SOME ASPECTS OF FREE FIELD RESOLUTIONS IN 2D CFT
WITH APPLICATION TO THE QUANTUM DRINFELD-SOKOLOV REDUCTION*

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Abstract
We review some aspects of the free field approach to two-dimensional conformal field theories. Specifically, we discuss the construction of free field resolutions for the integrable highest weight modules of untwisted affine Kac-Moody algebras, and extend the construction to a certain class of admissible highest weight modules. Using these, we construct resolutions of the completely degenerate highest weight modules of $\mathcal{W}$-algebras by means of the quantum Drinfeld-Sokolov reduction. As a corollary we derive character formulae for these degenerate highest weight modules.

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1. Introduction

Free field techniques have been widely used in the study of two-dimensional conformal field theories. Realizations of the chiral algebra by free fields, resolutions which project onto its irreducible representations from the Fock space modules, and chiral vertex operators intertwining between resolutions, form the tools of this approach (see the review [1], and references therein).

In this paper we intend to highlight several points which have not been emphasized in the literature, and refer the reader to [1] for a more complete discussion of the basics. The main theme will be the application to quantum Drinfeld-Sokolov reduction, which allows one to investigate virtually all properties of $\mathcal{W}$-algebras using corresponding properties of Kac-Moody algebras. Our presentation of this application is very much inspired by, in particular, references [2–8] in which most of the results can be found.

The paper is organized as follows. In Section 2 we review the free field realizations of affine Kac-Moody algebras, and their associated screening operators. The construction of intertwining operators between free field Fock spaces, and how these can be used to build a complex of free field Fock spaces yielding a resolution of an irreducible highest weight module, is explained in Section 3. Here we will not restrict ourselves to integrable highest weights – relevant for the corresponding WZW-model – but also treat a class of admissible weights introduced by Kac and Wakimoto [9–11]. This is the class of weights relevant for the quantum Drinfeld-Sokolov reduction since they lead to completely degenerate highest weight modules for the corresponding $\mathcal{W}$-algebra. This will be discussed in Section 4. Finally, in Section 5, we apply the results of Section 4 to derive character formulae for the completely degenerate $\mathcal{W}$-algebra modules, and compare them to the characters obtained in the coset model approach to $\mathcal{W}$-algebras.

For basic notations used throughout this paper the reader is invited to consult Section 1.1 in [1].

2. Free field realizations of affine Kac-Moody algebras

In this section we briefly outline the construction of realizations of the (untwisted) affine Kac-Moody algebra $\widehat{g}$ on free field Fock spaces. The first example of such a realization was constructed by Wakimoto for $\widehat{sl}(2)$ [12]. In general one proceeds as follows [13,14]. Let $G$ be the (complex) group corresponding to $g$ and let $B_-$ be a Borel subgroup. A character $\chi_\Lambda : B_- \to \mathbb{C}^*$ defines a holomorphic line bundle over the flag manifold $B_- \backslash G$. The group $G$, and thus also $g$, act on (local) sections of this line bundle. Upon introducing complex coordinates $z_\alpha, \alpha \in \Delta_+$, on a maximal cell of $B_- \backslash G$ this becomes a realization of $g$ in terms of linear differential operators. Restricting this realization to polynomials in $z_\alpha$.
This realization can be lifted to a realization of the affine Kac-Moody algebra \( \hat{\mathfrak{g}} \) at level \( k \) \cite{13}.

Specifically, introduce a set of bosonic first order fields \((\beta^\alpha(z), \gamma^\alpha(z)), \alpha \in \Delta_+\), of conformal dimension \((1,0)\) (corresponding to \( \frac{\partial}{\partial z^\alpha} \) and \( z_\alpha \), respectively), and a set of rank \( \mathfrak{g} = \ell \) scalar fields \( \phi^i(z) \), with operator product expansions \( \gamma^\alpha(z) \beta^\alpha(w) \sim \delta_{\alpha\alpha'}/(z-w) \) and \( \phi^i(z) \phi^j(w) \sim -\delta^{ij} \ln(z-w) \). Introduce furthermore the Fock spaces \( \mathcal{F}_\Lambda = \mathcal{F}^\Lambda \otimes \mathcal{F}^{\beta\gamma} \) that are freely generated from the vacuum \( |\Lambda\rangle \) by the oscillators \( \beta^\alpha_n, a^i_n \) for \( n < 0 \), \( \gamma^\alpha_n \) for \( n \leq 0 \), and labeled by the momentum zero mode of the scalar field \( p^i|\Lambda\rangle = \alpha_+ \Lambda^i|\Lambda\rangle \). Here \( \alpha_+^{-1} = \sqrt{k + h^\gamma} \), and we will also use the notation \( \alpha_- \), where \( \alpha_+ \alpha_- = -1 \).

The following expressions for the Chevalley generators define a realization of \( \hat{\mathfrak{g}}_k \) on \( \mathcal{F}_\Lambda \) of highest weight \( \Lambda \)

\[
\begin{align*}
    e_i(z) &= \beta^{\alpha_i}(z) + \ldots , \\
    h_i(z) &= -\alpha_- (\alpha_i^\vee \cdot i\partial \phi(z)) + \sum_{\alpha \in \Delta_+} (\alpha, \alpha_i^\vee) : \gamma^\alpha(z) \beta^\alpha(z) : , \\
    f_i(z) &= \alpha_- \gamma^{\alpha_i}(z) (\alpha_i^\vee \cdot i\partial \phi(z)) + \ldots .
\end{align*}
\]

Suppose we introduce a degree by \( \text{deg}(\beta^\alpha) = -1, \text{deg}(\gamma^\alpha) = 1, \text{deg}(\phi^i) = 0 \). Then in \( e_i(z) \) the dots stand for terms of \( \text{deg} \geq 0 \) in terms of \( \beta\gamma \)-fields only. The term of \( \text{deg} = -1 \) is basis independent while the higher degree terms depend on the particular choice of basis. The dots in \( f_i(z) \) stand for terms of \( \text{deg} \leq 1 \) in terms of \( \beta\gamma \)-fields only, and are basis dependent (unfortunately even the terms of highest degree). They are however completely fixed once a particular choice for \( e_i(z) \) has been made.

The Virasoro algebra acts on \( \mathcal{F}_\Lambda \) by means of the Sugawara construction, which in the realization (2.1) takes the form

\[
T(z) = -\frac{1}{2} : \partial \phi(z) \cdot \partial \phi(z) : -\alpha_+ \rho \cdot i\partial \phi(z) - \sum_{\alpha \in \Delta_+} : \beta^\alpha(z) \partial \gamma^\alpha(z) : .
\]

The \( L_0 \)-eigenvalue of the Fock space vacuum \( |\Lambda\rangle \), \textit{i.e.} its conformal dimension, is given by

\[
h_\Lambda = \frac{1}{2} \alpha_+^2 (\Lambda, \Lambda + 2\rho).
\]

Finally, we need operators acting between Fock spaces labeled by \textit{different} weights \( \Lambda \) – the so-called screening operators – which will be used to build operators that intertwine with the action of \( \hat{\mathfrak{g}}_k \). In the finite-dimensional analogue problem they arise from the (left) action of \( \mathfrak{n}_+ \) on \( B_- \setminus G \), giving differential operators which clearly commute with \( \mathfrak{n}_+ \) in the realization. Lifting these to \( \hat{\mathfrak{g}}_k \) gives operators of the form

\[
s_i(z) = - (\beta^{\alpha_i}(z) + \ldots) e^{-i \alpha_+ \alpha_i \cdot \phi(z)}, \quad i = 1, \ldots, \ell ,
\]

\footnote{See [1] for an algebraic equivalent of this construction.}
where the dots stand for terms of deg ≥ 0 in terms of βγ-fields only. The operator products of \( s_i(z) \) with \( e_i(z) \) and \( h_i(z) \) are regular, while

\[
f_i(z)s_j(w) \sim -\delta_{ij} \frac{2(k + h^\vee)}{2(\alpha_i, \alpha_i)} \partial_w \left( \frac{1}{z - w} e^{-i\alpha_i, \alpha_i, \phi(w)} \right). \tag{2.4}
\]

For \( \hat{g}_k = sl(n)_k \), (2.4) can be verified directly, using the explicit realization of the currents \( f_i(z) \) and \( s_i(z) \) [15,14]. In the general case this result, conjectured in [1], can be proved by observing that an algebraic field redefinition reduces the computation to an \( sl(2) \) subalgebra [7].

3. Intertwiners and resolutions

The operator product expansions of \( s_i(z) \) with \( e_i(z) \), \( h_i(z) \) and \( f_i(z) \) discussed in the previous section provide the starting point for constructing intertwiners, which are mappings between Fock spaces that commute with the action of the current algebra. In particular (2.4) suggests we consider the (formal) operators \( Q_{\Lambda,\Lambda'}(C) : F_{\Lambda} \to F_{\Lambda'} \),

\[
Q_{\Lambda,\Lambda'}(C) = \left[ s_{i_1} \ldots s_{i_n} \right] = \int_C dz_1 \ldots dz_n s_{i_1}(z_1) \ldots s_{i_n}(z_n),
\]

where \( \Lambda' = \Lambda - \beta \) (note that \( \beta = \sum_j \alpha_j \) is a sum of positive roots), and \( C \) is a suitably closed multi-contour as discussed below. More precisely, when acting on a vector \( v \in F_{\Lambda} \),

\[
Q_{\Lambda,\Lambda'}(C)v = \sum_{l_1, \ldots, l_n} \int_C dz_1 \ldots dz_n \prod_{k<l}(z_k - z_l)^{\alpha_i^2(\alpha_i, \alpha_i)} \prod_k z_k^{-\alpha_i^2(\Lambda, \alpha_i)}
\times \prod_k z_k^{-l_k-1} : \hat{s}_{i_1,l_1} \ldots \hat{s}_{i_n,l_n} : T_{\Lambda'}^\Lambda v. \tag{3.2}
\]

Here, \( \hat{s}_i \) denotes the screening operator with the scalar zero modes removed and \( \hat{s}_i(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} \). The translation operator \( T_{\Lambda'}^\Lambda \) maps \( |\Lambda\rangle \) to \( |\Lambda'\rangle \). Finally, \( C \) is a contour in \( H_n(M, S) \) where \( M = \{(z_1, \ldots, z_n) \in \mathcal{F}^n \mid z_i \neq 0, z_i \neq z_j (i \neq j)\} \) and \( S \) is the (rank one) local system associated to the multivalued integrand of (3.2) [16]. In other words \( C \) must be a closed multi-contour after we take into account phase factors arising from non-trivial monodromies of the integrand.

To determine whether such a \( C \) exists, it is convenient to introduce new coordinates \((\zeta_1, \ldots, \zeta_n)\) on \( M \) defined by \( \zeta_1 = z_1 \) and \( \zeta_i = z_i/z_1 \) for \( i = 2, \ldots, n \). This gives a decomposition of \( M \) into the product \( \mathcal{F}^* \times M' \), where \( M' = \{ (\zeta_2, \ldots, \zeta_n) \in \mathcal{F}^{n-1} \mid \zeta_i \neq 0, \zeta_i \neq 1, \zeta_i \neq \zeta_j (i \neq j) \} \). The corresponding factorization in homology (the Eilenberg-Zeiler theorem) gives \( H_n(M, S) = H_1(\mathcal{F}^*, S') \otimes H_{n-1}(M', S'') \) (compare [17]). Here \( S' \).
and $S''$ are the induced local systems determined by the change of variables, $z_i \to \zeta_i$, in the integral. Since $H_1(\mathcal{C}^*, S')$ is non-trivial only if $S'$ is constant, which just means that the integrand is single valued in $\zeta_1$, we find that all terms in (3.2) vanish unless $\Lambda$ and $\beta$ satisfy the following condition

$$-\alpha^2_+ ((\Lambda + \rho, \beta) - \frac{1}{2} (\beta, \beta)) \in \mathbb{Z}.$$  \hspace{1cm} (3.3)

To summarize, we have shown that (3.3), with $\beta$ being a sum of positive roots, gives a necessary condition for the operator $Q_{\Lambda, \Lambda'}(\mathcal{C})$ to be a nontrivial intertwiner between two Fock spaces. In fact, this condition also appears to be sufficient. The latter requires a more detailed analysis of the integrals in (3.2), or, equivalently, of the homology classes $H_{n-1}(M', S'')$. For $\hat{sl}(2)_k$ it follows directly from the results in [18,17], while arguments along these lines are given for the general case in [15].

For a given $\Lambda$, all $\beta \in \Delta_+$ satisfying (3.3) can be determined quite easily, after we rewrite (3.3) more concisely in terms of affine roots and weights. Let $l_1, \ldots, l_n$ be integers for which the corresponding residue in (3.2) is non-zero. Since the $L_0$ eigenvalue of $\hat{s}_{i,l}$ equals $-l$, it is natural to associate to $\hat{s}_{i,l}$ an affine root $\hat{\alpha}_{i,l} = l\delta + \alpha_i$, where $\delta$ is the (imaginary) root dual to $-L_0$. Thus we define

$$\hat{\Lambda} = \Lambda + k\Lambda_0, \quad \hat{\rho} = \rho + h^\vee \Lambda_0,$$

$$\hat{\alpha}_{i_k,l_k} = l_k\delta + \alpha_{i_k}, \quad \hat{\beta} = \sum_k \hat{\alpha}_{i_k,l_k}.$$ \hspace{1cm} (3.4)

Clearly, $\hat{\beta}$ (which is an extension of $\beta$ to the affine root lattice $\hat{P}$) lies on the (half-)lattice

$$\hat{\beta} \in \mathbb{Z}_+ \cdot \left( \hat{\Delta}^{(+)}_+ \cup (-\hat{\Delta}^{(-)}_+) \right),$$ \hspace{1cm} (3.5)

where

$$\hat{\Delta}^{(+)}_+ = \{ \hat{\alpha} = n\delta + \alpha \mid \alpha \in \Delta_+, n \geq 0 \},$$

$$\hat{\Delta}^{(-)}_+ = \{ \hat{\alpha} = n\delta - \alpha \mid \alpha \in \Delta_+, n > 0 \}.$$ \hspace{1cm} (3.6)

In this notation (3.3) simply becomes

$$\left( \hat{\Lambda} + \hat{\rho}, \hat{\beta} \right) - \frac{1}{2} (\hat{\beta}, \hat{\beta}) = 0,$$ \hspace{1cm} (3.7)

which we recognize as the Kac-Kazhdan equation [19].

Recall that the Kac-Kazhdan equation (3.7) is the starting point for the analysis of the decomposition structure of Verma modules, i.e. of the singular vector structure and the embedding pattern for the corresponding Verma modules. There, however, $\hat{\beta} \in \mathbb{Z}_+ \cdot \hat{\Delta}_+$. In the present case of Fock space modules, as a consequence of (3.5), some of the arrows
in the embedding diagram will get inverted, but the main structure nevertheless remains the same.

In particular, setting \( \hat{\Lambda}' = \hat{\Lambda} - \hat{\beta} \), equation (3.7) leads to

\[
|\hat{\Lambda}' + \hat{\rho}|^2 = |\hat{\Lambda} + \hat{\rho}|^2 ,
\]

implying

\[
\hat{\Lambda}' = w * \hat{\Lambda} \equiv w(\hat{\Lambda} + \hat{\rho}) - \hat{\rho} ,
\]

for some element \( w \) of the affine Weyl group \( \hat{W} \) of \( \hat{g} \). Projecting this equation onto the weight lattice of \( g \), we identify the set of weights \( \Lambda' \) that satisfy (3.3) with the orbit of \( \Lambda \) under the shifted action of \( \hat{W} \). [Recall that if we represent \( w = t_\gamma w_0 \), where \( w_0 \) is an element of the Weyl group \( W \) of \( g \), \( t_\gamma \) is the translation by a vector \( \gamma \) in the long root lattice of \( g \) and \( \Lambda \) is a weight of level \( k \), then \( w * \Lambda = w_0(\Lambda + \rho) - \rho + (k + h^\vee)\gamma \).]

In view of (3.9) we are now left to consider the problem of determining the existence of an intertwiner of the form (3.2) between two Fock spaces \( F_{w*\Lambda} \) and \( F_{w'*\Lambda} \), where \( w, w' \in \hat{W} \). In analogy with the similar analysis in the case of Verma modules [19], one would like to characterize the solution by some special property of the corresponding elements of \( \hat{W} \). Indeed, (3.5) motivates the following definition of a “twisted length” on \( \hat{W} \) (see e.g. [20,1])

\[
\tilde{l}(w) = |\Phi_w^{(+)}| - |\Phi_w^{(-)}| , \quad \Phi_w^{(\pm)} = \hat{\Delta}_+^{(\pm)} \cap w(\hat{\Delta}_-) .
\]

Further, writing \( w \rightarrow w' \) if there exists an \( \alpha \in \hat{\Delta}_+ \) such that \( w = r_\alpha w' \) and \( \tilde{l}(w) = \tilde{l}(w') + 1 \), we may define a partial ordering (“twisted Bruhat ordering”) by \( w \preceq w' \) iff there exist \( w_1, \ldots, w_k \in \hat{W} \) such that \( w \rightarrow w_1 \rightarrow \ldots \rightarrow w_k \rightarrow w' \). We find that for integrable weights \( \Lambda \) of level \( k \), \( \Lambda \in P^+_k \),

\[
\text{Hom}_{\mathcal{U}(\hat{g})}(F_{w*\Lambda}, F_{w'*\Lambda}) \neq 0 \quad \text{iff} \quad w \preceq w' .
\]

Explicit construction of the intertwiners (3.2) requires a more detailed understanding of the closed multi-contours \( C \). Recently, insight into the structure of \( H_n(M, S) \) has been gained by the observation that the space of (suitable) relative cycles carries a representation of the quantum group \( \mathcal{U}_q(g) \) at \( q = \exp(\pi i \alpha^2_+) \), which is, in fact, isomorphic to a Verma module. The absolute cycles correspond to the singular vectors in this Verma module [21–23]. The “dual” of this statement, which we will briefly discuss below, was proved in [14] (see also [24,25]).

Instead of varying the contour \( C \) one may take a fixed contour and instead vary the analytic continuation of the integrand. For the contour \( C \) we take a set of nested contours
$|z_1| > \ldots > |z_n|$ (for $z_i \neq 1$) where $z_i$ is integrated counterclockwise from 1 to 1 around the origin $z = 0$. The ambiguity in the phase of the integrand is fixed by analytic continuation from the region on the positive real half axis where $0 < z_n < \ldots < z_1$. With these conventions the operators $[s_{i_1} \ldots s_{i_n}]$ and $[s_{\pi(i_1)} \ldots s_{\pi(i_n)}]$ related by a permutation $\pi$ are not necessarily proportional. We will show however that within $[\ldots]$ the $s_i$’s satisfy the Serre relations of $U_q(n)$, thus clarifying, in this “dual” context, that the space of potential intertwiners is isomorphic to a quantum group Verma module.

Observe that upon interchanging two screening operators $s_i$ and $s_j$ one picks up a phase factor $q^{(\alpha_i, \alpha_j)}$. This motivates the following definition of the adjoint operator

$$(\text{ad } s_i) s_j = s_i s_j - q^{(\alpha_i, \alpha_j)} s_j s_i. \quad (3.12)$$

More generally

$$(\text{ad } s_i)x = s_i x - q^{(\beta, \alpha_i)} x s_i, \quad (3.13)$$

if $\beta = \sum \alpha_{i_j}$ is the weight of the string of screening currents $x = s_{i_1} \ldots s_{i_n}$. With these notations the Serre relations for $s_i$ can be written as [26]

$$(\text{ad } s_i)^{1-\alpha_{i_j}} s_j = 0, \quad (3.14)$$

where $\alpha_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$ is the Cartan matrix of $g$. Now imagine writing the integral over the contour $C$ to a sum of integrals where all the variables are located on the unit circle and have a specified ordering. Denote by $I_{i_1 \ldots i_n}$ the integral with ordering $0 < \arg z_1 < \ldots < \arg z_n < 2\pi$ [14]. We claim that for arbitrary $N \geq 1$

$$(\text{ad } s_i)^N s_j = B_N I_{i \ldots ij}, \quad (3.15)$$

where

$$B_N = \prod_{\kappa = 0}^{N-1} \left( \frac{1 - q^{b_{ii} + 2b_{ij}}(1 - q^{(\kappa+1)b_{ii}})}{1 - q^{b_{ii}}} \right), \quad b_{ij} \equiv (\alpha_i, \alpha_j). \quad (3.16)$$

Clearly this implies the Serre relations (3.14).²

Equation (3.16) is proved by induction. One easily checks the assertion for $N = 1$. The induction step goes as follows

$$(\text{ad } s_i) ((\text{ad } s_i)^{N-1} s_j) = s_i ((\text{ad } s_i)^{N-1} s_j) - q^{(N-1)b_{ii} + b_{ij}} ((\text{ad } s_i)^{N-1} s_j) s_i$$

$$= \left( \frac{1 - q^{Nb_{ii}}}{1 - q^{b_{ii}}} \right) B_{N-1} I_{i \ldots ij} + q^{(N-1)b_{ii} + b_{ij}} B_{N-1} I_{i \ldots iji}$$

$$- q^{(N-1)b_{ii} + b_{ij}} \left( B_{N-1} I_{i \ldots iji} + \left( \frac{1 - q^{Nb_{ii}}}{1 - q^{b_{ii}}} \right) B_{N-1} I_{i \ldots ij} \right). \quad (3.17)$$

² We would like to emphasize that the proof holds for arbitrary generalized Cartan matrices.
It is now quite straightforward to determine which of the operators (3.1) actually commute with $\hat{g}_k$. This requires evaluating the commutator

$$[ f_i(z), [ P(s) ]] ,$$

where $P(s)$ is some polynomial in screening operators. Using (2.4) we find that the commutator consists of boundary terms at $\arg z_i = 0$ and $\arg z_i = 2\pi$ only. By carefully keeping track of the phase factors that one picks up by crossing the variables it can be established that the commutator (3.18) vanishes precisely when the polynomial $P(s)$ corresponds to a singular vector in the aforementioned quantum group Verma module. For details we refer to [14]. It is also clear that, in the language of homology with local coefficients, the vanishing of (3.18) is equivalent to the corresponding contour $C$ being closed. We conclude that the set of intertwiners can be determined by analysing the structure of singular vectors in the corresponding quantum group Verma module.

The final step in the construction of the resolution is to combine the set of intertwiners and Fock spaces into a complex. We refer to [14,1] for details. This leads to the following complex (i.e. $d(n)d(n-1) = 0$)

$$\ldots \xrightarrow{d(-2)} \mathcal{F}_\Lambda^{(-1)} \xrightarrow{d(-1)} \mathcal{F}_\Lambda^{(0)} \xrightarrow{d(0)} \mathcal{F}_\Lambda^{(1)} \xrightarrow{d(1)} \ldots$$

(3.19)

where for the case of $\Lambda$ integrable

$$\mathcal{F}_\Lambda^{(n)} = \bigoplus_{\{w \in \hat{W} | l(w) = n\}} \mathcal{F}_{w*\Lambda} .$$

(3.20)

This complex provides a resolution of the irreducible module $L_\Lambda$, i.e.

$$H^d_{(n)}(\mathcal{F}_\Lambda) = \begin{cases} L_\Lambda & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} .$$

(3.21)

The cohomology of (3.19) was computed rigorously for $\hat{sl}(2)_k$ [27,15]. In general case it is believed that a proof may be given by showing, similarly as in the case of finite-dimensional Lie algebras, an isomorphism between (3.19) and the so-called weak resolution of $L_\Lambda$ constructed in [20].

For the purpose of obtaining the completely degenerate highest weight modules of $\mathcal{W}$-algebras using the quantum Drinfeld-Sokolov reduction (Section 4) it will turn out that considering only integrable weights is not sufficient. Rather, we have to consider fractional levels, say

$$k + h^\vee = \frac{p}{p'}, \quad \gcd(p,p') = 1, \quad \gcd(p',r^\vee) = 1, \quad p \geq h^\vee, \quad p' \geq h ,$$

(3.22)
and weights of the form

$$\Lambda = \Lambda^{(+)} - (k + h^\vee)\Lambda^{(-)},$$  \hspace{1cm} (3.23)

with $\Lambda^{(+)} \in P_+^{p-h^\vee}$ and $\Lambda^{(-)} \in P_+^{p'-h^\vee}$. Here, $r^\vee$ is the “dual tier number” of $\hat{g}$. We recall that in the simply laced case $h = h^\vee$, $P_+ = P_+^{\vee}$ and $r^\vee = 1$.

Weights of the form (3.23) are a subset of the class of so-called admissible weights [9]. In fact, for reasons that will become clear in Sections 4 and 5, these weights are such that $\Lambda^{(-)} - (k + h^\vee)\rho^\vee$ is a so-called nondegenerate principal admissible weight [11].

It turns out that the entire discussion of intertwiners between Fock spaces $F_{w\ast\Lambda}$, with $\Lambda$ integrable, can easily be extended to this more general class of admissible weights. For $\Lambda$ of the form (3.23), condition (3.3) simply becomes

$$-\frac{p'}{(p - h^\vee) + h^\vee}((\Lambda^{(+)} + \rho, \beta) - \frac{1}{2}(\beta, \beta)) + (\Lambda^{(-)}, \beta) \in \mathbb{Z}. \hspace{1cm} (3.24)$$

However, since $p$ and $p'$ are relatively prime and $\Lambda^{(-)}$ is integral, this is equivalent to (3.3) with $\Lambda \to \Lambda^{(+)}$ and $k \to p - h^\vee$. Similarly, the vanishing of (3.18) translates into the requirement that $P(s)$ corresponds to a singular vector in the quantum group Verma module of highest weight $\Lambda^{(+)}$. Thus, once more restricting to the intertwiners of the form (3.2), we obtain immediately that

$$\text{Hom}_{U(\hat{g})}(F_{w\ast\Lambda^{(+)}-(k+h^\vee)\Lambda^{(-)}}, F_{w'\ast\Lambda^{(+)}-(k+h^\vee)\Lambda^{(-)}}) \neq 0 \quad \text{iff} \quad w \preceq w'. \hspace{1cm} (3.25)$$

Moreover, using these homomorphisms we can build a complex as in (3.19), but with

$$F^{(n)}_{\Lambda} = \bigoplus_{\{w \in \hat{W} | \hat{l}(w) = n\}} F_{w\ast\Lambda^{(+)}-(k+h^\vee)\Lambda^{(-)}}. \hspace{1cm} (3.26)$$

Since $d^{(i)}$ are precisely the same as in the resolution for the integrable weight $\Lambda^{(+)}$, we have indeed constructed a complex. In fact we expect that this complex provides a Fock space resolution of the irreducible module $L_{\Lambda}$. For $\hat{sl}(2)$ this has been proved in [27]. However, contrary to the integrable case, the corresponding weak resolution for arbitrary admissible representations has not yet been constructed, so it is not clear how a general proof should proceed.

One piece of evidence supporting the validity of (3.21) is that by computing the character of $L_{\Lambda}$, using the Euler-Poincaré principle under the assumption that (3.21) holds, one recovers the character formulae of Kac and Wakimoto [9,10].

In the subsequent sections we therefore assume the validity of (3.21) and examine its consequences.

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\footnote{Moreover, up to Weyl reflections $w \in \hat{W}$, these exhaust the class of nondegenerate principal admissible weights. For simplicity we ignore these Weyl reflected weights. They lead to isomorphic $\mathcal{W}$-algebra modules anyhow.}
4. Quantum Drinfeld-Sokolov reduction

Consider the following 1-dimensional representation of \( \hat{n} \equiv n_+ \otimes \mathcal{O}(t) \)

\[
\chi_{DS}(e^\alpha(z)) = \begin{cases} 
1 & \text{if } (p^\vee, \alpha) = 1, \text{ i.e. if } \alpha \text{ is a simple root} \\
0 & \text{otherwise} 
\end{cases},
\]

(4.1)

The quantum Drinfeld-Sokolov reduction amounts to enforcing the constraint \( e^\alpha(z) \sim \chi_{DS}(e^\alpha(z)) \) through a BRST procedure (see e.g. [4,6]). The BRST operator corresponding to this constraint is given by

\[
Q = Q_0 + Q_1,
\]

(4.2)

where

\[
Q_0 = \oint \frac{dz}{2\pi i} \left( \sum_{\alpha \in \Delta_+} c^\alpha(z)e^\alpha(z) - \frac{1}{2} \sum_{\alpha \beta \gamma \in \Delta_+} f^{\alpha \beta \gamma}(z)c^\alpha(z)c^\beta(z) : \right),
\]

and

\[
Q_1 = -\oint \frac{dz}{2\pi i} \sum_{\alpha \in \Delta_+} (c^\alpha(z)\chi_{DS}(e^\alpha(z))).
\]

They satisfy

\[
Q^2 = Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0.
\]

(4.4)

Let \( C^*(\hat{\mathfrak{g}}_k, \hat{n}) \) denote the completion of \( \mathcal{U}(\hat{\mathfrak{g}}_k) \otimes \mathcal{Cl} \) by infinite series (\( \mathcal{Cl} \) is the Clifford algebra of the fermionic ghost system). This space is graded by ghost number \( (\text{gh}) \). We put \( (\text{gh})(c^\alpha) = 1 = (\text{gh})(b^\alpha) \). The differential \( Q \) acts on \( C^*(\hat{\mathfrak{g}}_k, \hat{n}) \) by (super) commutation and its cohomology, which is again a (graded) Lie algebra, will be denoted by \( H_Q(\hat{\mathfrak{g}}_k, \hat{n}, \chi_{DS}) \). Roughly speaking this is just the operator cohomology of \( Q \) restricted to the space of operators built from the modes of the current algebra. The algebra \( \mathcal{W}_k[\mathfrak{g}] \equiv H_Q^{(0)}(\hat{\mathfrak{g}}_k, \hat{n}, \chi_{DS}) \) is called the \( \mathcal{W} \)-algebra associated to \( \mathfrak{g} \). Indeed, an examination of the spectral sequence associated to the double complex with differentials \( (Q_0, Q_1) \) shows that for generic \( k \) (i.e. \( k \not\in -1 + Q_+ \)) the algebra \( \mathcal{W}_k[\mathfrak{g}] \) can be identified with the commutant of a set of screening charges [6,7], thus making contact with the definition of a \( \mathcal{W} \)-algebra given by Fateev and Lukyanov [2,3].

It has been conjectured that \( H_Q^{(n)}(\hat{\mathfrak{g}}_k, \hat{n}, \chi_{DS}) \simeq 0 \) for \( n \neq 0 \) [6,7].

The Drinfeld-Sokolov reduction procedure clearly at the same time provides the representation spaces for the symmetry algebra. For, by taking the cohomology of \( Q \) on modules \( M \otimes \mathcal{F}^{bc} \), where \( M \) is any \( \hat{\mathfrak{g}}_k \)-module from the category \( \mathcal{O} \) [30,31], we obtain a

\[\text{This is true for simply laced } \mathfrak{g}. \text{ For non-simply laced the algebra } \mathcal{W}[B_n] \text{ of Fateev-Lukyanov, for example, rather corresponds to } \mathcal{W}[B(0, n)] \text{ in the terminology of this paper (see e.g. [28,29]).}\]
functor sending \( \mathfrak{g}_k \)-modules \( M \in \mathcal{O} \) to modules \( H_Q(\mathfrak{n}, \chi_{DS}, M) \) of the “Hecke algebra” \( H_Q(\mathfrak{g}_k, \mathfrak{n}, \chi_{DS}) \). In particular, \( H_Q^{(n)}(\mathfrak{n}, \chi_{DS}, M) \) becomes a \( H_Q^{(0)}(\mathfrak{g}_k, \mathfrak{n}, \chi_{DS}) \)-module for each \( n \in \mathbb{Z} \).

The algebra \( \mathcal{W}_k[\mathfrak{g}] \) will, at least, contain the Virasoro algebra. An explicit representative is given by

\[
T(z) = T^{Sug}(z) + \rho^\vee \cdot \partial h(z) + T^{gh}(z),
\]

where \( T^{Sug} \) is the conventional Sugawara operator, and

\[
T^{gh}(z) = \sum_{\alpha \in \Delta_+} (((\rho^\vee, \alpha) - 1) : b^\alpha \partial c^\alpha : (z) + (\rho^\vee, \alpha) : \partial b^\alpha c^\alpha : (z)).
\]

Of course, the “improvement terms” in (4.5) just amount to requiring that the BRST current \( j(z) = j_0(z) + j_1(z) \) becomes a primary field of conformal dimension one so that \( Q \) and \( T(z) \) commute. The central charge of this Virasoro algebra is

\[
c = \frac{k \dim \mathfrak{g}}{k + h^\vee} - 12k|\rho^\vee|^2 - 2 \sum (6(\rho^\vee, \alpha)^2 - 6(\rho^\vee, \alpha) + 1)
\]

\[
= \ell - 12(\alpha_+ + \alpha_-)\rho^\vee^2,
\]

where we have introduced \( \alpha_+ \alpha_- = -1 \), and \( \alpha_+^{-2} = k + h^\vee \), as before. The conformal dimension of a \( \mathfrak{g}_k \)-module \( M \) with highest weight \( \Lambda \) under (4.5) becomes

\[
h_\Lambda = \frac{1}{2} \alpha_+^2 (\Lambda, \Lambda + 2 \rho) - (\Lambda, \rho^\vee) = \frac{1}{2} (\Lambda \alpha_+ + \Lambda \alpha_- + 2(\alpha_+ \rho + \alpha_- \rho^\vee)).
\]

For simply laced Lie algebras \( \mathfrak{g} \), levels \( k \) as in (3.22) and weights \( \Lambda \) as in (3.23), the equations (4.7) and (4.8) can be recognized as the central charges and conformal dimensions of the completely degenerate highest weight modules of the \( \mathcal{W} \)-algebra minimal models [3,32] (remember \( \rho = \rho^\vee \), and identify \( \alpha_0 = \alpha_+ + \alpha_- \)).\(^5\) This, ultimately, is the explanation of our choice (3.23). Note that (4.8) can be written as \( \frac{1}{2} \alpha_+^2 (\Lambda', \Lambda' + 2 \rho) \) where \( \Lambda' = \Lambda - (k + h^\vee)\rho^\vee \). Thus (4.8) can be interpreted as the (untwisted) conformal dimension of the nondegenerate principal admissible weight \( \Lambda' \).

Now suppose we take \( M = L_\Lambda \). From the above it is reasonable to expect that, for weights \( \Lambda \) as in (3.23), we have \( H_Q^{(n)}(\mathfrak{n}, \chi_{DS}, L_\Lambda) \simeq 0 \) for \( n \neq 0 \) while \( H_Q^{(0)}(\mathfrak{n}, \chi_{DS}, L_\Lambda) \simeq L_\Lambda^W \), where \( L_\Lambda^W \) is an irreducible highest weight module of the algebra \( \mathcal{W}_k[\mathfrak{g}] \).\(^6\) We will provide some evidence in support of this conjecture in Section 5.

\(^5\) In the case of \( \mathcal{W} \)-algebras it is conventional to parametrize highest weight modules through \( \Lambda \alpha_+ = \Lambda^{(+)} \alpha_+ + \Lambda^{(-)} \alpha_- \).

\(^6\) For weights \( \Lambda \) not of the form (3.23) this is not necessarily true. One can show, for instance, that for dominant integer weights \( H_Q^{(n)}(\mathfrak{n}, \chi_{DS}, L_\Lambda) \simeq 0 \) for all \( n \in \mathbb{Z} \) [15,6]. Also, the fact that \( H_Q^{(0)}(\mathfrak{n}, \chi_{DS}, L_\Lambda) \) is an irreducible \( \mathcal{W} \)-algebra highest weight module is not a priori obvious but nevertheless appears to be true for weights of the form (3.23).
Assuming then that \( H_Q^{(n)}(\hat{n}, \chi_{DS}, L_\Lambda) \simeq L_\Lambda^\mathcal{W} \delta_{n,0} \), there exists a natural candidate for a resolution of \( L_\Lambda^\mathcal{W} \), obtained simply by applying the functor \( H_Q(\hat{n}, \chi_{DS}, \cdot) \) to a resolution of the \( \mathfrak{g}_k \)-module \( L_\Lambda \). So suppose, following Section 3, that we are given a resolution of the \( \mathfrak{g}_k \)-module \( L_\Lambda \) where the differential \( d \) is constructed out of the screening operators \( s_i(z) \) and the terms are free field Fock spaces \( \mathcal{F}_A^{(n)} = \mathcal{F}_A^{(n)\phi} \otimes \mathcal{F}_A^{\beta\gamma} \). Then, using the fact that \([Q, d] = 0\), as well as the result (see e.g. [4–6])

\[
H_Q^{(n)}(\hat{n}, \chi_{DS}, \mathcal{F}_A^{\phi} \otimes \mathcal{F}_A^{\beta\gamma}) \simeq \delta_{n,0} \mathcal{F}_A^{\phi},
\]

we have

\[
L_\Lambda^\mathcal{W} \simeq H_Q(\hat{n}, \chi_{DS}, L_\Lambda) \simeq H_Q(\hat{n}, \chi_{DS}, H_d(\mathcal{F}_A^{(\phi)} \otimes \mathcal{F}_A^{\beta\gamma})) \simeq H_d(H_Q(\hat{n}, \chi_{DS}, \mathcal{F}_A^{(\phi)} \otimes \mathcal{F}_A^{\beta\gamma})) \simeq H_d(\mathcal{F}_A^{(\phi)}). \tag{4.10}
\]

That is, we have constructed a resolution of the \( \mathcal{W} \)-algebra module \( L_\Lambda^\mathcal{W} \) in terms of free field Fock spaces \( \mathcal{F}_A^{(n)\phi} \).

In fact, one can show that in \( Q \)-cohomology [4,5]

\[
s_i(z) \sim : e^{-i\alpha_+ \cdot \phi(z)} : \equiv \tilde{s}_i(z), \tag{4.11}
\]

as well as

\[
T(z) \sim -\frac{1}{2} : \partial\phi(z) \cdot \partial\phi(z) : - (\alpha_+ \rho + \alpha_- \rho^\vee) \cdot i\partial^2 \phi(z). \tag{4.12}
\]

Equation (4.11) implies that the differential \( d \) of the resolution (4.10) can be replaced by the differential \( \tilde{d} \) obtained from \( d \) by replacing the screening operators \( s_i(z) \) by their \( Q \)-cohomologous counterparts \( \tilde{s}_i(z) \) in terms of scalar fields \( \phi(z) \) only. Furthermore, since the \( \mathcal{W} \)-algebra is exactly the commutant of the screening charges \( \hat{f} \tilde{s}_i(z) \) as remarked before, this confirms that the \( \mathcal{W} \)-algebra commutes with \( \tilde{d} \) and thus acts on all the terms in the resolution. For \( \hat{g} = \mathfrak{sl}(2) \) the differential \( \tilde{d} \) can be recognized as the differential of Felder’s complex that provides a resolution of the Virasoro degenerate highest weight modules [33]. This observation constitutes a proof of (4.10) for \( \hat{\mathfrak{sl}}(2) \), as has been discussed in [4].

Equation (4.12) gives an explicit expression for the Virasoro part of this commutant and confirms equations (4.7) and (4.8). For \( g = \mathfrak{a}_n \) one can find a generating function for the other generators \( W_k(z) \) of the \( \mathcal{W} \)-algebra by means of the so-called quantum Miura transformation [2,3]

\[
\sum_{k=0}^{n+1} W_k(z)(\alpha_0 \partial)^{n+1-k} = : (\alpha_0 \partial - \epsilon_1 \cdot i\partial \phi(z))(\alpha_0 \partial - \epsilon_2 \cdot i\partial \phi(z)) \cdots (\alpha_0 \partial - \epsilon_{n+1} \cdot i\partial \phi(z)) : \tag{4.13}
\]

where \( \{\epsilon_i, i = 1, \ldots, n+1\} \) is the set of weights of the vector representation of \( \mathfrak{a}_n \), normalized according to \( \epsilon_i \cdot \epsilon_j = \delta_{ij} - \frac{1}{n+1} \), and such that the simple roots of \( \mathfrak{a}_n \) are given by \( \alpha_i = \epsilon_i - \epsilon_{i+1} \).
5. Some character formulae

In this section we apply the Euler-Poincaré principle to the resolution of Section 4, to derive a character formula for the (completely degenerate) irreducible highest weight modules \( L^W_\Lambda \). We compare the result to the formulae obtained from the coset approach to \( \mathcal{W} \)-algebras [32]. For notational simplicity we restrict ourselves to simply laced Lie algebras \( g \).

So, we take \( \Lambda = \Lambda^{(+)} - (k + h^\vee)\Lambda^{(-)} \) as in (3.23). Recall

\[
\frac{c}{24} = \frac{1}{24}(\ell - 12a_0^2|\rho|^2) = \frac{\ell}{24} - \frac{(p - p')^2}{2pp'}|\rho|^2, \tag{5.1}
\]

\[
h_{w*\Lambda^{(+)} - (k + h^\vee)\Lambda^{(-)}} - \frac{c}{24} = -\frac{\ell}{24} + \frac{1}{2pp'}|p'w(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho)|^2. \tag{5.2}
\]

Under the assumption \( H_Q(\tilde{\mathfrak{g}}, \chi_{DS}, L_\Lambda) \simeq L^W_\Lambda \) the character \( \text{ch}_{L^W_\Lambda} \) of \( L^W_\Lambda \) can be calculated using (5.2) and the Euler-Poincaré principle (as usual \( q = \exp(2\pi i\tau) \)) \(^7\)

\[
\text{ch}_{L^W_\Lambda}(\tau) = \text{Tr}_{L^W_\Lambda} q^{L_0-c/24} = \sum_{n \in \mathbb{Z}} (-1)^n \text{Tr}_{H_Q^{(n)}(L_\Lambda)} q^{L_0-c/24} = \sum_{n \in \mathbb{Z}} (-1)^n \text{Tr}_{X^{(n)}_\Lambda} q^{L_0-c/24} = \sum_{n \in \mathbb{Z}} (-1)^n \text{Tr}_{X^{(n)}_\Lambda} q^{L_0-c/24} = \frac{1}{\eta(\tau)} \sum_{w \in \tilde{\mathcal{W}}} \epsilon(w) q^{2pp'|p'w(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho)|^2}. \tag{5.3}
\]

We now briefly compare the above to the results that follow from the coset approach to \( \mathcal{W} \)-algebras. We recall that for \( \Lambda \in P_+^1 \) we have a \( \mathcal{W}[g] \) algebra realized on \( L_\Lambda \otimes M \) for any \( \tilde{\mathfrak{g}} \)-module \( M \) from the category \( \mathcal{O} \) (see [32] for \( \mathcal{W}[A_2] \)). The generators of \( \mathcal{W}[g] \) can be expressed in terms of the generators of \( \tilde{\mathfrak{g}}_1 \oplus \tilde{\mathfrak{g}}_k \). This \( \mathcal{W}[g] \)-algebra commutes with the diagonal action of \( \tilde{\mathfrak{g}}_{k+1} \).

In particular, taking \( M = L_\Lambda' \) with principal admissible weight \( \Lambda' = \Lambda^{(+)} - (u - 1)(k + h^\vee)\Lambda_0 \), \( \Lambda^{(+)} \in P_+^{p-h^\vee} \), where we have parametrized \( k + h^\vee = p/(p' - p) = p/u \) leads to the following branching function for the occurrence of \( L^{\Lambda''}_\Lambda \), \( \Lambda'' = \Lambda^{(-)} - (u - 1)(k + h^\vee + 1)\Lambda_0 \), \( \Lambda^{(-)} \in P_+^{p'-h^\vee} \) in the decomposition of \( L_\Lambda \otimes L_{\Lambda'} \) under the diagonal \( \tilde{\mathfrak{g}}_{k+1} \) action \([9,11]\)

\[
b_{\Lambda^{''}}^{\Lambda^{''}} = \frac{1}{\eta(\tau)} \sum_{w \in \tilde{\mathcal{W}}} \epsilon(w) q^{2pp'|p'w(\Lambda^{(+)} + \rho) - p(\Lambda^{(-)} + \rho)|^2}. \tag{5.4}
\]

\(^7\) For this character to be nonvanishing it is essential that the admissible weight \( \Lambda' \equiv \Lambda - (k + h^\vee)\rho \) be nondegenerate, \( i.e. \) that there do not exist roots \( \alpha \in \Delta_+ \) such that \( (\Lambda', \alpha) \in \mathbb{Z}_+ \).
This formula is in agreement with (5.3). For the unitary case $p' = p + 1$ formula (5.4) was derived in [34] from a free field resolution for coset models.

The agreement of (5.3) and (5.4) can be considered not only as an indication of the correctness of the assumption (4.10), but at the same time as an indication that the $W$-algebras derived from the coset $\hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_{k-h^\vee}/\hat{\mathfrak{g}}_{k-h^\vee+1}$ and the quantum Drinfeld-Sokolov reduction of $\hat{\mathfrak{g}}_k$ are, in fact, isomorphic.

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7. References

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