Scaling properties of the pairing problem in the strong coupling limit

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Abstract

We study the excited states of the pairing Hamiltonian providing an expansion for their energy in the strong coupling limit. To assess the role of the pairing interaction we apply the formalism to the case of a heavy atomic nucleus. We show that only a few statistical moments of the level distribution are sufficient to yield an accurate estimate of the energy for not too small values of the coupling $G$ and we give the analytic expressions of the first four terms of the series. Further, we discuss the convergence radius $G_{\text{sing}}$ of the expansion showing that it strongly depends upon the details of the level distribution. Furthermore $G_{\text{sing}}$ is not related to the critical values of the coupling $G_{\text{crit}}$, which characterize the physics of the pairing Hamiltonian, since it can exist even in the absence of these critical points.

1 Introduction

The problem of the pairing interaction in a Fermi system, like e.g. an atomic nucleus, has been dealt with long time ago in the case of $n$ pairs living in a single level and the solution is well-known.
Instead, the case of \( n \) pairs distributed over a set of \( L \) levels, each one with a pair degeneracy \( \Omega_\mu \) and energy \( \epsilon_\mu \), is treated with the Richardson’s equations [1] (in short RE), to be described below, but no explicit solution can be given in a closed form: hence for a finite system one has to resort to numerical methods [2].

The RE, assuming that \( n \) pairs are distributed, in the absence of interaction, over \( L \) levels, read

\[
\sum_{\mu=1}^{L} \frac{\Omega_\mu}{2\epsilon_\mu - E_i} - 2 \sum_{k \neq i}^{n} \frac{1}{E_k - E_i} = \frac{1}{G} \tag{1}
\]

with \( i = 1, \ldots, n \), \( G \) being the strength of the interaction. The \( E_i \), namely the unknowns, are sometimes viewed as the energies of the single \((i^{th})\) pairs, although this statement is questionable: in fact they have no direct physical meaning and could be complex. The true observable, namely the energy of the system in a given state, in the Richardson framework turns out to be

\[
E = \sum_{i=1}^{n} E_i \tag{2}
\]

and is of course real. In [1] the \( n \) pairs are set up by two fermions in time reversal states coupled to zero momentum or angular momentum.

The space of the parameters in the pairing problem is wide, being generated by the set of the unperturbed s.p.e. (single particle energies) \( \{\epsilon_\mu\} \), by their pair degeneracies \( \{\Omega_\mu\} \) and by the coupling constant \( G \). We remind that \( L \) can be of the order of, say, 10 or less in the nuclear case, but is of the order of the Avogadro number in the case of a band in a metal.

In the strong coupling limit, however, the dependence upon the whole set of parameters occurs only through some simple combination of them. Actually here the RE reduce to a system of equations essentially parameter-free, whose solutions, namely the \( E_i \), can be obtained by exploiting a scaling property, hence an analytic solution for the total energy can be given, although not in a closed form, but as an expansion in inverse powers of \( G \).

This topic has been addressed in some recent papers [3, 4, 5] in the context of solid state physics (actually superconducting metallic grains) where a major simplification occurs since the unperturbed levels are assumed to be equally spaced \( (\epsilon_\mu = \hbar \omega_0 \mu \) with \( \mu = 1, 2, \cdots L \)) and to host only one pair \( (\Omega_\mu = 1) \).
The case of nuclear physics requires an extension of this approach since the nuclear levels are distributed inside a major shell with various energies and degeneracies. The pairing problem for \( n \) pairs living in any number of degenerate levels has been recently addressed in the framework of pseudodeformed quasispin \( SU(2) \) algebra [6] and an exact solution has been provided in Ref. [7] for an orbit-dependent interaction in the special case of two non-degenerate energy levels, but an analytic solution to the general problem is not presently available.

As we shall see, however, in the strong coupling limit the pairing energy depends only upon the statistical properties of the level distribution.

In this paper we propose a further derivation of the strong coupling expansion which, extending the one presented in Ref. [8], applies not only to the ground state, but to the excited states energies as well and neatly displays in its coefficients the renormalization of the statistical moments of the levels distribution, of the strength of the interaction and of the number of pairs prevented to take an active part into the dynamics induced by the interaction with the trapped pairs. Moreover, and importantly, we succeed as in Ref.[9] in yielding analytic expressions for the “pair energies” \( E_i \). This we do in Sections 2 and 3. In Section 4 and 5 we address a specific nuclear problem to illustrate how the method works and in Section 6 we compare our results with the exact numerical solution of the RE.

## 2 The strong coupling expansion

Let us first define the “strong coupling limit”. As already discussed in [10,11] and [8], we introduce the average

\[
\bar{\epsilon} = \frac{1}{\Omega} \sum_{\mu=1}^{L} \Omega_\mu \epsilon_\mu
\]

(3)

and the variance

\[
\sigma = \sqrt{\frac{1}{\Omega} \sum_{\mu=1}^{L} \Omega_\mu (\epsilon_\mu - \bar{\epsilon})^2},
\]

(4)

with

\[
\Omega = \sum_{\mu=1}^{L} \Omega_\mu ,
\]

(5)
of the levels distribution. Then, since $\bar{\epsilon}$ is an intrinsically irrelevant parameter, as it depends upon the choice of the zero point of the energy, we safely assume $\bar{\epsilon} = 0$. Thus the two energy scales entering into the pairing problem will be set by $\sigma$ and $G$. When the condition $\sigma \ll G$ is met, then the single particle levels (s.p.l.) span a very narrow energy range and the well-known solution

$$E = -G \left( n - \frac{v}{2} \right) \left( \Omega - n - \frac{v}{2} + 1 \right)$$

is expected to be a very good approximation ($v$ denotes the seniority).

It was found in [10] that a convenient expansion parameter is

$$\alpha = \frac{2\sigma}{G\Omega},$$

the strong coupling limit corresponding to

$$\alpha \ll 1.$$  

It is known that in the strong coupling limit for a given state some of the $E_i$ are large (those contributing to the collectivity of the state) and of the order of $G$, while the others remain trapped between the unperturbed levels and are consequently of the order of $2\sigma$. Actually the collectivity is associated with the existence or not of broken pairs. The state with zero seniority is the collective state and has the lowest energy. The other states, with $v = 2, 4, \ldots$, correspond to larger energies, but the broken pairs still contribute to the energy through the Pauli principle.

Hence, having chosen $\bar{\epsilon} = 0$ it is natural to split, introducing an integer $k$, the solutions $E_i$ into two subsets $\{E_i, i = 1, \ldots, k\}$ and $\{E_j, j = k+1, \ldots, n\}$ with the condition $|E_i| \ll |E_j|$ $\forall i, j$. Thus in this partition the first $k$ pairs are trapped while the others take part in setting up the collective excitations of the system. Clearly $k$ just corresponds to the Gaudin number $N_G$ [12], which in turn is related to the “like-seniority” $v_l$ introduced in [11] according to $N_G = v_l/2$.

Consider then the equations for the $E_j$ with $j = k+1, \ldots, n$ (the “large” energies): by expanding in powers of the small quantities $E_i$ and $\epsilon_{\mu}$ we get

$$-\frac{1}{E_j} \sum_{\mu=1}^{L} \Omega_{\mu} \sum_{m=0}^{\infty} \left( \frac{2\epsilon_{\mu}}{E_j} \right)^m + \frac{2}{E_j} \sum_{i=1}^{k} \sum_{m=0}^{\infty} \left( \frac{E_i}{E_j} \right)^m - 2 \sum_{p=k+1}^{n} \sum_{\substack{p=k+1 \\ p\neq j}}^{n} \frac{1}{E_p - E_j} = \frac{1}{G}.$$  

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Likewise the first $k$ equations, related to the trapped solutions, can be expanded as follows

$$\sum_{\mu=1}^{L} \frac{\Omega_{\mu}}{2\epsilon_{\mu} - E_i} - 2 \sum_{p=1 \atop p \neq i}^{k} \frac{1}{E_p - E_i} = \frac{1}{G} + 2 \sum_{p=k+1}^{n} \frac{1}{E_p} \sum_{m=0}^{\infty} \left( \frac{E_i}{E_p} \right)^m. \quad (10)$$

The equations (9) and (10) are of course exact providing the above expansions converge.

We introduce next the statistical moments of the levels distribution according to

$$M^{(n)} = \sigma^n m^{(n)} = \frac{1}{\Omega} \sum_{\mu=1}^{L} \Omega_{\mu} \epsilon_{\mu}^{n} \quad (11)$$

(note that $m^{(1)} \propto \bar{\epsilon} = 0$, $m^{(2)} \equiv 1$ by definition and, of course, $m^{(0)} = 1$) and rewrite the unknown $E_i, E_j$ in terms of the new dimensionless variables $z_i, y_j$ as follows

$$E_i = 2\sigma z_i(\alpha) \quad \quad i = 1, \ldots, k \quad (12)$$
$$E_j = \frac{2\sigma}{\alpha} y_j(\alpha) \quad \quad j = k + 1, \ldots, n \quad (13)$$

Then eqs. (9) and (10) become, respectively,

$$\frac{1}{y_j} \sum_{m=0}^{\infty} \left( m^{(m)} - 2 \frac{\Omega}{\sigma} \sum_{p=1}^{k} z_p^m \right) \left( \frac{\alpha}{y_j} \right)^m + \frac{2}{\Omega} \sum_{p=k+1}^{n} \frac{1}{y_p - y_j} + 1 = 0 \quad (14)$$

and

$$\frac{1}{\Omega} \sum_{\mu=1}^{L} \frac{\Omega_{\mu}}{z_i - \sigma} - 2 \frac{\Omega}{\sigma} \sum_{p=1 \atop p \neq i}^{k} \frac{1}{z_p - \sigma} = \alpha + 2\alpha \sum_{m=0}^{\infty} \left( \sum_{p=k+1}^{n} \frac{1}{y_p^{m}} \right) (\alpha z_i)^m. \quad (15)$$

In Eqs. (12) and (13) $z_i$ and $y_j$ are assumed to be regular functions of $\alpha$ in some neighborhood of the origin, to have a finite, non-vanishing limit when $\alpha \to 0$ and to admit a Taylor expansion.

$^1$Note that the present definition differs from the one of Ref. 8 by the factor $\sigma^n$. 

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Eq. (14) can be conveniently rewritten as

\[ F_j \equiv \frac{1}{y_j} \sum_{m=0}^{\infty} \tilde{m}^{(m)}(\{z_i(\alpha)\}) \left( \frac{\alpha}{y_j} \right)^m + \frac{2}{\Omega} \sum_{p=k+1}^{n} \frac{1}{y_p - y_j} + 1 = 0, \tag{16} \]

having defined

\[ \tilde{m}^{(m)}(\{z_i(\alpha)\}) = m^{(m)} - \frac{2}{\Omega} \sum_{p=1}^{k} z_p^{m}, \tag{17} \]

a form explicitly displaying the renormalization of the moments of the level distribution induced by the dynamics of the trapped pairs.

From (16) then it clearly follows

\[ K_q = \sum_{j=k+1}^{n} y_j^2 \left\{ \frac{1}{y_j} \sum_{m=0}^{\infty} \tilde{m}^{(m)}(z_i) \left( \frac{\alpha}{y_j} \right)^m + \frac{2}{\Omega} \sum_{p=k+1}^{n} \frac{1}{y_p - y_j} + 1 \right\} = 0, \tag{18} \]

an expression which will turn out to be useful later on.

Before examining explicitly the expansion in powers of \( \alpha \) let us briefly discuss eqs. (14) and (15). We observe first of all that the case of 0 likeness (with no renormalization of the moments) coincides with the findings of Ref. [8] and is already a generalization of the case handled in ref. [3] since in (14) the moments of the level distribution are generic whereas in ref. [3] the choice \( \Omega_\mu = 1 \) is made, which is appropriate for a system of electrons, but not of nucleons. Moreover we shall show in the following that it is possible to write recursively (but not in a closed form!) the energy of the system associated with the untrapped pairs at a given order in \( \alpha \) and eq. (16) clearly shows that at a given order \( p \) only the first \( p \) moments of the level distribution contribute to this energy. At the leading order we expect of course to recover the result of the degenerate case, the first order is absent because \( m^{(1)} = 0 \) while the second is meaningful and so on.

Furthermore it turns out that the impact of the trapped pairs on the energy of the collective state (see the eq. (16)) amounts to a renormalization of the moments of the level distribution.

Likewise, for the trapped solutions at leading order a similar effect occurs. Indeed, at leading order (namely, \( m = 0 \) in eq. (15)) the collective component of the state (namely the untrapped energies) renormalizes the
coupling constant acting in the sector of the $k$ trapped solutions according to the replacement

$$\alpha \rightarrow \alpha + \frac{2\alpha}{\Omega} (n - k) .$$

(19)

Thus we surmise the following iterative procedure: we first solve eq. (14) at the leading order ($\alpha^{-1}$), next we determine the rhs of eq. (15) at the order $\alpha^0$ and solve the equation (eventually numerically), then we come back to eq. (14) and so on.

In the next section we shall deal at leading order (namely in the very large coupling limit) with the collective component of the state energy.

3 The collective sector

It is clear that when a collective state develops with a large binding energy, then $\sigma$ has to be quite small and, accordingly, the energy of the degenerate case, namely

$$E_{\text{degenerate}} = -G \left( n - \frac{v}{2} \right) \left( \Omega - n - \frac{v}{2} + 1 \right) ,$$

(20)

should be recovered. We want now to show that a similar formula (i.e., up to the replacement $v \rightarrow v_l$) holds at the leading order in the strong coupling expansion for the Gaudin excited states as well.

To this purpose we go back to eq. (14) keeping only the term $m = 0$. Expanding $y_i(\alpha)$ as follows

$$y_i(\alpha) = \sum_{h=0}^{\infty} \frac{1}{h!} \alpha^h y_i^{(h)} ,$$

(21)

to leading order, eq. (14) then reads

$$\left( 1 - \frac{v_l}{\Omega} \right) \frac{1}{y_j^{(0)}} + \frac{2}{\Omega} \sum_{m=k+1}^{n} \frac{1}{y_m^{(0)} - y_j^{(0)}} = -1 ,$$

(22)

where only the three quantities $v_l$, $n$ and $\Omega$ (expressed by integer numbers) appear, while the dependence upon the coupling constant is embedded in the rescaling of eq. (13).
The further rescaling
\[ y_j^{(0)} = \left( 1 - \frac{v_l}{\Omega} \right) \bar{y}_j \]  
leads to the system
\[ \frac{1}{y_j} + \frac{2}{\Omega - v_l} \sum_{m=k+1}^{n} \frac{1}{y_m - y_j} + 1 = 0, \]  
which, redefining \( \Omega \) and \( n \) according to the prescriptions
\[ \Omega \rightarrow \Omega + v_l \]  
\[ n \rightarrow n + \frac{v_l}{2}, \]  
can be recast as follows
\[ f_j \equiv \frac{1}{Y_j(\Omega; n)} + \frac{2}{\Omega} \sum_{m=1, m \neq j}^{n} \frac{1}{Y_m(\Omega; n) - Y_j(\Omega; n)} + 1 = 0, \]  
having set for later convenience \( \bar{y}_i \equiv Y_i(\Omega - v_l; n - v_l/2) \). Note that at leading order from (16) it follows the relation \( f_j = F_j(v_l = 0) \). Eqs. (27) represent the key ingredient in describing the dynamics of the strong coupling limit at the leading and at the higher order as well.

To further proceed consider the equation
\[ g_q \equiv \sum_{j=1}^{n} f_j Y_j^q = K_q (v_l = 0) = 0, \]  
whose properties are extensively described in appendix A. Here we first recall that, as found out in ref. [3], the solutions of (28) are given by the zeros of a Laguerre polynomial and that the sum \( \sum_{i=1}^{n} Y_i \) can be analytically expressed. For this purpose we specify eq. (28) to the case \( q = 1 \) and use (27). Thus the first term of the sum yields \( n \). Then the contributions to the sum stemming from the second term can be collected pairwise to get
\[ \frac{2}{\Omega} \frac{Y_i}{Y_k - Y_i} + \frac{2}{\Omega} \frac{Y_k}{Y_i - Y_k} = -\frac{2}{\Omega} \]  
\[ 8 \]
and since the number of such pairs is \( n(n-1)/2 \) they sum up to \(-n(n-1)/\Omega\).

Finally the third term yields the required quantity. Thus we end up with

\[
\sum_{i=1}^{n} Y_i = -\frac{n(\Omega - n + 1)}{\Omega}.
\]

The above sum fully determines the behaviour of all the energies of the collective pairs for any state of any system (i.e. with any \( n \) and \( v_l \)) in the strong coupling limit, yielding (we recall that \( v_l = 2k \))

\[
E_i = \frac{2\sigma}{\alpha} y_j^{(0)} = \frac{2\sigma}{\alpha} \left( 1 - \frac{v_l}{\Omega} \right) Y_i \left( \Omega - v_l; n - \frac{v_l}{2} \right).
\]

The total energy of the ground state is of course the sum of the \( E_i \) and owing to (30) it turns out to be

\[
E = \frac{2\sigma}{\alpha} \left( 1 - \frac{v_l}{\Omega} \right) \sum_{i=1}^{n-v_l/2} Y_i \left( \Omega - v_l; n - \frac{v_l}{2} \right)
\]

\[
= -G \left( n - \frac{v_l}{2} \right) \left( \Omega - n - \frac{v_l}{2} + 1 \right),
\]

coinciding with (20) up to the replacement \( v_l \to v \).

This result reflects the meaning of like-seniority. We are dealing in fact with \( n \) pairs all coupled to \( J = 0 \), hence with a zero seniority state, but the physics of the collective component of the state is not ruled by \( n \), but instead by those pairs that take part in the setting up of the collectivity, i.e., that are not trapped. The trapped pairs turn out to be irrelevant to the energy of the system at this order in \( \alpha \) and play essentially the same role of the broken pairs.

These are the pairs coupled to an angular momentum \( J \neq 0 \), which set up the seniority. They do not interact with the other ones and therefore are simply accounted for by

1. adding their unperturbed energies \( 2\epsilon_{\mu} \) to the total energy,

2. reducing each \( \Omega_{\mu} \) by one unit each time a pair coupled to \( J \neq 0 \) lives in the \( \mu^{th} \) level (blocking effect) providing the partners of the pair live on the same s.p.l.
We conclude that the dynamics of the collective component of the states of a system with \( n \) pairs, whatever the degree of collectivity might be, is ruled in the strong coupling limit by the equations (27), which is free of parameters, but for the integers \( \Omega \), and by the scaling law (31). Also worth recalling is that the Richardson’s equations for \( n \) pairs are based on the Bethe ansatz

\[
|\Psi_n\rangle = \prod_{k=1}^{n} \left( \sum_{\mu=1}^{L} \frac{C_k}{2\epsilon_\mu - E_k} \hat{A}_\mu^\dagger \right) |0\rangle ,
\]

where

\[
C_k = \frac{1}{\sqrt{\sum_{\mu=1}^{L} \Omega_\mu (2\epsilon_\mu - E_k)^2}}
\]

is a normalization factor and

\[
\hat{A}_\mu^\dagger = \sum_{m_\mu = -j_\mu}^{j_\mu} (-1)^{j_\mu - m_\mu} \hat{a}^\dagger_{j_\mu, m_\mu} \hat{a}^\dagger_{j_\mu, m_\mu}
\]

the quasi-spin operator. The Bethe ansatz represents an eigenstate of the pairing Hamiltonian if the parameters \( E_k \) fulfill the RE. As a consequence the scaling properties of the pair energies \( E_k \) above discussed entail analogous properties for the wave function of the system.

4 The trapped pairs: an example

In this Section we address the problem of computing the contribution of the energies of the trapped pairs to the total energy of the states. For these we have been unable to provide a strong coupling expansion, however we show that they fulfill a system of equations which, in leading order, decouples from the Richardson system for the untrapped pairs. Furthermore this system allows one to identify the unperturbed configuration from where each trapped contribution arises [13].

To see this we expand \( z_i(\alpha) \) as

\[
z_i(\alpha) = \sum_{h=0}^{\infty} \frac{1}{h!} z_i^{(h)} \alpha^h
\]
and using eq. (12) we can rewrite eqs. (15) at the leading order in the form

\[
\frac{1}{\Omega} \sum_{\mu=1}^{L} \frac{\Omega_{\mu}}{\sigma} - \frac{2}{\Omega} \sum_{\substack{p=1 \\ p \neq i}}^{k} \frac{1}{\zeta_{p}^{(0)} - \zeta_{i}^{(0)}} = 0 .
\]  

(37)

As above mentioned no closed form can be given for the solutions of (37), however the numerical solution now only concerns the \( k \) trapped pairs instead of the full set of \( n \) pairs.

We give here an example of how our approximation scheme works as compared with the exact solution by considering a schematic model of the lead isotope \(^{188}\text{Pb}\). In Table 1 we quote the experimental s.p.l. of the shell \( 5\hbar\omega \) and the associated energies taken from [14] (the zero of the energy is arbitrary). Observe that in the present case \( \bar{\epsilon} = -1.897 \text{ MeV} \), a quantity to be subtracted out from the single particle energies, and \( \sigma = 1.056 \text{ MeV} \).

We choose, as an example, an excited state by first switching off the interaction \( (\alpha \to \infty) \) and then by filling the two lowest levels (9 pairs) and placing two pairs in the level \( 3p_{3/2} \) and one in the \( 3p_{1/2} \).

In this case the system (37) contains only 3 equations. We solved it numerically getting (after the shift \( \epsilon_{i} \to \epsilon_{i} - \bar{\epsilon} \))

\[
\begin{align*}
\zeta_{1}^{(0)} & = 1.719 \\
\zeta_{2}^{(0)} & = 0.833 + 0.085i \\
\zeta_{3}^{(0)} & = 0.833 - 0.085i ,
\end{align*}
\]  

(38)

Table 1: Level structure of the highest neutron shell of lead

| \( 3p_{1/2} \) [2] | \( \Omega_{6} = 1 \) | \( \epsilon_{6} = 0 \) | \( \epsilon_{6} - \bar{\epsilon} = 1.897 \text{ MeV} \) |
| --- | --- | --- | --- |
| \( 2f_{5/2} \) [6] | \( \Omega_{5} = 3 \) | \( \epsilon_{5} = -0.57 \text{ MeV} \) | \( \epsilon_{5} - \bar{\epsilon} = 1.327 \text{ MeV} \) |
| \( 3p_{3/2} \) [4] | \( \Omega_{4} = 2 \) | \( \epsilon_{4} = -0.90 \text{ MeV} \) | \( \epsilon_{4} - \bar{\epsilon} = 0.997 \text{ MeV} \) |
| \( 1i_{13/2} \) [14] | \( \Omega_{3} = 7 \) | \( \epsilon_{3} = -1.64 \text{ MeV} \) | \( \epsilon_{3} - \bar{\epsilon} = 0.257 \text{ MeV} \) |
| \( 2f_{7/2} \) [8] | \( \Omega_{2} = 4 \) | \( \epsilon_{2} = -2.35 \text{ MeV} \) | \( \epsilon_{2} - \bar{\epsilon} = -0.453 \text{ MeV} \) |
| \( 1h_{9/2} \) [10] | \( \Omega_{1} = 5 \) | \( \epsilon_{1} = -3.47 \text{ MeV} \) | \( \epsilon_{1} - \bar{\epsilon} = -1.573 \text{ MeV} \) |
in turn yielding

\[ E_1^{(0)} = -0.164 \text{ MeV} \]
\[ E_2^{(0)} = (-2.035 + 0.179i) \text{ MeV} \]
\[ E_3^{(0)} = (-2.035 - 0.179i) \text{ MeV} \]

for the energies \( E_i \).

For later convenience we introduce also the quantity

\[ \zeta_q(\alpha) = \sum_{i=1}^{k} [z_i(\alpha)]^q \]

(40)

together with the expansion

\[ \zeta_q(\alpha) = \sum_{h=1}^{\infty} \frac{\alpha^h}{h!} \zeta^{(h)}_q . \]

(41)

For example in the present case

\[ \zeta_1^{(0)} = 3.384 . \]

(42)

5 The higher order corrections

In this section we use the iterative procedure previously discussed to get the higher order corrections for \( y_j(\alpha) \) and \( z_j(\alpha) \). These will be computed up to order \( \alpha^4 \).

We start, as in sec. 3, by examining the equation \( K_1 = 0 \) (see (18)), which reads

\[ \sum_{m=0}^{\infty} \tilde{m} \alpha^m \sum_{i=k+1}^{n} \frac{1}{y_i^m} - \frac{(n-k)(n-k-1)}{\Omega} + \sum_{i=k+1}^{n} y_i = 0 , \]

(43)

having again used the same procedure used in getting eq. (30). Next we expand in \( \alpha \). The 0th order is already known, while at first order, using the expansion (21), we get

\[ \sum_{i=k+1}^{n} y_i^{(1)} = \frac{2}{\Omega} \sum_{i=k+1}^{n} y_i^{(0)} \sum_{p=1}^{k} z_p^{(0)} = -\frac{2(n-k)}{\Omega - 2k} \zeta_1^{(0)} \]

(44)
that is sufficient for determining the correction to the total energy. In deriving the last expression use has been made of eq. (13) and (31) to connect the $Y_j$ with the $y_j$. Eq. (44) suggests to replace, as in (25) and (26), $\Omega$ and $n$ with

$$\tilde{\Omega} \equiv \Omega - 2k$$

$$\tilde{n} \equiv n - k.$$  \hspace{1cm} (45)

In term of these natural variables eq. (44) assumes the compact form

$$\sum_{i=k+1}^n y_i^{(1)} = -\frac{2\tilde{n}\zeta_1^{(0)}}{\tilde{\Omega}}.$$  \hspace{1cm} (47)

Thus the leading order correction to the state collective energy solely arises from the presence of the trapped pairs that merely renormalize $\bar{\epsilon}$.

For the higher order terms we need to solve the equations from $K_{-l+1} = 0$ to $K_1 = 0$: these fix the $l^{th}$ order. To lighten the notations we introduce the coefficient

$$C_p = \left(\frac{\Omega}{\Omega - 2k}\right)^{p+1} \frac{(n-k)(\Omega - n - k)}{\prod_{m=1}^p(\Omega - 2k - m)} \left(\frac{\tilde{\Omega}}{\Omega}\right)^{p+1} \frac{\tilde{n}(\tilde{\Omega} - \tilde{n})}{\prod_{m=1}^p(\Omega - m)}.$$  \hspace{1cm} (48)

We thus find

$$\sum_{i=k+1}^n y_i^{(2)} = -2C_1 \left\{ m^{(2)} - 2\frac{\zeta_2^{(0)}}{\Omega\zeta_2} - \frac{4}{\Omega\zeta_2} \left(\zeta_1^{(0)}\right)^2 \right\} - \frac{4\tilde{n}\zeta_1^{(1)}}{\Omega}$$  \hspace{1cm} (49)

$$\sum_{i=k+1}^n y_i^{(3)} = 6C_2(\Omega - 2n) \left\{ m^{(3)} - 2\frac{\zeta_3^{(0)}}{\Omega\zeta_3} + \frac{6}{\Omega\zeta_3} \left(\zeta_1^{(0)}\right) \right\} - \frac{16}{\Omega\zeta_2^2} \left[ \zeta_1^{(0)} \right]^3 + 12C_1 \frac{1}{\Omega} \left( \frac{\zeta_1^{(1)}}{\Omega} + \frac{4}{\Omega} \zeta_1^{(0)} \zeta_1^{(1)} \right) - \frac{6\tilde{n}\zeta_2^{(2)}}{\Omega}$$  \hspace{1cm} (50)

$$\sum_{i=k+1}^n y_i^{(4)} = -24C_3 \left\{ \tilde{\Omega}^2 - \frac{\tilde{n}(\tilde{\Omega} - \tilde{n})(5\tilde{\Omega} - 6)}{\Omega - 1} \right\} \times$$  \hspace{1cm} (51)

$$\times \left\{ m^{(4)} - 2\frac{\zeta_4^{(0)}}{\Omega\zeta_4} + \frac{8}{\Omega} \left[ m^{(3)} - 2\frac{\zeta_3^{(0)}}{\Omega\zeta_3} \right] \zeta_1^{(0)} \right\} - 192C_3 \left\{ 5 + \frac{5\tilde{n}(\tilde{n} - 6)}{\Omega} + 30 \frac{\tilde{n}^2}{\Omega^2} - \frac{(5\tilde{n} - 1)(\tilde{n} - 1)}{\Omega - 1} \right\}$$

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\[
-\frac{n(n-1)}{(\Omega-1)^2} \left\{ m^{(2)} - \frac{2\zeta^{(0)}}{\Omega}\right\} - \frac{2}{\Omega}\left[ \zeta^{(0)} \right]^2 - \frac{2}{\Omega} \left[ \zeta^{(0)} \right]^4 \right\}
\]

\[
+ 144C_3 \left( \frac{\tilde{\Omega} - 3}{\Omega} - 2n \right) m^{(2)} - \frac{2\zeta^{(0)}}{\Omega}\zeta_1^{(1)}
\]

\[
+ 24C_3 \tilde{\Omega} \left\{ 2\tilde{\Omega} - 9\tilde{n} - 1 + 12\tilde{n} - \frac{(3\tilde{n} - 1)(\tilde{n} - 1)}{\Omega - 1} \right\}
\]

\[
- \frac{n(n-1)}{(\Omega-1)^2} \left[ m^{(2)} - \frac{2\zeta^{(0)}}{\Omega}\right]^2
\]

\[
- 48C_2 \left( \frac{\Omega - 2n}{\Omega} \right) \left\{ \zeta_3^{(1)} + 6\frac{\zeta^{(0)}}{\Omega}\zeta_2^{(1)} + \frac{24}{\Omega^2}\zeta_1^{(1)} \left[ \zeta^{(0)} \right]^2 \right\}
\]

\[
+ 24C_1 \left\{ \frac{\Omega}{\Omega - 1} \right\} \left\{ \zeta_3^{(2)} + 4\frac{\zeta^{(0)}}{\Omega}\zeta_2^{(2)} + \frac{4}{\Omega}\left[ \zeta_1^{(1)} \right]^2 \right\} - \frac{8n}{\Omega} \zeta_1^{(3)}
\]

It is worth to point out the drastic simplification occurring for the unique state having all the pairs untrapped (vanishing like-seniority). Indeed in this case:

\[
\sum_{i=k+1}^{n} y_i^{(1)} \to 0 \quad (52)
\]

\[
\sum_{i=k+1}^{n} y_i^{(2)} \to -2\frac{n(\Omega - n)}{\Omega - 1} m^{(2)} \quad (53)
\]

\[
\sum_{i=k+1}^{n} y_i^{(3)} \to 6\frac{n(\Omega - n)(\Omega - 2n)}{(\Omega - 1)(\Omega - 2)} m^{(3)} \quad (54)
\]

\[
\sum_{i=k+1}^{n} y_i^{(4)} \to 24 \frac{n(\Omega - n)}{(\Omega - 2)(\Omega - 3)} \left\{ \left[ \frac{n(\Omega - n)(5\Omega - 6)}{(\Omega - 1)^2} - \frac{\Omega^2}{\Omega - 1} \right] m^{(4)} \right\}
\]

\[
+ \left[ 2\Omega + 1 + \frac{n(4 - 9(\Omega - n))}{\Omega - 1} - \frac{(4n - 1)(n - 1)}{(\Omega - 1)^2} \right. 
\]

\[
- \frac{n(n - 1)}{(\Omega - 1)^3} \right\} \left[ m^{(2)} \right]^2 \right\},
\]

which coincides with the findings of ref. [8].

Two comments are now in order. First in the above we have explicitly inserted the second moment of the level distribution, although its value is
1 by definition, in order to explicitly follow how the moments of the s.p.l. distribution are renormalized order by order. Next we recall that $m^{(3)}$ coincides with the skewness of the distribution and $m^{(4)}$ is linked to the kurtosis $c$ by the relation $c + 3 = m^{(4)}$. We thus see that the coefficients of the strong coupling expansion of the ground state energy reflect finer and finer details of the levels distribution as the order grows. The same occurs for the excited states, but here the connection is much more cumbersome.

Now we switch to the trapped states and we evaluate numerically, order by order, the unknown quantities $z^{(m)}_i$. We have already determined, in the previous section, the zero order $z^{(0)}_i$ by solving numerically eq. (37). At the next-to-leading order eq. (15) reads

$$\frac{1}{\Omega} \sum_{\mu=1}^{L} \left( \frac{\epsilon_{\mu}}{\sigma} - z^{(0)}_i \right)^2 + \sum_{p=1}^{k} \sum_{p \neq i} \left( \frac{z^{(1)}_p - z^{(1)}_i}{(z^{(0)}_p - z^{(0)}_i)^2} \right) = 1 + \frac{2}{\Omega} \sum_{j=k+1}^{m} n \frac{1}{y^{(0)}_j}$$

where in the second line use has been made of eq. (72) of Appendix B. The above is now a linear sistem. We solved it within our model, getting the results collected in Table 2, where also the terms up to the fourth order are reported.

In accord with the previous discussion, the 0-th order contribution to the $z_i$ relates to $G = \infty$ ($\alpha = 0$) and the higher order terms describe the evolution with $G$ of the trapped energies. For these the impact of the untrapped pairs is felt.

Concerning the range of validity of the expansion (36), it should be set by the critical values of $G$ (or $\alpha$) which are specific of each state of the pairing Hamiltonian.

6 Comparison with the exact results

In this Section we test the efficiency of the formalism previously developed by comparing its predictions with the exact results obtained by numerically solving the RE in the specific example of our toy model for $^{188}$Pb.

We also search for the range of values of the coupling constant $\alpha$ (or $G$) where our strong coupling expansion holds valid. This we do by discussing the analytic properties in $G$ of the solutions $E_i$ ($i = 1 \cdots n$) (and hence of the system’s total energy $E$) of the RE.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & $i = 1$ & $i = 2$ & $i = 3$ \\
\hline
$z_i^{(1)}$ & -0.015 & $-0.025 + 0.012i$ & $-0.025 - 0.012i$ \\
\hline
$z_i^{(2)}/2!$ & 0.181 & $0.178 + 0.081i$ & $0.178 - 0.081i$ \\
\hline
$z_i^{(3)}/3!$ & 0.124 & $0.076 + 0.013i$ & $0.076 - 0.013i$ \\
\hline
$z_i^{(4)}/4!$ & -0.747 & $-0.251 + 0.034i$ & $-0.251 - 0.034i$ \\
\hline
\end{tabular}
\caption{The results of the numerical calculation for the higher order coefficients $z_i^{(p)}$ in the expansion of $z_i$. The index $p$ refers to the order of the correction. The index $i$ runs over the three trapped pairs living in the $3p_{1/2}$ $(i = 1)$ and $3p_{3/2} (i = 2, 3)$ s.p.l.}
\end{table}

6.1 The singularities in $G$ of the pairs and of the total energies

It was discovered by Richardson [1] that some critical values of $G$ may exist where a level with pair degeneracy $\Omega_\mu$ tries to host $\Omega_\mu + 1$ pairs. These critical points necessarily appear in the case of the metals. In fact here for $G \to 0$ the pair energies $E_i$ tend to the unperturbed values $2\epsilon_i$, which are real. But to reach the collective state at high $G$ all the $E_i$, but the lowest one, must escape from the grid set up by the poles displayed by the RE, which are placed at the unperturbed single particle energies. As it is well-known, the escaping mechanism relates to the evolution with $G$ of the pair energies. Considering a specific $E_i$, note that it starts from the real value $2\epsilon_i$ at $G = 0$ and then merges with the lower neighbour solution $E_{i-1}$ at the energy $2\epsilon_{i-1}$ for a particular critical value of $G$. Beyond this critical point the two pair energies $E_i$ and $E_{i-1}$ become complex conjugate and their imaginary part enable them to overcome all the other obstacles to their evasion from the grid. This mechanism is shown in figs. 1 for a typical case of equally spaced unperturbed levels with unit pair degeneracy.

One would expect these critical values of $G$ to play a crucial role in determining the convergence domain of the strong coupling expansion for the system’s energy. Actually the situation turns out to be more involved since the singularities of the pair energies $E_i$ cancel out in the sum yielding the total energy of the system (see Ref. [8] for a discussion of this point).

Clearly the situations occurring in nuclear physics are drastically different from the metallic situation since the s.p.l. energies are different and must be
Figure 1: Evolution with $G$ of the real parts of the single particle energies for a system with equally spaced unperturbed levels with unit pair degeneracy, with 12 levels and with 4 pairs (left panel) and 5 pairs (right panel). $G$ and $E_i$ in arbitrary units.

examined case by case.

Sticking to our example of $^{188}$Pb we have drawn in fig. 2 the real parts of the exact solutions of the RE.

In this connection we remind that our model of $^{188}$Pb has 12 pairs in the $5\hbar\omega$ shell. For sake of illustration we consider of this nucleus the excited state with $N_G = 3$ (or $\nu_l = 6$). Hence 3 pairs remain trapped: they arise from the $3p_{3/2}$ and $3p_{1/2}$ levels. Of the remaining 9, 5 pairs arise from the $1h_{9/2}$ level and 4 from the $2f_{7/2}$ one. It is found that at very small $G$ the energies of the former are proportional to the fifth roots of the unity. Thus 4 of them are complex and the 5th is real. Since all of them decrease as $G$ increases, they reach their asymptotic values at high $G$ without encountering
Concerning the energies of the pairs stemming from the $2f_{7/2}$ level, they are two by two complex conjugate and hence not affected by the trapping mechanism. Thus no singularities in $G$ arise and one would expect the expansion in $\alpha$ for the collective part of the energy to converge everywhere. Note however that this analysis refers to $G$ positive and real.

To better illustrate the subtleties of the escaping mechanism we next consider another case (not realistic) by interchanging the levels $2f_{7/2}$ and $1h_{9/2}$, as shown in fig. 3. Now the lowest level may host only 4 pairs that could reach their asymptotic value with continuity, but the next one has 5
pairs and one of the associated $E_i$ has to be real. Thus it cannot escape the trapping, unless through a critical point, that must necessarily exist since the solutions of the RE in the strong coupling regime has 9 pairs taking part to the collectivity. Thus at this critical value this single solution must meet the four lower ones exactly at the lowest unperturbed level as fig. 3 indeed shows to happen (note that for $G > G_{cr}$ the four lowest solutions are accordingly complex and hence only two lines appear in our figure, hardly distinguishable, however, because they are very close to each other). One could be tempted to conclude that the validity of the power expansion in $\alpha$ ends at $\alpha = \frac{2 \sigma}{G_{cr} \Omega}$, but, as we shall see in the next Section, this is not so.
Actually precise statements about the domain of convergence of the strong coupling series are hard to make (see, however, Ref. [8]) and in fact each case should be separately examined. Concerning the existence of critical values of $G$, they depend crucially upon the occupation number of the levels.

Figure 4: The exact solution for the pairing energy of $^{188}Pb$ (solid line), compared with the order $G$ (dashed line), the order $G^0$ (dotted line), the order $G^{-1}$ (dash-dotted line), the order $G^{-2}$ (long-dashed line) and the order $G^{-3}$ (solid line again). The last three contributions are divergent at the origin.

6.2 Testing our approach

In this subsection we test our approach against the exact solution of the RE. We display in fig. 4 the exact result for the case of the excited state of $^{188}Pb$
and compare it with the expansion in powers of $\alpha$ up to the order $\alpha^3$ (or $G^{-3}$).

In the figure we have accounted for an overall energy shift since in our model $\bar{\epsilon} \neq 0$. First we observe that the order next to the leading is not vanishing owing to the interaction between the collective mode and the three trapped pairs, but the effect appears to be very small (indeed the two lines representing the $0^{th}$ and $1^{st}$ order are almost superimposed). Next it is seen that a very good accord between the RE exact solution and our approach is obtained up to $G \cong 0.35$: for lower values of $G$ it appears that higher order terms in the expansion are required.

However, and importantly, for $G \simeq 0.3$ the terms of the expansion diverge, thus possibly signalling the occurrence of a singularity (see Ref. [8]). Note that the highest order in the expansion is the most sensitive to the occurrence of this possible singularity.

This outcome might be related to the well-known result for the energy of a pair living in two levels, a case where obviously critical values of $G$ cannot exist, which reads

$$E = -d(\lambda + \sqrt{1 + \lambda^2}) = -G - \sqrt{d^2 + G^2},$$  \hspace{1cm} (57)$$

being $G = \lambda d$ and $d$ the distance between the two levels. Clearly (57) can be expanded only for $\lambda > 1$. Since in our model the average $d$ is also approximately 1, one may conclude that for the excited state of $^{188}\text{Pb}$ the strong coupling expansion holds valid till values of $G$ much smaller than in the case of one pair living in two levels (assuming the same $G$ and $d$ in both situations). Of course it should be kept in mind that we deal with an excited (and not the ground) state of $^{188}\text{Pb}$ where we have six (and not one) pairs contributing to the collective part of the energy and where the pair degeneracy of the s.p.l. is not one.

In conclusion while we cannot make a precise statement about the radius of convergence of the strong coupling series (each case requires to be separately examined) our results confirm that a singularity exists in the complex
$G$-plane, limiting the validity of the strong coupling expansion to values of $G > G^{\text{sing}}$, being $G^{\text{sing}}$ the smallest element of the set of all the singular points. Notably, $G^{\text{sing}}$ is unrelated to the $G^{\text{crit}}$, in fact existing even when there are no $G^{\text{crit}}$, and, furthermore, that our value of $G^{\text{sing}}$ appears to be in accord with the finding of Ref. [8].

7 Conclusions

In this paper the RE, which solve the pairing Hamiltonian problem for a system of $n$ pairs, are reduced, in the strong coupling limit, to a parameter-free set of equations (but for the total pair degeneracy of the s.p.l., $\Omega$), namely the set (27).

The eigenvalues of this system are obtained by solving an algebraic equation of order $n$, whose coefficients are explicitly given in eq. (63) of Appendix A.

Once the solutions of the system are known, the physical unknown $E_j$ are determined to leading order up to a rescaling. In the case of states of 0 like-seniority ($N_G = 0$) this rescaling is indeed what one would naturally expect, but the rescaling is quite more involved for states of finite $v_l$. Finally, and remarkably, the total energy of the state can be determined to leading order without solving explicitly the system.

Addressing the higher order corrections, we have explicitly derived their analytic expressions in the simpler case $v_l = 0$. We have shown, in accord with ref. [8], that at the order $p$ only $p$ parameters are involved in their determination, namely the first $p$ moments of the s.p.l. distribution. When $v_l \neq 0$ the trapped states renormalize dynamically these moments.

Concerning the numerical aspect of our approach, in the general case only one substantial calculation is required, the remaining steps to get the energies involving the solution of simple linear systems. Thus we point out that, although an exact analytic expression for the expansion of the whole energy of the excited states cannot be provided because of the coupling of the trapped and untrapped solutions, however order by order in our expansion the two systems of equations yielding the energies of the trapped and untrapped pairs can be decoupled. As a consequence on the one side the collective part of the energy of any state can indeed be expressed as an expansion and on the other this occurrence offers numerical advantages, especially when $n$ is large, and also a better insight on the nature of the excited states of the pairing.
Finally the convergence of the series is discussed. Its radius of convergence is set by a singularity lying in the complex plane of the coupling constant, whose exact location crucially depends upon the distribution of the levels. However the modulus of this singularity must be smaller than the lowest $G_{\text{crit}}$. Its physical meaning represents the minimum value of $G$ at which the single particle aspect of the problem can be treated as a perturbation.

### A Properties of the functions $Y_i(\Omega; n)$

We study the properties of the system (28):

$$
\sum_{i=1}^{n} [Y_i(\Omega; n)]^{n-1} + \frac{2}{\Omega} \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{[Y_i(\Omega; n)]^{n}}{Y_k(\Omega; n) - Y_i(\Omega; n)} + \sum_{i=1}^{n} [Y_i(\Omega; n)]^{n} = 0 .
$$

**Property I:** The system (28) is algebraic.

In fact for each term of the second sum, let it be $\frac{2}{\Omega} \frac{Y_i^m}{Y_k - Y_i}$ another term in the sum exists with the indices interchanged, namely $\frac{2}{\Omega} \frac{Y_k^m}{Y_i - Y_k}$ and their sum is clearly a polynomial. The whole second term in (28) is thus a symmetric function of order $m - 1$ of the variables $Y_i$.

**Property II:** each equation of the system (28) can be expressed in terms of the symmetric polynomials

$$
S_k(\Omega, n) = \sum_{p_1 < p_2 < \cdots < p_k} Y_{p_1} Y_{p_2} \cdots Y_{p_k} .
$$

(58)

**Property III:**

$$
\sum_{i=1}^{n} Y_i = -\frac{n(\Omega - n + 1)}{\Omega} .
$$

(59)

**Property IV:** if $n = \Omega - 1$ the system (28) admits the solution $Y_i = 0 \ \forall i$.

In fact only the equation $g_1$ contain a constant term that reads (see eq. (30)) $-n(\Omega - n + 1)/\Omega$. 

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**Property V:** if \( k \) solution coincide they vanish and further must be \( \Omega = k - 1 \); conversely if \( \Omega = k - 1 \) there exist \( k \) vanishing solutions.

To prove this we rewrite the system (27) labelling from 1 to \( k \) the vanishing \( Y_i \) and we separate the system \( f_i = 0 \) into two subsystems. The first reads

\[
\frac{1}{Y_i} + \frac{2}{\Omega} \sum_{m=1, m \neq i}^{k} \frac{1}{Y_m - Y_i} + \frac{2}{\Omega} \sum_{m=k+1}^{n} \frac{1}{Y_m} = -1, \quad i = 1, \ldots, k \tag{60}
\]

and the second is

\[
\frac{1 - \frac{2k}{\Omega}}{Y_j} + \frac{2}{\Omega} \sum_{p=k+1, p \neq j}^{n} \frac{1}{Y_m - Y_j} = -1, \quad j = k + 1, \ldots, n \tag{61}
\]

Imagine now we solve in some way, numerically for instance, the set (61). The quantity \( \sum_{m=k+1}^{n} \frac{1}{Y_m} \) in (60) will thus be a finite, known term. Rescaling then the \( Y_i \) in (60) according to

\[
Y_i = \frac{\tilde{Y}_i}{1 + \frac{2}{\Omega} \sum_{m=k+1}^{n} \frac{1}{Y_m}} \tag{62}
\]

the equations for the \( \tilde{Y}_i \) will now keep exactly the form (27), but we shall have to deal with only \( k \) of them. Thus repeating the derivation of property V we conclude that these (and consequently the \( Y_i \)) can vanish only if

\( \Omega = k - 1 \).

**Property VI:** the solutions \( Y_i \) of the system (28) are the roots of the equation (in \( x \))

\[
\sum_{p=0}^{n} \binom{n}{p} (\frac{\Omega - n + 1}{\Omega})_p x^{n-p} = 0. \tag{63}
\]

In fact we know that solving the system (28) amounts to find the roots of the algebraic equation

\[
\sum_{p=0}^{n} (-1)^p S_p x^{n-p} = 0 \tag{64}
\]
with the $S_p$ defined by (58) and $S_0 = 1$. Owing to Property II each $g_m$ can be expressed in terms of the $S_p$ with $p = 1, \ldots, m$ and $S_m$ is contained linearly. Thus $g_1$ can only contain linearly $S_1$, that is immediately determined, and all the other $S_k$ can be obtained recursively by solving first order equations, thus getting the $S_p$ as functions of $\Omega$ and $n$.

Further, it is easily seen by induction that

$$S_p = \frac{P_p(\Omega)}{\Omega^p} \quad (65)$$

where $P_p$ is a polynomial (to be determined later) of order $p$ in $\Omega$.

To further determine $P_p(\Omega)$, property VI tells us that if $\Omega = k - 1$ then $k$ solutions are vanishing and thus the first $k$ coefficients of the equation (64) must vanish.

For instance for $\Omega = n - 1$ the equation must have the form $x^n = 0$, entailing the vanishing of all the polynomial. Thus all of them must factorize a term $(\Omega - n + 1)$. This fixes $P_1$ up to a constant. At $\Omega = n - 2$ all the polynomial but the first must vanish in order to have $n - 1$ vanishing solutions, and so on. Thus they ultimately take the form

$$P_k(\Omega) = t^n_k (\Omega - n + 1)(\Omega - n + 2) \ldots (\Omega - n + k) = (\Omega - k + 1)_k \quad (66)$$

(having introduced the Pochhammer symbol). In the above the $t^n_k$ are numerical (rational) coefficients that can depend upon $n$, but not upon $\Omega$.

To fix them we exploit their independence from $\Omega$ and take the limit $\Omega \to \infty$. Then from (65) and (66) it follows

$$S_k(\Omega \to \infty, n) = t^n_k . \quad (67)$$

On the other hand in this limit the system (64) is trivially solved, because the Pauli terms vanish, and yields $Y_i(\Omega \to \infty, n) = -1$. Thus in this limit eq. (64) becomes

$$\sum_{p=0}^{n} (-1)^p t^n_p x^{n-p} = (x + 1)^n = \sum_{p=0}^{n} {n \choose p} x^{n-k} = 0 , \quad (68)$$

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that immediately provides

\[ t_p^n = (-1)^p \binom{n}{p} . \tag{69} \]

Thus (64) can be rewritten in compact form as

\[ \sum_{p=0}^{n} \binom{n}{p} \frac{(\Omega - n + 1)p}{\Omega^p} \Delta^{n-p} = 0 . \]

This completes the proof of the property.

**Property VII**: the functions \( Y_i(\Omega; n) \) display branch points for the integers \( \Omega = 1, 2, \ldots, n - 1 \) and eventually a pole for \( \Omega = 0 \). Near \( \Omega = k - 1 \) they behaves like \( Y_i(\Omega; k - 1) \propto \sqrt{\Omega} \). If we put \( \Omega = k - 1 + \epsilon k \) with \( k \to 0 \) we find that the solutions \( Y_i \) have the behaviour

\[ Y_i(k - 1 + \epsilon k; n) = \sum_{p=1}^{k} C_p^{(k)} \left[ e_{i}^{(k)} \epsilon \right]^p \tag{70} \]

where the \( e_{i}^{(k)} \) are the roots of the unity, namely

\[ e_{i}^{(k)} = e^{2i\pi m/k} . \tag{71} \]

**B Useful relations**

We display here a list of properties of the functions \( y_i(\alpha = 0) \) derived from the equations

\[ g_m = 0 \]

(with \( m \), if it is the case, \( < 0 \)). We recall that the functions \( y_i \) – with like-seniority (Gaudin number) \( \neq 0 \) – are obtained from

\[ y_i = \left( 1 - \frac{2k}{\Omega} \right) Y_i(\Omega - 2k; n - k) . \]

They read

\[ \sum_{i=1}^{n} \frac{1}{Y_i(\Omega; n)} = -n \tag{72} \]
\[
\sum_{i=1}^{n} \frac{1}{Y_{i}^{2}(\Omega; n)} = \frac{n(\Omega - n)}{\Omega - 1} \tag{73}
\]
\[
\sum_{i=1}^{n} \frac{1}{Y_{i}^{3}(\Omega; n)} = \frac{n(\Omega - n)(\Omega - 2n)}{(\Omega - 1)(\Omega - 2)} \tag{74}
\]
\[
\sum_{i=1}^{n} \frac{1}{Y_{i}^{4}(\Omega; n)} = \frac{n(\Omega - n)}{(\Omega - 1)(\Omega - 2)(\Omega - 3)} \left[ \Omega^{2} - \frac{n(\Omega - n)(6\Omega - 5n)}{\Omega - 1} \right] \tag{75}
\]
\[
\sum_{i=1}^{n} Y_{i}(\Omega; n) = -\frac{n(\Omega - n + 1)}{\Omega} \tag{76}
\]
\[
\sum_{i=1}^{n} Y_{i}^{2}(\Omega; n) = -\frac{n(\Omega - 2n + 2)(\Omega - n + 1)}{\Omega^{2}} \tag{77}
\]
\[
\sum_{i=1}^{n} Y_{i}^{3}(\Omega; n) = -\frac{1}{\Omega^{3}} n(\Omega - n + 1) \times \left[ 6 - 11n + 5n^{2} + \Omega(5 - 5n + \Omega) \right] \tag{78}
\]

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