Boson Realization of the $su(3)$-Algebra. II

Holstein-Primakoff Representation for the Lipkin Model

Constança Providência, João da Providência, Yasuhiko Tsue and Masatoshi Yamamura

1Departamento de Física, Universidade de Coimbra, 3004-516 Coimbra, Portugal
2Physics Division, Faculty of Science, Kochi University, Kochi 780-8520, Japan
3Faculty of Engineering, Kansai University, Suita 564-8680, Japan

Abstract

On the basis of the Schwinger boson representation for the Lipkin model developed in (I), the Holstein-Primakoff representation for the $su(3)$-algebra is presented. Including the symmetric case, the representation obtained in this paper contains all the cases.
§1. Introduction

It is well known that we have two forms for the boson realization of the $su(2)$-algebra. One is called the Schwinger boson representation\(^1\) and the other the Holstein-Primakoff boson representation.\(^2\) Each has its own characteristic and merit. In the Schwinger representation, the $su(2)$-generators can be expressed in terms of bilinear form with respect to two kinds of boson operators. In the Holstein-Primakoff representation, they are expressed in terms of one kind of boson operator with the square root type operator.

When we describe fluctuation around the equilibrium, boson operators are helpful. In the case of the Holstein-Primakoff representation, the equilibrium is introduced from the outside and the boson operator plays a role of describing only fluctuation. Contrarily, in the case of the Schwinger representation, the equilibrium itself is treated by boson operators, and then, the separation of the boson operators into two roles may be devised in each problem under investigation not only for the equilibrium but also for the fluctuation. Therefore, if the fluctuation is not so large, the Holstein-Primakoff representation permits us to describe the fluctuation more transparently than the case of the Schwinger boson representation.

In the previous paper,\(^3\) which, hereafter, is referred to as (I), we presented the Schwinger boson representation for the $su(3)$-algebra.\(^3\) This form is a concrete development of the $su(3)$-algebra in the case of the $su(M + 1)$-algebra for the Lipkin model.\(^4\) With the aid of the framework given in this form, we can describe the $su(3)$-Lipkin model in terms of the Schwinger boson representation. From the reason mentioned in the introductory part of this section, it may be interesting to present the Holstein-Primakoff representation in a complete form. The case of the symmetric representation is well known.\(^5\)

A main aim of this paper is to present the Holstein-Primakoff representation of the $su(3)$-algebra as a disguised form of the Schwinger representation developed in (I). Usually, the Holstein-Primakoff representation is obtained in terms of certain ordering of boson operators after applying the MYT boson mapping method proposed by Marumori, Yamamura and Tokunaga.\(^6\) This idea was demonstrated by Marshalek.\(^5\) However, in this paper, we adopt another idea. Regarding the intrinsic state in the Schwinger representation as representing the equilibrium, the fluctuations around the equilibrium are described by boson operators. In the case of the $su(2)$-algebra, the intrinsic state is specified by one quantum number which expresses the magnitude of the $su(2)$-spin. Then, the Holstein-Primakoff representation of the $su(2)$-algebra is characterized by the magnitude of the $su(2)$-spin. In the case of the $su(3)$-algebra, the intrinsic state is characterized by two quantum numbers, and then, the Holstein-Primakoff representation is characterized by two quantum numbers. Of course, the $su(3)$-generators are expressed in terms of these two quantum numbers.
In §2, the basic idea is demonstrated in the case of well-known $su(2)$-algebra. Section 3 is a central part of this paper. Following basic idea developed in §2, the Holstein-Primakoff representation for the $su(3)$-algebra is presented. In §4, various properties of the representation obtained in §3 are discussed. Especially, the physical space, which is a certain subspace in the whole boson space, is given. Finally, the case of the simplest approximation is discussed in relation to the RPA method in the fermion space.

§2. An illustrative example for the basic idea — the case of the $su(2)$-algebra

For the preparation for our final aim, we will sketch the case of the $su(2)$-algebra. Through this sketch, we can demonstrate our basic viewpoint. The $su(2)$-algebra consists of three generators $\hat{S}_\pm, \hat{S}_0$, which can be expressed in terms of two kinds of bosons ($\hat{a}, \hat{a}^*$) and ($\hat{b}, \hat{b}^*$):

$$\hat{S}_+ = \hat{a}^* \hat{b} , \quad \hat{S}_- = \hat{b}^* \hat{a} , \quad \hat{S}_0 = (1/2)(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) .$$  \hspace{1cm} (2.1)

In the representation (2.1), an operator which commutes with $\hat{S}_\pm, \hat{S}_0$ can be defined as follows:

$$\hat{S} = (1/2)(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) .$$  \hspace{1cm} (2.2)

The intrinsic state $|s\rangle$ obeys the condition

$$\hat{S}_- |s\rangle = 0 , \quad \hat{S}_0 |s\rangle = -s |s\rangle .$$  \hspace{1cm} (2.3)

Explicitly, $|s\rangle$ is given as

$$|s\rangle = \left(\frac{1}{\sqrt{(2s)!}}\right)^{-1} (\hat{b}^*)^{2s} |0\rangle . \quad (s = 0, 1/2, 1, \cdots)$$  \hspace{1cm} (2.4)

In (A), we used the notation

$$m_0 = 2s . \quad (m_0 = 0, 1, 2, \cdots)$$  \hspace{1cm} (2.5)

Since $\hat{S}_+$ plays a role of the excited state generating operator, the excited state denoted as $|s, s_0\rangle$ is obtained in the form

$$|s, s_0\rangle = \sqrt{\frac{1}{(2s)! (s + s_0)!}} (\hat{S}_+)^{s+s_0} |s\rangle . \quad (s_0 = -s, -s + 1, \cdots, s - 1, s)$$  \hspace{1cm} (2.6)

Operation of $\hat{S}$ on the state $|s, s_0\rangle$ gives us

$$\hat{S} |s, s_0\rangle = s |s, s_0\rangle .$$  \hspace{1cm} (2.7)
The definition of $\hat{S}$ leads us

$$(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) |s, s_0\rangle = 2s |s, s_0\rangle ,$$

(2.8a)

i.e.,

$$\hat{b}^* \hat{b} |s, s_0\rangle = \left(\sqrt{2s - \hat{a}^* \hat{a}}\right)^2 |s, s_0\rangle .$$

(2.8b)

From the relation (2.8), we have

$$\hat{b}^* \hat{b} |c(m_0)\rangle = \left(\sqrt{m_0 - \hat{a}^* \hat{a}}\right)^2 |c(m_0)\rangle .$$

(2.9)

Here, $|c(m_0)\rangle$ is an arbitrary superposition of the states $|s, s_0\rangle$ ($s_0 = -s, -s+1, \cdots, s-1, s)$:

$$|c(m_0)\rangle = \sum_{s_0} c_{s_0}(s) |s, s_0\rangle . \quad (m_0 = 2s)$$

(2.10)

For the relation (2.9), we can give the following interpretation: The intrinsic state $|s\rangle$, which is expressed only in terms of the boson $\hat{b}^*$, plays the same role as that in the free vacuum of the Hartree-Fock theory $|f\rangle$. It shows the equilibrium which is characterized by the eigenvalue of the density matrix, i.e., \(\rho_n = 1\) and \(0\). The state $|s\rangle$ is characterized by the eigenvalue of $\hat{b}^* \hat{b}$, i.e., \(2s\). Fluctuations around $|f\rangle$ can be described in terms of the operations of the particle-hole pairs on $|f\rangle$ and for the sake of the fluctuations, the value of \(\rho_n\) decreases from \(1\). In the present system, fluctuations around $|s\rangle$ can be described in the framework of the operations of $\hat{S}_+$ on $|s\rangle$, i.e., the operations of $\hat{a}^*$ and $\hat{b}$. The operations of $\hat{a}^*$ make the fluctuations increase and the operations of $\hat{b}$ make the effect of $m_0$ decrease.

The relation (2.9) teaches us the above interpretation.

Concerning the above interpretation, the Holstein-Primakoff representation may be superior to the Schwinger representation. We introduce a new boson space in which, in this paper, the boson operator is denoted as $(\hat{\alpha}, \hat{\alpha}^*)$ and the vacuum $|0\rangle$ plays the same role as that of $|s\rangle$ in the Schwinger representation. The fluctuations around $|0\rangle$ can be described in terms of the operations of $\hat{\alpha}^*$ and the decrease from the value $m_0$ is given by the operations of the operator $\sqrt{m_0 - \hat{\alpha}^* \hat{\alpha}}$. This is suggested by the relation (2.9). Then, the correspondence to the Schwinger representation is summarized as follows:

$$|s\rangle \sim |0\rangle ,$$

(2.11a)

$$(\hat{a}, \hat{a}^*) \sim (\hat{\alpha}, \hat{\alpha}^*) ,$$

(2.11b)

$$\hat{b} \text{ and } \hat{b}^* \sim \sqrt{m_0 - \hat{\alpha}^* \hat{\alpha}} . \quad (m_0 = 2s)$$

(2.11c)
Under the correspondence (2.11), the form (2.1) is replaced with the form
\[
\hat{S}_+ \sim \bar{S}_+ = \hat{\alpha}^* \sqrt{m_0 - \hat{\alpha}^* \hat{\alpha}} ,
\hat{S}_- \sim \bar{S}_- = \sqrt{m_0 - \hat{\alpha}^* \hat{\alpha}} ,
\hat{S}_0 \sim \bar{S}_0 = \hat{\alpha}^* \hat{\alpha} - m_0 / 2 .
\] (2.12)

The above is identical with the Holstein-Primakoff representation and we can prove
\[
[ \bar{S}_+ , \bar{S}_- ] = 2 \bar{S}_0 , \quad [ \bar{S}_0 , \bar{S}_\pm ] = \pm \bar{S}_\pm .
\] (2.13)

In the strict sense, the form (2.12) is valid in the subspace spanned by
\[
|n\rangle = \left( \sqrt{n!} \right)^{-1} (\hat{\alpha}^*)^n |0\rangle . \quad (n = 0, 1, \cdots , m_0)
\] (2.14)

The space spanned by \( \{|n\rangle ; n = 0, 1, \cdots , m_0 \} \) is called the physical space.

\section{§3. The case of the \textit{su}(3)-algebra}

Following the basic viewpoint for constructing the Holstein-Primakoff representation from the Schwinger representation, we present the form of the \textit{su}(3)-algebra. First, we note the relations (I-6-4a) and (I-6-4b):
\[
\begin{align*}
(\hat{b}^* \hat{b} + \hat{\alpha}_+^* \hat{\alpha}_+ + \hat{\alpha}_-^* \hat{\alpha}_-)|I^1 I_0^0 , I I_0^0 ; T\rangle &= m_0 |I^1 I_0^0 , I I_0^0 ; T\rangle , \quad (3.1a) \\
(\hat{\alpha}^* \hat{\alpha} + \hat{b}_+^* \hat{b}_+ + \hat{b}_-^* \hat{b}_-)|I^1 I_0^0 , I I_0^0 ; T\rangle &= m_1 |I^1 I_0^0 , I I_0^0 ; T\rangle . \quad (3.1b)
\end{align*}
\]

The relation (3.1) corresponds to the relation (2.8a) in the \textit{su}(2)-algebra. Taking into account that the intrinsic state is expressed in terms of \( \hat{b}^* \) and \( \hat{b}_-^* \), we rewrite the relation (3.1) in the form
\[
\begin{align*}
\hat{b}^* \hat{b} |I^1 I_0^0 , I I_0^0 ; T\rangle &= \left( \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} \right)^2 |I^1 I_0^0 , I I_0^0 ; T\rangle , \quad (3.2a) \\
\hat{b}_-^* \hat{b}_- |I^1 I_0^0 , I I_0^0 ; T\rangle &= \left( \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{b}_+^* \hat{b}_+} \right)^2 |I^1 I_0^0 , I I_0^0 ; T\rangle . \quad (3.2b)
\end{align*}
\]

Of course, the relation (3.2) gives us
\[
\begin{align*}
\hat{b}^* \hat{b}|c(m_0 , m_1)\rangle &= \left( \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} \right)^2 |c(m_0 , m_1)\rangle , \quad (3.3a) \\
\hat{b}_-^* \hat{b}_-|c(m_0 , m_1)\rangle &= \left( \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{b}_+^* \hat{b}_+} \right)^2 |c(m_0 , m_1)\rangle . \quad (3.3b)
\end{align*}
\]

Here, \( |c(m_0 , m_1)\rangle \) denotes an arbitrary superposition of the state \( |I^1 I_0^0 , I I_0^0 ; T\rangle \):
\[
|c(m_0 , m_1)\rangle = \sum_{I^1 I_0^0} C_{I^1 I_0^0} |I^1 I_0^0 , I I_0^0 ; T\rangle . \quad (3.4)
\]
It should be noted that, as was shown in the relation (I-6.2), \((m_0, m_1)\) is related to \((T, I^0)\) as follows:

\[
m_0 = 2|(T - 3/2) - I^0|, \quad m_1 = 2I^0.
\] (3.5)

The relation (3.3) parallels the relation (2.3) in the interpretation. The fluctuations around the state \(|m_0, m_1\rangle\) can be described in terms of the boson operators \((\hat{a}_+, \hat{a}_+^\dagger), (\hat{a}_-, \hat{a}_-^\dagger), (\hat{a}, \hat{a}^\dagger)\) and \((\hat{b}_+, \hat{b}_+^\dagger)\), and by the fluctuations, the effects of \(m_0\) and \(m_1\) decrease.

Under the above argument, we complete the Holstein-Primakoff representation for the \(su(3)\)-algebra. For this purpose, a new boson space is introduced in terms of bosons \((\hat{a}_+, \hat{a}_-^\dagger), (\hat{a}_-, \hat{a}_+^\dagger), (\hat{a}, \hat{a}^\dagger)\) and \((\hat{\beta}_+, \hat{\beta}_+^\dagger)\). The vacuum \(|0\rangle\) of this boson space plays the same role as that of \(|m_0, m_1\rangle\). In the same way as that in \(\S 2\), we set up the correspondences

\[
|m_0, m_1\rangle \sim |0\rangle, \quad (\hat{a}_+, \hat{a}_+^\dagger), (\hat{a}_-, \hat{a}_-^\dagger), (\hat{a}, \hat{a}^\dagger) \quad \text{and} \quad (\hat{b}_+, \hat{b}_+^\dagger)
\]

\[
\sim (\hat{\alpha}_+, \hat{\alpha}_+^\dagger), (\hat{\alpha}_-, \hat{\alpha}_-^\dagger), (\hat{\alpha}, \hat{\alpha}^\dagger) \quad \text{and} \quad (\hat{\beta}_+, \hat{\beta}_+^\dagger),
\]

\[
\hat{b} \quad \text{and} \quad \hat{b}^\dagger \sim \sqrt{|m_0 - \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\alpha}_-^\dagger \hat{\alpha}_-|},
\]

\[
\hat{b}_- \quad \text{and} \quad \hat{b}_-^\dagger \sim \sqrt{|m_1 - \hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}_+^\dagger \hat{\beta}_+|}. \quad \text{(3.6c)}
\]

The expression (I-2.10) gives us the following form for the \(su(3)\)-generators:

\[
\hat{I}_+ \sim \hat{T}_+ = \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\beta}_+^\dagger \sqrt{|m_1 - \hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}_+^\dagger \hat{\beta}_+|},
\]

\[
\hat{I}_- \sim \hat{T}_- = \hat{\alpha}_-^\dagger \hat{\alpha}_- - \sqrt{|m_1 - \hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}_+^\dagger \hat{\beta}_+|},
\]

\[
\hat{I}_0 \sim \hat{T}_0 = (1/2)(\hat{\alpha}_+^\dagger \hat{\alpha}_+ + \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \hat{\alpha}) - m_1/2,
\]

\[
\hat{M}_0 \sim \hat{M}_0 = (3/2)(\hat{\alpha}_+^\dagger \hat{\alpha}_+ + \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \hat{\alpha}) - (m_0 + m_1/2),
\]

\[
\hat{D}_- \sim \hat{D}_- = \hat{\alpha}^\dagger \sqrt{|m_0 - \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \hat{\alpha}|},
\]

\[
\hat{D}_+ \sim \hat{D}_+ = \hat{\alpha}_+^\dagger \sqrt{|m_0 - \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \hat{\alpha}|},
\]

\[
\hat{D}_- \sim \hat{D}_- = \sqrt{|m_0 - \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \hat{\alpha}|},
\]

\[
\hat{D}_+ \sim \hat{D}_+ = \sqrt{|m_1 - \hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}_+^\dagger \hat{\beta}_+ |}. \quad \text{(3.7c)}
\]

The \(su(1, 1)\)-generators (I-2.11) and \(\tilde{R}_0\) shown in the relation (I-2.14) lead to

\[
\tilde{T}_+ \sim \tilde{T}_+ = \hat{\alpha}^\dagger \sqrt{|m_0 - \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \sqrt{|m_1 - \hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}_+^\dagger \hat{\beta}_+ |}},
\]

\[
\tilde{T}_- \sim \tilde{T}_- = \sqrt{|m_0 - \hat{\alpha}_+^\dagger \hat{\alpha}_+ - \hat{\alpha}_-^\dagger \hat{\alpha}_- + \hat{\alpha}^\dagger \hat{\alpha}|} - \sqrt{|m_1 - \hat{\alpha}^\dagger \hat{\alpha} - \hat{\beta}_+^\dagger \hat{\beta}_+ |},
\]

\[
\tilde{T}_0 \sim \tilde{T}_0 = (1/2)(m_1 + m_0 + 3),
\]

\[
\tilde{R}_0 \sim \tilde{R}_0 = (1/2)(m_1 - m_0). \quad \text{(3.8c)}
\]
Thus, we obtained the Holstein-Primakoff representation of the $su(3)$-algebra. In the next section, we will discuss various features of the representation.

§4. Various properties of the representation

In the previous section, we developed the Holstein-Primakoff representation of the $su(3)$-algebra. As can be seen in the form (3.7), it may be better to call the mixed representation. First of all, we must stress that the representation (3.7) satisfies the commutation relations of the $su(3)$-algebra, which are listed in the relation (I.2.2). Through the straightforward calculation, we can confirm it. However, it should be noted that the expression (3.8) for $(\tilde{T}_-0)$ does not satisfy the commutation relation for the $su(1,1)$-algebra. The forms (3.8b) and (3.8c) are natural, because we construct the form (3.7) under the condition that the eigenvalues of $\tilde{T}_0$ and $\tilde{R}_0$ are $(1/2)(m_1 + m_0 + 3)$ and $(1/2)(m_1 - m_0)$, respectively. But, we will show the importance of the expression (3.8a) for defining the physical space.

Next, our interest is related to the physical space. First, we note the existence of the operators $\sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} \hat{\alpha}^* \hat{\alpha} + \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+}$, which means that $(m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-) \text{ and } (m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+)$ should be positive definite. Then, the physical space should be spanned by the states with the condition

\[
\hat{\alpha}_+^* \hat{\alpha}_+ + \hat{\alpha}_-^* \hat{\alpha}_- = 0, 1, 2, \ldots, m_0,
\]

\[
\hat{\alpha}^* \hat{\alpha} + \hat{\beta}_+^* \hat{\beta}_+ = 0, 1, 2, \ldots, m_1.
\]

From the reason mentioned below, only the condition (4.1) is not sufficient for determining the physical space. The states, which compose the physical space, correspond to the states $|I^1 I^0, I I_0; T\rangle$ shown in the relation (I.4.4), i.e.,

\[
|I^1 I^0, I I_0; T\rangle \sim |I^1 I^0, I I_0; T\rangle.
\]

Under the correspondences (3.6a) and (3.7), the state $|I^1 I^0, I I_0; T\rangle$ is obtained. In order to show $(m_0, m_1)$ explicitly, hereafter, $|I^1 I^0, I I_0; T\rangle$ is abbreviated as

\[
|I^1 I^0, I I_0; T\rangle = |I^1 I^0; m_0 m_1\rangle.
\]

Our present boson space consists of four kinds of the boson operators. However, even if obeying the condition (4.1), $|I^1 I I_0; m_0 m_1\rangle$ is expressed in terms of three quantum numbers, $(I^1, I, I_0)$. Therefore, in order to specify the physical space definitely, a condition additional to restriction (4.1) is necessary. In order to find this condition, we pay attention to the form (3.8a), from which the following relation is derived:

\[
\tilde{T}_-|I^1 I I_0; m_0 m_1\rangle = 0.
\]
In the space where two operators \( \frac{\sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_- - \hat{\alpha}_-^* \hat{\alpha}_-}}{\sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+}} \) and \( \frac{\sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+}}{\sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-}} \) can be defined, we are able to prove the relations

\[
[ \hat{\tau}_-, T_+ ] = [ \hat{\tau}_-, D_\pm ] = 0 ,
\]

\[
\hat{\tau}_- |m_0, m_1\rangle = 0 .
\]  

(4.5a)

(4.5b)

Here, \( \hat{\tau}_- \) is defined as

\[
\hat{\tau}_- = \left( \frac{\sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-}}{\sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+}} \right)^{-1} \hat{T}_- .
\]  

(4.6)

With the use of the relation (4.5), we have

\[
\hat{T}_- |I^1 I^0_0; m_0 m_1\rangle = \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+} \times \hat{\tau}_- |I^1 I^0_0; m_0 m_1\rangle = 0 .
\]  

(4.7)

The relation (4.4) gives us

\[
(I^1 I^0_0; m_0 m_1) |\hat{T}_+ = 0 ,
\]

(4.8a)

namely,

\[
(I^1 I^0_0; m_0 m_1) (\hat{T}_+)^n = 0 . \quad (n = 1, 2, 3, \ldots)
\]  

(4.8b)

Then, we have

\[
(I^1 I^0_0; m_0 m_1) |n I^1 I' I'^0_0; m_0 m_1\rangle = 0 ,
\]

(4.9)

\[
|n I^1 I' I'^0_0; m_0 m_1\rangle = (\hat{T}_+)^n |I^1 I' I'^0_0; m_0 m_1\rangle .
\]  

(4.10)

Since \( \hat{T}_+ \) is also defined in the space obeying the condition (4.1), \( n \) can run in the region governed by the condition. Further, we can prove

\[
\hat{T}_- |n I^1 I' I'^0_0; m_0 m_1\rangle \neq 0 . \quad (n \neq 0)
\]  

(4.11)

The above consideration gives us the conclusion that the physical space is defined as follows:

In all the states composing the bosons \( \hat{\alpha}^*, \hat{\alpha}_+^*, \hat{\alpha}_-^* \) and \( \hat{\beta}_+^*, \) the state \( |ph\rangle \) belonging to the physical space obeys the condition (4.1), and further, the condition \( \hat{T}_- |ph\rangle = 0 . \)

Finally, we will discuss the symmetric representation. In the case \( m_1 = 0 , \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+} \) is defined only in the condition \( \hat{\alpha}^* \hat{\alpha} = \hat{\beta}_+^* \hat{\beta}_+ = 0 . \) Then, the form (3.17) reduces to the following:

\[
\hat{T}_+ = \hat{\alpha}_+^* \hat{\alpha}_- , \quad \hat{T}_- = \hat{\alpha}_-^* \hat{\alpha}_+ , \quad \hat{T}_0 = (1/2)(\hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-) ,
\]

(4.12a)

\[
\overline{M}_0 = (3/2)(\hat{\alpha}_+^* \hat{\alpha}_+ + \hat{\alpha}_-^* \hat{\alpha}_-) - m_0 .
\]

(4.12b)

\[
\overline{D}_- = \hat{\alpha}_-^* \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} , \quad \overline{D}_+ = \hat{\alpha}_+^* \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} ,
\]

(4.12c)

\[
\overline{D}_- = \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} \hat{\alpha}_- , \quad \overline{D}_+ = \sqrt{m_0 - \hat{\alpha}_+^* \hat{\alpha}_+ - \hat{\alpha}_-^* \hat{\alpha}_-} \hat{\alpha}_+ ,
\]

(4.12c)
On the other hand, the case \( m_0 = 0 \) gives us \( \hat{\alpha}^*_+ \hat{\alpha}^- = \hat{\alpha}^*_- \hat{\alpha}^+ = 0 \). Therefore, the form (5.1) reduces to the following form:

\[
\begin{align*}
\hat{T}_+ &= -\hat{\beta}^*_+ \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}^*_+ \hat{\beta}^-_+}, \\
\hat{T}_- &= -\sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}^*_+ \hat{\beta}^-_+}, \\
\hat{T}_0 &= \hat{\beta}^*_+ \hat{\beta}^-_+ - (1/2)(m_1 - \hat{\alpha}^* \hat{\alpha}) , \\
\hat{M}_0 &= (3/2)\hat{\alpha}^* \hat{\alpha} - m_1/2 , \\
\hat{D}^- &= \hat{\alpha}^* \hat{\beta}^-_+ , \\
\hat{D}^+ &= \hat{\alpha}^* \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}^*_+ \hat{\beta}^-_+} , \\
\hat{D}^-_+ &= \hat{\beta}^*_+ \hat{\alpha} , \\
\hat{D}^+_+ &= \sqrt{m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}^*_+ \hat{\beta}^-_+} \hat{\alpha} .
\end{align*}
\]

The above is identical with well-known Holstein-Primakoff representation in the case of the symmetric representation.

\section{5. The simplest approximation}

In order to illustrate that the Holstein-Primakoff representation presented in this paper is reasonable, we treat the simplest approximation which corresponds to the RPA method. Usually, the Hamiltonian of the \( su(3) \)-Lipkin model consists of two parts: \( \hat{H} = \hat{H}_0 + \hat{H}_1 \). First part \( \hat{H}_0 \) is composed of the linear combination of \( \hat{I}_0 \) and \( \hat{M}_0 \) and the second is expressed in terms of the bilinear form with respect to \( \hat{I}_\pm, \hat{D}^*_\pm \) and \( \hat{D}_\pm \). In this paper, mainly we focus on \( \hat{H}_0 \), which can be expressed as

\[
\hat{H}_0 = (1/3)(\epsilon_0 + \epsilon_1 + \epsilon_2)\hat{N} + (1/3)[(\epsilon_2 - \epsilon_0) + (\epsilon_1 - \epsilon_0)] \hat{M}_0 + (\epsilon_2 - \epsilon_1) \hat{I}_0 .
\]

(5.1)

Here, \( \hat{N} \) denotes a \( c \)-number which corresponds to fermion number in the original fermion space and \( \epsilon_0, \epsilon_1 \) and \( \epsilon_2 \) represent the single-particle energies of the levels 0, 1 and 2, respectively. On the basis of the correspondence between the fermion and the Schwinger boson space discussed in Ref.4), the form (5.1) is derived from the fermion Hamiltonian \( \tilde{\hat{H}}_0 \):

\[
\tilde{\hat{H}}_0 = \epsilon_0 \tilde{\hat{N}}_0 + \epsilon_1 \tilde{\hat{N}}_1 + \epsilon_2 \tilde{\hat{N}}_2 .
\]

(5.2)

Here, \( \tilde{\hat{N}}_0, \tilde{\hat{N}}_1 \) and \( \tilde{\hat{N}}_2 \) denote the fermion number operators of the levels 0, 1 and 2, respectively. With the use of the relation (5.1), \( \hat{H}_0 \) can be rewritten in the Holstein-Primakoff representation as follows:

\[
\begin{align*}
\hat{H}_0 &= U_0 + \hat{K}_0 , \\
U_0 &= (1/3)(\epsilon_0 + \epsilon_1 + \epsilon_2)\hat{N} - (1/3)[(\epsilon_2 - \epsilon_0) + (\epsilon_1 - \epsilon_0)] m_0 - (1/3)[(\epsilon_2 - \epsilon_1) + (\epsilon_2 - \epsilon_0)] m_1 , \\
\hat{K}_0 &= (\epsilon_2 - \epsilon_0)(\hat{\alpha}^*_+ \hat{\alpha}^- + \hat{\alpha}^*_- \hat{\alpha}^+) + (\epsilon_1 - \epsilon_0)\hat{\alpha}^*_+ \hat{\alpha}^- + (\epsilon_2 - \epsilon_1)\hat{\beta}^*_+ \hat{\beta}^-_+ .
\end{align*}
\]

(5.3)
We can see in the Hamiltonian $\overline{K}_0$ that "formally" there exist four types of the excitations.

Next, let us turn our eyes to the second part $\hat{H}_1$. In this paper, we do not contact this part concretely. The part $\hat{H}_1$ is expressed in terms of the bilinear form with respect to the operators $\hat{D}_\pm$ and $\hat{D}_\pm^*$. The equilibrium of our present system is characterized in terms of two parameters $m_0$ and $m_1$, and the fluctuations around the equilibrium are treated by the boson operators. As the simplest approximation, we pick up only the linear terms in the expressions (3.7) and (3.8) for the boson operators:

$$
\begin{align*}
\hat{T}_+ &= -\sqrt{m_1} \hat{\beta}_+^*, \\
\hat{T}_- &= -\sqrt{m_1} \hat{\beta}_+,
\end{align*}
$$

$$
\begin{align*}
\hat{D}_+^* &= \sqrt{m_0} \hat{\alpha}_+^*, \\
\hat{D}_-^* &= \sqrt{m_0} \hat{\alpha}_-^*, \\
\hat{D}_+ &= \sqrt{m_0} \hat{\alpha}_+ + \sqrt{m_1} \hat{\alpha}, \\
\hat{D}_- &= \sqrt{m_0} \hat{\alpha}_- + \sqrt{m_1} \hat{\alpha}_+.
\end{align*}
$$

(5.6a) (5.6b) (5.7)

Since $\overline{K}_0$ is of the bilinear with respect to the fluctuations, in our approximation, $\overline{H}_1$ is also of the bilinear with respect to the fluctuations. This is in a form similar to the RPA method.

First, we consider the case $m_1 = 0$. In this case, we have

$$
\begin{align*}
\hat{T}_+ &= 0, \\
\hat{T}_- &= 0,
\end{align*}
$$

$$
\begin{align*}
\hat{D}_+^* &= \sqrt{m_0} \hat{\alpha}_+^*, \\
\hat{D}_-^* &= \sqrt{m_0} \hat{\alpha}_-^*, \\
\hat{D}_+ &= \sqrt{m_0} \hat{\alpha}_+ + \sqrt{m_1} \hat{\alpha}, \\
\hat{D}_- &= \sqrt{m_0} \hat{\alpha}_- + \sqrt{m_1} \hat{\alpha}_+.
\end{align*}
$$

(5.8a) (5.8b) (5.9)

The above relations tell us that the second part $\overline{H}_1$ is expressed in terms of the bilinear form with respect to $(\hat{\alpha}_+^*, \hat{\alpha}_\pm)$. Further, since $\hat{T}_+ \hat{T}_- = m_0 \hat{\alpha}^* \hat{\alpha}$ and $\langle \hat{T}_- ph \rangle = 0$, $\langle ph \hat{T}_+ \rangle = 0$, we can treat the system in the space with the number $\hat{\alpha}^* \hat{\alpha} = 0$. Therefore, $\overline{K}_0$ given in the relation (5.5) can be expressed as

$$
\overline{K}_0 = (\epsilon_2 - \epsilon_0) \hat{\alpha}_+^* \hat{\alpha}_+ + (\epsilon_1 - \epsilon_0) \hat{\alpha}_-^* \hat{\alpha}_- + (\epsilon_2 - \epsilon_1) \hat{\beta}_+^* \hat{\beta}_+.
$$

(5.10)

For the expression (5.10), we can give the interpretation that the first and the second terms represent the excitation from the level 0 to 2 and 0 to 1, respectively, which appear in the conventional RPA. The third also appears in the conventional treatment and it troubles us. However, in our present case, the operator $(m_1 - \hat{\alpha}^* \hat{\alpha} - \hat{\beta}_+^* \hat{\beta}_+)$ appearing in the square root operator in the relation (3.7) should be positive definite, and then, in the case $m_1 = 0$, not only $\hat{\alpha}^* \hat{\alpha}$ but also $\hat{\beta}_+^* \hat{\beta}$ should vanish. Thus, we have

$$
\overline{K}_0 = (\epsilon_2 - \epsilon_0) \hat{\alpha}_+^* \hat{\alpha}_+ + (\epsilon_1 - \epsilon_0) \hat{\alpha}_-^* \hat{\alpha}_-.
$$

(5.11)

In the case $m_0 = 0$, the treatment is in a form similar to the case $m_1 = 0$. We list up the relations in the following form:

$$
\begin{align*}
\hat{T}_+ &= -\sqrt{m_1} \hat{\beta}_+^*, \\
\hat{T}_- &= -\sqrt{m_1} \hat{\beta}_+,
\end{align*}
$$

$$
\begin{align*}
\hat{D}_+^* &= \sqrt{m_1} \hat{\alpha}_+^*, \\
\hat{D}_-^* &= \sqrt{m_1} \hat{\alpha}_-^*, \\
\hat{D}_+ &= \sqrt{m_1} \hat{\alpha}_+ + \sqrt{m_1} \hat{\alpha}, \\
\hat{D}_- &= \sqrt{m_1} \hat{\alpha}_- + \sqrt{m_1} \hat{\alpha}_+.
\end{align*}
$$

(5.12a) (5.12b)

10
\[
\hat{T}_+ = -\sqrt{m_1} \hat{\alpha}_+^*, \quad \hat{T}_- = -\sqrt{m_1} \hat{\alpha}_+, \quad (5.13)
\]
\[
\overline{K}_0 = (\epsilon_2 - \epsilon_0)\hat{\alpha}^*\hat{\alpha} + (\epsilon_2 - \epsilon_1)\hat{\beta}_+^*\hat{\beta}_+. \quad (5.14)
\]

In the case \((m_0 \neq 0, m_1 \neq 0)\), it may be convenient to introduce the following operators:

\[
\hat{\beta}_+^* = \left(\sqrt{m_0 + m_1}\right)^{-1} \left(\sqrt{m_0} \hat{\alpha}_+^* + \sqrt{m_1} \hat{\alpha}_+^*\right),
\hat{\beta}_+ = \left(\sqrt{m_0 + m_1}\right)^{-1} \left(\sqrt{m_0} \hat{\alpha}_+ + \sqrt{m_1} \hat{\alpha}_+\right),
\hat{\beta}_- = \left(\sqrt{m_0 + m_1}\right)^{-1} \left(\sqrt{m_0} \hat{\alpha}_- - \sqrt{m_1} \hat{\alpha}_+\right),
\hat{\beta}_-^* = \left(\sqrt{m_0 + m_1}\right)^{-1} \left(\sqrt{m_0} \hat{\alpha}_-^* - \sqrt{m_1} \hat{\alpha}_+^*\right),
\hat{\delta}_+ = \left(\sqrt{m_0 + m_1}\right)^{-1} \left(\sqrt{m_0} \hat{\alpha}_+ - \sqrt{m_1} \hat{\alpha}_+\right). \quad (5.15a)
\]

Further, in this case, the relations (5.6b) and (5.7) can be rewritten as

\[
\hat{\delta}_+^* = \frac{\hat{\beta}_+^* - \hat{\beta}_-^*}{\sqrt{m_0 + m_1}}, \quad \hat{\delta}_+ = \frac{\hat{\beta}_+ - \hat{\beta}_-}{\sqrt{m_0 + m_1}}, \quad \hat{\delta}_- = \frac{\hat{\beta}_- - \hat{\beta}_+}{\sqrt{m_0 + m_1}}, \quad \hat{\delta}_-^* = \frac{\hat{\beta}_-^* - \hat{\beta}_+^*}{\sqrt{m_0 + m_1}}. \quad (5.15b)
\]

Then, we have

\[
\hat{\alpha}_+^*\hat{\alpha}_+ + \hat{\alpha}^*\hat{\alpha} = \hat{\delta}_+^*\hat{\delta}_+ + \hat{\delta}_-^*\hat{\delta}_-. \quad (5.16)
\]

Therefore, we describe the system in terms of \((\hat{\alpha}_-^*, \hat{\alpha}_-), (\hat{\beta}_+^*, \hat{\beta}_+), (\hat{\delta}_+^*, \hat{\delta}_+)\) and \(\overline{K}_0\) is expressed in the form

\[
\overline{K}_0 = (\epsilon_2 - \epsilon_0)\hat{\delta}_+^*\hat{\delta}_+ + (\epsilon_1 - \epsilon_0)\hat{\alpha}_-^*\hat{\alpha}_- + (\epsilon_2 - \epsilon_1)\hat{\beta}_+^*\hat{\beta}_+. \quad (5.19)
\]

We can see that the case \(m_1 = 0\) and the case \(m_0 = 0\) are described in terms of two kinds of the excitations, but, the case \((m_0 \neq 0, m_1 \neq 0)\) in terms of three kinds of the excitations.

Let us investigate the difference of the mechanism of these excitations. For this investigation, we must remember Ref.4); the discussion on the correspondence between the original fermion and the Schwinger boson representation for the Lipkin model. The intrinsic state \(|m\rangle\) in the original fermion space is specified by the occupation numbers for the level 0, 1 and 2, that is, \(n_0 = \Omega - n\), \(n_1\) and \(n_2\), respectively. Here, \(\Omega\) denotes the degeneracy of each level. In Ref.4), we showed the relation

\[
n_1 = n_0 - m_0, \quad n_2 = n_0 - m_0 - m_1. \quad (5.20)
\]

Then, if \(m_1 = 0, n_1 = n_2 < n_0\) and \(m_0\) fermions can excite from the level 0 to 1 or 2. If \(m_0 = 0, n_2 < n_1 = n_0\) and \(m_1\) fermions can excite from the level 0 or 1 to 2. Therefore, these two cases can be described in terms of two kinds of bosons. In the case \((m_0 \neq 0, m_1 \neq 0)\), \(m_0\) fermions, \(m_1\) fermions and \((m_0 + m_1)\) fermions can excite from the level 0 to 1, the level 1 to
2 and the level 0 to 2, respectively. Therefore, three kinds of boson operators are necessary. The above situation is interpreted in the original fermion space, in which the commutation relations are approximated as follows:

\[
[\tilde{I}_+, \tilde{I}_-] = 2\tilde{I}_0 \approx 2\langle m|\tilde{I}_0|m\rangle = -m_1 , \\
[\tilde{D}^*_+, \tilde{D}^-_] = \tilde{M}_0 - \tilde{I}_0 \approx \langle m|\tilde{M}_0 - \tilde{I}_0|m\rangle = -m_0 , \\
[\tilde{D}^*_+, \tilde{D}^*_+] = \tilde{M}_0 + \tilde{I}_0 \approx \langle m|\tilde{M}_0 + \tilde{I}_0|m\rangle = -(m_0 + m_1) .
\]

(5.21a) (5.21b) (5.21c)

Here, \(\tilde{I}_\pm, \tilde{D}^*_\pm\) and \(\tilde{D}_\pm\) denote the operators in the original fermion space which correspond to \(\hat{I}_\pm, \hat{D}^*_\pm\) and \(\hat{D}_\pm\), respectively. In the present approximation, the other commutation relations vanish. The relations (5.21a) \sim (5.21c) correspond to the relations (5.6a) and (5.17). The above argument may support that our present formalism is reasonable and it may serve in the study of the higher order corrections.

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