On the minimal feedback arc set of $m$-free Digraphs *

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Abstract
For a simple digraph $G$, let $\beta(G)$ be the size of the smallest subset $X \subseteq E(G)$ such that $G - X$ has no directed cycles, and let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in $G$. A digraph $G$ is called $m$-free if $G$ has no directed cycles of length at most $m$. This paper proves that $\beta(G) \leq \frac{1}{m-2} \gamma(G)$ for any $m$-free digraph $G$, which generalized some known results.

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1 Introduction

Let $G = (V, E)$ be a digraph without loops and parallel edges, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set.

It is well known that the cycle rank of an undirected graph $G$ is the minimum number of edges that must be removed in order to eliminate all of the cycles in the graph. That is, if $G$ has $v$ vertices, $\varepsilon$ edges, and $\omega$ connected components, then the minimum number of edges whose deletion from $G$ leaves an acyclic graph equals the

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cycle rank (or Betti number) $\rho(G) = \varepsilon - \nu + \omega$ (see Xu [15]). However, the same problem for a digraph is quite difficulty. In fact, the Betti number for a digraph was proved to be NP-complete by Karp in 1972 (see the 8th of 21 problems in [9]).

A digraph $G$ is called to be $m$-free if there is no directed cycle of $G$ with length at most $m$. We say $G$ is acyclic if it has no directed cycles. For a digraph $G$, let $\beta(G)$ be the size of the smallest subset $X \subseteq E(G)$ such that $G - X$ is acyclic, here $X$ is called a minimal feedback arc-set of $G$. Let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in $G$, called the number of missing edges of $G$.

Chudnovsky, Seymour, and Sullivan [4] proved that $\beta(G) \leq \gamma(G)$ if $G$ is a 3-free digraph and gave the following conjecture.

**Conjecture 1.1** If $G$ is a 3-free digraph, then $\beta(G) \leq \frac{1}{2} \gamma(G)$.

Concerning this conjecture, Dunkum, Hamburger, and Pór [5] proved that $\beta(G) \leq 0.88 \gamma(G)$. Very recently, Chen et al. [3] improved the result to $\beta(G) \leq 0.8616 \gamma(G)$. Conjecture 1.1 is closely related to the following special case of the conjecture proposed by Caccetta and Häggkvist [2].

**Conjecture 1.2** Any digraph on $n$ vertices with minimum out-degree at least $n/3$ contains a directed triangle.

Short of proving the conjecture, one may seek as small a value of $c$ as possible such that every digraph on $n$ vertices with minimum out-degree at least $cn$ contains a triangle. This was the strategy of Caccetta and Häggkvist [2], who obtained the value $c \leq 0.3819$. Bondy [1] showed that $c \leq 0.3797$, and Shen [11] improved it to $c \leq 0.3542$. By using a result of Chudnovsky, Seymour and Sullivan [4] related to Conjecture 1.1, Hamburger, Haxell, and Kostochka [6] further improved this bound to 0.35312. Namely, any digraph on $n$ vertices with minimum out-degree at least 0.35312$n$ contains a directed triangle.

More generally, Sullivan [14] proposed the following conjecture, and gave an example showing that this would be best possible if this conjecture is true. Conjecture 1.1 is the special case when $m = 3$.

**Conjecture 1.3** If $G$ is an $m$-free digraph with $m \geq 3$, then

$$\beta(G) \leq \frac{2}{(m+1)(m-2)} \gamma(G).$$

Sullivan proved partial results of Conjecture 1.3 and showed that $\beta(G) \leq \frac{1}{m-2} \gamma(G)$ for $m = 4, 5$. In this article, we prove the following theorem, which extends Sullivan’s result to more general $m$-free digraphs for $m \geq 4$.

**Theorem 1.4** If $G$ is an $m$-free digraph with $m \geq 4$, then $\beta(G) \leq \frac{1}{m-2} \gamma(G)$.
2 Some Lemmas

Let $G$ be a simple digraph. For two disjoint subsets $A, B \subseteq V(G)$, let $E(A, B)$ denote the set of directed edges from $A$ to $B$, that is, $E(A, B) = \{(a, b) | a \in A, b \in B\}$. Let $\bar{E}(A, B)$ be the missing edges between $A$ and $B$. It follows that

\[ |\bar{E}(A, B)| = |\bar{E}(B, A)| = |A| \cdot |B| - |E(A, B)| - |E(B, A)|. \]

A directed $(v_0, v_k)$-path $P$ in $G$ is a sequence of distinct vertices $(v_0, v_1, \cdots, v_{k-1}, v_k)$, where $(v_i, v_{i+1})$ is a directed edge for each $i = 0, \cdots, k - 1$, its length is $k$. Clearly, the subsequence $(v_1, \cdots, v_{k-1})$ is a $(v_1, v_{k-1})$-path, denoted by $P'$. We can denote $P = (v_0, P', v_k)$. A directed path $P$ is said to be induced if every edge in the subgraph induced by vertices of $P$ is contained in $P$.

For $v \in V(G)$, let $N^+_i(v)$ be the set of vertices $u$ such that the shortest directed $(v, u)$-path has length $i$. Similarly, let $N^-_i(v)$ be the set of vertices whose shortest directed path to $v$ has length $i$. An induced directed $(v_0, v_k)$-path is called to be shortest if $v_k \in N^+_k(v_0)$. From definition, we immediately have the following result.

**Lemma 2.1** If $(v_0, v_1, \cdots, v_{k-1}, v_k)$ is a shortest induced directed $(v_0, v_k)$-path, then for any $i$ and $j$ with $0 \leq i < j \leq k$,

\[ v_j \in N^+_{j-i}(v_i) \quad \text{and} \quad v_i \in N^-_{j-i}(v_j). \]

Let $\mathcal{P}(G)$ be the set of shortest induced directed paths of $G$, and $m$ be a positive integer with $m \geq 4$. Let $v \in V(G)$ and $k$ be an integer with $1 \leq k \leq m - 3$. For any $P \in \mathcal{P}(G)$ of length $k - 1$ and $x, y, z \in V(G)$, set

\[ P_k(v) = \{(x, y, z)| (x, P, y, z) \in \mathcal{P}(G), x = v\} \quad \text{and} \quad p_k(v) = |P_k(v)|, \]

\[ Q_k(v) = \{(x, y, z)| (x, P, y, z) \in \mathcal{P}(G), y = v\} \quad \text{and} \quad q_k(v) = |Q_k(v)|, \]

\[ R_k(v) = \{(x, y, z)| (x, P, y, z) \in \mathcal{P}(G), z = v\} \quad \text{and} \quad r_k(v) = |R_k(v)|. \]

\[ P'_k(v) = \{(x, y, z)| (x, y, P, z) \in \mathcal{P}(G), x = v\} \quad \text{and} \quad p'_k(v) = |P'_k(v)|, \]

\[ Q'_k(v) = \{(x, y, z)| (x, y, P, z) \in \mathcal{P}(G), y = v\} \quad \text{and} \quad q'_k(v) = |Q'_k(v)|, \]

\[ R'_k(v) = \{(x, y, z)| (x, y, P, z) \in \mathcal{P}(G), z = v\} \quad \text{and} \quad r'_k(v) = |R'_k(v)|. \]

**Lemma 2.2** For any integer $k$ with $1 \leq k \leq m - 3$ and $P \in \mathcal{P}(G)$ of length $k - 1$,

\[ \sum_{v \in V(G)} p_k(v) = \sum_{v \in V(G)} q_k(v) = \sum_{v \in V(G)} r_k(v), \quad (2.1) \]

and

\[ \sum_{v \in V(G)} p'_k(v) = \sum_{v \in V(G)} q'_k(v) = \sum_{v \in V(G)} r'_k(v). \quad (2.2) \]

**Proof:** For each integer $k$ with $1 \leq k \leq m - 3$ and $P \in \mathcal{P}(G)$ of length $k - 1$,

\[ \sum_{v \in V(G)} p_k(v), \sum_{v \in V(G)} q_k(v), \sum_{v \in V(G)} r_k(v) \]

are all equal to the number of triples $(x, y, z)$ of distinct vertices such that $(x, P, y, z) \in \mathcal{P}(G)$ for $P \in \mathcal{P}(G)$. Thus (2.1) holds. The proof of (2.2) is similar. \[ \blacksquare \]
Lemma 2.3 If $G$ is an $m$-free digraph, then for any $v \in V(G)$ and any integer $k$ with $1 \leq k \leq m - 3$,

$$
\begin{align*}
    p_k(v) &= |E(N_{k+1}^+(v), N_{k+2}^+(v))|, \\
    q_k(v) &\leq |E(N_{k+1}^+(v), N_1^+(v))|, \\
    r_k(v) &\leq |E(N_1^-(v), N_{k+2}^+(v))|, \\
    p_k'(v) &\leq |E(N_1^+(v), N_{k+2}^+(v))|, \\
    q_k'(v) &\leq |E(N_{k+1}^+(v), N_1^-(v))|, \\
    r_k'(v) &\leq |E(N_{k+2}^-(v), N_{k+1}^+(v))|.
\end{align*}
$$

Proof: By definition, for each edge $(u, w) \in E(N_{k+1}^+(v), N_{k+2}^+(v))$, there exists $v_i \in N_i^+(v)$, for each $i = 1, 2, \cdots k$, such that $(v, v_1, \cdots, v_k, u, w)$ is a directed $(v, w)$-path of length $k + 2$. Since $G$ is $m$-free and $1 \leq k \leq m - 3$, it is easy to see that $(v, v_1, \cdots, v_k, u, w)$ is a shortest induced directed path. It follows that $(v, u, w) \in P_k(v)$ and

$$
p_k(v) \geq |E(N_{k+1}^+(v), N_{k+2}^+(v))|. \tag{2.3}
$$

On the other hand, for each $(v, u, w) \in P_k(v)$, from the definition of $P_k(v)$ and Lemma 2.1 we have $u \in N_{k+1}^+(v)$ and $w \in N_{k+2}^+(v)$. Thus $(u, w) \in E(N_{k+1}^+(v), N_{k+2}^+(v))$. It follows that

$$
p_k(v) \leq |E(N_{k+1}^+(v), N_{k+2}^+(v))|. \tag{2.4}
$$

Combining (2.3) and (2.4), we have that $p_k(v) = |E(N_{k+1}^+(v), N_{k+2}^+(v))|$. The proof of $r_k'(v) = |E(N_{k+2}^-(v), N_{k+1}^+(v))|$ is similar.

For each $(u, w) \in Q_k(v)$, from the definition of $Q_k(v)$ and Lemma 2.1 we have $u \in N_{k+1}^+(v)$, $w \in N_1^+(v)$ and $uw \notin E(G)$. Since $G$ is $m$-free, we have $(w, u) \notin E(G)$. If not, there exists a directed cycle $(v, w, u, \cdots)$ with length $l = k + 3 \leq m$, a contradiction. So $(u, w) \in |E(N_{k+1}^-(v), N_1^+(v))|$. Thus, $q_k(v) \leq |E(N_{k+1}^- (v), N_1^+(v))|$. The proof of $q_k'(v) \leq |E(N_{k+1}^+(v), N_1^-(v))|$ is similar.

For each $(u, w, v) \in R_k(v)$, from the definition of $R_k(v)$ and Lemma 2.1 we have $u \in N_{k+2}^-(v)$, $w \in N_1^-(v)$ and $(u, w) \notin E(G)$. Since $G$ is $m$-free, $(w, u) \notin E(G)$. Otherwise, there exists a directed cycle $(w, u, \cdots, w)$ with length $l = k + 2 \leq m - 1$, a contradiction. Thus we have $(u, w) \in |E(N_1^-(v), N_{k+2}^+(v))|$. It derives that $r_k(v) \leq |E(N_1^-(v), N_{k+2}^+(v))|$. The proof of $p_k'(v) \leq |E(N_{k+1}^+(v), N_{k+2}^+(v))|$ is similar.

For any $v \in V(G)$ and any integer $k$ with $1 \leq k \leq m - 3$, set

$$
\alpha_k(v) = \frac{p_k(v)}{s_k(v)} \quad \text{and} \quad \beta_k(v) = \frac{r_k'(v)}{t_k(v)}.
$$

Here

$$
\begin{align*}
    s_k(v) &= \sum_{i=k}^{m-3} p_i'(v) + \sum_{i=1}^{k} q_i'(v) \quad \text{and} \quad t_k(v) = \sum_{i=k}^{m-3} r_i(v) + \sum_{i=1}^{k} q_i(v). \tag{2.5}
\end{align*}
$$

The result is obvious.
Lemma 2.4 If \( a_i \geq 0, b_i \geq 0 \) for each \( i = 1, 2, \cdots, n \), and \( \sum_{i=1}^{n} b_i > 0 \), then  

\[
\min_{1 \leq i \leq n} \left\{ \frac{a_i}{b_i} \right\} \leq \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}.
\]

Let  

\[
\alpha = \min_{v \in V(G), 1 \leq k \leq m-3} \{\alpha_k(v)\} \quad \text{and} \quad \beta = \min_{v \in V(G), 1 \leq k \leq m-3} \{\beta_k(v)\}.
\]

Applying Lemma 2.4 we obtain the following bound about \( \alpha \) and \( \beta \).

Lemma 2.5 If \( G \) is a \( m \)-free digraph, then  

\[
\min\{\alpha, \beta\} \leq \frac{1}{m-2}.
\]

Proof: By Lemma 2.4, we have  

\[
\alpha = \min_{v \in V(G), 1 \leq k \leq m-3} \{\alpha_k(v)\} = \min_{v \in V(G), 1 \leq k \leq m-3} \left\{ \frac{p_k(v)}{s_k(v)} \right\} \leq \frac{\sum_{k=1}^{m-3} \sum_{v \in V(G)} p_k(v)}{\sum_{k=1}^{m-3} \sum_{v \in V(G)} s_k(v)},
\]

and  

\[
\beta = \min_{v \in V(G), 1 \leq k \leq m-3} \{\beta_k(v)\} = \min_{v \in V(G), 1 \leq k \leq m-3} \left\{ \frac{r_k'(v)}{t_k(v)} \right\} \leq \frac{\sum_{k=1}^{m-3} \sum_{v \in V(G)} r_k'(v)}{\sum_{k=1}^{m-3} \sum_{v \in V(G)} t_k(v)}.
\]

It follows that  

\[
\min\{\alpha, \beta\} \leq \frac{\sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} p_k(v) + \sum_{v \in V(G)} r_k'(v) \right)}{\sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} s_k(v) + \sum_{v \in V(G)} t_k(v) \right)},
\]

Summing \( s_k(v) \) and \( t_k(v) \) over all \( v \in V(G) \) and noting (2.5), we have  

\[
\sum_{k=1}^{m-3} \sum_{v \in V(G)} s_k(v) = \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} p_i'(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^{k} \sum_{v \in V(G)} q_i'(v) \right)
\]

\[
= \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} r_i'(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^{k} \sum_{v \in V(G)} r_i'(v) \right)
\]

\[
= \sum_{k=1}^{m-3} \left( \sum_{i=1}^{m-3} \sum_{v \in V(G)} r_i'(v) \right) + \sum_{v \in V(G)} r_k'(v)
\]

\[
= (m-2) \sum_{k=1}^{m-3} \sum_{v \in V(G)} r_k'(v)
\]
and
\[
\sum_{k=1}^{m-3} \sum_{v \in V(G)} t_k(v) = \sum_{k=1}^{m-3} \left( \sum_{i=k}^{m-3} \sum_{v \in V(G)} r_i(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^{k} \sum_{v \in V(G)} q_i(v) \right) = \sum_{k=1}^{m-3} \left( \sum_{i=1}^{m-3} \sum_{v \in V(G)} p_i(v) \right) + \sum_{k=1}^{m-3} \left( \sum_{i=1}^{k} \sum_{v \in V(G)} p_i(v) \right) = \sum_{k=1}^{m-3} \sum_{i=1}^{m-3} p_i(v) + \sum_{v \in V(G)} p_k(v) = (m-2) \sum_{k=1}^{m-3} \sum_{v \in V(G)} p_k(v).
\]

It follows that
\[
\sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} s_k(v) + \sum_{v \in V(G)} t_k(v) \right) = (m-2) \sum_{k=1}^{m-3} \left( \sum_{v \in V(G)} p_k(v) + \sum_{v \in V(G)} r'_k(v) \right).
\]

Substituting this equality into (2.7) yields
\[
\min \{\alpha, \beta\} \leq \frac{1}{m-2}.
\]

The lemma follows.

\[ \blacksquare \]

3 Proof of Theorem 1.4

Clearly Theorem 1.4 holds for \(|V(G)| \leq m\). We proceed the proof by induction on \(|V(G)|\) under the assumption that Theorem 1.4 holds for all digraphs with \(|V(G)| \leq n\), here \(n > m\). Now let \(G\) be an \(m\)-free digraph with \(|V(G)| = n\), we may assume that for any \(v \in V(G)\), \(N^+_1(v) \neq \emptyset\) and \(N^-_1(v) \neq \emptyset\). Otherwise, if there exists \(v \in V(G)\) such that \(N^+_1(v) = \emptyset\) or \(N^-_1(v) = \emptyset\), then \(v\) is not in a directed cycle. From the inductive hypothesis, we can choose \(X \subseteq E(G-v)\) with \(|X| \leq \frac{1}{m-2} \gamma(G-v)\) such that \((G-v) - X\) is acyclic, then \(G - X\) has no directed cycles. It follows that

\[ \beta(G) \leq |X| \leq \frac{1}{m-2} \gamma(G-v) \leq \frac{1}{m-2} \gamma(G). \]

From Lemma 2.3, we have that \(\alpha \leq \frac{1}{m-2}\) or \(\beta \leq \frac{1}{m-2}\). For each case, we prove that there exists \(X \subseteq E(G)\) satisfying \(|X| \leq \frac{1}{m-2} \gamma(G)\) and \(G - X\) has no directed cycles.

**Case 1.** \(\alpha \leq \frac{1}{m-2}\).

By (2.6), there exist a vertex \(v \in V(G)\) and an integer \(k\) with \(1 \leq k \leq m-3\) such that
\[ \alpha = \alpha_k(v) = \frac{p_k(v)}{s_k(v)} \leq \frac{1}{m-2}. \]
We consider the partition \( \{V_1, V_2\} \) of \( V(G) \), where
\[
V_1 = \bigcup_{i=1}^{k+1} N_i^+(v), \quad V_2 = V(G) \setminus V_1.
\]

Clearly, \( N_1^-(v) \subset V_2 \) and \( \bigcup_{i=k+2}^{m-1} N_i^+(v) \subset V_2 \). Since \( G \) is an \( m \)-free digraph, we claim
\[
N_1^-(v) \cap \bigcup_{i=1}^{m-1} N_i^+(v) = \emptyset.
\]

Otherwise, let \( u \in N_1^-(v) \cap \bigcup_{i=1}^{m-1} N_i^+(v) \). Then \( (u, v) \in E(G) \) and there exists a directed \((v, u)\)-path \( P \) with length \( l_1 \leq m - 1 \). Then \( P + (u, v) \) is a directed cycle with length \( l_1 + 1 \leq m \), a contradiction.

Thus the number of missing edges between \( V_1 \) and \( V_2 \) satisfies
\[
|\bar{E}(V_1, V_2)| \geq |\bar{E}(\bigcup_{i=1}^{k+1} N_i^+(v), N_1^-(v) \cup (\bigcup_{i=k+2}^{m-1} N_i^+(v)))| \\
\geq \sum_{i=k+2}^{m-1} |\bar{E}(N_i^+(v), N_i^+(v))| + \sum_{i=2}^{k+1} |\bar{E}(N_i^+(v), N_i^-(v))| \\
\geq \sum_{i=k}^{m-1} p_i(v) + \sum_{i=1}^{k} q_i(v) \\
= s_k(v).
\]

It follows that
\[
\gamma(G) = \gamma(G_1) + \gamma(G_2) + |\bar{E}(V_1, V_2)| \geq \gamma(G_1) + \gamma(G_2) + s_k(v). \tag{3.1}
\]

Let \( G_i \) be the induced subgraph by \( V_i \) for each \( i = 1, 2 \). Since \( |V_1| < n \) and \( |V_2| < n \), from the inductive hypothesis, we have \( \beta(G_1) \leq \frac{1}{m-2} \gamma(G_1) \) and \( \beta(G_2) \leq \frac{1}{m-2} \gamma(G_2) \). We choose \( X_i \subseteq E(G_i) \) with
\[
|X_i| \leq \frac{1}{m-2} \gamma(G_i) \quad \text{for each } i = 1, 2 \tag{3.2}
\]
such that \( G_i - X_i \) is acyclic.

Let \( X_3 = E(V_1, V_2) \). Then \( X_3 = E(N_{k+1}^+(v), V_2) = E(N_{k+1}^-+v), N_{k+2}^+(v)) \), and
\[
|X_3| = |E(N_{k+1}^+(v), N_{k+2}^+(v))| = p_k(v). \tag{3.3}
\]

Let \( X = X_1 \cup X_2 \cup X_3 \). Then \( G - X \) has no directed cycles and, by \((3.1) \sim (3.3)\),
\[
|X| = |X_1| + |X_2| + |X_3| \\
= |X_1| + |X_2| + p_k(v) \\
\leq \frac{1}{m-2} \gamma(G_1) + \frac{1}{m-2} \gamma(G_2) + \frac{1}{m-2} s_k(v) \\
= \frac{1}{m-2} \left( \gamma(G_1) + \gamma(G_2) + s_k(v) \right) \\
\leq \frac{1}{m-2} \gamma(G).
\]

Case 2. \( \beta \leq \frac{1}{m-2} \).
By (2.7), there exist a vertex \( v \in V(G) \) and an integer \( k \) with \( 1 \leq k \leq m - 3 \) such that
\[
\beta = \beta_k(v) = \frac{r_k(v)}{t_k(v)} \leq \frac{1}{m - 2}.
\]

We consider the partition \( \{V_1, V_2\} \) of \( V(G) \), where
\[
V_1 = \bigcup_{i=1}^{k+1} N_i^-(v), \quad V_2 = V(G) \setminus V_1.
\]

Clearly, \( N_1^+(v) \subset V_2 \), \( \bigcup_{i=k+2}^{m-1} N_i^-(v) \subset V_2 \) and \( N_1^+(v) \cap \bigcup_{i=k+2}^{m-1} N_i^-(v) = \emptyset \). The number of missing edges between \( V_1 \) and \( V_2 \) satisfies
\[
|\bar{E}(V_1, V_2)| \geq |\bar{E}(\bigcup_{i=1}^{k+1} N_i^-(v), N_1^+(v) \cup (\bigcup_{i=k+2}^{m-1} N_i^-(v)))| \geq \sum_{i=k+2}^{m-1} |\bar{E}(N_i^-(v), N_i^+(v))| + \sum_{i=2}^{k+1} |\bar{E}(N_i^-(v), N_i^+(v))| - \sum_{i=1}^{k} r_i(v) - \sum_{i=1}^{k} q_i(v) = t_k(v).
\]

Then
\[
\gamma(G) = \gamma(G_1) + \gamma(G_2) + |\bar{E}(V_1, V_2)| \geq \gamma(G_1) + \gamma(G_2) + t_k(v).
\]

Let \( G_i \) be the induced subgraph by \( V_i \) for each \( i = 1, 2 \). For \( i = 1, 2 \), from the inductive hypothesis, \( \beta(G_1) \leq \frac{1}{m-2} \gamma(G_1) \) and \( \beta(G_2) \leq \frac{1}{m-2} \gamma(G_2) \), we can choose \( X_i \subseteq E(G_i) \) with \( |X_i| \leq \frac{1}{m-2} \gamma(G_i) \) such that \( G_i - X_i \) is acyclic. Let \( X_3 = (V_2, V_1) \), we have \( X_3 = E(V_2, N_{k+1}^-(v)) = E(N_{k+2}^-(v), N_{k+1}^+(v)) \), and \( |X_3| = r_k(v) \). Let \( X = X_1 \cup X_2 \cup X_3 \). Then \( G - X \) has no directed cycles. Hence
\[
|X| = |X_1| + |X_2| + |X_3| = |X_1| + |X_2| + r_k(v) \leq \frac{1}{m-2} \gamma(G_1) + \frac{1}{m-2} \gamma(G_2) + \frac{1}{m-2} t_k(v) = \frac{1}{m-2} (\gamma(G_1) + \gamma(G_2) + t_k(v)) \leq \frac{1}{m-2} \gamma(G).
\]

For each case, there exists \( X \subseteq E(G) \) satisfying \( |X| \leq \frac{1}{m-2} \gamma(G) \) and \( G - X \) has no directed cycles. This implies that \( \beta(G) \leq |X| \leq \frac{1}{m-2} \gamma(G) \), and Theorem 1.3 follows.

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