ALGEBRAS ASSOCIATED WITH BLASCHKE PRODUCTS OF TYPE $G$

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Abstract. Let $\Omega$ and $\Omega_{\text{fin}}$ be the sets of all interpolating Blaschke products of type $G$ and of finite type $G$, respectively. Let $E$ and $E_{\text{fin}}$ be the Douglas algebras generated by $H^\infty$ together with the complex conjugates of elements of $\Omega$ and $\Omega_{\text{fin}}$, respectively. We show that the set of all invertible inner functions in $E$ is the set of all finite products of elements of $\Omega$, which is also the closure of $\Omega$ among the Blaschke products. Consequently, finite convex combinations of finite products of elements of $\Omega$ are dense in the closed unit ball of the subalgebra of $H^\infty$ generated by $\Omega$. The same results hold when we replace $\Omega$ by $\Omega_{\text{fin}}$ and $E$ by $E_{\text{fin}}$.

1. Introduction

Let $D$ be the open unit disk and $T$ be the unit circle on the complex plane. Let $H^\infty$ be the Banach algebra of bounded analytic functions on the open unit disk $D$. Via radial limits we can consider $H^\infty$ as a closed subalgebra of $L^\infty$, where $L^\infty$ is the family of all essential bounded measurable functions on $T$. Any function $h$ in $H^\infty$ with $|h| = 1$ a.e. on $T$ is called an inner function. Let $\{z_n\}$ be a sequence in $D$ with $\sum_n (1 - |z_n|) < \infty$. Then the function

$$b(z) = \prod_n \frac{\bar{z}_n}{|z_n|} \frac{z_n - z}{1 - \bar{z}_n z} \quad \text{for } z \in D,$$

is called a Blaschke product with roots $\{z_n\}$. Let

$$\delta(b) = \inf_k \prod_{n \neq k} \frac{|z_k - z_n|}{1 - \bar{z}_n z_k}.$$

If $\delta(b) > 0$, then $b$ and $\{z_n\}$ are called interpolating. By [Ca], if $\delta(b) > 0$, then for every bounded sequence $\{a_n\}$, there exists $f$ in $H^\infty$ such that $f(z_n) = a_n$ for every $n$. If

$$\lim_{k \to \infty} \prod_{n \neq k} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| = 1,$$

then $b$ and $\{z_n\}$ are called thin or sparse.

We denote by $M(H^\infty)$ the maximal ideal space of $H^\infty$. A closed subalgebra $B$ between $H^\infty$ and $L^\infty$ is called a Douglas algebra. We denote by $M(B)$ the maximal ideal space of the Douglas algebra $B$. For an interpolating Blaschke product $b$, we
denote by $H^\infty[b]$ the Douglas algebra generated by $H^\infty$ and the complex conjugate of $b$. For a function $f$ in $H^\infty$, let

$$Z(f) = \{ x \in M(H^\infty) : f(x) = 0 \}$$

and for $0 < c \leq 1$,

$$\{|f| < c\} = \{ x \in M(H^\infty) : |f(x)| < c \}.$$ 

For a point $x$ in $M(H^\infty)$, there is a representing measure $\mu_x$ on $M(L^\infty)$, that is,

$$f(x) = \int_{M(L^\infty)} f \, d\mu_x$$

for every $f \in H^\infty$. We denote by $\text{supp}\, \mu_x$ the support set for the representing measure $\mu_x$.

By the Corona Theorem, $D$ can be considered as a dense subset of $M(H^\infty)$. For points $x, y$ in $M(H^\infty)$, let

$$\rho(x, y) = \sup\{|f(y)| : f \in H^\infty, \|f\|_\infty \leq 1, f(x) = 0 \}$$

and put

$$P(x) = \{ m \in M(H^\infty) : \rho(m, x) < 1 \}.$$ 

The set $P(x)$ is called the Gleason part containing $x$. For $z, w \in D$, we have

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|$$

and $P(0) = D$. We call $x \in M(H^\infty)$ a trivial part if $P(x) = \{ x \}$. Let

$$G = \bigcup \{ P(x) : x \in M(H^\infty), P(x) \neq \{ x \} \}.$$ 

Then $G$ is an open subset of $M(H^\infty)$.

By Hoffman’s work [Ho], $Z(b) \subset G$ for every interpolating Blaschke product $b$ and for each $x$ in $G$, there exists an interpolating Blaschke product $b$ such that $b(x) = 0$. Also by [Ho], for each $x \in G$ there exists a one-to-one and onto map $L_x : D \to P(x)$ such that $f \circ L_x \in H^\infty$ for every $f \in H^\infty$. The map $L_x$ is given as follows. Let $\{z_\alpha\}$ be a net in $D$ with $z_\alpha \to x$ and let $L_{z_\alpha}(z) = (z + z_\alpha)/(1 + \bar{z_\alpha}z)$. Then

$$(f \circ L_x)(z) = \lim_{\alpha}(f \circ L_{z_\alpha})(z) \text{ for } f \in H^\infty \text{ and } z \in D.$$ 

A Blaschke product $b$ is of type $G$ if it is interpolating and $\{|b| < 1\} \subset G$. It is of finite type $G$ if it is of type $G$ and for every $x \in Z(b)$ the set $Z(b) \cap P(x)$ is finite. A Blaschke product is locally thin if for each $x \in Z(b)$ there is an interpolating Blaschke product $q$ such that

$$\lim_{\alpha}(1 - |z_\alpha|^2)|q'(z_\alpha)| = 1$$

whenever $\{z_\alpha\}$ is a subnet of the root sequence $\{z_\alpha\}$ of $q$ that converges to $x$. Note $q$ may be different from $b$. In fact, by [Go-Li-Mo], if $b = q$ for every $x \in z(b)$, then
$b$ is a thin Blaschke product. Blaschke products of type $G$, finite type $G$ and locally thin Blaschke products are very important in the studies of Douglas algebras (see for example, [Go-Li-Mo], [Gu], [Gu-Iz-1] and [Gu-Iz-2]).

Let $\Omega$ be the family of all interpolating Blaschke products of finite type $G$ and $A$ be the closed subalgebra of $H^\infty$ generated by $\Omega$. Let $B = [A, \overline{A}]$ be the smallest (closed) $C^*$-subalgebra of $L^\infty$ containing $A$. That is, $B$ is generated by the ratio of interpolating Blaschke products of type $G$. Then $E = [H^\infty, \overline{A}]$ will be the Douglas algebra generated by $H^\infty$ and the complex conjugate of elements of $\Omega$.

Our main results are that every inner function $u$ in $E$ is a finite product of interpolating Blaschke products of type $G$, from which we are able to identify the closure of the interpolating Blaschke products of type $G$ among the Blaschke products. As a consequence, we get $B = C_E$, where $C_E$ denotes the $C^*$-subalgebra of $L^\infty$ generated by the invertible inner functions in $E$ and their complex conjugates.

Another consequence is that the finite products of interpolating Blaschke products of type $G$ are the only inner functions that are in $B \cap H^\infty$. Hence by Theorem 4.1 of [Ch-Ma], $A = B \cap H^\infty$ and finite convex combinations of finite products of interpolating Blaschke products of type $G$ are dense in the closed unit ball of $A$. For Blaschke product of finite type $G$, we obtain similar results. In obtaining these results, we follow the approach in [He], but our proofs rely heavily on the results about type $G$ and finite type $G$ developed in [Gu-Iz-1] and [Gu-Iz-2].

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2. Results For Type $G$

We begin with a few useful propositions concerning basic properties of interpolating Blaschke product of type $G$.

**Proposition 1.** If $B$ is of type $G$ and $b$ is a subproduct of $B$, then $b$ is of type $G$. If $b_1, b_2$ are of type $G$ and $b_1b_2$ is an interpolating Blaschke product, then $b_1b_2$ is of type $G$. If $b$ is of type $G$ and $b_\lambda = (b - \lambda)/(1 - \overline{\lambda}b)$ is an interpolating Blaschke product, then $b_\lambda$ is of type $G$. The statements also hold if type $G$ is replaced by finite type $G$.

**Proof.** For type $G$, the first statement follows from $\{|b| < 1\} \subset \{|B| < 1\} \subset G$. The second statement follows from $\{|b_1b_2| < 1\} = \{|b_1| < 1\} \cup \{|b_2| < 1\} \subset G$. The third statement follows from $\{|b_\lambda| < 1\} = \{|b| < 1\} \subset G$.

For finite type $G$, the first statement follows from $Z(b) \cap P(m) \subset Z(B) \cap P(m)$. The second statement follows from $Z(b_1b_2) \cap P(m) = (Z(b_1) \cap P(m)) \cup (Z(b_2) \cap P(m))$. The third statement follows from Theorem 3.2 (iii) of [Gu-Iz-2] because $H^\infty[b_\lambda] = H^\infty[b]$ by considering their maximal ideal spaces. 

We remark that not all of the statements in Proposition 1 are true for the family of thin Blaschke products.
Proposition 2. Suppose \( b \) is an interpolating Blaschke product of type \( G \) with roots \( \{ z_n \} \) in \( D \). Let \( q \) be an interpolating Blaschke product with roots \( \{ w_n \} \) in \( D \) such that \( \rho(w_n, z_n) \leq r \) for all \( n \) and for some \( r < 1 \). Then \( q \) is of type \( G \).

Proof. Suppose \( 0 < \lambda < 1 \) and \( z \in D \) such that \( |q(z)| < \lambda \). Then, by Lemma 1.4 and Corollary 1.3 on page 4 of [Gar],

\[
|b(z)| = \prod_{n=1}^{\infty} \rho(z, z_n) \leq \prod_{n=1}^{\infty} \left( \frac{\rho(z, w_n) + r}{1 + r \rho(z, w_n)} \right) \leq |q(z)| + r \frac{|q(z)|}{1 + r |q(z)|} < \lambda + r \frac{1}{1 + r \lambda} = \lambda' < 1.
\]

So we have

\[
\{ |q| < 1 \} = \bigcup_{0 < \lambda < 1} \{ |q| < \lambda \} \subset \bigcup_{0 < \lambda' < 1} \{ |b| < \lambda' \} = \{ |b| < 1 \} \subset G.
\]

Proposition 3. Let \( \mathcal{F} = \{ x \in M(H^\infty) : x \) is in the closure of some interpolating sequences in \( D \) whose Blaschke products are of type \( G \} \). Then \( \mathcal{F} \) is the union of a family of nontrivial Gleason parts.

Proof. By the definition of \( \mathcal{F} \), every point in \( \mathcal{F} \) belongs to a nontrivial Gleason part. So let \( m_0 \in \mathcal{F} \) and \( m \in P(m_0) \). Then \( m_0 \in \overline{\{ z_n \}} \) for some interpolating sequence \( \{ z_n \} \) whose Blaschke product \( b(z) \) is of type \( G \). So there is a subnet \( \{ z_{n_\alpha} \} \) of \( \{ z_n \} \) converging to \( m_0 \). Since \( m \in P(m_0) \), there is \( \zeta \in D \) such that

\[
\lim_{\alpha} L_{z_{n_\alpha}}(\zeta) = L_{m_0}(\zeta) = m.
\]

Let

\[
\zeta_n = L_{z_n}(\zeta) = \frac{\zeta + z_n}{1 + z_n \overline{\zeta}}.
\]

Then \( \rho(\zeta_n, z_n) = |\zeta| < 1 \) for all \( n \). By Corollary 1.6 on page 407 of [Gar], there is a factorization \( b = b_1 b_2 \cdots b_k \) with

\[
\delta(b_j) > \frac{2|\zeta|}{1 + |\zeta|^2} \text{ for } j = 1, 2, \ldots, k.
\]

By Lemma 5.3 on page 310 of [Gar], each \( Z(b_j) \cap D = \{ z_{j,n} \} \) is interpolating. Since

\[
\overline{\{ z_n \}} = \bigcup_{j=1}^{k} \overline{Z(b_j)}
\]

and the \( Z(b_j) \)'s have disjoint closures [Gar, p. 422], it follows that \( m_0 \in \overline{Z(b_j)} = \{ z_{j,n} \} \) for some \( j \) and the net \( \{ z_{n_\alpha} \} \) is eventually in \( Z(b_j) \) because

\[
M(H^\infty) \setminus \bigcup_{i \neq j} \overline{Z(b_i)}
\]
is an open neighborhood of $\overline{Z(b_j)}$ and $m_0 \in \overline{Z(b_j)}$.

By Proposition 1, each $b_j(z)$ is of type $G$. By Proposition 2 (and the above argument), the Blaschke product with roots $\{\zeta_{j,n}\}$ is of type $G$. Finally,
\[
\lim_{\alpha} \alpha \zeta_{j,\alpha} = \lim_{\alpha} L_{z_{j,\alpha}}(z) = L_{m_0}(\zeta) = m
\]
and our assertion follows.

**Proposition 4.** An interpolating Blaschke product $b$ of type $G$ has modulus 1 on those Gleason parts of $M(H^\infty)$ that do not contain a point in $Z(b)$. Since $Z(b)$ equals the closure of $Z(b) \cap D$ and $\mathcal{F}$ is the union of a family of Gleason parts, $b$ has in particular modulus 1 on $M(H^\infty) \setminus \mathcal{F}$.

**Proof.** By Lemma 1.1 of [Gu-Iz-2] (or Theorem 1 of [Gu-Iz-1]) we have
\[
\{|b| < 1\} = \bigcup_{m \in Z(b)} P(m).
\]
Thus
\[
|b| = 1 \text{ on } m(H^\infty) \setminus \bigcup_{m \in Z(b)} P(m).
\]
The second statement follows from the proof of Proposition 3.

**Corollary 5.** $M(E) = M(H^\infty) \setminus \mathcal{F}$.

**Proof.** By Theorem 1.3 on page 375 of [Gar],
\[
M(E) = \{m \in M(H^\infty) : |b(m)| = 1 \text{ for all } b \in \Omega\}.
\]
Now the results follows immediately from Proposition 4.

**Corollary 6.** $A$ is a proper subalgebra of $H^\infty$.

**Proof.** This follows because $M(H^\infty) \setminus (\mathcal{F} \cup M(L^\infty))$ is not empty.

**Theorem 7.** Every invertible inner function in $E$ is a finite product of interpolating Blaschke products of type $G$.

**Proof.** Let $u$ be an arbitrary invertible inner function in $E$, then $\bar{u} = u^{-1}$ in $E \subset L^\infty$. So $H^\infty[\bar{u}] \subset E$. By Corollary 5 above and Theorem 1.3 on page 375 of [Gar]
\[
M(H^\infty) \setminus \mathcal{F} = M(E) \subset M(H^\infty[\bar{u}]) = \{|u| = 1\}.
\]
Hence $\{|u| < 1\} \subset \mathcal{F} \subset G$. This implies $Z(u)$ cannot contain any trivial parts. By Corollary 24 of [McD-Su], $u = b_1b_2\cdots b_n$, where each $b_j$ is an interpolating Blaschke product. Finally, for each $j$,
\[
\{|b_j| < 1\} \subset \bigcup_{k=1}^n \{|b_k| < 1\} = \{|u| < 1\} \subset G.
\]
Corollary 8. $B = C_E$, the $C^*$-subalgebra of $L^\infty$ generated by the inner functions invertible in $E$ and their complex conjugates.

Corollary 9. Let $b$ be a finite product of interpolating Blaschke products of type $G$. If $f \in H^\infty$ is such that $\|f\|_\infty < 1$ and $\bar{f}b$ equals on $H^\infty$ a function $g$ almost everywhere on $T$, then the function
\[ b_f(z) = \frac{b(z) - f(z)}{1 - g(z)}, \quad \text{for } z \in D, \]
is a finite product of interpolating Blaschke products of type $G$.

Proof. Just observe that $b_f$ is an invertible inner function in $E$. \qed

In [Ch-Ma], Chang and Marshall showed that for an arbitrary Douglas algebra $J$, the closed unit ball of $H^\infty \cap C_J$ is the norm-closed convex hull of the Blaschke products in $H^\infty \cap C_J$, where $C_J$ is the $C^*$-subalgebra of $L^\infty$ generated by the invertible inner functions in $J$ and their complex conjugates. They also showed that $J = H^\infty + C_J$ and that $D$ is dense in the maximal ideal space of $H^\infty \cap C_J$. In our case $J = E$, $C_J = B$ and we have the following corollary. (Note that an inner function in $B \cap H^\infty$ is invertible in $E$.)

Corollary 10. (a) $A = B \cap H^\infty$, and finite convex combinations of finite products of interpolating Blaschke products of type $G$ are dense in the closed unit ball of $A$.
(b) $E = H^\infty + B$.
(c) $D$ is dense in the maximal ideal space $M(A)$ of $A$.

3. Results For Finite Type $G$

Next we will turn to the main results for interpolating Blaschke product of finite type $G$ analogous to those established in section 2. Let $\Omega_\text{fin}$ be the family of all interpolating Blaschke products of finite type $G$ and $A_\text{fin}$ be the closed subalgebra of $H^\infty$ generated by $\Omega_\text{fin}$. Let $E_\text{fin} = [H^\infty, A_\text{fin}]$, then $E_\text{fin}$ is the Douglas algebra generated by $H^\infty$ and the complex conjugate of elements of $\Omega_\text{fin}$. We will show that Theorem 7 holds if $E$ is replaced by $E_\text{fin}$.

Theorem 11. Every invertible inner function in $E_\text{fin}$ is a finite product of interpolating Blaschke products of finite type $G$.

Proof. We will first show that the analog of Proposition 2 is true for interpolating Blaschke products of finite type $G$. Let $b$ be an interpolating Blaschke product of finite type $G$ with zeros $\{z_n\}$ in $D$, and let $q$ be an interpolating Blaschke product with zeros $\{w_n\}$ in $D$ such that $\rho(z_n, w_n) \leq r$ for all $n$ and some $r < 1$. The proof of Proposition 2
shows that \( \{|q| < 1\} \subset \{|b| < 1\} \). Since \( b \) is of finite type \( G \), by Theorem 2.1 of [Gu-Iz-2], there is a subproduct \( b_0 \) of \( b \) such that \( \{|b_0| < 1\} = \{|q| < 1\} \). We will show that \( q \) is of finite type \( G \).

For \( x \in Z(q) \), \( |b_0(x)| < 1 \). By Lemma 1.1 of [Gu-Iz-2], there is an \( x_0 \in Z(b_0) \) such that \( x \in P(x_0) \). Suppose the set \( Z(q) \cap P(x) = Z(q) \cap P(x_0) \) is infinite. Then, by Theorem 3.1(i) of [Gu-Iz-2], there exist \( y \) and \( y_0 \) in \( Z(q) \) such that \( \text{supp} \mu_y \subset \text{supp} \mu_{y_0} \). Hence there are \( m \) and \( m_0 \) in \( Z(b_0) \) such that \( y \in P(m) \) and \( y_0 \in P(m_0) \), but then \( \text{supp} \mu_m \subset \text{supp} \mu_{m_0} \) (because by page 143 of [Gam], \( \text{supp} \mu_y = \text{supp} \mu_m \) and \( \text{supp} \mu_{y_0} = \text{supp} \mu_{m_0} \)). Since \( b_0 \) is of finite type \( G \), this contradicts Theorem 3.2(ii) of [Gu-Iz-2]. Thus \( Z(q) \cap P(x_0) = Z(q) \cap P(x) \) must be finite.

Next we remark that the analogs of Propositions 3, 4 and Corollaries 5, 6 for finite type \( G \) also hold by the same reasoning because of the analog of Proposition 2 for finite type \( G \).

Now let \( u \) be an invertible inner function in \( E_{\text{fin}} \subset E \). By Theorem 7, \( u = u_1 u_2 \cdots u_m \), where each \( u_i \) is of type \( G \). Observe that if \( \mathcal{F}_{\text{fin}} \) is the analog of \( \mathcal{F} \) for finite type \( G \), then

\[
Z(u_i) \subset \{|u| < 1\} \subset \mathcal{F}_{\text{fin}} = \bigcup_{b \in \Omega_{\text{fin}}} \{|b| < \frac{1}{2}\}.
\]

Let \( \delta_i = \inf \{\rho(w, z) : w, z \in Z(u_i) \cap D, w \neq z\} > 0 \). Since \( Z(u_i) \) is compact,

\[
Z(u_i) \subset \bigcup_{j=0}^{n_i} \{|b_j| < \frac{1}{2}\},
\]

for some \( b_1, b_2, \ldots, b_{n_i} \) of finite type \( G \). Let

\[
S_{ij} = Z(u_i) \cap \{|b_j| < \frac{1}{2}\} \cap D;
\]

then

\[
Z(u_i) \cap D = \bigcup_{j=1}^{n_i} S_{ij}.
\]

By removing overlapping elements, we may assume the \( S_{ij} \)'s are disjoint.

Since \( b_j \) is of type \( G \), by Lemma 2.1 of [Gu-Iz-2], there is \( \delta < 1 \) such that

\[
S_{ij} \subset \{|b_j| < \frac{1}{2}\} \subset \{z \in D : \rho(z, \{z_{j,n}\}) \leq \delta\},
\]

where \( \{z_{j,n}\} \) is the root sequence of \( b_j \) in \( D \). For each disk \( B(z_{j,n}, \delta) = \{z \in D : \rho(z, z_{j,n}) \leq \delta\} \), there is at most \( k_i \) elements of \( S_{ij} \) in \( B(z_{j,n}, \delta) \), where \( k_i \) depends only on \( \delta_i \). So \( S_{ij} \) is the union of at most \( k_i \) sequences, each of which has at most one element in each \( B(z_{j,n}, \delta) \). By the analog of Proposition 2 for finite type \( G \), proved above, the Blaschke product with root sequence \( S_{ij} \) is a product of at most \( k_i \) interpolating Blaschke product of finite type \( G \). So \( u_i \) is a finite product of at most
In general, it is a difficult problem to determine the closure of an infinite set of interpolating Blaschke products among the family of all Blaschke products (see for example [Li]). However, for Blaschke products of type \( G \) and finite type \( G \), their closures can be identified because of Proposition 1 and Theorems 7, 11.

**Theorem 12.** Let \( B \) be the family of all Blaschke products with essential sup-norm. The closure of all interpolating Blaschke products of type \( G \) in \( B \) is the set of all finite products of interpolating Blaschke products of type \( G \). Also, the closure of all interpolating Blaschke products of finite type \( G \) in \( B \) is the set of all finite products of interpolating Blaschke products of finite type \( G \).

**Proof.** Suppose \( B \) is in the closure of type \( G \) products. Take \( B \) of type \( G \) such that \( \|B^0 - b\|_\infty = \|1 - Bb\|_\infty < 1 \). It follows that \( Bb \) is invertible in \( E \) and so is \( B = (Bb)b \). By Theorem 7, \( B \) is a finite product of interpolating Blaschke products of type \( G \).

For the converse, it suffices to show for \( B_1, B_2 \) of type \( G \) and \( \varepsilon > 0 \), there is \( B \) of type \( G \) such that \( \|B_1B_2 - B\|_\infty < \varepsilon \). Let the root sequences of \( B_1 \) and \( B_2 \) be \( \{z_n\} \) and \( \{w_n\} \), respectively. Since each of these sequences are separated,

\[
\delta_I = \inf\{\rho(z_m, z_n), \rho(w_m, w_n) : m \neq n\} > 0.
\]

By Hoffman’s lemma (see Lemma 1.4 on pages 404-5 of [Gar]), there are \( \delta_0, \varepsilon_0 < \delta_I/3 \) such that

\[
V_n \subset \{z \in D : \rho(z, z_n) < \varepsilon_0\},
\]

\[
W_n \subset \{z \in D : \rho(z, w_n) < \varepsilon_0\}
\]

and

\[
\left\| \frac{z - z - (\delta_0/2)}{1 - (\delta_0/2)z} \right\|_\infty < \varepsilon,
\]

where \( V_n \) and \( W_n \) are the components of \( \{z \in D : |B_1(z)| < \delta_0\} \) and \( \{z \in D : |B_2(z)| < \delta_0\} \) containing \( z_n \) and \( w_n \), respectively. Since \( 3\varepsilon_0 < \delta_I \), we have

\[
\rho(V_n, V_m) > \delta_I/3 \text{ and } \rho(W_n, W_m) > \delta_I/3 \text{ for } n \neq m.
\]

Factor \( B_2 = B_3B_4 \) so that \( w_n \in Z(B_3) \) if \( \rho(w_n, Z(B_1)) \geq \delta_0/4 \) and \( w_n \in Z(B_4) \) if \( \rho(w_n, Z(B_1)) < \delta_0/4 \). Let

\[
B_5 = \frac{B_4 - (\delta_0/2)}{1 - (\delta_0/2)B_4}
\]

and \( B = B_1B_3B_5 \), then \( \|B_1B_2 - B\|_\infty = \|B_4 - B_5\|_\infty < \varepsilon \).

Next we will show \( B \) is an interpolating Blaschke product. Since \( \rho(Z(B_1), Z(B_3)) \geq \delta_0/4 \), \( B_1B_3 \) is an interpolating Blaschke product. By Lemma 1.4 on pages 404-5 of [Gar], \( B_5 \) is an interpolating Blaschke product.
To see \( \rho(Z(B_1B_3), Z(B_5)) > 0 \), let \( w \in Z(B_5) \) and \( z \in Z(B_1B_3) \). Then \( w \in W_n \) for some \( n \) and \( w_n \in Z(B_4) \). If \( m \neq n \), then
\[
\frac{\delta_0}{2} = \rho(B_5(w), B_5(w_n)) \leq \rho(w, w_n) < \epsilon_0 < \rho(w, w_m).
\]

Since \( w_n \in Z(B_4) \), we have \( \rho(w_n, z_k) < \delta_0/4 \) for some \( z_k \in Z(B_1) \).

In the case \( z \in Z(B_1) \) and \( \rho(z, Z(B_4)) < \epsilon_0/4 \), there is \( w_m \in Z(B_4) \) such that \( \rho(z, w_m) < \delta_0/4 \) and so
\[
\rho(w, z) \geq \rho(w, w_m) - \rho(w_m, z) \geq \frac{\delta_0}{2} - \frac{\delta_0}{4} = \frac{\delta_0}{4}.
\]

In the case \( z \in Z(B_1) \) and \( \rho(z, Z(B_4)) \geq \delta_0/4 \), we have \( z \neq z_k \), hence \( \rho(z, z_k) \geq \delta_I \). So
\[
\rho(w, z) \geq \rho(z, z_k) - \rho(z_k, w_n) - \rho(w_n, w) \geq \delta_I - \frac{\delta_0}{4} - \epsilon_0 \geq \frac{\delta_I}{2}.
\]

In the case \( z \in Z(B_3) \), we have
\[
\rho(w, z) \geq \rho(z, w_n) - \rho(w_n, w) \geq \delta_I - \epsilon_0 \geq \frac{2\delta_I}{3}.
\]

So \( \rho(Z(B_1B_3), Z(B_5)) > 0 \). Therefore, \( B = B_1B_3B_5 \) is an interpolating Blaschke product.

Finally since \( B_1, B_2 \) are of type \( G \), by Proposition 1, \( B \) is of type \( G \). This completes the proof of the statement for the closure of type \( G \). For the closure of finite type \( G \), use Theorem 11 instead of Theorem 7 and repeat the above proof. \( \square \)

4. Questions

Let \( A_{\text{loc}} \) be the closed subalgebra of \( H^\infty \) generated by the locally thin Blaschke product and let \( E_{\text{loc}} = [H^\infty; A_{\text{loc}}] \).

1. Does Theorem \( \Box \) hold if \( E \) is replaced with \( E_{\text{loc}} \)?
2. Does \( E = E_{\text{loc}} \) or does \( E_{\text{loc}} = E_{\text{fin}} \), or neither?

If we let \( A^* \) be the closed subalgebra of \( H^\infty \) generated by the thin Blaschke products, and set \( E^* = [H^\infty; A^*] \), the main result of Hedenmalm [He, Theorem 2.6] asserts that Theorem \( \Box \) holds for \( E^* \). By Proposition 1.1 and Example 3.1 of [Gu-Iz-2], there exists a Blaschke product \( b \) of finite type \( G \), which is not a finite product of thin Blaschke products. It follows that \( E^* \not\subseteq E_{\text{fin}} \) because otherwise \( E_{\text{fin}} = E^* \) would contain \( H^\infty[b] \) forcing \( b \) to be a finite product of thin Blaschke products by [He, Theorem 2.6]. Also, by the proof of Theorem 2 of [Gu-Iz-1], there exists a Blaschke product of type \( G \), but not of finite type \( G \). So we have \( E_{\text{fin}} \not\subseteq E \). By Lemma 1 of [Gu], it can be shown that \( E_{\text{fin}} \subseteq E_{\text{loc}} \), but it is not clear whether \( E_{\text{loc}} \subseteq E \) or \( E_{\text{loc}} \geq E \).
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