Decomposing the $SU(N)$ connection and the Wu-Yang potential

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Abstract

Based on the decomposition of $SU(2)$ gauge field, we derive a generalization of the decomposition theory for the $SU(N)$ gauge field. We thus obtain the invariant electro-magnetic tensors of $SU(N)$ groups and the extended Wu-Yang potentials. The sourceless solutions are also discussed.

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I. INTRODUCTION

Recently Faddeev et al. proposed a decomposition of the four dimensional $SU(N)$ Yang-Mills field $A^a_\mu$. In their paper with some ansatz the new variables were given for studying the knot theory and the QCD. One of our author (Duan) pointed out that the gauge potential should be decomposed in terms of the gauge covariant [4], i.e. $A_\mu = a_\mu + b_\mu$, which the $a_\mu$ satisfies the gauge transformation $a'_\mu = ga_\mu g^{-1} + \partial_\mu gg^{-1}$ and the $b_\mu$ satisfies the adjoint transformation $b'_\mu = gb_\mu g^{-1}$. The $a_\mu$ part may show the geometry property of system and the $b_\mu$ part may be looked upon as vector boson which would be massive. Based on the decomposition theory of the $SU(2)$ gauge field [4], the gauge potential of the $SU(N)$ gauge field is decomposed in terms of local bases corresponding to the Cartan subalgebra without any hypothesis. With this decomposition the extended 't Hooft electromagnetic tensor is derived and the Wu-Yang potential is given. At last, we discuss the sourceless solution of the gauge field equation of group $SU(N)$.

II. SU(2) GAUGE FIELD AND THE 'T HOOFT MONOPOLE

In this section we introduce the decomposition theory of $SU(2)$ gauge field and corresponding Wu-Yang potential. In terms of the $SU(2)$ gauge theory, the covariant derivation of a unit gauge field $n^a(x)$ is

$$D_\mu n^a(x) = \partial_\mu n^a(x) + \varepsilon^{abc} B^b_\mu n^c,$$

where $B^a_\mu$ is the gauge potential of the $SU(2)$ gauge theory. By virtue of this definition we can give the expression of the gauge potential $B^a_\mu$

$$B^a_\mu = A_\mu n^a + \varepsilon^{abc} \partial_\mu n^b n^c - \varepsilon^{abc} D_\mu n^b n^c,$$  \hspace{1cm} (1)

where $A_\mu = B^a_\mu n^a$ is the Abelian projection of the $SU(2)$ gauge potential. Thus the gauge potential is decomposed formally, one will find that the decomposition is very useful in the 't Hooft monopole theory. We calculate the corresponding field strength
\[ F_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu} - [B_{\mu}, B_{\nu}], \]

from the Eq. (1) we give

\[ F_{\mu\nu}^a n^a = (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) - \varepsilon^{abc} n^a \partial_{\mu} n^b \partial_{\nu} n^c + \varepsilon^{abc} n^a D_{\mu} n^b D_{\nu} n^c. \]  \( (2) \)

Since the left hand \( F_{\mu\nu}^a n^a \) and the last term of right hand are the gauge invariant term, we find that the following quantity is a gauge invariant

\[ f_{\mu\nu} = (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) - K_{\mu\nu}, \]

where \( K_{\mu\nu} = \varepsilon^{abc} n^a \partial_{\mu} n^b \partial_{\nu} n^c \). One can find that \( f_{\mu\nu} \) is just the gauge field tensor defined by \'t Hooft which is fundamental for the magnetic monopole

\[ f_{\mu\nu} = F_{\mu\nu}^a n^a - \varepsilon^{abc} n^a D_{\mu} n^b D_{\nu} n^c. \]  \( (3) \)

Let \( \vec{e}_a(x) \) \( (a = 1, 2) \) are the vierbein perpendicular to \( \vec{n}(x) \), it is easy to verify that there exists a \( U(1) \) potential

\[ a_\mu = \varepsilon^{ab} (\vec{e}_a \cdot \partial_\mu \vec{e}_b), \]  \( (4) \)

which satisfies \( K_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \). One can find that this potential is just the Wu-Yang potential of the magnetic monopole system.

### III. Local Basis and the SU(N) Connection

Let \( T_a \) \( (a = 1, 2, ..., r) \) be Lie algebraic bases of the \( SU(N) \) group \( G \), and \( H_i(i = 1, 2, ..., L) \) the Cartan subalgebra, i.e.

\[ [T_a, T_b] = i \varepsilon^{abc} T_c, \quad [H_i, H_j] = 0. \]  \( (5) \)

The local basis of Cartan subalgebra is defined as

\[ n_i(x) = U(x) H_i U^\dagger(x), \]  \( (6) \)
where $U(x)$ is a unitary matrix on manifold $M$. The covariant derivative of the local basis $n_i(x)$ is

$$D_\mu n_i = \partial_\mu n_i - ig[A_\mu, n_i]$$

(7)

where $A_\mu$ is a $su(N)$ Lie algebra vector

$$A_\mu = A^a_\mu T_a.$$ 

The curvature is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].$$

(8)

In terms of the relation

$$f_{abl} f_{acm} n_i^b n_j^c + n_i^l n_j^m = \delta^{lm},$$

(9)

one can find that a $su(N)$ Lie algebra vector $V$ can be decomposed with the local basis

$$V = (V, n_i) n_i + [[V, n_i], n_i], \quad \forall V \in su(N).$$

(10)

Since it is a $su(N)$ Lie algebra vector, with the Eq. (7) the $SU(N)$ connect $A_\mu$ can be decomposed as

$$A_\mu = (A_\mu, n_i) n_i + \frac{1}{ig}[\partial_\mu n_i, n_i] - \frac{1}{ig}[D_\mu n_i, n_i].$$

(11)

Similiar to the 't Hooft electromagnetic tensor in the $SU(2)$ gauge field, we can define the extended 't Hooft electromagnetic tensor

$$f^i_{\mu\nu} = (F_{\mu\nu}, n_i) + \frac{i}{g}(n_i, [D_\mu n_j, D_\nu n_j]),$$

(12)

which is gauge invariant. By make using of the relation

$$(n_i, [n_j, A]) = 0,$$

(13)

one can prove that
\[ f_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + \frac{i}{g} (n_i, [\partial_\mu n_j, \partial_\nu n_j]), \] 

(14)

in which \( A_\mu^i \) is the projection of \( A_\mu \) on the \( n_i \)

\[ A_\mu^i = (A_\mu, n_i). \]

From the above discussion, one can find that this decomposition is different from the work of Faddeev et al. With the idea that the connection should possess its inner structure thus can be decomposed, we express the connection \( A_\mu \) with the covariant part \( b_\mu \) and the non-covariant part \( a_\mu \), i.e. \( A_\mu = a_\mu + b_\mu \). In the case of SU(N), we have

\[ a_\mu = (A_\mu, n_i) + \frac{1}{ig}[\partial_\mu n_i, n_i], \quad b_\mu = -\frac{1}{ig}[D_\mu n_i, n_i]. \]

(15)

In the section I, we discussed the non-covariant part \( a_\mu \) should give the geometry information of system. In the following section we can find that this part corresponds to the Wu-Yang potential. In the paper of Faddeev et al., the decomposition of SU(N) was given by introducing the new variables and they focused on the covariant part \( b_\mu \) to consider the effective Lagrangian.

IV. WU-YANG POTENTIAL AND THE SOURCELESS SOLUTION OF THE SU(N) GAUGE FIELD

Since it was proposed, the existence of Wu-Yang potential has been doubted for a long time. As a class of important gauge field configurations for QCD, Wu-Yang monopoles are fundamental for confinement and compete with the instanton-like configurations which are responsible for chiral symmetry breaking [8]. In this section we will be sure that the Wu-Yang potential exist in the \( SU(N) \) gauge theory.

From Eq. (3) one have

\[ \partial_\mu n_i = ig[\tilde{A}_\mu, n_i], \]

(16)

where \( \tilde{A}_\mu = \frac{1}{ig} \partial_\mu U U^\dagger \), is obviously a flat connection, i.e.,
∂µAν − ∂νAµ − ig[Aµ, Aν] = 0 \quad (17)

In terms of Eq. (14) and (17) we can obtain

\[ \frac{1}{ig}(n_i, [\partial_\mu n_j, \partial_\nu n_j]) = \partial_\mu a^i_\nu \, - \, \partial_\nu a^i_\mu, \]

where

\[ a^i_\mu = (\tilde{A}_\mu, n_i) \quad (18) \]

are the Wu-Yang potential. Thus we show that Wu-Yang potentials are Abelian projection of the flat connection in the SU(N) gauge theory. Furthermore, since there are \((N - 1)\) Abelian projection directions, one can find \((N - 1)\) Wu-Yang potentials in SU(N) gauge theory.

Finally, from Eq. (12) we note that when \(D_\mu n_i = 0\),

\[ f^i_{\mu\nu} = (F_{\mu\nu}, n^i). \quad (19) \]

But from the relation that

\[ (D_\mu D_\nu - D_\nu D_\mu)n_i = [F_{\mu\nu}, n_i], \quad (20) \]

we find that \([F_{\mu\nu}, n_i] = 0 \, (i = 1, 2, \ldots, L)\). That is to say, when \(D_\mu n_i = 0\), \(F_{\mu\nu}\) must commute with \(n_i\). Therefore using Eq. (19), \(F_{\mu\nu}\) can be expressed as

\[ F_{\mu\nu} = f^i_{\mu\nu} n_i. \quad (21) \]

Then from \(D_\mu n_i = 0\), we have

\[ D_\nu F_{\mu\nu} = (\partial_\nu f^i_{\mu\nu})n_i. \]

Hence the solutions of equation

\[ \partial_\nu f^i_{\mu\nu} = 0, \quad i = 1, 2, \ldots, L, \quad (22) \]

corresponding to the solutions of the sourceless equation
\[ D_\nu F_{\mu\nu} = 0. \]

If we adopt the Lorentz condition \( \partial_\nu A_i^\nu = 0 \) \((i = 1, 2, \ldots, L)\), from Eqs. (14) and (22) one can get

\[ \nabla^2 A_\mu^i = \partial_\nu K_{\mu\nu}^i, \quad (23) \]

where

\[ K_{\mu\nu}^i = \frac{i}{g} (n_i, [\partial_\mu n_j, \partial_\nu n_j]). \]

The solution of Eq. (23) is

\[ A_\mu^i = -\frac{1}{4\pi} \int G(x - x') \partial_\nu K_{\mu\nu}^i(x') d^4 x', \quad (24) \]

where \( G(x - x') \) is the retarded Green’s function

\[ \nabla^2 G(x - x') = -4\pi \delta^4(x - x'). \]

Then, it follows from Eq. (11) that the equation

\[ A_\mu = -\frac{1}{4\pi} \left[ \int G(x - x') \partial_\nu K_{\mu\nu}^i(x') d^4 x' \right] n_i + \frac{1}{ig} [\partial_\mu n_i, n_i], \quad (25) \]

is the sourceless solution of the gauge field equation.

**V. CONCLUSION**

In this paper, we give the decomposition of SU(2) and SU(N) connection with the idea that connection should possess their inner structures. One can find that the decomposition in this paper have no any hypothesis or ansatz since the expressions of connections are obtained with their definitions directly. In terms of the decomposition of SU(2) connection, we give the ’t Hooft invariant gauge field tensor which can describe the magnetic monopole correctly. And with the expression of SU(N) connection we consider the Wu-Yang potential which is responsible for the confinement in QCD.
VI. ACKNOWLEDGEMENT

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