Entropy is a consequence of a discrete time

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Abstract. While the basic microscopic physical laws are time reversible, the arrow of time and time irreversibility appears only at the macroscopic physical laws by the second law of thermodynamics with its entropy term $S$. It is the attempt of the present work to bridge the microscopic physical world with its macroscopic one with an alternative approach than the statistical mechanics theory of Gibbs and Boltzmann. For simplicity a “classical”, single particle in a one dimensional space is selected. In addition, it is assumed that time is discrete with constant step size. As a consequence time irreversibility at the microscopic level is obtained if the present force is of complex nature ($F(r) \neq \text{const}$). In order to compare this discrete time irreversible mechanics with its classical Newton analog, time reversibility is reintroduced by scaling the time steps for any given time step $n$ by the variable $s_n$ leading to the Nosé-Hoover Lagrangian comprising a term $N_d f_k B T \ln s_n$ ($k_B$ the Boltzmann constant, $T$ the temperature, and $N_d$ the number of degrees of freedom) which is defined as the microscopic entropy $S_n$ at time point $n$ multiplied by $T$. Upon ensemble averaging of the microscopic entropy in a many particles system in thermodynamic equilibrium it approximates its macroscopic counterpart known from statistical mechanics. The presented derivation with the resulting analogy between the ensemble averaged microscopic entropy and its statistical mechanics analog suggests that the entropy term itself has its root not in statistical mechanics but rather in the discreteness of time.

1. Introduction
The second law of thermodynamics claims that the entropy $S$ of an isolated system increases over time until the system is in its thermodynamic equilibrium. Following the systematic formulation of statistical mechanics by Gibbs and Boltzmann, entropy $S^G$ is a measure of the number of accessible micro-states of the system of interest in its thermodynamic equilibrium. For a micro canonical ensemble the entropy is given by $S^G = k_B \ln \Omega$ with $k_B$ the Boltzmann constant, and $\Omega$ the number of accessible micro-states [1][2]. Interestingly and somewhat strangely, this probability-based entity exerts its presence as being part of the total free energy with $G = U + pV - TS$ with $U$ the inner energy, $p$ the pressure, $V$ the volume, and $T$ the temperature [1][2]. While it is not obvious that a pure statistical number times $k_B T$ is part of the total free energy of the system, the statistical mechanics argument is sound, that a system has the tendency to evolve towards its most probable state, which is the equilibrium state. However, also for the latter argument there are powerful protests such as the Loschmidt and Zermelo’s reversal and reoccurrence objections [3][4]. At the heart of these objections is that a single event does not follow statistics and thus the ensemble-based entropy is not relevant and thus can not be calculated. Another oddity is that microscopic physics is time reversible, but that in accordance to statistical mechanics an arrow of time comes apparently only into play at...
the macroscopic level with ensemble averaging over many repeated experiments. Consequently, it is the notion of the author and many other scientists (for example reference [5] and [6]) that the arrow of time is such an important physical measure that it has likely its root deep in physics at the microscopic level and not just at the macroscopic statistical level.

But from where could the arrow of time, time irreversibility, and entropy be originated from? As we shall see, it is our attempt to derive the arrow of time and the entropy part of the total free energy from the assumption that time is discrete. The discreteness of time is sound because experimentally measured time is composed of an array of events, which can be exemplified formally from the assumption that time is discrete. The discreetness of time is sound because of the uncertainty principle between energy and time (∆E>h/2, with h being the Planck constant). Furthermore, upon energy quantization for quantum mechanics the introduction of a discrete time appears to be straightforward (see for example references [7-13]). It is however not a very popular concept, because the continuity of time and space enables powerful mathematical tools.

To describe the evolution of a non-relativistic classical particles system in presence of a discrete time, we follow here in part the approach by Lee [10], who introduced time as a dynamical discrete variable yielding a scaling of time that depends on the potential present. The consequences of scaling time onto the law of energy conservation requests a reformulation of the Lagrangian as established by Nosé for isothermal molecular dynamics simulations of macroscopic systems [14-18]. There is a logarithmic term in the Nosé-Hoover Lagrangian which is dependent on the scaling of time. As we shall see, this term is defined as the microscopic entropy and if ensemble-averaged is equivalent to the macroscopic entropy in thermodynamic equilibrium. After the introduction of the discrete time (3.1), and the Nosé-Hoover Lagrangian (3.2), the Newton equation is derived (3.3), and time irreversibility in a microscopic system is discussed (3.4), followed by the definition of the microscopic entropy both for a single particle (3.5), as well as a many non-interacting particles system (3.6). It is then further shown that the averaged microscopic entropy corresponds to the Boltzmann entropy (3.7). Thus, the presented work is following in part our original article on this subject [19].

2. Theory

In classical physics time t is a continuous parameter. A non-relativistic particle with mass m in a one-dimensional space at the position r(t) at a given time t in a potential V(r) has the following Lagrangian $L = \frac{1}{2}m \dot{r}^2 - V(r)$ with $\dot{r} = \frac{dr}{dt}$ being the velocity. The equation of motion, i.e. the Newtonian law, can then be derived from the Lagrangian equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = -\frac{\partial L}{\partial r}$.

If time is a discrete variable $t_n$ the continuous function r(t) is replaced by a sequence of discrete values [10]:

$$(r_0, t_0), (r_1, t_1), ..., (r_n, t_n), ..., (r_{N+1}, t_{N+1}) (1)$$

with $(r_0, t_0)$ the initial and $(r_{N+1}, t_{N+1})$, the final position. In this description $r_n$ is still continuous, while $t_n$ is discrete and as requested a dynamic variable. The dynamic part of the time can be described by a time scaling variable $s_n$ defined by

$$s_n \Delta t_n = t_n - t_{n-1} (2)$$

with $\Delta t_n = \text{const.}$ (Please note, that the $s_n$ is defined as the corresponding variable $s$ of Nosé with $dt = \frac{ds}{s}$ in the Nosé-Hoover thermostat, see below; [2, 14-16,17,18,20]) yielding a description of the evolution of the particle with $t_0 = 0$ by the following sequence of discrete values:

$$(r_0, t_0 = 0), (r_1, s_1 \Delta t), ..., (r_n, \sum_{i=1}^{n} s_i \Delta t), ..., (r_{N+1}, \sum_{i=1}^{N+1} s_i \Delta t) (3)$$
In discrete mechanics there are many possible definitions of the velocity $\dot{r}_n$. Since it is the attempt of the present work to bridge the gap between microscopic and macroscopic physics, also the microscopic physics should have the property to be time irreversible under certain conditions. Thus, the velocity at time point $n$ $\dot{r}_n$ is defined time asymmetric. Furthermore, it is defined backward in time permitting a forward progressing description of the system from past and present information. This approach is in line with our daily experience that the presence is determined by the past and presence (i.e. to describe a present state of a system information can experimentally-derived only from the past and presence):

$$\dot{r}_n = r_n(r_n, r_{n-1}, t_n, t_{n-1}) = \frac{r_n - r_{n-1}}{t_n - t_{n-1}} \quad (4)$$

### 2.1. The Nosé-Hoover Lagrangian

In presence of a discrete dynamic time variable described by the time scaling variable $s_n$ a Lagrangian must be derived that is able to describe adequately the system. Such a Lagrangian under an artificial continuous scaled time $s$ has been introduced by Nosé and Hoover in the so called Nosé-Hoover thermostat [2, 14-16,17,18,20]. It allows the molecular dynamics simulation at constant temperature of a system and samples the configurations of a canonical ensemble. Hence, the Nosé-Hoover thermostat introduces a Lagrangian that describes a system on the microscopic level, which yields upon ensemble averaging a macroscopic thermodynamic description of the system bridging the microscopic with the macroscopic world, which is the scope of the present work [14,21].

The discrete analog of the Nosé-Hoover Lagrangian (in real space) is given by the following expression:

$$L^N_n = L^N_n(r_n, r_{n-1}; t_n, t_{n-1}) = L^N_n(r_n, r_{n-1}; \dot{s}_n, s_n) L_n$$

where $k_B$ is the Boltzmann constant, $Q$ is a constant, which has been described as a “mass”-like term for the motion of $s_n$, with $Q > 0$ and with actual units of energy time squared, and $N_{df}$ is the degree of freedom of the single particle system. The first two terms of the Lagrangian represent the kinetic energy minus the potential energy of the system, while the third and fourth terms represent the kinetic energy minus the potential energy associated with the $s_n$ variable. The third term can be interpreted as a kinetic energy of a heat bath coupled to the system of interest, and the fourth term describes the heat transfer between the heat bath and the system of interest. The heat bath mimics a bath composed of an infinite collection of harmonic oscillators and thus shows an infinite heat capacity [22]. The heat or temperature bath is however of another nature than usually defined in thermodynamics because it is a heat bath of the unitless time scaling variable $s_n$ with a mass-like $Q$ with units $Js^2$ (energy*seconds*seconds) and a velocity $\dot{s}_n = \frac{s_{n+1} - s_n}{t_n - t_{n-1}}$ with the unit $s^{-1}$. Nonetheless, it enables the use of a temperature as highlighted by the explicit presence of the temperature $T$ in the fourth term of the Lagrangian although the system described is of microscopic nature.

### 2.2. The Newton equation for a discrete time

If the discrete analog of the Lagrangian equation is defined as

$$\frac{1}{\Delta t_n} \left( L^N_{n+1}(r_{n+1}, \dot{r}_{n+1}) \right) - \frac{\partial L^N_n(r_n, \dot{r}_n)}{\partial \dot{r}_n} = - \frac{\partial L^N_n}{\partial r_n} \quad (6)$$
the discrete Newton’ law can be obtained using the discrete Lagrangian from eq. 5

\[ \frac{1}{\Delta t_n} (m \ddot{r}_{n+1}s_{n+1} - m \dot{r}_n s_n) = s_n F(r_n) \] (7)

\[ \frac{1}{\Delta t_n} (\dot{r}_{n+1} - \dot{r}_n + \dot{r}_n - \frac{s_n s_{n+1}}{m s_{n+1}}) = \frac{s_n}{m s_{n+1}} F(r_n) \] (8)

\[ \frac{\dot{r}_{n+1} - \dot{r}_n}{\Delta t_n} = \frac{s_n}{m s_{n+1}} F(r_n) - \dot{r}_n (1 - \frac{s_n}{s_{n+1}}) \frac{1}{\Delta t_n} \] (9)

\[ \dot{r}_n = \frac{s_n}{m s_{n+1}} F(r_n) - \dot{r}_n (1 - \frac{s_n}{s_{n+1}}) \frac{1}{\Delta t_n} \] (10)

\[ \ddot{r}_n = \frac{s_n}{m s_{n+1}} F(r_n) - \gamma_n \dot{r}_n \] (11)

with \( \gamma_n = \frac{s_n}{s_{n+1}} \) and \( s_n = \frac{s_{n+1} - s_n}{\Delta t_n} \).

An alternative derivation used below is

\[ \frac{1}{\Delta t_n} (\dot{r}_{n+1}s_{n+1} - \dot{r}_{n+1} + \dot{r}_{n+1} - \dot{r}_n) = \frac{1}{m} F(r_n) \] (12)

\[ \dot{r}_n = \frac{1}{\Delta t_n} (\dot{r}_{n+1} - \dot{r}_n) = \frac{1}{m} F(r_n) - \dot{r}_{n+1} (\frac{s_{n+1}}{s_n} - 1) \frac{1}{\Delta t_n} \] (13)

Eqs. 11 and 13, respectively, describe the Newton’s law under a discrete variable time.

If there is no scaling of time (i.e. \( s_i = 1 \) for all \( i = 1...N + 1 \)) the Newton’s law under a discrete time with constant time steps is of the form

\[ m \ddot{r}_n = F(r_n) \] (14)

which resembles its continuous analog. In presence of a scaling of time unequal to 1 however, the Newton’s law has an additional term, which can be regarded a friction term (see also below).

2.3. Time reversibility/irreversibility in a one dimensional space

Having established the Newton’s law (eqs. 11 and 13) in presence of a discrete dynamic time, we would like to pursue the issue of time reversibility or irreversibility. Time reversibility can be described by a two step process having one step forward followed by a step backward. Let us consider the evolution of the discrete Newton’s law of a single particle with two step forwards in a one dimensional space. Following eq. 11

(i)

\[ \dot{r}_{n+1} = \dot{r}_n + \frac{s_n}{m s_{n+1}} \Delta t_n F(r_n) - (1 - \frac{s_n}{s_{n+1}}) \dot{r}_n \] (15)

For the second time step we take the second expression of the discrete Newton’s law (eq. 13)

(ii)

\[ \dot{r}_{n+2} = \dot{r}_{n+1} + \frac{1}{m} \Delta t_{n+1} F(r_{n+1}) - \dot{r}_{n+2} (\frac{s_{n+2}}{s_{n+1}} - 1) \] (16)
If the second step is now backward in time

\[
\dot{r}_{n+2} = \dot{r}_{n+1} - \frac{1}{m} \Delta t_{n+1} F(r_{n+1}) - \dot{r}_{n+2}(\frac{s_{n+2}}{s_{n+1}} - 1)
\]

(17)

and if time reversibility is requested (i.e. \( \dot{r}_n = \dot{r}_{n+2} \) and \( s_{n+2} = s_n \))

\[
\dot{r}_n = \dot{r}_{n+2} = \dot{r}_{n+1} - \frac{1}{m} \Delta t_{n+1} F(r_{n+1}) - \dot{r}_{n}(\frac{s_n}{s_{n+1}} - 1)
\]

(18)

\[
\dot{r}_n = \dot{r}_n + \frac{s_n}{m s_{n+1}} \Delta t_n F(r_n) - (1 - \frac{s_n}{s_{n+1}})\dot{r}_n - \frac{1}{m} \Delta t_{n+1} F(r_{n+1}) - \dot{r}_n(\frac{s_n}{s_{n+1}} - 1)
\]

(19)

\[
\frac{s_n}{m s_{n+1}} \Delta t_n F(r_n) = \frac{1}{m} \Delta t_{n+1} F(r_{n+1})
\]

(20)

and since \( \Delta t_n = \Delta t_{n+1} = const \)

\[
\frac{s_n}{s_{n+1}} = \frac{F(r_{n+1})}{F(r_n)}
\]

(21)

This equation is defined as the reversibility axiom. Following the request from the reversibility axiom the discrete time mechanics is time reversible if time is a dynamic variable. Time reversibility is gained by the use of the scaling of time (i.e. \( s_i \)), which introduces a friction term into the discrete Newtonian law such, that if in the time forward step the force increases \( F(r_{n+1}) > F(r_n) \) this force is scaled down by the friction term in the step backward in order to fulfill reversibility.

If however the time steps are constant in size (i.e. \( s_i = 1 \) for all \( i = 1...N+1 \)) the microscopic system evolves time irreversible (unless \( F(r_n) = const = F(r_{n+1}) \)) because the reversibility axiom (eq. 21) is not fulfilled. That complex processes with many particles under a complex force (i.e. \( F(r_n) \neq const \)) are time irreversible is in line with our daily experiences (such as a glass vessel that falls from a table and breaks into pieces), and thus sound. It is therefore the view of the author that the presented discrete time mechanics with \( \Delta t_n = const \) = \( \Delta t \) (with \( \Delta t \) probably equal to the Planck time \( 5.4 \times 10^{-44} \) s and \( s_i = 1 \)) may well describe nature, while the continuous, time reversible Newton mechanics is only an approximation. Most importantly, under this assumption time reversibility of a system under a complex force is lost and hence an arrow of time at the microscopic physics level is introduced automatically without the need to introduce a macromolecular ensemble description that follows thermodynamics.

If the presented discrete time mechanics is a more profound theory than the Newton’s mechanics, on the one hand it should under certain boundary conditions asymptotically approximate the Newton’s mechanics and thus time reversibility (which easily can be shown by \( \lim \Delta t \to 0 \) in eq. 14) and on the other hand it should reflect irreversible properties (i.e. the entropy) of a thermodynamic state in its thermodynamic equilibrium which is described next.

2.4. The microscopic entropy of a single particle

Following Nosé and Hoover [2, 14, 17,18 20] from the discrete Lagrangian (eq. 5) a constant of motion evaluating to the total energy of the entire system is given by

\[
H_n^N = \frac{1}{2} m \dot{r}_n^2 + V(r_n) + \frac{1}{2} Q \frac{s_n^2}{s_{n+1}^2} + N q k_B T \ln s_n
\]

(22)

This constant of motion is composed of the inner energy of the system \( U \) described by the first two terms followed by the energy of the bath and the exchange energy (i.e. the latter two
terms having the variable $s_n$). Because of the resemblance of this term with the free energy $A = U - TS$ a microscopic entropy of a single particle at time point $t_n$ is defined to be

$$S_n = -N_{df} k_B \ln s_n$$

(23)

yielding

$$H_n^N = \frac{1}{2} m_i r_{i,n}^2 + V(r_n) + \frac{1}{2} Q_i \frac{s_{n}^2}{s_{i,n}^2} - TS_n = U + \frac{1}{2} Q_i \frac{s_{n}^2}{s_{i,n}^2} - TS_n$$

(24)

Please note that this microscopic entropy $S_n$ is a non ensemble-averaged term and concomitantly is in general of reversible character and thus not a monotonously increasing quantity. Furthermore, it can be calculated for a single particle system for a single event without the boundary of the system to be in thermodynamic equilibrium.

2.5. The microscopic entropy of a system with many particles

To extend the system from one particle to a system with $Z$ non interacting particles the potential $V_i(r_{i,n})$ for each particle $i$ and a particle individual scaling of time denoted $s_{i,n}$, following the reversibility axiom eq. 21: $\frac{s_{i,n+1}}{s_{i,n}} = \frac{F_i(r_{i,n})}{F_i(r_{i,n+1})}$, are introduced yielding a constant of motion evaluating to the total energy of the $Z$ particles system of type

$$H_n = \sum_{i=1}^{Z} \frac{1}{2} m_i r_{i,n}^2 + V_i(r_n) + \frac{1}{2} Q_i \frac{s_{n}^2}{s_{i,n}^2} + N_{i,df} k_B T \ln s_{i,n}$$

(25)

In accordance with eq. 23, the microscopic entropy of the $Z$ particles system is defined as

$$S_n = -\sum_{i=1}^{Z} N_{i,df} k_B \ln s_{i,n}$$

(26)

In an attempt to simplify eqs. 25 and 26 by reducing amongst others the number of scaling factors $s_{i,n}$ it is assumed that there are $J$ groups of particles and within each group the particles have undistinguishable properties but are still distinct since the system is microscopically described (i.e. within a group each particle has the same degrees of freedom, the same response to the potential, the same scaling factor, and the same velocity and mass). We define $p_j = \frac{\# of particles in group j}{Z}$ with $\sum_{j=1}^{J} p_j = 1$. For very large systems (i.e. at the thermodynamic limit) $p_j$ is also the probability of a particle to be in the $j$ group. Thus, the derived microscopic entropy of the $Z$ particles system at time point $n$ is given by

$$S_n = -Z \sum_{j=1}^{J} p_j N_{i,df} k_B \ln s_{j,n} = -k_B g \ln s_n = k_B N_{df} <\ln s_n>$$

(27)

with $N_{df} = Z N_{i,df}$ requesting for simplicity reasons that all particles have the same degree of freedom. $N_{df}$ describes thus the degree of freedom of the entire system.

In the next step, it is assumed that the scaling of time is very small (i.e. $s_{j,n}$ is very close to 1). This assumption is called the “slow changing force limit”. It is valid if the change of the force from one time step to the next is very small (i.e. $F(r_{n+1}) \approx F(r_n) + \Delta$ with $\Delta$ small). Under this assumption the $\ln$ of eq. 27 can be described by a Taylor expansion of first order and the averaging can be put inside the $\ln$, which results in
\begin{align*}
S_n &= -k_B N_d f \ln s_n \approx -k_B N_d f \sum_{j=1}^{J} p_j (s_{j,n} - 1) = -k_B N_d f \left[ \left( \sum_{j=1}^{J} p_j s_{j,n} \right) - 1 \right] \quad (28) \\
S_n &\approx -k_B N_d f \ln \sum_{j=1}^{J} p_j s_{j,n} = -k_B N_d f \ln <s_{n}> \quad (29)
\end{align*}

Correspondingly, the microscopic entropy difference between two time points \( n \) and \( m \) for a \( Z \) particles system is given by

\[ \Delta S = S_m - S_n = -k_B N_d f (\ln s_m - \ln s_n) \approx -k_B N_d f \ln \frac{<s_m>}{<s_n>} \quad (30) \]

It is interesting to note that this description can be used to calculate for a \( Z \) particles system at the thermodynamic limit the microscopic entropy difference between two time points by using averaged time scaling factors. Furthermore it is reiterated, that in contrast to the macroscopic thermodynamic entropy, the microscopic entropy can also be calculated for a system with a single particle or a few particles at any given time point \( n \). In addition, the microscopic entropy difference can be calculated without the request that the system in both states is in thermodynamic equilibrium. Furthermore, the averaging is not over all possible states as in statistical mechanics, but over the observed state.

### 2.6. Boltzmann entropy versus the microscopic entropy of a many particles system

In the following it is the aim to show for a large system of \( Z \) independent, identical, indistinguishable particles (such as the ideal gas or diluted gas) a profound relationship between the microscopic entropy and the corresponding macroscopic one, i.e. the Boltzmann entropy. For the system of interest each micro-state of the macroscopic system in thermodynamic equilibrium has the same probability yielding the Boltzmann entropy to be

\[ S^B = -k_B \ln p_j = -k_B N_{df} \sum_{j=1}^{J} p_j \ln p_j \quad (31) \]

with \( p_j = \frac{Z_j}{Z} \) being the probability of a single particle to be in the state \( j \) and the average is a single particle average taken over all the possible states \( J \) of the particle.

In comparison, the microscopic entropy of this \( Z \) particles system is given by \( S_n = -k_B N_{df} \sum_{j=1}^{J} p_j \ln s_{j,n} \). Because the description is microscopic, each particle is distinct and thus requests its own scaling of time (i.e. \( s_{j,n} \)). In the “slow exchanging force limit” the averaging can be put inside the \( \ln \)

\[ S_n \approx -k_B Z \ln <s_n> = -k_B N_{df} \ln \sum_{j=1}^{J} p_j s_{j,n} \quad (32) \]

Since the Boltzmann entropy is only defined for an ensemble, for a comparison between the microscopic and the Boltzmann entropy, single particle averaging of the microscopic entropy is necessary. This can be obtained as follows

\[ <S_n> \approx -k_B N_{df} \ln \sum_{j=1}^{J} p_j s_{j,n} \approx -k_B N_{df} \sum_{j=1}^{J} p_j \ln (p_j s_{j,n}) \quad (33) \]
A classical thermodynamic system in equilibrium for which the entropy is calculated can be assumed to be in the “slow changing force” limit with $s_{l,n}$ close to 1, yielding

$$< S_n > \approx -k_B N_d f \sum_{l=1}^{J} p_l \ln(p_l s_{l,n}) \approx -k_B N_d f \sum_{l=1}^{J} p_l \ln p_l = S^B$$  \hspace{1cm} (34)

Thus, for a system in the “slow changing force limit” with independent non-interacting particles, the average microscopic entropy in equilibrium approximates the Boltzmann entropy of the system. By setting the scaling factor to 1, also the time reversibility property of the microscopic entropy is lost (i.e. the reversibility axiom of eq. 21 is not fulfilled anymore, please also note, that time irreversibility of the microscopic entropy can also be obtained by setting $Q$ to infinity, for more details see reference [19]).

3. Conclusion

An alternative microscopic derivation of entropy is given, which originates from the quantization of time, which has only occasionally been investigated in the past. This theory introduces an arrow of time already at the microscopic physics level. Furthermore, it connects the microscopic physics with the macroscopic one without a statistical argument and can thus be calculated for any system (and thus also for a system with a small number of particles), which is not possible for the Boltzmann entropy. In contrast to the thermodynamic entropy, the Boltzmann entropy gets extensive and thus a state function only for a system at the thermodynamic limit (with number of particles $\rightarrow$ infinity) [23]. It also resolves the many objections that have been raised for the Boltzmann entropy, since it can be calculated for a single system without ensemble averaging and without the request that the system under study must be in thermodynamic equilibrium.

In addition, in the presented derivation the presence of the entropy term being part of the total energy of the system is a consequence of a friction term in the discrete Newton’s equation that is a consequence of the request for time reversibility, while in the Boltzmann approach oddly a statistical argument is part of the total energy of the system.

In summary, it is suggested that the original description of the entropy by Boltzmann is just an ensemble averaging of the time scaling variable $s_n$ which is in equilibrium close to 1, but that the entropy term itself has its root not in statistical mechanics but rather in the discreteness of time.

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References

[1] Landau L D and Lifshitz E M 1980 Statistical Physics 5 (3 ed) (Oxford: Pergamon Press) ISBN 0-7506-3372-7, translated by J B Sykes and M J Kearsley
[2] Hoover W G 1999 Adv. Series in nonlinear dynamics 13 1 (ISBN 981-02-4073-2)
[3] Loschmidt J 1876 Sitzungsber. Kais. Akad. Wiss. Wien, Math. Naturwiss. Classe 73 128
[4] Poincaře H 1890 Acta Math. 13 1
[5] Prigogine I 1997 The End of Certainty: Time, Chaos and the New Laws of Nature (New York: The Free Press)
[6] Przeź D 1996 Time’s arrow and Archimedes’ Point: New directions for the physics of time ISBN 0-19-510095-6
[7] Farias R A H and Recami E 2007 Preprint arXiv:quant-ph/9706059
[8] Jaroszkiewicz G and Norton K 1997 J. Phys. A.: Math. Gen 30 3115
[9] Jaroszkiewicz G and Norton K 1998 J. Phys. A.: Math. Gen 31 977
[10] Lee T D 1983 Phys. Lett. B 122 217
[11] Levi R 1927 Journal de Physique et le Radium 8 182
[12] Thomson J J 1925 Proc. Roy. Soc. of Edinburgh 46 90
[13] Yang C N 1947 Phys. Rev. 72 874
[14] Nosè S 1984 J. Chem. Phys. 81 511
[15] Nosè S 1984 Mol. Phys. 52 255
[16] Nosè S 1986 Mol. Phys. 57 187
[17] Hoover W G 1985 Phys. Rev. A 31 1695
[18] Hoover W G 2007 Mol. Simulation 33 13.
[19] Rieck R 2014 Entropy 16 3149
[20] Martyna G J, Klein M L and Tuckerman M 1992 J. Chem. Phys. 97 2635
[21] Hüenberger P H 2005 Adv. Polym. Sci. 173 105
[22] Campisi M and Hänggi P 2013 J. Phys Chem. B 117 12829
[23] Styer DF 2004 Am. J. Phys. 72 25