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Zero bias transformation and asymptotic expansions II:
the Poisson case

Ying Jiao

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Abstract

We apply a discrete version of the methodology in [12] to obtain a recursive asymptotic expansion for $E[h(W)]$ in terms of Poisson expectations, where $W$ is a sum of independent integer-valued random variables and $h$ is a polynomially growing function. We also discuss the remainder estimations.

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Key words: Poisson approximation, zero bias transformation, Stein’s method, asymptotic expansions, discrete reverse Taylor formula.

1 Introduction and main result

It should be noted in the first place that the notation used in this paper is similar as in [12], however, their meanings are different since we here consider discrete random variables. Stein’s method for Poisson approximation has been introduced by Chen [8]. Let $Z$ be an $\mathbb{N}$-valued random variable ($\mathbb{N}$-r.v.), then $Z$ follows the Poisson distribution with parameter $\lambda$ if and only if the equality $E[Zf(Z)] = \lambda E[f(Z + 1)]$ holds for any function $f : \mathbb{N} \to \mathbb{R}$ such that both sides of the equality are well defined. Based on this observation, Chen has proposed the following discrete Stein’s equation:

\begin{equation}
xf(x) - \lambda f(x + 1) = h(x) - \mathcal{P}_\lambda(h), \quad x \in \mathbb{N}
\end{equation}

where $\mathcal{P}_\lambda(h)$ is the expectation of $h$ with respect to the $\lambda$-Poisson distribution. If $X$ is an $\mathbb{N}$-r.v., one has $E[h(X)] - \mathcal{P}_\lambda(h) = E[xf_h(X) - f_h(X + 1)]$ where $f_h$ is a solution of (1) and is given as

\begin{equation}
f_h(x) = \frac{(x - 1)!}{\lambda^x} \sum_{i=x}^{\infty} \frac{\lambda^i}{i!} (h(i) - \mathcal{P}_\lambda(h)).
\end{equation}
The value $f_h(0)$ can be arbitrary and is not used in calculations in general.

Stein’s method has been adopted for Poisson approximation problems since $[8]$ in a series of papers such as $[3], [4], [5]$ among many others, one can also consult the monograph $[6]$ and the survey paper $[10]$. In particular, Barbour $[3]$ has developed, in parallel with the normal case $[2]$, asymptotic expansions for sum of independent $N$-r.v.s and for polynomially growing functions. The asymptotic expansion problem has also been studied by using other methods such as Lindeberg method (e.g. $[7]$).

In this paper, we address this problem by the zero bias transformation approach. Similar as in Goldstein and Reinert $[11]$, we introduce a discrete analogue of zero bias transformation (see also $[9]$). Let $X$ be an $N$-r.v. with expectation $\lambda$. We say that an $N$-r.v. $X^*$ has Poisson $X$-zero biased distribution if the equality

$$\mathbb{E}[X f(X)] = \lambda \mathbb{E}[f(X^* + 1)]$$

holds for any function $f : \mathbb{N} \rightarrow \mathbb{R}$ such that the left side of (3) is well defined. The distribution of $X^*$ is unique: one has $P(X^* = x) = \frac{x + 1}{\lambda} P(X = x + 1)$. Combining Stein’s equation (1) and zero bias transformation (3), the error of the Poisson approximation can be written as

$$\mathbb{E}[h(X)] - \mathcal{P}_\lambda(h) = \lambda \mathbb{E}[f_h(X^* + 1) - f_h(X + 1)].$$

A first order correction term for the Poisson approximation has been proposed in $[8]$ by using the Poisson zero bias transformation.

Recall the difference operator $\Delta$ defined as $\Delta f(x) = f(x + 1) - f(x)$. For any $x \in \mathbb{N}_* := \mathbb{N} \setminus \{0\}$ and any $n \in \mathbb{N}$, one has $\Delta \binom{x}{n} = \binom{x}{n-1}$. If $f$ and $g$ are two functions on $\mathbb{N}$, then

$$\Delta(f(x)g(x)) = f(x + 1)\Delta g(x) + g(x)\Delta f(x).$$

We have the Newton’s expansion ([$4$, Thm 5.1]), which can be viewed as an analogue of the Taylor’s expansion in the discrete case. For all $x, y \in \mathbb{N}$ and $N \in \mathbb{N}$,

$$f(x + y) = \sum_{j=0}^{N} \binom{y}{j} \Delta^j f(x) + \sum_{0 \leq j_1 < \cdots < j_{N+1} < y} \Delta^{N+1} f(x + j_1).$$

Let us introduce the following quantity, where we use the same notation as in $[12]$, but its meaning is changed. For any $N$-r.v. $Y$ and any $k \in \mathbb{N}$ such that $\mathbb{E}(|Y|^k) < +\infty$, denote by

$$m_Y^{(k)} := \mathbb{E} \left[ \binom{Y}{k} \right] = k! [Y]_k$$

where $[Y]_k$ is the $k^{th}$ factorial moment of $Y$. Let $X$ and $Y$ be two independent $N$-r.v.s and $f : \mathbb{N} \rightarrow \mathbb{R}$ such that $\Delta^k f(X)$ and $\Delta^k f(X + Y)$ are both integrable, then

$$\mathbb{E}[f(X + Y)] = \sum_{k=0}^{N} m_Y^{(k)} \mathbb{E}[\Delta^k f(X)] + \delta_N(f, X, Y).$$
where
\[
\delta_N(f, X, Y) = \mathbb{E} \left[ \sum_{0 \leq j_1 < \cdots < j_{N+1} < Y} \Delta^{N+1} f(X + j_1) \right].
\]

We introduce the discrete reverse Taylor formula. Once again, the following result is very similar with [12, Pro1.1], however, with different significations of notation.

**Proposition 1.1 (discrete reverse Taylor formula)** With the above notation, we have
\[
\mathbb{E}[f(X)] = \sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_+, |J| \leq N} m_{X^*}^{(J)} \mathbb{E}[\Delta^{|J|} f(X + Y)] + \varepsilon_N(f, X, Y)
\]

where
\[
\varepsilon_N(f, X, Y) := -\sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_+, |J| \leq N} m_{X^*}^{(J)} \delta_{N-|J|}(\Delta^{|J|} f, X, Y),
\]

for any integer \( d \geq 1 \) and any \( J = (j_1, \cdots, j_d) \) with expectations \( \lambda_i \), which are “sufficiently good” in a sense we shall precise later. Let \( W = X_1 + \cdots + X_n \) and denote \( \lambda_W := \mathbb{E}[W] = \lambda_1 + \cdots + \lambda_n \). Let \( W^{(i)} = W - X_i \) and \( X_i^* \) be an \( \mathbb{N} \)-r.v., independent of \( W^{(i)} \) and which has the Poisson \( X_i \)-zero biased distribution. Finally, let \( I \) be a random index valued in \( \{1, \cdots, n\} \) which is independent of \( (X_1, \cdots, X_n, X_1^*, \cdots, X_n^*) \) and such that \( \mathbb{P}(I = i) = \lambda_i / \lambda_W \) for any \( i \). Then, similar as in [11], the random variable \( W^* := W^{(i)} + X_i^* \) follows the Poisson \( W \)-zero biased distribution.

We give below the asymptotic expansion formula in the Poisson case.

**Theorem 1.2** Let \( N \in \mathbb{N} \) and \( p \geq 0 \). Let \( h : \mathbb{N} \to \mathbb{R} \) be a function which is of \( O(x^p) \) at infinity and \( X_i \) \( i = 1, \cdots, n \) be a family of independent \( \mathbb{N} \)-r.v.s having up to \( (N + p + 1)^{th} \) order moments. Let \( W = X_1 + \cdots + X_n \) and \( \lambda_W = \mathbb{E}[W] \). Then \( \mathbb{E}[h(W)] \) can be written as the sum of two terms \( C_N(h) \) and \( e_N(h) \) such that \( C_0(h) = \mathcal{P}_{\lambda_W}(h) \) and \( e_0(h) = \mathbb{E}[h(W)] - \mathcal{P}_{\lambda_W}(h) \), and recursively for any \( N \geq 1 \),
\[
C_N(h) = C_0(h) + \sum_{i=1}^{n} \lambda_i \sum_{d \geq 1} (-1)^{d-1} \sum_{J \in \mathbb{N}_+, |J| \leq N} m_{X_i^*}^{(J)} (m_{X_i}^{(J)} - m_{X_i^*}^{(J)}) C_{N-|J|}(\Delta^{|J|} f_h(x+1)),
\]

\[
e_N(h) = \sum_{i=1}^{n} \lambda_i \left[ \sum_{d \geq 1} (-1)^{d-1} \sum_{J \in \mathbb{N}_+, |J| \leq N} m_{X_i^*}^{(J)} (m_{X_i}^{(J)} - m_{X_i^*}^{(J)}) e_{N-|J|}(\Delta^{|J|} f_h(x+1)) \right]
\]
\[
+ \sum_{k=0}^{N} m_{X_i^*}^{(k)} e_{N-k}(\Delta^k f_h(x+1), W^{(i)} , X_i) + \delta_N(f_h(x+1), W^{(i)} , X_i^*),
\]

where for any integer \( d \geq 1 \) and any \( J \in \mathbb{N}_+, J^\dagger \in \mathbb{N}_+ \) denotes the last coordinate of \( J \), and \( J^\circ \) denotes the element in \( \mathbb{N}_+^{d-1} \) obtained from \( J \) by omitting the last coordinate.
Remark 1.3 In view of the similarity between the above theorem and [12, Thm.1.2], which has also been shown by the two papers [2, 3] of Barbour, the following question arises naturally: can we generalize the result to any infinitely divisible distribution?

2 Several preliminary results

In this section, we are interested in some properties concerning the function $h$ and the associated function $f_h$. Compared to the normal case, we no longer need differentiability conditions on $h$ in Theorem 1.2 and shall concentrate on its increasing speed at infinity. This makes the study much simpler.

We begin by considering the modified Stein’s equation on $\mathbb{N}_*$:

\[ x\tilde{f}(x) - \lambda \tilde{f}(x+1) = h(x), \quad x \in \mathbb{N}_*. \]

The above equation may have many solutions, one of which is given by

\[ \tilde{f}_h(x) := \frac{(x-1)!}{\lambda x} \sum_{i=x}^{\infty} \frac{\lambda^i}{i!} h(i). \]

A general solution of (12) can be written as $\tilde{f}_h(x) + C(x-1)!/\lambda x$, where $C$ is an arbitrary constant. However, when $h$ is of polynomial increasing speed at infinity, $\tilde{f}_h$ is the only solution of (12) which has polynomial increasing speed at infinity.

In order that the function $\tilde{f}_h$ is well defined, we need some condition on $h$. Denote by $\mathcal{E}_\lambda$ the space of functions $h$ on $\mathbb{N}_*$ such that, for any polynomial $P$, we have

\[ \sum_{i \geq 1} \frac{\lambda^i}{i!} |h(i)P(i)| < +\infty. \]

Clearly $\mathcal{E}_\lambda$ is a linear space. We list below some properties of $\mathcal{E}_\lambda$.

Proposition 2.1 The following assertions hold:

1) for any $Q \in \mathbb{R}[x, x^{-1}]$ and any $h \in \mathcal{E}_\lambda$ where $\mathbb{R}[x, x^{-1}]$ denotes the set of Laurent polynomials on $\mathbb{R}$, we have $Qh \in \mathcal{E}_\lambda$;

2) for any $h \in \mathcal{E}_\lambda$, $\Delta h \in \mathcal{E}_\lambda$ and $\tilde{f}_h \in \mathcal{E}_\lambda$;

Proof. 1) is obvious by definition.

2) Let $h_1$ be the function defined as $h_1(x) := h(x+1)$. If $P$ is a polynomial, then

\[ \sum_{i \geq 1} \frac{\lambda^i}{i!} |P(i)h_1(i)| = \sum_{j \geq 2} \frac{\lambda^{j-1}}{(j-1)!} |P(j-1)h(j)| = \lambda^{-1} \sum_{j \geq 2} \frac{\lambda^j}{j!} |jP(j-1)h(j)| < +\infty \]
since $h \in \mathcal{E}_\lambda$. Therefore, $\Delta h \in \mathcal{E}_\lambda$. We next prove the second assertion. For any arbitrary polynomial $P$, there exists another polynomial $Q$ such that, for any integer $i \geq 1$, $Q(i) \geq \sum_{j=1}^{\left\lfloor \frac{P(j)}{i} \right\rfloor} h(j)$. Therefore

$$
\sum_{a \geq 1} \frac{\lambda^a}{a!} \left| P(a) \frac{(a-1)!}{\lambda^a} \sum_{i=a}^\infty \frac{\lambda^i}{i!} h(i) \right| \leq \sum_{a \geq 1} \frac{|P(a)|}{a} \sum_{i=a}^\infty \frac{\lambda^i}{i!} |h(i)|
$$

$$
= \sum_{i \geq 1} \left( \sum_{a=1}^i \frac{|P(a)|}{a} \frac{\lambda^i}{i!} |h(i)| \right) \leq \sum_{i \geq 1} Q(i) \frac{\lambda^i}{i!} |h(i)| < +\infty,
$$

which implies that $\tilde{f}_h \in \mathcal{E}_\lambda$. \hfill \square

For any function $h \in \mathcal{E}_\lambda$, we define $\tau(h) : \mathbb{N} \to \mathbb{R}$ such that

$$
\tau(h)(x) = h(x+1)/(x+1).
$$

Note that for any integer $k \geq 1$, one has $\tau^k(h)(x) = x!h(x+k)/(x+k)!$. The proof of Proposition 2.1 shows that $\tau$ is actually an endomorphism of $\mathcal{E}_\lambda$.

**Lemma 2.2** Let $h \in \mathcal{E}_\lambda$. Then

$$
\tilde{f}_{\tau(h)}(x) = \tilde{f}_h(x+1)/x.
$$

**Proof.** Let $u(x) = \tilde{f}_h(x+1)/x$. Dividing both sides of (12) by $x$ and then replacing $x$ by $x+1$, we obtain $\tilde{f}_h(x+1) - \lambda \tilde{f}_h(x+2)/(x+1) = \tau(h)(x)$, or equivalently,

$$
xu(x) - \lambda u(x+1) = \tau(h)(x).
$$

(14)

Since $\tilde{f}_{\tau(h)}$ is the only solution of (14) in $\mathcal{E}_\lambda$, the lemma is proved. \hfill \square

**Corollary 2.3** Let $p \in \mathbb{R}$. If $h(x) = O(x^p)$, then $\tilde{f}_h(x) = O(x^{p-1})$.

**Proof.** First of all,

$$
0 \leq \frac{x!}{\lambda^x} \sum_{i \geq x} \frac{\lambda^i}{i!} = \sum_{i \geq x} \frac{\lambda^{i-x}}{i!} \frac{x!}{x!} \leq \sum_{i \geq x} \frac{\lambda^{i-x}}{(i-x)!} = e^{-\lambda}.
$$

Therefore, when $p \leq 0$, one has

$$
x \tilde{f}_h(x) = \frac{x!}{\lambda^x} \sum_{i \geq x} \frac{\lambda^i}{i!} h(i) = O(x^p)
$$

since $i^p \leq x^p$ if $x \leq i$. Hence $\tilde{f}_h(x) = O(x^{p-1})$. The general case follows by induction on $p$ by using (14). \hfill \square
We now introduce the function space: for any $p \in \mathbb{R}$, denote by $H^p$ the space of all functions $h : \mathbb{N} \to \mathbb{R}$ such that $h(x) = O(x^p)$ when $x \to \infty$. In the following are some simple properties of $H^p$, their proofs are direct.

**Proposition 2.4**

1) For any $p \geq 0$ and any $h \in H^p$, the restriction of $h$ on $\mathbb{N}^*$ lies in $\bigcap_{\lambda > 0} E_\lambda$.

2) If $h \in H^p$, then also are $h(x + 1)$ and $\Delta h$.

3) If $h \in H^p$ and $g \in H^q$, then $gh \in H^{p+q}$.

The following proposition is essential for applying the recursive estimation procedure.

**Proposition 2.5**

Let $p \geq 0$. If $h \in H^p$, then $f_h \in H^p$.

**Proof.** Note that $f_h$ coincides with $\tilde{f}_h$ on $\mathbb{N}^*$ where $\tilde{h} = h - \mathcal{P}_\lambda(h)$. Since $h \in H^p$, also is $\tilde{h}$. Then Corollary 2.3 implies $f_h(x) = O(x^{p-1}) = O(x^p)$. \[\square\]

## 3 Proof of the main result

In this section, we give the proof of Proposition [1.1] and of Theorem [1.2], which are essentially the same with the ones of [12, Prop1.1, Thm1.2] in a discrete setting.

**Proof of Proposition 1.1**

We replace $E[\Delta^{|J|}f(X + Y)]$ on the right side of (8) by

$$
\sum_{k=0}^{N-|J|} m_Y^{(k)} E[\Delta^{|J|+k}f(X)] + \delta_{N-|J|}(\Delta^{|J|}f, X, Y)
$$

and observe that the sum of terms containing $\delta$ vanishes with $\varepsilon_N(f, X, Y)$. Hence the right side of (8) equals

$$
\sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_d^p, |J| \leq N} \sum_{k=0}^{N-|J|} m_Y^{(J)} E[\Delta^{|J|+k}f(X)]
$$

If we split the terms for $k = 0$ and for $1 \leq k \leq N - |J|$ respectively, the above formula can be written as

$$(15) \quad \sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_d^p, |J| \leq N} m_Y^{(J)} E[\Delta^{|J|}f(X)] + \sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_d^p, |J| \leq N} m_Y^{(J)} \sum_{k=1}^{N-|J|} m_Y^{(J)} E[\Delta^{|J|+k}f(X)].$$

We make the index changes $J' = (J, k)$ and $u = d + 1$ in the second part of (15) and find that it is nothing but

$$
\sum_{u \geq 1} (-1)^{u-1} \sum_{J' \in \mathbb{N}_u^p, |J'| \leq N} m_Y^{(J')} E[\Delta^{|J'|}f(X)].
$$
By taking the sum, it only remains the term of index \( d = 0 \) in the first part of (13), which is equal to \( \mathbb{E}[f(X)] \). So the lemma is proved. 

\[ \square \]

**Proof of Theorem 1.2** We prove the theorem by induction on \( N \). The case where \( N = 0 \) is trivial. Assume that the assertion holds for \( 0, \cdots, N - 1 \).

Since \( h \in \mathcal{H}_p \), by Lemma 2.5 and Proposition 2.4, for any \( k \in \{1, \cdots, N\} \), \( \Delta^k f_h(x + 1) \in \mathcal{H}_{p-1} \subset \mathcal{H}_p \). Therefore \( C_{N-k}(\Delta^k f_h(x + 1)) \) and \( e_{N-k}(\Delta^k f_h(x + 1)) \) are well defined and

\[
\mathbb{E}[\Delta^k f_h(W + 1)] = C_{N-k}(\Delta^k f_h(x + 1)) + e_{N-k}(\Delta^k f_h(x + 1)).
\]

We now prove the equality \( \mathbb{E}[h(W)] = C_N(h) + e_N(h) \). Recall that for any \( i \in \{1, \cdots, n\} \), \( X_i^* \) follows the Poisson \( X_i \)-zero biased distribution and is independent of \( W^{(i)} = W - X_i \).

\( I^* \) is an independent random index such that \( \mathbb{P}(I = i) = \lambda_i/\lambda \), and \( W^* = W^{(i)} + X_i^* \). So \( \mathbb{E}[h(W)] - C_0(h) \) is equal to

\[
\lambda W \mathbb{E}[f_h(W^* + 1) - f_h(W + 1)] = \sum_{i=1}^n \lambda_i \left( \mathbb{E}[f_h(W^{(i)} + X_i^*)] - \mathbb{E}[f_h(W + 1)] \right),
\]

where, by using (3),

\[
\mathbb{E}[f_h(W^{(i)} + X_i^*)] = \sum_{k=0}^N m_X^{(k)} \mathbb{E}[\Delta^k f_h(W^{(i)} + 1)] + \delta_N(f_h(x + 1), W^{(i)}, X_i^*).
\]

By replacing \( \mathbb{E}[\Delta^k f_h(W^{(i)} + 1)] \) in the above formula by its \((N-k)\)th order reverse Taylor expansion, we obtain that \( \mathbb{E}[f_h(W^{(i)} + X_i^*)] \) equals

\[
\sum_{k=0}^N m_X^{(k)} \left[ \sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_d, |J| \leq N-k} m_X^{(J)} \mathbb{E}[\Delta^{|J|+k} f_h(W + 1)] + \delta_N(f_h(x + 1), W^{(i)}, X_i) \right] + \delta_N(f_h(x + 1), W^{(i)}, X_i^*). \]

Note that the term of indices \( k = d = 0 \) in the sum is \( \mathbb{E}[f_h(W + 1)] \). Therefore, \( \mathbb{E}[f_h(W^{(i)} + X_i^*)] - \mathbb{E}[f_h(W + 1)] \) is the sum of the following three terms

\[
\sum_{k=1}^N \sum_{d \geq 0} (-1)^d \sum_{J \in \mathbb{N}_d, |J| \leq N-k} m_X^{(J)} \mathbb{E}[\Delta^{|J|+k} f_h(W + 1)],
\]

\[
\sum_{d=1}^N (-1)^d \sum_{J \in \mathbb{N}_d, |J| \leq N} m_X^{(J)} \mathbb{E}[\Delta^{|J|} f_h(W + 1)],
\]

\[
\sum_{k=0}^N m_x^{(k)} \mathbb{E}[\Delta^k f_h(x + 1), W^{(i)}, X_i] + \delta_N(f_h(x + 1), W^{(i)}, X_i^*). \]

By interchanging summations and then making the index changes \( K = (J, k) \) and \( u = d + 1 \), we obtain

\[
(13) = \sum_{u \geq 1} (-1)^{u-1} \sum_{K \in \mathbb{N}_u} m_X^{(K)} m_X^{(K)} \mathbb{E}[\Delta^K f_h(W + 1)].
\]
As the equality $m_{X_i}^{(J)} = m_{X_i}^{(J')}m_{X_i}^{(J)}$ holds for any $J$, (12) + (17) simplifies as

$$\sum_{d \geq 1} (-1)^{d-1} \sum_{J \in \mathbb{N}^d, |J| \leq N} m_{X_i}^{(J')} \left( m_{X_i}^{(J)} - m_{X_i}^{(J)} \right) \mathbb{E}[\Delta^{[J]}f_h(W + 1)].$$

By the hypothesis of induction, we have

$$\mathbb{E}[\Delta^{[J]}f_h(W + 1)] = C_{N-|J|}(\Delta^{[J]}f_h(x + 1)) + e_{N-|J|}(\Delta^{[J]}f_h(x + 1)),$$

so the equality $\mathbb{E}[h(W)] = C_N(h) + e_N(h)$ holds with $C_N(h)$ and $e_N(h)$ being defined in (10) and (11).

\[\square\]

4 Error estimations

In this section, we concentrate on the remainder $e_N(h)$ in the asymptotic expansion. The following quantity will be useful. Let $p \geq 0$. For $h \in \mathcal{H}_p$ and $N \in \mathbb{N}$, we define

$$\|h\|_{N,p} := \sup_{x \in \mathbb{N}^*} \left| \frac{\Delta^{N+1}h(x)}{x^p} \right|,$$

which is finite by Proposition (2.4 2).

**Lemma 4.1** Let $N \in \mathbb{N}$, $k \in \{0, \cdots, N\}$ and $p \geq 0$. Let $X$ be an $\mathbb{N}$-r.v. with $p$th order moment, $Y$ be an $\mathbb{N}$-r.v. independent of $X$ and having $(N - k + 1 + p)$th order moment. Then, for any $f \in \mathcal{H}_p$, the following inequalities hold:

$$|\delta_{N-k}(\Delta^k f(x + 1), X, Y)| \leq \max(2^{p-1}, 1) f \|h\|_{N,p} \left( \mathbb{E}[X^p]m_Y^{(N-k+1)} + m_Y^{(N-k+1), p} \right),$$

where

$$m_Y^{(N-k+1), p} := \mathbb{E}\left[ \left( \frac{Y}{N-k+1} \right)^p \right].$$

The discrete reverse Taylor remainder satisfies

$$|\delta_{N-k}(\Delta^k f(x + 1), X, Y)| \leq \max(2^{p-1}, 1) f \|h\|_{N,p} \sum_{d \geq 0} \sum_{J \in \mathbb{N}^d, |J| \leq N-k} m_Y^{(J)} \left( \mathbb{E}[X^p]m_Y^{(N-k-|J|+1)} + m_Y^{(N-k-|J|+1), p} \right).$$

**Proof.** By definition (7) and (13),

$$|\delta_{N-k}(\Delta^k f(x + 1), X, Y)| \leq \mathbb{E}\left[ \sum_{0 \leq j_1 < \cdots < j_{N-k+1} < Y} |\Delta^{N+1}f(X + 1 + j_1)| \right]$$

$$\leq \|f\|_{N,p} \mathbb{E}\left[ \sum_{0 \leq j_1 < \cdots < j_{N-k+1} < Y} (X + j_1 + 1)^p \right] \leq \|f\|_{N,p} \mathbb{E}\left[ \left( \frac{Y}{N-k+1} \right)^p (X + Y)^p \right]$$

$$\leq \max(2^{p-1}, 1) f \|h\|_{N,p} \left( \mathbb{E}[X^p]m_Y^{(N-k+1)} + m_Y^{(N-k+1), p} \right),$$
where we have used in the last inequality the estimations \((X + Y)^p \leq 2^{p-1}(X^p + Y^p)\) if \(p > 1\) and \((X + Y)^p \leq X^p + Y^p\) if \(p \leq 1\). Thus (21) is proved. The inequality (22) follows from (11) and (21). \(\square\)

**Proposition 4.2** Let \(N \in \mathbb{N}, p \geq 0\) and \(h \in H_p\). Let \(X_i (i = 1, \ldots, n)\) be a family of independent \(N\)-r.v.s with mean \(\lambda_i > 0\) and up to \((N + p + 1)\)th order moments; \(W = X_1 + \ldots + X_n\). Let \(X_i^*\) be an \(N\)-r.v. having Poisson \(X_i\)-zero biased distribution and independent of \(W(i) := W - X_i\). Then the following estimations hold.

1) When \(N = 0\),

\[
|e_0(h)| \leq \max(2^{p-1}, 1)\|f_h\|_{0,p} \sum_{i=1}^{n} \left( E[(W(i))^p] (E[X_i^2] + \lambda_i^2 - \lambda_i) + \lambda_i (E[(X_i^*)^p+1]) + E[(X_i)^p+1]) \right).
\]

2) When \(N \geq 1\), one has the recursive estimation:

\[
|e_N(h)| \leq \sum_{i=1}^{n} \lambda_i \left[ \sum_{d \geq 1} \sum_{|J| \leq N} m_{X_i}^{(N)} (m_{X_i}^{(J)} + m_{X_i}^{(J^*)}) |e_{N-|J|}(\Delta|J|f_h(x + 1))| \\
+ \max(2^{p-1}, 1)\|f_h\|_{N,p} \sum_{k=0}^{N} \sum_{|J| \leq N-k} m_{X_i}^{(J)} (E[(W(i))^p] m_{X_i}^{(N-k-|J|+1)} + m_{X_i}^{(N-k-|J|+1)p}) \\
+ \max(2^{p-1}, 1)\|f_h\|_{N,p} (E[(W(i))^p] m_{X_i}^{(N+1)} + m_{X_i}^{(N+1,p)}) \right],
\]

**Proof.** We begin by the case when \(N = 0\). By (11),

\[
e_0(h) = \sum_{i=1}^{n} \lambda_i \left( E[f_h(W(i) + X_i^* + 1)] - E[f_h(W(i) + X_i + 1)] \right) \\
= \sum_{i=1}^{n} \lambda_i (\delta_0(f_h(x + 1), W(i), X_i^*) + \varepsilon_0(f_h(x + 1), W(i), X_i)) \\
\leq \max(2^{p-1}, 1)\|f_h\|_{0,p} \sum_{i=1}^{n} \lambda_i \left( E[(W(i))^p] (m_{X_i}^{(1)} + m_{X_i}^{(J)}) + (m_{X_i}^{(1,p)} + m_{X_i}^{(J,p)}) \right)
\]

where the last inequality is by estimations (23) and (22), so (22) follows. Combining in addition the recursive formula (11), we obtain the inequality (23). \(\square\)
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