COOPERATIVE IRREDUCIBLE SYSTEMS
OF ORDINARY DIFFERENTIAL EQUATIONS
WITH FIRST INTEGRAL

JANUSZ MIERCZYŃSKI

Proceedings of the Second Marrakesh International Conference on Differential Equations, 1995

Abstract. This note considers cooperative irreducible systems of ordinary
differential equations admitting a $C^1$ first integral with positive gradient. We
prove that all forward or backward nonwandering points are equilibria. We
obtain also some results on the global phase portrait of such systems. The
main tool of proof is a canonically defined Finsler structure with respect to
which the derivative skew-product dynamical system is contractive.

1. Introduction

A system of ordinary differential equations (ODEs)

$$\dot{x}^i = f^i(x), \quad x = (x^1, \ldots, x^n),$$

where $f = (f^1, \ldots, f^n): X \to \mathbb{R}^n$ is a $C^1$ vector field on an open set $X \subset \mathbb{R}^n$, is
called cooperative if $(\partial f^i/\partial x^j)(x) \geq 0$ for $i \neq j$ and all $x \in X$.

Let $\phi(t; x_0)$ denote the nonextendible solution of system (1) with the initial
condition $\phi(0; x_0) = x_0$. We write $\phi_k x_0$ instead of $\phi(t; x_0)$. For each $x \in X$
the mapping $t \mapsto \phi_k x$ (called the trajectory of $x$) is defined on an open interval
$(\sigma(x), \tau(x))$ containing 0. The restriction of the trajectory of $x$ to $(0, \tau(x))$ is called the backward [resp. forward] semitrajectory of $x$. The images of
(semi)trajectories are referred to as (semi)orbits. We say $x \in X$ is an equilibrium if
$\phi_k x = x$ for all $t \in \mathbb{R}$, or, equivalently, if $f(x) = 0$. A point $y \in X$ is an $\omega$-limit point
of $x \in X$ if there is a sequence $t_k \to \infty$ as $k \to \infty$, such that $\lim_{k \to \infty} \phi_{t_k} x = y$.
Notice that $\tau(x) = \infty$, while it is possible that $\tau(y) < \infty$. The definition of an
$\alpha$-limit point is analogous. The set of $\omega$-limit [resp. $\alpha$-limit] points is called the
$\omega$-limit set [$\alpha$-limit set] of $x$, and denoted by $\omega(x)$ [$\alpha(x)$].

The symbol $\| \cdot \|$ stands for the Euclidean norm in $\mathbb{R}^n$. We say $x \in X$ is forward nonwandering if for each $\epsilon > 0$ and each $0 < t < \tau(x)$ there are $y \in X$ with $\tau(y) > t$
and $t < \theta < \tau(y)$ such that $\| x - y \| < \epsilon$ and $\| x - \phi_{\theta} y \| < \epsilon$. A point $x \in X$ is
backward nonwandering if for each $\epsilon > 0$ and each $\sigma(x) < t < 0$ there are $y \in X$
with $\sigma(y) < t$, and $\sigma(y) < \theta < 0$ such that $\| x - y \| < \epsilon$ and $\| x - \phi_{\theta} y \| < \epsilon$. A point
that is forward nonwandering or backward nonwandering is called nonwandering.
It is straightforward that an $\omega$-limit point is forward nonwandering. Notice that in
the above definitions we do not assume the forward [backward] semitrajectory of
either $x$ or $y$ to be defined on the whole half-line $[0, \infty) \cup (-\infty, 0]$.

Key words and phrases. Cooperative system of ordinary differential equations. First integral.
Positive gradient. Nonwandering point.
For two points \( x, y \in \mathbb{R}^n \) denote
\[
\begin{align*}
x \leq y & \text{ if } x^i \leq y^i \text{ for each } i, \\
x < y & \text{ if } x \leq y \text{ and } x \neq y, \\
x \ll y & \text{ if } x^i < y^i \text{ for each } i.
\end{align*}
\]

For \( x \leq y \) we define a closed order interval as
\[
[x, y] := \{ z \in \mathbb{R}^n : x \leq z \leq y \},
\]
and for \( x \ll y \) we define an open order interval as
\[
[[x, y]] := \{ z \in \mathbb{R}^n : x \ll z \ll y \}.
\]

A set \( X \subset \mathbb{R}^n \) is said to be \( p \)-convex if the line segment with endpoints \( x \) and \( y \) is contained in \( X \) for each \( x, y \in X, x < y \), and \( p \)-order-convex if \( [x, y] \subset X \) for each \( x, y \in X, x < y \).

The next result gives an important property of cooperative systems of ODEs.

**Theorem 1.** Assume \( (1) \) is a cooperative system of ODEs on a \( p \)-convex open set \( X \subset \mathbb{R}^n \). Let \( x \leq y \). Then \( \phi_t x \leq \phi_t y \) for each \( t \in [0, \min(\tau(x), \tau(y))] \).

The above theorem was proved in [Müller (1927)] and [Kamke (1932)]. Some gaps in the earlier proofs were filled in [Ważewski (1950)]. The property is referred to as monotonicity of the local flow generated by \( (1) \).

An important feature of cooperative systems of \( n \) ordinary differential equations is that the limiting behavior of a point whose forward semi-orbit has compact closure is at most so complicated as that in a general system of \( n-1 \) ODEs. More precisely, the following result holds (see Theorem A in [Hirsch (1982)]):

**Theorem 2.** Let \( (1) \) be a cooperative system of ODEs on a \( p \)-convex open set \( X \subset \mathbb{R}^n \). Assume that the forward [resp. backward] semi-orbit of \( x \in X \) has compact closure in \( X \). Put \( L = \omega(x) [\text{resp. } L = \alpha(x)] \). Then the restricted flow \( \{ \phi_t | L \} \) is topologically equivalent to the flow of a Lipschitz system of ODEs on \( \mathbb{R}^{n-1} \). Moreover, no two points in \( L \) are related by \( \ll \).

An important class of cooperative systems is formed by cooperative irreducible systems of ODEs, that is, cooperative systems such that for each \( x \in X \) the matrix \( ([\partial f^i / \partial x^j](x)) \) is irreducible. A system of ODEs is called strongly cooperative if \( (\partial f^i / \partial x^j)(x) > 0 \) for \( i \neq j \).

For cooperative irreducible systems Theorem\( (1) \) can be strengthened to:

**Theorem 3.** Assume \( (1) \) is a cooperative irreducible system of ODEs on a \( p \)-convex open set \( X \subset \mathbb{R}^n \). Let \( x < y \). Then \( \phi_t x \ll \phi_t y \) for each \( t \in (0, \min(\tau(x), \tau(y))] \).

The above property is called strong monotonicity of the local flow generated by \( (1) \).

For cooperative irreducible systems one can say much more about their behavior (see Theorem 2.4 in [Smith and Thieme (1991)]; for earlier results see [Hirsch (1988)] and [Poláčik (1989)]):

**Theorem 4.** Assume \( (1) \) is a cooperative irreducible system of ODEs on a \( p \)-convex open set \( X \subset \mathbb{R}^n \) such that each forward semi-orbit has compact closure in \( X \). Then there exists an open dense set \( Y \subset X \) such that \( \omega(x) \) is a singleton for each \( x \in Y \).
Generally, in cooperative irreducible systems there are no restrictions on $\alpha$-limit sets. More precisely, according to [Smale (1976)], any dynamics on the standard $n$-dimensional simplex can be embedded as a repeller in an $(n + 1)$-dimensional strongly cooperative system of ODEs.

For a recent monograph on cooperative systems of ODEs see [Smith (1995)].

2. Cooperative Irreducible Systems with First Integral

By a first integral for (1) we mean a continuous function $H : X \to \mathbb{R}$ which is constant on orbits of (1). A first integral is nontrivial if it is not constant on any open set.

The existence of nontrivial first integrals puts severe restrictions on cooperative irreducible systems. Namely, it was proved in [Hirsch (1985)] that if the set of equilibria is countable and all points have forward semi-orbit closure compact in $X$ then each first integral is trivial.

In the three-dimensional case much more can be proved, even without assuming the abundance of points with compact forward semi-orbit closure (see [Mierczyński (1995)]):

**Theorem 5.** Assume that (1) is a cooperative irreducible system of ODEs on $\mathbb{R}^n$ admitting a $C^1$ first integral with nonzero gradient. Then each limit set is either empty or a singleton.

3. Cooperative Irreducible Systems with Monotone First Integral

3.1. Limiting Behavior. When one assumes that a first integral for a cooperative irreducible system of ODEs is strongly monotone, that is, from $x < y$ it follows $H(x) < H(y)$, then, under the assumption that all forward semi-orbits have compact closure in $X$, all $\omega$-limit sets are singletons (compare [Mierczyński (1987)]).

This result carries over to the case of abstract strongly monotone semidynamical systems with monotone first integral defined on strongly ordered Banach spaces ([Arino (1991)]).

Further, for some classes of cooperative periodic [resp. almost periodic] (in time) systems of ODEs admitting a first integral with appropriate monotonicity properties it was proved that each solution with compact forward semi-orbit closure converges to a periodic [resp. almost periodic] solution ([Nakajima (1979), Sell and Nakajima (1980), Tang et al (1993), Jiang (1995)]). Moreover, in the almost periodic case in the corresponding nonlinear skew-product (local) flow the image of that solution intersects each fiber at precisely one point (in the case where the system is periodic with period $T$ this simply means that the limiting solution has period $T$).

In many of the proofs of the results mentioned above the idea was to use a kind of Lyapunov function.

In the present note we do not assume any compactness of forward (or backward) semi-orbits. On the other hand, we make extensive use of the fact that a cooperative irreducible system of ODEs generates a linear skew-product dynamical system on the tangent bundle of $X$ possessing some monotonicity properties.

Let us introduce some notation. For $t \in (\sigma(x), \tau(x))$ the derivative $D\phi_t(x)$ of $\phi_t(x)$ with respect to $x$ is a linear isomorphism from the tangent space at $x$ into
the tangent space at \( \phi_t x \), satisfying the nonautonomous linear matrix ODE
\[
M' = Df(\phi_t x)M
\]
with initial condition \( M(0) = \text{Id} \), where \( Df := [(\partial f^i/\partial x^j)] \). The local linear skew-product dynamical system
\[
(x, v) \mapsto (\phi_t x, D\phi_t(x)v), \quad x \in X, \quad v \in \mathbb{R}^n, \quad \sigma(x) < t < \tau(x),
\]
will be referred to as the derivative local flow.

The order relations \( \leq, < \) and \( \ll \) are defined in a natural way on tangent vectors. We will refer to vectors \( v \geq 0 \) as nonnegative, and to vectors \( v \gg 0 \) as positive. The set of all nonnegative (free) vectors is called the (nonnegative) cone.

The derivative flow enjoys the following strong monotonicity property (see Hirsch (1985)):

**Theorem 6.** Assume that (1) is a cooperative irreducible system of ODEs on an open set \( X \subset \mathbb{R}^n \). Then for each \( x \in X \), each \( t \in (0, \tau(x)) \) and each \( v > 0 \) one has \( D\phi_t(x)v \gg 0 \).

Let \( \mathcal{H}(x) \) stand for the level set of the first integral \( H \) passing through \( x \). If \( H \) is of class \( C^1 \) and \( \text{grad} \ H(x) \neq 0 \) for all \( x \in X \) then \( \mathcal{H}(x) \) is a \( C^1 \) submanifold of codimension 1.

The main result of the present note is

**Theorem 7.** Let (1) be a cooperative irreducible system of ODEs on an open \( X \subset \mathbb{R}^n \). Assume that (1) admits a first integral \( H \) of class \( C^1 \) with positive gradient. Then each nonwandering point is an equilibrium.

**Proof.** In Mierczyński (1991) it was proved that for a cooperative irreducible system of ODEs admitting a \( C^1 \) first integral with positive gradient there is a canonical Finsler on the foliation of \( X \) into level sets of \( H \) under which the derivative (local) flow is contractive. By a Finsler we understand a continuous mapping \((x, v) \mapsto |v|_x \) such that for each \( x \) the assignment \( v \mapsto |v|_x \) is a norm on the tangent space at \( x \) of \( \mathcal{H}(x) \).

This canonical Finsler is constructed in the following way: For each \( x \in X \), we translate the tangent space of \( \mathcal{H}(x) \) at \( x \) by a positive vector \( v_x \) such that \( \langle \text{grad} \ H(x), v_x \rangle = 1 \). The intersection of the resulting hyperplane with the nonnegative cone is a compact convex set \( A_x \) linearly isomorphic to the standard \((n-1)\)-dimensional simplex. Finally, we take the compact convex balanced set \( A_x - A_x \) to be the unit ball in the tangent space of \( \mathcal{H}(x) \) at \( x \).

Proposition 2 in Mierczyński (1991) states that for each \( x \in X \), each nonzero vector \( v \) tangent at \( x \) to \( \mathcal{H}(x) \) and each \( t \in (0, \tau(x)) \) one has \( |D\phi_t(x)v|_{\phi_t(x)} < |v|_x \).

(As a matter of fact, that proposition is stated for strongly cooperative systems, but its proof carries over verbatim to the case of cooperative irreducible systems.)

Suppose for contradiction that \( x \) is a forward nonwandering point not being an equilibrium. Let \( L \subset X \) be a \( C^1 \) embedded \((n-1)\)-dimensional disk transverse to \( f(x) \) and having \( x \) in its relative interior. Pick \( s > 0 \), \( s < \min \{ \tau(z) : z \in L \} \).

Denote
\[
\lambda := \max\{|D\phi_t(z)v|_{\phi_t(z)} : z \in L, \ v \in T_z\mathcal{H}, \ |v|_z = 1\},
\]
where \( T_z\mathcal{H} \) denotes the tangent space of \( \mathcal{H}(z) \) at \( z \). It is obvious that \( 0 < \lambda < 1 \). Now, take a \( C^1 \) embedded \((n-1)\)-dimensional disk \( M \subset L \) transverse to \( f(x) \),
having \( x \) in its relative interior and such that
\[
(1 - \mu)|f(x)|_x < |f(z)|_z < (1 + \mu)|f(x)|_x \quad \text{for all} \quad z \in M,
\]
where \( \mu := (1 - \lambda)/(1 + \lambda) \).

As \( x \) is, by assumption, a forward nonwandering point, there exists a point \( z \in M \) and \( t > s \) such that \( \phi_t z \in M \). But
\[
|f(\phi_t z)|_{\phi_t z} = |D\phi_{t-s}(\phi_s z)f(\phi_s z)|_{\phi_t z} < |f(\phi_s z)|_{\phi_t z} \leq \lambda|f(z)|_z < (1 - \mu)|f(x)|_x,
\]
a contradiction.

For a backward nonwandering point \( x \) not being an equilibrium, let \( L \subset X \) be a \( C^1 \) embedded \((n - 1)\)-dimensional disk transverse to \( f(x) \) and having \( x \) in its relative interior. Pick \( s < 0 \), \( s < \max\{\sigma(z) : z \in L\} \). Denote
\[
\lambda := \min\{|D\phi_s(z)v|_{\phi_s(z)} : z \in L, v \in T_z H, |v|_z = 1\}.
\]
It is obvious that \( \lambda > 1 \). Now, take a \( C^1 \) embedded \((n - 1)\)-dimensional disk \( M \subset L \) transverse to \( f(x) \), having \( x \) in its relative interior and such that
\[
(1 - \mu)|f(x)|_x < |f(z)|_z < (1 + \mu)|f(x)|_x \quad \text{for all} \quad z \in M,
\]
where \( \mu := (\lambda - 1)/(\lambda + 1) \).

As \( x \) is, by assumption, a backward nonwandering point, there exists a point \( z \in M \) and \( t < s \) such that \( \phi_t z \in M \). But
\[
|f(\phi_t z)|_{\phi_t z} = |D\phi_{t-s}(\phi_s z)f(\phi_s z)|_{\phi_t z} > |f(\phi_s z)|_{\phi_t z} \geq \lambda|f(z)|_z > (1 - \mu)|f(x)|_x,
\]
a contradiction. \( \square \)

As a corollary we obtain (see [Mierczyński (1991)])

**Theorem 8.** Let (I) be a cooperative irreducible system of ODEs on an open \( X \subset \mathbb{R}^n \) admitting a first integral \( H \) of class \( C^1 \) with positive gradient. Then

(i) Each \( \omega \)-limit set is either a singleton or empty.

(ii) If \( \alpha(x) \) is nonempty then \( x \) is an equilibrium.

**Proof.** Part (i) follows by the fact that each \( \omega \)-limit point is forward nonwandering. Also, each \( \alpha \)-limit point is backward nonwandering. Assume that for some \( x \), \( \alpha(x) = \{y\} \). We have \( 0 = |f(y)|_y = \lim_{t \to -\infty} |f(\phi_t y)|_{\phi_t y} = \sup\{|f(\phi_t x)|_{\phi_t x} : t \in (-\infty, \tau(x))\} \) (see Proposition 2 in [Mierczyński (1991)]), hence \( |f(x)|_x = 0 \), and \( x \) is an equilibrium, therefore \( x = y \). \( \square \)

### 3.2. Global Picture
In the present subsection we shall obtain some insight into the global nature of the dynamical system restricted to a level set \( H(x) \) of \( H \). The standing assumption will be:

\( X \subset \mathbb{R}^n \) is an open order-convex set such that for any pair \( x, y \in X \) their maximum \( x \vee y \) and minimum \( x \wedge y \) are in \( X \).

For instance, \( X = \mathbb{R}^n \), or \( X = [a, b] \) for some \( a, b \in \mathbb{R}^n \), \( a \ll b \), or else \( X \) is the positive orthant \( \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x^i > 0 \quad \text{for all} \quad i\} \).

We begin with an auxiliary

**Lemma 9.** Let \( H \) be a \( C^1 \) first integral with positive gradient for a cooperative irreducible system (I). Then for each \( x \in X \), \( H(x) \) is connected.
Lemma 9

Proof. Fix $x \in X$, and write $\mathcal{H} := \mathcal{H}(x)$. Take $y, z \in \mathcal{H}, y \neq z$. As $y + z = y \wedge z + y \vee z$, it follows that the points $y, z, y \wedge z$ and $y \vee z$ belong to a two-dimensional affine subspace $V$. We have $H(y \wedge z) < H(x)$ and $H(y \vee z) > H(x)$. For each $\lambda \in [0, 1]$ denote by $A_\lambda$ the union of the line segment joining $y \vee z$ with $\lambda y + (1 - \lambda) z$ and the line segment joining $y \wedge z$ with $\lambda y + (1 - \lambda) z$. The set $A := \bigcup_{\lambda \in [0, 1]} A_\lambda$ is a two-dimensional (analytic) submanifold-with-corners contained in $V$. It is apparent that the gradient of the restriction $H|A$ is everywhere nonzero. The implicit function theorem yields that the set $A \cap \mathcal{H}$ is $C^1$ diffeomorphic to the real interval $[0, 1]$. This finishes the proof. 

As in the case of a Riemannian metric, for a $C^1$ curve $\gamma$ contained in $\mathcal{H}$ define the length of $\gamma$ as

$$\ell(\gamma) := \int_a^b |\gamma'(s)|_{\gamma(s)} ds,$$

where $\gamma : [a, b] \to \mathcal{H}$ is a parametrization of the curve. It is straightforward that the length does not depend on a parametrization. We define the Finsler distance $d(x, y)$ between two points $x, y \in \mathcal{H}$ as the infimum of the lengths of all $C^1$ curves with endpoints $x$ and $y$. The $C^1$ manifold $\mathcal{H}$ together with the Finsler distance $d(\cdot, \cdot)$ is a metric space.

Theorem 10. Let a cooperative irreducible system \[1\] admit a first integral $H$ of class $C^1$ with positive gradient. Assume that $\mathcal{H}$ is a level set of $H$ such that $\tau(x) = \infty$ for all $x \in \mathcal{H}$. Then either

(a) There is precisely one equilibrium $y$ in $\mathcal{H}$, and $y$ is a global attractor in $\mathcal{H}$;

or

(b) There is no equilibrium in $\mathcal{H}$, and for each $x \in \mathcal{H}$ one has $\omega(x) = \emptyset$.

Proof. As a consequence of Theorem\[7\] if there is no equilibrium in $\mathcal{H}$ then $\omega(x) = \emptyset$ for all $x \in \mathcal{H}$. So, assume $y \in \mathcal{H}$ is an equilibrium. First, we claim that $y$ is a unique equilibrium in $\mathcal{H}$. Suppose per contra that there is another equilibrium $y_1$. Put $z := y \vee y_1$. We have $z > y, z > y_1$. By strong monotonicity, for each $t, 0 < t < \tau(z)$, one has $\phi_t z \gg \phi_t y = y$ and $\phi_t z \gg \phi_t y_1 = y_1$. But this implies that $\phi_t z \gg z$, that is, in the level set of $H$ passing through $z$ there are two points being in the $\ll$ relation, which is impossible.

By Main Theorem in \[Mierczyński (1991)\], $y$ is (locally) exponentially asymptotically stable relative to $\mathcal{H}$. Consequently, the set $A := \{ x \in \mathcal{H} : \omega(x) = \{ y \} \}$ is relatively open in $\mathcal{H}$. Suppose by way of contradiction that $A \neq \mathcal{H}$. Since by Lemma\[3\] $\mathcal{H}$ is connected, the relative boundary $\partial_{\mathcal{H}} A$ is nonempty. Pick a point $z \in \partial_{\mathcal{H}} A$. As $z \notin A$, we have $\omega(z) = \emptyset$ by Theorem\[7\].

Because $y$ is asymptotically stable in $\mathcal{H}$, by \[Conley (1978)\] there is a compact relative neighborhood $B$ of $y$ in $\mathcal{H}$ such that $B \subset A$ and $\phi_0 B \subset B$ for all $t \geq 0$. Take $\varepsilon > 0$ so small that $\{ x \in \mathcal{H} : d(x, y) \leq 2 \varepsilon \}$ is contained in the relative interior of $B$. Pick $x_1 \in A$ with $d(x_1, z) < \varepsilon$. Let $T > 0$ be such that $d(\phi_T x_1, y) < \varepsilon$. As $\phi_T z$ exists, we must have $d(\phi_T x_1, \phi_T z) < \varepsilon$. Consequently, $d(\phi_T z, y) < 2 \varepsilon$. But from this it follows that $\phi_T z \in B \subset A$, hence $\omega(z) = \{ y \}$. This contradiction completes the proof. \[\square\]
If $X = \mathbb{R}^n$, a well-known condition guaranteeing $\tau(x) = \infty$ for each $x$ is the existence of positive constants $C_1$ and $C_3$ such that $\|f(x)\| \leq C_1 \|x\| + C_3$. For another condition see the following result.

**Theorem 11.** Let a cooperative irreducible system defined on $\mathbb{R}^n$ admit a first integral $H$ of class $C^1$, such that all the coordinates of $\text{grad } H(x)$ are positive, bounded and bounded away from zero, uniformly in $x \in X$. Then we have either

(a) For each $x \in \mathbb{R}^n$, $\omega(x)$ is a singleton. Moreover, the set of equilibria is simply ordered by $\ll$;

or

(b) For each $x \in \mathbb{R}^n$, $\omega(x) = \emptyset$.

**Proof.** We begin by showing that there is a constant $C > 0$ such that $\|v\| \leq C \|x\|_x$ for each $x \in \mathbb{R}^n$ and each vector $v$ tangent at $x$ to $H(x)$. Fix $x \in \mathbb{R}^n$, and put

$$G(x) := \frac{\text{grad } H(x)}{\|\text{grad } H(x)\|^2}.$$  

Recall that in the construction of the Finsler $|\cdot|$ the unit ball $B_x$ is defined as $A_x - A_x$, where $A_x := \{v \geq 0 : \langle G(x), v \rangle = 1\}$. In particular, $G(x) \in A_x$. Fix $v \in B_x$, that is, $\|v\|_x \leq 1$. Write $v = v_1 - v_2$, where $v_1, v_2 \in A_x$. For $i = 1, 2$, put $w_i := v_i - G(x)$. Of course, $v = w_1 - w_2$. As $v_1 \geq 0$, we have $-w_i = G(x) - v_i \leq G(x)$. On the other hand, $\langle G(x), -w_i \rangle$ is easily seen to be zero. Write $G(x) = (a_1, \ldots, a_n)$, $-w_1 = (b_1, \ldots, b_n)$. We have $a_j > 0$, $b_j \leq a_j$ and $\sum_{j=1}^n a_j b_j = 0$. Define

$$c_j := \frac{1}{a_j \|\text{grad } H(x)\|^2} - a_j.$$  

It is straightforward that all $c_j$’s are positive and bounded uniformly in $x \in \mathbb{R}^n$. We claim that

$$-c_j \leq b_j \leq c_j$$  

for each $1 \leq j \leq n$. Indeed, suppose $b_k > c_k$ for some $k$. We have then

$$a_k b_k > \frac{1}{\|\text{grad } H(x)\|^2} - a_k^2 = \|G(x)\|^2 - a_k^2 = \sum_{j=1}^n a_j^2 - \sum_{j=1}^n a_j b_j,$$  

hence $\sum_{j=1}^n a_j b_j > 0$, a contradiction. The other inequality is proved by a similar argument. We have thus obtained that $\|v\| \leq \|w_1\| + \|w_2\|$ does not exceed a constant independent of $x \in \mathbb{R}^n$.

Now suppose by way of contradiction that $T := \tau(x) < \infty$ for some $x \in \mathbb{R}^n$. Then the improper integral $\int_0^T |f(\phi_t x)| |d\phi_t|$ is convergent, from which it follows in a standard way that the finite limit $\lim_{t \to T} \phi_t x$ exists, a contradiction.

Assume that there exists an equilibrium $y \in \mathbb{R}^n$. We may assume $y = 0$ and $H(0) = 0$. Take a positive real number $r$. Let $\epsilon > 0$ be such that for each $x$ with $\|x\| \leq \epsilon$ one has $H(x) < r/2$. Since the coefficients of grad $H$ are positive and bounded away from zero, for each $x > 0$, $\|x\| = \epsilon$, the half line $\{sx : s \geq 1\}$ intersects the level set $H^{-1}(r) := \{z \in \mathbb{R}^n : H(z) = r\}$ at precisely one point $M(x)$. It is easy to see that the mapping $M$ is a homeomorphism of $\{x \in \mathbb{R}^n : x \geq 0, \|x\| = \epsilon\}$ onto $\{z \in H^{-1}(r) : z \geq 0\}$. As the latter set is forward invariant, a well-known application of Brouwer’s fixed point theorem implies that there exists an equilibrium in $H^{-1}(r)$. An analogous argument applies to the case $r < 0$. From Theorem 10 we
deduce that $\omega(x)$ is a singleton for any $x \in \mathbb{R}^n$. The fact that the set of equilibria is simply ordered follows by Proposition 2.1 in [Mierczyński (1987)].

References

[Arino (1991)] O. Arino, *Monotone semi-flows which have a monotone first integral*, in: “Delay differential equations and dynamical systems (Claremont, California, 1990),” Lecture Notes in Math., 1475, Springer, Berlin, pp. 64–75.

[Conley (1978)] C. C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conf. Ser. in Math., 38, Amer. Math. Soc., Providence, R.I., 1978.

[Hirsch (1982)] M. W. Hirsch, *Systems of differential equations which are competitive or cooperative. I. Limit sets*, SIAM J. Math. Anal., 13 (1982), pp. 167–179.

[Hirsch (1985)] M. W. Hirsch, *Systems of differential equations that are competitive or cooperative. II. Convergence almost everywhere*, SIAM J. Math. Anal., 16 (1985), pp. 423–439.

[Hirsch (1988)] M. W. Hirsch, *Stability and convergence in strongly monotone dynamical systems*, J. Reine Angew. Math. 383 (1988), pp. 1–53.

[Jiang (1995)] Jiang Ji-fa, *Periodic monotone systems with an invariant function*, preprint.

[Kamke (1932)] E. Kamke, *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II*, Acta Math., 58 (1932), pp. 57–85.

[Mierczyński (1987)] J. Mierczyński, *Strictly cooperative systems with a first integral*, SIAM J. Math. Anal., 18 (1987), pp. 642–646.

[Mierczyński (1991)] J. Mierczyński, *A class of strongly cooperative systems without compactness*, Colloq. Math., 62 (1991), pp. 43–47.

[Mierczyński (1995)] J. Mierczyński, *Three-dimensional cooperative systems that admit a first integral with non-zero gradient*, submitted for publication.

[Müller (1927)] M. Müller, *Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen*, Math. Z., 26 (1927), pp. 619–645.

[Nakajima (1979)] F. Nakajima, *Periodic time-dependent gross-substitute systems*, SIAM J. Math. Anal., 36 (1979), pp. 421–427.

[Poláčik (1989)] P. Poláčik, *Convergence in smooth strongly monotone flows defined by semilinear parabolic equations*, J. Differential Equations, 79 (1989), pp. 89–110.

[Sell and Nakajima (1980)] G. R. Sell and F. Nakajima *Almost periodic gross-substitute dynamical systems*, Tôhoku Math. J. (2), 32 (1980), pp. 255–263.

[Smale (1976)] S. Smale, *On the differential equations of species in competition*, J. Math. Biol., 3 (1976), pp. 5–7.

[Smith (1995)] H. L. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*, Math. Surveys Monogr., 41, Amer. Math. Soc., Providence, R.I., 1995.

[Smith and Thieme (1991)] H. L. Smith and H. R. Thieme, *Convergence for strongly order-preserving semiflows*, SIAM J. Math. Anal., 22 (1991), pp. 1081–1101.

[Tang et al (1993)] B. Tang, Y. Kuang and H. Smith, *Strictly nonautonomous cooperative system with a first integral*, SIAM J. Math. Anal., 24 (1993), pp. 1312–1330.

[Ważewski (1950)] T. Ważewski, *Systèmes des équations et des inégalités différentielles ordinaires aux deuxièmes membres monotones et leurs applications*, Ann. Soc. Polon. Math., 23 (1950), pp. 112–166.

Institute of Mathematics, Wroclaw University of Technology, Wybrzeże Wyspianskiego 27, PL-50-370 Wroclaw, Poland