Semi-analytical Solution of Dirac equation in Schwarzschild Geometry

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Separation of the Dirac equation in the spacetime around a Kerr black hole into radial and angular coordinates was done by Chandrasekhar in 1976. In the present paper, we solve the radial equations in a Schwarzschild geometry semi-analytically using Wentzel-Kramers-Brillouin approximation (in short WKB) method. Among other things, we present analytical expression of the instantaneous reflection and transmission coefficients and the radial wave functions of the Dirac particles. Complete physical parameter space was divided into two parts depending on the height of the potential well and energy of the incoming waves. We show the general solution for these two regions. We also solve the equations by a Quantum Mechanical approach, in which the potential is approximated by a series of steps and found that these two solutions agree. We compare solutions of different initial parameters and show how the properties of the scattered wave depend on these parameters.

04.20.-q, 04.70.-s, 04.70.Dy, 95.30.Sf

I. INTRODUCTION

The spacetime around an isolated black hole is flat and Minkowskian at a large distance where usual quantum mechanics is applicable, while the spacetime closer to the singularity is curved and no satisfactory quantum field theory could be developed as yet. However, occasionally, it is useful to look into an intermediate situation when a weak perturbation (due to, say, gravitational, electromagnetic or Dirac waves) originating from infinity scatters from a black hole. The resulting wave is partially transmitted into the black hole through the horizon and partially scatters off from it to infinity. In the linearized ('test field') approximation this problem has been attacked in the past by several authors [1-4]. These methods are mostly numerical and most of the solutions obtained so far is for particles of integral spin only. Chandrasekhar [3-4] separated the Dirac equation in Kerr geometry into radial and angular parts. These works were extended to other spacetimes, such as in Kerr-Newman geometry [5], and around dyon black holes [6]. Subsequently, Chakrabarti [7] solved the angular part of the Dirac equation in Kerr geometry and gave the eigenvalues of the equation. These and the present works mostly concern scattering off tiny black holes and thus changing the incoming solution appreciably into an outgoing solution. Scattering effects from larger black holes could be studied by phase shift analysis and these has also been done recently [8].

In the present paper, we attack a simpler problem to have a ‘feel’ for the complete solution when the black hole is non-rotating. In the next Section, we present the basic equations. In §3, we classify the parameter space in terms of the physical and unphysical regions and present the method we adopt to solve the equations. In §4, we present a complete solution. In §5, we present solutions using a classical method in which the potential is approximated by a series of steps and then compare solutions of these two methods. In §6, we also compare solutions of various parameters and show how a Schwarzschild black hole distinguishes incoming particles of various masses. Finally, in §7, we draw our conclusions.

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II. BASIC EQUATIONS OF THE PROBLEM

Following Chandrasekhar [4], the radial part of the Dirac equation is easily reduced into a Schrödinger like equation. However, because the spin-spin coupling term is absent in the Schwarzschild geometry, the radial equation is much simpler to deal with. The eigenvalue of the angular equation for spin $\pm \frac{1}{2}$ is trivially obtained as $(l + 1/2)^2$ [7, 9-10] where $l$ is the orbital quantum number. In what follows, we choose $l = 1/2$ throughout for concreteness. This eigenvalue turns out to be the separation constant $\lambda$ of the original Dirac equation [4]. Here we solve the equation for one possible value of separation constant $\lambda$ (for $l = 1/2$, $\lambda$ is unity). In future we plan to explore the nature of the solutions for other orbital quantum numbers.

Presently, we need to solve only the following coupled radial equations [4]:

\[ \Delta^{\frac{1}{2}} D_0 R_{-\frac{1}{2}} = (1 + i m_pr) \Delta^{\frac{1}{2}} R_{+\frac{1}{2}}, \]  
\[ \Delta^{\frac{1}{2}} D_1^{\dagger} \Delta^{\frac{1}{2}} R_{+\frac{1}{2}} = (1 - i m_pr) R_{-\frac{1}{2}}, \]

where,

\[ D_n = \partial_r + \frac{iK}{\Delta} + 2n \frac{(r - M)}{\Delta}, \]
\[ \Delta = r^2 - 2Mr, \]
\[ K = r^2 \sigma. \]

Here $n = \text{integer}$, $\sigma = \text{frequency of incoming Dirac wave}$, $M = \text{mass of the black hole}$, $m = \text{azimuthal quantum number}$, $m_p = \text{rest mass of the Dirac particle}$; $p$ indicates particle, $R_{+\frac{1}{2}}(R_{-\frac{1}{2}})$ = radial wave function for spin up (down) particles. $D_1^{\dagger}$ is the complex conjugate operator. It is to be noted that the dimensionless unit is chosen, so that $G = h = c = 1$. The radial equation here is in coupled form. We can decouple it and express the equation either in terms of spin up or spin down wave function. However, it is more convenient to follow Chandrasekhar’s [4] approach by which the basis was changed along with the independent variable $r$. That way, the coupled equation was reduced into two independent one dimensional wave equations since they are easier to solve.

We first define

\[ r_* = r + 2M \log |r - 2M|, \]  
\[ \frac{d}{dr_*} = \Delta \frac{d}{r^2 dr}, \]

where, $r > r_+(= 2M)$,

\[ \frac{d}{dr_*} = \Delta \frac{d}{r^2 dr}, \]

and choose $\Delta^{\frac{1}{2}} R_{+\frac{1}{2}} = P_{+\frac{1}{2}}$, $R_{-\frac{1}{2}} = P_{-\frac{1}{2}}$.

In terms of $r_*$, the operators take the form:

\[ D_0 = \frac{r^2}{\Delta} \left( \frac{d}{dr_*} + i\sigma \right) \]
and

\[ D_1^{\dagger} = \frac{r^2}{\Delta} \left( \frac{d}{dr_*} - i\sigma \right). \]

We choose $\theta = \tan^{-1}(m_p r)$ which yields,

\[ \cos \theta = \frac{1}{\sqrt{(1 + m_p^2 r^2)}}, \quad \sin \theta = \frac{m_p r}{\sqrt{(1 + m_p^2 r^2)}}. \]
and 

\((1 \pm im_pr) = exp(\pm i\theta)\sqrt{(1 + m_p^2 r^2)}.\)

Following exactly Chandrasekhar’s [4] approach we write

\[
P_{+ \pm} = \psi_{+ \pm} exp\left[-\frac{1}{2}i\tan^{-1}(m_pr)\right]
\]

and

\[
P_{- \pm} = \psi_{- \pm} exp\left[+\frac{1}{2}i\tan^{-1}(m_pr)\right].
\]

Finally, a choice of \(\hat{r}_* = r_* + \frac{1}{\sigma}\tan^{-1}(m_pr)\) yields \(d\hat{r}_* = \left(1 + \frac{\Delta}{r_*^2 \sigma, 1 + m_p^2 r^2}\right) dr_*\).

With these definitions, the differential equations (1-2) are re-written as

\[
\left(\frac{d}{d\hat{r}_*} - W\right) Z_+ = i\sigma Z_-
\]

and

\[
\left(\frac{d}{d\hat{r}_*} + W\right) Z_- = i\sigma Z_+,
\]

where, \(Z_\pm = \psi_{+ \pm} \pm \psi_{- \pm}\), and

\[
W = \frac{\Delta^\frac{1}{2}(1 + m_p^2 r^2)^{3/2}}{r^2(1 + m_p^2 r^2) + m_p\Delta/\sigma}\).
\]

One important point to note: the transformation of spatial coordinate \(r\) to \(r_*\) (and \(\hat{r}_*\)) is taken not only for mathematical simplicity but also for a physical significance. When \(r\) is chosen as the radial coordinate, the decoupled equations for independent waves show diverging behaviour. However, by transforming those in terms of \(r_*\) (and \(\hat{r}_*\)) we obtain well behaved functions. The horizon is shifted from \(r = r_+\) to \(\hat{r}_* = -\infty\).

From the above set of equations, we readily obtain a pair of independent one-dimensional wave-equations,

\[
\left(\frac{d^2}{d\hat{r}_*^2} + \sigma^2\right) Z_\pm = V_\pm Z_\pm
\]

where,

\[
V_\pm = W^2 \pm \frac{dW}{d\hat{r}_*}
\]

\[
= \frac{\Delta^\frac{1}{2}(1 + m_p^2 r^2)^{3/2}}{r^2(1 + m_p^2 r^2) + m_p\Delta/\sigma}\left[\Delta^\frac{1}{2}(1 + m_p^2 r^2)^{3/2} \pm \left((r - M)(1 + m_p^2 r^2) + 3m_p^2 r \Delta\right)\right]
\]

\[
\mp \frac{\Delta^\frac{1}{2}(1 + m_p^2 r^2)^{3/2}}{r^2(1 + m_p^2 r^2) + m_p\Delta/\sigma}\left[2r(1 + m_p^2 r^2) + 2m_p^2 r^3 + m_p(r - M)/\sigma\right].
\]

III. PARAMETER SPACE AND METHOD TO SOLVE EQUATIONS

We obtain solutions by employing WKB [11-12] method and then imposing strict boundary conditions on the horizon, so that the reflection coefficient is zero and transmission coefficient is unity at the horizon. After establishing the general solution, we present here the solution of eq. (9) for two sets of parameters as illustrative examples.
It is advisable to choose the parameters in such a way that there is a significant interaction between the particle and the black hole. This is possible when the Compton wavelength of the incoming wave is of the same order as the Schwarzschild radius of the black hole, i.e.,

\[
\frac{2GM}{c^2} \sim \frac{\hbar}{m_p c},
\]

here, we are choosing \( G = \hbar = c = 1 \), so

\[
m_p \sim \frac{1}{2M}.
\]

Again, in the case of Schwarzschild geometry, frequency of the incoming particle (or, wave) will be of the same order as inverse of time. So,

\[
\frac{c^3}{2GM} \sim \sigma.
\]

Using the units as before, one can write,

\[
m_p \sim \sigma \sim (2M)^{-1}.
\]  

In principle, however, one can choose any values of \( \sigma \) and \( m_p \) for a particular black hole and the corresponding solution is possible but we shall concentrate the region of parameter space where the solution is expected to be interesting as pointed out above, namely, region close to \( m_p = \sigma \). In Fig. 1a, we draw this line. The parameter space is spanned by the frequency \( \sigma \) and the rest mass of the incoming particle \( m_p \). It is clear that 50% of total parameter space where \( \sigma < m_p \) is unphysical, and one need not study this region. Rest of the parameter space \( (\sigma > m_p) \) is divided into two regions – I: \( E > V_m \) and II: \( E < V_m \), where \( V_m \) is the maximum of the potential. While in Region I, the wave is locally sinusoidal because the wave number \( k \) is real for the entire range of \( \hat{r}_* \). In Region II, on the other hand, the wave is decaying in some region when \( E < V \), i.e., where the wave ‘hits’ the potential barrier and in the rest of the region, the wave is propagating. We shall show solutions in these two regions separately. In Region-I whatever be the physical parameters, energy of the particle is always greater than the potential energy and WKB approximation is generally valid in the whole range (i.e. \( \frac{1}{\hat{r}_*^2} \frac{d^2}{dr_*^2} \ll 1 \)). In cases of Region-II, energy of the particle is always less than the maximum height of potential barrier. Thus, at two points (where, \( k = 0 \)) total energy matches with the potential energy and in the neighbour of those two points WKB approximate method is not valid. They have to be dealt separately. In Fig. 1b, we show contours of constant \( w_{max} = \max (\frac{1}{\hat{r}_*^2} \frac{d^2}{dr_*^2}) \) for a given set \((\sigma, m_p)\) of parameters. The labels show the actual values of \( w_{max} \). Clearly, except for parameters very close to the boundary of Regions I and II, WKB approximation is safely valid for any value of \( \hat{r}_* \). One has to employ different method (such as using Airy Functions, see below) to find solutions in this region.

**IV. THE COMPLETE SOLUTION**

**Solutions of Region I**

In this region, for any set of parameter, energy of the particle is always greater than the corresponding potential energy. We first re-write equation (9) as,

\[
\frac{d^2Z_+}{d\hat{r}_*^2} + (\sigma^2 - V_+) Z_+ = 0.
\]  

This is nothing but Schrödinger equation corresponding to the total energy of the wave \( \sigma^2 \). This can be solved by regular WKB method [11-12]. Let,

\[
k(\hat{r}_*) = \sqrt{\sigma^2 - V_+}, \tag{14}
\]

\[
u(\hat{r}_*) = \int k(\hat{r}_*) d\hat{r}_* + \text{constant.} \tag{15}
\]
Here, \( k \) is the wavenumber of the incoming wave and \( u \) is the Eiconal. The solution of the equation (13) is,

\[
Z_+ = \frac{A_+}{\sqrt{k}} \exp(iu) + \frac{A_-}{\sqrt{k}} \exp(-iu).
\]  

(16)

with

\[
A_+^2 + A_-^2 = k. \tag{17}
\]

In this case \( \sigma > V_+ \) and also \( \frac{1}{k} \frac{dk}{d\hat{r}_*} \ll k \), so WKB approximation is generally valid in the whole region. The quantity \( \frac{1}{k} \frac{dk}{d\hat{r}_*} \) falls off rapidly with distance. Thus, WKB is strictly valid at long distance only.

It is clear that a standard WKB solution where \( A_+ \) and \( A_- \) are kept constants throughout should not be accurate, since the physical inner boundary condition on the horizon must be that the reflected component is negligible there. Thus WKB approximation requires a slight modification in which the spatial dependence of \( A_\pm \) is allowed. On the other hand, at a large distance, where WKB is strictly valid, \( A_+ \) and \( A_- \) should tend to be constants, and hence their difference is also a constant:

\[
A_+ - A_- = c. \tag{18}
\]

Here, \( c \) is determined from the WKB solution at a large distance. This along with (17) gives,

\[
A_\pm(r) = \pm \frac{c}{2} + \frac{\sqrt{2k(r) - c^2}}{2}. \tag{19}
\]

This spatial variation, strictly valid at large distances only, should not be extendible to the horizon without correcting for the inner boundary condition. These values are to be shifted by, say, \( A_{\pm h} \) respectively, so that on the horizon one obtains physical \( R \) and \( T \). We first correct reflection coefficient on the horizon as follows: Let \( A_{-h} \) be the value of \( A_- \) on the horizon (see, equation (19)),

\[
A_{-h} = -\frac{c}{2} + \frac{\sqrt{2k(r_+) - c^2}}{2}. \tag{21}
\]

It is appropriate to use \( A_- = A_- - A_{-h} \) rather than \( A_- \) since \( A_- \) vanishes at \( r = r_+ \).

Incorporating these conditions, the solution (16) becomes,

\[
Z_+ = \frac{A_+}{\sqrt{q}} \exp(iu) + \frac{A_+}{\sqrt{q}} \exp(-iu). \tag{20}
\]

with the usual normalization condition

\[
A_+^2 + A_-^2 = q. \tag{21}
\]

where, \( A_+ = A_+ - A_{+h} \). Here, \( q \) is to be determined self-consistently by equating the asymptotic behaviour of this reflection coefficient with that obtained using WKB method. This \( q \) in turn is used to compute \( A_+ = A_+ - A_{+h} \), and therefore the transmission coefficient \( T \) from eq. (21). In this way, normalization of \( R + T = 1 \) is assured.

Normalization factor \( q \rightarrow k \) as \( \hat{r}_* \rightarrow \infty \) and the condition \( \frac{1}{k} \frac{dk}{d\hat{r}_*} \ll q \) is found to be satisfied whenever \( \frac{1}{k} \frac{dk}{d\hat{r}_*} \ll k \) is satisfied. This is the essence of our modification of the WKB. In a true WKB, \( A_\pm \) are constants and the normalization is with respect to a (almost) constant \( k \). However, we are using it as if WKB is instantaneously valid everywhere. Our method may therefore be called ‘Instantaneous’ WKB approximation or IWKB for short. Using the new notations, the instantaneous values (i.e., local values) of the reflection and transmission coefficients are given by (see, eq. 20),

\[
R = A_-^2 \tag{22a}
\]

\[
T = A_+^2 \tag{22b}
\]

 Determination of \( A_{+h} \) is done by enforcing \( R \) obtained from eq. (22a) as the same as that obtained by actual WKB method at infinity.
To be concrete, we choose one set of parameters from Region I. (A large number of solutions are compared in §6 below.) Here, total energy of the incoming particle is greater than the potential barrier height for all values of $\hat{r}_*$. We use mass of the black hole, $M = 1$; mass of the particle, $m_p = 0.8$, orbital quantum number, $l = \frac{1}{2}$, azimuthal quantum number, $n = -\frac{1}{2}$, frequency of the incoming wave, $\sigma = 0.8$.

\textit{From eq. (9) we observe that there are two wave equations for two potentials $V_+$ and $V_-$. The nature of potentials are shown in Fig. 2. It is clear that potentials $V_\pm$ are well behaved. They are monotonically decreasing as the particle approaches the black hole, and the total energy chosen in this case ($\sigma^2$) is always higher compared to $V_\pm$. For concreteness, we solve using potential $V_+$. Similar procedure can be adopted using potential $V_-$ to compute $Z_-$ and its form would be}

\begin{equation}
Z_- = \frac{A'_+ - A'_{+h} \exp(iu')}{\sqrt{q'}} + \frac{A'_- - A'_{-h} \exp(-iu')}{\sqrt{q'}}.
\end{equation}

\text{Note the occurrence of the negative sign in front of the reflected wave. This is to satisfy the asymptotic property of the wave functions which must conserve the Wronskian [4]. Since the coefficients should not change sign between infinity and the horizon (as that would tantamount to having zero amplitude, i.e., unphysical, absence of either the forward or the backward component) the same sign convention is followed throughout the space. Local values of the reflection and transmission coefficients could also be calculated in the same manner. In the solution (eq. 20 and 20'), first term represents the incident wave and the second term represents the reflected wave.}

\text{In Fig. 3 we show the nature of $V_+$ (solid curve), $k$ (dashed curve) and $E(= \sigma^2)$ (short-dashed curve). The difference of $E$ and $V_+$ and therefore $k$ goes up as the particle approaches the black hole.}

\text{In Fig. 4, variation of ‘local’ reflection and transmission coefficients are shown. It is observed that as matter comes close to the black hole, the barrier height goes down. As a result, the penetration probability increases resulting in the rise of the transmission coefficients. At the same time, the reflection coefficient tends to be zero. It is to be noted, that, strictly speaking, the terms ‘reflection’ and ‘transmission’ coefficients are traditionally defined with respect to the asymptotic values. The spatial dependence that we show are to be interpreted as the instantaneous values. This is consistent with the spirit of IWKB approximation that we are using.}

\text{The behaviour of the solutions with $V_-$ is not very different from what were shown in Figs. (3-4) except in a region very close to the black hole horizon where $V_+$ and $V_-$ differs slightly (see, Fig. 2).}

\text{Using the solutions of equations with potential $V_+$ and $V_-$, the radial wave functions $R_{\pm \frac{1}{2}}$ and $R_{- \frac{1}{2}}$ for spin up and spin down particles respectively of the original Dirac equation are given below,}

\begin{align}
\text{Re}\left(R_{\pm \frac{1}{2}}\right) &= \frac{a_+ \cos(u - \theta) + a_- \cos(u + \theta)}{2\sqrt{k}} + \frac{a'_+ \cos(u' - \theta) - a'_- \cos(u' + \theta)}{2\sqrt{k'}} \tag{23a} \\
\text{Im}\left(R_{\pm \frac{1}{2}}\right) &= \frac{a_+ \sin(u - \theta) - a_- \sin(u + \theta)}{2\sqrt{k}} + \frac{a'_+ \sin(u' - \theta) + a'_- \sin(u' + \theta)}{2\sqrt{k'}} \tag{23b} \\
\text{Re}\left(R_{- \frac{1}{2}}\right) &= \frac{a_+ \cos(u + \theta) + a_- \cos(u - \theta)}{2\sqrt{k}} - \frac{a'_+ \cos(u' + \theta) - a'_- \cos(u' - \theta)}{2\sqrt{k'}} \tag{23c} \\
\text{Im}\left(R_{- \frac{1}{2}}\right) &= \frac{a_+ \sin(u + \theta) - a_- \sin(u - \theta)}{2\sqrt{k}} - \frac{a'_+ \sin(u' + \theta) + a'_- \sin(u' - \theta)}{2\sqrt{k'}} \tag{23d}
\end{align}

\text{Here, $a_+ = (A_+ - A_{+h})/\sqrt{(q/k)}$ and $a_- = (A_- - A_{-h})/\sqrt{(q/k)}$. Here, we have brought back $k$ and $k'$ so that these may resemble the original solution (eq. 16) using WKB approximation. $\frac{a'_+}{\sqrt{k'}}$ and $\frac{a'_-}{\sqrt{k'}}$ are the transmitted and reflected amplitudes respectively for the wave of corresponding potential $V_-$.}

\text{Figure 5 shows the resulting wave functions for both the spin +$\frac{1}{2}$ and spin $-\frac{1}{2}$ particles respectively. The eiconals used in plotting these functions (see, eq. 23\textit{a-d}) have been calculated by approximating $V_\pm$ in terms of polynomials (This was done since $V_\pm$ as presented in eq. 10 is not directly integrable.) and using the definition $u(\hat{r}_*) = \int \sqrt{(\sigma^2 - V_\pm)}d\hat{r}_*$. Note that the amplitude as well as the wavelength remain constants in regions where $k$ is also a constant. As discussed before, the wave functions are almost sinusoidal close to the horizon and at a very large distance (albeit with different frequencies). Since the net current ($|P_{\frac{1}{2}}|^2 - |P_{-\frac{1}{2}}|^2$) is conserved, probability of spin $+\frac{1}{2}$ is complimentary to the probability of spin $-\frac{1}{2}$ particles respectively.}
Here we study the second region where for any set of physical parameter total energy of the incoming particle is less than the maximum height of the potential barrier. Thus, the WKB approximation is not valid in the whole range of \( \hat{r}_s \). In such regions, the solutions will be a linear combination of Airy functions because the potential is approximately linear in \( \hat{r}_s \) in those intervals. At the junctions one has to match the solutions with Airy functions along with the solution obtained by the WKB method. In the region where the WKB approximation is valid, local values of reflection and transmission coefficients and the wave functions can be calculated easily by following the same method described in Case I. In other regions, the equation reduces to

\[
\frac{d^2 Z_+}{d\hat{r}_s^2} - x Z_+ = 0, \tag{24}
\]

where, \( x = \beta^+(\hat{r}_s - p) \), \( \beta \) is chosen to be positive and \( p \) is the critical point where the total energy and potential energy are matching.

Let \( Z_+(x) = x^{\frac{1}{3}}Y(x) \) and considering region \( x > 0 \) the equation (24) reduces to

\[
x^2 \frac{d^2 Y}{dx^2} + x \frac{dY}{dx} - \left( x^3 + \frac{1}{4} \right) Y(x) = 0. \tag{25}
\]

By making yet another transformation,

\[
\xi = \frac{2}{3}x^{\frac{1}{2}}, \tag{26}
\]

we obtain,

\[
\xi^2 \frac{d^2 Y}{d\xi^2} + \xi \frac{dY}{d\xi} - \left( \xi^2 + \frac{1}{9} \right) Y(\xi) = 0. \tag{27}
\]

This is the modified Bessel equation. The solution of this equation is \( I_{+\frac{1}{3}}(\xi) \) and \( I_{-\frac{1}{3}}(\xi) \). Hence, the solution of eq. (27) will be,

\[
Z_+(x) = x^{\frac{1}{3}}[C_1 I_{+\frac{1}{3}}(\xi) + C_2 I_{-\frac{1}{3}}(\xi)]. \tag{28}
\]

When \( x < 0 \) the corresponding equation is,

\[
\xi^2 \frac{d^2 Y}{d\xi^2} + \xi \frac{dY}{d\xi} + \left( \xi^2 - \frac{1}{9} \right) Y(\xi) = 0, \tag{29}
\]

which is the Bessel equation. The corresponding solution is

\[
Z_+(x) = |x|^{\frac{1}{3}}[D_1 J_{+\frac{1}{3}}(\xi) + D_2 J_{-\frac{1}{3}}(\xi)], \tag{30}
\]

where \( J_\pm \) and \( I_\pm \) are the Bessel functions and the modified Bessel functions of order \( \frac{1}{3} \) respectively.

The Airy functions are defined as

\[
Ai(x) = \frac{1}{3}x^{\frac{1}{3}}[I_{-\frac{1}{3}}(\xi) - I_{+\frac{1}{3}}(\xi)], \quad x > 0, \tag{31}
\]

\[
Ai(x) = \frac{1}{3}|x|^{\frac{1}{3}}[J_{-\frac{1}{3}}(\xi) + J_{+\frac{1}{3}}(\xi)], \quad x < 0, \tag{32}
\]

\[
Bi(x) = \frac{1}{\sqrt{3}}x^{\frac{1}{3}}[I_{-\frac{1}{3}}(\xi) + I_{+\frac{1}{3}}(\xi)], \quad x > 0, \tag{33}
\]

\[
Bi(x) = \frac{1}{\sqrt{3}}|x|^{\frac{1}{3}}[J_{-\frac{1}{3}}(\xi) - J_{+\frac{1}{3}}(\xi)], \quad x < 0. \tag{34}
\]
In terms of Airy functions, the solutions (28) and (30) can be written as

\[
Z_+ = \frac{3}{2} (C_2 - C_1) Ai(x) + \frac{\sqrt{3}}{2} (C_2 + C_1) Bi(x) \quad \text{for } x > 0,
\]

\[
Z_+ = \frac{3}{2} (D_2 + D_1) Ai(x) + \frac{\sqrt{3}}{2} (D_2 - D_1) Bi(x) \quad \text{for } x < 0.
\]

By matching boundary conditions it is easy to show that the solution corresponding \( x > 0 \) and that corresponding \( x < 0 \) are continuous when \( C_1 = -D_1 \) and \( C_2 = D_2 \).

To have an explicit solution, we choose the following set of parameters: \( M = 1 \), \( m_p = 0.1 \), \( l = \frac{1}{2} \), \( m = -\frac{1}{2} \) and \( \sigma = 0.15 \).

In Fig. 6, we show the nature of \( V_+ \) and \( V_- \). However, while solving, we use the equation containing \( V_+ \) (eq. 9).

Unlike the case in the previous Section, here \( \sigma^2 \) is no longer greater than \( V_\pm \) at all radii. As a result, \( k^2 \) may attain negative values in some region. In Fig. 7, nature of \( V_+ \) (solid curve), parameter \( k \) (dashed curve) and energy \( E \) (short-dashed curve) are shown. Here, WKB approximation can be applied in regions other than \( \hat{r}_* \sim -6 \) to \(-1 \) and \( 4 \) to \( 8 \) where \( k \) is close to zero and the condition \( \frac{1}{\sqrt{r}} \frac{dr}{d\hat{r}_*} \ll k \) is not satisfied. In the region \( \hat{r}_* \sim 8 \) to \( 4 \) around the turning point \( \hat{r}_* \sim 5.6088 \) the solutions turns out as [13]

\[
Z_+ = 1.858386 Ai(x) + 0.600610914 Bi(x).
\]

Similarly, the solution from \(-1 \) to \(-6 \) i.e. around the turning point \( \hat{r}_* = -3.0675 \) can be calculated as [13]

\[
Z_+ = 1.978145 Ai(x) + 0.7168807 Bi(x).
\]

It is to be noted that in the region \( \hat{r}_* \sim 4 \) to \(-1 \), even though the potential energy dominates over the total energy, WKB approximation method is still valid. Here the solution will take the form \( \frac{\exp(-u)}{\sqrt{k}} \) and \( \frac{\exp(+u)}{\sqrt{k}} \). Asymptotic values of the instantaneous reflection and the transmission coefficients (which are traditionally known as the ‘reflection’ and ‘transmission’ coefficients respectively) are obtained from the WKB approximation. This yields the integral constant \( c \) as in Case I. 

\[\text{From eq. 22(a-b) local reflection and transmission coefficients are calculated, behaviour of which are shown in Fig. 8. The constants } A_{-h} \text{ and } A_{+h} \text{ are calculated as before. Note the decaying nature of the reflection coefficient inside the potential barrier.}
\]

\[\text{V. SOLUTION OF THE EQUATIONS BY STEP-POTENTIAL METHOD}
\]

In the above sections we presented our semi-analytical solutions by WKB method with an appropriate boundary condition at the horizon. A numerical approach would be to replace the potential \( V(\hat{r}_*) \) by a collection of step function as shown in Fig. 9a. Here, the solid steps approximate the dashed potential for \( m_p = 0.8 \) and \( \sigma = 0.8 \). The standard junction conditions of the type,

\[
Z_{+,n} = Z_{+,n+1}
\]

where,

\[
Z_{+,n} = A_n \exp[i k_n \hat{r}_{*,n}] + B_n \exp[-i k_n \hat{r}_{*,n}]
\]

and

\[
\frac{dZ_+}{d\hat{r}_*}|_n = \frac{dZ_+}{d\hat{r}_*}|_{n+1}
\]

where,

\[
\frac{dZ_+}{d\hat{r}_*}|_n = i k_n A_n \exp[i k_n \hat{r}_{*,n}] - i k_n B_n \exp(-i k_n \hat{r}_{*,n})
\]

at each of the \( n \) steps were used to connect solutions at successive steps. As before, we use the inner boundary condition, to be \( R \to 0 \) at \( \hat{r}_* \to -\infty \). In reality, we used as many as 12000 steps to accurately follow the shape of the potential. Smaller step sizes were used whenever \( k \) varies faster. Fig. 9b shows the comparison of the instantaneous reflection coefficients in both the methods. The solid curve is from the WKB method of previous section and the dotted curve is from the step-potential method as we described here. The agreement is clearly excellent.
VI. BLACK HOLE: A MASS SPECTROGRAPH?

In order to show that the black hole scatters incoming waves of different rest masses ($m_p$) and of different energies ($\sigma^2$) quite differently, we show a collection of solutions in Figs. 10(a-d). In Figs. 10a, we show reflection and transmission coefficients for waves with parameters $\sigma = 0.8$ (solid), 0.85 (dotted) and 0.90 (dashed) respectively with the same $m_p = 0.8$. As the energy of the particle rises comparable to the height of the potential (which is solely dependent on $m_p$ at a large distance), the reflection coefficient goes down and the transmission coefficient goes up. In Fig. 10b, the real part of the wave $Z_+$, corresponding to these three cases are shown. At $\hat{r}_+ = 0$, the wave pattern is independent of $\sigma$ as the phase factor is trivially the same in all the cases. The dispersal of the wave with frequency is clear. Waves with smaller energy and longer wavelength are scattered with higher amplitude of $\text{Re}(Z_+)$ as the fraction of the reflected wave goes up when energy is reduced. This behaviour is valid till $R < 0.5$ since the amplitude of $\text{Re}(Z_+) = (1 + \sqrt{TR})^{1/2}$. For $R > 0.5$, amplitude of $\text{Re}(Z_+)$ goes down with energy. In Fig. 10(c-d), solutions are shown with varying the rest mass of the particles while keeping $\sigma$ fixed at 0.8. The solid curve, dotted curve and the dashed curves are for $m_p = 0.8$, 0.76, 0.72 respectively. Most interesting aspect is that close to the black hole $\hat{r}_+ \lesssim 0$, the reflection and transmission coefficients as well as the nature of the wave are quite independent of the rest mass. This is understandable, as just outside the horizon, the potential is insensitive to $m_p$. However, farther out, amplitude of $\text{Re}(Z_+)$ goes up as before when $m_p$ is raised as larger fraction of the wave is reflected from the potential (Fig.10c).

This interesting behaviour, for the first time shows that one could scatter a mixed wave off a black hole and each of the constituent wave would behave differently as in a prism or a mass spectrograph.

VII. CONCLUDING REMARKS

In this paper, we analytically studied scattering of spin-half particles from a Schwarzschild black hole. In particular, we presented the nature of the radial wave functions and the reflection and transmission coefficients. Our main motivation was to give an analytical expression of the solution which could be useful for further study of interactions among Hawking radiations, for instance. We verified that these analytical solutions were indeed correct by explicitly solving the same set of equations numerically using step-potential approach as described in Section V. We classified the entire parameter space in terms of the physical and unphysical regions and the physical region was further classified into two regions, depending on whether the particle ‘hits’ the potential barrier or not. We chose one illustrative example in each of the regions. We emphasize that the most ‘interesting’ region to study would be close to $m_p \sim \sigma$. However we pointed out (Fig. 1b) that for $m_p \leq 0.3$, WKB solutions cannot be trusted, and other methods (such as those using Airy functions) must be employed.

We used the well known WKB approximation method as well as the step-potential method of quantum mechanics to obtain the spatial dependence of the coefficients of the wave function. This in turn, allowed us to determine the reflection and transmission coefficients and the nature of wave functions. The usual WKB method with constant coefficients and (almost) constant wave number $k$ is successfully applied even when the coefficients and wave number are not constant everywhere. Solution from this ‘instantaneous’ WKB (IWKB for short) method agrees fully with that obtained from a purely classical numerical method where the potential is replaced by a collection of steps (simple quantum mechanical approach). The resulting wave forms satisfy the inner and the outer boundary conditions. Our method of obtaining solutions should be valid for any black hole geometry which are asymptotically flat so that radial waves could be used at a large distance. This way we ensure that the analytical solution is closer to the exact solution. In Region II, in some regions, WKB method cannot be applied and hence Airy function approach or our step-potential approach could be used.

In the literature, reflection and transmission coefficients are defined at a single point. These definitions are meaningful only if the potential sharply changes in a small region while studies are made from a large distance from it. In the present case, however, the potential changes over a large distance and we are studying in these regions as well. Although we used the words ‘reflection’ and ‘transmission’ coefficients, in this paper very loosely, our definitions are very rigorous and well defined. These quantities are simply the instantaneous values. In our belief these are more physical. The problem at hand is very similar to the problem of reflection and transmission of acoustic waves from a strucured string of non-constant density where reflection and transmission occurs at each point.

Unlike in the case of a Kerr black hole, the solution of the angular equation around a Schwarzschild black hole is independent of the azimuthal or meridional angles [5-7]. This is expected because of symmetry of the space-time. However, otherwise, the nature of the reflection and transmission coefficients were found to strongly distinguish solutions of different rest masses and different energies as illustrated in Figs. 10(a-d). For instance, when the energy of the wave is increased for a given mass of the particle, reflected component goes down. In regions where $R > 0.5$,
Re(Z+) goes down with energy, but where R < 0.5, Re(Z+) goes up with energy. In any case, the wave scattered off are distinctly different. In a way, therefore, black holes can act as a mass spectrograph! For instance a mixture of waves should be splitted into its components by the black hole. Our method is quite general and should be used to study outgoing Hawking radiations also. This is beyond the scope of this paper and would be dealt with in future.

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FIGURE CAPTIONS

Fig. 1a: Classification of the parameter space in terms of the energy and rest mass of the particles. The physical region \( \sigma > m_p \) is further classified in terms of whether the particle actually ‘hits’ the barrier or not.

Fig. 1b: Contours of constant \( w_{\text{max}} = \max \left( \frac{1}{k} \frac{dk}{d \hat{r}_*} \right) \) are shown to indicate that generally \( w << 1 \) and therefore WKB approximation is valid in most of the physical region. Labels indicate values of \( w \).

Fig. 2: Behaviour of potentials \( V_+ \) (solid curve) and \( V_- \) (dashed curve), as a function of \( \hat{r}_* \). The parameters are chosen from Region I of Fig. 1.

Fig. 3: Behaviour of \( V_+ \) (solid curve), \( k \) (dashed curve), total energy \( E \) (short-dashed curve), as functions of \( \hat{r}_* \).

Fig. 4: Behaviour of local transmission (\( T \), solid curve) and reflection (\( R \), dashed curve) coefficients, as functions of \( \hat{r}_* \). Close to the horizon, transmission is a hundred percent and reflection is almost zero.

Fig. 5: Behaviour of (a) \( \text{Re}(R_{1/2} \Delta^{1/2}) \), (b) \( \text{Im}(R_{1/2} \Delta^{1/2}) \), (c) \( \text{Re}(R_{-1/2}) \), (d) \( \text{Im}(R_{-1/2}) \), as a function of \( \hat{r}_* \). Note the complimentary nature wave functions of the spin \( \frac{1}{2} \) and spin \( -\frac{1}{2} \) particles. This is because the current is conserved.
Fig. 6: Behaviour of $V_+$ (solid curve) and $V_-$ (dashed curve), as a function of $\hat{r}_*$. The parameters are chosen from Region II of Fig. 1(a-b).

Fig. 7: Descriptions are same as in Fig. 3. See text for the choice of parameters.

Fig. 8: Descriptions are same as in Fig. 4. See text for the choice of parameters.

Fig. 9a: Steps (solid) approximating a potential (dotted) thus reducing the problem to that of a quantum mechanics. The parameters are $m_p = 0.8$ and $\sigma = 0.8$.

Fig. 9b: Comparison of variation of instantaneous reflection coefficient $R$ with the radial coordinate $\hat{r}_*$ using analytical WKB method (solid) and numerical step-potential method (dotted). The parameters are $m_p = 0.8$ and $\sigma = 0.8$.

Fig. 10(a-d): Comparison of (a) reflection and transmission coefficients and (b) real amplitude of the wave function $Z_+$ for $m_p = 0.8$ and $\sigma = 0.80$ (solid), 0.85 (dotted) and 0.90 (dashed) respectively. (c-d) Similar quantities for $m_p = 0.80$, (solid) 0.76 (dotted) and 0.72 (dashed) respectively keeping $\sigma = 0.8$ fixed. Higher reflection component enhances the wave amplitude, thus differentiating the incoming waves very clearly.
