EFFECTIVELY COMPUTING INTEGRAL POINTS ON THE MODULI OF SMOOTH QUARTIC CURVES

ARIYAN JAVANPEYKAR

ABSTRACT. We prove an effective version of the Shafarevich conjecture (as proven by Faltings) for smooth quartic curves in $\mathbb{P}^2$. To do so, we establish an effective version of Scholl’s finiteness result for smooth del Pezzo surfaces of degree at most four.

1. INTRODUCTION

We show that the set of integral points on the moduli of smooth quartic hypersurfaces in $\mathbb{P}^2$ is finite and effectively computable.

In [16] Faltings proved Shafarevich’s conjecture for smooth proper curves: for a number field $K$, a finite set of finite places $S$ of $K$, and an integer $g \geq 2$, the set of $K$-isomorphism classes of smooth proper genus $g$ curves over $\mathcal{O}_{K,S}$ is finite. In other words, Faltings established that the set of $\mathcal{O}_{K,S}$-points of the stack of smooth proper genus $g$ curves $\mathcal{M}_g$ is finite.

Faltings’s aforementioned finiteness result for smooth proper curves over $\mathcal{O}_{K,S}$ of fixed genus is not known to be effective. That is, there is currently no algorithm that, on input given a number field $K$ and a finite set of finite places $S$ of $K$, computes as output the finite set of $\mathcal{O}_{K,S}$-points of the stack $\mathcal{M}_g$.

An effective resolution of Shafarevich’s conjecture would have deep consequences in Diophantine geometry as, for example, an effective resolution of Shafarevich’s conjecture for smooth proper curves would imply an algorithmic version of the Mordell conjecture; see Section 6 for a discussion (cf. [36, 41, 42]).

In this paper we investigate the open substack $\mathcal{C}_{(4,1)}$ of smooth quartic curves in the stack of smooth proper genus 3 curves $\mathcal{M}_3$. More precisely, our main result says that the finite set of integral points on the open substack $\mathcal{C}_{(4,1)}$ of non-hyperelliptic curves of $\mathcal{M}_3$ can be effectively computed (see Theorem 5.1). A more down-to-earth version of our result reads as follows.

**Theorem 1.1.** Let $K$ be a number field and let $S$ be a finite set of finite places of $K$. Then the set of $\mathcal{O}_{K,S}$-isomorphism classes of smooth quartic hypersurfaces in $\mathbb{P}^2_{\mathcal{O}_{K,S}}$ is finite and effectively computable.

Note that the analogous statement for smooth cubic hypersurfaces in $\mathbb{P}^2_{\mathcal{O}_{K,S}}$ follows from effective versions of Shafarevich’s finiteness theorem for elliptic curves [12, 17, 38].

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Moreover, effective versions of Shafarevich’s conjecture have also been obtained for cyclic curves of prime degree \([13, 23, 43]\). The effectiveness of our finiteness result (Theorem 1.1) follows, as in the work of Fuchs-von Känel-Wüstholz [17], von Känel [43], and de Jong-Rémond [13] from the effective resolution of the S-unit equation in \(K\). The latter was achieved by Győry-Yu [19] (see also [15]) and relies on the theory of linear forms in logarithms [2, 3].

The geometric idea behind our proof of Theorem 1.1 is quite simple. Indeed, a smooth quartic curve in \(\mathbb{P}^2\) induces a smooth del Pezzo surface of degree two by taking a double covering of \(\mathbb{P}^2\) ramified precisely along the quartic. Moreover, the isomorphism class of the obtained smooth del Pezzo surface determines the isomorphism class of the corresponding quartic curve. This construction is a special case of what is sometimes called the cyclic covering trick. We show that it reduces Theorem 1.1 to a finiteness statement about del Pezzo surfaces.

In [37] Scholl proved the finiteness of all smooth del Pezzo surfaces over a number field \(K\) with good reduction outside a fixed set of finite places \(S\) of \(K\). In particular, the set of \(O_{K,S}\)-isomorphism classes of smooth del Pezzo surfaces over \(O_{K,S}\) of degree at most four is finite. To prove Theorem 1.1 we establish an effective version of the latter finiteness statement.

**Theorem 1.2.** Let \(K\) be a number field and let \(S\) be a finite set of finite places. Then the set of \(O_{K,S}\)-isomorphism classes of smooth del Pezzo surfaces over \(O_{K,S}\) of degree at most 4 is finite and effectively computable.

In Section 6 we discuss applications of Theorem 1.1 to an effective version of the Mordell conjecture. We emphasize that the results we obtain in Section 6 form a first step towards an effective version of the Mordell conjecture, as we prove an effective version of the Mordell conjecture for some class of complete curves, assuming there is an algorithm that, on input a number field \(K\) and a finite set of finite places \(S\) of \(K\), computes as output a finite set of finite places \(S'\) of \(K\) containing \(S\) such that all smooth quartic curves \(C\) in \(\mathbb{P}^2_K\) with a smooth proper model over \(O_{K,S}\) have non-hyperelliptic reduction outside \(S'\) (see Corollary 6.4 for a precise statement).

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**Conventions.** For \(S\) a scheme, a curve over \(S\) is a flat proper finitely presented morphism \(X \to S\) whose geometric fibres are connected schemes of dimension one. We denote by \(|S|\) the cardinality of an arbitrary finite set \(S\). By \(\log\) we mean the principal value of the natural logarithm. We define the product taken over the empty set as 1.
2. Del Pezzo surfaces

Let $k$ be an algebraically closed field. A smooth projective connected surface $X$ over $k$ is a (smooth) del Pezzo surface if $\omega_{X/k}^\vee$ is ample. The degree $d(X)$ of a del Pezzo surface $X$ over $k$ is defined to be the self-intersection $(\omega_{X/k},\omega_{X,k})$ of the canonical line bundle $\omega_{X/k}$. Note that $1 \leq d(X) \leq 9$. The automorphism group of a smooth del Pezzo surface $X$ over $k$ is finite if and only if $d(X) \leq 5$. Recall that, if $d(X) \leq 7$, a line on $X$ is defined to be a $(-1)$-curve. In general, a curve $L$ on $X$ is a line if the following holds.

1. If $d(X) = 9$, then $L \cdot (−K_X) = 3$.
2. If $X \cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, then $L \cdot (−K_X) = 2$.
3. If $d(X) \neq 9$ and $X \not\cong \mathbb{P}^1_k \times_k \mathbb{P}^1_k$, then $L \cdot (−K_X) = 1$.

Let $S$ be a scheme. Recall that a smooth proper morphism of schemes $X \to S$ is a (smooth) del Pezzo surface (over $S$) if its geometric fibres are del Pezzo surfaces. Note that $X \to S$ is a smooth del Pezzo surface if and only if it is a Fano scheme of relative dimension two [22, §2]. If $S$ is a connected scheme, then the degree of a del Pezzo surface $X \to S$ is constant in the fibres (this follows from [22, Lem. 3.3]).

A smooth del Pezzo surface over a field $k$ is split, or standard, if all lines are defined over $k$ (see [37]).

Let $X \to S$ be a del Pezzo surface over $S$. We define $\mathcal{L}_{X/S} = \mathcal{L}_{−1,0}$ to be the Hilbert scheme of lines in $X$ [37, §3.4]. Note that $\mathcal{L}_{X/S} \to S$ is a morphism of schemes, and that $\mathcal{L}_{X/S}$ parametrizes the lines (i.e. exceptional curves) on $X$ over $S$.

Lemma 2.1. Let $A$ be a principal ideal domain and let $S = \text{Spec}(A)$. Let $X \to S$ be a smooth del Pezzo surface of degree $d \leq 7$ over $S$. If $\mathcal{L}_{X/S} \to S$ is constant (i.e., $\mathcal{L}_{X/S}$ is a disjoint union of copies of $S$), then there exist $9 − d$ points $x_1,\ldots,x_{9−d}$ in $\mathbb{P}^2(S)$ and an $S$-isomorphism of schemes from $X$ to the blow-up of $\mathbb{P}^2_S$ in $x_1,\ldots,x_{9−d}$.

Proof. This follows from [37] Prop. 3.7].

Let $S$ be a Dedekind scheme (i.e., an integral noetherian normal one-dimensional scheme) with function field $K$. Let $X$ be a smooth del Pezzo surface over $K$. We say that $X$ has good reduction over $S$ if there exist a smooth del Pezzo surface $X \to S$ over $S$ and an isomorphism $X_K \cong X$ over $K$.

3. The unit equation

Let $K$ be a number field. For $a$ in $K$, let $h(a)$ be the usual absolute logarithmic Weil height of $a$, as defined in [9, 1.6.1]. Write $d_K = [K : \mathbb{Q}]$ for the degree of $K$ over $\mathbb{Q}$. Define $D_K$ to be the absolute value of the discriminant of $K$ over $\mathbb{Q}$.

Let $S$ be a finite set of finite places of $K$. Let $\mathcal{O}_{K,S}$ be the ring of $S$-integers in $K$. Write $N_S = \prod_{v \in S} N_v$ for the norm of $S$, where $N_v$ denotes the number of elements in the residue field of $v$. Also, we let $h_S = |\text{Pic}(\mathcal{O}_{K,S})|$ be the class number of $\mathcal{O}_{K,S}$.

Lemma 3.1 (H.W. Lenstra jr.). Let $K$ be a number field. Then

$$|\text{Pic}(\mathcal{O}_K)| \leq (d_K + D_K)^{d_K}$$
Proof. This follows from a result of Lenstra [27, Thm. 6.5]. Indeed, as $|\text{Pic}(\mathcal{O}_K)|$ is the class number of $K$ and $\frac{2}{\pi} < 1$, Lenstra’s result implies that

$$|\text{Pic}(\mathcal{O}_K)| \leq D_K^{\frac{1}{2}} \left( \frac{d_K - 1 + \frac{1}{2} \log(D_K)}{(d_K - 1)!} \right)^{d_K - 1} \leq D_K^{\frac{1}{2}} \frac{(d_K + D_K)^{d_K}}{(d_K - 1)!(d_K + D_K)}.$$ 

This clearly implies that

$$|\text{Pic}(\mathcal{O}_K)| \leq (d_K + D_K)^{d_K}. \quad \square$$

The following lemma is a consequence of a result of von Känel [44] which in turn is a direct consequence of Győry-Yu’s theorem [19] and builds on the theory of linear forms in logarithms [2, 3].

Lemma 3.2 (Győry-Yu, von Känel). Let $K$ be a number field and let $S$ be a finite set of finite places of $K$. Let $L/K$ be a finite field extension of degree $l$ over $K$ which ramifies only over $S$. Let $S_L$ be the finite set of finite places of $L$ lying over $S$. Then the following statements hold.

1. There is a finite set of finite places $S'_L$ of $L$ which satisfies the following properties.
   - The finite set $S'_L$ contains $S_L$.
   - The ring $\mathcal{O}_{L,S'_L}$ is a principal ideal domain.
   - The inequality $N_{S'_L} \leq N_{S_L}D_K^{h_{S'_L}}$ holds.
   - The inequality $|S'_L| \leq |S_L| + h_{S_L}$ holds.

2. Let $S'_L$ be as in (1). If $a$ is an element of $L$ such that $a \in \mathcal{O}_{L,S'_L}^\times$ and $1 - a \in \mathcal{O}_{L,S'_L}$, then the inequality

$$h(a) \leq (12ld_KN_SD_K)^{20000^{\log l}D_K^{|S|}(d_K + D_K)|S|}$$

holds.

3. We have

$$d_L = ld_K, \quad D_L \leq D_K^{lN_{S'_L}^{ld_K^{|S|}}}, \quad N_{S_L} \leq N_{S'_L}^{l}. \quad (1)$$

Proof. Note that (1) follows from [43, Lem. 4.1].

Let $S'_L$ be as in (1). To prove (2) and (3), we apply [44, Proposition 6.1.(ii)] as follows. Let $S'$ be the places of $K$ lying under $S'_L$. Note that $L/K$ only ramifies over $S'$ (as $S \subset S'$). Let $m = \max(6, l)$ and $T = S'$. Also, let $U$ be the finite set of finite places of $L$ lying over $S'$ (so that $S'_L \subset U$). We now apply loc. cit. with the above choices of $T$ and $U$ to see that the inequality

$$h(a) \leq \left( 2md_KN_{S'}^{\log(m)} \right)^{15md_K - 1} D_K^{-m - 1}$$

holds. Note that $m \leq 6l$. In particular,

$$h(a) \leq \left( 12ld_KN_{S'}^{6l} \right)^{90ld_K} D_K^{6l^3}. \quad (2)$$

Note that $N_{S_L} \leq N_{S'_L}^{l}$. Moreover, by our choice of $S'_L$ and (1), the inequality

$$N_{S'} \leq N_{S'_L} \leq N_{S_L}D_{L,S'_L}^{h_{S'_L}} \leq N_{S}D_{L,S'_L}^{h_{S'_L}} \leq N_{S'_L}D_{L,S'_L}^{h_{S'_L}}$$

holds. Therefore, by [43, Proposition 6.1.(ii)],

$$h(a) \leq (12ld_KN_{S'}^{6l})^{20000^{\log l}D_K^{|S|}(d_K + D_K)|S|}.$$
holds. Now, as $L$ only ramifies over $S$, it follows from Dedekind’s discriminant theorem (see [44, Lem. 6.2]) that

$$D_L \leq D^t_K N^l_S l^d_K |S|.$$  

This concludes the proof of (3). By Lenstra’s upper bound for the class number (Lemma 3.1), the inequality

$$h_{SL} \leq |\text{Pic}(O_L)| \leq (ld_K + D_L)^{ld_K}$$

holds. We now use the upper bound $D_L \leq D^t_K N^l_S l^d_K |S|$ and obtain

$$h_{SL} \leq (ld_K + D_L)^{ld_K} \leq (ld_K + D^t_K N^l_S l^d_K |S|)^{ld_K}.$$  

Combining our inequalities, we obtain

$$h(a) \leq (12ld_K N^l_S)^{90ld_K} D^d_K \leq \left(12ld_K N^6_S D_L^{6h_{SL}}\right)^{90ld_K} D^d_K \leq \left(12ld_K N^6_S \left(D^t_K N^l_S l^d_K |S|\right)^{6h_{SL}}\right)^{90ld_K} D^d_K \leq \left(12ld_K N^6_S \left(D^t_K N^l_S l^d_K |S|\right)^{6l(ld_K + D^t_K N^l_S l^d_K |S|)^{ld_K}}\right)^{90ld_K} D^d_K \leq (12ld_K N^6_S D^t_K)^{6l^2 ld_K |S| - 6l(ld_K + D^t_K N^l_S l^d_K |S|)^{ld_K} - 90ld_K - 6d \leq (12ld_K N^6_S D^t_K)^{20000d^2 |S|(ld_K + D^t_K N^l_S l^d_K |S|)^{ld_K}}.$$  

This concludes the proof of (2). \qed

4. An effective Shafarevich theorem for del Pezzo surfaces

We now prove an effective version of Scholl’s finiteness theorem for del Pezzo surfaces of degree at most four. As in Scholl’s paper [37], we reduce to a finiteness statement about the unit equation.

The unit equation appears via a consideration of the coordinates of certain points on $\mathbb{P}^2$ which are in general position. Here by general position, we mean the following. Let $1 \leq d \leq 9$ and let $P_1, \ldots, P_{9-d}$ be points on $\mathbb{P}^2$, where $k$ is a field. We say that these points are (geometrically) in general position if no three lie on a line in $\mathbb{P}_k^2$, no six lie on a conic in $\mathbb{P}_k^2$, and no eight lie on a cubic in $\mathbb{P}_k^2$ which is singular at one of the points.

**Theorem 4.1.** Let $K$ be a number field, let $S$ be a finite set of finite places of $K$, and let $1 \leq d \leq 4$. Let $X$ be a smooth del Pezzo surface of degree $d$ over $K$ with good reduction outside $S$. There exist a finite field extension $L/K$ of degree at most $240!$ which ramifies only over $S$, and there exist $5 - d$ points

$$P_5 = (a_5 : b_5 : 1), \ldots, P_{9-d} = (a_{9-d} : b_{9-d} : 1)$$

in $\mathbb{P}^2(L)$ such that the following statements hold.
(1) The surface $X_L$ is the blow-up of $\mathbb{P}^2_L$ in

$$(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 1), P_5, \ldots, P_{9-d},$$

and these points are in general position.

(2) Define $l_d$ to be the order of the Weyl group of the root system of $E_{9-d}$. Then the inequality

$$\max_{i=5, \ldots, 9-d} (h(a_i), h(b_i)) \leq (12l_dD_K \nu S) (l_dD_K + l_dD_K \nu S)^{200000^{d_K} l_dD_K (12l_dD_K + l_dD_K \nu S)} l_dD_K$$

holds.

(3) Write $S_L$ for the set of finite places of $L$ lying over $S$. Then

$$d_L \leq l_dD_K, \quad D_L \leq l_dD_K \nu S, \quad N_{S_L} \leq l_dD_K \nu S.$$

Proof. Let $L$ be the smallest number field such that all lines of $X$ are defined over $L$. Note that the number field $L$ is of degree at most $l_d$ over $K$ and ramifies only over $S$. (This follows from Scholl’s [37, Prop. 3.6] and standard facts about lines on smooth del Pezzo surfaces [14 §8.2].) Let $S_L$ be the finite set of finite places of $L$ lying over $S$.

By the first part of Lemma 3.2, there exists a finite set $S'_L$ of finite places of $L$ containing $S_L$ such that the following statements hold.

- The ring $O_{L,S'_L}$ is a principal ideal domain.
- The inequality $N_{S'_L} \leq N_{S_L}D_K$ holds.
- The inequality $|S'_L| \leq |S_L| + h_{S_L}$ holds.

To prove the theorem, we now follow the proof of [37 Prop. 4.2]. Since $X$ has good reduction outside $S$, we see that $X_L$ has good reduction outside $S_L$ (and thus $S'_L$). Let $\mathcal{X} \to \text{Spec} \ O_{L,S'_L}$ be a smooth del Pezzo surface such that $\mathcal{X}_L$ is isomorphic to $X_L$. Since $O_{L,S'_L}$ is a principal ideal domain and $\mathcal{L}_{\mathcal{X}/O_{L,S'_L}}$ is constant over $\text{Spec} \ O_{L,S'_L}$, it follows from Lemma 2.1 that there are points $P_5, \ldots, P_{9-d}$ in $\mathbb{P}^2(O_{L,S'_L})$ such that $\mathcal{X}$ is isomorphic to the blow-up of $\mathbb{P}^2_{O_{L,S'_L}}$ in the $O_{L,S'_L}$-points

$$(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 1), P_5, \ldots, P_{9-d}.$$ 

In particular, as $\mathcal{X}$ is smooth over $O_{K,S'_L}$, the points

$$(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 1), P_5, \ldots, P_{9-d},$$

are in general position. For $i = 5, \ldots, 9 - d$, write $P_i = (\alpha_i : \beta_i : \gamma_i)$ with $\alpha_i, \beta_i, \gamma_i \in O_{L,S'_L}^\times$. Now, as no three of any collection of four of these points are collinear, we see that $\alpha_i\beta_i\gamma_i \in O_{L,S'_L}^\times$. For $i = 5, \ldots, 9 - d$, we define $a_i = \frac{\alpha_i}{\gamma_i}$ and $b_i = \frac{\beta_i}{\gamma_i}$. Now,

$$(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0), (1 : 1 : 1), P_5, \ldots, P_{9-d},$$

are in general position and $\mathcal{X}$ is isomorphic to the blow-up of $\mathbb{P}^2_{O_{L,S'_L}}$ in these $O_{L,S'_L}$-points. Again, as no three of any collection of four of these points are collinear, it follows that $(1 - a_i)(1 - b_i) \in O_{L,S'_L}^\times$. Therefore, for all $i = 5, \ldots, 9 - d$, the algebraic
numbers $a_i$ and $b_i$ are solutions to the $S'_L$-unit equation in $L$. In particular, by the second part of Lemma 3.2, the inequality
\[
\max_{i=5,\ldots,9-d} (h(a_i), h(b_i)) \leq (12l_dK N_S D_K)^{20000d^2d_K |S| (t d K + D d K N_S t d K |S|)^{ld K}}
\]
holds. This concludes the proof of (1) and (2). Note that (3) follows from the third part of Lemma 3.2. \qed

**Remark 4.2.** We note that, with notation as in (2) of Theorem 4.1,
\[
l_1 = 696729600, \quad l_2 = 2903040, \quad l_3 = 51840, \quad l_4 = 1920;
\]
see [14, Cor. 8.2.16].

**Proof of Theorem 4.2** Let $K_{\text{split}}$ be the compositum of all number fields $L/K$ such that $L$ is ramified only over $S$ and of degree at most $240!$. Then, by Theorem 4.1, any smooth del Pezzo surface over $K$ of degree $1 \leq d \leq 4$ with good reduction outside $S$ is split over the number field $K_{\text{split}}$. It follows from Theorem 4.1 that the set of $K_{\text{split}}$-isomorphism classes of smooth del Pezzo surfaces over $K$ of degree $1 \leq d \leq 4$ with good reduction outside $S$ is finite and effectively computable. (Indeed, the heights of the coefficients of the coordinates of the points we require to blow-up in $\mathbb{P}^2(K_{\text{split}})$ are bounded explicitly.) As the automorphism group of a smooth del Pezzo surface of degree at most four is finite (and effectively computable using methods as in [7, 8]), a standard Galois cohomological argument now concludes the proof (see part (a) of the proof of [37, Thm. 4.5]). \qed

5. An effective Shafarevich theorem for smooth quartic curves

Let $H_3$ be the stack of hyperelliptic curves in $M_3$. The morphism $H_3 \to M_3$ is a closed immersion and the coarse moduli space of $H_3$ is affine.

Let $M_3^{\text{nh}}$ be the complement of $H_3$ in $M_3$. Note that $M_3^{\text{nh}}$ parametrizes smooth proper non-hyperelliptic curves of genus 3, and that $M_3^{\text{nh}}$ is an open substack of $M_3$.

Let Hilb be the Hilbert scheme of smooth quartic curves in $\mathbb{P}^2$, i.e., Hilb is the affine scheme over $Z$ given by the complement of the discriminant divisor in $\mathbb{P}(H^0(\mathbb{P}^2_Z, O_{\mathbb{P}^2_Z}(4)))$. Note that $\text{PGL}_{3,Z}$ acts on Hilb. Let $C_{(4,1)} := [\text{PGL}_{3,Z}\backslash \text{Hilb}]$ be the stack of smooth quartic curves in $\mathbb{P}^2$ (see [5, 21]). Note that the natural morphism from $C_{(4,1)}$ to the complement $M_3^{\text{nh}}$ of $H_3$ in $M_3$ is an isomorphism of stacks over $Z$. (The only subtle point here is that the automorphism group of a smooth quartic curve in $\mathbb{P}^2$ equals its “linear” automorphism group as a hypersurface; see [11, 35].)

If $S$ is a scheme, then an $S$-object of $C_{(4,1)}$ is not necessarily isomorphic to a smooth quartic hypersurface in $\mathbb{P}^2_S$. There are obstructions (coming from the Brauer group of $S$ and $H^1(S, GL_3,S)$ [21, §2.1.2]) to an $S$-object of $C_{(4,1)}$ being a smooth quartic curve in $\mathbb{P}^2_S$ (and not only in a non-trivial Brauer-Severi scheme or projective bundle over $S$). On the other hand, if $k$ is a field, then any $k$-object of $C_{(4,1)}(k)$ is in fact a smooth quartic curve in $\mathbb{P}^2_k$ (cf. [31]).
We now prove an effective version of Faltings’s theorem (quondam Shafarevich’s conjecture) for smooth quartic curves (i.e., non-hyperelliptic smooth proper genus 3 curves).

**Theorem 5.1.** Let $K$ be a number field and let $S$ be a finite set of finite places of $K$. The essential image of the functor $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S}) \to \mathcal{C}_{(4;1)}(K)$ is finite and effectively computable.

*Proof.* The set of isomorphism classes of rank three vector bundles on $\text{Spec } \mathcal{O}_{K,S}$ is finite and effectively computable, as it is given by $H^1_{\text{et}}(\mathcal{O}_{K,S}, \text{GL}_3, \mathcal{O}_{K,S})$. (The latter cohomology set can be computed explicitly as follows. Firstly, Borel’s finiteness theorem [10, Thm. 5.1] is effective. Therefore, the set $c(\mathcal{O}_{K,S}, \text{GL}_3, \mathcal{O}_{K,S})$ (with notation as in [18, §5]) is finite and effectively computable. Finally, there is a natural bijection $c(\mathcal{O}_{K,S}, \text{GL}_3, \mathcal{O}_{K,S}) \cong H^1_{\text{et}}(\mathcal{O}_{K,S}, \text{GL}_3, \mathcal{O}_{K,S})$.)

Let $S'$ be a finite set of finite places of $K$ with the following properties.

1. All elements of $H^1_{\text{et}}(\mathcal{O}_{K,S}, \text{GL}_3, \mathcal{O}_{K,S})$ are trivial over $\text{Spec } \mathcal{O}_{K,S'}$.
2. The set $S'$ contains all places lying over 2, and
3. $\text{Pic}(\mathcal{O}_{K,S'}) = 0$.

Note that, by Lemma 3.2 and the fact that $H^1_{\text{et}}(\mathcal{O}_{K,S}, \text{GL}_3, \mathcal{O}_{K,S})$ is effectively computable, we can indeed effectively determine such a finite set $S'$.

Note that the set of $K$-isomorphism classes of smooth del Pezzo surfaces of degree two over $K$ with good reduction outside $S'$ is finite and effectively computable (Theorem 1.2). Let $D_1, \ldots, D_n$ be smooth del Pezzo surfaces of degree two over $K$ such that any smooth del Pezzo surface of degree two over $K$ with good reduction outside $S'$ is isomorphic to some $D_i$ with $i$ in $\{1, \ldots, n\}$.

Note that, all objects of $\mathcal{C}_{(4;1)}(K)$ are smooth quartic curves in $\mathbb{P}_K^2$, all vector bundles of rank three over $\mathcal{O}_{K,S}$ trivialize over $\mathcal{O}_{K,S'}$, and the Picard group of $\text{Spec } \mathcal{O}_{K,S'}$ is trivial. Therefore, if $Y$ is in the essential image of the functor $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S}) \to \mathcal{C}_{(4;1)}(K)$, then there is a smooth quartic $Y$ in $\mathbb{P}_K^2_{\mathcal{O}_{K,S'}}$ whose generic fibre $Y_K$ is $K$-isomorphic to $Y$ (use [21] Lem. 4.8.(2)) and the explicit description of the functor of points of $\mathcal{C}_{(4;1)}$ given in [4, §2.3.2]).

Let $f \in \mathcal{O}_{K,S'}[x_0, x_1, x_2]$ be a homogeneous polynomial such that $Y$ is isomorphic to the zero locus of $f$ in $\mathbb{P}_K^2_{\mathcal{O}_{K,S'}}$. Since a double cover $\mathcal{D}$ of $\mathbb{P}_K^2_{\mathcal{O}_{K,S'}}$ ramified along $Y$ can be written as the zero locus of $x_3^2 = f$ in a suitable weighted projective space, it follows from the Jacobian criterion for smoothness that $\mathcal{D}$ is smooth over $\mathcal{O}_{K,S'}$ (here we use that $S'$ contains all the places lying over 2). Therefore, a double cover $\mathcal{D}$ of $\mathbb{P}_K^2_{\mathcal{O}_{K,S'}}$ ramified precisely along $Y$ is a smooth del Pezzo surface of degree two over $\mathcal{O}_{K,S'}$. Note that the isomorphism class of $\mathcal{D}_K$ determines the isomorphism class of $Y_K$.

There exists an integer $i$ in $\{1, \ldots, n\}$ such that $\mathcal{D}_K$ is $K$-isomorphic to $D_i$. Therefore, as the automorphism group of a smooth proper genus three curve is finite, up to a standard Galois cohomological argument, we conclude that the set of isomorphism classes of $K$-objects of $\mathcal{C}_{(4;1)}(K)$ which come from an $\mathcal{O}_{K,S'}$-point of $\mathcal{C}_{(4;1)}$ is finite and effectively computable. \hfill $\square$
Remark 5.2. To explain the idea behind our proof of Theorem 5.1, let $\mathcal{DP}_2$ be the stack of smooth del Pezzo surfaces of degree two over $\mathbb{Z}$. Note that $\mathcal{DP}_2$ is a Deligne-Mumford separated algebraic stack of finite type over $\mathbb{Z}$. The cyclic covering trick exhibits $\mathcal{DP}_{2, \mathbb{Z}[1/2]}$ as a (non-neutral) $\mu_2$-gerbe $\mathcal{DP}_{2, \mathbb{Z}[1/2]} \to \mathcal{C}_{(4;1), \mathbb{Z}[1/2]}$ over the stack $\mathcal{C}_{(4;1), \mathbb{Z}[1/2]}$. More precisely, given a smooth del Pezzo surface $X$ over a ring $A$ with $2 \in A^\times$, the anti-canonical map exhibits $X$ as a double cover of some (twisted) projective space of relative dimension two over $A$. The branch locus of this double cover is a (twisted) smooth quartic curve over $A$. We refer the reader to [1] for a further discussion of the stack of smooth del Pezzo surfaces of degree two as a certain stack of cyclic covers.

It is the structure of $\mathcal{DP}_{2, \mathbb{Z}[1/2]}$ as a $\mu_2$-gerbe over $\mathcal{C}_{(4;1), \mathbb{Z}[1/2]}$ that we have exploited in our proof of Theorem 5.1.

It seems worthwhile noting that the stacks $\mathcal{DP}_2$ and $\mathcal{C}_{(4;1)}$ are not isomorphic (not even over $\mathbb{C}$). Indeed, all smooth del Pezzo surfaces of degree two over $\mathbb{C}$ have a non-trivial automorphism of order two, whereas the general quartic curve in $\mathbb{P}^2_\mathbb{C}$ has no non-trivial automorphisms.

On the other hand, the induced morphism on coarse moduli spaces $\mathcal{DP}_{2, \mathbb{C}} \to \mathcal{C}_{(4;1), \mathbb{C}}^{\text{coarse}}$ is an isomorphism of complex algebraic affine varieties [30, §7.2].

Corollary 5.3. Let $K$ be a number field and let $S$ be a finite set of finite places of $K$. The set of isomorphism classes of the groupoid $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S})$ is finite and effectively computable.

Proof. As the functor $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S}) \to \mathcal{C}_{(4;1)}(K)$ is injective on the underlying sets of isomorphism classes, this follows from Theorem 5.1. $\square$

Proof of Theorem 1.1. As the set of $\mathcal{O}_{K,S}$-isomorphism classes of smooth quartic curves over $\mathcal{O}_{K,S}$ is a subset of the set of isomorphism classes of the groupoid $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S})$, this follows from Corollary 5.3. $\square$

Remark 5.4. Note that the Shafarevich conjecture is effective for hyperelliptic curves [13, 23, 43]. In particular, the finite set of integral points on $\mathcal{H}_3$ can be computed effectively.

Nonetheless, we are not able to infer an effective version of the Shafarevich conjecture for all smooth proper genus three curves by combining the results of loc. cit. with Theorems 1.1 and 5.1. On the other hand, we are able to reduce the effective Mordell conjecture for some complete curves to a statement about effectively bounding primes of hyperelliptic reduction on a smooth quartic curve over a fixed $\mathcal{O}_{K,S}$ (see Theorem 6.2).

An effective version of the Shafarevich conjecture for all smooth proper genus three curves would imply an effective version of the Mordell conjecture for some class of curves; see Section 6.
6. Towards Algorithmic Mordell for Some Class of Curves

We give an application of our main result on computing integral points on the stack $C_{1(4)} = \mathcal{M}_3^{\text{nh}}$ (Theorem 5.1) to computing rational points on certain complete hyperbolic curves.

Our aim is to provide a criterion for a complete curve to satisfy a version of the Mordell conjecture in which one can also algorithmically determine the set of rational points; see Corollary 6.4 for a precise statement. Let us start with stating this conjecture.

**Conjecture 6.1** (Algorithmic Mordell). *There exists an algorithm that, on input given a number field $K$, a smooth projective geometrically connected curve $X$ over $K$ of genus at least two, and a number field $L$ over $K$, computes as output the finite set $X(L)$.*

To make this conjecture mathematically precise, let us note that by “algorithm” we mean Turing machine (as defined in [20]). We refer the reader to [34] for a useful discussion.

The finiteness of the set $X(L)$ follows from Faltings’s theorem (*quondam* Mordell’s conjecture) [16, 40]. Note that before Faltings’s theorem, there was not a single example known of a number field $K$ and a smooth proper geometrically connected curve of genus at least two over $K$ such that, for all number fields $L$ over $K$, the set $X(L)$ was provably finite.

Conjecture 6.1 is clearly different from *Mordell effectif* as stated in [33, §4]. Indeed, Conjecture 6.1 implies that the height of an $L$-rational point of $X$ is bounded by some effectively computable real number depending only on $X$, $K$, and $L$, but it does not infer any linear or polynomial dependence on the discriminant of $L$ nor the height of $X$.

Faltings’s proof of the Mordell conjecture exploits the fact that, for all smooth projective curves $Y$ over $\mathbb{C}$ of genus at least two, there exist a finite étale cover $X \to Y$, an integer $g \geq 3$, and a finite morphism $X \to \mathcal{M}_{g,\mathbb{C}}$; see [32, 40]. An effective version of the Shafarevich conjecture for all smooth proper genus $g$ curves would therefore imply Conjecture 6.1; see [36] for a precise statement.

Levin has shown that effective Shafarevich theorems for “special” classes of curves have applications to effectively computing integral points on affine curves [28]. The main result of this section (see Theorem 6.2 and Corollary 6.4 below) aims at showing a similar result for rational points on complete curves $X$ mapping non-trivially to $\mathcal{M}_3$.

In [15] Zaal has explicitly constructed complete curves $X$ which map finitely to $\mathcal{M}_3$. The aim of this section is to prove Algorithmic Mordell for such complete curves $X$, under suitable assumptions.

**Theorem 6.2.** Let $K$ be a number field and let $S$ be a finite set of finite places of $K$. Let $X$ be a complete curve over $\mathcal{O}_{K,S}$ which maps finitely to $\mathcal{M}_3,\mathcal{O}_{K,S}$.

Suppose that there is an effectively computable finite set of finite places $S'$ of $K$ containing $S$ such that all smooth quartic curves in the image of $X(K)$ in $\mathcal{M}_3(K)$ have no hyperelliptic reduction over $\mathcal{O}_{K,S'}$. 

Then the set $X(K)$ is finite and effectively computable.

Proof. By assumption, we are given explicitly a finite morphism $p : X \to \mathcal{M}_{3, \mathcal{O}_{K,S}}$, where $\mathcal{M}_3$ is the stack of smooth proper genus three curves.

Let $\mathcal{X}'$ be the image of $p$ in $\mathcal{M}_{3, \mathcal{O}_{K,S}}$. Note that $\mathcal{X}'$ is closed in $\mathcal{M}_{3, \mathcal{O}_{K,S}}$ and proper over $\mathcal{O}_{K,S}$. As $X_K$ is proper and $\mathcal{H}_{3,K}$ has affine coarse space, the intersection $\mathcal{X}'_K \times_{\mathcal{M}_{3,K}} \mathcal{H}_{3,K}$ of $\mathcal{X}'_K$ and $H_X := \mathcal{H}_{3,K}$ in $\mathcal{M}_{3,K}$ is a finite scheme over $K$. To prove the theorem, we may and do assume that all closed points of $H_X$ lie in $\mathcal{M}_{3}(K)$.

Let $S'$ be as in the statement of the theorem. By Theorem 5.1 the essential image of the functor

$$\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S'}) \to \mathcal{C}_{(4;1)}(K)$$

is finite and effectively computable. Let $y_1, \ldots, y_n$ in $\mathcal{C}_{(4;1)}(K)$ be representatives for the essential image of $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S'}) \to \mathcal{C}_{(4;1)}(K)$.

By the defining property of $S'$, the image of $X(K) \to \mathcal{M}_3(K)$ is contained in the union of $H_X$ with the image of $\mathcal{C}_{(4;1)}(\mathcal{O}_{K,S'}) \to \mathcal{C}_{(4;1)}(K) \to \mathcal{M}_3(K)$. In particular, the image of $X(K)$ in $\mathcal{M}_3(K)$ lies in the effectively computable finite set of $K$-rational points $H_X \cup \{y_1, \ldots, y_n\}$ of $\mathcal{M}_3(K)$. As $X_K \to \mathcal{M}_{3,K}$ is a finite morphism and the image of $X(K)$ is finite, we can now conclude that $X(K)$ is finite.

Let $y$ be in $H_X \cup \{y_1, \ldots, y_n\} \subset \mathcal{M}_3(K)$. Note that the fibre of $X \to \mathcal{M}_{3,K}$ over $y$ with respect to $X_K \to \mathcal{M}_{3,K}$ is either empty or a zero-dimensional subscheme of $X$. The theory of Gröbner bases allows one to effectively compute whether the fibre over $y$ is empty. (Here we use that the morphism $p$ is given explicitly, so that one can effectively compute equations for the closed subscheme $p^{-1}(y_i)$.) In particular, to conclude the proof, we may and do assume that the fibre over $y$ is non-empty. Now, to conclude the computation of $X(K)$, it suffices to show that the set of points on a non-empty zero-dimensional finite scheme over $\mathcal{O}$ can be effectively computed. This can be done using elimination theory and factoring of polynomials over number fields.

Remark 6.3. Let us discuss the assumption in Theorem 6.2. To do so, let $K$ be a number field, let $S$ be a finite set of finite places of $K$, and let $X \to \text{Spec} \mathcal{O}_{K,S}$ be a smooth proper curve of genus 3. If $X_K$ is not a hyperelliptic curve, then the set of primes $p \subset \mathcal{O}_{K,S}$ such that the fibre of $X$ over $p$ is hyperelliptic is finite. Moreover, if $X$ is fixed, then the set of “hyperelliptic reductions” of $X$ is effectively computable, as it is given by the intersection product of the hyperelliptic locus in $\mathcal{M}_3$ with the $\mathcal{O}_{K,S}$-section of $\mathcal{M}_3$ corresponding to $X$ (pulled-back to $\mathcal{O}_{K,S}$). Now, the hypothesis in Theorem 6.2 says that, for all number fields $K$ and all finite sets of finite places $S$ of $K$, we can effectively compute a finite set of finite places $S'$ of $K$ such that all smooth proper genus 3 curves $X$ over $\mathcal{O}_{K,S}$ with $X_K$ not a hyperelliptic curve have non-hyperelliptic reduction at all $p \notin S'$.

Note that the Shafarevich conjecture (as proven by Faltings) implies that the set $S'$ exists (but might not be effectively computable). However, an effective version of the Shafarevich conjecture for genus three curves would imply that this assumption holds. On the other hand, the assumption made in Theorem 6.2 is a priori weaker than an effective version of the Shafarevich conjecture.
Corollary 6.4. Let $X$ be a proper curve over a number field $K$ and let $X \to \mathcal{M}_{3,K}$ be a finite morphism.

Suppose there is an algorithm that, on input a number field $L$ over $K$, computes as output a finite set of finite places $S$ of $L$ such that all smooth quartic curves $C$ in the image of $X(L) \to \mathcal{M}_{3}(L)$ lie in the essential image of $\mathcal{M}_{3}^{\text{nh}}(\mathcal{O}_{L,S}) \to \mathcal{M}_{3}(L)$.

Then Algorithmic Mordell holds for $X$, i.e., there is an algorithm that, on input a number field $L$, computes as output the finite set $X(L)$.

Proof. This follows directly from Theorem 6.2. \hfill \square

Remark 6.5. Note that the Bogomolov-Miyaoka-Yau inequality implies that a smooth proper genus two curve over $\mathbb{C}$ does not map finitely to $\mathcal{M}_{3,\mathbb{C}}$; see [21]. It seems reasonable to suspect that there are complete hyperbolic complex algebraic curves $X$ which do not map finitely to $\mathcal{M}_{3,\mathbb{C}}$, even after passing to a finite étale cover. On the other hand, any complete hyperbolic curve maps finitely, up to a finite étale cover, to $\mathcal{M}_{g,\mathbb{C}}$ for some $g \geq 3$; see [32].

Remark 6.6. We emphasize that our proof of Theorem 6.2 gives a non-efficient algorithm for several reasons. For instance, part of the algorithm consists of writing down all solutions to the unit equation in some number ring. Moreover, one has to work with number fields of high degree.

Corollary 6.7. Let $X \to Y$ be a finite étale morphism of smooth proper geometrically connected curves over a number field $K$. Let $X \to \mathcal{M}_{3,K}$ be a finite morphism.

Suppose there is an algorithm that, on input a number field $L$ over $K$, computes as output a finite set of finite places $S$ of $L$ such that all smooth quartic curves $C$ in the image of $X(L) \to \mathcal{M}_{3}(L)$ have non-singular non-hyperelliptic reduction over $\mathcal{O}_{L,S}$.

Then there is an algorithm that, on input a number field $L$, computes the finite set $Y(L)$.

Proof. This follows from Corollary 6.4 and the quantitative version of the Chevalley-Weil theorem for curves [9]. (Note that a smooth quartic curve over $L$ has non-singular non-hyperelliptic reduction over $\mathcal{O}_{L,S}$ if and only if it lies in the essential image of $\mathcal{M}_{3}^{\text{nh}}(\mathcal{O}_{L,S}) \to \mathcal{M}_{3}(L)$. In other words, a smooth quartic curve $X$ over $K$ has non-singular non-hyperelliptic reduction over $\mathcal{O}_{L,S}$ if, and only if, its minimal regular proper model $\mathcal{X} \to \text{Spec} \mathcal{O}_{L,S}$ (as defined in [20, Defn. 9.3.12]) is smooth over $\mathcal{O}_{L,S}$ and, for all $b$ in $\text{Spec} \mathcal{O}_{L,S}$, the fibre $\mathcal{X}_b$ is a non-hyperelliptic curve.) \hfill \square

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