Gravitational Collapse

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ABSTRACT

We review here some recent developments on the issue of final fate of gravitational collapse within the framework of Einstein theory of gravity. The structure of collapsed object is discussed in terms of either a black hole or a singularity having causal connection with outside universe. Implications for cosmic censorship are discussed.

1. Introduction

When a massive star, more than a few solar masses, has exhausted its internal nuclear fuel, it is believed to enter the stage of an endless gravitational collapse without having any final equilibrium state. According to the Einstein theory of gravitation, the star goes on
shrinking in its radius, reaching higher and higher densities. What would be the final fate of such an object according to the general theory of relativity? This is one of the central questions in relativistic astrophysics and gravitation theory today. It has been suggested that the ultra-dense object that forms as a result of collapse could be a black hole in the space and time from which not even light rays can escape. Alternatively, if the event horizon of gravity fails to cover the final crunch, it could be a visible singularity which can causally interact with the outside universe and from which emissions of light and matter may be possible.

An investigation of this nature is of importance from both the theoretical as well as observational point of view. At the theoretical level, working out the final fate of collapse in general relativity is crucial to the problem of cosmic censorship and asymptotic predictability [1]; namely, whether the singularities forming at the end point of collapse will be necessarily covered by the event horizons of gravity. Such a cosmic censorship hypothesis remains fundamental to the theoretical foundations of entire black hole physics, and its numerous astrophysical applications which have been invoked in past decades (e.g. the area theorem for black holes, laws of black hole thermodynamics, Hawking radiation effect, predictability; and on observational side, accretion of matter by black holes, massive black holes at the center of galaxies etc). On the other hand, existence of visible or naked singularities would offer a new approach on these issues requiring modification and reformulation of our usual theoretical conception on censorship.

Our purpose here is to discuss some of the recent developments in this direction, exam-
ining the possible final fate of gravitational collapse in general relativity. To investigate this issue, dynamical collapse scenarios have been examined in the past decade or so for many cases such as clouds composed of dust, radiation, perfect fluids, or also of matter compositions with more general equations of state (for references and details, see e.g. [2]). We try to discuss some of these developments and the implications of this analysis towards a possible formulation of cosmic censorship are pointed out. Finally, the open problems in the field are discussed and some concluding remarks try to summarize the overall situation.

In Section 2, some features of spherically symmetric collapse and basic philosophy on cosmic censorship are discussed. It is the spherical collapse of a homogeneous dust cloud, as described by the Oppenheimer–Snyder model [3], which led to the general concept of trapped surfaces and black hole. The expectation is, even when collapse is inhomogeneous or non-spherical, black holes must always form covering the singularity implied by the trapped surface. Some available versions of censorship hypothesis are discussed, pointing out formidable difficulties for any possible proof. It is concluded that the first major task is to formulate a provable version of censorship conjecture; and that to achieve this, a detailed and careful analysis of available gravitational collapse scenarios is essential where the possible occurrence and physical nature of the naked singularity forming is to be analyzed. It is only such an examination of collapse situations which could tell us what features are to be avoided, and which ones to look for, while formulating and proving any reasonable version of censorship. This would also lead us to a better and more effective understanding of the nature and occurrence of naked singularities in gravitational collapse.
Towards this purpose, Section 3 reviews the phenomena of black hole and naked singularity formation for gravitational collapse of several different forms of matter. The discussion begins with the radiation collapse models which provide an explicit and clear idea on the nature and structure of the singularity forming. Further generalizations of these results include forms of matter such as perfect fluids, dust, and also general matter fields satisfying the weak energy condition. The emerging pattern shows that for a rather general form of equation of state and collapsing matter, it is possible for a non-zero measure set of non-spacelike trajectories to escape from the naked singularity which could form from a regular initial data, and which could be powerfully strong for the growth of curvatures. Non-spherical collapse and scalar field collapse are also discussed here. Section 4 gives some concluding remarks and future directions.

2. Spherically Symmetric Collapse and Cosmic Censorship

To understand the possible final fate of a massive gravitationally collapsing object, we first outline here the spherically symmetric collapse situation. Though this represents an idealization, the advantage is one can solve this case analytically to get exact results when matter is taken in the form of a homogeneous dust cloud. In fact, the basic motivation for the idea and theory of black holes comes from this collapse model.

Consider a gravitationally collapsing spherical massive star. We need to consider the interior solution for the object which will depend on the properties of matter, equation of
state, and the physical processes taking place within the steller interior. However, assuming the matter to be pressureless dust allows to solve the problem analytically, providing many important insights. Here the energy-momentum tensor is given by $T^{ij} = \rho u^i u^j$, and one needs to solve the Einstein equations for the spherically symmetric metric. This determines the metric potentials, and the interior geometry of the collapsing dust ball is given by,

$$ds^2 = -dt^2 + R^2(t) \left[ \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\Omega^2 \right]$$

(1)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric on two-sphere. The geometry outside is vacuum Schwarzschild space-time. The interior geometry of the dust cloud matches at the boundary $r = r_b$ with the exterior Schwarzschild space-time.

![Figure 1](image)

Fig. 1: The standard picture of spherically symmetric homogeneous dust collapse.
The basic features of such a collapsing, spherical, homogeneous dust cloud configuration are given in Fig. 1. The collapse is initiated when the star surface is outside its Schwarzschild radius \( r = 2m \), and a light ray emitted from the surface of the star can escape to infinity. However, once the star has collapsed below \( r = 2m \), a black hole region of no escape develops in the space-time, bounded by the event horizon at \( r = 2m \). Any point in this empty region represents a trapped surface (which is a two-sphere in space-time) in that both the outgoing and ingoing families of null geodesics emitted from this point converge and hence no light comes out of this region. Then, the collapse to an infinite density and curvature singularity at \( r = 0 \) becomes inevitable in a finite proper time as measured by an observer on the surface of the star. The black hole region in the resulting vacuum Schwarzschild geometry is given by \( 0 < r < 2m \), the event horizon being the outer boundary. On the event horizon, the radial outwards photons stay where they are, but all the rest are dragged inwards. No information from this black hole can propagate outside \( r = 2m \) to observers far away. We thus see that the collapse gives rise to a black hole in the space-time which covers the resulting space-time singularity. The ultimate fate of the star undergoing such a collapse is then an infinite curvature singularity at \( r = 0 \), which is completely hidden within the black hole. No emissions or light rays from the singularity could go out to observer at infinity and the singularity is causally disconnected from the outside space-time.

The question now is whether one could generalize these conclusions on the occurrence of a space-time singularity in collapse for non-spherical situation, and whether these are valid at least for small perturbations from exact spherical symmetry. It is known [4], using the
stability of Cauchy development in general relativity, that the formation of trapped surfaces is indeed a stable property when departures from spherical symmetry are taken into account.

The argument essentially is the following: Considering a spherical collapse evolution from given initial data on a partial Cauchy surface \( S \), we find the formation of trapped surfaces \( T \) in the form of all the spheres with \( r < 2m \) in the exterior Schwarzschild geometry. The stability of Cauchy development then implies that for all initial data sufficiently near to the original data in the compact region \( J^+(S) \cap J^-(T) \), the trapped surfaces still must occur. Then, the curvature singularity of spherical collapse also turns out to be a stable feature, as implied by the singularity theorems, which show that the closed trapped surfaces always imply the existence of a space-time singularity under reasonable general conditions.

There is no proof available, however, that such a singularity will continue to be hidden within a black hole and remain causally disconnected from outside observers, even when the collapse is not exactly spherical or when the matter does not have the form of exact homogeneous dust. Thus, in order to generalize the notion of black holes to gravitational collapse situations other than exact spherically symmetric homogeneous dust case, it becomes necessary to rule out such naked or visible singularities by means of an explicit assumption. This is stated as the cosmic censorship hypothesis, which essentially states that if \( S \) is a partial Cauchy surface from which collapse commences, then there are no naked singularities to the future of \( S \), that is, which could be seen from the future null infinity. This is true for the spherical homogeneous dust collapse, where the resulting space-time is future asymptotically predictable and the censorship holds. Thus, the breakdown of physical theory at the space-time...
time singularity does not disturb prediction in future for the outside asymptotically flat region.

What will be the corresponding scenario for other collapse situations, when inhomogeneities, non-sphericity etc are allowed for? The answer to this question is not known either in the form of a proof for the future asymptotic predictability for general space-times, or of any other suitable version of cosmic censorship hypothesis. It is clear that the assumption of censorship in a suitable form is crucial to basic results in black hole physics. In fact, when one considers the gravitational collapse in a generic situation, the very existence of black holes requires this hypothesis [1].

If one is to establish the censorship by means of a rigorous proof, that of course requires a much more precise formulation of the hypothesis. The statement that result of a complete gravitational collapse must always be a black hole and not a naked singularity, or all singularities of collapse must be hidden in black holes, causally disconnected from observers at infinity, is not rigorous enough. This is because, under completely general circumstances, the censorship or asymptotic predictability is false as one could always choose a space-time manifold with a naked singularity which would be a solution to Einstein’s equations if we define $T_{ij} \equiv (1/8\pi)G_{ij}$. Thus, at the minimum, certain conditions on the stress-energy tensor are required, e.g. an energy condition. However, it turns out that to obtain an exact characterization of the restrictions one should require on matter fields in order to prove a suitable version of censorship hypothesis is an extremely difficult task and no such specific conditions are available as yet.
The requirements in the black hole physics and general predictability requirements in gravitation theory have led to several different formulations of cosmic censorship hypothesis. The version known as the *weak cosmic censorship* refers to the asymptotically flat space-times and has reference to the null infinity. Weak censorship, or asymptotic predictability, effectively postulates that the singularities of gravitational collapse cannot influence events near the future null infinity $I^+$. If $S$ is the partial Cauchy surface on which the regular initial data for collapse is defined, this is the requirement that $I^+$ is contained in the closure of $D^+(S)$, which is future development from $S$. Thus, the data on $S$ predicts the entire future for far away observers. The other version, called the *strong cosmic censorship*, is a general predictability requirement on any space-time, stating that all physically reasonable space-times must be globally hyperbolic (see e.g. Penrose (1979) in Ref. 1). In effect, the weak cosmic censorship states that the region of space-time outside a black hole must be globally hyperbolic. A precise formulation of this version of censorship will consist in specifying exact conditions under which the space-time would be strongly asymptotically predictable. In its weak form the censorship conjecture does not allow causal influences from singularity to asymptotic regions in space-time, that is, singularity cannot be globally naked. However, it could be locally naked in the sense that an observer within the event horizon and in the interior of the black hole could possibly receive particles or photons from the singularity. Clearly, one needs to sharpen such a formulation. For example, the metric on $S$ should approach that of Euclidian three-space at infinity and matter fields should satisfy suitable fall off conditions at spatial infinity; also one might want the null generators of $I^+$ to be complete, and one has to specify what exactly is meant by ‘physically reasonable’
matter fields. In fact, as far as the cosmic censorship hypothesis is concerned, it is a major problem in itself to find a satisfactory and mathematically rigorous formulation of what is physically desired to be achieved. Developing a suitable formulation would probably be a major advance towards the solution of the main problem. It should be noted that presently no general proof is available for any suitably formulated version of the weak censorship. The main difficulty appears to be that the event horizon is a feature depending on the whole future behavior of the solution over an infinite time period, whereas the present theory of quasi-linear hyperbolic equations guarantee the existence and regularity of the solutions over a finite time internal only. In this connection, the results of Christodoulou [5] on spherically symmetric collapse of a massless scalar field are relevant, where it is shown using global existence theorems on partial differential equations that global singularity free solutions can exist for weak enough initial data. In any case, even if it is true, the proof for a suitable version of the weak censorship conjecture would seem to require much more knowledge on general global properties of Einstein’s equations and solutions than is known presently.

It is now possible to summarize the overall situation as follows. Clearly, the cosmic censorship hypothesis is a crucial assumption underlying all of the black hole physics and gravitational collapse theory, and several important related areas. Whereas no proof for this conjecture is available, the first major task to be accomplished here is in fact to formulate rigorously a satisfactory version of the hypothesis. The proof of cosmic censorship would confirm the already widely accepted and applied theory of black holes, while its overturn would throw the black hole dynamics into serious doubt. Thus, cosmic censorship turns
out to be one of the most important open problem of considerable significance for general relativity and gravitation theory today. Even if it is true, a proof for this conjecture does not seem possible unless some major theoretical advances by way of mathematical techniques and understanding the global structure of Einstein equations are made. In fact, the direction of theoretical advances needed is not quite clear.

This situation leads us to conclude that the first and foremost task at the moment is to carry out a detailed and careful examination of various gravitational collapse scenarios to examine them for their end states. It is such an investigation, of more general collapse situations, which could indicate what theoretical advances to expect for a proof and what features to avoid while formulating the cosmic censorship.

3. Final Fate of Gravitational Collapse

It would seem from the previous considerations that we still do not have sufficient data and information available on the various possibilities for gravitationally collapsing configurations so as to decide one way or other on the issue of censorship hypothesis. What appears really necessary is a detailed investigation of different collapse scenarios, and to examine the possibilities arising, in order to have insights into the issue of the final fate of gravitational collapse. It is with such a purpose that we would like to discuss now several collapse situations involving different forms of matter to understand the final fate of collapse.

Since we are interested in collapse scenarios, we require that the space-time contains a
regular initial spacelike hypersurface on which the matter fields, as represented by the stress-energy tensor $T_{ij}$, have a compact support and all physical quantities are well-behaved on this surface. Also, we require the matter to satisfy a suitable energy condition and that the Einstein equations are satisfied. We say that the space-time contains a naked singularity if there is a future directed non-spacelike curve which reaches a far away observer or infinity in future, and in the past it terminates at the singularity.

We will be mainly interested in the nature of singularities arising as the final end product of collapse, rather than for example, the *shell-crossing* naked singularities which have been shown to occur in spherical collapse of dust [6] where shells of matter implode in such a way that fast moving outer shells overtake the inner shells, producing a globally naked singularity outside the horizon. These are singularities where shells of matter pile up to give two-dimensional caustics and the density and some curvature components blow up. The general point of view is, however, such singularities need not be treated as serious counter-examples to censorship hypothesis because these are merely consequent to intersection of matter flow lines. This gives a distributional singularity which is gravitationally weak in the sense that curvatures and tidal forces remain finite near the same.

On the other hand, there are *shell-focusing* naked singularities occurring at the center of spherically symmetric collapsing configurations of dust or perfect fluid or radiation shells, as we shall consider here. These are more difficult to ignore. One can rule them out only by saying that the dust or perfect fluids are not really ‘fundamental’ forms of matter. However, if censorship is to be established as a rigorous theorem, such objections have to be made
precise in terms of a clear and simple restriction on the stress-energy tensor, because these are forms of matter which otherwise satisfy reasonability conditions such as the dominant energy condition (provided there are no large negative pressures) or a well posed initial value formulation for the coupled Einstein-matter field equations. Further, these forms of matter are widely used in discussing various astrophysical processes.

When should one regard a naked singularity forming in gravitational collapse as a serious situation which must guide the formulation and proof of the censorship hypothesis, or which must be regarded as an important counter-example? The following could be imposed as a minimum set of conditions for this purpose. Firstly, the naked singularity has to be visible at least for a finite period of time to any far away observer. If only a single null geodesic escaped, it would provide only an instantaneous exposure by means of a single wave front. In order to yield any observable consequences, a necessary condition is that a family of future directed non-spacelike geodesics should terminate at the naked singularity in past. Next, this singularity must not be gravitationally weak, creating a mere space-time pathology, but must be a strong curvature singularity in a suitable sense where densities and curvatures diverge sufficiently fast. This would ensure that the space-time does not admit any continuous extension through the singularity in any meaningful manner, and hence such a singularity cannot be avoided. The physical effects due to strong fields should then be important near such a strong curvature singularity. Finally, the form of matter should be reasonable in that it must satisfy a suitable energy condition ensuring the positivity of energy, and the collapse ensues from an initial spacelike surface with a well-defined non-singular initial data.
We first consider the phenomena of gravitational collapse of a spherical shell of radiation in this context and examine the nature and structure of resulting singularity with special reference to censorship, and the occurrence of black holes and naked singularities. The main motivation to discuss this situation is this provides a clear picture in an explicit manner of what is possible in gravitational collapse.

3.1 The Vaidya-Papapetrou Radiation Collapse Models

These are inflowing radiation space-times which represent the situation of a thick shell of directed radiation collapsing at the center of symmetry in an otherwise empty universe which is asymptotically flat far away. The imploding radiation is described by the Vaidya space-time, given in $(v, r, \theta, \phi)$ coordinates as

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right)dv^2 + 2dvdr + r^2d\Omega^2$$

(2)

The radiation collapses at the origin of coordinates, $v = 0, r = 0$. Throughout the discussion here the null coordinate $v$ denotes the advanced time and $m(v)$ is an arbitrary but non-negative increasing function of $v$. The stress-energy tensor for the radial flux of radiation is

$$T_{ij} = \rho k_i k_j = \frac{1}{4\pi r^2} \frac{dm}{dv} k_i k_j$$

(3)

with $k_i = -\delta^v_i$, $k_i k^i = 0$, which represents the radially inflowing radiation along the world lines $v = \text{const}$. Note that $dm/dv \geq 0$ implies that the weak energy condition is satisfied.
The situation is that of a radially injected radiation flow into an initially flat and empty region, which is focused into a central singularity of growing mass by a distant source. The source is turned off at a finite time $T$ when the field settles to the Schwarzschild space-time. The Minkowski space-time for $v < 0, m(v) = 0$ here is joined to a Schwarzschild space-time for $v > T$ with mass $m_0 = m(T)$ by way of the Vaidya metric above.

The question is, will there be families of future directed non-spacelike geodesics which might possibly terminate at the singularity $v = 0, r = 0$ in the past, thus producing a naked singularity of the space-time; and if so, under what conditions this phenomena is ruled out to produce a black hole (see e.g. [2] and references therein). An interesting case, where many details can be explicitly worked out to yield considerable insight into the final fate of these collapse models is when the mass function $m(v)$ is chosen to be a linear function, $2m(v) = \lambda v$, with $\lambda > 0$. This is the Vaidya–Papapetrou space-time. In this case, $2m(v) = 0$ for $v < 0$, $2m(v) = \lambda v$ for $0 < v < T$ and $m(v) = m_0$ for $v > T$. Then, the mass for the final Schwarzschild black hole is $m_0$ and the causal structure of the space-time depends on the values chosen for the constants $m_0, T$, and $\lambda$. In this case the space-time admits a homothetic Killing vector field and it is possible to work out all the families of non-spacelike geodesics; and also to determine when such future directed families will terminate at the singularity in past, thus creating a naked singularity in the space-time [7]. It turns out that it is the time rate of collapse, as characterized by the value of the parameter $\lambda$, which determines the formation of either a naked singularity or a black hole as the end product of collapse in this case. For the range $0 < \lambda \leq 1/8$, the singularity turns out to be naked with families
of infinitely many geodesics escaping away from the singularity. For the range $\lambda > 1/8$ the event horizon fully covers the singularity, giving rise to a black hole.

The nature of this singularity, when it is naked, can be further explored by examining the curvature growth along the families of geodesics in the limit of approach to the singularity in the past. One considers the behavior of scalars such as $\psi = R_{ab}K^aK^b$, where $R^{ab}$ is the Ricci curvature and $K^a$ is the tangent vector to the nonspacelike geodesics. If $k$ is the affine parameter along the outgoing trajectories with $k = 0$ at the singularity, it is seen that $\psi$ diverges as $1/k^2$ in the limit of approach to the singularity along all the families of nonspacelike geodesics coming out of the singularity. This makes it the most powerful curvature singularity as characterized in the literature (see e.g. Clarke in Ref. 1), which is as strong as the big bang singularity of the Friedmann models. Another important invariant that characterizes the strength of the singularity is the Kretschmann scalar near the naked singularity, which is given by

$$\alpha = R^{ijkl}R_{ijkl} = \frac{48M(v)^2}{r^6} = \frac{12\lambda^2X^2}{r^6} \quad (4)$$

Examining the behavior of $\alpha$ along the families of the non-spacelike geodesics joining the singularity, it is verified that this scalar always diverges, thus establishing a scalar polynomial singularity as expected. However, an interesting directional behavior is revealed by the singularity as far as the scalar $\alpha$ is concerned, which was not the case for the scalar $\psi$ above. Unlike $\psi$ whose behavior was independent of direction of approach to singularity, the Kretschmann scalar not only shows directional dependence but also a dependence on
the parameter $\lambda$ which characterizes the rate at which the null dust is imploding. Such a situation has been referred to as a ‘directional singularity’, where the singularity strength varies with direction. Ellis and King [8] discussed such a directional property within the cosmological scenario of Friedman models where the strength depends on the direction along which the geodesic enters the singularity. Thus we see that similar property is exhibited by the naked singularity resulting from gravitational collapse.

To summarize, in the Vaidya–Papapetrou radiation collapse, not just isolated radial null geodesics but entire families of future directed non-spacelike geodesics escape from the naked singularity in the past, which forms at the origin of coordinates. The structure of these families and the curvature growth along such trajectories illustrates that this is a strong curvature visible singularity in a very powerful sense and curvatures diverge very rapidly along all the families of non-spacelike geodesics meeting the singularity in the past. Thus, this is not a removable singularity. Further, these results also generalize to the case when the mass function is not linear and hence space-time is not self-similar. Thus, the occurrence of naked singularity cannot be attributed to the assumption of linearity of mass function.

### 3.2 Inhomogeneous Dust Collapse

How are the conclusions given in Section 2 for the homogeneous dust collapse modified when the inhomogeneities of matter distribution are taken into account? It is important to include effects of inhomogeneities because typically a realistic collapse would start from a
very inhomogeneous initial data with a centrally peaked density profile. This problem can be investigated using the Tolman–Bondi-Lemaitre models [9], which describe gravitational collapse of an inhomogeneous spherically symmetric dust cloud, that is, a perfect fluid with equation of state $p = 0$. This is an infinite dimensional family of asymptotically flat solutions of Einstein’s equations, which is matched to the Schwarzschild space-time outside the boundary of the collapsing star. The Oppenheimer and Snyder [3] homogeneous dust ball collapse is a special case of this class of solutions.

This question of inhomogeneous dust collapse has attracted attention of many researchers [10] and it is again seen that the introduction of inhomogeneities leads to a rather different picture of gravitational collapse. Shell-crossing naked singularities occur in Tolman-Bondi-Lemaitre models when shells of dust cross one another at a finite radius, which could be locally and even globally naked. We shall not consider these for the reasons discussed earlier. More important are the shell-focusing singularities occurring on the central world line which we discuss below.

To discuss the structure of singularity forming in a general class of Tolman-Bondi-Lemaitre models, the metric for spherically symmetric collapse of inhomogeneous dust, in comoving coordinates $(t, r, \theta, \phi)$, is given by,

$$ds^2 = -dt^2 + \frac{R'^2}{1 + f} dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (5)

$$T^{ij} = \epsilon \delta^i_l \delta^j_l, \quad \epsilon = \epsilon(t, r) = \frac{F'}{R^2 R'}$$  \hspace{1cm} (6)
where $T^{ij}$ is the stress-energy tensor, $\epsilon$ is the energy density, and $R$ is a function of both $t$ and $r$ given by

$$\dot{R}^2 = \frac{F}{R} + f$$

(7)

Here the dot and prime denote partial derivatives with respect to the parameters $t$ and $r$ respectively. As we are considering collapse, we require $\dot{R}(t, r) < 0$. The quantities $F$ and $f$ are arbitrary functions of $r$ and $4\pi R^2(t, r)$ is the proper area of the mass shells. The area of such a shell at $r = \text{const.}$ goes to zero when $R(t, r) = 0$. For gravitational collapse situation, we take $\epsilon$ to have compact support on an initial spacelike hypersurface and the space-time can be matched at some $r = \text{const.} = r_c$ to the exterior Schwarzschild field with total Schwarzschild mass $m(r_c) = M$ enclosed within the dust ball of coordinate radius of $r = r_c$. The apparent horizon in the interior dust ball lies at $R = F(r_c)$.

With the integration of equation for $\dot{R}$ above we have in all three arbitrary functions of $r$, namely $f(r)$, $F(r)$, and $t_0(r)$ where the last indicates the time along the singularity curve. One could, however, use the coordinate freedom left in the choice of scaling of $r$ to reduce the number of arbitrary functions to two. Thus, rescaling $R$ such that $R(0, r) = r$ leaves us with only two free functions $f$ and $F$. The time $t = t_0(r)$ corresponds to $R = 0$ where the area of the shell of matter at a constant value of the coordinate $r$ vanishes. It follows that the singularity curve $t = t_0(r)$ corresponds to the time when the matter shells meet the physical singularity. Thus, the range of coordinates is given by $0 \leq r < \infty$, $-\infty < t < t_0(r)$. It follows that unlike the collapsing Friedmann case, or the homogeneous dust case, where the
physical singularity occurs at a constant epoch of time (say, at \( t = 0 \)), the singular epoch now
is a function of \( r \) as a result of inhomogeneity in the matter distribution. One could recover
the Friedmann case from the above if we set \( t_0(r) = t'_0(r) = 0 \). The function \( f(r) \) classifies
the space-time as bound, marginally bound, or unbound depending on the range of its values
which are \( f(r) < 0, f(r) = 0, f(r) > 0 \), respectively. The function \( F(r) \) is interpreted as the
weighted mass within the dust ball of coordinate radius \( r \). For physical reasonableness the
weak energy condition is assumed, that is, \( T_{ij}V^iV^j \geq 0 \) for all non-spacelike vectors \( V^i \). This
implies that the energy density \( \epsilon \) is everywhere positive, \( (\epsilon \geq 0) \) including the region near
\( r = 0 \). From the scaling above, the energy density \( \epsilon \) on the hypersurface \( t = 0 \) is written as
\( \epsilon = F'/r^2 \). Then the weak energy condition implies that \( F' \geq 0 \) throughout the space-time.

Using the above framework, the nature of the singularity \( R = 0 \) can be examined. In
particular, the problem of nakedness or otherwise of the singularity can be reduced to the
existence of real, positive roots of an algebraic equation, constructed out of the free functions
\( F \) and \( f \) and their derivatives [11], which constitute the initial data of this problem. We call
the singularity to be a central singularity if it occurs at \( r = 0 \). Partial derivatives \( R' \) and \( \dot{R}' \)
can be written as,

\[
\left( \frac{\partial R(t, r)}{\partial r} \right)_{t=\text{const.}} = R' = (\eta - \beta)P - \left[ \frac{1 + \beta - \eta}{\sqrt{\lambda + f}} + (\eta - \frac{3}{2}\beta)\frac{t}{r} \right] \dot{R} \quad (8)
\]

\[
\left( \frac{\partial R'(t, r)}{\partial t} \right)_{r=\text{const.}} = \frac{\beta}{2r} \dot{R} + \frac{\lambda}{2rP^2} \left[ \frac{1 + \beta - \eta}{\sqrt{\lambda + f}} + (\eta - \frac{3}{2}\beta)\frac{t}{r} \right] \quad (9)
\]
where we have used the notation,

\[ R(t, r) = rP(t, r), \quad \eta = \eta(r) = r \frac{F'}{F}, \quad \beta = \beta(r) = r \frac{f'}{f}, \quad F(r) = r \lambda(r) \quad (10) \]

To focus the discussion, we restrict to functions \( f(r) \) and \( \lambda(r) \) which are analytic at \( r = 0 \) such that \( \lambda(0) \neq 0 \).

The tangents \( K^r = dr/dk \) and \( K^t = dt/dk \) to the outgoing radial null geodesics, with \( k \) as the affine parameter, satisfy

\[ \frac{dK^t}{dk} + \frac{\dot{R}}{\sqrt{1 + f}} = 0, \quad K^r K^t = 0, \quad \frac{dt}{dr} = \frac{K^t}{K^r} = \frac{R'}{\sqrt{1 + f}} \quad (11) \]

Our purpose is to find whether these geodesics terminate in the past at the central singularity \( r = 0, t = t_0(0) \). The exact nature of this singularity \( t = 0, r = 0 \) could be analyzed by the limiting value of \( X \equiv t/r \) at \( t = 0, r = 0 \). If the geodesics meet the singularity with a definite value of the tangent then using l’Hospital rule we get

\[ X_0 = \lim_{t \to 0, r \to 0} \frac{t}{r} = \lim_{t \to 0, r \to 0} \frac{dt}{dr} = \lim_{t=0, r=0} \frac{R'}{\sqrt{1 + f}} \quad (12) \]

where the notation is, \( \lambda_0 = \lambda(0), \beta_0 = \beta(0), f_0 = f(0) \) and \( Q = Q(X) = P(X, 0) \). Using the expression for \( R' \) earlier, the above can be written as \( V(X_0) = 0 \), where

\[ V(X) \equiv (1 - \beta_0)Q + \left( \frac{\beta_0}{\sqrt{\lambda_0 + f_0}} + (1 - \frac{3}{2} \beta_0)X \right) \sqrt{\frac{\lambda_0}{Q}} + f_0 - X \sqrt{1 + f_0} \quad (13) \]
Hence if the equation \( V(X) = 0 \) has a real positive root, the singularity could be naked. In order to be the end point of null geodesics at least one real positive value of \( X_0 \) should satisfy the above. Clearly, if no real positive root of the above is found, the singularity \( t = 0, r = 0 \) is not naked. It should be noted that many real positive roots of the above equation may exist which give the possible values of tangents to the singular null geodesics terminating at the singularity. However, such integral curves may or may not realize a particular value \( X_0 \) at the singularity. Suppose now \( X = X_0 \) is a simple root to \( V(X) = 0 \). To determine whether \( X_0 \) is realized as a tangent along any outgoing singular geodesics to give a naked singularity, one can integrate the equation of the radial null geodesics in the form \( r = r(X) \) and it is seen that there is always at least one null geodesic terminating at the singularity \( t = 0, r = 0 \), with \( X = X_0 \). In addition there would be infinitely many integral curves as well, depending on the values of the parameters involved, that terminate at the singularity. It is thus seen [11] that the existence of a positive real root of the equation \( V(X) = 0 \) is a necessary and sufficient condition for the singularity to be naked. Finally, to determine the curvature strength of the naked singularity at \( t = 0, r = 0 \), one may analyze the quantity \( k^2 R_{ab} K^a K^b \) near the singularity. Standard analysis shows that the strong curvature condition is satisfied, in that the above quantity remains finite in the limit of approach to the singularity.

The assumption of vanishing pressures here, which could be important in the final stages of the collapse, may be considered as the limitation of dust models. On the other hand, it is also argued sometimes that in the final stages of collapse, the dust equation of state could be relevant and at higher and higher densities the matter may behave more and more like dust.
Further, if there are no large negative pressures (as implied by the validity of the energy conditions), then the pressure also might contribute gravitationally in a positive manner to the effect of dust and may not alter the conclusions. The considerations given to the case with non-zero pressure are described below.

3.3 Collapse with Non-zero Pressures

It is clearly important to consider collapse situations which consider matter with non-zero pressures and with reasonable equations of state. It is possible that pressures may play an important role for the later stages of collapse and one must investigate the possibility if pressure gradients could prevent the occurrence of naked singularity.

We now discuss these issues, namely, the existence, the termination of non-spacelike geodesic families, and the strength of such a singularity for collapse with non-zero pressure. Presently we consider only self-similar collapse models and generalization of this will be discussed next. A numerical treatment of self-similar perfect fluid was given by Ori and Piran [12] and the analytic consideration was given by Joshi and Dwivedi [13].

A self-similar space-time is characterized by the existence of a homothetic Killing vector field. A spherically symmetric space-time is self-similar if it admits a radial area coordinate $r$ and an orthogonal time coordinate $t$ such that for the metric components we have $g_{tt}(ct, cr) = g_{tt}(t, r)$, $g_{rr}(ct, cr) = g_{rr}(t, r)$, for all $c > 0$. Thus, along the integral curves of the Killing vector field all points are similar. For the self-similar case, the Einstein equations reduce to
ordinary differential equations.

The spherically symmetric metric in comoving coordinates is given by,

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + r^2 S^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2) \quad (14)$$

Self-similarity implies that all variables of physical interest are expressed in terms of the similarity parameter $X = t/r$ and so $\nu, \psi$ and $S$ are functions of $X$ only. The pressure and energy density in comoving coordinates are ($u^a = e^{-\nu} \delta^a_t$)

$$P = \frac{p(X)}{8\pi r^2}, \quad \rho = \frac{\rho(X)}{8\pi r^2} \quad (15)$$

The field equations in this case for a perfect fluid have been discussed in [14], and after suitable integrations [13] these can be written as,

$$e^{2\psi} = \alpha \eta^{-\frac{1}{a+1}} S^{-4}, \quad e^{2\nu} = \gamma (\eta X^2)^{\frac{2a}{a+1}} \quad (16)$$

$$\dot{V}(X) = X e^{2\nu} [\eta + p] e^{2\psi} = X e^{2\nu} [H - 2] \quad (17)$$

$$\left( \frac{\dot{S}}{S} \right)^2 V + \left( \frac{\dot{S}}{S} \right) (\dot{V} + 2X e^{2\nu}) + e^{2\nu + 2\psi} \left( -\eta - e^{-2\nu} + \frac{1}{S^2} \right) = 0 \quad (18)$$

where the dot denotes differentiation with respect to the similarity parameter $X$. The quantities $V$ and $H$ here are defined by

$$V(X) \equiv e^{2\psi} - X^2 e^{2\nu}, \quad H \equiv (\eta + p)e^{2\psi} \quad (19)$$
Here the assumption is that the collapsing fluid is obeying an adiabatic equation of state
\[ p(X) = a\eta(X), \]
with \( 0 \leq a \leq 1 \). The special case \( a = 0 \) describes dust and \( a = 1/3 \) gives the
equation of state for radiation.

The point \( t = 0, r = 0 \), is a singularity where the density necessarily diverges. Such a
divergence is also observed when we approach the singularity along any line of self-similarity
\( X = X_0 \). This leads to the divergence of curvature scalars such as \( R^i_j R^i_j \) and also of the
Ricci scalar \( R = \rho - 3p \) if \( a \neq 1/3 \). It is to be examined whether this singularity could be
possibly naked, and if so whether families of non-spacelike geodesics would terminate at the
same in the past.

For this purpose, the first requirement is, the families of non-spacelike geodesics are to
be integrated. This is possible in this case using the existence of a homothetic Killing vector
field [13]. The geodesic equations are then,

\[ \frac{dt}{dr} = \frac{X \pm e^{2\psi}Q}{1 \pm Xe^{2\psi}Q} \]  \hspace{1cm} (20)

For a specific discussion, let us choose the function \( Q \) to be positive throughout and \( \pm \) signs
represent outgoing or ingoing solutions. The point \( t = 0, r = 0 \), is a singular point of the
above differential equation. The nature of the limiting value of \( X = t/r \) plays an important
role in the analysis of non-spacelike curves that terminate at the singularity in past and
reveals the exact structure of the singularity. From geodesic equations, using l’Hospital’s
rule we get

\[ X_0 = \lim_{t \to 0, r \to 0} \frac{t}{r} = \lim_{t \to 0, r \to 0} \frac{dt}{dr} = \frac{X_0 \pm e^{2\psi(X_0)}Q(X_0)}{1 \pm X_0 e^{2\nu(X_0)}Q(X_0)} \] (21)

Thus we see that if trajectories are to go out of the singularity, then there exists a real positive value \( X_0 \) such that

\[ V(X_0) \equiv e^{2\psi(X_0)} - X_0^2 e^{2\nu(X_0)} = 0 \] (22)

If \( V(X_0) = 0 \) has no real positive roots then geodesics clearly do not terminate at the singularity with a definite tangent. In case when \( V(X_0) = 0 \) has a positive real root, a single radial null geodesic would escape from the singularity. Existence of positive real roots of \( V(X) = 0 \) is therefore a necessary and sufficient condition for the singularity to be naked and at least one single null geodesic in the \((t, r)\) plane would escape from the singularity.

In order to find whether a family of null or timelike geodesics would terminate at the singularity in the present case, one can consider the equation of geodesics \( r = r(X) \) in \((r, X)\) plane, restricting to positive sign solutions which are outgoing geodesics. Suppose \( V(X) = 0 \) has one simple real positive root \( X = X_0 \). Then using the equation for \( \dot{V}(X) \) given above, we can write near the singularity \( V(X) = (X - X_o)X_o e^{2\nu(X_o)}(H(X_o) - 2) \), and using the fact that \( Q \) is positive, geodesics can be integrated near the singularity to get

\[ r = D(X - X_o)^{\frac{2}{n_o - 2}} \] (23)
where \( H_0 = H(X_0) \). When \( H_0 > 2 \) it is seen that an infinity of integral curves will meet the singularity in the past with tangent \( X = X_0 \); different curves being characterized by different values of the constant \( D \). It follows that this singularity is at least locally naked from which an infinity of non-spacelike curves are ejected. In the case \( H_0 < 2 \) but \( H_0 > 0 \), the singularity would be a node in the \((r,t)\) plane and infinity of curves will still escape. However, if \( H_0 < 0 \), in \((r,t)\) plane the integral curves move away from the singularity and never terminate there. Thus, we see that infinitely many integral curves would terminate at the singularity as long as

\[
\infty > H_0 = H(X_0) = (\eta + p)e^{2\psi} > 0
\]  

(24)

The above will be satisfied provided the weak energy condition holds and further that the energy density as measured by any timelike observer is positive in the collapsing region near the singularity. One then examines the curvature strength of the singularity along the trajectories coming out and it is again seen that the strong curvature condition is satisfied.

The results could be summarized as follows. If in a self-similar collapse a single null radial geodesic escapes the singularity, then an entire family of non-spacelike geodesics would also escape provided the positivity of energy density is satisfied as above. It also follows that no families of non-spacelike geodesics would escape the singularity, even though a single null trajectory might, if the weak energy condition is violated.

3.4 Gravitational Collapse with General Form of Matter
Consideration of matter forms above such as directed radiation, dust, perfect fluids etc imply some general pattern emerging about the final outcome of gravitational collapse. Hence one could ask the question whether the final fate of collapse would be independent of the form of matter under consideration. An answer to this is important because it has often been conjectured in the literature (see e.g. [1,2]) that once a suitable form of matter with an appropriate equation of state, and satisfying energy condition, is considered then there may be no naked singularities. After all, there is always a possibility that during the final stages of collapse matter may not have any of the forms considered above, because such relativistic fluids are phenomenological and one must treat matter in terms of fundamental fields, such as for example, a massless scalar field.

Some efforts in this direction are worth mentioning where the above results on perfect fluid were generalized to matter forms without any restriction on the form of $T_{ij}$, which was supposed to satisfy the weak energy condition only [15]. A consideration to a general form of matter was also given by Lake; and by Szekeres and Iyer [16] who do not start by assuming an equation of state but a class of metric coefficients is considered with a certain power law behavior. The main argument of the results such as [15] is along the following lines. In the discussion above it was pointed out that naked singularities could form in the gravitational collapse from a regular initial data, from which non-zero measure families of non-spacelike trajectories come out. The criterion for the existence of such singularities was characterized in terms of the existence of real positive roots of an algebraic equation constructed out of the field variables. A similar procedure was developed now for general form of matter. In
comoving coordinates, the general matter can be described by three functions, namely the energy density and the radial and tangential pressures. The existence of naked singularity is again characterized in terms of the real positive roots of an algebraic equation, constructed from the equations of non-spacelike geodesics which involve the three metric functions. The field equations then relate these metric functions to the matter variables and it is seen that for a subspace of this free initial data in terms of matter variables, the above algebraic equation will have real positive roots, producing a naked singularity in the space-time.

It thus follows that the occurrence of naked singularity is basically related to the choice of initial data to the Einstein field equations, and would therefore occur from regular initial data within the general context considered, subject to the matter satisfying weak energy condition. The condition on initial data which leads to the formation of black hole is also characterized in these works. It then appears that the occurrence of naked singularity or a black hole is more a problem of choice of the initial data for field equations rather than that of the form of matter or the equation of state. This has important implication for cosmic censorship in that in order to preserve the same one has to avoid all such regular initial data causing naked singularity, and hence a deeper understanding of the initial data space is required in order to determine such initial data and the kind of physical parameters they would specify. This would, in other words, classify the range of physical parameters to be avoided for a particular form of matter. More importantly, it would also pave the way for the black hole physics to use only those ranges of allowed parameter values which would produce black holes, thus putting black hole physics on a more firm footing.
3.5 Scalar Field Collapse

Much attention has been devoted in past years to analyze the collapse of a scalar field, both analytically [5, 18], as well as more recently numerically [17]. We now discuss this case briefly. This is a model problem of a single massless scalar field which is minimally coupled to gravitational field and it provides possibly one of the simplest scenarios to investigate the nonlinearity effects of general relativity. On the analytic side, the results by Christodoulou show that when the scalar field is sufficiently weak, there exists a regular solution, or global evolution for an arbitrary long time of the coupled Einstein and scalar field equations. During the collapse, there is a convergence towards the origin, and after a bounce the field disperses to infinity. For strong enough field, the collapse is expected to result into a black hole. For self-similar collapse, he has also obtained results to show that the collapse will result into a naked singularity. However, he claimed that the initial conditions resulting into a naked singularity are a set of measure zero and hence the naked singularity formation may be an unstable phenomenon.

Such an approach helps study the cosmic censorship problem as the evolution problem in the sense of examining the global Cauchy development of a self-gravitating system outside an event horizon. A ‘dynamical’ version of cosmic censorship will demand that given reasonable initial data which is asymptotically flat, and assuming some reasonable energy conditions, there exists a global Cauchy evolution of the system outside the event horizon in the sense that the solution exists for arbitrary large times for an asymptotic observer. For a discussion of such an approach in the context of self-gravitating scalar fields we refer to Malec in Ref. 18.
He considers the problem of global existence of solutions and also finds an explicit example of an initial configuration that results into a naked singularity at the center of symmetry.

The problem of scalar field collapse has been numerically studied by Choptuik and others [17]. Choptuik considered a family of scalar field solutions where a parameter \( p \) characterized the strength of the scalar field. His numerical calculations showed that for black hole formation, there is a critical limit \( p \to p^* \) and the mass of the resulting black holes satisfy a power law \( M_{bh} \propto (p - p^*)^{\gamma} \), where the critical exponent \( \gamma \) has value of about 0.37. It was then conjectured that such a critical behavior may be a general property of gravitational collapse, because similar behavior was found by Abraham and Evans for imploding axisymmetric gravitational waves. Also, Evans and Coleman considered the collapse of radiation with equation of state \( p = \rho/3 \), assuming self-similarity for solutions. It is still not clear if the critical parameter \( \gamma \) will have the same value for all forms of matter chosen and some further investigation may be required to determine this issue. As the parameter \( p \) moves from the weak to strong range, very small mass black holes can form. This has relevance to censorship, because in such a case one can probe and receive messages from arbitrarily near to the singularity and this is naked singularity like behavior. Attempts have also been made to construct models analytically which may reproduce such a critical behavior [18], assuming self-similarity. In particular, Brady has constructed solutions which have dispersal, and also solutions with black holes or naked singularities.

3.6 Non-spherical Collapse
What will be the final fate of gravitational collapse which is not spherically symmetric? 

The main phases of spherical collapse of a massive star would be typically instability, implosion of matter, and subsequent formation of an event horizon and a space-time singularity of infinite density and curvature with infinite gravitational tidal forces. This singularity may or may not be fully covered by the horizon as we have discussed above. Again, small perturbations over the spherically symmetric situation would leave the situation unchanged \cite{19} in the sense that an event horizon will continue to form in the advanced stages of the collapse.

The next question is, do horizons still form when the fluctuations from the spherical symmetry are high and the collapse is highly non-spherical? It was shown by Thorne \cite{20}, for example, that when there is no spherical symmetry, the collapse of infinite cylinders do give rise to naked singularities in general relativity, which are not covered by horizons. This situation motivated Thorne to propose the following *hoop conjecture* for finite systems in an asymptotically flat space-time, which characterizes the final fate of non-spherical collapse: The horizons of gravity form when and only when a mass $M$ gets compacted in a region whose circumference in every direction obeys $C \leq 2\pi(2GM/c^2)$. Thus, unlike the cosmic censorship conjecture, the hoop conjecture does not rule out all the naked singularities but only makes a definite assertion on the occurrence of the event horizons in gravitational collapse. We also note that the hoop conjecture is concerned with the formation of event horizons, and not with naked singularities. Thus, even when event horizons form, say for example in the spherically symmetric case, it does not rule out the existence of naked singularities, i.e. it
does not imply that such horizons must always cover the singularities.

When the collapse is sufficiently aspherical, with one or two dimensions being sufficiently larger than the others, the final state of collapse could be a naked singularity, according to the hoop conjecture. Such a situation is inspired by the Lin, Mestel and Shu [21] instability in Newtonian gravity where a non-rotating homogeneous spheroid collapses, maintaining its homogeneity and sphericity but its deformations grow. If the initial condition is that of a slightly oblate spheroid, the collapse results into a pancake singularity through which the evolution could proceed. However, for a slightly prolate spheroidal configuration, the matter collapses to a thin thread which ultimately results into a spindle singularity. This is more serious in nature in that the gravitational potential and the tidal forces blow up as opposed to only density blowing up in a shell-crossing singularity. Even in the case of an oblate collapse, the passing of matter through the pancake causes prolateness and subsequently a spindle singularity again results without the formation of any horizon.

It was indicated by the numerical calculations of Shapiro and Teukolsky [22] that a similar situation maintains in general relativity as well in conformity with the hoop conjecture. They evolved collisionless gas spheroids in full general relativity which collapse in all cases to singularities. When the spheroid is sufficiently compact a black hole may form, but when the semimajor axis of the spheroid is sufficiently large, a spindle singularity results without an apparent horizon forming. One could treat these only as numerical results, as opposed to a full analytic treatment, which need not be in contradiction to a suitably formulated version of cosmic censorship. However, this gives rise to the possibility of occurrence of
naked singularities in collapse of finite systems in asymptotically flat space-times, which could be in violation of weak cosmic censorship but in conformity with the hoop conjecture.

Apart from such numerical simulations, some analytic treatments of aspherical collapse are also available. For example, the aspherical Szekeres models for irrotational dust without any Killing vectors, generalizing the spherical Tolman-Bondi-Lemaitre collapse, were recently studied [23] to deduce the existence of strong curvature central naked singularities. While this indicates that naked singularities are not necessarily confined to spherical symmetry, it must be noted that dynamical evolution of a non-spherical collapse still remains a largely uncharted territory.

4. Concluding Remarks and Open Issues

We have discussed here several gravitational collapse scenarios and the following pattern appears to be emerging. In the first place, singularities not covered fully by the event horizon, i.e. naked singularities, could occur in several collapsing configurations from regular initial data, with reasonable equations of state such as describing radiation, dust or a perfect fluid with a non-zero pressure, or also for general forms of matter. Secondly, families of a non-zero measure set of photons or particles, escape from such a naked singularity (which is a region of extreme densities in the space-time) to reach far-away observers. Finally, the singularity is physically significant in that densities and curvatures diverge powerfully near such a naked singularity. It would appear that such results regarding the final fate of gravitational collapse,
generated from study of different physically reasonable collapse scenarios, may provide useful insights into black hole physics and may be of help for any possible formulation of the cosmic censorship hypothesis.

One possible insight, for example, could be that the final state of a collapsing star, in terms of either a black hole or a naked singularity, may not really depend on the form or equation of state of collapsing matter, but is actually determined by the physical initial data in terms of the initial density profiles and pressures. As a specific example, the role played by the initial density and velocity distributions was examined [24] for the gravitational collapse of spherically symmetric inhomogeneous dust cloud. The collapse can end in either a black hole or a naked singularity depending on the values of initial parameters. In the marginally bound case, the collapse ends in a naked singularity if the leading nonvanishing derivative at the center is either the first one or the second one. If the first two derivatives are zero and the third derivative non-zero, the singularity is naked and strong, or covered, depending on a quantity determined by the third derivative and the central density. If the first three derivatives are zero, the collapse ends in a black hole. Analogous results are found when the cloud is not marginally bound, and also for a cloud for which the collapse begins from rest. There is a transition from the naked singularity phase to the black hole phase as the initial density profile is made more and more homogeneous near the center. As one progresses towards more homogeneity (and hence towards a stronger gravitational field), there first occurs a weak naked singularity, then a strong naked singularity, and finally a black hole.

The important question then is the genericity and stability of such naked singularities
arising from regular initial data. Will the initial data subspace, which gives rise to naked singularity as end state of collapse, have zero measure in a suitable sense? In that case, one would be able to reformulate more suitably the censorship hypothesis, based on a criterion that naked singularities could form in collapse but may not be generic.

It has also been proposed (see e.g. [25]) that one may try to evolve some kind of a physical formulation for cosmic censorship, where the available studies on various gravitational collapse scenarios such as above may provide a useful guide. The idea here would be to study the various properties of naked singularities collectively as they emerge from the studies so far. One would then argue that objects with such properties are not physical and hence would not matter in any manner even if they occurred in nature. We list some of the possibilities in this direction below:

(i) One may, for example, ask about the ‘mass’ of such a naked singularity. If they turned out to be massless with a suitably well-defined meaning of mass, then they may not be physically treated as important. One should keep in view here the difficulties associated with specifying notions such as mass and energy in general relativity.

(ii) Another possibility is to argue that the redshift from such a naked singularity would be infinite and hence it would not be observable any way. Here also, we have to define redshift carefully, as we are no longer dealing with a regular event of the space-time, but with the redshift of a singularity where densities and curvatures diverge.

(iii) It is possible that the use of a different system, such as the Einstein-Vlasov equations,
in order to trace the evolution of collapse might help avoid naked singularities, or for that matter singularities of all kinds. Some investigations in this direction have been reported [26] and this alternative needs to be pursued further. Studying exact solutions of the Einstein-Maxwell equations describing moving black holes, or further investigation of available static exact solutions also might yield interesting insights on the nature and properties of naked singularities and may provide a test for censorship [27].

(iv) Still another possibility is to invoke quantum effects and quantum gravity. While naked singularities may form in classical general relativity, quantum gravity should presumably remove them. So, why bother about them? The point here is that even though the final singularity may be removed in this way, still there would be very high density and curvature regions in the classical regime, which would be causally communicating with outside observers, as opposed to the black hole situation where this is not the case. Some interesting efforts have been made recently to show that quantum effects could remove the naked singularity [28]; this would then be a quantum cosmic censorship.

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