LOSS OF CONTINUITY OF THE SOLUTION MAP FOR THE
EULER EQUATIONS IN $\alpha$-MODULATION AND HÖLDER
SPACES

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Abstract. We study the incompressible Euler equations in the $\alpha$-modulation
$M_{p,q}^{s,\alpha}$ and Hölder $C^{1+\sigma}$ spaces on the plane. We show that for these spaces
the associated data-to-solution map is not continuous on bounded sets.

1. Introduction

In this paper we study the Cauchy problem for the non-periodic Euler equations
of incompressible hydrodynamics
\[
\begin{align*}
\begin{aligned}
&u_t + u \cdot \nabla u + \nabla p = 0, & t \geq 0, \quad x \in \mathbb{R}^n \\
&\text{div } u = 0 \\
&u(0) = u_0
\end{aligned}
\end{align*}
\]
(E)

with initial data in the $\alpha$-modulation spaces. In particular, our results apply to
the Besov spaces including the classical Hölder-Zygmund spaces. According to the
standard notion of well-posedness due to Hadamard a Cauchy problem is said to be
locally (in time) well-posed in a Banach space $X$ if given any initial data $u_0$ in $X$
there is a time $T > 0$ and a unique solution $u$ in a Banach space $Y \subset C([0,T),X)$
which depends continuously on the initial data. Otherwise the Cauchy problem is
said to be locally ill-posed.

Ill-posedness results establishing loss of continuity of the solution map $u_0 \rightarrow u$
for the Euler equations in the $C^1$ space and the borderline Besov space $B_{\infty,1}^1$
have been proved recently in [18]. Here we refine the techniques of that paper to obtain
ill-posedness results of this type in $\alpha$-modulation spaces $M_{p,q}^{s,\alpha}$ for $1 < s < 2$ and in
$C^{1+\sigma}$ for $0 < \sigma < 1$. More precisely, following the approach of Bourgain and Li [3]
we construct a Lagrangian flow with a large gradient and then choose a suitable
high-frequency perturbation of the initial vorticity to show that the assumption of
continuity of the solution map $u_0 \rightarrow u$ in the above spaces necessarily leads to a
contradiction with the results of Kato and Ponce [13, 14]. We will work with the
vorticity equations which in two dimensions assume the form
\[
\begin{align*}
\begin{aligned}
&\omega_t + u \cdot \nabla \omega = 0, & t \geq 0, \quad x \in \mathbb{R}^2 \\
&u = K \ast \omega = \nabla \perp \Delta^{-1} \omega \\
&\omega(0) = \omega_0
\end{aligned}
\end{align*}
\]  
(1.1)
where
\[
K(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2} \quad \text{and} \quad \nabla^\perp = \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right)
\]
denote the Biot-Savart kernel and the symplectic gradient, respectively.

The first rigorous results on the Cauchy problem for the incompressible Euler equations go back to Gyunter [10], Lichtenstein [15] and Wollibner [23]. A survey of those and numerous further results can be found for example in Majda and Bertozzi [16], Constantin [5] or Bahouri, Chemin and Danchin [1]. For the most recent progress on local ill-posedness in borderline spaces such as \(C^1\), \(W^{n/p+1,p}\), \(B^{n/p+1}_{p,q}\) as well as \(C^k\), \(C^{k-1,1}\) for integer \(k \geq 1\) we refer to the papers of Bourgain and Li [3, 4], Elgindi and Masmoudi [8] and the authors [18]. For earlier results in \(C^\sigma\) with \(0 < \sigma < 1\), \(B^s_{p,\infty}\) for \(s > 0\), \(p > 2\) and \(s > n(2/p - 1)\), \(1 \leq p \leq 2\) or the logarithmic Lipschitz spaces \(\log \text{Lip}^\alpha\) for \(0 < \alpha < 1\) we refer to Bardos and Titi [2], Cheskidov and Shvydkoy [6] and the authors [17].

Our main goal is to prove the following result.

**Main Theorem.** Let \(0 < \sigma < 1\), \(0 < \alpha \leq 1\), \(2 \leq p \leq \infty\), \(1 \leq q \leq \infty\) and suppose that \(M^{1+\sigma,\alpha}_{p,q}\) is continuously embedded in \(C^1\). Then the solution map of the incompressible Euler equations (E) is not continuous on bounded subsets of \(M^{1+\sigma,\alpha}_{p,q}\).

Thus, the Euler equations are in general locally ill-posed in \(\alpha\)-modulation spaces in the sense of Hadamard given above. For the definition of the \(\alpha\)-modulation spaces see Section 2 below.

**Remark 1.** Observe that \(M^{1,\infty}_{p,q}\) coincides with the usual Besov space \(B^s_{p,q}\). Therefore, somewhat surprisingly, Theorem 1 also yields ill-posedness (in the sense that the data-to-solution map loses its continuity properties) even in the classical Hölder spaces \(B^{1+\sigma}_{\infty,\infty} = C^{1+\sigma}\) for \(0 < \sigma < 1\).

Continuous dependence results for the Euler equations (in the strong topology) have been obtained for initial data in Sobolev spaces \(H^s\) and more generally \(W^{s,p}\) with \(s > n/p + 1\) for example in Ebin and Marsden [8], Kato and Lai [12] and Kato and Ponce [14]. However, this is a rather difficult part of the local well-posedness theory which has not yet been satisfactorily resolved.

**Remark 2.** A different mechanism involving a gradual loss of regularity of the solution map is described by Morgulis, Shnirelman and Yudovich [19].

**Remark 3.** In this context it is also worth pointing out that neither for the critical Besov space \(B^{1}_{\infty,1}(\mathbb{R}^n)\) nor for the space \(B^{1+p/n}_{p,1}(\mathbb{R}^n)\) are the Euler equations strongly ill-posed in the sense of Bourgain and Li [3]. This can be seen by examining the arguments given in Pak and Park [20] and Vishik [22].

Our general strategy will be similar to that employed in [18] which we will use as the main reference. The remainder of the paper is organized as follows. In Section 2 we describe the general set up and prove several technical lemmas. The whole of Section 3 is then devoted to the proof of Theorem 1. Although the constructions in Sections 2 and 3 will be carried out in 2D they can be readily adapted to the 3D case. Rather than doing that in Section 4 we give a direct proof of ill-posedness in \(C^{1+s}\) by using a 3D shear flow argument.

\[\text{For example, we have } M^{1+\sigma,\alpha}_{\infty,q} \subset C^1 \text{ whenever } \sigma > 2(1-\alpha)(1-1/q).\]
2. Basic Setup: Vorticity and Lagrangian Flow

We first recall the definition of $\alpha$-modulation spaces. For a more detailed account the reader is referred for example to [11]. A countable set $Q$ of subsets $Q \subset \mathbb{R}^n$ is called an admissible covering if $\mathbb{R}^n = \bigcup_{Q \in Q} Q$ and if there is $n_0 < \infty$ such that 
\[# \{Q' \in Q : Q \cap Q' \neq \emptyset \} \leq n_0 \text{ for all } Q \in Q. \#
\]
Let
\[
\begin{align*}
r_Q &= \sup\{r \in \mathbb{R} : B(c_r, r) \subset Q, c_r \in \mathbb{R}^n\} \\
R_Q &= \inf\{R \in \mathbb{R} : Q \subset B(c_R, R), c_R \in \mathbb{R}^n\}.
\end{align*}
\]
Given $0 \leq \alpha \leq 1$, an admissible covering is an $\alpha$-covering of $\mathbb{R}^n$ if $|Q| \sim (1 + |x|^2)^{\alpha n/2}$ (uniformly) for all $Q \in Q$ and all $x \in Q$ and where $\sup_{Q \in Q} R_Q/r_Q \leq K$ for some $K < \infty$. Let $Q$ be an $\alpha$-covering of $\mathbb{R}^n$. A bounded admissible partition of unity of order $p$ (abbreviated $p$-BAPU) corresponding to $Q$ is a family of smooth functions $\{\psi_Q\}_{Q \in Q}$ satisfying
\[
\psi_Q : \mathbb{R}^n \to [0, 1], \quad \text{supp}\ \psi_Q \subset Q,
\]
\[
\sum_{Q \in Q} \psi_Q(\xi) = 1, \quad \xi \in \mathbb{R}^n,
\]
\[
\sup_{Q \in Q} |Q|^{1/p - 1} \|\mathcal{F}^{-1}\psi_Q\|_{L^p} < \infty
\]
where $\mathcal{F}$ denotes the Fourier transform.

For any $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $0 \leq \alpha \leq 1$ the $\alpha$-modulation space $M_{s,\alpha}^p(\mathbb{R}^n)$ is the space of all tempered distributions $f$ for which the following norm
\[
\|f\|_{M_{s,\alpha}^p} = \left\{ \begin{array}{ll}
\left( \sum_{Q \in Q} (1 + |\xi_Q|^2)^{qs/2} \|\mathcal{F}^{-1}\psi_Q\mathcal{F}f\|_{L^p(\mathbb{R}^n)}^q \right)^{1/q} & \text{if } 1 \leq q < \infty \\
\sup_{Q \in Q} (1 + |\xi_Q|^2)^{qs/2} \|\mathcal{F}^{-1}\psi_Q\mathcal{F}f\|_{L^p} & \text{if } q = \infty
\end{array} \right.
\]
is finite, where $\{\xi_Q \in Q : Q \in Q\}$ is an arbitrary sequence. One shows that this definition is independent of an $\alpha$-covering $Q$ and of $p$-BAPU.

The following embedding results for $\alpha$-modulation spaces are known. Suppose that $\alpha_1 < \alpha_2 < 1$ and $1 \leq p \leq \infty$. Then
\[
M_{s,\alpha_1}^p \subset M_{s,\alpha_2}^p
\]
and, in particular, for any $\alpha < 1$ and $1 \leq p \leq \infty$ we have
\[
M_{s,\alpha}^p \subset B_{s,1}^p
\]
The proofs of these results can be found e.g. in [11]; see Thm. 4.1 and Thm. 4.2.

We next proceed to choose the initial vorticity $\omega_0$ in (1.1). Given any radial bump function $0 \leq \varphi \leq 1$ with support in $B(0, 1/4)$ define
\[
\varphi_0(x_1, x_2) = \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \varphi(x_1 - \varepsilon_1, x_2 - \varepsilon_2).
\]
Fix a positive integer $N_0 \in \mathbb{Z}_+$ and for any $M \gg 1$ let
\[
\omega_0(x) = M^{-2} N^{-\frac{1}{2}} \sum_{N_0 \leq k \leq N_0 + N} \varphi_k(x),
\]
where $1 < p < \infty$, $N = 1, 2, 3, \ldots$ and where
\[
\varphi_k(x) = 2^{(-1 + \frac{2}{p})k} \varphi_0(2^k x).
\]
Clearly, the function \( \varphi_0 \) is odd in both variables \( x_1, x_2 \) and for any \( k \geq 1 \) the supports of \( \varphi_k \) are disjoint and contained in a bounded set
\[
\text{supp } \varphi_k \subset \bigcup_{\varepsilon_1, \varepsilon_2 = \pm 1} B((\varepsilon_1 2^{-k}, \varepsilon_2 2^{-k}), 2^{-(k+2)}).
\]
It is easy to check that \( \omega_0 \in W^{1,r}(\mathbb{R}^2) \cap C_c^\infty(\mathbb{R}^2) \). In fact, we have

**Lemma 4.** For any \( 2 < r < \infty \) and any positive integer \( N \), we have
\[
\| \omega_0 \|_{W^{1,r}} \lesssim c M^{-2}
\]

**Proof.** A straightforward calculation is omitted. \( \square \)

Let \( u = \nabla^\perp \Delta^{-1} \omega \in W^{2,r} \cap C^\infty \) be the associated velocity field and consider its Lagrangian flow \( \eta(t) \), i.e., the solution of the initial value problem
\[
\frac{d\eta}{dt}(t, x) = u(t, \eta(t, x))
\]
\( \eta(0, x) = x. \)

It can be checked that \( \eta(t) \) is smooth and preserves the coordinate axes \( x_1, x_2 \) as well as the symmetries of the initial vorticity \( \omega_0 \) in (2.2). In fact, the flow is hyperbolic near the origin (a stagnation point) and we have the following

**Proposition 5.** Given \( M \gg 1 \) we have
\[
\sup_{0 \leq t \leq M^{-3}} \| D\eta(t) \|_\infty \geq M
\]
for any sufficiently large integer \( N > 0 \) in (2.2) and any \( 2 < r < \infty \) sufficiently close to 2.

**Proof.** See [18]; Prop. 6. \( \square \)

In order to proceed we need the following simple comparison result for the derivatives of solutions of the Lagrangian flow equations.

**Lemma 6.** Let \( u \) and \( v \) be smooth divergence-free vector fields on \( \mathbb{R}^2 \). If \( \eta \) and \( \tilde{\eta} \) are the solutions of (2.5) corresponding to \( u \) and \( u + v \) respectively, then
\[
\sup_{0 \leq t \leq 1} \sum_{i=0,1} \| D^i \eta(t) \|_\infty - \| D^i \tilde{\eta}(t) \|_\infty \leq C \sup_{0 \leq t \leq 1} \sum_{i=0,1} \| D^i v(t) \|_\infty
\]
for some \( C > 0 \) depending only on the \( L^\infty \) norms of \( u \) and its derivatives.

**Proof.** Follows at once by applying Gronwall’s inequality to the equation satisfied by the difference \( \eta(t) - \tilde{\eta}(t) \). \( \square \)

3. **The Proof of the main Theorem**

As in the previous section let \( \omega(t) \in W^{1,r} \cap C^\infty \) be the solution of the vorticity equations (1.1) with the initial condition (2.2) and let \( \eta(t) \) be the Lagrangian flow of the velocity field \( u = \nabla^\perp \Delta^{-1} \omega \) as above. Our main goal in this section will be to prove

**Theorem 7.** Let \( r > 2 \). Assume that the incompressible Euler equations are well-posed in the \( \alpha \)-modulation space \( M^{1+\sigma,\alpha}_{p,q}(\mathbb{R}^2) \) for any \( p \geq 2, q \geq 1, 0 < \alpha \leq 1 \) and \( 0 < \sigma < 1 \) in the sense of Hadamard. Moreover, assume that \( M^{1+\sigma,\alpha}_{p,q} \) is topologically embedded in \( C^1(\mathbb{R}^2) \). Then there exist a \( T > 0 \) and a sequence \( \omega_{0,n} \) in \( C^\infty_c \) with the following properties.
1. There is a constant $C > 0$ such that $\|\omega_{0,n}\|_{W^{1,r}} \leq C$ for sufficiently large positive integers $n$.
2. For any $M \gg 1$ there is $0 < t_0 \leq T$ such that $\|\omega_n(t_0)\|_{W^{1,r}} > M^{1/3}$ for sufficiently large $n$ and for all $r$ sufficiently close to 2.

Since Hadamard's notion entails continuity of the data-to-solution map we deduce from Theorem 4 that continuity cannot hold in $M_{p,q}^{1+s,0}(\mathbb{R}^2)$ or else we get a contradiction with the following result.

**Theorem (Kato-Ponce [13]).** Let $1 < r < \infty$ and $s > 1 + 2/r$. For any $\omega_0 \in W^{s-1,r}(\mathbb{R}^2)$ and any $T > 0$ there exists a constant $K = K(T, \|\omega_0\|_{W^{s-1,r}}) > 0$ such that

$$\sup_{0 \leq t \leq T} \|\omega(t)\|_{W^{s-1,r}} \leq K.$$  

Our Main Theorem will be a direct consequence of Theorem 7.

**Proof of Theorem 7.** Given any large number $M \gg 1$ pick $T \leq M^{-3}$. Observe that if $\|\omega_n(t_0)\|_{W^{1,r}} > M^{1/3}$ for some $0 < t_0 \leq M^{-3}$ then there is nothing to prove and therefore we may assume that

$$\|\omega(t)\|_{W^{1,r}} \leq M^{1/3}, \quad 0 \leq t \leq M^{-3}. \quad (3.1)$$

Next, by Proposition 5 we can find $0 \leq t_0 \leq M^{-3}$ and a point $x^* = (x_1^*, x_2^*)$ in $\mathbb{R}^2$ for which the absolute value of one of the entries in the Jacobi matrix $D\eta(t_0, x^*)$ is at least as large as $M$. Because the velocity field $u$ is in $W^{2,r}$ so is the associated Lagrangian flow $\hat{\chi}$ and hence by continuity in some sufficiently small $\delta$-neighbourhood of $x^*$ we have e.g.,

$$|\frac{\partial \eta_2}{\partial x_2}(t_0, x)| > M \quad \text{for all } |x - x^*| < \delta. \quad (3.2)$$

We proceed to construct a sequence of high-frequency perturbations of the initial vorticity in $W^{1,r}$. To this end we choose a smooth bump function $0 \leq \tilde{\chi} \leq 1$ with compact support in the unit ball $B(0,1)$ in the Fourier space and normalized by $\int_{\mathbb{R}^2} \tilde{\chi}(\xi) \, d\xi = 1$. Using this function we set

$$\hat{\rho}(\xi) = \tilde{\chi}(\xi - \xi_0) + \tilde{\chi}(\xi + \xi_0), \quad \xi \in \mathbb{R}^2, \quad \xi_0 = (2,0) \quad (3.3)$$

so that $\text{supp} \, \hat{\rho} \subset B(-\xi_0,1) \cup B(\xi_0,1)$ with

$$\rho(0) = \int_{\mathbb{R}^2} \hat{\rho}(\xi) \, d\xi = 2 \quad (3.4)$$

and observe that for any $a > 4$ we have

$$\text{supp} \, \hat{\rho}(\cdot \pm a, \cdot) \cap B(0,1) = \emptyset. \quad (3.5)$$

Define

$$\hat{\rho}_{k, \lambda}^{\tilde{\alpha}, \tau}(x) = \frac{\lambda^{-1+\frac{\tau}{k-1}}}{k!} \sum_{\xi_1, \xi_2 = \pm 1} \varepsilon_1 \varepsilon_2 \rho(\lambda(x - x_1^*)) \sin k\xi_1, \quad k \in \mathbb{Z}^+, \quad \lambda > 0 \quad (3.6)$$

where $x_1^* = (\varepsilon_1 x_1^*, \varepsilon_2 x_2^*)$ and $\lambda > 0$ and $0 < \tilde{\alpha} < 1$ will be further specified below.

\[2\text{E.g., by the wellposedness theory of [14].}\]
Remark 8. Note that the parameter $\lambda$ in (3.6) relates to the speed with which the support of the function $\rho$ is spreading in the Fourier space while $k$ expresses the speed of its translation. In the standard modulation space $M^{1+\sigma,0}(\mathbb{R}^2)$ one would need to set $\lambda = 1$ but in that case the spreading speed of the support of $\rho$ (its shrinking speed in physical space) is zero and hence the arguments we apply in the present paper break down. Therefore, the case of the standard modulation space $M^{1+\sigma,0}(\mathbb{R}^2)$ remains an open problem.

Before defining a suitable perturbation of $\omega_0$ we need to derive several estimates for $\tilde{\beta}^r_{k,\lambda}$ which we collect in the following lemma.

Lemma 9. Let $2 \leq p \leq \infty$, $2 < r < \infty$ and $0 < \sigma < 1$. For any $k \in \mathbb{Z}^+$ and $\lambda > 0$ sufficiently large, we have

1. $\|\tilde{\beta}^r_{k,\lambda}\|_{W^{1,r}} \lesssim k^{-1+\alpha} + k^{\delta} \lambda^{-1}$

2. $\|\Delta^{1/2+\sigma} \partial_j \Delta^{-1} \hat{\beta}^r_{k,\lambda}\|_{L^p} \lesssim k^{-1+\alpha} \lambda^{-1+2(1/r-1/p)}(\lambda^{\sigma} + k^{\sigma})$

3. $\|\partial_j \Delta^{-1} \hat{\beta}^r_{k,\lambda}\|_{L^p} \lesssim k^{-1+\alpha} \lambda^{-1+2(1/r-1/p)}$

where $j = 1, 2$.

Proof. We need to compute the $L^r$ norms of $\tilde{\beta}^r_{k,\lambda}$ and its first derivative $\partial_1 \tilde{\beta}^r_{k,\lambda}$. By the triangle inequality and the fact that $\tilde{\rho}$ has compact support we have

$$\|\tilde{\beta}^r_{k,\lambda}\|_{L^r} \lesssim k^{-1+\alpha} \lambda^{-1} \sum_{\varepsilon_1, \varepsilon_2} \left( \int_{\mathbb{R}^2} |\rho(\lambda(x - x^\varepsilon))| \lambda^2 \, dx \right)^{1/r} \lesssim k^{-1+\alpha} \lambda^{-1}.$$ 

For the first derivatives, we have

$$\left\| \frac{\partial \tilde{\beta}^r_{k,\lambda}}{\partial x_1} \right\|_{L^r} \lesssim k^{-1+\alpha} \lambda^{2/r} \sum_{\varepsilon_1, \varepsilon_2} \left\| \frac{\partial \rho}{\partial x_1}(\lambda(\cdot - x^\varepsilon)) \right\|_{L^r} + k^{\delta} \lambda^{-1+2/r} \sum_{\varepsilon_1, \varepsilon_2} \left\| \rho(\lambda(\cdot - x^\varepsilon)) \right\|_{L^r} \approx k^{-1+\alpha} \left\| \frac{\partial \rho}{\partial x_1} \right\|_{L^r} + k^{\delta} \lambda^{-1} \lesssim k^{-1+\alpha} + k^{\delta} \lambda^{-1}$$

and similarly

$$\left\| \frac{\partial \tilde{\beta}^r_{k,\lambda}}{\partial x_2} \right\|_{L^r} \lesssim k^{-1+\alpha} \left\| \frac{\partial \rho}{\partial x_2} \right\|_{L^r} \lesssim k^{-1+\alpha}.$$ 

Combining these estimates gives the bound for $\|\tilde{\beta}^r_{k,\lambda}\|_{W^{1,r}}$.

In order to derive the estimates in the remaining cases it will be convenient to use the Fourier transform

$$\tilde{\beta}^r_{k,\lambda}(\xi) = \frac{\lambda^{-1+2/r}}{k^{\sigma} \alpha} \sum_{\varepsilon_1, \varepsilon_2=\pm 1} \sum_{j=1,2} \xi_1 \xi_2 \xi_j^{j+1} \frac{\hat{\rho}(\lambda^{-1} \xi_j)}{2\pi \lambda^2} e^{-2\pi i (\xi_1 x_1^{\varepsilon_1} + \xi_2 x_2^{\varepsilon_2})}$$

where $\xi_j = (\xi_1 + \frac{(1-j)^1}{2\pi} k, \xi_2)$. Let $p'$ be the conjugate exponent to $p$. Applying the Hausdorff-Young inequality we obtain

$$\|\Delta^{1+\sigma} \partial_j \Delta^{-1} \hat{\beta}^r_{k,\lambda}\|_{L^p} \lesssim \left\| \left| \xi^{p'} \hat{\beta}^r_{k,\lambda} \right| \right\|_{L^{p'}} \lesssim k^{-1+\alpha} \lambda^{-1+2/r} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \lambda^{-2p'} |\xi|^{p'} |\hat{\rho}(\lambda^{-1} \xi_j)| \, d\xi \right)^{1/p'}$$
and changing the variables we further estimate by

\[ \lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{\tilde{\alpha}}{2}} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \left( \left( \gamma_1 - \frac{(1-j)}{2\pi} k \right)^2 + \eta_2^2 \right)^{\frac{\sigma}{p'}} |\hat{\rho}(\gamma)|^{p'} d\gamma \right) \]

\[ \lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{\tilde{\alpha}}{2}} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \left( \left( \lambda_1 - \frac{(1-j)}{2\pi} k \right)^2 + (\lambda_2)^2 \right)^{\frac{\sigma}{p'}} |\hat{\rho}(\gamma)|^{p'} d\gamma \right) \]

\[ \lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+2(\frac{1}{p'}-\frac{1}{p})} \]

for any \( \sigma \geq 0 \).

By the same calculation as above, we also have

\[ \| \partial_j \Delta^{-1} \hat{\beta}_{k,\lambda} \|_{L^p} \lesssim \| \cdot \|_{L^{p'}} \]

\[ \lesssim k^{-1+\tilde{\alpha}} \lambda^{-1+\frac{\tilde{\alpha}}{2}} \sum_{j=1,2} \left( \int_{\mathbb{R}^2} \left( \left( \lambda_1 - \frac{(1-j)}{2\pi} k \right)^2 + (\lambda_2)^2 \right)^{\frac{\sigma}{p'}} |\hat{\rho}(\gamma)|^{p'} d\gamma \right) \]

\[ \lesssim k^{-1+\tilde{\alpha}} \lambda^{-2+2(\frac{1}{p'}-\frac{1}{p})} \]

for sufficiently large \( k \) and \( \lambda \). \( \square \)

From now on we will restrict to the case

(3.8) \( \lambda = k^{\tilde{\alpha}}, \ k = n \) and \( 0 < \tilde{\alpha} \leq \alpha < 1 \)

and observe that it is possible to choose the integers \( n \) are sufficiently large so that, in particular, the assumptions of the previous lemma hold. Let \( \beta_n = \hat{\beta}_{k,\lambda} \) and define a sequence of initial vorticities by

(3.9) \( \omega_{0,n}(x) = \omega(x) + \beta_n(x), \ n \gg 10. \)

Combining the first part of Lemma 9 with equation (2.4) of Lemma 4 we find that \( \omega_{0,n} \) belong to \( W^{1,r} \), namely

(3.10) \( \| \omega_{0,n} \|_{W^{1,r}} \leq \| \omega_0 \|_{W^{1,r}} + \| \beta_n \|_{W^{1,r}} \lesssim 1 \)

for any sufficiently large \( n \). This proves the first assertion of Theorem 7.

Denote by \( \omega_n(t) \) the sequence of vorticity solutions of (1.1) with initial data \( \omega_{0,n} \) and as before let \( \eta_n(t) \) be the Lagrangian flows of the corresponding velocity fields \( u_n = \nabla^\perp \Delta^{-1} \omega_n \). The following lemma will be crucial in what follows.

**Lemma 10.** Let \( 0 < \sigma < 1, \ 0 < \alpha \leq 1 \) and \( 2 \leq p < \infty \). For any \( 1 \leq q \leq \infty \) we have

\[ \| \nabla^\perp \Delta^{-1} \beta_n \|_{M^{1+\sigma,\alpha}_{q,p}} \lesssim \| \nabla^\perp \Delta^{-1} \beta_n \|_{L^p} + \| \Delta^{1+\sigma} \nabla^\perp \Delta^{-1} \beta_n \|_{L^p} \]

for any sufficiently large \( n \in \mathbb{Z}^+ \).

**Proof.** From (3.5) and (3.7) we see that for any sufficiently large integer \( n \in \mathbb{Z}^+ \) the subsets \( \text{supp} \beta_n \) and \( B(0,1) \) are disjoint. Thus, it suffices to consider the case \( \sigma = 0 \), that is

\[ \| \beta_n \|_{L^p} \lesssim \| \beta_n \|_{M^{1,\alpha}_{q,p}}. \]
Let $Q$ be an admissible $\alpha$-covering by sets of size $|Q| \sim (1 + |x|^2)^\alpha$. Using (3.7), (3.8) and the fact that $\text{supp} \hat{\beta} \subset B(0, 3)$ we find

$$\text{supp} \hat{\beta}_{n=2^j} \subset B((2^j, 0), 2^{\tilde{\alpha}j}) \subset B((2^j, 0), 2^{\alpha j})$$

so that for any $j \in \mathbb{Z}_+$ there is a $Q \in Q$ with $\text{supp} \hat{\beta}_{2^j} \subset Q$ and we have

$$\| \beta_{2^j} \|_{M_{p,q}^{\alpha,\sigma}} = \left( \sum_{Q \in Q} \| F^{-1} \psi_Q F \beta_{2^j} \|_p^q \right)^{1/q} \simeq \| \beta_{2^j} \|_{L^p}$$

for $q < \infty$. Note that the case $q = \infty$ is analogous. \hfill \Box

Suppose now that the data-to-solution map for the Euler equations (13) is continuous from bounded subsets in $M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2)$ into $C([0, 1], M_{p,q}^{1+\sigma,\alpha}(\mathbb{R}^2))$. Choose $0 < \tilde{\alpha} \leq \alpha$ so that

$$-1 + \sigma + 2\tilde{\alpha}(1/r - 1/p) < 0.$$ 

Then, from estimates 2 and 3 of Lemma 9 we have

$$\| \nabla \Delta^{-1} \beta_n \|_{L^p} + \| \Delta^{1/2} \nabla \Delta^{-1} \beta_n \|_{L^p} \to 0 \quad \text{as} \ n \to \infty$$

where $\beta_n$ is the perturbation sequence defined in (3.6) and combining (3.9) with Lemma 10 we obtain

(3.11) \quad \| \nabla \Delta^{-1}(\omega_{0,n} - \omega_0) \|_{M_{p,q}^{1+\sigma,\alpha}} \to 0 \quad \text{as} \ n \to \infty.

The continuity assumption on the solution map and (3.11) now imply

(3.12) \quad \sup_{0 \leq t \leq T} \| \nabla \Delta^{-1}(\omega_n(t) - \omega(t)) \|_{M_{p,q}^{1+\sigma,\alpha}} \to 0 \quad \text{as} \ n \to \infty

from which, using the embedding assumption $M_{p,q}^{1+\sigma} \subset C^1$ and Lemma 10 we obtain

(3.13) \quad \sup_{0 \leq t \leq T} \sum_{i=0,1} \| D^i \eta_n(t) - D^i \eta(t) \|_{\infty} \to 0 \quad \text{as} \ n \to \infty

where as before $\eta(t)$ is the Lagrangian flow of the (smooth) divergence-free vector field $u = \nabla \Delta^{-1} \omega$ of Proposition 5 and $\eta_n(t)$ is the flow of $u_n = \nabla \Delta^{-1} \omega_n$ whose initial vorticities are given by (2.2) and (3.9) respectively. The rest of the argument is completely analogous to the proof of Theorem 3 in [13]. Thus we omit the details. \hfill \Box

4. A DIRECT PROOF FOR $C^{1+\sigma}(\mathbb{R}^3)$ BASED ON SHEAR FLOW

In this section we present a short and direct argument showing the loss of continuity of the data-to-solution map of (13) in the classical Hölder space $C^{1+\sigma}$ with $0 < \sigma < 1$. It is inspired by conversations with A. Shnirelman and C. Bardos from whom we learned about the DiPerna-Majda shear flow techniques.

Consider two 3D shear flows $u(t, x) = (f(x_2), 0, h(x_1 - tf(x_2)))$ and $v(t, x) = (g(x_2), 0, h(x_1 - tg(x_2)))$.

Let $f$, $g$ and $h$ be bounded functions in $C^{1+\sigma}(\mathbb{R}^3)$ with $h$ chosen so that in addition its derivative satisfies

$$h'(x_1) = |x_1|^\sigma \quad \text{for} \quad -2a \leq x_1 \leq 2a$$
where $a = \max\{\sup_{x,y} |f(x)|, \sup_{x,y} |g(x)|\}$. It is easy to verify that both $u(t)$ and $v(t)$ solve the 3D Euler equations with initial conditions

$$u_0(x) = (f(x_2), 0, h(x_1)) \quad \text{and} \quad v_0(x) = (g(x_2), 0, h(x_1)).$$

Now, given any $\epsilon > 0$ adjust $f$ and $g$ so that

$$\|u_0 - v_0\|_{C^{1+\sigma}} = \|f - g\|_{C^{1+\sigma}} < \epsilon$$

and consider at any time $0 \leq t \leq 1$ the norm of the difference of the corresponding solutions

$$\|u(t) - v(t)\|_{C^{1+\sigma}} = \|f - g\|_{C^{1+\sigma}} + \|h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot))\|_{C^{1+\sigma}} \geq \|\nabla (h(\cdot - tf(\cdot)) - h(\cdot - tg(\cdot)))\|_{C^\sigma} \geq \|h'(-tf(\cdot)) - h'(-tg(\cdot))\|_{C^\sigma}$$

which can be further bounded from below by

$$\geq \sup_{x,y \in [-b, b]^2} \frac{|(|x_1 - tf(x_2)^\sigma - |x_1 - tg(x_2)|^\sigma) - (|y_1 - tf(y_2)^\sigma - |y_1 - tg(y_2)|^\sigma)|}{|x - y|^\sigma}$$

where $b = \min\{\sup_{x_1} |f(x_1)|, \sup_{x_1} |g(x_1)|\}$.

Finally, pick $x_2 = y_2 = c$ for some arbitrary constant $c$ so that the expression above becomes

$$\geq \sup_{x_1, x_1 \in [-b, b]^2} \frac{|(|x_1 - tf(x)|^\sigma - |x_1 - tg(x)|^\sigma) - (|y_1 - tf(y)|^\sigma - |y_1 - tg(y)|^\sigma)|}{|x_1 - y_1|^\sigma}$$

and evaluate it once again from below by choosing $x_1 = tg(c)$ and $y_1 = tf(c)$ to get the bound

$$\geq \frac{t^\sigma |g(c) - f(c)|^\sigma + t^\sigma |f(c) - g(c)|^\sigma}{t^\sigma |f(c) - g(c)|^\sigma} = 2.$$
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