On the Structure of Modular Categories

Michael Müger*
Faculteit Wiskunde en Informatica, Universiteit Utrecht, Netherlands
and Korteweg-de Vries Institute, Amsterdam, Netherlands
email: mmueger@science.uva.nl

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Abstract

For a braided tensor category \( \mathcal{C} \) and a subcategory \( \mathcal{K} \) there is a notion of centralizer \( \mathcal{C}(\mathcal{K}) \), which is a full tensor subcategory of \( \mathcal{C} \). A pre-modular tensor category \([7]\) is known to be modular in the sense of Turaev iff the center \( \mathcal{Z}_2(\mathcal{C}) \equiv C_2(\mathcal{C}) \) (not to be confused with the center \( \mathcal{Z}_1 \) of a tensor category, related to the quantum double) is trivial, i.e. consists only of multiples of the tensor unit, and \( \dim \mathcal{C} \neq 0 \). Here \( \dim \mathcal{C} = \sum_i d(X_i)^2 \), the \( X_i \) being the simple objects.

We prove several structural properties of modular categories. Our main technical tool is the following double centralizer theorem. Let \( \mathcal{C} \) be a modular category and \( \mathcal{K} \) a full tensor subcategory closed w.r.t. direct sums, subobjects and duals. Then \( \mathcal{C}(\mathcal{C}(\mathcal{K})) = \mathcal{K} \) and \( \dim \mathcal{K} \cdot \dim C_2(\mathcal{K}) = \dim \mathcal{C} \).

We give several applications. (1) If \( \mathcal{C} \) is modular and \( \mathcal{K} \) is a full modular subcategory, then also \( \mathcal{L} = \mathcal{C}(\mathcal{K}) \) is modular and \( \mathcal{C} = \mathcal{K} \otimes \mathcal{L} \). Thus every modular category factorizes (non-uniquely, in general) into prime ones. We study the prime factorizations of the categories \( D(G) \mod \). (2) If \( \mathcal{C} \) is a modular \(*\)-category and \( \mathcal{K} \) is a full tensor subcategory then \( \dim \mathcal{C} \geq \dim \mathcal{K} \cdot \dim \mathcal{Z}_2(\mathcal{K}) \). We give examples where the bound is attained and conjecture that every pre-modular \( \mathcal{K} \) can be embedded fully into a modular category \( \mathcal{C} \) with \( \dim \mathcal{C} = \dim \mathcal{K} \cdot \dim \mathcal{Z}_2(\mathcal{K}) \). (3) For every finite group \( G \) there is a braided tensor \(*\)-category \( \mathcal{C} \) such that \( \mathcal{Z}_2(\mathcal{C}) \simeq \text{Rep} G \) (thus \( \text{Gal}(\mathcal{C}) \cong G \)) and the modular closure/modularization \([21, 7]\) is non-trivial.

1 Introduction

Braided tensor categories \([19]\) play a central rôle in the representation theory of quantum groups \([14, 20]\), of Kac-Moody algebras \([22]\) and of quantum field theories \([15, 23]\). They also serve as input data for the construction of invariants of knots, links and 3-manifolds.

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In both areas – representation theory and low dimensional topology – a particular subclass of braided tensor categories is distinguished, that of modular categories. First formalized in \cite{38}, they are semisimple rigid ribbon categories that have finitely many isomorphism classes of simple objects and satisfy a non-degeneracy condition. Modular categories derive their name from the fact that they define \cite{39} a (projective) finite dimensional representation of $SL(2,\mathbb{Z})$. At first sight mysterious, this modular representation is best understood in topological terms. Viz., a modular category gives rise, for every closed oriented surface $M$, to a finite dimensional projective representation of the mapping class group of $M$. The modular representation associated with a modular category is then just the representation of the mapping class group of the torus. In the important special case of $*$-categories (or unitary categories) arising in quantum field theory this modular representation had been known earlier and rigorously studied in \cite{36}.

The non-degeneracy condition mentioned above amounts to non-degeneracy of a certain matrix, which is just the collection of invariants of the Hopf link for the possible labelings of the two components. Also the meaning of the non-degeneracy condition in the construction of the 3-manifold invariant is quite transparent. Yet, it is clearly desirable to have a more intrinsic understanding in purely algebraic terms. In the special case of unitary categories, it has long been known \cite{36} that the modularity condition is equivalent to the absence of ‘degenerate’ objects. The latter property has a very satisfactory interpretation in terms of triviality of the center $Z_2(C)$ of the braided category. This center is a canonical full symmetric subcategory of $C$ and must not be confused with another notion of ‘center’ which is defined for any – not necessarily braided – tensor category. We denote the latter by $Z_1(C)$. In the more general situation of a pre-modular category \cite{7}, the equivalence between modularity and triviality of the center has been proven only recently \cite{9,6}. (In \cite{29, Corollary 7.11} this proof is obtained as a byproduct.)

Since symmetric tensor categories are precisely those braided tensor categories which coincide with their center, modular categories may be seen as braided tensor categories diametrically opposed to the symmetric ones: Modular categories are related to symmetric tensor categories like groups with trivial center to abelian groups, or like factors to commutative von Neumann algebras. Now, under some additional conditions, symmetric tensor categories are just representation categories of groups \cite{11,3}, thus they should be considered as very basic algebraic objects. Our point of view is that modular categories merit to be perceived similarly and to be subjected to detailed scrutiny and, as far as feasible, classification. So far, very little was known in the way of a general theory, our Theorem \ref{4.5}, according to which every modular category is (equivalent to) a finite direct product of prime objects, apparently being one of the first structural results.

On the other hand, many different constructions of modular categories are known, which we briefly review. There is a large class of constructions which go under the heading of quantum doubles, all of which yield modular categories under suitable assumptions. For the definition of the quantum double of finite dimensional Hopf algebras (in particular, group algebras) and of tensor categories (the ‘center’ $Z_1$ mentioned above) we refer to \cite{20}. Proofs of modularity were given in \cite{3} for quantum doubles of finite groups and in \cite{3} for the twisted versions of \cite{12}, and in \cite{14} and \cite{29, Appendix} for semisimple cosemisimple Hopf algebras. These cases are subsumed by quantum groupoids and by semisimple spherical categories, for which modularity of the quantum double was proven.
in [31] and [29], respectively. (Actually, by results of Hayashi [17] and Ostrik [33], the categories considered in [29] are always representation categories of quantum groupoids. Thus the proof of the main theorem of [29] could also be deduced from the results of [31] with some additional effort. Conceptually, however, the direct proof seems more satisfactory.) Quantum groups at roots of unity give rise to modular categories if one considers appropriate quotients of their representation categories [40]. These categories are in fact ∗-categories [11] (or unitary categories [39]). (The categories obtained in this way and from quantum doubles, respectively, are not completely unrelated since the universal $R$-matrix of a deformed enveloping algebra $A$ is computed by expressing $A$ as a quotient of a quantum double. On the categorical side, every modular category $\mathcal{C}$ is a full tensor subcategory of its own quantum double $\mathcal{Z}_1(\mathcal{C})$, cf. [29]. This is remarkable insofar as the definition of the latter does not refer to the braiding of $\mathcal{C}$.) For some of the above modular categories a beautiful purely combinatorial construction is known [6]. Furthermore, there is an operation [7, 26] which, heuristically, amounts to dividing a pre-modular category $\mathcal{C}$ by its center and which can be interpreted as a Galois completion [29, 8]. This procedure is applicable whenever the objects of the center $\mathcal{Z}_2(\mathcal{C})$ have positive integer dimensions (automatic for ∗-categories) and trivial twists, and it yields a modular category that is non-trivial whenever $\mathcal{C}$ is not symmetric. Finally, for a suitable class of rational chiral conformal field theories, axiomatized using operator algebras, one can prove [21] that the category of representations is a modular ∗-category. (In WZW- and orbifold models the representation categories are those of [41] and of [12], respectively.) For a review of these results we refer to [27], where also some of the results of the present paper were announced.

In the next section we briefly review some of the formalism of modular categories, restricting ourselves to the facts that are needed in this paper. With the exception of Lemma 2.13 the results are well known, but we emphasize the rôle of centralizers in braided categories. In Section 3 we prove our main technical result, a double centralizer theorem in modular categories. Section 4 gives several applications, the most important of which is that every modular subcategory of a modular category is a direct factor. This implies that every modular category factorizes as a finite direct product of prime ones. Here, a modular category $\mathcal{C}$ is prime if every full modular subcategory is either trivial or equivalent to $\mathcal{C}$. In Section 5 we give some preliminary results about full embeddings of pre-modular categories into modular categories and conclude with a remark about the Galois theory for braided tensor categories developed in [26, 8].

2 Preliminaries on Modular Categories

2.1 Notation

We assume known the standard definitions of (braided, symmetric) tensor categories, cf. [24, 19]. All categories in this paper are supposed small and all tensor categories strict. (A tensor category is strict if the tensor product satisfies associativity $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ ‘on the nose’ and the unit object $\mathbf{1}$ satisfies $X \otimes \mathbf{1} = \mathbf{1} \otimes X = X \forall X$. By the coherence theorems, every tensor category is equivalent to a strict one.) Our notation is fairly standard, except that we omit the $\otimes$-symbol for the product of objects:
XY \equiv X \otimes Y. \text{ Unit objects and unit morphisms in tensor categories are denoted by } 1 \text{ and } \text{id}_X \in \text{End}(X), \text{ respectively. Our categories will be } \mathbb{F}\text{-linear over a field } \mathbb{F}. \text{ We use capital letters } X, Y, \ldots \text{ to denote both objects and isomorphism classes of simple objects. What is meant should be obvious from the context. As usual, we denote } N^Z_{XY} = \dim \text{Hom}(Z, XY).

A *-category \cite{16, 23} or unitary category \cite{39} is a } \mathbb{C}\text{-linear category equipped with an antilinear involutive and contravariant endofunctor } * \text{ that leaves the objects fixed and such that } s^* \circ s = 0 \text{ implies } s = 0.

For the definitions of (left) rigid and ribbon categories see \cite{19, 39}. \text{ Every left rigid ribbon category is spherical } \cite{3}, \text{ i.e. every object } X \text{ has a two-sided dual } X^\vee \text{ and left and right traces coincide. (Conversely, every spherical category with braiding automatically has a compatible ribbon structure } \cite{12}. \text{ We denote the twist by } \{\theta_X, X \in C\}. \text{ To every simple object } X \text{ in a ribbon category we assign } \omega_X \in \mathbb{F} \text{ by } \omega_X \text{id}_X = \theta_X. \text{ One has } \omega_X = \omega_{X^\vee} \text{ for all simple } X, \text{ and in a *-category, } |\omega_X| = 1 \text{ for all } X. \text{ A pre-modular category } \cite{7} \text{ is a semisimple } \mathbb{F}\text{-linear rigid ribbon category with finitely many isomorphism classes of simple objects and tensor unit satisfying } \text{End} 1 \cong \mathbb{F}.

A subcategory } S \subset C \text{ is full iff } \text{Hom}_C(X, Y) = \text{Hom}_C(X, Y) \forall X, Y \in S, \text{ thus it is determined by } \text{Obj } S. \text{ A subcategory } S \text{ is replete iff } X \in S \text{ implies } Y \in S \text{ for all } Y \in C \text{ isomorphic to } X. \text{ By a semisimple tensor subcategory of a semisimple spherical category } C \text{ we mean a full subcategory which is stable w.r.t. direct sums, subobjects (thus in particular replete) and duals. If } \Gamma \text{ denotes the set of isomorphism classes of simple objects of } C, \text{ the semisimple subcategories are in one-to-one correspondence with the subsets } \Gamma' \subset \Gamma \text{ which are closed under duals and satisfy } X, Y \in \Gamma', N^Z_{XY} \neq 0 \Rightarrow Z \in \Gamma'.

### 2.2 Monodromies in braided tensor categories

**Definition 2.1** The monodromy of two objects } X, Y \text{ in a tensor category with braiding } c \text{ is defined by}

\[ c_M(X, Y) = c(Y, X) \circ c(X, Y) \in \text{End}(XY). \]

**Definition 2.2** For a braided spherical category } C \text{ over } \mathbb{F} \text{ and } X, Y \in C \text{ define } S(X, Y) \in \mathbb{F} \text{ by}

\[ S(X, Y) \text{id}_1 = \text{Tr}_{XY}(c_M(X, Y)) = \begin{pmatrix} X & \downarrow \\ Y & \uparrow \end{pmatrix} \]

**Remark 2.3** 1. } S(X, Y) \text{ depends only on the isomorphism classes } [X], [Y].

2. Note that we did not assume } X, Y \text{ to be simple. \hfill \square
**Lemma 2.4** Let $\mathcal{C}$ be a $\mathbb{F}$-linear semisimple rigid ribbon category with $\text{End}1 \cong \mathbb{F}$. The following identities hold:

1. $S(UX, Y) = \frac{1}{d(Y)} S(U, Y)S(X, Y)$ $\forall Y$ simple, $\forall U, X$.  
2. $X \cong \bigoplus_i X_i$, $Y \cong \bigoplus_j Y_j \implies S(X, Y) = \sum_{i,j} S(X_i, Y_j)$.  
3. $\frac{1}{d(X)}S(X, Y)S(X, Z) = \sum_W N^W_{YZ} S(X, W)$ $\forall X, Y, Z$ simple.

**Proof.** The first claim follows from $\text{Tr}_{UX} = \text{Tr}_X \circ (\text{Tr}_U \otimes \text{id})$ together with the consequence $(\text{Tr}_U \otimes \text{id})(c_M(U, Y)) = d(Y)^{-1}S(U, Y)\text{id}_Y$ of simplicity of $Y$. The second claim is immediate by cyclic invariance of the trace. Part (iii) follows by applying (i) and (ii) to $S(UV, Y)$.

The next result is valid without the restriction to unitary (i.e. $\ast$-) categories, cf. [3, 8]. The proof uses a general result on handle slides [1], which we do not wish to enlarge upon. Rather than referring the result away completely, we give a simple argument for unitary categories which does not require $\mathcal{C}$ to be finite.

**Proposition 2.5** Let $\mathcal{C}$ be a $\mathbb{F}$-linear semisimple rigid ribbon category with $\text{End}1 \cong \mathbb{F}$ (equivalently, a tensor $\ast$-category with direct sums, subobjects and conjugates). Let $X, Y$ be simple. Then $S(X, Y) = d(X)d(Y)$ iff $c_M(X, Y) = \text{id}_{XY}$.

**Proof.** The ‘if’ statement is obvious from the definition of $S$. Thus assume $S(X, Y) = d(X)d(Y)$. Using the well known equations [39, 36]

\[
d(X)d(Y) = \sum Z N^Z_{XY} d(Z),
\]

\[
S(X, Y) = \sum Z N^Z_{XY} \frac{\omega_Z}{\omega_X\omega_Y} d(Z),
\]

this implies

\[
\sum Z N^Z_{XY} d(Z) \frac{\omega_Z}{\omega_X\omega_Y} = \sum Z N^Z_{XY} d(Z).
\]

Restricting the summation to those $Z$ for which $N^Z_{XY} \neq 0$ we have

\[
\sum Z' N^Z_{XY} d(Z) \left(1 - \frac{\omega_Z}{\omega_X\omega_Y}\right) = 0. \tag{2.1}
\]

Since the $\omega'$s have absolute value one, we have $\text{Re}(1 - \omega_Z/\omega_X\omega_Y) \geq 0$, with equality iff $\omega_Z/\omega_X\omega_Y = 1$. In view of $N^Z_{XY} d(Z) > 0$ in (2.1) we conclude that $\omega_Z = \omega_X\omega_Y$ whenever $N^Z_{XY} > 0$. Let $Z$ be simple and consider $s : XY \to Z$, $t : Z \to XY$. Then with the ribbon condition $\theta_{XY} = \theta_X \otimes \theta_Y \circ c_M(X, Y) = c_M(X, Y) \circ \theta_X \otimes \theta_Y$ we have

\[
c_M(X, Y) \circ t \circ s = \theta_{XY} \circ (\theta_X \otimes \theta_Y)^{-1} \circ t \circ s = (\omega_X\omega_Y)^{-1} \theta_{XY} \circ t \circ s = (\omega_X\omega_Y)^{-1} t \circ \theta_Z \circ s = \frac{\omega_Z}{\omega_X\omega_Y} t \circ s.
\]
Since \( \text{End}(XY) \) is unital and spanned by morphisms \( t \circ s \) as above, the fact that \( \omega_Z = \omega_X \omega_Y \) for all \( Z \) contained in \( XY \) implies \( c_M(X,Y) = \text{id}_{XY} \).

**Definition 2.6** Let \( \mathcal{C} \) be a braided tensor category and \( \mathcal{K} \) a set of objects in \( \mathcal{C} \), equivalently, a full subcategory of \( \mathcal{C} \). Then we define the centralizer \( C_{\mathcal{C}}(\mathcal{K}) \) of \( \mathcal{K} \) in \( \mathcal{C} \) (or relative commutant \( \mathcal{C} \cap \mathcal{K}' \)) as the full subcategory defined by

\[
\text{Obj} C_{\mathcal{C}}(\mathcal{K}) = \{ X \in \mathcal{C} \mid c_M(X,Y) = \text{id}_{XY} \ \forall Y \in \mathcal{K} \}.
\]

**Remark 2.7** 1. In [26, Subsection 5.2] the subcategory \( C_{\mathcal{C}}(\mathcal{K}) \) was called \( C_{\mathcal{K}} \). In subfactor theory, a related notion appears under the name ‘permutant’ in [34].

2. If there is no danger of confusion concerning the ambient category \( \mathcal{C} \), we will occasionally write \( \mathcal{K}' \) instead of \( C_{\mathcal{C}}(\mathcal{K}) \).

If \( \mathcal{K}_1, \mathcal{K}_2 \) are full subcategories of \( \mathcal{C} \), by \( \mathcal{K}_1 \cap \mathcal{K}_2 \) we denote the smallest replete full subcategory of \( \mathcal{C} \) containing \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) and stable under tensor products, direct sums and retracts.

**Lemma 2.8** For \( \mathcal{C}, \mathcal{K} \) as above, \( C_{\mathcal{C}}(\mathcal{K}) \) is replete and monoidal. If \( \mathcal{C} \) is a semisimple category with direct sums and subobjects, then the same holds for \( C_{\mathcal{C}}(\mathcal{K}) \). If \( \mathcal{C} \) has duals for all objects then also \( C_{\mathcal{C}}(\mathcal{K}) \) has duals. If \( \mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{C} \) then \( C_{\mathcal{C}}(\mathcal{K}_1 \cap \mathcal{K}_2) = C_{\mathcal{C}}(\mathcal{K}_1) \cap C_{\mathcal{C}}(\mathcal{K}_2) \).

**Proof.** Repleteness of \( C_{\mathcal{C}}(\mathcal{K}) \) follows from naturality of the braiding \( c \). It is easy to see that \( c_M(X_i, Y) = \text{id}_{X_iY} \) for \( i = 1,2 \) implies \( c_M(X_1X_2, Y) = \text{id}_{X_1X_2Y} \), thus \( C_{\mathcal{C}}(\mathcal{K}) \) is closed under tensor products. In a semisimple category with \( X = \bigoplus_{i \in I} X_i \) one has \( c_M(X,Y) = \text{id}_{XY} \) iff \( c_M(X_i,Y) = \text{id}_{X_iY} \) for all \( i \in I \). This implies that \( C_{\mathcal{C}}(\mathcal{K}) \) is closed under direct sums and subobjects. The statement concerning duals follows by the same argument as in the proof of [26, Proposition 2.7]. As to the last claim, the inclusion \( C_{\mathcal{C}}(\mathcal{K}_1 \cap \mathcal{K}_2) \subset C_{\mathcal{C}}(\mathcal{K}_1) \cap C_{\mathcal{C}}(\mathcal{K}_2) \) is obvious. If \( X \) is in \( C_{\mathcal{C}}(\mathcal{K}_1) \) and \( C_{\mathcal{C}}(\mathcal{K}_2) \) then it also has trivial monodromy \( c_M \) with all tensor products of objects in \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), as well as direct sums and retracts of such. \( \square \)

**Definition 2.9** The center of a braided tensor category \( \mathcal{C} \) is

\[
\mathcal{Z}_2(\mathcal{C}) = C_{\mathcal{C}}(\mathcal{C})
\]

We say a semisimple braided tensor category has trivial center, if every object of \( \mathcal{Z}_2(\mathcal{C}) \) is a direct sum of copies of the tensor unit \( 1 \) or if, equivalently, every simple object in \( \mathcal{Z}_2(\mathcal{C}) \) is isomorphic to \( 1 \).

**Remark 2.10** 1. Clearly, a braided tensor category is symmetric iff it coincides with its center.

2. The objects of the center have previously been called degenerate [34, 26], transparent [6] and pseudotrivial [37]. Yet, calling them central seems the most natural terminology, since the above definition is the correct analogue for braided tensor categories of the center of a monoid, as can be seen appealing to the theory of \( n \)-categories.

3. Proposition 2.5 now has the interpretation that the simple object \( X \) is central iff \( S(X,Y) = d(X)d(Y) \) for all simple \( Y \in \mathcal{C} \). \( \square \)
2.3 Finite dimensional categories

In our considerations so far we did not make finiteness assumptions on $\mathcal{C}$. From now on we will work with categories which have finite dimension in the following sense.

**Definition 2.11** Let $\mathcal{C}$ be a semisimple $\mathbb{F}$-linear spherical category with $\text{End}(\mathbf{1}) \cong \mathbb{F}$. If the set $\Gamma$ of isomorphism classes of simple objects is finite, the dimension of $\mathcal{C}$ is defined by

$$\dim \mathcal{C} = \sum_{X \in \Gamma} d(X)^2,$$

otherwise it is $\infty$.

**Remark 2.12**
1. The sum over the squared dimensions appeared in [39], where its square roots are called ranks of the category. In subfactor theory [32, 33] this number is called the ‘global index’. The designation ‘dimension’ for this number has also been used in [5]. It is vindicated by the fact that $\dim \text{Rep}(H) = \dim H$ for a finite dimensional semisimple Hopf algebra, in particular group algebras. That $\dim \mathcal{C}$ is the correct generalization of $\dim H$ it is corroborated by its behavior under various constructions like the quantum double, where $\dim \mathcal{Z}_1(\mathcal{C}) = (\dim \mathcal{C})^2$ [29].

2. The dimension of a semisimple $\mathbb{F}$-linear category can be defined unambiguously whenever the category has two-sided duals, cf. [28, Subsect. 2.2]. A sovereign/spherical or $*$-structure is not needed.

If $\mathcal{K}$ is a subcategory of $\mathcal{C}$, let $\chi_\mathcal{K}$ be the characteristic function of $\text{Obj} \mathcal{K}$. Viz., $\chi_\mathcal{K}(X) = 1$ if $X \in \mathcal{K}$ and $\chi_\mathcal{K}(X) = 0$ otherwise.

**Lemma 2.13** Let $\mathcal{C}$ be a pre-modular category and $\mathcal{K}$ a semisimple tensor subcategory. Then for all $X \in \mathcal{C}$ we have

$$\sum_{Y \in \mathcal{K}} d(Y) S(X, Y) = d(X) \dim \mathcal{K} \chi_\mathcal{C}(\mathcal{K})(X). \quad (2.2)$$

**Proof.** If $X \in \mathcal{C}(\mathcal{K})$ then $S(X, Y) = d(X)d(Y)$ for all $Y \in \mathcal{K}$, and (2.2) follows immediately. Thus it remains to show that the left hand side of (2.2) vanishes if $c_M(X, Y) \neq \text{id}_{XY}$ for some $Y \in \mathcal{K}$. To this purpose, consider Lemma 2.4 (c) with $Y, Z \in \mathcal{K}$. Since $\mathcal{K}$ is a sub-tensor category, the summation runs only over isomorphism classes of simple $W \in \mathcal{K}$. Multiplication with $d(Y)$ and summation over $Y \in \mathcal{K}$ yields

$$\frac{S(X, Z)}{d(X)} \sum_{Y \in \mathcal{K}} d(Y) S(X, Y) = \sum_{Y, W \in \mathcal{K}} d(Y) N^{W}_{YZ} S(X, W).$$

Now,

$$\sum_{Y} d(Y) N^{W}_{YZ} = \sum_{Y} d(Y) N^{\Sigma}_{YZ} = d(W)d(Z) \quad \text{and we obtain}$$

$$\left( \sum_{Y \in \mathcal{K}} d(Y) S(X, Y) \right) [S(X, Z) - d(X)d(Z)] = 0 \quad \forall X \in \mathcal{C}, Z \in \mathcal{K}.$$
If now $X \not\in C_C(K)$ then there exists a $Z \in K$ which has non-trivial monodromy with $X$, and by Proposition 2.5 we have $S(X, Z) \neq d(X)d(Z)$. Thus the expression in the big brackets vanishes and we are done. ■

For $K = C$, the lemma reduces to a known result, and together with Remark 2.10 it implies the following corollary, cf. e.g. [3]. But for our purposes the generalization to arbitrary tensor subcategories $K$ will be essential.

**Corollary 2.14** Let $C$ be a pre-modular category with $\dim C \neq 0$ and let $X$ be a simple object. Then the following are equivalent:

(i) $X$ is central.

(ii) $S(X, Y) = d(X)d(Y)$ for all simple $Y \in C$.

(iii) $\sum Y S(X, Y)d(Y) \neq 0$.

**Lemma 2.15** Let $C$ be pre-modular and $Y, Z$ simple. Then

$$\sum X S(X, Y)S(X, Z) = \dim C \sum W \in Z_2(C) N_{YZ}^W d(W).$$

If $Z_2(C)$ is trivial then $S^2 = \dim C C$, where $C = (C_{XY}), C_{XY} = \delta_{X,Y}$.

**Proof.** Multiplying (iii) of Lemma 2.4 with $d(X)$ and summing over $X$ we obtain

$$\sum X S(X, Y)S(X, Z) = \sum X,W N_{YZ}^W S(X, W)d(X) = \dim C \sum W \in Z_2(C) N_{YZ}^W d(W),$$

where we have used Lemma 2.13 with $K = C$. The last claim follows from $N_{XY}^0 = \delta_{X,Y}$. ■

**Corollary 2.16** Let $C$ be pre-modular with $\dim C \neq 0$. Then the following are equivalent:

(i) The center $Z_2(C)$ is trivial.

(ii) The matrix $(S(X, Y))$, indexed by isomorphism classes of simple objects, is invertible.

**Proof.** The implication (ii)$\Rightarrow$(i) is obvious. Conversely, if $Z_2(C)$ is trivial, the lemma gives $S^2 = \dim C C$. (ii) then follows from invertibility of $C$ and $\dim C \neq 0$. ■

**Remark 2.17** 1. Alternatively, the statement of the corollary can be obtained directly from a handle slide argument, cf. [3, 29].

2. If $C$ is a $*$-category, one easily shows $\overline{S(X,Y)} = S(X,Y) = S(X,Y)$. In the case where $Z_2(C)$ is trivial, Lemma 2.15 implies that the columns (or rows) of $S$ are
mutually orthogonal, thus $\tilde{S} = (\dim C)^{-1/2}S$ is unitary. Without modularity one can still show that for simple $X, Y$ the columns $S_X, S_Y$ (or rows) of the $S$-matrix are either orthogonal or parallel, cf. [36]. If $\overline{C}$ is the modularization and $F(X)$ the image of $X$ in $\overline{C}$, t.f.a.e.: (i) $S_X \parallel S_Y$, (ii) there exists $Z \in \mathbb{Z}_2(C)$ such that $\text{Hom}(X, ZY) \neq \{0\}$, (iii) $\text{Hom}_{\overline{C}}(F(X), F(Y)) \neq \{0\}$, (iv) $F(X), F(Y)$ have the same simple summands, cf. [26]. □

3 Centralizers in Modular Categories

**Definition 3.1** [38] A modular category is a pre-modular category satisfying the (equivalent) conditions of Corollary 2.16.

We are now in a position to prove our first main result.

**Theorem 3.2** Let $C$ be a modular category and let $\mathcal{K} \subset C$ be a semisimple tensor subcategory. Then we have

(i) $C_C(C_C(\mathcal{K})) = \mathcal{K}$.

(ii) $\dim \mathcal{K} \cdot \dim C_C(\mathcal{K}) = \dim C$.

(We also simply write $\mathcal{K}' = \mathcal{K}$ and $\dim \mathcal{K} \cdot \dim \mathcal{K}' = \dim C$.)

**Proof.** We apply Lemma 2.13 to compute the characteristic function of $C_C(C_C(\mathcal{K}))$:

$$\chi_{C_C(C_C(\mathcal{K}))}(X) = \frac{1}{d(X) \dim C_C(\mathcal{K})} \sum_{Z \in C_C(\mathcal{K})} S(X, Z)d(Z) = \frac{1}{d(X) \dim C_C(\mathcal{K})} \sum_{Z \in \mathcal{C}} \chi_{C_C(\mathcal{K})}(Z) S(X, Z)d(Z).$$

We use the lemma once again to compute $\chi_{C_C(\mathcal{K})}(Z)$, obtaining

$$\chi_{C_C((C_C(\mathcal{K})))}(X) = \frac{1}{d(X) \dim C_C(\mathcal{K})} \sum_{Z \in \mathcal{C}} S(X, Z)d(Z) \frac{1}{d(Z) \dim \mathcal{K}} \sum_{U \in \mathcal{K}} S(Z, U)d(U) = \frac{1}{d(X) \dim C_C(\mathcal{K}) \dim \mathcal{K}} \sum_{U \in \mathcal{K}} d(U) \sum_{Z \in \mathcal{C}} S(X, Z)S(Z, U).$$

The summation over $Z \in \mathcal{C}$ can be performed using Lemma 2.15, and since the center of $\mathcal{C}$ is trivial, by Corollary 2.16 we have

$$\sum_{Z \in \mathcal{C}} S(X, Z)S(Z, U) = \dim \mathcal{C} \delta_{[X], [U]}.$$

Using $d(U) = d([U])$ and the fact that $\mathcal{K}$ is closed w.r.t. duals we obtain

$$\chi_{C_C(C_C(\mathcal{K}))}(X) = \frac{\dim \mathcal{C}}{d(X) \dim \mathcal{K} \dim C_C(\mathcal{K})} \sum_{U \in \mathcal{K}} d(U) \delta_{[X], [U]} = \frac{\dim \mathcal{C}}{\dim \mathcal{K} \dim C_C(\mathcal{K})} \chi_{\mathcal{K}}(X). \quad (3.1)$$
Since the tensor unit $1$ is contained in any tensor subcategory we have $\chi_K(1) = \chi_{K''}(1) = 1$. Thus for $X = 1$, (3.1) proves claim (ii), and plugging this back into (3.1), claim (i) ensues.

Remark 3.3 1. Using (i), one easily verifies the following: If $K$ is any subcategory closed w.r.t. duals then $C_K(C_K(K))$ is the semisimple tensor subcategory generated by $K$, i.e. the completion w.r.t. direct sums and subobjects.

2. In a subfactor context, the double centralizer property $K'' = K$ was stated by A. Ocneanu [34] without published proof. Subfactor analogues of both (i) and (ii) were proved by Izumi [18] using considerable machinery. By contrast, the above proof uses only well known properties modular categories.

Corollary 3.4 Let $C$ be a modular category and let $K \subset C$ be a semisimple tensor subcategory. Then

$$Z_2(C_K(K)) = Z_2(K).$$

Proof. We compute

$$Z_2(C_K(K)) = C_{C_K(K)}(C_K(K)) = (C \cap K') \cap (C \cap K')' = (C \cap (C \cap K'))' \cap K' = C_{C_K(K)}(K) = C_K(K) = Z_2(K),$$

where we have used $C_K(C_K(K)) = K$.

The two most interesting cases are where $K$ is modular or symmetric.

Corollary 3.5 Let $C$ be a modular category and let $K \subset C$ be a semisimple tensor subcategory that is modular. Then $L \equiv C_K(K)$ is modular, too.

Proof. By Corollary 2.16 modularity of $K$ is equivalent to triviality of $Z_2(K) = C_K(K)$. By Corollary 3.4 also $C_L(L) = C_K(K)$ is trivial, thus $L$ is modular.

Corollary 3.6 Let $C$ be a modular category and let $K \subset C$ be a semisimple tensor subcategory that is symmetric. Let $L \equiv C_K(K)$. Then $Z_2(L) = K$.

Proof. Obvious: $Z_2(K) = K$.

4 On the Structure of Modular Categories

4.1 Prime factorization of modular categories

Our first application of the double centralizer theorem in modular categories is also the most striking one. It illustrates how different modular categories are from their opposite extreme case, viz. the symmetric categories, the group duals (at least in characteristic zero [13, 13]).
The following was proved in [29] as Corollary 7.8. (The proof uses only two preceding results and is independent of the rest of [29].) Here \( A \boxtimes B \) is the completion w.r.t. direct sums of the product of \( A \) and \( B \) as \( \mathbb{F} \)-linear categories. \( A \boxtimes B \) has an obvious tensor structure if \( A, B \) do.

**Proposition 4.1** Let \( \mathcal{C} \) be a braided tensor category that is \( \mathbb{F} \)-linear semisimple with \( \text{End} \mathbf{1} \cong \mathbb{F} \) and two-sided duals. Let \( \mathcal{K} \subset \mathcal{C} \) be a semisimple tensor subcategory which has trivial center \( \mathbb{Z}_2(\mathcal{K}) \). Then we have the equivalence

\[
\mathcal{K} \boxtimes_{\mathcal{C}} \mathcal{C}(\mathcal{K}) \simeq \mathcal{K} \vee \mathcal{C}(\mathcal{K})
\]

of braided tensor categories. If \( \mathcal{C} \) is spherical then by restriction also \( \mathcal{K} \) and \( \mathcal{C}(\mathcal{K}) \) are spherical, and the above equivalence is one of spherical categories. Similarly, if \( \mathcal{C} \) is a \( * \)-category.

The question arises naturally, whether \( \mathcal{K} \vee \mathcal{L} = \mathcal{C} \), since this would imply \( \mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{L} \). If \( \mathcal{C} \) is modular, the double centralizer theorem provides the missing step.

**Theorem 4.2** Let \( \mathcal{C} \) and \( \mathcal{K} \) be modular categories where \( \mathcal{K} \) is identified with a full (tensor) subcategory of \( \mathcal{C} \). Let \( \mathcal{L} = \mathcal{C}(\mathcal{K}) \). Then there is an equivalence of ribbon categories:

\[
\mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{L}.
\]

**Proof.** Modularity of \( \mathcal{L} \) has been proved in Lemma 3.5. If we can show that the full subcategories \( \mathcal{K} \) and \( \mathcal{L} \) of \( \mathcal{C} \) generate \( \mathcal{C} \), Proposition 4.1 provides an equivalence \( \mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{L} \) of braided tensor categories. The equivalence is automatically an equivalence of ribbon categories since \( \mathcal{K} \) and \( \mathcal{L} \) commute. (Alternatively, one appeals to the compatibility of the spherical structures.) For the remaining fact \( \mathcal{K} \vee \mathcal{L} = \mathcal{C} \) we give two proofs, the first of which works only for unitary modular categories.

**Unitary categories.** By Proposition 4.1, we have \( \dim \mathcal{K} \vee \mathcal{L} = \dim \mathcal{K} \cdot \dim \mathcal{L} \), which by Theorem 3.2 coincides with \( \dim \mathcal{C} \). Since \( \mathcal{K} \vee \mathcal{L} \) is a full subcategory of \( \mathcal{C} \) and the numbers \( d(X)^2 \in \mathbb{R} \) are non-negative, the equality

\[
\dim \mathcal{C} = \sum_{X \in \mathcal{C}} d(X)^2 = \sum_{X \in \mathcal{K} \vee \mathcal{L}} d(X)^2 = \dim \mathcal{K} \vee \mathcal{L}
\]

implies that all simple objects of \( \mathcal{C} \) are contained in \( \mathcal{K} \vee \mathcal{L} \) and therefore \( \mathcal{C} = \mathcal{K} \vee \mathcal{L} \simeq \mathcal{K} \boxtimes \mathcal{L} \), as desired.

**General case.** The preceding argument to the effect that \( \mathcal{K} \vee \mathcal{L} \) exhausts \( \mathcal{C} \) does not work if we are not dealing with \( * \)-categories, since \( \dim \mathcal{A} = \dim \mathcal{B} \) does not imply that a replete full inclusion \( \mathcal{A} \subset \mathcal{B} \) is an identity. Yet, we can argue as follows:

\[
\mathcal{K} \vee \mathcal{L} = \mathcal{C}(\mathcal{C}(\mathcal{K} \vee \mathcal{L})) = \mathcal{C}(\mathcal{C}(\mathcal{K}) \cap \mathcal{C}(\mathcal{L})) = \mathcal{C}(\mathcal{C}(\mathcal{K}) \cap \mathcal{C}(\mathcal{C}(\mathcal{K})))
\]

\[
= \mathcal{C}(\mathcal{C}(\mathcal{K}) \cap \mathcal{K}) = \mathcal{C}(\mathcal{C} \cap \mathcal{K} \cap \mathcal{K}) = \mathcal{C}(\mathcal{C}(\mathcal{K})) = \mathcal{C}(\mathbb{Z}_2(\mathcal{K})) = \mathcal{C}.
\]

Here we used Lemma 2.8 and the fact that \( \mathbb{Z}_2(\mathcal{K}) \) is trivial. ■
Remark 4.3 1. That \( \mathcal{L} \) is modular can be derived alternatively from the easy fact that a direct product \( \mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{L} \) is modular iff both \( \mathcal{K} \) and \( \mathcal{L} \) are modular.

2. An interesting special case is provided by the quantum double of a modular category \( \mathcal{K} \). If we put \( \mathcal{C} = \mathbb{Z}_1(\mathcal{K}) \), there exists a braided monoidal embedding functor \( I : \mathcal{K} \hookrightarrow \mathcal{C} \). The above theorem implies \( \mathcal{C} \simeq \mathcal{K} \boxtimes \mathcal{C} \mathcal{C}(I(\mathcal{K})) \). In this situation, \( \mathcal{C}(I(\mathcal{K})) \) can be computed explicitly and is given by \( \hat{I}(\mathcal{K}) \), where \( \hat{\mathcal{K}} \) coincides with \( \mathcal{K} \) as a tensor category, but has the opposite braiding, and \( \hat{I} \) is a braided monoidal embedding. Thus \( \mathcal{C} \simeq \mathcal{K} \boxtimes \hat{\mathcal{K}}, \) which is the statement of \([29, \text{Theorem 7.9}]\). Even if \( \mathcal{K} \) is not modular, \( I(\mathcal{K}) \) and \( \hat{I}(\hat{\mathcal{K}}) \) are each other’s centralizers in \( \mathcal{C} \), and we have \( I(\mathcal{K}) \cap \hat{I}(\hat{\mathcal{K}}) = I(\mathbb{Z}_2(\mathcal{K})) \). See \([29, \text{Section 7}]\) for the details.

3. Finally, it is interesting to note the analogy with the following result from classical non-commutative algebra. Let \( A \subset B \) be an inclusion of matrix algebras. Then (i) \( C_B(A) \) is a matrix algebra, (ii) \( C_B(C_B(A)) = A \), and (iii) \( B \cong A \otimes C_B(A) \). These results also hold in the infinite dimensional case if \( A, B \) are type I factors (i.e. isomorphic to the algebras of bounded linear operators on some Hilbert spaces).

Definition 4.4 A modular category \( \mathcal{C} \) is prime if every semisimple modular subcategory is equivalent either to \( \mathcal{C} \) or the trivial modular category \( \text{Vect}_\mathbb{F} \).

Now we can state our second main result.

Theorem 4.5 Every modular category is equivalent to a finite direct product of prime ones.

Proof. Obvious consequence of Theorem 4.2 since modular categories have finitely many (equivalence classes of) simple objects and proper replete full subcategories have strictly fewer simple objects.

Remark 4.6 1. A. Bruguières \([9]\) first observed that the above factorization in general is non-unique. Yet, uniqueness does hold if every simple object except the unit has dimension unequal to one. In the next subsection we will find many examples for the non-uniqueness.

2. For the classification of modular categories, Theorem 4.5 has the obvious consequence that it is sufficient to consider prime ones. As in other respects, modular categories are better behaved than finite groups since there are no non-trivial exact sequences to be considered.

4.2 Quantum doubles of finite abelian groups

Quantum doubles of finite groups provide a large and relatively easy-to-analyze class of modular categories, cf., e.g., \([3]\). As an exploratory step towards a classification of modular \((\ast,\ast)\)-categories it is natural to find the prime factorizations of the categories \( D(G) - \text{mod} \) (\( \equiv D(\mathbb{C}G) - \text{mod} \)). If \( G \) is a direct product of subgroups \( K, L \) then it is easy to see that \( D(G) \cong D(K) \otimes D(L) \) as quasitriangular Hopf algebra, thus \( D(G) - \text{mod} \simeq D(K) - \text{mod} \boxtimes D(L) - \text{mod} \). Therefore, in order for \( D(G) - \text{mod} \) to be prime, \( G \) must
be prime (not a direct product of non-trivial subgroups). This condition is, however, not sufficient, as is shown by the complete analysis of the abelian case given below.

If $G$ is abelian, the simple objects of $C = D(G) \mod$ have dimension one and are invertible, the isomorphism classes of simple objects and their tensor product being given by the abelian group $\Gamma(C) = G \times \hat{G}$. There is an obvious one-to-one correspondence between replete full tensor subcategories $K \subset C$ and subgroups $K \subset \Gamma(C)$. Apart from this we only need the fact that $S((g, \chi), (h, \sigma)) = \langle \sigma, g \rangle \langle \chi, h \rangle$.

It is well known that every finite abelian group is isomorphic to a direct product of cyclic groups of prime power order,

$$G \cong \mathbb{Z}/p^1_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/p^k_k \mathbb{Z},$$

where the pairs $(p_i, n_i)$ are unique up to permutation. By the above remark we can therefore restrict ourselves to a consideration of $G = \mathbb{Z}/p^n \mathbb{Z}$.

**Theorem 4.7** Let $p$ be prime, $G = \mathbb{Z}/p^n \mathbb{Z}$ and $C = D(G) \mod$.

(i) If $p = 2$ then $C$ is prime.

(ii) If $p$ is odd, there is a one-to-one correspondence between isomorphisms $\alpha : G \to \hat{G}$ and replete full modular subcategories $K_\alpha \subset C$ given by $K_\alpha = \{ (g, \alpha(g)) \mid g \in G \}$. The categories $K_\alpha$ are prime, and $C C(C_\alpha) = C_{\hat{\alpha}}$, where $\overline{\alpha(\cdot)} = \alpha(\cdot)^{-1}$. The prime factorizations of $C$ are thus given by $C \cong C_\alpha \boxtimes C_{\hat{\alpha}}$, $\alpha \in \text{Isom}(G, \hat{G})$.

**Proof.** Let $K \subset C$ be a replete full tensor subcategory corresponding to the subgroup $K \subset \Gamma$. If $K$ is modular then, by Theorem 4.2, $\Gamma \cong K \times L$, where $L$ corresponds to the centralizer $L = C C(K)$. By the uniqueness result for the factorization of finite abelian groups, a (non-trivial) factorization $\Gamma \cong K \times L$ of $\Gamma \cong (\mathbb{Z}/p^n \mathbb{Z})^2$ is possible only if $K \cong L \cong \mathbb{Z}/p^n \mathbb{Z}$. Identifying $G$ and $\hat{G}$ with $\mathbb{Z}/p^n \mathbb{Z}$, the $S$-matrix is given by

$$S((a, b), (c, d)) = e^{\frac{2\pi i}{p^n}(ad + bc)}.$$ 

Furthermore, every cyclic subgroup $K \subset \Gamma$ of order $p^n$ is of the form $\{ (ja, jb) \mid j \in \mathbb{Z}/p^n \mathbb{Z} \}$, where $a, b$ are not both multiples of $p$. $K$ is modular iff $j \neq 0$ implies $(ja, jb) \notin \mathbb{Z}_2(K)$, which by Proposition 2.14 is equivalent to

$$\sum_{k=0}^{p^n-1} e^{\frac{2\pi i}{p^n} 2j knab} = 0.$$ 

In view of $\sum_{k=0}^{N-1} a^k = (a^N - 1)/(a - 1)$ for $a \neq 1$ and $N$ otherwise, this is the case iff $e^{\frac{2\pi i}{p^n} 2jab} \neq 1$ and $e^{\frac{2\pi i}{p^n} 2jap^n} = 1$. The latter condition is always satisfied, and the former leads to

$$(ja, jb) \in \mathbb{Z}_2(K) \iff 2jab \in p^n \mathbb{Z}. \quad (4.1)$$

If $p = 2$ then (4.1) is satisfied by $j = p^{n-1}$ irrespective of $a, b$. Thus $(ja, jb) \in \mathbb{Z}_2(K)$ and $K$ is not modular, proving (i). From now on let $p$ be odd. If $p \mid a$ or $p \mid b$ then again $j = p^{n-1}$ solves (4.1) and $K$ is not modular. If $p \nmid a$ and $p \nmid b$ then (4.1) is satisfied iff
Let $j \equiv 0 \pmod{p^n}$, which implies modularity of $\mathcal{K}$. Rephrasing this in an invariant fashion leads to the first part of statement (ii). Proper subgroups of $\mathcal{K}_\alpha$ have order smaller than $p^n$ and therefore cannot give rise to modular subcategories by the argument at the beginning of the proof. Thus the $\mathcal{K}_\alpha, \alpha \in \text{Isom}(G, \hat{G})$ are prime, and they exhaust the prime factors of $\mathcal{C}$. With $\alpha, \alpha' \in \text{Isom}(G, \hat{G})$, Proposition 2.3 finally implies that $\mathcal{K}'_\alpha = \mathcal{K}_{\alpha'}$ iff

\[ S((g, \alpha(g)), (h, \alpha'(h))) = \langle \alpha(g), h \rangle \langle \alpha'(h), g \rangle = 1 \quad \forall g, h \in G. \]

For $G = \mathbb{Z}/p^n\mathbb{Z}$ and $\alpha \in \text{Isom}(G, \hat{G})$ one easily shows $\langle \alpha(g), h \rangle = \langle \alpha(h), g \rangle \forall g, h$, which evidently entails $\mathcal{K}'_\alpha = \mathcal{K}_\alpha$.

**Remark 4.8**

1. In view of $\#\text{Isom}(G, \hat{G}) = p^n - p^n - 1$, the theorem nicely illustrates to which extent the factorization of modular categories into primes can be non-unique.

2. The table

| $G$ | $G$ simple? | $D(G) - \text{mod prime?}$ |
|-----|-------------|--------------------------|
| $\mathbb{Z}/2\mathbb{Z}$ | Yes | Yes |
| $\mathbb{Z}/p\mathbb{Z}$, $p \neq 2$ | Yes | No |
| $\mathbb{Z}/2^n\mathbb{Z}$, $n \geq 2$ | No | Yes |
| $\mathbb{Z}/p^n\mathbb{Z}$, $p \neq 2, n \geq 2$ | No | No |

makes clear that already for an abelian group $G$ there is no relation between simplicity of $G$ and primality of $D(G) - \text{mod}$.

## 5 Further Applications

### 5.1 Modular extensions of braided categories

The theory of Galois extensions of braided tensor categories developed in [7, 26] provides a means to construct a modular category $\overline{\mathcal{C}}$ from a given pre-modular category $\mathcal{C}$. In the language of [26], this modular closure is given by $\overline{\mathcal{C}} = \mathcal{C} \rtimes \mathbb{Z}_2(\mathcal{C})$, i.e. by adding morphisms which turn the objects in $\mathbb{Z}_2(\mathcal{C})$ into multiples of the tensor unit $1$. The dimension of the modular closure is given by

\[ \dim \overline{\mathcal{C}} = \frac{\dim \mathcal{C}}{\dim \mathbb{Z}_2(\mathcal{C})}, \tag{5.1} \]

cf. [6, 30]. Thus $\overline{\mathcal{C}}$ is, in a sense, a quotient of $\mathcal{C}$ by $\mathbb{Z}_2(\mathcal{C})$, in fact it is trivial iff $\mathcal{C}$ is symmetric.

On the other hand one may wish to find a modular category related to $\mathcal{C}$ without modifying the latter. More precisely, the problem is to find a modular category $\check{\mathcal{C}}$, into which $\mathcal{C}$ embeds as a full subcategory. If we restrict ourselves to $*$-categories, for which the dimensions take values in $\mathbb{R}_+$, we obtain a lower bound on the dimension of $\check{\mathcal{C}}$ as an immediate corollary of Theorem 3.2.
**Proposition 5.1** Let \( \mathcal{C} \) be a unitary modular category and \( \mathcal{K} \subseteq \mathcal{C} \) a semisimple tensor subcategory. Then

\[
\dim \mathcal{C} \geq \dim \mathcal{K} \cdot \dim \mathbb{Z}_2(\mathcal{K}).
\]

(5.2)

Equality holds iff \( \mathcal{C}_C(\mathcal{K}) = \mathcal{C}_K(\mathcal{K}) = \mathbb{Z}_2(\mathcal{K}) \).

**Proof.** The obvious inclusion (of replete full tensor subcategories) \( \mathcal{C}_C(\mathcal{K}) \supseteq \mathcal{C}_K(\mathcal{K}) = \mathbb{Z}_2(\mathcal{K}) \) implies \( \dim \mathcal{C}_C(\mathcal{K}) \geq \dim \mathbb{Z}_2(\mathcal{K}) \) and therefore \( \dim \mathcal{C} = \dim \mathcal{K} \cdot \dim \mathcal{C}_C(\mathcal{K}) \geq \dim \mathcal{K} \cdot \dim \mathbb{Z}_2(\mathcal{K}) \). Equality in (5.2) is equivalent to \( \mathcal{C}_C(\mathcal{K}) = \mathcal{C}_K(\mathcal{K}) = \mathbb{Z}_2(\mathcal{K}) \).

It is natural to expect that this bound is optimal.

**Conjecture 5.2** Let \( \mathcal{C} \) be a unitary pre-modular category. Then there exists a unitary modular category \( \hat{\mathcal{C}} \) and a full and faithful tensor functor \( I : \mathcal{C} \to \hat{\mathcal{C}} \) such that

\[
\dim \hat{\mathcal{C}} = \dim \mathcal{C} \cdot \mathbb{Z}_2(\mathcal{C}).
\]

Such a category \( \hat{\mathcal{C}} \) is called a minimal modular extension of \( \mathcal{C} \).

**Remark 5.3**

1. It is instructive to compare this equation with (5.1). In fact, the appearance of the factor \( \dim \mathbb{Z}_2(\mathcal{C}) \) in the conjecture and in (5.1) is not accidental. This is evident from the orbifold construction in conformal field theory, where the Galois closure \( \overline{\mathcal{C}} \) and the minimal modular extension \( \hat{\mathcal{C}} \) of a pre-modular category \( \mathcal{C} \) both appear naturally, cf. [27].

2. The conjecture makes sense also without the unitarity assumption. But without the latter it is not clear in which sense \( \hat{\mathcal{C}} \) is minimal.

3. If \( \mathcal{C} \) is symmetric, i.e. \( \mathcal{C} = \mathbb{Z}_2(\mathcal{C}) \), (5.2) reduces to \( \dim \hat{\mathcal{C}} \geq (\dim \mathcal{C})^2 \). If \( \omega_X = 1 \) for all simple \( X \), by [13] we have a finite group \( G \) such that \( \mathcal{C} \simeq G - \text{mod} \) and \( \dim \mathcal{C} = |G| \). Since the modular category \( D(G) - \text{mod} \) satisfies \( \dim D(G) - \text{mod} = |G|^2 \) and contains \( G - \text{mod} \) as a full subcategory we see that pre-modular *-categories that are symmetric with trivial twists in fact admit a minimal modular extension.

4. The example \( \mathcal{C} = G - \text{mod} \) also shows that \( \hat{\mathcal{C}} \), if it exists, will in general not be unique. The modular categories \( D^\omega(G) - \text{mod} \), where \( \omega \in \mathbb{Z}^3(G, \mathbb{T}) \), are minimal modular extensions of \( G - \text{mod} \) which are inequivalent for different \( [\omega] \in H^3(G, \mathbb{T}) \).

5. In a subfactor context, the existence (and ‘essential uniqueness’!) of such a \( \mathcal{C} \) is claimed by Ocneanu [34]. To the best of our knowledge a proof has not appeared. \( \square \)

Despite the fact that we do not know how to construct a minimal modular extension for a given pre-modular category, using our results [29] on the categorical quantum double we can construct many examples of unitary pre-modular categories which do possess a minimal modular extension.

Let \( \mathcal{C} \) be a pre-modular category with \( \dim \mathcal{C} \neq \infty \) and let \( \mathbb{Z}_1(\mathcal{C}) \) be is quantum double. It is known [29] that \( \mathcal{C} \) is modular and satisfies \( \dim \mathbb{Z}_1(\mathcal{C}) = (\dim \mathcal{C})^2 \). There are fully faithful braided tensor functors \( I : \mathcal{C} \to \mathbb{Z}_1(\mathcal{C}) \) and \( \tilde{I} : \hat{\mathcal{C}} \to \mathbb{Z}_1(\mathcal{C}) \), where \( \hat{\mathcal{C}} \) equals \( \mathcal{C} \) as a tensor category, the braiding given by \( c(X, Y) = c(Y, X)^{-1} \). Let

\[
\mathcal{E} = I(\mathcal{C}) \vee \tilde{I}(\hat{\mathcal{C}})
\]

15
be the replete full subcategory of $Z_1(C)$ generated by $I(C)$ and $\tilde{I}(\tilde{C})$.

**Proposition 5.4** The subcategory $\mathcal{E} \subset Z_1(C)$ satisfies

(i) $Z_2(\mathcal{E}) = I(Z_2(C)) = \tilde{I}(Z_2(\tilde{C}))$.

(ii) $\dim \mathcal{E} = \frac{(\dim C)^2}{\dim Z_2(C)}$.

**Proof.** We compute the centralizer of $\mathcal{E}$ in $Z_1(C)$:

$$C_{Z_1(C)}(\mathcal{E}) = C_{Z_1(C)}(I(C) \lor \tilde{I}(\tilde{C})) = C_{Z_1(C)}(I(C)) \cap C_{Z_1(C)}(\tilde{I}(\tilde{C})) = \tilde{I}(\tilde{C}) \cap I(C) = I(Z_2(C)).$$

(We have used Lemma 2.8 and the observations from [29, Section 4] that $I(C)$ and $\tilde{I}(\tilde{C})$ are each other’s centralizer in $Z_1(C)$ and satisfy $I(C) \cap \tilde{I}(\tilde{C}) = I(Z_2(C))$.) This computation obviously implies $Z_2(\mathcal{E}) = \mathcal{E} \cap \mathcal{E}' = I(Z_2(C))$. Furthermore, we have $\dim C_{Z_1(C)}(\mathcal{E}) = \dim Z_2(C)$, and using part (b) of the double centralizer theorem we obtain

$$\dim \mathcal{E} = \frac{(\dim C)^2}{\dim Z_2(C)}.$$

**Remark 5.5** Fact (ii) is intuitively clear: If $Z_2(C)$ is trivial then $\mathcal{E}$ is equivalent to the direct product of $I(C) \cong C$ and $\tilde{I}(\tilde{C}) \cong \tilde{C}$, cf. [29, Theorem 7.10]. Thus for non-trivial $Z_2(C)$, $\mathcal{E}$ should be a product of $I(C)$ and $\tilde{I}(\tilde{C})$ amalgamated over the common subcategory $I(Z_2(C))$.  

**Corollary 5.6** $Z_1(C) \supset \mathcal{E}$ is a minimal modular extension of $\mathcal{E}$.

**Proof.** By construction, $Z_1(C)$ is modular and contains $\mathcal{E}$ as replete full braided subcategory. In view of the preceding results and the known facts on the double we have

$$\dim Z_1(C) = (\dim C)^2 = \frac{(\dim C)^2}{\dim Z_2(C)} \dim Z_2(C) = \dim \mathcal{E} \cdot \dim Z_2(\mathcal{E}),$$

thus the bound in Proposition 5.1 is attained.

**5.2 The inverse problem for Galois extensions of premodular categories**

In this subsection we will solve the inverse problem of the Galois theory for braided tensor categories which was developed in [26], see also [8]. To begin with, given an arbitrary compact group $G$, we have $\text{Gal}(\text{Rep}(G)) \cong G$. Thus every compact group is the Galois group of some braided tensor category. This is, however, not very interesting since symmetric tensor categories have trivial modular closure. We will therefore show that for every finite group there is a finite dimensional unitary pre-modular category
with $\text{Gal}(C) \cong G$ and which is not symmetric, thus has non-trivial modular closure. To this purpose we will construct modular categories $C$ containing $S = \text{Rep}(G)$ as a full subcategory. Defining $D = C_C(S)$, Corollary 3.6 implies $Z_2(D) = S$, thus we have $\text{Gal}(D) \cong G$ as desired. Non-triviality of $D \rtimes Z_2(D) = D \rtimes S$ is equivalent to $D \supseteq S$, thus $\dim C > (\dim S)^2$.

We exhibit two ways of constructing such a $C$, both of which involve the center $Z_1(C)$ or quantum double of a tensor category $C$, which was already referred to in Subsection 5.1. See [20] for a nice discussion of the center construction and [29] for additional results which we will need.

The first construction is: $C = Z_1(Z_1(S))$. Since $S$ is braided, $Z_1(S)$ and $Z_1 \circ Z_1(S)$ contain $S$ as a full subcategory. By the results of [29], $C$ is modular and $\dim C = (\dim S)^4$. In particular, $\dim C > (\dim S)^2$ as required. Since we obtain

$$\dim(C_C(S) \rtimes S) = (\dim S)^2 = |G|^2,$$

it is natural to conjecture that the modular closure of $Z_1 = C_C(S)$ is equivalent to $Z_1(S) \cong D(G) - \text{mod}$.

The other procedure is as follows. Let

$$1 \rightarrow H \rightarrow K \rightarrow G \rightarrow 1$$

be an exact sequence of finite groups with $H \neq \{e\}$. With $C = Z_1(\text{Rep}(K))$, $C$ is modular and $\dim C = |K|^2$. Furthermore, $S = \text{Rep}(G)$ is contained as a full subcategory in $\text{Rep}(K)$ and thus in $Z_1(\text{Rep}(K))$. We obtain

$$\dim(C_C(S) \rtimes S) = \frac{|K|^2}{|G|^2} = |H|^2.$$

We therefore expect that $D \rtimes S = C_C(S) \rtimes S$ is equivalent to $Z_1(\text{Rep}(H)) \cong D(H) - \text{mod}$.

Thus we have given two different proofs of the following

**Theorem 5.7** Let $G$ be a finite group. Then there is a unitary pre-modular category $C$ such that $\text{Gal}(C) \cong G$ and such that $C$ is not symmetric, thus having a non-trivial modular closure.

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**References**

[1] D. Altschuler & A. Bruguières: Sliding property in sovereign categories. Preprint.

[2] D. Altschuler & A. Coste: Invariants of 3-manifolds from finite groups. In: Proc. XXth Int. Conf. Diff. Geom. Meth. in Theor. Phys., New York, 1991, pp. 219–233, World Scientific, 1992.

[3] D. Altschuler & A. Coste: Invariants of three manifolds from finite group cohomology. J. Geom. Phys. 11, 191-203 (1993).
[4] B. Bakalov & A. Kirillov Jr.: Lectures on Tensor Categories and Modular Functors. AMS, 2001.

[5] J. W. Barrett & B. W. Westbury: Spherical categories. Adv. Math. 143, 357-375 (1999).

[6] A. Beliakova & C. Blanchet: Modular categories of types B, C and D. Comment. Math. Helv. 76, 467-500 (2001).

[7] A. Bruguières: Catégories prémonodulaires, modularisations et invariants de variétés de dimension 3. Math. Annalen. 316, 215-236 (2000).

[8] A. Bruguières: Galois theory for braided tensor categories. In preparation.

[9] A. Bruguières: Private communication.

[10] V. Chari & A. Pressley: A Guide to Quantum Groups. Cambridge University Press, 1995.

[11] P. Deligne: Catégories tannakiennes. In: P. Cartier et al. (eds.): Grothendieck Festschrift, vol. II, pp. 111–195. Birkhäuser Verlag, 1991.

[12] R. Dijkgraaf, V. Pasquier & P. Roche: Quasi Hopf algebras, group cohomology and orbifold models. Nucl. Phys. B(Proc. Suppl.)18B, 60-72 (1990).

[13] S. Doplicher & J. E. Roberts: A new duality theory for compact groups. Invent. Math. 98, 157-218 (1989).

[14] P. Etingof & S. Gelaki: Some properties of finite dimensional semi-simple Hopf algebras. Math. Res. Lett. 5, 191-197 (1998).

[15] K. Fredenhagen, K.-H. Rehren & B. Schroer: Superselection sectors with braid group statistics and exchange algebras I. General theory. Commun. Math. Phys. 125, 201-226 (1989).

[16] P. Ghez, R. Lima & J. E. Roberts: W*-categories. Pac. J. Math. 120, 79-109 (1985).

[17] T. Hayashi: A canonical Tannaka duality for finite semisimple tensor categories. Preprint math.QA/9904073.

[18] M. Izumi: The structure of sectors associated with Longo-Rehren inclusions I. General theory, Commun. Math. Phys. 213, 127-179 (2000).

[19] A. Joyal & R. Street: Braided tensor categories. Adv. Math. 102, 20-78 (1993).

[20] C. Kassel: Quantum Groups. Springer Verlag, New York-Berlin, 1995.

[21] Y. Kawahigashi, R. Longo & M. Müger: Multi-interval subfactors and modularity of representations in conformal field theory. Commun. Math. Phys. 219, 631-669 (2001).

[22] D. Kazhdan & G. Lusztig: Tensor structures arising from affine Lie algebras I & II. J. Amer. Math. Soc. 6, 905-947, 949-1011 (1993).

[23] R. Longo & J. E. Roberts: A theory of dimension. K-Theory 11, 103-159 (1997).

[24] S. Mac Lane: Categories for the Working Mathematician. 2nd ed. Springer-Verlag, 1998.

[25] G. Moore & N. Seiberg: Classical and quantum conformal field theory. Commun. Math. Phys. 123, 177-254 (1989).
[26] M. Müger: Galois theory for braided tensor categories and the modular closure. Adv. Math. 150, 151-201 (2000).

[27] M. Müger: Conformal field theory and Doplicher-Roberts reconstruction. In: R. Longo (ed.): *Mathematical Physics in Mathematics and Physics: Quantum and Operator Algebraic Aspects*. Fields Inst. Commun. 20, 297-319 (2001).

[28] M. Müger: From subfactors to categories and topology I. Frobenius algebras in and Morita equivalence of tensor categories. Preprint [math.CT/0111204](http://arxiv.org/abs/math.CT/0111204).

[29] M. Müger: From subfactors to categories and topology II. The quantum double of tensor categories and subfactors. Preprint [math.CT/0111205](http://arxiv.org/abs/math.CT/0111205).

[30] M. Müger: Galois theory of braided tensor categories II. In preparation.

[31] D. Nikshych, V. Turaev & L. Vainerman: Invariants of knots and 3-manifolds from quantum groupoids. Preprint [math.QA/0006078](http://arxiv.org/abs/math.QA/0006078).

[32] A. Ocneanu: Quantized group string algebras and Galois theory for algebras. In: D. E. Evans & M. Takesaki (eds.): *Operator Algebras and Applications*, Vol. 2. London Math. Soc. Lect. Notes 136, CUP 1988.

[33] A. Ocneanu: Quantum symmetry, differential geometry of finite graphs and classification of subfactors. Lectures given at Tokyo Univ. 1990, notes taken by Y. Kawahigashi.

[34] A. Ocneanu: Chirality for operator algebras (Recorded by Y. Kawahigashi). In: H. Araki, Y. Kawahigashi & H. Kosaki (eds.): *Subfactors*. World Scientific, 1994

[35] V. Ostrik: Module categories, weak Hopf algebras and modular invariants. Preprint [math.QA/0111138](http://arxiv.org/abs/math.QA/0111138).

[36] K.-H. Rehren: Braid group statistics and their superselection rules. In: D. Kastler (ed.): *The Algebraic Theory of Superselection Sectors. Introduction and Recent Results*. World Scientific, 1990. See also the unpublished addendum at [http://www.Theorie.Physik.UNI-Goettingen.DE/~rehren/oldpubl.html](http://www.Theorie.Physik.UNI-Goettingen.DE/~rehren/oldpubl.html)

[37] S. F. Sawin: Jones-Witten invariants for nonsimply-connected Lie groups and the geometry of the Weyl alcove. Preprint [math.QA/9905010](http://arxiv.org/abs/math.QA/9905010).

[38] V. G. Turaev: Modular categories and 3-manifold invariants. Int. J. Mod. Phys. B6, 1807-1824 (1992).

[39] V. G. Turaev: *Quantum Invariants of Knots and 3-Manifolds*. Walter de Gruyter, 1994.

[40] V. G. Turaev & H. Wenzl: Quantum invariants of 3-manifolds associated with classical simple Lie algebras. Intern. Journ. Math. 4, 323-358 (1993).

[41] H. Wenzl: C*-Tensor categories from quantum groups. J. Amer. Math. Soc. 11, 261-282 (1998).

[42] D. N. Yetter: Framed tangles and a theorem of Deligne on braided deformations of tannakian categories. Contemp. Math. 134, 325-350 (1992).