LOWER BOUNDS FOR NORMS OF PRODUCTS OF POLYNOMIALS ON $L_p$ SPACES

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Abstract. For $1 < p < 2$ we obtain sharp inequalities for the supremum of products of homogeneous polynomials on $L_p(\mu)$, whenever the number of factors is no greater than the dimension of these Banach spaces (a condition readily satisfied in the infinite dimensional settings). The results also holds for the Schatten classes $S_p$. For $p > 2$ we present some estimates on the involved constants.

Introduction

This work is framed in what is sometimes called the factor problem for homogeneous polynomials. Given homogeneous polynomials $P_1, \ldots, P_n$ defined on $(\mathbb{C}^N, \| \cdot \|_p)$, our aim is to find the best constant $M$ such that

\begin{equation}
\| P_1 \cdots P_n \| \geq M \| P_1 \| \cdots \| P_n \|.
\end{equation}

The constant will necessarily depend on $p$ and on the degrees of the polynomials, but not on the number of variables $N$. And, of course, we must set what the norm of a polynomial is.

Recall that a mapping $P : X \to \mathbb{C}$ is a (continuous) $k$-homogeneous polynomial if there exists a (continuous) $k$-linear map $T : X \times \cdots \times X \to \mathbb{C}$ such that $P(x) = T(x, \ldots, x)$ for all $x \in X$. The space of continuous $k$-homogeneous polynomials on a Banach space $X$ is denoted by $\mathcal{P}(^kX)$. It is a Banach space under the uniform norm

\[ \| P \|_{\mathcal{P}(^kX)} = \sup_{\|z\|_X = 1} |P(z)|. \]

Considering this norm, inequality (1) was studied for polynomials defined on finite and infinite dimensional Banach spaces. For instance R. Ryan and B. Turett
gave bounds for the special case where the polynomials \( \{P_i\}_{i=1}^n \) are actually continuous linear forms on \( X \). Moreover, C. Benítez, Y. Sarantopoulos and A. Tonge [2] proved that if \( P_i \) has degree \( k_i \) for \( 1 \leq i \leq n \), then inequality (1) holds with constant

\[
M = \frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{(k_1 + \cdots + k_n)}}
\]

for any complex Banach space. The authors also showed that this is the best universal constant, since there are polynomials on \( \ell_1 \) for which equality prevails. However, for many spaces it is possible to improve this bound. For instance, for complex Hilbert spaces, the second named author proved in [4] that the optimal constant is

\[
M = \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{(k_1 + \cdots + k_n)}}}.
\]

In this work we establish the best constant for complex \( L_p(\Omega, \mu) \) spaces whenever \( 1 < p < 2 \). We show that in this case inequality (1) holds with constant

\[
M = \sqrt{\frac{k_1^{k_1} \cdots k_n^{k_n}}{(k_1 + \cdots + k_n)^{(k_1 + \cdots + k_n)}}}.
\]

The constant is optimal provided the involved spaces have enough dimension at least \( n \). This constant also works (and is optimal) for polynomials on the Schatten classes \( \mathcal{S}_p \). For the remaining values of \( p \), we obtain some estimates of the optimal constants.

1. Main results

We begin with some definitions. If \( E \) and \( F \) are isomorphic Banach spaces, their Banach-Mazur distance (see [5] Chapter 1, [8]) is defined as

\[
d(E, F) = \inf \{\|u\|\|u^{-1}\| \mid u: E \to F \text{ isomorphism}\}.
\]

Given a Banach space \( X \) and \( n \in \mathbb{N} \), we define

\[
D_n(X) := \sup \{d(E, \ell^n_2) : E \text{ subspace of } X \text{ with } \dim E = n\}.
\]

From Corollary 5 in [3], we obtain

\[
D_n(L_p(\Omega, \mu)) \leq n^{1/p - 1/2},
\]

whenever \( L_p(\Omega, \mu) \) has dimension at least \( n \).
The proof of the following lemma is inspired by Proposition 1 in [6].

**Lemma 1.1.** Let $X$ be a Banach space and let $P_1, \ldots, P_n : X \to \mathbb{C}$ be homogeneous polynomials of degree $k_1, \ldots, k_n$ respectively. Then

\[
\|P_1 \cdots P_n\|_{\mathcal{P}(kX)} \geq \sqrt[n]{\prod_{i=1}^{n} \frac{k_i}{k}} D_n(X)^{-k} \|P_1\|_{\mathcal{P}(k_1X)} \cdots \|P_n\|_{\mathcal{P}(k_nX)},
\]

where $k = \sum_{i=1}^{n} k_i$.

**Proof.** Given $\varepsilon > 0$, we can take a set of norm one vectors \(\{x_1, \ldots, x_n\} \subset X\) such that \(\|P_j(x_j)\| > (1 - \varepsilon) \|P_j\|_{\mathcal{P}(k_jX)}\), for $1 \leq j \leq n$. Let $E \subset X$ be any $n$–dimensional subspace containing the subspace spanned by \(\{x_1, \ldots, x_n\}\) and let $T : \ell^2_2 \to E$ be a norm one isomorphism with \(\|T^{-1}\| \leq D_n(X)\). We have

\[
\|P_1 \cdots P_n\|_{\mathcal{P}(kX)} \geq \|P_1 \cdots P_n\|_{\mathcal{P}(k_E)} \geq \|T^{-1}\|^k \|P_1 \cdots P_n\|_{\mathcal{P}(k_1E)} \cdots \|P_n\|_{\mathcal{P}(k_nE)},
\]

where (5) follows from (2). \qed

**Remark 1.2.** If we restrict ourselves to the spaces $L_p(\Omega, \mu)$ and polynomials with the same degree, we can combine Lemma 1.1 with Lewis’ result (4) to obtain

\[
\|P_1 \cdots P_n\|_{\mathcal{P}(k L_p(\Omega, \mu))} \geq \frac{1}{n^{nk/p}} \|P_1\|_{\mathcal{P}(k L_p(\Omega, \mu))} \cdots \|P_n\|_{\mathcal{P}(k L_p(\Omega, \mu))}
\]

for $1 \leq p \leq 2$. For $2 \leq p \leq \infty$ we have

\[
\|P_1 \cdots P_n\|_{\mathcal{P}(k L_p(\Omega, \mu))} \geq \frac{1}{n^{nk/q}} \|P_1\|_{\mathcal{P}(k L_p(\Omega, \mu))} \cdots \|P_n\|_{\mathcal{P}(k L_p(\Omega, \mu))},
\]

where $q$ is the conjugate exponent of $p$.

Note that (6) is precisely (1) with the constant $M$ given in (3). In order to extend this result to a general case, where the polynomials have arbitrary degrees, it is convenient to consider another particular case. In the sequel we
will say that \( P, Q : \ell^N_p \to \mathbb{C} \) depend on different variables if it is possible to find disjoint subsets \( I, J \subset \{1, 2, \ldots, N\} \), such that \( P(\sum_{i=1}^N a_i e_i) = P(\sum_{i \in I} a_i e_i) \) and \( Q(\sum_{i=1}^N a_i e_i) = Q(\sum_{i \in J} a_i e_i) \), for all \( \{a_i\}_{i=1}^N \subset \mathbb{C} \).

For polynomials depending on different variables, (1) becomes an equality when \( M \) is given by (3), as the following lemma shows.

**Lemma 1.3.** Let \( P_1, \ldots, P_n \) be homogeneous polynomials of degrees \( k_1, \ldots, k_n \) respectively, defined on \( \ell^N_p \), depending on different variables. If \( k = k_1 + \cdots + k_n \), then we have

\[
\|P_1 \cdots P_n\|_{\mathcal{P}(k, \ell^N_p)} = \sqrt{\prod_{i=1}^n \frac{k_i^{k_i}}{k^k}} \|P_1\|_{\mathcal{P}(k_1, \ell^N_p)} \cdots \|P_n\|_{\mathcal{P}(k_n, \ell^N_p)}.
\]

**Proof.** First, we prove this lemma for two polynomials \( P \) and \( Q \) of degrees \( k \) and \( l \). We may suppose that \( P \) depends on the first \( r \) variables and \( Q \) on the last \( N - r \) ones. Given \( z \in \ell^N_p \), we can write \( z = x + y \), where \( x \) and \( y \) are the projections of \( z \) on the first \( r \) and the last \( N - r \) coordinates respectively. We then have

\[
|P(z)Q(z)| = |P(x)Q(y)| \leq \|P\|_{\mathcal{P}(k, \ell^N_p)} \|Q\|_{\mathcal{P}(l, \ell^N_p)} \|x\|^k_p \|y\|^l_p.
\]

Since \( \|z\|^p_p = \|x\|^p_p + \|y\|^p_p \), we can estimate the norm of \( PQ \) as follows

\[
\|PQ\|_{\mathcal{P}(k+l, \ell^N_p)} = \sup_{\|z\|=1} |P(z)Q(z)|
\]

\[
\leq \sup_{|a|^p+|b|^p=1} |a|^k |b|^l \|P\|_{\mathcal{P}(k, \ell^N_p)} \|Q\|_{\mathcal{P}(l, \ell^N_p)}
\]

\[
= \sqrt{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k, \ell^N_p)} \|Q\|_{\mathcal{P}(l, \ell^N_p)},
\]

the last equality being a simple application of Lagrange multipliers. In order to see that this inequality is actually an equality, take \( x_0 \) and \( y_0 \) norm-one vectors where \( P \) and \( Q \) respectively attain their norms, each with nonzero entries only in the coordinates in which the corresponding polynomial depends. If we define

\[
z_0 = \sqrt{\frac{k}{k+l}} x_0 + \sqrt{\frac{l}{k+l}} y_0,
\]

then \( z_0 \) is a norm one vector which satisfies

\[
|P(z_0)Q(z_0)| = \sqrt{\frac{k^k l^l}{(k+l)^{k+l}}} \|P\|_{\mathcal{P}(k, \ell^N_p)} \|Q\|_{\mathcal{P}(l, \ell^N_p)}.
\]
We prove the general statement by induction on \( n \). We assume the result is valid \( n-1 \) polynomials and we know that it is also valid for two. We omit the subscripts in the norms of the polynomials to simplify the notation. We then have

\[
\left\| \prod_{i=1}^{n} P_i \right\| = \sqrt{\frac{\left( \sum_{i=1}^{n-1} k_i \right)^{\sum_{i=1}^{n-1} k_i}}{\left( \sum_{i=1}^{n} k_i \right)^{\sum_{i=1}^{n} k_i}}} \left\| \prod_{i=1}^{n-1} P_i \right\| \left\| P_n \right\|
\]

Now we are ready to prove our main result.

**Theorem 1.4.** Let \( P_1, \ldots, P_n \) be homogeneous polynomials of degrees \( k_1, \ldots, k_n \), respectively on \( \ell^N_p \), \( 1 \leq p \leq 2 \). If \( k = k_1 + \cdots + k_n \), then we have

\[
\left\| P_1 \cdots P_n \right\|_{L^p(\ell^N_p)} \geq \sqrt{\prod_{i=1}^{n} \frac{k_i^{k_i}}{k^k}} \left\| P_1 \right\|_{L^p(\ell^N_p)} \cdots \left\| P_n \right\|_{L^p(\ell^N_p)}.
\]

The constant is optimal provided that \( N \geq n \).

**Proof.** We first prove the inequality for two polynomials: we take homogeneous polynomials \( P \) and \( Q \) of degree \( k \) and \( l \). If \( k = l \), the result follows from Remark 1.2. Let us suppose \( k > l \). Moving to \( \ell^N_{p+1} \) if necessary, we take a norm one polynomial \( S \), of degree \( d = k-l \), depending on different variables than the polynomials \( P \) and \( Q \). An example of such a polynomial is \( (e^{N+1}_p)^d \). In the following, we identify \( \ell^N_p \) with a subspace of \( \ell^N_{p+1} \) in the natural way. We use Lemma 1.3 for equalities (8) and (10), and inequality (6) for inequality (9) to...
obtain:

$$\|PQ\|_{\mathcal{P}(k\ell_p^N)} = \|PQ\|_{\mathcal{P}(k\ell_p^{N+1})} \|S\|_{\mathcal{P}(d\ell_p^{N+1})}$$

(8)

$$\geq \sqrt{\frac{(k+l+d)(k+l+1)}{(k+l)(k+l+1)d}} \|P\|_{\mathcal{P}(k\ell_p^{N+1})} \|QS\|_{\mathcal{P}(d\ell_p^{N+1})}$$

(9)

$$\geq \sqrt{\frac{(k+l+d)(k+l+1)}{(k+l)(k+l+1)d}} \|P\|_{\mathcal{P}(k\ell_p^{N+1})} \|QS\|_{\mathcal{P}(d\ell_p^{N+1})}$$

(10)

$$= \sqrt{\frac{(2k)^2}{(k+l)(k+l+1)d^2} \frac{1}{4^k/p}} \|P\|_{\mathcal{P}(k\ell_p^{N+1})} \|Q\|_{\mathcal{P}(d\ell_p^{N+1})} \|S\|_{\mathcal{P}(d\ell_p^{N+1})}$$

The proof of the general case continues by induction on $n$ as in the previous lemma.

To see that the constant is optimal whenever $N \geq n$, consider for each $i = 1, \ldots, n$ the polynomial $P_i = (\epsilon_i')^k$. From Lemma 1.3 we obtain equality in (7). □

Theorem 1.4 holds also for polynomials on $\ell_p$. This is a consequence of the following: if $P \in \mathcal{P}(k\ell_p)$ then

$$\|P\|_{\mathcal{P}(k\ell_p)} = \lim_{N \to \infty} \|P \circ i_N\|_{\mathcal{P}(k\ell_p^N)}$$

where $i_N$ is the canonical inclusion of $\ell_p^N$ in $\ell_p$. The proof of this fact is rather standard. Anyway, in the next section we will show that Theorem 1.4 holds for spaces $L_p(\mu)$, which comprises $\ell_p$ as a particular case.

2. Spaces $L_p$ and Schatten classes

In this section we show that the results obtained for $\ell_p$ can be extended to spaces $L_p(\Omega, \mu)$ and to the Schatten classes $S_p$ for $1 \leq p \leq 2$. We will sometimes omit parts of the proofs which are very similar to those in the previous section.

Let $(\Omega, \mu)$ be a measure space. From now on, the notation $\Omega = A_1 \sqcup \ldots \sqcup A_n$ will mean that it is possible to decompose the set $\Omega$ as the union of measurable
Proof. Given \( f \in L_p(\Omega, \mu) \) we write it as \( f = f\mathcal{X}_{A_1} + f\mathcal{X}_{A_2} \) and then
\[
\|P(f)Q(f)\| = |P(f\mathcal{X}_{A_1})Q(f\mathcal{X}_{A_2})| \\
\leq \|P\|_{\mathcal{P}(k,lP_p(\Omega,\mu))} \|Q\|_{\mathcal{P}(lP_p(\Omega,\mu))} \|f\mathcal{X}_{A_1}\|^p \|f\mathcal{X}_{A_2}\|^p.
\]
Since \( \|f\|^p = \|f\mathcal{X}_{A_1}\|^p + \|f\mathcal{X}_{A_2}\|^p \) the proof continues as in Lemma 1.3. \( \square \)

Combining this lemma with the fact that \( D_n(L_p(\mu)) = n^{1/p - 1/2} \) we obtain the next result.

Theorem 2.2. Let \( P_1, \ldots, P_n \) be homogeneous polynomials of degrees \( k_1, \ldots, k_n \) respectively on \( L_p(\Omega, \mu) \), \( 1 \leq p \leq 2 \). If \( k = k_1 + \cdots + k_n \), then we have
\[
\|P_1 \cdots P_n\|_{\mathcal{P}(kL_p(\Omega,\mu))} \geq \sqrt[p]{\prod_{i=1}^n \frac{k_i^{k_i}}{k^k}} \|P_1\|_{\mathcal{P}(k_1L_p(\Omega,\mu))} \cdots \|P_n\|_{\mathcal{P}(k_nL_p(\Omega,\mu))}.
\]
If \( \Omega \) admits a decomposition as \( \Omega = A_1 \cup \cdots \cup A_n \), then the constant is optimal.

Proof. We prove the result for two polynomials. Let \( P \) and \( Q \) be homogeneous polynomials of degree \( k \) and \( l \). If \( k = l \), the result follows from Remark 1.2. Then, we can suppose \( k > l \). Let us define an auxiliary measure space \((\Omega', \mu')\) by adding an additional point \( \{c\} \) to \( \Omega \). The measure \( \mu' \) in \( \Omega' \) is given by \( \mu'(U) = \mu(U) \) if \( U \subseteq \Omega \), and \( \mu'(U) = \mu(U \cap \Omega) + 1 \) whenever \( c \in U \). It is clear that we have \( \Omega' = \Omega \cup \{c\} \). Let us consider the polynomials \( P', Q' \) and \( S \) of degree \( k, l \) and \( d = k - l \) respectively, defined on \( L_p(\Omega', \mu') \) by \( P'(f) = P(f|_{\Omega}) \), \( Q'(f) = Q(f|_{\Omega}) \) and \( S(f) = (f|_c)^d \). Observe that \( \|S\|_{\mathcal{P}(dL_p(\Omega',\mu'))} = 1 \). The polynomials \( P'Q' \) and \( S \) are in the conditions of Lemma 2.1. Proceeding as in the proof of Theorem 1.4.
Moreover, for each responding results for the Schatten classes. Let

\[ \| P Q \|_{\mathcal{P}^{(k+1)P}(\Omega, \mu)} = \| P' Q' \|_{\mathcal{P}^{(k+1)P}(\Omega', \mu')} \| S \|_{\mathcal{P}^{(d)P}(\Omega, \mu')} \]

\[ \geq \varepsilon \left( \frac{(k + l) + d}{(k + l)^{d+1}} \right) \| P' Q' S \|_{\mathcal{P}^{(2k)P}(\Omega', \mu')} \]

\[ = \varepsilon \left( \frac{k^{1/l}}{(k + l)^{k+1}} \right) \| P' \|_{\mathcal{P}^{(k)P}(\Omega, \mu')} \| Q' \|_{\mathcal{P}^{(k)P}(\Omega', \mu')} \| S \|_{\mathcal{P}^{(d)P}(\Omega, \mu')} \]

The general case follows by induction exactly as in the proof of Lemma 1.3 and the optimality of the constant is analogous to that of Theorem 1.4. □

Now we show how the previous proofs can be adapted to obtain the corresponding results for the Schatten classes. Let \( P_1, \ldots, P_n : \mathcal{S}_p(H) \to \mathbb{C} \) be \( k \)-homogeneous polynomials on \( \mathcal{S}_p = \mathcal{S}_p(H) \), the \( p \)-Schatten class of operators on the Hilbert space \( H \). In Corollary 2.10 of [5], Tomczak-Jaegermann proved that \( D_n(\mathcal{S}_p) \leq n^{1/p - 1/2} \). Then, by Lemma 1.1, we have

\[ \| P_1 \cdots P_n \|_{\mathcal{P}^{(k)S_p}} \geq \frac{1}{n^{k/p}} \| P_1 \|_{\mathcal{P}^{(k)S_p}} \cdots \| P_n \|_{\mathcal{P}^{(k)S_p}}. \]

Suppose that \( H = H_1 \oplus H_2 \) (an orthogonal sum) and let \( \pi_1, \pi_2 : H \to H \) be the orthogonal projections onto \( H_1 \) and \( H_2 \) respectively. If the homogeneous polynomials \( P, Q : \mathcal{S}_p(H) \to \mathbb{C} \) satisfy \( P(s) = P(\pi_1 \circ s \circ \pi_1) \) and \( Q(s) = Q(\pi_2 \circ s \circ \pi_2) \) for all \( s \in \mathcal{S}_p \), we can think of \( P \) and \( Q \) as depending on different variables. Moreover, for each \( s \in \mathcal{S}_p \), it is rather standard to see that

\[ \| \pi_1 \circ s \circ \pi_1 \|_{\mathcal{S}_p} + \| \pi_2 \circ s \circ \pi_2 \|_{\mathcal{S}_p} = \| \pi_1 \circ s \circ \pi_1 + \pi_2 \circ s \circ \pi_2 \|_{\mathcal{S}_p}. \]

Also, we have

\[ \pi_1 \circ s \circ \pi_1 + \pi_2 \circ s \circ \pi_2 = \frac{1}{2} \left( s + (\pi_1 - \pi_2) \circ s \circ (\pi_1 - \pi_2) \right). \]

By the ideal property of Schatten norms, the last operator has norm (in \( \mathcal{S}_p \)) not greater than \( \| s \|_{\mathcal{S}_p} \). We then have

\[ \| \pi_1 \circ s \circ \pi_1 \|_{\mathcal{S}_p} + \| \pi_2 \circ s \circ \pi_2 \|_{\mathcal{S}_p} \leq \| s \|_{\mathcal{S}_p}. \]
Now, with (11) and (12) at hand, we can follow the proof of Lemma 1.3 to obtain the analogous result for Schatten classes.

Finally, the trick of adding a variable in Theorem 1.4 or a singleton in Theorem 2.2 can be performed for Schatten classes just taking the orthogonal sum of $H$ with a (one dimensional) Hilbert space. As a consequence, mimicking the proof of Theorem 1.4 we obtain the following.

**Theorem 2.3.** Let $P_1, \ldots, P_n$ be polynomials of degrees $k_1, \ldots, k_n$ respectively on $S_p(H)$ with $1 \leq p \leq 2$. If $k = k_1 + \cdots + k_n$, then we have

$$
\|P_1 \cdots P_n\|_{P(kS_p(H))} \geq \sqrt[p]{\prod_{i=1}^{n} \frac{k_{i}^{k_{i}}}{k}} \|P_1\|_{P(k_1S_p(H))} \cdots \|P_n\|_{P(k_nS_p(H))}.
$$

The constant is optimal provided that $\dim(H) \geq n$.

3. Remarks on the case $p > 2$

We end this note with some comment on the constant for $p > 2$. If $X$ is a Banach space, let $M(X, k_1, \ldots, k_n)$ be the largest value of $M$ such that

$$
\|P_1 \cdots P_n\|_{P(kX)} \geq M \|P_1\|_{P(k_1X)} \cdots \|P_n\|_{P(k_nX)}
$$

for any set of homogeneous polynomials $P_1, \ldots, P_n$ on $X$, of degrees $k_1, \ldots, k_n$ respectively. From [2], [4] and Theorem 1.4 we know that

$$
M(\ell_p^N, k_1, \ldots, k_n) = \sqrt[p]{\prod_{i=1}^{n} \frac{k_{i}^{k_{i}}}{k}}
$$

provided that $1 \leq p \leq 2$ and $N \geq n$. In [6] Proposition 8, the authors show that the best constant for products of linear functionals on an infinite dimensional Banach space is worse than the corresponding one for Hilbert spaces. In our notation, they show that

$$
M(\ell_2, 1, \ldots, 1) \geq M(X, 1, \ldots, 1)
$$

for every infinite dimensional Banach space $X$. Next theorem, together with Theorems 1.4 and 2.2, show that the same holds for products of homogeneous polynomials in $\ell_p^N$ and $L_p$ spaces, provided that the dimension is greater than or equal to the number of factors. That is, the constant for Hilbert spaces is
better than the constant of any other $L_p$ space for homogeneous polynomials of any degree, even in the finite dimensional setting.

**Theorem 3.1.** For $N \geq n$ and $2 \leq p \leq \infty$, we have

$$M(\ell_2^N, k_1, \ldots, k_n) \geq M(\ell_p^N, k_1, \ldots, k_n) \geq \left( n^{k_1 + \cdots + k_n} \right)^{\frac{1}{p} - \frac{1}{2}} M(\ell_2^N, k_1, \ldots, k_n).$$

The same holds for $L_p(\Omega, \mu)$ whenever $\Omega$ admits a decomposition as in Theorem 2.2.

**Proof.** The second inequality is a direct consequence of Lemma 1.1, so let us show the first one. Consider the linear forms on $\ell_p^N$ defined by the vectors

$$g_j = \left( 1, e^{2\pi i k_1/N}, e^{2\pi i k_2/N}, \ldots, e^{2\pi i (N-1)/N} \right) \text{ for } j = 1, \ldots, n.$$  

These are orthogonal vectors in $\ell_2^N$. We can choose an orthogonal coordinate system such that the $g_i$’s depend on different variables (we are in $\ell_2^N$). So by Lemma 1.3, inequality (1) holds as an equality with the constant for Hilbert spaces given in (2):

$$M(\ell_2^N, k_1, \ldots, k_n) \|g_1^{k_1} \|_{\mathcal{P}(k_1 \ell_2^N)} \cdots \|g_n^{k_n} \|_{\mathcal{P}(k_n \ell_2^N)} = \|g_1^{k_1} \cdots g_n^{k_n} \|_{\mathcal{P}(k \ell_2^N)}.$$  

For products of orthogonal linear forms this equality was observed in (1) and for the general case (with arbitrary powers) in Remark 4.2 of [4].

On the other hand, we have $\|g_j^{k_j} \|_{\mathcal{P}(k_j \ell_2^N)} = (N^{1/2})^{k_j}$, and $\|g_i^{k_i} \|_{\mathcal{P}(k_i \ell_2^N)} = (N^{1/2})^{k_i}$. Combining all this we obtain the following:

$$M(\ell_2^N, k_1, \ldots, k_n) = \frac{\|g_1^{k_1} \cdots g_n^{k_n} \|_{\mathcal{P}(k \ell_2^N)}}{\|g_1^{k_1} \|_{\mathcal{P}(k_1 \ell_2^N)} \cdots \|g_n^{k_n} \|_{\mathcal{P}(k_n \ell_2^N)}}$$

$$= \frac{\|g_1^{k_1} \cdots g_n^{k_n} \|_{\mathcal{P}(k \ell_2^N)}}{(N^{1/2(k_1 + \cdots + k_n)})^{k_1} \cdots (N^{1/2(k_1 + \cdots + k_n)})^{k_n}}$$

$$\geq \frac{\|g_1^{k_1} \cdots g_n^{k_n} \|_{\mathcal{P}(k \ell_2^N)}}{(N^{1/2(k_1 + \cdots + k_n)})^{k_1} \cdots (N^{1/2(k_1 + \cdots + k_n)})^{k_n}}$$

$$= M(\ell_p^N, k_1, \ldots, k_n).$$
This shows the statement for $\ell_p^N$. Since the space $L_p(\Omega, \mu)$, with our assumptions on $\Omega$, contains a 1-complemented copy of $\ell_p^n$, the statement for $L_p(\Omega, \mu)$ readily follows. \qed

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