Generalized Logotropic Models and their Cosmological Constraints

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Abstract. We propose a new class of cosmological unified dark sector models called “Generalized Logotropic Models”. They depend on a free parameter $n$. The original logotropic model [P.H. Chavanis, Eur. Phys. J. Plus 130, 130 (2015)] is a special case of our generalized model corresponding to $n = 1$. In our scenario, the Universe is filled with a single fluid, a generalized logotropic dark fluid (GLDF), whose pressure $P$ includes higher order logarithmic terms of the rest-mass density $\rho_m$. The total energy density $\epsilon$ is the sum of the rest-mass energy density $\rho_m c^2$ and the internal energy density $u$ which play the role of dark matter energy density $\epsilon_m$ and dark energy density $\epsilon_{de}$, respectively. We investigate the cosmological behavior of the generalized logotropic models by focusing on the evolution of the energy density, scale factor, equation of state parameter, deceleration parameter and squared speed of sound. Low values of $n \leq 3$ are favored. We also study the asymptotic behavior of the generalized logotropic models. In particular, we show that the model presents a phantom behavior and has three distinct ways of evolution depending on the value of $n$. For $n \leq 2$, it leads to a little rip and for $n > 2$ to a big rip. We predict the value of the big rip time as a function of $n$ without any free (undetermined) parameter.

Keywords: dark energy theory, dark matter theory, cosmology, unified dark sector, logotropic model
1 Introduction

Cosmological observations from various independent research teams show that the current Universe is accelerating [1–5]. This accelerating expansion is due to an unknown component called "dark energy" which works against gravity. The nature of the dark sector is still unknown and several alternative models of dark matter (DM) and dark energy (DE) have been proposed to account for the observation of the present cosmic acceleration. The standard cold dark matter (ΛCDM) model is the simplest DE model and relies on a cosmological constant to drive the current acceleration of the Universe and on the existence of a pressureless DM to explain the observed properties of the large-scale structures of the Universe [6, 7]. However, the ΛCDM model suffers from the cosmological constant problem [8, 9], namely why the value of the cosmological constant is so tiny, and the cosmic coincidence problem [10, 11], namely why DM and DE are of similar magnitudes today although they scale differently with the universe’s expansion. The CDM model also faces important problems at the scale of DM halos such as the core-cusp problem [12], the missing satellite problem [13–15], and the “too big to fail” problem [16]. This leads to the so-called small-scale crisis of CDM [17]. Among the wide range of alternative DE models that have been proposed in the literature (quintessence, k-essence, Chaplygin gas, tachyons, phantom fields, holography...), an interesting class of dynamical DE models considers DM and DE as different manifestations of a single-component underlying fluid, often assumed to be a perfect fluid. The unified dark matter and dark energy
UDM models have the remarkable feature of describing the dark sector of the Universe as a single component which behaves as DM at early times and as DE at late times [18–37]. The cosmological aspects of these UDM models have been studied recently in refs. [34, 36–40, 42]. In particular, the logotropic model is a candidate for unifying DE and DM [30]. The logotropic model is able to account for the transition between a DM era and a DE era and is indistinguishable from the $\Lambda$CDM model, for what concerns the evolution of the cosmological background, up to 25 billion years in the future when it becomes phantom [30, 31, 33, 39]. Remarkably, the logotropic model implies that DM halos should have a constant surface density and it predicts its universal value $\Sigma_{\text{th}}^0 = 0.01955c\sqrt{\Lambda}/G = 133 M_\odot/pc^2$ [30, 31, 33, 39] without adjustable parameter. This theoretical value is in good agreement with the value $\Sigma_{\text{obs}}^0 = 141^{+83}_{-52} M_\odot/pc^2$ obtained from the observations [41]. The logotropic model also predicts the values of the present proportion of DM and DE $\Omega_{\text{dm},0} = \frac{1}{1+e}(1-\Omega_{b,0}) = 0.256$ and $\Omega_{\text{de},0} = \frac{1}{1+e}(1-\Omega_{b,0}) = 0.695$ [33], in good agreement with the observed values $\Omega_{\text{dm},0} = 0.259$, $\Omega_{\text{de},0} = 0.691$ and $\Omega_{b,0} = 0.0486$.

In this work, we introduce a new class of UDM models called “Generalized Logotropic Models” characterized by a single fluid equation of state (EoS) of the form

$$P = \sum_{i=0}^{N} A_i \ln \left( \frac{\rho_m}{\rho_P} \right), \tag{1.1}$$

where $P$ is the pressure, $\rho_m$ is the rest-mass density, $A_i$ are constants with the dimension of an energy density and $\rho_P$ is a constant with the dimension of a mass density. The above EoS returns the standard $\Lambda$CDM model for $N = 0$ and the original logotropic model [30] for $N = 1$. Following [30], we shall identify $\rho_P$ with the Planck density $\rho_P = c^5/\hbar^2 G^2 = 5.16 \times 10^{99} \text{g} \text{m}^{-3}$. The single fluid which obeys Eq. (1.1) will be called the generalized logotropic dark fluid (GLDF).

In this paper, we are interested in studying the dynamical evolution of various generalized logotropic models. In particular, we examine in detail various forms of generalized logotropic EoS and investigate how the Universe evolution is affected by the corresponding EoS. In section 2, we provide an approach to motivate the logotropic model. In section 3, we consider an isotropic, homogeneous and spatially flat Universe and introduce the main equations characterizing generalized logotropic models. In section 4, we consider a subclass of models with $A_i = A_n \delta_{i,n}$ that are more easily tractable. In section 5, we investigate the cosmological behavior of the generalized logotropic models and focus on the evolution of the energy density, scale factor, EoS parameter, decceleration parameter and squared speed of sound. We also derive analytically the asymptotic behavior of the generalized logotropic models and distinguish three types of evolution depending on the value of $n$. Finally, section 6 is devoted to remarks and conclusions. In Appendix A, we show that the GLDF is asymptotically equivalent to a form of Modified Chaplygin Gas (MCG). In Appendix B, we consider the two-fluid model associated with the GLDF and determine the EoS of DE. In Appendix C, we derive the present proportion of DM and DE by taking into account the presence of baryons.

2 Generalized Logotropic Equation of State

We consider the scenario where both DM and DE originate from a single dark fluid. We first recall the justification of the standard logotropic equation of state given in [30]. In
the Newtonian regime, the condition of hydrostatic equilibrium, which describes the balance between the gravitational force and the pressure gradient, is given by

\[ \nabla P + \rho \nabla \Phi = 0. \quad (2.1) \]

Here \( P \) and \( \rho \) denote the pressure and mass density of the fluid, respectively. Moreover, \( \Phi \) is the gravitational potential. We assume one of the simplest direct relations between the mass density \( \rho \) and the pressure \( P \) of the fluid given by the polytropic EoS

\[ P = K \rho^\gamma, \quad (2.2) \]

where \( \gamma \) is the polytropic index and \( K \) is a constant. Using the above equation, the condition of hydrostatic equilibrium becomes

\[ K \gamma \rho^{\gamma-1} \nabla \rho + \rho \nabla \Phi = 0. \quad (2.3) \]

Considering the limit \( \gamma \to 0, K \to +\infty \) with \( A = K \gamma \) fixed [30], we get

\[ A \frac{\nabla \rho}{\rho} + \rho \nabla \Phi = 0. \quad (2.4) \]

By comparing equations (2.4) and (2.1), we find that the EoS of the logotropic gas reads [30]

\[ P = A \ln \left( \frac{\rho}{\rho_*} \right), \quad (2.5) \]

where \( \rho_* \) is an integration constant. In this paper, we consider a generalization of this EoS of the form

\[ P = \sum_{i=0}^{N} A_i \ln^i \left( \frac{\rho}{\rho_*} \right), \quad (2.6) \]

where \( A_i \) are arbitrary constants. This generalized logotropic EoS is interesting in its own right and, as we shall see, possesses interesting properties. However, the above derivation suggests that the standard logotropic EoS (2.5) plays a special role in the problem.

3 Generalized Logotropic Cosmology

In this section, we consider a flat homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) universe filled with a perfect single fluid, having an energy density \( \epsilon \), rest-mass density \( \rho_m \), and pressure \( P \). The expansion dynamics is governed by the Friedmann equations [43]

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \epsilon, \quad (3.1) \]

\[ \dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\epsilon + 3P), \quad (3.2) \]
where \( a(t) \) is the scale factor and \( H = \dot{a}/a \) is the Hubble parameter. Combining Eqs. (3.1) and (3.2), we obtain the energy conservation equation
\[
\frac{d\epsilon}{dt} + 3H(\epsilon + P) = 0. \tag{3.3}
\]

For a relativistic fluid with an adiabatic evolution (or at \( T = 0 \)), the first law of thermodynamics reduces to
\[
d\epsilon = P + \epsilon \rho_m \, d\rho_m. \tag{3.4}
\]

By integrating this equation for a given EoS, \( P = P(\rho_m) \), a relation between the energy density and the rest-mass density can be obtained as \([30]\)
\[
\epsilon = \rho_m c^2 + \sum_{i=0}^{N} A_i I_i(\rho_m), \tag{3.5}
\]
with the integral \( I_i(\rho_m) \) given by
\[
I_i(\rho_m) = \rho_m \int_{\rho_m}^{\infty} \ln^i \left( \frac{\rho_m}{\rho_P} \right) \frac{d\rho_m'}{\rho_m'^2}. \tag{3.7}
\]

With the change of variables \( t = \ln(\rho_m'/\rho_P) \), we obtain
\[
I_i(\rho_m) = \frac{\rho_m}{\rho_P} \int_{\ln(\rho_m/\rho_P)}^{+\infty} t^i e^{-t} \, dt. \tag{3.8}
\]

In terms of the incomplete gamma function
\[
\Gamma(\alpha + 1, x) = \int_{x}^{+\infty} t^\alpha e^{-t} \, dt \tag{3.9}
\]
the integral \( I_i(\rho_m) \) can be rewritten as
\[
I_i(\rho_m) = \frac{\rho_m}{\rho_P} \Gamma \left[ i + 1, \ln \left( \frac{\rho_m}{\rho_P} \right) \right] = i! \sum_{k=0}^{i} \frac{1}{k!} \ln^k \left( \frac{\rho_m}{\rho_P} \right). \tag{3.10}
\]

To get the second equality, we have used the relation
\[
\Gamma(n + 1, x) = n! e^{-x} \sum_{k=0}^{n} \frac{1}{k!} x^k, \tag{3.11}
\]
which can be obtained by computing \( \partial \Gamma(n + 1, x) / \partial x \) from Eqs. (3.9) and (3.11). For future purposes, we note the asymptotic behavior\(^1\)
\[
\Gamma(n + 1, x) \sim x^n e^{-x} \quad (x \to \infty). \tag{3.12}
\]

\(^1\)We have assumed that \( n \) is an integer. However, it can be shown that this asymptotic behavior remains valid when \( n \) is a real number.
The energy density can then be written as
\[ \epsilon = \rho_m c^2 - \sum_{i=0}^{N} A_i \frac{\rho_m}{\rho_P} \Gamma \left[ i + 1, \ln \left( \frac{\rho_m}{\rho_P} \right) \right]. \]  
(3.13)

Equations (1.1) and (3.13) determine the EoS \( P(\epsilon) \) in parametric form. By combining the energy conservation equation (3.3) with the first law of thermodynamics (3.4) one finds that the rest-mass density evolves as
\[ \frac{d\rho_m}{dt} + 3H\rho_m = 0 \quad \Rightarrow \quad \rho_m = \frac{\rho_{m,0}}{a^3}, \]  
(3.14)
where \( \rho_{m,0} = \rho_m(a_0 = 1) \) is the present value of the rest-mass density and \( a_0 = 1 \) is the present value of the scale factor. Equation (3.14) expresses the conservation of the rest-mass.

On the other hand, the internal energy is given by
\[ u = -\sum_{i=0}^{N} A_i I_i(\rho_m) = -\sum_{i=0}^{N} A_i I_i\left( \frac{\rho_{m,0}}{a^3} \right), \]  
(3.15)
As argued in [30], the rest-mass energy density \( \rho_m c^2 \) plays the role of pressureless DM (\( \epsilon_m = \rho_m c^2 \)) and the internal energy \( u \) plays the role of DE (\( \epsilon_{de} = u \)).\(^2\) The energy density \( \epsilon = \epsilon_m + \epsilon_{de} \) of the generalized logotropic model is therefore the sum of two terms where the first term \( \epsilon_m \) can be interpreted as the DM energy density given by
\[ \epsilon_m = \frac{\epsilon_{m,0}}{a^3}, \]  
(3.16)
and the second term \( \epsilon_{de} \) can be interpreted as the DE energy density given by
\[ \epsilon_{de} = -\frac{e^{-1-1/B}}{a^3} \sum_{i=0}^{N} A_i \Gamma \left( i + 1, -1 - \frac{1}{B} - 3\ln a \right), \]  
(3.17)
where, following [30], we have defined the dimensionless parameter \( B \) through the relation
\[ \frac{\rho_P}{\rho_{m,0}} = e^{1+1/B}. \]  
(3.18)
The Friedmann equation (3.1) can be written as
\[ H^2 = H_0^2 \left( \frac{\epsilon_m}{\epsilon_0} + \frac{\epsilon_{de}}{\epsilon_0} \right), \]  
(3.19)
where \( \epsilon_0 = 3H_0^2 c^2/8\pi G \) is the critical energy density and \( H_0 \) is the value of the Hubble parameter at the present time.

The pressure and the energy density of the generalized logotropic model can be expressed in terms of the scale factor as
\[ P = \sum_{i=0}^{N} A_i \left( -1 - \frac{1}{B} - 3\ln a \right)^i, \]  
(3.20)
\(^2\)The notation \( \rho_m \) stands either for pressureless DM density or for rest-mass density.
\[ \epsilon = \frac{\Omega_{m,0}\epsilon_0}{a^3} - \frac{e^{-1-1/B}}{a^3} \sum_{i=0}^{N} A_i \Gamma \left( i + 1, -1 - \frac{1}{B} - 3 \ln a \right). \] (3.21)

The present proportion of DM and DE, denoted \( \Omega_{m,0} \) and \( \Omega_{de,0} \), are given by

\[ \Omega_{m,0} = \frac{\epsilon_{m,0}}{\epsilon_0}, \] (3.22)

\[ \Omega_{de,0} = \frac{\epsilon_{de,0}}{\epsilon_0} = -\frac{e^{-1-1/B}}{\epsilon_0} \sum_{i=0}^{N} A_i \Gamma \left( i + 1, -1 - \frac{1}{B} \right) = 1 - \Omega_{m,0}. \] (3.23)

For given values of \( \Omega_{m,0} \) and \( \epsilon_0 \) (through \( H_0 \)), which can be obtained from the observations, Eq. (3.18) determines \( B \) and Eq. (3.23) provides a constraint on the coefficients \( A_i \). Using \( \rho_P = \dot{\epsilon}^2/(8\pi G^2) = 5.16 \times 10^9 \text{g m}^{-3} \) and \( \rho_{m,0} = \Omega_{m,0}\epsilon_0/c^2 \) with \( \Omega_{m,0} = 0.309 \) and \( \epsilon_0/c^2 = 8.62 \times 10^{-24} \text{g m}^{-3} \), we get \( B = 3.53 \times 10^{-3} \). We have taken the values of \( \Omega_{m,0} \) and \( \epsilon_0 \) from the \( \Lambda \)CDM model but since \( B \) is defined by a logarithm, its value is very insensitive to the precise values of \( \Omega_{m,0} \) and \( \epsilon_0 \). Therefore, the value \( B = 3.53 \times 10^{-3} \) is very robust [31].

In the late Universe (\( a \to \infty \) and \( \rho \to 0 \)), the DE dominates and, using Eq. (3.12), we get \( \epsilon \sim -A_N(-3 \ln a)^N \) and \( P \sim A_N(-3 \ln a)^N \), which implies that the EoS \( P(\epsilon) \) behaves asymptotically as \( P/\epsilon \to -1 \). We note that in order to have \( \epsilon > 0 \) when \( a \to +\infty \), the parameter \( A_N \) must be negative if \( N \) is even and positive if \( N \) is odd. Below we present some interesting UDM models derived from the generalized logotropic model, focusing on the EoS \( P(\epsilon) \), the energy density \( \epsilon \) and the cosmological implications of these models. We also investigate the era of DE dominance that drives the accelerated expansion of the Universe.

### 4 Particular models

#### 4.1 The case \( A_i = A_n \delta_{i,n} \)

This case focuses on just the \( n \)-th order term of the finite series leading to the EoS

\[ P = A_n \ln^n \left( \frac{\rho_m}{\rho_P} \right). \] (4.1)

The energy density is given by

\[ \epsilon = \rho_m c^2 - A_n \frac{\rho_m}{\rho_P} \Gamma \left[ n + 1, \ln \left( \frac{\rho_m}{\rho_P} \right) \right] = \rho_m c^2 + u = \epsilon_m + \epsilon_{de}, \] (4.2)

where the first term is the rest-mass energy density (DM) and the second term is the internal energy density (DE). The pressure and total energy density as a function of the scale factor reduce to

\[ P = A_n \left( -1 - \frac{1}{B} - 3 \ln a \right)^n, \] (4.3)

\[ \epsilon = \frac{\Omega_{m,0}\epsilon_0}{a^3} - A_n \frac{e^{-1-1/B}}{a^3} \Gamma \left( n + 1, -1 - \frac{1}{B} - 3 \ln a \right). \] (4.4)
At the present time (i.e. $a = 1$), the above equation leads to

$$A_n = -\frac{(1 - \Omega_{m,0}) \epsilon_0 e^{1+1/B}}{\Gamma (n + 1, -1 - \frac{1}{B})},$$  \hspace{1cm} (4.5)

which determines $A_n$ as a function of the measured values of $\Omega_{m,0}$ and $\epsilon_0$. The dimensionless constant $B$ is also determined by the measured values of $\Omega_{m,0}$ and $\epsilon_0$ (see above). As a result, there is no free parameter in this model. Note that

$$\frac{A_n}{(1 - \Omega_{m,0}) \epsilon_0} = -\frac{e^{1+1/B}}{\Gamma (n + 1, -1 - \frac{1}{B})},$$  \hspace{1cm} (4.6)

is a dimensionless constant depending only on $n$ and $B$. Furthermore, $\rho_\Lambda \equiv (1 - \Omega_{m,0}) \epsilon_0/e^2 = 5.96 \times 10^{-24} \text{ g m}^{-3}$ is the present value of the DE density (it is equal to the cosmological density $\rho_\Lambda = \Lambda/(8\pi G)$ in the $\Lambda$CDM model). From Eq. (4.6), we find that $A_n < 0$ for $n$ even and $A_n > 0$ for $n$ odd.

On the other hand, from Eq. (4.1), the pressure $P$ vanishes when $\rho_m = \rho_P$. If $n$ is even, the pressure $P$ is always negative. If $n$ is odd, the pressure $P$ is positive for $\rho_m > \rho_P$ and negative for $\rho_m < \rho_P$. In practice, we consider a regime where $\rho_m \ll \rho_P$ because the logotropic model does not describe the early inflation. In that case, the pressure is always negative. The EoS $P(\epsilon)$ is given in the reversed form $\epsilon(P)$ by

$$\epsilon = e^{\frac{\rho_P}{\rho_m} e^{1/n} - A_n e^{\frac{\rho_P}{\rho_m} e^{1/n}} \Gamma (n + 1, + \frac{P}{A_n} e^{1/n})},$$  \hspace{1cm} (4.7)

where the upper sign corresponds to the most relevant case $\rho_m < \rho_P$ and the lower sign corresponds to $\rho_m > \rho_P$. Substituting Eq. (4.5) into Eqs. (4.3) and (4.4), we obtain after simple manipulations

$$P = -\frac{(1 - \Omega_{m,0}) \epsilon_0}{\Gamma (n + 1, -1 - \frac{1}{B})} e^{1+1/B} \left(-1 - \frac{1}{B} - 3 \ln a\right)^n,$$  \hspace{1cm} (4.8)

$$\epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} + \frac{(1 - \Omega_{m,0}) \epsilon_0}{a^3} \frac{\Gamma (n + 1, -1 - \frac{1}{B} - 3 \ln a)}{\Gamma (n + 1, -1 - \frac{1}{B})}.$$  \hspace{1cm} (4.9)

In the early Universe ($a \to 0$, $\rho_m \to +\infty$) the DM $\epsilon_m$ dominates and we have $\epsilon \sim \Omega_{m,0} \epsilon_0/a^3$, $P \sim A_n(-3 \ln a)^n$ such that $P/\epsilon \to 0$ whereas, in the late Universe ($a \to \infty$, $\rho_m \to 0$), the DE $\epsilon_{de}$ dominates and we have $\epsilon \sim -A_n(-3 \ln a)^n$, $P \sim A_n(-3 \ln a)^n$ such that $P/\epsilon \to -1$. Furthermore, in order to have $\epsilon > 0$ for $a \to +\infty$, the parameter $A_n$ must be negative if $n$ is even and positive if $n$ is odd. As we have seen, this is guaranteed by Eq. (4.5). Finally, it is worth mentioning that for $B \to 0$ (corresponding to a form of semiclassical limit $\rho_P \to +\infty$ or $h \to 0$), the $\Lambda$CDM model is recovered [31]. Indeed, using Eq. (3.12), we find that Eqs. (4.8) and (4.9) reduce to

$$P = -(1 - \Omega_{m,0}) \epsilon_0,$$  \hspace{1cm} (4.10)

$$\epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} + (1 - \Omega_{m,0}) \epsilon_0.$$  \hspace{1cm} (4.11)
4.2 The case \( A_i = A_0 \delta_{i,0} \)

For \( n = 0 \), we recover the \( \Lambda \)CDM model (interpreted as a UDM model) corresponding to the EoS

\[ P = A_0. \]  

(4.12)

The energy density is given by

\[ \epsilon = \rho_m c^2 - A_0, \]  

(4.13)

where the first term is DM and the second term is DE. The total energy density as a function of the scale factor reads

\[ \epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} - A_0. \]  

(4.14)

At the present time (i.e. \( a = 1 \)), Eq. (4.14) leads to

\[ A_0 = -(1 - \Omega_{m,0}) \epsilon_0. \]  

(4.15)

We note that \( A_0 < 0 \). Numerically, \( A_0/c^2 = -\rho_\Lambda = -5.96 \times 10^{-24} \text{ g m}^{-3} \), where \( \rho_\Lambda \) is the cosmological density. The pressure \( P \) and the energy density \( \epsilon \) are then given by Eqs. (4.10) and (4.11). The pressure \( P = -\rho_\Lambda c^2 \) is constant and negative. As the universe expands, starting from \( +\infty \), the energy density \( \epsilon \) decreases and tends to a constant value \( \rho_\Lambda c^2 \). The DE density \( \epsilon_{de} = \rho_\Lambda c^2 \) is constant.

4.3 The case \( A_i = A_1 \delta_{i,1} \)

For \( n = 1 \), we recover the original logotropic model \([30]\) corresponding to the EoS

\[ P = A \ln \left( \frac{\rho_m}{\rho_P} \right), \]  

(4.16)

where we have noted \( A_1 = A \). The energy density is given by

\[ \epsilon = \rho_m c^2 - A \left[ 1 + \ln \left( \frac{\rho_m}{\rho_P} \right) \right], \]  

(4.17)

where the first term is DM and the second term is DE. The EoS \( P(\epsilon) \) is given in the reversed form \( \epsilon(P) \) by

\[ \epsilon = e^{P/A} \rho_P c^2 - A - P. \]  

(4.18)

The pressure and total energy density as a function of the scale factor read

\[ P = -A \left( 1 + \frac{1}{B} + 3 \ln a \right), \]  

(4.19)

\[ \epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} + A \left( \frac{1}{B} + 3 \ln a \right). \]  

(4.20)

At the present time (i.e. \( a = 1 \)), Eq. (4.20) leads to

\[ A = B (1 - \Omega_{m,0}) \epsilon_0. \]  

(4.21)
We note that $A > 0$. Numerically, $A/c^2 = B\rho_A = 3.53 \times 10^{-3}\rho_A = 2.10 \times 10^{-26}\text{g m}^{-3}$. The pressure $P$ and the energy density $\epsilon$ become

\[
P = -(1 - \Omega_{m,0})\epsilon_0 (B + 1 + 3B\ln a),
\]

\[
\epsilon = \frac{\Omega_{m,0}\epsilon_0}{a^3} + (1 - \Omega_{m,0})\epsilon_0 (1 + 3B\ln a).
\]

The pressure $P$ is positive when $\rho_m > \rho_P$ and negative when $\rho_m < \rho_P$. It vanishes at $\rho_m = \rho_P$. As the universe expands, the pressure decreases from $+\infty$ to $-\infty$. Starting from $+\infty$, the energy density $\epsilon$ first decreases, reaches a minimum $\epsilon_{\text{min}} = A\ln(\rho_P c^2/A) > 0$ at $\rho_m = A/c^2$, then increases to $+\infty$. The DE density $\epsilon_{de}$ increases from $-\infty$ to $+\infty$. The DE is negative when $\rho_m > \rho_P/e$ and positive when $\rho_m < \rho_P/e$ (its value at $\rho_m = \rho_P$ is $\epsilon_{de} = -A < 0$). In the logotropic model, since the DE density corresponds to the internal energy density $u$ of the LDF, it can very well be negative as long as the total energy density $\epsilon$ is positive. Note, however, that in the regime of interest $\rho_m \ll \rho_P$, the DE density $\epsilon_{de}$ is positive.

**4.4 The case** $A_i = A_2 \delta_{i,2}$

For $n = 2$, we obtain the EoS

\[
P = A_2 \ln^2 \left( \frac{\rho_m}{\rho_P} \right).
\]

The energy density is given by

\[
\epsilon = \rho_m c^2 - 2A_2 \left[ 1 + \ln \left( \frac{\rho_m}{\rho_P} \right) + \frac{1}{2} \ln^2 \left( \frac{\rho_m}{\rho_P} \right) \right],
\]

where the first term is DM and the second term is DE. The EoS $P(\epsilon)$ is given in the reversed form $\epsilon(P)$ by

\[
\epsilon = e^{\mp \sqrt{\frac{P}{A_2}}} \rho_P c^2 - 2A_2 \left( 1 \mp \sqrt{\frac{P}{A_2}} \frac{\rho_P}{2A_2} \right),
\]

where the upper sign corresponds to the most relevant case $\rho_m < \rho_P$ and the lower sign corresponds to $\rho_m > \rho_P$. The pressure and total energy density as a function of the scale factor read

\[
P = A_2 \left( 1 + \frac{1}{B} + 3\ln a \right)^2,
\]

\[
\epsilon = \frac{\Omega_{m,0}\epsilon_0}{a^3} - 2A_2 \left[ -\frac{1}{B} - 3\ln a + \frac{1}{2} \left( 1 + \frac{1}{B} + 3\ln a \right)^2 \right].
\]

At the present time (i.e. $a = 1$), Eq. (4.28) leads to

\[
A_2 = -\frac{1 - \Omega_{m,0}}{1 + \frac{1}{B^2}} \epsilon_0.
\]
We note that $A_2 < 0$. Numerically, $A_2/c^2 = -B^2/(1 + B^2)\rho_\Lambda = -1.25 \times 10^{-5} \rho_\Lambda = 7.43 \times 10^{-29} \text{g m}^{-3}$. The pressure $P$ and the energy density $\epsilon$ become

$$
P = -\frac{(1 - \Omega_{m,0}) \epsilon_0}{1 + B^2} \frac{1}{1 + B^2},
$$

$$(4.30)$$

$$
\epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} + (1 - \Omega_{m,0}) \epsilon_0 \frac{1 + B^2 + 6B \ln a + 9B^2 \ln^2 a}{1 + B^2}.
$$

$$(4.31)$$

The pressure $P$ is always negative and vanishes at $\rho_m = \rho_P$. As the universe expands, starting from $-\infty$, the pressure first increases, vanishes at $\rho = \rho_m$, then decreases to $-\infty$. Starting from $+\infty$, the energy density $\epsilon$ first decreases, reaches a minimum $\epsilon_{\text{min}} = -A_2(\rho_m c^2/2A_2 - 1)^2 > 0$ at $\rho_m$ solution of $\rho_m c^2 = 2A_2[1 + \ln(\rho_m/\rho_P)]$, then increases to $+\infty$. Starting from $+\infty$, the DE density $\epsilon_{\text{de}}$ first decreases, reaches a minimum $\epsilon_{\text{de}}^{\text{min}} = -A_2 > 0$ at $\rho_m = \rho_P/e$, then increases to $+\infty$ (its value at $\rho_m = \rho_P$ is $\epsilon_{\text{de}} = -2A_2 > 0$).

Cases with $n = 1, 2$ are the simplest logotropic models. Since $n$ is a free parameter, cases with $n > 2$ lead to a collection of generalized logotropic models.

### 4.5 The case $N = 2$

This generalized logotropic model is obtained by considering the first three terms $A_0$, $A_1$ and $A_2$. This leads to the EoS

$$
P = A_0 + A_1 \ln \left( \frac{\rho_m}{\rho_P} \right) + A_2 \ln^2 \left( \frac{\rho_m}{\rho_P} \right).
$$

$$(4.32)$$

The energy density is given by

$$
\epsilon = \rho_m c^2 - A_0 - A_1 \left[ 1 + \ln \left( \frac{\rho_m}{\rho_P} \right) \right] - 2A_2 \left[ 1 + \ln \left( \frac{\rho_m}{\rho_P} \right) + \frac{1}{2} \ln^2 \left( \frac{\rho_m}{\rho_P} \right) \right],
$$

$$(4.33)$$

where the first term is DM and the other terms are DE. The EoS $P(\epsilon)$ can be obtained in the reversed form $\epsilon(P)$ by eliminating $\rho_m$ between Eqs. (4.32) and (4.33).$^3$ The pressure and the energy density evolve with the scale factor as

$$
P = A_0 + A_1 \left( -1 - \frac{1}{B} - 3 \ln a \right) + A_2 \left( -1 - \frac{1}{B} - 3 \ln a \right)^2,
$$

$$(4.34)$$

$$
\epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} - A_0 + A_1 \left( \frac{1}{B} + 3 \ln a \right) - 2A_2 \left[ -\frac{1}{B} - 3 \ln a + \frac{1}{2} \left( 1 + \frac{1}{B} + 3 \ln a \right)^2 \right].
$$

$$(4.35)$$

At the present time (i.e. $a = 1$), the above equation gives

$$
(1 - \Omega_{m,0}) \epsilon_0 = -A_0 + \frac{A_1}{B} - A_2 \left( 1 + \frac{1}{B^2} \right),
$$

$$(4.36)$$

which constrains one of the three free parameters. As a result, the model has only two free parameters. As expected, in the late Universe ($a \to \infty$, $\rho \to 0$), the DE density $\epsilon_{\text{de}}$ dominates and we have $P/\epsilon \to -1$. It is worth mentioning that in order to have $\epsilon > 0$, $A_2$ must be negative.

$^3$We can note that Eq. (4.32) is a second degree equation in $\ln(\rho_m/\rho_P)$.
5 Evolution of the generalized logotropic model

In the following we consider models with $A_i = A_n \delta_{i,n}$. The evolution of the energy density as a function of the scale factor is plotted in Fig. 1, where we have used the best-fit values of the parameters $\Omega_{m,0} = 0.309$ and $B = 3.53 \times 10^{-3}$ obtained from the $\Lambda$CDM model (they are consistent with the cosmological analysis of the original logotropic model [39, 40]). It shows that the Universe starts at $a = 0$ with an infinite energy density. The energy density $\epsilon$ first decreases with the increase of the scale factor $a$, reaches a minimum, then increases with the scale factor characterizing a phantom Universe [44, 45].

In the generalized logotropic model, the Friedmann equation (3.19) takes the form

$$H = \frac{\dot{a}}{a} = H_0 \sqrt{\frac{\Omega_{m,0}}{a^3} + \frac{1 - \Omega_{m,0}}{a^3} \frac{\Gamma(n + 1, -1 - \frac{1}{B} - 3 \ln a)}{\Gamma(n + 1, -1 - \frac{1}{B})}}. \quad (5.1)$$

The evolution of the scale factor as a function of the time is given by

$$H_0 t = \int_0^a \frac{da'}{a' \sqrt{\frac{\Omega_{m,0}}{a'^3} + \frac{1 - \Omega_{m,0}}{a'^3} \frac{\Gamma(n+1, -1 - \frac{1}{B} - 3 \ln a')}}{\Gamma(n+1, -1 - \frac{1}{B})}}. \quad (5.2)$$

The $\Lambda$CDM model is recovered from Eq. (5.2) for $n = 0$ or $B = 0$. In that case, Eq. (5.2) can be integrated analytically yielding

$$a = \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2} \sqrt{1 - \Omega_{m,0}} H_0 t\right), \quad (5.3)$$

$$\frac{\epsilon}{\epsilon_0} = \frac{1 - \Omega_{m,0}}{\tanh^2\left(\frac{3}{2} \sqrt{1 - \Omega_{m,0}} H_0 t\right)}, \quad (5.4)$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{Energy density $\epsilon/\epsilon_0$ as a function of the scale factor $a$ for $\Omega_{m,0} = 0.309$, $B = 3.53 \times 10^{-3}$ and $n = 0, 1, 2, 3, 5, 10$.}
\end{figure}
whereas for $B \neq 0$ it can only be integrated numerically. Figure 2 shows the behavior of the scale factor $a$ as a function of $H_0 t$ for $n = 0, 1, 2, 3, 5, 10$. The age of the universe is given by

$$t_{\text{age}} = \frac{1}{H_0} \int_0^1 \left( \frac{\Omega_{m,0} a'}{a^{3n}} + \frac{1-\Omega_{m,0}}{a^{3n}} \right) \frac{\Gamma(n+1, -\frac{1}{3} - 3 \ln a')}{\Gamma(n+1, -\frac{1}{3} - 3 \ln a')} \, da'$$

(5.5)

with $H_0 = 2.195 \times 10^{-18}$ s$^{-1}$, i.e., $H_0^{-1} = 14.4$ Gyrs. For the $\Lambda$CDM model ($n = 0$) we recover the well-known result $t_{\text{age}} = 0.956 H_0^{-1} = 13.8$ Gyrs. A difference larger than 0.1% (the typical error bar on the age of the universe) occurs in models with $n > 3$ so these models should be rejected. For example, we get $t_{\text{age}} = 0.966 H_0^{-1} = 13.9$ Gyrs for $n = 20$, $t_{\text{age}} = 0.998 H_0^{-1} = 14.4$ Gyrs for $n = 100$, and $t_{\text{age}} = 1.12 H_0^{-1} = 16.1$ Gyrs for $n = 1000$.

The asymptotic behavior of the generalized logotropic model can be obtained analytically. For $a \to 0$, we have a matter dominated Universe which corresponds to Einstein-deSitter solution given by

$$a \sim \left( \frac{3}{2} \sqrt{\Omega_{m,0} H_0 t} \right)^{2/3},$$

(5.6)

$$\frac{\epsilon}{\epsilon_0} \sim \frac{4}{9 H_0^2 t^2}.$$  

(5.7)

For $a \to +\infty$, the energy density behaves as

$$\epsilon \sim 3^n |A_n| (\ln a)^n.$$  

(5.8)

In this asymptotic regime, the Friedmann equation can be written as

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3c^2} \epsilon} \sim K_n (\ln a)^{n/2},$$

(5.9)

\footnote{We recall that the logotropic model is not aimed at describing the early inflation.}
where we define the constant $K_n$ as

$$K_n = 3^{n/2}H_0 \sqrt{\left( \frac{1 - \Omega_{m,0}}{H_0^2} \right) e^{1+1/B} \Gamma\left( n + 1, -1 - \frac{1}{B} \right)}.$$  

(5.10)

The above equation can be integrated into

$$K_n t \sim \int^a_{a'} \frac{da'}{a'(\ln a')^{n/2}} \sim \int^{\ln a}_{s} \frac{ds}{s^{n/2}}.$$  

(5.11)

where we have made the change of variables $s = \ln a'$ to get the second equality. It is interesting to note that the asymptotic evolution of the scale factor as a function of time depends mainly on the value of the parameter $n$. We can distinguish three relevant types of evolution which correspond to $n < 2$, $n = 2$, and $n > 2$.

(i) For $n < 2$: this solution, which includes the original logotropic model [30, 31, 39], describes a super de Sitter evolution of the form

$$a \propto \exp \left\{ \left( \frac{1}{2} (2 - n) K_n t \right)^{\frac{2}{n-2}} \right\}, \quad \epsilon \propto t^n,$$  

(5.12)

where the scale factor grows super exponentially rapidly with cosmic time causing an algebraic divergence of the energy density. For $n = 1$ we recover the original logotropic model where $a \sim e^t$ and $\epsilon \sim t$ [30, 31, 39]. Since the scale factor and the density increase indefinitely with time, this is called Little Rip [46]. For $n = 0$ we recover the $\Lambda$CDM model presenting an exponential (de Sitter) expansion $a \sim e^t$ and a constant energy density $\epsilon \sim 1$.

(ii) For $n = 2$: we get a double exponential evolution of the form

$$a \propto \exp[K_2 t], \quad \epsilon \propto \exp(2K_2t),$$  

(5.13)

where the scale factor grows hyper exponentially rapidly with cosmic time causing an exponential divergence of the energy density (Little Rip).

(iii) For $n > 2$: this case represents a situation in which the Universe ends up with a finite-time future singularity. One finds

$$a \propto \exp \left\{ \left[ \frac{1}{2} (n - 2) K_n (t_s - t) \right]^{-\frac{2}{n-2}} \right\}, \quad \epsilon \propto \left[ \frac{1}{2} (n - 2) K_n (t_s - t) \right]^{-\frac{2n}{n-2}}.$$  

(5.14)

The singularity at $t = t_s$ corresponds to a Big Rip [47] characterized by the divergence of the scale factor and energy density in finite time. The big rip time is given by

$$t_{BR} = \frac{1}{H_0} \int^{\infty}_0 \frac{da'}{a'} \sqrt{\frac{\Omega_{m,0}}{a'^4} + \frac{1 - \Omega_{m,0}}{a'^3} \frac{\Gamma(n+1,-1-\frac{1}{B} - 3\ln a')}{\Gamma(n+1,-1-\frac{1}{B})}}.$$  

(5.15)

For measured values of $\Omega_{m,0}$ and $\epsilon_0$ (hence $H_0$), this is just a function of $n$. There is no other free (undetermined) parameter in our model. We get $t_{BR} = 3240$ Gyrs for $n = 3$, $t_{BR} = 1650$ Gyrs for $n = 4$, $t_{BR} = 422$ Gyrs for $n = 10$, $t_{BR} = 47.1$ Gyrs for $n = 100$, and $t_{BR} = 19.2$ Gyrs for $n = 1000$. However, we recall that the models with $n > 3$ are excluded from the observations.
Using Eqs. (4.8) and (4.9), the EoS parameter $w = P/\epsilon$ can be expressed in terms of the scale factor as

$$w(a) = \frac{-(1 - \Omega_{m,0}) e^{1+1/B} a^{3 \left(-1 - \frac{1}{B} - 3 \ln a\right)^n}}{\Omega_{m,0} \Gamma(n + 1, -1 - \frac{1}{B}) + (1 - \Omega_{m,0}) \Gamma(n + 1, -1 - \frac{1}{B} - 3 \ln a)}.$$  

(5.16)

For $a = 1$, we get

$$w_0 = \frac{-(1 - \Omega_{m,0}) e^{1+1/B} \left(-1 - \frac{1}{B}\right)^n}{\Gamma(n + 1, -1 - \frac{1}{B})}.$$  

(5.17)

For the $\Lambda$CDM ($n = 0$) we recover the well-known value $w_0 = - (1 - \Omega_{m,0}) = -0.691$. We get $w_0 = -0.693$ for $n = 1$, $w_0 = -0.696$ for $n = 2$, $w_0 = -0.698$ for $n = 3$, $w_0 = -0.701$ for $n = 4$, $w_0 = -0.715$ for $n = 10$, $w_0 = -0.935$ for $n = 100$, and $w_0 = -3.12$ for $n = 1000$. For $n > 3$ we are out of the error bars (typically 1\% for the value of $w_0$) so these models should be rejected. The behavior of the EoS parameter $w(a)$ as function of the scale factor $a$ is plotted in Fig. 3. It shows that the GLDF behaves asymptotically as a superposition of two non-interacting fluids (i.e. DM and DE) with the total pressure and total energy given by the sum

$$P = P_m + P_{de}, \quad \epsilon = \epsilon_m + \epsilon_{de}.$$  

(5.18)

Their EoS parameters are

$$w_m = \frac{P_m}{\epsilon_m} = 0, \quad w_{de} = \frac{P_{de}}{\epsilon_{de}} = \frac{-e^{1+1/B} a^{3 \left(-1 - \frac{1}{B} - 3 \ln a\right)^n}}{\Gamma(n + 1, -1 - \frac{1}{B} - 3 \ln a)}.$$  

(5.19)

In a flat Universe, the deceleration parameter $q = -\ddot{a}/\dot{a}^2 = -\dot{H}/H^2 - 1$ is related to the EoS parameter by

$$q = \frac{1 + 3w}{2}.$$  

(5.20)
The Universe is undergoing an accelerating expansion if \( q < 0 \) (i.e. \( w < -1/3 \)). In Fig. 4, we show the behavior of \( q(a) \) as a function of the scale factor for different values of \( n \).

From Eq. (4.9) the transition scale factor \( a_t \), corresponding to \( \epsilon_m = \epsilon_{de} \), is obtained by solving the transcendental equation

\[
\Gamma \left( n + 1, -1 - \frac{1}{B} - 3 \ln a_t \right) = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} \Gamma \left( n + 1, -1 - \frac{1}{B} \right).
\]  

(5.21)

Another parameter to study is the squared speed of sound \( c_s^2 \), which is a key ingredient to investigate the stability of any model. In particular, the sign of \( c_s^2 \) plays a crucial role for determining classical stability. It is defined by

\[
c_s^2 = \frac{dP}{d\epsilon} c^2 = \frac{dP}{d\rho_m} c^2.
\]  

(5.22)

Differentiating Eqs. (1.1) and (3.6) with \( A_i = A_n \delta_{i,n} \), we get

\[
\frac{c_s^2}{c^2} = \frac{n \ln^{n-1} \left( \frac{\rho_m}{\rho_p} \right)}{\rho_{m} c^2 - n \frac{\rho_m}{\rho_p} \Gamma \left[ n, \ln \left( \frac{\rho_m}{\rho_p} \right) \right]}.
\]  

(5.23)

In Fig. 5 we plot the squared speed of sound \( c_s^2/c^2 \) as a function of the scale factor for different values of \( n \). The causality and classical stability conditions are satisfied if the speed of sound varies in the range \( 0 \leq c_s^2/c^2 \leq 1 \). One sees from Fig. 5 that, in fact, there are regions where the causality and classical stability conditions are satisfied but the extent of these regions decreases as \( n \) increases.\(^5\)

\(^5\)The speed of sound is real in the normal regime \( \rho_{m}/\rho_p > 0 \) and imaginary in the phantom regime \( \rho_{m}/\rho_p < 0 \). It becomes infinite (before becoming imaginary) when we enter into the phantom regime, i.e., when \( \rho_{m}/\rho_p = 0 \).
In this work, we have proposed a new class of cosmological unified dark sector models “Generalized Logotropic Models”. These models are a generalization of the logotropic model [30] by considering the pressure $P$ as a sum of higher logarithmic terms of the rest-mass density $\rho_m$. The pressure can naturally take negative values in these cosmological models. In this scenario, the Universe is filled with a single fluid without the need of a cosmological constant. Our generalized model depends on a set of free parameters $A_i$ with $i = 1, 2, \cdots, N$. In particular, we have considered a special class of generalized logotropic models where $A_i = A_n \delta_m$, which depends on two free parameters $A_n$ and $n$. The usual logotropic model [30] corresponds to $n = 1$ and $A_1$. We have also presented the model with $N = 2$, which contains the first two terms $A_1$ and $A_2$ of the series. We have highlighted the most relevant properties of these generalized logotropic models. To fix bounds on the free parameters of our models, we employed the best fit of the parameters $\Omega_{m,0}$ and $B$ obtained from the cosmological analysis carried on the original logotropic model (i.e. $n = 1$) [39]. After fixing the free parameters, we investigated the cosmological behavior of the generalized logotropic models by focusing on the evolution of the DE density, scale factor, EoS parameter, deceleration parameter and squared speed of sound. We showed the asymptotic behavior of these models and noticed three distinct ways of evolution depending on the value of $n$. In all the analyzed cases, we established that generalized logotropic models lead to realistic cosmological models in which the dark sector is represented by a unique fluid. The deviation of the generalized models from the standard $\Lambda$CDM model depends on the value of the parameter $n$. At later times, higher values of $n$ lead to higher deviations from $\Lambda$CDM. This implies that the generalized logotropic models can be used as realistic background cosmological models to describe our Universe with a free parameter $n$. We estimated that only models with $n \leq 3$ are consistent with the observations. To find out the most suitable values of $n$, it will be necessary to perform a fitting procedure by using, for instance, the Monte Carlo method and a detailed comparison with cosmological data like SN, BAO, and CMB surveys. We expect to perform this analysis in future works.
A further interesting generalization of the logotropic model is to consider a single fluid described by the EoS
\begin{equation}
P = A \ln^\alpha \left( \frac{\rho_m}{\rho P} \right),
\end{equation}
where $A$ is a real number given by
\begin{equation}
A = -\frac{(1 - \Omega_{m,0}) \epsilon_0 e^{1+1/B}}{\Gamma (\alpha + 1, -1 - \frac{1}{B})},
\end{equation}
and $\alpha$ is a real positive parameter which can be constrained by the cosmological data. There are two ways to recover the $\Lambda$CDM, either by taking $\alpha = 0$ or $B = 0$ which makes the model rich and particularly more appealing in describing DE especially for $0 < \alpha < 1$. Such a model interpolates between the $\Lambda$CDM for values of $\alpha$ close to zero and to the usual logotropic model for values of $\alpha$ close to one.

In summary, single fluids with generalized logotropic EoS may have interesting cosmological features and, thus, they represent a good candidate to describe the DE sector. The investigation and analysis of these models will be carried out in detail in future works.

Finally, we would like to mention that the logotropic model and the generalization presented in this work are not free of intrinsic problems, as all the cosmological models known in the literature. In fact, the speed of sound in logotropic models has the unpleasant property of increasing with the scale factor, leading, like for the Chaplygin gas model, to oscillations in the mass power-spectrum that are not detected in observations at the cosmological level [48]. However, there are several possibilities to solve this problem by considering additional effects such as non-linear and non-adiabatic perturbations, among others (see the discussion in [49]). In any case, UDM models constitute a subject of intensive research as possible alternative scenarios to the popular and generally accepted $\Lambda$CDM model. Our paper provides a class of models exhibiting a transition between a normal behavior and a phantom behavior governed by a single equation of state. In addition, depending on the value of the parameter $n$, we can have different types of late evolution: no singularity, little rip, big rip... It is very interesting to note that all these models are consistent with the $\Lambda$CDM model up to the present time but will differ in the future. A virtue of our model is to show that it is very difficult to predict the future evolution of the universe based on present observations.

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A Asymptotic equation of state

In this Appendix, we establish the asymptotic EoS $P(\epsilon)$ of the GLDF and make the connection with the MCG model [19, 26–28].
For \( a \to +\infty \), using Eqs. (3.12) and (3.21), we find that the energy density evolves with the scale factor as

\[
\epsilon \sim |A_N| \left( 1 + \frac{1}{B} + 3 \ln a \right)^N. \tag{A.1}
\]

Let us determine the corresponding asymptotic EoS from the energy conservation equation (3.3) which can be rewritten as

\[
\frac{d\epsilon}{da} + \frac{3}{a}(\epsilon + P) = 0. \tag{A.2}
\]

From Eqs. (A.1) and (A.2), we get

\[
P = -\epsilon - \frac{a}{3} \frac{d\epsilon}{da} = -\epsilon - N|A_N| \left( 1 + \frac{1}{B} + 3 \ln a \right)^{N-1} = -\epsilon - N|A_N| \left( \frac{\epsilon}{|A_N|} \right)^{(N-1)/N} = -\epsilon \left[ 1 + N \left( \frac{|A_N|}{\epsilon} \right)^{1/N} \right]. \tag{A.3}
\]

Therefore, the asymptotic EoS of the GLDF reads

\[
P = -\epsilon - N|A_N|^{1/N} \epsilon^{1-1/N}. \tag{A.4}
\]

This is a particular case of the generalized polytropic EoS (or MCG model) \[19, 26–28\]

\[
P = \alpha \epsilon + K \left( \frac{\epsilon}{\epsilon_0} \right)^\gamma \tag{A.5}
\]

corresponding to \( \alpha = -1 \), \( K = -N(|A_N|\epsilon^2)^{1/N} \) and \( \gamma = 1 - 1/N \). Therefore, the GLDF is asymptotically equivalent to the MCG with \( \alpha = -1 \). Since \( w = P/\epsilon < -1 \), the EoS (A.4) leads to a phantom behavior in agreement with the results of Sec. 5. For \( N = 1 \), the EoS (A.4) reduces to

\[
P = -\epsilon - A. \tag{A.6}
\]

For \( N = 2 \), it reduces to

\[
P = -\epsilon - 2\sqrt{|A_2|} \epsilon^{1/2}. \tag{A.7}
\]

Let us check that we recover the asymptotic EoS (A.4) directly from the generalized logotropic EoS defined by Eqs. (3.20) and (3.21). For \( a \to +\infty \), we have

\[
P \simeq A_N x^N + A_{N-1} x^{N-1} + ..., \tag{A.8}
\]

\[
\epsilon \simeq -A_N x^N - A_N N x^{N-1} - A_{N-1} x^{N-1} + ..., \tag{A.9}
\]

where

\[
x \equiv -1 - \frac{1}{B} - 3 \ln a \to -\infty. \tag{A.10}
\]
We stress that it is necessary to account for the first order correction to the leading term $x^N$ in Eqs. (A.8) and (A.9). From

$$P \simeq A_N x^N \left( 1 + \frac{A_{N-1}}{A_N} \frac{1}{x} \right), \quad (A.11)$$

$$\epsilon \simeq -A_N x^N \left( 1 + N \frac{1}{x} + \frac{A_{N-1}}{A_N} \frac{1}{x} \right), \quad (A.12)$$

we get

$$\frac{P}{\epsilon} \simeq - \left( 1 + \frac{A_{N-1}}{A_N} \frac{1}{x} \right) \left( 1 - N \frac{1}{x} - \frac{A_{N-1}}{A_N} \frac{1}{x} \right)$$

$$\simeq - \left( 1 - \frac{N}{x} \right)$$

$$= - \left[ 1 + N \left( \frac{|A_N|}{\epsilon} \right)^{1/N} \right], \quad (A.13)$$

which returns Eq. (A.3).

### B The two-fluid model

In this Appendix, we determine the two-fluid model corresponding to the GLDF with $A_i = A_n \delta_{i,n}$. In particular, we establish the EoS of the DE in the two-fluid model.

The GLDF is a one-fluid model (i.e. a UDM model) unifying DM and DE. The pressure and the energy density are given by

$$P = A_n \ln \left( \frac{\rho_m}{\rho_P} \right), \quad (B.1)$$

$$\epsilon = \rho_m c^2 - A_n \frac{\rho_m}{\rho_P} \Gamma \left[ n + 1, \ln \left( \frac{\rho_m}{\rho_P} \right) \right] = \rho_m c^2 + u = \epsilon_m + \epsilon_{de}. \quad (B.2)$$

Concerning the evolution of the homogeneous background, this one-fluid model is equivalent to a two-fluid model made of pressureless DM with an EoS $P_m = 0$ giving $\epsilon_m = \Omega_{m,0} \epsilon_0 / a^3$ and DE with an EoS $P_{de}(\epsilon_{de})$ giving $\epsilon_{de} = u(a)$. Noting that $P = P_m + P_{de} = P_{de}$, the EoS $P_{de}(\epsilon_{de})$ of DE is determined in parametric form by the equations

$$P_{de} = A_n \ln^n \left( \frac{\rho_m}{\rho_P} \right), \quad (B.3)$$

$$\epsilon_{de} = -A_n \frac{\rho_m}{\rho_P} \Gamma \left[ n + 1, \ln \left( \frac{\rho_m}{\rho_P} \right) \right]. \quad (B.4)$$

Eliminating $\rho_m$ between these two expressions we get the EoS of DE under the reversed form $\epsilon_{de}(P_{de})$ as

$$\epsilon_{de} = -A_n c \left| \frac{P_{de}}{A_n} \right|^{1/n} \Gamma \left[ n + 1, + \left| \frac{P_{de}}{A_n} \right|^{1/n} \right], \quad (B.5)$$

---

The equivalence between the one-fluid model and the two-fluid model is lost when we consider the formation of structures.
where the upper sign corresponds to the most relevant case $\rho_m < \rho_P$ and the lower sign corresponds to $\rho_m > \rho_P$.

Let us check that the two-fluid model returns the results of the one-fluid model for the homogeneous background. The evolution $\epsilon_{de}(a)$ of the DE density with the scale factor can be obtained from the energy conservation equation

$$\frac{d\epsilon_{de}}{da} + \frac{3}{a} (\epsilon_{de} + P_{de}) = 0 \quad \Leftrightarrow \quad \int \frac{d\epsilon_{de}}{\epsilon_{de} + P_{de}} = -3 \ln a \quad (B.6)$$

with the EoS $P_{de}(\epsilon_{de})$ defined by Eqs. (B.3) and (B.4). At this stage, $\rho_m$ is just a dummy variable. It is easy to establish that

$$\epsilon'_{de}(\rho_m) = -A_n n \frac{\rho_P}{\rho_m^2} \Gamma \left[ n, \ln \left( \frac{\rho_m}{\rho_P} \right) \right] \quad (B.7)$$

and

$$P_{de}(\rho_m) + \epsilon_{de}(\rho_m) = -A_n n \frac{\rho_P}{\rho_m} \Gamma \left[ n, \ln \left( \frac{\rho_m}{\rho_P} \right) \right]. \quad (B.8)$$

Therefore, we have the identity

$$P_{de}(\rho_m) + \epsilon_{de}(\rho_m) = \rho_m \epsilon'_{de}(\rho_m), \quad (B.9)$$

and the energy conservation equation (B.6) becomes

$$\frac{d\rho_m}{da} + \frac{3}{a} \rho_m = 0 \quad \Leftrightarrow \quad \int \frac{d\rho_m}{\rho_m} = -3 \ln a, \quad (B.10)$$

implying that $\rho_m \propto a^{-3}$. This leads to results consistent with those of Sec. 4. However, we cannot establish that $\rho_m$ is the rest-mass (or DM) density (they could differ by a multiplicative constant). This implies that we cannot determine $A_n$ in the two-fluid model contrary to the one-fluid model. More explicit results are given below.

**B.1 The case $n = 1$**

For $n = 1$, Eq. (B.5) reduces to the affine EoS

$$P_{de} = -\epsilon_{de} - A \quad (B.11)$$

with $A > 0$. This is the EoS of DE corresponding to the original logotropic gas [39]. It coincides with the asymptotic EoS (A.6) of the LDF seen as a one-fluid (UDM) model. The affine EoS (B.11) was first introduced and studied in [28].

Let us check that the two-fluid model returns the results of the one-fluid model. Integrating the energy conservation equation (B.6) for DE with the EoS (B.11) we obtain

$$\epsilon_{de} = 3A \ln \left( \frac{a}{a_s} \right), \quad (B.12)$$

where $a_s$ is a constant of integration. If we add the contribution of pressureless DM, we find that the total energy density is given by

$$\epsilon = \frac{\Omega_{m,0} \epsilon_0}{a^3} + 3A \ln \left( \frac{a}{a_s} \right). \quad (B.13)$$
Using the condition (at $a = 1$)

$$
\epsilon_0(1 - \Omega_{m,0}) = -3A \ln a_s,
$$

(B.14)

we can rewrite Eq. (B.13) under the form

$$
\epsilon = \frac{\Omega_{m,0}\epsilon_0}{a^3} + 3A \ln a + \epsilon_0(1 - \Omega_{m,0}).
$$

(B.15)

This expression is consistent with Eq. (4.23) of the one-fluid model if we set $A = B\epsilon_0(1 - \Omega_{m,0})$, except that the two-fluid model does not determine the value of the constant $A$, contrary to the one-fluid model. Indeed, the Planck scale $\rho_P$ does not occur in Eq. (B.11), unlike Eq. (4.16). This is a huge advantage of the one-fluid model [30].

B.2 The case $n = 2$

For $n = 2$, Eq. (B.5) reduces to

$$
\epsilon_{de} = -2A_2 \left( 1 \mp \sqrt{\frac{P_{de}}{A_2}} + \frac{P_{de}}{2A_2^2} \right)
$$

(B.16)

with $A_2 < 0$. This yields a second degree equation for $\sqrt{P_{de}/A_2}$ of the form

$$
\frac{P_{de}}{A_2} \mp 2 \sqrt{\frac{P_{de}}{A_2}} + 2 + \frac{\epsilon_{de}}{A_2} = 0.
$$

(B.17)

When $\rho_m > \rho_P$, Eq. (B.17) with the lower sign has just one solution

$$
\sqrt{-\frac{P_{de}}{|A_2|}} = -1 + \sqrt{\frac{\epsilon_{de}}{|A_2|} - 1}.
$$

(B.18)

As $\rho_m$ decreases from $+\infty$ to $\rho_P$, the DE density $\epsilon_{de}$ decreases from $+\infty$ to $2|A_2|$ and the pressure $P_{de}$ increases from $-\infty$ to $0$ (see Sec. 4.4). This leads to the EoS

$$
P_{de} = -|A_2| \left( -1 + \sqrt{\frac{\epsilon_{de}}{|A_2|} - 1} \right)^2
$$

(B.19)

or, equivalently,

$$
P_{de} = -\epsilon_{de} + 2|A_2| \sqrt{\frac{\epsilon_{de}}{|A_2|} - 1},
$$

(B.20)

which is valid for $\epsilon_{de} \geq 2|A_2|$. For $\epsilon_{de} \to +\infty$, Eq. (B.20) reduces to

$$
P_{de} \simeq -\epsilon_{de} + 2|A_2|^{1/2} \sqrt{\epsilon_{de}}.
$$

(B.21)

When $\rho_m < \rho_P$, the solutions of Eq. (B.17) with the upper sign are

$$
\sqrt{-\frac{P_{de}}{|A_2|}} = 1 \pm \sqrt{\frac{\epsilon_{de}}{|A_2|} - 1}.
$$

(B.22)
As \( \rho_m \) goes from \( \rho_P \) to 0, the DE density \( \epsilon_{de} \) first decreases from \( 2|A_2| \) to \( |A_2| \) then increases to \( +\infty \) while the pressure \( P_{de} \) decreases from 0 to \( -\infty \) (see Sec. 4.4). Equation (B.17) has two solutions for \( |A_2| \leq \epsilon_{de} \leq 2|A_2| \), one solution (with the upper sign) for \( \epsilon_{de} \geq 2|A_2| \) and no solution for \( \epsilon_{de} \leq |A_2| \). This leads to the EoS

\[
P_{de} = -|A_2| \left( 1 \pm \sqrt{\frac{\epsilon_{de}}{|A_2|} - 1} \right)^2
\]

or, equivalently,

\[
P_{de} = -\epsilon_{de} \mp 2|A_2| \sqrt{\frac{\epsilon_{de}}{|A_2|} - 1},
\]

which is valid with the two signs for \( |A_2| \leq \epsilon_{de} \leq 2|A_2| \) and with the upper sign for \( \epsilon_{de} \geq 2|A_2| \). This is the EoS of DE corresponding to the GLDF in the case \( n = 2 \). For \( \epsilon_{de} \to +\infty \), Eq. (B.24) reduces to

\[
P_{de} \simeq -\epsilon_{de} - 2|A_2|^{1/2} \sqrt{\epsilon_{de}},
\]

which coincides with the asymptotic EoS (A.7) of the GLDF seen as a one-fluid (UDM) model.

Let us check that the two-fluid model returns the results of the one-fluid model. Integrating the energy conservation equation (B.6) for DE with the EoS (B.20) or (B.24) we obtain

\[
\frac{\epsilon_{de}}{|A_2|} = 1 + 9 \ln^2 \left( \frac{a}{a_*} \right),
\]

where \( a_* \) is a constant of integration. If we add the contribution of pressureless DM, we find that the total energy density is given by

\[
\epsilon = \frac{\Omega_m \epsilon_0}{a^3} + |A_2| \left[ 1 + 9 \ln^2 \left( \frac{a}{a_*} \right) \right].
\]

Using the condition (at \( a = 1 \))

\[
\epsilon_0(1 - \Omega_{m,0}) = |A_2| \left( 1 + 9 \ln^2 a_* \right),
\]

we can rewrite Eq. (B.27) under the form

\[
\epsilon = \frac{\Omega_m \epsilon_0}{a^3} + \epsilon_0(1 - \Omega_{m,0}) \left[ 1 + \frac{9|A_2| \ln^2 a}{\epsilon_0(1 - \Omega_{m,0})} + \frac{6|A_2| \ln a}{\epsilon_0(1 - \Omega_{m,0})} \sqrt{\frac{\epsilon_0(1 - \Omega_{m,0})}{|A_2|}} - 1 \right]
\]

This expression is consistent with Eq. (4.31) of the one-fluid model if we set \( |A_2|/|\epsilon_0(1 - \Omega_{m,0})| = B^2/(1 + B^2) \), except that the two-fluid model does not determine the value of the constant \( A_2 \), contrary to the one-fluid model. This is a huge advantage of the one-fluid model [30].
C Present proportion of dark matter and dark energy

In this Appendix we recall the argument of [33] leading to a prediction of the present proportion of DM and DE in the universe and take into account the presence of baryons. The original logotropic model [30, 31, 33, 39] is based on the EoS

\[ P = A \ln \left( \frac{\rho_{dm}}{\rho_P} \right), \]

where \( \rho_{dm} \) is the rest-mass density of the LDF, \( A \) is a new fundamental constant of physics (superseding the cosmological constant) and \( \rho_P \) is the Planck density. This EoS provides a unification of DM and DE. It is very interesting that the Planck density appears in this EoS in order to make the variable in the logarithm dimensionless. This implies that quantum effects play a certain role in the late universe where the logotropic model applies. The rest-mass density evolves as [see Eq. (3.14)]

\[ \rho_{dm} = \frac{\Omega_{dm,0} \epsilon_0}{a^3}, \]

and it plays the role of DM. As a result \( \Omega_{dm,0} \) is interpreted as the present proportion of DM in the universe. The energy density of the LDF is [see Eq. (4.17)]

\[ \epsilon_{df} = \rho_{dm} c^2 - A \left[ 1 + \ln \left( \frac{\rho_{dm}}{\rho_P} \right) \right] = \epsilon_{dm} + \epsilon_{de}, \]

where the first term (rest-mass) is interpreted as DM and the second term (internal energy) as DE. We must also include the contribution of baryons which form a pressureless gas (\( P_b = 0 \)). Their energy density evolves as

\[ \epsilon_b = \frac{\Omega_{b,0} \epsilon_0}{a^3}, \]

where \( \Omega_{b,0} \) is the present proportion of baryons in the universe. The total energy density is therefore

\[ \epsilon = \rho_{dm} c^2 - A \left[ 1 + \ln \left( \frac{\rho_{dm}}{\rho_P} \right) \right] + \epsilon_b. \]

Substituting Eq. (C.2) into Eq. (C.5), we get

\[ \epsilon = \frac{\Omega_{dm,0} \epsilon_0}{a^3} - A \left[ 1 + \ln \left( \frac{\Omega_{dm,0} \epsilon_0}{\rho_P c^2 a^3} \right) \right] + \frac{\Omega_{b,0} \epsilon_0}{a^3}. \]

Applying this relation at the present time (\( a = 1 \)) and introducing the present proportion of DE in the universe \( \Omega_{de,0} = 1 - \Omega_{dm,0} - \Omega_{b,0} \) we get

\[ A = \frac{\Omega_{de,0} \epsilon_0}{\ln \left( \frac{\rho_P c^2}{\rho_{dm,0} a^3} \right) - 1}. \]

Introducing the present DE density \( \rho_\Lambda \equiv \Omega_{de,0} \epsilon_0/c^2 \), we can rewrite the foregoing equation as

\[ A = \frac{\rho_\Lambda c^2}{\ln \left( \frac{\rho_P}{\rho_\Lambda} \right) + \ln \left( \frac{\Omega_{de,0}}{1-\Omega_{de,0}-\Omega_{b,0}} \right) - 1}. \]
We now postulate that [33]

\[ A = \frac{\rho_{\Lambda} c^2}{\ln \left( \frac{\rho_P}{\rho_{\Lambda}} \right)} \]  

(C.9)

or, equivalently, that \( B = \ln(\rho_P/\rho_{\Lambda}) \). This implies

\[ \frac{\Omega_{de,0}}{1 - \Omega_{de,0} - \Omega_{b,0}} = e, \]

(C.10)

determining the present proportion of DM and DE [33]

\[ \Omega_{de,0}^{th} = \frac{e}{1 + e}(1 - \Omega_{b,0}), \quad \Omega_{dm,0}^{th} = \frac{1}{1 + e}(1 - \Omega_{b,0}). \]

(C.11)

If we neglect baryonic matter \( \Omega_{b,0} = 0 \) we obtain the pure numbers \( \Omega_{de,0}^{th} = \frac{e}{1 + e} = 0.731059... \) and \( \Omega_{dm,0}^{th} = \frac{1}{1 + e} = 0.268941... \) which give the correct proportions 70% and 25% of DE and DM [33]. If we take baryonic matter into account and use the measured value of \( \Omega_{b,0} = 0.0486 \pm 0.0010 \), we get \( \Omega_{de,0} = 0.6955 \pm 0.0007 \) and \( \Omega_{dm,0} = 0.2559 \pm 0.0003 \) which are very close to the observed values \( \Omega_{de,0} = 0.6911 \pm 0.0062 \) and \( \Omega_{dm,0} = 0.2589 \pm 0.0057 \) within the error bars. The postulate from Eq. (C.9) means that the fundamental constant \( A \) is equal to the present DE energy density (more precisely \( \rho_{\Lambda} c^2/\ln(\rho_P/\rho_{\Lambda}) \)). This can be viewed as a strong cosmic coincidence [33] giving to our epoch a central place in the history of the universe.

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