Construction of Boundary Conditions for Navier–Stokes Equations from the Moment System

Ruo Li1 · Yichen Yang2 · Yizhou Zhou2

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Abstract
This work concerns with boundary conditions (BCs) of the linearized moment system for rarefied gases. Under the acoustic scaling, we analyze the boundary-layer behaviors of the moment system by resorting to a three-scale asymptotic expansion. The asymptotic analysis casts the flows into the outer solution, the viscous layer, and the Knudsen layer. Starting from the BCs of the moment system, we construct BCs for the Navier–Stokes equations by a matching requirement. The obtained BCs contain the effect of second-order terms on the velocity slip and temperature jump. For the illustrative case of the Rayleigh problem, we prove the validity of the constructed BCs through error estimates. Meanwhile, numerical tests are presented to show the performance of the constructed BCs.

Keywords Moment system · Boundary conditions · Navier–Stokes equations · Boundary layers

1 Introduction

For gases in the low-density regime and microscales, one should consider kinetic theory of gases, described by the Boltzmann equation [12]. Since the Boltzmann equation is a problem in high (seven) dimensions, the direct simulation will lead to much more computational cost than the hydrodynamic equations. Grad proposed the famous moment method [20] to reduce the kinetic equation into low-dimensional moment systems. These systems are first-order partial differential equations, which may be regarded as intermediate models between...
the Boltzmann equation and hydrodynamic equations. Recently, with the development of hyperbolic regularization [5–7, 17, 28], the moment method has attracted more attention and become a powerful tool in the simulation of gas flow.

In this paper, we mainly focus on the acoustic scaling behavior of the moment system. The rarefaction effects of the gas are characterized by the Knudsen number \( \varepsilon = \lambda / L \) with \( \lambda \) the mean free path length and \( L \) the relevant characteristic length. Under the acoustic scaling, the dimensionless Boltzmann equation and its associated moment system contain \( 1 / \varepsilon \) in the collision term. The Euler equations and the Navier–Stokes (NS) equations can be formally derived from the moment system for small \( \varepsilon \) [52]. For the initial value problems, the rigorous proofs of the derivations are given in [16, 35, 62] by resorting to the structural stability criterion [60]. It was proved that the \( \varepsilon \)-dependent solution of the moment system converges to the solution of the Euler equations as \( \varepsilon \) goes to zero. Moreover, the error estimates [35] indicate that the error between the solution of the moment system and the solution of the NS equations is of \( o(\varepsilon) \), i.e., the error goes to zero faster than \( \varepsilon \) as \( \varepsilon \to 0 \). In this sense, the NS equations are called the first-order approximation of the moment system.

Considering the simulation in a bounded domain, proper boundary conditions (BCs) should be prescribed, which is rather challenging for moment equations [52]. Following the basic idea in [20], the work [9] provides a method to derive BCs for the moment closure system from the Maxwell BCs [37] for the Boltzmann equation. As far as we know, most works for BCs of the moment system are about numerical simulations [21, 24, 29, 56] while theoretical results seem quite rare. It was shown in [8, 11] that the number of derived BCs for the moment system equals the number of positive characteristic speeds. In [61], the authors discuss the BCs for a linearized four moments system and show the existence of relaxation limit. In the present work, our goal is to further investigate BCs of the general moment system and connect them with BCs of the NS equations.

For the initial-boundary value problems (IBVPs) with small parameters, it is necessary to consider the effect of boundary layers in the asymptotic analysis. At this point, we would like to remark that the asymptotic theory of the Boltzmann equation (with boundaries) for small Knudsen number has been systematically developed since the pioneer works of Sone in 1960s for steady flows [47–50]. For the time-dependent problem, by resorting to Sone’s slip flow theory, a complete formal asymptotic derivation has been given in [27] for the acoustic scaling and in [55] for the diffusive scaling. Particularly, under the acoustic scaling, the work [27] uses a three-scale asymptotic solution to the Boltzmann equation by separating the whole field into the acoustic zone, the viscous boundary layer, and the Knudsen layer. Our derivation for the moment system refers to the theory of IBVPs for hyperbolic relaxation systems [59, 63, 64] and exploits the three-scale expansion as well.

For the general linearized moment system, we present a modification of the Grad BCs such that they are maximal positive, which is important for the well-posedness of the symmetric hyperbolic system [31, 38, 41]. By performing an asymptotic analysis, we show the boundary-layer behaviors of the linearized moment system with Maxwell-type BCs. Specifically, we assume that the domain is the half-space \( \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0 \} \) with the boundary \( \{ x_2 = 0 \} \). Then the asymptotic solution \( W_\varepsilon \) is constructed by

\[
W_\varepsilon (t, x^w; x_2) = \overline{W} (t, x^w; x_2) + \hat{W} \left( t, x^w; \frac{x_2}{\sqrt{\varepsilon}} \right) + \tilde{W} \left( t, x^w; \frac{x_2}{\varepsilon} \right), \quad x^w = (x_1, x_3).
\]

Here \( \overline{W} \) is the outer (bulk) solution while \( \hat{W} \) and \( \tilde{W} \) are boundary-layer corrections. Through a standard formal asymptotic analysis, we show how to determine each coefficient in the asymptotic solution of the moment system. The procedure is parallel to the expansions in
Sone’s generalized slip flow theory \cite{50} and the coefficients computed from the moment system match those in \cite{27,50}. Although the moment method can be regarded as the approximate model of the Boltzmann equation, there seems rare work on the asymptotic analysis of the moment equations. The present work helps to understand the time-dependent behavior of the moment equations around the boundary under the acoustic scaling. In particular, the asymptotic analysis indicates that the moment system with proper BCs inherits the boundary-layer structure of the original Boltzmann equation (summarized in \cite{50}).

Furthermore, we connect BCs for the moment system with those for the hydrodynamics models (the Euler or NS equations). Under the acoustic scaling, the Euler equations are the limiting equations of the moment system as $\varepsilon \to 0$. Thus we require their BCs to be the so-called reduced BCs (BCs satisfied by the relaxation limit) of the moment system. The derivation of reduced BCs has been developed in \cite{59,63,64} for general relaxation systems.

As for the NS equations—the first-order approximation to the moment system, we construct the BCs by:

- **Matching requirement**: the solution to the NS equations with constructed BCs approximates $\bar{W}(t, x^w; x_2) + \hat{W}(t, x^w; x_2/\sqrt{\varepsilon})$ with an error of $o(\varepsilon)$ for sufficiently small $\varepsilon$.

The matching requirement means that we expect the NS equations together with their BCs are satisfactory approximations for the moment system in the $x_2$ and $x_2/\sqrt{\varepsilon}$ spatial scales. Our resultant BCs for the NS equations recover those in \cite{27}, which include terms with second-order spatial derivatives. This kind of BCs is useful in considering some physical phenomena with relatively large Knudsen numbers, while the first-order slip BCs may lose accuracy (see e.g. \cite{12,15,37,51}). Besides \cite{27}, we also review other works for BCs of the NS equations. The slip BCs for the NS equations have been extensively studied both in theory and experiment \cite{12,57}. The representative work \cite{14} derived the slip BCs for steady flows with stationary boundaries. For time-dependent problems, the work \cite{2} presents a method to derive the slip BCs with explicit values of the slip coefficients. This method is mainly based on the Chapman-Enskog expansion and the analysis of the Knudsen layer developed in \cite{50}. More related works include \cite{25,30}.

The main objective of the present paper is the moment system. We give three remarks to illustrate our motivation in considering the moment method: (1) Theoretically, the derivation through the moment method may provide more possibilities to present rigorous proofs. Take the Rayleigh problem as an illustrative example. In this simplified case, we prove the well-posedness of BCs for the moment system by checking the strictly dissipative condition \cite{3} and show the validity of asymptotic solutions by the energy method. Through an error estimate, we rigorously prove that the constructed Robin-type BCs for the NS equations satisfy the aforementioned matching requirement. (2) Numerically, our derivation automatically gives a discretized version of the Knudsen layer equation \cite{50} by the Grad moment method. Actually, we describe the Knudsen layer behavior by half-space problems for the moment equations. For such problems, the well-posedness and solvability have been shown in \cite{33} and effective algorithms can be applied easily. (3) Note that the moment method is not restricted to the Boltzmann equation but provides a general framework for the kinetic equation. The present paper tries to identify the issues (e.g. the boundary-layer analysis) in learning IBVPs for moment systems. The results are expected to be generalized to other moment systems in future work.

The rest of the paper is organized as follows. As a preparation, Sect. 2 is devoted to reviewing the necessary background and introducing basic notations. In Sect. 3, by considering the Rayleigh problem, we illustrate our basic idea through a simplified model. For the general linearized moment system, the asymptotic equations are derived in Sect. 4. Based on this,
the BCs for the NS equations are constructed in Sect. 5. The numerical tests are presented in Sect. 6. At last, the conclusion is given in Sect. 7.

2 Background

2.1 The Linearized Boltzmann Equation

We consider the non-dimensional linearized Boltzmann equation (LBE) under the acoustic scaling [50, Chapter 1],

\[ \partial_t f + \xi \cdot \nabla_x f = \mathcal{L}(f)/\varepsilon, \tag{2.1} \]

where \( t \in \mathbb{R}^+ \) is the non-dimensional time, \( x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3 \) the non-dimensional spatial coordinates, \( \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \) the non-dimensional microscopic velocity, and \( f = f(t, x, \xi) \) denotes the perturbed distribution function around the Maxwellian at rest:

\[ \mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp \left( -\frac{1}{2} \right). \tag{2.2} \]

The linearized collision operator \( \mathcal{L} \) describes the interaction between particles and its explicit form is given in Appendix A for the Shakhov model and hard-sphere (HS) gas. Here \( \varepsilon \) represents the Knudsen number and we assume that \( \varepsilon \) is small.

Then the macroscopic variables can be defined as

\[
\begin{align*}
\rho &= \langle f \rangle, & u_i &= \langle \xi_i f \rangle, & p_{ij} &= \langle \xi_i \xi_j f \rangle, & p &= \frac{1}{3} \sum_{i=1}^{3} p_{ii}, \\
\theta &= p - \rho = \left( \frac{1}{3} - \frac{1}{3} \right) f, & \sigma_{ij} &= p_{ij} - p \delta_{ij}, & q_i &= \left( \frac{1}{2} \right) \xi_i f,
\end{align*}
\] (2.3)

where

\[ \langle \cdot \rangle = \int_{\mathbb{R}^3} \cdot \, d\xi. \]

We call \( \rho = \rho(t, x) \) the density, \( u = u(t, x) = (u_1, u_2, u_3) \) the macroscopic velocity, \( \theta = \theta(t, x) \) the temperature, \( p_{ij} \) the pressure tensor, \( p \) the pressure, \( \sigma_{ij} \) the stress tensor and \( q = (q_1, q_2, q_3) \) the heat flux.

At a simple boundary, i.e., there is no mass flow across it, the classical Maxwell specular-diffuse BC for the non-dimensional LBE (2.1) reads as

\[ f(t, x, \xi) = \begin{cases} f(t, x, \xi), & \xi - u^w \cdot n \geq 0, \\
\chi f^w(t, x, \xi) + (1 - \chi) f(t, x, \xi^w), & \xi - u^w \cdot n < 0,
\end{cases} \tag{2.4} \]

where \( n = n(t, x) = (n_1, n_2, n_3) \) is the normal vector pointing out the domain and \( \chi \in [0, 1] \) is the (tangential momentum) accommodation coefficient (AC). When \( \chi = 0 \), the BC turns to the specular-reflection BC, with

\[ \xi^w = \xi - 2((\xi - u^w) \cdot n)n. \]

When \( \chi = 1 \), it is the diffuse-reflection BC for the non-dimensional LBE, determined by

\[ f^w(t, x, \xi) = \mathcal{M}(\xi) \left( \rho^w + u^w \cdot \xi + \theta^w \frac{1}{2} \xi^2 \right), \]

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where \( u^w = u^w(t, x) \) and \( \theta^w = \theta^w(t, x) \) are (macro) velocity and temperature of the wall. The density \( \rho^w \) is chosen such that the no mass flow condition holds at the wall:

\[
\int_{\mathbb{R}^3} [(\xi - u^w) \cdot n] f \, d\xi = 0.
\]

For simplicity, here and hereafter we consider the planar boundary \( \partial \Omega = \{ x_2 = 0 \} \) with \( n = (0, -1, 0) \) for all time and

\[ u^w \cdot n = 0. \]

### 2.2 The Grad Moment System

In Grad’s framework [20], one can take moments on both sides of the LBE to obtain an infinite moment system

\[
\frac{\partial}{\partial t} \langle f \phi_\alpha \rangle + \sum_{d=1}^{3} \frac{\partial}{\partial x_d} \langle \xi_d f \phi_\alpha \rangle = \frac{1}{\varepsilon} \langle L[f] \phi_\alpha \rangle, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \tag{2.5}
\]

where \( \phi_\alpha = \phi_\alpha(\xi) \) is the orthonormal Hermite polynomial [19] with the weight function \( \mathcal{M} \) given in (2.2). Namely,

\[
\{ \mathcal{M} \phi_\alpha \phi_\beta \} = \delta_{\alpha, \beta}.
\]

If we let \( \phi_0 = 1 \), then the above relation will [19] lead to \( \phi_{e_i} = \xi_i \) and

\[
\xi_d \phi_\alpha = \sqrt{\alpha_d} \phi_{\alpha - e_d} + \sqrt{\alpha_d + 1} \phi_{\alpha + e_d}, \quad d = 1, 2, 3, \tag{2.6}
\]

where \( e_1 = (1, 0, 0) \in \mathbb{N}^3 \), etc., and \( \phi_\alpha \) is regarded as zero if \( \alpha \) has negative components. The obtained Hermite polynomials are isotropic, i.e.,

\[
\phi_\alpha = \prod_{i=1}^{3} \phi_{\alpha_i e_i},
\]

For example, if we take \( d = 2 \) and \( \alpha = e_1 \), the relation (2.6) gives

\[
\xi_2 \phi_{e_1} = \sqrt{0} \phi_{e_1 - e_2} + \sqrt{1} \phi_{e_1 + e_2} \quad \Rightarrow \quad \phi_{e_1 + e_2} = \xi_1 \xi_2.
\]

If we take \( d = 1 \) and \( \alpha = e_1 \), the relation (2.6) gives

\[
\xi_1 \phi_{e_1} = \sqrt{1} \phi_0 + \sqrt{2} \phi_{2e_1} \quad \Rightarrow \quad \phi_{2e_1} = \frac{1}{\sqrt{2}} (\xi_1^2 - 1).
\]

Similarly, one can get the other Hermite polynomials recursively.

Denote the moment variables by

\[
w_\alpha = \langle f \phi_\alpha \rangle.
\]

By definition, we can immediately relate the low-order moment variables with the macroscopic variables defined in (2.3) as

\[
\rho = w_0, \quad u_i = w_{e_i}, \quad \theta = \frac{\sqrt{2}}{3} \sum_{i=1}^{3} w_{2e_i}, \quad \sigma_{ij} = \sqrt{1 + \delta_{ij}} w_{e_i + e_j} - \theta \delta_{ij}, \quad q_i = \frac{1}{2} \sum_{j=1}^{3} \sqrt{(e_i + 2e_j)!} w_{e_i + 2e_j}, \tag{2.7}
\]

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where $\alpha! = \alpha_1! \alpha_2! \alpha_3!$.

To close the infinite system (2.5), Grad’s moment method considers the ansatz

$$f \in \text{span}\{M\phi_\alpha, |\alpha| \leq M\} \Rightarrow f = M \sum_{|\alpha| \leq M} w_\alpha \phi_\alpha,$$

where $M$ is a given integer called the moment order and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. Substituting the ansatz into (2.5) yields Grad’s $M$-th order moment system

$$\frac{\partial W}{\partial t} + \sum_{d=1}^{3} A_d \frac{\partial W}{\partial x_d} = -\frac{1}{\varepsilon} QW,$$

(2.8)

where $W \in \mathbb{R}^N$ with $N = \#\{\alpha \in \mathbb{N}^3 : |\alpha| \leq M\}$. Following the notations in [6], the vector $W$ and the matrices $A_d, Q$ can write as

$$W[N(\alpha)] = w_\alpha, \quad A_d[N(\alpha), N(\beta)] = \langle M\xi_d \phi_\alpha \phi_\beta \rangle = \sqrt{\alpha_d} \delta_{\alpha,\beta} - e_d + \sqrt{\alpha_d + 1} \delta_{\beta,\alpha} + e_d,$$

$$Q[N(\alpha), N(\beta)] = -\langle L[M\phi_\beta] \phi_\alpha \rangle.$$

(2.9)

Here the notation $W[N(\alpha)]$ represents the $N(\alpha)$-th element of the vector $W \in \mathbb{R}^N$, where

$$N : \{\alpha \in \mathbb{N}^3 : |\alpha| \leq M\} \rightarrow \{1, 2, \ldots, N\}$$

is a one-to-one mapping given in Appendix B. Analogously, $A_d[N(\alpha), N(\beta)]$ represents the $N(\alpha)$-th row and $N(\beta)$-th column of the matrix $A_d \in \mathbb{R}^{N \times N}$. As an illustrative example, we see that if only $\alpha = e_1 + ke_2$ ($k = 0, 1, \cdots, M - 1$) are considered and $M$ is even, then

$$A_2 = \begin{pmatrix} 0 & M_c^T \\ M_c & 0 \end{pmatrix}, \quad M_c = \begin{pmatrix} \sqrt{2} & \sqrt{3} & \sqrt{4} & \sqrt{5} & \cdots & \sqrt{M-2} & \sqrt{M-1} \end{pmatrix}.$$

(2.10)

In general, $A_d$ is also symmetric and we have

$$A_2 = \begin{pmatrix} 0 & M_o \\ M_o^T & 0 \end{pmatrix},$$

(2.11)

where $M_o \in \mathbb{R}^{m \times n}$ is of full column rank (see Appendix B), $m = \#I_e, n = N - m = \#I_o$ with

$$I_e = \{\alpha \in \mathbb{N}^3 : |\alpha| \leq M, \alpha_2 \text{ even}\}, \quad I_o = \{\alpha \in \mathbb{N}^3 : |\alpha| \leq M, \alpha_2 \text{ odd}\}.$$

The matrix $Q$ is assumed to be symmetric positive semi-definite and its explicit form is given in Appendix A for the Shakhov model and HS gas. In the case of (2.10), the matrix $Q$ for the BGK model reads as

$$Q = Q_c = \text{diag}(0, 1, \cdots, 1).$$

(2.12)

Now we turn to consider the BCs for the moment system (2.8). Following Grad’s idea [20], to ensure the continuity of BCs when $\chi \rightarrow 0$, one can test the Maxwell BC (2.4) by odd polynomials (with respect to $\xi_2$) to construct BCs for moment variables. One classical choice [9, 18, 20] is $\phi_\alpha$ with $\alpha \in I_o$. On the other hand, as shown in [34], we can equivalently
choose the test polynomials as $\xi_2^\alpha$ with $\alpha \in \mathbb{Z}_+$ and $|\alpha| \leq M - 1$. The procedure directly gives the Grad BCs for (2.8) in the even-odd parity form:

$$E \left( \hat{\chi}, S, M_0 \right)(W - b) = 0, \quad u \cdot \mathbf{n} = 0,$$

where $M_0 \in \mathbb{R}^{m \times n}$ as in (2.11), $S \in \mathbb{R}^{m \times m}$ symmetric positive definite and $E = (I_n, 0) \in \mathbb{R}^{n \times m}$ with $I_n$ the $n$-th order identity matrix. The explicit forms of $M_0$ and $S$ are put in Appendix B. Here

$$\hat{\chi} = \frac{2\chi}{(2 - \chi)\sqrt{2\pi}}$$

and $b[\mathcal{N}(\alpha)] = b_\alpha$, with $b_0 = \rho^w$, $b_e = u^w$, $b_{2e} = \theta^w/\sqrt{2}$, otherwise $b_\alpha = 0$.

Note that $\rho^w$ is determined by the no mass flow condition, so we must add the condition $u \cdot \mathbf{n} = 0$ in (2.13). For the symmetric hyperbolic system (2.8), the correct number of BCs should coincide with the number of negative eigenvalues of the boundary matrix $\sum_{i=1}^{3} A_i n_i = -A_2$. It was shown in [20] that the boundary matrix $-A_2$ has $n$ positive eigenvalues, $n$ negative eigenvalues and $m - n$ zero eigenvalues. Thus, the Grad BCs (2.13) contain the correct number of BCs for the initial boundary value problem (IBVP). Despite that, the Grad BCs have rare well-posed results and shown unstable in [40]. In Sect. 4, we will introduce a modification of the Grad BCs such that the obtained BCs are maximal positive, which plays a key role in the well-posedness of the linear symmetric hyperbolic system [31, 36, 38, 41].

### 3 Rayleigh Problem: An Illustrative Example

#### 3.1 Simplified Moment System

In this section, we illustrate the basic idea to analyze the boundary layers and to construct BCs for the NS equations through a simplified model. To this end, we consider the following additional technical assumptions:

**Assumption 3.1 (Simplified model)**

(i) The flows are confined in two infinite flat plates, which are initially given in equilibrium and then driven by the motion of the lower plate at $\{x_2 = 0\}$. The velocity of the plate is in the $x_1$-direction, i.e., $u^w = (u^w(t), 0, 0)$, and is initiated by a slow change. The gradient in the $x_2$-direction is dominant and terms $\partial/\partial x_1, \partial/\partial x_3$ are omitted. The upper plate at $\{x_2 = L\}$ is at rest and the considered time is relatively short such that the motion of the lower plate has almost no influence on the fluid near the upper plate.

(ii) The flows are depicted by the linear moment system (2.8) with even $M$. The collision term is described by the BGK model. The initial data of the system are prescribed as zero.

**Remark 3.1** Assumption (i) is similar to the Couette flow, where the flows are bounded by two plates [50]. Yet under the standard framework, the plane Couette flow often indicates the time-independent case where the steady equations are considered. Here, we consider the instantaneous flow driven by a single plate in a relatively short time. Thus, after an appropriate scaling, the flows studied above equal the half-space problem with a single plate called the Rayleigh problem in the literature [13, 46].

Assumption (ii) is made since the BGK collision term is relatively simple and we want to avoid the characteristic boundaries for the moment system which are difficult to deal
with. Notice that Assumption 3.1 is only used in Sect. 3 while the general case without these assumptions is considered in Sects. 4 and 5. Thanks to Assumption 3.1, the equations for the moments \( w_\alpha \) with \( \alpha = e_1 + k e_2 \) \( (k = 0, 1, \cdots, M - 1) \) can be decoupled from the whole moment system (2.8) (see also [18, 22]). For simplicity of notations, we denote \( x = x_2 \) and \( w_k = w_k(t, x) = w_{e_1 + k e_2} \) for \( k = 0, 1, \cdots, M - 1 \) in this section. Then the equations for the moments

\[
W_c = \begin{pmatrix} W_e \\ W_o \end{pmatrix}
\]

with \( W_e = (w_0, w_2, \cdots, w_{M-2})^T \), \( W_o = (w_1, w_3, \cdots, w_{M-1})^T \) can be written as

\[
\begin{align*}
\partial_t W_c + A_c \partial_x W_c &= -\frac{1}{\varepsilon} Q_c W_c, \\
W_c(0, x) &= 0, \quad x \in \mathbb{R}^+, \\
W_c(t, +\infty) &= 0,
\end{align*}
\]

(3.1)

where \( A_c \) is given in (2.10) and \( Q_c \) in (2.12).

**Remark 3.2** From (2.7), we know that \( w_0 = w_{e_1} \) is the velocity \( u_1 \) and \( w_1 = w_{e_1 + e_2} \) is the stress \( \sigma_{12} \). In this section, we denote \( u = u_1 \) and \( \sigma = \sigma_{12} \) for short.

On the other hand, the BCs for \( W_c \) can also be decoupled from (2.13) as

\[
B_c W_c(t, 0) = b_c(t).
\]

(3.2)

Here the coefficient matrix \( B_c = (\hat{\chi} S_c, M_c) \). Note that \( S_c \) is one part of the matrix \( S \) in (2.13). Since we only use the fact that \( S_c \) is a symmetric positive definite matrix, the specific expression of \( S_c \) is omitted. The right-hand side term in (3.2) is

\[
b_c(t) = \hat{\chi} S_c(u^w(t), 0, \cdots, 0)_{M/2-1}^T.
\]

The initial data and BCs are assumed to be compatible at \((t, x) = (0, 0)\). Namely,

\[
B_c W_c(0, 0) = b_c(0),
\]

which means \( u^w(0) = 0 \).

As to the IBVP (3.1)–(3.2), we claim that the BCs (3.2) are strictly dissipative. According to the classical theory of IBVPs for hyperbolic systems [3], the strictly dissipative condition ensures well-posedness. Moreover, from [59, 63] we know that this condition also guarantees the existence of the zero relaxation limit.

**Proposition 3.1** Suppose the coefficient \( 0 < \chi \leq 1 \). Then the equations (3.1) with BCs (3.2) are well-posed. Moreover, the solution admits a zero relaxation limit as \( \varepsilon \) goes to zero.

To prove this proposition, we recall the definition

**Definition 3.1** (Strictly dissipative condition) The BCs (3.2) for symmetric hyperbolic system (3.1) are referred to as strictly dissipative if there is a positive constant \( c \) such that

\[
y^T A_c y \leq -c |y|^2 + c^{-1} |B_c y|^2
\]

for all \( y \in \mathbb{R}^{M/2} \).
Proof of Proposition 3.1  It suffices to check that the following symmetric matrix is positive definite:

$$B_c^T B_c - cA_c - c^2 I = \left( \begin{array}{cc} \hat{S}^2 - c^2 I & (\hat{S} - c I)M_c \\ M_c^T (\hat{S} - c I) & M_c^T M_c - c^2 I \end{array} \right).$$

Here $\hat{S} = \hat{\chi} S_c$. By a congruent transformation, we only need to discuss the positiveness of

$$\begin{pmatrix} \hat{S}^2 - c^2 I & 0 \\ 0 & K \end{pmatrix}$$

with $K = M_c^T M_c - c^2 I - M_c^T (\hat{S} - c I) (\hat{S}^2 - c^2 I)^{-1} (\hat{S} - c I) M_c$.

For sufficiently small $c > 0$, we notice that $(\hat{S}^2 - c^2 I)^{-1} = \hat{S}^{-1} - O(c^2)$ and thereby

$$K = 2c M_c^T \hat{S}^{-1} M_c + O(c^2).$$

For $\hat{\chi} > 0$, the matrix $\hat{S}$ is positive definite, and thereby $K$ is positive definite for sufficiently small $c$. Consequently, we verify the strictly dissipative condition for the BCs.

\[\square\]

3.2 Asymptotic Analysis

We need to analyze the boundary-layer behavior of (3.1) with BCs (3.2) at $x = 0$ for sufficiently small $\varepsilon$. To this end, we recall the theory of hyperbolic relaxation system [59, 64] and consider the ansatz:

$$W_c(t, x) = \bar{W}_c(t, x) + \hat{W}_c \left( t, \frac{x}{\sqrt{\varepsilon}} \right) + \tilde{W}_c \left( t, \frac{x}{\varepsilon} \right).$$

(3.3)

Here $\bar{W}_c$ is the outer solution representing the quantities far away from the boundary, the others are boundary-layer corrections which satisfy

$$\tilde{W}_c(t, \infty) = \hat{W}_c(t, \infty) = 0.$$  

We expand these three variables as

$$\bar{W}_c = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \bar{W}^{(j)}_c, \quad \hat{W}_c = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \hat{W}^{(j)}_c, \quad \tilde{W}_c = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \tilde{W}^{(j)}_c.$$  

For each $j$, the component-wise form for the coefficient reads as

$$\bar{W}^{(j)}_c = \begin{pmatrix} \bar{u}^{(j)}_0, \bar{w}^{(j)}_2, \ldots, \bar{w}^{(j)}_{M-2}, \bar{\sigma}^{(j)}_0, \bar{w}^{(j)}_3, \ldots, \bar{w}^{(j)}_{M-1} \end{pmatrix}^T,$n

$$\hat{W}^{(j)}_c = \begin{pmatrix} \hat{u}^{(j)}_0, \hat{w}^{(j)}_2, \ldots, \hat{w}^{(j)}_{M-2}, \hat{\sigma}^{(j)}_0, \hat{w}^{(j)}_3, \ldots, \hat{w}^{(j)}_{M-1} \end{pmatrix}^T,$n

$$\tilde{W}^{(j)}_c = \begin{pmatrix} \tilde{u}^{(j)}_0, \tilde{w}^{(j)}_2, \ldots, \tilde{w}^{(j)}_{M-2}, \tilde{\sigma}^{(j)}_0, \tilde{w}^{(j)}_3, \ldots, \tilde{w}^{(j)}_{M-1} \end{pmatrix}^T.$n

Notice that we have used the notations $u = w_0$ and $\sigma = w_1$ (see Remark 3.2).
3.2.1 Equations for Coefficients

(1) Outer solution

The outer solution \( \hat{W}_e \) should approximately satisfy the equation (3.1). We substitute \( \hat{W}_e \) into (3.1) and require the system to be satisfied up to \( O(\varepsilon^{1/2}) \). By comparing coefficients of each order of \( \varepsilon \), we obtain

\[
\begin{align*}
O(\varepsilon^{-1}) : & \quad \tilde{\sigma}^{(0)} = 0, \quad \hat{w}_0^{(0)} = 0, \quad (k = 2, 3, \ldots, M - 1), \\
O(\varepsilon^{-1/2}) : & \quad \tilde{\sigma}^{(1)} = 0, \quad \hat{w}_0^{(1)} = 0, \quad (k = 2, 3, \ldots, M - 1), \\
O(\varepsilon^0) : & \quad \partial_t \hat{u}^{(0)} = 0, \quad \sigma^{(2)} = -\partial_x \hat{w}^{(0)}, \quad \hat{w}_k^{(2)} = 0, \quad (k = 2, 3, \ldots, M - 1), \\
O(\varepsilon^{1/2}) : & \quad \partial_t \hat{w}_1^{(1)} = 0, \quad \sigma^{(3)} = -\partial_x \hat{w}_1^{(1)}, \quad \hat{w}_k^{(3)} = 0, \quad (k = 2, 3, \ldots, M - 1).
\end{align*}
\]

Besides, we require the first equation in (3.1) to be satisfied up to \( O(\varepsilon) \) which gives

\[
O(\varepsilon) : \quad \partial_t \hat{u}^{(2)} + \partial_x \sigma^{(2)} = 0.
\]

The initial data of the outer solution \( \hat{W}_e(0, x) \) should be given according to the initial data of (3.1), which equal to zero by Assumption 3.1. Then it is not difficult to see from the above equations that \( \hat{W}_e(t, x) \equiv 0 \).

(2) Viscous layer solution

Similarly, we require the boundary-layer correction term \( \hat{W}_c \) to satisfy the system (3.1) up to \( O(\varepsilon^{1/2}) \). Let \( y = x/\sqrt{\varepsilon} \). Comparing coefficients of each order of \( \varepsilon \) yields

\[
\begin{align*}
O(\varepsilon^{-1}) : & \quad \tilde{\sigma}^{(0)} = 0, \quad \hat{w}_k^{(0)} = 0, \quad (k = 2, 3, \ldots, M - 1), \\
O(\varepsilon^{-1/2}) : & \quad \tilde{\sigma}^{(1)} = -\partial_y \hat{u}^{(0)}, \quad \hat{w}_k^{(1)} = 0, \quad (k = 2, 3, \ldots, M - 1), \\
O(\varepsilon^0) : & \quad \partial_t \hat{u}^{(0)} + \partial_x \tilde{\sigma}^{(1)} = 0, \quad \tilde{\sigma}^{(2)} = -\partial_x \hat{u}^{(1)}, \quad \hat{w}_2^{(2)} = -\sqrt{2} \partial_y \tilde{\sigma}^{(1)}, \\
O(\varepsilon^{1/2}) : & \quad \partial_t \hat{u}_1^{(1)} + \partial_x \tilde{\sigma}^{(2)} = 0, \quad \tilde{\sigma}^{(3)} = -\partial_x \tilde{\sigma}^{(1)} - \partial_y \hat{u}^{(2)} - \sqrt{2} \partial_y \hat{w}_2^{(2)}, \\
& \quad \hat{w}_2^{(3)} = -\sqrt{2} \partial_x \tilde{\sigma}^{(2)}, \quad \hat{w}_3^{(3)} = -\sqrt{3} \partial_y \hat{w}_2^{(2)}, \quad \hat{w}_k^{(3)} = 0, \quad (k = 4, 5, \ldots, M - 1).
\end{align*}
\]

Moreover, we require the first equation in (3.1) to be satisfied up to \( O(\varepsilon) \) which gives

\[
O(\varepsilon) : \quad \partial_t \hat{u}^{(2)} + \partial_y \tilde{\sigma}^{(3)} = 0.
\]

Now we show how to solve the viscous layer solutions from the above equations. Firstly, we solve two parabolic equations

\[
\partial_t \hat{u}^{(0)} - \partial_{yy} \hat{u}^{(0)} = 0, \quad \partial_t \hat{u}^{(1)} - \partial_{yy} \hat{u}^{(1)} = 0
\]

(3.5)

to obtain \( \hat{u}^{(0)} \) and \( \hat{u}^{(1)} \). Then we can obtain \( \tilde{\sigma}^{(1)}, \tilde{\sigma}^{(2)} \) and \( \hat{w}_2^{(2)} \) by algebraic relations in (3.4). Having these, we solve a parabolic equation

\[
\partial_t \hat{u}^{(2)} - \partial_{yy} \hat{u}^{(2)} = \partial_y \tilde{\sigma}^{(1)} + \sqrt{2} \partial_y \hat{w}_2^{(2)}
\]

(3.6)

to obtain \( \hat{u}^{(2)} \). At last, by certain algebraic equations in (3.4) we can determine \( \tilde{\sigma}^{(3)}, \hat{w}_2^{(3)} \) and \( \hat{w}_3^{(3)} \). Clearly, the equations (3.5)–(3.6) need BCs for \( \hat{u}^{(0)}, \hat{u}^{(1)} \) and \( \hat{u}^{(2)} \).

(3) Knudsen layer solution

Let \( z = x/\varepsilon \). We require \( \hat{W}_c \) to satisfy the equation (3.1) up to \( O(\varepsilon^{1/2}) \), which yields

\[
\begin{align*}
O(\varepsilon^{(j-2)/2}) : & \quad \frac{\partial_t \tilde{\sigma}^{(j-2)}}{\varepsilon} = -\partial_t \tilde{w}^{(j-2)}, \quad \partial_x \tilde{\sigma}^{(j)} + \sqrt{2} \partial_x \tilde{w}^{(j)} = -\tilde{\sigma}^{(j)} - \partial_t \tilde{\sigma}^{(j-2)}, \\
(\hat{w}_o^{(j)})_{0} - \frac{\partial_t \hat{w}_o^{(j)}}{\varepsilon} & = -\frac{\partial_t \hat{w}_o^{(j)}}{\varepsilon} - \partial_t \tilde{w}_o^{(j)}
\end{align*}
\]

(3.7)
Here the notations are defined by

\[
\tilde{w}_e^{(j)} = \begin{pmatrix} \tilde{w}_e^{(j)} \\ \tilde{w}_o^{(j)} \\ \vdots \\ \tilde{w}_M^{(j)} \\ \tilde{w}_{M-2}^{(j)} \\ \vdots \\ \tilde{w}_{M-1}^{(j)} \end{pmatrix}, \quad \tilde{w}_o^{(j)} = \begin{pmatrix} \tilde{w}_o^{(j)} \\ \tilde{w}_M^{(j)} \end{pmatrix}, \quad \tilde{\Lambda}_e = \begin{pmatrix} 0 & \tilde{M}_e \\ \tilde{M}_e^T & 0 \end{pmatrix}, \quad \tilde{M}_e = \begin{pmatrix} \sqrt{3} \\ \sqrt{4} \\ \sqrt{5} \\ \sqrt{M-2} \\ \sqrt{M-1} \end{pmatrix}.
\]

Next, we show the procedure to obtain Knudsen layer solutions from the above equations. For \( j = 0, 1 \), we need to solve the ODE systems

\[
\tilde{\Lambda}_e \partial_z \begin{pmatrix} \tilde{w}_e^{(0)} \\ \tilde{w}_o^{(0)} \end{pmatrix} = -\begin{pmatrix} \tilde{w}_e^{(0)} \\ \tilde{w}_o^{(0)} \end{pmatrix}, \quad \tilde{\Lambda}_e \partial_z \begin{pmatrix} \tilde{w}_e^{(1)} \\ \tilde{w}_o^{(1)} \end{pmatrix} = -\begin{pmatrix} \tilde{w}_e^{(1)} \\ \tilde{w}_o^{(1)} \end{pmatrix}.
\] (3.8)

Then it is easy to see that

\[
\tilde{\sigma}^{(j)} = 0, \quad \tilde{u}^{(j)} = -\sqrt{2}\tilde{w}_2^{(j)}, \quad j = 0, 1.
\] (3.9)

Having these, we proceed to solve

\[
\tilde{\Lambda}_e \partial_z \begin{pmatrix} \tilde{w}_e^{(2)} \\ \tilde{w}_o^{(2)} \end{pmatrix} = -\begin{pmatrix} \tilde{w}_e^{(2)} \\ \tilde{w}_o^{(2)} \end{pmatrix} - \partial_t \begin{pmatrix} \tilde{w}_e^{(0)} \\ \tilde{w}_o^{(0)} \end{pmatrix}, \quad \tilde{\Lambda}_e \partial_z \begin{pmatrix} \tilde{w}_e^{(3)} \\ \tilde{w}_o^{(3)} \end{pmatrix} = -\begin{pmatrix} \tilde{w}_e^{(3)} \\ \tilde{w}_o^{(3)} \end{pmatrix} - \partial_t \begin{pmatrix} \tilde{w}_e^{(1)} \\ \tilde{w}_o^{(1)} \end{pmatrix}.
\] (3.10)

At last, \( \tilde{\sigma}^{(2)}, \tilde{u}^{(2)} \) and \( \tilde{\sigma}^{(3)}, \tilde{u}^{(3)} \) are determined by relations

\[
\partial_z \tilde{\sigma}^{(2)} = -\partial_t \tilde{u}^{(0)}, \quad \partial_z \tilde{u}^{(2)} = -\sqrt{2}\partial_z \tilde{w}_2^{(2)} - \tilde{\sigma}^{(2)}
\] (3.11)

and

\[
\partial_z \tilde{\sigma}^{(3)} = -\partial_t \tilde{u}^{(1)}, \quad \partial_z \tilde{u}^{(3)} = -\sqrt{2}\partial_z \tilde{w}_2^{(3)} - \tilde{\sigma}^{(3)}.
\] (3.12)

To obtain bounded solutions to the ODE systems (3.8) and (3.10), we state

**Proposition 3.2** There is an orthogonal matrix \( R \) satisfying

\[
\tilde{\Lambda}_e R = \begin{pmatrix} \tilde{\Lambda}_+ & 0 \\ 0 & -\tilde{\Lambda}_+ \end{pmatrix}, \quad R = \begin{pmatrix} R_e & R_e \\ R_o & -R_o \end{pmatrix}.
\]

Here \( R_o \) and \( R_e \) are \( \left( \frac{M-2}{2} \times \frac{M-2}{2} \right) \)-invertible matrices, \( \tilde{\Lambda}_+ \) is a positive diagonal matrix.

Thanks to this proposition, we can express the moments by characteristic variables

\[
\begin{pmatrix} \tilde{w}_e^{(k)} \\ \tilde{w}_o^{(k)} \end{pmatrix} = \begin{pmatrix} R_e & R_e \\ -R_o & R_o \end{pmatrix} \begin{pmatrix} \tilde{w}_+^{(k)} \\ \tilde{w}_-^{(k)} \end{pmatrix}.
\]

For bounded solutions, we require \( \tilde{w}_+^{(k)} \equiv 0 \) and get

\[
\tilde{w}_e^{(k)} = R_e \tilde{w}_-^{(k)}, \quad \tilde{w}_o^{(k)} = R_o \tilde{w}_-^{(k)}.
\] (3.13)

The value of \( \tilde{w}_-^{(k)} \) should be determined from the BCs.
3.2.2 Boundary Conditions

Substituting the asymptotic solution into the BCs (3.2) and matching each order of $\varepsilon$ yield

$$O(\varepsilon^{j/2}) \ (j = 0, 1, 2): \ (\hat{\chi} S_c, M_c)[\hat{W}_c^{(j)}(t, 0) + \tilde{W}_c^{(j)}(t, 0)] = b_c^{(j)}$$

with

$$b_c^{(0)} = \hat{\chi} S_c (u^w, 0, \ldots, 0)^T, \quad b_c^{(1)} = b_c^{(2)} = 0.$$  

For further discussions, we denote

$$M_c = \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{g}} \tilde{M}_c \end{pmatrix}, \quad \tilde{g} = (\sqrt{2}, 0, \ldots, 0)^T, \quad \hat{\omega}_c^{(j)} = (\hat{\omega}_2^{(j)}, \ldots, \hat{\omega}_{M-2}^{(j)})^T, \quad \tilde{\omega}_c^{(j)} = (\tilde{\omega}_3^{(j)}, \ldots, \tilde{\omega}_{M-1}^{(j)})^T.$$  

Then the above equations can be rewritten as

$$O(\varepsilon^{j/2}) \ (j = 0, 1, 2): \ \hat{\chi} S_c \begin{pmatrix} \hat{u}_c^{(j)} + \tilde{u}_c^{(j)} \\ \hat{w}_c^{(j)} + \tilde{w}_c^{(j)} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{g}} \tilde{M}_c \end{pmatrix} \begin{pmatrix} \tilde{\sigma}^{(j)} + \hat{\sigma}^{(j)} \\ \tilde{\omega}_c^{(j)} + \tilde{\omega}_c^{(j)} \end{pmatrix} = b_c^{(j)}.$$  

(1) Order $\varepsilon^0$:

Recall that $\tilde{\sigma}^{(0)} = 0$ and $\hat{\sigma}^{(0)} = 0$. From (3.9), we know that $\tilde{u}^{(0)} = -\sqrt{2} \tilde{w}_2^{(0)} = -\tilde{g}^T \tilde{w}_c^{(0)}$. Then it follows from (3.13) that

$$\hat{\chi} S_c \begin{pmatrix} \hat{u}_c^{(0)} - \tilde{g}^T R_e \tilde{w}_c^{(0)} \\ R_e \tilde{w}_c^{(0)} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \frac{1}{\tilde{g}} \tilde{M}_c \end{pmatrix} \begin{pmatrix} 0 \\ R_o \tilde{w}_c^{(0)} \end{pmatrix} = b_c^{(0)}.$$

The unknowns are $\hat{u}_c^{(0)}$ and $\tilde{w}_c^{(0)}$. Thus we rewrite the above equation as

$$\begin{pmatrix} \hat{\chi} S_c \begin{pmatrix} 1 & -\tilde{g}^T R_e \\ 0 & R_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}_c R_o \end{pmatrix} \end{pmatrix} \begin{pmatrix} \hat{u}_c^{(0)} \\ \tilde{w}_c^{(0)} \end{pmatrix} = \hat{\chi} S_c \begin{pmatrix} u^w \\ 0 \end{pmatrix}.$$  

To show the solvability of this equation, we state

**Lemma 3.1** For $\hat{\chi} > 0$, the matrix

$$H_M(\chi) := \hat{\chi} S_c \begin{pmatrix} 1 & -\tilde{g}^T R_e \\ 0 & R_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}_c R_o \end{pmatrix}$$

is invertible.

**Proof** Proposition 3.2 implies that $\tilde{M}_c R_o = R_e \tilde{\Lambda}_+$ and $2R_e R_e^T = I$. Then we multiply $H_M(\chi)$ with

$$\begin{pmatrix} 1 & \tilde{g}^T \\ 0 & 2R_e^T \end{pmatrix}$$

from the right to obtain

$$\begin{pmatrix} \hat{\chi} S_c \begin{pmatrix} 1 & -\tilde{g}^T R_e \\ 0 & R_e \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \tilde{M}_c R_o \end{pmatrix} \end{pmatrix} \begin{pmatrix} 1 & \tilde{g}^T \\ 0 & 2R_e^T \end{pmatrix} = \hat{\chi} S_c + \begin{pmatrix} 0 & 0 \\ 0 & 2R_e \tilde{\Lambda}_+ R_e^T \end{pmatrix}.$$  

Clearly, the matrix on the right-hand side is positive definite since $S_c$ is positive definite and $\hat{\chi} > 0$. Consequently, we get the invertibility stated in the lemma. \(\square\)
Thanks to this lemma, the unique solution to (3.15) is
\[ \hat{u}^{(0)}(t, 0) = u^w(t), \quad \hat{w}_-^{(0)}(0, t) = 0. \]  
(3.16)

It follows from (3.9) and (3.13) that \( \tilde{W}^{(0)}_c \equiv 0 \).

(2) Order \( \epsilon^{1/2} \):

Similarly, we can derive the BCs for \( j = 1 \). Recall that \( \tilde{\sigma}^{(1)} = -\partial_y \hat{u}^{(0)}, \tilde{\sigma}^{(1)} = 0 \) and \( \hat{\sigma}^{(1)} = -\sqrt{2} \tilde{w}_2^{(1)} = -\tilde{g}^T \hat{w}_c^{(1)} \). Substituting into (3.14) and using (3.13) yields
\[
\begin{bmatrix}
\tilde{\chi} \hat{S}_c 
\begin{pmatrix}
1 & \tilde{g}^T R_e \\
0 & 2 R_e 
\end{pmatrix} 
\end{bmatrix} \begin{bmatrix}
\hat{u}^{(1)} \\
\hat{w}_-^{(1)} 
\end{bmatrix} = \partial_y \hat{u}^{(0)} \left( \frac{1}{\tilde{g}} \right).
\]

Due to Lemma 3.1, this equation is uniquely solvable and we have
\[ \hat{u}^{(1)}(t, 0) = K_M(\chi) \partial_y \hat{u}^{(0)}(t, 0) \quad \text{with} \quad K_M(\chi) := (1, 0) H_M^{-1} \left( \frac{1}{\tilde{g}} \right). \]  
(3.17)

By resorting to the proof of Lemma (3.1), we find that
\[
K_M(\chi) = (1, 0) \begin{bmatrix}
1 & \tilde{g}^T R_e \\
0 & 2 R_e 
\end{bmatrix} \begin{bmatrix}
\tilde{\chi} S_c + \begin{pmatrix}
0 & 0 \\
0 & 2 \tilde{\Lambda} - R_e 
\end{pmatrix} 
\end{bmatrix}^{-1} \left( \frac{1}{\tilde{g}} \right) > 0
\]

since \( S_c, \tilde{\Lambda} \) are positive definite and \( \tilde{\chi} > 0 \).

(3) Order \( \epsilon^1 \):

At last, we derive the BCs for \( \hat{u}^{(2)} \). Since \( \tilde{w}^{(0)} = 0 \), it follows from (3.11) that \( \tilde{\sigma}^{(2)} = 0 \) and \( \hat{\sigma}^{(2)} = -\sqrt{2} \tilde{w}_2^{(2)} = -\tilde{g}^T \hat{w}_c^{(2)} \). Besides, recall that \( \hat{\sigma}^{(2)} = -\partial_y \hat{u}^{(1)}, \hat{w}_2^{(2)} = -\sqrt{2} \partial_y \hat{\sigma}^{(1)} = \sqrt{2} \partial_{yy} \hat{u}^{(0)} \). Due to the equation (3.5) and (3.16), it follows that
\[ \hat{w}_2^{(2)}(t, 0) = \sqrt{2} \partial_{yy} \hat{u}^{(0)}(t, 0) = \sqrt{2} \partial_y u^w(t). \]  
(3.18)

Moreover, by the definition of \( \hat{w}_c^{(2)} \) and \( \tilde{g} \) we can write \( \hat{w}_c^{(2)} = \tilde{g} \partial_y u^w(t) \). Substituting these relations into (3.14) and using (3.13) yields
\[
\begin{bmatrix}
\tilde{\chi} S_c 
\begin{pmatrix}
1 & \tilde{g}^T R_e \\
0 & 2 R_e 
\end{pmatrix} 
\end{bmatrix} \begin{bmatrix}
\hat{u}^{(2)} \\
\hat{w}_-^{(2)} 
\end{bmatrix} = \begin{bmatrix}
\partial_y \hat{u}^{(1)} - \tilde{\chi} S_c 
\begin{pmatrix}
0 & 0 \\
0 & \tilde{g} 
\end{pmatrix} 
\end{bmatrix} \partial_y u^w(t).
\]

Thanks to Lemma 3.1, we can solve the above algebraic equation to obtain
\[ \hat{u}^{(2)}(t, 0) = K_M(\chi) \partial_y \hat{u}^{(1)}(t, 0) + J_M(\chi) \partial_y u^w(t). \]  
(3.19)

Here the constant \( K_M(\chi) \) is defined in (3.17) and
\[ J_M(\chi) := -(1, 0) H_M(\chi)^{-1} \tilde{\chi} S_c \left( \frac{0}{\tilde{g}} \right). \]

3.2.3 Validation

Construct an approximate solution to (3.1):
\[ W_{c^{app}} = W_{c^{app}}^{app} + \hat{W}_{c^{app}} + \tilde{W}_{c^{app}}. \]
Here $\hat{W}_c^{app} = 0$ and

$$\tilde{W}_c^{app} = \hat{W}_c^{(0)} + \sqrt{\varepsilon} \hat{W}_c^{(1)} + \varepsilon \hat{W}_c^{(2)} + \varepsilon^{3/2} \left( 0, \hat{w}_2^{(3)}, \ldots, \hat{w}_{M-2}^{(3)}, \hat{w}_3^{(3)}, \ldots, \hat{w}_{M-1}^{(3)} \right)^T$$

where the coefficients are determined by the equations (3.4) with BCs (3.16)–(3.19). Moreover, the term $\hat{W}_c^{app}$ is constructed by

$$\hat{W}_c^{app} = \hat{w}_0^{(0)} \varepsilon + \sqrt{\varepsilon} \hat{w}_1^{(1)} + \varepsilon \hat{w}_2^{(2)} + \varepsilon^{3/2} \hat{w}_3^{(3)},$$

where the coefficients are determined by (3.8)–(3.12). The above approximate solutions are truncated from $\hat{W}_c$ and $\tilde{W}_c$. Notice that the term $\tilde{\sigma}_c^{(3)}$ is not involved in the expression of $\hat{W}_c^{app}$ since it cannot be determined from (3.4).

Substituting the approximate solution $W_c^{app}$ into the equation (3.1), we have

$$\partial_t W_c^{app} + A_c \partial_x W_c^{app} = -\frac{1}{\varepsilon} Q_c W_c^{app} + \left( \begin{array}{c} 0 \\ \varepsilon \hat{R} \end{array} \right) + \varepsilon \tilde{R}$$

by our construction (3.4) and (3.7). Here the residual $\hat{R} = \hat{R}(x/\sqrt{\varepsilon})$ has $M - 1$ components and $\tilde{R} = \tilde{R}(x/\varepsilon)$ has $M$ components. Moreover, there exists a constant $C$ independent of $\varepsilon$ such that

$$\int_{R^+} \left| \hat{R} \left( \frac{x}{\sqrt{\varepsilon}} \right) \right|^2 dx = \sqrt{\varepsilon} \int_{R^+} \left| \tilde{R}(y) \right|^2 dy \leq C \varepsilon^{1/2}$$

and

$$\int_{R^+} \left| \tilde{R} \left( \frac{x}{\varepsilon} \right) \right|^2 dx = \varepsilon \int_{R^+} \left| \hat{R}(z) \right|^2 dz \leq C \varepsilon.$$

On the other hand, according to the discussion in Subsect. 3.2.2, we substitute $W_c^{app}$ into the BCs (3.2) and obtain

$$B_c W_c^{app}(t, 0) = \varepsilon^{3/2} h(t),$$

where $h(t)$ satisfies $\int_0^T h(t)^2 dt \leq C$. At last, the initial data of $W_c^{app}$ are given as zero.

Next, we show the validity of the approximate solution $W_c^{app}$ by the following theorem:

**Theorem 3.2** The approximate solution $W_c^{app}$ and the exact solution $W_c$ to the equation (3.1) satisfy the following error estimate for any $t \in [0, T]$:

$$\| (W_c^{app} - W_c)(t, \cdot) \|_{L_2(R^+)} + \left( \int_0^T |W_c^{app}(t, 0) - W_c(t, 0)|^2 \, dt \right)^{1/2} \leq C(T) \varepsilon^{3/2}.$$  

**Proof** Denote the error $E(t, x) = W_c^{app}(t, x) - W_c(t, x)$. It satisfies the IBVP:

$$\begin{cases}
\partial_t E + A_c \partial_x E = -\frac{1}{\varepsilon} Q_c E + \left( \begin{array}{c} 0 \\ \varepsilon \hat{R} \end{array} \right) + \varepsilon \tilde{R}, \\
B_c E(t, 0) = \varepsilon^{3/2} h(t), \\
E(0, x) = 0.
\end{cases}$$

$$\left(3.21\right)$$
Multiplying $E^T(t, x)$ on the left side and integrating over $x \in [0, \infty)$ yields

$$
\frac{d}{dt} \|E(t, \cdot)\|^2_{L^2(\mathbb{R}^+)} - E^T(t, 0)A_c E(t, 0) = -\frac{2}{\varepsilon} \|E^{II}\|^2_{L^2(\mathbb{R}^+)} + 2\varepsilon \int_{\mathbb{R}^+} (E^{II})^T \tilde{R} dx + 2\varepsilon \int_{\mathbb{R}^+} E^T \tilde{R} dx,
$$

$$
\leq -\frac{2}{\varepsilon} \|E^{II}\|^2_{L^2(\mathbb{R}^+)} + \varepsilon^3 \int_{\mathbb{R}^+} |\tilde{R}|^2 dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^+} |E^{II}|^2 dx + \varepsilon^2 \int_{\mathbb{R}^+} |\tilde{R}|^2 dx + \int_{\mathbb{R}^+} |E|^2 dx
\leq C\varepsilon^3 + \|E(t, \cdot)\|^2_{L^2(\mathbb{R}^+)}.
$$

Here the vector $E$ has $M$ components and $E^{II}$ represents the last $M - 1$ components of $E$. Since the BCs satisfy the strictly dissipative condition, it follows that

$$
E^T(t, 0)A_c E(t, 0) \leq -c\|E(t, 0)\|^2 + c^{-1}\varepsilon^3 |h(t)|^2.
$$

At last, by using Gronwall’s inequality, we obtain the estimate stated in the theorem. \qed

### 3.3 Construction of BCs for NS Equations

In this subsection, we consider the simplified NS equation under Assumption 3.1:

$$
\partial_t u_{ns} - \varepsilon \partial_{xx} u_{ns} = 0, \quad x \in \mathbb{R}^+,
$$

(3.22)

with zero initial values and $u_{ns}(t, \infty) = 0$. To construct BCs at $x = 0$ for the NS equation, we collect the coefficients of $\tilde{u}^{(j)}$ and $\tilde{u}'^{(j)}$ in (3.3) and compare them with the solution $u_{ns}$ to the NS equation. Since $\tilde{u}^{(j)} \equiv 0$, we construct an approximate solution to the NS equation by

$$
u_{ns}^a = \tilde{u}^{(0)} + \sqrt{\varepsilon} \tilde{u}^{(1)} + \varepsilon \tilde{u}^{(2)}.
$$

According to the equations of $\tilde{u}^{(j)}$ in (3.4), we check that $u_{ns}^a$ satisfies

$$
\partial_t u_{ns}^a - \varepsilon \partial_{xx} u_{ns}^a = \varepsilon R_{ns} \left( \frac{x}{\sqrt{\varepsilon}} \right).
$$

Here the residual $R_{ns} = R_{ns}(x/\sqrt{\varepsilon})$ satisfies

$$
\int_{\mathbb{R}^+} |R_{ns}|^2 \left( \frac{x}{\sqrt{\varepsilon}} \right) dx = \sqrt{\varepsilon} \int_{\mathbb{R}^+} |R_{ns}|^2 (y) dy \leq C\sqrt{\varepsilon}.
$$

On the other hand, we aim to provide BCs for the NS equation so that $u_{ns}^a$ approximately satisfies this BC as well. To this end, from (3.16), (3.17) and (3.19), we construct

$$
u_{ns}(t, 0) - \varepsilon K_M(\chi) \partial_x u_{ns}(t, 0) = \nu(t) + \varepsilon J_M(\chi) \partial_t \nu(t).
$$

(3.23)

Substituting $u_{ns}^a$ into this condition yields

$$
u_{ns}^a(t, 0) - \varepsilon K_M(\chi) \partial_x u_{ns}^a(t, 0)
= \tilde{u}^{(0)}(t, 0) + \sqrt{\varepsilon} \tilde{u}^{(1)}(t, 0) + \varepsilon \tilde{u}^{(2)}(t, 0) - \varepsilon K_M(\chi) \left( \frac{1}{\sqrt{\varepsilon}} \partial_y \tilde{u}^{(0)}(t, 0) + \partial_y \tilde{u}^{(1)}(t, 0) \right)
+ \varepsilon^{3/2} b_{ns}(t)
= \nu(t) + \varepsilon J_M(\chi) \partial_t \nu(t) + \varepsilon^{3/2} b_{ns}(t).
$$

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where \( \int_0^T b_{ns}^2(t) dt \leq C \). To show that the constructed BCs (3.23) fulfill the matching requirement stated in the introduction, we claim

**Theorem 3.3** The solution \( u_{ns} \) to the NS equation (3.22) and the approximate solution \( u_{ns}^a \) satisfy the following error estimate for any \( t \in [0, T] \):

\[
\| (u_{ns} - u_{ns}^a)(t, \cdot) \|_{L^2(\mathbb{R}^+)} + \left( \int_0^T |u_{ns}(t, 0) - u_{ns}^a(t, 0)|^2 dt \right)^{1/2} \leq C(T)e^{5/4}.
\]

**Proof** Let \( E_{ns} = u_{ns} - u_{ns}^a \). The above discussion implies that \( E_{ns} \) satisfies the IBVP

\[
\begin{align*}
\partial_t E_{ns} - \varepsilon \partial_{xx} E_{ns} &= \varepsilon R_{ns}, \\
E_{ns}(t, 0) - \varepsilon K_M \partial_x E_{ns}(t, 0) &= \varepsilon^{3/2} b_{ns}(t), \\
E_{ns}(0, x) &= 0, \quad E_{ns}(t, \infty) = 0,
\end{align*}
\]

Multiplying \( E_{ns} \) on the left of the equation yields

\[
\frac{1}{2} \frac{d}{dt} \| E_{ns} \|^2_{L^2(\mathbb{R}^+)} - \varepsilon (E_{ns} E_{ns}^x)_x + \varepsilon |E_{ns}^x|^2 = \varepsilon E_{ns} R_{ns}.
\]

Integrating over \( x \in [0, \infty) \) and using the BC, we have

\[
\frac{1}{2} \frac{d}{dt} \| E_{ns} \|^2_{L^2(\mathbb{R}^+)} = -\varepsilon E_{ns}(t, 0) E_{ns}^x(t, 0) - \varepsilon \int_{\mathbb{R}^+} |E_{ns}^x|^2 dx + \varepsilon \int_{\mathbb{R}^+} E_{ns} R_{ns} dx
\]

\[
= - \frac{1}{K_M} E_{ns}(t, 0) \left[ E_{ns}(t, 0) - \varepsilon^{3/2} b_{ns}(t) \right] - \varepsilon \int_{\mathbb{R}^+} |E_{ns}^x|^2 dx + \varepsilon \int_{\mathbb{R}^+} E_{ns} R_{ns} dx
\]

\[
\leq - \frac{1}{2K_M} |E_{ns}(t, 0)|^2 + \frac{\varepsilon^3}{2K_M} b_{ns}^2(t) - \varepsilon \int_{\mathbb{R}^+} |E_{ns}^x|^2 dx + \frac{1}{2} \int_{\mathbb{R}^+} |E_{ns}|^2 dx
\]

\[
+ \frac{\varepsilon^2}{2} \int_{\mathbb{R}^+} |R_{ns}|^2 dx.
\]

(3.24)

Since \( K_M > 0 \), it follows that

\[
\frac{d}{dt} \| E_{ns} \|^2_{L^2(\mathbb{R}^+)} \leq \frac{\varepsilon^3}{K_M} b_{ns}^2(t) + \int_{\mathbb{R}^+} |E_{ns}|^2 dx + \varepsilon^2 \int_{\mathbb{R}^+} |R_{ns}|^2 dx \leq C \varepsilon^{5/2} + \| E_{ns} \|^2_{L^2(\mathbb{R}^+)}. \]

By Gronwall’s inequality, we get the estimate

\[
\| E_{ns}(t, \cdot) \|_{L^2(\mathbb{R}^+)} \leq C(T)e^{5/4}.
\]

At last, we integrate (3.24) over \( t \in [0, T] \) to obtain the estimate

\[
\int_0^T |E_{ns}(t, 0)|^2 dt \leq C(T)e^{5/2}.
\]

This completes the proof. \( \square \)
4 Asymptotic Analysis: General Case

4.1 Moment Equations

Now we consider the 3D time-dependent linear moment system

$$\frac{\partial W_\varepsilon}{\partial t} + \sum_{d=1}^{3} A_d \frac{\partial W_\varepsilon}{\partial x_d} = -\frac{1}{\varepsilon} Q W_\varepsilon,$$

with the BCs at \(\{x_2 = 0\}\),

$$B(W_\varepsilon - b) = 0, \quad (4.1)$$

where \(b \in \mathbb{R}^N\) is given by (2.13) and \(B \in \mathbb{R}^{n \times N}\). We assume that the initial value is compatible with the boundary condition and there is no initial layer. Then we can focus on the boundary-layer behavior of the moment system.

In the Grad BCs (2.13), the matrix

$$B = E[\hat{\chi} S, M_o],$$

which has rare theoretical properties as mentioned in Sect. 2. To ensure the well-posedness of linear moment equations, since \(S\) is symmetric positive definite, we can modify the BCs as \([34, 40]\)

$$[\hat{\chi} M_o^T, M_o^T S^{-1} M_o](W_\varepsilon - b) = 0, \quad (4.2)$$

which means

$$B = [\hat{\chi} M_o^T, M_o^T S^{-1} M_o].$$

We can check that the modified BCs with the no mass flow condition are maximal positive, which is important for the well-posedness of linear hyperbolic equations [31, 38, 41].

**Theorem 4.1** The modified BCs (4.2) with \(u \cdot n = -u_2 = 0\) are maximal positive. Namely,

(i) If \(u^w = 0\) and \(\theta^w = 0\), the BCs can determine a linear space \(\mathcal{N}\) with

$$\dim \mathcal{N} = m,$$

where \(m\) is the number of nonnegative eigenvalues of \(-A_2\) counting the multiplicity.

(ii) For any \(v \in \mathcal{N}\),

$$-v^T A_2 v \geq 0.$$

**Proof** By definition, we have

$$A_2 = \begin{bmatrix} 0 & M_o \\ M_o^T & 0 \end{bmatrix}.$$ 

The no mass flow condition says that if \(v \in \mathcal{N}\), then

$$v[\mathcal{N}(e_2)] = u_2 = 0.$$

Note that \(M_o^T S^{-1} M_o > 0\). We divide the vector into two parts, as

$$v = \begin{bmatrix} v_e \\ v_o \end{bmatrix}, \quad v_e \in \mathbb{R}^m, \quad v_o \in \mathbb{R}^n.$$
If $\hat{\chi} = 0$, we have

$$v \in \mathcal{N} \iff v_o = 0.$$  

So $\dim \mathcal{N} = m$ and $v^T A_2 v = 0$.

If $\hat{\chi} > 0$, since $M_o^T$ is upper triangular by definition, $\rho^w$ will only occur in the first line of (4.2) and we have

$$a_{11}(\rho - \rho^w) + a^T v_e^* = v_o[\mathcal{N}(e_2)] = 0,$$

where we write

$$v_e = \begin{bmatrix} \rho \\ v_e^* \end{bmatrix}, \quad (M_o^T S^{-1} M_o)^{-1} M_o^T = \begin{bmatrix} a_{11} & a^T \\ * & * \end{bmatrix}.$$  

Since $M_o^T S^{-1} M_o > 0$ and $M_o$ is lower triangular with full column rank, $a_{11}$ equals the product of the first row and first column of $(M_o^T S^{-1} M_o)^{-1}$ and $M_o^T$, which is not zero. So for any $v_e \in \mathbb{R}^m$, the condition (4.2) with $u_2 = 0$ can determine the unique $v_o$ and $\rho^w$, which means that

$$\dim \mathcal{N} = m.$$  

Next, because of the same reason, we can write

$$M_o(M_o^T S^{-1} M_o)^{-1} M_o^T = \begin{bmatrix} c_0 & c_1^T \\ c_1 & S^* \end{bmatrix} \geq 0,$$

where $c_0 > 0$ and $S^* \geq 0$. Now $\rho^w$ should also satisfy

$$c_0(\rho - \rho^w) + c_1^T v_e^* = 0,$$

which means that

$$M_o(M_o^T S^{-1} M_o)^{-1} M_o^T (v_e - b_e) = \begin{bmatrix} 0 & 0 \\ 0 & S^* - c_0^{-1} c_1 c_1^T \end{bmatrix} v_e := \tilde{S} v_e.$$  

Here $\tilde{S}$ is symmetric positive semi-definite, too. Thus, for any $v \in \mathcal{N}$, we have

$$-\frac{1}{2} v^T A_2 v = -v_e^T M_o v_o$$

$$= \hat{\chi} v_e^T M_o (M_o^T S^{-1} M_o)^{-1} M_o^T (v_e - b_e)$$

$$= \hat{\chi} v_e^T \tilde{S} v_e \geq 0.$$  

This completes the proof.  

\[\square\]

**4.2 Hilbert Solutions**

We consider the three scales ansatz:

$$W_e (t, x^w; x_2) = \overline{W} (t, x^w; x_2) + \widehat{W} (t, x^w; y) + \widetilde{W} (t, x^w; z).$$  

(4.3)

Here the variables are

$$y = \frac{x_2}{\sqrt{\epsilon}}, \quad z = \frac{x_2}{\epsilon}, \quad x^w = (x_1, x_3),$$

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\( \bar{W} \) is the outer solution while \( \hat{W} \) and \( \tilde{W} \) are boundary-layer corrections satisfying

\[
\hat{W}(t, x^w; \infty) = \tilde{W}(t, x^w; \infty) = 0.
\]

We expand these solutions in a power series of \( \sqrt{\varepsilon} \):

\[
\bar{W} = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \bar{W}^{(j)}, \quad \hat{W} = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \hat{W}^{(j)}, \quad \tilde{W} = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \tilde{W}^{(j)}.
\]

Next we will derive the equations for each coefficient \( \bar{W}^{(j)} \), \( \hat{W}^{(j)} \) and \( \tilde{W}^{(j)} \).

### 4.2.1 Outer Solution

At the scale of bulk flow, the outer solution \( \bar{W} \) should satisfy the linear moment equations. Plugging the expansion of \( \bar{W} \) into (2.8) and matching the order of \( \varepsilon \), we have

\[
\begin{align*}
O(\varepsilon^{-1}) : & \quad \varepsilon = -Q \bar{W}^{(0)}, \\
O(\varepsilon^{-1/2}) : & \quad \varepsilon = -Q \bar{W}^{(1)}, \\
O(\varepsilon^{1/2}) : & \quad \frac{\partial \bar{W}^{(j)}}{\partial t} + \sum_{d=1}^{3} A_d \frac{\partial \bar{W}^{(j)}}{\partial x_d} = -Q \bar{W}^{(j+2)}, \quad j \geq 0.
\end{align*}
\]

We can analyze these linear equations by the null space method, which will divide the equations into two parts according to the orthogonal projection onto the null space of \( Q \). In the language of matrices, we assume \( G \in \mathbb{R}^{N \times p} \) is an orthonormal basis matrix of the null space of \( Q \), and \( H \in \mathbb{R}^{N \times (N-p)} \) is an orthogonal complement of \( G \). Then we have

\[
I_N = GG^T + HH^T,
\]

where \( I_N \in \mathbb{R}^{N \times N} \) is the identity matrix. One can multiply (4.4) from the left by \( G^T \) to obtain

\[
\frac{\partial G^T \bar{W}^{(j)}}{\partial t} + \sum_{d=1}^{3} G^T A_d \frac{\partial \bar{W}^{(j)}}{\partial x_d} = 0, \quad j \geq 0,
\]

which are called equilibrium equations. Multiplying (4.4) from the left by \( H^T \) and noting that

\[
H^T Q = H^T Q(GG^T + HH^T) = H^T Q H H^T
\]

with \( H^T Q H > 0 \), we have

\[
\begin{align*}
O(\varepsilon^{-1}) : & \quad 0 = H^T \bar{W}^{(0)}, \\
O(\varepsilon^{-1/2}) : & \quad 0 = H^T \bar{W}^{(1)}, \\
O(\varepsilon^{1/2}) : & \quad \frac{\partial H^T \bar{W}^{(j)}}{\partial t} + \sum_{d=1}^{3} H^T A_d \frac{\partial \bar{W}^{(j)}}{\partial x_d} = -(H^T Q H) H^T \bar{W}^{(j+2)}, \quad j \geq 0,
\end{align*}
\]

which are known as the constitutive relations.

Following this way, we can successively get the equilibrium equations about \( G^T \bar{W}^{(j)} \), where \( H^T \bar{W}^{(j)} \) is seen as given from the algebraic relations (4.6). We put the details of the
routine computation in Appendix C. In conclusion, the resulting equilibrium equations are linearized Euler-type equations

$$\frac{\partial \rho^{(j)}}{\partial t} + \sum_d \frac{\partial \bar{u}^{(j)}}{\partial x_d} = 0,$$

$$\frac{\partial \bar{u}_i^{(j)}}{\partial t} + \frac{\partial \left( \rho^{(j)} + \bar{u}^{(j)} \right)}{\partial x_i} = -\sum_d \frac{\partial \sigma^{(j)}_{id}}{\partial x_d}, \quad i = 1, 2, 3,$$

$$\frac{3}{2} \frac{\partial \bar{u}^{(j)}}{\partial t} + \sum_d \frac{\partial \bar{u}^{(j)}}{\partial x_d} = -\sum_d \frac{\partial \bar{q}^{(j)}_d}{\partial x_d},$$

where $\sigma^{(j)}_{id}$ and $\bar{q}^{(j)}_d$ are derived from (4.6). When $j = 0$ or $j = 1$, we have $\sigma^{(j)}_{id} = 0$, $\bar{q}^{(j)}_d = 0$.

When $j = 2$, we have

$$\sigma^{(j)}_{id} = -\gamma_1 \left( \frac{\partial \bar{u}^{(j-2)}}{\partial x_d} + \frac{\partial \bar{u}^{(j-2)}}{\partial x_i} - \frac{2}{3} \delta_{id} \nabla \cdot \bar{u}^{(j-2)} \right), \quad i, d = 1, 2, 3,$$

$$\bar{q}^{(j)}_d = -\frac{5}{2} \gamma_2 \frac{\partial \bar{u}^{(j-2)}}{\partial x_d},$$

where the constants $\gamma_1$ and $\gamma_2$ are defined by the $M$-th order moment equations in (C.1), (C.2). In the BGK model, we have $\gamma_1 = \gamma_2 = 1$. For the hard sphere gas, the constants $\gamma_1$ and $\gamma_2$ given by the moment approach vary with the moment order $M$. When $M$ is sufficiently large, we have $[39] \gamma_1 = 1.270042$ and $\gamma_2 = 1.922284$, which are the same as the asymptotic theory of the Boltzmann equation [50].

According to the classical theory of the linear hyperbolic system [3], the linearized Euler-type equations (4.7) need and only need one boundary condition at the wall, i.e., the value of $\bar{u}^{(j)}_2$ should be prescribed at $x_2 = 0$.

### 4.2.2 Viscous Layer Solution

We assume there exists a viscous layer solution $\hat{W}$ which changes dramatically at the normal direction to the boundary and vanishes outside the viscous layer with a thickness of $O(\sqrt{\varepsilon})$. To compare with the nonlinear boundary layer theory [50], we collect the outer solution and the viscous layer correction together:

$$W = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j W^{(j)}, \quad W^{(j)}(t, \mathbf{x}) = \overline{W}^{(j)}(t, \mathbf{x}) + \hat{W}^{(j)}(t, \mathbf{x}^w; y),$$

with $y = x_2/\sqrt{\varepsilon} \geq 0$ and $\mathbf{x}^w = (x_1, x_3)$. The outer solution $\overline{W}$ and $W$ should both satisfy the moment system (2.8). Since the system is linear, we find that $\hat{W} = W - \overline{W}$ also satisfies (2.8). Then plugging $\hat{W}$ into the equation and comparing each order of $\varepsilon$, we have

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Utilizing an analogous null space method, the equilibrium variables $G^T \hat{W}^{(j)}$ satisfy two degenerate algebraic relations and three linear parabolic equations, which are similar to the linearized Prandtl boundary layer equations [50]. Multiplying (4.11) from the left by $H^T$, we obtain the constitutive relations about $H^T \hat{W}^{(j)}$ (details can be found in Appendix C).

When $j = 0$, the viscous layer solutions satisfy the two algebraic relations

$$\hat{u}_2^{(0)} = 0, \quad \hat{p}^{(0)} = 0$$

(4.12)

with

$$\hat{p}^{(j)} := \hat{p}^{(j)} + \hat{\xi}^{(j)}$$

and the following three parabolic equations

$$\frac{\partial \hat{u}_i^{(0)}}{\partial t} = \gamma_1 \frac{\partial^2 \hat{u}_i^{(0)}}{\partial y^2}, \quad i \neq 2,$$

(4.13a)

$$\frac{\partial \hat{\xi}^{(0)}}{\partial t} = \gamma_2 \frac{\partial^2 \hat{\xi}^{(0)}}{\partial y^2},$$

(4.13b)

where $\gamma_1$ and $\gamma_2$ are constants defined in (C.1), (C.2).

When $j = 1$, we have the equations

$$\hat{u}_2^{(1)} = \int_y^\infty \left( \frac{\partial \hat{p}^{(0)}}{\partial t} + \sum_{d \neq 2} \frac{\partial \hat{u}_d^{(0)}}{\partial x_d} \right) (t, x^w; s) \, ds,$$

(4.14a)

$$\hat{p}^{(1)} = 0,$$

(4.14b)

$$\frac{\partial \hat{u}_i^{(1)}}{\partial t} = \gamma_1 \frac{\partial^2 \hat{u}_i^{(1)}}{\partial y^2}, \quad i \neq 2,$$

(4.14c)

$$\frac{\partial \hat{\xi}^{(1)}}{\partial t} = \gamma_2 \frac{\partial^2 \hat{\xi}^{(1)}}{\partial y^2},$$

(4.14d)

which contain two algebraic relations determining $\hat{u}_2^{(1)}$, $\hat{p}^{(1)}$, and three parabolic equations.

For the higher-order constitutive relations determining $\hat{u}_2^{(2)}$, $\hat{p}^{(2)}$, and three parabolic equations.

$$\hat{\sigma}_{id}^{(2)} = -\gamma_1 \left( \frac{\partial \hat{u}_d^{(0)}}{\partial x_i} + \frac{\partial \hat{u}_i^{(0)}}{\partial x_d} - \frac{2}{3} \frac{\delta_{id}}{\delta_{id}} \left( \frac{\partial \hat{u}_1^{(0)}}{\partial x_1} + \frac{\partial \hat{u}_3^{(0)}}{\partial x_3} + \frac{\partial \hat{u}_2^{(1)}}{\partial y} \right) \right) - \frac{2}{3} \gamma_2 \delta_{id} \frac{\partial^2 \hat{\xi}^{(0)}}{\partial y^2}, \quad i, d \neq 2,$$

$$\hat{\sigma}_{22}^{(2)} = -\gamma_1 \left( 2 \frac{\partial \hat{u}_2^{(1)}}{\partial y} - 2 \frac{3}{2} \left( \frac{\partial \hat{u}_1^{(0)}}{\partial x_1} + \frac{\partial \hat{u}_3^{(0)}}{\partial x_3} + \frac{\partial \hat{u}_2^{(1)}}{\partial y} \right) \right) + \frac{4}{3} \gamma_2 \frac{\partial^2 \hat{\xi}^{(0)}}{\partial y^2},$$

$$\hat{\sigma}_{2d}^{(2)} = -\gamma_1 \frac{\partial \hat{u}_d^{(1)}}{\partial y}, \quad d \neq 2.$$
\[ q_i^{(2)} = -\frac{5}{2} \gamma_2 \frac{\partial \theta^{(0)}}{\partial x_i} + \gamma_3 \frac{\partial^2 \theta^{(0)}}{\partial y^2}, \quad i \neq 2, \]
\[ q_2^2 = -\frac{5}{2} \gamma_2 \frac{\partial \theta^{(1)}}{\partial y}. \]

where the constant \( \gamma_3 \) is given by (C.3) and equals one in the BGK case. For the hard sphere gas, the value of \( \gamma_3 \) is 1.947906 according to the high-order moment approach, which is the same as the results of the LBE [50].

According to the theory of parabolic equations in half-space, every order of the viscous layer solutions exactly need three boundary conditions at \( y = 0 \), i.e., the values of \( \tilde{u}_d^{(j)}(0), d \neq 2 \), and \( \tilde{\theta}^{(j)}(0) \).

### 4.2.3 Knudsen Layer Solution

The IBVP of the moment system needs \( n \) boundary conditions at the wall (cf. Sect. 2). So generally we should insert a Knudsen layer \( \tilde{W} \) to match the BCs up to the higher order. The Knudsen layer solution is assumed to change dramatically at the normal direction to the boundary and vanishes outside the Knudsen layer with a thickness of \( O(\varepsilon) \).

The asymptotic solution \( W_\varepsilon \) defined in (4.3) and \( W = W + \tilde{W} \) should satisfy the moment system (2.8). Since the equations are linear, we find that the Knudsen layer solution \( \tilde{W} = W_\varepsilon - W \) satisfies (2.8). Substituting the ansatz

\[ \tilde{W}(t, x^w; z) = \sum_{j=0}^{\infty} (\sqrt{\varepsilon})^j \tilde{W}^{(j)}(t, x^w; z) \]

into the moment system and matching the order of \( \varepsilon \), we have

\[ A_2 \frac{\partial \tilde{W}^{(j)}}{\partial z} = -Q \tilde{W}^{(j)}, \quad j = 0, 1, \]
\[ A_2 \frac{\partial \tilde{W}^{(j)}}{\partial z} = -Q \tilde{W}^{(j)} - \frac{\partial \tilde{W}^{(j-2)}}{\partial t} - \sum_{d \neq 2} \frac{\partial \tilde{W}^{(j-2)}}{\partial x_d}, \quad j \geq 2. \]

The Knudsen layer solution satisfies a system of linear homogeneous (or non-homogeneous) ODEs in half-space. According to the results by Bobylev and Bernhoff [4] (see also [33]), the Knudsen layer solution needs and only needs \( n - 4 \) boundary conditions, where \( n \) is exactly the number of boundary conditions required by the linear moment equations, and 4 coincides with the number required by the outer solution and viscous layer solution.

### 4.3 Knudsen-Layer Analysis and Slip BCs

To successively determine the asymptotic solutions, it remains to derive slip BCs for the outer solution and viscous layer solution. We plug the asymptotic expansion into the BCs (4.1) and match the order of \( \varepsilon \). Analogously as in Sone’s generalized slip flow theory, the solvability conditions of the Knudsen layer solution determine those slip BCs. The solvability theorem of half-space problems (4.16) for the moment system is studied in [33], which can be stated as
Theorem 4.2 Consider the half-space problem for $\tilde{W} = \tilde{W}(z)$:

$$A_2 \frac{\partial ^2 \tilde{W}}{\partial z^2} = -Q \tilde{W}, \quad z \geq 0,$$

$$B(\tilde{W}(0) - h) = 0, \quad \tilde{W}(\infty) = 0,$$

(4.17)

where $A_2$ and $Q$ are given by (2.9), $B$ given by (4.2), and $h \in \mathbb{R}^N$. We can decompose any $h$ as

$$h = (GG^T + HH^T)h = G_e(G_e^T h) + \varphi_2(\varphi_2^T h) + H(HT h),$$

where $G_e = [\varphi_0, \varphi_1, \varphi_3, \varphi_4] \in \mathbb{R}^{N \times 4}$ defined in Appendix C. Then, for any given $\varphi_2^T h$ and $HT h$, the system (4.17) can uniquely solve $\tilde{W}$ and $G_e^T h$.

When $j = 0$, the Knudsen layer solution satisfies

$$A_2 \frac{\partial \tilde{W}(0)}{\partial z} = -Q \tilde{W}(0),$$

$$B(W(0) + \tilde{W}(0) - b(0)) = 0, \quad \text{at } z = 0,$$

(4.18)

where we expand $b$ in (2.13) analogously with

$$b(0)[N(\theta)] = \rho^{w,0}, \quad b(0)[N(e_i)] = u_i^{w,0}, \quad b(0)[N(2e_i)] = \theta^{w,0}/\sqrt{2}.$$

Here we can calculate (see Appendix C) that

$$G_e^T(W(j) - b(j)) = \left[ \rho^{(j)} - \rho^{w,(j)}, u_1^{(j)} - u_1^{w,(j)}, u_3^{(j)} - u_3^{w,(j)}, \frac{\sqrt{6}}{2} \left( \theta^{(j)} - \theta^{w,(j)} \right) \right]^T.$$

By Theorem 4.2, there is only the zero solution $\tilde{W}(0) = 0$ and $G_e^T(W(0) - b(0)) = 0$ since $HT(W(0) - b(0)) = 0$ from (4.6). This shows that there is no Knudsen layer when $j = 0$ and we have the no-slip BCs

$$u_i^{(0)} = u_i^{w,0}, \quad i = 1, 2, 3,$$

$$\theta^{(0)} = \theta^{w,0}.$$

(4.19a)

(4.19b)

Analogously, when $j = 1$, we regard $HT W(1)$ as the driven term and $G_e^T(W^{(1)} - b^{(1)})$ can be solved from (4.17). From (4.11), we have $HT \tilde{W}^{(1)} = 0$ and $HT \tilde{W}^{(1)}$ represented by derivatives of $G_e^T \tilde{W}^{(0)}$. Thus, we have the following slip BCs for the outer solution and viscous layer solution:

$$u_2^{(1)} = 0,$$

$$u_1^{(1)} - u_i^{w,1} = \sqrt{2} k_0 \frac{\partial \tilde{u}_i^{(0)}}{\partial y}, \quad i = 1, 3,$$

$$\theta^{(1)} - \theta^{w,1} = \sqrt{2} t_1 \frac{\partial \tilde{\theta}^{(0)}}{\partial y},$$

(4.20a)

(4.20b)

(4.20c)

where $k_0$ and $t_1$ are constants solved from the elemental problems defined below.

When $j = 2$, since $\tilde{W}(0) = 0$, the Knudsen layer solution also satisfies the linear homogeneous equations in half-space. From (4.11), now $HT W^{(2)}$ can be represented by derivatives...
of $G^T W^{(0)}$, $G^T W^{(1)}$ and $H^T W^{(1)}$. According to the solvability condition, after a careful calculation, we have the slip BCs

$$u^{(2)}_2 = 0,$$

$$u^{(2)}_i - u^{w,(2)}_i = \sqrt{2}k_0 \left( \frac{\partial \tilde{u}_i^{(0)}}{\partial x_2} + \frac{\partial \tilde{u}_2^{(0)}}{\partial x_i} + \frac{\partial \tilde{u}_1^{(1)}}{\partial y} \right) + 2t_0 \frac{\partial \tilde{\theta}^{(0)}}{\partial x_i} + 2k_2 \frac{\partial^2 \tilde{u}_i^{(0)}}{\partial y^2}, \quad i = 1, 3,$$

$$\theta^{(2)} - \theta^{w,(2)} = \sqrt{2}t_1 \left( \frac{\partial \tilde{\theta}^{(1)}}{\partial y} + \frac{\partial \tilde{\theta}^{(0)}}{\partial x_2} \right) + 2t_2 \frac{\partial^2 \tilde{\theta}^{(0)}}{\partial y^2} + k_1 \left( \frac{\partial \tilde{u}_2^{(1)}}{\partial y} + \frac{\partial \tilde{u}_2^{(0)}}{\partial x_2} \right),$$

where the constants $k_i$ and $t_i$ are solved from the following elemental problems.

The elemental problems arise from the linear superposition principle. For example, $H^T \tilde{W}^{(1)}$ is a linear combination of derivatives of $G^T \tilde{W}^{(0)}$. So we can decompose the vector $H^T \tilde{W}^{(1)}$ into several parts, which have different “driven” terms such as the gradient of velocity and the gradient of temperature. In essence, we replace $h$ in (4.17) by different driven terms to obtain the elemental problems. Here we have six elemental problems in the form

$$A \frac{\partial \tilde{W}}{\partial z} = -Q \tilde{W}, \quad \tilde{W} = \tilde{W}(z), \quad z \geq 0, \quad \tilde{W}(\infty) = 0,$$

$$B \left( G_e G^T_e h + \tilde{W} \right) = B H (H^T Q H)^{-1} A, \quad \text{at } z = 0.$$

Note that we let $\varphi_T^2 h = 0$ in (4.17) and $A$ can be seen as the given driven term. Here $G^T_e h$ and $\tilde{W}$ are unknowns. More precisely, using the vectors $r_{id}$ and $s_d$ defined in (C.1) (C.2), we have

- Velocity slip problem.
  $$A = r_{12}, \quad k_0 = \frac{\sqrt{2}}{2} \varphi_1^T h.$$

Note that $k_0$ is part of $G^T_e h$, solved from the system.

- Temperature jump problem.
  $$A = s_2, \quad t_1 = \frac{\sqrt{3}}{3} \varphi_4^T h.$$

- Thermal creep problem.
  $$A = s_1, \quad t_0 = \frac{1}{2} \varphi_4^T h.$$

- The fourth problem.
  $$A = \sqrt{2}r_{22}, \quad k_1 = \frac{\sqrt{6}}{3} \varphi_4^T h.$$

- Second order viscous slip problem.
  $$A = -H^T A_2 H (H^T Q H)^{-1} r_{12}, \quad k_2 = \frac{1}{2} \varphi_1^T h.$$
Second order temperature jump problem.

\[ A = -H^T A_2 H (H^T Q H)^{-1} s_2, \quad t_2 = \frac{\sqrt{6}}{6} \varphi_4^T h. \]

Till now, we have derived the asymptotic equations and their slip boundary conditions from the linear moment system by Hilbert expansion. The procedure to determine the asymptotic solutions are concluded as follows:

1. Solve \( \hat{u}_2^{(j)} \) from the algebraic relation of the viscous layer solution.
2. Determine \( \hat{u}_2^{(j)} = -\hat{u}_2^{(j)} \) at the boundary and solve the linearized Euler-type equations (4.7) to get \( \hat{W}^{(j)} \).
3. Determine \( \hat{u}_i^{(j)}, i \neq 2, \) and \( \hat{\theta}^{(j)} \) from the slip boundary conditions. Then solve the parabolic equations to get \( \hat{W}^{(j)} \).
4. Solve the Knudsen layer solution \( \tilde{W}^{(j)} \) from the half-space problem.
5. Let \( j = j + 1 \) and return to the first step.

### 4.4 Discussion

As mentioned in the introduction, it is well-known that the Euler equations or NS equations could be derived from the moment equations by Chapman-Enskog expansion or Maxwell iteration [10, 39]. These methods have been originally developed for the Boltzmann equation and then applied to the moment system to validate the asymptotic limit. Considering the problem with boundaries, the systematic asymptotic analysis was established for the Boltzmann equation by Sone’s generalized slip flow theory originally for steady flows [47, 48, 50]. However, as far as we know, there is no similar asymptotic analysis for the IBVPs of moment equations in the literature. Thus, our work is distinguished from some relevant papers about the boundary layers of the moment equations [18, 53] due to the formal but systematic Knudsen-layer analysis for general cases.

In Sone’s generalized slip flow theory, the fluid-dynamic equations and slip BCs for the time-dependent LBE under the acoustic scaling have been established in [27]. Considering the planar boundary, we find that the moment approach gives the same governing equations and slip BCs in the bulk region and viscous layer as those in [27]. Specifically, in the bulk region, Eqs. (31)–(33), Eqs. (34)–(36), and Eqs. (40)–(42) in [27] correspond to (4.7)–(4.9). Equations (81)–(83) in [27] coincide with the BCs (4.19a) (4.19b). In the viscous layer, Eqs. (64)–(67) and Eqs. (68)–(71) in [27] correspond to the equilibrium equations (4.12)–(4.13b) and (4.14). The slip BCs (98)-(100) and Eqs. (101)–(103) in [27] coincide with (4.20a)–(4.20c) and (4.21a)–(4.21c).

From the numerical aspect, there are two differences between the results in this section and [27]. Firstly, the constitutive relations (4.8), (4.9) and (4.15) have the same form as those in [27, 50]. But here the coefficients \( \gamma_i \) are calculated by the algebraic relations given in Appendix C, instead of the integral representations in the slip flow theory [50]. Secondly, the Knudsen layer is described by half-space problems for the moment equations here, which is different from the kinetic layer equations in [50]. The moment approach automatically gives a numerical method for the kinetic layer equation. In the linear case, if we approximate the integral representations of \( \gamma_i \) by appropriate quadrature points and approximate kinetic layer equations by the Grad moment method, we can recover the asymptotic equations obtained in this section.
In a word, the section has established the relations between arbitrary order moment equations and slip BCs for the fluid-dynamic equations. The obtained boundary-layer structure helps us understand the time-dependent behavior of the moment equations around the boundary under the acoustic scaling.

5 NS Equations and Slip BCs

5.1 Construction

The NS equations never directly appear in the Hilbert expansion [12]. However, we may formally retrieve the NS equations and construct their BCs by collecting the coefficients of $\bar{W}^{(j)}$ and $\hat{W}^{(j)}$ (outer solutions and the viscous layer solutions) based on the matching requirement.

Inspired by the simplified model in Sect. 3, we consider

$$W_m = \sum_{j=0}^{2} (\sqrt{\varepsilon})^j \bar{W}^{(j)} + \sum_{j=0}^{2} (\sqrt{\varepsilon})^j \hat{W}^{(j)}.$$ 

Then we can write equations about $W_m$ from (4.4) and (4.11). For example, we have

$$\frac{\partial \rho_m}{\partial t} + \sum_d \frac{\partial u_{d,m}}{\partial x_d} = 0 + \frac{\partial}{\partial t} \sum_{j=0}^{2} (\sqrt{\varepsilon})^j \hat{\rho}^{(j)} + \sum_d \frac{\partial}{\partial x_d} \sum_{j=0}^{2} (\sqrt{\varepsilon})^j \hat{u}^{(j)}_d$$

$$\triangleq R_0(t, x^w; y),$$

where $R_0$ is of the order $O(\varepsilon)$ and vanishes when $y \to +\infty$. Treating the other equilibrium equations in the same way, we have

$$\frac{\partial \rho_m}{\partial t} + \sum_d \frac{\partial u_{d,m}}{\partial x_d} = 0 + R_0(t, x^w; y), \quad (5.1a)$$

$$\frac{\partial u_{i,m}}{\partial t} + \frac{\partial (\rho_m + \theta_m)}{\partial x_i} = \varepsilon \sum_d \frac{\partial}{\partial x_d} \left( y_1 \left( \frac{\partial u_{i,m}}{\partial x_d} + \frac{\partial u_{d,m}}{\partial x_i} - \frac{2}{3} \delta_{id} \nabla \cdot \mathbf{u}_m \right) \right)$$

$$+ R_1(t, x^w; y) + T_i(t, x^w; x_2), \quad i = 1, 2, 3, \quad (5.1b)$$

$$\frac{3}{2} \frac{\partial \theta_m}{\partial t} + \sum_d \frac{\partial u_{d,m}}{\partial x_d} = \frac{5}{2} \varepsilon \sum_d \frac{\partial}{\partial x_d} \left( \gamma_2 \frac{\partial \theta_m}{\partial x_d} \right) + R_4(t, x^w; y) + T_4(t, x^w; x_2), \quad (5.1c)$$

where $R_1$, $R_3$ and $R_4$ are of the order $O(\varepsilon)$, $R_2$ of the order $O(\sqrt{\varepsilon})$, all vanishing when $y \to +\infty$, and $T_i$ is of the order $O(\varepsilon^{3/2})$. At the same time, from the BCs (4.19a)–(4.19b), (4.20a)–(4.20c) and (4.21a)–(4.21c), the BCs for $W_m$ should be

$$u_{2,m} = 0, \quad (5.2a)$$

$$u_{i,m} - u_i^w = \sqrt{2} k_0 \varepsilon \left( \frac{\partial u_{i,m}}{\partial x_2} + \frac{\partial u_{2,m}}{\partial x_i} \right) + 2 t_0 \varepsilon \frac{\partial \theta_m}{\partial x_i}$$

$$+ 2 k_2 \varepsilon^2 \frac{\partial^2 u_{i,m}}{\partial x_2^2} + O(\varepsilon^{3/2}), \quad i = 1, 3, \quad (5.2b)$$
\[ \theta_m - \theta^w = \sqrt{2t_1 \epsilon} \frac{\partial \theta_m}{\partial x_2} + 2t_2 \epsilon^2 \frac{\partial^2 \theta_m}{\partial x_2^2} + k_1 \epsilon \frac{\partial u_{2,m}}{\partial x_2} + O(\epsilon^{3/2}). \]  

**NS equations.** Discarding the residual terms \( R_i \) and \( T_i \) in (5.1a)–(5.1c), we formally obtain the (linearized dimensionless) Navier–Stokes equations

\[
\frac{\partial \rho}{\partial t} + \sum_d \frac{\partial u_d}{\partial x_d} = 0, \tag{5.3a}
\]

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (\rho + \theta)}{\partial x_i} = \epsilon \sum_d \frac{\partial}{\partial x_d} \left( \gamma_1 \left( \frac{\partial u_i}{\partial x_d} + \frac{\partial u_d}{\partial x_i} - \frac{2}{3} \delta_{id} \nabla \cdot \mathbf{u} \right) \right), \quad i = 1, 2, 3, \tag{5.3b}
\]

\[
3 \frac{\partial \theta}{2 \partial t} + \sum_d \frac{\partial u_d}{\partial x_d} = \frac{5}{2} \epsilon \sum_d \frac{\partial}{\partial x_d} \left( \gamma_2 \frac{\partial \theta}{\partial x_d} \right). \tag{5.3c}
\]

**Slip BCs for the NS equations.** Moreover, discarding \( O(\epsilon^{3/2}) \) terms in (5.2a)–(5.2c), we obtain the constructed BCs at \( x_2 = 0 \):

\[
u_2 = 0, \tag{5.4a}
\]

\[
u_i - u_i^w = \sqrt{2k_0 \epsilon} \left( \frac{\partial u_i}{\partial x_2} + \frac{\partial u_2}{\partial x_i} \right) + 2t_0 \epsilon \frac{\partial \theta}{\partial x_i} + 2k_2 \epsilon^2 \frac{\partial^2 u_i}{\partial x_2^2}, \quad i = 1, 3, \tag{5.4b}
\]

\[
\theta - \theta^w = \sqrt{2t_1 \epsilon} \frac{\partial \theta}{\partial x_2} + 2t_2 \epsilon^2 \frac{\partial^2 \theta}{\partial x_2^2} + k_1 \epsilon \frac{\partial u_2}{\partial x_2}. \tag{5.4c}
\]

Assume there is no initial layer and the initial values are same for equations of \( W_m \) and the NS equations. Subtracting the NS equations (5.3a)–(5.3c) by the truncated moment equations (5.1a)–(5.1c), we shall find that the error functions \( \rho - \rho_m, u_i - u_{i,m} \) and \( \theta - \theta_m \) satisfy (5.1a)–(5.1c) with the zero initial value and the BCs (5.2a)–(5.2c) where \( u_i^w = \theta^w = 0 \). In the case of the Rayleigh problem, we have proved that

\[ \|u_1 - u_{1,m}(t, x^w; \cdot)\|_{L^2(\mathbb{R}^+)} \leq C(T)\epsilon^{5/4} \]

for \( t \in [0, T] \). For the general situation, we have no rigorous proof but expect that the \( L^2 \) errors of these five field variables would be higher order terms about \( \epsilon \) in some norm.

**Remark 5.1** Note that the second-order spatial derivatives at the normal direction appear in the above BCs. As to the constructed BCs in Sect. 3, the term with second-order derivatives is transferred to the temporal derivative (see (3.18)). For the general system, it seems not straightforward to use the same technique to validate the constructed BCs.

In conclusion, we have formally constructed the linearized NS equations with slip BCs from the linear moment equations. The obtained BCs contain not only the first-order derivatives but also the second-order derivatives at the normal direction to the wall. Besides the formal derivation, our analysis on the Rayleigh problem exhibits that the second-order terms are necessary for instantaneous flows to get a first-order approximation solution.

**5.2 Remarks on the Obtained Slip BCs**

Due to the rarefaction effects around the solid wall, the model accuracy of the NS equations is extremely affected by their BCs [23]. As shown in [1], the first-order slip models are only suitable for about \( \epsilon \lesssim 0.1 \). To simulate flows in the transition regime by the NS equations,
there are classical second-order slip BCs in the form of (5.4a)–(5.4c) derived by, e.g., Maxwell [37], Sreekanth [51], Deissler [15] and Cercignani [12]. These derivations are both from the physical and mathematical insights. Note that the second-order slip BCs for the NS equations have been widely used in reality [1].

In Sone’s generalized slip flow theory, the fluid-dynamic equations and their second-order slip BCs also appear. For the linear case, the Hilbert expansion is exploited in [26, 50] and asymptotic equations are given explicitly up to second order. The slip BCs are proposed for the corresponding fluid-dynamic equations (asymptotic equations) instead of the NS equations. Compared with the second-order slip BCs Eqs. (3.42a)–(3.42c) in [50], our second-order terms only contain the normal derivatives, not including the tangential derivatives or mixed derivatives. In our BCs, the second-order derivatives arise from the consideration of the viscous layer and \( O(\sqrt{\varepsilon}) \) expansions. Due to the existence of the viscous layer, the second-order normal derivatives, e.g., \( \varepsilon^2 \frac{\partial^2 \theta}{\partial x_2^2} \), are in essence of the magnitude \( O(\varepsilon) \), not \( O(\varepsilon^2) \), while the second-order tangential or mixed derivatives are of the magnitude \( O(\varepsilon^2) \), ignored as the higher-order quantities.

Under the acoustic scaling, the paper [27] has already established fluid-dynamic equations and their slip BCs for the LBE (2.1) by considering the same ansatz (4.3) and power series of \( \sqrt{\varepsilon} \). To handle the secular terms in the Hilbert expansion, the work [27] constructs the linearized NS equations and the second-order slip BCs. In comparison, we successfully recover the results in [27] through the moment approach. In fact, Eqs. (118)–(120) in [27] coincide with (5.4a)–(5.4c) in this paper. In our derivations, the slip coefficients in the BCs can be explicitly solved from the half-space moment equations, which have analytical solutions [33].

Next we give some comments on the time scaling in the LBE. We have emphasized that the LBE (2.1) is prescribed under the acoustic scaling, where the viscous layer with a thickness of \( O(\sqrt{\varepsilon}) \) occurs. For the steady problem, the viscous layer would vanish in the linear case. Actually, we can see this from the Rayleigh problem (3.5). Meanwhile, the second-order terms in the BCs would be higher-order quantities in the linear steady problem. Notice that the viscous layer may appear for the non-linear steady problem [50, Chapter 3].

The diffusive scaling considers the time evolution longer than the acoustic scaling. In this case, the corresponding fluid-dynamic equations and their BCs have been established in [55] for the time-dependent LBE. Next, we give a brief discussion about the derivation through the moment approach. In the diffusive scaling, the linearized moment equations are

\[
\varepsilon \frac{\partial W_\varepsilon}{\partial t} + \sum_{d=1}^{3} A_d \frac{\partial W_\varepsilon}{\partial x_d} = - \frac{1}{\varepsilon} Q W_\varepsilon.
\]  

(5.5)

To match the equations and BCs both up to the order of \( \varepsilon \), we have the following ansatz:

\[
W_\varepsilon(t, x^w; x_2) = \overline{W}(t, x^w; x_2) + \tilde{W}(t, x^w; z),
\]

where \( z = x_2/\varepsilon \), \( \tilde{W}(t, x^w; +\infty) = 0 \) and these solutions are expanded in power series of \( \varepsilon \):

\[
\overline{W} = \sum_{j=0}^{\infty} \varepsilon^j \overline{W}^{(j)}, \quad \tilde{W} = \sum_{j=0}^{\infty} \varepsilon^j \tilde{W}^{(j)}.
\]
Multiplying (5.5) left by $G^T$ gives the equilibrium equations

$$\varepsilon \partial_t (G^T W) + \sum_d G^T A_d \partial_{x_d} W = 0,$$

and multiplying (5.5) left by $H^T$ gives constitutive relations. Substituting the outer solutions $\tilde{W}$ into the constitutive relations and matching the order of $\varepsilon$, we have $H^T \tilde{W}^{(0)} = 0$. Then the equilibrium equations give

$$\sum_d G^T A_d G \partial_{x_d} \left( G^T W^{(0)} \right) = 0,$$

(5.6)

$$\partial_t (G^T \tilde{W}^{(m)}) + \sum_d G^T A_d \left( G \partial_{x_d} (G^T \tilde{W}^{(m+1)}) + H \partial_{x_d} (H^T \tilde{W}^{(m+1)}) \right) = 0, \quad m \geq 0.$$  

(5.7)

Here $H^T \tilde{W}^{(1)} = -(H^T QH)^{-1} \sum_d H^T A_d \partial_{x_d} \tilde{W}^{(0)}$ is given by constitutive relations and other terms $H^T \tilde{W}^{(m)}$ with $m \geq 2$ can be derived similarly. The detailed calculation is similar to that in (4.7) and (4.11). We have checked that, under the diffusive scaling, the moment approach would also give the same governing equations and slip BCs as those in [55] (our slip coefficients are calculated from the half-space moment systems). Noted that the second-order terms in the BCs vanish in this case.

Here are some comments on the derivations for the diffusive scaling, which are analogous as in [55]. (i) In the $m$-th order equations, the velocity $\tilde{u}^{(m)}$ is coupled with the $(m + 1)$-th order pressure $\tilde{p}^{(m+1)}$. (ii) One should modify $\tilde{p}^{(2)}$ to construct the NS equations (cf. Equations (41), (42), and (44) in [55]). (iii) The Knudsen-layer correction $\tilde{W}$ should satisfy (5.5). According to Theorem 4.2, we know that $\tilde{W}^{(0)} = 0$ and the BCs for $\tilde{W}^{(0)}$ are the no-slip BCs. Moreover, $\tilde{W}^{(1)}$ satisfy the linear homogeneous half-space problem and the BCs for $\tilde{W}^{(1)}$ should be the first-order slip BCs (cf. Equation (45) in [55]). (iv) The terms like $\partial_{x_d} u_2$ in (5.4c) would vanish because of the divergence-free condition given by (5.6) and the rigid-body motion of the boundary.

Our work constructs slip BCs based on the Hilbert expansion. Alternatively, the work [2] uses the first-order Chapman-Enskog expansion and Knudsen-layer analysis for the Boltzmann equation (under the acoustic scaling) to derive the compressible NS equations and their slip BCs. Putting aside the second-order derivatives, the BCs (5.4a)–(5.4c) in our work coincide with the linearized version of the first-order slip BCs for the NS equations in [2]. It seems possible to add a Knudsen layer correction of the magnitude $O(\sqrt{\varepsilon})$, i.e., some term like $\sqrt{\varepsilon} f_K^{(1)}(t, x^w, \xi; x_2/\varepsilon)$, to the ansatz in [2] and derive slip BCs for the compressible NS equations. Nevertheless, we have not checked the details of this idea.

### 5.3 Comparison of the Slip Coefficients

To compare the obtained slip coefficients, we refer to Sone’s generalized slip flow theory [26, 50], where the slip coefficients solved from kinetic layer equations provide a good reference for the solutions to the elemental problems (4.22) in this paper.

Numerically, we can see that the solved constants $k_0$, $t_0$, $t_1$, $k_1$, $k_2$ and $t_2$ individually converge to the coefficients $b_1^{(1)}$, $b_2^{(1)}$, $c_1^{(0)}$, $c_5^{(0)}$, $b_4^{(1)}$ and $c_6^{(0)}$ in [26] when the moment order $M$ goes larger. This is because we both use the Hilbert expansion and introduce the Knudsen layer correction, while [26] starts from the Boltzmann equation and we start from the moment
We have used this method in [33] to calculate slip coefficients for the Shakhov model in the literature.

Incidentally, [26, 50] linearize the Boltzmann equation around BGK equations. The coefficients \( M \) rather than \( \chi \) could be approximated

| \( k_0 \) | \( t_0 \) | \( k_2 \) | \( t_0 \) | \( k_2 \) | \( t_0 \) | \( k_2 \) | \( t_0 \) | \( k_2 \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| BGK | 1.01619 | 0.99247 | 1.00360 | 1.00772 | 1.00984 | 1.01112 |
| HS  | 1.25395 | 1.22327 | 1.23904 | 1.24418 | 1.24669 | 1.24819 |

Table 1 The coefficients \( k_0, t_0, k_2 \) compared with [26] when \( \chi = 1 \). (\( k_0 \) is compared with \( b_1^{(1)} \), \( t_0 \) with \( b_2^{(1)} \), \( k_2 \) with \( b_4^{(1)} \), where the latter coefficients appearing in [26])

| \( k_0 \) | \( t_1 \) | \( k_2 \) | \( t_1 \) | \( k_2 \) | \( t_1 \) | \( k_2 \) |
|-----|-----|-----|-----|-----|-----|-----|
| BGK | 0.44046 | 0.42763 | 0.43922 | 0.44019 | 0.44040 | 0.44046 |
| HS  | 0.45957 | 0.54720 | 0.46917 | 0.46236 | 0.46111 | 0.46069 |

Table 2 The coefficients \( k_1, t_1, t_2 \) compared with [26] when \( \chi = 1 \). (\( k_1 \) is compared with \( c_5^{(0)} \), \( t_1 \) with \( c_1^{(0)} \), \( t_2 \) with \( c_6^{(0)} \), where the latter coefficients appearing in [26])

Incidentally, [26, 50] linearize the Boltzmann equation around \( \pi^{-3/2} \exp(-|\xi|^2) \) rather than \( M \). So for the convenience of comparison, there are some scaling constants in (5.4a)–(5.4c) before the slip coefficients \( t_i \) and \( k_i \).

The numerical method to solve elemental problems is briefly stated in Appendix D. We have used this method in [33] to calculate slip coefficients for the Shakhov model with Maxwell diffuse-specular BCs. In [34], the same method is utilized to calculate slip coefficients for inverse-power-law intermolecular potentials with Cercignani-Lampis BCs. However, slip coefficients according to the moment method for the HS gas with Maxwell BCs have not been presented, especially for the second-order slip and jump problems. The data about the non-classical slip coefficients, e.g., \( k_1 \), \( k_2 \) and \( t_2 \), for varying \( \chi \) are also rare in the literature.

We first compare the calculated coefficients with results in [26] for the BGK model and HS gas when the accommodation coefficient \( \chi = 1 \). Tables 1 and 2 lists the corresponding results. The parity of \( M \) would affect the solution behavior for different problems. To make the convergence trends clear, we put the results for \( M \) with the same parity in one table. We can see that the relative error is less than 1% when \( M > 10 \). So we may deduce that moment equations with mild moments can model the half-space problems well.

Due to the discrete essence of the moment method, we can write analytical solutions of these slip coefficients to the moment equations (see Appendix D). The procedure needs to solve a linear algebraic system. When \( M = 3 \) or \( M = 4 \), the explicit expressions of slip coefficients in terms of the accommodation coefficient \( \chi \) could be given. However, when \( M \) becomes larger, the explicit formulae are complicated and these coefficients should be calculated numerically. On the other hand, the coefficients with large \( M \) could be approximated.
by simple fitting formulae, which have similar forms as in the case when \( M = 3 \) or \( M = 4 \). These explicit formulae may be useful for some engineering applications.

For example, when \( M = 4 \), we can write the explicit formula of \( k_0 \) for the BGK model by solving the moment system analytically, as

\[
k_0 = \frac{\sqrt{\pi}}{2} \frac{2 - \chi}{\chi} + \frac{1}{4\sqrt{2\hat{\chi}} + 2\sqrt{6}},
\]

where \( \chi \) is the accommodation coefficient and \( \hat{\chi} \) is defined in (2.14). When \( M \) tends to infinity, the accurate analytical formula of \( k_0 \) as a function of \( \chi \) is too complicated. However, we may propose the approximative formula

\[
k_0 = \left( \frac{\sqrt{\pi}}{2} \frac{2 - \chi}{\chi} + \frac{1}{a_0 \hat{\chi} + b_0} \right) \gamma_1,
\]

(5.8)

where \( a_0 \) and \( b_0 \) are fitting parameters depending on the collision model and moment order \( M \), but not depending on \( \chi \). Here we choose the accurate \( \gamma_1 = 1.270042 \) for the HS gas. Note that \( \lim_{\chi \to 0} \frac{\chi}{2 - \chi} k_0 \) can be analytically obtained and the fitting parameters \( a_0, b_0 \) can be obtained from the linear least square method.

Analogously, in the temperature jump problem, we can easily get \( \lim_{\chi \to 0} \frac{\chi}{2 - \chi} t_1 \) and the approximative formula may be

\[
t_1 = \left( \frac{5\sqrt{\pi}}{8} \frac{2 - \chi}{\chi} + \frac{1}{c_1 \hat{\chi} + d_1} \right) \gamma_2,
\]

(5.9)

where \( \gamma_2 = 1.922284 \) for the HS gas and \( c_1, d_1 \) are fitting parameters determined by the linear least square method. For slip coefficients from other elemental half-space problems, we first calculate their limits when \( \chi \to 0 \) numerically and then determine the fitting parameters \( a_i, b_i, c_i \) and \( d_i \) by the linear least square method, with the formulae

\[
t_0 = \left( \alpha_0 + \frac{\hat{\chi}}{c_0 \hat{\chi} + d_0} \right) \gamma_3,
\]

(5.10)

\[
k_2 = - \left( \beta_2 + \frac{\hat{\chi}}{a_2 \hat{\chi} + b_2} \right) \gamma_3,
\]

(5.11)

\[
k_1 = \left( \beta_1 + \frac{\hat{\chi}}{a_1 \hat{\chi} + b_1} \right) \gamma_1,
\]

(5.12)

\[
t_2 = - \left( \alpha_2 + \frac{\hat{\chi}}{c_2 \hat{\chi} + d_2} \right) \gamma_3.
\]

(5.13)

Here \( \gamma_3 = 1.947906 \) for the HS gas. We calculate the coefficients \( k_0, t_0 \) and \( k_2 \) when \( M = 50 \) for the BGK model and HS gas, with \( \chi \) varying from 0.05 to 1 stepped by 0.05. The coefficients \( k_1, t_1 \) and \( t_2 \) are calculated when \( M = 51 \). The fitting parameters determined by these data are given in Table 3.

Tables 4 and 5 validate the fitting formulae. The reference values when \( \chi = 1 \) are from [26]. Note that we have not found reference values of the coefficients \( k_1, k_2 \) and \( t_2 \) for \( \chi \neq 1 \). In the tables, the column “\( M = 50 \)” or “\( M = 51 \)” lists the numerical results of the moment system, and the column “Formula” lists the coefficients obtained by the corresponding formulae. We can see that these formulae are rather accurate.
moving plate and regard it as a half-space problem. When \( t \) is relatively short such that we can focus on the boundary-layer behavior around the plate, we only consider the fully diffuse BCs, i.e., \( \chi \) As shown in Remark 3.2, the simulation time is assumed to be relatively short such that we can focus on the boundary-layer behavior around the moving plate and regard it as a half-space problem. When \( k_0 = 0 \) and \( k_2 = 0 \), it is the

| \( k_0 \) (5.8) | \( a_0 = 5.183, b_0 = 3.621 \) | \( a_0 = 6.804, b_0 = 4.428 \) |
| \( t_0 \) (5.10) | \( a_0 = 0.25, c_0 = 4.007, d_0 = 2.821 \) | \( a_0 = 0.25683, c_0 = 7.153, d_0 = 4.307 \) |
| \( k_2 \) (5.11) | \( \beta_2 = 0.5, a_2 = 2.003, b_2 = 1.411 \) | \( \beta_2 = 0.33310, a_2 = 4.206, b_2 = 2.735 \) |
| \( k_1 \) (5.12) | \( \beta_1 = 1/3, a_1 = 5.01, b_1 = 3.453 \) | \( \beta_1 = 0.277, a_1 = 6.807, b_1 = 4.055 \) |
| \( t_1 \) (5.9) | \( c_1 = 3.496, d_1 = 2.373 \) | \( c_1 = 5.045, d_1 = 3.030 \) |
| \( t_2 \) (5.13) | \( a_2 = 11/12, c_2 = 1.055, d_2 = 0.725 \) | \( a_2 = 1.16877, c_2 = 1.221, d_2 = 0.744 \) |

Table 3 Fitting parameters in the formulae

| \( \chi \) | Ref | \( M = 50 \) | Formula |
| --- | --- | --- | --- |
| \( k_0 \) (5.8) | 0.1 | 17.103* | 17.099 |
| | 0.5 | 2.8612 | 2.8587 |
| | 0.7 | 1.8187 | 1.8169 |
| | 1.0 | 1.0162 | 1.0152 |
| \( t_0 \) (5.10) | 0.1 | 0.26418* | 0.26405 |
| | 0.5 | 0.31889 | 0.31843 |
| | 0.7 | 0.34512 | 0.34458 |
| | 1.0 | 0.38316 | 0.38258 |
| \( k_2 \) (5.11) | 0.1 | -0.52810 | -0.52809 |
| | 0.5 | -0.63685 | -0.63683 |
| | 0.7 | -0.68917 | -0.68914 |
| | 1.0 | -0.76632 | -0.76523 |

** Results of the BGK model are from [45]

*** From the recent paper [54]. The results of [54] are multiplied by \( \gamma_1 \) due to the different definitions of the mean free path

* From [44], multiplied by \( \gamma_2 \)

### 6 Rayleigh Problem: Numerical Verification

In this section, we explore the velocity profile of the Rayleigh problem numerically. The setting is given in Sect. 3, where the gas is driven by the motion of an infinite plate. Here we only consider the fully diffuse BCs, i.e., \( \chi = 1 \) in (2.13). The simplified NS equations (3.22) and (5.4b) for this problem are

\[
\frac{\partial u_1}{\partial t} = \varepsilon \frac{\partial^2 u_1}{\partial x^2},
\]

\[
u_1(0, x) = 0,
\]

\[
u_1(t, 0) - u_{1w}(t, 0) = \sqrt{2} k_0 \varepsilon \frac{\partial u_1}{\partial x}(t, 0) + 2 k_2 \varepsilon^2 \frac{\partial^2 u_1}{\partial x^2}(t, 0),
\]

where \( x > 0 \) and \( t \in [0, T] \). As shown in Remark 3.2, the simulation time is assumed to be relatively short such that we can focus on the boundary-layer behavior around the moving plate and regard it as a half-space problem. When \( k_0 = 0 \) and \( k_2 = 0 \), it is the
The coefficients \( k_1 \), \( t_1 \) and \( t_2 \) for different accommodation coefficients \( \chi \)

| \( \chi \) | BGK M = 51 | HS M = 51 |
|---|---|---|
| \( k_1 \) | \( (5.12) \) | 0.1 | 0.44064 | 0.36407 | 0.36409 |
| | | 0.5 | 0.38989 | 0.38989 | 0.40942 | 0.40939 |
| | | 0.7 | 0.40999 | 0.40999 | 0.43017 | 0.42998 |
| | | 1.0 | 0.44046 | 0.44046 | 0.45957 | 0.45914 | 0.45862 |
| \( t_1 \) | \( (5.9) \) | 0.1 | 21.450* | 21.445 | 21.445 | 41.039* | 41.054 | 41.053 |
| | | 0.5 | 3.6291 | 3.6261 | 3.6261 | 6.8212 | 6.8278 | 6.8281 |
| | | 0.7 | 2.3175 | 2.3154 | 2.3154 | 4.3201 | 4.3244 | 4.3246 |
| | | 1.0 | 1.3027 | 1.3015 | 1.3015 | 2.4002 | 2.4021 | 2.4019 |
| \( t_2 \) | \( (5.13) \) | 0.1 | -0.97127 | -0.97125 | -2.3795 | -2.3795 |
| | | 0.5 | -1.1812 | -1.1811 | -2.7612 | -2.7614 |
| | | 0.7 | -1.2813 | -1.2813 | -2.9367 | -2.9364 |
| | | 1.0 | -1.4276 | -1.4261 | -1.4259 | -3.1800 | -3.1839 | -3.1812 |

* From [43]. Due to the different definitions of the mean free path, the results of [43] should be multiplied by \( r^2 \).

no-slip case and we label it “no-slip” in the figures below. The label “1st-slip” corresponds to \( k_0 = 1.01619, \ k_2 = 0 \) and “2nd-slip” corresponds to \( k_0 = 1.01619, \ k_2 = -0.76632 \). In the numerical test, we choose

\[ u^w_1(t, 0) = 1 - \cos(2\pi t). \]

We use an implicit finite difference method to solve this parabolic equation. Assume \( x_i = ih \) where \( 0 \leq i \leq N \) and \( h = 5\sqrt{\varepsilon}/N, t_n = n\Delta t, u^n_i := u_1(t_n, x_i) \), then the numerical scheme is

\[ \frac{u^{n+1}_i - u^n_i}{\Delta t} = \frac{1}{\varepsilon} \left( u^{n+1}_{i+1} - 2u^{n+1}_i + u^{n+1}_{i-1} \right). \]

For the boundary conditions, we assume that \( u^N_1 = 0 \) and the derivatives are approximated by the one-sided difference:

\[ u^{n+1}_0 - u^{n+1}_{w,n+1} = \sqrt{2k_0 \varepsilon} \frac{u^{n+1}_1 - u^{n+1}_0}{h} + 2k \varepsilon^2 \frac{u^{n+1}_2 - 2u^{n+1}_1 + u^{n+1}_0}{h^2}. \]

It is well-known that the implicit scheme enlarges the feasible time step and the above scheme is at least first-order accurate in space. To validate it, we take \( t = 0.1 \) and choose a sufficiently small time step \( \Delta t = 10h^2/\varepsilon \). The solutions when \( N = 2^{13} = 8192 \) are chosen as the reference \( U_{ref} \). We consider the 2—norm relative errors between the solutions \( U_N \) when \( N = 2^i, \ 6 \leq i \leq 10 \) and \( U_{ref}, \ i.e., e = \| U_N - U_{ref} \| / \| U_{ref} \|. \) Tables 6 and 7 validate the spatial accuracy of the scheme for \( \varepsilon = 0.1 \) and \( \varepsilon = 0.001 \).

For the BGK model, the linear moment equations are

\[ \frac{\partial W_c}{\partial t} + A_c \frac{\partial W_c}{\partial x} = -\frac{1}{\varepsilon} \bar{Q}_c W_c, \]

\[ W_c(0, x) = 0, \]

\[ B_c W_c(t, 0) = b_c(t), \]
where $W_c$, $A_c$, $Q_c$, $B_c$, $b_c$ are defined in Sect. 3.

This is a linear hyperbolic IBVP with constant coefficients. We use the first order upwind scheme to approximate the convection term and implicitly deal with the source term:

$$\frac{W_{i+1}^n - W_i^n}{\Delta t} + \mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n = -\frac{1}{\varepsilon} Q_c W_i^{n+1},$$

where $W_i^n := W_c(t_n, x_i)$ and $\mathcal{F}_{i+1/2}^n$ is the upwinding numerical flux [32]. Assume the decomposition $A_c = R \Lambda R^{-1}$ with $\Lambda = \Lambda^+ + \Lambda^-$ where $\Lambda^+$ is diagonal with positive entries and $\Lambda^-$ is diagonal with negative entries. Then the upwinding numerical flux is defined as

$$\mathcal{F}_{i+1/2}^n = A_c^+ W_i^n + A_c^- W_{i+1}^n,$$

where $A_c^+ = R \Lambda^+ R^{-1}$ and $A_c^- = R \Lambda^- R^{-1}$. We also deal with the boundary condition according to the characteristic information, which is illustrated in Sect. 3. Precisely, due to the block structure of $A_c$, as in Proposition 3.2, the matrix $R$ can be chosen as

$$R = \begin{pmatrix} R_e & R_e \\ R_o & -R_o \end{pmatrix} \in \mathbb{R}^{M \times M}, \quad R_e, R_o \in \mathbb{R}^{\frac{M}{2} \times \frac{M}{2}}.$$

Denote the characteristic variable by $V_c = R^T W_c = (V_c^T, V_c^T)^T$. Then at $x = 0$, we can calculate $V_-(t_n, 0)$ from the known value $W^n_0$ and obtain $V_+(t_{n+1}, 0)$ via the algebraic relations given by the boundary conditions. The procedure belongs to a simple extrapolation. The time step is determined by the CFL condition.

Table 8 shows that the scheme is first order in space. Here we take $t = 0.1$. The reference solutions are solved in the case $N = 8192$ and the relative error $e$ is the same as in Table 6. Figure 1 exhibits the effect of the moment order $M$ of the moment equations. We can see that when $M \gtrsim 8$, the moment order $M$ has very little influence on the solutions. Thus, in the rest of this section, we only consider the moment equations with $M = 8$ and label the corresponding solutions “MoM”.

\[\text{Table 6} \quad \text{Relative errors and spatial convergence order of the scheme for the NS equations when } \varepsilon = 0.1\]

| $N$  | no-slip | 1st-slip | 2nd-slip |
|------|---------|----------|----------|
|      | $e$     | order    | $e$      | order    | $e$      | order    |
| 64   | 0.6193  | 1.15     | 0.04917  | 1.15     | 0.02211  | 1.15     |
| 128  | 0.2513  | 1.30     | 0.1091   | 1.20     | 0.04917  | 1.15     |
| 256  | 0.1091  | 1.20     | 0.04917  | 1.15     | 0.02211  | 1.15     |
| 512  | 0.04917 | 1.15     | 0.02211  | 1.15     |
| 1024 | 0.02211 | 1.15     |

\[\text{Table 7} \quad \text{Relative errors and spatial convergence order of the scheme for the NS equations when } \varepsilon = 0.001\]

| $N$  | no-slip | 1st-slip | 2nd-slip |
|------|---------|----------|----------|
|      | $e$     | order    | $e$      | order    | $e$      | order    |
| 64   | 0.6193  | 0.7296   | 0.7566   |
| 128  | 0.2513  | 0.2907   | 0.3006   |
| 256  | 0.1091  | 0.1242   | 0.1281   |
| 512  | 0.04917 | 0.05531  | 0.05695  |
| 1024 | 0.02211 | 0.02471  | 0.02541  |
Table 8  Relative errors and spatial convergence order of the scheme for the moment equations when $M = 8$

| $N$ | $\varepsilon = 0.001$ | $\varepsilon = 0.05$ | $\varepsilon = 0.1$ |
|-----|-----------------|-----------------|-----------------|
|     | $e$ | order | $e$ | order | $e$ | order |
| 64  | 0.4554 |          | 0.4951 |          | 0.6991 |          |
| 128 | 0.2304 | 0.98   | 0.2096 | 1.24   | 0.2744 | 1.35   |
| 256 | 0.1166 | 0.98   | 0.09633 | 1.12  | 0.1228 | 1.16   |
| 512 | 0.05759 | 1.02  | 0.04523 | 1.09  | 0.05714 | 1.10   |
| 1024 | 0.02722 | 1.08  | 0.02082 | 1.12  | 0.02622 | 1.12   |

Fig. 1  The velocity profile for different $M$ when $t = 0.1$ and $\varepsilon = 0.1$

Fig. 2  The velocity profile when $\varepsilon = 0.1$. Left: $t = 0.1$. Right: $t = 0.25$

Fig. 3  The velocity profile when $\varepsilon = 0.05$. Left: $t = 0.1$. Right: $t = 0.25$
Fig. 4 The velocity profile when $\varepsilon = 0.001$. Left: $t = 0.1$. Right: $t = 0.25$

Fig. 5 The log-log diagram of the velocity errors. Left: $t = 0.1$. Right: $t = 0.25$

Figures 2, 3 and 4 show the velocity profile when $\varepsilon = 0.1$, 0.05 and 0.001, where we choose $N = 1024$. In these figures, the x-axis is $x/\sqrt{\varepsilon}$, which means zooming in to observe the boundary-layer behavior of the solutions.

From Fig. 2, we can see that the “MoM” solution deviates from the NS solutions when $x$ is close to zero, and agrees with the “2nd-slip” solution when $x$ has a distance from zero. The phenomena coincide with the theory: firstly, the NS equations can not capture the Knudsen layer but the moment equations can, so when $x = O(\varepsilon)$, the “MoM” solution would deviate from the NS solutions. Secondly, when $x = O(\sqrt{\varepsilon})$, the moment equations can capture the viscous layer, which is also well approximated by the NS equations with second-order BCs. So the “MoM” solution agrees with the “2nd-slip” solution when $x = O(\sqrt{\varepsilon})$.

From Figs. 2, 3 and 4, we can see that when $\varepsilon$ goes smaller, the difference between the “1st-slip” and “2nd-slip” solutions tends to vanish. To show the error between NS solutions with different BCs, we exhibit Fig. 5.

In Fig. 5, the label “+1st-slip” means the $L^2$ error of the “no-slip” solution and “1st-slip” solution while the label “+2nd-slip” representing the error between the “1st-slip” and “2nd-slip” solution. Here we also choose $N = 1024$. We can see that the “+1st-slip” error roughly has a convergence rate of the order 1/2 while the “+2nd-slip” error is about the first order. The fact implies that if the $L^2$ error between the viscous layer solution of the moment equations and the “2nd-slip” solution is $o(\varepsilon)$, then the error between the moment solution and the “1st-slip” solution should be at most $O(\varepsilon)$. Thus, for instantaneous flows, the second-order slip BCs for the NS equations are necessary to obtain a first-order approximation solution to the moment equations.
In conclusion, the well-designed numerical example of the Rayleigh problem verifies the theoretical results in Sect. 3.

7 Conclusions

This work is concerned with time-dependent behaviors of the linearized moment system around a planar solid wall under the acoustic scaling. We first applied the Hilbert expansion and boundary-layer analysis to the linearized moment system, which is known as Sone’s generalized slip flow theory [50] for the Boltzmann equation. It seems that rare works have been devoted to the asymptotic analysis for the boundary-value problems of moment systems. This procedure gives the same governing equations and slip boundary conditions as Sone’s theory in the bulk region and viscous boundary layer, while the Knudsen layer equations are replaced by the half-space moment equations. In comparison, the moment approach automatically provides a simple but effective numerical method to solve the Knudsen layer equation. Moreover, slip coefficients can be easily calculated with different collision terms. In fact, explicit expressions in terms of the accommodation coefficient are given for some cases.

Under the acoustic scaling, we formally obtained the linearized NS equations as well as their second-order slip boundary conditions by summing up asymptotic expansions of the moment system. This procedure is parallel to the asymptotic analysis for the Boltzmann equation in [27]. For the Rayleigh problem, we rigorously prove the error estimate for the asymptotic expansion. Moreover, for this simple case, the slip boundary conditions for the NS equations are validated by both energy estimates and numerical experiments. As for the general moment system, it seems difficult to give a rigorous error estimate and this is left for future study.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

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Appendix A: The Linearized Collision Operator

The linearized Boltzmann collision operator \( \mathcal{L}[f] \) in (2.1) is defined as

\[
\mathcal{L}[f](t, x, \xi) = \mathcal{M}(\xi) \int_{\mathbb{R}^3} \int_{S^2} K[f, \mathcal{M}] \mathcal{M}(\xi_*) B(|\xi - \xi_*|, \Theta) \, d\Theta 
\]

where \( B = B(\cdot, \cdot) \) is a differential cross-section depending on the potential between gas molecules. The function \( K[\cdot] \) is defined as

\[
K[\psi](\xi, \xi_*, \Theta) = \psi(\xi_*) + \psi(\xi') - \psi(\xi_*) - \psi(\xi),
\]
where $\xi'$ and $\xi'_*\Sigma$ are determined by $\xi, \xi_*$ and $\Theta$ from the elastic collision process. In the three-dimensional HS model, we can write

$$B = |\xi - \xi_*\Sigma|,$$

where $\Sigma$ is a constant. Much more results can be found in [12, Chapter III]. In (2.9), the elements of the matrix $Q$ for the HS gas can be calculated numerically by the method proposed in [58].

The Shakhov model [42] is a simplified collision model. In our case, the linearized operator $L[f]$ for the Shakhov case can write as

$$L[f] = L_{BGK}[f] = M(\xi)\left(\rho + u \cdot \xi + \theta \frac{|\xi|^2 - 3}{2}\right) - f,$$

$$L[f] = L_{Sh}[f] = L_{BGK}[f] + \frac{2}{5}(1 - \text{Pr})q \cdot \xi \frac{|\xi|^2 - 5}{2} M(\xi).$$

Here $\text{Pr}$ is the Prandtl number and when $\text{Pr} = 1$ the Shakhov model becomes the BGK model. Note that the macroscopic variables are defined as (2.3). Incidentally, we linearize the Boltzmann equation around $M(\xi)$ (see (2.2)) rather than $\pi - \frac{3}{2} \exp(-|\xi|^2)$ as in [50] because $M(\xi)$ is commonly used in the moment method [6, 20]. Thus, some coefficients in the obtained LBE are different from the classical form in [50]. In (2.9), for the BGK model, the direct calculation gives the only non-zero entries

$$Q[\mathcal{N}(2e_i), \mathcal{N}(2e_j)] = \delta_{ij} - \frac{1}{3},$$

$$Q[\mathcal{N}(\alpha), \mathcal{N}(\alpha)] = 1, \quad |\alpha| \geq 2, \quad \alpha \neq 2e_i.$$

For the Shakhov model, we only need to modify the matrix $Q$ in the BGK case such that

$$Q[\mathcal{N}(e_i + 2e_j), \mathcal{N}(e_i + 2e_k)] = \delta_{jk} - \frac{1 - \text{Pr}}{5} \sqrt{1 + 2\delta_{ij}} \sqrt{1 + 2\delta_{ik}}.$$

### Appendix B: Details About the Moment System

As in [34], the multi-index $\alpha \in \mathbb{N}^3$ is first ordered by the parity of $\alpha_2$, then by the norm $|\alpha|$, and finally by the anti-lexicographic order. The isomorphism $\mathcal{N}$ is defined according to the above ordering.

**Definition B.1** For $\alpha, \beta \in \mathbb{N}^3$, we define $<$ as follows:

- When $\alpha_2$ is even and $\beta_2$ is odd, $\alpha < \beta$.
- When $\alpha_2$ and $\beta_2$ have the same parity, but $|\alpha| < |\beta|$, then $\alpha < \beta$.
- When $\alpha_2, \beta_2$ have the same parity and $|\alpha| = |\beta|$, but there exists $1 \leq i \leq 3$ such that $\alpha_i > \beta_i$ and $\alpha_j = \beta_j$ for all $j < i$, then $\alpha < \beta$.

The isomorphism $\mathcal{N} : \{\alpha \in \mathbb{N}^3 : |\alpha| \leq M\} \rightarrow \{1, 2, ..., N\}$ is naturally defined such that

$$\mathcal{N}(\alpha) < \mathcal{N}(\beta) \iff \alpha < \beta.$$
\[ \int_{\mathbb{R}^2} \int_0^{+\infty} \xi_2 \phi_\alpha f(t, x, \xi) \, d\xi = \chi \int_{\mathbb{R}^2} \int_0^{+\infty} \xi_2 \phi_\alpha f^w(t, x, \xi) \, d\xi + (1 - \chi) \int_{\mathbb{R}^2} \int_0^{+\infty} \xi_2 \phi_\alpha f(t, x, \xi^+) \, d\xi. \]

Then we will substitute \( f = M \sum_{|\alpha| \leq M} w_{\alpha} \phi_\alpha \) into the above relation. The half-space integral will be transformed into the whole space integral according to the even-odd parity. After some manipulations, the Grad BCs read as

\[ \left( 1 - \frac{\chi}{2} \right) \sum_{\beta \in I_o} \langle \xi_2 M \phi_\alpha \phi_\beta \rangle w_\beta = -\frac{\chi}{2} \sum_{\beta \in I_e} \langle |\xi_2| M \phi_\alpha \phi_\beta \rangle (w_\beta - b_\beta), \] (B.1)

where \( \alpha \in I_e \) and \( |\alpha| \leq M - 1 \). To rewrite (B.1) into a matrix form, we further define two mappings

\[ N_1 : I_e \to \{1, 2, \ldots, m\}, \quad N_2 : I_o \to \{1, 2, \ldots, n\} \]

by the relations

\[ N_1(\alpha) < N_1(\beta) \iff \alpha < \beta, \quad \text{for } \alpha, \beta \in I_e, \]
\[ N_2(\alpha) < N_2(\beta) \iff \alpha < \beta, \quad \text{for } \alpha, \beta \in I_o. \]

Having these, we define \( M_o \in \mathbb{R}^{m \times n} \) and \( S \in \mathbb{R}^{m \times m} \) by

\[ M_o[N_1(\alpha), N_2(\beta)] = \langle \xi_2 M \phi_\alpha \phi_\beta \rangle, \quad \alpha \in I_e, \ \beta \in I_o. \]
\[ S[N_1(\alpha), N_1(\beta)] = \frac{2\pi}{\sqrt{2 \pi}} \langle |\xi_2| M \phi_\alpha \phi_\beta \rangle, \quad \alpha, \ \beta \in I_e. \]

It was shown in [33] that \( M_o \) is of full column rank and \( S \) is symmetric positive definite.

\section*{Appendix C: Details of the Asymptotic Analysis}

\subsection*{C.1: The Matrix \( G \) and \( H \)}

The choices of \( G \) and \( H \) in Sect. 4.2.1 are not unique. Apparently, the different choices will give equivalent equations in the linear case. Here we will give a specific choice of \( G \) and \( H \).

Due to the collision invariants of the linearized Boltzmann operator [12], the null space of \( Q \) always has a constant dimension for any moment order \( M \geq 3 \), i.e.,

\[ p = \dim \text{Null}[Q] = 5, \]

which corresponds to the conservation of mass, momentum and energy. Thus, we can choose \( G \in \mathbb{R}^{N \times p} \) as

\[ G = [\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4], \]

where \( \varphi_i \in \mathbb{R}^N \) is given with the non-zero entries

\[ \varphi_0[N(0)] = 1, \]
\[ \varphi_i[N(e_i)] = 1, \quad i = 1, 2, 3, \]
\[ \varphi_4[N(2e_i)] = \sqrt{3}/3. \]
Incidentally, we can define
\[ G_e = [\varphi_0, \varphi_1, \varphi_3, \varphi_4]. \]

The equilibrium variables \(G^T W\) is
\[ G^T W = \left[ \rho, u_1, u_2, u_3, \frac{\sqrt{6}}{2} \theta \right]. \]

Due to the structure of \(G\), we can choose \(H \in \mathbb{R}^{N \times (N-p)}\) such that the columns of \(H\) are all unit vectors with only one component being one, i.e.,
\[ H[\mathcal{N}(\alpha), \mathcal{N}(\beta) - 5] = \delta_{\alpha, \beta}, \quad |\beta| > 1, \quad \beta \neq 2e_1, 2e_2, 2e_3. \]

except for two columns which will form an orthogonal matrix with \(\varphi_4\), i.e.,
\[ H[\mathcal{N}(\alpha), \mathcal{N}(2e_2) - 5] = \delta_{\alpha, 2e_1} \frac{\sqrt{3}}{3} + \delta_{\alpha, 2e_2} \frac{-3 - \sqrt{3}}{6} + \delta_{\alpha, 2e_3} \frac{3 - \sqrt{3}}{6}, \]
\[ H[\mathcal{N}(\alpha), \mathcal{N}(2e_3) - 5] = \delta_{\alpha, 2e_1} \frac{\sqrt{3}}{3} + \delta_{\alpha, 2e_2} \frac{3 - \sqrt{3}}{6} + \delta_{\alpha, 2e_3} \frac{-3 - \sqrt{3}}{6}. \]

C.2: Determination of \(\gamma_i\)

We first focus on the constitutive relations (4.8) (4.9) of outer solutions. From the general iteration in (4.6), utilizing the symmetry properties of \(Q\) as mentioned in the Maxwell iteration of moment equations [39], one can define
\[ \gamma_1 = r_{12}^T (H^T QH)^{-1} r_{12}, \]  
\[ \gamma_2 = \frac{1}{5} s_1^T (H^T QH)^{-1} s_1, \]

where non-zero entries of \(r_{id} \in \mathbb{R}^{N-p}\) and \(s_d \in \mathbb{R}^{N-p}\), \(d = 1, 2, 3\), are
\[ r_{id}[\mathcal{N}(e_i + e_d) - 5] = 1, \quad i \neq 1 \text{ or } d \neq 1, \]
\[ s_d[\mathcal{N}(3e_d) - 5] = \sqrt{3/2}, \quad s_d[\mathcal{N}(e_d + 2e_i) - 5] = \sqrt{1/2}, \quad i \neq d. \]

Then in the viscous layer, multiplying (4.11) left by \(H^T\) gives
\[ H^T \hat{W}^{(0)} = 0, \]
\[ -(H^T QH)H^T \hat{W}^{(1)} = H^T A_2 (GG^T + HH^T) \frac{\partial \hat{W}^{(0)}}{\partial y} = H^T A_2 GG^T \frac{\partial \hat{W}^{(0)}}{\partial y}, \]
\[ -(H^T QH)H^T \hat{W}^{(j+2)} = H^T \left( \frac{\partial \hat{W}^{(j)}}{\partial t} + \sum_{d \neq 2} A_d \frac{\partial \hat{W}^{(j)}}{\partial x_d} \right) + H^T A_2 \frac{\partial \hat{W}^{(j+1)}}{\partial y} \]
\[ = H^T \sum_{d \neq 2} A_d GG^T \frac{\partial \hat{W}^{(j)}}{\partial x_d} \]
\[ + H^T A_2 (GG^T + HH^T) \frac{\partial \hat{W}^{(j+1)}}{\partial y}, \quad j \geq 0. \]
The first formula shows that \( \hat{\sigma}_{id}^{(0)} = \hat{q}_{d}^{(0)} = 0 \). From the second formula, the non-equilibrium variable \( H^T \hat{W}^{(1)} \) is represented by derivatives of \( G^T \hat{W}^{(0)} \). From the third formula, the variable \( H^T \hat{W}^{(j+2)} \) can be represented by derivatives of \( G^T \hat{W}^{(j+1)} \). By induction, we conclude that \( H^T \hat{W}^{(j)} \) can be represented by derivatives of \( G^T \hat{W}^{(s)} \), \( s < j \).

For example, \( \hat{\sigma}_{11}^{(2)} \) can be represented by the linear combination of components of \( H^T \hat{W}^{(2)} \). Since all matrices are known, with the aid of the computer algebra system, we can calculate that

\[
\hat{\sigma}_{11}^{(2)} = -\gamma_3 \left( 2 \frac{\partial \hat{u}_1^{(0)}}{\partial x_1} - \frac{2}{3} \left( \frac{\partial \hat{u}_1^{(0)}}{\partial x_1} + \frac{\partial \hat{u}_3^{(0)}}{\partial x_3} + \frac{\partial \hat{u}_2^{(1)}}{\partial y} \right) \right) - \frac{2}{3} \gamma_3 \frac{\partial^2 \hat{\theta}^{(0)}}{\partial y^2},
\]

where the constant

\[
\gamma_3 = r_{12}^T (H^T Q H)^{-1} (H^T A_2 H)(H^T Q H)^{-1} s_1.
\]

Due to the symmetry of \( Q \), the same \( \gamma_3 \) appears in other relations [50], e.g., \( \hat{\sigma}_{id}^{(2)} \).

**Appendix D: Numerical Method of the Half-Space Problem**

We use the numerical method mentioned in [33] to solve the half-space problem (4.17):

\[
A_2 \frac{\partial \tilde{W}}{\partial z} = -Q \tilde{W}, \quad \tilde{W} = \tilde{W}(z), \quad z \geq 0,
\]

\[
B(\tilde{W}(0) - h) = 0, \quad \tilde{W}(\infty) = 0.
\]

The numerical method is briefly stated as follows. Since \( A_2 \) is symmetric and \( Q \) is symmetric positive semi-definite, we can solve a generalized eigenvalue problem to get

\[
A_2 x_i = \lambda_i Q x_i, \quad x_i \in \mathbb{R}^N, \quad \lambda_i \in \mathbb{R} \cup \{\infty\}, \quad i = 1, 2, \ldots, N.
\]

Because \( \tilde{W} \) vanishes at infinity, the characteristic variables corresponding to non-positive eigenvalues should be zero, i.e.,

\[
x_i^T Q \tilde{W} = 0, \quad \lambda_i \leq 0 \text{ or } \lambda_i = \infty.
\]

The above formula gives a relation between components of \( \tilde{W} \), i.e., only part of components of \( \tilde{W} \) is independent.

Substitute the relation into BCs and write \( h \) as \( h = h_+ + h_- \), where \( h_- \) represents the given driven term, then we may solve a linear algebraic system to determine \( h_+ \). The solvability of this problem is ensured by Theorem 4.2 when \( h_- \) is appropriately chosen.

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