ON QUASI $\kappa$-METRIZABLE SPACES

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Abstract. The aim of this paper is to investigate the class of quasi $\kappa$-metrizable spaces. This class is invariant with respect to arbitrary products and contains Schepin’s $\kappa$-metrizable spaces as a proper subclass.

1. Introduction

Recall that a $\kappa$-metric on a space $X$ is a non-negative function $\rho(x,C)$ on a space $X$ is a non-negative function $\rho(x,C)$ of two variables, a point $x \in X$ and a regularly closed set $C \subset X$, satisfying the following conditions:

$K1)$ $\rho(x,C) = 0$ iff $x \in C$;
$K2)$ If $C \subset C'$, then $\rho(x,C') \leq \rho(x,C)$ for every $x \in X$;
$K3)$ $\rho(x,C)$ is continuous function of $x$ for every $C$;
$K4)$ $\rho(x,\bigcup C_\alpha) = \inf_\alpha \rho(x,C_\alpha)$ for every increasing transfinite family $\{C_\alpha\}$ of regularly closed sets in $X$.

A $\kappa$-metric on $X$ is said to be regular if it satisfy also next condition

$K5)$ $\rho(x,C) \leq \rho(x,C') + \overline{\rho}(C',C)$ for any $x \in X$ and any two regularly closed sets $C,C'$ in $X$, where $\overline{\rho}(C',C) = \sup\{\rho(y,C') : y \in C\}$.

We say that a function $\rho(x,C)$ is an quasi $\kappa$-metric (resp., a regular quasi $\kappa$-metric) on $X$ if it satisfies the axioms $K2) - K4)$ (resp., $K2) - K5)$ and the following one:

$K1)^*$ For any $C$ there is a dense open subset $V$ of $X \setminus C$ such that $\rho(x,C) > 0$ iff $x \in V$.

Obviously, we can assume that $\rho(x,C) \leq 1$ for all $x$ and $C$, in such a case we say that $\rho$ is a normed quasi $\kappa$-metric.

Quasi $\kappa$-metrizable spaces were introduced in [8]. Our interest of this class was originated by Theorem 1.4 from [8] stating that a compact space is quasi $\kappa$-metrizable if and only if it is skeletally generated.

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Unfortunately, the presented proof of the implication that any skeletally generated compactum is quasi $\kappa$-metrizable is not correct. Despite of this incorrectness, the class of quasi $\kappa$-metrizable is very interesting. It is closed with respect to arbitrary products and contains as a proper subclass the $\kappa$-metrizable spaces. The aim of this paper is to investigate the class of quasi $\kappa$-metrizable spaces, and to provide a correct characterization of skeletally generated spaces.

Let me note that skeletally generated spaces were introduced in [9] and, by [8, Theorem 1.1], a space is skeletally generated iff it is $I$-favorable in the sense of [2]. Recall that a map $f : X \to Y$ is skeletal if $\text{Int}_f(U) \neq \emptyset$ for every open $U \subset X$. A space $X$ is skeletally generated [9] if there is an inverse system $S = \{X_\alpha, p_\beta^\alpha\}$ of separable metric spaces $X_\alpha$ and skeletal surjective bounding maps $p_\beta^\alpha$, satisfying the following conditions: (1) the index set $A$ is $\sigma$-complete (every countable chain in $A$ has a supremum in $A$); (2) for every countable chain $\{\alpha_n\}_{n \geq 1} \subset A$ with $\beta = \sup\{\alpha_n\}_{n \geq 1}$ the space $X_\beta$ is a (dense) subset of $\lim\left\langle X_\alpha, p_\alpha^{\alpha_n+1}\right\rangle$; (3) $X$ is embedded in $\lim S$ and $p_\alpha(X) = X_\alpha$ for each $\alpha$, where $p_\alpha : \lim S \to X_\alpha$ is the $\alpha$-th limit projection. If in the above definition all bounding maps $p_\alpha^\beta$ are open, we say that $X$ is openly generated.

All topological spaces are Tychonoff and the single-valued maps are continuous. The paper is organized as follows: Section 2 contains the proof that any product of quasi $\kappa$-metrizable spaces is also quasi $\kappa$-metrizable. In Section 3 we provide some more properties of quasi $\kappa$-metrizable spaces. For example, it is shown that this property is preserved by open and perfect surjections, and that the Čech-Stone compactification of any pseudocompact quasi $\kappa$-metrizable space is quasi $\kappa$-metrizable. It also follows from the result from Section 3 that there exist quasi $\kappa$-metrizable spaces which are not $\kappa$-metrizable. In Section 4 we introduce a similar wider class of spaces, the weakly $\kappa$-metrizable spaces, and proved that a compact space is skeletally generated if and only if it is weakly $\kappa$-metrizable. This implies that every skeletally generated space is also weakly $\kappa$-metrizable.

2. Products of quasi $\kappa$-metrizable spaces

Let $\mathcal{B}$ be a base for a space $X$ consisting of regularly open sets. A real-valued function $\xi : X \times \mathcal{B} \to [0, 1]$ will be called a $\pi$-capacity if it satisfies the following conditions:

E1) $\xi(x, U) = 0$ for $x \notin U$, and $0 \leq \xi(x, U) \leq 1$ for $x \in U$.

E2) For any $U \in \mathcal{B}$ the set $\{x \in U : \xi(x, U) > 0\}$ is dense in $U$. 

E3) For any \( U \) the function \( \xi(x, U) \) is lower semi-continuous, i.e if \( \xi(x_0, U) > a \) for some \( x_0 \in X \) and \( a \in \mathbb{R} \), then there is a neighborhood \( O_{x_0} \) with \( \xi(x, U) > a \) for all \( x \in O_{x_0} \).

E4) For any two mappings \( U : T \to \mathcal{B} \) and \( \lambda : T \to X \), where \( T \) is a set with an ultrafilter \( \mathcal{F} \), such that the limit \( \bar{\lambda} = \lim_{\mathcal{F}} \lambda(t) \) exists and \( \lim_{\mathcal{F}} \xi(\lambda(t), U(t)) > a > 0 \), then there exists \( \tilde{U} \in \mathcal{B} \) such that \( \tilde{U} \subseteq \lim_{\mathcal{F}} U(t) \) and \( \xi(\bar{\lambda}, \tilde{U}) > a \). Here, \( \lim_{\mathcal{F}} U(t) = \bigcap_{F \in \mathcal{F}} \bigcup_{t \in F} U(t) \).

A capacity is called regular if it satisfies also the following condition:

E5) If \( \xi(x, U) > a > 0 \), there exists \( U_a \in \mathcal{B} \) such that \( \xi(x, U_a) \geq a \) and \( \xi(y, U) \geq \xi(x, U) - a \) for all \( y \in U_a \).

Our definition of a \( \pi \)-capacity is almost the same as the Schepin’s definition [6] of capacity, the only difference is that Schepin requires \( \xi(x, U) > 0 \) for all \( x \in U \).

**Lemma 2.1.** Suppose \( \xi : X \times \mathcal{B} \to [0, 1] \) is a (regular) \( \pi \)-capacity on \( X \). Then the function \( \rho_{\xi}(x, C) \), \( \rho_{\xi}(x, C) = 0 \) if \( x \in C \) and \( \rho_{\xi}(x, C) = \sup\{\xi(x, U) : U \cap C = \emptyset\} \) otherwise, is a (regular) quasi \( \kappa \)-metric on \( X \).

**Proof.** Suppose \( \xi \) is a \( \pi \)-capacity on \( X \). Clearly, \( \rho_{\xi} \) satisfies condition K2). According to the proof of [7] Proposition 6, chapter 3 \( \rho_{\xi} \) also satisfies conditions K3) – K4). To check condition K1)*, let \( C \) be a proper regularly closed subset of \( X \). Then there is a subfamily \( \{U_\alpha : \alpha \in \mathcal{A}\} \) of \( \mathcal{B} \) covering \( X \setminus C \). For every \( \alpha \) the set \( V_\alpha = \{x \in U_\alpha : \xi(x, U_\alpha) > 0\} \) is dense in \( U_\alpha \). So \( V = \bigcup_{\alpha \in \mathcal{A}} V_\alpha \) is dense in \( X \setminus C \) and \( \rho_{\xi}(x, C) > 0 \) for all \( x \in V \).

Let show that if \( \xi \) is a regular \( \pi \)-capacity, then \( \rho_{\xi} \) satisfies condition K5). Suppose \( C, C' \) are two regularly closed subsets of \( X \) and \( x \in X \). Obviously, \( \rho_{\xi}(x, C) \leq \rho_{\xi}(x, C') \) implies \( \rho_{\xi}(x, C) \leq \rho_{\xi}(x, C') + \overline{\rho_{\xi}(C', C)} \). So, let \( \rho_{\xi}(x, C) > \rho_{\xi}(x, C') \), and choose an integer \( m \) such that \( \rho_{\xi}(x, C) > \rho_{\xi}(x, C') + 1/n \) for all \( n \geq m \). Hence, there is \( U \in \mathcal{B} \) such that \( U \cap C = \emptyset \) and \( \xi(x, U) > a_n = \rho_{\xi}(x, C') + 1/n \). So, according to condition E5), there is \( U_{a_n} \in \mathcal{B} \) such that \( \xi(x, U_{a_n}) \geq a_n \) and \( \xi(x, U) \leq \xi(y, U) + a_n \) for all \( y \in U_{a_n} \). Since \( \xi(x, U_{a_n}) \geq a_n \), \( U_{a_n} \cap C' \neq \emptyset \) (otherwise we would have \( \rho_{\xi}(x, C') \geq \rho_{\xi}(x, C') + 1/n \)). Hence, \( \xi(x, U) \leq \xi(z, U) + a_n \) for every \( z \in U_{a_n} \cap C' \), which yields \( \xi(x, U) \leq \overline{\rho_{\xi}(C', C)} + \rho_{\xi}(x, C') + 1/n \) for all \( n \geq m \) and \( U \in \mathcal{B} \) with \( U \cap C = \emptyset \). Therefore, \( \rho_{\xi}(x, C) \leq \rho_{\xi}(x, C') + \overline{\rho_{\xi}(C', C)}. \)

**Lemma 2.2.** Let \( \rho \) be a (regular) normed quasi \( \kappa \)-metric on \( X \) and \( \mathcal{B} \) be a base for \( X \) consisting of regularly open sets. Then the formula
\[ \xi(x, U) = \sup \{ \rho(x, C) : C \cup U = X \} \text{ defines a (regular) } \pi \text{-capacity on } X. \]

**Proof.** It is easy to show that \( \xi \) satisfies conditions E1) and E3). Condition E4) was established in [8, Lemma 3] in the case \( \rho \) is a \( \kappa \)-metric, but the same proof works for quasi \( \kappa \)-metrics as well. Let show condition E2). We fix \( U \in \mathcal{B} \) and consider the family \( \mathcal{B}_U = \{ G \in \mathcal{B} : G \subseteq U \} \). For every \( G \in \mathcal{B}_U \) the set \( V_G = \{ x \in G : \rho(x, C_G) > 0 \} \) is open and non-empty, where \( C_G = X \setminus G \). Hence, \( V = \bigcup_{G \in \mathcal{B}_U} V_G \) is dense in \( U \).

Moreover, \( \xi(x, U) \geq \rho(x, C_G) \) for every \( G \in \mathcal{U} \) and \( x \in V_G \) because \( U \cup C_G = X \). So, \( \xi(x, U) > 0 \) for all \( v \in V \).

It remains to show that \( \xi \) is regular, i.e. it satisfies E5), provided \( \rho \) is regular. Let \( U \in \mathcal{B} \) and \( \xi(x, U) > a > 0 \) for some \( x \in U \). Then, according to our definition, there is a regularly closed set \( C \subset X \) such that \( C \cup U = X \) and \( \rho(x, C) > \max \{ a, \xi(x, U) - a \} \).

The set \( W = \{ y \in U : \rho(y, C) > \xi(x, U) - a \} \) is open and non-empty because \( \rho(., C) \) is continuous and \( x \in W \). Choose \( U_a \in \mathcal{B} \) with \( U_a \subset W \), and let \( C' = X \setminus U_a \). Then, by K5), \( \rho(x, C) \leq \rho(x, C') + \bar{\rho}(C', C) \). Consequently, \( \rho(x, C') \geq \rho(x, C) - \bar{\rho}(C', C) \). On the other hand, \( \bar{\rho}(C', C) = \sup_{y \in C'} \rho(y, C) = \sup_{y \in C' \cap W} \rho(y, C) \) because \( \rho(y, C) = 0 \) for all \( y \in C \). Observe also that \( C' \setminus C \subset U \), and since \( \rho(y, C) > \xi(x, U) - a \) for all \( y \in W \) and \( \rho(y, C) \leq \xi(x, U) - a \) for all \( y \in U \setminus W \), we have \( \bar{\rho}(C', C) = \sup_{y \in C' \cap W} \rho(y, C) \). Therefore, \( -\bar{\rho}(C', C) = -\sup_{y \in C' \cap W} \rho(y, C) \leq -\xi(x, U) + a \leq -\rho(x, C) + a \), so \( \rho(x, C') \geq a \). Since \( U_a \cup C' = X \), the last inequality implies \( \xi(x, U_a) \geq a \). Finally, \( \xi(y, U) \geq \rho(y, C) \geq \xi(x, U) - a \) for all \( y \in U_a \) because \( U_a \subset W \). Hence, \( \xi \) satisfies E5). \( \square \)

Let consider the following condition, where \( \rho(x, C) \) is a non-negative function with \( C \) being a regularly closed subset of \( X \):

**K1)** For any regularly closed \( C \subseteq X \) there is \( y \not\in C \) with \( \rho(y, C) > 0 \) and \( \rho(x, C) = 0 \) for all \( x \in C \).

**Remark 2.3.** Observe that in the previous lemma we actually proved the following more general statement: Suppose \( \rho \) satisfies conditions K1)** and K2) - K4), and \( \rho(x, C) \leq 1 \) for all \( x \in X \) and all regularly closed sets \( C \subset X \). Then \( \xi(x, U) = \sup \{ \rho(x, C) : C \cup U = X \} \) defines a \( \pi \)-capacity on \( X \). Moreover, \( \xi \) is regular if \( \rho \) satisfies also K5).

**Corollary 2.4.** Suppose there is a function \( \rho \) on \( X \) satisfying conditions K1)** and K2) - K4). Then there is a quasi \( \kappa \)-metric \( d \) on \( X \). Moreover, \( d \) is regular if \( \rho \) satisfies also condition K5).

**Proof.** We can suppose that \( \rho \) is normed. Then, by Lemma 2.2, there is a \( \pi \)-capacity \( \xi \) on \( X \). Finally, Lemma 2.1 implies the existence of a
quasi $\kappa$-metric $d$ on $X$. Moreover, if $\rho$ satisfies condition $K5$), then $\xi$ is regular, so is $d$.

**Theorem 2.5.** Any product of (regularly) quasi $\kappa$-metrizable spaces is (regularly) quasi $\kappa$-metrizable.

**Proof.** Suppose $X = \prod_{\alpha \in A} X_{\alpha}$ and for every $\alpha$ there is a normed (regular) quasi $\kappa$-metric $\rho_{\alpha}$ on $X$. Following the proof of [6, Theorem 2], for every $\alpha$ we fix a base $B_{\alpha}$ on $X_{\alpha}$, and let $B$ be the standard base for $X$ consisting of sets of the form $U = \bigcup_{i=1}^{n} \pi_{\alpha^{-1}}(U_{i})$ with $U_{i} \in B_{\alpha}$ and $U_{i} \neq X_{\alpha_{i}}$, where $\pi_{\alpha} : X \rightarrow X_{\alpha}$ is the projection. Denote by $v(U)$ the collection $\{\alpha_{1}, ..., \alpha_{n}\}$. According to Lemma 2.2, for every $\alpha$ there exists a (regular) $\pi$-capacity $\xi_{\alpha}$ on $X_{\alpha}$. Consider the function $\xi : X \times B \rightarrow \mathbb{R}$ defined by $\xi(x, U) = \inf_{\alpha \in v(U)} \xi_{\alpha}(\pi_{\alpha}(x), \pi_{\alpha}(U))/|v(U)|$.

Obviously, condition $E1$) is satisfied. Moreover, since for each $\alpha_{i}$ the set $W_{i} = \{z \in X_{\alpha_{i}} : \xi_{i}(z, U_{i}) > 0\}$ is open and dense in $U_{i}$, the set $W = \bigcup_{i=1}^{n} \pi_{\alpha_{i}^{-1}}(W_{i})$ is dense in $U$. Schepin has shown that the function $\xi$ is a (regular) capacity provide each $\xi_{\alpha}$ is so, see the proof of [7, Theorem 15] and [6, Theorem 2]. The same arguments show that $\xi$ also satisfies conditions $E2) – E4)$, and condition $E5)$ in case each $\xi_{\alpha}$ is regular. Therefore, $\xi$ is a (regular) $\pi$-capacity. Finally, by Lemma 2.1 there exists a (regular) quasi $\kappa$-metric on $X$. $\Box$

3. SOME MORE PROPERTIES OF QUASI $\kappa$-METRIZABLE SPACES

**Proposition 3.1.** Let $X$ be a quasi $\kappa$-metrizable space and $Y \subset X$. The $Y$ is also quasi $\kappa$-metrizable in each of the following cases: (i) $Y$ is dense in $X$; (ii) $Y$ is regularly closed in $X$; (iii) $Y$ is open in $X$.

**Proof.** If $\rho$ is a quasi $\kappa$-metric on $X$ and $Y \subset X$ is dense, the equality $d(y, \overline{Y}) = \rho(y, \overline{X})$, where $U \subset Y$ is open defines a quasi $\kappa$-metric on $Y$. The second case follows from the observation that every regularly closed subset of $Y$ is also regularly closed in $X$. The third case follows from the first two because every open subset of $X$ is dense in its closure. $\Box$

Let consider the following condition.

$K4)^* \rho(x, \bigcup C_{n}) = \inf_{n} \rho(x, C_{n})$ for every increasing sequence $\{C_{n}\}$ of regularly closed sets in $X$.

**Lemma 3.2.** Suppose $X$ is a space admitting a non-negative function $\rho(x, C)$ satisfying conditions $K1)^*$, $K2)$, $K3)$ and $K4)^*$. Then $X$ is quasi $\kappa$-metrizable provided $X$ has countable cellularity. In particular, every compact space admitting such a function $\rho$ is quasi $\kappa$-metrizable.
Proof. It suffices to show that \( \rho \) satisfies condition \( K4) \) if \( X \) is of countable cellularity. So, let \( \{ C_\alpha \} \) be an increasing transfinite family of regularly closed sets in \( X \). Then \( \bigcup \alpha C_\alpha = \bigcup \alpha U_\alpha \) and \( \{ U_\alpha \} \) is also increasing, where \( U_\alpha \) is the interior of \( C_\alpha \). Since \( X \) has countable cellularity, there are countably many \( \alpha_i \) such that \( \bigcup_{i \geq 1} U_{\alpha_i} \) is dense in \( \bigcup \alpha U_\alpha \). We can assume that the sequence \( \{ \alpha_i \} \) is increasing, so is the sequence \( \{ U_{\alpha_i} \} \). Because \( \rho \) satisfies condition \( K4) \)*, we have \( \rho(x, \bigcup C_{\alpha_i}) = \inf_i \rho(x, C_{\alpha_i}) \). This implies that \( \rho(x, \bigcup C_{\alpha_i}) = \inf \alpha \rho(x, C_{\alpha_i}) \). Indeed, since \( \bigcup C_{\alpha_i} = \bigcup C_{\alpha_i} \), \( \inf \alpha \rho(x, C_{\alpha_i}) < \inf_i \rho(x, C_{\alpha_i}) \) for some \( x \in X \) would imply the existence of \( \alpha_0 \) with \( \rho(x, C_{\alpha_0}) < \inf_i \rho(x, C_{\alpha_i}) \) for all \( i \). Because any two elements of the family \( \{ C_{\alpha_i} \} \) are comparable with respect to inclusion, the last inequality means that \( C_{\alpha_0} \) contains all \( C_{\alpha_i} \). Hence, \( C_{\alpha_0} = \bigcup \alpha C_{\alpha_i} \) and \( \rho(x, C_{\alpha_0}) \) would be equal to \( \inf_i \rho(x, C_{\alpha_i}) \), a contradiction.

It was shown in [3, Theorem 1.4] that every compact space \( X \) admitting a non-negative function \( \rho(x, C) \) satisfying conditions \( K1) \)*, \( K2) \), \( K3) \) and \( K4) \)* is skeletonally generated, and hence \( X \) has countable cellularity. Therefore, any such compactum is quasi \( \kappa \)-metrizable.

It was shown by Chigogidze [11] that the Čech-Stone compactification of every pseudocompact \( \kappa \)-metrizable space is \( \kappa \)-metrizable. We have a similar result for quasi \( \kappa \)-metrizable spaces.

**Theorem 3.3.** If \( X \) is a pseudocompact (regularly) quasi \( \kappa \)-metrizable space, then \( \beta X \) is (regularly) quasi \( \kappa \)-metrizable.

**Proof.** Suppose \( \rho(x, C) \) is a quasi \( \kappa \)-metric on \( X \). We can assume that \( \rho(x, \overline{X}) \leq 1 \) for all \( x \in X \) and all open \( U \subseteq X \) (\( \overline{U} \) denotes the closure of \( U \) in \( X \)). For every open \( W \subseteq \beta X \) consider the function \( f_W \) on \( X \) defined by \( f_W(x) = \rho(x, \overline{W \cap X}) \). Let \( \tilde{f}_W : \beta X \to \mathbb{R} \) be the continuous extension of \( f_W \), and define \( d(y, W) = \tilde{f}_W(y), y \in \beta X \). Obviously, \( d(y, \overline{W}) = 0 \) if \( y \in W \cap X \). Since \( W \cap X \) is dense in \( \overline{W} \), \( d(y, \overline{W}) = 0 \) for all \( y \in \overline{W} \). Moreover, if \( \overline{W} \neq \beta X \), then \( \overline{W \cap X} \neq X \). So, there is an open dense subset \( V \) of \( X \setminus \overline{W} \) with \( \rho(x, \overline{W \cap X}) > 0 \) for all \( x \in V \). Since \( f_W \) is continuous, the set \( \tilde{V} = \{ y \in \beta X : f_W(y) > 0 \} \) is open in \( \beta X \) and disjoint from \( \overline{W} \). Finally, because \( V \subset \tilde{V} \) and \( V \) is dense in \( X \setminus \overline{W} \), \( \tilde{V} \) is dense in \( \beta X \setminus \overline{W} \). So, \( d \) satisfies condition \( K1) \)*. Conditions \( K2) \) and \( K3) \) also hold. Hence, by Lemma 3.2, it suffices to show that \( d \) satisfies \( K4) \)*. To this end, let \( \{ \overline{W_n} \} \) be an increasing sequence of regularly closed subsets of \( \beta X \) and \( W = \bigcup_{n \geq 1} W_n \). We have \( d(y, \overline{W}) \leq \inf_n d(y, \overline{W_n}) \) for all \( y \in \beta X \). Moreover, since \( \rho \) satisfies
K4), \(d(y, \overline{W}) = \inf_n d(y, \overline{W}_n)\) if \(y \in X\). Suppose there is \(y_0 \in \beta X \setminus X\) with \(d(y_0, \overline{W}) < \inf_n d(y_0, \overline{W}_n)\). Consequently, for every \(n\) there exists a neighborhood \(V_n\) of \(y_0\) in \(\beta X\) such that \(\delta < d(y, \overline{W}_n)\) for all \(y \in V_n\), where \(d(y_0, \overline{W}) < \delta < \inf_n d(y_0, \overline{W}_n)\). We also choose a neighborhood \(V_0\) of \(y_0\) with \(d(y, \overline{W}) < \delta\) for all \(y \in V_0\). This implies that \(d(y, \overline{W}) < \delta \leq \inf_n d(y, \overline{W}_n)\) provided \(y \in V = \bigcap_{n>1} V_0 \cap V_n\). But \(V \cap X \neq \emptyset\) because \(X\) is pseudocompact. Thus, \(d(y, \overline{W}) < \inf_n d(y, \overline{W}_n)\) for any \(y \in V \cap X\), a contradiction.

It follows from the definition of \(d\) that it satisfies condition K5) provided \(\rho\) is regular. \(\square\)

**Corollary 3.4.** Every pseudocompact quasi \(\kappa\)-metrizable space \(X\) is skeletally generated.

**Proof.** We already noted that every quasi \(\kappa\)-metrizable compactum is skeletally generated, see [8]. So, by Theorem 3.3, \(\beta X\) is skeletally generated. Finally, by [2] and [8], every dense subset of a skeletally generated space is also skeletally generated. \(\square\)

**Proposition 3.5.** Suppose \(f : X \to Y\) is a perfect open surjection and \(X\) is (regularly) quasi \(\kappa\)-metrizable. Then \(Y\) is also (regularly) quasi \(\kappa\)-metrizable.

**Proof.** Let \(\rho\) be a quasi \(\kappa\)-metric on \(X\). Since \(f\) is open, \(f^{-1}(\overline{U}) = \overline{f^{-1}(U)}\) for any open \(U \subset Y\). So, \(f^{-1}(\overline{U})\) is regularly closed set in \(X\) and we define

\[
d(y, \overline{U}) = \sup\{\rho(x, f^{-1}(\overline{U})) : x \in f^{-1}(y)\}.
\]

One can check that \(d\) satisfies conditions K2) and K4), and condition K5) in case \(\rho\) is regular. Moreover, \(\overline{U} \neq Y\) implies \(f^{-1}(\overline{U}) \neq X\). So, there is a dense open subset \(V \subset X \setminus f^{-1}(\overline{U})\) such that \(\rho(x, f^{-1}(\overline{U})) > 0\) iff \(x \in V\). Then \(f(V)\) is a dense and open subset of \(Y \setminus \overline{U}\) such that \(f^{-1}(y) \cap V \neq \emptyset\) for all \(y \in f(V)\). Hence, \(d(y, \overline{U}) > 0\) if \(y \in f(V)\). If \(y \notin f(V)\), then \(f^{-1}(y) \cap V = \emptyset\). Thus, \(d(y, \overline{U}) > 0\) iff \(y \in f(V)\). Finally, let check continuity of the functions \(d(., \overline{U})\). Suppose \(d(y_0, \overline{U}) < \varepsilon\) for some \(y_0\) and \(U\). Then \(\rho(x, f^{-1}(\overline{U})) < \varepsilon\) for all \(x \in f^{-1}(y)\). Consequently, there is a neighborhood \(W\) of \(f^{-1}(y)\) with \(\rho(x, f^{-1}(\overline{U})) < \varepsilon\) for all \(x \in W\). Since, \(f\) is closed, \(y_0\) has a neighborhood \(G\) such that \(f^{-1}(G) \subset W\). This implies that \(d(y, \overline{U}) < \varepsilon\) for all \(y \in G\). Now, let \(d(y_0, \overline{U}) > \delta\) for some \(\delta \in \mathbb{R}\). So, there exists \(x_0 \in f^{-1}(y_0)\) with \(\rho(x_0, f^{-1}(\overline{U})) > \delta\). Choose a neighborhood \(O\) of \(x_0\) such that \(\rho(x, f^{-1}(\overline{U})) > \delta\) for all \(x \in O\). Then, \(f(O)\) is a neighborhood of \(y_0\) and \(d(y, \overline{U}) > \delta\) for any \(y \in f(O)\). Therefore, each \(d(., \overline{U})\) is continuous. \(\square\)
Proposition 3.6. Let $f : X \to Y$ be a proper irreducible surjection, and $Y$ is (regularly) quasi $\kappa$-metrizable. Then $X$ is also (regularly) quasi $\kappa$-metrizable.

Proof. Suppose $\rho$ is a quasi $\kappa$-metric on $Y$. For every regularly closed $C \subset X$ define $d(x, C) = \rho(f(x), f(C))$. This definition is correct because $f(C)$ is regularly closed in $Y$. Indeed, let $C = \bigcup_{\alpha} C_{\alpha}$, where $U_{\alpha} = \{y \in Y : f^{-1}(y) \subset U\} = Y \setminus f(X \setminus U)$ is open in $Y$. It is easily seen that $d$ satisfies conditions $K2)$ and $K3)$. Condition $K4)$ follows from the equality $f(\bigcup_{\alpha} C_{\alpha}) = \bigcup_{\alpha} f(C_{\alpha})$ for any family of regularly closed sets in $X$. To see that $d$ satisfies also condition $K1)^*$, we observe that for every regularly closed $C \subset X$ there is a dense open subset $V \subset Y \setminus f(C)$ such that $\rho(y, f(C)) > 0$ iff $y \in V$. Then $W = f^{-1}(V)$ is open in $X$ and disjoint from $C$. Moreover, $d(x, C) > 0$ iff $x \in W$. It remains to show that $W$ is dense in $X \setminus C$. And that is really true because for every open $O \subset X \setminus C$ the set $O_{\sharp}$ is a non-empty open subset of $Y \setminus f(C)$. So, $O_{\sharp} \cap V \neq \emptyset$, which implies $W \cap O \neq \emptyset$.

One can also see that $d$ is regular provided so is $\rho$. \hfill \Box

Corollary 3.7. The absolute of any (regularly) quasi $\kappa$-metrizable space is (regularly) quasi $\kappa$-metrizable.

Remark 3.8. The last corollary shows that the class of $\kappa$-metrizable spaces is a proper subclass of the quasi $\kappa$-metrizable spaces. Indeed, let $X$ be a $\kappa$-metrizable compact infinite space. Then its absolute $aX$ is quasi $\kappa$-metrizable. On the other hand, $aX$ being extremally disconnected can not be $\kappa$-metrizable (otherwise, it should be discrete by [7, Theorem 11]).

Corollary 3.9. Every compact space co-absolute to a quasi $\kappa$-metrizable space is skeletally generated.

Proof. Let $X$ and $Y$ be compact spaces having the same absolute $Z$. So, there are perfect irreducible surjections $g : Z \to Y$ and $f : Z \to X$. If $Y$ is quasi $\kappa$-metrizable, then so is $Z$, see Proposition 3.6. Hence, $Z$ is skeletally generated, and by [4, Lemma 1], $X$ is also skeletally generated. \hfill \Box

Recall that the hyperspace $\exp X$ consists of all compact non-empty subsets $F$ of $X$ such that the sets of the form

$$[U_1, ..., U_k] = \{H \in \exp X : H \subset \bigcup_{i=1}^{k} U_i \text{ and } H \cap U_i \neq \emptyset \text{ for all } i\}$$
form a base $B_{\exp}$ for $\exp X$, where each $U_i$ belongs to a base $B$ for $X$, see [5].

**Proposition 3.10.** If $X$ is (regularly) quasi $\kappa$-metrizable, so is $\exp X$.

**Proof.** Let $B$ be a base for $X$ and $\rho$ be a (regular) quasi $\kappa$-metric on $X$. Then $\rho$ generates a (regular) $\pi$-capacity $\xi_\rho : X \times B \to \mathbb{R}$ on $X$. Following the proof of [6, Theorem 3], we define a function $\xi : \exp X \times B_{\exp} \to \mathbb{R}$ by

$$\xi(F, [U_1, \ldots, U_k]) = \frac{1}{n} \min \{ \inf_{x \in F} \max_i \xi_\rho(x, U_i), \min_i \sup_{x \in F} \xi_\rho(x, U_i) \}.$$ 

It was shown in [6] that $\xi$ satisfies conditions $E1), E3) and E4)$, and that $\xi$ is regular provided $\xi_\rho$ is regular. Let show that $\xi$ satisfies condition $E2)$. Since $\xi$ satisfies $E3)$, it suffices to prove that for every $[U_1, \ldots, U_k]$ there is a dense subset $V_{\exp} \subset [U_1, \ldots, U_k]$ with $\xi(F, [U_1, \ldots, U_k]) > 0$ for all $F \in V_{\exp}$. To this end, for each $i$ fix an open dense subset $V_i$ of $U_i$ such that $\xi_\rho(x, U_i) > 0$ if $x \in V_i$. Let $V_{\exp}$ consists of all finite sets $F \subset X$ such that $F \subset \bigcup_{i=1}^n V_i$ and $F \cap V_i \neq \emptyset$ for all $i$. Then $V_{\exp}$ is dense in $[U_1, \ldots, U_k]$ and $\xi(F, [U_1, \ldots, U_k]) > 0$ for all $F \in V_{\exp}$. Hence, by Lemma 2.1, $\exp X$ is (regularly) quasi $\kappa$-metrizable.

Schepin [6] Theorem 3a] has shown that if $\exp X$ is $\kappa$-metrizable, then so is $X$. We don’t know if a similar result is true for quasi $\kappa$-metrizable spaces.

### 4. Skeletally generated spaces

In this section we provide a characterization of skeletally generated compact spaces in terms of functions similar to quasi $\kappa$-metrics. We say that a non-negative function $d : X \times C \to \mathbb{R}$ is a weak $\kappa$-metric, where $C$ is the family of all regularly closed subsets of $X$, if it satisfies conditions $K1)^*, K2) - K3)$ and the following one:

$K4)$ For every increasing transfinite family $\{C_\alpha\}_\alpha \subset C$ the function $f(x) = \inf_\alpha d(x, C_\alpha)$ is continuous.

**Theorem 4.1.** A compact space is skeletally generated if and only if it is weakly $\kappa$-metrizable.

**Proof.** First, let show that every skeletally generated compactum $X$ is weakly $\kappa$-metrizable. We embed $X$ as a subset of $\mathbb{R}^\tau$ for some cardinal $\tau$. Then, according to [8 Theorem 1.1], there is a function $e : T_X \to T_{\mathbb{R}^\tau}$ between the topologies of $X$ and $\mathbb{R}^\tau$ such that: (i) $e(U) \cap e(V) = \emptyset$
provided \( U \) and \( V \) are disjoint; (ii) \( e(U) \cap X \) is dense in \( U \). We define a new function \( e_1 : T_X \to T_{\mathbb{R}^\tau} \),

\[
e_1(U) = \bigcup \{ e(V) : V \in T_X \text{ and } \nabla \subseteq U \}.
\]

Obviously \( e_1 \) satisfies conditions (i) and (ii), and it is also monotone, i.e. \( U \subseteq V \) implies \( e_1(U) \subseteq e_1(V) \). Moreover, for every increasing transfinite family \( \gamma = \{ U_\alpha \} \) of open sets in \( X \) we have \( e_1(\bigcup_\alpha U_\alpha) = \bigcup_\alpha e_1(U_\alpha) \). Indeed, if \( z \in e_1(\bigcup_\alpha U_\alpha) \), then there is an open set \( V \in T_X \) with \( \nabla \subseteq \bigcup_\alpha U_\alpha \) and \( z \in e(V) \). Since \( \nabla \) is compact and the family is increasing, \( V \) is contained in some \( U_\alpha \). Hence, \( z \in e(V) \subseteq e_1(U_\alpha) \). Consequently, \( e_1(\bigcup_\alpha U_\alpha) \subseteq \bigcup_\alpha e_1(U_\alpha) \). The other inclusion follows from monotonicity of \( e_1 \).

Because \( \mathbb{R}^\tau \) is \( \kappa \)-metricizable (see [8]), there is a \( \kappa \)-metric \( \rho \) on \( \mathbb{R}^\tau \). For every regularly closed \( C \subseteq X \) and \( x \in X \) we can define the function \( d(x, C) = \rho(x, \overline{e_1(\text{Int}C)}) \), where \( e_1(U) \) is the closure of \( e_1(U) \) in \( \mathbb{R}^\tau \). It is easily seen that \( d \) satisfies conditions \( K2 \) - \( K3 \). Let show that it also satisfies \( K4_0 \) and \( K1^* \). Indeed, assume \( \{ C_\alpha \} \) is an increasing transfinite family of regularly closed sets in \( X \). We put \( U_\alpha = \text{Int}C_\alpha \) for every \( \alpha \) and \( U = \bigcup_\alpha U_\alpha \). Thus, \( e_1(U) = \bigcup_\alpha e_1(U_\alpha) \). Since \( \{ e_1(U_\alpha) \} \) is an increasing transfinite family of regularly closed sets in \( \mathbb{R}^\tau \), for every \( x \in X \) we have

\[
\rho(x, \bigcup_\alpha e_1(U_\alpha)) = \inf_\alpha \rho(x, e_1(U_\alpha)) = \inf_\alpha d(x, C_\alpha).
\]

Hence, the function \( f(x) = \inf_\alpha d(x, C_\alpha) \) is continuous on \( X \) because so is \( \rho(\cdot, \bigcup_\alpha e_1(U_\alpha)) \). To show that \( K1^* \) also holds, observe that \( d(x, C) = 0 \) if and only if \( x \in X \cap \overline{e_1(\text{Int}C)} \). Because \( e_1(\text{Int}C) \cap X \) is dense in \( C \), \( C \subseteq e_1(\text{Int}C) \). Hence, \( V = X \setminus \overline{e_1(\text{Int}C)} \) is contained in \( X \setminus C \) and \( d(x, C) > 0 \) if \( x \in V \). To prove \( V \) is dense in \( X \setminus C \), let \( x \in X \setminus C \) and \( W_x \subseteq X \setminus C \) be an open neighborhood of \( x \). Then \( W \cap \text{Int}C = \emptyset \), so \( e_1(W) \cap e_1(\text{Int}C) = \emptyset \). This yields \( e_1(W) \cap X \subseteq V \). On the other hand, \( e_1(W) \cap X \) is a non-empty subset of \( W \), hence \( W \cap V \neq \emptyset \). Therefore, \( d \) is a weak \( \kappa \)-metric on \( X \).

The other implication was actually established in the proof of Theorem 1.4] from [8], and we sketch the proof here. Suppose \( d \) is a weak \( \kappa \)-metric on \( X \) and embed \( X \) in a Tychonoff cube \( \mathbb{I}^4 \) with uncountable \( A \), where \( I = [0, 1] \). For any countable set \( B \subseteq A \) let \( \mathcal{A}_B \) be a countable base for \( X_B = \pi_B(X) \) consisting of all open sets in \( X_B \) of the form \( X_B \cap \prod_{\alpha \in B} V_{\alpha} \), where each \( V_{\alpha} \) is an open subinterval of \( I \) with rational end-points and \( V_{\alpha} \neq \mathbb{I} \) for finitely many \( \alpha \). Here \( \pi_B : \mathbb{I}^A \to \mathbb{I}^B \) denotes the projection, and let \( p_B = \pi_B|X \). For any open \( U \subseteq X \) denote by \( f_U \)
the function \( d(x, \overline{U}) \). We also write \( p_B \prec g \), where \( g \) is a map defined on \( X \), if there is a map \( h : p_B(X) \to g(X) \) such that \( g = h \circ p_B \). Since \( X \) is compact this is equivalent to the following: if \( p_B(x_1) = p_B(x_2) \) for some \( x_1, x_2 \in X \), then \( g(x_1) = g(x_2) \). We say that a countable set \( B \subset A \) is \( d \)-admissible if \( p_B \prec f_{p_B^{-1}(V)} \) for every \( V \in \mathcal{A}_B \). Denote by \( \mathcal{D} \) the family of all \( d \)-admissible subsets of \( A \). We are going to show that all maps \( p_B : X \to X_B, B \in \mathcal{D} \), are skeletal and the inverse system \( S = \{ X_B : p_B^B : D \subset B, D, B \in \mathcal{D} \} \) is \( \sigma \)-continuous with \( X = \lim S \).

It was shown in [8] that for any countable set \( B \subset A \) there is \( D \in \mathcal{D} \) with \( B \subset D \), and the union of any increasing sequence of \( d \)-admissible sets is also \( d \)-admissible. So, we need to show only that \( p_B : X \to X_B \) is a skeletal map for every \( B \in \mathcal{D} \).

Suppose there is an open set \( U \subset X \) such that the interior in \( X_B \) of \( \overline{p_B(U)} \) is empty. Then \( W = X_B \setminus \overline{p_B(U)} \) is dense in \( X_B \). Let \( \{ W_m \}_{m \geq 1} \) be a countable cover of \( W \) with \( W_m \in \mathcal{A}_B \) for all \( m \). Since \( \mathcal{A}_B \) is finitely additive, we may assume that \( W_m \subset W_{m+1}, m \geq 1 \). Because \( B \) is \( d \)-admissible, \( p_B \prec f_{p_B^{-1}(W_m)} \) for all \( m \). Hence, there are continuous functions \( h_m : X_B \to \mathbb{R} \) with \( f_{p_B^{-1}(W_m)} = h_m \circ p_B, m \geq 1 \). Recall that \( f_{p_B^{-1}(W_m)}(x) = d(x, \overline{p_B^{-1}(W_m)}) \) and \( p_B^{-1}(W) = \bigcup_{m \geq 1} p_B^{-1}(W_m) \).

Therefore, \( f_{p_B^{-1}(W)}(x) = d(x, \overline{p_B^{-1}(W)}) \leq f(x) = \inf_m f_{p_B^{-1}(W_m)}(x) \) for all \( x \in X \). Moreover, \( f \) is continuous and \( f_{p_B^{-1}(W_{m+1})}(x) \leq f_{p_B^{-1}(W_m)}(x) \) because \( W_m \subset W_{m+1} \). The last inequalities together with \( p_B \prec f_{p_B^{-1}(W_m)} \) yields that \( p_B \prec f \). So, there exists a continuous function \( h \) on \( X_B \) with \( f(x) = h(p_B(x)) \) for all \( x \in X \). But \( f(x) = 0 \) for all \( x \in p_B^{-1}(W) \), so \( f(\overline{p_B^{-1}(W)}) = 0 \). This implies that \( h(\overline{W}) = 0 \). Since \( p_B(\overline{p_B^{-1}(W)}) = \overline{W} = X_B \), we have that \( h \) is the constant function zero. Consequently, \( f(x) = 0 \) for all \( x \in X \). Finally, the inequality \( d(x, \overline{p_B^{-1}(W)}) \leq f(x) \) yields that \( d(x, \overline{p_B^{-1}(W)}) = 0 \) for all \( x \in X \). On the other hand, \( \overline{p_B^{-1}(W)} \cap U = \emptyset \). So, according to K1\(^{*}\), there is an open subset \( U' \) of \( U \) with \( d(x, \overline{p_B^{-1}(W)}) > 0 \) for each \( x \in U' \), a contradiction. \( \square \)

Because any compactification of a skeletally generated space is skeletally generated (see [8]) and the weakly \( \kappa \)-metrizability is a hereditary property with respect to dense subsets, we have the following

**Corollary 4.2.** Every skeletally generated space is weakly \( \kappa \)-metrizable.

All results in Section 3, except Proposition 3.10, remain valid for weakly \( \kappa \)-metrizable spaces. Theorem 4.4 and a result of Kucharski-Plewik [4, Theorem 6] imply that Proposition 3.10 is also true for
weakly $\kappa$-metrizable compacta. But the following questions are still open.

**Question 4.3.** Is any product of weakly $\kappa$-metrizable spaces weakly $\kappa$-metrizable?

**Question 4.4.** Is any weakly $\kappa$-metrizable space with a countable cellularity skeletally generated?

**References**

[1] A. Chigogidze, *On $\kappa$-metrizable spaces*, Uspehi Mat. Nauk **37** (1982), no. 2, 241–242 (in Russian).

[2] P. Daniels, K. Kunen and H. Zhou, *On the open-open game*, Fund. Math. **145** (1994), no. 3, 205–220.

[3] A. Kucharski and S. Plewik, *Skeletal maps and I-favorable spaces*, Acta Univ. Carolin. Math. Phys. **51** (2010), 67–72.

[4] A. Kucharski and S. Plewik, *Game approach to universally Kuratowski-Ulam spaces*, Topology Appl. **154** (2007), no. 2, 421–427.

[5] K. Kuratowski, *Topology, vol. I*, Academic Press, New York; PWN-Polish Scientific Publishers, Warsaw 1966.

[6] E. Shchepin, *On $\kappa$-metrizable spaces*, Math. USSR Izvestija **14** (1980), no. 2, 1–34.

[7] E. Shchepin, *Topology of limit spaces of uncountable inverse spectra*, Russian Math. Surveys **315** (1976), 155–191.

[8] V. Valov, *I-favorable spaces: Revisited* Topology Proc. **51** (2018), 277–292.

[9] V. Valov, *External characterization of I-favorable spaces*, Mathematica Balkanica **25** (2011), no. 1-2, 61–78

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