TWO FORMULAS FOR $F$-POLYNOMIALS

FEIYANG LIN*, GREGG MUSIKER†, AND TOMOKI NAKANISHI‡

Abstract. We discuss a product formula for $F$-polynomials in cluster algebras, and provide two proofs. One proof is inductive and uses only the mutation rule for $F$-polynomials. The other is based on the Fock-Goncharov decomposition of mutations. We conclude by expanding this product formula as a sum and illustrate applications. This expansion provides an explicit combinatorial computation of $F$-polynomials in a given seed that depends only on the $c$-vectors and $g$-vectors along a finite sequence of mutations from the initial seed to the given seed.

1. Introduction

Cluster algebras are certain commutative algebras with a rich combinatorial structure, first introduced in [FZ02]. Their inception was motivated by the study of semicanonical bases of Lie algebras; but since then, researchers have made deep connections between cluster algebras and many other areas of math and physics, including discrete dynamical systems, Poisson geometry, higher Teichmüller spaces, commutative and non-commutative algebraic geometry, string theory, and quiver representation theory; see [Kel13] for specific references.

The generators of a cluster algebra, called cluster variables, can be naturally described using two pieces of data which were introduced in [FZ07], namely $F$-polynomials and $g$-vectors. More detail is given in Section 2. Thus, $F$-polynomials play a key part in the study and applications of cluster patterns. In this paper, we draw attention to a less-known product formula for $F$-polynomials applicable to all skew-symmetrizable cluster algebras (Theorem 3.1) that we refer to as Gupta’s formula. Gupta’s formula can also be expanded using the generalized binomial theorem into a summation formula indexed over integer tuples, giving an explicit formula for the coefficients of $F$-polynomials (Theorem 6.2).

We begin this paper with some basics on cluster algebras, $F$-polynomials, $c$-vectors, and $g$-vectors in Sections 2.1 and 2.2. We then recall for the reader the Fock-Goncharov decomposition of mutations [FG09] in Section 2.3 and reformulate mutations in terms of two automorphisms of the field of rational functions in the initial cluster variables in Section 2.4. The main theorem follows in Section 3 with two proofs in Sections 4 and 5. These results hold for an arbitrary initial skew-symmetrizable matrix $B_{t_0}$. Finally, we conclude with a second manifestation of our main theorem in Section 6 this time as a formula involving an infinite sum, rather than as a product formula.

* Department of Mathematics, University of California, Berkeley, CA 94720, USA. fylin@berkeley.edu.
† School of Mathematics, University of Minnesota, 206 Church St. SE, Minneapolis, MN, 55455 USA. musiker@math.umn.edu.
‡ Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8602, Japan. nakanisi@math.nagoya-u.ac.jp.
Let us give a brief account of the short history of Gupta’s formula, justifying its name. As part of her REU project [Gup18] advised by the second author, Meghal Gupta discovered both the product and summation versions of this $F$-polynomial formula, and gave an elementary proof for it, up to the use of sign-coherence. She then used these two formulas to obtain explicit formulas for $F$-polynomials for cluster patterns associated to the $r$-Kronecker quiver, the affine $A_{n,1}$ quiver and Gale Robinson quivers, and resolved Eager and Franco’s stabilization conjecture [EF12, Sec. 9.5] for several special cases. Comparing with [Gup18], the reader will notice that our current formulation of Gupta’s formula is very different from the original formulation. For example, instead of her usage of $a_{i,j}$, $b_{i,j}$, and $W(n, w_1, w_2, \ldots, w_k)$ in [Gup18, Def. 2.15 and Def. 2.16], we use dot products and sums involving more familiar cluster algebraic objects such as $c$-vectors and $g$-vectors. Gupta’s proof for the product version of her formula (Theorem 3.1) is embedded in her proof towards the summation version (Theorem 6.2), but it proceeds by induction in essentially the same manner as the first proof we give, utilizing the same recurrence relation for $F$-polynomials as we do. In particular, Equation (2) from [Gup18, Sec. 3.2] translates to Theorem 6.2 herein. Our second proof is dissimilar to her approach, relying on the Fock-Goncharov decomposition of mutations into a tropical and a non-tropical part.

**Funding.** This work was supported in part by the Japan Society for the Promotion of Science [16H03922]; and the National Science Foundation Research Training Group [DMS-174563].

**Acknowledgements.** The authors thank Bernhard Keller and Man-Wai Cheung for useful comments. It is inspired by Meghal Gupta’s work in the 2018 REU at University of Minnesota in Combinatorics and Algebra, as well as the first author’s work during the 2020 REU as well as her work supported by Harvey Mudd College. We also thank the anonymous referees who provided numerous suggestions that have improved the exposition of our paper.

## 2. Background

Before delving into the statement of Gupta’s formula and its proofs, we will establish essential notation and provide some background on the Fock-Goncharov decomposition of mutations and related automorphisms, closely following [Nak23]. We assume that the reader is familiar with the basics of cluster algebras, such as the definitions of $F$-polynomials, $c$-vectors, $g$-vectors in [FZ07], but begin by reviewing some of the salient points.

### 2.1. The basics (Part 1).

Let $T_n$ be the $n$-regular tree graph where the $n$ edges attached to each vertex are distinctly labeled by $1, \ldots, n$. We say that a pair of vertices $t$ and $t'$ in $T_n$ are $k$-adjacent if they are connected with an edge labeled by $k$.

Let $\mathbb{T} = \text{Trop}(y)$ be the tropical semifield generated by $y_1, \ldots, y_n$. The tropical semifield comes equipped with two binary operations, namely *tropical addition* which is denoted as $\oplus$, and ordinary multiplication of Laurent monomials. The tropical addition of two Laurent monomials in $\mathbb{T}$ is defined as

$$y_1^{d_1}y_2^{d_2}\cdots y_n^{d_n} \oplus y_1^{e_1}y_2^{e_2}\cdots y_n^{e_n} = y_1^{\min(d_1,e_1)}y_2^{\min(d_2,e_2)}\cdots y_n^{\min(d_n,e_n)}$$

for all integers $d_i$ and $e_i$ for $1 \leq i \leq n$. It is straightforward to show that these two operations turn $\mathbb{T}$ into a semifield, meaning that the commutative, associative, and distributive
laws are satisfied. Let \( \mathbb{Z} \mathbb{P} \) be the group ring of the multiplicative group of \( \mathbb{P} \) over \( \mathbb{Z} \), and let \( \mathbb{Q} \mathbb{P} \) be its fraction field. In this paper, we consider cluster algebras of geometric type, meaning that their coefficients lie in \( \mathbb{P} \).

A cluster algebra (of rank \( n \)) is defined by an initial seed with principal coefficients, which is a tuple \( \Sigma_{t_0} = (x_{t_0}, y_{t_0}, B_{t_0}) \) where \( t_0 \) is some vertex in \( T_n \), \( x_{t_0} = (x_1, \ldots, x_n) \) is an \( n \)-tuple of algebraically independent elements in the rational field \( \mathbb{Q} \mathbb{P}(x_{t_0}) \), \( y_{t_0} = (y_1, \ldots, y_n) \) is the \( n \)-tuple of tropical generators of \( \mathbb{P} \), and \( B_{t_0} \) is the initial exchange matrix, which is an \( n \)-by-\( n \) skew-symmetrizable integer matrix, meaning that there exists a diagonal matrix \( D \) with positive entries such that \( DB_{t_0} = -B_{t_0}^T D \). The principal extension of the initial exchange matrix \( B_{t_0} \) is denoted as \( B_{t_0} \) and it is the \( 2n \)-by-\( n \) matrix whose top \( n \) rows is \( B_{t_0} \) and whose bottom \( n \) rows is an \( n \)-by-\( n \) identity matrix.

From the initial seed \( \Sigma_{t_0} \), we obtain seeds \( \Sigma_t \) for all \( t \in T_n \) by an iterative process known as mutation: given two \( k \)-adjacent vertices \( t, t' \in T_n \), and a seed \( \Sigma_t = (x_t, y_t, B_t) \) with extended exchange matrix \( \tilde{B}_t \), by mutating \( \Sigma_t \) in direction \( k \), we obtain a seed \( \Sigma_{t'} = (x_{t'}, y_{t'}, B_{t'}) \) and an extended exchange matrix \( \tilde{B}_{t'} \). This assignment of a seed to each \( t \in T_n \) is a cluster pattern, written \( \Sigma = \{ \Sigma_t = (x_t, y_t, B_t) \}_{t \in T_n} \). In this process, we also obtain two more families of matrices \( C_t \) and \( G_t \), known as \( C \), \( \mathcal{G} \) matrices, as well as a collection of polynomials in \( \mathbb{Z}[y_1, \ldots, y_n] \), known as \( F \)-polynomials. Write \( A = (a_i)_{i=1}^n \) to mean that \( a_i \) is the \( i \)-th column of the \( n \)-by-\( n \) matrix \( A \). Then given a seed \( \Sigma_t = (x_t, y_t, B_t) \in \Sigma \), we write

\[
\begin{align*}
x_t &= (x_{t;1}, \ldots, x_{n;t}), \\
y_t &= (y_{1;t}, \ldots, y_{n;t}), \\
B_t &= (b_{i;t})_{i=1}^n = (b_{ij;t})_{i,j=1}^n, \\
C_t &= (c_{i;t})_{i=1}^n = (c_{ij;t})_{i,j=1}^n, \\
G_t &= (g_{i;t})_{i=1}^n = (g_{ij;t})_{i,j=1}^n.
\end{align*}
\]

For the initial seed \( \Sigma_{t_0} = (x_{t_0}, y_{t_0}, B_{t_0}) \), we often drop the index \( t_0 \):

\[
\begin{align*}
x_{t_0} &= x = (x_1, \ldots, x_n), \\
y_{t_0} &= y = (y_1, \ldots, y_n), \\
B_{t_0} &= B = (b_{ij})_{i,j=1}^n.
\end{align*}
\]

Let us say a bit more about how \( x_t, y_t, B_t, C_t, G_t \) and \( F \)-polynomials are obtained via mutations. Let \( \mu_{k;t} \) denote the mutation map applied to the seed \( \Sigma_t \) in direction \( k \). To mutate the exchange matrix \( B_t = (b_{ij;t}) \), we write \( [n]_+ = \max(n, 0) \) and define \( B_{t'} = (b_{ij;t'}) \) as follows:

\[
b_{ij;t'} = \begin{cases} 
-b_{ij;t} & \text{if } i = k \text{ or } j = k \\
b_{ij;t} + b_{ik;t}[b_{kj;t}]_+ + [-b_{ik;t}]_+ b_{kj;t} & \text{otherwise.}
\end{cases}
\]

By applying mutation to \( B_t \), we obtain \( B_{t'} \) whose the top \( n \) rows yield \( B_{t'} \), and the bottom \( n \) rows yield \( C_{t'} \), the \( C \)-matrix associated to \( \Sigma_{t'} \). Like our initial choice of \( B_{t_0} \), each exchange matrix \( B_t \) is skew-symmetrizable. We may take a common skew-symmetrizer \( D = (d_1^{-1}, \ldots, d_n^{-1}) \), meaning that \( DB_t = -B_t^T D \) for all \( t \in T_n \), and such that \( d_1, \ldots, d_n \) are positive integers. We fix such a \( D \) from now on. Note that the integers \( d_i \)'s match the ones in [Gro+18] in the scattering diagram formalism.
The mutation of $y_t = (y_{1,t}, \ldots, y_{n,t})$, as elements of $\mathbb{P}$, is defined as

$$y_{i;t'} = \begin{cases} y_{k;t}^{-1} & \text{if } i = k \\ y_{i;t} y_{k;t}^{-1} (1 + y_{k;t})^{-b_{ki,t}} & \text{if } i \neq k. \end{cases}$$

A related set of auxiliary variables, the $\hat{y}$-variables are defined by

$$\hat{y}_{i;t} = y_{i;t} \prod_{j=1}^{n} x_{j,t}^{b_{ji,t}}.$$  

Again for the initial seed, we often drop the index $t_0$ and write

$$\hat{y}_{i;t_0} = \hat{y}_i = y_i \prod_{j=1}^{n} x_{j}^{b_{ji}}.$$  

The effect of $\mu_{k;t}$ on $x_t$ can be understood as an isomorphism of fields of rational functions. If we fix a choice of $\varepsilon \in \{-1, 1\}$, then the map $\mu_{k;t}$ is defined as follows:

$$\mu_{k;t} : \mathbb{Q} \mathbb{P}(x_t') \rightarrow \mathbb{Q} \mathbb{P}(x_t)$$

$$x_{i;t'} \mapsto \begin{cases} x_{k;t}^{-1} \left( \prod_{j=1}^{n} x_{j,t}^{\varepsilon b_{jk,t}} \right) (1 + \hat{y}_{k,t})^{-1} \varepsilon^{b_{ik,t}} & \text{if } i = k, \\ x_{i;t} \varepsilon^{b_{ik,t}} & \text{if } i \neq k. \end{cases}$$

(1)

It is easy to see that the definition is independent of the choice of $\varepsilon$. This map allows us to understand each $x_{i;t}$ as an element of $\mathbb{Q} \mathbb{P}(x_t)$. Repeated mutations give rise to a map $\mathbb{Q} \mathbb{P}(x_t) \rightarrow \mathbb{Q} \mathbb{P}(x_{t_0})$, which allows us to understand each $x_{i;t}$ as an element of $\mathbb{Q} \mathbb{P}(x_{t_0})$. Since $\mathbb{P} = \text{Trop}(y_1, \ldots, y_n)$, we may also view $x_{i;t}$ as a rational function (in fact a Laurent polynomial) in $\mathbb{Q}(x_{t_0}, y_{t_0})$ such that only monomials in $x_{t_0}$ appear in the denominator. Note that when iterating mutations, the order of the composition of the maps is opposite to the order of the mutations due to the domain and image of the map in (1).

We define the associated $F$-polynomials $F_{i;t} \in \mathbb{Z}[y_{t_0}]$ as the specialization of $x_{i;t}|_{x_{t_0}=1}$. Based on equation (1), i.e. see Prop. 5.1 of [FZ07], the $F$-polynomials satisfy the recurrence

$$F_{k;t'} = F_{k;t}^{-1} \left( \frac{y_{k;t} \prod_{j=1}^{n} F_{j;t}^{b_{jk,t}}}{1 + y_{k;t}} + \frac{1}{1 + y_{k;t}} \prod_{j=1}^{n} F_{j;t}^{b_{jk,t}} \right).$$  

(2)

Furthermore, we obtain the $g$-vector $g_{i;t}$ as the exponent vectors of the Laurent monomial arising from the specialization $x_{i;t}|_{x_{t_0}=0}$. The $G$-matrix $G_t$ is defined by concatenating the $g$-vectors $g_{1;t}$ through $g_{n;t}$ together so that they form its columns.

2.2. The basics (Part 2). We remind the reader of some well-known properties that will be useful later in our proofs.
Proposition 2.1 ([FZ07], Prop. 6.6). Let \( t', t \in \mathbb{T}_n \) be \( k \)-adjacent. Then \( \mathbf{g}_{i,t'} = \mathbf{g}_{i,t} \) if \( i \neq k \) and

\[
\mathbf{g}_{k,t'} = -\mathbf{g}_{k,t} - \sum_{j=1}^{n} [c_{jk,t}] + b_{j,t_0} + \sum_{j=1}^{n} [b_{jk,t}] + \mathbf{g}_{j,t} = -\mathbf{g}_{k,t} - \sum_{j=1}^{n} [-c_{jk,t}] + b_{j,t_0} + \sum_{j=1}^{n} [-b_{jk,t}] + \mathbf{g}_{j,t}.
\]

Theorem 2.2. The following holds for any \( t \in \mathbb{T}_n \).

(First duality. [FZ07], Eq. (6.14))

\[
G_t B_t = B_{t_0} C_t;
\]

(Second duality. [NZ12], Eq. (3.11).)

\[
D^{-1} G_t^T D C_t = I;
\]

(Sign-coherence. [Gro+18], Cor. 5.5) for all \( 1 \leq k \leq n \), either \( c_{ik,t} \geq 0 \) for all \( 1 \leq i \leq n \), or \( c_{ik,t} \leq 0 \) for all \( 1 \leq i \leq n \). In other words, there exists \( \varepsilon_{k,t} \in \{-1, 1\} \) such that \( \mathbf{c}^+_{k,t} = \varepsilon_{k,t} \mathbf{c}_{k,t} \) consists of only non-negative entries.

The first duality motivates the following definition.

Definition 2.3. Let \( \tilde{C}_t = G_t B_t = B_{t_0} C_t \), and let \( \tilde{C}_t = (\tilde{c}_{j,t})_{j=1}^{n} \). Let \( \tilde{c}^+_{j,t} = \varepsilon_{j,t} \tilde{c}_{j,t} \).

Contrary to the case of the \( \mathbf{c}^+ \)-vectors, as we will see in Example 3.3 we do not expect the entries of \( \tilde{c}^+ \) to be all non-negative.

Additionally, the presence of the skew-symmetrizer \( D \) in the second duality motivates us to consider the inner product \( (\mathbf{u}, \mathbf{v})_D = \mathbf{u}^T D \mathbf{v} \). We observe that \( (\mathbf{u}, \mathbf{v})_D = (\mathbf{v}, \mathbf{u})_D \), and

\[
(\mathbf{u}, B_t \mathbf{v})_D = \mathbf{u}^T D B_t \mathbf{v} = -\mathbf{u}^T B_t^T D \mathbf{v} = -(B_t \mathbf{u}, \mathbf{v})_D
\]

for all \( t \in \mathbb{T}_n \).

Using this inner product, the second duality can be equivalently stated as follows in terms of individual \( \mathbf{g} \)-vectors and \( \mathbf{c} \)-vectors: for \( 1 \leq i, j \leq n \) and any \( t \in \mathbb{T}_n \),

\[
(\mathbf{g}_{i,t}, d_j \mathbf{c}_{j,t})_D = \delta_{i,j}.
\]

Sign-Coherence allows us to rewrite Proposition 2.1 in a simpler form.

Corollary 2.4. Let \( t', t \in \mathbb{T}_n \) be \( k \)-adjacent. Then \( \mathbf{g}_{i,t'} = \mathbf{g}_{i,t} \) if \( i \neq k \) and

\[
\mathbf{g}_{k,t'} = -\mathbf{g}_{k,t} + \sum_{j=1}^{n} [-\varepsilon_{k,t} b_{jk,t}] + \mathbf{g}_{j,t}.
\]

2.3. Fock-Goncharov decomposition for principal coefficients. Following [Nak23], which builds off [FG09] in the \( \varepsilon = 1 \) case, we consider the following decomposition of the map \( \mu_{k,t} \):

\[
\mu_{k,t} = \rho_{k,t} \circ \tau_{k,t},
\]
where the map \( \tau_{k:t} \) is the following isomorphism,
\[
\tau_{k:t} : \mathbb{Q}^P(x_{i;t'}) \to \mathbb{Q}^P(x_i),
\]
where
\[
\begin{align*}
\tau_{k:t}(x_{i;t'}) &= \begin{cases} 
\frac{1}{x_{k:t}^{-1}} \prod_{j=1}^n x_{j;t}^{-|\varepsilon_{k:t}b_{jk:t}|^+} & i = k, \\
 x_{i;t} & i \neq k,
\end{cases}
\end{align*}
\]
while the map \( \rho_{k:t} \) is the following automorphism,
\[
\rho_{k:t} : \mathbb{Q}^P(x_i) \to \mathbb{Q}^P(x_i),
\]
\[
\rho_{k:t}(x_{i;t}) = x_{i;t}(1 + \varepsilon_{k:t} y_{k:t}^{-\delta_{i,k}}).
\]

We call the decomposition \( \tau_{k:t} \) the Fock-Goncharov decomposition of a mutation \( \mu_{k:t} \) (with tropical sign) with respect to the initial vertex \( t_0 \). We also call \( \tau_{k:t} \) and \( \rho_{k:t} \) the tropical part and the nontropical part of \( \mu_{k:t} \), respectively. See Section 4 of [Nak23] for more information. Comparing this decomposition with our definition of \( \mu_{k:t} \) in (1), we note that here we are choosing \( \varepsilon \) to be the tropical sign \( \varepsilon_{k:t} \) and hence the denominator \( 1 \oplus \varepsilon_{k:t} = 1 \).

The involution property of the mutations of cluster variables \( x_i \) and \( g \)-vectors \( g_{i;t} \) is restated as follows.

**Proposition 2.5.** Let \( t, t' \in \mathbb{T}_n \) be vertices which are \( k \)-adjacent. Then, the following relations hold.
\[
\mu_{k:t'} \circ \mu_{k:t} = \text{id},
\]
\[
\tau_{k:t'} \circ \tau_{k:t} = \text{id}.
\]

**Proof.** Here, the first equality is nothing but the involution of the mutations of \( x \)-variables. The second equality simply relies on the two identities \( \varepsilon_{k:t'} = -\varepsilon_{k:t} \) and \( b_{jk:t'} = -b_{jk:t} \). \( \square \)

For any \( t \in \mathbb{T}_n \), consider a sequence of vertices \( t_0, t_1, \ldots, t_{r+1} = t \in \mathbb{T}_n \) such that they are sequentially adjacent with edges labeled by \( k_0, \ldots, k_r \). Then, we define isomorphisms
\[
\begin{align*}
\mu^{t_0}_t &= \mu_{k_0:t_0} \circ \mu_{k_1:t_1} \circ \cdots \circ \mu_{k_r:t_r} : \mathbb{Q}^P(x_i) \to \mathbb{Q}^P(x), \\
\tau^{t_0}_t &= \tau_{k_0:t_0} \circ \tau_{k_1:t_1} \circ \cdots \circ \tau_{k_r:t_r} : \mathbb{Q}^P(x_i) \to \mathbb{Q}^P(x),
\end{align*}
\]
where we apply the automorphisms from right-to-left, and we have set \( x = x_{t_0} \). Thanks to Proposition 2.5, \( \mu^{t_0}_t \) and \( \tau^{t_0}_t \) depend only on \( t_0 \) and \( t \). Namely, we do not have to care about any instances of the redundancy \( k_{r+1} = k_s \) in the sequence \( k_0, \ldots, k_r \).

The following proposition tells that the tropical parts \( \tau^{t_0}_t \) are nothing but the mutations of the tropical parts of cluster variables (\( g \)-vectors).

**Proposition 2.6.** The following formulas hold:
\[
\begin{align*}
\mu^{t_0}_t(x_{i;t}) &= x^{g_{i;t}} F_{i;t}(\tilde{y}), \\
\tau^{t_0}_t(x_{i;t}) &= x^{g_{i;t}}, \\
\tau^{t_0}_t(y_{i;t}) &= \tilde{y}^{g_{i;t}}.
\end{align*}
\]

**Proof.** Equation (9) is the separation formula, i.e. Proposition 3.13 and Corollary 6.3 of [FZ07]. Note that the denominator of the separation formula is 1 due to Prop 5.2 of [FZ07] since we assume that our cluster pattern \( \Sigma \) has principal coefficients at the initial seed
t_0$. We next note that (7) follows from Corollary 2.4. Finally, (8) follows from (7) and Theorem 2.2. See Proposition 4.3 of [Nak23] for more details.

2.4. Nontropical parts and \(-P\)-polynomials. Observing Proposition 2.6, it is clear that the nontropical parts \(\rho_{k,t}\) are responsible for generating and mutating \(-P\)-polynomials. Let us make this statement more manifest, following [Nak23].

Let us introduce the following automorphisms \(q_{k,t}\) of \(\mathbb{Q}P(x)\) for the initial \(x\)-variables \(x\).

\[
q_{k,t} : \mathbb{Q}P(x) \to \mathbb{Q}P(x),
\]

\[
x^m \mapsto x^m(1 + \hat{y}^{+}_{k,t})^{-\left(m, d_k c_{k,t}\right)} D, \quad m \in \mathbb{Z}^n,
\]

where \(\hat{y}_i\) are the initial \(\hat{y}\)-variables. Recall that \(B = B_{t_0}\). For \(n \in \mathbb{Z}^n\), we have

\[
q_{k,t}(\hat{y}^n) = q_{k,t}(y^n x^B) = y^n x^B (1 + \hat{y}^{+}_{k,t})^{-\left(Bn, d_k c_{k,t}\right)} D = \hat{y}^n (1 + \hat{y}^{+}_{k,t})^{\left(n, d_k c_{k,t}\right)} D,
\]

where we have used (3) at the last equality. But \(c_{k,t} = Bc_{k,t}\), so

\[
q_{k,t}(\hat{y}^n) = \hat{y}^n (1 + \hat{y}^{+}_{k,t})^{\left(n, d_k c_{k,t}\right)} D.
\]

One can easily confirm the following properties.

**Proposition 2.7** ([Nak23], Prop. 4.5). The following facts hold:

(a). We have the formula

\[
q_{k,t}(x^{g_{k,t}}) = x^{g_{k,t}} (1 + \hat{y}^{+}_{k,t})^{-\delta_{i,k}}.
\]

(b). The following relation holds:

\[
\tau_t^{t_0} \circ \rho_{k,t} = q_{k,t} \circ \tau_t^{t_0}.
\]

(c). If \(t'\) and \(t\) are \(k\)-adjacent, we have

\[
q_{k,t'} \circ q_{k,t} = \text{id}.
\]

Similar to our definitions of \(\mu_t^{t_0}\) and \(\tau_t^{t_0}\), given any \(t \in T_n\), we consider a sequence of vertices \(t_0, t_1, \ldots, t_{r+1} = t \in T_n\), which are sequentially adjacent by edges labelled by \(k_0, \ldots, k_r\). Then we define the automorphism

\[
q_{t_0}^{t_0} := q_{k_0, t_0} \circ q_{k_1, t_1} \circ \cdots \circ q_{k_r, t_r} : \mathbb{Q}P(x) \to \mathbb{Q}P(x).
\]

Again, by Proposition 2.7(c), it depends only on \(t_0\) and \(t\).

One can separate the tropical and the nontropical parts of \(\mu_t^{t_0}\) as follows:

**Proposition 2.8** ([Nak23], Prop. 4.6). The following decomposition holds:

\[
\mu_t^{t_0} = q_{t_0}^{t_0} \circ \tau_t^{t_0} : \mathbb{Q}P(x_t) \to \mathbb{Q}P(x).
\]

**Proof.** This can be proved by the successive application of Proposition 2.7(b). For simplicity, we omit the symbol \(\circ\) for the composition of maps below. First, we note that

\[
\rho_{k_0, t_0}(x_i) = x_i (1 + \hat{y}_{k_0})^{-\delta_{i,k_0}} = q_{k_0, t_0}(x_i).
\]
Then, by Proposition \[\text{(2.7)}\] we have (since \(\rho_{k_0:t_0} = q_{k_0:t_0}, \tau_{t_1} = \tau_{k_0:t_0}, \tau_{t_2} = \tau_{k_0:t_0} \tau_{k_1:t_1}, \text{etc.}\))

\[
\mu_{t_0}^{t_0} = \mu_{k_0:t_0} \mu_{k_1:t_1} \cdots \mu_{k_{r-1}:t_{r-1}} \\
= \rho_{k_0:t_0} \tau_{k_0:t_0} \rho_{k_1:t_1} \tau_{k_1:t_1} \cdots \rho_{k_{r-1}:t_{r-1}} \tau_{k_{r-1}:t_{r-1}} \\
= q_{k_0:t_0} \tau_{t_1} \rho_{k_1:t_1} \tau_{k_1:t_1} \cdots \rho_{k_{r-1}:t_{r-1}} \tau_{k_{r-1}:t_{r-1}} \\
= q_{k_0:t_0} q_{k_1:t_1} \tau_{t_1} \rho_{k_2:t_2} \cdots \rho_{k_{r-1}:t_{r-1}} \tau_{k_{r-1}:t_{r-1}} \\
= \cdots \\
= q_{k_0:t_0} q_{k_1:t_1} \cdots q_{k_{r-1}:t_{r-1}} \tau_{t_0} \\
= q_{t_0}^{t_0} t_{t_0}.
\]

where \(t = t_{r+1}\). We record this argument as a commutative diagram as illustrated below.

![Diagram](image)

We conclude that the automorphisms \(q_t^{t_0}\) generate the nontropical parts of cluster variables (namely, \(F\)-polynomials) in the following manner.

**Theorem 2.1** (\[\text{Nak23}\], Thm. 4.7). The following formula holds:

\[
q_t^{t_0}(x_{g_{i:t}}) = x_{g_{i:t}} F_{i:t}(\hat{y}).
\]

**Proof.** This follows from Propositions \[\text{(2.6)}\] and \[\text{(2.8)}\] as

\[
q_t^{t_0}(x_{g_{i:t}}) = q_t^{t_0}(\tau_{t}^{t_0}(x_{i:t})) = \mu_t^{t_0}(x_{i:t}) = x_{g_{i:t}} F_{i:t}(\hat{y}).
\]

\[\square\]

### 3. Statement of Gupta’s Formula

We are now ready to state the main theorem.

**Theorem 3.1** (Gupta’s Formula). Given a mutation sequence \(\mu_{i_1}, \mu_{i_2}, \ldots\), let \(t_f\) be the seed obtained by applying the mutations \(\mu_{i_1}, \mu_{i_2}, \ldots, \mu_{i_j}\) to the initial seed \(t_0\) in the cluster pattern defined by the initial exchange matrix \(B_{t_0}\), starting with the application of \(\mu_{i_1}\), and proceeding left-to-right. Here, \(B_{t_0}\) is an arbitrary initial skew-symmetrizable matrix.
Let
\[ d_{(j)} = d_{ij}, \quad c_{(j)} = c_{ij; t_j}, \quad c_{(j)}^+ = c_{ij; t_j}, \quad \tilde{c}_{(j)}^+ = \tilde{c}_{ij; t_j}, \quad g_{(j)} = g_{ij; t_j}, \quad z_j = \tilde{y}^c_{(j)}. \]
Then the \( \ell \)-th \( F \)-polynomial along the mutation sequence is
\[
F_{i; t; t}(\tilde{y}) = \prod_{j=1}^{\ell} L_{j}^{\left(g_{(j)}, d_{(j)} c_{(j)}\right)} \quad \text{where} \quad L_1 = 1 + z_1, L_k = 1 + z_k \prod_{j=1}^{k-1} L_{j}^{\left(c_{(j)}^+, d_{(j)} c_{(j)}\right)}.
\]

**Remark 3.2.** The formula can be restated in a different manner using ordinary dot products. We write \( C_t^{B; t_0} \) denote the \( C \)-matrix at seed \( t \) in the cluster pattern defined by an initial exchange matrix \( B \), when the choice of \( B \) is not clear from the context. Let \( \tilde{C}_t = C_t^{-B^T; t_0} = (\tilde{c}_{i; t})_{i=1}^{n} \), and let \( \tilde{c}_{(j)} = \tilde{c}_{ij; t_j} \). Then
\[
F_{i; t; t}(\tilde{y}) = \prod_{j=1}^{\ell} L_{j}^{\tilde{c}_{(j)} \cdot g_{(j)}} \quad \text{where} \quad L_1 = 1 + z_1, L_k = 1 + z_k \prod_{j=1}^{k-1} L_{j}^{\tilde{c}_{(j)} \cdot \tilde{c}_{(j)}}.
\]
To see that these two formulations agree, we recall the following equality ([NZ12, Eq. (2.7)]):
\[
C_t^{-B^T; t_0} = D C_t^{B; t_0} D^{-1}.
\]

It follows that \( \tilde{c}_{(j)} = D d_{(j)} c_{(j)} \). Thus
\[
\begin{align*}
(g_{(j)}, d_{(j)} c_{(j)})_D &= g_{(j)}^T D d_{(j)} c_{(j)} = g_{(j)}^T \tilde{c}_{(j)} = \tilde{c}_{(j)} \cdot g_{(j)}, \\
(\tilde{c}_{(k)}^+, d_{(j)} c_{(j)})_D &= \tilde{c}_{(k)}^{+T} D d_{(j)} c_{(j)} = \tilde{c}_{(k)}^{+T} \tilde{c}_{(j)} = \tilde{c}_{(j)} \cdot \tilde{c}_{(k)}^+.
\end{align*}
\]

**Example 3.3.** Let \( \mu = \mu_1 \mu_2 \mu_1 \) and let
\[
B_{t_0} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.
\]
Then
\[
D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix},
\]
and \( d_{(1)} = d_1 = 1, d_{(2)} = d_2 = 4, \) and \( d_{(3)} = d_1 = 1 \). We will compute the \( F \)-polynomial \( F_{1, t_3}(y_1, y_2) \). The \( c, \tilde{c}^+ \), \( g \)-vectors involved are as follows:
\[
\begin{align*}
c_{(1)} &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & c_{(2)} &= \begin{bmatrix} -1 \\ -1 \end{bmatrix}, & c_{(3)} &= \begin{bmatrix} -3 \\ -4 \end{bmatrix}, \\
\tilde{c}_{(1)}^+ &= \begin{bmatrix} 0 \\ -4 \end{bmatrix}, & \tilde{c}_{(2)}^+ &= \begin{bmatrix} 1 \\ -4 \end{bmatrix}, & \tilde{c}_{(3)}^+ &= \begin{bmatrix} 4 \\ -12 \end{bmatrix}, \\
g_{(1)} &= \begin{bmatrix} -1 \\ 4 \end{bmatrix}, & g_{(2)} &= \begin{bmatrix} -1 \\ 3 \end{bmatrix}, & g_{(3)} &= \begin{bmatrix} -3 \\ 8 \end{bmatrix}.
\end{align*}
\]
So \( z_1 = \tilde{y}_1, z_2 = \tilde{y}_1 \tilde{y}_2, z_3 = \tilde{y}_1^3 \tilde{y}_2^4 \). The relevant inner products are
\[
\begin{align*}
(g_{(3)}, d_{(1)} c_{(1)})_D &= 3, & (g_{(3)}, d_{(2)} c_{(2)})_D &= 4, & (g_{(3)}, d_{(3)} c_{(3)})_D &= 1, \\
(\tilde{c}_{(2)}^+, d_{(1)} c_{(1)})_D &= -1, & (\tilde{c}_{(3)}^+, d_{(1)} c_{(1)})_D &= -4, & (\tilde{c}_{(3)}^+, d_{(2)} c_{(2)})_D &= -4.
\end{align*}
\]
Therefore,

\[
L_1 = 1 + z_1 = 1 + \hat{y}_1, \\
L_2 = 1 + z_2 L_1^{(\hat{e}_{(2)}^+, d_{(1)}^+ c_{(1)}) D} = 1 + \hat{y}_1 \hat{y}_2 (1 + \hat{y}_1)^{-1}, \\
L_3 = 1 + z_3 L_1^{(\hat{e}_{(3)}^+, d_{(1)}^+ c_{(1)}) D} L_2^{(\hat{e}_{(3)}^+, d_{(2)}^+ c_{(2)}) D} \\
= 1 + \hat{y}_1^2 \hat{y}_2^2 (1 + \hat{y}_1)^{-4} (1 + \hat{y}_1 \hat{y}_2 (1 + \hat{y}_1)^{-1})^{-4} \\
= 1 + \hat{y}_1^3 \hat{y}_2^4 (1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2)^{-4}.
\]

Applying Gupta’s formula then gives us

\[
F_{1,t_3}(\hat{y}_1, \hat{y}_2) = L_1^{(g_{(3)}^+, d_{(1)}^+ c_{(1)}) D} L_2^{(g_{(3)}^+, d_{(2)}^+ c_{(2)}) D} L_3^{(g_{(3)}^+, d_{(3)}^+ c_{(3)}) D} \\
= L_1^3 L_2^3 L_3 \\
= (1 + \hat{y}_1)^3 (1 + \hat{y}_1 \hat{y}_2 (1 + \hat{y}_1)^{-1})^4 (1 + \hat{y}_1^2 \hat{y}_2^4 (1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2)^{-4}) \\
= \frac{(1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2)^4 + \hat{y}_1 \hat{y}_2^4}{1 + \hat{y}_1} \\
(11) \\
= 1 + 3\hat{y}_1 + 3\hat{y}_1^2 + \hat{y}_1^3 + 4\hat{y}_1 \hat{y}_2 + 8\hat{y}_1^2 \hat{y}_2 + 4\hat{y}_1^3 \hat{y}_2 + 6\hat{y}_1^2 \hat{y}_2^2 + 6\hat{y}_1 \hat{y}_2^2 + 4\hat{y}_1^3 \hat{y}_2^2 + \hat{y}_1^3 \hat{y}_2^4.
\]

4. First Proof of Theorem 3.1

Recall that \([n]_+ = \max(n, 0)\). Given a vector \(u = (u_1, \ldots, u_n)\), let \([u]_+ = ([u_1]_+, \ldots, [u_n]_+)\).

Proof. We shall prove the following formula for each \(F\)-polynomial at the seed \(t_\ell\), which specializes to Theorem 3.1 when \(i = i_k\):

\[
F_{i,t_\ell} = \prod_{j=1}^{\ell} L_j^{(g_{i,t_j}^+, d_{j}^+ c_{(j)}) D} \text{ where } L_1 = 1 + z_1, L_k = 1 + z_k \prod_{j=1}^{k-1} L_j^{(\hat{e}_{(k)}^+, d_{(j)}^+ c_{(j)}) D}.
\]

We proceed by induction on \(\ell\). The base case is \(\ell = 0\), where the formula above reduces to the empty product, which we interpret to be 1 by convention. Since we are at the initial seed, the \(F\)-polynomial for each cluster variable is just 1. So the \(F\)-polynomials at \(t_0\) agree with the formula.

Now suppose that the formula is correct for some \(\ell \geq 0\), and let \(k = i_{\ell+1}\), \(t = t_\ell\) and \(t' = t_{\ell+1}\). By (3), \((g_{i,t'}^+, d_k c_{(\ell+1)}) D = \delta_{i,k}\). (Note here that \(d_k = d_{(\ell+1)}\).) So if \(i \neq k\), since \(g_{i,t'} = g_{i,t}\), for all \(1 \leq j \leq \ell\),

\[
(g_{i,t'}^+, d_{(j)}^+ c_{(j)}) D = (g_{i,t}^+, d_{(j)}^+ c_{(j)}) D,
\]

and

\[
(g_{i,t'}^+, d_k c_{(\ell+1)}) D = 0.
\]

Therefore, when \(i \neq k\),

\[
F_{i,t'} = F_{i,t} = \prod_{j=1}^{\ell} L_j^{(g_{i,t_j}^+, d_{(j)}^+ c_{(j)}) D} = \prod_{j=1}^{\ell+1} L_j^{(g_{i,t_j}^+, d_{(j)}^+ c_{(j)}) D}. 
\]
It remains to prove (12) for $F_{k,t'}$. By the recurrence of $F$-polynomials, i.e. equation (2), we know that

$$F_{k,t'} = \frac{\tilde{\gamma}[c_{k,t}] + \prod_{j=1}^{n} F_{t,j}^{[b_{j,k,t}]} + \tilde{\gamma}[-c_{k,t}] + \prod_{j=1}^{n} F_{t,j}^{[-b_{j,k,t}]}}{F_{k,t}}.$$ 

Assuming that (12) is true for $t_\ell$, we can substitute $F_{t,j}$ in the numerator and $F_{k,t}$ in the denominator by the appropriate products of the $L_j$’s, obtaining

$$F_{k,t'} = \tilde{\gamma}[c_{k,t}] + \prod_{j=1}^{\ell} L_j \left( \sum_{i=1}^{n} [b_{i,k,t}] + g_{i,t} - g_{k,t} \right) \times D + \tilde{\gamma}[-c_{k,t}] + \prod_{j=1}^{\ell} L_j \left( \sum_{i=1}^{n} [-b_{i,k,t}] + g_{i,t} - g_{k,t} \right) \times \times D,$$

where the $-g_{k,t}$ comes from $F_{k,t}$ in the denominator.

By Proposition 2.1,

$$\sum_{i=1}^{n} [b_{i,k,t}] + g_{i,t} - g_{k,t} = g_{k,t'} + \sum_{j=1}^{n} [c_{j,k,t}] + b_{j,t_0} = g_{k,t'} + B_{t_0} [c_{k,t}] +$$

and

$$\sum_{i=1}^{n} [-b_{i,k,t}] + g_{i,t} - g_{k,t} = g_{k,t'} + \sum_{j=1}^{n} [-c_{j,k,t}] + b_{j,t_0} = g_{k,t'} + B_{t_0} [-c_{k,t}] +.$$

So

$$F_{k,t'} = \tilde{\gamma}[c_{k,t}] + \prod_{j=1}^{\ell} L_j \left( g_{k,t'} + B_{t_0} [c_{k,t}] \times D \right) + \tilde{\gamma}[-c_{k,t}] + \prod_{j=1}^{\ell} L_j \left( g_{k,t'} + B_{t_0} [-c_{k,t}] \times D \right).$$

If $c_{k,t} = 1$,

$$F_{k,t'} = \tilde{\gamma}[c_{k,t}] \prod_{j=1}^{\ell} L_j \left( g_{k,t'} + B_{t_0} c_{k,t} \times D \right) + \tilde{\gamma}[-c_{k,t}] \prod_{j=1}^{\ell} L_j \left( g_{k,t'} + B_{t_0} [c_{k,t}] \times D \right),$$

and if $c_{k,t} = -1$,

$$F_{k,t'} = \prod_{j=1}^{\ell} L_j \left( g_{k,t'} \times D \right) + \tilde{\gamma}[-c_{k,t}] \prod_{j=1}^{\ell} L_j \left( g_{k,t'} + B_{t_0} c_{k,t} \times D \right).$$

We can combine these two cases:

$$F_{k,t'} = \prod_{j=1}^{\ell} L_j \left( g_{k,t'} \times D \right) \times (1 + \tilde{\gamma}[c_{k,t}] \prod_{j=1}^{\ell} L_j \left( B_{t_0} c_{k,t} \times D \right))$$

$$= \prod_{j=1}^{\ell} L_j \left( g_{k,t'} \times D \right) \times \left( 1 + z_{\ell+1} \prod_{j=1}^{\ell} L_j \left( c_{k,t} \times D \right) \right)$$

$$= \left( \prod_{j=1}^{\ell} L_j \left( g_{k,t'} \times D \right) \right) L_{\ell+1}.$$
Recall that \((g_{k,t'}, d_k c_{(\ell+1)}) = \delta_{c,k}\), so \((g_{k,t'}, d_k c_{(\ell+1)})_D = 1\). Hence,
\[
F_{k,t'} = \prod_{j=1}^{\ell+1} L_j (g_{k,t'}, d_j c_{(j)})_D
\]

as desired.

5. SECOND PROOF OF THEOREM 3.1

Proof. By Theorem 2.1, it suffices to show that
\[
q^t_0 (x^{g(t)}) = x^{g(t)} \prod_{j=1}^{\ell} L_j (g_j, d_j c_{(j)})_D.
\]

We first prove that for \(0 \leq m \leq k-1\),
\[
(q^t_0 (z_k) = z_k \prod_{j=1}^{m} L_j (\hat{c}_{(j)}^+ d_j c_{(j)})_D.
\]

We prove (13) for a fixed \(k\) by induction on \(m\). The automorphism \(q^t_0\) is the identity and the empty product is understood to be 1, so the claim follows when \(m = 0\). Now suppose that it is true for some \(0 \leq m < k-1\). We wish to prove it for \(m + 1\). Recall from (9) that
\[
q^t_0 (y^n) = \hat{y}^n (1 + y^{c_{(i,t)}}(n_d c_{(i,t)})_D.
\]

Therefore
\[
q^t_0 (z_{m+1}) = z_{m+1} (1 + y^{c_{(m+1,t)}}(c_{(m+1)}\tilde{e}_{(m+1,t_m)})_D.
\]

Since \(c_{(m+1,t_m)} = -c_{(m+1,t_m+1)}\), we have \(c_{(m+1,t_m)}^+ = c_{(m+1)}^+ \) and \(\tilde{e}_{(m+1,t_m)} = -\tilde{e}_{(m+1)}\). So
\[
(c_{(m+1)}^+, d_{(m+1)}\tilde{e}_{(m+1,t_m)})_D = (c_{(m+1)}^+, d_{(m+1)}B_{t_0} c_{(m+1)})_D
\]
\[
= (B_{t_0} c_{(m+1)}^+, d_{(m+1)} c_{(m+1)})_D
\]
\[
= (\hat{c}_{(k)}^+, d_{(m+1)} c_{(m+1)})_D.
\]

which allows us to write
\[
q^t_0 (z_{m+1}) = z_{m+1} (1 + z_{m+1})^+(\hat{c}_{(k)}^+, d_{(m+1)} c_{(m+1)})_D.
\]

Using the induction hypothesis and substituting in our calculation of \(q^t_0 (z_{m+1})\),
\[
q^t_0 (z_{m+1}) = q^t_0 (q^t_0 (z_{m+1}))
\]
\[
= q^t_0 (z_{m+1} (1 + z_{m+1})^+(\hat{c}_{(k)}^+, d_{(m+1)} c_{(m+1)})_D
\]
\[
= z_k \prod_{j=1}^{m} L_j (\hat{c}_{(k)}^+, d_j c_{(j)})_D (1 + z_{m+1} \prod_{j=1}^{m} L_j (\hat{c}_{(m+1)}^+, d_j c_{(j)})_D
\]
\[
= z_k \prod_{j=1}^{m+1} L_j (\hat{c}_{(k)}^+, d_j c_{(j)})_D
\]
This concludes the proof of (13). With it, we compute that for all $1 \leq j \leq \ell$,

$$q_{t_j}^{t_0} \left( \frac{x \mathcal{G}(t)}{\mathcal{G}(t)} \right) = q_{t_{j-1}}^{t_0} \left( \frac{q_{i_j,t_j-1}^{t_0} \left( x \mathcal{G}(t) \right)}{\mathcal{G}(t)} \right)$$

$$= q_{t_{j-1}}^{t_0} \left[ (1 + \hat{Y}^{t_j,t_{j-1}})^{-1} \left( \mathcal{G}(t), \mathcal{G}(j) \right) D \right]$$

$$= q_{t_{j-1}}^{t_0} \left[ (1 + z_j) \left( \mathcal{G}(t), \mathcal{G}(j) \right) D \right]$$

$$= (1 + q_{t_{j-1}}^{t_0} (z_j)) \left( \mathcal{G}(t), \mathcal{G}(j) \right) D$$

$$= \left( 1 + z_j \prod_{k=1}^{j-1} L_k^{\mathcal{G}(t), \mathcal{G}(j)} D \right)$$

$$= L_j^{\mathcal{G}(t), \mathcal{G}(j)} D.$$

It follows that

$$q_{t_j}^{t_0} \left( \frac{x \mathcal{G}(t)}{\mathcal{G}(t)} \right) = \prod_{j=1}^{\ell} q_{t_j}^{t_0} \left( \frac{x \mathcal{G}(t)}{\mathcal{G}(t)} \right)$$

$$= \prod_{j=1}^{\ell} \left( \mathcal{G}(t), \mathcal{G}(j) \right) D$$

$$= \prod_{j=1}^{\ell} L_j^{\mathcal{G}(t), \mathcal{G}(j)} D$$

as desired. □

**Remark 5.1.** After Gupta’s work [Gup18] was posted on the arXiv, Man-Wai Cheung and Bernhard Keller separately reached out to Gupta and the second author, both pointing out that Gupta’s formula was related to some known results in the literature.

Keller pointed out that Gupta’s formula can be obtained from Theorem 6.4 of [Kel13], which was first proven in [Nag13]. Cheung noticed that after translating the relevant notation arising in scattering diagrams, Gupta’s formula can be deduced from the formula for $F$-polynomials in terms of the path-ordered products of [Gro+18].

In fact, in the skew-symmetric case, the formula (10) is immediately obtained by setting $q = 1$ in Theorem 6.4 of [Kel13], which is the quantum version of the formula (10). Also, the automorphism $q_{t_j}^{t_0}$ is identified with a path-ordered product in the corresponding scattering diagram in [Gro+18]. Then, the formula (10) is immediately obtained from Theorem 4.9 of [Gro+18]. (See also [Rea20], [Nak23].) Thus, both approaches prove Gupta’s formula via the formula (10), and the proof presented here, which is based on the formula (10), should coincide with their suggestions, while we emphasize that we proved the formula (10) without referring to the quantum case or Theorem 4.9 of [Gro+18].

Meanwhile, Gupta’s formula itself and our proof of the formula (10) crucially depend on the sign-coherence property proved by [Gro+18]. However, in [Nak23], it was also clarified that the sign-coherence is proved without Theorem 4.9 of [Gro+18]. Thus, our proof here together with the proof of the formula (10) is still in a different perspective to the above ones.
6. Expansion of Gupta’s Formula

One may wish to expand the product formula given in Theorem 3.1 into a sum. To do so, we first prove a lemma.

Lemma 6.1. Given integers \(h_1, \ldots, h_\ell\) and the same setup as Theorem 3.1,

\[
\prod_{j=1}^\ell L_j^{h_j} = \sum_{(m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^\ell \left( h_j + \sum_{k=j+1}^\ell m_k (\bar{c}_k, d_j c(j)) D \right) \tilde{y}^{\sum_{j=p}^\ell m_j c^+_j}.
\]

Proof. We prove the following claim by induction: for all \(1 \leq p \leq \ell\),

\[
(14)
\prod_{j=p}^\ell L_j^{h_j} = \sum_{(m_p, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=p}^\ell L_j^{m_j} \left( h_j + \sum_{k=j+1}^\ell m_k (\bar{c}_k, d_j c(j)) D \right) \tilde{y}^{\sum_{j=p}^\ell m_j c^+_j}.
\]

When \(p = 1\), this claim specializes to our lemma.

First note that by the Generalized Binomial Theorem, for any \(h \in \mathbb{Z}\), we can write

\[
L_k^h = \left( 1 + \frac{h}{L_k} \prod_{j=1}^{k-1} L_j^{\bar{c}_j} (d_j c(j)) D \right)^h
\]

\[
= \sum_{m \in \mathbb{Z}_{\geq 0}} \frac{h^m}{m!} \prod_{j=1}^{k-1} L_j^{m_j} \left( h_j + \sum_{k=j+1}^\ell m_k (\bar{c}_k, d_j c(j)) D \right) \tilde{y}^{m c^+_k}.
\]

If we let \(k = \ell\) and \(h = h_\ell\), the above is precisely the base case \(p = \ell\) of our claim.

Now suppose that (14) is true for \(p + 1\). Multiplying both sides by \(L_p^{p_1}\), we get

\[
(16)
\prod_{j=p+1}^\ell L_j^{h_j} = \sum_{(m_{p+1}, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \left( \prod_{j=p+1}^\ell L_j^{m_j} \left( h_j + \sum_{k=j+1}^\ell m_k (\bar{c}_k, d_j c(j)) D \right) \tilde{y}^{m_j c^+_j} \right).
\]

Letting \(k = p\) and \(h = h_p + \sum_{k=p+1}^\ell m_k (\bar{c}_k, d(p) c(p)) D\), our computation in Eq. (15) allows us to expand the middle term of (16)

\[
L_p^{h_p + \sum_{k=p+1}^\ell m_k (\bar{c}_k, d(p) c(p)) D} = \sum_{m_p \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{p-1} L_j^{m_j} \left( h_p + \sum_{k=p+1}^\ell m_k (\bar{c}_k, d(p) c(p)) D \right) \tilde{y}^{m_p c^+_p}.
\]

Therefore, Eq. (16) reorganizes into

\[
\prod_{j=p}^\ell L_j^{h_j} = \sum_{(m_p, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}} \prod_{j=p}^\ell L_j^{m_j} \left( h_j + \sum_{k=j+1}^\ell m_k (\bar{c}_k, d_j c(j)) D \right) \tilde{y}^{\sum_{j=p}^\ell m_j c^+_j}
\]

as desired. This completes our proof. \(\square\)
Applying Lemma 6.1 to Theorem 3.1 yields the following alternative form of Gupta’s formula.

**Theorem 6.2.** Under the same conditions as Theorem 3.1,

\[
F_{i;\ell} (\hat{y}) = \sum_{(m_1, m_2, m_3) \in \mathbb{Z}_{\geq 0}} \prod_{j=1}^{\ell} \left( (g_{(j)}, d_{(j)} c_{(j)}) D + \sum_{k=j+1}^{\ell} m_k (c^+_{(k)}, d_{(j)} c_{(j)}) D \right) \hat{y}^{m_1 + m_2 + 3m_3} \hat{y}_2^{m_2 + 4m_3}.
\]

**Remark 6.3.** An expression resembling Theorem 6.2 also appears in Gupta’s original formulation, in particular, see Theorem 3.1 of [Gup18]. However one then has to use Definitions 2.16 and 2.17 of [Gup18], as well as translation into the language of \( c \)- and \( g \)-vectors, as well as algebraic manipulation of factorials, to result in the current expression.

**Example 6.4.** We now try to reproduce the result of Example 3.3 using Theorem 6.2.

Back in Example 3.3, we have already done the relevant computations involved in the statement of the new formula, giving us

\[
F_{1;3} (\hat{y}) = \sum_{(m_1, m_2, m_3) \in \mathbb{Z}_{\geq 0}} \left( \binom{3 - m_2 - 4m_3}{m_1} \binom{4 - 4m_3}{m_2} \right) \binom{1}{m_3} \hat{y}_1^{m_1 + m_2 + 3m_3} \hat{y}_2^{m_2 + 4m_3}.
\]

Since \( \binom{N}{2} = 0 \) when \( N > 0 \) and \( s < 0 \), or when \( N > 0 \) and \( s > N \), we may restrict our attention to certain tuples. Table 1 captures ten tuples and their contributions to \( F_{1;3} (\hat{y}) \).

For example, the first row considers all the tuples of the form \((m_1, 0, 0)\). Since the first binomial evaluates to \( \binom{3}{m_1} \), there is only a nonzero contribution when \( m_1 = 0, 1, 2, 3 \), yielding four terms that collect into \((1 + \hat{y}_1)^3\). However, this table is not exhaustive. It leaves two slightly different cases that we will consider separately. When \( m_3 = 0 \), the second binomial evaluates to \( \binom{4}{m_2} \), therefore \( m_2 \) may also take the value \( m_2 = 4 \). The contribution of tuples of the form \((m_1, 4, 0)\) is

\[
(17) \quad \hat{y}_1^{m_1} \hat{y}_2^4 \sum_{m_1 \geq 0} \binom{-1}{m_1} \hat{y}_1^{m_1} = \hat{y}_1^{m_1} \hat{y}_2^4 \sum_{m_1 \geq 0} (-1)^{m_1} \hat{y}_1^{m_1}.
\]

At this point we have considered all tuples where \( m_3 = 0 \). If \( m_3 = 1 \), in order for the second binomial to be nonzero, we must have \( m_2 = 0 \), and the contribution of tuples of the form \((m_1, 0, 1)\) is

\[
(18) \quad \hat{y}_1^1 \hat{y}_2^4 \sum_{m_1 \geq 0} \binom{-1}{m_1} \hat{y}_1^{m_1} = \hat{y}_1^1 \hat{y}_2^4 \sum_{m_1 \geq 0} (-1)^{m_1} \hat{y}_1^{m_1}.
\]

Summing (17) and (18) leaves a single term \( \hat{y}_1^3 \hat{y}_2^4 \). Since \( m_3 \) can only take the values 0 or 1, we have now finished enumerating all tuples with a nonzero contribution to the formula,
allowing us to conclude that

\[ F_{1,t_3}(\tilde{y}) = (1 + \tilde{y}_1)^3 + 4\tilde{y}_1\tilde{y}_2(1 + \tilde{y}_1)^2 + 6\tilde{y}_1^2\tilde{y}_2^2(1 + \tilde{y}_1) + 4\tilde{y}_1^3\tilde{y}_2^3 + \tilde{y}_1^3\tilde{y}_2^4, \]

which agrees with Eq. (11).

References

[EF12] Richard Eager and Sebastián Franco. “Colored BPS pyramid partition functions, quivers and cluster transformations”. In: *Journal of High Energy Physics* 2012.9 (2012), pp. 1–44.

[FG09] Vladimir V. Fock and Alexander B. Goncharov. “Cluster ensembles, quantization and the dilogarithm”. In: *Annales Scientifiques de l’École Normale Supérieure* Ser. 4, 42.6 (2009), pp. 865–930.

[FZ02] Sergey Fomin and Andrei Zelevinsky. “Cluster algebras I: foundations”. In: *Journal of the American Mathematical Society* 15.2 (2002), pp. 497–529.

[FZ07] Sergey Fomin and Andrei Zelevinsky. “Cluster algebras IV: coefficients”. In: *Compositio Mathematica* 143.1 (2007), pp. 112–164.

[Gro+18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. “Canonical bases for cluster algebras”. In: *Journal of the American Mathematical Society* 31.2 (2018), pp. 497–608.

[Gup18] Meghal Gupta. *A formula for F-polynomials in terms of c-vectors and stabilization of F-polynomials*. Dec. 2018. arXiv: [1812.01910 [math.CO]].

[Kel13] Bernhard Keller. “Cluster algebras and derived categories”. In: *Derived Categories in Algebraic Geometry* (2013).

[Nag13] Kentaro Nagao. “Donaldson–Thomas theory and cluster algebras”. In: *Duke Mathematical Journal* 162.7 (May 2013), pp. 1313–1367. ISSN: 0012-7094. DOI: [10.1215/00127094-2142753].

[Nak23] Tomoki Nakanishi. “Cluster algebras and scattering diagrams, Part II”. In: *Cluster patterns and scattering diagrams*. MSJ Memoirs 41 (2023).

[NZ12] Tomoki Nakanishi and Andrei Zelevinsky. “On tropical dualities in cluster algebras”. In: *Contemporary Mathematics* 565 (2012), pp. 217–226. ISSN: 978-0-8218-5317-7.

[Rea20] Nathan Reading. “A combinatorial approach to scattering diagrams”. In: *Algebraic Combinatorics* 3.3 (2020), pp. 603–636.