Electron internal energy and internal motion (Zitterbewegung) as consequence of local U(1) gauge invariance in two-spinor language

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Abstract

Starting with the results obtained in a previous paper in which classical local U(1) gauge invariance in terms of the electromagnetic field strenghts instead of the usual formulation mediated by the four potential was introduced it is shown that using the gauge freedom associated with the third component of the magnetic field, previously obtained spinor equations of motion describe the internal dynamics of a free 1/2 spin particle suggesting a kinematic origin of its rest mass and helicity. The controversial Zitterbewegung (trembling motion) appears in a natural way as internal motion with the velocity of light. Such an interpretation is in contrast with the usual quantum mechanical explanation of transitions between positive and negative energy states.

1 Introduction

The main purpose of this article is to present the Zitterbewegung (trembling motion) phenomena from a different perspective and other outlook than the usual quantum mechanical interpretation based in the Dirac equation. As far as I know, the only study leading to the same results and similar interpretation dates back to 1990 (see [2]). Curiously enough both approaches being conceptually and technically quite different converge to the same results. From the technical and conceptual standpoint, the present work is
based on a previous paper [1] presenting a classical $U(1)$ local gauge invariance formulation via a lagrangian with an interaction in terms of the electric and magnetic field strengths. This approach is certainly different to the usual lagrangian leading to the Dirac Equation interacting with an external electromagnetic field described by the four potential $A^\mu$. As the present study is strongly based on the mentioned article, I have tried to find a compromise between continuous referencing to [1] and a certain self consistency while avoiding too much duplication.

The starting point of [1] (see also [5]) are the following linear first order differential spinor equations (the derivative is taken respect to the proper time $\tau$):

$$\begin{align*}
\frac{d\eta^A}{d\tau} &= \frac{e}{m} \phi^{AB} \eta_B \\
\frac{d\pi^A}{d\tau} &= -\frac{e}{m} \phi_{AB} \pi^B
\end{align*}$$

(1)

(A short derivation of these equations is included at the beginning of next section) These coupled spinor equations (natural units $\hbar = c = 1$ will be used) are equivalent to the Lorentz Force describing the motion of a $1/2$ spin particle of mass $m$ and charge $e$ (typically an electron) under an electromagnetic field described by the symmetric second-rank spinor $\phi_{AB}$, explicitely given by

$$\phi_{AB} = \frac{1}{2} \begin{pmatrix} [E_1 - i B_2] - i [E_2 - i B_2] & -E_3 + i B_3 \\
-E_3 + i B_3 & [-E_1 + i B_2] - i [E_2 - i B_2] \end{pmatrix}.$$  

(2)

In turn, $\phi_{AB}$ and its complex conjugate $\phi_{A'B'}$ form the antisymmetric four-rank electromagnetic field spinor in its standard form that can be found in [2] and any other book dealing with this subject.

$$F_{ABA'B'} = \epsilon_{AB} \bar{\phi}_{A'B'} + \epsilon_{A'B'} \phi_{AB},$$

(3)

where $\epsilon_{AB} = \epsilon^{AB}$ is the spinor metric

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(4)

(Capital indices are lowered (raised) by means of the metric spinor $\epsilon_{AB}$ already defined above. See Appendix 1)

The solution of equations [1] determine the four momentum of the particle given by the hermitian spinor defined as superposition of the two null
directions $\pi^A \bar{\pi}^{A'}$ and $\eta^A \bar{\eta}^{A'}$ as \[3\]

$$p^{AA'} = \frac{1}{\sqrt{2}} \left[ \pi^A \bar{\pi}^{A'} + \eta^A \bar{\eta}^{A'} \right]. \quad (5)$$

Since $p^{AA'}$ is to represent the four-momentum of a massive particle, must be time-like and certainly fulfill the condition:

$$p^{AA'} p_{AA'} = m^2. \quad (6)$$

On the other hand, following the standard representation, the different components of $p^{AA'}$ are labeled according to

$$p^{AA'} = \begin{pmatrix} p^{00'} & p^{01'} \\ p^{10'} & p^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} p^0 + p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 - p^3 \end{pmatrix}. \quad (7)$$

This last expression must be used when solving an specific case, via the spinor equations, to identify the components of $p^{AA'}$ in the solution.

2 \textbf{\textit{U(1) Local gauge transformation of field strength quantities and Zitterbewegung}}

As I have mentioned in the introduction, the particle field interaction, will not involve the four potential $A^\mu$ as in the traditional approach. Instead, the coupling will be to the electric and magnetic field strengths. To this end and to emphasize the geometrical origin of this formulation (see also \[5\]) consider an infinitesimal transformation of $\eta^A$ mediated by an element of the group $\text{SL}(2,C)$:

$$\delta \eta^A = \exp \left[ \frac{1}{2} (\tilde{\omega} \bar{\sigma} + i \tilde{\theta} \bar{\sigma}) \right] \eta^A, \quad (8)$$

being $\tilde{\omega}$ and $\tilde{\theta}$ the infinitesimal parameters of boosts and rotations respectively and $\bar{\sigma}$ the usual Pauli matrices. Since the transformation is infinitesimal, we can write

$$\delta \eta^A = \left[ I + \frac{1}{2} (\tilde{\omega} \bar{\sigma} + i \tilde{\theta} \bar{\sigma}) \right] \eta^A \quad (9)$$

3
According to the interpretation given in [5] associating the fields \( \vec{E} \) and \( \vec{B} \) with infinitesimal boosts and rotations, and via a classical detour, we would say that any change in the dynamic state of the particle should be proportional to the force field acting on the particle and the lapse of proper time. Accordingly

\[
\delta \omega = K \epsilon(x) \delta \tau \tag{10}
\]

\[
\delta \theta = K \beta(x) \delta \tau \tag{11}
\]

By substitution of \( \epsilon \) and \( \beta \) by \( \vec{E} \) and \( \vec{B} \) and expanding the term associated with the Pauli matrices we get a second rank spinor (coincident with the second rank electromagnetic field spinor \( \phi_{AB} \) ):}

\[
\phi_{AB}^A = \frac{1}{2} \begin{bmatrix}
E_3 & E_1 + iE_2 \\
E_1 - iE_2 & -E_3
\end{bmatrix} + i \begin{bmatrix}
B_3 & B_1 + iB_2 \\
B_1 - iB_2 & -B_3
\end{bmatrix} \tag{12}
\]

It is only necessary to lower the first upper index, following the rules given in the first appendix, to find the symmetric form (2) of the field spinor \( \phi_{AB} \).

On the other hand, from equation (9) and subsequents equations, making \( K = e/m \), it is immediate to obtain

\[
\frac{d}{d\tau} \eta^A = \frac{e}{m} \phi_{AB}^A \eta^B. \tag{13}
\]

As was done in [1] and repeated here for self-consistency, the lagrangian density, for a free particle, along the classical path of the particle (with dimension energy per unit length) is

\[
\mathcal{L} = \pi_A \dot{\eta}^A, \tag{14}
\]

(dot denoting derivative respect to proper time \( \tau \)) together with the Euler Lagrange equation

\[
\frac{d}{d\tau} \frac{\partial \mathcal{L}}{\partial \dot{\eta}^A} - \frac{\partial \mathcal{L}}{\partial \eta^A} = 0. \tag{15}
\]

With a similar equation for the spinor \( \pi^A \). The former equation leads to

\[\dot{\pi}_A = 0 \Rightarrow \pi_A = \text{const.}\]

\(^1\)In the definition of the four-momentum \( p^{AA'} \), the spinors \( \pi^A \) and \( \eta^A \) enter on equal footing. It is then clear that the lagrangian could also be defined swapping both spinors.
We consider now the consequences of imposing invariance under local \((\text{along the classical path parametrized by } \tau)\) phase transformations

\[
\begin{align*}
\eta^A &\rightarrow e^{i\alpha(\tau)}\eta^A \\
\pi^A &\rightarrow e^{i\xi(\tau)}\pi^A.
\end{align*}
\tag{16}
\]

The phase parameters \(\alpha(\tau)\) and \(\xi(\tau)\) cannot be independent as the spinors \(\eta^A\) and \(\pi^A\) are also not independent since from (5), they are related by the condition

\[
p^{AA'}p_{AA'} = |\pi^A\eta_A|^2 = m^2.
\tag{17}
\]

In consequence \(\eta^A\pi_A = \text{const.}\), leading to the constraint \(\xi(\tau) = -\alpha(\tau)\).

As the classical trajectory should not be affected by any phase transformation, it is apparent that local gauge transformations leaves invariant the four-momentum of the particle:

\[
p^{AA'} = \frac{1}{\sqrt{2}} \left[ \pi^A\bar{\pi}^{A'} + \eta^A\bar{\eta}^{A'} \right].
\]

However, the free lagrangian (14) transforms to

\[
\mathcal{L} \rightarrow i\dot{\alpha}\eta^A\pi_A + \dot{\eta}^A\pi_A = i\dot{\alpha}\epsilon_{AB}\eta^B\pi^A + \dot{\eta}^A\pi_A.
\tag{18}
\]

To find a gauge invariant lagrangian we have to add a term

\[
-\frac{e}{m}\phi_{AB}\eta^B\pi^A,
\tag{19}
\]

and impose the condition for the new field \(\phi_{AB}\) of transforming, under local phase transformations, as \(\phi_{AB} \rightarrow \phi_{AB} + \frac{m}{e}\delta\epsilon_{AB}\)

\[
\phi_{AB} \rightarrow \phi_{AB} + \frac{m}{e}\delta\epsilon_{AB},
\tag{20}
\]

\footnotemark[2]From a pure mathematical point of view, the validity of transformation (20) is consequence of the following theorem applied to valence-2 spinors (see Stewart J. \textit{Advanced General Relativity}. 1991 Cambridge Univ. Press. Page 69): “Any spinor \(\tau_{A...F}\) is the sum of the totally symmetric spinor \(\tau_{(A...F)}\) and (outer) products of \(\epsilon\)'s with totally symmetric spinors of lower valence.
then, the new lagrangian
\[ \mathcal{L} = \dot{\eta}^A \pi_A - \frac{e}{m} \phi_{AB} \eta^B \pi^A, \] (21)
is invariant under \( U(1) \) local-phase transformations. The transformation that holds for the conjugate second-rank spinor \( \bar{\phi}_{A'B'} \), is given by
\[ \bar{\phi}_{A'B'} \rightarrow \bar{\phi}_{A'B'} - \frac{im}{e} \dot{\alpha} \epsilon_{A'B'}. \] (22)

These kind of transformations leave however invariant the associated four-rank spinor of the Maxwell field strength \( F_{A'B'A'B'} \)
\[ F_{A'B'A'B'} = \epsilon_{AB} \bar{\phi}_{A'B'} + \epsilon_{A'B'} \phi_{AB}. \]

From the Euler Lagrange equations applied to the lagrangian given by (21) it is immediate to obtain
\[ \dot{\pi}_A = -\frac{e}{m} \phi_{AB} \eta^B. \] (23)

This equation and those of (1) are gauge invariant. If the invariance of the four-rank spinor \( F_{A'B'A'B'} \) follows from the transformation rules of the field spinors \( \phi_{AB} \) and \( \bar{\phi}_{A'B'} \) a simple look at (2) and (4) reveal that only the components \( \phi_{01} \) and \( \phi_{10} \) (together with their complex conjugates) are affected. Furthermore since the transformation is purely imaginary, there is only one field quantity (i.e. \( B_3 \)) affected by the transformations. Consequently, the spinor equations are modified in much the same way as a gauge change in the electromagnetic potential \( A^\mu \) modifies the equations and their solutions (for example: the Coulomb gauge in QED). It is clear that this peculiarity of the third component of the magnetic field deserves further attention. In what follows, we shall examine the consequences of this local gauge invariance for \( B_3 \). For simplicity the case of \( B_3 = \text{const.} \) will be studied. Given the equation
\[ \dot{\eta}_A = -\frac{e}{m} \phi_{AB} \eta^B, \] (24)
and applying the gauge transformations:
\[ \phi_{01} \rightarrow \tilde{\phi}_{01} = \phi_{01} + \frac{m}{e} \dot{\alpha} \epsilon_{01} \] (25)
\[ \phi_{10} \rightarrow \tilde{\phi}_{10} = \phi_{10} + \frac{im}{e} \hat{\alpha}_{10} \]  

(26)

The transformed equations for the physical components (upper indices) are

\[ \dot{\eta}^0 = \frac{e}{m} \left( \frac{-B_3}{2} + \frac{me}{\hat{\alpha}} \right) \eta^0 \]  

(27)

\[ \dot{\eta}^1 = \frac{e}{m} \left( \frac{B_3}{2} + \frac{me}{\hat{\alpha}} \right) \eta^1. \]  

(28)

As in the last equations we cannot gauge away simultaneously both effective fields, let us choose

\[-\frac{B_3}{2} + \frac{me}{\hat{\alpha}} = 0,\]  

(29)

then

\[ \dot{\eta}^0 = 0 \implies \eta^0 = \text{const.} = \sqrt{\hat{m}} \]  

The integration constant \( \sqrt{\hat{m}} \) needs to have the dimension of an energy squared. Under condition (29), (28) reduce to

\[ \dot{\eta}^1 = \frac{e}{m} \left( \frac{2m}{e} \hat{\alpha} \right) \eta^1 = i2\hat{\alpha} \eta^1 \]  

(30)

As both charge and mass of the particle as well as magnetic field have disappeared, the last relation should be valid for any 1/2 spin particle. Doing the integration:

\[ \eta^1 = \sqrt{\hat{m}} e^{i2\hat{\alpha} \tau} \]  

(31)

The spinor \( \eta^A \) is

\[ \eta^A = \sqrt{\hat{m}} \left( \frac{1}{e^{i2\hat{\alpha} \tau}} \right). \]  

(32)

As for the other spinor and proceeding in the same way:

\[ \pi^A = \sqrt{\hat{m}} \left( \frac{1}{e^{i2\hat{\alpha} \tau}} \right). \]  

(33)

Denoting \( \hat{\alpha} = \omega \), from the four-momentum definition (5) and (7)

\[ p^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2\hat{m} & 2\hat{m} e^{i2\omega \tau} \\ 2\hat{m} e^{-i2\omega \tau} & 2\hat{m} \end{pmatrix}. \]  

(34)

Remembering that \( Det[p^{AA'}] \) equals 1/2 the Lorentz norm, we have a null path in momentum spacetime. After some lengthy calculations to solve for the components:

\[ p^0 = E = 2\hat{m} \]
\[ p^3 = 0 \]
\[ p^1 = 2 \dot{m} \cos 2 \omega \tau \]
\[ p^2 = -2 \dot{m} \sin 2 \omega \tau . \]

2\(mc^2/h\) (in conventional units) is the so-called Zitterbewegung (trembling motion), found in the Dirac Theory and subject of many controversial interpretations. From the last and following relations, we shall find a classical interpretation as an internal circular motion taking place inside of the particle (electron). It makes sense now to identify \(\dot{m}\) as the electron mass. Since \(p^i = 2mu^i, (i = 1, 2)\)

\[ (p^1)^2 + (p^2)^2 = 4m^2 \implies (u^1)^2 + (u^2)^2 = 1. \quad (35) \]

(Circular motion in the x-y plane with the velocity of light).

Let us now take the other alternative (20), namely

\[ \frac{B_3}{2} + \frac{m}{e} \dot{\alpha} = 0 \quad (36) \]

The solutions are now

\[ \pi^A = \sqrt{m} \begin{pmatrix} e^{i2\alpha \tau} \\ 1 \end{pmatrix} \quad (37) \]
\[ \eta^A = \sqrt{m} \begin{pmatrix} e^{i2\alpha \tau} \\ 1 \end{pmatrix}. \quad (38) \]

Performing the same calculations as in the other case

\[ p^1 = 2m \cos 2 \omega \tau \]
\[ p^2 = 2m \sin 2 \omega \tau \]

The spatial trajectories in the x-y plane can be immediately obtained:

\[ x = \frac{1}{2\omega} \sin 2 \omega \tau \]
\[ y = -\frac{1}{2\omega} \cos 2 \omega \tau \]

and

\[ x = \frac{1}{2\omega} \sin 2 \omega \tau \]
\[ y = \frac{1}{2\omega} \cos 2 \omega \tau \]

Being clock and counter clock-wise respectively.

The previous results concerning the momentum and position of the electron behavior are in agreement with those obtained in [2] (see equation 64 in the cited work).
3 Discussion

A fundamental question that, perhaps, we have some reasons to ask now is what is the origin of the electric charge. Previously the charge “e” disappeared in (23) and did not appear again. However, the results seem independent of the charge. Since the neutrinos have a certain mass, and possibly a magnetic moment, it may be possible to explain the opposite helicities of neutrinos within the three families. As early as 1930, Schrodinger [4] made the first analysis of what was to be called afterwards “Zitterbewegung” (Trembling Motion). The frequency $2\omega$ also appears in the solutions of the Dirac Equation for the propagation of a free packet. He interpreted $\omega_0 = \frac{2mc^2}{\hbar}$ as a fluctuation in the position of the electron with radius

$$\Delta r = \frac{c}{\omega_0} = \frac{\hbar}{2mc}.$$ 

Assuming a velocity of the electron equal to the velocity of light about some mean position inducing an spin angular momentum

$$\Delta r.mc = \frac{\hbar}{2}.$$ 

A relatively modern interpretation [8] is that there are unavoidably cross terms between the positive and negative energy solutions which oscillate rapidly in time with frequencies

$$\frac{2p_0c}{\hbar} \geq \frac{2mc^2}{\hbar}.$$ 

Perhaps the reader will ask why $B_3$? The answer has to do with the choice of the third Pauli matrix $\sigma_3$. There is nothing special about the $z$-axis, but once we choose this axis for $\sigma_3$, the $z$-axis has, in practice, a special relevance. (More technically, magnetic fields are related to rotations described by the SO(3) group having SU(2) as covering group (see [9], [1], and [10]).

Finally, and as already mentioned, the results in this work seems to be in agreement with those obtained in [2].

Appendix 1: A Short Introduction to Spinor Calculus

With the aim of making this article accessible to potential readers not familiar with two-spinor formalism, I consider that a very simple and basic introduction to spinor calculus would be particularly helpful for the obvious reason that the formalism originally developed by Penrose and Rindler
in their books “Spinors and space-time”, [7] that I have cited in this and
previous papers, is not widely used or very familiar to a large number of
physicists.

Spinors like $\eta^A$ or $\pi^A$ belong to a simplectic complex two-dimensional
vector space $S$. The complex conjugate vector space $S'$ has elements $\bar{\eta}^{A'}$. We also need to consider the two dual spaces $S^*, S'^*$ with elements $\xi_A, \bar{\xi}_{A'}$.

$$
\eta^A = \begin{pmatrix} \eta^0 \\ \eta^1 \end{pmatrix} \in S, \quad \bar{\eta}^{A'} = \begin{pmatrix} \bar{\eta}^{0'} \\ \bar{\eta}^{1'} \end{pmatrix} \in S'
$$

$$
\xi_A = (\xi_0, \xi_1) \in S^*, \quad \bar{\xi}_{A'} = (\bar{\xi}_0, \bar{\xi}_1) \in S'^* \quad (39)
$$

Just as the familiar metric tensor $\eta_{\mu\nu}$ of Minkowski space, in $S$ we also have a metric spinor

$$
\epsilon_{AB} = \epsilon^{A'B'} = \epsilon_{AB} = \epsilon_{A'B'} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (40)
$$

relating any spinor $\eta^A$ with $\eta_A$ according to the rules:

$$
\eta^A = \epsilon^{AB} \eta_B
$$

and

$$
\xi_A = \epsilon_{BA} \xi_B,
$$

with similar rules for the complex conjugate quantities. It follows that the components of $\eta^A$ are related to the components of $\eta_A$ by

$$
\eta^0 = \eta_1, \quad \eta^1 = -\eta_0
$$

Accordingly, for any spinor

$$
\eta^A \eta_A = \eta^0 \eta^1 - \eta^1 \eta^0 = 0
$$

Care is needed with the index ordering because $\epsilon_{AB}$ is skew:

$$
\eta^A = \epsilon^{AB} \eta_B = -\epsilon_{BA} \eta_B.
$$

Just as in ordinary tensor calculus, spinors of higher rank, like those used in this work: $p^{AA'}$, $\phi_{AB}$ and $F_{ABA'B'}$, can be defined. In particular, there is an isomorphism between real four vectors in Minkowskian space-time and hermitian second rank spinors:

$$
V^\alpha \rightarrow V^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} V^0 + V^3 & V^1 + iV^2 \\ V^1 - iV^2 & V^0 - V^3 \end{pmatrix}. \quad (41)
$$
The Lorentz norm equals one half the determinant of the above matrix. In particular, if the vector is null, like $\pi^A \bar{\pi}^{A'}$, it may be written as the outer product of a complex two-dimensional vector and its complex conjugate:

$$\pi^A \bar{\pi}^{A'} = \begin{pmatrix} \pi^0 \bar{\pi}^0' & \pi^0 \bar{\pi}^{1'} \\ \pi^1 \bar{\pi}^0' & \pi^1 \bar{\pi}^{1'} \end{pmatrix}. \quad (42)$$

Appendix 2: Classical Weyl-spinor lagrangian and quantum Weyl and Dirac lagrangians

As we have seen the rather simple choice, in spinor language, of the lagrangian density leading to the spinor equation of motion is

$$L = \pi_A \dot{\eta}^A, \quad (14b)$$

in contrast, by comparison, with the familiar expression of the Dirac quantum lagrangian for a free particle

$$L = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi. \quad (43)$$

One thing to note is that while the spinor lagrangian density has dimension energy per unit length, the dimension of the classical lagrangian is energy thus emphasizing the deep difference between the standard approach and the spinorial one (note that the mass is also absent in the spinor definition of the four momentum). In any case it is apparent that classical lagrangians for a free particle only contain one term.

If we try to find the Weyl version of the Dirac Equation (DE) via the usual lagrangian approach we need to add a mass term. In [1] the Weyl 2-spinor version of the DE is found in a simple way starting from

$$p^{AA'} = \frac{1}{\sqrt{2}} [\pi^A \bar{\pi}^{A'} + \eta^A \bar{\eta}^{A'}]$$

(see (22) and subsequent equations in [1]).

Following now the standard approach, in the DE case the Euler-Lagrange equations to obtain the DE from the lagrangian (43) are

$$\partial_\mu \frac{\partial L}{\partial \psi^\mu} - \frac{\partial L}{\partial \psi} = 0, \quad (44)$$

and a similar one for $\bar{\psi}$. As it is standard material found in many books, we shall not develop further the steps leading to the DE. However, in our
Weyl two-spinor approach things are, if not substantially more involved, less familiar. Let us start with the free lagrangian (14b). Spinors $\pi^A$ and $\eta^A$ are now no longer defined along any classical path parameterized by proper time but instead should be regarded as functional or distributions in four dimensional spacetime. In consequence, the derivative with respect to proper time should be replaced by general derivatives in the context of two-spinor calculus:

$$\dot{\pi}^A \rightarrow \nabla^{AA'}\pi_{A'}$$

$$\dot{\eta}_{A'} \rightarrow \nabla_{AA'}\eta^A.$$  

The appropriate Euler-Lagrange equations in two-spinor calculus (generalization of (15)) is

$$\nabla^{AA'} \frac{\partial L}{\partial (\nabla^{AA'}\pi_{A'})} - \frac{\partial L}{\partial \pi_{A'}} = 0$$  

and

$$\nabla_{AA'} \frac{\partial L}{\partial (\nabla_{AA'}\eta^A)} - \frac{\partial L}{\partial \eta^A} = 0.$$  

The two coupled lagrangians are defined as

$$L = \pi_A \nabla^{AA'}\pi_{A'} + \frac{m}{\sqrt{2}} \eta^{A'}\pi_{A'}$$  

and

$$L = \bar{\eta}^{A'} \nabla_{AA'}\eta^A - \frac{m}{\sqrt{2}} \eta^A \pi_A.$$  

Taking the first

$$\frac{\partial L}{\partial (\nabla^{AA'}\pi_{A'})} = \pi_A$$

$$\frac{L}{\partial \pi_A} = \frac{m}{\sqrt{2}} \eta_{A'},$$

and from (50):

$$\nabla^{AA'}\pi_A = \frac{m}{\sqrt{2}} \bar{\eta}_{A'}.$$  

Performing the same calculations with the second we arrive at the 2-spinor version of the DE in Weyl representation as they appear in [7] :

$$\nabla^{AA'}\pi_A = \frac{m}{\sqrt{2}} \bar{\eta}_{A'}$$

$$\nabla_{AA'}\eta^A = -\frac{m}{\sqrt{2}} \pi_A.$$
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