Here we show how to design phase-shifting algorithms (PSAs) for nonuniform phase-shifted fringe patterns using their frequency transfer function (FTF). Assuming that the nonuniform/nonlinear (NL) phase-steps are known, we introduce the desired zeroes in the FTF to obtain the specific NL-PSA formula. The advantage of designing NL-PSAs based on their FTF is that one can reject many distorting harmonics of the fringes. We can also estimate the signal-to-noise ratio (SNR) for interferograms corrupted by additive white Gaussian noise (AWGN). Finally, for non-distorted noiseless fringes, the proposed NL-PSA retrieves the modulating phase error-free, just as standard/linear PSAs do.

Our contribution. Here we present explicitly N-step NL-PSA formulas with a desired FTF spectral response. The NL-PSA's FTF reject the highest number of fringe harmonics for a given number of phase steps. For noiseless, non-distorted data, our NL-PSA formulas give the exact modulating phase, not just a good approximation (as other NL-PSAs do [6-10]). In simple terms, the phase recovered with our NL-PSA equals the error-free phase obtained by standard/linear PSAs for noiseless fringes [11]. Moreover, using the designed FTF, one can easily estimate the SNR from basic stochastic process theory [11,12]. This contrasts with G-PSA, AIA, and PCA-PSA which do not give any SNR figure-of-merit [1-10]. For amplitude-distorted fringes, our NL-PSA gives much better results than G-PSA, AIA, or PCA-PSA because our NL-PSA explicitly rejects many harmonics. The only constrain to our FTF-based NL-PSA design is that one needs a previous estimate of the nonlinear phase-steps. But this is not difficult for spatial linear-carrier fringes, for which the Fourier method can be used, or by the use of the Carré nonlinear phase-step formula for temporal fringes with no spatial carrier [11].

Linear phase-step PSAs. Before going to the main contribution of this work, we briefly review the concept of the FTF for spectral analysis of linear phase-step PSAs. Let us start by the standard mathematical form for continuous phase-shifted fringes,

\[ I(t; \varphi_{x,y}) = a_{x,y} + b_{x,y} \cos \left( \varphi_{x,y} + \omega_0 t \right). \]  \hspace{1cm} (1)

The measuring phase is \( \varphi_{x,y} \); the background of the fringes is \( a_{x,y} \); the contrast is \( b_{x,y} \), and the angular frequency is \( \omega_0 \). For notation economy, we will use \( (\varphi, a, b) \) instead of \( (\varphi_{x,y}, a_{x,y}, b_{x,y}) \). If our interferograms have amplitude distortion, then one must include the fringe harmonics as,

\[ I(t; \varphi) = a + \sum_{k=1}^{\infty} b_k \cos \left( k \left( \varphi + \omega_0 t \right) \right). \]  \hspace{1cm} (2)

It is usual to express the \( n \)th sample \( I(n; \varphi) \) as,

\[ I(n; \varphi) = \int_{-\infty}^{\infty} I(t; \varphi) \delta(t-n) \, dt; \quad n \in \{0,1,\ldots,N-1\}. \]  \hspace{1cm} (3)
Using $\delta(t-n)$ the sampling Dirac delta function. Then a linear phase-stepped PSA may be written as,

$$Ae^{\phi} = \sum_{n=0}^{N-1} c_n^* I(n; \phi); \quad (c_n \in \mathbb{C}); \quad i = \sqrt{-1}. \quad (4)$$

The asterisk denotes the complex conjugate. This system has the following impulse response,

$$h(t) = \sum_{n=0}^{N-1} c_n \delta(t-n). \quad (5)$$

Taking the Fourier transform of $h(t)$ one obtains the spectral response of the PSA (the FTF) as,

$$H(\omega) = F[h(t)] = \sum_{n=0}^{N-1} c_n e^{-i\omega n}. \quad (6)$$

If the fringe data is corrupted by AWGN, the SNR-gain ($G_{\text{SNR}}$) for a linear $N$-step PSAs is given by [11],

$$G_{\text{SNR}} = \frac{\left| \sum_{n=0}^{N-1} c_n e^{-i\omega_0 n} \right|^2}{\sum_{n=0}^{N-1} |c_n|^2} \leq N. \quad (7)$$

The $G_{\text{SNR}}$ numerator is proportional to the energy of the demodulated signal at $\omega = \omega_0$, while the denominator is proportional to the filtered noise energy. The highest $G_{\text{SNR}}$ is obtained only for LS-PSA in which $\omega_0 = 2\pi / N$; otherwise $G_{\text{SNR}} < N$ [11]. For example, the 7-step linear least-squares PSA (LS-PSA) has the following FTF [11],

$$H(\omega) = \sum_{n=0}^{6} e^{i\omega n} e^{-i\omega_0 n}; \quad (\omega_0 = 2\pi / 7). \quad (8)$$

And the plot of the periodic $|H(\omega)|$ is shown in Fig. 1.

**Nonlinear phase-steps PSA.** We describe nonuniform temporal samples from Eq. (1) as,

$$I(t_n; \phi) = \int_I \left[ a + b \cos(\phi + \omega t) \right] \delta(t-t_n) dt. \quad (9)$$

Being $t_n$ nonuniform sampling times. It is common practice to label the fringe samples by their nonlinear phase-steps as,

$$I(\theta_n; \varphi) = a + b \cos(\varphi + \theta_n); \quad (\theta_n = \omega t_n). \quad (10)$$

Note that the angular frequency and sampling times are irrelevant. Therefore, from now on we will work with normalized frequency $\omega_0=1.0$ (radians/second). We then use $\theta_n$ instead of ($\omega t_n / 1.0$). We remark that $\theta_n$ are known. Figure 2 shows a possible realization of 9 nonuniform sampled fringe (red dots), and the Fourier spectra of the continuous-time fringe.

**FFT for nonuniform phase-stepped PSAs.** Our specific goal is to find a NL-PSA as a linear combination of the nonuniform phase-stepped interferograms as,

$$Ae^{\phi} = \sum_{n=0}^{N-1} c_n^* I(\theta_n; \varphi); \quad (c_n \in \mathbb{C}). \quad (11)$$

Following the same receipt as linear PSAs [11], the NL-PSA’s impulse response is given by,

$$h(t) = \sum_{n=0}^{N-1} c_n \delta(t-\theta_n / 1.0). \quad (12)$$

And its FTF is given by,

$$H(\omega) = F[h(t)] = \sum_{n=0}^{N-1} c_n e^{-i\omega \theta_n}. \quad (13)$$

If the fringe data is corrupted by AWGN, the SNR-gain ($G_{\text{SNR}}$) for a $N$-step NL-PSAs is given by,

$$G_{\text{SNR}} = \frac{\left| \sum_{n=0}^{N-1} c_n e^{-i\omega \theta_n} \right|^2}{\sum_{n=0}^{N-1} |c_n|^2} \leq N. \quad (14)$$

The equality is obtained if, and only if, $\theta_n = 2\pi n / N$; reducing to the standard linear LS-PSA (see Eq. (7)).

**Three step NL-PSA.** The minimum (normalized) quadrature conditions are,

$$H(-1) = 0; \quad H(0) = 0; \quad H(1) = 1. \quad (15)$$

According to these $H(\omega)$ constraints, one needs to solve for ($c_0,c_1,c_2$), for the known phase-steps ($\theta_n = 0, \theta_1, \theta_2$) as,
\[ \sum_{n=0}^{2} c_n e^{i \theta_n} = 0; \quad \sum_{n=0}^{2} c_n = 0; \quad \sum_{n=0}^{2} c_n e^{-i \theta_n} = 1. \]  

(16)

Which can be rewritten in matrix form as,

\[
\begin{bmatrix}
1 & e^{i \theta_1} & e^{i \theta_2} \\
1 & e^{i \theta_1} & e^{i \theta_2} \\
1 & e^{-i \theta_1} & e^{-i \theta_2}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

(17)

For \( \theta_1 \neq \theta_2 \), this 3-by-3 matrix is nonsingular, and one finds \((c_0, c_1, c_2)^T\). The explicit NL-PSA formula is then,

\[ A e^{i \hat{\varphi}} = c_0^2 I(\theta_0; \varphi) + c_1^2 I(\theta_1; \varphi) + c_2^2 I(\theta_2; \varphi). \]

(18)

This 3-step NL-PSA is error-free \((\hat{\varphi} = \varphi)\) for noiseless, non-distorted, temporal fringes. Figure 3(a) shows 3 nonuniform phase samples, and the FTF which phase demodulate them.

**Five phase-steps NL-PSA**

With more than 3 nonuniform phase-stopped fringes, one may reject more fringe harmonics. For example if we want the FTF to have the following constraints,

\[ H(-2) = 0; H(-1) = 0; H(0) = 0; H(1) = 1; H(2) = 0. \]

(19)

One would need 5 coefficients \((c_n)\) as,

\[
\begin{bmatrix}
1 & e^{i \theta_1} & e^{i \theta_2} & e^{i \theta_3} & e^{i \theta_4} \\
1 & e^{i \theta_1} & e^{i \theta_2} & e^{i \theta_3} & e^{i \theta_4} \\
1 & e^{-i \theta_1} & e^{-i \theta_2} & e^{-i \theta_3} & e^{-i \theta_4} \\
1 & e^{-i \theta_1} & e^{-i \theta_2} & e^{-i \theta_3} & e^{-i \theta_4} \\
1 & e^{-2i \theta_1} & e^{-2i \theta_2} & e^{-2i \theta_3} & e^{-2i \theta_4}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

(20)

One may also change the desired FTF’s constraints to,

\[ H(-1) = 0; H(0) = 0; H(1) = 1; H(2) = 0; H(3) = 0. \]

(21)

Then the five coefficients \((c_n)\) change to,

\[
\begin{bmatrix}
1 & e^{i \theta_1} & e^{i \theta_2} & e^{i \theta_3} & e^{i \theta_4} \\
1 & e^{i \theta_1} & e^{i \theta_2} & e^{i \theta_3} & e^{i \theta_4} \\
1 & e^{2i \theta_1} & e^{2i \theta_2} & e^{2i \theta_3} & e^{2i \theta_4} \\
1 & e^{2i \theta_1} & e^{2i \theta_2} & e^{2i \theta_3} & e^{2i \theta_4} \\
1 & e^{3i \theta_1} & e^{3i \theta_2} & e^{3i \theta_3} & e^{3i \theta_4}
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3 \\
c_4
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \]

(22)

Figures 3(b) and 3(c) show 5 nonuniform samples of a phase-shifted fringe, and two different FTFs which phase-demodulate the 5 fringe-samples. The resulting 5-sample NL-PSA formula is given by,

\[ A e^{i \hat{\varphi}} = \sum_{n=0}^{4} c_n^2 I(\theta_n; \varphi). \]

(23)

These two 5-step NL-PSAs are also error-free \((\hat{\varphi} = \varphi)\) for noiseless, non-distorted, fringes.
worst the AIA and the PCA-PSA must pre-filter its background signal at \( \omega = 0 \) [5,9,10].

**Computer simulation.** We simulated 7 fringe patterns with the aforementioned phase-steps \((\theta_i) = (0, 0.78, 1.81, 3.11, 4.54, 5.93, 7.24)\) and designing the FTF to have the zeroes in Eq. (24). The resulting FTF’s complex coefficients are \(c_i = (-0.06, 0.21-0.21i, -0.05-0.2i, -0.22, -0.04-0.22i, 0.23+0.08i, -0.07-0.1i)\). The resulting NL-PSA is then,

\[
Ae^{i\phi} = \sum_{n=0}^{6} c_n f \left( \theta_n; \phi \right). \tag{24}
\]

The first sample has the lowest weight \(|c_0|=0.06\), meaning that its information is taken less into account. Two out of seven noiseless fringes, and its demodulated phase are shown in Fig. 5.

![Fig. 5](image)

Panels (a)-(b) show 2 out of 7 noiseless, nonlinear phase-stopped fringes. Panel (c) shows the error-free estimated phase.

We remark that the estimated phase for our FTF based NL-PSAs, is mathematically error-free \((\phi = \phi)\), whenever the phase-steps \((\theta_i)\) among the interferograms are known accurately. However, error-free phase estimation is not mathematically guaranteed in PCA-PSA [10]. In the case of AIA and noiseless fringes, it normally takes many iterations to reach an error-free phase estimation [10]. In other words, for noiseless fringes, the estimated phase recovered by our NL-PSAs is as good as the one obtained by standard linear PSAs [11]. Of course, for fringes corrupted by AWGN and harmonics, we obtain a distorted demodulated phase, as it is the case for standard linear PSAs [11].

**SNR gain for our specific NL-PSA.** For our specific case with \((\theta_i) = (0, 0.78, 1.81, 3.11, 4.54, 5.93, 7.24)\), the SNR-gain is given by,

\[
G_{SNR} = \left( \frac{\sum_{n=0}^{6} |c_n|^2}{\sum_{n=0}^{6} |c_n|^2} \right) = 5.142. \tag{25}
\]

Resulting in a 27% SNR-gain reduction with respect to a 7 samples linear LS-PSA \((G_{SNR}=7)\).

**Comparison against PCA-PSA.** Before concluding we show a comparison of our FTF based NL-PSA and the PCA-PSA. It is well known that the PCA-PSA does not give, in general, an error-free phase estimation, even for noiseless nonlinear phase-stopped fringes [9,10]; this can be seen in Fig. 6. However the approximate PCA-PSA’s solution may be used as initial condition for the AIA. Then the AIA, after several iterations, converges to an almost error-free phase estimation, for noiseless fringes [10].

![Fig. 6](image)

Panel (a) shows the PCA-PSA’s estimated phase from the noiseless fringe-data in Fig. 5. Panel (b) shows the phase estimation error. Panel (c) shows a central cut of the phase estimation error in radians. The phase error of our NL-PSA is about \(10^{-15}\) (a numerical zero).

**Conclusions.** The herein proposed NL-PSA theory is a key contribution to nonlinear phase-steps interferometry in the sense that:

1) As far as we know, the spectral response (the FTF) for NL-PSAs was obtained for the first time. This FTF is in turn used to find the corresponding \(N\)-step NL-PSA formula.

2) For a given number of nonlinear phase-steps \((\theta_i)\), our NL-PSA has the highest fringe harmonics rejection. In contrast, the G-PSA, AIA or PCA-PSA do not reject, by design, higher order harmonics of the fringe data [1-10].

3) Our FTF-based NL-PSA give us as a bonus, the signal-to-noise ratio gain \((G_{SNR})\) for fringes corrupted by AWGN. This contrast with G-PSA, AIA, and PCA-PSA which do not give a SNR estimate [1-10] from basic stochastic process theory [11,12].

4) Finally, our FTF-based NL-PSA design recovers the demodulated phase error-free for noiseless, non-distorted fringes. This not being the case for PCA-PSA [9,10].

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