ON PALEY-TYPE 
AND HAUSDORFF–YOUNG–PALEY-TYPE INEQUALITIES 
FOR GENERALIZED GEGENBAUER EXPANSIONS 

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Abstract. We establish Paley-type and Hausdorff–Young–Paley-type inequalities for generalized Gegenbauer expansions.

Key words and phrases: orthogonal polynomials, generalized Gegenbauer polynomials, Paley-type inequality, Hausdorff–Young–Paley-type inequality

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1. Introduction and preliminaries

In this section, we introduce some classes of orthogonal polynomials on $[-1, 1]$, including the so-called generalized Gegenbauer polynomials. For a background and more details on the orthogonal polynomials, the reader is referred to [2–4, 7].

Let $\mathbb{N}_0$ denote the set of non-negative integers. Let $\alpha, \beta > -1$. The Jacobi polynomials, denoted by $P_{n}^{(\alpha,\beta)}(\cdot)$, $n \in \mathbb{N}_0$, are orthogonal with respect to the Jacobi weight function $w_{\alpha,\beta}(t) = (1-t)^\alpha (1+t)^\beta$ on $[-1,1]$, namely,

$$
\int_{-1}^{1} P_{n}^{(\alpha,\beta)}(t) P_{m}^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) \, dt = \begin{cases} 
\frac{2^{\alpha+\beta+1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) \Gamma(n + 1) \Gamma(n + \alpha + \beta + 1)}, & n = m, \\
0, & n \neq m.
\end{cases}
$$

Here, as usual, $\Gamma$ is the gamma function.

For $\mu \geq 0$, and $n \in \mathbb{N}_0$, the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}(\cdot)$ are defined by

$$
C_{2n}^{(\lambda,\mu)}(t) = a_{2n}^{(\lambda,\mu)} P_n^{(\lambda-1/2,\mu-1/2)}(2t^2 - 1), \\
C_{2n+1}^{(\lambda,\mu)}(t) = a_{2n+1}^{(\lambda,\mu)} t P_n^{(\lambda-1/2,\mu+1/2)}(2t^2 - 1),
$$

where $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_n = 1, \quad (\lambda)_n = \lambda(\lambda + 1) \cdots (\lambda + n - 1) \quad \text{for} \quad n = 1, 2, \ldots.
$$

They are orthogonal with respect to the weight function

$$
v_{\lambda,\mu}(t) = |t|^{2\mu} (1-t^2)^{\lambda-1/2}, \quad t \in [-1,1].
$$

For $\mu = 0$, these polynomials, denoted by $C_n^{(\lambda)}(\cdot)$, are called the Gegenbauer polynomials:

$$
C_n^{(\lambda)}(t) = C_n^{(\lambda,0)}(t) = \frac{(2\lambda)_n}{(\lambda + 1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(t).
$$
For a function $f(x)$ the maximum of two real numbers $x$ and $y$ is defined as $\max(x, y)$. For brevity, we will omit “$n \to \infty$” in the asymptotic notation.

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The aim of this paper is to establish Paley-type and Hausdorff–Young–Paley-type inequalities for generalized Gegenbauer expansions in Sections 2 and 3, respectively.

The generalized Gegenbauer polynomials play an important role in Dunkl harmonic analysis (see, for example, [3, 4]). So, the study of these polynomials and their applications is very natural.

The notation $f(n) \sim g(n)$, $n \to \infty$, means that there exist positive constants $C_1$, $C_2$, and a positive integer $n_0$ such that $0 \leq C_1 g(n) \leq f(n) \leq C_2 g(n)$ for all $n \geq n_0$. For brevity, we will omit “$n \to \infty$” in the asymptotic notation.

Define the uniform norm of a continuous function $f$ on $[-1, 1]$ by

$$\|f\|_\infty = \max_{-1 \leq t \leq 1} |f(t)|.$$ 

The maximum of two real numbers $x$ and $y$ is denoted by $\max(x, y)$.

In [3], we prove the following result.

**Theorem 1.** Let $\lambda > -\frac{1}{2}$, $\mu > 0$. Then

$$\|\tilde{C}^{(\lambda, \mu)}\|_\infty \sim n^{\max(\lambda, \mu)}.$$ 

Given $1 \leq p \leq \infty$, we denote by $L_p(v_{\lambda, \mu})$ the space of complex-valued Lebesgue measurable functions $f$ on $[-1, 1]$ with finite norm

$$\|f\|_{L_p(v_{\lambda, \mu})} = \left( \int_{-1}^1 |f(t)|^p v_{\lambda, \mu}(t) \, dt \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\|_{L_\infty} = \esssup_{x \in [-1, 1]} |f(x)|, \quad p = \infty.$$ 

For a function $f \in L_p(v_{\lambda, \mu})$, $1 \leq p \leq \infty$, the generalized Gegenbauer expansion is defined by

$$f(t) \sim \sum_{n=0}^{\infty} \hat{f}_n \tilde{C}^{(\lambda, \mu)}_n(t), \quad \text{where} \quad \hat{f}_n = \int_{-1}^1 f(t) \tilde{C}^{(\lambda, \mu)}_n(t) v_{\lambda, \mu}(t) \, dt.$$ 

For $1 < p < \infty$, we denote by $p'$ the conjugate exponent to $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

The aim of this paper is to establish Paley-type and Hausdorff–Young–Paley-type inequalities for generalized Gegenbauer expansions in Sections 2 and 3, respectively.
2. Paley-type inequalities for generalized Gegenbauer expansions

Using well-known techniques and the following obvious lemma, we establish in Theorem 2 Paley-type inequalities for generalized Gegenbauer expansions.

**Lemma 1.** Suppose \( A > 0 \), \( \gamma \geq 1 \), \( \varphi \) is a positive function on \( \mathbb{N}_0 \), \( \psi \) is a non-negative function on \( \mathbb{N}_0 \), and

\[
\sup_{t \geq 0} \left\{ t^{\gamma-1} \left( \sum_{t \leq \varphi(n) \leq A} \psi(n) \right) \right\} < \infty.
\]

Then

\[
\sum_{\varphi(n) \leq A} (\varphi(n))^{\gamma} \psi(n) = \int_0^A \left( \sum_{t \leq (\varphi(n))^{\gamma} \leq A^{\gamma}} \psi(n) \right) dt = \gamma \int_0^A t^{\gamma-1} \left( \sum_{t \leq \varphi(n) \leq A} \psi(n) \right) dt.
\]

**Theorem 2.** (a) If \( 1 < p \leq 2 \), \( f \in L_p(v_{\lambda,\mu}) \), \( \omega \) is a positive function on \( \mathbb{N}_0 \) such that

\[
M_{\omega} = \sup_{t > 0} \left\{ t \left( \sum_{\omega(n) \geq t} (n + 1)^{2\max(\lambda,\mu)} \right) \right\} < \infty,
\]

then

\[
\left\{ \sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{p} - \frac{1}{q} \right) \max(\lambda,\mu) \omega(n)^{\frac{1}{p} - \frac{1}{q}} |\hat{\phi}(n)| \right)^q \right\}^{1/q} \leq A_p M_{\omega}^{\frac{1}{p} - \frac{1}{q}} \| f \|_{L_p(v_{\lambda,\mu})}.
\]  

(b) If \( 2 \leq q < \infty \), \( \omega \) is a positive function on \( \mathbb{N}_0 \) satisfying \( M_{\omega} \) and \( \phi \) is a non-negative function on \( \mathbb{N}_0 \) such that

\[
\sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{q} - \frac{1}{q} \right) \max(\lambda,\mu) \omega(n)^{\frac{1}{q} - \frac{1}{q}} |\phi(n)| \right)^q < \infty,
\]

then the algebraic polynomials

\[
\Phi_N(t) = \sum_{n=0}^{N} \phi(n) \tilde{C}^{(\lambda,\mu)}_n(t)
\]

converge in \( L_q(v_{\lambda,\mu}) \) to a function \( f \) satisfying \( \hat{f}_n = \phi(n) \), \( n \in \mathbb{N}_0 \), and

\[
\| f \|_{L_q(v_{\lambda,\mu})} \leq A_q M_{\omega}^{\frac{1}{q} - \frac{1}{q}} \left\{ \sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{q} - \frac{1}{q} \right) \max(\lambda,\mu) \omega(n)^{\frac{1}{q} - \frac{1}{q}} |\phi(n)| \right)^q \right\}^{1/q}.
\]

**Proof.** Let \( \sigma = \max(\lambda, \mu) \).

(a) To prove (3), we note that for \( p = 2 \) the Parseval identity implies equality in (3) with \( A_2 = 1 \). We now consider (3) as the transformation from \( L_p(v_{\lambda,\mu}) \) into the sequence \( \{(n + 1)^{-\sigma}(\omega(n))^{-1} \hat{f}_n\}_{n=0}^{\infty} \) in the \( \ell_p \) norm with the weight \( \{(n + 1)^{2\sigma}(\omega(n))^2\}_{n=0}^{\infty} \) and show...
that this transformation is of weak type \((1,1)\). By Theorem 11, Lemma 11 and (2), we have \(|\hat{f}_n| \leq C\|f\|_{L_1(v_{\lambda,\mu})}(n+1)^a\) and
\[
m\{n: (n+1)^{-\sigma}(\omega(n))^{-1}|\hat{f}_n| > t\} \leq m\{n: C(\omega(n))^{-1}\|f\|_{L_1(v_{\lambda,\mu})} > t\} \leq \sum_{\omega(n) \leq A} (n+1)^{2\sigma}(\omega(n))^2 = 2 \int_0^A t \left( \sum_{t \leq \omega(n) \leq A} (n+1)^{2\sigma} \right) dt \leq 2AM_\omega,
\]
where \(A = \frac{\|f\|_{L_1(v_{\lambda,\mu})}}{t}\).

The last estimate is a weak \((1,1)\) estimate which, using the Marcinkiewicz interpolation theorem, implies (3).

(b) We have \(1 < q' \leq 2\). For brevity, write \(\psi_n\) in place of \(\left((n+1)(\frac{1}{\lambda} - \frac{1}{\mu})\sigma(\omega(n))\frac{1}{\psi} - \frac{1}{\phi(n)}\right)^q\). Suppose that \(g \in L_{q'}(v_{\lambda,\mu})\) and that \(N < N'\) are positive integers. Applying H"older’s inequality and (a), we find that
\[
\left| \int_{-1}^1 \Phi_N(t)g(t)v_{\lambda,\mu}(t)\,dt \right| = \left| \sum_{n=0}^N \phi(n)\hat{g}_n \right| = \left| \sum_{n=0}^N \left((n+1)(\frac{1}{\lambda} - \frac{1}{\mu})\sigma(\omega(n))\frac{1}{\psi} - \frac{1}{\phi(n)}\right) \left((n+1)(\frac{1}{\lambda} - \frac{1}{\mu})\sigma(\omega(n))\frac{1}{\psi} - \frac{1}{\phi(n)}\right)^q \hat{g}_n \right| \leq \left( \sum_{n=0}^N \psi_n \right)^{1/q} \left( \sum_{n=0}^N \left((n+1)(\frac{1}{\lambda} - \frac{1}{\mu})\sigma(\omega(n))\frac{1}{\psi} - \frac{1}{\phi(n)}\right)^q \hat{g}_n \right)^{1/q'} \leq \left( \sum_{n=0}^N \psi_n \right)^{1/q} A_{q'}M_\omega^{\frac{1}{q'} - \frac{1}{q}} \|g\|_{L_{q'}(v_{\lambda,\mu})}. \tag{6}
\]
Similarly,
\[
\left| \int_{-1}^1 \left( \Phi_N(t) - \Phi_{N'}(t) \right) g(t)v_{\lambda,\mu}(t)\,dt \right| \leq \left( \sum_{n=N+1}^{N'} \psi_n \right)^{1/q} A_{q'}M_\omega^{\frac{1}{q'} - \frac{1}{q}} \|g\|_{L_{q'}(v_{\lambda,\mu})}. \tag{7}
\]
Hence, by [5] Theorem (12.13), the inequalities (6) and (7) lead respectively to the estimates
\[
\|\Phi_N\|_{L_q(v_{\lambda,\mu})} \leq \left( \sum_{n=0}^N \psi_n \right)^{1/q} A_{q'}M_\omega^{\frac{1}{q'} - \frac{1}{q}} \tag{8}
\]
and
\[
\|\Phi_N - \Phi_{N'}\|_{L_q(v_{\lambda,\mu})} \leq \left( \sum_{n=N+1}^{N'} \psi_n \right)^{1/q} A_{q'}M_\omega^{\frac{1}{q'} - \frac{1}{q}}. \tag{9}
\]
The last inequality combined with (11) shows that the sequence \(\{\Phi_N\}_{N=1}^\infty\) is a Cauchy sequence in \(L_q(v_{\lambda,\mu})\) and therefore convergent in \(L_q(v_{\lambda,\mu})\); let \(f\) be its limit. Then, by mean convergence,
\[
\hat{f}_n = \lim_{N \to \infty} \left( \hat{\Phi}_N \right)_n, \quad n \in \mathbb{N}_0,
\]
which is easily seen to equal $\phi(n)$. Moreover, the defining relation

$$f = \lim_{N \to \infty} \Phi_N \quad \text{in} \quad L_q(v_{\lambda,\mu})$$

and the inequality (8) show that (5) holds and so complete the proof.

□

3. Hausdorff–Young–Paley-type inequalities for generalized Gegenbauer expansions

In [9], we prove the following Hausdorff–Young-type inequalities for generalized Gegenbauer expansions.

**Theorem 3.** (a) If $1 < p \leq 2$ and $f \in L_p(v_{\lambda,\mu})$, then

$$\left\{ \sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{p} - \frac{1}{p'} \right) \max(\lambda,\mu) \left| \hat{f}_n \right| \right)^{p'} \right\}^{1/p'} \leq B_p \|f\|_{L_p(v_{\lambda,\mu})}.$$  

(b) If $2 \leq q < \infty$ and $\phi$ is a function on non-negative integers satisfying

$$\sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{q} - \frac{1}{q'} \right) \max(\lambda,\mu) \left| \phi(n) \right| \right)^{q'} < \infty,$$

then the algebraic polynomials

$$\Phi_N(t) = \sum_{n=0}^{N} \phi(n) \tilde{C}_n^{(\lambda,\mu)}(t)$$

converge in $L_q(v_{\lambda,\mu})$ to a function $f$ satisfying $\hat{f}_n = \phi(n)$, $n \in \mathbb{N}_0$, and

$$\|f\|_{L_q(v_{\lambda,\mu})} \leq B_q \left\{ \sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{q} - \frac{1}{q'} \right) \max(\lambda,\mu) \left| \phi(n) \right| \right)^{q'} \right\}^{1/q'}.$$

**Theorem 5** contains the Paley-type and the Hausdorff–Young-type inequalities for the expansions by orthonormal polynomials with respect to the weight function $v_{\lambda,\mu}$ (see (1)). To prove it, we need Stein’s modification of the Riesz–Thorin interpolation theorem (see [6, Theorem 2, p. 485]) given below.

**Theorem 4 (Stein).** Suppose $\nu_1$ and $\nu_2$ are $\sigma$-finite measures on $M$ and $S$, respectively, and $T$ is a linear operator defined on $\nu_1$-measurable functions on $M$ to $\nu_2$-measurable functions on $S$. Let $1 \leq r_0, r_1, s_0, s_1 \leq \infty$ and $\frac{1}{r} = \frac{1-t}{r_0} + \frac{t}{r_1}$, $\frac{1}{s} = \frac{1-t}{s_0} + \frac{t}{s_1}$, where $0 \leq t \leq 1$. Suppose further that

$$\|(Tg) \cdot v_i\|_{L_s(S,\nu_2)} \leq L_i \|g \cdot u_i\|_{L_{r_i}(M,\nu_1)}, \quad i = 0, 1,$$

where $u_i$ and $v_i$ are non-negative weight functions. Let $u = u_0^{1-t} \cdot u_1^t$, $v = v_0^{1-t} \cdot v_1^t$.

Then

$$\|(Tg) \cdot v\|_{L_s(S,\nu_2)} \leq L \|g \cdot u\|_{L_{r}(M,\nu_1)}$$

with $L = L_0^{1-t} \cdot L_1^t$. 

Theorem 5. Let $\sigma = \max(\lambda, \mu)$.

(a) If $1 < p \leq 2$, $f \in L_p(v_{\lambda,\mu})$, $\omega$ is a positive function on $\mathbb{N}_0$ satisfying the condition (2), and $p \leq s \leq p'$, then

$$\left\{ \left( n + 1 \right)^{\frac{2}{p} - 1} \sigma(\omega(n))^{\frac{1}{s} - \frac{1}{p'}} |\tilde{f}_n| \right\}^{1/s} \leq C_p(s) M_{\omega}^{\frac{1}{s} - \frac{1}{p'}} \| f \|_{L_p(v_{\lambda,\mu})}. \quad (9)$$

(b) If $2 \leq q < \infty$, $q' \leq r \leq q$, $\omega$ is a positive function on $\mathbb{N}_0$ satisfying the condition (2) and $\phi$ is a non-negative function on $\mathbb{N}_0$ such that

$$\sum_{n=0}^{\infty} \left( (n + 1)^{1 - \frac{2}{q'}} \sigma(\omega(n))^{\frac{1}{s} - \frac{1}{q'}} |\phi(n)| \right)^{r'} < \infty,$$

then the algebraic polynomials

$$\Phi_N(t) = \sum_{n=0}^{N} \phi(n) \tilde{C}_n^{(\lambda,\mu)}(t)$$

converge in $L_q(v_{\lambda,\mu})$ to a function $f$ satisfying $\tilde{f}_n = \phi(n)$, $n \in \mathbb{N}_0$, and

$$\| f \|_{L_q(v_{\lambda,\mu})} \leq C_q'(r) M_{\omega}^{\frac{1}{s} - \frac{1}{p'}} \left\{ \sum_{n=0}^{\infty} \left( (n + 1)^{1 - \frac{2}{q'}} \sigma(\omega(n))^{\frac{1}{s} - \frac{1}{q'}} |\phi(n)| \right)^{r'} \right\}^{1/r'}.$$

Proof. (a) This part was proved for $s = p$ (with $C_p(p) = A_p$) and $s = p'$ (with $C_p(p') = B_p$) in Theorems 2 and 3 respectively. So for $p = 2$, we obtain the equality in (9) with $C_2(2) = 1$.

Consider now the case that $1 < p < 2$. To prove (9), we set in Theorem 3: $M = [-1, 1]$, $\nu_1$ the Lebesgue measure, $S = \{ n \}_{n=0}^{\infty}$, $\nu_2$ the counting measure, $g = f$, $Tg = \{ \tilde{f}_n \}_{n=0}^{\infty}$, $r = r_0 = r_1 = p$, $u = u_0 = u_1 = v_{\lambda,\mu}$, $s_0 = p'$, $s_1 = p$, $v_0 = \{ (n + 1)^{\frac{2}{p} - 1} \sigma(\omega(n))^{\frac{1}{s} - \frac{1}{p'}} \}_{n=0}^{\infty}$, $v_1 = \{ (n + 1)^{\frac{2}{q'} - 1} \sigma(\omega(n))^{\frac{1}{s} - \frac{1}{q'}} \}_{n=0}^{\infty}$, $L_0 = B_p$, $L_1 = A_p M_{\omega}^{\frac{1}{s} - \frac{1}{p'}}$, and $\frac{1}{s} = \frac{1-t}{p'} + \frac{t}{p}$. As $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{s} - \frac{1}{p} = (1-t)(\frac{1}{p} - \frac{1}{p'})$, $\frac{1}{s} - \frac{1}{s} = t(\frac{1}{p} - \frac{1}{p'})$, the proof of (9) is concluded.

Because of

$$1 - t = \frac{\frac{1}{s} - \frac{1}{p}}{\frac{1}{p'} - \frac{1}{p}}, \quad t = \frac{\frac{1}{p'} - \frac{1}{s}}{\frac{1}{p'} - \frac{1}{p}},$$

it is clear that $C_p(s) = B_p^{1-t} A_p^t$ and $M_{\omega}^{\frac{1}{s} - \frac{1}{p'}} = M_{\omega}^{\frac{1}{s} - \frac{1}{p'}}$.

(b) Taking into account the previously given proof (see part (b) in Theorem 2), the proof is obvious and left to the reader. \qed

4. Conclusion

As an application of the results above, we are going to obtain sufficient condition for a special sequence of positive real numbers to be a Fourier multiplier.
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