On the Independent Double Roman Domination in Graphs

Doost Ali Mojdeh 1 · Zhila Mansouri 1

Received: 20 March 2019 / Revised: 13 September 2019 / Accepted: 14 September 2019 / Published online: 11 October 2019
© Iranian Mathematical Society 2019

Abstract
An independent double Roman dominating function (IDRDF) on a graph \( G = (V, E) \) is a function \( f : V(G) \to \{0, 1, 2, 3\} \) having the property that if \( f(v) = 0 \), then the vertex \( v \) has at least two neighbors assigned 2 under \( f \) or one neighbor \( w \) assigned 3 under \( f \), and if \( f(v) = 1 \), then there exists \( w \in N(v) \) with \( f(w) \geq 2 \), such that the set of vertices with positive weight is independent. The weight of an IDRDF is the value \( \sum_{u \in V} f(u) \). The independent double Roman domination number \( i_{dR}(G) \) of a graph \( G \) is the minimum weight of an IDRDF on \( G \). We continue the study of the independent double Roman domination and show its relationships to both independent domination number (IDN) and independent Roman \{2\}-domination number (IR2DN).

We present several sharp bounds on the IDRDN of a graph \( G \) in terms of the order of \( G \), maximum degree and the minimum size of edge cover. Finally, we show that, any ordered pair \((a, b)\) is realizable as the IDN and IDRDN of some non-trivial tree if and only if \( 2a + 1 \leq b \leq 3a \).

Keywords Independent double Roman domination · Independent Roman \{2\}-domination · Independent domination · Graphs

Mathematics Subject Classification 05C69 · 05C5

1 Introduction and Terminologies

Let \( G \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). For any vertex \( v \in V \), the open neighborhood of \( v \) is the set \( N(v) = \{u \in V \mid uv \in E\} \) and the closed neighborhood of \( v \) is the set \( N[v] = N(v) \cup \{v\} \). For a set \( S \subseteq V \), the

Communicated by Hamid Reza Ebrahimi Vishki.

Doost Ali Mojdeh
damojdeh@umz.ac.ir
Zhila Mansouri
mansoury.zh@yahoo.com

1 Department of Mathematics, University of Mazandaran, Babolsar, Iran
open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. Given a set $S \subseteq V$, the private neighborhood $pn[v, S]$ of $v \in S$ is defined by $pn[v, S] = N[v] - N[S - \{v\}]$, equivalently, $pn[v, S] = \{u \in V : N[u] \cap S = \{v\}\}$. Each vertex in $pn[v, S]$ is called a private neighbor of $v$. From the definition of $pn[v, S]$, there may be $v \in pn[v, S]$. We use [10] as a reference for terminology and notation which are not defined here.

A set $S \subseteq V$ in $G$ is called a dominating set if $N[S] = V$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set in $G$. A set $S \subseteq V$ in $G$ is called an independent dominating set if $S$ is a dominating set and the induced subgraph by $S$, $G[S]$, has no edge. The independent domination number $i(G)$ of $G$ is the minimum cardinality of an independent dominating set in $G$ [5].

A function $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function (an RDF) on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ with $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$. The Roman domination number $\gamma_R(G)$ is the minimum weight of an RDF on $G$. The Roman domination was introduced by Cockayne et al. [4]. Since 2004, so many papers have been published on this topic, where several new variations were introduced: weak Roman domination, maximal Roman domination, mixed Roman domination, and recently, Roman [2]-domination [3] and double Roman domination [2,11].

A Roman [2]-dominating function (R2DF) is a function $f : V(G) \to \{0, 1, 2\}$ with the property that, for every vertex $v \in V$ with $f(v) = 0$, we have $f(N(v)) \geq 2$, that is, there is at least one vertex $u \in N(v)$, with $f(u) = 2$, or there are at least two vertices $x, y \in N(v)$ with $f(x) = f(y) = 1$. The weight of an R2DF is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$, and the minimum weight of an R2DF is called the Roman [2]-domination number and denoted by $\gamma_{[R2]}(G)$. An independent Roman [2]-dominating function (IR2DF) is a Roman [2]-dominating function $f$ with the property that the vertices with positive weights are independent. The weight of an IR2DF $f$ is the value $f(V(G)) = \sum_{v \in V(G)} f(v)$, and the minimum weight of an IR2DF is called the independent Roman [2]-domination number and denoted by $i_{[R2]}(G)$ [9].

A double Roman dominating function (DRDF) on a graph $G$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex $v$ has at least two neighbors assigned 2 under $f$ or one neighbor $w$ with $f(w) = 3$, and if $f(v) = 1$, then there exists $u \in N(v)$, such that $f(u) \geq 2$. The weight of a DRDF is the value $f(V(G)) = \sum_{u \in V} f(u)$. The double Roman domination number (DRDN) $\gamma_{DR}(G)$ of a graph $G$ is the minimum weight of a DRDF on $G$. For simplicity, a DRDF $f$ on a graph $G$ may be represented by the ordered 4-tuple $(V_0, V_1, V_2, V_3)$ of $V(G)$ induced by $f$, where $V_i = \{u \in V(G) | f(u) = i\}$ for $0 \leq i \leq 3$.

A DRDF $f = (V_0, V_1, V_2, V_3)$ is called independent if $V_1 \cup V_2 \cup V_3$ is an independent set in $G$. The independent double Roman domination number (IDRDN) $i_{dR}(G)$ is the minimum weight of an independent double Roman dominating function (IDRDF) on $G$, see also [7].

In this work, we mainly present lower and upper bounds on IDRDN of graphs, as, for example, by the well-known result of Gallai (concerning the maximum matching and the minimum edge cover [10]), we prove that $i_{dR}(G) \leq i_{[R2]}(G) + \beta'(G)$ in which $G$ is a graph of order $n$ with no isolated vertices and $\beta'(G)$ is the minimum size of an edge cover of $G$. We also prove that $2i(T) + 1 \leq i_{dR}(T) \leq 3i(T)$ for all.
trees $T$ of order $n \geq 2$ and show that all values between the lower and upper bounds are realizable.

2 Preliminary Results

In this section, we obtain some basic results and give the exact formulas for the IDRDNs for some well-known graphs. We first show $i_{dR}$ is well-defined for all graphs.

**Proposition 2.1** Every graph $G$ has an IDRDF.

**Proof** Let $S$ be a maximal independent set of $G$. Then, every vertex in $V - S$ has at least one neighbor in $S$. Now, the function $f : V(G) \rightarrow \{0, 1, 2, 3\}$ which assigns three to the vertices in $S$ and 0 to the other ones is an IDRDF of $G$. \hfill $\square$

In fact, Proposition 2.1 guarantees that the IDRDF and, therefore, the IDRDN $i_{dR}(G)$ exist for all graphs $G$.

Since in any IDRDF $f = (V_0, V_1, V_2, V_3)$, the set $V_1 \cup V_2 \cup V_3$ is an independent set in $G$, so by the definition, $V_1 = \emptyset$ for any $i_{dR}(G)$-function. It turns out to be useful in dealing with some results in this paper.

**Observation 2.2** In any IDRDF $f : V(G) \rightarrow \{0, 1, 2, 3\}$, $f(v) \neq 1$ for all $v \in V(G)$.

The DRDNs of path $P_n$ and cycle $C_n$ were given in [1], and they showed: for $n \geq 1$, $\gamma_{dR}(P_n) = n$, if $3 \mid n$ and $\gamma_{dR}(P_n) = n + 1$ otherwise, also, for $n \geq 3$, $\gamma_{dR}(C_n) = n + 1$, if $n \equiv 1, 5 \pmod{6}$ and $\gamma_{dR}(C_n) = n$ otherwise.

The IDRDNs of $P_n$ and $C_n$ have been proven by Maimani et al. [7] and we leave their proofs.

**Proposition 2.3**

1. For $n \geq 1$, $i_{dR}(P_n) = \gamma_{dR}(P_n)$.
2. For $n \geq 3$, $i_{dR}(C_n) = \gamma_{dR}(C_n)$.

In what follows the IDRDNs of the complete graphs and complete $r(\geq 2)$-partite graphs are given, with straightforward proofs.

**Observation 2.4**

(i) Let $G = K_{m_1, \ldots, m_r}$ be a complete $r(\geq 2)$-partite with $m_1 \leq \cdots \leq m_r$. Then

$$i_{dR}(G) = \begin{cases} 3 & \text{if } m_1 = 1, \\ 2m_1 & \text{otherwise}. \end{cases} \tag{2.1}$$

(ii) $i_{dR}(K_n) = 3$ for $n \geq 2$ and $i_{dR}(K_n) = 2$ for $n=1$.

**Observation 2.5** Let $G$ be a graph of order $n \geq 2$. Then

(i) $i_{dR}(G) = 3$ if and only if $\Delta(G) = n - 1$.
(ii) $i_{dR}(G) \in \{4, 5\}$ if and only if $\Delta(G) = n - 2$.

**Proof** (i) Is trivial.
(ii) Let \( i_{\text{dR}}(G) = 4 \). Then, there exist two non-adjacent vertices \( u, v \) assigned with value 2, so that every other vertex of \( G \) is adjacent to both of them. Therefore, \( \Delta(G) = n - 2 \).

Let \( i_{\text{dR}}(G) = 5 \). Then, there exist two non-adjacent vertices \( u, v \), where one of them assigned with value 3 and the other assigned with value 2, so that every other vertex of \( G \) should be adjacent to the vertex with value 3. Therefore, \( \Delta(G) = n - 2 \) too.

Conversely, let \( \Delta(G) = n - 2 \). Then, there exist two non-adjacent vertices \( u, v \), so that all other vertices of \( G \) are adjacent to \( u \), or all other vertices of \( G \) are adjacent to \( v \), or both of them. If \( \text{deg}(v) = \text{deg}(u) = n - 2 \), then we should assign value 2 to both of them, and therefore, \( i_{\text{dR}}(G) = 4 \). If \( \text{deg}(v) = n - 2 \) and \( \text{deg}(u) \leq n - 3 \), or vice versa, then we should assign 3 to the vertex with degree \( n - 2 \) and value 2 to the other, and therefore, \( i_{\text{dR}}(G) = 5 \). \( \Box \)

3 Independent Double Roman and Independent Roman \( \{2\} \)-Domination

In this section, we establish some relationships between the IDRDN and IR2DN in graphs.

Proposition 3.1 For any graph \( G \)

\[
\frac{3}{2} i_{(R2)}(G) \leq i_{\text{dR}}(G) \leq 2 i_{(R2)}(G)
\]

and these bounds are sharp.

Proof Let \( f = (V_0, V_1, V_2) \) be an \( i_{(R2)}(G) \)-function with \( i_{(R2)}(G) = |V_1| + 2|V_2| \). Then, \( g = (V'_0 = V_0, V'_2 = V_1, V'_3 = V_2) \) is an IDRDF of \( G \). Therefore

\[
i_{\text{dR}}(G) \leq 2|V_1| + 3|V_2| \leq 2(|V_1| + 2|V_2|) = 2i_{(R2)}(G).
\]

The graph \( K_n \) and the graph of order at least four with two independent vertices \( u \) and \( v \), such that all the other vertices are adjacent to both \( u \) and \( v \), achieve the upper bound.

To prove the lower bound, let \( f = (V_0^f, V_2^f, V_3^f) \) be an \( i_{\text{dR}}(G) \)-function. If \( g = (V'_0 = V_0^f, V'_1 = V_2^f, V'_2 = V'_3 = V_3^f) \), then \( g \) is an IR2DF of \( G \) with

\[
w(g) = |V_2^f| + 2|V_3^f| \leq \frac{2}{3} \left(2|V_2^f| + 3|V_3^f|\right) = \frac{2}{3} i_{\text{dR}}(G).
\]

Using Part (i) of Observation 2.5, we have \( \frac{3}{2} i_{(R2)}(G) = i_{\text{dR}}(G) = 3 \) for all graphs \( G \) with \( \Delta(G) = n - 1 \). Therefore, the lower bound is sharp. \( \Box \)

Since for any graph of order \( n \geq 1 \), \( i_{(R2)}(G) \) and \( i_{\text{dR}}(G) \) are positive, as an immediate result from Proposition 3.1, we have:
For every graph $G$

$$i_{[R^2]}(G) < i_{dR}(G).$$

**Theorem 3.2** Let $G$ be a connected graph. Then

$$i_{dR}(G) \geq i_{[R^2]}(G) + i(G)$$

and this bound is sharp.

**Proof** The bound clearly holds for $K_1$ and $K_2$. Therefore, we assume that $G$ is of order $n \geq 3$. In view of Observation 2.2, we let $f = (V_0^f, V_2^f, V_3^f)$ be an $i_{dR}(G)$-function. We observe that $V_2^f \cup V_3^f$ is an independent dominating set in $G$, and therefore

$$i(G) \leq |V_2^f| + |V_3^f|. \quad (3.1)$$

We define $g: V(G) \rightarrow \{0, 1, 2\}$ by

$$g(v) = \begin{cases} f(v) - 1 & \text{if } v \in V_2^f \cup V_3^f, \\ f(v) & \text{if } v \in V_0^f. \end{cases}$$

Clearly, $g$ is an IR2DF. Moreover

$$i_{[R^2]}(G) \leq \sum_{v \in V(G)} g(v) = |V_2^f| + 2|V_3^f|. \quad (3.2)$$

Inequalities (3.1) and (3.2) imply that $i(G) + i_{dR}(G) \leq 2|V_2^f| + 3|V_3^f| = i_{dR}(G)$.

For seeing the sharpness of the bound, it suffices to consider the complete graph $K_n$ or any graph $G$ with $\Delta(G) = |V(G)| - 1$ or the $r$-partite graph $G = K_{m_1, \ldots, m_r}$ with $m_1 \leq \cdots \leq m_r$, where $m_1 \geq 2$. \hfill \Box

We recall that a matching $M$ of a graph $G$ is a set of non-loop edges with no shared endpoints; in the other words, one can say that no two edges in $M$ have a common vertex. A maximum matching is a matching of maximum size among all matchings in graph. We also recall that an edge cover of $G$ is a set $L$ of edges, such that every vertex of $G$ is incident to some edge of $L$. A minimum edge cover is an edge cover of minimum size among all edge covers in graph. By $\alpha'(G)$ and $\beta'(G)$, we mean the maximum cardinality of a matching and the minimum cardinality of an edge cover in $G$, respectively.

We make use of the following classic result due to Gallai.

**Lemma 3.3** ([10]) If $G$ is a graph of order $n$ with no isolated vertices, then $\alpha'(G) + \beta'(G) = n$. 

\begin{flushright}
\text{Springer}
\end{flushright}
Theorem 3.4 Let $G$ be a graph with no isolated vertices. Then

$$i_{dR}(G) \leq i_{\{R2\}}(G) + \beta'(G)$$

and this bound is sharp.

Proof Let $f : V(G) \rightarrow \{0, 1, 2\}$ be an $i_{\{R2\}}(G)$-function. We define $g : V(G) \rightarrow \{0, 2, 3\}$ by

$$g(v) = \begin{cases} f(v) + 1 & \text{if } v \in V_1^f \cup V_2^f, \\ 0 & \text{if } v \in V_0^f. \end{cases}$$

It is easy to verify that $g$ is an IDRDF of $G$. Therefore

$$i_{dR}(G) \leq \sum_{v \in V(G)} g(v) = 2|V_1^f| + 3|V_2^f| = i_{\{R2\}}(G) + |V_1^f| + |V_2^f| = i_{\{R2\}}(G) + n - |V_0^f|. \tag{3.3}$$

Since $f$ is an $i_{\{R2\}}(G)$-function, each edge of a maximum matching has an end point in $V_0^f$. Therefore, $\alpha'(G) \leq |V_0^f|$. Now, Lemma 3.3 and the inequality (3.3) imply that

$$i_{dR}(G) \leq i_{\{R2\}}(G) + n - \alpha'(G) = i_{\{R2\}}(G) + \beta'(G).$$

For seeing the sharpness of the bound, consider the complete bipartite graph $K_{p,p}$ in which $p \geq 2$. Then $i_{dR}(K_{p,p}) = 2p = p + p = i_{\{R2\}}(K_{p,p}) + \beta'(K_{p,p}).$ \hfill \Box

4 Independent Double Roman Domination and Independent Domination

We give some lower and upper bounds on the IDRDN in terms of the independent domination number.

Proposition 4.1 For any graph $G$, $2i(G) \leq i_{dR}(G) \leq 3i(G)$. These bounds are sharp.

Proof The upper bound follows by the proof of Proposition 2.1. The lower bound is a trivial consequence of Theorem 3.2.

For seeing the sharpness of the upper bound, let $\mathcal{F}$ be the family of graphs $G$ with $k$ independent vertices $v_1, v_2, \ldots, v_k$, such that every other vertices of $G$ is adjacent to at least one of $v_i$ and each $v_i$ has at least one private neighbor. Then, $i(G) = k$ and $i_{dR}(G) = 3k$.

For seeing the sharpness of the lower bound, let $\mathcal{H}$ be a family of graphs $G$, such that $G$ has a minimum maximal independent set with at least 2 vertices like $U$, such that every vertices of $G - U$ are adjacent to 2 vertices of $U$. Then, $i_{dR}(G) = 2i(G)$. For example, consider the complete $r(\geq 2)$-partite graph $G = K_{m_1, \ldots, m_r}$ with $2 \leq m_1 \leq \cdots \leq m_r$. It is easy to see that $i_{dR}(G) = 2i(G)$. \hfill \Box
Recall that a set $R \subseteq V(G)$ is a packing set of $G$ if $N[x] \cap N[y] = \emptyset$ for any two distinct vertices $x, y \in R$. The packing number $\rho(G)$ is the maximum cardinality of a packing set in $G$. Let $\delta$ denote the minimum degree of the graph $G$. A classical result shows that: for any graph $G$, $\rho(G) \leq \gamma(G) \leq i(G)$ [5].

**Proposition 4.2** If $G$ is a connected graph of order $n$, then

$$i_{dR}(G) + (2\delta - 1)\rho(G) \leq 2n$$

and this bound is sharp.

**Proof** Let $R$ be a maximum packing set of $G$ and $A = N(R)$. Let $B = V(G) - (A \cup R)$. Each vertex in $A$ has exactly one neighbor in $R$ and each vertex in $R$ has at least $\delta$ neighbors in $A$. Therefore, $\delta|R| \leq |[R, V - R]| = |A|$. Therefore

$$|B| = n - |A \cup R| = n - |A| - |R| \leq n - (\delta + 1)|R|.$$ 

Now, we define $f:V(G) \to \{0, 1, 2, 3\}$ by

$$f(v) = \begin{cases} 
3 & \text{if } v \in R \\
0 & \text{if } v \in A \\
2 & \text{if } v \in B.
\end{cases}$$

It is easy to see that $f$ is an IDRDF of $G$. Therefore

$$i_{dR}(G) \leq w(f) = 3|R| + 2|B| = 3|R| + 2n - (2\delta + 2)|R| = 2n - (2\delta - 1)\rho(G).$$

This bound is sharp for the complete graph $K_n$, for $n \geq 2$. $\Box$

The following result has been proven by Maymani et al. [7] that we leave its proof, but we mention that, the bound is also sharp for the cycles $C_n$, where $n \equiv 3 \pmod{6}$.

**Proposition 4.3** For any graph $G$ of order $n$, without isolated vertices and with maximum degree $\Delta$

$$i_{dR}(G) \geq \frac{2n}{\Delta} + \frac{\Delta - 2}{\Delta}i(G).$$

This bound is sharp for the cycles $C_n$, where $n \equiv 0, 2, 3, 4 \pmod{6}$, and for the paths $P_n$, where $n \equiv 0 \pmod{3}$.

### 5 Trees

A family of trees has been characterized by double Roman domination approach [8]. We want to characterize trees with a new approach. We make use the following result to show that the IDRDNs of trees can be bounded from below and above just in terms of the independent domination number. The result may be important in its own right.
Theorem 5.1 For any tree $T$ of order $n \geq 2$, $i_{[R_2]}(T) \geq i(T) + 1$.

Proof If $T$ is a star, then $i_{[R_2]}(T) = 2 = i(T) + 1$. Let $T$ be a double star $T_{r,s}$ in which $1 \leq r \leq s$. If $r = 1$, then $i_{[R_2]}(T) = 3 = 2 + 1 = i(T) + 1$. If $r \geq 2$, then $i_{[R_2]}(T) = 2 + r = i(T) + 1$. Therefore, we assume from now on that $diam(T) \geq 4$. Let $P$ be a diametral $r$, $s$-path of $T$. We root the tree $T$ at $r$. Let $f$ be an $i_{[R_2]}(T)$-function of $T$. We now deal with two cases depending on $f$.

Case 1. Suppose that there exists a vertex $x$ for which $f(x) = 2$. It is easy to observe that $S = \{v \in V(T) : f(v) \neq 0\}$ is an independent dominating set in $T$. Therefore, $i(T) \leq |S| \leq w(f) - 1 = i_{[R_2]}(T) - 1$.

Case 2. Suppose that $f(x) = 0$ or 1 for all $x \in V(T)$. Since $f$ is an IR2DF, it follows that $f$ assigns 1 to all leaves and 0 to all support vertices. Let $u$ be a support vertex and $L_u$ be the set of all leaves adjacent to $u$. If $x \in N(u) - L_u$ has the weight $f(x) = 1$, then every vertex $v \in N(x) - \{u\}$ has a neighbor $w \neq u$ with $f(w) = 1$ and $f(v) = 0$ by the properties of the IR2DF $f$. Let $A = \{x \in N(u) - L_u : f(x) = 1\}$. Since $f(u) = 0$, $|L_u \cup A| \geq 2$. If now, we define $f':V(G) \rightarrow \{0, 1\}$ by

$$f'(z) = \begin{cases} 0 & \text{if } z \in (L_u \cup A \cup V_0^f \setminus \{u\}) \\ 1 & \text{otherwise.} \end{cases}$$

it follows that $S' = \{v \in V(T) : f'(v) \neq 0\}$ is an independent dominating set of $T$. Thus, $i(T) \leq |S'| \leq w(f') \leq w(f) - 1 = i_{[R_2]}(T) - 1$. \hfill \Box

In what follows, for the sake of completeness, we characterize the family of all trees for which the lower bound in Theorem 5.1 holds with equality. To this aim, we begin with the following lemma.

Lemma 5.2 Let $T$ be a tree and $f = (V_0, V_1, V_2)$ be a $\gamma_{[R_2]}(T)$-function. If $w(f) = \gamma_{[R_2]}(T) = \gamma(T) + 1$, then $V_1 \cup V_2$ is an independent set.

Proof Since $f = (V_0, V_1, V_2)$ be a $\gamma_{[R_2]}(T)$-function, it follows that $|V_1| + 2|V_2| = \gamma_{[R_2]}(T) = \gamma(T) + 1 \leq |V_1| + |V_2| + 1$. Therefore, $|V_2| \leq 1$. Suppose on the contrary, $V_1 \cup V_2$ is not independent set. Let two vertices $v$ and $u$ in $V_1 \cup V_2$ be adjacent. Since $T$ is a tree, no vertex in $V_0$ is adjacent to both $u$ and $v$. Let $u \in V_1$ and $v \in V_2$. Then, each vertex $w$ in $V_0 \cap N(u)$ has another neighbor in $V_1$. Therefore, $V_1 \setminus \{u\} \cup \{v\}$ is a dominating set in $T$ of cardinality $|V_1| < |V_1| + 2|V_2| - 1 = \gamma(T)$, a contradiction. Similarly, if $u \in V_2$ and $v \in V_1$, then we achieve the same contradiction. Let $\{u, v\} \subseteq V_1$. If $z \in N(u) \cap V_1$ and $z \neq v$, then $z$ and $v$ are not adjacent, and each vertex in $V_0 \cap N(z)$ ($V_0 \cap N(v)$) is adjacent to another vertex in $(V_1 \setminus \{u, z, v\}) \cup V_2$; therefore, $V_1 \setminus \{z, v\} \cup V_2$ is a dominating set in $T$ of cardinality at most $|V_1| - 1 < |V_1| + 2|V_2| - 1 = \gamma(T)$, a contradiction. Similarly, for $z \in N(v) \cap V_1$ and $z \neq u$, we achieve the same contradiction. Now, let every vertex $z \in N(u) \cap N(v) \setminus \{u, v\}$ is in $V_0$, then there are $z_1 \in N(u) \cap V_0$ and $z_2 \in N(v) \cap V_0$, such that $z_1$ has a neighbor $w_1$ other than $u$ and $z_2$ has a neighbor $w_2$ other than $v$ with positive weights (otherwise, the proof is clear). Now, if $f(w_1) = f(w_2) = 1$, then the set $V_1 \setminus \{u, v, w_1, w_2\} \cup \{z_1, z_2\} \cup V_2$ is a dominating set of cardinality $\gamma_{[R_2]}(T) - 2$, a contradiction. If $f(w_1) = 1$ and $f(w_2) = 2$, then the set $V_1 \setminus \{u, v, w_1\} \cup \{z_1, z_2\} \cup V_2$ is a dominating set of the size
at most $|V_1| - 1 < |V_1| + 2|V_2| - 1 = \gamma(T)$, which is also a contradiction. Thus, $V_1 \cup V_2$ is an independent set.

From Lemma 5.2, we have the following.

**Corollary 5.3** Let $T$ be a tree and $f = (V_0, V_1, V_2)$ be a $\gamma_{\{R_2\}}(T)$-function. If $\gamma_{\{R_2\}}(T) = \gamma(T) + 1$, then $i_{\{R_2\}}(T) = \gamma_{\{R_2\}}(T)$ and $i(T) = \gamma(T)$.

**Proof** In Lemma 5.2, it has been shown that $V_1 \cup V_2$ is independent. If $V_2 \neq \emptyset$, then $V_1 \cup V_2$ is an independent dominating set and so $i_{\{R_2\}}(T) = \gamma_{\{R_2\}}(T).$ If $V_2 = \emptyset$, then there exists a vertex $v \in V_0$ for which $v$ has exactly two neighbors in $V_1$ like $u, w$. We define the function $g: V \to \{0, 1\}$ by

$$g(z) = \begin{cases} 
0 & \text{if } z \in (V_0 \cup \{v\} \setminus \{u, w\}) \\
1 & \text{otherwise.}
\end{cases}$$

It follows that $S = \{v \in V(T): g(v) = 1\}$ is a minimum dominating set of $T$ of cardinality $\gamma_{\{R_2\}}(T) - 1.$ Since $S$ is an independent set, it also follows that $|S| = i(T)$.

In [6], Henning et al. characterized all trees $T$ of order $n \geq 2$ for which $\gamma_{\{R_2\}}(T) = \gamma(T) + 1.$ To this aim, they introduced two families of trees. For positive integers $r$ and $s$, let $F_{r,s}$ be the tree obtained from a double star $S_{r,s}$ by subdividing every edge exactly once. For example, $P_7 = F_{1,1}.$ The tree $F_{4,4}$ is shown in Fig. 1. Let $\mathcal{F}$ be the family of all such trees $F_{r,s}$, that is, $\mathcal{F} = \{F_{r,s}: r, s \geq 1\}.$

Let $\mathcal{T}$ be the family of trees $T_{k,j}$ of order $k \geq 2$ where $k \geq 2j + 1$ and $j \geq 0,$ obtained from a star by subdividing $j$ edges exactly once. The tree $T_{12,4}$ is shown in Fig. 2.

They proved that:

**Theorem 5.4** ([6], Theorem 6) Let $T$ be a non-trivial tree. Then

$$\gamma_{\{R_2\}}(T) = \gamma(T) + 1 \text{ if and only if } T \in \mathcal{T} \cup \mathcal{F}.$$

Now, we characterize all trees $T$ of order $n \geq 2$ for which $i_{\{R_2\}}(T) = i(T) + 1.$ We deduce this result from Lemma 5.2, Corollary 5.3 and Theorem 5.4.
Theorem 5.5 Let $T$ be a non-trivial tree. Then

$$i_{[R2]}(T) = i(T) + 1 \text{ if and only if } T \in \mathcal{T} \cup \mathcal{F}.$$ 

Theorem 3.2, Proposition 4.1 and Theorem 5.1 yield the following for trees.

Corollary 5.6 For any tree $T$ of order $n \geq 2$, $2i(T) + 1 \leq i_{dR}(T) \leq 3i(T)$.

Our final result in this section shows that every value in the range of Corollary 5.6 is realizable for trees, that is, all values between the lower and upper bounds of Corollary 5.6 are realizable. We first recall that the corona $G \circ K_1$ of a graph $G$ is formed from $G$ by adding a new vertex $w$ and edge $vw$ for each vertex $v \in V(G)$.

Theorem 5.7 An ordered pair $(a, b)$ is realizable as the independent domination number and independent double Roman domination number of some non-trivial tree if and only if $2a + 1 \leq b \leq 3a$.

Proof Let $T$ be a tree with $i(T) = a$ and $i_{dR}(T) = b$. By Corollary 5.6, $2a + 1 \leq b \leq 3a$. Next, we show that each ordered pair is realizable. For $b = 2a + 1$, consider the corona of the star $K_{1,t}$, for $t \geq 1$. Then, the set of all support vertices other than the center of $K_{1,t}$ and the leaf neighbor of the center is an $i(K_{1,t} \circ K_1)$-set. It is straightforward to check that $i(K_{1,t} \circ K_1) = t + 1$, and if we assign the value 2 to all leaves other than the leaf neighbor of the center of $K_{1,t}$ and the value 3 to the center, then it is easy to see that $i_{dR}(K_{1,t} \circ K_1) = 2t + 3 = 2i(K_{1,t} \circ K_1) + 1$. Assume now that $b \geq 2a + 2$. Let $T$ be the tree formed from a subdivided star $K_{1,a}^*$ by choosing $b - (2a + 2)$ support vertices of $K_{1,a}^*$ and adding another vertex as a neighbor of leaf of each of them. Thus, $T$ has $b - 2a - 2$ vertices other than the center, which are neither support vertices nor leaves. Again, it is straightforward to check that all support vertices are an $i(T)$-set and so $i(T) = a$. For seeing $i_{dR}(T) = b$, note that each of the $b - 2a - 2$ new support vertices must be assigned with value 3 under any $i_{dR}(T)$-function. Assigning the value 2 to the any leaf of non-adjacent to the new support vertices and the center of $T$, and 0 to the other vertices. It is easy to check that this function is in fact of the minimum weight. Hence, we have $i_{dR}(T) = 3(b - 2a - 2) + 2(a - (b - (2a + 2))) + 2 = b$, as desired. This completes the proof. \hfill $\square$

Acknowledgements The authors sincerely thank the referees for their careful review of this paper and some useful comments and valuable suggestions.

References

1. Ahangar, H.A., Chellali, M., Sheikholeslami, S.M.: On the double Roman domination in graphs. Discret. Appl. Math. 232, 1–7 (2017)
2. Beeler, R.A., Haynes, T.W., Hedetniemi, S.T.: Double Roman domination. Discret. Appl. Math. 211, 23–29 (2016)
3. Chellali, M., Haynes, T.W., Hedetniemi, S.T., MacRae, A.: Roman [2]-domination. Discret. Appl. Math. 204, 22–28 (2016)
4. Cockayne, E.J., Dreyer, P.A., Hedetniemi, S.M., Hedetniemi, S.T.: Roman domination in graphs. Discret. Math. 278, 11–22 (2004)

Springer
5. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: Fundamentals of domination in graphs. Marcel Dekker, New York (1998)
6. Henning, M.A., Klostermeyer, W.F.: Italian domination in trees. Discret. Math. 217, 557–564 (2017)
7. Maimani, H.R., Momeni, M., Nazari-Moghaddam, S., Rahimi Mahid, F., Sheikholeslami, S.M.: Independent double Roman domination in graphs. Bull. Iran. Math. Soc. (2019). https://doi.org/10.1007/s41980-019-00274-8
8. Mojdeh, D.A., Parsian, A., Masoumi, I.: Characterization of double Roman trees, to appear in Ars Combinatoria
9. Rahmouni, A., Chellali, M.: Independent Roman [2]-domination in graphs. Discret. Math. 236, 408–414 (2018)
10. West, D.B.: Introduction to graph theory, 2nd edn. Prentice Hall, New Jersey (2001)
11. Zhang, X., Li, Z., Jiang, H., Shao, Z.: Double Roman domination in trees. Inf. Process. Lett. 134, 31–34 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.