A full fuzzy method for solving fuzzy fractional differential equations based on the generalized Taylor expansion

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Abstract In many mathematical researches, in order to solve the fuzzy fractional differential equations we should transform these problems to crisp corresponding problems and by solving them the approximate solution can be obtained. The aim of this paper is to present a new direct method to solve the fuzzy fractional differential equations without this transformation. In this work, the fuzzy generalized Taylor expansion by using the sense of fuzzy Caputo fractional derivative for fuzzy-valued functions is presented. For solving fuzzy fractional differential equations, the fuzzy generalized Euler’s method is applied. In order to show the accuracy and efficiency of presented method, the local and global truncation errors are determined. Moreover, the consistence, the convergence and the stability of the generalized Euler’s method are proved in details. Eventually, the numerical examples specially in switching point case show the exibility and the capability of the presented method.

Keywords Generalized Hukuhara derivative · Fuzzy fractional initial value problems · Fuzzy generalized Taylor expansion · Fuzzy generalized Euler’s method · Global truncation error · Local truncation error.

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1 Introduction

Fuzzy set theory is a powerful tool for modeling uncertain problems. Therefore, large varieties of natural phenomena have been modeled using fuzzy concepts. Particularly, fuzzy fractional differential equation is a common model in different science, such as population models, evaluating weapon system, civil engineering and modeling electro-hydraulic. Hence the concept of fractional derivative is a very important topic in fuzzy calculus. Therefore, fuzzy fractional differential equations have attracted lots of attention in mathematics and engineering researches [6, 27]. First work devoted to the subject of fuzzy fractional differential equations is the paper by Agarwal et al. [2]. They have defined the Riemann-Liouville differentiability concept under the Hukuhara differentiability to solve fuzzy fractional differential equations.

In recent years, fractional calculus has introduced as an applicable topic to produce the accurate results of mathematical and engineering problems such as aerodynamics and control systems, signal processing, bio-mathematical problems and others [2, 10, 17, 25, 28, 37].

Furthermore, fractional differential equations in the fuzzy case [2] have studied by many authors and they have solved by various methods [11, 26]. In [15, 16] Hoa et al. studied the fuzzy fractional differential equations under Caputo-Katugampola fractional derivative and Caputo gH-differentiability and in [1] Agarwal et al. had a survey on mentioned problem to show the its relation with optimal control problems. Also, Long et al. [18] illustrated the solvability of fuzzy fractional differential equations and Salahshour et al. [34] applied the fuzzy Laplace transforms to solve this problem.

There are many numerical methods to solve the fuzzy fractional differential equations by transforming to crisp problems [3, 20, 22]. In this paper, a new direct method is introduced to solve the mentioned problem without changing to crisp form. Taylor expansion method is one of the famous and applicable methods to solve the linear and non-linear problems [21, 30, 38].

In this paper, the fuzzy generalized Taylor expansion based on the fuzzy Caputo fractional derivative is expanded. Then the Euler’s method is applied to solve the fuzzy fractional differential equations. Also, the local and global truncation errors are considered and finally the consistence, the convergence and the stability of the generalized Euler’s method are demonstrated. Furthermore, some examples with switching point problem are solved by using the presented method. The numerical results show the precision of the generalized Euler’s method to solve the fuzzy fractional differential equations.

2 Basic Concepts

At first, the brief summary of the fuzzy details and some preliminaries are revisited [8, 9, 14, 19, 23, 24, 29, 35].

**Definition 1** Set $\mathcal{F} = \{ u : \mathbb{R}^{n} \rightarrow [0, 1] \text{ such that } u \text{ satisfies in the conditions I to IV } \}$
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I. $u$ is normal: there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$,

II. $u$ is fuzzy convex: for $0 \leq \lambda \leq 1$, $u(\lambda x_1 + (1-\lambda x_2)) \geq \min\{u(x_1), u(x_2)\}$,

III. $u$ is upper semi-continuous: for any $x_0 \in \mathbb{R}^n$, it holds that $u(x_0) \geq \lim_{x \to x_0} u(x)$,

IV. $[u]_0 = \text{supp}(u) = \text{cl}\{x \in \mathbb{R}^n \mid u(x) > 0\}$ is a compact subset,

is called the space of fuzzy numbers or the fuzzy numbers set. The $r$-level set is $[u]_r = \{x \in \mathbb{R}^n \mid u(x) \geq r, 0 < r \leq 1\}$. Then from I to IV, it follows that, the $r$-level sets of $u \in \mathbb{R}_F$ are nonempty, closed and bounded intervals.

**Definition 2** A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_F$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $\underline{u}(r) = a + (b - a)r$ (or lower bound of $u$) and $\overline{u}(r) = c - (c - b)r$ (or upper bound of $u$) are the endpoints of $r$-level sets for all $r \in [0, 1]$.

A crisp number $k$ is simply represented by $\underline{u}(r) = u(r) = k$, $0 \leq r \leq 1$ and called singleton. For arbitrary $u, v \in \mathbb{R}_F$ and scalar $k$, we might summarize the addition and the scalar multiplication of two fuzzy numbers by

- **addition**: $[u \oplus v]_r = u(r) + v(r)$,
- **scalar multiplication**: $[k \odot u]_r = \begin{cases} k\underline{u}(r), & k \geq 0, \\ k\overline{u}(r), & k < 0. \end{cases}$

The Hausdorff distance between fuzzy numbers is given by $H : \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}^+ \cup \{0\}$ as:

$$H(u, v) = \sup_{0 \leq r \leq 1} \max\{|\underline{u}(r) - u(r)|, |\overline{u}(r) - \overline{u}(r)|\},$$

where $[u]_r = [\underline{u}(r), \overline{u}(r)]$, $[v]_r = [\underline{v}(r), \overline{v}(r)]$. The metric space $(\mathbb{R}_F, H)$ is complete, separable and locally compact where the following conditions are valid for metric $H$:

I. $H(u \oplus w, v \oplus w) = H(u, v)$, $\forall u, v, w \in \mathbb{R}_F$.

II. $H(\lambda u, \lambda v) = |\lambda| H(u, v)$, $\forall \lambda \in \mathbb{R}$, $\forall u, v \in \mathbb{R}_F$.

III. $H(u \oplus v, w \oplus z) \leq H(u, w) + H(v, z)$, $\forall u, v, w, z \in \mathbb{R}_F$.

**Definition 3** Let $u, v \in \mathbb{R}_F$, if there exists $w \in \mathbb{R}_F$, such that $u = v + w$, then $w$ is called the Hukuhara difference (H-difference) of $u$ and $v$, and it is denoted by $u \ominus v$. Furthermore, the generalized Hukuhara difference (gH-difference) of two fuzzy numbers $u, v \in \mathbb{R}_F$ is defined as follows

$$u \ominus_{gH} v = w \iff \begin{cases} (i) & u = v + w, \\ \text{or} & (ii) \ v = u + (-1)w. \end{cases}$$

It is easy to show that conditions (i) and (ii) are valid if and only if $w$ is a crisp number. The conditions of the existence of $u \ominus_{gH} v \in \mathbb{R}_F$ are given in [9].
In this paper, the meaning of fuzzy-valued function is a function \( f : A \rightarrow \mathbb{R}_F \), \( A \in \mathbb{R} \) where \( \mathbb{R} \) is the set of all real numbers and \([f(t)]_r = [f^-(t;r), f^+(t;r)]\) so called the \( r \)-cut or parametric form of the fuzzy-valued function \( f \).

**Definition 4** A fuzzy-valued function \( f : [a, b] \rightarrow \mathbb{R}_F \) is said to be continuous at \( t_0 \in [a, b] \) if for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( H(f(t), f(t_0)) < \varepsilon \), whenever \( t \in [a, b] \) and \( |t - t_0| < \delta \). We say that \( f \) is fuzzy continuous on \([a, b]\) if \( f \) is continuous at each \( t_0 \in [a, b] \).

Throughout the rest of this paper, the notation \( C_f([a, b], \mathbb{R}_F) \) is called the set of fuzzy-valued continuous functions which are defined on \([a, b]\).

If \( f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}_F \) be continuous by the metric \( H \) then \( \int_a^b f(s)ds \) is a continuous function in \( t \in [a, b] \) and the function \( f \) is integrable on \([a, b]\).

Furthermore it holds

\[
\left[ \int_a^b f(s)ds \right]_r = \left[ \int_a^b f^-(s;r)ds, \int_a^b f^+(s;r)ds \right].
\]

**Definition 5** Let \( f : [a, b] \rightarrow \mathbb{R}_F \), \( t_0 \in (a, b) \) with \( f^-(t;r) \) and \( f^+(t;r) \) both differentiable at \( t_0 \) for all \( r \in [0, 1] \) and \( D_{gH}(gH\text{-derivative}) \) exists:

I. The function \( f \) is \( F[(i)-gH] \)-differentiable at \( t_0 \) if \( [D_{gH}f(t_0)]_r = [Df^-(t_0;r), Df^+(t_0;r)] \).

II. The function \( f \) is \( F[(ii)-gH] \)-differentiable at \( t_0 \) if \( [D_{gH}f(t_0)]_r = [Df^+(t_0;r), Df^-(t_0;r)] \).

### 3 Definitions and Properties of Fractional \( gH \)-Differentiability

In this section, let us focus on some definitions and properties related to the fuzzy fractional generalized Hukuhara derivative which are useful in the sequel of this paper.

**Definition 6** [7] Let \( f(t) \) be a fuzzy Lebesgue integrable function. The fuzzy Riemann-Liouville fractional (for short \((F.RL)\)-fractional) integral of order \( \alpha > 0 \) is defined as follows

\[
F^{RL}_{[a, t]} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds.
\]

**Definition 7** [7] Let \( f : [a, b] \rightarrow \mathbb{R}_F \). The fuzzy fractional derivative of \( f(t) \) in the Caputo sense is in the following form

\[
F^{C}_{[a, t]} f(t) = F^{RL}_{[a, t]} \left( D^m_{gH}f(t) \right) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} D^m_{gH}f(s)ds,
\]

\( m - 1 < \alpha < m, \; m \in \mathbb{N}, \; t > a, \)
where \( \forall m \in \mathbb{N}, D^mf_{gH}(s) \) (\( gH \)-derivatives of \( f \)) are integrable. In this paper, we consider fuzzy Caputo generalized Hukuhara derivative (for short \( FC[gH] \)-derivative) of order \( 0 < \alpha \leq 1 \), for fuzzy-valued function \( f \), so the \( FC[gH] \)-derivative will be expressed by

\[
FC^\alpha Df(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} Df_{gH}(s) \, ds, \quad t > a.
\]

(1)

**Lemma 1** Let \( f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}_\mathcal{F} \) be continuous. Then \( FC^\alpha f(t) \), for \( 0 < \alpha \leq 1 \) and \( t \in [a, b] \) is a continuous function.

**Proof.** Under assumptions of the continuous functions, \( f(s) \) is a fuzzy Lebesgue integrable function. On the other hand, since \( \forall 0 < \alpha \leq 1, (t-s)^{\alpha-1} \geq 0 \) is continuous, so \( \int_a^t (t-s)^{\alpha-1} f(s) \, ds \) is a continuous function and as a result \( FC^\alpha f(t) \) is a continuous function in \( t \in [a, b] \).

**Lemma 2** Let \( f \in C_f(\mathbb{R}, \mathbb{R}_\mathcal{F}), m \in \mathbb{N} \). Then the fuzzy Riemann-Liouville fractional integrals \( FC^\alpha \int_{[a,t]}(FC^\alpha \int_{[a,t_1]}...(FC^\alpha \int_{[a,t_{m-1}]} f(t_{m-1}))...)(t_{m-1}) \) for \( 0 < \alpha \leq 1 \), are continuous functions in \( t_{m-1}, t_{m-2}, ..., t \), respectively. Here \( t_{m-1}, t_{m-2}, ..., t \geq a \) and they are real numbers.

**Proof** This lemma is a fairly straightforward generalization of Lemma 1. The proof will be done by introducing on \( m \in \mathbb{N} \). Assume that the lemma holds for \( m \)-times applying operator \( FC \)-fractional integrating for function \( f \), we will prove it will correct for \( (m+1) \)-times applying operator \( FC \)-fractional integrating for function \( f \). By Lemma 1 since \( f \in C_f(\mathbb{R}, \mathbb{R}_\mathcal{F}) \) thus \( FC^\alpha \int_{[a,t_{m-1}]} f(t_{m-1}) \) is a continuous function in \( t_{m-1} \). Furthermore, under the hypothesis of induction,

\[
(FC^\alpha \int_{[a,t_{m-1}]}(FC^\alpha \int_{[a,t_{m-2}]}...) f(t_{m-1})...)(t_{m-1}) \}
\]

are continuous functions in \( t_{m-1}, t_{m-2}, t_{m-3}, ..., t \), respectively. It follows easily that

\[
(FC^\alpha \int_{[a,t_{m-1}]}(FC^\alpha \int_{[a,t_{m-2}]}...) f(t_{m-1})...)(t_{m+1}) \}
\]

is a continuous function in \( t_{m+1} \), which is our claim.

**Definition 8** Let \( f : [a, b] \to \mathbb{R}_\mathcal{F} \) be the fuzzy Caputo generalized Hukuhara differentiable (for short \( FC[gH] \)-differentiable) at \( t_0 \in [a, b] \) if for \( 0 \leq r \leq 1 \)

\[
FC^\alpha D^rf_{gH}(t_0) = [C^\alpha D^r f^-(t_0; r), C^\alpha D^r f^+(t_0; r)],
\]

and that \( f \) is \( FC([i]) - gH \)-differentiable at \( t_0 \) if

\[
[FC^\alpha D^r f_{gH}(t_0)]_r = [C^\alpha D^r f^+(t_0; r), C^\alpha D^r f^-(t_0; r)],
\]
where
\[ C D_\alpha^a f^-(t_0; r) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_0} (t_0-s)^{-\alpha} Df^-(s;r) ds, \]
\[ C D_\alpha^a f^+(t_0; r) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t_0} (t_0-s)^{-\alpha} Df^+(s;r) ds. \]

**Definition 9** [2] Let \( f : [a, b] \to \mathbb{R}_\mathcal{F} \) be a fuzzy-valued function on \([a, b]\). A point \( t_0 \in [a, b] \) is said to be a switching point for the \( FC[gH] \)-differentiability of \( f \), if in any neighborhood \( V \) of \( t_0 \) there exist points \( t_1 < t_0 < t_2 \) such that

(type I) \( f \) is \( FC[(i)gH] \)-differentiable at \( t_1 \) while \( f \) is not \( FC[(ii)gH] \)-differentiable at \( t_1 \), and \( f \) is \( FC[(ii)gH] \)-differentiable at \( t_2 \) while \( f \) is not \( FC[(i)gH] \)-differentiable at \( t_2 \), or

(type II) \( f \) is \( FC[(ii)gH] \)-differentiable at \( t_1 \) while \( f \) is not \( FC[(i)gH] \)-differentiable at \( t_1 \), and \( f \) is \( FC[(i)gH] \)-differentiable at \( t_2 \) while \( f \) is not \( FC[(ii)gH] \)-differentiable at \( t_2 \).

**Theorem 31** [2] If \( f : [a, b] \to \mathbb{R}_\mathcal{F} \), \([f(t)]_r = [f^-(t;r), f^+(t;r)]\) and \( f \) is integrable for \( 0 \leq r \leq 1 \), \( t \in [a, b] \) and \( \alpha, \beta > 0 \) then we have
\[ FRLI_{[a,t]}^\alpha (FRLI_{[a,t]}^\beta f)(t) = FRLI_{[a,t]}^{\alpha+\beta} f(t). \]

**Lemma 3** [2] Suppose that \( f : [a, b] \to \mathbb{R}_\mathcal{F} \) be a fuzzy-valued function and \( Df_{gH} \) is exist, then for \( 0 < \alpha \leq 1 \),
\[ DRLI_{[a,t]}^\alpha (FC D_\alpha^a f)(t) = f(t) \odot_{gH} f(a), \quad 0 \leq r \leq 1. \]

The principal significance of this lemma is in the following theorem:

**Theorem 32** [2] Let \( f : [a, b] \to \mathbb{R}_\mathcal{F} \) be the fractional \( gH \)-differentiable such that type of Caputo differentiability \( f \) in \([a, b]\) does not change. Then for \( a \leq t \leq b \) and \( 0 < \alpha \leq 1 \),

I. If \((f(s) \in FC[(i)gH] \)-differentiable then \( FC D_\alpha^a f_i,gH(t) \) is \((F.RL)\)-integrable over \([a, b]\) and
\[ f(t) = f(a) \oplus FRLI_{[a,t]}^\alpha (FC D_\alpha^a f_i,gH)(t), \]

II. If \((f(s) \in FC[(ii)gH] \)-differentiable then \( FC D_\alpha^a f_{ii},gH(t) \) is \((F.RL)\)-integrable over \([a, b]\) and
\[ f(t) = f(a) \ominus (-1)^k FRLI_{[a,t]}^\alpha (FC D_\alpha^a f_{ii},gH)(t). \]

**Lemma 4** Suppose that \( f : [a, b] \to \mathbb{R}_\mathcal{F} \) is the fractional \( gH \)-differentiable and \( FC D_\alpha^a f_{gH}(t) \in C_f([a, b], \mathbb{R}_\mathcal{F}) \) then for \( 0 < \alpha \leq 1 \),
\[ FRLI_{[a,t]}^\alpha (FC D_\alpha^a f_{gH})(t) = (-1) \oplus FRLI_{[a,t]}^\alpha (FC D_\alpha^a f_{ii},gH)(t), \]
Proof. Since $FC^\alpha f_{i, gH}(t)$ is continuous, it follows that $FC^\alpha f_{i, gH}(t)$ is the Riemann-Liouville integrable, and by using Lemma 3 for $0 \leq r \leq 1$

$$\int [FC^\alpha f_{i, gH}(t)]_r = [FC^\alpha f_{i, gH}(t)]_r - [FC^\alpha f_{i, gH}(t)]_r$$

Moreover,

$$\int [FC^\alpha f_{i, gH}(t)]_r = [FC^\alpha f_{i, gH}(t)]_r - [FC^\alpha f_{i, gH}(t)]_r$$

By combining Eqs (2) with (3) the lemma is proved.

Theorem 33 Let $FC^\alpha f : [a, b] \rightarrow \mathbb{R}_f$ and $FC^\alpha f \in C_f([a, b], \mathbb{R}_f)$. For all $t \in [a, b]$ and $0 < \alpha \leq 1$,

I. Let $FC^\alpha f_{i, gH}(t)$ is continuous, then

$$FC^\alpha f_{i, gH}(t) = FC^\alpha f_{i, gH}(a) \circ \frac{d^\alpha}{dt^\alpha} f_{i, gH}(t).$$

II. If $FC^\alpha f_{i, gH}(t)$, where $i = 1, ..., n$, are the $FC^\alpha f_{i, gH}(t)$ and the type of differentiability does not change in the interval $[a, b]$, then

$$FC^\alpha f_{i, gH}(t) = FC^\alpha f_{i, gH}(a) \circ \frac{d^\alpha}{dt^\alpha} f_{i, gH}(t).$$

III. Assume that $FC^\alpha f_{i, gH}(t)$, where $i = 2k - 1$, $k \in \mathbb{N}$, are the $FC^\alpha f_{i, gH}(t)$ and they are $FC^\alpha f_{i, gH}(t)$, for $i = 2k$, $k \in \mathbb{N}$

$$FC^\alpha f_{i, gH}(t) = FC^\alpha f_{i, gH}(a) \circ \frac{d^\alpha}{dt^\alpha} f_{i, gH}(t).$$

IV. Suppose that $FC^\alpha f_{i, gH}(t)$, where $i = 2k - 1$, $k \in \mathbb{N}$, are the $FC^\alpha f_{i, gH}(t)$ and they are $FC^\alpha f_{i, gH}(t)$ for $i = 2k$, $k \in \mathbb{N}$, so

$$FC^\alpha f_{i, gH}(t) = FC^\alpha f_{i, gH}(a) \circ \frac{d^\alpha}{dt^\alpha} f_{i, gH}(t).$$

Proof. By assuming $FC^\alpha f \in C_f([a, b], \mathbb{R}_f)$, $i = 0, ..., n$ we give the proof only for parts II and III. Proving the other parts are similar.

II. Our proof starts with the observation that $FC^\alpha f_{i, gH}(t)$, where $i = 1, ..., n$, are $FC^\alpha f_{i, gH}(t)$ and they are $FC^\alpha f_{i, gH}(t)$ for $i = 2k$, $k \in \mathbb{N}$, so

$$FC^\alpha f_{i, gH}(t) = FC^\alpha f_{i, gH}(a) \circ \frac{d^\alpha}{dt^\alpha} f_{i, gH}(t).$$
Thus, we obtain

\[
FC D_s^{(i-1)\alpha} f_{i, gH}(t) = FC D_s^{(i-1)\alpha} f_{i, gH}(a) \oplus F.RL I_{[a,t]}^{\alpha}(FC D_s^{\alpha} f_{i, gH})(t).
\]

III. Under the conditions stated in the part III, \( FC D_s^{\alpha} f \) is \( FC[(i) – gH] \)-differentiable for \( i = 2k-1, \) \( k \in \mathbb{N} \) and it is \( FC[(ii) – gH] \)-differentiable for \( i = 2k, \) \( k \in \mathbb{N} \). In the sense of Section 2 and by Theorem [32] we get

\[
\begin{align*}
[FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r &= [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r \\
&= [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r + [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r - [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r \\
&= [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r + [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r - [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r \\
&= [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r + [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r - [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r \\
&= [FC D_s^{(i-1)\alpha} f_{i, gH}(t)]_r,
\end{align*}
\]

which completes the proof.

4 Fuzzy Generalized Taylor Theorem

Theorem 41 Let \( T = [a, a+\beta] \subset \mathbb{R} \), with \( \beta > 0 \) and \( FC D_s^{\alpha} f \in \mathcal{C}_f([a, b], \mathbb{R}_f) \), \( i = 1, \ldots, n \). For \( t \in T \), \( 0 < \alpha \leq 1 \)

I. If \( FC D_s^{\alpha} f, \ i = 0, 1, \ldots, n-1 \) are \( FC[(i) – gH] \)-differentiable, provided that type of fuzzy Caputo differentiability has no change. Then

\[
f(t) = f(a) \oplus FC D_s^{\alpha} f_{i, gH}(a) \odot \frac{(t-a)^{\alpha}}{\Gamma (\alpha + 1)} \oplus FC D_s^{2\alpha} f_{i, gH}(a) \odot \frac{(t-a)^{2\alpha}}{\Gamma (2\alpha + 1)} \oplus \ldots \oplus FC D_s^{(n-1)\alpha} f_{i, gH}(a) \odot \frac{(t-a)^{(n-1)\alpha}}{\Gamma ((n-1)\alpha + 1)} \odot R_n(a, t),
\]

where \( R_n(a, t) := F.RL I_{[a,t]}^{\alpha}(F.RL I_{[a,t]}^{\alpha}(F.RL I_{[a,t]}^{\alpha}(F.C D_s^{\alpha} f_{i, gH})(t_n))\ldots) \).

II. If \( FC D_s^{\alpha} f, \ i = 0, 1, \ldots, n-1 \) are \( FC[(ii) – gH] \)-differentiable, provided that type of fuzzy Caputo differentiability has no change. Then

\[
f(t) = f(a) \odot (-1) FC D_s^{\alpha} f_{i, gH}(a) \odot \frac{(t-a)^{\alpha}}{\Gamma (\alpha + 1)} \oplus (-1) FC D_s^{2\alpha} f_{i, gH}(a) \odot \frac{(t-a)^{2\alpha}}{\Gamma (2\alpha + 1)} \oplus \ldots \oplus (-1) FC D_s^{(n-1)\alpha} f_{i, gH}(a) \odot \frac{(t-a)^{(n-1)\alpha}}{\Gamma ((n-1)\alpha + 1)} \odot (-1) R_n(a, t),
\]

where \( R_n(a, t) := F.RL I_{[a,t]}^{\alpha}(F.RL I_{[a,t]}^{\alpha}(F.RL I_{[a,t]}^{\alpha}(F.C D_s^{\alpha} f_{i, gH})(t_n))\ldots) \).
III. If $F^C D^\alpha_{f, i} f, i = 2k - 1, k \in \mathbb{N}$ are $F^C[(i) - gH]-\text{differentiable}$ and $F^C D^\alpha_{f, i} f, i = 2k, k \in \mathbb{N} \cup \{0\}$ are $F^C[(ii) - gH]-\text{differentiable}$, then

\[ f(t) = f(a) \odot (-1)\ F^C D^\alpha_{f, i} f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{\Gamma(\alpha + 1)} \odot F^C D^{2\alpha}_{f, i} f_{i,gH}(a) \odot \frac{(t-a)^{2\alpha}}{\Gamma(2\alpha + 1)} \]

\[ \odot (-1) \odot (-1) F^C D^\alpha_{f, i} f_{i,gH}(a) \odot \frac{(t-a)\left(\frac{1}{2}\right)^\alpha}{\Gamma(\frac{\alpha}{2} + 1)} \odot (-1) \odot (-1) R_n(a,t), \]

where $R_n(a,t) := F.RL I^\alpha_{[a,t]}(F.RL I^\alpha_{[a,t_1]}(F.RL I^\alpha_{[a,t_{n-1}]}(F^C D^\alpha_{f, i} f_{i,gH})(t_n))...)$.

IV. For $F^C D^\alpha_{f, i} f \in C_f([a,b], \mathbb{R}_f), n \geq 3$, suppose that $f$ on $[a,\xi]$ is $F^C[(ii) - gH]-\text{differentiable}$ and on $[\xi, b]$ is $F^C[(i) - gH]-\text{differentiable}$, in fact $\xi$ is switching point (type II) for $\alpha$-order derivative of $f$. Moreover, for $t_0 \in [a, \xi]$, let $2\alpha$-order derivative of $f$ in $\xi_i$ of $[t_0, \xi]$ have switching point (type I). On the other hand, the type of differentiability for $F^C D^\alpha_{f, i} f, i \leq n$ on $[\xi, b]$ does not change. So

\[ f(t) = f(t_0) \odot (-1) F^C D^\alpha_{f, i} f_{i,gH}(t_0) \odot \frac{(\xi - t_0)^\alpha}{\Gamma(\alpha + 1)} \odot (-1) F^C D^{2\alpha}_{f, i} f_{i,gH}(t_0) \odot \frac{(t_0 - \xi)^{2\alpha}}{\Gamma(2\alpha + 1)} \]

\[ \odot F^C D^\alpha_{f, i} f_{i,gH}(\xi) \odot \frac{(t-\xi)^\alpha}{\Gamma(\alpha + 1)} \odot F^C D^{2\alpha}_{f, i} f_{i,gH}(\xi) \odot \frac{(t-\xi)^{2\alpha}}{\Gamma(2\alpha + 1)} \]

\[ F.RL I^\alpha_{[t_0,\xi]} F.RL I^\alpha_{[\xi_1,\xi]} F.RL I^\alpha_{[t_1,\xi_2]}(F^C D^{3\alpha}_{f, i} f_{i,gH})(t_4) \]

\[ \odot (-1) F.RL I^\alpha_{[t_0,\xi]} F.RL I^\alpha_{[\xi_1,\xi]} F.RL I^\alpha_{[t_1,\xi_2]}(F^C D^{3\alpha}_{f, i} f_{i,gH})(t_5) \]

\[ F.RL I^\alpha_{[t_1,\xi]} F.RL I^\alpha_{[\xi_1,\xi]} F.RL I^\alpha_{[t_2,\xi_1]}(F^C D^{3\alpha}_{f, i} f_{i,gH})(t_6). \]

Proof. Under the assumptions that $F^C D^\alpha_{f, i} f \in C_f([a,b], \mathbb{R}_f), i = 1, ..., n$, we conclude that $F^C D^\alpha_{f, i} f$ are (F.RL)-fractional integrable on $T$.

I. Since $f$ is a continuous function and $F^C[(i) - gH]-\text{differentiable}$, by Theorem 32 we get

\[ f(t) = f(a) \odot F.RL I^\alpha_{[a,t]}(F^C D^\alpha_{f, i} f_{i,gH})(t_1), \]

and Theorem 33 yields

\[ F^C D^\alpha_{f, i} f_{i,gH}(t_1) = F^C D^\alpha_{f, i} f_{i,gH}(a) \odot F.RL I^\alpha_{[a,t]}(F^C D^{2\alpha}_{f, i} f_{i,gH})(t_2). \]

Therefore

\[ F.RL I^\alpha_{[a,t]}(F^C D^\alpha_{f, i} f_{i,gH})(t_1) = F^C D^\alpha_{f, i} f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{\Gamma(\alpha + 1)} \odot F.RL I^\alpha_{[a,t]} F.RL I^\alpha_{[a,t_1]}(F^C D^{2\alpha}_{f, i} f_{i,gH})(t_2). \]
Since the last double (F.R.L)-fractional integral belongs to $\mathbb{R}_f$ and by using Lemma 2 we have

$$f(t) = f(a) \oplus \text{FC} D^\alpha f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{T(\alpha + 1)} \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_1) \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_2).$$

By similar argument,

$$\text{FC} D^\alpha f_{i,gH}(t_2) = \text{FC} D^\alpha D^\alpha f_{i,gH}(a) \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_3).$$

Applying operator (F.R.L)-fractional integral to $(\text{FC} D^\alpha f_{i,gH})(t_2)$, we obtain

$$\text{F.R.L} I^\alpha_{[a,t]}(\text{FC} D^\alpha f_{i,gH})(t_2) = \text{F.R.L} I^\alpha_{[a,t]}(\text{FC} D^\alpha f_{i,gH})(a) \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_3),$$

thus

$$\text{F.R.L} I^\alpha_{[a,t]}(\text{FC} D^\alpha f_{i,gH})(t_2) = \text{FC} D^\alpha f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{T(\alpha + 1)} \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_3),$$

furthermore

$$\text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_2) = \text{FC} D^\alpha f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{T(\alpha + 1)} \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_3).$$

The last triple integral belongs to $\mathbb{R}_f$. By Lemma 2 we get

$$f(t) = f(a) \oplus \text{FC} D^\alpha D^\alpha f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{T(\alpha + 1)} \oplus \text{FC} D^\alpha f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{T(2\alpha + 1)} \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_1) \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_2).$$

The high order of the last formula by Lemma 2 is a continuous function in terms of $t$ so it belongs to $\mathbb{R}_f$. With the same manner, we can demonstrate that part I is satisfied.

**II.** Let $f$ is $\text{FC}[(ii) - gH]$-differentiable, we have

$$f(t) = f(a) \odot (-1) \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_1).$$

Under the hypotheses of Theorem, type of differentiability does not change, so by Theorem 3 and by attention to (F.R.L)-integrability of $\text{FC} D^\alpha f_{i,gH}$ on $T$, we obtain

$$\text{FC} D^\alpha f_{i,gH}(t_1) = \text{FC} D^\alpha f_{i,gH}(a) \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_2).$$

Applying operator $\text{F.R.L} I^\alpha_{[a,t]}$ to $\text{FC} D^\alpha f_{i,gH}(t_1)$, gives

$$\text{F.R.L} I^\alpha_{[a,t]}(\text{FC} D^\alpha f_{i,gH})(t_1) = \text{FC} D^\alpha f_{i,gH}(a) \odot \frac{(t-a)^\alpha}{T(\alpha + 1)} \oplus \text{F.R.L} I^\alpha_{[a,t]} \text{FC} D^\alpha f_{i,gH}(t_2).$$
Lemma 2 implies that the last double (F.RL)–fractional integral belongs to \( \mathbb{R}_f \). So

\[
f(t) = f(a) \circ (-1)^\alpha F.C D_\alpha^f_{i.i.g.H}(a) \circ \frac{(t-a)\alpha}{T(\alpha + 1)} \circ (-1)^\alpha F.RL I_{[a,t]}(FC D_\alpha^2 f_{i.i.g.H})(t_2).
\]

By repeating the above argument, we get

\[
FC D_\alpha^2 f_{i.i.g.H}(t_2) = FC D_\alpha^2 f_{i.i.g.H}(a) \oplus F.RL I_{[a,t_2]}(FC D_\alpha^2 f_{i.i.g.H})(t_3).
\]

Therefore, we find that

\[
F.RL I_{[a,t]}(FC D_\alpha^2 f_{i.i.g.H})(t_2) = FC D_\alpha^2 f_{i.i.g.H}(a) \circ \frac{(t_1 - a)\alpha}{T(\alpha + 1)} \oplus F.RL I_{[a,t_1]}(FC D_\alpha^2 f_{i.i.g.H})(t_3).
\]

Moreover

\[
F.RL I_{[a,t]} F.RL I_{[a,t]}(FC D_\alpha^2 f_{i.i.g.H})(t_2) = FC D_\alpha^2 f_{i.i.g.H}(a) \circ \frac{(t-a)2\alpha}{T(\alpha + 1)} \oplus F.RL I_{[a,t]} F.RL I_{[a,t_2]}(FC D_\alpha^2 f_{i.i.g.H})(t_3).
\]

By Lemma 2, the last triple (F.RL)–fractional integral belongs to \( \mathbb{R}_f \). Therefore, substituting above equation into Eq. (4), we find that

\[
f(t) = f(a) \circ (-1)^\alpha F.C D_\alpha^f_{i.i.g.H}(a) \circ \frac{(t-a)\alpha}{T(\alpha + 1)} \oplus (-1)^\alpha FC D_\alpha^2 f_{i.i.g.H}(a)
\]

\[
\circ \frac{(t-a)2\alpha}{T(\alpha + 1)} \oplus (-1)^\alpha F.RL I_{[a,t]} F.RL I_{[a,t]} F.RL I_{[a,t_2]}(FC D_\alpha^2 f_{i.i.g.H})(t_3).
\]

The rest of the proof runs as before.

III. Suppose that \( f \) is \( FC[[i] - gH] \)-differentiable. Using Theorem 32 we have

\[
f(t) = f(a) \circ (-1)^\alpha F.RL I_{[a,t]}(FC D_\alpha^f_{i.i.g.H})(t_1).
\]

Under the hypothesis of theorem, since \( f \) is \( FC[[i] - gH] \)-differentiable, \( FC D_\alpha^f \) is \( FC[[i] - gH] \)-differentiable. So, by Theorem 33 we get

\[
FC D_\alpha^f f_{i.i.g.H}(t_1) = FC D_\alpha^f f_{i.i.g.H}(a) \circ (-1)^\alpha F.RL I_{[a,t]}(FC D_\alpha^2 f_{i.i.g.H})(t_2).
\]

Now, applying operator (F.RL)–integral to \( FC D_\alpha^f f_{i.i.g.H}(t_1) \) gives

\[
F.RL I_{[a,t]}(FC D_\alpha^f f_{i.i.g.H})(t_1) = F.RL I_{[a,t]}(FC D_\alpha^f f_{i.i.g.H})(a) \circ (-1)^\alpha F.RL I_{[a,t]} F.RL I_{[a,t_1]}(FC D_\alpha^2 f_{i.i.g.H})(t_2)
\]

\[
= FC D_\alpha^f f_{i.i.g.H}(a) \circ \frac{(t-a)\alpha}{T(\alpha + 1)} \oplus (-1)^\alpha F.RL I_{[a,t]} F.RL I_{[a,t]}(FC D_\alpha^2 f_{i.i.g.H})(t_2).
\]

Lemma 2 now leads to the last double (F.RL)–fractional integral belongs to \( \mathbb{R}_f \). So

\[
f(t) = f(a) \circ (-1)^\alpha FC D_\alpha^f f_{i.i.g.H}(a) \circ \frac{(t-a)\alpha}{T(\alpha + 1)} \oplus F.RL I_{[a,t]} F.RL I_{[a,t]}(FC D_\alpha^2 f_{i.i.g.H})(t_2).
\]
Similarly, since \( FC D^{\alpha}_{+} f \) is \( FC[(i) - gH] \)-differentiable, \( FC D^{2\alpha}_{+} f \) is \( FC[(ii) - gH] \)-differentiable and we get
\[
FC D^{2\alpha}_{+} f_{i_{ii}gH}(t_2) = FC D^{2\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{2\alpha} / (2\alpha + 1) \oplus (t_2 - a)^{2\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(t_3).
\]
Thus
\[
F.RL I^\alpha_{[a,t]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(t_2) = FC D^{2\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{2\alpha} / (2\alpha + 1) \oplus (t_2 - a)^{2\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(t_3).
\]
Now, applying operator \( F.RL I^\alpha_{[a,t]} \) gives
\[
F.RL I^\alpha_{[a,t]} \cdot FC D^{2\alpha}_{+} f_{i_{ii}gH}(t_2) = FC D^{2\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{2\alpha} / (2\alpha + 1) \oplus (t_2 - a)^{2\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(t_3).
\]
Since satisfies all the other conditions for the Lemma\[I\] the last triple \( (F.RL) \)-fractional integral belongs to \( RF_j \). Then
\[
f(t) = f(a) \oplus (t - a)^{2\alpha} / (2\alpha + 1) \oplus FC D^{2\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{2\alpha} / (2\alpha + 1) \oplus (t_2 - a)^{2\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(t_3),
\]
with simple and similar method, the proof for this type of differentiability will be completed.

**IV.** Since \( f \) is \( FC[(ii) - gH] \)-differentiable in \( [t_0, \xi_1] \), Theorem\[II\] leads to
\[
f(\xi) = f(t_0) \oplus (t - a)^{2\alpha} / (2\alpha + 1) \oplus FC D^{2\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{2\alpha} / (2\alpha + 1) \oplus (t_2 - a)^{2\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(t_3),
\]
and in the interval \( [\xi, b] \), \( f \) is \( FC[(i) - gH] \)-differentiable, so for \( t \in [\xi, b] \)
\[
f(t) = f(\xi) \oplus F.RL I^\alpha_{[\xi,t]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(s_1).
\]
According to the hypothesis, we know that \( \xi \) is a switching point for differentiability \( f \), thus by substituting Eq.\[I\] into Eq.\[II\] we obtain
\[
f(t) = f(t_0) \oplus (t - a)^{2\alpha} / (2\alpha + 1) \oplus FC D^{2\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{2\alpha} / (2\alpha + 1) \oplus (t_2 - a)^{2\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{2\alpha}_{+} f_{i_{ii}gH})(s_1).
\]
Consider the first \( (F.RL) \)-fractional integral on the right side of the Eq.\[II\]. By noting the hypothesis of theorem, the fuzzy Caputo derivative of the function \( f \) has the switching point \( \xi_1 \) of type I. So, \( FC D^{\alpha}_{+} f_{i_{ii}gH} \) is \( FC[(i) - gH] \)-differentiable on \( [t_0, \xi_1] \), then type of differentiability can be changed. By these conditions, the Theorem\[III\] admits that
\[
FC D^{\alpha}_{+} f_{i_{ii}gH}(\xi_1) = FC D^{\alpha}_{+} f_{i_{ii}gH}(t_0) \oplus (t - a)^{\alpha} / (\alpha + 1) \oplus FC D^{\alpha}_{+} f_{i_{ii}gH}(a) \oplus (t_1 - a)^{\alpha} / (\alpha + 1) \oplus (t_2 - a)^{\alpha} F.RL I^\alpha_{[a,t_2]}(FC D^{\alpha}_{+} f_{i_{ii}gH})(t_3).
\]
On the other hand, we know that \( FC D^{\alpha}_{+} f_{i_{ii}gH} \) is \( FC[(ii) - gH] \)-differentiable on \( [\xi, t_1] \) and the type of differentiability does not change. Thus, for \( t_1 \in [\xi, t_1] \)
from Theorem 33 it follows that
\[ F.C D^2_\alpha f_{i.i.gH}(t_1) = F.C D^2_\alpha f_{i.i.gH}(\xi_1) \oplus F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_3). \] (9)
Substituting Eq. (8) into Eq. (5) gives
\[ F.C D^2_\alpha f_{i.i.gH}(t_1) = F.C D^2_\alpha f_{i.i.gH}(t_0) \oplus (-1)^{F.R.L I^\alpha_{[t_0,\xi_1]}(F.C D^2_\alpha f_{i.i.gH})(t_2)} \oplus F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_3), \] (10)
that
\[ F.C D^2_\alpha f_{i.i.gH}(t_1) = F.C D^2_\alpha f_{i.i.gH}(t_0) \oplus F.R.L I^\alpha_{[t_0,t_2]}(F.C D^2_\alpha f_{i.i.gH})(t_4), \]
\[ \Rightarrow F.R.L I^\alpha_{[t_0,\xi_1]}(F.C D^2_\alpha f_{i.i.gH})(t_2) \]
\[ = F.C D^2_\alpha f_{i.i.gH}(t_0) \circ (\xi_1 - t_0)^\alpha / \Gamma(\alpha + 1) \oplus F.R.L I^\alpha_{[t_0,\xi_1]}(F.C D^2_\alpha f_{i.i.gH})(t_4) \]
follows from Theorem 33 and also
\[ F.C D^2_\alpha f_{i.i.gH}(t_3) = F.C D^2_\alpha f_{i.i.gH}(\xi_1) \oplus F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_5), \]
\[ \Rightarrow F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_3) \]
\[ = F.C D^2_\alpha f_{i.i.gH}(\xi_1) \circ (t_1 - \xi_1)^\alpha / \Gamma(\alpha + 1) \oplus F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_5). \] (11)
The insertion of the Eqs. (11) and (12), in Eq. (10) leads to obtain
\[ F.C D^2_\alpha f_{i.i.gH}(t_1) = F.C D^2_\alpha f_{i.i.gH}(t_0) \circ (t_0 - \xi_1)^\alpha / \Gamma(\alpha + 1) \oplus F.C D^2_\alpha f_{i.i.gH}(\xi_1) \]
\[ \oplus (t_1 - \xi_1)^\alpha / \Gamma(\alpha + 1) \oplus (-1)^{F.R.L I^\alpha_{[t_0,\xi_1]}(F.C D^2_\alpha f_{i.i.gH})(t_4)} \oplus F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_5). \] (13)
Finally, the first (F.R.L) - fractional integral on the right side of the Eq. (7) obtains as follows
\[ F.R.L I^\alpha_{[t_0,\xi]}(F.C D^2_\alpha f_{i.i.gH})(t_1) = F.C D^2_\alpha f_{i.i.gH}(t_0) \circ (\xi - t_0)^\alpha / \Gamma(\alpha + 1) \oplus F.C D^2_\alpha f_{i.i.gH}(t_0) \circ (t_0 - \xi_1)^\alpha / \Gamma(\alpha + 1) \]
\[ \oplus (\xi - \xi_1)^\alpha / \Gamma(\alpha + 1) \oplus F.C D^2_\alpha f_{i.i.gH}(\xi_1) \circ (\xi - \xi_1)^2\alpha / \Gamma(2\alpha + 1) \oplus (t_0 - \xi_1)^2\alpha / \Gamma(2\alpha + 1) \]
\[ \oplus (-1)^{F.R.L I^\alpha_{[t_0,\xi_1]}(F.R.L I^\alpha_{[t_0,\xi]}(F.C D^2_\alpha f_{i.i.gH})(t_4)} \oplus F.R.L I^\alpha_{[t_1,t_3]}(F.R.L I^\alpha_{[t_1,t_3]}(F.C D^2_\alpha f_{i.i.gH})(t_5). \] (13)
The only point remaining concerns the behaviour of the second (F.R.L) - fractional integral on the right side of the Eq. (7). We can now proceed analogously to the first (F.R.L) - fractional integral:

By noting the hypothesis of theorem, \( F.C D^i_\alpha f_{i.i.gH}, i = 2, 3 \) are \( F.C[(i) - gH] - \)differentiable
Having disposed of this preliminary step, we can now return to the Eq. (7) following fuzzy fractional initial value problem (17),

\[ F \ C \ D^\alpha \ f_{1,gH}(s_1) = F \ C \ D^\alpha \ f_{1,gH}(\xi) \oplus F.RL I_{[\xi,s_1]}^\alpha (F \ C \ D^{2\alpha} \ f_{1,gH})(s_2), \quad (14) \]

and

\[ F \ C \ D^{2\alpha} \ f_{1,gH}(s_2) = F \ C \ D^{2\alpha} \ f_{1,gH}(\xi) \oplus F.RL I_{[\xi,s_2]}^\alpha (F \ C \ D^{3\alpha} \ f_{1,gH})(s_3). \]

\[ \Rightarrow F.RL I_{[\xi,s_1]}^\alpha (F \ C \ D^{2\alpha} \ f_{1,gH})(s_2) = F \ C \ D^{2\alpha} \ f_{1,gH}(\xi) \circ \frac{(s_1 - \xi)^\alpha}{(\alpha + 1)} \oplus F.RL I_{[\xi,s_1]}^\alpha \ F.RL I_{[\xi,s_2]}^\alpha (F \ C \ D^{3\alpha} \ f_{1,gH})(s_3). \quad (15) \]

Substituting (15) into (14) we obtain

\[ F \ C \ D^\alpha \ f_{1,gH}(s_1) = F \ C \ D^\alpha \ f_{1,gH}(\xi) \oplus F \ C \ D^{2\alpha} \ f_{1,gH}(\xi) \circ \frac{(s_1 - \xi)^\alpha}{(\alpha + 1)} \oplus F.RL I_{[\xi,s_1]}^\alpha \ F.RL I_{[\xi,s_2]}^\alpha (F \ C \ D^{3\alpha} \ f_{1,gH})(s_3). \]

Thus, the second (F.RL)–fractional integral on the right side of the Eq. (6) is as following

\[ F.RL I_{[\xi,t]}^\alpha (F \ C \ D^\alpha \ f_{1,gH})(s_1) = F \ C \ D^\alpha \ f_{1,gH}(\xi) \circ \frac{(t - \xi)^\alpha}{(\alpha + 1)} \oplus F \ C \ D^{2\alpha} \ f_{1,gH}(\xi) \circ \frac{(t - \xi)^{2\alpha}}{(2\alpha + 1)} \]

\[ \oplus F.RL I_{[\xi,t]}^\alpha \ F.RL I_{[\xi,s_1]}^\alpha \ F.RL I_{[\xi,s_2]}^\alpha (F \ C \ D^{3\alpha} \ f_{1,gH})(s_3). \quad (16) \]

Having disposed of this preliminary step, we can now return to the Eq. (7). By substituting Eq. (13) and Eq. (16), in Eq. (7), the desired result is achieved.

5 Fuzzy Generalized Euler’s method

In this section, we will touch only a few aspects of the fuzzy generalized Taylor theorem and restrict the discussion to the fuzzy generalized Euler’s method. This case is important enough to be stated separately. We consider, the following fuzzy fractional initial value problem

\[ \begin{cases} F \ C \ D^\alpha \ y_{gH}(t) = f(t, y(t)), \quad t \in [0, T], \\ y(0) = y_0 \in \mathbb{R}_F, \end{cases} \quad (17) \]

where \( f : [0, T] \times \mathbb{R}_F \rightarrow \mathbb{R}_F \) is continuous and \( y(t) \) is an unknown fuzzy function of crisp variable \( t \). Furthermore, \( F \ C \ D^\alpha \ y_{gH}(t) \) is the fuzzy fractional derivative \( y(t) \) in the Caputo sense of order \( 0 < \alpha \leq 1 \), with the finite set of switching points. Now, by dividing the interval \([0, T]\) with the step length of \( h \), we have the partition \( \tilde{I}_N = \{0 = t_0 < t_1 < \ldots < t_N = T\} \) where \( t_k = kh \) for \( k = 0, 1, 2, \ldots, N \).

**Case I.** Unless otherwise stated we assume that the unique solution of the fuzzy fractional initial value problem (17), \( F \ C \ D^{2\alpha} \ y(t) \in \mathcal{C}_f([0, T], \mathbb{R}_F) \cap \mathcal{L}^s([0, T], \mathbb{R}_F) \) is \( F \ C \{(i - gH)\}-differentiable such that the type of differentiability does not
we conclude that and, $y(t)$ satisfies in problem (5.1), so

Thus, for sufficiently small $k$ for some points $k$ lie between $t_k$ and $t_{k+1}$. Since $h = t_{k+1} - t_k$, we have

$$y(t_{k+1}) = y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot FC D_\alpha^\alpha y_{i,gH}(t_k) \oplus \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k),$$

and, $y(t)$ satisfies in problem (5.1), so

$$y(t_{k+1}) = y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)) \oplus \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k),$$

$$\mathcal{H}(y(t_{k+1}), y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)) \oplus \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k)) \leq \mathcal{H}(y(t_{k+1}), y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)) + \mathcal{H}(0, \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k)), $$

as $h \to 0$ since

$$\mathcal{H}(y(t_{k+1}), y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)) \rightarrow 0,$$

$$\mathcal{H}(0, \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k)) \rightarrow 0,$$

we conclude that

$$\mathcal{H}(y(t_{k+1}), y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)) + \mathcal{H}(0, \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k)) \rightarrow 0,$$

$$\Rightarrow \mathcal{H}(y(t_{k+1}), y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)) \oplus \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D_\alpha^{2\alpha} y_{i,gH}(\eta_k)) \rightarrow 0.$$

Thus, for sufficiently small $h$ we find that

$$y(t_{k+1}) \approx y(t_k) \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y(t_k)),$$

and finally we get

$$\begin{cases} y_0 = y_0, \\ y_{k+1} = y_k \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, y_k), \quad k = 0, 1, ..., N - 1. \end{cases} \quad (18)$$

**Case II.** Assume that $FC D_\alpha^{2\alpha} y(t) \in C_f([0, T], \mathbb{R}_\mathbb{F})$ is $FC [\{ii\} - gH]$-differentiable such that the type of differentiability does not change on $[0, T]$. So the fractional Taylor’s series expansion of $y(t)$ about the point $t_k$ at $t_{k+1}$ is

$$y(t_{k+1}) = y(t_k) \odot (\eta_{k+1} - t_k) \odot FC D_\alpha^{\eta_{k+1} - t_k} y_{ii,gH}(t_k) \odot (\eta_{k+1} - t_k) \odot FC D_\alpha^{2\alpha} y_{ii,gH}(\eta_k).$$
According to the process described in Case I, the generalized Euler’s method takes the form
\[
\begin{align*}
    y_0 &= y_0, \\
    y_{k+1} &= y_k \odot (-1) \frac{h}{T(\alpha + 1)} \odot f(t_k, y_k), \quad k = 0, 1, ..., N - 1.
\end{align*}
\] (19)

**Case III.** Let us suppose that \( t_0, t_1, ..., t_{j}, \zeta, t_{j+1}, ..., t_N = T \) is a partition of interval \([0, T]\) and \( y(t) \) has a switching point in \( \zeta \in [0, T] \) of type I. So according to Eqs. (18) and (19), we have
\[
\begin{align*}
    y_0 &= y_0, \\
    y_{k+1} &= y_k \odot \frac{h}{T(\alpha + 1)} \odot f(t_k, y_k), \quad k = 0, 1, ..., j. \\
    y_{k+1} &= y_k \odot (-1) \frac{h}{T(\alpha + 1)} \odot f(t_k, y_k), \quad k = j + 1, j + 2, ..., N - 1.
\end{align*}
\] (20)

**Case IV.** Consider \( y(t) \) has a switching point type II in \( \zeta \in [0, T] \) such that \( t_0, t_1, ..., t_{j}, \zeta, t_{j+1}, ..., t_N \) is a partition of interval \([0, T]\). Hence by Eqs. (18) and (19), we conclude that
\[
\begin{align*}
    y_0 &= y_0, \\
    y_{k+1} &= y_k \odot \frac{h}{T(\alpha + 1)} \odot f(t_k, y_k), \quad k = 0, 1, ..., j. \\
    y_{k+1} &= y_k \odot \frac{h}{T(\alpha + 1)} \odot f(t_k, y_k), \quad k = j + 1, j + 2, ..., N - 1.
\end{align*}
\] (21)

Our next concern will be the behavior of the fuzzy generalized Euler method.

### 6 Analysis of the Fuzzy Generalized Euler’s method

In this section, the local and the global truncation errors of the fuzzy generalized Euler’s method are illustrated. So by applying them the consistence, the convergence and the stability of the presented method are proved. Also, several definitions and concepts of the fuzzy generalized Euler’s method are presented under \( FC[gH] \)-differentiability [5,13].

#### 6.1 Local Truncation Error, Consistent

Consider the unique solution of the fuzzy fractional initial value problem (17):

**Definition 10** If \( y(t) \) is \( FC[(i) - gH] \)-differentiable on \([0, T]\) and the type of differentiability does not change, now we define the residual \( R_k \) as
\[
R_k = y(t_{k+1}) \odot gH \left( y(t_k) \odot \frac{h}{T(\alpha + 1)} \odot f(t_k, y(t_k)) \right),
\]
and if \( y(t) \) is \( FC[(ii) - gH] \)-differentiable on \([0, T]\), we have
\[
R_k = y(t_{k+1}) \odot gH \left( y(t_k) \odot (-1) \frac{h}{T(\alpha + 1)} \odot f(t_k, y(t_k)) \right).
\]
On the other hand, the local truncation error (L. T. E.) \( \tau_k \) is defined as
\[
\tau_k = \frac{1}{h} R_k,
\]
and the fuzzy generalized Euler’s method is said to be consistent if
\[
\lim_{h \to 0} \max_{t_k \leq T} \mathcal{H}(\tau_k, 0) = 0.
\]
Therefore, due to the type of differentiability of \( y(t) \) for \( \eta_k \in [t_k, t_{k+1}] \), the residual \( (R_k) \) and the L. T. E. \( (\tau_k) \) are defined as follows:

- **\( FC[[i] - gH] - differentiability \)** \Rightarrow
  \[
  R_k = \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D^2_\alpha y_{i.gH}(\eta_k),
  \]
  \[
  \tau_k = \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \odot FC D^2_\alpha y_{i.gH}(\eta_k),
  \]

- **\( FC[[ii] - gH] - differentiability \)** \Rightarrow
  \[
  R_k = \ominus (-1) \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot FC D^2_\alpha y_{ii.gH}(\eta_k),
  \]
  \[
  \tau_k = \ominus (-1) \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \odot FC D^2_\alpha y_{ii.gH}(\eta_k).
  \]

\(< \) Investigating the **consistence** of the fuzzy generalized Euler’s method:
For this purpose, assume that \( \mathcal{H}(FC D^2_\alpha y_{ii.gH}(\eta_k), 0) \leq M \). We have two following steps:

**step I.** If \( y(t) \) be \( FC[[i] - gH] - differentiable \), then
\[
\lim_{h \to 0} \max_{t_k \leq T} \mathcal{H}(\tau_k, 0) = \lim_{h \to 0} \max_{t_k \leq T} \mathcal{H}(\frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \odot FC D^2_\alpha y_{i.gH}(\eta_k), 0)
\]
\[
= \lim_{h \to 0} \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \max_{t_k \leq T} \mathcal{H}(FC D^2_\alpha y_{i.gH}(\eta_k), 0)
\]
\[
\leq \lim_{h \to 0} \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} M = 0.
\]

**step II.** The same conclusion can be drawn for the \( FC[[ii] - gH] - differentiability \)
of \( y(t) \), so
\[
\lim_{h \to 0} \max_{t_k \leq T} \mathcal{H}(\tau_k, 0) = \lim_{h \to 0} \max_{t_k \leq T} \mathcal{H}(\ominus (-1) \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \odot FC D^2_\alpha y_{ii.gH}(\eta_k), 0)
\]
\[
= \lim_{h \to 0} \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \max_{t_k \leq T} \mathcal{H}(\ominus FC D^2_\alpha y_{ii.gH}(\eta_k), 0)
\]
\[
= \lim_{h \to 0} \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} \mathcal{H}(FC D^2_\alpha y_{ii.gH}(\eta_k), 0) \leq \lim_{h \to 0} \frac{h^{\alpha}}{\Gamma(2\alpha + 1)} M = 0.
\]

Thus, note that we have actually proved that the fuzzy generalized Euler’s method is consistent as long as the solution belongs to \( C_f([0, T], \mathbb{R}_F) \).
6.2 Global Truncation Error, Convergence

**Lemma 5** \[\text{If } z \in \mathbb{R}, \text{ then } 1 + z \leq e^z.\]

**Definition 11** \[\text{The global truncation error is the agglomeration of the local truncation error over all the iterations, assuming perfect knowledge of the true solution at the initial time step.}\]

In the fuzzy fractional initial value problem (17), assume that $y(t)$ is $FC[[ii] - gH]$-differentiable, then the global truncation error is

$$
e_{k+1} = y(t_{k+1}) \odot_{gH} y_{k+1} = y(t_{k+1}) \odot_{gH} \left[ y_0 \odot \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_0, y_0) \right.
\left. + \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_1, y_1) + \ldots + \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_k, y_k) \right],$$

and for the $FC[[ii] - gH]$-differentiability of $y(t)$, we have

$$
e_{k+1} = y(t_{k+1}) \odot_{gH} y_{k+1} = y(t_{k+1}) \odot_{gH} \left[ y_0 \odot (-1) \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_0, y_0) \right.
\left. + (-1) \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_1, y_1) + \ldots + (-1) \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_k, y_k) \right].$$

**Definition 12** If global truncation error leads to zero as the step size goes to zero, the numerical method is convergent, i.e.

$$\lim_{h \to 0} \max_k H(e_{k+1}, 0) = 0, \Rightarrow \lim_{h \to 0} \max_k H(y(t_{k+1}), y_{k+1}) = 0.$$  

In this case, the numerical solution converges to the exact solution.

<< Investigating the convergence of the fuzzy generalized Euler’s method: To suppose that $FC[2\alpha] y(t)$ exists and $f(t, y)$ satisfies in Lipschitz condition on the $\{ (t, y) \mid t \in [0, p], y \in \mathcal{B}(y_0, q), p, q > 0 \}$, the research on this subject will be divided into two steps:

**Step 1.** Suppose that $y(t)$ is $FC[[ii] - gH]$-differentiable, now by using Eq. \[\text{and assumption } r_k = \frac{h^{2\alpha}}{T(2\alpha + 1)} \circ FC[2\alpha] y_{gH}(t_k) \text{ the exact solution of the fuzzy fractional initial value problem (17) satisfies}\]

$$y(t_{k+1}) = y(t_k) \odot \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_k, y(t_k)) \odot r_k.$$ 

Subtracting the above equation from Eq. \[d\text{, deduces}\]

$$H(y(t_{k+1}), y_{k+1}) = H(y(t_k), y_k) + \frac{h^\alpha}{T(\alpha + 1)} H\left( f(t_k, y(t_k)), f(t_k, y_k) \right) + H(r_k, 0).$$

Since $f(t, y(t))$ satisfies in Lipschitz condition

$$H\left( f(t_k, y(t_k)), f(t_k, y_k) \right) \leq \ell_k H_{\text{d}} (y(t_k), y_k),$$
and this inequality obey,

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell_k) \mathcal{H}(y(t_k), y_k) + \mathcal{H}(r_k, 0).$$

(22)

From now on let

$$\ell = \max_{0 \leq k \leq N} \ell_k, \quad r = \max_{0 \leq k \leq N} \mathcal{H}(r_k, 0),$$

thus, the Eq. (22) can be written as

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell) \mathcal{H}(y(t_k), y_k) + r.$$

Since the inequality holds for all \(k\), it follows that

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^k \mathcal{H}(y(t_0), y_0) + r \left[1 + (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell) + \ldots + (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^k\right].$$

In the same trend, finds

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^k \mathcal{H}(y(t_0), y_0) + r \left[1 + (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell) + \ldots + (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^k\right].$$

Using the formula for the sum of a geometric series, we obtain

$$\sum_{i=0}^{k} (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^i = \frac{(1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^{k+1} - 1}{\frac{h^\alpha}{\Gamma(\alpha + 1)} \ell},$$

which leads to the following inequality

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell)^{k+1} \mathcal{H}(y(t_0), y_0) + \frac{r \Gamma(\alpha + 1)}{h^\alpha \ell} \left[1 + \frac{h^\alpha}{\Gamma(\alpha + 1)} \ell\right]^{k+1} - 1].$$

On the other hand, Lemma 5 and \(0 \leq (k+1) \cdot \frac{h^\alpha}{\Gamma(\alpha + 1)} \leq T\) (for \(k+1 \leq (N-1)\)) yields

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq e^{\ell T} \mathcal{H}(y(t_0), y_0) + \frac{r \Gamma(\alpha + 1)}{h^\alpha \ell} [e^{\ell T} - 1],$$

given that \(\mathcal{H}(y(t_0), y_0) = 0\) and

$$r = \max_{0 \leq k \leq N-1} \mathcal{H}(r_k, 0) = \frac{h^{2\alpha}}{T(2\alpha + 1)} \max_{0 \leq t \leq T} \mathcal{H}(FC D_{t}^{\alpha} y_{i,gH}(t), 0),$$

immediately concludes that

$$\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq \frac{h^\alpha \Gamma(\alpha + 1)}{\ell T(2\alpha + 1)} [e^{\ell T} - 1] \max_{0 \leq t \leq T} \mathcal{H}(FC D_{t}^{\alpha} y_{i,gH}(t), 0).$$
So, \( \lim_{k \to 0} \mathcal{H}(y(t_{k+1}), y_{k+1}) \to 0 \) and in this step, the fuzzy generalized Euler’s method is convergent.

**Step II.** To estimate the step II, consider \( y(t) \) is \( FC[(ii) - g\mathcal{H}] \)-differentiable, by using Eq. (19) and let

\[
r_k = \ominus(-1) \frac{h^{2\alpha}}{T(2\alpha + 1)} \odot F C D^{2\alpha}_{\ast} y_{i\ast \mathcal{H}}(t_k),
\]

the exact solution of the Eq. (17) satisfies

\[
y(t_{k+1}) = y(t_k) \ominus (-1) \frac{h^\alpha}{T(\alpha + 1)} \odot f(t_k, y(t_k)) \oplus r_k.
\]

\[\Rightarrow \mathcal{H}(y(t_{k+1}), y_{k+1}) = \mathcal{H}(y(t_k), y_k) + \frac{h^\alpha}{T(\alpha + 1)} \Theta(\mathcal{F} f(t_k, y_k) \ominus g\mathcal{H} f(t_k, y(t_k)), 0] + \mathcal{H}(r_k, 0).\]

The inequality

\[
\mathcal{H}(f(t_k, y_k) \ominus g\mathcal{H} f(t_k, y(t_k)), 0) = \mathcal{H}(f(t_k, y_k), f(t_k, y(t_k))) \leq \ell_k \mathcal{H}(y(t_k), y_k)
\]

which is the conclusion of Lipschitz condition, implies that

\[
\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell_k) \mathcal{H}(y(t_k), y_k) + \mathcal{H}(r_k, 0).
\]

(23)

Now, assume that

\[
\ell = \max_{0 \leq k \leq N-1} \ell_k, \quad r = \max_{0 \leq k \leq N-1} \mathcal{H}(r_k, 0),
\]

and rewrite Eq. (23) as

\[
\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell) \mathcal{H}(y(t_k), y_k) + r.
\]

Since the inequality holds for all \( k \), we get

\[
\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell) \left( 1 - \frac{h^\alpha}{T(\alpha + 1)} \ell \right) \mathcal{H}(y(t_{k-1}), y_{k-1}) + r
\]

\[
= (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell)^2 \mathcal{H}(y(t_{k-1}), y_{k-1}) + r \left[ 1 + (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell) \right].
\]

Repeated application of the above inequality enables us to write

\[
\mathcal{H}(y(t_{k+1}), y_{k+1}) \leq (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell)^{k+1} \mathcal{H}(y(t_0), y_0) + r \left[ 1 + (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell) + \ldots + (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell)^k \right].
\]

Obviously, this sum is a geometric series, so we have

\[
\sum_{i=0}^{k} (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell)^i = \frac{1 - (1 - \frac{h^\alpha}{T(\alpha + 1)} \ell)^{k+1}}{\frac{h^\alpha}{T(\alpha + 1)} \ell}.
\]
that resulted to
\[
H(y(t_{k+1}), y_{k+1}) \leq (1 - \frac{h^\alpha}{\Gamma(\alpha + 1)} \cdot \ell)^{k+1} H(y(t_0), y_0) + \frac{r\Gamma(\alpha + 1)}{h^\alpha \ell} \left[ 1 - (1 - \frac{h^\alpha}{\Gamma(\alpha + 1)} \cdot \ell)^{k+1} \right].
\]

With \( z = -\frac{h^\alpha}{\Gamma(\alpha + 1)} \cdot \ell \) in Lemma 5 concludes that
\[
(1 - \frac{h^\alpha}{\Gamma(\alpha + 1)} \cdot \ell)^{k+1} \leq e^{-\frac{h^\alpha}{\Gamma(\alpha + 1)} \cdot \ell} \leq e^{-\ell T},
\]
where \( 0 \leq (k + 1) \cdot \frac{h^\alpha}{\Gamma(\alpha + 1)} \leq T \) for \( (k + 1) \leq (N - 1) \). Thus in Eq. (24), we obtain
\[
H(y(t_{k+1}), y_{k+1}) \leq e^{-\ell T} H(y(t_0), y_0) + \frac{r\Gamma(\alpha + 1)}{h^\alpha \ell} \left[ 1 - e^{-\ell T} \right].
\]

Moreover
\[
r = \max_{0 \leq k \leq N-1} H(r_k, 0) = \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \max_{0 \leq t \leq T} H^{(FC)} D^{2\alpha}_{\ast y_{i=0} H}(t, 0),
\]
and the accuracy of the initial value, concludes that \( H(y(t_0), y_0) = 0 \), so
\[
H(y(t_{k+1}), y_{k+1}) \leq \frac{h^\alpha \Gamma(\alpha + 1)}{\ell \Gamma(2\alpha + 1)} \left[ 1 - e^{-\ell T} \right] \max_{0 \leq t \leq T} H^{(FC)} D^{2\alpha}_{\ast y_{i=0} H}(t, 0).
\]

Now, letting \( h \to 0 \) then \( H(y(t_{k+1}), y_{k+1}) \to 0 \) which is the desired conclusion and we can say that the fuzzy generalized Euler’s method is convergent.

6.3 Stability

Now, the stability of the presented method is illustrated. For this aim, the following definition is presented as

**Definition 13** Assume that \( y_{k+1}, k+1 \geq 0 \) is the solution of fuzzy generalized Euler’s method where \( y_0 \in \mathbb{R}_F \) and also \( z_{k+1} \) is the solution of the same numerical method where \( z_0 = y_0 \oplus \delta_0 \in \mathbb{R}_F \) shows its perturbed fuzzy initial condition. The fuzzy generalized Euler’s method is stable if there exists positive constant \( \hat{h} \) and \( K \) such that
\[
\forall (k + 1) \frac{h^\alpha}{\Gamma(\alpha + 1)} \leq T, \; k \leq N - 1, \; h \in (0, \hat{h}) \Rightarrow H(z_{k+1}, y_{k+1}) \leq K\delta
\]
whenever \( H(\delta_0, 0) \leq \delta \).

\( \triangleright \) Investigating the stability of the fuzzy generalized Euler’s method:

The proof falls naturally into two steps:

**step I.** If \( y(t) \) is \( FC[(i)-gH] \)-differentiable, by using Eq. (18) the perturbed problem is in the following form
\[
z_{k+1} = z_k \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot f(t_k, z_k), \quad z_0 = y_0 \oplus \delta_0.
\]
So, considering the Eqs. (18) and (25), gets
\[ H(z_{k+1}, y_{k+1}) \leq H(z_k, y_k) + \frac{h^\alpha}{I(\alpha + 1)} \mathcal{H}(f(t_k, z_k), f(t_k, y_k)). \]

By using the properties of Hausdorff metric and the Lipschitz condition that expressed in Section 2, we have
\[ H(z_{k+1}, y_{k+1}) \leq (1 + \frac{h^\alpha}{I(\alpha + 1)} \ell) H(z_k, y_k). \]

Continuing this process and iterating the inequality, leads to the following relation
\[ H(z_{k+1}, y_{k+1}) \leq (1 + \frac{h^\alpha}{I(\alpha + 1)} \ell)^{k+1} H(z_0, y_0). \]

Now, the Lemma 5 implies
\[ H(z_{k+1}, y_{k+1}) \leq e^{\frac{h^\alpha}{I(\alpha + 1)} \ell^{(k+1)}} H(z_0 \ominus_H y_0, 0), \]
and finally
\[ H(z_{k+1}, y_{k+1}) \leq e^{-\frac{h^\alpha}{I(\alpha + 1)} \ell} H(z_0 \ominus_H y_0, 0) \leq K \delta, \]
where \( K = e^{-\ell T} \).

**step II.** The same proof obtains when we consider the assumption \( FC[(ii) - gH]\)-differentiability of \( y(t) \). The numerical method (19) is applied to perturbation problem. So we get
\[ z_{k+1} = z_k \ominus (-1) \frac{h^\alpha}{I(\alpha + 1)} \ominus f(t_k, z_k), \quad z_0 = y_0 \oplus \delta_0. \quad (26) \]

According to the Eqs. (19) and (26), we have
\[ H(z_{k+1}, y_{k+1}) \leq H(z_k, y_k) - \frac{h^\alpha}{I(\alpha + 1)} \mathcal{H}(f(t_k, z_k), f(t_k, y_k)), \]
which we have been working under the assumption that specifications of the Hausdorff metric are satisfied. Using the Lipschitz condition can be concluded that
\[ H(z_{k+1}, y_{k+1}) \leq (1 - \frac{h^\alpha}{I(\alpha + 1)} \ell) H(z_k, y_k). \]

Repeating with the inequality and applying Lemma 5 lead us to the following inequality
\[ H(z_{k+1}, y_{k+1}) \leq (1 - \frac{h^\alpha}{I(\alpha + 1)} \ell)^{k+1} H(z_0, y_0) \leq e^{-\frac{h^\alpha}{I(\alpha + 1)} \ell^{(k+1)}} H(z_0 \ominus_H y_0, 0) \leq e^{-\ell T} H(\delta_0, 0) \leq K \delta, \]
where \( K = e^{-\ell T} \). For the general case, above analysis, just amounts to the fact that the fuzzy generalized Euler method is a stable approach.
7 Numerical Simulations

In this section, several examples of the fractional differential equations are solved by using the full fuzzy generalized Euler method. Also, the numerical results are demonstrated on some tables for different values of $h$ and $x$.

**Example 1** Let us consider the following initial value problem

$$ F^C D^*_x y(x) = (0, 1, 1.5) \odot \Gamma(\alpha + 1), \quad 0 \leq x \leq 1, $$

where $y(0) = 0$ and $y(x) = (0, 1, 1.5) \odot x^\alpha$ is the exact $F^C\{i-gH\}$-differentiable solution of problem. In order to find the numerical results we should construct the following iterative formula as

$$ y_{k+1} = y_k \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \odot [(0, 1, 1.5) \odot \Gamma(\alpha + 1)], \quad k = 0, 1, \cdots, N - 1. $$

In Table 1 the numerical results for different values of $x, \alpha$ and $h$ are demonstrated. In Fig. 1 the exact solution and the Caputo $gH$-derivative for $\alpha = 0.6$ are demonstrated.

![Fig. 1](image)

**Table 1** Numerical results of Example 1 for various $x, \alpha$ and $h$.

| $x$ | $\alpha$ | $h$ | $y_k$ |
|-----|----------|-----|-------|
| 0   | 0.4      | 0.2 | (0, 0.309249, 0.463874) |
| 0.2 | 0.4      | 0.2 | (0, 0.309249, 0.463874) |
| 0.4 | 0.4      | 0.2 | (0, 0.309249, 0.463874) |
| 0.6 | 0.4      | 0.2 | (0, 0.309249, 0.463874) |
| 0.8 | 0.4      | 0.2 | (0, 0.309249, 0.463874) |
| 1.0 | 0.4      | 0.2 | (0, 0.309249, 0.463874) |
Example 2 Consider the following problem
\[ FCD_\alpha^\gamma y(x) = (-1) \odot y(x), \quad 0 \leq x \leq 1, \]
where \( y(0) = (0, 1, 2) \) and the exact \( FCD_{[ii-gH]} \)-differentiable solution of problem is in the form \( y(x) = (0, 1, 2) \odot E_\alpha(-x^\alpha) \). In order to solve the mentioned problem the following formula should be applied as

\[ y_0 = (0, 1, 2), \quad y_{k+1} = y_k \ominus h^{\alpha} \frac{\Gamma(\alpha + 1)}{\alpha} y_k, \quad k = 0, 1, \ldots, N - 1. \]

The numerical results based on the presented method are obtained in Table 2 for various \( x, \alpha = 0.3, 0.6, 0.9 \) and \( h = 0.2, 0.02 \). The figures of exact solution and the Caputo gH-derivative are shown in Fig. 2.

![Fig. 2](a) The exact solution (b) the Caputo gH-derivative of Example 2 for \( \alpha = 0.6 \).

### Table 2 Numerical results of Example 2 for various \( x, \alpha \) and \( h \).

| \( x \) | \( \alpha = 0.3 \) | \( \alpha = 0.6 \) | \( \alpha = 0.9 \) |
|-------|----------------|----------------|----------------|
| 0.1   | (0,0.00000865,0.00001706) | (0,0.00000753,0.00001516) | (0,0.00000637,0.00001340) |
| 0.2   | (0,0.00000941,0.00001882) | (0,0.00000860,0.00001694) | (0,0.00000778,0.00001507) |
| 0.3   | (0,0.00001050,0.00001994) | (0,0.00001003,0.00001809) | (0,0.00000927,0.00001619) |
| 0.4   | (0,0.00001179,0.00002112) | (0,0.00001136,0.00001925) | (0,0.00001057,0.00001738) |
| 0.5   | (0,0.00001326,0.00002290) | (0,0.00001269,0.00002100) | (0,0.00001187,0.00001919) |
| 0.6   | (0,0.00001493,0.00002456) | (0,0.00001411,0.00002253) | (0,0.00001303,0.00002064) |
| 0.7   | (0,0.00001682,0.00002604) | (0,0.00001518,0.00002398) | (0,0.00001388,0.00002160) |
| 0.8   | (0,0.00001888,0.00002740) | (0,0.00001658,0.00002483) | (0,0.00001463,0.00002262) |
| 0.9   | (0,0.00002112,0.00002865) | (0,0.00001798,0.00002557) | (0,0.00001537,0.00002355) |
| 1.0   | (0,8.8746 + 10^{-4},0.00001739) | (0,0.00001926,0.00002670) | (0,0.00002004,0.00002459) |

Example 3 Let us to consider the following problem
\[ FCD_\alpha^\gamma y(x) = \frac{\pi^2 x^{2-\alpha} \alpha^2}{(2-3\alpha + \alpha^2)\Gamma(1-\alpha)}F_\alpha\left(1; \left[\frac{3}{2} - \frac{\alpha}{2}, \frac{3}{2} - \frac{\alpha}{2}\right]; -\frac{1}{4} \pi^2 x^2 \alpha^2\right) \odot \left(0, \frac{1}{2}, 1 \right), \quad 1 \leq x \leq 2, \]

where \( y(1) = (0, \frac{1}{2}, 1) \odot \cos(\alpha \pi) \) and the exact solution is \( y(x) = (0, \frac{1}{2}, 1) \odot \cos(\alpha \pi x) \). We know that this problem has the switching point at \( x = 1.40426 \).
According to Eq. (20) we should divide the interval \([1, 2]\) to the two subinterval then the following iterative formulas are applied as

\[
y_{k+1} = y_k \oplus \frac{h^\alpha}{\Gamma(\alpha + 1)} \left( -\frac{\pi^2 x^2 - \alpha^2}{(2 - 3\alpha + \alpha^2) \Gamma(1 - \alpha)} F_q \left( a; b; z x_k^2 \right) \odot \left( \frac{1}{2}, 1 \right) \right),
\]

where \(a = 1, b = \left[ \frac{3}{2} - \frac{\alpha}{2}, 2 - \frac{\alpha}{2} \right]\) and \(z = -\frac{1}{2} x^2 \alpha^2\). Numerical results are demonstrated in Table 3 for \(\alpha = 0.8\) and \(h = 0.2, 0.02, 0.002\). In Fig. 3 the graphs of the exact solution and the Caputo gH-derivative are presented for \(\alpha = 0.8\).

| Table 3 | Numerical results of Example 3 for \(\alpha = 0.8\), \(h = 0.2, 0.02, 0.002\) and various \(x\). |
|---------|-------------------------------------------------|
| \(x\)  | \(h = 0.2\)                                      | \(h = 0.02\)                           | \(h = 0.002\)                          |
| 1.1     | (0, -0.452376, -0.918699)                        | (0, -0.463549, -0.928996)             | (0, -0.464888, -0.929776)             |
| 1.2     | (0, -0.489654, -0.984721)                        | (0, -0.495423, -0.991834)             | (0, -0.496057, -0.992115)             |
| 1.3     | (0, -0.496554, -0.984721)                        | (0, -0.495423, -0.991834)             | (0, -0.496057, -0.992115)             |
| 1.4     | (0, -0.452376, -0.918699)                        | (0, -0.463549, -0.928996)             | (0, -0.464888, -0.929776)             |
| 1.5     | (0, -0.401121, -0.800241)                        | (0, -0.404397, -0.808832)             | (0, -0.404568, -0.809017)             |
| 1.6     | (0, -0.307754, -0.628003)                        | (0, -0.317689, -0.637143)             | (0, -0.318712, -0.637424)             |
| 1.7     | (0, -0.208788, -0.417784)                        | (0, -0.21233, -0.425584)              | (0, -0.21289, -0.425779)              |
| 1.8     | (0, -0.092756, -0.176232)                        | (0, -0.0935876, -0.187196)            | (0, -0.0936907, -0.187381)            |
| 1.9     | (0, 0.0034523, 0.0618529)                        | (0, 0.0331271, 0.0627734)             | (0, 0.0313953, 0.0627905)             |
| 2.0     | (0, 0.146488, 0.301241)                          | (0, 0.15359, 0.308805)                | (0, 0.154508, 0.309017)               |

Fig. 3 (a) The exact solution (b) the Caputo gH-derivative of Example 3 for \(\alpha = 0.8\).
8 Conclusions

Fractional differential equations are one of the important topics of the fuzzy arithmetic which have many applications in sciences and engineering. Thus finding the numerical and analytical methods to solve these problems is very important. This paper was presented based on the two main topics. Firstly, proving the generalized Taylor series expansion for fuzzy valued function based on the concept of generalized Hukuhara differentiability. Secondly, introducing the fuzzy generalized Euler’s method as an applications of the generalized Taylor expansion and applying it to solve the fuzzy fractional differential equations. The capabilities and abilities of presented method were showed by presenting several theorem about the consistence, the convergence and the stability of the generalized Euler’s method. Also, accuracy and efficiency of method were illustrated by considering on the local and global truncation errors. The numerical results specially in the switching point case showed the precision of the generalized Euler’s method to solve the fuzzy fractional differential equations.

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