Einstein’s first gravitational field equation 101 years latter

Juan Betancort Rijo

Instituto de Astrofísica de Canarias,
E-38200 La Laguna, Tenerife, Spain.

Departamento de Astrofísica, Universidad de La Laguna,
E-38205 La Laguna, Tenerife, Spain.

Felipe Jiménez Ibarra

Universidad de La Laguna, E-38205 La Laguna, Tenerife, Spain.

(Dated: January 15, 2014)

Abstract

We review and strengthen the arguments given by Einstein to derive his first gravitational field equation for static fields and show that, although it was ultimately rejected, it follows from General Relativity (GR) for negligible pressure. Using this equation and considerations following directly from the equivalence principle (EP), we show how Schwarzschild metric and other vacuum metrics can be obtained immediately. With this results and some basic principles, we obtain the metric in the general spherically symmetric case and the corresponding hydrostatic equilibrium equation. For this metrics we obtain the motion equations in a simple and exact manner that clearly shows the three sources of difference (implied by various aspects of the EP) with respect to the Newtonian case and use them to study the classical tests of GR. We comment on the origin of the problems of Einstein first theory of gravity and discuss how, by removing it the theory could be made consistent and extended to include rotations, we also comments on various conceptual issues of GR as the origin of the gravitational effect of pressure.
I. INTRODUCTION

In the development of General Relativity (GR) Einstein (1907[1]) started by using the equivalence principle (EP) to analyse the “apparent” gravitational field in an uniformly accelerated system. In this analysis he found that the synchronized time (coordinate time) could not agree with the local proper time, a fact from which it followed immediately the gravitational redshift, the bending of light trajectories and the dependence of light velocity on position when measured using coordinate time. In 1911[2] he reviewed and expanded these ideas and conjectured that the effects found in the “apparent” gravitational field of an accelerated system, should also be present in a genuine gravitational field as that of the earth or the sun. In particular, he predicted a deflection of light rays grazing the sun of roughly one arc second. However, after a lengthy and contorted path he will come to know that, unlike the gravitational redshift and the dependence of the velocity of light on position, which hold exactly in a general static field, the deflection of light ray had a hidden assumption on the flatness of space. Depending on the curvature of space this deflection could be zero (conformally flat space-time, corresponding to a negative spatial curvature) or twice the quoted value (full GR, corresponding to positive spatial curvature around the sun). In 1912[3] he assumed that a general static gravitational field could be described by a single scalar field, namely, the position dependent velocity of light, $c(\vec{x})$, that he has found to be the case for the field in an accelerated system. Again, the implicit assumption here was that space remained flat and, as we shall discuss latter, this was bound to give rise to difficulties. Einstein obtained the equation satisfied by the field $c(\vec{x})$ in an accelerated system and through some considerations extended it to the case of a general static field. The equation was:

$$\nabla^2 c(\vec{x}) = 4\pi G c(\vec{x})\rho(\vec{x}),$$

(1)

where $\rho(\vec{x})$ is the matter density distribution. The problem was that one can not arrange (through suitable definition of mass and force within a gravitational field) for the simultaneous conservation of energy and momentum. This difficulties led Einstein to advance a modified equation fully consistent with energy-momentum conservation, but not exactly consistent with the EP, which would be satisfied only by sufficiently weak fields. Einstein considered the EP to be the happiest thought of his life, thus having to forsake this princ-
ple was rather unpalatable to him. In this state of inner dissatisfaction with his latest field equation, he started to muse on the plausibility of his assumption concerning the Euclidean character of space. Analysing the intrinsic geometry of a rotating disk, he realized that, even in the absence of a genuine gravitational field (i.e. flat space-time), in the co-rotating system, where an “apparent” gravitational field exist, the geometry of space can not be Euclidean. The EP then led him to believe that the same would be true for an actual gravitational field. It seems that it was at this point that he fully realized that the EP implied the tensorial character of the gravitational field, which is essential even in the case of static fields. From this moment his research took a brisk turn. Once he realized that the appropriate formalism for his problem was that of “absolute calculus” (the name given in those day to de differential geometry) he set himself to command that calculus and initiated a long an extremely arduous route culminating three and a half year latter (in November 1915) with the fully-fledged theory of GR. The story of these development is described in some detail in books like Abraham Pais’s, Einsteins’s scientific biography or Howard & Stachel’s history of the development of GR and it represented for Einstein a novel style of research, not only with respect to GR, but with respect to all his previous scientific career. Up to this time physical intuition played the guiding role; in the future that role will be played mostly by formal considerations. It was after GR was obtained that the intuitions for the new physics was gained by extracting its implications. However, the question of which portion of GR could have been anticipated by simple consideration based on EP have, to our knowledge, not been fully clarified. More precisely, the questions of the validity of Eq. (1) and whether the derivation Einstein gave for it was correct, or which relevant results may be inferred from it, are not found in the most widely used text on the conceptual development of GR.

In this work we want to point out that the Eq. (1) is correct (for negligible pressure) and that the arguments given by Einstein to prove it from the EP are, with some qualifications, correct, being some other assumption that Einstein took that led to contradictions. The goals that we pursue are three-fold: from the conceptual point of view we want to explore how much can one learn about a static gravitational field from simple (pre- Riemannian) considerations concerning the EP, as it was the aim of the interrupted Einstein GR research program. From a technical point of view we want to stress that Eq. (1) is a very useful one for static field, and that in the case of spherical symmetry it leads immediately to rather
interesting results. From the historic point of view we want to remark that by a slight twist of events a theory of gravitation accounting for most relevant facts (including the three classical test) would have been available in 1912.

The work is organized as follows: In section 1 we review Einstein derivation of Eq. (1) weighing and strengthening the arguments he gave and show that, in the case of negligible pressure, it follows from Einstein equations of GR. In section 2 we use Eq. (1) together with a simple argument relating the $c$ field to the spatial metric to derive de Schwarzschild metric. We also consider some other interesting results following from Eq. (1). In section 3 we discuss Einstein’s pre-Riemannian theories of gravitation reviewing the arguments and the difficulties that he found. We also consider which could have been the course of development of GR if Einstein had been able to continue his initial program, which involved considering particularly meaningful physical situation of increasing complexity.

II. EINSTEIN’S FIRST GRAVITATIONAL FIELD EQUATION

In a work published in February 1912[3] Einstein obtained the gravitational field which according to de EP must exist in an accelerated system with sufficient accuracy to show that the Laplacian of the corresponding $c$-field must vanish. He did not elaborate on the properties of that field beyond this result because he felt some uneasiness about the fact that that gravitational field was not homogeneous. In a work by Born[7] concerning the motion of rigid bodies in special relativity it have been shown that in order for a body to be moved rigidly, so that every portion of it retain its size in the comoving systems, different parts of the body must experience different accelerations. An uniformly accelerated system must have a position dependent acceleration (although at a given point it is constant in time) so that the gravitational field implied by de EP is not homogeneous. Einstein knew this result in 1912 and its implication that the EP can only be applied locally, but he was so confused by it that he preferred to avoid mentioning it explicitly. Instead, he just said that the field under consideration is of “some specific kind”, implicitly acknowledging that it is not an uniform field, as he had assumed previously.

The structure of that field may be investigated using the simple methods used by Einstein, but for a modern reader it may be easier to use Rindler[8] metric for an uniformly accelerated system. Using Cartesian spatial coordinates and synchronized time coincident with proper
time at the origin we have:

\[ d^2 s = g_{00} d^2 t - d^2 x - d^2 y - d^2 z; \]
\[ g_{00} = c_o^2 \left( 1 + \frac{a_0 x}{c_o^2} \right)^2 \]
\[ c(x) \equiv g_{00}^{1/2}; \]
\[ \vec{g} = -\nabla \phi = c_o^2 \frac{\partial}{\partial \vec{x}} \ln g_{00}^{1/2} = -\frac{a_0}{1 + a_0 c_o^{-2} x} \hat{i}; \]
\[ \phi = c_0^2 \ln g_{00}^{1/2}; \]
\[ c(x) = c_0 e^{\phi/c_0^2}. \]  

(2)

where \( \vec{g} \) is the gravitational field strength (i.e. the force on the unit mass at rest) and \( \phi \) the corresponding potential. We have considered an acceleration in the \( x \) direction (\( \hat{i} \) being the unit vector in this direction). The relationship between \( \phi \) and \( g_{00} \) is a well-known one for static field (Landau-Lifshitz\[9\]), that could be derived, anyway, from simple considerations. \( a_0 \) is the acceleration at the origin (\( x = 0 \)) and \( c_0 \) is the speed of light in vacuum measured with local proper time (i.e. it is a constant). At \( x \) the absolute value of the gravitational field strength is \( a_0 (1 + a_0 c_o^{-2} x)^{-1} \), in particular at \( x = -a_0^{-1} c_o^2 \) the gravitational field goes to infinity and is obvious that a solidly moving body can not extend beyond that point.

According to the EP, this result implies that the proper acceleration (i.e. that measured in the comoving system) is \( a_0 (1 + a_0 c_o^{-2} x)^{-1} \), that depends on \( x \). This is the result that stunned Einstein and that is confusing in at least two respects: first it is the question of how is it possible that the different portions of the system accelerate at different rates. At any one time an inertial system (with coordinate time given by local proper time synchronized as usual in special relativity) may be found where all portions of the accelerating system are instantaneously at rest. This have been assumed by Einstein in his treatment of the accelerating system, and it seems to follow necessarily from the very definition of a bodily accelerated system. It is actually true, but it is in apparent contradiction with different part of the system having different accelerations. Secondly, it is puzzling that different parts of a system can accelerate at different rates relatively to the instantaneously comoving inertial system without deformations. The answer to the last question is as follows: assume that two portions of the accelerated system, the one at \( x = 0 \) and the one at \( x = l \), accelerate at the same rate. If at some time both point are at rest with respect to an inertial system, at
time \( t \) each of these portion would have displaced with respect to this system by exactly the same amount, since both portions experience the same hyperbolic motion (constant proper acceleration). So the distance between both portions as judged from the inertial system would always be \( l \). But as \( t \) increases the velocity of those portions with respect to the inertial system increases and if in the inertial system the distance between the two portions remains equal to \( l \), this means, due to Lorentz-Fitzgerald contraction, that in the accelerating system, or in the inertial system where the accelerating system is instantaneously at rest, the distance between those portion must increase. We then see that in order for the system to be “rigidly” accelerated, so that the distance between both portion remains constant in it, the trailing portion must experience a somewhat larger acceleration than the leading one, so that as both portion gain speed with respect to an inertial system the distance between them as measured in this system show the contraction corresponding to an intrinsically constant size. The first questions may be answered as follows: if the system is initially at rest with respect to certain inertial one and at some time start to accelerate and after some time has gained certain speed, it is possible to find another inertial system in which the accelerating system is instantaneously at rest because, although the trailing portion of the accelerating system accelerate at a larger rate, this portion started its motion latter than the leading one as judged from this latter inertial system, due to the difference in simultaneity with respect to the former inertial system.

It is convenient to discuss these questions because they are interesting in themself and because it help understanding Einstein reluctance to discuss the global properties of the gravitational field under consideration.

What he did to avoid these difficulties was to considerer the local transformation from the comoving inertial to the accelerating system around \( t = 0, x = 0 \) at second order in \( t \). He found for small values of \( x \) (i.e. \( x << c_0^2 a^{-1} \))

\[
c \simeq c_0 + \frac{a}{c_0} x
\]

\( a \) is, in fact, the proper acceleration at \( x = 0 \) (what we called \( a_0 \) in Eq. (2)), but Einstein avoided the question of the dependence of \( a \) on \( x \). From this result he concluded that for this field the Laplacian of \( c \) vanishes. This conclusion seems rigorous, because the same arguments can be given for any value of \( x \). In fact, the exact expression for \( c \), given in Eq. (2), has vanishing Laplacian.
Having studied the gravitational field in an accelerated system he went on to consider the general static case. To this end he made the crucial assumption that the geometry of space was Euclidean so that the situation could be described with just one function, namely, the c-field. It is interesting to note that although this preliminary theory is sometime referred to as “scalar”, $c(x)$ it is in fact a component of a second rank tensor in four dimensions ($c(x) = g^{1/2}_0(x)$) and, although Einstein did not possess this language at the time, he was well aware of the fact that $c(x)$ it is an scalar only with respect to purely spatial transformation.

For a general gravitational field Einstein assumed that, as for the accelerated system, the Laplacian of $c(x)$ vanished outside the matter distribution. This may seem a big jump, and Einstein did not comment on its plausibility, but it may be supported by the following considerations: in an accelerated system the Laplacian of $c$ is equal to zero even for arbitrarily large fields (as $x$ goes to $-a_0/c^2_0$ the strength of the field goes to infinity). For a genuine gravitational field, we know that in the weak field limit ($\phi/c^2_0 \ll 1$) Newtonian gravitation hold, so that we have:

$$\nabla^2 \phi \simeq 0; \quad \nabla^2 c \simeq 0$$

By transforming the weak field generated by a distribution of masses to an accelerated system it could be shown that the resulting field, which is non trivial and non-weak also satisfies that $\nabla^2 c = 0$.

From a formal point of view one could say that apart from $c$ itself the only other scalar (spatial) that can be formed with $c$ is the modulus of its gradient, but the presence of any of this two terms on the field equation in vacuum would be in contradiction with the EP, since we would not have the correct field equation in the case of an accelerating system. We can therefore say that, if not rigorously provable, the equation $\nabla^2 c = 0$ for vacuum is at least very highly plausible.

In the presence of matter it is clear that in the weak field limit we must have:

$$c \simeq c_0(1 + \frac{\phi_0}{c^2_0}); \quad \nabla^2 c \simeq \frac{4\pi G \rho}{c_0}.$$  \hspace{1cm} (3)

where $\rho$ stands for energy density (all through the rest of this work). For the general static case Einstein proposed the equation:
\[ \nabla^2 c = kcp, \quad \text{where} \quad k = \frac{4\pi G}{c^2_0}. \quad (4) \]

The derivation he provided for it is incorrect. Einstein asserted that both sides of Eq. (4) should be homogeneous in \( c \) because “it follows immediately from the meaning of \( c \) that \( c \) is determined only up to a constant factor that depends on the constitution of the clock with which one measures \( t \) at the origin of \( K \) (the reference system)”. What strictly follows from this observation is merely the trivial fact that both sides of Eq. (4) must have the same dimensions so that the equation hold for any choice of the time unit. However, Eq. (4) may be inferred correctly from the fact that any point can be taken as origin of the system so that at that point coordinate time agrees with proper time and, therefore, \( c = c_0 \). The function \( c \) is defined up to a constant factor not because time units can be chosen arbitrarily (which it is obviously true, but lack physical implications) as the quotation say, but because the origin of the system can be chosen arbitrarily (which lastly depend on the EP), as the text most probably was intended to mean. In fact, interpreting “the constitution of clock” as concerning not the physical clock at the origin, but the procedure used to establish the time coordinate (that involves choosing an origin), this would just be the meaning of the quotation. Here we see a simple instance of the ubiquitous and confusing statement that general covariance is an important component of GR. In fact, the physical content of this theory lays entirely on the EP in its strong version, that concerns also the gravity field equations and not merely the matter motion (weak version). General covariance is a property of the formalism and lack physical meaning. What is physically meaningful is the fact that that general covariance is displayed by expressions containing only the metric tensor and quantities describing matter. The meaningful fact is that GR equations do not contain elements (i.e. fields) not associated with matter or gravity that could imply that the equations for matter and gravity did not have the same form (locally) in all freely falling systems. In other words, the content of “general covariance” is simply the absence of foreign elements in the equations for matter and gravity, which is a fact implied by EP (strong).

In the presents case, as we have said, the relevant fact to prove that Eq. (4) must have the stated form (for negligible pressure) it is not the conventional one that this equation must hold for any choice of units, but the fact, implied by the EP (strong), that \( c \) field can be obtain using a coordinate time that may be chosen to agree with the proper time at any
point.

In summary, we have seen that the arguments given by Einstein to obtain Eq. (4) are rather cogent, which is not the case for the arguments that led him to drop this equation, as we shall latter see.

The arguments leading to Eq. (4) are strong enough to leave almost no doubt about its validity. Now we shall see that this equation followes from the equations of GR (Einstein’s equations) when pressure is negligible. These equations can be written in the form

\[ R_{ij} = \frac{8\pi G}{c^4} \left( T_{ij} - \frac{1}{2} g_{ij} T \right) \]  

(5)

where \( R_{ij} \) is the Ricci Tensor (i, j run from 0 to 3), \( T_{ij} \) is the energy-momentum tensor, and \( T \) is its trace \( (T \equiv T^i_i) \). Using now the following expression for the Riemann tensor:

\[ R_{iklm} = \frac{1}{2} \left( \frac{\partial^2 g_{lm}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l} \right) + g_{np} \left( \Gamma^m_{nk} \Gamma^p_{lm} - \Gamma^m_{nm} \Gamma^p_{kl} \right) \]  

(6)

and choosing locally Cartesian coordinates at any given point \((g_{\mu\nu} = -\delta_{\mu\nu}; \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = 0, \text{with } \mu, \nu, \lambda \text{ from 1 to 3})\) we have for \( R_{00} \):

\[ R_{00} = g^{ij} R_{0000} = -\frac{1}{2} g^{ij} \left( \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} \right) + g_{np} g^{ij} \left( \Gamma^m_{0n} \Gamma^p_{ij} - \Gamma^m_{0m} \Gamma^p_{ij} \right) = \]

\[ = -\frac{1}{2} g^{ij} \frac{\partial^2 g_{00}}{\partial x^i \partial x^j} + g_{mn} g^{ij} \left( \Gamma^m_{0i} \Gamma^n_{0j} - \Gamma^m_{0j} \Gamma^n_{0i} \right) \]  

(7)

where we have used the fact that \( g_{ij} = 0 \) for \( i \neq j \). From the definition of \( \Gamma \)’s we have:

\[ \Gamma^\mu_0 = \Gamma^0_\mu = 0; \quad \Gamma^0_\mu = \Gamma^0_0 = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^\mu} \]

where use has been made of the static character of the metric (i.e. \( \frac{\partial g_{ij}}{\partial x^\mu} = 0 \)). We then have for \( R_{00} \):

\[ R_{00} = -\frac{1}{2} g^{\mu\nu} \frac{\partial^2 g_{00}}{\partial x^\mu \partial x^\nu} + g^{\mu\nu} g_{00} \left( \Gamma^0_\mu \right)^2 \]  

(8)

But in the locally Cartesian spatial coordinates that we are using the first term is simply one half of the Laplacian of \( g_{00} \) and the second:
\[ g^{\mu\nu}g_{00} \left( \Gamma^0_{\mu\nu} \right)^2 = -\frac{1}{4g_{00}} \left( \frac{\partial g_{00}}{g_{\mu\nu}^{1/2} \partial x^\mu} \right)^2. \]

where we have use the fact that \( g^{\mu\nu} = (g_{\mu\nu})^{-1} \) (note that this expression is valid for any orthogonal system), is simply \(-1/(4g_{00})\) times the square of the gradient (this involves a definition of gradient which is not the most widely used, but it has the most direct meaning and is the one used through this work). In the usual three-dimensional notation:

\[ R_{00} = \nabla^2 g_{00} - \frac{1}{4g_{00}} |\nabla g_{00}|^2 \quad (9) \]

Using the relationship:

\[ \nabla^2 f(\phi) = \frac{d^2 f}{d\phi^2} |\nabla \phi|^2 + \frac{d^2 f}{d\phi^2} |\nabla \phi|^2. \]

We finally obtain for \( R_{00} \)

\[ R_{00} = g_{00}^{1/2} \nabla^2 g_{1/2}^{00}. \quad (10) \]

We have carried out this derivation in a locally Cartesian system, but, since \( R_{00} \) is a scalar for spatial transformations, it is clear that Eq. (10) must hold in any system of spatial coordinates.

Assuming that the source is a perfect fluid with energy density \( \rho \) and pressure \( P \), we have for \( T \):

\[ T = (\rho - 3P). \]

If the fluid was not perfect we would only need to change \( P \) by its average value in the three principal direction. We then have:

\[ T_{00} - \frac{1}{2}g_{00}T = g_{00}\rho - \frac{1}{2}g_{00}(\rho - 3P) = \frac{g_{00}}{2}(\rho + 3P). \]

Inserting this and de Eq. (10) into Eq. (5), we find:

\[ \nabla^2 g_{00}^{1/2} = \frac{4\pi G}{c_0^4} g_{00}^{1/2}(\rho + 3P) \quad (11) \]

Noting that Einstein c field is just \( g_{00}^{1/2} \), we see that in the case of negligible pressure this equation reduces to Eq. (4), that had been obtained by Einstein from simple considerations.
The pressure term is related to the delicate issue of the gravitational effect of pressure. We shall latter comment on its origin in GR and on the fact that it doesn’t seem possible to derive it from simple consideration concerning static fields.

It is interesting to note that if a $\lambda$ term was considered, that is, if we added to the left hand side of Eq. (5) a term of the form $\lambda g_{ij}$ ($\lambda$ being a constant), in Eq. (11) we should have $\rho - \frac{\lambda c^2}{4\pi G}$ in place of $\rho$.

We may summarize this section by saying that the arguments given by Einstein in 1912 to derive Eq. (11) were quite solid and that, as a matter of fact, this equation is valid in GR when pressure is negligible, as it is apparent by comparing Eq. (11) with the exact equation for $g_{00}$ in an static field in GR, given by Eq. (11). This equation is quite meaningfull both because it is instrumental in the explanation of several basics facts, as we shall show, and because of its close relationship with an equation that played a role in the development of GR (Eq. (11)). The fact that this equation appear neither on chapters on statics fields in the best known textbook, nor in the books known to us on the historic and conceptual development of GR is quite surprising. This is still more difficult to understand noting that it is a well-known result that Eq. (10) holds for a metric where $g_{00}$ it is the only non trivial coefficients. As we have seen, this could have been easily generalized to any static metric and with it obtain Eq. (11).

III. SPHERICALLY SYMMETRIC FIELD: DERIVATION OF SCHWARSZCHILD METRIC

Shortly after obtaining Eq. (4) and then rejecting it, Einstein started questioning the assumption that in a gravitational field space remains flat. After some groping in the dark he came to the conclusion that the spatial metric must be integrated into a common structure with the “proper” gravitational field, $g_{00}$. This structure is the four dimensional tensor $g_{ij}$, that in a sense represent the gravitational field in GR. Therefore, in this theory there are ten different “gravitational potentials”, namely, the ten algebraically independent component of a symmetric second rank tensor in four dimension, and to determine them ten equations are needed. This can not be achieved if the field source was only the matter density; a tensorial source is needed. It is then obvious that this source must be the energy momentum tensor, $T_{ij}$ and the ten needed equation are those given in Eq. (5).
The “proper” gravitational potential is still given by $g_{00}$ (more precisely $c^2_0/2 \ln g_{00}$), in the sense that the gravitational field strength, $\vec{g}$, is given by minus its gradient. But this quantity only gives us the force that must be exerted upon the unit of mass to keep it at a fixed position within the field, while all $g_{ij}$ are needed to obtain particles trajectories. Furthermore, even if we were only interested in $g_{00}$, to obtain it by integrating Eq. (4) (or rather, its exact counterpart Eq. (11)) the spatial metric coefficients, $g_{\mu\nu}$, are needed, since they enter in the explicit expression for the Laplacian. Thus, Eq. (4), although it is exact (for negligible pressure) in any static field it is not of much use without integrating it simultaneously with all the other equation entering in Eq. (5).

For spherically symmetric fields, however, we shall show that Eq. (4) determines uniquely the whole metric (with a simple additional assumption in the non-vacuum case). Choosing appropriate polar coordinates, a metric with spherical symmetry can be reduced to the form:

$$d^2 s = g_{00} d^2 t - g_{rr} d^2 r - r^2 \left( d^2 \theta + \sin^2 \phi d^2 \phi \right)$$  \hspace{1cm} (12)

where $\theta, \phi$ are the usual polar angles and $r$ is a radial coordinate defined so that the area of a sphere of radius $r$ is $4\pi r^2$. There are only two undetermined metric coefficients, $g_{00}$ and $g_{rr}$, the others being predetermined by symmetry and the choice of coordinate.

If we could find a relationship between this two coefficients, we could use Eq. (11) to completely determine the metric. We shall now see how to obtain a relationship between $g_{00}$ and $g_{rr}$ in the vacuum case (Schwarzschild metric) by means of the following argument:

Consider the flat time-space (i.e. Minkowskian) and the usual Galilean reference system, $K$, within it. Now consider another system, $K'$, that can be described as a continuous set of locally Galilean systems, $k'(\vec{x})$, so that at each point, $\vec{x}$, (Cartesian coordinates with respect to $K$), the local system $k'(\vec{x})$ at time $t_0$ is moving with respect to $K$ radially from the origin with speed $V(r) \ (r \equiv |\vec{x}|, \vec{V}(\vec{x})$ is parallel to $\vec{x}$).Let us study the spatial geometry in system $K'$. In system $K$ space is flat, so the length of a circumference with radius $r$ in, let us say, plane $z = 0$ and centred at the origin is simply $2\pi r$. In system $K'$ the length measurements are carried out with rods comoving with the local Galilean system $k'(\vec{x})$. To measure the length of the circumference this rods must be set at point $\vec{x}$ perpendicularly to the velocity of $k'(\vec{x})$ with respect to $K$, therefore, rods measuring the circumference have the same length in $K$ and in $K'$ and, consequently, the length of the circumference must also be $2\pi r$ in this last system. However the radius of the circumference in $K'$ is now longer than
Because in measuring it the rods must be parallel to the velocity of \( k'(\vec{x}) \) with respect to \( K \) so that in this system those rods are affected by the Lorenz-Fitzgerald contraction. Therefore, more rods are necessary to cover the radius in \( K' \) than in \( K \). More precisely, the proper radius, \( \chi \), in \( K' \), due to the contraction is given by:

\[
\chi(r) = \int_0^r \frac{dr'}{\sqrt{1 - \left(\frac{V(r')}{c_0}\right)^2}}.
\]

(13)

From the differential relationship between \( \chi \) and \( r \) we have immediately:

\[
g_{rr}(r) = \left[1 - \left(\frac{V(r)}{c_0}\right)^2\right]^{-1/2}.
\]

On the other hand, the relationship between proper time, \( \tau \), and the time coordinate in system \( K, t \), is given by:

\[
d\tau \equiv \frac{g_{00}^{1/2}}{c_0} dt = \left[1 - \left(\frac{V(r)}{c_0}\right)^2\right]^{1/2} dt.
\]

So we have:

\[
g_{rr} = c_0^2 g_{00}^{-1}.
\]

(14)

It must be noted that the time coordinate in system \( K' \) is also \( t \), the Galilean time in \( K \). So, by the geometry of the space in system \( K' \) we mean the metric of hypersurface \( t = t_0 \). This metric is not constant, because as the local systems move away from the origin the relationship between \( V \) and \( r \) changes. However, to our purposes it will suffice to obtain the metric at \( t_0 \); a metric that we will compare with the static metric that we are going to discuss next.

In the initial formulation of the EP the effect of a constant gravitational field are consider equivalent to the non-inertial effects in an uniformly accelerated system. In this system, although different parts of it are in some sense at rest with respect to each other, when a particle, for example a photon, move from A to B, by the time the photon arrives at B the inertial system in which B is instantaneously at rest is moving with respect to that in which A was at rest at the emission time. This fact determines the metric in the accelerated system (\( g_{00} \) being the only non-trivial coefficient). In a similar manner, one way interpret
the spherically symmetric metric in vacuum as the metric in $K'$ at $t_0$, which is not constant, forced to become constant by the presence of the gravitational field, which in the process turns space-time from Minkowskian to a curved one. It must be noted that for this interpretation to work $V(r)$ must be equal to the velocity attained by a particle that start falling at infinity with zero velocity by the time it reach radii $r$, although this is immaterial with respect to relationship (14).

In system $K'$, although the metric is instantaneously equal at time $t_0$ to that in the gravitational field, the redshift experienced by light when moving within the system it is not equal to that for the gravitational field (not even for infinitesimal time of flight), because this redshift depends not just on the metric in $K'$ at $t_0$, but also on its evolution. For example, for the redshift experienced between $r$ (at $t_0$) and infinity in $K'$ we have (with $c_0 = 1$):

$$1 + \frac{\Delta \lambda}{\lambda} = \frac{1 + V(r)}{\sqrt{1 - V(r)^2}} \equiv \gamma(r) (1 + V(r))$$

where the factor $1 + V(r)$ is the classical Doppler effect associated with the fact that in $K'$ the local systems are actually moving. The $\gamma$ factor correspond to the transversal Doppler effect, associated with the difference between the time at the local system and that at infinity, which corresponds to the redshift in the gravitational field.

It is possible to select a “reference system”, $K''$, in Minkowski space whose metric is in closer relationship with that in a gravitational field. The system $K''$ can be conceived as a discrete set of local systems of $K'$ with small but finite size that for a short period of time, $\Delta t$, move with velocity $V(r)$ or $-V(r)$ and then instantaneously change its velocity to $-V(r)$ or $V(r)$. The metric in this system and the redshift of light propagating between any two points averaged over times larger than $\Delta t$ (the classical Doppler averages out) is the same as in the gravitational field.

What follows from this interpretation is that in vacuum the effect of the gravitational field on space and time are both implied by a single quantity (interpreted here as $V(r)$) and, therefore, $g_{rr}$ and $g_{00}$ are related, being Eq. (14) the necessary form of this relationship. We shall see that this depends on the fact that in vacuum both $g_{rr}$ and $g_{00}$ at $r$ depends on the enclosed mass, which is a constant. On the other hand, in the general spherically symmetric case, $g_{rr}$ depends on the enclosed mass but $g_{00}$ (normalized to the center of the cloud) depends on the enclosed mass and pressure-volume (furthermore, even without
pressure \( g_{rr} \) depends just on \( M(r) \) (see Eq. (19)), while \( g_{00} \) depends on \( M(r') \) for \( r' \leq r \) and no general relationship between \( g_{rr} \) and \( g_{00} \) can exist, as we shall see by showing that the contrary assumption leads to inconsistencies. In the vacuum case, however, Eq. (14) must hold for any value of the cosmological constant and in the case of a charged mass “point” (REF Reissner-Nordstron metric).

Note that for this relationship to be valid it is implicit that coordinate time have been chosen so that \( g_{00} \) goes to one at infinity.

Using Eq. (14) in Eq. (4), for vacuum (with \( \lambda = 0 \)), we may immediately derive the whole metric Eq. (12), which has now been reduced to just one independent quantity, \( g_{00} \):

\[
\nabla^2 c \equiv \nabla^2 g_{00}^{1/2} = \frac{1}{r^2 g_{rr}^{1/2}} \frac{\partial}{\partial r} \left( r^2 g_{00}^{1/2} \frac{\partial g_{00}^{1/2}}{\partial r} \right) = \frac{1}{r^2} \frac{g_{00}^{1/2}}{c_0^2} \frac{\partial}{\partial r} \left( r^2 g_{00}^{1/2} \frac{\partial g_{00}^{1/2}}{\partial r} \right) = 0. \tag{15}
\]

where the first equality come from the expression for Laplacian in spherical coordinates for a quantity, \( g_{00}^{1/2} \), that is spherically symmetric, and the last comes from using Eq. (4).

A first integration of Eq. (15) give:

\[
r^2 g_{00}^{1/2} \frac{\partial g_{00}^{1/2}}{\partial r} = r^2 \frac{1}{2} \frac{\partial g_{00}}{\partial r} = A
\]

where \( A \) is a constant. Integrating again:

\[
g_{00} = -\frac{2A}{r} + B.
\]

With the usual convention of taking the coordinate time to agree with proper time at infinity (which is necessary for Eq. (14) to hold), we must have \( B \) equal to \( c_0^2 \).

At large values of \( r \) the Newtonian limit is valid, so:

\[
g_{00}(r) \simeq c_0^2 \left( 1 - \frac{2\phi}{c_0^2} \right) = c_0^2 \left( 1 - \frac{2GM}{c_0^2 r} \right).
\]

We must then have \( A = GM \), so that we have for \( g_{00}, g_{rr} \):

\[
g_{00} = c_0^2 \left( 1 - \frac{2GM}{c_0^2 r} \right); \quad g_{rr} = \frac{c_0^2}{g_{00}} \tag{16}
\]

which are the well-known values of these coefficients for Schwarzschild metric.

We have said before that the quantity that can be called the “gravitational potential” in GR is \( c_0^2 \ln g_{00} \), in the sense that minus its gradient is equal to the force on a static unit of
mass. To study the difference of this potential with respect to the Newtonian counterpart, it is convenient to express \( \vec{g} \) in the following form:

\[
\vec{g} = -\frac{1}{g_{r r}} \frac{\partial}{\partial r} \ln g_{0 0}^{1/2} e_r = c_0 \frac{\partial g_{0 0}^{1/2}}{\partial r} e_r
\]  

(17)

where \( e_r \) is the unit vector in the \( r \) direction and where Eq. (14) has been used. We see from Eq. (15) that if \( g_{rr} \) was not affected by the gravitational field, remaining equal to 1, we would obtain:

\[
g_{0 0}^{1/2} = c_0 \left( 1 - \frac{G M}{c_0^2 r} \right)
\]

and \( \vec{g} \) would be given by the Newtonian expression. It is then clear that what causes \( \vec{g} \) to differ from the Newtonian case is the fact that the gravitational field affect the value of \( g_{rr} \), causing it to be , in the present case, equal to \( g_{0 0}^{-1/2} \).

For the vacuum case with non-zero cosmological constant, we need using Eq. (11), that, noting that a \( \lambda \) term can formally be treated like a homogeneous fluid with equation of state \( p = -\rho \), takes the form:

\[
\nabla^2 g_{0 0}^{1/2} = -g_{0 0}^{1/2} \lambda
\]

Using again relationship (14), we find:

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial g_{0 0}}{\partial r} \right) = -2 \lambda c_0^2 r^2.
\]

Integrating this equation with the same conditions as in \( \lambda = 0 \) case, one obtains immediately:

\[
g_{0 0} = c_0^2 \left( 1 - \frac{2 M G}{c_0^2 r} - \frac{\lambda r^2}{3} \right), \quad g_{rr} = c_0^2 g_{0 0}^{-1}
\]

where \( M \) is a constant. This is the solution for the metric of a “point” mass in a non-zero \( \lambda \) vacuum, that was first obtained by Eddington in 1923 [15].

The Reissner-Nordstrom metric can be derived in an equally simple manner as we shall show in a future work.

Assuming in the general spherical case that relationship (14) holds, leads, together with Eq. (11), amongst other inconsistencies, to the fact that the mass of the distribution (for bounded distributions) depends on the enclosed mass and pressure-volume, that as we shall show in next section it is in contradiction with basics principles. It is therefore clear that
this relationship it is not valid, a result that could have been anticipated by more direct (but more involved) argument.

To obtain the metric in the general case we shall use the results obtained in the vacuum case together with the assumption that to obtain the field at $r$ only the matter distribution up to $r$ is needed. This is a consequence of Birkhoff Theorem (REFERENCIA), derived from G.R. This fact can be proved using Eq. (11) and some general considerations, but we will simply take it here as a plausible assumption. This assumption, that is simply the supposition that the full gravitational theory will also satisfy Newton’s iron sphere theorem, it is so natural that a derivation containing it can still be considered to be based on basic principles. With the mentioned assumption and using Eq. (16) it is clear that we must have for $g_{rr}$:

$$g_{rr}(r) = \left(1 - \frac{2GM(r)}{c^20r}\right)^{-1}$$  \hspace{1cm} (18)

where $M(r)$ is the mass enclosed within $r$. $M(r_0)$ is defined so that if there were no matter beyond $r_0$, the asymptotic Newtonian potential would be:

$$\frac{GM(r_0)}{r}.$$ 

In next section we shall show from basics considerations that $M(r)$ is given by:

$$M(r) = \frac{4\pi}{c^2} \int_0^r \rho(r')r'^2 dr'$$  \hspace{1cm} (19)

where $\rho$ is the energy density. Using Eq. (19) in Eq. (18) the known result for $g_{rr}$ is recovered. Inserting this expression for $g_{rr}$ in Eq. (11) we obtain an equation for $g_{00}$

$$\frac{1}{g_{rr}^{1/2}} \frac{\partial}{\partial r} \left( \frac{r^2}{g_{rr}^{1/2}} \frac{\partial g_{00}^{1/2}}{\partial r} \right) = \frac{4\pi G}{c^4} (\rho + 3P) g_{00}^{1/2}$$  \hspace{1cm} (20)

with the conditions:

$$\frac{\partial g_{00}^{1/2}}{\partial r} \big|_{r=0} = 0; \quad g_{00}(\infty) = c_0^2$$

However, rather than integrating this equation, we shall obtain the solution at the end of next section by a more physically meaningful procedure.

Once we have the metric, the trajectories of free falling particles can be obtained using the geodesic equation in space-time for this metric, which is the standard procedure in textbook
on GR. However, to end this section, we shall indicate how this can be done in a simple manner and in much closer keeping with classical mechanics.

GR implies very small corrections for planetary orbits and we think it is instructive to describe the effects by mean of small correcting terms added to the Newtonian equations and explain their origin. To say that in GR gravity is not a force, that planets merely follow space-time geodesics and, in consequence, use a qualitatively different treatment to deal with quantitatively very small effects, does not seem to us the most convenient presentation.

In fact, the very description of gravity as the space-time geometry is an adventitious interpretation and not at the essence of Einstein’s theory, as pointed out by Weinberg[16]. This interpretation may be interesting and compelling for those who already know the theory, but to use it to convey to the layman the meaning of GR does not seem the most expedient way.

The Lagragian for a particle in a static gravitational field may be written in the form:

\[ L = -m \frac{ds}{dt} = -mc_0^2 \sqrt{g_{00}} - \left( \frac{dl}{dt} \right)^2; \tag{21} \]

\[ d^2 l \equiv g_{\mu\nu} dx^\mu dx^\nu \]

where \( s \) is the invariant relativistic interval and \( l \) is the arc length (\( \mu, \nu \) run from 1 to 3). If space was Euclidean, one can easily show that the equation of motion derived from Eq. (21) would be:

\[ \frac{d\vec{v}}{dt} = \left( -\frac{1}{2} \nabla \right) \nabla g_{00} \parallel + \left( -\frac{1}{2} \nabla \right) \nabla g_{00} \perp \left( 1 - 2 \left( \frac{v}{c} \right)^2 \right) = -\frac{1}{2} \nabla g_{00} + \frac{1}{c^2} \left( \vec{v} \cdot \nabla g_{00} \right) \vec{v}; \tag{22} \]

\[ \vec{v} \equiv \frac{d\vec{x}}{dt} \]

where the suffixes \( \perp, \parallel \) denotes respectively the components perpendicular and parallel to \( \vec{v} \). In this expression both \( \vec{v} \) and its derivative are with respect to coordinate time \( t \).

Using the local proper time, \( \tau \) (not to be confused with particle proper time), we have:

\[ \frac{d\vec{v}}{d\tau} = \vec{g}_\perp + \left( 1 - \left( \frac{v}{c} \right)^2 \right) \vec{g}_\parallel \]

\[ \vec{g} = -\frac{c^2}{2g_{00}} \nabla g_{00}; \quad \vec{v} \equiv \frac{d\vec{x}}{d\tau} \tag{23} \]
where $\vec{g}$ is the “gravitational field” (i.e. the force on a unit mass at rest). This equations can be obtained immediately using the fact that in a locally inertial system (i.e. free falling) the acceleration vanishes. Therefore, both Eq. (22) and Eq. (23) merely express the simple (non-Riemannian) EP, which implies a velocity dependent gravitational force. Note that from Eq. (23) it is obvious that $v$ will always remain smaller than $c_0$, as it must be when time $\tau$ is used. However Eq. (22) do not imply such restriction, because in time $t$ $v$ can be larger than $c_0$, although it have to be smaller than $c(\vec{x})$.

In general, space it is not Euclidean, and an extra term must be included in Eq. (23). In the spherically symmetric case we have:

$$\left(\frac{dl}{dt}\right)^2 = \left(\frac{d\chi}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2, \quad (24)$$

where, since it is obvious that orbits must remain in a plane, we are considering only two coordinates $\chi$ (or $r$) and $\theta$, that determine the position within that plane.

Using Eq. (24) in Eq. (21) one obtains the corresponding Lagrangian from which the equations of evolution for $\chi$, $\theta$ can be obtained in the standard manner. But it is simpler and more enlightening to recall the classical derivation and remark the differences. Classically we may write:

$$\frac{d\vec{v}}{dt} = \frac{d}{dt} \left(\dot{r} \vec{e}_r + r \dot{\theta} \vec{e}_\theta\right) = \vec{F} \quad (25)$$

where $\vec{e}_r$, $\vec{e}_\theta$, are the unit vector along the radial and tangential directions, and $\vec{F}$ is the total force per unit mass, that in the Newtonian case is given simply by $\vec{g}$. Using the following relationship:

$$\dot{\vec{e}}_r = \dot{\theta} \vec{e}_\theta; \quad \dot{\vec{e}}_\theta = -\dot{\theta} \vec{e}_r, \quad (26)$$

where the dot denotes derivation with respect to $t$, we have:

$$\left(\ddot{r} - r \dot{\theta}^2\right) \vec{e}_r + \frac{d}{dt} \left(r \dot{\theta} \vec{e}_\theta\right) = \vec{g}.$$

To obtain the evolution equations in the present case we only need to note that the underlying reasons for Eq. (26) are the relationships satisfied in the Euclidean plane. The
relationships satisfied in a non-Euclidean plane with radial symmetry around \( r = 0 \) can be obtained by equally simple geometrical considerations:

\[
\dot{e}_r = (\sin \rho) \dot{\theta} e_\theta, \quad \dot{e}_\theta = -(\sin \rho) \dot{\theta} e_r.
\]

\[
\sin \rho \equiv \frac{d r}{d \chi} = g_{rr}^{-1/2}. \tag{27}
\]

Representing the actual non-Euclidean plane as an axially symmetric curved plane embedded in Euclidean space and being tangent to the plane \( z = 0 \) at the origin, \( \rho \) is the angle between the radial direction within the curved plane and the \( z \) axis.

In the present case it also holds that the derivative of \( \vec{v} \) with respect to coordinate time \( t \) is equal to the total force per unit mass, \( \vec{F} \), but now \( \vec{v} \) is given by:

\[
\vec{v} = \dot{\chi} \vec{e}_r + r \dot{\theta} \vec{e}_\theta,
\]

and \( \vec{F} \) is given by the left hand side of Eq. (22). Using Eq. \( \text{27} \) for the derivative of the base vector, we finally find

\[
\left( \ddot{\chi} - g_{rr}^{-1/2} \dot{\theta}^2 \right) \vec{e}_r + \left[ \dot{\theta} + \frac{d}{dt} \left( r \dot{\theta} \right) \right] \vec{e}_\theta = -\frac{1}{2} \vec{\nabla} g_{00} + \frac{1}{g_{00}} \left( \vec{\nabla} g_{00} \cdot \vec{v} \right) \vec{v}, \tag{28}
\]

\[
\vec{v} \equiv \dot{\chi} \vec{e}_r + r \dot{\theta} \vec{e}_\theta.
\]

From the radial and the tangential part of these equation we obtain respectively:

\[
g_{rr}^{-1/2} \frac{d}{dt} \left( g_{rr}^{3/2} \dot{r} \right) - r^2 \dot{\theta}^2 = -\frac{1}{2} \frac{\partial g_{00}}{\partial r}; \quad \frac{d}{dt} \left( \frac{r^2 \dot{\theta}}{g_{00}} \right) = 0
\]

where Eq. \( \text{14} \) has been used, so, at variance with Eq. \( \text{28} \), that is valid for any spherically symmetric metric, the first of these equations is only valid for vacuum metrics. These equation can be integrated exactly, but we will only discuss simple cases that can easily be treated approximately in a more transparent manner.

For almost circular orbits, neglecting terms proportional to \( \dot{r} \), we have:

\[
g_{rr}^{-1/2} \dot{r} - g_{rr}^{-1/2} r \dot{\theta}^2 = -\frac{1}{2} \vec{\nabla} g_{00} \tag{29}
\]
\[
d \left( \frac{r^2 \dot{\theta}}{g_{00}} \right) \frac{dt}{dt} = 0
\]

The differences with respect to the classical case are three fold: First there is the fact that the force per unit mass at rest, \( \vec{g} \), is not exactly equal to the Newtonian one. Then we have the velocity dependence of the gravitational force and, finally we have the non-Euclidean character of space, which is fully described by the function \( r(\chi) \). However, even when \( \vec{g} \) (related to the acceleration in time \( \tau \)) differs from the Newtonian one, what appears in the right hand side of Eq. \((28)\) (which correspond to time \( t \)) is exactly equal to the Newtonian case, on the other hand, as we have said, for almost circular orbits the radial velocity dependence is negligible, but not the tangential one, that renders the presence of \( g_{00} \) in the last of Eq. \((29)\). So, the difference with respect to the Newtonian case are due to the velocity dependence of \( \vec{F} \) and to the non-Euclideanity of space.

The precession of the perihelion of almost circular orbit can easily be obtained. Integrating the last of Eq. \((29)\) and inserting it in the first, we find:

\[
\ddot{r} = g_{rr}^{-1} \left( r \dot{\theta}^2 - \frac{1}{2} \frac{\partial}{\partial r} g_{00} \right) = g_{rr}^{-1} \left( \frac{J^2 g_{00}^2}{r^3} - \frac{GM}{r^2} \right)
\]

where \( J \) is an integration constant. For a small displacement, \( \Delta r \), from the value of \( r \) at which the last parentheses vanishes, \( r_0 \), we have:

\[
\ddot{r} \simeq -g_{rr}^{-1} \left( \frac{GM}{r_0^3} \left( 1 - \frac{4GM}{r_0 c_0^2} \right) \right) \Delta r = -\omega_r^2 \Delta r
\]

where \( \omega_r \) is the angular frequency of small radial oscillations. For the angular motion we have:

\[
\omega_\theta \equiv \dot{\theta} = \left( \frac{GM}{r_0^3} \right)^{1/2}.
\]

Therefore, the angular frequency of the perihelion, \( \Omega \), is given by:

\[
\Omega = \omega_r - \omega_\theta \simeq \frac{3GM}{r_0 c_0^2} \omega_\theta.
\]

It must be noticed that when the orbit has finite eccentricity, \( r_0 \) is not the mean radius. From its definition:
where \( a \) is the semi-major axis and \( e \) is the eccentricity. For finite radial oscillations higher order terms on \( \Delta r \) must be included in Eq. (31), but we know that without the relativistic corrections \( \omega_r \) and \( \omega_\theta \) (that for general orbits is given by the above expression with \( r_0 = a \)) remain equal. Therefore, with this value of \( r_0 \), the above expression for \( \Omega \) is exact to first order in the potential. Notice that a negative value of \( \Omega \) means that the perihelion moves in the direction of the revolution. From the above computation it is clear that 2/3 of \( \Omega \) are due to the velocity dependence of \( \vec{F} \), and this is the value of \( \Omega \) corresponding to Einstein’s 1912 theory. The remaining third comes from the presence of \( g_{rr} \) in Eq. (30), obviously related to the non-Euclidianity of space. It is interesting to note that in Nordstrom theory, where space-time is conformally flat (therefore, \( g_{rr} \) is the inverse of that for GR), the spatial effect has the opposite sign to that for GR, while the effect of the velocity dependence of \( \vec{F} \) is the same. However, in that theory, the gravitational potential does not enter Eq. (28) as in the Newtonian theory. The sum of all effects render a value of \( \Omega \) that is in magnitude 7/3 of that for GR, but with the opposite sign.

The deflection of light rays can also be easily computed. From the derivation of Eq. (28) it is clear that in quasi-Cartesian coordinates and to first order on \( G \) we have:

\[
\frac{d\vec{V}}{dt} = -\frac{1}{2} \nabla g_{00} + \frac{1}{2} \left( \nabla g_{00} \right)_\parallel - \left[ \dot{\theta} \dot{\chi} \left( g_{00}^{-1/2} - 1 \right) e_\theta - r \dot{\theta}^2 \left( g_{00}^{-1/2} - 1 \right) e_r \right] \tag{32}
\]

where we have used the fact that for light \( v = c \) and where the last parentheses comes from the fact that replacing Eq. (26) by (27) is equivalent to adding the apparent force given by the parentheses while retaining the Euclidean affinity structure. We have expressed the parentheses in polar coordinates because it is simpler to derive, but now must be transformed to Cartesians. Considering a ray with impact parameter \( r_0 \) and taking the coordinate system so that the ray moves in the \( x, y \) plane along the straight line (almost) \( y = r_0 \), we then have:

\[
\frac{dV_y}{dt} = -\frac{GM}{r^2} \frac{r_0}{r} - \frac{GM}{r^2} \frac{r_0}{r}
\]

where the first term on the right hand side corresponds to the homologous term in Eq. (32) and the last one corresponds to the parenthesis. The second term in Eq. (32) does not contribute because it goes in the \( x \) direction. Taking \( t = 0 \) at the moment of maximum
approach and assuming that light traces a straight line at constant speed, \( c_0 \) (zeroth order on \( G \)), it is obvious that the deflection, \( \Delta \phi \), is given by:

\[
\Delta \phi = \frac{2}{c_0} \int_{\text{mustbe}0}^{\infty} \frac{dV_y}{dt} dt = \frac{2}{c_0} \int_{0}^{\infty} \frac{2GM}{(r_0^2 + c_0^2 t^2)^{3/2}} dt = \frac{4GM}{r_0 c_0^2}
\]

Which is the well-known result, that is usually derived in a rather different manner. We have seen that one half of the effect comes from the non-Riemannian EP (first term of the right in Eq. (32)) while the other half comes from the non-Euclideanity of space, (last parentheses in Eq. (32)).

**IV. HISTORIC OVERTONES AND OTHER TOPICS**

We have said before that the argument that led Einstein to reject Eq. (4) was incorrect. The argument [17] is based in the demonstration of the non-conservation the momentum of a distribution of particle interacting through a gravitational field satisfying Eq. (4) and following the dynamic that he had previously derived [18]. Met with this contradiction Einstein chose to reject Eq. (4), but the problem lied not with this equation, that we have seen that is correct (for negligible pressure), but with the assumption that space can remain flat in a gravitational field. We have mentioned this fact before; here we shall explicitly show it in a simple manner. To this end, instead of the general case, we shall consider the case of two interacting point masses. Furthermore, we assume that both masses are instantaneously at rest. To obtain the acceleration of the masses we may use Eq. (23). This expression does not fully incorporate non-Euclidianity, since Eq. (26) rather than Eq. (27) are implicit, but this is irrelevant for motion in a radial direction.

\[
\frac{d\vec{v}}{dt} = -\frac{1}{2} \nabla g_{00} \text{ for } \vec{v} = 0
\]

Now from Lagrangian Eq. (21) it is clear that the linear momentum is given by:

\[
\vec{p} = \frac{mc_0 \vec{v}}{c \sqrt{1 - (\frac{\vec{v}}{c})^2}}
\]

This agrees with the expression derived by Einstein [18]. It is then clear that the change of momentum of mass 1 and 2 are respectively:
\[
\frac{dp_1}{dt} = \frac{m_1 c_0}{c_1} \frac{dv_1}{dt} = -\frac{m_1 c_0}{2c_1} \tilde{\nabla} c_1^2; \quad \frac{dp_2}{dt} = -\frac{m_2 c_0}{2c_2} \tilde{\nabla} c_2^2
\]

where \(c_1, c_2\) are the values of the \(c\) field (\(\equiv g_{00}^{1/2}\)) at mass 1 and mass 2 respectively. For the gradient we have:

\[
\tilde{\nabla} c_1^2 = \frac{1}{g_{rr_1}} \frac{d c_1^2}{d r} \tilde{e}_r, \quad \tilde{\nabla} c_2^2 = -\frac{1}{g_{rr_2}} \frac{d c_2^2}{d r} \tilde{e}_r,
\]

where \(\tilde{e}_r\) is a unitary vector in the direction of \(\tilde{r} (\equiv \tilde{r}_2 - \tilde{r}_1)\). Using Eq. (16) for \(c (\equiv g_{00}^{1/2})\) and \(g_{rr} (\equiv c_0^2/g_{00})\), we find:

\[
\frac{dp_1}{dt} = m_1 \frac{d c_1^2}{d r} \tilde{e}_r = \frac{m_1 m_2 G}{r^2} \tilde{e}_r; \quad \frac{dp_2}{dt} = -\frac{m_1 m_2 G}{r^2} \tilde{e}_r.
\]

The change of the total momentum, therefore vanishes. The analysis of this simple example makes it clear what was wrong with Einstein’s 1912 theory. In the above expression \(\tilde{r}\) is the actual “metric” velocity (i.e. \(v = \frac{dl}{dt}\), \(l\) being the length of arc). In polar coordinates in the plane of motion:

\[
v = \sqrt{g_{rr} \dot{r}^2 + (r \dot{\theta})^2}
\]

In Einstein’s theory space was flat, so in the present case (\(\dot{\theta} = 0\)) \(v\) is identical to \(\dot{r}\). But in this case instead of Eq. (33) we would have

\[
g_{rr}^{1/2} \frac{d \vec{v}_E}{dt} = -\frac{1}{2} \tilde{\nabla} g_{00}
\]

where \(\vec{v}_E\) is \(\vec{v}\) in Einstein theory (i.e. \(|\vec{v}| = \dot{r}\)). Then we would have:

\[
\frac{dp_1}{dt} \bigg|_E = \frac{1}{g_{rr_1}^{1/2}} \frac{c_0 m_1 m_2 G}{r^2} \tilde{e}_r
\]

and a similar expression for mass 2. This is clearly incompatible with momentum conservation.

As we have seen, confronted with this contradiction Einstein chose to retain the Euclidianity of space and to modify Eq. (4) so as to make it compatible with momentum conservation (and Euclidianity) although it was then incompatible with the strict (i.e. for any field strength) EP.
When sometime latter he realized that space could not remain Euclidean in a gravitational field, he retook the EP, that he had rejected very reluctantly, and started a new path through differential geometry. No discussion and revision of the previous work at the new light can be found in the literature. This may seem quite reasonable: since the non-Euclidianity of space in a gravitational field turns the spatial metric coefficients, $g_{\mu\nu}$, into some sort of gravitational potentials, obtaining an equation for just $g_{00}$ does not look very interesting. But before going after the full theory Einstein could as well have clarified the situation with respect to Eq. (4) and see how far he could have gone with it in some simple cases. In fact, as we have shown here, if he had followed that course and realized of the argument relating $g_{rr}$ and $g_{00}$ in the vacuum case, Schwarzschild’s metric and its consequences would have been around since 1912. The mentioned argument is akin to the one given by Einstein for showing the non-Euclidianity of space in a rotating system, only that somewhat more sophisticated because it involves comparing spatial geometries in flat space-time and curved space-time. Thus, we think that this could have been a realistic course of events. Had this been the case, not only would have been available several of the most relevant results of GR for astrophysics since that early date, but substantial insight would have been added to the full theory when it became available, by showing its direct connection with basic principle through the analysis of simple cases. Furthermore, we think that, after realizing that space could not remain flat in a gravitational field, Einstein initial project of dealing with the static and stationary case could have been fulfilled. It is true that the task then would has exceeded that of the original Einstein intentions, since equations must be obtain to determine all metric coefficients, not just $g_{00}$. However, that task would have been still much smaller than obtaining the full G.R. equations and, again, would have contributed substantial insight into it. We have not so far pursued that course, but we now briefly describe here how we think it could be done.

We have said that Einstein found that Eq. (4) can not be derived from a variational principle (so that energy-momentum conservation is not guaranteed), but we know that that equation is correct (for negligible pressure) and, of course, momentum is conserved. How are this two fact made compatible by a curved space?.. Once it is recognized that the gravitational field affect the space metric, the corresponding metric coefficients become dynamical quantities and, therefore, must be included in the Lagrangian for matter and the gravitational field. The additional term in the Lagrangian must be invariant under spatial
coordinate transformation, so it must be formed out of \( c \) (that is constant under spatial transformations) and the Ricci scalar of the spatial metric, \( R \). Simple considerations lead to a term of the form \( ARc \), with \( A \) a constant. This constant must be chosen so that Eq. (4) is obtained through variation of the Lagrangian with respect to \( c \). Variation with respect to spatial metric coefficients should provide the other equations needed. On the other hand, from the consideration of the flat space-time metric in a rotating system it may readily be shown that the term:

\[
-2\frac{\Omega^2}{c^4}
\]

where \( \Omega \) stand for the rotation speed, must be added to the right hand side of Eq. (4) in the presence of rotation (i.e., in a non globally synchronizable system). Now, the rotation velocity \( \vec{\Omega} \) is related to the coefficients \( g_{0\mu} \) (\( \mu \) from 1 to 3) by

\[
\vec{\Omega} = \frac{c_0}{2} \text{curl}\vec{g}; \quad \vec{g}_{\mid \mu} \equiv -\frac{g_{0\mu}}{c_0^2}.
\]  

(34)

Inferring the term that must be added to the Lagrangian to obtain the modified Eq. (4) and variating the Lagrangian with respect to all metric coefficients, the ten needed equations could be obtained.

This line of development might have difficulties, in particular, it must be noted that the spatial metric coefficients enter the Lagrangian through derivatives of up to the second order (this is true in GR for all metric coefficients). However, it seem, in principle, a reasonable path to follow towards GR and a clarifying analysis once the theory is available. If this approach worked, space-time geometrical formalism would only be needed to deal with time dependent gravitational fields.

We have seen that Einstein obtained Eq. (4) from simple considerations and that GR leads to Eq. (11) which agrees with Eq. (4) only when pressure is negligible. The question now is what was missing in the arguments leading to Eq. (4) and whether Eq. (11) could be derived through some simple consideration that provides that missing element. This question is related to the interesting and very delicate issue of the gravitational effect of pressure and given its relevance for this work we consider it convenient to pay some attention to it.

The source of the gravitational field in GR is the energy-momentum tensor, \( T_{ij} \) (with \( i, j \) from 0 to 3), whose purely spatial components corresponds to pressure (the stress tensor). It
is then no surprise that pressure affect the metric tensor, $g_{ij}$, which are in some sense a set of ten gravitational potentials (they are all needed to determine particle trajectories). However, as we stated before, the coefficient $g_{00}$ is the one that provides the force on a unit mass at rest, so it is in close relationship with the Newtonian potential and the fact that pressure contribute to its sources, as seen in Eq. (11), seems rather baffling. In one of his 1912 papers[17], Einstein discussed the possibility of the gravitational action of stresses (i.e. its ability to weight in a given gravitational field) and rejected it with an argument that involved the EP and the equivalence of mass and energy. He seems to have assumed that what he has proved correctly for the passive coupling of matter to gravity was also true for the active coupling (i.e. the ability of matter to generate a gravitational field). In the Newtonian theory the superposition principle holds and this implies that momentum conservation can only be satisfied if it holds for any couple of interacting elements (i.e. mass points or volume elements, for extended systems), that implies the action and reaction law. In this case it is clear that the active and passive coupling must be equal. In the present case, however, the superposition principle does not hold and, therefore, the equality of action and reactions is not satisfied. This explains that we may have a passive coupling proportional to $\rho$ while the active coupling is proportional to $\rho + 3P$. But for stable and bounded objects interacting at large distances, so that the superposition principle holds, Einstein’s argument applies also to the active coupling, implying a net gravitational effect proportional to the total energy of those objects.

Einstein considered a box filled with radiation, which is an stable bounded system and, therefore, as he had shown, its gravitational effect at distances much larger than its size depends solely on the sum of the masses in the box (for a small box i.e. negligible binding energy). But, having concluded correctly that the net gravitational effect of stresses has to vanish for this system, he arrived at the wrong conclusion (at least implicitly) that the gravitational effect of stresses must vanish (which is exactly true only for its passive role), because he only considered the stresses in the containing walls and not the negative stresses within it, associated with the radiation pressure. In fact, Einstein argument, that we shall see later in more detail, is compatible with stresses contributing to $g_{00}$, because in any stable bound system positive an negative stresses cancel each other. In self gravitating systems with positive pressure the stabilizing stresses are provided by the gravitational field (the energy-momentum tensor of gravity is another delicates issue, but we can not address it
Knowing that there is not a clear argument to exclude pressure from contributing to the gravitational field strength, the question is whether a simple and direct argument can be given to derive Eq. (11). The answer to this question could be that, after all, the simplest version of GR (with only linear term on the Ricci scalar on the Lagrangian and without cosmological term) follows entirely from the strong EP and Eq. (11) follows from it. Thus, the train of thought leading to the presence of the term $3P$ in the Eq. (11) could be considered to be the argument asked for. Because of its interest, we shall review that train of thought. However this scarcely can be considered a direct argument like that leading to Eq. (4), because it goes all the way through GR. We know of a simple argument showing the gravitational effect of pressure, but it involves a non-static field and we prefer to present it in a future work. Einstein, on the other hand, had no reason in 1912 to search for that argument, and, to our knowledge, it was only on completing GR that he realized of the gravitational effect of pressure in the sense discussed here.

To understand the origin of the $3P$ term in Eq. (11) it is interesting to note that in a preliminary theory, where the field equations were:

$$R_{ij} = \frac{8\pi G}{c^2}T_{ij}$$

the equation for $g_{00}$ in the static field would be given by Eq. (4) rather than by Eq. (11). Therefore, it is clear that the key to the questions that we are analysing lies with the origin of the extra term in the right hand side of Eq. (5). This equation is equivalent to:

$$R^*_{ij} \equiv R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi G}{c^2}T_{ij}. \quad (36)$$

Our question has now been reduced to explaining why it is $R^*_{ij}$ rather than $R_{ij}$ that must appear in the left hand side of the field equations. This equations follows outright by variating the simplest GR Lagrangian, but in order to see the connection of our question with physical principles we shall follow the usual explanation, although with a somewhat different presentation.

Both the metric coefficients and the components of the energy-momentum tensor are sets of ten algebraically independent quantities. However, since the choice of coordinates is arbitrary and fixing a coordinate system involves four independent functions (the coordinates here).
conditions), only six of the \( g_{ij} \) and \( T_{ij} \) are functionally independent. That is, once the coordinates have been fixed, all possible energy-momentum distributions can be described by six independent functions. Any sets of ten functions can describe an a priori possible \( T_{ij} \) distribution, but it would correspond to certain intrinsic distribution as represented in certain coordinate system. When the latter has been fixed, not all set of ten functions are possible \( T_{ij} \) distribution. In other words, the intrinsic structural possibilities of both \( g_{ij} \) and \( T_{ij} \) are span by sets of six independent functions. The same applies to \( R_{ij} \), which can be expressed in term of the \( g_{ij} \). In consequence, we should have just six quantities intrinsically characterizing \( R_{ij} \) and \( T_{ij} \), but since on any given system of coordinates there are ten \( R_{ij} \) and \( T_{ij} \), the field equations must be ten equations, although only six must be functionally independent. However, since the \( g_{ij} \) and \( T_{ij} \) are different structures (in fact \( T_{ij} \) is not even specified in terms of the constituting field), the only way in which the number of independent equation can be reduced to six is by having the side of the equation corresponding to gravity, which is geometrical in character and explicitly specified in terms of its constituting fields (\( g_{ij} \)), satisfying four identities. But from Bianchi’s identities it is known that:

\[
R^{*i}_{\ j;i} = 0.
\]

\( R^{*i}_{\ ij} \) must then be the “geometrical” side of the field equations, as shown in Eq. (36). Taking the four divergence on both sides of Eq. (36) we immediately have:

\[
T^{i}_{\ j;\ i} = 0 \quad (37)
\]

that expresses energy-momentum conservation, that appears now as a direct consequence of general covariance (i.e. the laws of physics has the same form in all coordinated systems). We remind, however, that general covariance has no physical content, which is provided by the fact that general covariance, a conventional requirement that should be met by any theory, can be achieved with just matter and gravity, without foreign elements, a fact that follows from the strong EP. If this principle did not hold and certain additional fields caused the laws of physics (matter and gravity) not to be the same in all free falling systems, the argument given above would also be valid, but instead of Eq. (37) we would have the divergence of \( T_{ij} \) plus a tensor build up from the foreign fields set equal to zero. In this case, the conservation that followed from general covariance (that holds by construction) would
not imply the local conservation of energy-momentum (ordinary) alone.

In some works one can see the above argument shortened to saying that, given the empirical fact described by Eq. (37), the left hand side of the field equations must have identically vanishing divergence, so it must be $R^i_{ij}$. This is a complete reversal of the logic of the argument given here, in which the fact of having $R^i_{ij}$ on the left hand side of Eq. (36) followed from general covariance, while energy-momentum conservation follows from Eq. (36), and it is not a valid reasoning. If energy-momentum conservation were merely an empirical fact or consequence of another symmetry (i.e. other than general covariance), Eq. (35) could be valid. We should then have:

$$R^i_{jii} = 0$$

which is not an identity, but there would be no logical impediment to it. Eq. (35), although it satisfies general covariance, it does not follow from a variational principle and, therefore, no conservation law is implied in general by that property. There is a local conservation of matter energy-momentum, that was already there, but not an ordinary conservation of “total” energy-momentum (including gravity). In fact, it was demanding that this be the case (which follows automatically from a variational principle) that Einstein obtained Eq. (34)

We have seen that the $3P$ term in Eq. (11) is a consequence of general covariance, or rather, of the fact, implied by the strong EP, that it can be achieved with only matter and gravity. Now we shall use Einstein argument showing that only mass gravitates (valid for stable bounded objects) to obtain an expression for the mass of an static and bounded spherical distribution of energy-momentum.

Considerer an static and bounded distribution of mass and pressure. It is clear that at large enough distances the gravitational potential takes the form:

$$\phi \sim \frac{GM}{r} \quad (38)$$

The constant $M$ we call the mass of the distribution. By considering the acceleration experienced by the whole system within a uniform gravitational field and using the EP Einstein showed that the gravitational mass of the system, $M$, must be equal to the inertial mass, that is, it is the latter that couples passively to the gravitational field ($g_{00}$) and the
same is true for the active coupling when computing weak fields. Note that this argument does not apply to a portion of the system. A volume element, \( dV \), couples actively to \( g_{00} \) not through its mass, but through its mass plus \( 3P \) times \( dV \), as we discussed earlier. Now, as pointed out by Einstein, it is clear that the total energy and momentum of the system must depend on the velocity of its center of mass like a mass point in special relativity. This is true even when the latter theory does no hold in a region around the system where the field is sufficiently strong and it can be proved rigorously by considering energy-momentum conservation for a set of systems that interact weakly (the field on any system due to all other systems being weak) without changing their inner structure. From this fact, the proportionality between mass (inertial) and total energy follows immediately. In consequence, \( M \) in Eq. (38) must be given by the total energy of the system. It would be tempting to write:

\[
M = \int \rho dV + \frac{1}{2c_0^2} \int \left( \frac{g_{00}^{1/2}}{c_0} - 1 \right) \rho dV
\]  

(39)

where the second term is the gravitational binding energy. However, writing the binding energy as one half of the total gravitational energy (sum of the gravitational energy of all mass elements) is only an approximation, since the presence of \( c \) on the right hand side of Eq. (11) (or Eq. (4)) makes it clear that the superposition principle does not hold (in consequence, the energy of mass \( i \) due to mass \( j \) is not symmetric in \( i, j \)). But we shall see now that in the case of a spherically symmetric cloud \( M(r) \) can be obtained exactly using the vacuum case solution. This expression for \( M(r) \) can be used to obtain the full solution (\( g_{00} \) and \( g_{rr} \)) in the non-homogeneous spherical case, as we have shown in the previous section. If matter and pressure distributions beyond radii \( r \) do not affect the solution within \( r \), the value of \( g_{rr} \) at \( r \) must be given by Eq. (18)

\[
g_{rr}(r) = \left( 1 - \frac{2GM(r)}{c_0^2r} \right)^{-1}
\]

with \( M(r) \) the mass enclosed within \( r \). The value of \( g_{00}(r) \) when normalized so that \( g_{00}(0) = c_0^2 \) is also unaffected by matter and pressure outside \( r \), but when, as usual, is normalized so that \( g_{00}(\infty) = c_0^2 \), there is a dependence on outside matter. To obtain \( M(r) \), we shall assume that there is no matter beyond \( r \) (which would be irrelevant under our assumption) and compute the total work, \( W \), that must be exerted on the cloud to disperse
it by taking layer after layer to infinity (obviously, \( W \) is numerically equal to the binding energy). \( W \) is clearly given by:

\[
W(r) = -\int_0^r \left( \frac{g_{00}(r')^{1/2}}{c_0} - 1 \right) \rho(r') 4\pi r'^2 g_{rr}^{1/2}(r') \, dr'
\]

(40)

where \( g_{00}^{1/2}(r)/c_0 \) is the total energy of a unit mass at \( r \), including the rest mass, while \( g_{00}^{1/2}(r)/c_0 - 1 \) is its gravitational energy. We have seen that \( M(r) \) must be equal to the total energy (divided by \( c_0^2 \)), which, in turn, must be equal to the energy of the infinitely dispersed cloud minus \( W \). But the former is simply the rest energy of the cloud:

\[
\int_0^r \rho(r') 4\pi r'^2 g_{rr}^{1/2}(r') \, dr.
\]

(41)

Subtracting Eq. (40) from this quantity (and dividing by \( c_0^2 \)), we have for \( M(r) \):

\[
M(r) = \frac{1}{c_0^2} \int_0^r g_{00}^{1/2}(r') \rho(r') 4\pi r'^2 g_{rr}^{1/2}(r') \, dr'.
\]

(42)

It must be noted that \( g_{00}(r'),g_{rr}(r') \) in this expression are not the actual ones within the cloud but the ones that will exist after all layers above \( r' \) had been removed, that is, the ones corresponding to the vacuum case solution with mass \( M(r') \). This is so because layer \( r' \) is carried to infinity not through the actual field but through the field generated by \( M(r') \). But for the vacuum case solution the product: \( g_{00}^{1/2} g_{rr}^{1/2}/c_0 \), is equal to one. Thus, we must have:

\[
M(r) = \frac{1}{c_0^2} \int_0^r \rho 4\pi r'^2 \, dr'
\]

(43)

which agrees with the result obtained within the full GR[13].

It must be noted that Eq. (43) it is not the sum of the masses, which is given by Eq. (41). It is the sum of the mass elements multiplied by their corresponding values of \( g_{00}^{1/2}(r') \). Note also that it is not the sum of the total energies of the mass elements, because \( g_{00} \) is not the actual one within the cloud extending up to \( r \), but the vacuum solution with mass \( M(r') \).

It must be remarked that the meaning of \( M(r) \) is the mass that must enter the vacuum solution that would exist if all matter beyond \( r \) was removed and that it must be equal to the total energy within \( r \). But for the latter to be true, the remaining system (the matter within \( r \)) must be stable. This can only be achieved by enclosing the system in an spherical
container with radius \( r \) to withstand the pressure, \( P(r) \), after the outside material, has been removed. It is only in this circumstances that there will be a vacuum solution beyond \( r \) with \( M \) given by Eq. (42). The question now is: does the presence of the container affect the solution at \( r \)? If the presence of the container, that is massless and arbitrarily thin, was immaterial, \( g_{00}(r) \) (normalized to the center of the cloud), \( g_{rr}(r) \), must be equal to the corresponding value in the actual solution (before removing the outer matter), because the outer matter does no affect them. But we know that this cannot be, because otherwise \( g_{00} \) would not depend on pressure, in contradiction with Eq. (11).

One may discard the possibility of \( g_{rr} \) depending on the enclosed pressure-volume by the following consideration: take an arbitrarily rigid spherically symmetric solid. It is possible to redistribute the tension (remaining in equilibrium) increasing it in the outer parts and diminishing it in the inner ones spending an arbitrarily small amount of energy. This can be done, for example, by heating the outer layers, since for given thermal properties the energy needed diminishes as the elastic modulus increases. But if \( g_{rr} \) depended on the enclosed pressure-volume, its value in the inner parts would change, because of the outwardly transferred pressure-volume. This would imply an arbitrarily large increase of the elastic energy not compensated by a diminishing gravitational energy (which increases slightly due to the small expansion of the body). This is in contradiction with energy conservation, although the reader may also note another inconsistency hidden in this argument that is independent of energy conservation.

Consequently, \( g_{rr} \) must be equal at both sides of the arbitrarily thin container and, therefore, must be given by Eq. (18), that, consistently, tell us that \( g_{rr} \) depends only on the enclosed mass. On the other hand, for \( g_{00} \) Eq. (11) shows the relevance of the enclosed pressure-volume.

From Eq. (11) we have:

\[
\vec{n} g_{00}^{1/2}(2) - \vec{n} g_{00}^{1/2}(1) = \frac{4\pi G}{c_0^2} \frac{g_{00}^{1/2}(r) \int PdV}{r^2} \epsilon_r
\]

(44)

here 2, 1 distinguish respectively the solution just outside and just inside the container and where the volume integral is over the walls of the container. \( \vec{P} \) stand for the pressure in this wall (in fact a tension) no to be confused with the pressure, \( P(r) \), positive for usual fluids, which is exerted by the fluid upon the wall. Simple equilibrium considerations lead to the following relationship between the tension, \( T \) on surface elements whose vectors are
within the plane tangent to the container walls and $P$.

$$T = \frac{rP(r)}{2\Delta r}; \quad 3\bar{P} = \frac{rP(r)}{\Delta r}$$

where $\Delta r$ is the physical thickness of the container walls and where the last equality follows from the fact that $P$ is the average of the pressure in 3 orthogonal directions, and in the direction perpendicular to the walls the tension is zero. Inserting this in Eq. (44) we find:

$$\frac{1}{g_0^{1/2}} \frac{\partial g_0^{1/2}}{\partial r}(1) = \frac{1}{g_0^{1/2}} \frac{\partial \bar{g}_0}{\partial r}(2) + \frac{4\pi G}{c_0^4} g_0^{1/2} r P(r).$$

For $g_0$ itself it is no necessary to distinguish between the inner and the outer value, because it is continuous. It is convenient to write this equation in the form:

$$\frac{1}{2g_0^{1/2}} \frac{\partial \ln g_0}{\partial r} = \frac{1}{2g_0^{1/2}} \frac{\partial \ln \bar{g}_0}{\partial r} + \frac{4\pi G}{c_0^4} g_0^{1/2} r P(r)$$

(45)

which gives the relationship between the inner (actual solution) field strength and the outer one given by the vacuum solution, $\bar{g}_0$

$$\bar{g}_0(r) = c_0^2 \left( 1 - \frac{2GM(r)}{c_0^2 r} \right).$$

It must be noticed that in derivating $\bar{g}_0$ in Eq. (45) $M(r)$ must be hold fixed, because there is no matter beyond $r$ in the hypothetical situation that we are considering. However, on integrating this expression to obtain the solution for the actual distribution, $M(r)$ is not a constant. We then have for $g_0$:

$$\ln g_0 = -\frac{1}{c_0^2} \int_r^\infty \left( \frac{2GM(r')}{r'^2} + \frac{8\pi Gr'}{c_0^2} P(r') \right) \left( 1 - \frac{2GM(r')}{c_0^2 r'^2} \right)^{-1} dr'$$

(46)

with the normalization $g_0(\infty) = c_0^2$. This is equivalent to the expression derived by Weinberg (REF: Weinberg pag 302) with the standard procedure. In this procedure the equation of hydrostatic equilibrium is hidden amongs the three independent equations. Here we can derive that equation from direct considerations, obtaining:

$$\frac{1}{\sqrt{g_0}} \frac{\partial P \sqrt{g_0}}{\partial r} = -\rho \frac{\partial \ln g_0^{1/2}}{\partial r}$$

(47)
where the presence of the factor $\rho$ is due to the fact that the weight of a volume element in a gravitational field is proportional to $\rho$. The presence of $g_{00}$ in the left hand side comes from the fact that the total energy (including gravitational energy) transferred by a unit force when the body upon which is exerted is displaced by a unit of length it is not one unit of energy, but $\frac{g_{00}^{1/2}}{c_0}$.

Using Eq. (43) in Eq. (45) and if $P$ is a unique function of a second order differential equation for $M(r)$ is obtain. Also, for a polttropic model ($\rho = A P^\alpha$, $\alpha < 1$), we can integrate Eq. (45) to obtain a relationship between $P$ or $\rho$ and $g_{00}$:

$$P = \left(A(g_{00}^{\frac{1}{2\alpha}} - 1)\right) \frac{1}{1 - \alpha}.$$  (48)

It is clear, however, that these models can not correspond exactly to finite mass object: for $\alpha < 3/4$ as well as for the case $\alpha = 1$, which must be treated separately.

V. SUMMARY AND CONCLUSIONS

We have reviewed the arguments given by Einstein in the derivation of his first, non-Riemmanian, theory of gravitation (1912). We have weighed carefully these argument and strengthened some points that could look fragile in the original presentation. We concluded that, except for the argument proving that only matter gravitates, these arguments are cogent enough as to leave no serious doubt about the validity of Einstein first field equation, at least for negligible pressure, and that its ultimate rejection by Einstein was due to a misidentification of the culprit of the contradiction he was finding. Once we concluded that Einstein first gravity field equation has to be correct (almost), we proceeded to demonstrate that this equation follows from the field equations of GR when pressure is negligible. In general we find that the source of “the gravitational field” ($g_{00}$) is mass density plus three times the pressure, rather than mass density alone. We have shown that with Einstein first field equation (Eq. (4)) and a basic argument relating $g_{rr}$ and $g_{00}$ the spherically symmetric vacuum case (called Schwarzschild solution when treated within the full GR) can be obtained immediately both with and without cosmological constant, and have pointed out that the metric for a charged mass “point” can easily be obtained in the same manner. We noted that in the static case the field equation remains formally equal to that in the Newtonian case.
(Laplacian of the field equal to zero), only that replacing the usual gravitational potential, $\phi$, by $c$:

$$c \equiv c_0 e^{\phi/c_0^2}.$$ 

But we also noted that there is an important difference between both equations hidden in the Laplacian, because while in the Newtonian case, being space flat, $g_{rr}$ is equal to one, in the relativistic one $g_{rr}$ depends on $g_{00}$. It have been shown that, although in the case of a general spherically symmetric distribution no algebraic relationship between $g_{00}$ and $g_{rr}$ exist, still it may be solved on basic principles. To this end we used results obtained in the vacuum case, an expression for the mass enclosed within radii $r$ obtained from first principles and the assumption that $g_{rr}(r)$ only depends on matter within $r$. We have written particle trajectories in a form that can easily be compared with its Newtonian counterpart, in order to be able to see in a conceptual manner (i.e. not merely studying the differences at various orders in $v$ and $\phi$) the origins of the differences of particles trajectories equations around a “point” mass in Newtonian and Einsteinian theories. We showed that the differences are three fold. First it is the fact that the gravitational field strength is different from the Newtonian one, a difference whose origin has been explained before. However, it enter in the motion equation in such a manner that the corresponding term agrees exactly with it Newtonian counterpart. Then there is a dependence of the gravitational “force” on velocity that followes immediately from the EP (non-Riemmanian), with a term proportional to $v^2$. This dependence is obviously contained in Einstein’s 1912 theory. Finally it is the fact that the geometry of a plane containing the central point it is not Euclidean. We have shown that this effect can be reduced to replacing some simple relationships valid in the Euclidean plane by the corresponding ones in the actual plane, that can be derived from elementary geometrical considerations. Within this analysis we have computed the angular velocity of the perihelion of quasi-Keplerian orbits and found that $2/3$ of it are due to the velocity dependence (i.e. to EP), the other third being due to non-Euclideanity. We have also used the same approach for computing the deflection of light rays, showing in a simple manner the well known result that $1/2$ of the effect is “Newtonian” and the other half comes from non-Euclidianity, with the velocity dependent term playing no role.

By considering the gravitational interaction of two “point” masses, we have made patent how the assumption of flat space causes the non-conservation of momentum that plagued
Einstein first gravitational theory. We have then discussed a possible path of development of GR that could have happened if, after realizing that the flatness of space was untenable, Einstein had removed the error from his first gravitational theory and continued his initial research program.

We have discussed Einstein arguments showing that pressure cannot gravitate and why it fails. Then we reviewed the argumental line leading to the presence of pressure in Eq. (11), showing that it is a direct consequence of “general covariance” and, therefore, closely related to energy-momentum conservation. We have concluded that of all questions treated in this work this is, arguably, the most difficult to have been anticipated in 1912 and, although we have thought of an argument proving this fact, we have not presented it here because it involves non-static fields. Finally, we have used Einstein argument relating gravitational mass and total energy to obtain an expression for the mass enclosed within radii \( r \) in a general spherically symmetric distribution, which is instrumental in deriving the metric in this case. This we have done in two manner, first using the expression for \( g_{rr} \) obtained with the mentioned argument in Eq. (11) and then by extending this argument to obtain \( g_{00} \) through direct physical considerations. We also give a direct derivation of the equation of hydrostatic equilibrium and discuss how to use it together with the expression for \( g_{00} \) and \( P(\rho) \) to obtain the equilibrium configuration of a self-gravitating non-rotating cloud. In the standard procedure, using Einstein equations (GR), this is obtained in a non very transparent manner.

The general aim of this work has been to try to understand the most elementary result of GR in terms of basic principles. To this end Einstein’s 1912 gravity equation has been instrumental along with the EP, energy-momentum conservation and the mass energy relationship. The results that we have derived are not only of heuristic value, which has been our main goal, but are amongst the most relevant result of GR for astrophysics.

In all other fields of physics we are used to dealing with trivial cases in a simple manner, identifying the relevant basic principles in a problem to anticipate the aspect of it that can quickly be learned and combining elements of different solutions (not necessarily superposing them) to gain some insight into the actual problem. In GR, however, this usual heuristic is almost entirely lacking. Confronted with a particular problem it is the geometrical symmetry that determine its treatment and then we switch on our elegant mathematical formalism to obtain the solution, gaining little insight in the process. Then, when extracting the physical
meaning of the solution, it is not done, in our opinion, in the most enlightening manner, even in the simple cases in which more clarifying procedures seems possible.

Take the case of the spherically symmetric vacuum solution. In the homologous case in electromagnetism we know in advance that there is just one quantity, the electrostatic potential, to be determined. How could a point endowed with just one type of charge determine two different (not a function of each other) radial dependences? In the present case, however, we have two equations for \( g_{rr} \) and \( g_{00} \) and only after integrating them it is found, in an nontransparent manner, that they are simply related, a relationship that could has been advanced based on the EP.

In the case of a spherical cloud in equilibrium we have, in GR, three independent equations. Certain combination of them must give the hydrostatic equilibrium equation, but this is not clear at all in most expositions.

For the spherically symmetric vacuum solution particle trajectories can be determined exactly in the most elegant manner using Hamilton-Jacobi formalism. But then the origin of the relativistic effects is completely masked. Approximate treatments as that given by Einstein[4] are somewhat more transparent, but still, in our opinion, insatisfactory. General three-dimensional treatments[12] are much more helpful, but still, having to compute explicitly the three-dimensional affine connections seems without proportions with the intrinsic simplicity of the problem. Here we have shown how simply can this problem be treated exactly, and not for the reason of the economy of work but to gain in transparency and insight.

The questions we have treated here are not specialized ones, but very basic and of interest for any physicist with an eye on GR. The fact that at least some of them has not been treated, or rather, that they are not treated where obviously they should have been, seems to us a sort of anomaly, since there is no question here that could not has been treated and be widely known at least 80 years ago.

It is true that in GR we are deprived of our most valued source of intuition, the “solidity” of space, that time is not what it used to be and then we have the non-linearities, tensorial field and source, etc. But this should be an stimulus to work harder and not something to make us stagger back and stick to an infallible but obscure formalism renouncing to develop a detailed physical intuition of GR for fears of being betrayed by it too often.

The terrain is treacherous, and we may atest to it. The simple and, we hope, very easy to
follow derivations given here were (some of them) not easy to obtain. When trying to reduce the result searched for to a combination of simple ideas, the above mentioned complexities of GR made progress difficult, but now that we have managed to treat the questions presented here in a consistent manner, we can only hope that this questions enhance the understanding of GR of the reader as much as it did with ours.

In the presentations of GR the dependence of the relative rate of two clocks on a gravitational field on the difference of potential between them and the related gravitational redshift are directly derived in a simple manner from the the EP. Some of them even discuss the non-Euclidianity of space in a rotating system in flat time-space. But beyond this point direct derivations from basic principles are no longer given, not even heuristics ones, being replaced by mathematical considerations. This mimic the development of GR: when Einstein met with overwhelming difficulties in 1912, he switched to a mathematically guided line of research. In consequence, elementary facts and considerations, as how do genuine gravitational static gravitational fields affect the space geometry, how and why the passive and active couplings of matter to gravity (the field $g_{00}$) do differ and why the latter include pressure, or how the implications of the EP in the case of flat space combine with the effect of the spatial curvature in determining particle trajectories, are not discussed. It is the aim of this work to contribute to ease this situation, that would have been accepted in no other field of physics.

We are aware of the fact that, although some of the simple derivations given here could be used as a base for an elementary exposition of GR, such as it stands it is rather an advanced one, being addressed to those who know well the standard presentations of the facts treated here. Our goal has been to make patent to those readers the concepts and principles involved in these facts and how simply could they be derived. We also want to remark that, even when we make some comments on the history of GR, this is not even partially a work on the history of physics. This is a work on physics, and those comments have been made to stress underlying conceptual issues. Nevertheless, we think that the historical facts that we comment are basically correct, although when we refer to Einsteins conceptions with respect to some question at a given date it may be only our best interpretation of what we actually
know.

[1] Jahrbuch der Radioaktivitt und Elektronik 4, p. 411. English version Swartz, H.M., (1977 a,b,c), Am. J. Phys 45, 5512-517, 811-817 899-902)

[2] H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, with notes by A. Sommerfeld, tr. by W. Perrett and G. B. Jeffery.

The principle of relativity, a collection of original memoirs on the special and general theory of relativity. (Dover Publications, Inc., 1923), p. 97–108.

[3] Albert Einstein, Trans. Anna Beck, Cons. Peter Havas, The collected papers of Albert Einstein, Vol. 4: The Swiss Years: Writings, 1912-1914. (Princeton University Press, 1994), p. 95–106.

[4] Albert Einstein, Trans. Edwin Plimpton Adams, The Meaning of Relativity. Four lectures delivered at Princeton University, May, 1921. (Princeton University Press, Princeton, 1923).

[5] Abraham Pais, Subtle is the Lord: The Science and the Life of Albert Einstein, first edition. (Clarendon Press-Oxford University Press, New York-Oxford, 1982).

[6] Howard d., Stachel J., Einstein studies Vol. 1: Einstein and the history of General Relativity (Birkhauser, 1986).

[7] Die Theorie des starren Elektrons in der Kinematik des Relativittsprinzips, (The Theory of the Rigid Electron in the Kinematics of the Principle of Relativity). Annalen der Physik 335 (11), 1-56.

[8] Wolfgang Rindler, Essential Relativity: Special, General and Cosmological. (Springer; 2nd edition, 1997), p. 49–51.

[9] L.D. Landau, E.M. Lifshitz, The classical theory of fields, Vol. 2. (Morton Hamermesh; 4th edition, 1987).

[10] L.D. Landau, E.M. Lifshitz, The classical theory of fields, Vol. 2. (Morton Hamermesh; 4th edition, 1987), p. 302.

[11] L.D. Landau, E.M. Lifshitz, The classical theory of fields, Vol. 2. (Morton Hamermesh; 4th edition, 1987), p. 254.

[12] L.D. Landau, E.M. Lifshitz, The classical theory of fields, Vol. 2. (Morton Hamermesh; 4th edition, 1987), p. 253.

[13] L.D. Landau, E.M. Lifshitz, The classical theory of fields, Vol. 2. (Morton Hamermesh; 4th
[14] Jose M. Sanchez Ron, *Origen y desarrollo de la Relatividad*. (Aliaza universidad, S.A., Madrid, 1983), p.150.

[15] A.Eddington, *The Mathematical Theory of Gravitation*. (Cambridge University Press, Cambridge, 1923).

[16] S. Weinberg, *Gravitation & Cosmology*. (Wiley & sons Cambridge, New York, 1972).

[17] Lichtgeschwindigkeit und Statik des Gravitationsfeldes, (The Speed of Light and the Statics of the Gravitational Field). Annalen der Physik 38, section 4, p. 443-458.

[18] Lichtgeschwindigkeit und Statik des Gravitationsfeldes, (The Speed of Light and the Statics of the Gravitational Field). Annalen der Physik 38, section 2, p. 355-69.