Hyperelliptic Integrals and Mirrors of the Johnson–Kollár del Pezzo Surfaces

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Abstract

For all \( k > 0 \) integer, we show explicitly that the hypergeometric function

\[
\tilde{I}_k(\alpha) = \sum_{j=0}^{\infty} \frac{((8k+4)j)!j!}{(2j)!((2k+1)j)!(4k+1)j)!} \alpha^j
\]

is a period of a pencil of curves of genus \( 3k + 1 \). The function \( \tilde{I}_k \) is the regularised \( I \)-function of the family of anticanonical del Pezzo hypersurfaces \( X = X_{8k+4} \subset \mathbb{P}(2, 2k + 1, 2k + 1, 4k + 1) \) and the pencil we construct is a candidate LG mirror of the elements of the family. The surfaces \( X \) were first constructed by Johnson and Kollár [17]. The main feature of these surfaces, which makes the mirror construction especially interesting, is that \( |-K_X| = |O_X(1)| = \emptyset \); thus, there is no way to form a Calabi–Yau pair \( (X, D) \) out of \( X \). We also discuss the connection between our constructions and the work of Beukers, Cohen and Mellit [1] on hypergeometric functions.

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1 Introduction

We begin by stating our results; then we briefly comment on the context and on the methods of our proofs; we conclude with a few open questions.

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1.1 Results

We state our results; more detail will be given in Section 2 and in Section 4.

For all $k > 0$ integer, consider the hypergeometric function defined by the power series:

$$
\hat{I}_k(\alpha) = \sum_{j=0}^{\infty} \frac{((8k+4)j)!j!}{(2j+1)(4k+1)j!^2} \alpha^j.
$$

(1)

The function $\hat{I}_k$ satisfies a hypergeometric differential equation of order $6k+2$, see Remark 13.

Conjecturally $\hat{I}_k$ is (a shift of) Givental’s $\hat{G}$-function — a generating function of certain Gromov–Witten invariants, see Remark 8 — of the family of anticanonical del Pezzo hypersurfaces

$$X_{8k+4} \subset \mathbb{P}(2,2k+1,2k+1,4k+1)
$$

(2)

These surfaces form the main series of the classification of anticanonical quasi-smooth and well-formed two-dimensional weighted hypersurfaces [17], see Section 2.2.

In this paper we give positive answers to the following questions which, as explained in Section 2.1, for us are equivalent:

**Question 1.** Is $\hat{I}_k$ a period of a pencil of curves?

**Question 2.** Does the family of anticanonical del Pezzo hypersurfaces of Equation (2) have a Landau–Ginzburg (LG) mirror?

For all $k > 0$ integer, our answer to both questions is given by the one-parameter family of hyperelliptic curves:

$$Y_k = (\alpha y^2 - h_{k,\alpha}(t_0,t_1) = 0) \subset \mathbb{P}(1,1,3k+2) \times \mathbb{C}^\times
$$

(3)

where $t_0, t_1, y$ are coordinates on $\mathbb{P}(1,1,3k+2)$, $\alpha$ is a coordinate on $\mathbb{C}^\times$ and

$$h_{k,\alpha}(t_0,t_1) = t_1(4t_1^{2k+1} + \alpha t_0^{2k+1})(-64t_1^{4k+2} + t_0 t_1^{4k+1} - 32\alpha t_0^{2k+1} t_1^{2k+1} - 4\alpha^2 t_0^{4k+2})
$$

(4)

together with the projection $w_k : Y_k \to \mathbb{C}^\times$ to the second factor.

**Remark 3.** Let

$$\alpha_{k,0} = \frac{(4k+1)^{4k+1}}{4^{8k+3}(2k+1)^2(2k+1)}
$$

(5)

For $\alpha \neq \alpha_{k,0}$, the fibre $Y_{k,\alpha} = w_k^{-1}(\alpha)$ is a nonsingular hyperelliptic curve of genus $3k+1$.

Our first result is:

**Theorem 4** (Main Theorem). For all $k > 0$ integer, $\hat{I}_k(\alpha) = \pi_k(\alpha)$, where

(A) $\hat{I}_k(\alpha)$ is the hypergeometric function of Equation (1) above, and

(B) $\pi_k(\alpha)$ is the period of the family $Y_k$ (3) given by:

$$\pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \frac{t^{2k} dt}{y}
$$

(6)

where $\gamma_{k,\alpha} \subset Y_{k,\alpha}$ is the explicit cycle described in Section 3 and $t = t_1/t_0$. 

2
The solutions of the differential equation satisfied by \( \tilde{I}_k \) are sections of an irreducible complex local system on \( U_k = \mathbb{P}^1 \setminus \{ 0, \alpha_k, 0, \infty \} \), which we denote by \( \mathbb{H}^\text{red}_k \), see Remark 13.

Write \( Y_{U_k} = w_k^{-1}(U_k) \) and denote by \( w_{U_k} : Y_{U_k} \to U_k \) the restriction.

By the main Theorem and Remark 12 \( \mathbb{H}^\text{red}_k \) is a subquotient of the local system \( R^1 w_{U_k} \ast \mathbb{C} \). Since the two local systems have the same rank, we obtain:

**Corollary 5.** For all \( k > 0 \) integer, \( \mathbb{H}^\text{red}_k \) is isomorphic to \( R^1 w_{U_k} \ast \mathbb{C} \).

Our next result connects our construction to the work of Beukers, Cohen and Mellit [1]. More detail on this discussion can be found in Section 4.2. For all \( k > 0 \) integer, consider the manifold:

\[
W_k = \left( \alpha \cdot (u_1 + u_2 + u_3 + u_4 - 1) - u_1^2 u_2^{2k+1} u_3^{2k+1} u_4^{4k+1} = 0 \right) \subset \mathbb{T}^4 \times \mathbb{C}^X
\]

where \( \mathbb{T}^4 \cong (\mathbb{C}^\times)^4 \) is a 4-dimensional torus with coordinates \( u_1, \ldots, u_4 \) and \( \alpha \) is a coordinate on \( \mathbb{C}^\times \). Denote by \( v_k : W_k \to \mathbb{C}^\times \) the second projection. The 3-dimensional hypersurface \( W_{k, \alpha} = v_k^{-1}(\alpha) \subset \mathbb{T}^4 \) is nonsingular if and only if \( \alpha \neq \alpha_{k,0} \).

A very special case of the work [1] — which we summarise in Section 4.1 — relates point counting in characteristic \( p \) on \( W_{k, \alpha} \) to finite analogs of the function \( \tilde{I}_k \) of Equation (1). In Section 4 we prove:

**Theorem 6.** For all \( k > 0 \) integer, write \( W_{U_k} = v_k^{-1}(U_k) \) and denote by \( v_{U_k} : W_{U_k} \to U_k \) the restriction. Then:

\[
\text{gr}_3^W R^3 v_{U_k} \ast \mathbb{Q} (1) = R^1 w_{U_k} \ast \mathbb{Q}
\]

This gives an interpretation of a special case of the results in [1] in terms of mirror symmetry.

1.2 Context and a few words on our proofs

Our mirrors of the surfaces \( X = X_{8k+4} \subset \mathbb{P}(2,2k+1,2k+1,4k+1) \) are not covered by any mirror construction known to us. Indeed, since \( H^0(X, -K_X) = H^0(X, \mathcal{O}_X(1)) = 0 \), there is no Calabi–Yau pair \((X,D)\), and hence the intrinsic mirror symmetry program [2] does not apply to this context.

By [18] Theorem 5.4.4] the local systems \( \mathbb{H}^\text{red}_k \) support a canonical rational variation of Hodge structures (VHS). By the criterion of [4] [8], the variation has Hodge weight one. Thus, it is natural to ask — even as there is no reason to expect it — if \( \mathbb{H}^\text{red}_k \) is a (direct summand of) the variation of \( H^1 \) of a one-parameter family of curves: this motivates our Question 1. Since an irreducible local system supports at most one rational VHS, see [5] Proposition 2.1], our main Theorem implies that \( \mathbb{H}^\text{red}_k = R^1 w_{U_k} \ast \mathbb{Q} \) as VHS.

The general shape of the Fano/LG correspondence suggests that the mirror of the family of anticanonical del Pezzo hypersurfaces \( X_{8k+4} \) is a function \( w_k : Y_k \to \mathbb{C}^\times \) with one-dimensional fibres, smooth over \( U_k \), together with an identification of \( R^1 w_{U_k} \ast \mathbb{C} \) with the hypergeometric local system of solutions of the differential equation satisfied by \( \tilde{I}_k \). This motivates our Question 2.

Our proof of the main Theorem is elementary: we expand the period in power series with the help of the residue theorem. The key difficulty is to find the equation of \( Y_k \) (and the integration cycles \( \gamma_k \)).

The work [1] and in particular the pencil of 3-folds \( v_k : W_k \to \mathbb{C}^\times \) of Equation (7) are the starting point for our investigations. Morally, this work identifies the local system \( \mathbb{H}^\text{red}_k \) with \( \text{gr}_3^W R^3 v_{U_k} \ast \mathbb{C} \) — more detail on this point can be found in Sections 4.1 and 4.2 — and thus provides a mirror
of the wrong dimension. Then it is natural to ask if there is a morphism \( w_k : Y_k \to \mathbb{C}^\times \) with one-dimensional fibres such that \( \text{gr}_W^3 R^3 w_{Y_k} \otimes \mathbb{Q}(1) = R^1 w_{Y_k} \otimes \mathbb{Q} \). Our Theorem \( 6 \) states that our family of hyperelliptic curves indeed has this property. The constructions in the proof of the Theorem in Section \( 5 \) make it clear how the curve \( Y = Y_{k,\alpha} \) arises naturally from a study of the geometry of the 3-fold \( W = W_{k,\alpha} \) for \( \forall \alpha \neq \alpha_{k,0} \).

The idea of the proof of Theorem \( 6 \) is to construct a partial compactification \( W \subset \hat{W} \) with a del Pezzo fibration \( \phi : \hat{W} \to \mathbb{C}^\times \). The del Pezzo fibration becomes visible after a monomial substitution of coordinates, see Equation \( 26 \). In the Appendix, we explain how we discovered the del Pezzo fibration by running a minimal model program for a partial resolution of a compactification of \( W \). Once we have the del Pezzo fibration, the proof of Theorem \( 6 \) is an exercise in mixed Hodge theory for which models exist in the literature. The key point is Lemma \( 24 \) which constructs an algebraic cycle \( Z \subset \text{CH}_2(\mathcal{Y} \times_{\mathbb{C}^\times} \hat{W}) \) inducing an isomorphism \( p_* \mathbb{Q}_Y \to R^2 \phi_* \mathbb{Q}_{\hat{W}}(1) \).

It is natural to ask if our computation of the cohomology of \( W \) in terms of the del Pezzo fibration \( \phi : \hat{W} \to \mathbb{C}^\times \) really is the easiest way to prove Theorem \( 6 \) We think that it is. In Section \( \Box 3 \) we construct a compactification \( W \subset W' \) and a conic bundle structure \( \psi : W' \to F_1 \) birational to \( W \); this gives another, more complicated, way to understand the geometry of \( W \).

1.3 Further questions

It may be possible to determine all hypergeometric functions that are periods of a pencil of curves, and describe the pencils explicitly.

There are several contexts where a construction of mirror symmetry is available that produces a mirror of the wrong dimension: one of these is general toric complete intersections \([11, 1]\); another is the abelian/nonabelian correspondence \([2]\). Our ideas here can form the basis of a systematic method to extract from these constructions mirrors of the correct dimension. In particular it would be extremely attractive to eliminate the assumptions of Hori–Vafa \([11]\) and obtain mirrors for all toric complete intersections.

It will be interesting to see that our mirrors satisfy Homological Mirror Symmetry.

1.4 Structure of the paper

All sections in this paper logically depend on the Introduction; other than that, they are logically mutually independent and can be read (or not read, as the case may be) in any order.

The paper is structured as follows. In Section \( 2 \) we expand on the notion of mirror symmetry for Fano anticanonical weighted hypersurfaces. The families of surfaces \( X_k \) are an example of such hypersurfaces; our main Theorem can be interpreted as giving mirrors of these families. In Section \( 3 \) we prove the main Theorem. In Section \( 4 \) we show how our Equation \( 7 \) arises from the work \([1]\) as a mirror of the wrong dimension. In Section \( 5 \) we prove Theorem \( 6 \). In the Appendix we explain how we discovered the del Pezzo fibration \( \phi : \hat{W} \to \mathbb{C}^\times \). The Appendix ends with the construction of a compactification \( W \subset W' \) with a conic bundle structure \( \psi : W' \to F_1 \).

1.5 Acknowledgments

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2 Anticanonical weighted hypersurfaces and mirror symmetry

In this Section we expand on the notion of mirror symmetry for Fano anticanonical weighted hypersurfaces. The families of surfaces $X = X_d \subset \mathbb{P}(a_0, \ldots, a_m)$ of the Introduction are an example of such hypersurfaces; our main Theorem can be interpreted as giving mirrors of these families.

We denote by $\mathbb{P}(a_0, a_1, \ldots, a_m)$ the weighted projective $m$-space with weights $a_0 \leq a_1 \leq \cdots \leq a_m$; we simply write $\mathbb{P}$ when $m$ and the weights are clear from the context.

For $X = X_d \subset |O_{\mathbb{P}}(d)|$ a quasismooth [12, 6.3] and wellformed [12, 6.10] hypersurface of degree $d$, the adjuction formula for the canonical sheaf $K_X [12, 6.14]$ states that

$$-K_X = O_X (d - \sum_{i=0}^m a_i).$$

By definition $X$ is Fano if and only if $-K_X$ is ample, i.e., if and only if $d < \sum_{i=0}^m a_i$. We say that $X$ is anticanonical if $-K_X = O_X (1)$, that is, $d = \sum_{i=0}^m a_i - 1$.

2.1 Our notion of mirror symmetry

We state what we mean by mirror for a quasismooth wellformed Fano anticanonical weighted hypersurface $X = X_d \subset \mathbb{P}(a_0, \ldots, a_m)$ of dimension $n = m - 1$.

Definition 7. The regularised $I$-function of $X$ is defined as the hypergeometric series

$$\mathcal{I}_X (\alpha) = \sum_{j=0}^{\infty} \frac{(d j)! j!}{(a_0 j)! \cdots (a_m j)!} \alpha^j \quad (\alpha \in \mathbb{C}) \quad (9)$$

Remark 8. The paper [3] defines the regularised $G$-function $\mathcal{G}_X$ of $X$, a generating series for certain Gromov–Witten invariants of $X$. Conjecturally, $\mathcal{G}_X (\alpha) = e^{-\alpha c_1} \mathcal{I}_X (\alpha)$, where $c$ is the only rational number such that the right-hand side has the form $1 + O(\alpha)$.

Remark 9. Our functions $\mathcal{I}_X$ satisfy a hypergeometric differential equation on $\mathbb{P}^1$, nonsingular outside $\Sigma = \{0, a_0, \infty\}$ where $a_0 = \prod a_i$, whose solutions are the sections of an irreducible complex local system on $\mathbb{P}^1 \setminus \Sigma$ which we denote by $H^\text{red}_X$. To be a little more specific, write

$$P_0 (j) = -\prod_{i=0}^n (a_i j)(a_i j - 1) \cdots (a_i j - a_i + 1) \quad \text{and} \quad P_1 (j - 1) = j (d j)(d j - 1) \cdots (d j - d + 1)$$

Consider the differential operator $H_X = P_0 (D) + \alpha P_1 (D) \in \mathbb{Z}[\alpha, D]$ where $D = \alpha \frac{d}{d\alpha}$, and denote by $H^\text{red}_X$ the operator obtained removing from both $P_0 (j)$ and $P_1 (j - 1)$ a copy of every common factor. By [18, Corollary 3.2.1] $H^\text{red}_X$ is irreducible and it is easy to see that $H^\text{red}_X \cdot \mathcal{I}_X = 0$.

Definition 10. A Landau–Ginzburg (LG) model is a tuple $(Y^n, w, \omega, \gamma)$ where:

(i) $Y$ is a smooth algebraic manifold of dimension $n$;

(ii) $w: Y \to \mathbb{C}^\times$ is a quasi-projective morphism. For $\alpha \in \mathbb{C}^\times$ we denote by $Y_\alpha = w^{-1} (\alpha)$ the fibre.

Denote by $U \subset \mathbb{C}^\times$ the set of regular values of $w$, and by $w_U: Y_U = w^{-1} (U) \to U$ the restriction.

\footnote{In [3] $X$ is a smooth variety. Here we think of a quasi-smooth well-formed weighted hypersurface as a smooth Deligne–Mumford stack. The definition of $\mathcal{G}_X$ makes sense in this more general context.}
(iii) \( \omega \in \Gamma(U, w_\ast \Omega^{n-1}_{Y_Y/U}). \) For \( \alpha \in U, \) we write \( \omega_\alpha \in H^0(Y_\alpha, \Omega^{n-1}_{Y_\alpha}) \) the corresponding form;

(iv) \( \gamma \in \Gamma(D^\times, R^{n-1}w_\ast\Omega_Y) \) where \( 0 \in D \subset \mathbb{C} \) is a small disk and \( D^\times = D \setminus \{0\}. \) For \( \alpha \in D^\times \) we denote by \( \gamma_\alpha \in H_{n-1}(Y_\alpha, \mathbb{Q}) \) the corresponding cycle.

The period of the LG model is the function

\[
\pi(\alpha) = \int_{\gamma_\alpha} \omega_\alpha \quad (\alpha \in D^\times)
\]

Definition 11. Let \( X = X_d \subset \mathbb{P}(a_0, \ldots, a_m) \) be a quasismooth wellformed Fano anticanonical weighted hypersurface of dimension \( n = m - 1. \) A LG model \( (Y^n, w, \omega, \gamma) \) is mirror of \( X \) if for all \( \alpha \)

\[
\hat{I}(\alpha) = \pi(\alpha)
\]

Remark 12. It follows directly from Definition 11 that \( \mathbb{H}^\text{red}_X \) is a subquotient of \( R^{n-1}w_\ast\mathbb{C} \) and this fact could be taken as a weak version of mirror symmetry.

Indeed, write

\[
\mathcal{E} = (R^{n-1}w_\ast\mathbb{Q}) \otimes \mathcal{O}_U.
\]

The sheaf \( \mathcal{E} \) carries an algebraic connection \( \nabla : \mathcal{E} \to \Omega^1_U \otimes \mathcal{E} \) (the Gauss–Manin connection) by means of which we can regard it as a \( \mathcal{D}_U \)-module, where \( \mathcal{D}_U \) denotes the sheaf of differential operators on \( U, \) and

\[
R^{n-1}w_\ast\mathbb{C} = \text{Hom}_{\mathcal{D}_U}(\mathcal{O}_U, \mathcal{E}), \quad \text{and} \quad R^{n-1}w_\ast\mathbb{Q} = \text{Hom}_{\mathcal{D}_U}(\mathcal{E}, \mathcal{O}_U)
\]

are the local systems of flat sections and of solutions of \( \mathcal{E}. \) Now \( w_\ast\Omega^{n-1}_{Y_Y/U} \subset \mathcal{E} \) as the last piece of the Hodge filtration and we regard \( \omega \) as a section of \( \mathcal{E} \) by means of this inclusion. On the other hand we regard \( \gamma \) as a solution of \( \mathcal{E} \) and recover the period as \( \pi = \gamma(\omega). \) We have an inclusion and a surjection of \( \mathcal{D}_U \)-modules:

\[
\mathcal{D}_U \cdot \omega \subset \mathcal{E}, \quad \text{and} \quad \gamma : \mathcal{D}_U \cdot \omega \to \mathcal{D}_U \cdot \pi
\]

So we have \( \mathbb{H}^\text{red}_X = \text{Hom}_{\mathcal{D}_U}(\mathcal{D} \cdot \pi, \mathcal{O}_U) \subset \text{Hom}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \omega, \mathcal{O}_U) \) and \( R^{n-1}w_\ast\mathbb{C} = \text{Hom}_{\mathcal{D}_U}(\mathcal{E}, \mathcal{O}_U) \to \text{Hom}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \omega, \mathcal{O}_U). \)

2.2 Anticanonical del Pezzo hypersurfaces

In this paper we call a Fano surface a del Pezzo surface.

Johnson and Kollár [17] classify all anticanonical quasismooth wellformed del Pezzo surfaces in weighted projective 3-spaces. Their classification consists of 22 sporadic cases and the series [2], where \( k \in \mathbb{N}, \) \( k > 0. \) The 22 sporadic cases are all listed in [17, Theorem 8].

By (9) for all \( k > 0 \) integer the regularised \( I \)-function of any surface of the family [2] is given by Equation (1).

Remark 13. By Remark 9 the function \( \hat{I}_k \) [1] satisfies an hypergeometric differential equation on \( \mathbb{P}^1 \) which is singular on \( \Sigma_k = \{0, \alpha_k, \infty\}, \) where \( \alpha_k \) is as in Equation (5). The reduced differential operator associated to \( \hat{I}_k, \) which we denote by \( H^\text{red}_k, \) has order \( 6k + 2, \) thus the local system \( \mathbb{H}^\text{red}_k \) given by its solutions has rank \( 6k + 2. \)

Remark 14. In light of the definitions of Section [21], the data at the beginning of Section 3 define a LG model and our main Theorem can be interpreted as stating that this LG model is the mirror to the corresponding family of Johnson and Kollár. Also, by Remarks 12 and 13 \( \mathbb{H}^\text{red}_k \simeq R^1w_{U_k} \mathbb{C}, \) as stated in Corollary [3].
3 Proof of the main Theorem

In this Section, we prove the main Theorem stated in the Introduction. We begin by giving data to construct the period integral.

The period integral  Fix an integer \( k > 0 \). We define data \((Y_k, w_k, \omega_k, \gamma_k)\) as follows:

(i) \( Y_k \subset \mathbb{P}(1, 1, 3k + 2) \times \mathbb{C}^\times \) is the 2-dimensional manifold given by (3) and (4);
(ii) \( w_k: Y_k \to \mathbb{C}^\times \) is the projection on the second factor.

Let \( \alpha_{k,0} \) be as in Equation (5) and consider \( \alpha \neq \alpha_{k,0} \).

(iii) \( \omega_{k,\alpha} = \frac{1}{2\pi i} \int_{Y_{k,\alpha}} \omega_k \), where \( t = t_1/t_0 \);
(iv) \( \gamma_{k,\alpha} \) is the cycle that we describe next.

From this data we construct the period integral:

\[
\pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \omega_{k,\alpha}
\]

This is the period of \( Y_k \) of Equation (6).

Remark 15. This data defines a LG model, according to Definition 10, and the period (6) is the period of the LG model (10). Our main Theorem can be interpreted as stating that this LG model is a mirror of the family of surfaces \( X_k \).

The cycle of integration  Let us denote by \( p_{k,\alpha}: Y_{k,\alpha} \to \mathbb{P}^1 \) the 2 : 1 cover; \( p_{k,\alpha} \) is branched at the \( 6k + 4 \) roots of the polynomial

\[
h_{k,\alpha}(t) = t \left( 4t^{2k+1} + 2 \right) \left( -64t^{4k+2} + 4t^{4k+1} - 32\alpha t^{2k+1} - 4\alpha^2 \right)
\]

Lemma 16. For \( |\alpha| \ll 1 \) the polynomial \( h_{k,\alpha} \) has:

- A root at \( t = 0 \);
- \( 4k + 1 \) roots of norm \( \sim |\alpha|^2/(4k+1) \);
- \( 2k + 1 \) roots of norm \( \sim |\alpha|^{1/(4k+1)} \);
- A root of norm \( \sim \frac{1}{64} \).

Proof. Clearly \( h_{k,\alpha} \) has a root at \( t = 0 \) and \( 2k + 1 \) roots of norm \( |\alpha|^{2/(4k+1)} \). Consider the polynomial \( g_{k,\alpha}(t) = -64t^{4k+2} + 4t^{4k+1} - 32\alpha t^{2k+1} - 4\alpha^2 \); then as \( \alpha \to 0 \) \( g_{k,\alpha}(t) \) has a root \( t_\alpha \sim \frac{1}{64} \) and \( 4k + 1 \) roots \( t_\alpha \) of norm \( \sim |\alpha|^{2/(4k+1)} \). Indeed

\[
\lim_{\alpha \to 0} g_{k,\alpha}(t) = t^{4k+1}(-64t + 1)
\]

hence for \( |\alpha| \ll 1 \) \( g_{k,\alpha} \) has a root that tends to \( \frac{1}{64} \) and \( 4k + 1 \) roots that tend to 0. Now \( t_\alpha \) is a root if and only if

\[
t_\alpha^{4k+1} = 4(t_\alpha^{2k+1} + \alpha)^2
\]

and if \( t_\alpha \to 0 \), then \( |t_\alpha|^{4k+1} \sim |\alpha|^2 \).
Choose a continuous function \( \rho_k : (0, 1) \to \mathbb{R} \) such that for \( r \ll 1 \):

\[
 r^{\frac{2}{4k+1}} \ll \rho_k(r) \ll r^{\frac{1}{2k+1}}
\]

For \( |\alpha| \ll 1 \) let \( \gamma_{k,\alpha} \) be the circle of radius \( \rho_k(|\alpha|) \) around the origin in \( \mathbb{C} \) starting at \( t_0 = \rho_k(|\alpha|) \). By Lemma 16 this circle divides \( \mathbb{P}^1 \) into two regions each containing an even number of branch points of \( p_{k,\alpha} \), see Figure 1. Hence \( p_{k,\alpha}^{-1}(\gamma_{k,\alpha}) \subset Y_{k,\alpha} \) consists of two disjoint circles: we take \( \gamma_{k,\alpha} \) to be the lift along which the power series expansion in Equation (11) is valid, see Figure 2.

\[\text{Figure 1: The circle } \gamma_{k,\alpha} \subset \mathbb{C}. \text{ The 10 marked points represent the roots of } h_{k,\alpha} \text{ for } k = 1.\]

\[\text{Figure 2: The circle } \gamma_{k,\alpha} \text{ on } Y_{k,\alpha}. \text{ The gray sheet of the cover represents the one where (11) is valid. The 5 line segments on each sheet indicate a choice of branch cuts for } k = 1.\]

\[\text{Alternatively we choose the base point for our lift such that } y > 0.\]
Proof of the main Theorem  An immediate consequence of the next two lemmas.

Lemma 17. Let \( \pi_k(\alpha) \) be the period integral described above. Then, setting \( m = (2k + 1)j \):

\[
\pi_k(\alpha) = \sum_{j=0}^{\infty} \left( \frac{-1/2}{m} \right) (-1)^m 4^{3m-j} \sum_{p=j}^{2m} \left( \frac{-1/2}{p-j} \right) \left( \frac{2m}{p} \right) \alpha^j
\]

(11)

Proof. By a small manipulation we write the period as:

\[
\pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \frac{1}{\sqrt{\left( 1 + \frac{4t^{2k+1}}{\alpha} \right)}} \frac{1}{\sqrt{\left( 1 - t \left( 8 + \frac{2\alpha}{t^{2k+1}} \right)^2 \right)}} \frac{dt}{t}. \tag{12}
\]

By the defining inequalities of the function \( \rho_k \), and our choice of \( \gamma_{k,\alpha} \), both the following power series expansions hold:

\[
\left( 1 + \frac{4t^{2k+1}}{\alpha} \right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \left( \frac{-1/2}{n} \right) \frac{4^n t^{(2k+1)n}}{\alpha^n}
\]

\[
\left( 1 - t \left( 8 + \frac{2\alpha}{t^{2k+1}} \right)^2 \right)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} \left( \frac{-1/2}{m} \right) \left( -1 \right)^m 4^m t^m \left( 1 + \frac{\alpha}{4t^{2k+1}} \right)^{2m}
\]

Plugging the two power series in (12), switching the series and the integral signs and using the binomial theorem we obtain:

\[
\pi_k(\alpha) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{2m} \left( \frac{-1/2}{n} \right) \left( -1 \right)^m 4^m \left( \frac{2m}{p} \right) \left( -1 \right)^m 4^{3m-n-p} \alpha^{p-n} \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \frac{dt}{t^{m-(2k+1)(p-n)}}.
\]

Finally, applying the residue theorem around \( \alpha = 0 \) and setting \( j = p - n \), we obtain the result.

Lemma 18. For all \( k > 0 \) and \( j \geq 0 \) integers, setting \( m = (2k + 1)j \):

\[
\left( \frac{-1/2}{m} \right) (-1)^m 4^{3m-j} \sum_{p=j}^{2m} \left( \frac{-1/2}{p-j} \right) \left( \frac{2m}{p} \right) = \frac{((8k + 4)j)!j!}{(2j)!((2k+1)j)!^2((4k+1)j)!}
\]

(13)

Proof. Clearly (13) holds when \( j = 0 \), thus we assume \( j \geq 1 \). In what follows we repeatedly use the identity:

\[
(2l - 1)!! = \frac{(2l)!}{2^l \cdot l!} \quad (l > 0 \text{ integer}) \tag{14}
\]

We have:

\[
\left( \frac{-1/2}{m} \right) = \frac{(-1)^m (2m - 1)!!}{2^m m!} = \frac{(-1)^m}{4^m} \left( \frac{2m}{m} \right)
\]

(15)

We set \( i = p - j \) and we write the finite sum on the left hand side of (13) as:

\[
\sum_{p=j}^{2m} \left( \frac{-1/2}{p-j} \right) \left( \frac{2m}{p} \right) = \sum_{i=0}^{2m-j} \left( \frac{-1/2}{i} \right) \left( \frac{2m}{j+i} \right) = \sum_{i=0}^{2m-j} \left( \frac{-1/2}{i} \right) \left( \frac{2m}{2m-j-i} \right) = \left( \frac{2m-1/2}{2m-j} \right)
\]

(16)

where the last equality in (16) follows from the Chu–Vandermonde formula for generalised binomial coefficients:

\[
\sum_{i=0}^{n} \binom{\beta}{i} \binom{\alpha}{n-i} = \binom{\beta + \alpha}{n} \quad (\alpha, \beta \in \mathbb{C}, \ n \in \mathbb{N})
\]

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Plugging (15) and (16) in (13) and using that $m = (2k + 1)j$ we can rewrite (13) as:

$$4^{(4k+1)j} ((4k + 2)j)! \left( \frac{(4k + 2)j - 1/2}{(4k + 1)j} \right) = \frac{((8k + 4)j)!}{(2j)!((4k + 1)j)!} j!$$

(17)

Now we note that

$$\left( \frac{(4k + 2)j - 1/2}{(4k + 1)j} \right) = \frac{((8k + 4)j - 1)!!}{(2j - 1)!!} 2^{4k+1} ((4k + 1)j)!$$

Using this equality combined with (14) for $l = j$, we simplify (17) as:

$$2^{4k+2j} ((4k + 2)j)! ((8k + 4)j - 1)!! = ((8k + 4)j)!$$

(18)

Since (18) manifestly holds (it is (14) for $l = (4k + 2)j$), the result is proved.

4 Relations to the work of Beukers, Cohen, Mellit

In this Section we show how our Equation (7) arises from the work [1] as a mirror of the wrong dimension.

4.1 Finite hypergeometric functions and point counting

We summarise the main result in [1].

Let $v, w \in \mathbb{Q}^d$ be such that $\forall i, j \in \{1, \ldots, d\}$ $v_i \neq w_j$ mod $Z$ and the polynomials $\prod_{j=1}^d (x - e^{2\pi i v_j})$ and $\prod_{j=1}^d (x - e^{2\pi i w_j})$ are products of cyclotomic polynomials. Then there exist natural numbers $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$, with $p_1 + \cdots + p_s = q_1 + \cdots + q_s$, such that

$$\prod_{j=1}^d \frac{x - e^{2\pi i v_j}}{x - e^{2\pi i w_j}} = \prod_{j=1}^r x^{p_j} - 1 = \prod_{j=1}^s x^{q_j} - 1$$

and the analytic hypergeometric function

$$dF_d-1(v, w|\lambda) = \sum_{n=0}^{\infty} \frac{(v_1)_n \cdots (v_d)_n}{(w_1)_n \cdots (w_d)_n} \lambda^n$$

where $(x)_n = \begin{cases} x(x+1)\cdots(x+n-1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$

can be rewritten in the form:

$$dF_d-1(v, w|\lambda) = \sum_{j=0}^{\infty} \frac{(p_1 j)! \cdots (p_r j)!}{(q_1 j)! \cdots (q_s j)!} M^{-j} \lambda^j, \quad M = \frac{p_1^{p_1} \cdots p_r^{p_r}}{q_1^{q_1} \cdots q_s^{q_s}}$$

(19)

Similarly, the finite hypergeometric function $H_q(v, w|\lambda)$ — where $q$ is a prime power coprime with $\text{hcf}(v, w)$ — can be written in terms of $p_i, q_j$ only [1] Definition 1.1, Theorem 1.3].

For all $\alpha \in \mathbb{F}_q^\times$, Beukers, Cohen and Mellit consider the quasiprojective (in fact affine) variety $W_\alpha$ given by the homogeneous equations:

$$\left\{ \begin{array}{l} y_1 + y_2 + \cdots + y_r - x_1 - \cdots - x_s = 0 \\ \alpha \cdot y_1^{p_1} \cdots y_r^{p_r} = x_1^{q_1} \cdots x_s^{q_s} \end{array} \right. \text{ (for all } j, x_j, y_j \neq 0)$$

and prove the following, see [1] Theorem 1.5] for the precise statement and [1] Section 5] for its proof:

**Theorem 19.** If $\gcd(p_i, q_j) = 1$ for all $i, j \in \{1, \ldots, d\}$ and $M \cdot \alpha \neq 1$, there exists a nonsingular completion $\overline{W}_\alpha$ of $W_\alpha$ such that $|\overline{W}_\alpha(\mathbb{F}_q)| = H_q(v, w, M \cdot \alpha)$ up to factors depending only on $p_i, q_j$. 

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4.2 The family of 3-folds of Theorem 6

We provide some additional context for our statement of Theorem 6. Specifically, we build a precise connection with Theorem 19.

Fix $k > 0$ integer. We specialise the above discussion to the pair of vectors $(v_k, w_k)$ in $\mathbb{Q}^{d_k}$, $d_k = 6k + 2$:

$$v_k = \left( \frac{j}{8k + 4} \right)_{j \in \{1, \ldots, 8k + 3\} \setminus \{4i \in \{1, \ldots, 2k\} \cup \{4k+2\}\} \setminus \{1, \ldots, 2k+1\}, m \in \{1, \ldots, 4k+1\}}$$

$$w_k = \left( \frac{l}{2k + 1}, \frac{m}{4k + 1} \right)_{l \in \{1, \ldots, 2k+1\}, m \in \{1, \ldots, 4k+1\}}$$

This leads to

$$p_k = (8k + 4, 1) \quad q_k = (2, 2k + 1, 2k + 1, 4k + 1)$$

For $\lambda = M_k \cdot \alpha$, by [19]

$$d_k F_{d_k-1}(v_k, w_k|\lambda) = \sum_j \frac{((8k + 4)j)!j!}{(2j)!((2k + 1)j)!^2((4k + 1)j)!} M_k^{-j} \chi^j = \hat{I}_k(\alpha)$$

that is, the regularised $I$-function of Equation 1.

Still following the above discussion, for all $\alpha$ in $\mathbb{C}^\times$ consider the quasiprojective (in fact, affine) 3-fold $W_{k, \alpha}$ given by the homogeneous equations:

$$\left\{ \begin{array}{l}
y_1 + y_2 - (x_1 + x_2 + x_3 + x_4) = 0 \\
\alpha \cdot y_1^{8k+4} y_2 = x_1^2 x_2^{2k+1} x_3^{2k+1} x_4^{4k+1} \\
\end{array} \right. \text{ (for all } j, x_j, y_j \neq 0) \quad (20)$$

Next, we manipulate the Equations (20) to rewrite them as in the Introduction. By solving the first Equation for $y_2$ and writing $x_0 = y_1$, for all $\alpha$ in $\mathbb{C}^\times$ the system of Equations (20) leads to the Equation of a 3-fold in $\mathbb{P}^4$:

$$\alpha \cdot x_0^{8k+4} (x_1 + x_2 + x_3 + x_4 - x_0) = x_1^2 x_2^{2k+1} x_3^{2k+1} x_4^{4k+1}$$

The 3-fold $W_{k, \alpha}$ is the intersection of the projective threefold defined by the Equation above with the torus $\mathbb{T}^4 \subset \mathbb{P}^4$ and, in the affine chart $\{x_0 \neq 0\}$ with coordinates $u_i = x_i/x_0$, it is described by the Equation:

$$\alpha \cdot (u_1 + u_2 + u_3 + u_4 - 1) - u_1^2 u_2^{2k+1} u_3^{2k+1} u_4^{4k+1} = 0 \quad (\forall i \ u_i \neq 0)$$

This is the same as Equation (7); hence the meaning of the symbol $W_{k, \alpha}$ is the same as in the Introduction.

Theorem 19 computes $|W_{k, \alpha}(\mathbb{F}_q)|$ in terms of the finite analog of $\hat{I}_k$. The statement can be interpreted as asserting that the manifold $W_k$ of Equation (7) is a sort of LG mirror of the family of del Pezzo surfaces $X_{8k+4} \subset \mathbb{P}(2, 2k + 1, 2k + 1, 4k + 1)$ of the wrong dimension. Indeed, by [6] the only interesting cohomology group of $W_{k, \alpha}$ occurs in degree 3, hence by the Weil conjectures the main contribution to $|W_{k, \alpha}(\mathbb{F}_q)|$ comes from $H^3(W_{k, \alpha})$. Presumably for this completion we also have $H^3(W_{k, \alpha}, \mathbb{Q}) = g_3^W H^3(W_{k, \alpha}, \mathbb{Q})$ — as we do in Lemma 21.

5 Proof of Theorem 6

In this Section we prove Theorem 6 stated in the Introduction.

Fix $k > 0$ integer and let $U_k = \mathbb{C}^\times \setminus \{\alpha_{k,0}\}$, as in the Introduction. We prove that for all $\alpha \in U_k$ there is an isomorphism of pure Hodge structures:

$$g_3^W H^3_c(W_{k, \alpha}, \mathbb{Q})(1) = H^1(Y_{k, \alpha}, \mathbb{Q}) \quad (21)$$
Notation. For convenience in what follows we set \( a = 1/\alpha \). We write Equation (7) as
\[
(u_1 + u_2 + u_3 + u_4 - 1 - au_4^2u_2^{2k+1}u_3^{2k+1}u_4^{4k+1} = 0) \subset \mathbb{T}^4 \times \mathbb{C}^\times
\] (22)
and Equation (3) as
\[
(a^2y^2 - a^3h_{k,a}(t_0, t_1) = 0) \subset \mathbb{P}(1, 1, 3k + 2) \times \mathbb{C}^\times
\] (23)
where \( h_{k,a} \) is as in Equation (4). We denote by \( W_{k,a} \) the fibres of (22) and (23) over \( a \), so \( W_{k,a} = W_{k,a} \) and \( Y_{k,a} = Y_{k,a} \) where \( \alpha = 1/a \).

For the rest of this Section we fix \( k > 0 \) and \( a \neq a_{k,0} \), where \( a_{k,0} = 1/\alpha_{k,0} \), and we will omit all reference to \( k \) and \( a \). Also, we write \( p_{\mathbb{C}^\times} : Y_{\mathbb{C}^\times} = p^{-1}(\mathbb{C}^\times) \to \mathbb{C}^\times \) for the restriction of the \( 2:1 \) cover \( p : Y \to \mathbb{P}^1 \).

Proof of Theorem 6. Let \( W \subset \widehat{W} \) be the partial compactification of Equation (28) and \( \phi : \widehat{W} \to \mathbb{C}^\times \) the projective degree 2 del Pezzo fibration of Lemma 22. Applying in sequence Lemmas 21, 23, 24 we see:
\[
gr_3^W H^3_c(W, \mathbb{Q}) = gr_3^W H^3_c(\widehat{W}, \mathbb{Q}) = gr_3^W H^1_c(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}) = (gr_3^W H^1_c(\mathbb{C}^\times, p_{\mathbb{C}^\times}\mathbb{Q})) (-1)
\]
and the latter group is \( H^1(Y, \mathbb{Q}) (-1) \).

The partial compactification. We construct a partial compactification \( \widehat{W} \subset \mathbb{P}(1, 1, 2, 2)/\mu_2 \times \mathbb{C}^\times \) of \( W \) (where \( \mu_2 \) acts on \( \mathbb{P}(1, 1, 2, 2) \) as described below). Lemma 21 states that \( gr_3^W H^3_c(W, \mathbb{Q}) = gr_3^W H^3_c(\widehat{W}, \mathbb{Q}) \) hence for our purpose we might as well work with \( \widehat{W} \) in place of \( W \). The advantage of working with \( \widehat{W} \) is that, as stated in Lemma 22, the second projection \( \phi : \widehat{W} \to \mathbb{C}^\times \) is a fibration with fibres del Pezzo surfaces, and the point of the substitution (26) is precisely to make this structure manifest. (In Appendix A we explain how we discovered the del Pezzo fibration structure, and the substitution, by the methods of the minimal model program.)

Consider the weighted projective space \( \mathbb{P}(1, 1, 2, 2) \) with weighted homogeneous coordinates \( x_1, x_2, y_1, y_2, \) and the quotient \( \mathbb{P}(1, 1, 2, 2)/\mu_2 \) where the group \( \mu_2 \) acts on the affine coordinates \( x = x_2/x_1, y = y_1/x_1^2, z = y_2/x_1^3 \) by
\[
(x, y, z) \mapsto (-x, -y, -z)
\] (24)
Here and in what follows we write
\[
G = \mathbb{P}(1, 1, 2, 2)/\mu_2 \times \mathbb{C}^\times
\] (25)
and we denote by \( t \) the coordinate on \( \mathbb{C}^\times \). Note that \( G \) is a (noncompact) toric variety and the 4-dimensional torus \( T_G \subset G \) is the locus \( (x, y, z, t \neq 0) \) in the affine piece \( x_1 \neq 0 \). The substitution
\[
u_1 = x^{-1}y \quad u_2 = x^3z^{-1}t \quad u_3 = x^{-1}z^{-1} \quad u_4 = x^{-1}z
\] (26)
identifies \( W \) with
\[
\left(-at^{2k+1}y^2 + yz + z^2 + 1 - xz + tx^4 = 0\right) / \mu_2 \subset \mathbb{T}_G
\] (27)
We denote by \( \widehat{W} \) the closure of \( W \) in \( G \), given by the weighted homogeneous equation:
\[
\left(-at^{2k+1}y_1^2 + y_1y_2 + y_2^2 + x_1^4 - x_1x_2y_2 + tx_2^4 = 0\right) / \mu_2 \subset G
\] (28)
The 3-fold \( \widehat{W} \) is a partial compactification of \( W \).
Remark 20. The 3-fold $\hat{W}$ is quasismooth and we think at it as a smooth orbifold. More precisely $\hat{W}$ has nonisolated quotient singularities: although it is singular, it is a rational homology manifold. Because of this, for the purpose of cohomological computations, we can pretend that $\hat{W}$ is smooth. Below we take the convention that the set of regular values of the map $\phi: \hat{W} \to \mathbb{C}^\times$ is the set of values $t \in \mathbb{C}^\times$ such that the corresponding fibre $\hat{W}_t$ is quasismooth.

Lemma 21. There is an identity of pure Hodge structures:

$$\gr_3^W H^3_c(W, \mathbb{Q}) = \gr_3^W H^3_c(\hat{W}, \mathbb{Q})$$

Proof. Note first that the 3-fold $W$ is nonsingular but noncompact, thus $H^3_c(W, \mathbb{Q})$ is a mixed Hodge structure with weights $\leq 3$, and so is $H^3_c(\hat{W}, \mathbb{Q})$. Consider the divisor $D = \hat{W} \setminus W$, and denote by $i: D \hookrightarrow \hat{W}$ and $j: W \hookrightarrow \hat{W}$ the inclusions. We have a long exact sequence of mixed Hodge structures

$$\cdots \to H^3_c(D, \mathbb{Q}) \to H^3_c(W, \mathbb{Q}) \to H^3_c(\hat{W}, \mathbb{Q}) \to H^3_c(D, \mathbb{Q}) \to \cdots$$

To prove (29), we check that $\gr_3^W H^3_c(D, \mathbb{Q}) = \gr_3^W H^3_c(D, \mathbb{Q}) = (0)$. To this end, we study the geometry of the surface $D$; $D$ is the union $D = \bigcup_{i=1}^4 D_i$, where:

$$D_1 = \hat{W} \cap (x_1 = 0), \quad D_2 = \hat{W} \cap (x_2 = 0), \quad D_3 = \hat{W} \cap (y_1 = 0), \quad D_4 = \hat{W} \cap (y_2 = 0)$$

One can check that $D \subset \hat{W}$ is (locally the quotient of) a simple normal crossing divisor with no 0-dimensional strata. By setting

$$D^{[1]} = \bigcup_i D_i \quad \text{and} \quad D^{[2]} = \bigcup_{i < j} D_{ij}, \quad \text{with} \quad D_{ij} = D_i \cap D_j$$

we get a strict simplicial resolution $D^{[2]} \Rightarrow D^{[1]} \to D$ and the long exact sequence:

$$\cdots \to \bigoplus_i H^{m-1}_c(D_i, \mathbb{Q}) \to \bigoplus_{i < j} H^{m-1}_c(D_{ij}, \mathbb{Q}) \to H^m_c(D, \mathbb{Q}) \to \bigoplus_i H^m_c(D_i, \mathbb{Q}) \to \cdots$$

(30)

It follows that $H^2_c(D, \mathbb{Q})$ has weights $\leq 2$. Now choose $m = 3$ in (30) and examine $H^3_c(D, \mathbb{Q})$. On the left hand side, $\bigoplus_{i < j} H^3_c(D_{ij}, \mathbb{Q})$ has weights $\leq 2$. On the right hand side, $\bigoplus_i H^3_c(D_i, \mathbb{Q})$ a priori has weights $\leq 3$. To conclude, we next show that, in fact, for $i = 1, 2, 3, 4$, $H^3_c(D_i, \mathbb{Q})$ has weights $< 3$. Consider first the surface $D_1$, given by

$$\left(-at^{2k+1}y_1^2 + y_2^2 + y_1y_2 + tx_2^4 = 0\right) / \mu_2 \subset \mathbb{P}(1, 2, 2) / \mu_2 \times \mathbb{C}^\times$$

This is the same as the surface

$$\left(-at^{2k+1}y_1^2 + y_2^2 + y_1y_2 + tz_2^2 = 0\right) / \mu_2 \subset \mathbb{P}^2 / \mu_2 \times \mathbb{C}^\times$$

(31)

where $z_2, y_1, y_2$ are homogeneous coordinates of $\mathbb{P}^2$ and $\mu_2$ acts as $(y_1, y_2) \mapsto (-y_1, -y_2)$ on the affine piece ($z_2 = 1$). Note that the quotient of $\mathbb{P}^2$ by the $\mu_2$-action is the weighted projective space $\mathbb{P}(1, 1, 2)$ with homogeneous coordinates $y_1, y_2, w_2 = z_2^2$. In $\mathbb{P}(1, 1, 2)$ (31) becomes

$$\left(at^{2k+1}y_1^2 - y_2^2 - y_1y_2 + tw_2\right) \subset \mathbb{P}(1, 1, 2)$$

thus, since $t \neq 0$, we conclude that $D_1 \simeq \mathbb{P}^1 \times \mathbb{C}^\times$ and then $H^3_c(D_1, \mathbb{Q})$ is a pure Hodge structure of weight 2. An almost identical argument holds for $D_2$. The surface $D_4$ is given by

$$\left(-at^{2k+1}y_1^2 + x_1^4 + tx_2^4 = 0\right) / \mu_2 \subset \mathbb{P}(1, 1, 2) / \mu_2 \times \mathbb{C}^\times$$
By means of the substitution of (32) is the surface \( z = \frac{y^2 + x^4}{z_1} \) write this as

\[
\left( w_2^2 = z_1z_2(-z_1^2 + \frac{z_1z_2}{4} - tz_2^2) \right) \subset \mathbb{P}(1, 1, 2) \times \mathbb{C}^\times
\]

where \( z_1, z_2, w_2 = y_2^2 \) are homogeneous coordinates on \( \mathbb{P}(1, 1, 2) \). A natural compatification of (32) is the surface \( \mathcal{D}_3 \) given by

\[
\left( w_2^2 = z_1z_2(-t_0^2z_1^2 + t_1^2z_1z_2 - t_0t_1z_2^2) \right) \subset \mathbb{P}(1, 1, 2) \times \mathbb{P}^1
\]

Note that \( \mathcal{D}_3 \) is a 2 : 1 cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) branched along a divisor in \( |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 2)| \), thus \( \mathcal{D}_3 \) is a rational surface and hence \( H^3(\mathcal{D}_3, \mathbb{Q}) = (0) \). Then \( H^3(D_3, \mathbb{Q}) \) has weights \( \leq 2 \), since, by setting \( \Gamma = \mathcal{D}_3 \setminus D_3 \), we have the long exact sequence of mixed Hodge structures

\[ \cdots \rightarrow H^2_c(\Gamma, \mathbb{Q}) \rightarrow H^3_c(D_3, \mathbb{Q}) \rightarrow H^3(\mathcal{D}_3, \mathbb{Q}) \rightarrow \cdots \]

and \( \text{gr}^W_3 H^2(\Gamma, \mathbb{Q}) = (0) \).

**Lemma 22.** Let \( \phi: \hat{W} \rightarrow \mathbb{C}^\times \) be the projection onto the second factor. Denote by \( \Delta \) the set of critical values of \( \phi \) and write \( \Omega = \mathbb{C}^\times \setminus \Delta \) for the set of regular values. Let \( j: \Omega \hookrightarrow \mathbb{C}^\times \) be the natural inclusion and denote by \( \phi_\Omega : \hat{W}_\Omega = \phi^{-1}(\Omega) \rightarrow \Omega \) the induced morphism. Let \( \delta_1, \delta_2, \delta \in \mathbb{C}[t] \) be the polynomials:

\[
\delta_1(t) = 4at^{2k+1} + 1 \quad \delta_2(t) = a^2t^{4k+2} - 4t \left( 4at^{2k+1} + 1 \right)^2 \quad \delta = \delta_1 \cdot \delta_2
\]

Write \( \Omega_1 = \mathbb{C}^\times \setminus \{ \delta_1 = 0 \}, \hat{W}_{\Omega_1} = \phi^{-1}(\Omega_1) \). Denote by \( K \) the function field \( k(\mathbb{C}^\times) = \mathbb{C}(t) \). Then:

\( (t) \) \( \Delta = \{ \delta = 0 \} \). If \( t \) is a root of \( \delta_1, \hat{W}_t \) has a unique non quasismooth point \( p_t = (0 : 0 : -2 : 1) \); if \( t \) is a root of \( \delta_2, \hat{W}_t \) has a unique non quasismooth point

\[
q_t = \left( 1 : \sqrt{\frac{2\delta_1(t)}{at^{2k+1}}} : \frac{1}{\delta_1(t)} : \frac{2at^{2k+1}}{\delta_1(t)} \right)
\]

In both cases the non quasismooth point is an ordinary double point.
(2) After the change of coordinates:

\[ y_1 \mapsto \frac{y_1}{2} - \frac{x_1x_2}{2(4at^{2k+1} + 1)} \quad \text{and} \quad y_2 \mapsto \frac{y_1}{2} + y_2 - \frac{x_1x_2}{2} \]  

(36)

the equation of the fibre \( \hat{W}_t \) over \( t \in \Omega_1 \) is:

\[
\left( -(4at^{2k+1} + 1)y_1^2 + y_2^2 + x_1^4 + tx_1^2 - \frac{at^{2k+1}}{4at^{2k+1} + 1}(x_1x_2)^2 = 0 \right) / \mu_2 \subset \mathbb{P}(1, 1, 2)/\mu_2
\]

(35)

(3) For all \( t \in \Omega \) the fibre \( \hat{W}_t \) is a quasismooth del Pezzo surface of degree 2 with two \( \frac{1}{4} \cdot (1, -1) \) points \( p_t^+, p_t^- \), on \( (x_1 = x_2 = 0)/\mu_2 \), and intersecting \( (y_1 = y_2 = 0)/\mu_2 \) in two points \( q_t^+ \) and \( q_t^- \). In the coordinates of Equation (36),

\[
p_t^\pm = \left( 0 : 0 : 1 : \pm \frac{\sqrt{\delta_1(t)}}{2} \right) \quad \text{and} \quad q_t^\pm = \left( \frac{\sqrt{at^{2k+1} + \delta_2(t)}}{2 \cdot \delta_1(t)} : 1 : 0 : 0 \right) \]

(37)

(4) For all \( t \in \Omega \) the fibre \( \hat{W}_t \) has Picard rank \( r = h^2(\hat{W}_t, \mathbb{Q}) = 2 \). More specifically, \( \hat{W}_t \) contains a configuration of lines as pictured in Figure 3, and a basis of \( \text{Pic}(\hat{W}_t)_\mathbb{Q} \) is given the classes of the curves \( C_{t,1} \) and \( C_{t,2} \).

(5) In the variables of Equation (36), the restriction \( \phi_1 : Y_1 = \hat{W} \cap (x_1 = x_2 = 0)/\mu_2 \to \mathbb{C}^* \) is a 2 : 1 branched cover with branch locus \( \{ \delta_1 = 0 \} \), and the restriction \( \phi_2 : Y_2 = \hat{W}_{\Omega_1} \cap (y_1 = y_2 = 0)/\mu_2 \to \Omega_1 \) is a 2 : 1 branched cover with branch locus \( \{ \delta_2 = 0 \} \). In particular,

\[
\phi_1^{-1}(t) = \begin{cases} 
    p_t^\pm & \text{if } \delta_1(t) \neq 0 \\
    p_t & \text{if } \delta_1(t) = 0
\end{cases} \quad \text{and} \quad \phi_2^{-1}(t) = \begin{cases} 
    q_t^\pm & \text{if } \delta_2(t) \neq 0 \\
    q_t & \text{if } \delta_2(t) = 0
\end{cases}
\]

(38)

(6) The Picard rank of \( \hat{W}_K \) is \( \rho = 1 \).

(7) \( R^3 \phi_* \mathbb{Q} = R^1 \phi_* \mathbb{Q} = (0) \).

(8) The natural homomorphism \( R^2 \phi_* \mathbb{Q} \to j_*j^* R^2 \phi_* \mathbb{Q} = j_* R^2 \phi_{\Omega,1} \mathbb{Q} \) is an isomorphism.

![Figure 3: A quasismooth fibre \( \hat{W}_t \) and the four points \( p_t^\pm, q_t^\pm \) in \( \hat{W}_t \). Numerically equivalent curves on \( \hat{W}_t \) are marked with the same symbol.](image)
Proof. To prove (1), fix \( t \in \mathbb{C}^* \) and compute the Jacobian of the polynomial in Equation (28). To prove that for all \( t \in \Delta \) the non quasismooth point of \( \hat{W}_t \) is an ordinary double point, one can check that the Hessian of (28) at that point is invertible.

Assertion (2) is a simple substitution.

To prove (3), fix \( t \in \Omega \) and consider the quotient map \( \sigma: \mathbb{P}(1,1,2,2) \to \mathbb{P}(1,1,2,2)/\mu_2 \). Let \( \hat{V}_t = \sigma^{-1}(\hat{W}_t) \) be the preimage. Note that:

\[
-K_{\hat{V}_t} = \sigma^* (-K_{\hat{W}_t}) \quad \text{and} \quad K_{\hat{V}_t}^2 = 2 \cdot K_{\hat{W}_t}^2
\]

since \( \sigma \) is \( 2 : 1 \) and étale in codimension 1. We have \(-K_{\hat{V}_t} = \mathcal{O}(2) \) and \(-K_{\hat{V}_t}^2 = 4 \), thus \( \hat{V}_t \subset \mathbb{P}(1,1,2,2) \) is a del Pezzo surface of degree 4. The orbifold points of \( \hat{V}_t \), of type \( 1/2 \cdot (1,1) \), are the two points \( P^\pm_t \) of \( \hat{V}_t \) on the line \((x_1 = x_2 = 0)\); in the coordinates of Equation (36),

\[
P^\pm_t = \left( 0 : 0 : 1 : \pm \frac{\sqrt{\delta_1(t)}}{2} \right)
\]

Note that, by (24), on the affine piece \((y_1 = 1)\) the group \( \mu_2 \) acts by

\[
(x_1, x_2, y_2) \to (ix_1, -ix_2, y_2)
\]

Hence \( \hat{W}_t \subset \mathbb{P}(1,1,2,2)/\mu_2 \) is a del Pezzo surface of degree 2 with two \( 1/4 \cdot (1,-1) \) points \( p^\pm_t = \sigma(P^\pm_t) \), as in (37); these are the only orbifold points of \( \hat{W}_t \) since \((1 : 0 : 0 : 0)\) and \((0 : 1 : 0 : 0)\) do not satisfy (36). Setting \( y_1 = y_2 = 0 \) in Equation (36), one finds that \( \hat{W}_t \cap (y_1 = y_2 = 0) / \mu_2 \) is given by the two points \( q^\pm_t \) in (37).

To prove (4), note that the crepant resolution \( \hat{W}_t \) of \( \hat{W}_t \) is a smooth weak del Pezzo surface of degree 2. By Demazure’s Theorem this is the blow-up of \( \mathbb{P}^2 \) in 7 nongeneral points, thus it has Picard rank \( r = 8 \). Then the surface \( \hat{W}_t \), obtained by blowing down 6 exceptional curves on \( \hat{W}_t \), has Picard rank \( r = 2 \). We explain how the geometry of \( \hat{W}_t \) singles out a distinguished basis of generators of \( \text{Pic}(\hat{W}_t)_\mathbb{Q} \). Setting

\[
\nu_\pm(t) = \sqrt{\frac{at^{2k+1} \pm \sqrt{\delta_2(t)}}{2 \cdot \delta_1(t)}}
\]

on \( \mathbb{P}(1,1,2,2) \) Equation (36) factors as:

\[
(y_2 - \sqrt{\delta_1(t)}y_1)(y_2 + \sqrt{\delta_1(t)}y_1) + (x_1 - \nu_+(t)x_2)(x_1 + \nu_+(t)x_2)(x_1 - \nu_-(t)x_2)(x_1 + \nu_-(t)x_2) = 0
\]

This exhibits eight lines on \( \hat{V}_t \), four of which passing through \( P^+_t \), the other four passing through \( P^-_t \), as pictured on the left of Figure 4. The four lines on \( \hat{V}_t \) through \( P^+_t \) correspond to two orbits under the \( \mu_2 \) action, and the same holds for the four lines through \( P^-_t \). Namely, in \( \mathbb{P}(1,1,2,2)/\mu_2 \) Equation (36) can only be factored as:

\[
(y_2 - \sqrt{\delta_1(t)}y_1)(y_2 + \sqrt{\delta_1(t)}y_1) + (x_1^2 - \nu_+^2(t)x_2^2)(x_1^2 - \nu_-^2(t)x_2^2) = 0
\]

This exhibits four lines on the surface \( \hat{W}_t \), as pictured in the middle of Figure 4, two of which passing through \( p^+_t \):

\[
C_{t,1} : \left( y_2 - \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_+^2(t)x_2^2 = 0 \right) \quad C_{t,2} : \left( y_2 - \sqrt{\delta_1(t)}y_1 = x_1^2 \nu_-^2(t)x_2^2 = 0 \right)
\]

(39)
the other two passing through \( p_t^- \):

\[
C_{t,3} : \left( y_2 + \sqrt{\delta_1(t)} y_1 = x_1^2 - \nu_2^2(t) x_2^2 = 0 \right) \quad C_{t,4} : \left( y_2 + \sqrt{\delta_1(t)} y_1 = x_1^2 - \nu_2^2(t) x_2^2 = 0 \right)
\]

(40)

To see that \( C_{t,1} \sim C_{t,4} \) note that, on \( \widehat{W}_t \), the union of the strict transform \( \tilde{C}_{t,1} \) and the three exceptional curves above \( p_t^+ \), as pictured on the right of Figure 4, supports a fibre of a conic bundle where another fibre is supported on the union of the strict transform \( \tilde{C}_{t,4} \) and the three exceptional curves above \( p_t^- \). Taking direct image this implies indeed that \( C_{t,1} \sim C_{t,4} \). Similarly, \( C_{t,2} \sim C_{t,3} \). Then the two curves \( C_{t,1} \) and \( C_{t,2} \) form a basis of \( \text{Pic}(\widehat{W}_t)\mathbb{Q} \).

![Figure 4](image-url)

**Figure 4**: The surfaces \( \widehat{V}_t, \widehat{W}_t \) and \( \widetilde{W}_t \). The eight lines on \( \widehat{V}_t \) descend to four lines on \( \widehat{W}_t \). The two sets of bold lines on \( \widetilde{W}_t \) correspond to two distinct fibers of a conic bundle over \( \mathbb{P}^1 \).

Statement (5) follows immediately from Equation (36), and by (1) and (3). In particular, to prove (38) one can check by (35) that, in the variables of Equation (28), if \( t \) is a root of \( \delta_1 \) then \( \phi_1^{-1}(t) = (0 : 0 : 2 : -1) = p_t \), and if \( t \) is a root of \( \delta_2 \) then \( \phi_2^{-1}(t) \) is given by

\[
\left( \sqrt{\frac{at^{2k+1}}{2\delta_1(t)}} : \sqrt{\frac{at^{2k+1}}{2\delta_1(t)}} : \sqrt{\frac{2(at^{2k+1})^3}{\delta_1(t)}} \right)
\]

which is the point \( q_t \) in (34).

We prove (6) as follows. Note that the function fields of \( Y_1 \) and \( Y_2 \) are \( k(Y_1) = K(\sqrt{\delta_1}) \) and \( k(Y_2) = K(\sqrt{\delta_2}) \). Set \( L = K(\sqrt{\delta_1}, \sqrt{\delta_2}) \) and consider the lattice of Galois field extensions:

\[
K(\sqrt{\delta_1}) \quad K(\sqrt{\delta_2}) \quad K(\sqrt{\delta_1}\sqrt{\delta_2})
\]

The Galois group \( \text{Gal}(L/K) \simeq C_2 \times C_2 \) is generated by the two authomorphisms \( \sigma_1 \) and \( \sigma_2 \), where:

\[
\sigma_1|_{K(\sqrt{\delta_1})} = \text{id} \quad \sigma_1(\sqrt{\delta_2}) = -\sqrt{\delta_2} \quad \text{and} \quad \sigma_2|_{K(\sqrt{\delta_2})} = \text{id} \quad \sigma_2(\sqrt{\delta_1}) = -\sqrt{\delta_1}
\]

By (4), \( \text{Pic}(\widehat{W}_L) \) is generated by the classes of the curves \( C_1 \) and \( C_2 \), of Equation (39). Then by Galois descent \( \text{Pic}(\widehat{W}_K) \) has only one generator, given by the class of the curve \( C_1 + C_2 + C_3 + C_4 \).
\( \hat{\mathcal{W}}_K \), since \( C_1 + C_2 + C_3 + C_4 \) is the only curve in \( \hat{\mathcal{W}}_L \) which is invariant with respect to the induced action of \( \text{Gal}(L/K) \).

Assertion (7) follows from the fact that, by (1) and (4), \( \forall t \in \mathbb{C}^\times \), \( H^1(\hat{\mathcal{W}}_t, \mathbb{Q}) = H^3(\hat{\mathcal{W}}_t, \mathbb{Q}) = (0) \).

To prove (8), consider a singular value \( s \in \Delta \) and the fibre \( \hat{\mathcal{W}}_s \). To check that \( R^2\phi_\Omega \rightarrow j_* R^2 \phi_\Omega \cdot \mathbb{Q} \) is an isomorphism in a neighbourhood of \( s \), it is enough to show that the natural homomorphism

\[
H^2(\hat{\mathcal{W}}_s, \mathbb{Q}) \rightarrow H^2(\hat{\mathcal{W}}_t, \mathbb{Q})^{T_s}
\]

is an isomorphism, where \( t \in \mathbb{C}^\times \) is near \( s \), \( T_s : H^2(\hat{\mathcal{W}}_t, \mathbb{Q}) \rightarrow H^2(\hat{\mathcal{W}}_t, \mathbb{Q}) \) is the local monodromy operator at \( s \), and \( H^2(\hat{\mathcal{W}}_t, \mathbb{Q})^{T_s} \) denotes the group of monodromy invariants. Indeed on the one hand, by the proper base change theorem, \( H^2(\hat{\mathcal{W}}_s, \mathbb{Q}) \) is the fibre at \( s \) of \( R^2 \phi_\Omega \cdot \mathbb{Q} \), and on the other hand \( H^2(\hat{\mathcal{W}}_t, \mathbb{Q})^{T_s} \) is the fibre at \( s \) of \( j_* R^2 \phi_\Omega \cdot \mathbb{Q} \).

We have an exact triangle of constructible complexes on \( \hat{\mathcal{W}}_s \):

\[
\mathbb{Q} \rightarrow \psi \mathbb{Q} \rightarrow \varphi \mathbb{Q} \xrightarrow{+1}
\]

where \( \psi \) and \( \varphi \) are the nearby and vanishing cycle functors \([10, \text{Exposé I}]\). Since \( \hat{\mathcal{W}}_s \) has isolated hyperquotient singularities — in fact by (4) it only has one non quasismooth point — \( \varphi \mathbb{Q} \) is supported at the non quasismooth point of \( \hat{\mathcal{W}}_s \) and is concentrated in degree 2 \([10]\). Thus from the exact sequence:

\[
(0) = H^1(\hat{\mathcal{W}}_s, \varphi \mathbb{Q}) \rightarrow H^2(\hat{\mathcal{W}}_s, \mathbb{Q}) \rightarrow H^2(\hat{\mathcal{W}}_s, \psi \mathbb{Q}) = H^2(\hat{\mathcal{W}}_t, \mathbb{Q}) \rightarrow \cdots
\]

we conclude that the natural homomorphism \( H^2(\hat{\mathcal{W}}_s, \mathbb{Q}) \rightarrow H^2(\hat{\mathcal{W}}_t, \mathbb{Q}) \) is injective. By the local invariant cycle Theorem \([7, \text{Theorem 1.4.1}]\) then \( H^2(\hat{\mathcal{W}}_s, \mathbb{Q}) \) is the group of monodromy invariant cycles in \( H^2(\hat{\mathcal{W}}_t, \mathbb{Q}) \).

\[\square\]

**Lemma 23.** There is an identity of mixed Hodge structures:

\[
H^3_c(\hat{\mathcal{W}}, \mathbb{Q}) = H^1_c(\mathbb{C}^\times, R^2 \phi_* \mathbb{Q})
\]

(41)

**Proof.** Note that the two functors \( \phi_* \) and \( \phi_{\Omega} \) coincide, since the map \( \phi \) is proper. Consider the Leray spectral sequence of the morphism \( \phi \) with second page \( E_2^{pq} = H^p_c(R^q \phi_* \mathbb{Q}) \Rightarrow H^{p+q}(\hat{\mathcal{W}}, \mathbb{Q}) \), which is known to degenerate at the second page. The groups contributing to \( H^*_c(\hat{\mathcal{W}}, \mathbb{Q}) \) are:

\[
H^*_c(\mathbb{C}^\times, R^3 \phi_* \mathbb{Q}), \ H^*_c(\mathbb{C}^\times, R^2 \phi_* \mathbb{Q}), \ H^*_c(\mathbb{C}^\times, R^1 \phi_* \mathbb{Q})
\]

Identity \((41)\) holds if and only if \( H^*_c(\mathbb{C}^\times, R^3 \phi_* \mathbb{Q}) = H^*_c(\mathbb{C}^\times, R^1 \phi_* \mathbb{Q}) = (0) \), and indeed this is so, by Lemma \([22, 7]\). \[\square\]

**Lemma 24.** There is an isomorphism of mixed sheaves on \( \mathbb{C}^\times \):

\[
p_{\mathbb{C}^\times} \cdot Q_{Y_{\mathbb{C}^\times}} \rightarrow R^2 \phi_* Q_{\hat{\mathcal{W}}}(1)
\]

(42)

**Proof.** The sketch of the proof is as follow: let \( \Omega \subset \mathbb{C}^\times \) and \( j : \Omega \hookrightarrow \mathbb{C}^\times \) be as in Lemma \([22]\) and let \( p_{\Omega} : Y_\Omega = p^{-1}(\Omega) \rightarrow \Omega \) be the induced morphism; we first construct a homomorphism

\[
z : p_{\Omega} \cdot Q_{Y_\Omega} \rightarrow R^2 \phi_{\Omega} \cdot Q_{\hat{\mathcal{W}}_\Omega}(1)
\]

(43)

and then we show that it is an isomorphism. This concludes the proof since on the one hand it is obvious that \( j_* p_{\Omega} \cdot Q_{Y_\Omega} = p_{\mathbb{C}^\times} \cdot Q_{\mathbb{C}^\times} \), on the other hand by Lemma \([22, 8]\) \( j_* R^2 \phi_{\Omega} \cdot Q_{\hat{\mathcal{W}}_\Omega} = R^2 \phi_* Q_{\hat{\mathcal{W}}} \).
Below we construct a cycle

\[ Z \in \text{CH}_2 \left( Y_\Omega \times_\Omega \hat{W}_\Omega \right) \]

inducing the homomorphism \( z \) via the natural maps:

\[
\text{CH}_2 \left( Y_\Omega \times_\Omega \hat{W}_\Omega \right) \to H^4_{BM} \left( Y_\Omega \times_\Omega \hat{W}_\Omega \right) = \text{Hom}_{D_{\text{et}}} \left( R\rho_{\Omega,\ast}Q_{Y_\Omega}, R\phi_{\Omega,\ast}Q_{\hat{W}_\Omega}[2](1) \right) \to \text{Hom} \left( \rho_{\Omega,\ast}Q_{Y_\Omega}, R^2\phi_{\Omega,\ast}Q_{\hat{W}_\Omega}(1) \right)
\]

Consider the diagram

\[
\begin{array}{ccc}
Y_\Omega \times_\Omega \hat{W}_\Omega & \xrightarrow{\phi_{\Omega}} & \hat{W}_\Omega \\
\downarrow Y_\Omega & & \downarrow \Omega \\
\Omega & \xrightarrow{\rho_{\Omega}} & \hat{W}_\Omega
\end{array}
\]

Let \( \overline{Z} \subset \hat{W}_\Omega \) be the union over all \( t \in \Omega \) of the four curves \( C_{t,1}, \ldots, C_{t,4} \) in the fibre \( \hat{W}_t \) described in Lemma 22, that is:

\[
Z = \left( -(4at^{2k+1} + 1)y_1^2 + y_2^2 = x_1^4 - \frac{a^{2k+1}}{4at^{2k+1} + 1} (x_1 x_2)^2 + tx_2^2 = 0 \right) / \mu_2 \subset \hat{W}_\Omega
\]

We have that:

\[
Y_\Omega \times_\Omega \overline{Z} = Z_1 + Z_2 \subset Y_\Omega \times_\Omega \hat{W}_\Omega
\]

is the sum of two irreducible components and we take \( Z \) to be one of these components. In order to see this, let \( K = \mathbb{C}(t) \) and \( L = K(\sqrt{\delta_1}, \sqrt{\delta_2}) \), as in the proof of Lemma 22, and consider the field extensions \( K \subset K(\sqrt{\delta}) \subset L \). The key point is to notice that \( Y_\Omega \) is a 2 : 1 cover of \( \Omega \) with function field \( k(Y_\Omega) = K(\sqrt{\delta}) \). Indeed, by [23] \( Y \) is a 2 : 1 cover of \( \mathbb{P}^1 \) branched at the \( 6k + 4 \) roots of the polynomial

\[
h(t) = t \left( 4at^{2k+1} + 1 \right) \left( -64a^2t^{2k+1} + a^2t^{4k+1} - 32at^{2k+1} - 4 \right)
\]

and by [33] we have \( h(t) = \delta_1(t) \cdot \delta_2(t) = \delta(t) \). Then, since the generator of the Galois group \( \text{Gal}(L/K(\sqrt{\delta})) \) exchanges \( C_1 \) with \( C_4 \) and \( C_2 \) with \( C_3 \) on \( \hat{W}_L \), by Galois descent the cycle \( C_1 + C_2 + C_3 + C_4 \) on \( \hat{W}_{k(Y_\Omega)} \) splits into two components \( C_1 + C_4 \) and \( C_2 + C_3 \), each defined over \( k(Y_\Omega) \) and corresponding to the two irreducible components \( Z_1 \) and \( Z_2 \). Note also that \( \rho_{\Omega,\ast}Z = \overline{Z} \).

It remains to show that the induced homomorphism \( z \) is an isomorphism. This can be checked at the generic point \( \eta \), or indeed at any point \( t \in \Omega \). This is precisely the statement in Lemma 22(4) that the set \( \{ C_1 + C_4, C_2 + C_3 \} \) is a basis of \( \text{Pic}(\hat{W}_t)_Q = H^2(\hat{W}_t, Q) \).

**A Toric MMP**

Let \( W \) and \( \hat{W} \) be the fibre of (22) over \( a \) and its partial compactification (28) in \( G = \mathbb{P}(1, 1, 2, 2)/\mu_2 \times \mathbb{C}^\times \), as in Section [5].

In this Appendix, we explain how we discovered the del Pezzo fibration \( \phi : \hat{W} \to \mathbb{C}^\times \), and the substitution (26) by the methods of the minimal model program for toric hypersurfaces.
The space $\hat{W}$ has canonical but not terminal singularities. In Section A.3 we construct a birational model $W'$ with terminal singularities and a Mori fibration $\psi: W' \to F_1$.

As in Section (5), we fix once and for all an integer $k > 0$ and a value $a \neq a_{k,0}$, and we omit all reference to $k$ and $a$ in what follows.

**Notation** We set up our notation for toric varieties. For a lattice $L$, we denote by $L_\mathbb{R} = L \otimes \mathbb{Z} \mathbb{R}$ the associated real vector space. For a torus $T$, we denote by $M = \text{Hom}(T, \mathbb{C}^\times)$ the character lattice and let $N = \text{Hom}(M, \mathbb{Z})$.

For a fan $\Sigma \subset N_\mathbb{R}$, we denote by $F_\Sigma$ the associated toric variety. We denote by $\rho_i$ the primitive generators of the 1-dimensional cones of $\Sigma$, and by $D_i \subset F_\Sigma$ the corresponding divisors.

If $F$ is a proper toric variety and $D = \sum a_i D_i$ is a Weil divisor on $F$, then we denote by $P_D = \bigcap \{m : \langle \rho_i, m \rangle \geq -a_i\} \subset M_\mathbb{R}$ the associated polytope; it is well known that $P_D \cap M$ is a basis of $H^0(F, D)$.

Given a rational polytope $P \subset M_\mathbb{R}$, we denote by $\Sigma_P$ the normal fan of $P$. If $P$ is not full dimensional $\Sigma_P$ is a generalised fan, that is, all the cones of $\Sigma_P$ contain a fixed vector subspace $\sigma_0$ with associated lattice $N_0 = N \cap \sigma_0$. We denote by $F_P$ the toric variety for the fan $\Sigma_P/\sigma_0 \subset (N/N_0)_\mathbb{R}$.

We recall two well-known facts about toric varieties which we use repeatedly:

**Fact 1** There is a 1-to-1 correspondence between:

(I) the set of polarised toric varieties $(F, D)$, that is, pairs of a proper toric variety $F$ and (torus invariant) ample divisor $D$, and

(II) the set of full-dimensional rational polytopes $P \subset M_\mathbb{R}$.

Given a pair $(F, D)$, the corresponding polyhedron is $P = P_D$. Conversely, given $P$, we let $F = F_P$ and $D = \sum b_i D_i$ where $P = \bigcap \{m : \langle \rho_i, m \rangle \geq -b_i\}$ is the unique facet presentation of $P$.

**Fact 2** Suppose that $D$ is a nef divisor on a proper toric variety $F$. Let $P = P_D$ be the polytope of $D$ and $(F_P, D_P)$ the corresponding polarised pair. Then $(F_P, D_P)$ is an ample model of $(F, D)$, in other words:

(i) There is a proper toric morphism $f: F \to F_P$, and

(ii) $D = f^* D_P$.

**A.1 The toric variety associated to $W$**

Let $M = \text{Hom}(T^4, \mathbb{C}^\times)$ be the group of characters of the torus $T^4$ in Equation (7) and let $N$ be its dual lattice. Denote by $\{e_1, e_2, e_3, e_4\}$ the basis of $N$ dual to the basis of $M$ consisting of the coordinate functions $u_1, u_2, u_3, u_4$ on $T^4$. In this Section we study the Newton polytope $P \subset M_\mathbb{R}$ of the polynomial in Equation (22), defining the variety $W \subset T^4$, and the compactification of $W$ in the toric variety $F = F_P$. 
The polytope $P$ is a 4-dimensional lattice polytope with six vertices $u_0 = 0$, $u_i$ for $i \in \{1, 2, 3, 4\}$ and $u_5 = (2, 2k + 1, 2k + 1, 4k + 1)$, and facet presentation:

$$P = \bigcap_{i=1}^{8} \{ m \in M_{\mathbb{R}} : \langle \rho_i, m_i \rangle \geq -a_i \} \subset M_{\mathbb{R}}$$

where $(\rho_i, a_i) \in N \times \mathbb{Z}$ are:

$$\begin{align*}
(\rho_1, a_1) &= (e_1, 0) \quad \text{for} \quad i = 1, 2, 3, 4 \\
(\rho_5, a_5) &= ((4k + 1, -1, -1, -1)) \\
(\rho_6, a_6) &= ((-1, 3, -1, -1)) \\
(\rho_7, a_7) &= ((-1, -1, 3, -1)) \\
(\rho_8, a_8) &= ((-4k + 1, -4k + 1, -4k + 3), (4k + 1))
\end{align*}$$

Hence the fan $\Sigma = \Sigma_P$ is a complete fan with six 4-dimensional cones and eight rays generated by the vectors $\rho_i$, and $F$ is a compact 4-dimensional toric variety. Note that $P$ does not depend on $a$ and so neither do $\Sigma$ and $F$. The compactification of $W$ in $F$ corresponds to an element of the linear system $|O_F(D)|$, where $D = \sum_{i=1}^{8} a_i D_i$ is the ample divisor of $F$ associated to $P$ (as in Fact 1 above).

**Figure 5:** On the left, a picture of the polytope $P \subset M_{\mathbb{R}}$; on the right, a picture of the normal fan $\Sigma \subset N_{\mathbb{R}}$ with rays $\rho_i$ in $N_{\mathbb{R}}$.

**Remark 25.** The variety $F$ is not $\mathbb{Q}$-Gorenstein. Let $F' \to F$ be any small birational morphism such that $K_{F'}$ is $\mathbb{Q}$-Cartier. Then $F'$ has noncanonical singularities. Indeed, consider the affine open subsets $F'_{\tau_1}$ and $F'_{\tau_2}$ corresponding to the cones

$$\tau_1 = \langle \rho_4, \rho_8 \rangle_+ \quad \text{and} \quad \tau_2 = \langle \rho_5, \rho_8 \rangle_+$$

of the fan of $F'$. Then

$$F'_{\tau_1} \cong \mathbb{C}^{\times 2} \times \frac{1}{4k + 1} (4k - 1, 1) \quad \text{and} \quad F'_{\tau_2} \cong \mathbb{C}^{\times 2} \times \frac{1}{4k + 2} (1, 1)$$

The key point here is that the surface quotient singularities $\frac{1}{4k + 1} (4k - 1, 1)$ and $\frac{1}{4k + 2} (1, 1)$ are not canonical: the vectors $\frac{1}{4k + 1} (4k - 1, 1)$ and $\frac{1}{4k + 2} (1, 1)$ correspond to valuations with discrepancies $-\frac{1}{4k + 1}$ and $-\frac{4k}{4k + 2}$. The weighted blow ups with weights $\frac{1}{4k + 1} (4k - 1, 1)$ and $\frac{1}{4k + 2} (1, 1)$ are the
minimal canonical partial resolutions of these singularities. Our aim is to construct the minimal canonical resolution \( f: \tilde{F} \to F \) (see Definition 26). By what we just said, at the very least, we need to insert the two vectors:

\[
\rho_9 = \frac{1}{4k+1}((4k-1)\rho_4 + \rho_8) = (-1, -1, -1, 2) \quad \text{and} \quad \rho_{10} = \frac{1}{4k+2}(\rho_5 + \rho_8) = (0, -1, -1, 1) \tag{45}
\]

In other words, it is clear that the centres of \( \rho_9 \) and \( \rho_{10} \) are divisors on \( \tilde{F} \); below we see that in fact these are the only vectors we need to add.

A.2 A Mori fibre space structure

The plan In this Section our plan is to:

(i) construct the minimal canonical resolution \( f: \tilde{F} \to F \) of \( F \);

(ii) run a minimal model program with scaling for \( (\tilde{F}, \tilde{D}) \), where \( \tilde{D} = f^*(D) \);

(iii) study the final product \( (F, D) \) to determine the transform of \( W \) in \( F \).

A.2.1 The minimal canonical resolution

**Definition 26.** Let \( X \) be a normal projective variety. The minimal canonical resolution of \( X \) is a projective birational morphism \( f: Y \to X \) where:

\( K_Y \) is \( f \)-ample and \( Y \) has canonical singularities.

For \( i \in \{1, \ldots, 8\} \) let \( a_i \) be as in (44) and set \( a_9 = a_{10} = 1 \). For \( \varepsilon \geq 0 \) consider the 4-dimensional polytope:

\[
P(\varepsilon) = \bigcap_{i=1}^{10} \{ m \in M_{\mathbb{R}} : \langle \rho_i, m \rangle \geq -a_i + \varepsilon \} \subset M_{\mathbb{R}} \tag{46}
\]

For \( 0 < \varepsilon \ll 1 \), the presentation in Equation (46) is the facet presentation of \( P(\varepsilon) \), and the normal fan of \( P(\varepsilon) \) is independent of \( \varepsilon \); denote it by \( \tilde{\Sigma} \) and let \( \tilde{F} = F_{P(\varepsilon)} \) be the corresponding toric variety.

Let \( \tilde{D} = \sum_{i=1}^{10} a_i D_i \); by Fact 1 above the divisor

\[
\tilde{D}(\varepsilon) = \sum_{i=1}^{10} (-\varepsilon + a_i)D_i = \varepsilon K_{\tilde{F}} + \tilde{D} \tag{47}
\]

is ample on \( \tilde{F} \).

**Lemma 27.** (a) There is a morphism \( f: \tilde{F} \to F \) which is the minimal canonical resolution, and

(b) \( \tilde{D} = f^*(D) \)

**Sketch of proof.** Note that \( \tilde{D} \) is nef and \( \tilde{D} = \tilde{D}(0) \). For \( \varepsilon = 0 \) we have that

\[
P(0) = \bigcap_{i=1}^{10} \{ m \in M_{\mathbb{R}} : \langle \rho_i, m \rangle \geq -a_i \} = \bigcap_{i=1}^{8} \{ m \in M_{\mathbb{R}} : \langle \rho_i, m \rangle \geq -a_i \} = P
\]

where the second equality above follows from (45) and the choice \( a_9 = a_{10} = 1 \); thus by Fact 2 above \((F, D)\) is the ample model of \( (\tilde{F}, \tilde{D}) \); in other words there is a morphism \( f: \tilde{F} \to F \) and \( \tilde{D} = f^*(D_P) \). Since for \( 0 < \varepsilon \ll 1 \) the divisor \( \tilde{D}(\varepsilon) = \varepsilon K_{\tilde{F}} + \tilde{D} \) is ample on \( \tilde{F} \), \( K_{\tilde{F}} \sim_f 1/\varepsilon \tilde{D}(\varepsilon) \) is \( f \)-ample. Finally, one can check explicitly that \( \tilde{F} \) has canonical singularities. \( \Box \)
A.2.2 The minimal model program

For $0 < \varepsilon \ll 1$ the divisors $\tilde{D}(\varepsilon)$ are ample on $\tilde{F}$, since $K_{\tilde{F}}$ is $f$-ample and $D \subset F$ is ample. We recover $\tilde{F}$ as the toric variety whose spanning fan is the normal fan of the polytopes $P(\varepsilon)$ for small values of $0 < \varepsilon$.

The minimal model program with scaling consists of an inductively defined finite sequence of toric varieties $F_j$, divisors $D_j$, and a strictly increasing sequence of rational numbers $\varepsilon_j$, and birational maps:

$$ F_0 \rightarrow \cdots \rightarrow F_j \rightarrow F_{j+1} \rightarrow \cdots \rightarrow F_r = \mathcal{F} $$

where:

1. We start with $F_0 = \tilde{F}$, $D_0 = \tilde{D}$, and $\varepsilon_0 = \max\{\varepsilon > 0 \mid \varepsilon K_{\tilde{F}} + \tilde{D} \text{ is nef on } \tilde{F}\}$;
2. For $j \geq 0$, $t_j : F_j \rightarrow F_{j+1}$ is the divisorial contraction or flip of the face $R_j \subset \text{NE}(F_j)$ with $(\varepsilon_j K_j + D_j)|_{R_j} = 0$ and $(K_j + D_j)|_{R_j} < 0$;
3. For $j > 0$ we denote by $D_j$ the proper transform on $F_j$, $K_j = K_{F_j}$, and:

$$ \varepsilon_j = \max\{\varepsilon > \varepsilon_{j-1} \mid \varepsilon K_j + D_j \text{ is nef on } F_j\} $$

4. The program ends at $F_r = \mathcal{F}$ where either:

The pair $(\tilde{F}, \tilde{D})$ is a minimal model that is $\varepsilon = 1$ and $K_{\tilde{F}} + \tilde{D}$ is nef; or

The pair $(\tilde{F}, \tilde{D})$ is a Mori fibre space (Mfs) that is $\varepsilon < 1$, the contraction $\psi : \tilde{F} \rightarrow S$ of $R_r$ has relative dimension $> 0$ and $K_{\tilde{F}} + \tilde{D}$ is $\psi$-ample. In this case $\varepsilon$ is the quasi-effective threshold of the pair $(\tilde{F}, \tilde{D})$:

$$ \varepsilon = \sup\{t \mid t K_{\tilde{F}} + \tilde{D} \in \text{Eff}(\tilde{F})\} $$

By Fact 1, we recover $F_j$ as the toric variety whose spanning fan is the normal fan of the polytopes $P(\varepsilon)$ for $\varepsilon \in (\varepsilon_{j-1}, \varepsilon_j)$, since the divisors $\varepsilon K_j + D_j$ are ample on $F_j$; the threshold values $\cdots < \varepsilon_j < \varepsilon_{j+1} < \cdots$ are those where the polytope $P(\varepsilon)$ changes shape.

A.2.3 The final product

Returning to the polytopes of Equation (46), $P(1) = \emptyset$, thus the minimal model program just described will end with a Mfs. The sequence of threshold values is:

$$ \frac{1}{2}, \frac{4k - 1}{8k - 3}, \frac{2k + 1}{3k}, \frac{2}{3} $$

Thus $\varepsilon = 2/3$,

$$ P\left(\frac{2}{3}\right) = \left[\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1\right), \left(\frac{2}{3}, \frac{2k + 1}{3}, \frac{2k + 1}{3}, \frac{4k + 1}{3}\right)\right] $$

We computed the threshold values as follows. Consider the 5-dimensional lattice $N' = N \oplus \mathbb{Z}$ and the dual lattice $M'$, whose elements are pairs $(m, r)$ with $m \in M$ and $r \in \mathbb{Z}$; also denote by $q : M' \rightarrow \mathbb{Z}$ the projection to the second factor. Let $\rho_i = (\rho_i, -1) \in N', i = 1, \ldots, 10$ and let $P'$ be the 5-dimensional polyhedron:

$$ P' = \bigcap_{i=1}^{10} \{m' \in (M')_R : \langle m', \rho_i \rangle \geq -a_i \} \subset (M')_R $$

Then $\forall \varepsilon P(\varepsilon) = q^{-1}(\varepsilon) \cap P'$ and the threshold values occur at the vertices of $P'$.
and $S = \mathbb{P}^1$. More precisely, the normal fan of $P(2/3)$ has two maximal cones:

$$C_1 = \{ n \in N_R : n_2 + n_3 + 2n_4 \geq 0 \} \quad \text{and} \quad C_2 = \{ n \in N_R : n_2 + n_3 + 2n_4 \leq 0 \}$$

intersecting along the hyperplane $C_0 = \{ n \in N_R : n_2 + n_3 + 2n_4 = 0 \} \subset N_R$.

For all $\varepsilon \in ((2k+1)/(3k+2), 2/3)$, $P(\varepsilon)$ is a 4-dimensional polytope with 9 vertices and 7 facets, whose normals are generated by the vectors $\rho_1, \rho_2, \rho_3, \rho_5, \rho_6, \rho_7, \rho_{10};$ thus its normal fan $\Sigma$ has 7 rays and 9 maximal cones, which are listed in Figure 6. One can check that the 4-dimensional toric variety $\overline{F}$ is $\mathbb{Q}$-Gorenstein but not $\mathbb{Q}$-factorial, and has canonical singularities.

The toric morphism $\psi: \overline{F} \to \mathbb{P}^1$ is induced by the quotient map $N \to N/N_0$ where $N_0 = N \cap C_0$, sending 

$$(n_1, n_2, n_3, n_4) \mapsto n_2 + n_3 + 2n_4$$

![Figure 6: A picture of the fan $\Sigma \subset N_k$ and of the cones $C_1 \ni \rho_2, \rho_3$ and $C_2 \ni \rho_5$. The vectors $\rho_1, \rho_6, \rho_7, \rho_{10}$ span the hyperplane $C_0 \subset N_k$, represented by the area coloured in gray. The cone $C_1$ is the union of the 5 maximal cones of $\Sigma$ containing $\rho_2$ or/and $\rho_3,$ listed on the right of the picture. The cone $C_2$ is the union of the 4 maximal cones of $\Sigma$ containing $\rho_5$, listed on the left.](image)

The preimage via $\psi: \overline{F} \to \mathbb{P}^1$ of the torus $\mathbb{C}^\times \subset \mathbb{P}^1$ is the non-compact toric variety $G$ associated to the subfan $\Delta \subset \Sigma$ given by the cones of $\Sigma$ contained $C_0$. The fan $\Delta$ has four 3-dimensional cones (see Figure 6):

$$\langle \rho_6, \rho_1, \rho_{10} \rangle + \langle \rho_7, \rho_1, \rho_{10} \rangle + \langle \rho_6, \rho_7, \rho_1 \rangle + \langle \rho_6, \rho_7, \rho_{10} \rangle$$

Since the quotient $N/N_0$ is torsion-free, we may choose a splitting $N = N_0 \oplus N_1$ and regard $\Delta$ as a fan $\Delta_0$ in $N_0$; we have $\Delta = \Delta_0 \oplus \{ 0 \}$, thus $G = G_0 \times \mathbb{C}^\times$, where $G_0$ is the 3-dimensional toric variety with spanning fan $\Delta_0 \subset N_0$. To determine $G_0$, note that the vectors $\rho_1, \rho_6, \rho_7, \rho_{10}$ spanning the hyperplane $C_0$ satisfy the linear relation:

$$\rho_6 + \rho_7 + 2\rho_1 + 2\rho_{10} = 0$$

However $G_0$ is not the weighted projective space $\mathbb{P}(1,1,2,2)$, as $1/2 \cdot (\rho_6 + \rho_1 + \rho_{10}) \in N_0$. Then write:

$$N_{00} = \mathbb{Z}\rho_6 + \mathbb{Z}\rho_1 + \mathbb{Z}\rho_{10} \quad \text{and} \quad N_0 = N_{00} + 1/2 \cdot (1,1,1)\mathbb{Z}$$

We have $M_{00} \supset M_0 = \{ m \in M_{00} : m_1 + m_2 + m_3 = 0 \ \text{mod} \ 2 \}$ and $\mathbb{C}[M_0]$ is the ring of invariants $\mathbb{C}[M_{00}]^{\mu_2}$, where the group where $\mu_2$ acts on $\mathbb{C}[M_{00}]$ by $(\xi, \chi^m) = \xi^{m_1+m_2+m_3} \chi^m$. Equivalently $G_0$ is the quotient variety:

$$G_0 = \mathbb{P}(1,1,2,2)/\mu_2$$

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Consider the toric variety \( \text{Construction} \) group \( \psi \) fibration of Lemma 22(3). The del Pezzo fibration \( \phi \): \( \hat{W} \rightarrow \mathbb{C}^\times \) of Section 5 is the restriction of the toric morphism \( \varphi: F \rightarrow \mathbb{P}^1 \).

### A.3 A conic bundle structure

The del Pezzo fibration \( \hat{W} \rightarrow \mathbb{C}^\times \) is not a Mori fibre space in the Mori category: \( \hat{W} \) has strictly canonical singularities along two sections of \( \phi \) corresponding to the orbifold points \( p_t^\pm \in \hat{W}_t \) (all \( t \)) of Lemma 22(3).

In this Section we construct a birational model \( W' \) with terminal singularities and a Mori fibration \( \psi: W' \rightarrow F_1 \) in the Mori category. In our view, \( \hat{W} \) is a better model in which to find the group \( H^1(Y, \mathbb{Q}) \).

**Construction** Consider the toric variety \( G' \) with weight matrix\(^4\)

\[
\begin{array}{ccccccc}
t_1 & t_2 & v_1 & v_2 & z_1 & z_2 & z \\
1 & 1 & 0 & -1 & 0 & -k & -k \\
0 & 0 & 1 & 1 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
\]

and irrelevant ideal \((t_1, t_2)(v_1, v_2)(z, z_1, z_2)\). Note the morphism \( G' \rightarrow F_1 \). Set

\[
\tilde{\delta}_1(t_1, t_2) = 4at_1^{2k+1} + t_2^{2k+1}
\]

and consider the hypersurface \( W' \subset G' \) given by

\[
- z_1^2 + t_2 \tilde{\delta}_1 z_2^2 + v_1 v_2 z^2 \left( \tilde{\delta}_1 v_1^2 - at_1^{2k+1} t_2 v_1 v_2 + t_1 t_2 \tilde{\delta}_1 v_2^2 \right) = 0 \quad (48)
\]

**Theorem 28.** (A) \( W' \) has terminal singularities and the morphism \( \psi: W' \rightarrow F_1 \) is a conic bundle Mfs;

(B) There is a commutative diagram:

\[
\begin{array}{ccc}
\hat{W} & \xrightarrow{\phi} & W' \\
\downarrow & & \downarrow \psi \\
\mathbb{C}^\times & \xrightarrow{\psi} & F_1
\end{array}
\]

\(^4\)This matrix defines an action of \( G_1^a \) on \( \mathbb{A}^7 \) and \( G' \) is a GIT quotient.
Proof. (A) It is possible, and not too hard, to verify explicitly that \( W' \) has isolated cA singularities.

(B) We start by exhibiting an explicit birational map from a chart of \( G \) to a chart of \( G' \).

Consider the chart \((t_2 = v_2 = z = 1) \subset G'\); this is isomorphic to \( \mathbb{C}^4 \) with coordinate functions \( t_1, v_1, z_1, z_2 \). In this chart, \( W' \) is given by the equation:

\[-z_1^2 + \delta_1 z_2^2 + v_1 \left( \delta_1 v_1^2 - a t_1^{2k+1} v_1 + t_1 \delta_1 \right) = 0\]

Next consider the chart \((x_2 = 1) \subset G'\); this is isomorphic to \( U \mathbb{C}^4 \times U (1, 1, 1) \) with coordinate ring \( \mathbb{C}[t] \otimes \mathbb{C}[x_1, y_1, y_2]^{\mu_2} \) generated by the functions \( t, x_1^2, x_1 y_1, \; \& c. \) In this chart, \( \hat{W} \) is given by the equation:

\[-\delta_1 y_1^2 + y_2^2 + x_1^4 - \frac{a t_1^{2k+1}}{\delta_1} x_1^2 + t = 0\]

We define a rational map from the chart in \( G \) to the chart in \( G' \) by:

\[ v_1, z_1, z_2 \mapsto x_1^2, \delta_1 x_1 y_1, x_1 y_2 \]

This map is in fact birational with inverse given by:

\[ x_1^2, x_1 y_1, y_1^2, x_1 y_2, y_1 y_2 \mapsto v_1, \frac{z_1}{\delta_1}, \frac{z_1^2}{v_1 \delta_1}, z_2, \frac{z_1 z_2}{v_1 \delta_1}, \frac{z_2^2}{v_1} \]

It is easy to see that the equation for \( \hat{W} \) is transformed into the equation for \( W' \). \( \square \)

Remark 29. The variety \( W' \) has many singular points (more precisely, above \( \tilde{v}_1 = v_j = 0, t_1 = v_1 = 0, t_2 = v_j = 0, j = 1, 2 \); thus the conic bundle \( \psi: W' \to \mathbb{F}_1 \), although it is a Mori fibre space, is not standard (in the birational geometry literature, a 3-fold \( Mfs: X \to S \) is a standard conic bundle if \( \dim S = 2 \) and \( X \) is nonsingular, which implies that \( S \) is also nonsingular, all fibres are conics, and the discriminant is normal crossing). Hence the conjectural rationality criteria \([16,14,15,13]\) do not directly apply, though they suggest that \( W' \) is nonrational (and possibly even birationally rigid). It would be interesting to study the geometry of \( W' \) further.

Remark 30. The conic bundle \( \psi: W' \to \mathbb{F}_1 \) is an analog of a construction that in the singularity theory and mirror symmetry literature is called a “double suspension.” It is possible, of course, to find \( H^1(Y, \mathbb{Q}) \) in \( H^3(W', \mathbb{Q}) \), but it is easier to find it in \( H^3(\hat{W}, \mathbb{Q}) \).

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