Evolution equation for the structure function $g_2(x,Q^2)$

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Abstract:

We perform an extensive study of the scale dependence of flavor-singlet contributions to the structure function $g_2(x,Q^2)$ in polarized deep-inelastic scattering. We find that the mixing between quark-antiquark-gluon and three-gluon twist-3 operators only involves the three-gluon operator with the lowest anomalous dimension and is weak in other cases. This means, effectively, that only those three-gluon operators with the lowest anomalous dimension for each moment are important, and allows to formulate a simple two-component parton-like description of $g_2(x,Q^2)$ in analogy with the conventional description of twist-2 parton distributions. The similar simplification was observed earlier for the nonsinglet distributions, although the reason is in our case different.
1 Introduction

Twist-three parton distributions in the nucleon are attracting constant interest as unique probes of quark-gluon correlations in hadrons. Quantitative studies of twist-three effects are becoming possible with the increasing precision of experimental data at SLAC and RHIC, and can constitute an important part of the future spin physics program on high-luminosity accelerators like ELFE, eRHIC, etc.

The structure function \( g_2(x, Q^2) \) of polarized deep inelastic scattering received most attention in the past. The experimental studies at SLAC \([1, 2, 3]\) have confirmed theoretical expectations about the shape of \( g_2(x, Q^2) \) and provided first evidence on the most interesting twist-3 contribution. On the theoretical side, a lot of effort was invested to understand the physical interpretation of twist-3 distributions (see e.g. \([4, 5, 6]\) for the review of various aspects) and their scale dependence \([7, 8, 9, 10, 11, 12]\). One-loop corrections to the coefficient functions have been calculated \([13, 14, 15]\).

In spite of the significant progress that has been achieved, understanding of the scale dependence of \( g_2(x, Q^2) \) still poses an outstanding theoretical problem. The difficulty is due to the well-known fact \([4, 5, 6]\) that the structure function \( g_2(x, Q^2) \) presents by itself only one special projection of a more general three-particle quark-antiquark-gluon correlation function in the nucleon that depends, generally, on two variables - the momentum fractions carried by partons.

In deep inelastic scattering with a transversely polarized target, only this special projection can be measured. On the other hand, the scale dependence of the parent quark-antiquark-gluon correlation function involves the “full” function in a nontrivial way \([8]\) and the knowledge of one particular projection \( g_2(x, Q_0^2) \) at a given value of \( Q_0^2 \) does not allow to predict \( g_2(x, Q^2) \) at different momentum transfers: a DGLAP-type evolution equation for \( g_2(x, Q^2) \) in QCD does not exist or, at least, is not warranted. The reason is simply that inclusive measurements do not provide complete information on the relevant three-particle parton correlation function.

From the phenomenological point of view this conclusion is not satisfactory since it would mean that one cannot relate results of the measurements of \( g_2(x, Q^2) \) at different values of \( Q^2 \) to one another without model assumptions. The theoretical challenge is, therefore, to find out whether the complicated pattern of quark-gluon correlations can be reduced to a few effective degrees of freedom. In this case one will be able to find a meaningful approximation to the scale dependence that introduces a minimum amount of nonperturbative parameters.

Such an approximation is known \([10]\) for the flavor-nonsiglet (NS) contribution to the structure function. To explain this result, it is convenient to use the language of the Operator Product Expansion (OPE), see Sect. 2 for more details. The statement of the OPE is that odd moments \( n = 3, 5, \ldots \) of the structure function \( g_2(x, Q^2) \) can be expanded in contributions of multiplicatively renormalized local quark-antiquark-gluon operators\(^3\)

\[
\int_0^1 dx \, x^{n-1} g_2^{NS}(x, Q^2) = \sum_{k=0}^{n-3} C_{n-3}^k \left( \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right)^{\gamma_{n-3}^k/b} \langle O_{n-3}^k(\mu) \rangle, \tag{1.1}
\]

\(^3\) We neglect the Wandzura-Wilczek twist-2 contributions throughout this paper.
where $C_{k}^n$ are the coefficient functions and $\langle \langle O_{k}^n(\mu) \rangle \rangle$ are reduced matrix elements normalized at the scale $\mu$; $\gamma_{k}^n$ are the corresponding anomalous dimensions that we assume are ordered with $k$: $\gamma_{0}^n < \gamma_{1}^n < \ldots < \gamma_{n}^n$ for each $n$, and $b = 11/3N_c - 2/3n_f$. Note that the number of contributing operators rises linearly with $n$ in the r.h.s. of (1.1). This should be compared with the familiar case of leading twist-2 distributions. There, a single operator (flavor-nonsinglet) exists for each moment. A measurement of the moment of $g_2(x, Q^2)$ cannot separate between contributions of different operators and is, therefore, not sufficient to predict the scale dependence.

The situation simplifies drastically, however, in the large $N_c$ limit. It turns out that the tree-level coefficient functions, $C_{k}^n$, of all operators other than the one with the lowest anomalous dimension for each $n$ are suppressed by powers of $1/N_c^2$ so that to this accuracy one can approximate the sum in (1.1) by the first term $k = 0$. The corresponding anomalous dimension $\gamma_{0}^n$ can be calculated analytically and result can be reformulated as a DGLAP-type evolution equation

\[
Q^2 \frac{d}{dQ^2} g_{NS}(x, Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dz}{z} P_{NS}(x/z) g_{NS}^2(z, Q^2),
\]

where $P_{NS}(z) = \left[ 4C_F \left( 1 - z \right) \right]_+ + \delta(1- z) \left[ C_F + \frac{1}{N_c} \left( 2 - \frac{\pi^2}{3} \right) \right] - 2C_F$, (1.2)

The present paper is devoted to the extension of this analysis to the flavor-singlet sector in which case twist-3 composite local three-gluon operators have to be included. Coefficient functions of three-gluon operators vanish at tree level, so that gluon contributions appear entirely through the evolution. The number of independent three-gluon operators is, roughly speaking, half of the number of quark-gluon operators and, similarly, is rising with the moment $n$. The subject of this work is to find out whether the whole set of gluon operators contributes significantly, or one can reduce the gluon contribution to a certain single degree of freedom. We find that such a reduction is indeed possible and formulate a two-channel DGLAP-type evolution equation for the structure function $g_2(x, Q^2)$ that presents our main result.

In physical terms, the approximation constructed in this paper corresponds to the introduction of quark and gluon transverse spin parton distributions which are identified with the particular components of quark-antiquark-gluon and three-gluon parton correlation functions that possess the lowest anomalous dimension. We will find that, first, the structure function $g_2(x, Q^2)$ is dominated by contributions of these two distributions at large scales (including the $O(\alpha_s)$ gluon contribution calculated in [15]), and, second, the ‘leakage’ of transverse spin to genuine three-particle degrees of freedom at lower scales is small due to the specific pattern of the QCD evolution. We would like to emphasize that importance of this result is not so much in the possibility to calculate the scale dependence, but in the identification of important transverse spin degrees of freedom that are preserved by QCD interactions.

The outline of the paper is as follows. Sect. 2 is mainly introductory. We introduce necessary notation and present a summary of the existing calculations of the coefficient functions in the operator product expansion. Sect. 3 is devoted to the general formal-
ism of the renormalization of twist-3 gluon operators. We emphasize importance of the conformal symmetry and introduce a convenient framework that allows to treat renormalization as a quantum mechanical problem with hermitian Hamiltonian. This section is necessarily rather technical and a reader who is only interested in the applications may prefer to skip this discussion and go over directly to Sect. 4 where we collect our results. The main result of this paper is the generalization of the DGLAP evolution equation (1.2) to the flavor-singlet channel; it is repeated in Sec. 5 which also contains a summary and conclusions.

In Appendix A we present a detailed calculation of the relevant anomalous dimensions. In Appendix B we collect necessary formulae for the Racah 6j-symbols of the $SL(2,\mathbb{R})$ group. The conformal basis representation of the QCD evolution kernels is discussed in Appendix C.

2 The Operator Product Expansion

The hadronic tensor which appears in the description of deep inelastic scattering of polarized leptons on polarized nucleons, involves two structure functions
\[
W^{(A)}_{\mu\nu} = \frac{1}{p\cdot q} \varepsilon_{\mu \nu \alpha \beta} q^\beta \left\{ s^\beta g_1(x_B, Q^2) + \left[ s^\beta - \frac{s \cdot q}{p \cdot q} p^\beta \right] g_2(x_B, Q^2) \right\} 
\] (2.1)

and is related to the antisymmetric part of the Fourier-transform of the T-product of two electromagnetic currents:
\[
W^{(A)}_{\mu\nu} = \frac{1}{\pi} \text{Im} T^{(A)}_{\mu\nu} ,
\]
\[
i T^{(A)}_{\mu\nu} = \frac{i}{2} \int d^4 x \ e^{i q x} \langle p, s | T\{ j_\mu(x/2) j_\nu(x/2) - j_\nu(x/2) j_\mu(x/2) \} | p, s \rangle .
\] (2.2)

The light-cone expansion of (2.2) at $x^2 \to 0$ goes in terms of nonlocal light-cone operators of increasing twist, schematically
\[
\frac{i}{2} T\{ j_\mu(x) j_\nu(-x) - j_\nu(x) j_\mu(-x) \} \xrightarrow{x^2 \to 0}
\]
\[
\frac{i \varepsilon_{\mu \nu \alpha \beta}}{16 \pi^2} \frac{\partial}{\partial x_\alpha} \left\{ [C \ast O_\beta]^{\text{tw}=2} + [C \ast O_\beta]^{\text{tw}=3} + (\text{higher twists}) \right\} ,
\] (2.3)

where $C \ast O_\beta$ stands for the product (convolution) of the coefficient functions and operators of the corresponding twist. This expression is explicitly $U(1)$-gauge invariant, i.e.
\[
\partial/\partial x_\mu T\{ j_\mu(x) j_\nu(-x) \} = 0.
\]

2.1 Twist-2

For the sake of completeness and in order to facilitate the comparison to twist-3, we collect here the relevant portion of the results for the leading-twist.

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4We define the nucleon spin vector as $s_\mu = \bar{u}(p, s) \gamma_\mu \gamma_5 u(p, s)$ where $u(p, s)$ is the nucleon spinor $\bar{u}(p, s) u(p, s) = 2M$, so that $s^2 = -4M^2$. 
Retaining the tree-level quark contribution and the leading gluon correction that starts at order $O(\alpha_s)$ one obtains \cite{15}

$$
\left[ C \ast O_\beta \right]^{\text{tw-2}} = \frac{x \beta}{x^2} \sum_{q=u,d,s,...} \epsilon_q \int_0^1 du \left\{ \left[ \bar{q}(ux) \not\!\gamma_5 q(-ux) + (x \rightarrow -x) \right] \mu_{\text{MS}}^2 \right\}
$$

$$
+ \frac{4\alpha_s}{\pi} \left( \ln(-x^2\mu_{\text{MS}}^2) + 2\gamma_E \right) \left( u \ln u + u(1-u) \right) \text{Tr} \left\{ G_{x\xi}(ux) \tilde{G}_{x\xi}(-ux) \right\}
$$

where $G_{x\xi} = x_\mu G_{q_\xi}^{a\mu}$ and the subscript $[\ldots]_{\mu^2}$ indicates the normalization scale of the operator. Here and in what follows it is implied that the gauge invariance of nonlocal light-cone operators is restored by including the gauge factors $P \exp(i \int dz A^\mu(z))$. Note simplicity of the answer \eqref{4.3}; the entire gluon contribution can be eliminated by choosing the proper scale of the quark operator $\mu_{\text{MS}}^2 = 1/(-x^2 e^{2\gamma_E})$. This property is lost in the momentum space since after the Fourier transformation contributions of different light-cone separations get mixed.

Going over to the matrix elements, one introduces the quark and gluon helicity distributions

$$
\langle p, s | \bar{q}(x) \not\!\gamma_5 q(-x) | p, s \rangle = (sx) \int_{-1}^1 d\xi e^{2i\xi px} \Delta q(\xi, \mu^2),
$$

$$
\langle p, s | \text{Tr} \left\{ G_{x\alpha}(x) \tilde{G}_{x\alpha}(-x) \right\} | p, s \rangle = \frac{i}{4} (sx)(px) \int_{-1}^1 d\xi e^{2i\xi px} \xi \Delta g(\xi, \mu^2). \quad (2.5)
$$

In the first case, positive (negative) $\xi$ correspond to the contribution of quarks (antiquarks) $\Delta q(x_B) = q^\dagger(x_B) - \bar{q}^\dagger(x_B)$, $\Delta q(-x_B) \equiv \Delta \bar{q}(x_B) = \bar{q}^\dagger(x_B) - q^\dagger(x_B)$, respectively. For gluons, $\Delta g(\xi) = \Delta g(-\xi)$.

Ignoring the gluon contribution for a moment, making a Fourier transformation of \eqref{4.4}, taking imaginary part and comparing with the definition of structure functions in \eqref{2.1} one obtains

$$
g_1(x_B, Q^2) = \frac{1}{2} \sum_q e_q^2 \left[ \Delta q(x_B, \mu^2 = Q^2) + \Delta q(-x_B, \mu^2 = Q^2) \right], \quad (2.6)
$$

where $x_B = Q^2/(2pq)$ is the Bjorken variable, and twist-2 contribution to $g_2(x_B, Q^2)$

$$
[g_2(x_B, Q^2)]^{\text{tw-2}} = g_2^{WW}(x_B, Q^2) = -g_1(x_B, Q^2) + \int_{x_B}^1 \frac{dy}{dy} g_1(y, Q^2). \quad (2.7)
$$

Equation \eqref{2.6} tells that the structure function $g_1(x, Q^2)$ is a measure of the quark helicity distribution in the nucleon, as well known. Equation \eqref{2.7} is the familiar Wandzura-Wilczek relation \cite{16}. We would like to stress that despite the fact that the relation in \eqref{2.7} is affected by higher order perturbative radiative corrections, the Wandzura-Wilczek relation between the structure functions, Eq. \eqref{2.7}, is exact to all orders of perturbation theory and is only modified by twist-3 contributions to $g_2(x_B, Q^2)$ that are subject of this paper. The reason for this is that the relation \eqref{2.7} follows from particular (and unique) form of the Lorentz structure for the antisymmetric part of the T-product in \eqref{2.3} which in turn is dictated by the $U(1)$ gauge invariance. Although the coefficient
function in front of the twist-2 operator in (2.3) has a nontrivial perturbative expansion, it affects both structure functions $g_1(x, Q^2)$ and $g_2(x, Q^2)$ simultaneously and in the same way. Hence the Wandzura-Wilczek relation holds true.

For most of the subsequent discussion it will be convenient to go over to the moments space:

$$f(n, Q^2) = \int_0^1 dx_B \, x_B^{n-1} \, f(x_B, Q^2),$$

for any function $f$. In particular, restoring the gluon contribution we get \[15\]

$$g_1(n, Q^2) = \frac{1}{2} \sum_{q=u,d,s,...} \frac{e_q^2}{\alpha_s} \left\{ \Delta q(n, \mu_{\text{MS}}^2) + \Delta \bar{q}(n, \mu_{\text{MS}}^2) \right\} + \frac{\alpha_s}{2\pi} \frac{n-1}{n(n+1)} \Delta g(n, \mu^2) \left[ \ln \frac{Q^2}{\mu_{\text{MS}}^2} - \psi(n) - 1 - \gamma_E \right],$$

$$(2.8)$$

$$[g_2(n, Q^2)]^{\text{tw}-2} = -\frac{n-1}{n} g_1(n, Q^2).$$

\[2.9\]

2.2 Twist-3

The twist-3 contribution to the T-product in (2.3) is more complicated. One obtains the following expression\[15\]

$$[C * O_\beta]^{\text{tw}-3} =$$

\[
\frac{i}{2x^2} \sum_{q=u,d,s,...} e_q^2 \int_0^1 du \int_{-u}^u dv \left\{ (u+v) S_\beta(u, v, -u) + (u-v) S_\beta(-u, v, u) \right. \\
+ \frac{\alpha_s}{\pi} \left[ (u+v) \bar{O}_\beta(v, u, -u) + 2u \bar{O}_\beta(u, v, -u) + (u-v) \bar{O}_\beta(u, -u, v) \right] \\
+ \frac{4\alpha_s}{\pi} \left( \ln(-x^2\mu_{\text{MS}}^2) + 2\gamma_E + 1 \right) \left[ (\bar{u}u + \frac{1}{4} \bar{u}^2 + u \ln u) \left[ v \bar{O}_\beta(v, u, -u) + u \bar{O}_\beta(u, v, -u) \right] \\
- v \bar{O}_\beta(u, -u, v) \right] - \frac{1}{12} \bar{u}^2(u+2) \left[ \bar{O}_\beta(u, -u, v) + \bar{O}_\beta(u, v, -u) + \bar{O}_\beta(v, u, -u) \right] \right\} \mu_{\text{MS}}^{-2},
\]

where $\bar{u} = 1 - u$ and we have introduced the C-even nonlocal quark-gluon operator

$$S_\beta(u, v, -u) = S_\beta^+(u, v, -u) + S_\beta^-(u, v, -u),$$

$$S_\beta^\pm(a, b, c) = \frac{1}{2} \bar{q}(ax) [ig \tilde{G}_{\beta a}(bx) \pm g G_{\beta a}(bx) \gamma_5] \not{q}(cx)$$

\[2.11\]

and the nonlocal three-gluon operator

$$\bar{O}_\beta(u, v, w) = \frac{i q}{2} f^{abc} \tilde{G}_{x a}(u x) \tilde{G}_{x b}(v x) \tilde{G}_{x c}(w x).$$

\[2.12\]

5To avoid misunderstanding, note that this result does not include the $O(\alpha_s)$ correction to the coefficient function of quark-antiquark-gluon operators. This correction has been calculated recently in \[13, 14\] using a different operator basis.
The relatively complicated expression in the last two lines in (2.10) reflects quark-gluon mixing and reproduces the corresponding term in the renormalization group equation for the twist-3 operator $S_\beta(u, v, -u)$ \[3, 4, 7\]

\[
[S_\beta(u, v, -u)]_{\mu_2} - [S_\beta(u, v, -u)]_{\mu_1}^2 - \frac{\alpha_s}{2\pi} \ln \frac{\mu_2^2}{\mu_1^2} \int_{-u}^u ds \int_{-u}^s dt (2u)^{-3} \times \left\{ [2u(s - t) + 4(u - s)(t + s)] \tilde{O}_\beta(s, v, t) - 2u(s - t)[\tilde{O}_\beta(s, t, v) - \tilde{O}_\beta(v, s, t)] \right\}.
\]

(2.13)

Similar to the twist-2 contribution in Eq. (2.4), this contribution can be eliminated by appropriate choice of the normalization scale of the quark operator. Finally, the expression in the second line in (2.10) is not affected by the scale choice and defines a ‘genuine’ twist-3 gluon coefficient function.

Nucleon matrix elements of the nonlocal operators in (2.11), (2.12) define parton correlation functions

\[
\langle p, s | S_\mu^\pm(u, v, -u) | p, s \rangle = 2i(px) [s_\mu(px) - p_\mu(sx)] \int_{-1}^1 D\xi e^{ipx[\xi_1 u + \xi_2 v - \xi_3 u]} D_q^\pm(\xi_1, \xi_2, \xi_3)
\]

(2.14)

and

\[
\langle p, s | \tilde{O}_\mu(u, v, -u) | p, s \rangle = -2(2\pi)^2 [s_\mu(px) - p_\mu(sx)] \int_{-1}^1 D\xi e^{ipx[\xi_1 u + \xi_2 v - \xi_3 u]} D_q(\xi_1, \xi_2, \xi_3)
\]

(2.15)

where the integration measure is given by

\[
\int_{-1}^1 D\xi \equiv \int_{-1}^1 d\xi_1 d\xi_2 d\xi_3 \delta(\xi_1 + \xi_2 + \xi_3).
\]

(2.16)

The quark correlation functions $D_q^\pm(\xi_i)$ have the following symmetry property:

\[
D_q^\pm(\xi_1, \xi_2, \xi_3) = (D_q^\pm)^*(\xi_3, \xi_2, \xi_1).
\]

(2.17)

They are in general complex functions, but the imaginary parts do not contribute to the structure functions and can be omitted. In turn, the gluon correlation function $D_g(\xi_i)$ is real and antisymmetric to the interchange of the first and the third argument:

\[
D_g(\xi_1, \xi_2, \xi_3) = -D_g(\xi_3, \xi_2, \xi_1).
\]

(2.18)

Substituting (2.10) into (2.3) and taking the Fourier transform one obtains the moments of the structure function for positive odd $n$

\[
|g_2(n, Q^2)|^{tw-3} = \frac{1}{2} \sum_{q=u,d,s,...} e_q^2 \frac{4}{n} \int_{-1}^1 D\xi \left\{ D_q(\xi_i, \mu_{\text{MS}}^2) \Phi_n^q(\xi_1, \xi_3) + \alpha_s \frac{D_g(\xi_i, \mu_{\text{MS}}^2)}{n + 1} \left[ \Phi_n^g(\xi_i) + \Omega_n^{gg}(\xi_i) \left( \ln \frac{Q^2}{\mu_{\text{MS}}^2} - \psi(n) - \gamma_E - 1 \right) \right] \right\}.
\]

(2.19)

Here $D_q(\xi_i)$ is the distribution function corresponding to the C–even combination of quark-gluon operators (2.11)

\[
\langle p, s | S_\mu(u, v, -u) | p, s \rangle = 4i(px) [s_\mu(px) - p_\mu(sx)] \int_{-1}^1 D\xi e^{ipx[\xi_1 u + \xi_2 v - \xi_3 u]} D_q(\xi_1, \xi_2, \xi_3)
\]

(2.20)
so that
\[
D_q(\xi_1, \xi_2, \xi_3) = \text{Re } D^+_q(\xi_1, \xi_2, \xi_3) = \text{Re } D^-_q(-\xi_3, -\xi_2, -\xi_1),
\]  
(2.21)
the quark coefficient function is defined as
\[
\Phi^q_n(\xi_1, \xi_3) = -\frac{\partial}{\partial \xi_3} \frac{\xi_3^{n-1} - (-\xi_3)^{n-1}}{\xi_1 + \xi_3}
\]  
(2.22)
and the gluon coefficient functions can be expressed in terms of \( \Phi^q_n \) as
\[
\Phi^g_n(\xi_i) = [\Phi^g_{n-1}(\xi_1, \xi_3) + \Phi^g_{n-1}(-\xi_1 - \xi_3)] + (\xi_i \leftrightarrow -\xi_3),
\]  
(2.23)
\[
\Omega^{gg}_n(\xi_i) = \left(1 + \frac{2}{n(n-2)}\right) \Phi^g_n(\xi_i) + \frac{2(n-1)}{n(n-2)} [\Phi^g_{n-1}(\xi_1, -\xi_3) + \Phi^g_{n-1}(-\xi_3, \xi_1 + \xi_3)].
\]  
(2.24)
Explicit expressions for \( \Phi^q_n \) and \( \Phi^g_n \) for the lowest moments \( n = 3, 5, 7 \) can be found in [15].

Equivalently, one may choose to expand moments of the structure function in contributions of local operators, for example
\[
[S^±]_N^k = \frac{1}{2}(\bar{D} \cdot x)^k [ig\hat{G}_{\mu\nu} \pm gG_{\mu\nu}\gamma_5] \not x \not x (\bar{D} \cdot x)^N-k q,
\]  
\[
[G_{\mu}]_N^k = \frac{i}{2}gf^{abc}G^a_{x\alpha} (\bar{D} \cdot x)^k \bar{G}^b_{x\mu} (\bar{D} \cdot x)^N-1-k G_{x\alpha}.
\]  
(2.25)
The reduced matrix elements \( \langle \ldots \rangle \) of local operators
\[
\langle p, s | [S^±]_N^k | p, s \rangle = 2(ipx)^{N+1} [s_{\mu}(px) - p_{\mu}(sx)] \langle [S^±]_N^k \rangle,
\]  
\[
\langle p, s | [G_{\mu}]_N^k | p, s \rangle = 2(ipx)^{N+1} [s_{\mu}(px) - p_{\mu}(sx)] \langle [G]_N^k \rangle
\]  
(2.25)
are equal to moments of the three-particle distributions
\[
\langle [S^±]_N^k \rangle = \int_{-1}^1 d\xi\xi^k \xi_3^{N-k} D^+_q(\xi), \quad \langle [G]_N^k \rangle = \int_{-1}^1 d\xi\xi^k \xi_3^{N-1-k} D_g(\xi).
\]  
(2.26)
The symmetry relations (2.17) and (2.18) imply that \( \langle [S^±]_N^k \rangle^* = (-1)^N \langle [S^±]_N^{N-k} \rangle \) and \( \langle [G]_N^k \rangle = -\langle [G]_N^{N-k} \rangle \). Therefore, the number of independent quark and gluon matrix elements contributing to a given \( n = (N+3) \)th moment is equal to \( \ell_q = N + 1 \) and \( \ell_g = [N/2] \), respectively. Finally, we define the reduced matrix elements corresponding to the C-even quark-gluon operator (2.11) as
\[
\langle [S]_N^k \rangle = \text{Re } \langle [S^+]_N^k \rangle = (-1)^N \text{Re } \langle [S^-]_N^{N-k} \rangle = \int_{-1}^1 d\xi\xi^k \xi_3^{N-k} D_q(\xi).
\]  
(2.27)
\[\footnote{In difference to Ref. [12] we prefer to define the quark contribution in terms of \( \text{Re } D^+_q(\xi) \) instead of \( \text{Re } D^-_q(\xi) \). Because of this, the quark coefficient function in (2.22) differs from the one given in [12] by the replacement \( \xi_3 \leftrightarrow -\xi_3 \). The gluon coefficients are symmetric under this transformation and are not affected.}
\[\footnote{The distinction between ‘plus’ and ‘minus’ distributions is delicate since it is affected by the convention used to define the \( \gamma_5 \) matrix. We use \( \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = -i\gamma_0\gamma_1\gamma_2\gamma^3 \) and \( \epsilon^{0123} = -\epsilon_{0123} = 1 \) [19]. A sign change in the definition of the \( \gamma_5 \) matrix results in the replacement \( S^+ \leftrightarrow S^- \) and, for matrix elements \( \langle [S]_N^k \rangle \to (-1)^N \langle [S]_N^{N-k} \rangle \). The coefficient functions in the OPE are changed accordingly, so that the results for the physical observables remain intact.}
2.3 Transverse spin parton densities: why and why not

In order to pursue a parton model-like interpretation, one can introduce transverse spin distributions as the specific projections of general quark-antiquark-gluon and three-gluon operators. Their definition is suggested by the explicit form of the contribution of these operators to the operator product expansion (2.10)

\[
\int_{-u}^{u} dv \langle p, s | [(u + v) S^+_{\mu}(u, v, -u) + (u - v) S^-_{\mu}(u, v, -u)] | p, s \rangle = -i \left[ s_\mu - p_\mu \left( \frac{sx}{px} \right) \right] \int_{-1}^{1} d\xi e^{2i\xi u px} \Delta q_T(\xi, \mu^2) \tag{2.28}
\]

and

\[
\int_{-u}^{u} dv \langle p, s | [(u + v) \tilde{O}_\mu(u, v, -u) + 2u \tilde{O}_\mu(u, v, -u) + (u - v) \tilde{O}_\beta(u, -u, v)] | p, s \rangle = 2 \left[ s_\mu(p x) - p_\mu(s x) \right] \int_{-1}^{1} d\xi e^{2i\xi u px} \xi \Delta g_T(\xi, \mu^2) \tag{2.29}
\]

so that

\[
\int_{-1}^{1} d\xi \xi^{n-1} \Delta q_T(\xi) = 4 \int_{-1}^{1} d\xi \Phi^q_n(\xi) D_q(\xi_i) ,
\]

\[
\int_{-1}^{1} d\xi \xi^{n-1} \Delta g_T(\xi) = 2 \int_{-1}^{1} d\xi \Phi^g_n(\xi) D_g(\xi_i) . \tag{2.30}
\]

The functions \( \Delta q_T(x_B, Q^2) \) and \( \Delta g_T(x_B, Q^2) \) describe the momentum fraction distribution of the transverse spin of the proton and have the same support property as the parton distribution in (2.3). Note that \( \Delta g_T(\xi) = \Delta g_T(-\xi) \). However, in contrast with (2.5), they do not have any probabilistic interpretation but rather can be expressed through the more general three parton correlation functions \( D_q(\xi_1, \xi_2, \xi_3) \) and \( D_g(\xi_1, \xi_2, \xi_3) \) integrating out the dependence on one momentum fraction. The explicit expressions for the lowest moments (2.30) look as follows

\[
\int_{0}^{1} dx (\Delta q_T(x) + \Delta q_T(-x)) = 0, \tag{2.31}
\]

\[
\int_{0}^{1} dx x^2(\Delta q_T(x) + \Delta q_T(-x)) = 4 \int_{-1}^{1} dx D_q(x_i) = 4 \left\langle [S^0_{10}] \right\rangle, \tag{2.32}
\]

\[
\int_{0}^{1} dx x^4(\Delta q_T(x) + \Delta q_T(-x)) = 4 \int_{-1}^{1} dx (x_i^2 - 2x_i x_3 + 3x_3^2) D_q(x_i) \tag{2.33}
\]

\[
= 4 \left[ \left\langle [S^2_{12}] \right\rangle - 2\left\langle [S^1_{12}] \right\rangle + 3\left\langle [S^3_{12}] \right\rangle \right]
\]

for the quark distribution and

\[
\int_{0}^{1} dx \Delta g_T(x) = \int_{0}^{1} dx x^2 \Delta g_T(x) = 0, \tag{2.34}
\]

\[
\int_{0}^{1} dx x^4 \Delta g_T(x) = 10 \int_{0}^{1} dx x_1 D_g(x_i) = -10 \left\langle [G^0_{12}] \right\rangle \tag{2.35}
\]
for the gluon distribution.

At tree level, neglecting the $O(\alpha_s)$-correction to (2.19), one obtains using (2.30)

$$[g_2^{\text{Born}}(x_B, Q^2)]^{\text{tw}-3} = \frac{1}{2} \sum q e_q^2 \int_{x_B}^1 \frac{dy}{y} [\Delta q_T(y) + \Delta q_T(-y)]$$

(2.36)

that looks very similar to the leading twist expression (2.6) and (2.7). The two contributions in the square brackets can be interpreted as the contributions of quarks and antiquarks, respectively. Following the analogy with the leading twist, it is convenient to introduce the combinations of definite signature

$$\Delta q_T^\pm(y, Q^2) = \Delta q_T(y, Q^2) \pm \Delta q_T(-y, Q^2).$$

(2.37)

The distribution $\Delta q_T^+(y, Q^2)$ corresponds to the even signature and can be obtained by the analytic continuation from even moments $N$ of the OPE. The distribution $\Delta q_T^-(y, Q^2)$ can be obtained by the analytic continuation from odd moments $N$ and defines the valence quark contribution. The gluon contribution enters into (2.19) through $O(\alpha_s)$ corrections and, according to (2.30), its ‘genuine’ twist-3 part is parameterized by the gluon distribution $\Delta g_T(x)$. This suggests that similar to the leading twist expressions (2.6) and (2.7), the twist-3 structure function $g_2^{\text{tw}-3}(x)$ can be described in terms of the quark and gluon distributions, $\Delta q_T^+(x)$ and $\Delta g_T(x)$, respectively.

A deficiency of this interpretation is, however, that it does not go through beyond the leading order. This is seen explicitly on the gluon contribution in Eqs. (2.19) and (2.23): The coefficient function $\Omega_{qg}^n(\xi_i)$ that is responsible for the mixing with quark-gluon operators does not coincide with $\Phi_{qg}^n(\xi_i)$ and, therefore, this mixing brings in gluon contributions that are not expressed entirely in terms of $\Delta g_T(\xi)$ defined in (2.30). Another reason is that the scale dependence of the distributions introduced in (2.28), (2.29) involves the full three-particle functions $D_q$ and $D_g$ in a nontrivial way and, again, brings in additional degrees of freedom.

Aim of this paper is to analyse the effects of QCD evolution in some detail. We will find that although the above mentioned difficulties do exist, their numerical impact is likely to be minimal. We will then be able to write an approximate effective two-channel evolution equation involving the two distributions $\Delta q_T^+(y, \mu^2)$ and $\Delta g_T(y, \mu^2)$ in full analogy with the flavor-singlet DGLAP evolution equations in the leading twist.

3 Hamiltonian approach to the three-particle evolution equations

Choosing $\mu_{\text{MS}}^2 = Q^2$ one can eliminate large logarithmic corrections to the gluon coefficient function in (2.19). To the leading logarithmic accuracy ($LL$) and retaining the flavor-singlet ($S$) contribution to the structure function $g_2(x)$ we write

$$g_2^{\text{LL}}(n, Q^2) = \langle e_q^2 \rangle \frac{2}{n} \int_{-1}^1 D\xi \Phi_n^q(\xi_1, \xi_3) D_q^S(\xi_i; Q^2),$$

(3.1)

where

$$\langle e_q^2 \rangle = \frac{1}{n_f} \sum_{q=u,d,s,...} e_q^2, \quad D_q^S(\xi_i) = D_u(\xi_i) + D_d(\xi_i) + D_s(\xi_i) + \ldots.$$  

(3.2)
The gluon distribution is not present explicitly (to this accuracy) but reappears through the evolution of the quark distribution to lower scales. To see how this happens, expand (3.1) in contributions of flavor-singlet local operators \( \langle S_k^N(Q^2) \rangle \) defined in (2.27). Using (2.22) and (2.26) one finds

\[
g_{2}^{LL}(n, Q^2) = \left( e_q^2 \right) \frac{2}{n} \sum_{k=0}^{N} (-1)^{N-k} (N - k + 1) \langle S_k^N(Q^2) \rangle. \tag{3.3}
\]

The scale dependence of the reduced matrix elements is described by the system of coupled evolution equations

\[
Q^2 \frac{d}{dQ^2} \langle S_k^N(Q^2) \rangle = -\frac{\alpha_s}{4\pi} \left[ [V_{N}^{qq}]_{kk'} \langle S_{k'}^N(Q^2) \rangle + [V_{N}^{qq}]_{km'} \langle G_{m'}^N(Q^2) \rangle \right],
\]

\[
Q^2 \frac{d}{dQ^2} \langle G_{m}^N(Q^2) \rangle = -\frac{\alpha_s}{4\pi} \left( [V_{N}^{qg}]_{mk} \langle S_{k}^N(Q^2) \rangle + [V_{N}^{qg}]_{mn} \langle G_{m}^N(Q^2) \rangle \right) \tag{3.4}
\]

with \([V_{N}^{AB}]\) being the known matrices of anomalous dimensions \([3], k, k' = 0, ..., N \) and \( m, m' = 0, ..., [N/2] - 1. \) (Here \( N = n - 3 \) is the number of derivatives in the quark-gluon operator, \([\ldots]\) stands for an integer part.) Solving these equations one defines \([3N/2] + 1 \) linear combinations of the matrix elements

\[
\langle \mathcal{O}_{N, \alpha} \rangle = \sum_{0 \leq k \leq N} C_{ak}^q(N) \langle [S]^k_N \rangle + \sum_{0 \leq m \leq [N/2]-1} C_{am}^q(N) \langle [G]^m_N \rangle, \quad \alpha = 0, ..., [3N/2] \tag{3.5}
\]

that are renormalized multiplicatively and obtain the moments of the structure function in the standard form (1.1). The corresponding anomalous dimensions \( \gamma_N^\alpha \) can be found by diagonalizing the full matrix of the anomalous dimensions entering (3.4)

\[
(C_{\alpha}^q, C_{\alpha}^g) \left[ \gamma_N^\alpha \mathbb{I} \right] = 0, \quad \gamma_N^\alpha = \left( \begin{array}{cc} V_{N}^{qq} & V_{N}^{qg} \\ V_{N}^{qg} & V_{N}^{gg} \end{array} \right). \tag{3.6}
\]

The left eigenstates of the mixing matrix define the vector of the coefficient functions \((C_{ak}^q(N), C_{am}^q(N))\) entering (3.3).

For lowest values of the moments the mixing matrix \( \gamma_N \) looks as follows. For \( N = 0 \) the matrix consists of only one element

\[
\gamma_0 = \frac{17}{6} N_c + \frac{1}{6} N_c + \frac{2}{3} n_f, \tag{3.7}
\]

while for \( N = 2 \) it has the following form

\[
\gamma_2 = \left[ \begin{array}{cccc}
\frac{17}{4} N_c & -\frac{7}{6} N_c & n_f & \frac{7}{4} N_c & \frac{2}{3} n_f & -\frac{1}{12} N_c & \frac{1}{2} N_c & \frac{1}{5} n_f & -\frac{17}{30} n_f \\
\frac{1}{2} N_c & +\frac{7}{12} N_c & \frac{2}{3} n_f & \frac{59}{12} N_c & -\frac{3}{2} N_c & \frac{4}{15} n_f & \frac{1}{3} N_c & \frac{1}{12} N_c & \frac{1}{5} n_f & \frac{7}{30} n_f \\
-\frac{3}{20} N_c & \frac{3}{20} N_c & \frac{1}{5} n_f & \frac{3}{5} N_c & \frac{23}{20} N_c & \frac{2}{3} n_f & \frac{287}{60} N_c & -\frac{37}{60} N_c & \frac{1}{5} n_f & \frac{1}{18} n_f \\
-\frac{37}{120} N_c & \frac{7}{40} N_c & \frac{23}{120} N_c & \frac{2}{3} n_f & +\frac{307}{60} N_c 
\end{array} \right]. \tag{3.8}
\]
Going through an explicit calculation of (3.5) and (3.6) and putting $N_c = n_f = 3$ one finds for the two lowest moments (3.3)

\[
\frac{3}{2} \langle \epsilon_q^2 \rangle^{-1} g_2^{LL}(3, Q^2) = \langle S_0^0 \rangle_L
\]

\[
\frac{5}{2} \langle \epsilon_q^2 \rangle^{-1} g_2^{LL}(5, Q^2) = L^{\gamma_0^0/b} \left[ 0.415 \langle S_2^0 \rangle - 2.558 \langle S_2^1 \rangle + 2.776 \langle S_2^2 \rangle + 0.966 \langle G_2^0 \rangle \right]
\]

\[
+ L^{\gamma_2^0/b} \left[ 0.340 \langle S_2^2 \rangle - 0.152 \langle S_2^1 \rangle - 0.261 \langle S_2^0 \rangle - 0.114 \langle G_2^0 \rangle \right]
\]

\[
+ L^{\gamma_2^0/b} \left[ 0.134 \langle S_2^2 \rangle + 0.496 \langle S_2^1 \rangle + 0.404 \langle S_2^1 \rangle - 0.909 \langle G_2^0 \rangle \right]
\]

\[
+ L^{\gamma_2^0/b} \left[ 0.111 \langle S_2^2 \rangle + 0.214 \langle S_2^1 \rangle + 0.080 \langle S_2^0 \rangle + 0.057 \langle G_2^0 \rangle \right],
\]

(3.9)

where $L = \alpha_s(Q^2)/\alpha_s(\mu^2)$, all reduced matrix elements on the r.h.s. are normalized at the scale $\mu^2$ and the flavor-singlet anomalous dimensions are equal to

\[
\gamma_0^0 = 10.5556, \quad \gamma_2^0 = 10.7393, \quad \gamma_2^1 = 13.5155, \quad \gamma_2^2 = 17.6794, \quad \gamma_2^3 = 18.1714.
\]

(3.10)

If $\mu^2 = Q^2$, then $L = 1$ and the coefficients in front of $\langle S_2^0 \rangle$, $\langle S_2^1 \rangle$, $\langle S_2^2 \rangle$, $\langle G_2^0 \rangle$ coincide with their tree-level values 1, -2, 3 and 0, respectively, as expected from (3.3). Note that the largest coefficients occur in the contribution of the operator with the lowest anomalous dimension, that is similar to flavor-nonsinglet case [14], and the most important correction is apparently associated with the operator with the second-largest anomalous dimension. Aim of this work is to explain this structure and understand how it extends for arbitrary moment $n$. To this end, we develop a new framework for solving the three-particle evolution equations, dubbed Hamiltonian approach in what follows. A short account of the same technique is presented in our letter [12], where it was used to calculate $1/N_c^2$ correction to the evolution in flavor-nonsinglet sector.

The basic idea of our approach can be explained as follows. As it follows from (3.8), the mixing matrices $V_N$ is (3.4) do not have any obvious symmetry and, in general, are quite complicated. In particular, they are not hermitian and their eigenvectors are not orthogonal to each other. On the other hand, by a numerical diagonalization, Eq. (3.6), one finds that all eigenvalues of these matrices (anomalous dimensions) are real for arbitrary $N$. This property is not obvious and allows to suspect some hidden symmetry of the problem, which is not manifest in the particular representation of the evolution equations (3.4) involving only forward matrix elements of the operators. We will argue that this symmetry indeed exists and is nothing else as the familiar conformal symmetry of the QCD Lagrangian.

The conformal symmetry manifest itself through the $SL(2, \mathbb{R})$ invariance of the renormalization group equations describing the evolution of the local twist-3 operators including operators with the total derivatives. The $SL(2, \mathbb{R})$ symmetry of these evolution equations is obscured by the restriction to forward matrix elements of the operators in (3.4) (or, equivalently, the condition that momentum fractions of the partons sum to zero in (3.1), cf. (2.10), $\xi_1 + \xi_2 + \xi_3 = 0$). Since the operators containing total derivatives have vanishing forward matrix elements, it seems natural to neglect their mixing with the twist-3 operators (2.24) in the discussion of deep inelastic scattering. But it is this
reduction that complicates the structure of the evolution equations if it is imposed from the beginning.

Our approach relies on the conformal symmetry of the evolution equations and can be illustrated by the following scheme indicating a chain of transformations on the matrix of the evolution kernels $V_N$:

\[
\begin{align*}
\text{forward} & \quad \begin{pmatrix} \text{non-hermitian kernels} \end{pmatrix} \quad \ell_q + \ell_g \\
\text{non-forward} & \quad \begin{pmatrix} \text{hermitian kernels} \end{pmatrix} \quad \frac{1}{2}[\ell_q(\ell_q - 1) + \ell_g(\ell_g - 1)] \\
\text{hermitian kernels} & \quad \text{in conformal basis} \quad \ell_q + \ell_g
\end{align*}
\]

Instead of dealing with the non-hermitian “forward” mixing matrices $V_N$ of dimension $\ell_q + \ell_g = (N + 1) + [N/2]$, with $\ell_q$ and $\ell_g$ being the total number of quark-antiquark-gluon and three-gluon forward matrix elements, we choose to consider much bigger matrices (but hermitian with respect to the so-called conformal scalar product) of dimension $[\ell_q(\ell_q - 1) + \ell_g(\ell_g - 1)]/2$ that take into account the mixing with the operators containing total derivatives. Diagonalizing the thus defined “non-forward” evolution kernels we expand its eigenstates over the basis of “spherical harmonics” of the conformal $SL(2, \mathbb{R})$ group and obtain much simpler matrix equation of the coefficients in this expansion (see Appendix C). Thanks to the conformal invariance, the corresponding matrix, defining the non-forward evolution kernels in the conformal basis representation, has smaller dimension, $\ell_q + \ell_g$, and is now hermitian. We then make a forward projection at the very end.

### 3.1 Coefficient functions of local operators

An arbitrary local three-particle operator $O_{N, \alpha}$ is defined by the set of coefficients in the expansion over the standard basis of operators built from the elementary fields $\Phi_k$ and covariant derivatives (cf. (3.1)), schematically

\[
O_{N, \alpha} = \sum_{k_1+k_2+k_3=N} w_{k_1k_2k_3} (D^{k_1} \Phi_1)(D^{k_2} \Phi_2)(D^{k_3} \Phi_3),
\]  
(3.11)

or, equivalently, by a characteristic homogenous polynomial of three variables

\[
\Psi_{N, \alpha}(x_1, x_2, x_3) = \sum_{k_1+k_2+k_3=N} w_{k_1k_2k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3},
\]  
(3.12)

which we shall refer to as the \textit{coefficient function of the operator} (not to be mixed with the coefficient functions in the OPE). The rationale for the name is that with the help of the coefficient function the local operator can be “projected out” of the corresponding nonlocal operator, in our case

\[
O_{N, \alpha} = \left[ 2N \Psi_{N, \alpha}^q (\partial_{z_1}, \partial_{z_2}, \partial_{z_3}) S^0(z_1, z_2, z_3) + n_f \Psi_{N-1, \alpha}^q (\partial_{z_1}, \partial_{z_2}, \partial_{z_3}) \tilde{O}_\mu(z_1, z_2, z_3) \right] \bigg|_{z_i=0}
\]

\[
= \left( 2N \Psi_{N, \alpha}^q (\partial_{z_i}), n_f \Psi_{N-1, \alpha}^q (\partial_{z_i}) \right) \left( S^0_{\mu}(z_i) \tilde{O}^\mu_{\mu}(z_i) \right) \bigg|_{z_i=0},
\]
(3.13)

where $\Psi_{N}^q(x_i)$ and $\Psi_{N-1}^q(x_i)$ are homogenous polynomials of degree $N_{qqq} = N$ and $N_{ggg} = N - 1$, respectively.\footnote{The difference in the number of derivatives is compensated by the different dimensions of the quark and gluon field.}
(3.13) is equivalent to finding of the appropriate coefficient functions. The normalization color factors have been included in (3.13) for later convenience (see Eq. (3.63) below).

To expose the symmetries of the problem, it is convenient to start with the evolution equations for the corresponding nonlocal operators \( S_\mu(z_i) \) and \( \hat{O}_\mu(z_i) \) defined in (2.11) and (2.12), respectively. They have the following general form

\[
Q^2 \frac{d}{dQ^2} S_\mu(z_1, z_2, z_3) = -\frac{A_s}{4\pi} \left[ \hat{H}_{gq} S_\mu(z_1, z_2, z_3) + \hat{H}_{gg} \hat{O}_\mu(z_1, z_2, z_3) \right], \\
Q^2 \frac{d}{dQ^2} \hat{O}_\mu(z_1, z_2, z_3) = -\frac{A_s}{4\pi} \left[ \hat{H}_{gq} S_\mu(z_1, z_2, z_3) + \hat{H}_{gg} \hat{O}_\mu(z_1, z_2, z_3) \right],
\]

(3.14)

where \( z_i \) stand for the light-cone coordinates of the quarks and gluons, and \( \hat{H}_{ab} \) \((a, b = q, g)\) are integral operators acting on quark (gluon) coordinates and describing the interaction between quarks and gluons on the light-cone, \( \hat{H}_{gq} S_\mu(z_i) = \int dz_{i}^\prime \hat{H}_{gq}(z_i | z_i^\prime) S_\mu(z_i^\prime) \) and similar for the other kernels. Note that the short-distance expansion of the nonlocal operators \( S_\mu^\pm \) and \( \hat{O}_\mu \) gives rise to the local twist-3 operators (2.24) as well as operators with the total derivatives. In difference to the previous discussion we do not assume the translation invariance, \( S_\mu(z_1, z_2, z_3) \neq S_\mu(z_1 + \delta, z_2 + \delta, z_3 + \delta) \), so that the mixing with operators containing total derivatives is included in (3.14). The explicit expressions for the kernels \( \hat{H}_{ab} \) can be found in [8]. They will not be needed for what follows.

The evolution equation (3.14) has the form of the Schrödinger equation with the \( 2 \times 2 \) matrix of the evolution kernels \( \hat{H}_{ab} \) playing the rôle of the Hamiltonian. Let \( \hat{\Psi}_{N,\alpha}^q(z_i) \) and \( \hat{\Psi}_{N-1,\alpha}^q(z_i) \) be homogenous polynomials in the light-cone coordinates of quarks and gluons of degree \( N \) and \( N - 1 \), respectively, satisfying the Schrödinger equation

\[
\begin{pmatrix}
\hat{H}_{gq} \\
\hat{H}_{gg}
\end{pmatrix}
\begin{pmatrix}
\hat{\Psi}_{N,\alpha}^q(z_i) \\
\hat{\Psi}_{N-1,\alpha}^q(z_i)
\end{pmatrix} = \mathcal{E}_{N,\alpha}
\begin{pmatrix}
\hat{\Psi}_{N,\alpha}^q(z_i) \\
\hat{\Psi}_{N-1,\alpha}^q(z_i)
\end{pmatrix},
\]

(3.15)

with the same evolution kernels as in (3.14). The nonlocal operators \( S_\mu(z_i) \) and \( \hat{O}_\mu(z_i) \) can be expanded in terms of these functions with certain operator-valued coefficients

\[
\begin{pmatrix}
S_\mu(z_i) \\
\hat{O}_\mu(z_i)
\end{pmatrix}_{Q^2} = \sum_{N,\alpha} \begin{pmatrix}
\hat{\Psi}_{N,\alpha}^q(z_i) \\
\hat{\Psi}_{N-1,\alpha}^q(z_i)
\end{pmatrix} \mathcal{O}_{N,\alpha}(Q^2),
\]

(3.16)

where the subscript in the l.h.s. stands for the normalization scale. It follows from the evolution equation that the local operators \( \mathcal{O}_{N,\alpha} \) that appear in this expansion are renormalized multiplicatively and their anomalous dimensions are determined by the corresponding energy eigenvalues \( \mathcal{E}_{N,\alpha} \)

\[
\gamma^\alpha_N = \mathcal{E}_{N,\alpha},
\]

(3.17)

with the subscript \( \alpha = 0, 1, ..., [3N/2] \) enumerating different solutions to (3.15). The scale dependence of \( \mathcal{O}_{N,\alpha} \) takes, therefore, the standard form

\[
\mathcal{O}_{N,\alpha}(Q^2) = \mathcal{O}_{N,\alpha}(\mu^2) \left( \frac{A_s(Q^2)}{A_s(\mu^2)} \right)^{\gamma^\alpha_N/b},
\]

(3.18)
So far we have introduced two different sets of polynomials: \((\hat{\Psi}_N^q(z_i), \hat{\Psi}_N^g(z_i))\) and \((\Psi_N^q(x_i), \Psi_N^g(x_i))\). The former set defines the expansion (3.16) of nonlocal operators \(S_\mu(z_i)\) and \(\hat{O}_\mu(z_i)\) over the complete set of local multiplicatively renormalizable operators \(O_{N,\alpha}\), while the latter determines the particular form of the multiplicatively renormalizable operators \(O_{N,\alpha}\), cf. (3.11), (3.12). It follows from the definition (3.13) and (3.16) that

\[
2N_c \Psi^q_{N,\alpha}(\partial_{z_i}) \hat{\Psi}_{N',\alpha'}^q(z_i) + n_f \Psi^g_{N-1,\alpha}(\partial_{z_i}) \hat{\Psi}_{N'-1,\alpha'}^g(z_i) \bigg|_{z_i=0} = \delta_{NN'}\delta_{\alpha\alpha'}, \tag{3.19}
\]

and in this sense the polynomials \(\hat{\Psi}\) are dual to the coefficient functions \(\Psi\). In what follows we shall refer to \(\hat{\Psi}\)–functions as coefficient functions of a local operator in the dual representation. One can determine the functions \(\Psi^q_{N,\alpha}\) and \(\Psi^g_{N-1,\alpha}\) from the orthogonality condition (3.13) provided the complete set of eigenfunctions \(\hat{\Psi}_N^q\) and \(\hat{\Psi}_N^g\) of the Schrödinger equation (3.16) is given. This task seems complicated, but in fact is not. Using conformal symmetry, we will be able to find an explicit expression connecting the “direct” \(\Psi(x_i)\) and dual \(\hat{\Psi}(z_i)\) coefficient functions of a multiplicatively renormalizable operator. The answer is given in Eq. (3.37) below.

### 3.2 Conformal symmetry

A remarkable property of the evolution kernels in (3.14) is that they are invariant under the projective transformations of the light-cone coordinates of quark and gluons, \(z_k\),

\[
z_k \to \frac{az_k + b}{cz_k + d}, \quad ad - bc = 1. \tag{3.20}
\]

This invariance has its roots in the conformal symmetry of the QCD Lagrangian and the transformations (3.20) form the \(SL(2,\mathbb{R})\) (collinear) subgroup of the full conformal group acting on the fields “living” on the light-cone. As well known, the conformal symmetry of QCD is broken by quantum corrections. However, since the leading-order renormalization group equations are driven by tree-level counterterms, they have to respect the symmetry of the QCD Lagrangian.

The action of the \(SL(2,\mathbb{R})\) transformations (3.20) on (quantum) fields \(\Phi_a(z)\), where \(a = \tilde{q}, q, g, \tilde{g}\) corresponds to \(\tilde{\phi}(z), q(z), G_{x\perp}(z), G_{x\perp}(z)\), respectively, is defined as

\[
\Phi_a(z) \to (cz + d)^{-2j} \Phi_a \left( \frac{az + b}{cz + d} \right) \tag{3.21}
\]

and is described by three generators \(\hat{L}_+, \hat{L}_-\) and \(\hat{L}_0\) that can be realized as first-order differential operators acting on the field coordinates:

\[
\hat{L}_a \Phi_a(z) = -\partial_z \Phi_a(z), \quad \hat{L}_a^+ \Phi_a(z) = (z^2 \partial_z + 2j_a z) \Phi_a(z), \quad \hat{L}_a^0 \Phi_a(z) = (z \partial_z + j_a) \Phi_a(z). \tag{3.22}
\]

Here \(j_a = (l_a + s_a)/2\) is the conformal spin of the field \(\Phi_a(z)\), with \(l_a\) being a canonical dimension (3/2 for quarks and 2 for gluons) and \(s_k\) the spin projection on the light-cone
direction, $\Sigma_{px} \Phi_a = i s_a \Phi_a$. In the case at hand the spin projections have their maximum values $s_q = s_g = 1/2$, $s_\bar{q} = s_\bar{g} = 1$, leading to

$$j_q = j_\bar{q} = 1, \quad j_g = j_\bar{g} = 3/2.$$  \hspace{1cm} (3.23)

In order to unify the notation, we introduce two $2 \times 2$ matrices of the evolution kernels $\hat{H}$ and the generators of conformal transformations $\hat{L}_k$ ($k = \pm, 0$) in the quark-antiquark-gluon and three-gluon channels

$$\hat{H} = \begin{pmatrix} \hat{H}_{qq} & \hat{H}_{qg} \\ \hat{H}_{gq} & \hat{H}_{gg} \end{pmatrix}, \quad \hat{L}_k = \begin{pmatrix} \hat{L}_{qq} & 0 \\ 0 & \hat{L}_{gg} \end{pmatrix},$$  \hspace{1cm} (3.24)

where $\hat{L}_{qq}$ and $\hat{L}_{gg}$ ($k = \pm, 0$) are the total three-particle $SL(2, \mathbb{R})$ generators acting on antiquark-quark-gluon and three-gluon coordinates, respectively:

$$\hat{L}_{qq} = \hat{L}_q + \hat{L}_{\bar{q}} + \hat{L}_g, \quad \hat{L}_{gg} = \hat{L}_g + \hat{L}_{\bar{g}} + \hat{L}_\bar{g}.$$  \hspace{1cm} (3.25)

The conformal invariance of the evolution equation is stated as

$$[\hat{H}, \hat{L}_k] = [\hat{H}, \hat{L}^2] = [\hat{L}^2, \hat{L}_k] = 0$$  \hspace{1cm} (3.26)

where

$$\hat{L}^2 = \frac{1}{2}(\hat{L}_+ \hat{L}_- + \hat{L}_- \hat{L}_+) + \hat{L}^0$$  \hspace{1cm} (3.27)

is the three-particle quadratic Casimir operator.

Thanks to the conformal invariance (3.26) the solutions to (3.15) can be classified according to the representations of the $SL(2, \mathbb{R})$ group. Namely, we can impose additional constraints on the eigenfunctions

$$\left[ \hat{L}_0 - J \right] \begin{pmatrix} \hat{\Psi}_N^q (z_i) \\ \hat{\Psi}_{N-1}^q (z_i) \end{pmatrix} = \hat{L}_- \begin{pmatrix} \hat{\Psi}_N^q (z_i) \\ \hat{\Psi}_{N-1}^q (z_i) \end{pmatrix} = \left[ \hat{L}^2 - J(J - 1) \right] \begin{pmatrix} \hat{\Psi}_N^q (z_i) \\ \hat{\Psi}_{N-1}^q (z_i) \end{pmatrix} = 0,$$  \hspace{1cm} (3.28)

where the $SL(2, \mathbb{R})$ generators are given by the same expressions as in (3.22). Since the eigenfunctions are homogenous polynomials in the light-cone coordinates $z_k$, the first equation in (3.28) is automatically satisfied provided

$$J = \frac{7}{2} + N_{qq} = \frac{9}{2} + N_{gg},$$  \hspace{1cm} (3.29)

where $N_{qq}$ and $N_{gg}$ count the total number of covariant derivatives in the corresponding local operator, $N_{qq} = N_{gg} + 1 = N$. Notice that for $J = 7/2$, or equivalently $N_{qq} = 0$ there exists no three-gluon contribution. The second condition ensures that $\hat{\Psi}_N^q$ and $\hat{\Psi}_{N-1}^q$ are invariant under translations of the light-cone coordinates. This requirement defines the so-called highest weight of the discrete series representation of the $SL(2, \mathbb{R})$ group, labeled by the conformal spin $J = N + 7/2$. An infinite tower of solutions to (3.15) can be obtained from the highest weight by the repeated application of the ‘step-up’ operator $\hat{L}_+$:

$$\begin{pmatrix} \hat{\Psi}_N^q (z_i) \\ \hat{\Psi}_{N-1}^q (z_i) \end{pmatrix} = \hat{L}_+^n \begin{pmatrix} \hat{\Psi}_N^q (z_i) \\ \hat{\Psi}_{N-1}^q (z_i) \end{pmatrix}.$$  \hspace{1cm} (3.30)
Since \( \hat{L}_+ \) commutes with the Hamiltonian, Eq. (3.26), all states \( n^\pm \Psi_N \), with different \( n = 0, 1, 2, \ldots \) have the same energy. As we will show in a moment (see Eq. (3.36)), the corresponding operators \( n^\pm \hat{O}_N \) are just those obtained from the highest weight state operator by adding the \( n \)-th power of the total derivative and, therefore, they do not survive upon taking a forward matrix element. Note that the expansion in (3.10) formally includes operators with arbitrary powers of total derivatives, but we can ignore their contribution and concentrate on studying the properties of the highest weights (3.28) only.

Going over from the dual coefficient functions \( \hat{\Psi}(z_i) \) to the coefficient functions \( \Psi(x_i) \) defined in (3.13) corresponds to going over to a different (non-standard) representation of the conformal group. Using the relation (3.19) and requiring

\[
[L_k^\pm, \hat{\Psi}(z_1, z_2, z_3)] = 0 \quad (3.32)
\]

one finds the following representation of the \( SL(2, \mathbb{R}) \) generators on the space of the coefficient functions \( \Psi(x_1, x_2, x_3) \)

\[
L_k^0 \Psi(x_k) = (x_k \partial_{x_k} + j_k) \Psi(x_k),
\]

\[
L_k^+ \Psi(x_k) = -x_k \Psi(x_k),
\]

\[
L_k^- \Psi(x_k) = (x_k \partial_{x_k}^2 + 2j_k \partial_{x_k}) \Psi(x_k)
\]

with the conformal spins \( j_k \) defined in (3.23). Note that these expressions are more complicated compared to the standard expressions (3.22). Eqs. (3.32) and (3.22) define the \( SL(2) \) generators in two different representations and, as such, they are related to each other through a transformation

\[
L_k^\pm = -T^{-1} \hat{L}_k^\pm T, \quad L_k^0 = T^{-1} \hat{L}_k^0 T, \quad \hat{\Psi}(z_i) = [T \Psi](z_i) \quad (3.33)
\]

that maps into each other the coefficient functions in two different representations. The explicit form of the \( T \)-transformation is given by

\[
\hat{\Psi}(z_i) = \Psi(\partial_{z_i}) \prod_{i=1}^3 (1 - x_i z_i)^{-2j_i} \bigg|_{x_i=0} = \prod_{i=1}^3 \int_0^\infty \frac{dt_i}{\Gamma(2j_i)} e^{-t_i} \Psi(z_i t_i). \quad (3.34)
\]

To verify this relation, use (3.22) and (3.32) to check that \( \hat{L}_k^\pm \hat{\Psi}(z_i) = -[T(L_k^\pm \Psi)](z_i) \) and \( \hat{L}_k^0 \hat{\Psi}(z_i) = [T(L_k^0 \Psi)](z_i) \). The conformal constraints (3.28) on the coefficient functions corresponding to the highest weight look exactly as before:

\[
[L_0 - J] \begin{pmatrix} \Psi_N^q(x_i) \\ \Psi_{N-1}^q(x_i) \end{pmatrix} = \mathcal{L} \begin{pmatrix} \Psi_N^q(x_i) \\ \Psi_{N-1}^q(x_i) \end{pmatrix} = [L^2 - J(J - 1)] \begin{pmatrix} \Psi_N^q(x_i) \\ \Psi_{N-1}^q(x_i) \end{pmatrix} = 0, \quad (3.35)
\]

with the \( SL(2) \) generators defined in the representation (3.32). Note that the “step-up” operator \( L_+ \) has become very simple and its action consists of adding the sum of derivatives acting on each of the three fields, as we anticipated:

\[
\begin{pmatrix} n \Psi_N^q(x_i) \\ n \Psi_{N-1}^q(x_i) \end{pmatrix} = \hat{L}_+^n \begin{pmatrix} \Psi_N^q(x_i) \\ \Psi_{N-1}^q(x_i) \end{pmatrix} = (x_1 + x_2 + x_3)^n \begin{pmatrix} \Psi_N^q(x_i) \\ \Psi_{N-1}^q(x_i) \end{pmatrix}. \quad (3.36)
\]
Finally, applying the transformation (3.34) to the coefficient function (3.12) we obtain the following expression for the dual coefficient function

$$\hat{\Psi}(z_i) = \sum_{k_1+k_2+k_3=N} u_{k_1k_2k_3} z_1^{k_1} z_2^{k_2} z_3^{k_3} \frac{\Gamma(k_1+2j_1) \Gamma(k_2+2j_2) \Gamma(k_3+2j_3)}{\Gamma(2j_1) \Gamma(2j_2) \Gamma(2j_3)}. \quad (3.37)$$

This relation establishes the one-to-one correspondence between the coefficient functions (3.12) and their dual counterparts (3.37).

Much of the following discussion is based on the fact that the coefficient functions of multiplicatively renormalizable operators $\Psi_{N,\alpha}(x_i)$ satisfying the highest weight condition (3.35) are orthogonal with respect to the so-called conformal scalar product. This property becomes crucial in establishing the hermiticity of the evolution Hamiltonian in (3.15). The hermiticity property will be quite helpful in the further analysis and as we argue below is a direct consequence of the conformal symmetry.

Eq. (3.19) suggests to define the following scalar product on the space of coefficient functions given by (3.12)

$$\langle \Psi_{N,\alpha} | \Psi_{N,\beta} \rangle \sim \Psi_{N,\alpha}^\dagger(\partial_{z_i}) \Psi_{N,\beta}(z_i) \bigg| \begin{array}{c} z_i = 0 \\ \end{array} = \Psi_{N,\alpha}(0) [T \Psi]_{N,\beta}(z_i) \bigg| \begin{array}{c} z_i = 0 \\ \end{array}. \quad (3.38)$$

For the coefficient functions of local operators without total derivatives that satisfy the constraints (3.25) (i.e. those that we are interested in) one can equivalently rewrite the definition in (3.38) in a more familiar integral form:

$$\langle \Psi_{N,\alpha} | \Psi_{N,\beta} \rangle = \frac{\Gamma(2j_1+2j_2+2j_3+2N)}{\Gamma(2j_1) \Gamma(2j_2) \Gamma(2j_3)} \int_0^1 [dx] x_1^{2j_1-1} x_2^{2j_2-1} x_3^{2j_3-1} \Psi_{N,\alpha}(x_i) \Psi_{N,\beta}(x_i),$$

where $j_k$ are the conformal spins of the operators entering (3.11). Here, the integration goes over the region $0 \leq x_k \leq 1$, $x_1 + x_2 + x_3 = 1$ and the integration measure is defined as

$$[dx] = dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1). \quad (3.40)$$

The $SL(2)$ generators (3.32) are (anti)self-adjoint operators on the space of the coefficient functions endowed with the scalar product (3.38). Indeed using Eqs. (3.31), (3.32) and (3.33) it is straightforward to verify that

$$\langle \Psi_{N,\alpha} | L_k^\pm \Psi_{N,\beta} \rangle = -\langle L_k^\mp \Psi_{N,\alpha} | \Psi_{N,\beta} \rangle, \quad \langle \Psi_{N,\alpha} | L_k^0 \Psi_{N,\beta} \rangle = \langle L_k^0 \Psi_{N,\alpha} | \Psi_{N,\beta} \rangle. \quad (3.41)$$

As a consequence, the two-particle Casimir operators $L_{ik}^2 = (L_i + L_i)^2$ defined as (3.27) are the self-adjoint operators.

### 3.3 Helicity basis

As we will see in the next section, the evolution Hamiltonians (3.24) can be written in terms of the two-particle Casimir operators $\tilde{L}_{ik}^2 = (\tilde{L}_i + \tilde{L}_i)^2$ in the dual representation. This property ensures that the Hamiltonians inherit hermiticity properties of the generators and are self-adjoint operators as well. As a consequence, their eigenvalues alias the anomalous dimensions (3.17) are real and the corresponding eigenfunctions are orthogonal to each other

$$\langle \Psi_{N,\alpha} | \Psi_{N,\beta} \rangle \sim \delta_{\alpha\beta}. \quad (3.42)$$
In fact, for three-gluon operators there is a complication that the solution to the Schrödinger equation (3.15) has to be found on the subspace of functions \( \tilde{\Psi}^q_{N,\alpha}(z_i) \) that are antisymmetric under the exchange of the first and third gluon \( z_1 \leftrightarrow z_3 \). The permutation operator \( P_{13} \) that projects onto the states with correct symmetry is not a self-adjoint operator so that one has to be careful\(^9\). In physical terms, the problem arises because, as we will see in a minute, the three-gluon operator \( \tilde{\mathcal{O}}_\mu \) contains a sum of contributions of gluons with opposite helicity, antisymmetrized because of the crossing symmetry. The way out\(^20\) is therefore to write down the evolution equation for the helicity eigenstates and restore the crossing symmetry at the end.

The construction of the helicity basis is based on the decomposition of the quark and gluon fields entering the definition of the nonlocal operators \( S_\mu(z_i) \) and \( \tilde{\mathcal{O}}_\mu(z_i) \) into the components of different chirality

\[
q^\pm(z) = \frac{1 \mp \gamma_5}{2} q(z), \quad \tilde{q}^\pm(z) = \tilde{q}(z) \frac{1 \mp \gamma_5}{2}, \quad G^\pm_\mu(z) = \frac{1}{2} \left[ G_{\mu,n} \pm i \epsilon^\pm_{\mu\nu} G_{\nu,n} \right],
\]

where \( \epsilon^\pm_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} p_\alpha n_\beta / (p \cdot n) \). The fields defined in this way satisfy the conditions

\[
\gamma_5 q^\pm(z) = \pm q^\pm(z), \quad i \epsilon^\pm_{\mu\nu} G^\pm_\nu(z) = \pm G^\pm_\mu(z)
\]

and describe quark, antiquark and gluon of a definite helicity. Making this decomposition one obtains the expressions for the nonlocal operators \( S_\mu(z_i) \) and \( \tilde{\mathcal{O}}_\mu(z_i) \) in terms of the chiral fields. We remind that \( \mu = 1, 2 \) is a transverse index and in order to construct the three-particle states with definite overall helicity one has to take particular linear combinations, projecting onto the two complex vectors in the transverse plane:

\[
w_\mu = e^{(1)}_\perp, \mu + i e^{(2)}_\perp, \mu, \quad \overline{w}_\mu = e^{(1)}_\perp, \mu - i e^{(2)}_\perp, \mu.
\]

We find for the quark-antiquark-gluon operator

\[
S_w(z_1, z_2, z_3) = \tilde{q}^- (z_1) G^+_w(z_2) \not\!q^+(z_3) + \tilde{q}^+(z_3) G^+_w(z_2) \not\!q^- (z_1),
\]

\[
S_{\overline{w}}(z_1, z_2, z_3) = \tilde{q}^+(z_1) G^-_{\overline{w}}(z_2) \not\!q^-(z_3) + \tilde{q}^-(z_3) G^-_{\overline{w}}(z_2) \not\!q^+(z_1).
\]

Notice that the quark and the antiquark have opposite helicity in both cases, and helicity of the gluon ±1 coincides with the total helicity of the system. (We tacitly imply that the momenta of the three partons are aligned along the same light-cone direction defined by the proton momentum \( p \).) The similar decomposition of the three-gluon operator looks as follows

\[
\tilde{\mathcal{O}}_w(z_1, z_2, z_3) = \frac{1}{2} \left[ T_w(z_1, z_2, z_3) - T_{\overline{w}}(z_3, z_2, z_1) \right],
\]

\[
\tilde{\mathcal{O}}_{\overline{w}}(z_1, z_2, z_3) = \frac{1}{2} \left[ T_{\overline{w}}(z_1, z_2, z_3) - T_w(z_3, z_2, z_1) \right],
\]

where the notation was introduced

\[
T_w(z_1, z_2, z_3) = \frac{ig}{2} f^{abc} G^{a,-}_{w}(z_1) G^{b,+}_{w}(z_2) G^{c,+}_{w}(z_3),
\]

\[
T_{\overline{w}}(z_1, z_2, z_3) = \frac{ig}{2} f^{abc} G^{a,+}_{w}(z_1) G^{b,-}_{w}(z_2) G^{c,-}_{w}(z_3).
\]

\(^9\)The situation is in fact very similar to the Hartree-Fock construction of the completely antisymmetric fermionic wave functions in quantum mechanics.
The operators $T_w(z_i)$ and $T_{\overline{m}}(z_i)$ describe the state of three gluons with the total helicity
$\pm 1$, respectively.

It follows from (3.47) and (3.48) that the operators $\tilde{O}_w(z_1, z_2, z_3)$ and $T_w(z_1, z_2, z_3)$ are
antisymmetric under the interchange of gluons with the same and the opposite helicity,

$\tilde{O}_w(z_1, z_2, z_3) = -\tilde{O}_w(z_3, z_2, z_1) = -P_{31}\tilde{O}_w(z_1, z_2, z_3),$

$T_w(z_1, z_2, z_3) = -T_w(z_1, z_3, z_2) = -P_{23}T_w(z_1, z_2, z_3)$

(3.49)

with $P_{tk}$ being the permutation operators. Using this, we can invert (3.47) to get

$T_w(z_1, z_2, z_3) = \tilde{O}_w(z_1, z_2, z_3) + \tilde{O}_w(z_2, z_3, z_1) - \tilde{O}_w(z_3, z_1, z_2)$

$= (1 + P_{12}P_{31} - P_{23}P_{31}) \tilde{O}_w(z_1, z_2, z_3).$

(3.50)

The same relation holds between the operators $T_{\overline{m}}$ and $\tilde{O}_w$.

The relations (3.47) and (3.50) allow one to rewrite the evolution equation (3.14) for
the three-gluon operator $\tilde{O}(z_i)$ in terms of $T(z_i)$ with definite helicity. The difference
amounts to the following redefinition of the evolution kernels (3.24) $\hat{H} \rightarrow \hat{H}^h$

$\hat{H}^h = \begin{pmatrix} \hat{H}_{qq} & \hat{H}_{qh} \\ \hat{H}_{qg} & \hat{H}_{hh} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + P_{12} - P_{23} \end{pmatrix} \begin{pmatrix} \hat{H}_{qq} & \hat{H}_{qg} \\ \hat{H}_{gq} & \hat{H}_{gg} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2}(1 - P_{31}) \end{pmatrix}.$

(3.51)

Note that the kernel $\hat{H}_{qq}$ is not affected by this transformation. Despite the fact that the
two evolution equations are obviously equivalent, the premium in dealing with helicity
operators $T_w(z_i)$ (or $T_{\overline{m}}(z_i)$) is that, as we will show below, the evolution kernel \( \hat{H}^h \)
becomes hermitian on the space of the coefficient functions. The original kernel in (3.24)
is not hermitian due to the presence of additional permutation operators in (3.31). The explicit expressions for the operator $\hat{H}^h$ will be given below.

According to (3.47), going over from $\tilde{O}(z_i)$ to the helicity states $T(z_i)$ is equivalent to
the following ansatz for the dual coefficient function of the three-gluon operator:

$\hat{\Psi}^h_{N-1}(z_1, z_2, z_3) = \frac{1}{2} \left[ \hat{\Psi}^h_{N-1}(z_1, z_2, z_3) - \hat{\Psi}^h_{N-1}(z_3, z_2, z_1) \right].$

(3.52)

where the coefficient functions $\hat{\Psi}^h_{N-1}(z_i)$ are defined by the short distance expansion of
$T_w(z_i)$. Applying the equivalence relation (3.50) one finds

$\hat{\Psi}^h_{N-1}(z_1, z_2, z_3) = \hat{\Psi}^h_{N-1}(z_1, z_2, z_3) + \hat{\Psi}^h_{N-1}(z_2, z_3, z_1) - \hat{\Psi}^h_{N-1}(z_3, z_1, z_2).$

(3.53)

Substituting (3.47) into (3.13), one rewrites the multiplicatively renormalizable operators as

$O_{N,\alpha} = [2N c \Psi_{N,\alpha} \partial z_i S_\mu (z_i) + n_f \Psi^h_{N-1,\alpha} \partial z_i T_\mu (z_i)]_{z_i = 0},$

(3.54)

and obtains similar relations between thus defined coefficient functions $\Psi^h_{N-1,\alpha}(x_i)$ and
$\Psi^q_{N-1,\alpha}(x_i)$. We find that the local three-gluon operators are projected out of the $T(z_i)$
by the following coefficient function

$\Psi^h_{N-1,\alpha}(x_1, x_2, x_3) = \frac{1}{2} \left[ \Psi^q_{N-1,\alpha}(x_1, x_2, x_3) - \Psi^q_{N-1,\alpha}(x_1, x_3, x_2) \right].$

(3.55)
that has the same symmetry (3.49) as the helicity operator itself:

\[ \Psi_{N-1}^h(x_1, x_3, x_2) = P_{23} \Psi_{N-1}^h(x_2, x_1, x_3) = -\Psi_{N-1}^h(x_1, x_2, x_3). \] (3.56)

The inverse relation looks as

\[ \Psi_{N-1,\alpha}^q(x_1, x_2, x_3) = \Psi_{N-1,\alpha}^h(x_1, x_2, x_3) + \Psi_{N-1,\alpha}^h(x_3, x_1, x_2) - \Psi_{N-1,\alpha}^h(x_2, x_3, x_1). \] (3.57)

Notice that the relations between the two sets of the coefficient functions are different in the direct and dual representations. Nevertheless, it follows from (3.52) and (3.57) that the coefficient functions \( \Psi_{N-1,\alpha}^h(z_i) \) and \( \Psi_{N-1}^h(x_i) \) satisfy the same conformal constraints (3.28) and (3.35) as \( \Psi_{N-1,\alpha}^q(z_i) \) and \( \Psi_{N-1}^q(x_i) \), respectively.

3.4 Conformal basis

Solution of the evolution equations (3.58) can be simplified significantly by expanding the coefficient functions \( \Psi_N^q(x_i) \) and \( \Psi_{N-1}^h(x_i) \) over the basis of “spherical harmonics” consistent with conformal symmetry constraints (3.28).

3.4.1 General construction

For a generic three-particle operator, (3.11) and (3.12), such a ‘conformal basis’ can be constructed as follows [21]. Conformal symmetry allows to fix the total three-particle conformal spin \( J = j_1 + j_2 + j_3 + N \) of the state that translates to the condition that the corresponding coefficient function satisfies Eqs (3.28). We define the set of functions \( Y_{j_1}^{(31)} \) by requiring that they obey Eq. (3.28) and, in addition, have a definite value of the conformal spin in one of the two-particle channels, (31) for definiteness. The latter condition leads to

\[ L_{31}^2 Y_{j_1}^{(31)}(x_i) = j(j - 1)Y_{j_1}^{(31)}(x_i), \] (3.60)

where \( j = j_1 + j_3 + n \) and \( n = 0, ..., N \).

Taken together, Eqs. (3.64), (3.28) determine the functions \( Y_{j_1}^{(31)} \) uniquely and have the following solution:

\[ Y_{j_1}^{(31)}(x_i) = r_{j_1}(x_1 + x_2 + x_3)^{j_1-j_3}(x_1 + x_2)^{j_3-j_1-j_2} \times P_{j_1-j_2-j_3}^{(2j_2-1,2j_3-1)} \left( \frac{x_1 - x_2 + x_3}{x_1 + x_2 + x_3} \right) P_{j_1-j_2}^{(2j_3-1,2j_1-1)} \left( \frac{x_1 - x_3}{x_1 + x_3} \right). \] (3.61)
Here \( P_n^{(a,b)}(x) \) is the Jacobi polynomial and \( r, j \) an arbitrary normalization factor. The basis functions \( Y_j^{(31)2}(x_i) \) are orthogonal with respect to the scalar product \( (3.39) \). Requiring that \( \| Y_j^{(31)2} \|^2 = 1 \) we fix the normalization to be
\[
\begin{align*}
  r_{j_1}^{-2} &= \frac{\Gamma(2J)}{\Gamma(2j_1)\Gamma(2j_2)\Gamma(2j_3)} \frac{\Gamma(j + j_1 - j_3)\Gamma(j - j_1 + j_3)}{\Gamma(j - j_1 - j_3 + 1)\Gamma(j + j_1 + j_3 - 1)(2j - 1)} \\
  &\times \frac{\Gamma(J - j + j_2)\Gamma(J + j - j_2)}{\Gamma(J - j - j_2 + 1)\Gamma(J + j + j_2 - 1)(2J - 1)}. \quad (3.62)
\end{align*}
\]

The above construction of the conformal basis involves an obvious ambiguity in which order the spins of partons are coupled to the total spin \( J \). Choosing in \( (3.60) \) a different two-particle channel one obtains a different conformal basis related to the original one through the matrix \( \Omega \) of Racah 6j-symbols of the \( SL(2, \mathbb{R}) \) group
\[
Y_{f_j}^{(31)2}(x_i) = \sum_{j_1 + j_2 \leq j' \leq J - j_3} \Omega_{jj'}(J) Y_{j_j'}^{(12)3}(x_i). \quad (3.63)
\]

Properties of the Racah 6j-symbols as well as their explicit expressions in terms of the generalized hypergeometric series \( _4F_3(1) \) are summarized in Appendix A.

### 3.4.2 Quark-gluon conformal basis

We now specify the above general construction to the particular cases of quark-antiquark-gluon and three-gluon operators and define
\[
Y_{N,k}^{\text{q}}(x_1, x_2, x_3) = Y_{N+7/2,k+2}^{(31)2}(x_1, x_2, x_3) \bigg|_{j_1 = j_3 = 1, j_2 = 3/2} \quad (3.64)
\]
\[
Y_{N-1,k-1}^{\text{h}}(x_1, x_2, x_3) = Y_{N+7/2,n+2}^{(31)2}(x_1, x_2, x_3) \bigg|_{j_1 = j_2 = j_3 = 3/2},
\]

where the prefactors have been chosen for later convenience. The three-particle conformal spin is given by \( J = N + 7/2 = j_q + j_g + j_q + N = j_q + j_g + j_q + N - 1 \) in both cases, and the conformal spin in the \( (31) \)-subchannel is equal to \( j = k + 2 = j_q + j_g + k = j_q + j_g + k - 1 \). Notice that \( j \geq 3 \) and, therefore, the basis functions \( Y_{N,k-1}^{\text{h}}(x_i) \) in the gluon-gluon subchannel are well-defined for \( k - 1 \geq 0 \). In what follows we assume that \( Y_{N-1,k-1}^{\text{h}}(x_i) = 0 \).

Each of the two sets of functions in \( (3.64) \) forms an orthonormal basis with respect to the conformal scalar product \( (3.39) \) whose explicit form depends on the conformal spin of the particles and, therefore, is different for the quark-antiquark-gluon and three-gluon systems. This suggests to define the scalar product on the space of two-dimensional vectors of coefficient functions in the following form
\[
\langle \Psi_1 | \Psi_2 \rangle = 2N_c \langle \Psi_1^q | \Psi_2^q \rangle + n_f \langle \Psi_1^h | \Psi_2^h \rangle, \quad \Psi_a = \begin{pmatrix} \Psi_a^q \\ \Psi_a^h \end{pmatrix}, \quad (3.65)
\]

where the scalar product in each sector, \( \langle \Psi_1^q | \Psi_2^q \rangle \) and \( \langle \Psi_1^h | \Psi_2^h \rangle \), is obtained from \( (3.39) \) by substituting \( j_1 = j_3 = 1, j_2 = 3/2 \) and \( j_1 = j_2 = j_3 = 3/2 \), respectively. As we show below such choice of the scalar product ensures the hermiticity of the evolution kernels.
We shall look for the solutions to the evolution equation (3.58) in the following form

\[
\begin{pmatrix}
\Psi^q_N(x_i) \\
\Psi^h_{N-1}(x_i)
\end{pmatrix} = \sum_{k=0}^{N} \begin{pmatrix}
\frac{u^{q}_{N,k}}{\sqrt{N_c}} Y^q_{N,k}(x_i) \\
\frac{u^{h}_{N,k}}{\sqrt{n_f}} Y^h_{N-1,k-1}(x_i)
\end{pmatrix}
\] (3.66)

with \(u^{q}_{N,k}\) and \(u^{h}_{N,k}\) being the expansion coefficients. In this way, the evolution kernels finally become symmetric and real matrices acting on the vector of the expansion coefficients. We fix the normalization of the coefficients by requiring the eigenstates (3.66) to have the unit norm

\[
\|\Psi\|^2 = 2N_c\|\Psi^q_{N,\alpha}\|^2 + n_f\|\Psi^h_{N-1,\alpha}\|^2 = \sum_{k=0}^{N} |u^{q}_{N,k}|^2 + |u^{h}_{N,k}|^2 = 1 \quad (3.67)
\]

where \(u^{h}_{N,0} = 0\).

### 3.5 QCD evolution kernels

To one-loop accuracy, the evolution kernels are given \[20\] by the sum of two-particle Hamiltonians describing the pair-wise interaction between quarks and gluons on the light-cone: \(\hat{H} = \hat{H}_{12} + \hat{H}_{23} + \hat{H}_{31}\), or, equivalently \(H = H_{12} + H_{23} + H_{31}\). Conformal invariance (3.26) then implies that each pair-wise Hamiltonian \(H_{ik}\) only depends on the sum of conformal spins of the interacting particles, e.g. in the “direct” representation \(H_{ik} = H(J_{ik})\), where

\[
L^2_{ik} = (L_i + L_k)^2 = J_{ik}(J_{ik} - 1) \quad (3.68)
\]

and the \(SL(2)\) generators \(L^\alpha_i\) \((\alpha = \pm, 0)\) are given in Eq. (3.22). The explicit form of this dependence can most easily be obtained by comparing the eigenvalues. To this end it is sufficient to calculate the one-gluon exchange diagrams in a simplified situation when momenta of the contributing partons sum up to zero. Alternatively, one can start with the known integral representation for the evolution Hamiltonian \[9\] and project it onto the conformal basis (3.64).

#### 3.5.1 Diagonal evolution kernels

The diagonal quark evolution kernel is given by

\[
H_{qq}|Y^q_{N,n}\rangle = \left[ H_{S^+} + \frac{2n_f}{3} \delta_{J_{13},2} \right] |Y^q_{N,n}\rangle, \quad (3.69)
\]

where \(b = 11/3N_c + 2/3n_f\) is the lowest-order coefficient of the QCD \(\beta\)-function. The first term in brackets stands for the flavour-nonsinglet evolution kernel (see below) and the second term proportional to \(\delta_{J_{13},2}\) comes from the additional Feynman diagram in which the quark and the antiquark annihilate to produce a gluon that splits again into the \(q\bar{q}\)-pair. Since the quark and the antiquark are produced in this way in one spatial point, contribution of this diagram is different from zero only for \(J_{13} = j_q + j_{\bar{q}} = 2\). The flavour-nonsinglet Hamiltonian \(H_{S^+}\) can be represented as \[20, 22, 23, 24\]:

\[
H_{S^+} = N_c \tilde{H}^{(0)} - \frac{2}{N_c} \tilde{H}^{(1)}, \quad (3.70)
\]
where

\[ H^{(0)} = V^{(0)}_{qg}(J_{12}) + U^{(0)}_{qg}(J_{23}) \]  

\[ H^{(1)} = V^{(1)}_{qg}(J_{12}) + U^{(1)}_{qg}(J_{23}) + U^{(1)}_{qq}(J_{13}) . \]  

Here, the notation was introduced for the two-particle quark-quark and quark-gluon kernels

\[ V^{(0)}_{qg}(J) = \psi(J + 3/2) + \psi(J - 3/2) - 2\psi(1) - 3/4 , \]  

\[ U^{(0)}_{qg}(J) = \psi(J + 1/2) + \psi(J - 1/2) - 2\psi(1) - 3/4 , \]  

\[ V^{(1)}_{qg}(J) = \frac{(-1)^{J-5/2}}{(J - 3/2)(J - 1/2)(J + 1/2)} , \quad U^{(1)}_{qg}(J) = -\frac{(-1)^{J-5/2}}{2(J - 1/2)} , \]  

\[ U^{(1)}_{qq}(J) = \frac{1}{2} \left[ \psi(J - 1) + \psi(J + 1) \right] - \psi(1) - 3/4 , \]

where \( \psi(x) = d\ln \Gamma(x)/dx \).

The diagonal gluon kernel is defined as

\[ H_{hh}|Y_{N-1,k-1}^h⟩ = N_c [H_{3/2} - V_{3/2}] |Y_{N-1,k-1}^h⟩ \]  

where

\[ H_{3/2} = 2[U_{gg}(J_{12}) + U_{gg}(J_{23}) + U_{gg}(J_{31})] - b/N_c , \]  

\[ V_{3/2} = V_{gg}(J_{12}) + V_{gg}(J_{31}) \]

and

\[ U_{gg}(j) = \psi(j) - \psi(1) , \]  

\[ V_{gg}(j) = \frac{2}{j(j - 1)} + \frac{3(1 + (-1)^j)}{(j - 2)(j - 1)j(j + 1)} . \]

We notice that the kernel \( H_{hh} \) is invariant under permutations of the two gluons of the same helicity

\[ [H_{hh}, P_{23}] = 0 . \]  

Note that the pair-wise diagonal Hamiltonians \( \tilde{H}_{ik} \) and \( H_{ik} \) have the same functional dependence on the Casimir operators, i.e. \( \tilde{H}_{ik} = h(J_{ik}) \) and \( H_{ik} = h(J_{ik}) \) with the same function \( h \).

### 3.5.2 Off-diagonal evolution kernels

The off-diagonal kernels describe the mixing between quark-antiquark-gluon and three-gluon states. The conformal symmetry implies that the conformal spin of both states should be the same, and this applies both to the total conformal spin of the three-parton system and the conformal spin of the parton pair involved in the mixing. It follows that, therefore, acting on the basis function in the quark sector, \( |Y_{N,k}^q⟩ \), the evolution kernel \( H_{hq} \) transforms it into the basis function in the gluon sector, \( |Y_{N-1,k-1}^h⟩ \), with the same
Comparing this expression with the expansion is the standard operator basis (3.5) and as a consequence, the off-diagonal kernels can be written down in the following form

\[ H_{qh} = W_{qh}(J_{31}) \frac{1-P_{23}}{2}, \quad H_{hq} = \frac{1-P_{23}}{2} W_{hq}(J_{31}), \]  

(3.80)

where the operators \( W_{qh} \) and \( W_{hq} \) is defined as

\[ W_{qh}|Y^h_{N-1,k-1}\rangle = -n_f \left[ \frac{J_{13}^2 - J_{13} + 2(-1)^{J_{13}}}{J_{13} (J_{13} - 1) \sqrt{(J_{13} + 1)(J_{13} - 2)}} \right] |Y^q_{N,k}\rangle, \]

\[ W_{hq}|Y^q_{N,k}\rangle = -2N_c \left[ \frac{J_{13}^2 - J_{13} + 2(-1)^{J_{13}}}{J_{13} (J_{13} - 1) \sqrt{(J_{13} + 1)(J_{13} - 2)}} \right] |Y^h_{N-1,k-1}\rangle. \]  

(3.81)

The following comments are in order. The off-diagonal kernels \( H_{qh} \) and \( H_{hq} \) originate from the Feynman diagrams in which quark and antiquark annihilate into two gluons of opposite helicity. As a consequence, these kernels depend on the conformal spin in the quark-antiquark subchannel, \( J_{13} \), and the additional projector \( (1 - P_{23})/2 \) takes into account antisymmetry of the three-gluon state under the interchange of gluons with the same helicity, Eq. (3.48).

It follows from (3.81) that the off-diagonal kernels are adjoint to each other, \( H_{gh} = H_{hq}^\dagger \), with respect to the scalar product (3.66). Indeed, calculating the matrix elements of the off-diagonal kernels between the states \( \Psi^q_N \) and \( \Psi^h_{N-1} \) defined in (3.66) and taking into account their symmetry properties (3.56) one finds

\[ \langle \Psi^q_N|H_{qh}\Psi^h_{N-1}\rangle = \langle H_{hq}\Psi^q_N|\Psi^h_{N-1}\rangle \]

\[ = -\sqrt{2N_c} n_f \sum_{k=1}^N \frac{(k+1)(k+2) + 2(-1)^k}{(k+1)(k+2) \sqrt{k(k+3)}} (u^q_{N,k})^* u^h_{N-1,k-1}, \]  

(3.82)

where we have substituted \( J_{31} = k+2 \). Repeating the similar calculation for the matrix elements of the diagonal kernels, \( \langle \Psi^q_N|H_{qq}\Psi^q_N\rangle \) and \( \langle \Psi^h_{N-1}|H_{hh}\Psi^h_{N-1}\rangle \), one makes sure that \( H_{qq}^\dagger = H_{qq} \) and \( H_{hh}^\dagger = H_{hh} \). We conclude that the matrix of the evolution kernels entering the Schrödinger equation (3.58) is a hermitian operator on the space of the coefficient functions endowed with the scalar product (3.67). As a consequence, its eigenvalues \( \mathcal{E}_N \) are real and the corresponding eigenfunctions are orthogonal to each other.

### 3.6 Expansion in conformal operators

Once the Schrödinger equation in the helicity basis (3.58) is solved, we can easily restore the gluon coefficient function \( \Psi^q_{N,\alpha}(\xi) \), using the symmetry relation (3.57) and reconstruct the multiplicatively renormalizable operators (3.13). Their reduced forward matrix elements can be expressed in terms of the multiparton distributions as follows:

\[ \langle \mathcal{O}_{N,\alpha}(Q^2) \rangle = \frac{1}{||\Psi_{N,\alpha}||^2} \int_{-1}^1 D\xi \left[ 4N_c \Psi^q_{N,\alpha}(\xi) D_q(\xi, Q^2) + n_f \Psi^q_{N-1,\alpha}(\xi) D_g(\xi, Q^2) \right]. \]  

(3.83)

Notice that in contrast with (3.39) the integration goes here over the region \( \xi_1 + \xi_2 + \xi_3 = 0 \). Comparing this expression with the expansion is the standard operator basis (3.3) and
assuming that the eigenvectors $\Psi_{N,\alpha}$ is normalized to unity one gets the equivalence relations

$$4N_c \Psi_{N,\alpha}^q (\xi_i) \bigg|_{\sum \xi_i = 0} = \sum_{0 \leq k \leq N} C_{\alpha k}^q (N) \xi_1^k \xi_3^{N-k},$$

$$n_f \Psi_{N-1,\alpha}^q (\xi_i) \bigg|_{\sum \xi_i = 0} = \sum_{0 \leq m \leq [N/2]-1} C_{\alpha m}^q (N) \left( \xi_1^m \xi_3^{N-1-m} - \xi_1^{-1-m} \xi_3^m \right). \quad (3.84)$$

Next, expanding the OPE coefficient function $\Phi^q_{n}(\xi_1, \xi_3)$ (3.1) over the eigenstates of the evolution kernels

$$\left( \begin{array}{c} \Phi^q_{N+3}(\xi_1, -\xi_1 - \xi_3, \xi_3) \\ 0 \end{array} \right) = \sum_{\alpha} w_{N,\alpha} \left( \begin{array}{c} 2N_c \Psi_{N,\alpha}^q (\xi_i) \\ n_f \Psi_{N-1,\alpha}^q (\xi_i) \end{array} \right) \bigg|_{\xi_1+\xi_2+\xi_3=0} \quad (3.85)$$

we can decompose moments of the structure function $g_2(x, Q^2)$ (3.3) in multiplicatively renormalizable contributions (3.83) as

$$g_{2L}(N+3, Q^2) = \frac{1}{2} (\epsilon_q^2) \frac{4}{N+3} \sum_{\alpha=0}^{[N/2]} w_{N,\alpha} \langle \mathcal{O}_{N,\alpha}(Q^2) \rangle. \quad (3.86)$$

The expansion coefficients $w_{N,\alpha}$ can be expressed in terms of $C_{\alpha k}^q (N)$ and $C_{\alpha m}^q (N)$ by comparing the coefficients in front of different powers of $\xi_1$ and $\xi_3$ in the both sides of (3.85).

### 3.6.1 Quark coefficient function

There exists, however, a more efficient way of finding the same coefficients. To this end, observe that the OPE coefficient function $\Phi^q_{N+3}(\xi_1, -\xi_1 - \xi_3, \xi_3)$ can uniquely be continued from the hyperplane $\xi_1 + \xi_2 + \xi_3 = 0$ to arbitrary values of $\xi_1$ by requiring that the thus defined function $\Phi^q_{N+3}(\xi_1, \xi_2, \xi_3)$ satisfies the conformal constraints (3.33) (and coincides with $\Phi^q_{N+3}(\xi_1, -\xi_1 - \xi_3, \xi_3)$ at $\xi_2 = -\xi_1 - \xi_3$). To find an explicit expression, let us expand $\Phi^q_{N+3}(\xi_1, \xi_2, \xi_3)$ over the conformal basis (3.64)

$$\Phi^q_{N+3}(x_1, x_2, x_3) = \sum_{k=0}^{N} \phi^q_{N,k} Y^q_{N,k}(x_i) \quad (3.87)$$

with $\phi^q_{N,k} = \langle \Phi^q_{N+3} | Y^q_{N,k} \rangle$. Projecting out the hyperplane $\sum_i x_i = 0$, we find using (3.64) that the basis functions are reduced to

$$Y^q_{N,k}(x_i) \bigg|_{\sum_i x_i = 0} \sim (x_1 + x_3)^N P^{(1,1)}_k \left( \frac{x_1 - x_3}{x_1 + x_3} \right) \quad (3.88)$$

and form an orthogonal basis on the subspace $x_1 + x_3 = 1$ and $0 \leq x_1, x_3 \leq 1$. This property allows us to calculate the expansion coefficients $\phi^q_{N,k}$ as

$$\phi^q_{N,k} \sim \int_0^1 dx_1 dx_3 \delta(1 - x_1 - x_3) x_1 x_3 P^{(1,1)}_k \left( \frac{x_1 - x_3}{x_1 + x_3} \right) \Phi^q_{N+3}(x_1, -x_1 - x_3, x_3). \quad (3.89)$$
Substituting the actual expression for $\Phi_{N+3}(\xi_i)$ and performing the integration one arrives at

$$
\phi_{N,k}^q = \left[ (-1)^{N-k} + 1 \right] \frac{N + k + 5}{N - k + 1} + \left[ (-1)^{N-k} - 1 \right] \frac{N + k + 4}{N - k + 2} \times \left( \frac{(k + 1)(k + 2)(2k + 3)(N - k + 1)(N - k + 2)}{8(N + 3)(N + k + 4)(N + k + 5)} \Gamma^4(N + 3) \right)^{1/2} . \tag{3.90}
$$

Using these coefficients one can easily calculate the norm of the quark coefficient function

$$
\|\Phi_{N+3}^q\|^2 = \sum_{k=0}^{N} (\phi_{N,k}^q)^2 = \frac{\Gamma^4(n)}{\Gamma(2n)} n^4 \left[ 1 + \frac{1}{n^2} (1 - 4\psi(n) - 4\gamma_E) - \frac{2}{n^3} \right] , \tag{3.91}
$$

where in the r.h.s. $n = N + 3$.

Finally, using (3.87) and (3.90) one finds the expansion coefficients for moments of the structure function

$$
\omega_{N,\alpha} = \langle \Phi_{N+3}^q | \Psi_{N,\alpha}^q \rangle = \frac{1}{\sqrt{2N_c}} \sum_{k=0}^{N} \phi_{N,k}^q u_{N,k}^q \cdot \tag{3.92}
$$

The coefficients $u_{N,k}^q$ stand for the expansion of a multiplicatively renormalizable operator in the conformal basis, cf. Eqs. (3.66) and (3.67). Their dependence on the particular operator (index $\alpha$) is tacitly assumed.

### 3.6.2 Gluon coefficient functions

The similar procedure can be used to calculate the expansion coefficients of the gluon coefficient functions $\Phi_{k}^g$ and $\Omega_{k}^{qg}$ defined in (2.19) and (2.23). The corresponding conformal harmonics in the helicity basis are given in the forward direction by

$$
Y_{N-1,k}^h(x_i) \Big|_{\sum_i x_i = 0} \sim (x_1 + x_3)^{N-1} P_k^{(2,2)} \left( \frac{x_1 - x_3}{x_1 + x_3} \right) . \tag{3.93}
$$

In order to decompose the gluon coefficient function (2.23) over this basis we first construct the same coefficient function in the helicity representation

$$
\Phi_{N+3}^h(x_i) = \frac{1}{2} \left[ \Phi_{N+3}^q(x_1, x_2, x_3) - \Phi_{N+3}^q(x_1, x_3, x_2) \right] \tag{3.94}
$$

so that $P_{23} \Phi_{N+3}^h(x_i) = -\Phi_{N+3}^h(x_i)$ and then define the expansion coefficients $\phi_{N-1,k}^h$ as

$$
\Phi_{N+3}^h(x_i) = \sum_{k=0}^{N-1} \phi_{N-1,k}^h Y_{N-1,k}^h(x_i) . \tag{3.95}
$$

Using orthogonality of the Jacobi polynomials we can calculate these coefficients as

$$
\phi_{N-1,k}^h \sim \int_0^1 dx_1 dx_3 \delta(1 - x_1 - x_3) (x_1 x_3)^2 P_k^{(2,2)} \left( \frac{x_1 - x_3}{x_1 + x_3} \right) \Phi_{N+3}^h(x_1, -x_1 - x_3, x_3) , \tag{3.96}
$$

where $\delta$ is the Dirac delta function.
Substituting the gluon coefficient function (2.23) into (3.94) and (3.96) and going through the calculation one finds the following explicit expressions for the expansion coefficients for even conformal spins \( N \)

\[
\phi_{N-1,k}^h = -\frac{(N + k + 5)(N + 7 + 5/2 k + 1/2 k^2)}{(k + 1)(N - k + 1)} h_{N,k}
\]

(3.97)

for even \( k \) and

\[
\phi_{N-1,k}^h = \frac{(k + 4)(N + k + 6)}{2(N - k)} h_{N,k}
\]

(3.98)

for odd \( n \). Here, the normalization constant \( h_{N,k} \) is given by

\[
h_{N,k}^2 = \frac{(k + 1)(2k + 5)(N - k)(N - k + 1)(N + 3)^2}{8(k + 3)(k + 4)(k + 2)(N + 5 + k)(N + k + 6)} \Gamma(2N + 6).
\]

(3.99)

Using (3.95) we can calculate the norm of the gluon coefficient function

\[
\|\Phi^h_{N+3}\|^2 = \sum_{k=0}^{N-1} (\phi_{N-1,k}^h)^2 = \frac{\Gamma^4(n) n^3(n-1)}{\Gamma(2n)} \left[ 1 + \frac{2(n - 4)(n + 1)}{(n - 1)^2 n} (\psi(n) + \gamma_E) \right],
\]

(3.100)

where \( n = N + 3 \), as above.

The second gluon coefficient function, \( \Omega^{gq}_{n}(\xi_i) \), appears in the expression for the moments of the structure function (2.19) due to mixing between quark-antiquark-gluon and three-gluon distribution amplitudes. Similar to (3.94) and (3.95), one can transform this function into the helicity representation and decompose it over the conformal basis

\[
\Omega_{N+3}^{h}(x_i) = \frac{1}{2} \left[ \Omega_{N+3}^{gq}(x_1, x_2, x_3) - \Omega_{N+3}^{gq}(x_1, x_3, x_2) \right] = \sum_{k=0}^{N-1} \omega_{N-1,k}^h Y_{N-1,k}^h(x_i).
\]

(3.101)

Substituting \( \Omega^{gq}_{n}(\xi_i) \) by its explicit expression (2.23) it becomes straightforward to calculate the coefficients \( \omega_{N-1,k}^h \) using (3.96). The resulting explicit expression is cumbersome and will not be displayed here. Comparing the expansions (3.99) and (3.101) one finds, however, [12] that the two coefficient functions \( \Phi^g_{n}(\xi_i) \) and \( \Omega^{gq}_{n}(\xi_i) \) are to a good accuracy proportional to each other

\[
\Omega^{gq}_{n}(\xi_i) \approx c(n) \Phi^g_{n}(\xi_i).
\]

(3.102)

Applying the transformation (3.94) and (3.101) to the both sides of this relation and using the orthogonality of the conformal basis we can calculate the coefficient \( c(n) \) as

\[
c(n) = \langle \Omega^{gq}_{n} | \Phi^g_{n} \rangle / \| \Phi^g_{n} \|^2 = \sum_{k=0}^{N-1} \phi_{N-1,k}^h \omega_{N-1,k}^h / \| \Phi^g_{n} \|^2
\]

(3.103)

with the norm \( \| \Phi^g_{n} \| \) given in (3.100). Going through the calculation one finds that \( c(n) \) is given at large \( n \) by the following expansion:

\[
c(n) = 1 + \frac{1}{n^2} \left[ 4 \ln n + 4 \gamma_E - 6 \right] + \mathcal{O}(1/n^3).
\]

(3.104)
Figure 1: The spectrum of anomalous dimensions of flavor-singlet twist-3 operators with the mixing between quark-antiquark-gluon (crosses) and three-gluon operators (open circles) switched off, cf. (4.1).

4 Results

In this section we present detailed results on the solution of the Schrödinger equation in Eq. (3.58). The main advantage of the Hamiltonian approach described in the previous section is that it allows to understand qualitative features of the solutions using the intuition and a wealth of analytical tools well known from quantum mechanics. For this analysis, we decompose the full Hamiltonian in (3.58) in two parts:

$$H^h = H_0 + V, \quad H_0 = \begin{pmatrix} H_{qq} & 0 \\ 0 & H_{hh} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & H_{qh} \\ H_{hq} & 0 \end{pmatrix}. \quad (4.1)$$

The Hamiltonian $H_0$ governs the scale-dependence separately in the ‘quark’ and ‘gluon’ sectors, whereas the off-diagonal kernel $V$ describes the mixing between the two sectors. Our strategy throughout this section will be first to solve the Schrödinger equation for $H_0$ and then examine the deformation of the spectrum induced by $V$ that we, somewhat imprecisely, refer to as the quark-gluon mixing. The rationale for such a two-step procedure is that properties of the individual Hamiltonians $H_{qq}$ and $H_{hh}$ have been studied in detail in recent works [22, 23, 24] using their relation to completely integrable models. We will, therefore, be able to use these results.

4.1 Diagonal evolution kernels

The spectrum of eigenvalues of $H_0$ is obviously given by the superposition of the two independent spectra of quark-antiquark-gluon and three-gluon operators

$$H_{qq} \Psi^{q,(0)}_{N,\alpha} = E_{N,\alpha}^q \Psi^{q,(0)}_{N,\alpha}, \quad H_{hh} \Psi^{h,(0)}_{N,\beta} = E_{N,\beta}^h \Psi^{h,(0)}_{N-1,\beta} \quad (4.2)$$

and the corresponding eigenfunctions are given by

$$\Psi^{(0)}_{N,\alpha}(x_i) = \begin{pmatrix} \Psi^{q,(0)}_{N,\alpha}(x_i) \\ 0 \end{pmatrix}, \quad \Psi^{(0)}_{N,\beta+N+1}(x_i) = \begin{pmatrix} 0 \\ \Psi^{h,(0)}_{N-1,\beta}(x_i) \end{pmatrix}. \quad (4.3)$$
Figure 2: The dependence of the anomalous dimensions of flavor-singlet quark-antiquark-gluon (crosses) and three-gluon operators (open circles) on the number of light quark flavors \( n_f \). The dotted line corresponds to the linear dependence \( \sim (2/3)n_f \), see text. The quark-gluon mixing is switched off, cf. (4.1).

The superscript \((0)\) stands to remind that the off-diagonal mixing terms are omitted and the subscripts \(\alpha\) and \(\beta\) enumerate different ‘quark’ and ‘gluon’ eigenstates corresponding to multiplicatively renormalizable operators with \(N\) (quark) or \(N-1\) (gluon) covariant derivatives with the same canonical dimension (and the same conformal spin \(J = N + 7/2\)). In addition, we require that the ‘gluon’ eigenfunctions satisfy the symmetry property (3.56). For a given \(N\) the total number of the ‘quark’ and ‘gluon’ eigenstates is equal to \(\ell_q = N + 1\) and \(\ell_g = \lfloor N/2 \rfloor\), respectively, so that \(\alpha = 0, \ldots, N\) and \(\beta = 0, \ldots, \lfloor N/2 \rfloor - 1\).

The results of the numerical calculation of the spectrum for \(N < 20\) and \(n_f = 3\) are shown in Fig. 1. Note that the gluon eigenstates (open circles) and the quark eigenstates (crosses) occupy two bands that lie on the top of each other. For large \(N\) the eigenvalues (anomalous dimensions) rise logarithmically

\[
4C_F \ln N \leq \mathcal{E}_{N,\alpha}^q \leq 4N_c \ln N,
\]

\[
4N_c \ln N \leq \mathcal{E}_{N,\beta}^g \leq 6N_c \ln N
\]

(4.4)

and the coefficients in front of \(\ln N\) are related to the color charges of the corresponding (classical) parton configurations. Since \(\sim \mathcal{O}(N)\) levels have to fit within the band-width \(\sim \mathcal{O}(\ln N)\) (1.4), the distance between the neighboring levels in general goes to zero. The analysis of the ‘fine structure’ of the spectra [23, 24] reveals, however, that in the limit \(N \to \infty\) three levels remain separated from the rest of the spectrum by a finite gap. These three special levels are: the lowest quark level, the highest quark level and the lowest gluon level; they will play a decisive rôle in what follows.

4.1.1 The highest quark-antiquark-gluon state

The eigenvalues \(\mathcal{E}_{N,\alpha}^q\) and \(\mathcal{E}_{N,\beta}^g\) depend on the number of light quark flavors \(n_f\). This dependence is shown in Fig. 2 for \(N = 8\) and \(0 \leq n_f \leq 10\) and reveals the following
Figure 3: The coefficients of the expansion (4.4) of $Y^q_{N,k=0}(x_i)$ over the eigenstates of the quark Hamiltonian $H_{qq}$. $N = 8$ and $N = 20$ for the left and the right panel, respectively.

remarkable pattern: In the gluon sector the $n_f$–dependence of all energy levels is linear, $E^g_{N,\beta} \sim 2n_f/3$, and it can be traced to the additive $b$–correction to $[3.75]$. In contrast to this, in the quark sector all energy levels except the highest one vary very slowly with $n_f$. At the same time, the $n_f$–dependence of the highest quark level is almost identical to that of the gluon levels, $E^q_{N,N} \sim 2n_f/3$. To understand this property, notice that the $n_f$–dependence of the diagonal quark kernel in Eq. (3.69) comes entirely from the annihilation term $(2n_f/3)\delta_{J_{13}}$. At large $n_f$ this term dominates and the corresponding eigenstate is given by

$$\Psi^{q,(0)}_{N,N}(x_i) = Y^q_{N,k=0}(x_i) + O(1/n_f).$$

(4.5)

Remarkably enough, this relation holds true with high accuracy for small values of $n_f$ as well, including $n_f \to 0$. To illustrate this, we show in Fig. 3 the coefficients in the expansion of $Y^q_{N,k=0}(x_i)$ over the complete set of the quark eigenstates:

$$Y^q_{N,k=0}(x_i) = \sum_{\alpha=0}^{N} \langle Y^q_{N,k=0} | \Psi^{q,(0)}_{N,\alpha} \rangle \Psi^{q,(0)}_{N,\alpha}(x_i)$$

(4.6)

for $n_f = 3$ and two different values of $N$. Note that the normalization is such that $||\Psi^{q,(0)}_{N,\alpha}|| = 1$ and, as a consequence, the sum of the coefficients squared is equal to one. It is seen that the overlap with the exact wave function of the highest ‘quark’ state $\alpha = N$ is very close to unity.

As a yet another test of the approximation for the wave function (4.3) one evaluates the corresponding energy (see Appendix A for details)

$$E^q_{N,N} \simeq \langle Y^q_{N,k=0} | H_{qq} | Y^q_{N,k=0} \rangle = N_c \left[ 2\Psi(N+3) + 2\Psi(N+4) + 4\gamma_E - \frac{6}{(N+3)^2} - \frac{19}{6} \right]$$

$$+ \frac{1}{N_c} \frac{6}{(N+3)^2(N+4)} + \frac{2}{3}n_f.$$  

(4.7)

The few first terms of the expansion of the matrix element at large $N$ are

$$E^q_{N,N} \simeq N_c \left[ 4\ln(N+3) + 4\gamma_E - \frac{19}{6} - \frac{1}{3(N+3)^2} \right] + \frac{2}{3}n_f + O\left(\frac{1}{(N+3)^3}\right).$$

(4.8)
Table 1: Exact numerical results for the energy of the three ‘special’ levels (see text) compared with the calculation using the approximations for the corresponding eigenfunctions. The results for the lowest quark-antiquark-gluon, the highest quark-antiquark-gluon and the lowest three-gluon eigenstates are shown in the two left, two middle and two right columns, respectively.

The approximation (4.7) is compared with the exact numerical calculations of the energy in Table 1 (two middle columns). The accuracy turns out to be very good, better than 1% for all $N$.

Last but not least, we note that according to (3.88) the wave function entering (4.5), $Y_{N,0}^q(x_i) \sim (x_1 + x_3)^N = (-x_2)^N$, depends on a single (gluon) momentum fraction only and, therefore, it defines a local operator (3.11) and (3.12) that contains derivatives acting on the gluon but not on the quark fields. All such operators can be obtained from a Tailor expansion of the nonlocal operator (2.11) in which the quark and the antiquark are located at the same space-time point and which can be rewritten as a two-gluon operator using the equations of motion:

$$S^q_{\beta}(0,v,0) = i\bar{q}(0)g\tilde{G}_{\beta x}(vx)\not{q}(0) = -i\tilde{G}^a_{\beta x}(vx)D^{ab}_\xi G^b_{\xi x}(0).$$

Here $a, b$ are color indices and $D^{ab}$ is the covariant derivative in the adjoint representation. Thus, to a good accuracy, the quark-antiquark-gluon three-particle operator with the highest anomalous dimension is in fact a two-particle gluon operator!

### 4.1.2 The lowest quark-antiquark-gluon state

Since eigenfunctions of the hermitian Hamiltonian $H_{qq}$ are orthogonal to each other and since the eigenfunction corresponding to the highest anomalous dimension turns out to be very close to the eigenfunction of the $n_f$-dependent contribution of the annihilation diagram that gives rise to the second term in the Hamiltonian (3.69), it follows that the wave functions of all other levels have a negligible overlap with this term and, therefore, are to a good accuracy $n_f$-independent, in agreement with Fig. 2.

This means that within the accuracy of (3.13), the eigenfunctions and eigenvalues of all quark levels other than the highest one coincide with those of the flavor-nonsinglet quark Hamiltonian $H_{S+}$ that was studied in \cite{21, 23, 24}. In particular, it has been shown that the wave function corresponding to the lowest anomalous dimension coincides, in the large-$N_c$ limit, with the tree-level coefficient function in the OPE (2.22) continued from the hyperplane $\xi_1 + \xi_2 + \xi_3 = 0$ to arbitrary values of $\xi_i$ using the conformal symmetry,
Figure 4: The coefficients of the expansion of $\Phi^q_{N+3}(x_i)/\|\Phi^q_{N+3}\|$ (4.10) over the eigenstates $\psi^q_{N,\alpha}^{(0)}$ of the quark Hamiltonian $H_{qq}$.

cf. (3.87):

$$\psi^q_{N,\alpha=0} = \frac{\Phi^q_{N+3}(x_i)}{\|\Phi^q_{N+3}\|} + O(1/N^2).$$

(4.10)

The additional factor in the r.h.s. takes into account the different normalization of the states. In Fig. 4 we show the coefficients of the expansion of $\Phi^q_{N+3}(x_i)/\|\Phi^q_{N+3}\|$ over the eigenstates of the quark Hamiltonian $H_{qq}$ for $N = 8$ and $N = 20$. It is seen that the expansion is dominated by the lowest quark state $\alpha = 0$. The corresponding eigenvalue (anomalous dimension) can be calculated as (see Appendix A for details)

$$\mathcal{E}^q_{N,0} \simeq \frac{\langle \Phi^q_{N+3}|H_{qq}|\Phi^q_{N+3}\rangle}{\|\Phi^q_{N+3}\|^2} = N_c E^{(0)}_N + \frac{1}{N_c} E^{(1)}_N$$

(4.11)

where [10]

$$E^{(0)}_N = 2\Psi(N + 3) + \frac{1}{N + 3} - \frac{1}{2} + 2\gamma_E$$

(4.12)

and [12]

$$E^{(1)}_N = -2 \left[ \ln(N + 3) + \gamma_E + \frac{3}{4} - \frac{\pi^2}{6} + O \left( \frac{\ln^2(N + 3)}{(N + 3)^2} \right) \right].$$

(4.13)

The quality of this approximation is illustrated in Table 1 (compare the numbers in the first two columns).

4.1.3 The lowest three-gluon state

Last but not least, we have to work out an approximate description of the lowest three-gluon state. As noticed in [12], the corresponding eigenfunction appears to be close to the one-loop gluon coefficient function $\Phi^g(\xi_i)$ (2.23) transformed to the helicity representation (3.94) (and continued to arbitrary values of $\xi_i$ using the conformal symmetry):

$$\psi^{h,(0)}_{N-1,\beta=0} \simeq \frac{\Phi^h_{N+3}(x_i)}{\|\Phi^h_{N+3}\|} = \frac{1}{2\|\Phi^h_{N+3}\|} \left( \Phi^g_{N+3}(x_1, x_2, x_3) - \Phi^g_{N+3}(x_1, x_3, x_2) \right).$$

(4.14)
In difference to the above, this approximation cannot be justified in any formal limit, but is no less accurate, as illustrated in Fig. 5 where we show the corresponding (normalized) expansion coefficients in the basis of exact three-gluon eigenstates.

The corresponding approximation for the energy can be calculated as (see Appendix A for details)

\[
E_g^N,0 \simeq \frac{\langle \Phi_{N+3}^h | H_{hh} | \Phi_{N+3}^h \rangle}{\| \Phi_{N+3}^h \|^2} = N_c E_N^g + \frac{2}{3} n_f
\]

with

\[
E_N^g = 4 \ln(N+3) + 4\gamma_E + \frac{1}{3} - \frac{\pi^2}{3} + \frac{1}{N+3} \left[ (\ln(N+3) + \gamma_E) \left( \frac{2\pi^2}{3} - 6 \right) - \frac{\pi^2}{3} \right] + O\left( \frac{\ln^2(N+3)}{(N+3)^2} \right).
\]

(4.15)

(4.16)

It is compared with the exact result in Table [I] (the two last columns). Note that the energies of the lowest gluon and the highest quark states are very close to each other, as seen also from Fig. 5:

\[
[E_g^q_{N,N} - E_g^g_{N,0}] / E_g^q_{N,N} \sim \frac{10^{-2}}{\ln(N+3)}.
\]

(4.17)

In fact, the difference is so small that one can treat these two levels as degenerate for most purposes.

### 4.2 The quark-gluon mixing

The mixing of quark-antiquark gluon and three-gluon operators is governed by the off-diagonal part \( \mathcal{V} \) of the Hamiltonian (4.4). It turns out that the mixing is generally rather weak and for most practical purposes can be taken into account using the standard perturbation theory. To the leading order, the mixing-induced corrections to the ‘pure’
The exact energy of the three ‘special’ levels (see text) with the mixing-induced corrections taken into account compared with the energy of the corresponding eigenstates with the mixing effects neglected. The results for the lowest quark-antiquark-gluon, the highest quark-antiquark-gluon and the lowest three-gluon eigenstates are shown in the two left, two middle and two right columns, respectively.

Table 2: The exact energy of the three ‘special’ levels (see text) with the mixing-induced corrections taken into account compared with the energy of the corresponding eigenstates with the mixing effects neglected. The results for the lowest quark-antiquark-gluon, the highest quark-antiquark-gluon and the lowest three-gluon eigenstates are shown in the two left, two middle and two right columns, respectively.

| N  | $\mathcal{E}_{N,0}$ | $\mathcal{E}_{N,0} - \mathcal{E}_{N,0}^q$ | $\mathcal{E}_{N,N}$ | $\mathcal{E}_{N,N} - \mathcal{E}_{N,N}^q$ | $\mathcal{E}_{N,N+1}$ | $\mathcal{E}_{N,N+1} - \mathcal{E}_{N,N}^q$ |
|----|---------------------|---------------------------------|---------------------|---------------------------------|---------------------|---------------------------------|
| 2  | 10.739              | -0.193                          | 17.679              | -0.428                          | 18.171              | 0.821                           |
| 4  | 12.571              | -0.058                          | 22.324              | -0.262                          | 22.648              | 0.561                           |
| 6  | 13.912              | -0.027                          | 25.637              | -0.148                          | 25.872              | 0.397                           |
| 8  | 14.980              | -0.015                          | 28.195              | -0.090                          | 28.420              | 0.312                           |
| 10 | 15.867              | -0.009                          | 30.278              | -0.065                          | 30.530              | 0.270                           |
| 30 | 20.823              | -0.001                          | 41.600              | -0.034                          | 42.056              | 0.209                           |

It is easy to see that $\langle \Psi^{(0)}_N | V | \Psi^{(0)}_N \rangle = 0$ and, therefore, the energy eigenvalues do not receive any corrections to the first order of the perturbative expansion. This explains why the mixing-induced corrections to the anomalous dimensions are very small (see Table 2). The numerical results presented below are in all cases based on the exact diagonalization of the mixing matrix, and the first-order expressions in (4.18) only serve to illustrate the picture. Note that the perturbation theory cannot be used to describe the mixing between the highest quark-antiquark gluon and the lowest three-gluon states because of the vanishing energy denominators; in this case the explicit diagonalization is mandatory.

The mixing matrix $V_{\beta \alpha}$ has a rather peculiar wing-like shape as illustrated in Fig. 6. We remind that the index $\alpha$ along the horizontal axis serves to numerate different quark-antiquark-gluon states, starting with the one with the lowest anomalous dimension at $\alpha = 0$, and the index $\beta$ enumerates the three-gluon eigenstates in the similar fashion. Dark regions in Fig. 6 correspond to large absolute values of $V_{\beta \alpha}$ and light regions indicate small matrix elements. Note the complicated ‘chess-board’ pattern with alternating large and small entries.

The most important feature that is seen in Fig. 6 is that the lowest quark eigenstate $\alpha = 0$ mixes significantly only with the lowest gluon eigenstate $\beta = 0$. In fact, we find a (roughly) exponential hierarchy of the matrix elements

$$|V_{0,0}| \gg |V_{1,0}| \gg ... \gg |V_{N/2-1,0}|.$$
that is valid for all \( N \), see Fig. 6. Higher quark-antiquark-gluon states get mixed with gluons more heavily and involve many three-gluon states in an essential way. For a few highest quark states, \( \alpha \sim N \), the mixing is again simplified somewhat, but still involves several (in general \( \sim \ln N \)) gluon states, see Fig. 6. It can be shown that the mixing coefficient of the highest quark-antiquark gluon state (4.13) with the lowest three-gluon state (4.14) \( V_{0,\beta}^{\alpha} \) vanishes in the large-\( N \) limit and this smallness overcomes the enhancement due the small energy denominator in (4.18) in the same limit, cf. (4.17).

Since the Hamiltonian (4.1) is hermitian, the same mixing matrix describes the mixing of the gluon states with the quark states. It is seen that the lowest gluon states \( \beta = 0, 1, \ldots \) get mixed with many quark states while the highest gluon states \( \beta \sim N/2 \) only mix with quark states with \( \alpha \sim 2/3N \).

As a yet another illustration of the mixing pattern, we show in Fig. 8 the exact numerical results for the expansion coefficients of two different exact eigenstates of the 'full' Hamiltonian (4.1), \( \Psi_{N,0}(x_i) \) and \( \Psi_{N,N+1}(x_i) \), over the 'pure' quark and gluon eigenstates. The chosen states are those that reduce to the lowest quark and the lowest gluon states if the mixing is 'switched off'. For the lowest quark state (which is the lowest eigenstate in the whole spectrum) the single significant contribution comes from the lowest gluon state (\( \sim 10^{-3} \)) while all other contributions are suppressed by another order of magnitude. For the gluon state the mixing is much larger (\( \leq 10^{-1} \)) and involves many quark states. This is the same structure that we observed earlier in Fig. 6. Another conclusion is that since the mixing-induced corrections are at most of the order of 10% (for large \( N \sim 100 \)), the perturbative expansion in (4.18) should be rather accurate and this indeed can be verified by the direct numerical calculation.
Figure 7: The mixing coefficients $V_{\beta 0}$ of the different three-gluon states $\beta = 0, 1, \ldots, \lfloor N/2 \rfloor - 1$ with the lowest quark-antiquark-gluon state, for four different values of $N$: $N = 8$ (triangles), $N = 20$ (squares), $N = 50$ (open circles) and $N = 100$ (full circles).

4.3 Reduction of the mixing matrix: The two-channel DGLAP equation

We have defined the transverse spin quark and gluon distributions $\Delta q_T(x)$ and $\Delta g_T(x)$ as the specific projections of the generic quark-antiquark-gluon and three-gluon operators, Eqs. (2.28) and (2.29), respectively. They correspond to the leading order quark and gluon contributions to the operator product expansion of the T-product of the two electromagnetic currents (2.10) and determine the odd moments of the structure function (2.19) at a high scale $Q^2$

\[
[g_2(n, Q^2)]^{tw-3} = \frac{1}{2} \sum_{q=u,d,s,\ldots} e_q^2 \left[ \frac{1}{n} \Delta q_T^+(n, Q^2) + \frac{\alpha_s}{\pi} C_g^T(n) \Delta g_T(n, Q^2) \right].
\] (4.22)

Here, we took into account the relation between the gluon coefficient functions (3.102) and introduced a notation for the overall gluon coefficient function $C_g^T$

\[
C_g^T(n) = \int_0^1 dx \, x^{n-1} C_g^T(x) = \frac{1}{n(n+1)} (1 - c(n) [\psi(n) + \gamma_E + 1])
\] (4.23)

with $c(n)$ given by (3.104). It is therefore natural to exercise a model for the structure function in which $g_2(x, Q^2)$ can be expressed in terms of these two distributions defined at a low scale $\mu^2$. Such a model cannot be exact, since it is not theoretically consistent: Other partonic degrees of freedom are generated through the QCD evolution of $\Delta q_T(x, Q^2)$ and $\Delta g_T(x, Q^2)$ down to a low scale. Assuming the absence of their contribution at two different scales $Q^2$ and $\mu^2$ simultaneously is, therefore, not possible. We shall argue,
Figure 8: Exact numerical results for the expansion coefficients of the exact eigenstates of the Hamiltonian $H$ over the eigenvectors of the diagonal blocks: a) - the exact lowest eigenstate, $\langle \Psi_{N,0}|\Psi_{N,k}^{(0)} \rangle$; b) - the exact lowest eigenstate in 'gluon' sector, $\langle \Psi_{N,N+1}|\Psi_{N,k}^{(0)} \rangle$. The index $k$ enumerates the operators ordered according to their anomalous dimension; $0 \leq k = \alpha \leq N$ corresponds to quark operators, $k = N + 1 + \beta \geq N + 1$ to gluons, cf. (4.3). The ‘quark’ eigenstates are shown by crosses, the ‘gluon’ eigenstates by open circles.

however, that the admixture of the additional degrees of freedom turns out to be small numerically, at least for large $N$. This means that the two-component model for the structure function $g_2(x,Q^2)$ based on the parton distributions $\Delta q_T(x,Q^2)$ and $\Delta g_T(x,Q^2)$ has a good numerical accuracy and can eventually be made more sophisticated (involve more states) when and if high accuracy experimental data become available.

In order to unravel the qualitative structure of the evolution it is necessary to go over to large moments $N$ so that the number of contributing operators becomes large and possible systematic effects more transparent. To this end we take $N = 100$ and, as a first step, examine the coefficients in the expansion of $\Delta q_T^+(N + 3) = \int_0^1 dx x^{N+2} \Delta q_T^+(\xi,Q^2)$ and $\Delta g_T(N + 3) = \int_0^1 dx x^{N+2} \Delta g_T(\xi,Q^2)$ in contributions of the multiplicatively renormalizable operators (3.83). We recall that the moments $\Delta q_T^+(n)$ and $\Delta g_T(n)$ are given by reduced matrix elements of the local composite quark-antiquark-gluon and three-gluon operators defined as

$$
\Delta q_T^+(n) = 2 \langle \Phi_n^{\alpha}(\partial_i) S_\mu(z_i) \rangle \bigg|_{z_i=0}, \quad \Delta g_T(n) = \langle \Phi_n^{\alpha}(\partial_i) \tilde{O}_\mu(z_i) \rangle \bigg|_{z_i=0}.
$$

(4.24)

Replacing the nonlocal operators by the expansion over the multiplicatively renormalizable operators (3.16) one gets

$$
\Delta q_T^+(N + 3, Q^2) = 2 \sum_{\alpha=0}^{3N/2} \langle \Phi_{N+3}^{\alpha}|\Psi_{N,\alpha}^{(0)} \rangle \langle \mathbf{O}_{N,\alpha}(\mu^2) \rangle \mathcal{L}_{N,\alpha}/b,
$$

where
Figure 9: Exact numerical results for the coefficients: a) $-|\langle \Psi_{q,N,k}^N | \Phi_{q,N+3}^N \rangle|/\|\Phi_{q,N+3}^N\|$ and b) $-|\langle \Psi_{h,N-1,k}^N | \Phi_{h,N+3}^N \rangle|/\|\Phi_{h,N+3}^N\|$ of the expansion of the $N=100$th moment of quark and gluon transverse spin distributions, respectively, in the contributions of multiplicatively renormalized operators. The index $k$ enumerates the operators ordered according to their anomalous dimension; $0 \leq k = \alpha \leq N$ corresponds to quark operators, $k = N + 1 + \beta \geq N + 1$ to gluons, cf. (4.3). The ‘quark’ eigenstates are shown by crosses, the ‘gluon’ eigenstates by open circles.

\[
\Delta g_T(N+3,Q^2) = \sum_{\alpha=0}^{3N/2} \langle \Phi_{h,N+3}^N | \Psi_{h,N-1,\alpha}^N \rangle \langle \langle O_{N,\alpha}(\mu^2) \rangle \rangle L_{E_{N,\alpha}}^{N,\beta}, \tag{4.25}
\]

where $\Psi_{h,N,\alpha}^N$ and $\Psi_{h,N-1,\alpha}^N$ are ‘quark’ and ‘gluon’ components of the exact eigenstate of the evolution kernel in the helicity basis.

The expansion coefficients $\langle \Phi_{h,N+3}^N | \Psi_{h,N,\alpha}^N \rangle$ for $N=100$ are shown in Fig. 9a. It is seen that $\Delta g_T(N+3)$ is grossly dominated by the contribution of the local operator with the lowest anomalous dimension, which means that all other contributions to the evolution are small. Among the other contributions, noticeable corrections come from a few quark operators with the low anomalous dimensions, and a single gluon operator with the lowest anomalous dimension in the gluon sector. The contributions of the quark operators with the anomalous dimensions close the the lowest one are less interesting than the gluon contribution for the following two reasons:

- Since the anomalous dimensions of important quark operators are all close to that of the leading quark operator, the corresponding contributions are barely distinguishable by the evolution.

- The admixture of quark operators with the next-to-the-lowest etc. anomalous dimensions is roughly the same in the flavor-nonsiglet and the flavor-singlet channels. Because of this, one does not expect any qualitative effects. On the other hand, the appearance of the gluon operator is a new feature of the singlet evolution.

Neglecting the higher quark and gluon states in (4.25) we find that for arbitrary (large) $N$ the moments of the quark distribution receive the dominant contribution from only


two states – the lowest quark ($\alpha = 0$) and the lowest gluon ($\alpha = N + 1$) levels and, therefore, depend on only two nonperturbative parameters. As a consequence, thus defined moments $\Delta q^+(n, Q^2)$ satisfy the two-channel evolution equation which can be written in the standard DGLAP form

$$Q^2 \frac{d}{dQ^2} \Delta q^+_T(x; Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} \left[ P^T_{qq}(x/y) \Delta q^+_T(y; Q^2) + P^T_{qg}(x/y) \Delta g_T(y; Q^2) \right]. \quad (4.26)$$

In order to determine the splitting functions one can take moments so that the convolution in (4.26) gets replaced by a product and identify the moments of the splitting functions with the corresponding anomalous dimensions (with sign minus).

Since moments of the quark distributions are obtained from the nonlocal quark-antiquark-gluon operators by projecting onto the quark coefficient function (4.24), one can calculate the anomalous dimensions by applying $\Phi^q\Phi^\dagger_l$ to the evolution equation (3.14). Since the coefficient functions $\Phi^q_{N+3}$ and $\Phi^g_{N+3}$ of the quark and gluon transverse spin distributions $\Delta q_T(N+3)$ and $\Delta g_T(N+3)$ (2.30) turn out to be very close to the lowest eigenstates of the ‘pure’ quark and ‘pure’ gluon Hamiltonian $H_0$ (1.1), we may invert (4.24) and expand the nonlocal operators $S_\mu(z_i)$ and $\tilde{O}_\mu(z_i)$ as

$$\langle S_\mu(z_i) \rangle = \sum_n \frac{\tilde{\Phi}^q_n(z_i)}{2\| \Phi^q_n \|^2} \Delta q^+_T(n, Q^2) + \ldots,$$

$$\langle \tilde{O}_\mu(z_i) \rangle = \sum_n \frac{\tilde{\Phi}^g_n(z_i)}{\| \Phi^h_n \|^2} \Delta g_T(n, Q^2) + \ldots, \quad (4.27)$$

where $\tilde{\Phi}^q_n$ and $\tilde{\Phi}^g_n$ are related to the coefficient functions through the transformation (3.34) and the dots denote the contribution of higher quark and gluon eigenstates. Then, substituting (4.27) into the evolution equation (3.14), one calculates the corresponding anomalous dimensions (for even $N$) as

$$\int_0^1 dy y^{N+2} P^T_{qq}(y) = -\frac{\langle \Phi^q_{N+3} | H_{qq} | \Phi^q_{N+3} \rangle}{\| \Phi^q_{N+3} \|^2} = -\mathcal{E}^q_{N,0},$$

$$\int_0^1 dy y^{N+2} P^T_{qg}(y) = -2\frac{\langle \Phi^q_{N+3} | H_{qh} | \Phi^h_{N+3} \rangle}{\| \Phi^q_{N+3} \| \| \Phi^h_{N+3} \|} = -\mathcal{E}^q_{N,0} \frac{\| \Phi^h_{N+3} \|}{\| \Phi^q_{N+3} \|}, \quad (4.28)$$

where the last factor in the second expression serves to correct for the different normalization of the quark and gluon coefficient functions. The first matrix element has been calculated in [10, 12] and the answer for the large-$N$ expansion of $\mathcal{E}^q_{N,0}$ is given in (4.12), (4.13).

$$\int_0^1 dy y^{n-1} P^T_{qq}(y) = -4C_F \left[ (n+1) + \frac{1}{2n} + \gamma_E \right] + C_F + \frac{1}{N_c} \left( 2 - \frac{\pi^2}{3} \right) + O \left( \frac{\ln^2 n}{n^2} \right). \quad (4.29)$$

For the second matrix element we have

$$\langle \Phi^q_{N+3} | H_{qh} | \Phi^h_{N+3} \rangle = -n_f \sum_{k=1}^N \frac{(k + 1)(k + 2) + 2(-1)^k}{(k + 1)(k + 2)\sqrt{k(k + 3)}} \phi^{q}_N \phi^{h}_{N-1,k-1}. \quad (4.30)$$
Using the explicit expressions for the coefficients $\phi_{N,k}^q$ (3.90) and $\phi_{N-1,k-1}^h$ (3.97) one gets

$$
\langle \Phi_n^q|H_{qh}|\Phi_n^h \rangle = n_f \frac{\Gamma^4(n)n}{2\Gamma(2n)} \left[ \frac{(n-1)(n^3-2n^2-6n-12)}{4(n+1)(n-2)} + \frac{(n^2-2n+4)(\psi(n)+\gamma_E)}{2(n-1)} \right].
$$

Combining together Eqs. (4.31), (3.100) and (3.91) and expanding at large $n = N + 3$ we obtain

$$
\int_0^1 dy y^{n-1} P_{Tqg}(y) = -4n_f \left[ \frac{1}{n} - \frac{1}{n^2} + \frac{-5 + 4\gamma_E + 4\ln n}{n^3} + O \left( \frac{1}{n^4} \right) \right],
$$

which agrees well with the exact expression for all $n > 3$.

In order to get a closed system of equations, we have to consider the evolution of $\Delta g_T(x)$ as well. The coefficients of the expansion of the corresponding coefficient function over the basis of multiplicatively renormalizable operators, Eq. (4.25), are shown in Fig. 9b for $N = 100$. We see that the gluon distribution is dominated by the contribution of the gluon operator with the lowest anomalous dimension in the gluon sector, and contributions of all other gluon operators with higher anomalous dimensions is strongly suppressed. Notice, however, that the contribution of the quark operator with the lowest anomalous dimension is small compared to the contributions of a large number $\sim N$ of other quark operators with larger anomalous dimensions. This means that although we can formally write

$$
Q^2 \frac{d}{dQ^2} \Delta g_T(x; Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} \left[ P_{Tg}(x/y) \Delta g_T(y; Q^2) + P_{Tq}(x/y) \Delta q_T^+(y; Q^2) + \ldots \right],
$$

with

$$
\int_0^1 dy y^{n+2} P_{Tg}(y) = -\frac{\langle \Phi_{N+3}^h|H_{gh}|\Phi_{N+3}^h \rangle}{\|\Phi_{N+3}^h\|^2} = -\mathcal{E}_{N,0}^g,
$$

$$
\int_0^1 dy y^{n+2} P_{Tq}(y) = -\frac{1}{2} \frac{\langle \Phi_{N+3}^h|H_{hq}|\Phi_{N+3}^q \rangle}{\|\Phi_{N+3}^h\|\|\Phi_{N+3}^q\|},
$$
where $E_{N,0}$ is given in (4.13), taking into account the mixing term $\sim P^T_{qq}(x/y)$ is not justified to the accuracy that the mixing with other quark degrees of freedom is omitted. We suggest, therefore, to neglect the mixing of quarks to the gluon distribution altogether. It is this mixing that sets the limitations for the accuracy the two-component transverse spin model. This accuracy is in fact rather high since the sum of squares of the coefficients of all quark operators in Fig. 9b is less than 2%.

For smaller values of $N$, the hierarchy of different contributions does not look so convincing, see Fig. 10, but qualitatively remains the same. Most importantly, contributions of all gluon operators other than the one with the lowest anomalous dimension remain negligible.

The DGLAP equations (4.26) and (4.33) can easily be solved going over to moments. Neglecting the quark mixing in (4.33) we find using (4.22)

$$g_{2L}^{2n}(n, Q^2) = \frac{(e^2_q)^{2n}}{2n} \left[ \Delta q^{+}(n, \mu^2) L^{S_q(n)/b} \right.$$}

$$+ \Delta g_T(n, \mu^2) \left( L^{S_g(n)/b} - L^{S_q(n)/b} \right) \frac{\gamma_T(n)}{\gamma_{gg}(n) - \gamma_{qq}(n)} \right], \quad (4.35)$$

where $\gamma_T(n) = - \int_0^1 dx x^{-1} P^T_{ab}(x)$ and $L = \alpha_s(Q^2)/\alpha_s(\mu^2)$. For the two lowest moments $n = 3$ and $n = 5$, expanding moments of the parton distributions at a low normalization scale $\Delta q^{+}(n, \mu^2)$ and $\Delta g_T(n, \mu^2)$ over the reduced matrix elements of local operators $\langle [S]_k \rangle$ and $\langle [G]_k \rangle$, Eqs. (2.31) – (2.34), and using the explicit expressions for the evolutions kernels, one obtains

$$\frac{3}{2} (e^2_q)^{-1} g_{2L}^{2}(3, Q^2) = L^{10.556/b} \langle [S]_0 \rangle,$$

$$\frac{5}{2} (e^2_q)^{-1} g_{2L}^{2}(5, Q^2) = L^{10.897/b} \left[ \langle [S]_2 \rangle - 2 \langle [S]_1 \rangle + 3 \langle [S]_0 \rangle + 0.974 \langle [G]_1 \rangle \right]$$

$$+ L^{17.350/b} \left[ - 0.974 \langle [G]_2 \rangle \right]. \quad (4.36)$$

These results have to be compared with the exact expressions in (3.9). We see that the approximation advocated in this work corresponds to taking into account (for $n = 5$) the two multiplicatively renormalizable operators with a large gluon contribution (the first and the third lines in (3.9)) and neglecting the other terms. The coefficients in front of the three quark local operators in the first line in (3.9) are reproduced reasonably well. Note that values of the anomalous dimensions come out to be very close to the exact results.

5 Summary and conclusions

Based on our systematic analysis of the evolution pattern of twist-three operators we arrive at the following approximate two-channel evolution equation for the flavor-singlet quark and gluon transverse spin distributions:

$$Q^2 \frac{d}{dQ^2} \Delta q^{+}q(x; Q^2) = \frac{\alpha_s}{4\pi} \int_0^1 \frac{dy}{y} \left[ P^T_{qq}(x/y) \Delta q^{+}q(y; Q^2) + P^T_{qq}(x/y) \Delta g_T(y; Q^2) \right],$$

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\[ Q^2 \frac{d}{dq^2} \Delta g_T(x; Q^2) = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} P_{gg}^T(x/y) \Delta g_T(y; Q^2), \] (5.1)

where the flavor-singlet quark distribution is defined similar to (3.2)

\[ \Delta q_T^+(x) = \Delta u_T(x) + \Delta \bar{u}_T(x) + \Delta d_T(x) + \Delta \bar{d}_T(x) + \ldots. \] (5.2)

The Hamiltonian approach developed in this paper can be used to determine the asymptotic expansion of the moments of the splitting functions at large \(N\) that translates to the expansion in powers of \(1 - x\) at large momentum fractions \(x\). Since quark and gluon distributions in the nucleon appear to be decreasing strongly at large \(x\), this contribution to the splitting functions is the most important one numerically. Using the results for the moments given in the text, we derive the following expressions for the splitting functions:

\[
\begin{align*}
P_{qq}^T(x) &= \left[ \frac{4C_F}{1-x} \right]_+ + \delta(1-x) \left[ C_F + \frac{1}{N_c} \left( 2 - \frac{\pi^2}{3} \right) \right] - 2C_F, \\
P_{gg}^T(x) &= \left[ \frac{4N_c}{1-x} \right]_+ + \delta(1-x) \left[ N_c \left( \frac{\pi^2}{3} - \frac{1}{3} \right) - \frac{2}{3} n_f \right] \\
&\quad + N_c \left( \frac{\pi^2}{3} - 2 \right) + N_c \ln \frac{1-x}{x} \left( \frac{2\pi^2}{3} - 6 \right), \\
P_{qg}^T(x) &= -4n_f \left[ x - 2(1-x)^2 \ln(1-x) \right].
\end{align*}
\] (5.3)

Here, the first two expressions are accurate up to corrections of order \(O(1-x)\) for \(x \to 1\) and the third expression has the accuracy \(O((1-x)^3)\). Note that the quark splitting function \(P_{qq}^T(x)\) is the same in the flavor singlet and flavor-nonsinglet channels, cf. (1.2). Remarkably enough, the obtained the twist-3 evolution kernels turn out to be very similar to the well-known expressions for the twist-2 DGLAP kernels. Moreover, the leading \(x \to 1\) asymptotics of the diagonal kernels is the same for twist-2 and twist-3 and the difference occurs at the level of subleading \(\sim \delta(1-x)\) corrections. One can argue following [25] that this property is rather general and it holds to all orders of perturbation theory.

To the leading logarithmic accuracy, the structure function \(g_2(x, Q^2)\) is expressed through the quark distribution

\[ g_{2}^{LL}(x, Q^2) = g_{2}^{WW}(x, Q^2) + \frac{1}{2} \sum_q e_q^2 \int_x^1 \frac{dy}{y} \Delta q_T^+(y, Q^2), \] (5.4)

where \(g_{2}^{WW}(x, Q^2)\) is the Wandzura-Wilczek contribution (2.7) and the gluon contribution arises entirely through the evolution equations (5.1) (after the separation of the flavor-singlet part). Note that both the gluon and the quark distributions are defined by analytic continuation from the odd moments \(n = 1, 3, \ldots\) and must satisfy the constraints

\[ \int_0^1 dx \Delta q_T^+(x, Q^2) = \int_0^1 dx \Delta g_T(x, Q^2) = 0 \] (5.5)

that follow from properties of the coefficient functions \(\Phi^q, \Phi^g\) in (2.30). The Burkhardt-Cottingham sum rule [26] \(\int_0^1 dx g_2(x, Q^2) = 0\) then follows from (5.4) which derivation
involves an additional implicit assumption about the absence of subtraction constants in
the dispersion relation for the spin-dependent Compton amplitude, see [4].

The gluon distribution is subject to the additional constraint

$$\int_0^1 dx x^2 \Delta g_T(x, Q^2) = 0,$$

(5.6)

which ensures that the gluon contribution vanishes for the third moment.

To the next-to-leading logarithmic accuracy, the twist-3 gluon contribution at large
scales can be calculated as a finite part of the box diagram in the background gluon field
and the result [15] projected onto the gluon transverse spin distribution has the form

$$g_{2N}^T(x, Q^2) = \ldots + \frac{1}{2} \sum_q e_q^2 \frac{\alpha_s}{\pi} \int_x^1 \frac{dy}{y} C_g^T(x/y) \Delta g_T(y, Q^2),$$

$$C_g^T(x) = -(1 - x) \left[ 1 + \ln \frac{x}{1 - x} \right] - (1 - x)^3 \left[ \frac{2}{3} \ln^2(1 - x) - \frac{19}{9} \ln(1 - x) + \frac{83}{54} - \frac{\pi^2}{9} \right] + O((1 - x)^4),$$

(5.7)

where the dots stand for the leading-order quark contribution and the $O(\alpha_s)$ quark corrections. We have argued in [15] and in this paper that this projection has a high accuracy
for all integer moments (i.e. all contributions left out by this projection are very small).

To summarize, the set of formulas given in this Section presents a theoretically mo-
tivated approximation for the QCD description of deep inelastic scattering from a trans-
versely polarized nucleon. Importance of this approximation is not so much in the possi-
bility to calculate the scale dependence, but in the identification of important transverse
spin degrees of freedom that are preserved by QCD interaction. The formalism devel-
oped in this paper is very general and can be applied to the study of other higher-twist
distributions. It is based on the operator product expansion and conformal symmetry of
the QCD Lagrangian. This technique turns out to be very effective in dealing with the
operator renormalization for large $N$ and in many cases a WKB-type expansion in $1/N$
can be constructed analytically. Going over from the moments to the momentum fraction
space involves analytic continuation and in general may be quite complicated, see [12] for
a discussion. Because of this, the small$-x$ behavior of twist-3 (and higher-twist) parton
distributions presents a nontrivial problem and deserves further study.

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Appendices
A Appendix: Calculation of the anomalous dimensions

In this Appendix we describe the calculation of the energy of the lowest quark and gluon levels, Eqs. (4.11) and (4.16), respectively, and the energy of the highest quark level, Eq. (4.7).

The energy of the lowest gluon level is defined by Eq. (4.15) and involves the expectation value of the Hamiltonian \( H_{hh} \) over the gluon state \( \Phi_n^h(x_i) \) and its norm. To calculate the both, we shall use the expansion of the wave function over the conformal basis. An arbitrary three-gluon state can be characterized by total conformal spin \( J = n + 1/2 = N + 7/2 \) and the conformal spin in a certain two-particle channel. Depending on the particular choice of the two-particle channel one can define three different sets of basis functions \( Y^{(12)3}_{n+1/2,k+3}(x_i), Y^{(23)1}_{n+1/2,k+3}(x_i) \) and \( Y^{(31)2}_{n+1/2,k+3}(x_i) \). The functions belonging to each basis are linear independent whereas the ones belonging to two different sets are related to each other through a linear transformation

\[
Y^{(31)2}_{n+1/2,k+3}(x_i) = \sum_{m} \Omega^{(h)}_{km} Y^{(12)3}_{n+1/2,m+3}(x_i) = \sum_{m} (-1)^{m+k} \Omega^{(h)}_{km} Y^{(23)1}_{n+1/2,m+3}(x_i), \tag{A.1}
\]

with \( \Omega^{(h)}_{km} = \langle Y^{(31)2}_{n+1/2,k+3}|Y^{(12)3}_{n+1/2,m+3}\rangle \) being the Racah 6j-symbols.

The gluon coefficient function in the helicity representation can be expanded in any one of the three conformal basis so that one gets three sets of the expansion coefficients:

\[
\Phi_n^h(x_i) = \sum_{k=0}^{n-4} \phi^{(31)2}_{n,k} Y^{(31)2}_{n+1/2,k+3}(x_i) = \sum_{k=0}^{n-4} \phi^{(12)3}_{n,k} Y^{(12)3}_{n+1/2,k+3}(x_i) = \sum_{k=0}^{n-4} \phi^{(23)1}_{n,k} Y^{(23)1}_{n+1/2,k+3}(x_i). \tag{A.2}
\]

These coefficients can be calculated similar to (3.96) and are given by

\[
\phi^{(12)3}_{n,k} = -\phi^{(31)2}_{n,k} = h_{n,k} \left[ \frac{(n+2+k)(n+4+k^2/2+5k/2)}{(k+1)(n-2-k)} \right], \quad \phi^{(23)1}_{n,k} = 0 \tag{A.3}
\]

for even \( k \), and

\[
\phi^{(12)3}_{n,k} = \phi^{(31)2}_{n,k} = -\frac{1}{2} \phi^{(23)1}_{n,k} = h_{n,k} \frac{(k+4)(n+3+k)}{2(n-3-k)} \tag{A.4}
\]

for odd \( k \). Here, the normalization factor \( h_{n,k} \) is defined as

\[
h_{n,k}^2 = \frac{(k+1)(-n+3+k)(-n+2+k)(2k+5)n^2}{8(k+2)(k+3)(k+4)(n+2+k)(n+3+k)\Gamma(2n)}. \tag{A.5}
\]

The relations between the coefficients respect the symmetry properties of \( \Phi_n^h(x_i) \). The coefficients \( \phi^{(23)1}_{n,k} \) vanish for even \( k \) to ensure the antisymmetry of the state under the interchange of gluons with the same helicity, \( x_2 \leftrightarrow x_3 \). The sum of the coefficients in the three different channels vanishes for arbitrary \( k \) since the three-gluon state is annihilated by the operator \( 1 + P + P^2 = (P^3 - 1)/(P - 1) \) with \( P \) being the operator of cyclic permutations.

The ‘spherical harmonics’ \( Y^{(31)2}_{n,k}(x_i) \) form an orthonormal basis on the space of the coefficient functions endowed with the scalar product (3.39), \( \langle Y^{(31)2}_{n,k}|Y^{(31)2}_{n,m}\rangle = \delta_{k,m} \). The
same is true for the states $Y_{n,k}^{(12)3}(x_i)$ and $Y_{n,k}^{(23)1}(x_i)$. Using this property we may calculate the norm of the gluon state in one of the three equivalent forms

$$\|\Phi_n^h\|^2 = \sum_{k=0}^{n-4} \left[ \phi_{n,k}^{(12)3} \right]^2 = \sum_{k=0}^{n-4} \left[ \phi_{n,k}^{(23)1} \right]^2 = \sum_{k=0}^{n-4} \left[ \phi_{n,k}^{(31)2} \right]^2.$$  

(A.6)

Substituting the expressions for the expansion coefficients (A.3) and (A.4), one gets a finite sum over the kernel $U_{gg}$, whose evaluation leads to Eq. (3.100). Expanding the result at large $n$ one obtains

$$\|\Phi_n^h\|^2 = \frac{\Gamma^4(n)}{\Gamma(2n)} \frac{n^4}{16} \left[ 1 + \frac{1}{n} \left( 2 \ln n + 2 \gamma_E - 1 \right) + O(1/n^2) \right],$$  

(A.7)

where we factored out the ratio of the $\Gamma$–functions for later convenience.

In order to calculate the expectation value of the evolution kernel defined in (3.74), we notice that $H_{hh}$ has a two-particle structure and can be split in three contributions each of which only depends on the operator of the conformal spin $J_{ab}$ in a given two-particle channel

$$\langle \Phi_n^h | H_{hh} | \Phi_n^h \rangle = N_c \langle \Phi_n^h | 2U_{gg}(J_{12}) - V_{gg}(J_{12}) | \Phi_n^h \rangle$$

$$+ N_c \langle \Phi_n^h | 2U_{gg}(J_{31}) - V_{gg}(J_{31}) | \Phi_n^h \rangle$$

$$+ 2N_c \langle \Phi_n^h | U_{gg}(J_{23}) | \Phi_n^h \rangle - b \| \Phi_n^h \|^2.$$  

(A.8)

Since the operator $J_{12}$ is diagonal in the conformal basis $Y_{n,k}^{(12)3}(x_i)$ (by construction), it is natural to evaluate the first term in (A.8) by expanding the gluon state $\Phi_n^h$ over this particular basis. By the same token, the second and the third terms in (A.8) are most easily calculated using the expansion over $Y_{n,k}^{(31)2}(x_i)$ and $Y_{n,k}^{(23)1}(x_i)$, respectively. In this way one arrives at

$$\langle \Phi_n^h | H_{hh} | \Phi_n^h \rangle = 2N_c \sum_{k=0}^{n-4} U_{gg}(k+3) \left( [\phi_{n,k}^{(12)3}]^2 + [\phi_{n,k}^{(31)2}]^2 + [\phi_{n,k}^{(23)1}]^2 \right)$$

$$- N_c \sum_{k=0}^{n-4} V_{gg}(k+3) \left( [\phi_{n,k}^{(12)3}]^2 + [\phi_{n,k}^{(31)2}]^2 \right) - b \| \Phi_n^h \|^2,$$  

(A.9)

where we have used that the conformal spin in the two-gluon channel is equal to $2j_g + k = k + 3$. Using explicit expressions for the kernels $U_{gg}$ and $V_{gg}$ defined in (3.77) and the expansion coefficients (A.3) and (A.4), one can rewrite (A.9) as a finite sum over the two-particle spin $k$. The sum involving $V_{gg}$ can be calculated analytically while the sum involving $U_{gg}$ can be expanded in inverse powers of $1/n$ at large $n$. After some algebra one arrives at

$$\sum_{k=0}^{n-4} V_{gg}(k+3) \left( [\phi_{n,k}^{(12)3}]^2 + [\phi_{n,k}^{(31)2}]^2 \right) = \frac{\Gamma^4(n)}{\Gamma(2n)} \frac{n^4}{4} \left[ \frac{\pi^2}{6} - \frac{3}{2} + \frac{1}{n} + O\left( \frac{\ln n}{n^2} \right) \right]$$  

(A.10)

and

$$\sum_{k=0}^{n-4} U_{gg}(k+3) \left( [\phi_{n,k}^{(12)3}]^2 + [\phi_{n,k}^{(31)2}]^2 + [\phi_{n,k}^{(23)1}]^2 \right)$$

$$= \frac{\Gamma^4(n)}{\Gamma(2n)} \frac{n^4}{8} \left[ \ln n + \gamma_E + \frac{\pi^2}{12} - \frac{1}{2} + \frac{1}{n} \left( 2(\ln n + \gamma_E)^2 - \frac{1}{2}(\ln n + \gamma_E) \right) + O\left( \frac{\ln^2 n}{n^2} \right) \right].$$  

(A.11)
Substituting these expressions into (A.9) and combining them with (A.7) one arrives at the expression for the energy of the lowest gluon state given in the text, Eq (4.15).

Calculation of the energy of the lowest quark level, Eq. (4.11), goes along the same lines. The quark coefficient function $\Phi_n^q$ is defined in Eq. (2.22) on the hyperplane $x_1 + x_2 + x_3 = 0$ corresponding to the kinematics of the forward scattering, and can be continued to arbitrary values of the momentum fractions $x_i$ using the conformal symmetry. Its expansion over the conformal ‘spherical harmonics’ in the three different two-particle channels looks like

$$
\Phi_n^q(x_i) = \sum_{k=0}^{n-3} \varphi_{n,k}^{(12)3} Y_{n+1/2,k+5/2}^{(12)3}(x_i) = \sum_{k=0}^{n-3} \varphi_{n,k}^{(23)1} Y_{n+1/2,k+5/2}^{(23)1}(x_i) = \sum_{k=0}^{n-3} \varphi_{n,k}^{(31)2} Y_{n+1/2,k+2}^{(31)2}(x_i),
$$

(A.12)

where e.g. $Y_{n+1/2,k+5/2}^{(12)3}(x_i)$ is defined by the general expression (3.61) with $j_1 = j_3 = 1$, $j_2 = 3/2$, the total conformal spin equal to $J = n = N + 3$ and the two-particle conformal spin in the (12) channel given by $j_{12} = j_1 + j_2 + k = 3 + k$.

The explicit expressions for the expansion coefficients can be obtained using the representation similar to (3.89). The result is:

$$
\varphi_{n,k}^{(12)3} = 2 (-1)^k \frac{n(n + 2 + k)}{k+1} q_{n,k},
$$

$$
\varphi_{n,k}^{(23)1} = -2 (-1)^n (k + 3) (n + 2 + k) q_{n,k},
$$

$$
\varphi_{n,k}^{(31)2} = - (3 + 2 k) (k + 2) \tilde{q}_{n,k}
\times \left[ \frac{(-1)^{n-k} - 1}{2} \frac{n + 2 + k}{n - 2 - k} + \frac{(-1)^{n-k} + 1}{2} \frac{n + 1 + k}{n - 1 - k} \right],
$$

(A.13)

with the normalization factors

$$
q_{n,k}^2 = \frac{(k+1)(n-2-k)}{4(n+2+k)(k+2)(k+3)} \frac{\Gamma^4(n)}{\Gamma(2n)},
$$

$$
\tilde{q}_{n,k}^2 = 2 \frac{(n-1-k)(k+3)}{(3+2k)(n+1+k)} q_{n,k}^2.
$$

(A.14)

Using the expansion in (A.12) one finds three equivalent representations for the norm of the quark state

$$
\|\Phi_n^q\|^2 = \sum_{k=0}^{n-3} [\varphi_{n,k}^{(12)3}]^2 = \sum_{k=0}^{n-3} [\varphi_{n,k}^{(23)1}]^2 = \sum_{k=0}^{n-3} [\varphi_{n,k}^{(31)2}]^2.
$$

(A.15)

Substituting explicit expressions for the expansion coefficients and performing the summation one arrives at Eq. (3.91). The expansion of the norm at large $n$ reads

$$
\|\Phi_n^q\|^2 = \frac{\Gamma^4(n) n^4}{\Gamma(2n)} \left[ 1 + \frac{1}{n^2} (1 - 4 \ln n - 4 \gamma_E) + O(1/n^2) \right].
$$

(A.16)

The expectation value of the diagonal quark evolution kernel $H_{qq}$ defined in (3.69) can be written in the form analogous to (A.8)

$$
\langle \Phi_n^q | H_{qq} | \Phi_n^q \rangle = \langle \Phi_n^q | N_c V^{(0)}_q (J_{12}) - \frac{2}{N_c} V^{(1)}_q (J_{12}) | \Phi_n^q \rangle
$$
Expanding the quark state (A.12) over the suitable conformal basis one obtains

\[ \langle \Phi_n^q | H_{qq} | \Phi_n^q \rangle = \sum_{k=0}^{n-3} [\varphi_{n,k}^{(12)}]^2 \left( N_c V_{qq}^{(0)}(k + 5/2) - \frac{2}{N_c} V_{qq}^{(1)}(k + 5/2) \right) + \sum_{k=0}^{n-3} [\varphi_{n,k}^{(23)}]^2 \left( N_c U_{qq}^{(0)}(k + 5/2) - \frac{2}{N_c} U_{qq}^{(1)}(k + 5/2) \right) + \sum_{k=0}^{n-3} [\varphi_{n,k}^{(31)}]^2 \left( \frac{2n_f}{3} \delta_{k,0} - \frac{2}{N_c} U_{qq}^{(1)}(k + 2) \right), \tag{A.18} \]

where we have replaced the operators of the two-particle conformal spins by the corresponding eigenvalues \( J_{ab} = j_a + j_b + (n - 3) \). The necessary expansion coefficients and explicit expressions for the evolution kernels entering this expression are given in (A.13) and (3.73), respectively.

It turns out that the part of the sum in (A.18) proportional to \( N_c \) can be calculated exactly

\[ \sum_{k=0}^{n-3} [\varphi_{n,k}^{(12)}]^2 V_{qq}^{(0)}(k + 5/2) + [\varphi_{n,k}^{(23)}]^2 U_{qq}^{(0)}(k + 5/2) = ||\Phi_n^q||^2 \left( \psi(n) + \frac{1}{n} - \frac{1}{2} + 2\gamma_E \right). \tag{A.19} \]

The \( 1/N_c \) correction to the sum (A.18) is given by a ratio of rather complicated sums and can easily be expanded at large \( n \) leading to

\[ \sum_{k=0}^{n-3} [\varphi_{n,k}^{(12)}]^2 V_{qq}^{(1)}(k + 5/2) = ||\Phi_n^q||^2 \left( \frac{7}{4} - \frac{\pi^2}{6} + O(\ln n/n^2) \right), \tag{A.20} \]
\[ \sum_{k=0}^{n-3} [\varphi_{n,k}^{(23)}]^2 U_{qq}^{(1)}(k + 5/2) = ||\Phi_n^q||^2 (0 + O(1/n^2)), \]
\[ \sum_{k=0}^{n-3} [\varphi_{n,k}^{(31)}]^2 U_{qq}^{(1)}(k + 2) = ||\Phi_n^q||^2 (\ln n + \gamma_E - 1 + O(\ln^2 n/n^2)) \].

Finally, the \( n_f \)-dependent contribution to the (A.18) is given for odd \( n \) by

\[ [\varphi_{n,0}^{(31)}]^2 = \frac{3(n - 1)(n + 2)}{(n + 1)(n - 2)} \Gamma^4(n) \Gamma(2n) = ||\Phi_n^q||^2 (0 + O(1/n^4)) \tag{A.21} \]

Combining (A.19), (A.20), (A.21) and (A.17) we obtain the expression for the energy of the lowest quark level given in Eq. (4.11).

The eigenstate of the highest quark level can be approximated by the expression in (4.6). Calculating the corresponding energy (4.7) and using (3.69), one gets

\[ E_{N,N}^q = \frac{2}{3} n_f + \langle Y^q_{N,k=0} | H_{S^+} | Y^q_{N,k=0} \rangle. \tag{A.22} \]
Using the explicit expression (3.70) for the Hamiltonian \( H_{S+} \) and taking into account that \( Y_{N,k=0}^q \) is symmetric with respect to the interchange of the quarks, \((P_{13} - 1)Y_{N,k=0}^q = 0\), one obtains

\[
\mathcal{E}_{N,N}^q = \frac{2}{3} n_f - \frac{2}{N_c} U_{qq}^{(1)}(J_{13} = 2) + \langle Y_{N,0}^q | V(J_{12}) | Y_{N,0}^q \rangle, \tag{A.23}
\]

where the notation was introduced for the linear combination of the kernels (3.73)

\[
V(J) = N_c \left( V_{qq}^{(0)}(J) + U_{qq}^{(0)}(J) \right) - \frac{2}{N_c} \left( V_{qq}^{(1)}(J) + U_{qq}^{(1)}(J) \right). \tag{A.24}
\]

In order to calculate the matrix element entering (A.23) we use the Racah decomposition (3.1) to expand \( Y_{N,k=0}^q \) over the conformal basis in the quark-gluon channel

\[
Y_{N,0}^q(x_i) = \sum_m \Omega_{2,m+5/2}(N + 7/2) Y_{N+7/2,m+5/2}^{(12)3}(x_i). \tag{A.25}
\]

Taking into account that \( U_{qq}^{(1)}(2) = 0 \) one finds

\[
\mathcal{E}_{N,N}^q = \frac{2}{3} n_f + \sum_{m=0}^N V(m + 5/2) [\Omega_{2,m+5/2}(N + 7/2)]^2. \tag{A.26}
\]

Finally, using the explicit expressions for the Racah symbols (3.3) and the two-particle evolution kernels (3.73) and and performing the summation in (A.26) one arrives at the result given in Eq. (4.7).

## B  Appendix: Racah symbols for the \( SL(2, R) \) group

In this Appendix we derive the explicit expression for the Racah 6\(j\)—symbols \(\Omega_{jj'}\) defined in (3.63)

\[
Y_{jj}^{(31)2}(x_i) = \sum_{j_1+j_2+j_3\leq j} \Omega_{jj}(J) Y_{jj'}^{(12)3}(x_i), \tag{B.1}
\]

where the basis functions \(Y_{jj}^{(31)2}\) are given in Eq. (3.61). In order to find the coefficients \(\Omega_{jj}(J)\) it is sufficient to set the three variables \(x_i\) in the Eq. (B.1) to the following values: \(x_1 = -x, x_2 = x\) and \(x_3 = 1\). Using the explicit expressions for the basis functions (3.61) one gets for (B.1):

\[
(-1)^{J-j-j_2} r_{jj} \frac{\Gamma(j+j_3-j_1) \Gamma(J-j+j_2)}{\Gamma(2j_3) \Gamma(2j_2) (j-j_1-j_3)! (J-j-j_2)!} \times \tag{B.2}
\]

\[
x \times {}_2 F_1 (j_1+j_3-j_1, 1+j_3-j_1-j_2; j_2; x) \times {}_2 F_1 (-J+j+j_2, J+j+j_2-1, 2j_2; x) \]

\[
= \sum_{j' = j_1+j_2}^{J-j_3} \Omega_{jj'}(J) \frac{r_{jj'} (-x)^{j-j_2}}{(j'-j_1-j_2)! (J-j'-j_3)! (2j'-1)! (1+j+j_2-1)}.
\]

The product of the two hypergeometric functions in the l.h.s. of this identity is nothing else but the generating function for the Racah polynomials (see [27] for details). Expanding the hypergeometric functions and comparing the terms in the l.h.s. and the r.h.s.
of (3.2) with the same power of $x$ one derives after some algebra the following explicit expression:

$$
\Omega_{jj'}(J) = (-1)^{j-j'} \frac{\Gamma(J-j_1-j_2-j_3+1)}{\Gamma(2j_1)\Gamma(J+j_1+j_2+j_3-1)} f(J,j',j_1,j_2,j_3) f(J,j,j_1,j_2,j_3)
\times {}_4F_3 \left( \begin{array}{c}
-j'+j_1+j_2, j'+j_1+j_2-1, -j+j_1+j_3, j+j_1+j_3-1 \\
2j_1, -J+j_1+j_2+j_3, J+j_1+j_2+j_3-1
\end{array} \right| 1 \right), \quad (B.3)
$$

where

$$
f(J,j',j_1,j_2,j_3) = \left[ (2j'-1) \frac{\Gamma(j'+j_1-j_2)\Gamma(j'+j_1+j_2-1)\Gamma(J-j'+j_3)\Gamma(J+j'+j_3-1)}{\Gamma(j'+j_2-j_1)\Gamma(j'-j_1-j_2+1)\Gamma(J-j'-j_3+1)\Gamma(J+j'-j_3)} \right]^{1/2}. \quad (B.4)
$$

\section{Appendix: Evolution kernels in the conformal basis}

To solve the Schrödinger equation (3.58) we expand the eigenstates over the conformal basis in the quark-antiquark-gluon and three-gluon sectors, Eq. (3.66). In this representation the evolution kernels are given by real and symmetric matrices acting on the vector of the expansion coefficients $(u^q_{N,k}, u^{\bar{q}}_{N,k})$ defined in (3.66). In this appendix we work out the explicit form of these matrices and discuss their properties.

The diagonal quark evolution kernel (3.69) is described in the conformal basis $Y^q_{N,k}$ by a square matrix of dimension $\ell_q = N+1$

$$
[H_{qq}]_{km} = \langle Y^q_{N,k}|H_{qq}|Y^q_{N,m} \rangle = \langle Y^q_{N,k}|H_{S+}|Y^q_{N,m} \rangle + \frac{2n_f}{3} \delta_{k,m} \delta_{k,0} \quad (C.1)
$$

with $0 \leq k, m \leq N$. In turn, the Hamiltonian $H_{S+}$ is given by the sum of pair-wise kernels (3.73) depending on the conformal spins in the different two-particle channels. We recall that the operators $J_{31}$ are diagonal by definition on the space of the states $Y^q_{N,k}$ so that

$$
\langle Y^q_{N,k}|V(J_{31})|Y^q_{N,m} \rangle = V(k+2) \delta_{km} \quad (C.2)
$$

for an arbitrary $V$. On the other hand, contributions to the Hamiltonian that are functions of $J_{12}$ can be expanded over the conformal basis in the two-particle $(12)-$channel

$$
\langle Y^q_{N,k}|V(J_{12})|Y^q_{N,m} \rangle = \sum_{l=0}^{N} V(l+5/2) \langle Y^q_{N,k}|Y^{(12)3}_{N+7/2,l+5/2}\rangle \langle Y^{(12)3}_{N+7/2,l+5/2}|Y^q_{N,m} \rangle. \quad (C.3)
$$

The similar representation exists for $V(J_{23})$. The scalar product of the $Y$-functions belonging to different conformal basis is given by the Racah $6j-$symbols defined in (B.4) and (B.3). In particular

$$
\langle Y^{(12)3}_{N+7/2,l+5/2}|Y^q_{N,m} \rangle = \Omega_{m+2,l+5/2}(N + 7/2) \equiv \Omega^{(q)}_{ml}, \quad (C.4)
$$

$$
\langle Y^{(23)1}_{N+7/2,l+5/2}|Y^q_{N,m} \rangle = (-1)^{l+m} \Omega_{m+2,l+5/2}(N + 7/2) \equiv (-1)^{l+m} \Omega^{(q)}_{ml}. \quad (C.5)
$$
Taking into account these relations one finds
\[
\langle Y_{N,k}^q | V(J_{12}) | Y_{N,m}^q \rangle = \sum_{l=0}^{N} V(l + 5/2) \Omega_{ml}^{(q)} \Omega_{kl}^{(q)},
\]  
(C.6)

\[
\langle Y_{N,k}^q | V(J_{23}) | Y_{N,m}^q \rangle = \sum_{l=0}^{N} (-1)^{l+m} V(l + 5/2) \Omega_{ml}^{(q)} \Omega_{kl}^{(q)}.
\]  
(C.7)

Using these identities and the explicit expression for \(H_S^+\) given in Sect. 3.5.1 one finds the following matrix representation of the diagonal quark kernel
\[
[H_{qq}]_{km} = \left[ -\frac{2}{N_c} U_{qq}^{(1)} (k + 2) + \frac{2n_f}{3} \delta_{k,0} \right] \delta_{k,m} + \sum_{l=0}^{N} \Omega_{ml}^{(q)} \Omega_{kl}^{(q)}
\]  
(C.8)

\[
\times \left( N_c V_{gg}^{(0)} (l+5/2) - \frac{2}{N_c} V_{gg}^{(1)} (l+5/2) + (-1)^{k+m} \left[ N_c U_{gg}^{(0)} (l+5/2) - \frac{2}{N_c} U_{gg}^{(1)} (l+5/2) \right] \right).
\]  
(C.8)

The explicit expressions for the Racah symbols and the evolution kernels are given in Eqs. (B.3) and (3.73), respectively.

The gluon diagonal kernel in the helicity representation (3.74) is given in the conformal basis \(Y_{N-1,k}^g\) by a square matrix of dimension \(\ell_g = N/2\)
\[
[H_{hh}]_{km} = \langle Y_{N-1,k}^h | H_{hh} | Y_{N-1,m}^h \rangle = N_c \langle Y_{N-1,k}^h | H_{3/2} - V_{3/2} | Y_{N-1,m}^h \rangle
\]  
(C.9)

with \(0 \leq k, m \leq [N/2] - 1\). To obtain the explicit expression one has to follow the steps similar to that in the quark case. Introducing the notation for the relevant Racah symbols
\[
\langle Y_{N+7/2, l+3}^{(12)} | Y_{N-1,m}^g \rangle = \Omega_{m+3,l+3}(N + 7/2) \equiv \Omega_{ml}^{(h)}
\]  
(C.10)

and using the explicit expressions for the gluon kernels (3.77) one finds
\[
[H_{hh}]_{km} = \delta_{k,m} [N_c (2 U_{gg} (k + 3) - V_{gg} (k + 3)) - b]
\]  
(C.11)

\[
+ N_c \sum_{l=0}^{[N/2]} \Omega_{ml}^{(h)} \Omega_{kl}^{(h)} [2 U_{gg} (l + 3) (1 + (-1)^{k+m}) - V_{gg} (l + 3)].
\]  
(C.11)

The explicit expressions for the Racah symbols and the functions \(U_{gg}, V_{gg}\) are given in Eqs. (B.3) and (3.77), respectively.

Finally, the explicit expressions for the off-diagonal kernels \(H_{qh}\) and \(H_{hq}\) are given in (3.82).

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