Cheeger Inequalities for Submodular Transformations

Yuichi Yoshida*
National Institute of Informatics
yyoshida@nii.ac.jp
August 30, 2017

Abstract

The Cheeger inequality for undirected graphs, which relates the conductance of an undirected graph and the second smallest eigenvalue of its normalized Laplacian, is a cornerstone of spectral graph theory. The Cheeger inequality has been extended to directed graphs and hypergraphs using normalized Laplacians for those, that are no longer linear but piecewise linear transformations.

In this paper, we introduce the notion of a submodular transformation $F : \{0, 1\}^n \rightarrow \mathbb{R}^m$, which applies $m$ submodular functions to the $n$-dimensional input vector, and then introduce the notions of its Laplacian and normalized Laplacian. With these notions, we unify and generalize the existing Cheeger inequalities by showing a Cheeger inequality for submodular transformations, which relates the conductance of a submodular transformation and the smallest non-trivial eigenvalue of its normalized Laplacian. This result recovers the Cheeger inequalities for undirected graphs, directed graphs, and hypergraphs, and derives novel Cheeger inequalities for mutual information and directed information.

Computing the smallest non-trivial eigenvalue of a normalized Laplacian of a submodular transformation is NP-hard under the small set expansion hypothesis. In this paper, we present a polynomial-time $O(\log n)$-approximation algorithm for the symmetric case, which is tight, and a polynomial-time $O(\log^2 n + \log n \cdot \log m)$-approximation algorithm for the general case.

We expect the algebra concerned with submodular transformations, or submodular algebra, to be useful in the future not only for generalizing spectral graph theory but also for analyzing other problems that involve piecewise linear transformations, e.g., deep learning.

*Supported by JST ERATO Grant Number JPMJER1305 and JSPS KAKENHI Grant Number JP17H04676.
## Contents

1 Introduction .......................................................... 1
   1.1 Background ......................................................... 1
   1.2 Our contributions ................................................. 1
   1.3 Proof sketch ....................................................... 5
   1.4 Discussions ....................................................... 7
   1.5 Organization ...................................................... 8

2 Preliminaries ......................................................... 8

3 Submodular Transformations and their Laplacians ......................... 9
   3.1 Submodular Laplacians ............................................ 9
   3.2 Normalized submodular Laplacians ............................... 11

4 Cheeger Inequalities for Submodular Transformations ....................... 11
   4.1 Lower bound on conductance ..................................... 11
   4.2 Upper bound on conductance ..................................... 12
      4.2.1 Rounding .................................................... 12
      4.2.2 Proof of Theorem 1.3 ....................................... 14

5 Covering Number of Base Polytopes ..................................... 17

6 Approximating the Smallest Non-trivial Eigenvalue in the Symmetric Case .... 19
   6.1 SDP relaxation and rounding ..................................... 19
   6.2 Analysis ........................................................... 22

7 Approximating the Smallest Non-trivial Eigenvalue in the General Case ........ 23
   7.1 SDP relaxation and rounding ..................................... 23
   7.2 Analysis ........................................................... 25
      7.2.1 Denominator of Rayleigh quotients ......................... 25
      7.2.2 Numerator of Rayleigh quotients .......................... 27
      7.2.3 Consolidation of results .................................. 30

8 Non-trivial Eigenvalues of Submodular Laplacians .......................... 30
   8.1 Diffusion process .................................................. 31
   8.2 Non-trivial eigenpairs ............................................ 32

A Facts on Normal Distributions ........................................ 35

B Inequalities .......................................................... 36
1 Introduction

1.1 Background

Spectral graph theory is concerned with the relations between the properties of a graph and the eigenvalue/vectors of matrices associated with the graph (refer to [7] for a book). One of the most seminal results in spectral graph theory is the Cheeger inequality [1, 2], which we briefly review below. Let $G = (V, E)$ be an undirected graph. The conductance of a vertex set $\emptyset \subsetneq S \subsetneq V$ is defined as

$$\phi_G(S) = \frac{\text{cut}_G(S)}{\min\{\text{vol}_G(S), \text{vol}_G(V \setminus S)\}},$$

where the cut size of $S$, denoted by $\text{cut}_G(S)$, is the number of edges between $S$ and $V \setminus S$, and the volume of $S$, denoted by $\text{vol}_G(S)$, is the sum of degrees of the vertices in $S$. The conductance $\phi_G$ of $G$ is the minimum conductance of a vertex set $\emptyset \subsetneq S \subsetneq V$. The problem of finding a vertex set of a small conductance has been intensively studied because such a set can be regarded as a tight community [9, 16]. Although computing $\phi_G$ is an NP-hard problem, we can well approximate it using the Cheeger inequality, which relates $\phi_G$ and an eigenvalue of a matrix constructed from $G$ known as the normalized Laplacian. Here, the Laplacian of $G$ is the matrix $L_G = D_G - A_G$, where $D_G \in \mathbb{R}^{V \times V}$ is the diagonal matrix consisting of the degrees of vertices and $A_G \in \mathbb{R}^{V \times V}$ is the adjacency matrix, and the normalized Laplacian of $G$ is the matrix $L_G = (D_G^{-1/2} G D_G^{-1/2}) = I - D_G^{-1/2} A_G D_G^{-1/2}$. Then, the Cheeger inequality [1, 2] states that

$$\frac{\lambda_G}{2} \leq \phi_G \leq \sqrt{2\lambda_G},$$

where $\lambda_G$ is the second smallest eigenvalue of $L_G$ (note that the smallest eigenvalue is zero with the corresponding trivial eigenvector $D_G^{1/2} \mathbf{1}$, where $\mathbf{1}$ is the all-one vector). Indeed, the second inequality of (1) yields an algorithm, which computes a set $\emptyset \subsetneq S \subsetneq V$ of conductance at most $\sqrt{2\lambda_G} = O(\sqrt{\phi_G})$ from an eigenvector corresponding to $\lambda_G$. Moreover, the Cheeger inequality is tight in the sense that computing a set with a conductance $o(\sqrt{\phi_G})$ is NP-hard [25] assuming the small set expansion hypothesis (SSEH) [24].

Extensions of the Cheeger inequality were recently proposed for directed graphs [28] and hypergraphs [5, 18] by using modified notions of conductance and a normalized Laplacian. We note that normalized Laplacians for directed graphs and hypergraphs are no longer linear but piecewise linear transformations. We can show that those normalized Laplacians always have the eigenvalue of zero associated with a trivial eigenvector, and that they also have a non-trivial eigenvalue in the sense that the corresponding eigenvector is orthogonal to the trivial eigenvector. Then, the extended Cheeger inequalities [5, 18, 28] relate the conductance of a directed graph or a hypergraph with the smallest non-trivial eigenvalue of its normalized Laplacian. However, as those normalized Laplacians are no longer linear transformations, computing its smallest non-trivial eigenvalue becomes NP-hard under the SSEH [5, 18]. Although a polynomial-time $O(\log n)$-approximation algorithm is known for hypergraphs on $n$ vertices, which is tight under the SSEH [5, 18], no non-trivial polynomial-time approximation algorithm is known for directed graphs.

1.2 Our contributions

In this paper, we unify and extend the existing Cheeger inequalities discussed above by introducing the notions of a submodular transformation and its normalized Laplacian. A set function $F : \Omega \rightarrow \mathbb{R}$
\{0,1\}^V \to \mathbb{R} is called submodular if \(F(S) + F(T) \geq F(S \cap T) + F(S \cup T)\) for every \(S, T \subseteq V\).
We note that the cut function \(\text{cut}_G : \{0,1\}^V \to \mathbb{R}\) associated with an undirected graph, a directed graph, or a hypergraph \(G\) is submodular, where \(\text{cut}_G(S)\) for a vertex set \(S\) represents the number of edges, arcs, or hyperedges leaving \(S\) and entering \(V \setminus S\). We say that a function \(F : \{0,1\}^V \to \mathbb{R}^E\) is a submodular transformation if \(F_e : S \mapsto F(S)(e)\) is a submodular function for every \(e \in E\).

To derive a Cheeger inequality for a submodular transformation \(F : \{0,1\}^V \to \mathbb{R}^E\), we need to define the conductance of a set with respect to \(F\) and the normalized Laplacian associated with \(F\). First, we define the degree \(d_F(v)\) of \(v \in V\) as the number of \(F_e\)'s to which \(v\) is relevant. (See Section 2 for the formal definition.) For a set \(S \subseteq V\), we define the volume of \(S\) as \(\text{vol}_F(S) = \sum_{v \in S} d_F(v)\) and the cut size of \(S\) as \(\text{cut}_F(S) = \sum_{e \in E} F_e(S)\). Then, we define the conductance \(\phi_F(S)\) of a set \(\emptyset \subseteq S \subseteq V\) as
\[
\phi_F(S) = \min \{\text{cut}_F(S), \text{cut}_F(V \setminus S)\} \div \min \{\text{vol}_F(S), \text{vol}_F(V \setminus S)\}.
\]
We define the conductance of \(F\) as \(\phi_F = \min_{\emptyset \subseteq S \subseteq V} \phi_F(S)\).

**Example 1.1.** Let \(G = (V, E)\) be an undirected graph. Now, we consider a submodular transformation \(F : \{0,1\}^V \to \mathbb{R}^E\), where \(F_e\) is the cut function of the undirected graph with a single edge \(e\). Then, \(d_F(v)\) for a vertex \(v \in V\) coincides with the usual degree of \(v\), and \(\text{cut}_F(S)\) for a vertex set \(S \subseteq V\) coincides with the usual cut size of \(S\). As \(F_e\) is symmetric, that is, \(\text{cut}_F(S) = \text{cut}_F(V \setminus S)\) holds for every vertex set \(S \subseteq V\), \(\phi_F(S)\) coincides with the conductance of \(S\) in the graph sense.

Using a submodular transformation \(F : \{0,1\}^V \to \mathbb{R}^E\), we can define its Laplacian \(L_F : \mathbb{R}^V \to \{0,1\}^{R^V}\). We defer its definition to Section 3 as we need several other notions to define it. Here, we note that \(L_F\) is set-valued and \(L_F(x)\) forms a convex polytope in \(R^V\). However, the measure of the set consisting of \(x \in X\) with \(L_F(x)\) not being a single point is zero. Hence, we can almost always regard \(L_F\) as a function that maps a vector in \(R^V\) to another vector in \(R^V\). Moreover, for \(x \in R^V\) with \(L_F(x)\) consisting of a single point, \(L_F\) acts as a linear transformation. Hence, we can basically regard \(L_F\) as a piecewise linear function.

Next, we define the normalized Laplacian \(\mathcal{L}_F : R^V \to R^V\) as \(\mathcal{L}_F(x) = D_F^{-1/2}L_F(D_F^{-1/2}x)\), where \(D_F \in R^{V \times V}\) is a diagonal matrix with \((D_F)_{vv} = \delta_F(v)\) \((v \in V)\). We say that \(\lambda \in \mathbb{R}\) is an eigenvalue of \(\mathcal{L}_F\) if there exists a non-zero vector \(v \in \mathbb{R}^V\) such that \(\mathcal{L}_F(x) \geq \lambda v\). As with the normalized Laplacian for an undirected graph, when \(F(\emptyset) = F(V) = 0\), we can show that \(\mathcal{L}_F\) is positive-semidefinite, that is, all the eigenvalues are non-negative, and that \(\mathcal{L}_F(D_F^{-1/2}1) \geq 0\), that is, \(0\) is the smallest eigenvalue of \(\mathcal{L}_F\) with the corresponding trivial eigenvector \(D_F^{-1/2}1\). Then, we can also show that there exists a non-trivial eigenvalue in the sense that the corresponding eigenvector is orthogonal to \(D_F^{-1/2}1\). We denote by \(\lambda_F\) the smallest non-trivial eigenvalue of \(\mathcal{L}_F\).

**Example 1.2.** For an undirected graph \(G = (V, E)\), we define a submodular transformation \(F : \{0,1\}^V \to \mathbb{R}^E\) as in Example 1.1. Then, \(\mathcal{L}_F\) essentially equals to the usual normalized Laplacian \(\mathcal{L}_G\) for \(G\) because \(\mathcal{L}_F(x)\) consists of a single vector \(L_Gx\). (See Example 3.3 for details.) Moreover, \(\lambda_F\) is equal to the second smallest eigenvalue of \(\mathcal{L}_G\).

We show the following Cheeger inequality that relates \(\phi_F\) and \(\lambda_F\):

**Theorem 1.3.** Let \(F : \{0,1\}^V \to \mathbb{R}^E\) be a submodular transformation with \(F(\emptyset) = F(V) = 0\) and \(F(S) \in [0,1]\) for every \(S \subseteq V\). Then, we have
\[
\frac{\lambda_F}{2} \leq \phi_F \leq 2\sqrt{\lambda_F}.
\]
We now see several instantiations of Theorem 1.3.

**Example 1.4** (Undirected graphs). For an undirected graph \( G = (V, E) \), we define a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) as in Example 1.1. Then, Theorem 1.3 reduces to the Cheeger inequality for undirected graphs (with a slightly worse coefficient in the right inequality, that is, 2 instead of \( \sqrt{2} \)).

**Example 1.5** (Directed graphs). Let \( G = (V, E) \) be a directed graph. Then, we define a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) so that, for each arc \( e \in E, F_e : \{0, 1\}^V \rightarrow \mathbb{R} \) is the cut function of the directed graph with a single arc \( e \). Then, \( d_F(v) \) for a vertex \( v \in V \) is the number of arcs to which \( v \) is incident as a head or a tail, and \( \text{cut}_F(S) \) for a vertex set \( S \subseteq V \) is the number of arcs leaving \( S \) and entering \( V \setminus S \). Then, the Cheeger inequality derived from Theorem 1.3 coincides with that in [28].

**Example 1.6** (Hypergraphs). Let \( G = (V, E) \) be a hypergraph. Then, we define a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) so that, for each hyperedge \( e \in E, F_e : \{0, 1\}^V \rightarrow \mathbb{R} \) is the cut function of the hypergraph with a single hyperedge \( e \). Then, \( d_F(v) \) for a vertex \( v \in V \) is the number of hyperedges incident to \( v \), and \( \text{cut}_F(S) \) for a vertex set \( S \subseteq V \) is the number of hyperedges containing a vertex in \( S \) and another vertex in \( V \setminus S \). Then, the Cheeger inequality derived from Theorem 1.3 coincides with that in [5, 18].

Theorem 1.3 also derives some novel Cheeger inequalities for joint distributions.

**Example 1.7** (Mutual information). Let \( V \) be a set of Boolean random variables with \( |V| = n \). Then, it is known that the mutual information \( I(S; V \setminus S) \) as a function of \( S \) satisfies submodularity. From the fact that the random variables are Boolean, \( I \) is bounded by \( n \). Now, we define a submodular transformation (or, function) \( F : \{0, 1\}^V \rightarrow \mathbb{R}^V \) as \( I \), divided by \( n \) for normalization. Then, \( d_F(v) = 1 \) for \( v \in V \), and \( \text{cut}_F(S) = I(S; V \setminus S)/n \). Since \( I(S; V \setminus S) \) is symmetric, we have \( \phi_F = \min_{S \subseteq V} I(S; V \setminus S)/\min\{|S|, n - |S|\} \). Intuitively speaking \( \phi_F \) is small when there is a partition of \( V \) into large sets \( S \) and \( V \setminus S \) such that we obtain little information on \( V \setminus S \) by observing \( S \), and vice versa. We can bound \( \phi_F \) from below and above by Theorem 1.3 using \( \lambda_F \).

**Example 1.8** (Directed information). Let \( V \) be a finite set with \( |V| = n \) and for each \( v \in V \), we consider a sequence \((v_1, \ldots, v_\tau)\) of Boolean random variables, where we regard \( v_t \) as the random variable associated with \( v \) at time \( t \in \{1, \ldots, \tau\} \). Then, for a set \( S \subseteq V \) and \( t \in \{1, \ldots, \tau\} \), we define \( S_t = \{v_t \mid v \in S\} \) as the set of random variables associated with \( S \) available at time \( t \), and define \( S_{\leq t} = \{S_1, \ldots, S_t\} \). For two sets \( S, T \subseteq V \), the directed information from \( S \) to \( T \), denoted by \( I(S \rightarrow T) \), is defined as \( \sum_{t=1}^\tau I(S_{\leq t}; T_t \mid T_{\leq t-1}) \), which measures the amount of information that flows from \( S_{\leq \tau} \) to \( T_{\leq \tau} \). Directed information has many applications in causality analysis [20, 21, 22]. The directed information \( I(S \rightarrow V \setminus S) \) as a function of \( S \) is known to be submodular but is unnecessarily symmetric [29].

As in Example 1.7, we define a submodular transformation (or, function) \( F : \{0, 1\}^V \rightarrow \mathbb{R}^V \) as \( I \), divided by \( n\tau \) for normalization. Then, we can bound \( \phi_F \) from below and above by Theorem 1.3 using \( \lambda_F \).

We note that we can easily generalize Examples 1.7 and 1.8 to the case with multiple joint distributions.
The right inequality in Theorem 1.3 is algorithmic in the following sense: Given a vector \( x \in \mathbb{R}^V \) orthogonal to \( D_F^{1/2} \mathbf{1} \), we can compute in polynomial time a set \( \emptyset \subseteq S \subseteq V \) such that 
\[
\phi_F(S) \leq 2\sqrt{R_F(x)},
\]
where \( R_F(x) \) is the Rayleigh quotient of \( L_F \) defined as
\[
R_F(x) = \frac{\langle x, L_F(x) \rangle}{\|x\|_2^2}.
\]
Here, we can show that \( \langle x, y \rangle \) has the same value for any \( y \in \mathcal{L}_F(x) \), and hence we denote it by \( \langle x, L_F(x) \rangle \) by abusing the notation. We can show that \( \lambda_F \) is the minimum of \( R_F(x) \) subject to \( x \neq 0 \) and \( x \) being orthogonal to the trivial eigenvector, that is, \( D_F^{1/2} \mathbf{1} \).

Example 1.9. For a submodular transformation \( F : \{0,1\}^V \to \mathbb{R}^E \) associated with a directed graph \( G = (V, E) \) (see Example 1.1), we have 
\[
\langle x, L_F(x) \rangle = \sum_{\{u,v\} \in E} \left( \frac{x(u)}{d_F(u)} - \frac{x(v)}{d_F(v)} \right)^2.
\]
For a submodular transformation \( F : \{0,1\}^V \to \mathbb{R}^E \) associated with a hypergraph \( G = (V, E) \) (see Example 1.6), we have 
\[
\langle x, L_F(x) \rangle = \sum_{e \in E} \max_{u,v \in e} \left( \frac{x(u)}{d_F(u)} - \frac{x(v)}{d_F(v)} \right)^2.
\]

As opposed to the matrix case, it is NP-hard to compute \( \lambda_F \) under the SSEH. Hence, we consider approximating \( \lambda_F \). First, we provide the following approximation algorithm for symmetric submodular transformations. Here, we say that a submodular transformation \( F : \{0,1\}^V \to \mathbb{R}^E \) is symmetric if \( F(S) = F(V \setminus S) \) for every \( S \subseteq V \).

Theorem 1.10. There is an algorithm that, given \( \epsilon > 0 \) and (a value oracle of) a non-negative symmetric submodular transformation \( F : \{0,1\}^V \to \mathbb{R}^E \) with \( F(\emptyset) = 0 \), computes a non-zero vector \( x \in \mathbb{R}^V \) such that \( \langle x, D_F^{1/2} \mathbf{1} \rangle = 0 \) and
\[
\lambda_F \leq R_F(x) \leq O\left( \frac{\log n}{\epsilon^2} \lambda_F + \epsilon B \right),
\]
with a probability of at least \( 9/10 \) in \( \text{poly}(nm)^{\text{poly}(1/\epsilon)} \) time, where \( n = |V|, m = |E| \), and \( B \) is the maximum Euclidean norm of a point in the base polytopes of \( F_e \)’s.

The definition of the base polytope is deferred to Section 2. We do not need the condition \( F(V) = 0 \) because it follows from \( F(\emptyset) = 0 \) and the symmetry of \( F \). The left inequality is trivial because \( \lambda_F \) is the minimum of \( R_F(x) \) subject to \( x \neq 0 \) and \( \langle x, D_F^{1/2} \mathbf{1} \rangle = 0 \). We note that the approximation ratio of \( O(\log n) \) is tight \([5, 18]\) under the SSEH even when the submodular transformation \( F \) is constructed from a hypergraph as in Example 1.6.

For general submodular transformations, we give the following algorithm:

Theorem 1.11. There is an algorithm that, given \( \epsilon > 0 \) and (a value oracle of) a non-negative submodular transformation \( F : \{0,1\}^V \to \mathbb{R}^E \) with \( F(\emptyset) = F(V) = 0 \), computes a non-zero vector \( x \in \mathbb{R}^V \) such that \( \langle x, D_F^{1/2} \mathbf{1} \rangle = 0 \) and
\[
\lambda_F \leq R_F(x) \leq O\left( \frac{\log n \log(n^{1/2}m)}{\epsilon^2} \lambda_F + \epsilon B \right) = O\left( \frac{\log^2 n}{\epsilon^4} + \frac{\log n \log m}{\epsilon^2} \right) \lambda_F + \epsilon B),
\]

4
with a probability of at least $9/10$ in \(\text{poly}(nm)^{\text{poly}(1/\epsilon)}\) time, where \(n = |V|, m = |E|\), and \(B\) is the maximum Euclidean norm of a point in the base polytopes of \(F_e\)’s.

Again, the left inequality is trivial. Although the approximation ratio for the general case is slightly worse than that for the symmetric case, it remains polylogarithmic in \(n\) and \(m\).

Now, we provide concrete bounds on \(B\) for some specific cases. For the cut functions explained in Examples 1.4, 1.5, and 1.6, we have \(B \leq 1\), and hence the approximated eigenvalue \(\hat{\lambda}_F\) satisfies \(\lambda_F \leq \hat{\lambda}_F \leq O(\log n/\epsilon^2 \cdot \lambda_F + \epsilon)\). Then, we have \(\Omega(\epsilon^2(\hat{\lambda}_F - \epsilon)/\log n) \leq \phi_F \leq O(\sqrt{\lambda_F})\) by Theorem 1.3. Hence, the lower bound is meaningful when \(\hat{\lambda}_F = \Omega(1)\), which always holds when \(\phi_F = \Omega(1)\).

For the mutual and directed information explained in Example 1.7 and 1.8, we have \(B \leq 1/n\), and hence we have \(\Omega(\epsilon^2(\hat{\lambda}_F - 1/n)/\log n) \leq \phi_F \leq O(\sqrt{\lambda_F})\) by Theorem 1.3. Hence, the lower bound is meaningful when \(\hat{\lambda}_F = \Omega(1/n)\).

### 1.3 Proof sketch

The proof of our Cheeger inequality for submodular transformations (Theorem 1.3) is similar to those of the existing Cheeger inequalities [1, 2, 5, 18, 28], although we have to use some specific properties of submodular functions.

In order to prove Theorem 1.10 and 1.11, that is, to approximate the smallest non-trivial eigenvalue of the normalized Laplacian of a submodular transformation, we use semidefinite programming (SDP). To this end, we first rephrase its Rayleigh quotient using Lovász extensions. For a set function \(F : \{0, 1\}^V \to \mathbb{R}\), we define its Lovász extension \(f : \mathbb{R}^V \to \mathbb{R}\) as \(f(x) = \max_{w \in B(F)} \langle w, x \rangle\), where \(B(F) \subseteq \mathbb{R}^V\) is the base polytope of \(F\) (see Section 2 for the definition). Then, for a submodular transformation \(F : \{0, 1\}^V \to \mathbb{R}^F\), the numerator of \(\mathcal{R}_F(x)\) can be written as

\[
\langle x, \mathcal{L}_F(x) \rangle = \sum_{e \in E} f_e(x)^2 = \sum_{e \in E} \left( \max_{w \in B(F_e)} \langle w, x \rangle \right)^2,
\]

where \(f_e : \mathbb{R}^V \to \mathbb{R}\) is the Lovász extension of \(F_e\). Now the goal is to minimize this numerator \((2)\) subject to \(\|x\|^2 = 1\) and \(\langle x, D_e^{-1/2} \rangle = 0\).

In the symmetric case, we can show that it is possible to further rephrase the numerator of \(\mathcal{R}_F(x)\) as

\[
\langle x, \mathcal{L}_F(x) \rangle = \sum_{e \in E} f_e(x)^2 = \sum_{e \in E} \max_{w \in B(F_e)} \langle w, x \rangle^2.
\]

A problem here is that \(B(F_e)\) is a polytope and we cannot express the maximum over \(B(F_e)\) in an SDP. Although it is not difficult to show that we only have to take the maximum over extreme points of \(B(F_e)\), the number of extreme points can be \(n!\) in general, which is prohibitively large.

To address this issue, we replace \(B(F_e)\) with an \(\epsilon B\)-cover \(C_e \subseteq B(F_e)\) (see Theorem 1.10 for the definition of \(B\)), which is a set of points such that for any \(w \in B(F_e)\), there exists a point \(p \in C_e\) with \(\|p - w\|_2 \leq \epsilon B\) Using the properties of submodular functions, we can show that there is an \(\epsilon B\)-cover of size roughly \(O(n^{1/\epsilon^2})\) (instead of being exponential in \(n\)), and we can efficiently compute it by exploiting Wolfe’s algorithm [27], which is useful for judging whether a given point is close to a base polytope. Then, we can solve the resulting SDP in polynomial time in \(n\) and \(m\). The additive error of \(\epsilon B\) in Theorem 1.10 (and Theorem 1.11 as well) occurs when replacing \(B(F_e)\) by its \(\epsilon B\)-cover \(C_e\).
For each variable $x(v)$ in the Rayleigh quotient, we introduce an SDP variable $x_e \in \mathbb{R}^N$ for a large $N \geq n$. Then after solving the obtained SDP, we round the obtained solution $\{x_e\}_{e \in V}$ using the Gaussian rounding, that is, $x_e \mapsto \langle x_e, g \rangle =: z(v)$, where $g \in \mathbb{R}^N$ is sampled from a standard normal distribution $\mathcal{N}(0, I_N)$. Then, we can show that the value of $\sum_{e \in E} f_e(x)^2 = \sum_{e \in E} \max_{w \in \mathcal{C}_e} \langle w, z \rangle^2$ is roughly equal to $\sum_{e \in E} \max_{w \in \mathcal{C}_e} \langle w, z \rangle^2$. Note that, as each $z(v) (v \in V)$ is normally distributed, $\langle w, z \rangle$ for each $w \in \mathcal{C}_e$ acts as a normal random variable. Then, the value $\sum_{e \in E} \max_{w \in \mathcal{C}_e} \langle w, z \rangle^2$ is larger than the SDP value by a factor of $O(\max_{e \in E} \log |\mathcal{C}_e|) = O((\log n)/\epsilon^2)$, caused when taking the maximum of $|\mathcal{C}_e|$ many squared normal variables for each $e \in E$. We can also show that the denominator is at least half and the constraint $\langle z, D_F^{1/2} \rangle = 0$ is satisfied with high probability, and hence we establish Theorem 1.10.

The general case is more involved as we should stick to the numerator of the form (2). To see the difficulty, suppose that the numerator of the Rayleigh quotient is zero in the SDP relaxation, that is, we obtained an SDP solution $\{x_e\}_{e \in V}$ satisfying $\max_{w \in \mathcal{C}_e} \sum_{v \in V} w(v) \langle x_v, v_1 \rangle \leq 0$ for every $e \in E$, where $v_1 \in \mathbb{R}^N$ is a unit vector representing the value of one. Here, this value is supposed to represent $f_e(x) = \max_{w \in \mathcal{C}_e} \langle w, x \rangle$. Hence for the vector $z \in \mathbb{R}^V$ obtained by rounding $\{x_e\}_{e \in V}$, we expect that $f_e(z) \leq 0$. However, if we adopt the Gaussian rounding as with the symmetric case, then $\langle w, z \rangle$ for each $w \in \mathcal{C}_e$ acts as a normal random variable. This means that, with a high probability, we have $f(z) = \max_{w \in \mathcal{C}_e} \langle w, z \rangle > 0$, and hence the approximation ratio can be arbitrarily large.

The above-mentioned problem is avoided by decomposing $x_e$ as $\langle x_e, v_1 \rangle v_1 + P_{v_1} x_e$, where $P_{v_1} \in \mathbb{R}^{N \times N}$ is the projection matrix to the subspace orthogonal to $v_1$. Then, we construct two vectors $z_+ \in \mathbb{R}^V$ and $z_- \in \mathbb{R}^V$ such that $z_+(v) = \langle x_e, v_1 \rangle + \delta \langle P_{v_1} x_e, g \rangle$ and $z_-(v) = \langle x_e, v_1 \rangle - \delta \langle P_{v_1} x_e, g \rangle$ for each $v \in V$, where $\delta = 1/(\log(\sqrt{mn}/\epsilon^2))$ and $g \in \mathbb{R}^N$ is sampled from the standard normal distribution $\mathcal{N}(0, I_N)$. This rounding procedure places more importance on the direction $v_1$ than on other directions. Then, with an additional constraint in the SDP, we can show that the Rayleigh quotient of at least one of them achieves $O((\log n \log(\sqrt{mn}/\epsilon^2))/\epsilon^2)$-approximation.

We have mentioned that the smallest non-trivial eigenvalue $\lambda_F \geq 0$ of the normalized Laplacian $L_F: \{0, 1\}^V \rightarrow \mathbb{R}^V$ of a submodular transformation $F: \{0, 1\}^V \rightarrow \mathbb{R}^F$ is obtained as the minimum of the Rayleigh quotient $R_F(x)$ subject to $x \neq 0$ and $\langle x, D_F^{1/2} \rangle = 0$. As opposed to symmetric matrices, the relation between the eigenvalues of $L_F$ and the Rayleigh quotient $R_F$ is not immediate because $L_F$ is not a linear transformation. Indeed, it is not clear whether $L_F$ has a non-trivial eigenvalue at all. To show this, consider the following diffusion process associated with $L_F$: $dx \in -L_F(x)dt$, that is, at each moment we move the current vector $x \in \mathbb{R}^V$ to a direction chosen from $-L_F(x)$. The idea of using such a diffusion process was already mentioned in [5, 18, 28]. The fact that $L_F(x)$ is not continuous in $x$ means that $x$ may not proceed beyond a certain point. For example, after moving $x$ to $x'$ along the direction $v \in -L_F(x)$ for an infinitesimal time, it could be that the direction $-v$ is in $-L_F(x')$ and $x'$ returns to $x$ by moving along the direction $-v$ for an infinitesimal time. In this problem, we need to choose a direction $v \in -L_F(x)$ at each moment so that the direction $v$ also exists in $-L_F(x')$, where $x'$ is the vector obtained by moving $x$ along the direction $v$ for an infinitesimal time. Fortunately, we can show that such a direction always exists by using Kakutani’s fixed point theorem [10]. Then, analyzing the point at which $x$ converges in the diffusion process, we can guarantee that there exists a small non-trivial eigenvalue of $L_F$ and it is achieved by the minimum of the Rayleigh quotient $R_F(x)$ subject to $x \neq 0$ and $\langle x, D_F^{1/2} \rangle = 0$. 






1.4 Discussions

For undirected graphs, several extensions of the Cheeger inequality have been proposed. For a graph $G = (V, E)$, the order-$k$ conductance of $k$ disjoint vertex sets $S_1, \ldots, S_k \subseteq V$ is defined as their maximum conductance, and the order-$k$ conductance of a graph is the minimum order-$k$ conductance of $k$ disjoint vertex sets taken from the graph. Then, the higher order Cheeger inequality [15, 19] bounds the order-$k$ conductance of a graph from below and above by the $k$-th smallest eigenvalue of its normalized Laplacian. The standard conductance is also analyzed using the $k$-th smallest eigenvalue [13, 14]. In [26], it is argued that the largest eigenvalue of a normalized Laplacian can be used to bound from below and above the bipartiteness ratio, which measures the extent to which the graph is approximated by a bipartite graph. Its higher order version is also studied [17]. It would be interesting to generalize these extended Cheeger inequalities for submodular transformations.

We believe that the notion of a submodular transformation will be useful not only for generalizing spectral graph theory but also for analyzing various problems that involve piecewise linear functions. To see this, we introduce the notion of a Lovász transformation, which is a function of the form $f : \mathbb{R}^V \rightarrow \mathbb{R}^E$ such that $f_e : x \mapsto f(x)(e)$ is the Lovász extension of some submodular function for each $e \in E$.

Lovász transformations are piecewise linear in general, and can express any linear transformation coordinate-wise: $x \mapsto \max\{x, 0\}$. Then, in the regression setting with the $\ell_2$-norm loss, given training examples $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^n \times \mathbb{R}$, we aim to find $W_1, \ldots, W_L$ that minimizes the loss function $\sum_{i=1}^m \|f(x_i) - y_i\|_2^2$. As the loss function is non-convex, we cannot hope to obtain the global minimum in polynomial time, and hence we want to analyze the structure of local minima. When ReLUs are not applied in a neural network, every local minimum is known to be a global minimum (under a plausible assumption) [11]. However, the proof heavily relies on elegant properties of linear transformations and it does not generalize to the case with ReLUs.

Note that the function $\max\{x - y, 0\}$ is the Lovász extension of the cut function of the directed graph consisting of a single arc $(x, y)$. Using this fact, we can express the feed-forward neural network $f : \mathbb{R}^n \rightarrow \mathbb{R}$ used in deep learning is of the following form:

$$f(x) = W_L(\sigma_{L-1}(\cdots \sigma_2(W_2\sigma_1(W_1x)))),$$

where $W_1 \in \mathbb{R}^{d_{L-1} \times d_{L-1}}$ ($\ell \in \{1, \ldots, L\}$) is a matrix with $d_0 = n$ and $d_L = 1$ and $\sigma_{\ell} : \mathbb{R}^{d_\ell} \rightarrow \mathbb{R}^{d_\ell}$ ($\ell \in \{1, \ldots, L - 1\}$) is a rectified linear unit (ReLU), which applies the following operation coordinate-wise: $x \mapsto \max\{x, 0\}$. Then, in the regression setting with the $\ell_2$-norm loss, given training examples $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^n \times \mathbb{R}$, we aim to find $W_1, \ldots, W_L$ that minimizes the loss function $\sum_{i=1}^m \|f(x_i) - y_i\|_2^2$. As the loss function is non-convex, we cannot hope to obtain the global minimum in polynomial time, and hence we want to analyze the structure of local minima. When ReLUs are not applied in a neural network, every local minimum is known to be a global minimum (under a plausible assumption) [11]. However, the proof heavily relies on elegant properties of linear transformations and it does not generalize to the case with ReLUs.

Indeed, the smallest non-trivial eigenvalue of the Laplacian $L_F$ of a submodular transformation $F : \{0, 1\}^V \rightarrow \mathbb{R}^E$ is equal to $\min_{x, 1} \|f(x)\|_2^2/\|x\|_2^2$ for the corresponding Lovász transformation $f : \mathbb{R}^V \rightarrow \mathbb{R}^E$, which can be regarded as the smallest non-trivial singular value of $f$. (The connection
will become clear in Section 3.) Hence, this work can be seen as the first step toward extending linear algebra to the algebra with Lovász transformations, or submodular algebra.

1.5 Organization
In Section 2, we review basic properties of submodular functions. In Section 3, we formally define submodular transformation and its Laplacian, and observe their basic properties. We prove the Cheeger inequality for submodular transformations in Section 4. We consider the covering number of the base polytope of a submodular function in Section 5. Then, we provide polynomial-time approximation algorithms for the smallest non-trivial eigenvalue of a normalized submodular Laplacian for the symmetric and general cases in Sections 6 and 7, respectively. In Section 8, we show that the (normalized) Laplacian of a submodular transformation has a non-trivial eigenvalue and it can be obtained by minimizing the Rayleigh quotient.

2 Preliminaries
For an integer \( n \in \mathbb{N} \), we define \([n]\) as the set \( \{1, 2, \ldots, n\} \). For a subset \( S \subseteq V \), we define \( \mathbf{1}_S \in \mathbb{R}^n \) as the indicator vector of \( S \), that is, \( \mathbf{1}_S(v) = 1 \) if \( v \in S \) and \( \mathbf{1}_S(v) = 0 \) otherwise. When \( S = V \), we simply write \( \mathbf{1} \). For a vector \( \mathbf{x} \in \mathbb{R}^V \) and a subset \( S \subseteq V \), we define \( \mathbf{x}|_S \in \mathbb{R}^V \) as the vector such that \( \mathbf{x}|_S(v) = x(v) \) for every \( v \in S \) and \( \mathbf{x}|_S(v) = 0 \) for every \( v \in V \setminus S \). The support of a vector \( \mathbf{x} \in \mathbb{R}^n \), denoted by \( \text{supp}(\mathbf{x}) \), is defined as the set \( \{v \in V \mid x(v) \neq 0\} \). For a polyhedron \( P \subseteq \mathbb{R}^V \), \( \mathbf{p} \in \mathbb{R}^V \), and \( r > 0 \), we define \( \mathbf{p} + P = \{\mathbf{p} + \mathbf{x} \mid \mathbf{p} \in P\} \) and \( rP = \{r\mathbf{x} \mid \mathbf{x} \in P\} \). For a polytope \( P \), we define \( \|\mathbf{p}\|_H = \max_{\mathbf{p} \in P} \|\mathbf{p}\|_2 \) as the maximum \( \ell_2 \)-norm of a point in \( P \).

For a set function \( F : \{0,1\}^V \to \mathbb{R} \), we define \( \|F\|_\infty = \max_{S \subseteq V} F(S) \). For a set function \( F : \{0,1\}^V \to \mathbb{R} \), a set \( S \subseteq V \), and an element \( v \in V \setminus S \), we define \( f(v \mid S) \) as the marginal gain \( f(S \cup \{v\}) - f(S) \).

A function \( F : \{0,1\}^V \to \mathbb{R} \) is referred to as submodular if

\[
\begin{align*}
f(S) + f(T) & \geq f(S \cup T) + f(S \cap T)
\end{align*}
\]

for every \( S, T \subseteq V \). We say that a function \( F : \{0,1\}^V \to \mathbb{R} \) is symmetric if \( F(S) = F(V \setminus S) \) for every \( S \subseteq V \). A submodular function \( F : \{0,1\}^V \to \mathbb{R} \) is referred to as normalized if \( F(\emptyset) = 0 \). In this work, we only consider normalized submodular functions.

We consider a variable \( v \in V \) relevant in \( F : \{0,1\}^V \to \mathbb{R} \) if adding (or removing) \( v \) from the input set may change the value of \( F \), that is, there exists some \( S \subseteq V \setminus \{v\} \) such that \( F(S) \neq F(S \cup \{v\}) \). We consider \( v \) irrelevant otherwise. The support of \( F \), denoted by \( \text{supp}(F) \), is the set of relevant variables of \( F \).

Let \( F : \{0,1\}^V \to \mathbb{R} \) be a submodular function. The submodular polyhedron \( P(F) \) and the base polytope \( B(F) \) of \( F \) are defined as

\[
P(F) = \left\{ \mathbf{x} \in \mathbb{R}^V \mid \sum_{v \in S} x(v) \leq F(S) \ \forall S \subseteq V \right\} \quad \text{and} \quad B(F) = \left\{ \mathbf{x} \in P(F) \mid \sum_{v \in V} x(v) = F(V) \right\}.
\]

As the name suggests, it is known that the base polytope is bounded (Theorem 3.12 of [8]).

The Lovász extension \( f : \mathbb{R}^V \to \mathbb{R} \) of a submodular function \( F : \{0,1\}^V \to \mathbb{R} \) is defined as

\[
f(x) = \max_{w \in B(F)} \langle w, x \rangle.
\]
We note that \( f(1_S) = F(S) \) for every \( S \subseteq V \) and hence we can uniquely recover a submodular function from its Lovász extension.

We define \( \partial f(x) = \arg\max_{w \in B(F)} \langle w, x \rangle \) as the set of vectors \( w \in B(F) \) that attains \( f(x) \). The following is well known:

**Lemma 2.1** (Theorem 3.22 of [8]). Let \( f : \mathbb{R}^V \rightarrow \mathbb{R} \) be the Lovász extension of a submodular function. Then, every extreme point \( w \) of \( \partial f(x) \) is obtained as follows: Let \( v_1, \ldots, v_n \) be an ordering of \( V \) with \( |V| = n \) such that \( x(v_1) \geq \cdots \geq x(v_n) \). Then, \( w(v_i) = f(v_i \mid \{v_1, \ldots, v_{i-1}\}) \) for every \( i \in [n] \).

In particular, every extreme point of \( \partial f(0) = B(F) \) can be obtained by following this approach by setting \( x = 0 \).

The algorithm for computing \( w \in \partial f(x) \) based on the ordering of values in \( x \) is known as Edmonds’ algorithm in the literature. By Lemma 2.1, as long as the ordering of values \( x(v) (v \in V) \) does not change, we can use the same \( w \in \mathbb{R}^V \) for computing \( f(x) \).

### 3 Submodular Transformations and their Laplacians

In this section, we introduce the notion of a submodular transformation and its Laplacian and normalized Laplacian.

For a function \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) and \( e \in E \), let \( F_e : \mathbb{R}^V \rightarrow \mathbb{R} \) be the \( e \)-th component of \( F \), that is, \( F_e : x \mapsto F(x)(e) \). Then, we define a submodular transformation as follows:

**Definition 3.1** (Submodular transformation). We say that \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) is a submodular transformation if the function \( F_e : \mathbb{R}^V \rightarrow \mathbb{R} \) is a submodular function for every \( e \in E \).

For a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \), we always use the symbols \( n \) and \( m \) to denote \( |V| \) and \( |E| \). We say that a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) is symmetric if \( F(S) = F(V \setminus S) \) for every \( S \subseteq V \). The Lovász extension \( f : \mathbb{R}^V \rightarrow \mathbb{R}^E \) of a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) is such that \( f_e : x \mapsto f(x)(e) \) is the Lovász extension of \( F_e \) for each \( e \in E \). The Lovász extensions of submodular transformations are collectively referred to as Lovász transformations. For a submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R} \), we will use symbols \( f \) and \( f_e (e \in E) \) to denote those functions.

In Section 3.1, we define the Laplacian of a submodular transformation, which we collectively refer to as a submodular Laplacian, and study its basic spectral properties. In Section 3.2, we discuss the normalized version of a submodular Laplacian.

#### 3.1 Submodular Laplacians

We define the Laplacian associated with a submodular transformation as follows:

**Definition 3.2** (Submodular Laplacian). Let \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) be a submodular transformation. Then, the Laplacian \( L_F : \mathbb{R}^V \rightarrow 2^{\mathbb{R}^V} \) of \( F \) is defined as

\[
L_F(x) = \left\{ \sum_{e \in E} w_e \langle w_e, x \rangle \mid w_e \in \partial f_e(x) (e \in E) \right\} = \left\{ WW^T x \mid W \in \prod_{e \in E} \partial f_e(x) \right\},
\]

\(^1\)We adopted the notation \( \partial f(x) \) because each vector in \( \partial f(x) \) is a subgradient of \( f \) at \( x \) [8]. However, we do not use this property in the work presented in this paper.
where \(f_e\) is the Lovász extension of \(F_e\) for each \(e \in E\).

We can verify that, for every \(z \in L_F(x)\), we have \(\langle x, z \rangle = \sum_{e \in E} f_e(x)^2\), and hence we write \(\langle x, L_F(x) \rangle\) to denote \(\sum_{e \in E} f_e(x)^2\) by abusing the notation. Let \(f : \mathbb{R}^V \rightarrow \mathbb{R}^E\) be the Lovász extension of \(F\). Then, we have \(f(x) = W^T x\) for any \(W \in \prod_{e \in E} \partial f_e(x)\). Hence, we can symbolically understand \(L_F\) as \(f^T f\) because \(\langle x, L_F(x) \rangle = \sum_{e \in E} f_e(x)^2 = \|f(x)\|^2_2\), and this is the intuition behind the definition of \(L_F\).

**Example 3.3.** For an undirected graph \(G = (V, E)\), we define a submodular transformation \(F : \{0,1\}^V \rightarrow \mathbb{R}^E\) as in Example 1.1. Then for an edge \(e = \{u, v\} \in E\), we have \(w_e = (-1,1)\) if \(x(u) < x(v)\), \(w_e = (1,-1)\) if \(x(u) > x(v)\), and \(w_e\) is of the form \((a,-a)\) for \(a \in [-1,1]\) if \(x(u) = x(v)\). Then, we can verify that \(L_F(x) = (D_G - A_G)x = L_Gx\), where \(L_G \in \mathbb{R}^{V \times V}\) is the usual Laplacian of \(G\).

A pair \((\lambda, x) \in \mathbb{R} \times \mathbb{R}^V\) is called an **eigenpair** of a submodular Laplacian \(L_F : \mathbb{R}^V \rightarrow 2^{\mathbb{R}^V}\) if \(L_F(x) \ni \lambda x\). Such \(\lambda\) and \(x\) are called **eigenvalue** and **eigenvector** of \(L_F\), respectively. When a submodular transformation \(F : \{0,1\}^V \rightarrow \mathbb{R}^E\) satisfies \(F(V) = 0\), its Laplacian satisfies the following elegant spectral properties:

**Lemma 3.4.** Let \(F : \{0,1\}^V \rightarrow \mathbb{R}^E\) be a submodular transformation with \(F(V) = 0\). Then, \((0,1)\) is an eigenpair of \(L_F\).

**Proof.** We have \(L_F(1) = \{\sum_{e \in E} w_e f_e(1) \mid w_e \in \partial f_e(1) (e \in E)\} = \{0\} \ni 0 \cdot 1\).

**Lemma 3.5.** Let \(F : \{0,1\}^V \rightarrow \mathbb{R}^E\) be a submodular transformation. Then, \(L_F\) is positive-semidefinite, that is, all the eigenvalues of \(L_F\) are non-negative.

**Proof.** Let \((\lambda, x)\) be an eigenpair of \(L_F\). Then, we have \(\langle x, L_F(x) \rangle = \lambda \|x\|^2_2\). On the other hand, we have \(\langle x, L_F(x) \rangle = \sum_{e \in E} f_e(x)^2 \geq 0\). Hence, \(\lambda\) should be non-negative.

From Lemmas 3.4 and 3.5, the value 0 is the smallest eigenvalue of \(L_F\) with the corresponding eigenvector \(1\). Hence, we call \(1\) the **trivial eigenvector** of \(L_F\) and call \((0,1)\) the **trivial eigenpair** of \(L_F\).

The **Rayleigh quotient** \(R_F : \mathbb{R}^V \rightarrow \mathbb{R}\) of the Laplacian of a submodular transformation \(F : \{0,1\}^V \rightarrow \mathbb{R}^E\) is defined as

\[
R_F(x) = \frac{\langle x, L_F(x) \rangle}{\langle x, x \rangle} = \frac{\sum_{e \in E} f_e(x)^2}{\|x\|^2_2} = \frac{\|f(x)\|^2_2}{\|x\|^2_2}.
\]

When \(L_F\) is a matrix, the minimum of \(R_F(x)\) subject to \(x \neq 0\) and \(x \perp u\) provides the smallest non-trivial eigenvalue and the minimizer is the corresponding eigenvector of \(L_F\). In Section 8, we show the following relation for general submodular transformations:

**Theorem 3.6.** For a submodular transformation \(F : \{0,1\}^V \rightarrow \mathbb{R}^E\) with \(F(V) = 0\), the Laplacian \(L_F\) has a non-trivial eigenpair, that is, there exist \(\gamma \in \mathbb{R}_+\) and a non-zero vector \(z \in \mathbb{R}^V\) such that \(z \perp 1\) and \(L_F(z) \ni \gamma z\). Furthermore, each such \(\gamma\) and \(z\) satisfies \(\gamma = R_F(z)\).
3.2 Normalized submodular Laplacians

Let $F : \{0, 1\}^V \to \mathbb{R}^E$ be a submodular transformation. We define the degree vector $d_F \in \mathbb{R}^V$ of $F$ as $d_F(v) = |\{e \in E \mid v \in \text{supp}(F_e)\}|$. We say that $d_F(v)$ is the degree of $v \in V$ with respect to $F$. Let $D_F \in \mathbb{R}^{V \times V}$ be the diagonal matrix with $(D_F)_{vv} = d_F(v)$. Then, we define the normalized Laplacian $L_F : \mathbb{R}^V \to 2\mathbb{R}^V$ of $f$ as $L_F(x) = D_F^{-1/2}L_F(D_F^{-1/2}x)$, or more formally, $L_F(x) = \{D_F^{-1/2}z \mid z \in L_F(D_F^{-1/2}x)\}$. When we consider normalized Laplacians, we always assume that every element of $d_F$ is positive as otherwise we cannot define $D_F^{-1/2}$.

We define an eigenpair/value/vector of the normalized Laplacian of a submodular transformation as with the Laplacian of a submodular transformation. Then, using the same argument as in Lemmas 3.4 and 3.5, we can show that, for any submodular transformation $F : \{0, 1\}^V \to \mathbb{R}^E$ with $F(V) = 0$, its normalized Laplacian $L_F$ has an eigenpair $(0, D_F^{1/2}1)$ and that $L_F$ is positive-semidefinite. We call $D_F^{1/2}1$ the trivial eigenvector of $L_F$ and call $(0, D_F^{1/2}1)$ the trivial eigenpair of $L_F$. We define $\mathcal{R}_F : \mathbb{R}^V \to \mathbb{R}$ as the Rayleigh quotient of the normalized Laplacian of $f$, that is,

$$\mathcal{R}_F(x) = \frac{\langle x, L_F(x) \rangle}{\langle x, x \rangle} = \frac{\sum_{e \in E} f_e(D_F^{-1/2}x)^2}{\|x\|^2} = \frac{\|f(D_F^{-1/2}x)\|^2}{\|x\|^2}.$$ 

We have the following, which is a counterpart of Theorem 3.6 for normalized Laplacians.

**Theorem 3.7.** For a submodular transformation $F : \{0, 1\}^V \to \mathbb{R}^E$, the normalized Laplacian $L_F$ has a non-trivial eigenvector, that is, there exist $\gamma \in \mathbb{R}_+$ and a non-zero vector $z \in \mathbb{R}^V$ such that $x \perp D_F^{1/2}1$ and $L_F(z) \ni \gamma z$. Furthermore, each such $\gamma$ and $z$ satisfies $\gamma = \mathcal{R}_F(z)$.

4 Cheeger Inequalities for Submodular Transformations

In this section, we prove our Cheeger inequality for submodular transformations, that is, Theorem 1.3. We prove the left and right inequalities of Theorem 1.3 in Sections 4.1 and 4.2, respectively.

The following fact is useful in this section.

**Proposition 4.1.** Let $F : \{0, 1\}^V \to \mathbb{R}^E$ be a submodular transformation with $F(V) = 0$ and let $f : \mathbb{R}^V \to \mathbb{R}^E$ be its Lovász extension. Then, we have $f(x + c1) = f(x)$ for any $c \in \mathbb{R}^V$.

**Proof.** Fix $e \in E$. Note that any $w \in B(F_e)$ satisfies $w(V) = 0$ because $F_e(V) = 0$. Then, we have $f_e(x + c1) = \max_{w \in B(F_e)} \langle w, x + c1 \rangle = \max_{w \in B(F_e)} \langle w, x \rangle = f_e(x)$. \(\square\)

4.1 Lower bound on conductance

**Proof of the left inequality of Theorem 1.3.** Let $\emptyset \subseteq S \subseteq V$ be a subset that achieves $\phi_F = \phi_F(S)$ with $\text{vol}_F(S) \leq \text{vol}_F(V \setminus S)$. Let $x \in \mathbb{R}^V$ be the vector obtained from $D_F^{1/2}1_S$ by projecting it to the subspace orthogonal to $D_F^{1/2}1$. Then, we can write $x = D_F^{1/2}1_S + cD_F^{1/2}1/\|D_F^{1/2}1\|_2$, where

$$c^2 = \frac{\langle D_F^{1/2}1_S, D_F^{1/2}1 \rangle^2}{\|D_F^{1/2}1\|_2^2} = \frac{(\sum_{v \in S} d_F(v))^2}{\sum_{v \in V} d_F(v)} = \frac{\text{vol}_F(S)^2}{\text{vol}_F(V)} \leq \frac{1}{2} \text{vol}_F(S).$$
Then by the Pythagorean theorem, we have
\[
\|x\|_2^2 = \|D_F^{1/2}1_S\|_2^2 - c^2 \geq \text{vol}_F(S) - \frac{1}{2}\text{vol}_F(S) = \frac{1}{2}\text{vol}_F(S).
\]

Further, we have
\[
x^\top L(x) = \sum_{e \in E} f_e(D_F^{-1/2}x)^2 = \sum_{e \in E} f_e(1_S + c1/\|D_F^{1/2}1\|)^2 = \sum_{e \in E} f_e(1_S)^2 = \sum_{e \in E} F_e(S)^2,
\]
where we used Proposition 4.1 in the third equality.

As \(F_e(S) \in [0, 1]\) holds for every \(e \in E\) and \(S \subseteq V\), we have
\[
\lambda_F \leq R_F(x) = \frac{x^\top L(x)}{\|x\|_2^2} = \frac{2\sum_{e \in E} F_e(S)^2}{\text{vol}_F(S)} \leq \frac{2\sum_{e \in E} F_e(S)}{\text{vol}_F(S)}.
\]
Similarly, by considering \(-x\), we can show that \(\lambda_F \leq 2\sum_{e \in E} F_e(V \setminus S)/\text{vol}_F(S)\), and hence we obtain \(\lambda_F \leq 2\phi_F\).

\[\square\]

### 4.2 Upper bound on conductance

In this section, we first provide an extension of the rounding known as *sweep rounding*, which is used in the proof of the Cheeger inequality for undirected graphs (Section 4.2.1). Then, we prove the right inequality of Theorem 1.3 (Section 4.2.2).

#### 4.2.1 Rounding

We start with the following equivalent definition of Lovász extension:

**Lemma 4.2** (See, e.g., Definition 3.1 of [3]). *Let \(F : \{0, 1\}^V \to \mathbb{R}\) be a submodular function and \(f : \mathbb{R}^V \to \mathbb{R}\) be its Lovász extension. Then, we have*

\[
f(x) = \int_0^\infty F([v \in V \mid x(v) \geq r}) \, dr + \int_{-\infty}^0 \left(F([v \in V \mid x(v) \geq r}) - F(V)\right) \, dr.
\]

For \(\tau \in [0, 1]\), we define the threshold function \(\text{thr}_\tau : [0, 1] \to \{0, 1\}\) as \(\text{thr}_\tau(x) = 1\) if \(x \geq \tau\) and \(\text{thr}_\tau(x) = 0\) otherwise. For a vector \(x \in [0, 1]^V\), we define \(\text{thr}_\tau(x) \in \{0, 1\}^V\) as the vector obtained from \(x\) by applying \(\text{thr}_\tau(\cdot)\) coordinate-wise. Then, we can rephrase \(f(x)\) using the threshold function as follows:

**Lemma 4.3.** *Let \(F : \{0, 1\}^V \to \mathbb{R}\) be a submodular function. Then, we have*

\[
f(x) = \int_0^1 f(\text{thr}_\tau(x)) \, d\tau
\]

*for any \(x \in [0, 1]^V\).*

**Proof.** By Lemma 4.2, we have

\[
f(x) = \int_0^\infty F([v \in V \mid x(v) \geq \tau}) \, d\tau + \int_{-\infty}^0 \left(F([v \in V \mid x(v) \geq \tau}) - F(V)\right) \, d\tau
\]

\[
= \int_0^1 F([v \in V \mid x(v) \geq \tau}) \, d\tau = \int_0^1 f(\text{thr}_\tau(x)) \, d\tau,
\]

where in the last equality, we used the fact that \(f(1_S) = F(S)\) for \(S \subseteq V\).

\[\square\]
Next, we provide two rounding methods, one for the case \( x \in [0,1]^V \) and the other for the case \( x \in [-1,0]^V \).

**Lemma 4.4.** Let \( F : \{0,1\}^V \rightarrow \mathbb{R}^E \) be a submodular transformation and \( f : \mathbb{R}^V \rightarrow \mathbb{R}^E \) be its Lovász extension. For any \( x \in [0,1]^V \), there exists a set \( \emptyset \subset S \subset \text{supp}(x) \) such that

\[
\frac{\text{cut}_F(S)}{\text{vol}_F(S)} \leq \frac{\sum_{e \in E} f_e(x)}{\sum_{v \in V} d_F(v) x(v)}.
\]

Moreover, we can compute such a set \( S \) in \( O(n \log n + nm) \) time.

**Proof.** By Lemma 4.3, we have

\[
\int_0^1 \sum_{e \in E} f_e(\text{thr}_\tau(x)) \, d\tau = \int_0^1 \sum_{v \in V} d_F(v) \text{thr}_\tau(x(v)) \, d\tau = \sum_{e \in E} f_e(x).
\]

Therefore, there exists \( \tau^* \in [0,1] \) such that

\[
\frac{\sum_{e \in E} f_e(\text{thr}_{\tau^*}(x))}{\sum_{v \in V} d_F(v) \text{thr}_{\tau^*}(x(v))} \leq \frac{\sum_{e \in E} f_e(x)}{\sum_{v \in V} d_F(v) x(v)}.
\]

Let \( S \) be the support of the vector \( \text{thr}_{\tau^*}(x) \). Note that we can always choose \( S \) to be non-empty. Since \( \text{thr}_{\tau^*}(x) \) is a \( \{0,1\} \)-vector, we have \( \text{thr}_{\tau^*}(x) = 1_S \). Then, we have

\[
\frac{\sum_{e \in E} f_e(\text{thr}_{\tau^*}(x))}{\sum_{v \in V} d_F(v) \text{thr}_{\tau^*}(x(v))} = \frac{\sum_{e \in E} F_e(S)}{\sum_{v \in V} d_F(v)} = \frac{\text{cut}_F(S)}{\text{vol}_F(S)}.
\]

Therefore, we have

\[
\frac{\text{cut}_F(S)}{\text{vol}_F(S)} \leq \frac{\sum_{e \in E} f_e(x)}{\sum_{v \in V} d_F(v) x(v)} \quad \text{and} \quad \emptyset \subset S \subset \text{supp}(x).
\]

We can find this set \( S \) as follows. First, let \( v_1, \ldots, v_n \) be the ordering of \( V \) such that \( x(v_1) \geq \cdots \geq x(v_n) \). Then, we consider sets of the form \( \{v_1, \ldots, v_k\} \) for \( k \in [n] \) and then return the set with the smallest conductance. The running time of this algorithm is \( O(n \log n + nm) \). \(\Box\)

**Corollary 4.5.** Let \( F : \{0,1\}^V \rightarrow \mathbb{R}^E \) be a submodular transformation with \( F(V) = 0 \) and \( f : \mathbb{R}^V \rightarrow \mathbb{R}^E \) be its Lovász extension. For any \( x \in [-1,0]^V \), there exists a set \( \emptyset \subset S \subset \text{supp}(x) \) such that

\[
\frac{\text{cut}_F(V \setminus S)}{\text{vol}_F(S)} \leq \frac{\sum_{e \in E} f_e(x)}{\sum_{v \in V} d_F(v) x(v)}.
\]

Moreover, we can compute such a set \( S \) in \( O(n \log n + nm) \) time.
Proof. Define a submodular transformation $F' : \{0,1\}^V \to \mathbb{R}^E$ as $F'(S) = F(V \setminus S)$, and let $f' : \mathbb{R}^V \to \mathbb{R}^E$ be its Lovász extension. Then, we have $f'(z) = f(1-z) = f(-z)$ for any $z \in \mathbb{R}^V$ by Proposition 4.4.1.

We apply Lemma 4.4 on $F'$ and $-x$. Then, we obtain a set $\emptyset \subseteq S \subseteq \text{supp}(x)$ such that

$$\frac{\text{cut}_f(V \setminus S)}{\text{vol}_f(S)} = \frac{\sum_{e \in E} F'_e(S)}{\text{vol}_f(S)} = \frac{\text{cut}_{f'}(S)}{\text{vol}_{f'}(S)} \leq \frac{\sum_{e \in E} f'_e(-x)}{\text{vol}_{f'}(S)} - \sum_{v \in V} d_{f'}(v)x(v) = \frac{\sum_{e \in E} f_e(x)}{\text{vol}_{f'}(S)} - \sum_{v \in V} d_{f'}(v)x(v).$$

4.2.2 Proof of Theorem 1.3

We start proving Theorem 1.3. To this end, we need several auxiliary lemmas. For a vector $x \in \mathbb{R}^V$, we define $x_+ \in \mathbb{R}^V$ and $x_- \in \mathbb{R}^V$ as

$$x_+(v) = \begin{cases} x(v) & \text{if } x(v) \geq 0, \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad x_-(v) = \begin{cases} x(v) & \text{if } x(v) \leq 0, \\ 0 & \text{otherwise}, \end{cases}$$

Lemma 4.6. Let $f : \mathbb{R}^V \to \mathbb{R}$ be the Lovász extension of a submodular function $F : \{0,1\}^V \to \mathbb{R}$. If $f$ is non-negative, then we have

$$f(x_+)^2 + f(x_-)^2 \leq f(x)^2.$$

Proof. Recall that

$$f(x) = \max_{w \in B(F)} \langle w, x \rangle$$

Let $w^* \in \mathbb{R}^V$ be the maximizer of this maximization problem. Then, by Lemma 2.1, we can calculate $w^*$ as follows: First, let $v_1, \ldots, v_n$ be an arbitrary ordering of indices in $V$, such that $x(v_1) \geq x(v_2) \geq \cdots \geq x(v_n)$. Now, we obtain $w^*(v_k) = F(v_k | \{v_1, \ldots, v_{k-1}\})$ for each $k \in [n]$.

The value of $f(x_+)$ and $f(x_-)$ can also be determined by the following maximization problems:

$$f(x_+) = \max_{w \in B(F)} \langle w, x_+ \rangle \quad \text{and} \quad f(x_-) = \max_{w \in B(F)} \langle w, x_- \rangle$$

Let $w_+$ and $w_-$ be the maximizers for $f(x_+)$ and $f(x_-)$, respectively. Then, as we can use the same ordering $v_1, \ldots, v_n$ to determine $w_+$ and $w_-$, we can assume $w_+ = w_- = w^*$. Now, we have

$$f(x)^2 = (w_+, x_+)^2 \geq f(x_+)^2 + f(x_-)^2 + 2 \langle w^*, x_+ \rangle \langle w^*, x_- \rangle = f(x_+)^2 + f(x_-)^2 + 2f(x_+)f(x_-) \geq f(x_+)^2 + f(x_-)^2,$$

where we used the non-negativity in the inequality. □

The last component we use for proving Theorem 1.3 is the following equivalent definition of the Lovász extension:
Lemma 4.7 (See, e.g., Definition 3.1 of [3]). Let $F : \{0, 1\}^V \to \mathbb{R}$ be a submodular function and $f : \mathbb{R}^V \to \mathbb{R}$ be its Lovász extension. For $x \in \mathbb{R}^V$, let $v_1, \ldots, v_n$ be an ordering of $V$, such that $x(v_1) \geq x(v_2) \geq \cdots \geq x(v_n)$. Let $S_k = \{v_1, \ldots, v_k\}$ ($k \in \{0, \ldots, n\}$). Then, we have

$$f(x) = \sum_{k \in [n-1]} F(S_k)(x(v_k) - x(v_{k+1})) + F(V)x(v_n).$$

In particular when $F(V) = 0$, we have

$$f(x) = \sum_{k \in [n-1]} F(S_k)(x(v_k) - x(v_{k+1})).$$

Lemma 4.8. Let $F : \{0, 1\}^V \to \mathbb{R}^E$ be a non-negative submodular transformation and $F(V) = 0$, and let $x \in \mathbb{R}^V$ be a vector $\langle x, D_F^{1/2}1 \rangle = 0$. Then, there exists a set $\emptyset \subseteq S \subseteq V$ such that

$$\phi_F(S) \leq 2\sqrt{R_F(x)}.$$

Moreover, we can find such a set $S$ in $O(n \log n + nm)$ time.

Proof. Let $\bar{x} = D_F^{-1/2}x$. Note that we have assumed $d_F(v)$ is positive for every $v \in V$. Then, we have

$$R_F(x) = \frac{\sum_{v \in V} f_e(\bar{x})^2}{\sum_{v \in V} d_F(v)\bar{x}(v)^2},$$

where $f : \mathbb{R}^V \to \mathbb{R}^E$ is the Lovász extension of $F$. Let $\bar{y} = \bar{x} + c1$ for some appropriate $c \in \mathbb{R}$ such that $\text{vol}_F(\text{supp}(\bar{y})) \leq \text{vol}_F(V)/2$ and $\text{vol}_F(\text{supp}(\bar{y})) \leq \text{vol}_F(V)/2$ hold. Let $y = D_F^{1/2} \bar{y}$. Then, as $f_e(\bar{x}) = f_e(\bar{y})$ by Proposition 4.1 and $\|D_F^{1/2}\bar{y}\|_2 \geq \|D_F^{1/2}\bar{x}\|_2$ by the Pythagorean theorem, we have

$$R_F(x) \geq \min \left\{ \frac{\sum_{v \in V} f_e(\bar{y}_+)^2}{\sum_{v \in V} d_F(v)\bar{y}_+(v)^2}, \frac{\sum_{v \in V} f_e(\bar{y}_-)^2}{\sum_{v \in V} d_F(v)\bar{y}_-(v)^2} \right\}$$

(By Lemma 4.6)

(3)

Suppose the term for $\bar{y}_+$ achieves the minimum in (3). Let $\bar{y}_+^2 \in \mathbb{R}^V$ be the vector defined as $\bar{y}_+^2(v) = \bar{y}_+(v)^2$ for each $v \in V$. Let $v_1, \ldots, v_n$ be the ordering of $V$, such that $\bar{y}_+^2(v_1) \geq \cdots \geq \bar{y}_+^2(v_n)$. For each $e \in E$, we take the subsequence $v_{e,1}, \ldots, v_{e,n_e}$ of this ordering consisting of elements relevant to $F_e$, preserving the order. Note that $\text{supp}(F_e) = \{v_{e,1}, \ldots, v_{e,n_e}\}$ and that the ordering $v_{e,1}, \ldots, v_{e,n_e}$ can be used to compute $f_e(\bar{y}_+^2)$ as well as $f_e(\bar{y}_+)$. As $F_e(V) = F_e(\{v_{e,1}, \ldots, v_{e,n_e}\}) = 0$ for every $e \in E$, we have

$$\sum_{e \in E} f_e(\bar{y}_+)^2 = \sum_{e \in E} \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\})(\bar{y}(v_{e,k})^2 - \bar{y}(v_{e,k+1})^2)$$

(By Lemma 4.7)
\[
\begin{align*}
&= \sum_{e \in E} \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\}) (\bar{y}_+(v_{e,k}) - \bar{y}_+(v_{e,k+1})) \bar{y}_+(v_{e,k}) \\
&\quad + \sum_{e \in E} \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\}) (\bar{y}_+(v_{e,k}) - \bar{y}_+(v_{e,k+1})) \bar{y}_+(v_{e,k+1}).
\end{align*}
\]

We now analyze the first term.
\[
\sum_{e \in E} \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\}) (\bar{y}_+(v_{e,k}) - \bar{y}_+(v_{e,k+1})) \bar{y}_+(v_{e,k})
\leq \sqrt{\sum_{e \in E} \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\})^2 (\bar{y}_+(v_{e,k}) - \bar{y}_+(v_{e,k+1}))^2} \sqrt{\sum_{e \in E} \sum_{k \in [n_e-1]} \bar{y}_+(v_{e,k})^2}
\leq \sqrt{\sum_{e \in E} \left( \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\}) (\bar{y}_+(v_{e,k}) - \bar{y}_+(v_{e,k+1})) \right)^2} \sqrt{\sum_{e \in E} \sum_{k \in [n_e-1]} \bar{y}_+(v_{e,k})^2}
\leq \sqrt{\sum_{e \in E} f_e(\bar{y}_+)^2} \sqrt{\sum_{v \in V} d_F(v) \bar{y}_+(v)^2}.
\]

In the second inequality, we used the fact that \( F_e \) is non-negative for every \( e \in E \).

Similarly, we have
\[
\sum_{e \in E} \sum_{k \in [n_e-1]} F_e(\{v_{e,1}, \ldots, v_{e,k}\}) (\bar{y}_+(v_{e,k}) - \bar{y}_+(v_{e,k+1})) \bar{y}_+(v_{e,k+1}) \leq \sqrt{\sum_{e \in E} f_e(\bar{y}_+)^2} \sqrt{\sum_{v \in V} d_F(v) \bar{y}_+(v)^2}.
\]

Combining (4), (5), (6), for \( y_+ = D_F^{1/2} \bar{y}_+ \) we have
\[
\frac{\sum_{e \in E} f_e(\bar{y}_+)^2}{\sum_{v \in V} d_F(v) \bar{y}_+(v)^2} \leq 2 \sqrt{\frac{\sum_{e \in E} f_e(\bar{y}_+)^2}{\sum_{v \in V} d_F(v) \bar{y}_+(v)^2}} \leq 2 \sqrt{R_F(y_+)} \leq 2 \sqrt{R_F(x)}.
\]

Now, we apply Lemma 4.4 on \( \bar{y}_+^2 \). Then, we obtain a set \( \emptyset \subset S \subset \text{supp}(\bar{y}_+^2) \) with \( \text{vol}_F(S) \leq \text{vol}_F(\text{supp}(\bar{y}_+^2)) = \text{vol}_F(\text{supp}(\bar{y}_+)) \leq \text{vol}_F(V)/2 \). Moreover, we have \( \text{cut}_F(S)/\text{vol}_F(S) \leq 2 \sqrt{R_F(x)} \), which means \( \phi_F(S) \leq 2 \sqrt{R_F(x)} \).

Now, we consider the case that the term for \( \bar{y}_- \) achieves the minimum in (3). This time, we define \( \bar{y}_-^2 \in \mathbb{R}^V \) as the vector such that \( \bar{y}_-^2(v) = \bar{y}_-(v)^2 \) for each \( v \in V \). By an argument similar to the previous case, we can show that
\[
\frac{\sum_{e \in E} f_e(-\bar{y}_-)^2}{\sum_{v \in V} d_F(v) \bar{y}_-(v)^2} \leq 2 \sqrt{\frac{\sum_{e \in E} f_e(-\bar{y}_-)^2}{\sum_{v \in V} d_F(v) \bar{y}_-(v)^2}} \leq 2 \sqrt{R_F(y_-)} \leq 2 \sqrt{R_F(x)}.
\]

Here, we apply Corollary 4.5 on \(-\bar{y}_-^2\). Then, we obtain a set \( \emptyset \subset S \subset \text{supp}(\bar{y}_-^2) \) with \( \text{vol}_F(S) \leq \text{vol}_F(\text{supp}(\bar{y}_-^2)) = \text{vol}_F(\text{supp}(\bar{y}_-)) \leq \text{vol}_F(V)/2 \). Moreover, we have \( \text{cut}_F(V \setminus S)/\text{vol}_F(S) \leq 2 \sqrt{R_F(x)} \), which means \( \phi_F(S) \leq 2 \sqrt{R_F(x)} \).

In both cases, we have \( \phi_F(S) \leq 2 \sqrt{R_F(x)} \).
Proof of the right inequality of Theorem 1.3. For each \( e \in E \), as \( F_e(V) = 0 \), we have \( \mathbf{0} \in B(F_e) \). It follows that \( f_e \) is non-negative because \( f_e(\mathbf{x}) = \max_{\mathbf{w} \in B(F_e)} \langle \mathbf{w}, \mathbf{x} \rangle \geq \langle \mathbf{0}, \mathbf{x} \rangle = 0 \).

Now, we obtain \( \phi_F \leq 2\sqrt{\mathcal{R}_F(\mathbf{x})} \) by invoking Lemma 4.8 with the eigenvector \( \mathbf{x} \in \mathbb{R}^V \) corresponding to \( \lambda_F \). The theorem follows because \( \mathcal{R}_F(\mathbf{x}) = \lambda_F \) by Theorem 3.7.

5 Covering Number of Base Polytopes

For a set \( S \subseteq \mathbb{R}^V \) and \( \epsilon > 0 \), we say that a set of points \( C \) in \( S \) is an \( \epsilon \)-cover of \( S \) if, for any \( \mathbf{x} \in S \), there exists a point \( \mathbf{p} \in C \) with \( \| \mathbf{x} - \mathbf{p} \|_2 \leq \epsilon \). The \( \epsilon \)-covering number of \( S \), denoted by \( N(\epsilon, S) \), is the smallest size of an \( \epsilon \)-cover of \( S \). In this section, we show that the \( \epsilon \|B(F)\|_H \)-covering number of the base polytope \( B(F) \) of a submodular function \( F : \{0,1\}^V \rightarrow \mathbb{R} \) is small and provides an efficient method to construct such a cover.

The following lemma states that the base polytope of a submodular function is contained in a small \( \ell_1 \)-ball.

**Lemma 5.1.** Let \( F : \{0,1\}^V \rightarrow \mathbb{R}_+ \) be a non-negative submodular function. Then, we have

\[
\max_{\mathbf{w} \in B(F)} \| \mathbf{w} \|_1 \leq 2\|F\|_{\infty}.
\]

**Proof.** As \( B(F) \) is a convex polytope, the maximum \( \ell_1 \)-norm of a point in \( B(F) \) is attained at an extreme point \( \mathbf{w}^* \) of \( B(F) \). By Lemma 2.1, there exists an ordering \( v_1, \ldots, v_n \) of \( V \), such that \( \mathbf{w}^*(v_k) = F(v_k \mid S_{k-1}) \) \((k \in [n])\), where \( S_k = \{v_1, \ldots, v_k\} \).

We now lower bound \( \|F\|_{\infty} \) by using \( \|\mathbf{w}^*\|_1 \). Let \( v_1^+, \ldots, v_n^+ \) be the sequence obtained from the ordering \( v_1, \ldots, v_n \) by extracting \( v_k \)'s such that \( \mathbf{w}^*(v_k) > 0 \), preserving the order, and let \( S_k^+ = \{v_1^+, \ldots, v_k^+\} \) \((k \in [n^+])\). Then based on the submodularity, for any \( k \in [n^+] \), we have \( F(v_k^+ \mid S_{k-1}^+) \geq F(v_k^+ \mid S_{k-1}^+) = \mathbf{w}^*(v_k^+) \), where \( k' \geq k \) is such that \( v_k^+ = v_{k'} \). This means that

\[
F(S_{n^+}^+) = \sum_{k \in [n^+]} F(v_k^+ \mid S_{k-1}^+) \geq \sum_{k \in [n^+]} \mathbf{w}^*(v_k^+) \geq \frac{1}{2\|\mathbf{w}^*\|_1},
\]

where we used the fact that \( \mathbf{w}^*(V) = f(V) \geq 0 \) in the last inequality. Then, we have \( \|\mathbf{w}^*\|_1 \leq 2\|F\|_{\infty} \).

The above lemma suggests that, when \( \|F\|_{\infty} \leq 1/2 \), the base polytope is contained in the \( \ell_1 \)-ball \( B_1^V := \{ \mathbf{x} \in \mathbb{R}^V \mid \| \mathbf{x} \|_1 \leq 2 \} \). The following covering number of \( B_1^V \) is known to be obtained by using Maurey’s empirical method (see, e.g., [23]).

**Lemma 5.2.** For every \( \epsilon > 0 \), we have

\[
N(\epsilon, B_1^V) \leq U_{5.2}(\epsilon, B_1^V) := \left(1 + 2\epsilon^2 n \right)^{1/\epsilon^2},
\]

where \( n = |V| \). Moreover, we can compute an \( \epsilon \)-cover of \( B_1^V \) of size \( U_{5.2}(\epsilon, B_1^V) \) in \( O(nU_{5.2}(\epsilon, B_1^V)) \) time.

This lemma states that the \( \epsilon \)-covering number of the \( B_1^V \) is polynomial in \( n \) (as long as \( \epsilon \) is constant), which will be crucial when bounding the time complexity and the approximation ratio.
Algorithm 1 Construction of an $\epsilon$-cover of the base polytope of a submodular function.

Require: a submodular function $F : \{0,1\}^{V} \to [0,1]$, $r \geq 0$, and $\epsilon > 0$.
1: Construct an $(\epsilon/3)$-cover of $rB_V^1$, where we identify $V$ with $[n]$.
2: $C \leftarrow \emptyset$.
3: for each $p \in P$ do
   4: Define $F_p : \{0,1\}^{V} \to \mathbb{R}$ so that $F_p(S) = F(S) - p(S)$ ($S \subseteq V$).
   5: Run Wolfe’s algorithm on $B(F_p) \cap (-p + rB^1_V)$ and $\epsilon/3$, and let $w_p$ be the returned vector.
   6: if $\|w_p\|_2 \leq 2\epsilon/3$ then
      7: $C \leftarrow C \cup \{p + w_p\}$.
8: return $C$.

of our algorithms for approximating eigenvalues in Sections 6 and 7. In contrast, the $\epsilon$-covering number of the $\ell_2$-ball $B(0,1)^V := \{x \in \mathbb{R}^{V} \mid \|x\|_2 \leq 1\}$ is exponential in $n$ (see, e.g., [23]).

Lemmas 5.1 and 5.2 implies that we can compute a polynomial-size set $P$ of points in $\mathbb{R}^V$ such that any point in the base polytope $B(F)$ of a submodular function $F : \{0,1\}^{V} \to \mathbb{R}$ has a close point in $P$. Obtaining an $\epsilon$-cover of $B(F)$ from $P$ requires us to eliminate the points outside of $B(F)$. To this end, we use Wolfe’s algorithm [27], which computes the minimum $\ell_2$-norm point in a polytope. The following theoretical guarantee is known for Wolfe’s algorithm:

Lemma 5.3 ([4]). Let $F : \{0,1\}^{V} \to \mathbb{R}$ be a submodular function, and let $p \in \mathbb{R}^{V}$ and $r > 0$. Wolfe’s algorithm computes a point $w^* \in B(F) \cap (p + rB^1_V)$ such that $\|w^*\|_2^2 \leq \min_{w \in B(F) \cap (p + rB^1_V)} \{\|w\|_2^2 + 2\epsilon^2\}$ in $O(n^4\|B(F) \cap (p + rB^1_V)\|_H^2/\epsilon^2)$ time, where $n = |V|$.

We remark that [4] considers the case that the given polytope is $B(F)$ instead of $B(F) \cap (p + rB^1_V)$. However, their argument relies only on the fact that the given polytope is convex and we can solve a linear programming over the polytope, which is true for $B(F) \cap (p + rB^1_V)$.

Now, we show that we can construct a small cover for a base polytope restricted to a small $\ell_1$-ball.

Lemma 5.4. Let $F : \{0,1\}^{V} \to \mathbb{R}$ be a non-negative submodular function. For every $\epsilon > 0$, we can construct an $\epsilon$-cover $C$ of $B(F) \cap rB^1_V$ of size $O(U_{5.2}(\frac{\epsilon}{3r}, B^1_V))$ in $O(r^2 n^4 U_{5.2}(\frac{\epsilon}{3r}, B^1_V)/\epsilon^2)$ time, where $n = |V|$.

Proof. Our algorithm for constructing an $\epsilon$-cover $C$ is summarized in Algorithm 1. It first constructs an $(\epsilon/3)$-cover of $rB^1_V$. Then, for each $p \in P$, we compute a minimum-norm point $w_p$ in $B(F_p) \cap (-p + rB^1_V)$ for $F_p = F - p$ by running Wolfe’s algorithm with an error parameter $\epsilon/3$. Then, if $\|w_p\|_2$ is sufficiently small, or more specifically, $\|w_p\|_2 \leq 2\epsilon/3$, then we add $p + w_p$ to $C$.

Note that $p + w_p$ belongs to $B(F) \cap rB^1_V$ as $B(F_p) = \{w - p \mid w \in B(F)\}$. Hence, we need to check that any point in $B(F) \cap rB^1_V$ has a close point in the constructed set $C$.

For every $w \in B(F) \cap rB^1_V$, there exists a point $p \in P$ such that $\|w - p\|_2 \leq \epsilon/3$. Then, by Lemma 5.3, we have

\[
\|w_p\|_2^2 \leq \arg\min_{w' \in B(F_p) \cap (-p + rB^1_V)} \|w'\|_2^2 + 2\left(\frac{\epsilon}{3}\right)^2 = \arg\min_{w' \in B(F) \cap (-p + rB^1_V)} \|w' - p\|_2^2 + \frac{2\epsilon^2}{9} \leq \frac{\epsilon^2}{3} \leq \left(\frac{2\epsilon}{3}\right)^2.
\]
Hence, \( p + w_p \in B(F) \cap rB^V_1 \) will be added to \( C \). Note that
\[
\|w - (p + w_p)\|_2 \leq \|w - p\|_2 + \|w_p\|_2 \leq \epsilon,
\]
which implies the returned set \( C \) is an \( \epsilon \)-cover of \( B(F) \cap rB^V_1 \).

Now, we analyze the time complexity of the algorithm. By Lemma 5.2, we need \( O(nU_{n,2}(\epsilon/3, rB^V_1)) = O(nU_{n,2}(\frac{\epsilon}{6}, B^V_1)) \) time to compute an \((\epsilon/3)\)-cover \( P \) of \( rB^V_1 \). For each point \( p \in P \), we run Wolfe’s algorithm. We have
\[
\frac{\max_{w \in B(F) \cap (-p + rB^V_1)} \|w\|_2}{\max_{w \in B(F) \cap (-p + rB^V_1)} \|w\|_1} = \frac{\max_{w \in B(F) \cap rB^V_1} \|w - p\|_1}{\max_{w \in B(F) \cap rB^V_1} \|w\|_1} \leq \frac{2r}{\|p\|_1} \leq 2r.
\]
Hence, the running time of Wolfe’s algorithm is \( O(r^2 n^4/\epsilon^2) \) by Lemma 5.3. Then, the total running time is \( O(r^2 n^4 U_{n,2}(\frac{\epsilon}{6}, B^V_1)/\epsilon^2) \).

**Theorem 5.5.** Let \( F : \{0,1\}^V \rightarrow \mathbb{R} \) be a non-negative submodular function. For every \( \epsilon > 0 \), we can construct an \( \epsilon \|B(F)\|_H \)-cover \( C \) of \( B(F) \) of size \( O(\log_2 n \cdot U_{n,2}(\epsilon/6, B^V_1)) \) in \( O(n^4 \log n \cdot U_{n,2}(\epsilon/6, B^V_1)/\epsilon^2) \) time, where \( n = |V| \).

**Proof.** Let \( K = \max_{v \in V} F(\{v\}) \). Then, it is easy to check \( K \leq \|B(F)\|_H \leq nK \). We define \( r_i = 2^i K \) for \( i \in \{0, \ldots, L\} \), where \( L = \lceil \log_2 n \rceil \). For each \( i \in \{0, \ldots, L\} \), we construct an \( \epsilon/2 \)-cover \( C_i \) by invoking Lemma 5.4 on \( B(F)/r_i \cap B^V_1 \), and then we return the union \( C := \bigcup_{i=0}^L r_i C_i \). The size of \( C \) and the time complexity for constructing \( C \) are as claimed.

Now, we show that \( C \) is an \( \epsilon \|B(F)\|_H \)-cover of \( B(F) \). Let \( w \in B(F) \) be an arbitrary vector in the base polytope. If \( \|w\|_2 \leq r_0 \), then there is a point \( p \in C_0 \) such that \( \|w/r_0 - p\|_2 \leq \epsilon/2 \), which means that \( r_0 p \in C_0 \subseteq C \) satisfies \( \|w - r_0 p\| \leq r_0 \epsilon/2 \leq \epsilon K \leq \epsilon \|B(F)\|_H \). Otherwise, let \( i \in \{1, \ldots, L\} \) be such that \( r_{i-1} < \|w\|_2 \leq r_i \). Such \( i \) always exists because \( r_0 < \|w\|_2 \leq nK \). Then, there exists a point \( p \in C_i \) such that \( \|w/r_i - p\|_2 \leq \epsilon/2 \), which means that \( r_i p \in r_i C_i \subseteq C \) satisfies \( \|w - r_i p\| \leq \epsilon r_i/2 \leq \epsilon r_{i-1} \leq \epsilon \|w\|_2 \leq \epsilon \|B(F)\|_H \). \( \square \)

## 6 Approximating the Smallest Non-trivial Eigenvalue in the Symmetric Case

In this section, we prove Theorem 1.10, that is, we provide a polynomial-time algorithm that approximates the smallest non-trivial eigenvalue of the normalized Laplacian of a symmetric submodular transformation to within a factor of \( O(\log n) \) and a small additive error. We explain our SDP relaxation and rounding method in Section 6.1 and then provide an approximation guarantee in Section 6.2.

### 6.1 SDP relaxation and rounding

Our algorithm is based on SDP relaxation, and our SDP formulation is based on the following simple observation, which exploits the symmetry:

**Proposition 6.1.** For a symmetric submodular function \( F : \{0,1\}^V \rightarrow \mathbb{R} \), we have \( B(F) = -B(F) \), that is, \( -w \in B(F) \) for every \( w \in B(F) \).
Proof. As $B(F)$ is a convex polytope, it suffices to check whether $-w \in B(F)$ holds for each extreme point $w$ of $B(F)$.

Let $w \in B(F)$ be an extreme point of $B(F)$. By Lemma 2.1, there exists an ordering $v_1, \ldots, v_n$ of $V$, where $n = n$, such that $w(v_k) = F(v_k | \{v_1, \ldots, v_{k-1}\})$ ($k \in [n]$). Consider the ordering $v'_1, \ldots, v'_n$ of $V$ such that $v'_k = v_{n-k+1}$ ($k \in [n]$). Again by Lemma 2.1, the vector $w' \in \mathbb{R}^V$ with $w'(v'_k) = F(v'_k | \{v'_1, \ldots, v'_{k-1}\})$ is an extreme point of $B(F)$. For every $k \in [n]$, we have

$$w'(v_{n-k+1}) = w'(v'_k) = F(v'_k | \{v'_1, \ldots, v'_{k-1}\})$$
$$= F(\{v_n, \ldots, v_{n-k+1}\}) - F(\{v_n, \ldots, v_{n-k+2}\})$$
$$= F(\{v_1, \ldots, v_{n-k}\}) - F(\{v_1, \ldots, v_{n-k+1}\}).$$

Hence, we have $-w \in B(F)$. 

Then, we can rephrase $f(x)^2$ as follows:

**Corollary 6.2.** For the Lovász extension $f : \mathbb{R}^V \rightarrow \mathbb{R}$ of a symmetric submodular function $F : \{0,1\}^V \rightarrow \mathbb{R}$, we have

$$f(x)^2 = \max_{w \in B(F)} \langle w, x \rangle^2$$

for every $x \in \mathbb{R}^V$.

Proof. Let $w^* \in \arg\max_{w \in B(F)} \langle w, x \rangle^2$. By Proposition 6.1, we can also assume that $w^* \in \arg\max_{w \in B(F)} \langle w, x \rangle$; otherwise, we can replace $w^*$ with $-w^* \in B(F)$ to achieve this. Then, we have

$$f(x)^2 = \left( \max_{w \in B(F)} \langle w, x \rangle \right)^2 = \langle w^*, x \rangle^2 = \max_{w \in B(F)} \langle w, x \rangle^2.$$
Algorithm 2 Approximation of the smallest non-trivial eigenvalue of the normalized Laplacian of a symmetric submodular transformation.

Require: a symmetric submodular transformation $F : \{0, 1\}^V \to \mathbb{R}^E$ and $\epsilon > 0$.
1: Solve SDP (9).
2: Let $g \in \mathbb{R}^N$ be a random vector sampled from the standard normal distribution $N(0, I_N)$.
3: Define $z \in \mathbb{R}^V$ as $z(v) = \langle x_v, g \rangle$ for each $v \in V$.
4: return $D_F^{1/2} z$.

where $N \geq n$ is a sufficiently large integer. Then, for a matrix $X = (x_v)_{v \in V} \in \mathbb{R}^{N \times V}$, our SDP relaxation is the following:

\[
\text{SDP}(F) := \begin{align*}
\text{minimize} & \sum_{e \in E} \| \eta_e \|^2_2, \\
\text{subject to} & \| Xw \|^2_2 \leq \| \eta_e \|^2_2, \quad \forall e \in E, w \in B(F_e), \\
& \sum_{v \in V} d_F(v)\| x_v \|^2_2 = 1, \\
& \sum_{v \in V} d_F(v) x_v = 0.
\end{align*}
\]

The value $\| Xw \|^2_2 = \| \sum_{v \in V} w(v)x_v \|^2_2$ is supposed to represent the value $\langle w, x \rangle^2$ in (7).

Unfortunately, for each $e \in E$, there are infinitely many choices for $w \in B(F_e)$, and hence we cannot efficiently write down SDP (8). One observation is that we only have to consider extreme points of $B(F_e)$ because the maximum of $\| Xw \|^2_2$ over the base polytope $B(F_e)$ is attained at its extreme point. However, we are still prevented from efficiently writing down SDP (8) because the number of extreme points of a base polytope can be $n!$ in general.

To address the above-mentioned problem, we consider replacing base polytopes $B(F_e)$ by their $\epsilon\|B(F_e)\|_H$-covers, where $\epsilon > 0$ is an error parameter. For each $e \in E$, let $C_e$ be the $\epsilon\|B(F_e)\|_H$-cover of $B(F_e)$ given in Theorem 5.5. We consider the following SDP obtained from SDP (8) by replacing $B(F_e)$ with $C_e$ for each $e \in E$:

\[
\text{SDP}_\epsilon(F) := \begin{align*}
\text{minimize} & \sum_{e \in E} \| \eta_e \|^2_2, \\
\text{subject to} & \| Xw \|^2_2 \leq \| \eta_e \|^2, \quad \forall e \in E, w \in C_e, \\
& \sum_{v \in V} d_F(v)\| x_v \|^2_2 = 1, \\
& \sum_{v \in V} d_F(v) x_v = 0.
\end{align*}
\]

As $C_e \subseteq B(F_e)$, it is clear that $\text{SDP}_\epsilon(F) \leq \text{SDP}(F)$, and hence $\text{SDP}_\epsilon(F)$ is a relaxation of (7). Moreover, as the size of $C_e$ is polynomial (as long as $\epsilon$ is constant), we can solve SDP (9) in polynomial time.

After solving SDP (9), we sample $g \in \mathbb{R}^N$ from the standard normal distribution $N(0, I_N)$ and then we round the SDP solution to a vector $z \in \mathbb{R}^V$ with $z(v) = \langle x_v, g \rangle$ ($v \in V$). Our algorithm is summarized in Algorithm 2.
6.2 Analysis

Now, we provide an approximation guarantee of Algorithm 2. The following lemma is useful to analyze the error caused by replacing \( B(F_e) \) with \( C_e \).

**Lemma 6.3.** Let \( F : \{0, 1\}^V \rightarrow \mathbb{R} \) be a submodular function and let \( C \subseteq B(F) \) be an \( \epsilon \)-cover of \( B(F) \) for \( \epsilon > 0 \). Then, for any vector \( x \in \mathbb{R}^V \), we have

\[
\max_{w \in B(F)} \langle w, x \rangle^2 \leq \max_{w \in C} \langle w, x \rangle^2 + 2\epsilon \|x_{\text{supp}(F)}\|^2 \cdot \|B(F)\|_H.
\]

**Proof.** Let \( w^* \) be the maximizer of \( \max_{w \in B(F)} \langle w, x \rangle^2 \). Then, there exists \( w' \in C \) such that \( \|w^* - w'\|_2 \leq \epsilon \). By using the fact that \( w(v) = 0 \) for every \( w \in B(F) \) and \( v \in V \setminus \text{supp}(F) \), we have

\[
\max_{w \in B(F)} \langle w, x \rangle^2 - \max_{w \in C} \langle w, x \rangle^2 = \langle w^* - w', x \rangle \cdot \langle w^* + w', x \rangle = \epsilon \|x_{\text{supp}(F)}\|_2 \cdot \|B(F)\|_H \cdot \|x_{\text{supp}(F)}\|_2 \\
= 2\epsilon \|x_{\text{supp}(F)}\|^2 \cdot \|B(F)\|_H.
\]

**Lemma 6.4.** Let \( z \in \mathbb{R}^V \) be the output of Algorithm 2 on a symmetric submodular transformation \( F : \{0, 1\}^V \rightarrow \mathbb{R}^E \) and \( \epsilon > 0 \). Then, we have

\[
\mathcal{R}_F(D_F^{1/2} z) = O\left(\frac{\log n}{\epsilon^2} \lambda_F + \epsilon \max_{v \in E} \|B(F_e)\|^2 \right),
\]

with a probability of at least \( 1/24 \), where \( \lambda_F \geq 0 \) is the smallest non-trivial eigenvalue of \( \mathcal{L}_F \).

**Proof.** For the expected numerator of \( \mathcal{R}_F(D_F^{1/2} z) \), we have

\[
\mathbb{E}_z \left[ \sum_{e \in E} f_e(z)^2 \right] = \mathbb{E}_z \left[ \sum_{e \in E} \max_{w \in B(F_e)} \langle w, z \rangle^2 \right] \\
= \mathbb{E}_z \left[ \sum_{e \in E} \max_{w \in C_e} \langle w, z \rangle^2 + O \left( \epsilon \sum_{e \in E} \|z_{\text{supp}(F_e)}\|^2 \cdot \|B(F_e)\|^2 \right) \right] \quad \text{(By Lemma 6.3)} \\
= \mathbb{E}_z \left[ \sum_{e \in E} \max_{w \in C_e} \langle w, z \rangle^2 + O \left( \epsilon \max_{e \in E} \|B(F_e)\|^2 \cdot \sum_{v \in V} d_F(v) z(v)^2 \right) \right]. \quad \text{(10)}
\]

First, we analyze the first term in the expectation of (10). For \( w \in C_e \), \( \langle w, z \rangle \) is a normal distribution with mean 0 and variance \( \|Xw\|^2 \). Hence, Proposition A.1, which bounds the maximum of squared normal random variables, and Theorem 5.5 imply that

\[
\mathbb{E}_z \left[ \sum_{e \in E} \max_{w \in C_e} \langle w, z \rangle^2 \right] \leq 4 \log \max_{e \in E} |C_e| \cdot \sum_{e \in E} \max_{w \in C_e} \|Xw\|^2 = \frac{K \log n}{\epsilon^2} \cdot \text{SDP}_\epsilon(F), \quad \text{(11)}
\]

for some constant \( K \in \mathbb{R}_+ \).

Next, we analyze the second term in the expectation of (10). Using the linearity of expectation, we obtain

\[
\mathbb{E}_z \left[ \sum_{v \in V} d_F(v) z(v)^2 \right] = \sum_{v \in V} d_F(v) \mathbb{E}_z \left[ (x_v, g)^2 \right] = \sum_{v \in V} d_F(v) \|x_v\|^2 = 1. \quad \text{(12)}
\]
From (11) and (12), by Markov’s inequality, we have
\[
\Pr \left[ \sum_{e \in E} f_e(z)^2 \leq \frac{24K \log n}{\epsilon^2} \cdot \text{SDP}_\epsilon(F) + 24\epsilon \max_{e \in E} \|B(F_e)\|^2_H \right] \geq 1 - \frac{1}{24}.
\] (13)

Now, we analyze the denominator of \( R_F(D_{1/2}^1 z) \). By Proposition A.3, we have
\[
\Pr \left[ \sum_{v \in V} d_F(v)z(v)^2 \geq \frac{1}{2} \right] \geq \frac{1}{12}.
\] (14)

From (13) and (14), by the union bound, we have
\[
\Pr \left[ R_F(D_{1/2}^1 z) \leq \frac{48K \log n}{\epsilon^2} \cdot \text{SDP}_\epsilon(F) + 48\epsilon \max_{e \in E} \|B(F_e)\|^2_H \right] \geq \frac{1}{24}.
\]

Proof of Theorem 1.10. Let \( z \in \mathbb{R}^V \) be the output of Algorithm 2 on \( F \) and \( \epsilon > 0 \). Because of the constraint \( \sum_{v \in V} d_F(v)x_v = 0 \), we have \( \langle D_{1/2}^1 z, D_{1/2}^1 1 \rangle = \sum_{v \in V} d_F(v)\langle x_v, g \rangle = \langle 0, g \rangle = 0 \). Hence \( z \) is always feasible. The approximation guarantee is given by Lemma 6.4. The total time complexity is dominated by the time complexity for solving SDP (9), which is \( \text{poly}(nm)^{\text{poly}(1/\epsilon)} \).

Note that we can augment the success probability to \( 9/10 \) by running this algorithm a constant number of times and by outputting the vector with the smallest Rayleigh quotient.

7 Approximating the Smallest Non-trivial Eigenvalue in the General Case

In this section, we prove Theorem 1.11, that is, we provide a polynomial-time algorithm that approximates the smallest non-trivial eigenvalue of the normalized Laplacian of a general submodular transformation to within a factor of \( O(\log^2 n + \log n \log m) \) and a small additive error. We explain our SDP relaxation and rounding method in Section 7.1 and then provide an approximation guarantee in Section 7.2.

For a technical reason, we assume that the input submodular transformation \( F : \{0,1\}^V \to \mathbb{R}^E \) satisfies \( F(S) \in [0,1/100]^E \) (instead of \( [0,1]^E \)) for \( S \subseteq V \). This can be obtained by dividing the input function by \( 100 \max_{e \in E} \|F_e\|_\infty \), which preserves the approximation guarantee.

7.1 SDP relaxation and rounding

Our SDP formulation is based on the following observation:

**Proposition 7.1.** Let \( f : \mathbb{R}^V \to \mathbb{R} \) be the Lovász extension of a submodular function \( F : \{0,1\}^V \to \mathbb{R} \). If \( f \) is non-negative, then we have
\[
f(x)^2 = \frac{1}{2} \max_{w \in B(F)} \left( \langle w, x \rangle^2 + \langle w, x \rangle |\langle w, x \rangle| \right).
\]

**Proof.** We have
\[
f(x)^2 = \left( \max_{w \in B(F)} \langle w, x \rangle \right)^2 = \max_{w \in B(F)} \max \{ \langle w, x \rangle, 0 \}^2
\]
and Proposition 7 and the assumption that \( \sum_{i \in V} v_i = 0 \). We avoid this problem, as in the symmetric case, by choosing a sufficiently large integer. In addition, for each \( e \) as follows:

\[
\frac{1}{2} \sum_{e \in E} \max_{w \in B(F_e)} \left( \langle w, D^{-1/2}_F x \rangle^2 + \langle w, D^{-1/2}_F \rangle \langle w, D^{-1/2}_F \rangle \right)
\]

subject to \( \|x\|_2^2 = 1 \) and \( \langle x, D^{-1/2}_F 1 \rangle = 0 \). By replacing \( x \) with \( D^{1/2}_F x \), the minimum can be written as follows:

\[
\text{minimize} \quad \frac{1}{2} \sum_{e \in E} \eta_e^2,
\]

subject to \( \langle w, x \rangle^2 + \langle w, x \rangle \langle w, x \rangle \leq \eta_e^2 \quad \forall e \in E, \forall w \in B(F_e), \)

\[
\sum_{v \in V} d_F(v) x(v)^2 = 1, \quad \sum_{v \in V} d_F(v) x(v) = 0.
\]

To derive an SDP relaxation, we introduce vectors \( \eta_e \in \mathbb{R}^N (e \in E) \) and \( x_v \in \mathbb{R}^N (v \in V) \) that are supposed to represent \( \eta_e \) (\( e \in E \)) and \( x(v) \) (\( v \in V \)), respectively, where \( N \geq n \) is a sufficiently large integer. In addition, for each \( e \in E \) and \( w \in B(F_e) \), we introduce vectors \( v_{\langle w, x \rangle} \in \mathbb{R}^N \) (\( e \in E, w \in B(F_e) \)) that are supposed to represent \( \|\langle w, x \rangle\| \). Then, for a matrix \( X = (x_v)_{v \in V} \in \mathbb{R}^{N \times V} \), our SDP relaxation is the following:

\[
\text{SDP}(f) := \text{minimize} \quad \frac{1}{2} \sum_{e \in E} \|\eta_e\|_2^2,
\]

subject to \( \|X w\|_2^2 + \langle X w, v_{\langle w, x \rangle}\rangle \leq \|\eta_e\|_2^2 \quad \forall e \in E, w \in B(F_e), \)

\[
\|v_{\langle w, x \rangle}\|_2^2 = \|X w\|_2^2 \quad \forall e \in E, w \in B(F_e), \quad \langle v_{\langle w, x \rangle}, v_1 \rangle \geq \|v_{\langle w, x \rangle}\|_2^2 \quad \forall e \in E, w \in B(F_e),
\]

\[
\sum_{v \in V} d_F(v) x(v)^2 = 1, \quad \sum_{v \in V} d_F(v) x(v) = 0.
\]

As in the symmetric case, the value \( \|X w\|_2^2 = \|\sum_{v \in V} w(v) x_v\|_2^2 \) is supposed to represent the value \( \langle w, x \rangle^2 \) in (15). The vector \( v_1 \in \mathbb{R}^N \) is a fixed unit vector that represents the value of one. The constraint \( \|v_{\langle w, x \rangle}\|_2^2 = \|X w\|_2^2 \) is supposed to represent \( \|\langle w, x \rangle\| = \|X w\|_2 \). The constraint \( \langle v_{\langle w, x \rangle}, v_1 \rangle \geq \|v_{\langle w, x \rangle}\|_2^2 \) is supposed to represent \( \|\langle w, x \rangle\| \geq \|\langle w, x \rangle\|_2^2 \), which is valid because \( |\langle w, x \rangle| \leq \|w\|_1 \cdot \max |x(v)| \leq 2/100 \leq 1 \) by Lemma 5.1 and the assumption that \( \|F_e\|_\infty \leq 1/100 \) for every \( e \in E \) discussed in the beginning of Section 7.

We cannot efficiently solve the SDP relaxation (16) because the numbers of the vectors \( v_{\langle w, x \rangle} \) and constraints are uncountably many. We avoid this problem, as in the symmetric case, by
replacing each $B(F_e)$ ($e \in E$) with its $\epsilon$-cover $C_e$ provided in Theorem 5.5:

$$\text{SDP}_\epsilon(f) := \text{minimize } \frac{1}{2} \sum_{e \in E} \| \eta_e \|_2^2, \quad \text{subject to }$$

\[
\begin{align*}
&\|Xw\|_2^2 + \langle Xw, v_{(w,x)} \rangle \leq \| \eta_e \|_2^2 \quad \forall e \in E, w \in C_e, \\
&\|v_{(w,x)}\|_2^2 = \|Xw\|_2^2 \quad \forall e \in E, w \in C_e, \\
&\langle v_{(w,x)}, v_1 \rangle \geq \|v_{(w,x)}\|_2^2 \quad \forall e \in E, w \in C_e \\
&\sum_{v \in V} d_F(v) \| x_v \|_2^2 = 1, \\
&\sum_{v \in V} d_F(v) x_v = 0.
\end{align*}
\]

As $C_e \subseteq B(F_e)$, it is clear that $\text{SDP}_\epsilon(f) \leq \text{SDP}(f)$, and hence $\text{SDP}_\epsilon(f)$ is a relaxation of (15). Further, as the size of $C_e$ is polynomial (as long as $\epsilon$ is constant), we can solve $\text{SDP}_\epsilon(f)$ in polynomial time.

After solving $\text{SDP}_\epsilon(f)$, we sample $g \in \mathbb{R}^V$ from the standard normal distribution $\mathcal{N}(0, I_N)$ and then define $z_+ \in \mathbb{R}^V$ as $z_+(v) = \langle x_v, v_1 \rangle + \delta \langle P_{v_1} x_v, g \rangle$ ($v \in V$) and $z_- \in \mathbb{R}^V$ as $z_-(v) = \langle x_v, v_1 \rangle - \delta \langle P_{v_1} x_v, g \rangle$ ($v \in V$). Here, $\delta = O(1/\sqrt{\log(n^2/c^2 \epsilon)})$ and $P_{v_1}$ is the projection matrix to the subspace orthogonal to $v_1$. Then, we return the one with the smaller Rayleigh quotient. Intuitively, this rounding procedure places more importance on the direction $v_1$ than on other directions. Our algorithm is summarized in Algorithm 3.

### 7.2 Analysis

Now, we provide an approximation guarantee of Algorithm 3.

#### 7.2.1 Denominator of Rayleigh quotients

We analyze the maximum denominator of $\mathcal{R}_F(D_F^{1/2} z_+)$ and $\mathcal{R}_F(D_F^{1/2} z_-)$.
Lemma 7.2. Let $z_+, z_- \in \mathbb{R}^V$ be the vectors obtained in Algorithm 3 on a submodular transformation $F : \{0, 1\}^V \to \mathbb{R}^E$ (and some $\epsilon > 0$). Then, we have

$$\frac{\delta^2}{2} \leq \max \left\{ \mathbb{E} \left[ \sum_{v \in V} d_F(v)z_+(v)^2 \right], \mathbb{E} \left[ \sum_{v \in V} d_F(v)z_-(v)^2 \right] \right\} \leq 24 + 10\delta,$$

with a probability of at least $1/50$.

Proof. For the later convenience, we define $\alpha = \sqrt{\sum_{v \in V} d_F(v)\langle x_v, v_1 \rangle^2}$ and $\beta = \sqrt{\sum_{v \in V} d_F(v)\|P_{v_1}^x x_v\|^2}$. We have

$$\max \left\{ \sum_{v \in V} d_F(v)z_+(v)^2, \sum_{v \in V} d_F(v)z_-(v)^2 \right\} = \max_{\sigma \in \{-1, 1\}} \sum_{v \in V} d_F(v) \left( \langle x_v, v_1 \rangle + \sigma \delta \langle P_{v_1}^x x_v, g \rangle \right)^2$$

$$= \sum_{v \in V} d_F(v)\langle x_v, v_1 \rangle^2 + \delta^2 \sum_{v \in V} d_F(v)\langle P_{v_1}^x x_v, g \rangle^2 + \delta \left| \sum_{v \in V} d_F(v)\langle x_v, v_1 \rangle \langle P_{v_1}^x x_v, g \rangle \right|$$

$$= \alpha^2 + \delta^2 \sum_{v \in V} d_F(v)\langle P_{v_1}^x x_v, g \rangle^2 + \delta \left| \sum_{v \in V} d_F(v)\langle x_v, v_1 \rangle \langle P_{v_1}^x x_v, g \rangle \right|$$

(18)

For the second term of (18), as $\mathbb{E} \left[ \sum_{v \in V} d_F(v)\langle P_{v_1}^x x_v, g \rangle^2 \right] = \|P_{v_1}^x x_v\|^2_2$, by Proposition A.3, we have

$$\mathbb{P} \left[ \sum_{v \in V} d_F(v)\langle P_{v_1}^x x_v, g \rangle^2 \geq 1 \right] \geq \frac{1}{12}. \tag{19}$$

By Markov’s inequality, we have

$$\mathbb{P} \left[ \sum_{v \in V} d_F(v)\langle P_{v_1}^x x_v, g \rangle^2 \leq 24\beta^2 \right] \geq 1 - \frac{1}{24}. \tag{20}$$

For the third term of (18), by Mill’s inequality, we have

$$\mathbb{P} \left[ \left| \sum_{v \in V} d_F(v)(\langle x_v, v_1 \rangle \langle P_{v_1}^x x_v, g \rangle \right) \geq t \right] = \mathbb{P} \left[ \left| \langle \sum_{v \in V} d_F(v)\langle x_v, v_1 \rangle P_{v_1}^x x_v, g \rangle \right| \geq t \right] \leq \frac{1}{t} \sqrt{\frac{2}{\pi}} \exp \left( -\frac{t^2}{2\sigma^2} \right),$$

where $\sigma = \| \sum_{v \in V} d_F(v)\langle x_v, v_1 \rangle P_{v_1}^x x_v \|_2$. Note that $\sigma \leq \alpha \beta$ by the vector version of the Cauchy-Schwarz inequality (see Lemma B.1). Hence, by setting $t = 10\alpha \beta \log \frac{1}{\alpha \beta}$, where we regard $t = 0$ when $\alpha \beta = 0$, we have

$$\mathbb{P} \left[ \left| \sum_{v \in V} d_F(v)(\langle x_v, v_1 \rangle \langle P_{v_1}^x x_v, g \rangle \right) \geq 10\alpha \beta \log \frac{1}{\alpha \beta} \right] \leq \frac{1}{10\alpha \beta \log \frac{1}{\alpha \beta}} \sqrt{\frac{2}{\pi}} \exp \left( -5 \log \frac{1}{\alpha \beta} \right) \leq \frac{1}{100}. \tag{21}$$

By the union bound on (19), (20), and (21), with a probability of at least $1/50$, we have $\beta^2/2 \leq \sum_{v \in V} d_F(v)\langle P_{v_1}^x x_v, g \rangle^2 \leq 24\beta^2$ and $\left| \sum_{v \in V} d_F(v)(\langle x_v, v_1 \rangle \langle P_{v_1}^x x_v, g \rangle \right| \leq 10\alpha \beta \log \frac{1}{\alpha \beta}$. In what follows, we assume this happened.

For the upper bound, from (18) and our assumptions, we have

$$\max \left\{ \sum_{v \in V} d_F(v)z_+(v)^2, \sum_{v \in V} d_F(v)z_-(v)^2 \right\} \leq \alpha^2 + 24\delta^2 \beta^2 + 10\delta \alpha \beta \log \frac{1}{\alpha \beta} \leq 24 + 10\delta,$$

26
Next, we analyze the maximum numerator of
where we used the fact that $\alpha \beta = 1$ and the maximum of $\alpha \beta \log(1/\alpha \beta)$ subject to $\alpha^2 + \beta^2 = 1$ is $\log(2)/2 \leq 1$.

For the lower bound, from (18) and our assumptions, we have

$$\max \left\{ \sum_{v \in V} d_F(v)z_+(v)^2, \sum_{v \in V} d_F(v)z_-(v)^2 \right\} \geq \alpha^2 + \frac{\delta^2 \beta^2}{2} \geq \frac{\delta^2}{2},$$

where we used the fact that $\alpha^2 + \beta^2 = 1$. Note that the third term of (18) does not appear because we take the maximum of $\sum_{v \in V} d_F(v)z_+(v)^2$ and $\sum_{v \in V} d_F(v)z_-(v)^2$. \hfill \qed

### 7.2.2 Numerator of Rayleigh quotients

Next, we analyze the maximum numerator of $R_F(D_F^{1/2}z_+)$ and $R_F(D_F^{1/2}z_-)$.

The following lemma is useful to bound the error that occurred by replacing the base polytope with its $\epsilon$-cover.

**Lemma 7.3.** Let $F : \{0,1\}^V \to \mathbb{R}$ be a submodular function and let $C \subseteq B(F)$ be an $\epsilon$-cover of $B(F)$ for $\epsilon > 0$. Then, we have

$$\max_{w \in B(F)} \max \{ \langle w, x \rangle, 0 \}^2 \leq \max_{w \in C} \{ \langle w, x \rangle, 0 \}^2 + \max \{ \epsilon^2, 2\epsilon \max_{w \in B(F)} \| w \|_2 \} \cdot \| x \|_{\text{supp}(F)}^2$$

for any $x \in \mathbb{R}^V$.

**Proof.** Because $w(v) = 0$ for every $v \in V \setminus \text{supp}(F)$, it suffices to show the inequality for which $x$ is replaced with $x|_{\text{supp}(F)}$.

Let $w^*$ be the maximizer of $\max_{w \in B(F)} \{ \langle w, x|_{\text{supp}(F)} \rangle, 0 \}^2$. If $\langle w^*, x|_{\text{supp}(F)} \rangle < 0$, then the inequality clearly holds. Hence, we assume $\langle w^*, x|_{\text{supp}(F)} \rangle \geq 0$.

From the definition of $\epsilon$-cover, there exists $w' \in C$ with $\| w^* - w' \|_2 \leq \epsilon$. Our goal is showing that $\langle w^*, x|_{\text{supp}(F)} \rangle^2 \leq \max \{ \langle w', x|_{\text{supp}(F)} \rangle, 0 \}^2 + \max \{ \epsilon^2, 2\epsilon \max_{w \in B(F)} \| w \|_2 \} \cdot \| x|_{\text{supp}(F)}^2$.

If $\langle w', x|_{\text{supp}(F)} \rangle < 0$, then we have

$$\langle w^*, x|_{\text{supp}(F)} \rangle = \langle w', x|_{\text{supp}(F)} \rangle + \langle w^* - w', x|_{\text{supp}(F)} \rangle < \epsilon \| x|_{\text{supp}(F)} \|_2,$$

which implies $\langle w^*, x|_{\text{supp}(F)} \rangle^2 \leq \epsilon^2 \| x|_{\text{supp}(F)} \|_2^2$.

Otherwise, we have

$$\langle w^*, x|_{\text{supp}(F)} \rangle^2 - \max \{ \langle w', x|_{\text{supp}(F)} \rangle, 0 \}^2 = \langle w^*, x|_{\text{supp}(F)} \rangle^2 - \langle w', x|_{\text{supp}(F)} \rangle^2$$

$$= \langle w^* - w', x|_{\text{supp}(F)} \rangle \cdot \langle w^* + w', x|_{\text{supp}(F)} \rangle \leq \epsilon \| x|_{\text{supp}(F)} \|_2 \cdot 2 \max_{w \in B(F)} \| w \|_2 \cdot \| x|_{\text{supp}(F)} \|_2$$

$$= 2\epsilon \| x|_{\text{supp}(F)} \|_2^2 \max_{w \in B(F)} \| w \|_2.$$

In what follows, we fix a submodular transformation $F : \{0,1\}^V \to \mathbb{R}^E$ and $\epsilon > 0$, and let $X = (x_e) \in \mathbb{R}^{N \times V}$ be the SDP solution and let $z_+, z_- \in \mathbb{R}^V$ be the vectors obtained by rounding $X$. Now, we divide $w \in C_e (e \in E)$ into two classes by the value of $\langle Xw, v_1 \rangle$.

$$W^+_e = \left\{ w \in C_e \mid \langle Xw, v_1 \rangle > -\frac{1}{2} \right\}, \quad W^-_e = C_e \setminus W^+_e,$$
\[ W^+ = \bigcup_{e \in E} W^+_e, \quad W^- = \bigcup_{e \in E} W^-_e. \]

We will see that, although \( w \in W^-_e (e \in E) \) makes no contribution to the SDP value, it also does not contribute to in \( f_e(z^+) \) and \( f_e(z^-) \), and hence no loss is incurred for such \( w \) by rounding. On the other hand, although \( w \in W^+_e (e \in E) \) may make a large contribution to the SDP value, we can specify its lower bound by using \( \|Xw\|^2 \) (instead of \( \max\{\langle Xw, v_1 \rangle, 0\}^2 \) ), and hence we can use an argument similar to the symmetric case.

First, we analyze the contribution of \( w \in W^- \).

**Lemma 7.4.** With a probability of at least 99/100, we have
\[
\max \left\{ \langle w, z_+ \rangle, \langle w, z_- \rangle \right\} \leq 0
\]
for every \( w \in W^- \).

**Proof.** We have
\[
\max_{w \in W^-} \max \left\{ \langle w, z_+ \rangle, \langle w, z_- \rangle \right\} = \max_{w \in W^-} \max_{\sigma \in \{-1, 1\}} \sum_{v \in V} w(v) \left( \langle x_v, v_1 \rangle + \sigma \delta \langle v_1^w x_v, g \rangle \right)
\]
\[
\leq -\frac{1}{2} + \delta \max_{w \in W^-} \left| \sum_{v \in V} w(v) P_{v_1^w} x_v, g \right|.
\]

By Lemma A.1 and Markov's inequality, with a probability of 99/100, we have
\[
\delta \max_{w \in W^-} \left| \sum_{v \in V} w(v) P_{v_1^w} x_v, g \right| \leq 100\delta \sqrt{\frac{\log 2 \sum_{e \in E} |C_e| \max_{w \in W^-} \left| \sum_{v \in V} w(v) P_{v_1^w} x_v \right|^2}{2}}.
\]

Note that \( \|x_v\|_2 \leq 1 \) for any \( v \in V \) and \( \|w\|_1 \leq 2/100 = 1/50 \) for any \( w \in C_e \subseteq B(F_e) \) from Lemma 5.1. Hence, we have (23) \( \leq 1/50 \) by choosing the hidden constant in \( \delta \) to be sufficiently small. Then by (22), we have
\[
\max_{w \in W^-} \max \left\{ \langle w, z_+ \rangle, \max_{w \in C_e} \langle w, z_- \rangle \right\} \leq -\frac{1}{2} + \frac{1}{50} \leq 0
\]
with a probability of at least 99/100.

We next show that we can bound the SDP value from below by using \( \|Xw\|^2 \) for \( w \in W^+_e \).

**Lemma 7.5.** For every \( e \in E \), we have
\[
\max_{w \in W^+_e} \|Xw\|^2 \leq 2\|\eta_e\|^2.
\]

**Proof.** Take an arbitrary vector \( w \) in \( W^+_e \), and let \( \theta \in [0, \pi] \) be the angle between \( \langle w, x \rangle \) and \( Xw \).

Then, we have
\[
\|\eta_e\|^2 = \|Xw\|^2 + \langle Xw, v_{\langle w, x \rangle} \rangle = (1 + \cos \theta)\|Xw\|^2.
\]

28
Hence, we want to provide a lower bound for \( \cos \theta \).

Let \( \theta' \) be the angle between \( \mathbf{v}_{(w,x)} \) and \( \mathbf{v}_1 \), and let \( \theta'' \) be the angle between \( X\mathbf{w} \) and \( \mathbf{v}_1 \). From the constraints in (17), we have

\[
\langle \mathbf{v}_{(w,x)}, \mathbf{v}_1 \rangle \geq \|\mathbf{v}_{(w,x)}\|^2_2 = \|X\mathbf{w}\|^2_2,
\]

which implies that \( \cos \theta' \geq \|X\mathbf{w}\|^2_2 \). On the other hand, as \( \mathbf{w} \in W_c^+ \), we have \( \cos \theta'' \geq \max\{-1/(2\|X\mathbf{w}\|^2_2), -1\} = -\min\{1/(2\|X\mathbf{w}\|^2_2), 1\} \).

We note that \( \|X\mathbf{w}\|^2_2 \leq \sum_{e \in \mathbf{v}} \|\mathbf{w}(v)\|_2 \leq 2/100 = 1/50 \) by Lemma 5.1. Then, we have

\[
\cos \theta \geq \cos(\theta' + \theta'') = \cos \theta' \cos \theta'' - \sin \theta' \sin \theta'' 
\geq -\|X\mathbf{w}\|^2_2 \cdot \min\left\{\frac{1}{2\|X\mathbf{w}\|^2_2}, 1\right\} - \sqrt{1 - \|X\mathbf{w}\|^2_2} \sqrt{1 - \min\left\{\frac{1}{4\|X\mathbf{w}\|^2_2}, 1\right\}} 
= -\min\left\{\frac{1}{2}, \|X\mathbf{w}\|^2_2\right\} - \sqrt{1 + \min\left\{\frac{1}{4\|X\mathbf{w}\|^2_2}, 1\right\} - \|X\mathbf{w}\|^2_2 - \min\left\{\frac{1}{4\|X\mathbf{w}\|^2_2}, 1\right\}} 
= -\frac{1}{50} - \sqrt{1 + \frac{1}{2500} - 0 - 1} = -\frac{1}{25}.
\]

Then, we have (24) \( \geq 24/25 \cdot \|X\mathbf{w}\|^2_2 \), and the claim holds.

Now, we show that the maximum numerator of \( \mathcal{R}_F(z_+) \) and \( \mathcal{R}_F(z_-) \) is roughly at most \( O(\log n) \) times the SDP value with a certain probability. We start with the following:

**Lemma 7.6.** We have

\[
\mathbb{E}\left[ \max\left\{ \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle \mathbf{w}, z_+ \rangle^2, \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle \mathbf{w}, z_- \rangle^2 \right\} \right] = O\left( \frac{\log n}{\epsilon^2} \text{SDP}_\epsilon(f) \right).
\]

**Proof.** We have

\[
\mathbb{E}\left[ \max\left\{ \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle \mathbf{w}, z_+ \rangle^2, \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle \mathbf{w}, z_- \rangle^2 \right\} \right] \leq \mathbb{E}\left[ \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle \mathbf{w}, z_+ \rangle^2, \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle \mathbf{w}, z_- \rangle^2 \right] 
= \mathbb{E}\left[ \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \sum_{\sigma \in \{-1,1\}} \langle \mathbf{w}(v), \mathbf{v}_1 \rangle^2 \left( \langle \mathbf{x}_v, \mathbf{v}_1 \rangle + \sigma \delta \langle P_{v_1} \mathbf{x}_v, \mathbf{g} \rangle \right)^2 \right] 
= \mathbb{E}\left[ \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \sum_{\sigma \in \{-1,1\}} \left( \langle X\mathbf{w}, \mathbf{v}_1 \rangle + \sigma \delta \langle P_{v_1} X\mathbf{w}, \mathbf{g} \rangle \right)^2 \right] 
\leq 4 \max_{e \in E} \log(2|C_e|) \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \langle X\mathbf{w}, \mathbf{v}_1 \rangle^2 + \delta^2 \langle P_{v_1} X\mathbf{w}, \mathbf{g} \rangle^2 \quad \text{(By Proposition A.2)} 
\leq 4 \max_{e \in E} \log(2|C_e|) \sum_{e \in E} \max_{\mathbf{w} \in W_c^+} \|X\mathbf{w}\|^2_2 
\leq 16 \max_{e \in E} \log(2|C_e|) \sum_{\mathbf{e} \in E} \|\eta_\mathbf{e}\|^2_2 \quad \text{(By Lemma 7.5)} 
= O\left( \frac{\log n}{\epsilon^2} \text{SDP}_\epsilon(f) \right). \]

**Lemma 7.7.** Let \( z_+, z_- \in \mathbb{R}^V \) be the vectors obtained in Algorithm 3 on a submodular transformation \( F : \{0,1\}^V \rightarrow \mathbb{R}^E \) (and some \( \epsilon > 0 \)). Then, we have

\[
\max\left\{ \sum_{e \in E} f_\mathbf{e}(z_+)^2, \sum_{e \in E} f_\mathbf{e}(z_-)^2 \right\} = O\left( \frac{\log n}{\epsilon^2} \text{SDP}_\epsilon(f) + \epsilon \max_{e \in E} \|B(F_e)\|^2_H \right).
\]

29
with a probability of at least 1/100.

Proof. We only show the bound for $z_+$ as the analysis for $z_-$ is the same. We have

$$\sum_{e \in E} f_e(z_+)^2 = \sum_{e \in E} \max_{w \in B(f_e)} \max \{ \langle w, z_+ \rangle, 0 \}^2$$

$$= \sum_{e \in E} \max_{w \in C_e} \max \{ \langle w, z_+ \rangle, 0 \}^2 + O(\epsilon \sum_{e \in E} \|z_+|_{\text{supp}(f_e)}\|^2 \cdot \|B(F_e)\|^2_{H}) \quad \text{(By Lemma 7.3)}$$

$$= \sum_{e \in E} \max_{w \in C_e} \max \{ \langle w, z_+ \rangle, 0 \}^2 + O(\epsilon \max_{e \in E} \|B(F_e)\|^2_{H} \cdot \sum_{v \in V} d_F(v) z_+(v)^2).$$

The second term is bounded by $O(\epsilon)$ with a probability of at least 1/50 by Lemma 7.2.

Now, we analyze the first term. Note that

$$\sum_{e \in E} \max_{w \in C_e} \max \{ \langle w, z_+ \rangle, 0 \}^2$$

$$\leq \sum_{e \in E} \max_{w \in W_e} \max \{ \langle w, z_+ \rangle, 0 \}^2 + \sum_{e \in E} \max_{w \in W_e} \max \{ \langle w, z_+ \rangle, 0 \}^2$$

$$\leq \sum_{e \in E} \max_{w \in W_e} \langle w, z_+ \rangle^2 + \sum_{e \in E} \max_{w \in W_e} \max \{ \langle w, z_+ \rangle, 0 \}^2$$

$$= O\left(\frac{\log n}{\epsilon^2} \text{SDP}_\epsilon(f)\right) \quad \text{(By Lemmas 7.4 and 7.6)}$$

with a probability of at least 99/100.

By the union bound, we have the claim. \qed

7.2.3 Consolidation of results

Proof of Theorem 1.11. Let $z_+, z_- \in \mathbb{R}^V$ be the output of Algorithm 2 on $f$ and $\epsilon > 0$. As with the proof of Theorem 1.10, we can show that both $z_+$ and $z_-$ are feasible. By considering the one with the larger denominator in the Rayleigh quotient, we have the desired approximation guarantee by combining Lemmas 7.2 and 7.7. The total time complexity is dominated by the time complexity for solving SDP (17), which is $\text{poly}(nm)^{\text{poly}(1/\epsilon)}$.

Note that we can augment the success probability to 9/10 by running this algorithm a constant number of times and by outputting the vector with the minimum Rayleigh quotient. \qed

8 Non-trivial Eigenvalues of Submodular Laplacians

In this section, we prove Theorem 3.6. We omit the proof of Theorem 3.7 as it is obtained by replacing $L_F$ and $R_F$ by $L_F$ and $L_F$ in the proof of Theorem 3.6.

In Section 8.1, we introduce a diffusion process associated with a submodular Laplacian. In Section 8.2, we use this diffusion process to show that the submodular Laplacian has a non-trivial eigenvector.
8.1 Diffusion process

We study the non-trivial eigenpairs of a submodular Laplacian by considering a diffusion process defined as follows:

**Definition 8.1.** Let $F : \{0, 1\}^V \rightarrow \mathbb{R}^E$ be a submodular transformation with $F(V) = 0$. We stipulate that the time evolution of $x \in \mathbb{R}^V$ obeys the following equation:

$$\frac{dx}{dt} \in -L_F(x)dt.$$  \hspace{1cm} (25)

The initial condition is given by an arbitrary vector $x_0 \in \mathbb{R}^V$. Let $x_t$ denote $x$ at time $t \in \mathbb{R}_+$. The process with the Laplacian of an undirected graph is referred to as the **heat equation** in the literature [6, 12]. We do not write $\frac{dx}{dt}$ because $x$ is not differentiable in general.

We note that the diffusion process (25) is not a priori well-defined for the following reason: Let $x \in \mathbb{R}^V$ be a vector. Then, after an infinitesimal time $dt$, $x$ is moved to $x' = x - L_F(x)dt$. After another infinitesimal time $dt$, $x'$ is moved to $x'' = x' - L_F(x')dt$. Here, $x''$ may coincide with $x$ by canceling the previous move. This means that the diffusion process does not proceed beyond the vector $x$.

We avoid the above-mentioned problem by carefully choosing $dx$. We say that $W \in \mathbb{R}^{V \times E}$ is **valid** at $x$ if $W \in \prod_{e \in E} \partial f_e(x)$. Our goal is to ensure that $W$ remains valid after an infinitesimal time. If this were true, $W$ continues to remain valid for $\epsilon$ unit of time for some $\epsilon > 0$ because the ordering of values in $x$ does not change after an infinitesimal amount of time has passed, and hence we can simulate the diffusion process (25).

To enable us to work with an infinitesimal amount of time, we introduce some definitions. For a vector $x \in \mathbb{R}^V$, we define a relation $\leq_x$ over $V$ such that $v \leq_x v'$ if and only if $x(v) \leq x'(v)$. We define relations $<_x$, $>_x$, and $=_x$, similarly. For vectors $x, w \in \mathbb{R}^V$ and an infinitesimal value $dt$, we define a relation $\leq_{x+wdt}$ over $V$ such that $v \leq_{x+wdt} v'$ if and only if $v <_x v'$ or $v =_x v'$ and $w(v) \leq w'(v)$. For each $e \in E$, we define $\partial f_e(x + wdt)$ as the convex hull of the points obtained by invoking Lemma 8.3 with $f_e$ and the ordering of $V$ induced by $\leq_{x+wdt}$. We say that $W \in \mathbb{R}^{V \times E}$ is **infinitesimally valid** at $x \in \mathbb{R}^V$ if $W$ is valid at $x$ and $W \in \prod_{e \in E} \partial f_e(x - \mathbb{R}^{V \times E} x dt)$, that is, $W$ is valid after simulating the process (25) along the direction $-\mathbb{R}^{V \times E} x$ for an infinitesimal time.

The remaining issue is whether we can find infinitesimally valid $W \in \mathbb{R}^{V \times E}$ for the given vector $x \in \mathbb{R}^V$. Here, we use Kakutani’s fixed-point theorem [10] to find such $W$. A set-valued function $\varphi : X \rightarrow 2^Y$ is said to have a closed graph if the set $\{(x, y) | y \in \varphi(x)\}$ is a closed subset of $X \times Y$ in the product topology, that is, for all sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ such that $x_n \rightarrow x$, $y_n \rightarrow y$ and $y_n \in \varphi(x_n)$ for all $n \in \mathbb{N}$, we have $y \in \varphi(x)$. For a set-valued function $\varphi : X \rightarrow 2^X$, $x \in X$ is called a fixed point of $\varphi$ if $x \in \varphi(x)$. Now, the following holds:

**Theorem 8.2** (Kakutani’s fixed-point theorem [10]). **Let $S$ be a non-empty, compact and convex subset of $\mathbb{R}^n$. Let $\varphi : S \rightarrow 2^S$ be a set-valued function with a closed graph such that $\varphi(x)$ is non-empty and convex for all $x \in S$. Then $\varphi$ has a fixed point.**

**Lemma 8.3.** Let $F : \{0, 1\}^V \rightarrow \mathbb{R}^E$ be a submodular transformation. For any $x \in \mathbb{R}^V$, there is $W \in \mathbb{R}^{V \times E}$ that is infinitesimally valid at $x$.

**Proof.** We define a set-valued function $\varphi : \prod_{e \in E} \partial f_e(x) \rightarrow 2\prod_{e \in E} \partial f_e(x)$ as follows. Given $W \in \prod_{e \in E} \partial f_e(x)$, define $\varphi(W) = \prod_{e \in E} \partial f_e(x - \mathbb{R}^{V \times E} x dt) \subseteq \prod_{e \in E} \partial f_e(x)$. We can easily confirm that $\varphi$ has a closed graph. As the polytope $\prod_{e \in E} \partial f_e(x)$ is compact and convex, by Kakutani’s fixed-point theorem, there exists $W \in \prod_{e \in E} \partial f_e(x)$ such that $W \in \prod_{e \in E} \partial f_e(x - \mathbb{R}^{V \times E} x dt)$. This means that $W$ is infinitesimally valid at $x$. \hfill $\Box$
8.2 Non-trivial eigenpairs

In this section, we show that the Laplacian \( L_F \) of a submodular transformation \( F : \{0,1\}^V \rightarrow \mathbb{R}^E \) with \( F(V) = 0 \) has a non-trivial eigenpair and each non-trivial eigenpair \((\gamma, z)\) satisfies \( \gamma = R_F(z) \).

Our strategy is to observe the value of the Rayleigh quotient in the diffusion process (25). Note that, as we choose one vector from \( L_F(x) \) at each time \( t \), we can represent \( L_F \) at time \( t \) as a matrix and denote it by \( L_t \in \mathbb{R}^{V \times V} \). In this section, the norm \( \| \cdot \| \) always represents the \( l_2 \)-norm.

**Lemma 8.4.** We have \( \frac{d\|x\|^2}{dt} = -2R_F(x)\|x\|^2dt \).

*Proof.*

\[
\frac{d\|x\|^2}{dt} = 2\langle x, dx \rangle = -2\langle x, L_txdt \rangle = -2R_F(x)\|x\|^2dt.
\]

We define \( \overline{x} = x/\|x\| \). Then, we have the following.

**Lemma 8.5.** We have \( \frac{d\overline{x}}{dt} = \left(R_F(\overline{x})\overline{x} - L_t\overline{\overline{x}}\right)dt \) and \( dR_F(x) = 2(R_F(\overline{x})^2 - \|L_t\overline{x}\|^2)dt \).

*Proof.* From Lemma 8.4, we have \( \frac{d\|x\|}{dt} = -R_F(x)\|x\|dt \). Then, we have

\[
\frac{d\overline{x}}{dt} = \frac{\|x\|dx - xd\|x\|}{\|x\|^2} = \frac{R_F(x)\|x\| - L_tx\|x\|}{\|x\|^2}dt
\]

\[
= \left(\frac{R_F(x)x}{\|x\|} - L_tx\right)\frac{dt}{\|x\|} = \left(R_F(\overline{x})\overline{x} - L_t\overline{\overline{x}}\right)dt.
\]

Let \( Q_F(x) = \langle x, L_F(x) \rangle \) and \( Q_t(x) = \langle x, L_tx \rangle \). Note that \( Q_F(x) \) does not depend on the choice of \( W \in \prod_{e \in E} \partial f_e(x) \) used in Definition 3.2. Hence, we have

\[
dR_F(x) = dQ_F(\overline{x}) = dQ_t(\overline{x}) = \langle \frac{dQ_t(\overline{x})}{d\overline{x}}, d\overline{x} \rangle = 2\langle L_t\overline{\overline{x}}, (R_F(\overline{x})\overline{x} - L_t\overline{\overline{x}})dt \rangle = 2(R_F(\overline{x})^2 - \|L_t\overline{x}\|^2)dt.
\]

**Corollary 8.6.** \( R_F(x) \) is non-increasing in \( t \).

*Proof.* Note that \( \|L_t\overline{x}\| \geq \langle \overline{x}, L_t\overline{x} \rangle = R_F(\overline{x}) \). The inequality holds because \( \overline{x} \) is a unit vector. From Lemma 8.5, we have \( dR_F(x) \leq 0 \). Since \( R_F \) is a continuous function of \( t \), we have the desired result.

**Theorem 8.7.** Suppose that we initiate a simulation of the diffusion process (25) with a non-zero vector \( x_0 \perp 1 \). Then, as \( t \to \infty \), \( x \) and \( R_F(x) \) converge to some \( z \in \mathbb{R}^V \) and \( \gamma \in \mathbb{R}_+ \), respectively, such that

\[
z \perp 1, \quad L_F(z) \ni \gamma z, \quad \text{and} \quad \gamma = R_F(z).
\]

*Proof.* Note that \( R_F(x) \) is bounded from below by 0 from Lemma 3.5. Since \( R_F(x) \) is non-increasing from Corollary 8.6, \( R_F(x) \) converges to some non-negative value as \( t \to \infty \).

Let \( \gamma \in \mathbb{R}_+ \) be the limit. We have \( \lim_{t \to \infty} \|L_t\overline{x}\|^2 = \gamma^2 \) by Lemma 8.5. It follows that \( \lim_{t \to \infty} \langle L_t\overline{x} - \gamma\overline{x}, L_t\overline{x} \rangle = 0 \). Since \( \lim_{t \to \infty} \|L_t\overline{x}\| = \gamma \) and \( \|\overline{x}\| = 1 \), we must have \( \lim_{t \to \infty} L_t\overline{x} - \gamma\overline{x} = 0 \) or \( \lim_{t \to \infty} L_t\overline{x} = 0 \).
However, the latter implies that $\gamma = 0$. Hence, we have $\lim_{t \to \infty} L_t \bar{x} - \gamma \bar{x} = 0$ in both cases. In particular, this means $\lim_{t \to \infty} d \bar{x} \to 0$ by Lemma 8.5. As $\bar{x}$ is bounded, $\bar{x}$ converges to a vector $z$, which is an eigenvector of $L_F$ with the eigenvalue $\gamma = R_F(z)$.

It is clear that $z \perp 1$ because we always have $d \bar{x} \perp 1$ when we start the diffusion process with a vector $x_0 \perp 1$.

Theorem 8.7 immediately implies Theorem 3.6.

Acknowledgments

The authors would like to thank Tasuku Soma for many useful discussions.

References

[1] N. Alon. Eigenvalues and expanders. Combinatorica, 6(2):83–96, 1986.

[2] N. Alon and V. D. Milman. $\lambda_1$, isoperimetric inequalities for graphs, and superconcentrators. Journal of Combinatorial Theory, Series B, 38(1):73–88, 1985.

[3] F. Bach. Learning with submodular functions: A convex optimization perspective. Foundations and Trends in Machine Learning, 6:145–373, 2013.

[4] D. Chakrabarty, P. Jain, and P. Kothari. Provable submodular minimization using wolfe’s algorithm. In Proceedings of the 28th Advances in Neural Information Processing Systems (NIPS), pages 802–809, 2014.

[5] T.-H. H. Chan, A. Louis, Z. G. Tang, and C. Zhang. Spectral properties of hypergraph Laplacian and approximation algorithms. 2016, arXiv:1605.01483.

[6] F. Chung. The heat kernel as the pagerank of a graph. Proceedings of the National Academy of Sciences, 104(50):19735–19740, 2007.

[7] F. R. K. Chung. Spectral Graph Theory. CBMS Regional Conference Series. American Mathematical Society, 1997.

[8] S. Fujishige. Submodular functions and optimization, volume 58 of Annals of Discrete Mathematics. Elsevier, 2nd edition, 2005.

[9] D. F. Gleich and C. Seshadhri. Vertex neighborhoods, low conductance cuts, and good seeds for local community methods. In Proceedings of the 18th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD), pages 597–605, 2012.

[10] S. Kakutani. A generalization of brouwer’s fixed point theorem. Duke Mathematical Journal, 8(3):457–459, 1941.

[11] K. Kawaguchi. Deep learning without poor local minima. In Proceedings of the 30th Annual Conference on Neural Information Processing Systems (NIPS), pages 586–594, 2016.
[12] K. Kloster and D. F. Gleich. Heat kernel based community detection. In Proceedings of the 20th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD), pages 1386–1395, 2014.

[13] T. C. Kwok, L. C. Lau, and Y. T. Lee. Improved cheeger’s inequality and analysis of local graph partitioning using vertex expansion and expansion profile. SIAM Journal on Computing, 46(3):890–910, 2017.

[14] T. C. Kwok, L. C. Lau, Y. T. Lee, S. Oveis Gharan, and L. Trevisan. Improved cheeger’s inequality: analysis of spectral partitioning algorithms through higher order spectral gap. In Proceedings of the 45th annual ACM symposium on Theory of Computing (STOC), pages 11–20, 2013.

[15] J. R. Lee, S. O. Gharan, and L. Trevisan. Multiway spectral partitioning and higher-order cheeger inequalities. Journal of the ACM, 61(6):37–30, 2014.

[16] J. Leskovec, K. J. Lang, and M. Mahoney. Empirical comparison of algorithms for network community detection. In Proceedings of the 19th International Conference on World Wide Web (WWW), pages 631–640, 2010.

[17] S. Liu. Multi-way dual cheeger constants and spectral bounds of graphs. Advances in Mathematics, 268:306–338, 2015.

[18] A. Louis. Hypergraph markov operators, eigenvalues and approximation algorithms. In Proceedings of the 47th Annual ACM on Symposium on Theory of Computing (STOC), pages 713–722, 2015.

[19] A. Louis, P. Raghavendra, P. Tetali, and S. Vempala. Many sparse cuts via higher eigenvalues. In Proceedings of the 44th Annual ACM Symposium on Theory of Computing (STOC), pages 1131–1140, 2012.

[20] J. Massey. Causality, feedback and directed information. Proceedings of the International Symposium on Information Theory Applications (ISITA), 1990.

[21] H. H. Permuter, Y.-H. Kim, and T. Weissman. Interpretations of Directed Information in Portfolio Theory, Data Compression, and Hypothesis Testing. IEEE Transactions on Information Theory, 57(6):3248–3259, 2011.

[22] H. H. Permuter, T. Weissman, and A. J. Goldsmith. Finite state channels with time-invariant deterministic feedback. IEEE Transactions on Information Theory, 55(2):644–662, 2009.

[23] G. Pisier. The Volume of Convex Bodies and Banach Space Geometry. Cambridge University Press, 1999.

[24] P. Raghavendra and D. Steurer. Graph expansion and the unique games conjecture. In Proceedings of the 42nd Annual ACM Symposium on Theory of Computing (STOC), pages 755–764, 2010.

[25] P. Raghavendra and D. Steurer. Reductions between expansion problems. In Proceedings of the IEEE 27th Annual Conference on Computational Complexity (CCC), pages 64–73, 2012.
A Facts on Normal Distributions

We review several facts on normal distributions.

**Proposition A.1** (Fact 8.6 of [5]). Suppose $X_1, X_2, \ldots, X_n$ are normal random variables that are not necessarily independent such that $E[X_i] = 0$ ($i \in [n]$) and $E[X_i^2] = \sigma_i^2$ ($i \in [n]$). Then, we have $E[\max_{i \in [n]} X_i^2] \leq 4\sigma^2 \log n$ and $E[\max_{i \in [n]} X_i] \leq 2\sigma\sqrt{\log n}$, where $\sigma := \max_{i \in [n]} \sigma_i$.

By slightly changing the proof of Proposition A.1, we can show a similar bound for biased normal random variables.

**Proposition A.2**. Suppose $X_1, X_2, \ldots, X_n$ are normal random variables that are not necessarily independent such that $E[X_i] = \mu_i$ ($i \in [n]$) and $E[X_i^2] = \sigma_i^2$ ($i \in [n]$). Then, we have $E[\max_{i \in [n]} X_i^2] \leq 4\lambda^2 \log n$ and $E[\max_{i \in [n]} X_i] \leq 2\lambda\sqrt{\log n}$, where $\lambda := \max_{i \in [n]} \sqrt{\mu_i^2 + \sigma_i^2}$.

**Proof.** For $i \in [n]$, we write $X_i = \mu_i + \sigma_i Z_i$, where $Z_i$ has the standard normal distribution $\mathcal{N}(0, 1)$. Observe that, for any real numbers $x_1, x_2, \ldots, x_n$ and positive integer $p$, we have $\max_{i \in [n]} x_i^2 \leq (\sum_{i \in [n]} x_i^{2p})^{1/p}$. Hence, we have

$$
E[\max_{i \in [n]} X_i^2] \leq E \left[ \left( \sum_{i \in [n]} X_i^{2p} \right)^{1/p} \right]
\leq \left( E \left[ \sum_{i \in [n]} X_i^{2p} \right] \right)^{1/p} 
\leq \lambda^2 \left( E \left[ \sum_{i \in [n]} Z_i^{2p} \right] \right)^{1/p}
= \lambda^2 \left( \sum_{i \in [n]} \frac{(2p)!}{p!2^p} \right)^{1/p}
\leq \sigma^2 pd^{1/p}
$$

Selecting $p = \lfloor \log n \rfloor$ provides the first result $E[\max_{i \in [n]} X_i^2] \leq 4\lambda^2 \log n$. Moreover, the inequality $E[|X|] \leq \sqrt{E[X^2]}$ immediately provides the second result. \hfill \Box
Proposition A.3 (Fact 8.7 of [5]). Let $X_1, \ldots, X_n$ be normal random variables that are not necessarily independent $E[X_i] = 0$ ($i \in [n]$) and $E[\sum_{i \in [n]} X_i^2] = 1$ ($i \in [n]$). Then, we have
\[
\Pr\left[\sum_{i \in [n]} X_i^2 \geq \frac{1}{2}\right] \geq \frac{1}{12}.
\]

B Inequalities

The following vector version of the Cauchy-Schwarz inequality holds:

Lemma B.1. For $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ and $v_1, \ldots, v_k \in \mathbb{R}^n$, we have
\[
\left\|\sum_{i \in [k]} \alpha_i v_k\right\|_2^2 \leq \sum_{i \in [k]} \alpha_i^2 \cdot \sum_{i \in [k]} \|v_i\|_2^2.
\]

Proof.
\[
\left\|\sum_{i \in [k]} \alpha_i v_k\right\|_2^2 = \sum_{j \in [n]} \left(\sum_{i \in [k]} \alpha_i v_k(j)\right)^2 \leq \sum_{j \in [n]} \left(\sum_{i \in [k]} \alpha_i^2 \cdot \sum_{i \in [k]} v_k(j)^2\right) = \sum_{j \in [n]} \left(\sum_{i \in [k]} \alpha_i^2\right) \left(\sum_{i \in [k]} v_k(j)^2\right)
\]
\[
= \left(\sum_{i \in [k]} \alpha_i^2\right) \left(\sum_{j \in [n]} \sum_{i \in [k]} v_k(j)^2\right) = \sum_{i \in [k]} \alpha_i^2 \cdot \sum_{i \in [k]} \|v_i\|_2^2. \qed
\]