Abstract
A knotted surface in $S^4$ may be described by means of a hyperbolic diagram that captures the 0-section of a special Morse function, called a hyperbolic decomposition. We show that every hyperbolic decomposition of a knotted surface $K$ defines a projection of $K$ onto a 2-sphere $\Sigma$, whose set of critical values is the hyperbolic diagram of $K$. We apply such projections, called flattenings, to define three invariants of knotted surfaces: the layering, the trunk, and the partition number. The basic properties of flattenings and their derived invariants are obtained. Our construction is used to study flattenings of satellite 2-knots.

Keywords Embedded surface · Hyperbolic decomposition · Flattening · Width · Layering · Trunk · Partition number

Mathematics Subject Classification 57K45

1 Introduction

Width of knotted surfaces was first considered by Carter and Saito [2] in the context of charts. A chart of a knotted surface $K$ in $\mathbb{R}^4$ is obtained by taking a projection $p : \mathbb{R}^4 \to \mathbb{R}^3$ that is generic with respect to $K$, and then projecting the singular set of the associated immersion $p(K)$ onto a 2-plane. Takeda studied the width of surface knots using generic planar projections of embedded surfaces and considering possible images of such projections that consist of fold curves and cusp points [20].

In this paper, we study width–related invariants of embedded surfaces from a different perspective. Instead of considering generic projections, we apply hyperbolic decompositions of a knotted surface $K$ to define its layering, trunk and partition number. We show that each hyperbolic decomposition of $K$ induces a projection of $K$ to a 2-sphere, and the set of critical values of this projection may be identified with a ch-diagram of the surface $K$, defined by Yoshikawa [23]. Such diagrams and closely related banded link diagrams are often used in the study of surfaces inside 4-manifolds, thus we hope our description might lead to new results concerning width and related invariants of knotted surfaces.
The idea behind our invariants comes from the classical knot theory. The bridge number of classical knots was first considered by Schubert [16], while the more general notion of width was defined by Gabai [6]. The trunk of 1-knots was defined by Ozawa [15]. All three invariants might be interpreted in terms of Morse functions as follows [25]. Let $K$ be a knot in the 3-sphere $S^3$. Denote by $\mathcal{M}(K)$ the collection of all Morse functions $h: S^3 \to \mathbb{R}$ with exactly two critical points, such that the restriction $h|_K$ is also Morse. Given a function $h \in \mathcal{M}(K)$, denote by $c_0 < c_1 < \ldots < c_n$ the critical values of $h|_K$ and choose regular values $r_i \in (c_{i-1}, c_i)$ for $i = 1, 2, \ldots, n$. Each function $h$ defines three values

$$w(h) = \sum_{i=1}^n |K \cap h^{-1}(r_i)|, \quad b(h) = \# \text{of maxima of } h|_K, \quad \text{trunk}(h) = \max_{1 \leq i \leq n} |K \cap h^{-1}(r_i)|$$

that give rise to three knot invariants: the width $w(K) = \min_{h \in \mathcal{M}(K)} w(h)$, the bridge number $b(K) = \min_{h \in \mathcal{M}(K)} b(h)$ and the trunk of a knot: trunk$(K) = \min_{h \in \mathcal{M}(K)}$ trunk$(h)$.

Our aim is to apply a similar construction one dimension higher. First we need to find a suitable family of functions that will do the trick for knotted surfaces. To describe the general setting, we use the following standard results.

**Theorem 1.1** [11] Let $M$ and $N$ be smooth manifolds, and let $f: M \to N$ be a smooth map with constant rank $k$. Each level set of $f$ is a closed embedded submanifold of codimension $k$ in $M$.

**Lemma 1.2** (Ehresmann fibration lemma [4]) Let $M$ and $N$ be smooth manifolds, and let $f: M \to N$ be a proper submersion. Then $M$ is a fiber bundle over $N$ with projection given by $f$.

**Corollary 1.3** Let $K$ be a smoothly embedded surface in a smooth 4-manifold $X$ (the embedding being proper at the boundary if necessary), and let $\Sigma$ be a 2-manifold. Consider a smooth map $f: X \to \Sigma$ and denote by $A \subset K$ the set of critical points of $f|_K$. Then cardinality $|K \cap f^{-1}(x)|$ is constant on each connected component of $\Sigma \setminus f(A)$.

**Proof** For each regular value of $f|_K$, the fiber is a finite discrete set of points by Theorem 1.1. Moreover, the restriction $f|_K\setminus A : K\setminus A \to f(K\setminus A) \subset \Sigma$ is a fiber bundle by Lemma 1.2, so the cardinality of its fibers is constant on each connected component of $\Sigma \setminus f(A)$.

Under the setting described in Corollary 1.3, denote by $U_0, U_1, \ldots, U_n$ the connected components of $\Sigma \setminus f(A)$ and choose regular values $r_i \in U_i$ for $i = 0, 1, \ldots, n$. The Corollary implies that the values

$$\text{lay}(f) = \sum_{i=1}^n |K \cap f^{-1}(r_i)|, \quad p(f) = n, \quad \text{trunk}(f) = \max |K \cap f^{-1}(r_i)|$$

are well defined. Summing up over a suitable collection of maps (or fixing a map and summing up over all equivalent embeddings of $K$ in $X$), we might arrive at three invariants of knotted surfaces inside $X$.

The paper is organized as follows. Section 2 contains the basic material about presentations of knotted surfaces that we will need. In Sect. 3, we present a projection of an embedded surface $K$ called a flattening, which is associated to a marked graph diagram of $K$. We discuss its critical values and introduce the terminology needed to describe a flattening. Multiplicities of the flattening map and their basic properties are discussed. In Sect. 4, we define three invariants of knotted surfaces based on flattenings: the layering, the trunk and the partition...
Some basic results regarding these invariants are obtained. In Subsect. 4.1, we study flattenings and the associated invariants of satellite 2-knots. Subsection 4.2 concludes the paper by offering several ideas for further study.

2 Preliminaries

For the remainder of this paper, we restrict our attention to embedded surfaces in $S^4$. Throughout the paper, we denote by $K$ a smoothly embedded, connected closed surface in $S^4$. In this Section we briefly recall the basic descriptions of embedded surfaces that we will work with. A good introductory overview of the subject is offered in [3].

**Definition 2.1** A Morse function $h : S^4 \to \mathbb{R}$ is called a hyperbolic splitting of an embedded surface $K$ if it satisfies the following conditions:

1. $h$ has exactly two critical points on $S^4$,
2. $h_K$ is also Morse,
3. all minima of $h_K$ occur in the level $h^{-1}(-1)$,
4. all maxima of $h_K$ occur in the level $h^{-1}(1)$,
5. all hyperbolic points of $h_K$ occur in the level $h^{-1}(0)$.

It is well known that every embedded surface in $S^4$ admits a hyperbolic splitting [12]. We will denote by $K_t = K \cap h^{-1}(t)$ the $t$-section of the knotted surface, induced by $h$. Similarly, we will denote $S^4_t = h^{-1}(t)$ and $S^4_I = h^{-1}(I)$ for any interval $I \subset \mathbb{R}$.

A quite illuminating presentation of an embedded surface may be given by its movie. A movie of a surface $K$ with a hyperbolic splitting $h : S^4 \to \mathbb{R}$ is a sequence of diagrams of sections $K_t \subset S^4_t$ for $t \in [-1, 1]$. See Fig. 1 for a simple movie of a projective plane.

Instead of following the whole movie, the information about a hyperbolic splitting of a surface may be compressed in a single diagram. Such presentations have been used in the early studies of knotted surfaces (see for example [12, 23]), and they remain an important tool in the recent literature (see [8, 9, 13]). The basic idea is that a hyperbolic splitting, by definition, induces a handle decomposition of the embedded surface. This handle decomposition may be conveniently presented by either a marked graph or a banded link.

![Fig. 1 A movie of a projective plane](image-url)
Following [13], we define a **band** for a link \( L \) in \( S^3 \) as an embedding \( b : I \times I \to S^3 \) such that \( b(\partial I \times I) = b \cap L \). We denote a new link \( L_b = (L \setminus b(\partial I \times I)) \cup b(I \times \partial I) \) and call it the link that results from **resolving** the band \( b \). Similarly, if \( b \) denotes a collection of pairwise disjoint bands for \( L \), we denote by \( L_b \) the link that results from \( L \) by resolving all bands in \( b \). A pair \((L, b)\) is called a **banded link** if \( L \) is a link in \( S^3 \), \( b \) is a band for \( L \) and both \( L \) and \( L_b \) are unlinks.

Let \( h : S^4 \to \mathbb{R} \) be a hyperbolic splitting of a knotted surface \( \mathcal{K} \). It follows from Definition 2.1 that for a small positive \( \epsilon \), both \( \mathcal{K}_{-\epsilon} \) and \( \mathcal{K}_\epsilon \) are unlinks. Moreover, at every hyperbolic point of \( h_\mathcal{K} \), a 1-handle (a band) is added to the boundary of the 0-skeleton of \( \mathcal{K} \) along two intervals, embedded in \( \mathcal{K}_{-\epsilon} \). The attachment of all 1-handles changes the boundary of the resulting surface, which is obtained from \( \mathcal{K}_{-\epsilon} \) by resolving all bands: \( \mathcal{K}_\epsilon = (\mathcal{K}_{-\epsilon})_b \). Thus, the hyperbolic splitting \( h \) defines a banded link \((\mathcal{K}_{-\epsilon}, b)\).

Conversely, given a banded link \((L, b)\), the condition that \( L \) and \( L_b \) are unlinks provides a construction of a knotted surface \( \mathcal{K} = \mathcal{K}(L, b) \) as follows. View the 4-sphere as \( S^4 = S^3 \times [-2, 2]/(S^3 \times [-2], S^3 \times [2]) \) and let \( h : S^4 \to \mathbb{R} \) be the projection to the second component. Define

1. \( \mathcal{K}_{-\epsilon} = L \),
2. \( \mathcal{K} \cap S^4_{[-1, -\epsilon)} \) are disks, capping off every component of \( L \),
3. \( \mathcal{K} \cap S^4_{[-\epsilon, 0)} = L \times [-\epsilon, 0] \),
4. \( \mathcal{K}_0 = L \cup b \), for each component of \( b \), add the band \( b(I \times I) \) to \( L \) along \( b(\partial I \times I) \),
5. \( \mathcal{K} \cap S^4_{[0, \epsilon]} = L_b \times (0, \epsilon] \),
6. \( \mathcal{K} \cap S^4_{[\epsilon, 1]} \) are disks, capping off every component of \( L_b \).

By [13, Proposition 2.4], the disks capping off the components of \( L \) and \( L_b \) are unique up to isotopy. Thus, an embedded surface \( \mathcal{K} \) with a hyperbolic splitting \( h \) is completely determined by the banded link \((L, b)\), defined by \( h \), since \( \mathcal{K} = \mathcal{K}(L, b) \).

Alternatively, a hyperbolic splitting \( h \) of a knotted surface \( \mathcal{K} \) may be presented by a **marked graph diagram**. By Definition 2.1, the 0-section \( \mathcal{K}_0 \) defines an embedded 4-valent graph, with vertices corresponding to saddles (critical points of index 1 of the Morse function \( h_\mathcal{K} \)). A regular projection \( p : S^3_1 \to \Sigma \) takes \( \mathcal{K}_0 \) to its diagram, a 4-valent graph \( \Gamma = p(\mathcal{K}_0) \) in the 2-sphere \( \Sigma \). In this diagram, the vertices corresponding to crossings include the information about the overcrossing and undercrossing strands, while vertices corresponding to saddles are endowed with markers. A marker at a vertex determines the corresponding resolutions below and above the critical point, see Fig. 2. We call \( \Gamma \) a **marked graph diagram** of \( \mathcal{K} \) with the hyperbolic splitting \( h \). Such diagrams were introduced by Yoshikawa [23]; they are also called ch-diagrams.

By the following theorem, an embedded surface with a hyperbolic splitting is completely defined by its marked graph diagram.

**Theorem 2.2** [10] Let \( \mathcal{K}_i \) be embedded surfaces with hyperbolic splittings \( h_i \), and let \( \Gamma_i \) be a marked graph diagram of \( h_i^{-1}(0) \cap \mathcal{K}_i \) for \( i = 1, 2 \). If \( \Gamma_1 = \Gamma_2 \), then \( \mathcal{K}_1 \) is isotopic to \( \mathcal{K}_2 \).

By [9], each banded link \((L, b)\) defines a marked graph as follows. Apply an ambient isotopy of \( S^3 \) to shorten the bands of \( b \), until each band is contained in a small disk, then
Let $\Gamma_1$ denote a smooth map $S : K$ that defines a smooth map $\Phi_1$ of $\Sigma_1 \to K$ to each hyperbolic splitting of a surface $K$, perspectives on its embedding.

In other words, the flow line of the vector field grad $\nabla K$ is a projection of $K$ vertical isotopy of $\Phi_1$. When $G$ is given by a marked graph diagram $\Gamma$, the result of this transformation will also be called a banded link, associated with $\Gamma$.

3 The flattening of an embedded surface

In this Section, we show that each marked graph diagram of an embedded surface $K$ defines a projection of $K$ to a 2-sphere. A description of such “flattened surface” provides a new perspective on its embedding.

To each hyperbolic splitting of a surface $K$, one may associate two particularly nice families of isotopies of $K$. Let $f : F \to S^4$ be a smooth embedding of a surface $K = f(F)$. Choose a hyperbolic splitting $h$ of $K$ and let $f : F \times I \to S^4 \times I$ be a smooth isotopy of $K$ so that $K = f(F \times [0])$. Following [8], we say that $f$ is horizontal with respect to $h$ if $h(pr_1(f(x,t)))$ is independent of $t$ for all $x \in F$. We say that $f$ is vertical with respect to $h$ if for each $x \in F$, the image of $\{x\} \times I$ under $pr_1 \circ f$ is contained in a single orbit of the flow of grad$\nabla h$. Thus, a horizontal isotopy of $K$ moves $K_t$ within $S^4_t$, preserving $h_K$. A vertical isotopy of $K$ changes $h_K$, but preserves the projection of $K$ onto each level set $S^4_t$.

Let $\Gamma$ be a marked graph diagram of an embedded surface $K$ in $S^4$. It follows from our discussion in Sect. 2 that $\Gamma$ determines a hyperbolic splitting $h : S^4 \to \mathbb{R}$ of $K$. Denote by $c(h)$ the two critical points of $h$, and let $v \subseteq K_0$ denote the union of all hyperbolic points of $h_K$. Let $p : S^4_0 \to \Sigma$ be the projection to a 2-sphere $\Sigma$ which is regular on $K_0 \setminus v$ and for which $p(K_0) = \Gamma$. Define a projection $h^\perp : S^4 \setminus c(h) \to \Sigma$ as follows. Denote by $\Phi : \mathbb{R} \times S^4 \to S^4$ the flow of the vector field $\text{grad}(h)$. For any point $x \in S^4 \setminus c(h)$, set

$$h^\perp(x) = p(\Phi(t,x) \cap S^4_0).$$

In other words, the flow line of the vector field grad$\nabla h$ running through a point $x \in S^4 \setminus c(h)$ intersects the 0-section $S^4_0$ in a single point; projection of this point onto $\Sigma$ is the image $h^\perp(x)$.

Lemma 3.1 The map $h^\perp : S^4 \setminus c(h) \to \Sigma$ is smooth.

Proof Since grad$\nabla h$ is a smooth vector field on a compact smooth manifold $S^4$, it generates a smooth flow $\Phi : \mathbb{R} \times S^4 \to S^4$ by [19, page 147, Theorem 6]. For every $x \in S^4 \setminus c(h)$, there exists a unique value $t_x \in \mathbb{R}$ such that $h(\Phi(t_x,x)) = 0$ and thus $h^\perp(x) = p(\Phi(t_x,x))$. This defines a smooth map $S^4 \setminus c(h) \to \mathbb{R}$, $x \mapsto t_x$. The projection $p : S^4_0 \to \Sigma$ is also smooth. $\square$

Denote by $h_K$ (resp. $h^\perp_K$) the restriction of $h$ (resp. $h^\perp$) to $K$. The map $h^\perp_K : K \to \Sigma$ will be called the flattening map that corresponds to the marked graph diagram $\Gamma$ of $K$. If we think of the hyperbolic splitting as a height function on $K$, then the flattening map flattens...
\( \mathcal{K} \) against \( \Sigma \), and as \( \mathcal{K} \) itself is not flat (but might be knotted) we obtain creases along some curves in \( \Sigma \). In our case, these creases are simple to describe.

**Proposition 3.2** Let \( \Gamma \) be a marked graph diagram of an embedded surface \( \mathcal{K} \). If \( h \) denotes the hyperbolic splitting of \( \mathcal{K} \) defined by \( \Gamma \), then the set of critical values of the flattening map \( h_{\mathcal{K}}^{-1}: \mathcal{K} \to \Sigma \) equals \( \Gamma \).

**Proof** It follows from Definition 2.1 that the Morse function \( h \) has no critical points in a 4-ball containing \( \mathcal{K} \). By [19, page 148, Theorem 7], we may choose local coordinates \((x_1, x_2, x_3, x_4)\) so that \( \text{grad}(h) = \frac{\partial h}{\partial x_4} \). Thus, the flow lines of \( \text{grad}(h) \) are parallel vertical lines.

The complement of the 0-section \( \mathcal{K}_0 \) of the surface \( \mathcal{K} \) consists of two components \( \mathcal{K}(0,1) = \mathcal{K} \cap S_{(0,1)}^4 \) and \( \mathcal{K}(-1,0) = \mathcal{K} \cap S_{(-1,0)}^4 \). After applying a horizontal isotopy of \( \mathcal{K} \) that fixes \( \mathcal{K}_0 \), we may assume that at every point \( x \in \mathcal{K}_0 \) (resp. \( x \in \mathcal{K}_0 \)), the vector field \( \text{grad}(h) \) is transverse to \( \mathcal{K} \) and thus \( x \) has a neighbourhood \( \mathcal{U} \) such that \( h_{\mathcal{K}}^{-1}|_{\mathcal{U}} : \mathcal{U} \to h_{\mathcal{K}}^{-1}(\mathcal{U}) \) is a diffeomorphism. It follows that all critical points of \( h_{\mathcal{K}}^{-1} \) are contained in \( \mathcal{K}_0 \), and the set of critical values is contained in \( \Gamma = p(\mathcal{K}_0) \).

Let \( y \in \Gamma \), then \( y = p(x) \) for some \( x \in \mathcal{K}_0 \). First suppose that \( x \) is not a hyperbolic point of \( h_{\mathcal{K}} \). Then the gradient flows of \( \text{grad}(h) \) and \( \text{grad}(h_{\mathcal{K}}) \) coincide and \( \text{grad}(h) \) spans a 1-dimensional linear subspace of \( T_x \mathcal{K} \) that lies in the kernel of \( Dh_{\mathcal{K}}^{-1}(x) \), therefore \( y = h_{\mathcal{K}}^{-1}(x) \) is a critical value of \( h_{\mathcal{K}}^{-1} \).

In case \( x \) is a hyperbolic point of \( h_{\mathcal{K}} \), then \( x = \Phi(0,x) \) is a critical point of the projection \( p \), since \( \mathcal{K}_0 \) fails to be a manifold at \( x \). Thus \( y = h_{\mathcal{K}}^{-1}(x) \) is a critical value of \( h_{\mathcal{K}}^{-1} \).

In order to describe flattenings of knotted surfaces, we introduce the following terminology. Let \( \Gamma \) be an embedded 4-valent graph in a 2-sphere \( \Sigma \). Each connected component of \( \Sigma \setminus \Gamma \) will be called a **region** of \( \Gamma \). Let \( \mathcal{K} \) be a smoothly embedded surface in a 4-manifold \( M \) (the embedding being proper at the boundary if necessary), and let \( f: M \to \Sigma \) be a smooth map, such that the set of critical values of \( f|_{\mathcal{K}} \) is contained in \( \Gamma \). We define the **multiplicity** of \( f|_{\mathcal{K}} \) in a region \( \mathcal{U} \) as \( m_{f|_{\mathcal{K}}}(\mathcal{U}) = |\mathcal{K} \cap f^{-1}(\mathcal{U})| \) for any \( \mathcal{U} \subset \Sigma \). It follows from Corollary 1.3 that the multiplicity of \( f|_{\mathcal{K}} \) in a region is well defined.

It is often convenient to identify sections \( \mathcal{K}_t \) for different values of \( t \). Denote by \( v \subset \mathcal{K}_0 \) the set of all hyperbolic points of \( h_{\mathcal{K}} \), and let \( \mathcal{V} \subset \mathcal{K} \) denote the union of all ascending and descending manifolds of these critical points. For each \( t \in (-1,1) \setminus \{0\} \), there exists a diffeomorphism \( \rho_{t,0}: \mathcal{K}_t \setminus \mathcal{V} \to \mathcal{K}_0 \setminus \mathcal{V} \), induced by the gradient flow of the restriction \( h_{\mathcal{K}} \). Moreover, the same flow induces a map \( \mathcal{K}_t \cap \mathcal{V} \to v \) that is two-to-one, and thus the diffeomorphism \( \rho_{t,0} \) may be extended to a continuous map \( \overline{\rho}_{t,0}: \mathcal{K}_t \to \mathcal{K}_0 \).

Let \( \Gamma \) be a marked graph diagram that defines a hyperbolic splitting \( h \) of an embedded surface \( \mathcal{K} \). Denote by \( \Gamma_+ \) (resp. \( \Gamma_- \)) the two resolutions of \( \Gamma \), defined by the markers: \( \Gamma_- \) is a diagram of \( \mathcal{K}_{-}\epsilon \) and \( \Gamma_+ \) represents a diagram of \( \mathcal{K}_{+}\epsilon \). The gradient flow of \( h_{\mathcal{K}} \) induces diffeomorphisms \( \rho_{\pm\epsilon,0}: \mathcal{K}_{+\epsilon} \setminus \mathcal{V} \to \mathcal{K}_0 \setminus \mathcal{V} \). The diffeomorphism \( \rho_{\pm\epsilon,0} \) may be extended to a continuous map \( \overline{\rho}_{\pm\epsilon,0}: \mathcal{K}_{\pm\epsilon} \to \mathcal{K}_0 \), whose restriction to \( \mathcal{K}_{\pm\epsilon} \setminus \mathcal{V} \) is two-to-one. This induces maps between the diagrams \( \rho_{\pm\epsilon}: \mathcal{K}_{\pm\epsilon} \to \Gamma \). Diagrams \( \Gamma_- \) and \( \Gamma_+ \) will be called the “lower half diagram” and the “upper half diagram” of \( \Gamma \).

**Example 3.3** Figure 4 depicts a marked graph diagram of the spin of the trefoil knot (the construction of spun knots is described on page 15). The hyperbolic splitting \( h \), corresponding to this diagram, induces the flattening map \( h_{\mathcal{K}}^{-1} \), whose multiplicities in the respective regions are shown.

Suppose \( U \) is a region of \( \Gamma \), then its boundary \( \partial U \subset \Sigma \) is a subset of \( \Gamma \). Denote by \( U_+ \) the region of \( \Gamma_+ \) for which \( \rho_+^{-1}(\partial U) \subset \partial U^+ \). Similarly, denote by \( U^- \) the region of \( \Gamma_- \) for
Fig. 4 A marked graph diagram of the spin of the trefoil knot with its lower and upper half diagrams and nonzero multiplicities of the flattening map in their respective regions

which $\rho^- (\partial U) \subset \partial U^-$. We will say that the region $U$ of $\Gamma$ is associated with the region $U^-$ of $\Gamma_-$ and the region $U^+$ of $\Gamma_+$. Now $\Gamma_-$ (resp. $\Gamma_+$) is an unlink, and the disks capping off the components of this unlink represent the 0-handles (resp. 2-handles) of $\mathcal{K}$. Denote $\mathcal{K}^+ = \mathcal{K} \cap S^4_{[\epsilon,1]}$ and $\mathcal{K}^- = \mathcal{K} \cap S^4_{[-1,-\epsilon]}$. Applying the same reasoning as in the proof of Proposition 3.2, the set of critical values of the restriction $h_{\mathcal{K}^+} : \mathcal{K}^+ \to \Sigma$ may be identified with $\Gamma_+$, while the set of critical values of the restriction $h_{\mathcal{K}^-} : \mathcal{K}^- \to \Sigma$ may be identified with $\Gamma_-$. It follows that the multiplicity of $h_{\mathcal{K}^-}$ in the region $U$ may be computed as

$$m_{h_{\mathcal{K}^-}} (U) = m_{h_{\mathcal{K}^+}} (U^-) + m_{h_{\mathcal{K}^+}} (U^+) + 1.$$

Multiplicities of the flattening map $h_{\mathcal{K}^-}$ in the regions of $\Gamma$ are guided by some simple properties that we describe below. Two regions of $\Gamma$ (resp. $\Gamma_\pm$) are called adjacent if their boundaries in $\Sigma$ share the same edge of $\Gamma$ (resp. $\Gamma_\pm$).

**Lemma 3.4** Let $\Gamma$ be the marked graph diagram of an embedded surface $\mathcal{K}$ with a hyperbolic splitting $h$. The multiplicities of $h_{\mathcal{K}^+}$ in any two adjacent regions of $\Gamma_+$ differ by 1. Also, the multiplicities of $h_{\mathcal{K}^-}$ in any two adjacent regions of $\Gamma_-$ differ by 1.

**Proof** Let $U$ and $U'$ be two regions of $\Gamma_+$, whose boundaries in $\Sigma$ share a common edge $a \subset \Gamma_+$. In the handle decomposition of $\mathcal{K}$ induced by $h$, the arc $a$ represents a part of the boundary of a disk that is a 2-handle of $\mathcal{K}$. The addition of this 2-handle causes a splitting into regions along $a$, and an increase of multiplicity in one of these regions (above which the 2-handle lies) by 1. It follows that $m_{h_{\mathcal{K}^+}} (U') = m_{h_{\mathcal{K}^+}} (U) + 1$. For the lower half diagram, the proof is analogous. □

**Corollary 3.5** Let $\Gamma$ be the marked graph diagram of an embedded surface $\mathcal{K}$ with a hyperbolic splitting $h$. The multiplicities of the flattening map $h_{\mathcal{K}^-}$ in any two adjacent regions of $\Gamma$ differ by 0 or 2.
Fig. 5 The surface above a vertex of $\Gamma$: a fold crossing (left), a branch point (middle) and a cusp (right)

**Proof** Let $U_1$ and $U_2$ be two regions of $\Gamma$, whose boundaries in $\Sigma$ share a common edge $a \subset \Gamma$. The region $U_i$ is associated to a region $U_i^+$ of $\Gamma_+$ and to a region $U_i^-$ of $\Gamma_-$ for $i = 1, 2$. The preimage of $a$ under the map $\rho_\pm : \Gamma_\pm \to \Gamma$ is an edge $\rho_\pm^{-1}(a)$ of $\Gamma_\pm$, that is common to the boundaries of $U_1^\pm$ and $U_2^\pm$ in $\Gamma_\pm$. Thus, $U_1^+$ and $U_2^+$ are adjacent regions of $\Gamma_+$, and $U_1^-$ and $U_2^-$ are adjacent regions of $\Gamma_-$. Let us denote $m_{h_\pm}^K(U_1^+) = k$ and $m_{h_\pm}^K(U_1^-) = l$, then by Lemma 3.4 we have $m_{h_\pm}^K(U_2^+) = k \pm 1$ and $m_{h_\pm}^K(U_2^-) = l \pm 1$. It follows that $m_{h_\pm}^K(U_1) = k + l$, while the multiplicity $m_{h_\pm}^K(U_2)$ is either $k + l$ or $k + l \pm 2$.

**Corollary 3.6** Let $\Gamma$ be the marked graph diagram of an embedded surface $K$ with a hyperbolic splitting $h$. For any region $U$ of $\Gamma$, the multiplicity $m_{h_\pm}^K(U)$ is even.

**Proof** This follows directly from the Corollary 3.5 and the fact that every marked graph diagram contains a region where the flattening map has multiplicity 0.

Mappings of the plane into the plane were first thoroughly analysed by Whitney, who accomplished that a generic map between two 2-dimensional manifolds may have singular points lying along smooth non-intersecting curves called the “folds”, and isolated “cusp” points on the folds [22]. In our case, the flattening map $h_\pm^K : K \to \Sigma$ is not quite generic, as $K_0$ contains all hyperbolic points of $h_K$. The marked graph diagram $\Gamma$ represents the image of the fold curves. Its vertices may be of three different types:

1. a fold crossing represents a crossing of two fold curves,
2. a branch point where two cusps meet (a non-generic situation),
3. a cusp.

A part of the surface above each of type of vertex is shown in Fig. 5. The local multiplicities of the flattening map $h_\pm^K$ in the regions surrounding a vertex $v$ of $\Gamma$ are either

- of the form $n, n + 2, n + 4, n + 2$, if $v$ is a fold crossing,
- of the form $n, n + 2, n, n + 2$, if $v$ is a branch point, or
- of the form $n, n, n, n \pm 2$, if $v$ is a cusp

for some even $n \geq 0$, see Fig. 6.

**Example 3.7** In Fig. 7, a marked graph diagram of the projective plane is given. Both vertices of the diagram $\Gamma$ represent cusps of the flattening map $h_K^K$.

Multiplicities of the map $h_{K_+}^\pm$ (resp. $h_{K_-}^\pm$) in the regions of the upper (resp. lower) half diagram of $K$ may also be determined by considering the movie of the knotted surface $K$. A movie of $K$ captures diagrams of sections $K_t \subset S^4_t$ for $t \in [-1, 1]$. Two successive diagrams
Fig. 6 Local multiplicities of the flattening map around a vertex of $\Gamma$: a fold crossing (left), a branch point (middle) and a cusp (right).

Fig. 7 A marked graph diagram of the projective plane with its lower and upper half diagrams.

Fig. 8 Change of the local multiplicities of the map $h_{K_{\pm}}$ during elementary string interactions of movies, defined in [2]. From left to right: Reidemeister moves I, II, III, the birth of a simple closed curve, and a saddle in this sequence will differ at most by a Reidemeister move of type I, II or III, the birth or death of some simple closed curves (at $t = \pm 1$), or some saddle points (at $t = 0$). Figure 8 depicts the change of local multiplicities of $h_{K_{\pm}}$ during each of these interactions.

**Lemma 3.8** Let $\Gamma$ be a marked graph diagram of a closed surface $K$, smoothly embedded in $S^4$. Denote by $h$ the hyperbolic splitting of $K$, defined by $\Gamma$. The multiplicity of $h_{K_{\pm}}$ in any region $U$ of $\Gamma$ is completely determined by the marked graph diagram $\Gamma$.

**Proof** The region $U$ is associated with a region $U^+$ of $\Gamma_+$ and with a region $U^-$ of $\Gamma_-$. The resolution $\Gamma_+$ is the diagram of an unlink $K_\epsilon$ in the 3-sphere $S^3_\epsilon$. By [13, Proposition 2.4], the collection of 2-disks, capping off the components of this unlink, is unique up to isotopy. We may thus choose any sequence of moves from Fig. 8 to deform $\Gamma_+$ into a diagram of a split unlink. Tracing the change of multiplicities backwards in this sequence yields the multiplicity $m_{h_{K_+}}(U^+)$. The same reasoning may be applied on the resolution $\Gamma_-$ to obtain the multiplicity $m_{h_{K_-}}(U^-)$, and adding up both, we obtain $m_{h_{K}}(U)$. \qed
3.1 Relationship between flattenings and generic planar projections

Takeda studied embedded surfaces in \( \mathbb{R}^4 \) by using generic projections to the plane [20]. In this Subsection, we discuss the relationship between his perspective and flattenings, coming from hyperbolic splittings.

**Definition 3.9** [20] Let \( f : F \to \mathbb{R}^4 \) be an embedding of a closed connected surface and let \( \pi : \mathbb{R}^4 \to \mathbb{R}^2 \) be an orthogonal projection. We say that \( \pi \) is **generic** with respect to \( f \) if \( \pi \circ f \) is a \( C^\infty \) stable mapping.

**Proposition 3.10** [20] Let \( f : \mathbb{R}^4 \to \mathbb{R}^2 \) be a smooth mapping of a closed connected surface to the plane. Denote by \( S(f) \) the set of singular points of \( f \). Then \( f \) is \( C^\infty \) stable iff \( S(f) \) consists merely of fold points and cusps, if its restriction to the set of fold points is an immersion with normal crossings and if for each cusp \( q \) we have

\[
 f^{-1}(f(q)) \cap S(f) = \{q\}.
\]

If \( f : \mathbb{R}^4 \to \mathbb{S}^4 \) is an embedding of a surface with a hyperbolic decomposition \( h : \mathbb{S}^4 \to \mathbb{R} \), then the flattening \( h^\perp : \mathbb{S}^4 \to \Sigma \) does not represent a generic projection with respect to \( f \). However, the flattening map can be slightly disturbed to obtain a generic projection.

**Proposition 3.11** Let \( f : \mathbb{R}^4 \to \mathbb{R}^4 \) be an embedding of a closed connected surface and denote by \( j : \mathbb{R}^4 \to \mathbb{S}^4 \) the compactification map. For each marked graph diagram \( \Gamma \) of \((j \circ f)(F)\), there exists a projection \( \pi : \mathbb{R}^4 \to \mathbb{R}^2 \) that is generic with respect to \( f \), so that the set of critical values of \( \pi \circ f \) is isotopic to the upper half diagram \( \Gamma_+ \).

**Proof** Denote \( \mathcal{K} = (j \circ f)(F) \) and let \( \Gamma \) be a marked graph diagram of \( \mathcal{K} \) that defines a hyperbolic splitting \( h : \mathbb{S}^4 \to \mathbb{R} \) of \( \mathcal{K} \). Choose a small positive number \( 0 < \varepsilon < 1 \). Let \( p : \mathbb{S}^4 \to \mathbb{R}^2 \) be the projection which is regular on \( \mathcal{K}_r \) and for which \( p(\mathcal{K}_r) = \Gamma_+ \). Denote by \( \Phi : \mathbb{R} \times \mathbb{S}^4 \to \mathbb{S}^4 \) the flow of the vector field \( \nabla h \) and define a projection \( \hat{h} : \mathbb{S}^4 \setminus c(h) \to \mathbb{R}^2 \) by

\[
\hat{h}(x) = p\left(\Phi(t, x) \cap \mathbb{S}^4_r\right),
\]

where \( c(h) \) denotes the two critical points of \( h \). Using the same reasoning as in the proofs of Lemma 3.1 and Proposition 3.2, we show that \( \hat{h} \) is a smooth projection whose set of critical values equals \( \Gamma_+ \).

Choose a 4-ball neighborhood \( U \) of \( \mathcal{K} \) so that \( \mathcal{K} \subset U \subset \mathbb{S}^4 \setminus c(h) \) and a diffeomorphism \( \psi : U \to \mathbb{R}^4 \). After applying a horizontal isotopy of \( \mathcal{K} \) if necessary, we may assume that the set of singular points of the composition \( \hat{h} \circ \psi^{-1} : F \to \mathbb{R}^2 \) consists merely of fold points immersed with normal crossings. It follows by Proposition 3.10 that \( \hat{h} \circ \psi^{-1} \circ f : \mathbb{R}^4 \to \mathbb{R}^2 \) is generic with respect to \( f \). \( \square \)

4 Invariants of embedded surfaces that arise from flattenings

In this Section, we apply flattenings to define three invariants of knotted surfaces. Denote by \( \mathcal{G}(\mathcal{K}) \) the collection of all marked graph diagrams of an embedded surface \( \mathcal{K} \). Each marked graph diagram \( \Gamma \in \mathcal{G}(\mathcal{K}) \) defines a hyperbolic splitting \( h : \mathbb{S}^4 \to \mathbb{R} \) and a smooth flattening map \( h^\perp : \mathbb{S}^4 \setminus c(h) \to \Sigma \), where \( \Sigma \) denotes a 2-sphere inside the 0-section \( \mathbb{S}^4_0 \). By Proposition
the set of critical values of $h^\perp_K$ equals $\Gamma$. A vertex of $\Gamma$ is called \textbf{inessential} if it is a marked vertex that represents a branch point of the flattening map $h^\perp_K$. Any vertex of $\Gamma$ that is not inessential is called \textbf{essential}. Two regions $U$ and $U'$ of $\Gamma$ will be called \textbf{equivalent} if there exists a chain of regions $U_0 = U, U_1, U_2, \ldots, U_k = U'$ such that the boundaries $\partial U_i$ and $\partial U_{i+1}$ in $\Sigma$ share the same inessential vertex and their associated regions in $\Gamma_+$ coincide: $U^+_i = U^+_i$ in $\Gamma_+$ for $i = 0, 1, \ldots, k$. It is easy to see this defines an equivalence relation on the set of regions of $\Gamma$. Moreover, in two equivalent regions, the flattening map $h^\perp_K$ has the same multiplicity, see Fig. 9.

Denote by $U_0, U_1, \ldots, U_n$ the regions of $\Gamma$. Define an equivalence relation $\sim$ on the index set $\{0, 1, \ldots, n\}$ by

$$i \sim j \iff U_i \text{ is equivalent to } U_j.$$ 

Denote by $[i]$ the equivalence class of an index $i$ and let $\mathcal{I} = \{[i] | i \in \{0, 1, \ldots, n\}\}$. Now define

$$\text{lay}(\Gamma) = \sum_{[i] \in \mathcal{I}} m_{h^\perp_K}(U_i), \quad \text{trunk}(\Gamma) = \max_{[i] \in \mathcal{I}} m_{h^\perp_K}(U_i), \quad p(\Gamma) = \# \left\{ [i] \in \mathcal{I} | m_{h^\perp_K}(U_i) > 0 \right\},$$

$$\text{lay}(K) = \min_{\Gamma \in \mathcal{G}(K)} \text{lay}(\Gamma), \quad \text{trunk}(K) = \min_{\Gamma \in \mathcal{G}(K)} \text{trunk}(\Gamma), \quad p(K) = \min_{\Gamma \in \mathcal{G}(K)} p(\Gamma).$$

The values in the last line will be called the \textbf{layering} of $K$, the \textbf{trunk} of $K$ and the \textbf{partition number} of $K$ respectively. Clearly, these invariants are related to some extent:

**Proposition 4.1** For any smoothly embedded closed surface $K$ we have

$$\text{lay}(K) \geq 2p(K) + \text{trunk}(K) - 2. \quad (1)$$

**Proof** Let $\Gamma \in \mathcal{G}(K)$ be any marked graph diagram of $K$, and let $h^\perp_K$ be the corresponding flattening map. By Corollary 3.6, any nonzero multiplicity of $h^\perp_K$ in a region of $\Gamma$ is $\geq 2$, thus the sum over all equivalence classes of regions gives

$$2(p(\Gamma) - 1) + \text{trunk}(\Gamma) \leq \text{lay}(\Gamma).$$

Now choose a marked graph diagram $\Gamma_1 \in \mathcal{G}(K)$ for which $\text{lay}(\Gamma_1) = \text{lay}(K)$, then $\text{lay}(K) \geq 2(p(\Gamma_1) - 1) + \text{trunk}(\Gamma_1) \geq 2p(K) + \text{trunk}(K) - 2. \quad \square$

**Remark 4.2** Takeda defined similar invariants of embedded surfaces using generic projections to the plane [20]. Let $f : F \to \mathbb{R}^4$ be an embedding of a closed connected surface and let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be an orthogonal projection that is generic with respect to $f$. Then the set of singular points $S(\pi \circ f)$ consists of folds and cusps, and $(\pi \circ f)(S(\pi \circ f))$ divides the plane into several regions. The value $|(\pi \circ f)^{-1}(x)|$ for an element $x$ in a given region is called the \textit{local width}, while $w(f, \pi)$ denotes the maximum of the local widths over all the regions and $t w(f, \pi)$ denotes the sum of the local widths over all the regions. The \textbf{width} $w(f(\vec{F}))$ of an embedded surface $f(F)$ is the minimum of $w(f, \vec{F})$, where $\vec{F}$ runs over all the embeddings isotopic to $f$ and $\vec{F}$ runs over all orthogonal projections which are generic with respect to $\vec{F}$. The \textbf{total width} $t w(f(\vec{F}))$ of an embedded surface $f(F)$ is the minimum of $t w(f, \vec{F})$, 

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where \( \hat{f} \) runs over all the embeddings isotopic to \( f \) and \( \hat{\pi} \) runs over all orthogonal projections which are generic with respect to \( f \).

**Corollary 4.3** For any embedded surface \( K \) in \( S^4 \), its width (as defined by Takeda) and its trunk are related by \( w(K) \leq \text{trunk}(K) \). Moreover, its total width (as defined by Takeda) and its layering are related by \( tw(K) \leq \text{lay}(K) \).

**Proof** Let \( f : F \to S^4 \) be an embedding with \( f(F) = K \). Suppose that \( \text{trunk}(K) = k \) and let \( \Gamma \) be a marked graph diagram with \( \text{trunk}(\Gamma) = k \). Denote by \( h : S^4 \to \mathbb{R} \) the hyperbolic splitting of \( K \), associated with the marked graph \( \Gamma \). By Proposition 3.11, there exists a projection \( \pi : \mathbb{R}^4 \to \mathbb{R}^2 \) that is generic with respect to \( f \), such that the set of critical values of \( \pi \circ f \) is isotopic to \( \Gamma_+ \). Each region \( U \) of \( \Gamma \) is associated with a region \( U^+ \) of \( \Gamma_+ \) and the multiplicity of the flattening map \( m_{h|_K} : (U) \to \mathbb{R} \) equals the local width of the projection \( \pi \circ f \) in the region \( U^+ \). Moreover, for any two regions \( U_1 \) and \( U_2 \) we have \( U_1^+ = U_2^+ \) if and only if \( U_1 \sim U_2 \). It follows that \( w(f, \pi) = k \) and consequently \( w(K) \leq \text{trunk}(K) \).

Similarly, let \( \text{lay}(K) = m \) and let \( \Gamma \) be a marked graph diagram with \( \text{lay}(\Gamma) = m \). Denote by \( h : S^4 \to \mathbb{R} \) the hyperbolic splitting of \( K \), associated with the marked graph \( \Gamma \). By Proposition 3.11, there exists a projection \( \pi : \mathbb{R}^4 \to \mathbb{R}^2 \) that is generic with respect to \( f \), such that the set of critical values of \( \pi \circ f \) is isotopic to \( \Gamma_+ \). By similar reasoning as in the previous paragraph, it follows that \( tw(f, \pi) = m \) and thus \( tw(K) \leq \text{lay}(K) \).

Let us consider the simplest class of embedded surfaces: those which are unknotted. Recall that an orientable surface \( K \) in \( S^4 \) is **unknotted** if it bounds a handlebody. By [7, Theorem 1.2], a surface \( K \) in \( S^4 \) is unknotted if and only if it is isotopic to a surface in \( S^3 \subset S^4 \).

**Lemma 4.4** Let \( K \) be an orientable closed surface in \( S^4 \). The following statements are equivalent:

(i) \( K \) is unknotted.
(ii) \( p(K) = 1 \)
(iii) \( \text{lay}(K) = 2 \)
(iv) \( \text{trunk}(K) = 2 \)

**Proof** (i) \( \Rightarrow \) (ii) If \( K \) is an unknotted sphere, it admits a marked graph diagram without any vertices (a circle), thus \( p(K) = 1 \). Suppose \( K \) is an unknotted orientable closed surface of genus \( g > 0 \), then it admits a marked graph diagram \( \Gamma \) with \( 2g \) inessential vertices, see Fig. 10. All regions of \( \Gamma \) in which \( h_{K}^{\uparrow} \) has nonzero multiplicity, belong to the same equivalence class, thus \( p(\Gamma) = 1 \) and consequently \( p(K) = 1 \).

(ii) \( \Rightarrow \) (iii) Suppose \( p(K) = 1 \). Then \( K \) admits a marked graph diagram in which all regions where \( h_{K}^{\uparrow} \) has nonzero multiplicity belong to the same equivalence class, and by Corollary 3.5 this multiplicity equals 2. It follows that \( \text{lay}(K) = 2 \).

(iii) \( \Rightarrow \) (iv) The implication is obvious.

(iv) \( \Rightarrow \) (i) By Corollary 4.3, an embedded orientable surface \( K \) with \( \text{trunk}(K) = 2 \) has \( w(K) = 2 \), and it follows by [20, Theorem 3.3] that \( K \) is unknotted. \( \square \)
Next, we examine how our invariants behave under connected sum of surfaces.

**Proposition 4.5** Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be closed connected smoothly embedded surfaces in \( S^4 \), then

\[
p(\mathcal{K}_1 \# \mathcal{K}_2) \leq p(\mathcal{K}_1) + p(\mathcal{K}_2) - 1, \quad \text{lay}(\mathcal{K}_1 \# \mathcal{K}_2) \leq \text{lay}(\mathcal{K}_1) + \text{lay}(\mathcal{K}_2) - 2,
\]

and \( \text{trunk}(\mathcal{K}_1 \# \mathcal{K}_2) \leq \max\{\text{trunk}(\mathcal{K}_1), \text{trunk}(\mathcal{K}_2)\} \).

**Proof** Let \( \Gamma_i \in \mathcal{G}(\mathcal{K}_i) \) be a marked graph diagram of the embedded surface \( \mathcal{K}_i \), and let \( h_i \) be their corresponding hyperbolic splittings for \( i = 1, 2 \). Choose a 2-disk \( B_i \) that contains \( \Gamma_i \) for \( i = 1, 2 \), then disjointly embed these disks into a common 2-sphere \( \Sigma \) by a map \( j : B_1 \cup B_2 \to \Sigma \). Denote by \( U \) the common region of \( j(\Gamma_1) \) and \( j(\Gamma_2) \), then \( \Sigma \setminus (j(B_1) \cup j(B_2)) \subset U \). Choose a region \( U_i \) of \( j(\Gamma_i) \) that is adjacent to \( U \) and has \( m_{h_i \Gamma_i}^{-1}(U_i) = 2 \) for \( i = 1, 2 \). Choose two arcs \( a_i \subset \partial U_i \cap \partial U \) for \( i = 1, 2 \) and join the regions \( U_1 \) and \( U_2 \) by adding a band along \( a_1 \cup a_2 \), then replace this band with a marked vertex as in Fig. 3 to obtain the connected sum of graphs \( \Gamma = j(\Gamma_1) \# j(\Gamma_2) \). Then \( \Gamma \) represents a marked graph diagram for \( \mathcal{K}_1 \# \mathcal{K}_2 \). Every region of \( \Gamma \) where \( h_{\mathcal{K}_1 \# \mathcal{K}_2}^{-1} \) has nonzero multiplicity is either a region of \( j(\Gamma_1) \), a region of \( j(\Gamma_2) \) or the region coming from \( U_1 \) and \( U_2 \), therefore \( p(\Gamma) = p(\Gamma_1) + p(\Gamma_2) - 1 \). If \( W \) is any region of \( \Gamma_i \), different from \( U_1 \) and \( U_2 \), then \( m_{h_i \mathcal{K}_1 \# \mathcal{K}_2}^{-1}(W) = m_{h_i \Gamma_i}^{-1}(W) \).

The multiplicity of \( h_{\mathcal{K}_1 \# \mathcal{K}_2}^{-1} \) in the region arising from \( U_1 \) and \( U_2 \) equals 2. It follows that \( \text{lay}(\Gamma) = \text{lay}(\Gamma_1) + \text{lay}(\Gamma_2) - 2 \) and \( \text{trunk}(\Gamma) = \max\{\text{trunk}(\Gamma_1), \text{trunk}(\Gamma_2)\} \).

Thus, for every pair of marked graph diagrams \( \Gamma_i \in \mathcal{G}(\mathcal{K}_i) \), there exists a marked graph diagram \( \Gamma \in \mathcal{G}(\mathcal{K}_1 \# \mathcal{K}_2) \) such that \( p(\Gamma) = p(\Gamma_1) + p(\Gamma_2) - 1, \text{lay}(\Gamma) = \text{lay}(\Gamma_1) + \text{lay}(\Gamma_2) - 2 \) and \( \text{trunk}(\Gamma) = \max\{\text{trunk}(\Gamma_1), \text{trunk}(\Gamma_2)\} \). Choosing the diagrams \( \Gamma_i \) so that \( p(\Gamma_1) = p(\mathcal{K}_1) \) and \( p(\Gamma_2) = p(\mathcal{K}_2) \), it follows that \( p(\mathcal{K}_1 \# \mathcal{K}_2) \leq p(\mathcal{K}_1) + p(\mathcal{K}_2) - 1 \). Similarly, we may conclude that \( \text{lay}(\mathcal{K}_1 \# \mathcal{K}_2) \leq \text{lay}(\mathcal{K}_1) + \text{lay}(\mathcal{K}_2) - 2 \) and \( \text{trunk}(\mathcal{K}_1 \# \mathcal{K}_2) \leq \max\{\text{trunk}(\mathcal{K}_1), \text{trunk}(\mathcal{K}_2)\} \). \( \square \)

**Remark 4.6** Observe that the inequality (1) from Proposition 4.1 is an equality in the case of an unknotted orientable surface \( \mathcal{K} \) by Lemma 4.4. Likewise, all three inequalities from Proposition 4.5 are equalities when both \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) are unknotted orientable surfaces.

Spun knots, introduced by Artin, represent the oldest known examples of knotted spheres [1]. Let \( K \) be a 1-knot in \( S^3 \). A 1-tangle \((B^3, K^0)\) with endpoints on two antipodal points \( p_1, p_2 \in \partial B^3 \) is obtained by removing a small open ball neighborhood of a point on \( K \). View the 4-sphere as \( S^4 = B^4 \cup S^3 \) and decompose the equatorial 3-sphere as \( S^3 = B^3 \cup_{S^2} B^3 \). Let \((B^3, K^0)\) be a 1-tangle in the first 3-ball. Choose a regular neighborhood \( \nu S^2 \cong S^2 \times B^2 \), then spin the pair \((B^3, K^0)\) around \( S^2 \times \{0\} \) in the complement of \( \nu S^2 \). This gives a decomposition

\[
(B^3, K^0) \times S^1 \cup (S^2, \{p_0, p_1\}) \times B^2 = (S^4, S(K)) .
\]

Capping off the annulus \( K^0 \times S^1 \) by the two disks \( \{p_0, p_1\} \times B^2 \), we obtain the knotted sphere \( S(K) \), called the spin of \( K \). Since our definitions of width, trunk and partition number
of embedded surfaces originate from similar invariants of 1-dimensional knots, an obvious question arises whether the values of 1-dimensional invariants of a knot $K$ are in any way connected with the corresponding 2-dimensional invariants of the surface knot $S(K)$.

**Theorem 4.7** Let $K$ be a 1-knot with bridge number $b(K)$. Then

$$\text{trunk}(S(K)) \leq 2b(K).$$

**Proof** Suppose $K$ is a 1-knot with $b(K) = k$. Then $K$ may be given as the plat closure of a braid $\beta$ on $2k$ strands. In [13], the authors constructed a banded link diagram for $S(K)$ that is shown on the left of Fig. 12. Denote by $\Gamma \in \mathcal{G}(S(K))$ the corresponding marked graph diagram of $K$, depicted in the right of Fig. 12. Let $\Sigma$ be the 2-sphere, containing the diagram $\Gamma_1$.

Choose any region $U$ of the diagram $\Gamma$ where the flattening map $h_\Sigma^+: S(K) \to \Sigma$ has nonzero multiplicity. Since $\beta$ is a braid on $2k$ strands, there exists a horizontal path $\alpha : I \to \Sigma$ from $x \in U$ to a point $y \in V$, where $V$ is a region of $\Gamma$ where $h_\Sigma^+|_{S(K)}$ has multiplicity zero, so that $\alpha$ crosses $\Gamma$ at most $k$ times. By Corollary 3.5, the multiplicities of $h_\Sigma^+|_{S(K)}$ in any two adjacent regions of $\Gamma$ differ by at most 2, therefore $m_{h_\Sigma^+|_{S(K)}}(U) \leq 2k$. It follows that $\text{trunk}(\Gamma) \leq 2k$ and consequently $\text{trunk}(S(K)) \leq 2k$. \qed

**Corollary 4.8** If $K$ is a 2-bridge knot, then $\text{trunk}(S(K)) = 4$.

**Proof** By [20, Proposition 3.8], an $n$-twist spun 2-bridge knot $K$ has $w(K) = 4$ for any $n \neq \pm 1$. When $n = 0$, Corollary 4.3 implies that $\text{trunk}(S(K)) \geq w(S(K)) = 4$ and by Theorem 4.7, the equality follows. \qed

Beside topological properties of flattenings, such as partition number and region multiplicities, one might consider their geometric properties, such as the shape of regions and the number of vertices of given type. For a flattening $h_\Sigma^+: K \to \Sigma$, the closure of each region in $\Sigma$ is a curved edge polygon, whose vertices are the vertices of $\Gamma$. For any region $U$ of $\Gamma$, denote by $\widehat{U} = \bigcup_{U_i \sim U} U_i$ the subset of $\Sigma$ containing all regions equivalent to $U$, and let $s(U)$ denote the number of essential vertices in $\partial \widehat{U}$. Denoting by $U_0, U_1, \ldots, U_n$ the regions of $\Gamma$ and by $\mathcal{I} = \{[i] | i \in \{0, 1, \ldots, n\}\}$ the set of equivalence classes of indices, we may define

$$s(\Gamma) = \frac{\sum_{[i] \in \mathcal{I}} s(U_i)}{p(\Gamma)} \quad \text{and} \quad s(K) = \min_{\Gamma \in \mathcal{G}(K)} s(\Gamma).$$
We call this invariant the **shape** of a knotted surface $\mathcal{K}$. Shape distinguishes the unknotted orientable surfaces:

**Proposition 4.9** Let $\mathcal{K}$ be an orientable closed surface in $S^4$. Then $\mathcal{K}$ is unknotted if and only if $s(\mathcal{K}) = 0$.

**Proof** If $\mathcal{K}$ is an unknotted surface of genus $g$, it admits a marked graph diagram $\Gamma$ in Fig. 10 with $2g$ marked vertices. Since all the vertices in this diagram are inessential, we have $s(\Gamma) = 0$ and consequently $s(\mathcal{K}) = 0$. Suppose $\mathcal{K}$ is a surface with $s(\mathcal{K}) = 0$, then there exists a marked graph diagram $\Gamma \in \mathcal{G}(\mathcal{K})$ with no essential vertices. It follows that $\Gamma$ has no crossings (all its vertices are saddles), thus $\mathcal{K}_{-}\epsilon$ and $\mathcal{K}_\epsilon$ are unlinks without crossings. In the section $S^4_{\epsilon}$ of the hyperbolic decomposition corresponding to $\Gamma$, we may cap off the components of $\mathcal{K}_{-}\epsilon$ by disks, add the bands that correspond to 1-handles and obtain a surface whose boundary is an unlink without crossings (equivalent to $\mathcal{K}_\epsilon$), and may thus be capped off by disks inside the same section. We obtain a surface $\mathcal{K}' \subset S^4_{\epsilon}$ that is isotopic to $\mathcal{K}$ by [13, Proposition 2.4]. Since $\mathcal{K}$ is isotopic to a surface inside $S^3$, it is unknotted. \hfill \Box

### 4.1 Flattenings of satellite 2-knots

Flattenings of surfaces are a useful tool for the study of satellite 2-knots. Let $\mathcal{K}_P$ be a 2-sphere embedded in $S^2 \times D^2$, and let $\mathcal{K}_C$ be a 2-sphere embedded in $S^4$ with a tubular neighborhood $\nu(\mathcal{K}_C)$. If $f : S^2 \times D^2 \rightarrow \nu(\mathcal{K}_C)$ is a diffeomorphism, then $f(\mathcal{K}_P)$ is called a satellite knot with pattern $\mathcal{K}_P$ and companion $\mathcal{K}_C$. We recall the construction of banded link diagrams of satellite knots, described in [8].

Suppose $\mathcal{K}$ is a satellite knot with pattern $\mathcal{K}_P$ and companion $\mathcal{K}_C$. View the 4-sphere as $S^4 = S^3 \times [-2, 2]/(S^3 \times \{-2\}, S^3 \times \{2\})$, with a Morse function $h : S^4 \rightarrow \mathbb{R}$ that projects to the second factor. Choose an embedding of $\mathcal{K}_C$ for which $\mathcal{K}_C \cap S^3_4$ is a 1-knot, while the saddles of $\mathcal{K}_C$ lie in the sections $S^4_{\delta \{1\}}$; this is the normal form of [10].

For the ambient manifold of the pattern, we choose $V = S^2 \times D^2 \subset S^4$ which intersects the 0-section in a solid torus $W = V \cap S^3_0 \cong S^1 \times D^2$, while $V \cap S^4_{\{-2,0\}} = V \cap S^4_{\{0,2\}} \cong D^2 \times D^2$. Draw a banded link diagram for $\mathcal{K}_P$ inside $W$ (that lies in the 0-section $S^4_0$). Choose a meridian disk $D$ of $W$ that is disjoint from all bands in the diagram for $\mathcal{K}_P$; the number $\omega$ of (unsigned) intersection points $\mathcal{K}_P \cap D$ is called the geometric winding of $\mathcal{K}_P$ (this number depends on the choice of $D$).

We draw a banded link diagram for $\mathcal{K}_C$ and move it by isotopy so that it lies inside $W' = \nu(\mathcal{K}_C) \cap S^3_0$. Choose a meridian disk $D'$ for $W'$ that intersects the banded link transversely in one point, then isotopes the diffeomorphism $f : V \rightarrow \nu(\mathcal{K}_C)$ so that $f(W) = W'$ and $f(D \times I) = W' \setminus (D' \times I)$. A banded link diagram for $\mathcal{K}$ is obtained by drawing the 0-framed satellite of $\mathcal{K}_P \cap S^4_0 \subset W$ around $\mathcal{K}_C \cap S^4_0$, attaching the bands corresponding to $\mathcal{K}_P$ and attaching $\omega$ copies of each band corresponding to $\mathcal{K}_C$ (the bands that lie below $S^4_0$ need to be pushed above, which is done by taking their dual bands).

**Example 4.10** Let $\mathcal{K}_C$ be the spin of the figure eight knot, whose marked graph diagram is given in the middle of Fig. 13. Take a simple pattern $\mathcal{K}_P$ that winds around the 2-sphere $S^2 \times \{0\}$ in $S^2 \times D^2$ three times; its marked graph diagram is given on the left of Fig. 13. Satellite $\mathcal{K}$ with pattern $\mathcal{K}_P$ and companion $\mathcal{K}_C$ admits a banded link diagram that is shown on the right of Fig. 14. The corresponding marked graph diagram with multiplicities of $h^\perp_{\mathcal{K}}$ in most of its regions is shown in Fig. 16.
The above example may lead to the following observations. Denote by $\Gamma$ (resp. $\Gamma_C$ and $\Gamma_P$) the marked graph diagrams of the satellite $K$ (resp. companion $K_C$ and pattern $K_P$), described above. Let $h$ (resp. $h_C$ and $h_P$) be the hyperbolic decompositions of $K$ (resp. $K_C$ and $K_P$), corresponding to these diagrams. The regions of $\Gamma$ consist of four different types:

1. regions in the complement of $f(W)$ correspond to the regions of $\Gamma_C$,
2. rectangular regions that correspond to the edges of $\Gamma_C$,
3. regions that correspond to the vertices of $\Gamma_C$,
4. regions that correspond to the regions of $\Gamma_P$.

Denote by $\omega$ the geometric winding of $K_P$. Every region of $\Gamma_C$ induces one region of type (1), and every region of $\Gamma_P$ induces one region of type (4). There is only one region of $\Gamma$ that belongs to both type (1) and type (4); in the diagram on Fig. 16, this is the lowest region with multiplicity 6. Every edge of $\Gamma_C$ induces $(\omega - 1)$ regions of type (2), and every vertex (either crossing or marked vertex) of $\Gamma_C$ induces $(\omega - 1)^2$ regions of type (3). Every marked vertex of $\Gamma_C$ gives rise to $\omega$ marked vertices of $\Gamma$, and each of these vertices identifies two regions. It follows that

$$p(\Gamma) = p(\Gamma_P) + p(\Gamma_C) - 1 + (\omega - 1)e(\Gamma_C) + (\omega - 1)^2 (c(\Gamma_C) + v(\Gamma_C)) - (\omega - 1)v(\Gamma_C),$$

where $e(G)$ (resp. $c(G)$ and $v(G)$) denote the number of edges (resp. the number of crossings and marked vertices) of a marked graph $G$. Recall that the minimal number of vertices over all marked graph diagrams of a knotted surface $K$ is called the ch-index of $K$ and denoted by $\text{ch}(K)$ [23].

Using the diagram of a satellite 2-knot, described above, we obtain an upper bound for its trunk.
To the geometric winding of \( \omega K \) the highest multiplicity of \( \omega \) at \( \omega \) the highest multiplicity of \( h \). Let \( K \) be a satellite 2-knot with companion \( K_C \) and pattern \( K_P \). Denote by \( \omega \) the geometric winding of \( K_P \). Then

\[
\text{trunk}(K) \leq \max\{\omega \text{trunk}(K_C), \text{trunk}(K_P)\}
\]  \hspace{1cm} (2)

**Proof** Denote by \( \Gamma \) (resp. \( \Gamma_C \) and \( \Gamma_P \)) the marked graph diagrams of the satellite \( K \) (resp. companion knot \( K_C \) and pattern \( K_P \)), obtained by the procedure, described above. Let \( h \) (resp. \( h_C \) and \( h_P \)) be the hyperbolic decompositions of \( K \) (resp. \( K_C \) and \( K_P \)), corresponding to these diagrams.

Recall the diffeomorphism \( f : S^2 \times D^2 \to v(K_C) \) maps the pattern \( K_P \) onto the satellite knot \( K \). Let \( U \) be a region of \( \Gamma \) of type (1) that corresponds to the region \( U' \) of \( \Gamma_C \), then the fiber of \( h_K^+ \) above \( U \) consists of \( \omega \) copies of the fiber of \( (h_C^+)_K \) over \( U' \), and thus \( m_{h_K^+}(U) = \omega m_{(h_C^+)_K}(U') \). The multiplicity of \( h_K^+ \) in a region of type (4) agrees with the multiplicity of \( (h_P^+)_K \) in its corresponding region of \( \Gamma_P \).

Let \( e \) be an edge of \( \Gamma_C \) that separates two regions \( U_1 \) and \( U_2 \) of \( \Gamma_C \). Then \( e \) gives rise to \( (\omega - 1) \) regions of type (2), and multiplicities of \( h_K^+ \) in those regions interpolate between \( \omega m_{(h_C^+)_K}(U_1) \) and \( \omega m_{(h_C^+)_K}(U_2) \) (where multiplicity in each successive region jumps by 2).

Any vertex \( v \) of \( \Gamma_C \) is incident to four edges of \( \Gamma_C \) and gives rise to \( (\omega - 1)^2 \) regions of type (3). The multiplicities of \( h_K^+ \) in these regions depend on the type of the vertex \( v \), see Fig. 15. In case \( v \) is a fold crossing, the multiplicities in the regions of type (3) interpolate between the multiplicities of \( h_K^+ \) in the regions of type (2) corresponding to the edges incident at \( v \) (where multiplicity in each successive region jumps by 2). In case \( v \) is a branch point, the highest multiplicity of \( h_K^+ \) in a region of type (3), corresponding to \( v \), coincides with the highest multiplicity of \( h_K^+ \) in a region of type (1) that comes from a region of \( \Gamma_C \), incident to \( v \).

It follows from the above discussion that the highest multiplicity of \( h_K^+ \) is either attained in a region of type (1) or in a region of type (4) and therefore

\[
\text{trunk}(\Gamma) = \max\{\omega \text{trunk}(\Gamma_C), \text{trunk}(\Gamma_P)\}.
\]

By choosing the diagrams \( \Gamma_C \) and \( \Gamma_P \) so that \( \text{trunk}(\Gamma_C) = \text{trunk}(K_C) \) and \( \text{trunk}(\Gamma_P) = \text{trunk}(K_P) \), we obtain the desired inequality.

**Remark 4.12** If the geometric winding of \( K_P \) equals 1, then the satellite with pattern \( K_P \) and companion \( K_C \) is in fact the connected sum \( K_P \# K_C \). In the special case when \( K_P \) is unknotted, we have \( K_P \# K_C = K_C \) and since \( \text{trunk}(K_C) \geq 2 \), the inequality (2) becomes an equality.
In a recent paper [5], Freedman and Hillman use the satellite construction together with some intricate topological machinery to show that there exist \( n \)-dimensional knots in \( \mathbb{R}^n \) of arbitrarily large width for each \( n \geq 1 \). Their definition of width is slightly different, but conceptually similar to our trunk invariant in dimension 2. In order to distinguish between Takeda’s width and the width defined by Freedman and Hillman, we will denote the latter by \( w_{FH} \).

**Definition 4.13** [5] Given a smooth embedding \( K : S^2 \hookrightarrow \mathbb{R}^4 \), let \( \pi : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) be any composition \( \mathbb{R}^4 \xrightarrow{d} \mathbb{R}^4 \xrightarrow{p} \mathbb{R}^2 \), where \( d \) is any diffeomorphism and \( p \) denotes the projection onto the last 2 coordinates. The width of \( K \) is denoted by \( w_{FH}(K) \) and defined as

\[
w_{FH}(K) = \min_{\pi} \left\{ \max \{|\pi^{-1}(p)| : p \in \mathbb{R}^n \text{ a regular value of the composition } \pi \circ K \} \right\},
\]

where the minimum is taken over all product projections \( \pi \) specified above.

Comparing this definition with Takeda’s definition of width (see Remark 4.2 and Definition 3.9), we may conclude the following:

**Lemma 4.14** For any 2-knot \( K \), we have

\[
w_{FH}(K) \leq w(K) \leq \text{trunk}(K).
\]

**Proof** Suppose \( K : S^2 \hookrightarrow \mathbb{R}^4 \) is a smooth embedding. Any orthogonal projection \( \pi : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) that is generic with respect to \( K \), possibly precomposed by an isotopy of \( K(S^2) \), defines a product projection as specified in Definition 4.13. Therefore \( w_{FH}(K) \leq w(K) \), while the second inequality is provided by Corollary 4.3.

Using the above relationship, we may establish the following.

**Proposition 4.15** There exist knotted spheres in \( S^4 \) with arbitrarily large trunk. There exist knotted spheres for which the difference between the left-hand side and the right-hand side of the inequality (1) is arbitrarily large.

**Proof** By [5, Theorem 4], there exist smooth knots \( K : S^2 \hookrightarrow \mathbb{R}^4 \) with arbitrarily large width \( w_{FH}(K) \). It follows by Lemma 4.14 that the corresponding knots in \( S^4 \) have arbitrarily large trunk. To verify the second statement, suppose that a knotted sphere \( K \) has trunk \( (K) = d \) for some even \( d \gg 2 \). Then Corollary 3.5 implies that for any marked graph diagram \( \Gamma \) of \( K \), there exist regions \( U_i \) of \( \Sigma \setminus \Gamma \) with \( m_{K}(U_i) = 2 + 2i \) for \( i = 1, 2, \ldots, \frac{d-4}{2} \). It follows that \( \text{lay}(\Gamma) \geq 2p(\Gamma) + \text{trunk}(\Gamma) - 2 + \frac{1}{4}(d^2 - 6d + 8) \) for any marked graph diagram \( \Gamma \). Choose \( \Gamma \) for which \( \text{lay}(\Gamma) = \text{lay}(K) \) and we obtain

\[
\text{lay}(K) - (2p(K) + \text{trunk}(K) - 2) \geq \text{lay}(\Gamma) - (2p(\Gamma) + \text{trunk}(\Gamma) - 2) \geq \frac{1}{4}(d^2 - 6d + 8).
\]

The first example of a 2-knot in \( S^4 \) that becomes unknotted when connect summing with a standard real projective plane was found by Viro [21]. Using the satellite construction, Kim constructed another infinite family of such examples which are not ribbon 2-knots [18]. His examples, together with Freedman and Hillman’s results, provide the following.

**Theorem 4.16** There exist knotted surfaces \( K_1 \) and \( K_2 \) in \( S^4 \) for which the difference between the right-hand side and the left-hand side of the inequality

\[
\text{trunk}(K_1 \# K_2) \leq \max \{\text{trunk}(K_1), \text{trunk}(K_2)\}
\]

from Proposition 4.5 is arbitrarily large.
Proof In [18], the author constructed an infinite number of satellite 2-knots which become unknotted by connected summing with a standard real projective plane $\mathbb{P}^2$. Specifically, these are obtained as $2n$-cables of the 2-twist spin of any 2-bridge knot. For the definition of twist-spinning, see Zeeman [24]. Let $k$ be a 2-bridge knot, denote by $\tau_2(k)$ the 2-twist spin of $k$ and by $K_n$ its $2n$-cable, where $n$ is a positive integer. By [5], a 2-twist spin of a nontrivial classical knot has a positive homological width $w_H(\tau_2(k))$ and by [5, Theorem 2], its $2n$-cable has width $w_{FH}(K_n) \geq 2n w_H(\tau_2(k)) \geq 2n$. It follows by Lemma 4.14 that trunk$(K_n) \geq 2n$.

The standard real projective plane $\mathbb{P}^2$ admits a marked graph diagram $\Gamma$ with trunk$(\Gamma) = 2$, see Fig. 7. It follows that trunk$(\mathbb{P}^2) \leq 2$ and since $w(\mathbb{P}^2) = 2$ by [20], Corollary 4.3 implies trunk$(\mathbb{P}^2) = 2$. By [18, Theorem 2.9] we have $K_n \# \mathbb{P}^2 = \mathbb{P}^2$, therefore max{trunk$(K_n)$, trunk$(\mathbb{P}^2)$} − trunk$(K_n \# \mathbb{P}^2) = 2n - 2$. \hfill $\square$
4.2 Directions for further study

(1) The width of classical knots is closely related with the bridge number. Bridge decompositions of knotted surfaces were studied by Meier and Zupan, who obtained several results about the bridge number of surfaces inside the 4-sphere and in other 4-manifolds [13, 14]. An important goal would be to understand the relationship between the bridge number, the layering and the partition number of an embedded surface.

(2) In the same direction, we would like to investigate the connection between the flattenings of surfaces and trisections of surfaces, presented in [13, 14]. It is an interesting question whether some information, obtained by flattening a surface $\mathcal{K}$, could actually be read from a suitable trisection diagram of $\mathcal{K}$.

(3) It might be fruitful to explore the shape of regions in a surface diagram. Are there typical shape structures occurring in the diagrams of some families of knotted surfaces? What does the shape of regions in a diagram of a surface $\mathcal{K}$ tell us about its properties? What can we learn by investigating the values of the shape invariant $s(\mathcal{K})$?

(4) Several results concerning the bridge number and width of classical satellite knots have been established [16, 17, 25]. As we demonstrate in Subsect. 4.1, flattenings offer a suitable way to study the layering and other invariants of satellite 2-knots yet unexplored.

(5) Our construction could be generalized to surfaces in an arbitrary four manifold. Flattenings thus obtained could increase our means of presenting and understanding embedded surfaces.

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References

1. Artin, E.: Zur Isotopie zweidimensionaler Flächen im $\mathbb{R}^4$. Abh. Math. Sem. Univ. Hamburg 4(1), 174–177 (1925)
2. Carter, J.S., Saito, M.: Knotted Surfaces and their Diagrams. Mathematical Surveys and Monographs, vol. 55. American Mathematical Society, Providence, RI (1998)
3. Carter, J.S., Kamada, S., Saito, M.: Surfaces in 4-space. Encyclopaedia of Mathematical Sciences, vol. 142. Springer-Verlag, Berlin (2004)
4. Ehresmann, C.: Les connexions infinitésimales dans un espace fibré différentiable, Coloque de topologie (espaces fibres), Bruxelles, 29–55, Georges Thone. Liège; Masson et Cie, Paris (1950)
5. Freedman, M., Hillman, J.: Width of codimension two knots. J. Knot Theory Ramif. (2020). https://doi.org/10.1142/S0218216519500949
6. Gabai, D.: Foliations and the topology of 3-manifolds. J. Diff. Geom. 18, 445–503 (1983)
7. Hosokawa, F., Kawauchi, A.: Proposals for unknotted surfaces in four spaces. Osaka J. Math. 16(1), 233–248 (1979)
8. Hughes, M., Kim, S., Miller, M.: Isotopies of surfaces in 4-manifolds via banded unlink diagrams. Geom. Topol. 24, 1519–1569 (2020)
9. Jablonowski, M.: On a banded link presentation of knotted surfaces, J. Knot Theory Ramif. 25, no. 3 (2015)
10. Kawauchi, A., Shibuya, T., Suzuki, S.: Descriptions on surfaces in four space, I; normal forms. Math. Sem. Notes Kobe Univ. 10, 72–125 (1982)
11. Lee, J.M.: Introduction to smooth manifolds. Springer, New York (2003)
12. Lomonaco, S. J.: The homotopy groups of knots I. How to compute the algebraic 2-type. Pacific J. Math. 95, no. 2 (1981)
13. Meier, J., Zupan, A.: Bridge trisections of knotted surfaces in $S^4$. Trans. Amer. Math. Soc. 369(10), 7343–7386 (2017)
14. Meier, J., Zupan, A.: Bridge trisections of knotted surfaces in 4-manifolds. In: Proceedings of the National Academy of Sciences 115. https://doi.org/10.1073/pnas.1717171115 (2017)
15. Ozawa, M.: Waist and trunk of knots. Geometriae Dedicata 149, 85–94 (2010)
16. Schubert, H.: Über eine numerische Knoteninvariante. Math. Z. 61, 245–288 (1954)
17. Schultens, J.: Additivity of bridge numbers of knots. Math. Proc. Camb. Philos. Soc. 135, 539–544 (2003)
18. Kim, S.: Gluck twist and unknotting of satellite 2-knots. arXiv e-prints arXiv:2009.07353 (2020)
19. Spivak, M.: A Comprehensive Introduction to Differential Geometry, vol. 1, 3rd edn. Publish or Perish, Berkeley (1979)
20. Takeda, Y.: Widths of surface knots. Algebr. Geom. Topol. 6, 1831–1861 (2006)
21. Viro, O.J.: Local knotting of submanifolds. Math. USSR-Sbornik 19(2), 166 (1973)
22. Whitney, H.: On singularities of mappings of Euclidean spaces. I. Mappings of the plane into the plane. Ann. Math. Vol 62, No. 3 (1955)
23. Yoshikawa, K.: An enumeration of surfaces in four-space. Osaka J. Math. 31, 497–522 (1994)
24. Zeeman, E.C.: Twisting spun knots. Trans. Am. Math. Soc. 115, 471–495 (1965)
25. Zupan, A.: A lower bound on the width of satellite knots. Top. Proc. 40, 179–188 (2012)

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