Preparation contextuality: the ground of quantum communication advantage

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Where does quantum advantage spring from? Such an investigation necessitates invoking an ontology on which non-classical features of quantum theory are explored. One such non-classical ontic-feature is preparation contextuality (PC) and advantage in oblivious communication tasks is its operational signature. This letter primarily addresses quantum advantage in communication complexity (CC). We demonstrate that quantum advantage in one-way CC operationally reveals PC. Specifically, we construct oblivious communication tasks tailored to given CC problems. The bound on classical success probability in the oblivious communication tasks forms our preparation non-contextual inequalities. We use the same states along with their orthogonal mixtures and the same measurements responsible for advantage in CC problems to orchestrate an advantageous protocol for the oblivious communication tasks and the violation of the associated inequalities. Further, we find a criterion for unbounded violation of these inequalities and demonstrate the same for two widely studied CC problems. Additionally, the tools thus developed enables the complete proof of the fact that (spatial and temporal) Bell-inequality violation implies an advantage in oblivious communication tasks, thereby revealing PC. Along with the implications of this work, we discuss other known indications towards our assertion that PC is the principal non-classical feature underlying quantum advantage.

Introduction.— Quantum resources paired up with ingenious quantum algorithms and protocols have outshone their classical counterparts in a plethora of computation, communication, and information processing tasks. While quantum theory has an operational formalism, any ontological framework that seeks to explain it must have certain non-classical features [1, 2]. These non-classical ontic-features provide fundamental insights into the subtleties of the quantum formalism responsible for this overwhelming feat. In this letter, we seek to identify the pivotal ontic-feature that powers quantum advantage. We require that such an ontic-feature be revealed operationally in the set-up displaying quantum advantage. We claim that no ontic-feature of quantum theory that discards any preparation non-contextual (PNC) models (introduced by Spekkens [3], the notion discards any preparation non-contextual (PNC) models as viable ontological descriptions of quantum theory. A model is PNC if it assigns identical distributions over the ontic-states to preparations that are operationally indistinguishable. Following the initial observation that PC is responsible for advantage in parity oblivious multiplexing [4], it was recently shown that quantum advantage in any oblivious-communication (OC) task reveals PC [5–7].

Quantum resources and strategies have extensive implications and applications in the field of communication complexity (CC) [8–11]. A typical CC problem entails two parties Alice and Bob, who are provided with inputs \( x \in \{0, 1, \ldots, n_x - 1\} \), \( y \in \{0, 1, \ldots, n_y - 1\} \) respectively. Their task is to compute the value of a binary output bivariate function, \( f(x, y): \{1, \ldots, n_x\} \times \{1, \ldots, n_y\} \to \{0, 1\} \) by exchanging messages. There are two interchangeable metrics to gauge their performance: (1) maximal achievable probability of success (denoted by \( p_{C_d} \) for classical resources and \( p_{Q_d} \) for quantum resources) given a bounded amount of communication (say bounded dimension \( d \) of the communicated system), and (2) amount of communication, usually quantified in bits \( C(f, p_S) \) or qubits \( Q(f, p_S) \), required to achieve a specified probability of success \( p_S \). Quantum advantage implies \( p_{Q_d} > p_{C_d} \) or alternatively \( Q(f, p_S) < C(f, p_S) \).

In this letter, we demonstrate that almost all one-way CC quantum advantage can be operationally attributed to PC. Given a CC task along with an advantageous quantum strategy employing a set of states and measurements, we construct an OC task and orchestrate an advantageous quantum strategy utilizing the same set-up and additionally the orthogonal mixtures of these states. Specifically, we construct PNC inequalities tailored to given one-way CC tasks and show that quantum advantage therein implies a violation of these inequalities. Moreover, we obtain a criterion for unbounded quantum violation of these PNC inequalities and demonstrate it for two widely studied CC problems. To illustrate the validity of our thesis in general operational theories, we provide an alternative construction of the OC task and a protocol thereof which utilizes the duals of states and measurements responsible for advantage in the given CC problem and retrieve the same results. Further, we use the machinery thus developed to provide the complete proof of the fact that of any Bell inequality violation (BV) implies PC. Towards the end of this letter, we discuss our implications and gather evidence in support of candidature of PC as the most fundamental non-classical ontic-feature of quantum theory. We conclude by discussing avenues for future research.

One-way CC task.— A prevalent subclass of communication complexity problems wherein only Alice is allowed.
Quantum theory tackles one-way communication. The sender Alice is provided an input \(a\) and she is to guess the value of a function \(f(x, y)\) contained in a bit \(z \in \{0, 1\}\). The expression for maximal classical average guessing probability \(p_{\text{cd}}\) is,

\[
p_{\text{cd}} = \sum_{x, y} p(x, y) p(z = f(x, y)|x, y). \tag{1}
\]

As this expression is the average of success probability over all possible settings \(x, y\), shared randomness yields no advantage. More specifically in a classical protocol Alice prepares the message \(m\) employing an encoding scheme \(E\) which comprises of a probability distribution \(p_E(m|x)\). Bob yields an output \(z\) based on his input \(y\) and the message \(m\) using a decoding scheme \(D\) entailing a probability distribution \(p_D(z|y, m)\). This leads us to the following expression for \(p_{\text{cd}}\),

\[
p_{\text{cd}} = \max_{\{E\}, \{D\}} \left\{ \sum_{m=0}^{d-1} \sum_{y} p(y) \left( \sum_{x} p(x|y) p_E(m|x) p_D(f(x, y)|y, m) \right) \right\}. \tag{2}
\]

Quantum theory tackles one-way \(OC\) problems using two non-equivalent [14, 15] classes of strategies:

- **Prepare and measure**: Alice’s state \((\rho_x, \text{the communicated system})\) preparation protocol is followed by a binary outcome measurement \((\{M^y_x\})\) at Bob’s end. We denote the quantum average guessing probability as \(p_{\text{Qd}}\) as follows,

\[
p_{\text{Qd}} = \sum_{x, y} p(x, y) \text{tr}(\rho_x M^y_x|z = f(x, y)). \tag{3}
\]

- **Entanglement assisted classical communication**: Alice and Bob share an entangled state \(\rho_{AB}\) (a density operator on \(\mathcal{H}_A \otimes \mathcal{H}_B\)). Alice performs a \(d\) outcome measurement \((\{M^y_m\})\) and sends her outcome \(m\) as the message. Upon receiving the message \(m\), Bob performs a binary outcome measurement \((\{M^y|m\})\). Here the expression for \(p_{\text{Qd}}\) is,

\[
p_{\text{Qd}} = \sum_{x, y} p(x, y) \sum_{m=0}^{d-1} \text{tr}(\rho_{AB} M^y_m \otimes M^y|z = f(x, y)). \tag{4}
\]

**\(OC\) task.** — A general \(OC\) task involves two parties [5]. The sender Alice is provided an input \(a \in \{0, 1, \ldots, n_a - 1\}\) and the receiver Bob gets an input \(b \in \{0, 1, \ldots, n_b - 1\}\) and yields an output \(c \in \{0, 1, \ldots, n_c - 1\}\). Their task is to guess the value of a function \(g(a, b) : \{0, 1, \ldots, n_a - 1\} \times \{0, 1, \ldots, n_b - 1\} \to \{0, 1, \ldots, n_c - 1\}\). In contrast to the \(CC\) task defined above, there is no restriction on the amount of communication. Instead, communication must be oblivious to (not reveal any information about) the value of a specific function of Alice’s input, say \(O(a)\).

For this letter, we need-only deal with a subclass of the general \(OC\) task wherein Alice’s input comprises of a pair \(a = (a_1, a_2)\) with \(a_1 \in \{0, \ldots, n_{a_1} - 1\}, a_2 \in \{0, \ldots, n_{a_2} - 1\},\) and the communication is constrained to be oblivious to the value of \(a_1\) i.e. \(O(a) = a_1\). The oblivious constraint for classical encoding strategies, referred as \(E\), is expressed as,

\[
\forall m, a_1, \ p_E(m) := p_E(m|a_1) = \sum_{a_2} p(a_2|a_1)p_E(m|a_1, a_2). \tag{5}
\]

This condition simply ensures that the same classical mixture is prepared for all values of \(a_1\). The maximal classical success probability here is,

\[
p_{\text{NC}} = \max_{\{E\}, \{D\}} \left\{ \sum_{m} \sum_{b} p(b) \left( \sum_{a} p(a|b)p_E(m|a)p_D(g(a, b)|b, m) \right) \right\}. \tag{6}
\]

where \(D\) denotes Bob’s decoding strategy and \(m\) can take arbitrary number of distinct values. The symbol \(p_{\text{NC}}\) is used to reflect the fact that in an \(OC\) task the optimal classical success probability is the same as the optimal non-contextual success probability [5]. In order to obtain an insight into the optimal non-contextual strategies we make the following observations.

**Observation 1. Decoding in an \(OC\) task.** In-order to attain the maximal success probability, Bob’s decoding strategy is to output the most probable value \(g(a, b)\) given Alice’s message \(m\) pertaining to an encoding \(E\) and his input \(b\). This in-turn implies that for a fixed encoding strategy Bob’s optimal decoding strategy \(D^*\) is deterministic i.e.,

\[
p_{D^*}(c = i|b, m) = \begin{cases} 1, & \text{if } p(g(a, b) = i|m) \geq p(g(a, b) \neq i|m), \\ 0, & \text{else}, \end{cases} \tag{7}
\]

where

\[
p(g(a, b) = i|m) = \sum_{a|g(a, b) = i} p(a|b)p_E(m|a). \tag{8}
\]

This allows us to re-express (6) as,

\[
p_{\text{NC}} = \max_{E} \left\{ \sum_{m} \sum_{b} p(b) \max_{i} \left( \sum_{a|g(a, b) = i} p(a|b)p_E(m|a) \right) \right\}. \tag{9}
\]

**Observation 2. Encoding in an \(OC\) task.** For any classical encoding strategy \(E\) define a parameter
\( q_{E,m}(a_1, a_2) := \frac{p(a_2|a_1)p_E(m|a_1,a_2)}{p_E(m|x)}, \) It follows from the oblivious constraint (5) that,

\[
\forall m, a_1, \sum_{a_2} q_{E,m}(a_1, a_2) = 1. \tag{10}
\]

We may now re-write (9) in terms of \( q_{E,m}(a_1, a_2) \) as,

\[
p_{NC} = \max_E \left\{ \sum_m p_E(m) \sum_b p(b) \max_i \left\{ \sum_{a_1,a_2} p(a_1|b) q_{E,m}(a_1, a_2) \right\} \right\}
\leq \max_{q_{E}(a_1,a_2)} \left\{ \sum_b p(b) \max_i \left\{ \sum_{a_1,a_2} p(a_1|b) q_{E}(a_1, a_2) \right\} \right\}. \tag{11}
\]

The last inequality is implied by the fact that \( \sum_m p_E(m) = 1 \). Specifically, the last inequality states that in-order to obtain an upper bound on \( p_{NC} \) its enough to find the optimal encoding strategy \( E^* \) for a single level of the message, which justifies the use of the symbol \( q_E(a_1, a_2) \).

The constraint (10) along with the fact that \( \forall a_1, a_2, q_{E}(a_1, a_2) \geq 0 \) implies that the set of all valid instances of \( q_{E}(a_1, a_2) \) and hence the set of all encoding strategies for a single level of the message form a convex polytope. The extremal points of this polytope resemble deterministic distributions i.e. they have for each \( a_1, q(a_1,a_2) = 0 \) for all values of \( a_2 \) except a specific \( a_2^{\text{xt}} \) for which \( q(a_1,a_2^{\text{xt}}) = 1 \) (see Lemma 1 in supplementary material for the proof). Hence with regard to find a upper bound on \( p_{NC} \) it is sufficient to evaluate the expression (11) at these deterministic extremal points and find the optimal. \( \square \)

The quantum strategy for the \( OC \) task is straightforward and involves Alice preparing a state of arbitrary dimension \( \rho_a \) and the oblivious constraint simply implies that the same mixed state \( \rho \) is prepared for all values of \( a_1 \) i.e. \( \forall a_1, \sum_{a_2} p(a_2|a_1)\rho_{a_1,a_2} = \rho \).

Quantum advantage in \( CC \) implies advantage in \( OC \).—

Given an instance of the general one-way \( CC \) problem we construct the following \( OC \) task,

\[
a = (x, a_2), b = y, c = z, O(a) = x \\quad p(a,b) = p(x,a_2, b) = p(y)p(x|y)p(a_2|x),
\]

where \( a_2 \in \{0,1\}, p(a_2|x) = \begin{cases} \frac{1}{2}, & \text{if } a_2 = 0 \\ \frac{d-1}{d}, & \text{if } a_2 = 1 \end{cases} \]

and \( g(a,b) = f(x,y) \oplus a_2 \). \tag{12}

**Proposition 1.** The non-contextual success probability of the \( OC \) task described above is upper bounded by the optimal classical success probability of the \( CC \) problem,

\[
p_{NC} \leq \sum_y p(y) \sum_{x,a_2^*} p(x|y)\delta_{i^*, f(x,y)} + \sum_y p(y) \sum_{x,a_2^*=1} p(x|y)\delta_{i^*, f(x,y)+a_2^*}. \tag{14}
\]

To complete the proof we demonstrate the following classical protocol employing a two-leveled message \( m' \) for the \( CC \) problem,

\[
if \ q_{E'}(x,a_2^*) = 1, \ p_{E'}(m' = a_2^*|x) = 1; \tag{15}
\]

\[
if \ p_{D^*}(c = i^*|y) = 1, \ p_{D}(z,y,m) = \delta_{x,i^* \oplus m'}.
\]

Inserting this strategy in (2), one obtains the same success probability in \( CC \) problem as given in the right side of (14). Hence the desired thesis. \( \square \)

We remark that the result (13) forms our primary \( P_{NC} \) inequality. Moving on, we orchestrate a quantum protocol for the \( OC \) task violating this inequality,
based on the quantum protocol responsible for the advantage in the CC problem. Recall that a prepare and measure quantum CC protocol entails Alice preparing a qudit $\rho_x$ followed by Bob performing the measurement $\{M_z^x\}$ on the state. The quantum strategy for the OC task described in (12) involves Alice preparing the same states $\rho_{x,a_2=0} = \rho_x$ along with their orthogonal mixture $\rho_{x,a_2=1} = \frac{1}{d-1} I$ while Bob’s measurements remain the same as in the CC protocol. Alice’s preparations are therefore oblivious to $x$, as $\forall x \sum_{a_2} p(a_2|x) \rho_{x,a_2} = \frac{1}{2} I$. This setup enables us to lower bound the quantum success probability in the OC task in the following way,

$$p_{Q^*} = \sum_{x,a_2=1,y} p(x,a_2,y) \text{Tr}(\rho_x M_z^{x} = f(x,y))$$

$$+ \sum_{x,a_2=2,y} p(x,a_2,y) \text{Tr}\left(\frac{I - \rho_x}{d-1} M_z^{x} = f(x,y)\right)$$

$$= \frac{1}{d} \left(2p_{Q^*} + d - 1 - \chi\right), \quad (16)$$

where $\chi = \sum_{x,y} q(x,y) \text{Tr}\left(M_z^{x} = f(x,y)\right)$. Given quantum advantage in CC problem ($p_{Q^*} > p_{C_0}$) and (16), an advantage is obtained in the OC task ($p_{Q^*} > p_{C_0}$) described in (12) whenever the following condition holds,

$$\frac{1}{d} \left(2p_{C_2} + d - 1 - \chi\right) \geq p_{C_2}, \quad (17)$$

First, one can infer $\chi \leq dp_G$ where $p_G$ is the guessing probability in CC without any communication (see Lemma 2 in supplementary material). Further, we present the following lemma to further insight into the above condition. The proof is contained in the supplementary material.

**Lemma 3.** Given a CC problem and a protocol using a two-leveled classical message with a success probability $p_{C_2}$, the success probability of a protocol using a d-leveled classical message is lower bounded in the following way,

$$p_{C_2} \geq 1 - \exp\left(-\frac{1}{2p_{C_2}} \log d(p_{C_2} - \frac{1}{2})^2\right). \quad (18)$$

Imposing Lemma 3, the primary condition (17) can be re-expressed in terms of only $p_{C_2}$ and $d$.

$$d(p_{C_2} + p_G - 1) + 2 \exp\left(-\frac{\log d}{2p_{C_2}} \left(p_{C_2} - \frac{1}{2}\right)^2\right) \leq 1. \quad (19)$$

Recall that whenever the above condition is met for a CC task the quantum advantage thereof forms an instance of violation of PNC inequality (13). Furthermore, (17) trivially holds in the case of CC tasks wherein the dimension of the communicated system is restricted to two and any quantum advantage thereof implies PC.

A quantum strategy for the OC task based on advantageous entanglement assisted classical communication protocol and the condition for an advantage thereof is presented in the supplementary material. An equivalent alternative construction of the OC task is given in the supplementary material [13]. Intriguingly, all results remain intact. The difference lies in the quantum protocol for the OC task which utilizes the exact duals of the states and measurements used in the CC task.

**Unbounded violation of PNC inequality.**— Let us rewrite the PNC inequality (13) as $\alpha_{NC} \leq \alpha_{C_2}$, where $\alpha_{NC} = p_{NC} - \frac{1}{2}, \alpha_{C_2} = p_{C_2} - \frac{1}{2}$. Then a quantum advantage in a CC problem adhering to the condition (17) implies that there exists quantum protocol with $\alpha_{Q^*} \geq \alpha_{NC} - \frac{1}{2} = \frac{1}{2} (2p_{Q^*} + d - 1 - \chi) - \frac{1}{2}$. Before going further, we remark that quantum advantage in a CC problem is prevalently reported in terms of the amount of communication required to achieve a bounded probability of success $p_S$, i.e., $Q(f,p_S) < C(f,p_S)$. To relate the innumerable instances of quantum advantage in above mentioned form with our results, we present the following lemma, proof of which is provided in the supplementary material.

**Lemma 4.** Given a CC problem and a protocol which achieves a success probability $p_S$ using $C(f,p_S)$ bits, the success probability of a protocol using a two-leveled classical message is upper bounded in the following way,

$$p_{C_2} \leq \frac{1}{2} + \sqrt{\frac{2p_S}{C(f,p_S)}}. \quad (20)$$

Using Lemma 4, the ratio of quantum and preparation non-contextual values of $\alpha$ (denoted by $\beta$) has the expression,

$$\beta \geq \frac{\alpha_{Q^*}}{\alpha_{NC}} \geq \frac{1}{d} \left(2p_{Q^*} + d - 1 - \chi\right) - \frac{1}{2}$$

$$\geq \sqrt{\frac{C(f,p_S)}{2p_S}} \left(2p_{Q^*} + d(2 - d) p_G - 1\right). \quad (21)$$

To obtain an unbounded violation of the PNC inequality $\alpha_{NC} \leq \alpha_{C_2}$, it suffices to show that $\beta$ could be arbitrarily large ($> 1$) [16]. We demonstrate the same for two widely studied CC problems [17, 18] with exponential quantum advantage.

1. **Vector in subspace:** Alice is given an $n$-dimensional unit vector $u$ and Bob is given a subspace of dimension $n/2, S$ with the promise that either $u \in S$ or $u \in S^\perp$. Their goal is to decide which is the case. Here $p_{Q^*} = 1, Q(f,1) = d = \log n, C(f,p_S = \frac{2}{3}) = \Omega(\sqrt{n})$ (Theorem 4.2 in [17]) and a simple calculation yields $\chi = \frac{\log n}{2}, p_G = \frac{1}{2}$. Inserting these into (21) one obtains an arbitrarily large lower bound for the ratio $\beta \geq \Omega(\sqrt{\frac{1}{n}})$.}

2. **Hidden matching:** Alice is given a bit string $x \in \{0,1\}^n$ of length $n$ and Bob is given $y \in M_n$ ($M_n$ denotes the family of all possible perfect matchings on $n$ nodes). Their goal is to output a tuple $(i,j,t)$ such that the edge $(i,j)$ belongs to the matching $y$ and $t = x_i \oplus x_j$. Clearly
the hidden matching problem is not a typical \( CC \) problem, specifically it is a relational problem. Nevertheless, we find that the machinery developed so far including Proposition 2, still holds for relational \( CC \) problem (see Lemma 5 in supplementary material). It is seen that for Hidden matching \( p_{Q_n} = 1, Q(f, 1) = d = \log n, p_{G} = \frac{1}{n^2} \), and \( C(R, \frac{n}{2}) = \Omega(\sqrt{n}) \) [18]. Inserting into (21) one obtains an even larger violation as the lower bound on \( \beta \) grows faster, i.e., \( \beta \geq \Omega\left(\frac{\sqrt{n}}{\log n}\right) \).

Any quantum violation of a Bell inequality implies advantage in a \( OC \) task. — While \( BV \) which employ space-like separation between Alice and Bob reveal Bell non-locality (\( NL \)) of the underlying ontology \((NT)\) there is an operational time-symmetry [21]. Heuristically, for any Bell experiment an \( OC \) task can be constructed porting \( BV \) to an advantage in the \( OC \) task. For the space-like separated scenario the collapsed state on Bob’s end is prepared and sent in the \( OC \) task and for the time-like separated case the pre-measurement state at Bob’s end is prepared and sent in the \( OC \) task. This would make all \( BV \) operationally reveal \( PC \). However, while deterministic encoding strategies yield the bound on Bell inequalities, the \( PC\) bound on the success parameter of the \( OC \) task might spring from probabilistic encoding schemes [5]. A rather inadequate attempt to prove the above thesis was made in [6], as it explicitly assumed deterministic encoding schemes for the constructed \( OC \) task. We use the tools developed in this letter, including Observation 1 and 2 to provide the complete proof for the thesis (see supplementary material for details.). While we deal with space like separated \( BV \), the same can be easily extended to time-like separated case.

Implications. — Although quantum theory has revolutionized the fields of computation, communication and information processing, there is little insight into what makes quantum theory stand out. Any serious pursuit into this question must arguably invoke a ground common to both classical and quantum theories, on which non-classical features of the later could be discussed [3]. Regardless of the approach, answers to this question are of immense significance, as they carry the potential of directing the search for tasks with quantum advantage. For instance, device independent information processing emerged from \( NL \). This letter paves the way for identification of the ontic-feature(s) from which quantum advantage springs. Quantum communication advantage stands on two widely studied pillars namely, \( CC \) with quantum resources \([8, 9]\), and device-independent information processing based on quantum \( NL \) [20]. In a nutshell (see Fig. 2), the letter converts almost all existing (theoretical and experimental) evidence of quantum \( CC \) advantage to a proof of \( PC \). Furthermore, we provide a complete proof of the fact any quantum \( BV \) (space-like and time-like) reveals \( PC \), thereby attributing all of quantum communication advantage to \( PC \). Our implications are strict, i.e. the very set-up (state and measurements) for quantum advantage in \( CC \) with minimal additions reveals \( PC \), unlike the weak implication of [22] that reveals \( NL \) of completed unrelated entangled states used for port-based teleportation of Alice’s preparations. Kochen-Specker contextuality (\( KS \)) [23, 24] is known to be intimately linked with many aspects of quantum computation [25, 26] and [5] ports all state independent witnesses of \( KS \) operationally reveal \( PC \) [5]. This leads us to our tentative assertion that \( PC \) forms one of (if not) the most fundamental feature of quantum theory. Finally, we point out the minimal condition on a general operational theory for our implications to hold and present an intuitive counterexample based on random access code [27] (see supplementary material).

![FIG. 2: A pictorial depiction of our operational (the dotted box and small arrows) and ontological (outside the dotted box and large arrows) implications. Entanglement assisted, prepare and measure quantum communication complexity advantage (\( CC \)) implies advantage in oblivious communication task \( OC \) subject to the conditions (17),(23). These implications inherently assume operational quantum formalism.](image)
with \( \mathcal{SD} \) (or some other set of features) ensures a \( \mathcal{CC} \) advantage remains to be addressed. Given the significance of \( \mathcal{OC} \) task, a natural direction for future research is to look for information theoretic principles [29] that restrict success in \( \mathcal{OC} \) tasks to quantum maximum.

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[7] To avoid confusion, note that, 'communication tasks' referred in [6] are solely based on oblivious constraints. In this letter, we call the same as 'oblivious communication tasks'. While, the relation between other communication tasks and preparation contextuality is posed as an open problem in [6]. Here we provide an answer to that question.
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SUPPLEMENTARY MATERIAL

Lemmas and proofs

Lemma 1. The set of valid assignments of \( q(a_1, a_2) \) satisfying the following linear constraints,

\[
\forall a_1, a_2, \; q(a_1, a_2) \geq 0, \sum_{a_2} q(a_1, a_2) = 1
\]

where \( a_1 \in \{0, ..., n_{a_1} - 1\}, a_2 \in \{0, ..., n_{a_2} - 1\} \) for arbitrary \( n_{a_1}, n_{a_2} \), form a convex polytope. All extremal points of this polytope resemble deterministic probability distributions, i.e., for any extremal point \( q(a_1, a_2) \) is 1 or 0.

Proof. Let us represent the variables by \( n_{a_1} \times n_{a_2} \) matrix whose \((a_1, a_2)\)-th element is \( q(a_1, a_2) \). This is column stochastic matrix. The extremal points are described as follows. We consider a string \( (e_0, e_1, ..., e_{n_{a_1}}) \) where \( e_0 \in \{0, ..., n_{a_2} - 1\} \). Each extremal matrix is defined this string such that \( q(a_1, a_2) = \delta_{a_2, e_0} \). There are \( n_{a_1} \) number of such strings and each corresponds to extremal points. One can check that, any arbitrary matrix whose element \( q(a_1, a_2) \) can be obtained by the convex combination of these extremal points, in which the coefficient of the matrix corresponds to \( e_0, e_1, ..., e_{n_{a_1}} \) is 18 \( \sum_{a_2=0}^{n_{a_2}-1} q(i, e_i)/n_{a_2} \).

Lemma 2. For a given quantum prepare and measure communication complexity protocol the following holds,

\[
\chi \leq d p_G,
\]

where \( \chi = \sum_{x,y} p(x,y) \text{Tr} (M^y_{z=f(x,y)}) \), \( d \) is dimension of the communicated system and \( p_G \) is guessing probability without communication.

Proof. It is straightforward to see that, when there is no communication, given \( y \) the best strategy for Bob would be to output \( f(x,y) \) which is more likely according to the prior probability of the inputs, i.e.,

\[
p_G = \sum_y p(y) \max \left( \frac{\sum_x p(x|y)}{d}, \frac{\sum_x p(x|y)}{d} \right)
\]

By denoting \( \chi^y_z = \text{Tr}(M^y_z) \), and imposing the fact \( \chi^y_0 + \chi^y_1 = d \), one obtains,

\[
\chi = \sum_{x,y} p(x,y) \chi^y_{z=f(x,y)}
\]

\[
= d \sum_y p(y) \left( \frac{\sum_x p(x|y)}{d} \chi^y_0 + \frac{\sum_x p(x|y)}{d} \chi^y_1 \right)
\]

\[
\leq d \sum_y p(y) \max \left( \frac{\sum_x p(x|y)}{d}, \frac{\sum_x p(x|y)}{d} \right)
\]

\[
= d p_G.
\]

Proof of Lemma 3. We have a communication complexity protocol \( P \) which uses a bit of communication to obtain a success probability of \( p_{C_2} \). Now we shall use the pumping argument to discern the desired thesis (18). Consider yet another protocol \( P' \) wherein Alice and Bob repeat protocol \( P \log d \) times. They produce as their final outcome the majority of outcomes obtained in \( \log d \) runs of \( P \). If \( \lceil \log d \rceil \) is even they succeed if \( P \) succeeds \( \lceil \log d \rceil + 1 \) times and if \( \lceil \log d \rceil \) is odd they succeed if \( P \) succeeds \( \lceil \log d \rceil \) times. Let us denote the event that the protocol \( P' \) succeeds by \( E \) and the number of simultaneous occurrence of \( E \) is captured in the variable \( S \). This allows us to lower bound \( p_{C_4} \) as,

\[
p_{C_2} \geq p(S > \lceil \frac{\log d}{2} \rceil)
\]

\[
= \sum_{i=\lceil \frac{\log d}{2} \rceil + 1}^\infty \left( \lceil \frac{\log d}{2} \rceil \right) p_{C_2}^i (1 - p_{C_2})^{\lceil \frac{\log d}{2} \rceil - i}.
\]

The right hand side of the above equation is further lower bounded based on Chernoff’s inequality as,

\[
p(S > \lceil \frac{\log d}{2} \rceil) \geq 1 - \exp \left( -\frac{1}{2p_{C_2}} \log d (p_{C_2} - \frac{1}{2}) \right).
\]

Proof of Lemma 4. We have a communication complexity protocol which achieves success probability \( p_S \) using \( C(f, p_S) \) bits of communication. We know from the pumping argument used in the proof for Lemma 2,

\[
p_S \geq 1 - \exp \left( -\frac{1}{2p_{C_2}} C(f, p_S) (p_{C_2} - \frac{1}{2}) \right).
\]

Now expanding the above exponential term in the above inequality and taking the first two terms one retrieves,

\[
p_S \geq \left( \frac{1}{2p_{C_2}} C(f, p_S) (p_{C_2} - \frac{1}{2}) \right).
\]

This is conveniently re-expressed as,

\[
\frac{2p_S}{C(f, p_S)} \geq \frac{(p_{C_2} - \frac{1}{2})^2}{p_{C_2}} \geq (p_{C_2} - \frac{1}{2})^2,
\]

where the second inequality follows from the observation that \( 0 \leq p_{C_2} \leq 1 \) and subsequently yields the desired thesis (20).

Lemma 5. For Hidden matching problem an \( O(1) \) task can be constructed with a success probability \( p_{NC} \), such that \( p_{NC} \leq p_{C_2} \).

Proof. In the hidden matching task, Alice is given a bit string \( x \in \{0, 1\}^n \) of length \( n \) and Bob is given \( y \in M_n \) where \( M_n \) denotes the family of all possible perfect matchings on \( n \) nodes. Their goal is to output a tuple \( z = \)
(i, j, t) such that the edge (i, j) belongs to the matching y and t = xi ⊕ xj. Being a relational problem, given an input (x, y), Bob’s task is to return z from a set of possible relation, i.e., R(x, y) = {(i, j, t)}. Subsequently, one infers from (2),

\[ p_{\text{C}_2} = \max_{\{x\} \in \{D\}} \sum_{m=1}^{d} \sum_{y=1}^{n_y} \sum_{p(x|y)p_E(m|x)p_D(z \in R(x, y)|y, m)} \]  \hspace{.5cm} (22)

We follow the same construction of the OC task described in Fig. 1. The corresponding OC is also relational problem in which g(a, b) = R(x, y) for a2 = 0, and g(a, b) = R(x, y) for a2 = 1 where R(x, y) = {(i, j, 1 ⊕ t)}. Let \( p_{\text{D}^*}(c = (i^*, j^*, t^*)|b) = 1 \) be the optimal decoding strategy corresponds to the optimal extremal point \( q_{\text{C}}*(a_1, a_2^*) = 1 \). This allows us to conveniently upper bound \( p_{\text{NC}} \) as follows,

\[ p_{\text{NC}} \leq \sum_{b} \sum_{a_1|a_2^* = 0, (i^*, j^*, t^*) \in R(a_1, b)} p(a_1|b) \]

\[ + \sum_{b} \sum_{a_1|a_2^* = 1, (i^*, j^*, t^*) \in R(a_1, b)} p(a_1|b). \]

Further, consider the following classical strategy employing two-leveled message \( m' \),

if \( q_{\text{C}'}(x, a_2^*) = 1 \), \( p_E(m' = a_2^*|x) = 1 \);

if \( p_{\text{D}^*}(i^*, j^*, t^*|y) = 1 \), \( p_D(i, j, t|y, m) = \delta(i, j, t, (i^*, j^*, m^*)), \)

Inserting this strategy in (2), and using the fact in hidden matching problem that,

\[ \forall y, \sum_{x|(i, j^*, 1 ⊕ t^*) \in R(x, y)} p(x|y) = \sum_{x|(i, j^*, t^*) \in R(x, y)} p(x|y), \]

one obtains the same expression of success probability in CC problem as given in the right side of (22),

\[ p_{\text{C}_2} \geq \sum_{y} \sum_{x|m' = 0, (i^*, j^*, t^*) \in R(x, y)} \sum_{y} p(x|y) \]

\[ + \sum_{y} \sum_{x|m' = 1, (i^*, j^*, 1 ⊕ t^*) \in R(x, y)} p(x|y) \]

\[ = \sum_{y} \sum_{x|m' = 0, (i^*, j^*, t^*) \in R(x, y)} p(x|y) \]

\[ + \sum_{y} \sum_{x|m' = 1, (i^*, j^*, t^*) \in R(x, y)} p(x|y) \]

\[ \geq p_{\text{NC}}. \]

\[ \square \]

**Construction of a quantum strategy for OC task based on an advantageous entanglement assisted classical communication protocol for the CC problem**

Next up, we consider the entanglement assisted classical communication protocol for the CC problem with success probability \( p_{\text{Q}_2} > p_{\text{C}_2} \). In order to utilize the machinery developed above we first construct a quantum prepare and measure protocol deploying a d’ dimensional communcated system but with the same probability of success \( p_{\text{Q}_2} \). Suppose that reduced density matrix of Bob \( \rho_B \) is of dimension e i.e. \( \text{dim}(H_B) = e \). Upon receiving x Alice prepares the state \( \rho_x = |m⟩⟨m| \otimes \rho_B \) where the state \( |m⟩⟨m| \) is simply the quantum encoding of the classical message m into d orthogonal states. She accomplishes this feat by measuring \( \{M_m \} \) on the entangled state \( \rho_{\text{AB}} \) to which we assume she has access to. The communicated system is of dimension \( d' = d + e \). Bob first retrieves the message by performing the measurement \( \{M_m \} \) on the appropriate subsystem of the communicated system and depending on it performs the measurement \( \{M_m^x \} \) on rest of the communicated system, captured conveniently in a joint measurement \( \{M_m^x = M_m \otimes M_m^x \} \). This yields the same success probability \( p_{\text{Q}_2} \). Now using the machiney described above we convert this prepare and measure protocol into an OC protocol and lower bound its quantum success probability, \( p_{\text{Q}_2} \geq \frac{1}{d'} (2p_{\text{Q}_2} + d' - 1 - \chi) \). This leads us to the condition for quantum advantage in the OC task,

\[ \frac{1}{d'} (2p_{\text{Q}_2} + d' - 1 - \chi) \geq p_{\text{C}_2}. \]  \hspace{.5cm} (23)

Notice that in a rather predominant subclass of entanglement assisted classical communication protocols Bob applies a CPTP map \( \phi_m \) on \( \rho_B \) and performs the measurement \( \{M_m^x \} \) on \( \phi_m(\rho_B) \). In such cases Alice having access to the message m sends \( \rho_x = \phi_m(\rho_B) \) effectively reducing the dimension of the communicated system in the prepare measure protocol to \( d' = e \), thereby improving the feasibility of the quantum advantage in the OC task.

**Alternative equivalent OC task**

Given a general one-way CC problem with \( p_{\text{Q}_2} \geq p_{\text{C}_2} \), we construct the following OC task (shown is Fig. 3),

\[ a = (y, z), \quad b = x, \quad c \in \{0, 1\}, \quad \mathcal{O}(a) = y \]

\[ p(a, b) = p(y, z, x) = p(x)p(y|x)p(z|y), \]

where \( p(z|y) = \frac{\chi^2}{\chi^2 + d'} \),

\[ g(a, b) = f(x, y) ⊕ z. \]

**Proposition 2.** The non-contextual success probability of the OC task described above is upper bounded by the
optimal classical success probability of the CC problem, i.e. \( p_{NC} \leq p_{C_2} \).

**Proof.** We follow the same method given in the proof of Proposition 1. Based on Observation 1-2, we know there exists a single level encoding scheme \( q_{C_2}^e(y, z^*) = 1 \) and a decoding scheme \( p_{D_2}(c^*|x) = 1 \) and the maximum they can achieve is, 

\[
B = \sum_{u,v,x,y} c_{x,y}(u,v)p(x,y)p(u,v|x,y),
\]

where \( c_{x,y}(u,v) \geq 0 \). In this task they are allowed to share correlations which essentially provide advice of the form \( p(u,v|x,y) \). If Alice and Bob share a local-realist (classical) correlation, the maximum they can achieve is,

\[
B_C = \sum_{\lambda,u,v,x,y} c_{x,y}(u,v)p(x,y)p(\lambda)p_\lambda(u|x)p_\lambda(v|y).
\]

This fact is captured in Bell inequalities.

Consider any quantum strategy which violates a Bell inequality. The probability is getting outcome \( u \) when measurement \( x \) is performed on the shared quantum state is \( p_{Q}(u|x) \) and the reduced quantum state on Bob’s subsystem is denoted by \( \rho_{Q,\lambda}^{B,Y} \). Let the quantum value \( B_Q > B_C \). We construct a OC task as described below,

\[
a = (a_1, a_2) = (x, u), \quad b = y, \quad c = v, \quad \mathcal{O}(a) = x
\]

\[
p(a, b) = P(x, u, y) = p(y)p(x|y)p_{Q}(u|x).
\]

The figure of merit in the OC is, \( s_{NC} = \sum_{a,b,x,y} c_{x,y}(a,b)p(x,a,y)p(b|x,a,y) \).

**Proposition 3:** The non-contextual success probability of the OC task is upper bounded by the optimal local-realist value of Bell expression, i.e. \( s_{NC} \leq B_C \).

**Proof.** Let us denote the extremal points (encoding) by \( g(x, a_1^*) \) for each \( x \) and \( p_{D}(b^*_y|y), m^* \) 1 (decoding) corresponds to the optimal value. Now we can upper bound \( s_{NC} \) as follows,

\[
s_{NC} \leq \sum_{x,y} p(x,y) c_{x,y}(a,b)\delta_{a_1^*,a}\delta_{b^*_y,b}
\]

We propose a hidden variable as follows,

\[
\begin{align*}
&\text{if } q(x, a_1^*) = 1, \quad p_\lambda(a) = a_1^*|x) = 1; \\
&\text{if } p_{D}(b^*_y|y, m^*) = 1, \quad p_\lambda(b = b^*_y|y) = 1
\end{align*}
\]

Using this local strategy, one obtains the same value for the expression \( B \) as in the expression (25).

A quantum strategy for the OC task can be easily constructed from the state and measurement responsible for \( BV \): Alice sends \( \rho_{Q,\lambda}^{B,Y} \) for input \( (x, u) \) and Bob’s measurement settings are the same as in Bell scenario. It satisfy the oblivious condition due to no-signalling (see [6] for details). Thus, we conclude \( s_Q \geq s_{Q^*} = B_Q > B_C \geq p_{NC} \).
Does CC advantage in any operational theory imply an advantage in OC?—

Our results for CC advantage in a prepare and measure protocol still hold for any operational theory with an additional feature, self-duality of preparation states and measurement effects (SD) [1]. This is due to the fact that the states and measurements that reveals PC in the alternative OC task are just the dual of the measurement effects and states employed in the CC. The property of SD emerges from a set of natural postulates in the framework of general probabilistic theories [28]. However, the implication is not true in any operational theory. Here, we demonstrate a toy-theory and an ontic-model with CC advantage but no possibility of PC.

Consider a well-known CC task, the (2 → 1) random access code [27] wherein Alice receives two random input bits $x_1, x_2$ to be encoded into a two dimensional system and sends it to Bob. Bob receives a random input bit $y$ along with the message from Alice and is required to guess $x_y$. Let the theory, having three ontic states and two measurements, is a fragment of quantum theory. The three ontic states with the encoding $\psi_{x_1 x_2}$ correspond to pure quantum preparations as, $\psi_{11,10} = |1\rangle, \psi_{00} = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle, \psi_{01} = \cos(\theta)|0\rangle - \sin(\theta)|1\rangle$ where $\theta = \frac{\pi}{8}$ and two binary-outcome measurements corresponding to $y = 0,1$ are $\sigma_z, \sigma_x$ respectively. This fragment of quantum theory doesn’t adhere to SD. Clearly the theory admits advantage in this task as the average success probability $p \approx 0.8 > p_{c_2} = 0.75$. However since the theory has only three ontic states, any mixed state in this theory has a unique decomposition, thus ruling out the possibility of PC.