Multi-pion Bose-Einstein correlations in high energy heavy-ion collisions

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I. INTRODUCTION

The main goal of colliding heavy nuclei at high energy is to produce quark-gluon plasma (QGP). One of the key quantities related to QGP is the energy density. To determine the energy density, we need to know the space-time information of the source which is the main objective of Hanbury-Brown-Twiss correlations. Two-pion Bose-Einstein (BE) correlations are widely used in high energy heavy-ion collisions to provide information on the space-time structure, degree of coherence and dynamics of the region where the pions were produced.

In principle, the extension of two-pion interferometry to multi-pion interferometry is straightforward. Experimentally, ultrarelativistic hadronic and nuclear collisions provide the environment for creating dozens, and in some cases hundreds, of pions. Therefore, it is necessary to consider multi-pion correlations in these processes. The bosonic nature of the pion should affect the single and two-pion spectra and distort the two-pion correlation function. There is a kind of coherence length which corresponds to the wave packet length of the emitter, which causes pions to concentrate at low momenta. Thus, it is very interesting to analyze the effects of the multi-pion correlations and wavepacket length on these variables.

The arrangement of this paper is as follows: The derivation of the multi-pion correlation function for a totally chaotic source is given in section 2. In section 3, the effects of multi-pion correlations on the multiplicity distribution are discussed. In section 4, the effects of multi-pion correlations on two-pion correlation function are analyzed for n-pion events and for events with all possible multiplicities. In section 5, our results for a source distribution are given. Conclusions are given in section 6.

II. PION CORRELATION FUNCTION FOR A CHAOTIC SOURCE

The general definition of the pure n pion correlation function \( C_n(\vec{p}_1, \ldots, \vec{p}_n) \) is

\[
C_n(\vec{p}_1, \ldots, \vec{p}_n) = \frac{P_n(\vec{p}_1, \ldots, \vec{p}_n)}{\prod_{i=1}^n P_i(\vec{p}_i)},
\]

where \( P_n(\vec{p}_1, \ldots, \vec{p}_n) \) is the probability of observing n-pions with momenta \( \{\vec{p}_i\} \) all in the same n-pion event.

A state created by a classical pion source is described by

\[
|\phi> = \exp(i \int d\vec{p} \int d^4x j(x) \exp(i p x) c^+(\vec{p})|0>,
\]

where \( c^+(\vec{p}) \) is the pion-creation operator. \( j(x) \) is the pion current, which can be expressed as

\[
j(x) = \int d^4x' d^4p j(x', p) \nu(x') \exp(-ip(x - x')).
\]

Here \( j(x', p) \) is the probability amplitude of finding a pion with momentum \( p \), emitted by the emitter at \( x' \). \( \nu(x') = \exp(i \phi(x')) \) is a random phase factor which has been extracted from \( j(x', p) \). All emitters are uncorrelated in coordinate space when assuming

\[
\{\nu^*(x')\nu(x)\} = \delta^4(x' - x).
\]

Here \( \{\cdots\} \) means phase average which is defined as

\[
\{\prod_{i=1}^n \nu^*(x_i) \prod_{j=1}^m \nu(y_j)\} \sim \int_0^{2\pi} \prod_{i=1}^n \prod_{j=1}^m d\phi(x_i) d\phi(y_j) \frac{2\pi}{2\pi} \nu^*(x_i) \nu(y_j).
\]

Eq.(4) is the ideal case. In more realistic cases, each chaotic emitter has a small coherent wave packet length, and Eq.(4) can be replaced by

\[
\{\nu^*(x')\nu(x)\} = \frac{1}{\delta^4} \exp\left[- \frac{(x_1 - x_1')^2}{\delta^2} - \frac{(x_2 - x_2')^2}{\delta^2} - \frac{(x_3 - x_3')^2}{\delta^2} - \frac{(x_0 - x_0')^2}{\delta^2}\right].
\]

Here \( \delta \) is a parameter which determines the coherent length (time) scale of the emitter. For simplicity, the

\[\footnote{\delta \text{ is smaller than the total size } (R) \text{ of the source which consists of all emitters}}\]
same coherent scale is taken for both space and time at the moment. The above formula shows that two emitters within the range of $\delta$ can be seen as one emitter, while two-emitters out of this range are incoherent. From Eq.(5) we have

$$\prod_{i=1}^{n} \nu^*(x_i) = \prod_{j=1}^{m} \nu(x_j) = 0.$$  \hspace{1cm} (7)

The coherent state can be expanded in Fock-Space as

$$|\phi> = \sum_{n=0}^{\infty} (i \int j(x)e^{ipx}c^+(p)d\phi^4x)^n |0> = \sum_{n=0}^{\infty} |n>,$$  \hspace{1cm} (8)

with

$$|n> = (i \int j(x)e^{ipx}c^+(p)d\phi^4x)^n |0>.$$  \hspace{1cm} (9)

Here $|n>$ is an $n$-pion state. According to the definition of Eq.(1), the pure $n$-pion correlation function can be re-expressed as

$$C_n(p_1, \cdots p_n) = \frac{\{<n|c^+(p_1)\cdots c^+(p_n)c(p_n)\cdots c(p_1)|n>\}}{\prod_{i=1}^{n} (1<1\nu^+c(p_i)c(p_i)|1>)}.$$  \hspace{1cm} (10)

From above definition, the pure two-pion correlation function reads (see Appendix)

$$C_2(p_1, p_2) = \frac{\{<2|c^+(p_1)c^+(p_2)c(p_2)c(p_1)|2>\}}{\prod_{i=1}^{n} (1<1\nu^+c(p_i)c(p_i)|1>)} = 1 + \frac{\{<1\nu^+(p_1)c(p_2)|1>\}}{\prod_{i=1}^{n} (1<1\nu^+c(p_i)c(p_i)|1>)}.$$  \hspace{1cm} (11)

Using the relationship

$$c(p)|n> = i \int d^4x j(x) \exp(ipx)|n-1>$$  \hspace{1cm} (12)

we have

$$\{<1\nu^+(p_1)c(p_2)|1>\} = \{ \int d^4x_1 d^4x_2 j^*(x_1) j(x_2)\exp(-i(p_1 x_1 - p_2 x_2)) \}$$

$$= \{ \int d^4y d^4t j^*(Y + y/2) \exp(-iky - iqY) \}$$

$$= \int d^4Y g_w(Y, k) \exp(-iq \cdot Y).$$  \hspace{1cm} (13)

Here $Y = \frac{x_1 + x_2}{2}$ and $y = x_1 - x_2$ are four dimensional coordinates, while $k = \frac{p_1 + p_2}{2}$ and $q = p_1 - p_2$ are the corresponding four dimensional momenta. $g_w(Y, k)$ is the Wigner function, which can be understood as the probability of finding a pion at $Y$ with momentum $k$. It is defined as

$$g_w(Y, k) = \{ \int d^4y j^*(Y + y/2) j(Y - y/2) \exp(-ik \cdot y) \}.$$  \hspace{1cm} (14)

Inserting Eq.(3) into the above equation we have

$$g_w(Y, k) = \{ \int d^4x' \exp(-ik \cdot y)$$

$$\int d^4x' j^*(x', p_1)d^4p_1 \exp(ip_1(Y + y/2 - x'))$$

$$\nu^*(x') \nu(x'')} \delta^4(k - \frac{p_1 + p_2}{2})$$

$$e^{ip_1(Y-x')} e^{-ip_2(Y-x'')} \}.$$  \hspace{1cm} (15)

so the pure single pion inclusive distribution $P_1(\vec{p})$ can be expressed as:

$$P_1(\vec{k}) = \int g_w(Y, k)d^4Y$$

$$= \{ \int d^4x' d^4x'' d^4p_1 d^4p_2 j^*(x', p_1) j(x'', p_2)$$

$$\nu^*(x') \nu(x'') \delta^4(k - \frac{p_1 + p_2}{2})$$

$$e^{ip_1(Y-x')} e^{-ip_2(Y-x'')} \}.$$  \hspace{1cm} (16)

with $k_0 = \sqrt{k^2 + m_0^2}$. The two-pion correlation function reads

$$C_2(p_1, p_2) = 1 +$$

$$\frac{\{ \int d^4x d^4x' g_w(x, k) g_w(x', k) \exp(iq \cdot (x - x')) \}}{\int d^4x d^4x' g_w(x, p_1) g_w(x', p_2)}.$$  \hspace{1cm} (17)

Similarly, the pure $n$-pion correlation function can be expressed as

$$C_n(p_1, p_2, \cdots, p_n) = \sum_{\sigma} \chi_{1,\sigma(1)} \chi_{2,\sigma(2)} \cdots \chi_{n,\sigma(n)}$$  \hspace{1cm} (18)

with

$$\chi_{i,j} = \chi(p_i, p_j) = \frac{\{ \int d^4x g_w(x, \frac{p_i + p_j}{2}) \exp(ip_i - p_j) \cdot x \}}{\sqrt{\int d^4x d^4y g_w(x, p_i) g_w(y, p_j)}}.$$  \hspace{1cm} (19)

Here $\sigma(i)$ denotes the ith element of a permutation of the sequence $1, 2, 3, \cdots, n$, and the sum over $\sigma$ denotes the sum over all $n!$ permutations of this sequence.
In the following, we shall often use the expression of the pure $n$-pion momentum probability distribution $P_n(\vec{p}_1, \cdots, \vec{p}_n)$ given by

$$P_n(\vec{p}_1, \cdots, \vec{p}_n) = \sum_{\sigma} \rho_{1, \sigma(1)} \rho_{2, \sigma(2)} \cdots \rho_{n, \sigma(n)}$$  \hspace{1cm} \text{(20)}$$

with

$$\rho_{i,j} = \rho(p_i, p_j) = \frac{1}{n_0} \int d^4x g_0(x, \frac{p_i + p_j}{2}) \exp(i(p_i - p_j) \cdot x).$$  \hspace{1cm} \text{(21)}$$

Here $n_0$ is the mean multiplicity without BE correlations, defined by

$$n_0 = \int g_0(x, \vec{k}) d^4x d\vec{k},$$  \hspace{1cm} \text{(22)}$$

Once the source distribution $g_0(x, k)$ is known, the pure multi-pion correlation function can be calculated from Eq.(18).

### III. EFFECTS OF MULTI-PION CORRELATIONS ON PION MULTIPlicity DISTRIBUTION

From the above definition of $|n>$ we have

$$\omega(n) = \{< n|n > \} = \frac{n^n}{n!} \int P_n(p_1, \cdots, p_n) \prod_{i=1}^{n} d\vec{p}_i$$  \hspace{1cm} \text{(23)}$$

and

$$\{< \phi|\phi > \} = \sum_k \{< k|k > \}$$

$$= \sum_k \frac{n_0^n}{k!} \int d\vec{p}_1 \cdots d\vec{p}_k \prod_{i=1}^{k} P_k(p_1, \cdots, p_k)$$

$$= \sum_k \omega(k).$$  \hspace{1cm} \text{(24)}$$

Then the pion-multiplicity distribution becomes

$$P(n) = \{< n|n > \} = \frac{\omega(n)}{\sum_k \omega(k)}.$$  \hspace{1cm} \text{(25)}$$

For particles without BE correlations, we have

$$\rho_{i,j} = \delta_{i,j} \rho_{i,i}$$  \hspace{1cm} \text{(26)}$$

and

$$\int P_n(p_1, \cdots p_n) \prod_{i} d\vec{p}_i = 1.$$  \hspace{1cm} \text{(27)}$$

therefore

$$P(n) = \frac{n_0^n}{n!} e^{-n_0}.$$  \hspace{1cm} \text{(28)}$$

That is $P(n)$ reduces to the usual Poisson distribution. If we take into account multi-pion Bose-Einstein correlations, the pion multiplicity distribution is not a Poisson distribution anymore (see Fig.1). From the above equation, we can also obtain the mean multiplicity $<M>$

$$<M> = \sum_n n \cdot P(n).$$  \hspace{1cm} \text{(29)}$$

### IV. EFFECTS OF MULTI-PION CORRELATIONS ON TWO-PION CORRELATION FUNCTION

#### A. Two-pion correlation function for $n$-pion events

For $n(n \geq 2)$ events, the two-pion correlation function can be defined as

$$C_2^n(p_1, p_2) = \frac{P^n_i(\vec{p}_1, \vec{p}_2)}{P^n_i(\vec{p}_1) P^n_i(\vec{p}_2)},$$  \hspace{1cm} \text{(30)}$$

where $P^n_i(\vec{p}_1, \cdots, \vec{p}_i)$ is the modified $i$-pion inclusive distribution in $n$ pion events. The definition of $P^n_i(\vec{p}_1, \cdots, \vec{p}_i)$ is

$$P^n_i(\vec{p}_1, \cdots, \vec{p}_i) = \frac{\{< n|c^+ (\vec{p}_1) \cdots c^+ (\vec{p}_i) c(\vec{p}_i) \cdots c(\vec{p}_n)|n > \}}{\{< n|n > \} n \cdots (n-i+1)}.$$  \hspace{1cm} \text{(31)}$$

For an event with multiplicity $n$, the modified two-pion inclusive and single-pion inclusive distributions can be expressed as

$$P^n_2(\vec{p}_1, \vec{p}_2) = \frac{\int \prod_{i=1}^{n} d\vec{p}_i P_n(\vec{p}_1 \cdots \vec{p}_n)}{\int \prod_{i=1}^{n} d\vec{p}_i P_n(\vec{p}_1 \cdots \vec{p}_n)}$$  \hspace{1cm} \text{(32)}$$

and

$$P^n_1(\vec{p}_1) = \frac{\int \prod_{i=1}^{n} d\vec{p}_i P_n(\vec{p}_1 \cdots \vec{p}_n)}{\int \prod_{i=1}^{n} d\vec{p}_i P_n(\vec{p}_1 \cdots \vec{p}_n)}$$  \hspace{1cm} \text{(33)}$$

respectively.

As $n$ increases, the computation of the above intergrals becomes more and more complex. Now we define the function

$$G_i(p, q) = n_0^i \int \rho(p, p_1) \rho(p_1, p_2) \cdots \rho(p_{i-2}, p_{i-1}) d\vec{p}_2 \cdots d\vec{p}_i$$  \hspace{1cm} \text{(34)}$$

From the expression of $P_n(\vec{p}_1, \cdots, \vec{p}_n)$ (Eq. (20)), the two-pion inclusive distribution can be expressed as

\[3\text{This definition takes into account the higher order Bose-Einstein correlations.}\]

\[4\text{It is easily checked that } \int d\vec{p}_1 \cdots d\vec{p}_i P^n_i(\vec{p}_1, \cdots, \vec{p}_i) = 1.\]
with

$$\omega(n) = \frac{n^n}{n!} \int \prod_{k=1}^{n} dp_k P_n(p_1, \ldots, p_n) .$$

(36)

The single-pion distribution is given as

$$P_1^n(\vec{p}) = \frac{1}{n \omega(n)} \sum_{i=1}^{n} G_i(p, p) \omega(n - i).$$

(37)

Now the main problem is to extract the expression of \(\omega(n)\). From Eq.(37) we have

$$\omega(n) = \frac{1}{n} \sum_{i=1}^{n} C(i) \omega(n - i),$$

(38)

with

$$C(i) = \int dp G_i(p, p).$$

(39)

Using the above method, the two-pion and single pion inclusive distributions can be calculated for n-pion events. \(G_i(p, q)\) and \(C(i)\) can be calculated using Monte-Carlo or analytical integration.

**B. Two-pion correlation function for all events**

For a state \(\ket{\phi}\) which contains all possible multiplicities, the modified single pion spectrum distribution can be expressed as\(^5\)

$$P_1^\phi(\vec{p}) = \frac{\{< \phi | c^+(\vec{p}) c(\vec{p}) | \phi >\}}{\{< \phi | \phi >\} < M >}$$

$$= \frac{\sum_n P_1^n(\vec{p}) \cdot n \cdot \omega(n)}{\{< \phi | \phi >\} < M >}$$

$$= \frac{\sum_i G_i(p, p) \omega(n - i)}{\{< \phi | \phi >\} < M >} = \frac{\sum_i G_i(p, p)}{< M >}$$

(40)

where Eq.(24) is applied. The two-pion inclusive distribution can be expressed as

$$P_2^\phi(\vec{p}, \vec{q}) = \frac{\{< \phi | c^+(\vec{p}) c^+(\vec{q}) c(\vec{q}) c(\vec{p}) | \phi >\}}{\{< \phi | \phi >\} < M >}$$

$$= \frac{\sum_n P_2^n(\vec{p}, \vec{q}) \cdot n \cdot (n-1) \omega(n)}{\{< \phi | \phi >\} < M >}$$

$$= \frac{1}{\{< \phi | \phi >\} < M >} \sum_n$$

$$\frac{\sum_{i,j} G_i(p, p) G_j(q, q) + G_i(p, q) \cdot G_j(q, p)}{< M >} .$$

(41)

The two-pion correlation function for all multiplicity distribution is

$$C_2^\phi(\vec{p}, \vec{q}) = \frac{P_2^\phi(\vec{p}, \vec{q})}{P_1^\phi(\vec{q}) P_1^\phi(\vec{p})} .$$

(42)

**V. MULTI-PION CORRELATIONS FOR A CHAOTIC SOURCE**

In this section, we will give an example to investigate the wave packet length and the multi-pion correlations effects on pion multiplicity, single pion and two-pion distributions.

We assume that the chaotic emitter amplitude distribution is

$$j(x, k) = \sqrt{n_0} \exp\left(-\frac{x_1^2 - x_2^2 - x_3^2}{2 R_0^2}\right) \exp\left(-\frac{k_1^2 + k_2^2 + k_3^2}{2 \Delta^2}\right).$$

(43)

Where \(R_0\) and \(\Delta\) are the parameters which represents the radius of the chaotic source and the momentum range of pions respectively. By bringing Eq.(6) and Eq.(43) into Eqs.(15-17), we can easily get the source distribution function \(g_w(x, K)\)

$$g_w(x, K) = n_0 \cdot \left(\frac{1}{\pi R_0^2}\right)^{\frac{3}{2}} \exp\left(-\frac{x^2}{2 R_0^2}\right) \delta(x_0)$$

$$\exp\left(-\frac{k_1^2 + k_2^2 + k_3^2}{2 \Delta^2}\right).$$

(44)

the normalized pure pion single particle spectrum distribution

$$P_1(p) = \left(\frac{1}{\Delta^2}\right)^{\frac{3}{2}} \exp\left(-\frac{p^2}{\Delta^2}\right),$$

(45)

and the pure two-pion correlation function

$$C_2(p_1, p_2) = 1 + \exp\left(-\frac{q^2}{2} \frac{4 R_0^2}{\Delta^2 + 4 R_0^2}\right).$$

(46)
with

$$R_G^2 = R_0^2 + \frac{1}{\Delta^2}, \quad \frac{1}{\Delta^2} = \frac{1}{\Delta^2} + \frac{R_0^2 \delta^2}{\delta^2 + 4R_0^2}. \quad (47)$$

In the above formula, the definition of \( \vec{q} \) is \( \vec{q} = \vec{p}_1 - \vec{p}_2 \). Then the derived radius through pure two-pion interferometry is

$$R^2 = \frac{4R_0^2}{\delta^2 + 4R_0^2}. \quad (48)$$

It is clear that as the coherence length \( \delta \) increases, the mean momentum of pions (\( \Delta_0 \)) becomes smaller (Eq.(47)). It can be seen clearly from Eq.(48) that as the coherence length \( \delta \) increases, the apparent radius derived from pure two-pion interferometry also becomes smaller. 

$$g_w(x, \frac{p+q}{2}) = n_0 \cdot \frac{1}{(\pi R_G^2)^{3/2}} e^{-\frac{x^2}{R_G^2}}$$

$$= \frac{1}{(2\pi \Delta_0^2)^{3/2}} e^{-\frac{(x-q)^2}{4\Delta_0^2}} \frac{e^{-\frac{(x+q)^2}{4\Delta_0^2}}}{\Delta_0^2}. \quad (49)$$

then

$$\rho(p, q) = \frac{1}{n_0} \int g_w(x, \frac{p+q}{2}) e^{-i(p-q)x} dx$$

$$= \frac{1}{(2\pi \Delta_0^2)^{3/2}} e^{-\frac{(p-q)^2}{4\Delta_0^2}} \cdot \frac{e^{-\frac{(p+q)^2}{4\Delta_0^2}}}{\Delta_0^2}. \quad (50)$$

Defining

$$G_n(p, q) = n_0^n \int \rho(p, \vec{p}_1) \prod_{i=1, n-2} d\vec{p}_i \rho(\vec{p}_i, \vec{p}_{i+1})$$

$$d\vec{p}_{n-1} \rho(\vec{p}_{n-1}, q) \quad (51)$$

and using Eq.(50), we easily obtain

$$G_n(p, q) = n_0^n \cdot \alpha_n e^{-a_n(p^2+q^2)} + g_n p q, \quad (52)$$

where

$$a_{n+1} = a_1 - \frac{g_1^2}{4b_n}, \quad b_n = a_n + a_1, \quad g_{n+1} = \frac{g_n g_1}{2b_n}, \quad (53)$$

and

$$\alpha_{n+1} = \alpha_n \left( \frac{1}{\Delta_0^2} \right)^{3/2} \left( \frac{1}{b_n} \right)^{3/2}, \quad (54)$$

with

$$a_1 = \frac{R_G^2}{4} + \frac{1}{4\Delta_0^2}, \quad g_1 = \frac{R_G^2}{2} - \frac{1}{2\Delta_0^2}, \quad \alpha_1 = \frac{1}{(\pi \Delta_0^2)^{3/2}}. \quad (55)$$

Now we can calculate the pion multiplicity, single particle and two-particle pion distributions according to the formula given in the preceding sections. The pion multiplicity distribution is shown in Fig.1. It is clear that the probability with high pion multiplicity is greater than that for a conventional Poisson distribution. This is consistent with the nature of bosons, that is, the emission of bosons encourages the emission of more bosons. We also find that as the radius and temperature decrease, the effect of BE correlations on the multiplicity distribution becomes larger. This can be explained by the fact that as the radius and the temperature decrease the quantum effect become larger. It is also clear that as the wavepacket \( \delta \) increases, the effect of BE correlations on the multiplicity distribution becomes larger. It can be seen clearly from Eq.(47) that as \( \delta \) increases, \( \Delta_0 \) (the effective temperature) becomes smaller therefore the quantum effect becomes larger.

![FIG. 1. Pion multiplicity distribution with and without (solid cure) Bose-Einstein correlations. The input value of \( \Delta \) and \( n_0 \) are 0.25GeV and 13. The dashed, dotted and dot-dashed line corresponds to \( R_0 = 3 \text{fm}, \delta = 0 \text{fm}; R_0 = 3 \text{fm}, \delta = 0.5 \text{fm} \) and \( R_0 = 2.8 \text{fm}, \delta = 0 \text{fm} \) respectively.](image-url)

In Fig.2, the mean multiplicity \( < M > \) is plotted against the parameter \( n_0 \), the mean multiplicity without BE correlations. When \( n_0 \) is not very large, \( < M > \) increases with increasing \( n_0 \). However, at a critical point \( n_c \), the mean multiplicity \( < M > \) becomes to be infinity. From Eqs. (39-40), the mean multiplicity can be expressed as:

$$< M > = \sum_i C(i) = \sum_i n_i^0 \alpha_i \left( \frac{\pi}{2a_i - g_i} \right)^{3/2}. \quad (56)$$

It is clear that the above equation will remain convergence provided that

$$\lim_{n \to \infty} \frac{C(n+1)}{C(n)} = \lim_{n \to \infty} n_0 \cdot \left( \frac{1}{\alpha_n} \right)^{3/2} \left( \frac{1}{b_n} \right)^{3/2} \left( \frac{2a_{n+1} - g_{n+1}}{2a_n - g_n} \right)^{3/2} < 1. \quad (57)$$

From Eqs.(53-55), we have
\[
\lim_{n \to \infty} a_n = \frac{R_G}{2\Delta_0}, \\
\lim_{n \to \infty} b_n = \left(\frac{R_G}{2} + \frac{1}{2\Delta_0}\right)^2, \\
\lim_{n \to \infty} g_n = 0.
\]

Including the multipion BE correlations, the single particle momentum distribution for events with fixed multiplicity is given in Fig.4. It is shown that, as the multiplicity of pion increases, the number of pions at low momentum becomes larger. This has been observed in many experiments \textsuperscript{[33,34,35]}. Many different explanations of the experimental results have been proposed \textsuperscript{[31,32,34]}. Some of those \textsuperscript{[33,35]} stated that the enhancement of single pion spectrum at low momentum spectrum is due to the two-pion Bose-Einstein correlations. Our results show that the enhancement of single pion spectrum at low momentum caused by multi-pion correlations is more pronounced than that caused by two-particle correlations. So a detailed analysis of the reason for the enhancement of single particle spectrum at low momentum is still necessary. The effects of the wavepacket on the pion spectrum distribution are also shown in Fig.4. It is clear that the effects of the wavepacket on the pion spectrum distribution becomes larger for larger pion multiplicity. The single particle momentum distributions for total events are shown in Fig.5. Similar features as figure 4 are obtained.

\[
\lim_{n \to \infty} a_n = \frac{R_G}{2\Delta_0}, \\
\lim_{n \to \infty} b_n = \left(\frac{R_G}{2} + \frac{1}{2\Delta_0}\right)^2, \\
\lim_{n \to \infty} g_n = 0.
\]

FIG. 2. The mean multiplicity < \( M > \) vs. \( n_0 \). The solid line corresponds to the case without BE correlation \((n_0 = < M >)\). The dashed and dotted lines corresponds to \( \delta = 0 \) fm and \( \delta = 0.5 \) fm respectively. The input value of \( R_0 \) and \( \Delta \) are 5 fm and 0.36 GeV.

Eq. (56) remains convergence if
\[
\begin{align*}
n_0 &< \frac{1}{8} (1 + R_G \Delta_0)^3 \\
&= \frac{1}{8} \left(1 + \sqrt{1 + \frac{4R_G^2\Delta_0^2}{4 + (R_G^2\Delta_0^2 + 1)\delta^2/R_0^2}}\right)^3 = n_c.
\end{align*}
\]

We denote this critical pion multiplicity as \( n_c \). It is very interesting to notice that the critical multiplicity \( n_c \) depends only on \( R_0\Delta_0 \) and \( \delta/R_0 \). The critical multiplicity \( n_c \) vs. \( R_0\Delta_0 \) for different values of \( \delta/R_0 \) are shown in Fig.3. It is clear that as \( \delta/R_0 \) increases the corresponding critical multiplicity becomes smaller.

FIG. 3. The critical multiplicity \( n_c \) vs. \( R_0\Delta \). The solid line and dashed line corresponds to \( \delta/R_0 = 0 \) and \( \delta/R_0 = 0.2 \) respectively.

FIG. 4. The single particle momentum distribution for different multiplicities. The wider solid line and wider dotted line corresponds to multiplicity \( M = 20 \), \( \delta = 0 \) fm and \( M = 80 \), \( \delta = 0.5 \) fm respectively. The wider dashed line and wider dot-dashed line corresponds to multiplicity \( M = 20 \), \( \delta = 0.5 \) fm and \( M = 80 \), \( \delta = 0.5 \) fm respectively. The thin solid line and the thin dashed line corresponds to the input momentum distribution with \( \delta = 0 \) fm and \( \delta = 0.5 \) fm respectively. The input value of \( R_0 \) and \( \Delta \) are 5 fm and 0.36 GeV.

The effects of multi-pion correlations on the two-pion correlation function are shown in Fig.6, where we fixed the multiplicity of the events. It can clearly be seen that as the multiplicity of the event increases, the two-pion correlation function has a lower chaoticity, though the actual source is totally chaotic. The apparent radius derived from two-pion interferometry also becomes smaller. Two-pion correlation functions for different mean multiplicity are shown in Fig.7. A similar property to that in Fig.6 can be seen. It is clear that the effects of the
wavepacket on two-pion correlation becomes larger for larger multiplicity.

![FIG. 5. The single particle momentum distribution for different $n_0$. The wider solid line and wider dotted line corresponds to $n_0 = 8$, $\delta = 0\,\text{fm}$ and $n_0 = 8$, $\delta = 0.5\,\text{fm}$ respectively. The wider dashed line and wider dot-dashed line corresponds to $n_0 = 50$, $\delta = 0\,\text{fm}$ and $n_0 = 50$, $\delta = 0.5\,\text{fm}$ respectively. The thin solid line and thin dashed line corresponds to the input momentum distribution with $\delta = 0\,\text{fm}$ and $\delta = 0.5\,\text{fm}$. The input value of $R_0$ and $\Delta$ are $5\,\text{fm}$ and $0.36\,\text{GeV}$.](image1.png)

![FIG. 6. Two-pion correlation function for fixed multiplicity events. The thin solid line correspond to multiplicity $M = 2$ and $\delta = 0\,\text{fm}$. The thin dashed line and thin dotted line corresponds to multiplicity $M = 20$, $\delta = 0\,\text{fm}$ and $M = 20$, $\delta = 0.5\,\text{fm}$ respectively. The wider solid line and the wider dashed line corresponds to $M = 80$, $\delta = 0\,\text{fm}$ and $M = 80$, $\delta = 0.5\,\text{fm}$ respectively. The input value of $R_0$ and $\Delta$ are $3\,\text{fm}$ and $0.36\,\text{GeV}$.](image2.png)

![FIG. 7. Two-pion correlation function for all events. The solid line corresponds to $n_0 = 10$, $\delta = 0\,\text{fm}$. The dashed line and the dotted line corresponds to $n_0 = 98$, $\delta = 0\,\text{fm}$ and $n_0 = 98$, $\delta = 0.5\,\text{fm}$ respectively. The input value of $R_0$ and $\Delta$ are $5\,\text{fm}$ and $0.36\,\text{GeV}$.](image3.png)

VI. CONCLUSIONS

Early in the next decade two heavy-ion accelerators, the Relativity Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC), will provide the environment for creating hundreds of pions. So the multi-pion Bose-Einstein correlations effect in those processes should be studied. In this paper, the multi-pion Bose-Einstein correlations and wavepacket effects on pion multiplicity distribution, pion spectrum distribution and two-pion interferometry have been studied. It has been shown that multi-pion Bose-Einstein correlations and wavepacket cause an abundance of pions at the low momentum, increase the pion’s mean multiplicity and decrease both the apparent radius of the source and the coherent source parameter derived from two-pion interferometry. For larger pion multiplicity, the effects of the wavepacket on the pion multiplicity distribution, pion spectrum distribution and two-pion interferometry become larger.

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Appendix
The derivation of Eq. (11) will be given in this appendix. From Eq. (10) we have
\[ C_2(\vec{p}_1, \vec{p}_2) = \frac{\langle 2 | c^+(\vec{p}_1) c^+(\vec{p}_2) c(\vec{p}_2) c(\vec{p}_1) | 2 \rangle}{\langle 1 | c^+(\vec{p}_1) c(\vec{p}_1) | 1 \rangle \{ \langle 1 | c^+(\vec{p}_2) c(\vec{p}_2) | 1 \rangle \}}. \] (60)

with
\[ | n > = \left( \frac{i \int j(x) e^{ip \cdot x} c(\vec{p}) d\vec{p} dx}{n!} \right)^n | 0 > . \] (61)

Using the relationship
\[ c(\vec{p}) | n > = i \int d^4 x j(x) \exp(ip \cdot x)| n - 1 > , \] (62)
we have
\[ c(\vec{p}_1) c(\vec{p}_2) | 2 > = i \int d^4 x j(x) \exp(ip \cdot x) \int d^4 y j(y) \exp(ip \cdot y)| 0 > . \] (63)

Then the two pion spectrum distribution reads
\[ \{ \langle 2 | c^+(\vec{p}_1) c^+(\vec{p}_2) c(\vec{p}_2) c(\vec{p}_1) | 2 \rangle \} = \{ \int d^4 x j^*(x_1) \exp(ip \cdot x_1) \int d^4 y j^*(y_1) \exp(ip \cdot y_1) \int d^4 x j(x_2) \exp(-ip \cdot x_2) \int d^4 y j(y_2) \exp(-ip \cdot y_2) \} . \] (64)

Here \{ \cdots \} means the phase average(Eq.(5)). Using Eq.(3), Eq.(5) and Eq.(6), we have the following relationship
\[ \{ j^*(x_1) \cdots j^*(x_n) j(y_1) \cdots j(y_m) \} = 0 \quad n \neq m, \] (65)
then
\[ \{ j^*(x_1) j^*(y_1) j(x_2) j(y_2) \} = \{ j^*(x_1) j(x_2) \} \{ j^*(y_1) j(y_2) \} + \{ j^*(x_1) j(y_2) \} \{ j^*(y_1) j(x_2) \} . \] (66)

Eq.(64) can be re-expressed as
\[ \{ \langle 2 | c^+(\vec{p}_1) c^+(\vec{p}_2) c(\vec{p}_2) c(\vec{p}_1) | 2 \rangle \} = \{ \langle 1 | c^+(\vec{p}_1) c(\vec{p}_1) | 1 \rangle \} \{ \langle 1 | c^+(\vec{p}_2) c(\vec{p}_2) | 1 \rangle \} + \{ \langle 1 | c^+(\vec{p}_1) c(\vec{p}_2) | 1 \rangle \} \{ \langle 1 | c^+(\vec{p}_2) c(\vec{p}_1) | 1 \rangle \} . \] (67)

Bringing Eq. (67) into Eq. (60) we obtain Eq. (11).

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Figure Captions

1. Pion multiplicity distribution with and without (solid curve) Bose-Einstein correlations. The input value of $\Delta$ and $n_0$ are 0.25 GeV and 13. The dashed, dotted and dot-dashed line corresponds to $R_0 = 3 \text{ fm}$, $\delta = 0 \text{ fm}$; $R_0 = 3 \text{ fm}$, $\delta = 0.5 \text{ fm}$ and $R_0 = 2.8 \text{ fm}$, $\delta = 0 \text{ fm}$ respectively.

2. The mean multiplicity $< M >$ vs. $n_0$. The solid line corresponds to the case without BE correlation ($n_0 =< M >$). The dashed and dotted lines corresponds to $\delta = 0 \text{ fm}$ and $\delta = 0.5 \text{ fm}$ respectively. The input value of $R_0$ and $\Delta$ are 5 fm and 0.36 GeV.

3. The critical multiplicity $n_c$ vs. $R_0 \Delta$. The solid line and dashed line corresponds to $\delta/R_0 = 0$ and $\delta/R_0 = 0.2$ respectively.

4. The single particle momentum distribution for different multiplicities. The wider solid line and wider dotted line corresponds to multiplicity $M = 20$, $\delta = 0 \text{ fm}$ and $M = 20$, $\delta = 0.5 \text{ fm}$ respectively. The wider dashed line and wider dot-dashed line corresponds to multiplicity $M = 80$, $\delta = 0 \text{ fm}$ and $M = 80$, $\delta = 0.5 \text{ fm}$ respectively. The thin solid line and thin dashed line corresponds to the input momentum distribution with $\delta = 0 \text{ fm}$ and $\delta = 0.5 \text{ fm}$ respectively. The input value of $R_0$ and $\Delta$ are 5 fm and 0.36 GeV.

5. The single particle momentum distribution for different $n_0$. The wider solid line and wider dotted line corresponds to $n_0 = 8$, $\delta = 0 \text{ fm}$ and $n_0 = 8$, $\delta = 0.5 \text{ fm}$ respectively. The wider dashed line and wider dot-dashed line corresponds to $n_0 = 50$, $\delta = 0 \text{ fm}$ and $n_0 = 50$, $\delta = 0.5 \text{ fm}$ respectively. The thin solid line and thin dashed line corresponds to the input momentum distribution with $\delta = 0 \text{ fm}$ and $\delta = 0.5 \text{ fm}$. The input value of $R_0$ and $\Delta$ are 5 fm and 0.36 GeV.

6. Two-pion correlation function for fixed multiplicity events. The thin solid line correspond to multiplicity $M = 2$ and $\delta = 0 \text{ fm}$. The thin dashed line and thin dotted line corresponds to multiplicity $M = 20$, $\delta = 0 \text{ fm}$ and $M = 20$, $\delta = 0.5 \text{ fm}$ respectively. The wider solid line and the wider dashed line corresponds to $M = 80$, $\delta = 0 \text{ fm}$ and $M = 80$, $\delta = 0.5 \text{ fm}$ respectively. The input value of $R_0$ and $\Delta$ are 3 fm and 0.36 GeV.

7. Two-pion correlation function for all events. The solid line corresponds to $n_0 = 10$, $\delta = 0 \text{ fm}$. The dashed line and the dotted line corresponds to $n_0 = 98$, $\delta = 0 \text{ fm}$ and $n_0 = 98$, $\delta = 0.5 \text{ fm}$ respectively. The input value of $R_0$ and $\Delta$ are 5 fm and 0.36 GeV.