Complex modes in optical fibers and silicon waveguides

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At a fixed frequency, a general wave field in an open optical waveguide is a sum of finite number of guided modes and an integral of radiation modes. Most studies on open lossless dielectric waveguides are concerned with regular guided modes which are confined around the waveguide core, have a real propagation constant, and propagate along the waveguide axis without attenuation. However, there may be guided modes with a complex propagation constant even though the waveguide consists of lossless isotropic dielectric materials, and they are the so-called complex modes. The complex modes are proper modes confined around the waveguide core, and are different from the leaky modes which have divergent wave fields in the transverse plane. For waveguides with discontinuities and defects, the complex modes can be excited and must be included in eigenmode expansions. However, existing studies on complex modes in open lossless dielectric waveguides are very limited. In this Letter, we consider circular fibers and silicon waveguides, study the formation mechanism of complex modes, and calculate the dispersion relations for several complex modes in each waveguide. For circular fibers, we also determine the minimum refractive-index ratio for the existence of complex modes. Our study fills a gap in optical waveguide theory and provides a basis for realizing potential applications of complex modes.

I. INTRODUCTION

For open lossless dielectric waveguides, a regular guided mode at a given real frequency is a proper eigenmode with a field confined around the waveguide core [1, 2]. It has a real propagation constant, carries a finite power, and propagates along the waveguide axis without attenuation. If the waveguide is lossy, any guided mode must have a complex propagation constant due to absorption loss, and it propagates forward with an exponentially decaying amplitude. A leaky mode [1–3], on the other hand, is an improper mode with a divergent field in the transverse plane. It radiates out power in lateral directions, has a complex propagation constant due to radiation loss, and also decays exponentially as it propagates forward. The so-called complex modes also have a complex propagation constant, but not because of absorption and radiation losses [4]. Instead, the appearance of complex propagation constant is related to the non-self-adjoint linear operators in full-vector waveguide eigenvalue formulations for a given real frequency. Complex modes were first discovered in closed waveguides supporting backward waves [5–7]. The existence of complex modes in open lossless dielectric waveguides was first demonstrated by Jabłoński for a high-index-contrast optical fiber [8]. It has been shown that complex modes also exist in silicon waveguides and they are related to the numerical instability of the full vectorial paraxial beam propagation method [9]. In addition, complex modes also exist in periodic waveguides [10] and in waveguides with anisotropic media or metamaterials [11, 12]. Guided modes with complex propagation constants also appear in PT-symmetric waveguides where the real and imaginary parts of the complex refractive index profile are even and odd (in a transverse variable) respectively [13–14], but they are different from the complex modes in lossless waveguides. Unlike the leaky modes, a complex mode in an open waveguide is a proper eigenmode with a field confined around the core. A unique property of complex modes is that they do not transport power along the waveguide axis. For waveguides with discontinuities or a local defects, complex modes can be excited and must be included in eigenmode expansions.

The existing studies on complex modes in open lossless dielectric waveguides are very limited. The example of Jabłoński is for a circular fiber with a very high index ratio (\(r = n_1/n_2 = 7.81\), where \(n_1\) and \(n_2\) are the refractive indices of the core and the cladding, respectively) [8]. For silicon waveguides, complex modes are only calculated for a single frequency [9]. In this Letter, we carry out a more detailed study on complex modes in circular fibers and silicon waveguides. We show that complex modes form bands, and each band is given by a propagation constant being a complex-valued function of the real frequency. For circular fibers, we also determine the minimum values for index ratio \(r = n_1/n_2\) beyond which complex modes of different azimuthal order appear.

II. THEORY

We consider a circular step-index optical fiber with a homogeneous cladding and a silicon waveguide on a silica substrate, and depict their cross sections in Fig. [1]. Let \(z\) be the variable along the waveguide axis, \(x\) and \(y\) be the transverse variables, we consider eigenmodes that depend on \(z\) and time \(t\) as \(e^{i(\beta z - \omega t)}\), where \(\omega\) is the angular frequency and \(\beta\) is the propagation constant. We are concerned with the eigenvalue formulation where \(\omega\) is real and given, and \(\beta\) (or \(\beta^2\)) is the eigenvalue [1, 2]. An eigenmode is designated as a proper one, if the field decays exponentially to zero as \(r = \sqrt{x^2 + y^2} \to \infty\). For a lossless waveguide, the regular guided modes are proper
eigenmodes with a real $\beta$. They form bands with starting points on the light line $\omega = (c/n_2)\beta$ and are given by their dispersion relations $\omega = \omega_j(\beta)$, where $c$ is the speed of light in vacuum and $j$ is an integer index. For the two waveguides shown in Fig. 1 it appears that all dispersion curves of the regular guided modes have a positive slope for a positive $\beta$, i.e., $d\omega_j/d\beta > 0$ for $\beta > 0$.

A leaky mode in an open waveguide must satisfy an outgoing radiation condition and decay to zero as it propagates forward (to $z = +\infty$). Therefore, the complex propagation constant $\beta$ of a leaky mode (depending on $z = e^{i\beta z}$) should satisfy $\text{Re}(\beta) > 0$ and $\text{Im}(\beta) > 0$. For both waveguides shown in Fig. 1 in the unbound regime with refractive index $n_2$, all electromagnetic field components of the eigenmode satisfy the same Helmholtz equation

$$\nabla^2 u + \beta^2 u = 0,$$

where $k_0 = \omega/c$ is the freespace wavenumber. For the complex $\beta$ of a leaky mode, $k_0^2 n_2^2 - \beta^2$ is in the lower half of the complex plane, $\kappa_2 = \sqrt{k_0^2 n_2^2 - \beta^2}$, by the standard complex square root function with a branch cut along the negative real axis, is in the fourth quadrant, then the outgoing radiation condition implies that

$$u \sim \frac{C_0(\theta)}{r} e^{i\kappa_2 r}, \quad r \to \infty,$$

where $\theta$ is the polar angle of $(x, y)$ and $C_0$ is a function depending on $\theta$. Since $\text{Im}(\kappa_2) < 0$, $u$ diverges as $r \to \infty$, thus a leaky mode is an improper mode with a divergent field in the transverse plane.

The complex modes found in closed waveguides many decades ago emerge from local maxima on the dispersion curves of regular guided modes [5, 7]. Some complex modes in periodic waveguides (with a periodicity along the waveguide axis) also emerge from such points where the group velocity is zero [11]. This implies that the associated dispersion curve (of regular guided modes) must contain a region with a negative slope corresponding to backward waves [11]. However, backward waves do not seem to exist in open lossless $z$-invariant waveguides consisting of ordinary isotropic dielectric materials. Therefore, a different mechanism is needed to realize complex modes in open lossless waveguides. It turns out that a leaky mode simply turns to a complex mode when the real part of the propagation constant $\beta$ passes zero. More specifically, if the originally positive $\text{Re}(\beta)$ of a leaky mode becomes negative as $\omega$ is varied, and $\text{Im}(\beta)$ remains positive, then $k_0^2 n_2^2 - \beta^2$ moves to the upper half plane, $\kappa_2$ moves to the first quadrant, and the field component $u$, satisfying condition (3), decays to zero as $r \to \infty$. Therefore, a leaky mode simply becomes a complex mode when $\text{Re}(\beta)$ changes sign. Since the propagation constant $\beta$ of the leaky mode is in the first quadrant, that of the complex mode is in the second quadrant. An open optical waveguide typically has an infinite sequence of leaky modes, and each leaky mode satisfies a dispersion relation with $\beta$ being a complex-valued function of $\omega$ [3]. If on a particular dispersion curve, there is an interval of $\omega$ in which $\text{Re}(\beta) < 0$, then, the eigenmode on that interval is actually a complex mode.

From the reciprocity principle, if the waveguide has a mode with propagation constant $\beta$, it must have another one with propagation constant $-\beta$. This implies that the propagation constants of the leaky and complex modes can also lie in the third and fourth quadrants, respectively. For a lossless waveguide with a real refractive index profile, we can take a complex conjugate for a complex mode (with propagation constant $\beta = \beta' + i\beta''$) satisfying the governing Maxwell’s equations, and obtain another complex mode with propagation constant $\beta = \beta' - i\beta''$. Together with the reciprocity, we conclude that complex modes appear in a group of four, with propagation constants $\beta' \pm i\beta''$ and $-\beta' \pm i\beta''$. If we take complex conjugate for a leaky mode (with propagation constant $\beta$) satisfying the Maxwell’s equations, we obtain an eigenmode with propagation constant $\beta$, but it is not a leaky mode, since the outgoing radiation condition is reversed. The obtained mode may be called absorbing mode, since it gains power from lateral incoming waves and propagates forward with an exponentially increasing amplitude. In summary, a pair of leaky modes (with propagation constants in first and third quadrants, respectively) give rise to two complex modes with propagation constants in the second and fourth quadrants, and a pair of absorbing modes give rise to the other two complex modes (with propagation constants in first and third quadrants).

### III. CIRCULAR FIBER

For the step-index circular fiber shown in Fig. 1a, the eigenmodes are classified into groups depending on the azimuthal order $\nu$. For a leaky or complex mode with a propagation constant $\beta$ in the upper half plane, and if the angular dependence is $\sin(\nu \theta)$ or $\cos(\nu \theta)$, then the field components in the cladding are proportional to $H_\nu^{(1)}(\kappa_2 r) \sin(\nu \theta)$ or $H_\nu^{(1)}(\kappa_2 r) \cos(\nu \theta)$, where $H_\nu^{(1)}$ is $\nu$th
order Hankel function of first kind. The above choice ensures that the asymptotic condition (2) is satisfied.

For \( \nu = 0 \), \( \beta \) satisfies the dispersion equation

\[
\frac{\kappa_2 H_{1}^{(1)}(\kappa_2 a)}{H_{1}^{(1)}(\kappa_2 a)} = \eta \frac{J_1(\kappa_1 a)}{J_1(\kappa_1 a)},
\]

(3)

where \( \kappa_1 = (k_0 a_1^2 - \beta^2)^{1/2} \), \( J_0 \) and \( J_1 \) are Bessel functions of first kind, \( \eta = 1 \) and \( \eta = \rho^2 \) for the TE and TM polarizations, respectively. It can be proved that Eq. (3) does not have any solution with Re\((\beta)\) = 0. If this is not true, i.e., there is a \( \beta \) such that Re\((\beta)\) = 0, then \( \beta^2 \leq 0 \), the right hand side of Eq. (3) is real, but the imaginary part of the left hand side is nonzero, because

\[
\text{Im} \left[ \frac{H_0^{(1)}(\kappa_2 a)}{H_{1}^{(1)}(\kappa_2 a)} \right] = \frac{Y_0(\kappa_2 a)J_1(\kappa_2 a) - J_0(\kappa_2 a)Y_1(\kappa_2 a)}{J_1^2(\kappa_2 a) + Y_1^2(\kappa_2 a)} \neq 0,
\]

where \( Y_0 \) and \( Y_1 \) are Bessel functions of second kind. In fact, for \( \nu = 0 \), the eigenmodes are separately in the TE and TM polarizations, the related eigenvalue problems are self-adjoint, and complex modes do not exist.

For \( \nu \neq 0 \), \( \beta \) can be solved from the following dispersion equation

\[
(\rho^2 J^- - H^-)(J^+ - H^+) = (H^+ - \rho^2 J^+)(J^- - H^-),
\]

(4)

where

\[
J^\pm = \frac{J_{\nu+1}(\kappa_1 a)}{ak_1 J_\nu(\kappa_1 a)}, \quad H^\pm = \frac{H_{\nu+1}^{(1)}(\kappa_2 a)}{ak_2 H_\nu^{(1)}(\kappa_2 a)}.
\]

Jablonski found a complex mode for a very large index ratio \( \rho = n_1/n_2 = 7.81 \). However, complex modes do exist in optical fibers with lower index contrast. In Fig. 2(a), (b), (c) and (d), we show the dispersion curves [hmax wavenumber \( k_0 \) vs. Re\((\beta)\)] of some leaky and complex modes for \( \nu = 1, 2, 3 \) and 4, respectively. The results are obtained for a fixed \( n_2 = 1 \) and \( n_1 = 2, 2.5, 3 \) and 3.5. The leaky and complex modes are shown together in each curve for positive and negative Re\((\beta)\), respectively. It can be seen that complex modes exist when \( \nu = 3 \) and 3.5 for \( \nu = 1 \), when \( \nu = 2, 2.5, 3 \) and 3.5 for \( \nu = 2 \), and when \( \nu = 2, 2.5, 3 \) and 3.5 for \( \nu = 3 \) and 4. In Fig. 3 we show the z component of the electric field for four complex modes marked as A, B, C and D in Fig. 2. The azimuthal dependence is assumed to be \( \cos(\nu \theta) \).

The above numerical results suggest that complex modes can exist in the optical fiber for each \( \nu \neq 0 \) whenever the index ratio \( \rho \) is larger than a critical value \( \rho_c \). For \( \rho = \rho_c \), Re\((\beta)\), as a function of \( k_0 \), has a minimum which is exactly zero. We can calculate the critical index ratio \( \rho_c \), \( k_0 \) and Im\((\beta)\) at the minimum from a system consisting of Eq. (4) and

\[
\text{Re} \left( \frac{d\beta}{dk_0} \right) = 0.
\]

(5)

The results are listed in Table I. In Fig. 4 we show four dispersion curves of the leaky modes for \( \rho = \rho_c \) and \( \nu = 1, 2, 3 \) and 4, respectively.
TABLE I. Critical values of index ratio $\rho = n_1/n_2$ for leaky mode branches reaching a zero minimum of $\text{Re}(\beta)$.

| $\nu$ | $\rho_\nu$ | $n_2k_0a$ | $\text{Im}(\beta)a$ |
|-------|-----------|------------|-----------------|
| 1     | 2.9474    | 0.5827     | 1.1513          |
| 2     | 2.2396    | 1.2753     | 1.8171          |
| 3     | 1.9574    | 2.0060     | 2.4092          |
| 4     | 1.7985    | 2.7620     | 2.9626          |

IV. SILICON WAVEGUIDE

In [9], the silicon waveguide is assumed to have a rectangular cross section with width $w = 500 \text{ nm}$, height $h = 250 \text{ nm}$, and refractive index $n_1 = 3.479$, and is placed on a substrate with refractive index $n_2 = 1.445$ and covered by air ($n_0 = 1$). A complex mode was calculated for free-space wavelength $\lambda_0 = 1.52 \mu\text{m}$. Similar to the case of the optical fiber, a complex mode is formed when a leaky mode reaches $\text{Re}(\beta) = 0$ at some frequency. In Fig. 5 we show three dispersion curves of leaky and complex modes, where $\text{Re}(\beta)$ passes through zero. For each dispersion curve, the complex modes exist in an interval of $k_0$ in which $\text{Re}(\beta) < 0$. The example obtained by Xie et al. [9] corresponds to point X in Fig. 5. In Fig. 5 we show field patterns for three complex modes marked as A, B, C in Fig. 5.

V. CONCLUSION

In summary, a complex mode in an open lossless dielectric waveguide is a proper eigenmode with a field confined around the waveguide core, has a complex propagation constant due to the non-self-adjoint nature of the waveguide eigenvalue problem formulated at a given real frequency, and it does not carry a power along the waveguide axis. We have calculated the dispersion curves for some complex modes in circular optical fibers and silicon waveguides, and found the minimum index ratio for
the existence of complex modes of various azimuthal order in circular fibers. Unlike the complex modes in closed waveguides that emerge from local maxima on dispersion curves of regular guided modes, the complex modes studied in this Letter are obtained from leaky modes when the real part of the complex propagation constant changes sign. While the study of optical waveguides has a long history, complex modes in open lossless dielectric waveguides are seldom mentioned in existing literature. Our study fills a gap in optical waveguide theory. It is worthwhile to further analyze the properties of the complex modes and explore their potential applications.

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