RANDOM GRAPHS OF FREE GROUPS CONTAIN SURFACE
SUBGROUPS

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Abstract. A random graph of free groups contains a surface subgroup.

1. Introduction

Gromov’s Surface Subgroup Question asks whether every one-ended hyperbolic group contains a subgroup isomorphic to the fundamental group of a closed surface with $\chi < 0$.

In this paper we show that a random graph of free groups contains many closed surface subgroups, with probability going to 1 as a certain parameter in the model of randomness goes to infinity.

Graphs of free groups are a very important special case for Gromov’s question, for several reasons. For example, let $G$ be a one-ended hyperbolic group which is isomorphic to the fundamental group of a non-positively curved cube complex (informally, $G$ is said to be cubulated). Agol [1] showed that $G$ is virtually special, and therefore $G$ contains a subgroup isomorphic to a one-ended graph of free groups (see Theorem A.5). Many classes of hyperbolic groups are known to be cubulated, including

(1) $C'(1/6)$ groups (Wise, [11]);
(2) random groups at density $< 1/6$ (Ollivier–Wise, [8]);

and others. Such groups are all now known to contain graphs of free groups, so our main result makes it plausible that they all contain surface subgroups.

1.1. Precise statement of main theorem. Let $F_k$ and $F_l$ be free groups of rank $k$ and $l$, and suppose we have chosen a free generating set for each group. A random homomorphism of length $n$ is a homomorphism $\phi : F_k \to F_l$ which takes each of the generators of $F_k$ to a reduced word of length $n$ in the generators of $F_l$ (and their inverses), independently and randomly with the uniform distribution. See §5.1 for more details.

If we fix a finite graph, and for each edge and vertex in the graph we fix a free group of finite (nonzero) rank and a free generating set, we can define a random graph of free groups (with the given edge and vertex groups and the given generators) by taking each homomorphism of an edge group to a vertex group to be a random homomorphism of length $n$. Informally we call the result a random graph of free groups of length $n$. Evidently, to show that a random (nontrivial) graph of free groups contains a surface subgroup, it suffices to prove it in the case of an amalgamated free product or an HNN extension. With this terminology, our main theorem is the following:

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Random Surface Subgroup Theorem 5.3.2. Let $H := F_1 *_G F_2$ or $H := F*G$ be obtained by amalgamating two free groups over random subgroups of rank $k \geq 1$ of length $n$, or by taking an HNN extension over two random subgroups of rank $k \geq 1$ of length $n$. Then $H$ contains a closed surface subgroup with probability $1 - O(e^{-Cn})$.

Note that for any $\lambda > 0$ a random graph of free groups of length $n$ satisfies the small cancellation condition $C'(\lambda)$ with probability $1 - O(e^{-Cn})$ (this echoes an observation made by Button [2]) and therefore random graphs of free groups are hyperbolic and virtually special.

The proof of the main theorem is constructive; that is, given a graph of groups there is an explicit procedure (guaranteed to work with very high probability) to construct a surface subgroup, which is certified as injective by local combinatorial conditions. A major step in the construction depends on being able to build a folded fatgraph with prescribed boundary satisfying certain equidistribution properties (namely $(T, \epsilon)$-pseudorandomness — see §5.1); this step is carried out in Calegari–Walker [3], Thm. 8.9. and was used there to construct injective surface subgroups in certain ascending HNN extensions of free groups. The methods from [3] can easily be adapted to certify the existence of surface subgroups in particular graphs of free groups using the program scallop [4]. Experiments suggest that such surface subgroups are extremely easy to find. The proof of the main theorem also produces not one but infinitely many surface subgroups; see Remark 5.3.4.

2. Free groups

2.1. Standard rose. Let $F$ be a finitely generated free group. We fix an identification of $F$ with $\pi_1(X)$ where $X$ is a rose — i.e. a wedge of finitely many circles. Informally, we say $X$ is a rose for $F$. The oriented circles of $X$ determine a (free) generating set for $F$ which we denote $a, b, c, \ldots$. Inverses are denoted by upper case letters, so $A := a^{-1}, B := b^{-1}$ and so on.

Definition 2.1.1. A graph $Y$ over $X$ is a graph together with a map $f : Y \to X$ taking edges of $Y$ to reduced simplicial paths in $X$. A graph $Y$ over $X$ is folded if the map $f$ is an immersion (i.e. if it is locally injective).

If $Y$ is a graph over $X$, each oriented edge of $Y$ is labeled with a reduced word in $F$ in such a way that reversing the orientation gives the inverse label. If $Y$ is folded, immersed loops in $Y$ are labeled with cyclically reduced words in $F$.

2.2. Core associated to a subgroup. Let $G$ be a finitely generated subgroup of $F$, and let $X_G$ be the cover of $X$ associated to $G$. We think of $X_G$ as a graph with a basepoint.

There is a compact core $Y_G \subset X_G$, defined to be the minimal subgraph of $X_G$ containing the basepoint so that the inclusion $Y_G \to X_G$ is a homotopy equivalence. We think of $Y_G$ as a graph with a basepoint.

Stallings showed how to obtain $Y_G$ algorithmically by starting with a rose whose edges are labeled by reduced words in $F$ (the generators of $G$) and then folding the rose until it immerses in $X$ [10].

2.3. Core associated to a conjugacy class of subgroup. Let $Z_G \subset Y_G$ be the minimal subgraph of $Y_G$ so that the inclusion $Z_G \to Y_G$ is a homotopy equivalence.
Informally, $Y_G$ is obtained from $Z_G$ by connecting it to the basepoint (in $X_G$). We think of $Z_G$ as a graph without a basepoint. The graph $Z_G$ depends only on the conjugacy class of $G$ in $F$.

3. Amalgams and HNN extensions

In this section, we recall the standard construction of an Eilenberg–Mac Lane space $a$ for graph of free groups, and give a criterion for a map from a surface to be $\pi_1$-injective.

3.1. Mapping cylinder. Let $F$ be a free group, and let $G$ be a finitely generated subgroup. If $X$ is a rose for $F$, and $Z_G$ is the core graph associated to the conjugacy class of $G$, there is an immersion $f : Z_G \to X$ and we can build the mapping cylinder

$$C_f := Z_G \times [0,1] \cup X/(z,1) \sim f(z)$$

More generally, let $G_1, \ldots, G_n$ be a finite collection of finitely generated subgroups of $F$ and, for each $i$, let $f_i : Z_{G_i} \to X$ be the corresponding immersion of core graphs. Then we may consider the coproduct immersion

$$f = \coprod_i f_i : \coprod_i Z_{G_i} \to X$$

and build the mapping cylinder $C_f$ in the same manner.

3.2. Amalgams. Let $F_1$ and $F_2$ be free groups with roses $X_1$ and $X_2$, and let $G$ be a finitely generated free group with inclusions $\phi_1 : G \to F_1$ and $\phi_2 : G \to F_2$. We can form the amalgamated free product

$$H := F_1 *_G F_2$$

There are core graphs $Z_{G,1}$ and $Z_{G,2}$ associated to $G$, and immersions $f_i : Z_{G,i} \to X_i$ for $i = 1,2$ giving rise to mapping cylinders $C_{f_1}$ and $C_{f_2}$. The map $\phi = \phi_2 \circ \phi_1^{-1}$ gives rise to a canonical homotopy class of homotopy equivalence $Z_{G,1} \to Z_{G,2}$.

Let $W_\phi$ be the space obtained from the mapping cylinders $C_{f, i}$ by gluing $Z_{G,1}$ to $Z_{G,2}$ by a homotopy equivalence representing $\phi$. Then $\pi_1(W_\phi) = H$, and $W_\phi$ contains subgraphs $Z$, $X_1$, $X_2$ with fundamental groups corresponding to the subgroups $G$, $F_1$, $F_2$.

3.3. HNN extensions. Let $F$ be a free group with rose $X$, and let $G$ be a finitely generated free group with two inclusions $\phi_1, \phi_2 : G \to F$. Let $\phi = \phi_2 \circ \phi_1^{-1}$. We can form the HNN extension

$$H := F*_\phi$$

similarly. Again, there are core graphs $Z_{G,1}$ and $Z_{G,2}$, where $Z_{G,i}$ is the core of the covering space of $X$ associated to the conjugacy class of $\phi_i(G)$. These are equipped with immersions $f_i : Z_{G,i} \to X$ and the coproduct $f = f_1 \cup f_2$ defines a mapping cylinder $C_f$.

In this case, we let $W_\phi$ be the space obtained from $C_f$ by identifying the two copies of $Z_{G,i}$ by a homotopy equivalence representing $\phi$. As before, $\pi_1(W_\phi) = H$, and $W_\phi$ contains subgraphs $Z$ and $X$ with fundamental groups corresponding to the subgroups $G$ and $F$, respectively.
Remark 3.3.1. Of course, one can similarly construct Eilenberg–Mac Lane spaces for any graph of free groups. However, as the fundamental group of a graph of free groups contains an amalgamated product or HNN extension as a subgroup, we can restrict ourselves to these cases without loss of generality.

3.4. Maps of surfaces. In this section we state a criterion for a continuous map from a surface into a graph of spaces to be injective.

Let $W_\phi$ be one of the spaces constructed above and let $\sigma : S \to W_\phi$ be a continuous map from a closed, oriented surface to $W_\phi$, transverse to $Z$. Let $\alpha = \sigma^{-1}(Z) \subset S$. Cutting along $\alpha$ decomposes $S$ into a finite set of compact subsurfaces with boundary $S_1, \ldots, S_n$. Each comes equipped with a map of pairs

$$\sigma_i : (S_i, \partial S_i) \to (C_f, Z)$$

where $f : Z \to X$ is an immersion of graphs and $C_f$ is the corresponding mapping cylinder. Furthermore, there is an orientation-reversing involution of the disjoint union of the boundaries of the $S_i$, coming from how they were glued up in $S$ along $\alpha$.

Definition 3.4.1. Consider a map of pairs

$$\sigma_i : (S_i, \partial S_i) \to (C_f, Z)$$

where $S$ is a compact surface with boundary. A compressing bigon for $\sigma_i$ is the continuous image of a bigon $B \to C_f$ with $\partial B$ equal to the union of two arcs $\alpha \cup \beta$ such that:

1. there is a proper essential embedding of $\alpha$ in $S_i$ so that the restriction of $B \to C_f$ to $\alpha$ equals $\sigma_i$; and
2. $\beta$ is mapped into $Z$.

Lemma 3.4.2. If $\sigma$ is not $\pi_1$-injective then some $\sigma_i$ admits a compressing bigon.

Proof. Let $\gamma$ be an immersed essential loop in $S$ mapping to an inessential loop in $W_\phi$. Suppose that $\gamma$ is chosen transverse to $\alpha$, and intersecting it in the least number of components (so that every arc of intersection is essential in some $S_i$). Let $D$ be a disk with $\partial D = \gamma$, and let $\sigma : D \to W_\phi$ extend $\sigma(\gamma)$. Make $D$ transverse to $Z$, and remove loops of intersection by a homotopy.

An outermost bigon in $D - \sigma^{-1}(Z)$ has one arc in $\gamma$ and the other mapping to $Z$. Since every arc of $\gamma$ is essential and proper in $S_i$, it is a compressing bigon. $\square$

Lemma 3.4.3. Suppose $S_i$ admits a compressing bigon. Then there is an essential non-boundary parallel loop in $S_i$ whose image under $\sigma_i$ is freely homotopic into $Z$.

Proof. Let $B$ be a compressing bigon, let $\alpha$ be the arc of $\partial B$ proper in $S_i$, and let $\beta$ be the other arc, mapping to $Z$.

If the two vertices of $B$ are on the same component $\delta$ of $\partial S_i$, let $\delta'$ be a subarc of $\partial S_i$ joining these vertices. Then, for any $n$, the concatenation $\delta^n \delta' \alpha$ is a loop in $S_i$ homotopic to $\delta^n \delta' \beta$ which is in $Z$, so $\delta^n \delta' \alpha$ is conjugate into $G$. Moreover, $\delta^n \delta' \alpha$ is essential, and is not boundary parallel in $S_i$ for all but at most finitely many values of $n$, or else $\alpha$ would be properly homotopic into $\delta$.

If the two vertices of $B$ are on different components $\delta$, $\delta'$ of $\partial S_i$, for any nonzero $n, m$ the loop $\delta^n \alpha (\delta')^m \alpha^{-1}$ is essential in $S_i$ and homotopic into $Z$, and for all but finitely many $n, m$ it is not homotopic into $\partial S_i$. $\square$
4. Folded fatgraphs

4.1. Fatgraphs.

**Definition 4.1.1.** A fatgraph is a graph $Y$ together with a cyclic order on the edge incident to each vertex.

A fatgraph admits a canonical fattening to a compact, oriented surface $S(Y)$ in such a way that $Y$ includes in $S(Y)$ as a spine, and there is a canonical deformation retraction of $S(Y)$ to $Y$.

Pulling back the simplicial structure of $Y$ gives $\partial S(Y)$ the structure of a graph.

**Definition 4.1.2.** A fatgraph over $X$ is a fatgraph $Y$ whose underlying graph is a graph over $X$. A fatgraph $Y$ over $X$ is folded if the underlying graph is folded.

Suppose $Y$ is a folded fatgraph over $X$. Then the composition $S(Y) \to Y \to X$ is an injection on $\pi_1$.

4.2. Relative fatgraphs. Let $F$ be a free group, and suppose we have fixed a rose $X$ with $\pi_1(X) = F$. Let $f_i : Z_i \to X$ be finitely many immersions of finite graphs and let $f : Z \to X$ be their coproduct. Equivalently, we may fix a finite set of subgroups $G_1, \ldots, G_n$ and set $Z$ to be the disjoint union of the corresponding core graphs $Z_{G_i}$.

**Definition 4.2.1.** A folded fatgraph $Y$ over $X$ has boundary in $Z$ if for every component $\partial_i$ of $\partial S(Y)$ the image of this loop in $\pi_1(X)$ lifts to $Z$.

The immersion of $Y$ into $X$ extends naturally to a map $\sigma$ of $S(Y)$ into the mapping cylinder $C_f$ such that the boundary $\partial S(Y)$ immerses into the natural copy of $Z$. That is, we have the following commutative diagram

$$
\begin{array}{ccc}
Y & \longrightarrow & S(Y) \\
\downarrow & & \downarrow \sigma \\
X & \longrightarrow & C_f \\
\end{array}
$$

where the inclusions $Y \to S(Y)$ and $X \to C_f$ are both deformation retracts.

**Definition 4.2.2.** A folded fatgraph $Y$ over $X$ with boundary in $Z$ is boundary incompressible if every essential loop in $S(Y)$ whose image lifts to $Z$ is already freely homotopic (in $S(Y)$) into $\partial S(Y)$.

It is evident that the condition of being boundary incompressible rules out the existence of a compressing bigon, by Lemma 3.4.3. Our next result strengthens this criterion. To state the result cleanly, we will make use of the fibre product.

Let $p_i : \hat{C}_i \to C_f$ be the covering space of the mapping cylinder $C_f$ corresponding to the conjugacy class of the subgroup $G_i$. The covering space $\hat{C}_i$ can be constructed as follows: if $\hat{X}_i$ is the covering space of $X$ corresponding to the conjugacy class of $G_i$ then the immersion $f_i : Z_i \to X$ lifts to an embedding $\hat{f}_i : Z_i \to \hat{X}_i$, which identifies $Z_i$ with the core of $\hat{X}_i$; $\hat{C}_i$ is the mapping cylinder of $\hat{f}_i$.

**Definition 4.2.3.** The space

$S(Y) \times_{C_f} \hat{C}_i = \{(x, y) \in S(Y) \times \hat{C}_i \mid \sigma(x) = p_i(y)\}$,

is called the fibre product of the maps $\sigma$ and $p_i$. 
It is a standard exercise to check that the projection $S(Y) \times_{C_f} \hat{C}_i \to S(Y)$ is a covering map.

**Proposition 4.2.4.** Let $Y$ be a folded fatgraph with boundary in $Z$. If $S(Y)$ is boundary compressible then there is:

1. a homotopically non-trivial loop $\gamma$ in the boundary of $S(Y)$,
2. an essential, non-boundary-parallel loop $\gamma'$ in $S(Y)$, and
3. a connected component $\hat{S}(Y)$ of one of the fibre products $S(Y) \times_{C_f} \hat{C}_i$,

such that both $\gamma$ and $\gamma'$ lift to $\hat{S}(Y)$.

**Proof.** Suppose $S(Y)$ is boundary compressible, so there is some compressing bigon $B$ as in Lemma 3.4.3 with one boundary arc $\alpha$ proper in $S$. We will give a proof in the case that both endpoints of $\alpha$ lie on the same boundary component $\gamma$; the proof in the other case is similar.

Fix a base point $* \in \gamma \cap \alpha$. We may then take $\sigma(*)$ as a compatible base point in some $Z_i$, so $\pi_1(Z_i, \sigma(*))$ becomes a natural choice of representative for $G_i$ inside its conjugacy class. This in turn defines a base point $\hat{*}$ in the covering space $\hat{C}_i$. Since $\sigma(*) = p(\hat{*})$, the pair $(*, \hat{*})$ defines a point in the fibre product $S(Y) \times_{C_f} \hat{C}_i$. Let $\hat{S}(Y)$ be the component of $S(Y) \times_{C_f} \hat{C}_i$ that contains $(*, \hat{*})$. It follows from standard covering-space theory that

$$\pi_1(\hat{S}(Y), (*, \hat{*})) = \sigma_*^{-1} \pi_1(\hat{C}_i, \hat{*})$$

when thought of as a subgroup of $\pi_1(S(Y), *)$. In particular, $\gamma$ lifts to $\hat{S}(Y)$.

Let $\delta$ be the subarc of $\gamma$ with the same endpoints as $\alpha$ and let $\gamma'' = \delta \cup \alpha$. Since $\delta \subseteq Z_G$ and $\alpha$ bounds one half of a bigon whose other side is in $G$, we see that $\sigma(\gamma'')$ is contained in $G$, and so $\gamma''$ also lifts to $\hat{S}(Y)$. Indeed, the entire subgroup of $\pi_1(S(Y), *)$ generated by $\gamma$ and $\gamma''$ also lifts to $\hat{S}(Y)$.

Because $\gamma''$ is not homotopic into $\gamma$, this subgroup is free of rank two, and so contains elements represented by curves which are essential and not boundary parallel. Let $\gamma'$ be any such curve. \qed

### 5. Random subgroups

**5.1. Random homomorphisms.** Fix integers $k, l \geq 2$ and let $F_k$, $F_l$ be free groups on $k$, $l$ generators respectively.

**Definition 5.1.1.** A random homomorphism of length $n$ is a homomorphism $\phi : F_k \to F_l$ which takes each generator of $F_k$ to a random reduced word in $F_l$ of length $n$, independently and with the uniform distribution.

If we fix $F$ free and $X$ a rose for $F$ (and therefore, implicitly, a free generating set for $F$) then if we pick $k$, it makes sense to define a random $k$-generator subgroup of $F$ of length $n$ to be the image of $F_k \to F$ under a random homomorphism of length $n$. We suppose $G \subseteq F$ is a random $k$-generator subgroup of length $n$.

The construction of a random subgroup chooses for us a generating set for $G$. The core graph $Y_G$ is obtained from a rose $R_G$ for $G$ (with edges corresponding to the given generating set) by writing new edge labels which are reduced words in $F$, and then folding the result.

**Lemma 5.1.2.** Let $\phi : F_k \to F_l$ be a random homomorphism of length $n$. Then with probability $1 - O(e^{-Cn})$ the result of folding the rose $R_G \to Y_G$ is a homotopy
equivalence, and is an isomorphism away from a neighborhood of the vertex of $R_G$ of diameter $O(\log n)$.

Proof. This lemma is simply the observation that two random reduced words in $F_l$ of length $n$ have a common prefix or suffix of length at most $O(\log n)$ with probability $1 - O(e^{-Cn})$.

Definition 5.1.3. Fix a free group $F$ of rank $l$ and a free generating set. A cyclically reduced word $w$ in the generators is $(T, \epsilon)$-pseudorandom if for every reduced word $\sigma$ in $F$ of length $T$, the number of copies of $\sigma$ in $w$ (denoted $C_\sigma(w)$) satisfies

$$1 - \epsilon \leq \frac{C_\sigma(w)}{\text{length}(w)} \cdot (2l)(2l - 1)^{T - 1} \leq 1 + \epsilon$$

In a free group of rank $l$ there are $(2l)(2l - 1)^{T - 1}$ reduced words of length $T$. So informally, a cyclically reduced word is $(T, \epsilon)$-pseudorandom if its distribution of subwords of length $T$ is distributed as in a random word, up to an error of order $\epsilon$.

We also extend the definition of pseudorandom to finite collections of cyclically reduced elements. Moreover, by abuse of notation, if we fix a rose $X$ for $F$, we say that an immersed 1-manifold $\Gamma \to X$ is $(T, \epsilon)$-pseudorandom if the corresponding collection of cyclic words in $F$ is $(T, \epsilon)$-pseudorandom.

Lemma 5.1.4. Fix any positive integer $T$, and $\epsilon > 0$. Then if $\phi : F_k \to F_l$ is a random homomorphism of length $n$, with probability $1 - O(e^{-Cn})$ the cyclically reduced representative of $\phi(g)$ is $(T, \epsilon)$-pseudorandom for every nontrivial $g$ in $F_k$.

Proof. A random word in $F_l$ of length $n$ is $(T, \epsilon)$-pseudorandom with probability $1 - O(e^{-Cn})$. Each generator of $F_k$ gets taken by $\phi$ to a random word in $F_l$ of length $n$, and non-inverse generators get taken to words with at most $O(\log n)$ cancellation, with probability $1 - O(e^{-Cn})$. The proof follows.

5.2. Combinatorial Rigidity. A reduced word in $F$ is represented by an immersed path $\delta : [0, 1] \to X$. A cyclically reduced word in $F$ is represented by an immersed loop $\gamma : S^1 \to X$. Every element of $F$ is uniquely represented by a reduced word, and every conjugacy class is uniquely represented by a cyclically reduced word.

Definition 5.2.1. Let $f : Z \to X$ be an immersion of finite graphs. A loop $\gamma : S^1 \to X$ is combinatorially rigid in $Z$ if it admits a unique lift $\gamma : S^1 \to Z$.

Let $G_1, \ldots, G_n$ be a finite collection of subgroups of $F$, represented by an immersion $f : Z \to X$. A conjugacy class represented by an element $g$ in $F$ is combinatorially rigid in the $G_i$ if the unique geodesic loop $\gamma : S^1 \to X$ that represents it is combinatorially rigid in $Z$. It is fully combinatorially rigid in the $G_i$ if the conjugacy classes of $g$ and all its (nontrivial) powers are combinatorially rigid in the $G_i$.

Note that a loop $\gamma$ which is (fully) combinatorially rigid in the $G_i$ is necessarily conjugate into a unique $G_i$, by definition. In fact, combinatorial rigidity corresponds to a well known algebraic condition.

Definition 5.2.2. Let $G$ be a group. A subgroup $H$ of $G$ is malnormal if $H \cap H^g = 1$ for every $g \in G - H$. More generally, a family of subgroups $\{H_i\}$ is malnormal if $H_i \cap H_j^g = 1$ whenever $i \neq j$ or $g \in G - H_i$. 


Proposition 5.2.3. Let \( \{G_i\} \) be a family of subgroups of \( F \). Then \( \{G_i\} \) is malnormal if and only if for every \( j \), every nontrivial conjugacy class in \( G_j \) is fully combinatorially rigid in the \( G_i \).

Proof. First, suppose that the \( G_i \) are malnormal, and let \( f : Z \to X \) be the corresponding immersion of finite graphs. Fix a base point on the image of \( \gamma \) and let \( g \) be the corresponding element of \( G_j \). Replacing \( g \) by \( g^n \), it suffices to show that \( \gamma \) is combinatorially rigid in \( Z \). Consider a lift \( \gamma_1 : S^1 \to Z_k \subseteq Z \). A second such lift \( \gamma_2 \) contradicts malnormality immediately unless \( j = k \), in which case it corresponds to an element \( h \in F \setminus G_j(g) \) such that \( hgh^{-1} \in G_i \) for some \( i \), which also contradicts malnormality.

Conversely, suppose the \( G_i \) are not malnormal, so that \( g^h \in G_k \) and either \( j \neq k \) or \( h \in F - G_j \). Then the immersed loop \( \gamma : S^1 \to X \) representing the conjugacy class of \( g \) has two distinct lifts to \( Z \), corresponding to the elements \( g \) and \( g^h \).

Lemma 5.2.4. Let \( N \) be fixed and finite, and let \( \phi_i : G = F_k \to F_i \) for \( 1 \leq i \leq N \) be a finite collection of random homomorphisms of length \( n \). Then the family of images \( \{\phi_i(G)\} \) is malnormal with probability \( 1 - O(e^{-Cn}) \). In particular, the same holds for a single random homomorphism.

Proof. Let \( \sigma \) be any reduced word of length \( n/2 \) in \( F_i \). If \( w \) is a random reduced word of length \( n \), the probability that \( w \) contains a copy of \( \sigma \) is \( O(e^{-Cn}) \). Moreover, if \( w \) is a random reduced word of length \( n \) conditioned to contain a copy of \( \sigma \), the probability that it contains more than one copy of \( \sigma \) is \( O(e^{-Cn}) \).

The homomorphism \( \phi_i \) assigns a random word of length \( n \) to each edge of the rose \( R_G \), and with probability \( 1 - O(e^{-Cn}) \), adjacent edges are folded at most \( O(\log n) \) in the corresponding folded graph \( Y_{\phi_i} \). For each edge \( e \) of \( R_G \), let \( w_{e,i} \) be the subword of length \( n/2 \) in \( Y_{\phi_i} \) starting at some fixed location in the interior. Then with probability \( 1 - O(e^{-Cn}) \), for each \( e \), this is the unique copy of \( w_e \) in \( \sqcup_i Y_{\phi_i} \). It follows that every nontrivial loop in \( \sqcup_i Y_{\phi_i} \) is combinatorially rigid, so \( \{\phi_i(G)\} \) is malnormal.

Now let’s suppose that we have chosen a finite collection of elements \( g_i \in G \) whose union is homologically trivial in \( G \) and such that each \( g_i \) is fully combinatorially rigid. As a collection of cyclic words in the generators \( a, b, c \), etc. of \( F \), there are as many as \( A \), as many \( B \) as \( B \) and so on.

Definition 5.2.5. Let \( S(Y) \) be a folded fatgraph whose boundary components \( \partial_i S(Y) \) are conjugate to \( g_i \in G \) cyclically reduced. The \( f \)-vertices on \( \partial S(Y) \) are the vertices corresponding to the (valence > 2) vertices of \( Z \). We say \( S(Y) \) is \( f \)-folded if every \( f \)-vertex maps to a 2-valent vertex of \( Y \), and distinct \( f \)-vertices map to distinct vertices of \( Y \).

Notice that we need the \( g_i \) to be combinatorially rigid in order to unambiguously identify where the \( f \)-vertices lie on each \( \partial_i S(Y) \).

The \( f \)-folded condition just means that the vertices of \( Y \) which have valence at least 3 all correspond to interior points on edges of \( Z_G \), after identifying \( \partial S(Y) \) with its image in \( Z_G \) in the unique manner guaranteed by combinatorial rigidity.

Proposition 5.2.6. Let \( S(Y) \) be \( f \)-folded. Then \( S(Y) \) is boundary incompressible.

Proof. Let \( \gamma_i : S^1 \to Z \) be the geodesic representative of the conjugacy class of \( g_i \in G \) and let \( \tilde{\gamma}_i : S^1 \to \tilde{C}_G \) be the unique lift of \( \gamma_i \). By full combinatorial rigidity,
the union of the lifts of the boundary components of $S(Y)$ to the fibre product

$$S(Y) \times_{C_f} \hat{C}_i \subseteq S(Y) \times \hat{C}_i$$

is equal to the intersection $(S(Y) \times_{C_f} \hat{C}_i) \cap (\partial S(Y) \times \bigcup \hat{C}_i(S^1))$.

Let $\hat{S}(Y)$ be a component of some $S(Y) \times_{C_f} \hat{C}_j$ that contains a lift $\hat{\delta}$ of some $\gamma_i$. Then $\hat{S}(Y)$ deformation retracts to a spine $\hat{Y}$, which is a covering of $Y$.

The $f$-folded condition precisely means that, if $y$ is branch point of $Y$ and $z$ is a branch point of $Z$ that lies on some $\gamma_i$ then $\sigma(x) \neq f(y)$.

The claim now is that the core of $\hat{Y}$ consists of precisely the lift $\hat{\delta}$. Indeed, Stallings showed that the core of the fibre product is contained in the fibre product of the cores [10, Theorem 5.5]. Any branch vertex of the core of $\hat{Y}$ contained in $\hat{\delta}$ maps on the one hand to a branch vertex of $Y$ and, on the other hand, to a branch vertex of $Z_G$ that lies on $\gamma_i$. This contradicts the $f$-folded hypothesis.

We have shown that every component of every $S(Y) \times_{C_f} \hat{C}_j$ that contains a lift of a component of $\partial S(Y)$ has cyclic fundamental group. Therefore, $S(Y)$ is boundary-incompressible by Proposition 12.4.

5.3. The Random Surface Subgroup Theorem. We are now in a position to prove the main result in this section, the Random Surface Subgroup Theorem. The most involved part of the argument is to show that a pseudo-random homologically trivial chain bounds an $f$-folded surface; but actually, this is already proved in [3], in the course of the proof of Thm. 8.9.

**Proposition 5.3.1.** Let $\phi : F_k \to F_l$ be a random homomorphism of length $n$. Then with probability $1 - O(e^{-Cn})$ the image $\phi(F_k)$ is malnormal (so that every nontrivial element is combinatorially rigid) and for every finite collection of nontrivial elements $g_i$ in $F_k$ whose image in $F_l$ is homologically trivial, there is an $f$-folded fatgraph $Y$ with $\partial S(Y)$ equal to the union of the $\phi(g_i)$.

**Proof.** The image is malnormal with the desired probability by Lemma 5.2.4 so it makes sense to talk about $f$-vertices on $\partial S(Y)$. By Lemma 5.1.4 for any fixed $T, \epsilon$, and for any finite collection $g_i$, we can assume each individual $\phi(g_i)$ is $(T, \epsilon)$-pseudorandom, also with the desired probability.

Now, Calegari–Walker [3] Theorem 8.9 (the Random $f$-folded Surface Theorem), prove that for $(T, \epsilon)$ sufficiently big (depending on the rank $l$), for any $(T, \epsilon)$-pseudorandom homologically trivial immersed 1-manifold $\Gamma$ in $X$, and any subset $V$ of vertices in $\Gamma$ so that no two vertices in $V$ are closer than $N$ for some fixed $N \geq T$ (also depending on $l$) there is a folded fatgraph $Y$ with $\partial S(Y) = \Gamma$ and such that every vertex in $V$ maps to a 2-valent vertex of $Y$, with distinct vertices of $V$ mapping to distinct vertices of $Y$. The first step of the construction (in place of 8.3.2 in [3]) is to take for each vertex $v \in V$, an arc $\sigma$ of length 2 in $\Gamma$ containing $v$ as the midpoint, and glue it to a disjoint copy of $\sigma^{-1}$ in $\Gamma$ not containing any point in $V$. By $(T, \epsilon)$-pseudorandomness, and the sparsity of $V$ in $\Gamma$, this pairing can be done, producing what is called in [3] a chain with tags. Then the remainder of the proof of [3] applies verbatim (actually, the proof is even easier than in [3] since the last step 8.3.7 is unnecessary).

In our context, taking $V$ to be the $f$-vertices on the $\phi(g_i)$, we have $N = O(n)$, and $S(Y)$ will be $f$-folded, as claimed. □
Theorem 5.3.2 (Random Surface Subgroup). Let $H := F_1 *_{G} F_2$ or $H := F *_{G}$ be obtained by amalgamating two free groups over random subgroups of rank $k \geq 1$ of length $n$, or by taking an HNN extension over two random subgroups of rank $k \geq 1$ of length $n$. Then $H$ contains a closed surface subgroup with probability $1 - O(e^{-Cn})$.

Proof. Let $g_i$ be any finite collection of nontrivial elements in $G$ which is homologically trivial. For example, we could pick any nontrivial element $g \in G$ and take for our collection the union of $g$ and $g^{-1}$.

By Proposition 5.3.1 for each of the inclusion maps $\phi_j$ of the edge group, the image $\phi_j(g_i)$ bounds an $f$-folded surface. By Proposition 5.2.6 these surfaces are injective and boundary incompressible in their respective factors, so the closed surface obtained by gluing them along their boundary is injective in $H$. □

Remark 5.3.3. We may weaken the hypotheses of Theorem 5.3.2 in some circumstances. If $G \to F$ is an inclusion of free groups, and $\ker : H_1(G) \to H_1(F)$ is nontrivial, [3, Proposition 6.3] says that we can find a surface (group) in $F$ with boundary representing some nontrivial class in the kernel which is an absolute minimizer for the scl norm (i.e. relative 2-dimensional Gromov norm). Such a minimizer is necessarily incompressible and boundary incompressible. So if we build $H = F_1 *_{G} F_2$ (for example) where $G \to F_1$ has $\ker : H_1(G) \to H_1(F_1)$ nontrivial, and $G \to F_2$ is random, then $H$ contains a closed surface subgroup, with probability $1 - O(e^{-Cn})$.

Remark 5.3.4. Notice that our argument gives rise to many surface subgroups; at least one for every homologically trivial collection of conjugacy classes in $G$ (and actually many more than that, since there are many choices in the construction of an $f$-folded surface). Hence the number of surface subgroups of genus $g$ in a random graph of free groups should grow at least like $g^{C_g}$. The homologically trivial collection of conjugacy classes may be recovered from the surface by seeing how it splits in the graph-of-groups structure, so these surfaces are really distinct.

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Appendix A. Graphs of free groups in virtually special groups

In this appendix, we explain why many families of word-hyperbolic groups are known to contain one-ended fundamental groups of graphs of free groups with quasiconvex edge groups.

Recall that a group is special if it is the fundamental group of a compact, non-positively curved, special cube complex in the sense of Haglund and Wise [7]. The reader is referred to that paper for the definition; we will only need the fact that the codimension-one hyperplanes of a special cube complex are embedded. Agol proved that any word-hyperbolic group which is also the fundamental group of a non-positively curved cube complex is virtually special. Hence, we have the following families of examples (among others).
Example A.1 (Random groups). By a theorem of Dahmani–Guirardel–Przytycki [5], a random group (in the density model) is never special. However, at densities less than $1/6$, a random group is cubulated [8] and hence virtually special.

Example A.2 (Small-cancellation groups). All $C'(1/6)$ groups are cubulated [11] and hence virtually special.

Definition A.3. Let $X$ be a (nonpositively curved) special cube complex and let $\Gamma = \pi_1(X)$. A codimension-1 hyperplane subgroup of $\Gamma$ is the image of the fundamental group of a hyperplane under the map induced by inclusion. Because hyperplanes are convex, the induced map is injective. More generally, a codimension-$(k + 1)$ hyperplane subgroup is the intersection of two codimension-$k$ hyperplane subgroups, where a base point is fixed on the intersection of the hyperplanes being considered. Note that, if $X$ is compact, then there are only finitely many non-trivial codimension-$k$ subgroups as $k$ varies.

Remark A.4. Because hyperplanes are embedded, each codimension-$k$ hyperplane subgroup is a graph of groups over the codimension-$(k + 1)$ hyperplane subgroups it contains, with quasiconvex edge groups.

The following is well known to the experts, but as far as we are aware does not appear in the literature. We include it here for completeness.

Theorem A.5. Let $X$ be a compact, non-positively curved cube complex in which every codimension-one hyperplane is embedded, and suppose that $\pi_1(X)$ is word-hyperbolic and one-ended. Then $\pi_1(X)$ has a subgroup $H$ which is one-ended, word-hyperbolic and the fundamental group of a graph of free groups in which the edge groups are quasiconvex.

In particular, the conclusion of the theorem holds for virtually special groups.

To prove the theorem we make use of the following lemma, which is fundamental to the work of Diao and Feighn [6, p. 1837] and goes back to a theorem of Shenitzer [9].

Lemma A.6. Let $G$ be finitely generated and the fundamental group of a graph of groups $\mathcal{S}$ with non-trivial, finitely generated edge groups and suppose that $G$ splits freely. Then $G$ is the fundamental group of a graph of groups $\mathcal{S}'$ in which every edge group of $\mathcal{S}'$ is a finitely generated subgroup of an edge group of $\mathcal{S}$ and some vertex group of $\mathcal{S}'$ splits freely relative to the incident edge groups.

Proof. Let $T$ be the Bass–Serre tree for the graph of groups $\mathcal{S}$ and let $S$ be the Bass–Serre tree for some non-trivial free splitting of $G$. Because $G$ is finitely generated, we may assume that the actions of $G$ on $S$ and $T$ are both cocompact. Consider the diagonal action of $G$ on $S \times T$; the quotient $Q = (S \times T)/G$ naturally has the structure of a complex of groups with fundamental group $G$.

Consider the action of $G_v$, a vertex group of $\mathcal{S}$, on $S$. If $G_v$ fixes a point then the conclusion of the lemma already holds. Therefore, we may assume that $G_v$ does not fix a point. Because $\mathcal{S}$ is finitely generated (which follows from the fact that $G$ and the edge groups are finitely generated), there is a unique minimal, $G_v$-invariant subtree $S_v$, on which $G_v$ acts cocompactly.

Similarly, every edge group $G_e$ of $\mathcal{S}$ acts cocompactly on a unique minimal invariant subtree $S_e \subseteq S$: either $G_e$ fixes a vertex of $S$, which must be unique because
$G_e$ is non-trivial but edge stabilizers of $S$ are trivial, or otherwise $S_e$ exists because $G_e$ is finitely generated.

Now, $Q$ has a compact core $K$, which can be described as follows. Consider the subcomplex $\tilde{K} \subseteq S \times T$ consisting of all pairs $(s, t)$ such that $s \in S_x$ where $x$ is the vertex or edge of $T$ that contains $t$. Then $\tilde{K}$ is a contractible, $G$-invariant subcomplex of $S \times T$ and the quotient $K = \tilde{K}/G$ is the required compact core.

The complex $K$ is a square complex; call the 1-cells that are images of edges of $S$ horizontal and the images of edges $T$ vertical. The edge stabilizers of the action of $G$ on $S$ are realized by horizontal subgraphs contained in the middle of the squares of $K$. Because the edge stabilizers of $S$ are trivial these subgraphs are trees, and a square of $K$ containing a leaf can be collapsed onto three of its sides. Proceeding inductively, we may collapse such a tree to a single point $y$ in a subcomplex $K'$.

Cutting vertically down the middles of the squares of $K'$ gives a new graph of groups $G'$ for $G$ in which the edge groups are subgroups of the edge groups of $G$. The vertical component of the 1-skeleton of $K'$ that contains $y$ is a vertex group of $G'$ that splits freely relative to the incident edge groups, as required.

Combining Lemma A.6 with Grushko’s Theorem, we quickly obtain the following.

**Lemma A.7.** Suppose $G$ is word-hyperbolic and the fundamental group of a graph of free groups in which every edge group is quasiconvex. Then

$$G \cong F_r \ast G'$$

where $F_r$ is a free group and $G'$ does not split freely and is the fundamental group of a graph of free groups in which every edge group is quasiconvex.

**Proof.** By Lemma A.6 and induction (invoking Grushko’s theorem), we have that

$$G \cong F_r \ast G'$$

where $G'$ is the fundamental group of a graph of free groups in which every edge group is a finitely generated subgroup of an edge group of $G$. Necessarily, $G'$ is word-hyperbolic and the edge groups of $G'$ are quasiconvex because free groups are locally quasiconvex.

**Proof of Theorem A.5.** Consider a descending chain of subgroups

$$\Gamma = H_0 \supseteq H_1 \supseteq H_2 \supseteq \ldots \supseteq H_k \supseteq \ldots$$

where $H_k$ is a codimension-$k$ hyperplane subgroup. Let $k$ be maximal such that $H_k$ is non-free. Then $H_k$ is word-hyperbolic, non-free and the fundamental group of a graph of free groups with quasiconvex edge groups. By Lemma A.7 we can write

$$H_k = F_r \ast H$$

where $H$ is as required; note that $H$ is one-ended because $H_k$ is torsion-free and non-free.

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