Interplay of Friction and Structural Asymmetry in Non-Reciprocal Dynamics of a Rolling Prism

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Abstract

We investigate how the dynamics of a hexagonal prism rolling on a floor change with friction and structural asymmetry. It is shown that the asymmetry induces the non-reciprocal dynamics where the prism rolls only in one direction regardless of the direction of the initial velocity. While such behavior is primarily suppressed with increasing friction, we observe that there is a certain range of friction coefficient where the non-reciprocity is restored. This suggests that the friction not only diminishes the dynamics, but also assists it by exerting a torque to the prism.
When a physical system has asymmetry in its structure or potential landscape, it gives rise to anisotropy in its dynamics. For example, a ratchet makes use of its asymmetric potential landscape to allow a motion in one direction [1-3], and the electric current applied to a diode flows almost only in one direction due to the asymmetry at the interface of the p-type and n-type semiconductor. These are referred to as “non-reciprocal” phenomena, and the understanding and control of them are not only of scientific interest (e.g. functions of molecular motors [4,5], non-reciprocal electronic response of a condensed matter [6,7], and so on) but also connected to industrial application.

While structural asymmetry can potentially trigger wide varieties of non-reciprocal dynamics, energy dissipation such as friction is also a factor which governs the behavior of the system. For example, if a particle in a one-dimensional ratchet-like potential landscape feels a friction force, its dynamics will ultimately diminish unless there is an external source which injects energy into the system. In this case, the non-reciprocity as well as the dynamics itself disappears as the friction is increased. Thus, one might expect that the dissipative force such as friction always diminishes the dynamics of the system. However, this is not necessarily the case when the kinetic energy of the system has more than one component, e.g. translational motion, and rotational motion. Here, classical mechanical systems such as rigid body dynamics is one of the simplest systems to examine such problems.

In this Letter, we numerically investigate the rotational dynamics of a hexagonal prism [8,9] with asymmetric structure. It is shown that the asymmetry induces the non-reciprocal dynamics where the prism rolls only in one direction regardless of the direction of the initial velocity. We observe that there is a certain range of friction
coefficient where the non-reciprocity is restored. This suggests that the friction not only diminishes the dynamics, but also assists it by exerting a torque to the prism.

Figure 1 (a) shows the free body diagram of the hexagonal prism. First, we describe the geometrical and physical properties of the prism. In this system, asymmetry is introduced by assuming that one triangular part of the hexagon (shaded triangle in Fig.1 (a)) is slightly heavier \( \frac{m}{6} + \delta m \) than the other 5 parts \( \frac{m}{6} - \frac{\delta m}{5} \) where \( m \) is the total mass of the prism. This makes the position of the center of mass (black point in Fig.1 (a)) shift by \( \delta = \frac{2\sqrt{3}\delta m}{5} a_0 \), assuming \( \frac{\delta}{a_0} \ll 1 \). Here, \( a_0 \) is the arm length from the geometric center to each corner. Each corner of the hexagon is labeled as corner 1~6.

We define \( a_1 \sim a_6 \) as the length from the center of mass to each corner \( (a_1 = a_2 = a_6 = a_0 - \frac{\sqrt{3}}{2}\delta, a_3 = a_0, \text{ and } a_4 = a_5 = a_0 + \frac{\sqrt{3}}{2}\delta) \). The moment of inertia around the center of mass is \( I = \frac{3}{8} ma_0^2 \). Here we assume that the length of the prism is unity. The tilting angle between the vertical axis and the center of mass is defined as \( \theta \). We also name \( \theta_1 \) to \( \theta_6 \) as the angle when the face of the corner completely touches the floor.

The dynamics of the prism is formulated as below. As shown in Fig.1 (a), \( N, F, \) and \( mg \) is the normal force, friction force, and gravity (\( g \) is the gravitational acceleration). The angular velocity is defined as \( \omega = \frac{d\theta}{dt} \). The initial condition is that the bottom face in Fig.1 (a) is completely touching the floor (and the prism rolls around the corner 4 or 3) so as \( \theta = \theta_4 = \frac{\pi}{6} + \frac{1}{2}\frac{\delta}{a_0} \) or \( \theta = -\frac{\pi}{6} + \frac{\delta}{a_0} \). The initial angular velocity of the prism \( \omega_0 \) is given at time \( t = 0 \).

Figure 1 (b) shows the schematic of the potential energy \( U \) around \( \theta = \theta_4 \).
as a function of $\theta$. The potential has one local minimum at $\theta = \theta_4$, and two local maxima. The total energy of the prism consists of three parts, potential energy $U$, kinetic energy of the center of mass $K_{com}$ and rotational kinetic energy $K_{roll}$. The prism can go over the potential maxima if $U + K_{roll}$ is greater than $mg a_4$ or $mg a_3$. Here, the friction affects both translational and rotational motion (relation between $K_{com}$ and $K_{roll}$), as well as the energy dissipation. Thus, the behavior of the prism changes according to the value of friction even though the initial total energy is fixed.

Below we consider the case $\omega_0 < 0$. The equation of motion of the body is given as below.

$$m \frac{dv_x}{dt} = F \ldots (1).$$

$$m \frac{dv_y}{dt} = N - mg \ldots (2).$$

$$I \frac{d\omega}{dt} = Na_4 \sin \theta - Fa_4 \cos \theta \ldots (3).$$

Here, $v_x$ and $v_y$ are the velocity of the center of mass in $x$ and $y$ direction. These equations are analogous to the case of a rod falling on the floor [10,11]. When the prism does not slip on the floor (i.e. hinged), $v_x = a_4 \omega \cos \theta$ and $v_y = -a_4 \omega \sin \theta$. By substituting these relations to Eqs. (1)-(3), we obtain $\frac{d\omega}{dt} = \frac{mg a_4}{I + ma_4^2} \sin \theta$. $F$ and $N$ are also given as a function of $\theta$, $\omega$, and $\frac{d\omega}{dt}$. The prism starts to slip when $|F| \geq \mu N$. Here, $\mu$ is the kinetic friction coefficient. We assume that the kinetic and static friction coefficient are the same. This approximation does not affect the general feature of the results presented below. When the prism is slipping towards negative direction of $x$, $F$ in Eqs. (1) and (3) can be replaced as $F = \mu N$ and we obtain

$$\frac{d\omega}{dt} = \frac{ma_4 (\sin \theta - \mu \cos \theta)(g - a_4 \omega^2 \cos \theta)}{I + ma_4^2 \sin \theta (\sin \theta - \mu \cos \theta)} \ldots (4).$$

The sign of $\mu$ in the above equations are reverted when the direction of slipping is
When \( \theta \) becomes greater or smaller than a certain value, the corner of the prism touching the floor changes. For example, when \( \theta < -\theta_4 \), the corner with which the prism rolls switches from the corner 4 to corner 5, and we reset \( \theta \rightarrow \theta_5 \). We also set the coefficient of the restitution \( \varepsilon \) which determines the magnitude ratio of \( v_y \) before and after the corner touches the floor.

The trajectory of the center of mass is calculated by numerically integrating Eqs. (1)-(4). We fix \( m = 5 \) g, \( a_0 = 5 \) mm, \( \delta = 0.25 \) mm, \( \varepsilon = 0.9 \), and \( \mu = 0.3 \). These values are reasonable for a commercial pencil made of plywood and desk. We calculated the trajectory of 2 seconds, which is long enough for the prism to be in the stationary state.

Fig.2 (a) shows the trajectory in the symmetric case (\( \delta = 0 \)). The behavior of the prism can be categorized into three patterns. One is “damp” (\( \omega_0 = \pm 15 \) rad/sec), where the prism stays around the initial position. Here, the corner of the prism touching the floor alternate between the corners 4\( \rightarrow \)3\( \rightarrow \)4\( \rightarrow \)3…(4\( \rightarrow \)5\( \rightarrow \)6…). The other patterns are “roll positive (negative)”, where the prism continues to roll \( x \) positive (negative) directions when \( \omega_0 \) is positive (negative) as shown for \( \omega_0 = \pm 20, \pm 25 \) rad/sec. When the prism rolls positive (negative), the corner touching the floor changes from the corner 4\( \rightarrow \)3\( \rightarrow \)2… (4\( \rightarrow \)5\( \rightarrow \)6…). Note that the dynamics is reciprocal as the rolling direction reverses when the sign of \( \omega_0 \) is reversed.

On the other hand, the non-reciprocity appears when the asymmetry is introduced to the prism. As shown in Fig.2 (b), the prism exhibits “roll positive” in both
\( \omega_0 = 20 \) and \(-20\) rad/sec. When the magnitude of \( \omega_0 \) is further increased to \( \omega_0 = \pm 25 \) rad/sec, the prism again rolls positive/negative, restoring the reciprocity of the dynamics. Thus, it is shown that the rotational dynamics of the asymmetric prism becomes non-reciprocal when \( \omega_0 \) and \( \mu \) satisfies a certain condition. This is the central finding in this Letter.

To clarify the condition for the non-reciprocity, the dynamics is systematically categorized as a function of friction coefficient \( \mu \) and the initial angular velocity \( \omega_0 \) as shown in Figure 3(a) and (b). Conditions for numerical calculations are the same as in Fig.2. The trajectory for a run time of 0.5 sec is calculated to focus only on which direction the prism rolls. Here, we define a value ranging from -1 to 1 according to the dynamics ("damp" = 0, "roll positive" = 1, "roll negative" = -1). As shown in Fig.3 (a), in the symmetric case (\( \delta = 0 \)), the prism either damps or rolls in the same direction as the initial angular velocity, making the diagram symmetric (reciprocity).

As shown in Fig.3 (b), the prism rolls positive even when \( \omega_0 < 0 \) (\( 0 \leq \mu < 0.03 \), \( \omega_0 \sim -30 \) rad/sec, or \( \mu \geq 0.18, -24 \leq \omega_0 \leq -17 \) rad/sec). This is the region where the non-reciprocity appears. The diagram can be interpreted as below. When there is no friction, whether the prism can roll positive or not is determined only by the structural asymmetry and initial energy. As \( \mu \) is increased (\( 0.03 \leq \mu < 0.18 \)), non-reciprocity is suppressed, and the "damp" region directly transits into "roll negative". This behavior shows that the energy dissipation due to friction inhibits the peculiar dynamics induced by the asymmetry. However, the non-reciprocity is restored when the friction is further increased (\( \mu \geq 0.18 \)), suggesting that the friction does not only diminish but also assist the rotational dynamics of the prism.
To gain further insight into how the friction affects the dynamics, we evaluated the energy dissipation rate as shown in Fig.4. The energy dissipation rate $\Delta E \equiv \left[1 - \frac{E(t=\tau)}{E(t=0)}\right]$ is defined as the ratio of the total energy at $t = 0$ and $t = \tau$. Here, $\Delta E$ is calculated from the trajectory in 0.1 sec by fixing $\omega_0 = -20$ rad/sec (see the black dotted line in Fig.3 (b)). As shown in Fig.4, the energy dissipation rate $\Delta E$ takes its maximum at $\mu = 0.18$, and $\Delta E = 0$ in $\mu \geq 0.35$. The dynamics of the prism switches from “damp” to “roll positive” at $\mu = 0.24$ (dotted line in Fig.4). When $\mu$ is smaller than 0.24, the prism does not roll positive because of two reasons. One is that the friction is so small that the kinetic energy of the translational motion dominates the dynamics rather than the rotational motion. In addition, even though the translation motion is suppressed with increasing $\mu$, the friction dissipates the energy of the prism. In both cases, the energy of the rotational motion becomes less than the threshold to go over the potential barrier ($K_{roll} < mg a_{3,4}$).

On the other hand, when $\mu \geq 0.24$, $\Delta E$ gradually decreases and the prism exhibits “roll positive”. Especially, $\Delta E$ becomes 0 when $\mu \geq 0.35$. This is because the prism is hinged to the floor and it does not slip. Thus, there is no energy dissipation and translational motion, lowering the threshold energy to roll positive. The most distinctive part is the intermediate condition ($0.24 \leq \mu \leq 0.35$), where the prism rolls positive even under energy dissipation (see the gray region in Fig.4). Here, the friction suppresses the translational motion and decreases $\Delta E$, restoring the “roll positive” (non-reciprocity). Here, one can also say that the torque exerted by the friction assists the prism to roll. Note that such effect becomes possible because the kinetic energy consists of more than one component (translational motion and rolling motion). From above discussions, it is
shown that the rotational dynamics of the asymmetric hexagonal prism exhibits non-reciprocity thanks to the interplay between the friction and the structural asymmetry.

Finally, we note other possible dynamics which can be observed in the system. In Fig.3 (b), we calculate the trajectory for up to 0.5 sec to focus only on in which direction the prism rolls and neglect the proceeding dynamics. However, if we calculate the full trajectory until the prism stops (~ 2 sec), a region where the prism switches its rolling direction from positive to negative appears. This appears at around $\omega_0 \sim -22$ rad/sec and $\mu \geq 0.3$. In this condition, the prism is initially “roll-positive” because the rotational kinetic energy $K_{roll}$ is greater than the potential barrier $mg a_3 < K_{roll} < mg a_4$. Considering that $a_{1,2,6} < a_3 < a_{4,5}$, it rolls from the corner $4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 6$ unless the energy dissipation breaks the condition $K_{roll} > mga_3$. However, the prism cannot roll positive ($6 \rightarrow 5$) because $a_5 < a_6$. This is the point that the prism stops to “roll positive”, and it switches to “roll negative” from the corner $6 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$. Such oscillating behavior can also be observed when one throws a pencil on a desk.

In conclusion, we have investigated the rotational dynamics of a hexagonal prism with asymmetric structure. It is shown that if the friction coefficient and the initial angular velocity satisfies a certain condition, the prism exhibits the non-reciprocal dynamics where it rolls only in one direction regardless of the direction of the initial velocity. The systematic investigation of the dynamics and the energy dissipation suggests that the friction not only diminishes the dynamics, but also assists it by exerting a torque to the prism. This work demonstrates that the interplay of dissipative force and
the structural asymmetry enables the non-reciprocal dynamics of a system with multiple degrees of freedom.

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**Data Availability**

The data that support the findings of this study are available from the corresponding author upon reasonable request.
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Figure Captions

**Figure 1:** (a) Free body diagram of the hexagonal prism (cross section). (b) Potential energy landscape of the system.

**Figure 2:** The center of mass trajectories of a symmetric (a) and asymmetric (b) prism \(a_0 = 5.0\) mm, \(\delta = 0.25\) mm, \(m = 5.0\) g, \(\mu = 0.30\). Unit of \(\omega_0\) (rad/sec) is omitted in the Figure.

**Figure 3:** Categorization of the dynamics as a function of friction coefficient and the initial angular velocity of a symmetric (a) and asymmetric (b) prism. Color scale is given according to the categorization of the dynamics (“damp” = 0, “roll positive (+)” = 1, “roll negative (-)” = -1). Black dotted line indicates \(\omega_0 = -20\) rad/sec.

**Figure 4:** Energy dissipation rate of the asymmetric prism calculated from the trajectory up to 0.1 sec \(\omega_0 = -20\) rad/sec, \(\mu = 0.30\). Black dotted line indicates \(\mu = 0.24\). Gray region shows \(0.24 \leq \mu \leq 0.35\).
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