The Use of Nuclear $\beta$-decay as a Test of Bulk Neutrinos in Extra Dimensions

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Abstract

Theories which include neutrinos in large extra dimensions can be constrained by nuclear beta decay experiments. We examine universality of $\beta$ decay strengths of Fermi transitions. From this we find that the extra dimensional Yukawa coupling for a higher dimensional scale of 10 TeV, and two extra dimensions can be constrained to $y \lesssim 1$. Kinematic implications are also discussed. In particular, an extra dimensional scenario will produce a tritium decay endpoint spectrum with a different shape than that for just one massive state.

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1 Introduction

It is well known that consistent string theory can only be formulated in space time higher than four dimensions, specifically 10 or 11 dimensions. Until recently the predominant view was that since the extra dimensions have not been detected, they must be compactified in small volumes with radii of Planck size. This was supported by the study of the heterotic string where the string scale $M_S$ is related to the Planck scale $M_P$ via

$$M_S = M_P \sqrt{k \alpha_G}$$

(1)

where $\alpha_G$ denotes the unified gauge coupling constant and $k$ is an integer of order unity which labels the level of the Kac-Moody algebra. Here the string scale is close to the Planck scale and this renders string effects unobservable. Recently, Witten [2] observed that in nonperturbative string theory $M_S$ can be much lower. Lykken [3] and Antoniadis [4] independently suggested the possibility that $M_S$ can be as low as a few TeV. In this case, the fundamental string scale is not constrained theoretically as above but instead is constrained phenomenologically. This gains support from breakthroughs in understanding D-brane constructions in string theory [5]. In the D-brane scenario the number of extra dimensions scanned by the Standard Model (SM) particles and the graviton can be very different.

Buoyed by these developments intense efforts are made to understand the main features of low scale string theories. A popular construction is to assume a factorizable geometry. The gravitational field is taken to propagate in the full 10 or 11-dimensional bulk volume whereas the SM particles are localized in the 3-brane and hence are not sensitive to the extra dimensions [2, 3, 5, 7]. The crucial observation [6] is that the weakness of gravity in four dimension is due to its spreading in the $n$ extra dimensions and Eq. (1) is replaced by

$$M_P^2 = M_*^{n+2} V_n$$

(2)

where $V_n$ is the volume of the compactified extra space and $M_*$ is the higher dimensional Planck scale. The two scales $M_S$ and $M_*$ are related but the exact relation will not be needed in this paper. It is simplest to consider toroidal compactification, for which the volume is

$$V_n = (2\pi)^n R_1 R_2 R_3 \ldots R_n$$

(3)

and $R_i (i = 1, 2, \ldots n)$ are the radii of the extra dimensions. For symmetric compactification one takes all the radii to be equal and denoted it by $R$.

This formulation has dramatic effects that lead to a new perspective on the mass hierarchy problem [6] and the gauge coupling constants unification [8]. Furthermore, these theories predict a deviation from the Newtonian $1/r$ law of gravitational potential at submillimeter range. The fact that this law is well tested above a distance of 1mm leads to the limit that $n \geq 2$ for $M_* = 1$ TeV, for the symmetric case. New experiments
currently underway will probe smaller distance scales \[4\]. Besides the direct search for a deviation of the Newtonian gravitational law the above scenario can also be indirectly constrained by astrophysical considerations. Although somewhat model dependent these considerations suggest that \( n \geq 2 \) and \( M_* \geq 30 \) TeV \[10\].

A different scenario to explain the weak gravitational force is proposed by Randall and Sundrum \[11\]. This mechanism does not require large extra dimensions but instead invokes a non-factorizable geometry. Here the extra dimension has the effect of modifying the space-time metric. Experiments that probe gravity at small distances are then not interpreted in terms of the scaling law of Eq. \(2\). The phenomenology of the graviton and its Kaluza-Klein (KK) excitations is different from the case of factorizable geometry and has been studied in Ref.\[12\]. Given the differences of the these two scenarios and their extensions, and the richness of the phenomenology for each case it is clear that probes other than the graviton and its KK spectrum will be important additions to the study of higher dimensional physics.

It has been suggested recently \[13, 14\] that an extra right-handed neutrino that is a standard model gauge singlet may also be a probe of the extra dimensions. When a field is allowed to extend into the compactified extra dimensions it has associated with it Kaluza-Klein excitations which can have detectable effects in low energy experiments. Neutrinos propagating in the extra dimension will be referred to as the bulk neutrinos. The couplings of the bulk neutrino states with the active neutrino will give rise to a small neutrino mass for the active neutrino due to the spreading in the bulk volume. Some effects of these neutrinos in various low energy observables have been suggested \[15\]. A solution to the solar neutrino problem in terms of matter enhanced flavor transformation of \( \nu_{eL} \) into bulk neutrino states was studied in \[16\]. Bulk neutrinos can induce energy loss in stars \[17\] and the possible connection to the supernova collapse phase is studied in \[18\].

In this paper we concentrate on the constraints on the extra dimensional scenarios from universality tests of nuclear \( \beta \)-decays. Such studies have been used to provide a strong basis for the SM and more recently their usage is mainly in the test of unitarity of the quark mixings. We point out that the difference in Q-values for the nuclei which are very well measured can be used to constrain the properties of the bulk neutrinos such as the Yukawa couplings, the higher dimensional scale, \( M_* \), and the compactification radius \( R \). These are very general properties of neutrinos in extra dimensions and one does not need to employ the scaling relation between the Planck scale and \( M_* \) as given in Eq.(2). To illustrate the effects of the bulk neutrinos we give the Kurie plots of two nuclei and the nuclear recoil spectrum of a Fermi transition. The simplest model of bulk neutrinos given below may allow a signal to be detected in the \(^3\)H beta decay spectra. Other models which modify the tritium beta decay endpoint are discussed in \[17\]. However, in searches for a small admixture of a more massive neutrino, the features of the extra dimensional model are not within currently detectable limits. However more realistic models with larger
mixings with the active neutrinos may be constructed which can be probed by these experiments as well. We concentrate only on one family of bulk neutrino and the active $\nu_{eL}$. However, it is straightforward to extend the model to include three bulk neutrinos; one for each family. This implies further model assumptions and will take us into the realm of models for fermion families in extra dimensions which is beyond the scope of this study.

This paper is organized as follow. In section 2 we briefly outline the model of bulk neutrinos that we use and the mixings with $\nu_{eL}$. This serves the purpose of fixing of our notation. Sec. 3 details the study of universality using the data from Q-values of several $\beta$-decays. Next we present the impact of bulk neutrinos on the beta spectrum, and look at the recoil momentum for Fermi transitions. We give our conclusion in Sec 4.

## 2 A Simple Model of Bulk Neutrinos

The basic idea of producing neutrino masses using extra dimensions is given in Ref. \[16\]. In the interest of being self contained and to establish our notation we give some of the construction of a simple model of bulk neutrinos. The simplest model assumes that the fermions charged under the standard model gauge group as well as the gauge bosons and the Higgs boson are localized on a 3-brane embedded in the bulk of larger dimensions. If we assume only one extra dimension then space-time is labelled by $(x^{\mu}, z)$ where $\mu = 0, 1, 2, 3$. The extra coordinate $z$ is assumed to compactify into a circle of radius $R$. The brane scenario stipulates that the left-handed SM lepton doublet $L = (\nu_{eL}, e_L)$ is given by $L(x^{\mu}, z = 0)$. The fields $\nu_{eL}$ and $e_L$ are Weyl spinors. Next we assume that a SM singlet fermion, denoted by $\nu(x, z)$ exists and it propagates in the full five dimensional bulk. Its right- and left-handed projections are labelled by $\nu_R$ and $\nu_L$ respectively. It is distinguished from the SM neutrino by not carrying a flavor index. In this simple model gravity is weak because there are additional dimensions in which it propagates.

In five dimensions there are five gamma matrices $\Gamma^\mu$ and $\Gamma^5$ where $\mu = 0, 1, 2, 3$. A convenient representation of the Clifford algebra in five dimensions is to choose $\Gamma^\mu = \gamma^\mu$ and $\Gamma^5 = i\gamma^5$ with the usual $\gamma^\mu$ of Minkowski space-time. The effective Lagrangian for generating a neutrino mass is given by

$$\mathcal{L} = \int_0^{2\pi R} dz \bar{\nu} (i\gamma^\mu \partial_\mu + i\Gamma_5 \partial_z) \nu + y_s \int_0^{2\pi R} dz \delta(z) LH \nu_R + h.c.$$  \hspace{1cm} (4)

where $y_s$ is the dimensionful Yukawa coupling and $H$ is the Higgs doublet and it is related to the dimensionless coupling $y$ via

$$y_s = \frac{y}{M_s^{n/2}} \hspace{1cm} (5)$$

and $n = 1$ for one extra dimension.
For simplicity we have neglected a higher dimensional bare Dirac mass term. This can be naturally implemented under $\mathbb{Z}_2$ orbifold compactification \cite{14}. Following the Kaluza-Klein ansatz we Fourier expand $\nu_R$ and $\nu_L$ as follows:

\begin{align}
\nu_R(x, z) &= \frac{1}{\sqrt{2\pi R}} \sum_{k=-\infty}^{\infty} \nu_{kR} \exp \left( \frac{i k z}{R} \right) \\
\nu_L(x, z) &= \frac{1}{\sqrt{2\pi R}} \sum_{k=-\infty}^{\infty} \nu_{kL} \exp \left( \frac{i k z}{R} \right)
\end{align}

Substituting this into Eq.(4) and integrating over $z$ yields the effective Lagrangian in four dimensions:

$$
\mathcal{L} = \sum_{k=-\infty}^{\infty} \left[ \bar{\nu}_{kL} i \gamma^\mu \partial_\mu \nu_{kL} + \bar{\nu}_{kR} i \gamma^\mu \partial_\mu \nu_{kR} \right] + \sum_{k=-\infty}^{\infty} m_k \bar{\nu}_{kL} \nu_{kR} + \frac{y_s}{\sqrt{2\pi R}} \sum_{k=-\infty}^{\infty} \bar{L}H \nu_{kR} + h.c.
$$

where

$$m_k = \frac{k}{R} \quad (9)$$

is the mass of the $k$-th KK tower state. The mass splitting between each adjacent tower state is $1/R$. As seen in Eq.(8) the coupling between the KK-tower neutrino states and the active $\nu_{eL}$ given by the Yukawa term. After spontaneous electroweak symmetry breaking a Dirac mass term is generated for the active $\nu_{eL}$ and is given by

$$m_D = \frac{y v}{\sqrt{4\pi R M_*}} \quad (10)$$

where $v = 247$ GeV. This mass is suppressed by a bulk volume factor as first noticed in Ref. \cite{13} and \cite{14}.

Rewriting all the mass terms in Eq.(8) we have

$$m_D \bar{\nu}_{eL} \nu_{0R} + m_D \sum_{k=1}^{\infty} \bar{\nu}_{eL} (\nu_{kR} + \nu_{-kR}) + \sum_{k=1}^{\infty} m_k (\bar{\nu}_{kL} \nu_{kR} - \bar{\nu}_{-kL} \nu_{-kR}) + h.c. \quad (11)$$

We find it useful to define the following orthogonal states

\begin{align}
\nu'_{kR} &= \frac{1}{\sqrt{2}} (\nu_{kR} + \nu_{-kR}) & \nu''_{kR} &= \frac{1}{\sqrt{2}} (\nu_{kR} - \nu_{-kR}) \\
\nu'_{kL} &= \frac{1}{\sqrt{2}} (\nu_{kL} - \nu_{-kL}) & \nu''_{kL} &= \frac{1}{\sqrt{2}} (\nu_{kL} + \nu_{-kL})
\end{align}

and

$$m_D \bar{\nu}_{eL} \nu_{0R} + \sqrt{2} m_D \sum_{k=1}^{\infty} \bar{\nu}_{eL} \nu'_{kR} + \sum_{k=1}^{\infty} m_k (\bar{\nu}'_{kL} \nu'_{kR} + \bar{\nu}''_{kL} \nu''_{kR}) + h.c. \quad (14)$$
The states with double prime superscripts have no low energy interactions and will be ignored. The mass terms can now be written in the familiar form of $\bar{\nu}_L M \nu_R$ in the bases of $\nu_L = (\nu_{eL}, \nu'_{1L}, \nu'_{2L}, \ldots)$ and $\nu_R = (\nu'_{0R}, \nu'_{1R}, \nu'_{2R}, \ldots)$. The mass matrix $M$ for $k + 1$ states looks as follows

$$M = \begin{pmatrix}
  m_D & \sqrt{2}m_D & \sqrt{2}m_D & \ldots & \sqrt{2}m_D \\
  0 & \frac{1}{R} & 0 & \ldots & 0 \\
  0 & 0 & \frac{2}{R} & \ldots & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  0 & 0 & 0 & 0 & \frac{k}{R}
\end{pmatrix}$$

(15)

To find the left mass eigenstates, we consider the matrix $MM^\dagger$. Explicitly,

$$MM^\dagger = \frac{1}{R^2} \begin{pmatrix}
  (k + \frac{1}{2})\zeta^2 & \zeta & 2\zeta & \ldots & k\zeta \\
  \zeta & 1 & 0 & \ldots & 0 \\
  2\zeta & 0 & 4 & \ldots & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  k\zeta & 0 & 0 & 0 & k^2
\end{pmatrix}$$

(16)

where $\zeta = \sqrt{2m_D R}$. The mass eigenvalues $\lambda$ are given by the characteristic equation of Eq.(14) which is $\det[MM^\dagger - \lambda^2]$. After some algebra it reduces to the transcendental equation [14]:

$$\pi \zeta^2 \cot(\pi \lambda R) = 2\lambda R$$

(17)

The solution of this equation yields the lowest mixed mass eigenstate $\simeq m_D$. For small $\zeta$ this is also the maximally mixed state. It is reasonable to assume that experiments that directly measures the neutrino mass will be probing this state. The current experimental limit on the mass of $\nu_e$ is $m_{\nu_e} < 2.5$ eV [20]. When combined with this, Eq. (10) gives the following constraint

$$y \lesssim 3.2 \times 10^{-3} \left( \frac{M_*}{10 \text{TeV}} \right)^{\frac{1}{2}} \left( \frac{R}{1 \text{mm}} \right)^{\frac{1}{2}} \left( \frac{m_D}{1 \text{eV}} \right)$$

(18)

The diagonalization of Eq. (10) leads to the mixing of the $k$-th state to the lowest mass eigenstate $\nu_{0L}^m$ given by [16]

$$\tan 2\theta_k = \frac{2k\zeta}{k^2 - (k + \frac{1}{2})\zeta^2}$$

(19)

This equation diagonalizes the submatrix between the lowest state $\nu_{0L}$ and the state $\nu_{kL}$. Where $\zeta$ is small, this is a good approximation to the exact mixing angle, obtainable by complete diagonalization of the matrix $MM^\dagger$. In this limit, $\zeta$ can be used as an expansion parameter in the diagonalization process. For example, for $M_* = 10 \text{TeV}$, $y = 10^{-3}$ and $R \sim 10^{-7}$ mm we find $\zeta \sim 10^{-1}$. Hence, for small $\zeta$ we can see from above that the
mixing of the KK states with the lowest mass neutrino eigenstate becomes progressively smaller as the value of $k$ increases. Furthermore, these KK states have no direct gauge interactions with other SM particles and their presence can only be probed through their mixing with the state $\nu_{0L}^\nu$.

In the small $\zeta$ limit, the vacuum oscillation survival probability for electron neutrinos is a particularly simple expression,

$$P(\nu_e \to \nu_e) \approx 1 - \frac{\pi^2 \zeta^2}{3} + 2 \sum_k \frac{\zeta^2}{k^2} \left[ 1 - 2 \sin^2 \left( \frac{\delta m^2_{0k} t}{2 E_\nu} \right) \right] + \mathcal{O}(\zeta^4) \quad (20)$$

where $\delta m^2_{0k} \approx m_D^2 - m_k^2$. The data from a reactor neutrino experiment can be used to find a limit on the Yukawa coupling. CHOOZ [21] quotes a limit on two neutrino mixing of $\sin^2 2\theta < 10\%$ for $\delta m^2 > 10^{-3}\text{eV}^2$. This translates into a limit on the extra dimensional parameters of

$$y < 4.4 \times 10^{-7} \left( \frac{1\text{mm}}{R} \right)^{1/2} \left( \frac{M_*}{10\text{TeV}} \right)^{1/2}. \quad (21)$$

If combined with the limit from the neutrino mass, Eq (18), we find that the least stringent limit on $y$ occurs at $R = 7 \times 10^{-6}\text{mm}$.

However, Eq. (19) reveals a different phenomenon if $\zeta$ is large; i.e. $\zeta \gg 1$. Now it is the higher KK states with $k \approx \zeta$ that will have the largest mixing with the the lowest eigenstate; whereas states with lower $k$ values will have small mixing of order $\frac{1}{\zeta}$. In order for this to happen in $n = 1$ case we require $R \geq 10^{-3}\text{mm}$. Obviously this cannot happen with $\zeta \leq 1$ and a small compactification radius. When relatively large values of $R$ are considered, masses of these KK states can be in the $\sim 1\text{eV}$ range.

The above considerations can be generalized to higher extra dimensions. For $n = 2$ and different compactification radii $R_1$ and $R_2$ the scaling equation for $m_D$ becomes

$$m_D = \frac{y v}{2\pi M_* \sqrt{2R_1 R_2}} \quad (22)$$

and yields the following constraint the on the Yukawa coupling

$$y \lesssim 1.8 \times 10^6 \left( \frac{M_*}{10\text{TeV}} \right) \left( \frac{R_1 R_2}{1\text{mm}^2} \right)^{1/2} \left( \frac{m_D}{1\text{eV}} \right). \quad (23)$$

For $R_1 = R_2$ and a comparison with Eq.(18) shows that this is a much less stringent constraint on the parameters of the theory compared to the case of only one extra dimension. This is expected since the bulk neutrino state has a larger volume in which to spread and the Yukawa coupling is inversely proportional to the square root of this volume. An interesting case occurs for asymmetric compactification where one radius is much smaller than the other. Take for example $R_2 = 10^{-11}\text{mm}$ and $R_1 = 0.1\text{mm}$ then Eq.(23) approaches the case of $n = 1$ and gives a limit on $y$ similar to that of Eq.(18).
The masses of the KK tower states are generalized to

\[ m_{k,l} = \sqrt{\left( \frac{k^2}{R_1^2} + \frac{l^2}{R_2^2} \right)} \]  

(24)

where \( k \) and \( l \) denote the KK level corresponding to radii \( R_1 \) and \( R_2 \) respectively. It is seen that the smaller compactification radius \( R_2 \) will give rise to higher mass KK state in the asymmetric compactification scenario. For example if \( R_2 = 10^{-7} \text{mm} \) one can have keV bulk neutrinos mixing into the active \( \nu_e \) state that come from the small radius (i.e. for \( k = 0 \) and small \( l \) values). When both radii are large such neutrinos come only from the high KK values. Next we examine the mixings of these bulk states.

For small mixing the electron neutrino state can written as

\[ \nu_e = \frac{1}{N} \left( \nu_0 + \sum_{k=1}^{\infty} \frac{1}{k} \nu_{k,0} + \sum_{k, l \geq 1}^{\infty} \frac{1}{\sqrt{k^2 + (\frac{R_1}{R_2})^2 l^2}} \nu_{k,l} \right) \]  

(25)

where \( \zeta_1 = \sqrt{2} m_D R_1 \ll 1 \) and \( N \) is the normalization constant. It is seen that for the asymmetric case the KK states which come solely from the small radius may have larger masses than those from the larger radius; moreover, they are accompanied by correspondingly smaller mixings. We conclude that high mass states will have small mixing, regardless of the size of the compactification radii mainly due to the constraint on \( m_D \) from the direct limit on the electron neutrino mass. Since the direct limits on the mu and tau neutrino masses are much less strict, large mixing may occur in these families for considerably smaller KK masses.

For two extra dimensions, the constant, \( N \), depends on the cutoff scale \( M_* \), as \( N \sim 1 + \zeta^2 \log(M_* R) \), if \( R_1 = R_2 = R \). Limits derived from universality of Fermi transitions which are presented in the next section do not depend on this normalization and therefore on the cutoff procedure. However, other calculations such as vacuum oscillation survival probabilities will depend on the normalization. In particular, this would become important if one were to compare the theory with the reactor neutrino data. In this case, the constraint on \( y \) and \( M_* \) from universality could be used together with survival probability data to limit the size, \( R \).

In the next section we study the constraint \( \beta \) decay universality places on the parameters \( y, M_* \) and \( R \).

### 3 Universality

In this section we study the consequences for beta decay of the model for bulk neutrinos outlined in section 2. Although bulk neutrinos do not have gauged interactions with SM particles, the allowed Yukawa coupling enables them to generate mass for the active
neutrino. As seen in the last section the smallness of the neutrino mass is due to the spreading of the state in the higher dimensions. The KK excitations are also seen to have mixing effects with the active neutrino. In a given weak decay, the number of KK states that can be excited will depend on the energy released in the reaction. If $1/R << Q$, where $Q$ is the nuclear Q-value, then many states contribute. For example, for $n = 1$, if $Q = 1\text{MeV}$ and $R = 10^{-7}\text{mm}$ then around 500 states contribute. In previous studies of such classical weak interactions one concentrates on one or two massive neutrino states and their kinematic effect. In the present scenario, towers of KK neutrinos ranging in mass from eV to a few MeV are involved in a nuclear $\beta$ decay. They have to be taken into account in the study of universality, electron spectra and nuclear recoil spectra. We shall examine these issues separately to see if they provide useful constraints on the Yukawa coupling, $y$ and the size $R$, of the extra dimension. We begin with universality.

The rate of beta decay from a single state in the parent nucleus to another state in the daughter nucleus is given by,

$$\lambda = \frac{\ln 2}{ft} P(m_\nu)$$

The $ft$-value is proportional to the inverse of the matrix element. $P(m_\nu)$ is the phase space factor, which depends on the mass of the emitted neutrino, $m_\nu$. We define the phase space factor as

$$P(m_\nu) = \frac{1}{m_e^5} \int_{m_e}^{Q + m_e - m_\nu} F(Z, E_e) E_e p_e (Q + m_e - E_e) [(Q + m_e - E_e)^2 - m_\nu^2]^{1/2} dE_e \quad (27)$$

The Q-value is the energy difference between the initial and final nuclear states, while $E_e$, and $p_e$ are the electron energy and momentum respectively. The Coulomb wave correction factor is denoted by $F(Z, E)$. We use the approximation of this factor found in [22].

For pure Fermi, $0^+ \rightarrow 0^+$ transitions and zero mass electron neutrinos, the corrected $ft$-values, $Ft = ft(1 + \delta_R)(1 - \delta_C)$ for all nuclei should be the same under the assumption of universality in the standard model. Here, $\delta_R$ is the nucleus dependent part of the radiative correction and $\delta_C$ is the isospin symmetry breaking correction. These corrections contribute at the 1% level. Results of Ft-values for 10 nuclei are listed in [23].

The bulk neutrino scenario outlined in the previous section would cause an apparent deviation from the constant $Ft$ values predicted by universality in the standard model. This is because the actual phase space factor in an extra dimensional scenario is more complicated than Eq.(27). In general $ft$-values are found by using Eq. [26], with a measured rate and a calculated phase space factor. The transition rate for the case of one dimension is given in the small mixing limit, $\zeta << 1$, by

$$\lambda = \frac{\ln 2}{ft_{xd}} \left[ \left( 1 - \frac{\pi^2 \zeta^2}{6} \right) P(m_\nu \approx 0) + \sum_{k=1}^{k_{max}} P(m_{\nu_k} \approx \frac{k}{R} \zeta^2) \right] \quad (28)$$
The maximum number of neutrinos to be summed over is determined by the maximum mass of a neutrino that can be released in the beta decay, $k_{\text{max}} = Q R$. Transitions with a higher Q value will have more neutrino states that can contribute to the decay. Note that the $ft$ value in an extra dimensional scenario will not necessarily take on the same value as given in the standard model. (Although, the ‘true’ $ft$-values will be the same for all $0^+ \rightarrow 0^+$ transitions.) This will change the calculated value of the quark mixing angle $V_{ud}$ as discussed below. Because of this uncertainty, it is best to employ a normalization when looking for the effects of bulk neutrinos. In this example we will compare the relative difference in beta decays from two different nuclei.

Assuming that bulk neutrinos exist in one extra dimension, we can compare the $ft$-value for the extra dimensional scenario with the apparent $ft$-values for the standard model for the nucleus, $A_1$, by taking a ratio, 

$$\frac{ft(A_1)_{\text{sd}}} {ft(A_1)_{\text{SM, apparent}}} \approx (1 - \frac{\pi^2 \zeta^2}{6}) + \sum_{k=1}^{k_{\text{max}}} \frac{P(A_1, m_{\nu_k}) \zeta^2}{P(A_1, 0) k^2},$$

where we have used $\lambda_{\text{SM}}/\lambda_{\text{sd}} = 1$. Since in the extra dimensional scenario, the $ft$-values for two nuclei are the same up to the isospin and radiative corrections, the apparent Standard Model values will not be the same,

$$\frac{ft(A_2)_{\text{SM, apparent}}} {ft(A_1)_{\text{SM, apparent}}} \approx 1 + \sum_{k=1}^{k_{\text{max}}=Q_1 R} \frac{P(A_1, m_{\nu_k}) \zeta^2}{P(A_1, 0) k^2} - \sum_{k=1}^{k_{\text{max}}=Q_2 R} \frac{P(A_2, m_{\nu_k}) \zeta^2}{P(A_2, 0) k^2} - [\delta_R(A_1) - \delta_R(A_2)] + [\delta_C(A_1) - \delta_C(A_2)].$$

The maximum value of the second two terms may be obtained from the experimental uncertainty in the $Ft$-values from [23]. The sums in the above expression are very insensitive to lower bound on the sum, so it is only necessary to consider the highest modes in the sum, where there are relatively large differences in the phase space factor ratio. The mixing of the low energy KK states is not relevant to universality considerations.

The above formula is valid for large $\zeta$, provided that $\zeta^2/k^2 << 1$, if the term $(\zeta^2/k^2)$ is replaced by $f(\zeta)^2(\zeta^2/k^2)$. Here $f(\zeta)$ is a function which must be calculated for each $\zeta$ by explicit diagonalization of the mass matrix given by Eq. [16]. This explicit diagonalization shows that the mixing of state with $k >> m_D R$ is approximately $\zeta'/k = f(\zeta)/k$. For $\zeta = 100$, $f(\zeta) = 0.007$. For larger $\zeta$, $f(\zeta)$ decreases, while for $\zeta << 1$, $f(\zeta) = 1$.

We use $\beta^+$ decay data from $^{14}$O and $^{54}$Co with a combined error of 0.22%. These nuclei have Q values of 1.81 MeV and 7.22 MeV respectively. This can be translated into a limit on the Yukawa coupling, $y$ and the size of the extra dimension, R,

$$0.0022 \gtrsim 1.9 \times 10^{-7} \text{MeV}^2 y^2 f(\zeta)^2 \left(\frac{10 \text{TeV}}{M_e}\right) \left(\frac{1 \text{mm}}{R}\right) \sum_{k=1}^{Q_1 R} \left(\frac{P(A_1, m_{\nu_k})}{P(A_1, 0)} - \frac{P(A_2, m_{\nu_k})}{P(A_2, 0)}\right) \frac{1}{m_{\nu_k}^2} - \sum_{k=Q_1 R}^{Q_2 R} \left(\frac{P(A_2, m_{\nu_k})}{P(A_2, 0)}\right) \frac{1}{m_{\nu_k}^2}$$

(31)
Converting the sums in the above equation into integrals and integrating over all the available neutrino states produces a limit on the Yukawa coupling. As long as the small mixing limit applies, the limit on $y$ has little $R$ dependence and remains fairly constant for example, at $y < 10^{-3}$ for a scale of $M_* = 10$ TeV. The limit becomes less strict for a larger high dimensional scale. The lack of $R$ dependence can be seen when converting the sum in Eq. (31) to an integral, and making the change of variables $dk \to R dm_\nu$. In the case of large mixing, the limit is less strict, due to the factor $f(\zeta)$.

When comparing with Eqs. (18, 21), one can see that the limit on the Yukawa coupling from universality is always less stringent than the limit on the Yukawa coupling derived from the maximum neutrino mass and mixing determined by tritium beta decay and reactor neutrino experiments, respectively. This latter limit is shown as the bottom curve in Figure 1. Therefore, one does not obtain a useful constraint from universality when considering only one extra dimension.

One can perform the same exercise for two extra dimensions which both have the same size $R$. In the perturbative limit, the mass eigenstates are $m_{\nu_{k,n}}^2 \approx \left( k^2 + n^2 \right)/R^2$. In this case the sum ranges over two indices, $k$ and $n$. An equation similar to Eq. (31) may be derived for this case,

\[
0.0022 \gtrsim 6 \times 10^{-25} \text{MeV}^2 y^2 \left( \frac{10 \text{ TeV}}{M_*} \right)^2 \left( \frac{\text{mm}}{R} \right)^2 \sum_{k,n=1}^{k^2+n^2=Q_1 R} \left( \frac{P(A_1, m_{\nu_{k,n}})}{P(A_1, 0)} - \frac{P(A_2, m_{\nu_{k,n}})}{P(A_2, 0)} \right) m_{\nu_{k,n}}^2 - \sum_{k,n=Q_1 R}^{k^2+n^2=Q_1 R} \left( \frac{P(A_2, m_{\nu_{k,n}})}{P(A_2, 0)} \right) \left( \frac{1}{m_{\nu_{k,n}}^2} \right)
\] (32)

All two dimensional cases considered here, $M_* > 1$ TeV, fulfill the condition $\zeta^2 \ll 1$. We solve the above equation by converting the sums to integrals, and find that the limit is again insensitive to $R$. Figure 1 also shows the constraint on $M_*$ and $y$ for the two dimensional case. It can be seen that greater precision is needed to probe scales greater than 10 TeV, since above this scale the Yukawa coupling is not constrained to be less than of order 1.

The Standard Model value of $V_{ud}$ can be obtained from measured $Ft$ values as in [23] with the additional input of $G_F$, the weak coupling constant from muon decay. Using these quantities, the unitarity sum $V_{ud}^2 + V_{us}^2 + V_{ub}^2$ is calculated to be smaller than one by two standard deviations. The extra dimensional scenario would effect both the muon decay measurement as well as the $Ft$ values from beta decay. This introduces additional parameters and model uncertainties, such as the coupling of the muon neutrino to the same or another Kaluza Klein tower of neutrinos. If, for example, the muon neutrinos did not couple to any bulk neutrinos, then the calculated value of $V_{ud}$ should appear too large and an extra dimensional scenario could not account for the shortfall in the unitarity sum through the determination of $V_{ud}$ from beta decay.
Next we make contact with another much discussed test of universality using charged pion decays [24]. The pion decays, $\pi \rightarrow e \nu_e$ and $\pi \rightarrow \mu \nu_\mu$ can be probed kinematically. Since the electron in the decay is relativistic, while the muon is not, the contributions of a massive neutrino differ considerably in the two decays; a massive electron neutrino will make a larger difference than a massive muon neutrino. In the case of massless neutrinos the standard model ratio is $\Gamma(\pi \rightarrow e \nu) / \Gamma(\pi \rightarrow \mu \nu) = 1.233 \times 10^{-4}$. The experimentally measured quantity is $R^{\pi\mu}_{\pi\mu} = (1.230 \pm 0.004) \times 10^{-4}$ [25]. Therefore, the extradimensional contribution to these decays can not exceed 0.3%. The contribution has been calculated explicitly by [24] and for the $n = 2$ case with two families of bulk neutrinos. The contribution is approximately given by

$$7.52 \pi \times 10^2 y_e^2 \left( \frac{1\text{TeV}}{M_*} \right)^2 - 6.23 \pi \times 10^{-1} \left( \frac{1\text{TeV}}{M_*} \right)^2 y_\mu^2 \lesssim 0.003.$$ (33)

With this reaction alone, the only way to obtain a constraint on either $y_e$ or $y_\mu$ as a function of $M_*$ is to make an assumption about one of the couplings such as $y_e \sim 1$ or $y_\mu \sim 1$. Ideally, one would obtain a separate constraint on one of the couplings and use the pion decay data to limit the other. Taking the constraint on $y_e$ from beta decay universality discussed previously we find that another two orders of magnitude in precision in the beta decay data would be required in order to probe $y_\mu$. However, a complete discussion must include an analysis of neutrino oscillations which is beyond the scope of this paper; however, see [26] for a discussion.

4 Kinematic searches

In addition to changing the apparent $ft$ values, bulk neutrino scenarios can in principle also be probed kinematically. Here we demonstrate the consequences of the extra dimensional scenario on nuclear recoil momenta, and electron spectra. Massive neutrino searches in beta decay have focused on examining beta spectra [20, 27, 28, 29] and on examining the nuclear recoil spectra [30, 31]. A tower of Kaluza-Klein bulk states will produce a somewhat different signal than just one massive neutrino. We examine the signature in both of these types of spectra from the KK tower of states. Larger mixing scenarios may be probed by tritium beta decay searches for eV range neutrinos. However, as we shall see below, the model of extra dimensions cannot currently be constrained by searches for heavy (keV-MeV) neutrinos.

With a massive electron neutrino, the number of electron neutrinos as a function of momentum would be,

$$N(p_e) = \frac{1}{p_e^2} \frac{1}{F(Z, p_e)} \left[ \left( Q - E_e + m_e \right)^2 - m_\nu^2 \right]^{1/2} (Q - E_e + m_e)$$ (34)
We show the normalized Kurie plot, \( (N(p_e)/(p_e^2 F(Z,p_e)))^{1/2} \) vs. \( E_e \) for \( ^{38}\text{mK} \) at the top of Figure 3, for the case of \( m_\nu = 0 \). The Q-value for this decay is 5.02 MeV.

In the case of one extra dimension, the function is modified to

\[
\frac{N(p_e)}{p_e^2 F(Z,p_e)} = \frac{1}{m_e^2} \frac{1}{f_{txd}} \sum_{k=0}^{k_{max}} |U_{ke}|^2 \left[ \left( Q - E_e + m_e \right)^2 - m_{\nu k}^2 \right]^{1/2} (Q - E_e + m_e). \tag{35}
\]

Here \( |U_{ke}|^2 \) is the mixing of the kth mass eigenstate with the electron neutrino. For heavy massive neutrino searches (keV-MeV) very small mixing is relevant as discussed below. However, for massive neutrino searches in the eV range, such as tritium beta decay, a slightly larger mixing is applicable. Several states will give a different signature than just one or two massive states in a kinematic study. We illustrate this in the lower panel of Figure 2. This panel shows the beta endpoint spectrum for tritium for three cases. The dashed line shows the spectrum for a single massive neutrino of 2.3 eV. The solid line shows the result for a scenario with one extra dimensions, and \( \zeta = .13 \), \( 1/R = 25 \) eV giving \( m_D = 2.3 \) eV. The first ten mass eigenstates were calculated by explicit diagonalization of the matrix, Eq. \( (16) \). The first three occur at 2.3 eV, 25 eV and 50 eV, with mixings \( |U_{ke}|^2 \) of 0.974, 0.017 and 0.004 respectively. As can be seen from the figure, the Kaluza-Klein tower produces several bumps as well as less counts slightly away from the endpoint than a single massive neutrino. For comparison a two neutrino mixing scenario is shown as the dot-dashed line. The mass eigenstates are at 2.3 eV and 25 eV with a mixing of \( \sin^2 \theta \approx 2.5\% \).

We now turn to searches for massive neutrinos in the keV and MeV range. In the case of one extra dimension and small mixing, the function is modified to

\[
\frac{N(p_e)}{p_e^2 F(Z,p_e)} = \frac{1}{m_e^2} \frac{1}{f_{txd}} \sum_{k=1}^{k_{max}} \frac{\zeta^2}{k^2} \left[ \left( Q - E_e + m_e \right)^2 - m_{\nu k}^2 \right]^{1/2} (Q - E_e + m_e). \tag{36}
\]

We have normalized to the number of counts in order to eliminate the uncertainty in \( f_{txd} \). Therefore, we replace \( N(p_e) \rightarrow N(p_e)/N \), where for one extra dimension, \( N = \int N(p_e)dp_e \). We plot the results for the extra dimensional scenario as the ratio,

\[
\mathcal{R} = \left[ \frac{N(p_e)}{Np_e^2 F(Z,p)} \right]_{xd}^{1/2} / \left[ \frac{N(p_e)}{Np_e^2 F(Z,p)} \right]_{SM}^{1/2}. \tag{37}
\]

Two cases are shown in Figure 3. The panel in the bottom left hand corner shows a situation where many modes can contribute to the decay; \( R = 10^{-8}\text{mm} \), \( m_D = 1 \) eV. The mass of the lightest mode is at about 0.02 MeV and has a mixing of around \( \zeta^2 = 5 \times 10^{-9} \). Nuclei with smaller Q values will have a slightly larger signature for the continuum case.
The experimental limits on the mixing angle for a single 17 keV neutrino, for example, are of order $10^{-3}$. Since the mixing of the $\sim 20$ keV modes from this extra dimensional scenario are considerably weaker, these modes are not likely to be detectable with present experimental data. The mixing of a $\sim 1$ MeV neutrino is smaller still, since the mixing goes as $\zeta/k = \sqrt{2} m_D R/k$. If the lowest KK state occurs around 1 MeV, this implies a size $R \approx 2 \times 10^{-10}$ mm and a mixing of $\zeta^2 \sim 3 \times 10^{-13}$. The point here is that the mixing of the high mass KK mode is proportional to $m_D$ which is constrained by tritium beta decay experiments. Therefore, in this simple model the mixing of keV and MeV neutrinos will always be small. The bottom right hand corner of Figure 3 shows contributions from three separate neutrino masses, 1, 2 and 3 MeV. The bumps shown in the lower right panel are unobservable, since current constraints on the mixing of an MeV neutrino are many orders of magnitude less ($\sin^2 2\theta \sim 10^{-3}$).

We also consider the method of detecting massive neutrinos by measuring nuclear recoil spectra. In the presence of a massive neutrino, the nuclear recoil spectrum for a pure Fermi transition has the shape,

$$P_r(Z, r) = \frac{1}{2} \int_{E_{min}}^{E_{max}} F(Z, E_e) \left[ r E_e E_\nu + r \frac{a}{2} (r^2 - p_e^2 - p_\nu^2) \right] dE_e$$

(38)

Here, $r$ is the recoil momentum of the nucleus and $a = -1/3$. For a pure Gamow-Teller transition, $a = 1$. The limits of integration depend on the mass of the neutrino and are given by

$$E_{max, min} = \frac{1}{2} \left[ \frac{E_0 (E_0^2 - r^2 + m_e^2 - m_\nu^2) \pm r \sqrt{(E_0^2 - r^2 - m_e^2 - m_\nu^2)^2 - 4m_e^2 m_\nu^2}}{E_0^2 - r^2} \right]$$

(39)

Here $E_0 = Q + m_e$ is the total energy available for the decay. We again normalize the distribution by integrating over all possible recoil energies and finding $P_r(m_\nu)$, for each given neutrino mass. The distribution for $m_\nu = 0$ is shown on the top panel of Figure 4. The normalized probability distribution for the scenario with one extra dimension looks like,

$$P_r(r) = \frac{(1 - \frac{\pi^2 \zeta^2}{6}) P_r(r, m_\nu \approx 0) + \sum_k \frac{\zeta^2}{k^2} P_r(r, m_\nu \approx k/R)}{P_r}$$

(40)

where the normalization is

$$P_r = (1 - \frac{\pi^2 \zeta^2}{6}) \int P_r(r, m_\nu \approx 0) dr + \sum_k \frac{\zeta^2}{k^2} \int P_r(r, m_\nu \approx k/R) dr$$

(41)

To see the effect of the bulk neutrino states on recoil spectra, we again plot normalized ratios, $(P_r(r)/P_r)_{SM} / (P_r(r)/P_r)_{SM}$ The results are shown in Figures 4 for the Fermi transition in $^{38m}$K. The top panels show the normalized spectra for a single zero mass neutrino. The bottom left panels show the effect when many neutrino modes can contribute to the decay, while the bottom right panels show the case where only a few neutrinos can contribute. However, as in the case with the Kurie plots, the mixing of the KK modes is several orders of magnitude too small to be detected with current experimental measurements.
5 Conclusions

We have studied the effects of bulk neutrinos in nuclear $\beta$ decays in a simple model which only allows the SM singlet state to propagate in one or more compactified extra dimensions. All other SM particles are confined to a 3-brane and have no KK excitations. There are many well studied nuclei with different Q values that allow us to use universality to place constraints on the Yukawa coupling and the fundamental scale $M_\ast$. This information is complementary to the various studies of physics of higher dimensions using the gravitational force or KK graviton probes.

Our study shows that for $n = 1$ the most stringent constraint on the parameters of the bulk neutrinos arise from the direct measurement of the mass of $\nu_e$ and from reactor neutrino oscillation experiments. Universality tests do not give additional information. On the other hand for $n = 2$ the universality test restricts the Yukawa coupling to be less than unity for $M_\ast \leq 10\text{ TeV}$. Future experiments with radioactive beams measuring decays with Q values of order 10 MeV and improved accuracy with current measurements that can push this by a factor of ten will be most welcome.

The Kurie plot and the recoil nuclear spectrum in some selected decays are also given. For probes of massive Kaluza-Klein states we find the nuclei with smaller Q values are more sensitive. Currently, for medium and heavy nuclei, the sensitivity of experiments designed for massive neutrino searches are several orders of magnitude below what is needed to probe mixing of the massive KK neutrinos in these reactions. This may be due to the simplicity of the model we have considered and hopefully such studies will prove useful for more realistic models with enhanced mixings that also address the family problem. We also point out that our study may be used in connection with other data to constrain parameters of second and third generation mixing with bulk neutrinos. These scenarios are more model dependent and contain more parameters. We give an illustration of this point for charged pion decays.

On the other hand we find that the tritium $\beta$ decay spectrum can be significantly altered by the presence of KK bulk neutrino states in the eV range. They can reveal themselves in the shape of endpoint region of the Kurie plot if the mixings are favorable. It is useful to compare the results from universality limits to the ones derived from astrophysical constraints which are indirect. The latter relies on the energy loss carried away by bulk neutrinos. For example with limits derived from the neutrino magnetic moment for $n = 2$ there is essentially no constraint on the neutrino magnetic moment induced from the sort of extra dimensional scenario presented here, while the limit on the Yukawa coupling is of order 1 for $M_\ast \sim 10\text{ TeV}$ from universality. For $n = 1$ one obtains a constraint from the mu and tau neutrino magnetic moment whereas universality gives no useful constraints. As mentioned before in this case for electron neutrinos the strongest constraint comes from the electron neutrino mass limit and oscillation experiments.
6 Acknowledgements

We wish to thank J. Behr and M. Trinczek for helpful discussions. One of us (J.N.N.) would like to thank Prof. T.K. Lee of the National Center for theoretical Science for his hospitality.

This research is partly supported by the Natural Science and Engineering Council of Canada.
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Figure 1: The upper curve shows the limit on the extra dimensional Yukawa coupling from nuclear beta decay as a function of higher dimensional scale $M_*$. This constraint is for a number of extra dimensions, $n=2$. The lower curve is for $n=1$. It shows the limit on the extra dimensional Yukawa coupling from the tritium beta decay neutrino mass limit and the CHOOZ oscillation probability limit. This lower curve is a tighter constraint than the universality test for one extra dimension.

Figure 2: The top panel shows the endpoint of a Kurie plot for tritium beta decay, for which the $Q$ value is 18.6 keV. The horizontal axis is electron kinetic energy - $Q$. The bottom panel shows the same region with three different neutrino mass scenarios. This plot shows the subtraction of zero mass neutrino Kurie plots from the massive neutrino Kurie plots. The dashed line shows the spectrum for a single mass neutrino of 2.3 eV. The solid line shows the spectrum for one extra dimension with $\zeta = .13$ and $1/R = 25$ eV. The dot-dashed line shows a two neutrino mixing scenario with states at 2.3 eV and 25 eV and a mixing of 2.5%.

Figure 3: The top panel shows a normalized Kurie plot for $^{38m}$K. The horizontal axis is total electron energy. The dashed line corresponds to a zero mass neutrino, while the solid line corresponds to a neutrino with mass 2.3 eV. The bottom two panels show ratios of the normalized Kurie plot in the standard model to the normalized Kurie plot in the extra dimensional scenario. The bottom left panel, shows the effect for $R = 10^{-8}$ mm and a Dirac mass term of $m_D = 1$ eV. The bottom right panel was produced with the parameters $R = 2 \times 10^{-10}$ mm and $m_D = 1$ eV.

Figure 4: The top panel shows a normalized nuclear recoil spectrum for $^{38m}$K. The bottom two panels show ratios of the normalized spectrum in the standard model to the normalized spectrum in the extra dimensional scenario. The bottom left panel, shows the effect for $R = 10^{-8}$ mm and a Dirac mass term of $m_D = 1$ eV. The bottom right panel was produced with the parameters $R = 2 \times 10^{-10}$ mm and $m_D = 1$ eV.
