Stochastic Taylor Expansions for Functionals of Diffusion Processes

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Abstract

In the present paper, a stochastic Taylor expansion of some functional applied to the solution process of an Itô or Stratonovich stochastic differential equation with a multi-dimensional driving Wiener process is given. Therefore, the multi-colored rooted tree analysis is applied in order to obtain a transparent representation of the expansion which is similar to the B-series expansion for solutions of ordinary differential equations in the deterministic setting. Further, some estimates for the mean-square and the mean truncation errors are given.

Key words: stochastic Taylor expansion, stochastic differential equation, multi-colored rooted tree analysis, strong approximation, mean-square approximation

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ fulfilling the usual conditions. Since each non-autonomous stochastic differential equation (SDE) can be written as an autonomous SDE system with one additional equation representing time, we consider without loss of generality autonomous SDE systems only. Thus, for some $0 \leq t_0 < T < \infty$ let $(X_t)_{t \in [t_0, T]}$ be the solution of the $d$–dimensional autonomous SDE system

\[ dX_t = a(X_t) \, dt + b(X_t) \ast dW_t \] (1)
with an $m$–dimensional driving Wiener process $(W_t)_{t \geq 0}$ w.r.t. $(\mathcal{F}_t)_{t \geq 0}$. Then, SDE (1) can be written in integral form

$$X_t = X_{t_0} + \int_{t_0}^t a(X_s) \, ds + \sum_{j=1}^m \int_{t_0}^t b^j(X_s) \, dW^j_s$$

(2)

for $d, m \geq 1$ and $t \in [t_0, T]$, where we write $\ast dW^j_s = dW^j_s$ in the case of an Itô stochastic integral and $\ast dW^j_s = \circ dW^j_s$ for a Stratonovich stochastic integral. Here, we suppose that $a : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are measurable functions which are sufficiently smooth and we denote by $b^j$ for $j = 1, \ldots, m$ the $j$th column of the $d \times m$-matrix function $b = (b^j)$. Let $X_{t_0} \in \mathbb{R}^d$ be the $\mathcal{F}_{t_0}$-measurable initial value with $X_{t_0} \in L^2(\Omega)$. In the following, we suppose that the conditions of the Existence and Uniqueness Theorem [4] are fulfilled for SDE (2) and we denote by $\| \cdot \|$ the Euclidean norm.

The aim of the present paper is to give an expansion of $f(X_t)$ for some functional $f : \mathbb{R}^d \to \mathbb{R}$. Therefore, we define for $j = 1, \ldots, m$ the operators

$$\hat{L}^0 = \sum_{k=1}^d a^k \frac{\partial}{\partial x^k}, \quad \hat{L}^j = \sum_{k,l=1}^d b^{k,j} b^{l,j} \frac{\partial^2}{\partial x^k \partial x^l}, \quad L^j = \sum_{k=1}^d b^{k,j} \frac{\partial}{\partial x^k},$$

(3)

and $L^0 = \hat{L}^0 + \frac{1}{2} \sum_{j=1}^m \hat{L}^j$. Considering now the Itô SDE (2), we obtain for sufficiently smooth $f$ by recursive application of Itô’s formula

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^t L^0 f(X_s) \, ds + \sum_{j=1}^m \int_{t_0}^t L^j f(X_s) \, dW^j_s$$

$$= f(X_{t_0}) + \int_{t_0}^t L^0 f(X_s) \, ds + \sum_{j_1=1}^m \int_{t_0}^t \left( L^{j_1} f(X_{t_0}) + \int_{t_0}^s L^0 L^{j_1} f(X_u) \, du + \sum_{j_2=1}^m \int_{t_0}^s L^{j_2} L^{j_1} f(X_u) \, dW^j_{u}\right) \, dW^{j_2}_s$$

(4)

Repetition of this procedure by applying Itô’s formula to $L^0 f(X_s)$ yields a further expansion, and so on. As a result of this, we obtain the Itô-Taylor-expansion due to Platen and Wagner [5,10] with a remainder of integral type. In the following, we always assume that all necessary derivatives and multiple stochastic integrals exist. If SDE (2) is considered in the Stratonovich sense, then the same expansion as in (4) applies however with $L^0$ replaced by $\hat{L}^0$.

In the present paper, we develop an expansion of $f(X_t)$ based on multi–colored rooted trees. This turns out to be an extension of the rooted tree approach for ordinary differential equations due to Butcher [3] in the deterministic setting. Burrage and Burrage [2] developed the expansion of $f(X_t)$ by multi–colored rooted trees for the special case when $X_t$ is the solution of a Stratonovich SDE and when $f(X_t) = X_t$. In contrast to this, we give an expansion not only for
the Stratonovich solution $X_t$ but also for solutions of Itô SDEs and additionally for arbitrary functionals $f(X_t)$ of the solution process. Therefore, we follow the approach proposed in [8,12] and make use of an additional node corresponding to the functional $f$ for the root of the considered trees. Further, we have to take into account the more complex structure of the operator $L^0$ for Itô SDEs compared to $\hat{L}^0$ for Stratonovich SDEs. As the main advantage of the rooted tree expansion of $f(X_t)$, we obtain a clear and simple expansion with equal elementary differentials pooled together. Compared to the approach based on hierarchical sets by Kloeden and Platen [5], each elementary differential can be determined directly by the corresponding rooted tree. Further, expansions based on rooted trees allow a systematic development of higher order derivative free approximations similar to the deterministic setting [2,3,11].

2 Colored Rooted Tree Analysis

Following the approach in [11,12], we give a definition of colored trees which will be suitable for SDEs w.r.t. a multi-dimensional Wiener process.

**Definition 2.1** A monotonically labelled S-tree (stochastic tree) $t$ with $l = l(t) \in \mathbb{N}$ nodes is a pair of maps $t = (t', t'')$ with

$t' : \{2, \ldots, l\} \to \{1, \ldots, l - 1\}$

$t'' : \{1, \ldots, l\} \to A$

so that $t'(i) < i$ for $i = 2, \ldots, l$. Unless otherwise noted, we choose the set $A = \{\gamma, \tau_j : j \in \{0, 1, \ldots, m\}\}$. Let $LTS$ denote the set of all monotonically labelled S-trees w.r.t. $A$.

Then $t'$ defines a father son relation between the nodes, i.e. $t'(i)$ is the father of the son $i$. Furthermore the color $t''(i)$, which consists of one element of the set $A$, is added to the node $i$ for $i = 1, \ldots, l(t)$. Here, $\tau_0 = \bullet$ is a deterministic node, $\tau_j = \bigcirc_j$ is a stochastic node with $j \in \{1, \ldots, m\}$ and $\gamma = \otimes$ can be the root of a tree. The variable index $j$ is associated with the $j$th component of the corresponding $m$-dimensional Wiener process of the considered SDE. As

![Fig. 1. Three elements of LTS with $j_1, j_2, j_3, j_4 \in \{1, \ldots, m\}$]

an example Figure 1 presents three elements of $LTS$. 
In the following, we denote by \( d(t) = |\{ i : \gamma'(i) = \tau_0 \}| \) the number of deterministic nodes and by \( s(t) = |\{ i : \gamma''(i) = \tau_j, 1 \leq j \leq m \}| \) the number of stochastic nodes. The order \( \rho(t) \) of the tree \( t \) is defined as \( \rho(t) = d(t) + \frac{1}{2} s(t) \) with \( \rho(\gamma) = 0 \). The order of the trees presented in Figure 1 can be calculated as \( \rho(t_I) = 2.5 \) and \( \rho(t_{II}) = \rho(t_{III}) = 2 \).

Every labelled tree can be written by a combination of brackets: If \( t_1, \ldots, t_k \) are colored trees then we denote by \( [t_1, \ldots, t_k]_\gamma \) and \( [t_1, \ldots, t_k]_{\partial} \) the tree in which \( t_1, \ldots, t_k \) are each joined by a single branch to \( \gamma = \emptyset \) and \( \partial = \emptyset \) for \( j = 0, 1, \ldots, m \), respectively. Therefore proceeding recursively, for the three examples in Figure 1 we obtain \( t_I = [\tau_0^3, \tau_0^4]_\gamma, t_{II} = [\tau_3^3, \tau_4^4]_{\partial}^1 \) and \( t_{III} = [\tau_3^4, \tau_4^5]_{\partial}^1 \) for \( j_1, j_2, j_3, j_4 \in \{ 1, \ldots, m \} \).

Now, two labelled trees \( t, u \in LTS \) with \( l(t) = l(u) \) nodes are called equivalent, i.e. \( t \sim u \), if there exists a bijective map \( \pi : \{ 1, \ldots, l \} \rightarrow \{ 1, \ldots, l \} \) with \( t'(i) = \pi^{-1}(u'(\pi(i))) \) for \( i = 2, \ldots, l \) and \( t''(i) = u''(\pi(i)) \) for \( i = 1, \ldots, l \). The set of all equivalence classes under the relation \( \sim \) is denoted by \( TS = LTS/\sim \).

We denote by \( \alpha(t) \) the cardinality of \( t \), i.e. the number of possibilities of monotonically labelling the nodes of \( t \) with numbers \( 1, \ldots, l(t) \). For example, the labelled trees \( [\tau_0^3, \tau_0^4]_\gamma, [\tau_0^4, \tau_0^4]_\gamma \) and \( [\tau_0^4, \tau_0^4]_\gamma \) with \( j_1 \in \{ 1, \ldots, m \} \) belong to the same equivalence class as \( t_I \) in the example above. Thus, we have \( \alpha(t_I) = 3 \). For \( j_1, j_2 \in \{ 1, \ldots, m \} \) with \( j_1 \neq j_2 \), we obtain the two different labelled trees \( [\tau_3^3, \tau_4^4]_{\partial}^1 \) and \( [\tau_3^3, \tau_4^4]_{\partial}^1 \) belonging to the same equivalence class as \( t_{II} \) and we get \( \alpha([\tau_3^3, \tau_4^4]_{\partial}^1) = 2 \). However, if we choose \( j_1 = j_2 \), then there exists only one labelled tree \( [\tau_3^3, \tau_4^4]_{\partial}^1 \) and we obtain \( \alpha([\tau_3^3, \tau_4^4]_{\partial}^1) = 1 \).

For every rooted tree \( t \in TS \), there exists a corresponding element differential. The elementary differential is defined recursively by \( F(\gamma)(x) = f(x), F(\tau_0)(x) = a(x) \) and \( F(\tau_j)(x) = b^j(x) \) for single nodes and by

\[
F(t)(x) = \begin{cases} 
 f^{(k)}(x) \cdot (F(t_1)(x), \ldots, F(t_k)(x)) & \text{for } t = [t_1, \ldots, t_k]_\gamma \\
 a^{(k)}(x) \cdot (F(t_1)(x), \ldots, F(t_k)(x)) & \text{for } t = [t_1, \ldots, t_k]_0 \\
 b^{(k)}(x) \cdot (F(t_1)(x), \ldots, F(t_k)(x)) & \text{for } t = [t_1, \ldots, t_k]_j 
\end{cases}
\tag{5}
\]

for a tree \( t \) with more than one node and \( j \in \{ 1, \ldots, m \} \). Here \( f^{(k)} \), \( a^{(k)} \) and \( b^{(k)} \) define a symmetric \( k \)-linear differential operator, and one can choose the sequence of labelled S-trees \( t_1, \ldots, t_k \) in an arbitrary order. For example, the \( I \)th component of \( a^{(k)} \cdot (F(t_1), \ldots, F(t_k)) \) can be written as

\[
(a^{(k)} \cdot (F(t_1), \ldots, F(t_k)))^I = \sum_{J_1, \ldots, J_k=1}^{d} \frac{\partial^k a^I}{\partial x^{J_1} \cdots \partial x^{J_k}} (F^{J_1}(t_1), \ldots, F^{J_k}(t_k))
\]

where the components of vectors are denoted by superscript indices, which are chosen as capitals. As a result of this we calculate for the trees in Figure 1 the
elementary differentials

\[ F(t_I) = f''(b^{j_1}(a), a) = \sum_{J_1, J_2 = 1}^{d} \frac{\partial^2 f}{\partial x^{J_1} \partial x^{J_2}} \left( \sum_{K_1 = 1}^{d} \frac{\partial b^{J_1}_{i_1}}{\partial x^{K_1}} a_{K_1} \cdot a_{j_2} \right), \]

\[ F(t_{II}) = f'(a''(b^{j_1}, b^{j_2})) = \sum_{J_1 = 1}^{d} \frac{\partial f}{\partial x^{J_1}} \left( \sum_{K_1, K_2 = 1}^{d} \frac{\partial^2 a^{J_1}_{i_1}}{\partial x^{K_1} \partial x^{K_2}} b_{K_1}^{i_1} \cdot b_{K_2}^{j_2} \right), \]

\[ F(t_{III}) = f''(b^{j_1}(b^{j_3}), b^{j_2}(b^{j_4})) = \sum_{J_1, J_2 = 1}^{d} \frac{\partial^2 f}{\partial x^{J_1} \partial x^{J_2}} \left( \sum_{K_1, K_2 = 1}^{d} \frac{\partial b^{J_1}_{i_1}}{\partial x^{K_1}} b_{K_1}^{i_1} \cdot \frac{\partial b^{J_2}_{i_2}}{\partial x^{K_2}} b_{K_2}^{i_2} \right). \]

Next, we assign to every tree a corresponding multiple stochastic integral. For \( t \in TS \) and an adapted right continuous stochastic process \( (Z_t)_{t \geq t_0} \) the corresponding multiple stochastic integral is recursively defined by

\[ I_{t_0, t}[Z] = \begin{cases} 
(\prod_{i=1}^{k} I_{t_i; t_0, t})[Z] & \text{if } t = [t_1, \ldots, t_k] \\
\left( \int_{t_0}^{t} \prod_{i=1}^{k} I_{t_i; t_0, s} * dW^j_s \right)[Z] & \text{if } t = [t_1, \ldots, t_k] 
\end{cases} \tag{6} \]

with \( *dW^0_s = ds \), \( I_{\gamma; t_0, t}[Z] = \int_{t_0}^{\gamma} Z_s * dW^j_s \), \( I_{\gamma; t_0, t}[Z] = Z_t \), \( I_{t_0; t_0, t} = I_{t_0, t}[1] \) and with the notation

\[ \left( \int_{t_0}^{t} \int_{t_0}^{s_1} \cdots \int_{t_0}^{s_n} *dW^j_{s_1} * dW^j_{s_2} \cdots * dW^j_{s_n} \right)[Z] = I_{(j_1, j_2, \ldots, j_n)}[Z]_{t_0, t} \tag{7} \]

in (6). The product of two stochastic integrals can be written as a sum

\[ \int_{t_0}^{t} X_s * dW^i_s \int_{t_0}^{t} Y_s * dW^j_s = \int_{t_0}^{t} X_s Y_s 1_{\{i = j \neq 0 \wedge \neq 0 \}} \, ds \]

\[ + \int_{t_0}^{t} X_s \int_{t_0}^{s} Y_u * dW^j_u * dW^j_s + \int_{t_0}^{t} \int_{t_0}^{s} X_u * dW^j_u Y_s * dW^j_s \tag{8} \]

for \( 0 \leq i, j \leq m \) [5], where the first summand on the right hand side appears only in the case of Itô calculus. For example, we calculate for \( t_I \)

\[ I_{t_I; t_0, t}[Z] = (I_{t_{0; t_0, t}} I_{t_{[\gamma]; t_0, t}})[Z] = (\int_{t_0}^{t} ds \int_{t_0}^{t} I_{t_0; t_0, s} * dW^j_s)[Z] \]

\[ = (\int_{t_0}^{t} \int_{t_0}^{s} I_{t_0; t_0, u} * dW^j_u \, ds + \int_{t_0}^{t} ds \int_{t_0}^{s} I_{t_0; t_0, s} * dW^j_s)[Z] \]

\[ = (\int_{t_0}^{t} \int_{t_0}^{s} dv * dW^j_i \, ds + \int_{t_0}^{t} \int_{t_0}^{s} du * dW^j_i)[Z] \]

\[ = I_{(0, j_1, 0)}[Z]_{t_0, t} + 2I_{(0, 0, j_1)}[Z]_{t_0, t}. \tag{9} \]
For the tree $t_{II}$ we obtain

$$I_{t_{II} ; t_{0} , t}[Z] = \left(\int_{t_{0}}^{t} I_{\tau, t} \cdot d\tau \right) [Z]$$

$$= \left(\int_{t_{0}}^{t} \int_{t_{0}}^{s} I_{\tau, t} \cdot ds \right) [Z]$$

$$= \left(\int_{t_{0}}^{t} \int_{t_{0}}^{s} I_{\tau, t} \cdot ds \right) [Z]$$

$$+ \int_{t_{0}}^{t} \int_{t_{0}}^{s} 1_{\{j_{1} = j_{2} \neq 0 \wedge \neq \emptyset\}} \cdot ds \right) [Z]$$

$$= I_{(j_{2}, j_{1} , 0)}[Z]_{t_{0} , t} + I_{(j_{1} , j_{2} , 0)}[Z]_{t_{0} , t} + I(0,0)[Z \cdot 1_{\{j_{1} = j_{2} \neq 0 \wedge \neq \emptyset\}}]_{t_{0} , t}.$$ (10)

Let $t \in TS$ with $t = [t_{1} , \ldots , t_{1} , t_{2} , \ldots , t_{k} , \ldots , t_{k}]_{j} = [t_{1}^{n_{1}} , t_{2}^{n_{2}} , \ldots , t_{k}^{n_{k}}]_{j}$, $j \in \{ \gamma , 0 , 1 , \ldots , m \}$, where $t_{1} , \ldots , t_{k}$ are distinct subtrees with multiplicities $n_{1} , \ldots , n_{k}$, respectively. Then we recursively define the symmetry factor by

$$\sigma(t) = \prod_{i=1}^{k} n_{i}! \cdot \sigma(t_{i})^{n_{i}}.$$ (11)

Next, we define the density of a tree which is a measure of its non-bushiness. For $t \in TS$ let the density $\gamma(t)$ be recursively defined by $\gamma(t) = 1$ if $l(t) = 1$ and

$$\gamma(t) = l(t) \prod_{i=1}^{k} \gamma(t_{i})$$ (12)

if $t = [t_{1} , \ldots , t_{k}]_{j}$ with some $j \in \{ \gamma , 0 , 1 , \ldots , m \}$. Then, for $t \in TS$ we obtain

$$\alpha(t) = \frac{l(t)!}{\gamma(t) \cdot \sigma(t)}$$ (13)

see also [3]. For example, we calculate for $t_{I}$ that $\sigma(t_{I}) = 1$ and with $l(t_{I})! = 24$ and $\gamma(t_{I}) = 8$ we obtain $\alpha(t_{I}) = 3$. For the tree $t_{II}$ we have to consider two cases: if $j_{1} \neq j_{2}$ we have $\sigma(t_{II}) = 1$ and we get with $l(t_{II})! = 24$ and $\gamma(t_{II}) = 12$ that $\alpha(t_{II}) = 2$. However, in the case of $j_{1} = j_{2}$ we have some symmetry and thus $\sigma(t_{II}) = 2$ which results in $\alpha(t_{II}) = 1$. Analogously, we consider for $t_{III}$ the case of $j_{1} = j_{2}$ and $j_{3} = j_{4}$ with some symmetry where $\sigma(t_{III}) = 2$ and with $l(t)! = 120$ and $\gamma(t_{III}) = 20$ follows $\alpha(t_{III}) = 3$. For all other cases, we have no symmetry and thus $\sigma(t_{III}) = 1$ where we obtain $\alpha(t_{III}) = 6$ different monotonically labelled trees in the equivalence class of $t_{III}$.

3 Stochastic Taylor Expansion

In order to give a stochastic Taylor expansion of the solution of the considered SDE (2) with some remainder term, we have to introduce the sets of descendant trees. For $t \in TS$ and $j \in \{0,1,\ldots,m\}$ let $H^{j}(t)$ denote the set of all
trees in $TS$ which are obtained from $t$ by adding one node $\tau_j$. Further, let $H^j(t)$ denote the set of all trees in $TS$ which are obtained from $t$ by adding the two nodes $\tau_j^a$ and $\tau_j^b$ where both nodes have the same $j \in \{1, \ldots, m\}$, $a = l(t) + 1$, $b = l(t) + 2$ and where neither of them is father of the other. For example, if $t = [\tau_{j_1}, \tau_{j_2}]_\gamma$ for some arbitrarily fixed $j_1, j_2 \in \{1, \ldots, m\}$ then we obtain

$$H^0(t) = \{[\tau_{j_1}, \tau_{j_2}, \tau_0]_\gamma, [	au_{j_1}, [	au_0]_{j_1}, \tau_{j_2}]_\gamma\}$$

$$H^1(t) = \{[\tau_{j_1}, \tau_{j_2}, \tau_{j_3}]_\gamma, [[\tau_{j_3}^a, \tau_{j_2}^a]_{j_1}, [\tau_{j_2}^a, \tau_{j_3}^a]_{j_2}, [\tau_{j_3}^a, \tau_{j_3}^b]_{j_2}, [\tau_{j_3}^b, \tau_{j_3}^a]_{j_2}, [\tau_{j_3}^b, \tau_{j_3}^b]_{j_2}, [\tau_{j_3}^b, \tau_{j_3}^b]_{j_2}]_\gamma, [\tau_{j_1}, [\tau_{j_1}^a, \tau_{j_2}^a]_{j_2}, [\tau_{j_1}^a, \tau_{j_3}^a]_{j_2}, [\tau_{j_1}^a, \tau_{j_3}^b]_{j_2}, [\tau_{j_1}^b, \tau_{j_3}^b]_{j_2}, [\tau_{j_1}^b, \tau_{j_3}^b]_{j_2}]_\gamma, [\tau_{j_2}^a, [\tau_{j_2}^a, \tau_{j_3}^a]_{j_2}, [\tau_{j_2}^a, \tau_{j_3}^b]_{j_2}, [\tau_{j_2}^b, \tau_{j_3}^b]_{j_2}, [\tau_{j_2}^b, \tau_{j_3}^b]_{j_2}]_\gamma, [\tau_{j_3}^a, [\tau_{j_3}^a, \tau_{j_3}^a]_{j_2}, [\tau_{j_3}^a, \tau_{j_3}^b]_{j_2}, [\tau_{j_3}^b, \tau_{j_3}^b]_{j_2}, [\tau_{j_3}^b, \tau_{j_3}^b]_{j_2}]_\gamma, j_3 = j_3^b \in \{1, \ldots, m\} \}}$$

(14)

independently whether $j_1 = j_2$ or $j_1 \neq j_2$. In the following, let $\frac{1}{2} \mathbb{N}_0 = \{p : 2p \in \mathbb{N}_0\}$. Then, based on the introduced multi-colored rooted trees, we obtain the following stochastic Taylor expansion for the solution of the Itô SDE (2):

**Theorem 3.1** For the solution process $(X_t)_{t \in [t_0, T]}$ of the Itô SDE (2) and for $p \in \frac{1}{2} \mathbb{N}_0$, $f : \mathbb{R}^d \to \mathbb{R}$ with $f, a^i, b^{ij} \in C^{2p+2}(\mathbb{R}^d, \mathbb{R})$ for $i = 1, \ldots, d, j = 1, \ldots, m$, we obtain the expansion

$$f(X_t) = \sum_{t \in TS_{\rho(t) \leq p}} F(t)(X_{t_0}) \frac{I_{t_0, t} I_{t, t_0}}{\sigma(t)} + \mathcal{R}_p(t, t_0)$$

(15)

P-a.s. with remainder term

$$\mathcal{R}_p(t, t_0) = \sum_{t \in TS_{\rho(t) = p + 1/2}} \frac{I_{t_0, t} F(t)(X_t)}{\sigma(t)} + \sum_{t \in TS_{\rho(t) = p}} \sum_{u \in H^0(t)} \frac{I_{t_0, t} [\int_{t_0}^t F(u)(X_s) ds]}{\sigma(t)}$$

$$+ \sum_{t \in TS_{\rho(t) = p}} \sum_{u \in H^1(t)} \frac{I_{t_0, t} [\int_{t_0}^t F(u)(X_s) ds]}{2 \sigma(t)}$$

(16)

provided all of the appearing multiple Itô integrals exist.

**Proof.** First, we assign to every $t \in LTS$ a corresponding multiple stochastic integral. Therefore, let $\Gamma = \{t \in LTS : t''(l(t)) = \tau_0\}$, $\Lambda^j = \{t \in LTS : t''(l(t)) = t''(l(t) - 1) = \tau_j \wedge t''(l(t)) \neq l(t) - 1\}$ and $\Sigma^j = \{t \in LTS : (t''(l(t)) = \tau_j \wedge t''(l(t) - 1) \neq \tau_j) \vee (t''(l(t)) = t''(l(t) - 1) = \tau_j \wedge t''(l(t)) = l(t) - 1)\}$ for $j = 1, \ldots, m$. Then, define for $t \in LTS$ with $l = l(t)$ nodes and for an adapted right continuous stochastic process $(Z_t)_{t \geq t_0}$ the corresponding
multiple Itô integral recursively by

$$
\hat{I}_t[Z]_{t_0,t} = \begin{cases} 
Z_t & \text{if } \rho(t) = 0 \\
\hat{I}_t[\int_{t_0}^t Z_s \, ds]_{t_0,t} & \text{if } t \in \Gamma \\
\hat{I}_t[\int_{t_0}^t Z_s \, dW_s^j]_{t_0,t} & \text{if } t \in \Lambda^j \\
\hat{I}_t[\int_{t_0}^t Z_s \, dW_s]_{t_0,t} & \text{if } t \in \Sigma^j
\end{cases}
$$

(17)

where $t_-$ is the tree which is obtained from $t$ by removing the last node
with label $l(t)$ and $t_- = (t_-)$ denotes the tree where the last two
nodes with labels $l(t)$ and $l(t) - 1$ are removed. Following the notation in \cite{5},
for a multi-index $\alpha = (j_1, \ldots, j_l) \in \{0, 1, \ldots, m\}^l$ let $l(\alpha) = l$
be the length with $l(\nu) = 0$ for the multi-index $\nu$ of length 0. Further,
let $\mathcal{M}$ be the set of all multi-indices and $n(\alpha)$ be the number
of components of $\alpha$ which are equal to 0. For $j \in \{0, 1, \ldots, m\}$
let $(j) * \alpha = (j, j_1, \ldots, j_l)$. Then, define $\mathcal{I}(j_1, \ldots, j_l)[Z]_{t_0,t} =
\int_{t_0}^t \mathcal{I}(j_1, \ldots, j_{l-1})[Z_s]_{t_0,s} \, dW^j_s$ if $l \geq 1$
and $\mathcal{I}_\nu[Z]_{t_0,t} = Z_t$. Finally, let $f_{(j_1, \ldots, j_l)} =
L^j f_{(j_2, \ldots, j_l)}$ and $f_\nu = f$. In the following, we consider the hierarchical set
$\mathcal{A}_p = \{ \alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2p \}$ and define $\mathcal{A}_p = \{ \alpha \in \mathcal{M} : l(\alpha) + n(\alpha) = 2p \}$
for $p \in \frac{1}{2} \mathbb{N}_0$ with $\mathcal{A}_p = \bigcup_{i \leq p} \mathcal{A}_i$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ if $i \neq j$. Then, a recursive
application of the Itô-formula yields the Itô-Taylor expansion \cite{5}

$$
f(X_t) = \sum_{\alpha \in \mathcal{A}_p} \mathcal{I}_\alpha[f_\alpha(X_{t_0})]_{t_0,t} + \sum_{\alpha \in \mathcal{B}(\mathcal{A}_p)} \mathcal{I}_\alpha[f_\alpha(X_{t_0})]_{t_0,t}
$$

(18)

P-a.s. with $\mathcal{B}(\mathcal{A}_p) = \{ \alpha \in \mathcal{M} \setminus \mathcal{A}_p : \alpha = (j) * \alpha, j \in \{0, 1, \ldots, m\}, \alpha \in \mathcal{A}_p \}$. Clearly,
$\mathcal{A}_p = \{ (0) * \alpha \in \mathcal{M} : \alpha \in \mathcal{A}_{p-1} \} \cup \{ (j) * \alpha \in \mathcal{M} : \alpha \in \mathcal{A}_{p-1}, j \in \{1, \ldots, m\} \}$
for $p \geq 1$. As a result of this, we have to prove

$$
\sum_{\alpha \in \mathcal{A}_p} \mathcal{I}_\alpha[f_\alpha(X_{t_0})]_{t_0,t} = \sum_{t \in \text{LTS}_{\mathcal{A}}} F(t)(X_{t_0}) \hat{I}_t[1]_{t_0,t}
$$

(19)

for $n \in \mathbb{N}_0$. Now, define a linear operator $K^j$ for $j = 0, 1, \ldots, m$
by

$$
K^j(\mathcal{I}(j_1, \ldots, j_l)[Z]_{t_0,t}) = \mathcal{I}(j_1, \ldots, j_l) \left[ \int_{t_0}^t L^j Z_s \, dW^j_s \right]_{t_0,t} = \mathcal{I}(j_1, \ldots, j_l)[L^j Z]_{t_0,t}.
$$

(20)

Then, for $n = 0$, we obtain

$$
\sum_{\alpha \in \mathcal{A}_0} \mathcal{I}_\alpha[f_\alpha(X_{t_0})]_{t_0,t} = \mathcal{I}_\nu[f_\nu(X_{t_0})]_{t_0,t} = f(X_{t_0}) = F(\gamma)(X_{t_0}) \hat{I}_\gamma[1]_{t_0,t}
$$

$$
= \sum_{t \in \text{LTS}} F(t)(X_{t_0}) \hat{I}_t[1]_{t_0,t}
$$

(21)

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and for $n = 1$, we get

$$
\sum_{\alpha \in A_{1/2}} I_{\alpha}[f_{\alpha}(X_{t_0})]_{t_0, t} = \sum_{j=1}^{m} I_{(j)}[f_{(j)}(X_{t_0})]_{t_0, t}
$$

$$
= \sum_{j=1}^{m} \sum_{k=1}^{d} \frac{\partial f}{\partial x^k}(X_{t_0}) b^{k,j}(X_{t_0}) \int_{t_0}^{t} 1 \, dW^j_s
$$

$$
= \sum_{j=1}^{m} F([\tau_1], (X_{t_0}) \hat{I}_{[\tau_1],1}^{1,1}]_{t_0, t}
$$

$$
= \sum_{t \in LTS} \sum_{\rho(t) = 12} F(t)(X_{t_0}) \hat{I}_{t}^{1,1}]_{t_0, t}
$$

(22)

Now, assume that (19) holds for some $n, n - 1 \geq 0$. For $t \in LTS$ and $j = 0, 1, \ldots, m$, we introduce the sets

$$
\hat{H}_1^1(t) = \{ u \in LTS : l(u) = l(t) + 1, u\{2, \ldots, l(t)\} = t', u'\{1, \ldots, l(t)\} = t''
$$

$$
\hat{H}_2^1(t) = \{ u \in LTS : l(u) = l(t) + 2, u\{2, \ldots, l(t)\} = t', u'\{1, \ldots, l(t)\} = t''
$$

$$
\hat{H}_1^2(t) = \{ u \in LTS : l(u) = l(t) + 1, u\{l(t) + 1\} = t', u'\{l(t) + 1\} = t''
$$

$$
\hat{H}_2^2(t) = \{ u \in LTS : l(u) = l(t) + 2, u\{l(t) + 2\} = t', u'\{l(t) + 2\} = t''
$$

(23)

and let $H_i^d = \hat{H}_i^d \cup \hat{H}_i^d$ for $i = 1, 2$. Then, with Lemma 2.7 and Lemma 2.8 in [12] we obtain

$$
\sum_{\alpha \in A_{1/2(n+1)}} I_{\alpha}[f_{\alpha}(X_{t_0})]_{t_0, t}
$$

$$
= K^0( \sum_{\alpha \in A_{1/2(n-1)}} I_{\alpha}[f_{\alpha}(X_{t_0})]_{t_0, t} ) + \sum_{j=1}^{m} K^j( \sum_{\alpha \in A_{1/2n}} I_{\alpha}[f_{\alpha}(X_{t_0})]_{t_0, t} )
$$

$$
= K^0( \sum_{t \in LTS} \hat{I}_{t}[F(t)(X_{t_0})]_{t_0, t} ) + \sum_{j=1}^{m} K^j( \sum_{t \in LTS} \hat{I}_{t}[F(t)(X_{t_0})]_{t_0, t} )
$$

$$
= \sum_{t \in LTS} \hat{I}_{t} \int_{t_0}^{t} \hat{F}(X_{t_0}) \, ds|_{t_0, t}
$$

$$
+ \sum_{j=1}^{m} \sum_{t \in LTS} \hat{I}_{t} \int_{t_0}^{t} \hat{F}_j(X_{t_0}) \, ds|_{t_0, t}
$$

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\[ + \sum_{j=1}^{m} \sum_{\substack{t \in LTS \\rho(t) = 1/2n \\nu'(l(t)) = \tau_j}} \hat{I}_t \left[ \int_{t_0}^{t} L^j F(t)(X_{t_0}) \, dW_s^j \right]_{t_0, t} \]
\[ + \sum_{j=1}^{m} \sum_{\substack{t \in LTS \\rho(t) = 1/2n \\nu'(l(t)) = \tau_j}} \hat{I}_t \left[ \int_{t_0}^{t} L^j F(t)(X_{t_0}) \, dW_s^j \right]_{t_0, t} \]
\[ = \sum_{t \in LTS} \sum_{\substack{\rho(t) = 1/2(n-1) \\nu(t) \neq \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ + \sum_{t \in LTS} \sum_{\substack{\rho(t) = 1/2(n-1) \\nu(t) = \tau_j}} \left( \sum_{u \in \bar{H}_1^2(t)} \hat{I}_t \left[ \int_{t_0}^{t} F(u)(X_{t_0}) \, dW_s^j \right]_{t_0, t} \right) \]
\[ + \sum_{t \in LTS} \hat{I}_t \left[ \int_{t_0}^{t} F(u)(X_{t_0}) \, dW_s^j \right]_{t_0, t} \]
\[ + \sum_{j=1}^{m} \sum_{\substack{t \in LTS \\rho(t) = 1/2n \\nu'(l(t)) = \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ = \sum_{t \in LTS} \sum_{\substack{\rho(t) = 1/2(n-1) \\nu(t) \neq \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ + \sum_{j=1}^{m} \sum_{\substack{t \in LTS \\rho(t) = 1/2n \\nu'(l(t)) = \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} + \sum_{u \in \bar{H}_1^2(t)} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ + \sum_{j=1}^{m} \sum_{\substack{t \in LTS \\rho(t) = 1/2n \\nu'(l(t)) = \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ = \sum_{t \in LTS} \sum_{\substack{\rho(t) = 1/2(n-1) \\nu(t) \neq \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ + \sum_{j=1}^{m} \sum_{\substack{t \in LTS \\rho(t) = 1/2n \\nu'(l(t)) = \tau_j}} \hat{I}_u [F(u)(X_{t_0})]_{t_0, t} \]
\[ = \sum_{t \in LTS} \hat{I}_t [F(t)(X_{t_0})]_{t_0, t} \]

because \( \{ t \in LTS : \rho(t) = 1/2(n + 1) \} = \bigcup_{j=1}^{m} \{ u \in H_1^j(t) \cup \bar{H}_1^j(t) : t \in LTS, \rho(t) = 1/2n \} \cup \{ u \in H_1^0(t) : t \in LTS, \rho(t) = 1/2(n - 1) \}. \) Thus, (19) holds for all \( n \in \mathbb{N}_0. \)
Due to \( \mathcal{B}(A_p) = \bigcup_{j=0}^{m} \{ (j) \ast \alpha \in \mathcal{M} : \alpha \in A_p \} \cup \{ (0) \ast \alpha \in \mathcal{M} : \alpha \in A_{p-1/2} \} \)
follows analogously for the remainder term with \( p \in \frac{1}{2} \mathbb{N} \)
\[
\sum_{\alpha \in \mathcal{B}(A_p)} I_{\alpha} [f_{\alpha}(X_t)]_{t_0, t} = \sum_{t \in \mathcal{T}_S} \hat{I}_t \left[ \int_{t_0}^{t} L^0 F(t)(X_s) \, ds \right]_{t_0, t}
+ \sum_{t \in \mathcal{T}_S} \sum_{u \in H^0(t)} \hat{I}_u [F(u)(X_t)]_{t_0, t} + \sum_{t \in \mathcal{T}_S} \sum_{u \in H^0(t)} \hat{I}_u [F(u)(X_t)]_{t_0, t}
= \sum_{t \in \mathcal{T}_S} \hat{I}_t \left[ \int_{t_0}^{t} (L^0 + \frac{1}{2} \sum_{j=1}^{m} \hat{L}^j) F(t)(X_s) \, ds \right]_{t_0, t} + \sum_{t \in \mathcal{T}_S} \sum_{u \in H^0(t)} \hat{I}_u \left[ \frac{1}{2} \int_{t_0}^{t} F(u)(X_s) \, ds \right]_{t_0, t}
+ \sum_{t \in \mathcal{T}_S} \hat{I}_t [F(t)(X_t)]_{t_0, t}.
\]

Taking into account the order of the symmetry group \( \sigma(t) \) for a rooted tree \( t \in \mathcal{T}_S \), we finally obtain the relationship
\[
\sum_{t \in \mathcal{T}_S} \hat{I}_t [F(t)(X_t)]_{t_0, t} = \sum_{t \in \mathcal{T}_S} \frac{I_{t} [F(t)(X_t)]_{t_0, t}}{\sigma(t)}
\tag{24}
\]
(see also [3]) for all \( n \in \mathbb{N}_0 \) which completes the proof.

A similar stochastic Taylor expansion for the Stratonovich SDE (2) can be obtained where the multiple stochastic integrals are defined with respect to Stratonovich calculus.

**Corollary 3.2** For the solution process \( (X_t)_{t \in [t_0, T]} \) of the Stratonovich SDE (2) and for \( p \in \frac{1}{2} \mathbb{N}_0 \), \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) with \( f, a^i, b^{i,j} \in C^{2p+2}(\mathbb{R}^d, \mathbb{R}) \) for \( i = 1, \ldots, d, j = 1, \ldots, m \), we obtain the expansion
\[
f(X_t) = \sum_{t \in \mathcal{T}_S} \frac{I_{t_0, t} [F(t)(X_t)]}{\sigma(t)} + \mathcal{R}_p(t, t_0)
\tag{25}
\]
P-a.s. with remainder term
\[
\mathcal{R}_p(t, t_0) = \sum_{t \in \mathcal{T}_S} \frac{I_{t_0, t} [F(t)(X_t)]}{\sigma(t)} + \sum_{t \in \mathcal{T}_S} \sum_{u \in H^0(t)} \frac{I_{t_0, t} [F(u)(X_s) \, ds]}{\sigma(t)}
\tag{26}
\]
provided all of the appearing multiple Stratonovich integrals exist.
We leave the proof of Corollary 3.2 to the reader since it is analogously to that of Theorem 3.1 however with the much simpler operator $\hat{L}^0$ instead of $L^0$.

Next, we give some results on the mean–square and mean convergence of the obtained stochastic Taylor expansion by an estimation for the remainder term. Therefore, we denote for $t \in [t_0, T]$ and some $p \in \frac{1}{2} \mathbb{N}_0$ by

$$Z_p(t) = \sum_{t \in TS, \rho(t) \leq p} F(t)(X_{t_0}) \frac{I_{t_0,t}}{\sigma(t)}$$

the truncated stochastic Taylor expansion for SDE (2). Further, let $[p]$ denote the largest integer not exceeding $p$.

**Proposition 3.3** Let $X_{t_0} \in L^2(\Omega)$ and let $(X_t)_{t \in [t_0, T]}$ be the solution of the Itô SDE (2). Suppose that for all $t \in TS$ with $\rho(t) = p + \frac{1}{2}$ and for all $t \in H^0(u) \cup H^1(u)$ with $\rho(u) = p$ exists some constant $C > 0$ such that

$$
\|F(t)(x)\|^2 \leq C(1 + \|x\|^2).
$$

Then there exists a constants $C_p > 0$ depending on $p$ and a constant $C_L > 0$ depending on the Lipschitz constant of the drift and diffusion and on $T$, such that for all $t \in [t_0, T]$

$$
E(\|f(X_t) - Z_p(t)\|^2) \leq C_p(1 + E(\|X_{t_0}\|^2)) \exp(C_L(t - t_0)) \frac{(t - t_0)^{2p+1}}{[p + \frac{1}{2}]!}
$$

for the mean–square truncation error and

$$
\|E(f(X_t) - Z_p(t))\| \leq C_p(1 + E(\|X_{t_0}\|^2))^{1/2} \exp(C_L(t - t_0)) \frac{(t - t_0)^{p+\kappa}}{(p + \kappa)!}
$$

with $\kappa = 1$ if $p \in \mathbb{N}_0$ and $\kappa = 1/2$ if $p \notin \mathbb{N}_0$ for the mean truncation error.

**Proof.** Due to the Existence and Uniqueness Theorem [1,4], for $T > t_0$ there exists a constant $C > 0$ which depends on $T$ and the Lipschitz constants of $a$ and $b^j$, $j = 1, \ldots, m$, such that for all $t \in [t_0, T]$

$$
E(\|X_t\|^2) \leq (1 + E(\|X_{t_0}\|^2)) \exp(C(t - t_0)).
$$

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Then, with (28) and due to the Itô isometry and with the Cauchy-Schwarz inequality follows

\[
\begin{align*}
\mathbb{E}(\|f(X_t) - Z_p(t)\|^2) &= \mathbb{E}(\|\mathcal{R}_p(t, t_0)\|^2) \\
&\leq C_{1,p} \sum_{t \in TS \atop \rho(t)=p+1/2} \mathbb{E}(\|I_{t:t_0,t}[F(t)(X_s)]\|^2) \\
&\quad + C_{2,p} \sum_{t \in TS \atop \rho(t)=p} \sum_{u \in H^0(t)} \mathbb{E}(\|I_{t:t_0,t}[^1_2 \int_{t_0}^t F(u)(X_s) \, ds]\|^2) \\
&\quad + C_{3,p} \sum_{t \in TS \atop \rho(t)=p} \sum_{u \in H^1(t)} \mathbb{E}(\|I_{t:t_0,t}[^1_2 \int_{t_0}^t F(u)(X_s) \, ds]\|^2) \\
&\leq C_{4,p} \sum_{t \in TS \atop \rho(t)=p+1/2} \frac{(t-t_0)^{2\rho(t)}}{[\rho(t)]!} (1 + \mathbb{E}(\|X_{t_0}\|^2)) \exp(C(t-t_0)) \\
&\quad + C_{5,p} \sum_{t \in TS \atop \rho(t)=p} \sum_{u \in H^0(t) \cup H^1(t)} \frac{(t-t_0)^{2\rho(t)+2}}{[\rho(t)+1]!} (1 + \mathbb{E}(\|X_{t_0}\|^2)) \exp(C(t-t_0)) \\
&\leq C_p (1 + \mathbb{E}(\|X_{t_0}\|^2)) \exp(C(t-t_0)) \frac{(t-t_0)^{p+1}}{p+1}.
\end{align*}
\]

Further, we obtain analogously with the Jensen inequality

\[
\begin{align*}
\|\mathbb{E}(f(X_t) - Z_p(t))\|^2 &= \|\mathbb{E}(\mathcal{R}_p(t, t_0))\|^2 \\
&\leq C_{6,p} \sum_{t \in TS \atop \rho(t)=p+1/2} \| \mathbb{E}(I_{t:t_0,t}[F(t)(X_s)])\|^2 \\
&\quad + C_{7,p} \sum_{t \in TS \atop \rho(t)=p} \sum_{u \in H^0(t)} \| \mathbb{E}(I_{t:t_0,t}[\int_{t_0}^t F(u)(X_s) \, ds])\|^2 \\
&\quad + C_{8,p} \sum_{t \in TS \atop \rho(t)=p} \sum_{u \in H^1(t)} \| \mathbb{E}(I_{t:t_0,t}[^1_2 \int_{t_0}^t F(u)(X_s) \, ds])\|^2 \\
&\leq C_{9,p} \sum_{t \in TS \atop \rho(t)=p+1/2} \frac{(t-t_0)^{2\rho(t)+2(\kappa-1/2)}}{((\rho(t) + \kappa - \frac{1}{2})!)^2} (1 + \mathbb{E}(\|X_{t_0}\|^2)) \exp(C(t-t_0)) \\
&\quad + C_{10,p} \sum_{t \in TS \atop \rho(t)=p} \sum_{u \in H^0(t) \cup H^1(t)} \frac{(t-t_0)^{2\rho(t)+2}}{([\rho(t) + 1]!)^2} (1 + \mathbb{E}(\|X_{t_0}\|^2)) \exp(C(t-t_0)) \\
&\leq C_p (1 + \mathbb{E}(\|X_{t_0}\|^2)) \exp(C(t-t_0)) \frac{(t-t_0)^{p+2\kappa}}{((p+\kappa)!)^2}.
\end{align*}
\]

The remainder term \(\mathcal{R}_p(t, t_0)\) from the Stratonovich expansion can be estimated analogously as in the proof of Proposition 3.3. This follows from Re-
mark 5.2.8 [5] because each multiple Stratonovich integral can be written as a sum of multiple Itô integrals of at least the same mean and mean-square orders.

**Corollary 3.4** Let $X_0 \in L^2(\Omega)$ and let $(X_t)_{t \in [t_0,T]}$ be the solution of the Stratonovich SDE (2). Suppose that for all $t \in TS$ with $\rho(t) = p + \frac{1}{2}$ or $\rho(t) = p + 1$ exists some constant $C > 0$ such that

$$\|F(t)(x)\|^2 \leq C(1 + \|x\|^2).$$

Then there exists a constant $C_p > 0$ depending on $p$ and a constant $C_L > 0$ depending on the Lipschitz constant of the drift and diffusion and on $T$, such that for all $t \in [t_0, T]$

$$E(\|f(X_t) - Z_p(t)\|^2) \leq C_p(1 + E(\|X_{t_0}\|^2)) \exp(C_L(t - t_0))(t - t_0)^{2p+1} \left[\frac{p + \frac{1}{2}}{p + \frac{1}{2}}\right]!$$

for the mean–square truncation error and

$$\|E(f(X_t) - Z_p(t))\| \leq C_p(1 + E(\|X_{t_0}\|^2))^{1/2} \exp(C_L(t - t_0))(t - t_0)^{p+\kappa} \left[\frac{p + \kappa}{p + \kappa}\right]!$$

with $\kappa = 1$ if $p \in \mathbb{N}_0$ and $\kappa = 1/2$ if $p \notin \mathbb{N}_0$ for the mean truncation error.

### 4 Example

We consider SDE (2) either with respect to Itô or Stratonovich calculus and give the corresponding stochastic Taylor expansions (15) and (25) based on rooted trees up to order $p = 1.5$. Taking into account the trees presented in Table 1 we obtain

$$f(X_t) = F(t_{0.1})(X_{t_0}) \frac{I_{t_{0.1};t_0,t}}{\sigma(t_{0.1})} + \sum_{1 \leq j_1 \leq m} F(t_{0.5.1})(X_{t_0}) \frac{I_{t_{0.5.1};t_0,t}}{\sigma(t_{0.5.1})}$$

$$+ F(t_{1.1})(X_{t_0}) \frac{I_{t_{1.1};t_0,t}}{\sigma(t_{1.1})} + \sum_{1 \leq j_1 \leq j_2 \leq m} F(t_{1.2})(X_{t_0}) \frac{I_{t_{1.2};t_0,t}}{\sigma(t_{1.2})} + \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m} F(t_{1.3})(X_{t_0}) \frac{I_{t_{1.3};t_0,t}}{\sigma(t_{1.3})}$$

$$+ \sum_{1 \leq j_1 \leq m} F(t_{1.5.1})(X_{t_0}) \frac{I_{t_{1.5.1};t_0,t}}{\sigma(t_{1.5.1})} + \sum_{1 \leq j_1 \leq m} F(t_{1.5.2})(X_{t_0}) \frac{I_{t_{1.5.2};t_0,t}}{\sigma(t_{1.5.2})} + \sum_{1 \leq j_1 \leq m} F(t_{1.5.3})(X_{t_0}) \frac{I_{t_{1.5.3};t_0,t}}{\sigma(t_{1.5.3})}$$

$$+ \sum_{1 \leq j_1 \leq j_2 \leq m} F(t_{1.5.4})(X_{t_0}) \frac{I_{t_{1.5.4};t_0,t}}{\sigma(t_{1.5.4})} + \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m} F(t_{1.5.5})(X_{t_0}) \frac{I_{t_{1.5.5};t_0,t}}{\sigma(t_{1.5.5})}$$

$$+ \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m} F(t_{1.5.6})(X_{t_0}) \frac{I_{t_{1.5.6};t_0,t}}{\sigma(t_{1.5.6})} + \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m} F(t_{1.5.7})(X_{t_0}) \frac{I_{t_{1.5.7};t_0,t}}{\sigma(t_{1.5.7})}$$

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with $R_{1.5}(t, t_0) = \mathcal{R}_{1.5}(t, t_0)$ in the case of the Itô SDE (2) and $R_{1.5}(t, t_0) = \mathcal{R}_{1.5}(t, t_0)$ in the case of the Stratonovich SDE (2). Now, we apply the indicator function $1_{\{* \neq 0\}}$ which vanishes in the case of Stratonovich calculus. Calculating the elementary differentials and the multiple integrals, we get

$$f(X_t) = f(X_{t_0}) + \sum_{1 \leq j_1 \leq m} \sum_{J, L = 1}^d \frac{\partial f}{\partial x^J} a^{J_1} (X_{t_0}) I_{(J_1)}[1]_{t_0, t}$$

$$+ \sum_{J = 1}^d \frac{\partial f}{\partial x^J} a^{J_1} (X_{t_0}) I_{(J_1)}[1]_{t_0, t}$$

$$+ \sum_{1 \leq j_1 \leq j_2 \leq m, J, K = 1}^d \sum \frac{\partial^2 f}{\partial x^J \partial x^K} b^{J_1 j_1} b^{K_j 2} (X_{t_0})$$

$$\times I_{(J_1, j_2)}[1]_{t_0, t} + I_{(j_1, j_2)}[1]_{t_0, t} + I_{(0)}[1_{j_1 = j_2 \wedge \neq 0}]_{t_0, t}$$

$$+ \sum_{1 \leq j_1 \leq m, J, K = 1}^d \sum \frac{\partial f}{\partial x^J} \frac{\partial b^{J_1 j_1}}{\partial x^K} b^{K_j 2} (X_{t_0}) I_{(j_2, j_1)}[1]_{t_0, t}$$

$$+ \sum_{1 \leq j_1 \leq m, J, K = 1}^d \sum \frac{\partial f}{\partial x^J} \frac{\partial b^{J_1 j_1}}{\partial x^K} a^K (X_{t_0}) I_{(j_1, 0)}[1]_{t_0, t}$$

$$+ \sum_{1 \leq j_1 \leq m, J, K = 1}^d \sum \frac{\partial^2 f}{\partial x^J \partial x^K} a^{J_1} b^{K_j 2} (X_{t_0}) (I_{(J_1, 0)}[1]_{t_0, t} + I_{(0, j_1)}[1]_{t_0, t})$$

$$+ \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m, J, K, L = 1}^d \frac{\partial^3 f}{\partial x^J \partial x^K \partial x^L} b^{J_1 j_1} b^{K_j 2} b^{L_j 3} (X_{t_0})$$

$$\times I_{(J_1, j_2, j_3)}[1]_{t_0, t} + I_{(j_2, j_3, j_1)}[1]_{t_0, t} + I_{(j_1, j_2, j_3)}[1]_{t_0, t} + I_{(0, j_3)}[1_{j_1 = j_2 \wedge \neq 0}]_{t_0, t}$$

$$+ I_{(j_3, 0)}[1_{j_1 = j_2 \wedge \neq 0}]_{t_0, t} + I_{(j_2, 0)}[1_{j_1 = j_2 \wedge \neq 0}]_{t_0, t} + I_{(j_1, 0)}[1_{j_2 = j_3 \wedge \neq 0}]_{t_0, t}$$

$$+ I_{(0, j_1)}[1_{j_2 = j_3 \wedge \neq 0}]_{t_0, t} + I_{(j_1, 0)}[1_{j_1 = j_3 \wedge \neq 0}]_{t_0, t}$$

$$+ I_{(1)}[1_{j_1 \neq j_2 \neq j_3 \neq 1}] + 2 \cdot 1_{j_1 = j_2 \neq 0} + 2 \cdot 1_{j_1 = j_2 \wedge \neq 0} + 6 \cdot 1_{j_1 = j_2 = j_3}$$

$$+ \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m, J, K = 1}^d \sum \frac{\partial^2 f}{\partial x^J \partial x^K} b^{J_1 j_1} b^{K_j 2} (X_{t_0})$$

$$\times I_{(J_1, j_2, j_3)}[1]_{t_0, t} + I_{(j_2, j_3, j_1)}[1]_{t_0, t} + I_{(j_1, j_2, j_3)}[1]_{t_0, t}$$

$$+ I_{(0, j_1)}[1_{j_2 = j_3 \wedge \neq 0}]_{t_0, t} + I_{(j_1, 0)}[1_{j_3 = j_2 \wedge \neq 0}]_{t_0, t}$$

$$+ \sum_{1 \leq j_1 \leq j_2 \leq j_3 \leq m, J, K, L = 1}^d \frac{\partial f}{\partial x^J} \frac{\partial^2 b^{J_1 j_1}}{\partial x^K \partial x^L} b^{K_j 2} b^{L_j 3} (X_{t_0})$$
Here, we obtain with Proposition 3.3 that

\begin{equation}
\begin{aligned}
&\times I_{(j_3,j_2,j_1)}[1]_{t_{0},t} + I_{(j_2,j_3,j_1)}[1]_{t_{0},t} + I_{(0,j_1)}[1_{\{j_2=j_3\wedge \ast \neq \emptyset\}}]_{t_{0},t} \\
+ &\sum_{1\leq j_1,j_2,j_3\leq m} \sum_{j,K,L=1}^{d} \left( \frac{\partial f}{\partial x^j} + \frac{\partial b_{j_1}^K}{\partial x^j} + \frac{\partial b_{j_2}^L}{\partial x^j} \right) (X_{t_{0}}) I_{(j_3,j_2,j_1)}[1]_{t_{0},t} \\
+ &R_{1.5}(t,t_{0}).
\end{aligned}
\end{equation}

As the main advantage, each elementary differential corresponds exactly to one tree. Further, all equal elementary differentials are pooled with a corresponding weight which may be a sum of some multiple stochastic integrals. For example, if we apply the calculated expansion of order \( p = 1.5 \) to the Itô SDE

\begin{equation}
dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_{t_0} = x_0, \tag{35}
\end{equation}

with \( d = m = 1 \) and \( \mu, \sigma \in \mathbb{R} \), then we obtain for \( f(x) = x \) the expansion

\begin{align*}
X_t &= x_0 + \sigma x_0 I_{(1)} + \mu x_0 I_{(0)} + \sigma^2 x_0 I_{(1,1)} + \mu \sigma x_0 I_{(1,0)} \\
&\quad + \sigma \mu x_0 I_{(0,1)} + \sigma^3 x_0 I_{(1,1,1)} + \mathcal{R}_{1.5}(t,t_{0}). \tag{36}
\end{align*}

Here, we obtain with Proposition 3.3 that \( (E(\|\mathcal{R}_{1.5}(t,t_{0})\|^2))^{1/2} = O((t-t_{0})^{3/2}) \) as an estimate for the mean–square error and \( \|E(\mathcal{R}_{1.5}(t,t_{0}))\| = O((t-t_{0})^{2}) \) for the mean error.

| Tree | Tree | Tree | Tree |
|------|------|------|------|
| \( t_{0.1} \) | \( \gamma \) | \( t_{0.5.1} \) | \( [\tau_{j_1}]_{\gamma} \) |
| \( t_{1.1} \) | \( [\tau_0]_{\gamma} \) | \( t_{1.2} \) | \( [\tau_{j_1}, \tau_{j_2}]_{\gamma} \) |
| \( t_{1.5.1} \) | \( [\tau_{j_1}, \tau_0]_{\gamma} \) | \( t_{1.5.2} \) | \( [\tau_{0}, j_1]_{\gamma} \) |
| \( t_{1.5.4} \) | \( [\tau_{j_1}, \tau_{j_2}, \tau_3]_{\gamma} \) | \( t_{1.5.5} \) | \( [\tau_{j_2}, \tau_{j_1}, \tau_3]_{\gamma} \) |
| \( t_{1.5.7} \) | \( [\tau_{j_3}, \tau_{j_2}, \tau_{j_1}]_{\gamma} \) | \( t_{1.5.6} \) | \( [\tau_{j_2}, \tau_{j_1}, \tau_{j_3}]_{\gamma} \) |
| \( t_{2.1} \) | \( [\tau_0]_{\gamma} \) | \( t_{2.2} \) | \( [\tau_0, \tau_0]_{\gamma} \) |
| \( t_{2.4} \) | \( [\tau_{j_1}, \tau_{j_2}]_{\gamma} \) | \( t_{2.5} \) | \( [\tau_{j_1}, [\tau_{j_2}]_{\gamma}] \) |
| \( t_{2.7} \) | \( [\tau_{j_1}, \tau_{j_2}, \tau_0]_{\gamma} \) | \( t_{2.8} \) | \( [\tau_{j_1}, [\tau_0]_{\gamma}] \) |
| \( t_{2.10} \) | \( [\tau_{j_2}, \tau_0]_{\gamma} \) | \( t_{2.11} \) | \( [\tau_{j_2}, \tau_{j_2}, \tau_3, \tau_4]_{\gamma} \) |
| \( t_{2.13} \) | \( [\tau_{j_1}, \tau_{j_3}, \tau_{j_4}]_{\gamma} \) | \( t_{2.14} \) | \( [\tau_{j_1}, [\tau_{j_3}]_{\gamma}]_{\gamma} \) |
| \( t_{2.16} \) | \( [\tau_{j_2}, \tau_{j_3}, \tau_{j_4}]_{\gamma} \) | \( t_{2.17} \) | \( [\tau_{j_2}, [\tau_{j_3}]_{\gamma}]_{\gamma} \) |
| \( t_{2.19} \) | \( [\tau_{j_4}, \tau_{j_3}, \tau_{j_2}, \tau_{j_1}]_{\gamma} \) | \( t_{2.20} \) | \( [\tau_{j_2}, [\tau_0]_{\gamma}] \) |

Table 1: Trees \( t \in TS \) of order \( \rho(t) \leq 2 \) with \( j_1, j_2, j_3, j_4 \in \{1, \ldots, m\} \).
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