Instanton Correction of Prepotential in Ruijsenaars Model Associated with $N = 2$ SU(2) Seiberg-Witten Theory

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Abstract

Instanton correction of prepotential of one-dimensional SL(2) Ruijsenaars model is presented with the help of Picard-Fuchs equation of Pakuliak-Perelomov type. It is shown that the instanton induced prepotential reduces to that of the SU(2) gauge theory coupled with a massive adjoint hypermultiplet.

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I. INTRODUCTION

The low energy effective action of $N = 2$ supersymmetric Yang-Mills theory is described by a prepotential and if this is obtained we can know (all) informations concerning to the effective theory, namely, in this case the theory becomes “solvable”. Actually, instantons have been known to contribute to the prepotential\textsuperscript{1} but not so much discussions were made. However, Seiberg and Witten\textsuperscript{2} pointed out several years ago that it was possible to determine the prepotential including instanton effects with the help of a Riemann surface and periods of a meromorphic 1-form on it. This approach using a Riemann surface is in general referred to Seiberg-Witten theory.

The effective action of gauge theory occasionally including massive hypermultiplets in the fundamental representation of the gauge group has been discussed in many view points, and accordingly, we have now much acquaintance with various properties of the prepotential and it’s related materials, such as Picard-Fuchs equations for periods\textsuperscript{3,4,5,6} renormalization group like equation for prepotential\textsuperscript{7,8,9,10} relation to integrable systems,\textsuperscript{11,12} appearance of WDVV equations\textsuperscript{13,14,15} flat coordinates\textsuperscript{16,17} and so on.

On the contrary, for a theory coupled with adjoint hypermultiplets\textsuperscript{21} not so much compared to the above cases including fundamental hypermultiplets are revealed, but there are strong supports that the relevant Seiberg-Witten solutions must be related with the integrable Calogero dynamical systems\textsuperscript{22} and in fact, a few examples of prepotential associated with spectral curves expected from Calogero systems showed a good prediction of the instanton correction\textsuperscript{23}

On the other hand, recently, Braden et al.\textsuperscript{24} analyzed a more general integrable system, called Ruijsenaars model, which can be thought as a “relativistic” version of Calogero system\textsuperscript{25} The Ruijsenaars model itself is considered as a candidate of integrable system related to five-dimensional gauge theory\textsuperscript{26} and it’s specific limits are known to recover Seiberg-Witten solutions in four and five dimensions. At the perturbative level, we can easily establish these correspondences, but we can not conclude that the Ruijsenaars model is in fact the integrable system relevant to these gauge theories unless instanton contribution is correctly taken into account. Therefore, the only necessary item to be discussed is now an exact treatment of this model including instanton effects. From this reason, we derive the instanton contribution to the prepotential of Ruijsenaars model in this paper.

The paper is organized as follows. In Sec.II, we briefly review the correspondence among the
Seiberg-Witten solution to the SU(2) gauge theory and Calogero and Ruijsenaars systems. In Sec.III, we consider the Picard-Fuchs equations for these integrable systems, but actually they are found to be available from Pakuliak-Perelomov equations. In Sec. IV, the period integrals and the effective coupling constant of the Ruijsenaars model are evaluated in the weak coupling regime. In Sec. V, a differential equation for the prepotential is constructed from the Pakuliak-Perelomov type Picard-Fuchs equation. One-instanton correction of the prepotential is presented. As a check, we consider a reduction to the prepotential of Calogero system and show that the instanton induced prepotential is consistently determined. Sec.VI is a brief summary.

II. CALOGERO AND RUIJSENAARS SYSTEMS

A. Calogero system

To begin with, notice that the Seiberg-Witten solution to the SU(2) Yang-Mills gauge theory coupled with a massive adjoint matter hypermultiplet is related to the spectral curve

$$\det(\mathcal{L}(\xi) - t) = 0$$

(2.1)

of one-dimensional SL(2) elliptic Calogero system with the $2 \times 2$ Lax matrix with Calogero coupling constant $g_0$

$$\mathcal{L}(\xi) = \begin{pmatrix} P & g_0 \frac{\sigma(Q + \xi)}{\sigma(\xi)\sigma(Q)} \\ g_0 \frac{\sigma(Q + \xi)}{\sigma(\xi)\sigma(Q)} & -P \end{pmatrix}, \quad P = p_1 = -p_2, \quad Q = q_1 - q_2, \quad (2.2)$$

where $p_i$ and $q_i$ are the canonical coordinate and momentum, respectively, $\sigma$ is the Weierstrass’s $\sigma$ function (see Appendix A) and $\xi$ is the spectral parameter associated with an elliptic curve

$$y^2 = \prod_{i=1}^{3} (x - e_i), \quad \sum_{i=1}^{3} e_i = 0. \quad (2.3)$$

In (2.3), branching points $e_i$ are functions in the modulus $\tau$ of (2.3) and have the expansion

$$e_1 = \frac{2}{3} (1 + 24q^2 + 24q^4 + \cdots),$$
$$e_2 = -\frac{1}{3} (1 + 24 + 24q^2 + 96q^3 + \cdots),$$
$$e_3 = -\frac{1}{3} (1 - 24q + 24q^2 - 96q^3 + \cdots). \quad (2.4)$$
where $q = e^{i\pi \tau}$. Since $\tau$ is identified with the bare effective coupling $\tau = 4\pi i/g^2 + \theta/2\pi$, where $g$ is the gauge coupling constant and $\theta$ is the vacuum angle, the factor $q^{2n}$ for $n \in \mathbb{N}$ corresponds to the instanton amplitude.

This spectral equation can be summarized into

$$g_0^2 \varphi(\xi) = t^2 - h, \quad h = P^2 + \varphi(Q) \quad (2.5)$$

where $\varphi(\xi)$ is the Weierstrass’s $\wp$ function and $h$ is the second Hamiltonian of this system.

Then the Seiberg-Witten differential $dS_{\text{Cal}}$ is given in the form of twice of a product of the eigenvalue $t$ of the Lax matrix and a holomorphic 1-form $d\omega = dx/y$ on (2.3), namely,

$$dS_{\text{Cal}} = 2td\omega = \frac{2\sqrt{h + g_0^2 x}}{y} dx, \quad (2.6)$$

where we have identified $x = \varphi(\xi)$. In general, Seiberg-Witten differential has the property such that it reduces to a holomorphic differential by a differentiation over moduli, and in fact for the case at hand, $\partial dS_{\text{Cal}} / \partial h \propto dx/y$. Below, we set $g_0 = 1$ for convenience.

For this differential, the Seiberg-Witten periods can be defined by

$$a = \oint_{\alpha} dS_{\text{Cal}}, \quad a_D = \partial \tilde{F} / \partial a = \oint_{\beta} dS_{\text{Cal}}, \quad (2.7)$$

where $\alpha$ and $\beta$ are the canonical basis of 1-cycles and $\tilde{F}$ is the prepotential.

Itoyama and Morozov\cite{Itoyama:1997} made a very interesting observation with respect to this Seiberg-Witten solution, which states that the data can be viewed as if they were given on the hyperelliptic curve

$$\tilde{y}^2 = (h + x) \prod_{i=1}^3 (x - e_i) \quad (2.8)$$

and the associated Seiberg-Witten differential

$$dS_{\text{Cal}} = \frac{2(h + x)}{\tilde{y}} dx \quad (2.9)$$

on (2.8). This observation is very important throughout the paper.

**B. Ruijsenaars model**

Next, let us discuss the case of one-dimensional SL(2) Ruijsenaars model\cite{Bourjaily:2010} whose Lax operator matrix is given by
\[ L(\xi) = \sqrt{\frac{\varphi(\mu) - \varphi(Q)}{\varphi(\mu) - \varphi(\xi)}} \begin{pmatrix} e^P & e^P \sigma(Q + \xi) \sigma(\mu) \\ e^{-P} \sigma(-Q + \xi) \sigma(\mu) & e^{-P} \sigma(\mu + \xi) \sigma(\xi) \end{pmatrix}, \] (2.10)

The Calogero model is recovered for small \( \mu \). Then the spectral equation

\[ \det(L(\xi) - t) = 0 \] (2.11)

determines the eigenvalue \( t \), and by this we can construct its corresponding Seiberg-Witten solution, but in contrast with the preceding Calogero model, the Seiberg-Witten differential in this case must take the form

\[ dS_{\text{Rui}} = \ln t d\omega, \quad t = \frac{H \pm \sqrt{H^2 - \varphi(\mu) + \varphi(\xi)}}{\sqrt{\varphi(\mu) - \varphi(\xi)}}, \quad H = \sqrt{\varphi(\mu) - \varphi(Q)} \cosh P, \] (2.12)

where \( H \) is the Hamiltonian of the system, because of the requirement such that \( \partial_H dS_{\text{Rui}} \) must be a holomorphic differential. Below, we take + sign for the eigenvalue \( t \) in (2.12).

The Seiberg-Witten differential of the form (2.12) is very characteristic, and takes just the same form with those arising in five-dimensional gauge theories. Accordingly, the Ruijsenaars model will be understood in the context of higher dimensional gauge theories. In fact, as is obvious from (2.12), we can see the relation between holomorphic differentials in “four” and five dimensions

\[ \frac{\partial dS_{\text{Rui}}}{\partial H} = \frac{\partial \tilde{S}_{\text{Cal}}}{\partial \tilde{h}}, \] (2.13)

where

\[ \tilde{S}_{\text{Cal}} = \frac{2 \sqrt{\tilde{h} + x}}{y} dx, \quad \tilde{h} = H^2 - \varphi(\mu). \] (2.14)

Notice that the difference between \( d\tilde{S}_{\text{Cal}} \) and \( dS_{\text{Cal}} \) is simply \( \tilde{h} \leftrightarrow h \).

For this \( dS_{\text{Rui}} \), we define the periods

\[ A = \frac{1}{2i\pi R} \oint \alpha dS_{\text{Rui}} = \frac{1}{i\pi R} \int_{e^3}^{e_2} dS_{\text{Rui}}, \quad A_D = \frac{\partial \mathcal{F}}{\partial A} = \frac{1}{2i\pi R} \oint \beta dS_{\text{Rui}} = \frac{1}{i\pi R} \int_{e_2}^{e_1} dS_{\text{Rui}}, \] (2.15)

where we have introduced the prepotential \( \mathcal{F} \). Furthermore, we have normalized such that these match with those used by Braden et al. therefore, \( R \) plays the role of the radius of \( S^1 \) when this model is regarded as the Seiberg-Witten solution related to the five-dimensional gauge theory compactified on \( S^1 \). Then the effective coupling constant is given by

\[ \tau_{\text{eff}} = \frac{\partial A_D}{\partial A}. \] (2.16)
III. PAKULIAK-PERELOMOV EQUATIONS AND PICARD-FUCHS EQUATIONS

To calculate the prepotential of Ruijsenaars model, the periods (2.13) should be evaluated, but
the periods of a Riemann surface are known to satisfy Picard-Fuchs equations. In the case at hand,
the Picard-Fuchs equation is available from that of the Calogero model by focusing on (2.13). As a
matter of fact, though the Seiberg-Witten Riemann surface (2.8) is hyperelliptic type represented by
branching points, since the Seiberg-Witten differential (2.9) is a linear sum of Abelian differentials
on (2.8), we can use a general technique to get Picard-Fuchs equations for periods of hyperelliptic
Riemann surfaces.\(^3\)

However, for the case at hand, since the curve (2.8) is given by using branching points, the idea
of derivation of Picard-Fuchs equations will naturally overlap to that of Pakuliak and Perelomov\(^27\)
provided the branching points also including \(x = -h\) are regarded as if they were independent
variables. Namely, Picard-Fuchs equation in the Calogero system, and thus that of the Ruijsenaars
model, are obtained from Pakuliak-Perelomov equations.

A. Pakuliak-Perelomov equations

In general, the hyperelliptic curve of genus \(r\) can be realized as a double cover of a polynomial
in \(x\)
\[y^2 = \prod_{i=1}^{2r+2} (x - e_i) = \sum_{i=0}^{2r+2} (-1)^i \sigma_i(e_j) x^{2r+2-i}, \tag{3.1}\]
where \(e_k\) are the branching points on the \(x\)-plane and we have expressed the coefficients of powers
in \(x\) by \(\sigma_i(e_j)\). Do not confuse them with the Weierstrass’s \(\sigma\) function.

We can define the periods of Abelian differentials on (3.1) by
\[K_j = \oint_{\gamma} \frac{x^j}{y} dx, \quad j = 0, 1, \cdots, 2r \tag{3.2}\]
where \(\gamma\) is an arbitrary non-contractible 1-cycle on (3.1). Pakuliak-Perelomov equations are the
equations of a system of first-order differential equations satisfied by \(K_j\) and the reduction method
to get such equations can be done in the following way.

First, notice that the first-order derivatives are given by
\[
\frac{\partial K_j}{\partial e_i} = \frac{1}{2} \oint_{\gamma} \frac{x^j dx}{y(x-e_i)}. \tag{3.3}\]
Here, defining the integrand as

\[ I_j = \frac{x^j}{y(x - e_i)}, \]  

we get the recursion relation

\[ I_j = K_{j-1} + e_i I_{j-1}, \]  

which indicates

\[ I_j = \sum_{n=0}^{j-1} e_i^{j-n-1} K_n + e_i^j I_0. \]  

Thus, from (3.3), we get

\[ 2 \frac{\partial K_j}{\partial e_i} = \sum_{n=0}^{j-1} e_i^{j-n-1} K_n + e_i^j I_0. \]  

Next, the relation

\[ \oint_\gamma \frac{d}{dx} \left( \frac{y}{x - e_i} \right) dx = 0 \]  

induces

\[ \frac{1}{2} \sum_{k=1}^{2r+2} \oint_\gamma P^{(i,k)}(x) y dx = \oint_\gamma P^{(i)}(x) y (x - e_i) dx, \]  

where

\[ P^{(i,k)}(x) = \prod_{i,k \neq j=1}^{2r+2} (x - e_j) = \sum_{j=0}^{2r} (-1)^j \hat{\sigma}^{(i,k)}_j x^{2r-j}, \]

\[ P^{(i)}(x) = \prod_{i \neq j=1}^{2r+2} (x - e_j) = \sum_{j=0}^{2r+1} (-1)^j \hat{\sigma}^{(i)}_j x^{2r+1-j}. \]  

In (3.10), we have expressed the coefficients by \( \hat{\sigma}^{(i)}_j \) and \( \hat{\sigma}^{(i,k)}_j \).

Again using (3.5), we can obtain

\[ \oint_\gamma \frac{P^{(i)}(x)}{y(x - e_i)} dx = \sum_{j=0}^{2r+1} (-1)^j \hat{\sigma}^{(i)}_j \left[ \sum_{n=0}^{2r-j} e_i^{2r-j-n} K_n + e_i^{2r+1-j} \oint_\gamma \frac{dx}{y(x - e_i)} \right], \]  

but from (3.7), the Pakuliak-Perelomov equations follow

\[ 2 \frac{\partial K_j}{\partial e_i} = \sum_{n=0}^{j-1} e_i^{j-n-1} K_n \]

\[ + \frac{e_i^j}{P^{(i)}(e_i)} \left[ \frac{1}{2} \sum_{k=1}^{2r+2} \sum_{n=0}^{2r} (-1)^{2r-n} \hat{\sigma}^{(i,k)}_{2r-n} K_n - \sum_{j=0}^{2r+1} \sum_{n=0}^{2r-j} (-1)^j \hat{\sigma}^{(i)}_j e_i^{2r-j-n} K_n \right]. \]  

The right hand side of (3.12) is a linear sum of various \( K_n \), but it would be easy to obtain the equations satisfied by a single \( K_i \) by repeating differentiations, and accordingly, such equations compose a Picard-Fuchs system.
B. Picard-Fuchs equation for Calogero model

In the Calogero model, if we focus only on $h$-derivatives, the Picard-Fuchs equation of the third-order

$$4\partial_h (\Delta_{\text{Cal}} \partial_h^2 dS_{\text{Cal}}) + 3h \partial_h dS_{\text{Cal}} = 0,$$

where

$$\Delta_{\text{Cal}}(h) = \prod_{i=1}^{3} (h + e_i),$$

follows from (3.12).

Since the Seiberg-Witten solution involves another parameter, the bare coupling constant, one more equation including $\tau$-derivatives like that discussed by Itoyama and Morozov may be expected. However, the derivation of prepotential from such equation requires technical problems, so we do not discuss it in this paper.

C. Picard-Fuchs equation for Ruijsenaars model

It is now easy to find Picard-Fuchs equation for Ruijsenaars model, if we notice the relation (2.13). Since $d\hat{S}_{\text{Cal}}$ satisfies the Picard-Fuchs equation (3.13) replaced $h$ by $\hat{h}$, we can obtain from (2.13) and (3.13)

$$\left[ \Delta \partial^3_{\hat{H}} + \partial \partial_{\hat{H}} \left( \ln \frac{\Delta}{H} \right) \partial^2_{\hat{H}} + 3H^2 \hat{h} \partial_{\hat{H}} \right] dS_{\text{Rui}} = 0,$$

where

$$\Delta = \prod_{i=1}^{3} (\hat{h} + e_i).$$

IV. PERIODS

To derive the instanton correction for the prepotential of Ruijsenaars model, some items should be prepared appropriately. The first one is the calculation of periods, but in contrast with usual cases, the evaluation of periods is very sensitive because the Seiberg-Witten differential of Ruijsenaars model involves a normalization factor differential depending on $q$, i.e., the second term of the right hand side of
\[dS_{Rui} = \ln \left[ H + \sqrt{\hat{h} + x} \right] \frac{dx}{y} - \ln[\varphi(\mu) + x] \frac{dx}{y}. \quad (4.1)\]

Since this term is independent of \(H\), the Picard-Fuchs equation can not detect the contribution of instantons arising from this term. In fact, this is because \(\varphi(\mu)\) can be expanded by \(q\). Therefore, it is necessary to calculate this in order to include instanton effects correctly. Otherwise, the prepotential will not reduce to any physical prepotential by scaling limits.\(^\text{24}\)

With this in mind, we can see that the period \(A\) in the weak coupling region \((i\tau \to \infty, q = e^{i\pi \tau} \to 0)\) behaves as

\[iRA = \ln \left[ H \sin \mu + \sqrt{H^2 \sin^2 \mu - 1} \right] - \frac{4H(H^2 \sin^2 \mu - \sin^2 \mu - 1) \sin^3 \mu q^2}{(H^2 \sin^2 \mu - 1)^{3/2}} + \ldots \quad (4.2)\]

and thus its inverse relation follows immediately

\[H = \frac{\cos RA}{\sin \mu} + \frac{2(2 - \cos 2\mu - \cos 2RA) \sin \mu \cos RA}{\sin^2 RA} q^2 + \ldots. \quad (4.3)\]

On the other hand, as for the dual period \(A_D\), it is enough to calculate it only at the perturbative level for a later convenience (see also Ref.\(24\)).

\[
\left. \frac{\partial A_D}{\partial H} \right|_{q \to 0} = -\frac{1}{\pi R} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{dx}{x \sqrt{x(H^2 + \frac{2}{3}) + \hat{h} - 1/3}} = -\frac{1}{\pi R} \frac{\sin \mu}{\sqrt{H^2 \sin^2 \mu - 1}} \left[ \ln \frac{H^2 \sin^2 \mu - 1}{H^2 \sin^2 \mu - \cos^2 \mu} - \ln \frac{\epsilon}{4} \right] \Big|_{\epsilon \to 0}. \quad (4.4)\]

Accordingly, extracting the finite part of (4.4) and with the help of (4.3), we get the perturbative effective coupling constant

\[
\left. \tau_{eff} \right|_{q \to 0} = \frac{1}{i\pi} \ln \frac{\sin^2 RA}{\sin^2 \mu - \sin^2 RA}. \quad (4.5)\]

This coincides with the perturbative calculus.\(^\text{26}\)

### V. Instanton Correction for Prepotential of Ruijsenaars Model

#### A. Differential equation for prepotential

In the case of SU(2) gauge group, we can give a differential equation for prepotential by using a familiar method using inversion relations of periods.\(^\text{12}\)
For our case, derivatives of periods are inverted by\[3\]

\[
\partial_H A = \frac{1}{H}, \quad \partial_H^2 A = -\frac{H''}{H^3}, \quad \partial_H^3 A = \frac{3H''^2}{H^5} - \frac{H'''}{H^4},
\]

\[
\partial_H A_D = \frac{\mathcal{F}'''}{H}, \quad \partial_H^2 A_D = \frac{\mathcal{F}'''}{H^2} - \frac{\mathcal{F}'' H''}{H^3}, \quad \partial_H^3 A_D = \frac{\mathcal{F}^{(4)}}{H^3} - 3\frac{H'' \mathcal{F}'''}{H^4} + \left(3\frac{H''^2}{H^5} - \frac{H'''}{H^4}\right)\mathcal{F}'',
\]

where \(\prime = \partial/\partial A\).

Then from (3.15), we have

\[
\mathcal{F}^{(4)} - \left(3\frac{H''}{H'} - \frac{\partial_H \Delta}{\Delta}\right) \mathcal{F}''' = 0,
\]

which is integrated to give

\[
\mathcal{F}''' = c \frac{H H'^3}{\Delta}.
\]

In (5.3), \(c\) is an integration constant to be fixed below. Versions of (5.3) can be found in several SU(2) gauge theories\[14\,28\]. Note that (5.3) agrees with the equation obtained from residue calculus [cf. (44) in Ref.24] in the perturbative limit.

### B. Instanton correction of prepotential

Next, substituting (4.3) into (5.3) and expanding it by small \(q\), we can obtain the third-order derivative of the prepotential

\[
\frac{\mathcal{F}'''}{cR^3} = -\frac{2 \cot RA \sin^2 \mu}{\cos 2RA - \cos 2\mu} + \frac{F_2 \cot RA}{(\cos 2RA - \cos 2\mu)^2} \left(\frac{\sin \mu}{\sin RA}\right)^4 q^2 + \cdots,
\]

where

\[
F_2 = -36 - 107 \cos 2RA - 14 \cos 4RA + 6 \cos 6RA - 38 \cos 4\mu + 100 \cos 2\mu
\]

\[+ 88 \cos 2RA \cos 2\mu + 20 \cos 4RA \cos 2\mu - 14 \cos 2RA \cos 4\mu.\]

The second term of the right hand side in (5.4) is the one-instanton contribution. Note that the prepotential is expanded by the invariant quantity \(q\) under the transformation \(\tau \rightarrow \tau + 1\).

To fix the constant \(c\), look at the perturbative part of (5.4). Furthermore, recalling \(\mathcal{F}''' = \partial \tau_{\text{eff}} / \partial A\), we get

\[
c = i \frac{2}{\pi R^2}
\]

from (4.3). In this way, we can arrive at the exact expression of the third-order derivative of prepotential.
C. Reduction to Calogero prepotential

As a check of our calculus, let us take a scaling limit $\mu = 2i\pi g_0 R, R \to 0$ for $F'''$. Then it is straightforward to see that

$$F''' = -\frac{8i\pi g_0^2}{A(A^2 + 4\pi^2 g_0^2)} - \frac{256i\pi^3 g_0^4(-135A^4 + 2520\pi^2 A^2 g_0^2 + 9360\pi^4 g_0^4)}{45A^5(A^2 + 4\pi^2 g_0^2)^2} q^2 + \cdots.$$  \hspace{1cm} (5.7)

The first term of the right hand side of (5.7) can be identified with the perturbative part of the Calogero model, but the second one does not seem to be that of the one-instanton contribution of the Calogero model. However, expanding it for a small $g_0$, we get

$$F''' = -\frac{8i\pi g_0^2}{A(A^2 + 4\pi^2 g_0^2)} + \left(\frac{768i\pi^3 g_0^4}{A^5} - \frac{20480i\pi^5 g_0^6}{A^7} + \cdots\right) q^2 + \cdots,$$  \hspace{1cm} (5.8)

whose first term in the brackets is nothing but the one-instanton correction of the Calogero model (see Appendix B), provided higher order terms in $g_0$ are ignored (this operation is necessary because the original reduction to the Calogero model is supplied by $\mu \to 0$, and then this is equivalent to the assumption of a small mass of the adjoint hypermultiplet). Actually, though the sign is different, this is due to the ambiguity of the Weyl reflection for $a$. Therefore, we can conclude that (5.8) coincides with the third-order derivative of the prepotential of the Calogero model. Other scaling limits can be treated in a similar way.

VI. SUMMARY

In this paper, we have shown that:

- The Picard-Fuchs equation in the Ruijsenaars system is generated from the Pakuliak-Perelomov equations.

- From the differential equation for prepotential, one-instanton prepotential of the Ruijsenaars model is obtained.

- Our prepotential is checked in the limit to the Calogero system, i.e., the $N = 2$ gauge theory with a massive adjoint hypermultiplet.
APPENDIX A: WEIERSTRASS FUNCTIONS

Suppose that $f(z)$ is a complex single valued function with the complex argument $z$. Then if $f(z + \omega) = f(z)$ holds for a complex number $\omega$, $f$ is called a periodic function with period $\omega$. If $f$ has two independent periods $\omega$ and $\tilde{\omega}$, $f$ is referred to double periodic. In addition, if $f$ is rational, $f$ is called elliptic function. The Weierstrass functions are the elliptic functions with the following properties.

- Definition:
  
  $$\wp(z) = \frac{1}{z^2} + \sum_{-\infty < m, n \neq 0 < \infty} \left[ \frac{1}{(z + m\omega + n\tilde{\omega})^2} - \frac{1}{(m\omega + n\tilde{\omega})^2} \right].$$  
  \hspace{1cm} (A1)

- Relation between $\wp$ and $\sigma$:
  
  $$\wp(z) = \frac{\partial_z \sigma(z)}{\sigma(z)}, \quad \wp(z) - \wp(\xi) = -\frac{\sigma(z + \xi)\sigma(z - \xi)}{[\sigma(z)\sigma(\xi)]^2}. \hspace{1cm} (A2)$$

- Parity:
  
  $$\wp(z) = \wp(-z), \quad \sigma(z) = -\sigma(z). \hspace{1cm} (A3)$$

- Differential equation:
  
  $$[\partial_z \wp(z)]^2 = 4 \prod_{i=1}^{3} \left[ \wp(z) - e_i \right]. \hspace{1cm} (A4)$$

- Laurent expansion:
  
  $$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \cdots, \quad -\frac{g_2}{4} = e_1e_2 + e_2e_3 + e_3e_1, \quad \frac{g_3}{4} = e_1e_2e_3. \hspace{1cm} (A5)$$

- $q = e^{i\pi \tau}$-expansion:
  
  $$\wp(z) = -\frac{1}{3} + \frac{1}{\sin^2 z} + 16q^2 \sin^2 z + \cdots. \hspace{1cm} (A6)$$
APPENDIX B: PREPOTENTIAL OF CALOGERO MODEL

In this appendix, we summarize some formulae and prepotential in Calogero model, but the derivation of prepotential is parallel to that of Ruijsenaars model, so only the necessary items are listed, and $g_0$ is identified as $g_0^2 = im^2/\pi$, where $m$ is the mass of the adjoint hypermultiplet. In this calculation, we present in the weak coupling region, and the 1-cycles are taken as the same ones with the Ruijsenaars model. In addition, note that due to our normalization there are several distinctions to the known results.\(^{22,23}\)

- Periods and inverse relation ($\tilde{h} = h/g_0^2$):

$$\frac{a}{g_0 \pi} = \frac{2}{3} \sqrt{9\tilde{h} - 3} + \frac{8\sqrt{3}(3\tilde{h} - 4)}{(3\tilde{h} - 1)^{3/2}} \tilde{q}^2 + \cdots,$$

$$\frac{\partial a_{\tilde{h}}}{\partial \tilde{h}} = -\frac{ig_0}{\sqrt{\tilde{h} - 1/3}} \left[ 2 \ln 2 + \ln \frac{3\tilde{h} - 1}{3\tilde{h} + 2} \right] + \cdots,$$

$$\tilde{h} = \frac{1}{3} + \frac{a^2}{4\pi^2 g_0^2} + \left( -8 + \frac{32\pi^2 g_0^2}{a^2} \right) \tilde{q}^2 + \cdots \quad (B1)$$

- Effective coupling constant:

$$\left. \frac{\partial^2 \tilde{F}}{\partial a^2} \right|_{q \to 0} = \frac{i}{\pi} \ln \left( \frac{1}{4} + \frac{\pi^2 g_0^2}{a^2} \right). \quad (B2)$$

- Third-order derivative of prepotential:

$$\frac{\partial^3 \tilde{F}}{\partial a^3} = -i\pi g_0^2 \frac{(\partial_{\tilde{h}} \tilde{h})^3}{\Delta_{\text{Cal}}(\tilde{h})}$$

$$= -\frac{8i\pi g_0^2}{a(a^2 + 4\pi^2 g_0^2)} - \frac{768i\pi^3 g_0^4}{a^5} \tilde{q}^2 + \cdots \quad (B3)$$
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