Perfect fluid and scalar field in the Reissner-Nordström metric

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We describe spherically symmetric steady-state accretion of perfect fluid in the Reissner-Nordström metric. We present analytic solutions for accretion of a fluid with the linear equations of state and of the Chaplygin gas. It is also shown that, under reasonable physical conditions, there is no steady-state accretion of a perfect fluid onto a Reissner-Nordström naked singularity. Instead, a static atmosphere of fluid is formed. We discuss a possibility of violation of the third law of black hole thermodynamics for a phantom fluid accretion.

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I. INTRODUCTION

The problem of matter accretion onto compact objects in Newtonian gravity was formulated in the self-similar manner by Bondi [1]. In the framework of General Relativity steady-state spherical symmetric flow of test gas onto a Schwarzschild black hole was investigated by Michel [2]. Detailed studies of spherically symmetric accretion of different types of fluids onto black holes were further undertaken in a number of works [3], see also a review [4].

In this paper, we study perfect fluids and scalar fields in the Reissner-Nordström (RN) metric. We describe spherically symmetric steady-state accretion of a test perfect fluid with a general equation of state onto a non-rotating charged black hole. We find analytic solutions for accretion of a perfect fluid with the linear equation of state and of the Chaplygin gas. A static atmosphere of fluids around a naked singularity is described in Sec. V. Approaching of a black hole to the extreme state by accretion of phantom fluid and a possibility of violation of the third law of thermodynamics is discussed in Sec. VI. We conclude in Sec. VII.

II. STEADY-STATE ACCRETION

In this section we study spherically symmetric steady-state accretion of a test perfect fluid with a general equation of state in the RN metric. Here we closely follow the approach of Michel [3] for a similar study performed for accretion of a gas in the Schwarzschild metric.

A similar result for Kerr naked singularity was found in [6] using numerical methods.
The RN metric reads,
\[ ds^2 = f dt^2 - f^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2), \]
where
\[ f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]
Here \( M \) is a black hole (or naked singularity) mass, and \( Q \) is its total charge. It is convenient to introduce dimensionless coordinates,
\[ \tau \equiv \frac{t}{M}, \quad x \equiv \frac{r}{M}, \]
and dimensionless electric charge of the black hole \( e \equiv Q/M \). In the case \( e^2 < 1 \) the equation \( f(x) = 0 \) has two roots,
\[ x_\pm = 1 \pm \sqrt{1 - e^2}. \]
The larger root, \( x = x_+ \), corresponds to the event horizon of the RN black hole, and \( x = x_- \) is the so-called Cauchy (or inner) horizon. In the opposite case, \( e^2 > 1 \), the RN metric describes a naked singularity without event horizon. The marginal case \( e^2 = 1 \) corresponds to an extreme black hole.

The energy-momentum of a perfect fluid reads,
\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu - pg_{\mu\nu}, \]
where \( \rho \) and \( p \) are the fluid energy density and pressure respectively, and \( u^\mu = dx^\mu/ds \) is the fluid four-velocity with normalization condition, \( u^\mu u_\mu = 1 \). We assume that the pressure is an arbitrary function of the density alone, \( p = p(\rho) \). To find integrals of motion, we use the projection of the equation for conservation of the energy-momentum tensor onto the 4-velocity, \( u_\mu T^{\,\mu\nu} = 0 \). This gives the continuity equation,
\[ u^\mu \rho_{,\mu} + (\rho + p) u^\mu \rho_{,\mu} = 0. \]
Integrating \( \rho \) once, we find the following integral of motion (the energy conservation):
\[ \mu x^2 n = -A, \]
where
\[ n = \exp \left[ \int_{\rho_{\infty}}^\rho \frac{d\rho'}{\rho' + p(\rho')} \right], \]
\( u = dr/ds < 0 \) in the case of inflow motion (accretion), and \( A > 0 \) is the constant of integration, which is related to the radial energy flux.

Integration of the time component of the conservation law, \( T^{\,\mu\nu}_{\,\mu\nu} = 0 \), gives another integral of motion (the relativistic Bernoulli equation):
\[ (\rho + p) (f + u^2)^{1/2} x^2 u = C_1, \]
where \( u = dr/ds \) and \( C_1 \) is the constant of integration. From \( \rho \) and \( \rho_{\infty} \) one can easily obtain:
\[ \frac{(\rho + p)}{n} (f + u^2)^{1/2} = C_2, \]
where
\[ C_2 \equiv \frac{-C_1}{A} = \frac{\rho_{\infty} + p(\rho_{\infty})}{n(\rho_{\infty})}, \]
here \( \rho_{\infty} \) is the energy density at infinity. Equations \( \rho \) and \( \rho_{\infty} \) along with the equation of state \( p = p(\rho) \) form a closed system for accretion onto a RN black hole (or naked singularity). This system is to be supplied by the appropriate boundary conditions. The obtained system of equations describes accretion of a perfect fluid with a general equation of state \( p = p(\rho) \), and may be applied, in particular, to accretion of the Chaplygin gas \( Q \) or dark energy described by the generalized linear equation of state \( A \).

The constant \( C_2 \) is fixed by the boundary condition at infinity. Fixing of \( A \) in \( \rho \) and, respectively, the flux is more tricky. It is provided by a physical requirement to have a smooth transition through the critical sound point (see details, e. g. in \( \rho \)). The resulting solution should be continuous from infinity down to the black hole horizon. Following \( \rho \), we find relations at the critical point,
\[ u^2 = \frac{x_s - e^2}{2x_s^2}, \quad c_s^2(\rho_s) = \frac{x_s - e^2}{2x_s^2 - 3x_s + e^2}, \]
where \( c_s(\rho) \equiv (\partial p/\partial \rho)^{1/2} \) is the sound speed, and the subscript ‘s’ indicates that the values are taken at the critical point. From \( \rho \) one can find,
\[ x_s^\pm = \frac{1 + 3c_s^2}{4c_s^2} \left\{ 1 \pm \left[ 1 - \frac{8c_s^2(1 + c_s^2)}{(1 + 3c_s^2)^2} \right]^{1/2} \right\}, \]
where \( c_s \equiv c_s(x_s) \). Critical points exist only if
\[ e^2 < \frac{(1 + 3c_s^2)^2}{8c_s^2(1 + c_s^2)}. \]
It is worthwhile to note that in contrast to the case of a Schwarzschild black hole, there are formally two different critical points, corresponding to the plus and the minus signs in \( \rho \). Note also that for \( e \to 0 \) we find \( x_s^- \to 0 \).

Depending on the values of \( e \) and \( c_s \) one can identify the following five cases:

- \( e < 1, e^2 < 1 \) \( (c_s^2 = 1) \). In this case the event and the Cauchy horizons exist, \( x^+ > x^- \), as well as both critical points; the outer critical point is outside the event horizon, \( x^+_s > x^+ \) \( (x^+_s = x^+) \), the inner critical point is between the event and the Cauchy horizons, \( x^- < x^-_s < x^+ \) \( (x^-_s = x^-) \).

- \( e < 1, e^2 > 1 \). Similar to the previous case the event and the Cauchy horizons, and both critical points
exist; however in this case the outer critical point is in between the event and the Cauchy horizons, \( x^{-} < x^{*} < x^{+} \) (\( x_{+} = x_{-} = x_{0} \)); the inner critical point is inside the Cauchy horizon, \( x^{*} < x^{+} \).

- \( e = 1 \). The event and the Cauchy horizons coincide, \( x^{+} = x^{-} = 1 \) and both critical points exist: in the subluminal case \( x_{+}^{*} > 1 \) and \( x_{-}^{*} = 1 \); for a stiff fluid, \( 2 = 1 \), we find \( x_{+}^{*} = 1 \); in the superluminal case, \( x_{+}^{*} = 1 \) and \( x_{-}^{*} < 1 \);

- \( 1 < e < 3/(2\sqrt{2}) \). The RN metric describes a naked singularity (the horizons are absent). Critical points exist for two different branches, namely, when

\[
\begin{align*}
2 &\leq \frac{-4e^{2} + 3 - 4e\sqrt{e^{2} - 1}}{8e^{2} - 9} \quad \text{(subluminal)}, \\
2 &\geq \frac{-4e^{2} + 3 + 4e\sqrt{e^{2} - 1}}{8e^{2} - 9} \quad \text{(superluminal)}.
\end{align*}
\]

- \( e \geq 3/(2\sqrt{2}) \). The RN metric describes a naked singularity. In contrast to the previous case, the critical points exist only for a subluminal branch \( \rho_{*} \).

In Fig. 1 the critical radii as functions of the sound speed are shown for several values of \( e \).

Substituting the value of \( x_{+}^{*} \) from (8) into the first relation in (7) and then, in turn, substituting \( x_{*} \) and \( u_{*} \) expressed in terms of \( c_{*} \) into (9) one finds the closed equation for \( \rho \) at the critical point,

\[
\frac{\rho_{s} + p_{s}}{\rho_{\infty} + p_{\infty}} \frac{n_{\infty}}{n_{s}} = \frac{1 + 3c_{s}^{2} + D}{\sqrt{2(1 + 3c_{s}^{2} + 4c_{s}^{2}(c_{s}^{2} - 1) + D)}}.
\]

For \( e = 0 \) Eqs. (9) and (10) reduces to the equation for the critical point in the case of the Schwarzschild black hole [3].

The black hole mass changes at rate \( \dot{M} = -4\pi r^{2}T_{0}^{r} \) due to fluid accretion. With the help of (11) and (10) this expression can be written as follows,

\[
\dot{M} = 4\pi AM^{2}[\rho_{\infty} + p_{\infty}].
\]

From this equation it is clear that accretion of phantom energy, defined by the condition \( \rho_{\infty} + p(\rho_{\infty}) < 0 \), is always accompanied with decrease of the black hole mass. This is in accordance with previous findings [5]. We would like to stress that the result is valid for any equation of state \( p = p(\rho) \) with \( \rho + p(\rho) < 0 \).

FIG. 1: The outer critical radius \( x^{+} \) (thick lines) and inner critical radius \( x^{*} \) (thin lines) are shown for several values of the electric charge \( e = Q/M \). Note that the outer critical radius coincides with the event horizon, \( x^{+} = 1 \), for the extreme black hole (\( e = 1 \)) in the case of \( c_{*} \geq 1 \).

III. PERFECT FLUID AS A SCALAR FIELD

It is well known that the dynamics of relativistic perfect fluid in the absence of vorticity can be described in terms of a scalar field. In particular, stiff fluid corresponds to a canonical massless scalar field. In order to describe more complicated equations of state one should introduce a generalized non-canonical scalar-field Lagrangian of the form,

\[
L = L(X), \quad X = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi.
\]

The energy-momentum tensor corresponding to the Lagrangian (11) is

\[
T_{\mu\nu} = L_{X}\nabla_{\mu}\phi\nabla_{\nu}\phi - g_{\mu\nu}L,
\]

where the subscript \( X \) denotes the derivative with respect to \( X \). The correspondence between scalar field and a perfect fluid with energy momentum tensor (2) is achieved by the following identifications (see, e. g. [10]):

\[
u_{\mu} = \frac{\nabla_{\mu}\phi}{\sqrt{2X}}.
\]

where the pressure \( p \) coincides with the Lagrangian density of the scalar field, \( p = L(X) \), and the energy density is

\[
\rho(X) = 2XL_{X} - L.
\]
The sound speed can be expressed as
\[ c_s^2 = \frac{\mathcal{L}_X}{\rho X} = \left(1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}\right)^{-1}. \]

Apart from the energy density \( \varepsilon \) and pressure \( p \) one can formally define “particle number density”,
\[ n \equiv \exp\left(\int \frac{d\rho}{\rho + p}\right) = \sqrt{X}\mathcal{L}_X. \]

and the enthalpy
\[ h \equiv \frac{\rho + p}{n} = 2\sqrt{X}. \]

Equations of motion following from (11) are
\[ \partial_\mu \left(\sqrt{-g}\mathcal{L}_X g^{\mu\nu} \partial_\nu \phi \right) = 0. \quad (12) \]

A steady-state flow is described by the ansatz,
\[ \phi(t,x) = a_\infty t + \psi(x), \quad (13) \]

where the constant \( a_\infty \) defines the “cosmological” value of \( \phi \) at spatial infinity. One can easily find that for the ansatz (13),
\[ X = \frac{1}{2} \left(\frac{a_\infty^2}{f} - f \psi'^2\right), \]

and the equation of motion (12) can be integrated once to give
\[ x^2 f \mathcal{L}_X \psi'(x) = \sqrt{2}A. \quad (14) \]

Equation (14) is in fact another form of (8), written in terms of the scalar field. Moreover, Eq. (13) is an algebraic equation on function \( \psi' \). Thus a general solution will contain \( A \), which should be determined via an analog of the critical point (17). From (12), one can find \( \psi'' \) in terms of \( \psi' \) (this expression also contains \( \mathcal{L}_X \) and \( \mathcal{L}_{XX} \)). The critical point is found then by equating of both theominator and the denominator of the obtained expression to zero. As a result, one obtains,
\[ \psi'^2 = \frac{a_\infty^2}{f^2(x_+ f_+^2 + 4f_+)} \cdot f_+ \psi'^2 \mathcal{L}_{XX} = \mathcal{L}_X. \quad (15) \]

which is another form of (17). Now, we have three equations (13), (15) which can be used to find \( \psi' \), \( x_+ \) and \( A \). This procedure is fully equivalent to the fixing of the critical point for the accretion of fluid. This description is very useful for some particular tasks.

In particular, let us analyze (14) in the limit \( x \to 0 \).
We have,
\[ 2X \sim \frac{x^2}{c^2} B^2 \sim \frac{c^2}{x^2} \psi'^2. \]

Since for the fluid \( X > 0 \), this leads to
\[ X \to 0, \psi'^2 \to 0, \quad x \to 0. \quad (16) \]

On the other hand, we find from (14),
\[ \mathcal{L}_X \psi' \to \text{const}, \quad x \to 0. \quad (17) \]

Combining (16) and (17) we conclude that a fluid reaches \( x = 0 \) during a steady-state accretion only if \( \mathcal{L}_X \to \infty \) for \( X \to 0 \). This means, in particular, that a fluid, described by the linear equation of state with \( \alpha \leq 1 \), does not reach the central singularity at \( x = 0 \), if \( \varepsilon \neq 0 \).

IV. ACCRETION ONTO BLACK HOLE

In this section we present and discuss several analytic solutions for steady-state accretion of a perfect fluid onto a charged black hole.

A. Linear equation of state

As the first example we consider the linear equation of state,
\[ p = \alpha (\rho - \rho_0), \quad (18) \]

where \( \alpha \) and \( \rho_0 \) are constants. This equation was introduced in [5] (see also [9]) to avoid hydrodynamical instability for a perfect fluid with the negative pressure. The constant \( \alpha \) in (18) determines square of the sound speed of small perturbations, \( \alpha = c_s^2 \), and it must be positive. Note, that (18) can be considered as the linear...
approximation to a general nonlinear equation of state $p = p(\rho)$ around some point $\rho = \rho_1$. Therefore, the results of this section can be applied to a generic equation of state, provided that $|\rho - \rho_1|$ is small enough.

Using (7) and (3), one can calculate from (3) the dimensionless constant $A$ for the linear equation of state,

$$A = \alpha^{1/2} x_+^2 \left( \frac{2\alpha x_+^2}{x_+ - e^2} \right)^{1-\alpha}.$$

(19)

The velocity and the energy density as functions of the radius is determined by solving (4) and (6),

$$f + u^2 = \left( - \frac{ux^2}{A} \right)^{2\alpha} \frac{\rho + p}{\rho_\infty + p_\infty} = \left( \frac{A}{ux^2} \right)^{1+\alpha}.$$  (20)

It is possible to express the solutions of the above equations through known analytical functions for specific values of $\alpha$, namely, $\alpha = 1/4, 1/3, 1/2, 2/3, 1, 3/2$ and 2. Below we present solutions corresponding to some particular values of $\alpha$.

Let us first consider the case of the stiff fluid: $\alpha = 1$. For the radial velocity and the energy density we find, respectively,

$$u^2 = \frac{(x-x_-)x_+^4}{(x+x_+)(x^2+x_+^2)x^2},$$

$$\rho = \frac{\rho_0}{2} + \left( \rho_\infty - \frac{\rho_0}{2} \right) \frac{(x+x_+)(x^2+x_+^2)}{(x-x_-)x^2}.$$  

The density at the horizon,

$$\rho_+ = \frac{\rho_0}{2} + \left( \rho_\infty - \frac{\rho_0}{2} \right) \frac{2x_+}{\sqrt{1-e^2}}.$$  (21)

Note, that the energy density diverges at the event horizon $x_+$ of an extreme black hole, $e = 1$.

The solutions for thermal photon gas: $\alpha = 1/3$ can be found accordingly. Indeed, radial distribution of the energy density in this case reads

$$\rho = \frac{\rho_0}{4} + \left( \rho_\infty - \frac{\rho_0}{4} \right) \left( \frac{1+2z}{3f} \right)^2,$$

where

$$z = \begin{cases} \cos \frac{2\pi - \beta}{3}, & x_+ \leq x \leq x_s; \\ \cos \frac{\beta}{3}, & x > x_s; \end{cases}$$

and

$$\beta = \arccos \left( 1 - \frac{27}{2} \frac{A^2 f^2}{x^4} \right).$$

Phantom energy in this particular case corresponds to the choice $\rho_0 > 4\rho_\infty$. At the event horizon $x = x_+$ we have,

$$\rho_+ = \rho(x_+) = \frac{\rho_0}{4} + \left( \rho_\infty - \frac{\rho_0}{4} \right) \frac{A^2}{x_+^4}.$$  

It is also worth to study the case of a superluminal fluid. As an example we take $\alpha = 2$. Now the inflow consists of two hydrodynamical branches:

$$u_{1,2} = \frac{A^2}{\sqrt{2} x^4} \sqrt{1 \pm \sqrt{1 + 4 \frac{x^3}{A^4}}}, \quad \rho_{1,2} = \left( \frac{A}{u_{1,2} x^2} \right)^3.$$  (22)

At the outer and inner horizons we find

$$u_1(x_\pm) = \frac{A^2}{x_\pm^4}, \quad u_2(x_\pm) = 0.$$  

The energy density diverges at $r_-$, and the solution does not exist for $r < r_-$. The behavior of superluminal fluids ($c_s > 1$) is quite unusual. Apart from the transonic solution (22), there is an infinite family of regular at $r > 0$ solutions, parameterized by $A$, with $A > A_c$. These solutions consist of the only one hydrodynamical branch, and the sonic horizon is absent. Using a solution with $A > A_c$ one can probe the singularity of a black hole with small perturbations. In fact, it is not clear how to choose the “correct” physical solution for a superluminal fluid.

Contrary to accretion of a superluminal fluid, for a subluminal fluid the solution exists only above some minimal radius $r_{\min}$, $0 < r_{\min} < r_-$, so that the inflowing fluid does not reach the central singularity (see Sec. III). The energy density of the fluid has the maximum at $r_{\min}$. For example, $r_{\min} = 2(\sqrt{2} - 1)M$ and $\rho(r_{\min}) = (8/3)^3(12\sqrt{2} + 17)\rho_\infty$ in the case of accretion of fluid with $\alpha = 1/3$ (thermal photon gas) onto the extremally charged black hole.

Note that similar behavior was found for geodesic motion of test particles with a nonzero mass [11, 12] in the RN metric. In particular, the radial component of the 4-velocity for parabolic radial geodesics (i.e. for particle with zero velocity at infinity) is,

$$u_p(x) = \pm \frac{\sqrt{2x - e^2}}{x}.$$  (24)

The particle bounces at $r_{\min} = Q^2/(2M)$ and $u_p(r_{\min}) = 0$ but $|u'_p(r_{\min})| = \infty$ according to (24).

The corresponding solutions for an accreting subluminal fluid are singular at $r = r_{\min}$, namely, $u'(r_{\min}) = \infty$.

2 One can argue, however, that all these problems are due to the unphysical choice of equation of state [13]. Note that $p \to 0$ as $x \to 0$. The equation of state (13) is unphysical for $\alpha \neq 1$ at $p \to 0$, due to the pathological behavior of equations of motion for $\psi$ in the limit $p \to 0$, as it was shown in [10]. To cure the model (13) with $\alpha \neq 1$ for small densities, one can modify equation of state, such that $p \to \rho$ as $\rho \to 0$. For example, in terms of the scalar filed the following Lagrangian

$$L = (\sigma + X)^{3/4} - \sigma,$$  (23)

with $\sigma$ being small, satisfies this requirement, giving also “superluminal” fluid $p = 2\rho$ for large densities.
FIG. 3: Radial 3-velocity \( v(x) \) for the inflowing fluid \((\alpha = 1/2, e = 0.999)\) with respect to the local static observers in the \(R\)-regions \( r_- < r < r_+ \) and \( 0 < r < r_- \). In the \(T\)-region, \( r_- < r < r_+ \), the local static observers do not exist, and thus the 3-velocity is undefined.

and \( \rho'(r_{\text{min}}) = -\infty \) (although both 4-velocity and the energy density are finite at \( r = r_{\text{min}} \)). As a result, the continuity equation (3) is ill-defined at \( r = r_{\text{min}} \). In the following we assume that (i) the fluid can have double-valued solutions, so that inflow and outflow solutions can coexist in the same point of the manifold and (ii) it passes through the singularity in solution at \( r = r_{\text{min}} \). Formally these assumptions imply that we can match solutions for inflow and outflow at \( r_{\text{min}} \), so that \( \rho_{\text{inflow}}(x) = \rho_{\text{outflow}}(x) \) and \( u_{\text{inflow}}(x) = -u_{\text{outflow}}(x) \). A physical interpretation then is as follows: the fluid accretes onto a black hole, then it bounces at \( r_{\text{min}} \) and flows outwards to the asymptotically flat internal spacetime. Since the inflow and the outflow are symmetric by construction, in the following we will present the results for the inflow only.

The resulting distribution for the energy density \( \rho(x) \) for the thermal photon gas is shown in Fig. 5. In Fig. 2 the corresponding distributions for the radial component of the 4-velocity is shown. In Fig. 3 we plot the radial 3-velocity \( v(x) \) with respect to the local static observers. Note, that the \( v(x) \) equals to the sound speed, \( v(r_{\text{min}}) = c_s \), at the minimal radius \( r_{\text{min}} \) for generic equation of state.

In Fig. 4 we depict a part of the Carter-Penrose diagram for the the Reissner-Nordström metric containing an accreting fluid. This diagram is symmetric and time-reversal due to the stationarity of the process. Note that for “astrophysical” black holes, formed by gravitational collapse of massive objects, the internal space-times are absent and one can expect that inflowing fluid modifies the metric inside the event horizon (see, e. g. [15, 25] and references herein).

In the Carter-Penrose diagram the streamlines of the outflowing fluid intersect with the inflowing ones in the region \( r_{\text{min}} < r < r_- \) (notice the intersecting dashed lines in Fig. 4). As we discussed before, we assume the inflow and outflow do not interact and they freely pass through each other (similar to the motion of test particles). If the fluid is viscous, the picture should be modified (at least for \( r < r_- \), but not for \( r > r_+ \)), since intersecting streamlines interact. The resulting flow may become time de-
pendent, turbulent or/and be accompanied by formation of shocks.

### B. Chaplygin gas

Another analytically solvable example we consider here is the Chaplygin gas,

\[ p = -\frac{\alpha}{\rho}, \]

where constant \( \alpha > 0 \) corresponds to a hydrodynamically stable fluid. The Chaplygin gas with \( \rho^2 < \alpha \) represents phantom energy with superluminal speed of sound. The opposite case, \( \rho^2 > \alpha \), corresponds to dark energy with \( \rho + p > 0 \) and \( 0 < c_s^2 < 1 \).

We find the following relations in the critical point:

\[
    f_s = \frac{\xi - 1}{\xi}, \quad x^\pm = \xi \left[ 1 \pm \sqrt{1 - \frac{e^2}{\xi}} \right], \quad A = \frac{x^2}{\sqrt{\xi}}, \quad (25)
\]

where \( \xi = \rho^2_{\infty}/\alpha \). The sonic point exists and the accretion is transonic for \( \xi \geq e^2 \), i.e., when square root is real in (25). Note that for the non-phantom Chaplygin gas this is always satisfied. On the other hand, in the phantom case the critical point is absent for some range of parameters, implying that physical solution does not exist. This, however, is merely a consequence of pathological behavior of Chaplygin gas in the phantom regime.

For radial dependence of the energy density and the radial 4-velocity \( u \) we find,

\[
    u = -\frac{A}{x^2} \sqrt{\frac{\xi - 1}{\xi(\rho/\rho_{\infty})^2 - 1}}, \quad (26)
\]

\[
    \frac{\rho}{\rho_{\infty}} = \sqrt{\frac{f - A^2(\xi - 1)x^{-4}}{\xi(f - 1) + 1}}, \quad (27)
\]

The value of the energy density at the event horizon is \( \rho(r_+)/\rho_{\infty} = A/x^2_{s} \). Solution (26) in the specific case \( \xi = 1 \) corresponds to the vacuum state with \( p = -\rho = -\rho_{\infty} \) and \( u = 0 \). The energy density of the non-phantom Chaplygin gas diverges at the inner critical point \( x_{\min} = x^+_s = \xi(1 - \sqrt{1 - e^2/\xi}) \).

### V. SOLUTIONS FOR NAKED SINGULARITY

As it was discussed in Sec. [II] only “superluminal” fluids reach a naked singularity in steady-state accretion. More precisely, when formulated in terms of a scalar field, a well-behaved at \( r > 0 \) solution exists only if the Lagrangian satisfies the relation \( dL/dX \to \infty \) as \( X \to 0^3 \). In this case one can specify the second boundary condition for accretion at the singularity, \( r = 0 \).

In the case of a “subluminal” fluid the critical solution for steady-state accretion exists not for all \( r \), but only for \( r > r_{\min} \). This is in fact similar to the case of RN black hole, when a fluid is bounced from the singularity, as it was discussed in Sec. [IV] The radial 4-velocity as a function of \( r \) is similar to the case of RN black hole, plotted in Fig. [2]. The 3-velocity, though, does not have a gap with undefined values, in contrast to the case of the black hole. If one thinks in terms of a superfluid, the solution for the critical flow can be interpreted as two physical solutions: the inflow and outflow, matched at the point \( r_{\min} \). Note though, that in the case of a black hole, the matching point \( r_{\min} \) (where the solution becomes singular) is hidden by the horizon, while in the case of RN naked singularity, the singular matching point is reachable by a static observer. One should expect that an arbitrarily small viscosity of the fluid drastically changes the solution, since the inflowing and outflowing components of the fluid interact in the whole space-time. Thus we may conclude that for any realistic fluid the steady-state accretion does not take place for the RN singularity.

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3 As it was discussed in Sec. [II] for the fluid to be non-pathological the condition \( dL/dX \to \text{const} \) as \( X \to 0 \) must be true. Therefore, strictly speaking, a “non-pathological” superluminal fluid also does not reach a naked singularity.
A. Static fluid atmosphere

It is interesting, however, that contrary to the black hole case, a static solution for naked singularity can be constructed. Such a solution describes a static light atmosphere with zero influx. Indeed, assuming $u = 0$ from [20] we find a static distribution of a test perfect fluid around RN naked singularity

$$\frac{\rho + p}{\rho_\infty + p(\rho_\infty)} \exp \left[ - \int_{\rho_\infty}^{\rho} \frac{dp'}{p' + p(p')} \right] = f^{-1/2}.$$ 

In the particular case of the linear equation of state (18) we obtain for static atmosphere

$$\rho(r) = \frac{\alpha \rho_0}{1 + \alpha} + \left( \rho_\infty - \frac{\alpha \rho_0}{1 + \alpha} \right) f^{-\frac{1 + \alpha}{2\alpha}}. \quad (28)$$

The energy density of ordinary matter (with $\rho_0 = 0$ and $\alpha > 0$) approaches zero at the singularity, $\rho \propto x^{1+1/\alpha}$ as $x \to 0$. In the case of phantom, the energy density is finite at $x = 0$, and so phantom fluid “overcomes” the naked singularity repulsiveness.

In the case $e^2 > 1$ by setting $u = A = 0$ in the equation [20] we find a static distribution of the Chaplygin gas around a naked singularity.

B. Static scalar field atmosphere

Note, that the solutions for static atmosphere of the fluid, considered above, in Sec. V A corresponds to the following solution in terms of a scalar field,

$$\frac{\partial \phi}{\partial t} = \text{const}, \quad \frac{\partial \phi}{\partial r} = 0.$$ 

One can notice, however, that zero energy flux, $T_{0}^t = -f L_X \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial r} = 0$ is also achieved by setting $\partial_\phi \phi = 0$. Then the equation of motion becomes

$$\frac{\partial}{\partial r} \left( r^2 L_X f \frac{\partial \phi}{\partial r} \right) = 0. \quad (29)$$

We restrict our study to the canonical scalar field, $L(X) = X$. The solutions of (29) in the case of RN black hole and a naked singularity are, respectively,

$$\phi(x) = \frac{\xi_1}{M(x_+ - x_-)} \ln \frac{x - x_+}{x - x_-} + \xi_2, \quad \phi(x) = \frac{\xi_1}{M \sqrt{e^2 - 1}} \arctan \left[ \frac{x - 1}{\sqrt{e^2 - 1}} \right] + \xi_2, \quad (30)$$

where $\xi_1$ and $\xi_2$ are constants. Note, that $\phi(1) = 0$ in (30) for any $e \neq 0$, but $\phi(0)$ is not necessarily zero. The energy density of the scalar field is $T_{0}^\phi = \xi_1^2 / (2r^4 f)$. In the case of a RN black hole it diverges at the horizon, while for a naked singularity the energy density is singular at $r = 0$. However this singularity is integrable and the mass of scalar field atmosphere is finite inside any finite $r$.

VI. APPROACH TO EXTREME STATE

A black hole can approach the extreme state by capturing particles with electric charge and/or angular momentum, but an infinite time is required to reach the extreme state \[7,21\]. This is a manifestation of the third law of the black hole thermodynamics \[7\]. Note, that during accretion of neutral phantom energy the electric charge of the RN black hole is unchanged, $Q = \text{const}$, while the black hole mass decreases. As a result the black hole approaches to near-extreme state due to the growing of the ratio $e = Q/M(t)$. In the test fluid approximation, the black hole reaches the extreme state in finite time $t = t_{NS}$, defined by the relation $Q = M(t_{NS})$. Indeed, using \[10\], the time $t_{NS}$ for a black hole with initial mass $M = M(0)$ and the electric charge $Q = \text{const}$ may be calculated from the following equation,

$$\int_0^{t_{NS}} \dot{m} \, dt = Q - M(0). \quad (31)$$

If we neglect the cosmological evolution of $\rho_\infty$, then from \[10\], \[19\] and \[31\] for the particular case of phantom fluid with the stiff equation of state ($\epsilon_s = 1$) we obtain,

$$t_{NS} = \frac{e_0^3 - 3e_0^2 + 2 - 2(1 - e_0^3)^{3/2}}{3e_0^4} \tau, \quad (32)$$

where $e_0 = Q/M(0)$ and $\tau = -\{4\pi [\rho_\infty + p(\rho_\infty)] M(0)\}^{-1}$ is the characteristic accretion time.

The finiteness of time $t_{NS}$ in (32) implies violation of the third law of black hole thermodynamics in the considered test fluid approximation.\footnote{Possibility for a black hole to be transformed into a naked singularity by phantom accretion was first discussed in \[23\].}

Notice, that in deriving the above result we assumed that the fluid does not back-react. This assumption, however, may not be valid for the near-extreme black holes/naked singularities. Indeed, in the case $\alpha \geq 1$, the energy density of the accreting fluid diverges at the horizon, as the black hole approaches the extreme state. This can be seen from \[19\], \[23\] and \[20\]. Similarly, violation of the test fluid approximation occurs at the radius $r = M$ for static atmosphere around near-extreme naked singularity due to divergence of the energy density, which can be verified from Eqs. (28). It is worth to note, that in the case of near-extreme Kerr-Newman naked singularity the energy density diverges at $r = M$ for an atmosphere of a fluid \[20\].

Meanwhile, when $0 < \alpha < 1$ the energy density of the accreting fluid remains finite even for the extreme black hole. Nevertheless, one can argue that the test fluid approximation is violated for the following reason. The test fluid approximation is valid if the back reaction of an accreting fluid is small. Consider, however, almost extreme
black hole, so that $|m - e| \ll m$. One can calculate the back reaction from the perturbed Einstein equations,

$$\delta G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

(33)

where $\delta G_{\mu\nu}$ is the deviation of the Einstein tensor due to the presence of the accreting fluid with the energy-momentum tensor $T_{\mu\nu}$. Even if perturbation of the metric calculated from (33) is small, in the limit $M \to Q$ a presence of the fluid may have drastic effect on the metric. Thus one should carefully consider the back reaction effects in the case of the near extreme black holes, even if the accreting fluid has a small energy-momentum tensor. The back reaction of the accretion flow may prevent conversion of a black hole into a naked singularity. This question, however, is beyond the scope of this paper, and we leave it for future investigation.

VII. CONCLUSION

In this paper we studied steady-state distribution of a test perfect fluid with a general equation of state, $p = \rho(p)$, and a scalar field in the Reissner-Nordstr"om metric. Similarly to the case of steady-state accretion of a perfect fluid onto a Schwarzschild black hole, the corresponding solution for the accretion exists also in the case of the RN black hole. On the other hand, no steady-state accretion of a perfect fluid exists onto the RN naked singularity, unless one introduces the double-valued velocity, energy density and the pressure of a fluid, in order to describe the inflow and the outflow occurring in the same points of space-time. Instead of steady-state accretion, a static atmosphere of the fluid is formed around a naked singularity. For both a black hole and naked singularity we found analytical solutions to the problem of the steady state configurations of perfect fluids with an arbitrary equation of state, $p = \rho(p)$. As particular cases, we studied a fluid with the linear equation of state, $p = \alpha(p - p_0)$ and the Chaplygin gas, $p = \alpha/p$. We also found a static distribution of a scalar field around a naked singularity.

When the accreting fluid is phantom, $\rho + p < 0$, the mass of the RN black hole decreases. This result is in the agreement with the previous findings [3, 31]. This poses a question, whether it is possible to convert a RN black hole into a naked singularity by accretion of phantom. Under the assumptions we made, such a conversion is possible, since the accreting phantom decreases the black hole mass, while the electric charge of the black hole remains the same. The conversion of a RN black hole into a naked singularity in the case of accretion of exotic matter with negative energy density $\rho < 0$ was already studied in [22, 32]. It is interesting to verify the possibility of similar conversion in the case of phantom fluid with a positive energy density $\rho > 0$ by taking into account back reaction, which, as we expect, plays an important role in the case of near-extreme states. We leave this question for future study.

Although the test fluid approximation seems to break down for the near-extreme state of the black hole/naked singularity, we would like to stress that for the far-from-the-extreme state of a black hole (in particular, for the Schwarzschild solution), the parameters of the perfect fluid and the boundary condition at the infinity can be tuned so, that the test fluid approximation describes well the accretion process.

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