Interpreting Distributional Reinforcement Learning: Regularization and Optimization Perspectives

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Abstract

Distributional reinforcement learning (RL) is a class of state-of-the-art algorithms that estimate the whole distribution of the total return rather than only its expectation. Despite the remarkable performance of distributional RL, a theoretical understanding of its advantages over expectation-based RL remains elusive. In this paper, we illuminate the superiority of distributional RL from both regularization and optimization perspectives. Firstly, by applying an expectation decomposition, the additional impact of distributional RL compared with expectation-based RL is interpreted as a risk-aware entropy regularization in the neural Z-fitted iteration framework. We also provide a rigorous comparison between the resulting entropy regularization and the vanilla one in maximum entropy RL. Through the lens of optimization, we shed light on the stability-promoting distributional loss with desirable smoothness properties in distributional RL. Moreover, the acceleration effect of distributional RL owing to the risk-aware entropy regularization is also provided. Finally, rigorous experiments reveal the different regularization effects as well as the mutual impact of vanilla entropy and risk-aware entropy regularization in distributional RL, focusing specifically on actor-critic algorithms. We also empirically verify that the distributional RL algorithm enjoys a more stable gradient behavior, contributing to its stable optimization and acceleration effect as opposed to classical RL. Our research paves a way towards better interpreting the superiority of distributional RL algorithms.

1 Introduction

The intrinsic characteristics of classical reinforcement learning (RL) algorithms, such as temporal-difference (TD) learning [Sutton and Barto, 2018] and Q-learning [Watkins and Dayan, 1992], are based on the expectation of discounted cumulative rewards that an agent observes while interacting with the environment. In stark contrast to the classical expectation-based RL, a new branch of algorithms called distributional RL estimates the full distribution of total returns and has demonstrated state-of-the-art performance in a wide range of environments [Bellemare et al., 2017a, Dabney et al., 2018b, Dabney et al., 2018b, Yang et al., 2019, Zhou et al., 2020, Nguyen et al., 2020]. Meanwhile, distributional RL also inherits other benefits in risk-sensitive control, policy exploration settings [Mavrin et al., 2019, Rowland et al., 2019] and robustness [Sun et al., 2021].

Despite the existence of numerous algorithmic variants of distributional RL with remarkable empirical success, we still have a poor understanding of what the effectiveness of distributional RL is stemming from and theoretical studies on advantages of distributional RL over expectation-based RL are less established. Existing works include [Lyle et al., 2021] that investigated the impact of distributional RL from the perspective of representation dynamics. Another recent work [Martin et al., 2020] mapped distributional RL problems to a Wasserstein gradient flow problem, treating the distributional
Bellman residual as a potential energy functional. Offline distributional RL [Ma et al., 2021] has also been proposed to investigate the efficacy of distributional RL in both risk-neutral and risk-averse domains. This is an encouraging trend for recent works towards closing the gap between theory and practice in distributional RL.

In this paper, we theoretically illuminate the superiority of distributional RL over expectation-based RL through the lens of regularization and optimization. Specifically, we simplify distributional RL as a \textit{neural Z-fitted iteration} framework, within which we establish an equivalence between distributional RL with a risk-aware entropy regularized maximum likelihood estimation (MLE) from the perspective of statistics. Then we elaborate the impact of our derived risk-aware entropy regularization by comparing with maximum entropy RL. From the perspective of optimization, we show the stable optimization property of distributional RL by leverage of the histogram density function approximation. In addition to stability blessing, the acceleration effect of distributional RL stemming from the risk-aware entropy regularization is also derived. Empirical results suggest different regularization effect as well as the mutual impact of risk-aware entropy and vanilla one in the actor-critic algorithms, and exhibit a more stable gradient behavior of distributional RL as opposed to classical RL.

2 Preliminary Knowledge

In classical RL, an agent interacts with an environment via a Markov decision process (MDP), a 5-tuple \((S, A, R, P, \gamma)\), where \(S\) and \(A\) are the state and action spaces, respectively. \(P\) is the environment transition dynamics, \(R\) is the reward function and \(\gamma \in (0, 1)\) is the discount factor.

**Action-value function vs Action-value distribution.** Given a policy \(\pi\), the discounted sum of future rewards is a random variable \(Z^\pi(s, a) = \sum_{t=0}^{\infty} \gamma^t R(s_t, a_t)\), where \(s_0 = s, a_0 = a, \) \(s_{t+1} \sim P(\cdot|s_t, a_t)\), and \(a_t \sim \pi(\cdot|s_t)\). In the control setting, expectation-based RL focuses on the action-value function \(Q^\pi(s, a)\), the expectation of \(Z^\pi(s, a)\), i.e., \(Q^\pi(s, a) = \mathbb{E}[Z^\pi(s, a)]\). Distributional RL, on the other hand, focuses on the action-value distribution, the full distribution of \(Z^\pi(s, a)\). Leveraging knowledge on the entire distribution can better capture the uncertainty of returns [Dabney et al., 2018a, Mavrin et al., 2019].

**Bellman operators vs distributional Bellman operators.** For the policy evaluation in expectation-based RL, the value function is updated via the Bellman operator \(\mathcal{T}^\pi Q(s, a) = \mathbb{E}[R(s, a)] + \gamma \mathbb{E}_{s' \sim p, \pi}[Q(s', a')]\). In distributional RL, the action-value distribution of \(Z^\pi(s, a)\) is updated via the distributional Bellman operator \(\mathcal{Z}^\pi\)

\[
\mathcal{Z}^\pi Z(s, a) = R(s, a) + \gamma Z(s', a'),
\]

where \(s' \sim P(\cdot|s, a)\) and \(a' \sim \pi(\cdot|s')\). From a theoretical perspective, both the Bellman operator \(\mathcal{T}^\pi\) in the policy evaluation setting and the Bellman optimality operator \(\mathcal{T}^{\pi_{\text{opt}}}\) in the control setting are contractive in the stationary policy case. In contrast, the distributional Bellman operator is contractive under certain distribution divergence metrics, but the distributional Bellman optimality operator \(\mathcal{T}^{\pi_{\text{opt}}}\) can only converge to a set of optimal non-stationary value distributions in a weak sense [Elie and Arthur, 2020].

3 Regularization Effect of Distributional RL

In this section, we analyze and then conclude that the superiority of distributional RL over classical RL is mainly stemming from its regularization effect.

3.1 Distributional RL: Neural Z-fitted Iteration

**Neural Q-Fitted Iteration.** In the function approximation setting, Deep Q Learning [Mnih et al., 2015] can be simplified into \textit{Neural Q-Fitted Iteration} [Fan et al., 2020] under tricks of experience replay and the target network \(Q_{\theta^*}\), where we update parameterized \(Q_{\theta}(s, a)\) in each iteration \(k\):

\[
Q_{\theta}^{k+1} = \arg\min_{Q_{\theta}} \frac{1}{n} \sum_{i=1}^{n} [y_i - Q_{\theta}^k(s_i, a_i)]^2,
\]

2
where the target $y_i = r(s_i, a_i) + \gamma \max_{a \in A} Q^k_{\theta^*}(s'_i, a)$ is fixed within every $T_{\text{target}}$ steps to update target network $Q_{\theta^*}$ by letting $\theta^* = \theta$. The experience buffer induces independent samples $\{(s_i, a_i, r_i, s'_i)\}_{i=1}^n$.

In an ideal case when we neglect the non-convexity and TD approximation errors, we have $Q^k_{\theta^{i+1}} = T^{opt} Q^k_{\theta^i}$, which is exactly the updating under Bellman optimality operator. From the perspective of statistics, the optimization problem in Eq. 2 can be viewed as Least Square Estimation (LSE) in a neural network parametric regression problem between the updating of target network $Q_{\theta^*}$.

**Neural Z-Fitted Iteration.** Analogous to neural Q-fitted iteration, we can also simplify value-based distributional RL methods based on a parameterized $Z_\theta$ into a Neural Z-fitted Iteration as

$$Z^{k+1}_\theta = \arg\min_{Z_\theta} \frac{1}{n} \sum_{i=1}^n d_p(Y_i, Z^k_\theta(s_i, a_i)), \quad (3)$$

where the target $Y_i = R(s_i, a_i) + \gamma Z^k_\theta(s'_i, \pi_Z(s'))$ with $\pi_Z(s') = \arg\max_{a'} \mathbb{E}[Z^k_\theta(s', a')]$ is fixed within every $T_{\text{target}}$ steps to update target network $Z_{\theta^*}$. $d_p$ is a divergence metric between two distributions. Notably, choices of representation manner on $Z_\theta$ and the metric $d_p$ are pivotal for the empirical success of distributional RL algorithms. For instance, QR-DQN [Dabney et al., 2018b] approximates Wasserstein distance $W_p$, which leverages quantiles to represent the distribution of $Z_{\theta^*}$. C51 [Bellemare et al., 2017a] represents $Z_{\theta^*}$ via a categorical distribution under the convergence of Cramér distance [Bellemare et al., 2017b, Rowland et al., 2018], a special case with $p = 2$ of the $\ell_p$ distance [Elie and Arthur, 2020], while Moment Matching [Nguyen et al., 2020] learns deterministic samples to represent the distribution of $Z_{\theta^*}$ based on Maximum Mean Discrepancy (MMD). Contractive properties under typical metrics $d_p$ can be summarized as follows

- $T^\infty$ is $\gamma$-contractive under the supreme form of Wassertein distance $W_p$.
- $T^\infty$ is $\gamma^{1/p}$-contractive under the supreme form of $\ell_p$ distance.
- $T^\infty$ is $\gamma^{\alpha/2}$-contractive under MMD$_\infty$ with the kernel $k_\alpha(x, y) = -\|x - y\|^\alpha, \forall \alpha \in \mathbb{R}$.

For the completeness, the definition of mentioned distances and the proof of contraction are provided in Appendix A. Being excessively addicted to these elusive metrics may hinder the progress towards a theoretical understanding of distributional RL. Instead, we shift our attention to Kullback–Leibler (KL) divergence, a widely-used metric to measure the difference between two distributions, to conduct our analysis. Note that the KL divergence is also applied in the typical distributional RL algorithm, C51 [Bellemare et al., 2017a]. Theoretically, we demonstrate in Proposition 1 that the KL divergence still enjoys desirable properties in distributional RL context, and can be reasonable for the theoretical analysis. Concretely, we assume $Z_\theta$ is absolutely continuous and has joint supports, under which the KL divergence is well-defined. Proof of Proposition 1 and the definition of supreme $D_{KL}$ are provided in Appendix B.

**Proposition 1.** Denote the supreme of $D_{KL}$ as $D_{KL}^\infty$, we have: (1) $T^\infty$ is a non-expansive operator under $D_{KL}^\infty$, i.e., $D_{KL}^\infty(\mathbb{E}^\infty Z_1, \mathbb{E}^\infty Z_2) \leq D_{KL}^\infty(Z_1, Z_2)$, (2) $D_{KL}^\infty(Z_n, Z) \rightarrow 0$ implies $W_p(Z_n, Z) \rightarrow 0$, (3) the expectation of $Z^\infty$ is still $\gamma$-contractive under $D_{KL}^\infty$, i.e., $\|\mathbb{E}^\infty Z_1 - \mathbb{E}^\infty Z_2\|_{KL} \leq \gamma \|\mathbb{E}Z_1 - \mathbb{E}Z_2\|_{\infty}$.

### 3.2 Distributional RL: A Novel Entropy-regularized MLE via Expectation Decomposition

The reasonable properties of KL divergence in Proposition 1 allows us to leverage it to conduct the theoretical analysis. To separate the impact of additional distribution information from the expectation of $Z^\infty$, we leverage the technique of gross error model from robust statistics [Huber, 2004], which is similar to the technique to analyze Label Smoothing [Müller et al., 2019] and Knowledge Distillation [Hinton et al., 2015]. Specifically, we denote the one-dimensional full distribution of $Z^\infty$ as $F$, and the distribution on the remaining support getting rid of $\mathbb{E}[Z^\infty]$ as $F_{\mu}$. Hence, we can obtain the distribution decomposition for $Z^\infty(s, a)$ as

$$F^{\infty, a}_{\mu}(x) = (1 - \epsilon) \mathbb{I}_{\{x \geq \mathbb{E}[Z^\infty(s, a)]\}}(x) + \epsilon F^{\mu, a}_{\mu}(x), \quad (4)$$

where $\epsilon$ controls the proportion of $F^{\mu, a}_{\mu}(x)$ and the indicator function $\mathbb{I}_{\{x \geq \mathbb{E}[Z^\infty(s, a)]\}} = 1$ if $x \geq \mathbb{E}[Z^\infty(s, a)]$, otherwise 0. We assume in advance $F^{\mu, a}_{\mu}$ is a continuous and differential distribution
function, and thus by definition $F^{s,a}$ is also a distribution, forming a class of distribution functions to express $Z^*(s,a)$. After taking derivatives on both sides, we attain their density function relationship as $p^{s,a}(x) = (1 - \epsilon)\delta_{(x \in \mathbb{E}[Z^*(s,a)] \cup \epsilon}) \mu^{s,a}(x)$, where $\mu^{s,a}(x)$ is the density function related to $Z^*(s,a)$ on remaining supports removing $\mathbb{E}[Z^*(s,a)]$. It is worth noting that the existence of $\mu^{s,a}(x)$ can be simply guaranteed by directly computing $\mu^{s,a}(x) = p^{s,a}(x)/\epsilon - (1 - \epsilon)\delta_{(x \in \mathbb{E}[Z^*(s,a)] \cup \epsilon}) / \epsilon$ as long as $p^{s,a}(x)$ and the expectation of $Z^*(s,a)$ exist. The definition of all distributions in Eq. 4 can also easily justified. Next, we use $p^{s,a}(x)$ to express the true target density function behind $\{Y_i\}_{i \in [n]}$, and $q^{s,a}_\theta(x)$ to denote the approximated one of $Z^*_\theta(s,a)$ in neural Z-fitted iteration via Eq. 3. Therefore, we can derive the following result in Proposition 2.

**Proposition 2.** Let $\mathcal{H}(P,Q)$ be the cross entropy between two probability measure $P$ and $Q$, i.e., $\mathcal{H}(P,Q) = -\int_{x \in \mathcal{X}} P(x) \log Q(x) \, dx$. Let $\alpha$ be a positive constant, and based on the decomposition in Eq. 4 and $D_{KL}$ as $d_p$, Neural Z-fitted iteration in Eq. 3 can be reformulated as

$$Z^{k+1}_\theta = \arg\min_{Z_\theta} \frac{1}{n} \sum_{i=1}^n \mathcal{H}(\delta_{(x \in \mathbb{E}[Z^*(s_i,a_i)])}, q^{s_i,a_i}_\theta) + \alpha \mathcal{H}(\mu^{s_i,a_i}, q^{s_i,a_i}_\theta).$$

We provide the proof in Appendix C. For the uniformity of notation, we still use $s, a$ in the following analysis instead of $s_i, a_i$ in Eq. 5. Importantly, the first term in Eq. 5 can be further simplified as $-\int_{x \in \mathcal{X}} \log q_\theta^*(x) \mathbb{E}[Z(s,a)]$. From the perspective of statistics, minimizing the first term can be viewed as a variant of Maximum Likelihood Estimation (MLE) on the expectation $\mathbb{E}[Z(s,a)]$ rather than the traditional MLE directly on observed samples. The cross entropy regularization in the second term pushes $q_\theta^*$ to approximate the density function $\mu^{s,a}$ in order to fully utilize the higher moments of distributional information, serving as the key to the superiority of distributional RL. This novel cross entropy regularization regarding $\mu^{s,a}$ and $q_\theta^*$, which we call risk-aware entropy regularization, is different from the classical entropy regularization used in maximum entropy RL. We further analyze their connection and difference in Section 3.3. In summary, distributional RL can be simplified as a novel entropy-regularized MLE within neural Z-fitted iteration framework in stark contrast to the Least-Square estimation of expectation-based RL in the neural Q-fitted iteration.

**Attainability of $\mu^{s,a}$.** The density function $\mu^{s,a}$ is computed by decomposing the expectation from the full distribution of $Z^*(s,a)$ via Eq. 4, which is normally attainable in practice as distributional RL algorithms use bootstrap, e.g., TD learning, to estimate the target distribution and thus obtain $\mu^{s,a}$. As such, we assume the availability of $\mu^{s,a}$ in the following risk-aware entropy regularization analysis.

### 3.3 Risk-aware Entropy Regularization

We focus on analyzing the effect of the derived novel cross entropy regularization in Eq. 5 in distributional RL by making a rigorous comparison with maximum entropy RL [Williams and Peng, 1991].

**Entropy Regularization in Maximum Entropy RL.** Maximum entropy RL, including Soft Q-Learning [Haarnoja et al., 2017], greedily maximizes the entropy of the policy $\pi(\cdot|s)$ in each state:

$$J(\pi) = \sum_{t=0}^T \mathbb{E}_{(s_t,a_t) \sim p_\pi} \left[ r(s_t,a_t) + \beta \mathcal{H}(\pi(\cdot|s_t)) \right],$$

where $\mathcal{H}(\pi_\theta(\cdot|s_t)) = -\sum_a \pi_\theta(a|s_t) \log \pi_\theta(a|s_t)$ and $p_\pi$ is the generated distribution following $\pi$. The temperature parameter $\beta$ determines the relative importance of the entropy term against the cumulative rewards, and thus controls the action diversity of the optimal policy. This maximum entropy regularization has various conceptual and practical advantages. Firstly, the learned policy is encouraged to visit states with high entropy, thus promoting the exploration over these states [Han and Sung, 2021]. Secondly, it considerably improves the learning speed [Mei et al., 2020] and therefore are widely used in state-of-the-art algorithms, e.g., Soft Actor-Critic (SAC) [Haarnoja et al., 2018]. Similar benefits of both distributional RL and maximum entropy RL motivate us to probe their mathematical connection.
Risk-aware Entropy Regularization in Distributional RL. As shown in Eq. 5, the cross entropy regularization for distributional RL encourages the learned policy to visit state and action pairs whose action-value distribution are far away from their expectation. This is owing to the fact that for these state and action pairs, \(\mu^{s,a}\) in Eq. 5, which contains second and higher moments of its distribution information, would be highly non-degenerated, and thus the cross entropy regularization plays a leading role. Given this analysis, we call the cross entropy regularization in distributional RL as a risk-aware one as it can stimulate the policy to explore state and action pairs with high degree of dispersion for their action-value distributions beyond only the expectation, therefore promoting the risk-aware exploration.

Next, with this risk-aware entropy regularization, we can further derive the convergence of Soft Distributional Policy Iteration, which is counterpart for Soft Policy Iteration [Haarnoja et al., 2018], i.e., the underpinning of SAC. In principle, Soft Distributional Policy Iteration replaces the vanilla Distributional Policy Iteration, which is a leading role. Given this analysis, we call the cross entropy regularization in distributional RL as risk-aware exploration. Raw dispersion for their action-value distributions beyond only the expectation, therefore promoting the risk-aware exploration.

\[
T_{sd}^\pi Q(s_t, a_t) \triangleq r(s_t, a_t) + \gamma E_{s_{t+1} \sim \rho^\pi} [V(s_{t+1})],
\]

where a new soft value function is defined by

\[
V(s_{t+1}) = E_{a_{t+1} \sim \pi, x \sim \mu^{s_{t+1}, a_{t+1}}} [Q(s_{t+1}, a_{t+1}) + \log q_{\theta}^{s_{t+1}, a_{t+1}}(x)].
\]

Note that in this specific tabular setting regarding \(s\) and \(a\), we particularly use \(q_{\theta}^{s,a}(x)\) to approximate the true density function of \(Z(s, a)\), and the distribution difference between \(\mu^{s,a}\) and \(q_{\theta}^{s,a}(x)\) determines our risk-aware entropy regularization. We show in Lemma 1 that our soft distributional Bellman operator still inherits the convergence property in the policy evaluation phase.

**Lemma 1.** (Soft Distributional Policy Evaluation) Consider the soft distributional Bellman operator \(T_{sd}^\pi\) in Eq. 7 and the behavior of expectation of \(Z(s, a)\), i.e., \(Q(s, a)\). Define \(Q^{k+1} = T_{sd}^\pi Q^k\), then \(Q^k\) will converge to a soft Q-value of \(\pi\) as \(k \to \infty\) with the new objective function defined as

\[
J'(\pi, \theta) = \sum_{t=0}^{T} E_{(s_t, a_t) \sim \rho^\pi} [r(s_t, a_t) - \gamma H(\mu^{s_t, a_t}, q_{\theta}^{s_t, a_t})].
\]

In soft distributional policy improvement for distributional RL, we modify the policy updating rule in the soft policy improvement via the cross-entropy term according to

\[
\pi_{new} = \arg \min_{\pi' \in \Pi} D_{KL}(H(\mu^{s_t, a_t}, q_{\theta}^{s_t, a_t}) \| \exp(Q^{\pi_{old}}(s_t, \cdot))/\Delta_{\pi_{old}}(s_t)),
\]

where the partition function \(\Delta_{\pi_{old}}(s_t)\) normalizes the distribution. In Lemma 2, we prove that for the risk-aware entropy regularization in distributional RL, the policy improvement can be guaranteed as well.

**Lemma 2.** (Soft Distributional Policy Improvement) Let \(\pi \in \Pi\) and a new policy \(\pi_{new}\) be updated via the policy improvement step. Then \(Q^{\pi_{new}}(s_t, a_t) \geq Q^{\pi_{old}}(s_t, a_t)\) for all \((s_t, a_t) \in S \times A\) with \(|A| \leq \infty\).

We can also immediately derive the soft distributional policy iteration algorithm that alternates between soft distributional policy evaluation and soft distributional policy improvement. It will provably converge to the optimal minimum risk-aware entropy policy among all policies in \(\Pi\) in Theorem 1.

**Theorem 1.** (Soft Distributional Policy Iteration) Assume \(H(\mu^{s_t, a_t}, q_{\theta}^{s_t, a_t}) \leq M\) for all \((s_t, a_t) \in S \times A\), where \(M\) is a constant. Repeatedly applying soft distributional policy evaluation and soft distributional policy improvement, the policy converges to a policy \(\pi^*\) such that \(Q^{\pi^*}(s_t, a_t) \geq Q^\pi(s_t, a_t)\) for all \(\pi \in \Pi\).

Please refer to Appendix D for the proof of Lemmas 1, 2 and Theorem 1.
Comparison between Maximum Entropy RL and Minimum Risk-aware Entropy Distributional RL. Firstly, the risk-aware entropy distributional RL replaces the vanilla policy entropy term in Eq. 6 with the cross entropy term between $\mu^{s,a}$ and $q_{\theta}^{s,a}$, yielding a new objective function $J'(\pi, \theta)$ in Lemma 1. By minimizing the risk-aware entropy regularization with respect to $\theta$, $q_{\theta}^{s,a}$ will gradually approximate the true action-value density function of $Z(s,a)$ by firstly approximating $\mu^{s,a}$. By further minimizing the risk-aware entropy regularization regarding $\pi$, the learned policy is encouraged to visit state and action pairs whose action-value distribution has higher degree of dispersion or is far beyond only its expectation, thus promoting risk-aware exploration for the uncertainty of environment as suggested in Figure 1. Another discrepancy is that the entropy in maximum entropy RL is state-wise, while our risk-aware regularization is state-action-wise, implying that it is a more fine-grained regularization to characterize the action-value distribution $Z(s,a)$.

Remark. A literally similar yet sharply different Bellman operator called distributional soft Bellman operator was proposed in distributional SAC [Ma et al., 2020] that directly adds the vanilla entropy term on distributional Bellman operator. We provide a more rigorous comparison among all related Bellman operators in Appendix L.

4 Optimization Properties of Distributional RL

We consider the function approximation setting with respect to $s$ and $a$. We reveal the stability property of distributional RL by means of histogram density approximation in Section 4.1, and we further analyze its acceleration effect based on the decomposition in Eq. 4 in Section 4.2.

4.1 Stable Optimization

We leverage the histogram function $f^{s,a}$ to parameterize the approximated density function $q_{\theta}^{s,a}$ of $Z(s,a)$. With the KL divergence as $d_p$, we can eventually derive the histogram distributional loss [Imani and White, 2018], which will be optimized within each update in the neural Z-fitted iteration in Eq. 3.

Denote $x(s)$ as the state feature on each state $s$, and we let the support of $Z(s,a)$ be uniformly partitioned into $k$ bins. The output dimension of $f^{s,a}$ can be $|A| \times k$, where we use the index $a$ to focus on the function $f^{s,a}$. Hence, the function $f^{s,a} : \mathcal{X} \rightarrow [0,1]^k$ provides a k-dimensional vector $f^{s,a}(x(s))$ of the coefficients, indicating the probability the target is in that bin given the state feature $x(s)$ and action $a$. Next, we use softmax based on the linear approximation $x(s)\mathbf{\theta}_i$ to express $f^{s,a}$, i.e., $f_i^{s,a}(x(s)) = \exp \left( x(s)^\top \mathbf{\theta}_i \right) / \sum_{j=1}^k \exp \left( x(s)^\top \mathbf{\theta}_j \right)$. For simplicity, we use $f^\theta_i(x(s))$ to replace $f_i^{s,a}(x(s))$. Note that the form of $f^{s,a}$ is similar to a multi-class Logistic function, but the difference is that the prediction probability $f_i^{s,a}$ is redefined as the probability in the $i$-th bin over the support of $Z(s,a)$, thus eventually serving as a density function. We contend that this is the most straightforward form to utilize the histogram function to express the density function $q_{\theta}^{s,a}$. Therefore, the resulting

![Figure 1: Impact of the risk-aware regularization in distributional RL. The risk-aware entropy regularization will push $q_{\theta}^{s,a}$ to approximate $\mu^{s,a}$ determined by the environment, thus promoting exploring the uncertainty of environment.](image-url)
histogram distributional loss $\mathcal{L}_\theta$ is formulated as:

$$
\mathcal{L}_\theta(s, a) = D_{KL}(p^{s,a}, q_\theta^{s,a}) \\
= \int_{x \in X} p^{s,a}(x) \log \left( \frac{p^{s,a}(x)}{q_\theta^{s,a}(x)} \right) dx \\
= -\sum_{i=1}^{k} p_i^{s,a} \log f_i^\theta(x(s))
$$

(8)

where $\theta = \{\theta_1, ..., \theta_k\}$ and $p_i^{s,a}$ is the probability in the $i$-th bin of the true density function $p^{s,a}(x)$ for $Z(s, a)$ defined in Eq. 4. The derivation of the histogram distributional loss is given in Appendix E. To attain the stable optimization property of distributional RL, we firstly derive the appealing properties of the histogram distribution loss in Lemma 3.

**Lemma 3.** *(Properties of Histogram Distributional Loss)* Assume $\|x(s)\| \leq l$ for each state $s$, then $\mathcal{L}_\theta$ is $kl^2$-smooth, convex and $kl$-Lipschitz continuous w.r.t. $\theta$.

Notably, due to the discrepancy between $\mathcal{L}_\theta$ with the multi-class logistic regression loss, the conclusions derived from Lemma 3 are not straightforward. Please refer to Appendix F for the proof. The derived Lipschitz properties of $d_\theta$ under histogram distributional loss plays an integral part in the stable optimization for distributional RL. In stark contrast, expectation-based RL, which optimizes a least-squared loss function in neural-Q-fitted iteration, does not inherit these smoothness properties. In particular, for histogram distributional loss we have $\|\nabla_\theta \mathcal{L}_\theta\| \leq kl$, while the gradient norm in expectation-based RL is $\|y_i - Q_\theta^k(s, a)\|\|x(s)\|$, which is likely to be arbitrarily large due to the unbounded $\|y_i - Q_\theta^k(s, a)\|$.

Next, we recap the definition of uniform stability for an algorithm while running Stochastic Gradient Descent (SGD) in Definition 1.

**Definition 1.** *(Uniform Stability)* [Hardt et al., 2016] Consider a function $g$, a randomized algorithm $\mathcal{M}$ is uniformly stable if for all data sets $S, S'$ such that $S, S'$ differ in at most one example, we have

$$
\sup_x \mathbb{E}_{\mathcal{M}} [g(\mathcal{M}(S); x) - g(\mathcal{M}(S'); x)] \leq \epsilon_{\text{stab}}.
$$

(9)

Given the definition of uniform stability, we show that distributional RL under the histogram distributional loss is $\epsilon_{\text{stab}}$-uniformly stable while running SGD in Theorem 2.

**Theorem 2.** *(Stable Optimization for Distributional RL)* Suppose that we run SGD under $\mathcal{L}_\theta$ in Eq. 8 with step sizes $\lambda_t \leq 2/kl^2$ for $T$ steps. Assume $\|x(s)\| \leq l$ for each state $s$, then we have $\mathcal{L}_\theta$ satisfies the uniform stability in Definition 1 with $\epsilon_{\text{stab}} \leq 4klT/n$.

Please refer to the proof of Theorem 2 in Appendix G. The stable optimization has multiple advantages. For examples, in deep learning literature [Hardt et al., 2016], stable optimization can guarantee the generalization with an $\epsilon_{\text{stab}}$-bounded generalization gap. In reinforcement learning, algorithms with more stability tend to achieve better final performance [Bjorck et al., 2021, Li and Pathak, 2021].

In summary, under the histogram density approximation, the optimization process within each update of neural Z-fitted iteration in Eq. 3 for distributional RL is uniformly stable with the stability errors shrinking at the rate of $O(n^{-1})$. This stability blessing is owing to the smooth and convex properties from histogram distributional loss, but we contend that similar conclusions can still hold in the non-convex case based on the conclusion in [Hardt et al., 2016]. By contrast, the least-squared loss in neural Q-fitted iteration in Eq. 2 for expectation-based RL is without these smooth properties and can not yield the stable optimization. The instability in the optimization process for classical RL can be used to explain its inferiority to distributional RL.

### 4.2 Acceleration Effect

We investigate the acceleration effect stemming from the risk-aware entropy regularization for distributional RL by comparing its sample complexity with classical RL when achieving the stationary optimization point.
To conduct this comparison, we still leverage expectation decomposition of \( F^{s,a}(x) \) proposed in Eq. 4, where \( \rho^{s,a}(x) = (1 - \epsilon)\delta_{\{x=\mathbb{E}[Z^\pi(s,a)]\}}(x) + \epsilon\mu^{s,a}(x) \). Within each update in the neural Z-fitted iteration, the target is to minimize \( \frac{1}{k}\sum_{i=1}^{k}L_\theta(s_i,a_i) \). We denote \( G_k(\theta) \) as the expectation of \( L_\theta \), i.e., \( G_k(\theta) = \mathbb{E}_{(s,a)\sim \rho^s}[L_\theta(s,a)] \). Based on the setting in Section 4.1, the convex and smooth properties in Lemma 3 still hold for \( G_k(\theta) \). We use \( G(\theta) \) for \( G_k(\theta) \) for simplicity. As the KL divergence enjoys the property of unbiased gradient estimates, we let the variance of its stochastic gradient over the expectation be bounded, i.e.,

\[
\mathbb{E}_{(s,a)\sim \rho^s} \left[ \| \nabla L_\theta(\delta_{\{x=\mathbb{E}[Z^\pi(s,a)]\}}, q_\theta^{s,a}) \| - \nabla G(\theta) \|^2 \right] = \sigma^2.
\]

Next, we characterize the approximation degree of \( q_\theta^{s,a} \) to the target distribution \( \mu^{s,a} \) by measuring its variance as \( \kappa\sigma^2 \):

\[
\mathbb{E}_{(s,a)\sim \rho^s} \left[ \| \nabla L_\theta(\mu^{s,a}, q_\theta^{s,a}) \| - \nabla G(\theta) \|^2 \right] = \kappa\sigma^2. \tag{10}
\]

Notably, \( \kappa \) can be used to measure the approximation error between \( q_\theta^{s,a} \) and \( \mu^{s,a} \). A favorable approximation of \( q_\theta^{s,a} \) to \( \mu^{s,a} \) would lead to a small \( \kappa \) that contributes to the acceleration effect of distributional RL as shown in Theorem 3. Based on Eq. 10, we immediately have Lemma 4.

**Lemma 4.** Based on the expectation decomposition in Eq. 4, and Eq. 10, we have:

\[
\mathbb{E}_{(s,a)\sim \rho^s} \left[ \| \nabla L_\theta(\rho^{s,a}, q_\theta^{s,a}) \| - \nabla G(\theta) \|^2 \right] \\
\leq (1 - \epsilon)\sigma^2 + \epsilon\kappa\sigma^2. \tag{11}
\]

Please refer to Appendix H for the proof. Before comparing the sample complexity in the optimization process in both expectation-based and distributional RL, we provide the definition of the first-order \( \tau \)-stationary point.

**Definition 2.** (First-order \( \tau \)-Stationary Point) While solving \( \min_\theta G(\theta) \), the updated parameters \( \theta_T \) after \( T \) steps is a first-order \( \tau \)-stationary point if \( \| \nabla G(\theta_T) \| \leq \tau \), where the small \( \tau \) is in \((0, 1)\).

Based on Definition 2, we formally characterize the acceleration effect determined by the risk-aware regularization for distributional RL in Theorem 3.

**Theorem 3.** (Acceleration Effect) While running SGD to minimize \( L_\theta \) with or without risk-aware regularization (\( \alpha = 0 \)) within neural Z-fitted iteration via Eq. 5, we assume the step size \( \lambda = 1/kt^2 \), \( \epsilon = 1/(1 + \tau) \), and the sample is uniformly drawn from \( T \) samples, then:

1. \( T = O\left( \frac{1}{\tau^2} \right) \) when we minimize \( L_\theta(\delta_{\{x=\mathbb{E}[Z^\pi(s,a)]\}}, q_\theta^{s,a}) \) in order to achieve \( \tau \)-stationary point for expectation-based RL.

2. When \( \kappa \leq \frac{\epsilon^2}{\lambda\tau^2} \) and let \( T = \frac{4G(\theta_0)}{\lambda\tau^2} = O\left( \frac{1}{\tau^2} \right) \), minimizing \( L_\theta(\rho^{s,a}, q_\theta^{s,a}) \) with risk-aware entropy regularization for distributional RL converges to a \( \tau \)-stationary point in expectation.

3. When \( \kappa > \frac{\epsilon^2}{\lambda\tau^2} \) and let \( T = \frac{G(\theta_0)}{\lambda\tau^2} = O\left( \frac{1}{\tau^2} \right) \), minimizing \( L_\theta(\rho^{s,a}, q_\theta^{s,a}) \) with risk-aware entropy regularization for distributional RL does not converge to a \( \tau \)-stationary point, but we have \( \mathbb{E} \left[ \| \nabla G(\theta_T) \| \right] \leq O(\kappa) \).

The proof is provided in Appendix I. Theorem 3 is inspired by the convergence analysis of label smoothing [Xu et al., 2020]. Importantly, Theorem 3 reveals two cases of acceleration effects for distributional RL algorithms. In the first common scenario ((1) in Theorem 3), when there is only a small approximation error between \( q_\theta^{s,a} \) and \( p^{s,a} \) (or \( \mu^{s,a} \)) with a small \( \kappa \). As such, a reduction of sample complexity from \( O\left( \frac{1}{\tau^2} \right) \) in expectation-based RL to \( O\left( \frac{1}{\tau^2} \right) \) can be obtained, therefore speeding up the convergence of RL algorithms. In the second scenario especially for some challenging tasks, where we can also attain a relatively large approximation error with a large \( \kappa \), distributional RL algorithms may fail to speed up the convergence, but still potentially achieve a reasonable performance as we can guarantee \( \mathbb{E} \left[ \| \nabla G(\theta_T) \| \right] \leq O(\kappa) \). Theses theoretical results also coincide with our past empirical observations [Dabney et al., 2018b], where distributional RL algorithms are even inferior to classical RL on a small proportion of Atari games.

**Extension of Representation Effect.** We also conduct some empirical analysis of distributional RL from the perspective of representation. In Appendix J, we find that distributional RL encourages state representation from the same action class classified by the policy in tighter clusters.
5 Experiments

We perform extensive experiments on eight continuous control MuJoCo games in OpenAI Gym to demonstrate both the regularization and optimization analysis in Sections 3 and 4. To evaluate the risk-aware entropy regularization effect of distributional RL, an ablation study is conducted based on Soft Actor Critic (SAC) [Haarnoja et al., 2018] and distributional Soft Actor Critic (dSAC) [Ma et al., 2020] to disentangle the impact of vanilla entropy in SAC that encourages the action diversity for high entropy exploration, and risk-aware entropy in distributional RL that leverage more value distribution knowledge for better risk-aware exploration. For the implementation, we leverage the quantiles generation strategy in Implicit Quantile Network (IQN) [Dabney et al., 2018a] in dSAC as demonstrated in the original paper [Ma et al., 2020]. Hyper-perameters and more implementation details are listed in Appendix K.

5.1 Regularization Effect for distributional RL

In Section 3, we analyze that the superiority of distributional RL can be attributed into the risk-aware regularization that encourages the policy to visit states whose value distribution is far beyond only its expectation. This resulting distributional regularization effect may be highly disjoint with the vanilla entropy regularization proposed in maximum entropy and SAC algorithm. To verify their different effects as well as mutual influence in the training process, we conduct an ablation study on SAC and distributional SAC by eliminating their vanilla entropy part, respectively. We denote their versions without vanilla entropy as SAC Noent and dSAC Noent.

From Figure 2, it manifests that the impacts of distributional RL regularization and vanilla entropy are highly divergent as each algorithm achieves vastly different performance compared with the others. Interestingly, we find SAC (red lines) is inferior to its no entropy version (blue lines) on some complex tasks, including Humanoid (which has a 17-dimensional action space) and BipealWalkerHardcore (hard for exploration). This may result from the less desired exploration in SAC analyzed in [Han and Sung, 2021], which can be further improved by their proposed max-min entropy framework. Remarkably, distributional SAC without entropy (dSAC Noent) in the cyan color outperforms other algorithms almost across all games especially on BipealWalkerHardcore, where the impact of risk-aware regularization in dSAC Noent significantly promotes a better exploration compared with the vanilla entropy. Moreover, it is also worthy of noting that the leverage of both risk-aware entropy regularization and vanilla entropy regularization may hurt each other, e.g., on Ant, Halfcheetah and

![Figure 2: Learning curve of SAC, distributional SAC (dSAC), SAC without vanilla entropy (SAC Noent) and distributional SAC without vanilla entropy (dSAC Noent) over 5 seeds with smooth size 5 across eight MuJoCo games.](image)


Swimmer games, where dSAC (black line) is significantly inferior to dSAC Noent (cyan line). This is owing to the fact that SAC encourages the policy to visit states with high entropy, while distributional RL promote the risk-aware exploration to visit states with more informative value distribution beyond its expectation. We argue that these two different regularization effects are likely to lead to divergent optimization directions, thus interfering with each other eventually.

**Acceleration effect of distributional RL.** If we consider the interference effect of vanilla entropy and risk-aware entropy regularization from distributional RL while jointly leveraging them, we can eliminate the vanilla entropy to investigate the acceleration effect of distributional RL. Figure 1 also suggests that distributional SAC without entropy (dSAC Noent) in the cyan line is consistently advantageous to SAC without entropy (SAC Noent) in the blue color, in terms of both better final performance and reduction of sample complexity. These empirical observations verified the acceleration effect of case (2) in Theorem 3. We leave a more rigorous demonstration of both cases regarding acceleration effect as future works.

### 5.2 Stable Optimization of distributional RL

To probe the training stability of distributional RL compared with its expectation-based counterpart, we plot the gradient norm of the critic and actor loss for all algorithms we considered on Ant, Humanoid and Walker games. Results are averaged over five runs as shown in Figure 3.

From Figure 3, it turns out that regardless of whether the vanilla entropy is leveraged or not, distributional RL algorithms, including dSAC and dSAC Noent, have a more stable gradient magnitude as opposed to their expectation-based algorithms across both actor and critic on the three games. Note that the stable optimization behavior of algorithms in Figure 3 seems to be uncorrelated with their performance as suggested in Figure 2. For example, dSAC (black line) is dramatically inferior to the other algorithms regarding the final performance, but it still enjoys the smallest gradient magnitude on the Ant game. In summary, distributional RL algorithms embraces more stable optimization process compared with the classical RL counterparts.
6 Discussions and Conclusion

Our interpretation of distributional RL from perspectives of regularization and optimization is mainly established on KL divergence as $d_p$, while a direct analysis based on Wasserstein distance is likely to render a more compelling analysis, albeit being theoretically tricky.

In this paper, we illuminate the superiority of distributional RL over expectation-based RL from the perspectives of regularization and optimization. We derive a risk-aware entropy regularization for distributional RL within Neural Z-fitted iteration framework, and make a rigorous comparison with vanilla entropy in maximum entropy RL. Moreover, stable optimization property and acceleration effect of distributional RL are also analyzed. Our research contributes to a deeper understanding of distributional RL algorithm.

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A Definition of distances and Contraction

Definition of distances. Given two random variables $X$ and $Y$, $p$-Wasserstein metric $W_p$ between the distributions of $X$ and $Y$ is defined as

$$W_p(X,Y) = \left( \int_0^1 |F_X^{-1}(\omega) - F_Y^{-1}(\omega)|^p d\omega \right)^{1/p} = \|F_X^{-1} - F_Y^{-1}\|_p,$$

which $F^{-1}$ is the inverse cumulative distribution function of a random variable with the cumulative distribution function as $F$. Further, $\ell_p$ distance [Elie and Arthur, 2020] is defined as

$$\ell_p(X,Y) := \left( \int_{-\infty}^{\infty} |F_X(\omega) - F_Y(\omega)|^p d\omega \right)^{1/p} = \|F_X - F_Y\|_p$$

The $\ell_p$ distance and Wasserstein metric are identical at $p = 1$, but are otherwise distinct. Note that when $p = 2$, $\ell_p$ distance is also called Cramér distance [Bellemare et al., 2017b] $d_C(X,Y)$. Also, the Cramér distance has a different representation given by

$$d_C(X,Y) = \mathbb{E}|X - Y| - \frac{1}{2}\mathbb{E}|X - X'| - \frac{1}{2}\mathbb{E}|Y - Y'|,$$

where $X'$ and $Y'$ are the i.i.d. copies of $X$ and $Y$. Energy distance [Szekely, 2003, Ziel, 2020] is a natural extension of Cramér distance to the multivariate case, which is defined as

$$d_E(X,Y) = \mathbb{E}\|X - Y\|_2 - \frac{1}{2}\mathbb{E}\|X - X'\| - \frac{1}{2}\mathbb{E}\|Y - Y'\|,$$

where $X$ and $Y$ are multivariate. Moreover, the energy distance is a special case of the maximum mean discrepancy (MMD), which is formulated as

$$\text{MMD}(X,Y;k) = (\mathbb{E}[k(X,X')] + \mathbb{E}[k(Y,Y')] - 2\mathbb{E}[k(X,Y)])^{1/2}$$

where $k(\cdot, \cdot)$ is a continuous kernel on $X$. In particular, if $k$ is a trivial kernel, MMD degenerates to energy distance. Additionally, we further define the supreme MMD, which is a functional $\mathcal{P}(X)^{S \times A} \times \mathcal{P}(X)^{S \times A} \to \mathbb{R}$ defined as

$$\text{MMD}_{\infty}(\mu,\nu) = \sup_{(x,a) \in S \times A} \text{MMD}_{\infty}(\mu(x,a),\nu(x,a))$$

Proof of Contraction.

- Contraction under supreme form of Wasserstein distance is provided in Lemma 3 [Bellemare et al., 2017a].
- Contraction under supreme form of $\ell_p$ distance can refer to Theorem 3.4 [Elie and Arthur, 2020].
- Contraction under MMD$_{\infty}$ is provided in Lemma 6 [Nguyen et al., 2020].

B Proof of Proposition 1

Proof. (1) We recap three crucial properties of a divergence metric. The first is scale sensitive (S) (of order $\beta$, $\beta > 0$), i.e., $d_p(cX,cY) \leq |c|^\beta d_p(X,Y)$. The second property is shift invariant (I), i.e., $d_p(A + X,A + Y) \leq d_p(X,Y)$. The last one is unbiased gradient (U). We use $p$ and $q$ to denote the density function of two random variables $X$ and $Y$, and thus $D_{\text{KL}}(X,Y)$ is defined as $D_{\text{KL}}(X,Y) = \int_{-\infty}^{\infty} p(x) \frac{p(x)}{q(x)} dx$. Firstly, we show that $D_{\text{KL}}(X,Y)$ is NOT scale sensitive:

$$D_{\text{KL}}(aX,aY) = \int_{-\infty}^{\infty} \frac{1}{a} p\left(\frac{x}{a}\right) \log \frac{\frac{1}{a} p\left(\frac{x}{a}\right)}{\frac{1}{aq}\left(\frac{x}{a}\right)} dx$$

$$= \int_{-\infty}^{\infty} p(y) \log \frac{p(y)}{q(y)} dy$$

$$= D_{\text{KL}}(X,Y), \text{ with } \beta = 0$$
We further show that $D_{KL}(X, Y)$ is shift invariant:

$$D_{KL}(A + X, A + Y) = \int_{-\infty}^{\infty} p(x - A) \log \frac{p(x - A)}{q(x - A)} \, dx$$

$$= \int_{-\infty}^{\infty} p(y) \log \frac{p(y)}{q(y)} \, dy$$

$$= D_{KL}(X, Y) \quad (19)$$

Moreover, it is well-known that KL divergence has unbiased sample gradients [Bellemare et al., 2017b]. The supreme $D_{KL}$ is a functional $\mathcal{P}(\mathcal{X})^{S \times A} \times \mathcal{P}(\mathcal{X})^{S \times A} \rightarrow \mathbb{R}$ defined as

$$D_{KL}^\infty(\mu, \nu) = \sup_{(s, a) \in S \times A} D_{KL}(\mu(x, a), \nu(x, a)) \quad (20)$$

Therefore, we prove $\mathfrak{T}^\pi$ is at best a non-expansive operator under the supreme form of $D_{KL}$:

$$D_{KL}^\infty(\mathfrak{T}^\pi Z_1, \mathfrak{T}^\pi Z_2) = \sup_{s, a} D_{KL}(\mathfrak{T}^\pi Z_1(s, a), \mathfrak{T}^\pi Z_2(s, a))$$

$$= D_{KL}(R_1(s, a) + \gamma Z_1(s', a'), R_2(s, a)) + \gamma Z_2(s', a'))$$

$$= D_{KL}(Z_1(s', a'), Z_2(s', a')) \quad (21)$$

$$\leq \sup_{s', a'} D_{KL}(Z_1(s', a'), Z_2(s', a'))$$

$$= D_{KL}^\infty(Z_1, Z_2)$$

Then we have $D_{KL}^\infty(\mathfrak{T}^\pi Z_1, \mathfrak{T}^\pi Z_2) \leq D_{KL}^\infty(Z_1, Z_2)$, implying that $\mathfrak{T}^\pi$ is a non-expansive operator under $D_{KL}^\infty$.

(2) By the definition of $D_{KL}^\infty$, we have $\sup_{s, a} \delta(Z_n(s, a), Z(s, a)) = 0$ implies $D_{KL}(Z_n, Z) \rightarrow 0$. $D_{KL}(Z_n, Z) \rightarrow 0$ implies the total variation distance $\delta(Z_n, Z) \rightarrow 0$ according to a straightforward application of Pinsker’s inequality

$$\delta(Z_n, Z) \leq \sqrt{\frac{1}{2} D_{KL}(Z_n, Z) \rightarrow 0} \quad (22)$$

$$\delta(Z, Z_n) \leq \sqrt{\frac{1}{2} D_{KL}(Z, Z_n) \rightarrow 0} \quad (23)$$

Based on Theorem 2 in WGAN [Arjovsky et al., 2017], $\delta(Z_n, Z) \rightarrow 0$ implies $W_p(Z_n, Z) \rightarrow 0$. This is trivial by recalling the fact that $\delta$ and $W$ give the strong an weak topologies on the dual of $(\mathcal{C}(\mathcal{X}), \|\cdot\|_{\infty})$ when restricted to $\text{Prob}(\mathcal{X})$.

(3) The conclusion holds because the $\mathfrak{T}^\pi$ degenerates to $T^\pi$ regardless of the metric $d_p$. Specifically, due to the linearity of expectation, we obtain that

$$\|E[\mathfrak{T}^\pi Z_1] - E[\mathfrak{T}^\pi Z_2]\|_{\infty} = \|T^\pi E[\mathcal{Z}_1] - T^\pi E[\mathcal{Z}_2]\|_{\infty} \leq \gamma \|E[\mathcal{Z}_1] - E[\mathcal{Z}_2]\|_{\infty}. \quad (24)$$

This implies that the expectation of $\mathcal{Z}$ under $D_{KL}$ exponentially converges to the expectation of $\mathcal{Z}^*$, i.e., $\gamma$-contraction.

C Proof of Proposition 2

**Proof.** Firstly, given a fixed $p(x)$ we know that minimizing $D_{KL}(p, q_\theta)$ is equivalent to minimizing $\mathcal{H}(p, q_\theta)$ by following

$$D_{KL}(p, q_\theta) = \int_{-\infty}^{+\infty} p(x) \log \frac{p(x)}{q_\theta(x)} \, dx$$

$$= -\int_{-\infty}^{+\infty} p(x) \log q_\theta(x) \, dx - (-\int_{-\infty}^{+\infty} p(x) \log p(x) \, dx) \quad (24)$$

$$= \mathcal{H}(p, q_\theta) - \mathcal{H}(p)$$

$$\propto \mathcal{H}(p, q_\theta)$$
Based on $\mathcal{H}(p, q_\theta)$, we can derive the objective function within each neural Z-fitted iteration as

$$
\frac{1}{n} \sum_{i=1}^{n} \mathcal{H}(p^{s_i,a_i}(x), q_\theta^{s_i,a_i}(x)) \\
= \frac{1}{n} \sum_{i=1}^{n} \left( \int_{-\infty}^{+\infty} p^{s_i,a_i}(x) \log q_\theta^{s_i,a_i}(x) \, dx \right) \\
= \frac{1}{n} \sum_{i=1}^{n} \left( \int_{-\infty}^{+\infty} \left( (1 - \epsilon) \delta_{x=E[Z^\pi(s,a)]}(x) + \epsilon \mu^{s_i,a_i}(x) \right) \log q_\theta^{s_i,a_i}(x) \, dx \right) \\
= \frac{1}{n} \sum_{i=1}^{n} \left[ (1 - \epsilon) \left( \int_{-\infty}^{+\infty} \delta_{x=E[Z^\pi(s,a)]}(x) \log q_\theta^{s_i,a_i}(x) \, dx \right) + \epsilon \mathcal{H}(\mu^{s_i,a_i}, q_\theta^{s_i,a_i}) \right] \\
= \frac{1}{n} \sum_{i=1}^{n} \left[ (1 - \epsilon) \mathcal{H}(\delta_{x=E[Z^\pi(s,a)]}, q_\theta^{s_i,a_i}) + \epsilon \mathcal{H}(\mu^{s_i,a_i}, q_\theta^{s_i,a_i}) \right] \\
\propto \frac{1}{n} \sum_{i=1}^{n} \mathcal{H}(\delta_{x=E[Z^\pi(s,a)]}, q_\theta^{s_i,a_i}) + \alpha \mathcal{H}(\mu^{s_i,a_i}, q_\theta^{s_i,a_i}), \text{ where } \alpha = \frac{\epsilon}{1 - \epsilon} > 0 \\
= \frac{1}{n} \sum_{i=1}^{n} \left( \int_{-\infty}^{+\infty} \log q_\theta^{s_i,a_i}(E[Z^\pi(s,a)]) \, dx + \alpha \mathcal{H}(\mu^{s_i,a_i}, q_\theta^{s_i,a_i}) \right)
$$

**D Proof of Convergence of Soft Distributional Policy Iteration in Theorem 1**

**D.1 Proof of Soft Distributional Policy Evaluaiton in Lemma 1**

*Proof.* Firstly, according the definition of our derived soft value function in Eq. 7, we can rearrange it as:

$$
V(s_{t+1}) = E_{\pi, s_{t+1}, a_{t+1}} \left[ Q(s_{t+1}, a_{t+1}) + \log q_\theta^{s_{t+1}, a_{t+1}}(x) \right] \\
= E_{\pi, a_{t+1}} \left[ Q(s_{t+1}, a_{t+1}) \right] + E_{\pi, s_{t+1}, a_{t+1}} \left[ \log q_\theta^{s_{t+1}, a_{t+1}}(x) \right] \\
= E_{\pi, a_{t+1}} \left[ Q(s_{t+1}, a_{t+1}) \right] - E_{\pi, a_{t+1}} \left[ \int_{-\infty}^{+\infty} \mu^{s_{t+1}, a_{t+1}}(x) \log q_\theta^{s_{t+1}, a_{t+1}}(x) \, dx \right] \\
= E_{\pi, a_{t+1}} \left[ Q(s_{t+1}, a_{t+1}) \right] - \mathcal{H} \left( \mu^{s_{t+1}, a_{t+1}}, q_\theta^{s_{t+1}, a_{t+1}} \right) \\
= E_{\pi, a_{t+1}} \left[ Q(s_{t+1}, a_{t+1}) \right] - \mathcal{H} \left( \mu^{s_{t+1}, a_{t+1}}, q_\theta^{s_{t+1}, a_{t+1}} \right)
$$

Further, we plug in $V(s_{t+1})$ into RHS of the iteration in Eq. 7, then we obtain

$$
\mathcal{T}_d^\pi Q(s_t, a_t) = r(s_t, a_t) + \gamma E_{s_{t+1} \sim \rho^\pi} [V(s_{t+1})] \\
= r(s_t, a_t) - \gamma E_{(s_{t+1}, a_{t+1}) \sim \rho^\pi} \left[ \mathcal{H} \left( \mu^{s_{t+1}, a_{t+1}}, q_\theta^{s_{t+1}, a_{t+1}} \right) \right] + \gamma E_{(s_{t+1}, a_{t+1}) \sim \rho^\pi} [Q(s_{t+1}, a_{t+1})] \\
\triangleq r^\pi(s_t, a_t) + \gamma E_{(s_{t+1}, a_{t+1}) \sim \rho^\pi} [Q(s_{t+1}, a_{t+1})],
$$

where $r^\pi(s_t, a_t) \triangleq r(s_t, a_t) - \gamma E_{(s_{t+1}, a_{t+1}) \sim \rho^\pi} \left[ \mathcal{H} \left( \mu^{s_{t+1}, a_{t+1}}, q_\theta^{s_{t+1}, a_{t+1}} \right) \right]$ is the entropy augmented reward we redefine. Applying the standard convergence results for policy evaluation [Sutton and Barto, 2018], we can attain that this Bellman updating under $\mathcal{T}_d^\pi$ is convergent under the assumption of $|A| < \infty$ and bounded entropy augmented rewards $r^\pi$. 

\[\square\]

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D.2 Proof of Soft Distributional Policy Improvement in Lemma 2

Our proof is based on soft policy improvement [Haarnoja et al., 2018]. Let $Q^{\pi_{\text{old}}}$ and $V^{\pi_{\text{new}}}$ be our action-value and state-value function, and let the optimized $\pi_{\text{new}}$ be defined as

$$
\pi_{\text{new}} = \arg \min_{\pi \in \Pi} D_{\text{KL}} \left( \mathcal{H}(\mu^{s_t,a_t}, q_0^{s_t,a_t}) \| \exp \left( \frac{Q^{\pi_{\text{old}}}(s_t, \cdot)}{\Delta^{\pi_{\text{old}}}(s_t)} \right) \right)
$$

$$
= \arg \min_{\pi \in \Pi} D_{\text{KL}} \left( \mathcal{H}(\mu^{s_t,a_t}, q_0^{s_t,a_t}) \| \exp \left( Q^{\pi_{\text{old}}}(s_t, \cdot) - \log \Delta^{\pi_{\text{old}}}(s_t) \right) \right),
$$

After the optimization, it must be the case that

$$
E_{a_t \sim \pi_{\text{new}}} \left[ \mathcal{H}(\mu^{s_t,a_t}, q_0^{s_t,a_t}) - Q^{\pi_{\text{old}}}(s_t, a_t) - \log \Delta^{\pi_{\text{old}}}(s_t) \right] \
\leq E_{a_t \sim \pi_{\text{old}}} \left[ \mathcal{H}(\mu^{s_t,a_t}, q_0^{s_t,a_t}) - Q^{\pi_{\text{old}}}(s_t, a_t) - \log \Delta^{\pi_{\text{old}}}(s_t) \right].
$$

As the partition function $\Delta^\pi$ only depends on the states, we can reduce the inequalities to

$$
E_{a_t \sim \pi_{\text{old}}} \left[ Q^{\pi_{\text{old}}}(s_t, a_t) - \mathcal{H}(\mu^{s_t,a_t}, q_0^{s_t,a_t}) \right] \
= E_{a_t \sim \pi_{\text{old}}} \left[ Q^{\pi_{\text{old}}}(s_t, a_t) - \mathcal{H}(\mu^{s_t,a_t}, q_0^{s_t,a_t}) \right].
$$

Furthermore, we consider the Bellman equation via the soft distributional Bellman operator $T^\pi_{sd}$:

$$
Q^{\pi_{\text{old}}}(s_t, a_t) \triangleq r(s_t, a_t) + \gamma E_{s_{t+1} \sim \rho} \left[ Q^{\pi_{\text{old}}}(s_{t+1}, \cdot) \right] \
\leq r(s_t, a_t) + \gamma E_{s_{t+1} \sim \rho} \left[ Q^{\pi_{\text{old}}}(s_{t+1}, a_{t+1}) - \mathcal{H}(\mu^{s_{t+1},a_{t+1}}, q_0^{s_{t+1},a_{t+1}}) \right] \
= r_{\pi_{\text{new}}}(s_t, a_t) + \gamma E_{s_{t+1} \sim \rho} \left[ Q^{\pi_{\text{old}}}(s_{t+1}, a_{t+1}) - \mathcal{H}(\mu^{s_{t+1},a_{t+1}}, q_0^{s_{t+1},a_{t+1}}) \right],
$$

where we have repeated expanded $Q^{\pi_{\text{old}}}$ on the RHS by applying the soft distributional Bellman operator. Convergence to $Q^{\pi_{\text{new}}}$ follows from Lemma 1.

D.3 Proof of Soft Distributional Policy Iteration in Theorem 1

The proof is similar to soft policy iteration [Haarnoja et al., 2018]. For the completeness, we provide the proof here. By Lemma 2, as the number of iteration increases, the sequence $Q^\pi$ at $i$-th iteration is monotonically increasing. Since we assume the risk-aware entropy is bounded by $M$, the $Q^\pi$ is thus bounded as the rewards are bounded. Hence, the sequence will converge to some $\pi^*$. Further, we prove that $\pi^*$ is in fact optimal. At the convergence point, for all $\pi \in \Pi$, it must be case that:

$$
D_{\text{KL}} \left( \mathcal{H}(\mu^{s_t,\pi^*(\cdot|s_t)}, q_0^{s_t,\pi^*(\cdot|s_t)}) \| \exp \left( \frac{Q^{\pi^*}(s_t, \cdot)}{\Delta^{\pi^*}(s_t)} \right) \right) \
\leq D_{\text{KL}} \left( \mathcal{H}(\mu^{s_t,\pi^*(\cdot|s_t)}, q_0^{s_t,\pi^*(\cdot|s_t)}) \| \exp \left( Q^{\pi^*}(s_t, \cdot) \right) \right).
$$

According to the proof in Lemma 2, we can attain $Q^{\pi^*}(s_t, a_t) > Q^\pi(s_t, a_t)$ for $(s_t, a_t)$. That is to say, the soft value of any other policy in $\Pi$ is lower than the converged policy, indicating that $\pi^*$ is optimal.

E Derivation of Histogram Distributional Loss

We show the derivation details of the Histogram distribution loss starting from KL divergence between $p$ and $q_0$. $p_i$ is the cumulative probability increment of target distribution $\{Y_i\}_{i \in [n]}$ within the $i$-th bin, and $q_0$ corresponds to a (normalized) histogram, and has density values $\frac{\ell_i(x(i))}{w_t}$ per bin. Thus,
we have:

\[
D_{\text{KL}}(p^{s,a}, q_0^{s,a}) = - \int_a^b p^{s,a}(y) \log q_0^{s,a}(y) dy
= - \sum_{i=1}^k \int_{l_i}^{l_i+w_i} p^{s,a}(y) \log \frac{f_0^\theta(x(s))}{w_i} dy
= - \sum_{i=1}^k \log \frac{f_0^\theta(x(s))}{w_i} \left( F^{s,a}(l_i + w_i) - F^{s,a}(l_i) \right)
\]

\[
= - \sum_{i=1}^k p_i^{s,a} \log f_0^\theta(x(s))
\]

where the last equality holds because the width parameter \(w_i\) can be ignored for this minimization problem.

**F Proof of Lemma 3**

**Proof.** For the histogram distributional loss below,

\[
\mathcal{L}_\theta(s,a) = - \sum_{i=1}^k p_i^{s,a} \log f_i^\theta(x(s)), \text{ where } f_i^\theta(x(s)) = \frac{\exp(x(s)\top \theta_i)}{\sum_{j=1}^k \exp(x(s)\top \theta_j)}
\]

we firstly prove its convexity. Note that \(- \log \frac{\exp(x(s)\top \theta_i)}{\sum_{j=1}^k \exp(x(s)\top \theta_j)} = \log \sum_{j=1}^k \exp(x(s)\top \theta_i) - x(s)\top \theta_i\), the first term is Log-sum-exp, which is convex (see Convex optimization by Boyd and Vandenberghe), and the second term is affine function. Thus, \(\mathcal{L}_\theta(s,a)\) is convex.

Secondly, we show that \(\mathcal{L}_\theta(s,a)\) is \(kl\)-Lipschitz. We compute the gradient of the Histogram distributional loss regarding \(\theta_i\):

\[
\frac{\partial}{\partial \theta_i} \sum_{j=1}^k p_j^{s,a} \log f_j^\theta(x(s))
= \sum_{j=1}^k p_j^{s,a} \frac{1}{f_j^\theta(x(s))} \nabla_{\theta_i} f_j^\theta(x(s))
= \sum_{j=1}^k p_j^{s,a} \frac{1}{f_j^\theta(x(s))} f_i^\theta(x(s))(\delta_{ij} - f_j^\theta(x(s)))x(s)
= \left( p_i^{s,a}(1 - f_i^\theta(x(s))) - \sum_{j \neq i}^k p_j^{s,a} f_j^\theta(x(s)) \right) x(s)
= (p_i^{s,a} - p_i^{s,a} f_i^\theta(x(s)) - (1 - p_i^{s,a}) f_i^\theta(x(s))) x(s)
= (p_i^{s,a} - f_i^\theta(x(s))) x(s)
\]

(30)
where $\delta_{ij} = 1$ if $i = j$, otherwise 0. Then, as we have $\|x(s)\| \leq l$, we bound the norm of its gradient

$$\| \frac{\partial}{\partial \theta} \sum_{j=1}^{k} p_j \log f_j^\theta(x(s)) \|$$

$$\leq \sum_{i=1}^{k} \| \frac{\partial}{\partial \theta_i} \sum_{j=1}^{k} p_j \log f_j^\theta(x(s)) \|$$

$$= \sum_{i=1}^{k} \| (p_i^{s,a} - f_i^\theta(x(s))) x(s) \|$$

$$\leq \sum_{i=1}^{k} |p_i^{s,a} - f_i^\theta(x(s))||x(s)||$$

$$\leq kl$$

The last equality satisfies because $|p_i - f_i^\theta(x(s))|$ is less than 1 and even smaller. Therefore, we obtain that $\mathcal{L}_\theta$ is $kl$-Lipschitz.

Lastly, we show that $\mathcal{L}_\theta$ is $kl^2$-Lipschitz smooth. A lemma is that $\log(1 + \exp(x))$ is $\frac{1}{4}$-smooth as its second-order gradient is bounded by $\frac{1}{4}$, and if $g(w)$ is $\beta$-smooth w.r.t. $w$, then $g(x, w)$ is $\beta\|x\|^2$-smooth. Based on this knowledge, we firstly focus on the 1-dimensional case of function $\log f_j^\theta(z)$. As we have derived, we know that $\frac{\partial}{\partial \theta_i} \log f_j^\theta(z) = \delta_{ij} - f_i^\theta(z)$. Then the second-order gradient is $\frac{\partial^2 \log f_j^\theta(z)}{\partial \theta_i \partial \theta_j} = -(\delta_{ik} - f_i^\theta(z)) = f_i^\theta(z) - 1$ if $i = k$, otherwise $f_i^\theta(z)$. Clearly, $|\frac{\partial^2 \log f_j^\theta(z)}{\partial \theta_i \partial \theta_j}| \leq 1$, which implies that $\log f_j^\theta(z)$ is 1-smooth. Thus, $\log f_j^\theta((x, \theta_i))$ is $\|x\|^2$-smooth, or $l^2$-smooth. Further, $\sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(x(s))$ is also $l^2$-smooth as we have

$$\| \nabla_{\theta_i} \sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(\mu) - \nabla_{\theta_i} \sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(\nu) \|$$

$$\leq \sum_{j=1}^{k} p_j^{s,a} \| \nabla_{\theta_i} \log f_j^\theta(\mu) - \nabla_{\theta_i} \log f_j^\theta(\nu) \|$$

$$\leq \sum_{j=1}^{k} p_j^{s,a} \cdot l^2 \| \mu - \nu \|$$

for each $\mu$ and $\nu$. Therefore, we further have

$$\| \nabla_{\theta} \sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(\mu) - \nabla_{\theta} \sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(\nu) \|$$

$$\leq \sum_{i=1}^{k} \| \nabla_{\theta_i} \sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(\mu) - \nabla_{\theta_i} \sum_{j=1}^{k} p_j^{s,a} \log f_j^\theta(\nu) \|$$

$$\leq \sum_{i=1}^{k} l^2 \| \mu - \nu \|$$

$$= kl^2 \| \mu - \nu \|$$

Finally, we conclude that $\mathcal{L}_\theta(s, a)$ is $kl^2$-smooth.

\[\square\]

**G  Proof of Theorem 2**

*Proof.* Consider the stochastic gradient descent rule as $G_{\lambda, \mathcal{L}}(\theta) = \theta - \lambda \nabla_{\theta} \mathcal{L}(\theta)$. Firstly, we provide two definitions about $\mathcal{L}_\theta$ for the following proof.
Definition 3. (σ-bounded) An update rule is σ-bounded if \( \sup_{\theta} \| \theta - \lambda \nabla_{\theta} \mathcal{L}(\theta) \| \leq \sigma \).

Definition 4. (η-expansive) An update rule is η-expansive if \( \sup_{v,w} \frac{\| G_{\lambda}(v) - G_{\lambda}(w) \|}{\| u - w \|} \leq \eta \).

Lemma 5. (Grow Recursion, Lemma 2.5 [Hardt et al., 2016]) Fix an arbitrary sequence of updates \( G_1, ..., G_T \) and another sequence \( G'_1, ..., G'_T \). Let \( \theta_0 = \theta'_0 \) be the starting point and define \( \delta_t = \| \theta'_t - \theta_t \| \), where \( \theta_t \) and \( \theta'_t \) are defined recursively through

\[
\theta_{t+1} = G_{\lambda,\mathcal{L}}(\theta_t), \quad \theta'_{t+1} = G'_{\lambda,\mathcal{L}}(\theta'_t)
\]

Then we have the recurrence relation:

\[
\delta_{t+1} \leq \begin{cases} 
\eta \delta_t & G_t = G'_t \text{ is } \eta \text{-expansive} \\
\min(\eta, 1) \delta_t + 2\sigma_t & G_t \text{ and } G'_t \text{ are } \sigma \text{-bounded}, G_t \text{ is } \eta \text{ expansive}
\end{cases}
\]

Lemma 6. (Lipschitz Continuity) Assume \( \mathcal{L}_\theta \) is \( L \)-Lipschitz, the gradient update \( G_{\lambda,\mathcal{L}} \) is \((\lambda L)\)-bounded.

Proof. \( \| \theta - G_{\lambda,\mathcal{L}}(\theta) \| = \| \lambda \nabla_{\theta} \mathcal{L}(\theta) \| \leq \lambda L \)

Lemma 7. (Lipschitz Smoothness) Assume \( \mathcal{L}_\theta \) is \( \beta \)-smooth, then for any \( \lambda \leq \frac{2}{\beta} \), the gradient update \( G_{\lambda,\mathcal{L}} \) is \( 1 \)-expansive.

Proof. Please refer to Lemma 3.7 in [Hardt et al., 2016] for the proof.

Based on all the results above, we start to prove Theorem 2. Our proof is largely based on [Hardt et al., 2016], but it is applicable in distributional RL setting as well as considering desirable properties of histogram distributional loss. According Lemma 3, we attain that \( \mathcal{L}_\theta \) is \( kl \)-Lipschitz as well as \( kl^2 \)-smooth, and thus based on Lemma 6, \( G_{\lambda,\mathcal{L}} \) is \((\lambda kl)\)-bounded, and \( 1 \)-expansive if \( \lambda \leq \frac{2}{kl^2} \). In the step \( t \), SGD selects samples that are both in \( S \) and \( S' \), with probability 1 − \( \frac{1}{n} \). In this case, \( G_t = G'_t \), and thus \( \delta_{t+1} \leq \delta_t \) as \( G_t \) is \( 1 \)-expansive based on Lemma 5. The other case is that samples selected are different with probability \( \frac{1}{n} \), where \( \delta_{t+1} \leq \delta_t + 2\lambda_k kl \) based on Lemma 5. Thus, if \( \lambda_t \leq \frac{2}{kl^2} \) we have:

\[
E [ \mathcal{L}(\theta_T; x) - \mathcal{L}(\theta'_T; x) ] \leq kl E [ \delta_T ], \text{ where } \delta_T = \| \theta_T - \theta'_T \|
\]

\[
\leq kl \left( 1 - \frac{1}{n} \right) E [ \delta_{T-1} ] + \frac{1}{n} E [ \delta_{T-1} ] + \frac{2\lambda_{T-1} kl}{n}
\]

\[
= kl \left( E [ \delta_{T-1} ] + \frac{2\lambda_{T-1} kl}{n} \right)
\]

\[
= kl \left( E [ \delta_0 ] + \sum_{i=0}^{T-1} \frac{2\lambda_i kl}{n} \right)
\]

\[
\leq \frac{2k^2 l^2}{n} T - \frac{1}{2} \sum_{i=0}^{T-1} \frac{2}{kl^2}
\]

\[
= \frac{4kT}{n}
\]

Since this bounds hold for all \( S, S' \) and \( x \), we attain the uniform stability in Definition 1 for our histogram distributional loss applied in distributional RL.

Define the population risk as:

\[
R[\theta] = E_x \mathcal{L}(\theta; x)
\]

and the empirical risk as:

\[
R_S[\theta] = \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(\theta; x_i)
\]

According to Theorem 2.2 in [Hardt et al., 2016], if an algorithm \( \mathcal{M} \) is \( \epsilon_{\text{stab}} \)-uniformly stable, then the generalization gap is \( \epsilon_{\text{stab}} \)-bounded, i.e.,

\[
\left| E_{S,A} [ R_S[\mathcal{M}(S)] - R[\mathcal{M}(S)] ] \right| \leq \epsilon_{\text{stab}}
\]

\[
\Box
\]
\section*{H Proof of Lemma 4}

\begin{equation}
E_{(s,a)\sim \rho^*} \left[ \| \nabla L_\theta(p^{s,a}, q^s_\theta) - \nabla G(\theta) \|^2 \right] \leq (1-\epsilon)\sigma^2 + \epsilon \kappa \sigma^2. \tag{35}
\end{equation}

\textbf{Proof.} As we know that \( p^{s,a}(x) = (1-\epsilon)\delta_{x=x(E[Z^s(a))]}(x) + \epsilon \mu^{s,a}(x) \), then we have:

\[
\nabla L_\theta(p^{s,a}, q^s_\theta) = (1-\epsilon)\nabla L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) + \epsilon \nabla L_\theta(\mu^{s,a}, q^s_\theta)
\]

Therefore,

\[
E_{(s,a)\sim \rho^*} \left[ \| \nabla L_\theta(p^{s,a}, q^s_\theta) - \nabla G(\theta) \|^2 \right]
\leq E_{(s,a)\sim \rho^*} \left[ (1-\epsilon)\| \nabla L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) - \nabla G(\theta) \|^2 + \epsilon \| \nabla L_\theta(\mu^{s,a}, q^s_\theta) - \nabla G(\theta) \|^2 \right]
\]

\[
= (1-\epsilon)\sigma^2 + \epsilon \kappa \sigma^2,
\]

where the first inequality uses the triangle inequality of norm, i.e., \( ||(1-\epsilon)a+\epsilon b||^2 \leq (1-\epsilon)||a||^2 + \epsilon ||b||^2 \), and the last equality uses the definition of the variance of \( L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) \) and \( L_\theta(\mu^{s,a}, q^s_\theta) \).

\section*{I Proof of Theorem 3}

\textbf{Proof.} (1) If we only consider the expectation of \( Z^s(a) \), the entropy-regularized MLE would degenerate to the pure MLE regarding \( \delta_{x=E[Z^s(a)]} \). As \( L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) \) is \( kl^2 \)-smooth, we have

\[
G(\theta_{t+1}) - G(\theta_t)
\leq \langle \nabla G(\theta_t), \theta_{t+1} - \theta_t \rangle + \frac{kl^2}{2} \| \theta_{t+1} - \theta_t \|^2
\]

\[
= -\lambda \langle \nabla G(\theta_t), \nabla L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) \rangle + \frac{kl^2\lambda^2}{2} \| \nabla L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) \|^2
\]

where the last first equation is according to the definition of Lipschitz-smoothness, and the last second one is based on the updating rule of \( \theta \). Next, we take the expectation on both sides,

\[
E \left[ G(\theta_{t+1}) - G(\theta_t) \right]
\leq -\lambda E \left[ \| \nabla G(\theta_t) \|^2 \right] + \frac{kl^2\lambda^2}{2} E \left[ \| \nabla L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) - \nabla G(\theta_t) \|^2 \right]
\]

\[
\leq -\lambda E \left[ \| \nabla G(\theta_t) \|^2 \right] + \frac{kl^2\lambda^2}{2} E \left[ \| \nabla L_\theta(\delta_{x=E[Z^s(a)]}, q^s_\theta) - \nabla G(\theta_t) \|^2 \right] + \frac{kl^2\lambda^2}{2} E \left[ \| \nabla G(\theta_t) \|^2 \right]
\]

\[
= \frac{\lambda (kl^2 - 2)}{2} E \left[ \| \nabla G(\theta_t) \|^2 \right] + \frac{kl^2\lambda^2}{2} \sigma^2
\]

\[
\leq -\frac{\lambda}{2} E \left[ \| \nabla G(\theta_t) \|^2 \right] + \frac{kl^2\lambda^2}{2} \sigma^2
\]

where the first two equation hold because \( \nabla G(\theta) = E \left[ \nabla L_\theta \right] \) and the last inequality comes from \( \lambda \leq \frac{1}{kl^2} \).

Through the summation, we obtain that

\[
E \left[ G(\theta_T) - G(\theta_0) \right] \leq -\frac{\lambda}{2} \sum_{t=0}^{T-1} E \left[ \| \nabla G(\theta_t) \|^2 \right] + \frac{kl^2\lambda^2 T}{2} \sigma^2
\]

We let \( E \left[ G(\theta_T) \right] = 0 \), we have

\[
\frac{1}{T} \sum_{t=0}^{T-1} E \left[ \| \nabla G(\theta_t) \|^2 \right] \leq \frac{2G(\theta_0)}{\lambda T} + kl^2 \lambda \sigma^2
\]

By setting \( \lambda \leq \frac{\tau^2}{2kl^2 \sigma^2} \) and \( T = \frac{4G(\theta_0)}{\lambda \tau^2} \), we can have \( \frac{1}{T} \sum_{t=0}^{T-1} E \left[ \| \nabla G(\theta_t) \|^2 \right] \leq \tau^2 \), implying that the degenerated MLE can achieve \( \tau \)-station point if the sample complexity \( T = O(\frac{1}{\tau^2}) \).
We are still based on the $kl^2$-smoothness of $\mathcal{L}(p^{s,a}, q_0^{s,a})$.

\[
G(\theta_{t+1}) - G(\theta_t) \\
\leq \langle \nabla G(\theta_t), \theta_{t+1} - \theta_t \rangle + \frac{kl^2}{2} \|\theta_{t+1} - \theta_t\|^2 \\
= -\lambda \langle \nabla G(\theta_t), \nabla \mathcal{L}_\theta(p^{s,a}, q_0^{s,a}) \rangle + \frac{kl^2\lambda^2}{2} \|\nabla \mathcal{L}_\theta(p^{s,a}, q_0^{s,a})\|^2 \\
= -\frac{\lambda}{2} \|\nabla G(\theta_t)\|^2 + \frac{\lambda}{2} \|\nabla G(\theta_t) - \nabla \mathcal{L}_\theta(p^{s,a}, q_0^{s,a})\|^2 + \frac{\lambda(kl^2\lambda - 1)}{2} \|\nabla \mathcal{L}_\theta(p^{s,a}, q_0^{s,a})\|^2 \\
\leq -\frac{\lambda}{2} \|\nabla G(\theta_t)\|^2 + \frac{\lambda}{2} \|\nabla G(\theta_t) - \nabla \mathcal{L}_\theta(p^{s,a}, q_0^{s,a})\|^2
\]

where the second equation is based on $\langle a, -b \rangle = \frac{1}{2} (\|a\|^2 - \|b\|^2)$, and the last inequality is according to $\lambda \leq \frac{1}{kl^2}$. After taking the expectation, we have

\[
\mathbb{E}[G(\theta_{t+1}) - G(\theta_t)] \\
\leq -\frac{\lambda}{2} \mathbb{E}[\|\nabla G(\theta_t)\|^2] + \frac{\lambda}{2} \mathbb{E}[\|\nabla G(\theta_t) - \nabla \mathcal{L}_\theta(p^{s,a}, q_0^{s,a})\|^2] \\
\leq -\frac{\lambda}{2} \mathbb{E}[\|\nabla G(\theta_t)\|^2] + \frac{\lambda}{2} ((1 - \epsilon)\sigma^2 + \epsilon\kappa\sigma^2)
\]

where the last inequality is based on Lemma 4. We take the summation, and therefore,

\[
\mathbb{E}[G(\theta_T) - G(\theta_0)] \leq -\frac{\lambda}{2} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla G(\theta_t)\|^2] + \frac{T\lambda}{2} ((1 - \epsilon)\sigma^2 + \epsilon\kappa\sigma^2)
\]

We let $\mathbb{E}[G(\theta_T)] = 0$ and $\epsilon = \frac{1}{1 + \kappa}$, then,

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla G(\theta_t)\|^2] \\
\leq \frac{2G(\theta_0)}{\lambda T} + (1 - \epsilon)\sigma^2 + \epsilon\kappa\sigma^2 \\
= \frac{2G(\theta_0)}{\lambda T} + \frac{2\kappa}{1 + \kappa}\sigma^2 \\
\leq \frac{2G(\theta_0)}{\lambda T} + 2\kappa\sigma^2
\]

If $\kappa \leq \frac{\sigma^2}{T\epsilon^2}$ and let $T = \frac{4G(\theta_0)}{\lambda\kappa\sigma^2}$, this leads to $\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla G(\theta_t)\|^2] \leq \tau^2$, i.e., $\tau$-stationary point, with the sample complexity as $O(\frac{1}{\tau^2})$. Thus, (2) has been proved. On the other hand, if $\kappa > \frac{\sigma^2}{T\epsilon^2}$, we set $T = \frac{G(\theta_0)}{\lambda\kappa\sigma^2}$. This implies that $\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla G(\theta_t)\|^2] \leq 4\kappa\sigma^2 = O(\kappa)$. Therefore, the degree of stationary point is determined the degree of distribution approximation measured by $\kappa$. Thus, we obtain (3).

\(\square\)

## J Representation Effect of Distribution RL

From the perspective of representation, we find that distributional RL encourages state representation from the same action class classified by the policy in tighter clusters.

The intrinsic characteristic of distributional RL is that it enables to learn a richer and more faithful representation of the environment, leading to more stable and efficient learning. To investigate the more informative representation resulting from distributional RL, we visualize how distributional RL changes the representation learned in the penultimate layer of the value network. We collect state features in the penultimate layer classified by the policy $\pi$ in different action classes, and perform t-SNE to reduce them to the two-dimensional space. In Figure 4, points in different colors represent state features classified into different action classes. It illustrates that both QR-DQN and C51 encourage the representation of state observations from the same action class to group in tighter clusters relative to expectation-based
Table 1: Hyper-parameters Sheet.

| Hyperparameter                              | Value   |
|---------------------------------------------|---------|
| **Shared**                                  |         |
| Policy network learning rate                | 3e-4    |
| (Quantile) Value network learning rate      | 3e-4    |
| Optimization                               | Adam    |
| Discount factor                             | 0.99    |
| Target smoothing                            | 5e-3    |
| Batch size                                  | 256     |
| Replay buffer size                          | 1e6     |
| Minimum steps before training               | 1e4     |
| **dSAC & dSAC Noent**                       |         |
| Number of quantile fractions \((N)\)        | 32      |
| Quantile fraction embedding size            | 64      |
| Huber regression threshold                  | 1       |

| Hyperparameter                          | Temperature Parameter \(\beta\) | Max episode length |
|-----------------------------------------|---------------------------------|-------------------|
| Walker2d-v2                             | 0.2                             | 1000              |
| Swimmer-v2                              | 0.2                             | 1000              |
| Reacher-v2                              | 0.2                             | 1000              |
| Ant-v2                                  | 0.2                             | 1000              |
| HalfCheetah-v2                          | 0.2                             | 1000              |
| Humanoid-v2                             | 0.05                            | 1000              |
| HumanoidStandup-v2                      | 0.05                            | 1000              |
| BipedalWalkerHardcore-v2               | 0.002                           | 2000              |

DQN. Interestingly, this phenomenon is similar to label smoothing [Müller et al., 2019], which leverages additional distributional knowledge in soft labels relative to hard labels. Therefore, we conclude that similar to the benefit of label smoothing, distributional RL also encourages the activations of the penultimate layer to be close to the template of the correct action class and distant to the templates of the incorrect action classes.

K Implementation Details

Our implementation is directly adapted from the source code in [Ma et al., 2020]. We consider the quantile regression for the distribution estimation. Instead of using fixed quantiles in QR-DQN [Dabney et al., 2018b], we leverage the quantile fraction generation based on IQN [Dabney et al., 2018a] that uniformly samples quantile fractions in order to approximate the full quantile function. In particular, we fix the number of quantile fractions as \(N\) and keep them in an ascending order. Besides, we adapt the sampling as \(\tau_0 = 0, \tau_i = \epsilon_i / \sum_{i=0}^{N-1} \), where \(\epsilon_i \in U[0, 1], i = 1, ..., N\).

Figure 4: Visualization on the penultimate layer of the value network on Breakout. Each color denotes one action class.
K.1 Hyper-parameters and Network structure.

We adopt the same hyper-parameters, which is listed in Table 1 and network structure as in the original distributional SAC paper [Ma et al., 2020].

L Comparison Between Different Operators

L.1 Soft Bellman Operator $T^\pi_s$

Soft Bellman Operator [Haarnoja et al., 2018] is defined in soft policy evaluation that incorporates a soft value function with an entropy term. Specifically, the soft Bellman operator $T^\pi_s$ is given by

$$T^\pi_s Q(s_t, a_t) \triangleq r(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1} \sim \rho^s}[V(s_{t+1})],$$

where $V(s_{t+1}) = \mathbb{E}_{a_{t+1} \sim \pi}[Q(s_{t+1}, a_{t+1}) - \log \pi(a_{t+1}|s_{t+1})]$ is state value function. By repeatedly applying soft Bellman operator, we can obtain the convergence guarantee in the soft policy evaluation, which is also the underpinning of soft policy iteration and SAC algorithm.

L.2 Distributional Soft Bellman Operator $\mathfrak{T}^\pi_{ds}$

Distributional Soft Bellman Operator is foundation of distributional SAC [Ma et al., 2020], which is defined by

$$\mathfrak{T}^\pi_{ds} Z(s, a) := R(s, a) + \gamma |Z(s', a') - \alpha \log \pi(a' | s')| | s' \sim \mathcal{P}(\cdot | s, a), a' \sim \pi(\cdot | s')$$

Distributional Soft Bellman Operator $\mathfrak{T}^\pi_{ds}$ additionally incorporates the entropy term into the vanilla distributional Bellman operator $\mathfrak{T}$, and still inherits the property of convergence from original operators.

L.3 Relationships between operators

Figure 5 suggests the mutual relationship between soft Bellman operator, distributional Bellman operator, distributional soft Bellman operation and our proposed soft distributional Bellman operator.

In summary, distributional soft Bellman operator incorporates the entropy (soft) part in soft Bellman operator into the vanilla distribution Bellman operator, thus eventually yielding a new variant of distributional Bellman operator. In contrast, our soft distributional Bellman operator starts from decomposing action-value distribution into a new soft term, i.e., risk-aware entropy regularization. This can be viewed as transforming distributional kind of Bellman operator into a new soft Bellman operator, which we call soft distributional Bellman operator in order to discriminate from distributional soft Bellman operator [Ma et al., 2020].