The generic model of General Relativity

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Abstract. We develop a generic spacetime model in General Relativity from which all existing model results are produced under specific assumptions, depending on the case. We classify each type of possible assumption, especially the role of observers and that of symmetries, and discuss their role in the development of a model. We apply the results in a step by step approach to the case of a Bianchi I spacetime and a string fluid.

1. Introduction

General Relativity is the first theory of Physics which uses geometry to a great extend. Newtonian Physics is also a geometric theory but the difference is that the 3-space geometry which is used has direct sensory representation therefore ‘couples’ much easier and direct with the description of physical phenomena. In General Relativity the space used (spacetime) and the geometry employed (Riemannian Geometry) do not allow for a direct comparison of physical phenomena with the mathematical description. Most frequently this results in a confusion as to where geometry starts and stops, how much Physics is used and done, what is the physical interpretation of the geometrically derived results and how it will be done etc. Even most important it does not seem to exist a concise exposition of the structure of General Relativity and a clear description of the various hypotheses made both at geometric and physical level.

In this article we shall develop systematically a generic model for GR. These model will have all necessary parameters which can be used to classify all existing spacetime models. Furthermore this approach can be used to employ physical and geometric assumptions consistently so that one will not conflict with the other.

2. Universal model of GR

Every model-theory of Physics has three constituents:

a. Fundamental variables and equations between these variables
b. Constraints expressed by relations among these variables
c. Interpretation of the mathematical results and assumptions in terms of (directly or indirectly) observational phenomena.

An additional assumption/ constraint on the theory is the economy one, that is, we demand that both the number of fundamental variables and the equations which relate them shall be of a minimum number and will satisfy an aesthetic ‘simplicity principle’, that is, they will have the simplest possible structure. This requirement is vague but it summarizes the feeling one attains with the years for GR.
The basic objects which are required to build up a theory of Physics are:
1. A background mathematical space into which the ‘motion’ or ‘evolution’ will be described.
2. Background symmetries which will simplify the fundamental geometric objects of the theory.
   b. A class of observers who will observe and describe motion.
   c. Matter which will make up the environment of the observers and modulate motion and evolution.
   d. Other physical assumptions.
These objects are related by the Relativity Principle of the Theory, which states that:

_The physical quantities will be modeled mathematically with the geometric objects defined by the transformation group of the theory._

In the following we comment briefly on each of the above ‘components’ of a physical theory.

3. The background space
Physics does not build its models in the ‘real’ space i.e. the space we live in. In Newtonian Physics we have a direct comparison of the two concepts of space due to our direct sensory conception of the world. In more recent models of Physics we have an ‘indirect sensory’ conception of the world by means of our measuring instruments. The first theory which did that has been Special Relativity and this is the reason people were unable to understand and anxious to disprove it by means of the paradoxes. It violated their ‘common sense’.

The trouble is the use of the word ‘space’ to describe both a mathematical entity (software) and a physical entity (hardware). The physical entity is what it is. The mathematical entity is a set with certain mathematical structures. Our intention here is not to enter into details about these delicate questions, which in any case are not first priority questions to a general physicist.

The mathematical space used by the theories of Physics is a manifold. A manifold is a structure on a set which locally is diffeomorphic to an open set in a flat space of some dimension (called the dimension of the manifold) therefore it can be covered by a set of coordinate patches form the flat space so that whenever two patches coincide there is a differentiable map which relates the patches either way. In General Relativity the set of points are the events which correspond to various facts in the real world, the flat space is the space $\mathbb{R}^4$ and the patches which cover the manifold are called coordinate systems. The differentiable maps which relate two coordinate systems are called coordinate transformations. This manifold is called space time$^1$. The coordinate transformations form a group under the action of composition of maps. This group is an infinite dimensional Lie group called the Manifold Mapping Group (MMG) ($\mathfrak{g}$). The mathematical quantities which if given in one coordinate patch can be computed in any other overlapping this patch by means of the appropriate coordinate transformation we call geometric objects. We say that the geometric objects form a representation of the MMG. The geometric objects which are related in a linear and homogeneous manner we call general tensors. This is as far as one can go with the assumption of structure of the mathematical space.

In such a space there is not much one can do because there is no structure to be associated with a physical entity. The main interest of General Relativity is to develop a theory for the gravitational field. Each structure in a space is realized by a geometric object (not necessarily a tensor e.g. the connection). Therefore in General Relativity one introduces a new structure in the background space, the metric which is a $(0, 2)$ tensor field and it is assumed that it geometrizes the gravitational field. Furthermore because at each point we want the theory to reduce to Special Relativity, it is postulated that the signature of the metric will be that of the

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$^1$ In addition to the local structure spacetime has a global structure described by its topology. However these topological properties do not interest us here.
Lorentz metric, that is $-2$. It is well known that having a metric on a manifold one can define a covariant derivative which is symmetric (torsion vanishes) and also the covariant derivative of the metric vanishes (zero metricity field). This covariant derivative is called the Riemannian derivative and it is defined by means of the geometric object (not a tensor!) which is the well known Christoffel symbols. The manifold structure together endowed with the fields of the metric and the Riemannian connection we call spacetime. Spacetime is the background space of General Relativity.

In spacetime all tensor fields must satisfy certain geometric identities. These identities together with other constraints which will follow will make up the framework within which General Relativity operates. In the following we classify these relations - constraints in various levels according to the wideness of applicability of their action.

**Level G1**

This level contains the geometric identities which result directly from the Riemannian structure of spacetime. These identities must be satisfied by whatever new fields will be introduced at later stages and are the following:

a. Ricci identity. For a tensor of type $(r, s)$ this identity is:

$$T_{j_1\ldots j_r \cdots l_1 \ldots l_s; h_1 \ldots h_s} - T_{j_1\ldots j_r \cdots l_1 \ldots l_s; h_1 \ldots h_s} = \sum_{a=1}^{r} R_{m_j a} h_k T_{j_1 \ldots j_{a-1} m; j_{a+1} \ldots j_r \cdots l_1 \ldots l_s} - \sum_{\beta=1}^{s} R_{m_j a} h_k T_{j_1\ldots j_r \cdots l_1 \ldots l_{a-1} m; l_{a+1} \ldots l_s}. \quad (1)$$

In particular for a vector field $X^{a}$ the Ricci identity reads:

$$X_{a;bc} - X_{a;cb} = R_{dabc} X^{d}. \quad (2)$$

b. Bianchi identities.

The Bianchi identities in Riemannian geometry are the following:

$$R_{abcd} [a;b] = 0,$$
$$C_{abcd} \ldots d = R^e[a;b] - \frac{1}{6} g^{e[a} R^{b]}. \quad (4)$$

In a four dimensional Riemannian space these two identities are equivalent \(^2\), therefore in spacetime they reduce to one identity. These identities imply the contracted Bianchi identity, which reads:

$$R^{ab}_{\ldots;a} - \frac{1}{2} R^{a} = 0 \leftrightarrow G^{ab}_{\ldots;b} = 0, \quad (5)$$

where:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \quad (6)$$

is the Einstein tensor.

**Level G2**

This level contains the symmetries (collineations) of the background spacetime. Symmetries are perhaps the most important aspect of physical theories. A symmetry is an additional constraint which we introduce mainly on some aesthetic or mathematical ground for simplifying the field equations or the geometry. Symmetries have various levels of effects.

a. Primary effects.

They are expressions of the form:

$$L_{\xi} T = A, \quad (7)$$

\(^2\) This equivalence holds for $n = 4$ only. See Kundt and Trumper Abh. Akad.Wiss. and Lit. Mains., Mat. Nat.Kl.,No 12 (1962).
where $T$ is a geometric object in spacetime and $A$ is a tensor field and $L_\xi$ stands for the Lie derivative wrt the vector $\xi^a$. In spacetime we can define many geometric objects by considering combinations of the partial derivatives of the metric $g_{ab}$ which do not vanish. The main such object is the connection coefficients (Christoffel symbols), the curvature tensor and its irreducible parts (Weyl tensor, Ricci tensor, curvature scalar). In general all geometric objects which are defined in terms of the metric and its derivatives we shall call metric geometric objects. A collineation is a symmetry of a metric geometric object, that is, a relation of the form (7) where $T$ is a metric geometric object. The following are examples of collineations:

\[
\begin{align*}
L_\xi g_{ab} &= 0, & \text{Killing Vector} \\
L_\xi g_{ab} &= 2c g_{ab}, & c = \text{const.}, & \text{Homothetic Killing Vector} \\
L_\xi g_{ab} &= 2\psi g_{ab}, & \psi \neq \text{const.}, & \text{Conformal Killing Vector} \\
L_\xi T^i_{jk} &= 0, & \text{Affine collineation Vector} \\
L_\xi R_{ab} &= 0, & \text{Ricci collineation} \\
L_\xi R^i_{jkl} &= 0, & \text{Curvature collineation} \\
g^{ab} L_\xi R_{ab} &= 0, & \text{Contracted Ricci collineation.} \\
L_\xi W^i_{jkl} &= 0, & \text{Contracted Ricci collineation.}
\end{align*}
\]

b. Secondary effects.

These are identities satisfied by the metric geometric objects and are due to primary collineations. For instance if $\xi^a$ is a conformal Killing vector (CKV) the a secondary effect is(see [22] Eqn.(5.1)):

\[
\left( R^a_{\xi^b} \right)_{;a} = -3\Box \psi,
\]

where $\Box \psi = g^{ab}\psi_{;ab}$.

c. Mathematical constraints

These are mathematical requirements introduced ‘by hand’ after the previous two sublevels of simplifying assumptions have been exhausted. The purpose of their introduction is to simplify further the equations obtained by the previous simplifying assumptions. Their form is indicated by the form of the equations they have to simplify. For example in the case of a CKV such a requirement is $\psi_{;ab} = 0$ because this condition removes $\psi_{;ab}$ from the field equations. A CKV which satisfies this condition is called a special CKV.

It is to be noted that up to now we discuss the background space only, therefore the results apply to all physical fields introduced and to be more specific to both the kinematics and the dynamics of the theory we shall develop on this space.

3.1. Some useful material about collineations

Each collineation has a different constraint ‘power’. The strongest is the $L_\xi g_{ab} = 0$ because if $\xi^a$ satisfies this (i.e. it is a Killing vector) then satisfies all the rest. A collineation is called proper if it cannot be reduced to a ‘simpler’ one. For example conformal Killing vectors (CKV) contain the homothetic Killing vectors (HKV) when the conformal function $\psi = \text{constant}$. A proper CKV is one for which $\psi$ is not constant.

The vectors which satisfy $L_\xi g_{ab} = 0$ are called Killing vectors and the equation $L_\xi g_{ab} = 0$ Killing equation. In an $n$-dimensional space with a non-degenerate metric there exist at most $\frac{n(n+1)}{2}$ KVs. If this is the case then the space is called a space of constant curvature\(^3\). The KVs serve all standard geometric symmetry concepts e.g. spherical symmetry, cylindrical symmetry etc.

The KVs form a finite dimensional Lie algebra. This algebra can be used to classify the metrics in various classes in the sense that one is able to write the metric in a form that takes

\(^3\) It should be called metric of constant curvature.
into account the symmetry and it is written in terms of a small number of parameters. An well
known example is the Freedman - Robertson - Walker (FRW) spacetime which is used in the
standard cosmology. The metric of this spacetime has the form:

\[ ds^2_{\text{FRW}} = -dt^2 + R(t)^2(dx^2 + dy^2 + dz^2), \tag{9} \]

where \((t, x, y, z)\) are Cartesian coordinates and \(R(t)\) is the only unknown function of \((t)\) time \(t\) and can be defined by means of the following symmetry assumptions \[2]:

1. Spacetime admits a timelike gradient Conformal Killing Vector (CKV) such that there
exists another reduced metric for which the CKV is a KV (coordinate) time \(t\) where \((t,x,y,z)\) are Cartesian coordinates and \(R(t)\) is the only unknown function of \((t)\) time \(t\) and can be defined by means of the following symmetry assumptions \[2]:

2. The 3-D hypersurfaces orthogonal to the CKV are spaces of constant curvature.

The collineations other than Killing vectors fix the metric to a lesser degree, however they
do act as constraints and can be used in any case as such.

3.2. The generic collineation

One can prove that the Lie derivative of every metric geometric object can be expressed in terms
of the quantity \(L_\xi g_{ab}\) and its derivatives. For example we have the relations:

\[
L_\xi \Gamma^c_{ab} = \frac{1}{2}g^{cd}\left[(L_\xi g_{ad})_{,b} + (L_\xi g_{bd})_{,a} - (L_\xi g_{ab})_{,d}\right], \tag{10}
\]

\[
L_\xi R^a_{bcd} = (L_\xi \Gamma^a_{bd})_{,c} - (L_\xi \Gamma^a_{bc})_{,d}, \tag{11}
\]

\[
L_\xi R_{ab} = (L_X \Gamma^c_{ab})_{,c} - (L_X \Gamma^c_{ca})_{,b}. \tag{12}
\]

This result leads us to introduce the concept of generic collineation by the identity:

\[
L_\xi g_{ab} = 2\psi g_{ab} + 2H_{ab}. \tag{13}
\]

where \(H_{ab}\) is a symmetric traceless tensor. Then it is possible to express every collineation in
terms of the symmetry variables \(\psi, H_{ab}\) and their derivatives. This approach greatly unifies and
simplifies the study of the effects of each collineation and reveals its relative significance.

As an example we refer the following general result:

\[
L_\xi R_{ab} = - (n - 2) \psi_{ab} - H_{(a;b)} - 2H_{,ab} - \Box H_{ab}, \tag{14}
\]

In terms of trace and traceless parts \(L_\xi R_{ab}\) is written as follows:

\[
L_\xi R_{ab} = - (n - 2) A_{ab} + 2K_{ab} - \Box H_{ab} + \frac{1}{n}g_{ab}\left[-2(n - 1)\Box \psi + 2H_{,ab}\right], \tag{15}
\]

where the traceless tensors:

\[
A_{ab} = \psi_{,ab} - \frac{1}{n}g_{ab}\Box \psi \quad \text{and} \quad K_{ab} = H^{(a;b)}_{,d} - \frac{1}{n}g_{ab}H^{,cd}_{,cd}. \tag{16}
\]

In spacetime \((n = 4)\) the above formulae read:

\[
L_\xi R_{ab} = - 2\psi_{ab} - g_{ab}\Box \psi + 2H^{(a;b)}_{,d} - \Box H_{ab}, \tag{17}
\]

\[
L_\xi R_{ab} = - 2A_{ab} + 2K_{ab} - \Box H_{ab} + \frac{1}{4}g_{ab}\left(-6\Box \psi + 2H_{,ab}\right), \tag{18}
\]

\[
A_{ab} = \psi_{,ab} - \frac{1}{4}g_{ab}\Box \psi \quad \text{and} \quad K_{ab} = H^{(a;b)}_{,d} - \frac{1}{4}g_{ab}H^{,cd}_{,cd}. \tag{19}
\]

This is as far as one can go with the spacetime structure (i.e. the metric tensor). It is possible to
continue making Geometry in spacetime but no Physics. To do the latter one has to introduce
more tensor fields which will describe / correspond to physical quantities.
4. The observers - Kinematics

To begin making Physics in spacetime one has to introduce observers. In accordance with Special Relativity the observers are introduced by the four-velocity i.e. a unit timelike vector field or, equivalently, by a timelike congruence of differentiable curves, which are the integral curves of the vector field. The Physics of the new timelike vector field we call the kinematics in spacetime. Obviously the Kinematics of a certain spacetime is different for different observers.

The standard approach is to consider the observers as a ‘fluid’ in spacetime and express the kinematics of these observers in terms of fluid flow quantities.

One point which must be emphasized is the following. In Special Relativity the curvature tensor vanishes, the geometry is absolute (that of flat space) and does not give conditions/constraints on four-velocity. In General Relativity the situation is different and the four-velocity one considers must satisfy Ricci’s identity. Concerning Bianchi’s second identity we note that it involves only the metric (geometry) therefore it does not restrict directly the four-velocity. However it does impose restrictions on the metric and consequently in the symmetries of the metric, hence possibly on the four-velocity. The conclusion is that when we introduce a four-velocity to describe the matter in a given spacetime and compute the kinematic variables (as we shall do below) we are led to a set of identities / constraints which the kinematic variables must satisfy.

The introduction of a non-null vector field in spacetime allows one to decompose a tensor equation and an individual tensor parallel and normal to the vector field. The vector field can be either timelike or spacelike. This decomposition, called 1+3 decomposition for obvious reasons, is of primary importance to Physics because it is covariant, in the sense that each irreducible part transforms independently of the other, therefore one can break the study of Physics (Kinematics or Dynamics) of a tensor field / equation by studying the simpler Physics of its irreducible parts. In the following we develop the basic mathematics of the 1+3 decomposition

\[4\] wrt \( P^a \).

4.1. 1+3 decomposition wrt a non-null vector field

Consider a non-null \( |P^i P_i| \neq 0 \) vector field \( P^i \) with signature \( \varepsilon(P) = P^i P_i / |P^i P_i| = \pm 1 \), where the + sign applies to a spacelike 4-vector and the – sign to a timelike 4-vector in a metric space with metric \( g_{ij} \). The projection tensor \( h_{ij}(P) \) associated with \( P^i \) is defined by the equation:

\[
h_{ij}(P) = g_{ij} - \frac{\varepsilon(P)}{P^2} P_i P_j,
\]

where \( P^2 = |P^i P_i| = \varepsilon(P) P^i P_i > 0 \). It is easy to prove the properties:

\[
h_{ij}(P) P^j = 0, h^{ij}(P) h_{jk}(P) = h^i_k(P), h^i_i(P) = 3.
\]

The tensor \( h_{ij}(P) \) projects normal to \( P^a \) and gives us the possibility to decompose any tensor field in irreducible parts in a direction parallel to \( P^a \) and another normal to \( P^a \).

**Proposition 1** Any vector field \( R^i \) can be 1+3 decomposed wrt \( P^i \) as follows:

\[
R^i = \alpha P^i + \beta^j h^i_j,
\]

where

\[
\alpha = \frac{\varepsilon(P)}{P^2} R^i P_i,
\]

\[
\beta^i = R^i.
\]

\[4\] The results apply to the \( 1 + (n - 1) \) decomposition by making the necessary adjustments e.g replace 3 with \( n - 1 \) etc.
**Proof.** We have $R^i = R^j g^i_j$. Using (20) this gives:

$$R^i = R^j g^i_j,$$

$$= R^i \left( \varepsilon (P) P_j P_i + h_i^j \right),$$

$$= \varepsilon (P) (R^j P_j) P^i + h_i^j R^j.$$

\[\blacksquare\]

**Proposition 2** A second rank tensor $Y_{ij}$ is 1+3 decomposed wrt the vector $P^i$ by means of the identity:

$$Y_{ij} = \alpha P_i P_j + \varepsilon (P) \beta_k h_i^k P_j + \varepsilon (P) \gamma_k h_i^k P_j + \epsilon_{ij},$$

(22)

where

$$\alpha = \frac{1}{P^4} Y_{ij} P^i P^j, \quad \beta_i = \frac{1}{P^2} Y_{ij} P^j, \quad \gamma_i = \frac{1}{P^2} Y_{ij} P^j, \quad \epsilon_{ij} = Y_{kr} h_i^k h_j^r.$$  

**Proof.** We write $Y_{ij} = Y_{kr} g^k_i g^r_j$ and using (20) we have:

$$Y_{ij} = Y_{kr} g^k_i g^r_j,$$

$$= Y_{kr} (\left( \frac{1}{P^2} \varepsilon (P) P^k P_i + h_i^k \right) \left( \frac{1}{P^2} \varepsilon (P) P^r P_j + h_j^r \right),$$

$$= \left( \frac{1}{P^4} Y_{kr} P^k P^r \right) P_i P_j + \frac{1}{P^2} \varepsilon (P) (Y_{kr} P^k) P_i h_j^r + \frac{1}{P^2} \varepsilon (P) (Y_{kr} P^r) h_i^k P_j + Y_{kr} h_i^k h_j^r,$$

$$= \alpha P_i P_j + \varepsilon (P) \beta_k h_i^k P_j + \varepsilon (P) \gamma_k h_i^k P_j + \epsilon_{ij}.$$

\[\blacksquare\]

We note that a symmetric 2-index tensor is specified (and specifies) in terms of five different quantities: one scalar, two vector and one projected second rank tensor.

5. **1+3 decomposition wrt a timelike unit vector field**

In the following we consider $P^i$ to be a normalized timelike vector field ($\varepsilon (P) = -1$) (e.g. a 4-velocity) and we denote it $u^i$. Then equations (20), (21), (22) give the 1+3 decomposition wrt $u^i$ ($u^i u_i = -1$):

$$h_{ij} (u) = g_{ij} + u_i u_j,$$

(23)

$$R^i = \left( R^{ij} u_j \right) u^i + R^j h^i_j (u),$$

(24)

$$Y_{ij} = \left( Y_{kr} u^k u^r \right) u_i u_j - \left( Y_{kr} u^k \right) u_i h_j^r - \left( Y_{kr} u^r \right) h_i^k u_j + Y_{kr} h_i^k h_j^r.$$  

(25)

For a **symmetric** tensor of type (0,2) decomposition (25) is written as follows:

$$Y_{ab} = \left( Y^{rs} u_r u_s \right) u_a u_b - Y^{rs} u_r h_{ab} u_a - Y^{rs} u_s h_{ra} u_b,$$

$$+ \frac{1}{3} (Y^{rs} h_{rs}) h_{ab} + (h^r_a h^s_b - \frac{1}{3} h_{ab} h^{rs}) Y_{rs},$$

(26)

that is we brake further the symmetric part in a trace and a traceless part.

We consider now various applications of the 1+3 decomposition wrt the vector field $u^a$. 

7
5.1. The kinematic variables of the four-velocity

We 1+3 decompose the tensor $u_{i,j}$. From (23) we have the following identity/decomposition:

$$u_{i;j} = (u_{k;r}u_k^r) u_{i;j} - (u_{k;r}u_k^r) u_i^r h_{j}^r - (u_{k;r}u_k^r) h_i^r u_j + u_{k;r} h_i^k h_j^r.$$  (27)

But $u_{i;j}u_i^j = \frac{1}{2} (u_i u^i)_{;j} = 0$ and $u_{i;j}u^i = \dot{u}_j = h_j^r \dot{u}_r$. Therefore equation (27) becomes:

$$u_{i;j} = -\dot{u}_i u_j + u_{k;r} h_i^k h_j^r.$$  (28)

We continue by decomposing the space-like part $u_{k;r} h_i^k h_j^r$ in an antisymmetric and a symmetric part as follows:

$$u_{k;r} h_i^k h_j^r = \omega_{ij} + \theta_{ij},$$  (29)
$$\omega_{ij} = u_{k;r} h_i^k h_j^r, \quad \theta_{ij} = u_{k;r} h_i^k h_j^r.$$  (30)

The symmetric part can be decomposed covariantly further to a trace and a traceless part, that is we write:

$$\theta_{ij} = \sigma_{ij} + \frac{1}{3} \theta h_{ij},$$  (32)

where

$$\theta_i = \theta_i^j = h^j_i u_{i,j}, \quad \sigma_{ij} = \theta_{ij} - \frac{1}{3} \theta h_{ij} = \left[h_i^r h_j^r - \frac{1}{3} h^r_k h_{ij} \right] u_{r;k}.$$  (33)

The term $\omega_{ij}$ is called the vorticity tensor of $u^i$, $\sigma_{ij}$ the shear tensor of $u^i$, $\theta$ the expansion of $u^i$ and $\dot{u}^i$ the four-acceleration of the timelike vector field $u^i$. These are the kinematic variables of $u^i$, considering $u^i$ to be the 4-velocity of a relativistic fluid.

We infer by their definition that the kinematic variables satisfy the properties:

$$\omega_{ij} = -\omega_{ji}, \quad \sigma_{ij} = \sigma_{ji}, \quad \sigma_i^i = \sigma_{ij} u^j = 0,$$
$$\omega_{ij} = h_i^k h_j^r \omega_{kr}, \quad \sigma_{ij} = h_i^k h_j^r \sigma_{kr}.$$  (34)

The geometric meaning of each kinematic term is obtained from the study of the integral curves of $u^a$. We shall not comment further on that at this point. The physical meaning of each term comes from the physical interpretation of $u^a$ as a four velocity. Therefore we interpret each quantity in terms of relative motion. In this respect we say that $\theta$ is expansion (isotropic strain), $\sigma_{ab}$ shear (anisotropic strain), $\omega_{ab}$ relative rotation and $\dot{u}_a$ 4-acceleration. These quantities (i.e. $\sigma_{ab}, \omega_{ab}, \dot{u}_a, \theta$) are the fundamental physical quantities of relativistic (and Newtonian) kinematics.

We introduce the two scalars:

$$\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab}, \quad \omega^2 = \frac{1}{2} \omega_{ab} \omega^{ab}.$$  (35)

Let us see an application.
**Proposition 3** Necessary and sufficient conditions for the vector field $u^i$ to be a Killing vector is:

$$\sigma_{ij} = 0, \theta = 0, \dot{u}^i = 0,$$

**Proof.** We have:

$$L_u g_{ij} = 2u_{(i;j)} = 2\sigma_{ij} + 2\frac{\theta}{3} h_{ij} - 2\dot{u}_{(i}u_{j)}, \quad (35)$$

In order $u^i$ to be a killing vector it should satisfy the condition $L_u g_{ij} = 0$. Combining these two relations we find:

$$\sigma_{ij} = 0, \theta = 0, \dot{u}^i = 0.$$

\[\square\]

### 5.2. The propagation and the constraint equations

As we remarked above the vector field of the four-velocity decomposes both the tensor fields defining the geometric objects of the theory as well as covariant equations between them. As we have seen in spacetime we have two main geometric identities: Ricci identity and Bianchi identities. It is possible that we 1+3 decompose these identities and obtain simpler relations which involve the kinematic quantities for the four-velocity. The resulting equations are constraint equations which must be satisfied by all the kinematic quantities involved.

The 1+3 decomposition of the Ricci identity leads to two sets of equations each set containing nine equations. The first set contains the derivatives of the kinematic quantities along $u^a$ which we call the propagation equations. The second set of equations is called the constraint equations. The two sets of equations have as follows:

**Propagation equations:**

$$h^{b}_{a} \omega^{a} = (\sigma^{a}_{d} - \frac{2}{3} \theta h^{a}_{d}) \omega^{d} + \frac{1}{2} \eta^{abcd} u_{b} \dot{u}_{c,d}, \quad (three \ equations) \quad (36)$$

$$\dot{\theta} + \frac{1}{3} \theta^{2} + 2(\sigma^{2} - \omega^{2}) = -R_{ab} u^{a} u^{b} + \dot{u}^{a} \sigma^{a}, \quad (one \ equation) \quad (37)$$

$$-E_{st} + \frac{1}{2} \left( h^{s}_{a} h^{t}_{b} - \frac{1}{3} h_{st} h^{ab} \right) R_{ab} = h^{a}_{b} h^{b}_{t} \left[ \sigma_{ab} - \dot{u}_{(a;b)} \right] + \sigma_{st} \sigma^{c}_{t} + (five \ equations)$$

$$+ \frac{2}{3} \sigma_{st} \theta + \omega_{s} \omega_{t} - \dot{u}_{s} \dot{u}_{t} - \frac{1}{3} \left( 2\sigma^{2} + \omega^{2} - \dot{u}_{b}^{b} \right) h_{st}. \quad (38)$$

**Constraint equations:**

$$h^{a}_{b} \omega_{\alpha} = \dot{u}^{a} \omega_{\alpha}, \quad (one \ equation) \quad (39)$$

$$h^{c}_{s} \left[ \frac{2}{3} \theta \omega - h^{a b} \sigma_{ca;b} - \eta_{a(m n} u^{a} \left( \omega^{m} u^{n} + 2\omega^{n} \dot{u}^{m} \right) \right] = -h^{c}_{s} R_{cd} u^{d}, \quad (three \ equations) \quad (40)$$

$$-h^{a}_{b} h^{b}_{t} \left[ \sigma_{c}^{b} \omega_{c}^{t} + \omega_{c}^{b} \sigma_{c}^{t} \right] \eta_{a(c d)} u^{c} + 2\dot{u}_{(s} \omega_{l)} = H_{st}. \quad (five \ equations) \quad (41)$$\hfill

Note that the propagation equation of $\omega_{ab}$ (36) contains only kinematic terms and it is independent of $R_{ab}$ hence of the dynamical variables to be introduced later on.
5.3. The 1+3 decomposition of the Bianchi identity (4)
We write the 1+3 decomposition of this geometric identity in terms of the Weyl tensor. It is well known that the Riemann tensor can be decomposed in the following irreducible parts:

\[ R_{abcd} = C_{abcd} + \frac{1}{2} (g_{ac}R_{bd} + g_{bd}R_{ac} - g_{ad}R_{bc} - g_{bc}R_{ad}) - \frac{1}{6} R g_{abcd}, \]  

(42)

where:

\[ g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}, \]  

(43)

and \( C_{abcd} \) is the Weyl tensor. Equation (42) is a mathematical identity. The Weyl tensor is decomposed further in terms of the electric part \( E_{ab} \) and the magnetic part \( H_{ab} \) wrt the vector field \( u^a \) as follows:

\[ C_{abcd} = (g_{abrs}g_{cdmt} - \eta_{abrs}\eta_{cdmt})u^r u^m E^{st} - (\eta_{abrs}g_{cdmt} + g_{abrs}\eta_{cdmt})u^r u^m H^{st}. \]  

(44)

We note the relations:

\[ E_{ac} = C_{abcd} u^b u^d, \quad H_{ac} = \frac{1}{2} \eta_{am, kl} C_{kld} u^m u^l. \]  

(45)

The tensors \( E_{ac}, H_{ac} \) are symmetric, traceless and satisfy the property:

\[ E_{ac}u^c = H_{ac}u^c = 0. \]  

(46)

If in (4) we substitute the Weyl tensor \( C^{abcd} \) in terms of the electric and the magnetic parts \( E^{ab}, H^{ab} \) defined in (45), we obtain four identities which have a form similar to Maxwell equations for the electric and the magnetic field:

\[ \nabla E = \begin{align*}
    h_a^i h_{s}^d E_{as}^i : d - \eta^{b pq} u_b \sigma_p^d H_{qd} + 3 H_t^t s \omega_s^i &= \frac{1}{3} h_a^i \mu_b - \frac{1}{2} h_a^k \pi_{cb}^k - \frac{1}{2} \theta \sigma_q^b + \frac{1}{2} \sigma_q^b u^b + \frac{1}{2} \sigma_q^b u^b \eta_{ac}(\omega_c^i + \sigma_c^i), \\
    \nabla H &= \begin{align*}
        h_a^i h_{s}^d H_{as}^i : d + \eta^{b pq} u_b \sigma_p^d E_{qd} - 3 E_s^t s \omega_s^i &= (\mu + p) \omega_t + \frac{1}{2} \eta^{b pq} u_b \eta_{aq}(\omega_q^i + \sigma_q^i), \\
        \dot{E}_{ab} &= \begin{align*}
            [h_a^m h_c^t E_{ac}^{(m)} + h_a^m (\eta^t)^{rsd} u_r H_s^{(m) \eta_{cd}} - 2 H_q^{(t \eta_m)^{b pq} u_b \dot{u}_p} + h_{m \theta} \sigma^{ab} &+ \theta E_{m t} - 3 E_s^{(m \sigma)} s - E_s^{(m \omega)} s ] \] &= -\frac{1}{2} (\mu + p) \theta^{m \theta} - \dot{u}^{(t \eta_m)} \\
            - \frac{1}{2} h_a^m \eta_{ac} \dot{\omega}_b^{(t \eta_m)} - \frac{1}{2} h_a^m \eta_{ac} \dot{\sigma}_b^{(t \eta_m)} \\
            - \frac{1}{2} \pi^{b(m) \dot{\omega}_b^t + \dot{\sigma}_b^t} - \frac{1}{2} \theta^{m \theta} = -\frac{1}{2} (h_{m \theta} \eta_a^{\theta} + \dot{u}_a \eta_a^{\theta} + \pi^{\theta \sigma} \sigma_{ab}), \; (49)
\end{align*}
\end{align*} \]

\[ \dot{H}_{ab} = \begin{align*}
    [h_m a h_c^{(m \eta)_r s d} E_{r s}^{(m) \eta_{cd}} + 2 E_{m}^{(t \eta_m)^{b pq} u_b \dot{u}_p} + h_{m \theta} \sigma^{ab} H_{ab} + \theta H_{m t} - 2 H_s^{(m \sigma) s} - H_s^{(m \omega) s} ] \] &= \frac{1}{2} \sigma^{(t \eta_m) b cf} u_b q_f - \frac{1}{2} h_{c \eta_{m \eta} \eta_{b c} \eta_{c \eta} \eta_{b \eta} \eta_{c \eta}} \\
    + \frac{1}{2} \left( H_{m t} \omega_c q_c - 3 \omega^{(m \eta) t} \right), \quad (50)
\end{align*} \]

The contracted Bianchi identity \( G^{ab} \) is contained in these identities.
6. Propagation of the kinematic quantities along the collineation vector

The propagation equations give the derivative of the kinematic quantities along the four-velocity $u^a$ of the observers. However when a collineation is present then one wants to know how the kinematic quantities propagate along the collineation vector. Because the collineation is defined in terms of the Lie derivative (Lie transport) one is interested in the quantities $L_\xi \{ u^a, \omega_{ab}, \sigma_{ab}, \theta, \dot{u}^a \}$. To do that we give the following general result.

**Proposition 4** Let $X^a$ be a non-null vector field with index $\varepsilon(X) = \pm 1$ i.e. $X^aX_a = \varepsilon(X)X^2$ ($X > 0$) and let $\xi^a$ be an arbitrary vector field with collineation parameters (see [6]) $\psi, H_{ab}$ that is:

$$\xi_{(a;b)} = \psi g_{ab} + H_{ab} = \frac{1}{2}L_\xi g_{ab}, \quad L_\xi g^{ab} = -2\psi g^{ab} - 2H^{ab}.$$  

Then $L_\xi X^a$ is 1+3 decomposed wrt the four-velocity $u^a$ as follows:

$$L_\xi X^a = \left[ (\ln X)_b \xi^b - \psi - \frac{\varepsilon(X)}{X^2} H_{cb} X^c X^b \right] X^a + V^a(X).$$  

where $V^a(X) = h^a_b(X)(L_\xi X^b)$ a vector field normal to $X^a$. For the covariant quantity $L_\xi X_a$ holds:

$$L_\xi X_a = \left[ (\ln X)_b \xi^b + \psi + \frac{\varepsilon(X)}{X^2} H_{cb} X^c X^b \right] X_a + \hat{V}_a(X),$$  

where:

$$\hat{V}_a(X) = 2h^c_b(X)H_{bc}X^c + V_a(X).$$  

Note that $\hat{V}_a(X) \neq V_a(X)$ !!

In case $X^2 = 1$ formulae (52) and (53) reduce to:

$$L_\xi X^a = -\left[ \psi + \varepsilon(X) H_{cb} X^c X^b \right] X^a + V^a(X),$$  

$$L_\xi X_a = \left[ \psi + \varepsilon(X) H_{cb} X^c X^b \right] X_a + \hat{V}_a(X).$$  

In the special case $X^a = u^a$ we find:

$$L_\xi u^a = -(\psi - H_{cd} X^c X^d) u^a + V^a(u),$$  

$$L_\xi u_a = (\psi - H_{cd} X^c X^d) u_a + \hat{V}_a(u).$$  

We have the following (easy to prove) result.

**Proposition 5** (i) $\hat{V}_a(X) = V^a(X)$ iff $h^{ab}(X)H_{bc}X^c = 0$ iff $H_{bc}X^c = aX_b$ i.e. $X^a$ is an eigenvector of $H_{bc}$.

(ii) $\hat{V}_a(X) = V^a(X)$ for all non-null $X^a$ iff $\xi^a$ is at most a CKV.

(iii) For $X^a = u^a$ we have $\hat{V}_a(u) = V^a(u)$ iff either $u^a$ is an eigenvector of $H_{bc}$ or $\xi^a$ is at most a CKV. Furthermore in that later case holds:

$$L_\xi u^a = -\psi u^a + V^a(u),$$  

$$L_\xi u_a = \psi u_a + V_a(u).$$
6.1. The case $X^2 = 1$

In the following we assume that $X^a$ is **UNIT** $(\varepsilon(X) = \pm 1)$ and compute the quantity $V_a(u)$ in terms of the “kinematic” quantities of $X^a$. We find:

$$L_\xi X_a = \left[2\omega(X)_{ab} + \varepsilon(X)(X^*_a X_b - X^*_b X_a)\right] \xi^b + (X_b \xi^b)_a$$  \hspace{1cm} (61)

where a star “*” over a symbol indicates derivation along $X^a$ i.e. $X^*_a = X_a, b X^b$. We also write $2\omega(X)_{ab} = h^c_a(X)h^d_b(X)(X_{cd} - X_{dc})$. We have according to the general 1+3 decomposition formula (note that $X^2 = 1$):

$$X^a_{;b} = h(X)^{c}_{a} h(X)^{d}_{b} X^{c}_{;d} + \varepsilon(X) X^*_a X_b.$$  \hspace{1cm} (62)

We 1+3 decompose $L_\xi X_a$ along $X^a$. For the parallel component we find:

$$X^a L_\xi X_a = -(X^*_b \xi^b) + (X_b \xi^b)^* = X^*_b \xi .$$  \hspace{1cm} (63)

This implies the formula\(^7\):

$$L_\xi X_a = \varepsilon(X) \left[-(X^*_b \xi^b) + (X_b \xi^b)^*\right] + \hat{V}_a(X).$$  \hspace{1cm} (64)

Comparing (61) and (64) we find:

$$\hat{V}_a(X) = 2\omega(X)_{ab} \xi^b + \varepsilon(X) X^*_a (X_b \xi^b) + (X_b \xi^b)_a - \varepsilon(X) (X_b \xi^b)^* X_a.$$  \hspace{1cm} (65)

Noting that:

$$(X_b \xi^b)_a - \varepsilon(X)(X_b \xi^b)^* X_a = h^b_a(X_c \xi^c),_b$$  \hspace{1cm} (66)

we obtain the final result:

$$\hat{V}_a(X) = 2\omega(X)_{ab} \xi^b + \varepsilon(X) X^*_a (X_b \xi^b) + h^b_a(X_c \xi^c),_b.$$  \hspace{1cm} (67)

It is possible to express the parallel component of the $L_\xi X_a$ in terms of the collineation components. Indeed, by Lie differentiating:

$$g^{ab} X_a X_b = \varepsilon(X),$$  \hspace{1cm} (68)

we find:

$$(L_\xi g^{ab}) X_a X_b + 2 g^{ab} X_a L_\xi X_b = 0 \Rightarrow$$

$$(\psi g^{ab} - 2 H^{ab}) X_a X_b + 2 g^{ab} X_a L_\xi X_b = 0 \Rightarrow$$

$$X^b L_\xi X_b = \varepsilon(X) \psi + H^{ab} X_a X_b.$$  \hspace{1cm} (69)

From (63) and (69) follows:

$$\varepsilon(X) \psi + H^{ab} X_a X_b = -(X^*_b \xi^b) + (X_b \xi^b)^* = X^*_b \xi .$$  \hspace{1cm} (70)

\(^6\) Note that $\xi^b$ does not in general coincides with $\xi^b$.

\(^7\) Compare with (56). Also note that $L_\xi X_a = \varepsilon(X)(X^*_a L_\xi X_b) + h^b_a L_\xi X_b$. 

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This relation implies the identity/decomposition:

\[ L_\xi X_a = \left[ \psi + \varepsilon(X) H^{cd} X_c X_d \right] X_a + 2\omega(X)_{ab} \xi^b + \varepsilon(X) \dot{X}_a (X_b \xi^b) + h^b_a (X_c \xi^c)_b. \quad (71) \]

Next, we compute the Lie derivative \( L_\xi X^a \). We find:

\[ L_\xi X^a = \left[ -\psi + \varepsilon(X) H^{cd} X_c X_d \right] X^a - 2H^{ab} X_b + 2\omega(X)_b \xi^b + \varepsilon(X) \dot{X}_a^b (X_b \xi^b) + h^{ab} (X_c \xi^c)_b. \quad (72) \]

Now \( (X^2 = 1) \):

\[ H^{ab} X_b = \varepsilon(X) (H^{cd} X_c X_d) X^a + h^a_0 H^{cd} X_d, \quad (73) \]

therefore:

\[ L_\xi X^a = \left[ -\psi - \varepsilon(X) H^{cd} X_c X_d \right] X^a - 2h^a_0 H^{cd} X_d + 2\omega(X)_b \xi^b + \varepsilon(X) \dot{X}_a^b (X_b \xi^b) + h^{ab} (X_c \xi^c)_b. \quad (74) \]

Comparing with (56) we find:

\[ V_a (X) = -2h^a_0 H^{cd} X_d + 2\omega(X)_b \xi^b + \varepsilon(X) \dot{X}_a^b (X_b \xi^b) + h^{ab} (X_c \xi^c)_b. \quad (75) \]

**Exercise 6** Using the results (71),(74) prove that \( L_\xi (X^a X_a) = 0 \).

6.2. The special case \( X^a = u^a \)

An important special case is \( X^a = u^a \), that is , \( X^a \) is unit and timelike (the four-velocity). In this case the general formulae (52) and (55) give:

\[ L_\xi u^a = (\psi(\xi) + H_{cd}(\xi) u^c u^d) u^a + V^a (u), \quad (76) \]

\[ L_\xi u_a = (\psi(\xi) - H_{cd}(\xi) u^c u^d) u_a + \dot{V}_a (u). \quad (77) \]

We write \( L_\xi u_a \) and \( L_\xi u^a \) in terms of the kinematic quantities of \( u^a \). From (71) follows:

\[ L_\xi u_a = \left[ \psi - H^{cd} u_c u_d \right] u_a + 2\omega_a \xi^b - \dot{u}_a (u_b \xi^b) + h^b_a (u_c \xi^c)_b, \quad (78) \]

and from (72):

\[ L_\xi u^a = - \left[ \psi - H^{cd} u_c u_d \right] u^a - 2h^a_0 H^{bc} u_c + 2\omega^a \xi^b - \dot{u}^a (u_b \xi^b) + h^{ab} (u_c \xi^c)_b. \quad (79) \]

These imply that:

\[ \dot{V}_a (u) = 2\omega_a \xi^b - \dot{u}_a (u_b \xi^b) + h_a^b (u_c \xi^c)_b, \quad (80) \]

\[ V^a (u) = -2h^a_0 H^{bc} u_c + 2\omega^a \xi^b - \dot{u}^a (u_b \xi^b) + h^{ab} (u_c \xi^c)_b. \quad (81) \]

From (78) we draw the following conclusions:

1. If \( u_a \xi^a \) is an acceleration potential (that is the acceleration is of the form \( a = -(u_b \xi^b)^{-1} \)) then for any collineation \( \xi^a \):

\[ L_\xi u_a = \left[ \psi - H^{cd} u_c u_d \right] u_a + 2\omega_a \xi^b. \quad (82) \]

One special case of this is when \( u_a \xi^a = 0 \) i.e. the symmetry vector is normal to the flow vector \( u^a \) (see [4]).
2. If $\xi^a$ is a KV we have:
\[ L_\xi u_a = 2 \omega_{ab} \xi^b - \dot{u}_a (u_b \xi^b) + h_a^b (u_c \xi^c)_{;b}, \]  
(83)
from which we conclude:
\[ L_\xi u_a = 0 \iff 2 \omega_{ab} \xi^b - \dot{u}_a (u_b \xi^b) + h_a^b (u_c \xi^c)_{;b} = 0. \]  
(84)

A CKV is inherited\(^8\) by the four-velocity field $u^a$ if $u_b \xi^b = 0$ and the fluid is either irrotational (i.e. $\omega_{ab} = 0$) or the vorticity vector $\omega^a \parallel \xi^a$ (see [4]).

3. For a CKV the inheritance of the symmetry by $u^a$ (i.e. $L_\xi u_a = \psi u_a$ or $L_\xi u^a = -\psi u^a$) is equivalent to the identity $h_a^b L_\xi dx^a = 0$ (see [22]). Obviously for a KV/HKV/SCKV the result remains true.

We note that the requirement of surface forming of $\xi^a$ with a unit timelike Killing vector (a static spacetime) results in the kinematic constraint (84) which is by no means trivial. For example when $\xi^a u_a = 0$ then $\omega_a \xi^a = 0$ i.e. $\omega_a / \xi^a$.

7. The 1+3 decomposition of the Lie derivative $L_{\xi^a} X^a$ wrt $X^a$
This decomposition is useful because allows us to compute the Lie derivative of the kinematic quantities along the collineation vector $\xi^a$. We have the following identity for any vector fields\(^9\) $X^a, \xi^a$:
\[ L_{\xi^a} X^a = (L_{\xi^a} X^a)_{;b} - (L_{\xi^a} \Gamma^c_{ab}) X^c, \]  
(85)
and also the identity:
\[ L_{\xi^a} \Gamma^c_{ab} = \frac{1}{2} g^{cd} [(L_{\xi^a} g_{da})_{;b} + (L_{\xi^a} g_{db})_{;a} - (L_{\xi^a} g_{ab})_{;d}]. \]  
(86)
Assuming $\xi^a$ to be a collineation the second identity gives:
\[ L_{\xi^a} \Gamma^c_{ab} = g^{cd} [\psi_{;b} g_{da} + \psi_{;a} g_{db} - \psi_{;d} g_{ab} + H_{da;b} + H_{db;a} - H_{ab;d}]. \]
Therefore:
\[ L_{\xi^a} X^a = (L_{\xi^a} X^a)_{;b} - [\psi_{;b} g_{da} + \psi_{;a} g_{db} - \psi_{;d} g_{ab} + H_{da;b} + H_{db;a} - H_{ab;d}] X^d. \]  
(87)

A different expression is found as follows:
\[ L_{\xi^a} \Gamma^c_{ab} = \frac{1}{2} g^{cd} [(L_{\xi^a} g_{da})_{;b} + (L_{\xi^a} g_{db})_{;a} - (L_{\xi^a} g_{ab})_{;d}], \]
\[ = g^{cd} [\xi_{(d;a)}_{;b} + \xi_{(d;b)}_{;a} - \xi_{(a;b)}_{;d}], \]
\[ = \frac{1}{2} g^{cd} [\xi_{d;ab} + \xi_{a;db} + \xi_{d;ba} + \xi_{b;da} - \xi_{a;bd} - \xi_{b;ad}], \]
\[ = g^{cd} [\xi_{d(ab)} + \xi_{a(db)} + \xi_{b(da)}], \]
\[ = g^{cd} \left[ \xi_{d(ab)} + \frac{1}{2} R_{tadb} \xi^t + \frac{1}{2} R_{tbd} \xi^t \right], \]
\[ = g^{cd} \left[ \xi_{d(ab)} + R_{t(a|d|b)} \xi^t \right]. \]  
(88)
This gives:
\[ L_{\xi^a} X^a = (L_{\xi^a} X^a)_{;b} - g^{cd} \left[ \xi_{d(ab)} + R_{t(a|d|b)} \xi^t \right] X^c, \]  
(89)
where we have defined the Riemann tensor with the Ricci identity:
\[ 2 \xi a_{;bc} = R_{tabc} \xi^t. \]  
(90)
\(^8\) The symmetry is inherited if $L_\xi u^a = \lambda u^a$.
\(^9\) This identity gives the commutation of the Lie and the covariant derivative in a Riemannian space. It can be found in [5].
7.1. The case \( X^2 = 1 \)

We consider the case \( X^2 = 1 \) and find from (55):

\[
(L_\zeta X_a)_{;b} = \left[ \psi + \varepsilon(X) H_{cd} X^c X^d \right] X_a + \hat{V}_a(X)_{;b},
\]

\[
= \left[ \psi + \varepsilon(X) H_{cd} X^c X^d \right] X_a + \left[ \varepsilon(X) H_{cd} X^c X^d \right] X_{a;b} + \hat{V}_a(X)_{;b}. \tag{91}
\]

where:

\[
\hat{V}_a(X) = 2 \omega(X)_{ab} \xi^b + \varepsilon(X) X^a a(X_b \xi^b) + h_a^b (X \xi^c)_{;b}. \tag{92}
\]

To save writing we set:

\[
K(X) = \psi + \varepsilon(X) H_{cd} X^c X^d, \tag{93}
\]

and have:

\[
(L_\zeta X_a)_{;b} = K(X)_{;b} X_a + K(X) X_{a;b} + \hat{V}_a(X)_{;b}. \tag{94}
\]

Then, using (91) we get:

\[
L_\zeta X_{a;b} = \left[ K(X)_{;b} X_a + K(X) X_{a;b} + \hat{V}_a(X)_{;b} \right] + \left[ \psi a gda + \psi^a gdb - \psi^d g_{ab} + H_{da;b} + H_{db;a} - H_{ab;d} \right] X^d, \tag{94}
\]

and also:

\[
L_\zeta X_{a;b} = \left[ K(X)_{;b} X_a + K(X) X_{a;b} + \hat{V}_a(X)_{;b} \right] - \left[ \xi_{d;ab} + R_{l(a;db)} \xi^l \right] X^d. \tag{95}
\]

7.1.1. The case \( X^a = u^a \)

For this special case we find from (94):

\[
L_\zeta u_{a;b} = K(u)_{;b} u_a + K(u) u_{a;b} + \hat{V}_a(u)_{;b} + \left[ \psi^a gda + \psi^a gdb - \psi^d g_{ab} + H_{da;b} + H_{db;a} - H_{ab;d} \right] u^d, \tag{96}
\]

\[
K = \psi - H_{cd} u^c u^d, \quad \hat{V}_a = \hat{V}_a(u). \tag{98}
\]

We consider the decomposition \( u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{2} \theta h_{ab} - \dot{u}_a u_b \) and the lhs becomes:

\[
L_\zeta \sigma_{ab} + L_\zeta \omega_{ab} + \frac{1}{2} L_\zeta \theta h_{ab} + \frac{1}{2} \theta L_\zeta h_{ab} - (L_\zeta \dot{u}_a) u_b - \dot{u}_a L_\zeta u_b,
\]

therefore we can compute the quantities \( L_\zeta \sigma_{ab}, L_\zeta \omega_{ab}, L_\zeta \theta, L_\zeta \dot{u}_a \) in terms of the collineation parameters \( \psi, H_{ab} \) by taking the irreducible parts of equation (96). Before we do that we need to calculate the Lie derivatives of the projection tensor \( h_{ab} \).

7.2. Calculation of \( L_\zeta h_{ab} \)

For the Lie derivative of the projection tensor we find the results:

\[
L_\zeta h_{ab}(X) = L_\zeta \left( g_{ab} - \frac{\varepsilon(X)}{X^2} X_a X_b \right) = 2 \psi h_{ab}(X) + 2 H_{ab} - 2 \frac{1}{X^4} \left( H_{cd} X^c X^d \right) X_{(a} X_{b)} + \left( X^2 \right)^3 \varepsilon(X) X_{(a} \dot{V}_{b)}(X) - 2 \frac{\varepsilon(X)}{X^2} \left[ (\ln X)_c \xi^c \right] X_{(a} X_{b)}, \tag{99}
\]
In the case where the vector field is unit ($X^2 = 1$), equations (99) and (100) reduce to:

$$L_{\xi}h_{ab}(X) = 2\psi h_{ab}(X) + 2H_{ab} - 2\left(H_{cd}X^cX^d\right)X_{(a}X_{b)} - 2\varepsilon(X)X_{(a}\dot{V}_b)(X),$$

(101)

$$L_{\xi}h^a_b(X) = -\varepsilon(X)[V^a(X)X_b + X^a\dot{V}_b(X)].$$

(102)

In the special case $X^a$ is the 4-velocity $u^a$ the above relations become:

$$L_{\xi}h_{ab} = 2\psi h_{ab} + 2H_{ab} - 2\left(H_{cd}u^c u^d\right)u_{(a}u_{b)} + 2u_{(a}\dot{V}_b),$$

(103)

$$L_{\xi}h^a_b = u^a\dot{V}_b + V^a u_b.$$  

(104)

Replacing $\dot{V}_b$, $V^a$ from (80), (81) we find:

$$L_{\xi}h_{ab} = 2\psi h_{ab} + 2H_{ab} - 2\left(H_{cd}u^c u^d\right)u_{(a}u_{b)} + 2u_{(a}[2\omega_{b)c}\xi^c - \dot{u}_b)(u_c\xi^c) + h^d_b(u_c\xi^c);d],$$

(105)

$$L_{\xi}h^a_b = u^a[2\omega_{b)c}\xi^c - \dot{u}_b)(u_c\xi^c) + h^d_b(u_c\xi^c);d] + [-2h^a_dH^d_{ce}u_c + 2\omega_c\xi^c - \dot{u}^a(u_c\xi^c) + h^{ad}(u_c\xi^c);d]u_b.$$  

(106)

Concerning $h^{ab}$ we find:

$$L_{\xi}h^{ab} = -g^{ac}g^{bd}L_{\xi}h_{cd},$$

$$= -2\psi h^{ab} - 2H^{ab}$$

$$- g^{ac}g^{bd}\left[-2(H_{mn}u^mu^n)u_{c}u_{d} + 2u_{(c}[2\omega_{d)m}\xi^m - \dot{u}_{d})(u_n\xi^n) + h^m_{d}(u_n\xi^n);m]\right].$$

(107)

**Exercise 7** Prove that for a spacelike CKV $\xi^a$ the following are true:

$$L_{\xi}h_{ab} = 2\psi h_{ab} + 4u_{(a}\omega_{b)c}\xi^c,$$

(108)

$$L_{\xi}h^a_b = 2(u^a\omega_{b)c}\xi^c + \omega^a_{c}u_b),$$

(109)

$$L_{\xi}h^{ab} = -2\psi h^{ab} - 2H^{ab} - 4g^{ac}g^{bd}u_{(c}\omega_{d)m}\xi^m.$$  

(110)
7.3. Calculation of $L_\xi \omega_{ab}, L_\xi \sigma_{ab}, L_\xi \theta, L_\xi \hat u_a$

We compute now the quantities $L_\xi \sigma_{ab}, L_\xi \omega_{ab}, L_\xi \theta, L_\xi \hat u_a$ in terms of the collineation parameters $\psi, H_{ab}$. We have\(^{10}\) $(\hat V = \hat V(u), V = V(u))$:

$$L_\xi \left(h_a^c h_b^d u_{c;d}\right) = h_a^c h_b^d \left(L_\xi u_{c;d}\right) + h_a^c (L_\xi h_b^d) u_{c;d} + h_b^d (L_\xi h_a^c) u_{c;d},$$

$$= h_a^c h_b^d \left[\psi - H_{mn} u^m u^n\right] u_c + \left[\psi - H_{mn} u^m u^n\right] u_{dc} + \hat V_{cd},$$

$$- h_a^c h_b^d \left[\psi_{de} g_{cd} + \psi_{cd} g_{ed} - \psi_{de} g_{cd} - H_{ec;cd} + H_{ede} - H_{edc}\right] u^e,$$

$$+ h_a^c \left[u^d \hat V_b + V^d u_b\right] u_{c;d} + h_b^d \left[u^c \hat V_a + V^c u_a\right] u_{c;d}.$$

We have

$$h_a^c h_b^d \left[\psi - H_{mn} u^m u^n\right] u_{cd} + h_a^c h_b^d \hat V_{cd} \tag{115}$$

For the symmetric part we find:

$$L_\xi \omega_{ab} = h_a^c h_b^d L_\xi u_{[c;d]} \Rightarrow$$

$$L_\xi \omega_{ab} = (\psi - H_{mn} u^m u^n) \omega_{ab} + h_a^c h_b^d \hat V_{[c;d]} + \hat u_a \hat V_b - 2u_{[a} \omega_{b]c} \hat V^c.$$ \hspace{1cm} \tag{116}

For the symmetric part $\theta_{ab} = u_{(a;b)}$ we find:

$$L_\xi \theta_{ab} = L_\xi \left(h_a^c h_b^d u_{[c;d]}\right) \Rightarrow$$

$$L_\xi \theta_{ab} = (\psi - H_{mn} u^m u^n) \theta_{ab} + h_a^c h_b^d \hat V_{[c;d]} + 2u_{[a}\theta_{b]c} V^c + \hat u_{[a} \hat V_{b]},$$

$$+ \hat h_{ab} - h_a^c h_b^d \left[H_{ec;cd} + H_{ede} - H_{edc}\right] u^e.$$ \hspace{1cm} \tag{117}

We decompose $\theta_{ab}$ in trace and traceless part as follows $\theta_{ab} = \sigma_{ab} + \frac{1}{3} \hat h_{ab} \theta$. For the trace we find:

$$L_\xi \theta = -(\psi - H_{mn} u^m u^n) \theta - 2H^{ab}\theta_{ab} + h_a^c h_b^d \hat V_{[c;d]} + 2\omega_{ab} \hat u^c - (u_a \omega^b) (u_b \xi^c)$$

$$+ \hat u^a (u_c \xi^b) + 3\hat \psi - h^{cd} \left[H_{ec;cd} + H_{ede} - H_{edc}\right] u^e.$$ \hspace{1cm} \tag{118}

We can simplify this expression further. The term:

$$2H^{ab}\theta_{ab} = 2H^{ab}\sigma_{ab} + \frac{2}{3} H^{ab} \hat h_{ab} \theta = 2H^{ab}\sigma_{ab} + \frac{2}{3} H^{mn} u_m u_n \theta,$$

\(^{10}\) Recall that $L_\xi u_a = \left[\psi - H^{cd} u_{c;d}\right] u_a + 2\omega_{ab} \xi^b - \hat u_a (u_c \xi^b) + h_a^b (u_c \xi^b), \tag{111}$

and from (72):

$$L_\xi u^a = - \left[\psi - H^{cd} u_{c;d}\right] u^a - 2h^a_b H^{cd} u_c + 2u^a_b \xi^b - \hat u^a (u_c \xi^b) + h^a_b (u_c \xi^b), \tag{112}$$

where

$$\hat V_a (u) = 2\omega_{ab} \xi^b - \hat u_a (u_c \xi^b) + h_a^b (u_c \xi^b), \tag{113}$$

$$V^a (u) = -2h^a_b H^{cd} u_c + 2u^a_b \xi^b - \hat u^a (u_c \xi^b) + h^a_b (u_c \xi^b). \tag{114}$$
Therefore:

\[ L_\xi \theta = - \left( \psi + \frac{5}{3} H_{mn} u^m u^n \right) \theta - 2H^{ab} \sigma_{ab} + h^{cd} \dot{V}_{(cd)} + 2\omega_{ab} \dot{u}^a \xi^b \\
- (u_a \dot{u}^a) (u_b \xi^b) + \dot{u}^a (u_c \dot{\xi}^c)_{;a} + 3 \dot{\psi} - h^{cd} [H_{ecd} + H_{edc} - H_{edc}] u^e. \tag{119} \]

Concerning the 4-acceleration we have:

\[ L_\xi \dot{u}_a = L_\xi \left( u_{a;b} u^b \right) = (L_\xi u_{a;b}) u^b + (L_\xi u^b) u_{a;b} \Rightarrow \]

\[ L_\xi \dot{u}_a = \dot{K}(u) u_a + \dot{V}_a (u;b) u^b + V^b (u) u_{a;b} + \psi_a - [H_{da;b} + H_{db;a} - H_{ab;d}] u^d u^b, \]

where \( K(u) = \psi - H_{mn} u^m u^n \).

There remains the shear \( \sigma_{ab} \). We find after a standard calculation:

\[ L_\xi \sigma_{ab} = K(u) \sigma_{ab} - \frac{1}{3} \left( \frac{2}{3} H_{mn} u^m u^n \right) \theta h_{ab} + h^c_{(a} h^d_{b)} \dot{V}_{(cd)} + 2\omega_{(a;b)} V^c + \dot{u}_{(a} \dot{V}_{b)} - h^c_{(a} h^d_{b)} [H_{ecd} + H_{edc} - H_{edc}] u^e - \frac{1}{3} \left[ -2H_{ab} - 2 \left( H_{edc} u^d \right)_{(a} u_{b)} + 2u_{(a} \dot{V}_{b)} \right] \theta \\
- \frac{1}{3} \left[ -2H^{mn} \sigma_{mn} + h^{mn} \dot{V}_{(m;n)} + 2\omega_{mn} \dot{u}^m \xi^n - (u_m \dot{u}^m) (u_n \dot{\xi}^n) \right] h_{ab}. \tag{120} \]

8. The matter - Dynamics

General Relativity approaches the gravitational field using the model of a fluid. That is one considers the stress tensor \( T_{ab} \) on the fluid to be determined by the gravitational field created by the material content of spacetime. This stress tensor determines the metric of spacetime via Einstein field equations:

\[ G_{ab} = T_{ab}. \tag{121} \]

where \( G_{ab} \) is the Einstein tensor. Subsequently the metric determines the timelike geodesics which when affinely parameterized become the four-velocity of a special class of observers called the intrinsic observers of the metric. The world lines of these observers are identified with the flow lines of the gravitational fluid. In that sense the particles of the fluid are the intrinsic observers.

This is the scenario of General Relativity [7]. We note that Einstein field equations (121) are independent of the observer and they relate the geometry of the background space with the matter content of the universe. Furthermore they cannot be solved because one neither knows the metric, therefore cannot compute the \( G_{ab} \) and one does not know the mater content in terms of some physical parameters. In other words equation (121) is used only to frame the general set up of the theory and cannot give any information until a class of observers is considered. The ideal choice would be the intrinsic observers but only `God' knows who are they. What it is left to us is to chose an arbitrary class of observers by means of a timelike unit vector field and adding some more simplifying assumptions write down (121) in terms of some set of differential equations whose solutions will give us an indication of spacetime as seen by these observers. The experiment and the observations will show us how close to the intrinsic observers we are and how close to `reality' our simplifying assumptions are. In conclusion we cannot solve the field equations but we can use them as the vehicle to make up scientific pictures of the gravitational field and get some answers which we hope they will be as close as possible to our measurements (=reality!). The ultimate truth is hidden within the intrinsic observers with whom we have no touch or communication.
The major question is:

*To what degree the matter content and the geometry of spacetime are interconnected as a result of the field equations (121)?*

The answer to this question has two parts. The first comes from the identities of Level G1 and the second from those of level G2. Indeed from level G1 we have that the symmetric tensor $G_{ab}$ satisfies the contracted Bianchi identity (5). From Einstein field equations (121) we infer that $T_{ab}$ is symmetric\(^{11}\) and must satisfy the constraint / identity:

$$T_{ab}^{;b} = 0.$$  \hspace{1cm} (122)

This identity gives the conservation equations, which are independent of the observers we choose!

Concerning the identities from level G2 we note that $L_\xi G_{ab}$ can be expressed in terms of the generic collineation $L_\xi g_{ab}$ and subsequently in terms of the collineation parameters $\psi, H_{ab}$. The exact expression in terms of trace and traceless parts is:

$$L_\xi G_{ab} = -(n-2) A_{ab} + 2 K_{ab} - \Box H_{ab} - RH_{ab} + \frac{1}{n} g_{ab} (n-1) (n-2) \Box \psi - (n-2) H_{;cd}^{;cd} + n R_{;cd} H^{;cd},$$  \hspace{1cm} (123)

where:

$$A_{ab} = \psi_{;ab} - \frac{1}{n} g_{ab} \Box \psi$$

and

$$K_{ab} = H_{(a;b)d}^d - \frac{1}{n} g_{ab} H^{;cd}_{;cd}.$$  \hspace{1cm} (124)

Then, Einstein field equations (121) relate $L_\xi T_{ab}$ in terms of the collineation parameters $\psi, H_{ab}$ which implies that the ‘symmetries’ of $T_{ab}$ as a two index tensor are related to the symmetries of the metric $g_{ab}$. In this manner when we make a symmetry assumption in spacetime we simultaneously impose a constraint as a symmetry of the energy momentum tensor, therefore we restrict the possible forms of matter this spacetime can support. In a sense by an assumption at the level G2 we constraint both the geometry and the matter content of spacetime.

To bring all the above at a level which can be used in practice and make Physics we must introduce a class of observers $u^a$. These observers are our choice and there is no guarantee whatsoever that they are or they will be the intrinsic observers of spacetime. However this is not to discourage us because it is the most we can do. The introduction of observers immediately introduces a kinematics in spacetime and all the considerations of the previous sections apply.

However the introduction of $u^a$ allows the 1+3 decomposition of the tensor objects and the covariant equations in the same way it did for the kinematics. The new tensor is the energy momentum tensor $T_{ab}$ which describes the matter content of spacetime and the new equation is the conservation equations (122).

8.1. The 1+3 decomposition of the energy momentum tensor: The dynamic variables

We apply the general formula (22) in the case $Y_{ab} = T_{ab}$ where $T_{ab} = T_{ba}$ is the energy momentum tensor. We define the irreducible parts (tensors):

$$\mu = T_{ab} u^a u^b,$$  \hspace{1cm} (125)

$$p = \frac{1}{3} h_{ab} T_{ab},$$  \hspace{1cm} (126)

$$q^a = h_{ab} T_{bc} u^c,$$  \hspace{1cm} (127)

$$\pi_{ab} = (h_{ab} - \frac{1}{3} h h r^s) T_{rs},$$  \hspace{1cm} (128)

\(^{11}\) Or at least only the symmetric part of $T_{ab}$ enters the field equations and determines the gravitational field.
and we have:
\[ T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab}. \] (129)

We note that in this decomposition \( T_{ab} \) is described by two scalar fields \((\mu, p)\) one spacelike vector \((q^a, g_a u^a = 0)\) and a traceless symmetric 2-tensor \((\pi_{ab}, g_{ab}\pi^{ab} = 0)\).

These quantities we call the physical variables and assume that they represent the mass density, the isotropic pressure, the heat flux and the traceless stress tensor respectively as measured by the observer \(u^a\). Following this decomposition and physical interpretation we consider the types of ‘gravitating fluids’ given in the following Table:

| \( \mu \) | \( p \) | \( q^a \) | \( \pi^{ab} \) | \( T_{ab} \) | Type of fluid |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | \( T_{ab} = \mu u_a u_b \) | Empty space |
| \( \neq 0 \) | 0 | 0 | 0 | \( T_{ab} = \mu u_a u_b + p h_{ab} \) | Perfect fluid |
| \( \neq 0 \) | \( \neq 0 \) | 0 | 0 | \( T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} \) | Isotropic non-perfect fluid |
| \( \neq 0 \) | \( \neq 0 \) | \( \neq 0 \) | \( \neq 0 \) | \( T_{ab} = \mu u_a u_b + p h_{ab} + \pi_{ab} \) | Anisotropic fluid without heat flux |
| \( \neq 0 \) | \( \neq 0 \) | \( \neq 0 \) | \( \neq 0 \) | \( T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} \) | General anisotropic fluid |

8.2. The 1+3 decomposition of the conservation equations

The conservation equations follow from the contracted Bianchi identity and Einstein field equations. By 1+3 decomposing \( T^{ab}_5 \) = 0 wrt \( u^a \) we find the conservation equations as a set of two equations, one set resulting form the projection along the direction \( u^a \) and one normal to \( u^a \). We replace \( T^{ab}_5 \) form (129) in terms of the physical variables and find the two conservation equations\(^{12}\):

\[ \dot{\mu} + (\mu + p)\theta + q_{ab}^h + q^a \dot{u}_a + \pi_{ab} \sigma_{ab} = 0, \] (130)

\[ (\mu + p)\dot{u}_a + h^a_{(b}(p_{c)} + \dot{q}_c + \pi_{d}^{(a} \sigma_{c)d}) + q^b_{(a}^{\theta} \omega_{ac} + \sigma_{ac} + \frac{4}{3} \theta h_{ac} = 0. \] (131)

9. The physical role of the propagation and the constraint equations

The propagation and the constraint equations have been written in terms of the kinematic variables and the Ricci tensor. However with the introduction of the physical variables by the 1+3 decomposition of the energy momentum tensor we can use Einstein equations and replace \( R_{ab} \) in terms of the physical variables. Then we have a complete physical role of the propagation and the constraint equations.

Einstein field equations give:

\[ R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T + \Lambda g_{ab}, \] (132)

where \( T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} \) by (129) and the trace is

\[ T = T^a_a = - \mu + 3p. \] (133)

Replacing in (132) we find \( R_{ab} \) in terms of the physical variables:

\[ R_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} + \frac{1}{2} g_{ab}(\mu - 3p + 2\Lambda), \] (134)

\(^{12}\) We write as usual the derivative along \( u^a \) by a dot, e.g. \( \dot{A}^a = A^a_{\mu} u^\mu. \)
We find:

9.1.1. The dynamical constraint equations

The only dynamic constraint equation is equation (40). We find:

\[ R_{ab} = (\mu + p)u_a u_b + \frac{1}{2}(\mu - p + 2\Lambda)g_{ab} + 2q_{(a}u_{b)} + \pi_{ab}. \]  

(135)

From (134) follows:

\[ R_{ab}u^a u^b = \mu - \frac{1}{2}(\mu - 3p) - \Lambda = \frac{1}{2}\mu + \frac{3}{2}p - \Lambda, \]  

(136)

\[ R_{ab}u^a = -\mu u_b - \frac{1}{2}q_b + \frac{1}{2}u_b(\mu - 3p + 2\Lambda), \]  

(137)

\[ R = R^a_a = \mu - 3p + 4\Lambda, \]  

(138)

where \( R = g^{ab} R_{ab} \) is the curvature scalar.

We turn now to the constraint and propagation equations and we divide them in two sets. One set contains the ones which do not contain the Curvature tensor and the Ricci tensor, therefore they are independent of the field equations (and the matter content of the universe) and they are kinematical equations. The second set contains the equations which depend on the Ricci tensor and the field equations and are dynamical equations.

9.1. The dynamical propagation equations

The dynamical propagation equations\(^\text{13}\) are equation (37) and equation (38). Replacing the Ricci tensor from (134) we find for the first:

\[ \dot{\theta} + \frac{1}{3}\theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a_{;a} = -\frac{1}{2}(\mu + 3p) + \Lambda. \]  

(139)

This is known as the Raychaudhuri equation.

Concerning the propagation equation (38) we have:

\[
\begin{align*}
&h^a_b h^t_c \left[ \dot{\sigma}_{ab} - \dot{u}_{(a;b)} \right] + \sigma_{(a|c|}\sigma^c_{t)} + \frac{2}{3}\sigma_{st}\theta + \omega_s \omega_t - \dot{u}_s \dot{u}_t - \frac{1}{3} \left( 2\sigma^2 + \omega^2 - \dot{u}^b_{;b} \right) h_{st} \\
&= -E_{st} + \frac{1}{2} \left( h^a_b h^b_a - \frac{1}{3} h_{st} h^{ab} \right) R_{ab}, \\
&= -E_{st} + \frac{1}{2} \left( h^a_b h^b_a - \frac{1}{3} h_{st} h^{ab} \right) \left( \mu u_a u_b + ph_{ab} + 2q_{(a}u_{b)} + \pi_{ab} + \frac{1}{2} g_{ab}(\mu - 3p + 2\Lambda) \right), \\
&= -E_{st} + \frac{1}{2} \left( h^a_b h^b_a - \frac{1}{3} h_{st} h^{ab} \right) \pi_{ab} = -E_{st} + \frac{1}{2} \pi_{st}. 
\end{align*}
\]

Therefore in terms of the physical variables the propagation equation (38) reads:

\[
\begin{align*}
&h^a_b h^t_c \left[ \dot{\sigma}_{ab} - \dot{u}_{(a;b)} \right] + \sigma_{(a|c|} \sigma^c_{t)} + \frac{2}{3} \sigma_{st} \theta + \omega_s \omega_t - \dot{u}_s \dot{u}_t - \frac{1}{3} \left( 2\sigma^2 + \omega^2 - \dot{u}^b_{;b} \right) h_{st} = -E_{st} + \frac{1}{2} \pi_{st}. \\
&= h^a_b h^t_c \left[ 2\theta_{ac} - h^{ab} \sigma_{acb} - \eta_{cmm} u^a \left( \omega^{mn}_c + 2\omega^m_i \dot{u}^n_i \right) \right] = q_b. 
\end{align*}
\]

(140)

This equation gives the propagation of shear.

9.1.1. The dynamical constraint equations

The only dynamic constraint equation is equation (40). We find:

\[
\begin{align*}
&h^a_s \left[ 2\theta_{ac} - h^{ab} \sigma_{acb} - \eta_{cmm} u^a \left( \omega^{mn}_c + 2\omega^m_i \dot{u}^n_i \right) \right] = q_b. 
\end{align*}
\]

(141)

\(^\text{13}\)The remaining propagation equation (36) giving the propagation of the vorticity, does not involve the Ricci tensor hence it is a kinematical equation.
10. The dynamical role of collineations

As we remarked in Section 8 the collineation parameters restrict the possible forms of $T_{ab}$. Indeed, Einstein field equations and (123) give:

$$L_\xi T_{ab} = -(n-2)A_{ab} + 2K_{ab} - \Box H_{ab} - RH_{ab} +$$

$$+ \frac{1}{n} g_{ab} \left[ (n-1)(n-2) \Box \psi - (n-2) H_{;cd}^c + n R_{cd} H^{cd} \right],$$

(142)

where the tensors $A_{ab}, K_{ab}$ are given in (124). From this equation it is possible to give the effect of collineation directly to the physical parameters of the observers $u^a$. Indeed if we replace in (142) $T_{ab}$ in $L_\xi T_{ab}$ using (129) and take the irreducible parts, we express the Lie derivative along the collineation vector of the dynamical variables in terms of the collineation parameters. This form of the field equations will relate directly the physical and the geometric variables thus enabling one to draw direct conclusions of the symmetry assumptions. Furthermore the approach is completely general and the results hold for any type of matter, all observers and all collineations.

From (135) we compute the $L_\xi R_{ab}$ in irreducible parts of the 1+3 decomposition wrt $u^a$:

$$L_\xi R_{ab} = \left[ \frac{1}{2} (\mu + 3p)^{;\alpha} - 2 \dot{\xi}_c u^c + \dot{\pi}_{cd} u^c u^d - (\mu + 3p - 2\Lambda) \left( \dot{\xi}_c u^c - 2 q_c \xi^c \right) u_a u_b +$$

$$- 2 \left[ - (\mu + p) \dot{u}_{h_c^{;\alpha}} - \dot{q}_{d;\alpha} u^d \right] - \frac{1}{2} (\mu + 3p - 2\Lambda) u_c \xi_{cd}^{;\alpha} u^d +$$

$$- q_c \xi_{cd}^{;\alpha} u^d + \frac{1}{2} (\mu - p + 2\Lambda) \dot{\xi}_{d;\alpha} u^d + (q_d u^d) \left( \dot{\xi}_c u_e - h_e^{;\alpha} \pi_{cd} \xi^c \right) h_{ab}$$

$$+ 2 \left[ \frac{3}{2} (\mu - p)^{;\alpha} + 2(q_{;\alpha} \dot{u}^c + \dot{\pi}_{cd} h^{cd}) + (\mu - p + 2\Lambda) (4 \psi + \dot{\xi}_c u_c) + 2 \pi_{cd} \xi_{cd} u^d + 2 \pi_{cd} H^{cd} \right] h_{ab}$$

$$+ 2 \left[ \frac{1}{4} (\mu - p)^{;\alpha} h_{e;j} + 2 q_{;\alpha} \dot{u}_{h_c^{;\alpha}} h^{cd} + \dot{\pi}_{cd} h_{e;j}^{;\alpha} h^d +$$

$$+ (\mu - p + 2\Lambda) \dot{\xi}_{d;\alpha} h_{e;j}^{;\alpha} + 2 \pi_{cd} \xi_{cd} u^d h_{e;j}^{;\alpha} h^d + 2 \pi_{cd} \xi_{cd} h_{e;j}^{;\alpha} h^d \right] h_{ab}$$

$$- \frac{1}{3} h_{ab} h_{e;j}^{;\alpha},$$

(143)

where $;\alpha$ means covariant derivation along $\xi^a$ e.g. $\dot{\mu} = \mu_{,a} \xi^a$. Also from (17) we have (for $n = 4!$):

$$L_\xi R_{ab} = - 2 \psi_{;ab} - g_{ab} \Box \psi + 2 H_{(a;b)d}^{d} - \Box H_{ab}.$$  

(144)

We 1+3 decompose this expression and equate with (143) from which we find the field equations in the form of Lie derivative of physical variables in terms of kinematic variables and the collineation parameters.

To do that we 1+3 decompose first $\psi_{;ab}$. We write:

$$\psi_{;ab} = \lambda_{;ab} u_a u_b + p_{\psi} h_{ab} + 2 q_{\psi} (a u_b) + \pi_{\psi ab},$$

(145)

where:

$$\mu_{\psi} = \psi_{;ab} u^a u^b, \ p_{\psi} = \frac{1}{3} \psi_{;ab} h_{ab}, \ q_{\psi a} = - \psi_{;bc} h^b_{a} u^c, \ \pi_{\psi ab} = (h^r_{a} h^s_{b} - \frac{1}{3} h_{ab} h^{rs}) \psi_{;rs}.$$ 

(146)
We also have:

$$\Box \psi = \psi_{;ab}g^{ab} = -\mu \psi + 3p \psi.$$  \hfill (147)

It is also possible to 1+3 decompose the collineation tensor $H_{ab}$ as follows:

$$H_{ab} = \mu_H u_a u_b + p_H h_{ab} - 2q_{H(a} u_{b)} + \pi_{Hab},$$  \hfill (148)

where:

$$\mu_H = H_{ab} u^a u^b, \quad p_H = \frac{1}{3} H_{ab} h^{ab}, \quad q_{Ha} = -H_{bc} h^b_a u^c, \quad \pi_{Hab} = (h^r_a h^s_b - \frac{1}{3} h_{ab} h^{rs}) H_{rs}.$$  \hfill (149)

We compute:

$$-2\psi_{;ab} - g_{ab} \Box \psi = -2 (\mu \psi u_a u_b + p \psi h_{ab} + 2q_{\psi(a} u_{b)} + \pi_{\psi ab}) + (\mu \psi - 3p \psi) (-u_a u_b + h_{ab}),$$

$$= -3(\mu \psi - p \psi) u_a u_b + (\mu \psi - 5p \psi) h_{ab} - 4q_{\psi(a} u_{b)} - 2\pi_{\psi ab}.$$  \hfill (150)

Concerning the second term in the rhs of (144) we define:

$$K_{ab} = 2H_{(a;bc)d}^{\ d} - \Box H_{ab},$$  \hfill (151)

and we write:

$$K_{ab} = \mu_K u_a u_b + p_K h_{ab} - 2q_{K(a} u_{b)} + \pi_{Kab},$$  \hfill (152)

where:

$$\mu_K = K_{ab} u^a u^b, \quad p_K = \frac{1}{3} K_{ab} h^{ab}, \quad q_{Ka} = -K_{bc} h^b_a u^c, \quad \pi_{Kab} = (h^r_a h^s_b - \frac{1}{3} h_{ab} h^{rs}) K_{rs}.$$  \hfill (153)

Then we have:

$$L_\xi R_{ab} = (-3\mu \psi + 3p \psi + \mu K) u_a u_b + (\mu \psi - 5p \psi + p_K) h_{ab} + 2(-2q_{\psi(a} + q_{K(a} u_{b)} - 2\pi_{\psi ab} + \pi_{Kab}).$$  \hfill (154)

Comparing with (143) we write the field equations in the following form:

$$\frac{1}{2} (\mu + 3p)^o - 2\dot{q}^c u^c + \dot{\pi}_{cd} u^c u^d - (\mu + 3p - 2\Lambda) \left( \dot{\xi}_c u^c \right) - 2q \dot{\xi}^c = -3\mu \psi + 3p \psi + \mu K,$$  \hfill (155)

$$\left[ - (\mu + p) \dot{u}_d h^d + \dot{q}_d h^d + \dot{\pi}_{dc} u^c h^d - \frac{1}{2} (\mu + 3p - 2\Lambda) u_c \xi_{cd} h^d \right] +$$

$$\left[ -q \dot{\xi}_{cd} h^d + \frac{1}{2} (\mu - p + 2\Lambda) \dot{\xi}_d h^d + (q_d h^d \left( \dot{\xi}^c u_c \right) - h^d \pi_{cd} \dot{\xi}^c \right] = -(-2q_{\psi c} + q_{Kc}),$$  \hfill (156)

$$\frac{3}{2} (\mu - p)^o + 2q \dot{\xi}^c + \dot{\pi}_{cd} h^{cd} + (\mu - p + 2\Lambda) (4\psi + \dot{\xi}^c u_c) + 2u_c \xi_{cd} q^d + 2\pi_{cd} H^{cd}$$

$$= \frac{3}{2} (\mu \psi - 5p \psi + p_K),$$  \hfill (157)
We simplify these equations. We note that:

\[
\psi_{ab} = 2 - \frac{1}{3} h_{ab} \phi^f
\]

\[
= -2 \psi_{ab} + \pi_{ab}. \quad (157)
\]

We simplify these equations. We note that:

\[
\xi^c u_c = \xi_{cd} u^c u^d = - \psi + H_{cd} u^c u^d,
\]

\[
q^d \left( u_c \xi^c_{cd} + \xi_d \right) = q^d \left( u^c \xi_{cd} + \xi_{dc} u^c \right) = 2 q^d u^c H_{cd}.
\]

Then, (154) is written as follows:

\[
LHS = \frac{1}{2} \left\{ (\mu + 3p)^0 - 4q_c u^c + 2\pi_{cd} u^c u^d - 2 (\mu + 3p - 2\Lambda) \left( \xi_c u^c \right) - 4q_c \xi_c \right\},
\]

\[
= \frac{1}{2} \left\{ (\mu + 3p)^0 + 4q_c u^c + 2\pi_{cd} u^c u^d - 2 (\mu + 3p - 2\Lambda) \left( -\psi + H_{cd} u^c u^d \right) - 4q_c \xi_c \right\},
\]

\[
= \frac{1}{2} \left\{ (\mu + 3p)^0 + 4q_c (\dot{u}^c - \xi^c) + 2\pi_{cd} u^c u^d - 2 (\mu + 3p - 2\Lambda) \left( -\psi + H_{cd} u^c u^d \right) \right\},
\]

\[
= \frac{1}{2} \left\{ (\mu + 3p)^0 + 4q_c \xi_{cd} u^c + 2\pi_{cd} u^c u^d - 2 (\mu + 3p - 2\Lambda) \left( -\psi + H_{cd} u^c u^d \right) \right\},
\]

hence (154) becomes:

\[
(\mu + 3p)^0 + 4q_c \xi_{cd} u^c + 2\pi_{cd} u^c u^d - 2 (\mu + 3p - 2\Lambda) \left( -\psi + H_{cd} u^c u^d \right) = 2(-3\mu_\psi + 3p_\psi + \mu_K). \quad (158)
\]

Similarly, for (156) we have:

\[
LHS = 2 \left[ \frac{3}{2} (\mu - p)^0 + 2q_c \dot{u}^c + \dot{\pi}_{cd} \dot{H}^{cd} + (\mu - p + 2\Lambda) (4\psi + \dot{\xi}^c u_c) + 2u_c \xi^c_{cd} q^d + 2\pi_{cd} H^{cd} \right],
\]

\[
= 2 \left[ \frac{3}{2} (\mu - p)^0 + 2q_c \dot{u}^c + \dot{\pi}_{cd} u^c u^d + (\mu - p + 2\Lambda) (3\psi + H_{cd} u^c u^d) + 2u^c q^d (\psi_{gcd} + H_{cd} - \xi_{dc}) \right.
\]

\[
+ 2\pi_{cd} H^{cd} \right],
\]

\[
= 2 \left[ \frac{3}{2} (\mu - p)^0 + 2q_c (\dot{u}^c - \xi^c) + \dot{\pi}_{cd} u^c u^d + (\mu - p + 2\Lambda) (3\psi + H_{cd} u^c u^d) + 2u^c q^d H_{cd} + 2\pi_{cd} H^{cd} \right],
\]

hence (156) becomes:

\[
3 (\mu - p)^0 + 4q_c \xi_{cd} u^c + 2\pi_{cd} u^c u^d + 2 (\mu - p + 2\Lambda) (3\psi + H_{cd} u^c u^d) + 4H_{cd} u^c q^d + 4\pi_{cd} H^{cd}
\]

\[
= 3 (\mu_\psi - 5p_\psi + p_K). \quad (159)
\]

Adding the two new equations we get:

\[
LHS = (\mu + 3p)^0 + 3 (\mu - p)^0 + 8q_c \xi_{cd} u^c + 4\pi_{cd} u^c u^d
\]

\[
+ 2 (3\mu - 3p + 6\Lambda + \mu + 3p - 2\Lambda) \psi
\]

\[
+ 2 (\mu - p + 2\Lambda - \mu + 3p + 2\Lambda) H_{cd} u^c u^d
\]

\[
+ 4H_{cd} u^c q^d + 4\pi_{cd} H^{cd},
\]

\[
= 4\mu + 8q_c \xi_{cd} u^c + 4\pi_{cd} u^c u^d + 8 (\mu + \Lambda) \psi
\]

\[
+ 8 (-\mu + \Lambda) H_{cd} u^c u^d + 4H_{cd} u^c q^d + 4\pi_{cd} H^{cd},
\]
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This equation expresses the derivative of $\mu$ along the collineation vector $\xi^a$. To find $\dot{p}$ we use (158) and replace $\dot{\mu}$. We have:

$$
3\dot{p} + \frac{1}{4} (-3\mu_\psi - 9p_\psi + 3p_K + 2\mu_K) - (2\mu + \Lambda)\psi \\
+ 2(p - \Lambda)H_{cd}u^e q^d - \pi_{cd}H^{cd} \\
+ 2q_c L_\xi u^c + \tilde{\pi}_{cd}u^e u^d - 2(\mu + 3p - 2\Lambda)(-\psi + H_{cd}u^e u^d) \\
= 2(-3\mu_\psi + 3p_\psi + \mu_K),
$$

This equation expresses the derivative of $\mu$ along the collineation vector $\xi^a$. To find $\dot{p}$ we use (158) and replace $\dot{\mu}$. We have:

$$
3\dot{p} - H_{cd}u^e q^d - \pi_{cd}H^{cd} \\
+ 2q_c L_\xi u^c + \tilde{\pi}_{cd}u^e u^d \\
+ 6(p - \Lambda)\psi - 2(2p + \mu - \Lambda)H_{cd}u^e u^d \\
= 2(-3\mu_\psi + 3p_\psi + \mu_K) - \frac{1}{4} (-3\mu_\psi - 9p_\psi + 3p_K + 2\mu_K),
$$

We concentrate now on (155):

$$
\text{LHS} = -(\mu + p) u_d h^d_e - \tilde{q}_d h^d_e + \tilde{\pi}_{cd}u^d h^c_e - \frac{1}{2} (\mu + 3p - 2\Lambda) u_c \xi_{cd} h^d_e \\
- q_c \xi_{cd} h^d_e + \frac{1}{2} (\mu - p + 2\Lambda) \xi_d h^d_e + (q_d h^d_e) \left( \dot{\xi}^c u_c \right) - h^d_e \pi_{cd} \dot{\xi}^c.
$$
The terms:
\[-\frac{1}{2} (\mu + 3p - 2\Lambda) u_c \xi_d^c h^d_e + \frac{1}{2} (\mu - p + 2\Lambda) \xi_d h^d_e = -\frac{1}{2} (\mu + 3p - 2\Lambda) H_{ed} u^c h^d_e + \frac{1}{2} (\mu + 3p - 2\Lambda) \xi_d h^d_e + \frac{1}{2} (\mu - p + 2\Lambda) \xi_d h^d_e,\]

hence:
\[LHS = -(\mu + p) h_{ed} L \xi u^d - \left[ \dot{q}_d h^d_e + q_e \xi_u^d h^d_e \right] + \pi_{de} u^d h^c_e - h^d_e \pi_{cd} \xi^c_e.\]

The terms:
\[
\dot{q}_d h^d_e + q_e \xi_u^d h^d_e = \dot{q}_d h^d_e + q_e (\psi \delta^d_a + H^d_a - \xi^d_u c) h^d_e, \\
= (\dot{q}_d - \xi^d_u c q_e) h^d_e + \psi q_e + H_{ed} q^e h^d_e, \\
= h^d_e L \xi q^d + \psi q_e + H_{ed} q^e h^d_e. 
\]

Also the Lie derivative:
\[L_\xi \pi_{cd} = \pi_{cd} \psi^f + \pi_{fd} \xi^f_c + \pi_{cf} \xi^f_d = \dot{\pi}_{cd} + \pi_{fd} \xi^f_c + \pi_{cf} \xi^f_d,\]

hence:
\[u^d h^d_e L_\xi \pi_{cd} = \pi_{cd} u^d h^c_e + \pi_{ef} \xi^f h^c_e.\]

Therefore:
\[LHS = -(\mu + p) h_{ed} L \xi u^d - h_{de} L_\xi q^d + u^d h^d_e L_\xi \pi_{cd} - \psi q_e - H_{ed} q^e h^d_e\]

and (155) becomes:
\[-(\mu + p) h_{ed} L \xi u^d - h_{de} L_\xi q^d + u^d h^d_e L_\xi \pi_{cd} - \psi q_e - H_{ed} q^e h^d_e = -(2q \dot{\psi} + qK),\]

or:
\[-h_{de} L_\xi q^d + u^d h^d_e L_\xi \pi_{cd} = (\mu + p) h_{ed} L \xi u^d + \psi q_e + H_{ed} q^e h^d_e = -(2q \dot{\psi} + qK).\]  

We continue with (157). We have:
\[LHS = 2 \left[ h^c_e \left( h^f_d - \frac{1}{3} h_{ab} h^f_d \right) \right] + \left[ \frac{1}{2} (\mu - p)^o h_{ef} + 2q \delta_{cd} h^c_e h^f_d + \pi_{cd} h^c_e h^f_d + (\mu - p + 2\Lambda) (\psi g_{dr} + H_{dr}) h^c_e h^f_d + 2q u_c \xi^c e_d h^d_e h^f_d + 2\pi_{cr} \xi^c_e d h^d_e h^f_d \right] 
\]

The terms:
\[2 \left[ h^c_e \left( h^f_d - \frac{1}{3} h_{ab} h^f_d \right) \right] \frac{1}{2} (\mu - p)^o h_{ef} = 0, \]

\[\left[ h^c_e \left( h^f_d - \frac{1}{3} h_{ab} h^f_d \right) \right] (\mu - p + 2\Lambda) \xi_d h^d_e h^f_d \]

\[= \left[ h^c_e \left( h^f_d - \frac{1}{3} h_{ab} h^f_d \right) \right] (\mu - p + 2\Lambda) (\psi g_{dr} + H_{dr}) h^d_e h^f_d, \]

\[= (\mu - p + 2\Lambda) \left[ h^c_e h^f_d - \frac{1}{3} h_{ab} h^f_d \right] H_{ef}, \]

\[= (\mu - p + 2\Lambda) \left[ h^c_e h^f_d - \frac{1}{3} h_{ab} h^f_d \right] H_{ef}, \]

\[= (\mu - p + 2\Lambda) \left[ h^c_e h^f_d - \frac{1}{3} h_{ab} h^f_d \right] H_{ef}, \]
i.e. the traceless part of the projection of the traceless tensor $H_{ab}$. The term:

$$
\left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] 2 q_{(a} \psi_{(g} c_{d)h^d_{e}h^f_{f}},
$$

$$
= \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] 2 q_{(a} H_{e}(c h^d_{e}h^f_{f}),
$$

$$
= \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] 2 q_{(a} u^e H_{e}h^d_{e},
$$

$$
= 2 q_{(a} h^d_{b)} H_{e}u^e - \frac{2}{3} \left( H_{e}u^e q^d \right) h_{ab}.
$$

The term:

$$
\left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] 2 p_{ca} \xi_{c} h^d_{e} h^f_{f},
$$

$$
= 2 \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] \pi_{c} \left( \psi g_{cd} + H_{e}h^d_{e}h^f_{f},
$$

$$
= 2 \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] (\psi_{c} h^d_{e} + \pi_{c} H_{e}h^d_{e}),
$$

$$
= 2 \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] \psi_{c} + 2 \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] \pi_{c} H_{e}h^d_{e},
$$

$$
= 2 \left[ \psi_{c} h^d_{e} - \frac{1}{3} \psi h^d_{e} \right] \pi_{c} H_{e}h^d_{e} + 2 \left[ \psi_{c} h^d_{e} - \frac{1}{3} \psi h^d_{e} \right] \pi_{c} H_{e}h^d_{e} - \frac{2}{3} h_{ab} (\pi_{cd} H_{cd}).
$$

Therefore we have:

$$
2 q_{(c} \psi_{d)} h^e_{c} h^f_{f} + \pi_{cd} h^e_{c} h^f_{f} + (\mu - p + 2A) \left[ h^e_{(a} h^f_{b)} - \frac{1}{3} h_{ab} h^{ef} \right] H_{e f}
$$

$$
+ 2 q_{(a} h^d_{b)} H_{c}u^e c - \frac{2}{3} \left( H_{e}u^e q^d \right) h_{ab}
$$

$$
+ 2 h^e_{(a} h^f_{b)} \pi_{c} H_{c}h^d_{e} - \frac{2}{3} h_{ab} (\pi_{cd} H_{cd})
$$

$$
= \frac{1}{2} \left[ \psi_{ab} + \pi_{Kab} \right].
$$

The gravitational field equations are Equations (158),(162),(159) and (163). With these field equations we have completed the scenario for the generic General Relativistic model.

**11. Application Example: The Bianchi I model**

We consider a model spacetime on which we make the following assumptions[3]:

**Geometric assumption:**

A diagonal Bianchi I space-time is a spatially homogeneous space-time which admits an Abelian group of isometries $G_3$, acting on spacelike hypersurfaces, generated by the spacelike KVs $\xi_1 = \partial_{x}$, $\xi_2 = \partial_{y}$, $\xi_3 = \partial_{z}$ and a timelike gradient KV the $u^a = \frac{\partial}{\partial t}$ which is normal to the homogeneous 3-d hypersurfaces.

In synchronous coordinates $\{t, x, y, z\}$ the above assumptions imply that the metric of this spacetime is:

$$
ds^2 = -dt^2 + A_2^2(t)(dx^a)^2,
$$

where the metric functions $A_1(t), A_2(t), A_3(t)$ are functions of the time coordinate only. When two of the functions $A_\mu(t)$ are equal (e.g. $A_2 = A_3$) the Bianchi I space-times reduce to
the important class of plane symmetric space-times (a special class of the Locally Rotational
Symmetric space-times [8],[9] which admit a $G_2$ group of isometries acting multiply transitively
on the spacelike hypersurfaces of homogeneity generated by the vectors $\xi_1, \xi_2, \xi_3$ and $\xi_4 = x^2\partial_3 - x^3\partial_2$.

For economy of writing in the following we write $A_{\mu}$ instead of $A^2_{\mu}(t)$. Furthermore we shall be
interested only in proper diagonal Bianchi I space-times (which in the following will be referred
for convenience simply as Bianchi I space-times), hence all metric functions are assumed to be
different and the dimension of the group of isometries acting on the spacelike hypersurfaces is
three.

The implications of the geometric assumption are:
1. The computation of the Ricci tensor

$$R_{tt} = -\frac{\ddot{A}_1 A_2 A_3 + \ddot{A}_2 A_1 A_3 + \ddot{A}_3 A_1 A_2}{A_1 A_2 A_3}, \quad (165)$$

$$R_{xx} = \frac{A_1 \left( \ddot{A}_1 A_2 A_3 + \ddot{A}_1 A_2 A_3 + \ddot{A}_1 A_3 A_2 \right)}{A_2 A_3}, \quad (166)$$

$$R_{yy} = \frac{A_2 \left( \ddot{A}_2 A_1 A_3 + \ddot{A}_2 A_2 A_3 + \ddot{A}_2 A_3 A_1 \right)}{A_1 A_3}, \quad (167)$$

$$R_{zz} = \frac{A_3 \left( \ddot{A}_3 A_1 A_2 + \ddot{A}_3 A_2 A_2 + \ddot{A}_3 A_3 A_1 \right)}{A_1 A_2}. \quad (168)$$

From the Ricci tensor we compute the Einstein tensor and then use Einstein field equations
to write the energy momentum tensor in terms of the metric functions $A_1(t), A_2(t), A_3(t)$ and
their derivatives. However that does not mean that we are able to discuss anything about the
physical variables (energy density, isotropic pressure etc.) because in order to do that we need
to have observers. The expression of $T_{ab}$ we find is the matter content of this spacetime, the
same for all observers.

2. The computation of the Weyl tensor

The Weyl tensor is important because it is involved in the second Bianchi identity. More
specifically form this tensor one computes the electric and the magnetic parts $E^{ab}, H^{ab}$ defined\(^{14}\)
in (45) which enter into the 1+3 decomposition of the second Bianchi identity given by equations
(47) - (50). We compute:

$$E_{xx} = \frac{A_1 \left( -2 \ddot{A}_1 A_2 A_3 + \ddot{A}_1 \ddot{A}_2 A_3 + \ddot{A}_1 \ddot{A}_3 A_2 + \ddot{A}_2 A_1 A_3 + \ddot{A}_3 A_1 A_2 - 2 \ddot{A}_2 \ddot{A}_3 A_1 \right)}{6 A_2 A_3}, \quad (169)$$

$$E_{yy} = \frac{A_2 \left( -2 \ddot{A}_2 A_1 A_3 + \ddot{A}_2 \ddot{A}_1 A_3 + \ddot{A}_2 \ddot{A}_3 A_1 + \ddot{A}_2 A_2 A_3 + \ddot{A}_3 A_1 A_2 - 2 \ddot{A}_1 \ddot{A}_3 A_2 \right)}{6 A_1 A_3}, \quad (170)$$

$$E_{zz} = \frac{-A_3 \left( 2 \ddot{A}_3 A_1 A_2 - \ddot{A}_1 \ddot{A}_3 A_2 - \ddot{A}_2 \ddot{A}_3 A_1 - \ddot{A}_1 \ddot{A}_2 A_3 - \ddot{A}_2 A_2 A_3 + 2 \ddot{A}_1 \ddot{A}_2 A_3 \right)}{6 A_1 A_2}. \quad (171)$$

\(^{14}\) $E_{ac} = C_{abcd} u^b u^d$, $H_{ac} = \frac{1}{2} \eta_{am,k} C_{kbc} u^m u^t$
and the magnetic part $H_{ab} = 0$.

**Observers.**

The choice of observers is open and independent of the choice of the symmetries (i.e. the model) spacetime. However as it has been noted the kinematic quantities they define must satisfy the propagation, the constraint equations and the Bianchi second identity.

We choose the observers (this is one choice, any other would do provided it satisfies the aforementioned identities) to be the ones defined by the time coordinate $t$ i.e. we take $u^a = \delta_0^a$ in the synchronous coordinate system.

The implications of this choice of observers are:

1. Kinematics

We 1+3 decompose $u_{a; b}$ and find the kinematic variables:

$$\theta = \ln(A_1 A_2 A_3),$$
$$\omega_{ab} = 0, \quad \sigma_{ab} = 0,$$

$$\sigma_{ab} = \frac{1}{3} \text{diag} \left( 0, A_1^2 \ln \left( \frac{A_1^2}{A_2 A_3} \right), A_2^2 \ln \left( \frac{A_2^2}{A_3 A_1} \right), A_3^2 \ln \left( \frac{A_3^2}{A_1 A_2} \right) \right).$$

a. The propagation equations give$^{15}$:

$$\dot{\theta} + \frac{1}{3} \dot{\theta}^2 + 2 \sigma^2 = \dot{A}_1 A_2 A_3 + \dot{A}_2 A_1 A_3 + \dot{A}_3 A_1 A_2, \quad (175)$$

$$-E_{st} = h_s^a h_t^b \dot{\sigma}_{ab} + \sigma_{sc} \sigma^c_t + \frac{2}{3} \sigma_{st} \dot{\theta} - \frac{2}{3} 2 \sigma^2 h_{st}. \quad (176)$$

b. The constraint equations give:

$$\frac{2}{3} h_s^c \dot{\theta}_{sc} = h_s^c h_t^b \sigma_{abc}, \text{(three equations)} \quad (177)$$

$$0 = -h_s^a h_t^b \sigma_{cd} \eta_{arc} u^r. \text{(five equations)} \quad (178)$$

Equations (175) - (178) must be satisfied identically by the kinematic quantities. It is easy to show that this is true for equations (175) and (176). Equation (177) is trivially satisfied because $\theta, \sigma_{ab}$ are functions of $t$ only. Equation (178) is also trivially satisfied because it contains derivatives of the components of $\sigma_{ab}$ along the space coordinates only (due to the term $\eta_{arc} u^r$).

We conclude that the propagation and the constraint equations do not give any conditions on the metric functions $A_1(t), A_2(t), A_3(t)$.

c. The Bianchi identities give the derivatives of $E_{st}, H_{st}$ therefore they do not add new constraints on the metric functions. They are only compatibility conditions.

d. The propagation of the kinematic quantities along the symmetry vectors.

From (76) and (77) and the fact the $\xi_{\mu}^a (\mu = 1, 2, 3)$ are Killing vectors (hence $\psi = 0, H_{ab} = 0$) we find:

$$V^a(u) = \dot{V}_a(u) = 0. \quad (179)$$

$^{15}$ Note that $\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab}$. For the case we are considering we calculate:

$$\sigma^2 = \frac{1}{3} \left[ \sum_{I=1}^3 \left( \dot{A}_I \dot{A}_I \right)^2 - \sum_{I \neq J=1}^3 \dot{A}_I \dot{A}_J \right]$$
From (119) and (120) we have taking into account the above results:
\[ L_\xi \theta = 0 \Rightarrow \theta = \theta(t), \]
\[ L_\xi \sigma_{ab} = 0 \Rightarrow \sigma_{ab} = \sigma_{ab}(t). \]
These equations give nothing new because we have already computed \( \theta, \sigma_{ab} \) and have found that they are functions of \( t \) only.

**Dynamics**

a. We compute the physical parameters for the chosen observers. We find:

\[ \mu = \frac{\dot{A}_1 \dot{A}_2 \dot{A}_3 + \ddot{A}_1 \dot{A}_2 A_2 + \ddot{A}_2 \dot{A}_3 A_3}{A_1 A_2 A_3}, \quad p = -\frac{2 \ddot{A}_2 A_1 A_3 + 2 \ddot{A}_3 A_1 A_2 + \ddot{A}_2 \dot{A}_3 A_1 + 2 \ddot{A}_1 A_2 A_3 + \ddot{A}_1 \dot{A}_3 A_2 + \ddot{A}_1 \dot{A}_2 A_3}{3 A_1 A_2 A_3}, \]

\[ \pi_{xx} = -\frac{A_1}{3 A_2 A_3} \left( -2 \ddot{A}_1 A_2 A_3 + \ddot{A}_2 A_1 A_3 + \ddot{A}_3 A_1 A_2 + 2 \ddot{A}_2 \dot{A}_3 A_1 - \ddot{A}_3 \dot{A}_1 A_2 - \ddot{A}_1 \dot{A}_2 A_3 \right), \]

\[ \pi_{yy} = -\frac{A_2}{3 A_1 A_3} \left( -2 \ddot{A}_2 A_1 A_3 + \ddot{A}_3 A_1 A_2 + \ddot{A}_1 A_2 A_3 + 2 \ddot{A}_1 \dot{A}_3 A_2 - \ddot{A}_2 \dot{A}_3 A_1 - \ddot{A}_1 \dot{A}_2 A_3 \right), \]

\[ \pi_{zz} = -\frac{A_3}{3 A_1 A_2} \left( -2 \ddot{A}_3 A_1 A_2 + \ddot{A}_2 A_1 A_3 + \ddot{A}_1 A_2 A_3 + 2 \ddot{A}_1 \dot{A}_3 A_2 - \ddot{A}_2 \dot{A}_3 A_1 - \ddot{A}_1 \dot{A}_2 A_3 \right). \]

The momentum transfer vector is \( q^a = 0 \). We note that the second equation can be written:

\[ \frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3} = -\frac{1}{2} (\mu + 3p). \]

We introduce the notation:

\[ I_2 = \frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} = -\frac{1}{2} (\mu + 3p), \]

\[ I_1 = \frac{\ddot{A}_1}{A_1 A_2} + \frac{\ddot{A}_1}{A_1 A_3} + \frac{\ddot{A}_2}{A_2 A_3} = \mu, \]

and we have:

\[ \mu = I_1, \]

\[ p = -\frac{1}{3} (2 I_2 + I_1), \]

\[ \pi_{xx} = \frac{A_1^2}{3} \left( \frac{\ddot{A}_1}{A_1} - \frac{\ddot{A}_2}{A_2 A_3} - I_2 + I_1 \right), \]

\[ \pi_{yy} = \frac{A_2^2}{3} \left( \frac{\ddot{A}_2}{A_2} - \frac{\ddot{A}_1}{A_1 A_3} - I_2 + I_1 \right), \]

\[ \pi_{zz} = \frac{A_3^2}{3} \left( \frac{\ddot{A}_3}{A_3} - \frac{\ddot{A}_1}{A_1 A_2} - I_2 + I_1 \right). \]

---

\(^{16}\)This is expected from the symmetries of the metric and the non-degeneracy of the \( T_{ab} \). Note that both \( R_{ab} \) and \( T_{ab} \) are of the same form as the metric. This is to be expected because they can be considered as metrics (a metric is a symmetric tensor of type (0, 2) and nothing more or less) and they admit the same KVs with the metric.
b. Conservation equations.

From (130) and (131) we have for our case:

\[ \dot{\mu} + (\mu + p)\theta + \pi^{ab}_{\sigma} = 0, \]
\[ h^c_\alpha (p_c + \pi^d_{c; d}) = 0. \]  

Equation (192) is trivially satisfied because all quantities are functions of \( t \). Equation (191) is also trivially satisfied if we replace the expressions of \( \mu, p, \theta, \pi_{ab} \) from the corresponding expressions.

We see that there are no field equations to solve! Indeed we have solved them in terms of three arbitrary functions which are the metric functions \( A_1(t), A_2(t), A_3(t) \)! Therefore for the observers \( u^a = \delta^a_0 \) we have solved the problem completely\(^{17}\).

We are free to select special solutions from the three parameter family of solutions we have found by imposing various requirements. In the following section we make one such requirement and consider those Bianchi I spacetime which for the observers we have chosen give rise to a special type of mater which we call string fluid. Needless to say that one could consider any other requirement and select other types of matter for the same spacetime and the same observers. Every specification / condition on the metric functions \( A_1(t), A_2(t), A_3(t) \) will produce a model (physical or not).

12. The string fluid

The connection between strings and vortices is well known [10], [11], [12], [13]. In particular a geometric or Nambu string is a two-dimensional timelike surface in spacetime. Letelier [14] has considered a fluid represented by a combination of geometric strings with particles attached to them so that both have the same four velocity. He called such a fluid a string fluid and he studied the gravitational field it produces in given spacetime backgrounds. In a series of papers various authors [15], [16], [19], [17], [18], [20] [21] have considered various types of collineations for a string fluid and derived the conditions which must be satisfied in order the string fluid to admit a given collineation.

In this work we consider the string fluid as an imperfect fluid and show that the energy momentum tensor of a string fluid corresponds to an anisotropic fluid with vanishing heat conduction. We use the 1+3 to show that the string fluid is in fact the simplest anisotropic fluid possible. Applying previous studies concerning the effects of collineations on anisotropic fluids we study many types of symmetries of a string fluid and give the results in a unified and geometric/systematic way. In this manner we recover, correct and systematize the results of the many previous works and most important we set the methodology for a similar study of other anisotropic fluids.

The energy momentum tensor for a string fluid with particles attached to the strings is given by the expression\(^{18}\) [12],[13]:

\[ T_{ab} = \rho (u_a u_b - n_a n_b) + q p_{ab}, \]  

where:

- \( \rho = \rho_p + \rho_s \) is the sum of the mass density of the strings (\( \rho_s \)) and the mass density of the particles (\( \rho_p \)),
- \( u^a \) is the common four velocity (\( u^a u_a = -1 \)) of the string and the attached particles
- \( n^a \) is a unit spacelike vector (\( n^a n_a = 1 \)) normal to \( u^a \) (\( u^a n_a = 0 \)), which specifies the direction of the string (and the direction of anisotropy of the string fluid)
- \( q \) is a parameter contributing to the dynamic and kinematic properties of the string

\(^{17}\) For another class of observers we could have more constraint equations which would have to be solved. Obviously in this case all dynamic and kinematic variables will (in general) be different.

\(^{18}\) This expression is found form equation (2.28a) of [12] if we set \( \rho = -\sigma \) and \( \pi \rightarrow -\pi \).
- \( p_{ab} = h_{ab} - n_a n_b \) is the screen projection operator defined by the vectors \( u^a, n^a \).

By rewriting the energy momentum tensor as:

\[
T_{ab} = \rho u_a u_b + \frac{1}{3} (2q - \rho) h_{ab} + (q + \rho) \left( \frac{1}{3} h_{ab} - n_a n_b \right),
\]

(194)

(or otherwise) we compute its 1+3 decomposition. It follows that for a string fluid:

\[
\mu = \rho, \quad p = \frac{1}{3} (2q - \rho), \quad q^a = 0, \quad \pi_{ab} = (q + \rho) \left( \frac{1}{3} h_{ab} - n_a n_b \right).
\]

(195)

We conclude that a string fluid is an anisotropic fluid with vanishing heat flux. Furthermore we note that \( q + \rho \neq 0 \) otherwise the string fluid reduces to a perfect fluid with energy momentum tensor \( T_{ab} = q g_{ab} \). This fluid has the unphysical equation of state \( \mu + p = 0 \).

From the above we note that the structure of the energy momentum tensor is compatible with the general expression (180) - (184) therefore the model of a string fluid is included in the model of Bianchi I spacetime with observers \( u^a = \delta_0^a \) we considered in the previous section. To find the specific metric functions \( A_1(t), A_2(t), A_3(t) \) which correspond (or select) the string fluid model we equate the dynamical parameters of (180) - (184) with those of (195). We assume the string direction to be along the \( x \)-axis i.e. we take \( A_1 \neq 0 \):

\[
n^a = \frac{1}{\sqrt{A_1}} \delta_1^a.
\]

(196)

From the mass density and the pressure we find:

\[
\mu + 3p = 2q \Rightarrow I_2 = -q,
\]

(197)

\[
\pi_{ab} = (q + \rho) \left( \frac{1}{3} h_{ab} - A_1^2 \delta_a^1 \delta_b^1 \right) = (\rho - I_2) \left( \frac{1}{3} h_{ab} - A_1^2 \delta_a^1 \delta_b^1 \right).
\]

In the coordinates we use:

\[
h_{ab} = diag(0, A_1^2, A_2^2, A_3^2).
\]

(198)

Replacing in the expression of \( \pi_{ab} \) we find:

\[
\pi_{ab} = \frac{1}{3} (\rho - I_2) diag(0, -2A_1^2, A_2^2, A_3^2).
\]

(199)

Equating the two expressions of \( \pi_{ab} \), (188)–(190) and (199) we find the field equations:

\[
\frac{\ddot{A}_1}{A_1} - \frac{\dot{A}_2 A_3}{A_2 A_3} = -\rho + I_2 = -(\rho + q),
\]

(200)

\[
\frac{\ddot{A}_2}{A_2} - \frac{\dot{A}_1 A_3}{A_1 A_3} = 0,
\]

(201)

\[
\frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1 \dot{A}_2}{A_1 A_2} = 0.
\]

(202)

The last three equations are dependent (one follows form the other two). Eventually we have the following system of four simultaneous equations for the five unknowns \( A_1(t), A_2(t), A_3(t), q, \rho \) :

\[
\frac{\dot{A}_1}{A_1} + \frac{\dot{A}_2}{A_2} + \frac{\dot{A}_3}{A_3} = \rho,
\]

(203)
\[
\begin{align*}
\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} &= -q, \\
\frac{\ddot{A}_2}{A_2} - \frac{\dot{A}_1\dot{A}_3}{A_1A_3} &= 0, \\
\frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1\dot{A}_2}{A_1A_2} &= 0.
\end{align*}
\]

This means that we still have the freedom to specify one more condition. This condition could be an equation of state (in the board sense).

Note that the kinematic variables are not effected. Therefore when we take the equation of state and we determine the functions \(A_1(t), A_2(t), A_3(t)\) then we can compute the kinematic variables \(\theta, \sigma_{ab}\) and draw conclusions on the kinematics of the string fluid.

### 12.1. An alternative energy momentum tensor

Instead of the energy momentum tensor (193) Letelier [14] considers the energy momentum tensor:

\[
T_{ab} = \rho u_a u_b - \lambda n_a n_b.
\]

This tensor reduces to the one we have considered if we set \(q = 0\) and \(\rho = \lambda\). In general it is different to the one we have discussed above. However our analysis applies the same. Considering the same observers and the same direction \(n^a\) we have the following 1+3 decomposition of \(T_{ab}\):

\[
T_{ab} = \rho u_a u_b - \frac{1}{3}\lambda h_{ab} + \lambda(\frac{1}{3}h_{ab} - n_a n_b),
\]

from which follows:

\[
\mu = \rho, p = -\frac{1}{3}\lambda, q_a = 0, \quad \sigma_{ab} = \frac{\lambda}{3}(0, -2A_1^2, A_2^2, A_3^2).
\]

Then:

\[
I_2 = -\frac{1}{2}(\mu + 3p) = -\frac{1}{2}(\rho - \lambda).
\]

Equating the two expressions of \(\sigma_{ab}\) (188)- (190) and (210) we find the field equations:

\[
\begin{align*}
\frac{\ddot{A}_1}{A_1} - \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} &= -\lambda + I_2 = -\frac{1}{6}(5\lambda + 3\rho), \\
\frac{\ddot{A}_2}{A_2} - \frac{\dot{A}_1\dot{A}_3}{A_1A_3} &= I_2 = \frac{1}{2}(\frac{\lambda}{3} - \rho), \\
\frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1\dot{A}_2}{A_1A_2} &= -\frac{\dot{A}_2}{A_2} - \frac{\dot{A}_1\dot{A}_3}{A_1A_3}.
\end{align*}
\]

The last three equations are dependent (one follows form the other two). Eventually we have again the following system of four simultaneous equations for the five unknowns \(A_1(t), A_2(t), A_3(t), \lambda(t), \rho(t)\):

\[
\begin{align*}
\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} &= 0, \\
\frac{\ddot{A}_2}{A_2} - \frac{\dot{A}_1\dot{A}_3}{A_1A_3} &= 0, \\
\frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1\dot{A}_2}{A_1A_2} &= 0.
\end{align*}
\]
\[
\begin{align*}
\frac{\ddot{A}_1}{A_1} + \frac{\ddot{A}_2}{A_2} + \frac{\ddot{A}_3}{A_3} &= \frac{1}{2}(\lambda - \rho), \\
\frac{\ddot{A}_2}{A_2} - \frac{\dot{A}_1}{A_1} \frac{\dot{A}_3}{A_3} &= \frac{1}{2}(\lambda - \rho), \\
\frac{\ddot{A}_3}{A_3} - \frac{\dot{A}_1}{A_1} \frac{\dot{A}_2}{A_2} &= \frac{1}{2}(\lambda - \rho).
\end{align*}
\]

We note that in this approach one has still the freedom to consider an extra condition / equation of state. The solution of the field equations will be the starting point of making Physics in this model.

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[3] Greek indices take the space values 1, 2, 3 and Latin indices the space-time values 0, 1, 2, 3. We define the sign of the curvature tensor form the identity
\[
A^a_{bc} - A^a_{cb} = R^d_{bcd} A^d
\]
or
\[
A^a_{bc} - A^a_{cb} = R^d_{dabc} A^d.
\]

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