ON THE SUPPORT FOR WEIGHT MODULES OF AFFINE KAC-MOODY-ALGEBRAS

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ABSTRACT. An irreducible weight module of an affine Kac-Moody algebra \( \mathfrak{g} \) is called dense if its support is equal to a coset in \( \mathfrak{h}^*/Q \). Following a conjecture of V. Futorny about affine Kac-Moody algebras \( \mathfrak{g} \), an irreducible weight \( \mathfrak{g} \)-module is dense if and only if it is cuspidal (i.e. not a quotient of an induced module). The conjecture is confirmed for \( \mathfrak{g} = A_2^{(1)}, A_3^{(1)} \) and \( A_4^{(1)} \) and a classification of the supports of the irreducible weight \( \mathfrak{g} \)-modules obtained. For all \( A_n^{(1)} \) the problem is reduced to finding primitive elements for only finitely many cases, all lying below a certain bound. For the left-over finitely many cases an algorithm is proposed, which leads to the solution of Futorny’s conjecture for the cases \( A_2^{(1)} \) and \( A_3^{(1)} \). Yet, the solution of the case \( A_4^{(1)} \) required additional combinatorics.

A new category of hypoabelian subalgebras, pre-prosolvable subalgebras, and a subclass thereof, quasicone subalgebras, is introduced and its objects classified.

1. Introduction

Survey. Let \( \mathfrak{g} \) be an affine Kac-Moody algebra with Cartan subalgebra \( \mathfrak{h} \), root system \( \Delta \) and center \( \mathbb{C}c \). A \( \mathfrak{g} \)-module \( V \) is called a weight module if \( V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda} \), where \( V_{\lambda} \) denote the weight subspaces (also called weight components) \( V_{\lambda} = \{ v \in V \mid \lambda(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{h} \} \). We will also assume \( V \) to have countable dimension. If \( V \) is an irreducible weight \( \mathfrak{g} \)-module then \( c \) acts on \( V \) as a scalar, called central charge of \( V \). For a weight \( g \)-module \( V \), the support is the set \( \text{supp} (V) = \{ \lambda \in \mathfrak{h}^* \mid V_{\lambda} \neq 0 \} \). The root lattice \( Q \) is the subgroup of \( \mathfrak{h}^* \) generated by \( \Delta \). If \( V \) is irreducible then \( \text{supp} (V) \subseteq \lambda + Q \) for some \( \lambda \in \mathfrak{h}^* \).

An irreducible weight \( g \)-module \( V \) is called dense if its support is equal to a coset in \( \mathfrak{h}^*/Q \) and non-dense if \( \text{supp} (V) \subsetneq \lambda + Q \); a point of the set \( \lambda + Q \setminus \text{supp} (V) \) will also be called a hole.

Another criterion to classify modules is according to the way they are constructed. There are two classes of irreducible weight \( g \)-modules, those parabolically induced from other modules and which are not; we call the latter cuspidal modules. A result of V. Futorny and A. Tsylke \([\text{FuT}01]\) reduces the classification of irreducible weight \( g \)-modules with finite-dimensional weight spaces to the classification of irreducible cuspidal modules over Levi subalgebras. Any such module is a quotient of a module induced from an irreducible cuspidal module over a finite-dimensional reductive Lie subalgebra. The pending conjecture that connects the two approaches is as follows.

Conjecture 1. An irreducible weight \( g \)-module is dense if and only if it is cuspidal \([\text{Fu}97]\).

The property of a weight module \( V \) with finite-dimensional weight spaces being cuspidal is, for a reductive Lie algebra \( \mathfrak{g} \), equivalent to the statement that all root operators act injectively on \( V \) (cf. \([\text{DMP}00\text{ Corollary 3.7}])

The classification of all simple weight modules with finite-dimensional weight spaces over affine Lie algebras is work-in-progress \([\text{DG}09]\). If we omit the requirement of finite-dimensional weight subspaces, the achievement of a complete classification is much more difficult, because the method of (twisted) localization, developed by V. V. Deodhar and T. Enright \([\text{De}80\text{ and references therein}]\) and successfully applied by O. Mathieu for the simple Lie algebra case \([\text{Ma}00]\) and by I. Dimitrov and D. Grantcharov in the affine case \([\text{DG}09]\), is not applicable.

The classification for non-dense irreducible \( A_n^{(1)} \)-modules with a finite-dimensional weight subspace has been completed by V. Futorny \([\text{Fu}96]\). The classification problem of non-zero central charge modules with all finite-dimensional weight subspaces is solved for all affine Kac-Moody algebras \([\text{FuT}01]\). In these cases, an irreducible module is either a quotient of a classical Verma module, or of a generalized

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Verma module, or of a loop module (induced from a Heisenberg subalgebra). An important tool is the concept of primitive vectors.

A primitive vector is a vector \( \nu \) of a weight \( g \)-module with the following property: there exists a parabolic subalgebra \( p \) with Levi decomposition \( p = L \oplus \mathcal{N} \) such that \( \mathcal{N} \) acts trivially on \( \nu \). This primitive vector generates an irreducible quotient of a classical Verma module, a generalized Verma type module or a generalized loop module, depending on the type of \( p \) [Fu94]. In the well-studied case of a classical Verma module, \( p \) is just a Borel subalgebra [Kac]. Equivalent to Conjecture 1, there is

Conjecture 1'. Every non-dense weight \( g \)-module \( V \) contains a primitive vector.

A proof of the conjectures is an important step towards the classification of irreducible weight \( g \)-modules.

Conventions. Denote by \( \mathbb{C} \) the complex numbers and by \( \mathbb{Z}_{\geq k} \) the set \( \{k, k+1, \ldots \} \), by \( \mathbb{Z}_+ = \mathbb{Z}_{\geq 1} \) and \( \mathbb{N}_0 \) for \( \mathbb{Z}_{\geq 0} \). The difference of sets is written \( A \setminus B = \{ x \in A \mid x \notin B \} \).

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2. Affine Lie Algebras

2.1. Prerequisites

Let \( g^o \) be a simple finite dimensional complex Lie algebra over \( \mathbb{C} \) with a non-degenerate invariant symmetric bilinear form \( \kappa : g^o \times g^o \rightarrow \mathbb{C} \). Let \( h^o \) be Cartan subalgebra and \( \Delta^o \) the root system with respect to \( h^o \). The loop algebra \( \hat{L}(g^o) = g^o \otimes k[t^\pm 1] \) has a \( \mathbb{Z} \)-grading and a double extension

\[
\hat{g} = \hat{L}(g^o) = (L(g^o) \oplus \omega_0 \mathbb{C}) \rtimes \hat{D} \mathbb{C} = L(g^o) \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

for \( \omega_D (x,y) = \tilde{k}(Dx,y), \tilde{k}(Dx \otimes t^n, y \otimes t^m) = \delta_{n+m,0} \kappa^r(x,y) \) and \( D(x \otimes t^n) := nx \otimes t^n \). The Lie bracket in \( \hat{g} \) is given by

\[
[x_1 \otimes t^{n_1} + a_1 c + \xi_1 d, x_2 \otimes t^{n_2} + a_2 c + \xi_2 d] = [x_1, x_2] \otimes t^{n_1 + n_2} + \xi_1 n_2 (x_2 \otimes t^{n_1}) - \xi_2 n_1 (x_1 \otimes t^{n_2}) + n_1 \kappa^r(x_1, x_2) \delta_{n_1 + n_2, 0} c.
\]

and the linear functionals \( n \delta (n \in \mathbb{Z}) \), with \( \delta(d) = 1 \) and \( \delta |_{h^o \oplus Cc} = \{0\} \) are roots [Kac, MP95]. The Cartan algebra of \( \hat{g} \) is \( h = h^o \oplus Cd \ominus Cd \).

The invariant form restricted to \( h \) is non-degenerate. There is thus an injective map \( \hat{b} : h \rightarrow h^\star, h \mapsto h^\star, \hat{b}^\star (x) = \kappa(x, h) \). For \( \alpha \in h^\star = b(h) \) we put \( \alpha^\sharp = b^{-1} (\alpha) \) and define a symmetric bilinear form on \( h^\star \) by \( (\alpha, \beta) = \kappa(\alpha^\sharp, \beta^\sharp) \).

The affine Weyl group \( \mathcal{W} \) is generated by the set of reflections

\[
r_\alpha (\lambda) = \lambda - \frac{2}{(\alpha, \alpha)} (\alpha, \alpha) \alpha, \alpha \in \Delta, \lambda = \lambda^o + zc + td \in h^\star, z, t \in \mathbb{C}.
\]

If \( r_\alpha \) is the identity on \( (h^\star)^\star \), then \( \alpha \) is an imaginary root, otherwise it is a real root. The set of roots is the disjoint union of imaginary and real roots, i.e. \( \Delta = \Delta_{im} \cup \Delta_{re} \). For \( \alpha \in (h^\star)^\star \) the translation with respect to \( \alpha \) is the operator \( t_\alpha \) acting on \( h^\star \) by

\[
t_\alpha (\lambda) = \lambda + \lambda (c) \alpha - \left( (\lambda, \alpha) + \frac{1}{2} (\alpha, \alpha) \lambda (c) \right) \delta (\lambda \in h^\star).
\]

A subset \( P \subset \Delta \) is called additively closed, if \( (P + P) \cap \Delta \subset P \). An additively closed subset is a parabolic system, if \( P \neq \Delta \) and \( P \cup -P = \Delta \). A subset \( P \subset \Delta \) is a positive system, if \( \text{span}_{\mathbb{Z}_0} P \cap -\text{span}_{\mathbb{Z}_0} P = \{0\} \) and \( P \cup -P = \Delta \). Two subsets are called equivalent if they lie in the same \( \mathcal{W} \times \{ \pm 1 \} \)-orbit.

From [Fu97, Ch. 2] we know that there exists a finite number of pairwise non-equivalent positive systems of the root system of \( g \). The positive systems \( P = \{ \alpha + Z \delta \mid \alpha \in \Delta^o \} \cap \Delta \cup Z_+ \delta \) and \( \Delta^+ = \{ \alpha + Z \delta \mid \alpha \in \Delta^o \} \cup \{ -\alpha + Z_+ \delta \mid \alpha \in \Delta^o \} \cup Z_+ \delta \) are non-equivalent.

Remark 1. In the literature \( P \) is also called imaginary parabolic partition of \( \Delta \), as related to the natural Borel subalgebra and imaginary Verma modules both introduced later. The set \( \Delta^+ \) is called standard (or classical) parabolic partition. Any other positive system that is not equivalent to \( P \) or \( \Delta^+ \) will be labeled mixed type.
Kac and Jacobson [KJ85], and independently the exposition of V. Futorny [Fu92] have determined a positive system of a finite-rank root system $\Delta$ uniquely by means of characteristic functionals. The latter exposition calls a positive system a parabolic partition.

Let $\Pi$ be a basis for the root system and $\Pi^*$ be the dual basis, defined by $\alpha^* (\beta) = \delta_{\alpha, \beta}$ for all $\alpha^* \in \Pi^*$ and all $\beta \in \Pi$. There exists a pair of bases $(\Pi_\delta, \Pi_\beta^0)$, where $\Pi_\delta^0$ contains $\delta^0$ and $\Pi_\delta$ contains $\delta$. We denote the coefficients of $\delta = \sum_{\alpha \in \Pi} k_\alpha \alpha$ with respect to such a base change by $k_\alpha \in \mathbb{N}_0$. $\alpha \in \Pi$. Define furthermore weights $\omega_\alpha : \mathfrak{h} \rightarrow \mathfrak{k}$ by $\omega_\alpha (\hat{\beta}) = \delta_{\alpha, \beta}$ for $\alpha, \beta \in \Pi$. Then the set $\{ \omega_\alpha \}_{\alpha \in \Pi}$ is the set of fundamental weights.

From the above it follows, that $2 \alpha^2 = (\alpha, \alpha) \alpha$. A weight $\lambda \in \mathfrak{h}^*$ is positive, if it is a positive linear combination of fundamental weights. Thus if $\lambda$ is positive with respect to $\Pi$, then $\lambda (\alpha^2)$ is also positive for all $\alpha \in \Pi$, unless $\alpha$ is an isotropic root. For any weight, $\ker (\lambda \circ \varphi) = (\ker \lambda)^\circ$.

Then we may define the weights $\lambda \pm X = \pm \sum_{\alpha \in X} \omega_\alpha$ for all $X \subset \Pi \setminus \{ \alpha_0 \} =: \Pi^0$ and for $X \subset \Pi^*$, set $\Delta_\pm (X) = \Delta_\pm (\phi_X, \phi_{\Pi^0})$.

Note that this is consistent with the definition of $\Delta_\pm (\Pi^0)$ as $\Delta_+ (\Pi^0) = \text{span}_{\mathbb{C}} \Pi^0 \cap \Delta$. The following theorem tells us that the equivalence classes of positive systems are parametrized by the sets $X \subset \Pi^0$.

**Theorem 2.** [Fu97] If $P$ is a positive system of an affine root system $\Delta$, then there is a set $X \subset \Pi^0$ such that $P = \Pi^0 \times \{ \pm 1 \}$-equivalent to $\Delta_+ (X)$.

### 2.2. Triangular decompositions and parabolic systems

If $P$ is a parabolic system for $\mathfrak{g}$ with Cartan algebra $\mathfrak{h}$ and root system $\Delta$, then we can define $P (P) = \mathfrak{h} + \sum_{\alpha \in P} \mathfrak{g}_\alpha = \mathfrak{g}_0 + \mathfrak{g}_P$ which is a subalgebra of $\mathfrak{g}$. It turns out to be a parabolic subalgebra in the commonly defined sense.

**Definition 3.** A triple $(\mathfrak{g}_0, \mathfrak{g}_\pm, \mathfrak{g}_\pm)$ of subalgebras of $\mathfrak{g}$ defines a split triangular decomposition, if there exist subsets $\Delta_\pm, \Delta_0 \subseteq \Delta$ such that (i) $\Delta = \Delta_+ \cup \Delta_0 \cup \Delta_-$ is a partition of $\Delta$, (ii) $\mathfrak{g}_\pm = \sum_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{g}_0 = \mathfrak{h} + \sum_{\alpha \in \Delta_0} \mathfrak{g}_\alpha$, (iii) $[\mathfrak{g}_0, \mathfrak{g}_\pm] \subseteq \mathfrak{g}_\pm$, and (iv): if $\alpha_1, \ldots, \alpha_n \in \Delta_-$, or $\alpha_1, \ldots, \alpha_n \in \Delta_-$, but $\sum_{i=1}^n \alpha_i \neq 0$ for $n > 0$.

A parabolic system $P$ is called principal, if there exists a split triangular decomposition such that $P = \Delta_+ \cup \Delta_0$ [DG09].

Every linear functional $\lambda \in \mathfrak{h}^*$ determines a split triangular decomposition by putting

$$\Delta_\pm (\lambda) = \left\{ \pm \alpha \in \Delta \mid \lambda (\alpha^2) > 0 \right\} \quad \text{and} \quad \Delta_0 (\lambda) = \left\{ \alpha \in \Delta \mid \lambda (\alpha^2) = 0 \right\}. \tag{2.4}$$

Clearly $\Delta_\pm (\lambda) \cup \Delta_0 (\lambda)$ is a parabolic system, which are all principal. Recall that

$$\Delta_\pm (\lambda) = (\lambda^{-1} (Z_\pm))^\circ \cap \Delta \quad \text{and} \quad \Delta_0 (\lambda) = (\ker \lambda)^\circ \cap \Delta.$$  

The scalar $\lambda (\delta^0) = \lambda (c)$ is called the central charge of $\lambda$.

The theory of parabolic systems of finite rank root systems is completely governed by a pair of subsets, $S$ and $X \subset \Pi^0$, and can be described in terms of three weights. To begin with, define

$$P (\lambda_1, \lambda_2, \lambda_3) = \Delta_+ (\lambda_1) \cup \left( \Delta_0 (\lambda_1) \cap \left( \Delta_+ (\lambda_2) \cup \Delta_+ (\lambda_3) \right) \right) \tag{2.5}$$

for a triple of weights $(\lambda_1, \lambda_2, \lambda_3)$. With (2.2), we can write

$$P (X, -S) = P (\phi_X, \lambda_{-S}, \phi_{\Pi^0}), \quad P_3 (\Pi^0) = P (\phi_{\Pi^0}, \lambda_{-S}, \lambda_{\{ \alpha_0 \}}) \quad \text{and} \quad P_3 = P (\lambda_{\Pi^0 \setminus S}, \phi_{\Pi^0}, \phi_{-\Pi^0}). \tag{2.6}$$

**Theorem 4.** Any parabolic system which is not a positive system is equivalent to either

- (i) $P (\phi_X, \lambda_{-S}, \lambda)$ with
  - (a) $\emptyset \subseteq S \subset X \subseteq \Pi^0$ and $\lambda = \phi_{\Pi^0}$ or with
  - (b) $S \subseteq X = \Pi^0$ and $\lambda = \lambda_{\{ \alpha_0 \}}$, or to
- (ii) $P (\lambda_{\Pi^0 \setminus S}, \phi_{\Pi^0}, \phi_{-\Pi^0})$ with $S \subset \Pi^0$. 

If the parabolic system does not contain $-\delta$, it falls in the classes (i), if it contains $-\delta$, in class (ii).

**Proof.** We need to show that the sets \([2,6]\) coincide with the ones defined in \([\text{Fu97} \text{Sec. 2.}]\):

\[
P(X, S) = \Delta_+ (X) \cup \text{add}_A (-S) \quad (S \subset X \subset \Pi^\circ, S \neq \emptyset)
\]

\[
P_3 (\Pi^\circ) = \text{add}_A (\Delta_+ (\Pi^\circ)) \cup (-S) \cup \{-\alpha_0\}
\]

\[
P_3 = \text{add}_A (\Delta_+ (\emptyset)) \cup (-S) \cup \{-\delta\}
\]

With tuples of weights, we have defined $\Delta_+ (\lambda_1, \lambda_2) = \Delta_+^o (\lambda_1) \cup (\Delta_0^o (\lambda_1) \cap \Delta_+^o (\lambda_2))$ in \([2,3]\) and inside the right hand side we find

\[
\Delta_+ (X) = \Delta_+ (\phi_X, \phi_{\Pi^\circ}) = \Delta_+^o (\phi_X) \cup \left(\Delta_0^o (\phi_X) \cap \Delta_+^o (\phi_{\Pi^\circ})\right).
\]

Plugging \([2.5]\) in \([2.6]\) for the left hand side

\[
P(X, S) = \Delta_+^o (\phi_X) \cup \left(\Delta_0^o (\phi_X) \cap \left(\Delta_0^o (\phi_{-S}) \cup \Delta_0^o (\phi_{\Pi^\circ})\right)\right),
\]

we obtain as the most inner term: $\Delta_+^o (\lambda_{-\delta}) \cup \Delta_+^o (\phi_{\Pi^\circ})$. This term contains both, $\text{add}_A (-S)$ and $\Delta_+^o (\phi_{\Pi^\circ})$.

Since $S \subset X$ implies $\text{add}_A (-S) \subset \Delta_+^o (\phi_X)$, the first coincidence follows.

For $P_3 (\Pi^\circ)$ it is sufficient to see that $\Delta_0^o (\phi_{\Pi^\circ}) \cap (\Delta_+^o (\lambda_{-\delta}) \cup \Delta_+^o (\lambda_{-\alpha_0}))$ contains $\text{add}_A (\Delta^\circ (S) \cup \{-\alpha_0\})$, taking into consideration the fact that $-\alpha_0 = \frac{1}{m_0} (\theta - \delta) \in \Delta_0^o (\phi_{\Pi^\circ})$.

For $P_3$, we have $\Delta_0^o (\lambda_{-\delta}) = \Delta_0^o (\sum_{a \in \Pi^\circ} \omega_a) = (\text{add}_A (\pm S) + Z\delta) \cup Z_{\neq 0}\delta$ and

\[
\Delta_+^o (\lambda_{-\delta}) = \left(\Delta_+^o \setminus \text{add}_A S + \Delta_0^o (\lambda_{-\delta}) \cup \{0\}\right) \cap \Delta
\]

\[
= \Delta_+^o \setminus \text{add}_A S + ((\text{add}_A (\pm S) + Z\delta) \cup Z_{\neq 0}\delta \cup \{0\}) \cap \Delta
\]

\[
= \left(\Delta_+^o \setminus \text{add}_A S + \Delta_0^o (\phi_{-S}) \cup \Delta^\circ + Z\delta, \right.
\]

Furthermore,

\[
\Delta_+^o (\phi_{\Pi^\circ}) \cup \Delta_+^o (\phi_{-\Pi^\circ}) = \Delta,
\]

because $\Delta_+^o (\phi_{\Pi^\circ})$ is a positive system by \([2]\). Thus, the left hand side reads

\[
\Delta_+^o (\lambda_{-\delta}) \cup \left(\Delta_0^o (\phi_{\Pi^\circ}) \cup \Delta_0^o (\phi_{-\Pi^\circ})\right)
\]

\[
= \left(\Delta_+^o \setminus \text{add}_A S + \Delta_0^o (\phi_{\Pi^\circ}) \cup \Delta_0^o (\phi_{-\Pi^\circ})\right) \cap \Delta
\]

\[
= \left(\Delta_+^o \setminus \text{add}_A S + \Delta_0^o (\phi_{\Pi^\circ}) \cup \Delta_0^o (\phi_{-\Pi^\circ})\right) \cap \Delta^\circ + Z\delta \cup (\text{add}_A (\pm S) + Z\delta) \cup Z_{\neq 0}\delta
\]

\[
= \left(\Delta_+^o \setminus \text{add}_A S + \Delta_0^o (\phi_{\Pi^\circ}) \cup \Delta_0^o (\phi_{-\Pi^\circ})\right) \cap \Delta^\circ + Z\delta \cup Z_{\neq 0}\delta,
\]

and the right hand side

\[
\text{add}_A (\Delta_+ (\emptyset) \cup (-S) \cup \{-\delta\}) = \text{add}_A \left(\Delta_+^o \cup Z\delta \cup Z_{\neq 0}\delta \cup \{-\delta\}\right) \cup \frac{1}{m_0} \sum_{\beta \in \Delta} g\beta.
\]

From the description it is also clear, that in the case $S = \Pi^\circ$, the parabolic systems $P_3 (\Pi^\circ)$ and $P (\Pi^\circ, S)$ coincide.

### 2.3. Parabolic subalgebras

Let $(\cdot, \cdot)$ be the standard form on $g$. The (standard) Hermitian form on $g$ is given by $(x, y)_g = (\sigma_0 (x), y)$. A unitary involution $\bar{\sigma}$ is given by the negative Chavelley involution and defined as $\bar{\sigma} (ax) = -a$ for $a \in \Delta, x_a \in g$ and $\bar{\sigma} (h) = h$ for $h \in h$. If $\sigma : \Delta \rightarrow \Delta$ is the linear involutive automorphism defined by $\sigma (b) = -b$ for any $b \in \Delta$, then both $\sigma_0$ and $\bar{\sigma}$ are functorial extensions to $g$.

**Definition 5.** A subalgebra $b$ is called Borel-type subalgebra if $\bar{\sigma} (b) \cap b = \emptyset$. It is called Borel subalgebra if there is a positive system $\Delta_+$ such that $b = h \oplus \sum_{\beta \in \Delta_+} g\beta$.

In contrast to Borel subalgebras of finite dimensional Lie algebras, the above definition admits Borel-type subalgebras that do not correspond to positive systems of $\Delta$ (cf. \([\text{BBFK13}]\)).

**Definition 6.** The subalgebra $p (P) = \emptyset \oplus \sum_{\beta \in P} g\beta$ for the parabolic system $P \subset \Delta$, is called parabolic subalgebra. Additionally, $p$ is called maximally parabolic, if it is maximal as a proper subalgebra.
Introduce the derived algebra \( g' = \left[ g, g \right] \) (cf. [Car, p. 335]) and the derived algebra related to the parabolic system \( P \), given by \( g'_P = \left[ g\cap\sigma(P), g\cap\sigma(P) \right] \). Let \( B \) be a basis for \( P \cap \sigma(P) \), then in root space decomposition this writes as
\[
g'_P = \text{span}_{C} \hat{B} \otimes C \left[t, t^{-1}\right] \oplus \sum_{\beta \in P^+ \cap \sigma(P) \cap \Delta} g_{\beta}.
\]
If \( \delta \in P \cap \sigma(P) \), then \( d = \delta \in \text{span}_{k} \hat{B} \). The Heisenberg subalgebra of \( g \) is the sum of the isotropic root spaces and the center,
\[
\mathcal{H} = \sum_{\alpha \in \mathcal{L} \cap \delta} g_{\alpha} \oplus Cc.
\]
If \( g \) is realized as affinization of a split simple Lie algebra \( (g^\circ, h^\circ, \kappa^\circ) \) (cf. Section 1.2), then \( \mathcal{H} = L(\kappa^\circ) \otimes_{\text{aff}} k \). With this we can introduce a subalgebra of the Heisenberg subalgebra \( \mathcal{H}_{\Pi \subseteq B} \subset \mathcal{H} \) defined by the relations \( [g'_P, \mathcal{H}_{\Pi \subseteq B}] = 0 \) and \( \mathcal{H}_{\Pi \subseteq B} + (g'_P \cap \mathcal{H}) = \mathcal{H} \). These relations are satisfied by
\[
\mathcal{H}_{\Pi \subseteq B} = \text{span}_{\kappa} (\Psi) \otimes k \left[t, t^{-1}\right] \oplus kc \text{ for } \Psi = \{ \psi \in \Pi \mid (\psi, B) = \{0\}\}.
\]

**Theorem 7.** [Fu97, Th. 3.3] Let \( g \) be the affine Lie algebra and \( \Delta \) its root system and \( P \) be an arbitrary parabolic system in \( \Delta \).

(i) The subalgebra \( p(P) \) of \( g \) has a decomposition
\[
p(P) = L \oplus \mathcal{N}, \text{ where } \mathcal{N} = g_{p \cap \sigma(P)}
\]
and \( L \) is one of the following types,

(a) a locally finite Lie algebra or

(b) \( L = (g'_P + \mathcal{H}_{\Pi \subseteq B}) + \mathfrak{h} \).

(ii) \( g \) has a split triangular decomposition associated to \( P \),
\[
g = \mathcal{N} \oplus L \oplus \mathcal{N}, \text{ where } \mathcal{N} = g_{\sigma(P) \cap p}.
\]

3. **Weight modules**

3.1. **Induced Representations** Consider a subset \( S \subset \Pi^\circ \) and define the subalgebra
\[
g(S) = cl_g (g_{\Phi} \oplus g_{-\Phi} \mid \Phi \in S) \subset g^\circ
\]

**Definition 8.** The Levi subalgebra associated with \( S \subset \Pi^\circ \) is the finite-dimensional reductive Lie algebra \( L_S = h + g(S) \). Denote for \( S \subset \Pi^\circ \)
\[
\mathcal{N}^\pm = \sum_{\varphi \in \Delta^\pm, \text{add}_{\delta}} g_{\varphi}, \quad \mathcal{P}^\pm = L_S \oplus \mathcal{N}^\pm,
\]

Start with a semisimple Levi component \( L_S = h + g(S) \) for a subset \( S \subset \Pi^\circ \). Because \( L_S \) is semisimple, it admits a triangular decomposition \( L_S = L_S^{+} \oplus L_S^{0} \oplus L_S^{-} \). Its universal enveloping algebra decomposes according to PBW Theorem, \( \mathcal{U}(L_S) = \mathcal{U}(L_S^{+}) \mathcal{U}(L_S^{0}) \mathcal{U}(L_S^{-}) \), with a non-trivial center \( \mathcal{Z} \subset \mathcal{U}(L_S) \).

Define the polynomial ring \( \mathcal{T}_S = \text{Sym}(h) \otimes \mathcal{Z} \). Here \( \text{Sym}(h) \cong \mathcal{U}(h) \otimes \mathbb{C}[\Pi, \mathfrak{c}] \). The generalized Harish-Chandra homomorphism associated to \( S \subset \Pi^\circ \) is the projection \( \phi_{HC}^{(S)} : \mathcal{U}(g) \to \mathcal{T}_S \) with respect to the decomposition
\[
\mathcal{U}(g) = (\mathcal{N}^\circ \mathcal{U}(g) + \mathcal{U}(g) \mathcal{N}^\circ) \oplus (L_S \mathcal{U}(L_S) + \mathcal{U}(L_S) L_S^{-}) \oplus \mathcal{T}_S.
\]

For a parabolic system \( P \) with subalgebra \( p(P) \) that is of type \( \delta \) according to Theorem 7 (\( L \) locally finite), there is a Levi decomposition \( p(P) = L \oplus \mathcal{N} \) that meets the condition for \( L \) to be a Levi subalgebra. Since the set \( P \cap -P \) is additively closed in \( \Delta \), we can designate a basis \( S \) for the positive root monoid inside \( P \cap -P \). Then \( L_S \) equals the Levi subalgebra \( L_S = h + g(S) \). We can construct an irreducible \( L_S \)-module as follows:

Fix \( \lambda \in h(S)^+ \) and \( \gamma \in \mathcal{C} \). Let \( \mathcal{C}_{\lambda, \gamma} \) be a 1-dimensional \( \mathcal{T}_S \)-module with the action \( (h \otimes z) v_{\lambda, \gamma} = \lambda(h) \gamma(z) v_{\lambda, \gamma} \) for \( h \in h(S) \) and \( z \in \mathcal{Z} \), and \( L_S v_{\lambda, \gamma} = 0 \). Then
\[
V_S(\lambda, \gamma) = \mathcal{U}(L_S) \otimes \mathcal{C}_{\lambda, \gamma}
\]
is a \( L_S \)-module, which has a unique irreducible quotient. Because \( g(S) \) is semisimple, the irreducible weight \( h + g(S) \)-modules with finite-dimensional weight spaces are classified by S. Fernando and O.
Mathieu (cf. [Ma00]) as being isomorphic to certain parabolically induced modules. Let $p$ be a parabolic subalgebra of $\mathfrak{g}(S)$ and $W$ be a cuspidal $p$-module (see introduction or Section 3.7 for definition), then the induced module

$$M_p(W) = \text{Ind}^H_W = \mathcal{U}(\mathfrak{g}(S)) \otimes_{\mathcal{U}(p)} W$$

has a unique irreducible quotient $L_p(W)$.

**Theorem 9.** [Fe90] If $V$ is an irreducible weight $\mathfrak{h} + \mathfrak{g}(S)$-module with only finite-dimensional weight components, then $V$ is isomorphic to $W$ or to $L_p(W)$ for some parabolic subalgebra $p$ and some cuspidal $p$-module $W$.

### 3.2. Parabolic induction

**Definition 10.** If $S_k$, $(k = 1, \ldots, N)$ are the connected components of the partition $S \subset \Pi^\circ$ with $S = \bigcup_{k=1}^N S_k$, then write

$$\hat{\mathfrak{g}}(S_k) = (\mathfrak{g}(S_k) \otimes \mathbb{C}[t^\pm t^{-1}] \otimes \mathbb{C}d) \times \mathbb{C}d$$

for the corresponding affine Lie algebra and denote the Lie subalgebra $\hat{\mathfrak{g}}(S) = \sum_{k=1}^N \hat{\mathfrak{g}}(S_k)$.

If $P \subset \Delta$ is a parabolic system, then $P \cap \sigma(P)$ is additively closed in $\Delta$. It is thus possible to choose $S$ such that $P \cap \sigma(P) = \text{span}_{\mathbb{Z}}S$ and meaningful to set $\mathcal{H}_P = \mathcal{H}(\hat{\mathfrak{g}}(S_k)) = \mathcal{H}(\hat{\mathfrak{g}} \mid \alpha \in S^\circ) \subset \mathcal{H}$.

Let $p = p(P) = L \oplus \mathcal{N}_P$ be a parabolic subalgebra corresponding to a parabolic system $P$. Given an irreducible weight $L$-module $V$, we extend the action to $\mathcal{N}$ trivially, obtaining an irreducible $p$-module $V$. Construct the $g$-module

$$M_p = \text{ind}^g(V) = \mathcal{U}(g) \otimes V.$$ 

Depending on $p$, it will be called **Generalized Verma Type module**, if $p$ is of type $(\alpha)$, and **generalized loop module**, if $p$ is of type $(\mu)$. As special cases, $M_p$ is a classical Generalized Verma Module if $p$ is a standard parabolic subalgebra and $V$ a finite-dimensional $p$-module. If $p$ is of type $(\alpha)$, denomination is further refined.

### 3.3. Standard Generalized Verma Modules

Standard Generalized Verma Modules (GVM) are induced from an irreducible module for a Levi subalgebra $L_S = \mathfrak{h} + \mathfrak{g}(S)$. First the general case:

If $S \subset \Pi^\circ$, then $\mathcal{P}_S^\pm = L_S \oplus \mathcal{N}_S^\circ$ and, for an irreducible $L_S$-module $V_S$, we set $\mathcal{N}_S^\circ V_S = 0$ and

$$M_S^\pm(\lambda, \gamma) = \text{ind}^g_{\mathcal{P}_S^\pm}(V_S) = \mathcal{U}(g) \otimes V_S.$$ 

(3.1)

It has a unique irreducible quotient $L_S^\pm(\lambda, \gamma)$. Notice that $V_S$ does not have to be finite-dimensional.

**Proposition 11.** [Fu97] Prop. 3.6. (iii)] Let $\mathcal{V}$ be an irreducible weight $g$-module and $0 \neq v \in \mathcal{V}_\lambda$ such that $\mathcal{N}_S^\circ v = 0$, then $\mathcal{V} \cong L_S^\pm(\lambda, V_S)$ for an irreducible $L_S$-module $V_S$.

### 3.4. Irreducible representations of the Heisenberg subalgebra

**Definition 12.** The **Heisenberg subalgebra** $\mathcal{H}(\alpha_1, \ldots, \alpha_k)$, $(k \leq n)$, is the image of $\sum_{i=1}^k \alpha_i \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}c$ under the quotient map $\mathfrak{g} \to \mathfrak{g}/(\mathfrak{h} \otimes 1)$.

The Heisenberg subalgebras admit the maximal abelian subalgebras $\mathcal{H}_\pm(\alpha_1, \ldots, \alpha_k) \oplus \mathbb{C}c$ with

$$\mathcal{H}_\pm(\alpha_1, \ldots, \alpha_k) = \sum_{i=1}^k \alpha_i \otimes t^\pm \mathbb{C}[t, t^{-1}].$$

Consider the full Heisenberg subalgebra and its maximal abelian subalgebra

$$\mathcal{H} = \mathcal{H}(\alpha_1, \ldots, \alpha_n) = \sum_{m \neq 0} \mathfrak{g}_{m\bar{m}} \oplus \mathbb{C}c \quad \mathcal{H}_+ = \mathcal{H}_+(\alpha_1, \ldots, \alpha_n) = \sum_{m > 0} \mathfrak{g}_{m\bar{m}}.$$ 

Let $a \in \mathbb{C}^*$ and $\mathbb{C}v_a$ be the the 1-dimensional $\mathcal{H}_+ \oplus \mathbb{C}c$-module for which $\mathcal{H}_+ v_a = 0$, $cv_a = av_a$. Consider the $\mathcal{H}$-module

$$M^+(\lambda) = \mathcal{U}(\mathcal{H}) \otimes_{\mathcal{U}(\mathcal{H}_+ \oplus \mathbb{C}c)} \mathbb{C}v_a.$$ 

(3.2)
It carries a natural $\mathbb{Z}$-grading with the $i$-th component $\sigma \left( \mathcal{U}(\mathcal{H}_{\lambda}) \right) v_i$. Define $M^+ (a)$ for $\mathcal{H}$ analogously.

Define another family of modules, so-called loop modules as in [ChP86]. Let $p: \mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H}) / \mathcal{U}(\mathcal{H}) c$ be the canonical projection. For $r > 0$, consider the $\mathbb{Z}$-graded ring $L_r = \mathbb{C}[r^{-r}, r^r]$. Denote by $P_r$ the set of graded ring epimorphisms $\Lambda: \mathcal{U}(\mathcal{H}) / \mathcal{U}(\mathcal{H}) c \to L_r$ with $\Lambda(1) = 1$. Define a $\mathcal{H}$-module structure on $L_r$ by the following action of any $e_k^{(i)} = \tilde{\alpha} \otimes t^k \in g_{k\delta}, \alpha \in \Pi^c$:

$$e_k^{(i)} t^{ir} = (\Lambda \circ p) \left( e_k^{(i)} \right) t^{ir} = t^{(k+s)r}, \Lambda \in P_r, k \in \mathbb{Z} \setminus \{0\}, ct^{ir} = 0, s \in \mathbb{Z}.$$

Denote this $\mathcal{H}$-module by $L_{r,\Lambda}$. Define $\Lambda_0$ the trivial homomorphism onto $\mathbb{C}$ with $\Lambda_0(1) = 1$, then $L_{0,\Lambda_0}$ is the trivial module.

**Proposition 13.** (i) [Fu96] Every irreducible $\mathbb{Z}$-graded $\mathcal{H}$-module $V$ with central charge $\lambda(c) = a \in \mathbb{C}$ with at least one finite-dimensional weight component $V_\phi$ is isomorphic to $M^+ (\lambda)$ up to a shifting of gradation.

(ii) [Ch86] Every irreducible $\mathbb{Z}$-graded $\mathcal{H}$-module with central charge zero is isomorphic to $L_{r,\Lambda}$ for some $r > 0, \Lambda \in P_r$ up to a shifting of gradation.

### 3.5. Generalized loop modules

Following [Ch86], we define the category $\tilde{\mathcal{O}} (\Pi^\circ)$ to be the category of weight $g$-modules $V$ satisfying the condition that there exist finitely many elements $\lambda_1, \ldots, \lambda_r \in \mathfrak{h}^*$ such that

$$\text{supp}(V) \subset \bigcup_{i=1}^r R(\lambda_i) \text{ where } R(\lambda_i) = \lambda_i - \text{span}_{\mathbb{Z}} \Pi^\circ + \mathbb{Z}\delta.$$

In $\Delta^\circ$, the set $\Phi = -\Delta^\circ_+ \cup \text{add}_{\Delta^\circ} (S)$ is a parabolic system for every $S \subset \Pi^\circ$. Thus, we can define $n_{\phi} = \text{g}_{\text{adP} \setminus \phi \text{P}}$ if $S$ is a basis for $P \setminus \text{adP} \setminus \phi \text{P}$ and $P$ is given by

$$P = \Delta^\circ \setminus \text{add}_{\Delta^\circ} (S) + \mathbb{Z}\delta.$$

Thus

$$n_{\phi} = \sum_{\phi \in F \setminus \phi \text{P}} \text{g}_{\phi}. \quad (3.3)$$

Back to the case where $S = \Pi^\circ$: Denote $n_+ = n_{\Pi^\circ}$ and $n_- = n_{-\Pi^\circ}$. Then $\text{g} = n_- \oplus (\mathfrak{h} + \mathcal{H}) \oplus n_+$ is a triangular decomposition with Borel subalgebra $\mathfrak{b} = (\mathfrak{h} + \mathcal{H}) \oplus n_+$. This $\mathfrak{b}$ is called the natural Borel subalgebra. Let $V$ be an irreducible $\mathbb{Z}$-graded $\mathcal{H}$-module with central charge $a \in \mathbb{C}$ and $\lambda \in \mathfrak{h}^*$ with $\lambda(c) = a$. Define a $\mathfrak{b}$-module structure on $V$ by the action $h v_k = (\lambda + k\delta)(h) v_k, n_+ v_k = 0$ for all $h \in \mathfrak{h}^\circ, v_k \in V_k, i \in \mathbb{Z}$. From this $\mathcal{H}$-module $V(\lambda)$ we can obtain a $\text{g}$-module by induction,

$$M_\text{b} (V(\lambda)) = \text{ind}_\text{b}^\text{g} (V(\lambda)) = \mathcal{U}(\text{g}) \otimes_{\mathcal{U}(\mathfrak{b})} V(\lambda).$$

This module is called imaginary Verma module. By definition, $M_\text{b} (V(\lambda))$ is $Q$-graded. Denote by $[\lambda]$ the image of $\lambda$ under the quotient map $\mathfrak{h}^* \to \mathfrak{h}^*/\mathbb{Z}\delta$. This image admits the classical Bruhat order and the relation $n_+ v_k = 0$ entails

$$M_\text{b} (V(\lambda)) = \bigoplus_{[\mu] \leq [\lambda]} V_\mu,$$

thus $\text{supp}(V) \subset \lambda - \text{span}_{\mathbb{Z}} \Pi^\circ + \mathbb{Z}\delta$.

The following facts hold for the above and an irreducible object $\tilde{V}$ in $\tilde{\mathcal{O}} (\Pi^\circ)$:

**Proposition 14.** (i) $M_\text{b} (V(\lambda))$ is $\mathcal{U}(n_-)$-free.

(ii) $M_\text{b} (V(\lambda))$ has a unique irreducible quotient $L_\text{b} (V(\lambda))$.

(iii) [Fu96] There exist $\lambda \in \mathfrak{h}^*$ and an irreducible $\mathcal{H}$-module $V$ such that $\tilde{V}$ is isomorphic to the unique irreducible quotient of $\text{ind}_\text{b}^\text{g} (V)$.

(iv) [ChP86] If $\tilde{V}$ has central charge zero, then $\tilde{V} \cong L_\text{b} (L_{r,\Lambda})$ for some $\lambda \in \mathfrak{h}^*, \lambda(c) = 0, \Lambda \in P_r$.

(v) [ChP86] If $\tilde{V}$ has central charge $a \in \mathbb{C}$ and $\dim \tilde{V}_\mu < \infty$ for at least one $\mu \in \text{supp}(\tilde{V})$, then $\tilde{V} \cong L_\text{b} (V(\lambda))$ for some $\lambda \in \mathfrak{h}^*, \lambda(c) = a$.

(vi) [ChP86] If $\tilde{V}$ is integrable then $\tilde{V}$ has central charge zero.
3.6. Non-standard or mixed modules  The modules presented in this section are called mixed type modules, because they are parabolically induced from irreducible modules for a Levi subalgebra that is a sum of a subalgebra of the Heisenberg subalgebra and (possibly several) affine Lie subalgebras. Recall $n_V^c = g_{\Lambda^c_{\phi}, \text{add}_{\Lambda^c_{\phi}}(X) + \mathbb{Z}}$ for $X \subset \Gamma'$.

Let $V$ be a $\mathbb{Z}$-graded $\mathcal{H}(\hat{S})$-module with central charge $a \in \mathbb{C}$ and $\lambda \in h^*$ with $\lambda(c) = a$. Define a $b_S$-module structure on $V$ by the action $h v_i = (\lambda + i\delta)(h)v_i$, $n_S v_i = 0$ for all $h \in h$, $v_i \in V$, $i \in \mathbb{Z}$.

Now we are able to state the central theorem for parabolic induction, which points out the fact that the parabolic induction functor “produces” irreducible representations for every parabolic subalgebra $p$ of $\mathfrak{g}$ and for every irreducible module of the Levi component of $p$. Therefore we have to see, that inducing from irreducible module of a subalgebra of type (II) (non-standard),

$$ p = \bigoplus_i \tilde{\mathfrak{g}}(S_i) \oplus \mathcal{H}(\hat{S}) \oplus n_S + \mathfrak{h} $$

with disconnected components $S_i$ with $\bigcup_i S_i = S$, is well behaved.

**Theorem 15.** Let $p = \bigoplus_i \tilde{\mathfrak{g}}(S_i) \oplus \mathcal{H}(\hat{S}) \oplus n_S + \mathfrak{h}$. If $V$ is an irreducible weight $p$-module such that $n_S$ acts zero, then the induction

$$ M_p = \text{ind}_p^g(V) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(p)} V $$

admits a unique irreducible quotient, $L_p$.

The proof is standard.

3.7. Classification problem for irreducible non-dense weight $\mathfrak{g}$-modules  We have shown already, that when inducing from the different parabolics, this yields different families of induced modules that have the same nice property of admitting unique irreducible quotients. The natural question then would be to ask in how far induction exhausts all irreducible modules. The main conjecture states that every non-dense module is induced. Our main theorem is

**Theorem 16.** Let $\mathfrak{g}$ be $A_1^{(1)}$, $A_2^{(2)}$, $A_2^{(1)}$, $A_1^{(1)}$ or $A_4^{(1)}$ and $V$ be an irreducible non-dense $\mathfrak{g}$-module, then there exists a vector $v \in V$ that is primitive with respect to the nilpotent part $\mathcal{N}$ of a parabolic subalgebra $p(P) = \mathcal{L} \oplus \mathcal{N}$.

If the affine root system has rank 2 and the module has only finite-dimensional weight subspaces, we are able to give a precise classification statement, because the non-trivial Levi subalgebras can only take a shape of a simple Lie algebra or a Heisenberg algebra (both of rank one). Whereas in general the Levi subalgebra itself could be a sum of affine Lie algebras, whose cuspidal modules still are not classified. Also nothing is known about the dimension of their weight spaces – although the latter we believe to be only infinite-dimensional.

**Theorem 17.** [Fu96, B09] If $\mathfrak{g}$ has rank 2, i.e. $\mathfrak{g} = A_1^{(1)}$ or $\mathfrak{g} = A_2^{(2)}$, and $V$ is an irreducible non-dense $\mathfrak{g}$-module with at least one finite-dimensional weight subspace, then $V$ is equivalent to one module out of the following pairwise non-equivalent classes:

$$ L_\alpha^\pm(\lambda, \gamma) \text{, if } \lambda(c) = 0, \quad L_\Phi(\lambda, \Lambda_r) \text{, if } \lambda(c) = a, \quad L_\delta( M^\pm(\lambda)) $$

for $\alpha \in \Delta^e$, $\lambda \in h^*$, $\gamma, a \in \mathbb{C}$, $r \in \mathbb{N}_0$, $\Lambda \in P_0$, where $\mathfrak{b} = h + \bigoplus_{\Phi \in (\mathfrak{g}^e + \mathbb{Z}) \setminus \mathfrak{a}} \mathfrak{g}_{\alpha}$ and $M^\pm(\lambda)$ the irreducible $\mathcal{H}(\tilde{\mathfrak{g}})$-module defined in 3.2.

If moreover $V$ has only finite-dimensional weight subspace and the central charge $\lambda(c) = a \neq 0$, then $V \cong L_\alpha^\pm(\lambda, \gamma)$ for some $\alpha \in \Delta^e$, $\lambda \in h^*$, $\gamma \in \mathbb{C}$.

Beyond the conjectures and Theorem [16], the following classification problems for irreducible weight $\mathfrak{g}$-modules are open problems:

(i) Modules of type $L_\delta(\lambda, V)$ where $V$ is a graded irreducible $\mathcal{H}$-module of non-zero central charge with all infinite-dimensional components. Here, recent progress has been made in [FK09] and [BBF13].

(ii) Dense $\mathfrak{g}$-modules of central charge zero.

(iii) Dense $\mathfrak{g}$-modules of non-zero central charge with an infinite-dimensional weight subspace.
4. Quasicone Arithmetics

4.1. Pre-prosolvable subalgebras Recall that a Lie algebra \(s\) is called solvable if the derived series yields \(\{0\}\) after finitely many steps. Fix a positive integer \(N\). The Lie algebra

\[
T_N g = g \otimes k[t] / t^{N+1} k[t]
\]

is called truncated current Lie algebra \([W11, Ta71]\). It inherits the triangular decomposition from \(g\).

Definition 20. A Lie subalgebra \(s \subset g\) is called

(P) perfect, if \([s,s] = s\),

(HA) hypoabelian, if its perfect radical (or perfect core), i.e. its largest perfect subalgebra, is trivial,

(PS) pre-prosolvable, if the completion of \(s/kc\) is isomorphic to the projective limit of an inverse system of solvable Lie algebras.

Note that the inverse limit \(\varprojlim g \otimes k[t] / t^{N+1} k[t] = g \otimes k[[t]]\) with respect to an appropriate topology, for instance the product topology on \(g^N\).

Construct a pre-prosolvable subalgebra \(s\) as follows: Let \(g^o\) be locally finite with Cartan algebra \(h^o\) and consider \(\{T_N g^o | N \in \mathbb{N}_0\}\) as a directed family of solvable Lie algebras with the obvious epimorphisms of Lie algebras \(\pi_{n,m}: T_m g^o \rightarrow T_n g^o, n \geq m\) and the limit \(\tau = g^o \otimes k[[t]] = \varprojlim g \otimes k[t] / t^{N+1} k[t]\). Now, \(s\) is the algebra of polynomials in \(\tau\).

Now, if the quotient of a locally affine Lie algebra \(g\) by its center admits a non-trivial Lie algebra homomorphism \(\phi: g/kc \rightarrow T_N g^o\), the image of a pre-prosolvable subalgebra \(s \subset g\) under the map \(\phi\) is solvable for every \(N \in \mathbb{Z}_+\).

Proposition 19. The pre-prosolvable subalgebra \(s\) is hypoabelian.

Proof. Assume \(g^o\) to be locally finite of arbitrary choice. Now the assumption is that the derived series becomes constant \(\phi(s)^{(m)} = \{0\}\) for some \(m \in \mathbb{Z}_+\). We want to show that the perfect radical of \(s\) is trivial.

Assume on the contrary, the perfect radical of \(s\) is non-trivial. Consequently the derived series becomes constant \(s^{(i)} = \tau \neq \{0\}\) for all \(i\) large enough. This is equivalent to saying \(s^{(i)}\) lies in the kernel of \(\phi\) for all \(i \geq m\). Since \(\tau\) is perfect itself, it must contain a subalgebra isomorphic to \(s l_2\). If \(\tau\) lies in the kernel of \(\phi\) for every solvable radical, then \(g^o\) must not contain a subalgebra isomorphic to \(s l_2\), which is a contradiction to \(g^o\) being locally finite of arbitrary choice.

For \(x \in H' = h^o \otimes k[t] \cap \{[t^a, s_a] | \alpha \in \Delta\}, (s_a = g_a \cap s)\), let \(n(x)\) be the greatest number such that \(t^{-n(x)} x\) is a non-zero element in \(h^o \otimes k[t] \cap s\). The number \(n = \max_{x \in h'} n(x)\) will be called the gap of \(H'\).

Proposition 20. (i) A conic subalgebra is a pre-prosolvable subalgebra that contains \(h^o \otimes t k[t]\).

(ii) The conic subalgebra \(s\) is a quasicone subalgebra if it has trivial intersection with \(h\) and \(\phi(\tau) = \{0\}\)

Remark 21. (i) The subalgebra of the locally finite Lie algebra of type \(A\), given by \(s = c_1 g \{e_{\alpha_1}, e_{\alpha_2}, \ldots\}\) is also hypoabelian, yet not pre-prosolvable according to our definition. With \(g^o = A_{\alpha}\) and the homomorphism \(\phi\) sending \(x \mapsto x \otimes 1\), the image of \(\phi\) is still not solvable.

(ii) Apart from conic subalgebras there are other pre-prosolvable subalgebras that are not nilpotent. The \(A_{\alpha}^{(1)}\)-subalgebra \(c_1 g \{e_{\alpha}, e_{\alpha \pm 1}, e_{\beta}, \ldots\} \otimes t^2 k[t]\) has a non-finite derived series with \(d^{(N)} = \{0\}\). A more exhaustive study of affine Kac-Moody subalgebras can be found in \([FRT08]\).

4.2. The tropical matrix algebra of quasicone subalgebras for \(A_{\alpha}^{(1)}\)

From now on, let \(g\) be \(A_{\alpha}^{(1)}\). We use the notations \(e_{\alpha}^{(k)} = h_k^\alpha \otimes k\) for \(k \in \mathbb{Z}\) and \(\alpha \in \Delta\), and \(e_{\alpha}^{(k_1, \ldots, k_n)} = \{e_{\alpha}^{(k_1)}, e_{\alpha}^{(k_2)}, \ldots | k \in I \subseteq \mathbb{Z}\}\).

Let \(\Pi^\alpha = \{\alpha_1, \ldots, \alpha_n\}\) be the standard root basis for \(A_{\alpha}^{(1)}\).
Notation. We denote a $g$-subspace

$$X = \text{span}_\mathbb{C} \left\{ \begin{array}{c} e_{a_1 + \cdots + a_i} \\ e_{-(a_1 + \cdots + a_i) + (A_{j+1} + s)} \end{array} \right\}_{i \leq j \in \{1, \ldots, n\}}$$

with $A_{ij}, \Omega_{ij} \subseteq \mathbb{Z}$ and the sum of sets being the Minkowski sum, i.e. $A + \Omega = \{a + \omega \mid a \in A, \omega \in \Omega\}$, by

$$\begin{pmatrix} * & A_{0,1} & A_{0,2} & \cdots & A_{0,n} \\ A_{1,0} & * & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n,0} & \cdots & A_{n,n-1} & * & \cdots & A_{n,n-1} \end{pmatrix} \begin{pmatrix} \mathbb{Z} & \Omega_{0,1} & \Omega_{0,2} & \cdots & \Omega_{0,n} \\ \Omega_{1,0} & \mathbb{Z} & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Omega_{n,0} & \cdots & \Omega_{n,n-1} & \mathbb{Z} \end{pmatrix}$$

Let $\bar{\alpha}_i$ be the standard basis of $\mathfrak{h}^\mathbb{N}$. Since $\mathfrak{c} (\alpha_i + \cdots + \alpha_j)^\mathbb{N} \otimes \mathbb{N}^l$, $i \leq j \in \{1, \ldots, n\}$, $l \in \mathbb{N}$, are vector spaces, the relations

$$\Omega_{i,j} = \Omega_{j,i} \quad \text{and} \quad \Omega_{i,j} \cap \Omega_{i,k} \subset \Omega_{i,k}$$

must hold for all $i, j, k \in \{0, \ldots, n\}$. Thus the omegas are determined by a selection of $n$ omegas, such that the other omegas can be generated by these relations, e.g. $\{\Omega_{i,j}\}_{i=1, \ldots, n}$.

For $X$ to be a subalgebra, the sets $A_{i,j}$ and $\Omega_{i,j}$, $(i \neq j \in \{0, \ldots, n\})$, have to satisfy the relations

$$A_{i,j} + A_{j,k} \subset \Omega_{i,k} \quad (i, j \in \{0, \ldots, n\}) \quad (4.3)$$

It follows that $\Omega_{i,j} \cap \Omega_{j,k} \subset \Omega_{i,k}$ for all $i \neq j \neq k \in \{0, \ldots, n\}$.

Definition 22. Consider a $g$-subspace as above. Denote a matrix $\mathcal{A} \in \mathcal{P}(\mathbb{Z})^{n \times n}$ as presentation matrix, if the matrix entries of

$$\mathcal{A} = \begin{pmatrix} A_0 & A_{01} & \cdots & A_{0n} \\ A_{10} & A_1 & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n0} & A_{n1} & \cdots & A_n \end{pmatrix}$$

satisfy $\Omega_{0,1} = A_0, \Omega_{n-1,n} = A_n, \Omega_{i-1,j} \cup \Omega_{i,i+1} = A_i, (0 < i < n)$ and $\Omega_{i,j} \cap \Omega_{j,k} = \Omega_{i,k}$ for all $i, j, k$ such that $i - 1 \leq j \leq k + 1$.

By the above considerations every presentation matrix can be associated with a subalgebra. For a presentation matrix to correspond to a subalgebra uniquely, we need more conditions to be satisfied.

Let $\mathcal{U}$ and $\cup$ be the matrix operations in $\mathcal{P}(\mathbb{Z})^{n \times n}$ inherited from the underlined set algebra $(\mathcal{P}(\mathbb{Z}), +, \cup)$. The additive identity matrix therein is given by

$$I = \begin{pmatrix} \{0\} & \emptyset & \cdots & \emptyset \\ \emptyset & \{0\} & \cdots & \emptyset \\ \vdots & \vdots & \ddots & \vdots \\ \emptyset & \emptyset & \cdots & \{0\} \end{pmatrix}$$

Proposition 23. If the presentation matrix $\mathcal{A} \in \mathcal{P}(\mathbb{Z})^{(n+1) \times (n+1)}$ satisfies the relation

$$\mathcal{A} \mathcal{U} (\mathcal{A} \cup I) = \mathcal{A}$$

then it identifies a subalgebra of $A_n^{(1)}$ uniquely.

Proof.
Let $\mathcal{A}$ be a presentation matrix. Relations $[4,3]$ are a sufficient condition for $\mathcal{X}$ to be a subalgebra, because they take the Lie algebra relations into account, in particular

\[
A_{i,j} + A_{j,i} \subseteq \Omega_{i,j} \iff [e_\alpha \otimes t^n, e_\alpha \otimes t^n] = \mathbb{C} \delta_{\alpha} \otimes t^{n_1 + n_2}
\]

\[
A_{i,j} + A_{l,k} \subseteq \Omega_{i,k} \iff [e_\alpha \otimes t^n, e_\beta \otimes t^n] = \mathbb{C} \delta_{\alpha + \beta} \otimes t^{n_1 + n_2}
\]

\[
A_{i,j} + \Omega_{k,l} \subseteq \Omega_{i,j} \iff [e_\alpha \otimes t^n, e_\beta \otimes t^n] = \mathbb{C} \delta_{\alpha \otimes t^n} \otimes t^{n_1 + n_2}
\]

for indices as above and corresponding roots $\alpha$ and $\beta$.

Denote $A_{ij} = A_i$. Then $\mathcal{A} \cup (\mathcal{A} \cup 1) = \mathcal{A}$ is equivalent to $A_{i,j} = A_{i,j} \cup \bigcup_{k=0}^{n} (A_{jk} + A_{k,i})$ for all $i,j \in \{1, \ldots, n\}$ or $A_{i,j} \supset \bigcup_{k=0}^{n} (A_{jk} + A_{k,i})$. The first relation from $[4,3]$ follows from

\[
A_{i,j} + A_{j,i} \subseteq \bigcup_k (A_{i,k} + A_{k,i}) \subseteq \bigcup_k \Omega_{i,k} = \Omega_{i,n-1,0}
\]

and $\Omega_{i-1,j} \cup \Omega_{i,j+1} = (\Omega_{i-1,j} \cap \Omega_{i,j}) \cup (\Omega_{i,j} \cap \Omega_{i,j+1}) = (\Omega_{i-1,j} \cup \Omega_{i,j+1}) \cap \Omega_{i,j} \subseteq \Omega_{i,j}$ for $i-1 \leq j \leq i+1$ (Definition $[22]$). The second is obvious. For the third, we observe

\[
A_{i,j} \supset (A_{i,j} + A_{j,i}) \cup (A_{i,j} + A_{i,j}) = A_{i,j} + (A_i \cup A_j)
\]

and herein the second term, $A_i \cup A_j = \bigcup_{m=i+1}^{j} \Omega_{m-1,0}$ since

\[
\Omega_{k,l} = \Omega_{k,r} \cap \Omega_{k,l} = \Omega_{k,r} \cap (\Omega_{k,r} \cap \Omega_{k,l}) = \Omega_{k,r} \cap \Omega_{k,l}
\]

and thus $\Omega_{k,l} \subseteq \Omega_{k,r}$ for $r$ such that $k-1 \leq r \leq l+1$ (Definition $[22]$). Repeating this step results in $\Omega_{k,r} \subseteq \Omega_{k,s}$ for $s$ such that $k-1 \leq s \leq r+1$. Consequently, $\Omega_{k,l} \subseteq \Omega_{k,s}$ for $k,l$ such that $r \leq l$. We observe $A_{i,j} \supset A_{i,j} + A_{j,i}$ and $A_{i,j} \supset A_{i,j} + A_{j,i} (i < j)$, for $i-1 \leq k \leq l \leq j+1$ as claimed.

For $\text{Sub}(\mathfrak{g})$ the category of subalgebras of $\mathfrak{g}$, this implies also that all coefficients that determine $\mathcal{X} \in \text{Sub}(A_n^{(1)})$ are uniquely determined.

Denote the category of presentation matrices $\mathcal{A} \in \mathcal{P}(\mathbb{Z})^{(n+1) \times (n+1)}$ that satisfy $\mathcal{A} \cup (\mathcal{A} \cup 1) = \mathcal{A}$ by $\mathcal{E}^*_n$ and the set of subalgebras of $\mathfrak{g} = A_n^{(1)}$ that contain $e$ and $d$ by $\text{Sub}(\mathfrak{g})$. With this, define a map

\[
\mathcal{E}^*_n \rightarrow \text{Sub}(\mathfrak{g}) : \mathcal{A} \mapsto \mathcal{X}
\]

according to Equation $[4,1]$. The preimage of an $A_n^{(1)}$-subalgebra $\mathcal{X}$ under this map will be called matrix presentation of $\mathcal{X}$.

**Notation 24.** We may also abuse the matrix notation to denote the Lie algebra closure

\[
\mathfrak{c}_B \left( \{ e_{\alpha_1 + \cdots + \alpha_j + A_{i,j}} \mid \alpha \in \Delta, \beta \in \mathfrak{c} \} \cup \{ e_{\alpha_1 + \cdots + \alpha_j + A_{i,j}} \mid \alpha \in \Delta, \beta \in \mathfrak{c} \} \right)
\]

if this is clear from context. In this case the denomination is not unique, e.g.

\[
\left\{ \begin{array}{ccc} * & * & * \\ * & \{ 1 \} & \{ 1 \} \end{array} \right\} = \left\{ \begin{array}{ccc} * & * & * \\ * & \mathbb{Z}_+ & \mathbb{Z}_+ \end{array} \right\}
\]

where the left hand side is not a presentation matrix.

**Definition 25.** If all of the sets $A_{i,j}$ and $A_i$ (i.e., $j = 1, \ldots, n$) are of type $\mathbb{Z}_{\geq k}$, then we use round parenthesis and write

\[
\left( \begin{array}{cccc} \mathbb{Z}_{\geq k_0} & \mathbb{Z}_{\geq k_1} & \mathbb{Z}_{\geq k_2} & \cdots \\ \mathbb{Z}_{\geq k_0} & \mathbb{Z}_{\geq k_1} & \mathbb{Z}_{\geq k_2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right) = \left( \begin{array}{cccc} k_0 & k_{01} & k_{02} & \cdots \\ k_{10} & k_1 & k_{12} & \cdots \\ k_{20} & k_{21} & k_2 & \cdots \end{array} \right)
\]

**Fact 26.** If $k_0 = k_1 = \cdots = k_n = 1$ then these subalgebras are quasicones, i.e. elements in $\mathfrak{C}$. 

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Denote \((\mathbb{Z}, \oplus, \odot) = (\mathbb{Z} \cup \{\pm \infty\}, \text{max}, +)\). The max-plus semi-ring. The identity \((\odot\)-multiplicative neutral element) in the corresponding max-plus matrix algebra is given by \(I = I_{\text{max}} = \text{diag}(0)\) (see [SpSt09] for an introduction to tropical mathematics).

**Lemma 27.** If \(B\) is a subset of \(g\) that contains \(h^0 \otimes t\mathbb{C}[t]\), but \(B \cap h^0 \otimes \mathbb{C}[t^{-1}] = \emptyset\), then its Lie-algebraic closure is given by

\[
\text{cl}_g(B) = B \odot' (B \odot' I).
\]

**Proof.** With the above, we have

\[
c_{i,j} = (B \odot' (B \odot' I))_{i,j} = \min_{k} (b_{i,k} + b_{k,j}, b_{i,j} + 0), \quad \text{for all } i, j = 0, \ldots, n,
\]

we need to show formula \([4.3]\), which is in this case simply

\[
c_{i,j} \leq c_{i,k} + c_{k,j} \text{ for all } k \in \{0, \ldots, n\} \setminus \{i, j\}, i \neq j. \quad (4.4)
\]

If \(b_{i,j} \leq b_{i,k} + b_{k,j}\) for all \(k\), then \(c_{i,j} \leq b_{i,j} + 0 \leq b_{i,k} + b_{k,j} \leq c_{i,k} + c_{k,j}\). If otherwise \(b_{i,j} > b_{i,k} + b_{k,j}\) for some \(k\), then \(c_{i,j} \leq b_{i,k} + b_{k,j}\) for that \(k\). Now \(b_{i,k} + b_{k,j} \leq \min_{k} (b_{i,k} + b_{k,j}, b_{i,j} + 0) = c_{i,j} \leq c_{i,k} + c_{k,j}\), as desired. \(\square\)

**Corollary 28.** The subalgebras containing \(h^0 \otimes \mathbb{C}[t]\) are exactly the idempotents in the tropical matrix rings \((\mathbb{C}, \odot')\) and \((\mathbb{C}, \odot_{\text{min}}')\).

**Remark 29.** (ii) Fix a basis \(\Pi^0 = \{\varphi_1, \ldots, \varphi_n\}\) and denote \(c_{2i} = c_{\varphi_i} (i = 1, \ldots, n)\). Denote by \(I_n = \{2^i + \cdots + 2^j \mid 0 \leq i \leq j \leq n\}\) the set of admissible indices for quasicone matrices for \(A_n^{(1)}\). The property of \(I_n\) to be a linearly ordered set of order \(q\) is required in the main algorithm at the end of the paper.

### 4.3. Defect of a quasicone

**Definition 30.** Define the defect function \(# : \mathbb{C} \rightarrow \mathbb{N}\) by

\[
#C = \sum_{\varphi \in \Delta^\circ} (c_{\varphi} + c_{-\varphi} - 2)_+.
\]

It aims to measure how much a quasicone fails to be a cone, and therefore the corresponding subalgebra fails to be a maximal parabolic.

**Remark 31.** A subalgebra \(C\) may only fail to be a quasicone if \(h^0 \otimes t\mathbb{C}[t]\) is not entirely contained or there exists a root \(\varphi \in \Delta^\circ\) such that either

\[
\max_{k \in \mathbb{Z}} \{e_{\varphi+k\delta} \notin C\} \text{ does not exist, or } \min_{k \in \mathbb{Z}} \{e_{\varphi+k\delta} \in C\} \text{ does not exist.}
\]

If \(\Pi^0 = \{\alpha_1, \ldots, \alpha_n\}\), then a change of basis \(\Pi \mapsto \Pi'\) of \(\Delta\) is accomplished by choosing linearly independent roots \(\alpha'_1, \ldots, \alpha'_n \in \text{add}_{\Delta^\circ} \Pi^0\) and extending it to \(\Pi'\) canonically. Then \((\Pi')^0 = \{\alpha'_1, \ldots, \alpha'_n\}\).

**Definition 32.** A quasicone matrix, respectively a quasicone subalgebra, is given in normal form or normal if \(c_{\varphi} = 1\) for all \(\varphi \in \Pi^0\) and \(c_{\kappa} + c_{-\kappa} \geq c_{\nu} + c_{-\nu}\) for all \(\kappa, \nu \in I_n\) with \(\kappa \prec \nu\). For the rest of the thesis we will generically refer to a quasicone matrix, a quasicone subalgebra or a quasicone of roots by quasicone if the structure is clear from the context.

**Lemma 33.** Any quasicone \(C\) is equivalent to a normal quasicone, i.e. there is an automorphism \(\varphi \in \text{Aut}(g)\) that induces a change of basis and thereby a map of quasicones \(\varphi : C \mapsto C'\) with \(C_{\kappa \kappa+1} = 1\) and \(c_{\kappa} + c_{-\kappa} \geq c_{\nu} + c_{-\nu}\) for all \(\kappa, \nu \in I_n\) with \(\kappa \prec \nu\).

**Proof.** Because any quasicone \(C \subset g\) is a subalgebra of \(g\), any automorphism of \(g\) induces an isomorphism of quasicones. The Weyl group \(W^0\) acts transitively and faithfully on the set of bases \(B\) for the root system \(\Delta^\circ\). First, we show that there is a \(w \in W^0\) such that \(w(C)\) satisfies \(c_{\kappa} + c_{-\kappa} \geq c_{\nu} + c_{-\nu}\) for all \(\kappa, \nu \in I_n\) with \(\kappa \prec \nu\). Select \(w_0 = \max_{\nu} (W^0)\) with respect to the Bruhat order \(\preceq\) on \(W^0\). The order of \([W^0]/(w_0)\) acts faithfully on the set

\[
\{\{\Pi, w_0 \Pi\} \mid \Pi \in B\} \subset B \times B,
\]

which is of order \(q!\), it acts by permutation on \(\Delta^\circ_+ (\Pi)\). This induces a canonical action on the ordered set \((c_{\kappa} + c_{-\kappa} \mid \kappa \in I_n)\), because \(I_n \cong \Delta^\circ_+ (\Pi)\). Eventually \([W^0]/(w_0)\) contains an element \(w\) such that \((c_{\kappa} + c_{-\kappa} \mid \kappa \in w(I_n))\) has the desired order.
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Now we show the existence of an isomorphism $\tau$ of quasicones that yields only ’ones’ on the super-diagonal in $\tau(C)$. Recall that $C = \langle \{ e_{\delta + \alpha} \mid \alpha \in \mathfrak{h}(I_n) \} \cup \mathfrak{h}^0 \otimes t\mathfrak{g}[t] \rangle$. The components for the desired map $\tau: (c_\kappa \mid \kappa \in \mathfrak{w}(I_n)) \mapsto (c_\kappa' \mid \kappa \in \mathfrak{w}(I_n))$ are ad hoc given by

$$(\tau | \kappa)(c_\kappa) = c_\kappa + \sum_{i=k_1}^{k_2} (1 - c_{2i})$$

if $\kappa = \sum_{i=k_1}^{k_2} 2i \in \mathfrak{w}(I_n), \ (0 \leq k_1 \leq k_2 < n).

\[ \square \]

4.4. Order relations on quasicones

Definition 34. Let’s define three partial orders on $C$ by

(i) $C \leq^{(i)} C'$ if $C_v = C_v'$ for all $v \in I_n$ and $C_\kappa < C_\kappa'$ for some $\kappa \in -I_n$ or
(ii) $C \leq^{(ii)} C'$ if $C_\kappa < C_\kappa'$ for some $\kappa \in -I_n$ and $C_v + C_{-v} = C_v' + C_{-v}'$ for all $v \in I_n$.
(iii) $\leq = \leq^{(i)} \cup \leq^{(ii)}$, i.e. $C \leq C'$ if $C \leq^{(i)} C'$ or $C \leq^{(ii)} C'$.

Remark 35. (i) The set of representatives of $C$ in normal form is equipped with the inclusion partial order forms a complete join-semilattice $(C, \subseteq, \vee)$ of subalgebras with infimum $\{0\}$ and the greatest upper bound $g$ itself. Thus, its order dual is a complete semilattice $(C, \sqcup, \wedge)$ (cf. [Na84]).

(ii) Since the union $\leq^{(i)} \cup \leq^{(ii)}$ is disjoint, there is a split exact sequence of posets

$$0 \to (C, \leq^{(ii)}) \to (C, \leq) \to (C, \leq^{(iii)}) \to 0.$$ (4.6)

Recall that the sequence $(C_v + C_{-v} \mid v \in I_n)$ is monotonically decreasing. Consider the set of monotonically decreasing positive integer sequences

$$\mathbb{N}_>^q = \{ k_v \mid k_v \geq k_{v'} \text{ if } v \leq v', v \in I_n \} \subset \mathbb{N}^q,$$

equipped with the natural partial order, which is equivalent to the lexical total order thereon. Define the map $\gamma: \mathbb{N}_>^q \to (C, \leq^{(i)})$ by

$$a \mapsto \gamma(a) = \bigwedge \{ C \in C \mid (C_v + C_{-v})_v = a \}.$$

In fact, $\gamma$ is well-defined. It is injective since two quasicones with a different defect for any of its sub-quasicones cannot be equal. Denote the vector $\gamma^{-1}(C) = (C_v + C_{-v} \mid v \in I_n)$ by gap of $C$,

$$\text{gap}(C) = \gamma^{-1}(C).$$ (4.7)

(iii) The gap of $C$ is closely related to the defect function, precisely $\#C = \sum_{v \in I_n} (\text{gap}(C)_v - 2)_+.$

(iv) The non-trivial representative that is the greatest lower bound in $C$ is the cone associated to the matrix

$$\bigwedge C \sim \begin{pmatrix}
1 & 1 & 2 & 3 & \cdots & n \\
-1 & 1 & 1 & 2 & \cdots & \\
-2 & -1 & 1 & 1 & \\
-3 & -2 & -1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 2 & \\
-n & \cdots & 1 & 1 & -1 & 1
\end{pmatrix} = \gamma(0^q).$$

Thus $(C, \leq)$ is a complete semilattice.

Definition 36. Select arbitrary elements $C^{\text{up}}, C^{\text{low}} \in C$. A quotient (complete) sublattice $C^{\text{up}}/C^{\text{low}} \subset C$ is defined as

$$C^{\text{up}}/C^{\text{low}} = \{ C \in C \mid C^{\text{low}} \leq C \leq C^{\text{up}} \}.$$
Consequently, every subset $C^{up}/C^{low} \subset C$ is a (complete) lattice. Now, consider the following quasicones

$$\tilde{C} \sim \begin{pmatrix} 1 & 1 & 2 & 3 & \cdots & n \\ c_{1,0} & 1 & 1 & 2 & \cdots & n-1 \\ c_{2,0} & c_{2,1} & 1 & 1 \\ c_{3,0} & c_{3,1} & \tilde{c}_{3,2} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 2 \\ \tilde{c}_{n,0} & \cdots & \tilde{c}_{n,n-1} & 1 & 1 \end{pmatrix}. \quad (4.8)$$

These are a lower bound with respect to $\leq (ii)$, because there are no normal quasicones $C$ such that $C_v > (\tilde{C})_v$ can be true for any $v > 0$. To define the matrices $\tilde{C}^{up}_d = \gamma((d+1)^{<q}), \tilde{d} \in \mathbb{Z}_{\geq -1}$, set

$$\tilde{c}_{-1} = \tilde{c}_{-2} = \cdots = \tilde{c}_{-2n-1} = \tilde{d}$$

$$\tilde{c}_{-3} = \tilde{c}_{-6} = \cdots = \tilde{c}_{-(2n+2n-1)} = \tilde{d} - 1$$

$$\vdots$$

$$\tilde{c}_{-(2n-1)} = \tilde{d} - n + 1$$

with the general rule \( \tilde{c}_v = \tilde{d} - \ell (v) + 1 \) for all $v < 0$. Then $\tilde{c}_v + \tilde{c}_{-v} = \tilde{d} + 1$ for all $v$ and all inequalities [4.4] are satisfied, so that this really represents a quasicone. Note that $\tilde{C}^{up}_d$ is the upper bound of the lattice.

### 4.5. Affine Weyl group actions and direct sums

The group of translations $T$ are the $\mathbb{Z}$-modules generated by rank two block matrices $t_j$ that act via common addition on $C$ (tropical Hadamard $\odot$-product). The Weyl group $W^0$ for the simple root system $\Delta^0$ is isomorphic to $S_n$, thus generated by transpositions which we will identify with the rank one matrices $s_{i,j}$:

$$t_i = \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ -1 & \cdots & -1 & 0 & \vdots \\ \vdots & \vdots & \ddots & -1 & \cdots & 0 \end{pmatrix}, \quad i = 1, \ldots, n, \quad s_{i,j} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & -1 \\ 0 & \cdots & -1 & 1 \end{pmatrix}.$$

$i, j = 0, \ldots, n, i \neq j$, where the $s_{i,j}$ with $j = i + 1$ form a minimal generating set. This group acts on $C$ via row-column permutations $s_{i,j} C s_{i,j}$. Thus the affine Weyl group $W = W^0 \ltimes T$ acts via $s_{i,j} (C + t_j) s_{i,j}$.

Because of matrix multiplication and the fact that the empty set $\emptyset$ serves as $\cup$-additive neutral element, we can build new pre-prosolvable presentation matrices, by taking direct sums.

For $A \in \mathcal{P}(\mathbb{Z})^{n \times n}$ and $B \in \mathcal{P}(\mathbb{Z})^{k \times k}$, define

$$A \boxplus B = \begin{cases} A & \text{if } 0 \times n \\ \emptyset_{0 \times n} & \text{if } 0 \times k \\ B & \text{if } k \times k \end{cases} \quad \text{and} \quad A \boxminus B = \begin{cases} A & \text{if } 0 \times k \\ \emptyset_{0 \times k} & \text{if } k \times k \end{cases}.$$

Let $\alpha_1, \ldots, \alpha_k \in \Pi^0$. Then denote $n_{\pm \{\alpha_1, \ldots, \alpha_k\}} := \sum_{i=1}^k \mathfrak{g}_{W^0}(\Pi^0 \setminus \alpha_i)(\pm \alpha_i) + \mathbb{Z} \delta$. The basic example would be

$$n_{\{\alpha_1, \ldots, \alpha_k\}} = \emptyset_{k \times k} \boxplus \emptyset_{(n-k) \times (n-k)} = \begin{pmatrix} \emptyset_{k \times k} & \emptyset_{n \times k} \\ \emptyset_{n \times k} & \emptyset_{(n-k) \times (n-k)} \end{pmatrix} = \begin{pmatrix} \emptyset_{k \times k} & \emptyset_{n \times k} \\ \emptyset_{n \times k} & \emptyset_{(n-k) \times (n-k)} \end{pmatrix}.$$

For a principal parabolic system $P = \Delta_+ \cup \Delta_-$, consider the corresponding ideal in the Lie algebra, i.e. $\mathfrak{g}_+ = \mathfrak{g}_{\Delta_+}$. Thanks to the matrix presentation [22] it is possible to describe those ideals by means of a block decomposition, each block representing a different type of ideal for a parabolic in a subalgebra $A_k^{(1)} \subset A_n^{(1)}$. 
5. Futorny’s Support Conjecture for $A_n^{(1)}$

5.1. Tropical Lie Actions on annihilating quasicones

From now on, $V$ is always an irreducible non-dense weight $A_n^{(1)}$-module. For all $n \geq 2$, we can reduce the problem of finding primitive elements when only finitely many quasicone subalgebras do not act trivially. We will give an explicite upper bound in the quasicone lattice $C$ for those cases to occur.

By Definition 2.2 a subalgebra $s \subseteq g$ is a quasicone subalgebra if it contains $H_+$ and has a trivial intersection with $h$. Therefore, any quasicone subalgebra of $g(S)$, for a partition $S \subseteq \Pi^0$ of a basis of $\Delta^0$, is equal to

$$C_S(k) = \left( \sum_{\alpha \in \Delta^0(S)} \sum_{n_\alpha \geq k_\alpha} g_\alpha + n_\alpha \delta \right) + H_+(S)$$

for integers $k = (k_\alpha \mid \alpha \in \Delta^0(S))$ with the property $k_\alpha + k_{-\alpha} \geq 1$ and $k_\alpha + k_\beta \geq k_{\alpha + \beta}$ whenever $\alpha, \beta$ and $\alpha + \beta$ are roots.

It is obvious that the integer vector $k$ defines the quasicone subalgebra $C_S(k)$ uniquely up to isomorphism. Denote the sets of quasicone subalgebras by $C = C_{\Pi^0}$ and $C_S$, respectively. The name quasicone subalgebra is justified, since in general $g_0 \cap C_S(k) = 0$ and the index set, where $U(h + C_S(k))$ is supported, is equal to the intersection of the root lattice with a cone. In other words, a quasicone subalgebra is a subalgebra over a blunt cone of roots.

Definition 37. Let $a \subseteq g$ be a subalgebra and $V$ a weight $g$-module.

(i) A vector $v \in V$ is called $a$-semiprimitive if $av = 0$.

(ii) $C_S(k)$ is a GVM-complete quasicone subalgebra or just complete, if $k_\alpha + k_{-\alpha} \in \{1, 2\}$ for all $\alpha \in \text{add}_\alpha(S)$.

(iii) If $a = C_{\Pi^0}$ is complete or $a = n_{\Pi^0}$ or $a = C_X \oplus n_{\Pi^0}$ for some $X \neq \Pi^0$ and $C_X$ is complete and $v \in V$ is $a$-semiprimitive, then $v$ is called $a$-primitive.

Fact 38. If $v$ is $a$-primitive then $v$ is primitive.

Proof. It is sufficient to show that for each subalgebra $a$, there exists a parabolic subalgebra $p = L \oplus \mathcal{N}$ according to Thm. 7 such that $a = \mathcal{N}$.

Let $A$ be such a subalgebra.

There exists a $g$-module $V$ and some $v \in V$, denote by $\text{Ann}(v) = \{g \in g \mid gv = 0\} \subseteq g$, and

$$\text{Ann}(v) = \{\varphi \in \Delta \mid \text{there is a } g \in \text{Ann}(v) \text{ that satisfies } g \in g_\varphi \} \subseteq \Delta.$$ As an immediately obvious matter of fact, $\text{Ann}(v)$ is closed under addition in $\Delta$.

Denote by $\mathcal{O} \subseteq 2^g$ the set of subalgebras of $g$. For the root operator $e_\varphi$ and $C = \text{Ann}(w), (\varphi \in \Delta^\text{re}, w \in V)$, define a map $e_\varphi : V \times \mathcal{O} \to V \times \mathcal{O}$ by $e_\varphi(w, C) = (e_\varphi w, C')$, such that $C' = \text{Ann}(e_\varphi w)$ for some given $w \in V$. Choose $\varphi \in \Delta$ such that $\text{Ann}(e_\varphi v) \in \mathcal{O}$, then this gives rise to an action

$$e_\varphi : Q \times \mathcal{O} \to Q \times \mathcal{O} : (\vartheta, C) \mapsto (\vartheta', C'),$$

where $\vartheta' = \vartheta + \varphi$ and

$$C' = \min_{V | v_\varphi = 0} \{\text{Ann}(e_\varphi v) \mid v \in V_{\mu + \vartheta} \text{ and } \text{Ann}(v) = C\},$$

$V$ going over all irreducible non-dense weight $g$-modules and the minimum refers to the partial order given by inclusion on the subalgebras. The lower bound in $\mathcal{O}$ is $\{0\}$. For this reason, Zorn’s lemma guarantees the existence of such a minimal subalgebra. The function is well-defined. In fact, the image of $C$ under $e_\varphi(\vartheta, -)$ is given by the Lie algebra closure

$$c_lg\langle \text{ade}_\varphi^{-1}(C) \cup \{g\} \rangle = c_lg\langle \{e_\varphi \mid [e_\varphi, e_\varphi] \in C\} \cup \{g\} \rangle,$$

where

$$g = \begin{cases} e_{-(\varphi + \vartheta)} & \text{if } \varphi + \vartheta \in \Delta^\text{re} \\ h_{\vartheta} \otimes t^{-k} & \text{if } \varphi + \vartheta = k\delta. \end{cases}$$
Lemma 39. Let \( w \in V \) and \( C = \text{Ann}(w) \) be a quasicone subalgebra. If \( \varphi \in \Delta^e \) and \( e_{\varphi + \delta} \in \text{Ann}(w) \), then \( \text{Ann}(e_{\varphi}w) \) contains \( h^\varphi \otimes tC[\ell] \).

**Proof.** Choose a basis \( \Pi \) for \( \Delta \) such that \( \varphi \in \Delta_+ (\Pi) \). Write \( e_{\varphi}^{(\varphi)} = (h_{\varphi} \otimes t^k) \). If \( e_{\varphi} \) does not act trivially, which would trivially meet the assertion, then
\[
e_{\varphi}^{(\varphi)} e_{\varphi}w = (\varphi (h_{\varphi}) e_{\varphi + \delta} + e_{\varphi} e_{\varphi}^{(\varphi)}) w = 0 \text{ for all } \varphi \in \Delta^o (\Pi) \text{ and } k > 0.
\]

\[\blacksquare\]

**Definition 40.** A strategy for \( v \) is a composition of operators \( s = e_{\varphi_1} \circ \cdots \circ e_{\varphi_m}, (m \in \mathbb{Z}_+) \), that satisfies

- \((S1)\) \( sv = e_{\varphi_m} \circ \cdots \circ e_{\varphi_1} v \neq 0 \) and
- \((S2)\) if \( C = \text{Ann}(v) \) is a quasicone, then \( \text{Ann}(sv) \) is also a quasicone.
- \((S3)\) \( \varphi_i + \cdots + \varphi_1 \in \Delta \) for \( i \in \{1, \ldots, m\} \).

Denote the set of strategies by \( \mathcal{S} \). The strategy is said to succeed (or to be successful) on the quasicone \( C = \text{Ann}(v) \) if and only if \( \#(\text{Ann}(sv)) < \#C \). A strategy is called circular if \( \varphi_1 + \cdots + \varphi_n \in \mathbb{Z}\delta \). We may say strategy for \( C \) assuming implicitly the existence of a \( v \) with the properties given above. The length of the strategy \( \ell(s) \) is the integer \( n \).

The length function and the function
\[
s : Q \times C \rightarrow Q \times C : (\vartheta, C) \mapsto s(\vartheta, C) = e_{\varphi_1} \circ \cdots \circ e_{\varphi_m} (\vartheta, C)
\]
are well-defined. We use the arrow ‘\( \triangleright \)’ and index notation to indicate this transformation as \( C_0 \triangleright s \rightarrow C_\varphi \) and omit the \( \vartheta \)-subscript if it is clear from the context.

**Conjecture 41.** There is a finite set of strategies \( S \subset \mathcal{S} \) such that the number of normal quasicones where no strategy succeeds is zero,
\[
\bigcap_{C \in \mathcal{C}} \{ \#\text{Ann}(sv) \geq \#C \text{ for all } s \in S \} = \emptyset.
\]

Assume \( C \) is given in normal form. Because of inequalities \((4.4)\), it follows that \( c_v \leq \ell(v) \). Given \( v \in \text{Ann}(C) \), there is a \( k \in \mathbb{N} \) such that
\[
s_k = e_{-\alpha_1 + \delta} \circ e_{\alpha_1}
\]
is a circular strategy for \( v \) because of Lemma [39]. It is of minimal length and called a shortest strategy. With the following lemma we establish the fact that \( k \) is determined by \( e \), and that only for a finite number of annihilating quasicones we cannot find a \( k \) such that the shortest strategy \( s_k \) succeeds.

**Lemma 42.** For \( e \in \{1, \ldots, n\} \), let \( v \in \text{Ann}(C) \) be an arbitrary vector, \( \text{Ann}(v) = \{0\} \) and the annihilating set of \( v \) be a quasicone \( \text{Ann}(v) = C \) in normal form. Then there exists a number \( k \in \{1, \ldots, n - 1\} \) such that the set \( \text{Ann}(e_{-\alpha_1 + \delta} \circ e_{\alpha_1} v) \) non-trivially contains a quasicone \( C' \) satisfying \( \#C' - \#C < 0 \), if the number \( c_{n-1} = c_{1,0} \) is greater than \( (n + 1) \cdot \varepsilon \).

**Proof.** The transforms are summarized schematically as
\[
\begin{pmatrix}
1 & 1 & c_{0,2} & c_{0,3} & \cdots & c_{0,n} \\
1 & 1 & c_{1,3} & c_{1,2} & \cdots & c_{1,m} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
c_{n,0} & c_{n,1} & & & & \vdots \\
\end{pmatrix}_{e_{-\alpha_1 + \delta}}
\begin{pmatrix}
1 & 1 & c_{0,2} & c_{0,3} & \cdots & c_{0,n} \\
0 & 1 & c_{1,2} & c_{1,3} & \cdots & c_{1,m} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & c_{n,1} & & & \vdots \\
\end{pmatrix}_{\alpha_{-\alpha_1 + \delta}}
\begin{pmatrix}
1 & 1 & c_{0,2}'' & c_{0,3}'' & \cdots & c_{0,n}'' \\
1 & 1 & c_{1,2}'' & c_{1,3}'' & \cdots & c_{1,m}'' \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
1 & 1 & c_{n,1}'' & & & \vdots \\
\end{pmatrix}_{-\delta}
\]
where \(\widehat{c}_{1,0} = \min(\varepsilon, c_{1,0})\), and the last step determines \(k = \widehat{c}_{1,0} - 1 = \min(\varepsilon - 1, c_{1,0} - 1)\). Without loss of generality, there are no additional trivial actions in the second step. For all \(i = 1, \ldots, n\),

\[
c'_{i,j} = \begin{cases} 
\min(\widehat{c}_{1,0} + c_{0,j}, c_{0,i}) & \text{if } c_{1,j} \leq c_{0,j} \\
\min(\widehat{c}_{1,0} + c_{0,j}, c_{1,j}) & \text{if } c_{1,j} > c_{0,j}
\end{cases}
\]

\[
= \min(\widehat{c}_{1,0} + c_{0,i}, \max(c_{0,i}, c_{1,j}))
\]

\[
c'_{i,0} = \begin{cases} 
\min(\widehat{c}_{1,0} + c_{1,i}, c_{1,1}) & \text{if } c_{1,0} \leq c_{1,1} \\
\min(\widehat{c}_{1,0} + c_{1,1}, c_{1,0}) & \text{if } c_{1,0} > c_{1,1}
\end{cases}
\]

\[
= \min(\widehat{c}_{1,0} + c_{1,1}, \max(c_{1,1}, c_{1,0}))
\]

\[
c''_{0,i} = \begin{cases} 
\min(1 + c'_{i,j}, c_{0,i}) & \text{if } c_{0,i} \geq c_{1,i} - k \\
\min(1 + c'_{i,j}, c'_{i,j} - k) & \text{if } c_{0,i} < c_{1,i} - k
\end{cases}
\]

\[
= \min(1 + c'_{i,j}, \max(c_{0,i}, c'_{i,j} - k))
\]

\[
c''_{i,1} = \begin{cases} 
\min(1 + c_{i,0}, c_{i,1}) & \text{if } c_{i,1} \geq c'_{i,0} - k \\
\min(1 + c_{i,0}, c''_{i,0} - k) & \text{if } c_{i,1} < c'_{i,0} - k
\end{cases}
\]

\[
= \min(1 + c_{i,0}, \max(c_{i,1}, c''_{i,0} - k)).
\]

First, \(c_{1,0} - \widehat{c}_{1,0} = -\min(\varepsilon, c_{1,0}) + c_{1,0} = \max(c_{1,0} - \varepsilon, 0)\). Further \(c'_{i,j} = \max(c_{0,i}, c_{1,j}), c'_{i,0} = \max(c_{1,1}, c_{1,0})\) and

\[
c''_{i,j} = \min(1 + \max(c_{0,i}, c_{1,j}), \max(c_{0,i}, \max(c_{0,i}, c_{1,j}) - k))
\]

\[
= \min(\max(c_{0,i} + k, c_{1,j} + k) - (k - 1), \max(c_{0,i} + k, c_{1,j}) - k)
\]

\[
= \begin{cases} 
c_{0,i} & \text{if } c_{0,i} + k > c_{1,j} \\
c_{1,j} - k & \text{else}
\end{cases}
\]

\[
= \max(c_{0,i}, c_{1,j} - k)
\]

\[
c''_{i,1} = \min(1 + c_{i,0}, \max(c_{i,1}, \max(c_{i,1}, c_{i,0}) - k))
\]

\[
= \begin{cases} 
c_{i,0} - k & \text{if } c_{i,0} - k > c_{i,1} \\
\min(1 + c_{i,0}, c_{i,1}) & \text{if } c_{i,0} - k \leq c_{i,1}
\end{cases}
\]

\[
= \max(c_{i,0} - k, \min(1 + c_{i,0}, c_{i,1})).
\]

Now in each expression, \((i = 1, \ldots, n)\),

\[
(c_{0,i} + c'_{i,j}) - (c_{0,i} + c_{1,i})
\]

\[
= \max(-c_{0,i}, -c_{1,i}) + \max(-c_{1,i}, -c_{0,i} - k)
\]

\[
= -\min(2c_{1,i}, c_{1,i} + c_{0,i}, 2c_{0,i} + k)
\]

\[
\leq -\min(2c_{1,i}, 2c_{1,i} - 1, 2c_{1,i} + k - 2), \text{ because } c_{0,i} + 1 \geq c_{1,i}
\]

\[
= -2c_{1,i} + \min(1, 2 - k)
\]

and

\[
(c''_{i,0} + c'_{i,1}) - (c_{i,0} + c_{i,1})
\]

\[
= \max(-c_{1,i}, -c_{0,i}) + \max(-c_{i,1}, -c_{0,i} - k)
\]

\[
= -\min(2c_{i,1}, c_{i,1} + c_{i,0} + k, \max(2c_{i,1} - 1, c_{i,0} + c_{i,1}), \max(c_{i,1} + c_{i,0} - 1, 2c_{i,0})
\]

\[
\leq -\min(2(c_{i,0} - 1) + k, 2c_{i,0} - 1 + k,
\]

\[
\max(2(c_{i,0} - 1) - 1, 2c_{i,0} - 1), \max(2(c_{i,0} - 1), 2c_{i,0}))
\]
Consequently, the sum
\[ (c'_{0,i} + c'_{1,i}) - (c_{0,i} + c_{1,i}) + (c''_{0,i} + c'_{1,i}) - (c_{i,0} + c_{i,1}) \]
\[ = -2c_{1,i} + \min(1, 2 - k) - 2c_{i,0} + \max(2 - k, 1) \]
\[ \leq -2((c_{0,i} - c_{1,i}) + c_{i,0}) + \min(1, 2 - k) + \max(1, 2 - k) \]
(because \( c_{0,i} \leq c_{1,i} + c_{i,0} \))
\[ \leq -2(1 + c_{1,0} - c_{1,0}) + 3 - k \] (because \( C \) is normal)
\[ = 1 - k. \]

Therefore, the gap decreases by at least
\[ \# C - \#(s_{k}(C)) \geq \max(c_{1,0} - \varepsilon, 0) + n \cdot (1 - k) \]
\[ = \max(c_{1,0} - \varepsilon, 0) + n \cdot (1 - \min(\varepsilon, c_{1,0}) - 1) \]
\[ = c_{1,0} - \varepsilon - n \cdot \varepsilon \] if \( c_{1,0} \geq n. \)

So, the balance is greater than zero if \( c_{1} = c_{1,0} > (n + 1) \cdot \varepsilon, \) as claimed. \( \square \)

**Corollary 43.** The number of annihilating quasicones for which \( s_{k} \) does not succeed for any \( k \in \mathbb{N} \) is finite.

### 5.2. Monoid of strategies

Recall the exponential root notation, \( \alpha_{i} \mapsto 2^{i-1}, (i = 1, \ldots, n) \). Consider the strategy
\[ s = e_{-(2^{i-1} + k_{0})} e_{\alpha_{2} + k_{0} + \delta} \cdots e_{2^{2} + k_{0}} e_{2^{1} + k_{0}} \]
for \( v \in V_{\delta} \) with \( \text{Ann}(v) = C \) and \( k = (k_{0}, k_{1}, \ldots, k_{n-1}, k_{n}) \in \mathbb{Z}^{n+1} \) recursively defined by
\[ k_{0} = 0 \]
\[ k_{1} = (e_{2^{1} + k_{i-1}} \cdots e_{2^{1} + k_{1}} e_{2^{0}} (-\delta, C))_{i, i+1} - 1, \] for \( i = 1, \ldots, n - 1 \)
\[ k_{n} = (e_{2^{n-1} + k_{n-1}} \cdots e_{2^{1} + k_{1}} e_{2^{0}} (-\delta, C))_{n, 0} + r(k) \]
where \( r(k) \) may be \(-1\) or such that \( k_{1} + \cdots + k_{n} = 0 \) (this way it is granted for \( h^{\rho} \otimes t \) to annihilate \( sv \)).

This \( s \) is a circular strategy and we call it the **shortest long strategy**.

**Proposition 44.** A strategy for \( C \) can be uniquely identified with a sequence of roots \( \varphi_{0}, \ldots, \varphi_{m} \in \Delta^{\circ} \), \( m \geq 0 \).

**Proof.** Let the strategy \( s \) for \( v \in V_{\delta} \) with \( \text{Ann}(v) = C \) be given by
\[ s = e_{\varphi_{0} + k_{0}} \cdots e_{\varphi_{i} + k_{i} \delta} e_{\varphi_{i} + k_{i} \delta} \]
with \( \varphi_{0}, \ldots, \varphi_{m} \in \Delta^{\circ} \) and the \( k_{0}, \ldots, k_{n} \in \mathbb{Z} \) recursively defined by
\[ k_{0} = c_{\varphi_{0}} - 1 \]
\[ k_{i+1} = (e_{\varphi_{i} + k_{i} \delta} \cdots e_{\varphi_{i} + k_{i} \delta} \cdots e_{\varphi_{0} + k_{0} \delta} (\varphi, C))_{\varphi_{i}} - 1, \] for \( \ell = 0, \ldots, m - 1 \).

Ad hoc, \( e_{\delta} e_{\varphi_{i} + k_{i} \delta} \) acts trivially on \( e_{\varphi_{i} + k_{i} \delta} \cdots e_{\varphi_{i} + k_{i} \delta} e_{\varphi_{0} + k_{0} \delta} (\varphi, C) \) for all \( \ell = 0, \ldots, m \) and thus \( h \otimes t \mathcal{C}[t] \) continues to do so. \( \square \)

Define a finite set of strategies for \( g = A_{n}^{(1)} \) by iteratively taking the set of strategies for \( A_{n-1}^{(1)} \) and certain strategies that comprise the corresponding root operators for all basic roots \( \alpha_{1}, \ldots, \alpha_{n} \).

**Definition 45.** A **simple basic strategy** is defined as
\[ s_{r} = e_{-r} \cdots e_{-2} \cdots e_{-2} \cdots e_{-2} \cdots e_{-2} \cdots e_{-2} \cdots e_{-r}, \quad e_{-r} = e_{-r} \cdots e_{-r} \cdots e_{-r}, \]
where \( r = (r_{k}, \ldots, r_{1}) \) is a monotonously ordered root partition of \( \theta = \alpha_{1} + \cdots + \alpha_{n} \), i.e. \( \alpha_{r_{1}} + \cdots + \alpha_{r_{k}} = \theta, r_{1} < \cdots < r_{k} \) or \( r_{1} > \cdots > r_{k} \), and \( \alpha_{r_{1}}, \ldots, \alpha_{r_{k}} \in \Delta^{\circ}. \)
By choice, all simple basic strategies are circular. The element corresponding to the partition with the ‘just vertical’ Young tableau is $s_0 = e_{-1}^{-2n} \circ e_{-2} \circ \cdots \circ e_1$.

**Example 46.** The simple basic strategies for $A_2^{(1)}$ are

$$\{ e_{-1} \circ e_1, e_{-3} \circ e_1, e_{-2} \circ e_1, e_{-1} \circ e_2 \circ e_1 \}.$$ 

**Remark 47.** The number of simple basic strategies for $A_n$ is $F_n = \sum_{i=0}^{n-1} 2^{i+1} - 1$, which we leave to the reader as an easy exercise.

**Conjecture.** The strategy that is successful for any quasicone lies in the monoid generated set of simple basic strategies.

**Answer.** The conjecture is wrong. There are $A_1^{(1)}$-quasicones – listed in Section 5.5 –, for which a successful strategy cannot be obtained by concatenating simple basic strategies.

The root graph is the graph of roots $\varphi \in \Delta^\vee$, with the edges between each two roots $\varphi_1, \varphi_2$ which satisfy $\varphi_1 - \varphi_2 \in \Delta^\vee$. The centered root graph $\Delta^v$ is the root graph with a center added and connected to all nodes of the root graph, i.e. the cone graph. By [44], a strategy is uniquely identified by an oriented path on the centered root graph of $\mathfrak{g}$. A path may contain circles and $n$-cycles but no loops. The corresponding path monoid is generated by circle-free paths and circles. For $A_n$, there are $(2n - 1)^{\ell - 1}$ paths of length $\ell$.

### 5.3. General Approach

**Lemma 48.** Let $v \in V_{\mu - \delta}$, $V_{\mu} = \{ 0 \}$, and $\ Ann (v)$ contain a pre-prosolvable subalgebra

$$Ann(v) \supset \left\{ \begin{array}{cccc}
0 & A_{0,1} & A_{0,2} & \cdots & A_{0,n} \\
A_{1,0} & 0 & \cdots & \\
A_{2,0} & \ddots & \ddots & \cdots \\
\vdots & \ddots & Z_{\geq c + c'} & Z_{\geq c} \\
A_{n,0} & \cdots & Z_{\geq c'} & Z_{\geq c + c'}
\end{array} \right\}$$

with $\min(A_{i,j}) + \min(A_{j,i}) > 0$ for all $i \neq j \in \{ 0, \ldots, n \}$. Then there exists a $k \in \mathbb{Z}_+$ and a vector $w \in V_{\mu - k\delta}$ such that $Ann(w)$ contains $\mathfrak{h} \otimes \mathfrak{t} \mathbb{C} [t]$.

**Proof.** We aim to prove by induction on $n$. Provided the statement is proved for $A_{n-1}^{(1)}$, it is also true for the submatrix/-algebra with index set $\{ 1, \ldots, n - 1 \}$ in $A_{n}^{(1)}$ by extending the action of $A_{n-1}^{(1)}$ canonically.

Induction beginning is $A_1^{(1)}$, which is well known [Fu96]. Assume, $Ann (v)$ does not contain $\mathfrak{g}_n \otimes \mathfrak{t} \mathbb{C} [t]$, i.e. $e_{k\delta}^{(\alpha_k)}v \neq 0$ for some $k \in \{ 2, \ldots, c + c' - 1 \}$, choose the maximal of such $k$ and set $w = e_{k\delta}^{(\alpha_k)}v$. Then $w \in V_{\mu - (k - 1)\delta}$ and

$$e_{(1-k)\delta}^{(\alpha_k)} \cdots e_{(k-1)\delta}^{(\alpha_k)} \in Ann (w).$$

But then, running through all $\kappa, \eta \in \mathbb{N}$, the operators

$$\left[ \left( \text{ad} x_\delta^{(\alpha_k)} \circ \text{ad} n^{(\alpha_k)} \right) e_{(\kappa + \eta)(1-k)\delta}, e_{-\alpha_k + A_n - 1 \delta}, e_{-\alpha_k + A_n - 1 \delta} \right] = e_{(\kappa + \eta)(1-k) + A_n - 1 \delta}$$

comprise $e_{(k-1)\delta}^{(\alpha_k)}$ for all integers $l \in \mathbb{Z}$, since $1 - k \leq -1$. They all act trivially on $w$, as desired. \hfill $\Box$

**Proposition 49.** Let $V$ be a non-dense weight $g$-module, then there exists a basis $\Pi = \{ \alpha_1, \ldots, \alpha_n, \delta \} \subset \Delta$ and a vector $v \in V$ that is primitive with respect to a quasicone $C$ or to $\mathfrak{n}_S$ for some $S \subset \Pi^\vee$.

**Remark 50.** We enumerate according to the following paradigm: Write 1. if the respective operator $e_{\varphi}$ acts injectively on the corresponding weight subspace, and 2. if all of the operators $\{ e_{\varphi + m \delta} \mid m \in \mathbb{Z} \}$ act trivially. This will represent a binary tree. At the end, we have to check if the resulting binary tree is complete, meaning that any leaf is one of the subalgebras of the claimed type or such that an appropriate lemma asserts the existence of a primitive element of the claimed type.

**Proof.** We proceed by induction on $n$. The induction start, for $n = 1$, we have two cases,
\[
\begin{align*}
\text{Ann}(v_{\alpha}) & \supset \left\{ 0, \{k\} \right\}_{\alpha-k\delta} \\
\text{Ann}(v_{-\delta}) & \supset \left\{ \{1\}, 0 \right\} -\delta
\end{align*}
\]

| \(\epsilon_{-\alpha-m\delta}\) & \(\epsilon_{-\alpha-(k-1)\delta}\) & \(\epsilon_{a-\alpha(m+m')\delta}\) & \(\epsilon_{a-(m+1)\delta}\) \\
| \(\epsilon_{-\alpha-m\delta}\) & \(\epsilon_{-\alpha-(k-1)\delta}\) & \(\epsilon_{a-\alpha(m+m')\delta}\) & \(\epsilon_{a-(m+1)\delta}\) \\
--- & --- & --- & --- \\
1. & \(\epsilon_{-\alpha-m\delta}\) & \(\epsilon_{-\alpha-(k-1)\delta}\) & \(\epsilon_{-\alpha-(m+1)\delta}\)

and we can continue with the argument 1. on the left side of this table.

Now, Lemma \([48]\) applies, giving us a quasicone

\[
\left\{ 0, 0 \right\}_\alpha = n-\alpha
\]

This finishes the induction start.

By induction assumption, for \(g(\Pi \setminus \{\alpha_n\}, \delta) \cong A_{n-1}^{(1)}\) there is a root \(\varphi \in \Delta(\Pi \setminus \{\alpha_n\})\) and a vector in \(V^{(n-1)} \in V_{\mu-\varphi}\) that is annihilated by a subalgebra equivalent to a quasicone \(C\) (case I) or to \(n_{\alpha}\) for some \(S' \subset \Gamma \setminus \{\alpha_n\}\) (case II) (cf. \([8]\) and \([3, 3]\)). By the non-density assumption, there exists a \(\mu \in h^* (\Pi \setminus \{\alpha_n\})\) with \(\mu \notin \text{supp} (V^{(n-1)}), \) i.e. \(U(g)_{\mu} v^{(n-1)} = 0, \) if \(v^{(n-1)} \in V_{\mu-\varphi}\).

Now let \(V\) be an irreducible \(A_{1}^{(1)}\)-module. The natural \(g\)-module monomorphism \(t: V^{(n-1)} \rightarrow V\) gives us an element \(v \in V\) that contains as annihilator \(\mathcal{Ann}(t(v^{(n-1)}))\), first of all, a quasicone (case I). Without restrictions \(v\) lies in the weight subspace \(V_{\mu-\delta}\).

1. \(e_{\alpha+\mu\delta} v_{-\delta} \neq 0\) for some \(m \in \mathbb{Z}\)

\[
\left. \begin{array}{cccc}
Z_{\geq 2} & * & \ldots & * \\
\vdots & \ddots & \ddots & \vdots \\
Z_{\geq s-m} & \ldots & Z_{\geq s-m} & \{1-m\} \end{array} \right\}_{\alpha_n+(m-1)\delta}
\]

(The upper left \(n \times (n-1)\)-submatrix remains unchanged)

1. \(e_{-\alpha+k\delta} \circ e_{-\alpha+j\delta}\) acts non-trivially for some \(k < j < 1 - m\), without restrictions, \(j = -m\) and \(k = j-1\)

\[
\left. \begin{array}{cccc}
Z_{\geq 2} & * & \ldots & * \\
\vdots & \ddots & \ddots & \vdots \\
Z_{\geq s-m} & \ldots & Z_{\geq s-m} & \{2m+1\} \end{array} \right\}_{-\alpha_n-(2m+1)\delta}
\]

Now, Lemma \([48]\) leads to the goal.

2. (assuming \(e_{(\alpha_j, Z_{\mu})} v_{-\delta} = 0\)

\[
\left. \begin{array}{ccc}
Z & \cdots & Z \\
\cdots & \ddots & \cdots \\
\{1\} & \cdots & \{1\} \end{array} \right\}_{-\delta}
\]

This is a reduction to \(A_{n-1}^{(1)}, \) because \(n_{\alpha}^{(1)}\) annihilates \(v\).

1.2. (all \(e_{-\alpha, Z_{\mu}}\) act zero)
This is a reduction to $A_{n-1}^{(1)}$ because $n_{\{\alpha_n\}}$ annihilates $v$.

Before continuing with the main proof we will need an auxiliary lemma.

**Lemma 51.** Let $\varphi \in \Delta^e$ such that $\varphi + \alpha_{n-1}$ is not a root. If

$$\text{Ann}(v_{\varphi}) \supset \begin{cases} \ldots \quad \ldots \quad 0 \quad 0 \\
\ldots \quad \ldots \quad 0 \quad 0 \\
Z \quad Z \quad Z \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \end{cases} \varphi$$

and $e_{-\varphi + \delta} \notin \text{Ann}(v_{\varphi})$ for some $k \in \mathbb{Z}$, then

$$\text{Ann}(v_{\varphi}) \supset n_{\{\alpha_n\}} \text{ or } \text{Ann}(v_{\varphi}) \supset s_n n_{\{\alpha_n\}}.$$

**Proof.** Since $e_{-\varphi - \delta} v_{\varphi} \neq 0$ for $k \in \mathbb{Z}$, then

1. $\begin{cases} \ldots \quad \ldots \quad 0 \quad 0 \\
Z \quad Z \quad Z \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \end{cases} \varphi$ \quad \begin{cases} \ldots \quad \ldots \quad 0 \quad 0 \\
Z \quad Z \quad Z \quad \{k\} \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \end{cases} \varphi$

Since $\varphi + \alpha_{n-1}$ is not a root the subspaces $g_{-(\alpha_{i+\ldots+\alpha_{n-1}})}$, $(i \in \{1, \ldots, n-1\})$ commute with $e_{-\varphi - \delta}$ and therefore annihilate $e_{-\varphi - \delta} v$.

1.1. $\begin{cases} \ldots \quad \ldots \quad \{k\} \quad 0 \\
Z \quad Z \quad Z \quad \{k\} \quad 0 \\
Z \quad Z \quad Z \quad \{k+k'\} \quad \{k\} \end{cases} \alpha_{n-(k+k')} \delta$

and $n_{\{\alpha_{n-1}\}}$ annihilates $v$ as claimed.

1.2. $\begin{cases} \ldots \quad \ldots \quad \{k\} \quad 0 \\
Z \quad Z \quad Z \quad \{k\} \quad Z \\
0 \quad 0 \quad 0 \quad \{k\} \end{cases} \varphi$ \quad \begin{cases} \ldots \quad \ldots \quad \{k\} \quad 0 \\
Z \quad Z \quad Z \quad \{k\} \quad Z \\
0 \quad 0 \quad 0 \quad \{k\} \quad 0 \end{cases} \varphi$

and $\text{Ann}(v_{\varphi}) \supset s_n n_{\{\alpha_n\}}$ as claimed.

Now, let’s turn our attention to case II. We aim to show that $\text{Ann}(v)$ contains $n_{\{\pm \alpha_k\}}$ for some $k \in \{0, \ldots n\}$. \qed
Without restrictions, we assume that \( \text{Ann}(v) \) contains \( n_{\{\alpha_k\}} \setminus n_{\{\alpha_n\}} \), \( (k \in \{0, \ldots, n-1\}) \), but no other \( n_{\{\alpha_{k'}\}} \) with \( k' > k \).

This time we must assume that \( v \) lies in the weight subspace \( V_{\mu + \alpha_n} \) in order to obtain a consistent induction step. Without loss of generality, we can always assume the worst case that all unspecified actions are non-zero (\( \emptyset \)).

We start with the special case \( n - k = 1 \) (induction start):

\[
\text{Ann}(v) \supseteq \left\{ \begin{array}{cccc}
* & * & 0 & \cdots & 0 \\
* & * & & & \\
0 & & & & \\
\vdots & & & & \\
* & \cdots & * & 0 & \\
Z & \cdots & Z & 0 & \\
\vdots & \vdots & \vdots & \vdots & \\
Z & \cdots & Z & 0 & 0 & 0 \\
0 & \cdots & Z & 0 & 0 & 0 \\
\end{array} \right\}_{\alpha_{n-1}}
\]

1. Assuming at first the worst case, that all unspecified actions are non-zero. For some \( m \in \mathbb{Z} \),

\[
e^{-(\alpha_{n-1} + \alpha_n) + m \delta}
\]

Now, Lemma 51 with \( \varphi = -\alpha_n \) applies to attain a \( n_{\{\alpha_{k'}\}} \)-primitive vector.

2.1.

\[
\left\{ \begin{array}{cccc}
* & * & 0 & \cdots & 0 \\
* & * & & & \\
0 & & & & \\
\vdots & & & & \\
0 & \cdots & * & 0 & \\
Z & \cdots & Z & 0 & \\
\vdots & \vdots & \vdots & \vdots & \\
0 & \cdots & 0 & Z & 0 \\
0 & \cdots & 0 & Z & 0 & \\
\end{array} \right\}_{\alpha_{n-1}}
\]

2.1.1.
e^{-(\alpha_{n-2} + \alpha_{n-3} + \alpha_k) + \delta} 
\begin{pmatrix}
* & * & 0 & \cdots & 0 \\
* & * & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
Z & \cdots & Z & \cdots & 0 \\
0 & \cdots & 0 & Z & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix} 
\Rightarrow 
\begin{pmatrix}
\alpha_{n-1} \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix} 
\alpha_{n-1} + \alpha_k

\begin{pmatrix}
* & \cdots & 0 \\
\vdots & \ddots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & Z & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix} 
\Rightarrow 
\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix} 
\alpha_{n-1} + \alpha_k + \delta

and again, Lemma 51 with \( \varphi = -\alpha_{n-2} \) applies to attain a \( \alpha_{n-1}^{\square} \)-primitive vector.

2.2.

\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix}
\Rightarrow 
\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix}

2.1.2.

\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix}
\Rightarrow 
\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix}

2.1.2.1. Let \( k \in \{1, \ldots, n-3\} \) be the largest number such that \( e^{-(\alpha_k + \cdots + \alpha_n) + j\delta} \) does not act zero for some \( j \in \mathbb{Z} \).

\begin{pmatrix}
* & \cdots & 0 \\
\vdots & \ddots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & Z & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix} 
\Rightarrow 
\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix} 
\alpha_{n-1} + \alpha_k + \delta

and Lemma 51 with \( \varphi = -(\alpha_k + \cdots + \alpha_n) \) applies to attain a \( \alpha_{n-1}^{\square} \)-primitive vector.

2.1.2.2.

\begin{pmatrix}
* & \cdots & 0 \\
\vdots & \ddots & \cdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & Z & 0 \\
0 & \cdots & 0 & \cdots & 0 \\
\end{pmatrix} 
\Rightarrow 
\begin{pmatrix}
\alpha_{n-1} + \alpha_k + \delta \\
\vdots \\
0 \\
0 \\
0 \\
Z \\
0 \\
Z \\
0 \\
\end{pmatrix} 

We show that the hypothesis is also true for \( n-k > 1 \), i.e when starting with a zero action of \( \alpha_{n-1}^{\square} \setminus \alpha_{n-2}^{\square} \), we can show that, by applying root operators, we can attain either a \( \alpha_{n-1}^{\square} \setminus \alpha_{n-2}^{\square} \)-primitive element or a primitive element. This way, we reduce inductively to the induction start where \( n-k = 1 \),
just above. The starting annihilating subalgebra is

\[
\begin{array}{cccc}
* & 0 & k & \cdots & 0 \\
0 & 0 & & & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
Z & \cdots & Z & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
Z & \cdots & Z & 0 & \cdots \\
\end{array}
\]

\(e_{-(\alpha_k + \cdots + \alpha_n)} \cdot \delta\)

1.

\[
\begin{array}{cccc}
* & 0 & k & \cdots & 0 \\
0 & 0 & & & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
Z & \cdots & Z & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
\end{array}
\]

\(e_{(\alpha_{k+1} + \cdots + \alpha_n)} \cdot \delta\)

1.1.

\[
\begin{array}{cccc}
* & 0 & k & \cdots & 0 \\
0 & * & & & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & * & 0 \\
Z & \cdots & Z & * & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots \\
\end{array}
\]

\(\delta\)

1.1.1.

\[
\begin{array}{cccc}
* & 0 & k & \cdots & 0 \\
0 & 0 & & & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
Z & \cdots & Z & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
Z & \cdots & Z & 0 & \cdots \\
Z & \cdots & Z & 0 & \cdots \\
\end{array}
\]

\(e_{\alpha_n} \cdot \delta\)

contains \(n_{-\alpha_k}\) as claimed.

2.
2.1. Now choose the largest \( p \in \{1, \ldots, k - 1\} \), such that \( e_{-(\alpha_p + \cdots + \alpha_k) - m\delta} \) does not act trivially for some \( m \in \mathbb{Z} \).

\[
\begin{pmatrix}
* & 0 & k & \cdots & 0 \\
0 & 0 & \ddots & & \\
\vdots & & \ddots & & \\
0 & \cdots & 0 & 0 & 0 \\
Z & Z & 0 & \cdots & \cdots \\
0 & \cdots & 0 & Z & 0 & \cdots & 0 \\
\end{pmatrix}_{a_k}
\]

2.1.1.

Now by induction assumption the lower right matrix is either a quasicone, with what \( C_{n, q} \neq \emptyset \) and therefore \( C_{n, q} = \mathbb{Z} \) for \( q \in \{1, \ldots, k - 1\} \), as desired, or it contains a subalgebra equivalent to \( n^{\square}_{-\{\alpha_r\}} \subset A_{n-k}^{(1)} \) for \( r \in \{k, \ldots, n\} \) which completes to \( n^{\square}_{-\{\alpha_k\}} \setminus n^{\square}_{-\{\alpha_r\}} \subset A_n^{(1)} \), proving the induction hypothesis.

1.2.

\[
\begin{pmatrix}
* & 0 & k & \cdots & 0 \\
0 & 0 & \ddots & & \\
\vdots & & \ddots & & \\
0 & \cdots & 0 & 0 & 0 \\
Z & Z & 0 & \cdots & \cdots \\
0 & \cdots & 0 & Z & 0 & \cdots & 0 \\
\end{pmatrix}_r
\]

contains \( n^{\square}_{-\{\alpha_{r+1}\}} \setminus n^{\square}_{-\{\alpha_r\}} \) as claimed.

1.1.2.
also contains $n^{\square}_{-\{\alpha_1\}} \setminus n^{\square}_-(\alpha_n)$.

2.1.2.1.

Now we can apply $e^{(a_\lambda+\cdots+a_{\lambda-1}+\delta)}$ and be in the situation of 2.1.1, which was solved, or we are in a situation analog to 2.1.2, but with a $p'<p$. Thus after finitely many steps, we arrive at a matrix equivalent to $n^{\square}_{-\{\alpha_1\}}$.

This proves the induction hypothesis and therefore the proposition.

5.4. Main Algorithm The algorithmic part of the proof is structured in form of a search forest. Let $S$ be the set of simple basic strategies. Applying strategies to a set of quasicones, after $N$ iterations we obtain an $|S|$-ary forest with $n$ trees of depth $N$. The image of the set of critical normal quasicones after $N$ steps is a proper subset $S^N \{C_i\}_{i=1,...,n} \setminus \{C_i\}_{i=1,...,n} \subseteq \{C_i\}_{j=1,...,n}$ but might become stationary.
SN \{ C_i \}_{i=1,\ldots,n} = \{ C_b \}_{k=1,\ldots,n'}, (n' < n), at some point. Assume there is an element \( C_j \in S^N \{ C_i \}_{i=1,\ldots,n} \cap \{ C_i \}_{i=1,\ldots,n} \). Then \( S^N C_j \subset S^N \{ C_i \}_{i=1,\ldots,n} \) so that applying strategies to \( C_j \) is obsolete because there is nothing new to attain. The following algorithm takes account to these cases.

**Algorithm. Concatenate Strategies**

**input**
- \{C_1, \ldots, C_n\} list quasicones with unknown successful strategy
- \{s_1, \ldots, s_k\} set of simple basic strategies

**output**
- list of quasicones where no concatenation of strategy was successful

**tools**
- dictionary dict, i.e. a list of key-value pairs \( \{key : value\}\);
- functions \text{Apply\_strategy()} and \text{Weyl\_normal\_form()}
- method \text{successful()} that returns True if the gap was reduced or a GVM-complete quasicone was achieved

```
# initialization
dict ← {1 : [{1}], \ldots, 1 : [{n}]}
# initialized with lists only containing one tuple, a one-tuple (start index)
λ ← −δ  # start weight
list_at_start ← [C_1, \ldots, C_n]
list_of_successful ← []
strategy_list ← \{s_1, \ldots, s_k\}

def Step():
    foreach C_i in list_at_start:
        foreach s_j in strategy_list:
            step ← \text{Apply\_strategy}(\lambda, C, s)
            C' ← \text{Weyl\_normal\_form}(step, C)
            if step.successful() or (C' in list_of_successful):
                list_of_successful.append(C_i if dict[i] ∈ step.path_to_successful
                                            for i ∈ \{1, \ldots, n\})
                break
        foreach tree_index in dict[i]:  # add new index:
            k ← list index of C'
            dict[k] ← dict[k].append(tree_index. append(j))

# main routine
old_list_of_successful ← list_of_successful
while list_of_successful != list_at_start :
    Step()
    if old_list_of_successful == list_of_successful :
        print (list_at_start \ list_of_successful)
        break
    else: print ‘successful strategies for all quasicones found’
```
The implementation of the functions `Apply_strategy()` and `Weyl_normal_form()` and the method `successful()` is straightforward from the definitions in the paper.

For certain anti-matroidal structures, greedy algorithms are unfeasible. The anti-matroidal structure of a quasicone subalgebra lattice $C$ suggests that the problem of finding successful strategies is a constraint satisfaction problem (CSP) but with rather “non-holonomic” constraints on the phase space $C \times (\Delta \cup \{0\})$, which is called arc consistency or path consistency in this context [Wa95, pp. 121–137]. This class of problems seems to be at least as complex as problems that can only be solved by integer linear programming.

![Figure 5.1](image.png)

**Figure 5.1.** Illustration of the algorithm: For $C_1$ a successful solution has been found in the second step. Therefore all the quasicones $C_{1[1][2]}$ are added to the list of solved cases. For $C_3$ the iteration $C_{3,2,1}$ turns out to be in that list and thus all quasicones $C_{3[2][1]}$ can also be considered solved.

5.5. **The Kac-Moody algebras $A_2^{(1)}, A_3^{(1)}$ and $A_4^{(1)}** In Table 3, the computational results for $A_2^{(1)}, A_3^{(1)}$ and $A_4^{(1)}$ are summarized. For $A_3^{(1)}$, almost all quasicones admit defect reduction or even yield defect 0 complete quasicones when applying simple basic strategies (cf. Definition 45). Only 8 quasicones $C_1, \ldots, C_8$ (having defects 2 and 3) admit no defect reduction. But this set is not stabilized by defect-invariant strategies: There exists at least one simple basic strategy such that $\#C_i - \#sC_i = 0$, for each $i \in \{1, \ldots, 8\}$. For those we compare output $sC_i$ and input $C_i$ and find that there is no pairwise equivalence between $sC_i$ and $C_j$ for $i, j \in \{1, \ldots, 8\}$. Consequently, further concatenation of strategies will lead to a defect reduction or yield a complete quasicone. Thus we showed that there is a concatenatable strategy that admits defect reduction in these cases either.

*full code available on http://github.com/thoma5B/Strategies-for-support-quasicones-of-affine-Lie-algebras-of-type-A*
TABLE 3. Computational results for algorithmic solution of finding successful strategies for $A_2^{(1)}$, $A_3^{(1)}$ and $A_4^{(1)}$: number of quasicones with no successful strategy in the approach set. The shortest strategy was defined below Definition 40 and the shortest long strategy is according to Definition 5.2.

| $\mathfrak{g}$ | total no. of considered quasicones | number of unsolved quasicones after applying ... | ... the shortest strategy | ... the shortest long strategy | ... a set of simple basic strategies | ... concatenations of simple basic strategies |
|-----------------|-------------------------------------|-----------------------------------------------|--------------------------|-------------------------------|---------------------------------|---------------------------------|
| $A_2^{(1)}$     | 48                                  | 32                                            | 0                        |                               |                                 |                                 |
| $A_3^{(1)}$     | 669                                 | 242                                           | 38                       | 8                             | 0                               |                                 |
| $A_4^{(1)}$     | 23431                               | 2747                                          | 536                      | 65                            | 8                               |                                 |

Recall the decomposition of the set of strategies $\mathfrak{g} = \bigcup_{\psi \in \mathfrak{g}} \mathfrak{g}_\psi$. In the $A_4^{(1)}$, concatenation of simple basic strategies do no result in a successful strategy. The eight left-over cases are solved manually according to the following paradigm,

1. for every root $\phi \in \Delta^\circ$ and $k \in \mathbb{Z}$, the root step $\sigma(\phi, k) : \mathfrak{g}_\psi \to \mathfrak{g}$ sending $s \mapsto s \circ e_\phi + k \delta$, seen as a partition of the highest root, gives rise to a new partition, namely minus the index of the component $\mathfrak{g}_{\psi + \phi + k \delta}$, i.e. $-(\psi + \phi + k \delta)$, which is supposed to be a root

2. $(\phi, k)$ is chosen such that $\text{gap}(C)_{\psi + \phi} = \max_{\psi \in \mathfrak{g}} \text{gap}(C)_\psi$ and $k = C_\psi - 1$ (cf. 4.7)

$$\begin{pmatrix}
* & 1 & 1 & 0 & -1 \\
2 & * & 1 & 1 & 0 \\
1 & 2 & * & 1 & 0 \\
2 & 1 & 1 & * & 1 \\
2 & 2 & 2 & 1 & *
\end{pmatrix} \xrightarrow{e_3 \circ e_{-1}} \begin{pmatrix}
* & 0 & 1 & 0 & -1 \\
2 & * & 1 & 1 & 0 \\
1 & 0 & * & 1 & 0 \\
2 & 1 & 1 & * & 1 \\
2 & 2 & 2 & 1 & *
\end{pmatrix}$$

$$\begin{pmatrix}
* & 1 & 0 & 1 & 1 \\
2 & * & 1 & 1 & 2 \\
1 & 1 & 1 & * & 1 \\
1 & 0 & 0 & 1 & *
\end{pmatrix} \xrightarrow{e_{-3} \circ e_{1}} \begin{pmatrix}
* & 1 & 0 & 1 & 1 \\
2 & * & 1 & 1 & 2 \\
1 & 1 & 1 & * & 1 \\
1 & 0 & 0 & 1 & *
\end{pmatrix}$$

$$\begin{pmatrix}
* & 1 & 1 & 0 & 1 \\
2 & * & 1 & 1 & 1 \\
1 & 2 & * & 1 & 1 \\
2 & 1 & 1 & * & 1 \\
1 & 1 & 1 & 1 & *
\end{pmatrix} \xrightarrow{e_{12} \circ e_{4} \circ e_{3} \circ e_{2} \circ e_{1} \circ e_{-3} \circ e_{-2} \circ e_{1}} \begin{pmatrix}
* & 1 & 2 & 0 & 1 \\
0 & * & 0 & -2 & -2 \\
0 & 2 & * & -1 & 1 \\
2 & 4 & 3 & * & 3 \\
1 & 1 & 3 & 1 & *
\end{pmatrix}$$

$$\begin{pmatrix}
* & 1 & 0 & 0 & 1 \\
2 & * & 1 & 1 & 1 \\
2 & 2 & * & 1 & 1 \\
2 & 1 & 1 & * & 1 \\
1 & 1 & 1 & 1 & *
\end{pmatrix} \xrightarrow{e_{-2} \circ e_{-3} \circ e_{-2} \circ e_{1}} \begin{pmatrix}
* & 1 & 0 & 0 & 1 \\
0 & \mathbb{Z} & 1 & -2 & 1 \\
0 & 2 & * & -1 & 1 \\
2 & 1 & 1 & \mathbb{Z} & 1 \\
1 & 1 & 1 & 1 & *
\end{pmatrix} \cong \mathfrak{g}$$

$$\begin{pmatrix}
* & 1 & 1 & 1 & 0 \\
2 & * & 1 & 1 & 1 \\
2 & 1 & 2 & * & 1 \\
1 & 1 & 1 & * & 1 \\
2 & 1 & 1 & 1 & *
\end{pmatrix} \xrightarrow{e_{-1} \circ e_{-1}} \begin{pmatrix}
* & 0 & 1 & 1 & 0 \\
2 & * & 1 & 1 & 1 \\
1 & 0 & * & 1 & 1 \\
2 & 1 & 1 & * & 1 \\
2 & 1 & 1 & 1 & *
\end{pmatrix}$$
Classification of irreducible non-dense modules of $A$

New Irreducible Modules for Heisenberg and Affine Lie Algebras

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