Series with Binomial-like Coefficients for the Investigation of Fractal Structures Associated with the Riemann Zeta Function

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Abstract: The paper continues the study of efficient algorithms for the computation of zeta functions over the complex plane. We aim to apply the modifications of algorithms to the investigation of underlying fractal structures associated with the Riemann zeta function. We discuss the computational complexity and numerical aspects of the implemented algorithms based on series with binomial-like coefficients.

Keywords: Riemann zeta function; fractal structures; numerical algorithms

1. Introduction

In this paper, we continue the study of efficient algorithms for computation of the Riemann zeta function over the complex plane, introduced by Borwein [1] and extended by Belovas et al., (see [2,3] and references therein). Šleževičienė [4], Vepštas [5], and Coffey [6] applied this methodology for the computation of Dirichlet $L$-functions, Hurwitz zeta function, and polylogarithm. Belovas et al. obtained limit theorems, which allowed the introduction of asymptotic approximations for the coefficients of the series of the algorithms. A preliminary presentation of computational aspects of the approach has been presented in [3]. Theoretical aspects of the approach (as well as more subtle proofs of the limit theorems) have been discussed in [7].

Fractal geography of the Riemann zeta function (and other zeta functions) was addressed by King [8]. Woon [9] and Tingen [10] computed Julia and Mandelbrot sets of the Riemann zeta function and Hurwitz zeta function, respectively, and studied the properties of these fractals. Recently Blankers et al. [11] investigated the analogs of Julia and Mandelbrot sets for dynamical systems over the hyperbolic numbers. In the present study, we enhance algorithms for the calculation of the Riemann zeta function, proposed in [2,3]. We specify the convergence rate to the limiting distribution for the coefficients of the series, identify the error term, and discuss computational complexity. The algorithms are compared against the recently proposed Zetafast algorithm [12] and are applied for the investigation of underlying fractal structures associated with the Riemann zeta function.

The paper is organized as follows. The first part is the introduction. In Section 2, we describe algorithms and present theoretical results. Section 3 is devoted to the visual investigation of the underlying fractal background of the Riemann zeta function. Pseudocodes of the algorithms for the computation and the visualization are given in Section 4. Sections 5 and 6 are devoted to presenting the results and conclusions, respectively.

Throughout this paper, $U \times V$ stands for the Cartesian product of sets $U$ and $V$. We denote by $\Phi(x)$ the cumulative distribution function of the standard normal distribution, and by $\Gamma(s)$ we denote the gamma function. Next, $[x]$ and $\lceil x \rceil$ stand for the floor function and the ceiling functions, respectively. All limits in the paper, unless specified, are taken as $n \to \infty$. 
2. MB- and BLC-Algorithms for the Computation of the Riemann Zeta Function

Let \( s = \sigma + it \) be a complex variable. The Riemann zeta-function is defined on the half-plane \( \sigma > 1 \) by the ordinary Dirichlet series or the Euler product formula,

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1},
\]
and by analytic continuation for other complex values. The Riemann zeta function is a meromorphic function (holomorphic on the whole complex plane except for a simple pole at \( s = 1 \) with residue 1). The Riemann zeta function satisfies the functional equation

\[
\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s),
\]

implying that \( \zeta(s) \) has simple zeros at \( s = -2n, \ n \in \mathbb{N} \), known as the trivial zeros. Other zeroes are called nontrivial. The famous Riemann hypothesis states that all the nontrivial zeros lie on the critical line \( s = 1/2 + it \). The hypothesis is closely related to the distribution of prime numbers, implying the best possible error term in the prime number theorem,

\[
\pi(x) = \int_{2}^{x} \frac{dt}{\log t} + O(\sqrt{x} \log x).
\]

Here \( \pi(x) \) is the prime-counting function, i.e., the number of primes less than or equal to \( x \), \( x \in \mathbb{R} \). A summary of the literature covering the problems related to the Riemann zeta function and its applications is presented in [13,14] and the references therein.

2.1. MB-Algorithm

In [3] Belovas et al., proposed a modification of Borwein’s efficient algorithm (MB-algorithm) for the computation of the Riemann zeta function [1]. The algorithm applies to complex numbers with \( \sigma \geq 1/2 \) and arbitrary \( t \). The Riemann zeta function is represented by the alternating series

\[
\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{k=0}^{n-1} \frac{(-1)^k \psi_{n,k}^{(j)}}{(k+1)^s} + \gamma_{n}^{(j)}(s). \tag{1}
\]

Here (case \( j = 1 \) in \( \psi_{n,k}^{(j)} \) corresponds MB-series), by Theorem 1 from [3], we have

\[
\psi_{n,k}^{(1)} = 1 - \frac{H_k}{H_n} \quad \text{and} \quad l_{\max} = \arg \max_{0 \leq k \leq n} \frac{(n + k - 1)!4^k}{(n-k)!(2k)!}, \quad n \in \mathbb{N}, \ 0 \leq k \leq n, \tag{2}
\]

while

\[
H_l = H_{l-1} + \exp(T_l - T_{l_{\max}} + (l - l_{\max}) \log 4), \quad H_0 = \exp(T_0 - T_{l_{\max}} - l_{\max} \log 4),
\]

\[
T_l = T_{l-1} + \log \frac{(n - l + 1)(n + l - 1)}{(2l - 1)(2l)}, \quad T_0 = -\log n, \quad 1 \leq l \leq n. \tag{3}
\]

The algorithm is nearly optimal in the sense that there is no sequence of \( n \)-term exponential polynomials that converge to the Riemann zeta function much faster than of the algorithm (see Theorem 3.1 in [1]).

2.2. BLC-Algorithm

This algorithm, introduced in [2], also uses series (1) (case \( j = 2 \) in \( \psi_{n,k}^{(j)} \) corresponds BLC-series), but with different binomial-like coefficients,

\[
\psi_{n,k}^{(2)} = I_{1/2}(k + 1, n - k + 1). \tag{4}
\]
Here \( I_x(a,b) \) stands for the regularized incomplete beta function,
\[
I_x(a,b) = \int_0^x t^{a-1} (1-t)^{b-1} dt / \int_0^1 t^{a-1} (1-t)^{b-1} dt.
\]

The error terms \( \gamma_n^{(j)}(s) \) of these methods are discussed in the following subsection.

### 2.3. Error Terms and Computational Complexity

First we formulate an auxiliary lemma, aiming to investigate the behaviour of the series in the neighbourhoods of critical points \( \tau_k \),
\[
\tau_k = 1 + 2\pi k / \log 2, \quad k \in \mathbb{N}_0.
\] (5)

Note that in (1) the denominator \( 1 - 2^{1-s} = 0 \) if and only if \( \Im s = 2\pi k / \log 2, \quad k \in \mathbb{Z} \) and \( \Re s = 1. \)

**Lemma 1.** Let \( \tau_k \) be defined by (5) and \( \omega_k \) be the circle
\[
\omega_k = \{s : |s - \tau_k| = \rho > 0\}.
\]

Then, for \( f(s) = 1/(1 - 2^{1-s}) \) and \( \rho \leq 3 / \log 2 \),
\[
\max_{s \in \omega_k} |f(s)| \leq \frac{1}{1 - 2^{-\rho}}.
\] (6)

**Proof of Lemma 1.** Parametrizing the complex function \( f(s) \) for the circle \( \omega_k \), we obtain
\[
g(\phi) := f(\tau_k + \rho e^{i\phi}) = 1 / (1 - 2 -2\pi k / \log 2 - \rho (\cos \phi + i \sin \phi)).
\]
\[
:= u(\phi)
\]

Next,
\[
|u(\phi)| = |1 - 2^{-\rho \cos \phi} (\cos(\rho \log 2 \sin \phi) - i \sin(\rho \log 2 \sin \phi))|
\]
\[
= (1 - 2^{1-\rho \cos \phi} \cos(\rho \log 2 \sin \phi) + 2^{-2\rho \cos \phi})^{1/2}.
\]
\[
:= v(\phi)
\]

The function \( v(\phi) \) is periodic with period \( 2\pi \) and symmetric with respect to \( \phi = \pi \) (indeed, \( v(\pi - \chi) = v(\pi + \chi) \)). Hence the statement of the lemma reduces to solving
\[
\min_{0 \leq \phi \leq \pi} v(\phi).
\]

Differentiating \( v(\phi) \), we get for \( 0 < \phi < \pi \)
\[
v'(\phi) = 2^{1-\rho \cos \phi} \rho \log 2 \cdot (2^{-\rho \cos \phi} \sin \phi - \sin \phi \cos(\rho \log 2 \sin \phi) + \cos \phi \sin(\rho \log 2 \sin \phi)) > 0.
\]
\[
:= w(\phi) > 0
\]

Indeed, with \( r = \rho \log 2 \) and
\( (r, \varphi) \in (0, 3) \times (0, \pi/2) \), we have
\[
\psi(\varphi) = e^{-r \cos \varphi} \sin \varphi - \sin \varphi \cos (r \sin \varphi) + \cos \varphi \sin (r \sin \varphi)
\]
\[
> \left( 1 - r \cos \varphi + \frac{1}{2} (r \cos \varphi)^2 - \frac{1}{6} (r \cos \varphi)^3 \right) \sin \varphi
\]
\[
- \left( 1 - \frac{1}{2} (r \sin \varphi)^2 + \frac{1}{24} (r \sin \varphi)^4 \right) \sin \varphi + \left( r \sin \varphi - \frac{1}{6} (r \sin \varphi)^3 \right) \cos \varphi
\]
\[
= \frac{1}{24} r^2 \sin \varphi \left( 12 - 4r \cos \varphi - r^2 \sin^4 \varphi \right) > 0.
\]

We have shown that \( \psi(\varphi) > 0 \) for \( (r, \varphi) \in ((0, 3) \times (0, \pi/2)) \cup ((0, 3) \times (\pi/2, \pi)) \). Note that \( \psi(\pi/2) > 0 \) for \( r \in (0, 3) \). Thus the function \( \psi(\varphi) \) is monotonically increasing and
\[
\psi_{\text{min}} = \min_{0 \leq \varphi \leq \pi} \psi(\varphi) = \psi(0) = (1 - 2^{-1})^2,
\]
with (7) and (8) yielding us the statement of the lemma.

The error term and the computational complexity are closely linked to the problem of the selection of the minimal number of terms in the series (1). Let us formulate the following theorem.

**Theorem 1.** Let \( \sigma \geq 1/2, t \geq 0, \epsilon > 0 \) and \( |s - \tau_\epsilon| \geq \epsilon \), then

(i) the error term of the series (1) is
\[
|\tau_n^{(i)} (s)| \leq \Theta_n^{(j)} (\cosh t)^{1/2} |1 - 2^{-1-\epsilon}|,
\]  
(ii) the series (1) to compute the Riemann zeta-function with \( d \) decimal digits of accuracy, require a number of terms
\[
n^{(j)} = \left[ D_1^{(j)} t + D_2^{(j)} d + D_{\epsilon}^{(j)} \right],
\]
with coefficients of expressions (9) and (10) presented in Table 1.

| Table 1. Coefficients of expressions (9) and (10). |
|-------|-------|-------|-------|-------|
| \( f \) | \( \Theta_n^{(j)} \) | \( D_1^{(j)} \) | \( D_2^{(j)} \) | \( D_{\epsilon}^{(j)} \) |
| 1 | \( \frac{2}{3 + \sqrt{8}} \) | \( \frac{\pi/2}{\log(3 + \sqrt{8})} \) | \( \log 10 \) | \( \log 2 \) |
| 2 | \( \frac{1}{2^{1/2}} \) | \( \frac{\pi/2}{\log 2} \) | \( \log 10 \) | \( \log 2 - \log(1 - 2^{-1}) \) |
Proof of Theorem 1. Let us start with MB-series. The error term of the series (1) is
(cf. Algorithm 2 in [1])

\[ |\gamma_n^{(1)}(s)| \leq \frac{2}{(3 + \sqrt{8})^n} \frac{1}{|1 - 2^{1-s}|} \frac{1}{|\Gamma(s)|} \int_0^1 \frac{(-\log x)^{\sigma-1}}{1 + x} \, dx. \]  

(11)

Considering the function \( I(\sigma) \), we have

\[ I(\sigma) \leq \int_0^1 (-\log x)^{\sigma-1} \, dx = \Gamma(\sigma). \]  

(12)

By a product representation of the gamma function (cf. 8.326.1 in [15]),

\[ \left| \frac{\Gamma(\sigma)}{\Gamma(s)} \right| = \prod_{n=0}^{\infty} \left(1 + \frac{t^2}{(\sigma + n)^2}\right). \]

The product is decreasing by \( \sigma \), hence (cf. 8.332.2 in [15]),

\[ \frac{\Gamma(\sigma)}{\Gamma(s)} \leq \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + it)} = \frac{\sqrt{\pi}}{\sqrt{\cosh \pi t}} = \sqrt{\cosh \pi t}. \]  

(13)

Hence,

\[ |\gamma_n^{(1)}(s)| \leq \frac{2}{(3 + \sqrt{8})^n} \frac{\sqrt{\cosh \pi t}}{|1 - 2^{1-s}|}. \]  

(14)

In view of (14), to compute the Riemann zeta-function with \( d \) decimal digits of accuracy, the approach requires a number \( n \) of terms not less than

\[ N_d(\sigma, t) = \frac{\log 2 + d \log 10 + \frac{1}{2} \log \cosh \pi t - \log |1 - 2^{1-s}|}{\log(3 + \sqrt{8})} \]

\[ = \frac{\pi t + \log(1 + e^{-2\pi t}) + 2 + 2d \log 10 - 2 \log |1 - 2^{1-s}|}{2 \log(3 + \sqrt{8})} \]

\[ \leq \frac{\pi t + 2d \log 10 - 2 \log |1 - 2^{1-s}| + 2 \log 2}{2 \log(3 + \sqrt{8})} \]  

(15)

1°. Let \(|\sigma - 1| > \varepsilon\). We have

\[ N_d(\sigma, t) \leq \frac{\pi t + 2d \log 10 - 2 \log |1 - 2^{1-\varepsilon}| + 2 \log 2}{2 \log(3 + \sqrt{8})} \]

\[ \leq \frac{\pi/2}{\log(3 + \sqrt{8})} t + \frac{\log 10}{\log(3 + \sqrt{8})} d + \frac{\log 2 - \log(1 - 2^{-\varepsilon})}{\log(3 + \sqrt{8})}. \]  

(16)

2°. Let \(|s - \sigma| \geq \varepsilon\) and \(|\sigma - 1| \leq \varepsilon\). By applying the maximum modulus principle and Lemma 1, we receive

\[ N_d(\sigma, t) \leq \frac{\pi t + 2d \log 10 - 2 \log |1 - 2^{-\varepsilon}| + 2 \log 2}{2 \log(3 + \sqrt{8})} \]

\[ = \frac{\pi/2}{\log(3 + \sqrt{8})} t + \frac{\log 10}{\log(3 + \sqrt{8})} d + \frac{\log 2 - \log(1 - 2^{-\varepsilon})}{\log(3 + \sqrt{8})} \]

\[ := D_1^{(1)} \leq D_2^{(1)} \leq D_3^{(1)}. \]  

(17)
thus concluding the proof. The deduction for BLC-series is analogical.

**Corollary 1.** Under the conditions of Theorem 1, for \( \varepsilon = 10^{-m}, m \in \mathbb{N} \), the series (1) to compute the Riemann zeta-function with \( d \) decimal digits of accuracy, requires the number of terms

\[
n^{(j)} = \left[ D_1^{(j)} t + D_2^{(j)} (d + m) \right] + 2 - j. \tag{18}\]

**Proof of Corollary 1.** The result (18) follows immediately, if we notice that for \( \varepsilon \to 0 \) we have

\[
\log(1 - 2^{-\varepsilon}) = \log \varepsilon + \log \log 2 + o(1).
\]

2.4. **NA-Modifications of MB- and BLC-Algorithms**

Limit theorems for coefficients of MB- and BLC-series enable us to derive a normal approximation for coefficients \( \psi_{n,k}^{(j)} \) (cf. (24) in [3]). We can formulate the following theorem.

**Theorem 2.** Coefficients \( \psi_{n,k}^{(j)} \) of the series (1) satisfy

\[
\psi_{n,k}^{(j)} = 1 - \Phi \left( \frac{k - \mu_{n}^{(j)}}{\sigma_{n}^{(j)}} \right) + O \left( \frac{1}{\sqrt{n}} \right). \tag{19}\]

Coefficients \( \mu_{n}^{(j)} \) and \( \sigma_{n}^{(j)} \) are presented in Table 2.

**Table 2.** Coefficients of the expression (19).

| \( j \) | \( \mu_{n}^{(j)} \) | \( \sigma_{n}^{(j)} \) |
|-------|---------------|---------------|
| 1     | \( \frac{n}{\sqrt{2}} \)   | \( \frac{\sqrt{\pi}}{\sqrt{8}} \) |
| 2     | \( \frac{n}{2} \)      | \( \frac{\sqrt{\pi}}{2} \)  |

**Proof of Theorem 2.** Let us start with MB-series coefficients. Suppose \( A_n \) is an integral random variable with the probability mass function

\[
P(A_n = k) = \frac{u_{n,k}}{\sum_{j=0}^{n} u_{n,j}}, \quad k = 0, \ldots, n. \tag{20}\]

Here (cf. (1) in [3])

\[
u_{n,k} = n \frac{(n + k - 1)4^k}{(n - k)!2^{2k}}, \quad n \in \mathbb{N}, \quad 0 \leq k \leq n.
\]

Thus,

\[
\psi_{n,k}^{(1)} = 1 - \frac{\sum_{j=0}^{k} u_{n,j}}{\sum_{j=0}^{n} u_{n,j}}. \tag{22}\]

Let \( F_n(x) \) be the cumulative distribution function of the random variable \( A_n \) (20), then (cf. Theorem 3 in [7])

\[
F_n(x + \mu_{n}^{(1)}) = \Phi(x) + O \left( \frac{1}{\sqrt{n}} \right), \quad x \in \mathbb{R}. \tag{23}\]
Note that the cumulative distribution function
\[ F_n\left( \sigma_n x + \mu_n \right) = \sum_{j \in \mathbb{Z}} \frac{\mu_{nj}}{\sigma_n x + \mu_n}. \]

Denoting \( k = [\sigma_n x + \mu_n] \) and taking into account (22) and (23), we obtain
\[ 1 - \Psi_{n,k}^{(1)} = \Phi\left( \frac{k - \mu_n^{(1)}}{\sigma_n^{(1)}} \right) + O\left( \frac{1}{\sqrt{n}} \right). \]

The first part of the theorem follows. Similar result for BLC-coefficients \( \Psi_{n,k}^{(2)} \) has been proven in [2].

Theorem 2 allows us to choose the number of terms \( n^{(j)} \) for the series (1),
\[ n^{(j)} = \left[ \mu_n^{(j)} + z_d \sigma_n^{(j)} \right], \tag{24} \]
for \( n \) large enough. Here \( z_d = \Phi^{-1}(1 - 10^{-d}) \). Note that
\[ n^{(1)} \sim \frac{\pi}{2\sqrt{2 \log(3 + \sqrt{8})}}, \quad n^{(2)} \sim \frac{\pi}{4\log 2}, \tag{25} \]
for fixed \( \sigma \) and \( d \). The refined version of NA-modification based methodology is summarized in Section 4.

2.5. Empirical Insights for NA-Modifications

While performing practical computations using NA-algorithms, we have noticed that the values produced were significantly more accurate than otherwise implied by \( d \) in the analytic estimate (10). In order to increase the performance and to have a clear course for future theoretical refinements, we propose empirical formulae for the minimum number of terms in the series (1) to compute the Riemann zeta-function with \( d \) decimal digits of accuracy.

Kuzma has proposed the following empirically-based estimate for the number of terms for the BLC-series \( \sigma = 1/2 \), \( \tau \in [1000, 1050] \),
\[ n^{(0)} = \left[ 0.67658827t + 113.26486067 \right]. \tag{25} \]

In the present section, we offer an improvement to this estimate.

Figure 1 displays the minimum \( n \) required to calculate the Riemann zeta function with \( d = 6 \) digits of accuracy using NA- and BLC-algorithms at \( \sigma = 1/2 \), \( t \in [1000, 1050] \) (the blue curve). The curves have clearly visible periodic peaks (marked by red vertical lines). The peaks have a period of \( \lambda = 2\pi / \log 2 \), which correspond to special points of Theorem 1. Since we are interested in the upper bound of this empirical curve, for the following calculations we use the points \( t = \lambda k, k \in \mathbb{N} \).
Figure 1. Periodic peaks of the minimum number of terms (the blue curves) in series (1) for \( d = 6 \) digits of accuracy at \((\sigma, t) \in 1/2 \times [1000, 1050]\). The curves have clearly visible periodic peaks, marked by red vertical lines.

Figure 2 shows regression models

\[ n^{(j)} = \left[ a^{(j)} t + b^{(j)} \sqrt{t} + c^{(j)} \right] \]  

(26)

derived for \( d \in [1, 10] \) using the points \((\sigma, t) \in 1/2 \times (0, 10,000)\). Each graph represents a fitted curve for a different \( d \) value.

Figure 3 illustrates fluctuations of the coefficients of the regression models (26) by \( d \). Here we can clearly see that \( a^{(1)} \) has no correlation with \( d \) while \( b^{(1)} \) and \( c^{(1)} \) does.

Fitting \( b^{(j)} \) with \( b^{(j)} = x \sqrt{d} + y \) and \( c^{(j)} \) with \( c^{(1)} = xd + y \) we obtain the following coefficients for (26) (see Table 3):

| \( j \) | \( a^{(j)} \) | \( b^{(j)} \) | \( c^{(j)} \) |
|-------|-------------|-------------|-------------|
| 1     | 0.451       | 1.407 \sqrt{d} − 0.245 | 0.371d + 0.195 |
| 2     | 0.637       | 2.026 \sqrt{d} − 0.272 | 1.602d − 0.026 |
3. Visualizations of Fractal Structures Associated with the Riemann Zeta Function

3.1. Methods of the Visualization

In this study we employ two methods to reveal the Riemann zeta function underlying nature. The first heuristic method (FH-method) calculates RGB colors of the graph of the Riemann zeta function, using a composition of special functions. Suppose we have a function \( f : (\mathbb{R}, \mathbb{C}) \rightarrow \mathbb{N}_0 \):

\[
f(x, z) = \begin{cases} 
    \lfloor x \log |z| \rfloor, & \text{if } z \neq 0, \\
    0, & \text{if } z = 0.
\end{cases}
\]  

(27)

Now we can define functions \( f_1, f_2, f_3 \):

\[
f_1(x, z) = f(\eta_1, \zeta(s)), \quad f_2(x, z) = f(\eta_2, \Re(\zeta(s))), \quad f_3(x, z) = f(\eta_3, \Im(\zeta(s))).
\]  

(28)

Next, we calculate \((R, G, B)\) colors of each pixel of the graph of the Riemann zeta function using polynomial functions of \( f_k \) (see Table 4):

\[
R = s_1(x, f_1, f_2, f_3) \mod 256, \\
G = s_2(x, f_1, f_2, f_3) \mod 256, \\
B = s_3(x, f_1, f_2, f_3) \mod 256.
\]

Table 4. List of \( s_k \) functions.

| \( l \) | \( s_1 \) | \( s_2 \) | \( s_3 \) |
|---|---|---|---|
| 1 | \( f_1 \) | \( f_2 \) | \( f_3 \) |
| 2 | \( 255 - f_1f_2f_3 \) | \( f_2f_3 \) | \( 255 - f_2 \) |
| 3 | \( f_1 \) | \( f_2 \) | \( f_3 \) |
| 4 | \( f_1f_3 \) | \( f_2 \) | \( f_3 \) |
| 5 | \( f_1f_3 \) | \( f_2f_3 \) | \( f_3 \) |

The second approach (second fractal heuristic (SFH) method) is based on the application of the Mandelbrot set to the visualization of the Riemann zeta function. Suppose we

Figure 3. Coefficients of the regression models \( a^{(1)}, b^{(1)} \) and \( c^{(1)} \) plotted against the decimal digits of accuracy.
aim to visualize $\zeta(\sigma + it)$ for $(\sigma, t) \in (\sigma_1, \sigma_2) \times (t_1, t_2)$. First, we introduce the log-transformation for each point $(x, y)$ of the graph,

\[
\begin{align*}
x &= L(\Re(\zeta(\sigma + it))), \\
y &= L(\Im(\zeta(\sigma + it))),
\end{align*}
\]

thus obtaining the set $Q = (x_{\text{min}}, y_{\text{min}}) \times (x_{\text{max}}, y_{\text{max}})$. Here

\[
L(x) = \begin{cases} 
\log |x|, & \text{if } x \neq 0, \\
0, & \text{if } x = 0.
\end{cases}
\]

Next, we linearly transform $Q$ into the subset $S$ of the complex plane, $(x, y) \in Q \rightarrow (x^*, y^*) \in S$.

We take $S = (-2, 0.47) \times (-1.12i, 1.12i)$, where the Mandelbrot set is defined. Then we use an algorithm to generate the Mandelbrot set, setting the start position at $z_0 = 0$ and $z^* = (x^*, y^*)$:

\[
z_{k+1} \leftarrow z_k^2 + z^*.
\]

Suppose that $k \in \mathbb{N}, k \leq v_{\text{max}}$ indicates the number of iterations (31), required to ascertain that $z^*$ does not belong to the Mandelbrot set, with

\[|z_{k+1}| \leq 2 \quad \text{and} \quad k < v_{\text{max}}.\]

For $k = v_{\text{max}}$, it is unclear if $z^*$ does not belong to the Mandelbrot set. Now let $k_0 = \lfloor 50k \rfloor$. We calculate RGB color for the $z^*$ point by the following rule:

\[
\text{RGB} = \begin{cases} 
(0, 0, 0), & \text{if } k = v_{\text{max}}, \\
(255, 255, k_0 \mod 256), & \text{if } 510 < k_0 < v_{\text{max}}, \\
(100, k_0 \mod 256, 255), & \text{if } 255 < k_0 \leq 510, \\
(0, 0, k_0 \mod 256), & \text{if } k_0 \leq 255.
\end{cases}
\]

3.2. Visual Investigations

The first visualization (see Figure 4) reveals the underlying structures in the “center” $S_1 \subset \mathbb{C}$ of the Riemann zeta function, received by two different methods (the color visualization and the fractal visualization). Here $S_1 = (-20, 8) \times (-14, 14)$. Figure 4a is obtained using $FH$-method with color parameters $\eta_1 = 100$ and $\eta_2 = \eta_3 = 8$. The color transform $g^{(1)}_k$ is linear (see Table 4). Figure 4b is obtained using $SFH$-method. Note small bright fractal feature on the right-hand side, calling for in-depth investigation.

Figure 5 presents zoom-in frames of $S_2$ region for the Riemann zeta function. Here $S_2 = (-5, 6) \times (\beta, \alpha + \beta)$, with four shifted in $\beta$ intervals (see Table 5 for the ranges). The frames were received using $FH$-method with color parameters $\eta_1 = 100$ and $\eta_2 = \eta_3 = 8$. The color transform $g^{(1)}_k$ is linear (see Table 4). Note nontrivial zeros of the Riemann zeta function (blue disks, marked with arrows in Figure 5a,b).
Figure 4. The structures of the “center” of the Riemann zeta function, \((\sigma, t) \in (-20, 8) \times (-14, 14)\), received by \(SH\) and \(SFH\) methods. Note small fractal feature on the right-hand side of (b).

Figure 5. \(FH\)-based zoomed-in frames of the Riemann zeta function (see Table 5 for the ranges). Note nontrivial zeros of the Riemann zeta function (blue disks, marked with arrows in Figure 5a,b).
Table 5. Ranges of the sets of Figure 5: \((\sigma, t) \in S_2, \alpha = 50\).

| Figure | \(\beta\) | \(S_2\) |
|--------|----------|---------|
| Figure 5a | 0         | \((-6, 5) \times (0, 50)\) |
| Figure 5b | 500      | \((-5, 6) \times (500, 550)\) |
| Figure 5c | 1000     | \((-5, 6) \times (1000, 1050)\) |
| Figure 5d | 5000     | \((-5, 6) \times (5000, 5050)\) |

Figure 6 (obtained by SFH-method) extends the investigation of the fractal feature, associated with the Riemann zeta function, observed in Figure 4b. The frame Figure 6a represents zoomed-in image of the feature in the range \((0.2, 2.2) \times (-1.6, 1.6)\). The frame Figure 6b is the next magnification step, belonging to the range \((0.95, 1.05) \times (-0.08, 0.08)\). Fractal structures received in Figure 6b are examined further in Figure 7.

![Figure 6](image)

(a) (b)

Figure 6. Fractal features of the Riemann zeta function in the pole area (see Table 6 for the ranges). (a) gives zoomed-in image of the feature observed in Figure 4b. (b) represents the next magnification step (red rectangle).

Table 6. Ranges of the sets of Figure 6, \((\sigma, t) \in S_3\).

| Figure | \(S_3\) |
|--------|---------|
| Figure 6a | \((0.20, 2.20) \times (-1.60, 1.60)\) |
| Figure 6b | \((0.95, 1.05) \times (-0.08, 0.08)\) |

Figure 7a displays zoomed-in frame of the fractal border presented in Figure 6b. The next five frames (each of them corresponds to a colored rectangle in Figure 7a) uncover some aesthetically pleasing features of fractal structures associated with the Riemann zeta function. Note snowflake-shaped fractals in Figure 7c, as well as pinwheel-shaped ones in Figure 7d,e, resembling discs of spiral galaxies. Clockwise spinning Figure 7e reminds us of the grand design spiral galaxy NGC 4254 in Coma Berenices. Counter-clockwise rotating Figure 7d resembles the Pinwheel Galaxy NGC 5457 in Ursa Major. Invariant features of fractal geometry generated from images provide a good set of descriptive values for the recognition of regions and objects, e.g., fractal signatures of galaxies are examined with the aim of classifying them (cf. [16]). Figure 7 is received by SFH-method. The ranges of the sets are given in Table 7.
Figure 7. Fractal structures associated with the near-pole region of the Riemann zeta function. Frames (b–f) are zoomed-in rectangles of (a). Ranges of the sets are given in Table 7.
Table 7. Ranges of the frames of Figure 6.

| Figure | $\sigma_1$     | $\sigma_2$     | $t_1$          | $t_2$          |
|--------|----------------|----------------|----------------|----------------|
| Figure 7a | 1.30000 | 1.04000 | -0.034000 | -0.024000 |
| Figure 7b | 1.03730 | 1.03925 | -0.029875 | -0.027925 |
| Figure 7c | 1.03730 | 1.03925 | -0.029875 | -0.027925 |
| Figure 7d | 1.03385 | 1.03550 | -0.033200 | -0.031550 |
| Figure 7e | 1.03410 | 1.03485 | -0.026000 | -0.025250 |
| Figure 7f | 1.03035 | 1.03150 | -0.032850 | -0.031700 |

Figure 8 illustrates other facets of the geography of the Riemann zeta function. Graphs for the range $(-30, 10) \times (-14, 16)$ are obtained using four different non-linear color transformations $g_k^{(l)}$, where $g_1^{(l)} \neq f_1$ or $g_2^{(l)} \neq f_3$ or $g_3^{(l)} \neq f_3$. Color parameters are given in Table 8.

Table 8. Color parameters of Figure 8.

| Figure | $\eta_1$ | $\eta_2$ | $\eta_3$ | $g_k^{(1)}$ |
|--------|----------|----------|----------|-------------|
| Figure 8a | 10       | 1        | 2        | $g_k^{(2)}$  |
| Figure 8b | 90       | 17       | 50       | $g_k^{(3)}$  |
| Figure 8c | 9        | 7        | 5        | $g_k^{(4)}$  |
| Figure 8d | 1        | 2        | 1        | $g_k^{(5)}$  |
Algorithm 1

This algorithm will return multiple values of the Riemann zeta function for fixed \( t \) and array \( \{r_j\} \). Note that \( L_k = \log k \) stand for precalculated logarithms.

1: procedure ZETA.M(\( \sigma : \text{array } [1..N] \) of real numbers; \( d, m, j : \text{natural numbers}; t : \text{real number} \)) ▷ (see Table 9)
2: \( n \leftarrow \begin{cases} \left\lceil \frac{(\pi/2)(t + (d + m)L_{10})}{\log(3 + \sqrt{8})} \right\rceil + 1, & j = 1, \\ \left\lceil \frac{(\pi/2)(t + (d + m)L_{10})}{L_2} \right\rceil, & j = 2, \\ a^{(j-4)t} + b^{(j-4)}T + c^{(j-4)}, & j = 5 \text{ or } j = 6 \end{cases} \)
3: if \( j \) is odd then ▷ MB\(^1\)- and EMB\(^5\)-block
4: \( T_0 \leftarrow -L_n, \ l_{\text{max}} \leftarrow \lfloor n/\sqrt{2} \rfloor \)
5: for \( l \in \{1..n\} \) do
6: \( T_l \leftarrow T_{l-1} + L_{n-l+1} + L_{n+l-1} - L_{2l-1} - L_{2l} \)
7: end for
8: \( H_0 \leftarrow \exp(T_0 - T_{\text{max}} - l_{\text{max}}L_4) \)
9: for \( l \in \{1..n\} \) do
10: \( H_l \leftarrow H_{l-1} + \exp(T_l - T_{\text{max}} + (l - l_{\text{max}})L_4) \)
11: end for
12: for \( k \in \{0..n\} \) do
13: \( \psi_{n,k}^{(j)} \leftarrow (1 - H_k/H_n)(\cos(tL_{k+1}) - i \sin(tL_{k+1})) \)
14: end for
15: else ▷ BLC\(^2\)- and EBLK\(^6\)-block
16: for \( k \in \{0..n\} \) do
17: \( \psi_{n,k}^{(j)} \leftarrow (\cos(tL_{k+1}) - i \sin(tL_{k+1})) \text{betainc}(k + 1, n - k + 1, 0.5) \)
18: end for
19: end if
20: \( \lambda \leftarrow 2(\cos(tL_2) - i \sin(tL_2)) \)
21: for \( r \in \{1..N\} \) do ▷ Calculation of MB- or BLC-series for the corresponding \( \sigma_r \)
22: \( S \leftarrow 0, \ p \leftarrow -1 \)
23: for \( k \in \{0..n\} \) do
24: \( p \leftarrow -p \)
25: \( S \leftarrow S + p\psi_{n,k}^{(j)} \exp(-\sigma_rL_{k+1}) \)
26: end for
27: \( S_r \leftarrow S/(1 - \lambda \exp(-\sigma_rL_2)) \)
28: end for
29: return \( S \) ▷ Returns the array \( S[1..N] \) of the Riemann zeta function values
30: end procedure
Algorithm 2 This algorithm will return values of the Riemann zeta function obtained by NA-modifications of MB- or BLC-method. Note that $L_k = \log k$ and $t > 10^3$.

1: function ZETA.NA($\sigma, t; d, m, j : \text{real numbers}; n : \text{natural number}$)
2: $n \leftarrow (\pi/2)t + (d + m)L_{10}$, $z \leftarrow \Phi^{-1}(1 - 10^{-d})$
3: if $j = 1$ then \(\triangleright\) NAMB-block
4: $n \leftarrow (n + L_2 - \log L_2)/\log(3 + \sqrt{8})$, $\mu_n \leftarrow n/\sqrt{2}$, $\sigma_n \leftarrow \sqrt{n}/\sqrt{32}$
5: else
6: $n \leftarrow (n - L_2 - \log L_2)/L_2$, $\mu_n \leftarrow n/2$, $\sigma_n \leftarrow \sqrt{n}/2$
7: end if
8: $k_0 \leftarrow \lfloor\mu_n + \sigma_n\rfloor$, $k_1 \leftarrow \mu_n - \sigma_n$
9: function $\psi(n,k)$ : nonnegative integers
10: if $k < k_1$ then
11: $\psi \leftarrow 1$
12: else
13: $\psi \leftarrow 1 - \Phi((k - \mu_n)/\sigma_n)$
14: end if
15: end function
16: $S \leftarrow 0$, $p \leftarrow -1$
17: for $k \in \{0, k_0\}$ do
18: $p \leftarrow -p$
19: $S \leftarrow S + p\psi(n,k)\exp(-\sigma L_{k+1})(\cos(t L_{k+1}) - i \sin(t L_{k+1}))$
20: end for
21: return $S/(1 - 2\exp(-\sigma L_2)(\cos(t L_2) - i \sin(t L_2)))$
22: end function

4.2. Visualization Algorithms

The third algorithm (Algorithm 3), corresponding the first heuristic method (FH-method), calculates RGB colors of the graph of the Riemann zeta function, using a composition of special functions.

Algorithm 3 This algorithm will return a colored image of Riemann zeta function for $(\sigma, t) \in (\sigma_{\min}, \sigma_{\max}) \times (t_{\min}, t_{\max})$. Other parameters: $\eta_1, \eta_2, \eta_3$—color parameters, $g_1, g_2, g_3$—polynomial functions of $f_1, f_2, f_3$ (see Table 4), $w$—width in pixels of output image $\text{img}$.

1: procedure FH($\sigma_{\min}, \sigma_{\max}, t_{\min}, t_{\max}, a, b, c : \text{real numbers}; w : \text{natural number}$)
2: $h \leftarrow w \cdot (t_{\max} - t_{\min})/(\sigma_{\max} - \sigma_{\min})$
3: $\text{img} \leftarrow []$
4: for $j \in \{0..h - 1\}$ do
5: $\text{row} \leftarrow []$
6: $t \leftarrow t_{\min} + j \cdot (t_{\max} - t_{\min})/(h - 1)$
7: for $k \in \{0..w - 1\}$ do
8: $\sigma \leftarrow \sigma_{\min} + k \cdot (\sigma_{\max} - \sigma_{\min})/(w - 1)$
9: $z \leftarrow \zeta(\sigma + it)$
10: $f_1 \leftarrow |\eta_1 \log |z||$
11: $f_2 \leftarrow |\eta_2 \log |R(z)||$
12: $f_3 \leftarrow |\eta_3 \log |\Im(z)||$
13: $g_1 \leftarrow g_1(f_1, f_2, f_3)$
14: $g_2 \leftarrow g_2(f_1, f_2, f_3)$
15: $g_3 \leftarrow g_3(f_1, f_2, f_3)$
16: $\text{RGB} \leftarrow [g_1 \text{ mod } 256, g_2 \text{ mod } 256, g_3 \text{ mod } 256]$
17: $\text{row} \leftarrow \text{row} + \text{RGB}$
18: end for
19: $\text{img} \leftarrow \text{img} + \text{row}$
20: end for
21: end procedure
The fourth algorithm (Algorithm 4), corresponding the second fractal heuristic method (SFH-method), employs the Mandelbrot set to visualize the Riemann zeta function.

Algorithm 4 This algorithm will return fractalized image of Riemann zeta function for \((\sigma, \tau) \in (\sigma_{\text{min}}, \sigma_{\text{max}}) \times (\tau_{\text{min}}, \tau_{\text{max}})\). Here \(m\) stands for max iterations to get more precise fractal image, \(w\)—width in pixels of output image \(img\). The output image utilizes yellow-black-blue color palette.

1: \textbf{procedure} SFH\((\sigma_{\text{min}}, \sigma_{\text{max}}, \tau_{\text{min}}, \tau_{\text{max}} : \text{real numbers}; w, m : \text{natural numbers})\)
2: \hspace{1em} \(h \leftarrow \lfloor w \cdot (\tau_{\text{max}} - \tau_{\text{min}})/(\sigma_{\text{max}} - \sigma_{\text{min}}) \rfloor\)
3: \hspace{1em} \(img \leftarrow [\]
4: \hspace{1em} \(w_1 \leftarrow 2.47/(\sigma_{\text{max}} - \sigma_{\text{min}})\)
5: \hspace{1em} \(w_2 \leftarrow (0.47\sigma_{\text{min}} + 2\sigma_{\text{max}})/(\sigma_{\text{min}} - \sigma_{\text{max}})\)
6: \hspace{1em} \(w_3 \leftarrow 2.24/(\tau_{\text{max}} - \tau_{\text{min}})\)
7: \hspace{1em} \(w_4 \leftarrow 1.12(\tau_{\text{min}} + \tau_{\text{max}})/(\tau_{\text{min}} - \tau_{\text{max}})\)
8: \hspace{1em} \textbf{for} \(j \in \{0..h - 1\} \textbf{ do}\)
9: \hspace{2em} \(row \leftarrow [\]
10: \hspace{3em} \(t \leftarrow \tau_{\text{min}} + j \cdot (\tau_{\text{max}} - \tau_{\text{min}})/(h - 1)\)
11: \hspace{3em} \textbf{for} \(k \in \{0..w - 1\} \textbf{ do}\)
12: \hspace{4em} \(\sigma \leftarrow \sigma_{\text{min}} + k \cdot (\sigma_{\text{max}} - \sigma_{\text{min}})/(w - 1)\)
13: \hspace{4em} \(z \leftarrow \zeta(\sigma + it)\)
14: \hspace{4em} \(z^* \leftarrow w_1\text{sign}(\Re(z)) \log |\Re(z)| + w_2 + (w_3\text{sign}(\Im(z)) \log |\Im(z)| + w_4)i\)
15: \hspace{4em} \(z \leftarrow 0\)
16: \hspace{4em} \(n \leftarrow 0\)
17: \hspace{4em} \textbf{while} \(|z| \leq 2 \text{ and } n < m \textbf{ do}\)
18: \hspace{5em} \(z \leftarrow z^2 + z^*\)
19: \hspace{5em} \(n \leftarrow n + 1\)
20: \hspace{4em} \textbf{end while}\)
21: \hspace{4em} \(RGB \leftarrow [0, 0, 0]\)
22: \hspace{4em} \textbf{if} \(|z| > 2 \text{ then}\)
23: \hspace{5em} \(l \leftarrow \lfloor 50n \rfloor\)
24: \hspace{5em} \textbf{if} \(l > 510 \text{ then}\)
25: \hspace{6em} \(RGB \leftarrow [255, 255, l \mod 256]\)
26: \hspace{5em} \textbf{else if} \(l > 255 \text{ then}\)
27: \hspace{6em} \(RGB \leftarrow [100, l \mod 256, 255]\)
28: \hspace{5em} \textbf{else}\n29: \hspace{6em} \(RGB \leftarrow [0, 0, l \mod 256]\)
30: \hspace{5em} \textbf{end if}\n31: \hspace{4em} \textbf{end if}\n32: \hspace{4em} \(row \leftarrow row + RGB\)
33: \hspace{4em} \textbf{end for}\n34: \hspace{1em} img \leftarrow img + row\)
35: \hspace{1em} \textbf{end for}\n36: \textbf{end procedure}\n
5. Numerical Experiments

We have performed numerical experiments with seven methods and modifications listed in Table 9.
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5.1. First Numerical Experiment

The first numerical experiment deals with normal approximation-based modifications (cf. Algorithm 2). Using NAMB ($j = 3$), NABLC ($j = 4$) and Zetafast ($j = 7$) methods we generate sequences of values of the Riemann zeta function $\{\zeta_{j,p}^{(l)}\}$, $1 \leq l \leq N$, $N = 10^5$, taking as arguments uniformly distributed $s_{l,p} \in S_p^{(1)}$. Here

$$S_p^{(1)} = (0.5, 1.5) \times (s_{kp} + \rho_1, s_{kp+1} - \rho_1), \quad (32)$$

where $s_{kp}$ stand for critical points (5) with $k_p = 2^{j+6}$, $1 \leq p \leq 3$, and $\rho_1 = 10^{-1}$. Thus we obtain 9 sequences overall (3 algorithms $\times$ 3 sets of arguments). Using Zetafast algorithm as a benchmark we calculate the accuracy $\delta_p^{(j)}$ and the relative performance $\theta_p^{(j)}$, $3 \leq j \leq 4$,

$$\delta_p^{(j)} = \max_{1 \leq l \leq N} |\zeta_{j,p}^{(l)} - \zeta_{j,p}^{(7)}|, \quad \theta_p^{(j)} = \tau_p^{(j)} / \tau_p^{(7)}, \quad 3 \leq j \leq 4, \quad (33)$$

where $\tau_p^{(j)}$ is the processing time of $j$th sequence $\{\zeta_{j,p}^{(l)}\}$, $1 \leq l \leq N$, for fixed $p$. The results of the first numerical experiment are presented in Table 10.

5.2. Second Numerical Experiment

The second numerical experiment aims to verify the accuracy of the algorithms on fixed horizontal lines, close to critical points. Using MB ($j = 1$) and BLC ($j = 2$) methods, their empirical modifications ($j = 5$ and $j = 6$) and Zetafast method ($j = 7$), we generate (cf. Algorithm 1) sequences of values of the Riemann zeta function $\{\zeta_{j,p}^{(l)}\}$, $1 \leq l \leq N$, $N = 10^3$, taking as arguments uniformly distributed $s_{l,p} \in S_p^{(2)}$. Here

$$S_p^{(2)} = (0.5, 1.5) \times t_p, \quad t_p = s_{kp} + \rho_1, \quad k_p = 2^{j+6}, \quad 1 \leq p \leq 3. \quad (34)$$

| Method | $j$ | $S_1^{(1)}$ | $S_2^{(1)}$ | $S_3^{(1)}$ |
|--------|-----|-------------|-------------|-------------|
| NAMB   | 3   | 1.80 × 10^{-11} | 1.60 × 10^{-11} | 2.90 × 10^{-11} |
|        |     | 0.088       | 0.12        | 0.18        |
| NABLC  | 4   | 1.82 × 10^{-11} | 1.74 × 10^{-11} | 3.35 × 10^{-11} |
|        |     | 0.22        | 0.32        | 0.45        |
| ZF     | 7   | 86.72       | 121.04      | 172.95      |

Table 10. Results of the first numerical experiment: accuracy $\delta_p^{(j)}$ and relative performance $\theta_p^{(j)}$, for $d = 6$, $m = 1$. The last line of the table shows the performance of ZF-algorithm (sec).
Thus we obtain 15 sequences overall (5 algorithms × 3 sets of arguments). Using the Zetafast algorithm as a benchmark we calculate the accuracy $\delta_p^{(i)}$ and the relative performance $\theta_p^{(i)}$ (cf. (33)). The results of the second numerical experiment are presented in Table 11.

The numerical experiments have been performed on Intel® Core™ i7-8750H 2.2 GHz (boosted to 4.0 GHz) processor with 16 GB DDR4 RAM. The code has been compiled with g++ 11.2.0 compiler using O3 optimization. C++ Boost library has been used for the implementation of the incomplete beta function for BLC-algorithm.

Table 11. Results of the second numerical experiment: accuracy $\delta_p^{(i)}$ and relative performance $\theta_p^{(i)}$ on fixed lines $t_p$, for $d = 6, m = 1$. The last line shows the performance of ZF-algorithm (sec).

| Method | $j$ | $S_1^{(2)}$ | $S_2^{(2)}$ | $S_3^{(2)}$ |
|--------|-----|-------------|-------------|-------------|
| MB     | 1   | $1.68 \times 10^{-11}$ | $1.46 \times 10^{-11}$ | $2.65 \times 10^{-11}$ |
|        |     | 0.04        | 0.055       | 0.078       |
| BLC    | 2   | $1.77 \times 10^{-11}$ | $1.55 \times 10^{-11}$ | $2.64 \times 10^{-11}$ |
|        |     | 0.1         | 0.15        | 0.2         |
| EMB    | 5   | $6.43 \times 10^{-7}$  | $5.62 \times 10^{-7}$  | $5.51 \times 10^{-7}$  |
|        |     | 0.024       | 0.032       | 0.044       |
| EBLC   | 6   | $7.07 \times 10^{-7}$  | $7.78 \times 10^{-7}$  | $7.84 \times 10^{-7}$  |
|        |     | 0.034       | 0.048       | 0.065       |
| ZF     | 7   | 86.64       | 121.29      | 173.14      |

6. Discussion and Concluding Remarks

6.1. Discussion of the Results

We have refined the error terms and the expressions for the minimal number of terms in MB- and BLC-series of efficient algorithms for the computation of the Riemann zeta function, taking into account the behavior of the series in the neighborhoods of critical points. Theorem 1 shows that MB-based algorithms converge faster than BLC-based algorithms. Indeed, the MB-coefficient of the error term $\Theta_n^{(1)} = O(0.172^n)$ while $\Theta_n^{(2)} = O(0.5^n)$ (cf. (9)). However, BLC-approach has its advantages that might be useful in analytical research (cf. (4)). Note that this deficiency of the MB-algorithm is solved by the introduction of NA-modification (19).

The results of the numerical experiments (see Tables 10 and 11) show that MB and BLC methods, along with their normal and empirical modifications, allow fast and accurate calculations of the Riemann zeta function for large values of argument $t$. The results demonstrate that the introduced modifications accelerate computations of the Riemann zeta function, compared to Zetafast method. These versions of algorithms are well-suited for distributed computations and grid computing.

6.2. Findings of Visual Investigations of Fractal Structures, Associated with the Riemann Zeta Function

The illustrations obtained using FH-method clearly show the arrangement of trivial and non-trivial zeros of the Riemann zeta function in the complex plane (see Figure 5a,b). In addition to these points, we can also see dark 2D curves that satisfy the conditions $\Re(\zeta(\sigma + it)) = 0$ and $\Im(\zeta(\sigma + it)) = 0$ (see Figure 4a). The SFH-method distributes deformed copies of the Mandelbrot set in the complex plane, thus relating the values of the Riemann zeta function to the fractal structure. This allows for a visual assessment of essential changes in the Riemann zeta function values. Next, SFH-approach reveals notable symmetric fractals characterizing the neighborhood of the pole of the Riemann zeta function (see Figures 6 and 7).
6.3. Future Research Directions

Numerical experiments with empirical formulas indicate that the theoretical selection of the number of terms of the series \( n \) can be reduced. Next, the accuracy of the normal approximation-based modifications of \( MB \) and \( BLC \) algorithms might be refined by employing the theory of large deviations. The figures presented in this work reveal areas of the complex plane where the modulus of the Riemann zeta function exhibits very volatile values. This allows us to investigate the complex plane regions of \( R_{\zeta}(s) = \Im(\zeta(s)) \), thus enabling us to locate non-trivial zeros' positions visually. In future works, these visual instruments could be refined.

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Abbreviations

The following abbreviations are used in this manuscript:

\( MB \) Modification of Borwein’s algorithm  
\( BLC \) Binomial-like coefficients  
\( NA \) Normal approximation  
\( FH \) First heuristic  
\( SFH \) Second fractal heuristic  
\( NGC \) New General Catalogue of Nebulae and Clusters of Stars

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