Casimir energy for spherical shell in Schwarzschild black hole background

1,2M.R. Setare∗and 3M.B. Altaie†

1Physics Dept. Inst. for Studies in Theo. Physics and Mathematics(IPM)
P. O. Box 19395-5531, Tehran, IRAN
2Department of Science, Physics group, Kordestan University, Sanandeg, Iran
3Department of Physics, Yarmouk University, Irbid-Jordan

March 24, 2022

Abstract

In this paper, we consider the Casimir energy of massless scalar field which satisfy Dirichlet boundary condition on a spherical shell. Outside the shell, the spacetime is assumed to be described by the Schwarzschild metric, while inside the shell it is taken to be the flat Minkowski space. Using zeta function regularization and heat kernel coefficients we isolate the divergent contributions of the Casimir energy inside and outside the shell, then using the renormalization procedure of the bag model the divergent parts are cancelled, finally obtaining a renormalized expression for the total Casimir energy.

∗E-mail: rezakord@ipm.ir
†E-mail: maltaie@yu.edu.jo
1 Introduction

The Casimir effect is one of the most interesting manifestations of nontrivial properties of the vacuum state in quantum field theory [1,2]. The Casimir effect can be viewed as the polarization of vacuum by the boundary conditions or geometry. Therefore, vacuum polarization induced by a gravitational field is also considered as Casimir effect. Since its first prediction by Casimir in 1948 [3], this effect has been investigated for different fields in different background geometries [4-7]. There is several methods for calculating Casimir energy. For instance, we can mention mode summation, Green’s function method [1], heat kernel method [6, 8] along with appropriate regularization schemes such as point separation [9], dimensional regularization [11], zeta function regularization [12, 4, 5]. Recently a general new methods to compute renormalized one–loop quantum energies and energy densities are given in [13, 14].

It has been shown [15, 16] that particle creation by black hole in four dimension is a consequence of the Casimir effect for spherical shell. Also it has been shown that the only existence of the horizon and of the barrier in the effective potential is sufficient to compel the black hole to emit black-body radiation with temperature that exactly coincides with the standard result for Hawking radiation. In [16], the results for the accelerated-mirror have been used to prove the above statement. To see more about relation between moving mirrors and black holes refer to [17]

Another relation between Casimir effect and Schwarzschild black hole thermodynamic is the thermodynamic instability. Widom et al [18, 19] showed that the black hole capacity is negative, then an increase in its energy decreases its temperature. They also showed that the electrodynamic Casimir effect can also produce thermodynamic instability.

The renormalized vacuum expectation value of the stress tensor of the scalar field in the Schwarzschild spacetime can be obtained by using different regularization methods.( see Refs. [20, 21, 22, 23, 24, 25]). \( <T_{\mu \nu}>_{\text{ren}} \) is needed, for instance, when we want to study back-reaction, i.e., the influence that the matter field in a curved background assert on the background geometry itself. This would be done by solving the Einstein equations with the expectation value of the energy-momentum tensor as source.

Regarding the Nugayev papers [15, 16], we would like to investigate the Casimir energy of massless scalar field which is conformally coupled to the Schwarzschild spacetime and satisfies Dirichlet boundary condition on a spherical shell. Casimir effect for spherical shells in the presence of the electromagnetic fields has been calculated several years ago [26, 27, 28]. The dependence of Casimir energy on the dimensions of the space for electromagnetic and scalar fields with Dirichlet boundary conditions in the presence of a spherical shell is discussed in [29, 30]. The Casimir energy for odd and even space dimensions and different fields, including the spinor field, and all the possible boundary conditions have been considered in [31]. There it is explicitly shown that although the Casimir energy for interior and exterior of a spherical shell are both divergent, irrespective of the number of space dimensions, the total Casimir energy of the shell remains finite for the case of odd space dimensions (see also [32]). Of some interest are cases where the field is confined to the inside of a spherical shell. This is sometimes called the bag boundary condition. The application of Casimir effect to the bag model is considered for the case of massive scalar field [34] and the Dirac field [35]. We will utilize the renormalization procedure used in the above cases for our problem.

The curvature effects in Schwarzschild background are well studied through various topics,
but the effects of boundaries do not seem to be so generally familiar. The Casimir energy for the massless scalar fields of two parallel plates in a two-dimensional Schwarzschild black hole with Dirichlet boundary conditions has been calculated in Ref. [36].

In this paper we would like to investigate the Casimir energy of massless scalar field for a spherical shell with Dirichlet boundary condition. Outside the shell, the spacetime is described by the Schwarzschild metric, while inside the shell is flat Minkowski space. The heat kernel and zeta function will be utilized to investigate the divergent parts of the vacuum energy. Heat kernel coefficients and zeta function of the Laplace operator on a manifold with different boundary conditions, both of them useful tools to calculate Casimir energies, have been calculated in [7, 6].

The paper is organized as follows: in the second section we briefly review the Casimir energy inside and outside of spherical shell in terms of zeta function. Then in section 3 we obtain the heat kernel coefficients for massless scalar field inside and outside of spherical shell, then we obtain the divergent part of Casimir energy inside and outside of shell separately. Section 4 is devoted to the conclusions.

2 Casimir energy inside and outside of spherical shell

In what follows as a boundary configuration we shall consider a spherical shell, outside the shell we consider the spacetime to be described by the Schwarzschild metric which has the form

\[ d^2s = -(1 - 2\frac{m}{r}) \, dt^2 + \left(1 - 2\frac{m}{r}\right)^{-1} \, dr^2 + r^2(d\theta^2 + \sin^2 \varphi \, d\varphi^2), \tag{1} \]

while inside the shell the spacetime is the flat Minkowski space. We shall consider the conformally coupled massless real scalar field \( \phi \), which satisfies

\[ \left( \Box + \frac{1}{6}R \right) \phi = 0, \quad \Box = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \right), \tag{2} \]

and propagates inside and outside the shell, \( R \) is scalar curvature which is zero in both the Schwarzschild and Minkowski backgrounds. For the point on spherical shell the scalar field obeys Dirichlet boundary condition

\[ \phi(r = a) = 0, \tag{3} \]

where \( a \) is the radius of spherical shell.

The quantization of the field described by Eq. (2) on the background of Eq. (1) is standard. Let \( \phi_{\alpha}^{(\pm)}(x) \) be complete set of orthonormalized positive and negative frequency solutions to the field equation (2), obeying boundary conditions (3). The canonical quantization can be done by expanding the general solution of Eq. (2) in terms of \( \phi_{\alpha}^{(\pm)} \),

\[ \phi = \sum_\alpha (\phi_{\alpha}^{+} a_{\alpha} + \phi_{\alpha} a_{\alpha}^{+}) \tag{4} \]

and declaring the coefficients \( a_{\alpha}, a_{\alpha}^{+} \) as operators satisfying the standard commutation relation for bosonic fields. The vacuum state \( |0> \) is defined as \( a_{\alpha}|0> = 0 \). This state is different from the vacuum state for black hole geometry without boundaries, \( |\bar{0}> \). A black
hole emits particles like a hot body at a temperature $\frac{\hbar}{\kappa}$ where $\kappa$ is a surface gravity of the black hole. Therefore, we have to consider the Hartle-Hawking state $|\bar{0}\rangle = |H\rangle$, this state is not empty at infinity, even in the absence of boundary conditions on the quantum field, but it corresponds to a thermal distribution of quanta at the Hawking temperature $T = \frac{\hbar}{8\pi m}$. In fact, the state $|H\rangle$ is related to a black hole in equilibrium with an infinite reservoir of black body radiation.

The quantum field has a ground state energy

$$E_0 = \frac{1}{2} \sum_k \lambda_k^{1/2},$$

(5)

where the $\lambda_k$'s are the one-particle energies with the quantum number $k$. The vacuum energy is divergent and we shall regularize it by

$$E_0 = \frac{1}{2} \sum_k \lambda_k^{1/2-s} \mu^{2s}, \quad Res > 2$$

(6)

where $\mu$ is an arbitrary mass parameter. It is similar to the subtraction point in the renormalization of perturbative quantum field theory. After renormalization the ground state energy will become independent of $\mu$. The one-particle energies are determined by the eigenvalue equation

$$-\Delta \varphi_k = \lambda_k \varphi_k.$$  

(7)

For the calculations we use the corresponding zeta function

$$\zeta_A(s) = \sum_k \lambda_k^{-s},$$

(8)

where operator $A$ is given by

$$A = -\Box.$$  

(9)

Therefore the regularized vacuum energy inside and outside the spherical shell are given by

$$E_{in}^{reg} = \frac{1}{2} \zeta_A^{in}(s - 1/2)\mu^{2s}, \quad E_{out}^{reg} = \frac{1}{2} \zeta_A^{out}(s - 1/2)\mu^{2s}.$$  

(10)

### 3 Zeta function and Heat-Kernel coefficients

The general structure of the ultraviolet divergencies can be obtained from the heat kernel expansion. For this reason one can represent the zeta function in Eq.(8) by an integral

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} K(t),$$

(11)

where

$$K(t) = (4\pi t)^{-3/2} \sum_k \exp(-\lambda_k t),$$

(12)

is the heat kernel. Now the ultraviolet divergencies of the vacuum energy are determined from the behaviour of the integrand in Eq.(11) at the lower integration limit and, hence, from the asymptotic expansion of the heat kernel for $t \to 0$

$$K(t) \sim \frac{1}{(4\pi t)^{3/2}} \sum_{k=0,1/2,1,...} B_k t^k.$$  

(13)
This expansion is known for a very general manifold, if the underlying manifold is without boundary, only coefficients with integer numbers enter, otherwise half integer powers of $t$ are present. The $B_k$ are given by

$$B_k = \int_M d\nu_k(x) + \int_{S^2} d\sigma_k(y),$$

(14)

the Seely-de Witt coefficients $a_k(x)$ vanish for half-odd integers, these coefficients are independent of the applied boundary condition, but the coefficients do depend on the spin of the field in question [7, 33]. The coefficients $c_k$ are functions of the second fundamental form of the boundary (extrinsic curvature), the induced geometry on the boundary (intrinsic curvature), and the nature of boundary conditions imposed. The simplest first of $a_k$ and $c_k$ coefficients for a manifold with boundary are given in [7]

$$a_0(x) = 1,$$

(15)

$$a_1(x) = \left(\frac{1}{6} - \xi\right)R,$$

(16)

$$a_2(x) = \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R^R_{\alpha\beta} - \frac{1}{6} (1/5 - \xi) \Box R + \frac{1}{2} (1/6 - \xi)^2 R^2,$$

(17)

where $\xi$ is the coupling constant between the scalar field and the gravitational field, for conformally coupling $\xi = 1/6$, $R_{\alpha\beta\gamma\delta}$, $R_{\alpha\beta}$ and $R$ are respectively, Riemann, Ricci and scalar curvature tensors. The $c_k$ coefficients for Dirichlet boundary condition are as follow [7]

$$c_0 = 0,$$

(18)

$$c_{1/2} = -\frac{\sqrt{\pi}}{2},$$

(19)

$$c_1 = \frac{1}{3} K - \frac{1}{2} f^{(1)},$$

(20)

$$c_{3/2} = \frac{\sqrt{\pi}}{2} \left(\frac{1}{6} \hat{R} - \frac{1}{4} R_{ik} N^i N^k + \frac{3}{32} (tr K)^2 - \frac{1}{16} tr K^2\right) + \frac{5}{16} tr K f^{(1)} - \frac{1}{4} f^{(2)},$$

(21)

$$c_2 = \frac{1}{3} \left(\frac{1}{6} - \xi\right) R (tr K) + \frac{1}{3} \left(\frac{3}{20} - \xi\right) \nabla_i N^i - \frac{1}{90} R_{ik} N^i N^k (tr K) + \frac{1}{30} R_{ij} N^i N^j K^k (22)$$

$$- \frac{1}{90} R_{il} K^{il} + \frac{1}{315} \left(\frac{5}{3} (tr K)^3 - 11 (tr K) (tr K^2) + \frac{40}{3} (tr K^3)\right) + \frac{1}{15} \Box (tr K)$$

(22)

where $f^{(i)}$ are the $i$'th normal derivative of the function $f$, $R_{ijkl}$ $R_{ik}$ and $\hat{R}$ are respectively Riemann, Ricci and the scalar curvature on the boundary, $K$ is extrinsic curvature tensor on the boundary

$$K_{ij} = \nabla_i N_j,$$

(23)

where $N_j$ is unit normal vector.

Given the expression in Eq.(11), it is easy to isolate the pole in $\zeta_A(s)$ since

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^1 dt t^{s-1} K(t) + \frac{1}{\Gamma(s)} \int_1^\infty dt t^{s-1} K(t)$$

(24)
Due to the exponential fall of $K(t)$ for large $t$, it is clear that the second term in the above expression is perfectly finite function of complex $s$. Observe that asymptotic expansion Eq.(11) implies that $\zeta_A(s)$ has a pole structure given by

$$\zeta_A(s) = \frac{1}{(4\pi)^{3/2}\Gamma(s)} \sum_{k=0,1/2,1,\ldots} B_k \frac{B_k}{k + s - 3/2} + \text{finite}$$  \hspace{1cm} (25)$$

Thus $\zeta_A(s)$ has a simple pole whenever $s = 3/2 - k$, except at $s = 0$ where any possible pole is cancelled by that of $\Gamma(s)$. The residue of the pole is given by

$$\frac{B_{3/2-s}}{(4\pi)^{3/2}\Gamma(s)}.$$  \hspace{1cm} (26)$$

However, $\zeta_A(s)$ is analytic at $s = 0$, and one can calculate simply the values of $\zeta$–function and its derivative at this point. Now, in order to determine the Casimir energy inside and outside the spherical shell, we must set $s = -1/2$ and we have a pole with nonzero residue if $B_2 \neq 0$. Then for the case of a massless free scalar field, the only remaining contributions are from $B_2$. These contribution for inside and outside the shell are divergent. Considering only the inner space, divergence appear and it is necessary to introduce contact term and perform a renormalization of its coupling. Result for massive scalar field contain new ultraviolet divergent terms in addition to that occurring in the massless case as has been discussed in [34]. However, when we consider both region of space, for free massless scalar field in flat space divergent part inside and outside the shell cancel out each other, then we do not need to introduce contact term, but for massless scalar field in curved space, similar to free massive case, when we add the interior and exterior energies to each other, there will be contributions which are divergent [37, 38]. For the case of a massless scalar field in curved space the divergent part of vacuum energy in zeta function regularization is proportional with $B_2^{\text{tot}}$ which is

$$B_2^{\text{tot}} = B_2^{\text{in}} + B_2^{\text{out}}.$$  \hspace{1cm} (27)$$

Now using Eqs.(10) and (25) we can write

$$E_{\text{div}}^{\text{in}} = \frac{\mu^{2s}}{2(4\pi)^{3/2}\Gamma(s - 1/2)} B_2^{\text{in}},$$  \hspace{1cm} (28)$$

Similarly for outside region

$$E_{\text{div}}^{\text{out}} = \frac{\mu^{2s}}{2(4\pi)^{3/2}\Gamma(s - 1/2)} B_2^{\text{out}}.$$  \hspace{1cm} (29)$$

Since the inside region is assumed to be flat and since the outside space is considered to be a Schwarzschild background, therefore

$$a_2^{\text{in}}(x) = 0,$$  \hspace{1cm} (30)$$

and

$$a_2^{\text{out}}(x) = \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}.$$  \hspace{1cm} (31)$$
The coefficients $c_2$ contains only odd powers of the second fundamental form $K$, if we consider infinitely thin boundary, which means that boundary consist of two oppositely oriented faces separated by an infinitesimal distance, then the second fundamental forms are equal and opposite on the two face of the boundary, and consequently we have \[ c_{2}^{in} + c_{2}^{out} = 0. \] (32)

Therefore
\[
B_2^{tot} = B_2^{in} + B_2^{out} = \int_M a_2^{out}(x)dv. \] (33)

Then the total divergent energy is given by
\[
E_{div}^{tot} = \frac{\mu^{2s}}{2(4\pi)^{3/2}\Gamma(s-1/2)}(B_2^{in} + B_2^{out}) = \frac{\mu^{2s}}{2(4\pi)^{3/2}\Gamma(s-1/2)} \int_M a_2^{out}(x)dv, \] (34)

Therefore the Casimir energy for this general case becomes divergent. At this stage we recall that $E_0$, as given by Eq.(6), is only one part of total energy. There is also a classical part. The total energy of the shell maybe written as
\[
E^{tot} = E_0 + E_{class} \] (35)

We can try to absorb $E_{div}$ into the classical energy. This technique of absorbing an infinite quantity into a renormalized physical quantity is familiar in quantum field theory and quantum field theory in curved space [7]. Here, we use a procedure similar to that of bag model [34, 35], there is some history of such notions going back to Milton paper [32], (to see application of this renormalization procedure in Casimir effect problem in curved space refer to [37, 38, 39, 40]). The classical energy of spherical shell may be written as,
\[
E_{class} = Pa^3 + \sigma a^2 + Fa + K + \frac{h}{a}, \] (36)

where $P$ is pressure, $\sigma$ is surface tension and $F, K, h$ do not have special names. The classical energy is expressed in a general dimensionally suitable form which depends on power of $a$, this definition is useful for its renormalization. In order to obtain a well defined result for the total energy, we have to renormalize only pressure of classical energy according to the below:
\[
P \rightarrow P = P - \frac{\mu^{2s}}{2(4\pi)^{3/2}a^3\Gamma(s-1/2)} \int_M a_2^{out}(x)dv. \] (37)

According to the renormalization procedure, we have to subtract from regularized expression for vacuum energy Eq.(10) the above divergent term. After subtracting this contribution from $E_0$ we denote it by
\[
E_0^{ren} = E_0 - E_{div}^{tot}, \] (38)

where
\[
E_0^{ren} = E_0^{(in)ren} + E_0^{(out)ren} \] (39)

The renormalized Casimir energy inside of the spherical shell for massless free scalar field with Dirichlet boundary conditions in flat Minkowski space is given by [31]
\[
E_0^{(in)ren} = \frac{0.008873}{2a}. \] (40)
But for outside the shell in our problem we have

\[ E^{(\text{out})\text{ren}}_0 = E^{\text{out}}_0 - E^{\text{out}}_{\text{div}} = \frac{1}{2} \zeta_a \gamma^{\text{out}}(s - 1/2) \mu^{2s} - \frac{\mu^{2s}}{2(4\pi)^{3/2}} \frac{1}{\Gamma(s - 1/2)} \int_M a^{\text{out}}_2(x) dv \]  

(41)

Therefore we can write the renormalized vacuum energy for the considered system as

\[ E^{\text{ren}}_0 = \frac{0.008873}{2a} + \frac{1}{2} \zeta_a \gamma^{\text{out}}(s - 1/2) \mu^{2s} - \frac{\mu^{2s}}{2(4\pi)^{3/2}} \frac{1}{\Gamma(s - 1/2)} \int_M a^{\text{out}}_2(x) dv. \]  

(42)

4 conclusions

In this paper we have developed a systematic approach to the calculation of the Casimir energy of a massless scalar field in the presence of a spherical shell as a boundary configuration. The spacetime outside the shell is described by the Schwarzschild metric, while inside the shell it is the flat Minkowski space. For the point on the spherical shell, the scalar field obeys Dirichlet boundary condition.

The renormalized vacuum expectation value of the stress tensor of the scalar field in the curved spacetime, is needed for instance, when we want to study back-reaction, i.e., the influence that the matter field in a curved background assert on the background geometry itself. It has been shown [15, 16] that particle creation by black hole in four dimension is as a consequence of the Casimir effect for spherical shell. It has been shown that the only existence of the horizon and of the barrier in the effective potential is sufficient to compel the black hole to emit black-body radiation with temperature that exactly coincides with the standard result for Hawking radiation. In [16], the results for the accelerated-mirror have been used to prove above statement. Regarding the Nugayev papers [15, 16], we have investigated the Casimir energy of massless scalar field which is conformally coupled to the Schwarzschild spacetime and satisfies Dirichlet boundary condition on a spherical shell.

Using zeta function regularization and heat kernel coefficients we obtain the divergent contributions for the Casimir energy inside and outside the shell. When we consider both region of space, for free massless scalar field in flat space, the divergent parts inside and outside cancel out each other, then we do not to introduce a contact term, but for massless scalar field in curved space, similar to free massive case, when we add the interior and exterior energies to each other, there are contributions which are divergent. For a massless scalar field the divergent part of vacuum energy in zeta function regularization is proportional to \( B^{\text{tot}} \), then the renormalization procedure become necessary in this situation. For this purpose one must introduce the classical energy and try to absorb the divergent part into it. In this paper we used a procedure similar to that of bag model [34, 35] for renormalization, according to which we have to subtract from regularized expression for vacuum energy in Eq.(10) the divergent term, consequently we obtained the renormalized vacuum energy for considered system given by Eq.(42).

References

[1] G. Plunien, B. Mueller, W. Greiner, Phys. Rep. 134, 87, (1986)
[2] V. M. Mostepanenko and N. N. Trunov. The Casimir Effect and its Applications. (Oxford Science Publications, New York, 1997)

[3] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793, (1948)

[4] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, Zeta Regularization Techniques with Applications (World Scientific, Singapore, 1994)

[5] E. Elizalde, Ten Physical Applications of Spectral Zeta Functions, Lecture Notes in Physics (Springer Verlag, Berlin, 1995)

[6] K. Kirsten, Spectral Functions in Mathematics and Physics, Chapman and Hall/CRC, Boca Raton, FL, 2001

[7] N. D. Birrell and P. C. W. Davies, Quantum Fields in Curved Space, (Cambridge University Press, 1986)

[8] K. Bormann and Antonsen "The Casimir effect of curved spacetime", hep-th/9608142.

[9] S. M. Christensen, Phys. Rev. D14, 2490(1976); 17, 946,(1978).

[10] S. L. Adler, J. Lieberman and Y. J. Ng, Ann. Phy. (N.Y) 106, 279(1977).

[11] S. Deser, M. J. Duff and C. J. Isham, Nucl. Phys B11, 45 (1976), see also D. M. capper and M. J. Duff, Nuovo Cimento 23A, 173(1974); Phys. Lett.53A, 361(1975).

[12] S. W. Hawking, Commun. Math. Phys. 55, 133(1977).

[13] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, Nucl.Phys.B645, 49, (2002).

[14] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra, H. Weigel, hep-th/0207205.

[15] R. M. Nugayev, V. I. Bashkov, Phys. Lett. 69A, 385, (1979)

[16] R. M. Nugayev, Phys. Lett. 91A, 216, (1982).

[17] P. C. W. Davies and S. A. Fulling, Proc. R. Soc. Lond. A.356, 237, (1977)

[18] A. Widom, E. Sassaroli, Y.N.Srivastava and J.Swain. The Casimir effect and thermodynamic instability. quant-ph/9803013

[19] E. Sassaroli, Y. N. Srivastava, J. Swain and A. Widom. The dynamical and static Csimir effect and the thermodynamic instability. hep-ph/9805479

[20] S. M. Christensen and S. A. Fulling, phys. Rev. D15, 2088, (1977)

[21] F. Antonsen, gr-qc/9710100.

[22] R. Balbinot and A. Fabbri, Phys. Lett. B459 112, (1999).
[23] R. Balbinot, A. Fabbri, V. Frolov, P. Nicolini, P. Sutton and A. Zelniko, Phys.Rev. 
D63 084029 (2001).

[24] J. Matyjasek, Acta Phys. Polon. B30 971, (1999).

[25] J. Matyjasek, Phys. Rev. D59 044002, (1999).

[26] T. H. Boyer, Phys. Rev. 174, 1764, (1968).

[27] R. Balian and B. Duplantier, Ann. Phys. (N.Y). 112, 165, (1978).

[28] K. A. Milton, L. L. DeRaad, and J. Schwinger, Ann. Phys. (N.Y)115, 338, (1978).

[29] C. M. Bender, K. A. Milton, Phys. Rev. D50, 6547, (1994)

[30] K. A. Milton. Phys. Rev. D55, 4940, (1997).

[31] G. Cognola, E. Elizalde, K. Kirsten, J.Phys. A34 7311-7327, (2001).

[32] K. A. Milton, Ann. Phys. (N.Y). 127, 49 (1980); Phys. Rev. D22, 1441, (1980).

[33] S. Blau, M. Visser, and A. Wipf, Nucl. Phys. B310, 163, (1988).

[34] M. Bordag, E. Elizalde, K. Kirsten and S. Leseduarte. Phys. Rev. D56, 4896, (1997).

[35] E. Elizalde, M. Bordag and K. Kirsten. J. Phys. A: Math. Gen. 31, 1743, (1998).

[36] M. R. Setare, Class. Quant. Grav. 18, 2097, (2001).

[37] M. R. Setare, and R. Mansouri. Class. Quant. Grav. 18, 2331, (2001).

[38] M. R. Setare. Class. Quant. Grav. 18, 4823, (2001).

[39] E.R. Bezerra de Mello , V.B. Bezerra , N.R. Khusnutdinov, J.Math.Phys. 42 562-581, 
(2001).

[40] Nail R. Khusnutdinov, Sergey V. Sushkov,Phys.Rev. D65 084028, (2002).