SCATTERING FOR DEFOCUSING GENERALIZED 
BENJAMIN-ONO EQUATION IN THE ENERGY SPACE 
$H^\frac{1}{2}(\mathbb{R})$

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Abstract. We prove the scattering for the defocusing generalized Benjamin-Ono equation in the energy space $H^\frac{1}{2}(\mathbb{R})$. We first establish the monotonicity formula that describes the unidirectional propagation. More precisely, it says that the center of energy moves faster than the center of mass. This type of monotonicity was first observed by Tao [24] in the defocusing gKdV equations.

We use the monotonicity in the setting of compactness-contradiction argument to prove the large data scattering in the energy space $H^\frac{1}{2}(\mathbb{R})$. On the way, we extend critical local theory of Vento [27] to the subcritical regime. Indeed, we obtain subcritical local theory and global well-posedness in the energy space.

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1. Introduction

We consider the *defocusing generalized Benjamin-Ono equation* (gBO):

\[
\begin{align*}
\partial_t u + \mathcal{H} \partial_{xx} u + \partial_x (u^{k+1}) &= 0, & (t, x) &\in \mathbb{R} \times \mathbb{R}, \\
u(0, x) &= u_0(x),
\end{align*}
\]

with \( u : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) a real-valued function, \( k \) an even natural number \( \geq 4 \), \( u_0 \) an initial data in homogeneous or inhomogeneous Sobolev space. We denote by \( \mathcal{H} \) the Hilbert transform on \( \mathbb{R} \), which acts on Schwartz functions \( f \) by

\[
\mathcal{H} f = \left( \frac{1}{\pi} \text{p.v.} \frac{1}{x} \right) \ast f.
\]

Equivalently, \( \mathcal{H} \) is a Fourier multiplier operator with multiplier \( m(\xi) = -i \text{sgn}(\xi) \):

\[
\hat{\mathcal{H}} f(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi).
\]

We let the nonlinearity \( F(u) \) by \(-\partial_x (u^{k+1})\), and rewrite (gBO) in the form

\[
\partial_t u + \mathcal{H} \partial_{xx} u = F(u).
\]

The linear part of (gBO) is called as the *linear Benjamin-Ono equation*. If the nonlinearity has the opposite sign, we call the equation *focusing*. The equation (gBO) is in fact a generalization of the Benjamin-Ono equation (BO)

\[
\partial_t u + \mathcal{H} \partial_{xx} u + \partial_x (u^2) = 0
\]

in its power of nonlinearity. The (gBO) with \( k = 1 \) is the above (BO), and the case when \( k = 2 \) is called as defocusing modified Benjamin-Ono equation (mBO). It is known that (BO) is completely integrable [10].

The equation (gBO) has scaling invariance. For any \( \lambda > 0 \) and a solution \( u(t, x) \) to (gBO) with initial data \( u_0(x) \), \( u_\lambda(t, x) := \lambda^{\frac{k}{2}} u(\lambda^2 t, \lambda x) \) is also a solution with initial data \( \lambda^{\frac{k}{2}} u_0(\lambda x) \). Thus, (gBO) is \( \dot{H}^{s_k} \)-critical with

\[
s_k = \frac{1}{2} - \frac{1}{k},
\]

in the sense that scalings preserve \( \dot{H}^{s_k} \) norm of the initial data.

Moreover, we have mass and energy conservation laws\footnote{In fact, there is another (formal) conservation law:}

\[
\int_\mathbb{R} u(t, x) dx = \int_\mathbb{R} u(0, x) dx
\]

for a solution \( u \) to (gBO), but this will not be used throughout the paper. Note that it is not positive-definite and may not even defined for \( H^\pm \) solutions.

The \( L^2 \)-mass (in short, *mass*) and energy of a function \( u(t, x) \) is defined by

\[
M(u)(t) := \int_\mathbb{R} u^2(t, x) dx, \\
E(u)(t) := \int_\mathbb{R} \left[ \frac{1}{2} \mathcal{H} u_x + \frac{1}{k+2} u^{k+2} \right](t, x) dx.
\]

Whenever \( u \) is a smooth solution, mass and energy are conserved:

\[
M(u)(t) = M(u)(0) \quad \text{and} \quad E(u)(t) = E(u)(0).
\]
The local theory of Benjamin-Ono type equations have been intensively studied. For readers who are interested in local theory of \((gBO)\) and \((mBO)\), we refer to a recent survey by Ponce [23]. We now want to pick up some works for \((gBO)\) with \(k \geq 4\). At first, by the energy method, Iório [11] showed local well-posedness (LWP) in \(H^s\) with \(s > \frac{1}{2}\). Using the method of gauge transform, Molinet and Ribaud [21] showed LWP in \(H^\frac{1}{2}\) for \(k \geq 5\) and in \(H^{\frac{4}{3}}\) for \(k = 4\). Moreover, they showed small data global well-posedness in the critical space \(H^s\). For \(k = 4\), the first large data critical LWP was obtained by Burq and Planchon [3]. Finally, Vento [27] obtained the critical LWP for all \(k \geq 4\) by modifying the linear propagator of \((gBO)\).

Throughout the paper, we rely on the critical LWP obtained by Vento, so we want to state the result in more detail. Denote the linear propagator of the linear Benjamin-Ono equation by \(V(t)\). A function \(u : I \times \mathbb{R} \to \mathbb{R}\) is called as a \(\dot{H}^s\) solution to \((gBO)\) if \(0 \in I\) is a time interval, \(u \in C^1_t \dot{H}^s \cap L^k_t L^{\infty}_x \cap X^s_T\) for any compact subinterval \(J \subseteq I\), (see Notations and Section 3.1 for definitions of these function spaces) and \(u\) satisfies the Duhamel formula

\[
u(t) = V(t)u_0 + \int_0^t V(t-s)F(u(s))ds
\]

for all \(t \in I\). Replacing \(H^s\) and \(X^s_T\) by \(H^s_r\) and \(X^s_r\) respectively, we similarly define \(H^s_r\) solutions to \((gBO)\). A solution is called global if \(I = \mathbb{R}\). We now state the critical LWP of \((gBO)\).

**Theorem 1.1** (Critical local theory, [27].) Let \(k \geq 4\).

1. For any \(u_\ast\) in \(H^s\), there exists \(T = T(u_\ast) > 0\) and \(r = r(u_\ast) > 0\) such that for any initial data \(u_0 \in B(u_\ast; r)\), there exists a unique \(H^s\) solution \(u\) in \(L^\infty_T \dot{H}^s \cap L^k_T L^\infty_x \cap X^s_T\). Moreover, the solution \(u\) indeed lies in \(\dot{H}^s\) and the solution map is locally Lipschitz.

2. For any \(s \geq s_k\), the \(H^s\)-version of the result holds. That is, we can replace \(L^\infty_T \dot{H}^s \cap L^k_T L^\infty_x \cap X^s_T\) by \(L^\infty_T H^s \cap L^k_T L^\infty_x \cap X^s_T\), \(H^s\), and \(Z^s_T\) by \(L^\infty_T H^s \cap L^k_T L^\infty_x \cap X^s_T\), \(H^s\), and \(Z^s_T\).

In Theorem 1.1, the lifespan of a solution depends on the profile of its initial data even for \(s > s_k\). However, a slight modification of the proof of Theorem 1.1 yields that for given \(u_0 \in H^5\) with \(s > s_k\), we can construct a \(H^s\) solution \(u \in C_T H^s\) whose lifespan \(T\) depends only on \(\|u_0\|_{H^s}\). Moreover, we use persistence of regularity (Proposition 3.9) to guarantee that this \(H^s\) solution is indeed a \(H^s\) solution. In particular, we obtain subcritical local well-posedness. When \(s \geq \frac{1}{2}\), by mass/energy conservation, we obtain the global well-posedness of \((gBO)\) in the energy space \(H^\frac{1}{2}\). This is our first result.

**Theorem 1.2** (Subcritical local theory and global well-posedness.) Let \(k \geq 4\).
1. (Subcritical LWP) Assume that $s > s_k$. For any $R > 0$, there exists $T = T(R) > 0$ such that if the initial data $u_0$ satisfies $\|u_0\|_{H^s} < R$, then there exists a unique $H^s$ solution $u$ in $L^\infty_T H^s_x \cap L^k_x L^\infty_T \cap X^s_T$. Moreover, the solution $u$ indeed lies in $Z^s_T$ and the solution map is continuous.

2. (GWP and conservation laws) Assume that $s \geq \frac{1}{2}$. For any $T > 0$ and initial data $u_0 \in H^s$, there exists a unique $H^s$ solution $u$ in $L^\infty_T H^s_x \cap L^k_x L^\infty_T \cap X^s_T$. Moreover, the solution $u$ indeed lies in $Z^s_T$ and the solution map is continuous. Furthermore, we have both mass and energy conservation.

Beyond the well-posedness theory, it is of great interest to study long-time dynamics of the solutions. It is widely believed that for the defocusing equations, the scattering holds (however, see also [25, 26] for supercritical equations.) The defocusing nature in general forces solutions to disperse in the physical space. As the nonlinearity contains the power of $u$, its effect becomes much weaker than the linear evolution. As a result, the linear evolution dominates the dynamics of $u$ and $u$ resembles some linear solution asymptotically. In mathematical terms, we say that a solution $u$ scatters forward (resp., backward) in time if there exists a scattering state $u_\pm$ satisfying

$$\lim_{t \to \pm \infty} \|V(t)u_\pm - u(t)\|_{H^s} = 0.$$  

It is well-known that finiteness of the solution norm implies the scattering. We now present our main result, that is, any $H^{\frac{1}{2}}$ solutions to (gBO) scatter.

**Theorem 1.3** (Scattering for defocusing gBO). For $k > 4$, any $H^{\frac{1}{2}}$ solution to (gBO) scatters both forward and backward in time. Moreover, there exists a function $L : [0, \infty) \to [0, \infty)$ such that

$$\|u\|_{X^s_T} + \|u\|_{L^k_x L^\infty_T} \leq L(M(u) + E(u)).$$

There have been a number of results addressing scattering for semilinear dispersive equations. In this paper, we will use concentration compactness argument. It originates from Lions [19, 20], and was first used in the dispersive equation by Bahouri and Gérard [1]. For the scattering problem, Kenig and Merle [12, 13] used it for the focusing energy-critical nonlinear Schrödinger and wave equations. In fact, their argument have a great generality and applied to many other equations. There are too extensive research on this subject to list them here. We refer, for example, [4, 5, 17, 18, 7] for the nonlinear Schrödinger equations (NLS). For the mass-critical defocusing generalized Korteweg-de Vries equation (gKdV), Dodson [6] proved the scattering. Recently, his argument is extended to supercritical defocusing (gKdV) [9].

The our case (gBO) shares a similar nature with the defocusing (gKdV)

$$\partial_t u + \partial_{xxx} u = \partial_x (|u|^{p-1} u).$$

---

2 For defocusing Benjamin-Ono type equations, by heuristic observations of time decay, it is believed that the linear scattering holds for $k > 2$.

3 Recall that $k$ is an even number, so $k = 6, 8, \ldots$.

4 As Kenig and Merle dealt with focusing equations, they showed scattering when solutions have energy less than that of the ground state.
in their nonlinearities and unidirectional propagation. It is expected that qualitative asymptotic behaviors of solutions are similar. In particular, we show that Tao’s monotonicity formula still holds in (gBO): 
\[ \partial_t (\langle x \rangle_E - \langle x \rangle_M) > 0. \]

Here, \( \langle x \rangle_M \) and \( \langle x \rangle_E \) denote mass and energy center, respectively. For details, see Section 2. Outline of proof of Theorem 1.3 closely follows that of (gKdV) [6]. However, we encounter several difficulties in (gBO). They are due to weaker dispersion, technical issues from the Hilbert transform, and their consequences, for example, trickier local theory. We will explain these issues in detail.

Outline of the Proof and Ideas. In this subsection, we explain our scheme of the proof and what difficulties arise in the setting of (gBO). As we already mentioned, the subcritical well-posedness (Theorem 1.2) follows by extending Vento’s argument [27]; see Section 3 for details. Henceforth, we focus on the scattering (Theorem 1.3).

We now explain how compactness-contradiction argument goes. Suppose that Theorem 1.3 fails. In the first step, we show the existence of the critical element, which does not scatter (both forward and backward in time). In this step, we start with the linear profile decomposition and then obtain nonlinear profiles. The first goal is to show that sum of nonlinear profiles becomes an approximate solution. This requires a long-time perturbation theory. As a result, the extremizing sequence converges to a critical element. Moreover, this critical element stays in a compact set modulo symmetries. In the next step, we use (a truncated version of) monotonicity formula to show that such a solution cannot exist. We now explain details and difficulties step by step.

The linear profile decomposition is used to obtain compactness property. To illustrate this in our case, we consider the local smoothing estimate
\[ \| V(t) u_0 \|_{L^k_x L^\infty_t} \lesssim \| u_0 \|_{H^{\frac{1}{2}}}. \]

This estimate has two non-compact symmetries: spatial translation and time translations. The linear profile decomposition says that lack of compactness of the estimate essentially comes from these symmetries.

Its rigorous statement and proof are fairly standard, we include the proof in Appendix A. However, the usual method cannot take care of the case when \( k = 4 \). This is because \( L^4_x L^\infty_t \) is indeed the endpoint exponents in the local smoothing estimates, so it cannot be obtained by interpolating other estimates. So we can only prove the case when \( k > 4 \), where \( L^k_x L^\infty_t \) norm can be interpolated with \( L^4_x L^\infty_t \) and \( L^\infty_{t,x} \). This is the only point where we should assume \( k > 4 \) in Theorem 1.3.

Each profile from the linear profile decomposition gives rise global nonlinear solutions, so called nonlinear profiles. To guarantee that the sum of nonlinear profiles approximates a nonlinear solution to (gBO), we need a

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Unidirectional propagation means that radiation waves propagate in one direction. See Section 2 for details.
long-time perturbation theory. In (gBO), we do not obtain a standard and strong long-time perturbation as in other contexts. This is principally due to a delicate local theory [27]. Its difficulty is well-explained in Vento [27], but we take it for beginning. At first, one may try a naive estimate

$$
\|D_x^{\frac{n_k}{2}} \partial_x (u^{k+1})\|_{L^5_t L^{10}_x} \lesssim \|D_x^{\frac{n_k}{2}} u\|_{L^\infty_t L^2_x} \|u\|_{L^6_t L^{\infty}_x}^k.
$$

Here, we do not know whether we can use Leibniz rule at endpoint. Moreover, we cannot guarantee that smallness of $T$ implies smallness of $L^k_x L^\infty_t$ norm because of $L^\infty$ factor. Vento resolves this difficulty by extracting out a dangerous high-low interaction in the nonlinearity, approximating it by suitable linear term $\pi(u_0, u)$, and rewrite (gBO) in distorted form:

$$(\partial_t + H \partial_{xx}) u + \pi(u_0, u) = [\pi(u_0, u) - \pi(u, u)] + g(u).$$

In other contexts, the long-time perturbation can be obtained in a general form by concatenating local theory on short time intervals. As an explicit example, in the (gKdV) setting [16], one uses $L^5_x L^{10}_t$ norm as a solution norm. If an interval $I$ (possibly unbounded) is a disjoint union of $I_j$’s, then by Minkowski’s inequality,

$$
\|u\|_{L^5_x L^{10}_t I} = \|u\|_{L^5_x L^{10}_t \ell_j} \geq \|u\|_{L^5_x L^{10}_t I_j} = \left( \sum_j \|u\|_{L^5_x L^{10}_t I_j}^k \right)^{\frac{1}{k}}.
$$

This says that one can subdivide $I$ into finitely many subintervals $I_j$ such that each $L^5_x L^{10}_t$ norm of $u$ is sufficiently small. In our case, however, we use solution norms $L^k_x L^\infty_t$ and $\dot{X}^k_t$, where both contains $L^\infty$ part. Thus, due to nature of $L^\infty$, the usual subdivision argument does not guarantee smallness of our solution norms.

Recalling the reason why we need long-time perturbation in the proof, we do not have to obtain its full power of perturbation. It only suffices to somehow obtain nonlinear version of the profile decomposition. For this purpose, we prove perturbation for a restricted class of approximate solutions, namely those explicitly constructed from the profiles obtained in the linear profile decomposition.

To obtain nonlinear profile decomposition, we use three kinds of perturbation theory, the small data global theory, local theory at time zero, and local theory at time $\pm \infty$. Note that the small data global theory and local theory at time $\pm \infty$ do not require modifying the linear propagator. This helps us to obtain perturbation lemmas at time $\pm \infty$ and safely ignore small data profiles constructed in profile decomposition. Thus, it only suffices to obtain perturbation on the remaining compact interval, on which we can concatenate local theory.

Since the orbit $\{u(t)\}_{t \in \mathbb{R}}$ is precompact in $H^\frac{1}{2}$ modulo spatial translation, we have only soliton-like enemies. See Remark [4.23] The monotonicity formula is the main tool to exclude soliton-like solutions. In (gKdV), Tao [24] established the monotonicity formula:

$$
\partial_t(\langle x \rangle_E - \langle x \rangle_M) > 0,
$$
where $\langle x \rangle_M$ and $\langle x \rangle_E$ denote mass and energy center of a solution, respectively. One can rewrite the monotonicity as an interaction form:

$$\partial_t \int_{\mathbb{R} \times \mathbb{R}} (y - x) \rho(t, x) e(t, y) dx dy > 0,$$

where $\rho$ and $e$ are mass density and energy density, respectively. See Section 2. Tao’s monotonicity formula is based on the following basic observations. Note that the group velocity $\frac{d}{d\xi} \omega(\xi) = -3\xi^2$ is sign definite and the higher frequency pieces travel faster. Moreover, the energy density is more weighted on higher frequency. Thus, the monotonicity formula quantitatively gives a clue that solutions disperse in the physical space. (gBO) also has the same property as the group velocity is $2|\xi|$.

Tao’s proof is nontrivial but surprisingly elementary. His argument seems to be applicable in (gBO). We can closely follow his argument, but there are technical issues arising from the Hilbert transform. More precisely, we have to show

$$s := \int_{\mathbb{R}} u^{k+1} \mathcal{H} u_x \geq 0.$$

In (gKdV), the corresponding statement is just $-\int_{\mathbb{R}} u^{k+1} u_{xx} \geq 0$, which is obvious. To show positivity of $s$ in (gBO), we use finite-dimensional approximation and the Lagrange multiplier method. We reduce it to the case of the circle $\mathbb{T}^1$. By a density argument in the frequency space, we reduce it to the finite-dimensional case $\mathbb{C}^{2N}$. Furthermore, by homogeneity of the functional $s$, we reduce it to the sphere case $S^{2N-1}$, which is compact. Then, in light of Lagrange multiplier, we can obtain a useful relation what an extremizer should satisfy. We then use Pohozaev type argument to show that $s$ is nonnegative.

Finally, to use monotonocity in practice, because we assume that $u$ is a $H^{1/2}$ solution, we consider the localized interaction functional

$$M(t) = \int_{\mathbb{R} \times \mathbb{R}} \Phi(y - x) \rho(t, x) e(t, y) dx dy,$$

where $\Phi(x) = \Phi_R(x)$ approximates $x$ but remains bounded by $R$ for large $R > 0$. The truncation creates a number of errors. In (gKdV), Dodson [6] utilized it in this concrete form (see also [22]). In our case, we want to point out two technical difficulties. At first, as opposed to other defocusing equations, the energy density

$$e = \frac{1}{2} u \mathcal{H} u_x + \frac{1}{k+2} u^{k+2}$$

is not pointwisely nonnegative. Due to this, one should interpret the integral involving $e$ in view of the Parseval identity to estimate various terms by the $H^{1/2}$ norm. Moreover, in error estimates, we run into terms containing derivatives of $\Phi$ and $\mathcal{H}$ that should be small as $C(\|u\|_{H^{1/2}}) \cdot o_R(1)$. Secondly, as the Hilbert transform does not satisfy a simple Leibniz rule, there are many commutator terms in computation such as $[\mathcal{H}, Q]$ for some smooth function $Q$. The calculations and error estimates are involved.

\[\text{As } u \text{ and } \mathcal{H} u \text{ cannot be localized simultaneously in the physical space, we work instead on the frequency space, where both } u \text{ and } \mathcal{H} u \text{ can be localized.}\]
Notations. We shall use the notation $A \lesssim B$ frequently. We say $A \lesssim B$ when there is some implicit constant $C$ that does not depend on $A$ and $B$ satisfying $A \leq CB$. For some parameter $r$, we write $A \lesssim_r B$ if the implicit constant $C$ depends on $r$. In this paper, $k$-dependence will be ignored; we shall abbreviate $A \lesssim_k B$ by $A \lesssim B$.

The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx.$$ 

We define the Sobolev norms for $s \in \mathbb{R}$ as

$$\|f\|_{H^s} := \|\langle D_x \rangle^s f\|_{L^2} \quad \text{and} \quad \|f\|_{\dot{H}^s} := \|D_x^s f\|_{L^2},$$

where $D_x^s$ and $\langle D_x \rangle^s$ are the Fourier multiplier operators with multiplier $|\xi|^s$ and $\langle \xi \rangle^s := (1 + |\xi|^2)^{-\frac{s}{2}}$, respectively.

We let $Q_j$ be the Littlewood-Paley projection to the frequency $\sim 2^j$ in $x$-variable. Given a function $u$, we write

$$u_{< j} = Q_{< j} u = Q_{< j - J} u,$$

$$u_{\sim j} = Q_{\sim j} u = \sum_{|r - j| \leq J} Q_r u$$

for some $J$ large. We remark that one should choose $J$ depending on $\|u_0\|_{\dot{H}^s}$ to obtain linear estimates for the distorted equation. In later sections, we still use same $J$ for global nonlinear solutions. Although $\dot{H}^s$ norm may change, but it is bounded by $H^s$. Due to conservation laws, our global solution has uniform $H^\frac{s}{2}$ bound. See [27, Lemma 3.5 and Proposition 3.2] or Section 3.1.

We use various mixed Lebesgue norms. For $1 \leq q, r \leq \infty$, an interval $I \subseteq \mathbb{R}$, and a Banach space $X$, we write

$$\|f(t)\|_{L^q_I L^r_I} := \left\| \left\| f(t) \right\|_X \right\|_{L^q_I (I)},$$

$$\|G(t, x)\|_{L^q_I L^r_I} := \left\| \left\| G \right\|_{L^r_I (I)} \right\|_{L^q_I (\mathbb{R})}.$$ 

In the situation of $q = r$, by Fubini’s theorem, we abbreviate $L^q_I L^r_I$ and $L^q_{t \times \mathbb{R}}$ by $L^q_{t, x}$. If $t \in I \mapsto G(t) \in X$ is continuous and bounded, we write $G \in C_I X$. If $I = \mathbb{R}$, $I = [-T, T]$, $I = [T, +\infty)$, or $I = (-\infty, -T]$, we replace subscript $I$ by $t$, $T, T_+$, or $T_-$, respectively.

We now define Besov-type spaces by following [27]. For $s \in \mathbb{R}$, $p, q, r \in [1, \infty]$, and an interval $I \subseteq \mathbb{R}$, we define

$$\|f\|_{\dot{B}^s_{p, r}(L^q_I)} := \left( \sum_{j \in \mathbb{Z}} 2^{js} \|Q_j f\|_{L^p_I L^r_I}^r \right)^{\frac{1}{r}},$$

where $Q_j$ is the Littlewood-Paley projection to the frequency $\sim 2^j$ in $x$-variable. In other words, $\dot{B}^s_{p, r}(L^q_I)$ sums up each $\|Q_j f\|_{L^p_I L^r_I}$ in $\ell^\frac{1}{r}$-sense.
Organization of the Paper. In Section 2, we establish the monotonicity formula for (gBO). In Section 3, we prove subcritical well-posedness in $H^s$ with $s > s_k$ and global well-posedness in $H^s$ with $s \geq \frac{1}{2}$. In Section 4, we derive the scattering criterion, linear profile decomposition, and nonlinear profile decomposition. We then prove existence of the critical element if Theorem 1.3 fails. In Section 5, we use the monotonicity formula to show that such critical element cannot exist, concluding Theorem 1.3.

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2. Monotonicity Formula

One of the crucial steps toward asymptotic control of the global solutions is a decay estimate, or monotonicity formula. Though most canonical nonlinear dispersive equations are Hamiltonian systems, it is well known that certain monotonicity phenomena occur and are formulated as for example, Morawetz inequality, Virial inequality, and so on.

In this section, we derive a monotonicity formula for the defocusing (gBO) as well as linear Benjamin-Ono flow. It is similar to the monotonicity formula in the defocusing generalized Korteweg-de Vries equation (gKdV). Tao constructed a monotonicity formula [24] for (gKdV). Tao’s argument is ingenious but elementary, and we observe that his argument similarly works for (gBO). However, we will meet a nontrivial technical difficulty, which does not appear in (gKdV).

Firstly, we recall Tao’s monotonicity formula in (gKdV). Consider the defocusing generalized Korteweg-de Vries equation (gKdV)

$$\partial_t u + \partial_x u = \partial_x |u|^{p-1} u$$

where $p$ is an integer $\geq 2$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. If a wave packet with frequency $\xi$ solves the Airy equation, i.e. the linear part of the (gKdV), then it propagates with group velocity $-3\xi^2$. This says that any solutions to the Airy equation essentially propagates to the left and the higher frequency piece travels faster. The defocusing (gKdV) preserves this phenomenon, as we shall see in the next paragraph.

Not only qualitatively, we can quantitatively capture the unidirectional propagation of (gKdV). Let us define mass density $\rho_{gKdV}$ and energy density $e_{gKdV}$ by

$$\rho_{gKdV} := u^2 \quad \text{and} \quad e_{gKdV} := \frac{1}{2}u_x^2 + \frac{1}{p+1}|u|^{p+1}.$$ 

Define the mass current $j_{gKdV}$ and energy current $k_{gKdV}$ by

$$j_{gKdV} := 2u_x^2 + \frac{2p}{p+1}|u|^{p+1} \quad \text{and} \quad k_{gKdV} := \frac{3}{2}u_x^2 + 2p|u|^{p-1}u_x^2 + \frac{1}{2}|u|^{2p}.$$ 

They satisfy the local conservation laws

$$\partial_t \rho_{gKdV} + \partial_x \rho_{gKdV} = \partial_x j_{gKdV},$$

$$\partial_t e_{gKdV} + \partial_x e_{gKdV} = \partial_x k_{gKdV}.$$
and we have mass and energy conservation laws,
\[ M_{\text{gKdV}}(u) := \int \rho_{\text{gKdV}} \quad \text{and} \quad E_{\text{gKdV}}(u) := \int e_{\text{gKdV}}. \]

We consider the center of mass and energy \( \langle x \rangle_M \) and \( \langle x \rangle_E \):
\[ \langle x \rangle_M := \frac{1}{M_{\text{gKdV}}(u)} \int x \rho_{\text{gKdV}} \quad \text{and} \quad \langle x \rangle_E := \frac{1}{E_{\text{gKdV}}(u)} \int x e_{\text{gKdV}}, \]
whenever the integrals are defined. We then have monotonicity formulae
\[ -\partial_t \langle x \rangle_M = \frac{1}{M_{\text{gKdV}}(u)} \int j_{\text{gKdV}} \gtrsim E_{\text{gKdV}}(u), \quad M_{\text{gKdV}}(u), \]
\[ -\partial_t \langle x \rangle_E = \frac{1}{E_{\text{gKdV}}(u)} \int k_{\text{gKdV}} > 0, \]

The formulae (2.3) and (2.4) tell that wave packets move to the left. From sign definiteness of the group velocity \(-3\xi^2\), we expect a bit more. Indeed, \( \langle x \rangle_E \) moves faster than \( \langle x \rangle_M \) because \( \langle x \rangle_E \) is more weighted on higher frequencies.

Tao \cite{24} obtained the following refined monotonicity (where we drop the subscript gKdV)
\[ \partial_t (\langle x \rangle_M - \langle x \rangle_E) \gtrsim \frac{M(u)}{E(u)} \left( \int |u|^{p+1} \right)^2. \]

Equivalently, this phenomenon of separation of mass center and energy center can be rewritten in an interaction form:
\[ \partial_t \int_{\mathbb{R} \times \mathbb{R}} (x - y)\rho(t,x)e(t,y)dxdy \]
\[ = \int e(t,y)dy \left( \partial_t \int x\rho(t,x)dx \right) - \int \rho(t,x)dx \left( \partial_t \int ye(t,y)dy \right) \]
\[ = M(u)E(u)\partial_t (\langle x \rangle_M - \langle x \rangle_E) \]
\[ \gtrsim M(u)^2 \left( \int |u|^{p+1} \right)^2. \]

Here, we assume good spatial decay of \( u \) to guarantee that integrals are finite. In \cite{6}, Dodson makes use of (2.6) to show the scattering of the defocusing mass-critical \((\text{gKdV})\) flow\footnote{In fact, the lower bound of (2.4) depends on \( \|u\|_{H^2} \), but it is not conserved under the flow.}. In practice, since one cannot assume that \( u \) has a good spatial decay, one uses a localized version of (2.6) replacing \( x - y \) by \( \Phi(x - y) \) where \( \Phi \) is a truncated version of \( x - y \).

We remark that the monotonicity formulae (2.3) and (2.4) are not sufficient to prove scattering. In order to show scattering, we should somehow preclude soliton-like solutions, which preserve their profile but move sufficiently fast to the left.

In the study of \((\text{gBO})\), we expect the similar unidirectional propagation as like \((\text{gKdV})\). In fact, a wave packet with frequency \( \xi \) propagates with group velocity \( 2|\xi| \) under the linear Benjamin-Ono flow. It is natural to \footnote{Dodson uses it to preclude soliton-like enemies.}
expect that the center of energy moves to the right faster than the center of mass. It turns out that Tao’s monotonicity holds in \((\text{gBO})\). Henceforth, we focus on obtaining analogous monotonicity formula of interaction form.

We define mass density \(\rho\) and energy density \(e\) of \(u\) by

\[
\rho[u] := u^2, \\
e[u] := \frac{1}{2} u \mathcal{H} u_x + \frac{1}{k + 2} u^{k+2}.
\]

We have mass and energy conservation laws

\[
M(u) := \int \rho \quad \text{and} \quad E(u) := \int e.
\]

We define mass current \(j\) and energy current \(k\) of \(u\) by

\[
j[u] := 2 u \mathcal{H} u_x + \frac{2(k + 1)}{k + 2} u^{k+2}, \\
k[u] := u_x^2 + \frac{3}{2} u^{k+1} \mathcal{H} u_x + \frac{1}{2} u^{2k+2}.
\]

They satisfy

\[
(2.7) \quad \partial_t \int x \rho = \int j \quad \text{and} \quad \partial_t \int x e = \int k.
\]

These namings come from analogy with the ones of \((\text{gKdV})\).

**Remark 2.1.** It is worth noticing that we do not have pointwise nonnegativity of \(e, j,\) and \(k\) now. This fact will cause problems when we estimate errors arising from localizing monotonicity formula. For instance, we cannot say that \(\|e[u]\|_{L^1}\) is equal to \(E(u)\), or even estimated in terms of \(\|u\|_{H^{\frac{1}{2}}}^2\). This is in contrast to the case of \((\text{gKdV})\), where we have \(\|e_{\text{gKdV}}[u]\|_{L^1} = E_{\text{gKdV}}(u)\).

We will come back to this issue in Section 5.

**Remark 2.2.** We are not sure whether local conservations laws such as \((2.1)\) and \((2.2)\) hold for \((\text{gBO})\). In the derivation what follows, we only use \((2.7)\). Note that when we prove mass/energy conservation, it suffices to use integration by parts and properties of the Hilbert transform.

Let us now state and prove the monotonicity formula for \((\text{gBO})\).

**Proposition 2.3** (Monotonicity formula). Let \(p \geq \sqrt{2}\) and \(u \in H^1\). Then, we have

\[
\int \rho \int k - \int j \int e \geq \frac{k^2}{2(k + 2)^2} M(u)^2 \left( \int u^{k+2} \right)^2.
\]

If \(u(t, x)\) is a classical solution to \((\text{gBO})\) satisfying, for example, \(\langle x \rangle u \in C_{t, \text{loc}} H^1\) then we have

\[
(2.8) \quad \partial_t \int_{\mathbb{R} \times \mathbb{R}} (y - x) \rho(x) e(y) dx dy \geq \frac{k^2}{2(k + 2)^2} M(u)^2 \left( \int u^{k+2} \right)^2.
\]

In particular,

\[
\partial_t (\langle x \rangle E - \langle x \rangle M) > 0.
\]

\[\footnote{In Section 5, we use a truncated version of the monotonicity formula instead of (2.8). Thus we do not need additional assumption on spatial decay of \(a\) for our later analysis.}
Proof. We closely follow Tao’s argument, but we encounter a technical difficulty. We will explain the difference on the way.

Note that the second and third assertion follow from the first assertion using mass and energy conservation and (2.7). The first assertion is elaborated as

\[
\left( \int u^2 \right) \left( \int u_x^2 + \frac{3}{2} \int u^{k+1} \mathcal{H} u_x + \frac{1}{2} \int u^{2k+2} \right) \\
- \left( 2 \int u \mathcal{H} u_x + \frac{2(k+1)}{k+2} \int u^{k+2} \right) \left( \frac{1}{2} \int u \mathcal{H} u_x + \frac{1}{k+2} \int u^{k+2} \right) \\
\geq \frac{k^2}{2(k+2)^2} \left( \int u^{k+2} \int u^2 \right)^2.
\]

Set real numbers \(a, b, q, r, s\) such that

\[
a^2 M(u) = \int u_x^2, \quad aq M(u) = \int u \mathcal{H} u_x,
\]
\[
b^2 M(u) = \int u^{2p}, \quad br M(u) = \int u^{k+2}, \quad abs M(u) = \int u^{k+1} \mathcal{H} u_x.
\]

It then suffices to show that

\[
a^2 (1 - q^2) + ab \left( \frac{3}{2} s - \frac{k+3}{k+2} qr \right) + b^2 \left( 1 - \frac{4(k+1)}{(k+2)^2} r^2 \right) \geq \frac{k^2}{2(k+2)^2} b^2 r^2,
\]

or equivalently,

\[
a^2 (1 - q^2) + ab \left( \frac{3}{2} s - \frac{k+3}{k+2} qr \right) + b^2 \left( 1 - r^2 \right) \geq 0.
\]

It is obvious that \(q\) and \(r\) are positive. However, it is not trivial whether \(s\) is positive or not. For a moment, we assume \(s > 0\) and proceed to complete the proof. Then, we will provide a proof of \(s > 0\) in Lemma 2.6.

Remark 2.4. In case of (gKdV), \(s\) corresponds to \(\int p |u|^{p-1} u^2_x\). So the positivity of \(s\) is obvious.

Lemma 2.5. The real symmetric matrix

\[
\begin{pmatrix}
1 & q & r \\
q & 1 & s \\
r & s & 1
\end{pmatrix}
\]

is positive semi-definite.

Proof. For any real numbers \(\alpha, \beta, \gamma\), a computation shows that

\[
\begin{pmatrix}
\gamma & \alpha & \beta \\
\alpha & 1 & s \\
\beta & s & 1
\end{pmatrix}
\begin{pmatrix}
1 & q & r \\
q & 1 & s \\
r & s & 1
\end{pmatrix}
\begin{pmatrix}
\gamma \\
\alpha \\
\beta
\end{pmatrix}
\]

\[
= \gamma^2 + \alpha^2 + \beta^2 + 2\gamma \alpha q + 2\gamma \beta r + 2\alpha \beta s
\]

\[
= \frac{1}{M(u)} \left( \int \gamma u + \frac{\alpha}{a} \mathcal{H} u_x + \frac{\beta}{b} u^{k+1} \right)^2
\]

is always nonnegative. \(\square\)
Taking determinants and minors, we have
\[ 0 < q, r, s \leq 1 \quad \text{and} \quad 1 - q^2 - r^2 - s^2 + 2qrs \geq 0. \]
Using discriminants, it suffices to show that
\[ \frac{k + 3}{k + 2} q r - \frac{3}{2} s \leq \sqrt{2 (1 - q^2) (1 - r^2)}. \]
Because \( k \geq 4 \), we have \( \frac{k + 3}{k + 2} \leq \sqrt{2} \). As \( qr \) is positive, it reduces to
\[ s \geq \frac{2 \sqrt{2}}{3} \left[ qr - \sqrt{(1 - q^2)(1 - r^2)} \right]. \]
On the other hand, we know \( (s - qr)^2 \leq (1 - q^2)(1 - r^2) \) by positive-definiteness of the matrix. This yields
\[ s \geq qr - \sqrt{(1 - q^2)(1 - r^2)}. \]
Hence, assuming that \( s \) is positive, this completes the proof.

The rest of this section is to show that \( s \) is positive. In other words, it suffices to show that
\[ \frac{2}{k + 1} \int_{\mathbb{R}} u^{k+1} \mathcal{H} u_x > 0. \]
It seems not easy to prove (2.9) directly. Indeed, we were not able to prove (2.9) using Fourier expressions or integration by parts.

The key observation is as follows. Using the transference principle between the Fourier series and transform, we change the problem defined on the torus \( \mathbb{T} \). We then use density argument to further reduce to the finite-dimensional setting. More precisely, we can treat (2.9) as a functional defined on some finite-dimensional Hilbert space. Moreover, as (2.9) is homogeneous in \( u \), it suffices to restrict ourselves on the finite-dimensional sphere, which is compact. We then use the Lagrange multiplier method and the relations satisfied by a minimizer. It turns out that (2.9) for a minimizer is positive.

For the later use in the truncated monotonicity formula, we need more general positivity lemma. (2.9), a direct consequence of substituting \( \chi = 1 \) into the following lemma (technically, one should mimic the proof).

**Lemma 2.6 (Positivity of \( s \)).** For \( \chi \in C_c^\infty \) and \( u \in H^1 \), we have
\[ \int_{\mathbb{R}} \chi^2 u^{k+1} \mathcal{H} u_x > -c \| \partial_x (\chi^2) \|_{L^\infty} \| u \|_{H^{\alpha}}^{k+2} \]
where \( \alpha = \frac{1}{2} - \frac{2}{k+2} \) and \( c \) is some implicit constant.

**Proof.** We observe that both \( u \) and \( \mathcal{H} u \) cannot be localized simultaneously in the physical space. So we are not able to directly use the density argument to reduce for \( u \) and \( \mathcal{H} u \in C_c^\infty \). Thus we localize them in the Fourier space instead. Lastly, we use the transference principle to finish the proof on the real line.

We first show our assertion on the torus \( \mathbb{T} \). More precisely, we show that
\[ \int_{\mathbb{T}} \chi^2 u^{k+1} D_x u > -c \| \partial_x (\chi^2) \|_{L^\infty(\mathbb{T})} \| u \|_{H^{\alpha}(\mathbb{T})}^{k+2}. \]
for all $u \in C^\infty(\mathbb{T})$ with $\hat{u}(0) = 0$. By density, we may assume that $\hat{u}$ is compactly supported. We consider a finite-dimensional Hilbert space $\mathcal{H}_N$ and a sphere $S$ in $\mathcal{H}_N$ as follows.

\[ \mathcal{H}_N := \{ u \in L^2(\mathbb{T}) : |\hat{u}(\xi)| = 0 \text{ if } \xi = 0 \text{ or } |\xi| > N \}, \]

\[ S := \{ u \in \mathcal{H}_N : \|u\|_{H^\alpha(\mathbb{T})} = 1 \}. \]

Compactness of $S$ in $\mathcal{H}_N$ will play a crucial role in what follows. Let us define a function $f : \mathcal{H}_N \to \mathbb{R}$ by

\[ f(u) := \int_{\mathbb{T}} \chi^2 u^{k+1} D_x u. \]

Then $f : \mathcal{H}_N \to \mathbb{R}$ is smooth and

\[ \nabla f(u) = Q_{\leq N}[(k+1)\chi^2 u^k D_x u + D_x(\chi^2 u^{k+1})] \]

with respect to the usual $L^2(\mathbb{T})$ inner product. On the other hand, the constraint function $g(u) := \|u\|_{H^\alpha(\mathbb{T})}$ satisfies

\[ \nabla g(u) = 2D_x^2 u. \]

Let $u_0$ be a point in $S$ that attains the minimum of $f$. By the Lagrange multiplier theorem, we have

\[ \nabla f(u_0) = \lambda D_x^2 u_0 \]

for some $\lambda \in \mathbb{R}$. In a spirit of Pohozaev identities, we compute

\[ \lambda = \langle \lambda D_x^2 u_0, u_0 \rangle = \langle \nabla f(u_0), u_0 \rangle = (k+2)f(u_0) \]

and

\[ \lambda \langle D_x^2 u_0, D_x u_0 \rangle = \langle \nabla f(u_0), D_x u_0 \rangle \]

\[ = (k+1) \int \chi^2 u_0^k[(D_x u_0)^2 + (\partial_x u_0)^2] - \frac{1}{k+2} \int \partial_xx(\chi^2)u_0^{k+2}. \]

Therefore,

\[ \lambda > \frac{1}{(k+2)^2} \|\partial_xx(\chi^2)\|_{L^\infty(\mathbb{T})} \|u_0\|^{k+2}_{L^{k+2}(\mathbb{T})} \|u_0\|^{k+2}_{H^{\alpha+\frac{1}{2}}(\mathbb{T})}. \]

Because $\|u_0\|_{H^{\alpha}} = 1$ and $\|u\|^{k+2}_{L^{k+2}} \lesssim \|u\|^2_{H^{\alpha+\frac{1}{2}}} \|u\|^k_{H^{\alpha}}$ for any $u \in C^\infty(\mathbb{T})$ having mean zero (see [2]), we have

\[ f(u_0) > -c\|\partial_xx(\chi^2)\|_{L^\infty(\mathbb{T})} \|u_0\|^{k+2}_{H^{\alpha}(\mathbb{T})} \]

for some $c > 0$. Therefore, we have

\[ \int_{\mathbb{T}} \chi^2 u^{k+1} D_x u > -c\|\partial_xx(\chi^2)\|_{L^\infty(\mathbb{T})} \|u\|^{k+2}_{H^{\alpha}(\mathbb{T})} \]

proving the claim on the torus $\mathbb{T}$.

We now show how to transfer the result on the torus $\mathbb{T}$ to that on the real line $\mathbb{R}$. Let $u$ be a function in $H^1(\mathbb{R})$. By density and scaling, we may assume that $\hat{\chi}$ and $\hat{u}$ are compactly supported on $[0,1]$ and continuous. Observe by Parseval’s identity that

\[ \int_{\mathbb{R}} \chi^2 u^{k+1} D_x u = \int_{\eta_1, \eta_2, \ldots, \xi_{k+1} \in [0,1]} \hat{\chi}(\eta_1) \hat{\chi}(\eta_2) \hat{\tilde{u}}(\xi_1) \cdots \hat{\tilde{u}}(\xi_{k+1}) |\xi|^k |\hat{\tilde{u}}(\xi) |\]
where \( \xi := \eta_1 + \eta_2 + \xi_1 + \cdots + \xi_{k+1} \). In light of Riemann sum, the right hand side in the above is expressed as

\[
\lim_{N \to \infty} \frac{1}{N^{k+4}} \sum_{\eta, \eta_2, \xi_1, \ldots, \xi_{k+1} \in A_N} \hat{\chi}(\frac{\eta}{N}) \hat{\chi}(\frac{\eta_2}{N}) \hat{\xi}(\frac{\xi_1}{N}) \cdots \hat{\xi}(\frac{\xi_{k+1}}{N}) \xi |\xi| \hat{u}(\frac{\xi}{N})
\]

where \( \xi = \eta_1 + \eta_2 + \xi_1 + \cdots + \xi_{k+1} \) and \( A_N = \{-N, -N+1, \ldots, -1, 1, \ldots, N-1, N\} \). Notice that \( A_N \) does not contain 0. The above display now equals

\[
\lim_{N \to \infty} \frac{1}{N^{k+4}} \int_T \chi_N u_{N}^{k+1} D_x u_N
\]

where \( \chi_N \) and \( u_N \) are functions defined on the torus \( T \) by

\[
\chi_N(x) := \sum_{\eta \in A_N} \hat{\chi}(\frac{\eta}{N}) e^{2\pi i \eta x} \quad \text{and} \quad u_N(x) := \sum_{\xi \in A_N} \hat{u}(\frac{\xi}{N}) e^{2\pi i \xi x}.
\]

Applying the result on the torus case, we have

\[
\int_R \chi^2 u^{k+1} H u_x > -c \limsup_{N \to \infty} \frac{1}{N^{k+4}} \| \partial_x \chi^2 u_N^{k+2} \|_{L^\infty(T)} \| u_N \|_{H^\alpha(T)}^{k+2}.
\]

In view of

\[
\lim_{N \to \infty} \frac{1}{N^{k+4}} \| \partial_x \chi^2 u_N^{k+2} \|_{L^\infty(T)} = \| \partial_x \chi^2 u \|_{L^\infty(R)}^{k+2},
\]

we conclude

\[
\int_R \chi^2 u^{k+1} H u_x > -c \| \partial_x \chi^2 u \|_{L^\infty(R)} \| u \|_{H^\alpha(R)}^{k+2}.
\]

\[\square\]

### 3. Well-posedness Theory

One of the main ingredients of compactness-contradiction argument is the perturbation theory. This is basically inherited from local well-posedness of the Cauchy problem. As like other canonical nonlinear dispersive equations, \((\text{gBO})\)’s local theory is based on local smoothing estimates. A satisfactory critical well-posedness was obtained by Vento \[27\]. Due to a full derivative in the nonlinearity, the local theory is more delicate than other equations such as nonlinear Schrödinger equations and generalized Korteweg-de Vries equation. In this section, we begin reviewing Vento’s proof of the critical local well-posedness. We then prove subcritical local theory and global well-posedness.

Denote by \( V(t) \) the linear propagator associated to the linear Benjamin-Ono flow. By Duhamel’s formula, a nonlinear solution \( u \) to initial data \( u_0 \) satisfies

\[
u(t) = V(t)u_0 + \int_0^t V(t-s)F(u(s))ds
\]

where \( F(u) = -\partial_x (u^{k+1}) \) is the nonlinearity of \((\text{gBO})\). Using the local smoothing estimates, one has

\[
\|V(t)u_0\|_{s} \lesssim \|u_0\|_{H^s},
\]

where \( s \) is determined by \( \alpha \) in the above formula.
where $\mathcal{S}$ is some space where the linear evolution lies and $\dot{H}^s$ is a Sobolev space where initial data lies. Usually, combined with Christ-Kiselev lemma, the nonlinear evolution part can be estimated by

$$\left\| \int_0^t V(t - s) F(s) ds \right\|_{\mathcal{S}} \lesssim \| F \|_{\mathcal{N}}$$

where $\mathcal{N}$ is the dual space of $\mathcal{S}$. The main goal of the local theory is to find suitable function spaces $\mathcal{S}$ and $\mathcal{N}$ such that a nonlinear estimate

$$\| F(u) \|_{\mathcal{N}} \lesssim \| u \|_{\mathcal{S}}^{k+1}$$

holds and $\mathcal{S}$ is embedded into $C_T \dot{H}^s$.

In Section 3.1, we review the critical local theory of $\text{(gBO)}$ by Vento [27]. In Section 3.2, we prove subcritical local well-posedness in $\dot{H}^s (s > s_k)$, and deduce global well-posedness in $\dot{H}^s (s \geq \frac{1}{2})$ by using conservation laws.

3.1. **Review of Vento’s argument.** In this subsection, we review Vento’s approach [27] in proving critical local well-posedness of $\text{(gBO)}$. Because our nonlinearity $F(u) = -\partial_x(u^{k+1})$ has one derivative, we should somehow recover one derivative. More precisely, in view of Duhamel’s formula, we want to estimate the integral part as

$$\left\| \int_0^t V(t - s) F(u(s)) ds \right\|_{\mathcal{S}} \lesssim \| \partial_x(u^{k+1}) \|_{\mathcal{N}} \lesssim \| u \|_{\mathcal{S}}^{k+1}.$$

In the linear Benjamin-Ono equation, it is well-known that local smoothing estimates can recover at most half derivative:

$$\| D_x^{s_k + \frac{1}{2}} V(t) u_0 \|_{L^\infty_x L^2_t} \lesssim \| u_0 \|_{\dot{H}^{s_k}}.$$ 

In a spirit of $TT^*$ formulation, one can recover the full derivative only when we use $L^\infty_x L^2_t$ norm for $\mathcal{S}$ and $L^1_x L^2_t$ norm for $\mathcal{N}$. However, in order to bound $L^1_x L^2_t$ norm by $L^\infty_x L^2_t$ (and other norms), we are forced to use maximal-type norms, $L^p_x L^{\infty}_t$. A naive choice would be as follows

$$\left\| D_x^{s_k - \frac{1}{2}} \partial_x(u^{k+1}) \right\|_{L^1_x L^2_t} \lesssim \left\| D_x^{s_k + \frac{1}{2}} u \right\|_{L^\infty_x L^2_t} \| u \|_{L^k_x L^\infty_t}^{\frac{k}{k-1}}.$$

However, for large data, since one takes $L^\infty_T$, it is not possible to make small $L^k_x L^\infty_t$ by shrinking $T$ small. This prohibits us from running contraction argument.

Vento [27] overcomes this difficulty by distorting the linear propagator. Consider a general form of a nonlinear evolution equation

$$\partial_t u + Lu = F(u)$$

for some linear operator $L$. In a spirit of paraproduct decomposition, we extract the strong interaction term $\mathcal{N}_1(u)$ of $F(u)$, for which one cannot establish a desired nonlinear estimate, and rewrite the equation by

$$\partial_t u + Lv - \mathcal{N}_1(u) = \mathcal{N}_2(u),$$

where $\mathcal{N}_2(u) := F(u) - \mathcal{N}_1(u)$. The strong interaction term $\mathcal{N}_1(u)$ is nonlinear in $u$, we approximate it by some linear operator $u \mapsto \mathcal{N}_1(u_0; u)$ using the initial data $u_0$. We then have the distorted equation

$$\partial_t u + \tilde{L}v = [\mathcal{N}_1(u) - \mathcal{N}_1(u_0; u)] + \mathcal{N}_2(u),$$
where $\tilde{\mathcal{L}} := \mathcal{L} - N_1(u_0)$. It turns out that $N_1(u_0; u)$ contains a strong low-high interaction, $u^k_{low} \partial_x u_{high}$. This phenomenon is universal in other nonlinear dispersive equations containing derivative nonlinearities such as KP-I, Benjamin-Ono, higher order KdV, and so on.

Vento’s observations are as follows. At first, the linear part to the above distorted equation still admits analogous linear estimates for the linear Benjamin-Ono equation, at least for short times. Second, even though we cannot avoid usage of $L_x^kL_t^{\infty}$, we have small $\|u - u_0\|_{L_x^kL_t^{\infty}}$ at least for short times. Indeed, this holds when $u$ is an exact solution to either Benjamin-Ono equation or distorted linear solution. This fact is well-exploited in $N_1(u) - N_1(u_0; u)$ estimate. Finally the good term $N_2(u)$, as its naming suggests, can be estimated small by shrinking $T$.

We first start with linear estimates for the linear Benjamin-Ono flow. A triplet $(\alpha, p, q) \in \mathbb{R}^3$ is said to be admissible if $(\alpha, p, q) = \left(\frac{1}{2}, \infty, 2\right)$ or

$$4 \leq p < \infty, \quad 2 < q \leq \infty, \quad \frac{2}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad \alpha = \frac{1}{p} + \frac{2}{q} - \frac{1}{2},$$

We then have linear estimates for these admissible triples as follows.

**Lemma 3.1.** Let $(\alpha, p, q)$ and $(\tilde{\alpha}, \tilde{p}, \tilde{q})$ be admissible triplets. We then have

$$\|D_x^\alpha V(t)\varphi\|_{L_x^pL_t^q} \lesssim \|\varphi\|_{L_x^2},$$

$$\left\|D_x^\alpha \int V(-s)G\right\|_{L_x^pL_t^{\tilde{q}'}} \lesssim \|G\|_{L_x^pL_t^{\tilde{q}'}}$$

$$\left\|D_x^{\alpha + \tilde{\alpha}} \int_0^t V(t-s)G(s)ds\right\|_{L_x^pL_t^{\tilde{q}'}} \lesssim \|G\|_{L_x^pL_t^{\tilde{q}'}}$$

where $\tilde{p}'$ and $\tilde{q}'$ are conjugate Lebesgue exponents of $\tilde{p}$ and $\tilde{q}$, respectively.

**Proof.** From [14] Theorem 2.5 and 4.1, one has

$$\|D_x^{\frac{1}{2}} V(t)\varphi\|_{L_x^pL_t^q} \sim \|\varphi\|_{L_x^2} \quad \text{and} \quad \|V(t)\varphi\|_{L_x^pL_t^q} \lesssim \|D_x^{\frac{1}{2}} \varphi\|_{L_x^2}.$$ 

We then apply Stein-Weiss interpolation to obtain the first estimate. The second estimate easily follows by duality. For the last one, we have by first two estimates that

$$\left\|D_x^{\alpha + \tilde{\alpha}} \int_0^t V(t-s)G(s)ds\right\|_{L_x^pL_t^{\tilde{q}'}} \lesssim \|G\|_{L_x^pL_t^{\tilde{q}'}}.$$ 

By the Christ-Kiselev lemma for reversed norm [3] Theorem B, we have

$$\left\|D_x^{\alpha + \tilde{\alpha}} \int_0^t V(t-s)G(s)ds\right\|_{L_x^pL_t^{\tilde{q}'}} \lesssim \|G\|_{L_x^pL_t^{\tilde{q}'}}$$

except the case where $(\alpha, p, q) = (\tilde{\alpha}, \tilde{p}, \tilde{q}) = \left(\frac{1}{2}, \infty, 2\right)$. In that case, we refer to [15] Theorem 2.1.}

We use Besov-type function spaces for $s \in \mathbb{R}$

$$\dot{S}^s_{\tilde{\ell}} = B^{s+\frac{3\gamma-1}{1-\gamma}}_{\frac{3}{2}}(L^2_{\tilde{\ell}}), \quad \dot{N}^s_{\tilde{\ell}} = B^{s-\frac{1}{2}}_{1}(L^2_{\tilde{\ell}}),$$
where \( \theta \in [0, 1] \). One can think that the space \( \hat{S}^{s, \theta} \) contains solutions to (gBO) and \( \hat{N}^{s} \) contains the nonlinearity of (gBO). We do not parametrize the space for nonlinearity because we will only consider \( \hat{N}^{s} \).

We define the space \( \hat{X}^{s}_{I} \) by

\[
\hat{X}^{s}_{I} := \hat{S}^{s,0}_{I} \cap \hat{S}^{s,1}_{I}.
\]

The parameter \( 0 < \epsilon \ll 1 \) only depends on \( k \) and is chosen small. For the choice of \( \epsilon \), see [27]. As \( \hat{X}^{s}_{I} \)-norm does not contain \( L^{\infty}_{T} \)-norm, it becomes smaller as \( T \) becomes smaller. This property is crucial in our argument. Of course, solutions to (gBO) should lie in \( C_{T}\hat{H}^{s}_{x} \). For this purpose, we finally work on the Banach space

\[
\hat{Z}^{s}_{I} := \{ u \in C_{I}\hat{H}^{s}_{x} \cap L^{1}_{x}L^{\infty}_{t} \cap \hat{S}^{s,0}_{I} \cap \hat{S}^{s,1}_{I} : u \text{satisfies (3.2) and (3.3)} \}.
\]

where \( (J \subseteq I \text{ is a subinterval of } I) \)

\[
(3.2) \text{ for any } t_{0} \in I, \quad \| u - u(t_{0}) \|_{L^{1}_{x}L^{\infty}_{t}} \to 0 \text{ whenever } t_{0} \in J \text{ and } |J| \to 0,
\]

\[
(3.3) \lim_{T \to +\infty} \| u \|_{L^{1}_{x}L^{\infty}_{[T, +\infty)}} = 0 = \lim_{T \to -\infty} \| u \|_{L^{1}_{x}L^{\infty}_{(-\infty, T)}}.
\]

The inhomogeneous form of the spaces \( X \) and \( Z \) for subcritical well-posedness theory are defined as follows. For \( s > 0 \), we define

\[
X^{s}_{I} := \hat{X}^{s}_{I} \cap \hat{X}^{s}_{I} \quad \text{and} \quad Z^{s}_{I} := \hat{Z}^{s}_{I} \cap \hat{Z}^{s}_{I}.
\]

**Proposition 3.2** (Linear estimates). Let \( k \geq 4 \) and \( s \in \mathbb{R} \). We then have

\[
\| V(t)\varphi \|_{\hat{S}^{s,0}_{I} \cap \hat{S}^{s,1}_{I} \cap L^{\infty}_{x} \hat{H}^{k}_{t}} \lesssim \| \varphi \|_{\hat{H}^{k}},
\]

\[
\| V(t)\varphi \|_{L^{1}_{x}L^{\infty}_{t}} \lesssim \| \varphi \|_{\hat{H}^{k}}.
\]

For retarded estimates, we have

\[
\left\| \int_{0}^{t} V(t - s)G(s)ds \right\|_{\hat{X}^{s}_{I} \cap L^{1}_{x}L^{\infty}_{t} \hat{H}^{k}_{x}} \lesssim \| G \|_{\hat{N}^{s,k}_{I}},
\]

\[
\left\| \int_{0}^{t} V(t - s)G(s)ds \right\|_{L^{1}_{x}L^{\infty}_{t}} \lesssim \| G \|_{\hat{N}^{s,k}_{I}}.
\]

**Proof.** Note that \( \hat{S}^{s,2}_{I} \) is equal to \( \hat{H}^{s} \) by definition. We use the linear estimates (Lemma 3.1) for each frequency piece \( Q_{j}\varphi \) or \( Q_{j}G \) and take \( \ell^{2}_{j} \) summation. For instance, the last estimate follows from

\[
\left\| V(t) \int_{\mathbb{R}} V(-s)G(s)ds \right\|_{L^{1}_{x}L^{\infty}_{t}} \lesssim \left\| \int_{\mathbb{R}} V(-s)G(s)ds \right\|_{\hat{S}^{s,2}_{I}} \lesssim \| G \|_{\hat{N}^{s,k}_{I}}.
\]

We then use Christ-Kiselev lemma [3 Theorem B] to conclude. \( \square \)

We now review the paraproduct decomposition of the nonlinearity \( F(u) = -\partial_{x}(u^{k+1}) \) following [27]. For simplicity of notations, we shall group similar frequency pieces when they satisfy the same estimates and number of them is bounded by a universal constant, not depending on \( J \). For instance, we
group $\sum_{\ell=0}^{k} u_{<r+1}^{k-\ell} u_{<r}^{\ell}$ by $u_{<r+1}^{k}$. We observe that

$$
\partial_x Q_j(u_{k+1}^{j+1}) = \partial_x Q_j \left( \lim_{r \to \infty} u_{<r}^{k+1} \right)
= \partial_x Q_j \left( \sum_{r \in \mathbb{Z}} u_{<r+1}^{k+1} - u_{<r}^{k+1} \right)
= \partial_x Q_j \left( \sum_{r \in \mathbb{Z}} u_{<r+1}^{k} \right)
+ \partial_x Q_j \left( \sum_{r \geq j + C_k} u_{<r}^{k-1} \right)
+ \partial_x Q_j \left( \sum_{|r-j| \leq C_k} u_{<r}^{k} \right)
= \pi_j(u, u) - g_j(u)
$$

where $C_k$ is some natural number only depending on $k$ and

$$
\pi_j(\psi, \phi) := \partial_x Q_j(\psi_{<j}^k \phi_{<j})
$$

$$
g_j(u) := -\partial_x Q_j \left( \sum_{r-j \geq -C_k} u_{<r}^{k-1} \right).
$$

Summing up in $j$, we have

$$
F(u) = -\pi(u, u) + g(u),
$$

where

$$
\pi(\psi, \phi) := \sum_j \pi_j(\psi, \phi),
$$

$$
g(\varphi) := \sum_j g_j(\varphi).
$$

Using the linear estimates, we can estimate each components of the nonlinearity.

**Lemma 3.3** (Nonlinear estimates, [27, Proposition 4.1]). We have

$$
\|\pi(\varphi, \psi)\|_{X_t^{s_k}} \lesssim \|\varphi\|_{L_t^k L_x^\infty} \|\psi\|_{X_t^{s_k}},
\|g(u)\|_{X_t^{s_k}} \lesssim \|u\|_{L_t^k L_x^\infty}^2 \|\psi\|_{X_t^{s_k}},
\|\pi(\varphi_1, \psi) - \pi(\varphi_2, \psi)\|_{X_t^{s_k}} \lesssim \|\varphi_1 - \varphi_2\|_{L_t^k L_x^\infty} (\|\varphi_1\|_{L_t^k L_x^\infty}^{k-1} + \|\varphi_2\|_{L_t^k L_x^\infty}^{k-1}) \|\psi\|_{X_t^{s_k}},
\|g(u_1) - g(u_2)\|_{X_t^{s_k}} \lesssim \|u_1 - u_2\|_{L_t^k L_x^\infty} \|u_1\|_{L_t^k L_x^\infty}^{k-2} + \|u_2\|_{L_t^k L_x^\infty}^{k-2} (\|u_1\|_{X_t^{s_k}} + \|u_2\|_{X_t^{s_k}})^2
+ \|u_1 - u_2\|_{X_t^{s_k}} (\|u_1\|_{L_t^k L_x^\infty}^{k-1} + \|u_2\|_{L_t^k L_x^\infty}^{k-1}) (\|u_1\|_{X_t^{s_k}} + \|u_2\|_{X_t^{s_k}}).
$$

**Proof.** We only prove first two estimates following [27, Proposition 4.1]. The remaining estimates can be shown in a similar manner. For the first
and global well-posedness of (gBO) in $H^s$ subsection, we prove subcritical well-posedness of (gBO) in the propagator $U$. Afterward, we obtain the persistence of regularity and combine with $V$ estimate, observe that the theory by Vento [27] guarantees profile dependent lifespans for $\dot{H}^s$ estimates [27, Proposition 3.2]. To do this, it suffices to obtain smallness of $\varphi$.

For the second estimate, observe that

$$
\|g(u)\|_{X^s_T} \lesssim \left( \sup_{r \geq j} \|u\|_{L^2_{x,t}}^{k-1} \right) \left( \sup_{r \geq j} \|u\|_{L^2_{x,t}} \right) \left( \sum_{r \geq j} \|u\|_{L^2_{x,t}}^{2}\right) \left( \sup_{r \geq j} \|u\|_{L^2_{x,t}} \right).
$$

where we used Young’s inequality in $j$. We then observe that

$$
\left\| \dot{u} \right\|_{L^2_{x,t}}^{2} = \left\| \dot{u} \right\|_{L^2_{x,t}}^{2}.
$$

Using the embedding $\ell^2_j \hookrightarrow \ell^1_j$ and the definition of $X^s_T$ norm, we have the assertion.$\square$

Vento’s proof of critical local well-posedness of (gBO) goes as follows. The paraproduct decomposition allows us to rewrite (gBO) as so called distorted equation

$$
(\partial_t + H\partial_x)u + \pi(u_0, u) = [\pi(u_0, u) - \pi(u, u)] + g(u).
$$

Let us denote by $U(t)$ the linear propagator associated to (3.4). As alluded to above, the propagator $U(t)$ still obeys analogous linear estimates what the propagator $V(t)$ satisfy, at least for short times (see Corollary 3.5 or [27] Proposition 3.2) where $T$ depends on $u_0 \in \dot{H}^s$ instead of $\|u_0\|_{H^s}$. Moreover, further shrinking $T$ if necessary, we may assume that $\|U(t)u_0 - u_0\|_{L^2_{x,t}}$ and $\|u\|_{X^s_T}$ are sufficiently small. Hence, using Duhamel’s formula to (3.4) and Lemma 5.3 to the nonlinearity of (3.4), one can iterate on the space $B_{X^s_T}(0; r) \cap B_{L^2_{x,t}}(u_0; r)$ for sufficiently small $r$.

3.2. Subcritical Local Theory and Global Well-posedness. In this subsection, we prove subcritical well-posedness of (gBO) in $H^s$ with $s > s_k$ and global well-posedness of (gBO) in $H^s$ with $s \geq \frac{1}{2}$. The critical local theory by Vento [27] guarantees profile dependent lifespans for $H^{s_k}$ and $H^s$ $(s \geq s_k)$ solutions, but does not answer the question of subcritical well-posedness. Here, we slightly modify Vento’s argument to obtain $H^s$ $(s > s_k)$ subcritical well-posedness, whose lifespan depends on $H^s$ norm of the initial data. Afterward, we obtain the persistence of regularity and combine with conservation laws to get the global well-posedness in $H^s$ $(s \geq \frac{1}{2})$.

We first obtain a subcritical version (Corollary 3.5) of the distorted linear estimates [27] Proposition 3.2. To do this, it suffices to obtain smallness of

$$
\|V(t)u_0 - u_0\|_{L^2_{x,t}}\quad \text{and} \quad \|D_x^0 V(t)u_0\|_{L^1_{x,t}}.
$$
by choosing \( T \) only depending on \( H^s \) norm of \( u_0 \).

**Lemma 3.4.** Let \( s > s_k \), \( u_0 \in H^s \), and \( \eta > 0 \). Then, there exists \( T = T(\|u_0\|_{H^s}, s, \eta) > 0 \) such that

\[
\|V(t)u_0 - u_0\|_{L^2_t L^\infty_x \cap L^\infty_t H^s_x} + \|D_x^0 V(t)u_0\|_{L^1_t L^\infty_x} < \eta.
\]

**Proof.** We proceed as in [27]. Observe that \( u := V(t)u_0 - u_0 \) solves

\[
(\partial_t + \mathcal{H}\partial_{xx})u = -\mathcal{H}\partial_{xx}u_0,
\]

\( u(0) = 0 \).

By Duhamel’s formula, we have

\[
V(t)Q_{<N}u_0 - Q_{<N}u_0 = -\int_0^t V(t-s)\mathcal{H}\partial_{xx}Q_{<N}u_0 ds
= -\int_0^t V(t)\mathcal{H}\partial_{xx}Q_{<N}u_0 ds.
\]

Therefore, we have

\[
\|V(t)Q_{<N}u_0 - Q_{<N}u_0\|_{L^2_t L^\infty_x \cap L^\infty_t H^s_x} \lesssim T^{2N}\|u_0\|_{H^s}.
\]

On the other hand, by the triangle inequality and linear estimates, we have

\[
\|V(t)Q_{\geq N}u_0 - Q_{\geq N}u_0\|_{L^2_t L^\infty_x \cap L^\infty_t H^s_x} \lesssim N^{-(s-s_k)}\|u_0\|_{H^s}.
\]

Only depending on \( H^s \) norm of \( u_0 \), choose \( N \gg 1 \) large and then \( T > 0 \) small to obtain the first estimate. For the second estimate, use Sobolev embedding in \( x \), Hölder’s inequality in \( t \), and linear estimates to have

\[
\|D_x^0 V(t)u_0\|_{L^1_t L^\infty_x} \lesssim T^{0+}\|u_0\|_{H^s} \lesssim T^{0+}\|u_0\|_{H^s}.
\]

We can choose \( T > 0 \) small to conclude. \( \square \)

**Corollary 3.5** \hspace{1em} (Distorted linear estimates). Let \( s > s_k \). For \( u_0 \in H^s \), there exists \( T = T(\|u_0\|_{H^s}, s) > 0 \) and a nondecreasing polynomial \( p_k \) such that whenever \( u \) is a solution to

\[
(\partial_t + \mathcal{H}\partial_{xx})u + \pi(u_0, u) = \tilde{f}
\]

\( u(0) = \tilde{u}_0 \)

for \( \tilde{u}_0 \in \dot{H}^{s_k} \) and \( \tilde{f} \in \mathcal{N}^{s_k}_T \), we have \( u \in \dot{Z}^{s_k}_T \) with the bound

\[
\|u\|_{\dot{Z}^{s_k}_T} \leq p_k(\|u_0\|_{H^{s_k}})(\|\tilde{u}_0\|_{H^{s_k}} + \|\tilde{f}\|_{\mathcal{N}^{s_k}_T}).
\]

If in addition \( \tilde{u}_0 \in H^s \) and \( \tilde{f} \in \mathcal{N}^{s}_T \), then we have \( u \in Z^s_T \) with the bound

\[
\|u\|_{Z^s_T} \leq p_k(\|u_0\|_{H^{s_k}})(\|\tilde{u}_0\|_{H^s} + \|\tilde{f}\|_{\mathcal{N}^s_T}).
\]

**Proof.** We only prove the first \( \dot{H}^{s_k} \)-critical estimate. The remaining inhomogeneous estimate easily follows by mimicking the proof. We proceed as in [27]. We start from the estimate presented in the proof of [27, Proposition 3.2]. With notations \( u_j = Q_j u \) and \( u_{L, < j} = V(t)u_{0, < j} \) for any fixed \( j \in \mathbb{Z} \), our starting point is

\[
\|u_j\|_{S_j^{s_k, 0}} \leq p_k(\|u_0\|_{H^{s_k}})\{\|\tilde{u}_0,j\|_{H^{s_k}} + \|\tilde{f}_j\|_{\mathcal{N}^{s_k}_T} + (A + B + C + D)\}
\]

for some

\[
A \leq \|u_0\|_{H^{s_k}}
\]

\[
B \leq \|u_0\|_{H^{s_k}}
\]

\[
C \leq \|u_0\|_{H^{s_k}}
\]

\[
D \leq \|u_0\|_{H^{s_k}}
\]
where
\[ A = \| \partial_x (u_0, x_j) \|^k u_j \| \chi^k_T, \]
\[ B = \| (u_0, x_j) \|^k u_j \| \chi^k_T, \]
\[ C = \| \partial_x [Q_j, u_0, x_j - u_L, x_j] u_{-j} \| \chi^k_T, \]
\[ D = \| \partial_x [Q_j, u_0, x_j] u_{-j} \| \chi^k_T. \]

We show that \( A + B \ll \| u_j \| \chi^k_T. \) By Bernstein estimates, we have
\[ \| \partial_x (u_0, x_j) \|^k u_j \| \chi^k_T \lesssim 2^{j (s_k + \frac{1}{2})} 2^{j - 2} \| (u_0, x_j) \|^k \| u_j \|_{L^2_T} \]
\[ \lesssim \frac{\| u_j \|_{L^2_T}}{\| u_0 \|_{L^2_T}} \| u_j \| \chi^k_T \]
and
\[ \| (u_0, x_j) \|^k u_j \| \chi^k_T \lesssim 2^{j (s_k + \frac{1}{2})} \| (u_0, x_j) \|^k \| u_j \|_{L^2_T} \]
\[ \lesssim \| u_j \|_{L^2_T} \| u_j \| \chi^k_T. \]

We choose sufficiently large \( J = J(\| u_0 \|_{L^k_T}) \) to obtain \( A + B \ll \| u_j \| \chi^k_T. \)

Therefore, we have
\[ \| u_j \| \chi^k_T \lesssim p_k(\| u_0 \|_{H^k_T}) \left\{ \| \tilde{u}_0, x_j \|_{H^k_T} + \| \tilde{f}_j \|_{\chi^k_T} + (C + D) \right\}. \]

From now on, we fix \( J \) determined as above and claim that \( C + D \ll \| u_{-j} \| \chi^k_T \) by choosing \( T = T(\| u_0 \|_{H^k_T}) > 0 \) sufficiently small. On the one hand, observe that
\[ C \lesssim 2^{j (s_k + \frac{1}{2})} \| u_0, x_j - u_L, x_j \|_{L^1_T} \| u_{-j} \|_{L^2_T} \]
\[ \lesssim \| u_0 - u_L \|_{L^2_T} \| u_{-j} \|_{\chi^k_T}. \]

On the other hand, by the commutator estimate [3, Lemma 2.4], we obtain
\[ D \lesssim 2^{j (s_k + \frac{1}{2})} \| \partial_x (u_0, x_j) \|^k \| u_{-j} \|_{L^2_T} \]
\[ \lesssim 2^{j (s_k + \frac{1}{2})} \| \partial_x u_0, x_j \|_{L^k_T} \| u_{-j} \|_{L^2_T} \]
\[ \lesssim \| D^+ u_0, x_j \|_{L^k_T} \| u_{-j} \|_{\chi^k_T}. \]

Applying Lemma [3,3] there is \( T = T(\| u_0 \|_{H^k_T}) > 0 \) such that \( C + D \ll \| u_{-j} \| \chi^k_T \).

This yields
\[ \| u_j \| \chi^k_T \lesssim p_k(\| u_0 \|_{H^k_T}) \left\{ \| \tilde{u}_0, x_j \|_{H^k_T} + \| \tilde{f}_j \|_{\chi^k_T} \right\}. \]

Taking \( L^2_T \) summation proves the assertion for \( \chi^k_T \) norm.

In order to estimate \( \tilde{Z}^k_T \) norm of \( u \), observe that \( u \) satisfies
\[ (\partial_t + H \partial_{xt}) u = -\pi (u_0, u) + \tilde{f}. \]

From Duhamel’s formula and Lemma [3,3] we obtain
\[ \| u \|_{L^2_T L^2_T \cap L^2_T \tilde{Z}^k_T} \lesssim \| V(t) \tilde{u}_0 \|_{L^2_T L^2_T \cap L^2_T \tilde{Z}^k_T} + \| u_0 \|_{L^2_T} \| u \|_{\chi^k_T} + \| \tilde{f} \|_{\chi^k_T} \]
\[ \lesssim p_k(\| u_0 \|_{H^k_T}) \left\{ \| \tilde{u}_0 \|_{H^k_T} + \| \tilde{f} \|_{\chi^k_T} \right\}. \]
To check the continuity condition (3.2), consider the estimate
\[ \|u - u(t_0)\|_{L^2_T L^\infty_x H^s_x} \]
\[ \lesssim \|V(t_0)u(t_0) - u(t_0)\|_{L^2_T L^\infty_x H^s_x} + \|\pi(u_0, u)\|_{N^s_I} + \|\tilde{f}\|_{N^s_I} \]
for any interval \( I \subset [-T, T] \) with \( t_0 \in I \). We then use Lemma 3.4 for the first term and Lebesgue’s DCT for the remaining terms.

\[ \square \]

**Remark 3.6.** If one merely assumes that \( u_0 \in H^{s_k} \) in Lemma 3.4 then \( T \) depends on the profile of \( u_0 \). Hence, the corresponding distorted estimate is only valid on the time interval whose length depends on the profile of \( u_0 \). More precisely, one has the following. Let \( u_0 \in H^{s_k} \) and \( \eta > 0 \) be fixed.

Then, there exists \( T = T(u_0, \eta) > 0 \) and \( r = r(u_0, \eta) > 0 \) such that
\[ \|V(t)u_0 - u_0\|_{L^2_T L^\infty_x H^{s_k}_x} + \|D_x^0 + V(t)u_0\|_{L^2_T L^\infty_x H^{s_k}_x} < \eta \]
holds for \( u_0 \in H^{s_k} \) with \( \|u_0 - u_0\|_{H^{s_k}} < r \). As a direct consequence, the distorted estimate for \( u_0 \in H^{s_k} \) holds with \( T = T(u_0) > 0 \). See [27] for details.

We now turn into obtaining \( H^s (s > s_k) \) norm dependent lifespan of a solution. Denote by \( U(t)u_0 \) the solution to the equation
\[ \partial_t u + \mathcal{H}\partial_x u + \pi(u_0, u) = 0, \]
\[ u(0) = u_0. \]

Following the argument in [27] Section 4.2, we should obtain smallness of
\[ \|U(t)u_0\|_{X_{T}^{s_k}} \]
and
\[ \|U(t)u_0 - u_0\|_{L^2_T L^\infty_x H^{s_k}_x} \]
by choosing \( T = T(\|u_0\|_{H^{s_k}} > 0 \)

**Lemma 3.7.** Let \( s > s_k, u_0 \in H^s \), and \( \eta > 0 \). Then, there exists \( T = T(\|u_0\|_{H^s}, s, \eta) > 0 \) such that
\[ \|U(t)u_0\|_{X_{T}^{s_k}} + \|U(t)u_0 - u_0\|_{L^2_T L^\infty_x H^{s_k}_x} < \eta. \]

**Proof.** Recall that \( X_{T}^{s_k} \) norm does not contain \( L^\infty_T \) component. Hence, we can use Sobolev embedding in space and Hölder’s inequality in time to have
\[ \|U(t)u_0\|_{X_{T}^{s_k}} \lesssim T^{0^+}\|U(t)u_0\|_{s^{s_k} + 1 + \gamma} \leq T^{0^+}p_k(\|u_0\|_{H^{s_k}})\|u_0\|_{H^{s_k}}. \]

Possibly taking \( T \) much smaller (only depending on \( \eta \) and \( H^s \) norm of \( u_0 \)), we have
\[ \|U(t)u_0\|_{X_{T}^{s_k}} < \eta, \]
which is the first assertion. In order to show the second assertion, we observe that
\[ \|U(t)u_0 - u_0\|_{L^2_T L^\infty_x H^{s_k}_x} \]
\[ \lesssim \|V(t)u_0 - u_0\|_{L^2_T L^\infty_x H^{s_k}_x} + \|U(t) - V(t)\|_{L^2_T L^\infty_x H^{s_k}_x} \]
\[ \lesssim \|V(t)u_0 - u_0\|_{L^2_T L^\infty_x H^{s_k}_x} + \|u_0\|_{L^2_T L^\infty_x H^{s_k}_x}. \]

Use the first assertion and Lemma 3.4.

\[ \square \]
Corollary 3.8. Let $u_0$ be a initial data in $H^s$. Then, there exists $T = T(\|u_0\|_{H^s}, s) > 0$ such that $u_0$ admits a unique $\dot{H}^{s_k}$-solution $u$ defined on $[0, T]$.

Proof. Consider the operator

$$[\Phi u](t) = U(t)u_0 + \int_0^t U(t - t')f(u(t'))dt',$$

where $f(u) = \pi(u_0, u) - \pi(u, u) + g(u)$. We will iterate in the space

$$B_{X_T^s}(0; \eta) \cap B_{L_T^\infty L_T^\infty}(u_0; \eta) \cap B_{L_T^\infty \dot{H}^{s_k}}(u_0; \eta)$$

with $0 < \eta \ll \min\{1, \|u_0\|_{\dot{H}^{s_k}}\}$ chosen later. For convenience, denote

$$\|u\|_Y := \|u\|_{X_T^s} + \|u - u_0\|_{L_T^\infty L_T^\infty} + \|u - u_0\|_{L_T^\infty \dot{H}^{s_k}}.$$

Taking $T = T(\|u_0\|_{H^s}, s, \eta) > 0$ sufficiently small, combining with Corollary 3.5 with Lemma 3.3 there exists $C = C(\|u_0\|_{\dot{H}^{s_k}})$ such that

$$\|\Phi u\|_Y \leq \|U(t)u_0\|_Y + C\|u\|_Y^2.$$

On the other hand, since

$$\|\Phi u_1 - \Phi u_2\|_{Z_T^{s_k}} \leq C\|f(u_1) - f(u_2)\|_{X_T^{s_k}}$$

with

$$f(u_1) - f(u_2) = [\pi(u_0, u_1 - u_2) - \pi(u_1, u_1 - u_2)] - [\pi(u_1, u_2) - \pi(u_2, u_2)] + [g(u_1) - g(u_2)],$$

we have

$$\|\Phi u_1 - \Phi u_2\|_{Z_T^{s_k}} \leq C\|u_1 - u_2\|_{Z_T^{s_k}}(\|u_1\|_{X_T^{s_k}} + \|u_2\|_{X_T^{s_k}}).$$

Therefore, we choose $\eta > 0$ small and then $T = T(\|u_0\|_{H^s}, s) > 0$ much smaller (to guarantee smallness of $\|U(t)u_0\|_Y$ by Lemma 3.7), we see that $\Phi$ becomes a well-defined contraction on $B_Y(\eta)$. By a fixed-point procedure, we can find a solution in $Z_T^{s_k}$. The remaining part of the proof is standard. \qed

So far, if the initial data $u_0$ lies in $H^s$ with $s > s_k$, we have constructed $\dot{H}^{s_k}$ solution with the lifespan only depending on $\|u_0\|_{H^s}$. It turns out that this $\dot{H}^{s_k}$ solution can be upgraded to the $H^s$ solution.

Proposition 3.9 (Persistence of regularity). Let $s > s_k$. Suppose that we have a $H^{s_k}$ solution $u \in \dot{Z}_T^{s_k}$ for some compact time interval $I$ with $u(t_0) \in H^s$ for some $t_0 \in I$. Then, $u \in Z_I^s$.

Proof. By time translation and reversing symmetries, we may assume that $I = [0, T]$ and $t_0 = 0$. Let $u_0 := u(t_0) \in H^s$. Applying the critical well-posedness (Theorem 1.1), we can construct $H^s$ solution $\tilde{u}$ with initial data $u_0 \in H^s$. Suppose that this $H^s$ solution $\tilde{u}$ is defined on $[0, T_1)$ where $T_1 > 0$ is chosen to be maximal. If $T_1 > T$, then $\tilde{u} = u$ on $[0, T]$ by uniqueness, so the conclusion follows easily.

We now consider the case when $T_1 \leq T$. In this case, uniqueness only tells us that $\tilde{u} = u$ on $[0, T_1]$ and we do not know whether $\tilde{u}$ can be defined at time $T_1$ or not. Due to continuity of $t \mapsto \tilde{u}(t) \in \dot{H}^{s_k}$ at $t = T_1$ and Remark 3.6.
there exists $\delta > 0$ such that for any $t_1 \in [T_1 - \delta, T_1]$ the following distorted linear estimate is valid on the interval $[T_1 - \delta, T_1]$: if $u$ solves

$$(\partial_t + \mathcal{H}\partial_{xx})u + \pi(u(t_1), u) = \tilde{f}$$

then we have the bound

$$\|u\|_{Z_{[t_1, T_1-\eta]}^{k}} \lesssim \|u(t_1)\|_{H^s} + \|\tilde{f}\|_{X_{[t_1, T_1-\eta]}^k},$$

for any $\eta \in (0, T_1 - t_1)$. We remark that the implicit constant above does not depend on choice of $t_1$ and $\eta$. In our situation, we substitute $\tilde{f} = \pi(u(t_1), u) - \pi(u, u) + g(u)$. If we mimic the proof of Lemma 3.3 then (where $I := [t_1, T_1 - \eta]$)

$$\|\tilde{f}\|_{X_{[t_1, T_1-\eta]}^k} \lesssim \|u - u(t_1)\|_{L_x^k L_t^\infty_{[t_1, T_1]}} + \|u\|_{L_x^k L_t^\infty_{[t_1, T_1]}} + \|\tilde{f}\|_{X_{[t_1, T_1-\eta]}^k}.$$ 

Because

$$\lim_{t_1 \uparrow T_1} \left(\|u - u(t_1)\|_{L_x^k L_t^\infty_{[t_1, T_1]}} + \|u\|_{L_x^k L_t^\infty_{[t_1, T_1]}}\right) = 0,$$

we have $\|\tilde{f}\|_{X_{[t_1, T_1-\eta]}^k} \ll \|u\|_{X_{[t_1, T_1-\eta]}^k}$ whenever $t_1$ is sufficiently close to $T_1$. Fixing such $t_1$, we have

$$\|u\|_{Z_{[t_1, T_1-\eta]}^{k}} \lesssim \|u(t_1)\|_{H^s}.$$

As $\eta \in (0, T_1 - t_1)$ is arbitrary, we obtain in particular

$$\sup_{t \in [0, T_1]} \|u(t)\|_{H^s} < \infty.$$

Since $u(t) \to u(T_1)$ in $\dot{H}^s$ as $t \to T_1$, we have

$$\|u(T_1)\|_{H^s} \leq \liminf_{t \to T_1} \|u(t)\|_{H^s} < \infty.$$

Therefore, $u(T_1)$ lies in $\dot{H}^s$ and we can construct $H^s$ solution at time $T_1$ both forward and backward in time. This should extend the solution $\bar{u}$, which yields a contradiction. \hfill \Box

**Proof of Theorem 1.3** (Subcritical LWP) Let $s > s_k$. For any $u_0 \in H^s$, by Corollary 3.3, there exists $T = T(\|u_0\|_{H^s}) > 0$ and $\dot{H}^s$ solution $u \in \dot{Z}^k_T$ with initial data $u_0$. By persistence of regularity (Proposition 3.3), the solution $u$ indeed lies in $Z^k_T$. This proves existence part. The uniqueness part and continuity of the solution map is standard.

(GWP and conservation laws) Note that persistence of regularity and local well-posedness guarantees that we can approximate any $H^{\frac{s}{2}}$ solutions by smooth solutions. Since the mass and energy functionals are $H^{s_k}$ continuous, we obtain mass and energy conservation laws. Combining with the subcritical well-posedness, the global well-posedness follows for $H^{\frac{s}{2}}$ solutions. If $s \geq \frac{1}{2}$, then we can use persistence of regularity to transfer the results for $H^{\frac{s}{2}}$ solutions to $H^s$ solutions. \hfill \Box

**Remark 3.10.** Even in case of the focusing (gBO) with $k \geq 4$, subcritical well-posedness part of Theorem 1.2 still holds. Therefore, one can have global well-posedness of focusing (gBO) with initial data satisfying certain mass, energy, and kinetic energy assumptions. See [8, Theorem 1.1].
4. Existence of Critical Element

From this section, we start proving Theorem 1.3. The scheme of the proof is a compactness-contradiction argument, which originates from the pioneering work of Kenig and Merle in the setting of energy-critical nonlinear Schrödinger and wave equation [12, 13]. The argument then successfully implemented in the scattering problem of various semilinear equations, such as nonlinear Schrödinger or wave equations. In case of mass-critical and mass-supercritical (gKdV), see [6] and [9].

Having the global well-posedness, we first derive a criterion that determines whether a solution scatters or not. It will be written in terms of finiteness of the spacetime norms $X^k_t$ and $L^k_x L^\infty_t$. Having established the criterion, suppose that Theorem 1.3 fails. Then, there exists a critical element that does not scatter and attains minimal mass/energy. A genuine property of this critical element is that it should stay in a compact set modulo symmetries of the equation (modulo spatial translations in our setting.) This section is devoted to obtain existence of such a critical element and its compactness property. To this end, as in other contexts, we use the profile decomposition and perturbation theory. Theorem 1.3 will be proved in Section 5 once we show that such a critical element indeed cannot exist.

In other literatures, the linear profile decomposition and long-time perturbation theory are used to obtain the nonlinear profile decomposition, that is, the sum of nonlinear profiles becomes an approximate solution. In our setting, however, we encounter a technical difficulty to deduce long-time perturbation theory from local well-posedness. It is because the iteration norms contain $L^\infty$-type norms $L^k_x L^\infty_t$ and $X^k_t$, in which subdivision of time interval does not guarantee the smallness of $L^k_x L^\infty_{I_j}$ for a short time interval $I_j$. Instead of obtaining a general form of long-time perturbation, we directly prove that nonlinear profile decomposition holds.

In Section 4.1, we derive the aforementioned criterion. In Section 4.2, we derive and discuss more about the linear profile decomposition. In Section 4.3, we prove the nonlinear profile decomposition. In Section 4.4, we finally construct a critical element. The last subsection (Section 4.5) is devoted to estimate the error term appeared in Section 4.3.

4.1. Scattering Criterion. In this subsection, we derive the scattering criterion in terms of spacetime norms. This is nothing but the local theory at time $\pm \infty$. In contrast to the usual local theory, we cannot make small $L^\infty_x L^2_{T,+}$ norm with choosing $T$ large. Fortunately, we can have small $L^k_x L^\infty_{T,+}$ norm and this allows us to achieve local well-posedness at time $+\infty$. This will be done for the original formulation of (gBO), not the distorted equation (3.4).

Lemma 4.1 (Vanishing of $L^k_x L^\infty_{T,+}$). Let $k \geq 4$. For any $\eta > 0$ and $\varphi \in \dot{H}^{4k}$, there exists $T = T(\eta, \varphi) < \infty$ such that
\[
\|V(t)\varphi\|_{L^k_x L^\infty_{T,+}} < \eta.
\]

\[\text{In view of the sharp local smoothing estimate } \|D^{1/2}_x V(t)u_0\|_{L^\infty_x L^2_T} \sim \|u_0\|_{L^2_x}, \text{ we do not expect that } L^\infty_x L^2_{(T,+\infty)} \text{ norm becomes small.}\]
Proof. From the linear estimates, we have \( \| V(t) \varphi \|_{L^2_x L^\infty_t} \lesssim \| \varphi \|_{H^s} \). By density argument, we may assume that \( \varphi \) is Schwartz. By DCT, it suffices to show that for each \( x \in \mathbb{R} \), \( \| V(t) \varphi(x) \|_{L^\infty_t} \) goes to zero as \( T \to +\infty \). This follows by the usual dispersive estimate.

As we shall use the formulation \([\text{gBO}]\) instead of \([3.4]\), we need to estimate the whole nonlinearity \( F(u) \) of \([\text{gBO}]\). The following nonlinear estimates are direct consequences of Lemma \([3.3]\).

**Corollary 4.2** (More estimates). Let \( k \geq 4 \) and \( I \) be an interval. We have

\[
\| F(u) \|_{X^s_{T+}} \lesssim \| u \|_{L^2_x L^\infty_t}^{k-1} \| u \|_{Z^s_{T+}}^2,
\]

\[
\| F(u) - F(v) \|_{X^s_{T+}} \lesssim \| u - v \|_{Z^s_{T+}} \left( \| u \|_{L^2_x L^\infty_t}^{k-2} + \| v \|_{L^2_x L^\infty_t}^{k-2} \right) \left( \| u \|_{Z^s_{T+}}^2 + \| v \|_{Z^s_{T+}}^2 \right).
\]

**Proof.** Apply Lemma \([3.3]\) for the following expressions.

\[
F(u) = -\pi(u, u) + g(u)
\]

\[
F(u) - F(v) = -\pi(u, u - v) - \pi(u, v) + \pi(v, v) + [g(u) - g(v)].
\]

**Proposition 4.3** (Existence of wave operator). Let \( k \geq 4 \).

1. For any \( \tilde{u}_+ \in \dot{H}^{s_k} \), there exists \( r > 0 \) and \( T = T(\tilde{u}_+) < +\infty \) such that any \( u_+ \in B_{\dot{H}^{s_k}}(\tilde{u}_+; r) \) admits a unique \( \dot{H}^{s_k} \) solution \( u \) in \( C_{T+} \dot{H}^{s_k} \cap L^k_x L^\infty_t \cap X^s_{T+} \) which scatters to \( V(t) u_+ \) in \( \dot{H}^{s_k} \) forward in time. Moreover, the solution \( u \) indeed lies in \( Z^s_{T+} \) and the solution map \( u_+ \mapsto u \in Z^s_{T+} \) is locally Lipschitz.

2. Let \( s \geq s_k \). Then, the \( \dot{H}^{s} \)-version of the result holds. That is, the same result holds for \( C_{T+} \dot{H}^{s} \cap L^k_x L^\infty_t \cap X^s_{T+} \) and \( Z^s_{T+} \).

**Proof.** We only prove the first statement, as the second one can be proven in a similar manner. Given \( u_+ \in \dot{H}^{s_k} \), consider the operator

\[
[F \Phi u](t) = V(t) u_+ - \int_t^\infty V(t-s) F(u(s))ds.
\]

Let us consider the norm

\[
\| u \|_\Lambda := \| u \|_{L^2_x L^\infty_t} + \delta \| u \|_{X^s_{T+} \cap L^\infty_t \dot{H}^{s_k}},
\]

where \( 0 < \delta \ll 1 \) to be chosen later. We shall apply the contraction mapping principle on \( B_\Lambda(0; \delta^{\frac{3}{4}}) \).

We present linear and nonlinear estimates. For the linear evolution, we estimate

\[
\| V(t) u_+ \|_\Lambda \lesssim \| V(t) \tilde{u}_+ \|_{L^2_x L^\infty_t} + \| \tilde{u}_+ - u_+ \|_{\dot{H}^{s_k}} + \delta \| u_+ \|_{\dot{H}^{s_k}}.
\]

For the nonlinear estimate, we apply Corollary \([1.2]\) to obtain

\[
\| F(u) - F(v) \|_{X^s_{T+}} \lesssim \| u - v \|_\Lambda \left( \| u \|_{L^2_x L^\infty_t}^{k-2} + \| v \|_{L^2_x L^\infty_t}^{k-2} \right) \left( \| u \|_\Lambda + \| v \|_\Lambda \right)
\]

\[
\lesssim \delta^{\frac{3}{4}-2} \| u - v \|_{Z^s_{T+}}^2.
\]

If in particular \( v = 0 \), then

\[
\| F(u) \|_{X^s_{T+}} \lesssim \delta^{\frac{3}{4}-2} \| u \|_\Lambda.
\]
Therefore,
\[
\|\Phi u\|_\Lambda \lesssim \|V(t)u_+\|_{L^k_T L^\infty_T} + r + \delta\|u_+\|_{H^s_k} + \delta^{3k-2}\|u\|_\Lambda,
\]
\[
\|\Phi u - \Phi v\|_\Lambda \lesssim \delta^{3k-2}\|u - v\|_\Lambda.
\]
If we choose \(\delta\) sufficiently small, \(r\) small, and choosing \(T = T(u_+)\) sufficiently large, then \(\Phi\) becomes a contraction.

We now show that any solution \(u \in C_{T+} \dot{H}^s \cap L^k_T L^\infty_T \cap \dot{X}^s_T\) indeed lies in \(\dot{Z}^s_T\). We only show (4.3), that is, \(\|u\|_{L^k_T L^\infty_T} \to 0\) as \(T \to +\infty\). To show this, by the Duhamel formula,
\[
\|u\|_{L^k_T L^\infty_T} \leq \|V(t)u_+\|_{L^k_T L^\infty_T} + \|F(u)\|_{\dot{X}^s_T}.
\]
As \(T \to +\infty\), we use Lemma 4.1 for the linear evolution and DCT for the nonlinear evolution to obtain \(u \in \dot{Z}^s_T\). \(\square\)

We now state the scattering criterion. In contrast to the case of mass-critical \((\text{gKdV})\), we should keep track of two norms \(L^k_T L^\infty_T\) and \(\dot{X}^s_T\). As we discussed in Section 3 we have one derivative in the nonlinearity, but the local smoothing estimate recovers at most half derivative. Thus we are forced to use \(L^\infty\) type spacetime norms. Consult Section 3.1.

**Proposition 4.4 (Scattering Criterion).** Let \(u\) be a global \(H^\frac{1}{2}\) solution satisfying \(\|u\|_{L^k_T L^\infty_T} + \|u\|_{\dot{X}^s_T} < \infty\) for some \(T < +\infty\). Then, \(u\) scatters in \(H^\frac{1}{2}\) forward in time. In particular, \(u\) belongs to \(Z^\frac{1}{2}_T\). The analogous statement holds for backward in time.

**Proof.** We first show that \(u\) scatters in \(H^s\). To this end, it suffices to show that
\[
t \mapsto \int_0^t V(-s)F(u(s))ds
\]
converges in \(H^s\) as \(t \to +\infty\). By linear estimates, observe that
\[
\left\| \int_0^{t_1} V(-s)F(u(s))ds \right\|_{\dot{H}^s} \lesssim \|F(u)\|_{\dot{X}^s_{[0,t_1]}}.
\]
Since \(\|F(u)\|_{\dot{X}^s_{[0,t_1]}} < \infty\), Lebesgue’s DCT implies \(H^s\) scattering. Denote the scattering state by \(u_+ \in H^s\).

To conclude that \(u\) scatters in \(H^\frac{1}{2}\), observe from the mass/energy conservation that \(u_+\) indeed belongs to \(H^\frac{1}{2}\). Applying Proposition 4.3 \(u\) indeed lies in \(Z^\frac{1}{2}_T\) and scatters to \(u_+\). \(\square\)

For small initial data, we have both global well-posedness and scattering.

**Theorem 4.5 (Small Data GWP and Scattering).** Let \(k \geq 4\).
1. There exists \(\eta > 0\) such that any initial data \(u_0 \in H^s\) with \(\|u_0\|_{\dot{H}^s} < \eta\) admits a unique \(H^s\)-solution \(u\) to \((\text{gBO})\) in \(Z^s_T\). Moreover, the solution map from \(B_{\dot{H}^s}(0; \eta)\) to \(Z^s_T\) is Lipschitz.
2. The solution \(u\) scatters in \(H^s\) both forward and backward in time.
3. For \(s \geq s_k\), the \(H^s\)-version of the result holds.
Proof. The proof is standard once one obtains Corollary 4.2 and exploit smallness of the initial data. Indeed, given \( u_0 \in \dot{H}^s \) with \( \| u_0 \|_{\dot{H}^s} \leq \eta \), we consider the operator \( \Phi \) defined by

\[
[\Phi u](t) := V(t)u_0 + \int_0^t V(t-s)F(s)ds.
\]

For some \( r \) chosen later, we iterate on the space \( B_{Z^s_k}(0;r) \). By the linear estimates and Corollary 4.2, we have

\[
\| \Phi u \|_{Z^s_k} \lesssim \| u_0 \|_{\dot{H}^s} + \| u \|_{k+1}^s_{Z^s_k} + \| v \|_{k+1}^s_{Z^s_k}.
\]

By choosing \( r \) small and then \( \eta \) small, \( \Phi \) becomes a contraction. The remaining parts of the proof are fairly standard. \( \square \)

Remark 4.6. In fact, the small data global well-posedness and scattering can be obtained using the usual local smoothing norms instead of Besov type norms. See \([21, \text{Appendix A.1}]\). There, one avoids paraproduct decomposition of the nonlinearity, but should use fractional Leibniz rules. In the following, we only need local theory in terms of \( \dot{Z}^s_k \) norm.

4.2. Linear Profile Decomposition. One of the main ingredients of compactness-contradiction argument is the profile decomposition. This type of results was intensively exploited in the study of critical dispersive equations for last decades. We start with the linear profile decomposition. Consider the linear estimate (for \( k \geq 4 \))

\[
\| V(t)u_0 \|_{L^k_x L^\infty_t} \lesssim \| u_0 \|_{H^{1/2}}.
\]

This embedding is not compact due to two noncompact symmetries:
1. time translations \( u_0 \mapsto V(t_0)u_0 \) for any \( t_0 \in \mathbb{R} \), and
2. spatial translations \( u_0 \mapsto u_0(\cdot - x_0) \) for any \( x_0 \in \mathbb{R} \).

Remark 4.7. If we replace \( \| u_0 \|_{H^{1/2}} \) by \( \| u_0 \|_{\dot{H}^s} \), then we have one more additional symmetry: the scaling symmetry \( u_0 \mapsto \lambda^{1/k}u_0(\lambda\cdot) \) for any \( \lambda > 0 \).

Roughly speaking, the linear profile decomposition says that these symmetries are essentially all the sources of lack of compactness for the linear estimate \( H^{1/2} \to L^k_x L^\infty_t \). Let us state the linear profile decomposition.

Proposition 4.8 (Linear Profile Decomposition). Let \( k > 4 \) and \( \{u_n\}_{n \in \mathbb{N}} \) be bounded in \( H^{1/2} \). After passing to a subsequence in \( n \) if necessary, there exist profiles \( \{\phi^j\}_{j \in \mathbb{N}} \subset H^{1/2} \), spatial parameters \( \{x^j_n\}_{n,j \in \mathbb{N}} \subset \mathbb{R} \), and time parameters \( \{t^j_n\}_{n,j \in \mathbb{N}} \subset \mathbb{R} \), and defining \( w^j_n \) for each \( J \in \mathbb{N} \) by

\[
u_n = \sum_{j=1}^J V(t^j_n)\phi^j(\cdot - x^j_n) + w^j_n,
\]
satisfying the following properties.

1. (Asymptotic orthogonality in $H^s$) For any $s \in [0, \frac{1}{2}]$ and $J \in \mathbb{N}$, we have

$$\lim_{n \to \infty} \left[ \|u_n\|_{H^s}^2 - \sum_{j=1}^{J} \|\phi_j\|_{H^s}^2 - \|w_n\|_{H^s}^2 \right] = 0. \tag{4.1}$$

2. (Asymptotic vanishing of the remainder) We have

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|V(t)w_n^J\|_{L_t^k L_x^\infty} = 0. \tag{4.2}$$

3. (Asymptotic vanishing of weak limit) For any $j \leq J < \infty$, we have

$$V(-t_j^j)w_n^J(\cdot + x_n^j) \rightharpoonup 0 \quad \text{weakly in } H^\frac{1}{2}. \tag{4.3}$$

4. (Asymptotic separation of parameters) For each $j \neq j'$, we have

$$|\mathbf{x}_{j_n}^j - \mathbf{x}_{j_n}^{j'}| + |t_{j_n}^j - t_{j_n}^{j'}| \to \infty. \tag{4.4}$$

Proof. The proof is standard, especially very similar to that in [7, Lemma 2.1]. For sake of completeness, we include it in Appendix A. □

Remark 4.9. Although the embedding $\|V(t)u_0\|_{L_t^k L_x^\infty} \lesssim \|u_0\|_{H^\frac{1}{2}}$ is true for $k \geq 4$, we do not know whether (4.2) is true for $k = 4$. The $L_t^k L_x^\infty$ estimate is at the genuine endpoint in local smoothing estimates. As one can see in the proof of Proposition 4.8, one has to use interpolation. If (4.2) were true for $k = 4$, Theorem 1.3 holds for $k = 4$.

Remark 4.10. One may ask whether (4.2) is true if we replace $L_t^k L_x^\infty$-norm by $X_t^{s_k}$-norm. However, this seems to be impossible. Indeed, the linear estimate $\|D_t^{1/2}V(t)u_0\|_{L_t^\infty L_x^2} \sim \|u_0\|_{L_x^2}$ says that $\|V(t)w_n^J\|_{X_t^{s_k}} \sim \|w_n^J\|_{H^{s_k}}$. Hence, asymptotic vanishing of $X_t^{s_k}$-norm is equivalent to asymptotic vanishing of $H^{s_k}$-norm, which seems to be impossible.

Let us discuss the statements of Proposition 4.8. There can be infinitely many profiles $\phi_j$. However, the time and spatial parameters associated to each profiles should be far enough, so that each (time and spatial translated) profile contributes to $u_n$ almost orthogonally. After deleting the contributions of profiles, the remainder term should asymptotically vanish in $L_t^k L_x^\infty$ norm. This is the heart of compactness modulo symmetries.

4.3. Nonlinear Profile Decomposition. As we are dealing with the nonlinear equation, we will obtain a nonlinear version of the linear profile decomposition. We refer to as the nonlinear profile decomposition (Theorem 4.11) holds if the sum of nonlinear profiles approximate nonlinear solutions to the original equation. In other contexts, this is achieved by the long-time perturbation theory. The long-time perturbation theory is basically a consequence of concatenating short-time perturbation theory, in which one exploits smallness of solution norms on each small time interval. In our case, however, the solution norm contains $L_t^k L_x^\infty$ norm that may not be small even for a short time interval $J$. Thus obtaining the long-time perturbation and hence nonlinear profile decomposition here becomes more delicate.
We now introduce the nonlinear profile decomposition in more concrete terms. Let \( \{u_n(0)\}_{n \in \mathbb{N}} \) be a bounded sequence in \( H^{\frac{1}{2}} \). We apply the linear profile decomposition (Proposition 4.8) to \( \{u_n(0)\}_{n \in \mathbb{N}} \) and follow the notations made in that proposition.

Possibly taking a further subsequence, we may assume that \( t_n^j \) converges in \([-\infty, +\infty)\). If \( t_n^j \to t^j \in (-\infty, +\infty) \), then we may replace \( \phi^j \) by \( V(t^j)\phi^j \) and \( t_n^j \) by \( t_n^j - t^j \) to assume that \( t_n^j \to 0 \). Moreover, replacing \( w_n^j \) by \( w_n^j + \sum_{j=1}^J [V(t_n^j)\phi^j - \phi^j] \), we may assume that \( t_n^j \equiv 0 \).

By global well-posedness of \( \text{gBO} \) (Theorem 1.2), there exist global \( H^{\frac{1}{2}} \) solutions \( v^j \) for each \( j \in \mathbb{N} \) such that \( v^j(0) = \phi^j \) if \( t_n^j \equiv 0 \), \( v^j \) scatters forward to \( V(t^j)\phi^j \) if \( t_n^j \to +\infty \), and \( v^j \) scatters to forward to \( V(t^j)\phi^j \) if \( t_n^j \to -\infty \). These \( v^j \)'s are referred to as nonlinear profiles associated to \( \{u_n(0)\}_{n \in \mathbb{N}} \). We define

\[
\begin{align*}
v_n^j(t) &:= v^j(t + t_n^j, \cdot - x_n^j), \\
u_n^j(t) &:= \sum_{j=1}^J u_n^j(t), \\
\hat{u}_n^j(t) &:= u_n^j(t) + V(t)w_n^j.
\end{align*}
\]

Note that \( \hat{u}_n^j \) and \( u_n^j \) are globally defined. The approximate solutions \( \{\hat{u}_n^j\}_{n,J \in \mathbb{N}} \) will be referred to as a nonlinear profile decomposition associated to \( \{u_n(0)\}_{n \in \mathbb{N}} \).

We now state the main theorem of this subsection.

**Theorem 4.11 (Nonlinear Profile Decomposition).** Consider a nonlinear profile decomposition \( \{\hat{u}_n^j\}_{n,J \in \mathbb{N}} \) associated to a bounded sequence \( \{u_n(0)\}_{n \in \mathbb{N}} \subset H^{\frac{1}{2}} \). Let \( I \subseteq \mathbb{R} \) be a fixed interval (possibly \( I = \mathbb{R} \)) containing 0. If \( \|v^j\|_{L^2(I)} < \infty \) for all \( j \in \mathbb{N} \), then

\[
\lim_{j \to \infty} \limsup_{n \to \infty} \|\hat{u}_n^j - u_n\|_{H^{\frac{1}{2}}} = 0.
\]

In particular, combining this with Lemma 4.13 below, we have

\[
\limsup_{n \to \infty} \|u_n\|_{L^2(I)} < \infty.
\]

Theorem 4.11 will be proved after we establish an appropriate perturbation theory. We now list some properties of the nonlinear profile decomposition associated to \( \{u_n(0)\}_{n \in \mathbb{N}} \). Namings of the lemmas presented below follow [18].

**Lemma 4.12 (Asymptotic agreement of initial data).** We have

\[
\sup_j \limsup_{n \to \infty} \|\hat{u}_n^j(0) - u_n(0)\|_{H^{\frac{1}{2}}^j} = 0.
\]

**Proof.** Observe that

\[
\|\hat{u}_n^j(0) - u_n(0)\|_{H^{\frac{1}{2}}} = \sum_{j \leq J: |t_n^j| \to \infty} \|v^j(t_n^j) - V(t_n^j)\phi^j\|_{H^{\frac{1}{2}}}.
\]

Here, we use the terminology “nonlinear profile decomposition” in two different ways. On the one hand, we call \( \{\hat{u}_n^j\}_{n,J \in \mathbb{N}} \) as a nonlinear profile decomposition. On the other hand, the statement of Theorem 4.11 is said that nonlinear profile decomposition holds.
The conclusion follows by the construction of \( v^j \).

**Lemma 4.13** (Uniform boundedness of approximate solutions). Let \( I \subseteq \mathbb{R} \) be a fixed interval (possibly \( I = \mathbb{R} \)). If \( \| v^j \|_{Z^\kappa} < \infty \) for all \( j \in \mathbb{N} \), we have

\[
\sup_j \limsup_{n \to \infty} (\| \tilde{u}_n^j \|_{Z^\kappa} + \| \tilde{u}_n \|_{L^\infty_t H_x^1}) < \infty.
\]

**Proof.** For notational simplicity, we only prove the case \( I = \mathbb{R} \). Fix \( 0 < \eta_0 \ll 1 \) such that we can apply the small data theory for \( \phi^j \) having \( \dot{H}^\kappa \)-norm less than \( \eta_0 \). Choose \( J_1 \in \mathbb{N} \) such that

\[
\left( \sum_{j > J_1} \| \phi^j \|^2_{\dot{H}^\kappa} \right)^{\frac{1}{2}} \leq \eta_0.
\]

We then see by Duhamel’s principle that

\[
\sup_j \limsup_{n \to \infty} \| \sum_{j > J_1} v^j_n \|_{Z^\kappa} \lesssim \sup_j \limsup_{n \to \infty} \| \sum_{j > J_1} \phi^j_n(0) \|_{\dot{H}^\kappa} + \sum_{j > J_1} \| \pi(v^j, v^j) \|_{\dot{H}^\kappa} + \| g(v^j) \|_{\dot{H}^\kappa} \lesssim \eta_0.
\]

Applying exactly the same method with replacing \( \dot{H}^\kappa \) by \( H^\kappa_x \), we have

\[
\sup_j \limsup_{n \to \infty} \| \sum_{j > J_1} v^j_n \|_{L^\infty_t H_x^1} \lesssim \eta_0.
\]

Since only finitely many profiles are left, (applying mass/energy conservation when we bound \( L^\infty_t H_x^1 \)) the conclusion follows. 

**Lemma 4.14** (Error estimation). Let \( I \subseteq \mathbb{R} \) be a fixed interval (possibly \( I = \mathbb{R} \)). If \( \| v^j \|_{Z^\kappa} < \infty \) for all \( j \in \mathbb{N} \), we have

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \| F(\tilde{u}_n^j) - \sum_{j=1}^J F(v^j_n) \|_{\dot{H}^\kappa} = 0.
\]

**Proof.** We postpone the proof in Section 4.5.

We now go back to our discussion of technical difficulties arising from long-time perturbation theory. It is instructive to compare with the mass-critical (gKdV) long-time perturbation theory [10], which we shall state as follows. Suppose that \( \tilde{u} \) solves

\[
(\partial_t + \partial_{xxx}) \tilde{u} = \partial_x(\tilde{u}^5) + e
\]

with initial data \( \tilde{u}_0 \) and satisfies

\[
\| \tilde{u}_0 \|_{L^\infty_t L^2_x} \leq M \quad \text{and} \quad \| \tilde{u} \|_{L^\infty_t L^{10}_x} \leq L
\]
for some finite $M$ and $L$. Then, there exists $\epsilon_0 = \epsilon_0(M,L) > 0$ such that whenever
\[ \| u_0 - \tilde{u}_0 \|_{L^2_x} + \| D^{-1} e \|_{L^2_t L^2_x} \leq \epsilon \]
for some $0 < \epsilon < \epsilon_0$, we have
\[ \| u - \tilde{u} \|_{L^5_x L^{10}_T} \lesssim_{M,L} \epsilon. \]
The case when $L$ is chosen sufficiently small constant is called the short-time perturbation theory.

The main step in proving the above perturbation theory is to subdivide the interval $I$ into $L$-dependently many subintervals (say $j = 1, \ldots, N(J)$) on which $\tilde{u}$ has sufficiently small $L^5_x L^{10}_{t,j}$ norm. Once $\tilde{u}$ has small $L^5_x L^{10}_{t,j}$ norm, by a short-time perturbation, one can conclude that $\tilde{u}$ is close to $u$. One then inductively applies the short-time perturbation, but only $L$-dependently many times to have the desired long-time perturbation.

In our case, $X_t^{s_k}$ and $L^k_x L^\infty_T$ are the iteration spaces in place of $L^5_x L^{10}_{t,j}$. The problem is that, as in many other literatures, short-time perturbation is merely a slight generalization of the local theory (Section 3). This still holds in our setting, so we can obtain short-time perturbation similar to that of (gKdV), which is written in terms of $M$ and $L$. Therefore, we shall not attempt to obtain such a general long-time perturbation theory. Instead, we directly prove the nonlinear profile decomposition (Theorem 4.11).

The first observation in proving the nonlinear profile decomposition is that, as in many other literatures, short-time perturbation is merely a slight generalization of the local theory (Section 3). This still holds in our setting, so we can obtain short-time perturbation similar to that of (gKdV). In other words, the short-time perturbation can be obtained by exploiting the smallness of $\| v^j - v^j(0) \|_{L^5_x L^{10}_T}$ and $\| v^j \|_{X_t^{s_k}}$ to the distorted equation (3.4) when $T$ is sufficiently small. This justifies approximating two close solutions on compact time intervals. However, this cannot take care of the case when we compare two solutions on unbounded intervals.

In order to take care of unbounded intervals, our second observation is to utilize Proposition 4.3. In Proposition 4.3, we saw that smallness of $L^5_x L^\infty_T$ is obtained for scattering solutions, so a perturbation theory on $[T, \infty)$ can be obtained.

Therefore, we must obtain two types of perturbation lemmas. The first one is associated to the usual local theory, which takes care of compact intervals. The second one is associated to the local theory at time $\pm \infty$, which takes care of remaining unbounded intervals.

**Lemma 4.15 (Perturbation lemma, I).** Suppose that $\tilde{u}$ and $u$ solve
\[
(\partial_t + H \partial_{xx}) u = F(\tilde{u}) + e
\]
\[
(\partial_t + H \partial_{xx}) u = F(u)
\]
and satisfy
\[ \| \tilde{u} \|_{Z_t^{s_k}} + \| u \|_{L^\infty_T H^1_x} \leq M. \]
By Corollary 4.2 and our hypothesis, we have

\[ \|\tilde{u} - u(t)\|_{L^L_x L^r_t} + \|\tilde{u}\|_{X^{r,k}_t} \leq \epsilon_0, \]
\[ \|u(t) - u(t)\|_{H^{r,k}} \leq \epsilon < \epsilon_0, \]
\[ \|e\|_{\mathcal{N}_{r,k}^k} \leq \epsilon < \epsilon_0 \]

for some \( t_0 \in I \) and \( \epsilon \in (0, \epsilon_0) \), we have

\[ \|\tilde{u} - u\|_{Z_t^r} \lesssim \epsilon. \]

**Proof.** Let \( w = \tilde{u} - u \). As we do not have smallness of \( \|\tilde{u}\|_{L^L_x L^r_t} \), we use

\[ (\partial_t + H\partial_{xx})u + \pi(\tilde{u}(t), u) = [\pi(\tilde{u}(t), \cdot) - \pi(u, \cdot)](\tilde{u}) + g(\tilde{u}) + \epsilon, \]
\[ (\partial_t + H\partial_{xx})u + \pi(\tilde{u}(t), u) = [\pi(\tilde{u}(t), \cdot) - \pi(u, \cdot)](u) + g(u). \]

Therefore, we have

\[ (\partial_t + H\partial_{xx})w + \pi(\tilde{u}(t), w) \]
\[ = [\pi(\tilde{u}(t), \cdot) - \pi(u, \cdot)](w) + [\pi(\tilde{u}, \cdot) - \pi(u, \cdot)](u) + g(\tilde{u}) - g(u). \]

As we are assuming that \( I \) is short enough, we apply Corollary 3.5 and Lemma 3.3 to obtain

\[ \|w\|_{Z^r_t} \lesssim p_k(M) [\|w(t_0)\|_{H^{r,k}} + \epsilon_0 M^{k-1} \|w\|_{Z^r_t} + \epsilon_0^2 M^{k-2} \|w\|_{Z^r_t} + p_{k,M}(\|w\|_{Z^r_t})] \]

for some higher order polynomial \( p_{k,M} \). By a continuity argument, we have

\[ \|w\|_{Z^r_t} \lesssim \|w(t_0)\|_{H^{r,k}} \lesssim \epsilon. \]

\[ \square \]

**Lemma 4.16** (Perturbation lemma, II). Suppose that \( \tilde{u} \) and \( u \) solve

\[ (\partial_t + H\partial_{xx})\tilde{u} = F(\tilde{u}) + e \]
\[ (\partial_t + H\partial_{xx})u = F(u) \]

and satisfy

\[ \|\tilde{u}\|_{Z^r_t} \leq M. \]

Then there exists \( \epsilon_0 = \epsilon_0(M) > 0 \) such that whenever

\[ \|\tilde{u}\|_{L^L_x L^r_t} \leq \epsilon_0, \]
\[ \|\tilde{u}(t) - u(t)\|_{H^{r,k}} \leq \epsilon < \epsilon_0, \]
\[ \|e\|_{\mathcal{N}_{r,k}^k} \leq \epsilon < \epsilon_0 \]

for some \( t_0 \in I \) and \( \epsilon \in (0, \epsilon_0) \), we have

\[ \|\tilde{u} - u\|_{Z^r_t} \lesssim \epsilon. \]

**Proof.** Let \( w = \tilde{u} - u \). By Duhamel’s formula,

\[ w(t) = V(t - t_0)w(t_0) + \int_{t_0}^t V(t - s)[F(\tilde{u}) - F(u) + e](s)ds. \]

By Corollary 4.2 and our hypothesis, we have

\[ \|w\|_{Z^r_t} \lesssim \|w(t_0)\|_{H^{r,k}} + \|F(\tilde{u}) - F(u)\|_{\mathcal{N}_{r,k}^k} + \|e\|_{\mathcal{N}_{r,k}^k} \]
\[ \lesssim \|w(t_0)\|_{H^{r,k}} + \epsilon_0 M^{k-2} \|w\|_{Z^r_t} + \|e\|_{\mathcal{N}_{r,k}^k} + p_{k,M}(\|w\|_{Z^r_t}), \]
for some higher order polynomial $p_{k,M}$. By a continuity argument, we have
\[ \|w\|_{Z^k_t} \lesssim \|w(t_0)\|_{H^k_t} \lesssim \epsilon. \]

With the above perturbation lemmas in hand, we explain how we handle the long-time control of $\tilde{\omega}_n^J$ and $u_n$. In view of uniform boundedness of initial data in $H^{\frac{1}{2}}$, there are only finitely many large profiles. Note that there can be infinitely many small profiles, but they will not cause problems due to almost orthogonality and small data global theory. We immediately estimate the sum of small profiles by a small $Z$ norm.

It remains to treat large profiles, which are finitely many! Because each nonlinear profile scatters by our hypothesis in Theorem 4.11, we can truncate intervals $(-\infty, -T]$ and $[T, +\infty)$ on which local theory at time $\pm\infty$ (Lemma 4.16) holds. Then, the remaining intermediate interval is compact. We then subdivide that compact interval into finitely many short subintervals so that we can apply Lemma 4.15. This is the main idea of Lemma 4.17 how we can chop the time interval into subintervals on which perturbation lemmas are applicable, and hence, prove Theorem 4.11.

**Proof of Theorem 4.11.** From Lemma 4.13, we fix $M < \infty$ satisfying
\[ \sup \limsup_{J \to \infty} \sup_{n \to \infty} \left( \|\tilde{\omega}_n^J\|_{Z^k_t} + \|\tilde{\omega}_n^J\|_{L_\infty^t H^k_x} \right) < M. \]

Choose $\eta = \epsilon_0(2M) > 0$ in Lemmas 4.16 and 4.15. We then apply the following lemma.

**Lemma 4.17 (Subdivision of $\mathbb{R}$).** Let $\{\tilde{\omega}_n^J\}_{n,J \in \mathbb{N}}$ and $M$ as above. For any $\eta > 0$, there exists $N \in \mathbb{N}$ such that for all large $J$ and $n > n(J)$, there exist intervals $I_{n,1}^J, I_{n,2}^J, \ldots, I_{n,N}^J$ partitioning $I$ such that every $I_{n,\ell}^J$ satisfies one of the following properties:
1. (type-I) we have $\|\tilde{\omega}_n^J(i_{n,\ell})\|_{L^\infty_t L^\infty_x I_{n,\ell}} + \|\tilde{\omega}_n^J\|_{L^\infty_t H^k_x I_{n,\ell}} \lesssim \eta$ and the length of $I_{n,\ell}^J$ is sufficiently short so that we can apply Corollary 3.5.
2. (type-II) we have $\|\tilde{\omega}_n^J\|_{L^\infty_t L^\infty_x I_{n,\ell}} \lesssim \eta$.

**Proof.** For notational simplicity, we only prove for the case of $I = \mathbb{R}$. Let us first consider small data part. Choose $J_1 \in \mathbb{N}$ such that
\[ \left( \sum_{j > J_1} \|\phi_j^J\|_{H^k_t}^2 \right)^{\frac{1}{2}} \lesssim \eta \]
and small data theory is applicable for each $v^j$, $j > J_1$. As in the proof of Lemma 4.13 we have
\[ \lim J \to \infty \limsup_{n \to \infty} \sum_{j > J_1} v^j_n \|\tilde{\omega}_n^J\|_{Z^k_t} \lesssim \eta. \]

We now consider large profiles $v^1_n, \ldots, v^{J_1}_n$. For each $v^j_n$, by Proposition 4.4 and definition of the space $Z^k_t$, we can partition $\mathbb{R}$ into
\[ \mathbb{R} = (-\infty, s^j_{n,1}] \cup [s^j_{n,1}, s^j_{n,2}] \cup \cdots \cup [s^j_{n,N_j-1}, s^j_{n,N_j}] \cup [s^j_{n,N_j}, +\infty) \]
so that \((-\infty, s_{1,1}^J]\) and \([s_{N_j}^J, \infty)\) are type-II intervals with \(v_n^J, \frac{\eta}{J_1}\) in place of \(u_n^J, \eta\), respectively and \([s_{1,2}^J, s_{1,2}^J]\), \([s_{1,3}^J, s_{1,3}^J]\), \ldots, and \([s_{N_j-1}^J, s_{N_j}^J]\) are type-I intervals with \(v_n^J, \frac{\eta}{J_1}\) in place of \(u_n^J, \eta\), respectively. We may further assume that \(|s_{1,\ell}^J - s_{1,\ell-1}^J| < \epsilon\) for all \(\ell = 2, \ldots, N_j\), where \(\epsilon > 0\) satisfies the following property:

\[
\sup_{1 \leq j \leq J_1} \left( \|v^J - \hat{v}(t_0)\|_{L^1_t L^\infty} + \|v^J\|_{X^{s_k}_t} \right) \lesssim \frac{\eta}{J_1}
\]

whenever \(\tilde{I}\) is a time interval whose length is shorter than \(\epsilon\) and \(t_0 \in \tilde{I}\). Note that such \(\epsilon\) exists because each \(v^J\) lies in \(Z^J_1\) (by Proposition 4.4) and we can use regularity. We then consider the refinement of the all intervals found above and use the triangle inequality.

For the remaining term, note that

\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|V(t)w_n^J\|_{L^1_t L^\infty} = 0,
\]

so it suffices to consider \(X^{s_k}_t\)-norm on type-I intervals (say \(I\)). By the Sobolev embedding and Hölder’s inequality, we have

\[
\|V(t)w_n^J\|_{X^{s_k}_t} \lesssim \epsilon^{0+} \|w_n^J\|_{H^{s_k}_t} \lesssim \epsilon^{0+} M.
\]

Therefore, it suffices to choose \(\epsilon = \epsilon(M, \eta) > 0\) small so that \(\epsilon^{0+} M \lesssim \eta\) and Corollary 3.5 is applicable on intervals having length less than \(\epsilon\). \(\square\)

From Lemma 4.17 we have finitely many (profile decomposition dependently, not \(n\) or \(J\) dependently) intervals partitioning \(\mathbb{R}\) on which Lemma 4.16 and 4.13 are applicable. Because we have asymptotic agreement of initial data (Lemma 4.12) and asymptotic vanishing of error (Lemma 4.14), we have the conclusion. \(\square\)

Local well-posedness theory guarantees that if two initial data are close, then corresponding solutions are close on a compact time interval measured in \(Z\) norm. Due to Theorem 4.13 if \(u\) is a scattering solution, we have a perturbation result in a neighborhood of \(u\), even on an unbounded time interval (say \([T, +\infty)\)). More precisely, we have the following corollary.

**Corollary 4.18** (Closeness of solutions on an unbounded interval). Let \(u\) and \(u_n\) be \(H^{s_k}_t\) solutions to \((gBO)\) such that \(\|u\|_{Z^{s_k}_t} < \infty\) and \(u_n(t_0) \to u(t_0)\) in \(H^{s_k}_t\) for some \(t_0 \in I\). Then, \(\|u_n - u\|_{Z^{s_k}_t} \to 0\).

**Proof.** Since \(u_n(t_0)\) converges to \(u(t_0)\) in \(H^{s_k}_t\),

\[
\tilde{u}_n(t) := u(t) + V(t - t_0)(u_n(t_0) - u(t_0))
\]

becomes a nonlinear profile decomposition associated to \(\{u_n(t_0)\}_{n \in \mathbb{N}}\) (with \(J = 1\)). As \(\|u\|_{Z^{s_k}_t} < \infty\), we can use the longtime perturbation theory to obtain

\[
\lim_{n \to \infty} \|\tilde{u}_n - u_n\|_{Z^{s_k}_t} = 0.
\]

Because \(\|\tilde{u}_n - u\|_{Z^{s_k}_t} \lesssim \|u_n(t_0) - u(t_0)\|_{H^{s_k}_t}\) goes to zero, we have

\[
\lim_{n \to \infty} \|\tilde{u}_n - u\|_{Z^{s_k}_t} = 0.
\]
This completes the proof. □

4.4. Existence of the Critical Element. Having established the non-linear profile decomposition, we now construct a critical element. Define functionals

\[ A(u) := M(u) + E(u), \]
\[ S_R(u) := \|u\|_{X^k_t} + \|u\|_{L_x^6 L_t^\infty}, \]
\[ L(A) := \sup\{S_R(u) : A(u) \leq A\}. \]

Suppose that Theorem 1.3 fails. Then, there exists some \( A_0 < \infty \) satisfying \( L(A_0) = +\infty \). We define a critical value

\[ A_c := \inf\{A : L(A) = +\infty\}. \]

By the small data theory (Theorem 4.5), we have \( 0 < A_c \leq A_0 < \infty \). In other words, \( A_c \) is the threshold in a sense that every solution having \( A \) less than \( A_c \) should scatter but a solution having \( A \) greater than \( A_c \) may not scatter.

Lemma 4.19. The function \( L \) is continuous at \( A = A_c \) and \( L(A_c) = +\infty \).

Proof. We first show that \( L(A_c) = +\infty \). Suppose not; we assume \( L(A_c) < +\infty \). Then, there exists a sequence \( \{u_n(0)\}_{n \in \mathbb{N}} \) in \( H^{\frac{5}{2}} \) such that \( A(u_n) \downarrow A_c \), but \( S_R(u_n) \uparrow +\infty \). Consider a nonlinear profile decomposition associated to \( u_n \). Note that all profiles have \( A \) less than or equal to \( A_c \). Therefore, we can apply longtime perturbation (Theorem 4.11) and conclude that \( \lim \sup_{n \to \infty} S_R(u_n) < \infty \). This is absurd.

We now show that \( L(A) \uparrow +\infty \) as \( A \uparrow A_c \). For any sufficiently large \( M \), we can choose \( u \) such that \( A(u) = A_c \) and \( S_R(u) > M \). Then, we can choose a compact interval \( I \subset \mathbb{R} \) such that \( S_I(u) > M \). Consider an initial data \( u_0, \epsilon = (1 - \epsilon)u_0 \). By local well-posedness, we have \( S_I(u_{0, \epsilon}) > M \) for a sufficiently small \( \epsilon \). This completes the proof. □

The next proposition establishes the compactness property of critical elements.

Proposition 4.20 (Palais-Smale Condition). Suppose that \( \{u_n\}_{n \in \mathbb{N}} \) is a sequence of \( H^{\frac{5}{2}} \)-solutions satisfying \( A(u_n) \leq A_c \) and

\[ \lim_{n \to \infty} S_{\geq t_n}(u_n) = \lim_{n \to \infty} S_{\leq t_n}(u_n) = +\infty \]

for some \( t_n \). Then, possibly taking a subsequence of \( u_n \), there exist spatial parameters \( \{x_n\}_{n \in \mathbb{N}} \) such that \( u_n(t_n, \cdot + x_n) \) converges in \( H^{\frac{5}{2}} \).

Proof. By time translation, we may assume that \( t_n \equiv 0 \). Consider a nonlinear profile decomposition associated to \( u_n \). We divide into two cases.

We first consider the case when \( \sup_j A(\phi^j) < A_c \). By definition of \( A_c \), every \( \phi^j \) has finite scattering norm. We apply Theorem 4.11 to obtain \( \lim \sup_{n \to \infty} S_R(u_n) < \infty \). This is absurd.

The next case is when \( \sup_j A(\phi^j) = A_c \). By asymptotic orthogonality in \( H^{\frac{5}{2}} \), we have only one profile \( u_n = v_n^1 + V(t)w_n^1 \) with \( w_n^1 \to 0 \) strongly in
If \( t_n^1 \to +\infty \), then \( u^1 \) scatters in forward by construction of \( v^1 \). We combine
\[
S_{\geq 0}(u_n) \leq S_{\geq t_n^1}(v^1) + \| w_n^1 \|_{H^s_k}
\]
with Proposition \ref{Prop:Existence} to obtain
\[
\limsup_{n \to \infty} S_{\geq 0}(u_n) \leq \limsup_{n \to \infty} S_{\geq t_n^1}(v^1) < \infty,
\]
which is absurd. Similarly, we can exclude the case \( t_n^1 \to -\infty \). Therefore, we conclude \( t_n^1 \equiv 0 \). In this case, \( u_n(0, \cdot + x_n) = \phi^1 + w_n^1(\cdot + x_n) \) converges to \( \phi^1 \) strongly in \( H^\frac{1}{2} \).

A global solution \( u \in C_t H^\frac{1}{2}_x \) to \([\text{BO}]\) is almost periodic modulo spatial translations (in short, almost periodic) if the set of cosets represented by \( \{ u(t) : t \in \mathbb{R} \} \) is precompact in the quotient topology of \( H^\frac{1}{2} \) modulo spatial translation. Equivalently, there is \( x(t) \in \mathbb{R} \) for each \( t \) such that the set
\[
\{ u(t, \cdot + x(t)) : t \in \mathbb{R} \}
\]
is precompact in \( H^\frac{1}{2} \).

**Theorem 4.21** (Existence of the critical element). *Suppose that Theorem \ref{Thm:Global} fails. Then, there exists an almost periodic solution \( u \in C_t H^\frac{1}{2}_x \) that does not scatter either forward nor backward in time.*

**Proof.** From continuity of \( L \) at \( A = A_c \) (Lemma \ref{Lem:Continuity}), we can choose \( u_n \) such that \( A(u_n) < A_c \), \( A(u_n) \uparrow A_c \), and \( S^R(u_n) \uparrow +\infty \). We claim that there exist \( t_n \in \mathbb{R} \) such that
\[
\lim_{n \to \infty} S_{\geq t_n}(u_n) = \lim_{n \to \infty} S_{\leq t_n}(u_n) = +\infty.
\]
Because \( A(u_n) < A_c \), the solutions \( u_n \) scatter in \( H^\frac{1}{2} \) both forward and backward in time. Therefore, we can choose \( \tilde{t}_n \in \mathbb{R} \) large such that \( S_{\geq \tilde{t}_n}(u_n) \lesssim A_c \). Hence, using continuity of \( \tilde{Z}_t^R \) norm, we can choose \( t_n \in \mathbb{R} \) for all large \( n \in \mathbb{N} \) such that \( S_{\geq t_n}(u_n) = \frac{1}{2} S^R(u_n) \). We then have \( S_{\leq t_n}(u_n) \geq S^R(u_n) - S_{\geq\tilde{t}_n}(u_n) = \frac{1}{2} S^R(u_n) \) for all large \( n \in \mathbb{N} \). The claim is now proved.

By Proposition \ref{Prop:Existence} (possibly passing to a subsequence) there exist \( x_n \in \mathbb{R} \) such that \( u_n(t_n, \cdot + x_n) \) converges strongly in \( H^\frac{1}{2} \), say \( u(0) \). Let \( u \) be a global \( H^\frac{1}{2} \) solution with initial data \( u(0) \). We show that \( u \) does not scatter forward in time. Suppose that \( S_{\geq 0}(u) < \infty \). Then, we have \( \lim_{n \to \infty} S_{\geq t_n}(u_n) = S_{\geq 0}(u) < \infty \) by Corollary \ref{Cor:Energy} which makes a contradiction. Similarly, \( u \) does not scatter backward in time.

We now show that \( u \) is almost periodic modulo spatial translations. As \( S_{\geq 0}(u) = S_{\leq 0}(u) = \infty \), we have \( S_{\geq t}(u) = S_{\leq t}(u) = \infty \) for all \( t \in \mathbb{R} \). Applying Proposition \ref{Prop:AlmostPeriodic} to \( \{ u(t) \}_{t \in \mathbb{R}} \), the conclusion follows. \( \square \)

Let us conclude this subsection by noting a quantitative formulation of almost periodic solutions. Using the Sobolev embedding \( H^\frac{1}{2} \to L^p \) for any \( 2 \leq p < \infty \) and the Arzela-Ascoli theorem in \( L^p \) space, we have the following.
Proposition 4.22 (Arzela-Ascoli for a.p. solutions). Suppose that \( u \in C_t \dot{H}^{\frac{1}{2}} \) is almost periodic modulo spatial translations. Then, there exist spatial parameters \( x(t) \in \mathbb{R} \) for each \( t \in \mathbb{R} \) such that for any \( p \in [2, \infty) \) and \( \eta > 0 \), there exists a modulus \( R = R(p, \eta, u) > 0 \) such that
\[
\sup_{t \in \mathbb{R}} \left( \int_{|x-x(t)| > R} |u(t, x)|^p dx \right) < \eta.
\]

Remark 4.23. In other contexts of critical setting, \( \{u(t)\}_{t \in \mathbb{R}} \) is almost periodic modulo symmetries other than spatial translations, such as scalings and frequency modulations. Thus, one introduces a frequency scale function \( N(t) \). In terms of behavior of \( N(t) \), it is decomposed into several scenarios. But in our case, as we work on inhomogeneous setting \( C_t \dot{H}^{\frac{1}{2}} \), we only have the case \( N(t) \equiv 1 \).

4.5. Proof of Lemma 4.14. In this subsection, we prove Lemma 4.14. We first introduce several lemmas, which account for decoupling of nonlinear profiles.

Lemma 4.24. For any \( s > 0 \), \( \ell = 1, 2, \ldots, k-2 \), and \( \varphi_1, \varphi_2, \psi_1, \psi_2 \in \dot{Z}_t^{2k} \), we have
\[
\left\| 2^{js} \left( \sum_{r \geq j} (Q_{\sim r} \varphi_1)(Q_{\geq r} \varphi_2)(Q_{\geq r} \psi_1)^\ell (Q_{\leq r} \psi_2)^{k-1-\ell} \right) \right\|_{L^2_t L^\infty_x} \lesssim_j \| \varphi_2 \|_{L^2_t L^\infty_x} \| \psi_1 \|_{L^2_t L^\infty_x} \| \psi_2 \|_{L^2_t L^\infty_x} \left\| 2^{js} ((Q_{\sim j} \varphi_1)(Q_{\leq j} \psi_1)) \right\|_{L^2_t L^2_x}.
\]

Proof. Observe that
\[
\left\| 2^{js} \left( \sum_{r \geq j} (Q_{\sim r} \varphi_1)(Q_{\geq r} \varphi_2)(Q_{\geq r} \psi_1)^\ell (Q_{\leq r} \psi_2)^{k-1-\ell} \right) \right\|_{L^2_t L^\infty_x} \leq \left\| \sum_{r \geq j} 2^{(j-r)s} \| (2^{r-s} Q_{\sim r} \varphi_1)(Q_{\geq r} \varphi_2)(Q_{\geq r} \psi_1)^\ell (Q_{\leq r} \psi_2)^{k-1-\ell} \|_{L^2_t L^2_x} \right\|_{L^2_x} \lesssim \| \varphi_2 \|_{L^2_t L^\infty_x} \| \psi_1 \|_{L^2_t L^\infty_x} \| \psi_2 \|_{L^2_t L^\infty_x} \left\| \sum_{r \geq j} 2^{(j-r)s} \| (2^{r-s} Q_{\sim r} \varphi_1)(Q_{\leq r} \psi_1) \|_{L^2_t L^2_x} \right\|_{L^2_x}.
\]

We then use Young’s inequality to obtain the conclusion. \( \square \)

Lemma 4.25. For any \( j \in \mathbb{N} \) and \( t \in \mathbb{R} \), we have
\[
v^j(t + t_n, \cdot + x_n) \to 0 \quad \text{weakly in } L^\frac{1}{2},
\]
whenever \( |t_n| + |x_n| \to \infty \).

Proof. It suffices to show that \( \langle \phi, v^j(t+t_n,\cdot+x_n) \rangle_{L^\frac{1}{2}} \to 0 \) for any \( \phi \in C_c^\infty (\mathbb{R}) \) with \( \| \phi \|_{H^\frac{1}{2}} = 1 \). We only consider the following two cases: 1. \( |t_n| \to \infty \) and 2. \( t_n \to t_0 \in \mathbb{R} \) and \( |x_n| \to \infty \). The remaining part of the proof is an easy exercise.

Let us consider the first case \( |t_n| \to \infty \). Because \( \| v^j \|_{L^{2k}_t} < \infty \), \( v^j \) scatters in \( H^\frac{1}{2} \) both forward and backward in time. Hence, we can approximate
$v^j(t + n, \cdot + x_n)$ by $V(t + n)\psi(\cdot + x_n)$ for some $\psi \in H^{1/2}$ for all large $n$. We then approximate $\psi$ by some $\tilde{\psi} \in C_c^\infty$ in $H^{1/2}$ sense. If we consider

$$|\langle \phi, v^j(t + n, \cdot + x_n) \rangle|_{H^{1/2}}|$$

$$\leq \|v^j(t + n) - V(t + n)\psi\|_{H^{1/2}} + \|\psi - \tilde{\psi}\|_{H^{1/2}} + \| \langle \nabla \phi \rangle \|_{L^1} \|V(t + n)\tilde{\psi}\|_{L^\infty}$$

and use the dispersive decay $\|V(t + n)\tilde{\psi}(\cdot + x_n)\|_{L^\infty} \lesssim (t + n)^{-\frac{3}{2}} \|\tilde{\psi}\|_{L^1}$, we get the conclusion.

We now consider the remaining case: $t_n \to t_0 \in \mathbb{R}$ and $|x_n| \to \infty$. Because $v^j \in C(H^{1/2}_x)$, we can approximate $v^j(t + n)$ by $v^j(t + t_0)$. We then approximate $v^j(t + t_0)$ by some $\psi \in C_c^\infty(\mathbb{R})$ in $H^{1/2}$ sense. More precisely, we consider

$$|\langle \phi, v^j(t + n, \cdot + x_n) \rangle|_{H^{1/2}}|$$

$$\leq \|v^j(t + n) - v^j(t + t_0)\|_{H^{1/2}} + \|v^j(t + t_0) - \psi\|_{H^{1/2}} + |\langle \phi, \psi(\cdot + x_n) \rangle|_{H^{1/2}}|.$$

We then apply Riemann-Lebesgue lemma to obtain $|\langle \phi, \psi(\cdot + x_n) \rangle|_{H^{1/2}} \to 0$. This completes the proof.

\begin{lemma}[Asymptotic decoupling of $L^2_t L^2_x$ norm, I] For each $m \in \mathbb{Z}$ and $j \neq j'$, we have

$$\lim_{n \to \infty} \|(Q_{\sim m} v_n^j)(Q_{\lesssim n} v_n^{j'})\|_{L^2_t L^2_x} = 0.$$

\end{lemma}

\begin{proof}
We may assume that $m = 0$. Using change of variables and the property (3), it suffices to show that

$$\lim_{n \to \infty} \|(Q_{\lesssim 0} v^j)(Q_{\sim 0} v^{j'}(\cdot + t_n)(\cdot + x_n))\|_{L^2_t L^2_x} = 0$$

whenever $T < \infty$ and $|t_n| + |x_n| \to \infty$. Here, the variables for $\cdot + t_n$ and $\cdot + x_n$ are $t$ and $x$, respectively. We shall show the assertion by DCT. Observe for any $t$ and $x$ that we have a pointwise bound

$$\sup_{n \in \mathbb{N}} |Q_{\sim 0} v^j(t + t_n)(x + x_n)| \lesssim \|v^j\|_{L^\infty L^2_x} < \infty$$

and hence

$$\left\| \|Q_{\lesssim 0} v^j\|_{L^\infty L^2_x} \right\|_{L^2_t L^2_x} \lesssim \sqrt{T} \|v^j\|_{L^\infty L^2_x} \|v^{j'}\|_{L^2_t L^2_x} < \infty.$$

This allows us to use DCT; we are now reduced to show that

$$Q_{\sim 0} v^j(t + t_n)(x + x_n) \to 0$$

for each fixed $t$ and $x$. Because $Q_{\sim 0}$ has a Schwartz convolution kernel, it suffices to show that

$$v^j(t + t_n, \cdot + x_n) \to 0$$

weakly in $H^{1/2}$ for each $t$. This follows from Lemma 4.26.

\end{proof}

\begin{lemma}[Asymptotic decoupling of $L^2_t L^2_x$ norm, II] For each $j \in \mathbb{Z}$ and $\ell' \leq J$, we have

$$\lim_{n \to \infty} \|(Q_{\sim j} V(t) w_n^j)(Q_{\lesssim j} w_n^{\ell'})\|_{L^2_t L^2_x} = 0.$$

\end{lemma}
Proof. We proceed similarly as in the proof of Lemma \ref{lem:4.26}. We may assume that $j = 0$. Using change of variables and $\lim_{T \to \infty} \|v^{t, t'}\|_{L^1_t L^\infty_x} = 0$, it suffices to show that

$$\lim_{n \to \infty} \|(Q \cdot v^{t, t'})(Q \cdot X)^n(X \cdot t^{t'}X^n + x^{t'})\|_{L^1_t L^2_x} = 0$$

whenever $T < \infty$. We shall show this assertion by DCT. Observe for any $t$ and $x$ that we have a pointwise bound

$$sup_{n \in N} |Q \cdot v^{t, t'}(x + x^{t'})| \lesssim sup_{n \in N} \|u^{t'}\|_{L^2} < \infty$$

and hence

$$\|Q \cdot v^{t, t'} \|_{L^1_t L^2_x} \lesssim \sqrt{T} \left( sup_{n \in N} \|u^{t'}\|_{L^2} \right) \|v^j\|_{L^k_x L^\infty_t} < \infty.$$}

This allows us to use DCT; we are now reduced to show that

$$[Q \cdot X^n(X \cdot X^{t'}) = 0$$

for each fixed $t$ and $x$. Because $Q \cdot X^n$ has a Schwartz convolution kernel and $t$ is fixed, it suffices to show that

$$[V(X^{t'})u^{t'}(X \cdot X^{t'}) \to 0$$

weakly in $H^\infty$. This follows from Lemma \ref{lem:4.25}.

Proof of Lemma \ref{lem:4.14}. We must show

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|F(\tilde{u}^J_n) - \sum_{j=1}^J F(v^j_n)\|_{X^\infty_t} = 0.$$}

We decompose

$$F(\tilde{u}^J_n) - \sum_{j=1}^J F(v^j_n)$$

$$= \left[ F(\tilde{u}^J_n) - (F(u^J_n) + F(V(t)w^J_n)) \right] + \left[ F(u^J_n) - \sum_{j=1}^J F(v^j_n) \right] + F(V(t)w^J_n)$$

Observe first that $F(\tilde{u}^J_n) - [F(u^J_n) + F(V(t)w^J_n)]$ is a linear combination of

$$\partial_x \left[ (u^J_n)^\ell (V(t)w^J_n)^{k+1-\ell} \right]$$

where $1 \leq \ell \leq k$. We then observe that $F(u^J_n) - \sum_{j=1}^J F(v^j_n)$ is a linear combination of

$$\partial_x \left[ (v^j_n)^\ell (v^{j'}_n)^{k+1-\ell} \right]$$

where $1 \leq \ell \leq k$ and $j \neq j'$ with $j, j' \leq J$. Therefore, in order to show our lemma, it suffices to show the following:

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|\partial_x [(u^J_n)^\ell (V(t)w^J_n)^{k+1-\ell}]\|_{X^\infty_t} = 0,$$

$$\lim_{n \to \infty} \|\partial_x [(v^j_n)^\ell (v^{j'}_n)^{k+1-\ell}]\|_{X^\infty_t} = 0,$$

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|F(V(t)w^J_n)\|_{X^\infty_t} = 0,$$

whenever $1 \leq \ell \leq k$ (and $j \neq j'$ for (4.6)).
For (4.7), combine (4.2) with the following estimate
\[ (4.7) \lesssim \lim_{J \to \infty} \lim_{n \to \infty} \|u_n^J\|_{H^{s_k}} \|V(t)w_n^J\|_{L^2_T L^\infty_x} = 0. \]

It suffices to show the remaining estimates (4.5) and (4.6).

We now show (4.5). Fix \( 1 \leq \ell \leq k \) and observe using paraproduct decomposition that
\[
\| \partial_x [u_n^J] (V(t)w_n^J)^{k+1-\ell} \|_{X^{s_k}} ^{\ell} \\
\lesssim (4.8) + (4.9) + (4.10) + (4.11) + (4.12)
\]
where
\[
(4.8) = \left\| 2^{j(s_k + \frac{1}{2})} \| (Q_{\sim J} V(t)w_n^J) (Q_{< J} u_n^J) \|_{L^1_t L^2_x} \right\| _{\ell^2_j}
\]
\[
(4.9) = \left\| 2^{j(s_k + \frac{1}{2})} \| (Q_{\sim J} u_n^J) (Q_{< J} u_n^J)^{\ell-1} (Q_{< J} V(t)w_n^J)^{k+1-\ell} \|_{L^1_t L^2_x} \right\| _{\ell^2_j}
\]
\[
(4.10) = \left\| 2^{j(s_k + \frac{1}{2})} \sum_{r \geq J} (Q_{\sim r} V(t)w_n^J) (Q_{> r} u_n^J)^{\ell-1} (Q_{< J} V(t)w_n^J)^{k+1-\ell} \|_{L^1_t L^2_x} \right\| _{\ell^2_j}
\]
\[
(4.11) = \left\| 2^{j(s_k + \frac{1}{2})} \sum_{r \geq J} (Q_{\sim r} V(t)w_n^J) (Q_{> r} u_n^J)^{\ell-2} (Q_{< J} V(t)w_n^J)^{k+1-\ell} \|_{L^1_t L^2_x} \right\| _{\ell^2_j}
\]
\[
(4.12) = \left\| 2^{j(s_k + \frac{1}{2})} \sum_{r \geq J} (Q_{\sim r} V(t)w_n^J) (Q_{> r} u_n^J)^{\ell-1} (Q_{< J} V(t)w_n^J)^{k+1-\ell} \|_{L^1_t L^2_x} \right\| _{\ell^2_j}.
\]

Of course, the terms (4.10) and (4.12) should be ignored if \( \ell = k \) and \( \ell = 1 \), respectively. In what follows, we separately estimate each term. For the terms (4.9)-(4.12), we again use asymptotic vanishing of the remainder (4.2). We apply Lemma 4.24 to obtain
\[
(4.9) \lesssim \| u_n^J \|_{X^{s_k}} \| u_n^J \|_{L^1_t L^\infty_x} \| V(t)w_n^J \|_{L^2_t L^\infty_x} ^{k+1-\ell},
\]
\[
(4.10) \lesssim \| u_n^J \|_{H^{s_k}} \| u_n^J \|_{L^1_t L^\infty_x} \| V(t)w_n^J \|_{L^2_t L^\infty_x} ^{k+1-\ell},
\]
\[
(4.11) \lesssim \| u_n^J \|_{X^{s_k}} \| u_n^J \|_{L^1_t L^\infty_x} \| V(t)w_n^J \|_{L^2_t L^\infty_x} ^{k+1-\ell},
\]
\[
(4.12) \lesssim \| u_n^J \|_{X^{s_k}} \| u_n^J \|_{L^1_t L^\infty_x} \| V(t)w_n^J \|_{L^2_t L^\infty_x} ^{k+1-\ell}.
\]
Recall that we assume \( 1 \leq \ell \leq k-1 \) for (4.10). We then apply (4.2) to conclude.

The term (4.8) is trickier. When \( \ell \neq k \), we can proceed as before and use asymptotic vanishing of the remainder (4.2). However, in case of \( \ell = k \), Lemma 4.24 only yields the bound
\[
\| u_n^J \|_{L^1_t L^\infty_x} \| V(t)w_n^J \|_{X^{s_k}} ^{k+1-\ell},
\]
where we do not have asymptotic vanishing of \( \| V(t)w_n^J \|_{X^{s_k}} \) (c.f. Remark 4.10). To resolve this difficulty, by the linear estimate, we use
\[
(4.8) \lesssim \| u_n^J \|_{H^{s_k}} \| u_n^J \|_{L^1_t L^\infty_x} \left\| 2^{j(s_k + \frac{1}{2})} \| (Q_{\sim J} V(t)w_n^J) (Q_{< J} u_n^J) \|_{L^1_t L^2_x} \right\| _{\ell^2_j}
\]
instead. Hence, it suffices to show for each $J \in \mathbb{N}$ that
\[
\lim_{n \to \infty} \left\| 2^{j_0(s_k+\frac{1}{2})} \|(Q_{\sim J} V(t))w_{n}^{j_0}(Q_{\ll j} u_{n}^{j} )\|_{L^2 \ell^2} \right\|_{2} = 0.
\]
By DCT in $j \in \mathbb{Z}$ and definition of $u_{n}^{j}$, it suffices to show that
\[
\lim_{n \to \infty} \|(Q_{\sim J} V(t))w_{n}^{j_0}(Q_{\ll j} u_{n}^{j} )\|_{L^2 \ell^2} = 0
\]
for each $j \in \mathbb{Z}$ and $1 \leq \ell' \leq J$. This follows from Lemma 4.24.

We now show (4.6). Using paraproduct decomposition and symmetric arguments in $j$ and $j'$, it suffices to show that
\[
\lim_{n \to \infty} (4.13) + (4.14) + (4.15) = 0
\]
for all $\ell = 1, 2, \ldots, k$, where
\[
(4.13) \quad \left\| 2^{m(s_k+\frac{1}{2})} \|(Q_{\sim m} v_{n}^{j'})(Q_{\ll m} v_{n}^{j} )\|_{L^2 \ell^2} \right\|_{2} = 0
\]
\[
(4.14) \quad \left\| 2^{m(s_k+\frac{1}{2})} \sum_{r \geq m} (Q_{\sim r} v_{n}^{j'})(Q_{\ll r} v_{n}^{j} )\|_{L^2 \ell^2} \right\|_{2} = 0
\]
\[
(4.15) \quad \left\| 2^{m(s_k+\frac{1}{2})} \sum_{r \geq m} (Q_{\sim r} v_{n}^{j'})(Q_{\ll r} v_{n}^{j} )\|_{L^2 \ell^2} \right\|_{2} = 0
\]
Of course, one should ignore (4.14) if $k = \ell$.

For (4.13), by Hölder’s inequality and Lebesgue’s DCT in $m \in \mathbb{Z}$, it suffices to show that
\[
\lim_{n \to \infty} \|(Q_{\sim m} v_{n}^{j'})(Q_{\ll m} v_{n}^{j} )\|_{L^2 \ell^2} = 0
\]
for each $m \in \mathbb{Z}$. This follows from Lemma 4.26.

For (4.14), recall that $1 \leq \ell \leq k - 1$. By Lemma 4.24, it suffices to show that
\[
\lim_{n \to \infty} \left\| 2^{m(s_k+\frac{1}{2})} \|(Q_{\sim m} v_{n}^{j'})(Q_{\ll m} v_{n}^{j} )\|_{L^2 \ell^2} \right\|_{2} = 0.
\]
By Lebesgue’s DCT in $m \in \mathbb{Z}$, it suffices to show that
\[
\lim_{n \to \infty} \|(Q_{\sim m} v_{n}^{j'})(Q_{\ll m} v_{n}^{j} )\|_{L^2 \ell^2} = 0
\]
for each $m \in \mathbb{Z}$. This follows from Lemma 4.26. For (4.15), one can argue similarly. This completes the proof of (4.6).

5. Preclusion of Minimal Blowup Solutions

We now fix a critical element $u$ found in Theorem 4.21. In this section, we assert that such $u$ cannot exist. This would yield a contradiction, so we conclude Theorem 1.3.

In order to preclude critical elements, we use the monotonicity formula (Proposition 2.3). Here, since $u \in C_t H^\frac{1}{2}_x$, we are forced to truncate the weight $(x - y)$ in the formula. This truncation was exploited by Dodson [6] in the defocusing (gKdV) and in many earlier works, for instance, [22]. As we work with the Hilbert transform, derivative operators, and their commutators, we are involved in several technical difficulties and tons of error terms.
In Section 5.1, we discuss how we localize the monotonicity formula and use it to prove Theorem 1.3. Estimates for the error terms appeared in Section 5.1 will be postponed to Section 5.2.

5.1. **Localized Interaction Functional.** Set large parameters $R \gg 1$ and $R_1 \ll R$ (e.g. $R_1 := R^{1-\epsilon}$ for some $0 < \epsilon \ll 1$). Define a smooth even cutoff function $\chi_R$ on $\mathbb{R}$ satisfying $0 \leq \chi \leq 1$, $\chi(x) = 0$ if $|x| \geq R + R_1$, and $\chi(x) = 1$ if $|x| \leq R$. We can further assume that $||\partial_x^j \chi_R||_{L^\infty} \lesssim_R R_1^{-j}$ and the support of $\partial_x^j \chi_R$ has measure $\leq 2R_1$ whenever $j \geq 1$. Define $\Phi(x) := \int_0^x (\chi_R^2 * \chi_R^2)(s) ds$. Then $\Phi$ is an odd function such that $\Phi(x) \approx x$ for $|x| \lesssim R$ and $|\Phi(x)| \approx R$ for $|x| \gg R$. Moreover, the convolution in $\Phi$ is well-suited for the monotonicity formula of interaction form. Note that

$$\frac{1}{R}(\chi_R^2 * \chi_R^2)(y-x) = \int_{\mathbb{R}} \chi_R^2(y-s)\chi_R^2(x-s) \frac{ds}{R}.$$  

Similarly to the work of Dodson [6], we define the **localized interaction functional** by

$$M(t) := \int_{\mathbb{R} \times \mathbb{R}} \Phi(y-x)\rho(t,x)e(t,y)dxdy. \quad (5.1)$$

If $u$ lies merely in $C_t H^{\frac{1}{2}}_x$, the integral $(5.1)$ should not be interpreted in absolute sense as we now explain. As in other defocusing equations, one may use a crude estimate

$$|M(t)| \lesssim \int_{\mathbb{R} \times \mathbb{R}} R|\rho(t,x)||e(t,y)||dxdy \lesssim_R \|u\|_{L^\infty} \int_{\mathbb{R}} |e(t,y)|dy.$$  

If $e(t,y)$ were pointwise nonnegative, then $\int_{\mathbb{R}} |e(t,y)|dy = E(u)$ so $M(t)$ can be well-defined for $u \in C_t H^{\frac{1}{2}}_x$. In our case, however,

$$e(t,y) = \frac{1}{2}u\hat{\mathcal{H}}u_x + \frac{1}{k+2}u^{k+2}(t,y)$$

is not a pointwise positive function, c.f. we only have $E(u) = \int_{\mathbb{R}} e(t,y)dy > 0$. The quantity $\int_{\mathbb{R}} |e(t,y)|dy$ does not seem to be estimated by $\|u\|_{L^\infty} H^{\frac{1}{2}}_x$. The following lemma says how we can interpret $(5.1)$ appropriately.

**Lemma 5.1** (**$H^{\frac{1}{2}}_x$ bound for localization**). Let $f \in H^1$ and $g \in C^\infty$ with $\|g\|_{W^{1,\infty}} := \|g\|_{L^\infty} + \|\partial_x g\|_{L^\infty} \lesssim H$. Then,

$$\left| \int gf\mathcal{H}f_x \right| + \left| \int gf^{k+2} \right| \lesssim H(\|f\|^2_{H^{\frac{1}{2}}_x} + \|f\|^{k+2}_{H^{\frac{1}{2}}_x}).$$

In particular, if $u \in C_t H^{\frac{1}{2}}_x$, then

$$\sup_{t \in \mathbb{R}} |M(t)| \lesssim_{A(u)} R. \quad (5.2)$$

**Proof.** By Hölder’s inequality and Sobolev embedding $H^{\frac{1}{2}}_x \hookrightarrow L^{k+2}$, we have

$$\left| \int gf^{k+2} \right| \lesssim H\|f\|^{k+2}_{H^{\frac{1}{2}}_x}.$$  

In order to estimate $\int gf\mathcal{H}f_x$, it suffices to establish the inequality

$$\|gf\|_{H^{\frac{1}{2}}_x} \lesssim H\|f\|_{H^{\frac{1}{2}}_x}$$
because
\[ \left| \int gfH_f x \right| = \left| \int D^{1/2}_x (gf)D^{1/2}_x f \right| \lesssim \|g\|_{H^{1/2}} \|f\|_{H^{1/2}}. \]
Applying the Stein-Weiss interpolation between
\[ \|gf\|_{L^2} \lesssim \|g\|_{L^\infty} \|f\|_{L^2} \lesssim H \|f\|_{L^2} \]
and
\[ \|gf\|_{H^1} \lesssim (\|g\|_{L^\infty} + \|\partial_x g\|_{L^\infty}) \|f\|_{L^2} + \|g\|_{L^\infty} \|\partial_x f\|_{L^2} \lesssim H \|f\|_{H^1}, \]
we get
\[ \|gf\|_{H^{1/2}} \lesssim H \|f\|_{H^{1/2}}. \]
To show the remaining assertion, pretend \( u(t) \in H^1 \) for each \( t \in \mathbb{R} \). For any fixed \( x \), note that the function \( y \mapsto \Phi(y-x) \) has \( W^{1,\infty} \) norm \( \lesssim R \). We then see that
\[ \sup_{t \in \mathbb{R}} |M(t)| = \sup_{t \in \mathbb{R}} \|\rho(t,x)\|_{L^1_\Phi} \int_y \Phi(y-x)e(t,y)\|_{L^\infty_y} \lesssim_{A(u)} R. \]
The argument allows us to use density argument for \( u(t) \in H^{1/2} \), so the assertion is true for \( u \in C_tH^{1/2}_x \).

In the following, a number of error terms appears and computations are involved, so we abbreviate the notations to present as neatly as possible. We write \( \chi_{R,s} := \chi_R(\cdot-s) \). For operators \( A \) and \( B \), we let \( [A,B] = AB - BA \) be the commutator. We then abbreviate various integrals as follows.
\[ \int_x := \int_{\mathbb{R}} dx, \quad \int_y := \int_{\mathbb{R}} dy, \quad \int_z := \int_{\mathbb{R}} dz, \]
\[ \int_s := \int_{\mathbb{R}} \frac{ds}{R}, \quad \int_r := \int_0^1 dr. \]
Notice that we integrate over \( \frac{ds}{R} \) in \( s \)-variable, and only on \([0,1]\) for \( r \)-variable. Finally, we use \( o_{A(u)}(R^1) \) as \( R \to \infty \) for a function decaying to zero. If we use \( R_1 = R^{1-\epsilon} \), then one can observe \( o_{A(u)}(R^1) \lesssim C_{A(u)}R^{-\epsilon'} \) for some \( \epsilon' > 0 \).

As a consequence of the monotonicity formula, we state the main proposition of this section.

**Proposition 5.2.** Let \( u \in C_tH^{1/2}_x \) be an almost periodic global solution to \((\text{gBO})\). If \( R \) is sufficiently large, then we have a lower bound
\[ |M(t_1) - M(t_2)| \gtrsim_u |t_1 - t_2| \]
for any \( t_1, t_2 \in \mathbb{R} \).

From a contradiction of \((\text{5.2})\) and \((\text{5.3})\) for the critical element constructed in Theorem \((\text{4.2})\) we deduce that Theorem \((\text{1.3})\) holds.

**Proof.** We rely on density argument and assume \( u(t,x) \) lies in \( C_{t,\text{loc}}^{\infty}H_x^{\infty} \). We assume that \( u \) is almost periodic modulo spatial translations, but we are not sure that \( u \) can be approximated by smooth almost periodic solutions \( u_n \). However, we shall use density argument to \((\text{5.10})\), which is true for arbitrary smooth solutions, and hence, for \( H^{1/2}_x \) solutions. We then apply almost periodicity of \( u \) to deduce \((\text{5.3})\). From now on, we assume that \( u \) is a global smooth solution.
In light of fundamental theorem of calculus, we shall calculate \( \partial_t M(t) \). When we calculate \( \partial_t M(t) \), as \( t \) is fixed, we drop the variable \( t \) for convenience. We start with

\[
\partial_t M(t) = (5.4) + (5.5).
\]

where

\begin{align*}
(5.4) & \quad \int_{x,y} \Phi(y - x) \rho(x) \partial_t e(y), \\
(5.5) & \quad \int_{x,y} \Phi(y - x) e(y) \partial_t \rho(x).
\end{align*}

We first estimate (5.4). Note that

\[
(5.4) = \int_{x,y} \Phi(y - x) \rho(x) \left[ \frac{1}{2} u_t H \partial_y u + \frac{1}{2} u H \partial_y u_t + u^{k+1} u_t \right](y).
\]

We rearrange the terms by their degree, namely, \( 2 \), \( k+2 \), and \( 2k+2 \) in \( u(y) \). More precisely, we decompose

\[
(5.4) = (5.6) + (5.7) + (5.8)
\]

where

\begin{align*}
(5.6) & \quad \int_{x,y} \Phi(y - x) \rho(x) \left[ \frac{1}{2} u_t H \partial_y u + \frac{1}{2} u H \partial_y u_t + u^{k+1} u_t \right](y), \\
(5.7) & \quad \int_{x,y} \Phi(y - x) \rho(x) \left[ -\frac{1}{2} \partial_y (u^{k+1}) (H \partial_y u) - \frac{1}{2} u H \partial_y (u^{k+1}) - u^{k+1} H \partial_y u \right](y), \\
(5.8) & \quad \int_{x,y} \Phi(y - x) \rho(x) \left[ -u^{k+1} \partial_y (u^{k+1}) \right](y).
\end{align*}

For the term, (5.6), we use integration by parts in \( y \)-variable to have

\[
(5.6) = \frac{1}{2} \int_{s} \rho[\chi_{R,s}] u(x) \int_{y} \frac{1}{4} (\chi_{R,s} H u_y)^2 + \frac{3}{4} (\chi_{R,s} u y)^2 - \frac{1}{2} u^2 \partial_y (\chi_{R,s})^2 \}(y) - \frac{1}{2} \int_{s} \rho(x) E_1(y),
\]

where the error term \( E_1 \) is defined by

\[
E_1(y) := \Phi''(y - x) u^2(y).
\]

Similarly, we observe that

\[
(5.8) = \frac{1}{2} \int_{s} \rho[\chi_{R,s}] u(x) \int_{y} \chi_{R,s} u^{2k+2}(y) \geq \int_{s} \rho[\chi_{R,s}] u(x) \int_{y} \frac{1}{2} \chi_{R,s} u^{2k+2}(y).
\]
We now consider (5.7). Let us temporarily write $\Phi(y - x)$ and $u^{k+1}(y)$ by $Q(y)$ and $v(y)$, respectively. Integration by parts in $y$-variable and self-adjointness of $\mathcal{H}\partial_y$ yield

\[
(5.7) = \int_{x,y} Q(y) \rho(x) \left( -\frac{1}{2} v_y \mathcal{H}\partial_y u - \frac{1}{2} u \mathcal{H}\partial_y v_y - v \mathcal{H}\partial_y u \right)(y)
\]
\[
= \int_{x,y} Q'(y) \rho(x) v_h \mathcal{H}\partial_y u(y) + \frac{1}{2} \int_{x,y} Q(y) \rho(x) [v_y \mathcal{H}\partial_y u] - u \mathcal{H}\partial_y v_y] \)(y)
\]
\[
= \int_{x,y} Q'(y) \rho(x) v_h \mathcal{H}\partial_y u(y) + \frac{1}{2} \int_{x,y} \rho(x) \left( u [\mathcal{H}\partial_y v] v(y) \right)(y).
\]

We then use

\[
[\mathcal{H}\partial_y, Q] v_y = \mathcal{H}(Q' v_y) + [\mathcal{H}, Q] \partial_y v
\]

\[
= \mathcal{H}\partial_y (Q' v) - \mathcal{H}(Q'' v) + [\mathcal{H}, Q] \partial_y v
\]

and self-adjointness of $\mathcal{H}\partial_y$ again to obtain

\[
(5.7) = \frac{3}{2} \int_{x,y} \Phi'(y - x) \rho(x) v_h \mathcal{H}\partial_y u(y) + \frac{1}{2} \int_{x,y} \rho(x) [E_2 + E_3](y)
\]

where error terms $E_2$ and $E_3$ are defined by

\[
E_2 := u [\mathcal{H}, \Phi(\cdot - x)] v_{yy},
\]

\[
E_3 := \Phi''(\cdot - x) u^{k+1} \mathcal{H} u.
\]

Therefore, collecting altogether,

\[
(5.12) \int_{x,y} \Phi(y - x) \rho(x) \partial_t e(y)
\]

\[
= \int_s \int_x \rho(\chi_{R,s} u)(x) \int_y \hat{y}^2 \mathcal{H} u_y \hat{y}^2 \frac{1}{4} (\mathcal{H} u_y )^2 + \frac{3}{4} u_y^2 + \frac{3}{2} \hat{u}^{k+1} \mathcal{H} u_y + \frac{1}{2} u^{2k+2}\](y)
\]

\[
(5.13) + \frac{1}{2} \int_{x,y} \rho(x) [-E_1 + E_2 + E_3](y)
\]

where the error terms $E_1, E_2, E_3$ are defined in (5.9), (5.10), and (5.11), respectively. In the next subsection (Lemmas 5.5 and 5.4), we prove that

\[
(5.14) \geq \int_s \int_x \rho(\chi_{R,s} u)(x) \int_y \hat{y}^2 \mathcal{H} u_y \hat{y}^2 \left( \mathcal{H} u_y \right)^2 - o_{\mathcal{A}(u), \mathcal{R}(1)},
\]

\[
(5.15) = o_{\mathcal{A}(u), \mathcal{R}(1)}.
\]

This takes care of (5.4).

We now compute (5.5).

\[
(5.5) = 2 \int_{x,y} \Phi(y - x) e(y) u(x) \mathcal{H} \partial_x u \partial_x u - \partial_x (u^{k+1}))(x)
\]

\[
(5.14) = - \int_s \int_x \hat{y}^2 \mathcal{H} u_y \hat{y}^2 \left( \mathcal{H} u_y \right)^2 (y)
\]

\[
+ 2 \int_{x,y} \Phi(y - x) [u_x \mathcal{H} u_x] e(y).
\]
We regard (5.14) as the main term and (5.15) as an error term. In the next subsection (Lemmas 5.7 and 5.6), we prove that
\[
\partial_t M(t) = (5.30) + (5.3) \\
\geq \int_s \int_x \rho[\chi_R u] u(x) \int_y k[\chi_R u](y) - \int_x j[\chi_R u](x) \int_y e[\chi_R u](y) - o_{\text{A(u)},R}(1).
\]

In conclusion, we have
\[
\partial_t M(t) = (5.30) + (5.3) \\
\geq \int_s \int_x \rho[\chi_R u] u(x) \int_y k[\chi_R u](y) - \int_x j[\chi_R u](x) \int_y e[\chi_R u](y) - o_{\text{A(u)},R}(1).
\]

Combining this with the monotonicity formula (Proposition 5.2), we obtain
\[
\partial_t M(t) \geq \frac{k^2}{2(k+2)^2} \int_s \left( \int_x (\chi_R u)^2(x) \right)^2 \left( \int_y (\chi_R u)^{k+2}(y) \right)^2 - o_{\text{A(u)},R}(1).
\]

By the fundamental theorem of calculus, we finally obtain
\[
|M(t_1) - M(t_2)| \geq \int_{t_1}^{t_2} \left( \int_x (\chi_R u)^2(t,x) \right)^2 \left( \int_y (\chi_R u)^{k+2}(t,y) \right)^2 dt - |t_1 - t_2| o_{\text{A(u)},R}(1)
\]
whenever \( t_1 > t_2 \). By density argument, (5.16) indeed holds whenever \( u \) is an arbitrary \( H \) solutions.

We now assume that \( u \) is almost periodic modulo spatial translations. In light of Proposition 4.22, there exist spatial center \( \{x(t)\}_{t \in \mathbb{R}} \) and \( R_0 \) depending on \( u \) such that whenever \( R > R_0 \) and \( |s - x(t)| \leq \frac{R}{2} \), we have
\[
\int_x (\chi_R u)^2(t,x) \geq \int_{|x-x(t)| \leq \frac{R}{2}} u^2(t,x) dx \geq \frac{M(u)}{2}
\]
and
\[
\int_y (\chi_R u)^{k+2}(t,y) \geq \int_{|y-x(t)| \leq \frac{R}{2}} u^{k+2}(t,y) dy \geq u 1.
\]

Therefore,
\[
(5.16) \geq u |t_1 - t_2| - |t_1 - t_2| o_{\text{A(u)},R}(1)
\]
for all \( R > R_0 \). Choosing \( R \) even larger, we finally have
\[
|M(t_1) - M(t_2)| \geq u |t_1 - t_2|.
\]

This completes the proof. \( \square \)

5.2. Various Error Estimates in proof of Proposition 5.2. In this subsection, we estimate various error terms appeared in the proof of Proposition 5.2. We have seen that many errors terms involve commutators of the Hilbert transform and a smooth function, say \([\mathcal{H}, Q]\) for some \( Q \in C^\infty \).

Using the formula of the Hilbert transform, we obtain
\[
([\mathcal{H}, Q]u)(y) = \frac{1}{\pi} \int_z Q(z) - Q(y) \frac{u(z)}{y-z} dz = -\frac{1}{\pi} \int_z \int_r Q'(rz + (1-r)y) u(z).
\]
Thus, by Fubini’s theorem,
\[
\int_y v[\mathcal{H}, Q]u = -\frac{1}{\pi} \int_r \int_{y,z} Q'(rz + (1 - r)y)v(y)u(z).
\]
If \( u \) or \( v \) contain derivatives, we can use integration by parts to move derivatives to \( Q \). For instance,
\[
\int_y v_y[\mathcal{H}, Q]u_y = -\frac{1}{\pi} \int_r \int_{y,z} r(1 - r)Q''(rz + (1 - r)y)v(y)u(z).
\]
In practice, we will use \( \Phi, \chi_{R,s}, \) and \( \chi'_{R,s} \) in place of \( Q \). Derivatives of \( Q \) decay in \( R \) as \( R \) grows. This says that error terms involving commutator should be well-estimated as \( R \) grows. Our main tool is Schur’s test, we state the exact formulation we use here, for reader’s convenience.

**Lemma 5.3** (Schur’s test). Let \( K(y, z) \) be a kernel satisfying the following three bounds:
1. (height) \( \|K\|_{L^\infty} \lesssim H \),
2. (size of \( y \)-support) \( \text{sup}_y |\{y : K(y, z) \neq 0\}| \lesssim R_1 \), and
3. (size of \( z \)-support) \( \text{sup}_y |\{z : K(y, z) \neq 0\}| \lesssim R_2 \).

Then,
\[
\left| \int_{y,z} K(y, z)u(y)v(z) \right| \lesssim H \sqrt{R_1 R_2} \|u\|_{L^2} \|v\|_{L^2}.
\]

**Proof.** The proof is fairly standard. \( \square \)

**Lemma 5.4** (Estimate of \( (5.13) \)). We have
\[
(5.13) = o_{A(u), R}(1).
\]

**Proof.** From Hölder’s inequality
\[
\left| \int_{x,y} \rho(x)E_i(x, y) \right| \lesssim \|\rho(x)\|_{L^1} \|E_i(x, y)\|_{L^\infty},
\]

it suffices to show
\[
\|E_i(x, y)\|_{L^\infty} = o_{A(u), R}(1)
\]
for each \( i = 1, 2, 3 \). When \( i = 1, 3 \), this follows easily by Hölder’s inequality in \( y \)-variable. When \( i = 2 \), observe that
\[
\int_y E_2(x, y) = -\frac{1}{\pi} \int_r \int_{y,z} r^2 \Phi''(rz + (1 - r)y - x)u(y)u^{k+1}(z).
\]
Whenever \( r \) is fixed, the kernel \( (y, z) \mapsto \Phi''(rz + (1 - r)y - x) \) has height \( \lesssim \frac{1}{R_1^2}, \) \( y \)-support size \( \lesssim \frac{R}{R_1^2} \), and \( z \)-support size \( \lesssim \frac{R}{r} \). Thus by Minkowski’s inequality in \( r \) and Schur’s test, we obtain
\[
\left| \int_y E_2(x, y) \right| \lesssim A(u) \int_r r^{\frac{2}{3}} (1 - r)^{-\frac{1}{3}} \frac{R}{R_1^2} = o_{A(u), R}(1).
\]
This completes the proof. \( \square \)

**Lemma 5.5** (Estimate of \( (5.12) \)). We have
\[
(5.12) \geq \int_s \int \rho(x)\chi_{R,s}(x) \int_y k(x_R,s)u(y) - o_{A(u), R}(1).
\]
Proof. Recall the expression (5.12)
\[
\int_s \int_x \rho(x, s, u)(x) \int_y \chi_{s, y}(y) \left[ \frac{1}{4} (\mathcal{H} u_y) \right]^2 + \frac{3}{4} u_y^2 + \frac{3}{2} u^{k+1} \mathcal{H} u_y + \frac{1}{2} u^{2k+2}(y).
\]

We first approximate \( \int_y (\chi_{s, y} \mathcal{H} u_y)^2 \) and \( \int_y (\chi_{s, y} u_y^2) \) by \( \int_y (\chi_{s, y} u_y)^2 \). Invoking Lemma 2.6, \( \int_y \chi_{s, y} u_y^{k+1} \mathcal{H} u_y \) is essentially greater than \( \int_y \chi_{s, y} u_y^{2k+2} \), with error \( \lesssim A(u) \|2 \mathcal{H} u^{(s)}\|_{L^\infty} \). We then approximate \( \int_y (\chi_{s, y} u_y)^{k+1} \mathcal{H} u_y \) by \( \int_y (\chi_{s, y} u_y)^{2k+2} \mathcal{H} (\chi_{s, y} u_y) \). Note that \( \int_y \chi_{s, y} u_y^{2k+2} \) is always greater than or equal to \( \int_y (\chi_{s, y} u_y)^{2k+2} \).

Moreover, from Hölder’s inequality and Fubini’s theorem, we have
\[
\left| \int_s \int_x \rho(x, s, u)(x) \int_y g(y, s) \right| \leq \| \int_s \int_x \rho(x, s, u)(x) \|_{L^2(\mathbb{R})} \| \int_y g(y, s) \|_{L^\infty(\mathbb{R})} \lesssim \| u \|_{L^\infty(\mathbb{R})} \| \int_y g(y, s) \|_{L^\infty(\mathbb{R})}.
\]

Therefore, to show our lemma, it suffices to show for each \( s \) that
\[
(5.17) \quad \int_y (\chi_{s, y} \mathcal{H} u_y)^2 = \int_y (\chi_{s, y} u_y)^2 + o_A(u), R(1)
\]
\[
(5.18) \quad \int_y (\chi_{s, y} u_y)^2 = \int_y (\chi_{s, y} u_y)^2 + o_A(u), R(1)
\]
\[
(5.19) \quad \int_y \chi_{s, y} u_y^{k+1} \mathcal{H} u_y = \int_y (\chi_{s, y} u_y)^{k+1} \mathcal{H} (\chi_{s, y} u_y) + o_A(u), R(1),
\]
and
\[
(5.20) \quad \sup_{s \in \mathbb{R}} \| (\chi_{s, y} u_y)^{2k+2} \|_{L^\infty(\mathbb{R})} \| u \|_{L^\infty(\mathbb{R})}^{k+2} = o_A(u), R(1).
\]

Since \( \| \partial_y \mathcal{H} (\chi_{s, y} u_y) \|_{L^\infty(\mathbb{R})} \lesssim 1 \), the (5.20) easily follows. We now establish three remaining estimates (5.17), (5.18), and (5.19).

For (5.17), observe that
\[
\chi_{s, y} (\mathcal{H} u_y)^2 = \chi_{s, y} \mathcal{H} u_y \mathcal{H} (\chi_{s, y} u_y) - (\chi_{s, y} \mathcal{H} u_y)[\mathcal{H} \partial_y, \chi_{s, y}] u
\]
\[= [\mathcal{H} (\chi_{s, y} u_y)]^2 - (\mathcal{H} (\chi_{s, y} u_y) + (\chi_{s, y} \mathcal{H} u_y))[\mathcal{H} \partial_y, \chi_{s, y}] u.
\]

We then use two different expressions of \([\mathcal{H} \partial_y, \chi_{s, y}]\)
\[
[\mathcal{H} \partial_y, \chi_{s, y}] u = [\mathcal{H}, \chi_{s, y}] u_y + \mathcal{H} (\chi_{s, y} u_y) = [\mathcal{H}, \chi_{s, y}] u_y + [\mathcal{H}, \chi_{s, y}] u + \chi_{s, y} \mathcal{H} u
\]
to obtain
\[
[\mathcal{H} (\chi_{s, y} u_y)]^2 - \chi_{s, y} (\mathcal{H} u_y)^2
\]
\[= \mathcal{H} (\chi_{s, y} u_y) ([\mathcal{H}, \chi_{s, y}] u_y + \mathcal{H} (\chi_{s, y} u_y))
\]
\[= (\chi_{s, y} \mathcal{H} u_y) ([\mathcal{H}, \chi_{s, y}] u_y + \mathcal{H} (\chi_{s, y} u_y)).
\]
Finally, we observe by integration by parts that

\[ \int_y \mathcal{H}(\chi_{R,s}u)_y[\mathcal{H}, \chi_{R,s}]u_y \]

\[ \leq \frac{1}{\pi} \int \frac{1}{r} (1-r) \left| \int_{y,z} \chi''_{R,s}(rz + (1-r)y) \chi_{R,s}(y) u(y) u(z) \right|, \]

Note that the kernel \( (y, z) \mapsto \chi''_{R,s}(rz + (1-r)y) \chi_{R,s}(y) \) has height \( \lesssim \frac{1}{R^3} \), \( y \)-support size \( \lesssim R \), and \( z \)-support size \( \lesssim \frac{R}{r} \). Therefore,

\[ \left| \int_y \mathcal{H}(\chi_{R,s}u)_y[\mathcal{H}, \chi_{R,s}]u_y \right| \lesssim A(u) \int_r \frac{R}{R^3} R^{1/2} (1-r) = o_A(u),R(1). \]

Next, we observe by integration by parts that

\[ \int_y \mathcal{H}(\chi_{R,s}u)_y \mathcal{H}(\chi'_{R,s}u) = \int_y (\chi_{R,s}u)_y (\chi'_{R,s}u) = \int_y (\chi'_{R,s})^2 u^2 \frac{1}{4} \int_y (\chi''_{R,s})^2 u^2 \]

and use Hölder’s inequality on each term. Next, again by integration by parts, we have

\[ \left| \int_y (\chi_{R,s}u_y)[\mathcal{H}, \chi_{R,s}]u_y \right| \]

\[ \leq \left| \int_y (\chi_{R,s}u)_y[\mathcal{H}, \chi_{R,s}]u_y \right| + \left| \int_y (\chi'_{R,s}u)[\mathcal{H}, \chi_{R,s}]u_y \right|. \]

and

\[ \left| \int_y (\chi_{R,s}u_y)[\mathcal{H}', \chi_{R,s}]u \right| \]

\[ \leq \left| \int_y (\chi_{R,s}u)_y[\mathcal{H}, \chi'_{R,s}]u \right| + \left| \int_y (\chi'_{R,s}u)[\mathcal{H}, \chi_{R,s}]u \right|. \]

On each term in the right hand sides, one can proceed similarly as before. Finally, we observe by integration by parts that

\[ \int_y [\chi_{R,s} \chi'_{R,s} \mathcal{H} u_y \mathcal{H} u](y) = -\frac{1}{4} \int_y [(\chi''_{R,s})^2 (\mathcal{H} u)^2](y) \]

This can be treated using Hölder’s inequality.

For (5.15), observe that

\[ \chi^2_{R,s} u^2_y = (\chi_{R,s} u)_y^2 - 2 \chi_{R,s} \chi'_{R,s} u_y u - (\chi'_{R,s})^2 u^2. \]

Using integration by parts, we obtain

\[ \int_y \chi^2_{R,s} u^2_y = \int_y (\chi_{R,s} u)_y^2 - \frac{1}{2} \int_y (\chi''_{R,s})^2 u^2 - \int_y (\chi'_{R,s})^2 u^2. \]

We then apply Hölder’s inequality to the last two terms in the right hand side.

For (5.19), observe that

\[ \chi^{k+2}_{R,s} u^{k+1} \mathcal{H} u_y = (\chi_{R,s} u)^{k+1} \mathcal{H}(\chi_{R,s} u)_y - (\chi_{R,s} u)^{k+1} [\mathcal{H} \partial_y, \chi_{R,s}] u. \]
Use $[\mathcal{H}\partial_y, \chi_{R,s}]u = [\mathcal{H}, \chi_{R,s}]u_y + \mathcal{H}(\chi'_{R,s}u)$, Schur’s test for the commutator term, and Hölder’s inequality for the remaining term. This completes the proof. □

**Lemma 5.6** (Estimate of (5.15)). We have

$$\tilde{o}_A(u), R(1).$$

**Proof.** Observe that

$$[\mathcal{H}, \Phi(y - \cdot)]u(x) = (1.15) = - \int_{x,y} (u_x[\mathcal{H}, \Phi(y - \cdot)]u_x)(x)e(y).$$

By Lemma 5.1 it suffices to show that the function

$$g(y) := \int_{x} (u_x[\mathcal{H}, \Phi(y - \cdot)]u_x)(x)$$

has $W^{1,\infty}$ norm $o_A(u), R(1)$. Observe that

$$g(y) = \frac{1}{\pi} \int_{x} r(1 - r)^{-\frac{3}{2}} \|u - (1 - r)xu\|^2 - (1 - r)xu(z).$$

Whenever $r$ is fixed, the kernel $(x, z) \mapsto \Phi''(y - rz - (1 - r)x)u(x)u(z)$. By Schur’s test, we have

$$||g||_{L^\infty} \lesssim_A (1 - r)^{-\frac{3}{2}} R^{-1} \lesssim_A \frac{1}{R}.$$

Similarly, we have $||g||_{W^{1,\infty}} \lesssim_A \frac{1}{R^2}$. □

**Lemma 5.7** (Estimate of (5.14)). We have

$$\tilde{\tilde{o}}_A(u), R(1).$$

**Proof.** Observe that

$$j[\chi_{R,s}u] = \chi_{R,s}^2 j[u] + 2(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) - \frac{2(k + 1)}{k + 2} \tilde{E}_4,$$

$$e[\chi_{R,s}u] = \chi_{R,s}^2 e[u] + \frac{1}{2}(\tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3) - \frac{1}{k + 2} \tilde{E}_4.$$

where

$$\tilde{E}_1(s, x) = (\chi_{R,s}u[\mathcal{H}, \chi'_{R,s}]u)(x),$$

$$\tilde{E}_2(s, x) = (\chi_{R,s}u[\mathcal{H}, \chi_{R,s}]u_x)(x),$$

$$\tilde{E}_3(s, x) = (\chi_{R,s}\chi'_{R,s}u\mathcal{H}u)(x),$$

$$\tilde{E}_4(s, x) = ((\chi_{R,s}^2 - \chi_{R,s}^{k+2})u^{k+2})(x).$$

Therefore, it suffices to show that

$$\tilde{E}_i(s, x) = o_A(u), R(1),$$

where $i = 1, 2, 3, 4$. □
whenever \(i, i' = 1, 2, 3, 4\).

Let us give a remark here. In the proof of Lemma 5.5, we used Hölder’s inequality in \(s\) to reduce ourselves into the situation when \(s\) is fixed. Indeed, because \(\rho(x)\) is nonnegative, we could use Fubini’s theorem to obtain

\[
\| \int_x \chi_{R,s}^2(x) \rho(x) \|_{L^1(\mathbb{R}^2)} = \| \chi_{R}^2(s) \|_{L^1(\mathbb{R}^2)} \int_x \rho(x) \lesssim A(u) 1.
\]

However, in the case where \(\rho\) is replaced by \(j\), we do not have almost everywhere nonnegativity of \(j\). If we take \(L^1(\mathbb{R}^2)\) norm, we only estimate

\[
\| \int_x \chi_{R,s}^2(x) j(x) \|_{L^1(\mathbb{R}^2)} \leq \| \chi_{R}^2(s) \|_{L^1(\mathbb{R}^2)} \int_x |j(x)| \lesssim \int_x |j(x)|,
\]

where the right hand side cannot be estimated in terms of the \(H^\frac{1}{2}\) norm.

Back to the proof, we now estimate (5.21) and (5.22). To avoid the aforementioned difficulty, we use Lemma 5.1 instead. Hence, it suffices to show that the function

\[ g_i(x) := \int_s \int_y \chi_{R}^2(x) \tilde{E}_i(s,y) \]

has \(W^{1,\infty}\) norm bounded by \(o_{A(u),R}(1)\) for each \(i = 1, 2, 3, 4\).

When \(i = 1\), observe that

\[ g_1(x) = -\frac{1}{\pi} \int_r \int_y K_1(x, r, y, z) u(y) u(z), \]

\[ K_1(x, r, y, z) := \int_s r \chi_{R}^2(x) \chi_{R}(y-s) \chi_{R}(rz + (1-r)y-s). \]

When \(x\) and \(r\) are fixed, the kernel \((y, z) \mapsto K_1(x, r, y, z)\) has height

\[ |K_1(x, r, y, z)| \leq \int_s \chi_{R}^n(rz + (1-r)y-s) \lesssim \frac{1}{R_1^2} \cdot \frac{R_1}{R} \lesssim \frac{1}{R R_1}. \]

It has \(y\)-support \(\lesssim R\) (by ignoring \(\chi_{R}\) part), and \(z\)-support \(\lesssim \frac{R}{R_1}\). By Schur’s test and Minkowski’s inequality in \(r\), we have

\[ \|g_1\|_{L^\infty} \lesssim A(u) \int_r \frac{1}{R_1} r^{-\frac{1}{2}} = o_{A(u),R}(1). \]

Note that the integral formula of \(\partial_z g_1\) has kernel \(\partial_z K_1\). Because \(\partial_z K_1\) obeys an even better estimate, we conclude that \(\|g_1\|_{W^{1,\infty}} = o_{A(u),R}(1)\).

When \(i = 2\), observe that

\[ g_2(x) = -\frac{1}{\pi} \int_r \int_y K_2(x, r, y, z) u(y) u(z), \]

\[ K_2(x, r, y, z) := \int_s r \chi_{R}^2(x) \chi_{R}(y-s) \chi_{R}(rz + (1-r)y-s). \]

One can proceed as exactly same way as when \(i = 1\).

When \(i = 3\), observe that

\[ g_3(x) = \int_y \int_s \chi_{R}^2(x) [\chi_{R} \chi'_{R}](y-s) [u \mathcal{H} u](y). \]
Using Hölder’s inequality in $y$,
\[
|g_3(x)| \leq \left\| \int_y R(x-s)[\chi Ry^2(y-s)] \right\|_{L^\infty_y} \|u\mathcal{H}u\|_{L^1_y} \lesssim \frac{1}{R} \|u\|_{L^2}^2.
\]
Similarly, we have a better estimate for $\partial_x g_3$, and so $\|g_3\|_{W^{1,\infty}} = o_{A(u),R}(1)$.

When $i = 4$, we have
\[
g_4(x) = \int_y \int_s \chi_R^2(x-s)[\chi_R^2 - \chi_R^{k+2}](y-s)u^{k+2}(y).
\]
Using Hölder’s inequality in $y$ as before,
\[
|g_4(x)| \leq \left\| \int_y \chi_R^2(x-s)[\chi_R^2 - \chi_R^{k+2}](y-s) \right\|_{L^\infty_y} \|u\|_{L^{k+2}} \lesssim \frac{R_1}{R} \|u\|_{L^{k+2}}.
\]
For $\partial_x g_4$, we have a better estimate. Thus $\|g_4\|_{W^{1,\infty}} = o_{A(u),R}(1)$.

Next, we turn to the proof of (5.23). By Hölder’s inequality in $s$, it suffices to show that
\[
\left\| \int_y \tilde{E}_1(s, y) \right\|_{L^2_y} = o_{A(u),R}(1).
\]
When $i = 1$, we observe that
\[
\int_y \tilde{E}_1(s, y) = \int_R \int_{y,z} \tilde{K}_1(r, s, y, z)u(y)u(z),
\]
\[
\tilde{K}_1(r, s, y, z) := \chi_R(y-s)\chi_R''(rz + (1-r)y - s)
\]
whenever $t$ and $s$ are fixed, the kernel $(y, z) \mapsto \tilde{K}_1(r, s, y, z)$ has height $\lesssim \frac{1}{Rt}$, $y$-support size $\lesssim R$ (ignoring $\chi_R''$), and $z$-support size $\lesssim \frac{R}{t}$. Therefore, by Schur’s test,
\[
\left\| \int_{y,z} \tilde{K}_1(r, s, y, z)u(y)u(z) \right\|_{L^\infty_y} \lesssim_{A(u)} R_1^{-\frac{3}{2}} R_1^{\frac{1}{2}} R^{-\frac{1}{2}}.
\]
On the other hand, whenever $r$ is fixed, the kernel $(y, z) \mapsto \|\tilde{K}_1(t, s, y, z)\|_{L^1_y}$ has height $\lesssim \frac{1}{Rt^2}$ and $z$-support size $\lesssim \frac{R}{t}$. The $y$-support of the kernel
\[
\|\tilde{K}_1(t, s, y, z)\|_{L^1_y} = \int_s \chi_R(y-s)\chi_R''(rz + (y-s) - ry)
\]
has $\lesssim \frac{R}{t}$. Therefore,
\[
\left\| \int_{y,z} \tilde{K}_1(r, s, y, z)u(y)u(z) \right\|_{L^1_y} \lesssim_{A(u)} \int_{y,z} \|\tilde{K}_1(r, s, y, z)\|_{L^1_y} |u(y)||u(z)| \lesssim_{A(u)} (Rt)^{-1} r^{-1}.
\]
Interpolating above two estimates, we have
\[
\left\| \int_y \tilde{E}_1(s, y) \right\|_{L^2_y} \lesssim \int_r \left\| \int_{y,z} \tilde{K}_1(r, s, y, z)u(y)u(z) \right\|_{L^2_y} \lesssim_{A(u)} \int_r R^{-\frac{1}{2}} R_1^{-\frac{1}{2}} R^{-\frac{1}{2}} \lesssim_{A(u)} R^{-\frac{1}{2}} R_1^{-\frac{1}{2}}.
\]
Note that the case of $i = 2$ can be treated in a similar manner.
When \( i = 3 \), observe that
\[
\left\| \int_y \tilde{E}_3(s,y) \right\|_{L^2(\frac{dy}{R^4})} \lesssim \left\| \tilde{E}_3(s,y) \right\|_{L^1_y(\frac{dy}{R^4})} \lesssim A(u) \frac{1}{R^4},
\]
\[
\left\| \int_y \tilde{E}_3(s,y) \right\|_{L^\infty(\frac{dy}{R^4})} \lesssim A(u) \frac{1}{R^4}.
\]
Interpolating the above two estimates, we have
\[
\left\| \int_y \tilde{E}_3(s,y) \right\|_{L^2(\frac{dy}{R^4})} \lesssim A(u) R^{-\frac{7}{4}} R_1 \frac{1}{R^4}.
\]
When \( i = 4 \), observe that
\[
\left\| \int_y \tilde{E}_4(s,y) \right\|_{L^2(\frac{dy}{R^1})} \lesssim \left\| \tilde{E}_4(s,y) \right\|_{L^1_y(\frac{dy}{R^4})} \lesssim A(u) \frac{1}{R^4},
\]
\[
\left\| \int_y \tilde{E}_4(s,y) \right\|_{L^\infty(\frac{dy}{R^1})} \lesssim A(u) 1.
\]
Interpolating above two estimates, we have
\[
\left\| \int_y \tilde{E}_4(s,y) \right\|_{L^2(\frac{dy}{R^1})} \lesssim A(u) R_1 \frac{1}{R^4} R^{-\frac{7}{4}}.
\]
This completes the proof. \( \square \)

**Appendix A. Proof of Proposition 4.8**

The linear profile decomposition (Proposition 4.8) is now fairly standard. In fact, it is very similar to that in Holmer-Roudenko [7, Lemma 2.1]. We include it here for sake of completeness. As a last remark, we are not sure whether Proposition 4.8 with \( k = 4 \) holds or not. The \( L^4_2 L^\infty_2 \) estimate is a genuine endpoint estimate among local smoothing estimates, so the usual interpolation argument does not work.

**Proof of Proposition 4.8**. As we can pass to a subsequence if necessary, combining with the usual diagonal trick, we may assume that all the limits what follow exist. Let \( C \) be the implicit constant for linear local smoothing estimates.

We proceed by induction. For notational convenience, let \( w^0_n := u_n \). Suppose that we have constructed \( \{\phi^j, x^j_n, t^j_n\} \) satisfying all the properties for \( j = 1, \ldots, J-1 \). We shall construct \( \{\phi^j, x^j_n, t^j_n\} \) as follows. Let
\[
A_J := \lim_{n \to \infty} \|w^{J-1}_n\|_{H^J} \quad \text{and} \quad c_J := \lim_{n \to \infty} \|V(t)w^{J-1}_n\|_{L^2_2 L^\infty_2}.
\]
If \( c_J = 0 \), then we set \( \phi^j = 0, t^j_n = x^j_n = 0 \) for all \( j \geq J \), and stop the procedure. Henceforth, we assume that \( c_J \neq 0 \). For some large \( R > 1 \) chosen later, consider a Schwartz function \( \chi \) such that \( |\hat{\chi}| \leq 1, \hat{\chi}(\xi) = 1 \) when \( \frac{1}{r} \leq |\xi| \leq r \), and \( \hat{\chi}(\xi) = 0 \) when \( |\xi| \leq \frac{1}{2r} \) or \( |\xi| \geq 2r \). Convolving \( w^{J-1}_n \) with \( \chi \), we have
\[
\|\chi \ast V(t)w^{J-1}_n\|_{L^2_2 L^\infty_2} \geq \|V(t)w^{J-1}_n\|_{L^2_2 L^\infty_2} - \|V(t)w^{J-1}_n - \chi \ast V(t)w^{J-1}_n\|_{L^2_2 L^\infty_2} \geq \|V(t)w^{J-1}_n\|_{L^2_2 L^\infty_2} - C\|w^{J-1}_n - \chi \ast w^{J-1}_n\|_{H^J}.
\]
As \( n \to \infty \), we have
\[
\lim_{n \to \infty} \| \chi * V(t) w_n^{J-1} \|_{L^2_x L^\infty_t} \geq c_J - C r^{-\frac{1}{4}} A_J.
\]

On the other hand, an easy interpolation yields (this is where the assumption \( k > 4 \) is required)
\[
\lim_{n \to \infty} \| \chi * V(t) w_n^{J-1} \|_{L^2_x L^\infty_t} \leq \lim_{n \to \infty} \left( \| \chi * V(t) w_n^{J-1} \|_{L^2_x L^2_t}^{\frac{4}{3}} \| \chi * V(t) w_n^{J-1} \|_{L^\infty_t}^{\frac{k}{3}} \right) ^{\frac{3}{k+4}}
\leq (C A_J)^{\frac{3}{k+4}} \lim_{n \to \infty} \| \chi * V(t) w_n^{J-1} \|_{L^2_x L^\infty_t}^{\frac{3}{k+4}}.
\]

Therefore,
\[
\lim_{n \to \infty} \| \chi * V(t) w_n^{J-1} \|_{L^\infty_t} \geq c_J (C A_J)^{-\frac{k}{k+4}} - (C A_J)^{\frac{1}{k+4}}
\]
If we choose \( r > 1 \) satisfying \((C A_J)^{-\frac{k}{k+4}} \leq \frac{1}{2} c_J (C A_J)^{-\frac{k}{k+4}}\), i.e.
\[
r \leq (C A_J)^k c_J^{-(k-4)},
\]
then we can choose \( t_n^J, x_n^J \) such that
\[
\| \chi * V(-t_n^J) w_n^{J-1} (x_n^J) \| \geq \frac{c_J}{2} (C A_J)^{-\frac{k}{k+4}}
\]
for all large \( n \). Let \( \phi^J \) be the weak \( H^\frac{1}{2} \)-limit of \([V(-t_n^J) w_n^{J-1}] (x_n^J)\) and \( w_n^J \) be \( w_n^{J-1} - V(t_n^J) \phi^J \cdot x_n^J \). Note that \( V(-t_n^J) w_n^J \cdot x_n^J \to 0 \) in \( H^\frac{1}{2} \). We can give a lower bound for \( H^\frac{1}{2} \)-norm of the profile \( \phi^J \). From
\[
c_J A_J^{-\frac{k}{k+4}} \leq \lim_{n \to \infty} \| \chi * V(-t_n^J) u_n(x_n^J) \| = \| \chi * \phi^J(0) \| \lesssim \| \chi \|_{H^{-s_k}} \| \phi^J \|_{H^{s_k}}
\]
we have
\[
\| \phi^J \|_{H^\frac{1}{2}} \geq c_J A_J^{-\frac{k}{k+4}} r^{-\frac{1}{4}} \geq c_J^4 A_J^{-\frac{k}{k+4}}.
\]

We now prove asymptotic orthogonality in \( \hat{H}^s \). By the definition of \( \phi^J \) and \( w_n^J \),
\[
\| w_n^{J-1} \|_{\hat{H}^s}^2 = \| w_n^J \|_{\hat{H}^s}^2 + \| V(t_n^J) \phi^J \cdot x_n^J \|_{\hat{H}^s}^2 + \langle w_n^J, V(t_n^J) \phi^J \cdot x_n^J \rangle_{\hat{H}^s} = \| w_n^J \|_{\hat{H}^s}^2 + \| \phi^J \|_{\hat{H}^s}^2 + \langle V(-t_n^J) w_n^J \cdot x_n^J, \phi^J \rangle_{\hat{H}^s}.
\]

As \( n \to \infty \), we have
\[
\lim_{n \to \infty} \left[ \| w_n^{J-1} \|_{\hat{H}^s}^2 - \| w_n^J \|_{\hat{H}^s}^2 - \| \phi^J \|_{\hat{H}^s}^2 \right] = 0.
\]
As the induction hypothesis tells
\[
\lim_{n \to \infty} \left[ \| u_n \|_{\hat{H}^s}^2 - \sum_{j=1}^{J-1} \| \phi^j \|_{\hat{H}^s}^2 - \| w_n^{J-1} \|_{\hat{H}^s}^2 \right] = 0,
\]
we conclude that
\[
\lim_{n \to \infty} \left[ \| u_n \|_{\hat{H}^s}^2 - \sum_{j=1}^{J} \| \phi^j \|_{\hat{H}^s}^2 - \| w_n^J \|_{\hat{H}^s}^2 \right] = 0.
\]
This completes the proof of asymptotic orthogonality in \( \hat{H}^s \).

We now show asymptotic separation of parameters. Combining with the induction hypothesis, it suffices to show that for each \( j < J \), we have \( |t_n^J -
t_n^j | + |x_n^j - x_n^j| \to \infty$. Suppose not; choose the maximal $j < J$ such that $t_n^j - t_n^j$ and $x_n^j - x_n^j$ converge for some subsequence. Then,

$$V(-t_n^j)w_n^j(\cdot + x_n^j) = V(-t_n^j)w_n^j(\cdot + x_n^j) + \sum_{j'=j+1}^{J-1} V(t_n^j - t_n^j)\phi^j(\cdot + x_n^j - x_n^j) + \phi^j.$$  

By the induction hypothesis, we know that $|t_n^j - t_n^j| + |x_n^j - x_n^j| \to \infty$ for all $j' < J$ so the summation part of the above display weakly converges to zero. By the construction, $V(-t_n^j)w_n^j(\cdot + x_n^j)$ converges weakly to zero. Moreover, since we assumed that $t_n^j - t_n^j$ and $x_n^j - x_n^j$ converge, $V(-t_n^j)w_n^j(\cdot + x_n^j)$ can be well approximated by $V(t_0)V(-t_n^j)w_n^j(\cdot + x_n^j - x_0)$ for some fixed $x_0$ and $t_0$, so it converges weakly to zero. Therefore, every term in the above display except $\phi^j$ weakly converges to zero. This yields a contradiction.

We now show asymptotic vanishing of the weak limit. Combining with the induction hypothesis, it suffices to show that for each $j = 1, \ldots, J$, we have $V(-t_n^j)w_n^j(\cdot + x_n^j) \to 0$. We express

$$V(-t_n^j)w_n^j(\cdot + x_n^j) = V(-t_n^j)w_n^j(\cdot + x_n^j) + \sum_{j'=j+1}^{J} V(t_n^j - t_n^j)\phi^j(\cdot + x_n^j - x_n^j).$$

The first term of the RHS vanishes by the construction. For the remaining term, we use separation of parameters.

We finally show asymptotic vanishing of remainder. Because

$$A_1^{2k} \sum_{j} c_j \phi_j \leq \sum_{j} c_j A_j^{2k} \leq \sum_{j} \|\phi_j\|_{H^k}^2 < \infty,$$

we should have $c_j \to 0$. \hfill \Box

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