$w_{1+\infty}$-type constraints in two–matrix and Kontsevich model–different approach

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ABSTRACT

The technique of $Q$-polinomials are used to derive the $w$- constraints in the two-matrix and Kontsevich-like model at finite $N$. These constraints are closed and form Lie algebra. They are associated with the matrices, $\lambda^n \partial^m$ with $n, m \geq 0$. In the case of two-matrix model they can be reduced to the $W$-constraints of [8]. For the case of Kontsevich-like model and two-matrix model with the finite polinomial potential, the number of constraints are limited by the power of the finite matrix potential i.e. the spin of $w$-s coincide with that power. This statement is the natural consequence of the form of constraints.
Recently the great deal of progress has been made in the nonperturbative formulation of low-dimensional toy models of string theory–two-dimensional gravity coupled to $c < 1$ matter–by formulating the theory in terms of large-$N$ matrix integrals [1, 2]. Soluble matrix integrals give the full partition function for string theory in simple backgrounds.

One of the prominent features of the two dimensional quantum gravity and string in less than one dimension is the appearance of the Virasoro or the $W$–algebra constraints for the partition function [3, 4]. The time-dependent partition function is identified with the corresponding $\tau$–function. The role of times is played by the coupling constants in the matrix potential. These constraints imposed on the $\tau$–function select a sub-class of $\tau$-functions from the entire space of generic $\tau$-functions and may be considered as equations of motion in the general string theory associated with the model.

In this letter we are deriving the $w_{1+\infty}$-constraints in Kontsevich-like(KL) and two-matrix model. Instead of orthogonal polynomial method we follow the technique formulated in [5]. The crucial point of this approach is that these type of constraints is generated by $\lambda^n\partial^m$ with $n.m \geq 0$ [6].

The partition function of Kontsevich model is proportional to the $N \times N$ (anti)Hermitian matrix integral

$$
\mathcal{F}[\Lambda] = \int DX \exp(-\text{tr}W[X] - \text{tr}\Lambda X)
$$

with $W[X] = X^3$ and $N \to \infty$.

The importance of Kontsevich-model appearing was the sprang possibility to describe the continuum limit of one-matrix models.

Using the technique of [5] we obtained the $w$–constraints in the most non-trivial way. It turns out, that in the case of KL-models i.e. with the matrix potential of finite dimension the $w$-constraints are depending merely on the power of potential and the spin of $w$ i.e. the number of constraints are limited by that power automatically. The two-matrix- case is very much akin of the KL-models when the potential of it is consist of two pieces: polinomial of finite degree and of infinite one.

### 1 One-Matrix model

Let us demonstrate the benefits of the method (the detailed description of that can be found in [5]) for the 1-matrix model case. The partition function of it is following

$$
Z_N(g_l) = \int d^N M \exp\left(\sum_{l=0}^{\infty} g_l \text{Tr}M^l\right)
$$

Rewriting this integral in terms of $M$-matrix eigenvalues, we obtain

$$
Z_N(g_l) = \int d^N \lambda \Delta^2(\lambda) \exp U(g_l)
$$

where

$$
U(g_l) = \sum_{l=0}^{\infty} \sum_{i=1}^{N} g_l \lambda_i^l
$$

Here the integration over the angular variables have been performed and $\Delta(\lambda) = \prod_{i} (\lambda_i - \lambda_j)$.
where \( n, m \in \mathbb{Z}_+ \). It is represented by the commutator

\[
[O_n^m, O_l^k] = \{(k + 1)n - (m + 1)l\}O_{n+l}^{m+k} \tag{6}
\]

The algebra, that is isomorphic to the \( w_{1+\infty} \) can be written in terms of eigenvalues \( \lambda_i \)

\[
D_{n,m} \equiv \sum_{i=1}^{n} \lambda_i^{n} \frac{\partial^{m}}{\partial \lambda_i^{m}} \tag{7}
\]

and conjugate one

\[
D_{n,m}^{+} = (-1)^{m} \sum_{i=1}^{N} \frac{\partial^{m}}{\partial \lambda_i^{m}} \lambda_i^{n} \tag{8}
\]

Let us define the operator

\[
D_m(p) \equiv \sum_{n=0}^{\infty} D_{n,m} \frac{1}{p^{n+1}} = \sum_{n=1}^{N} \sum_{i=1}^{\infty} \frac{\lambda_i^{n}}{P^{n+1}} \frac{\partial^{m}}{\partial \lambda_i^{m}} = \sum_{i=1}^{N} \frac{1}{P - \lambda_i} \frac{\partial^{m}}{\partial \lambda_i^{m}} \tag{9}
\]

It is, in some sense, the generating function for \( D_{n,m} \). The following formulae are useful to repeat

\[
\Delta^{-1}D_s(P)\Delta = \frac{1}{s+1}(\frac{\partial}{\partial P} + W(P))^{s+1}1, \quad s < N, \tag{10}
\]

where

\[
W(P) = \sum_{i=1}^{N} \frac{1}{P - \lambda_i} \tag{11}
\]

by definition.

For the further purposes we need a slightly different form of \( W(P) \). Let us mention here, that the representation \((\Pi)\) is singular at \( \lambda_i = P \). We need to reexpress the \( W(P) \) so as to be regular at that point. It can be achieved by writing \((\Pi)\) in terms of "time" \( \partial/\partial g_s \) differentiation. We find

\[
W(P) = \sum_{i=1}^{N} \frac{1}{P - \lambda_i} \tag{12}
\]

The \( w \)-constraints are obtained from the "partial differentiation" formula

\[
0 = \int d^{N} \lambda \lambda_{i} A D_{s}(P)B - (D_{s}^{+}(P)A)B \tag{13}
\]

where \( A = \Delta(\lambda) \) and \( B = \Delta(\lambda) \exp(U) \).

We can write the following expression for the high order differential operator

\[
D_{s}(P)(\Delta \exp(U)) := \sum_{m=0}^{s} (D_{m}(P)\Delta)(D_{s-m}(P)\exp(U))C_{s}^{m} \tag{14}
\]

where \( C_{s}^{m} \) are binomial coefficients.

Let us note here, that for the finite flat entries the formula \((14)\) is just the trivial Newton–Leibnitz formula. The left hand side is singular at \( \lambda_i = P \) and we define it by means of residue. On the right hand side, the first term turned out to be regular at that point in the representation \((\Pi)\) and the second one is singular. Hence, we can substitute the leading term on the right side.
where

\[ j(p) \equiv \sum_{l=0}^{\infty} lg_l P^{l-1} \]  

and \( Q_l \) is defined as

\[ Q_s[f] \equiv (\frac{\partial}{\partial P} + f(P))^s \]  

Thus, after all these artificial procedures formula (14) is transformed to the expression

\[ D_s(P)(\Delta \exp(U)) = \frac{\Delta}{m+1} : Q_{s+1}[j(P) + \frac{d}{dj}(P)] : - \exp(U) \]  

Here the sign :: – means the normal ordering i.e. \( \frac{d}{dj} \) standing before \( j \) and ” – “ script is the projection to the negative powers of \( P \). (The meaning of the latter will be cleared out below in considering the reduction of the constraints).

After all the preliminary procedures we can return to the (13) and inserting formulae (10), (18) into the (13) arrive to the following expression for the \( w_{1+\infty} \)-constraints

\[ w_{1+\infty}^{(l)} = \frac{1}{l} : Q_l[j(P) + \frac{d}{dj}(P)] : +(-1)^s Q_{l+1}^+ \frac{d}{dj}(P), \quad l < N, \]  

according to the [5] \( w^{(l+1)} \) are stated to be reducing to the Virasoo type ones. For example, for \( l = 2 \)

\[ w^{(2)} = (\frac{d}{dj})^2 + (j \frac{d}{dj})_+ = \sum_{l=1}^{\infty} \mathcal{L}_n P^{(-n-1)} \]  

where as it is easy to guess, the \( \mathcal{L}_n \) give the Virasoro algebra \[5\]. For \( l = 3 \) the expression is the following

\[ w^{(3)} = (j w^{(2)})_+ + \partial_P w^{(2)} \]  

and so on.

The important item to stress, is that the set of constraints (19) is closed. They form the Lie algebra; The commutator of \( lw^{(l)} \) with its conjugate is the following

\[ [lw^{(l)}(P), mw^{(m)}(P)] = (-1)^l (Q_{l+m}[f] - Q_{l+m}[g]) + (-1)^m (Q_{l+m}[-f] - Q_{l+m}[-g]) \]  

Taking into account (20) for \( l = m = 2 \) we obtained the Virasoro algebra \[6\]

2 Two–Matrix model case

Let us turn to the main result of the paper–to the two-matrix model. The partition function for the case is the following

\[ Z_{V,W} = \int DXDA \exp\{-trV[A] - trW[X] - trAX\} \]  

In terms of eigenvalues the integral will have the form

\[ Z_{V,W} = \int 2^{2K} \prod \lambda^{k_1}(1) \lambda^{k_2}(2) \cdots \{ \sum_{j=1}^{\infty} \lambda^{j}(1) \}_{1} \cdots \sum_{j=1}^{K} \lambda^{j}(2) \cdots \sum_{j=1}^{\infty} \lambda^{j}(2) \}_{2} \]
Here we denote by $\lambda_i(\alpha)$ the eigenvalues of accordingly–$X$ and $\Lambda$.

In the initial condition we set

$$A = \Delta(\lambda(1))\Delta(\lambda(2)) \exp\{-\sum_{m=1}^{\infty} t_m \lambda(1) - \sum_{m=0}^{K} g_m \lambda(1)\} \equiv \Delta(\lambda(1))\Delta(\lambda(2)) \exp(U)$$

and the operator $D_s$ is defined as follows:

$$D_S(P) \rightarrow D_l \equiv \sum D_{n,m}(\lambda(1)) \frac{1}{P_{m+1}} - \sum D_{m,n}(\lambda(2)) \frac{1}{P_{m+1}} \equiv D_l^{(1)} + D_l^{(2)}$$

Again, we have to derive the action of $D_l^{(1)}$ on the $\Delta$ and $\exp(U)$.

$$D_l^{(1)} \Delta = \Delta(\lambda(2))D_l^+(P)\Delta(\lambda(1)) \exp(U) = \Delta(\lambda(1))\Delta(\lambda(2)) \frac{(-1)^l}{l+1} : Q_{l+1}^+[j + \bar{j}] : \exp(U)$$

where by $j_1$ and $\bar{j}_1$ are denoted the expressions

$$j \equiv \sum_{l=1}^{\infty} l t_l P_l^{l-1} \quad (29)$$

$$j_1 \equiv \sum_{l=1}^{K} l g_l P_l^{l-1} \quad (30)$$

$$\bar{j} \equiv \frac{d}{dj_1} \equiv \sum_{l=1}^{\infty} P_l^{l-1} \frac{\partial}{\partial t_l} \quad (31)$$

$$\bar{j}_1 = \sum_{l=1}^{K} P_l^{l-1} \frac{\partial}{\partial g_l} \quad (32)$$

and

$$- D_l^{(2)} A = \Delta(\lambda(2))\Delta(\lambda(1)) : \frac{Q_{l+1}^+}{l+1} [\bar{j}_1 + j_1] : \exp(U) \quad (33)$$

Let us emphasize here that $D_l^{(2)}$ depends on $l$ through differentiation and its action on $\sum_{m=0}^{K} g_m P_m$ is zero for $l > K$. Hence, the power of $Q_{l,K}$ is limited by $K$.

The $w$–constraints in this case is found to be

$$w_{1+\infty}^{(l,K)} = \frac{1}{l} : Q_{l,K}^+ [\bar{j}_1 + j_1] : - + : Q_{l}^+[j + \bar{j}] : -$$

These constraints are closed (see the commutator (22) and form the Lie algebra. They can be written in terms of ”times” for spin-2 and-3. The spin-2 case constraints are the following

$$w_2 \propto \left(\frac{d}{dj}\right)^2 + (j \frac{d}{dj}) = \sum L_n P_n^{n-1} \quad (35)$$

(see [8] for the $L$ definition) and for spin-3...
3 The Kontsevich-like model

The KL-model is invented to define the continuum limit of the 1-matrix model. It is still fascinates the theoreticians, since it is describing the topological gravity in the case of $W[X] = X^3$. The partition function for this model has a form (1). Where

$$W[X] = \sum_{m=0}^{K} g_m X^m$$

(37)

In rewriting the integral in terms of eigenvalues, we need diagonalization of $X$

$$X \rightarrow M X_D M^+$$

(38)

where the $X_D$ is the diagonal matrix, with the eigenvalues $\lambda_1, \ldots, \lambda_N$ and $DX$ is the Haar measure

$$DX = DM \prod_{i} d\lambda_i \Delta^2(\lambda)$$

(39)

while $W[X]$ is invariant under the transformation $M$

$$MW[X]M^+ = W[X]$$

(40)

Using the expression for the Itsikson-Zuber integral we are arriving to the KL-model partition function in term of $\lambda_i$

$$Z_W = \int \prod d\lambda_i \frac{\Delta(\lambda_i(1))}{\Delta(\lambda_i(2))} \exp\{-\sum_{l=0}^{K} g_l \lambda_i^l - \sum_i \lambda_i(1)\lambda_i(2)\}$$

(41)

Inserting this equation into (13) and taking

$$A = \Delta(\lambda(1)) \Delta(\lambda(2)) \exp(W)$$

(42)

$$B = \exp(\sum \lambda_i(1)\lambda_i(2))$$

(43)

It is important to mention here, that the operator (27) is not producing the "multiloop color" operator i.e. the cross term like $\lambda(1)\lambda(2)$. Hence, the only remaining term in (13) will be ($D^+_l A)B$

$$D^+_l (P)_A = \frac{(-1)^l \Delta(\lambda(1))}{(l+1) \Delta(\lambda(2))} : Q^{(+)}_{l+1} [j_1(P) + \bar{j}_1(P)] : \exp(U)$$

(44)

(For the the details of the calculation see Appendix) and for the second part

$$-D^+_l A = \frac{\Delta(\lambda(1))}{\Delta(\lambda(2))} \left\{ \frac{1}{2} \partial_P^{(s-1)} Q_2 \bar{j}(P) \right\} \exp(U)$$

(45)

The $w$-constraints becoming as

$$w_{1+\infty}^{l+1} = \frac{(-1)^{l+1}}{l+1} : Q^{(+)}_{l+1} [j_1 + \bar{j}_1] + \frac{1}{4} \partial_P^{(s-2)} Q_2^2 [\bar{j}(P)] - \frac{1}{2} \partial_P^{(s-1)} Q_2 [\bar{j}(P)]$$

(46)

As a conclusion, we stress the main points of the article; We derive the $w$-constraints for the two-matrix model case for the two pieces of potential: infinite and finite. In this case within the approach, the constraints depend merely on the power of the finite -part potential, thus restricting automatically the number of them or equivalently, the spin of $W$-constraints. It is important to stress here once more, that the restriction of the spin of $w$ is straightforward, since $O(\lambda)$ depends only on the power of the potential. This approach was first put by E.Gava and K. Narain [7] and demonstrated later by A. Marshakov, A. Mironov and A. Morozov [8, 9]. The obtained constraints form the Lie algebra and can be reduced to that of [8], when written in terms of "time".
Appendix

We give here the method of calculating the action of $D_s(P)$ on the $\Delta_{-1}$. As an example, consider the case of $D_3(P)$

$$\Delta D_3(P)\Delta^{-1} = \text{res}\{\sum_{\lambda_j} \frac{1}{P - \lambda_j} (-\partial^2_{\lambda_j} (\Delta^{-1} \partial_{\lambda_j} \Delta) + \partial_{\lambda_j} (\Delta^{-1} \partial_{\lambda_j} \Delta)^2)\} \quad (A.1)$$

Here we have used the well-known relation

$$\Delta D_1(P)\Delta^{-1} = -\Delta^{-1}D_1(P)\Delta \quad (A.2)$$

It is important to stress here, that the action of $D$ on $\Delta^{-1}$ is defined through $\text{res}$ i.e. on the right and left hand side is assumed $\text{hboxres}$. Using the equation of [5] we obtain

$$\Delta^{-1}\partial P\Delta = \text{res}(\Delta^{-1}D_s(P)\Delta) = \frac{1}{s + 1}Q_s + 1\left[W(P)\right] \quad (A.3)$$

Thus, inserting (A3) into (A1) we obtain for the right hand side of (A1)

$$\Delta D_3(P)\Delta^{-1} = \frac{1}{4}Q_2^2 - \frac{1}{2}\partial P Q_2 \quad (A.4)$$

By induction we can find the expression for $\Delta D_s\Delta^{-1}$ for $\forall s$

$$\Delta D_s(P)\Delta^{-1} = \frac{1}{4}\partial P^{s-2}Q_2^2 - \frac{1}{2}\partial P^{s-1}Q_2 \quad (A.5)$$
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