The Motion of Out-of-Plane Equilibrium Points in the Elliptic Restricted Three-Body Problem at $J_4$

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Abstract. We have investigated the motion of the out-of-plane equilibrium points within the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at $J_4$ of the smaller primary in the field of stellar binary systems: Xi- Bootis and Sirius around their common center of mass in elliptic orbits. The positions and stability of the out-of-plane equilibrium points are greatly affected on the premise of the oblateness at $J_4$ of the smaller primary, semi-major axis and the eccentricity of their orbits. The positions $L_6, L_7$ of the infinitesimal body lie in the $xz$-plane almost directly above and below the center of each oblate primary. Numerically, we have computed the positions and stability of $L_6, L_7$ for the aforementioned binary systems and found that their positions are affected by the oblateness of the primaries, the semi-major axis and eccentricity of their orbits. It is observed that, for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane equilibrium points are unstable.

1. Introduction

Celestial bodies in the general restricted three-body problem are assumed to be spherical, but in nature, several celestial bodies have observed the significant effects of oblateness of their bodies [1,2,3,4,5], have observed the significant effects of oblateness of the bodies. The restricted three-body problem with oblateness of the primaries have received tremendous attention, specifically in the two and three-dimensional cases with respect to its five co-planar equilibrium points $L_i (i = 1, \ldots, 5)$: The points $L_1, L_2, L_3$ lying on the line joining the primaries are called collinear equilibrium points, while the points $L_4, L_5$ forming the triangle with the line joining the primaries are called triangular points. The collinear points have been shown to be generally unstable, while the triangular points are conditionally stable [3,6,7,8,9].

The elliptic restricted three-body problem (ER3BP) describes the three-dimensional motion of a small particle, called the third body (infinitesimal mass) under the gravitational attraction force of two finite bodies, called the primaries, which revolve on elliptic orbits in a plane around their common center of mass. The motions of an asteroid, a space probe or an artificial satellite under the gravitational attraction of the Sun-Jupiter or Earth-Moon systems are typical examples. The location and stability of triangular and collinear equilibrium points were seem to be affected by the oblateness of the primaries, semi-major axis and eccentricity of their orbits, which leads to a decrease in the size of the region of stability with an increase in the parameters involved [1,10,11,12].

The equation of motion of the three-dimensional restricted three-body problem with oblateness of the primaries allow the existence of out-of-plane equilibrium points. These points lie in the $xz$-plane symmetrically with respect to the $x$-axis along the curve almost directly above and below the center of each oblate primary. These points are denoted by $L_{6,7}$ [1,10,13,14,15].

Das et al. [14] observed that, in the photo-gravitational circular restricted three-body problem, the out-of-plane equilibrium points are of a passive micro size particle when their stability in the field of radiating binary systems are considered. Singh and Amuda [15] studied the out-of-plane
equilibrium points with Poynting-Robertson (P-R) drag, they found that due to the ratio of radiation to the gravitational force of the smaller primary and the expression of $\gamma_o$ coordinate with oblateness of the bigger primary. The out-of-plane equilibrium points exist but its stability analysis remain the same and are unstable.

Singh and Umar [1], examined the motion of a particle under the influence of an oblate dark degenerate primary and a luminous secondary and the stability of triangular points when both oblate primaries emit light energy simultaneously in the elliptic restricted three-body problem respectively. They found that, in the stellar systems, a planet moving in the field of a binary star system effectively constitutes a three-body system.

In our study, we have considered the motion of the out-of-plane equilibrium points within the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at $J_4$ of the smaller primary in the field of stellar binary systems: Xi-Booites and Sirius.

The paper is organized as follows: Sections 2 presents the equations of motion; Section 3 locates the positions of out-of-plane equilibrium points of the smaller primary; sections 4 examines their stability; section 5 explores numerical application and the discussions and conclusions are provided in section 6.

2. Equation of Motion

We adopt the equations of motion of an infinitesimal mass in the elliptic restricted three-body problem under the influence of the oblateness at $J_4$ of the smaller primary from [10] and are presented here in dimensionless-pulsating coordinate system $(\zeta, \eta, \xi)$ as follows;

$$\frac{\delta \zeta}{\delta \xi} - 2\eta' = \frac{\delta \Omega}{\delta \xi}, \quad \frac{\delta \zeta}{\delta \eta} + 2\xi' = \frac{\delta \Omega}{\delta \eta}, \quad \frac{\delta \zeta}{\delta \xi} = \frac{\delta \Omega}{\delta \xi} \tag{1}$$

with the force function

$$\Omega = (1 - e^2)^{-\frac{1}{2}} [\frac{1}{2} (\xi^2 + \eta^2) + \frac{1}{n^2} \{(\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu B_1}{2 r_2^3} - \frac{3 \mu B_2}{8 r_2^5} - \frac{3 \mu B_2 Z^2}{2 r_2^5} + \frac{9 \mu B_2 Z^2}{8 r_2^7}\}] \tag{2}$$

The mean motion, $n$, is given as

$$n^2 = \frac{(1+e^2)^{\frac{3}{2}}}{a(1-e^2)} \left[1 + \frac{3}{2} B_1 - \frac{15}{8} B_2\right] \tag{3}$$

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2, (i = 1, 2) \quad \xi_1 = -\mu, \quad \xi_2 = 1 - \mu, \quad \mu = \frac{m_2}{m_1 + m_2} \tag{4}$$

where, $m_1$, $m_2$ are the masses of the bigger and smaller primaries positioned at the points $(\xi_i, 0, 0), i = 1, 2$; $B_1 = J_2 / R_2^2 \ (i = 1, 2)$ characterize zonal harmonic oblateness of the smaller primary whose mean radius is $R_2$. $\mu = \frac{m_2}{m_1 + m_2}$ is the mass ratio, while $a$ and $e$ are the semi-major axis and eccentricity of the orbits, respectively.

3. Positions of Out-Of-Plane Equilibrium Points

The positions of the out-of-plane equilibrium points denoted by $L_{6,7}$, can be located by solving the solutions of $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$ that is;

$$\Omega_\xi = (1 - e^2)^{-\frac{1}{2}} \left[\xi - \frac{1}{n^2} \left(\frac{(1-\mu)(\xi+\mu)}{r_1^3} + \frac{\mu (\xi+\mu-1)}{r_2^3} + \frac{3 \mu (\xi+\mu-1) B_1}{2 r_2^5} - \frac{15 \mu (\xi+\mu-1) B_2}{8 r_2^7} - \frac{15 \mu (\xi+\mu-1) B_1 Z^2}{2 r_2^7} + \frac{63 \mu (\xi+\mu-1) B_2 Z^2}{8 r_2^9}\right)\right] \tag{5}$$

$$\Omega_\eta = (1 - e^2)^{-\frac{1}{2}} \left[\eta - \frac{1}{n^2} \left(\frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3 \mu B_1}{2 r_2^5} - \frac{15 \mu B_2}{8 r_2^7} - \frac{15 \mu B_2 Z^2}{2 r_2^7} + \frac{63 \mu B_2 Z^2}{8 r_2^9}\right)\right] \tag{6}$$

$$\Omega_\zeta = (1 - e^2)^{-\frac{1}{2}} \left[\zeta - \frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3 \mu B_1}{2 r_2^5} - \frac{15 \mu B_2}{8 r_2^7} - \frac{15 \mu B_2 Z^2}{2 r_2^7} + \frac{63 \mu B_2 Z^2}{8 r_2^9}\right] \tag{7}$$
For the equilibrium points, the solution of equations (5), (6) and (7) will be zero (\( \Omega_{\xi} = \Omega_{\eta} = \Omega_{\zeta} = 0 \))

\[
\xi - \frac{1}{n^2} \left( \frac{1-\mu}{r_1^3} \frac{\mu}{r_2^3} + \frac{\mu}{r_3^3} \right) + \frac{3\mu(\xi+\eta+1)B_1}{8r_2^7} - \frac{15\mu(\xi+\eta+1)B_2}{2r_2^7} + \frac{63\mu(\xi+\eta+1)B_2z^2}{8r_2^9} = 0
\]  

(8)

\[
\eta \left( 1 - \frac{1}{n^2} \left( \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} - \frac{15\mu B_2z^2}{2r_2^7} + \frac{63\mu B_2z^2}{8r_2^9} \right) \right) = 0
\]  

(9)

\[
-\zeta \left( \frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu B_1}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} - \frac{15\mu B_2z^2}{2r_2^7} + \frac{63\mu B_2z^2}{8r_2^9} \right) = 0
\]  

(10)

Now, for the solutions of equations (8) and (10) with \( \eta = 0 \) and \( \zeta \neq 0 \) equation (10) becomes

\[
\frac{1-\mu}{n^2r_3} + \frac{\mu}{n^2r_2} - \frac{3\mu B_1}{8n^2r_2^7} + \frac{15\mu B_2z^2}{8n^2r_2^9} - \frac{63\mu B_2z^2}{8n^2r_2^9} = 0
\]  

(11)

Multiplying equation (11) by \( \xi - \xi_1 \) and \( \xi - \xi_2 \) where \( \xi_1 = -\mu \) and \( \xi_2 = 1 - \mu \) we have respectively:

\[
\frac{1-\mu}{n^2r_3} - \frac{\mu}{n^2r_2} - \frac{3\mu B_1}{8n^2r_2^7} + \frac{15\mu B_2z^2}{8n^2r_2^9} - \frac{63\mu B_2z^2}{8n^2r_2^9} = 0
\]  

(12)

\[
\frac{1-\mu}{n^2r_3} - \frac{\mu}{n^2r_2} + \frac{3\mu B_1}{8n^2r_2^7} + \frac{15\mu B_2z^2}{8n^2r_2^9} - \frac{63\mu B_2z^2}{8n^2r_2^9} = 0
\]  

(13)

Subtracting equation (12) from (8) and substituting the value of \( n^2 = \frac{1}{a} \left[ 1 + \frac{3}{2} B_1 - \frac{15}{8} B_2 + \frac{3e^2}{2} \right] \) yields;

\[
\xi = -\mu \left[ 1 + \frac{3B_1}{2} - \frac{15B_2}{8} - \frac{3B_1}{2a^{2/3}} - \frac{15B_2}{8a^{2/3}} - \frac{15B_2z^2}{2a^2} - \frac{63B_2z^2}{2a^2} \right]
\]  

(14)

Now, from (4) we have

\[
\zeta^2 = r_1^2 - (\xi + \mu)^2, \ (i = 1, 2) \quad \xi_1 = -\mu, \ \xi_2 = 1 - \mu, \ \mu = \frac{m_2}{m_1 + m_2}
\]  

(15)

But from [1] we have;

\[
r_1^2 = a^{2/3} \left( 1 - e^2 - B_1 + \frac{5B_2}{4} \right) + B_2 \left( 1 - e^2 - B_1 + \frac{5B_2}{4} + B_1a^{2/3} - \frac{5B_2a^{-2/3}}{4} \right)
\]  

(16)

Considering equations (15) and (16) yields

\[
\zeta^2 = \left[ a^{2/3} \left( 1 - e^2 - B_1 + \frac{5B_2}{4} \right) - \mu^2 (1 + 3B_1 - \frac{15B_2}{4} - 3B_1a^{-2/3} - \frac{15B_2a^{-2/3}}{4} + 15B_1z^2a^{-2/3} - \frac{63B_2z^2a^{-2/3}}{4} \right) \right]
\]  

(17)

Equations (14) and (17) are the positions of the out-of-plane equilibrium points he smaller primary denoted by \( L_{6,7} \).

### 4. Stability of Out-of-Plane Equilibrium Points

To examine the stability of the motion of a body in the vicinity of any of the out-of-plane points, we established the characteristics equation of the system under consideration.

Now, let the location of any of the equilibrium point be denoted by \( (\xi_o, \eta_o, \zeta_o) \) and suppose the small displacement of the location are \( (\sigma, \beta, \alpha) \), then

\[
\xi = \xi_o + \sigma, \ \eta = \eta_o \quad \text{and} \quad \zeta = \zeta_o + \alpha
\]

Taking derivatives, we have

\[
\xi' = \sigma', \xi'' = \sigma'', \eta' = \beta', \eta'' = \beta'' \quad \text{and} \quad \zeta' = \alpha', \zeta'' = \alpha''
\]  

(18)

Given the equations of motion of the infinitesimal mass by [11] as;
\[
\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi}, \eta'' - 2\xi' = \frac{\partial \Omega}{\partial \eta} \text{ and } \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{19}
\]

We obtain the characteristics equation of the system as;

\[
\lambda^6 + (4 - \Omega^0_{\xi \xi} - \Omega^0_{\eta \eta} - \Omega^0_{\zeta \zeta}) \lambda^4 + (\Omega^0_{\xi \eta} \Omega^0_{\eta \eta} + \Omega^0_{\eta \zeta} \Omega^0_{\zeta \zeta} + \Omega^0_{\xi \zeta} \Omega^0_{\zeta \zeta} - 4 \Omega^0_{\xi \xi} - (\Omega^0_{\zeta \zeta})^2) \lambda^2 - 
\left( \Omega^0_{\xi \eta} \Omega^0_{\eta \xi} - (\Omega^0_{\zeta \zeta})^2 \right) \Omega^0_{\eta \eta} = 0 \tag{20}
\]

The superscripts 0 indicates that the partial derivatives are evaluated at the out-of-plane points under consideration. At the points under consideration ignoring products and higher order terms of very small parameters we have;

\[
\Omega^0_{\xi \xi} = (1 - e^2)^{-1/2} \left[ 1 - \frac{3(1 - \mu)}{4a^{7/3}} - \frac{3\mu}{4a^{7/3}} + \frac{21(1 - \mu)B_1}{8a^{7/3}} - \frac{3B_1}{4a^{7/3}} - \frac{3(1 - \mu)e^2}{4a^{7/3}} + \frac{3e^2}{4a^{7/3}} + \frac{105\mu B_2}{32a^2} + \frac{105\mu B_1 Z^2}{32a^2} - \frac{45(1 - \mu)B_2}{32a^{7/3}} - \frac{45\mu B_2}{8a^{7/3}} + \frac{39(1 - \mu)B_1}{8a^{7/3}} - \frac{39\mu B_1}{8a^{7/3}} - \frac{3(1 - \mu)e^2}{4a^{7/3}} + \frac{3e^2}{4a^{7/3}} + \frac{195(1 - \mu)B_2}{32a^2} + \frac{195\mu B_2}{32a^2} \right] \tag{21}
\]

\[
\Omega^0_{\eta \eta} = (1 - e^2)^{-1/2} \left[ \frac{3(1 - \mu)}{2a^{7/3}} - \frac{21(1 - \mu)B_1}{4a^{7/3}} + \frac{15\mu B_1}{16a^{7/3}} + \frac{45(1 - \mu)B_2}{16a^{7/3}} + \frac{45\mu B_2}{16a^{7/3}} - \frac{3(1 - \mu)e^2}{2a^{7/3}} + \frac{3e^2}{2a^{7/3}} - \frac{105\mu B_1 Z^2}{2a^2} + \frac{105\mu B_1 Z^2}{8a^2} + \frac{39\mu B_1}{8a^{7/3}} - \frac{39(1 - \mu)B_1}{8a^{7/3}} + \frac{39\mu B_1}{8a^{7/3}} - \frac{3(1 - \mu)e^2}{4a^{7/3}} + \frac{3e^2}{4a^{7/3}} + \frac{195(1 - \mu)B_2}{32a^2} + \frac{195\mu B_2}{32a^2} \right] \tag{22}
\]

\[
\Omega^0_{\zeta \zeta} = (1 - e^2)^{-1/2} \left[ \frac{3(1 - \mu)}{2a^{7/3}} - \frac{21(1 - \mu)B_1}{4a^{7/3}} + \frac{15\mu B_1}{16a^{7/3}} + \frac{45(1 - \mu)B_2}{16a^{7/3}} + \frac{45\mu B_2}{16a^{7/3}} - \frac{3(1 - \mu)e^2}{2a^{7/3}} + \frac{3e^2}{2a^{7/3}} - \frac{105\mu B_1 Z^2}{2a^2} + \frac{105\mu B_1 Z^2}{8a^2} + \frac{39\mu B_1}{8a^{7/3}} - \frac{39(1 - \mu)B_1}{8a^{7/3}} + \frac{39\mu B_1}{8a^{7/3}} - \frac{3(1 - \mu)e^2}{4a^{7/3}} + \frac{3e^2}{4a^{7/3}} + \frac{195(1 - \mu)B_2}{32a^2} + \frac{195\mu B_2}{32a^2} \right] \tag{23}
\]

\[
\Omega^0_{\xi \eta} = (1 - e^2)^{-1/2} \left[ \frac{3(1 - \mu)}{8a^{7/3}} + \frac{3(1 - \mu)e^2}{8a^{7/3}} + \frac{3e^2}{8a^{7/3}} - \frac{3\mu}{8a^{7/3}} + \frac{21(1 - \mu)B_1}{8a^{7/3}} + \frac{15\mu B_1}{8a^{7/3}} - \frac{45(1 - \mu)B_2}{8a^{7/3}} - \frac{45\mu B_2}{8a^{7/3}} - \frac{3(1 - \mu)e^2}{4a^{7/3}} + \frac{3e^2}{4a^{7/3}} + \frac{195(1 - \mu)B_2}{32a^2} + \frac{195\mu B_2}{32a^2} \right] \tag{24}
\]

Substituting equations (21)-(24) and neglecting higher order terms of very small parameters we have;

\[
\lambda^6 + P \lambda^4 + Q \lambda^4 - R = 0 \tag{25}
\]

Where;

\[
P = \frac{11}{2} + 2\mu + 3\mu a + \left\{ -\frac{9}{4} + \frac{3\mu}{2} \right\} B_1 + \left\{ -\frac{3}{4} + 4\mu \right\} e^2 + \left\{ \frac{75}{16} - \frac{225\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{4} \right\} B_1 Z^2 \tag{26}
\]

\[
Q = \frac{27}{4} - 9\mu + \left\{ -\frac{9}{2} - 6\mu \right\} a + \left\{ -\frac{165}{8} + \frac{276\mu}{8} \right\} B_1 + \left\{ -\frac{15}{2} - 12\mu \right\} e^2 + \left\{ \frac{135}{16} - \frac{15\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{8} \right\} B_1 Z^2 \tag{27}
\]

\[
Q = -\frac{81}{256} + \left\{ -\frac{81}{128} \right\} a + \left\{ \frac{3753}{512} - \frac{453\mu}{512} \right\} B_1 + \left\{ -\frac{513}{512} + \frac{540\mu^3}{256} \right\} e^2 + \left\{ -\frac{4185}{2048} \right\} B_2 \tag{28}
\]

Now, equation (25) becomes;

\[
\lambda^6 + \frac{11}{2} + 2\mu + 3\mu a + \left\{ -\frac{9}{4} + \frac{3\mu}{2} \right\} B_1 + \left\{ -\frac{3}{4} + 4\mu \right\} e^2 + \left\{ \frac{75}{16} - \frac{225\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{4} \right\} B_1 Z^2 \lambda^4 + \frac{27}{4} - 9\mu + \left\{ -\frac{9}{2} - 6\mu \right\} a + \left\{ -\frac{165}{8} + \frac{276\mu}{8} \right\} B_1 + \left\{ -\frac{15}{2} - 12\mu \right\} e^2 + \left\{ \frac{135}{16} - \frac{15\mu}{16} \right\} B_2 + \left\{ \frac{315\mu}{8} \right\} B_1 Z^2 \lambda^2 - \frac{81}{256} + \left\{ -\frac{81}{128} \right\} a + \left\{ \frac{3753}{512} - \frac{453\mu}{512} \right\} B_1 + \left\{ -\frac{513}{512} + \frac{540\mu^3}{256} \right\} e^2 + \left\{ -\frac{4185}{2048} \right\} B_2 = 0 \tag{29}
\]

5. Numerical Applications

Considering (14), (17) and (29), the locations and stability of the out-of-plane equilibrium points of the smaller primary are computed numerically using the software package MATHEMATICA for the systems Xi-Bootis and Sirius. The positions and stability of the out-of-plane equilibrium points considering the smaller primary for varying oblateness are given in Table 2 and Figures 1 & 2 for the system Xi-Bootis and Table 3 and Figures 3 & 4 to show the effects of the oblateness at $J_4$ of the smaller primary, mass ratio, eccentricity of the orbits and the semi-major axis. The effects by
varying the semi-major axis and eccentricity of the orbits on the out-of-plane equilibrium positions are examined in Tables 3, 5, 7 & 9 and Figures 5-10. While computing the stability of the out-of-plane equilibrium points of the smaller primary, we considered $a = 1 - \alpha$ where $\alpha \ll 1$. These computations are shown numerically in Tables 6-9. As evidenced in Tables 6-9, for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane equilibrium points are unstable. This result affirmed with that of [1, 4, 10, 11, 12, 14, 15].

Table 1: Numerical data

| Binary system | Masses | Mass ratio ($\mu$) | Semi-major axis ($a$) | Eccentricity ($e$) |
|---------------|--------|--------------------|----------------------|-------------------|
| Xi-Bootis     | 0.9000 | 0.660              | 0.4231               | 0.7304            | 0.5117            |
| Sirius        | 2.02   | 0.978              | 0.3262               | 2.8582            | 0.5942            |

Table 2: Locations of out-of-plane points for Xi-Bootis system for $Z = 0.01$

| Binary System | Mass ratio ($\mu$) | Semi-Major Axis ($a$) | Eccentricity ($e$) | Oblateness | Locations of the out-of-plane points |
|---------------|--------------------|-----------------------|-------------------|------------|-------------------------------------|
| Xi-Bootis     | 0.4231             | 0.7304                | 0.5117            | 0          | 0.00  0.00  -0.4231  0.647815        |
|               |                    |                       |                   | 0.02       | -0.01 -0.440152  0.631891           |
|               |                    |                       |                   | 0.04       | -0.02 -0.457203  0.615556           |
|               |                    |                       |                   | 0.06       | -0.03 -0.474255  0.598776           |
|               |                    |                       |                   | 0.08       | -0.04 -0.491306  0.581511           |
|               |                    |                       |                   | 0.10       | -0.05 -0.521709  0.616074           |

Table 3: Locations of out-of-plane points for Xi-Bootis system with varying Semi-major axis and Eccentricity for $Z = 0.01$

| Binary System | Mass ratio ($\mu$) | Semi-Major Axis ($a$) | Eccentricity ($e$) | Oblateness | Locations of the out-of-plane points |
|---------------|--------------------|-----------------------|-------------------|------------|-------------------------------------|
| Xi-Bootis     | 0.4231             | 0.005                 | 0.001             | 0.02       | -0.01 -9.81395  2.81632             |
|               |                    |                       |                   | 0.01       | -0.02 -3.98282  1.77906             |
|               |                    |                       |                   | 0.015      | -0.03 -2.43704  1.35634             |
|               |                    |                       |                   | 0.020      | -0.04 -1.7649  1.11763              |
|               |                    |                       |                   | 0.025      | -0.05 -1.40096  0.961747            |
|               |                    |                       |                   | 0.030      | -0.06 -1.17743  0.851251            |

Table 4: Locations of out-of-plane points for Sirius system for $Z = 0.01$

| Binary System | Mass ratio ($\mu$) | Semi-Major Axis ($a$) | Eccentricity ($e$) | Oblateness | Locations of the out-of-plane points |
|---------------|--------------------|-----------------------|-------------------|------------|-------------------------------------|
| Sirius        | 0.3262             | 2.8582                | 0.5942            | 0          | 0.00  0.00  -0.3262  1.09385        |
|               |                    |                       |                   | 0.02       | -0.01 -0.338751  1.06058           |
|               |                    |                       |                   | 0.04       | -0.02 -0.351302  1.02623           |
|               |                    |                       |                   | 0.06       | -0.03 -0.363854  0.990696           |
|               |                    |                       |                   | 0.08       | -0.04 -0.376405  0.953835           |
|               |                    |                       |                   | 0.10       | -0.05 -0.36679  1.01679             |
Table 5: Locations of out-of-plane points for Sirius system with varying Semi-major axis and Eccentricity for $Z = 0.01$

| Binary System | Mass ratio ($\mu$) | Semi-Major Axis (a) | Eccentricity (e) | Oblateness | Locations of the out-of-plane points |
|---------------|-------------------|---------------------|----------------|------------|-------------------------------------|
| Sirius        | 0.3262            | 0.005               | 0.001          | B$_1$ 0.02 | $\xi$ -0.01 7.56632 2.17396 |
|               |                   |                     |                | B$_2$ -0.02 | 0.07066 1.37824 |
|               |                   |                     |                |            | -1.8789 1.05705 |
|               |                   |                     |                |            | -1.3607 0.878285 |
|               |                   |                     |                |            | -1.08011 0.763772 |
|               |                   |                     |                |            | -0.907767 0.684549 |

Table 6: Stability of out-of-plane points for Xi-Bootis system for $\mu = 0.4231, \alpha = 0.2696, e = 0.5117$ and $Z = 0.01$

| Oblateness | Stability of out-of-plane points |
|------------|---------------------------------|
| B$_1$      | B$_2$                           |
| 0          | 0                               |
| 0.02       | -0.01                           |
| 0.04       | -0.02                           |
| 0.06       | -0.03                           |
| 0.08       | -0.04                           |
| 0.10       | -0.05                           |
| $\pm \lambda_{1,2}$ | $\pm \lambda_{3,4}$ | $\pm \lambda_{5,6}$ |
| 0          | 0                               |
| 0.02       | -0.01                           |
| 0.04       | -0.02                           |
| 0.06       | -0.03                           |
| 0.08       | -0.04                           |
| 0.10       | -0.05                           |

Table 7: Stability of out-of-plane points for Xi-Bootis system by varying Eccentricity for $\mu = 0.4231, \alpha = 0.2696, B_1 = 0.02, B_2 = -0.01$ and $Z = 0.01$

| Eccentricity (e) | Stability of out-of-plane points |
|------------------|---------------------------------|
| $\pm \lambda_{1,2}$ | $\pm \lambda_{3,4}$ | $\pm \lambda_{5,6}$ |
| 0.005            | 0 ± 0.446773i                  |
| 0.010            | 0 ± 0.446818i                  |
| 0.015            | 0 ± 0.446895i                  |
| 0.020            | 0 ± 0.447001i                  |
| 0.025            | 0 ± 0.447139i                  |
| 0.030            | 0 ± 0.447307i                  |

Table 8: Stability of out-of-plane points for Sirius system for $\mu = 0.3262, \alpha = -1.8582, e = 0.5942$ and $Z = 0.01$

| Oblateness | Stability of out-of-plane points |
|------------|---------------------------------|
| B$_1$      | B$_2$                           |
| 0          | 0                               |
| 0.02       | -0.01                           |
| 0.04       | -0.02                           |
| 0.06       | -0.03                           |
| 0.08       | -0.04                           |
| 0.10       | -0.05                           |
| $\pm \lambda_{1,2}$ | $\pm \lambda_{3,4}$ | $\pm \lambda_{5,6}$ |
| 0          | 0 ± 0.545066                    |
| 0.02       | -0.01                           |
| 0.04       | -0.02                           |
| 0.06       | -0.03                           |
| 0.08       | -0.04                           |
| 0.10       | -0.05                           |
**Table 9:** Stability of out-of-plane points for **Sirius** system by varying Eccentricity for $\mu = 0.3262, \alpha = -1.8582, B_1 = 0.02, B_2 = -0.01$ and $Z = 0.01$

| Eccentricity ($e$) | $\pm \lambda_{1,2}$ | $\pm \lambda_{3,4}$ | $\pm \lambda_{5,6}$ |
|-------------------|----------------------|----------------------|----------------------|
| 0.005             | $\pm 0.763191$       | $0 \pm 0.607147i$    | $0 \pm 2.12406i$    |
| 0.010             | $\pm 0.763159$       | $0 \pm 0.607152i$    | $0 \pm 2.12406i$    |
| 0.015             | $\pm 0.763105$       | $0 \pm 0.60716i$     | $0 \pm 2.12406i$    |
| 0.020             | $\pm 0.763029$       | $0 \pm 0.607172i$    | $0 \pm 2.12405i$    |
| 0.025             | $\pm 0.762931$       | $0 \pm 0.607187i$    | $0 \pm 2.12404i$    |
| 0.030             | $\pm 0.762812$       | $0 \pm 0.607205i$    | $0 \pm 2.12403i$    |

**Figure 1:** Effects of oblateness on $L_6$ ($\xi$) for **Xi Bootis** system with $\mu = 0.4231$, $a = 0.7304$, $e = 0.5117$, $Z = 0.01$

**Figure 2:** Effects of oblateness on $L_7$ ($\zeta$) for **Xi Bootis** system with $\mu = 0.4231$, $a = 0.7304$, $e = 0.5117$, $Z = 0.01$
Figure 3: Effects of oblateness on $L_6(\xi)$ for Sirius system with $\mu = 0.3262$, $a = 2.8582$, $e = 0.5942$, $Z = 0.01$.

Figure 4: Effects of oblateness on $L_7(\zeta)$ for Sirius system with $\mu = 0.3262$, $a = 2.8582$, $e = 0.5942$, $Z = 0.01$.

Figure 5: Effects of oblateness on $L_{6,7}(U)$ for Xi Bootis system for varying Semi-major axis ($a$) and Eccentricity ($e$).
Figure 6: Effects of oblateness on $L_{6,7}$ ($U$) for Sirius system for varying Semi-major axis ($a$) and Eccentricity ($e$)

Figure 7: Effects of oblateness on $L_{6}$ ($\xi$) for Xi-Bootis system for varying Semi-major axis ($a$) and Eccentricity ($e$)

Figure 8: Effects of oblateness on $L_{7}$ ($\zeta$) for Xi-Bootis system for varying Semi-major axis ($a$) and Eccentricity ($e$)
6 Discussion and Conclusion

We have investigated the motion of the out-of-plane equilibrium points in the field of stellar binary systems: Xi-Booitis and Sirius around their common centre of mass within the frame work of the Elliptic Restricted Three-Body Problem (ER3BP) at $J_4$. The motion is described by Equations 1-4. The location of the out-of-plane equilibrium points ($L_6, L_7$) are observed by Equations 14 & 17. Our results coincides with that of [10] in the absence of radiation pressures and $J_4$ oblateness. Neglecting $J_4$ oblateness, Eccentricity of the orbits, semi-major axis and $P - R$ Drag, our work agrees with [4, 14, 15].

For the aforementioned binary systems, we have perturbed the motion of the infinitesimal mass of the out-of-plane equilibrium points in the framework of the Elliptic Restricted Three-Body Problem (ER3BP) at $J_4$ to show the effects of oblateness up to zonal harmonic $J_4$, semi-major axis and eccentricity of the orbits. These effects are shown in Table 2 and Figures 1 & 2 for the system Xi-Booitis and Table 3 and Figures 3 & 4 for the system Sirius on the positions $L_6 (\xi)$ and $L_7 (\zeta)$. The positions are greatly affected by the aforementioned parameters. This is evidenced in the change of positions of the out-of-plane equilibrium points by varying oblateness $J_4$. For assume values of semi-major axis and eccentricity for some constants oblateness, the effects of the parameters on the out-of-plane equilibrium points are shown in Tables 4 & 5 and Figures 5-10 for Xi-Booitis and Sirius systems.

It is observed in Tables 6-9 that for each set of values, there exist at least one complex root with positive real part and hence in Lyapunov sense, the stability of the out-of-plane equilibrium points are unstable.
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