CMC Graphs With Planar Boundary in $\mathbb{H}^2 \times \mathbb{R}$

Ari J. Aiolfi, Patrícia Klaser
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Abstract

Let $\Omega \subset \mathbb{R}^2$ an unbounded convex domain and $H > 0$ be given, there exists a graph $G \subset \mathbb{R}^3$ of constant mean curvature $H$ over $\Omega$ with $\partial G = \partial \Omega$ if and only if $\Omega$ is included in a strip of width $1/H$ [7, 12]. In this paper we obtain results in $\mathbb{H}^2 \times \mathbb{R}$ in the same direction: given $H \in (0, 1/2)$, if $\Omega$ is included in a region of $\mathbb{H}^2 \times \{0\}$ bounded by two equidistant hypercycles $\ell(H)$ apart, we show that, if the geodesic curvature of $\partial \Omega$ is bounded from below by $-1$, then there is an $H$-graph $G$ over $\Omega$ with $\partial G = \partial \Omega$. We also present more refined existence results involving the curvature of $\partial \Omega$, which can also be less than $-1$.

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1 Introduction

Surfaces of constant mean curvature (cmc) in Riemannian manifolds are a classical subject in Differential Geometry. There are many existence results on closed cmc surfaces and on cmc surfaces with boundary (Plateau problem). The Plateau problem becomes a Dirichlet PDE problem if one searches surfaces that are graphs. In this paper we deal with existence results of cmc graphs with boundary included in a plane $\mathbb{H}^2 \times \{0\}$ in the manifold $\mathbb{H}^2 \times \mathbb{R}$.

It is quite natural that existence results for graphs with planar boundary can be obtained with weaker hypotheses than the ones for the Dirichlet problem for any continuous boundary data. In the Euclidean case, for example, given a bounded convex $C^{2, \alpha}$ domain $\Omega \subset \mathbb{R}^2$, there is a cmc graph with boundary $\partial \Omega$ if the curvature of $\partial \Omega$ ($k_{\partial \Omega}$) is greater than $H$, while the existence of a cmc $H$-graph over $\Omega$ taking any continuous boundary data is only guaranteed by $k_{\partial \Omega}$ bounded from below by $2H$. Besides, the a priori height estimate $|u| < a$ for some $a < 1/2H$ also guarantees the existence of cmc graphs with vanishing boundary data in bounded convex $C^{2, \alpha}$ domains (Theorem 3 of [12]). As a consequence of this result, we draw attention to the following result about unbounded convex domains:

Theorem 1.1 (Theorem 1.2 and 1.4 of [7] or Corollary 3 of [12]) The Dirichlet problem with zero boundary data associated to the cmc equation can be solved for convex domains included in a strip of width $1/H$. 

In [7], it is also proved that if \( \Omega \) is an unbounded convex domain and there exists a cmc \( H \) graph over \( \Omega \) with boundary \( \partial \Omega \), then \( \Omega \) must be included in a strip of width \( 1/H \). We are interested in generalize this results to \( \mathbb{H}^2 \times \mathbb{R} \), that is: Given \( \Omega \subset \mathbb{H}^2 \times \{0\} \), when is there a cmc \( H \) surface with boundary \( \partial \Omega \)?

The non existence result presented in [7] relies on the compactness of the cmc \( H \) sphere in \( \mathbb{R}^3 \) and on the structure of convex sets in the Euclidean plane. Unfortunately we could not obtain non existence results in \( \mathbb{H}^2 \times \mathbb{R} \).

An interesting point about Theorem 1.1 is that the lower bound on \( k_{\partial \Omega} \) does not depend on \( H \). Such a result also holds in \( \mathbb{H}^2 \times \mathbb{R} \), see Corollary 3.7, item (4): Suppose \( H \in (0,1/2) \) and let \( \Omega \subset \mathbb{H}^2 \) be a \( C^2 \) domain with \( k_{\partial \Omega} \geq -1 \) and \( \Omega \) contained in a region bounded by two hypercycles which are equidistant \( \ell = \ell(H) \) to a fixed geodesic, where \( \ell(H) \) is given by (25). Then there exists an \( H \)-graph with boundary \( \partial \Omega \).

Let \( \mathbb{H}^2 \) be the hyperbolic plane, \( \Omega \subset \mathbb{H}^2 \) a \( C^2,\alpha \) domain and \( H > 0 \), and consider the Dirichlet problem

\[
\begin{cases}
Q_H (u) := \text{div} \left( \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) + 2H = 0 \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega \text{ and } u \in C^2(\Omega) \cap C^0(\overline{\Omega}),
\end{cases}
\]

where \( \nabla \) and \( \text{div} \) are the gradient and divergent in \( \mathbb{H}^2 \), respectively. If \( u \) is a solution of the Dirichlet problem \( \textbf{1} \) then the graph of \( u \), denoted by \( G(u) \) and oriented with normal vector pointing downwards, is a cmc \( H \) surface in \( \mathbb{H}^2 \times \mathbb{R} \) with boundary \( \partial G(u) \subset \mathbb{H}^2 \times \{0\} = \mathbb{H}^2 \).

We show that, considering a (possibly unbounded) domain \( \Omega \) contained in a region of \( \mathbb{H}^2 \) bounded by two equidistant hypercycles, under hypotheses involving the distance between them, their curvatures, the curvature of \( \partial \Omega \) and assuming \( H \in (0,1/2) \), problem \( \textbf{1} \) is solvable (Corollaries 3.5 and 3.7).

These results are consequence of a technical theorem (Theorem 3.2), which was inspired in Theorem 3 of [12] and, essentially states that for \( H \in (0,1/2) \) and \( \Omega \) bounded, setting \( \kappa := \inf_{\partial \Omega} k_{\partial \Omega} \), if any solution \( u \) of \( \textbf{1} \) satisfies the \textit{a priori} height estimate \( |u| \leq a < F(\kappa,H) \) (see Remark 3.3 for an expression for \( F \)), with an additional hypothesis on \( \Omega \) in the case \( \kappa \leq -1 \), then \( \textbf{1} \) is solvable. We observe that the case \( \kappa > 2H \) for bounded domains is already considered in Theorem 1.5 of [14].

There are many height estimates for embedded compact \( H \)-surfaces \( S \subset \mathbb{M}^2 \times \mathbb{R} \), \( H > 0 \), with boundary in a slice \( \mathbb{M}^2 \times \{t_0\} \), given in terms of different hypotheses such as the area of \( S \cap \mathbb{M}^2 \times \{t \geq t_0\} \), the Gauss map of the immersion (see [8], [9] for the Euclidean case) or the Gauss curvature of \( \mathbb{M}^2 \) (see [4], [1] and, for Hadamard manifolds, see [6] and generalization for warped products in [2]). Beautiful existence results for \( H \)-graphs in \( \mathbb{R}^3 \) with boundary in a plane were obtained as a consequence of such estimates, with hypotheses on the length of \( \partial \Omega \) or the area of \( \Omega \) (see, for example, Corollary 4 of [8] and Corollary 5 of [12]).

We point out that, in the Euclidean case, the Dirichlet problem \( \textbf{1} \) for
arbitrary continuous boundary data can be reduced to the zero boundary data (see [13]) and it is quite natural to expect similar reductions to work for $\mathbb{H}^2 \times \mathbb{R}$.

2 Preliminaries

Given a domain $\Omega \subset \mathbb{R}^2$ with $k_{\partial \Omega} > H$, by using a hemisphere as a barrier one estimates the height of a solution to the Dirichlet problem [1]. Afterwards, moving the hemisphere slightly downwards and touching it at each point of $\partial \Omega$, one estimates the gradient at the boundary. These estimates and the classical PDE theory guarantee the existence of a constant mean curvature $H$ graph over $\Omega$ with boundary $\partial \Omega$. A natural way to improve this result would be to ask whether it holds for unbounded domains, and therefore the hypothesis $k_{\partial \Omega} > H$ cannot be required anymore. Hence a surface that works as barrier must have its boundary in $\mathbb{R}^2$ with curvature smaller than $H$. The simplest non spherical cmc surface is the cylinder, which if we consider having a horizontal axis is a bi-graph over a strip of width $1/H$. In [12] and in [7], it is used as a barrier to prove the result stated in the abstract.

Our main goal in this manuscript is to extend the result above to $\mathbb{H}^2 \times \mathbb{R}$. Nevertheless $\mathbb{H}^2 \times \mathbb{R}$ is less symmetric than $\mathbb{R}^3$ and so are the surfaces that play the role of the half cylinders. However, it is possible to find surfaces that work as barriers in our context, which is done in the following lemmas. None of the surfaces described here is new in the literature, see for instance [10] and [11] among many others. We recall them here in order to fix some notation and describe some important properties. The authors believe that the least known property is an estimate on the size of the set where the barriers are positive (inequalities (12) and (16) below), which was obtained in [5].

Lemma 2.1 Let $H \in (0, 1/2)$ and $l > 0$ be given. Let $\alpha_1$ and $\alpha_2$ be two hypercycles in $\mathbb{H}^2$, equidistant $l$ of a given geodesic of $\mathbb{H}^2$. There is an $H$-graph $G = G(l, H)$ over the connected region in $\mathbb{H}^2$ bounded by $\alpha_1 \cup \alpha_2$, contained in $\mathbb{H}^2 \times \{t \geq 0\}$, with $\partial G = \alpha_1 \cup \alpha_2$ and whose height is

$$h_H(l) := \frac{2H}{\sqrt{1 - 4H^2}} \ln \left[ \frac{\sqrt{1 - 4H^2} + \sqrt{1 - 4H^2 \tanh^2(l)}}{\sqrt{1 - 4H^2 + 1}} \cosh(l) \right]$$

Proof. Let $\gamma$ be the given geodesic. Set $d(z) = d_{\mathbb{H}^2}(z, \gamma)$, $z \in \mathbb{H}^2$. Let $\psi \in C^2([0, \infty))$ be given by

$$\psi (d) = 2H \int_{d}^{l} \frac{\tanh s}{\sqrt{1 - 4H^2 \tanh^2 s}} ds,$$

which is well-defined for $H \in (0, 1/2)$. Then observe that for a function of the form $u = \tilde{u} \circ d$ the PDE in [1] rewrites as

$$\left( \frac{\tilde{u}' (d)}{\sqrt{1 + [\tilde{u}' (d)]^2}} \right)' + \left( \frac{\tilde{u}' (d)}{\sqrt{1 + [\tilde{u}' (d)]^2}} \right) \tanh d + 2H = 0.$$
Therefore $u(z) = \psi(d(z))$ satisfies (1) in the domain bounded by $\alpha_1 \cup \alpha_2$ and it is non negative.  

**Definition 2.2** Given $H \in (0, 1/2)$, $r > \tanh^{-1}(-2H)$ and $\rho > 0$, we define 

$$c_H(r, t) = \frac{\cosh r + 2H (\sinh r - \sinh(r + t))}{\cosh(r + t)}, \quad t \geq 0,$$

$$s_H(\rho, t) = \frac{\sinh \rho + 2H (\cosh \rho - \cosh(\rho + t))}{\sinh (\rho + t)}, \quad t \geq 0,$$

$$z_H(r) = \sinh^{-1} \left( \frac{\sinh r + \cosh r}{2H} \right) - r,$$

$$Z_H(\rho) = \cosh^{-1} \left( \frac{\cosh \rho + \sinh \rho}{2H} \right) - \rho,$$

$$a_H(r) = \int_0^{z_H(r)} \frac{c_H(r, t)}{\sqrt{1 - |c_H(r, t)|^2}} dt,$$

$$A_H(\rho) = \int_0^{z_H(\rho)} \frac{s_H(\rho, t)}{\sqrt{1 - |s_H(\rho, t)|^2}} dt$$

and we denote by $\delta_H(r)$ and $\Delta_H(\rho)$ the positive numbers given by

$$\int_0^{\delta_H(r)} \frac{c_H(r, t)}{\sqrt{1 - |c_H(r, t)|^2}} dt = 0 \quad \text{and} \quad \int_0^{\Delta_H(\rho)} \frac{s_H(\rho, t)}{\sqrt{1 - |s_H(\rho, t)|^2}} dt = 0. \quad (10)$$

**Remark 2.3** The functions $c_H(r, \cdot)$ and $s_H(\rho, \cdot)$ are well defined in $[0, \infty)$, and satisfy $|c_H(r, t)| < 1, |s_H(r, t)| < 1$ for $t > 0$. Both are decreasing functions which assume the value zero at $z_H$ and $Z_H$, respectively. Despite taking the value 1 at $t = 0$, the integrals in (10) are well defined. Besides, we may define

$$a_H(\tanh^{-1}(-2H)) := \lim_{r \to \tanh^{-1}(-2H)} \int_0^{z_H(r)} \frac{c_H(r, t)}{\sqrt{1 - |c_H(r, t)|^2}} dt = +\infty. \quad (11)$$

**Lemma 2.4** Let $H \in (0, 1/2)$ and $r > \tanh^{-1}(-2H)$ be given. Let $\alpha$ be a hypercycle in $\mathbb{H}^2$ with geodesic curvature $k_{\alpha} = -\tanh r$. There are a hypercycle $\beta$ in $\mathbb{H}^2$, equidistant to $\alpha$, which satisfies

$$d(\alpha, \beta) = \delta_H(r) \geq 2z_H(r) \quad (12)$$

and an $H$-graph $G = G(r, H)$ over the connected component of $\mathbb{H}^2$ bounded by $\alpha \cup \beta$, contained in $\mathbb{H}^2 \times \{ t \geq 0 \}$ and such that $\partial G = \alpha \cup \beta$. The height of $G$ is $a_H(r)$.

**Proof.** Once again we use the distance function to turn the Dirichlet problem (1) into an ODE problem.

Let $A$ be the connected component of $\mathbb{H}^2 \setminus \alpha$ such that $k_{\beta A} = -\tanh r$ with the inner orientation. Set $d(z) = d_{\mathbb{H}^2}(z, \alpha)$, $z \in A$. For $\gamma$ being the geodesic
equidistant to $\alpha$, if the distance between $\gamma$ and $\alpha$ is $|r|$ and $\gamma \subset A$, observe that $r < 0$. Otherwise, if $\gamma \subset \mathbb{H}^2 \setminus A$, then $r > 0$.

Take $\tilde{u} \in C^2((0,\infty)) \cap C^0([0,\infty))$ satisfying $\tilde{u}(0) = 0$, $\tilde{u}'(d) \to +\infty$ when $d \to 0$ (see [5] for details), and such that the graph of $u = \tilde{u} \circ d \in C^2(A) \cap C^0(\overline{A})$ is an $H$-graph. Then

$$\tilde{u}(d) = \int_0^d \frac{c_H(r,t)}{\sqrt{1 - [c_H(r,t)]^2}} \, dt,$$

(13)

and clearly $u|_{\alpha} = 0$. It is immediate to see that $z_H(r)$ is the maximum point of the function $\tilde{u}$ and $\tilde{u}$ goes to $-\infty$ when $d \to +\infty$. Set

$$\beta := (G(u) \setminus \alpha) \cap (\mathbb{H}^2 \times \{0\}).$$

Notice that $\beta \subset A$ is a hypercycle which satisfies $d(\alpha,\beta) = \delta_H(r)$, where $\delta_H(r)$ is given by (10).

Now, we prove that $\tilde{u}$ is non-negative in $[0,2z_H(r)]$, that is $2z_H(r) \leq \delta_H(r)$. Notice that is enough to show that

$$|\tilde{u}'(z_H(r) + s)| \leq |\tilde{u}'(z_H(r) - s)|, \quad s \in (0, z_H(r)),$$

which is equivalent to $-\tilde{u}'(z_H(r) + s) \leq \tilde{u}'(z_H(r) - s)$.

Since

$$\tilde{u}'(s) = \frac{c_H(s)}{\sqrt{1 - [c_H(s)]^2}},$$

and $x \mapsto x \left(1 - x^2\right)^{-1/2}$ is increasing in the interval $(0,1)$, it is enough to show that

$$-c_H(z_H(r) + s) \leq c_H(z_H(r) - s).$$

(14)

We have $0 = \tilde{u}'(z_H(r)) = c_H(z_H(r))$ and, then, we can rewrite $c_H(t)$ as

$$c_H(t) = \frac{2H [\sinh(r + z_H(r)) - \sinh(r + t)]}{\cosh(r + t)}.$$  

(15)

Plugging (15) in (14), expanding cosh and sinh of sums and observing that

$$\sinh(r + z_H(r)) = \sinh r + \frac{\cosh r}{2H},$$

a straightforward computation gives us that (14) holds and the result follows.

As we will see in the next section, the graphs given by the lemmas above will provide our barriers relatively to the Dirichlet problem $\mathbb{H}$ in the case $-1 < \inf_{\partial \Omega} k_{\partial \Omega} \leq 2H$. Relatively to the case $\inf_{\partial \Omega} k_{\partial \Omega} \leq -1$, we also have barriers. They are pieces of $H$-nodoids, described in the following lemma:
Lemma 2.5 Let $H \in (0, 1/2)$ and $\rho > 0$ be given. Let $C_\rho \subset \mathbb{H}^2$ be a (hyperbolic) circle of radius $\rho$. There is a compact $H$-graph $G = G(\rho, H)$ contained in $\mathbb{H}^2 \times \{\text{boundary}\}$, over an annulus in $\mathbb{H}^2$ whose boundary is the union of the concentric circles $C_\rho$ and $C_{\rho^*}$, with

$$\rho^* := \rho + \Delta_\rho (\rho) \geq \rho + 2Z_\rho (\rho).$$

Moreover, the height of $G$ is $A_\rho (\rho)$. The quantities $\Delta_\rho (\rho)$, $Z_\rho (\rho)$ and $A_\rho (\rho)$ were presented in Definition 2.2.

Proof. Let $B_\rho \subset \mathbb{H}^2$ be the open disk of radius $\rho$ such that $\partial B_\rho = C_\rho$ and set $A = \mathbb{H}^2 \setminus B_\rho$ and $d(z) = d_{\mathbb{H}^2} (z, C_\rho)$, $z \in A$. The function $u = \tilde{u} \circ d \in C^2 (A) \cap C^0 (\overline{A})$ given by

$$\tilde{u} (d) = \int_0^d \frac{s_\rho (\rho, t)}{\sqrt{1 - [s_\rho (\rho, t)]^2}} \, dt$$

satisfies $Q_\rho (u) = 0$ in $A$, with $u|_{C_\rho} = 0$. Moreover, $\tilde{u}$ has its maximum point at $Z_\rho (\rho)$ and is non negative in $[0, 2Z_\rho (\rho)]$. The proof of these facts follows the same steps of the lemma above, just noting that, now, $\Delta d = \coth (\rho + d)$ and the conditions on the function $\tilde{u}$ are $\tilde{u} (0) = 0$ and $\tilde{u}' (d) \to +\infty$ when $d \to 0$ (see [5] for details).

The next result presents a relation between the barriers constructed in the previous lemmas.

Proposition 2.6 Let $H \in (0, 1/2)$ be given. The function $a_\rho$ is decreasing, the function $A_\rho$ is increasing and both have the same limit at infinity, which is

$$a_\rho (\infty) = \frac{\pi}{2} - \frac{4H}{\sqrt{1 - 4H^2}} \tanh^{-1} \left( \frac{1 - 2H}{\sqrt{1 - 4H^2}} \right).$$

Besides, if $\tanh^{-1} (-2H) \leq r < 0$, then $h_\rho (|r|) < a_\rho (r)$ and $|r| < z_\rho (r)$, where $h_\rho$ is given by (2) and the other quantities were presented in Definition 2.2.

Proof. First notice that straightforward computations give us that $Z_\rho$ and $z_\rho$ are increasing and decreasing functions, respectively, with the same limit at infinity

$$z_\rho (\infty) = \log \left( \frac{1}{2H} + 1 \right)$$

Also, we have $\partial_{\rho^*} (\rho, t) > 0$, $\partial_{\rho^*} (r, t) < 0$ and

$$\lim_{\rho \to \infty} \frac{s_\rho (\rho, t)}{\sqrt{1 - [s_\rho (\rho, t)]^2}} = \frac{-2H + (1 + 2H)e^{-t}}{\sqrt{1 - (-2H + (1 + 2H)e^{-t})^2}}$$

which coincides with the limit as $r$ goes to infinity of

$$\frac{c_\rho (r, t)}{\sqrt{1 - [c_\rho (r, t)]^2}}.$$
Then, both functions, \(a_H\) and \(A_H\) converge to
\[
\int_0^{\log(\frac{1}{d_H}))} \frac{-2H + (1 + 2H)e^{-t}}{\sqrt{1 - (-2H + (1 + 2H)e^{-t})^2}} \, dt
\]
at infinity. This integral results the expression presented in the statement. Moreover, for all \(\rho > 0\) and \(r > \tanh^{-1}(2H)\), we have that \(A_H(\rho) \leq a_H(r)\).

For the comparison with \(h_H\), take two hypercycles \(\alpha_1\) and \(\alpha_2\) equidistant \(|r|\) to a same geodesic \(\gamma\). Let \(A\) be the connected component of \(\mathbb{H}^2 \setminus \alpha_1\) which contains \(\gamma\). Then the distance, with sign, from \(\alpha_1\) to \(\gamma\) is \(r < 0\). From Lemma 2.3 there is a cmc \(H\)-graph \(G \subset \mathbb{H}^2 \times \{t \geq 0\}\) with boundary \(\alpha_1 \cup \beta\) in \(\mathbb{H}^2\), where \(\beta\) is a hypercycle which satisfies \(d(\alpha_1, \beta) \geq 2z_H(r)\) and, moreover, \(G\) is vertical at \(\alpha_1\). On the other hand, by Lemma 2.4 there is a cmc \(H\)-graph \(\Omega \subset \mathbb{H}^2 \times \{t \geq 0\}\) with boundary \(\alpha_1 \cup \alpha_2\) which is not vertical at \(\alpha_1\) (neither at \(\alpha_2\)). Now the tangent principle implies that \(h_H(|r|) < a_H(r)\).

The inequality \(|r| < z_H(r)\) follows from the definition of \(z_H(r)\).

### 3 Main results

Let us prove Theorem 3.2 which shows that an a priori height estimate is enough for the existence result in (1). After that we obtain the existence results.

**Definition 3.1** We say that a \(C^2\) domain \(\Omega \subset \mathbb{H}^2\) satisfies the exterior circle condition of radius \(\rho > 0\) if, for each \(p \in \partial \Omega\), there is hyperbolic circle \(C_\rho \subset \mathbb{H}^2 \setminus \Omega\) of radius \(\rho\) which is tangent to \(\partial \Omega\) at \(p\).

We observe that if \(-1 \leq \inf_{\partial \Omega} k_{\partial \Omega}\), then \(\Omega\) satisfies the exterior circle condition of radius \(\rho\) for all \(\rho > 0\).

**Theorem 3.2** Let \(\Omega \subset \mathbb{H}^2\) be a bounded \(C^{2,\alpha}\) domain and let \(H \in (0, 1/2)\) be given. Set \(\kappa = \inf_{\partial \Omega} k_{\partial \Omega}\).

(i) If \(\kappa > 2H\) then the Dirichlet problem (1) has a solution in \(C^{2,\alpha}(\Omega)\).

(ii) If \(\kappa \in (-1, 2H]\) and there exists \(0 < a < a_H(\tanh^{-1}(\kappa))\) such that any solution \(u\) of (1) satisfies the a priori height estimate \(\sup_{\Omega} |u| \leq a\), where \(a_H\) is given by (8), then there is a solution \(u \in C^{2,\alpha}(\Omega)\) to the Dirichlet problem (1).

(iii) If \(\kappa < -1\), \(\Omega\) satisfies the exterior circle condition of radius \(\coth^{-1}(\kappa))\) and there exists \(0 < b < A_H(\coth^{-1}(\kappa))\) such that any solution \(u\) to (1) satisfies the a priori height estimate \(\sup_{\Omega} |u| \leq b\), where \(A_H\) is given by (9), then the Dirichlet problem (1) has a solution \(u \in C^{2,\alpha}(\Omega)\).

(iv) If \(\kappa = -1\) and there exists \(0 < b < a_H(\infty)\) such that any solution \(u\) to (1) satisfies the a priori height estimate \(\sup_{\Omega} |u| \leq b\), where \(a_H(\infty)\) is given by (10), then the Dirichlet problem (1) has a solution \(u \in C^{2,\alpha}(\Omega)\).
Remark 3.3 As a consequence of the above result, the function \( F(\kappa, H) \) mentioned in the Introduction can be computed as
\[
F(\kappa, H) = \begin{cases} 
+\infty & \text{if } \kappa \geq 2H \\
\alpha_H(\tanh^{-1}(\kappa)) & \text{if } \kappa \in [-1, 2H) \\
\Lambda_H(\coth^{-1}(\kappa)) & \text{if } \kappa < -1.
\end{cases}
\]

Proof. The first item is a well-known result which holds also for continuous boundary data, see Theorem 1.5 in [14].

In order to prove the other cases we use barriers to obtain a priori estimates to \( \sup_{\partial \Omega} |\nabla u| \). Then, by standard elliptic PDE theory (see [3]), the existence result holds.

Our barriers are inspired in the quarters of cylinders in \( \mathbb{R}^3 \) which were used in this same way in [7] and [12].

Proof of (ii) We first consider the case \( \kappa \in (-1, 2H) \).

Here, the role of the cylinders will be played by parts of \( G = G(r, H) \), \( r > \tanh^{-1}(-2H) \), where \( G \) is described in Lemma 2.4. Nevertheless, since the sets \( (\mathbb{H}^2 \times \{t\}) \cap G, t \geq 0 \), are hypercycles, which do not all have the same geodesic curvature, moving them downwards requires some care. We start this proof by describing this movement.

Let \( \alpha \) be a curve in \( \mathbb{H}^2 \) oriented with normal vector \( \eta \) and of constant geodesic curvature \( \kappa \). Let \( r = \tanh^{-1}(-\kappa) \). Let \( \Lambda \) be the connected component of \( \mathbb{H}^2 \setminus \alpha \) for which \( \eta \) points and set \( d(z) = d_{\mathbb{H}^2}(z, \alpha), z \in \Lambda \). Let \( \Lambda \subset A \) be the strip of width \( z_H(r) \) bounded by \( \alpha \cup \alpha^* \), being \( z_H(r) \) given by (6). Then the graph of \( u = (\tilde{u} \circ d)|_\Lambda \), \( \tilde{u} \) given in (13), has cmc \( H \). Besides \( u \) vanishes on \( \alpha \) and, since \( \tilde{u}(z_H(r)) = a_H(r) > a \), \( u \) is greater than \( a \) on \( \alpha^* \).

Let \( r_1 > 0 \) be such that \( \tilde{u}(r_1) + a < a_H(r) \). Define \( w_1 : \Lambda \to \mathbb{R} \) by \( w_1(z) = (\tilde{u} \circ d)(z) - \tilde{u}(r_1) \) and let \( w \) be the restriction of \( w_1 \) to the subset of \( \Lambda \) given by \( \{ z \in \Lambda; w_1(z) \geq 0 \} \). Since \( u \) is an increasing function depending on the distance to \( \alpha \), the domain of \( w \) is bounded by two curves equidistant to \( \alpha \): \( \delta \) of distance \( r_1 \) (geodesic curvature \( -\tanh(r + r_1) = \kappa_1) \) on which \( w \) vanishes and \( \alpha^* \) of distance \( z_H(r) \) on which \( w \) is greater than \( a \). The graphs of functions \( w \) well located in \( \mathbb{H}^2 \) will work as our barriers. Since
\[
k_{\partial \Omega} \geq \kappa = -\tanh(r) > -\tanh(r + r_1) = \kappa_1,
\]
for each \( p \in \partial \Omega \) there is a curve \( \delta_p \) tangent to \( \partial \Omega \) at \( p \), contained in \( \Omega^C \) and of constant geodesic curvature \( \kappa_1 \) (oriented by \( \eta_\delta \) such that \( \eta_\delta(p) = \eta_{\partial \Omega}(p) \)). Let \( w_p : \Lambda_p \to \mathbb{R} \) be the barrier described above that depends on the distance to \( \delta_p \).

We claim that if \( u \in C^{2,\alpha}({\overline{\Omega}}) \) is a solution to (1), then \( u \leq w_p \) in the domain \( \Omega_p \), where both functions are defined.

Notice that \( \partial \Omega_p = \Gamma_1 \cup \Gamma_2 \), where \( \Gamma_1 = \partial \Omega_p \cap \partial \Omega \) and \( \Gamma_2 = \partial \Omega_p \cap \partial \Lambda_p \). We have \( u|_{\Gamma_1} = 0 \leq w_p|_{\Gamma_1} \) and, since \( \sup_{\Omega} |u| < a \) we have \( u|_{\Gamma_2} < a < w_p|_{\Gamma_2} \). Then \( u|_{\partial \Omega_p} \leq w_p|_{\partial \Omega_p} \). Now, since \( u(p) = w_p(p) \) and \( \sup_{\Omega_p} |\nabla w_p| \leq \tilde{u}(r_1) < \infty \), we conclude that \( w_p \) is supersolution relatively to the domain \( \Omega_p \). So, standard elliptic PDE theory guarantees the existence of a solution \( u \in C^{2,\alpha}({\overline{\Omega}}) \) to the Dirichlet problem (1).
Relatively to the case $\kappa = 2H$, assuming an a priori height estimate $a$, we see from [11] that there is $r > \tanh^{-1}(-2H)$ such that $0 < a < a_H(r)$. For such $r$, as $-\tanh r \leq 2H = \kappa$, we can proceed as above and this concludes the proof of item [iii].

**Proof of (iii)** Set $\rho_0 = \coth^{-1}(-\kappa)$. Since $\Omega$ satisfies the exterior circle condition of radius $\rho_0$, given $p \in \partial \Omega$ there is $q \in \mathbb{H}^2 \setminus \overline{\Omega}$ and a circle $C_{\rho_0}(q) \subset \mathbb{H}^2 \setminus \Omega$ such that $C_{\rho_0}(q)$ is tangent to $\partial \Omega$ at $p$.

Since $\rho \mapsto A_H(\rho)$ is a continuous increasing function, there are $\rho_1 < \rho_0$ and $s < 0$ satisfying $b < A_H(\rho_1) + s$ and $\rho_1 + \varepsilon < \rho_0$. The positive number $\varepsilon = \varepsilon(H, \rho_1)$ is given by $\varepsilon = \tilde{u}^{-1}(-s)$ for $\tilde{u} : [0, Z_H(\rho_1)] \to \mathbb{R}$ defined in (17):

$$
\tilde{u}(d) = \int_0^d \frac{\sqrt{1 - s_H(\rho_1, t)^2}}{s_H(\rho_1, t)} \, dt.
$$

By Lemma [26] setting $d(z) = d_{\mathbb{H}^2}(z, C_{\rho_1}(q))$, the graph $G$ of the function $u(z) = \tilde{u} \circ d(z) + s$ defined on the annulus

$$
A := \left\{ z \in [B_{\rho_1}(q)]^C, 0 \leq d(z) \leq Z_H(\rho_1) \right\},
$$

has cmc $H$ and is contained in $\mathbb{H}^2 \times \{t \geq s\}$. Besides $\mathbb{G} \cap (\mathbb{H}^2 \times \{0\})$ is a circle $C_{\rho_1+\varepsilon}(q)$ of radius $\rho_1 + \varepsilon < \rho_0$. The connected component of $\partial \mathbb{G}$ which is contained in $\mathbb{H}^2 \times \{t > 0\}$ is $\mathbb{G} \cap (\mathbb{H}^2 \times \{A_H(\rho_1) + s\}) \subset \mathbb{H}^2 \times \{t > b\}$.

Let $\Lambda = \{z \in A : \rho_1 + \varepsilon \leq d(z) \leq Z_H(\rho_1)\}$ be the annulus where $u$ is positive and set $w = u|_{\Lambda}$. We have $\sup_{\Lambda} |\nabla w| = \tilde{u}'(\varepsilon) < \infty$ and

$$
b < \sup_{\Lambda} w = A_H(\rho_1) + s.
$$

Now, consider the geodesic radius $\gamma_p$ linking $q$ to $p$, with $\gamma_p(0) = q$ and $\gamma_p(\rho_0) = p$ and translate the graph of $w$ along $\gamma_p$ until its lower boundary touches $p$. We name $w_p$ the function that has this translated surface as graph and $\Lambda_p$ be the domain of $w_p$.

Set $\Omega_p = \Lambda_p \cap \Omega$. Note that $\partial \Omega_p = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 = \partial \Omega_p \cap \partial \Omega$ and $\Gamma_2 = \partial \Omega_p \cap \partial \Lambda_p$. If $u$ is a solution of (11), then $u|_{\Gamma_1} = 0 \leq w_p|_{\Gamma_1}$. On the other hand, by hypothesis, $u|_{\Gamma_2} < b < w_p|_{\Gamma_2}$. It follows that $w_p$ is a supersolution relatively to the domain $\Omega_p$, that is, $u \leq w_p$ in $\Omega_p$. The result follows now, from the fact that $\sup_{\partial \Omega_p} |\nabla w_p| < \infty$.

**Proof of (iv)** Since $\kappa = -1$, $\Omega$ satisfies the exterior circle condition for all $\rho > 0$. From Proposition [23] it follows that there is $\rho$ large enough such that $b < A_H(\rho)$. Now, just proceed as in the case [iii].

We observe that in Theorem 3 of [12], the height estimate does not depend on $k_{\partial \Omega}$, it only requires $\Omega$ to be convex. Here, instead of assuming $\Omega$ convex, we require $k_{\partial \Omega} \geq -1$ and an analogous result holds.

**Corollary 3.4** Let $\Omega \subset \mathbb{H}^2$ be a bounded $C^{2,\alpha}$ domain and let $H \in (0, 1/2)$ be given. If $k_{\partial \Omega} \geq -1$ and any solution $u$ to (11) satisfies the a priori height estimate $\sup_{\Omega} |u| < a_H(\infty)$, where $a_H(\infty)$ is given by (18), then the Dirichlet problem (11) has a solution $u \in C^{2,\alpha}(\Omega)$.
Proof. From Proposition 2.6 $a_H(r)$ given by (8) is decreasing with $r$ and converges to $a_H(\infty)$ as $r \to \infty$. Set $r = \tanh^{-1}(-\kappa)$ if $\kappa > -1$. Then $\sup_{\Omega} |u| < a_H(r)$ and the result follows from item (iii) of Theorem 3.2. If $\kappa = -1$, the result is item (iv) of Theorem 3.2. ■

Corollary 3.5 Let $H \in (0, 1/2)$ and $\Omega \subset \mathbb{H}^2$ a $C^2$ domain be given. Set $\kappa = \inf_{\partial \Omega} k_{\partial \Omega}$. If $\kappa \in (-1, 2H)$, let $\delta_H(r)$ be as defined in (10), where $r = \tanh^{-1}(-\kappa)$. It follows that if $\overline{\Omega}$ is contained in a region bounded by two hypercycles $\delta_H(r)$ far apart, one of them of geodesic curvature $\kappa$ oriented with the normal vector pointing to $\Omega$, then the Dirichlet problem $\overline{\Omega}$ has a solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$.

Since an explicit expression is always nicer, we remark that Lemma 2.4 shows that $2z_H(r) < \delta_H(r)$ and therefore if $\overline{\Omega}$ is contained in a region bounded by two hypercycles

$$2 \left[ \sinh^{-1} \left( \sinh r + \frac{\cosh r}{2H} \right) - r \right]$$

far apart, one of them of geodesic curvature $\kappa = \tanh(-r)$ when oriented with the normal vector pointing to $\Omega$, then the existence result also holds.

Proof. Let $\alpha_1$ be the hypercycle equidistant to a geodesic $\zeta$ mentioned in the statement. It follows that the distance, with sign, from $\zeta$ to $\alpha_1$, is $r$. Let $d$ be the distance function to $\alpha_1$ in the connected component $A$ of $\mathbb{H}^2 \setminus \alpha_1$ which contains $\Omega$ and let $u_1 = \hat{u} \circ d$ be as defined in Lemma 2.4 [13]. In Lemma 2.4, we proved that for $\beta_1 \subset A$ the curve equidistant $\delta_H(r) > 2z_H(r)$ from $\alpha_1$, it holds that $\beta_1 = (G(u_1) \setminus \alpha_1) \cap (\mathbb{H}^2 \times \{0\})$ and $u_1 \geq 0$ in $\Lambda_1$, for $\Lambda_1$ the connected set bounded by $\alpha_1 \cup \beta_1$. From hypothesis, $\overline{\Omega} \subset \Lambda_1$, $u_1$ vanishes on $\partial \Lambda_1$ and depends on the distance to the boundary. Therefore there is $\varepsilon > 0$, such that $u_1 - \varepsilon > 0$ in $\Omega$. We denote by $u$ the function $u_1 - \varepsilon$ and by $\alpha$ and $\beta$ the curves that bound the set $\{u > 0\}$, which are equidistant to $\zeta$ and satisfy: $\alpha$ is in the region bounded by $\alpha_1 \cup \beta$ and $\beta$ is in the region bounded by $\alpha \cup \beta_1$. We observe that $\sup_{\Lambda_1} u = \sup_{\Lambda_1} (u_1 - \varepsilon) = a_H(r) - \varepsilon$.

Fix a point $p_0 = \zeta(0) \in \zeta$. Given $k \in \mathbb{N}$, let $\Gamma_{\pm k}$ be geodesics orthogonal to $\zeta$, intersecting $\zeta$ at $\zeta(\pm k)$. Notice that such geodesics are also orthogonal to $\alpha$ and $\beta$. For $k \in \mathbb{N}$, set $p_{\alpha,\pm k} := \alpha \cap \Gamma_{\pm k}$ and $p_{\alpha,k}$ the part of $\alpha$ from $p_{\alpha,-k}$ to $p_{\alpha,k}$. Analogously we define $p_{\beta,\pm k} \in \beta \cap \Gamma_{\pm k}$ and $p_{\beta,k}$. Let $R_k$ be the ‘rectangle’ with these four vertices. Consider the hyperbolic circles of diameter the segment $[p_{\alpha,k}, p_{\beta,k}]$, which are tangent to $\alpha$ and $\beta$ at $p_{\alpha,k}$ and $p_{\beta,k}$ and also the ones of diameter $p_{\alpha,-k}, p_{\beta,-k}$. Let $\lambda_k$ be the semicircles of such circles with extremes $p_{\alpha,k}$ and $p_{\beta,k}$, such that $\lambda_k \subset (R_k)^C$ and analogously define $\lambda_{-k}$. They have curvature greater than 1 when oriented with the normal pointing to $R_k$. Now consider the bounded $C^1$ domain $\Lambda_k$ whose boundary is the curve $\partial \Lambda_k = \lambda_k \cup \alpha_k \cup \lambda_{-k} \cup \beta_k$.

Given $p \in \partial \Lambda_k$, if $p \in \alpha \cup \beta$, define $w_p : \overline{\Lambda_k} \to \mathbb{R}$ by $w_p(z) = u(z)$, where $u$ is the function described in the beginning of this proof. If $p \in \lambda_k \cup \lambda_{-k}$,
since \( k_{\alpha_1} < 1 < \min \{ k_{\lambda_k}, k_{\lambda_{k-1}} \} \), we can take a curve \( \alpha_p \) tangent to \( \partial \Lambda_k \) at \( p \), of constant curvature \( k_{\alpha_p} = \tanh(r) \) when oriented with the normal pointing to \( \Lambda_k \). Let \( d_{\alpha_p} \) be the distance function to \( \alpha_p \) in the connected component of \( \mathbb{H}^2 \setminus \alpha_p \) that contains \( \Lambda_k \) and, using (13) set

\[
f(s) = \int_0^s \frac{C_H(r,t)}{\sqrt{1-[C_H(r,t)]^2}} dt \tag{21}
\]

\( 0 \leq s \leq d_{\alpha_p}(z) \leq z_H(r) \) and

\[
\Lambda_{k,p} = \{ z \in \Lambda_k; d_{\alpha_p}(z) \leq z_H(r) \}. \tag{22}
\]

Now, define \( w_p : \Lambda_k \to \mathbb{R} \) by

\[
w_p(z) = \begin{cases} 
\min \{ f(d_{\alpha_p}(z)), u(z) \}, & \text{if } z \in \Lambda_{k,p} \\
u(z), & \text{if } z \in \Lambda_k \setminus \Lambda_{k,p}.
\end{cases}
\]

Observe that, if \( y \in \partial \Lambda_{k,p} \cap \Lambda_k \), then \( f(d_{\alpha_p}(y)) = f(z_H(r)) = a_H(r) \geq u(y) \). Hence for any \( z \in \Lambda_k \), there is \( U \), a neighborhood of \( z \), such that either \( w_p \equiv \min \{ f \circ d_{\alpha_p}, u \} \) with both functions \( f \circ d_{\alpha_p} \) and \( u \) well defined in \( U \) or \( w_p \equiv u \) in \( U \). Since these functions have cmc graphs, we conclude that \( w_p \) is a supersolution relatively to the PDE

\[
\begin{align*}
Q_H(u) &= 0 \text{ in } \Lambda_k \\
u|_{\partial \Lambda_k} &= 0 \\
u &\in C^2(\Lambda_k) \cap C^0(\overline{\Lambda_k}).
\end{align*}
\tag{23}
\]

Since the constant function \( \phi = 0 \) in \( \Lambda_k \) is a subsolution to the Dirichlet problem \( \phi \in C^0(\overline{\Lambda_k}; \phi \text{ is a subsolution of } (23)) \), we obtain that

\[
v(z) = \sup \{ \phi(z) \mid \phi \in S \}, \quad z \in \Lambda_k \tag{24}
\]

is a solution of the Dirichlet problem (23) and, moreover, from the maximum principle, \( |v|_0 \leq a_H(r) - \varepsilon \).

We first suppose \( \Omega \) bounded.

Since \( \Omega \) is bounded, there is \( k \in \mathbb{N} \) such that \( \Omega \subset \Lambda_k \). Taking into account such \( \Lambda_k \), consider a sequence of bounded \( C^{2,\alpha} \) domains \( \Omega_n \), with \( \overline{\Omega_n} \subset \Omega_{n+1} \) and \( k_{\partial \Omega_n} \geq \kappa \) for all \( n \in \mathbb{N} \), satisfying

\[
\Omega = \bigcup_{n=1}^{+\infty} \Omega_n
\]

Since each \( \Omega_n \) is included in \( \Lambda_k \), if \( u \in C^2(\Omega_n) \) satisfies \( Q_H(u) = 0, u|_{\partial \Omega_n} = 0 \), it follows from the maximum principle that \( |u|_0 \leq a_H(r) - \varepsilon \). Then, by Theorem...
there is \( u_n \in C^2(\overline{\Omega}_n) \) such that \( Q_H(u_n) = 0 \) and \( u_n|_{\partial\Omega_n} = 0 \). It follows from standard compactness results (see [3]) the existence of a subsequence of \( (u_n) \), converging uniformly on compact subsets of \( \Omega \) to \( u \in C^2(\Omega) \) satisfying \( Q_H(u) = 0 \) in \( \Omega \). Besides, from the proof of Theorem 3.2, there is \( M > 0 \), such that \( |\nabla u_n| < M \) in \( \overline{\Omega}_n \) for all \( n \in \mathbb{N} \). Therefore \( u \in C^0(\overline{\Omega}) \) and \( u|_{\partial\Omega} = 0 \).

Now, suppose \( \Omega \) unbounded.

Consider a sequence of \( C^2 \) bounded domains \( \Omega_j \) such that \( \Omega_j \subset \Omega_{j+1} \) and

\[
\Omega = \bigcup_{j=1}^{+\infty} \Omega_j,
\]

Notice that, for each \( j \) there is \( k \) such that \( \Omega_j \subset \Lambda_k \). Thus, we obtain a subsequence of \( (\Lambda_k) \), which we denote by \( (\Lambda_{k_j}) \), with \( \Lambda_{k_j} \subset \Lambda_{k+1} \). Then, according to the bounded case, there is for each \( j \in \mathbb{N} \), a solution \( u_j \in C^2(\Omega_j) \cap C^0(\overline{\Omega_j}) \) of (1). Standard compactness results imply that \( (u_j) \) has a subsequence, that we denote \( (u_{\ell_j}) \) again, converging uniformly on compact subsets of \( \Omega \) to a solution \( u \in C^2(\Omega) \) to \( Q_H = 0 \) in \( \Omega \). By using the barriers as in the proof of Theorem 3.2, we conclude that the norm of the gradient of \( u_j \) is uniformly bounded and since \( u_j|_{\partial\Omega_j} = 0 \) for all \( j \), it follows that \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) and \( u|_{\partial\Omega} = 0 \).

**Lemma 3.6** Let \( H \in (0, 1/2) \), \( r > \tanh^{-1}(2H) \) and \( \rho > 0 \) be given. There are positive numbers \( R = R(H, r) \), \( \varrho = \varrho(H, \rho) \) and \( \ell = \ell(H) \), such that

\[
h_H(R) = a_H(r), \quad h_H(\varrho) = A_H(\rho) \quad \text{and} \quad h_H(\ell) = a_H(\infty)
\]

where \( h_H, a_H, A_H \) and \( a_H(\infty) \) are given by (2), (3), (9), and (11), respectively. Moreover \( \ell \) is the solution of

\[
\ln \left( \frac{\sqrt{1 - 4H^2} + \sqrt{1 - 4H^2 \tanh^2(\ell)}}{\sqrt{1 - 4H^2 + 1}} \right) \cosh(\ell) = \frac{\pi}{4H} \sqrt{1 - 4H^2} - 2 \tanh^{-1} \left( \frac{1 - 2H}{\sqrt{1 - 4H^2}} \right). \quad (25)
\]

**Proof.** The result follows immediately from the fact that \( h_H : [0, \infty) \to [0, \infty) \) in (2) is an increasing homeomorphism.

**Corollary 3.7** Let \( H \in (0, 1/2) \) and \( \Omega \subset \mathbb{H}^2 \) a \( C^2 \) domain be given. Set \( \kappa = \inf_{\partial\Omega} k_{\partial\Omega} \).

1. If \( \kappa \geq 2H \) and \( \overline{\Omega} \) is contained in a region bounded by two hypercycles equidistant to a fixed geodesic, then the Dirichlet problem (11) has a solution in \( C^2(\Omega) \cap C^0(\overline{\Omega}) \).
(2) If $\kappa \in (-1, 2H)$, for $r = \tanh^{-1}(-\kappa)$, let $R = R(H, r)$ be as defined in Lemma 3.6. If $\Omega$ is contained in a region bounded by two hypercycles equidistant $R$ to a fixed geodesic, then the Dirichlet problem (1) has a solution in $C^2(\Omega) \cap C^0(\Omega)$.

(3) If $\kappa < -1$, for $\rho = \coth^{-1}(-\kappa)$, let $\varrho = \varrho(H, \rho)$ be as defined in Lemma 3.6. If $\Omega$ is contained in a region bounded by two hypercycles equidistant $\varrho$ to a fixed geodesic and $\Omega$ satisfies the exterior circle condition of radius $\rho$, then the Dirichlet problem (1) has a solution in $C^2(\Omega) \cap C^0(\Omega)$.

(4) If $\kappa = -1$, suppose $\Omega$ contained in a region bounded by two hypercycles which are equidistant $\ell$ to a fixed geodesic, where $\ell = \ell(H)$ is given by (25), then the Dirichlet problem (1) has solution in $C^2(\Omega) \cap C^0(\Omega)$.

Proof.
The proofs of all cases in this corollary follow the same steps as the proof of Corollary 3.5. We start by constructing the domains $\Lambda_k, k \in \mathbb{N}$. Then using the graphs $\mathcal{G} = \mathcal{G}(l, H)$ described in Lemma 2.1 translated downwards, we construct supersolutions $w_p$ for all $p \in \partial \Lambda_k$. Then Theorem 3.2 implies the existence of a solution $v : \Lambda_k \to \mathbb{R}$, with $|v|_0 < F(\kappa, H)$, $F$ defined in Remark 3.3.

Then the same exhaustion $\Omega = \bigcup_n \Omega_n$ described in the proof of Corollary 3.5 must be done. We should only observe that in item (3) it is possible to assume that each $\Omega_n$ satisfies the same exterior circle condition as $\Omega$. With this exhaustion, the last steps of the previous proof work. ■

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Departamento de Matemática
Universidade Federal de Santa Maria
Av. Roraima 1000, Santa Maria RS, 97105-900, Brazil
Email: a.aiolfi@gmail.com; patricia.klaser@ufsm.br