PROJECTIVE CYCLIC GROUPS IN HIGHER DIMENSIONS

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Abstract. In this article we provide a classification of the projective transformations in $PSL(n+1, \mathbb{C})$ considered as automorphisms of the complex projective space $\mathbb{P}^n_\mathbb{C}$. Our classification is an interplay between algebra and dynamics. Just as in the case of isometries of $CAT(0)$-spaces, this is given by means of three types of transformations, namely: elliptic, parabolic and loxodromic. We describe the dynamics in each case, more precisely we determine the corresponding Kulkarni’s limit set, the equicontinuity region, the discontinuity region and in some cases we provide families of maximal regions where the corresponding cyclic group acts properly discontinuously. We also provide, in each case, some equivalent ways to classify the projective transformations.

Introduction

Discrete groups of projective transformations arise as monodromy groups of ordinary differential equations, see [13], or associated to Ricatti’s foliation, see [19], or as the monodromy groups of the so called orbifold uniformizing differential equations, see [23]. However outside the groups coming from complex hyperbolic geometry, a little is know about their dynamic, see [7]. Yet, as in the one dimensional case, one might expect interesting results. In this paper we deal with the basic problem of classifying the projective transformations.

When we look at elements in $PU(1,n)$, one has that they preserve a ball, then, as in the one dimensional case, this fact enables us classify the transformations in $PU(1,n)$ by means of their fixed points and their position in the closed complex ball. More precisely, an element is said to be: elliptic if it has a fixed point in the complex ball, parabolic if it has a unique fixed point in the boundary of the complex ball and finally the element is said to be loxodromic if it has exactly two fixed points in the boundary of the complex ball. Yet, when we think of automorphisms of $\mathbb{P}^n_\mathbb{C}$, this type of classification makes no sense, since in general there is not an invariant ball. So, to extend the previous classification to $PSL(n+1, \mathbb{C})$, we must think dynamically, more precisely we must look into the local behavior around the fixed points. The following definition captures this information.

Definition 0.1. Let $\gamma \in PSL(n+1, \mathbb{C})$ be a projective transformation, then

1. The element $\gamma$ is called elliptic if for each lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ of $\gamma$, one has that $\tilde{\gamma}$ is diagonalizable and each of its eigenvalues is unitary.
2. The element $\gamma$ is called loxodromic if for each lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ of $\gamma$, one has that $\tilde{\gamma}$ has a non-unitary eigenvalue.
3. The element $\gamma$ is parabolic, if for each lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ of $\gamma$, one has that $\tilde{\gamma}$ has only unitary eigenvalues and is non diagonalizable.

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Clearly this definition exhausts all the possibilities and coincides with the standard classification in the one and the two dimensional settings, as well as in the case of transformations in $PU(1,n)$, $n \geq 1$, see [10, 15, 17]. On the other hand, from our knowledge about complex Kleinian groups acting on $\mathbb{P}^2$, see [2], we know that the understanding of the dynamics of projective groups requires descriptions of the discontinuity set, the equicontinuity set, the Kulkarni’s limit set as well the maximal regions of discontinuity. One of the purposes of this article is to provide a description of the sets mentioned above for cyclic groups and their relation with the classification given in Definition 0.1. More precisely, in this article we show:

**Theorem 0.2 (The discontinuity set).** Let $\gamma \in PSL(n+1,\mathbb{C})$ be a projective transformation, then:

1. The element $\gamma$ is elliptic if and only if the set of accumulation points of orbits of points in $\mathbb{P}^n_\mathbb{C}$ under the action of $\langle \gamma \rangle$ either is empty or the whole space $\mathbb{P}^n_\mathbb{C}$, depending on whether $\gamma$ has finite order or not.
2. The element $\gamma$ is loxodromic if and only if the set of accumulation points of orbits of points in $\mathbb{P}^n_\mathbb{C}$ under the action of $\langle \gamma \rangle$ is a finite disjoint union of projective subspaces (see Theorem 2.7 for a detailed description).
3. The element $\gamma$ is parabolic if and only if the set of accumulation points of orbits of points in $\mathbb{P}^n_\mathbb{C}$ under the action of $\langle \gamma \rangle$ is a single proper projective subspace (see Theorem 2.7 for a detailed description).

**Theorem 0.3 (The equicontinuity set).** Let $\gamma \in PSL(n+1,\mathbb{C})$ be a projective transformation, then one has:

1. The element $\gamma$ is elliptic if and only if the equicontinuity set of $\langle \gamma \rangle$ is the whole space $\mathbb{P}^n_\mathbb{C}$.
2. The element $\gamma$ is loxodromic if and only if the equicontinuity set of $\langle \gamma \rangle$ can be described as the complement of union of two proper distinct projective subspaces $L_1, L_2$ of $\mathbb{P}^n_\mathbb{C}$ (see Theorem 2.9 for a precise description).
3. The element $\gamma$ is parabolic if and only if the equicontinuity set of $\langle \gamma \rangle$ is the complement of a projective subspace $L_1$ (see Theorem 2.9 for a precise description).

The Kulkarni’s discontinuity set was introduced in [14] as a way to construct regions where a group acts properly discontinuously and its complement, the so-called Kulkarni’s limit set, is where the dynamics concentrates (see the formal definitions below and see [14, 7] for a detailed discussion).

**Theorem 0.4 (The Kulkarni’s limit set).** Let $\gamma \in PSL(n+1,\mathbb{C})$ be a projective transformation, then:

1. If the element $\gamma$ is either parabolic or loxodromic, then the discontinuity set of $\langle \gamma \rangle$ coincides with the Kulkarni discontinuity region.
2. If the element $\gamma$ is elliptic, then the Kulkarni limit set of $\langle \gamma \rangle$ is either empty or the whole space $\mathbb{P}^n_\mathbb{C}$, depending on whether $\gamma$ has finite order or not.

From the one and two dimensional settings we know that Definition 0.1 can be given in terms of certain foliations (see [1, 17]). This provides a simple way to describe the global dynamics of cyclic groups. In this article we propose a generalization of such foliations, but before we state the analogous results, let us introduce some notation. Let $\mathbb{C}^{k,l}$, $k < l$, be a copy of $\mathbb{C}^{k+l}$ equipped with the
Theorem 0.5. Let $\gamma \in PSL(n+1, \mathbb{C})$ be a projective transformation. Then $\gamma$ is elliptic if and only if, up to conjugation, it preserves a foliation of $\mathbb{C}^n \setminus \{0\}$ by concentric $(1,n)$-spheres.

Theorem 0.6. Let $\gamma \in PSL(n+1, \mathbb{C})$ be a projective transformation. Then the element $\gamma$ is loxodromic if and only if there is a proper open set $W \subset \mathbb{P}_C^n$ such that $\gamma(W) \subset W$.

Theorem 0.7. Let $\gamma \in PSL(n+1, \mathbb{C})$ be a projective transformation. Then the element $\gamma$ is parabolic if and only if there are $k, l \in \mathbb{N}$ satisfying $k + l = n + 1$, a family $\mathcal{F}$ of $\gamma$-invariant $(k,l)$-spheres and $\gamma$-invariant, non-empty proper projective subspaces $Z, W \subset \mathbb{P}_C^n$ such that:

1. For every pair of different elements $T_1, T_2 \in \mathcal{F}$ one has $Z \subset T_1 \cap T_2 \subset W$.
2. One can check $\bigcup_{T \in \mathcal{F}} T \setminus W = \mathbb{P}_C^n \setminus W$.
3. If $\ell \subset \mathbb{P}_C^n$ is a $\gamma$-invariant line where the restriction of $\prec, \succ$ to $[\ell]^{-1}(\ell) \cup \{0\}$ has signature $(1,1)$, then there is a point $z \in \ell$ such that $\gamma z \neq z$.
4. The action of $\gamma$ restricted to $Z$ is a given by an elliptic element.

As corollary we get the following useful characterization given in terms of the fixed points:

Theorem 0.8. Let $\gamma \in PSL(n+1, \mathbb{C})$ be a projective transformation, then

1. The element $\gamma$ is loxodromic if and only if there are two distinct points $x, y \in Fix(\gamma)$ such that the action of $\gamma$ restricted to the complex line $\langle x, y \rangle$ is loxodromic.
2. The element $\gamma$ is parabolic if and only if every lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ is non-diagonalizable and for every couple of distinct points $x, y \in Fix(\gamma)$ the action of $\gamma$ restricted to the complex line $\langle x, y \rangle$ is elliptic.
The element \( \gamma \) is elliptic if and only if every lift \( \overline{\gamma} \in SL(n+1, \mathbb{C}) \) is diagonalizable and for every couple of distinct points \( x, y \in Fix(\gamma) \) the action of \( \gamma \) restricted to the complex line \( \langle x, y \rangle \) is elliptic.

The paper is organized as follows: in Section 1 we review some general facts and introduce the notation used along the text, in section 2 we describe the discontinuity set, the Kulkarni’s limit set and the equicontinuity region of projective cyclic groups, section 4 deals with the problem of classification but for groups of \( PU(k, l) \), which as we will see later is useful in the general setting, in sections 5, 6, 7, we describe completely the dynamic of elliptic, parabolic and loxodromic transformations respectively.

1. Preliminaries

1.1. Projective Geometry. The complex projective space \( P^n_\mathbb{C} \) is defined as:
\[
P^n_\mathbb{C} = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*,
\]
where \( \mathbb{C}^* \) acts by the usual scalar multiplication. This is a compact connected complex \( n \)-dimensional manifold, equipped with the Fubini-Study metric \( d_n \).

If \( | \cdot | : \mathbb{C}^{n+1} - \{0\} \to P^n_\mathbb{C} \) is the quotient map, then a non-empty set \( H \subset P^n_\mathbb{C} \) is said to be a projective subspace of dimension \( k \) if there is a \( \mathbb{C} \)-linear subspace \( \tilde{H} \) of dimension \( k + 1 \) such that \( |\tilde{H} \setminus \{0\}| = H \). The projective subspaces of dimension \( (n - 1) \) are called hyperplanes and the complex projective subspaces of dimension 1 are called lines.

Given a set of points \( P \) in \( P^n_\mathbb{C} \), we define:
\[
\langle\langle P \rangle\rangle = \bigcap \{ l \subset P^n_\mathbb{C} \mid \text{l is a projective subspace and } P \subset l \}.
\]
Clearly \( \langle\langle P \rangle\rangle \) is a projective subspace of \( P^n_\mathbb{C} \). On the other hand the points in \( P \) are said to be in general position if for each subset \( R \subset P \) with \( 1 \leq Card(R) \leq n + 1 \) we have that \( \langle\langle R \rangle\rangle \) has dimension \( Card(R) - 1 \).

1.2. The projective group \( PSL(n+1, \mathbb{C}) \). Consider the general linear group \( GL(n+1, \mathbb{C}) \). It is clear that every linear automorphism of \( \mathbb{C}^{n+1} \) defines a holomorphic automorphism of \( P^n_\mathbb{C} \), and it is well-known that every automorphism of \( P^n_\mathbb{C} \) arises in this way. Thus one has that the group of projective automorphisms is:
\[
PSL(n+1, \mathbb{C}) := GL(n+1, \mathbb{C})/(\mathbb{C}^*)^{n+1}
\]
where \( (\mathbb{C}^*)^{n+1} \) acts by the usual scalar multiplication. Then \( PSL(n+1, \mathbb{C}) \) is a Lie group whose elements are called projective transformations. We denote by \( [[\cdot]] : GL(n+1, \mathbb{C}) \to PSL(n+1, \mathbb{C}) \) the quotient map. Given \( \gamma \in PSL(n+1, \mathbb{C}) \) we say that \( \overline{\gamma} \in GL(n+1, \mathbb{C}) \) is a lift of \( \gamma \) if \( [[\gamma]] = \gamma \). Notice that \( PSL(n+1, \mathbb{C}) \) takes projective subspaces into projective subspaces.

Let us construct a completion of the Lie group \( PSL(n+1, \mathbb{C}) \), know as the space of pseudo-projective maps, see [6]. Let \( \tilde{M} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \) be a non-zero linear transformation which is not necessarily invertible. Let \( Ker(\tilde{M}) \) be its kernel and let \( Ker([[\tilde{M}]]) \) denote its projectivization. That is, \( Ker([[\tilde{M}]])) := [Ker(\tilde{M}) \setminus \{0\}] \). Then \( M \) induces a map \( [[\tilde{M}]] : P^n_\mathbb{C} \setminus Ker(M) \to P^n_\mathbb{C} \) given by:
\[
[[\tilde{M}]](v) = [\tilde{M}(v)].
\]
This is well defined because \( v \notin \ker(\tilde{M}) \). We call the map \( M = [[\tilde{M}]] \) a pseudo-projective transformation, and we denote by \( QP(n + 1, \mathbb{C}) \) the space of all pseudo-projective transformations of \( \mathbb{P}_\mathbb{C}^n \). Clearly \( QP(n + 1, \mathbb{C}) \) is a compactification of \( PSL(n + 1, \mathbb{C}) \). A linear map \( \tilde{M} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \) is said to be a lift of the pseudo-projective transformation \( M \) if \( [[\tilde{M}]] = M \).

**Proposition 1.1.** Let \((\gamma_m)_{m \in \mathbb{N}} \subset GL(n + 1, \mathbb{C})\) be a sequence of distinct elements. If there is a \( \gamma \in GL(n + 1, \mathbb{C}) \) such that \( \gamma_m \xrightarrow{m \to \infty} \gamma \) point wise, then \( [[\gamma_m]] \xrightarrow{m \to \infty} [[\gamma]] \) uniformly on compact sets of \( \mathbb{P}_\mathbb{C}^n \setminus \ker(\gamma) \).

In what follows we will say that the sequence \((\gamma_m)_{m \in \mathbb{N}} \subset PSL(n + 1, \mathbb{C})\) converges to \( \gamma \in PSL(n + 1, \mathbb{C}) \) in the sense of pseudo-projective transformations if \( \gamma_m \xrightarrow{m \to \infty} \gamma \) uniformly on compact sets of \( \mathbb{P}_\mathbb{C}^n \setminus \ker(\gamma) \).

**Definition 1.2.** The *equicontinuity region* for a family \( G \) of endomorphisms of \( \mathbb{P}_\mathbb{C}^n \), denoted \( \operatorname{Eq}(G) \), is defined to be the set of points \( z \in \mathbb{P}_\mathbb{C}^n \) for which there is an open neighborhood \( U \) of \( z \) such that \( G|_U \) is a normal family.

**Proposition 1.3** (See [6]). Let \( \Gamma \subset PSL(n + 1, \mathbb{C}) \) be a group and define

\[
\operatorname{Lim}(\Gamma) = \{ \gamma \in QP(n + 1, \mathbb{C}) : \text{there is } (\gamma_m)_{m \in \mathbb{N}} \subset \Gamma, \text{ with } \gamma_m \xrightarrow{m \to \infty} \gamma \},
\]

then

\[
\operatorname{Eq}(\Gamma) = \bigcup_{\gamma \in \operatorname{Lim}(\Gamma)} \ker(\gamma).
\]

### 1.3. Projective Unitary Groups

Let \( k < l \), in what follows \( \mathbb{C}^{k,l} \) is a copy of \( \mathbb{C}^{k+1} \) equipped with a Hermitian form of signature \((k, l)\) that we assume is given by:

\[
\langle u, v \rangle_{k,l} = -\sum_{j=1}^{k} u_j \overline{v}_j + \sum_{j=k+1}^{k+l} u_j \overline{v}_j,
\]

where \( u = (u_1, \ldots, u_{k+l}) \) and \( v = (v_1, \ldots, v_{k+l}) \). A vector \( v \) is called negative, null or positive depending (in the obvious way) on the value of \( \langle v, v \rangle \); we denote the set of negative, null or positive vectors by \( N^{k,l}_-, N^{k,l}_0 \) and \( N^{k,l}_+ \) respectively. We define \( \mathbb{H}^{k,l}_C \) as the image of \( N^{k,l}_- \) in \( \mathbb{P}_\mathbb{C}^n \) under the map \([\cdot]\).

If we let \( U(k, l) \subset GL(n + 1, \mathbb{C})\) be the subgroup consisting of the elements that preserve the above Hermitian form, then its projectivization \( [[U(k, l)]]_{n+1} \) is a subgroup of \( PSL(n + 1, \mathbb{C}) \) that we denote by \( PU(k, l) \).

Given \([v], [w] \in \mathbb{H}^{k,l}_C\) we define:

\[
d_{k,l}([v], [w]) = \arccosh \left( \sqrt{\frac{\langle v, w \rangle_{k,l}}{\langle v, v \rangle_{k,l} \langle w, w \rangle_{k,l}}} \right)
\]

is straightforward to check that this is a metric in \( \mathbb{H}^{k,l}_C \) compatible with its topology of \( \mathbb{H}^{k,l}_C \), see [12]. Now, the Arzelà-Ascoli theorem yields, see [13]:

**Theorem 1.4.** Let \( \Gamma \) be a subgroup of \( PU(k, l) \). The following three conditions are equivalent:

1. The subgroup \( \Gamma \subset PU(k, l) \) is discrete.
2. The region of discontinuity of \( \Gamma \) in \( \mathbb{H}^{k,l}_C \) is all of \( \mathbb{H}^{k,l}_C \).
1.4. The Grassmanians. A Grassmannian is a fancy way to provide a parametrization for the space of all linear subspaces of a vector space $V$ of a given dimension. More precisely, let $0 \leq k < n$, then we denote by $Gr(k, n)$ the Grassmanian of all $k$-dimensional projective subspaces of $\mathbb{P}^n_\mathbb{C}$ endowed with the Hausdorff metric induced by $d_n$, see [16]. One has that $Gr(k, n)$ is a compact, connected complex manifold of dimension $k(n - k)$. A method to realize the Grassmannian $Gr(k, n)$ as a subvariety of the projective space of the $k$th exterior power of $\Lambda^{k+1} \mathbb{C}^{n+1}$, in symbols $P(\Lambda^{k+1} \mathbb{C}^{n+1})$, is done by the so called Plücker embedding, which is given by:

$$\iota : Gr(k, n) \rightarrow P(\Lambda^{k+1} \mathbb{C}^{n+1})$$

$$\iota(V) \mapsto [v_1 \wedge \cdots \wedge v_{k+1}]$$

where $\langle v_1, \cdots, v_{k+1} \rangle = V$, clearly this is a well defined $PSL(n + 1, \mathbb{C})$-equivariant embedding. Moreover, it is possible to check that the topology on $Gr(k, n)$ induced by the Fubini-study metric $\Lambda^{k+1} d$ on $P(\Lambda^{k+1} \mathbb{C}^{n+1})$ agrees with the topology on $Gr(k, n)$ induced by the Hausdorff metric on the space of closed sets in $\mathbb{P}^n_\mathbb{C}$, which we will denote by $G(\mathbb{P}^n_\mathbb{C})$.

1.5. Complex Kleinian Groups. When we look at the action of a group acting on a general topological space, in general there is not a well-defined notion of limit set. There are several possible definitions of this concept, each with its own properties and characteristics, in this subsection we deal with the so called Kulkarni’s limit set.

**Definition 1.5** (see [14]). Let $\Gamma \subset PSL(n + 1, \mathbb{C})$ be a subgroup. We define

1. The set $\Lambda(\Gamma)$ as the closure of the set of cluster points of $\Gamma z$ where $z$ runs over $\mathbb{P}^n_\mathbb{C}$
2. The set $L_2(\Gamma)$ as the closure of cluster points of $\Gamma K$ where $K$ runs over all the compact sets in $\mathbb{P}^n_\mathbb{C} \setminus \Lambda(\Gamma)$.
3. The Kulkarni’s limit set of $\Gamma$ as:

$$\Lambda_{Kul}(\Gamma) = \Lambda(\Gamma) \cup L_2(\Gamma).$$

4. The Kulkarni’s discontinuity region of $\Gamma$ as:

$$\Omega_{Kul}(\Gamma) = \mathbb{P}^n_\mathbb{C} \setminus \Lambda_{Kul}(\Gamma).$$

We will say that $\Gamma$ is a Complex Kleinian Group if $\Omega_{Kul}(\Gamma) \neq \emptyset$, see [20]. The limit set in the Kulkarni’s sense enjoys the following properties, for a more detailed discussion on this topic in the 2 dimensional setting, see [2].

**Proposition 1.6** (See [17]). Let $\Gamma$ be a complex kleinian group. Then:

1. The sets $\Lambda_{Kul}(\Gamma), \Lambda(\Gamma), L_2(\Gamma)$ are $\Gamma$-invariant closed sets.
2. The group $\Gamma$ acts properly discontinuously on $\Omega_{Kul}(\Gamma)$.
3. Let $C \subset \mathbb{P}^n_\mathbb{C}$ be a closed $\Gamma$-invariant set such that for every compact set $K \subset \mathbb{P}^n_\mathbb{C} - C$, the set of cluster points of $\Gamma K$ is contained in $\Lambda(\Gamma) \cap C$, then $\Lambda_{Kul}(\Gamma) \subset C$. 


2. The Chaotic Sets

In the present section we describe the discontinuity set, the equicontinuity set, the Kulkarni’s limit set and maximal discontinuity set for the cyclic groups of $\text{PSL}(n+1, \mathbb{C})$ acting on $\mathbb{P}_n^2$, for a detailed discussion in this topic in the 2-dimensional case see [2, 7]. The following definition is useful.

**Definition 2.1.** Let $V$ be a $\mathbb{C}$-vector space and $T : V \to V$ be a $\mathbb{C}$-linear map such that each one of its eigenvalues is a unitary complex number. Let $k \in \mathbb{N}$; $V_1, \ldots, V_k \subset V$ be linear subspaces, $\{\lambda_1, \ldots, \lambda_k\}$ be unitary complex numbers, for each $j \in \{1, \ldots, k\}$ let $\beta_j = \{v_{j1}, \ldots, v_{j\text{dim}(V_j)}\}$ be a base of $V_j$ and $T_j : V_j \to V_j$, $1 \leq i \leq k$ be $\mathbb{C}$-linear maps satisfying:

- (1) $\bigoplus_{j=1}^k V_j = V$.
- (2) $\bigoplus_{j=1}^k \lambda_j T_j = T$.
- (3) For each $1 \leq j \leq k$, $(x - 1)^{\text{dim}V_j}$ is the characteristic polynomial of $\gamma_j$.
- (4) For each $1 \leq j \leq k$, either $\gamma_j$ is the identity or $[T_j]_{\beta_j}$ is a Jordan block.

Then $(k, \{V_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k, \{\lambda_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k)$, is called a Block decomposition for $T$.

With this in mind let us show:

**Lemma 2.2.** Let $V$ be a finite dimensional $\mathbb{C}$-linear and let $T : V \to V$ be a linear transformation such that each of its eigenvalues is a unitary number. Then for every $v \in V \setminus \{0\}$ there is a unique $k(v, T) \in \mathbb{N} \cup \{0\}$ such that the set of cluster points of $\{m \in \mathbb{N} \mid T^m(v) \in \text{span}(v)\}$ lies in $\langle\{x \in V : x \text{ is an eigenvector}\}\rangle \setminus \{0\}$.

**Proof.** Let $(k, \{V_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k, \{\lambda_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k)$ be a Block decomposition for $T$ and $v \in V \setminus \{0\}$, then $v = \sum_{i=1}^k v_i$ where $v_i \in V_i$. Set $v_j = \sum_{k=1}^{k_j} \alpha_{j,k} v_{jk}$ and define

\[
k(v_j, T_j) = \begin{cases} 
0 & \text{if } v_j = 0 \text{ or } T_j \text{ is the identity} \\
\max\{1 \leq k \leq k_j : \alpha_{j,k} \neq 0\} - 1 & \text{in other way}
\end{cases}
\]

\[
k(v_j, T_j) = \max\{k(v_j, T_j) : 1 \leq j \leq k\},
\]

\[
W_1 = \{j \in \{1, \ldots, k\} : k(v_j, T_j) < k(v, T)\},
\]

\[
W_2 = \{1, \ldots, k\} \setminus W_1.
\]

A straightforward calculation shows:

\[
(\begin{array}{c}
T^m(v) \\
(\begin{array}{c}
T^m(v_j)
\end{array})
\end{array}) &= \sum_{j \in W_1} \left(\begin{array}{c}
m \\
(\begin{array}{c}
k(v_j, T_j)
\end{array})
\end{array}\right) T^m(v_j) + \sum_{j \in W_2} \left(\begin{array}{c}
\lambda_j^m T^{\text{max}(v_j)}
\end{array}\right)
\]

Since the first part of the sum converges to 0, to conclude the proof is enough to consider the following equation

\[
(\begin{array}{c}
T^m(v_j)
\end{array}) = \begin{cases} 
\left(\begin{array}{c}
m \\
(\begin{array}{c}
k(v_j, T_j)
\end{array})
\end{array}\right)^{-1} v_j & \text{if } T_j \text{ is the identity} \\
\sum_{i=1}^{k_j} \left(\begin{array}{c}
\lambda_j \sum_{k=0}^{k_j} \left(\begin{array}{c}
m \\
(\begin{array}{c}
k(v_j, T_j)
\end{array})
\end{array}\right)^{-1} \alpha_{j,k} i
\end{array}\right) v_{ji} & \text{in other way}
\end{cases}
\]

\[
(\begin{array}{c}
T^m(v_j)
\end{array}) = \begin{cases} 
\left(\begin{array}{c}
m \\
(\begin{array}{c}
k(v_j, T_j)
\end{array})
\end{array}\right)^{-1} v_j & \text{if } T_j \text{ is the identity} \\
\sum_{i=1}^{k_j} \left(\begin{array}{c}
\lambda_j \sum_{k=0}^{k_j} \left(\begin{array}{c}
m \\
(\begin{array}{c}
k(v_j, T_j)
\end{array})
\end{array}\right)^{-1} \alpha_{j,k} i
\end{array}\right) v_{ji} & \text{in other way}
\end{cases}
\]
The following definition will enable us to detect the projective spaces which behave as attracting, repelling or indifferent sets for the dynamic of the group.

**Definition 2.3.** Let $\gamma \in SL(n+1, \mathbb{C})$ be a projective transformation, $k \in \mathbb{N}$; $V_1, \ldots, V_k \subset \mathbb{C}^{n+1}$ linear subspaces; $\gamma_i : V_i \to V_i$, $1 \leq i \leq k$, be $\mathbb{C}$-linear transformations and $r_1, \ldots, r_k \in \mathbb{R}$. The set $(k, \{V_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k, \{r_i\}_{i=1}^k)$ will be called a unitary decomposition for $\gamma$ if it is verified that:

1. $\bigoplus_{j=1}^k V_j = \mathbb{C}^{n+1}$.
2. For each $1 \leq i \leq k$, the eigenvalues of $\gamma_i$ are unitary complex numbers.
3. $0 < r_1 < r_2 < \ldots < r_k$.
4. $\bigoplus_{j=1}^k v_j \gamma_j = \gamma$.

2.1. **The discontinuity set.** Recall that a group action is called *discontinuous* if every point $x$ of $X$ has a neighborhood $U$ that meets $gU$ for only a finite number of elements $g$ of $G$. With this in mind we define

**Definition 2.4.** Let $\Gamma \subset PSL(n+1, \mathbb{C})$ be a group, then we define the discontinuity set of $\Gamma$ as the complement of the closure of the accumulation points of orbits $\Gamma z$ where $z \in \mathbb{P}^n_\mathbb{C}$, the later accumulation set is denoted by $\Lambda(\Gamma)$.

Clearly the discontinuity set is the largest open set where the group $\Gamma$ acts discontinuously. Before we state a proof the main result of this section we need to state the following definition.

**Definition 2.5.** Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-linear map and $\lambda_1, \ldots, \lambda_n$ be its eigenvalues, then we define the spectral radius $\rho(T)$ of $T$ as

$$\rho(T) = \max\{|\lambda_j| : j \in \{1, \ldots, n\}\}$$

The following lemma will be useful through the paper, see [4].

**Lemma 2.6.** Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a $\mathbb{C}$-linear map. If $\rho(T) < 1$, then $T^m \to \mathbf{0}$ point wise.

Finally we can state

**Theorem 2.7.** Let $\gamma \in PSL(n+1, \mathbb{C})$ be an element of infinite order. If $\tilde{\gamma} \in GL(n+1, \mathbb{C})$ is a lift of $\gamma$ and $(k, \{V_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k, \{r_j\}_{j=1}^k)$ is a unitary decomposition for $\tilde{\gamma}$, then:

$$\Lambda(\langle \gamma \rangle) = \bigcup_{j=1}^k (\{\langle x \in V : x \text{ is an eigenvector of } \gamma_j \rangle \} \setminus \{0\}).$$

**Proof.** Since $\bigcup_{j=1}^k (\{\langle x \in V : x \text{ is an eigenvector of } \gamma_j \rangle \} \setminus \{0\}) \subset \Lambda(\langle \gamma \rangle)$, thence it is enough to show that $\Lambda(\langle \gamma \rangle) \subset \bigcup_{j=1}^k (\{\langle PV(\gamma_j) \rangle \} \setminus \{0\})$. Let $v \in \mathbb{C}^n \setminus \{0\}$, then $v = \sum_{j=1}^k v_j$, where $v_j \in V_j$. Define $j_0 = \max\{j \in \{1, \ldots, k\} : v_j \neq 0\}$. A straightforward calculation shows

\begin{equation}
\left( \frac{m}{k(v_{j_0}, \gamma_{j_0})} \right)^{-1} \frac{\gamma_{j_0}^m(v)}{r_{j_0}^m} = \sum_{j=1}^k \left( \frac{m}{k(v_{j_0}, T_{j_0})} \right)^{-1} \frac{r_{j_0}^{m-1} \gamma_{j_0}^m(v_j)}{r_{j_0}^m}.
\end{equation}
Now, Lemma 2.2 yields that the set of cluster points of \( \{ \gamma^m(v) \}_{m \in \mathbb{N}} \) lies on \( \langle \{ x \in V : x \text{ is an eigenvector of } \gamma_j \} \rangle \setminus \{0\} \). Through a similar argument we conclude that the set of cluster points of \( \{ \gamma^{-m}(v) \}_{m \in \mathbb{N}} \) lies on
\[
\langle \{ x \in V : x \text{ is an eigenvector of } \gamma_j \} \rangle \setminus \{0\}.
\]
Which concludes the proof. \( \square \)

2.2. The equicontinuity set. The equicontinuity set is a remarkable open set where discrete projective group acts properly discontinuously, through this subsection we will describe the discontinuity region for the cyclic groups. Let's begin with a definition.

**Definition 2.8.** Let \( V \) be a \( \mathbb{C} \)-vector space and \( T : V \to V \) be \( \mathbb{C} \)-linear transformation such that each of its eigenvalues is a unitary complex number. Let \( (k, \{ V_j \}_{j=1}^k, \{ \gamma_j \}_{j=1}^k, \{ r_j \}_{j=1}^k) \) be a block decomposition for \( T \), then we define
\[
H(T) = \max \{ \{ \dim \mathbb{C} V_j : j \in \{1, \ldots, k\} \} \text{ and } \gamma_j \text{ is not the identity} \} \cup \{0\}
\]
\[
\Xi(T) = \begin{cases} 
\emptyset & \text{if } H(T) < 2 \\
\bigcup_{j} \beta_j \setminus \{ w \in \beta_j : \dim V_j = H(T) = \dim(\{ r_j^m(w) \}_{m \in \mathbb{Z}}) \} & \text{in other way}
\end{cases}
\]

Clearly \( H(T) \) and \( \Xi(T) \) does not depend on the choice of the block decomposition of \( T \).

**Theorem 2.9.** Let \( \gamma \in PSL(n+1, \mathbb{C}) \) be a projective transformation with infinite order. If \( \overline{\gamma} \in SL(n+1, \mathbb{C}) \) is a lift of \( \gamma \), \( (k, \{ V_j \}_{j=1}^k, \{ \gamma_j \}_{j=1}^k, \{ r_j \}_{j=1}^k) \) is a unitary decomposition for \( \overline{\gamma} \), then
\[
P^c_\mathbb{C} \setminus Eq(\Gamma) = \left[ \bigcup_{j>1} V_j \cup \Xi(\gamma_1) \right] \setminus \{0\} \cup \left[ \bigcup_{j<k} V_j \cup \Xi(\gamma_k) \right] \setminus \{0\}.
\]

**Proof.** Let \( T \in Lim(\langle \gamma \rangle) \), the there is a sequence \( (n_m) \subset \mathbb{Z} \) such that \( \gamma_{n_m} \xrightarrow{m \to \infty} T \).

After taking a subsequence, if it is necessary, we can assume that either \( (n_m) \) is negative or \( (n_m) \) is positive. Without loss of generality let us assume that \( n_m > 0 \).

Let \( (r, \{ U_j \}_{j=1}^r, \{ \beta_j \}_{j=1}^r, \{ \rho_j \}_{j=1}^r, \{ \kappa_j \}_{j=1}^r) \) be a block decomposition for \( \gamma_k \), define \( A_1 = \{ j \in \{1, \ldots, r\} : \dim U_j < H(\gamma_k) \} \) and \( A_2 = \{1, \ldots, r\} \setminus A_1 \), then a straightforward calculation shows:
\[
\gamma^m = \left[ \sum_{j<k} r_j^{n_m} \left( \frac{m}{H(\gamma_k)} - 1 \right)^{-1} r_j^{n_m} \gamma_j^m + \cdots \right] + \sum_{j \in A_1} \left( \frac{m}{H(\gamma_k)} - 1 \right)^{-1} \rho_j^{n_m} \kappa_j^m + \sum_{j \in A_2} \left( \frac{m}{H(\gamma_k)} - 1 \right)^{-1} \rho_j^{n_m} \kappa_j^m \right].
\]

Let \( (k_m) \) be a sequence such that for each \( j \in A_2 \) it holds that \( \rho_j^{m} \xrightarrow{m \to \infty} \beta_j \). For each \( l \in \{1, \ldots, r\} \), set \( \beta_l = \{ u_{jl} \} \) and \( S_l = U_l \to U_l \) be defined by
\[
S_l \left( \sum_{\nu=1}^{\dim U_l} a_{\nu} u_{\nu l} \right) = \begin{cases} 
0 & \text{if } l \in A_1 \\
\partial_l a_{\dim U_l} u_{1 l} & \text{if } l \in A_2
\end{cases}
\]
Finally for each \( j \in \{1, \ldots, k\} \) define \( T_j = V_j \to V_j \) by

\[
T_j = \begin{cases} 
0 & \text{if } j < k \\
\sum_{i=1}^k S_i & \text{if } j = k
\end{cases}
\]

Clearly \( \gamma^k \to \infty \left[ [\sum S_j] \right] \). Therefore \([\sum S_j] = T\). Now the result follows. \( \square \)

2.3. The Kulkarni’s Limit set. In this section we describe the Kulkarni’s limit set of cyclic groups. The following are useful lemmas:

Lemma 2.10. Let \( \gamma \in PSL(n+1, \mathbb{C}) \) be an element and \( \tilde{\gamma} \in SL(n+1, \mathbb{C}) \) be a lift of \( \gamma \) such that \( \tilde{\gamma} \) is diagonalizable and each of its eigenvalues is an unitary complex number. Then \( \Lambda_{Kul}(\Gamma) \) is either empty or the whole space \( \mathbb{P}_n^2 \) according \( \gamma \) has either finite or infinite order.

Proof. It follows directly from Theorem 2.9. \( \square \)

Lemma 2.11. Let \( \gamma \in PSL(n+1, \mathbb{C}) \) be an element such that \( \gamma \) has a lift \( \tilde{\gamma} \) such that \( \tilde{\gamma} \) is a \((n+1) \times (n+1)\)-Jordan block being 1 its unique eigenvalue. If \( \ell \) is an hyperplane not containing \([e_1]\), then

1. The action of \( \tilde{\gamma} \) on \( \wedge^n \mathbb{C}^{n+1} \) has an unique fixed point.
2. It is verified that \( \gamma^m(\ell) \to \infty \left[ [\{e_1, \ldots, [e_n]\}] \right] \).
3. It is verified that \( \Lambda_{Kul}(\langle \gamma \rangle) = [\{e_1, \ldots, [e_n]\}] \).

Proof. Let us show 1. Let us make the proof by induction on \( n \). For \( n = 1 \), we are considering the action of \( \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \) on \( \wedge^1 \mathbb{C}^{1+1} = \mathbb{C}^2 \), which trivially has a unique fixed point. Now let us proceed, to check the case \( n_0 + 1 \). At this step let us assume that there is a hyperplane \( \mathcal{L} \neq \langle \{e_1, \ldots, [e_n]\} \rangle \), such that \( \mathcal{L} \) is invariant under the action of \( \tilde{\gamma} \). Observe that \( \mathcal{L} \cap \langle \{e_1, \ldots, [e_n]\} \rangle \), is a hyperplane of \( \mathcal{L} \) invariant under \( \tilde{\gamma} \), by the inductive hypothesis we conclude that \( \langle \{e_1, \ldots, [e_n-1]\} \rangle \subset \mathcal{L} \). In consequence, if \( p \in \mathcal{L} \cap \langle \{e_1, \ldots, [e_n]\} \rangle \), the matrix of \( \tilde{\gamma} \) with respect the ordered base \( \{e_1, \ldots, [e_n], p\} \) is:

\[
T = \begin{pmatrix} 
1 & 1 & 0 & a_1 \\
0 & 1 & 1 & a_2 \\
0 & 0 & 1 & \ddots \\
\ddots & 1 & a_{n-1} \\
0 & 0 & 0 & \cdots & 1 & 0 
\end{pmatrix}
\]

A straightforward calculation shows that \((T - Id)^n = 0\), which is a contradiction since \( \tilde{\gamma} \) is a \((n_0 + 1) \times (n_0 + 1)\)-Jordan block. Which concludes the proof.

The proofs of 2, 3 follows easily from 1. \( \square \)

Lemma 2.12. Let \( A \in GL(k, \mathbb{C}) \) be a diagonal matrix such that each of its proper values is a unitary complex number, let \( B \) be a \( \ell \times \ell \)-Jordan block, \( \alpha \in \mathbb{C}^* \) and \( \gamma \in GL(k + l, \mathbb{C}) \) be given by

\[
\tilde{\gamma} = \begin{pmatrix} 
B & 0 & 0 \\
0 & 0 & \alpha A 
\end{pmatrix}.
\]

If \( \gamma = [\tilde{\gamma}] \), then \( \Omega_{Kul}(\langle \gamma \rangle) = Eq(\langle \gamma \rangle) \).
Proof. The proof is by induction on \( l \). For \( l = 2 \) we get that
\[
\tilde{\gamma} = \begin{pmatrix} B & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},
\]
which clearly shows the claim. Let us show the claim in the case \( l_0 + 1 \), clearly will be enough to show that given \( z \in \langle\{e_1, \ldots, [e_{l_0-1}]\}\rangle = L_1 \) it holds that \( \langle\{z, e_{l_0}\}\rangle \subseteq \Lambda_{Kul}(\gamma) \). Applying the inductive hypothesis to \( \gamma \) restricted to \( L = \langle\{e_1, \ldots, [e_{l_0}]\}\rangle \), we conclude that \( \Lambda_{Kul}(\gamma|_L) = L_1 \). Thus there is a sequence \( \langle z_m \rangle \subseteq L \) such that the cluster points of \( \{z_m\} \) lies on \( \mathbb{P}_C^{k+l_0} \setminus \Lambda(\langle\gamma\rangle) \), and
\[
\gamma^m(k_{1m}) \xrightarrow{m \to \infty} z.
\]
On the other applying Lemma 2.11 to \( \gamma \) restricted to \( \langle\{e_{k+1}, \ldots, [e_{l_0+1}]\}\rangle \) we conclude that there is a sequence \( \{z_m\} \subseteq \mathbb{P}_C^{k+l_0} \setminus L \) such that the cluster points of \( \langle z_m \rangle \) lies on \( \mathbb{P}_C^{k+l_0} \setminus \Lambda(\langle\gamma\rangle) \), and
\[
\gamma^m(k_{2m}) \xrightarrow{m \to \infty} z, e_{l_0}\langle\gamma\rangle\rangle.
\]
which concludes the proof.

Lemma 2.13. Let \( \gamma \in \text{PSL}(n+1, \mathbb{C}) \) be an element such that \( \gamma \) has a lift \( \tilde{\gamma} \) such that \( \tilde{\gamma} \) is non-diagonalizable having only unitary eigenvalues also let
\[
(k, \{V_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k, \{\lambda_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k)
\]
be a block decomposition for \( \tilde{\gamma} \). If for each \( j \in \{1, \ldots, k\} \) it verified that \( \gamma_j \) is a Jordan block, then \( \Omega_{Kul}(\langle\gamma\rangle) = Eq(\langle\gamma\rangle) \).

Proof. For each \( j \in \{1, \ldots, k\} \) let \( \ell_j \) be a hyperplane in \( V_j \) not containing eigenvectors of \( \gamma_j \). Define
\[
L = \left[ \left\langle \bigcup_{j=1}^k \ell_j \right\rangle \setminus \{0\} \right].
\]
Clearly \( L \cap \Lambda(\langle\gamma\rangle) = \emptyset \). By Lemma 2.11 we get
\[
\gamma^m(L) \xrightarrow{m \to \infty} \left[ \left\langle \bigcup_{j \in A_2 \setminus \{j_0\}} \Xi(\tilde{\gamma}_j) \right\rangle \setminus \{0\} \right] = \Lambda_{Kul}(\langle\gamma\rangle).
\]
Which concludes the proof.

Theorem 2.14. Let \( \gamma \in \text{PSL}(n+1, \mathbb{C}) \) be an element such that \( \gamma \) has a lift \( \tilde{\gamma} \) such that \( \tilde{\gamma} \) is non-diagonalizable and \( \tilde{\gamma} \) has only unitary eigenvalues, then
\[
\Omega_{Kul}(\langle\gamma\rangle) = Eq(\langle\gamma\rangle).
\]

Proof. Let \( (k, \{V_j\}_{j=1}^k, \{\beta_j\}_{j=1}^k, \{\lambda_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k) \) be a block decomposition for \( \tilde{\gamma} \). Define \( A_1 = \{j \in \{1, \ldots, k\} : \gamma_j \) is the identity \}, \( A_1 = \{1, \ldots, k\} \setminus A_1 \). In virtue of Lemma 2.12 we should consider the case \( \text{Card}(A_2) \geq 2 \). Let \( j_0 \in A_2 \), define
\[
V = \left[ \langle \bigcup_{j \in A_1 \cup \{j_0\}} V_j \rangle \setminus \{0\} \right], \quad V_1 = \left[ \langle \Xi(\gamma_{j_0}) \cup \bigcup_{j \in A_1} V_j \rangle \setminus \{0\} \right]
\]
\[
W = \left[ \langle \bigcup_{j \in A_2 \setminus \{j_0\}} V_j \rangle \setminus \{0\} \right], \quad W_1 = \left[ \langle \bigcup_{j \in A_2 \setminus \{j_0\}} \Xi(\gamma_j) \rangle \setminus \{0\} \right].
\]
To conclude the proof is enough to show that for every \( v \in V_1 \setminus \Lambda(\gamma) = V_1 \) and every \( w \in W_1 \setminus \Lambda(\gamma) = W_1 \), the line \( \langle v, w \rangle \) is contained in \( \Lambda_{Kul}(\langle\gamma\rangle) \). Let \( v \in V_1 \)
and \( w \in W_2 \), applying Lemma 2.12 to \( \gamma \) restricted to \( \mathcal{V} \) we conclude that there is a sequence \( (v_m) \subset \mathcal{V} \) such that the cluster points of \( (v_m) \) lies on \( \mathcal{V} \setminus \Lambda(\langle \gamma \rangle) \) and \( \gamma^m(v_m) \xrightarrow{m \to \infty} v \). Now, applying Lemma 2.13 to \( \gamma \) restricted to \( \mathcal{W} \) we conclude that there is a sequence \( (w_m) \subset \mathcal{W} \) such that the cluster points of \( (v_m) \) lies on \( \mathcal{W} \setminus \Lambda(\langle \gamma \rangle) \) and \( \gamma^m(w_m) \xrightarrow{m \to \infty} w \). Clearly the cluster points of \( (\ell_m = \langle v_m, w_m \rangle) \) do not lie in \( \Lambda(\langle \gamma \rangle) = \emptyset \) and \( \gamma^m(\ell_m) \xrightarrow{m \to \infty} \langle v, w \rangle \), which concludes the proof. □

**Lemma 2.15.** Let \( \gamma \in \text{PSL}(n+1, \mathbb{C}) \) be an element with infinite order, then for every \( z \in \mathbb{P}^n_\mathbb{C} \) there is a sequence \( (k_m(z))_{m \in \mathbb{N}} \subset \mathbb{P}^n_\mathbb{C} \) such that \( \gamma^m(k_m(z)) \xrightarrow{m \to \infty} z \).

**Proof.** Let \( z \in \mathbb{P}^n_\mathbb{C} \) be any element. Choose \( k_m(z) \in \gamma^{-m}B_{m^{-1}}(z) \), trivially \( \gamma^m(k_m(z)) \xrightarrow{m \to \infty} z \).

**Theorem 2.16.** Let \( \gamma \in \text{PSL}(n+1, \mathbb{C}) \) be an element such that \( \gamma \) has a lift \( \tilde{\gamma} \) such that there is a eigenvalue \( \lambda \) of \( \gamma \) such that \( |\lambda| \neq 1 \), then \( \Omega_{Kul}(\langle \gamma \rangle) = Eq(\langle \gamma \rangle) \).

**Proof.** Let \( (k, \{V_i\}_{i=1}^k, \{\gamma_i\}_{i=1}^k, \{r_i\}_{i=1}^k) \) be a unitary decomposition for \( \tilde{\gamma} \). Define

\[
\mathcal{V} = \left[ \left\langle \left\langle \Xi(\gamma_1) \cup \bigcup_{j=2}^k V_j \right\rangle \right\rangle \setminus \{0\} \right] ; \mathcal{V}_1 = \left[ \left\langle \left\langle \Xi(\gamma_1) \cup \bigcup_{j=2}^{k-1} V_j \right\rangle \right\rangle \setminus \{0\} \right]
\]

\[
\mathcal{W} = \left[ \left\langle \left\langle \Xi(\gamma_k) \cup \bigcup_{j=1}^{k-1} V_j \right\rangle \right\rangle \setminus \{0\} \right] ; \mathcal{W}_1 = \left[ \left\langle \left\langle \Xi(\gamma_k) \cup \bigcup_{j=2}^{k-1} V_j \right\rangle \right\rangle \setminus \{0\} \right]
\]

\[
\mathcal{S} = \left[ \left\langle \left\langle \bigcup_{j=2}^{k-1} V_j \right\rangle \right\rangle \setminus \{0\} \right]
\]

Clearly will be enough to show that for every \( x \in V_1 \) and every \( y \in V_1 \), the line \( \langle \langle x, y \rangle \rangle \subset \Lambda_{Kul}(\Gamma) \). Let \( x \in V_1 \) and \( y \in V_1 \). By Lemma 2.13 there is a sequence \( (x_m) \subset V_1 \) such that \( \gamma^m(x_m) \xrightarrow{m \to \infty} x \). Now, let \( z \in \mathcal{S} \) be a fixed point of \( \gamma \). Then for every \( m \) we can chose a sequence \( (\bar{x}_m) \subset \langle \langle z, x_m \rangle \rangle \) such that \( d_n(\bar{x}_m, [V_1]) = 2^{-1}d_n(z, [V_1]) \). A straightforward calculation shows \( \gamma^m(\bar{x}_m) \xrightarrow{m \to \infty} x \) and the cluster sets of \( (\bar{x}_m) \) lies on \( \mathbb{P}^n_\mathbb{C} \setminus \Lambda(\langle \gamma \rangle) \). Similarly Lemma 2.10 and Theorem 2.14 yields that there is a sequence \( (y_m) \subset \mathcal{V} \) such that \( \gamma^m(y_m) \xrightarrow{m \to \infty} y \) and the cluster sets of \( (y_m) \) lies on \( \mathbb{P}^n_\mathbb{C} \setminus \Lambda(\langle \gamma \rangle) \). To conclude observe that \( \gamma^m(\langle \langle y_m, \bar{x}_m \rangle \rangle) \xrightarrow{m \to \infty} \langle \langle x, y \rangle \rangle \) and the cluster sets of \( \langle \langle y_m, \bar{x}_m \rangle \rangle \) lies on \( \mathbb{P}^n_\mathbb{C} \setminus \Lambda(\langle \gamma \rangle) \). Which concludes the proof. □

### 3. Maximal Regions

As in the two dimensional case, see [14], the Kulkarni’s discontinuity region is not the largest open set where the cyclics groups acts properly discontinuously as the following example shows, we omit its proof here,
Lemma 3.1. Let $\gamma \in PSL(n+1, \mathbb{C})$ be an element, if $\gamma$ has a lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$.
If $(k, \{V_i\}_{i=1}^k, \{v_i\}_{i=1}^k, \{r_i\}_{i=1}^k)$ is an unitary decomposition of $\tilde{\gamma}$ such that $k > 3$, $r_1 < 1$, then:

$$
\Omega_1 = \mathbb{P}^n_\mathbb{C} \setminus \left( \{V_1 \setminus \{0\} \cup \langle \bigcup_{j>2} V_j \rangle \setminus \{0\} \right)
$$

$$
\Omega_2 = \mathbb{P}^n_\mathbb{C} \setminus \left( \{V_k \setminus \{0\} \cup \langle \bigcup_{j<k} V_j \rangle \setminus \{0\} \right)
$$

are maximal discontinuity regions.

4. The elements in $PU(k,l)$

Recall that elements in $PU(1,n)$ are classified in to three type, namely: hyperbolic, parabolic and elliptic. Such classification depends on the localization of the fixed points in $\mathbb{P}^n_\mathbb{C}$. More precisely, loxodromic elements are those elements with exactly two fixed points in $\partial \mathbb{P}^n_\mathbb{C}$, parabolic elements have exactly one fixed point in $\partial \mathbb{P}^n_\mathbb{C}$ and the elliptic ones have one fixed point in $\mathbb{P}^n_\mathbb{C}$, see [10]. This way to classify elements in $PU(1,n)$ makes hard to extent the classification to elements in $PSL(n+1, \mathbb{C})$, since not every element in $PSL(n+1, \mathbb{C})$ is conjugate to an element in $PU(1,n)$. In view of this problematic, our starting point to deal with this problem, will consist in provide an extension of the classification to $PU(k,l)$, which is a very "close" group to $PU(1,n)$, in the sense that we can provide a classification of elements be means of the fixed points and its relation with the "closed ball" induced by the Hermitian form. More precisely

Definition 4.1. Let $\gamma \in PU(k,l)$, then $\gamma$ is said to be:

1. Elliptic if $\gamma$ has at least one fixed point in $\mathbb{H}^{k,l}_\mathbb{C}$.
2. Loxodromic if the fixed points of $\gamma$ restricted to $\overline{\mathbb{H}^{k,l}_\mathbb{C}}$ lies on $\partial \mathbb{H}^{k,l}_\mathbb{C}$ and there are two points fixed points $x, y$ of $\gamma$ in $\partial \mathbb{H}^{k,l}_\mathbb{C}$ such that the action of $\gamma$ restricted to $\langle x, y \rangle$ is given by a loxodromic element.
3. Parabolic if the fixed points of $\gamma$ restricted to $\mathbb{H}^{k,l}_\mathbb{C}$, lies on $\partial \mathbb{H}^{k,l}_\mathbb{C}$ and for every pair of fixed points $x, y$ of $\gamma$ in $\partial \mathbb{H}^{k,l}_\mathbb{C}$ the action of $\gamma$ restricted to $\langle x, y \rangle$ is given by an elliptic element.

Clearly our definition is equivalent with the standard classification when $k = 1$, $l = n$. Now let us show that the previous definition exhaust all the possibilities

Lemma 4.2. Let $\gamma \in U(k,l)$ be a diagonalizable element such that each one of its eigenvalues is an unitary complex number, then $\gamma$ has an eigenvector in $N^{k+l}_\mathbb{C}$.

Proof. By induction on $k + l$. For $k + l = 2$ we get that the groups are either $U(1,1)$ or $U(2)$ which are known to satisfy the conclusion of the lemma. Now let us show the case $k + l = n + 1$. On the contrary let us assume that the eigenvalues of $\gamma$ lies on $\mathbb{C}^{n+1} \setminus N_{+}^{k+l}$. Let $v$ a eigenvalue then $\{v\}^\perp$ is an invariant hyperspace not containing $v$, in consequence the hermitian form $\langle , \rangle_{k,l}$ restricted to $\{v\}^\perp$ has signature $(k, l - 1)$, applying the inductive hypothesis to $\gamma$ restricted to $\{v\}^\perp$ endowed with the hermitian form induced by $\langle , \rangle_{k,l}$, we conclude that there is a eigenvalue of $\gamma$ in $N^{k+l}_\mathbb{C}$, which is a contradiction, which concludes the proof. □
Corollary 4.3. Let $\gamma \in PU(k,l)$ be an element, then $\gamma$ has a fixed point in $\mathbb{H}^{k,l}_C$.

Proof. Let $\gamma \in PU(k,l)$ if $\langle \gamma \rangle$ is not a discrete group then Lemma 4.2 yields the result. In other case, let $z \in \mathbb{H}^{k,l}_C$, then by Theorem 2.7 there is a sequence $(n_m) \subset \mathbb{Z}$ of distinct elements such that $\gamma^{n_m}(z)$ converges to a fixed point $p$ of $\gamma$, since $\mathbb{H}^{k,l}_C$ is invariant we conclude that $p \in \mathbb{H}^{k,l}_C$, which concludes the proof. □

Proposition 4.4. Each element in $PU(k,l)$ different from the identity belongs exactly to one of classes of our classification.

Proof. Let $\gamma \in PU(k,l)$ different from the identity then by Corollary 4.3 it yields that the set of fixed points of $\gamma$ restricted to $\mathbb{H}^{k,l}_C$ is non-empty. If at least one point is in $\mathbb{H}^{k,l}_C$ then $\gamma$ is elliptic otherwise since each element in $PSL(2,\mathbb{C})$ with two fixed points is either elliptic or loxodromic, we conclude that any element in $PU(k,l)$ different from the identity without fixed points in $\mathbb{H}^{k,l}_C$ is either parabolic or loxodromic. □

Lemma 4.5. Let $w, w \in V^k_l \setminus V^0_0$ be linearly independent elements, then the quadratic form restricted to $\langle\langle v, w \rangle\rangle$ is either identically 0 or has signature $(1,1)$.

Proof. By the theory of quadratic forms we know that $\langle \gamma, \gamma \rangle$ restricted to $\langle\langle v, w \rangle\rangle$ is either 0 or is equivalent to one of the following quadratic forms:

$$|z_1|^2; |z_1|^2 + |z_2|^2; -|z_1|^2; -|z_1|^2 - |z_2|^2; |z_1|^2 - |z_2|^2;$$

Since there are two null points, we conclude that $\langle \gamma, \gamma \rangle$ restricted to $\langle\langle v, w \rangle\rangle$ is either 0 or has signature $(1,1)$. □

Corollary 4.6. Let $\gamma \in U(k,l)$ be a diagonalizable element such that each one of its eigenvalues is an unitary complex number, then $\tilde{\gamma}$ has an eigenvector in $N^{k,l}_0 \cup N^{k,l}_+.$

Proof. On the contrary, let us assume that each eigenvector of $\gamma$ lies on $N^{k,l}_0$. Let $\beta = \{v_1, \ldots, v_{k+1}\}$ be a basis of eigenvectors. Let $v, w \in \beta$ be distinct points, then $\langle \gamma, \gamma \rangle$ restricted to $\langle\langle v, w \rangle\rangle$ cannot has signature $(1,1)$ (otherwise we get that $[\gamma]$ restricted to $[\langle\langle v, w \rangle\rangle \setminus \{0\}$ is a parabolic $PU(1,1)$ with two fixed points in $\partial \mathbb{H}^{1,1}_C$, then $[\gamma]$ restricted to $[\langle\langle v, w \rangle\rangle \setminus \{0\})$ is the identity. That is $\gamma$ has an eigenvector in $\mathbb{H}^{k,l}_C$ which is a contradiction). By Lemma 3.5 it yields that $\langle v, w \rangle \gamma \alpha_{k,l} = 0$. That is $\langle v, w \rangle \gamma \alpha_{k,l}$ is identically 0, which is a contradiction. □

Lemma 4.7. Let $\gamma \in PU(k,l)$ be an element with a diagonalizable lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ such that each one of the eigenvalues of $\tilde{\gamma}$ is an unitary complex number. If $\gamma$ is then there is a set $V_- \subset Fix(\gamma)$ with $k$ elements lying in $\mathbb{H}^{k,l}_C$ and a set $V_+$ of $l$ fixed points lying in $\mathbb{F}^{k+1}_C \setminus \mathbb{F}^{k,l}_C$ such that $\langle\langle V_+ \cup V_- \rangle\rangle = \mathbb{F}^{k+l}_C$.

Proof. By induction on $k + l$. For $k + l = 2$ we get that the groups are either $U(1,1)$ or $U(2)$ which are known to satisfy the conclusion of the lemma. Now let us show the case $k + l = n + 1$. By Corollary 4.0, there is a proper value $V$ of $\gamma$ such that $\langle v, v \rangle \gamma \alpha_{k,l} \neq 0$. Then $\langle v \rangle$ is an invariant hyperspace not containing $v$, in consequence the hermitian form $\langle \gamma, \gamma \rangle$ restricted to $\langle v \rangle$ has signature $(k-1, l)$ or $(k, l-1)$, depending whether $\langle v, v \rangle \gamma \alpha_{k,l} < 0$ or $\langle v, v \rangle \gamma \alpha_{k,l} > 0$. Applying the inductive hypothesis to $\gamma$ restricted to $\langle v \rangle$ endowed with the hermitian form induced by $\langle \gamma, \gamma \rangle$ the result follows. □
Proposition 4.8. Let $\gamma \in PU(k,l)$ be an element, then the following facts are equivalent:

1. The element $\gamma$ is elliptic.
2. For each lift $\tilde{\gamma} \in SL(k+l, \mathbb{C})$ of $\gamma$, it yields that $\tilde{\gamma}$ is diagonalizable and each one of its eigenvalues is an unitary complex number.

Proof. Clearly Lemma 4.7 yields that (2) implies (1). So lets show that (1) implies (2). Let $\gamma \in PU(k,l)$ be an elliptic element, since $\gamma$ has a fixed point in $\mathbb{H}^{k,l}_{\mathbb{C}}$ then Theorem 1.4 yields that $\langle \gamma \rangle$ is either non-discrete or finite. Now Jordan’s normal form theorem yield that any lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ is diagonalizable and each one of its eigenvalues is an unitary complex number. Which conclude the proof. □

Proposition 4.9. Let $\gamma \in PU(k,l)$ be an element, then the following facts are equivalent:

1. The element $\gamma$ is loxodromic.
2. For each lift $\tilde{\gamma} \in SL(k+l, \mathbb{C})$ of $\gamma$, it yields that at least one eigenvalue of $\tilde{\gamma}$ is a non-unitary complex number.

Proof. Lets show that (1) implies (2). On the contrary, let us assume that there is $\gamma \in PU(k,l)$ a loxodromic element and a lift $\tilde{\gamma}$ of $\gamma$ such that each one of its eigenvalues is an unitary complex number. Thus for every pair of fixed points $x, y$ of $\gamma$ it follows that $\langle \langle x, y \rangle \rangle$ is an invariant line, where the action is either the identity or an elliptic element. Which is a contradiction, since $\gamma$ is loxodromic.

Let $\gamma \in PU(k,l)$ and a lift $\tilde{\gamma} \in SL(k+l, \mathbb{C})$ with at least one non-unitary eigenvalue. Let $(\bar{n}, \{V_j\}_{j=1}^n, \{\gamma_j\}_{j=1}^k, \{r_j\}_{j=1}^l)$ be a decomposition for $\tilde{\gamma}$ and $v \in N^{k,l}_{\mathbb{C}}$, then $\bar{n} \leq 2$ and $v = \sum_{j=1}^n v_j$ where $v_j \in V_j$, then:

$\gamma^m([v]) = \left[ \sum_{j=1}^{\bar{n}} \frac{r_m}{r_j} T_j^m v_j \right]$;

$\gamma^{-m}([v]) = \left[ \sum_{j=1}^{\bar{n}} \frac{r_{-m}}{r_j} T_j^{-m} v_j \right]$;

Now by lemma 2.6 and Corollary 2.2 we conclude that there are sequences $(\bar{t}_m^+), (\bar{t}_m^-) \subset \mathbb{N}$, $v_+ \in V_{\bar{n}}$, $v_- \in V_1$ eigenvectors of $\tilde{\gamma}$ such that $\gamma^{\pm t_m^\pm}([v]) \xrightarrow{m \to \infty} [v_\pm]$.

By Proposition 4.8 it yields that $[v_{\pm}] \in \partial \mathbb{H}^{k,l}_{\mathbb{C}}$ are fixed points and the action of $\gamma$ in $\langle ([v_-], [v_+]) \rangle$ is given by a loxodromic element. Which conclude the proof. □

As corollary of the previous results we get:

Proposition 4.10. Let $\gamma \in PU(k,l)$ be an element, then the following facts are equivalent:

1. The element $\gamma$ is parabolic.
2. For each lift $\tilde{\gamma} \in SL(k+l, \mathbb{C})$ of $\gamma$, it holds that $\tilde{\gamma}$ is non-diagonalizable and each one of its eigenvalue is an unitary complex number.

5. Elliptic Transformations in $\text{PSL}(n+1, \mathbb{C})$

Recall that up to conjugation, an elliptic element $g \in PU(2,1)$ can be represented by a matrix of the form

$$
\bar{g} = \begin{pmatrix}
A & 0 \\
0 & \lambda
\end{pmatrix}
$$
Definition 5.1. The $(2n - 1)$-spheres in $\mathbb{P}^n_C$ are defined as the images of the set

$$T = \left\{ [z_1 : \ldots : z_{n+1}] \in \mathbb{P}^n_C : \sum_{j=1}^{n} |z_j|^2 = |z_{n+1}|^2 \right\}$$

under an element in $\text{PSL}(n+1, \mathbb{C})$.

Notice that if in the above discussion we take the origin of $\mathbb{C}^n$ as being the point $e_{n+1}$ and the hyperplane at infinity $\mathbb{P}^{n+1}_C$ as being $\mathcal{L} = \langle [e_1], \ldots, [e_n] \rangle$, then the above family of spheres actually provides a foliation of $\mathbb{P}^n_C \setminus (\mathcal{L} \cup \{e_{n+1}\})$, where each leaf is given by:

$$T(r) = \left\{ [z_1 : \ldots : z_{n+1}] \in \mathbb{P}^n_C : \sum_{j=1}^{n} |z_j|^2 = r|z_{n+1}|^2 \right\}, \quad r > 0.$$

Clearly each automorphism $h \in \text{PSL}(n+1, \mathbb{C})$ carries the above foliation into another family of $(2n - 1)$-spheres given by $h(T(r))$, $r > 0$, which is a foliation of $\mathbb{P}^n_C \setminus (h(\mathcal{L} \cup \{e_{n+1}\}))$.

Definition 5.2. A transformation $\gamma \in \text{PSL}(n+1, \mathbb{C})$ is called elliptic if it preserves each one of the leaves of a foliation as above. In other words, $\gamma \in \text{PSL}(n+1, \mathbb{C})$ is elliptic if and only if there exists $h \in \text{PSL}(n+1, \mathbb{C})$ such that $h^{-1}\gamma h(T(r)) = T(r)$ for every $r > 0$.

Proposition 5.3. The element $\gamma \in \text{PSL}(n+1, \mathbb{C})$ is elliptic if and only if it is conjugate to an elliptic element of $\text{PU}(n, 1)$.

Proof. Assume $\gamma \in \text{PSL}(n+1, \mathbb{C})$ is elliptic, then there is $h \in \text{PSL}(n+1, \mathbb{C})$ such that $h^{-1}\gamma h$ preserves every $2n - 1$-sphere $T(r)$, $r > 0$. It follows that $f := h^{-1}\gamma h \in \text{PU}(n, 1)$ and $[e_{n+1}]$ is a fixed point of $f$. Therefore $f$ is elliptic in $\text{PU}(n, 1)$.

Now let $\gamma \in \text{PU}(n, 1)$ be an elliptic element, then its a well known fact see [10] that, up to conjugation by a projective transformation, $\gamma$ has a lift $\tilde{\gamma} \in \text{SL}(3, \mathbb{C})$, which is a diagonal matrix where each of its eigenvalues is an unitary complex number. So we can assume that

$$\tilde{\gamma} = \left( \begin{array}{ccc} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{array} \right),$$

where $|\alpha_1| = \ldots = |\alpha_{n+1}| = 1$. Set $\mathcal{L} = \langle [e_1], \ldots, [e_n] \rangle$, thus the action on $\mathbb{P}^n_C \setminus \mathcal{L} = \mathbb{C}^n$ is given by $(z_1, \ldots, z_n) \mapsto (\alpha_1 \alpha_{n+1}^{-1} z_1, \ldots, \alpha_n \alpha_{n+1}^{-1} z_n)$, which clearly preserves each of the concentric $(2n - 1)$-spheres centered at 0. Which concludes the proof.

The next corollary follows easily from the proposition above and the fact that every element in $\text{U}(n)$ is diagonalizable and its eigenvalues are unitary complex numbers.
Corollary 5.4. An element $\gamma \in PSL(n+1, \mathbb{C})$ is elliptic if and only if $\gamma$ has a lift $\tilde{\gamma} \in SL(n+1, \mathbb{C})$ such that $\tilde{\gamma}$ is diagonalizable and every eigenvalue is an unitary complex number.

By definition an elliptic element in $PSL(n+1, \mathbb{C})$ preserves a foliation by “concentric” spheres. The proposition bellow says that such a transformation actually preserves $n+1$ foliations by “concentric” $(2n-1)$-spheres.

Proposition 5.5. If $\gamma \in PSL(n+1, \mathbb{C})$ is an elliptic transformation, then there are $n+1$ families of $\gamma$-invariant $(2n-1)$-spheres.

Proof. Let $\gamma$ be an elliptic transformation, then there exists $h \in PSL(n+1, \mathbb{C})$ such that $g = h^{-1}gh$ has a lift in $SL(n+1, \mathbb{C})$ which is diagonal matrix with unitary eigenvalues. Clearly for each $j \in \{1, \ldots, n+1\}$ the following is an invariant families of $(2n-1)$-spheres

$$T_j(r) = \left\{ [z_1 : \ldots : z_{n+1}] \in \mathbb{P}_{\mathbb{C}}^n : \sum_{i \neq j} |z_i|^2 = r |z_j|^2 \right\}, \quad r > 0;$$

\[ \square \]

6. PARABOLIC TRANSFORMATIONS IN $PSL(n+1, \mathbb{C})$

From the one and two dimensional setting we know that parabolic elements in the projective group correspond to those elements $\gamma$ which are conjugate to an element which preserve a complex ball $B$ an has an unique a fixed point $p$ on $\partial B$. Next we will see that this geometric point of view provide us a way to define parabolic elements in $PSL(n+1, \mathbb{C})$. We shall need the following definition:

Definition 6.1. Given $k, l \in \mathbb{N}$ we define the $(k, l)$-spheres in $\mathbb{P}_{\mathbb{C}}^n$ as the images of the set $[N_0^k]$ under an element in $PSL(n+1, \mathbb{C})$.

With this in mind let us define

Definition 6.2. The transformation $\gamma \in PSL(n+1, \mathbb{C})$ is called parabolic if there are $k, l \in \mathbb{N}$ satisfying $k + l = n + 1$, a family $\mathcal{F}$ of $\gamma$-invariant $(k, l)$-spheres and $\gamma$-invariant projective subspaces $Z, W \subset \mathbb{P}_{\mathbb{C}}^n$ such that:

1. For every pair of different elements $T_1, T_2 \in \mathcal{F}$ it follows that $Z \subset T_1 \cap T_2 \subset W$.

2. It yields that $\bigcup_{T \in \mathcal{F}} T \setminus W = \mathbb{P}_{\mathbb{C}}^n \setminus W$.

3. If $\ell \subset \mathbb{P}_{\mathbb{C}}^n$ is a $\gamma$-invariant line satisfying that the restriction of $\gamma \gamma_k \gamma$ to $[\gamma_k \gamma \gamma_k^{-1}(0)] \gamma_\gamma \gamma_k \gamma \gamma_k^{-1}(0)$ has signature $(1, 1)$, then there is a point $z \in \ell$ such that $\gamma z \neq z$.

4. The action $\circ \gamma$ restricted to $Z$ is a given by an elliptic element.

Lemma 6.3. Let $\gamma \in PSL(n+1\mathbb{C})$ be a parabolic element, then $\gamma$ cannot be elliptic.

Proof. On the contrary let us assume that there is an element $\gamma \in PSL(n+1, \mathbb{C})$ be an element which is simultaneously parabolic and elliptic. Let $k, l \in \mathbb{N}$, $\mathcal{F}$ a family of $\gamma$-invariant $(k, l)$-spheres and $Z$ a projective subspace of $\mathbb{P}_{\mathbb{C}}^n$ satisfying the items in Definition 6.2. Since $Z$ is $\gamma$-invariant and $Z \subset \partial \mathbb{P}_{\mathbb{C}}^k$, it yields that there is $z \in Z$ such that $\gamma x = x$. On the other hand, since $\gamma$ is elliptic,
Proposition (4.8) and Lemma (4.12) yield that \( \gamma \) there is a set \( V \subset \mathbb{H}^{k,l}_C \) with \( k \) elements and a set \( W \subset \mathbb{P}^n \setminus \mathbb{H}^{k,l}_C \) with \( l \) elements, such that each point in \( V \cup W \) is fixed by \( \gamma \) and \( \langle (V \cup W) \rangle = \mathbb{P}^n_C \). Then there is \( v \in V \) and \( w \in W \) such that \( z \in \ell = \langle (v, w) \rangle \). Therefore \( \ell \) is \( \gamma \)-invariant and \( \gamma \) restricted to \( \ell \) is the identity, which is a contradiction. \( \square \)

**Proposition 6.4.** If the element \( \gamma \in \text{PSL}(n+1, \mathbb{C}) \) is parabolic, then there are \( k, l \in \mathbb{N} \) such that \( k + l = n + 1 \) and \( \gamma \) is conjugate to a parabolic element in \( \text{PU}(k,l) \).

**Proof.** Let \( k, l \in \mathbb{N} \), \( \mathcal{F} \) a family of \( \gamma \)-invariant \((k,l)\)-spheres and \( \mathcal{Z} \) a projective subspace of \( \mathbb{P}^n_C \) satisfying the items in Definition (6.2). Then after conjugating with a projective transformation, if it is necessary, we get that \( \gamma \in \text{PU}(k,l) \). Therefore Lemma (6.3) yield that \( \gamma \) is either loxodromic or parabolic. Let us assume that \( \gamma \) is loxodromic, then there is a projective subspace \( W \subset \mathbb{H}^{k,l}_C \) such that \( W \) is invariant by \( \gamma \), \( \mathcal{Z} \cap W = \emptyset \) and for each point \( p \in \mathbb{H}^{k,l}_C \) the accumulation set of \( \{ \gamma^m p \} \) lies on \( W \). On the other hand, given \( p \in \mathbb{H}_C \), there is a leave of \( T \in \mathcal{F} \) such that \( p \in T \), since \( T \) is invariant, Theorem (1.4) yields that the accumulation set of \( \{ \gamma^m p \} \) lies on \( \mathcal{Z} \), which is a contradiction. Therefore \( \gamma \) is parabolic. \( \square \)

**Lemma 6.5.** Let \( A \in M(n, \mathbb{C}) \) be an invertible matrix, let us define

\[
C = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},
\]

then \( (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \) is an eigenvector of \( C_1 \) with eigenvalue \( \lambda \) if and only if \( x \) is an eigenvector of \( AA^t \) with eigenvalue \( \lambda^2 \) and \( y = \lambda Ax \).

The following lemmas will be useful

**Lemma 6.6** (Weyl, see [3]). Let \( A, B \in M(n, \mathbb{C}) \) be hermitian matrices with eigenvalues \( \alpha_1, \ldots, \alpha_n \) and \( \alpha_1, \ldots, \alpha_n \) respectively. Then

\[
\max |\alpha_j - \beta_j| \leq \| A - B \|.
\]

Where \( \| \cdot \| \) denotes the operator bound norm.

Let us define some notation

**Definition 6.7.** Given \( n > 2 \) and \( i, j \in \{1, \ldots, n\} \), let us define \( \ll i, j \gg^{(\pm)} : \mathbb{C}^n \to \mathbb{R} \) by

\[
\ll i, j \gg^{(\pm)} (z_1, \ldots, z_n) = z_i \overline{z_j} \pm \overline{z_i} z_j.
\]

**Lemma 6.8.** If \( n = 2k + 1 \) and \( \mathcal{Q} = \{ Q_r \}_{r \in \mathbb{R}} \) is the family of hermitian quadratic forms given by:

\[
r \ll n, n \gg^{(+)} + \sum_{k=0}^{k-1} (-1)^{i+k} \ll 1+j, n-j \gg^{(+)} + \sum_{j=0}^{k-2} \sum_{m=0}^{m-1} \sum_{i=0}^{i-1} (-1)^{m-i-k} \binom{m \quad -1}{i \quad I} \ll 2+j-l, m-n+m+1 \gg^{(+)}
\]

\[
+ \frac{1}{2} \sum_{m=1}^{k} \sum_{j=0}^{j-1} (-1)^{2k-m} \binom{m-1}{I} \ll k+1+j, k+1+m \gg^{(+)} + \frac{1}{2} \ll k+1, k+1 \gg^{(+)}.
\]

Then

1. It is verified that \( e_1 \in \bigcap_{Q \in \mathcal{Q}} Q^{-1}(0) \).
2. For each pair of distinct elements \( r, s \in \mathbb{R} \) it holds that

\[
Q_{r}^{-1}(0) \cap Q_{s}^{-1}(0) \subset \mathcal{L} = \langle \langle e_1, \ldots, e_{n-1} \rangle \rangle.
\]
(3) It yields that $\bigcup_{r \in \mathbb{R}} Q_r^{-1}(0) \setminus \mathcal{L} = \mathbb{C}^n \setminus \mathcal{L}$.  
(4) Each quadric in $Q$ has signature $(k, k+1)$.  
(5) If $A$ is the $n \times n$-Jordan block then $A(Q_r^{-1}(0)) = Q_r^{-1}(0)$ for each $r \in \mathbb{R}$.

Proof. The proofs of (1), (2) and (3) are straightforward calculations so we will omit it here.

Let us show (4). A simply inspection reveals that the matrix of coefficients $C_r$ of $Q_r$ has the form:

$$
\begin{pmatrix}
0 & 0 & A \\
0 & 1 & b \\
A^t & b^t & B_r
\end{pmatrix}
$$

where $A \in SL(k, \mathbb{C})$, $b \in \mathbb{C}^k$ and

$$
B_r = \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0 & r
\end{pmatrix}.
$$

Now let us consider the following hermitian matrix

$$
C_{0,0} = \begin{pmatrix}
0 & 0 & A \\
0 & 1 & 0 \\
A^t & 0 & 0
\end{pmatrix},
$$

then lemma 6.5 yields that $C_{0,0}$ signature $(k, k+1)$. Consider the family of hermitian matrices $\{C_{r,v} : (r, v) \in \mathbb{R} \times \mathbb{C}^k\}$ given by

$$
C_{r,v} = \begin{pmatrix}
0 & 0 & A \\
0 & 1 & v \\
A^t & v^t & B_r
\end{pmatrix}
$$

A straightforward calculation shows that $\det(C_{r,v}) \neq 0$, thus Lemma 6.6 yields that the sets

$$
U_{l,m} = \{(r, v) \in \mathbb{R} \times \mathbb{C}^k : C_{r,v} \text{ has signature } (l, m)\}
$$

form an open cover of disjoint sets for $\mathbb{R} \times \mathbb{C}^k$. Since $\mathbb{R} \times \mathbb{C}^k$ is connected we conclude that $U_{k,k+1} = \mathbb{R} \times \mathbb{C}^k$, which concludes the proof.

Let us show (5). Clearly it is enough to show that $H = (-1)^k Q_0$ is invariant under $A$. The proof is by induction on $k$. If $k = 1$, then

$$
AH = ((z_1 + z_2)\bar{z}_1 + z_3(z_1 + \bar{z}_3)) - |z_2 + z_3|^2 + \frac{i}{2}((z_2 + z_3)\bar{z}_3 + z_3(\bar{z}_2 + \bar{z}_3))
$$

$$
= (z_1\bar{z}_3 + z_3\bar{z}_1) + (z_2\bar{z}_3 + z_3\bar{z}_2) - |z_2|^2 - |z_3|^2 - (z_2z_3 + z_3z_2) + \frac{i}{2}(z_2\bar{z}_3 + z_3\bar{z}_2) + |z_3|^2
$$

$$
= H
$$

Let us prove the case $k_0 = k$. Trivially, we can write down $H = H_1 + H_2$ where:

$$
H_1 = \sum_{j=1}^{k-1} (-1)^j \begin{pmatrix} 1+j & n-j \end{pmatrix}^{(+)} + \sum_{j=1}^{k-2} \sum_{m=0}^{j-1} \sum_{l=0}^{m} (-1)^{m-j} \begin{pmatrix} m & l \end{pmatrix} \begin{pmatrix} 2+j-l & n-j-m \end{pmatrix}^{(+)}
$$

$$
+ \frac{1}{2} \sum_{m=1}^{k-1} \sum_{j=0}^{m-1} (-1)^{2k-m} \begin{pmatrix} m-1 & l \end{pmatrix} \begin{pmatrix} k+1-j & k+1+m \end{pmatrix}^{(+)} + \frac{1}{2} \begin{pmatrix} k+1 & k+1 \end{pmatrix}^{(+)}.
$$
Proof. Let \((Q, \gamma)\) be such that \(\gamma\) is non-diagonalizable and each of its eigenvalues is an unitary complex number, then \(\gamma\) is parabolic.

**Proof.** Let \((k, \{V_j\}_{j=1}^k, \{\gamma_j\}_{j=1}^k)\) be a Jordan decomposition for \(\gamma\). For each \(j\) let \(Q_{j,r}\) be the hermitian quadratic form in \(V_j\) given by

\[
Q_{j,r} = \begin{cases} 
\text{The standard quadric with signature } (0, k) \text{ if } \gamma_j \text{ is a diagonal matrix} \\
\text{The } r \text{ hermitian quadratic induced by Lemmas 6.8 or 6.9 in other case}
\end{cases}
\]

Let \((k_i, l_i)\) be the signature of the quadric \(Q_{j,0}\). Then \(T_r = \bigoplus_{j=1}^k H_{j,r}\) is a family of \(\gamma\)-invariant hermitian quadratics each of one has signature \(\sum_{j=1}^k k_j, \sum_{j=1}^k l_j\).

For each \(j\) define

\[
H_j = \begin{cases} 
\text{The unique fixed point of } \gamma_j, \text{ if } \gamma_j \text{ is non-diagonalizable} \\
\emptyset \text{ in other case}
\end{cases}
\]

\[
K_j = \begin{cases} 
\text{The unique fixed hyperplane of } \gamma_j, \text{ if } \gamma_j \text{ is non-diagonalizable} \\
v_j \text{ in other case}
\end{cases}
\]

Define \(H = \langle \bigcup_{j=1}^k H_j \rangle, K = \langle \bigcup_{j=1}^k K_j \rangle\). Clearly \(T_r, H, K\) and \(\gamma\) satisfy definition \([6.2]\) which concludes the proof.
Finally recall that in the one and two dimensional case is not hard to see that parabolic elements are simply those coming from \( PU(1,n) \), where \( k = 1,2 \), but as the following example shows, in higher dimensions there are parabolic elements which are not conjugated to elements in \( PU(1,n) \).

**Example 6.11.** Let \( n \geq 4 \) and \( \gamma \) be the projective transformation in \( PSL(n+1, \mathbb{C}) \) induced by the matrix

\[
\tilde{\gamma} = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & I_{n-4}
\end{pmatrix}
\]

where \( I_{n-4} \) is the identity matrix if \( n > 4 \) and nothing in other case, then \( \gamma \) is a parabolic element which cannot be conjugated to an element in \( PU(1,n) \).

**Proof.** A straightforward calculation shows that

\[
\gamma^{\pm m} = \begin{pmatrix}
1 & \pm m \\
0 & 1 \\
0 & 1 \\
I_{n-4}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
0_{n-4}
\end{pmatrix}
\]

in consequence \( \mathbb{P}^n \setminus Eq((\gamma)) \) is not an hyperplane, which is not possible for elements in \( PU(1,n) \), see [6]. Therefore, \( \gamma \) is not conjugate to an element in \( PU(1,n) \). \( \square \)

### 7. Loxodromic Transformations in \( PSL(n+1, \mathbb{C}) \)

Recall that a loxodromic element \( \gamma \) in \( PSL(2, \mathbb{C}) \) by definition has two fixed points in \( p,q \in \mathbb{P}^1_\mathbb{C} \). One of these points is repelling, the other attracting. Due to this fact one can always choose a small enough ball \( W \) with center at the attracting point such that \( \gamma(W) \subset W \). We will see bellow that this property characterizes the loxodromic elements.

**Definition 7.1.** Given \( \gamma \in PSL(n+1, \mathbb{C}) \) we will say that it is *loxodromic* if there is a proper open set \( W \) in \( \mathbb{P}^n_\mathbb{C} \) such that \( \gamma(W) \subset W \).

The following technical lemmas will help in the algebraic characterization of the loxodromic elements.

**Lemma 7.2.** Let \( \gamma \in PSL(n+1, \mathbb{C}) \) be a loxodromic element, then \( L_1(\gamma) \) is a non-empty disconnected set.

**Proof.** Let \( W \) be a proper open set as in Definition 7.1. Define

\[
\Omega = \bigcup_{n \in \mathbb{Z}} \gamma^n(\mathbb{P}^n_\mathbb{C} \setminus \gamma(\mathbb{P}^n_\mathbb{C})).
\]

Then \( \Omega \) is a non-empty set where \( (\gamma) \) acts properly discontinuously. In consequence

\[
L_0((\gamma)) \cup L_1((\gamma)) \subset W \cup \mathbb{P}^n_\mathbb{C} \setminus \overline{\mathbb{C}}.
\]

To conclude observe that \( L_1(\gamma) \cap W \neq \emptyset \) and \( L_1(\gamma) \cap \mathbb{P}^n_\mathbb{C} \setminus \overline{\mathbb{C}} \neq \emptyset \). \( \square \)

**Lemma 7.3.** Let \( T \in SL(n+1, \mathbb{C}) \) and \( \alpha_1, \ldots, \alpha_{n+1} \) be the eigenvalues of \( T \). If \( p \in \mathbb{C}^{n+1} \) is an eigenvector for \( \alpha_j \), \( |\alpha_j| \neq |\alpha_k| \) for \( k > 2 \) and \( |\alpha_1| = \max\{|\alpha_j| : j \in \{1, \ldots, n+1\} \} \), then \( [p] \) is an attracting fixed point for the action of \([T] \) in \( \mathbb{P}^n_\mathbb{C} \).
Lemma 7.4. Let $U \subset \text{Gr}(k, n)$ be an open set (resp. a closed set), then $\bigcup_{\ell \in U} \ell$ is an open set in $\mathbb{P}^n_\mathbb{C}$ (resp. closed).

Proof. Let us proof the case when $U$ is an open set. Clearly, is enough to assume that $U$ is an open ball in $\text{Gr}(k, n)$. Let us assume that $U = B_{\wedge^k d}(r, \ell)$, where $\wedge^k d$ is the Fubiny-Study metric on $\text{Gr}(k, n)$. Let $v_1, \ldots, v_{k+1} \in \mathbb{C}^n \setminus \{0\}$ be points in general position such that $[(v_1, \ldots, v_{k+1}) \setminus \{0\}] = \ell$, for each set $W = \{w_1, \ldots, w_k\} \subset \mathbb{C}^n$ of points in general position, consider the following function $\nabla_W : \mathbb{P}^n_\mathbb{C} \setminus [(W) \setminus \{0\}] \to \mathbb{R}^+$, given by

$$\nabla_W([z]) = \bigwedge^k d([w_1 \wedge \cdots \wedge w_k \wedge z], [v_1 \wedge \cdots \wedge v_{k+1}])$$

Clearly $\nabla_W$ is a well defined continuous function. To conclude is enough to observe that $U_{\ell \in U} \ell = U_{\ell \in U} \{z \in \mathbb{P}^n_\mathbb{C} \setminus [(W) \setminus \{0\}] : \nabla_W(z) < r\}$.

Let us proof the case when $U$ is a closed set. Let $(x_m) \subset U_{\ell \in U} \ell$ be a sequence converging to $x$. For each $m$ we can choose an element $\ell_m \in U$ such that $x_m \in \ell_m$. Since $\text{Gr}(k, n)$ is compact we can assume that there is $\ell_0 \in U$ such that $\ell_m \overset{m \to \infty}{\longrightarrow} \ell_0$, in the topology of $\text{Gr}(k, n)$. In consequence $\ell_m \overset{m \to \infty}{\longrightarrow} \ell_0$ as closed sets in the Hausdorff topology of $G(\mathbb{P}^n_\mathbb{C})$. Therefore $x \in \ell_0$, which concludes the proof.

Lemma 7.5. Let $T \in SL(n+1, \mathbb{C})$ and $1 \leq k \leq n + 1$. If $\alpha_1, \ldots, \alpha_{n+1}$ are the eigenvalues of $T$, then the eigenvalues of $\wedge^k T$ has the form $\alpha_{j_1} \cdots \alpha_{j_k}$ where $j_1, \ldots, j_k \in \{1, \ldots, n + 1\}$ and $j_i < j_\ell$, whenever $k < \ell$.

Lemma 7.6. Given $\mathcal{L} \in \text{Gr}(k, n)$, there is an open set $U \subset \text{Gr}(k, n)$ such that $\mathcal{L} \in U$ and $\bigcup_{\ell \in U} \ell \neq \mathbb{P}^n_\mathbb{C}$.

Proof. Let $W \subset \mathbb{P}^n_\mathbb{C} \setminus \mathcal{L}$ a non empty open set such that $\overline{W} \cap \mathcal{L} = \emptyset$. Define $\mathcal{W} = \{\ell \in \text{Gr}(k, n) : \ell \cap \overline{W} \neq \emptyset\}$, clearly $\mathcal{W} = \text{Gr}(n, k) \setminus \mathcal{W}$ is an open set containing $\mathcal{L}$ also satisfying $W \subset \mathbb{P}^n_\mathbb{C} \setminus \bigcup_{\ell \in \mathcal{W}} \ell$. Which concludes the proof.

Proposition 7.7. An element $\gamma \in \text{PSL}(n+1, \mathbb{C})$ is loxodromic if and only if it has a lift $\bar{\gamma} \in SL(n+1, \mathbb{C})$ with a non-unitary eigenvalue.

Proof. On the contrary let us assume there is $\bar{\gamma} \in SL(n+1, \mathbb{C})$ a lift of $\gamma$ whose eigenvalues are unitary complex numbers. Then Lemma 2.7 yields that $L_1(\gamma)$ is connected, which contradicts Lemma 7.2.
Conversely, let \( \gamma \in SL(n+1, \mathbb{C}) \) be a linear transformation with one non-unitary eigenvalue, then by the Normal Jordan form we can assume that \( \gamma \) can be written as

\[
\gamma = \begin{pmatrix}
    r_1 A_1 \\
r_2 A_2 \\
    \vdots \\
r_k A_k
\end{pmatrix}
\]

where \( r_k < r_{k-1} < \ldots < r_1 \) and each matrix \( A_k \) has only unitary eigenvalues. Now let \( \tilde{n} = \dim A_1 \) and \( \alpha_1, \ldots, \alpha_{\tilde{n}} \) be the eigenvalues of \( A_1 \). By Lemma 7.5 it follows that \( p = e_1 \wedge \cdots \wedge e_{\tilde{n}} \) is an eigenvalue of \( T = \bigwedge^k \gamma \) with eigenvalue \( \alpha = r^k_1 \alpha_1 \cdots \alpha_{\tilde{n}} \), \( \alpha \) is a simple root of \( \text{Det}(T - \lambda I) = 0 \) and \( r^k_1 = \max \{ |\beta| : \beta \text{ eigenvalue of } \bigwedge^k \gamma \} \). By Lemma 7.5 it yields that \( [p] \) is an attracting fixed point of \( [T] \) acting on \( P(\bigwedge^k \mathbb{C}^n) \).

Due to the Plücker embedding and Lemma 7.6 we conclude that there is an open set \( U \subset \text{Gr}(\tilde{n} - 1, n) \) such that \( [\gamma](U) \subset U \), \( \{ e_1, \ldots, e_{\tilde{n}} \} \notin U \) and \( \bigcup_{U \in U} U \neq \mathbb{P}^n \). To conclude observe that Lemma 7.3 yields that \( W = \bigcup_{U \in U} U \) is a proper open set which satisfy \( [\gamma](U) \subset U \). Which concludes the proof. \( \square \)

Remark 7.8.

1. Clearly the previous discussion shows our previous results.
2. In the case of projective parabolic transformation, our previous discussion shows that for elements in \( PU(k, l) \), \( k \geq 2 \), the Kulkarni’s discontinuity set is not longer the largest open set where the corresponding group acts properly discontinuously.
3. In the one, the two dimensional setting and in the case of transformations in \( PU(1, 3) \), see [11, 15, 17], transformations can be classified by the use of the trace, we do not know how to extent such result to the higher dimensional case.
4. There is also another classification of the projective transformations of \( PSL(3, \mathbb{C}) \) in terms of the fixed set given in [19], which is closely related to our classification but properly talking does not agree with the one exposed here.
5. Let \( X \) be the space, of all positive definite, symmetric \( 3 \times 3 \)-matrices with real coefficients, of determinant 1, there is a metric \( d \) such that \( X \) is a \( \text{CAT}(0) \)-space and the action of \( SL(3, \mathbb{R}) \) on \( X \) given by \( xf \), where \( x \in X \) and \( f \in SL(3, \mathbb{R}) \), is by isometries. Then by using the classification of isometries in \( \text{CAT}(0) \)-spaces one can classify elements of \( SL(3, \mathbb{R}) \) in to parabolic, loxodromic and elliptic however is not hard to show, see [8], that such classification does no agree with the one induced by definition 0.1.
6. From the the theory of \( \text{CAT}(0) \)-spaces one know that isometries can be classified in to three types namely elliptic, parabolic or hyperbolic. In virtue of the similarity of our results with the ones coming from \( \text{CAT}(0) \)-spaces, is natural to ask if it is possible to use this theory to deal with the classification of projective transformations. We got two naive partial answers: first since \( \mathbb{P}^n_\mathbb{C} \) is compact one cannot use directly the theory of \( \text{CAT}(0) \)-spaces to deal with the problem of classification of projective transformations, second, a result in [8] asserts that the fixed set of parabolic elements should be contractible in the Tits boundary of \( X \), in consequence for \( n > 1 \), the projective space \( \mathbb{P}^n_\mathbb{C} \) cannot be the tits boundary of a proper \( \text{CAT}(0) \)-space.
where the projective transformations are extensions of isometries of $X$. Unfortunately the authors do not know if it is possible to use the classification of elements of $CAT(0)$-spaces, in other way, to deal with the problem of classifying projective transformations.

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