Ordinarization Transform of a Numerical Semigroup and Semigroups with a Large Number of Intervals

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Abstract

A numerical semigroup is said to be ordinary if it has all its gaps in a row. Indeed, it contains zero and all integers from a given positive one. One can define a simple operation on a non-ordinary semigroup, which we call here the ordinarization transform, by removing its smallest non-zero non-gap (the multiplicity) and adding its largest gap (the Frobenius number). This gives another numerical semigroup and by repeating this transform several times we end up with an ordinary semigroup. The genus, that is, the number of gaps, is kept constant in all the transforms.

This procedure allows the construction of a tree for each given genus containing all semigroups of that genus and rooted in the unique ordinary semigroup of that genus. We study here the regularity of these trees and the number of semigroups at each depth. For some depths it is proved that the number of semigroups increases with the genus and it is conjectured that this happens at all given depths. This may give some lights to a former conjecture saying that the number of semigroups of a given genus increases with the genus.

We finally give an identification between semigroups at a given depth in the ordinarization tree and semigroups with a given (large) number of gap intervals and we give an explicit characterization of those semigroups.

Keywords: Numerical semigroup, semigroup tree, Freïman’s theorem, sumsets.

1 Introduction

Let \( \mathbb{N}_0 \) denote the set of non-negative integers. A numerical semigroup is a subset of \( \mathbb{N}_0 \) which is closed under addition, contains 0, and its complement in \( \mathbb{N}_0 \) is finite. One main reference for numerical semigroups is [14]. For a numerical semigroup \( \Lambda \) the elements in \( \mathbb{N}_0 \setminus \Lambda \) are called gaps and the number of gaps is the genus of the semigroup. The largest gap is called the Frobenius
number. The multiplicity $m$ of a numerical semigroup is its first non-zero non-gap. A numerical semigroup different than $\mathbb{N}_0$ is said to be ordinary if its gaps are all in a row.

It was conjectured in [2] that the number $n_g$ of numerical semigroups of genus $g$ asymptotically behaves like the Fibonacci numbers. More precisely, it was conjectured that $n_g \geq n_{g-1} + n_{g-2}$, that the limit of the ratio \( \frac{n_g}{n_{g-1} + n_{g-2}} \) is 1 and so that the limit of the ratio \( \frac{n_g}{n_{g-1}} \) is the golden ratio $\phi = \frac{1 + \sqrt{5}}{2}$.

Many other papers deal with the sequence $n_g$ [10, 11, 3, 5, 4, 6, 16, 1, 9] and recently Alex Zhai gave a proof for the asymptotic Fibonacci-like behavior of $n_g$ [15]. However, it has still not been proved that $n_g$ is increasing. In the present work we approach this problem.

All numerical semigroups can be organized in a tree $T$ whose root is the semigroup $\mathbb{N}_0$ and in which the parent of a numerical semigroup $\Lambda$ of genus $g$ is the numerical semigroup obtained by adjoining to $\Lambda$ its Frobenius number. So, the parent of a numerical semigroup of genus $g$ has genus $g-1$ and all numerical semigroups are in $T$, at a depth equal to its genus. In particular, $n_g$ is the number of nodes of $T$ at depth $g$. This construction was already considered in [13].

Here we will see that all numerical semigroups of a given genus $g$ can be organized in a tree $T_g$ rooted at the unique ordinary semigroup of genus $g$. One significant difference between $T_g$ and $T$ is that the first one has only a finite number of nodes, indeed, it has $n_g$ nodes, while $T$ is an infinite tree. We will see some relations between the trees $T_g$ and $T$. We conjecture that the number of numerical semigroups in $T_g$ at a given depth is at most the number of numerical semigroups in $T_{g+1}$ at the same depth. This conjecture would prove that $n_{g+1} \geq n_g$. Here we show this result for the lowest and largest depths.

We finally study semigroups with a large number of intervals. We prove that if \( \lfloor \frac{n}{2} \rfloor \geq \frac{g+2}{3} \), then the set of numerical semigroups of genus $g$ and $n$ intervals of gaps is empty if $n$ and $g$ have different parity and it is exactly the set of numerical semigroups of genus $g$ and depth $\lfloor \frac{n}{2} \rfloor$ in $T_g$ otherwise. Furthermore we give an explicit description of the form of these semigroups.

2 A tree with all semigroups of a given genus

2.1 Ordinarization transform and ordinarization number of a numerical semigroup

The ordinarization transform of a non-ordinary semigroup $\Lambda$ with Frobenius number $F$ and multiplicity $m$ is the set $\Lambda' = \Lambda \setminus \{m\} \cup \{F\}$. The ordinarization transform of an ordinary semigroup is itself. For instance, the ordinarization transform of the semigroup $\Lambda = \{0, 4, 5, 8, 9, 10, 12, \ldots \}$ is the semigroup $\Lambda' = \{0, 5, 8, 9, 10, 11, 12, \ldots \}$. Note that the genus of the ordinarization transform of a semigroup is the genus of the semigroup.

We can iterate the ordinarization transform and set $\Lambda'' = (\Lambda')'$ and in gen-
General $\Lambda^{(i)} = \Lambda^{(i-1)'}$. It is easy to check that if $\Lambda$ has genus $g$ then there exists an integer $i$ such that $\Lambda^{(i)} = \{0, g+1, g+2, \ldots\}$. If $\Lambda$ is non-ordinary then there exists a unique $i$ with this property and satisfying $\Lambda^{(i-1)} \neq \{0, g+1, g+2, \ldots\}$. We call this $i$ the ordinarization number of $\Lambda$. By extension, the ordinarization number of an ordinary semigroup is set to be 0. For instance, if $\Lambda = \{0, 4, 5, 8, 9, 10, 12, \ldots\}$ then $\Lambda' = \{0, 5, 8, 9, 10, 11, 12, \ldots\}$ which is not the ordinary semigroup while $\Lambda'' = \{0, 7, 8, 9, 10, 11, 12, \ldots\}$ which is the ordinary semigroup of genus 6. Thus the ordinarization number of $\Lambda$ is 2.

**Lemma 1.** The ordinarization number of a numerical semigroup of genus $g$ is the number of its non-zero non-gaps which are smaller than or equal to $g$.

**Proof.** A numerical semigroup of genus $g$ is non-ordinary if and only if its multiplicity is at most $g$. So, we can transform a numerical semigroup while its multiplicity is at most $g$. The number of times that we can transform a semigroup before getting the ordinary semigroup is thus the number of its non-zero non-gaps which are smaller than or equal to the genus.

Given a numerical semigroup $\Lambda$ it will be convenient to consider its enumeration $\lambda$ as the unique increasing bijective map between $\mathbb{N}_0$ and $\Lambda$. We will use $\lambda_i$ for $\lambda(i)$. By the previous lemma, if the ordinarization number of a semigroup is $r$ then the non-gaps which are at most $g$ are $\lambda_0 = 0, \lambda_1, \ldots, \lambda_r$.

**Lemma 2.** The maximum ordinarization number of a semigroup of genus $g$ is $\lfloor \frac{g}{2} \rfloor$.

**Proof.** Suppose that the ordinarization number of a semigroup is $r$. On one hand, since the Frobenius number $F$ is at most $2g - 1$, the total number of gaps from 1 to $2g - 1$ is $g$ and so the number of non-gaps from 1 to $2g$ is $g$. The number of those non-gaps which are larger than the genus is $g - r$. On the other hand $\lambda_r + \lambda_1, \lambda_r + \lambda_2, \ldots, 2\lambda_r$ are different non-gaps between $g + 1$ and $2g$. So, the number of non-gaps between $g + 1$ and $2g$ is at least $r$. Putting this altogether we get that $g - r \geq r$ and so $r \leq \frac{g}{2}$.

On the other hand, the bound is attained by the semigroup

\[ \{0, 2, 4, \ldots, 2 \left\lfloor \frac{g}{2} \right\rfloor, 2 \left( \left\lfloor \frac{g}{2} \right\rfloor + 1 \right), \ldots, 2g, 2g+1, 2g+2, \ldots \}. \] (1)

We want to see that the maximum ordinarization number as stated in the previous lemma is attained only by the semigroup in (1). For this purpose we need the next lemma. Its proof has been omitted but can easily be obtained.

**Lemma 3.** Consider a finite subset $A = \{a_1 < \cdots < a_n\} \subseteq \mathbb{N}_0$.

1. The set $A + A$ contains at least $2n - 1$ elements.

2. The set $A + A$ contains exactly $2n - 1$ elements if and only if $a_i = a_1 + \alpha i$ for all $i$ and for a given positive integer $\alpha$. 

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Lemma 4. Let $g > 0$. The unique numerical semigroup of genus $g$ and ordinarization number $\lfloor \frac{g^2}{2} \rfloor$ is \{0, 2, 4, \ldots, 2g, 2g + 1, 2g + 2, \ldots\}.

Proof. Suppose that the ordinarization number of $\Lambda$ is $\lfloor \frac{g^2}{2} \rfloor$. Since $\lambda_{\lfloor \frac{g^2}{2} \rfloor} \leq g$, we know that the set of all non-gaps between 0 and $2g$ must contain all the sums \{\lambda_i + \lambda_j : 0 \leq i, j \leq \lfloor \frac{g^2}{2} \rfloor\}. But the number of non-gaps between 0 and $2g$ is $g + 1$, while by Lemma 3 the set of sums above has at least $2\lfloor \frac{g^2}{2} \rfloor + 1$ elements.

If $g$ is even then by the second item of Lemma 3 we get that $\lambda_i = i\lambda_1$ for $i \leq \frac{g}{2}$. Now $\lambda_\frac{g}{2} = \frac{g}{2}\lambda_1 \leq g$ means that $\lambda_1 \leq 2$. If $\lambda_1 = 1$ this contradicts $g > 0$.

So, $\lambda_i = 2i$ for $0 \leq i \leq \frac{g}{2}$ and the remaining non-gaps between $g + 1$ and $2g$ are necessarily $\lambda_i = 2i$ for $i = \frac{g}{2} + 1$ to $i = g$.

If $g$ is odd and $g \notin \Lambda$ then we know that the set of all non-gaps between 0 and $2g - 1$ must contain all the sums \{\lambda_i + \lambda_j : 0 \leq i, j \leq \lfloor \frac{g^2}{2} \rfloor\}. But the number of non-gaps between 0 and $2g - 1$ is $g$, while by Lemma 3 the set of sums above has at least $2\lfloor \frac{g^2}{2} \rfloor + 1 = g$ elements and we can argue as before.

2.2 The tree $T_g$

The definition of the ordinarization transform of a numerical semigroup allows the construction of a tree $T_g$ on the set of all numerical semigroups of a given genus rooted at the unique ordinary semigroup of this genus, where the parent of a semigroup is its ordinarization transform and the descendants of a semigroup are the semigroups obtained by taking away a generator larger than the Frobenius number and adding a new non-gap smaller than the multiplicity in a licit place. To illustrate this construction with an example in Figure 1 we depicted $T_6$. The depth of a numerical semigroup of genus $g$ in $T_g$ is then its ordinarization number.

The next two lemmas show some relations between $T$ and $T_g$.

Lemma 5. If $\Lambda_1$ is a descendant of $\Lambda_2$ in $T$ then $\Lambda_1'$ is a descendant of $\Lambda_2'$ in $T$.

Proof. This is obvious when $\Lambda_1$ (and so $\Lambda_2$) is an ordinary semigroup. Suppose that $\Lambda_1$ is not ordinary. It is easy to check that in this case $\Lambda_1$ and $\Lambda_2$ have the same multiplicity $m$ and that $\Lambda_1 \setminus \{m\}$ is a descendant of $\Lambda_2 \setminus \{m\}$ in $T$. Now the lemma is a consequence of the fact that if we adjoin to a semigroup (in our case $\Lambda_1 \setminus \{m\}$) its Frobenius number we obtain its parent in $T$ (in our case $\Lambda_2 \setminus \{m\} = \Lambda_1'$) and if we repeat the same procedure (in our case obtaining $\Lambda_2'$) we obtain the parent of the parent in $T$ (in our case the parent of $\Lambda_1'$ in $T$).

Lemma 6. If two non-ordinary semigroups $\Lambda_1$ and $\Lambda_2$ with the same genus $g$ have the same parent in $T$ then they also have the same parent in $T_g$.

Proof. Indeed, the parent of $\Lambda_1$ and $\Lambda_2$ in $T_g$ is the parent they have in $T$ without its multiplicity.
Figure 1: $T_6$
2.3 Conjecture

Let \(n_{g,r}\) be the number of numerical semigroups of genus \(g\) and ordinarization number \(r\). For each genus \(g \leq 49\) we computed \(n_{g,r}\) for each ordinarization number \(r\) from 0 up to \(\lfloor \frac{g}{2} \rfloor\). The results are given in Table 1. One can observe that for each ordinarization number the number of semigroups of this ordinarization number increases with the genus or stays the same. By extending the definition of \(n_{g,r}\) for \(r > \lfloor \frac{g}{2} \rfloor\) by setting \(n_{g,r} = 0\) in this case, this leads to the next conjecture.

**Conjecture 7.** For each genus \(g \in \mathbb{N}_0\) and each ordinarization number \(r \in \mathbb{N}_0\), \(n_{g,r} \leq n_{g+1,r}\).

This is equivalent to the number of numerical semigroups at a given depth of \(T_g\) being at most the number of numerical semigroups at the same depth of \(T_{g+1}\). If the conjecture were true then the total number of nodes in \(T_g\) would be at most the total number of nodes in \(T_{g+1}\) proving that \(n_g\) increases with \(g\).

3 Partial proofs of the conjecture

We will prove the conjecture for particular values of the pair \(g, r\). We wrote these values in bold face in Table 1. It is obvious that for \(r = 0\) we always have \(n_{g,r} = 1\) since for any genus the ordinary semigroup is the unique numerical semigroup of ordinarization number 0. In the next subsections we will prove the conjecture for \(n_{g,1}\) and any \(g\) and for \(n_{g,r}\) and any \(g\), whenever \(r \geq \max\left(\frac{g}{4} + \frac{g}{2}, \frac{g+1}{2}\right) - 14\).

3.1 Ordinarization number 1

**Lemma 8.** Let \(g \in \mathbb{N}_0\). The number of semigroups of genus \(g\) and ordinarization number 1 is

\[
n_{g,1} = \left\lfloor \frac{g-1}{2} \right\rfloor \left\lceil \frac{g+1}{2} \right\rceil + \left\lfloor \frac{g-1}{2} \right\rfloor \left\lceil \frac{g+1}{2} \right\rceil = \begin{cases} \frac{3}{8}g^2 - \frac{1}{8}g & \text{if } g \text{ is even,} \\ \frac{3}{8}g^2 - \frac{3}{8} & \text{if } g \text{ is odd.} \end{cases}
\]

**Proof.** The semigroups of ordinarization number one are obtained from the ordinary semigroup \(\{0, g+1, g+2, \ldots\}\) by taking out one non-gap \(a\) and adding a non-zero non-gap \(b\) smaller than \(g+1\). Since any element in the ordinary semigroup larger than \(2g+1\) is a sum of two non-zero non-gaps it can not be taken out. So,

\[
g + 1 \leq a \leq 2g + 1.
\]

Fix \(a\) in the previous range. For \(b\) we have four necessary conditions which together become sufficient:

1. \(b \leq g\) since \(b\) must be a gap of the ordinary semigroup;
2. \(a - b \leq g\) since otherwise \(a = b + (a - b)\) and so \(a\) must be in the new semigroup;
3. $2b \geq g + 1$ because otherwise $b, 2b$ are two different non-gaps which are at most $g$ contradicting that the ordinarization number of the new semigroup is 1;

4. $2b \neq a$ because otherwise $a$ must be a non-gap.

From the first three conditions we deduce

$$\max \left( a - g, \left\lceil \frac{g + 1}{2} \right\rceil \right) \leq b \leq g.$$ 

Now, taking also the fourth condition into consideration we get that the number of options for the pair $a, b$ (and so $n_{g,1}$) is

$$n_{g,1} = \sum_{a=g+1}^{2g+1} (g + 1 - \max(a - g, \left\lceil \frac{g + 1}{2} \right\rceil)) - \#\{\text{even integers in } \{g + 1, \ldots, 2g + 1\}\}$$

$$= \sum_{a=g+1}^{\left\lceil \frac{g+1}{2} \right\rceil+g} (g + 1 - \left\lceil \frac{g + 1}{2} \right\rceil) + \sum_{a=g+1}^{\left\lceil \frac{g+1}{2} \right\rceil+g} (g + 1 - a + g) - \left\lceil \frac{g + 1}{2} \right\rceil$$

$$= \sum_{a=g+1}^{\left\lceil \frac{g+1}{2} \right\rceil+g} \left( g + 1 \right) + \sum_{a=g+1}^{\left\lceil \frac{g+1}{2} \right\rceil+g} (2g + 1 - a) - \left\lceil \frac{g + 1}{2} \right\rceil$$

$$= \left\lfloor \frac{g + 1}{2} \right\rfloor \left( g + 1 \right) + \sum_{i=0}^{\left\lfloor \frac{g+1}{2} \right\rfloor} i - \left\lceil \frac{g + 1}{2} \right\rceil$$

$$= \left\lfloor \frac{g + 1}{2} \right\rfloor \left( g + 1 \right) + \frac{\left\lfloor \frac{g + 1}{2} \right\rfloor}{2} \left( \left\lfloor \frac{g + 1}{2} \right\rfloor \right).$$

\[\square\]

**Corollary 9.** For each genus $g \in \mathbb{N}_0$, $n_{g,1} \leq n_{g+1,1}$.

**Proof.** If $g$ is even and $g + 1$ is odd then

$$n_{g,1} = \frac{3}{8} g^2 - \frac{1}{4} g$$

$$n_{g+1,1} = \frac{3}{8} (g + 1)^2 - \frac{3}{8} = \frac{3}{8} g^2 + \frac{3}{8} g + \frac{3}{4}$$

So, $n_{g+1,1} = n_{g,1} + g \geq n_{g,1}$.

On the other hand, if $g$ is odd and $g + 1$ is even then

$$n_{g,1} = \frac{3}{8} g^2 - \frac{3}{8}$$

$$n_{g+1,1} = \frac{3}{8} (g + 1)^2 - \frac{1}{4} (g + 1) = \frac{3}{8} g^2 + \frac{g}{2} + \frac{1}{8}$$

So, $n_{g+1,1} = n_{g,1} + \frac{g + 1}{2} \geq n_{g,1}$. \[\square\]
3.2 High ordinarization numbers

Next we will need Freiman’s Theorem [7, 8] as formulated in [12].

**Theorem 10 (Freiman).** Let $A$ be a set of integers such that $\# A = k \geq 3$. If $\#(A + A) \leq 3k - 4$, then $A$ is a subset of an arithmetic progression of length $\#(A + A) - k + 1 \leq 2k - 3$.

By means of Freiman’s Theorem we can prove the next lemma which tells that the semigroups of large ordinarization number have the first non-gaps even.

**Lemma 11.** If a semigroup $\Lambda$ of genus $g$ has ordinarization number $r$ with $\frac{g+2}{3} \leq r \leq \frac{g}{2}$ then all its non-gaps which are less than or equal to $g$ are even.

**Proof.** Suppose that $\Lambda$ is a semigroup of genus $g$ and ordinarization number $r \geq \frac{g+2}{3}$. This means in particular that $\lambda_0 = 0, \lambda_1, \ldots, \lambda_r \leq g$ and $\lambda_{r+1} \geq g + 1$.

Let $A = \Lambda \cap [0, g] = \{\lambda_0, \lambda_1, \ldots, \lambda_r\}$. We have that the elements in $A + A$ are upper bounded by $2g$ and so $A + A \subseteq \Lambda \cap [0, 2g]$. Then $\# (A + A) \leq \# (\Lambda \cap [0, 2g])$.

Since the Frobenius number of $\Lambda$ is at most $2g - 1$, $\# (\Lambda \cap [0, 2g]) = 2g + 1 - g = g + 1$. So, $\# A = r + 1$ while $\# (A + A) \leq g + 1$. Now, since $r \geq \frac{g+2}{3}$ we have $g + 1 \leq 3r - 1 = 3(r + 1) - 4$ and we can apply Theorem 10 with $k = r + 1$.

Thus we have that $A$ is a subset of an arithmetic progression of length at most $g + 1 - k + 1 = g - r + 1$. Let $d(A)$ be the common difference of this arithmetic progression.

Now, $d(A)$ can not be larger than or equal to three since otherwise $\lambda_r \geq r \cdot d(A) \geq 3r \geq 3\cdot \frac{g+2}{3} > g$, a contradiction with $r$ being the ordinarization number.

If $d(A) = 1$ then $A \subseteq [0, g - r]$ and we claim that in this case $A \subseteq \{0\} \cup [\left\lfloor \frac{g+1}{2} \right\rfloor, g - r]$. Indeed, suppose that $x \in A$. Then $2x$ satisfies either $2x \leq g - r$ or $2x \geq g + 1$. If the second inequality is satisfied then it is obvious that $x \in [0, g - r]$. If the first inequality is satisfied then we will prove that $mx \leq g - r$ for all $m \geq 2$ by induction on $m$ and this leads to $x = 0$. Indeed, if $mx \leq g - r$ then $x \leq \frac{g - r}{m} \leq \frac{g - \frac{g+2}{3}}{m} = \frac{2g - 2}{3m} < \frac{2g}{3m}$. Now $(m + 1)x < \frac{2g(m+1)}{(2m+1)3} = \frac{2g}{3m}$ and since $m \geq 2$, we have $(m + 1)x < \frac{2g(m+1)}{(2m+1)3} = g$. Since $(m + 1)x$ is in $\Lambda \cap [0, g] = A \subseteq [0, g - r]$ this means that $(m + 1)x \leq g - r$ and this proves the claim.

Now, $A \subseteq \{0\} \cup [\left\lfloor \frac{g+1}{2} \right\rfloor, g - r]$ implies that $r \leq g - r - \left\lfloor \frac{g+1}{2} \right\rfloor + 1 = \left\lfloor \frac{g+1}{2} \right\rfloor - r \leq \frac{g+1}{2} - \frac{g+2}{3} = \frac{-1}{6} < r$, a contradiction.

So, we deduce that $d(A) = 2$, leading to the proof of the lemma.

**Lemma 12.** Suppose that a numerical semigroup $\Lambda$ has $\omega$ gaps between 1 and $n - 1$ and $n \geq 2\omega + 2$ then

1. $n \in \Lambda$,

2. the Frobenius number of $\Lambda$ is smaller than $n$,

3. the genus of $\Lambda$ is $\omega$. 

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Proof. 1. The number of pairs $s, t$ with $1 \leq s \leq t \leq n - 1$ such that $s + t = n$ is \( \left\lfloor \frac{n}{2} \right\rfloor \geq \left\lfloor \frac{n + 1}{2} \right\rfloor = \omega + 1 \). Since there are $\omega$ gaps between 1 and $n - 1$ this means that in at least one of these pairs both $s$ and $t$ are non-gaps and so $n$ is a sum of two non-gaps and so a non-gap.

2. Using the same argument one can show by induction that $m \in \Lambda$ for all $m \geq n$. Thus, the Frobenius number must be smaller than $n$.

3. It is a consequence of the previous statements.

A consequence of this Lemma 12 is the well known fact that the Frobenius number of a numerical semigroup of genus $g$ is at most $2g - 1$. Indeed, otherwise take $\omega = g - 1$ and $n$ the Frobenius number of the semigroup and get a contradiction.

Let $\Lambda$ be a numerical semigroup. We say that a set $B \subset \mathbb{N}_0$ is $\Lambda$-closed if for any $b \in B$ and any $\lambda$ in $\Lambda$, the sum $b + \lambda$ is either in $B$ or it is larger than $\max(B)$. If $B$ is $\Lambda$-closed so is $B - \min(B)$. Indeed, $b - \min(B) + \lambda = (b + \lambda) - \min(B)$ is either in $B - \min(B)$ or it is larger than $\max(B) - \min(B) = \max(B - \min(B))$.

The new $\Lambda$-closed set contains 0. We will denote by $C(\Lambda, i)$ the $\Lambda$-closed sets of size $i$ that contain 0.

**Theorem 13.** Let $g \in \mathbb{N}_0$ and let $r$ be an integer with $\frac{g + 2}{3} \leq r \leq \left\lfloor \frac{g}{2} \right\rfloor$. Define $\omega = \left\lfloor \frac{g}{2} \right\rfloor - r - 1$.

1. If $\Omega$ is a numerical semigroup of genus $\omega$ and $B$ is a $\Omega$-closed set of size $\omega + 1$ and first element equal to 0 then

\[
\{2j : j \in \Omega\} \cup \{2j - 2\max(B) + 2g + 1 : j \in B\} \cup (2g + \mathbb{N}_0)
\]

is a numerical semigroup of genus $g$ and ordinarization number $r$.

2. All numerical semigroups of genus $g$ and ordinarization number $r$ can be uniquely written as

\[
\{2j : j \in \Omega\} \cup \{2j - 2\max(B) + 2g + 1 : j \in B\} \cup (2g + \mathbb{N}_0)
\]

for a unique numerical semigroup $\Omega$ of genus $g$ and a unique $\Omega$-closed set $B$ of size $\omega + 1$ and first element equal to 0.

3. The number of numerical semigroups of genus $g$ and ordinarization number $r$ depends only on $\omega$. It is exactly

\[
\sum_{\text{Semigroups } \Omega \text{ of genus } \omega} \#C(\Omega, \omega + 1).
\]

Proof. 1. Suppose that $\Omega$ is a numerical semigroup of genus $\omega$ and $B$ is a $\Omega$-closed set of size $\omega + 1$ and first element equal to 0. Let $X = \{2j : j \in \Omega\}$, $Y = \{2j - 2\max(B) + 2g + 1 : j \in B\}$, $Z = (2g + \mathbb{N}_0)$. 

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2. First of all notice that the complement $N_0 \setminus (X \cup Y \cup Z)$ has $g$ elements. Notice that all elements in $X$ are even while all elements in $Y$ are odd. So, $X$ and $Y$ do not intersect. Also the unique element in $Y \cap Z$ is $2g + 1$. The number of elements in the complement will be

$$\#N_0 \setminus (X \cup Y \cup Z) = 2g - \#\{x \in X : x < 2g\} - \#Y + 1 = 2g - \#\{s \in \Omega : s < g\} - \#B + 1 = 2g - \omega - \#\{s \in \Omega : s < g\}.$$ 

We know that all gaps of $\Omega$ are at most $2\omega - 1 < g$. So, $\#\{s \in \Omega : s < g\} = g - \omega$ and we conclude that $\#N_0 \setminus (X \cup Y \cup Z) = g$.

Before proving that $X \cup Y \cup Z$ is a numerical semigroup, let us prove that the number of non-zero elements in $X \cup Y \cup Z$ which are smaller than or equal to $g$ is $r$. Once we prove that $X \cup Y \cup Z$ is a numerical semigroup, this will mean, by Lemma 1, that it has ordinarization number $r$. On one hand, all elements in $Y$ are larger than $g$. Indeed, if $\lambda$ is the enumeration of $\Omega$ then $\max(B) \leq \lambda_\omega \leq 2\omega = (2\lfloor \frac{g}{2} \rfloor - 2r) \leq g - 2\lfloor \frac{g+2}{2} \rfloor - \frac{g}{2}$. Now, for any $j \in B$, $2j - 2\max(B) + 2g + 1 > 2g - 2\max(B) > g$. On the other hand, all gaps of $\Omega$ are at most $2\omega - 1 = 2\lfloor \frac{g}{2} \rfloor - 2r - 1 \leq g - 2\lfloor \frac{g+2}{2} \rfloor - 1 \leq \frac{g}{2} - 1$ and so all the even integers not belonging to $X$ are less than or equal to $g$. So, the number of non-zero non-gaps of $X \cup Y \cup Z$ smaller than or equal to $g$ is $\lfloor \frac{g}{2} \rfloor - \omega = r$.

To see that $X \cup Y \cup Z$ is a numerical semigroup we only need to see that it is closed under addition. It is obvious that $X + Z \subseteq Z$, $Y + Z \subseteq Z$, $Z + Z \subseteq Z$. It is also obvious that $X + X \subseteq X$ since $\Omega$ is a numerical semigroup and that $Y + Y \subseteq Z$ since, as we proved before, all elements in $Y$ are larger than $g$.

It remains to see that $X + Y \subseteq X \cup Y \cup Z$. Suppose that $x \in X$ and $y \in Y$. Then $x = 2i$ for some $i \in \Omega$ and $y = 2j - 2\max(B) + 2g + 1$ for some $j \in B$. Then $x + y = 2(i + j) - 2\max(B) + 2g + 1$. Since $B$ is $\Omega$-closed, we have that either $i + j \in B$ and so $x + y \in Y$ or $i + j > \max(B)$. In this case $x + y \in Z$. So, $X + Y \subseteq Y \cup Z$.

2. First of all notice that, since the Frobenius number of a semigroup $\Lambda$ of genus $g$ is smaller than $2g$, it holds

$$\Lambda \cap (2g + N_0) = (2g + N_0).$$

For any numerical semigroup the set $\Omega = \{\frac{g}{2} : j \in \Lambda \cap (2N_0)\}$ is a numerical semigroup. If $\Lambda$ has ordinarization number $r \geq \frac{4g^2}{3}$, then, by Lemma 11

$$\Lambda \cap [0, g] = (2\Omega) \cap [0, g].$$

The semigroup $\Omega$ has exactly $r + 1$ non-gaps between 0 and $\lfloor \frac{g}{2} \rfloor$ and $\omega = \lfloor \frac{g}{2} \rfloor - r$ gaps between 0 and $\lfloor \frac{g}{2} \rfloor$. We can use Lemma 12 with $n = \lfloor \frac{g}{2} \rfloor + 1$ since

$$2\omega + 2 = 2\lfloor \frac{g}{2} \rfloor - 2r + 2 \leq g - \frac{2(g+2)}{3} + 2 = \frac{g + 2}{3}.$$

10
which implies $2\omega + 2 \leq |\frac{4r+2}{3}| \leq |\frac{4r+2}{3}| = n$. Then the genus of $\Omega$ is $\omega$ and the Frobenius number of $\Omega$ is at most $\left\lfloor \frac{g}{2} \right\rfloor$. This means in particular that all even integers larger than $g$ belong to $\Lambda$.

Define $D = (\Lambda \cap [0,2g)) \setminus 2\Omega$. That is, $D$ is the set of odd non-gaps of $\Lambda$ smaller than $2g$. We claim that $B = \left\{ \frac{j-1}{2} : j \in D \cup \{2g+1\} \right\}$ is an $\Omega$-closed set of size $\omega + 1$. The size follows from the fact that the number of non-gaps of $\Lambda$ between $g+1$ and $2g$ is $g - r$ and that the number of even integers in the same interval is $\left\lceil \frac{g}{2} \right\rceil$. Suppose that $\lambda \in \Omega$ and $b \in \bar{B}$. Then $b = \frac{j-1}{2}$ for some $j$ in $D \cup \{2g+1\}$ and $b + \lambda = \frac{(j+2\lambda-1)}{2}$. If $\frac{(j+2\lambda-1)}{2} \geq \max(B) = \frac{(2g+1)-1}{2}$ we are done. Otherwise we have $j + 2\lambda \leq 2g$. Since $\Lambda$ is a numerical semigroup and both $j, 2\lambda \in \Lambda$, it holds $j + 2\lambda \in \Lambda \cap [0,2g]$. Furthermore, $j + 2\lambda$ is odd since so is $j$. So, $b + \lambda$ is either larger than $\max(B)$ or it is in $\bar{B}$. Then $B = \bar{B} - \min(B)$ is a $\Lambda$-closed set of size $\omega + 1$ and first element zero.

3. It is a consequence of the previous point.

Define the sequence $f_\omega$ by $f_\omega = \sum \text{Semigroups } \Omega \text{ of genus } \omega \# C(\Omega, \omega + 1)$. The first elements in the sequence, from $f_0$ to $f_{14}$ are

| $\omega$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|
| $f_\omega$ | 1 | 2 | 7 | 23 | 68 | 200 | 615 | 1764 | 5060 | 14626 | 41785 | 117573 | 332475 | 933891 | 2609832 |

From these first elements and Theorem 13 we can deduce $n_{g,r}$ for any $g$, whenever $r \geq \max\left(\frac{4r+2}{3}, \left\lfloor \frac{g}{2} \right\rfloor - 14\right)$. This is illustrated in Table 2. Since the sequence $f_\omega$ is increasing for $\omega$ between 0 and 14 we deduce the next corollary.

**Corollary 14.** For any $g \in \mathbb{N}$ and any $r \geq \max\left(\frac{4r+2}{3}, \left\lfloor \frac{g}{2} \right\rfloor - 14\right)$, it holds $n_{g,r} \geq n_{g+1,r}$.

If we proved that $f_\omega \leq f_{\omega+1}$ for any $\omega$, this would imply $n_{g,r} \leq n_{g+1,r}$ for any $r > \frac{g}{2}$.

**4 On numerical semigroups with a large number of gap intervals**

In this final section we present a bijection between semigroups at a given depth in the ordinarization tree and semigroups with a given (large) number of gap intervals.

**Lemma 15.** Suppose that a numerical semigroup $\Lambda$ has genus $g$ and ordinarization number $r \geq \frac{g+2}{3}$. Then it has $2r$ intervals of gaps if $g$ is even and $2r + 1$ intervals of gaps if $g$ is odd.

**Proof.** By Theorem 13 we deduce that, defining $\omega = \left\lfloor \frac{g}{2} \right\rfloor - r$, $\Lambda$ should be

$$\{2j : j \in \Omega\} \cup \{2j - 2 \max(B) + 2g + 1 : j \in B\} \cup (2g + \mathbb{N}_0)$$
for a unique numerical semigroup $\Omega$ of genus $\omega$ and a unique $\Omega$-closed set $B$ of size $\omega + 1$ and first element equal to 0. But this semigroup has $g - 2\omega$ intervals of gaps. This number is equal to $g - 2\left\lfloor \frac{\omega}{2} \right\rfloor + 2r$ which equals $2r$ if $g$ is even and $2r + 1$ if $g$ is odd.

Lemma 16. A numerical semigroup with $n$ intervals of gaps has ordinarization number at least $\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. The maximum number of gaps between 1 and $g$ is obtained for the semigroup (should it be a semigroup) that has $g - n + 1$ non-gaps in a row right after 0 and then $n - 1$ sequences of a non-gap and a gap and then no more gaps. In this case there would be $\left\lceil \frac{n - 1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ non-zero non-gaps in the same interval. □

Theorem 17. Suppose that a numerical semigroup $\Lambda$ has genus $g$ and $n$ intervals of gaps with $\left\lfloor \frac{n}{2} \right\rfloor \geq \frac{g + 2}{3}$. Then $g$ and $n$ have the same parity and $\Lambda$ has ordinarization number equal to $\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. By Lemma 16 $r \geq \frac{g + 2}{3}$, and Lemma 15 gives the result. □

Concluding, if $\left\lfloor \frac{n}{2} \right\rfloor \geq \frac{g + 2}{3}$, then the set of numerical semigroups of genus $g$ and $n$ intervals of gaps is empty if $n$ and $g$ have different parity and it is exactly the set of numerical semigroups of genus $g$ and ordinarization number $\left\lfloor \frac{n}{2} \right\rfloor$ otherwise.

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Table 1: Number of numerical semigroups of each ordinarization number for each genus $g \leq 49$. 

| $r$ | $g = 47$ | $g = 42$ | $g = 36$ | $g = 32$ | $g = 31$ | $g = 30$ | $g = 29$ | $g = 28$ | $g = 26$ | $g = 25$ | $g = 23$ | $g = 22$ | $g = 21$ | $g = 19$ | $g = 18$ | $g = 17$ |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 17  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 16  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 15  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 14  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 13  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 12  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 11  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 10  |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 9   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 8   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 7   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 6   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 5   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 4   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 3   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 2   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
| 1   |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
Table 2: Numbers $n_{g,r}$ deduced from Lemma $\S$ and Theorem $\S$ for each genus $g \leq 100$. 

| $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|---
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