^{cu}|_{\tilde{M}^{+}} = \tilde{W}^{cu}|_{\tilde{M}^{+}}.$$
By definition, for any point \( x \in \hat{M} \setminus \hat{A} \), \( h^s \) maps the connected component of \( \tilde{\mathcal{F}}^{cs}(x) \cap (\hat{M} \setminus \hat{A}) \) containing \( x \) to a connected component of \( \tilde{\mathcal{F}}^{cs}(x) \cap (\hat{M} \setminus \hat{A}) \). Notice that each connected component of \( \tilde{\mathcal{F}}^{cs}(x) \cap (\hat{M} \setminus \hat{A}) \) has non-empty interior in \( \tilde{\mathcal{F}}^{cs}(x) \) and its boundary consists of center leaves in \( \tilde{\mathcal{F}}^{cs}(x) \). Since \( h^s \) coincides with identity in a neighborhood of \( \tilde{L}^{-1}(c) \) restricted to \( \hat{M} \), by the continuous and injective property of \( h^s \), one has that \( h^s \) maps the connected component of \( \tilde{\mathcal{F}}^{cs}(x) \cap (\hat{M} \setminus \hat{A}) \) containing \( x \) into itself, for any \( x \in \hat{M} \setminus \hat{A} \).

One can prove the claim for \( \tau^s \) analogously, ending the proof of Claim 4.11. □

Claim 4.12. For any points \( x, y \in \hat{M} \), one has that
\[
y \in \tilde{\mathcal{F}}^{ss}(x) \iff y \in \tilde{W}^{ss}(x).
\]

Proof. Since \( f|_{\hat{M}} = \phi_r|_{\hat{M}} \), one can take a lift \( \tilde{f} \) of \( f \) such that the forward orbit of \( x \) under \( \tilde{f} \) coincides with the one under \( \tilde{\phi}_r \). If \( y \in \tilde{\mathcal{F}}^{ss}(x) \cap \hat{M} \), then the forward orbit of \( y \) under \( \tilde{f} \) coincides with the one under \( \tilde{\phi}_r \) since \( f|_{\hat{M}} = \phi_r|_{\hat{M}} \). Hence, the distance \( d(\tilde{f}^n(x), \tilde{f}^n(y)) = d(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) \) tends to zero exponentially. Since \( \tilde{\phi}_r \) has neutral behavior along the center, one has that \( y \in \tilde{W}^{ss}(x) \). One can argue for the other side analogously, concluding. □

By the definitions of \( h^s \) and \( \tau^s \), one has that \( h^s(x) \in \tilde{W}^{ss}(x) \) and \( \tau^s(x) \in \tilde{\mathcal{F}}^{ss}(x) \). By Claims 4.11, 4.12 and the definitions of \( h^s \), \( \tau^s \), one has that \( \tau^s \circ h^s = h^s \circ \tau^s = \text{Id} \). Hence, \( h^s \) is an orientation preserving homeomorphism on the set \( \hat{M} \setminus \hat{A} \). Moreover, \( h^s \) maps each leaf of \( \tilde{\mathcal{F}}^{ss}|_{\hat{M} \setminus \hat{A}} \) into itself surjectively as a homeomorphism, hence one can extend \( h^s \) to \( \hat{A} \) as \( \text{Id} \) such that \( h^s \) is continuous along each leaf of \( \tilde{\mathcal{F}}^{ss}|_{\hat{M} \setminus \hat{A}} \).

Lemma 4.13. The maps \( h^s, \tau^s : \hat{M} \mapsto \hat{M} \) are continuous.

Proof. One only needs to prove that the map \( h^s \) is continuous at \( \hat{A} \). The case for \( \tau^s \) follows analogously.

Assume, on the contrary, there exists a point \( x_0 \in \hat{A} \) where \( h^s \) is not continuous. Then there exists \( \epsilon_0 > 0 \) and a sequence of points \( \{x_n\}_{n>0} \subset \hat{M} \setminus \hat{A} \) such that
\[
\lim_{n \to \infty} x_n = x_0 \text{ and } d(h^s(x_n), h^s(x_0)) > \epsilon_0.
\]

To continue the proof, we need the following result:

Claim 4.14. On the universal cover \( \hat{M} \), for the lifts of \( f \), each center unstable leaf intersects a strong stable leaf in at most one point. The analogous property also holds for \( \phi_r \).

Proof. If there exists a strong stable leaf intersecting a center unstable leaf in two points, by a classical argument, one gets a closed transversal for the center unstable foliation, which implies that the center unstable foliation for the diffeomorphism on \( M \) admits a null-homotopy closed transversal. By Novikov’s theorem, the center unstable foliation for the diffeomorphism on \( M \) has compact leaves, contradicting to Theorem 2.7. □

Since \( x_n \) tends to \( x_0 \), by Claim 4.14 the center unstable leaf \( \tilde{W}^{cu}(x_n) \) intersects the strong stable leaf \( \tilde{W}^{ss}(x_0) = \tilde{F}^{ss}(x_0) \) in a unique point \( y_n \), for \( n \) large. Then
y_n tends to x_0. Moreover, since each connected component of \( \tilde{W}^{cu}(x_n) \cap \tilde{M}^- \) is the forward \( \tilde{\phi}_t \)-orbit of a connected component of \( \tilde{W}^{cu}(x_n) \cap \partial \tilde{M}^- \), the backward orbits of \( x_n \) and \( y_n \) under the flow \( \tilde{\phi}_t \) intersect the same connected component \( P_n \) of \( \partial \tilde{M}^- \cap \tilde{W}^{cu}(x_n) \) into unique points, and we denote them by \( p_n \) and \( q_n \) respectively, where \( P_n \) is diffeomorphic to \( \mathbb{R} \). Since \( \tilde{W}^{cu} \) coincides with \( \tilde{F}^{cu} \) in a neighborhood of \( \partial \tilde{M}^- \), one has that the forward orbits of \( p_n \) and \( q_n \) under the flow \( \tilde{\phi}_t \) are on the same center unstable leaf \( \tilde{F}^{cu}(q_n) \). By the choices of \( p_n, q_n \) and Claim 4.14 one has that the center unstable leaf \( \tilde{F}^{cu}(q_n) \) intersects \( \tilde{F}^{ss}(x_0) \) and \( \tilde{F}^{ss}(x_n) \) into unique points; remember that the orbits of the flow \( \tilde{\phi}_t \) also lie in \( \tilde{F}^{cu} \), by the definition of \( h^s \), one has that their intersections must be \( h^s(y_n) \) and \( h^s(x_n) \) respectively. By definition, \( h^s \) is continuous restricted to the strong stable leaf \( \tilde{F}^{ss}(x_0) \), hence \( d(h^s(y_n), x_0) \) tends to zero. By the continuity of center unstable foliation \( \tilde{F}^{cu} \), the center unstable plaque \( \tilde{F}^{cu}_{\epsilon/3}(h^s(y_n)) \subset \tilde{F}^{cu}(q_n) \) intersects the strong stable leaf \( \tilde{F}^{ss}(x_n) \) into a point \( z_n \) for \( n \) large, hence

\[
d(z_n, x_0) < d(z_n, h^s(y_n)) + d(h^s(y_n), x_0) < \epsilon_0, \quad \text{for } n \text{ large.}
\]

Once again, by Claim 4.14 one has that \( z_n = h^s(x_n) \), contradicting to \( d(h^s(x_n), x_0) < \epsilon_0 \).

Now, the map \( h^s : \tilde{M}^- \mapsto \tilde{M}^- \) is a homeomorphism. By the definition of \( h^s \), one has that \( h^s \) maps the orbits of \( \tilde{\phi}_t|_{\tilde{M}^-} \) to the orbits of \( \tilde{\theta}_t|_{\tilde{M}^-} \), and preserves the orientation of the orbits. Since \( h^s \) commutes with the automorphisms on \( \tilde{M} \) induced by \( \pi_1(M) \), the projection of \( h^s \) on the base manifold defines a homeomorphism of \( M^- \) satisfying the announced properties, ending the proof Proposition 4.7. \( \square \)

Ending the proof of Theorem 4. By applying Proposition 4.7 to the reversed dynamics on the set \( M^+ \), one gets a homeomorphism \( h^s : M^+ \mapsto M^+ \) satisfying the analogous properties. We define a homeomorphism \( h : M \mapsto M \) in the following way:

\[
h(x) = \begin{cases} 
h^s(x) & \text{if } x \in M^- \\
h^u(x) & \text{if } x \in M^+ 
\end{cases}
\]

The homeomorphism \( h \) coincides with \( \text{Id} \) on the set \( \mathcal{A} \cup \mathcal{R} \cup \mathcal{L}^{-1}(c) \). One can check that \( h \) sends the orbits of \( \phi_t \) to the orbits of \( \theta_t \) and preserves the orientation of the flows. This proves that \( \theta_t \) is topologically equivalent to Anosov flow \( \phi_t \). \( \square \)

5. The anomalous example in [BPP]: proof of Proposition 1.4

5.1. Construction of the example in [BPP]. In [BPP] Section 4], the authors built a 3-manifold \( N \) supporting a smooth non-transitive Anosov flow \( \psi_t \) having two transverse tori \( T_1 \) and \( T_2 \) with the properties:

(P1) \( T_1 \cup T_2 \) is far away from the non-wandering set of \( \psi_t \);
(P2) \( T_1 \cup T_2 \) separates \( N \) into two connected components \( N^+ \) and \( N^- \);
(P3) \( N^+ \) is a repelling region of \( \psi_t \) and the maximal invariant set of \( \psi_t \) in \( N^+ \) is a repeller \( \mathcal{R} \);
(P4). $N^-$ is an attracting region of $\psi_t$ and the maximal invariant set of $\psi_t$ in $N^-$ is an attractor $A$.

By [HP], one has the stable and unstable foliations of $\psi_t$ are $C^1$, hence on the transverse torus $T_i$, the stable manifold of $\psi_t$ and the unstable manifold of $\psi_t$ induce two $C^1$ foliations $F^s_i$ and $F^u_i$ respectively. For these foliations, one has the following property:

(P5). the induced foliations $F^s_i$ and $F^u_i$ consist of two Reeb components. The compact leaves of these Reeb components belong to the stable or the unstable manifolds of the periodic orbits of $\psi_t$, since these compact leaves are transverse to the flow in the stable or the unstable manifolds of the flow $\psi_t$. One can take a $C^1$ coordinate $\theta_i$ for each transverse torus so that under such coordinates, the induced foliations $F^s_i$ and $F^u_i$ on transverse torus $T_i$ are exactly as shown in Figure 3 (for details see Lemma 4.1 in [BPP]).

![Figure 3](image.png)

**Figure 3.** The real lines and the dash lines denote the leaves of the lifts of foliations $F^s_i$ and $F^u_i$ on the universal cover respectively.

Under this coordinate, one considers the Dehn twist on $T_1 \times [0,1]$ with the form

$$\tilde{\Psi} : (x,t) \mapsto (x + (0,\alpha(t)),t),$$

where $\alpha(t)$ is a smooth non-decreasing bump function supported on $[0,1]$ such that $\alpha(0) = 0$ and $\alpha(1) = 1$, and the circle $\{(0,t)\}_{t \in [0,1]}$ on $T_1$ is in the homotopy class of the compact leaves of $F^s_i$. We denote by $\Phi_t(x) = x + (0,\alpha(t))$ which is a smooth diffeomorphism on $T_1$. Under this coordinate, one has that $\Phi_t(F^u_i) \cap F^s_i$ for any $t \in [0,1]$. Consider the Dehn twist $\Theta_n = \Gamma_n^{-1} \circ \tilde{\Psi} \circ \Gamma_n$ defined on $\{\psi_t(T_1)\}_{t \in [0,n]}$, where $\Gamma_n : \{\psi_t(T_1)\}_{t \in [0,n]} \ni T_1 \times [0,1] \mapsto (x,\frac{t}{n})$. In [BPP], the authors prove that

**Theorem 5.1.** [BPP Theorem 8.1 and Lemma 9.1] For $n > 0$ large, the diffeomorphism $f_b = \Theta_n \circ \psi_n$ is partially hyperbolic with one dimensional neutral center.

By [BZ] Proposition 1.9], there exists a smooth Lyapunov function such that $\{T_1, T_2\}$ is a wandering regular level of this Lyapunov function. As a consequence, one can apply Proposition 4.3 to the diffeomorphism $f_b$. 
5.1.1. Action of $f_b$ on the space of center leaves intersecting $T_1$. We denote by $\mathcal{F}^l$ the $f_b$-invariant foliation tangent to $E^l$, for $l = ss, cs, c, cu, uu$. For each transverse torus $T_i$, we lift the foliations $\mathcal{F}_i^s, \mathcal{F}_i^u$ to the universal cover $\mathbb{R}^2$, and we denote them as $\tilde{\mathcal{F}}_i^s$ and $\tilde{\mathcal{F}}_i^u$ respectively. By transversality, a $\tilde{\mathcal{F}}_i^s$-leaf intersects a $\tilde{\mathcal{F}}_i^u$-leaf in at most one point. Since under the coordinate of $\theta_1$, the induced foliations on $T_1$ are shown as in Figure 3, one has the following result (as it is shown in Figure 4):

**Lemma 5.2.** Under the coordinate $\theta_1$ of $T_1$, we lift the induced foliations to the universal cover $\mathbb{R}^2$. Let $T \in \text{Diff}^1(\mathbb{R}^2)$ be the translation of the form $(t, s) \mapsto (t, s + 1)$. Then for any $x \in \mathbb{R}^2$, one has that the leaf $T(\tilde{\mathcal{F}}_i^u(x))$ intersects $\tilde{\mathcal{F}}_i^s(x)$ in a unique point.

Now, one can define a homeomorphism $\eta \in \text{Homeo}(\mathbb{R}^2)$ which maps the point $x$ to the point $T(\tilde{\mathcal{F}}_i^u(x)) \cap \tilde{\mathcal{F}}_i^s(x)$. One can check that $\eta$ induces a homeomorphism on $T_1$ and for notational convenience, we still denote it as $\eta$. By definition, one has that $\eta$ keeps every leaf of $\mathcal{F}_1^s$ and every leaf of $\mathcal{F}_1^u$ invariant. Moreover, the homeomorphism $\eta$ coincides with identity on union of the compact leaves of $\mathcal{F}_1^s$ and the union of the compact leaves of $\mathcal{F}_1^u$.

**Lemma 5.3.** The action of the diffeomorphism $f_b$ on the space of center leaves intersecting $T_1$ is equivalent to the homeomorphism $\eta^{-1}$.

**Proof.** By Proposition 4.1 in $W_1 = \{\psi_t(T_1)\}_{t \in [0, n]}$, the center unstable foliation $\mathcal{F}^{cu}$ is given by $\Theta_n(W^{cu})$ where $W^{cu}$ is the center unstable foliation of $\psi_n$. Consider the $C^1$ coordinate $\vartheta = (\theta_1, \text{Id}) \circ \Gamma_n$. Under this coordinate and restricted to $W_1$, the center unstable foliation $W^{cu}$ coincides with the product foliation $\mathcal{F}_1^u \times [0, 1]$. Then, by Lemma 5.2 for each point $x \in T_1$, the center leaf through $x$ intersects $\psi_n(T_1)$...
into $\psi_n(\eta(x))$; one can observe this by lifting $W_1$ to the universal cover $\tilde{W}_1$; and the lifts of the non-compact leaves of $\mathcal{F}^{cu}|_{W_1}$ intersect the lifts of the non-compact leaves of $\mathcal{F}^{cs}|_{W_1}$ in the way shown in Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The dash line denotes the center leaf obtained by the intersection of center stable and center unstable leaves.}
\end{figure}

Since $\Theta_n$ coincides with Id in a neighborhood of the boundary of $\{\psi_t(T_i)\}_{t\in[0,n]}$, the diffeomorphism $f_b$ sends the center leaf through $x \in T_1$ to the center leaf through $\psi_n(x)$, and the center leaf through the point $\psi_n(x)$ is the one through the point $\eta^{-1}(x) \in T_1$. \hfill \qed

For the transverse torus $T_2$, the diffeomorphism $f_b$ coincides with $\psi_n$ in $\{\psi_t(T_2)\}_{t\in\mathbb{R}}$, therefore, in $\{\psi_t(T_2)\}_{t\in\mathbb{R}}$, the center stable and center unstable foliations of $f_b$ coincide with the ones of $\psi_n$. Hence, the center leaves intersecting $T_2$ are invariant under $f_b$.

As a Corollary of Lemma 5.3, one has the following:

\textbf{Corollary 5.4.} For the partially hyperbolic diffeomorphism $f_b$, for any $j = 1, 2$ and any leaf $\mathcal{F}^s_j(y)$ (resp. $\mathcal{F}^u_j(y)$) of the foliation $\mathcal{F}^s_j$ (resp. $\mathcal{F}^u_j$), there exists a point $x \in \mathcal{F}^s_j(y)$ (resp. $\mathcal{F}^u_j(y)$) such that $f_b$ preserves the center leaf through $x$, that is, $f_b(\mathcal{F}^c(x)) = \mathcal{F}^c(x)$.

5.2. \textbf{Robust completeness of the center stable foliation of [BPP] example.}

By Theorem A, the center stable and center unstable foliations of $f_b$ are complete. Indeed, for this particular example, one can get the robust completeness of the center stable and center unstable foliations.

Now, we will use Theorem B to give the proof of Proposition 1.4.

\textit{Proof of Proposition 1.4.} Since $f_b$ has neutral behavior along the center, by Theorem 2.3 there exists a $C^1$ small neighborhood $\mathcal{V}$ of $f_b$ such that for any $g \in \mathcal{V}$, one has that

- $g$ is dynamically coherent;
there exists a homeomorphism $h_g : N \mapsto N$ such that for any $x \in N$ and $i = c, cs, cu$, one has
\[ h_g(F_i(x)) = F_i^t(h_g(x)) \text{ and } h_g(f_b(F_i(x))) = g(F_i^t(h_g(x))); \]
- the homeomorphism $h_g$ tends to identity in the $C^0$-topology when $g$ tends to $f$.

Recall that the maximal invariant sets of $f_b$ in $N^+$ and $N^-$ are $\mathcal{R}$ and $\mathcal{A}$ respectively. By Theorem 7A.1 in [HPS], one can choose a small enough neighborhood $U \subset V$ of $f_b$ such that for any $g \in U$, the maximal invariant set of $g$ in the region $N^-$ (resp. $N^+$) is $h_g(\mathcal{A})$ (resp. $h_g(\mathcal{R})$). Hence, the chain recurrent set of $g$ is contained in $h_g(\mathcal{A} \cup \mathcal{R})$.

We will first show the following:

**Lemma 5.5.** For any point $x \in h_g(\mathcal{R})$, one has that
\[ F^ss_g(F^c_g(x)) = F^c_g(x). \]

**Proof.** Notice that $F^cs_g(x) = h_g(F^cs(h_g^{-1}(x)))$. We lift the foliations to the universal cover $\tilde{V}$. Let $\tilde{F}^c_g(y)$ be a lift of $F^c_g(x)$. Notice that every center leaf in $h_g(\mathcal{R})$ is $g$-invariant.

**Claim 5.6.** There exists a lift $\tilde{g}$ of $g$ such that every center leaf in $\tilde{F}^c_g(y)$ is $\tilde{g}$-invariant.

**Proof.** By leaf conjugacy, there is at most one compact center leaf contained in $F^cs_g(x)$. We take a non-compact center leaf $L \subset F^cs_g(x)$, and we denote by $\tilde{L} \subset \tilde{F}^cs_g(y)$ a lift of $L$. Then there exists a unique lift $\tilde{g}$ of $g$ such that $\tilde{L}$ is $\tilde{g}$-invariant.

For any center leaf $\tilde{F}^c_g(z) \subset \tilde{F}^cs_g(y)$, by the fact that $\tilde{F}^c$ is topologically Anosov, there exists a strong stable curve $\tilde{\sigma}(t)_{t \in [0, 1]} \subset \tilde{F}^cs_g(y)$ such that $\tilde{\sigma}(0) \in \tilde{L}$ and $\tilde{\sigma}(1) \in \tilde{F}^c_g(z)$. Let $\sigma(t)$ be the projection of $\tilde{\sigma}(t)$ on $F^c_g(x)$, by the invariance of the center leaves, one has that there exists a continuous family of center curves $\gamma_t(s)$ joining the curve $\sigma(t)$ with $g(\sigma(t))$. Now, we lift the family of center curves to the leaf $\tilde{F}^c_g(y)$, then one gets a continuous family of center curves $\tilde{\gamma}_t(s)$ joining $\tilde{\sigma}(t)$ to a lift $\alpha(t)$ of $g(\sigma(t))$. By the uniqueness of $\tilde{g}$ and the non-compactness of $L$, the lift of $g(\sigma(0))$ can only be $\tilde{g}(\tilde{\sigma}(0))$ which implies that $\alpha(t)$ can only be $\tilde{g}(\tilde{\sigma}(t))$. Hence, one has that $\tilde{F}^c_g(z)$ is invariant under $\tilde{g}$. □

We only need to prove that, restricted to the leaf $\tilde{F}^cs_g(y)$, every strong stable leaf intersects every center leaf. Assume, on the contrary, that $\tilde{F}^ss_g(\tilde{F}^c_g(q))$ has boundary leaves for some $q \in \tilde{F}^cs_g(y)$. Let $\tilde{F}^ss_g(p)$ be one of the boundary leaf.

**Claim 5.7.** Let $\tilde{g}$ be the lift of $g$ given by Claim 5.6, then the leaf $\tilde{F}^ss_g(p)$ is $\tilde{g}$-invariant.

**Proof.** Since $\tilde{F}^c$ is topologically Anosov and $\tilde{F}^cs$ is the stable foliation of the center flow, the foliation $\tilde{F}^c_g$ has the same feature. The strong stable leaf $\tilde{F}^ss_g(p)$ separates the leaf $\tilde{F}^cs_g(y)$, which is a plane, into two connected components $P_1$ and $P_2$ such that
the center leaves converge in $P_1$ and separate in $P_2$. Once again by the topological Anosov property of center foliation, one has that $\mathcal{F}_g^{ss}(\mathcal{F}_g^c(q)) \subset P_1$. By Claim 5.6, every center leaf in $\mathcal{F}^{cs}(y)$ is $\tilde{g}$-invariant, hence $\mathcal{F}_g^{ss}(\mathcal{F}_g^c(q))$ is a $\tilde{g}$-invariant set and the center leaf $\mathcal{F}_g^c(p)$ is also $\tilde{g}$-invariant. Since $\tilde{g}$ sends a boundary leaf to a boundary leaf and $\mathcal{F}_g^{ss}(\mathcal{F}_g^c(q))$ is a path connected invariant set, one has that $\tilde{g}(\mathcal{F}_g^{ss}(p)) \subset P_1$. The boundary leaf $\mathcal{F}_g^{ss}(\tilde{g}(p))$ also separates the leaf $\mathcal{F}^{cs}(y)$ into two connected components $P_1'$ and $P_2'$ such that the center leaves converge in $P_1'$ and separate in $P_2'$. Hence, one has that $\mathcal{F}_g^{ss}(\mathcal{F}_g^c(q)) \subset P_1'$. By the invariance of $\mathcal{F}_g^c(p)$, one has that $\mathcal{F}_g^c(p)$ intersects $\mathcal{F}_g^{ss}(\tilde{g}(p))$, and by transversality, the intersection is unique and we denote it as $z$. If $\mathcal{F}_g^{ss}(p)$ is not $\tilde{g}$-invariant, then the connected component of $\mathcal{F}_g^c(p) \setminus \{z\}$ which does not contain $p$ is contained in $P_2'$. Also the connected component of $\mathcal{F}_g^c(p) \setminus \{p\}$ which does not contain $z$ is in $P_2$. As a consequence, on the leaf $\mathcal{F}^{cs}(y)$, one has that the center leaf $\mathcal{F}_g^c(p)$ is uniformly away from the center leaf $\mathcal{F}_g^c(q)$, which contradicts to the topologically Anosov property of the center foliation. \hfill $\square$

Since every center leaf in $\mathcal{F}_g^{cs}(y)$ is $\tilde{g}$-invariant and every center leaf intersects $\mathcal{F}_g^{ss}(p)$ in at most one point, one has that every point in $\mathcal{F}_g^{ss}(p)$ is a fixed point of $\tilde{g}$, contradicting to the fact that $\mathcal{F}_g^{ss}(p)$ is a strong stable leaf, ending the proof of Lemma 5.5. \hfill $\square$

Now, we consider the center stable leaves in the region $N \setminus h_g(R)$. Assume, on the contrary, that there exists a point $x$ such that $\mathcal{F}_g^{ss}(\mathcal{F}_g^c(x)) \subset \mathcal{F}_g^{cs}(x)$, then let $p$ be a point such that $\mathcal{F}_g^{ss}(p)$ is a boundary leaf of $\mathcal{F}_g^{ss}(\mathcal{F}_g^c(x))$. Since $h_g(A)$ is the maximal invariant set in $N^-$ and is saturated by center unstable leaves, there exists an integer $n$ large enough such that $\mathcal{F}_g^{ss}(g^n(p))$ intersects $h_g(A)$ in a point $q'$. Since every center leaf in $h_g(A)$ is $g$-invariant, the center leaf through $q'$ is contained in $\mathcal{F}_g^{cs}(x)$ and intersects $\mathcal{F}_g^{ss}(p)$. We denote by $q = g^{-n}(q')$, then $q \in \mathcal{F}_g^{ss}(p) \cap \mathcal{F}_g^c(q')$.

Now, we lift these leaves to the universal cover. Let $\tilde{x}$, $\tilde{p}$ and $\tilde{q}$ be the lifts of $x$, $p$ and $q$ respectively such that they are on the same center stable leaf $\mathcal{F}_g^{cs}(\tilde{x})$ and $\mathcal{F}_g^c(\tilde{q})$ intersects $\mathcal{F}_g^{ss}(\tilde{p})$.

Lemma 5.8. There exist a lift $\tilde{g}$ of $g$ and a center leaf $L \subset \mathcal{F}_g^{cs}(\tilde{x})$ such that

- the leaf $L$ is disjoint from $\mathcal{F}_g^{ss}(\tilde{p})$;
- the center leaves $\mathcal{F}_g^c(\tilde{q})$ and $L$ are $\tilde{g}$-invariant.

Proof. If $\mathcal{F}_g^c(x)$ is $g$-invariant, then at least one of the invariant center leaves $\mathcal{F}_g^c(x)$ and $\mathcal{F}_g^c(q)$ is not compact. Assume that $\mathcal{F}_g^c(q)$ is not compact (the other case follows analogously). Then there exists a unique lift $\tilde{g}$ of $g$ such that the center leaf $\mathcal{F}_g^c(\tilde{q})$ is $\tilde{g}$-invariant. By Theorem 13 and leaf conjugacy, one has the following:

- the strip bounded by $\mathcal{F}_g^c(\tilde{q})$ and $\mathcal{F}_g^c(\tilde{x})$ is trivially foliated by center leaves;
- there exists a $C^1$ curve $\ell \subset \mathcal{F}_g^c(\tilde{q})$ with infinity length such that for any point $z \in \ell$, the strong stable leaf $\mathcal{F}_g^{ss}(z)$ through $z$ intersects the center leaf $\mathcal{F}_g^c(\tilde{x})$. 


Since \( f_b \) preserves the orientation of the center foliation, by leaf conjugacy, there exists a point \( w \in \ell \) such that \( \tilde{g}(w) \in \ell \). We take the strong stable segment \( \tilde{\sigma}(t) \) through \( w \) whose two endpoints are contained in \( \tilde{\mathcal{F}}_g^c(\tilde{q}) \) and \( \tilde{\mathcal{F}}_g^c(\tilde{x}) \) respectively. Now, one can check that the arguments in Claim 5.6 can be applied and one gets that \( \tilde{\mathcal{F}}_g^c(\tilde{x}) \) is \( \tilde{g} \)-invariant. We only need to take \( L = \tilde{\mathcal{F}}_g^c(\tilde{x}) \).

If center leaf \( \mathcal{F}_g^c(x) \) is not \( g \)-invariant, then the center leaf \( h_g^{-1}(\mathcal{F}_g^c(x)) \) for \( f_b \) intersects the transverse torus \( T_1 \) and it is not \( f_b \)-invariant. Consider the the connected component \( \mathcal{P} \) of \( \mathcal{F}^{cs}(h_g^{-1}(x)) \backslash \mathcal{A} \) which contains the center leaf \( \mathcal{F}_g^c(h_g^{-1}(x)) \), then \( \mathcal{P} \) is a topological plane. Recall that on each transverse torus, the foliation \( \mathcal{F}^{cs} \) induces a foliation consisting of exactly two Reeb components. Since \( \mathcal{F}_g^c(h_g^{-1}(x)) \) is not \( f \)-invariant, by Lemma 5.3, the intersection between \( \mathcal{P} \) and \( T_1 \) can’t be a circle, hence the intersection is a line \( \ell_1 \); we identify \( \ell_1 \) with \( \mathbb{R} \), then \( \ell_1 \) accumulates to two circles \( S_1, S_2 \) when it tends to infinity; moreover, the circles \( S_1, S_2 \) are contained in the stable manifolds of two different periodic orbits of \( \psi_t \). Hence the boundary of \( \mathcal{P} \) restricted to \( \mathcal{F}_g^{cs}(h_g^{-1}(x)) \) consists of two center leaves belonging to the unstable manifolds of two different periodic orbits of \( \psi_t \) in \( \mathcal{A} \). Then, by leaf conjugacy, there exist two leaves \( L_1, L_2 \) of the foliation \( \tilde{\mathcal{F}}_g^c \) such that

- \( L_1 \cup L_2 \subset \tilde{\mathcal{F}}_g^{ss}(\tilde{x}) \);
- the leaves \( L_1 \) and \( L_2 \) bound a strip containing \( \tilde{\mathcal{F}}_g^c(\tilde{x}) \);
- the projections of the center leaves \( L_1, L_2 \) on the base manifold are \( g \)-invariant.

By the choices of \( L_1, L_2 \), neither \( \pi(L_1) \) nor \( \pi(L_2) \) is a compact leaf, hence one can apply the argument in the first case and one gets a lift \( \tilde{g} \) of \( g \) such that the center leaves \( L_1, L_2, \tilde{\mathcal{F}}_g^c(\tilde{q}) \) are \( \tilde{g} \)-invariant. Then there exists \( i \in \{1, 2\} \) such that \( \tilde{\mathcal{F}}_g^c(\tilde{x}) \) is contained in the strip \( S \) bounded by \( L_i \) and \( \tilde{\mathcal{F}}_g^c(\tilde{q}) \). Since \( \tilde{\mathcal{F}}_g^{ss}(\tilde{p}) \) does not intersect \( \tilde{\mathcal{F}}_g^c(\tilde{x}) \), the strong stable leaf \( \tilde{\mathcal{F}}_g^{ss}(\tilde{p}) \) does not intersect \( L_i \). We take \( L = L_i \), ending the proof of Lemma 5.8.

**Claim 5.9.** There exists a \( \tilde{g} \)-fixed point on \( \tilde{\mathcal{F}}_g^c(\tilde{q}) \) whose strong stable leaf does not intersect \( L \).

**Proof.** If \( \tilde{q} \) is a \( \tilde{g} \)-fixed point, we are done. Now, we assume that \( \tilde{q} \) is not a fixed point. Recall that \( \tilde{q} \in \tilde{h}_g(\mathcal{A}) \). We denote by \( I_q \) the connected component of \( \tilde{\mathcal{F}}_g^c(\tilde{q}) \setminus \{\tilde{q}\} \) such that under leaf conjugacy, it corresponds to the forward orbit of \( \tilde{h}_g^{-1}(q) \) under the Anosov flow \( \psi_t \). Up to replacing \( \tilde{g} \) by \( \tilde{g}^{-1} \), we can assume that \( \tilde{g}(\tilde{q}) \) is contained in the interior of \( I_q \). Since the leaf \( L \) is fixed by \( \tilde{g} \), the strong stable leaf through the orbit of \( \tilde{q} \) is disjoint from \( L \). We identify \( I_q \) with \( (0, +\infty) \), where \( q \) corresponds to \( 0 \). By Theorem 13 for the points on \( I_q \) tending to infinity, their strong stable leaves would intersect \( L \). Hence, one has that the forward orbit of \( \tilde{q} \) tends to a \( \tilde{g} \)-fixed point whose strong stable leaf is disjoint from \( L \).

By Claim 5.9 for notational convenience, one can assume that \( \tilde{q} \) is the fixed point of \( \tilde{g} \) in \( \tilde{\mathcal{F}}_g^c(\tilde{q}) \). Since \( \mathcal{F}_g^c(q) \) is contained in \( h_g(\mathcal{A}) \) and \( \mathcal{A} \) is an attractor of the Anosov flow \( \psi_t \), restricted to the metric on the center stable leaf \( \mathcal{F}_g^c(q) \), the leaf \( \mathcal{F}_g^c(q) \) is accumulated by \( g \)-invariant center leaves. Hence, there exist center leaves on \( \mathcal{F}_g^{cs}(\tilde{q}) \)
contained in $\tilde{h}(A)$ which intersect the strong stable leaf $\tilde{F}^{ss}(\tilde{q})$. Following the argument in Claim 5.6, one can check that these center leaves are invariant under $\tilde{g}$. Recall that in $\tilde{F}^{cs}(\tilde{q})$, each center leaf intersects each strong stable leaf in at most one point. Since these center leaves and the strong stable leaf $\tilde{F}^{ss}(\tilde{q})$ are $\tilde{g}$-invariant, one has that the intersection between these center leaves and $\tilde{F}^{ss}(\tilde{q})$ consists of $\tilde{g}$-fixed points which contradicts to the uniform contraction of $\tilde{g}$ along $\tilde{F}^{ss}(\tilde{q})$. Hence, the center stable foliation of $g$ is complete.

Similar argument applies for the center unstable foliation of $g$. □

We point out that we use [BPP] example to get robust completeness instead of the general [BZ] example, because we need the orbits (for Anosov flow) in the attracting set (resp. repelling set) are not isolated on it stable (resp. unstable) manifolds.

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