ON THE STRUCTURE OF COHEN-MACAULAY MODULES OVER HYPERSURFACES OF COUNTABLE COHEN-MACAULAY REPRESENTATION TYPE

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Abstract. Let $R$ be a complete local hypersurface over an algebraically closed field of characteristic different from two, and suppose that $R$ has countable Cohen-Macaulay (CM) representation type. In this paper, it is proved that the maximal Cohen-Macaulay (MCM) $R$-modules which are locally free on the punctured spectrum are dominated by the MCM $R$-modules which are not locally free on the punctured spectrum. More precisely, there exists a single $R$-module $X$ such that the indecomposable MCM $R$-modules not locally free on the punctured spectrum are $X$ and its syzygy $\Omega X$ and that any other MCM $R$-modules are obtained from extensions of $X$ and $\Omega X$.

1. Introduction

Let $R$ be a complete local hypersurface over an algebraically closed field. Suppose that the characteristic of $k$ is zero and that $R$ has finite Cohen-Macaulay (CM) representation type, namely, there exist only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay (MCM) $R$-modules. Then $R$ is isomorphic to the residue ring $k[[x_0, x_1, x_2, \ldots, x_d]]/(f)$ where $f$ is one of the following polynomials:

$$(A_n)(n \geq 1) \quad x_0^2 + x_1^{n+1} + x_2^2 + \cdots + x_d^2,$$
$$(D_n)(n \geq 4) \quad x_0^2 x_1 + x_1^{n-1} + x_2^2 + \cdots + x_d^2,$$
$$(E_6) \quad x_0^3 + x_1^4 + x_2^2 + \cdots + x_d^2,$$
$$(E_7) \quad x_0^3 + x_0 x_1^3 + x_2^2 + \cdots + x_d^2,$$
$$(E_8) \quad x_0^3 + x_1^5 + x_2^2 + \cdots + x_d^2.$$

In this case, all objects and morphisms in the category $\text{CM}(R)$ of MCM $R$-modules have been classified completely, that is, the Auslander-Reiten quiver of the stable category $\tilde{\text{CM}}(R)$ of $\text{CM}(R)$ has been obtained. For the details, see [2, 6, 11].

Now, assume that $k$ has characteristic different from two, and that $R$ has countable CM representation type, namely, there exist infinitely but only countably many isomorphism classes of indecomposable MCM $R$-modules. Then $R$ is isomorphic to $k[[x_0, x_1, x_2, \ldots, x_d]]/(f)$ where $f$ is either of the following [5 (1.6)]:

$$(A_d^\infty) \quad x_0^2 + x_2^2 + \cdots + x_d^2,$$
$$(D_d^\infty) \quad x_0^2 x_1 + x_2^2 + \cdots + x_d^2.$$

In this case, all objects in $\text{CM}(R)$ have been classified completely [2, 3, 6], but morphisms in $\text{CM}(R)$ have not. The purpose of this paper is to investigate the relationships among

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objects in CM$(R)$ by focusing on the objects that are locally free on the punctured spectrum of $R$. Modules locally free on the punctured spectrum have recently been studied in relation to whose nonfree loci; see [9, 10]. We also use nonfree loci to get our results.

Let us introduce here some notation. We denote by $\mathcal{P}(R)$ the full subcategory of CM$(R)$ consisting of all modules that are locally free on the punctured spectrum of $R$. Let $\mathcal{M}(R)$ be the set of nonisomorphic indecomposable MCM $R$-modules that are not locally free on the punctured spectrum of $R$, and let $\mathcal{V}(M)$ be the nonfree locus of a finitely generated $R$-module $M$. (Recall that the nonfree locus of $M$ is defined as the set of prime ideals $\mathfrak{p}$ of $R$ such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is nonfree.) Let $F_1 \xrightarrow{\partial} F_0 \to M \to 0$ be part of a minimal free resolution of $M$. Then the (first) syzygy of $M$ is by definition the image of the map $\partial$ and denoted by $\Omega M$. (Note that it is uniquely determined up to isomorphism.)

The main theorem of this paper is the following.

**Theorem 1.1.** Let $k$ be an algebraically closed field of characteristic different from 2. Let $R$ be a complete local hypersurface over $k$ of countable CM representation type. Then, as we stated above, $R$ is (isomorphic to) a residue ring $k[[x_0, x_1, x_2, \ldots, x_d]]/(f)$, where $f$ is either $(A^2_\infty)$ or $(D^d_\infty)$. Let $\mathfrak{p}_R = (x_0, x_2, \ldots, x_d)$ and $\mathfrak{m}_R = (x_0, x_1, x_2, \ldots, x_d)$ be ideals of $R$. The following hold.

1. There exists an $R$-module $X_R$ such that
   (a) $\mathcal{M}(R) = \{X_R, \Omega(X_R)\}$,
   (b) $\mathcal{V}(X_R) = \{\mathfrak{p}_R, \mathfrak{m}_R\} = \mathcal{V}(\Omega(X_R))$.
2. For each indecomposable $R$-module $M \in \mathcal{P}(R)$, there is an exact sequence
   \[ 0 \to L \to M \oplus R^n \to N \to 0 \]
   of $R$-modules with $L, N \in \mathcal{M}(R)$ and $n \geq 0$.

The first assertion of this theorem especially says that there exist at most two indecomposable MCM $R$-modules which are not locally free on the punctured spectrum of $R$. The second assertion especially says that the MCM $R$-modules which are locally free on the punctured spectrum of $R$ are dominated by the MCM $R$-modules which are not locally free on the punctured spectrum of $R$. On the other hand, the structure of the MCM modules locally free on the punctured spectrum has been clarified by Schreyer [8]:

**Theorem 1.2** (Schreyer). Let $n$ be a positive integer. The Auslander-Reiten quiver of the stable category of $\mathcal{P}(R)$ has the following form.

1. When $f = (A^{2n-1}_\infty)$:
   
   \[
   \begin{array}{cccccccc}
   \cdot & \longleftrightarrow & \cdot & \longleftrightarrow & \cdot & \longleftrightarrow & \cdot & \longleftrightarrow \ldots \\
   \end{array}
   \]

2. When $f = (D^{2n-1}_\infty)$:
   
   \[
   \begin{array}{cccccccc}
   \cdot & \longleftrightarrow & \cdot & \longleftrightarrow & \cdot & \longleftrightarrow & \cdot & \longleftrightarrow \ldots \\
   \end{array}
   \]

3. When $f = (A^{2n}_\infty)$:
Consequently, combining our Theorem \[\text{1.1}\] with this result due to Schreyer, we get a new understanding of the structure of the category of MCM modules over hypersurfaces of countable CM representation type.

We give two applications of our Theorem \[\text{1.1}\]. Using Theorem \[\text{1.1}\] we are able to calculate the dimension of the triangulated category \(\text{CM}(R)\) in the sense of Rouquier \[\text{7}\], and the Grothendieck groups of \(\text{CM}(R)\) and \(\text{CM}(R)\).

**Corollary 1.3.** With the notation of Theorem \[\text{1.1}\] the following hold.

1. The triangulated category \(\text{CM}(R)\) has dimension 1.
2. For \(m \geq 1\) one has:

\[
\begin{align*}
K_0(\text{CM}(R)) &= \langle [R], [X_R]\rangle \cong \begin{cases}
\mathbb{Z} & \text{if } f = (A_{\infty}^1), \\
\mathbb{Z}^2 & \text{if } f = (A_{\infty}^{2m}) \text{ or } f = (D_{\infty}^{2m-1}), \\
\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } f = (A_{\infty}^{2m+1}) \text{ or } f = (D_{\infty}^{2m}),
\end{cases} \\
K_0(\text{CM}(R)) &= \langle [X_R]\rangle \cong \begin{cases}
\mathbb{Z} & \text{if } f = (A_{\infty}^{2m}) \text{ or } f = (D_{\infty}^{2m-1}), \\
\mathbb{Z}/2\mathbb{Z} & \text{if } f = (A_{\infty}^{2m-1}) \text{ or } f = (D_{\infty}^{2m}).
\end{cases}
\end{align*}
\]

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## 2. One and two dimensional cases

In this section, we prove Theorem \[\text{1.1}\] in the cases where the ring \(R\) is of dimension 1 and 2. The following proposition includes Theorem \[\text{1.1}\] in the 1-dimensional case.

**Proposition 2.1.** Let \(S\) be 2-dimensional regular local ring and let \(x_0, x_1\) be a regular system of parameters of \(S\).

1. Let \(R = S/(x_0^2)\). Then \(R/(x_0) \cong \Omega(R/(x_0))\), and the following statements hold.
   (a) For every indecomposable MCM \(R\)-module \(M\), there is an exact sequence \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) of \(R\)-modules with \(L, N \in \{0, R/(x_0)\}\).
   (b) One has \(\mathcal{M}(R) = \{R/(x_0)\}\).
   (c) One has \(\mathcal{V}(R/(x_0)) = \{(x_0), (x_0, x_1)\}\).
2. Let \(R = S/(x_0 x_1)\). Then \(R/(x_0 x_1) \cong \Omega(R/(x_0))\), and the following statements hold.
   (a) For every indecomposable MCM \(R\)-module \(M\), there is an exact sequence \(0 \rightarrow L \rightarrow M \oplus R^n \rightarrow N \rightarrow 0\) with \(L, N \in \{0, R/(x_0), R/(x_0 x_1)\}\) and \(n = 0, 1\).
   (b) One has \(\mathcal{M}(R) = \{R/(x_0), R/(x_0 x_1)\}\).
   (c) One has \(\mathcal{V}(R/(x_0)) = \{(x_0), (x_0, x_1)\} = \mathcal{V}(R/(x_0 x_1))\).
Proof. (1) By \[\text{[2]}\text{ (4.1)], all the nonisomorphic indecomposable MCM } R\text{-modules are } R, R/(x_0) \text{ and Cok } \varphi_n \text{ } (n = 1, 2, \ldots), \text{where } \varphi_n = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix}. \text{We have the following short exact sequence of complexes.}

\[
\begin{array}{ccccccc}
0 & 0 & 0 & \\
\downarrow & & & \\
(\cdots & x_0 & R & x_0 & R & x_0 & R & \longrightarrow & 0) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
(\varphi_n & R^2 & \varphi_n & R^2 & \varphi_n & R^2 & \longrightarrow & 0) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
(\cdots & -x_0 & R & -x_0 & R & -x_0 & R & \longrightarrow & 0) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & \\
\end{array}
\]

Taking the long exact sequence of the homology modules, we get an exact sequence

\[0 \rightarrow R/(x_0) \rightarrow \text{Cok } \varphi_n \rightarrow R/(x_0) \rightarrow 0.\]

Also, the first row makes an exact sequence

\[0 \rightarrow R/(x_0) \rightarrow R \rightarrow R/(x_0) \rightarrow 0.\]

We have \(\mathcal{V}(R/(x_0)) = \{p, m\} = \text{Spec } R\), where \(p = (x_0)\) and \(m = (x_0, x_1)\) (cf. \[\text{[10]}\text{ (1.15(4))}\]). In particular, \(R/(x_0)\) belongs to \(\mathcal{M}(R)\). There is an equality \(\begin{pmatrix} -x_0 x_1^{-1} & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) of matrices over \(R_p\). Hence \((\text{Cok } \varphi_n)_p \cong R_p\) and \(\text{Cok } \varphi_n \notin \mathcal{M}(R)\). Therefore \(\mathcal{M}(R) = \{R/(x_0)\}\).

(2) By \[\text{[2]}\text{ (4.2)], all the nonisomorphic indecomposable MCM } R\text{-modules are } R, R/(x_0), R/(x_0 x_1), R/(x_0^2), R/(x_1), \text{Cok } \varphi^+_n, \text{Cok } \varphi^-_n, \text{Cok } \psi^+_n \text{ and Cok } \psi^-_n \text{ } (n = 1, 2, \ldots), \text{where } \varphi^+_n = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 \end{pmatrix}, \varphi^-_n = \begin{pmatrix} x_0 x_1 & x_1^{n+1} \\ 0 & -x_0 x_1 \end{pmatrix}, \psi^+_n = \begin{pmatrix} x_0 x_1 & x_1^n \\ 0 & -x_0 \end{pmatrix} \text{ and } \psi^-_n = \begin{pmatrix} x_0 & x_1^n \\ 0 & -x_0 x_1 \end{pmatrix}. \text{Setting } x_1^0 = 1, \text{for } n \geq 0 \text{ we have commutative diagrams}

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
R & x_0 & R & x_0 x_1 & R & x_0 & R & R & x_0 x_1 & R & R \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
R^2 & \varphi^+_n & R^2 & \varphi^+_n & R^2 & \varphi^+_n & R^2 & R^2 & \psi^+_n & R^2 & R^2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
R & -x_0 & R & -x_0 x_1 & R & -x_0 & R & R & -x_0 x_1 & R & R \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
with exact rows and columns. Also we have equalities of matrices over $R$:

$$
\begin{pmatrix}
  x_0 & 1 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  x_0 & 1 \\
  0 & -x_0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  -x_0 & 1
\end{pmatrix}
= \begin{pmatrix}
  x_0^2 & 0 \\
  0 & 1
\end{pmatrix},
$$

$$
\begin{pmatrix}
  1 & 0 \\
  x_0 & 1
\end{pmatrix}
\begin{pmatrix}
  x_0 x_1 & x_1 \\
  0 & -x_0 x_1
\end{pmatrix}
\begin{pmatrix}
  0 & 1 \\
  1 & -x_0
\end{pmatrix}
= \begin{pmatrix}
  x_1 & 0 \\
  0 & 0
\end{pmatrix}.
$$

Hence there are exact sequences

$$
0 \to R/(x_0 x_1) \to R \to R/(x_0) \to 0,
$$

$$
0 \to R/(x_0) \to R/(x_0^2) \to R/(x_0) \to 0,
$$

$$
0 \to R/(x_0 x_1) \to R/(x_1) \oplus R \to R/(x_0 x_1) \to 0,
$$

$$
0 \to R/(x_0) \to \text{Cok } \varphi_n^+ \to R/(x_0) \to 0,
$$

$$
0 \to R/(x_0 x_1) \to \text{Cok } \varphi_n^- \to R/(x_0 x_1) \to 0,
$$

$$
0 \to R/(x_0 x_1) \to \text{Cok } \psi_n^+ \to R/(x_0) \to 0,
$$

$$
0 \to R/(x_0) \to \text{Cok } \psi_n^- \to R/(x_0 x_1) \to 0.
$$

Put $p = (x_0)$, $q = (x_1)$ and $m = (x_0, x_1)$. Then we have $\text{Spec } R = \{p, q, m\}$, and easily see that the equalities $\mathcal{V}(R/(x_0)) = \mathcal{V}(R/(x_0 x_1)) = \{p, m\}$ and $\mathcal{V}(R/(x_0^2)) = \mathcal{V}(R/(x_1)) = \{m\}$ hold. Therefore $R/(x_0), R/(x_0 x_1) \in \mathcal{M}(R)$ and $R/(x_0^2), R/(x_1) \notin \mathcal{M}(R)$. Since $R_q$ is a field, all $R_q$-modules are free. There are equalities of matrices whose entries are in $R_p$:

$$
\begin{pmatrix}
  x_0 & 1 \\
  x_1 & 0
\end{pmatrix}
\begin{pmatrix}
  x_0 & x_1^n \\
  0 & -x_0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  -x_0 & 1
\end{pmatrix}
= \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix},
$$

$$
\begin{pmatrix}
  x_0 & 1 \\
  x_1 & 0
\end{pmatrix}
\begin{pmatrix}
  x_0 x_1 & x_1^n \\
  0 & -x_0
\end{pmatrix}
\begin{pmatrix}
  1 & 0 \\
  -x_0 x_1 & 1
\end{pmatrix}
= \begin{pmatrix}
  0 & 0 \\
  0 & 1
\end{pmatrix}.
$$

Hence $(\text{Cok } \varphi_n^+)_p \cong R_p \cong (\text{Cok } \psi_n^+)_p$. Note that $\text{Cok } \varphi_n^- \text{ and } \text{Cok } \psi_n^-$ are the syzygies of $\text{Cok } \varphi_n^+ \text{ and } \text{Cok } \psi_n^+$, respectively. Thus $\text{Cok } \varphi_n^+, \text{Cok } \varphi_n^-, \text{Cok } \psi_n^+ \text{ and } \text{Cok } \psi_n^-$ are not in $\mathcal{M}(R)$. Consequently, we have $\mathcal{M}(R) = \{R/(x_0), R/(x_0 x_1)\}$. \hfill \Box

Next, let us consider the case where the base ring has dimension 2.

**Proposition 2.2.** Let $k$ be an algebraically closed field.

1. Let $R = k[[x_0, x_1, x_2]]/(x_0 x_2)$. Then $R/(x_2) \cong \Omega(R/(x_0))$, and the following hold.
   - For any indecomposable MCM $R$-module $M$, there is an exact sequence $0 \to L \to M \to N \to 0$ with $L, N \in \{0, R/(x_0), R/(x_2)\}$.
   - One has $\mathcal{M}(R) = \{R/(x_0), R/(x_2)\}$.
   - One has $\mathcal{V}(R/(x_0)) = \{(x_0, x_2), (x_0, x_1, x_2)\} = \mathcal{V}(R/(x_2))$.

2. Let $R = k[[x_0, x_1, x_2]]/(x_0 x_1 - x_2^n)$. Then $R/(x_0, x_2) \cong \Omega(x_0, x_2)$, and the following hold.
   - For any indecomposable MCM $R$-module $M$, there is an exact sequence $0 \to L \to M \oplus R^n \to N \to 0$ with $L, N \in \{0, (x_0, x_2)\}$ and $n = 0, 1$.
   - One has $\mathcal{M}(R) = \{(x_0, x_2)\}$.
   - One has $\mathcal{V}((x_0, x_2)) = \{(x_0, x_2), (x_0, x_1, x_2)\}$.

**Proof.** (1) By [3, (5.3)], all indecomposable MCM $R$-modules are $R$, $R/(x_0)$, $R/(x_2)$, $\text{Cok } \varphi_n^+$ and $\text{Cok } \varphi_n^-$ ($n = 1, 2, \ldots$), where $\varphi_n^+ = \begin{pmatrix}
  x_2 & x_1^n \\
  0 & x_0
\end{pmatrix}$, $\varphi_n^- = \begin{pmatrix}
  x_0 & -x_1^n \\
  0 & x_2
\end{pmatrix}$. We have
There are exact sequences $0 \to R/(x_2) \to R \to R/(x_0) \to 0$, $0 \to R/(x_2) \to Cok \varphi_n^+ \to R/(x_0) \to 0$ and $0 \to R/(x_0) \to Cok \varphi_n^- \to R/(x_2) \to 0$. We have $\mathcal{V}(R/(x_0)) = \{(x_0, x_2), (x_0, x_1, x_2)\} = \mathcal{V}(R/(x_2))$, which implies $R/(x_0), R/(x_2) \in \mathcal{M}(R)$. Let $p$ be any nonmaximal prime ideal. Then one of $x_0, x_1, x_2$ is not in $p$, and we have:

$$
\begin{pmatrix}
0 & -x_1^n \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x_0 & x_2 \\
x_1 & 0
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
$$

if $x_0 \notin p$,

$$
\begin{pmatrix}
x_0 & -x_1^n \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_2 & x_1^n \\
x_0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
$$

if $x_1 \notin p$, and

$$
\begin{pmatrix}
0 & -x_1^n \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x_2 & x_1^n \\
x_0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
$$

if $x_2 \notin p$.

Over $R/p$. In each case $(Cok \varphi_n^+)_p$ is isomorphic to $R/p$. Since $Cok \varphi_n^+$ is the syzygy of $Cok \varphi_n^+$, we see that $Cok \varphi_n^+$ and $Cok \varphi_n^-$ are in $\mathcal{M}(R)$. Thus $\mathcal{M}(R) = \{R/(x_0), R/(x_2)\}$ holds.

(2) By [3 (5.7)] the following three assertions hold.

(i) All indecomposable MCM $R$-modules are $R$, $Cok \alpha^+$, $Cok \alpha^-$, $Cok \beta^+$, $Cok \beta^-$, $Cok \varphi_n^+$, $Cok \varphi_n^-$, $Cok \psi_n^+$ and $Cok \psi_n^-$ ($n = 1, 2, \ldots$), where $\alpha^+ = \begin{pmatrix} x_2 & x_0 x_1 \\ x_0 & x_2 \end{pmatrix}$, $\alpha^- = \begin{pmatrix} -x_2 & x_0 x_1 \\ x_0 & -x_2 \end{pmatrix}$, $\beta^+ = \begin{pmatrix} x_0 & x_2 \\ x_2 & x_1 \end{pmatrix}$, $\beta^- = \begin{pmatrix} x_1 & -x_2 \\ -x_2 & x_0 \end{pmatrix}$ and

$$
\varphi_n^+ = \begin{pmatrix} x_2 & x_0 x_1 & 0 & -x_1^{n+1} \\ x_0 & x_2 & x_1^n & 0 \\ 0 & 0 & x_2 & x_0 x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix},
\varphi_n^- = \begin{pmatrix} -x_2 & x_0 x_1 & 0 & -x_1^{n+1} \\ x_0 & -x_2 & x_1^n & 0 \\ 0 & 0 & -x_2 & x_0 x_1 \\ 0 & 0 & x_0 & -x_2 \end{pmatrix},
$$

$$
\psi_n^+ = \begin{pmatrix} x_2 & x_0 x_1 & -x_1^n & 0 \\ x_0 & x_2 & 0 & x_1^n \\ 0 & 0 & x_2 & x_0 x_1 \\ 0 & 0 & x_0 & x_2 \end{pmatrix},
\psi_n^- = \begin{pmatrix} -x_2 & x_0 x_1 & -x_1^n & 0 \\ x_0 & -x_2 & 0 & x_1^n \\ 0 & 0 & -x_2 & x_0 x_1 \\ 0 & 0 & x_0 & -x_2 \end{pmatrix}.
$$

(ii) There are isomorphisms $Cok \alpha^+ \cong Cok \alpha^-$, $Cok \beta^+ \cong Cok \beta^-$, $Cok \varphi_n^+ \cong Cok \varphi^-n$ and $Cok \psi_n^+ \cong Cok \psi_n^-$. 


(iii) The $R$-modules $\text{Cok} \beta^+$, $\text{Cok} \varphi^+_n$, $\text{Cok} \psi^+_n$ are locally free on the punctured spectrum.

We can easily check that the sequence $R^2 \xrightarrow{\alpha^+} R^2 \xrightarrow{x_0-x_2} R \to R/(x_0, x_2) \to 0$ is exact. Hence $\text{Cok} \alpha^+$ is isomorphic to the prime ideal $\mathfrak{p} := (x_0, x_2)$ of $R$. For an ideal $I$ of $R$, denote by $V(I)$ the set of prime ideals of $R$ containing $I$. It is easy to see that $V(\mathfrak{p})$ is contained in $V(p)$, while $\mathfrak{p}$ belongs to $V(p)$ because the local ring $R_p$ is not regular. Thus we obtain $V(p) = V(\mathfrak{p}) = \{\mathfrak{p}, m\}$, where $m = (x_0, x_1, x_2)$ is the maximal ideal of $R$, and we have $\mathcal{M}(R) = \{\mathfrak{p}\}$. Setting $x_1^0 = 1$, for $n \geq 0$ we have commutative diagrams

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
R^2 & \xrightarrow{\alpha^+} & R^2 & \xrightarrow{\alpha^+} & R^2 & \xrightarrow{\alpha^+} & R^2 & \\
(1 & 0 & 0 & 0)
\downarrow & (1 & 0 & 0 & 0)
\downarrow & (1 & 0 & 0 & 0)
\downarrow & (1 & 0 & 0 & 0)
\downarrow & (1 & 0 & 0 & 0)
\downarrow & (1 & 0 & 0 & 0)
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
R^4 & \xrightarrow{\varphi^+_n} & R^4 & \xrightarrow{\varphi^+_n} & R^4 & \xrightarrow{\varphi^+_n} & R^4 & \\
(0 & 0 & 1 & 0)
\downarrow & (0 & 0 & 1 & 0)
\downarrow & (0 & 0 & 1 & 0)
\downarrow & (0 & 0 & 1 & 0)
\downarrow & (0 & 0 & 1 & 0)
\downarrow & (0 & 0 & 1 & 0)
\end{array}
\]

with exact rows and columns. There is an equality of matrices over $R$

\[
\begin{pmatrix}
0 & -x_0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
x_0 & -x_1 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_2 & x_0 & 0 & -x_1 \\
x_0 & x_1 & 0 & 0 \\
x_2 & x_0 & 0 & x_1 \\
x_0 & x_2 & 0 & x_0
\end{pmatrix}
= \begin{pmatrix}
x_0^2 & x_2 & 0 & 0 \\
x_2 & x_1 & 0 & 0 \\
x_0 & -x_2 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

which gives an isomorphism $\text{Cok} \varphi^+_0 \cong \text{Cok} \beta^+ \oplus R$. Consequently, we obtain exact sequences $0 \to \text{Cok} \alpha^+ \to \text{Cok} \beta^+ \oplus R \to \text{Cok} \alpha^+ \to 0$, $0 \to \text{Cok} \alpha^+ \to \text{Cok} \varphi^+_n \to \text{Cok} \alpha^+ \to 0$ and $0 \to \text{Cok} \alpha^+ \to \text{Cok} \psi^+_n \to \text{Cok} \alpha^+ \to 0$. This completes the proof of the second assertion of the proposition.

\[\square\]

3. General case

In this section, we give a proof of Theorem \[\square\] in the general case. Throughout this section, we assume that $k$ is an algebraically closed field of characteristic different from two. First of all, let us recall the definition of a matrix factorization.

**Definition 3.1.** Let $S = k[[x_0, x_1, \ldots, x_n]]$ and $0 \neq f \in (x_0, x_1, \ldots, x_n)S$. A pair of square matrices $(A, B)$ with entries being in $S$ is called a **matrix factorization** of $f$ if it satisfies $AB = BA = fE$, where $E$ is an identity matrix.

Let $\text{MF}(f)$ be the category of matrix factorizations of $f$. Eisenbud \[\square\] (see also \[\square\] Chap. 7) proved that taking the cokernel induces a category equivalence $\text{MF}(f)/\langle(1, f)\rangle \xrightarrow{\cong} \text{CM}(S/(f))$, where $\langle(1, f)\rangle$ denotes the ideal of the category $\text{MF}(f)$ generated by $(1, f)$. We identify $\text{CM}(S/(f))$ with $\text{MF}(f)/\langle(1, f)\rangle$. 

The following lemma is called Knörrer’s periodicity (cf. [6] (3.1), [11] Chap. 12) and it will play a key role in this section.

**Lemma 3.2.** Let \( S = k[[x_0, x_1, \ldots, x_n]] \) and \( T = k[[x_0, x_1, \ldots, x_n, y, z]] \) be formal power series rings and \( f \in (x_0, x_1, \ldots, x_n)S \) a nonzero element. The functor \( \text{MF}(f) \to \text{MF}(f + yz) \) given by \( (A, B) \mapsto (\begin{pmatrix} A & yE \\ zE & -B \end{pmatrix}, \begin{pmatrix} B & xE \\ zE & -A \end{pmatrix}) \) induces a triangle equivalence between the stable categories \( F : \text{CM}(S/(f)) \cong \text{CM}(T/(f + yz)) \).

Here we recall some basic properties of the stable category of MCM modules. Let \( R \) be a Henselian Gorenstein local ring. For MCM \( R \)-modules \( M \) and \( N \), we have \( M \cong N \) in \( \text{CM}(R) \) if and only if \( M \oplus R^m \cong N \oplus R^n \) in \( \text{CM}(R) \) for some \( m, n \geq 0 \). Since \( R \) is Henselian, for any object \( M \in \text{CM}(R) \) there exists a unique object \( M_0 \in \text{CM}(R) \) such that \( M_0 \) has no nonzero free summand and that \( M_0 \cong M \) in \( \text{CM}(R) \). If \( M \) is an indecomposable object of \( \text{CM}(R) \), then \( M_0 \) is a nonfree indecomposable MCM \( R \)-module.

**Proposition 3.3.** Let \( S, T, f \) and \( F \) be as in Lemma 3.2. Put \( R = T/(f + yz) \) and \( R' = S/(f) \). For a nonfree MCM \( R' \)-module \( M \), the following statements hold.

1. One has an inclusion \( \mathcal{V}_R(FM) \subseteq \mathcal{V}_R(y, z) \).
2. One has a bijection \( \Phi : \mathcal{V}_R(FM) \to \mathcal{V}_R'(M) \) which sends \( \mathfrak{P} \) to \( \mathfrak{P}/(y, z) \).

**Proof.** (1) Let \( (A, B) \in \text{MF}(f) \) be a matrix factorization corresponding to \( M \). Then \( \text{Cok} (\begin{pmatrix} A & yE \\ zE & -B \end{pmatrix}, \begin{pmatrix} B & xE \\ zE & -A \end{pmatrix}) \) is isomorphic to \( FM \) up to free summand, and \( FM \) is a non-free MCM \( R \)-module. Let \( \mathfrak{P} \in \text{Spec} R \) with \( y \notin \mathfrak{P} \). Then there is an equality \( \begin{pmatrix} E & B \\ -y & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ z & -A \end{pmatrix} = \begin{pmatrix} E & B \\ -y & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) of matrices over \( R_\mathfrak{P} \). Hence \( (FM)_\mathfrak{P} \) is a free \( R_\mathfrak{P} \)-module, which implies \( \mathcal{V}_R(FM) \subseteq \mathcal{V}_R(y, z) \). Similarly, \( \mathcal{V}_R(FM) \subseteq \mathcal{V}_R(y, z) \) holds.

(2) The assignment \( \mathfrak{P} \mapsto \mathfrak{P}/(y, z) \) makes a bijection \( \mathcal{V}_R(y, z) \to \text{Spec} R' \). For \( \mathfrak{P} \in \mathcal{V}_R(y, z) \), set \( \mathfrak{p} = \mathfrak{P}/(y, z) \in \text{Spec} R' \). It is enough to show that \( \mathfrak{P} \in \mathcal{V}_R(FM) \) if and only if \( \mathfrak{p} \in \mathcal{V}_R(M) \). There are a ring isomorphism \( R/(y, z) \cong R' \) and an \( R' \)-module isomorphism \( FM/(y, z) \cong M \oplus \Omega_{R'} M \) up to free summand. We have \( R_\mathfrak{p} \)-module isomorphisms \( (FM)_\mathfrak{P}/(y, z)(FM)_\mathfrak{P} \cong (FM)/(y, z)(FM)_\mathfrak{P} \cong M_\mathfrak{P} \oplus (\Omega_{R'} M)_\mathfrak{P} \cong M_\mathfrak{p} \oplus (\Omega_{R'} M)_\mathfrak{p} \) if and only if \( \mathfrak{p} \in \mathcal{V}_R(M) \). Hence \( (FM)_\mathfrak{P} \) is \( R_\mathfrak{P} \)-free if and only if \( \mathfrak{p} \in \mathcal{V}_R(M) \).

Note that in general a nonfree finitely generated module \( M \) over a commutative local ring \( (R, \mathfrak{m}) \) is locally free on the punctured spectrum of \( R \) if and only if the equality \( \mathcal{V}_R(M) = \{ \mathfrak{m} \} \) holds. Thus Proposition 3.3 yields the corollary below.

**Corollary 3.4.** With the notation of Proposition 3.3, \( M \) is locally free on the punctured spectrum of \( R' \) if and only if \( FM \) is locally free on the punctured spectrum of \( R \).

Now we can prove Theorem 1.1 in the general case.

**Proof of Theorem 1.1** Let \( R = k[[x_0, x_1, x_2, \ldots, x_d]]/(f) \), where \( f \) is either \( (A_\infty^d) \) or \( (D_\infty^d) \). We induce on \( d = \dim R \). We may assume \( d \geq 3 \) thanks to Propositions 2.1 and 2.2. Put \( R' = k[[x_0, x_1, x_2, \ldots, x_{d-2}]]/(f') \) such that \( f = f' + x_3^2 + x_4' \). Then \( f' \) is either \( (A_{\infty}^{d-2}) \) or \( (D_{\infty}^{d-2}) \). As the characteristic of \( k \) is not 2, the ring \( R \) is isomorphic to \( k[[x_0, x_1, x_2, \ldots, x_{d-2}, y, z]]/(f' + yz) \), where \( y, z \) are indeterminates over \( k[[x_0, x_1, x_2, \ldots, x_{d-2}]] \). Let \( F : \text{CM}(R') \to \text{CM}(R) \) be the equivalence in Lemma 3.2.
(1) There is a nonfree indecomposable MCM $R'$-module $X_{R'}$ with $\mathcal{M}(R') = \{X_{R'}, \Omega R'(X_{R'})\}$ and $\mathcal{V}_R(X_{R'}) = \{(x_0, x_2, \ldots, x_{d-2})R', (x_0, x_1, x_2, \ldots, x_{d-2})R'\} = \mathcal{V}_R'(\Omega R'(X_{R'}))$ by the induction hypothesis. Let $X_R$ be the nonfree direct summand of $FX_{R'}$. Recall that the shift functor on the triangulated category $\underline{\text{CM}}(R)$ (respectively, $\underline{\text{CM}}(R')$) is the cosyzygy functor $\Omega^{-1}$ (respectively, $\Omega^{-1}$). Hence $F$ commutes with the syzygy functor, and we have $F(\Omega R'(X_{R'})) \cong \Omega R(FX_{R'}) \cong \Omega R(X_R)$ in $\underline{\text{CM}}(R)$. It follows from this that $\Omega R(X_R)$ is the nonfree direct summand of $F(\Omega R'(X_{R'}))$. By Proposition 3.3 and Corollary 3.4, we see that $\mathcal{M}(R) = \{X_R, \Omega R(X_R)\}$ and that $\mathcal{V}_R(X_R) = \{(x_0, x_2, \ldots, x_d)R, (x_0, x_1, x_2, \ldots, x_d)R\} = \mathcal{V}_R(\Omega R(X_R))$.

(2) Let $M \in \mathcal{P}(R)$ be an indecomposable $R$-module. Then there exists $M' \in \text{CM}(R')$ such that $FM'$ is isomorphic to $M$ up to free summand. Since the $R$-module $M$ is indecomposable, $M'$ can be chosen as an indecomposable $R'$-module. Corollary 3.4 implies that $M'$ is in $\mathcal{P}(R')$. By induction hypothesis, there is an exact sequence $0 \to L' \to M' \oplus R'^n \to N' \to 0$, where $L', N' \in \mathcal{M}(R')$ and $n \geq 0$. Then we obtain an exact triangle $L' \to M' \to N' \to \Omega^{-1} L'$ in $\underline{\text{CM}}(R')$, which gives an exact triangle $FL' \to M \to FN' \to \Omega^{-1} FL'$ in $\underline{\text{CM}}(R)$. Let $L$ and $N$ be the nonfree direct summands of $FL'$ and $FN'$, respectively. Then we have an exact triangle $L \to M \to N \to \Omega^{-1} L$, which gives a short exact sequence $0 \to L \to M \oplus R'^n \to N \to 0$ of $R$-modules. Since $L'$ and $N'$ belong to $\mathcal{M}(R')$, the modules $L$ and $N$ are in $\mathcal{M}(R)$ by Corollary 3.4.

4. Applications

In this section, we give some applications of Theorem 1.1. First, we calculate the dimension of the triangulated category $\underline{\text{CM}}(R)$. For the definition of the dimension of a triangulated category, see [7, (3.2)].

Proposition 4.1. The dimension of $\underline{\text{CM}}(R)$ is equal to 1.

Proof. Theorem 1.1(2) especially says that the dimension of $\underline{\text{CM}}(R)$ is at most 1. Note that our hypersurface $R$ is not of finite CM representation type and that every MCM $R$-module is isomorphic to its second syzygy up to free summand. Hence the dimension of $\underline{\text{CM}}(R)$ is nonzero. Now the conclusion follows.

Next, let us consider the Grothendieck group $K_0(\text{CM}(R))$ of $\text{CM}(R)$. Applying Proposition 2.1 and 2.2 we can calculate the Grothendieck group $K_0(\text{CM}(R))$ of $\text{CM}(R)$ for the hypersurfaces $R$ of types $(A_{\infty}^1)$, $(D_{\infty}^1)$, $(A_{\infty}^2)$ and $(D_{\infty}^2)$.

Proposition 4.2. With the notation of Theorem 1.1, we have

$$K_0(\text{CM}(R)) \cong \begin{cases} \mathbb{Z} & \text{if } f = (A_{\infty}^1), \\ \mathbb{Z}^2 & \text{if } f = (D_{\infty}^1) \text{ or } f = (A_{\infty}^2), \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } f = (D_{\infty}^2). \end{cases}$$

Proof. Let $R$ be as in Theorem 1.1 and $d = 1, 2$. By Propositions 2.1 and 2.2 there is an epimorphism $\mathbb{Z}^2 \to K_0(\text{CM}(R))$. Indeed, sending the canonical basis $\{(a_0, l)^t\}$ of $\mathbb{Z}^2$ to

$$\begin{align*} \{[R], [R/(x_0)]\} & \quad \text{if } f = (A_{\infty}^1), (D_{\infty}^1) \text{ or } (A_{\infty}^2), \\ \{[R], [(x_0, x_2)]\} & \quad \text{if } f = (D_{\infty}^2) \end{align*}$$

makes such a surjection. We get an exact sequence $0 \to \mathbb{Z}^{2-r} \to \mathbb{Z}^2 \to K_0(\text{CM}(R)) \to 0$, where $r$ is the rank of $K_0(\text{CM}(R))$. Let $Q$ be the total quotient ring of $R$. For
a commutative ring $A$, denote by $\operatorname{mod} A$ the category of finitely generated $A$-modules. Define homomorphisms $a : K_0(\operatorname{CM}(R)) \to K_0(\operatorname{mod} R)$ and $b : K_0(\operatorname{mod} R) \to K_0(\operatorname{mod} Q)$ by $a([M]) = [M]$ and $b([N]) = [N \otimes_R Q]$ for $M \in \operatorname{CM}(R)$ and $N \in \operatorname{mod} R$. For $N \in \operatorname{mod} R$, there is an exact sequence $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to N \to 0$ with $X_i \in \operatorname{CM}(R)$ for $0 \leq i \leq n$. (For instance, take a free resolution of $N$.) Then in $K_0(\operatorname{mod} R)$ the equality $[N] = \sum_{i=0}^n (-1)^i [X_i]$ holds, which shows that $a$ is surjective. Clearly, $b$ is also. Taking the composition, we have a surjection $K_0(\operatorname{CM}(R)) \to K_0(\operatorname{mod} Q)$. As $Q$ is Artinian, every finitely generated $Q$-module has finite length, and $K_0(\operatorname{mod} Q)$ is the free $\mathbb{Z}$-module with basis $\{[Q/\mathfrak{M}] | \mathfrak{M} \text{ is a maximal ideal of } Q\}$ (cf. [11, (1.7)]).

When $f = (A_1^1)$, the ring $Q$ has a unique maximal ideal. Hence $K_0(\operatorname{mod} Q) \cong \mathbb{Z}$. Since there is an exact sequence $0 \to R/(x_0) \to R \to R/(x_0) \to 0$, the equality $[R] = 2[R/(x_0)]$ holds in $K_0(\operatorname{CM}(R))$. Therefore we have $r = 1$. There is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \overset{(1,2)}{\longrightarrow} \mathbb{Z}^2 \longrightarrow K_0(\operatorname{CM}(R)) \longrightarrow 0 \\
\| & & \| \\
0 & \longrightarrow & \mathbb{Z} \overset{(1,0)}{\longrightarrow} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0
\end{array}
$$

with exact rows. Thus, the $\mathbb{Z}$-module $K_0(\operatorname{CM}(R))$ is isomorphic to $\mathbb{Z}$.

When either $f = (D_1^1)$ or $f = (A_2^\infty)$, the ring $Q$ has two maximal ideals, and we have $K_0(\operatorname{mod} Q) \cong \mathbb{Z}^2$. The $\mathbb{Z}$-module $K_0(\operatorname{CM}(R))$ is also isomorphic to $\mathbb{Z}^2$.

When $f = (D_2^\infty)$, the ring $Q$ is a field. Hence $K_0(\operatorname{mod} Q) \cong \mathbb{Z}$. With the notation of the proof of Proposition 2.2(2), we have isomorphisms $\operatorname{Cok} \alpha^- \cong \operatorname{Cok} \alpha^+ \cong (x_0, x_2)$ and an exact sequence $0 \to \operatorname{Cok} \alpha^- \to R^2 \to \operatorname{Cok} \alpha^+ \to 0$. Thus $K_0(\operatorname{CM}(R))$ has a relation $2[R] = 2[(x_0, x_2)]$, and we see that $r = 1$. There is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \overset{(2,2)}{\longrightarrow} \mathbb{Z}^2 \longrightarrow K_0(\operatorname{CM}(R)) \longrightarrow 0 \\
\| & & \| \\
0 & \longrightarrow & \mathbb{Z} \overset{(0,2)}{\longrightarrow} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
\end{array}
$$

with exact rows. Therefore $K_0(\operatorname{CM}(R))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. \hfill \Box

Before moving to the case of higher dimension, we verify that the following relationship exists between the Grothendieck group of $\operatorname{CM}(R)$ for a general Gorenstein complete local domain $R$ and that of $\operatorname{CM}(R)$.

**Lemma 4.3.** Let $R$ be a complete Gorenstein local domain. Then we have an exact sequence

$$
0 \to \langle [R] \rangle \overset{f}{\longrightarrow} K_0(\operatorname{CM}(R)) \overset{g}{\longrightarrow} K_0(\operatorname{CM}(R)) \to 0
$$

of $\mathbb{Z}$-modules, where $f, g$ are natural maps. Moreover, $\langle [R] \rangle \cong \mathbb{Z}$ holds.

**Proof.** It is trivial that $f$ is an injective map. As an exact sequence $0 \to X \to Y \to Z \to 0$ of MCM $R$-modules induces an exact triangle $X \to Y \to Z \to X[1]$ in $\operatorname{CM}(R)$, we have a well-defined map $g$. Clearly, $g$ is surjective and $gf = 0$. Let $X \to Y \to Z \to X[1]$ be an exact triangle in $\operatorname{CM}(R)$. Then we have an exact sequence $0 \to X \to Y \oplus R^n \to Z \to 0$ of MCM $R$-modules, which gives an equality $[X] - [Y] + [Z] = n[R]$ in $K_0(\operatorname{CM}(R))$. Thus we have the exact sequence in the lemma. As to the last statement, we have a map...
Let \( K_0(\text{CM}(R)) \to \mathbb{Z} \) given by \([M] \mapsto \text{rank}_R M\), where \( \text{rank}_R M \) denotes the rank of \( M \). The restriction of this map to \( \langle [R] \rangle \) is the inverse map of the natural surjection \( \mathbb{Z} \to \langle [R] \rangle \).

The Grothendieck group of \( \text{CM}(R) \) for a hypersurface \( R \) of countable CM representation type is described as follows.

**Proposition 4.4.** With the notation of Theorem 1.1, for \( m \geq 1 \) we have

\[
K_0(\text{CM}(R)) \cong \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & \text{if } f = (A_{\infty}^{2m-1}) \text{ or } f = (D_{\infty}^{2m}), \\
\mathbb{Z} & \text{if } f = (D_{\infty}^{2m-1}) \text{ or } f = (A_{\infty}^{2m}). 
\end{cases}
\]

**Proof.** By virtue of Knörrer’s periodicity (Lemma 3.2), we have only to deal with the cases \( \dim R = 1, 2 \). Using Propositions 2.1 and 2.2 we obtain the assertion.

**Proposition 4.5.** With the notation of Theorem 1.1, for \( m \geq 1 \) we have

\[
K_0(\text{CM}(R)) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } f = (A_{\infty}^{2m+1}) \text{ or } f = (D_{\infty}^{2m+2}), \\
\mathbb{Z}^2 & \text{if } f = (D_{\infty}^{2m+1}) \text{ or } f = (A_{\infty}^{2m+2}). 
\end{cases}
\]

**Proof.** Let \( R \) be as in Theorem 1.1. As \( m \geq 1 \), the ring \( R \) is an integral domain. Tensoring the quotient field \( Q \) of \( R \) induces a surjection \( K_0(\text{CM}(R)) \to K_0(\text{mod} Q) \) (this is nothing but the composition \( ba \) with the notation in the proof of Proposition 1.2), and \( K_0(\text{mod} Q) \cong \mathbb{Z} \) as \( Q \) is a field. Hence \( K_0(\text{CM}(R)) \) has a direct summand isomorphic to \( \mathbb{Z} \). We see from Theorem 1.1(2) that \( K_0(\text{CM}(R)) \cong \mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z} \) for some \( e \geq 0 \). Let \( F : \text{CM}(R') \to \text{CM}(R) \) be the \( m \)-th power of the triangle equivalence given in Lemma 3.2 where \( R' \) is the corresponding hypersurface of dimension 1 or 2.

When \( f \) is either \( D_{\infty}^{2m+1} \) or \( A_{\infty}^{2m+2} \), Proposition 4.4 and Lemma 4.3 imply the \( \mathbb{Z} \)-isomorphism \( K_0(\text{CM}(R)) \cong \mathbb{Z}^2 \).

Let \( f = (A_{\infty}^{2m+1}) \). Then, since there is an exact sequence \( 0 \to F((x_0), (x_0)) \to R^{2m} \to F((x_0), (x_0)) \to 0 \), the equality \( 2^m[R] = 2[F((x_0), (x_0))] \) holds in \( K_0(\text{CM}(R)) \). This gives a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \xrightarrow{(2^m)} \mathbb{Z}^2 \longrightarrow K_0(\text{CM}(R)) \longrightarrow 0 \\
\| & & \Downarrow \cong (0^{2m-1}) \\
0 & \longrightarrow & \mathbb{Z} \xrightarrow{(0)} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
\end{array}
\]

with exact rows. Thus, the \( \mathbb{Z} \)-module \( K_0(\text{CM}(R)) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

Let \( f = (D_{\infty}^{2m+2}) \). Then, with the notation of the proof of Proposition 2.2(2), we have an isomorphism \( F(\alpha^-, \alpha^+) \cong F(\alpha^+, \alpha^-) \) and an exact sequence \( 0 \to F(\alpha^-, \alpha^+) \to R^{2m+1} \to F(\alpha^+, \alpha^-) \to 0 \). Hence \( K_0(\text{CM}(R)) \) has a relation \( 2^{m+1}[R] = 2[F(\alpha^+, \alpha^-)] \). A commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z} \xrightarrow{(2^{m+1})} \mathbb{Z}^2 \longrightarrow K_0(\text{CM}(R)) \longrightarrow 0 \\
\| & & \Downarrow \cong (1^{2m-1}) \\
0 & \longrightarrow & \mathbb{Z} \xrightarrow{(0)} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow 0
\end{array}
\]

with exact rows exists, which shows that \( K_0(\text{CM}(R)) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). \( \Box \)
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