Two-dimensional steady-state temperature distribution of a thin circular plate due to uniform internal energy generation

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Abstract: This article is concerned with the determination of temperature, displacement, and thermal stresses in a thin circular plate due to uniform internal energy generation within it. The fixed circular edge \( (r = a) \) is kept at zero temperature and the upper \((z = h)\) and lower \((z = 0)\) surfaces are thermally insulated. The governing heat conduction equation has been solved using finite Hankel transform technique. The results are obtained in a series form in terms of Bessel's functions. The results for temperature, displacement, and stresses have been computed numerically and illustrated graphically.

Keywords: thermal stresses; thin circular plate; steady state; internal energy generation

AMS Subject classifications: 35B07; 35G30; 35K05; 44A10

1. Introduction

During the second half of the twentieth century, non-isothermal problems of the theory of elasticity became increasingly important. This is due mainly to their many applications in diverse fields. First, the high velocity of modern aircrafts give rise to an aerodynamic heating, which produce intense thermal stresses, reducing the strength of aircrafts structure. Secondly, in the nuclear field, the extremely high temperature and temperature gradients originating inside nuclear reactor influence their design and operations (Nowinski, 1978).

ABOUT THE AUTHOR

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PUBLIC INTEREST STATEMENT

Studying the two-dimensional steady-state temperature distribution of a thin circular plate due to uniform internal energy generation. The results obtained here are more useful in engineering problems, particularly in aerospace engineering for stations of a missile body not influenced by nose tapering. The missile skill material is assumed to have physical properties independent of temperature, so that the temperature \( T(r, z) \) is a function of radius and thickness only.
Nowacki (1957) determined the steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper surface with zero temperature on the lower surface and with the circular edge thermally insulated. The problem of the thermal deflection of an axisymmetrically heated circular plate in the case of fixed and simply supported edges have been considered by Boley and Weiner (1960). Roy Choudhary (1972) studied quasi-static thermal stresses in a thin circular plate due to transient temperature applied along the circumference of a circle on the upper face with the lower face at zero temperature and a fixed circular edge thermally insulated.

Furukawa, Noda, and Ashida (1991) studied a problem of finding the temperature and thermal stresses in an infinitely long cylinder within a generalized thermoelasticity with one relaxation time. Ishihara, Tanigawa, Kawamura, and Noda (1997) studied a heat conduction problem to determine the temperature distribution and discuss the thermoelastic deformation in thin circular plates subjected to partially distributed and axisymmetric heat supply on the outer curved surface. Gaikwad and Ghadle (2010) solved the quasi-static thermal stresses of an infinitely long circular cylinder having constant initial temperature under steady-state field. Parihar and Patil (2011) determined the transient heat conduction problem and analyzed thermal stresses in thin circular plate. Gaikwad and Ghadle (2012) solved non-homogeneous heat conduction problem and obtained thermal deflection due to internal heat generation in a thin hollow circular disk. Recently, Gaikwad (2013) analyzed thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply.

This article deals with the determination of temperature, displacement, and thermal stresses in a thin circular plate due to uniform internal energy generation within it. The fixed circular edge \( (r = a) \) is kept at zero temperature and the upper \((z = h)\) and lower \((z = 0)\) surfaces are thermally insulated, while the plate is also subjected to uniform internal energy generation \(g_0\) \((W/m^3)\). Under these realistic prescribed conditions, temperature, displacement, and thermal stresses in a thin circular plate due to uniform internal heat generation are required to be determined.

The mathematical formulation of this problem, using the general cylindrical heat equation is given as

\[
\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g_0}{k} = 0 \quad \text{in} \quad 0 \leq r \leq a, \quad 0 \leq z \leq h
\]  

(1)

with the boundary conditions,

\[
T = 0 \quad \text{at} \quad r = a
\]  

(2)

\[
\frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = 0
\]  

(3)

\[
\frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = h
\]  

(4)
where \( k \) is the thermal conductivity of the material of the circular plate.

Following Roy Choudhary (1972), we assume that a circular plate of small thickness \( h \) is in a plane state of stress. In fact “the smaller the thickness of the circular plate compared to its diameter, the nearer to a plane state of stress is the actual state”. Then the displacements equations of thermoelasticity have the form

\[
U_{ij,i} + \left( \frac{1 + \nu}{1 - \nu} \right) e_{j} = 2 \left( \frac{1 + \nu}{1 - \nu} \right) a_{i} T_{j} \\
\mathbf{e} = U_{k,i} ; \quad k, i = 1, 2,
\]

where \( U_{i} \) is the displacements component, \( e \) is the dilatation, \( T \) is the temperature, and \( \nu \) and \( a_{i} \) are, respectively, the Poisson ratio and linear coefficients of thermal expansions of the circular plate material.

Introducing \( U_{i} = \phi_{j,i} \quad i = 1, 2 \), we have

\[
\nabla^{2}_{r} \phi = (1 + \nu)a_{i} T, \\
\nabla^{2}_{z} \phi = \frac{\partial^{2} \phi}{\partial r^{2}} + \frac{1}{r} \frac{\partial \phi}{\partial r},
\]

\[
\sigma_{ij} = 2 \mu (\phi_{ij} - \delta_{ij} \phi_{kk}) \quad i, j, k = 1, 2,
\]

where \( \mu \) is Lamé constant and \( \delta_{ij} \) is the well-known Kronecker symbol.

In the axisymmetric case,

\[
\phi = \phi(r, z) \quad T = (r, z)
\]

and the differential equation governing the displacements potential function \( \phi(r, z) \) is given by

\[
\frac{\partial^{2} \phi}{\partial r^{2}} + \frac{1}{r} \frac{\partial \phi}{\partial r} = (1 + \nu)a_{i} T
\]  \( (5) \)

The stress components \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) of the circular plate are given by,
\[ \sigma_{rr} = -\frac{2\mu}{r} \frac{\partial \phi}{\partial r} \]  \hspace{1cm} (6)

\[ \sigma_{\theta \theta} = -2\mu \frac{\partial^2 \phi}{\partial r^2} \]  \hspace{1cm} (7)

with \( \phi = 0 \) at \( r = a \).

Also, in the planar state of stress within the circular plate,

\[ \sigma_{rz} = \sigma_{zr} = \sigma_{rz} = 0 \]  \hspace{1cm} (8)

Equations (1–8) constitute the mathematical formulation of the problem under consideration.

3. Solution of the heat conduction equation

We first reduce Poisson's equation (1) to the Laplace equation by defining a new dependent variable as described next:

A new dependent variable \( \theta(r, z) \) is defined as

\[ T(r, z) = \theta(r, z) + P(r, z) \]  \hspace{1cm} (9)

where the \( P(r, z) \) function in the cylindrical co-ordinate system is

\[ P(r, z) = -\frac{g_0 z^2}{k} \]  \hspace{1cm} (10)

Then Equation (9) becomes

\[ T(r, z) = \theta(r, z) - \frac{g_0 z^2}{k} \]  \hspace{1cm} (11)

Substituting Equation (11) into Equations (1–4), one obtains the Laplace equation with one inhomogeneous boundary condition,

\[ \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + \frac{\partial^2 \theta}{\partial z^2} = 0 \]  \hspace{1cm} (12)

with the boundary conditions,

\[ \theta = 0 \quad \text{at} \quad r = a \]  \hspace{1cm} (13)

\[ \frac{\partial \theta}{\partial z} = 0 \quad \text{at} \quad z = 0 \]  \hspace{1cm} (14)

\[ \frac{\partial \theta}{\partial z} = \frac{g_0 z}{2k} \quad \text{at} \quad z = h \]  \hspace{1cm} (15)

To obtain the expression of the function \( \theta(r, z) \), we develop the finite Hankel transform and its inverse transform over the variable \( r \) in the range \( 0 \leq r \leq a \) defined in Sneddon (1972) as

\[ H[\theta(r, z)] = \tilde{\theta}(a_m, z) = \int_{r=0}^{a} r' K_0(a_m, r') \theta(r', z) dr' \]  \hspace{1cm} (16)

\[ H^{-1}[\tilde{\theta}(a_m, z)] = \theta(r, z) = \sum_{m=1}^{\infty} K_0(a_m, r) \tilde{\theta}(a_m, z) \]  \hspace{1cm} (17)

where

\[ K_0(a_m, r) = \sqrt{\frac{2}{\pi}} \frac{I_0(a_m r)}{I_0(a_m a)} \]

and \( a_1, a_2, a_3, \ldots \) are the positive root of transcendental equation
\[ I_0(\alpha_m a) = 0 \quad (18) \]

This transform satisfies the relations,

\[
H \left[ \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right] = -a_m^2 \overline{\theta}(\alpha_m, z)
\]

\[
H \left[ \frac{\partial^2 \theta}{\partial z^2} \right] = \frac{d^2 \overline{\theta}}{dz^2}
\]

Applying the finite Hankel transform defined in Equation (16) to (12) and using the conditions (13–15), one obtains

\[
\frac{d^2 \overline{\theta}}{dz^2} - a_m^2 \overline{\theta} = 0 \quad (19)
\]

with

\[
\frac{\partial \overline{\theta}}{\partial z} = \frac{\sqrt{2} g_h h}{a} \frac{1}{2k} \frac{1}{I'_0(\alpha_m a)} \int_{r'=0}^a r' I_0(\alpha_m r') dr' \quad \text{at } z = h \quad (20)
\]

Solution of the differential Equation (19) is obtained as

\[
\overline{\theta}(\alpha_m, z) = \frac{\sqrt{2} g_h h}{a} \frac{1}{2k} \frac{1}{I'_0(\alpha_m a)} \frac{\cos(\alpha_m z)}{\alpha_m \sin(\alpha_m h)} \int_{r'=0}^a r' I_0(\alpha_m r') dr' \quad (21)
\]

On applying the inverse Hankel transform defined in Equation (17), one obtains

\[
\theta(r, z) = \frac{g_h h}{a^2 k} \sum_{m=1}^{\infty} I_0(\alpha_m r) \frac{\cos(\alpha_m z)}{\alpha_m \sin(\alpha_m h)} \int_{r'=0}^a r' I_0(\alpha_m r') dr' \quad (22)
\]

Substituting Equation (22) into Equation (9), one obtains the expression of the temperature distribution function as

\[
T(r, z) = \frac{g_h h}{a^2 k} \sum_{m=1}^{\infty} I_0(\alpha_m r) \frac{\cos(\alpha_m z)}{\alpha_m \sin(\alpha_m h)} \int_{r'=0}^a r' I_0(\alpha_m r') dr' - \frac{g_0 z^2}{4k} \quad (23)
\]

4. Displacement potential and thermal stresses

Using Equation (23) in (5), one obtains

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = (1 + \nu) \alpha \left[ \frac{g_h h}{a^2 k} \sum_{m=1}^{\infty} I_0(\alpha_m r) \frac{\cos(\alpha_m z)}{\alpha_m \sin(\alpha_m h)} \int_{r'=0}^a r' I_0(\alpha_m r') dr' - \frac{g_0 z^2}{4k} \right] \quad (24)
\]

Now suitable form of \( \phi \) satisfying (24) is given by

\[
\phi = (1 + \nu) \alpha \left[ \frac{g_h h}{a^2 k} \sum_{m=1}^{\infty} \frac{1}{\alpha_m^2 I_0^2(\alpha_m a)} \frac{\cos(\alpha_m z)}{\alpha_m \sin(\alpha_m h)} \int_{r'=0}^a r' I_0(\alpha_m r') dr' - \frac{g_0 z^2 r^2}{16k} \right] \quad (25)
\]

Using Equation (25) in Equations (6) and (7), one obtains the expressions of thermal stresses as
5. Numerical results and discussion

5.1. Dimensions

The constants associated with the numerical calculation are taken as

Radius of a circular plate $a = 1 \text{ m}$,

Thickness of a circular plate $h = 0.1 \text{ m}$.

5.2. Material properties

The numerical calculation has been carried out for a aluminum (pure) circular plate with the material properties as,

- Thermal diffusivity $\alpha = 84.18 \text{ m}^2/\text{s}$
- Thermal conductivity $k = 204 \text{ W/mK}$
- Density $\rho = 2707 \text{ kg/m}^3$
- Specific heat capacity $c_p = 896 \text{ J/kg K}$
- Poisson ratio $\nu = 0.35$
- Lamé constant $\mu = 26.67 \text{ GPa}$
- Coefficients of linear thermal expansion $a_t = 22.2 \times 10^{-6}$
- Young’s modulus elasticity of the material of the plate $E = 70 \text{ GPa}$.

The rate of internal energy generation is $g_0 = 1 \times 10^6 \text{ W/m}^3$.

5.3. Roots of the transcendental equation

The first five positive root of the transcendental equation $I_0(\alpha_m \alpha) = 0$ as defined in Ozisik (1968) are $\alpha_1 = 1.0025$, $\alpha_2 = 1.3262$, $\alpha_3 = 2.4463$, $\alpha_4 = 5.2945$, $\alpha_5 = 7.3663$.

For convenience, we set

$A = (1 + \nu) a_t$, $B = -2(1 + \nu)\mu a_t$.

Also noticed that

$$
\frac{\partial}{\partial r} (I_0(\alpha_m r)) = a_m I_1(\alpha_m r)
$$

$$
\frac{\partial^2}{\partial r^2} (I_0(\alpha_m r)) = -a_m^2 \left[ \frac{I_1(\alpha_m r)}{\alpha_m r} - I_0(\alpha_m r) \right].
$$

The numerical calculation has been carried out with the help of computational software MATLAB-2007 and the graphs are plotted with the help of MATLAB tools.

From Figure 2, it is observed that, near the centerline ($r \sim 0$), the temperature is increasing primarily by the internal energy generation and decreasing toward the outer surface in radial direction.
From Figure 3, it is observed that, due to the internal energy generation at a constant rate $g_0$ of a thin circular plate, the displacement function $\psi/A$ is maximum at the center and decreasing toward the outer surface in radial direction. It is proportional to the temperature.

From Figure 4, it can be observed that, due to internal heat generation at a constant rate $g_0$, the radial stress decreases from center $r = 0$ of a thin circular plate to the outer circular boundary $r = 1$ in radial direction.

From Figure 5, it is seen that, the angular stress function $\sigma_{\theta\theta}$ is maximum at center of a thin circular plate and decreasing toward the outer surface in radial direction.
6. Concluding remarks

In this paper, we analyzed the steady-state thermoelastic problem and determined the expressions of temperature, displacement, and thermal stresses of a thin circular plate due to uniform internal energy generation at a constant rate $g_0 = 1 \times 10^6 \text{W/m}^3$. The present method is based on the direct method, using the finite Hankel transform and using their inversion. As a special case, a mathematical model is constructed for aluminum (pure) a thin circular plate, with the material properties specified as above and examined the thermoelastic behaviors in the steady-state field for temperature change, displacement, thermal stresses in radial direction.

We conclude that, the displacement and the stress components occur near heat region. Due to the uniform internal energy generation of a thin circular plate at a constant rate $g_0 = 1 \times 10^6 \text{W/m}^3$, the radial stress and axial stress develops the tensile stresses in radial direction. Also, it can be observed from the figure, temperature and displacement, the direction of heat flow and direction of body displacement are the same and they are proportionate. In the plane state of stress, the stress...
components $\sigma_{xx}$, $\sigma_{zz}$, $\sigma_{xz}$ are zero. Also any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (23–27).

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