Trace Formulae of Characteristic Polynomial and Cayley-Hamilton’s Theorem, and Applications to Chiral Perturbation Theory and General Relativity

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By using combinatorics, we give a new proof for the recurrence relations of the characteristic polynomial coefficients, and then we obtain an explicit expression for the generic term of the coefficient sequence, which yields the trace formulae of the Cayley-Hamilton’s theorem with all coefficients explicitly given, and which implies a byproduct, a complete expression for the determinant of any finite-dimensional matrix in terms of the traces of its successive powers. And we discuss some of their applications to chiral perturbation theory and general relativity.

PACS numbers: 02.10.Ud, 02.10.Ox, 12.39.Fe, 04.20.-q.

Keywords: characteristic polynomial coefficients, Cayley-Hamilton’s theorem, chiral perturbation theory, general relativity.

I. INTRODUCTION

There is a famous theorem named in honor of Arthur Cayley and William Hamilton in linear algebra, which asserts that any $n \times n$ matrix $A$ is a solution of its associated characteristic polynomial $\chi_A$. In the popular $SU(2) \times SU(2)$ and $SU(3) \times SU(3)$ chiral perturbation theories, the Cayley-Hamilton theorem has been used to eliminate redundant terms and keep only independent pieces of the chiral Lagrangian. The trace formulae of the Cayley-Hamilton theorem for $2 \times 2$ and $3 \times 3$ matrices are respectively

\begin{equation}
A^2 - \text{tr}(A)A + \frac{1}{2}[\text{tr}^2(A) - \text{tr}(A^2)] = 0 ,
\end{equation}

\begin{equation}
A^3 - \text{tr}(A)A^2 + \frac{1}{2}[\text{tr}^2(A) - \text{tr}(A^2)]A - \frac{1}{6}[\text{tr}^3(A) - 3\text{tr}(A^2)\text{tr}(A) + 2\text{tr}(A^3)] = 0 .
\end{equation}

And a recursive algorithm to compute the characteristic polynomial coefficients of a generic $n \times n$ matrix as functions of the traces of its successive powers was given by Silva nearly ten years ago. The work of Ref. obtained the recurrence relations for the coefficient sequence by using an improved version of the remainder theorem and some additional results.

In this paper, with the knowledge of combinatorics, we give a new proof for these recurrence relations, and then we further obtain an explicit formula for the generic term of the coefficient sequence, which yields the trace formulae of Cayley-Hamilton’s theorem with all coefficients explicitly given, and which implies a byproduct, a complete expression for the determinant of any finite-dimensional matrix in terms of traces of its successive powers. We also discuss some applications to chiral perturbation theory and general relativity.

This paper is organized as follows. In Sec. II by using combinatorics, a new proof for the recurrence relations of the characteristic polynomial coefficients is given and the explicit expression for the generic term of the coefficient sequence is obtained, which improves the results of previous works. In Sec. III a detailed discussion of the applications of the Cayley-Hamilton theorem to $n$-flavor chiral perturbation theories up to $n = 5$ is presented. In Sec. IV as a further application, the determinant of the metric tensor in 10 dimensional spacetime is explicitly expressed by the traces of its successive powers. Sec. V is devoted to conclusions. A check for the Cayley-Hamilton relations using an alternative approach is presented in App. while a PASCAL code for the computation of the generic term of the coefficient sequence is given in App.

II. TRACE FORMULAE

Let $A$ be any complex $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ (possibly complex and identical). Then its trace is $\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, and the characteristic polynomial is $\chi_A(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$, where $I$ is the $n \times n$ unit matrix. And $A$ being similar to a Jordan normal form implies that $\text{tr}(A^m) = \lambda_1^m + \lambda_2^m + \cdots + \lambda_n^m$ for any positive integer $m$. Following the notation of Ref., the traces are denoted by $T_m \equiv \text{tr}(A^m)$ here and henceforth. Now we are ready to give a new proof for the theorem relating the coefficients of $\chi_A(\lambda)$ to the traces of $A^m$ ($m = 1, 2, \cdots, n$).
Theorem 1 If $\chi_\Delta(\lambda) = \lambda^n + D_1 \lambda^{n-1} + \cdots + D_{n-1} \lambda + D_n$ is the characteristic polynomial of the complex $n \times n$ matrix $A$ and $T_m = \text{tr}(A^m)$, then
\begin{equation}
md_m + D_{m-1}T_1 + D_{m-2}T_2 + \cdots + D_1T_{m-1} + T_m = 0, \quad m = 1, 2, \ldots, n.
\end{equation}

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the $n$ eigenvalues of the matrix $A$. Then $\chi_\Delta(\lambda) = \lambda^n + D_1 \lambda^{n-1} + \cdots + D_{n-1} \lambda + D_n = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$ yields
\begin{align*}
D_m &= (-)^m \sigma_m, \quad \sigma_m \equiv \sum_{i_1 < i_2 < \cdots < i_m} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_m}, \quad m = 1, 2, \ldots, n.
\end{align*}

Multiplying $\sigma_{m-1}$ by $(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$, we have
\begin{align*}
\sigma_{m-1}T_1 &= (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \cdot \sum_{i_1 < j_2 < \cdots < j_{m-1}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-1}} \\
&= \sum_{i=1}^n \lambda_i^2 \cdot \sum_{j_1 < j_2 < \cdots < j_{m-2}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-2}} + \sum_{i=1}^n \lambda_i \sum_{j_1 < j_2 < \cdots < j_{m-1}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-1}}.
\end{align*}

Now let us show that the second term of the last line of the above equation contains $m$ times of $\sigma_m$. For each specific term of $\sigma_m$, say $\lambda_1 \lambda_2 \cdots \lambda_m$, it is included $m$ times in the second term of the last line of the above equation, i.e., it is contained by the $m$ terms as follows
\begin{align*}
\lambda_{i_1} \cdot \sum_{j_1 < j_2 < \cdots < j_{m-1}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-1}}, \quad i = 1, 2, \ldots, m.
\end{align*}

Thus,
\begin{align*}
\sigma_{m-1}T_1 &= \sum_{i=1}^n \lambda_i^2 \cdot \sum_{j_1 < j_2 < \cdots < j_{m-2}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-2}} + m\sigma_m.
\end{align*}

Next, multiplying $\sigma_{m-2}$ by $(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2)$ gives
\begin{align*}
\sigma_{m-2}T_2 &= \sum_{i=1}^n \lambda_i^3 \cdot \sum_{j_1 < j_2 < \cdots < j_{m-3}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-3}} + \sum_{i=1}^n \lambda_i \lambda_i \sum_{j_1 < j_2 < \cdots < j_{m-2}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-2}}.
\end{align*}

Likewise,
\begin{align*}
\sigma_{m-3}T_3 &= \sum_{i=1}^n \lambda_i^4 \cdot \sum_{j_1 < j_2 < \cdots < j_{m-4}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-4}} + \sum_{i=1}^n \lambda_i^3 \cdot \sum_{j_1 < j_2 < \cdots < j_{m-3}} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_{m-3}}.
\end{align*}

\begin{align*}
&\vdots
\end{align*}

\begin{align*}
\sigma_1T_{m-1} &= \sum_{i=1}^n \lambda_i^m + \sum_{i=1}^n \lambda_i^{m-1} \sum_{j \neq i} \lambda_j = T_m + \sum_{i=1}^n \lambda_i^{m-1} \sum_{j \neq i} \lambda_j.
\end{align*}

Combining the above equations yields
\begin{align*}
\sigma_{m-1}T_1 - \sigma_{m-2}T_2 + \sigma_{m-3}T_3 - \sigma_{m-4}T_4 + \cdots + (-)^m \sigma_1T_{m-1} = m\sigma_m + (-)^m T_m,
\end{align*}
which results in
\begin{align*}
md_m + D_{m-1}T_1 + D_{m-2}T_2 + \cdots + D_1T_{m-1} + T_m = 0.
\end{align*}
From Theorem 1 the characteristic polynomials for $n \times n$ matrices can be recursively obtained as follows:

\[
\begin{align*}
n = 1, & \quad \chi_A(\lambda) = \lambda - T_1, \\
n = 2, & \quad \chi_A(\lambda) = \lambda^2 - T_1 \lambda + \frac{1}{2}(T_1^2 - T_2), \\
n = 3, & \quad \chi_A(\lambda) = \lambda^3 - T_1 \lambda^2 + \frac{1}{2}(T_1^2 - T_2) \lambda - \frac{1}{6}(T_1^3 - 3T_1 T_2 + 2T_3), \\
\ldots & 
\end{align*}
\]

It is interesting to note that the characteristic polynomial coefficients $D_m \ (1 \leq m \leq n)$ as polynomial functions of the traces $T_k \ (k = 1, 2, \ldots, n)$ are formally unchanged when $n$ increases, though $T_k$ are obviously different for distinct $n$. As a consequence, we can regard $D_m \ (m = 1, 2, \ldots)$ as an infinite sequence. Theorem 1 gives

\[
\begin{align*}
D_1 &= -T_1, \\
2D_2 &= -(T_1 D_1 + T_2), \\
3D_3 &= -(T_1 D_2 + T_2 D_1 + T_3), \\
\ldots & 
\end{align*}
\]

which leads to

\[
\begin{align*}
D_1 + 2x D_2 + 3x^2 D_3 + \ldots \\
&= -T_1 - x(T_1 D_1 + T_2) - x^2(T_1 D_2 + T_2 D_1 + T_3) - \ldots \\
&= -T_1(1 + x D_1 + x^2 D_2 + \cdots) + x T_2(1 + x D_1 + x^2 D_2 + \cdots) + \cdots .
\end{align*}
\]

If we define the generating functions $f(x)$ and $g(x)$ for the infinite sequences $\{D_n\}$ and $\{T_{k+1}\}$ respectively as follows

\[
f(x) \equiv \sum_{n=0}^{\infty} x^n D_n \quad (\text{with } D_0 \equiv 1), \quad g(x) \equiv \sum_{k=0}^{\infty} x^k T_{k+1},
\]

then

\[
f(0) = 1, \quad D_n = \frac{1}{n!} \frac{d^n}{dx^n} f(x) \Big|_{x=0}, \quad T_n = \frac{1}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} g(x) \Big|_{x=0}, \quad (n = 1, 2, \ldots)
\]

and Eq. (8) implies

\[
\frac{d}{dx} f(x) = -g(x) f(x),
\]

or

\[
f(x) = \exp \left[ - \int_0^x g(t) dt \right],
\]

which gives

\[
f(x) = \exp \left[ - \sum_{n=1}^{\infty} \frac{T_n}{n} x^n \right] = \prod_{n=1}^{\infty} \exp \left[ - \frac{T_n}{n} x^n \right] = \prod_{n=1}^{\infty} \prod_{p_n=0}^{\infty} \frac{1}{p_n!} \left[ - \frac{T_n}{n} x^n \right]^{p_n}
\]

\[
= \sum_{n=0}^{\infty} x^n \cdot \sum_{(p_1, p_2, \ldots, p_n) \in S_n} \prod_{m=1}^{n} \frac{1}{p_m!} \left[ - \frac{T_m}{m} \right]^{p_m},
\]

where the set $S_n$ is defined by all nonnegative-integer solutions $\{(p_1, p_2, \ldots, p_n)\}$ of the equation $p_1 + 2p_2 + \cdots + np_n = n$. Comparing Eq. (11) with Eq. (7), we obtain the expression for a generic term of the sequence $\{D_n\}$ as follows

\[
D_n = \sum_{(p_1, p_2, \ldots, p_n) \in S_n} \prod_{m=1}^{n} \frac{1}{p_m!} \left[ - \frac{T_m}{m} \right]^{p_m}.
\]
Example 1. Consider the computation of $D_4$. All nonnegative integer solutions $(p_1, p_2, p_3, p_4)$ for the equation $p_1 + 2p_2 + 3p_3 + 4p_4 = 4$ and their corresponding terms are shown in Table I. Adding up all these terms gives

$$D_4 = -\frac{T_4}{4} + \frac{T_1 T_3}{3} - \frac{T_2^2 T_2}{4} + \frac{T_2^2}{8} + \frac{T_1^4}{24}.$$  \hfill (13)

| \(p_1\) | \(p_2\) | \(p_3\) | \(p_4\) | corresponding terms |
|-----|-----|-----|-----|-------------------|
| 0   | 0   | 0   | 1   | $-\frac{T_4}{4}$ |
| 1   | 0   | 1   | 0   | $(-T_1)(-\frac{T_2}{2})$ |
| 2   | 1   | 0   | 0   | $\frac{1}{2}(-T_1)^2(-\frac{T_2}{2})$ |
| 0   | 2   | 0   | 0   | $\frac{1}{2}(-T_1)^2$ |
| 4   | 0   | 0   | 0   | $\frac{1}{2}(-T_1)^4$ |

TABLE I: All nonnegative integer solutions $(p_1, p_2, p_3, p_4)$ for the equation $p_1 + 2p_2 + 3p_3 + 4p_4 = 4$. For each solution, there is a corresponding term $\frac{1}{p_1}(\frac{1}{p_2}(-T_1)^p_1 \frac{1}{p_3}(-\frac{T_2}{2})^p_2 \frac{1}{p_4}(-\frac{3}{4})^p_3 \frac{1}{p_4}(-\frac{T_4}{4})^p_4)$. Summing of all the corresponding terms yields $D_4$.

Now we state without proof the well-known theorem in linear algebra:

**Theorem 2 (Cayley-Hamilton)** Any $n \times n$ matrix $\mathbf{A}$ obeys its own characteristic equation, that is, $\chi_\mathbf{A}(\mathbf{A}) = 0$.

Combining Eq. (12) with Theorems I and 2, we obtain the following improved results:

**Proposition 1** For any complex $n \times n$ matrix $\mathbf{A}$, its characteristic polynomial is $\chi_\mathbf{A}(\lambda) = \lambda^n + D_1 \lambda^{n-1} + \cdots + D_{n-1} \lambda + D_n$, and $\chi_\mathbf{A}(\mathbf{A}) = \mathbf{A}^n + D_1 \mathbf{A}^{n-1} + \cdots + D_{n-1} \mathbf{A} + D_n = 0$ where the coefficients are given by

$$D_k = \sum_{(p_1, p_2, \ldots, p_k) \in \mathcal{S}_k} \prod_{m=1}^{k} \frac{1}{p_m!} \left[ -\frac{T_m}{m} \right]^{p_m}, \quad (k = 1, 2, \ldots, n)$$  \hfill (14)

with $T_m$ standing for $\text{tr}(\mathbf{A}^m)$ and $\mathcal{S}_k$ being a set including all nonnegative-integer solutions \{(p_1, p_2, \ldots, p_k)\} of the equation $p_1 + 2p_2 + \cdots + kp_k = k$.

Note that the determinant of a matrix $\mathbf{A}$ is the product of all the eigenvalues $\lambda_m$ ($m = 1, 2, \ldots, n$), that is, $\det(\mathbf{A}) = \sigma_n = \prod_{m=1}^{n} \lambda_m$. Thus, Proposition 1 implies the following byproduct results:

**Proposition 2** For any complex $n \times n$ matrix $\mathbf{A}$, its determinant is given by

$$\det(\mathbf{A}) = (-)^n D_n = (-)^n \sum_{(p_1, p_2, \ldots, p_n) \in \mathcal{S}_n} \prod_{m=1}^{n} \frac{1}{p_m!} \left[ -\frac{T_m}{m} \right]^{p_m},$$  \hfill (15)

with $T_m$ standing for $\text{tr}(\mathbf{A}^m)$ and $\mathcal{S}_n$ being a set including all nonnegative-integer solutions \{(p_1, p_2, \ldots, p_n)\} of the equation $p_1 + 2p_2 + \cdots + np_n = n$.

### III. APPLICATIONS TO CHIRAL PERTURBATION THEORY

The trace formulae of the Cayley-Hamilton theorem can be used to build relations between traces of finite dimensional matrices. Let us study, case by case, the trace relations of $n \times n$ matrices for $n = 2, 3, 4, \text{ and } 5$. In the following, $\langle \mathbf{A} \rangle$ is employed to denote the trace of a matrix $\mathbf{A}$.

1. The $n = 2$ case

For any $2 \times 2$ matrix $\mathbf{A}$, Cayley-Hamilton theorem reads

$$0 = \chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{A}^2 - \langle \mathbf{A} \rangle \mathbf{A} + \frac{1}{2}(\langle \mathbf{A} \mathbf{A} \rangle - \langle \mathbf{A} \rangle^2)$$  \hfill (16)
Multiplying Eq. (10) by $A$ and taking the trace results in

$$0 = F(A) \equiv \langle A^3 \rangle - \frac{3}{2} \langle A^2 \rangle + \frac{1}{2} \langle A \rangle^3. \tag{17}$$

Substituting $A = \lambda_1 A_1 + \lambda_2 A_2$, where $\lambda_{1,2}$ are arbitrary parameters, into Eq. (17) one finds

$$0 = F(\lambda_1 A_1 + \lambda_2 A_2)
= \lambda_1^2 F(A_1) + \lambda_2^2 F(A_2) + \lambda_1^2 \lambda_2 F_{12}(A_1, A_2) + \lambda_2^2 \lambda_1 F_{21}(A_1, A_2)
= \lambda_1^2 \lambda_2 F_{12}(A_1, A_2) + \lambda_2^2 \lambda_1 F_{21}(A_1, A_2). \tag{18}$$

Now, substituting $A = \lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3$ into Eq. (17) leads to

$$0 = F(\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 A_3)
= \sum_{i=1}^3 \lambda_i^2 F(A_i) + \sum_{i<j} \lambda_i^2 \lambda_j F_{ij}(A_i, A_j) + \lambda_1 \lambda_2 \lambda_3 F_{123}(A_1, A_2, A_3)
= \lambda_1 \lambda_2 \lambda_3 F_{123}(A_1, A_2, A_3), \tag{19}$$

where the last equality comes from Eqs. (17) and (15). Thus, $F_{123}(A_1, A_2, A_3) = 0$, that is,

$$0 = \sum_{\text{2 perm}} \langle A_1 A_2 A_3 \rangle - \sum_{\text{3 perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle + \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle. \tag{20}$$

Likewise, multiplying Eq. (10) by $A^2$ and subsequently taking the trace gives

$$0 = \langle A^4 \rangle - \langle A^3 \rangle \langle A \rangle - \frac{1}{2} \langle A^2 \rangle^2 + \frac{1}{2} \langle A \rangle^2. \tag{21}$$

Then inserting $A = \sum_{i=1}^4 \lambda_i A_i$ into Eq. (21) and comparing the coefficients of $\lambda_1 \lambda_2 \lambda_3 \lambda_4$, one finds

$$0 = \sum_{\text{6 perm}} \langle A_1 A_2 A_3 A_4 \rangle - \frac{3}{4} \sum_{\text{8 perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle - \sum_{\text{3 perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle + \frac{1}{2} \sum_{\text{6 perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle. \tag{22}$$

In particular, if $\langle A_i \rangle = 0$, Eq. (22) becomes

$$0 = \sum_{\text{6 perm}} \langle A_1 A_2 A_3 A_4 \rangle - \sum_{\text{3 perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle, \quad \text{(for } \langle A_i \rangle = 0), \tag{23}$$

whose explicit form is

$$\langle A_1 A_2 A_3 A_4 \rangle + \langle A_1 A_4 A_2 A_3 \rangle + \langle A_2 A_1 A_3 A_4 \rangle + \langle A_2 A_3 A_1 A_4 \rangle + \langle A_3 A_1 A_2 A_4 \rangle + \langle A_3 A_2 A_1 A_4 \rangle
= \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle + \langle A_1 A_3 \rangle \langle A_2 A_4 \rangle + \langle A_1 A_4 \rangle \langle A_2 A_3 \rangle, \quad \text{(for } \langle A_i \rangle = 0). \tag{24}$$

Then further taking $A_1 = A_2 = A$ and $A_3 = A_4 = B$ in Eq. (24), one obtains

$$4 \langle A^2 B^2 \rangle + 2 \langle ABAB \rangle = \langle A^2 \rangle \langle B^2 \rangle + 2 \langle AB \rangle^2, \quad \text{(for } \langle A \rangle = \langle B \rangle = 0). \tag{25}$$

Eventually, multiplying Eq. (10) by $A^4$ and the same procedure as above results in

$$0 = \sum_{\text{120 perm}} \langle A_1 A_2 A_3 A_4 A_5 A_6 \rangle - \frac{5}{6} \sum_{\text{144 perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_6 \rangle - \frac{2}{3} \sum_{\text{90 perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 A_6 \rangle + \frac{2}{3} \sum_{\text{90 perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle. \tag{26}$$

Eqs. (20), (22) and (26) are useful relations for the construction of any 2-flavor chiral perturbation theory.

**Example 2.** Consider a 2-flavor chiral perturbation theory with the chiral (global) symmetry breaking pattern $[SU(2)_L \times SU(2)_R \times U(1)_Y]/[SU(2)_V \times U(1)_Y]$. The basic building block of the theory is usually chosen to be a
unitary unimodular $2 \times 2$ matrix field $U$, which transforms as $U \to RUL^\dagger$ under the chiral symmetry group with $R \in SU(2)_R$ and $L \in SU(2)_L$. If we define $X_\mu \equiv U^\dagger(D_\mu U)$ which transforms like $X_\mu \to LX_\mu L^\dagger$. As det $U = 1$, then

$$0 = \partial_\mu \det U = \partial_\mu (e^{\log \det U}) = e^{\log \det U} \partial_\mu (\log \det U) = \partial_\mu (\log \det U) = \partial_\mu (U^\dagger D_\mu U) = \partial_\mu (U^\dagger D_\mu U) = \partial_\mu (X_\mu) .$$

(27)

In the chiral Lagrangian, the chiral symmetric terms that consist of four $X_\mu$s are $\langle X_\mu X_\nu X_\mu X_\nu \rangle$, $\langle X_\mu X_\nu X_\mu X_\nu \rangle$, $\langle X_\mu X_\nu \rangle^2$, and $\langle X_\mu X_\nu \rangle/2$. Explicitly, the first two terms are

$$\langle X_\mu X_\nu X_\mu X_\nu \rangle = \sum_i g_{ii}^2 \langle X_i X_i X_i X_i \rangle + \sum_{i<k} g_{ik} g_{kk} \left(\langle X_i X_i X_k X_k \rangle + \langle X_k X_k X_i X_i \rangle \right) = \sum_i g_{ii}^2 \langle X_i X_i X_i X_i \rangle + 2 \sum_{i<k} g_{ik} g_{kk} \langle X_i X_i X_k X_k \rangle ,$$

(28)

$$\langle X_\mu X_\nu X_\mu X_\nu \rangle = \sum_i g_{ii}^2 \langle X_i X_i X_i X_i \rangle + 2 \sum_{i<k} g_{ik} g_{kk} \langle X_i X_i X_k X_k \rangle .$$

(29)

Note that there is no summation on repeated indices $i$ and $k$ unless specially stated by a summation symbol, $\sum$. Then, a linear combination of these two terms gives

$$2\langle X_\mu X_\nu X_\mu X_\nu \rangle + \langle X_\mu X_\nu X_\mu X_\nu \rangle = 3 \sum_i g_{ii}^2 \langle X_i X_i X_i X_i \rangle + \sum_{i<k} g_{ik} g_{kk} \left(4 \langle X_i X_i X_k X_k \rangle + 2 \langle X_i X_i X_k X_k \rangle \right) = \frac{3}{2} \sum_i g_{ii}^2 \langle X_i X_i \rangle^2 + \sum_{i<k} g_{ik} g_{kk} \left(\langle X_i X_i \rangle \langle X_k X_k \rangle + 2 \langle X_i X_i \rangle \langle X_k X_k \rangle \right) = \frac{1}{2} \langle X_\mu X_\nu \rangle^2 + \langle X_\mu X_\nu \rangle ,$$

(30)

where the second equality comes from Eqs. (24) and (25). From Eq. (30), one finds that at least one of the four 4-$X_\mu$ terms can be determined by the other three terms. A check for Eq. (30) by an alternative method is given in App. A.

2. The $n = 3$ case

For any $3 \times 3$ matrix $A$, Cayley-Hamilton theorem reads

$$0 = \chi_A(A) = A^3 - \langle A \rangle A^2 + \frac{1}{2} (\langle A \rangle^2 - \langle A^2 \rangle) A - \frac{1}{6} (\langle A \rangle^3 - 3 \langle A^2 \rangle \langle A \rangle + 2 \langle A^3 \rangle) ,$$

(31)

from which, the same procedure as the $n = 2$ case yields

$$0 = \sum_{6 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle - \sum_{8 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle - \sum_{3 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle + \sum_{6 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle - \langle A_1 \rangle \langle A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle ,$$

(32)

$$0 = \sum_{120 \text{ perm}} \langle A_1 A_2 A_3 A_4 A_5 A_6 \rangle - \frac{5}{6} \sum_{144 \text{ perm}} \langle A_1 A_2 A_3 A_4 A_5 A_6 \rangle - \frac{2}{3} \sum_{90 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 A_6 \rangle + \frac{2}{3} \sum_{90 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 A_6 \rangle - \frac{1}{2} \sum_{40 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 A_6 \rangle + \frac{1}{2} \sum_{120 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 A_6 \rangle .$$

(33)

Eqs. (32) and (33) are useful relations for the construction of any 3-flavor chiral perturbation theory, such as $[SU(3)_L \times SU(3)_R \times U(1)_V]/[SU(3)_V \times U(1)_V]$ and $SU(3)/SO(3)$ chiral perturbation theories.
3. The $n = 4$ case

For any $4 \times 4$ matrix $A$, Cayley-Hamilton theorem reads

$$0 = \chi_A(A) = A^4 - \langle A \rangle A^3 + \frac{1}{2} (\langle A \rangle^2 - \langle A^2 \rangle) A^2 - \frac{1}{6} (\langle A \rangle^3 - 3 \langle A^2 \rangle \langle A \rangle + 2 \langle A^3 \rangle) A$$

$$+ \left( - \frac{1}{4} \langle A^4 \rangle + \frac{1}{3} \langle A^3 \rangle \langle A \rangle + \frac{1}{8} \langle A^2 \rangle^2 - \frac{1}{4} \langle A^2 \rangle \langle A^2 \rangle + \frac{1}{24} \langle A \rangle^4 \right),$$  

(34)

from which, the same procedure as the $n = 2$ case yields

$$0 = \sum_{120 \text{ perm}} \langle A_1 A_2 A_3 A_4 A_5 A_6 \rangle - \frac{5}{6} \sum_{144 \text{ perm}} \langle A_1 A_2 A_3 A_4 A_5 \rangle \langle A_6 \rangle - \sum_{90 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 A_6 \rangle \langle A_6 \rangle$$

$$+ \frac{2}{3} \sum_{90 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle - \sum_{40 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 A_6 \rangle$$

$$+ \frac{5}{6} \sum_{120 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 \rangle \langle A_6 \rangle - \frac{1}{2} \sum_{40 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle$$

$$+ \sum_{15 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle \langle A_5 A_6 \rangle - \frac{2}{3} \sum_{45 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle$$

$$+ \frac{1}{3} \sum_{15 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle,$$  

(35)

which is a useful relation for the construction of any 4-flavor chiral perturbation theory, such as $[SU(4)_L \times SU(4)_R \times U(1) \nu]/[SU(4)_\nu \times U(1) \nu]$, $SU(4)/SO(4)$, and $SU(4)/Sp(4)$ chiral perturbation theories.

4. The $n = 5$ case

For any $5 \times 5$ matrix $A$, Cayley-Hamilton theorem reads

$$0 = \chi_A(A) = A^5 - \langle A \rangle A^4 + \frac{1}{2} (\langle A \rangle^2 - \langle A^2 \rangle) A^3 - \frac{1}{6} (\langle A \rangle^3 - 3 \langle A^2 \rangle \langle A \rangle + 2 \langle A^3 \rangle) A^2$$

$$+ \left( - \frac{1}{4} \langle A^4 \rangle + \frac{1}{3} \langle A^3 \rangle \langle A \rangle + \frac{1}{8} \langle A^2 \rangle^2 - \frac{1}{4} \langle A^2 \rangle \langle A^2 \rangle + \frac{1}{24} \langle A \rangle^4 \right) A$$

$$+ \left( - \frac{1}{5} \langle A^5 \rangle + \frac{1}{4} \langle A^4 \rangle \langle A \rangle + \frac{1}{6} \langle A^3 \rangle \langle A^2 \rangle - \frac{1}{6} \langle A^2 \rangle \langle A^3 \rangle - \frac{1}{8} \langle A^2 \rangle^2 \langle A \rangle$$

$$+ \frac{1}{120} \langle A^5 \rangle^2 - \frac{1}{120} \langle A \rangle^5 \right),$$  

(36)

from which, the same procedure as the $n = 2$ case yields

$$0 = \sum_{120 \text{ perm}} \langle A_1 A_2 A_3 A_4 A_5 A_6 \rangle - \sum_{144 \text{ perm}} \langle A_1 A_2 A_3 A_4 A_5 \rangle \langle A_6 \rangle - \sum_{90 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 A_6 \rangle$$

$$+ \sum_{90 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle - \sum_{40 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 A_6 \rangle$$

$$+ \sum_{120 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 A_5 \rangle \langle A_6 \rangle - \sum_{40 \text{ perm}} \langle A_1 A_2 A_3 \rangle \langle A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle$$

$$+ \sum_{15 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle \langle A_5 A_6 \rangle - \sum_{45 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle$$

$$+ \sum_{15 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 \rangle \langle A_4 \rangle \langle A_5 \rangle \langle A_6 \rangle,$$  

(37)

which is a useful relation for the construction of any 5-flavor chiral perturbation theory, such as $[SU(5)_L \times SU(5)_R \times U(1)_\nu]/[SU(5)_\nu \times U(1)_\nu]$ and $SU(5)/SO(5)$ chiral perturbation theories.
IV. APPLICATIONS TO GENERAL RELATIVITY

In general relativity, a second-rank covariant tensor field \( g_{\mu\nu} \) called the metric plays a crucial role. Its determinant, \( g \equiv \det(g_{\mu\nu}) \), is usually needed to be figured out in case one wants to handle the pseudoscalar volume element: \( \sqrt{-g}d^dx \), where the superscript \( d \) stands for the number of spacetime dimensions. Let the dimension number \( d \) be 10, which is required for the consistency of superstring theories. Using Eq. (19), one can express the determinant of the metric 10 \( \times \) 10 matrix \( g \) in terms of the traces of the successive powers of \( g \). Since there are 42 solutions \((p_1, p_2, \ldots, p_{10})\) of the equation \( p_1 + 2p_2 + \cdots + 10p_{10} = 10 \), there are also 42 terms in the expression of \( \det g \), which is given by

\[
\det(g) = -\frac{1}{10} T_{10} + \frac{1}{9} T_9 T_1 + \frac{1}{16} T_8 T_2 - \frac{1}{16} T_6 T_4^2 + \frac{1}{21} T_7 T_3 - \frac{1}{14} T_7 T_2 T_1 + \frac{1}{42} T_7 T_1^3 + \frac{1}{24} T_6 T_4 - \frac{1}{18} T_6 T_3 T_1 - \frac{1}{48} T_6 T_2^2 + \frac{1}{24} T_6 T_2 T_1^2 - \frac{1}{144} T_6 T_1^4 + \frac{1}{50} T_5^2 - \frac{1}{20} T_5 T_4 T_1 - \frac{1}{30} T_5 T_3 T_2 + \frac{1}{30} T_5 T_3 T_1^2 + \frac{1}{40} T_5 T_2^2 T_1 - \frac{1}{60} T_5 T_2 T_1^3 + \frac{1}{600} T_5 T_5^5 - \frac{1}{64} T_4 T_2^2 T_1 - \frac{1}{64} T_4^2 T_2 T_1^2 - \frac{1}{72} T_4 T_3 T_2 T_1 - \frac{1}{72} T_4 T_3 T_1^3 + \frac{1}{192} T_4 T_3^3 \nonumber
\]

\[
\cdots
\]

\[
-\frac{1}{36} T_3 T_2 T_1^4 + \frac{1}{144} T_3 T_2 T_1^3 - \frac{1}{720} T_3 T_2 T_1^5 + \frac{1}{15120} T_3 T_1^7 - \frac{1}{240} T_5^2 - \frac{1}{96} T_2 T_1^2 \nonumber
\]

\[
-\frac{1}{288} T_2 T_1^4 + \frac{1}{5760} T_2 T_1^6 - \frac{1}{80640} T_2 T_1^8 + \frac{1}{3628800} T_1^{10},
\]

where \( T_m \ (m = 1, 2, \ldots, 10) \) stand for \( \text{tr}(g^m) \). Eq. (38) shows that the determinant \( g = \det(g) \) is completely decided by the 10 traces, \( T_m \ (m = 1, 2, \ldots, 10) \), and this expression is fairly controllable and thus can be possibly used in future theoretical considerations. By using \textit{PASCAL} (a sample code is given in App. B), one easily finds that there are 56 trace terms among \( \det(g) \) for 11 dimensional spacetime corresponding to the M theory, while 2436 terms for 26 dimensional spacetime.

V. CONCLUSIONS

In this paper, we have given a new proof, by using combinatorics, for the recurrence relations of the characteristic polynomial coefficients, and, moreover, we have obtained an explicit expression for the determinant of any finite-dimensional matrix in terms of the traces of the successive powers of \( g \). For higher dimensional cases, however, we have obtained an explicit expression for the characteristic polynomial coefficients, and, moreover, we have obtained an explicit expression for the determinant of any finite-dimensional matrix in terms of the traces of the successive powers of \( g \). For higher dimensional spacetime, it is also easily obtained by using a \textit{PASCAL} program code, which is contained in the appendix.

APPENDIX A: A CHECK FOR THE CALEY-HAMILTON RELATIONS IN THE \( n = 2 \) CASE

In this appendix, we give a check for the Cayley-Hamilton relations in the \( n = 2 \) case by using the property of Pauli matrices. For definiteness, let us verify Eq. (31) in Example 2. Since \( X_\mu \) is traceless due to Eq. (27), then it can be decomposed, based on Pauli matrices, to \( X_\mu = \tau^a X_\mu^a \) with the coefficients \( X_\mu^a \ (a = 1, 2, 3) \) and \( \tau^a \ (a = 1, 2, 3) \) standing for Pauli matrices. Now let us show the following identity:

\[
\langle X_\mu X_\nu X_\rho X_\sigma \rangle = \frac{1}{2} \left( \langle X_\mu X_\nu \rangle \langle X_\rho X_\sigma \rangle + \langle X_\mu X_\sigma \rangle \langle X_\nu X_\rho \rangle - \langle X_\mu X_\nu \rangle \langle X_\rho X_\sigma \rangle \right),
\]

where \( \langle \cdots \rangle \) stands for taking the trace.

**Proof.** The well-known identities \( \tau^a \tau^b = \delta^{ab} + i\epsilon^{abc} \tau^c \) and \( \epsilon^{abc} \epsilon^{cde} \) imply that

\[
\langle \tau^a \tau^b \rangle = 2 \delta^{ab}, \quad \langle \tau^a \tau^b \tau^c \rangle = 2 \delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc}, \quad \langle \tau^a \tau^b \tau^c \tau^d \rangle = 2 \delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc}.
\]
Thus,
\[
\langle X_\mu X_\nu X_\rho X_\sigma \rangle = (r^{a'b'}r^{c'd'})X_\mu^a X_\nu^b X_\rho^c X_\sigma^d \\
= 2(\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc})X_\mu^a X_\nu^b X_\rho^c X_\sigma^d \\
= \frac{1}{2} \left( \langle X_\mu X_\nu \rangle \langle X_\rho X_\sigma \rangle + \langle X_\mu X_\rho \rangle \langle X_\nu X_\sigma \rangle - \langle X_\mu X_\sigma \rangle \langle X_\nu X_\rho \rangle \right).
\]
\[
\square
\]

Then, by contracting indices in Eq. (A1), we obtain
\[
\langle X_\mu X_\nu X_\rho X_\sigma \rangle = \frac{1}{2} (\langle X_\mu X_\nu \rangle)^2 - \frac{1}{2} (\langle X_\mu X_\rho \rangle)^2,
\]
(A2)
\[
\langle X_\mu X_\rho X_\nu X_\sigma \rangle = \frac{1}{2} (\langle X_\mu X_\nu \rangle)^2,
\]
(A3)
from which it is easily checked that
\[
2(\langle X_\mu X_\nu X_\rho X_\sigma \rangle + \langle X_\mu X_\rho X_\nu X_\sigma \rangle) = \frac{1}{2} (\langle X_\mu X_\nu \rangle)^2 + (\langle X_\mu X_\rho \rangle)^2,
\]
(A4)
which is exactly Eq. (30) obtained in Example 2. It should be kept in mind that Eq. (A1), or Eqs. (A2) and (A3) hold only for \( n = 2 \), though they are stronger than Eq. (A4) which comes from the Cayley-Hamilton theorem. However, from Eq. (32), one notes that the following relation holds as well in the \( n = 3 \) case.
\[
0 = \sum_{6 \text{ perm}} \langle A_1 A_2 A_3 A_4 \rangle - \sum_{3 \text{ perm}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle, \quad \text{(for } \langle A_4 \rangle = 0 \text{)}.
\]
(A5)
As a consequence, Eq. (A4) holds not only for \( 2 \times 2 \) but also for \( 3 \times 3 \) traceless matrices \( X_\mu \). Obviously, this relation is invalid for \( n \times n \) \((n \geq 4)\) matrices.

APPENDIX B: A PASCAL CODE FOR COMPUTATION OF \( D_n \)

```
Type splittype=array[1..40] of integer;
var num,sum,k:integer;
var spl:splittype;
var f1,f2: text;
procedure Output;
  var i,x,y:integer;
  var sp:splittype;
begin
  for i:=1 to 40 do sp[i]:=0;
  for i:=1 to k-2 do
    begin
      write(f1,spl[i],'+');
      sp[spl[i]]:=sp[spl[i]]+1;
    end;
  write(f1,spl[k-1],'=',num);sp[spl[k-1]]:=sp[spl[k-1]]+1;
  writeln(f1);
  for i:=1 to num do write(f2,sp[i], ' ');i:=1;
  while sp[i]=0 do i:=i+1;
  if sp[i]<1 then write(f2,'/','sp[i]','!')
  else write(f2,'*1/',sp[i],'!');i:=i+1;
  while i<=num do
    begin
      if sp[i]<>0 then
        begin
          if sp[i]=1 then write(f2,'*(-T_','i',i,'/',i,')')
          else write(f2,'*(-T_','i',i,'/',i,')^',sp[i],')');i:=i+1;
        end;
    end;
end;
```


i:=i+1;
end;
writeln(f2);
end;
procedure split(n,m:integer);
var i:integer;
begin
  if n=0 then begin Output; sum:=sum+1 end
  else if n>0 then
    begin
      for i:=m downto 1 do
        begin
          if n>=i then spl[k]:=i;
          k:=k+1;
          split(n-i,i);
          k:=k-1;
        end;
    end;
  end;
begin
  sum:=0;k:=1;
  assign(f1,'Output.txt');
  assign(f2,'Output2.txt');
  rewrite(f1);
  rewrite(f2);
  readln(num);
  split(num,num);
  writeln(f1,'Total is ',sum);
  writeln(f2,'Total is ',sum);
  close(f1);
  close(f2);
end.

ACKNOWLEDGMENTS

We would like to thank J. K. Parry, Zhan Xu and Ran Lu for helpful discussions. This work is supported in part by the National Natural Science Foundation of China.

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