On the center of affine Hecke algebras of type A

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0.1 Introduction. Let $G$ be a simple complex algebraic group. Let $W$ be its Weyl group and $\hat{W}$ the associated extended affine Weyl group. Let $\hat{H}$ be the Iwahori-Hecke algebra of $\hat{W}$. It is well-known that $\hat{H}$ admits two presentations: the Coxeter presentation which arises naturally when $\hat{H}$ is realized as the convolution algebra $L(\hat{G}, I)$ of compactly supported functions on a $p$-adic group $\hat{G} = G(\mathbb{Q}_p)$ which are bi-invariant under action of the Iwahori subgroup $I$ (see [IM]), and the Bernstein presentation, which arises when $\hat{H}$ is realized in the $G^\vee \times \mathbb{C}^*$-equivariant K-theory of the Steinberg variety associated to $G^\vee$ where $G^\vee$ is the Langlands dual group (see [Gi]). The interplay between these two presentations is central in the Deligne-Langlands correspondence for finite-dimensional irreducible representations of $\hat{H}$.

The center $Z(\hat{H})$ is easily described in the K-theoretic picture: it is spanned by the classes of the trivial (equivariant) bundles on $Z$. A geometric construction of this center in the convolution algebra presentation is given by Gaitsgory, [Ga]. This is in turn inspired by work of Beilinson, and Haines, Kottwitz and Rapoport in the framework of Shimura varieties, see [H1], [H2].

In this paper we give an explicit expression for the central elements of $\hat{H}$ in the Coxeter presentation when $G = GL(r)$ (Theorem 2.5). This expression generalizes those obtained by Haines in the minuscule case, [H2] and is in some sense more explicit than [Ga]. More generally, we obtain expressions for central elements in the “parabolic spherical” Hecke algebras $L(\hat{G}, P)$ where $P \supseteq I$ is a parahoric subgroup. In particular, taking $P = K$ to be a maximal compact open subgroup recovers Lusztig’s description [L1] of the Satake isomorphism between $Z(\hat{H})$ and the spherical algebra $\hat{H}_{sph}$ (in the case $G = GL(r)$).

Our method is based on the Hall algebra of a cyclic quiver, on Uglov’s higher-level Fock spaces and on the theory of canonical bases of Kashiwara and Lusztig. Namely, we use Ginzburg and Vasserot’s geometric description of quantum affine Schur-Weyl duality to construct an embedding of (half of) the center $Z(\hat{H})$ in the center of the Hall algebra $U_n^-$ of the quiver $\tilde{A}_{n-1}$ for $n \geq r$ (see [S]). This embedding is compatible with the canonical bases of $\hat{H}$ and $U_n^-$. To describe the center of $U_n^-$ we then consider the action on the Fock spaces $\Lambda_{\infty}^\leq$ recently introduced by Uglov [U], and use the fact that this action is again compatible with the canonical bases.

Finally, we give a simple alternate description of the center of $U_n^-$ in terms of a certain desingularization of orbit closures of representations of the quiver.
\(A\), introduced by Varagnolo and Vasserot \([V\!V]\). This can be seen as a cyclic analogue of the desingularization of orbit closures recently obtained by Reineke \([R\!e]\) for finite-type simply laced Dynkin quivers.

We note that the Fock spaces and their canonical bases appear to be a very fundamental object in type A representation theory: they describe Grothendieck groups and decomposition numbers of Hecke algebras of type A or B (or more generally cyclotomic Hecke algebras) at roots of unity (see \([L\!T]\), \([A\!\!M]\), \([G\!d]\)), and modular representations of symmetric groups (see \([D\!f]\), \([J\!\!J]\), \([G\!d]\)).

0.2 Notations. Set \(S = \mathbb{C}[v]\), \(A = \mathbb{C}[v, v^{-1}]\) and \(K = \mathbb{C}(v)\). We define a \(\mathbb{C}\)-linear ring involution \(u \mapsto \overline{u}\) on \(A\) by setting \(\overline{v} = v^{-1}\). Let \(F\) be a finite field with \(q^2\) elements. Let \(\mathbb{S}_r\) denote the symmetric group on \(r\) elements and let \(\{s_i\}_{i=1, \ldots, r-1}\) be the set of simple reflections. Let \(\mathbb{S}_r = \mathbb{S}_r \rtimes \mathbb{Z}'\) be the extended affine symmetric group and let \(s_0\) be the affine simple reflection. Let \(\Pi\) stand for the set of partitions and let \(\Pi_r\) be the set of partitions of length at most \(r\). Elements of \(\Pi^l\) for some \(l \in \mathbb{N}\) will be called \(l\)-multipartitions. Finally, we will denote by \(\overline{Y}\) the Zariski closure of any subset \(Y\) of an algebraic variety \(X\).

1 Affine Hecke algebras and canonical bases

1.1 Consider the Iwahori-Hecke algebra \(\widetilde{H}_r\) associated to \(\widetilde{\mathbb{S}}_r\), i.e. the \(A\)-algebra generated by elements \(T_\sigma\), \(\sigma \in \widetilde{\mathbb{S}}_r\) with relations

\[(T_{s_i} + 1)(T_{s_i} - v^{-2}) = 0\quad \text{for } i = 0, \ldots, r - 1,\]

\[T_\sigma T_\gamma = T_{\sigma\gamma}\quad \text{if } \ell(\sigma\gamma) = \ell(\sigma)\ell(\gamma).\]

We set \(\tilde{T}_\sigma = v^{l(\sigma)}T_\sigma\) for every \(\sigma \in \widetilde{\mathbb{S}}_r\).

It is well-known that \(\widetilde{H}_r\) admits another presentation (the Bernstein presentation) as the unital \(A\)-algebra generated by elements \(T_i^{\pm 1}, X_j^{\pm 1}\) where \(i \in [1, r-1]\), \(j \in [1, r]\) with the following relations

\[T_iT_i^{-1} = 1 = T_i^{-1}T_i,\]

\[(T_i + 1)(T_i - v^{-2}) = 0,\]

\[T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\]

\[|i - j| > 1 \Rightarrow T_iT_j = T_jT_i,\]

\[X_iX_j^{-1} = 1 = X_jX_i^{-1}X_i,\]

\[X_iX_j = X_jX_i,\]

\[T_iX_jT_i = v^{-2}X_{i+1},\]

\[j \neq i, i + 1 \Rightarrow X_jT_i = T_iX_j.\]

The isomorphism between the two presentations is such that \(T_{s_i} \mapsto T_i\) and \(\tilde{T}_\lambda^{-1} \mapsto X_1^{\lambda_1} \cdots X_r^{\lambda_r}\) if \(\lambda = (\lambda_1, \ldots, \lambda_r)\) is dominant. The center of \(\widetilde{H}_r\) is \(Z(\widetilde{H}_r) = \mathbb{A}[X_1^{\pm 1}, \ldots, X_r^{\pm 1}]^{\mathbb{S}_r}\).

Set \(Z_r^{-} = \mathbb{A}[X_1^{-1}, \ldots, X_r^{-1}]^{\mathbb{S}_r}\).}

1.2 For every \(t, s \in \mathbb{N}\) define the left (resp. right) representation of \(\widetilde{\mathbb{S}}_t\) on \(Z_t^l\) of level \(s\) by

\[s_j \cdot (i_1, \ldots, i_t) = (i_1, \ldots, i_{j+1}, i_j, \ldots, i_t),\]

\[1 \leq j < r,\]

\[\lambda \cdot (i_1, \ldots, i_t) = (i_1 + s\lambda_1, \ldots, i_t + s\lambda_t),\]

\[\lambda \in \mathbb{Z}_t.\]
and

\[(i_1, \ldots, i_t) \cdot s_j = (i_1, \ldots, i_{j-1}, i_j, \ldots, i_t), \quad 1 \leq j < r,
\]

\[(i_1, \ldots, i_t) \cdot \lambda = (i_1 + s\lambda_1, \ldots, i_t + s\lambda_t), \quad \lambda \in \mathbb{Z}^r\]

respectively. The set \(A^s_i = \{1 \leq i_1 \leq \cdots \leq i_s \leq s\}\) is a fundamental domain for both actions. For each \(i \in A^s_i\) we set \(G_i = Stab i \subset G_i\) and denote by \(\omega_i \in G_i\) the longest element. We also let \(G^i\) be the set of all minimal length elements of the cosets \(G_1 \setminus \hat{G}_i\).

1.3 Fix some \(n \in \mathbb{N}^+\). For any \(i, j \in A^n_i\) and any \(\sigma \in G_1 \setminus \hat{G}_i / \hat{G}_j\) we set \(T_{\sigma} = \sum_{\delta \in \sigma} T_\delta\) and we let \(\hat{H}_{ij} \subset \hat{H}_r\) be the \(K\)-linear span of the elements \(T_{\sigma}\) for \(\sigma \in G_1 \setminus \hat{G}_r / \hat{G}_j\). Set \(e_1 = \sum_{\delta \in \sigma_1} T_{\delta}\). Then \(\hat{H}_{ij} = e_1 \hat{H}_r e_1\). Put

\[\hat{S}_{n,r} = \bigoplus_{i,j \in A^n_i} \hat{H}_{ij}.\]

This space, equipped with the multiplication

\[e_i h e_j \cdot e_k h' e_1 = \delta_{jk} e_i h e_j h' e_1 \in \hat{H}_{ij} \quad \text{for all } h, h' \in \hat{H}_r\]

is called the affine q-Schur algebra. It is proved in \([GV]\), \([L3]\) that \(\hat{S}_{n,r}\) is a quotient of the modified quantum affine algebra \(\hat{U}_v(\mathfrak{gl}_n)\).

1.3 Set \(T_{n,r} = \bigoplus_{i,j \in A^n_i} e_i \hat{H}_r\). For any \(\sigma \in G_1 \setminus \hat{G}_r\) we put \(T_{\sigma} = \sum_{\delta \in \sigma} T_{\delta}\). Then \(\{T_{\sigma}\}, \sigma \in G_1 \setminus \hat{G}_r\) is an \(\hat{A}\)-basis of \(e_i \hat{H}_r\). It will be convenient to identify the element \(\sigma\) with \(i \cdot \sigma \in \mathbb{Z}^r\), so that \(\{T_p\}, p \in \mathbb{Z}^r\) is an \(\hat{A}\)-basis of \(T_{n,r}\).

The algebra \(\hat{H}_r\) acts on \(T_{n,r}\) by multiplication on the right, and \(\hat{S}_{n,r}\) acts on \(T_{n,r}\) on the left by

\[e_i h e_j \cdot e_k h' = \delta_{jk} e_i h e_j h' \in e_i \hat{H}_r \quad \text{for every } h, h' \in \hat{H}_r.\]

Let us denote these actions by \(\rho_r: \hat{S}_{n,r} \rightarrow \text{End} (T_{n,r})\) and \(\sigma_r : \hat{H}_r \rightarrow \text{End} (T_{n,r})\). It is obvious that these two actions commute. The following result is a quantum and affine analogue of Schur-Weyl duality.

**Theorem (\([GV]\)).** We have \(\hat{S}_{n,r} = \text{End}_{\hat{H}_r} (T_{n,r})\). Moreover, we have \(\hat{H}_r = \text{End}_{\hat{S}_{n,r}} (T_{n,r})\) if \(n \geq r\).

1.4 Let us now, following \([GV]\) and \([M]\), give the geometric realization of the above Schur-Weyl duality. Let \(L = \mathbb{F}(\mathbb{Z})\) and set \(G = GL_r(L)\). By definition, a lattice in \(L'\) is a free \(\mathbb{F}[\mathbb{Z}]\)-submodule of rank \(r\). Consider the variety \(X\) of sequences of lattices \((L_i)_{i \in \mathbb{Z}}\) such that

\[L_i \subset L_{i+1}, \quad \dim_{\mathbb{F}}(L_i / L_{i-1}) = 1, \quad L_{i+1} = z^{-1}L_i\]

(the affine flag variety of type \(GL_r\)). Consider also the variety \(Y\) of all \(n\)-step periodic flags in \(L'\), i.e. the set of all sequences of lattices \((L_i)_{i \in \mathbb{Z}}\) such that

\[L_i \subset L_{i+1}, \quad L_{i+n} = z^{-1}L_i\]

(the affine partial flags variety). The group \(G\) acts (transitively) on \(X\) and acts on \(Y\) in obvious ways. Consider the diagonal action of \(G\) on \(X \times X\) and \(Y \times Y\) respectively.
It is well-known that the set of \( G \)-orbits on \( X \times X \) is canonically identified with \( \hat{S}_r \). In order to describe these \( G \)-orbits we let \((e_1, \ldots, e_r)\) be a fixed \( L \)-basis of \( L^r \) and set \( e_{i+kr} = z^{-k}e_i \). Consider the right action of \( \hat{S}_r \) on \( Z^r \) of level \( r \). To any element \( x \) in the orbit of \( \rho_r = (1, 2, \ldots, r) \) we associate the flag \((L(x)_i)_{i \in \mathbb{Z}} \) defined by
\[
L(x)_i = \prod_{p(j) \leq i} \mathbb{F}e_j,
\]
where \( p : \mathbb{Z} \to \mathbb{Z} \) is the bijection uniquely defined by \( p(j) = x_j \) if \( 1 \leq j \leq r \) and \( p(j+r) = p(j) + r \). The \( G \)-orbit decomposition of \( X \times X \) reads
\[
X \times X = \bigsqcup_{\sigma \in \hat{S}_r} X_\sigma
\]
where \( X_\sigma = G \cdot (L(\rho_r \cdot \sigma), L(\rho_r)) \). Similarly, to each \( i \in \mathbb{Z}^r \) we associate the map \( p : \mathbb{Z} \to \mathbb{Z} \) uniquely defined by \( p(j) = i_j \) if \( 1 \leq j \leq r \) and \( p(j+r) = p(j) + n \). Consider the flag
\[
L(i)_i = \prod_{i(j) \leq i} \mathbb{F}e_j.
\]
Then \( Y = \bigsqcup_{i \in \mathbb{A}_n^r} Y_i \) where \( Y_i = G \cdot (L(i)) \) and
\[
Y_i \times Y_j = \bigsqcup_{\sigma \in S_{i,j} \setminus S_{y,z}} Y_\sigma
\]
where \( Y_\sigma = G \cdot (L(i \cdot \sigma), L(j)) \) and where the right action of \( \hat{S}_r \) on \( Z^r \) is now of level \( n \).

Let \( C_G(X \times X) \) (resp. \( C_G(Y \times Y) \)) be the space of complex-valued \( G \)-invariant functions on \( X \times X \) (resp. on \( Y \times Y \)) which are supported on finitely many orbits. The convolution product endows these spaces with an associative algebra structure. We let \( 1_{\mathcal{O}} \in C_G(X \times X) \) (resp. \( 1_{\mathcal{O}} \in C_G(Y \times Y) \)) be the characteristic function of a \( G \)-orbit \( \mathcal{O} \subset X \times X \) (resp. \( \mathcal{O} \subset Y \times Y \)).

**Theorem ([IM],[VV]).**

i) The linear map \( (\hat{H}_r)|_{v=q^{-1}} \to C_G(X \times X) \) defined by \( T_\sigma \mapsto 1_{X_\sigma} \) is an algebra isomorphism.

ii) The linear map \( (\hat{S}_{n,r})|_{v=q^{-1}} \to C_G(Y \times Y) \) such that \( T_\sigma \mapsto 1_{Y_\sigma} \) is an algebra isomorphism.

Now consider the diagonal action of \( G \) on \( Y \times X \). The collection of orbits are parametrized by \( \mathbb{Z}^r \); to \( i \in \mathbb{Z}^r \) corresponds the orbit \( \mathcal{O}_i \) of the pair \( (L(i), L(\rho_r)) \). The algebras \( C_G(X \times X) \) and \( C_G(Y \times Y) \) act by convolution on \( C_G(Y \times X) \) on the right and on the left respectively.

**Theorem ([VV]).** The map \( (T_{n,r})|_{v=q^{-1}} \to C_G(Y \times X) \) such that \( e_i \mapsto 1_{\mathcal{O}_i} \) for \( i \in \mathbb{A}_n^r \) extends uniquely to an isomorphism of \( (\hat{S}_{n,r})|_{v=q^{-1}} \times (\hat{H}_r)|_{v=q^{-1}} \) modules.
1.5 Let $u \mapsto \pi$ be the semilinear involution of $\hat{\mathbf{H}}_r$ defined by $T_\sigma = T_{\sigma^{-1}}^{-1}$ for all $\sigma$. For each $\sigma \in \hat{\Sigma}_r$ there exists a unique element $c_{\sigma} \in \hat{\mathbf{H}}_r$ such that
\[ c_{\sigma} = c_{\sigma}, \quad \text{ii) } c_{\sigma} = \hat{T}_\sigma + \sum_{\delta < \sigma} c_{\delta \sigma}(v) \hat{T}_\delta, \quad c_{\delta \sigma}(v) \in v\mathfrak{S}. \]

The polynomial $c_{\sigma, \delta}(v)$ is the affine Kazhdan-Lusztig polynomial of type $\hat{A}_{r-1}$ associated to $\sigma$ and $\delta$ (this polynomial is denoted by $h_{\sigma, \delta}$ in Soergel’s notation $[\text{Soe}]$).

For $\sigma \in \hat{\Sigma}_r$ and $L \in X$ let $X_{\sigma, L}$ be the fiber of the first projection $X_{\sigma} \to X$. Then $X_{\sigma, L}$ is the set of $L$-points of an algebraic variety of dimension $l(\sigma)$ whose isomorphism class is independent of $L$. Then
\[ c_{\sigma} = \sum_{i, \delta} v^{-i+y(\sigma)-y(\delta)} \dim \mathcal{H}^{i}_{X_{\sigma, L}}(IC_{X_{\sigma, L}}) \hat{T}_\delta \]
where $IC_{X_{\sigma, L}}$ denotes the intersection cohomology complex associated to $X_{\sigma, L}$ and where $\mathcal{H}^i$ stands for local cohomology.

Similarly, let $i, j \in \mathbb{A}^n_r$ and let $\sigma \in \hat{\Sigma}_r \setminus \hat{\Sigma}_r / \hat{\Sigma}_j$. Denote by $Y_{\sigma, i}$ the fiber above $(L(i))$ of the projection of $y_{\sigma} \to Y$ on the first component. This is the set of $L$-points of an algebraic variety of dimension $y(\sigma)$ (an explicit formula for $y(\sigma)$ can be found in [\text{L3}]). Put $\hat{T}_\sigma = v^{y(\sigma)} T_\sigma$. For every $\sigma \in \hat{\Sigma}_r \setminus \hat{\Sigma}_r / \hat{\Sigma}_j$ set
\[ c_{\sigma} = \sum_{i, \delta} v^{-i+y(\sigma)-y(\delta)} \dim \mathcal{H}^{i}_{Y_{\sigma, i}}(IC_{Y_{\sigma, i}}) \hat{T}_\delta. \]

It is clear that $\hat{\mathbf{H}}_{ij} = \hat{\mathbf{H}}_{ji}$. Define a semilinear involution $\tau : \hat{\mathbf{H}}_{ij} \to \hat{\mathbf{H}}_{ij}$ by $\tau(u) = v^{-2(l(\pi))} \pi$. The elements $\{c_{\sigma}\}$ for all $i, j \in \mathbb{A}^n_r$ form the canonical basis of $\mathfrak{S}_{n,r}$ and are characterized by the following two properties:
\[ \text{i) } \tau(c_{\sigma}) = c_{\sigma}, \quad \text{ii) } c_{\sigma} = \hat{T}_\sigma + \sum_{\delta < \sigma} c_{\delta, \sigma}(v) \hat{T}_\delta, \quad c_{\delta, \sigma}(v) \in v\mathfrak{S}. \]

1.6 Let $s, t \in \mathbb{N}^+$. For $i \in \mathbb{A}^n_r$ and $x \in i \cdot \hat{\Sigma}_i$, set $\langle x \rangle = e_i \hat{T}_a$, where $i \cdot a = x$ and $a \in \mathfrak{S}^1$. The set $\{(x), x \in i \cdot \hat{\Sigma}_i\}$ is an $\mathbb{A}$-basis of the space $e_i \hat{\mathbf{H}}_i$. Define a semilinear involution $u \mapsto \pi$ of $e_i \hat{\mathbf{H}}_i$ by $\pi x = e_i \pi$. There exists a unique $\mathbb{A}$-basis $\{c_x, x \in i \cdot \hat{\Sigma}_i\}$ of $e_i \hat{\mathbf{H}}_i$ such that
\[ \text{i) } \pi c_x = c_{\pi x}, \quad \text{ii) } c_x = \langle x \rangle + \sum_y P_{y, x} \langle y \rangle, \quad P_{y, x} \in v^{-1} \mathbb{Z}[v^{-1}]. \]

The polynomials $P_{y, x}$ are parabolic affine Kazhdan-Lusztig polynomials introduced by Deodhar [\text{D}] with $v_{a_y, a_x}$ in Soergel’s notation, where $a_x, a_y \in \mathfrak{S}^1$ are such that $x = i \cdot a_x$, $y = i \cdot a_y$.

2 The main result

2.1 Let $\Gamma$ be Macdonald’s ring of symmetric polynomial in the variables $y_i$, $i \in \mathbb{Z}$, defined over $\mathbb{A}$ (see [\text{Mac}]). Let $\Gamma_r = \mathbb{A}[y_1, \ldots, y_r]^{\hat{\Sigma}_r}$. Let $s_{\lambda} \in \Gamma_r$ be the Schur polynomial associated to $\lambda \in \Pi_r$.  

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Fix some \( n \in \mathbb{N} \) and let \( i \in \mathcal{A}_n \). From \( s_\lambda(X_1^{-1}, \ldots, X_r^{-1}) \in \hat{H}_r \) it follows that \( e_is_\lambda(X_1^{-1}, \ldots, X_r^{-1}) \in \hat{H}_r \). Define polynomials \( J_{i,\sigma}^\lambda \in \mathbb{Z}[v,v^{-1}] \) by the relation

\[
e_is_\lambda(X_1^{-1}, \ldots, X_r^{-1}) = (-v)^{(n-1)|\lambda|} \sum_{\sigma \in \mathcal{S}_i \setminus \mathcal{S}_r \setminus \mathcal{S}_i} J_{i,\sigma}^\lambda c_\sigma.
\]

In this section we give an explicit expression for \( J_{i,\sigma}^\lambda \) involving (parabolic) affine Kazhdan-Lusztig polynomials of type \( A \).

**Remark.** It is clear that (up to a power of \( v \)) \( J_{i,\sigma}^\lambda \) depends only on \( \mathcal{S}_i \) rather than on \( i \). In particular, any parabolic subgroup \( \mathcal{S}_{i_1} \times \cdots \times \mathcal{S}_{i_t} \) occurs as \( \mathcal{S}_i \) for some \( i \in \mathcal{A}_n \) as soon as \( n \geq t \).

**2.2** We first make some preliminary definitions. We will represent a partition \( \lambda \) by its associated Young diagram in the usual fashion. We will consider diagrams where the \((i,j)\)-box has content \( i - j + r_0 \mod n \) for some fixed \( r_0 \in \mathbb{Z}/n\mathbb{Z} \) and call the resulting tableau the partition \( \lambda \) with residue \( r_0 \). We will say that a box with content \( j \in \mathbb{Z}/n\mathbb{Z} \) can be added to the partition \( \lambda \) with residue \( r_0 \) if there exists a partition \( \lambda' \) with residue \( r_0 \) such that \( \lambda' / \lambda \) is a single box with content \( j \). For example, when \( n = 3 \), the partition \( \lambda = (421) \) with residue 1 is

and the dotted lines correspond to addable boxes.

To each \( p \in (\mathbb{Z}^+)^n \) and \( i \in \mathcal{A}_n \) we associate a multipartition (with residues) \( M_i(p) \). First, we attach a diagram (not a partition!)

\[
D_p = \{(i,j) \mid 0 < j \leq p_i\} \subset \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}^+,
\]

where we fill the \((i,j)\)-box with the content \( i_i + p_i - j \mod n \).

**Example 1.** Suppose \( r = n = 5 \), \( i = (1,2,3,4,5) \) and \( p = (4,3,4,3,5) \). Then \( D_p \) is

```plaintext
    5
   1 3
  2 2 4 4 2
 3 3 5 5 3
 4 4 1 1 4
```
Now consider the horizontal slices \( s_k = D_p \cap (\mathbb{Z}/r\mathbb{Z} \times \{k\}) \) and let \( k_0 \) be maximal such that \( s_{k_0} \neq \emptyset \). We construct the multipartition with residues \( \mathcal{M}_i(p) \) by successively adding the boxes from \( s_{k_0}, \ldots, s_1 \) in the following way. Set \( \mathcal{M}^{k_0+1} = \emptyset \).

Suppose \( \mathcal{M}^i = (\lambda^{(1)}_i, \ldots, \lambda^{(t)}_i) \) is known. Then \( \mathcal{M}^{i-1} = (\lambda^{(1)}_{i-1}, \ldots, \lambda^{(r)}_{i-1}) \) is obtained from \( \mathcal{M}^i \) by adding the boxes from \( s_i \) (possibly creating new partitions) in such a way that

i) For every \( 1 \leq v \leq r \), \( \lambda^{(v)}_{i-1}/\lambda^{(v)}_i \) is a skew tableau with at most one box in each row,

ii) \( \mathcal{M}^{i-1} \) is maximal for the following order :

\[
(\lambda^{(1)}_{i-1}, \lambda^{(2)}_{i-1}, \ldots) \geq (\mu^{(1)}_{i-1}, \mu^{(2)}_{i-1}, \ldots) \text{ if there exists } w \text{ such that } \\
\lambda^{(l)}_{i-1} = \mu^{(l)}_{i-1} \text{ for } 1 \leq l < w \text{ and } \lambda^{(w)}_{i-1} \geq \mu^{(w)}_{i-1},
\]

where \( \geq \) stands for the usual dominance order of partitions,

iii) If several new partitions appear in \( \mathcal{M}^{i-1} \) then they are in increasing order of their residue.

Set \( \mathcal{M}_i(p) = \mathcal{M}^1 \). We note that condition iii) above is not essential for the rest of the paper and here only to fix notations.

**Examples.** i) Let \( r = n \) and \( i = (1, \ldots, r) \). Suppose that \( p \) is antidominant up to cyclic permutation, i.e., there exists \( i \in \mathbb{Z}/r\mathbb{Z} \) such that \( p_i \geq p_{i-1} \geq \cdots \geq p_{i+1} \). Let \( \lambda \) be the associated partition. Then \( \mathcal{M}_i(p) \) consists of the single partition \( \lambda \) with residue \( i \).

ii) Consider \( r, n, i \) and \( p \) as in example 1. Then the algorithm for computing \( \mathcal{M}_i(p) \) runs as follows:
For \( \sigma \in \mathcal{G}/\mathcal{E}_r \) we let \( \sigma_0 \) be the longest element in \( \sigma \) and we set \( i \cdot \sigma = i \cdot \sigma_0 - i \in \mathbb{Z}^r \). For each \( \sigma \) such that \( i \cdot \sigma \in (\mathbb{Z}^+)^r \) we set \( M(\sigma) = M_1(i \cdot \sigma) \). Write \( M(\sigma) = (\sigma^{(1)}, \ldots, \sigma^{(l)}) \) where \( \sigma^{(l)} \neq \emptyset \), and \( r_\sigma = (r_1, \ldots, r_l) \) where \( r_i \in \mathbb{Z}/n\mathbb{Z} \) is the residue of \( \sigma^{(i)} \).

2.3 Let \( l \in \mathbb{N} \). Let \( (\sigma^{(1)}, \ldots, \sigma^{(l)}), (\mu^{(1)}, \ldots, \mu^{(l)}) \) be any \( l \)-multipartitions and let \( r = (r_1, \ldots, r_l) \in (\mathbb{Z}/n\mathbb{Z})^l \). Choose some \( s = (s_1, \ldots, s_l) \in \mathbb{Z}^l \) such that \( s_i \equiv r_i \pmod{n} \). For \( i = 1, \ldots, l \) and \( j \in \mathbb{N} \) we set \( v_j^{(i)} = s_i + \sigma_j^{(i)} + 1 - j \). Consider, for \( t \gg 0 \)

\[
\begin{align*}
 u &= (u_1^{(1)}, \ldots, u_1^{(l)}, u_1^{(2)}, \ldots, u_1^{(l)}), \\
 v &= (v_1^{(1)}, \ldots, v_1^{(l)}, v_1^{(2)}, \ldots, v_1^{(l)}).
\end{align*}
\]

Finally, we put

\[
P^{-,s}_{(\mu^{(1)}, \ldots, \mu^{(l)}), (\sigma^{(1)}, \ldots, \sigma^{(l)})} = P^{-,u}_{v, u}.
\]

Now let \( s \) be in the asymptotic range \( s_1 \gg s_2 \gg \cdots \gg s_l \) and set

\[
P^{-,r}_{(\mu^{(1)}, \ldots, \mu^{(l)}), (\sigma^{(1)}, \ldots, \sigma^{(l)})} = P^{-,u}_{v, u}.
\]

This polynomial is independent of the choices of \( s \) and \( t \) in the given asymptotic range (this follows for instance from [U] Section 4 and [S], Theorem 4.1). These can be thought of as some “stabilization” of polynomials \( P^{-,p,\sigma,\rho}_r \) of type \( A_r \) as \( r \) tends to infinity (see [P]). Moreover, it is easy to see that when \( l = 1 \), \( P^{-,r} \) is independent of \( r \) and we will omit it.

2.4 For any multipartition \( \mu = (\mu^{(1)}, \ldots, \mu^{(l)}) \) and \( r = (r_1, \ldots, r_l) \in (\mathbb{Z}/n\mathbb{Z})^l \) we set \( \mu' = ((\mu^{(1)})', \ldots, (\mu^{(l)})') \) and \( r' = (-r_1, \ldots, -r_1) \).

2.5 The following is the main result of this paper, and will be proved in Section 5.

**Theorem.** We have

\[
c_{\lambda} s_{\lambda}(X_1^{-1}, \ldots, X_r^{-1}) = (-\nu)^{(n-1)\lambda} \sum_{\sigma, i \cdot \sigma \in (\mathbb{Z}^+)^r} J_{\lambda, \sigma}^{i \cdot \sigma} c_{\sigma}
\]

where

\[
J_{\lambda, \sigma}^{i \cdot \sigma} = \sum_{\nu_1, \ldots, \nu_l, \mu_1, \ldots, \mu_l} c_{\lambda}^{\mu_1, \ldots, \mu_l} \nu^{(n-1)\mu} P_{\nu_1, \nu_2, \ldots}^{-,\lambda} P_{\nu_1, \nu_2, \ldots}^{-,\lambda} \cdots P_{\nu_1, \nu_2, \ldots}^{-,\lambda} \left( P_{\nu_1, \nu_2, \ldots}^{-,\lambda} \right)^r
\]

and \( \nu = (\nu_1, \ldots, \nu_l) \). Here \( c_{\lambda}^{\mu_1, \ldots, \mu_l} \) is the (generalized) Littlewood-Richardson coefficient.
Examples. i) Suppose that \( n = 1 \). Then \( i = (1^r) \) and \( \mathfrak{S}_i = \mathfrak{S}_r \). Moreover, \( \mathfrak{S}_r \setminus \mathfrak{S}_r / \mathfrak{S}_r = \Pi_r \) and for \( \sigma \in \Pi_r \), we have \( M(\sigma) = \sigma \) and \( l = 1 \). Hence the above theorem reduces to \( J^1_{\lambda, \sigma} = \sum_{\nu} P_{\nu, \lambda}^- P_{\nu, \sigma}^- = \delta_{\lambda, \sigma} \), i.e

\[
( \sum_{w \in \mathfrak{S}_r} T_w ) s_\lambda (X_1^{-1}, \ldots, X_r^{-1}) = c_\lambda,
\]

in accordance with \([L1]\).

ii) Let \( r = n \) and \( i = 1 \) (i.e. \( \mathfrak{S}_i = \{1\} \)). Let \( \lambda = (1^l), \ l \leq r \) be a minuscule weight. Then in the above expression for \( J^1_{\lambda, \sigma} \), the only nonzero terms correspond to the case when \( \mu_i \) is also minuscule for all \( i \). We obtain an expression for \( s_\lambda (X_1^{-1}, \ldots, X_r^{-1}) \) analogous to Theorem 1.1 in \([H2]\) for \( G = GL(r) \) (but which involves Kazhdan-Lusztig polynomials rather than \( R \)-polynomials). Note that \([H1]\), Proposition 5 also easily follows from the above theorem.

3 Hall algebra of a cyclic quiver

3.0 Notations. In this section we fix a positive integer \( n \). Let \((e_i), \ i \in \mathbb{Z}/n\mathbb{Z}\) be the canonical basis of \( \mathbb{N}^\mathbb{Z}/n\mathbb{Z}\). For \( i \in \mathbb{Z}/n\mathbb{Z} \) and \( l \in \mathbb{N}^* \), define the cyclic segment \([i;l]) to be the image of the projection to \( \mathbb{Z}/n\mathbb{Z} \) of the segment \([i_0, i_0 + l - 1] \subset \mathbb{Z} \) for any \( i_0 \equiv i \) (mod \( n \)). A cyclic multisegment is a linear combination \( m = \sum_i a_i^l [i;l]) \) of cyclic segments with coefficients \( a_i^l \in \mathbb{N} \). Let \( \mathcal{M} \) be the set of cyclic multisegments. For \( m \in \mathcal{M} \) we set \( \dim m = \sum a_i^l \). Note that \( \mathcal{M} \) is canonically isomorphic to \( \Pi^n : \ m = \sum a_i^l [i;l]) \) we associate the multipartition \((\lambda^{(1)}, \ldots, \lambda^{(n)}) \) with \( \lambda^{(i)} = (1^{a_i^l} 2^{a_i^l} \ldots) \).

3.1 Let \( Q \) be the quiver of type \( \tilde{A}_{n-1} \), i.e. the oriented graph with vertex set \( I = \mathbb{Z}/n\mathbb{Z} \) and edge set \( \Omega = \{(i, i + 1), i \in I\} \). For any \( I \)-graded \( \mathbb{F} \)-vector space \( V = \bigoplus_{i \in I} V_i \), let \( E_V \subset \bigoplus_{(i, j) \in \Omega} \text{Hom} (V_i, V_j) \) denote the space of nilpotent representations of \( Q \). The group \( G_V = \prod_{i \in I} GL(V_i) \) acts on \( E_V \) by conjugation. For each \( i \in I \) there exists a unique simple \( Q \)-module \( S_i \) of dimension \( e_i \), and for each pair \( (i, l) \in I \times \mathbb{N}^* \) there exists a unique (up to isomorphism) indecomposable \( Q \)-module \( S_{i;l} \) of length \( l \) and tail \( S_i \). Furthermore, every nilpotent \( Q \)-module \( M \) admits an essentially unique decomposition

\[
M \simeq \bigoplus_{i,l} a_i^l S_{i;l}.
\]

We denote by \( \overline{m} \) the isomorphism class of \( Q \)-modules corresponding (by \([8,11]\)) to the multisegment \( m = \sum a_i^l [i;l]) \). For \( m \in \mathcal{M} \) with \( \dim m = d \) and \( V_d \) an \( I \)-graded vector space of dimension \( d \), we let \( O_m \subset E_{V_d} \) be the \( G_{V_d} \)-orbit consisting of representations in the class \( \overline{m} \), and we let \( 1_m \in C_G(V_d) \) be the characteristic function of \( O_m \). Finally, we set \( f_m = q^{-\dim} 1_m \). We will write \( m < n \) if \( O_m \subset O_n \).

3.2 Set \( U_n^- = \bigoplus_d C_G(E_{V_d}) \). Note that, by definition, \((f_m)_{m \in \mathcal{M}} \) is a \( \mathbb{C} \)-basis of \( U_n^- \). The space \( U_n^- \) is endowed with the structure of a (Hall) algebra (see \([22]\)). We use the definitions of \([\mathcal{W} \mathcal{Y}], T\). Moreover, the structure constants for this algebra are polynomials in \( q \), and one can consider \( U_n^- \) as an \( \mathbb{A} \)-algebra with
$q = v^{-1}$. The algebra $\mathcal{U}_n$ is naturally $\mathbb{Z}/n\mathbb{Z}$-graded and we denote by $\mathcal{U}_n^{-}[d]$ the component of degree $d$. Let $\mathcal{U}_n(\widehat{\mathfrak{sl}_n})$ denote the Lusztig integral form of the quantum affine algebra of type $\widehat{A}_{n-1}$ and let $e_i^{(l)}, k_i^{(l)}, f_i^{(l)}, i \in I, l \in \mathbb{N}$ be the divided powers of the standard Chevalley generators. Let $\mathcal{U}_n^-(\widehat{\mathfrak{sl}_n})$ be the subalgebra of $\mathcal{U}_n(\widehat{\mathfrak{sl}_n})$ generated by $f_i^{(l)}, i \in I, l \in \mathbb{N}^*$. It is known that the map $f_i^{(l)} \mapsto f_{\ell_i}$ extends to an embedding of the algebras $\mathcal{U}_n^-(\widehat{\mathfrak{sl}_n}) \hookrightarrow \mathcal{U}_n^-$. 

3.3 For $m \in \mathcal{M}$, set

$$b_m = \sum_{i,n} v^{-i + \dim O_m - \dim O_n} \dim \mathcal{H}^j_{O_m}(\mathcal{I}C_{O_m}) \mathcal{F}_n, \quad (3.2)$$

where $\mathcal{H}^j_{O_m}(\mathcal{I}C_{O_m})$ is the stalk over a point of $O_m$ of the $j$th intersection cohomology sheaf of the closure $\overline{O}_m$ of $O_m$. Then $\mathcal{B} = \{b_m\}$ is the canonical basis of $\mathcal{U}_n^-$, introduced in [UY].

3.4 Let $L, L' \in Y$ be two $n$-step periodic flags in $\mathbb{L}$ satisfying $L' \subset L$. Following Lusztig (see [L2], [GV]) we associate to such a pair a nilpotent representation of $\widehat{A}_{n-1}$ of graded dimension $(\dim_{\mathbb{Y}}(L_i/L'_i))_{i \in \mathbb{Z}/n\mathbb{Z}}$. Let us denote by $L/L'$ this $\widehat{A}_{n-1}$-module. Set

$$a(L', L) = \sum_{i=1}^n \dim_{\mathbb{Y}}(L_i/L'_i)(\dim_{\mathbb{Y}}(L_{i+1}/L'_i) - \dim_{\mathbb{Y}}(L_i/L'_i)).$$

Define a map $\Theta : \mathcal{U}_n^- \rightarrow \mathfrak{s}_{n,r}$ by

$$\Theta(f)(L', L) = q^{-a(L', L)} f(L/L') \quad \text{if } L' \subset L$$

and $\Theta(f)(L, L') = 0$ if $L' \not\subset L$.

In order to describe $\Theta$, we consider the following parametrization of the collection of $\mathcal{G}$-orbits in $\mathbb{Y} \times \mathbb{Y}$. Let $M_{r,n}$ be the set of $\mathbb{Z} \times \mathbb{Z}$-matrices $s = (s_{ij})_{i,j \in \mathbb{Z}}$ with entries in $\mathbb{N}$ such that $s_{i+n,j+n} = s_{i,j}$ and $\sum_j s_{i,j} = r$. To each such $s \in M_{r,n}$ we associate the $\mathcal{G}$-orbit $Y_s$ whose elements are the pairs $(L, L')$ for which

$$s_{ij} = \dim_{\mathbb{Y}} \left( \frac{L_i \cap L'_j}{(L_i \cap L'_{j-1}) + (L_{i-1} \cap L'_j)} \right).$$

For $i, j \in \mathcal{A}_r^n$ we denote by $M_{ij}$ the set of all $s$ such that $Y_s \subset Y_i \times Y_j$. It is easy to see that

$$M_{ij} = \{s \in M_{r,n} \mid \sum_j s_{ij} = \#i^{-1}(i), \sum_i s_{ij} = \#j^{-1}(j) \}.$$
Let us associate to each \( m = \sum a_l[i; l] \) the matrix \((m_{i,j}) \in \bigcup_r M_{r,n}\) with \( m_{i,j} = a_j - i + 1 \). The set \( M^+ = \{(m_{i,j})_{i,j} \in \mathbb{Z} \mid m_{i+n,j+n} = m_{i,j}, \; i > j \Rightarrow m_{i,j} = 0\} \) is then identified with \( M \). If \( i \in A^n_\alpha \) and \( m \in M^+ \) we let \( m^i \in \bigcup_i M_{ij} \) be the matrix whose \((i, j)\)th entry is

\[
\delta_{ij}(\#i^{-1}(j + 1) - \sum_{k \leq j} m_{kj}) + (1 - \delta_{ij})m_{i+1,j}.
\]

**Proposition ([VV]).** The map \( \Theta : U_n^{-} \to \hat{S}_{n,r} \) is an algebra morphism satisfying \( \Theta(u) = \tau(\Theta(u)) \) for every \( u \in U_n^{-} \). Furthermore,

\[
\Theta(f_m) = \sum_{i \mid m^i \in M^+} \tilde{T}_m^i, \quad \Theta(b_m) = \sum_{i \mid m^i \in M^+} c_m^i.
\]

It follows from the above Proposition that \( T_{n,r} \) is endowed with a canonical \( U_n^{-}\)-module structure.

**Theorem ([S]).** The vector space \( R \) is a graded central subalgebra of \( U_n^{-} \) and the multiplication map induces an isomorphism \( U_n^{-}(\mathfrak{sl}_n) \otimes A R \cong U_n^{-} \). Moreover there exists surjective algebra morphisms \( i_r : R \to Z_r^{-} \) and an algebra isomorphism \( i : R \to \Gamma \) such that

\[
\rho_r \circ \Theta = \sigma_r \circ i_r, \quad i = \lim_{\leftarrow} i_r.
\]

Let \( s_\lambda \in \Gamma \) be the Schur polynomial associated to \( \lambda \in \Pi \), and set \( a_\lambda = i^{-1}(s_\lambda) \). Then \( i_r(a_\lambda) = s_\lambda(X_1^{-1}, \ldots, X_r^{-1}) \) for any \( r \geq t(\lambda) \). For \( m \in M \), define polynomials \( J^m_\lambda \in \mathbb{Z}[v, v^{-1}] \) by

\[
a_\lambda = \sum_m J^m_\lambda b_m. \tag{3.3}
\]

**Corollary.** For any \( r \in \mathbb{N} \) and \( i \in A^n_\alpha \) we have

\[
e_i s_\lambda(X_1^{-1}, \ldots, X_r^{-1}) = \sum_{m \mid m^i \in M^+} J^m_\lambda c_m^i. \tag{3.4}
\]

**Proof.** This follows by applying \( a_\lambda \) to \( e_i \cdot 1 \in T_{n,r} \), and using Theorem 3.5 and Proposition 3.4. \( \square \)

**Remarks.**

i) Let us consider the case \( n = 1 \) and \( i = (1^r) \). Then \( M = \Pi \) and \( U_1^{-} = R \cong \Gamma \), and it is known that \( i \) identifies the Poincaré-Birkhoff-Witt basis element \( f_\lambda \) with the Hall-Littlewood polynomial \( P_\lambda \) (see [Mac], Chap. III). In
particular, $K^λ μ(v)$ is the Kostka-Foulkes polynomial and from (3.4) we recover the well-known result of Lusztig ([L1]) concerning the Satake isomorphism

$$\sum_{σ ∈ S_r} T_σ s_λ(X_1^{−1}, \ldots , X_r^{−1}) = \sum_{μ ∈ Π} K^λ μ(v) T_{σr} μ_ρ r$$

ii) Define the following symmetric bilinear form on $U_n^−$ (the Green’s scalar product):

$$⟨f_m, f_{m'}⟩ = v^{−2 \dim Aut(m)} \frac{(1 − v^2)^{|m|}}{|Aut(m)|} δ_{m, m'},$$

where $Aut(m)$ stands for the group of automorphism of any representation in the orbit $O_m$ and $| \sum a_i[i; l]| = \sum i t a_i$. It is natural to consider the restriction of this scalar product $(,)$ on $U_n^−$ to $R \cong Γ$. Let $M_{per}$ denote the set of multisegments of the form $m = \sum a_i[i; l]$ such that $a_i = a_j$ for all $i, j$. By [S], Proposition 2.4 we have

$$R = \left( \bigoplus_{m \notin M_{per}} Ab_m \right)^{−1}.$$

Hence the restriction of $(,)$ to $R$ is nondegenerate. When $n = 1$ this restriction coincides, up to a constant, with the Hall-Littlewood scalar product. Let $(p_μ)_μ ∈ Π$ be the basis of power-sum symmetric functions and let $z(1^{m_1+2^{m_2}+\ldots}) = \prod_i m_i! 1^{−v^2 m_i}.$

**Conjecture.** The restriction of Green’s scalar product on $R ⊂ U_n^−$ is given by

$$(p_λ, p_μ) = δ_{λ, μ} z_λ v^{−2(n−1)|λ| (1 − v^2)^{|λ|} |μ|} \prod_{i = 1}^{l(λ)} \frac{1 − v^{−2λ_i}}{(1 − v^{−2λ_i})^2}.$$

This scalar product can be seen as a higher-rank analogue of the Hall-Littlewood scalar product.

## 4 Uglov’s Fock spaces

### 4.1

Let $n, l$ be positive integers and let $s_l ∈ Z^l$. Following [JMMO], Uglov attached to this data an integrable $U_v(\widehat{sl}_n)$-module $Λ_∞$ equipped with a distinguished basis $\{ |λ, s_l \}$, $λ_i ∈ Π^l$ (the higher-level Fock space, see [J], Section 1). The Fock space $Λ_∞$ is also endowed with an action of a Heisenberg algebra $H$ generated by operators $B_m$, $m ∈ Z^r$ (see [U], Sections 4.2, 4.3). Moreover, the $U_v(\widehat{sl}_n)$-action and the $H$-action commute.

**Remark.** When $l = 1$, Uglov’s Fock space coincides with the Fock space $Λ_∞$ introduced in [KMS].

### 4.2

We now extend the action of $U_v(\widehat{sl}_n)$ on $Λ_∞$ to an action of $U_∞^−$. We follow the method of Varagnolo-Vasserot [VV], Section 5. Let $U_∞^−$ be the Hall algebra of the quiver of type $A_∞$. It is known that $U_∞^− = U_v(\widehat{sl}_∞)$. Let $f_i$, $i ∈ Z$ be the standard generator corresponding to the vertex $i$.
We associate to each \( \lambda_l = (\lambda^{(1)}, \ldots, \lambda^{(l)}) \in \Pi^l \) an l-tuple of Young tableaux \((T_1, \ldots, T_l)\) such that

i) \( T_d \) is of shape \( \lambda^{(d)} \) for \( d = 1, \ldots, l \),

ii) The \((i, j)\)-box of \( T_d \) is filled with content \( s_d + i - j \).

If \( \lambda_l \) and \( \mu_l \) are two \( l \)-multipartitions such that \( \gamma = \mu_l \backslash \lambda_l \) corresponds to a box with content \( k \in \mathbb{Z} \), we say that \( \gamma \) is an addable \( k \)-box of \( \lambda_l \) and a removable \( k \)-box of \( \mu_l \). Let \( \gamma, \gamma' \) be two addable \( k \)-boxes of \( \lambda_l \). We say that \( \gamma < \gamma' \) if \( \gamma \) and \( \gamma' \) belong to \( T_d \) and \( T_d' \), respectively and \( d < d' \).

Let \( \lambda_l, \mu_l \in \Pi^l \) be such that \( \mu_l \backslash \lambda_l \) is a \( k \)-box. Define

\[
N^>(\mu_l, \lambda_l) = \# \{ \text{addable } k - \text{boxes } \gamma' \text{ of } \lambda_l \text{ such that } \gamma' > \gamma \} - \# \{ \text{removable } k - \text{boxes } \gamma' \text{ of } \lambda_l \text{ such that } \gamma' > \gamma \}.
\]

**Proposition.** The following endows \( \Lambda_{\infty}^\omega \) with a structure of a \( U_{\infty}^- \)-module:

\[
f_k \cdot |\lambda_l, s_l\rangle = \sum_{\mu_l} v^{N^>(\mu_l, \lambda_l)} |\mu_l, s_l\rangle
\]

where the sum ranges over all \( \mu_l \) for which \( \mu_l \backslash \lambda_l \) is a \( k \)-box.

**Proof.** Straightforward. \( \square \)

Define operators \( k_k \in \text{End} (\Lambda_{\infty}^\omega) \), \( k \in \mathbb{Z} \) by \( k_k \cdot |\lambda_l, s_l\rangle = v^{N_k(\lambda_l)} |\lambda_l, s_l\rangle \) where

\[
N_k(\lambda_l) = \# \{ \text{addable } k - \text{boxes of } \lambda_l \} - \# \{ \text{removable } k - \text{boxes of } \lambda_l \}.
\]

Now let \( d \in \mathbb{N}^{(2)} \) and set \( \overrightarrow{d} = (d_1, \ldots, d_n) \) where \( d_i = \sum_{j \equiv i \text{ (mod } n)} d_j \). Let \( V \) be a \( \mathbb{Z} \)-graded \( \mathbb{F} \)-vector space of dimension \( d \) and let \( \overrightarrow{V} \) be the \( \mathbb{Z}/n\mathbb{Z} \)-graded \( \mathbb{F} \)-vector space with \( \overrightarrow{V}_i = \bigoplus_{j \equiv i} V_j \). The collection of subspaces \( \overrightarrow{V}_i = \bigoplus_{j \equiv i} V_j \) defines a filtration of \( \overrightarrow{V} \) whose associated graded is \( V \). Set

\[
E_{\overrightarrow{V}} = \{ x \in E_{\overrightarrow{V}} \mid x(\overrightarrow{V}_{\geq i}) \subset \overrightarrow{V}_{\geq i+1} \text{ for all } i \}.
\]

Let \( p : E_{\overrightarrow{V}} \to E_V \) be the projection onto the graded. Let \( j : E_{\overrightarrow{V}} \subset E_V \) be the closed embedding. Following [VV], define a map \( \gamma_d : U_{\infty}^- \overrightarrow{d} \to U_{\infty}^- [d] \) by

\[
\gamma_d|_{v=q^{-1}} : \mathbb{C}[E_{\overrightarrow{V}}] \to \mathbb{C}[G_V(E_V)]
\]

\[
f \mapsto q^{-h(d)} p_n j^*(f)
\]

where \( h(d) = \sum_{i<j, i \equiv j} d_i (d_{j+1} - d_j) \).

For all \( \lambda_l \in \Pi^l \) and \( x \in U_{\infty}^- \) we put

\[
x \cdot |\lambda_l, s_l\rangle = \sum_{d} \left( \gamma_d(x) \prod_{j<i, j \equiv i} k_j^d \right) \cdot |\lambda_l, s_l\rangle.
\]

(4.1)

Then (see [VV] Section 6.2, and [AA])

**Proposition.** Formula (4.1) defines a representation \( \Xi : U_{\infty}^- \to \text{End} (\Lambda_{\infty}^\omega) \) which extends Uglov’s action of \( U_{\infty}^- (\mathfrak{sl}_n) \).
Remarks. i) The number $h(d)$ has the following interpretation. Let $\mathcal{F}_d$ be the variety of filtrations of $V$ whose associated graded is of dimension $d$. Then $\dim T^* \mathcal{F}_d = \dim G_T + h(d)$.

ii) The map $\gamma_d$ is “upper triangular” in the following sense. Let $x \in E_V$ and define $r(x) \in E_V$ by $r(x)_i = \bigoplus_j x_j$. Then $\gamma_d(f_m)(x) \neq 0 \Rightarrow r(x) \in O_m$.

4.3 Let $\mathcal{H}^- \subset H$ denote the subalgebra generated by $B_{-m}, m \in \mathbb{N}^*$. Define an algebra isomorphism $j: \Gamma \rightarrow H$ by setting $j(p_m) = B_{-m}$, where $p_m$ is the power-sum symmetric function. Recall the canonical map $i: \mathbb{R} \rightarrow \Gamma$ from Theorem 3.5.

Lemma. We have $\Xi|_{\mathbb{R}} = j \circ i$.

Proof (sketch). This is shown in a way similar to [VV]. We first consider the “limit” $\bigotimes \mathcal{T}^\infty$ of $T_{n,r}$ when $r \rightarrow \infty$ (see [VV], Section 10). Then $\Lambda^\infty_{\mathbb{R}}$ is naturally embedded in a certain quotient of $\bigotimes \mathcal{T}^\infty$ (see [U], Section 3.3). In particular, the $U^-$-action on $T_{n,r}$ induces an action on $\bigotimes \mathcal{T}^\infty$ and on $\Lambda^\infty_{\mathbb{R}}$. Let $\Xi'$ denote this last action. It follows from Theorem 3.5 and [U], Section 4 that $\Xi'|_{\mathbb{R}} = j \circ i$. Finally, an easy extension to the higher-level Fock space of the computation in [VV], Lemma 10.1 shows that $\Xi' = \Xi$. □

5 Canonical bases of Fock spaces

5.1 We keep the settings of the previous Section. Uglov has defined a semilinear involution $a \mapsto \overline{a}$ on $\Lambda^\infty_{\mathbb{R}}$ ([U], Section 4.4) and two canonical bases $\{b^\pm_{\lambda_i}\}_{\lambda_i \in \Pi^}$ characterized by the following properties:

$$b^\pm_{\lambda_i} = b^\pm_{\lambda_i},$$

$$b^+_{\lambda_i} \in \langle \lambda_i \rangle + v \bigoplus_{\mu_i} \mathbb{S}|\mu_i\rangle, \quad b^-_{\lambda_i} \in \langle \lambda_i \rangle + v^{-1} \bigoplus_{\mu_i} \mathbb{S}|\mu_i\rangle.$$

He furthermore computed the transition matrices $[b^\pm_{\lambda_i} : |\mu_i, s_i\rangle]$. In particular we have the following result.

Theorem ([U], 3.26).

$$b^\pm_{\lambda_i} = \sum_{\mu_i} P^\pm_{\mu_i, \lambda_i} |\mu_i, s_i\rangle.$$ 

Remark. When $l = 1$, Uglov’s canonical bases coincide with the canonical bases considered by Leclerc-Thibon ([LT]). In that setting, the transition matrices above were first obtained by Varagnolo and Vasserot ([VV]).

5.2 Let us now consider the nondegenerate scalar product $\langle , \rangle$ on $\Lambda^\infty_{\mathbb{R}_1}$ for which $\{\langle \lambda_i, s_i\rangle\}$ is orthonormal. Let $\{b^\pm_{\lambda_i}\}$ be the dual basis to $\{b^\pm_{\lambda_i}\}$ with respect to the scalar product $\langle , \rangle$.

Define a semilinear isomorphism $\Lambda^\infty_{\mathbb{R}_1} \rightarrow \Lambda^\infty_{\mathbb{R}_1}$, $u \mapsto u'$ by $|\lambda_i, s_i\rangle' = |\lambda_i', s_i'\rangle$.

Proposition ([U], 5.14). We have $(b_{\lambda_i}^{\pm\pm})' = b_{\lambda_i}^{\pm\pm}$. 

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5.3 Let $B_{si} = \{b^+_λ\}_{λ \in \Pi}$ be the (positive) canonical basis of $Λ^∞_i$.

**Theorem.** Let $m \in M$. Then $b_m \cdot |0, s_i⟩ = B_{si} \cup \{0\}$.

**Proof.** Lemma 4.3 implies that the $U^−_n$-action on $Λ^∞_i$ is the same as that considered in [3], Section 4. The result follows from [3], Theorem 4.2. □

5.4 Define a map $τ_{si} : M → Π^i \cup \{0\}$ by $τ_{si}(m) = 0$ if $b_m \cdot |0, s_i⟩ = 0$ and $b_m \cdot |0, s_i⟩ = b^+_τ_{si}(m)$ otherwise. This map is not easy to describe for a general $s_i$. Nevertheless we have the following result.

Let $m \in M$ and let $λ = (λ^{(1)}, \ldots, λ^{(n)})$ be the associated $n$-multipartition. Let $r = \sum_i l(λ^{(i)})$. Set $i = (1^{(λ^{(1)})}, 2^{(λ^{(2)})}, \ldots) \in A^+_n$ and

$$p = (λ^{(1)}, \lambda^{(1)}, λ^{(2)}, \ldots) \in (Z^+)^r.$$  

Finally, let $M_i(p) = (p^{(1)}, \ldots, p^{(l)})$ and let $r_i \in Z/nZ$ be the residue of $p^{(i)}$.

**Lemma.** Suppose that $s_1 ≫ s_2 ≫ \cdots ≫ s_l$ and that $s_i ≡ r_i \pmod{n}$ for $i = 1, \ldots, l$. Then $τ_{si}(m) = M_i(p)$.

**Proof.** See appendix. □

5.5 **Proof of Theorem 1.** Let $σ \in S_1 \setminus \mathcal{E}_r / S_1$. It follows from Corollary 3.5 that $J^1_{λ, σ} = J^1_{λ, m}$ if there exists $m \in M$ such that $m_i^1 = σ$ and $J^1_{λ, m} = 0$ otherwise. From Section 3.4 we see that

$$m_i^1 = σ ⇐⇒ i \cdot σ \in (Z^+)^r \text{ and } m = \sum_{j=1}^r |i_j; (i \cdot σ)_j⟩.$$  

Now we compute $J^1_{λ, m}$. Let $l, p, (r_i)_{i=1}^l$ be associated to $m$ as in Section 5.4. Let $s_i = (s_{i,1}, \ldots, s_{i,l})$ be in the asymptotic region $s_1 ≫ s_2 ≫ \cdots ≫ s_l$ and satisfy $s_i ≡ r_i \pmod{n}$ for all $i$. We evaluate both sides of (3.3) on $|0, s_i⟩ \in Λ^∞_i$. On the one hand, it follows from Lemma 4.3 and Uglov’s description of the action of the Heisenberg algebra [3], Proposition 5.3 that

$$a_λ \cdot |0, s_i⟩ = \sum_{µ_1, \ldots, µ_l} e_µ^i \cdot µ_1, \ldots, µ_l \sum_{j=0}^{(b-1)\mid µ_i \mid} e_{µ_1, µ_i} \cdots e_{µ_l, µ_i} |(µ_1, \ldots, µ_l), s_i⟩,$$

where $e_µ^{i, µ_i} \in Z[ν^{-1}]$ are defined by the relations $s_{i,µ} \cdot |0⟩ = \sum_{j=0}^{s_{i,µ_i}} e_µ^{i, µ_i} |µ_i⟩$ in the level $l=1$ Fock space representation of $U^−_n$. But by [3], Theorem 6.9 we have $s_{i,µ} \cdot |0⟩ = b_{µ_i}^−$ and thus $e_µ^{i, µ_i} = P_{i, µ_i}^−$. On the other hand, from Theorem 5.3 we have

$$\sum_{n} J_{λ, n} b_n \cdot |0, s_i⟩ = \sum_{n, τ_{si}(n) \neq 0} J_{λ, n} b^+_τ_{si}(n).$$

In particular, $J_{λ, m} = (b^+_τ_{si}(m))^* a_λ |0, s_i⟩$. But by Lemma 5.4 and Proposition 5.2,

$$(b^+_τ_{si}(m))^* = (b^+_τ_{si}(p))^* = (b^+_τ_{σ(p)})^* = (b^+_τ_{σ(σ)})^* = (b^+_τ_{σ(σ)})′.$$
Using the relations \((\mathbf{v}, \mathbf{v}) = (u', v')\) for any \(u, v \in \Lambda_\infty^+(\mathbb{U})\), Proposition 5.13) and \(a_\lambda \cdot |0, s_i⟩ = a_\lambda \cdot |0, s_i⟩\) (\(\mathbb{U}\), Proposition 4.2) we get

\[ J_{\lambda, m} = (b_{\lambda \mathcal{M}(\sigma)'}, a_\lambda \cdot |0, s_i⟩). \]

Now, from \(\mathbb{U}\), Theorem 7.13) we have \((b_{\mu \gamma})' = (-v)^{(\mu - 1)|\mu|} b_{\gamma \mu}\) in the level \(l = 1\) Fock space. Thus

\[ a_\lambda \cdot |0, s_i⟩ = (-v)^{(\mu - 1)|\mu|} \sum_{\mu_1, \ldots, \mu_l} c_{\mu_1, \ldots, \mu_l}^\lambda \prod_{k=1}^l (\mathbf{v}_{\mu_k, \mu_k}) |\mu, s_i⟩ \]

where \(\nu = (\nu_1, \ldots, \nu_l)\). The theorem follows. \(\square\)

6 On the Center of \(U_n\)

In this section we give a simple geometric characterization of the central subalgebra \(\mathbb{R} \subset U_n\) in terms of the maps \(\gamma_d : U_n^- \rightarrow U_\infty\) defined in Section 4.2.

6.1 Let \(d \in \mathbb{N}^{(2)}\) such that \(d_i \in \{0, 1\}\) for all \(i\). Then \(d\) is the dimension of a unique (noncyclic) multisegment \(n_d = \sum_{k=1}^l |i_k; l_k⟩\) in \(\mathbb{Z}\) satisfying the following condition:

\[ \forall j, k \quad [i_k, l_k] \cup [i_j, l_j] \text{ is not a segment.} \quad (6.1) \]

Let \(V_d\) be a \(\mathbb{Z}\)-graded \(\mathbb{F}\)-vector space of dimension \(d\). Set \(l(d) = \sum_k (l_k - 1)\).

Note that it follows from \(\mathbb{U}\) that \(E_{V_d}\) has a unique open \(G_{V_d}\)-orbit, say \(\mathcal{O}_d\).

**Lemma.** Suppose that \(i_1 \gg i_2 \gg \cdots \gg i_t\) and set \(\mathbf{i}_t = (i_1, \ldots, i_t)\). Then for any \(f \in U_\infty[d]\) we have

\[ f \cdot |0, \mathbf{i}_t⟩ = v^{-l(d)} f|_{\mathcal{O}_d} ([l_1], \ldots, [l_t]), \mathbf{i}_t⟩. \]

**Proof.** Note that \(E_{V_d} = \prod_{k=1}^l E_{V_d(k)}\) where \(V_d(k) = \bigoplus_{t=0}^{l_k-1} \mathbb{F} V_{i_k+i}t\). Let \(f_k \in E_{V_d(k)}\) for \(k = 1, \ldots, t\). From \(\mathbb{U}\) and Section 4.2 we deduce that

\[ f_1 \cdots f_t \cdot |0, \mathbf{i}_t⟩ = \sum_{\nu_1, \ldots, \nu_t} d_1(\nu_1) \cdots d_t(\nu_t) |(\nu_1, \ldots, \nu_t), \mathbf{i}_t⟩ \]

where \(f_k|0, i_k⟩ = \sum_{\nu} d_k(\nu)|\nu, i_k⟩\) in the level \(l = 1\) Fock space. But from \(\mathbb{U}\), Proposition 5., it is easy to see that \(f_k|0, i_k⟩ = v^{-(l_k-1)}(f_{k}|_{\mathcal{O}_d(k)}) |(l_k, i_k)⟩\) where \(\mathcal{O}_d(k) \subset E_{V_d(k)}\) is the open orbit. \(\square\)

6.2 Recall the element \(a_\lambda = i^{-1}(s_\lambda) \in \mathbb{R}\). For any \(\lambda, \mu \in \Pi\) let \(K_{\mu}^\lambda \in \mathbb{N}\) be the Kostka number.

**Theorem.** Let \(d \in \mathbb{N}^{(2)}\) such that \(d_i \in \{0, 1\}\). Then

\[ \gamma_d(a_\lambda) |_{\mathcal{O}_d} = v^{l(d)} b(h(d)) K_{(u_1, \ldots, u_t)}^\lambda \]

if there exists \(i_k, u_k \in \mathbb{Z}, k = 1, \ldots, t\) such that \(n_d = \sum_{k=1}^l |i_k; nu_k⟩\), and \(\gamma_d(a_\lambda) |_{\mathcal{O}_d} = 0\) otherwise.
Proof. Without loss of generality we may assume that \( n = \sum_{k=1}^l [i_k : l_k] \) where \( i_1 > i_2 > \cdots > i_t \). Choose \( d' = \bigcup_{k=1}^l [i_k' : l_k] \) where \( i_k' \equiv i_k \pmod{n} \) and \( i_1' \gg i_2' \gg \cdots \gg i_t' \). Let \( \xi : \mathbb{E}_{d'} \rightarrow \mathbb{E}_{d} \) be the obvious isomorphism. Then \( \xi \circ \gamma_{d'} = \gamma_{d} \). Now let us consider the Fock space \( \Lambda^\infty_{d'} \) where \( i'_t = (i'_1, \ldots, i'_t) \). Using [3], Proposition 5.3 we have

\[
(a_\lambda \cdot |0, i'_t \rangle, |(l_1, \ldots, l_t)\rangle |_{i'_t}) = \sum_{\mu_1, \ldots, \mu_t} \prod_{j=1}^t \gamma^\lambda_{(l_j)} P_{(l_j), n_{\mu_j}} \cdots P_{(l_t), n_{\mu_t}}
\]

Note that for any \( u_1, \ldots, u_t \in \mathbb{Z} \) we have \( \gamma^\lambda_{(u_1), \ldots, (u_t)} = K^\lambda_{\mu} \) where \( \mu \in \Pi \) is the partition with parts \( \{u_1, \ldots, u_t\} \).

On the other hand, by Lemma 6.1

\[
(a_\lambda \cdot |0, i'_t \rangle, |(l_1, \ldots, l_t)\rangle |_{i'_t}) = v^{e(d', i'_t) - (d, i'_t)} \gamma_{d'}(a_\lambda)\sigma_{d'}
\]

where

\[
e(d', i'_t) = \sum_{j=1}^t \sum_{j<i} d'_j.
\]

The result now follows from the easily checked identity

\[
e(d', i'_t) = \sum_b (b-1)|\mu_b| + h(d')
\]

when there exists \( u_k \in \mathbb{N}, k = 1, \ldots, t \) such that \( d' = \bigcup_k [i'_k : nu_k] \) and \( \mu_k = (u_k) \).

Remark. It follows from Remark 4.2 ii) that the previous theorem gives a characterization of the central element \( a_\lambda \).

7 Appendix

In this appendix we prove Lemma 5.4.

A.1 As in [3], Section 4, define a partial order on \( \Pi^l \) (depending on \( s_l \)) as follows. Let \( \mu = (\mu^{(1)}, \ldots, \mu^{(l)}) \in \Pi^l \). Set \( k^{(d)}_i = \mu^{(d)}_i + s_d + 1 - i \) for \( d = 1, \ldots, l \) and \( i \in \mathbb{N} \). Let us write \( k^{(d)}_i = c^{(d)}_i - nm^{(d)}_i \) where \( c_i \in \{1, \ldots, n\} \), and let \( k = (k_1 > k_2 > \cdots) \) be the ordered sequence whose underlying set is \( \{c_i^{(d)} + n(d-1) - nm^{(d)}_i | i \in \mathbb{N}, d = 1, \ldots, l\} \). Let \( s = s_1 + \cdots + s_l \). It is easy to see that \( k_i = s + 1 - i \) for \( i \gg 0 \) and we denote by \( \zeta(\mu) \) the partition such that \( \zeta(\mu)_i = k_i - s + i - 1 \). Now let \( \mu, \nu \in \Pi^l \). By definition, we set \( \mu \leq \nu \) if \( \zeta(\mu) \leq \zeta(\nu) \).

A.2 From now on we assume that \( v = 1 \).
It is more convenient to work with a different basis than \{f_n\}. Let \( n \in \mathcal{M} \) and let \( x \in \mathcal{O}_n \). Set \( V_k = \text{Ker} x^k \) and let \( \alpha^1, \ldots, \alpha^r \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}} \) be such that
\[
\dim V_k = \alpha^1 + \cdots + \alpha^k, \quad k = 1, \ldots, r
\]
and \( \dim V_r = \dim n \). Let \( f_{\alpha^i} \in U_n \) be the characteristic function of the trivial representation of the quiver \( \tilde{A}_{n-1} \) on \( V_{\alpha^i} \simeq V_i/V_{i-1} \).

**Lemma 1** ([VV], Section 13). We have \( f_{\alpha^1} \cdots f_{\alpha^r} \in f_n + \bigoplus_{l < n} Nf_l \).

Now let \( n, l \in \mathcal{M} \) such that \( \dim n = \dim l \). Let \((\beta^k)\) and \((\gamma^k)\) be the sequences of dimensions attached as above to \( n \) and \( l \) respectively. If \( u, v \in \mathbb{Z}/n\mathbb{Z} \) we write \( u \leq v \) if \( u_i \leq v_i \) for all \( i \in \mathbb{Z}/n\mathbb{Z} \).

**Lemma 2.** We have \( n \geq l \) if and only if \( \beta^1 + \cdots + \beta^k \leq \gamma^1 + \cdots + \gamma^k \) for all \( k \).

**Proof.** Straightforward. \( \square \)

We will write \((\beta^k) \leq (\gamma^k)\) if (a) holds and if \( \sum_k \beta^k = \sum_k \gamma^k \). Let \((\alpha^1, \ldots, \alpha^r)\) be the sequence attached to \( m \). We will first prove
\[
f_{\alpha^1} \cdots f_{\alpha^r} \cdot |0, s_i \rangle \in \mathbb{N}^*|\mathcal{M}_l(p), s_i \rangle + \bigoplus_{\mu \in \mathcal{M}_l(p)} \mathbb{N}|\mu, s_i \rangle
\]
and
\[
f_{\beta^1} \cdots f_{\beta^r} \cdot |0, s_i \rangle \in \bigoplus_{\mu \notin \mathcal{M}_l(p)} \mathbb{N}|\mu, s_i \rangle \quad \text{for all } (\beta^k) > (\alpha^k).
\]

**Lemma 3.** Let \( \mu = (\mu^{(1)}, \ldots, \mu^{(l)}) \in \Pi^l \) and let \( \beta \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}} \). We have
\[
f_{\beta} \cdot |\mu, s_i \rangle = \sum_{\nu} |\nu, s_i \rangle
\]
where the sum ranges over all multipartitions \( \nu = (\nu^{(1)}, \ldots, \nu^{(l)}) \) such that
i) \( \nu^{(i)} \setminus \mu^{(i)} \) is a skew diagram with at most one box in each row,
ii) The number of boxes in \( \cup_{i} \nu^{(i)} \setminus \mu^{(i)} \) with content \( j \mod n \) is \( \beta_j \).

**Proof.** Let \( d \in \mathbb{N}^{\mathbb{Z}} \) such that \( d \equiv \beta \) (mod \( n \)). Then \( \gamma_{d|\nu=1}(f_\beta) = \prod_i f_{\beta_i}^{(d_i)} \), where \( \prod \) denotes the ordered product from \(-\infty \) to \( \infty \) (see [VV], Remark 6.1) and where \( f_{\beta_i}^{(d_i)} \) is the divided power. Moreover, for any \( \sigma \in \Pi^l \),
\[
f_{\sigma} \cdot |\mu, s_i \rangle = \sum_{\gamma} |\gamma, s_i \rangle
\]
where the sum ranges over all \( \gamma \in \Pi^l \) such that \( \gamma \setminus \sigma \) is an \( i \)-box. The Lemma now follows from Section 4.2. \( \square \)
Finally, recall that \( s_1 \gg s_2 \gg \cdots \gg s_l \). It is clear from the definition that for \( \mu, \lambda \in \Pi^l \),
\[
\mu \geq \lambda \Rightarrow \exists k \text{ such that } \mu^{(i)} = \lambda^{(i)} \text{ for } i = 1, \ldots, k - 1 \text{ and } \mu^{(k)} \geq \lambda^{(k)}. \tag{d}
\]
Note that \( a_i^k \) is equal to the number of boxes with content \( i \) in the slice \( s_k \) of the diagram \( D_p \) associated to \( p \). Statements (b) and (c) now easily follow by Lemma 3 and by construction of \( \mathcal{M}_1(p) \).

**A.3** By [U], Theorem 2.4 it possible to choose \( s_l \gg s_{l+1} \gg \cdots \gg s_t \) for some \( t \gg 0 \) in such a way that \( b_m |s_t⟩ \neq 0 \), where \( s_t = (s_1, \ldots, s_t) \).

**Lemma 4.** We have \( b_m |0, s_t⟩ = b^+_{\mathcal{M}_1(p)}, \) where \( \mathcal{M}_1(p) = (\mathcal{M}_1(p), 0^f - 1) \).

**Proof.** By Lemma 1, we have
\[
f_{\alpha_1} \cdots f_{\alpha_r} |0, s_t⟩ \in |\tau_{s_t}(m), s_t⟩ + \bigoplus_{\mu < \tau_{s_t}(m)} \mathbb{Z} |\mu, s_t⟩.
\]
But from (b) and (f), it is clear that
\[
f_{\alpha_1} \cdots f_{\alpha_r} |0, s_t⟩ \in |\mathcal{M}_1(p), s_t⟩ + \bigoplus_{\mu \geq \mathcal{M}_1(p)} \mathbb{Z} |\mu, s_t⟩.
\]
Hence \( \tau_{s_t}(m) = \mathcal{M}_1(p) \). \( \square \)

In particular,
\[
b_m |0, s_t⟩ \in |\mathcal{M}_1(p), s_t⟩ + \bigoplus_{\mu < \mathcal{M}_1(p)} \mathbb{N} |\mu, s_t⟩.
\]
Consider the projection \( \pi : \Lambda_{\infty}^s \to \Lambda_{\infty}^s \) given by
\[
|\{\mu^{(1)}, \ldots, \mu^{(t)}\}, s_t⟩ \mapsto \begin{cases} |\{\mu^{(1)}, \ldots, \mu^{(l)}\}, s_l⟩ & \text{if } \mu^{(j)} = 0 \text{ for } j > l \\ 0 & \text{otherwise} \end{cases}
\]
It is clear from (4.1) that \( \pi(b_m |0, s_t⟩) = b_m |0, s_t⟩ \). Hence
\[
b_m |0, s_t⟩ \in |\mathcal{M}_1(p), s_t⟩ + \bigoplus_{\mu < \mathcal{M}_1(p)} \mathbb{N} |\mu, s_t⟩.
\]
This proves Lemma 5.4 \( \square \)

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