GENERALISATIONS OF CAPPARELLI’S AND PRIMC’S IDENTITIES, II: PERFECT $A_{n-1}^{(1)}$ CRYSTALS AND EXPLICIT CHARACTER FORMULAS

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Abstract. In the first paper of this series, we gave infinite families of coloured partition identities which generalise Primc’s and Capparelli’s classical identities.

In this second paper, we study the representation theoretic consequences of our combinatorial results. First, we show that the difference conditions we defined in our $n^2$-coloured generalisation of Primc’s identity, which have a very simple expression, are actually the energy function with values in $\{0,1,2\}$ for the perfect crystal of the tensor product of the vector representation and its dual in $A_{n-1}^{(1)}$.

Then we introduce a new type of partitions, grounded partitions, which allows us to retrieve connections between character formulas and partition generating functions without having to perform a specialisation.

Finally, using the formulas for the generating functions of our generalised partitions, we recover the Kac-Peterson character formula for the characters of all the irreducible highest weight $A_{n-1}^{(1)}$-modules of level 1, and give a new character formula as a sum of infinite products with obviously positive coefficients in the generators $e^{-\alpha_i} \ (i \in \{0,\ldots,n-1\})$, where the $\alpha_i$’s are the simple roots.

1. Introduction and statement of results

1.1. Background. A partition $\lambda$ of a positive integer $n$ is a non-increasing sequence of natural numbers $(\lambda_1, \ldots, \lambda_s)$ whose sum is $n$. The numbers $\lambda_1, \ldots, \lambda_s$ are called the parts of $\lambda$, and $|\lambda| = n$ is the weight of $\lambda$. For example, the partitions of 4 are $4, 3+1, 2+2, 2+1+1,$ and $1+1+1+1$.

The Rogers-Ramanujan identities [KK91] state that for $a = 0$ or $1$, the number of partitions of $n$ such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $1-a$ times is equal to the number of partitions of $n$ into parts congruent to $\pm(1+a)$ mod 5. In the 1980’s, Lepowsky and Wilson [LW84, LW85] gave an interpretation and proof of these identities in terms of characters for level 3 standard modules of the affine Lie algebra $A_1^{(1)}$ by using vertex operators. Since then, a very fruitful interaction between partition identities and representation theory has been developed, see for example [Cap93, MP87, MP99, MP01, Nan14, Pri94, PŠ16, Sil17]. More detail on the history of this field can be found in the first paper of this series [DK19].

In the present paper, we focus on the interaction between partition identities and crystal base theory. Crystal bases were introduced independently by Kashiwara [Kas90] and Lusztig [Lus90] to study representations of quantum algebras, which are $q$-deformations of universal enveloping algebras of classical Lie algebras. They have a nice combinatorial structure, and admit a particularly simple tensor product.

One of the most important questions in representation theory is finding nice explicit formulas for characters of representations. If $g$ is an affine Lie algebra, and $V$ an irreducible module of $g$ with highest weight $\Lambda$, then by definition, the character $\text{ch}(V)$ of $V$ multiplied by $e^{-\Lambda}$ can be expressed as a power series in $e^{-\alpha_0}, \ldots, e^{-\alpha_{n-1}}$ with positive coefficients, where $\alpha_0, \ldots, \alpha_{n-1}$ are the simple roots of $g$. However, finding explicit expressions for characters is not easy. The most famous example, the Weyl-Kac character formula [Kac90], gives a beautiful factorized expression for the character, but the coefficients of the monomials in $e^{-\alpha_i}$ in this expression are not obviously positive.

Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [KKM+92a, KKM+92b] introduced the theory of perfect crystals to find such nice expressions for characters via the so-called $(KMN)^2$ crystal base character formula. It allows one to construct crystals of irreducible highest weight modules for all classical weights of the same level. Then the crystal base character formula allows one to identify these perfect crystals with partitions satisfying certain difference conditions, which in certain cases gives rise to character formulas as partition generating functions. However, these formulas are in general obtained after doing a specialisation, for example replacing all the $e^{-\alpha_i}$’s by $q$ (which is the principal specialisation). In this paper,
we will prove a non-specialised character formula, with obviously positive coefficients, for all the irreducible 
highest weight $A_{n-1}^{(1)}$-modules of level 1.

But first, let us present our starting point, Primc’s partition identity (again, more detail can be found in 
our first paper [DK19]). In [Pri99], Primc used the (KMN)$^2$ crystal base character formula to study level 1 standard modules of $A_1^{(1)}$ and $A_2^{(1)}$. He computed an energy function for the perfect crystal of the tensor 
product of the vector representation and its dual in $A_1^{(1)}$ and $A_2^{(1)}$, and through the crystal base character 
formula, he gave the principal specialisation of the character formula in terms of partitions with difference 
conditions.

In the $A_1^{(1)}$ case, the energy matrix of the perfect crystal associated to the tensor product of the vector 
representation and its dual is the following:

\[
P_2 = \begin{pmatrix}
    a_1 b_0 & a_0 b_0 & a_1 b_1 & a_0 b_1 \\
    a_1 b_0 & 1 & 0 & 2 \\
    a_0 b_0 & 0 & 1 & 0 \\
    a_0 b_1 & 0 & 1 & 0
\end{pmatrix},
\]

(1.1)

and in $A_2^{(1)}$, the energy matrix is given by:

\[
P_3 = \begin{pmatrix}
    a_2 b_0 & a_2 b_1 & a_1 b_0 & a_0 b_0 & a_2 b_2 & a_1 b_1 & a_0 b_1 & a_1 b_2 & a_0 b_2 \\
    a_2 b_0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
    a_2 b_1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\
    a_1 b_0 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 \\
    a_0 b_0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
    a_0 b_1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \\
    a_1 b_1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
    a_0 b_1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 2 \\
    a_1 b_2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 2 \\
    a_0 b_2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 2
\end{pmatrix}.
\]

(1.2)

Consider coloured partitions satisfying the difference conditions of (1.1) (resp. (1.2)), where the coefficient 
$(i, j)$ in the matrix gives the minimal difference between consecutive parts coloured $i$ and $j$. Using the Weyl-
Kac character formula [Kac90], Primc proved that in both cases, when performing the principal specialisation 
(corresponding to some dilations on the variables in the generating function), the generating function for such partitions reduces to $\frac{1}{(q;q)_\infty}$, which is simply the generating function for partitions. Here we used, for 
$n \in \mathbb{N} \cup \{\infty\}$, the standard $q$-series notation

\[(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}).\]

In the first paper of this series [DK19], we gave a large family of coloured partition identities which 
generalise and refine Primc’s identities. To do so, we gave difference conditions which generalise both (1.1) 
and (1.2). Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be two sequences of colour symbols. For all $i, k, i', k' \in \mathbb{N}$, we defined the 
minimal difference $\Delta$ in the following way:

\[
\Delta(a_i b_k, a_i' b_{k'}) = \chi(i \geq i') - \chi(i = k = i'') + \chi(k \leq k') - \chi(k = i' = k'),
\]

(1.3)

where $\chi(prop)$ equals 1 if the proposition $prop$ is true and 0 otherwise.

Restricting $\Delta$ to colours $a_i b_j$ for $i, j \in \{0, 1\}$ gives (1.1), and restricting it to colours $a_i b_j$ for $i, j \in \{0, 1, 2\}$ 
gives (1.2).

Our general theorem in [DK19] gives the generating function for partitions $\lambda_1 + \cdots + \lambda_s$ into parts coloured 
$a_i b_j$ for all $i, j \in \{0, \ldots, n-1\}$, satisfying the difference conditions

\[\lambda_j - \lambda_{j+1} \geq \Delta(c(\lambda_j), c(\lambda_{j+1})),\]

where for all $j$, $c(\lambda_j)$ denotes the colour of the part $\lambda_j$. Let $P_n(m; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1})$ be the number of 
such $n^2$-coloured partitions of $m$ satisfying the difference conditions $\Delta$, such that for $i \in \{0, \ldots, n-1\}$,
the symbol $a_i$ (resp. $b_i$) appears $u_i$ (resp. $v_i$) times in the colour sequence. Defining the generating function
\[
G_n^P(q; b_0, \ldots, b_{n-1}) := \sum_{m, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1} \geq 0} P_n(n; u_0, \ldots, u_{n-1}; v_0, \ldots, v_{n-1}) q^m b_0^{u_0} \cdots b_{n-1}^{u_{n-1}} v_0^{v_0} \cdots v_{n-1}^{v_{n-1}},
\]
we showed the following.

**Theorem 1.1.** \cite{DK19} Theorem 1.27 Let $n$ be a positive integer. We have:
\[
G_n^P(q; b_0, \ldots, b_{n-1}) = [x^0] \prod_{i=0}^{n-1} (-b_i^{-1} x q)_{\infty} (-b_i x^{-1}; q)_{\infty}
\]
\[
= \frac{1}{(q; q)^n} \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}} \prod_{i=1}^{n-1} b_0^{s_i} b_i^{s_i+s_{i+1}} q^{s_i(s_i-s_{i+1})}
\]
\[
= \frac{1}{(q; q)^n} \sum_{i=1}^{n-1} \prod_{i=1}^{n-1} b_i^{-s_i-r_i+1} q^{r_i(r_i-r_i+1)}
\]
\[
\times \left( - \prod_{i=0}^{n-1} b_i b_i^{-1} q^{\frac{i(i+1)}{2} \sum_{j=1}^{n} r_j - r_i r_i+1; q} \right)_{\infty}
\]
\[
\times \left( - \prod_{i=0}^{n-1} b_i b_i^{-1} q^{\frac{i(i+1)}{2} \sum_{j=1}^{n} r_j + r_i r_i+1; q} \right)_{\infty}
\]

We can obtain a product formula for our generating function by doing the following dilations, which correspond to the principal specialisation that Primc considered in his paper:
\[
\begin{cases}
q & \mapsto q^n \\
b_i & \mapsto q^i \text{ for all } i \in \{0, \ldots, n-1\}.
\end{cases}
\]

**Corollary 1.2.** \cite{DK19} Corollary 1.26 By doing the transformations described in \cite{DK19}, we obtain the generating function for classical integer partitions:
\[
G_n^P(q^n; 1, \ldots, q^{n-1}) = [x^0] \prod_{i=0}^{n-1} (-q^{n-i} x; q^n)_{\infty} (-q^i x^{-1}; q^n)_{\infty}
\]
\[
= [x^0] (-qx; q)_{\infty} (-x^{-1}; q)_{\infty}
\]
\[
= \frac{1}{(q; q)_{\infty}}.
\]

The cases $n = 2$ and $n = 3$ in the corollary above recover Primc’s original results.

In \cite{DK19}, we also gave two generalisations of Capparelli’s identity \cite{Cap93}, another partition identity which arose from representation theory, via the theory of vertex operators. Let us also state these generalisations, as they give a different (but related) expression for the character formula.

For $i, k, i', k' \in \mathbb{N}$, define the minimal difference $\delta(a_ib_k, a_{i'}b_{k'})$ between a part coloured $a_ib_k$ and a part coloured $a_{i'}b_{k'}$ in the following way:
\[
\delta(a_ib_k, a_{i'}b_{k'}) = 1 \text{ for all } k \in \mathbb{N}^*,
\]
\[
\delta(a_ib_k, a_{i'}b_{k'}) = 1 \text{ for all } \ell < k,
\]
\[
\delta(a_{i'}b_{k'}, a_ib_k) = 1 \text{ for all } \ell < k,
\]
\[
\delta(a_{i'}b_{k'}, a_{i'}b_{k'}) = \Delta(a_{i'}b_{k'}, a_{i'}b_{k'}) \text{ in all the other cases.}
\]
Similarly, for $i, k, i', k' \in \mathbb{N}$, define the minimal difference $\delta'(a_kb_k, a_i b_i)$ between a part coloured $a_kb_k$ and a part coloured $a_i b_{i'}$ in the following way:

\[
\begin{align*}
\delta'(a_kb_k, a_kb_{k}) &= 1 \text{ for all } k \in \mathbb{N}^*, \\
\delta'(a_kb_k, a_kb_{k-1}) &= 1 \text{ for all } \ell \geq k \geq 1, \\
\delta'(a_{k-1}b_{\ell}, a_kb_k) &= 1 \text{ for all } \ell \geq k \geq 1, \\
\delta'(a_{i'}b_{i'}, a_i b_i) &= \Delta(a_i b_{i'}, a_i b_i) \text{ in all the other cases.}
\end{align*}
\]

Let $C_n(m; u_0, \ldots , u_{n-1}; v_0, \ldots , v_{n-1})$ (resp. $C_n'(m; u_0, \ldots , u_{n-1}; v_0, \ldots , v_{n-1})$) be the number of such $(n^2 - 1)$-coloured partitions of $m$, where the colour $a_0 b_0$ is not allowed, satisfying the difference conditions $\delta$ (resp. $\delta'$), such that for $i \in \{0, \ldots , n - 1\}$, the symbol $a_i$ (resp. $b_i$) appears $u_i$ (resp. $v_i$) times in the colour sequence. We define the generating functions

\[
\begin{align*}
G_n^C(q; b_0, \cdots , b_{n-1}) &:= \sum_{m,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1}\geq 0} C_n(m; u_0, \ldots , u_{n-1}; v_0, \ldots , v_{n-1}) q^{m} t_0^{u_0-u_0} \cdots t_{n-1}^{u_{n-1}-u_{n-1}}, \\
G_n^{C'}(q; b_0, \cdots , b_{n-1}) &:= \sum_{m,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1}\geq 0} C_n'(m; u_0, \ldots , u_{n-1}; v_0, \ldots , v_{n-1}) q^{m} t_0^{u_0-u_0} \cdots t_{n-1}^{u_{n-1}-u_{n-1}}.
\end{align*}
\]

We showed the following relation between our generalisations of Capparelli’s and Primc’s partitions with difference conditions.

**Theorem 1.3.** [DK19] Theorem 1.28 For every positive integer $n$, we have

\[
G_n^C(q; b_0, \cdots , b_{n-1}) = G_n^{C'}(q; b_0, \cdots , b_{n-1}) = (q;q)_\infty G_n^P(q; b_0, \cdots , b_{n-1}).
\]

Through Theorem 1.3, the generating function for the two types of generalised Capparelli partitions can also be written as a sum of infinite products.

In this paper, we use our results above to give new character formulas.

### 1.2. Statement of Results.

We will define all the necessary notions from crystal base theory in Section 2. For now, let us define a few notations which will allow us to state our main theorems.

Let $n$ be a positive integer, and consider the Cartan datum for the generalised Cartan matrix of affine type $A^{(1)}_{n-1}$. We denote by $\tilde{P} = \mathbb{Z} \Lambda_0 \oplus \cdots \oplus \mathbb{Z} \Lambda_{n-1}$ the lattice of the classical weights, where the elements $\Lambda_\ell (\ell \in \{0, \ldots , n - 1\})$ are the fundamental weights. The set of all the level 1 classical weights is given by $\tilde{P}_1^\vee = \{ \Lambda_\ell : \ell \in \{0, \ldots , n - 1\} \}$. The null root is denoted by $\delta$, and the simple roots by $\alpha_i, i \in \{0, \ldots , n - 1\}$.

Let $B = \{ v_i : i \in \{0, \ldots , n - 1\} \}$ be the crystal of the vector representation of $A^{(1)}_{n-1}$ and let $B^\vee = \{ v_i^\vee : i \in \{0, \ldots , n - 1\} \}$ be its dual. For all $v_i \in B$, we denote by $\overline{w} v_i \in \tilde{P}$ the classical weight of $v_i$. We finally set $\mathbb{B}$ to be the tensor product $B \otimes B^\vee$.

Given that [1.1] and [1.2] are energy matrices for perfect crystals coming from the tensor product of the vector representation and its dual in $A^{(1)}_1$ and $A^{(1)}_2$, respectively, it is natural to wonder whether our generalised difference conditions $\Delta$ define in 1.3) are also energy functions for certain perfect crystals. We answer this question in the affirmative by showing the following.

**Theorem 1.4.** Let $n$ be a positive integer, and let $B$ denote the crystal of the vector representation of $A^{(1)}_{n-1}$. The crystal $\mathbb{B} = B \otimes B^\vee$ is a perfect crystal of level 1. Furthermore, the energy function on $\mathbb{B} \otimes \mathbb{B}$ such that $H((v_0 \otimes v_0^\vee) \otimes (v_0 \otimes v_0^\vee)) = 0$ satisfies for all $k, \ell, k', \ell' \in \{0, \ldots , n - 1\}$,

\[
H((v_\ell \otimes v_\ell^\vee) \otimes (v_{k} \otimes v_k^\vee)) = \Delta(a_kb_k; a_{k'} b_{k'}),
\]

where $\Delta$ is the minimal difference for Primc generalised partitions defined in 1.3).

Prime showed Theorem 1.4 in the cases $n = 2$ and $n = 3$. The theorem is still true when $n = 1$, in which case the crystal $B$ has a single vertex and a loop 0, and the corresponding partitions are simply the classical partitions.

In [BFKL06], Benkart, Frenkel, Kang, and Lee gave another formulation of the energy function of certain level 1 perfect crystals of classical types, including the $A^{(1)}_{n-1}$-crystal studied in Theorem 1.4. However, they did not give a closed expression valid for all $k, \ell, k', \ell' \in \{0, \ldots , n - 1\}$ as we did in Theorems 1.4 and 1.3.
They used the fact that, when removing the 0-arrows from the crystal graph on Figure 4.4 (see more details on crystals and energy functions in Sections 2 and 3), the energy function \( H \) is constant on each connected component, and gave a table with the value of \( H \) for a representative of each connected component. The value of \( H \) for the other vertices can then be obtained by determining to which connected component they belong. Both their and our energy functions satisfy \( H((v_0 \otimes v_0) \otimes (v_0 \otimes v_0)) = 0 \), so they must be the same, even though their expressions differ. In this sense, Theorem 1.4 gives a simpler, more explicit and unified formula for the \( A_{n-1}^{(1)} \) energy function in \([BFKL06]\). 

Our proof of Theorem 1.4 in Sections 6 and 7 relies on explicitly building paths in the crystal graph. We only treat the case \( n \geq 3 \), as \( n = 1 \) and \( n = 2 \) give crystals with a slightly different shape, and we already know that the theorem is true in these cases. 

Theorem 1.4 gives a simple explicit expression for the energy function. Using the (KMN)\(^2 \) crystal base character formula of \([KKM+92]\), it allows us to relate the generating function \( G_n^P(q; b_0, \ldots, b_{n-1}) \) of generalised Prime partitions and the generating functions \( G_n^C(q; b_0, \ldots, b_{n-1}) \) and \( G_n^{C'}(q; b_0, \ldots, b_{n-1}) \) of the two types of generalised Capparelli partitions with the character of the irreducible highest weight module \( L(\Lambda_0) \). 

Unlike previous connections between character formulas and partition generating functions, where a specific specialisation (often the principal specialisation) was needed, here we give a non-dilated character formula.

**Theorem 1.5.** Let \( n \) be a positive integer, and let \( \Lambda_0, \ldots, \Lambda_{n-1} \) be the fundamental weights of \( A_{n-1}^{(1)} \). By setting \( e^{\Lambda_i} = b_i \) and \( e^{-\delta} = q \), we have the following identities:

\[
G_n^P(q; b_0, \ldots, b_{n-1}) = \frac{e^{-\Lambda_0} \operatorname{ch}(L(\Lambda_0))}{(q; q}_\infty),
\]

\[
G_n^C(q; b_0, \ldots, b_{n-1}) = G_n^{C'}(q; b_0, \ldots, b_{n-1}) = e^{-\Lambda_0} \operatorname{ch}(L(\Lambda_0)).
\]

This result gives an evaluation of the character of the irreducible highest weight module for the particular weight \( \Lambda_0 \), but we can extend our techniques to retrieve the characters for the other level 1 weights of \( \bar{P}_1^+ \).

**Theorem 1.6.** Let \( n \) be a positive integer, and let \( \Lambda_0, \ldots, \Lambda_{n-1} \) be the fundamental weights of \( A_{n-1}^{(1)} \). By setting \( e^{\Lambda_i} = b_i \) and \( e^{-\delta} = q \), we have the following identities for any \( \ell \in \{0, \ldots, n-1\} \):

\[
G_n^P(q; b_0q, \ldots, b_{\ell-1}q, b_{\ell}, \ldots, b_{n-1}) = \frac{e^{-\Lambda_\ell} \operatorname{ch}(L(\Lambda_\ell))}{(q; q}_\infty),
\]

\[
G_n^C(q; b_0q, \ldots, b_{\ell-1}q, b_{\ell}, \ldots, b_{n-1}) = G_n^{C'}(q; b_0q, \ldots, b_{\ell-1}q, b_{\ell}, \ldots, b_{n-1}) = e^{-\Lambda_\ell} \operatorname{ch}(L(\Lambda_\ell)).
\]

The case \( \ell = 0 \) of Theorem 1.6 gives Theorem 1.5.

As mentioned earlier, finding an expression of the character as a series with positive coefficients is an important problem. In \([KPS4]\), using modular forms and string functions, Kac and Peterson gave a formula for \( e^{-\Lambda} \operatorname{ch}(L(\Lambda)) \) for all the irreducible highest weight level 1 modules \( \Lambda \) of all classical types as a series in \( \mathbb{Z}[e^{-\alpha_0}, e^{-\alpha_1}, \ldots, e^{-\alpha_n-1}] \) with obviously positive coefficients. This built on earlier work of Kac \([Kac78]\), in which he proved the particular case where \( M = L(\Lambda_0) \) in \( A_2^{(1)} \), \( D_n^{(1)} \), and \( E_6^{(1)} \).

In \([BW15]\), Bartlett and Warnaar used Hall-Littlewood polynomials to give explicitly positive formulas for the characters of certain highest weight modules of the affine Lie algebras \( C_1^{(1)} \), \( A_{2n}^{(2)} \), and \( D_{n+1}^{(2)} \), which also led to generalisations for the Macdonald identities in types \( B_n^{(1)} \), \( C_n^{(1)} \), \( A_{2n-1}^{(2)} \), \( A_{2n}^{(2)} \), and \( D_{n+1}^{(2)} \). However their approach failed to give a formula for the case \( A_3^{(1)} \). Using Macdonald-Koornwinder theory, Rains and Warnaar \([RW]\) later found additional character formulas for these types, together with new Rogers-Ramanujan type identities.

In \([GOW16]\), Griffin, Ono, and Warnaar obtained a limiting Rogers-Ramanujan type identity for the principal specialisation of the character of some particular weights \( (m-k)\Lambda_0 + k\Lambda_1 \) in \( A_{n}^{(1)} \). On the other hand, Meurman and Primc \([MP99]\) treated the case of all levels of \( A_3^{(1)} \) via vertex operator algebras. 

Here, using our non-dilated character formula from Theorem 1.4 we recover, for all \( \ell \in \{0, \ldots, n-1\} \), the character formula of Kac-Peterson. Moreover, we give a new explicit expression for \( e^{-\Lambda_\ell} \operatorname{ch}(L(\Lambda_\ell)) \) as a
sum of \((n-1)!\) series with positive coefficients \(\mathbb{Z}[e^{-\alpha_0}, e^{-\alpha_1}, \cdots, e^{-\alpha_{n-1}}]\), each of which are infinite product generating functions for certain types of partitions.

**Theorem 1.7.** Let \(n\) be a positive integer, and let \(\Lambda_0, \ldots, \Lambda_{n-1}\) be the fundamental weights of \(A_{n-1}^{(1)}\). For all \(\ell \in \{0, \ldots, n-1\}\), we have

\[
e^{-\Lambda_\ell \text{ch}(L(\Lambda_\ell))} = \frac{1}{(e^{-\delta}; e^{-\delta})_\infty^{n-1}} \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}} e^{-s_\ell \delta} \prod_{i=1}^{n-1} e^{s_i \alpha_i} e^{s_i(s_{i+1} - s_i) \delta}
\]

(1.10)

\[
e^{-\Lambda_\ell \text{ch}(L(\Lambda_\ell))} = \left(\prod_{i=1}^{n-1} \frac{e^{-i(i+1)\delta}; e^{-i(i+1)\delta}}{(e^{-\delta}; e^{-\delta})_\infty}\right) \sum_{r_1, \ldots, r_{n-1}; \atop 0 \leq r_j \leq 1} e^{-r_\ell \delta} \prod_{i=1}^{n-1} e^{r_i \alpha_i} e^{r_i(r_{i+1} - r_i) \delta}
\]

(1.11)

where \(\delta = \alpha_0 + \cdots + \alpha_{n-1}\) is the null root.

The character formula (1.10) is, up to a change of variables, a reformulation of the Kac-Peterson character formula for the type \(A_{n-1}^{(1)}\) given in [KP84, p.217]. Thus, our partition identity Theorem 1.1 combined with theorem 1.6 makes the connection between the KMN crystal base character formula and the Kac-Peterson character formula.

The principal specialisation [Kac90, Chapter 10] for the affine type \(A_{n-1}^{(1)}\) consists in transforming the generators with

\[e^{-\alpha_i} \mapsto q \text{ for all } i \in \{0, \ldots, n-1\}.
\]

In that case, we have a natural transformation \(b_i := q^\ell b_0\) and a dilated version of the character formula can be deduced from Theorems 1.1 and 1.6.

**Corollary 1.8.** Let \(n\) be a positive integer, and let \(\Lambda_0, \ldots, \Lambda_{n-1}\) be the fundamental weights of \(A_{n-1}^{(1)}\). For all \(\ell \in \{0, \ldots, n-1\}\), the principal specialisation of \(e^{-\Lambda_\ell \text{ch}(L(\Lambda_\ell))}\), denoted by \(F_1(e^{-\Lambda_\ell \text{ch}(L(\Lambda_\ell))})\), is the generating function of the classical integer partitions with no parts divisible by \(n\):

\[
F_1(e^{-\Lambda_\ell \text{ch}(L(\Lambda_\ell))}) = (q^n; q^n) \times G_n \left( (q^n; q^n, q^{n+\ell-1}b_0, q^\ell, \ldots, q^{n-1}b_0) \right)
\]

\[
= (q^n; q^n) \times [x^0] \left( \prod_{i=0}^{\ell-1} (-q^{-i}b_0^{-1} x; q^n)_\infty (\sum_{j=1}^{n-1} q^j b_0^{-1} x; q^n)_\infty \right)
\]

\[
= (q^n; q^n) \times [x^0] (-q^{-\ell}b_0^{-1} x; q\infty) (q^\ell b_0^{-1} x; q\infty)
\]

In this particular case, we recover the principal specialisation of the Weyl-Kac character formula [Kac90].

The remainder of this paper is organised as follows. In Section 2 we recall the necessary definitions and theorems about representation theory and crystal bases. In Section 3 we define grounded partitions, which will play a key role in obtaining a non-specialised character formula. In Section 4 we define the \(A_{n-1}^{(1)}\) crystals related to our difference condition/energy function \(\Delta\). In Section 5 we prove our character formulas (Theorems 1.3, 1.4, and 1.7) assuming that \(\Delta\) is an energy function for our crystal. Finally, in Sections 6 and 7 we prove that this is indeed the case, by constructing some paths on the crystal graph.
2. Basics on Crystals

In this section, we recall the definitions and basic theorems from crystal base theory which are necessary for our purpose. We refer to the book [HK02], which we consider to be a good summary of the basic theory of Kac-Moody algebras [HK02 Chapter 2], quantum groups [HK02 Chapter 3] and crystal bases [HK02 Chapters 4, 10]. For a more combinatorial approach and more emphasis on the finite dimensional case, we refer the reader to [BS17].

Throughout this section, \( n \) is a fixed positive integer.

2.1. Cartan datum and quantum affine algebras. A square matrix \( A = (a_{i,j})_{i,j \in \{0, \ldots, n-1\}} \) is said to be a generalised Cartan matrix if \( A \) has the following properties:

- for all \( i \in \{0, \ldots, n-1\}, a_{i,i} = 2 \),
- for all \( i \neq j \in \{0, \ldots, n-1\}, a_{i,j} \in \mathbb{Z}_{\leq 0} \),
- \( a_{i,j} = 0 \) if and only if \( a_{j,i} = 0 \).

Moreover, if there exists a diagonal matrix \( D \) with positive integer coefficients such that \( DA \) is symmetric, then \( A \) is said to be symmetrisable. In addition, if the rank of the matrix \( A \) is \( n-1 \), then \( A \) is said to be of affine type. In this paper, we always assume that this is the case.

Let us consider such a matrix \( A \). Let \( P^\vee \) be a free abelian group of rank \( n+1 \) with \( \mathbb{Z} \)-basis \( \{h_0, \ldots, h_{n-1}, d\} \):

\[
P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_{n-1} \oplus \mathbb{Z}d.
\]

We will denote by \( h_i \) the restriction of \( h \) to \( P \). We call \( P^\vee \) the dual weight lattice. The complexification \( h = \mathbb{C} \otimes \mathbb{Z} P^\vee \) is called the Cartan subalgebra. The linear functionals \( \alpha_i \) and \( \Lambda_i \) \( (i \in \{0, \ldots, n-1\}) \) on \( h \) given by

\[
\langle h_j, \alpha_i \rangle := \alpha_i(h_j) = \delta_{i,j} = \delta_{i,0} \quad \langle d, \alpha_i \rangle := \alpha_i(d) = \delta_{i,0} \quad (i,j \in \{0, \ldots, n-1\})
\]

are respectively the simple roots and fundamental weights. Let \( h^* \) be the dual space of \( h \). We denote by \( \Pi = \{\alpha_i \mid i \in \{0, \ldots, n-1\}\} \subseteq h^* \) the set of simple roots, and define \( \Pi^\vee = \{h_i \mid i \in \{0, \ldots, n-1\}\} \subseteq h \) to be the set of simple coroots. We also set

\[
P = \{\lambda \in h^* \mid \lambda(P^\vee) \subseteq \mathbb{Z}\}
\]

to be the weight lattice. It contains the set of dominant integral weights

\[
P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in \{0, \ldots, n-1\}\}.
\]

The quintuple \((A, \Pi, \Pi^\vee, P, P^\vee)\) is said to be a Cartan datum for the Cartan matrix \( A \). The Kac-Moody affine Lie algebra \( \hat{g} \) attached to this datum is the Lie algebra with generators \( e_i, f_i \ (i \in \{0, \ldots, n-1\}) \) and \( h \in P^\vee \), with the following defining relations ([HK02 Definition 2.1.3]):

\begin{enumerate}
  \item \([h, h'] = 0 \text{ for all } h, h' \in P^\vee,\]
  \item \([e_i, f_j] = \delta_{ij} h_j,\]
  \item \([h, e_i] = \alpha_i(h) e_i \text{ for all } h \in P^\vee,\]
  \item \([h, f_i] = -\alpha_i(h) f_i \text{ for all } h \in P^\vee,\]
  \item \((\text{ad} e_i)^{1-a_{i,j}} e_j = (\text{ad} f_i)^{1-a_{i,j}} f_j = 0 \text{ for } i \neq j,\]
\end{enumerate}

where \( \text{ad} x : y \mapsto [x, y] \).

We also define the coroot lattice

\[
\hat{P}^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_{n-1},
\]

and its complexification \( \hat{h} = \mathbb{C} \otimes \mathbb{Z} \hat{P}^\vee \). The restriction of the \( \mathbb{Z} \)-submodule \( \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \cdots \oplus \mathbb{Z} \Lambda_{n-1} \)

of \( P \) to \( \hat{P}^\vee \) is called the lattice of classical weights and is denoted by \( \hat{P} \).

Remark. By (2.1), for all \( j \neq 0 \), we have

\[
\alpha_j = \sum_{i=0}^{n-1} a_{i,j} \Lambda_i \in \hat{P}.
\]

We will denote by \( \overline{\alpha}_0 \) the restriction of \( \alpha_0 \) to \( \hat{P} \).
Let $\tilde{P}^+ := \sum_{i=0}^{n} \mathbb{Z}_{\geq 0}\Lambda_i$ denote the corresponding set of dominant weights.

The center

$$\mathbb{Z}c = \{ h \in P^\vee : \langle h, \alpha_i \rangle = 0 \text{ for all } i \in \{0, \ldots, n-1\} \}$$

of the affine Lie algebra $\hat{g}$ is one-dimensional and generated by the canonical central element $c$, where $c = c_0h_0 + \cdots + c_{n-1}h_{n-1}$.

The space of imaginary roots

$$\mathbb{Z}\delta = \{ \lambda \in P : \langle h_i, \lambda \rangle = 0 \text{ for all } i \in \{0, \ldots, n-1\} \}$$

of $\hat{g}$ is also one-dimensional, generated by the null root $\delta$, where

$$\delta = d_0\alpha_0 + d_1\alpha_1 + \cdots + d_{n-1}\alpha_{n-1},$$

and the vector $t(d_0, d_1, \ldots, d_{n-1}) \in \mathbb{C}^n$ spans the kernel of the Cartan matrix $A$. The level $\ell$ of a dominant weight $\lambda \in P^+$ is given by the expression $\langle c, \lambda \rangle := \lambda(c) = \ell$.

For any $k \in \mathbb{Z}$ and an indeterminate $q$, let us set

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

We also set $[0]_q = 1$ and for $k \geq 1$, $[k]_q! = [k]_q[k-1]_q \cdots [1]_q$. For $m \geq k \geq 0$, define

$$\langle \frac{m}{k} \rangle_q = \frac{[m]_q!}{[k]_q! [m-k]_q!}.$$

We now have all the definitions necessary to introduce quantum affine Lie algebras.

**Definition 2.1.** [HK02, Definition 3.1.1] The quantum affine algebra $U_q(\hat{g})$ associated with the Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$ is the associative algebra with unit element over $\mathbb{C}(q)$ (where $q$ is an indeterminate) with generators $e_i, f_i (i \in \{0, \ldots, n-1\})$ and $q^h$ ($h \in P^\vee$), satisfying the defining relations:

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ for $h \in P^\vee, i \in \{0, \ldots, n-1\}$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ for $h \in P^\vee, i \in \{0, \ldots, n-1\}$,
4. $e_i f_j - f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in \{0, \ldots, n-1\}$,
5. $\sum_{k=0}^{1-a_{i,j}} \langle 1 - a_{i,j} \rangle_k e_i^{1-a_{i,j}-k} e_j^k e_j^k = 0$ for $i \neq j$,
6. $\sum_{k=0}^{1-a_{i,j}} \langle 1 - a_{i,j} \rangle_k f_i^{1-a_{i,j}-k} f_j^k f_j^k = 0$ for $i \neq j$.

Here $q_i = q^{s_i}$ and $K_i = q^{a_i h_i}$, where $D = \text{diag}(s_i : i \in \{0, \ldots, n-1\})$ is a symmetrising matrix of $A$.

**Definition 2.2.** The quantum affine algebra $U'_q(\hat{g})$ is the subalgebra of $U_q(\hat{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ ($i \in \{0, \ldots, n-1\}$).

Contrary to $U_q(\hat{g})$, the quantum affine algebra $U'_q(\hat{g})$ admits some non-trivial finite-dimensional irreducible modules.

2.2. **Integrable modules, highest weight modules and character formula.** We are now ready to define irreducible highest weight modules and characters.

**Definition 2.3.** Let $g$ be a Lie algebra with bracket $[\cdot, \cdot]$, and let $V$ be a vector space. Then $V$ is a $g$-module if there is a bilinear map $g \times V \to V$, denoted by $(x, v) \mapsto x \cdot v$, satisfying for all $x, y \in g$ and all $v \in V$:

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v).$$

A subspace $W$ of a $g$-module $V$ is called a submodule of $V$ if for all $x \in g$, $x \cdot W \subseteq W$. A $g$-module $V$ is said to be irreducible if its only submodules are $V$ and 0.

The notion of modules extends naturally from an affine Lie algebra $\hat{g}$ to its quantum affine algebra $U_q(\hat{g})$. 

Definition 2.4. A $U_q(\mathfrak{g})$-module $M$ is said to be integrable if it satisfies the following properties:

(a) $M$ has a weight space decomposition: $M = \bigoplus_{\lambda \in P} M_\lambda$, where $M_\lambda = \{ v \in M \mid q^h \cdot v = q^\lambda\langle h, v \rangle v \text{ for all } h \in P^v \}$;

(b) there are finitely many $\lambda_1, \ldots, \lambda_k \in P$ such that $\text{wt}(M) \subseteq \Omega(\lambda_1) \cup \cdots \cup \Omega(\lambda_k)$, where $\text{wt}(M) = \{ \lambda \in P \mid M_\lambda \neq 0 \}$ and $\Omega(\lambda_j) = \{ \mu \in P \mid \lambda_j + \sum_{i=0}^{n-1} \mathbb{Z}_{\leq 0} \alpha_i \}$;

(c) the elements $e_i$ and $f_i$ act locally nilpotently on $M$ for all $i \in \{ 0, \ldots, n-1 \}$.

We denote by $O^g_{\text{int}}$ the category of integrable $U_q(\mathfrak{g})$-modules.

For all $\lambda \in P$, a module of highest weight $\lambda$ is an integrable module such that:

(a) $\text{wt}(M) \subseteq \Omega(\lambda)$;

(b) $\text{dim} M_\lambda = 1$;

(b) $M = U_q(\mathfrak{g}) M_\lambda$.

For all $\lambda \in P$, up to isomorphism, there exists a unique highest weight module which is irreducible. We denote by $L(\lambda)$ the irreducible highest weight $U_q(\mathfrak{g})$-module of highest weight $\lambda$.

Definition 2.5. Let $M$ be an integrable module such that $\text{dim} M_\lambda < \infty$ for all $\lambda \in \text{wt}(M)$. The character of $M$ is defined by

\[ \chi(M) = \sum_{\lambda \in \text{wt}(M)} \text{dim} M_\lambda \cdot e^\lambda, \tag{2.2} \]

where the $e^\lambda$'s are formal basis elements of the group algebra $\mathbb{C}[h^*]$, with the multiplication defined by $e^\lambda e^\mu = e^{\lambda + \mu}$.

When $M$ is a highest weight module of highest weight $\lambda$, its character satisfies

\[ e^{-\lambda} \chi(M) = \sum_{\mu \in \text{wt}(M)} \text{dim} M_\mu \cdot e^{\mu - \lambda} \in \mathbb{Z}_{\geq 0}[[e^{-\alpha_i}, i \in \{0, \ldots, n-1\}]]. \]

All these definitions on modules also hold in the case of the $\mathfrak{g}$-modules $M'$, where the weight spaces are given by $M'_\lambda = \{ v \in M' \mid h \cdot v = \lambda \langle h, v \rangle v \text{ for all } h \in P^v \}$. Thus, looking at the generators of the weight spaces, for a fixed weight $\lambda \in P$, the irreducible highest weight $\mathfrak{g}$-module can be identified with the irreducible highest weight $U_q(\mathfrak{g})$-module, and we have equality of characters.

2.3. Crystal bases. The crystal base theory was developed independently by Kashiwara [Kas90] and Lusztig [Lus90] to study the category $O^q_{\text{int}}$ of integrable $U_q(\mathfrak{g})$-modules. If $M$ is a module in the category $O^q_{\text{int}}$, then for each $i \in \{0, \ldots, n-1\}$, a weight vector $u \in M_\lambda$ can be written uniquely in the form $u = \sum_{k=0}^{N} f_i^{(k)} u_k$, for some $N \geq 0$ and $u_k \in M_{\lambda + k\alpha_i} \cap \ker e_i$ for all $k = 0, 1, \ldots, N$, with $f_i^{(k)} = f_i^k / ([k]_q !)$. The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$, for $i \in \{0, \ldots, n-1\}$, are then defined as follows:

\[ \tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k. \tag{2.3} \]

Crystal bases will be seen as bases at $q = 0$. To do so, let us define the localisation of $\mathbb{C}[q]$ at $q = 0$ by $\mathbb{C}_0 = \{ f = g/h \mid g, h \in \mathbb{C}[q], \ h(0) \neq 0 \}$.

Definition 2.6. [HK02 Definition 4.2.2] Assume that $M$ is a $U_q(\mathfrak{g})$-module in the category $O^q_{\text{int}}$. A free $\mathbb{C}_0$-submodule $L$ of $M$ is a crystal lattice if

(i) $L$ generates $M$ as a vector space over $\mathbb{C}(q)$;

(ii) $L = \bigoplus_{\lambda \in P} L_\lambda$ where $L_\lambda = M_\lambda \cap \mathbb{C}(q)$;

(iii) $\tilde{e}_i L \subseteq L$ and $\tilde{f}_i L \subseteq L$, for all $i \in \{0, \ldots, n-1\}$.

Since the operators $\tilde{e}_i$ and $\tilde{f}_i$ preserve the lattice $L$, they also define operators on the quotient $L/qL$.

Definition 2.7. [HK02 Definition 4.2.3] A crystal base for a $U_q(\mathfrak{g})$-module $M \in O^q_{\text{int}}$ is a pair $(L, B)$ such that

(1) $L$ is a crystal lattice of $M$;

(2) $\mathbb{C}$ is a $\mathbb{C}$-basis of $L/qL \cong \mathbb{C} \otimes_{\mathbb{C}_0} L$;
(3) $\mathcal{B} = \bigcup_{\lambda \in P} \mathcal{B}_\lambda$, where $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathcal{L}/q\mathcal{L})$;

(4) $\tilde{e}_i \mathcal{B}_\lambda \subset \mathcal{B}_{\lambda - \alpha_i} \cup \{0\}$ and $\tilde{f}_i \mathcal{B}_\lambda \subset \mathcal{B}_{\lambda + \alpha_i} \cup \{0\}$ for all $i \in \{0, \ldots, n-1\}$;

(5) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$, for $b, b' \in \mathcal{B}$ and $i \in \{0, \ldots, n-1\}$.

To each module $M \in \mathcal{O}^\gamma_{\text{int}}$, one can associate a corresponding crystal base $(\mathcal{L}, \mathcal{B})$, which is unique up to isomorphism [HK02, Chapter 5]. Therefore, from now on, we will refer to “the” crystal base of $M$.

Furthermore, the crystal graph associated to $(\mathcal{L}, \mathcal{B})$ can be defined as follows. The set of vertices is $\mathcal{B}$, and the oriented edges are built as follows:

$$b \xrightarrow{\tilde{e}_i} b'$$ if and only if $\tilde{f}_i b = b'$ (or equivalently $\tilde{e}_i b' = b$).

**Remark 2.8.** When $\tilde{f}_i b = 0$ (resp. $\tilde{e}_i b = 0$), then there is no edge labelled $i$ coming out of $b$ (resp. arriving in $b$).

The crystal graph can be viewed as a combinatorial data of the module $M$.

For $i \in \{0, \ldots, n-1\}$, let us define functions $\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z}$ as follows:

$$\varepsilon_i(b) = \max \{k \geq 0 \mid \tilde{e}_i^k b \in \mathcal{B}\},$$

$$\varphi_i(b) = \max \{k \geq 0 \mid \tilde{f}_i^k b \in \mathcal{B}\}.$$

In other words, $\varepsilon_i(b)$ is the length of the longest chain of $i$-arrows ending at $b$ in the crystal graph, and $\varphi_i(b)$ is the length of the longest chain of $i$-arrows starting from $b$. Furthermore, we have $\varphi_i(b) - \varepsilon_i(b) = \lambda(h_i)$ for all $b \in \mathcal{B}_\lambda$. Thus, by setting $\text{wt} b = \lambda$,

$$\varepsilon(b) = \sum_{i=0}^{n-1} \varepsilon_i(b) \Lambda_i, \quad \text{and} \quad \varphi(b) = \sum_{i=0}^{n-1} \varphi_i(b) \Lambda_i,$$

we then have $\text{wt} b = \varphi(b) = \varepsilon(b)$ for all $b \in \mathcal{B}_\lambda$, where $\text{wt} b$ is the projection of $\text{wt} b$ on $\mathfrak{P}$. Also, by the definition of the weight vectors $u_k$ in the Kashiwara operators [2.3], we have for all $b \in \mathcal{B}$ such that $\tilde{e}_i b \neq 0$,

$$\text{wt} \tilde{e}_i b - \text{wt} b = \alpha_i.$$

Let us now introduce the notion of crystal.

**Definition 2.9.** [HK02, Definition 4.5.1] Let $A = (a_{i,j})_{0 \leq i,j \leq n-1}$ be a Cartan matrix with associated Cartan datum $(A, \Pi, \Pi'^\vee, P, P'^\vee)$. A crystal associated with $(A, \Pi, \Pi'^\vee, P, P'^\vee)$ is a set $\mathcal{B}$ together with maps

$$\text{wt} : \mathcal{B} \to P,$$

$$\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \cup \{0\} \quad (i \in \{0, \ldots, n-1\}),$$

$$\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \cup \{-\infty\} \quad (i \in \{0, \ldots, n-1\}),$$

satisfying the following properties for all $i \in \{0, \ldots, n-1\}$:

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$,
2. $\text{wt}(\tilde{e}_i b) = \text{wt} b + \alpha_i$ if $\tilde{e}_i b \in \mathcal{B}$,
3. $\text{wt}(\tilde{f}_i b) = \text{wt} b - \alpha_i$ if $\tilde{f}_i b \in \mathcal{B}$,
4. $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$ if $\tilde{e}_i b \in \mathcal{B}$,
5. $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $\tilde{e}_i b \in \mathcal{B}$,
6. $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ if $\tilde{f}_i b \in \mathcal{B}$,
7. $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $\tilde{f}_i b \in \mathcal{B}$,
8. $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in \mathcal{B}$,
9. if $\varphi_i(b) = -\infty$ for $b \in \mathcal{B}$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

In particular, if $(\mathcal{L}, \mathcal{B})$ is a crystal base, then $\mathcal{B}$ is a crystal.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two crystals. A crystal morphism between $\mathcal{B}_1$ and $\mathcal{B}_2$ is a map $\Psi : \mathcal{B}_1 \cup \{0\} \to \mathcal{B}_2 \cup \{0\}$ such that

- $\Psi(0) = 0$;
- $\Psi$ commutes with $\text{wt}, \varepsilon_i, \varphi_i$ for all $i \in \{0, \ldots, n-1\}$;
- for $b, b' \in \mathcal{B}_1$ such that $\tilde{f}_i b = b'$ and $\Psi(b), \Psi(b') \in \mathcal{B}_2$, we have $\tilde{f}_i \Psi(b) = \Psi(b')$, $\tilde{e}_i \Psi(b') = \Psi(b)$.
A morphism $\Psi$ is said to be strict if it commutes with $\tilde{e}_i$, $\tilde{f}_i$ for all $i \in \{0, \ldots n - 1\}$.

The theory of crystal bases behaves very nicely with respect to the tensor product of $\mathcal{O}_\text{int}^q$-modules, as can be seen in the next theorem.

**Theorem 2.10.** [HK02, Theorem 4.4.1] Let $M_1, M_2 \in \mathcal{O}_\text{int}^q$ and let $(\mathcal{L}_1, B_1), (\mathcal{L}_2, B_2)$ be the corresponding crystal bases. We set $\mathcal{L} = L_1 \otimes_{\mathcal{O}_\text{int}^q} L_2$ and $B = B_1 \otimes B_2 \cong B_1 \times B_2$. Then $(\mathcal{L}, B)$ is a crystal base of $M_1 \otimes_{\mathcal{O}_\text{int}^q} M_2$, with

\[
\begin{align*}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \epsilon_i(b_2), \\
 b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \epsilon_i(b_2),
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\check{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \epsilon_i(b_2), \\
 b_1 \otimes \check{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \epsilon_i(b_2),
\end{cases}
\end{align*}
\]  

(2.6)

where $b_1 \otimes 0 = 0 \otimes b_2 = 0$ for all $b_1 \in B_1$ and $b_2 \in B_2$. Furthermore, we have

\[
\begin{align*}
\text{wt}(b_1 \otimes b_2) &= \text{wt} b_1 + \text{wt} b_2, \\
\epsilon_i(b_1 \otimes b_2) &= \max\{\epsilon_i(b_1), \varphi_i(b_1) - \epsilon_i(b_2)\}, \\
\varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_2), \varphi_i(b_1) - \epsilon_i(b_2)\}.
\end{align*}
\]

The last but not the least tool we need in this paper is the notion of energy function, defined as follows.

**Definition 2.11.** [HK02, Definition 10.2.1] Let $M \in \mathcal{O}_\text{int}^q$ be a module, and $(\mathcal{L}, B)$ be the corresponding crystal base. An energy function on $B \otimes B$ is a map $H : B \otimes B \to \mathbb{Z}$ satisfying

\[
H(\tilde{e}_i(b_1 \otimes b_2)) = \begin{cases} 
H(b_1 \otimes b_2) & \text{if } i \neq 0, \\
H(b_1 \otimes b_2) + 1 & \text{if } i = 0 \text{ and } \varphi_0(b_1) \geq \varphi_0(b_2), \\
H(b_1 \otimes b_2) - 1 & \text{if } i = 0 \text{ and } \varphi_0(b_1) < \varphi_0(b_2),
\end{cases}
\]  

(2.7)

for all $i \in \{0, \ldots n - 1\}$ and $b_1, b_2$ with $\tilde{e}(b_1 \otimes b_2) \neq 0$.

By definition, in the crystal graph of $B \otimes B$, the value of $H(b_1 \otimes b_2)$, when it exists, determines all the values $H(b'_1 \otimes b'_2)$ for vertices $b'_1 \otimes b'_2$ in the same connected component as $b_1 \otimes b_2$. Note that the conditions (2.7) are equivalent to the following:

\[
\begin{align*}
H(\tilde{e}_i(b_1 \otimes b_2)) &= \begin{cases} 
H(b_1 \otimes b_2) + \chi(i = 0) & \text{if } \varphi_i(b_1) \geq \epsilon_i(b_2), \\
H(b_1 \otimes b_2) - \chi(i = 0) & \text{if } \varphi_i(b_1) < \epsilon_i(b_2),
\end{cases} \\
H(\check{f}_i(b_1 \otimes b_2)) &= \begin{cases} 
H(b_1 \otimes b_2) + \chi(i = 0) & \text{if } \varphi_i(b_1) \geq \epsilon_i(b_2), \\
H(b_1 \otimes b_2) - \chi(i = 0) & \text{if } \varphi_i(b_1) < \epsilon_i(b_2),
\end{cases}
\]  

(2.8)

Figure 2.1 gives the crystal graph $B$ of the vector representation of $A^{(1)}_1$ [HK02, 10.5.2], the tensor product $B \otimes B$, and an energy function $H$ on $B \otimes B$.

**Figure 2.1.**

\[
\text{B : } \begin{array}{ccc}
0 & 1 \\
0 & \rightarrow & 1
\end{array}
\]

\[
\text{B} \otimes \text{B : } \begin{array}{ccc}
0 & 0 & 1 \\
0 & \rightarrow & 1 & \otimes & 1
\end{array}
\]

$H : 1 \otimes 1 \rightarrow \chi(i \geq j)$
2.4. Perfect crystals. The theory of perfect crystals was developed by Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki [KKM+92a, KKM+92b] to study the irreducible highest weight modules over quantum affine algebras. Indeed, perfect crystals provide a construction of the crystal base $B(\lambda)$ of any irreducible $U_q(\widehat{\mathfrak{g}})$-module $L(\lambda)$ corresponding to a classical weight $\lambda \in \hat{P}^+$. We call affine crystal an abstract crystal associated with an affine Cartan datum $(A, \Pi, \Pi^\vee, \hat{P}, \hat{P}^\vee)$ (quantum algebra $U_q(\widehat{\mathfrak{g}})$), while the term classical crystal is used for an abstract crystal associated to the classical Cartan datum $(A, \Pi, \Pi^\vee, \hat{P}, \hat{P}^\vee)$ (quantum algebra $U_q(\mathfrak{g})$ defined in Definition 2.12).

All the theorems in this section are due to Kang, Kashiwara, Misra, Miwa, Nakashima, and Nakayashiki, but we give references to the book [HK02] for reader’s convenience.

Let us start by defining perfect crystals.

Definition 2.12. [HK02, Definition 10.5.1] For a positive integer $\ell$, a finite classical crystal $B$ is said to be a perfect crystal of level $\ell$ for the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ if

1. there is a finite-dimensional $U_q(\widehat{\mathfrak{g}})$-module with a crystal base whose crystal graph is isomorphic to $B$ (when the 0-arrows are removed);
2. $B \otimes B$ is connected;
3. there exists a classical weight $\lambda_0$ such that
   \[ \text{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i \quad \text{and} \quad |B_{\lambda_0}| = 1; \]
4. for any $b \in B$, we have
   \[ \langle c, \varepsilon(b) \rangle = \sum_{i=0}^{n-1} \varepsilon_i(b) \Lambda_i(c) \geq \ell; \]
5. for each $\lambda \in \hat{P}^+_\ell := \{ \mu \in \hat{P}^+ \mid \langle c, \mu \rangle = \ell \}$, there exist unique vectors $b^\lambda$ and $\bar{b}_\lambda$ in $B$ such that
   \[ \varepsilon(b^\lambda) = \lambda \quad \text{and} \quad \varphi(\bar{b}_\lambda) = \lambda. \]

In the remainder of this section, we fix a perfect crystal $B$.

The maps $\lambda \mapsto \varepsilon(b_\lambda)$ and $\lambda \mapsto \varphi(b_\lambda)$ then define two bijections on $\hat{P}^+_\ell$.

As a consequence of the last condition, for any $\lambda \in \hat{P}^+_\ell$, the vertex operator theory [HK02 (10.4.4)] leads to a natural crystal isomorphism

\[ \lambda \mapsto \varepsilon(b_\lambda), \quad \lambda \mapsto \varphi(b_\lambda), \quad \lambda \mapsto b_\lambda. \]

Definition 2.13. For $\lambda \in \hat{P}^+_\ell$, the ground state path of weight $\lambda$ is the tensor product

\[ p_\lambda = (g_k)_{k=0}^\infty = \cdots \otimes g_{k+1} \otimes g_k \otimes \cdots \otimes g_1 \otimes g_0, \]

where the elements $g_k \in B$ are such that

\[ \lambda_0 = \lambda \quad \lambda_{k+1} = \varepsilon(b_{\lambda_k}) \quad g_k = b_{\lambda_k+1} \quad \text{for all} \quad k \geq 0. \]

A tensor product $p = (p_k)_{k=0}^\infty = \cdots \otimes p_{k+1} \otimes p_k \otimes \cdots \otimes p_1 \otimes p_0$ of elements $p_k \in B$ is said to be a $\lambda$-path if $p_k = g_k$ for $k$ large enough.

Iterating the isomorphism (2.9), we obtain

\[ \lambda \mapsto \varepsilon(b_\lambda), \quad \lambda \mapsto \varphi(b_\lambda), \quad \lambda \mapsto b_\lambda. \]

and this gives a natural bijection, stated in the next theorem.

Theorem 2.14. [HK02, Theorem 10.6.4] Let $\lambda \in \hat{P}^+_\ell$. Then there is a crystal isomorphism

\[ \lambda \mapsto \mathcal{P}(\lambda) \]

between the crystal base $B(\lambda)$ of $L(\lambda)$ and the set $\mathcal{P}(\lambda)$ of $\lambda$-paths.
We describe the crystal structure of $\mathcal{P}(\lambda)$ as follows [HK02 (10.48)]. For any $p = (p_k)_{k=0}^\infty \in \mathcal{P}(\lambda)$, let $N \geq 0$ be the smallest integer such that $p_k = g_k$ for all $k \geq N$. We then set
\[
\overline{\text{wt}} p = \lambda_N + \sum_{k=0}^{N-1} \overline{\text{wt}} p_k,
\]
\[
\tilde{e}_i p = \cdots \otimes g_{N+1} \otimes \tilde{e}_i (g_N \otimes \cdots \otimes p_0),
\]
\[
\tilde{f}_i p = \cdots \otimes g_{N+1} \otimes \tilde{f}_i (g_N \otimes \cdots \otimes p_0),
\]
\[
\varepsilon_i (p) = \max \left( \varepsilon_i (p') - \varphi_i (g_N), 0 \right),
\]
\[
\varphi_i (p) = \varphi_i (p') + \max \left( \varphi_i (g_N) - \varepsilon_i (p'), 0 \right),
\]
where $p' := p_{N-1} \otimes \cdots \otimes p_1 \otimes p_0$, and $\overline{\text{wt}}$ is viewed as the classical weight of an element of $B$ or $\mathcal{P}(\lambda)$.

The explicit expression for the affine weight $\overline{\text{wt}} p$ in $P$ is given in the following theorem, which is known as the (KMN)$^2$ crystal base character formula, and plays a key role in connecting characters with partition generating functions.

**Theorem 2.15.** [HK02 Theorem 10.6.7] Let $\lambda \in \tilde{p}_l^+$, let $H$ be an energy function on $B \otimes B$, and let $p = (p_k)_{k=0}^\infty \in \mathcal{P}(\lambda)$. Then the weight of $p$ and the character of the irreducible highest weight $U_q (\hat{\mathbf{g}})$-module $L(\lambda)$ are given by the following expressions:
\[
\text{wt} p = \lambda + \sum_{k=0}^\infty \left( \overline{\text{wt}} p_k - \overline{\text{wt}} g_k \right) - \sum_{k=0}^\infty (k+1) \left( H(p_{k+1} \otimes p_k) - H(g_{k+1} \otimes g_k) \right),
\]
\[
\text{ch}(L(\lambda)) = \sum_{p \in \mathcal{P}(\lambda)} e^{\overline{\text{wt}} p}.
\]

A specialisation of Theorem 2.15 gives the following corollary.

**Corollary 2.16.** Suppose that $\lambda$ is such that $b_\lambda = b^\lambda = g$, and set $H(g \otimes g) = 0$. Then $\overline{\text{wt}} g = 0$, $g_k = g$ for all $k \in \mathbb{Z}_{\geq 0}$, and we have
\[
\text{wt} p = \lambda + \sum_{k=0}^\infty \overline{\text{wt}} p_k - \sum_{l=k}^\infty H(p_{l+1} \otimes p_l),
\]

This is the main result which we will use in the next section to connect crystal base theory to integer partitions.

### 3. Perfect crystals and grounded partitions

To make the connection between our combinatorial partition identities and character formulas, we introduce in this section a new type of coloured partitions: grounded partitions.

Let $C$ be a set of colours, and let $\mathbb{Z}_C = \{ k_c : k \in \mathbb{Z}, c \in C \}$ be the set of coloured integers. First, we relax the condition that parts of (coloured) partitions have to be in non-increasing order.

**Definition 3.1.** Let $\gg$ be a binary relation defined on $\mathbb{Z}_C$. A generalised coloured partition with relation $\gg$ is a finite sequence $(\pi_0, \ldots, \pi_s)$ of coloured integers, where for all $i \in \{0, \ldots, s-1\}$, $\pi_i \gg \pi_{i+1}$.

In the following, $c(\pi_i) \in C$ denotes the colour of the part $\pi_i$. The quantity $|\pi| = \pi_0 + \cdots + \pi_s$ is the weight of $\pi$, and $C(\pi) = c(\pi_0) \cdots c(\pi_s)$ is its colour sequence.

**Remark.** The binary relation is not necessarily an order. When $\gg$ is a strict total order, we can easily check that every finite set of coloured parts defines a classical coloured partition, by ordering the parts. In the same way, for a large total order, the generalised coloured partitions are finite multi-sets of coloured integers.

Let us choose a particular colour $c_g$. We now define grounded partitions, which are directly related to ground state paths.
Definition 3.2. A grounded partition with ground $c_g$ and relation $\gg$ is a non-empty generalised coloured partition $\pi = (\pi_0, \ldots, \pi_s)$ with relation $\gg$, such that $\pi_s = 0_{c_g}$, and when $s > 0$, $\pi_{s-1} \neq 0_{c_g}$.

Let $\mathcal{P}_{c_g}^{\gg}$ denote the set of such partitions.

In the following, we explicitly write $\pi = (\pi_0, \ldots, \pi_s, 0_{c_g})$. The trivial partition in $\mathcal{P}_{c_g}^{\gg}$ is then $(0_{c_g})$.

Example 3.3. For the set of classical integer partitions $\pi = (\pi_1, \ldots, \pi_s)$, where parts satisfy $\pi_1 \geq \cdots \geq \pi_s > 0$, the empty partition is such that $s = 0$. This set is in bijection with the set $\mathcal{P}_c$ of grounded coloured partitions with only one colour $c$, defined by the relation

$$k_c \gg l_c \text{ if and only if } k - l \geq 0.$$  

The bijection is given by

$$(\pi_1, \ldots, \pi_s) \mapsto ((\pi_1)_c, \ldots, (\pi_s)_c, 0_c),$$

where the empty partition $\emptyset$ corresponds to the grounded partition $(0_c)$.

In the remainder of this section, we make the connection between grounded partitions and crystal base theory. Let us fix a weight $\lambda \in P^+_\lambda$ such that $b_\lambda = b^\lambda = g$, and assume that $H(g \otimes g) = 0$. Let $C_B = \{c_b : b \in B\}$ be the set of colours indexed by $B$. We define the binary relation $\geq$ on $\mathbb{Z} C_B$ by

$$k_{c_s} \geq k'_{c_{s'}} \text{ if and only if } k - k' = H(b' \otimes b). \quad (3.1)$$

This relation leads to the following.

Proposition 3.4. Let $\phi$ be the map between $\lambda$-paths and grounded partitions defined as follows:

$$\phi : \ p \mapsto (\pi_0, \ldots, \pi_{s-1}, 0_{c_g}),$$

where $p = (p_k)_{k \geq 0}$ is a $\lambda$-path in $\mathcal{P}(\lambda)$, $s \geq 0$ is the unique non-negative integer such that $p_{s-1} \neq g$ and $p_k = g$ for all $k \geq s$, and for all $k \in \{0, \ldots, s - 1\}$, the part $\pi_k$ has colour $c_{p_k}$ and size

$$\sum_{l=k}^{s-1} H(p_{k+1} \otimes p_k).$$

Then $\phi$ is a bijection between $\mathcal{P}(\lambda)$ and $\mathcal{P}_{c_g}^{\gg}$. Furthermore, by taking $c_b = e^\text{wt}_{0_b}$, we have for all $\pi \in \mathcal{P}_{c_g}^{\gg}$,

$$e^{-\lambda \text{wt} \phi^{-1}(\pi)} = C(\pi)e^{-|\pi|}. \quad (3.2)$$

Proof. It is easy to see that $\phi(p)$ belongs to $\mathcal{P}_{c_g}^{\gg}$, since by [3.1] we have $\pi_k \geq \pi_{k+1}$ for $k \in \{0, \ldots, s - 1\}$, and $p_{s-1} \neq g$ implies that $\pi_{s-1} \neq 0_{c_g}$. Note that the ground state path $\cdots \otimes g \otimes g \otimes g \otimes g$ is associated to $(0_{c_g})$.

Let us now give the inverse bijection. Start with $\pi \in (\pi_0, \ldots, \pi_s, 0_{c_g}) \in \mathcal{P}_{c_g}^{\gg}$, different from $(0_{c_g})$, with colour sequence $c_{p_0} \cdots c_{p_{s-1}} c_g$. Recall that $\pi_s = 0_{c_g}$. We set $\phi^{-1}(\pi) = (p_k)_{k \geq 0}$, where $p_k = g$ for all $k \geq s$ and $p_k = p_k'$ for all $k \in \{0, \ldots, s - 1\}$.

- We first show that $p_{s-1} \neq g$. Assume for the purpose of contradiction that $p_{s-1} = g$. By [3.1], we know that $\pi_{s-1} \gg 0_{c_g}$ if and only if

$$\pi_{s-1} - 0_{c_g} = H(p_s \otimes p_{s-1}) = H(g \otimes g) = 0,$$

i.e. if and only if $\pi_{s-1} = 0_{c_g}$. This contradicts the fact that $\pi_{s-1} \neq 0_{c_g}$.

- By [3.1], we also have, for all $k \in \{0, \ldots, s - 1\}$, $\pi_k - \pi_{k+1} = H(p_{k+1} \otimes p_k)$. Therefore

$$\pi_k = \pi_k - 0_{c_g} = \sum_{l=k}^{s-1} \pi_l - \pi_{l+1} = \sum_{l=k}^{s-1} H(p_{l+1} \otimes p_l).$$

With what precedes, we have $\phi(\phi^{-1}(\pi)) = \pi$ and $\phi^{-1}(\phi(p)) = p$. We obtain [3.2] by Corollary 2.10 and by observing that

$$\pi_k = \sum_{l=k}^{s-1} H(p_{l+1} \otimes p_l) = \sum_{l=k}^{\infty} H(p_{l+1} \otimes p_l),$$

since $H(p_{l+1} \otimes p_l) = H(g \otimes g) = 0$ for all $l \geq s$. \qed
Example 3.5. Let us consider the energy matrix given by Primc for the case $A_1^{(1)}$. We refer to the notation given in Section 4, and we have the correspondence $c_{ij} \otimes v_i^j = a_{ij}b_j$ for all $i, j \in \{0, 1\}$. Let us set the ground state $g = v_0 \otimes v_0^0$ corresponding to the classical weight $A_0$, so that $c_g = a_{00}$.

- The ground state path $p_{\lambda_0} = \cdots \otimes (v_0 \otimes v_0^0) \otimes (v_0 \otimes v_0^0)$ corresponds to the partition $\phi(p_{\lambda_0}) = (0_{\alpha_0 \beta_0})$.
- For $p = \cdots \otimes (v_0 \otimes v_0^0) \otimes (v_0 \otimes v_0^0) \otimes (v_0 \otimes v_0^1) \otimes (v_1 \otimes v_1^1)$, we have

$$\phi(p) = (3_{\alpha_1 \beta_1}, 3_{\alpha_1 \beta_1}, 1_{\alpha_1 \beta_1}, 0_{\alpha_0 \beta_0}).$$

The next proposition allows us to describe the set $P_{c_g}^\gg$ of grounded partitions for the relation $\gg$ defined by

$$k_{c_g} \gg k'_{c_g},$$

if and only if $k - k' \geq H(b' \otimes b)$.

(3.3)

We refer to this relation as minimal difference conditions. One can view the partitions of $P_{c_g}^\gg$ as the partitions of $P \gg$ such that the differences between consecutive parts are minimal. Note that contrarily to $P_{c_g}^\gg$, the set $P_{c_g}^\gg$ has some partitions $\pi = (\pi_0, \ldots, \pi_{s-1}, 0_{c_g})$ such that $c(\pi_{s-1}) = c_g$. For this reason, the set $P_{c_g}^\gg$ is not exactly the set of all minimal partitions of $P_{c_g}^\gg$, but is still related to it.

Proposition 3.6. Recall that $P_{c_g}$ is the set of grounded partitions where all parts have colour $c_g$. There is a bijection $\Phi$ between $P_{c_g}^\gg$ and $P_{c_g} \times P_{c_g}$, such that if $\Phi(\pi) = (\mu, \nu)$, then $|\pi| = |\mu| + |\nu|$, and by setting $c_g = 1$, we have $C(\pi) = C(\mu)$.

Proof. We set $\Phi(0_{c_g}) = ((0_{c_g}), (0_{c_g}))$. Let us now consider any $\pi = (\pi_0, \ldots, \pi_{s-1}, 0_{c_g}) \in P_{c_g}^\gg$, different from $(0_{c_g})$, with colour sequence $c_{\pi_0}, \ldots, c_{\pi_{s-1}}, c_g$, and build $\Phi(\pi) = (\mu, \nu)$. Recall that $\pi_{s-1} \neq \pi_s = 0_{c_g}$. Let us set $p = (p_k)_{k \geq 0}$, with $p_k = g$ for all $k \geq s$ and $p_k = p'_{k}$ for all $k \in \{0, \ldots, s-1\}$, and set $r = \max\{k \in \{0, \ldots, s\} : p_{k-1} \neq g\}$.

Since $p_k = g$ for all $k \geq r$, with the convention $c_g = 1$, we obtain that $C(\pi) = c_{p_0} \cdots c_{p_{r-1}} = c_{p_0} \cdots c_{p_{r-1}}$. Note that $r = 0$ if and only if all the parts of $\pi$ have colour $c_g$. We set $\mu = (\mu_0, \ldots, \mu_{r-1}, 0_{c_g}) = \phi(p)$. By Proposition 3.4, for all $k \in \{0, \ldots, r-1\}$, the part $\mu_k$ is coloured by $c_{p_k}$ and has size

$$\sum_{l=k}^{r-1} H(p_{l+1}^0 \otimes p_l^0).$$

Let us now build $\nu = (\nu_0, \ldots, \nu_{t-1}, 0_{c_g}) \in P_{c_g}$, where $c(\nu_k) = c_g$ and $\nu_k > 0$ for all $k \in \{0, \ldots, t-1\}$. We distinguish two different cases.

- If $r < s$, then we set $t = s$ and $\nu = (\nu_0, \ldots, \nu_{s-1}, 0_{c_g})$, where

$$\begin{cases}
\nu_k = \pi_k - \mu_k & \text{for } k \in \{0, \ldots, r-1\}, \\
\nu_k = \pi_k & \text{for } k \in \{r, \ldots, s-1\}.
\end{cases}$$

By Proposition 3.4, the sequence $(\nu_k)_{k=0}^{s-1}$ is non-increasing. Moreover the fact that $H(g \otimes g) = 0$ and $\pi_{s-1} \neq 0_{c_g}$ implies that $\nu_{s-1} > 0$, and $(\nu_k)_{k=r}^{s-1}$ is a non-increasing sequence of positive integers. Finally, let us check that $\nu_{r-1} \geq \nu_t$. We have

$$\nu_{r-1} - \nu_t = \pi_{r-1} - \pi_t - \mu_{r-1} \geq H(p_r \otimes p_{r-1}) - H(p_r \otimes p_{r-1}) \text{ by } (3.3) \geq 0.$$

Thus $(\nu_k)_{k=0}^{s-1}$ is indeed a non-increasing sequence of positive integers.

- By definition, $r \leq s$, so the only other possible case is $r = s$. As before, $(\pi_k - \mu_k)_{k=0}^{s}$ is a non-increasing sequence of non-negative integers, now with $\pi_s - \mu_s = 0 - 0 = 0$. We then set $t = \min\{k \in \{0, \ldots, s\} : \pi_k = \mu_k\}$, and $\nu_k = \pi_k - \mu_k$ for all $k \in \{0, \ldots, t-1\}$.

Observe that for $\Phi(\pi) = (\mu, \nu)$, with $\pi = (\pi_0, \ldots, \pi_{s-1}, 0_{c_g})$, $\mu = (\mu_0, \ldots, \mu_{r-1}, 0_{c_g})$ and $\nu = (\nu_0, \ldots, \nu_{t-1}, 0_{c_g})$, we always have $s = \max\{r, t\}$, and by adding $s - \min\{r, t\}$ parts $0_{c_g}$ at the end of the shorter partition, we have $\pi_k = \mu_k + \nu_k$ and $c(\pi_k) = c(\mu_k)$ for all $k \in \{0, \ldots, s-1\}$. 

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The map $\Phi^{-1}$ from $\mathcal{P}_c^g \times \mathcal{P}_c^g$ to $\mathcal{P}_c^g$ simply consists in adding the parts of $\mu = (\mu_0, \ldots, \mu_{r-1}, 0_{c_g}) \in \mathcal{P}_c^g$ to those of $\nu = (\nu_0, \ldots, \nu_{t-1}, 0_{c_g}) \in \mathcal{P}_c^g$ to obtain a grounded partition $\pi \in \mathcal{P}_c^g$ in the following way:

- if $t \leq r$, then $\pi_k$ has size $\mu_k + \nu_k$ and colour $c(\mu_k)$, where we set $\nu_k = 0$ for all $k \in \{t, \ldots, r-1\}$, and we obtain the partition
  \[
  \pi = (\pi_0, \cdots, \pi_{t-1}, 0_{c_g}),
  \]
- if $t > r$, the first $r$ parts are defined as in the case $t \leq r$, and the remaining parts are $\pi_k = \nu_k$ for all $k \in \{r, \ldots, t-1\}$ with colour $c_g$, and we obtain the partition
  \[
  \pi = (\pi_0, \cdots, \pi_{t-1}, 0_{c_g}).
  \]

Examples 3.7. Let us consider the energy matrix given Primc for the case $A_1^{(1)}$, and let us set $c_g = 0_{a_0 b_0}$. We give three examples for the previous bijection, for the different cases $t < r$, $t = r$ and $t > r$.

- Case $t < r$: Let $\pi = (10_{a_0 b_0}, 7_{a_0 b_0}, 3_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0})$. Given the minimal difference conditions in $1.1$, our bijection gives
  \[
  \mu = (6_{a_0 b_0}, 5_{a_0 b_0}, 3_{a_0 b_0}, 3_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0}) \quad \text{and} \quad \nu = (4_{a_0 b_0}, 2_{a_0 b_0}, 2_{a_0 b_0}, 0_{a_0 b_0}).
  \]
- Case $t = r$: Let $\pi = (10_{a_0 b_0}, 7_{a_0 b_0}, 5_{a_0 b_0}, 4_{a_0 b_0}, 3_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0})$. We have
  \[
  \mu = (6_{a_0 b_0}, 5_{a_0 b_0}, 3_{a_0 b_0}, 3_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0}) \quad \text{and} \quad \nu = (4_{a_0 b_0}, 2_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0}).
  \]
- Case $t > r$: Let $\pi = (8_{a_0 b_0}, 5_{a_0 b_0}, 3_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0})$. We obtain
  \[
  \mu = (4_{a_0 b_0}, 3_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0}) \quad \text{and} \quad \nu = (4_{a_0 b_0}, 2_{a_0 b_0}, 2_{a_0 b_0}, 1_{a_0 b_0}, 1_{a_0 b_0}, 0_{a_0 b_0}).
  \]

We are now able to give a character formula in terms of generating functions for grounded partitions.

Theorem 3.8. Setting $q = e^{-\delta}$ and $c_b = e^{-\delta b}$ for all $b \in B$, we have $c_g = 1$, and the character of the irreducible highest weight $U_q(\mathfrak{g})$-module $L(\lambda)$ is given by the following expressions:

\[
\sum_{\pi \in \mathcal{P}_c^g} C(\pi)q^{\pi} = e^{-\lambda} \text{ch}(L(\lambda)),
\]

\[
\sum_{\pi \in \mathcal{P}_c^g} C(\pi)q^{\pi} = \frac{e^{-\lambda} \text{ch}(L(\lambda))}{(q; q)_\infty}.
\]

Proof. By Proposition 3.4 and (2.12),

\[
\sum_{\pi \in \mathcal{P}_c^g} C(\pi)q^{\pi} = \sum_{\pi \in \mathcal{P}(\lambda)} e^{-\lambda} e^{\text{wt} \pi} = e^{-\lambda} \text{ch}(L(\lambda)).
\]

By Corollary 2.16 $\text{wt} g = 0$. Thus $c_g = 0^0 = 1$, and Proposition 3.6 yields

\[
\sum_{\pi \in \mathcal{P}_c^g} C(\pi)q^{\pi} = \frac{1}{(q; q)_\infty} \sum_{\pi \in \mathcal{P}_c^g} C(\pi)q^{\pi} = \frac{e^{-\lambda} \text{ch}(L(\lambda))}{(q; q)_\infty}.
\]

By this theorem, the characters of irreducible highest weight modules of level $\ell$ can be computed as the generating functions of some grounded partitions. It is the key that connects the Primc generalised partitions to the characters of irreducible highest weight modules of level 1 for the affine Lie algebra $A_n^{(1)}$. 

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4. Perfect crystal of type $A_{n-1}^{(1)}$: tensor product of the vector representation and its dual

We now describe the perfect crystal $\mathcal{B}$ used in Theorem 1.4. Throughout this section, we fix an integer $n \geq 3$.

Consider the Cartan datum for the matrix $A = (a_{ij})_{i,j \in \{0, \ldots, n-1\}}$ where for all $i, j \in \{0, \ldots, n-1\}$,
\[ a_{ij} = 2\delta_{i,j} - \chi(i-j \equiv \pm 1 \mod n). \tag{4.1} \]

It corresponds to the affine type $A_{n-1}^{(1)}$ ([HK02, 10.1.1]). We then have the corresponding canonical central element $c$ and null root $\delta$, which are expressed in the following way:
\[ c = h_0 + h_1 + \cdots + h_{n-1}, \]
\[ \delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}. \tag{4.2} \]

Any dominant integral weight $\lambda = k_0\Lambda_0 + \cdots + k_{n-1}\Lambda_{n-1} \in \bar{P}^+$ has level $\langle c, \lambda \rangle = k_0 + \cdots + k_{n-1}$.

Thus, the set of classical weights of level 1 is exactly $\bar{P}_{1}^+ = \{ \Lambda_i : i \in \{0, \ldots, n-1\} \}$, the set of fundamental weights.

A perfect crystal of level 1 is given by the crystal graph in Figure 4.1 ([HK02, 11.1.1]).

**Figure 4.1.**

The $U'_q(\tilde{g})$-module corresponding to this crystal is called the vector representation of $A_{n-1}^{(1)}$. The most important property of this crystal is the order in which the arrows occur. The only purpose of labelling the vertices is to ease the calculations in the remainder of this paper. Noting that this crystal graph is cyclic, we identify $\{0, \ldots, n-1\}$ with the group $(\mathbb{Z}/n\mathbb{Z}, +)$. In this way, the crystal graph of $\mathcal{B}$ can be defined locally around each arrow $i$ as shown on Figure 4.2.

**Figure 4.2.**

**Remark.** For the type $A_1^{(1)}$, the Cartan matrix $A$ is defined differently and is given by
\[ \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \]

Nonetheless, the crystal graph of the vector representation behaves in the same way as in the case $n \geq 3$.

For all $i \in \{0, \ldots, n-1\}$, let $v_i$ be the element of $\mathcal{B}$ corresponding to the vertex labelled $i$. The functions of this crystal are given by the following relations:
\[ \overline{wt}v_i = \Lambda_{i+1} - \Lambda_i \quad \text{for all} \quad i \in \{0, \ldots, n-1\}, \tag{4.3} \]
\[ \begin{cases} \tilde{f}_iv_i = v_i \\ \varphi_i v_{i-1} = 1 \\ \tilde{f}_j v_j = \varphi_i v_j = 0 \quad \text{if} \quad j \neq i-1, \end{cases} \tag{4.4} \]
\[ \begin{cases} \tilde{e}_i v_i = v_{i-1} \\ \varepsilon_i v_i = 1 \\ \tilde{e}_j v_j = \varepsilon_i v_j = 0 \quad \text{if} \quad j \neq i. \end{cases} \tag{4.5} \]
We note that for this crystal, the unique maximal weight $\lambda_0$, as defined in Condition (3) of Definition 2.12 is attained in $v_0$ (i.e. $\lambda_0 = \overrightarrow{wt}v_0$). For all $i \in \{0, \ldots, n-1\}$, we have

$$\overrightarrow{wt}v_0 - \overrightarrow{wt}v_i = \sum_{j=1}^{i} \overrightarrow{wt}v_{j-1} - \overrightarrow{wt}v_j$$

$$= \sum_{j=1}^{i} \alpha_j \text{ by (2.5)}.$$

The fact that the null root vanishes on $\tilde{h}$ implies that in $\bar{P}$, $\overrightarrow{\alpha}_0 = -(\alpha_1 + \cdots + \alpha_{n-1})$. We also remark that the crystal $B$ has a unique minimal weight, attained in $v_{n-1}$:

$$\overrightarrow{wt}v_i - \overrightarrow{wt}v_{n-1} = \sum_{j=i+1}^{n-1} \overrightarrow{wt}v_{j-1} - \overrightarrow{wt}v_j$$

$$= \sum_{j=i+1}^{n-1} \alpha_j \text{ by (2.5)}.$$

Let us consider the dual $B^\vee$ of $B$, which is the crystal obtained from $B$ by reversing the edges in its graph, as shown on Figure 4.3.

Let $v^\vee$ denote the element of $B^\vee$ corresponding to $v$ in $B$. We then have the relations

$$\overrightarrow{wt}v^\vee = -\overrightarrow{wt}v,$$

$$\tilde{f}_i v^\vee = (\tilde{e}_i v)^\vee,$$

$$\varphi_i v^\vee = \varepsilon_i v,$$

$$\tilde{e}_i v^\vee = (\tilde{f}_i v)^\vee \text{ and } \varepsilon_i v^\vee = \varphi_i v. \quad (4.6)$$

Recall that the duality is an involution, since by the previous equalities, we have

$$(\tilde{f}_i[(v^\vee)^\vee], \tilde{e}_i[(v^\vee)^\vee], \varphi_i[(v^\vee)^\vee], \varepsilon_i[(v^\vee)^\vee]) = (\tilde{f}_i[(v^\vee)^\vee], \tilde{e}_i[(v^\vee)^\vee], \varphi_i v, \varepsilon_i v), \quad (4.7)$$

and the map $v \mapsto (v^\vee)^\vee$ is an isomorphism between $B$ and $(B^\vee)^\vee$. Thus $(B^\vee)^\vee$ can be identified with $B$.

The dual $B^\vee$ is also a perfect crystal of level 1, as its maximal weight is attained in the dual $v^\vee_{n-1}$ of the minimal vertex $v_{n-1}$ of $B$.

By Theorem 2.10 $B \otimes B^\vee$ is a crystal for the tensor product of the vector representation of $A_{n-1}^{(1)}$ and its dual, and the tensor rules (2.4) on $B \otimes B^\vee$ become

$$\tilde{e}_i(v_k \otimes v_i^\vee) = \begin{cases} \tilde{e}_i v_k \otimes v_i^\vee & \text{if } \varphi_i(v_k) \geq \varphi_i(v_i) \\ v_k \otimes \tilde{e}_i v_i^\vee & \text{if } \varphi_i(v_k) < \varphi_i(v_i) \end{cases},$$

$$\tilde{f}_i(v_k \otimes v_i^\vee) = \begin{cases} \tilde{f}_i v_k \otimes v_i^\vee & \text{if } \varphi_i(v_k) > \varphi_i(v_i) \\ v_k \otimes \tilde{f}_i v_i^\vee & \text{if } \varphi_i(v_k) \leq \varphi_i(v_i) \end{cases}.$$

Using (4.4) and (4.5), we can draw the corresponding crystal graph, given in Figure 4.4.
Again, the crystal graph of $\mathcal{B} \otimes \mathcal{B}^\vee$ can be defined locally by giving the vertices adjacent to the edges labelled $i$, as shown on Figure 4.5.

Figure 4.5.

\[
\begin{align*}
\mathcal{B} \otimes \mathcal{B}^\vee : & \quad v_0 \otimes v_{n-1}^\vee \rightarrow v_1 \otimes v_{n-1}^\vee \rightarrow v_2 \otimes v_{n-2}^\vee \rightarrow \cdots \rightarrow v_{n-2} \otimes v^\vee_2 \rightarrow v_{n-1} \otimes v^\vee_1 \\
k \not\in \{i-1, i\} & \quad v_k \otimes v_{i-1}^\vee \rightarrow v_{i-1} \otimes v_i^\vee \\
& \quad v_{i-1} \otimes v_k^\vee \rightarrow v_i \otimes v_{i-1}^\vee \\
& \quad v_i \otimes v_{i-1}^\vee \\
& \quad v_{i-1} \otimes v_k^\vee \\
& \quad v_i \otimes v_{i-1}^\vee
\end{align*}
\]

We obtain, for all $i$, the relations

\[
\begin{align*}
\varphi_i(v_{i-1} \otimes v_{i-1}^\vee) &= \varepsilon_i(v_{i} \otimes v_{i-1}^\vee) = 2 \\
\varphi_i(v_i \otimes v_{i-1}^\vee) &= \varepsilon_i(v_{i-1} \otimes v_i^\vee) = 0 \\
\varphi_i(v_i \otimes v_i^\vee) &= \varepsilon_i(v_i \otimes v_i^\vee) = 1 \\
\varphi_i(v_{i-1} \otimes v_{i-1}^\vee) &= \varepsilon_i(v_{i-1} \otimes v_{i-1}^\vee) = 0
\end{align*}
\]

\[
\begin{align*}
\varphi_i(v_k \otimes v_i^\vee) &= \varepsilon_i(v_i \otimes v_k^\vee) = 1 \\
\varphi_i(v_{i-1} \otimes v_k^\vee) &= \varepsilon_i(v_k \otimes v_{i-1}^\vee) = 1 \\
\varphi_i(v_k \otimes v_{i-1}^\vee) &= \varepsilon_i(v_i \otimes v_k^\vee) = 0
\end{align*}
\]

(4.8)

The local configurations for the vertices are given in Figure 4.6.

Figure 4.6.
The values of the functions $\varepsilon, \varphi$ defined in (2.4) are
\[
\begin{align*}
\varphi(v_{i-1} \otimes v_{j}^\vee) &= \varepsilon(v_{i} \otimes v_{j}^\vee) = 2\Lambda_i \\
\varepsilon(v_{i-1} \otimes v_{j}^\vee) &= \varphi(v_{i} \otimes v_{j}^\vee) = \Lambda_{i-1} + \Lambda_{i+1}, \\
\varphi(v_i \otimes v_{j}^\vee) &= \varepsilon(v_i \otimes v_{j}^\vee) = \Lambda_{i}.
\end{align*}
\]
(4.9)
where $k - l \notin \{0, \pm 1\}$. For all $k, l \in \{0, \ldots, n - 1\}$, the weight of $v_k \otimes v_l^\vee$ is given by
\[
\overline{\text{wt}}(v_k \otimes v_l^\vee) = \Lambda_{k+1} - \Lambda_{k} + \Lambda_l - \Lambda_{l+1}.
\]
(4.10)
We then observe that
\[
\langle c; \varepsilon(v_k \otimes v_l^\vee) \rangle = 1 + \chi(k \neq l).
\]
(4.11)

By [KKM+92a] Lemma 4.6.2, since $\mathcal{B}$ and $\mathcal{B}^\vee$ are perfect crystals of level 1, their tensor product $\mathcal{B}$ is also a perfect crystal of level 1. We observe that the potential grounds of $\mathcal{B}$ are the vertices $v_i \otimes v_j^\vee$, since by (4.9), for all $i \in \{0, \ldots, n - 1\}$, we have that
\[
\varepsilon(b^{\Lambda_i}) = \Lambda_i \text{ if and only if } b^{\Lambda_i} = v_i \otimes v_j^\vee \text{ and } \varphi(b_{\Lambda_i}) = \Lambda_i \text{ if and only if } b_{\Lambda_i} = v_i \otimes v_j^\vee.
\]

5. Proof of the character formulas

In this section, we prove our character formulas given in Theorems 1.5, 1.6, and 1.7 under the assumption that Theorem 1.4 is true. We will then prove Theorem 1.4 in the last two sections.

5.1. Proof of Theorem 1.5. We show that the set of grounded partitions $\mathcal{P}_{c_g}$, with $\gg$ defined in (3.3), grounded at $c_g$ for $g = (v_0 \otimes v_j^\vee)$, is in bijection with the set of Primc generalised partitions.

Let $(\pi_0, \ldots, \pi_{s-1}, 0_{c_g}) \in \mathcal{P}_{c_g}$, and set $b$ to be the vertex in $\mathcal{B}$ corresponding to the colour of $\pi_{s-1}$. Since $\pi_{s-1} \neq 0_{c_g}$, the minimal size of $\pi_{s-1}$ is $H((v_0 \otimes v_j^\vee) \otimes (v_i \otimes v_j^\vee))$ if $(i, j) \neq (0, 0)$, and 1 if $(i, j) = (0, 0)$. This corresponds to the size of minimal parts in Primc generalised partitions. By Theorem 1.4 for all $(i, j) \neq (0, 0)$,
\[
H((v_0 \otimes v_j^\vee) \otimes (v_i \otimes v_j^\vee)) = \Delta(a_{j} b_{i}, a_{0} b_{0}) = 1.
\]
Thus the Primc generalised partitions and the grounded partitions in $\mathcal{P}_{c_g}$ coincide in terms of minimal difference conditions and minimal part sizes, with the colour correspondence $c_{v_i \otimes v_j^\vee} \leftrightarrow a_{k} b_{l}$. Thus their generating functions are the same with the correspondence $e^{\overline{\text{wt}}(0)} = b_{i}$, since by (4.10),
\[
e^{\overline{\text{wt}}(v_0 \otimes v_j^\vee)} = e^{\overline{\text{wt}}(l) - \overline{\text{wt}}(k)} = b_{k}^{-1} b_{l}.
\]
Using the character formula of Theorem 3.8, this gives the desired result. \qed

5.2. Proof of Theorem 1.6. Let us now turn to the proof of Theorem 1.6. It uses some notions defined in our first paper [DK19], such as bound and free colours, reduced colour sequences, kernel, insertions, types. As they are only needed for this proof, we do not redefine them here, and refer the reader to Sections 1 and 2 of [DK19].

Let us fix $\ell \in \{0, \ldots, n - 1\}$ and recall that in the perfect crystal $\mathcal{B}$, we have $b^{\Lambda_{\ell}} = b_{\Lambda_{\ell}} = v_{\ell} \otimes v_{\ell}^\vee$. By assuming that Theorem 1.4 is true, we also have that $H((v_\ell \otimes v_j^\vee) \otimes (v_\ell \otimes v_j^\vee)) = \Delta(a_{\ell} b_{\ell}, a_{\ell} b_{\ell}) = 0$. Let us set $g = (v_\ell \otimes v_j^\vee)$ to be the ground in $\mathcal{B}$, and consider the set $\mathcal{P}_{c_g}$ of grounded partitions with ground $c_g$.

For $\pi = (\pi_0, \ldots, \pi_{s-1}, 0_{c_g}) \in \mathcal{P}_{c_g}$, write $c(\pi_k) = c(v_{i_k} \otimes v_{j_k}^\vee)$.

By (3.3), for all $k \in \{0, \ldots, s - 2\}$, the parts of $\pi$ satisfy the difference conditions
\[
\pi_k - \pi_{k+1} \geq H((v_{j_k} \otimes v_{i_{k+1}}^\vee) \otimes (v_{j_k} \otimes v_{i_{k+1}}^\vee)) = \Delta(a_{i_k} b_{j_k}, a_{i_{k+1}} b_{j_{k+1}}).
\]
Let us now study the size of the smallest part.

- If $(j_{s-1}, i_{s-1}) \neq (\ell, \ell)$, the minimal size of $\pi_{s-1}$ is
\[
\Delta(a_{i_{s-1}} b_{j_{s-1}}, a_{\ell} b_{\ell}) = \begin{cases} 
\chi(i_{s-1} > \ell) = \chi(i_{s-1} \geq \ell) + \chi(\ell > i_{s-1}) & \text{if } j_{s-1} = \ell \quad (i_{s-1} \neq \ell) \\
\chi(i_{s-1} \geq \ell) + \chi(\ell > i_{s-1}) & \text{if } j_{s-1} \neq \ell
\end{cases}
\]

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• Otherwise, we have that \( j_{s-1} = i_{s-1} = \ell \), and then \( c(\pi_{s-1}) = c_g \). In this case, \( \Delta(a_k b_\ell; a_k b_\ell) = 0 \) implies that the size of \( \pi_{s-1} \) must be at least 1 to have \( \pi_{s-1} \neq 0_c \). We observe that, in that case, we still have \( \chi(i_{s-1} \geq \ell) + \chi(\ell > j_{s-1}) \).

In both cases, our grounded partition \( \pi \), without the part \( 0_{c_g} \), is a partition satisfying the difference condition \( \Delta \) of Primc generalised partitions, but such that the minimal size for the last part, denoted by \( \Delta(a_{i_{s-1}}, b_{j_{s-1}}, a_\infty b_\infty) \) with our conventions from [DK19], is given by the expression

\[
\Delta(a_{i_{s-1}}, b_{j_{s-1}}, a_\infty b_\infty) = \chi(i_{s-1} \geq \ell) + \chi(\ell > j_{s-1}).
\]

Moreover we observe that this is always equal to 1 when \( a_{i_{s-1}}, b_{j_{s-1}} \) is a free color. Thus in the case \( \ell = 0 \), the minimal part always has size 1, independently of its colour. For larger \( \ell \), the minimal part may have size 0, 1, or 2 according to (5.1). Besides, we keep the convention \( \Delta(a_\infty b_\infty, c) = 1 \), as it is in our first paper.

The proof of Theorem 1.1 in [DK19] relies on a correspondence between Primc generalised partitions and coloured Frobenius partitions having the same kernel. In the case where the kernel ends with a free color \( a_k b_k \), the Primc generalised partition is also a partition grounded in \( c_g \) by adding \( 0_{c_g} \), and the type of the insertions inside the secondary pairs remain the same.

When the kernel ends with a bound color \( a_k b_{k'}, k \neq k' \), we modify the type of the insertion of \( a_{k'} b_{k'} \) to the right of \( a_k b_{k'} \), and it becomes

\[
T_\Delta(a_k b_{k'}) = \Delta(a_k b_{k'}, a_k b_{k'}) + \Delta(a_k b_{k'}, a_\infty b_\infty) - \Delta(a_k b_{k'}, a_\infty b_\infty) = 1 + \chi(k > k') - (\chi(k \geq \ell) + \chi(\ell > k')).
\]

Note that this value is still in \( \{0, 1\} \), since it can be rewritten as \( \chi(k > k') + \chi(k > k') - \chi(\ell > k') \). The types of the others insertions are the same as the types for the Primc generalised partitions in [DK19].

Recall from [DK19] that a \( n^2 \)-coloured Frobenius partition is a pair of coloured partitions

\[
\begin{pmatrix}
\lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} \\
\mu_0 & \mu_1 & \cdots & \mu_{s-1}
\end{pmatrix},
\]

where \( \lambda = \lambda_0 + \lambda_1 + \cdots + \lambda_{s-1} \) is a partition into \( s \) distinct non-negative parts, each coloured with some \( a_i \), \( i \in \{0, \ldots, n-1\} \), with the following order

\[
0_{a_{n-1}} < 0_{a_{n-2}} < \cdots < 0_{a_0} < 1_{a_{n-1}} < 1_{a_{n-2}} < \cdots < 1_{a_0} < \cdots,
\]

and \( \mu = \mu_0 + \mu_1 + \cdots + \mu_{s-1} \) is a partition into \( s \) distinct non-negative parts, each coloured with some \( b_i \), \( i \in \{0, \ldots, n-1\} \), with the order

\[
0_{b_0} < 0_{b_1} < \cdots < 0_{b_{n-1}} < 1_{b_0} < 1_{b_1} < \cdots < 1_{b_{n-1}} < \cdots.
\]

The colour sequence of such a partition is defined to be \( c(\lambda_0)c(\mu_0), \ldots, c(\lambda_{s-1})c(\mu_{s-1}) \). Here the size corresponding to the colour \( c(\lambda_i)c(\mu_i) \) is \( \lambda_i + \mu_i \).

We consider coloured Frobenius partitions such that the minimal size for \( \lambda_{s-1} + \mu_{s-1} \) is given by \( \Delta'(a_k b_{k'}, a_\infty b_\infty) = \Delta(a_k b_{k'}, a_\infty b_\infty) \), where \( c(\lambda_{s-1}) = a_k \), \( c(\mu_{s-1}) = b_{k'} \), and \( \Delta(a_k b_{k'}, a_\infty b_\infty) \) was defined in [5.1]. We say that such coloured Frobenius partitions are grounded at \( c_g \). We have \( \Delta'(a_k b_{k'}, a_\infty b_\infty) = 1 \) for any free color \( a_k b_k \). Note that the differences are the same as those defined in [DK19],

\[
\Delta'(a_{i'}, b_{j'}) = \chi(i \geq i') + \chi(j \leq j').
\]

Here we keep the convention \( \Delta'(a_\infty b_\infty, c) = 1 \). When the kernel of the coloured Frobenius partition ends with a bound color \( a_k b_{k'} \), the type of the insertion of the free color \( a_{k'} b_{k'} \) to its right, according to the differences \( \Delta' := 2 - \Delta' \), is given by

\[
T_{\Delta'}(a_k b_{k'}) = \Delta''(a_k b_{k'}, a_k b_{k'}) + \Delta''(a_k b_{k'}, a_\infty b_\infty) - \Delta''(a_k b_{k'}, a_\infty b_\infty)
\]

\[
= 2 - \left( \Delta(a_k b_{k'}, a_k b_{k'}) + \Delta'(a_k b_{k'}, a_\infty b_\infty) - \Delta'(a_k b_{k'}, a_\infty b_\infty) \right)
\]

\[
= 2 - \left[ 1 + \chi(k > k') + 1 - (\chi(k \geq \ell) + \chi(\ell > k')) \right]
\]

\[
= \chi(k \geq \ell) + \chi(\ell > k') - \chi(k > k').
\]
The types of all the insertions which are not at the right end of the kernel are the same as the types for $\Delta''$ of the coloured Frobenius partitions in [DK19]. Thus, (5.2) yields the relation

$$T_{\Delta}(a_k b_k') + T_{\Delta'}(a_k b_k') = 1.$$  

This means that an insertion has $\Delta$-type 1 if and only if it has $\Delta''$-type 0. Thus, by Theorem 3.1 of [DK19], the generating function for our grounded Primc generalised partitions with a fixed kernel is the same as the generating function for grounded coloured Frobenius partitions with the same kernel. Therefore, the generating function for Primc generalised partitions with minimal part size $\Delta(a_k b_k', a_\infty b_\infty)$ is the same as the generating function for coloured Frobenius partitions with minimal part size $\Delta'(a_k b_k', a_\infty b_\infty) = \chi(k \geq l) + \chi(l > k')$. The generating function for the latter, where for all $i \in \{0, \ldots, n - 1\}$, the power of $b_i$ counts the number of colours $b_i$ minus the number of colours $a_i$ in the colour sequence, is given by

$$[x^0] \prod_{i=0}^{l-1} (-b_i^{-1} x; q)_\infty (-b_i q x^{-1}; q)_\infty \times \prod_{i=l}^{n-1} (-b_i^{-1} x q; q)_\infty (-b_i x^{-1}; q)_\infty.$$  

In this product, the minimal size for $\lambda_{s-1}$ with colour $a_k$ is $\chi(k \geq \ell)$, while the minimal size for $\mu_{s-1}$ with colour $b_k'$ is $\chi(k' < \ell)$, so that the minimal size for $\lambda_{s-1} + \mu_{s-1}$ is indeed $\chi(k \geq \ell) + \chi(\ell > k')$. We conclude by noting that, by Theorem 1.1, this generating function is obtained by doing the changes of variables $b_i \mapsto b_i q^{\chi(i < \ell)}$ in

$$G_n^P(q; b_0, \ldots, b_{n-1}) = [x^0] \prod_{i=0}^{n-1} (-b_i^{-1} x q; q)_\infty (-b_i x^{-1}; q)_\infty,$$

which gives Theorem 1.6. \hfill $\Box$

5.3. **Proof of Theorem 1.7.** Finally, we turn to the proof of Theorem 1.7, which gives the expression of the character for $L(\Lambda_L)$ as a sum of series with positive coefficients.

By the definition of characters, the function $e^{-\Lambda_L} \text{ch}(L(\Lambda_L))$ can be expressed as a power series in $e^{-\alpha_i}$ for $i \in \{0, \ldots, n - 1\}$, or, by a change of variables, as a power series in $e^{-\delta}$ and $e^{\alpha_i}$ for $i \neq 0$. By definition of the crystal graph $B$, we have $f_i v_{i-1} = v_i$, so that by (2.5), we have $\overline{w_iv_i} - \overline{w_i} = \alpha_i$ for $i \in \{1, \ldots, n-1\}$ and $\overline{w_n v_n} - \overline{w_n} = \overline{e_0}$. The change of variables $e^\overline{w_i} = b_i$ then gives $e^{\alpha_i} = b_i^{-1} b_i^{-1}$ for $i \in \{1, \ldots, n-1\}$. It is then coherent to have

$$e^{\overline{e_0}} = b_{n-1} b_0^{-1} = \prod_{i=1}^{n-1} b_i b_i^{-1} = \prod_{i=1}^{n-1} e^{-\alpha_i}.$$  

The changes of variables are then natural, since for all $i \neq 0$, the weight $\alpha_i$ in $P$ is indeed a classical weight in $\overline{P}$. In addition, the series $G_n^P(b_0 q; b_\ell, b_{\ell-1} q, b_{\ell-2}, \ldots, b_{n-1})$ can be expressed in terms of summands of the form

$$\left( \prod_{i=0}^{n-1} b_i^{r_i} \right) q^m \quad \text{with} \quad \sum_{i=0}^{n-1} r_i = 0,$$

so that we can always retrieve the exponent of $b_i^{-1} b_i^{-1}$, for all $i \in \{1, \ldots, n - 1\}$, which corresponds to $\sum_{j=0}^{i-1} r_j$. Thus the identification

$$e^{-\delta} \leftrightarrow q$$

$$e^{\alpha_i} \leftrightarrow b_i^{-1} b_i^{-1}$$

is unique, and our generalisation of Primc’s identity allows us to retrieve the non-dilated version of the characters for all the irreducible highest weight modules with classical weight of level 1 for the type $A_{n-1}^{(1)}$. 22
Looking at Formula (1.5), we obtain the following correspondences (recall that $r_1 = 0 = r_n$)

$$\prod_{i=1}^{n-1} b_1^{-r_i} = \prod_{i=1}^{n-1} (b_{i-1} b_1^{-1})^{r_i} = \prod_{i=1}^{n-1} e^{r_i \alpha_i}$$

$$\prod_{j=0}^{i-1} b_j b_1^{-1} = \prod_{j=1}^{i} (b_{j-1} b_1^{-1})^j = e^{\sum_{j=1}^{i} j \alpha_j}$$

By doing these transformations in (1.5), we then obtain by Theorem 1.5 and Theorem 1.1 that

$$e^{-\Lambda_0 \chi(L(A_0))} = \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}, i=1}^{n-1} \prod_{s_{n}=0}^{n-1} e^{s_i \alpha_i} e^{s_i (s_{i+1} - s_i) \delta}$$

$$= \sum_{r_1, \ldots, r_{n-1} \geq 0} \prod_{r_i=0}^{r_i \leq r_{i+1}} e^{r_i \alpha_i} e^{r_i (r_{i+1} - r_i) \delta} \left( e^{-\sum_{i=1}^{n} i \alpha_i} e^{-\sum_{i=1}^{n} i \delta} \right) \infty$$

$$\times \left( e^{-\left(\sum_{i=1}^{n} i \alpha_i - \sum_{i=1}^{n} i \delta\right)} \right) \infty.$$

Note that for all $\ell \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, n-1\}$, the transformation $b_j \mapsto b_j q^{(j-\ell)}$ is equivalent to $b_{j-1} b_1^{-1} \mapsto q^{\chi(j-\ell)} b_{j-1} b_1^{-1}$. This corresponds to the transformations $e^\alpha \mapsto e^{-\chi(j-\ell) \delta + \alpha_j}$ for all $j \in \{1, \ldots, n-1\}$, and Theorem 1.7 follows. □

### 6. Tools for the Proof of Theorem 1.4

We already know that the crystal graph of $\mathbb{B} \otimes \mathbb{B}$ is connected, as $\mathbb{B}$ is a perfect crystal. However, here we reprove this by building the paths in this graph, as these paths will allow us to compute the energy function. First, let us define some tools that will help us simplify the construction of the paths.

#### 6.1. Symmetry in the crystal graph of $\mathbb{B} \otimes \mathbb{B}$

**Proposition 6.1.** Let $\mathcal{B}$ be a crystal, let $\mathcal{B}'$ be the dual of $\mathcal{B}$, and let us set $\mathcal{B} = \mathcal{B} \otimes \mathcal{B}'$. Denote by $\sigma'$ the element in $\mathcal{B}'$ corresponding to $\sigma \in \mathcal{B}$. Then for any $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \tau_1, \tau_2, \tau_3, \tau_4 \in \mathcal{B}$, we have the following equivalence in the crystal graph $\mathcal{B} \otimes \mathcal{B}$:

$$\tilde{\mathcal{F}}_i[(\sigma_1 \otimes \sigma_2') \otimes (\sigma_3 \otimes \sigma_4')] = (\tau_1 \otimes \tau_2') \otimes (\tau_3 \otimes \tau_4') \iff \tilde{\mathcal{E}}_i[(\sigma_4 \otimes \sigma_3') \otimes (\sigma_2 \otimes \sigma_1')] = (\tau_4 \otimes \tau_1') \otimes (\tau_2 \otimes \tau_3').$$

(6.1)

And an energy function $H$ on $\mathcal{B} \otimes \mathcal{B}$ satisfies

$$H[(\sigma_1 \otimes \sigma_2') \otimes (\sigma_3 \otimes \sigma_4')] - H[\tilde{\mathcal{F}}_i((\sigma_1 \otimes \sigma_2') \otimes (\sigma_3 \otimes \sigma_4'))] = H[(\sigma_4 \otimes \sigma_3') \otimes (\sigma_2 \otimes \sigma_1')] - H[\tilde{\mathcal{E}}_i((\sigma_4 \otimes \sigma_3') \otimes (\sigma_2 \otimes \sigma_1'))].$$

(6.2)

Furthermore, there exists a path between $(\sigma_3 \otimes \sigma_4') \otimes (\sigma_1 \otimes \sigma_2')$ and $(\tau_3 \otimes \tau_4') \otimes (\tau_1 \otimes \tau_2')$ if and only if there exists a path between $(\sigma_4 \otimes \sigma_3') \otimes (\sigma_2 \otimes \sigma_1')$ and $(\tau_4 \otimes \tau_1') \otimes (\tau_2 \otimes \tau_3')$. Moreover, in the case where $\tau_4 = \tau_1$ and $\tau_3 = \tau_2$, we have

$$H[(\sigma_1 \otimes \sigma_2') \otimes (\sigma_3 \otimes \sigma_4')] = H[(\sigma_4 \otimes \sigma_3') \otimes (\sigma_2 \otimes \sigma_1')].$$

(6.3)

The relevance of this proposition lies in the fact that if we find a path from $(v_0 \otimes v_0') \otimes (v_0 \otimes v_0')$ to $(v_1 \otimes v_{1}') \otimes (v_1 \otimes v_{1}')$, then we immediately have a path from $(v_0 \otimes v_0') \otimes (v_0 \otimes v_0')$ to $(v_k \otimes v_{k}') \otimes (v_k \otimes v_{k}')$ as well, by reversing the edges and taking the symmetric of the vertices in the path. By (6.3), this gives the following symmetry on the energy function:

$$H[(v_0 \otimes v_{k'}) \otimes (v_1 \otimes v_{1'})] = H[(v_k \otimes v_{k'}) \otimes (v_0 \otimes v_{0'})].$$
Besides, by \((1.3)\), we have
\[
\Delta(a_k b_l; a_{k'} b_{l'}) = \chi(k \geq k') - \chi(k = l = k') + \chi(l \leq l') - \chi(l = k' = l')
\]
\[
= \begin{cases} 
\chi(k > k') + \chi(l < l') & \text{if } l = k' \\
\chi(k \geq k') + \chi(l \leq l') & \text{if } l \neq k'
\end{cases}
\]
and then
\[
\Delta(a_k b_l; a_{k'} b_{l'}) = \Delta(a_{k'} b_{k'}; a_{b} b_{k}).
\]
Therefore, if we prove that \(H[(v_k \otimes v_{k'}) \otimes (v_l \otimes v_{l'})] = \Delta(a_k b_l; a_{k'} b_{l'})\), we equivalently have \(H[(v_k \otimes v_{k'}) \otimes (v_{k'} \otimes v_{l'})] = \Delta(a_{k'} b_{k'}; a_{b} b_{k})\). Thus, to prove Theorem \([1.3]\) in Section \([4]\) we will distinguish several cases according to some relations between \(k, k', l, l'\), and by interchanging \(k \equiv l'\) and \(k' \equiv l\), the symmetry will then imply the remaining cases.

**Proof of Proposition \([6.7]\)** First, let us recall \([4.6]\). For all \(v \in \mathcal{B}\) and \(i \in \{0, \ldots, n - 1\}\), we have:
\[
(f_i v^\vee, e_i v^\vee, \varphi_i v^\vee, \varepsilon_i v^\vee) = ((e_i v)^\vee, (f_i v)^\vee, \varepsilon_i v, \varphi_i v),
\]
so that \(\overline{w_i v^\vee} = -\overline{w_i v}\).

The tensor rules on \(\mathcal{B}\) are given by:
\[
\begin{align*}
\hat{e}_i(\sigma_1 \otimes \sigma_2^\vee) & = \begin{cases} 
\hat{e}_i \sigma_1 \otimes \sigma_2^\vee & \text{if } \varphi_i(\sigma_1) \geq \varphi_i(\sigma_2) \\
\sigma_1 \otimes \hat{e}_i \sigma_2^\vee & \text{if } \varphi_i(\sigma_1) < \varphi_i(\sigma_2),
\end{cases} \\
\hat{f}_i(\sigma_1 \otimes \sigma_2^\vee) & = \begin{cases} 
\hat{f}_i \sigma_1 \otimes \sigma_2^\vee & \text{if } \varphi_i(\sigma_1) > \varphi_i(\sigma_2) \\
\sigma_1 \otimes \hat{f}_i \sigma_2^\vee & \text{if } \varphi_i(\sigma_1) \leq \varphi_i(\sigma_2),
\end{cases}
\end{align*}
\]
or equivalently,
\[
\begin{align*}
\check{f}_i(\sigma_2 \otimes \sigma_1^\vee) & = \begin{cases} 
\check{f}_i \sigma_2 \otimes \sigma_1^\vee & \text{if } \varphi_i(\sigma_2) > \varphi_i(\sigma_1) \\
\sigma_2 \otimes \check{f}_i \sigma_1^\vee & \text{if } \varphi_i(\sigma_2) \leq \varphi_i(\sigma_1),
\end{cases} \\
\check{e}_i(\sigma_2 \otimes \sigma_1^\vee) & = \begin{cases} 
\check{e}_i \sigma_2 \otimes \sigma_1^\vee & \text{if } \varphi_i(\sigma_2) \geq \varphi_i(\sigma_1) \\
\sigma_2 \otimes \check{e}_i \sigma_1^\vee & \text{if } \varphi_i(\sigma_2) < \varphi_i(\sigma_1).
\end{cases}
\end{align*}
\]
Consider the involution \(\eta\) defined by
\[
\eta : \mathcal{B} \sqcup \{0\} \to \mathcal{B} \sqcup \{0\}
\]
\[
\begin{align*}
0 & \mapsto 0 \\
\sigma_1 \otimes \sigma_2^\vee & \mapsto \sigma_2 \otimes \sigma_1^\vee.
\end{align*}
\]
The tensor rules on \(\mathcal{B}\) give, for all \(i \in \{0, \ldots, n - 1\}\),
\[
(\eta \circ \hat{e}_i, \eta \circ \hat{f}_i) = (\check{f}_i \circ \eta, \check{f}_i \circ \eta),
\]
so that
\[
(\varepsilon_i \circ \eta, \varphi_i \circ \eta) = (\varepsilon_i, \varphi_i).
\]
By \([258]\), we obtain, for all \(\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in \mathcal{B}\),
\[
\varphi_i(\sigma_1 \otimes \sigma_2^\vee) > \varepsilon_i(\sigma_3 \otimes \sigma_4^\vee) \iff \check{f}_i((\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee)) = \check{f}_i(\sigma_1 \otimes \sigma_2^\vee) \otimes (\sigma_3 \otimes \sigma_4^\vee)
\]
By symmetry of the action of \(\eta\), we deduce
\[
\varphi_i(\sigma_1 \otimes \sigma_2^\vee) > \varepsilon_i(\sigma_3 \otimes \sigma_4^\vee) \iff \varphi_i(\eta(\sigma_3 \otimes \sigma_4^\vee)) < \varepsilon_i(\eta(\sigma_1 \otimes \sigma_2^\vee))
\]
\[
\iff \hat{e}_i(\eta(\sigma_3 \otimes \sigma_4^\vee) \otimes \eta(\sigma_1 \otimes \sigma_2^\vee)) = \eta(\sigma_3 \otimes \sigma_4^\vee) \otimes \hat{e}_i \circ \eta(\sigma_1 \otimes \sigma_2^\vee)
\]
\[
\iff \hat{e}_i(\eta(\sigma_3 \otimes \sigma_4^\vee) \otimes \eta(\sigma_1 \otimes \sigma_2^\vee)) = \eta(\sigma_3 \otimes \sigma_4^\vee) \otimes \eta \circ \hat{f}_i(\sigma_1 \otimes \sigma_2^\vee).
\]
We also obtain that
\[ \varphi_i(\sigma_1 \otimes \sigma_2^\prime) \geq \varepsilon_i(\sigma_3 \otimes \sigma_4^\prime) \quad \text{and} \quad \tilde{f}_i(\sigma_1 \otimes \sigma_2^\prime) \neq 0 \]
\[ \iff H[(\sigma_1 \otimes \sigma_2^\prime) \otimes (\sigma_3 \otimes \sigma_4^\prime)] = \chi(i = 0) \]
\[ \iff H[\eta(\sigma_3 \otimes \sigma_4^\prime) \otimes \eta(\sigma_1 \otimes \sigma_2^\prime)] = \chi(i = 0). \]

In the other case we have
\[ \varphi_i(\sigma_1 \otimes \sigma_2^\prime) \leq \varepsilon_i(\sigma_3 \otimes \sigma_4^\prime) \quad \text{and} \quad \tilde{f}_i(\sigma_3 \otimes \sigma_4^\prime) \neq 0 \]
\[ \iff H[(\sigma_1 \otimes \sigma_2^\prime) \otimes (\sigma_3 \otimes \sigma_4^\prime)] = -\chi(i = 0) \]
\[ \iff H[\eta(\sigma_3 \otimes \sigma_4^\prime) \otimes \eta(\sigma_1 \otimes \sigma_2^\prime)] = -\chi(i = 0), \]
and we obtain (6.1) and (6.2).

Let us now define the involution
\[ \zeta : \bigoplus \mathbb{B} \otimes \mathbb{B} \cup \{0\} \rightarrow \bigoplus \mathbb{B} \otimes \mathbb{B} \cup \{0\} \]
\[ (\sigma_1 \otimes \sigma_2^\prime) \otimes (\sigma_3 \otimes \sigma_4^\prime) \mapsto (\sigma_4 \otimes \sigma_3^\prime) \otimes (\sigma_2 \otimes \sigma_1^\prime). \]

By (6.1), we see that \( \tilde{e}_i \circ \zeta = \zeta \circ \tilde{f}_i \) and \( \tilde{f}_i \circ \zeta = \zeta \circ \tilde{e}_i \). Thus for all \( g_1, \ldots, g_s \in \{\tilde{e}_i, \tilde{f}_i : i \in \{0, \ldots, n - 1\}\} \), we have
\[ \zeta \circ g_1 \circ \cdots \circ g_s = \overline{g}_s \circ \cdots \circ \overline{g}_1 \circ \zeta, \]
where \( \overline{f}_i = \tilde{e}_i \) and \( \overline{e}_i = \tilde{f}_i \). Therefore, for \( b, b' \in \bigoplus \mathbb{B} \), we have
\[ g_1 \circ \cdots \circ g_s(b) = b' \iff \overline{g}_s \circ \cdots \circ \overline{g}_1(\zeta(b)) = \zeta(b'), \]
so that there is a path between two vertices if and only if there is a path between their images by \( \zeta \). By (6.2), we also observe that
\[ H(b) - H(b') = H(b) - H(g_s(b)) + H(g_s(b)) - H(g_{s-1} \circ g_s(b)) + \cdots + H(g_2 \circ \cdots \circ g_s(b)) - H(b') \]
\[ = H(\zeta(b)) - H(\overline{g}_s(\zeta(b))) + H(\overline{g}_s(\zeta(b))) - H(\overline{g}_{s-1} \circ \overline{g}_s(b)) + \cdots + H(\overline{g}_2 \circ \cdots \circ \overline{g}_s(\zeta(b)) - H(\zeta(b'))) \]
\[ = H(\zeta(b)) - H(\zeta(b')). \]

Choosing any \( b' \) such that \( b' = \zeta(b') \) (which means that the vertex \( b' \) is its own symmetric) gives (6.3). \( \square \)

6.2. Redefining the minimal differences \( \Delta \). To build a path from \((v_0 \otimes v_0') \otimes (v_0 \otimes v_0')\) to \((v_0 \otimes v_0') \otimes (v_0 \otimes v_0')\) and show that
\[ H[(v_0 \otimes v_0') \otimes (v_0 \otimes v_0')] = \Delta(a_k b_l), \]
we will distinguish the cases \( k' = l \) and \( k' \neq l \). But first, let us define a tool which will make our problem easier to solve.

**Definition 6.2.** Let us identify \( \{0, \ldots, n - 1\} \) with \( \mathbb{Z}/n\mathbb{Z} \), and consider the natural order on \( \{0, \ldots, n - 1\} \):
\[ 0 < 1 < \cdots < n - 2 < n - 1. \]
We also define, for all \( i, j \in \{0, \ldots, n - 1\} \), the intervals
\[ \text{int}(i, j) := \{i + 1, i + 2, \ldots, j - 1, j\}. \]

**Lemma 6.3.** For all \( i \in \{0, \ldots, n - 1\} \), we have the following:
\[ i < i - 1 \iff i = 0, \]
\[ \text{int}(i, i) = \{0, \ldots, n - 1\}, \]
\[ I \setminus \text{int}(i, j) = \text{int}(i, j) \iff i \neq j, \]
\[ 0 \notin \text{int}(j, i) \iff j < i, \]
\[ 0 \in \text{int}(i, j) \iff j \leq i. \]
The aim of this lemma is to rewrite the difference conditions \( \Delta \) according to the fact that 0 belongs to some interval or not. By (6.4), \( \Delta \) can be reformulated as follows:

\[
\Delta(a_k b_l; a_{k'} b_{l'}) = \begin{cases} 
\chi(0 \notin \text{int}(k', k)) + \chi(0 \notin \text{int}(l, l')) & \text{if } l = k' \\
\chi(0 \in \text{int}(k, k')) + \chi(0 \in \text{int}(l', l)) & \text{if } l \neq k'.
\end{cases}
\]  

(6.6)

**Proof of Lemma 6.3.** The first equivalence is straightforward, since \( i > i - 1 \) if and only if \( i \neq 0 \), and \( 0 < n - 1 = -1 \).

The second equality follows from the definition of \( \text{int} \), since we go around \( \{0, \ldots, n - 1\} \).

Note that

\[
\text{int}(i, j) = \{i + 1, i + 2, \ldots, j - 1, j\},
\]

while

\[
\text{int}(j, i) = \{j + 1, j + 2, \ldots, i - 1, i\},
\]

and if \( i \neq j \), these two sets are complementary in \( \{0, \ldots, n - 1\} \). Moreover, when \( i \neq j \), we have \( i \in \text{int}(j, i) \) and \( j \in \text{int}(i, j) \), so that both sets never equal \( \emptyset \) nor \( \{0, \ldots, n - 1\} \). Otherwise, when \( i = j \), they both equal \( \{0, \ldots, n - 1\} \). This gives the third equivalence.

For the fourth equivalence, the fact that \( 0 \in \{0, \ldots, n - 1\} \) gives

\[
0 \notin \text{int}(j, i) \iff 0 \notin \{j + 1, i + 2, \ldots, j - 1, i\},
\]

\[
\iff j \neq i \text{ and } \emptyset \neq \{j + 1, j + 2, \ldots, i - 1, i\} \subseteq \{1, \ldots, n - 1\}
\]

\[
\iff j < j + 1 \leq i.
\]

Finally, for the last equivalence, we note that

\[
\chi(j \leq i) = \chi(j < i) + \chi(j = i)
\]

\[
= \chi(j < i)\chi(j \neq i) + \chi(j = i)
\]

\[
= \chi(0 \notin \text{int}(j, i))\chi(i \neq j) + \chi(i = j)
\]

\[
= \chi(0 \in \text{int}(i, j))\chi(i \neq j) + \chi(i = j)\chi(0 \in \text{int}(i, i)).
\]

This concludes the proof. \( \square \)

7. **Proof of Theorem 1.4**

We are now ready to build the paths in \( \mathbb{B} \otimes \mathbb{B} \), and use them to compute the energy function \( H[(v_l \otimes v_{l'}) \otimes (v_k \otimes v_{k'})] \). We will use the relations in (4.8) and the local configurations of the vertices as defined in (4.6). The symmetric of \( (v_l \otimes v_{l'}) \otimes (v_k \otimes v_{k'}) \) is \( (v_k \otimes v_{l'}) \otimes (v_{k'} \otimes v_l) \), obtained by interchanging \( k' \equiv l, l' \equiv k \).

We distinguish several cases:

1. \( k' = l' \) and \( l = k \),
2. \( k' = l \neq k = l' \),
3. \( k' = l \) and \( k \neq l' \),
4. \( k' \neq k = l = l' \) (Symmetric: \( l \neq k = k' = l' \)),
5. \( l' \neq k' = k \neq l \) (Symmetric: \( k \neq l = l' \neq k' \)),
6. \( k \neq k', k' \neq l \) and \( l \neq l' \)
   - (a) \( k + 1, k' \notin \text{int}(l, l') \) (Symmetric: \( l' + 1, l \notin \text{int}(k', k) \)),
   - (b) \( k + 1 \in \text{int}(l, l') \) and \( k' \notin \text{int}(l, l') \) (Symmetric: \( l' + 1 \in \text{int}(k', k) \) and \( l \notin \text{int}(k', k) \)),
   - (c) \( k + 1 \notin \text{int}(l, l') \) and \( k' \in \text{int}(l, l') \) (Symmetric: \( l' + 1 \notin \text{int}(k', k) \) and \( l \in \text{int}(k', k) \)),
   - (d) \( k + 1, k' \in \text{int}(l, l') \) and \( l' + 1, l \in \text{int}(k', k) \).

7.1. **The case \( k' = l' \) and \( l = k \).** We build a path from \( (v_k \otimes v_{l'}) \otimes (v_k' \otimes v_{l'}) \) to \( (v_k \otimes v_{l'}) \otimes (v_l \otimes v_{l'}) \).

We consider the case \( k' \neq l \), as otherwise the two elements are the same. By (4.9), we have

\[
\varphi_i(v_k \otimes v_{l'}) = \varepsilon_i(v_k \otimes v_{l'}) = \chi(i = k').
\]
By the tensor rules (2.6), we then obtain the path
\[
(v_k' \otimes v'_k) \otimes (v_k' \otimes v'_L) \xrightarrow{k'} (v_k' \otimes v'_k) \otimes (v_k' \otimes v'_{k-1}) \xrightarrow{k-1} \cdots \xrightarrow{l+1} (v_k' \otimes v'_k) \otimes (v_k' \otimes v'_l) \xrightarrow{k'+1}
\]
This path is only made of forward moves \(\tilde{f}_i\), with \(i \in \text{int}(l, k') \cup \text{int}(k', l)\) appearing once, where we change the right side of the tensor products. By (2.8), we then have
\[
H[(v_k' \otimes v'_k) \otimes (v_k' \otimes v'_L)] - H[(v_k' \otimes v'_k) \otimes (v_k' \otimes v'_L)] = \chi(0 \in \text{int}(l, k')) + \chi(0 \in \text{int}(k', l)) = 1.
\]
By (6.3), we have the symmetry
\[
H[(v_l \otimes v'_l) \otimes (v_k' \otimes v'_L)] = H[(v_k' \otimes v'_k) \otimes (v_l \otimes v'_l)].
\]
We now build a path from \((v_l \otimes v'_l) \otimes (v_k' \otimes v'_L)\) to \((v_l \otimes v'_l) \otimes (v_l \otimes v'_L)\) and \((v_l \otimes v'_l) \otimes (v_l \otimes v'_L)\), and
\[
H[(v_k' \otimes v'_k) \otimes (v_l' \otimes v'_l)] = H[(v_l' \otimes v'_l) \otimes (v_l' \otimes v'_l)].
\]
Recall that by definition, \(H[(v_0 \otimes v'_0) \otimes (v_0 \otimes v'_0)] = 0\). Thus setting \(k' = 0\) yields by (6.6) that for all \(l \in \{0, \ldots, n-1\}\),
\[
H[(v_l \otimes v'_l) \otimes (v_l \otimes v'_L)] = 0 = 2\chi(0 \notin \text{int}(l, l)) = \Delta(a_l b_l; a_k b_k) - \Delta(a_k b_l; a_k b_k).
\]
Plugging this into (7.1) gives, for all \(k' \neq l\),
\[
H[(v_k' \otimes v'_k) \otimes (v_l \otimes v'_L)] = 1 = \chi(0 \in \text{int}(l, k')) + \chi(0 \in \text{int}(k', l')) = \Delta(a_l b_k; a_k b_l).
\]
### 7.2. The case \(k' = l \neq k' = l'\).

We now build a path from \((v_l \otimes v'_l) \otimes (v_k \otimes v'_k)\) to \((v_l \otimes v'_l) \otimes (v_k \otimes v'_k)\). By (4.19), we know that \(\varepsilon_i(v_k \otimes v'_k) = 1\) (\(i = k\)) and \(\varepsilon_i(v_k \otimes v'_k) = 0\) if \(i \not\in \{l, k\}\). Since \(k \neq l\), we have for all \(i \in \text{int}(l, k)\) that \((v_l \otimes v'_l) \neq (v_l \otimes v'_{l+1})\), and then \((v_l \otimes v'_l) \xrightarrow{k} (v_l \otimes v'_{l+1})\). We obtain the path
\[
(v_l \otimes v'_l) \otimes (v_k \otimes v'_k) \xrightarrow{k} (v_l \otimes v'_l) \otimes (v_k \otimes v'_{k-1}) \xrightarrow{k-1} \cdots \xrightarrow{l+1} (v_l \otimes v'_l) \otimes (v_k \otimes v'_l) \xrightarrow{k'=1+k}
\]
In the upper part of the path, we moved forward (by some \(\tilde{f}_i\)) by modifying the right side of the tensor product with arrows in \(\text{int}(l, k)\) appearing once. Then, in the lower part of the path, we moved forward by modifying the left side of the tensor product with arrows in \(\text{int}(k, l)\) appearing once. Using that \(k \neq l\), the energy function satisfies:
\[
H[(v_l \otimes v'_l) \otimes (v_k \otimes v'_k)] = H[(v_l \otimes v'_l) \otimes (v_k \otimes v'_k)] + \chi(0 \in \text{int}(l, k)) - \chi(0 \in \text{int}(k, l)) = 1 + 2\chi(0 \in \text{int}(l, k)) - 1 = 2\chi(0 \in \text{int}(l, k)).
\]
7.3. The case $k' = l$ and $k \neq l'$. The vertices $(v_{l'} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{k}^\prime)$ and $(v_{k} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{k}^\prime)$ are symmetric.

Since $k \neq l'$, we have that $\text{int}(k, l) \neq \text{int}(l', l)$. By symmetry, we can assume that $\text{int}(l', l) \nsubseteq \text{int}(k, l)$ and so $l' + 1 \notin \text{int}(k, l)$. In that case, we necessarily have $k \neq l$. Then, $\varphi_l(v_{l'} \otimes v_{l''}^\prime) = 1 = \varepsilon_{l}(v_{l'} \otimes v_{l''}^\prime)$ and $\varphi_{l'}(v_{l'} \otimes v_{l''}^\prime) = 0$ for all $i \in \text{int}(k, l) \setminus \{l\}$, and we have the path

$$
(v_{l} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{k}^\prime) \xleftarrow{\text{empty if } l = l'} v_{l-1} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{k}^\prime) \xleftarrow{\text{empty if } l = l'} \ldots \xleftarrow{\text{empty if } l = l'} v_{l+1} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{k}^\prime)
$$

and the energy function is given by

$$
H[(v_{l'} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{k}^\prime)] = \chi(l' \neq l) \chi(0 \in \text{int}(l', l)) + \chi(0 \in \text{int}(k, l))
$$

by (2.8)

$$
= \chi(0 \notin \text{int}(l', l)) + \chi(0 \notin \text{int}(l, k))
$$

by Lemma 6.3

$$
= \Delta(a_k b_l; a_l b_k)
$$

by (6.0).

This was the last case where $k' = l$. Also, we have already studied a special case where $k' \neq l$, which was the case $l' = k' \neq l = k$. We now study the other cases where $k' \neq l$.

7.4. The case $k' \neq k = l' = l'$. Since $l \notin \text{int}(l, k')$, we have the path

$$
(v_{l+1} \otimes v_{l+1}^\prime) \otimes (v_{l} \otimes v_{l''}^\prime) \xleftarrow{\text{empty if } k' = l+1} v_{l+1} \otimes v_{l+1}^\prime) \otimes (v_{l} \otimes v_{l''}^\prime) \xleftarrow{\text{empty if } k' = l+1} \ldots \xleftarrow{\text{empty if } k' = l+1} v_{l+k} \otimes v_{l+k}^\prime)
$$

Thus the energy function satisfies

$$
H[(v_{l} \otimes v_{l}^\prime) \otimes (v_{l} \otimes v_{k}^\prime)] = 1 + \chi(0 \in \text{int}(l, k'))
$$

by (2.8) and (7.3)

$$
= \chi(0 \in \text{int}(l, l')) + \chi(0 \in \text{int}(k', k'))
$$

by Lemma 6.3

$$
= \Delta(a_l b_{l'} b_k; a_k b_{l'})
$$

by (6.0).

7.5. The case $l' \neq k' = k \neq l'$. Since $l' \neq k'$, it means that

$$
\text{int}(l', k') \cup \text{int}(k', l) = \text{int}(l', l).
$$

Since $l' + 1$ and $k'$ do not belong to $\text{int}(k', l)$, we have by (1.9) that $\varphi_i(v_{l'} \otimes v_{l''}^\prime) = 0$ for all $i \in \text{int}(k', l)$.

This gives the path

$$
(v_{l' + 1} \otimes v_{l' + 1}^\prime) \otimes (v_{k'} \otimes v_{k' + 1}^\prime) \xleftarrow{\text{empty if } k' = l'+1} v_{l' + 1} \otimes v_{l' + 1}^\prime) \otimes (v_{k'} \otimes v_{k' + 1}^\prime) \xleftarrow{\text{empty if } k' = l'+1} \ldots \xleftarrow{\text{empty if } k' = l'+1} v_{k' + 1} \otimes v_{k' + 1}^\prime)
$$

We deduce the following formula for the energy function:

$$
H[(v_{l'} \otimes v_{l''}^\prime) \otimes (v_{l} \otimes v_{l''}^\prime)] = 1 + \chi(0 \in \text{int}(l', k')) + \chi(0 \in \text{int}(k', l))
$$

by (2.8) and (7.3)

$$
= \chi(0 \in \text{int}(k', k')) + \chi(0 \in \text{int}(l', l))
$$

by Lemma 6.3

$$
= \Delta(a_{k'} b_{l'} b_{k} b_{l'}; a_k b_{k'} b_{l'})
$$

by (6.0).

Let us now assume that $l' + 1 \in \text{int}(k', l)$. Since $\text{int}(k', l) \neq \emptyset$ and $l' \neq k'$, we necessarily have that $k' + 1 \neq l$ and $\text{int}(k', l') \subset \text{int}(k', l - 1)$, so that $l' \neq l$. Note also that, by (4.9),

$$
\varphi_{k'}(v_{l'} \otimes v_{l''}^\prime) = 0 = \varepsilon_{k'}(v_{k' - 1} \otimes v_{k'}^\prime),
$$

and
since \( k' \neq l' + 1 \), and \( \varphi_i(v_l \otimes v_{k'}^v) = 0 \) for all \( i \in \text{int}(l, k') \setminus \{ k' \} \). We then have the path
\[
(v_{k'} \otimes v_{k' - 1}^v) \otimes (v_{k'} \otimes v_{k' - 1}^v) \xrightarrow{k'} (v_{k'} \otimes v_{k' - 1}^v) \otimes (v_{k'} \otimes v_{k' - 1}^v) \xrightarrow{k' + 1} \cdots \xrightarrow{l'} (v_{k'} \otimes v_{k' - 1}^v) \otimes (v_{k'} \otimes v_{k' - 1}^v)
\]
nonempty since \( k' \neq l' + 1 \)

By the previous case \( (l' \neq k' = k \neq l) \), we obtain the energy function
\[
H[(v_{k'} \otimes v_{k' - 1}^v) \otimes (v_{k'} \otimes v_{k' - 1}^v)] = \chi(0 \in \text{int}(k', k')) + \chi(0 \in \text{int}(k' - 1, k' - 1)) = 2\chi(0 \in \text{int}(k', k')). \tag{7.4}
\]
In the computation of \( H \), by (2.8), the moves marked by \( \ast \) cancel each other, since it is the same arrow that operates backward consecutively on the right and on the left side of the tensor product. Besides, the moves marked by \( \bullet \) give \( \text{int}(l, k') \) and operate backward on the right side of the tensor product. As a consequence,
\[
H[(v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v)] \xrightarrow{l'} \cdots \xrightarrow{l' + 1} (v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v).
\]

By Case 7.4 and the symmetric of Case 7.5, we have
\[
H[(v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v)] = \chi(0 \in \text{int}(k', k')) + \chi(0 \in \text{int}(l', l')), \tag{7.5}
\]
and the energy function becomes
\[
H[(v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v)] = H[(v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v)] - \chi(0 \in \text{int}(l', l')) \tag{7.6}
\]
and (7.5) by Lemma 6.3
\[
= \chi(0 \in \text{int}(k', k')) + \chi(0 \in \text{int}(l', l')) \tag{7.6}
\]
and (7.5) by Lemma 6.3.

7.6. The case \( k \neq k' \), \( k' \neq l \) and \( l \neq l' \).

7.6.1. The sub-case \( k + 1, k' \notin \text{int}(l, l') \) (Symmetric case: \( l' + 1, l \notin \text{int}(k', k) \)). We have \( l' + 1, k' \notin \text{int}(l, l') \), so that \( \varphi_i(v_l \otimes v_{k'}^v) = 0 \) for all \( i \in \text{int}(l, l') \). Besides, \( k + 1 \notin \text{int}(l, l') \), so that \( \bar{c}_i(v_l \otimes v_{k'}^v) = (v_{l - 1} \otimes v_{k'}^v) \). We obtain the path
\[
(v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v) \xlongequal{l' + 1} \cdots \xrightarrow{l + 1} (v_l \otimes v_{k'}^v) \otimes (v_l \otimes v_{k'}^v).
\]

Note that the moves marked by \( \bullet \) cancel each other, and the moves marked by \( \ast \) give \( \text{int}(l, l') \), so that the calculation is the same as in the previous case.
7.6.3. The sub-case $k + 1 \notin \text{int}(l, l')$ and $k' \in \text{int}(l, l')$ (Symmetric case: $l' + 1 \notin \text{int}(k', k)$ and $l \in \text{int}(k', k)$). We have $l, k + 1 \notin \text{int}(l, l')$, so that $\varepsilon_i(v_l \otimes v_{k'}^j) = 0$ for all $i \in \text{int}(l, l')$. Note that $k' + 1 \in \text{int}(l, l')$, since $k' \in \text{int}(l, l')$ and $k' \neq l'$. This gives the path

\[
(v_l \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j) \xrightarrow{\ell + 1} \cdots \xrightarrow{k'} (v_{k'} \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j) \xrightarrow{k'} (v_{k'} \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j)
\]

As before, the moves marked by $\bullet$ cancel each other, and the moves $\ast$ give $\text{int}(l, l')$. We move with the $\tilde{f}_i$’s by changing the left side of the tensor product, and we get

\[
H[(v_l \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j)] = H[(v_l \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j)] - \chi(0 \in \text{int}(l, l'))
\]

by 2.6.

7.6.4. The sub-case $k + 1, k' \in \text{int}(l, l')$ and $l' + 1, l \in \text{int}(k', k)$. Note that this case overlaps with the case $k' = l' \neq k = l$ that we already checked in the first part. Omitting that case, we can assume by symmetry that $k \neq l$. We obtain the path

\[
(v_l \otimes v_{k'}^j) \otimes (v_k \otimes v_{k'}^j) \xrightarrow{\ell'} \cdots \xrightarrow{k' + 1} (v_{k'} \otimes v_{k'}^j) \otimes (v_k \otimes v_{k'}^j) \xrightarrow{k} \cdots \xrightarrow{l + 1} (v_l \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j).
\]

Since $k \neq l$, the fact that $l \in \text{int}(k', k)$ implies that $\text{int}(k', k) = \text{int}(k', l) \cup \text{int}(l, k)$, and the fact that $k + 1 \in \text{int}(l, l')$ implies that $\text{int}(l, l') = \text{int}(l, k) \cap \text{int}(k', l')$, so that $k', l' + 1 \notin \text{int}(l, k)$. Also, if $k' \neq l'$, then $l' + 1 \in \text{int}(k', k)$ implies that $\text{int}(k', l) = \text{int}(k', l') \cup \text{int}(l', k)$, so that $k \notin \text{int}(k', l')$. Since $l \neq l'$ and $k' \neq l$, the fact that $k' \in \text{int}(l, l')$ implies that

\[
\text{int}(l', k') = \text{int}(l', l) \cup \text{int}(l, k'),
\]

and the fact that $l \in \text{int}(k', k)$ and $l \neq k$ implies that

\[
\text{int}(l, k') = \text{int}(l, k) \cup \text{int}(k, k').
\]

Thus the computation of $H$ gives

\[
H[(v_l \otimes v_{k'}^j) \otimes (v_l \otimes v_{k'}^j)] = 1 - \chi(k' \neq l') \chi(0 \in \text{int}(k', l')) - \chi(0 \in \text{int}(l, k))
\]

by 2.8 and Lemma 6.3.

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