Distance from the Nucleus to a Uniformly Random Point in the Typical and the Crofton Cells of the Poisson-Voronoi Tessellation

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Abstract

Consider the distances $R_o$ and $\tilde{R}_o$ from the nucleus to a uniformly random point in the typical and Crofton cells, respectively, of the $d$-dimensional Poisson-Voronoi (PV) tessellation. The main objective of this paper is to characterize the exact distributions of $R_o$ and $\tilde{R}_o$. First, using the well-known relationship between the Crofton cell and the typical cell, we show that the random variable $\tilde{R}_o$ is equivalent in distribution to the contact distance of the Poisson point process. Next, we derive a multi-integral expression for the exact distribution of $R_o$. Further, we derive a closed-form approximate expression for the distribution of $R_o$, which is the contact distribution with a mean corrected by a factor equal to the ratio of the mean volumes of the Crofton and typical cells. An additional outcome of our analysis is a direct proof of the well-known spherical property of the PV cells having a large inball.

Index Terms

Poisson point process, Poisson-Voronoi tessellation, typical cell, Crofton cell, distance distribution.

I. INTRODUCTION

The Poisson point process (PPP) has found many applications in science and engineering due to its useful mathematical properties. Several of these applications specifically focus on the Poisson-Voronoi (PV) tessellation [1], which partitions space into disjoint cells whose nuclei are the points of the PPP. There is a rich literature focused on characterizing the statistical properties of the PV tessellation, such as the distributions of the contact and chord lengths [2], the distributions of the radii of the circumcircle and the incircle of the typical and Crofton cells [3], the distribution of the number of edges of the typical cell [4], the limiting shape of the Crofton and typical cells [5], and the relationship between the typical and Crofton cells [6]. However, it is quite surprising to note that the distributions of the distances from the nucleus to uniformly random points in the typical and Crofton cells of the $d$-dimensional PV tessellation have not yet been investigated, which is the main goal of this paper.

The motivation behind our investigation comes from wireless networks, where the PPP has been extensively used to model the locations of cell towers (also called base stations) in cellular networks such that the service region of each cell tower is simply the PV cell with the corresponding cell tower at its nucleus [7]–[10]. If one assumes mobile users to be distributed uniformly at random in the service region of each cell tower (a popular model used by the wireless networks community), one of the crucial steps towards the performance characterization of this network is to understand the distribution of the distance between a mobile user and its associated cell tower. In the PV tessellation, this corresponds to the distribution of the distance from the nucleus to uniformly random points in the typical and Crofton cells of the $d$-dimensional PV tessellation case.

Let $\Phi \triangleq \{x_1, x_2, \ldots\}$ be a homogeneous PPP with intensity $\lambda$ on $\mathbb{R}^d$. The PV cell with the nucleus at $x \in \Phi$ is defined as

$$V_x = \{y \in \mathbb{R}^d \mid \|y - x\| \leq \|x' - y\|, \forall x' \in \Phi\}, \quad x \in \Phi.$$
The set \( \{V_x\}_{x \in \Phi} \) is known as the PV tessellation. For any (deterministic) \( y \in \mathbb{R}^d \), almost surely there exists a unique nucleus \( x \in \Phi \) such that \( y \in V_x \). The PV cell containing the origin \( o \) is called the Crofton cell and is denoted by \( \tilde{V}_o \). The statistical properties of the typical cell can be characterized using Palm theory, which formalizes the notion of conditioning on the presence of a point at a specific location. Since by Slivnyak’s theorem, conditioning on a point is the same as adding a point to a PPP, we consider that the nucleus of the typical cell is the point process \( \Phi \cup \{o\} \) is \( o \), which is given by

\[
V_o = \{y \in \mathbb{R}^d | \|y\| \leq \|x - y\|, \forall x \in \Phi\}. \tag{2}
\]

Now, using \( \tilde{V}_o \) and \( V_o \), we define the main random variables of interest for this paper.

**Definition 1.** Let \( \tilde{R}_o \) denote the distance from the nucleus to a uniformly random point in the Crofton cell \( \tilde{V}_o \).

**Definition 2.** Let \( R_o \) denote the distance from the nucleus to a uniformly random point in the typical cell \( V_o \).

We focus on the statistical characterization of \( R_o \) and \( \tilde{R}_o \) for the PPP with intensity \( \lambda \). We derive the cumulative distribution function (CDF) of \( \tilde{R}_o \) and \( R_o \) in Sections III and III respectively. In Section III a closed-form expression for the exact CDF of \( \tilde{R}_o \) is derived based on the formula on the relationship between the typical and Crofton cells given in [6], [12]. It is well-known that the statistical properties of \( R_o \) are hard to characterize for the case of \( d > 1 \). Before going into the \( d > 1 \) case, we discuss the case of \( d = 1 \) in Section III-A for which the distribution of \( R_o \) is far easier to characterize. In Section III-B we present an analytical approach to derive the distribution of \( R_o \) for the \( d > 1 \) case based on the analysis of the temporal evolution of the PV structure presented in [13]. We also approximate the CDF of \( R_o \) using a simple expression in Section IV. Therein, we also characterize the distribution of \( R_o \) as \( d \) tends to infinity. In addition, based on the formulation developed in Section III we provide a simpler proof for the well-known spherical nature of large PV cells in Section V.

### II. DISTRIBUTION OF \( \tilde{R}_o \)

In this section, we derive a closed-form expression for the CDF of the distance from the nucleus to uniformly random point in the Crofton cell \( \tilde{V}_o \). It is well-known that the expected volume of the Crofton cell is greater than the expected volume of the typical cell. In fact, all the moments of the volume of the Crofton cell are known to be greater than the moments of the volume of the typical cell [6]. This is quite intuitive as the origin (or, for that matter, any fixed point) is more likely to lie in a bigger cell.

Before presenting the CDF of \( \tilde{R}_o \), we state the relationship of the distributions of typical cell and Crofton cell from [12, Corollary 4.2.4] as

\[
\mathbb{E}[f(\tilde{V}_o)] = \lambda \mathbb{E}^o[v_d(\tilde{V}_o)f(\tilde{V}_o)], \tag{3}
\]

where \( v_d \) is the Lebesgue measure in \( d \)-dimensions, \( \mathbb{E}^o \) is the expectation with respect to the Palm distribution, and \( f \) is any translation-invariant non-negative function on compact sets. We will use this expression along with an appropriately chosen function \( f \) to derive the CDF of \( \tilde{R}_o \) in Theorem I. Let \( B_r(x) \) represent the \( d \)-dimensional ball of radius \( r \) centered at \( x \). Let \( X \) be a random set in \( \mathbb{R}^d \). Using the results of [14] and [15], the \( n \)-th moment of the volume of \( X \) can be evaluated as

\[
\mathbb{E}[v_d(X)^n] = \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathbb{P}(x_1, \ldots, x_n \in X)dx_1 \cdots dx_n. \tag{4}
\]

Next, we restate a useful result from [16, Lemma 4.2] on the mean volume of \( B_r(o) \cap V_o \), which directly follows from (4).

**Lemma 1.** For the homogeneous PPP with intensity \( \lambda \) on \( \mathbb{R}^d \), the mean volume of the intersection of the ball \( B_r(o) \) with the typical cell \( V_o \) is given by

\[
\mathbb{E}^o[v_d(B_r(o) \cap V_o)] = \frac{1}{\lambda} \left(1 - \exp(-\lambda \kappa_d r^d)\right), \tag{5}
\]

where \( \kappa_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} + 1\right)} \) is the volume of the unit-radius ball in \( \mathbb{R}^d \).
\textbf{Proof.} Using (4), the first moment of the volume of intersection of \( B_r(o) \) with the typical cell \( V_o \) can be determined as
\[
\mathbb{E}^o[v_d(B_r(o) \cap V_o)] = \int_{\mathbb{R}^d} \mathbb{P}(x \in B_r(o) \cap V_o) \, dx = \int_{\mathbb{R}^d \cap B_r(o)} \mathbb{P}(x \in V_o) \, dx = (a) \int_0^r \exp(-\lambda \kappa_d v^d) v^{d-1} \, dv.
\]
where (a) follows by the change of Cartesian coordinates to polar coordinates and the void probability of the homogeneous PPP.

Now, we present the CDF of \( \tilde{R}_o \) using the result given in Lemma [1].

\textbf{Theorem 1.} For the homogeneous PPP with intensity \( \lambda \) on \( \mathbb{R}^d \), the CDF of the distance \( \tilde{R}_o \) from the nucleus to a uniformly random point in the Crofton cell \( \tilde{V}_o \) is
\[
F_{\tilde{R}_o}(r) = 1 - \exp \left( -\lambda \kappa_d r^d \right), \quad r \geq 0.
\]

\textbf{Proof.} Let \( x_o \) represent the nucleus of \( \tilde{V}_o \) and let \( y \) represent the uniformly distributed point in \( \tilde{V}_o \). We note that the distance \( \tilde{R}_o = ||x_o - y|| \) is less than \( r \) when \( y \) lies in the intersection of the ball \( B_r(x_o) \) and \( \tilde{V}_o \). Therefore, the CDF of \( \tilde{R}_o \) can be written as
\[
F_{\tilde{R}_o}(r) = \mathbb{P}(\tilde{R}_o \leq r) = \mathbb{E} \left[ \frac{v_d(B_r(x_o) \cap \tilde{V}_o)}{v_d(\tilde{V}_o)} \right].
\]

Now, we define the function \( f \) of the PV cell \( V_x \) as the ratio of the volumes of \( B_r(x) \cap V_x \) and \( V_x \). Thus, the function \( f \) for the Crofton cell and the typical cell, respectively, becomes
\[
f(\tilde{V}_o) = \frac{v_d(B_r(x_o) \cap \tilde{V}_o)}{v_d(\tilde{V}_o)} \quad \text{and} \quad f(V_o) = \frac{v_d(B_r(o) \cap V_o)}{v_d(V_o)}.
\]

By substituting the above function in (4), we obtain
\[
\mathbb{E} \left[ \frac{v_d(B_r(x_o) \cap \tilde{V}_o)}{v_d(\tilde{V}_o)} \right] = \lambda \mathbb{E}^o[v_d(B_r(o) \cap V_o)].
\]

Finally, we arrive at (6) by substituting the result of Lemma [1] in the above equation. \hfill \Box

Using Theorem 1, we can calculate the \( n \)-th moment of the distance \( \tilde{R}_o \).

\textbf{Corollary 1.} For the homogeneous PPP with intensity \( \lambda \) on \( \mathbb{R}^d \), the \( n \)-th moment of the distance \( \tilde{R}_o \) from the nucleus to uniformly random point in the Crofton cell \( \tilde{V}_o \) is
\[
\mathbb{E}[\tilde{R}_o^n] = \frac{\Gamma \left( 1 + \frac{d}{2} \right)}{(\lambda \kappa_d)^{\frac{n}{2}}}, \quad n \geq 0.
\]

\textbf{Remark 1.} Using the void probability, the distribution of the distance between the origin and the nucleus of \( \tilde{V}_o \), say \( x_o \), can be simply determined as \( \mathbb{P}(||x_o|| \leq r) = 1 - \exp(-\lambda \kappa_d r^d) \). However, it does not reveal any information about how the origin is distributed in the Crofton cell. While one can intuitively expect the origin to be uniformly distributed in \( \tilde{V}_o \), there does not appear to be a straightforward way to prove this. Using (4), we have presented a simple construction to establish that the distribution of the origin in \( \tilde{V}_o \) is in fact that of a uniformly random point in \( \tilde{V}_o \).

In the next section, we present our approach to the exact evaluation of the CDF of \( R_o \).

\textbf{III. DISTRIBUTION OF } R_o

We first characterize the CDF of \( R_o \) for \( d = 1 \) where the typical cell is completely characterized by the locations of the nearest points on either side of its nucleus. This allows us to explicitly describe the uniformly distributed point in the typical cell \( V_o \) and, in turn, determine the CDF of \( R_o \). In contrast, the structure of the typical cell for \( d > 1 \) is more complex, which makes the distribution of \( R_o \) far more difficult to determine, as will be demonstrated in Section III-B.
A. Distribution of $R_o$ for $d = 1$

Let $\Phi \triangleq \{x_1, x_2, \ldots\}$ be a homogeneous PPP with intensity $\lambda$ on $\mathbb{R}$. Let $x \in \Phi \cap \mathbb{R}^-$ and $y \in \Phi \cap \mathbb{R}^+$ be left and right neighboring points of the origin (i.e., nucleus of $V_o$), respectively. Since $\Phi$ is a PPP, $|x|$ and $|y|$ are i.i.d. random variables that follow an exponential distribution with mean $\lambda^{-1}$. Denote by $R_1 = \frac{1}{2}|x|$ and $R_2 = \frac{1}{2}|y|$ the distances to the boundary points of the typical cell $V_o$. Since $|x|$ and $|y|$ are i.i.d., $R_1$ and $R_2$ are also i.i.d. and follow exponential distribution with parameter $2\lambda$. Let $\tilde{R}_1 = \min(R_1, R_2)$ and $\tilde{R}_2 = \max(R_1, R_2)$. The joint probability density function (pdf) of $\tilde{R}_1$ and $\tilde{R}_2$ is \cite{17} Chapter 2

$$f_{\tilde{R}_1, \tilde{R}_2}(r_1, r_2) = 8\lambda^2 \exp(-2\lambda(r_1 + r_2)), \quad 0 \leq r_1 \leq r_2. \quad (8)$$

The distribution of the distance $R_o$ from the nucleus to a uniformly random point in the typical cell $V_o$ conditioned on $\tilde{R}_1$ and $\tilde{R}_2$ is

$$P(R_o \leq r | \tilde{R}_1 = r_1, \tilde{R}_2 = r_2) = \begin{cases} \frac{2r}{r_1 + r_2}, & 0 \leq r \leq r_1 \\ \frac{r + r_1}{r_1 + r_2}, & r_1 < r \leq r_2 \\ 1, & r_2 < r. \end{cases} \quad (9)$$

By deconditioning the above expression with respect to the joint distribution of $\tilde{R}_1$ and $\tilde{R}_2$, the CDF of $R_o$ is presented in the following theorem.

Theorem 1. For the homogeneous PPP with intensity $\lambda$ on $\mathbb{R}$, the CDF of the distance $R_o$ from the nucleus to a uniformly random point in the typical cell $V_o$ is

$$F_{R_o}(r) = 1 - \exp(-2\lambda r) + 2\lambda r \exp(-2\lambda r) - 4\lambda^2 r^2 E_1(2\lambda r), \quad r > 0, \quad (10)$$

where $E_1(z) = \int_z^\infty \frac{1}{t} \exp(-t) dt$ is the exponential integral function.

Proof. Using the expression for the conditional CDF of $R_o$ given in (9) and the joint pdf of $\tilde{R}_1$ and $\tilde{R}_2$ given in (8), the CDF of $R_o$ can be written as

$$F_{R_o}(r) = \int_0^r \int_0^{r_2} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1dr_2 + \int_r^\infty \int_0^{r_1} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1dr_2$$

$$+ \int_r^\infty \int_r^{r_2} \frac{2r}{r_1 + r_2} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1dr_2. \quad (11)$$

Further, using some mathematical simplifications, we obtain the result in (10). Please refer to Appendix A for more details on the manipulation of the integrals in (11).
In Fig. 1 we provide the plots for the CDFs of \( R_o \) and \( \tilde{R}_o \). From the figure, it can be seen that the distance \( \tilde{R}_o \) stochastically dominates the distance \( R_o \). In Section IV, we will demonstrate that this difference between the distributions of \( \tilde{R}_o \) and \( R_o \) diminishes with increasing \( d \).

### B. Distribution of \( R_o \) for \( d > 1 \)

Similar to the distribution of \( R_o \) for \( d = 1 \) being derived by conditioning on the nuclei of the neighboring PV cells in Section III-A, here we derive the distribution of \( R_o \) for \( d > 1 \) by conditioning on the points in a hypersphere centered at the origin such that it includes the nuclei of all neighboring PV cells of \( V_o \). We refer to the conditional positions of points in the sphere as the domain configuration. The domain configuration enables the characterization of the shape and size of the PV cell \( V_o \) which will be useful in the evaluation of the conditional distribution of \( R_o \). A similar construction is presented in [13], [18] to study the temporal evolution of the volume of the domain size and free boundary distributions for a PV transformation\(^1\) for \( d = \{1, 2, 3\} \). In the following subsection, we define the domain configuration and discuss its use for the conditional PV cell characterization.

1) **Domain Configuration:** First, we define the domain configuration and obtain its probability. Next, we discuss its connection with the conditional shape and size of the PV cell \( V_o \).

**Definition 3.** For \( \ell > 0 \), we define the set \( C^k_\ell \) as the set of \( k \) points with polar coordinates \((l_i, \theta_i)\) such that

\[
C^k_\ell \equiv \frac{1}{2} \{ \Phi \cap B_{2\ell}(o) \mid \Phi(B_{2\ell}(o)) = k \}.
\]

(12)

where \( l_i \) is the radial coordinate and \( \theta_i = [\theta_{i1}, \ldots, \theta_{(d-1)k}] \) are the angular coordinates.

Thus, the point \( \tilde{x}_i \triangleq (l_i, \theta_i) \in C^k_\ell \) bisects the line segment joining \( o \) and \( x_i \in \Phi \cap B_{2\ell}(o) \). By construction, \( l_i \in [0, \ell] \), \( \theta_{(d-1)k} \in [0, 2\pi) \) and \( \theta_{1k}, \ldots, \theta_{(d-2)k} \in [0, \pi] \). Henceforth, the set \( C^k_\ell \) is referred to as the domain configuration. Since \( \Phi \) is a PPP, conditioned on \( \Phi(B_{2\ell}(o)) = k \), the points \( x_i \in \Phi \cap B_{2\ell}(o) \), for \( i \in \{1, \ldots, k\} \), are distributed uniformly at random independently of each other in \( B_{2\ell}(o) \). Consequently, the \( k \) points \( \{\tilde{x}_i\}_{i=1}^k \) forming the domain configuration \( C^k_\ell \) are also distributed uniformly at random independently of each other in \( B_{2\ell}(o) \). Using this fact, we can express the pdf of the domain configuration as done next.

The differential volume element in \( d \) dimensions in polar coordinates is [19]

\[
\Delta = v^{d-1} \sin^{d-2}(\alpha_1) \cdots \sin(\alpha_{d-2})dv\alpha_1 \cdots d\alpha_{d-1}.
\]

Thus, the probability that a point distributed uniformly at random in \( B_{\ell}(o) \) lies in an infinitesimal region with volume \( \Delta_i \) such that \( v_i \leq \ell \) is equal to \( \frac{\Delta_i}{\kappa_d \ell^d} \). Now, we obtain the pdf of the configuration \( C^k_\ell \) conditioned on \( \Phi(B_l(o)) = k \) as

\[
P((l_1, \theta_1) \in \Delta_1, \ldots, (l_k, \theta_k) \in \Delta_k; \ell) \overset{(a)}{=} \prod_{i=1}^k P((l_i, \theta_i) \in \Delta_i)
\]

\[
\overset{(b)}{=} \prod_{i=1}^k \frac{1}{\kappa_d \ell^d} v_i^{d-1} \sin^{d-2}(\alpha_{1i}) \cdots \sin(\alpha_{(d-2)i})dv_i d\alpha_{1i} \cdots d\alpha_{(d-1)i}, \text{ for } 0 \leq v_i \leq \ell,
\]

(13)

where \( (a) \) follows from the independence of the elements of \( C^k_\ell \) and \( (b) \) follows from the uniform distribution of elements of \( C^k_\ell \) in \( B_{\ell}(o) \).

2) **Connections with the Typical Cell:** For an empty domain configuration \( C^0_\ell \), \( B_{\ell}(o) \) is contained in the typical cell \( V_o \). However, a non-empty domain configuration, i.e., \( C^k_\ell \) for \( k > 0 \), contains the mid-points of the chords of \( B_{\ell}(o) \) formed by the intersection of the edges of typical cell \( V_o \) with \( B_{\ell}(o) \). In addition, the line segments connecting these mid-points to the origin are perpendicular to the corresponding edges. Therefore, the domain configuration provides useful information about the structure of \( V_o \). We denote by \( V_o(C^k_\ell) \) the typical cell conditioned on the domain configuration \( C^k_\ell \). As \( k \to \infty \), it is easy to see that \( V_o(C^k_\ell) \) becomes deterministic. However, for any finite \( k \), \( V_o(C^k_\ell) \) is in general random because some of its edges may be defined by points of \( \Phi \) lying outside \( B_{2\ell}(o) \). That

\(\text{Footnote 1:}\) The simultaneously growing sets of randomly distributed nuclei (realized through PPP) at equal isotropic rate is referred to as the PV transformation. These sets eventually transform into the PV cells.
Figure 2. Illustration of $V_o(C^k \ell) \cap B_\ell(o)$ for $d = 2$.

said, conditioning on $C^k \ell$ is sufficient to uniquely determine the intersection of $V_o(C^k \ell)$ and the ball $B_\ell(o)$. Fig. 2 illustrates the intersection of the $B_\ell(o)$ with the cell $V_o(C^k \ell)$ for $d = 2$.

Let us define $H_x$ as the half-space formed by the points in $\mathbb{R}^d$ that are closer to the point $x \in \Phi$ than the origin, i.e.,

$$H_x \triangleq \{ y \in \mathbb{R}^d \mid \| y - x \| < \| y \| \}.$$  \hspace{1cm} (14)

Now, we denote by $L_i$ the surface (in $d - 1$ dimensions) of the spherical cap of $B_\ell(o)$ such that

$$L_i \triangleq H_{\tilde{x}_i} \cap \partial B_\ell(o),$$  \hspace{1cm} (15)

where $\partial B_\ell(o)$ is the boundary of $B_\ell(o)$. Note that the surface of the spherical cap is the arc of a circle for $d = 2$. From the above definition, it is clear that the point $\tilde{x}_i \in C^k \ell$ is the nearest equidistant point to the origin and $x_i$ that lies on the supporting hyperplane of $H_{\tilde{x}_i}$. Further, the point $\tilde{x}_i$ is also the center of the $(d - 1)$-dimensional chord that forms $L_i$. This is illustrated in Fig. 2 for $d = 2$. Now, since $\{\tilde{x}_i\}_{i=1}^k$ are distributed uniformly at random in $B_\ell(o)$ independently of each other, the corresponding surfaces of the spherical caps $\{L_i\}_{i=1}^k$ have i.i.d. surface areas\(^2\) and are placed uniformly at random on $\partial B_\ell(o)$. As will be evident in the sequel, this construction will allow us to establish useful conditional geometric properties of the PV cell such as the volume of the intersection of the ball with the PV cell, the conditional distribution of uniformly distributed points within the PV cell, and the shape of large PV cells. We will now use this construction to derive the distribution of $R_o$.

3) Distance Distribution: For a given domain configuration $C^k \ell$, we define

$$g_k(r; C^k \ell) = v_d(V_o(C^k \ell) \cap B_r(o)),$$  \hspace{1cm} (16)

for $0 \leq r \leq \ell$, as the volume of the intersection of $B_r(o)$ and cell $V_o(C^k \ell)$. As discussed before, $V_o(C^k \ell)$ is the typical cell conditioned on the domain configuration $C^k \ell$.

**Definition 4.** Let $R_\ell$ denote the distance from the nucleus of $V_o$ (i.e., the origin) to a uniformly random point in $V_o \cap B_\ell(o)$.

The first main goal is to characterize the CDF of $R_\ell$. Since for $\ell \to \infty$, $V_o \subset B_\ell(o)$, the CDF of $R_o$ will simply be

$$F_{R_o}(z) = \lim_{\ell \to \infty} \mathbb{P}(R_\ell \leq z).$$  \hspace{1cm} (17)

\(^2\)The surface area in this case is the Lebesgue measure in $d - 1$ dimensions.
We first characterize the CDF of $R_\ell$ conditioned on the domain configuration $C_\ell^k$. This conditional CDF of $R_\ell$ can be expressed as

$$F_{R_\ell}(r; C_\ell^k) = \frac{\nu_d(V_o(C_\ell^k) \cap B_r(o))}{\nu_d(V_o(C_\ell^k) \cap B_\ell(o))} = \frac{g_k(r; C_\ell^k)}{g_k(\ell; C_\ell^k)}, \quad 0 \leq r \leq \ell.$$  \hspace{1cm} (18)

Fig. 3 provides the visual interpretation of $g_k(r; C_\ell^k)$ and $g_k(\ell; C_\ell^k)$ for the typical cell for $d = 2$. The region $g_k(r; C_\ell^k)$ is shaded in green and the region $g_k(\ell; C_\ell^k)$ is shaded in brown for $k = 5$. Naturally, our next goal is to characterize $g_k(\cdot; C_\ell^k)$ for which we use $\{L_i\}_{i=1}^5$ given by (15).

Define the index set $\mathcal{I}(r)$ as the collection of indices $i$ for which $l_i \leq r$. This set points to the collection of the points $\tilde{x}_i$ of the domain configuration that lie inside $B_r(o)$. It is easy to see that $\bigcup_{i \in \mathcal{I}(r)} L_i$ represents the portion of $\partial B_r(o)$ that is outside the typical cell $V_o(C_\ell^k)$. This can be seen easily from Fig. 3 for $d = 2$, where the arcs on $B_r(o)$ corresponding to $\tilde{x}_1 = (l_1, \theta_1)$ and $\tilde{x}_2 = (l_2, \theta_2)$ do not lie in the cell. Using this insight, we will explicitly characterize the portion of $\partial B_r(o)$ that lies in $V_o(C_\ell^k)$, which will then be used to derive the CDF of $R_\ell$. This evaluation requires a careful consideration of the overlaps between the surfaces of the spherical caps $\{L_i\}_{i \in \mathcal{I}(r)}$.

Let $y \triangleq (r, \alpha)$ be the point on the $\partial B_r(o)$, where $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{d-1}]$. The Euclidean distance between $y \in \partial B_r(o)$ and $x_i \triangleq (2l_i, \theta_i) \in \Phi$ is

$$d_2(y, x_i) = \sqrt{\sum_{n=1}^{d} (y_n - x_{ni})^2},$$

where

$$x_{ni} = \begin{cases} 2l_i \cos(\theta_{1i}); & n = d, \\
2l_i \prod_{j=1}^{n} \sin(\theta_{ji}) \cos(\theta_{ni}); & 1 < n < d, \\
2l_i \prod_{j=1}^{n-1} \sin(\theta_{ji}); & n = d, \end{cases} \quad \text{and} \quad y_n = \begin{cases} r \cos(\alpha_1); & n = 1, \\
r \prod_{j=1}^{n} \sin(\alpha_j) \cos(\alpha_n); & n < d, \\
r \prod_{j=1}^{n-1} \sin(\alpha_j); & n = d. \end{cases}$$
Lemma 2. For given form the domain configuration but not on their locations. Let \( y \in B \) a uniformly random point in the typical cell \( V \) uniformly random point in the typical cell \( V \). Proof. Let \( C \) by (18), and the pdf of \( C \) gives the value of \( \int_{0}^{\ell} \int_{r=0}^{z} \prod_{i=1}^{k} D_i(l_i, \theta_i, y) \Delta(\alpha) \, d\alpha \, dr \, \Delta(\alpha) \, d\alpha \). where \( \Delta(\alpha) = \sin^{d-2}(\alpha_1) \cdots \sin(\alpha_{d-2}). \) Note that \( \prod_{i=1}^{k} D_i(l_i, \theta_i, y) \) is 1 at all points \( y \), such that \( 0 \leq r \leq z \), lying inside of \( B_z(o) \cap V_o(C_k^\ell) \), and 0 elsewhere. Thus, the integration of \( \prod_{i=1}^{k} D_i(l_i, \theta_i, y) \) over all the points \( y \in B_z(o) \) gives the value of \( g_k(z; C_k^\ell) \) for the given domain configuration, i.e.,

\[
g_k(z; C_k^\ell) = \int_{D} \int_{r=0}^{z} \prod_{i=1}^{k} D_i(l_i, \theta_i, y) r^{d-1} \Delta(\alpha) \, dr \, \Delta(\alpha) \, d\alpha. \tag{20}
\]

Using the above results, we present the distance distribution of a uniformly distributed point in \( V_o \cap B_z(o) \) conditioned on \( \Phi(B_{2\ell}(o)) = k \) in the following lemma. Note that in this lemma, we condition on the number of points that form the domain configuration but not on their locations. Let \( y_i = (u_i, \alpha_i) \) and \( D = [0, \ell] \times [0, \pi]^{d-2} \).

**Lemma 2.** For given \( \ell \), the CDF of \( R_{\ell} \) conditioned on \( \Phi(B_{2\ell}(o)) = k \) is

\[
F_{R_{\ell}}(z; k) = \int_{(D^d)^{k}} g_k(z; (u_1, \alpha_1), \ldots, (u_k, \alpha_k)) \prod_{i=1}^{k} \frac{1}{\kappa_d \ell^d} u_i^{d-1} \Delta(\alpha_i) \, dy_i. \tag{21}
\]

where \( g_k(z; (u_1, \alpha_1), \ldots, (u_k, \alpha_k)) \) is given by (20).

**Proof.** The CDF of \( R_{\ell} \) conditioned on \( \Phi(B_{2\ell}(o)) = k \) is \( F_{R_{\ell}}(z; k) = E_{C_k^\ell}[F_{R_{\ell}}(z; C_k^\ell)] \) where \( F_{R_{\ell}}(z; C_k^\ell) \) is given by (18), and the pdf of \( C_k^\ell \) is given in (13).

Using Lemma 2, we present the distance distribution of a uniformly distributed point in the typical cell in the following theorem.

**Theorem 2.** For the homogeneous PPP with intensity \( \lambda \) on \( \mathbb{R}^d \), the CDF of the distance \( R_o \) from the nucleus to a uniformly random point in the typical cell \( V_o \) is

\[
F_{R_o}(z) = \lim_{\ell \to \infty} \sum_{k=0}^{\infty} F_{R_{\ell}}(z; k) \mathbb{P}(\Phi(B_{2\ell}(o)) = k), \tag{22}
\]

where \( F_{R_{\ell}}(z; k) \) is given in Lemma 2.

**Proof.** The proof follows in two steps. We first take the expectation of the conditional CDF of \( R_{\ell} \), given in Lemma 2 over \( k \). We then take the limit \( \ell \to \infty \) under which this distance distribution of a uniformly distributed point in \( V_o \cap B_{\ell}(o) \) converges to that of a uniformly distributed point in \( V_o \) per (17).

**Corollary 2.** For the homogeneous PPP with intensity \( \lambda \) on \( \mathbb{R}^d \), the mean of the distance \( R_o \) from the nucleus to a uniformly random point in the typical cell \( V_o \) is

\[
\mathbb{E}[R_o] = \lim_{\ell \to \infty} \int_{0}^{\ell} (1 - \sum_{k=0}^{\infty} F_{R_{\ell}}(z; k) \mathbb{P}(\Phi(B_{2\ell}(o)) = k)) \, dz, \tag{23}
\]

where \( F_{R_{\ell}}(z; k) \) is given in Lemma 2.
Lemma 3. The second moment of the volume of the typical cell $V_o$ is

$$
\mathbb{E}[v_d(V_o)^2] = 4\pi C_{d,2} \int_0^\pi \int_0^\infty \int_0^\infty \exp(-\lambda U(v_1, v_2, u))(v_1 v_2)^{d-1}(\sin u)^{d-2}dv_2dv_1du,
$$

(24)
Following similar steps as in Lemma 3 completes the proof. Further, following the steps from the proof of \([16, \text{Theorem 3.1}]\), we can obtain (24). Let \(C_{d,2} = \frac{d!}{2^d (d-2)!} \kappa_d \kappa_{d-1} \kappa_1 \alpha_d \), \(C_{d,2} = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}) \Gamma(\frac{d}{2})} \), \(\psi_1 + \psi_2 = \pi - u\) and \(v_2 \sin^d \psi_1 = v_2^d \sin^d \psi_2\).

Note that \(U(v_1, v_2, u)\) represents the union of balls of radii \(v_1\) and \(v_2\) with centers at angle \(u\).

**Proof.** Using (4), we obtain the second moment of \(v_d(V_o)\) as

\[
\mathbb{E}[v_d(V_o)^2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{P}(x_1, x_2 \in V_o) dx_1 dx_2
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp(-\nu_d(\mathcal{B}_{\|x_1\|} \cup \mathcal{B}_{\|x_2\|}(x_2))) dx_2 dx_1.
\]

Further, following the steps from the proof of \([16, \text{Theorem 3.1}]\), we can obtain (24).

The \(n\)-th moment of the volume of the intersection of a ball of arbitrary radius with the typical cell is obtained in \([16, \text{Lemma 4.2}]\). Using this result, we present the first and second moments of \(v_d(\mathcal{B}_r(o) \cap V_o)\) in the following lemma.

**Lemma 4.** The first and second moments of the volume of the intersection of the ball \(\mathcal{B}_r(o)\) with the typical cell \(V_o\) are

\[
\mathbb{E}[v_d(\mathcal{B}_r(o) \cap V_o)] = \frac{1}{\lambda} \left( 1 - \exp(-\lambda \kappa_d r^d) \right)
\]

and

\[
\mathbb{E}[v_d(\mathcal{B}_r(o) \cap V_o)^2] = 4\pi C_{d,2} \int_0^\pi \int_0^r \exp(-\nu_d(\mathcal{B}_{\|x_1\|} \cup \mathcal{B}_{\|x_2\|}(x_2))) dx_2 dx_1 du.
\]

where \(U(v_1, v_2, u)\) is given by (25).

**Proof.** The first moment in (27) follows from Lemma 1. Similar to the second moment of the typical cell derived in Lemma 3, the second moment of the volume of \(\mathcal{B}_r(o) \cap V_o\) can be determined as

\[
\mathbb{E}[v_d(\mathcal{B}_r(o) \cap V_o)^2] = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{P}(x_1, x_2 \in \mathcal{B}_r(o) \cap V_o) dx_1 dx_2
\]

\[
= \int_{\mathbb{R}^d \cap \mathcal{B}_r(o)} \int_{\mathbb{R}^d \cap \mathcal{B}_r(o)} \exp(-\nu_d(\mathcal{B}_{\|x_1\|} \cup \mathcal{B}_{\|x_2\|}(x_2))) dx_1 dx_2.
\]

Following similar steps as in Lemma 3 completes the proof.

In \([22, \text{Lemma 3.1}]\), the correlation between the volume of the typical Stienen sphere and the volume of the typical cell is derived. Using the approach of \([22]\), we provide the covariance of the volumes of \(\mathcal{B}_r(o) \cap V_o\) and \(V_o\) in the following lemma.

**Lemma 5.** The covariance of the volume of the intersection of \(\mathcal{B}_r(o)\) with the typical cell \(V_o\) and the volume of the typical cell \(V_o\) is

\[
\text{Cov}[v_d(\mathcal{B}_r(o) \cap V_o), v_d(V_o)] = \frac{1}{2} \text{Var}[v_d(V_o)] - \frac{1}{2\lambda^2} \left( 1 - 2 \exp(-\lambda \kappa_d r^d) \right)
\]

\[
+ 2\pi C_{d,2} \int_0^\pi \int_0^r \int_0^r \exp(-\nu_d(\mathcal{B}_{\|x_1\|} \cup \mathcal{B}_{\|x_2\|}(x_2))) dx_2 dx_1 du
\]

\[
- 2\pi C_{d,2} \int_0^\pi \int_r^\infty \int_r^\infty \exp(-\nu_d(\mathcal{B}_{\|x_1\|} \cup \mathcal{B}_{\|x_2\|}(x_2))) dx_2 dx_1 du,
\]

where \(U(v_1, v_2, u)\) is given by (25).

**Proof.** Let \(\hat{V}_o(r) = V_o \setminus V_o \cap \mathcal{B}_r(o)\). The variance of the volume of \(\hat{V}_o(r)\) is

\[
\text{Var}[v_d(\hat{V}_o(r))] = \text{Var}[v_d(V_o) - v_d(\mathcal{B}_r(o) \cap V_o)]
\]
where (a) follows from the same steps as in Lemma 3 for the second moment of the volume of typical cell such that

Lastly, substituting (31), (32) and (33) in (30) completes the proof.

Proof. Using Lemma 5, we can write

Using (4), we can obtain the second moment as

Since we use the $\hat{\exp}$ expansion of the CDF includes the covariance term given in Lemma 5, we first provide its

Lemma 6. The $d$-th derivative of the second-order approximation of the CDF of $V_B$ and the volume of typical cell $V_o$ is determined by matching the $d$-th derivative of the second-order approximation of the CDF of $V_o$ at $r = 0$. As the second-order Taylor series expansion of the CDF includes the covariance term given in Lemma 5, we first provide its $d$-th derivative at $r = 0$ in the following lemma.

Lemma 6. The $d$-th derivative of the covariance of the volume of the intersection of $B_r(o)$ with the typical cell $V_o$ and the volume of typical cell $V_o$ w.r.t. $r$ is zero at $r = 0$.

Proof. Using Lemma 5 we can write

where

such that

Further,

Now, differentiating $f_1$ w.r.t. $r$, we obtain

$$
\frac{d}{dr} f_1(r) = \frac{d}{dr} \int_0^r \int_0^r g(r, v_1, v_2) dv_1 \, dv_2
$$

$$
= \int_0^r g(r, v_2) dv_2 + \int_0^r dv_1 \frac{d}{dr} \int_0^r g(r, v_1, v_2) dv_2
$$

This implies

Using Lemma 5, the variance of $v_d(B_r(o) \cap V_o)$ can be expressed as

Now, we obtain the mean and variance of $\hat{V}_o(r)$. Using (27), the first moment becomes

Using (4), we can obtain the second moment as

$$
\var[\hat{V}_o(r)] = \frac{1}{\lambda} \exp(-\lambda \kappa_d r^d).
$$

Using (32), we can obtain the second moment as

where (a) follows from the same steps as in Lemma 3 for the second moment of the volume of typical cell $V_o$. Lastly, substituting (31), (32) and (33) in (30) completes the proof. □

Since we use $1 - \exp(-\rho_d \lambda \kappa_d r^d)$ for the approximation of CDF of $R_o$, the c.f. $\rho_d$ is determined by matching the $d$-th derivative of the second-order approximation of the CDF of $R_o$ at $r = 0$. As the second-order Taylor series expansion of the CDF includes the covariance term given in Lemma 5, we first provide its $d$-th derivative at $r = 0$ in the following lemma.

Lemma 6. The $d$-th derivative of the covariance of the volume of the intersection of $B_r(o)$ with the typical cell $V_o$ and the volume of typical cell $V_o$ w.r.t. $r$ is zero at $r = 0$. Proof. Using Lemma 5 we can write

$$
\frac{d^d}{dr^d} \var[v_d(B_r(o) \cap V_o), v_d(V_o)] = \frac{d^d}{dr^d} \frac{1}{\lambda^2} \exp(-\lambda \kappa_d r^d) \Bigg|_{r=0} + \frac{d^d}{dr^d} (f_1(r) - f_2(r)) \Bigg|_{r=0},
$$

where

$$
f_1(r) = \int_0^r \int_0^r g(v_1, v_2) dv_1 dv_2,
$$

and

$$
f_2(r) = \int_r^\infty \int_r^\infty g(v_1, v_2) dv_1 dv_2,
$$

such that

$$
g(v_1, v_2) = 2\pi C_{d,2} \int_0^\pi \exp(-\lambda U(v_1, v_2, u)) (v_1 v_2)^{d-1} (\sin u)^{d-2} du.
$$

Further,

$$
\frac{d^d}{dr^d} \frac{1}{\lambda^2} \exp(-\lambda \kappa_d r^d) \Bigg|_{r=0} = -\frac{1}{\lambda} \ln \kappa_d = -\frac{1}{\lambda} \frac{2\pi^2}{\Gamma(d)} \Gamma\left(\frac{d}{2}\right).
$$

Now, differentiating $f_1$ w.r.t. $r$, we obtain

$$
\frac{d}{dr} f_1(r) = \frac{d}{dr} \int_0^r \int_0^r g(r, v_1, v_2) dv_1 \, dv_2
$$

$$
= \int_0^r g(r, v_2) dv_2 + \int_0^r dv_1 \frac{d}{dr} \int_0^r g(r, v_1, v_2) dv_2
$$

$$
= \int_0^r g(r, v_2) dv_2 + \int_0^r dv_1 \frac{d}{dr} \int_0^r g(r, v_1, v_2) dv_2
$$
\[ \frac{d^2}{dr^2} f_1(r) = \frac{d}{dr} \int_0^r g(r,v_2) dv_2 + \int_0^r \frac{d}{dr} g(v_1,r) dv_1, \]

where (a) and (b) are obtained using the successive application of Leibniz’s integral rule. Again differentiating, we obtain

\[ \frac{d^3}{dr^3} f_1(r) = 4 \frac{d}{dr} g(r,r) + \int_0^r \frac{d^2}{dr^2} g(r,v_2) dv_2 + \int_0^r \frac{d^2}{dr^2} g(v_1,r) dv_1, \]

\[ \frac{d^4}{dr^4} f_1(r) = 6 \frac{d^2}{dr^2} g(r,r) + \int_0^r \frac{d^3}{dr^3} g(r,v_2) dv_2 + \int_0^r \frac{d^3}{dr^3} g(v_1,r) dv_1, \]

Thus, in general, we have

\[ \frac{d^d}{dr^d} f_1(r) = (d-1) \frac{d^{(n-2)}}{dr^{(n-2)}} g(r,r) + \int_0^r \frac{d^{(n-1)}}{dr^{(n-1)}} g(r,v_2) dv_2 + \int_0^r \frac{d^{(n-1)}}{dr^{(n-1)}} g(v_1,r) dv_1. \]

Following similar steps, we obtain the \(d\)-fold derivative of \(f_2\) w.r.t. \(r\) as

\[ \frac{d^d}{dr^d} f_2(r) = (d-1) \frac{d^{(n-2)}}{dr^{(n-2)}} g(r,r) - \int_r^\infty \frac{d^{(n-1)}}{dr^{(n-1)}} g(r,v_2) dv_2 - \int_r^\infty \frac{d^{(n-1)}}{dr^{(n-1)}} g(v_1,r) dv_1. \]

Subtracting \(\frac{d^d}{dr^d} f_2(r)\) from \(\frac{d^d}{dr^d} f_1(r)\), we get

\[ \frac{d^d}{dr^d} (f_1(r) - f_2(r)) = \int_0^\infty \frac{d^{(d-1)}}{dr^{(d-1)}} g(r,v_2) dv_2 + \int_0^\infty \frac{d^{(d-1)}}{dr^{(d-1)}} g(v_2,r) dv_2. \tag{36} \]

Now, we obtain the \((d-1)\)-th derivative of \(g(r,v_2)\) at \(r = 0\) as

\[ \frac{d^{(d-1)}}{dr^{(d-1)}} g(r,v_2) \bigg|_{r=0} = 2\pi C_{d,2} \frac{d^{(d-1)}}{dr^{(d-1)}} \int_0^\pi \exp(-\lambda U(r,v_2,u))(v_2)^{d-1} (\sin u)^{d-2} du \bigg|_{r=0} \]

\[ = 2\pi C_{d,2} \int_0^\pi \frac{d^{(d-1)}}{dr^{(d-1)}} \exp(-\lambda U(r,v_2,u))(v_2)^{d-1} (\sin u)^{d-2} du \bigg|_{r=0} \]

\[ = 2\pi (d-1)! C_{d,2} \int_0^\pi \exp(-\lambda U(0,v_2,u))(v_2)^{d-1} (\sin u)^{d-2} du \]

\[ \equiv 2\pi (d-1)! C_{d,2} v_2^{d-1} \exp(-\lambda\kappa_d v_2^d) \int_0^\pi (\sin u)^{d-2} du \]

\[ \overset{(a)}{=} 2\pi (d-1)! C_{d,2} v_2^{d-1} \exp(-\lambda\kappa_d v_2^d), \]

where (a) follows due to \(U(0,v_2,u) = \kappa_d v_2^d\) and (b) follows using \(\int_0^\pi (\sin u)^{d-2} du = \sqrt{\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)}} \) [23, Eq. 3.62.5].

Now, using the above expression along with \(g(r,x) = g(x,r)\) and

\[ \int_0^\infty v^{d-1} \exp(-\lambda\kappa_d v^d) dv = \frac{1}{d\lambda\kappa_d} \int_0^\infty \exp(-t) dt = \frac{1}{d\lambda\kappa_d} = \frac{\Gamma\left(\frac{d}{2} + 1\right)}{d\lambda\pi^{\frac{d}{2}}}, \]

we can write (36) at \(r = 0\) as

\[ \frac{d^d}{dr^d} (f_1(r) - f_2(r)) \bigg|_{r=0} = \frac{1}{\lambda} \frac{2\pi \frac{d}{2}}{\Gamma\left(\frac{d}{2}\right)} \Gamma\left(\frac{d}{2} + 1\right). \tag{37} \]

Finally, the substitution of (35) and (37) in (34) completes the proof. \(\square\)
B. Approximate CDF of $R_o$

Now, in the following theorem we determine the c.f. of the approximated CDF of $R_o$, which is the main result of this section.

**Theorem 3.** For the homogeneous PPP with intensity $\lambda$ on $\mathbb{R}^d$, the approximate CDF of the distance $R_o$ from the nucleus to a uniformly random point in the typical cell $V_o$ is

$$F_{R_o}(r) \approx 1 - \exp(-\rho_d \lambda \kappa_d r^d),$$  

(38) where $\rho_d$ is the c.f. obtained by matching the $d$-th derivative of $f$ with that of the second-order Taylor series expansion of the exact CDF of $R_o$ at $r = 0$ and is given by

$$\rho_d = 1 + \frac{\text{Var}[v_d(V_o)]}{\mathbb{E}[v_d(V_o)]^2}. \quad (39)$$

**Proof.** The second order Taylor series expansion of the bivariate function $f(Z_1, Z_2) = \frac{Z_1}{Z_2}$ around the mean $(\bar{z}_1, \bar{z}_2)$ can be written as

$$f(Z_1, Z_2) \approx \frac{\bar{z}_1}{\bar{z}_2} + \frac{1}{\bar{z}_2^2} (Z_1 - \bar{z}_1) - \frac{\bar{z}_1}{\bar{z}_2^2} (Z_2 - \bar{z}_2) + \frac{1}{\bar{z}_2} (Z_1 - \bar{z}_1)(Z_2 - \bar{z}_2) + \frac{\bar{z}_1}{\bar{z}_2^2} (Z_2 - \bar{z}_2)^2.$$

Taking expectation of $f(Z_1, Z_2)$ w.r.t. $Z_1$ and $Z_2$, we get

$$\mathbb{E}[f(Z_1, Z_2)] \approx \frac{\bar{z}_1}{\bar{z}_2} - \frac{1}{\bar{z}_2^2} \text{Cov}[z_1, z_2] + \frac{\bar{z}_1}{\bar{z}_2^3} \text{Var}[z_2]. \quad (40)$$

The CDF of $R_o$ is

$$F_{R_o}(r) = \mathbb{E} \left[ \frac{v_d(B_r(o) \cap V_o)}{v_d(V_o)} \right].$$

Therefore, using (40), the second-order Taylor series expansion of $F_{R_o}(r)$ around the mean $(\mathbb{E}[v_d(B_r(o) \cap V_o)], \mathbb{E}[v_d(V_o)])$ can be written as

$$F_{R_o}(r) \approx \mathbb{E}[v_d(B_r(o) \cap V_o)] \left[ 1 + \frac{\text{Var}[v_d(V_o)]}{\mathbb{E}[v_d(V_o)]^2} \right] - \frac{\text{Cov}[v_d(B_r(o) \cap V_o), v_d(V_o)]}{\mathbb{E}[v_d(V_o)]^2}. \quad (41)$$

Using Lemma 3 and Lemma 4, we obtain

$$F_{R_o}(r) \approx \left( 1 - \exp(-\lambda \kappa_d r^d) \right) \left[ 1 + \frac{\text{Var}[v_d(V_o)]}{\mathbb{E}[v_d(V_o)]^2} \right] - \frac{\text{Cov}[v_d(B_r(o) \cap V_o), v_d(V_o)]}{\mathbb{E}[v_d(V_o)]^2}. \quad (41)$$

Now, as $1 - \exp(-\rho_d \lambda \kappa_d r^d)$ is considered for the approximation, we determine the c.f. $\rho_d$ by matching the $d$-th derivatives of $1 - \exp(-\rho_d \lambda \kappa_d r^d)$ and $F_{R_o}(r)$ at $r = 0$ as

$$\rho_d = \left. \frac{1}{d! \lambda \kappa_d} \frac{d^d}{dr^d} F_{R_o}(r) \right|_{r=0}. \quad (41)$$

Therefore, using (41) and Lemma 5, we have

$$\rho_d = 1 + \frac{\text{Var}[v_d(V_o)]}{\mathbb{E}[v_d(V_o)]^2}. \quad (42)$$

This completes the proof.

Before giving the numerical validation of the approximated CDF of $R_o$, we present the approximated $n$-th moment of the distance $R_o$ and some useful observations about the c.f. in the following corollaries.

**Corollary 3.** For the homogeneous PPP with intensity $\lambda$ on $\mathbb{R}^d$, the $n$-th moment of the distance $R_o$ from the nucleus to a uniformly random point in the typical cell $V_o$ is approximately

$$\mathbb{E}[R_o^n] \approx \Gamma \left( 1 + \frac{n}{d} \right) \left( \rho_d \lambda \kappa_d \right)^{\frac{n}{d}}. \quad (42)$$
Corollary 4. For the homogeneous PPP with intensity $\lambda$ on $\mathbb{R}^d$, the CDF of the distance $R_o$ from the nucleus to a uniformly random point in the typical cell $V_o$ can be approximated as $1 - \exp(-\lambda \rho_d \rho_d^d)$ where the c.f. $\rho_d$ is equal to the ratio of the mean volumes of the Crofton and typical cells, i.e.,

$$\rho_d = \frac{\mathbb{E}[v_d(\tilde{V}_o)]}{\mathbb{E}[v_d(V_o)]}. \quad (43)$$

Proof. From [6, Equation 2.5], we have

$$\mathbb{E}[v_d(\tilde{V}_o)] = \mathbb{E}[v_d(V_o)] + \frac{\text{Var}[v_d(V_o)]}{\mathbb{E}[v_d(V_o)]}.$$ Substituting the above expression in (39) gives (43).

Corollary 5. The c.f. $\rho_d$ approaches one as $d$ approaches infinity, i.e., $\lim_{d \to \infty} \rho_d = 1$.

Proof. Using [16, Theorem 3.1], we can write

$$\lim_{d \to \infty} \text{Var}[v_d(V_o)] = 0.$$ Since, the mean volume of the PV cell is $\lambda^{-1}$ for any $d$, the proof directly follows using (39) and above result.

Remark 2. From (7) and (42), it is clear that the ratio of the means of $\tilde{R}_o$ and $R_o$ is approximately $\sqrt[10]{d}$. Therefore, using Corollary 4, we can infer that the ratio of the means of $\tilde{R}_o$ and $R_o$ is approximately equal to the $d$-th root of the ratio of the mean volumes of the Crofton cell $\tilde{V}_o$ and the typical cell $V_o$. In other words, the distance from the nucleus to a uniformly random point in the typical cell scales with the distance from the nucleus to a uniformly random point in the Crofton cell by a factor equal to the $d$-th root of the ratio of the mean volumes of the Crofton cell $\tilde{V}_o$ and the typical cell $V_o$.

C. Numerical Comparisons

For the numerical evaluation of the approximated CDF of $R_o$, we obtain the c.f. $\rho_d$ using (39) for which the mean and variance of the volume of the typical cell are evaluated using Lemma 3. Fig. 5 validates the accuracy of the approximated CDF of $R_o$ by comparing it with the Monte Carlo simulations for the cases of $d \in \{1, \ldots, 10\}$. Fig. 5 clearly indicates that the CDF of $R_o$ gradually approaches that of $\tilde{R}_o$ as $d$ increases. Further, Table II verifies the accuracy of the approximated mean and variance of $R_o$ (obtained using Corollary 3) for $d \in \{1, \ldots, 10\}$. For $d = 2$, the obtained mean value of $R_o$ is 0.442 which is also close to the mean values 0.438 and 0.447 obtained using the curve-fitted c.f.s 13/10 and 5/4 of [20] and [21], respectively.
Aother in the annulus of noting that the conditioning on the k points of \( \Phi \) in the \( B_{2\epsilon}(0) \), defined as the domain configuration \( C_R^k \) (see (12)), allowed us to construct the set of surfaces of the spherical caps \( \{ L_i \}_{i=1}^k \) on the ball \( B_{\epsilon}(0) \) as in (15). This helps in determining the conditional volume of the typical cell \( V_o \) and thus the conditional CDF of \( R_o \). It is easy to observe that some points of the domain configuration \( C_R^k \) are the closest points on some boundaries of the typical cell \( V_o \) and thus the lines joining them to origin are perpendicular to the corresponding boundaries. Further, these points are also the midpoints of the chords formed by the corresponding spherical caps. This implies that these surfaces of spherical caps completely lie outside the typical cell \( V_o \) (see Fig. 2 for \( d = 2 \)). Therefore, it is quite straightforward to see that the typical cell is completely contained within \( B_{\epsilon}(0) \) only if the set \( \{ L_i \}_{i=1}^k \) completely covers the boundary of \( B_{\epsilon}(0) \). Using this fact, in this section, we provide an alternate proof to the well-known spherical property of \( d \)-dimensional PV cells containing a large inball.

Let the point \( \tilde{x}_0 \triangleq (R, \theta_0) \) denote the nearest point on the boundary of the typical cell \( V_o \) to its nucleus. Therefore, \( R \) is the radius of the largest ball \( B_R(o) \) contained within the typical cell \( V_o \), henceforth called the inradius of the cell. In this construction, it is evident that the nearest point \( x_0 \) in \( \Phi \) from the nucleus of \( V_o \) (i.e., the origin) is at \((2R, \theta_0)\) such that \([\|\tilde{x}_0\| = \frac{1}{2}\|x_0\| = R\). Note that the results presented in the following are conditioned on the inradius \( R \).

Let \( A(r, \epsilon) \) denote the annulus formed by two balls of radii \( r \) and \( r + \epsilon \) co-centered at the origin. Now, consider the domain configuration \( C_R^k = \{ \tilde{x}_i \}_{i=1}^k \) as the set containing the mid-point of lines joining the nucleus of \( V_o \) and the points in \( \Phi \setminus A(2R, 2\epsilon) \) given \( \Phi(A(2R, 2\epsilon)) = k \). Fig. 6 illustrates a potential configuration of \( C_R^k \) for the case of \( d = 2 \). By the Poisson property, the \( k \) points of \( C_R^k \) are distributed uniformly at random independently of each other in the annulus \( A(R, \epsilon) \) such that the CDF of \( [\|\tilde{x}_i\| = l_i] \), for \( \forall i \), conditioned on \( R \) is

\[
F_{l_i}(l) = \frac{l_i^d - R^d}{(R + \epsilon)^d - R^d}, \quad R \leq l \leq R + \epsilon.
\]

We define the set of \( k + 1 \) spherical caps \( \{ L_i \}_{i=0}^k \) corresponding to points \( \{ \tilde{x}_i \}_{i=0}^k = \{ \tilde{x}_0 \cup C_R^k \} \) on the \( B_{R+\epsilon}(o) \) with heights equal to \( \epsilon \) for \( i = 0 \) and \( R + \epsilon - l_i \) for \( i = 1, \ldots, k \). The surface area of the spherical cap \( L_i \) is [24]

\[
S_i = \begin{cases} \frac{1}{2} \chi_d(R + \epsilon)^{d-1} I_{1-\frac{\epsilon^2}{(R+\epsilon)^2}} \left( \frac{d-1}{2}, \frac{1}{2} \right), & \text{for } i = 0 \\ \frac{1}{2} \chi_d(R + \epsilon)^{d-1} I_{1-\frac{l_i^2}{(R+\epsilon)^2}} \left( \frac{d-1}{2}, \frac{1}{2} \right), & \text{for } i = 1, \ldots, k, \end{cases}
\]

where \( \chi_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \) is the surface area of the unit radius ball in \( \mathbb{R}^d \) and \( I_x(a,b) = \frac{B_x(a,b)}{B(a,b)} \) such that \( B(a,b) \) and \( B_z(a,b) \) are the beta function and the incomplete beta function, respectively. Note that \( 0 \leq S_i \leq S_0 \) \( \forall i \). Since the points in \( C_R^k \) are i.i.d. in \( A(R, \epsilon) \), the spherical caps \( \{ L_i \}_{i=1}^k \) of i.i.d. surface areas are placed uniformly at random independently of each other on \( B_{R+\epsilon}(o) \).

Now, we evaluate the probability that the uniformly chosen point \((R + \epsilon, \alpha)\) on the surface of \( B_{R_m+\epsilon}(o) \) belongs to the spherical cap \( L_i \), for \( i \in \{1, \ldots, k\} \), as

\[
p = P((R + \epsilon, \alpha) \text{ belongs to the cap } L_i \text{ of area } S_i) = \frac{1}{\chi_d(R + \epsilon)^{d-1} E[S_i]}
\]
the spherical cap $L$ to show that the exponential term in (48) tends to 0 as $R$ the probability that the point on the boundary of $V$ cell $B$. Also note that the probability that the uniformly chosen point $(R + \epsilon, \alpha)$ on the surface of $B_{R+\epsilon}(o)$ belongs to the spherical cap $L_0$ is

$$p_0 = \frac{1}{2} I_1 - \frac{\mu^2}{(R+\epsilon)^2} \left( \frac{d - 1}{2}, \frac{1}{2} \right).$$

(47)

Let $K = \Phi(A(2R, 2\epsilon))$. By definition, $K$ is Poisson with mean $\lambda\kappa_d((R + \epsilon)^d - R^d)$. Now to complete our argument, we evaluate the probability that the point on the boundary of $B_{R+\epsilon}(o)$ does not belong to $V_o$ as

$$Q_d(R, \epsilon) = \Pr((R + \epsilon, \alpha) \text{ belongs to at least one of the caps})$$

$$= 1 - (1 - p_0) \mathbb{E}[(1 - p)^K]$$

$$= 1 - \left( 1 - \frac{1}{2} I_1 - \frac{\mu^2}{(R+\epsilon)^2} \left( \frac{d - 1}{2}, \frac{1}{2} \right) \right) \exp \left( - \frac{1}{2} \lambda\kappa_d h(R, \epsilon) \right),$$

(48)

where

$$h(R, \epsilon) = \frac{(R + \epsilon)^d}{B \left( \frac{d-1}{2}, \frac{1}{2} \right)} B_1 - \frac{\mu^2}{(R+\epsilon)^2} \left( \frac{d - 1}{2}, \frac{d + 1}{2} \right) - R^d I_1 - \frac{\mu^2}{(R+\epsilon)^2} \left( \frac{d - 1}{2}, \frac{1}{2} \right),$$

(49)

and (a) directly follows using (46), (47) and the probability generating function of the Poisson distribution with mean $\lambda\kappa_d((R + \epsilon)^d - R^d)$. Now, in the following theorem we state the limiting case of (48).

**Theorem 4.** Given the inradius $R$, the probability that a point on the boundary of $B_{R+\epsilon}(o)$ does not belong to the PV cell $V_o$ approaches one as $R$ tends to infinity, i.e.,

$$\lim_{R \to \infty} Q_d(R, \epsilon) = 1, \quad \forall \epsilon > 0.$$

(50)

**Proof.** We note that, for $\epsilon > 0$, $I_1 - \frac{\mu^2}{(R+\epsilon)^2} \left( \frac{d - 1}{2}, \frac{1}{2} \right) \to 0$ as $R \to \infty$. Therefore, in order to prove (50), it is sufficient to show that the exponential term in (48) tends to 0 as $R \to \infty$ for $\epsilon > 0$, i.e.,

$$\lim_{R \to \infty} h(R, \epsilon) = \infty.$$
To this end, we multiply \( h(R, \epsilon) \) with \( B \left( \frac{d-1}{2}, \frac{1}{2} \right) \) to obtain

\[
\tilde{h}(R, \epsilon) = (R + \epsilon)^d B_1 - \frac{\epsilon^2}{(R + \epsilon)^2} \left( \frac{d - 1}{2}, \frac{d + 1}{2} \right) - R^d B_1 - \frac{\epsilon^2}{(R + \epsilon)^2} \left( \frac{d - 1}{2}, \frac{1}{2} \right). \tag{51}
\]

We have

\[
B_1 - \frac{\epsilon^2}{(R + \epsilon)^2} \left( \frac{d - 1}{2}, a \right) = \int_0^1 \left( 1 - \frac{\epsilon^2}{(R + \epsilon)^2} \right)^{d-1} (1-t)^{a-1} dt.
\]

Thus, using the binomial expansion of the term \((1-t)^{a-1}\), we get

\[
B_1 - \frac{\epsilon^2}{(R + \epsilon)^2} \left( \frac{d - 1}{2}, a \right) = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \prod_{l=0}^{k-1} (a - l) t^k dt
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \prod_{l=0}^{k-1} (a - l) \int_0^1 \left( 1 - \frac{\epsilon^2}{(R + \epsilon)^2} \right)^{k+\frac{d-1}{2}} dt
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \frac{\epsilon^2}{k!} \prod_{l=0}^{k-1} (a - l) \left( 1 - \frac{R^2}{(R + \epsilon)^2} \right)^{k+\frac{d-1}{2}}.
\]

Let \( A_k = \frac{1}{k!} \prod_{l=0}^{k-1} \left( \frac{d+1}{2} - 1 - l \right) \) and \( B_k = \frac{1}{k!} \prod_{l=0}^{k-1} \left( \frac{1}{2} - 1 - l \right) \). Using the above series expansion of the incomplete beta function, we can rewrite (51) as

\[
\tilde{h}(R, \epsilon) = \sum_{k=0}^{\infty} (-1)^k \left[ A_k (R + \epsilon)^d - B_k R^d \right] \left( 1 - \frac{R^2}{(R + \epsilon)^2} \right)^{k+\frac{d-1}{2}}
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \left[ \frac{(A_k - B_k) R^d}{k!} \prod_{l=0}^{d-1} \left( \frac{d}{n} \right) R^n e^{d-n} \right] \left( \frac{2Re + \epsilon^2}{(R + \epsilon)^{2k+d-1}} \right)^{k+\frac{d-1}{2}}
\]

\[
= \sum_{k=0}^{\infty} (-1)^k \left[ \frac{(A_k - B_k) R^d}{k!} \prod_{l=0}^{d-1} \left( \frac{d}{n} \right) R^n e^{d-n} \right] \left( \frac{2Re + \epsilon^2}{(R + \epsilon)^{2k+d-1}} \right)^{k+\frac{d-1}{2}}
\]

Now note that \( A_k - B_k \geq 0 \) for \( k \leq \frac{d+1}{2} \). Therefore, the terms in the above summation tend to infinity as \( R \) tends to infinity for \( k < \frac{d+1}{2} \). In addition, the terms converge to a constant for \( k = \frac{d+1}{2} \) (if \( d \) is odd) and to zero for \( k > \frac{d+1}{2} \). From this, it is clear that \( \tilde{h}(R, \epsilon) \to \infty \) as \( R \to \infty \). Therefore, we have \( h(R, \epsilon) \to \infty \) as \( R \to \infty \). □

From Theorem 4 it is easy to see that the boundary of a PV cell \( V_0 \) must be contained within the annulus \( A(R, \epsilon) \) as its inradius \( R \to \infty \) for an arbitrarily small \( \epsilon \). Hence PV cells with large inradii tend to be spherical. Therefore, the approach presented in this section provides an alternate proof for the well-known spherical nature of the PV cells having a large inball [3, 25, 26]. A realization of a PV cell \( V_0 \) with large inradius is shown in Fig. 7 for the case of \( d = 2 \).

APPENDIX A

SOLUTION OF INTEGRALS IN (11)

We have

\[
F_{R_1}(r) = \int_0^r \int_0^{r_1^2} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1 dr_2 + \int_r^\infty \int_0^{r_1 + r_2} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1 dr_2
\]

\[
+ \int_r^\infty \int_r^{r_2} \frac{2r}{r_1 + r_2} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1 dr_2.
\tag{52}
\]

First of all, it is easy to show that \( \text{Int}_1 \) reduces to

\[
\text{Int}_1 = 1 + \exp(-4\lambda r) - 2 \exp(-2\lambda r).
\tag{53}
\]
Now, we have
\[
\text{Int}_2 = \int_r^\infty \int_{r_1 + r_2}^{r} \frac{r}{r_1 + r_2} 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1dr_2 + \int_r^\infty \int_0^r 8\lambda^2 \exp(-2\lambda(r_1 + r_2))dr_1dr_2.
\]

By substituting \(r_1 + r_2 = y\), we solve \(\text{Int}_{21}\) as
\[
\text{Int}_{21} = 8\lambda^2r \int_r^\infty \int_{2r}^{r+y} \frac{1}{z} \exp(-2\lambda y)dzdy - 8\lambda^2 \int_r^\infty \int_{2(r+r_2)}^{r+r_2} \frac{1}{y} \exp(-2\lambda y)dydr_2 \]
\[
= 8\lambda^2 \int_r^\infty \int_{2r}^{r+y} \frac{1}{z} \exp(-2\lambda y)dzdy - 8\lambda^2r \int_r^\infty \int_{2(r+r_2)}^{r+r_2} \frac{1}{y} \exp(-2\lambda y)dydr_2 \]
\[
= 8\lambda^2 \int_r^\infty \int_{2r}^{r+y} \frac{1}{z} \exp(-2\lambda y)dzdy - 8\lambda^2r \int_r^\infty \int_{2(r+r_2)}^{r+r_2} \frac{1}{y} \exp(-2\lambda y)dydr_2 \]
\[
= 8\lambda^2 \int_r^\infty E_1(2\lambda r_2)dr_2 - 8\lambda^2r \int_r^\infty E_1(2\lambda(r + r_2))dr_2 \]
\[
= 8\lambda^2 \int_r^\infty E_1(2\lambda r_2)dr_2 - 8\lambda^2r \int_r^\infty E_1(2\lambda u)du,
\]
where \(E_1\) is an exponential integral function. From [23 Eq. 5.22.8], we have
\[
\int_x^\infty E_1(az)dz = \frac{1}{a} \exp(-ax) - xE_1(ax).
\]

Therefore, we get
\[
\text{Int}_{21} = 4\lambda r \exp(-2\lambda r) - 4\lambda r \exp(-4\lambda r) - 8\lambda^2 r^2 E_1(2\lambda r) + 16\lambda^2 r^2 E_1(4\lambda r).
\]

Similarly, by substituting \(r_1 + r_2 = y\), we solve \(\text{Int}_{22}\) as
\[
\text{Int}_{22} = 8\lambda^2 \int_r^\infty \int_{r_2}^{r+r_2} \frac{y-r_2}{y} \exp(-2\lambda y)dydr_2 \]
\[
= 8\lambda^2 \int_r^\infty \left( \int_{r_2}^{r+r_2} \exp(-2\lambda y)dy - r_2 \int_{r_2}^{r+r_2} \frac{1}{y} \exp(-2\lambda y)dy \right) dr_2 \]
\[
= 8\lambda^2 \int_r^\infty \frac{1}{2\lambda} \left( \exp(-2\lambda r_2) - \exp(-2\lambda(r + r_2)) \right) - r_2 (E_1(2\lambda r_2) - E_1(2\lambda(r + r_2))) dr_2
\]
\[ = 2\left(1 - \exp(-2\lambda r)\right) \exp(-2\lambda r) - 8\lambda^2 \int_{r}^{\infty} r_2 E_1(2\lambda r_2)\,dr_2 + 8\lambda^2 \int_{r}^{\infty} r_2 E_1(2\lambda(r + r_2))\,dr_2. \quad (56) \]

Now, using (54) and the integration by parts, we solve \( \text{Int}_{221} \) as

\[
\text{Int}_{221} = \int_{r}^{\infty} r_2 E_1(2\lambda r_2)\,dr_2 \\
= \frac{r}{2\lambda} \exp(-2\lambda r) - r^2 E_1(2\lambda r) - \int_{r}^{\infty} \left( r_2 E_1(2\lambda r_2) - \frac{1}{2\lambda} \exp(-2\lambda r_2) \right)\,dr_2 \\
= \frac{r}{2\lambda} \exp(-2\lambda r) - r^2 E_1(2\lambda r) + \frac{1}{4\lambda} \exp(-2\lambda r) - \text{Int}_{221} \\
= \frac{r}{4\lambda} \exp(-2\lambda r) + \frac{1}{8\lambda^2} \exp(-2\lambda r) - \frac{r^2}{2} E_1(2\lambda r). \quad (57)
\]

Now,
\[
\text{Int}_{222} = \int_{r}^{\infty} r_2 E_1(2\lambda(r + r_2))\,dr_2 \\
= \frac{1}{4\lambda^2} \int_{4\lambda r}^{\infty} (y - 2\lambda r)E_1(y)\,dy \\
= \frac{1}{4\lambda^2} \int_{4\lambda r}^{\infty} rE_1(y)\,dy - \frac{r}{2\lambda} \int_{4\lambda r}^{\infty} E_1(y)\,dy \\
= \frac{1}{4\lambda^2} \int_{4\lambda r}^{\infty} E_1(y)\,dy \big|_{4\lambda r}^{\infty} - \frac{1}{4\lambda^2} \int_{4\lambda r}^{\infty} \left( \int_{4\lambda r}^{\infty} E_1(y)\,dy \right)\,dy - B \\
= \frac{r}{\lambda} \exp(-4\lambda r) - 4r^2 E_1(4\lambda r) - \frac{1}{4\lambda^2} \int_{4\lambda r}^{\infty} (yE_1(y) - \exp(-y))\,dy - B \\
= \frac{r}{\lambda} \exp(-4\lambda r) - 4r^2 E_1(4\lambda r) + \frac{1}{4\lambda^2} \exp(-4\lambda r) - A - B \\
= \frac{r}{2\lambda} \exp(-4\lambda r) + \frac{1}{8\lambda^2} \exp(-4\lambda r) - 2r^2 E_1(4\lambda r) - \frac{r}{2\lambda} \left( \exp(-4\lambda r) - 4\lambda r E_1(4\lambda r) \right) \\
= \frac{1}{8\lambda^2} \exp(-4\lambda r), \quad (58)
\]

where step (a) follows by substituting \( A + B = \text{Int}_{222} + 2B \) and \( B = \frac{r}{2\lambda} \left( \exp(-4\lambda r) - 4\lambda r E_1(4\lambda r) \right) \). Substituting (57) and (58) in (56), we get

\[
\text{Int}_{22} = \exp(-2\lambda r) - 2\lambda r \exp(-2\lambda r) - \exp(-4\lambda r) + 4\lambda^2 r^2 E_1(2\lambda r). \quad (59)
\]

Now, adding (55) and (59), we get

\[
\text{Int}_2 = \exp(-2\lambda r) + 2\lambda r \exp(-2\lambda r) - \exp(-4\lambda r) - 4\lambda r \exp(-4\lambda r) - 4\lambda^2 r^2 E_1(2\lambda r) + 16\lambda^2 r^2 E_1(4\lambda r). \quad (60)
\]

Again substituting \( r_1 + r_2 = y \) and using (54), we evaluate \( \text{Int}_3 \) as

\[
\text{Int}_3 = 16\lambda^2 r \int_{r}^{\infty} \int_{r+2r_2}^{\infty} \frac{1}{y} \exp(-2\lambda y)\,dy\,dr_2 \\
= 16\lambda^2 r \int_{r}^{\infty} E_1(2\lambda(r + r_2))\,dr_2 - 16\lambda^2 r \int_{r}^{\infty} E_1(4\lambda r_2)\,dr_2 \\
= 8\lambda r \int_{4\lambda r}^{\infty} E_1(u)\,du - 16\lambda^2 r \left( \frac{1}{4\lambda} \exp(-4\lambda r) - r E_1(4\lambda r) \right) \\
= 8\lambda r \left( \exp(-4\lambda r) - 4\lambda r E_1(4\lambda r) \right) - 4\lambda r \exp(-4\lambda r) + 16\lambda^2 r^2 E_1(4\lambda r) \\
= 4\lambda r \exp(-4\lambda r) - 16\lambda^2 r^2 E_1(4\lambda r). \quad (61)
\]

Finally, adding (53), (60) and (61), we get (10).
Let \( a = \frac{d-1}{2} \) and \( b = \frac{1}{2} \). From step (a) of (46) and using \( I_x(a, b) = \frac{B_x(a, b)}{B(a, b)} \), we have

\[
p = \nu_R \int_R^{R+\epsilon} B_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b) \, l^{d-1} \, dl,
\]

where \( \nu_R = \frac{d(R+\epsilon)^{d-1} - R^{d-1}}{2B(a, b)} \). We solve the above integral using integration by parts as follows. Let \( u = l^{d-1} \) and \( u = B_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b) \). We have

\[
\frac{d}{dl}u = -\frac{2l}{(R+\epsilon)^2} \left[ \frac{l^2}{(R+\epsilon)^2} \right]^{b-1} \left( 1 - \frac{l^2}{(R+\epsilon)^2} \right)^{a-1},
\]

and thus

\[
p = -\nu_R \frac{R^d}{d} B_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b) + \nu_R \frac{(R+\epsilon)^d}{d} \int_0^{1} z^{b+\frac{d}{2} - 1} (1 - z)^{a-1} \, dz
\]

\[
= -\nu_R \frac{R^d}{d} B_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b) + \nu_R \frac{(R+\epsilon)^d}{d} \left[ B \left( b + \frac{d}{2}, 1 - a \right) - B \frac{\nu^2}{(R+\epsilon)^2} \left( b + \frac{d}{2} - 1, a \right) \right]
\]

\[
= -\nu_R \frac{R^d}{d} I_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b) + \nu_R \frac{(R+\epsilon)^d}{d} \left[ B \left( b + \frac{d}{2}, a \right) - B \frac{\nu^2}{(R+\epsilon)^2} \left( b + \frac{d}{2}, a \right) \right]
\]

\[
= \nu_R (R + \epsilon)^d B_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b) - \nu_R\frac{R^d}{d} I_1 - \frac{\nu^2}{(R+\epsilon)^2} (a, b)
\]

where \( \nu_R = \frac{1}{2(R+\epsilon)^{d-1} - R^{d-1}} \). The last equality follows using \( I_x(a, b) = 1 - I_{1-x}(b, a) \) and \( B(a, b) = B(b, a) \).

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